Some Problems on Jacobi Forms

By

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Certificate

This is to certify that the Ph.D. thesis titled "**Some Problems on Jacobi Forms**" by **Soumya Das** is a record of bona fide research work done under my supervision. It is further certified that the thesis represents independent and original work by the candidate.

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Declaration

The author hereby declares that the work in the thesis titled "**Some Problems on Jacobi Forms**", submitted for Ph.D. degree to the **Homi Bhabha National Institute** has been carried out under the supervision of Professor B. Ramakrishnan. Whenever contributions of others are involved, every effort is made to indicate that clearly, with due reference to the literature. The author attests that the work is original and has not been submitted in part or full by the author for any degree or diploma to any other institute or university.

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Abstract

We prove that under suitable conditions (both depending and independent of the weight), the Jacobi Poincaré series of exponential type of integer weight and matrix index does not vanish identically. For Jacobi forms of degree 1 (JF in short), a basis consisting of the "first" few Poincaré series is given. Equality of certain Kloosterman-type sums is proved. Also, a result on the non-vanishing of Jacobi Poincaré series is obtained when an odd prime divides the index.

We introduce a certain differential operator on the space of Hermitian Jacobi forms of degree 1 (HJF in short), to construct HJF from elliptic cusp forms. We construct HJF from Jacobi forms and also from differentiation of the variables. We determine the number of Fourier coefficients that determine a HJF and use it to embed a certain subspace of HJF into a direct sum of elliptic modular forms.

We compare the spaces of HJF of weight k and indices $1, 2$ with classical Jacobi forms (JF) of weight k and indices $1, 2, 4$. Using the embedding into JF, upper bounds for the order of vanishing of HJF at the origin is obtained. We compute the rank of HJF as a module over elliptic modular forms and prove the algebraic independence of the generators in case of index 1. Some related questions are discussed.

Synopsis

1. Introduction

The theory of Jacobi forms has been an interesting and fruitful area of research in the recent past, notably after the work of Martin Eichler and Don Zagier in the monograph, "The Theory of Jacobi Forms", in 1985. Jacobi forms are a cross between elliptic functions and modular forms in one variable. They arise as the Fourier-Jacobi coefficients of Siegel modular forms of degree 2 (and also for higher degree), as the Theta series attached to Bilinear forms; and have played an important role in the proof of the Saito-Kurokawa conjecture on the correspondence between Siegel modular forms of degree 2 and elliptic modular forms in [16]. Also one can construct Siegel modular forms (the so called Maass "Spezialschar") from Jacobi forms of index 1 explicitly. They are also widely used in Physics.

It is known that the space of Jacobi forms is the direct sum of spaces spanned by Eisenstein series and cusp forms. Moreover, by generalities, it is known that certain special cusp forms, namely the Poincaré series, span the space of cusp forms. However in the theory of modular forms and Jacobi forms, it is (in general) an open question to give concrete criterions for deciding which of the Poincaré series (indexed by the set of integers or by the set of symmetric, positive-definite, half-integral matrices) do not vanish identically. In one of my works, sufficient conditions are given, under which such a Poincaré series of integral weight and matrix index for higher degree Jacobi group over $\mathbb{Z}$, do not vanish identically.

If we replace the usual Jacobi group (over $\mathbb{Z}$) by the Hermitian Jacobi group (over the ring of integers of an imaginary quadratic field), then one obtains the notion of Hermitian Jacobi forms. They were first defined by K. Haverkamp in [20], where he studied the Theta decomposition and Hecke operators for Hermitian Jacobi forms and established a Trace formula for Hecke operators. Recently in [36], the structural properties of index 1 forms have been studied. In two of my works some analytic properties of Hermitian Jacobi forms analogous to classical Jacobi forms are studied.

Details of my thesis work are given below in Sections 2 and 3.

2. Higher degree Jacobi forms and Poincaré series

In 1980 R. A. Rankin proved that the *m-th* Poincaré series P_m^k of weight k, where k, m are positive integers, for the full modular group $SL(2,\mathbb{Z})$ is not identically zero for sufficiently large k and finitely many m depending on k. It is conjectured that $P_m^k \neq 0$ for all $m \geq 1$, when $\dim S_k \neq 0$ (the space of elliptic cusp forms). This is a hard problem in general; in particular when $k = 12$ this is equivalent to the Lehmer's conjecture on the non-vanishing of Ramanujan's τ function. C. J. Mozzochi extended Rankin's result to integral weight modular forms for congruence subgroups. In this thesis we prove similar results for higher degree Jacobi Poincaré series defined on the full Jacobi group $\Gamma_g^J = SL(2,\mathbb{Z}) \ltimes (\mathbb{Z}^g \times \mathbb{Z}^g)$, where g is a positive integer and is referred to as the degree of the Jacobi group. For $n \in \mathbb{N}$, $r \in \mathbb{Z}^g$ with $4n > m^{-1}[r^t]$, let $P_{n,r}^{k,m}$ be the (n,r)-*th* Poincaré series of weight k and index m (of exponential type) defined for $k > g+2$. It is well-known that the Poincaré series $P_{n,r}^{k,m}$ $(n \in \mathbb{Z}, r \in \mathbb{Z}^g)$ span $J_{k,m,g}^{cusp}$ (the space of Jacobi cusp forms of weight k, index m and degree g). We prove that under suitable conditions (both depending on the weight and independent of it), infinitely many $P_{n,r}^{k,m}$ do not vanish identically. One of the main results is the following theorem:

Let $D = \det \left(\begin{smallmatrix} 2n & r \\ r^t & 2m \end{smallmatrix}\right)$ and define $k' := k - g/2 - 1$.

Theorem 1. *Let k be even when $2r \equiv 0 \pmod{\mathbb{Z}^g \cdot 2m}$. Then there exist an integer k_0 and a constant $B > 3\log 2$ such that for all $k \geq k_0$ (depending only on g) and*

$$k' \leq \frac{\pi D}{\det(2m)} \leq {k'}^{1+\alpha(g)} \exp\left\{ -\frac{B \log k'}{\log\log k'} \right\},$$

the Jacobi Poincaré series $P_{n,r}^{k,m}$ does not vanish identically; where

$$\alpha(g) = \begin{cases} \frac{2}{3(g+2)} & \text{if} \quad 1 \leq g \leq 4, \\ \frac{2}{3g} & \text{if} \quad g \geq 5. \end{cases}$$

This is done by showing that the (n, r)-*th* Fourier coefficient of $P_{n,r}^{k,m}$ in positive. For this we use several compatible estimates of higher dimensional Kloosterman sums and that of classical Bessel functions. However, as in the case of elliptic modular forms, a complete description in this direction seems to be very difficult.

We also construct a basis of $J_{k,m,1}^{cusp}$ consisting of the "first" $\dim J_{k,m,1}^{cusp}$ Poincaré series. This is the analogue of the result of Petersson in the case of elliptic modular forms. This essentially follows from the dimension formula for classical Jacobi forms given in [16]. We also give conditions for non-vanishing of Poincaré series independent of the weight for classical Jacobi forms ($g = 1$). Also, a result on the non-vanishing of Jacobi Poincaré series is obtained when an odd prime divides the index, by considering the relation of one dimensional Kloosterman sums with the corresponding higher dimensional ones.

3. Hermitian Jacobi forms

As mentioned in the Introduction, we introduce a certain differential (heat) operator D_ν ($\nu \geq 0$) on the space of Hermitian Jacobi forms of degree 1, weight

$k \in \mathbb{Z}$ and index $m \in \mathbb{N}$ (denoted by $J_{k,m}(\mathcal{O}_K)$) constructed from the Taylor expansion of such a form after restricting to the "diagonal" subseries (which is invariant under the action of $SL(2,\mathbb{Z})$). Then a theory analogous to that of classical Jacobi forms is developed. We show that it commutes with certain Hecke operators and use it's adjoint to construct Hermitian Jacobi cusp forms from elliptic modular forms.

Let $f \in S_{k+2\nu}$ and $(,)$ be the Petersson inner product on $S_{k+2\nu}$. Let $\langle , \rangle$ be the Petersson inner product on $J_{k,m}^{cusp}(\mathcal{O}_K)$ and $D_\nu^* : S_{k+2\nu} \longrightarrow J_{k,m}^{cusp}(\mathcal{O}_K)$ be the adjoint of D_ν with respect to the above inner products.

Theorem 2. *With the above notations the Fourier development* of $D_\nu^* f$ *is given by*

$$D_\nu^* f\,(\tau, z_1, z_2) = \sum_{n=0}^{\infty} \sum_{\substack{r \in \mathcal{O}_K^\sharp \\ nm \geq N(r)}} c_{D_\nu^* f}(n,r) e^{2\pi i (n\tau + r z_1 + \bar{r} z_2)} \qquad \textit{where}$$

$$\begin{aligned} c_{D_\nu^* f}(n,r) = & \frac{\nu!(-1)^\nu (4\pi)^{2\nu-1}\Gamma(k+2\nu-1) m^{\nu-k+3}\left(nm - N(r)\right)^{k-2}}{\Gamma(k-2)(k-1)^{(\nu)}} \\ & \times \sum_{\lambda \in \mathfrak{O}_K} \frac{a\left(mN(\lambda) + r\lambda + \bar{r}\bar{\lambda} + n, f\right)}{\left(mN(\lambda) + r\lambda + \bar{r}\bar{\lambda} + n\right)^{k+\nu-1}} \\ & \times \sum_{j=0}^{\nu} \frac{(-1)^j (k-1)^{(2\nu-j)}}{(\nu-j)!^2 j!} \left(\frac{N(m\lambda + \bar{r})}{m\left(mN(\lambda) + r\lambda + \bar{r}\bar{\lambda} + n\right)} \right)^{\nu-j}, \end{aligned} \tag{0.0.1}$$

and $\quad f(\tau) = \sum_{n=1}^{\infty} a\,(n,f)\, e^{2\pi i n \tau}$.

Further, we construct Hermitian Jacobi forms as the image of the tensor product of two copies of Jacobi forms and also from differentiation of the variables. We determine the number of Fourier coefficients that determine a Hermitian Jacobi form and use it to embed a certain subspace of Hermitian Jacobi forms into a direct sum of modular forms for the full modular group. This is done using the Theta decomposition of such forms. We also prove that $J_{1,m}(\mathcal{O}_K) = 0$ for all $m \geq 1$ which is analogous to and proved by using N. P.

Skoruppa's result for classical Jacobi forms.

Next we treat classical Jacobi forms as an intermediate space between Hermitian Jacobi forms and elliptic modular forms. We present some of the structural properties of index 2 forms using the restriction maps $\pi_\rho\colon J_{k,m}(\mathcal{O}_K) \to J_{k,N(\rho)m}$ defined by $\pi_\rho\phi(\tau, z_1, z_2) = \phi(\tau, \rho z, \bar{\rho} z)$ ($\rho \in \mathcal{O}_K$, see [21]). Since we do not have (at present) the order of vanishing of a Hermitian Jacobi forms at the origin (which is known in the case of classical Jacobi forms), computations involving the Taylor expansions is not very fruitful for $m \geq 2$. The purpose of this work is to look at the structure of index 2 forms by comparing them with classical Jacobi forms. Among several other results, we mention the following theorem, which deals with the case of $k \equiv 0 \pmod 4$ for index 2 forms.

Theorem 3. *Let $k \equiv 0 \pmod 4$. We have the following exact sequence of vector spaces*

$$0 \longrightarrow J_{k,2}(\mathcal{O}_K) \xrightarrow{\pi_1 \times \pi_{1+i}} J_{k,2} \times J_{k,4} \xrightarrow{\Lambda(2)-\Lambda(4)} M_k \times S_{k+2} \longrightarrow 0 \qquad (0.0.2)$$

where $\Lambda(m) := D_0 + \frac{2}{m} D_2\colon J_{k,m} \to M_k \times S_{k+2}$; D_0 and D_2 *are well known differential operators on Jacobi forms given by,* $D_0\phi := \phi \mid_{z=0}$ and $D_2\phi := \left(\frac{k}{2\pi i}\frac{\partial^2}{\partial z^2}\phi - 2\frac{\partial}{\partial \tau}\phi\right)_{z=0}$.

We also compute the rank of index m forms of weight a multiple of 2 and 4 (denoted as $J_{n*,m}(\mathcal{O}_K)$, $n = 2, 4$) as a module over the algebra of elliptic modular forms. Unlike the classical Jacobi forms, the number of homogeneous products of degree m of the index 1 generators is less than the rank. Following the argument as in [16, p.97], we easily see that $J_{*,*}(\mathcal{O}_K)$ is free over M_*, and $J_{n*,m}(\mathcal{O}_K)$ is of finite rank $R_n(m)$ over M_*. We have the following:

Proposition 4. *(i)* $R_4(m) = m^2 + 2$, *(ii)* $R_2(m) = 2(m^2 + 1)$.

We also discuss several related questions on Hermitian Jacobi forms and propose a conjecture which states that $J_{k,m}(\mathcal{O}_K)$ can be embedded into a direct sum of n spaces of Jacobi forms, where n depends on the index m only. This is verified for $m = 1, 2$ in the thesis. We also propose a set of $m^2 + 2$ linearly independent forms in $J_{4*,2}(\mathcal{O}_K)$.

List of publications and preprints:

1. *Non-vanishing of Jacobi-Poincare series.* (submitted)
   http://arxiv.org/abs/0910.4303v2.
2. *Some aspects of Hermitian Jacobi forms.* (submitted)
   http://arxiv.org/abs/0910.4306v1.
3. *Note on Hermitian Jacobi forms.* (submitted)
   http://arxiv.org/abs/0910.4312v2.

To my Parents

Table of Contents

0 Introduction

In this thesis we will deal with several problems on Jacobi forms and to some extent relations and identities among certain Kloosterman-type sums. Jacobi forms have a rich and varying history in mathematics going back to Jacobi, who first studied the Theta series associated to a positive definite integer valued quadratic forms. They were also encountered as Fourier-Jacobi coefficients of Siegel modular forms of degree 2 (and also for higher degree). The first systematic exposition of the theory of Jacobi forms was given in the work of Martin Eichler and Don Zagier in the monograph, "The Theory of Jacobi Forms", in 1985. Jacobi forms are a cross between elliptic functions and modular forms in one variable. They are functions on $\mathcal{H} \times \mathbb{C}$ ($\mathcal{H}$ being the upper half plane) satisfying certain transformation formulas which are given in Chapter 1 and also has a particular type of Fourier expansion. We give some examples of Jacobi forms below in connection with the important role they have in the proof of the Saito-Kurokawa conjecture on the correspondence between Siegel modular forms of degree 2 and elliptic modular forms ([16]). Also one can construct Siegel modular forms (the so called `Maass ''Spezialschar''`) from Jacobi forms of index 1 explicitly. They are also widely used in Physics.

— *Example.* We let $F(Z)$ be a Siegel modular form of weight k and degree 2, i.e., a holomorphic function on the Siegel upper half plane of degree 2 consisting of matrices $\left(\begin{smallmatrix} \tau & z \\ z & \tau' \end{smallmatrix}\right)$ with $z \in \mathbb{C}$, $\tau, \tau' \in \mathcal{H}$ such that $\mathrm{Im}(z)^2 < \mathrm{Im}(\tau)\mathrm{Im}(\tau')$, invariant under the Siegel modular group $Sp_2(\mathbb{Z})$ consisting of 4×4 matrices M such that $M^tJM = J$, $J = \left(\begin{smallmatrix} 0 & -I_{2\times 2} \\ I_{2\times 2} & 0 \end{smallmatrix}\right)$.

Then it is true that F has a Fourier Jacobi expansion

$$F(Z) = \sum_{m=0}^{\infty} \phi_m(\tau, z) e(m\tau').$$

It turns out that for each $m \geq 0$, ϕ_m is a Jacobi form of weight k and index m.

It is known that the space of Jacobi forms is the direct sum of spaces spanned by Eisenstein series and cusp forms. Moreover, by generalities, it is known that certain special cusp forms, namely the Poincaré series, span the space of cusp forms. Poincaré series are constructed by averaging suitable functions over the action of relevant groups. However in the theory of modular forms and Jacobi forms, it is (in general) an open question to give concrete criterions for deciding which of the Poincaré series (indexed by the set of integers or by a lattice in the case of higher degree Jacobi forms) do not vanish identically. In Chapter 4, sufficient conditions are given, under which such a Poincaré series of integral weight, and matrix index for higher degree Jacobi group over $\mathbb{Z}$, do not vanish identically.

— *Higher degree Jacobi forms and Poincaré series*

In 1980 R. A. Rankin proved that the *m-th* Poincaré series P_m^k of weight k, where k, m are positive integers, for the full modular group $SL(2, \mathbb{Z})$ is not identically zero for sufficiently large k and finitely many m depending on k. It is conjectured that $P_m^k \neq 0$ for all $m \geq 1$, when $\dim S_k \neq 0$ (the space of elliptic cusp forms). This is a hard problem in general; in particular when $k = 12$ this is equivalent to the Lehmer's conjecture on the non-vanishing of Ramanujan's τ function. C. J. Mozzochi extended Rankin's result to integral weight modular forms for congruence subgroups. In this thesis we prove similar results for higher degree Jacobi Poincaré series defined on the full Jacobi group $\Gamma_g^J = SL(2, \mathbb{Z}) \ltimes (\mathbb{Z}^g \times \mathbb{Z}^g)$, where g is a positive integer and is referred to as the degree of the Jacobi group. For $n \in \mathbb{N}$, $r \in \mathbb{Z}^g$ with $4n > m^{-1}[r^t]$, let $P_{n,r}^{k,m}$ be the (n, r)-*th* Poincaré series of weight k and index m (of exponential type) defined for $k > g + 2$ (see Chapter 1 for details). It is well-known that the Poincaré series $P_{n,r}^{k,m}$ $(n \in \mathbb{Z}, r \in \mathbb{Z}^g)$ span the space of Jacobi cusp forms of weight k, index m and degree g. We prove that under suitable conditions (both depending on the weight and independent of it), infinitely many $P^{k,m}_{n,r}$ do not vanish identically. Let $D = \det \left(\begin{smallmatrix} 2n & r \\ r^t & 2m \end{smallmatrix}\right)$, $k' := k - g/2 - 1$. We decide whether $P^{k,m}_{n,r}$ is non-zero by comparing $\frac{D}{\det(2m)}$ with k' (see Theorem 4.1.3).

This is proved by showing that the (n, r)-*th* Fourier coefficient of $P^{k,m}_{n,r}$ in positive. For this we use several compatible estimates of higher dimensional Kloosterman sums and that of classical Bessel functions. However, as in the case of elliptic modular forms, a complete description in this direction seems to be very difficult.

It is well-known that the Fourier development of Poincaré series involves Kloosterman sums and Bessel functions. So, information about Kloosterman sums might give some results about the corresponding Poincaré series. We prove that the Kloosterman sums occuring in the Fourier development of index 1 Jacobi Poincaré series of degree 1 and those in half-integral weight Poincaré series in Kohnen's plus space are equal. This allows one to reduce the question about Jacobi Poincaré series to that of the corresponding object in half-integer weight modular forms via the Eichler-Zagier map.

We mention here another theorem by Petersson [32]. To the knowledge of the author, he was the first to give examples when certain Poincaré series does not vanish identically.

Theorem 0.0.1. *Let $n \geq 1$ and let $d := \dim S_k \neq 0$ (S_k is the space of elliptic cusp forms for $SL(2, \mathbb{Z})$). Then $P^k_1, \ldots, P^k_d$ form a basis for S_k. In particular they are non-zero.*

We construct an explicit basis of Jacobi cusp forms of weight k and index m (denoted $J^{cusp}_{k,m}$). This is the analogue of the result of Petersson in the case of elliptic modular forms. This essentially follows from the dimension formula for classical Jacobi forms given in [16]. There are several relations among the Jacobi Poincaré series, and following a result of R.C. Rhoades (see [35]) which describes all linear relations among the integer weight Poincaré series, it would be interesting to obtain such a result in the present case also.

We also give conditions for non-vanishing of Poincaré series independent of the weight for classical Jacobi forms ($g = 1$). Also, a result on the non-vanishing of Jacobi Poincaré series is obtained when an odd prime divides the index, by considering the relation of one dimensional Kloosterman sums with the corresponding higher dimensional ones.

— *Hermitian Jacobi forms*

If we replace the usual Jacobi group (over $\mathbb{Z}$) by the Hermitian Jacobi group (over the ring of integers of an imaginary quadratic field), then one obtains the notion of Hermitian Jacobi forms (of weight k, index m, denoted by $J_{k,m}(\mathcal{O}_K)$; see Chapter 1). They were first defined by K. Haverkamp in [20], where he studied the Theta decomposition and Hecke operators and established a Trace formula for Hecke operators. We touch upon various properties of Hermitian Jacobi forms in this thesis.

In Chapter 2, some analytic properties of Hermitian Jacobi forms analogous to classical Jacobi forms are studied. We introduce a certain differential (heat) operator D_ν ($\nu \geq 0$) on the space of Hermitian Jacobi forms of degree 1, weight $k \in \mathbb{Z}$ and index $m \in \mathbb{N}$ (denoted by $J_{k,m}(\mathcal{O}_K)$) constructed from the Taylor expansion of such a form after restricting to the "diagonal" subseries (which is invariant under the action of $SL(2, \mathbb{Z})$). Then a theory analogous to that of classical Jacobi forms is developed. We show that it commutes with certain Hecke operators and use it's adjoint to construct Hermitian Jacobi cusp forms from elliptic modular forms.

Further, we construct Hermitian Jacobi forms as the image of the tensor product of two copies of Jacobi forms and also from derivatives. We determine the number of Fourier coefficients that determine a Hermitian Jacobi form and use it to embed a certain subspace of Hermitian Jacobi forms into a direct sum of modular forms for the full modular group. This is done using the Theta decomposition of such forms. We also prove that $J_{1,m}(\mathcal{O}_K) = 0$ for all $m \geq 1$ which is analogous to and proved by using N. P. Skoruppa's result for classical Jacobi forms.

Recently in [36], the structural properties of index 1 forms have been studied by R. Sasaki. More precisely, he explicitly gave generators for index 1 forms in terms of their Theta decompositions (see Chapter 1 for details). Like in the case of classical Jacobi forms, where $J_{k,1}$ is Hecke-isomorphic with Kohnen's plus space, we have an analogue in this case as well.

Definition 0.0.2. *$M^+_{k-1}(4,\chi)$ is defined to be the space of modular forms on the congruence subgroup $\Gamma_0(4)$ with character χ whose Fourier expansion $\sum_{n=0}^{\infty} a(n)q^n$ have the property that $a(n) = 0$ for $n \equiv 1 \pmod 4$.*

Theorem 0.0.3 (cf. [36])**.** *With ϕ having a Theta decomposition as in Chapter 1 Theorem 1.2.4,*

$$\phi(\tau, z_1, z_2) = \sum_{s \in \mathcal{O}_K^\sharp / m\mathcal{O}_K} h_s(\tau) \cdot \theta^H_{m,s}(\tau, z_1, z_2) \tag{0.0.3}$$

$$\mapsto \frac{i}{2^{k-2}} \left(\sum_{s \in \mathcal{O}_K^\sharp / \mathcal{O}_K} h_s(4\tau) \right) \tag{0.0.4}$$

gives an isomorphism between $J_{k,1}(\mathcal{O}_K)$ and $M^+_{k-1}(4,\chi)$.

In Chapter 3 we treat classical Jacobi forms as an intermediate space between Hermitian Jacobi forms and elliptic modular forms. We present some of the structural properties of index 1, 2 forms using the restriction maps $\pi_\rho \colon J_{k,m}(\mathcal{O}_K) \to J_{k,N(\rho)m}$ defined by $\pi_\rho\phi(\tau, z_1, z_2) = \phi(\tau, \rho z, \bar{\rho} z)$ ($\rho \in \mathcal{O}_K$, see [21]). Since we do not have (at present) the order of vanishing of a Hermitian Jacobi forms at the origin (which is known in the case of classical Jacobi forms), computations involving the Taylor expansions is not very fruitful for $m \geq 2$. The purpose of this Chapter is to look at the structure of index 2 forms by comparing them with classical Jacobi forms. In particular, we have that in some cases the restriction maps give an isomorphism.

In [16] it is proved that (also see Chapter 1) the space of Jacobi forms of even weight and index m is a free module over M_* of rank $m+1$. We also compute the rank of index m Hermitian Jacobi forms of weight a multiple of 2 and 4 (denoted as $J_{n*,m}(\mathcal{O}_K)$, $n = 2, 4$) as a module over the algebra of elliptic modular forms. Unlike the classical Jacobi forms, the number of homogeneous products of degree m of the index 1 generators is less than the rank. Following the argument as in [16, p.97], we easily see that $J_{*,*}(\mathcal{O}_K)$ is free over M_*, and $J_{n*,m}(\mathcal{O}_K)$ is of finite rank $R_n(m)$ over M_*. We have the following:

Proposition 0.0.4. *(i)* $R_4(m) = m^2 + 2$, *(ii)* $R_2(m) = 2(m^2 + 1)$.

We also discuss several related questions on Hermitian Jacobi forms and propose a conjecture which states that $J_{k,m}(\mathcal{O}_K)$ can be embedded into a direct sum of n spaces of Jacobi forms, where n depends on the index m only. This is verified for $m = 1, 2$ in the thesis. A result in this direction should help in reducing some of the questions on Hermitian Jacobi forms to clasical Jacobi forms. We also propose a set of $m^2 + 2$ linearly independent forms in $J_{4*,2}(\mathcal{O}_K)$.

Chapter 1

Background and Preliminaries

In this chapter we give some basic definitions and results that will be used in the thesis. The first section focuses on the definition and several properties of Jacobi forms and the relevant Poincaré series. Generalities on higher degree Jacobi forms was dealt with by C. Ziegler in [44], and we follow his exposition closely in this chapter. We also summarize the results concerning classical Jacobi forms which will be needed later. The results about Poincaré series and estimates of relevant Kloosterman sums follow the work of Böcherer-Kohnen [7]. In the second section, we summarize basic facts about Hermitian Jacobi forms, most of which were established by K. Haverkamp in his thesis ([20]), and this section is mainly based on that. We do not give many proofs in this chapter, as the material is well known and easily available from the references.

1.1 Jacobi forms

1.1.1 Jacobi forms of higher degree

We begin with a general setting and define the Heisenberg group and the Jacobi group. However in later applications, we will need only some particular cases of the following; so we do not pursue the general approach too much.

Definition 1.1.1. *Let n, g be positive integers. Define*

$$H_{\mathbb{R}}^{(n,g)} = \left\{[(\lambda,\mu),\kappa] \mid \lambda,\mu \in \mathbb{R}^{(g,n)}, \kappa \in \mathbb{R}^{g,g}, (\kappa+\mu\lambda^t) \text{ symmetric}\right\}. \tag{1.1.1}$$

Then one can verify that $H_{\mathbb{R}}^{(n,g)}$ is a group with the following operation :

$$[(\lambda,\mu),\kappa] \circ [(\lambda',\mu'),\kappa'] := [(\lambda+\lambda',\mu+\mu'),\kappa+\kappa'+\lambda\mu'^t-\mu\lambda'^t]. \tag{1.1.2}$$

The symplectic group of degree n over $\mathbb{R}$ is denoted by $Sp(n,\mathbb{R})$ acts on $H_{\mathbb{R}}^{(n,g)}$ in the usual way :

$$[(\lambda,\mu),\kappa] \circ M := [(\lambda,\mu) \circ M, \kappa], \quad \text{where} \tag{1.1.3}$$

$(\lambda,\mu) \circ M = (A\lambda + C\mu, B\lambda + D\mu)$; $(M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathbb{R}^{2n,2n} \cap Sp(n,\mathbb{R})$).

Definition 1.1.2. *This allows one to define the semi-direct product $G_{\mathbb{R}}^{(n,g)} := Sp(n,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,g)}$, called the* ***`Jacobi group`*** *of degree (n,g) over $\mathbb{R}$.*

$G_{\mathbb{R}}^{(n,g)}$ acts on $\mathcal{H}_n \times \mathbb{C}^{(g,n)}$ as a group of automorphisms (bi-holomorphisms):

$$(M,[(\lambda,\mu),\kappa]) \circ (Z,W) := (M\langle Z\rangle, W + \lambda Z + \mu), \tag{1.1.4}$$

where $M\langle Z\rangle = (AZ+B)(CZ+D)^{-1}$.
Further, $G_{\mathbb{R}}^{(n,g)}$ acts on functions $\phi\colon \mathcal{H}_n \times \mathbb{C}^{(g,n)} \to \mathbb{C}$ preserving holomorphicity :

Definition 1.1.3. *Let $\mathcal{M}$ be a $g \times g$ symmetric, positive definite and half-integral matrix, i.e., $2\mathcal{M}_{i,j}, \mathcal{M}_{i,i} \in \mathbb{Z}$ and $k \in \mathbb{Z}$. Let $\zeta = [(\lambda,\mu),\kappa] \in H_{\mathbb{R}}^{(n,g)}$ and $M \in Sp(n,\mathbb{R})$. We define the following actions (stroke-operations with respect to $k, \mathcal{M}$) :*

$$(\phi \mid_{k,\mathcal{M}} M)(Z,W) := \det(CZ+D)^{-k} e\left(\operatorname{tr}(\mathcal{M}W(CZ+D)^{-1}CW^t)\right) \phi\left(M\langle Z\rangle, W(CZ+D)^{-1}\right), \tag{1.1.5}$$

$$(\phi \mid_{\mathcal{M}} \zeta)(Z,W) := e\left(\operatorname{tr}(\mathcal{M}(\lambda Z\lambda^t + 2\lambda W^t + \kappa\mu\lambda^t))\right) \phi(Z, W+\lambda Z+\mu). \tag{1.1.6}$$

where $e(z) := e^{2\pi i z}$. Finally, we define $\phi \mid_{k,\mathcal{M}} (M,\zeta) := \phi \mid_{k,\mathcal{M}} M \mid_{\mathcal{M}} \zeta$.

Remark 1.1.1. The above action is actually defined for a rational representation $\rho\colon GL(n,\mathbb{C}) \to GL(E)$ for a finite dimensional vector space E over $\mathbb{C}$ and for E-valued functions. Here we have taken $\rho(A) = (\det A)^k$ and $E = \mathbb{C}$.

Lemma 1.1.4. *The above actions are well defined, i.e., for $M, M' \in Sp(n,\mathbb{R})$ and $\zeta, \zeta' \in H_{\mathbb{R}}^{(n,g)}$ we have*

$$\phi \mid M \mid M' = \phi \mid (MM'), \tag{1.1.7}$$

$$\phi \mid \zeta \mid \zeta' = \phi \mid (\zeta \circ \zeta'), \tag{1.1.8}$$

$$\phi \mid \zeta \mid M = \phi \mid M \mid (\zeta \circ M). \tag{1.1.9}$$

Definition 1.1.5. *The Heisenberg groups and Jacobi groups can obviously be defined over any commutative ring $\mathcal{R}$; in particular, taking $\mathcal{R} = \mathbb{Z}$ we have*

$$H_{\mathbb{Z}}^{(n,g)} := \left\{ [(\lambda,\mu),\kappa] \in H_{\mathbb{R}}^{(n,g)} \mid \lambda,\mu \in \mathbb{Z}^{g,n},\ \kappa \in \mathbb{Z}^{(g,g)} \right\}, \tag{1.1.10}$$

$$G_{\mathbb{Z}}^{(n,g)} := Sp(n,\mathbb{Z}) \ltimes H_{\mathbb{Z}}^{(n,g)}, \quad (Sp(n,\mathbb{Z}) \text{ being the Siegel modular group of degree } n). \tag{1.1.11}$$

Now we can define the notion of a Jacobi form for the group $G_{\mathbb{Z}}^{(n,g)}$:

Definition 1.1.6. *A Jacobi form of weight $k \in \mathbb{Z}$ and index $\mathcal{M}$ (as in Definition 1.1.3) is a holomorphic function $\phi\colon \mathcal{H}_n \times \mathbb{C}^{(g,n)} \to \mathbb{C}$ satisfying the following :*

(i) $\phi \mid_{k,\mathcal{M}} M = \phi$ for all $M \in Sp(n,\mathbb{Z})$,

(ii) $\phi \mid_{\mathcal{M}} \zeta = \phi$ for all $\zeta \in H_{\mathbb{Z}}^{(n,g)}$.

(iii) ϕ has Fourier expansion of the form :

$$\phi(Z,W) = \sum_{\substack{T=T^t \geq 0 \\ T \text{ half-integral}}} \ \sum_{\substack{R \in \mathbb{Z}^{(n,g)} \\ \begin{pmatrix} T & \frac{R}{2} \\ \frac{R^t}{2} & \mathcal{M} \end{pmatrix} \geq 0}} c(T,R)e\left(\operatorname{tr}(TZ + RW)\right).$$

The vector space of Jacobi forms of degree (n,g) for the full Jacobi group is denoted by $J_{k,\mathcal{M}}$. We end the general setting by summarizing some of the salient properties of Jacobi forms in the following Proposition, whose proof can be found eg. in [44]:

Proposition 1.1.7. *(i) (Köecher Principle) Let $n \geq 2$ in the definition of Jacobi forms. Then condition (iii) in Definition 1.1.6 holds a priori.*
(ii) $J_{*,*} := \sum_{k,\mathcal{M}} J_{k,\mathcal{M}}$ *is naturally a bigraded ring. (Here $\mathcal{M}$ varies as in Definition 1.1.3).*
(iii) $\dim J_{k,\mathcal{M}} < \infty$*, and* $J_{k,\mathcal{M}} = 0$ *for* $k < 0$*.*

Remark 1.1.2. In this thesis, we will be concerned with the case $n = 1$ in the above formalism. This is mainly because, good estimates of Kloosterman sums are presently known to exist only in this case. In the general case, matrix modulus Kloosterman sums will occur, which are very difficult to handle, as remarked by Prof. W. Kohnen. Since the formulas in the case $n = 1$ will be used frequently, we write down the notations to be followed later and some of the (simplified) actions for further use:
The Jacobi group Γ_g^J of degree g is defined by $\Gamma_g^J := G_{\mathbb{Z}}^{(1,g)} = SL(2,\mathbb{Z}) \ltimes (\mathbb{Z}^g \times \mathbb{Z}^g)$; the group operation is given by

$$(\gamma, (x,y)) \cdot (\gamma', (x',y')) = (\gamma \cdot \gamma', (x,y) \cdot \gamma' + \gamma)\,.$$

where $\gamma, \gamma' \in SL(2,\mathbb{Z})$ and $x, x', y, y' \in \mathbb{Z}^g$ and $SL(2,\mathbb{Z})$ acts on the right on $\mathbb{Z}^g \times \mathbb{Z}^g$ in the usual way by $(x,y) \cdot \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = (ax + cy, bx + dy)$.
Γ_g^J operates on $\mathcal{H} \times \mathbb{C}^g$ in the usual way by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu) \right) \circ (\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, (c\tau + d)^{-1}(z + \lambda\tau + \mu) \right).$$

Let $k \in \mathbb{Z}$, m a symmetric, positive-definite, half-integral $(g \times g)$ matrix, i.e., $m = m^t, x^t m x > 0$ for all $x \in \mathbb{Z}^g$ and $m \in M(g, \mathbb{Q})$ (space of $g \times g$ matrices over $\mathbb{Q}$) with integer entries on the diagonal and half-integer entries on off-diagonal. Then we have the action of Γ_g^J on functions ϕ: $\mathcal{H} \times \mathbb{C}^g \to \mathbb{C}$ given by :

$$\phi|_{k,m}\gamma(\tau, z) := (c\tau + d)^{-k} e\left(-c(c\tau + d)^{-1} m[z + \lambda\tau + \mu] + m[\lambda]\tau + 2\lambda^t m z\right) \phi\left(\gamma \circ (\tau, z)\right).$$

(Here $A[B] = B^t A B$ for matrices A, B of appropriate sizes, B^t is the transpose of the matrix B.)

The vector space of Jacobi cusp forms of weight k, index m and degree g, denoted by $J^{cusp}_{k,m,g}$ is defined to be the space of holomorphic functions $\phi\colon \mathcal{H}\times\mathbb{C}^g\to\mathbb{C}$ satisfying $\phi|_{k,m}\gamma=\phi$ for all $\gamma\in\Gamma^J_g$ and having a Fourier expansion

$$\phi(\tau,z)=\sum_{n\in\mathbb{N},r\in\mathbb{Z}^g,4n>m^{-1}[r^t]} c_\phi(n,r)e(n\tau+rz).$$

— Henceforth we will deal with the case of Jacobi forms on $\mathcal{H}\times\mathbb{C}^{(g,1)}$, although the definitions are valid in the general case ($\mathcal{H}\times\mathbb{C}^{(g,n)}$).

1.1.2 Poincaré series

For $n\in\mathbb{N}$, $r\in\mathbb{Z}^g$ with $4n>m^{-1}[r^t]$, let $P^{k,m}_{n,r}$ be the (n,r)-*th* Poincaré series of weight k and index m (of exponential type) defined for $k>g+2$ by

$$P^{k,m}_{n,r}(\tau,z):=\sum_{\gamma\in\Gamma^J_{g,\infty}\backslash\Gamma^J_g} e(n\tau+rz)|_{k,m}\gamma(\tau,z)\qquad \tau\in\mathcal{H},\ z\in\mathbb{C}^g, \tag{1.1.12}$$

where $\Gamma^J_{g,\infty}:=\left\{\left(\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix},(0,\mu)\right)\mid n\in\mathbb{Z},\mu\in\mathbb{Z}^g\right\}$.

It is well known that $J^{cusp}_{k,m,g}$ is finite dimensional and the family of Poincaré series $P^{k,m}_{n,r}$ ($n\in\mathbb{N}$, $r\in\mathbb{Z}^g$ with $4n>m^{-1}[r^t]$) generate $J^{cusp}_{k,m,g}$. In [7, Lemma 1], S. Böcherer and W. Kohnen obtained the Fourier expansion of $P^{k,m}_{n,r}$:

Proposition 1.1.8. *(1) The function $P^{k,m}_{n,r}$ is in $J^{cusp}_{k,m,g}$. The Fourier expansion of the Poincaré series is given by*

$$P^{k,m}_{n,r}(\tau,z)=\sum_{n'\in\mathbb{N},r'\in\mathbb{Z}^g,4n'>m^{-1}[r'^t]} c^{k,m}_{n,r}(n',r')e(n'\tau+r'z),$$

where

$$\begin{aligned}c^{k,m}_{n,r}(n',r')&=\delta^{\pm}_m(n,r,n',r')+2\pi i^k\,det(2m)^{-1/2}\cdot(D'/D)^{k/2-g/4-1/2}\\&\quad\times\sum_{c\geq1}H^{\pm}_{m,c}(n,r,n',r')\,J_{k-g/2-1}\left(\frac{2\pi\sqrt{DD'}}{det(2m)\cdot c}\right)\end{aligned} \tag{1.1.13}$$

where $D' = \det \left(\begin{smallmatrix} 2n' & r' \\ r'^t & 2m \end{smallmatrix}\right)$, $\delta_m(n,r,n',r') := \begin{cases} 1 & \text{if } D = D', r \equiv r' \pmod{\mathbb{Z}^g \cdot 2m}, \\ 0 & \text{otherwise}, \end{cases}$

and $\delta_m^{\pm}(n,r,n',r') := \delta_m(n,r,n',r') + (-1)^k \delta_m(n,r,n',-r')$. *Further,*

$H_{m,c}(n,r,n',r') := c^{-g/2-1} \sum_{x(c),y(c)^*} e_c\left((m[x]+rx+n)\bar{y} + n'y + r'x\right) e_{2c}(r'm^{-1}r^t),$

where in the summation x *(resp.* y*) run over a complete set of representatives for* $\mathbb{Z}^{(g,1)}/c\mathbb{Z}^{(g,1)}$ *(resp.* $(\mathbb{Z}/c\mathbb{Z})^*$*),* $\bar{y}$ *denotes an inverse of* $y \pmod c$*, and* $H_{m,c}^{\pm}(n,r,n',r') := H_{m,c}(n,r,n',r') + (-1)^k H_{m,c}(n,r,n',-r')$. *Finally,* $J_{k-g/2-1}$ *denotes the Bessel function of order* $k - g/2 - 1$.

— Petersson inner product : Keeping the notations introduced above, let $\tau \in \mathcal{H}, z \in \mathbb{C}^{(g,1)}$ and write them in their real and imaginary parts: $\tau = u + iv$, $z = x + iy$. Then the volume element dV_g^J on $\mathcal{H} \times \mathbb{C}^{(g,1)}$ is defined by

$$dV_g^J = v^{-(g+2)} du \wedge dv \wedge dx \wedge dy \tag{1.1.14}$$

Then it follows that dV_g^J defines an invariant volume element for the Jacobi group Γ_g^J. Using this, one can define the Petersson inner product of two Jacobi forms ϕ, $\psi \in J_{k,m,g}$, at least one of which is a cusp form :

Definition 1.1.9. *(Petersson inner product)*

$$\langle \phi, \psi \rangle := \int_{\Gamma_g^J \backslash \mathcal{H} \times \mathbb{C}^{g,1}} \phi(\tau,z)\overline{\psi(\tau,z)} \exp\left(-4\pi i m[y] v^{-1}\right) dV_g^J. \tag{1.1.15}$$

Proposition 1.1.10. *Let* $\phi \in J_{k,m,g}$. *Then,* $\langle \phi, P_{n,r}^{k,m} \rangle = \lambda_{k,m,D}\, c_\phi(n,r)$, *where* $c_\phi(n,r)$ *denotes the* (n,r)*-th Fourier coefficient of* ϕ *and*

$$\lambda_{k,m,D} = 2^{(g-1)(k-g/2-1)-g} \cdot \Gamma(k-g/2-1) \cdot \pi^{-k+g/2+1} \cdot (\det m)^{k-(g+3)/2} \cdot D^{-k+g/2+1}.$$

Lemma 1.1.11. *The Poincaré series* $P_{n,r}^{k,m}$ *vanishes if* k *is odd and* $2r \equiv 0 \pmod{\mathbb{Z}^g \cdot 2m}$*.*

Proof. In fact the *(n,r)-th* coefficient $c(n,r)$ of a general Jacobi form of degree g is zero if k is odd when $2r \equiv 0 \pmod{\mathbb{Z}^g \cdot 2m}$. This is an easy consequence of the transformation properties of Jacobi forms. More precisely, $c(n,-r) = (-1)^k c(n,r)$ and

$$c(n,r) = c(n + r\lambda^t + m[\lambda^t]\,, r + 2m\lambda), \quad \text{where } \lambda \in \mathbb{Z}^g,$$

together imply the assertion easily. See for instance [16] for $g = 1$. $\square$

1.1.3 Jacobi forms of degree 1

We denote the space of Jacobi forms of degree 1 by $J_{k,m,1}$ or simply by $J_{k,m}$ depending on the context. We collect the main definitions and results that are freely used in this thesis. The reference for this subsection is [16]. As is customary, we denote the full Jacobi group of degree 1 by Γ^J.

1.1.3.1 Eisenstein series and cusp forms

Let $k > 2$ and s be any integer. Write $m = ab^2$, with a square-free. Then for each s, one can construct an Eisenstein series :

Definition 1.1.12. *(i) Let $\tau \in \mathcal{H}, z \in \mathbb{C}$, a, b, s as above. Define*

$$E_{k,m,s}(\tau, z) := \sum_{\gamma \in \Gamma^J_\infty \backslash \Gamma^J} e(as^2\tau + 2absz) \,|_{k,m}\, \gamma, \tag{1.1.16}$$

where $\Gamma^J_\infty := \Gamma^J_{1,\infty}$, which was defined in Remark 1.1.2.

(ii) The space spanned by $E_{k,m,s}$, $(s \in \mathbb{Z})$ is denoted as $J^{Eis}_{k,m}$.

Proposition 1.1.13. *Let $k > 2$. Then $E_{k,m,s} \in J_{k,m}$, depends only on $s \pmod b$, and we have $J_{k,m} = J^{cusp}_{k,m} \oplus J^{Eis}_{k,m}$, $J^{cusp}_{k,m}$ being the space of cusp forms defined in Remark 1.1.2. Moreover, the functions $E_{k,m,s}$, $0 \leq s \leq \frac{b}{2}$ (k even) or $0 < s < \frac{b}{2}$ (k odd) form a basis of $J^{Eis}_{k,m}$.*

Define $J_{2*,m} := \underset{k \in 2\mathbb{Z}}{\oplus} J_{k,m}$. It is easily seen that $J_{2*,m}$ is naturally a module over M_*. We denote $E_{k,m,0}$ by $E_{k,m}$. The next proposition states more :

Proposition 1.1.14. *(i) $E_{4,1}$ and $E_{6,1}$ are algebraically independent over M_*.*
(ii) $J_{2*,m}$ is free of finite rank $m+1$ over M_*, in particular $\left\{E_{4,1}^a E_{6,1}^b\right\}_{a+b=m}$ are $m+1$ linearly independent elements.

We have a complete description of $J_{k,1}$ in terms of the Eisenstein series :

Theorem 1.1.15. *(i) The map from*

$$M_{k-4} \times M_{k-6} \longrightarrow J_{k,1}, \quad \text{given by} \quad (f,g) \mapsto fE_{4,1} + gE_{6,1}$$

is an isomorphism. In particular, $J_{k,1}$ is a free module over M_* of rank 2 with basis $E_{4,1}$ and $E_{6,1}$.
(ii) Moreover, the map $D_0 + D_2 \colon J_{k,1} \longrightarrow M_k + S_{k+2}$ is an isomorphism, where the operators D_0 and D_2 are operators defined in Section 1.1.3.2.

We mention the structure of $J_{k,1}^{cusp}$ in more detail, as it will be used in the corresponding statement about Hermitian Jacobi forms.

Definition 1.1.16.

$$\phi_{10,1} = \frac{1}{144}(E_6 E_{4,1} - E_4 E_{6,1}) \in J_{10,1}^{cusp}, \tag{1.1.17}$$
$$\phi_{12,1} = \frac{1}{144}(E_4^2 E_{4,1} - E_6 E_{6,1}) \in J_{12,1}^{cusp}. \tag{1.1.18}$$

Incidentally, the first two weights for which we have non-zero forms of index 1 are $10, 12$ respectively. The first few terms of the Taylor development of them forms $\phi_{10,1}$ and $\phi_{12,1}$ will be needed later, so we mention them :

$$\phi_{10,1}(\tau, z) = (2\pi i)^2 \Delta(\tau) z^2 + O(z^4), \tag{1.1.19}$$

$$\phi_{12,1}(\tau, z) = 12\Delta(\tau) + O(z^2). \tag{1.1.20}$$

Further, the map

$$M_{k-10} \times M_{k-12} \longrightarrow J_{k,1}^{cusp}, \quad \text{given by} \quad (f,g) \mapsto f\phi_{10,1} + g\phi_{12,1}$$

is an isomorphism.

1.1.3.2 Taylor development of Jacobi forms

This section reviews the construction of differential operators on the space of Jacobi forms from their Taylor development around the origin. We first define the D_ν operators and state their properties. Consider the Taylor expansion of a Jacobi form $\phi \in J_{k,m}$ around $z = 0$:

$$\phi(\tau, z) = \sum_{\nu \geq 0} \chi_\nu(\tau)\, z^\nu, \text{ here } \chi_\nu(\tau) \text{ are the Taylor coefficients.}$$

$$D_\nu \phi(\tau) := A_{k,\nu} \sum_{0 \leq \mu \leq \frac{\nu}{2}} \frac{(-2\pi i m)^\mu\,(k+\nu-\mu-2)!}{(k+2\nu-2)!\,\mu!}\, \chi^{(\mu)}_{\nu-2\mu}(\tau), \tag{1.1.21}$$

where $A_{k,\nu} := (2\pi i)^{-\nu} \frac{(k+2\nu-2)!\,(2\nu)!}{(k+\nu-2)!}$.

Example 1.1.1. For $\nu = 0$, ξ_0 is nothing but χ_0. For $\nu = 2$, $\xi_2 = \chi_2 - \frac{2\pi i m}{k}\, \chi_0^{(\prime)}$.

Proposition 1.1.17. *Let $k, \nu \geq 0$. Then, ξ_ν is a modular form of weight $k+\nu$ for the full modular group $SL(2,\mathbb{Z})$. If $\nu > 0$, it is a cusp form.*

The next proposition is basic in most of the arguments and computations regarding Jacobi forms of degree 1. It follows from the periodicity properties of Jacobi forms and the argument principle in complex analysis.

Proposition 1.1.18. *Let $\phi \in J_{k,m}$. Then for fixed $\tau \in \mathcal{H}$, the function $z \mapsto \phi(\tau, z)$, if not identically zero, has exactly $2m$ zeros (counting multiplicity) in any fundamental domain for the action of the lattice $\mathbb{Z}\tau + \mathbb{Z}$ on $\mathbb{C}$.*

Therefore, a Jacobi form of index m is determined by the first $2m$ of it's Taylor coefficients. This fact is used in :

Theorem 1.1.19. *Let $\phi \in J_{k,m}$. Let $\left(\bigoplus_{0 \leq \nu \leq 2m} D_\nu\right)\phi = \sum_{0 \leq \nu \leq 2m} D_\nu \phi$. Then,*

$$\bigoplus_{0 \leq \nu \leq 2m} D_\nu \colon J_{k,m} \longrightarrow M_k \bigoplus_{1 \leq \nu \leq 2m} S_{k+\nu} \tag{1.1.22}$$

is an injection. This also shows (in one way) that $\dim J_{k,m}$ *is finite.*

1.1.4 The Eichler-Zagier map

In this section we define the Eichler-Zagier map, which was mentioned in the Introduction and will be used in the index 1 case, to reduce a problem on Jacobi forms to half-integral modular forms. First we define the relevant spaces of half-integral weight modular forms needed for our purpose.

— Half-integral weight modular forms : For k be an odd integer, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$, we define the automorphy factor

$$j(\gamma, \tau) := \Theta(\gamma\tau)/\Theta(\tau) = \left(\frac{c}{d}\right) \epsilon_d^{-1} \sqrt{c\tau + d}, \tag{1.1.23}$$

where $\Theta(\tau) = \sum\limits_{n\in\mathbb{Z}} q^{n^2}$ is the classical theta function. Here $\left(\frac{c}{d}\right)$ is the generalized quadratic residue symbol and $\epsilon_\delta = \begin{cases} 1 & \text{if } \delta \equiv 1 \pmod 4 \\ i & \text{if } \delta \equiv 3 \pmod 4 \end{cases}$. For technical reasons, in this case one needs to work on a *four*-sheeted covering of $GL_2^+(\mathbb{Q})$.

Definition 1.1.20.

$$(i)\, G = \left\{ (\alpha, \varphi(\tau)) \mid \alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2^+(\mathbb{Q}),\, \varphi(\tau)^2 = t\frac{c\tau + d}{\sqrt{\det \alpha}}, \textit{ for some } t = \pm 1 \right\}. \tag{1.1.24}$$

$$(ii)\, \widetilde{\Gamma_0(4)} := \{(\gamma, j(\gamma, \tau)) \in G \mid \gamma \in \Gamma_0(4)\} \subset G. \tag{1.1.25}$$

We note that the maps

$$L\colon \gamma \mapsto \tilde{\gamma} := (\gamma, j(\gamma, \tau)), \quad \text{and } P\colon (\gamma, j(\gamma, \tau)) \mapsto \gamma \tag{1.1.26}$$

are mutually inverse maps from $\Gamma_0(4)$ to $\widetilde{\Gamma_0(4)}$ and vice versa. We define next the *'stroke operation'*, i.e., the action of G on $\mathbb{C}$-valued functions on $\mathcal{H}$.

Definition 1.1.21. *For $\xi = (\alpha, \varphi(\tau)) \in G$, and any integer k, define*

$$f\mid_{k/2} \xi(\tau) := f(\alpha\tau)\varphi(\tau)^{-k}$$

Now we come to the definition of half-integral weight modular forms. We note that since the group of translations belong to $\Gamma_0(4)$, we can expand a function invariant under $\Gamma_0(4)$ into a Fourier series in the variable q.

Definition 1.1.22. *Let k be an odd integer. A function $f\colon \mathcal{H} \to \mathbb{C}$ is called a modular form of weight $k/2$ for $\Gamma_0(4)$ if the following hold :*
(i) f is holomorphic and $f\mid_{k/2}\gamma := j(\gamma,\tau)^k f(\gamma\tau) = f(\tau)$, $\forall\gamma \in \Gamma_0(4)$.
(ii) f is holomorphic at all the cusps of $\Gamma_0(4)$; i.e., at ∞ the q-expansion of f should have no negative powers of q. At any other cusp s, writing $s = \xi\infty$, $g_\xi := f\mid_{k/2}\xi$, we have $g_\xi(\tau+1) = t^k g_\xi(\tau)$, $t^k = e(r)$, $(r = 0, \frac{1}{4}, \frac{1}{2},$ or $\frac{3}{4})$. Therefore one can expand $g_\xi(\tau) = e(r\tau)\sum_n a_n e(n\tau)$. It is required that $a_n = 0$ for $n < 0$ for f to be holomorphic at the cusp s.
(iii) If in the Fourier expansion of g_ξ in (ii) above, $a_0 = 0$, for all cusps s, we call f a cusp form. (It is known that this definition does not depend on the choice of ξ.)

We denote the space of modular forms of weight $k/2$ for $\Gamma_0(4)$ by $M_{k/2}$ and that of cusp forms by $S_{k/2}$.
— Kohnen's + space : W. Kohnen defined the '+' space in the theory of half-integral weight modular forms in order to define the theory of newforms and for which some linear combination of the Shimura correspondence between half-integer and integer weight modular forms is an isomorphism. They also arise as the image of the Eichler-Zagier map on Jacobi forms.

Definition 1.1.23.

$$M^+_{k/2} := M^+_{k/2}(\Gamma_0(4)) = \left\{ f \in M_{k/2} \mid f(\tau) = \sum_{n\colon (-1)^{\frac{k-1}{2}} n \equiv 0,1 \pmod 4} a_n q^n \right\}. \tag{1.1.27}$$

Remark 1.1.3. The theory of half-integral weight modular forms can be defined for congruence subgroups of $\Gamma_0(4)$, but we do not need them in this thesis.

— The Eichler-Zagier map : Now we can define the Eichler-Zagier map on Jacobi forms. We define it for index 1 forms, though it is easily defined in the same way for any index m. Let $\phi(\tau,z) = \sum\limits_{\substack{D>0, r\in\mathbb{Z} \\ D\equiv -r^2 \pmod 4}} c(D,r)e\left(\frac{D+r^2}{4}\tau + rz\right) \in J_{k,1}$. From [16, Theorem 2.2], we know that the Fourier coefficients $c(D,r)$ do not depend on r. Hence we can define the following :

Definition 1.1.24. $Z_1 \colon J_{k,1} \to M^+_{k/2}$,

$$\sum_{\substack{D>0, r\in\mathbb{Z} \\ D\equiv -r^2 \pmod 4}} c(D)e\left(\frac{D+r^2}{4}\tau + rz\right) \mapsto \sum_{0<D\in\mathbb{Z}} c(D)e(D\tau). \tag{1.1.28}$$

Theorem 1.1.25. *Let k be even. Then Z_1 is a Hecke-invariant isomorphism between $J_{k,1}$ and $M^+_{k/2}$.*

1.2 Hermitian Jacobi forms

The theory of Hermitian Jacobi forms, as developed by Klaus Haverkamp in his thesis [20], is akin to that of classical Jacobi forms; however there are some notable differences as well. In this section we define Hermitian Jacobi forms over the Gaussian field $\mathbb{Q}(i)$ and collect some of their properties which will be used in the later chapters.

Let $K = \mathbb{Q}(i)$ and $\mathcal{O}_K = \mathbb{Z}[i]$ be it's ring of integers. The Hermitian Jacobi group over $\mathcal{O}_\mathbf{K}$ is $\Gamma^J(\mathcal{O}_K) = \Gamma^1(\mathcal{O}_K) \ltimes \mathcal{O}_K^2$, where $\Gamma^1(\mathcal{O}_K) = \left\{\epsilon M \mid M \in SL(2,\mathbb{Z}), \epsilon \in \mathcal{O}_K^\times\right\}$, $\mathcal{O}_K^\times$ being the unit group of $\mathcal{O}_K$.

The Hermitian Jacobi group acts on $\mathcal{H} \times \mathbb{C}^2$ as well on functions from $\mathcal{H} \times \mathbb{C}^2$ to $\mathbb{C}$. We give the definitions below, which lead to the definition of Hermitian Jacobi forms. Let $(\epsilon M, [\lambda,\mu]) \in \Gamma^J(\mathcal{O}_K)$, where $M = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ in $SL(2,\mathbb{Z})$, $M\tau = \frac{a\tau+b}{c\tau+d}$, $\epsilon \in \mathcal{O}_K^\times$.

Definition 1.2.1.

$$(i) \quad \epsilon M(\tau, z_1, z_2) := \left(M\tau, \frac{\epsilon z_1}{c\tau+d}, \frac{\bar{\epsilon} z_2}{c\tau+d}\right) \tag{1.2.1}$$

$$(ii) \quad [\lambda,\mu](\tau, z_1, z_2) := (\tau, z_1 + \lambda\tau + \mu, z_2 + \bar{\lambda}\tau + \bar{\mu}), \tag{1.2.2}$$

where $N\colon K \to \mathbb{Q}$ *is the norm map.*

Now let $\phi\colon \mathcal{H} \times \mathbb{C}^2 \to \mathbb{C}$ be a function. Then we define the '*stroke-operations*' on functions with respect to the action of $\Gamma^J(\mathcal{O}_K)$.

Definition 1.2.2.

$$(i) \quad \phi|_{k,m}\epsilon M(\tau, z_1, z_2) := \epsilon^{-k}(c\tau+d)^{-k} e^{\frac{-2\pi i m c z_1 z_2}{c\tau+d}} \phi\left(M\tau, \frac{\epsilon z_1}{c\tau+d}, \frac{\bar{\epsilon} z_2}{c\tau+d}\right) \tag{1.2.3}$$

$$(ii) \quad \phi|_m[\lambda, \mu] := e^{2\pi i m(N(\lambda)\tau + \bar{\lambda} z_1 + \lambda z_2)} \phi(\tau, z_1 + \lambda\tau + \mu, z_2 + \bar{\lambda}\tau + \bar{\mu}) \tag{1.2.4}$$

Now, we define Hermitian Jacobi forms.

Definition 1.2.3. *The space of Hermitian Jacobi forms for* $\Gamma^J(\mathcal{O}_K)$ *of weight* k *and index* m*, where* k*,* m *are positive integers, consists of holomorphic functions* ϕ *on* $\mathcal{H} \times \mathbb{C}^2$ *satisfying :*

$$(i) \quad \phi(\tau, z_1, z_2) = \phi|_{k,m}\epsilon M(\tau, z_1, z_2) \text{ for all } M \in SL(2, \mathbb{Z}),\ \epsilon \in \mathcal{O}_K^{\times}. \tag{1.2.5}$$

$$(ii) \quad \phi(\tau, z_1, z_2) = \phi|_{k,m}[\lambda, \mu] \text{ for all } \lambda, \mu \in \mathcal{O}_K. \tag{1.2.6}$$

(iii) *Such a form has the following form of Fourier expansion :*

$$\phi(\tau, z_1, z_2) = \sum_{n=0}^{\infty} \sum_{\substack{r \in \mathcal{O}_K^{\sharp} \\ nm \geq N(r)}} c_\phi(n, r) e^{2\pi i(n\tau + r z_1 + \bar{r} z_2)}, \tag{1.2.7}$$

where $\mathcal{O}_K^{\sharp} = \frac{i}{2}\mathcal{O}_K$ *(the inverse different of* $K \mid \mathbb{Q}$*).*

The complex vector space of Hermitian Jacobi forms of weight k and index m is denoted by $J_{k,m}(\mathcal{O}_K)$. We say that ϕ is a Hermitian Jacobi *cusp* form if it is a Hermitian Jacobi form such that $c_\phi(n, r) = 0$ for $nm = N(r)$. The space of Jacobi cusp forms of weight k and index m is denoted as $J_{k,m}^{cusp}(\mathcal{O}_K)$.

In ([20], [21]) Haverkamp showed that $c_\phi(n, r)$ depends only on $r \pmod{m\mathcal{O}_K}$ and $D(n, r) = nm - N(r)$. Therefore if we define

$$c_s(L) := \begin{cases} c_\phi(n, r) & \text{if } r \equiv s \pmod{m\mathcal{O}_K} \text{ and } L = 4D(n, r) \\ 0 & \text{otherwise,} \end{cases}$$

where $s \in \mathcal{O}_K^\sharp/m\mathcal{O}_K$ and $L \in \mathbb{Z}$, we can rewrite the Fourier expansion of ϕ as follows (known as the `Theta decomposition` for Hermitian Jacobi forms) :

Proposition 1.2.4 (*Theta decomposition*).

$$\phi(\tau, z_1, z_2) = \sum_{s \in \mathcal{O}_K^\sharp/m\mathcal{O}_K} h_s(\tau) \cdot \theta_{m,s}^H(\tau, z_1, z_2), \tag{1.2.8}$$

where

$$h_s(\tau) := \sum_{\substack{L=0 \\ N(s)+L/4 \in m\mathbb{Z}}}^{\infty} c_s(L) e^{\frac{2\pi i L \tau}{4m}}, \tag{1.2.9}$$

$$\theta_{m,s}^H(\tau, z_1, z_2) := \sum_{r \equiv s (mod\, m\mathcal{O}_K)} e\left(\frac{N(r)}{m}\tau + rz_1 + \bar{r}z_2\right). \tag{1.2.10}$$

Further, if we let $\Theta_m^H(\tau, z_1, z_2) := \left(\theta_{m,s}^H(\tau, z_1, z_2)\right)_{s \in \mathcal{O}_K^\sharp/m\mathcal{O}_K} \in \mathbb{C}^{4m^2}$, then we have [21, Theorem 2]:

Theorem 1.2.5. *For* $g = (g_s)_{s \in \mathcal{O}_K^\sharp/m\mathcal{O}_K}, g_s : \mathcal{H} \to \mathbb{C}$ *holomorphic, the following are equivalent :*

(i) ${}^t\Theta_m^H g \in J_{k,m}(\mathcal{O}_K)$.

(ii) $\parallel g(\tau) \parallel$ *is bounded as* $\mathrm{Im}(\tau) \to \infty$ *and* $g|_{k-1}M = \overline{U_m(M)}g$ *for all* $M \in \Gamma_1(\mathcal{O}_K) = \left\{\epsilon M | \epsilon \in \mathcal{O}_K^\times, M \in SL(2, \mathbb{Z})\right\}$,

where $\Theta_m^H |_{1,m} M = U_m(M) \cdot \Theta_m^H$ *is it's functional equation and* $U_m : \Gamma_1(\mathcal{O}_K) \to U(4m^2)$ *is a homomorphism defined by it (*$U(n)$ *is the group of* $n \times n$ *unitary matrices).*

Remark 1.2.1. (i) In the above decomposition, h_s are called the `Theta components` of ϕ.

(ii) We have $h_s \in M_{k-1}(\Gamma(4m))$. This follows from Theorem 1.2.5(ii) and the fact that $\Gamma(4m) \subset Ker(U_m)$ (see [21],[20] for a proof).

Proposition 1.2.6 ([20]). *The Theta components of h_s of $\phi \in J_{k,m}(\mathcal{O}_K)$ have the following transformation properties under $SL(2,\mathbb{Z})$ and $\mathcal{O}_K^\times$:*

$$h_s \mid_{k-1} T = e^{-2\pi i N(s)/m} h_s, \tag{1.2.11}$$

$$h_s \mid_{k-1} S = \frac{i}{4m} \sum_{s' \in \mathcal{O}_K^\sharp / m\mathcal{O}_K} e^{-4\pi i \mathrm{Re}(\bar{s}s')/m} h_{s'}, \tag{1.2.12}$$

$$h_s \mid_{k-1} \epsilon I = \epsilon h_{\epsilon s}, \qquad \epsilon \in \mathcal{O}_K^\times, \tag{1.2.13}$$

where $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

Remark 1.2.2. If we let $\vec{h}_{k,m} := (h_s)_{s \in \mathcal{O}_K^\sharp / m\mathcal{O}_K}$, then $\vec{h}_{k,m}$ is a vector-valued modular form for $\Gamma(4m)$ of weight k with transformation formulas as above.

Corollary 1.2.7. *As an obvious corollary of Theorem 1.2.5, we have the following :*

(i) Let $\phi \in J_{k,m}(\mathcal{O}_K)$. Then the map which assigns $\phi \mapsto \vec{h}_{k,m}$ is an isomorphism of vector spaces between $J_{k,m}(\mathcal{O}_K)$ and the space of vector-valued modular forms of weight k for $\Gamma(4m)$.

(ii) From Remark 1.2.1, we conclude that $J_{k,m}(\mathcal{O}_K) \hookrightarrow M_{k-1}\left(\Gamma(4m)\right)^{4m^2}$. In particular, $\dim J_{k,m}(\mathcal{O}_K) \leq 4m^2 \dim M_{k-1}\left(\Gamma(4m)\right)$.

— Petersson inner product : We have a notion of Petersson inner product for Hermitian Jacobi forms, corresponding to an invariant volume element on $\mathcal{H} \times \mathbb{C} \times \mathbb{C}$ for Γ^J. Let $\tau = u + iv \in \mathcal{H}$, $z_1 = x_1 + ix_2 \in \mathbb{C}$, $z_2 = x_2 + iy_2 \in \mathbb{C}$. Then the invariant volume element on $\mathcal{H} \times \mathbb{C} \times \mathbb{C}$ for Γ^J is given by

$$dV^J(\mathcal{O}_K) := v^{-4} du dv dx_1 dy_1 dx_2 dy_2.$$

We now define the Petersson inner product :

Definition 1.2.8 (*Petersson inner product*). *Let $\phi, \psi \in J_{k,m}(\mathcal{O}_K)$ with at least one of them being a cusp form. Define*

$$\langle \phi, \psi \rangle_H := \int_{dV^J(\mathcal{O}_K) \backslash \mathcal{H} \times \mathbb{C} \times \mathbb{C}} \phi(\tau, z_1, z_2) \overline{\psi(\tau, z_1, z_2)} e^{-\pi m N(z_1 - \overline{z_2})} v^k dV^J(\mathcal{O}_K). \tag{1.2.14}$$

where, $N \colon \mathbb{C} \to \mathbb{R}_{\geq 0}$ is the norm map.

1.2.1 Hermitian Jacobi forms of index 1

Recently R. Sasaki in [36] has obtained a characterization of $J_{k,1}(\mathcal{O}_K)$ in terms of an analogue of Kohnen's plus space in the context of integer-weight modular forms for a congruence subgroup and also in terms of the `Maass` subspace of Hermitian Modular forms of degree 2.

As a set of representatives of $\mathcal{O}_K^\sharp$ in $\mathcal{O}_K^\sharp/\mathcal{O}_K$ ($\cong \frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}$) we take $\mathcal{S}_1 := \{0, \frac{i}{2}, \frac{1}{2}, \frac{1+i}{2}\}$. In this section we denote the corresponding Theta components by $h_{i,j}$ and the Hermitian Theta functions of index 1 by $\theta^H_{i,j}$, where $\{i,j\} \in \{0,1\}$. We make the following definition following [36] for convenience of notation.

Definition 1.2.9.

$$x = \sum_{n\in\mathbb{Z}} e\left(\frac{n^2\tau}{2}\right), \quad y = \sum_{n\in\mathbb{Z}} (-1)^n e\left(\frac{n^2\tau}{2}\right), \quad z = \sum_{t\in\frac{1}{2}+\mathbb{Z}} e\left(\frac{t^2\tau}{2}\right) \tag{1.2.15}$$

are the so called "Theta constants".

It is classical, that $x^4 = y^4 + z^4$ (see [31], for example). This relation, along with the relations of x, y, z with $\theta^H_{i,j}(\tau, 0, 0)$, will be used in Chapter 3.

In [36], the following forms of index 1 (arising from the Fourier-Jacobi expansion of Hermitian modular forms of degree 2) were defined via their Theta decompositions, in order to determine the structure of $J_{k,1}(\mathcal{O}_K)$:

Definition 1.2.10. *In the following,* $\Phi_{k,1} \in J_{k,1}(\mathcal{O}_K)$ *for* $k = 4, 8, 12, 16$.

$$\Phi_{4,1} = (x^6 + y^6)\theta^H_{1,0} + z^6(\theta^H_{1,\frac{1}{2}} + \theta^H_{1,\frac{i}{2}}) + (x^6 - y^6)\theta^H_{1,\frac{1+i}{2}}, \tag{1.2.16}$$

$$\Phi_{8,1} = (x^{14} + y^{14})\theta^H_{1,0} + z^{14}(\theta^H_{1,\frac{1}{2}} + \theta^H_{1,\frac{i}{2}}) + (x^{14} - y^{14})\theta^H_{1,\frac{1+i}{2}}, \tag{1.2.17}$$

$$\Phi_{12,1} = (x^{22} + y^{22})\theta^H_{1,0} + z^{22}(\theta^H_{1,\frac{1}{2}} + \theta^H_{1,\frac{i}{2}}) + (x^{22} - y^{22})\theta^H_{1,\frac{1+i}{2}}, \tag{1.2.18}$$

$$\Phi_{16,1} = (x^{30} + y^{30})\theta^H_{1,0} + z^{30}(\theta^H_{1,\frac{1}{2}} + \theta^H_{1,\frac{i}{2}}) + (x^{30} - y^{30})\theta^H_{1,\frac{1+i}{2}}. \tag{1.2.19}$$

Also we define several *cusp* forms of index 1 for determining the structue of $J^{cusp}_{k,1}(\mathcal{O}_K)$:

Definition 1.2.11. *In the following,* $\Psi_{k,1} \in J_{k,1}^{cusp}(\mathcal{O}_K)$ *for* $k = 8, 12, 16.$

$$\Psi_{8,1} = E_4\phi_{4,1} - \phi_{8,1}, \tag{1.2.20}$$

$$\Psi_{12,1} = E_4\phi_{8,1} - \phi_{12,1}, \tag{1.2.21}$$

$$\Psi_{16,1} = E_4\phi_{12,1} - \phi_{16,1}. \tag{1.2.22}$$

— Differential operators from Taylor expansion : We consider the power series expansion of ϕ around $z_1 = z_2 = 0$ from the Fourier expansion (1.2.7) :

$$\phi(\tau, z_1, z_2) = \sum_{\alpha\geq 0, \beta\geq 0} \chi_{\alpha,\beta}(\tau) z_1^\alpha z_2^\beta. \tag{1.2.23}$$

Further from the Taylor expansion of Hermitian Jacobi forms, one can define the $D_\nu(\mathcal{O}_K)$ operators in the same way as for the case of Jacobi forms (see [13], [36]). Let $\phi(\tau, z_1, z_2) = \sum_{\alpha,\beta\geq 0} \chi_{\alpha,\beta}(\tau) z_1^\alpha z_2^\beta \in J_{k,1}(\mathcal{O}_K)$ be the Taylor expansion of ϕ around $z_1 = z_2 = 0$. Let $\xi_{1,1} := D_1(\mathcal{O}_K)\phi$, $\xi_{2,2} := D_2(\mathcal{O}_K)\phi$. Then

$$\xi_{1,1} := \chi_{1,1} - \frac{2\pi i}{k}\chi'_{0,0}, \quad \xi_{2,2} := \chi_{2,2} - \frac{2\pi i}{k+2}\chi'_{1,1} + \frac{(2\pi i)^2}{2(k+1)(k+2)}\chi''_{0,0} \tag{1.2.24}$$

define linear maps from $J_{k,1}(\mathcal{O}_K)$ to S_{k+2} and from $J_{k,1}(\mathcal{O}_K)$ to S_{k+4} respectively.

It is easy to see that $J_{k,1}(\mathcal{O}_K) = 0$, unless k is even. When $k \equiv 0 \pmod 4$ we have the following Theorem (cf. [36]) :

Theorem 1.2.12. *Let* $k \equiv 0 \pmod 4$. *Then the map*

$$\xi\colon J_{k,1}(\mathcal{O}_K) \to M_k \oplus S_{k+2} \oplus S_{k+4}, \qquad \phi \mapsto \chi_{0,0} + \xi_{1,1} + \xi_{2,2} - 6(\chi_{4,0} + \chi_{0,4}) \tag{1.2.25}$$

is an isomorphism; and the ***structure of index*** 1 ***forms*** *is given by the map*

$$\eta\colon M_{k-4} \oplus M_{k-8} \oplus M_{k-12} \to J_{k,1}(\mathcal{O}_K), \qquad (f, g, h) \mapsto f\phi_{4,1} + g\phi_{8,1} + h\phi_{12,1} \tag{1.2.26}$$

In particular, $\dim J_{k,1}(\mathcal{O}_K) = \frac{k}{4}$.

Remark 1.2.3. The structure result for $k \equiv 2 \pmod 4$ was done in [36], but it will be proved in Chapter 3 (see Corollary 3.2.3).

1.2.2 Hecke Operators and the dimension formula

We will need the Trace formula for Hecke operators in Chapter 3 to find the dimension of $J_{k,1}(\mathcal{O}_K)$. We first define the Hecke operators for Hermitian Jacobi forms :

Definition 1.2.13. *Let $l \in \mathbb{N}$, $\rho \in \mathcal{O}_K$ and $\phi \in J_{k,m}(\mathcal{O}_K)$. We have,*

$$(i)\, \phi \mid U_\rho(\tau, z_1, z_2) := \phi(\tau, \rho z_1, \bar{\rho} z_2) \in J_{k,N(\rho)m}(\mathcal{O}_K). \tag{1.2.27}$$

$$(ii)\, \phi \mid V_l := l^{\frac{l}{2}-1} \sum_{\substack{M:\, SL(2,\mathbb{Z})\backslash M(2,\mathbb{Z}) \\ \det M = l}} \phi \mid_{k,m} M \mid U_{\sqrt{l}} \in J_{k,ml}(\mathcal{O}_K). \tag{1.2.28}$$

$$(iii)\, \phi \mid T_l := l^{k-4} \sum_{\substack{M:\, SL(2,\mathbb{Z})\backslash M(2,\mathbb{Z}) \\ \det M = l^2}} \sum_{X \in \mathcal{O}_K / l\mathcal{O}_K} \phi \mid_{k,m} M \mid_m X \in J_{k,m}(\mathcal{O}_K). \tag{1.2.29}$$

The detailed theory of the properties of the Hecke operators and relations among them in the Hecke algebra can be found in [20]. We will use the U_ρ operator in Chapters 2 and 3 mainly to change the indices of the Hermitian Jacobi forms. The trace formula for T_l on the space $J^{cusp}_{k,m}(\mathcal{O}_K)$ is given in [20, Satz 8.10, p. 90]. We need the case $l = 1$ to read off the dimension of $J^{cusp}_{k,m}(\mathcal{O}_K)$.

— Dimension formula for cusp forms :

Theorem 1.2.14. *Let $k > 4$. Then,* $\dim J^{cusp}_{k,m}(\mathcal{O}_K) =$

$$\begin{aligned} &\frac{1}{8m}\Bigg(\frac{1}{2}\mathrm{Re}(i^k G_{-4}{}^\sharp(2,m)) - \frac{2}{3\sqrt{3}}\mathrm{Re}(\zeta^{k-2} G_{-4}{}^\sharp(1,m)) + \zeta^{2k+2} G_{-4}{}^\sharp(3,m) \\ &\qquad + (1+(-1)^k)m^2 - \frac{4}{3\sqrt{3}}\mathrm{Re}(i^k)\mathrm{Re}(\zeta^{2-k} G_{-4}{}^\sharp(1,m))\Bigg) \\ &+ \frac{k-2}{48}\left(4m^2 + 4((-1)^k + \mathrm{Re}(i^k)\right) \\ &- \frac{1}{8}\sum_{\epsilon \in \mathcal{O}_K{}^\times} \epsilon^{-k}\left(A_{ord(\epsilon)}(m) - \frac{1}{4m}\sum_{j=1}^{4m-1}(a_{m,1,4,\epsilon}(j) - a_{m,1,4,\epsilon}(-j))j\right), \end{aligned} \tag{1.2.30}$$

where $\zeta = \frac{1}{2}(1+\sqrt{3})$, $\epsilon \in {\mathcal{O}_K}^{\times}$ *and*

$$G_{-4}{}^{\sharp}(j,m) = \sum_{s \in {\mathcal{O}_K}^{\sharp}/m\mathcal{O}_K} e(jN(s)/m), \tag{1.2.31}$$

$$a_{m,1,4,\epsilon}(j) = \mathit{Card.}\left\{s \in {\mathcal{O}_K}^{\sharp}/m\mathcal{O}_K \mid s \equiv \epsilon s \pmod{m\mathcal{O}_K}, j \equiv 4N(s) \pmod{4m}\right\}, \tag{1.2.32}$$

$$A_{ord(\epsilon)} := a_{m,1,4,\epsilon}(0). \tag{1.2.33}$$

We will mainly need Theorem 1.2.14 for $m = 2$ and in that case, after a calculation we find,

$$G_{-4}{}^{\sharp}(1,2) = 4i, \quad G_{-4}{}^{\sharp}(2,2) = 8i, \quad G_{-4}{}^{\sharp}(3,2) = -4i.$$

$$\sum_{j=1}^{7}(a_{2,1,4,\epsilon}(j) - a_{2,1,4,\epsilon}(-j))j = \begin{cases} -32 & \text{if } \epsilon = 1 \\ 0 & \text{if } \epsilon = -1, \pm i \end{cases}.$$

From this, we easily conclude that for $k > 4$,

$$\dim J^{cusp}_{k,2}(\mathcal{O}_K) = \begin{cases} \frac{k-4}{2} & \text{if } k \equiv 0 \pmod 4 \\ \frac{k-1}{3} & \text{if } k \equiv 0 \pmod 4 \\ \frac{k-5}{4} & \text{if } k \equiv 1 \pmod 4 \\ \frac{k-3}{4} & \text{if } k \equiv 3 \pmod 4 \end{cases}. \tag{1.2.34}$$

— Dimension formula for the space of Eisenstein series : Let $k > 4$ and $s \in \mathcal{O}_K$. Then for each s such that $m \mid N(s)$, one can construct an Eisenstein series :

Definition 1.2.15. *(i) Let* $\tau \in \mathcal{H}, z_1, z_2 \in \mathbb{C}$, *define*

$$E^H_{k,m,s}(\tau, z) := \sum_{\gamma \in \Gamma^J_\infty(\mathcal{O}_K)\backslash \Gamma^J(\mathcal{O}_K)} e\left(\frac{N(s)}{m}\tau + sz_1 + \bar{s}z_2\right) |_{k,m}\, \gamma \tag{1.2.35}$$

where, $\Gamma^J_\infty(\mathcal{O}_K) := \left(\left(\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right), (0,\mu)\right), n \in \mathbb{Z}, \mu \in \mathcal{O}_K\right)$, *the stabilizer group of the function* $e\left(\frac{N(s)}{m}\tau + sz_1 + \bar{s}z_2\right)$ *under the action of* $\Gamma^J(\mathcal{O}_K)$.

(ii) The space spanned by $E^H_{k,m,s}$, *(*$s \in \mathbb{Z}$*) is denoted as* $J_{k,m}(\mathcal{O}_K)^{Eis}$.

It is easy to see that $E^{H}_{k,m,s} \in J_{k,m}(\mathcal{O}_K)$. For $k > 4$ from [20, Satz 2.5, p.25] we find the following result :

Proposition 1.2.16.

$$\dim J^{Eis}_{k,m}(\mathcal{O}_K) = \frac{1}{4}\sum_{j|4} \kappa_j(k) A_j(m), \tag{1.2.36}$$

where, $A_j(m) = Card.\{s \in \mathcal{O}_K/m\mathcal{O}_K \mid s \equiv e(1/j)s \pmod{m\mathcal{O}_K}\}$, $\kappa_1(k) = 1$, $\kappa_2(k) = (-1)^k$, $\kappa_4(k) = 2\cos\frac{k\pi}{2}$.

For $m = 2$, the above quantities are easy to calculate, and we find that

$$\dim J^{Eis}_{k,2}(\mathcal{O}_K) = \begin{cases} 2 & \text{if } k \equiv 0 \pmod 4 \\ 0 & \text{otherwise} \end{cases}.$$

Since $J_{k,m}(\mathcal{O}_K) = J^{cusp}_{k,m}(\mathcal{O}_K) \oplus J^{Eis}_{k,m}(\mathcal{O}_K)$, we get an explicit dimension formula for index 2 forms.

Chapter 2

Some aspects of Hermitian Jacobi forms

2.1 Introduction

In the theory of Jacobi forms, one of the use of differential operators has been to produce new Jacobi forms from a given one or to construct other classes of modular forms, eg. elliptic modular forms. Hermitian Jacobi forms has been introduced by Klaus Haverkamp in [21], [20]. He defined and studied Hecke operators on $J_{k,m}^{cusp}(\mathcal{O}_K)$ and obtained a trace formula for them. In this chapter we introduce a certain differential operator $D_\nu, \nu \in \mathbb{N}$ (see Proposition 2.2.5 for the precise definition) on the space of Hermitian Jacobi forms of weight k and index m (denoted $J_{k,m}(\mathcal{O}_K)$) of degree 1 for the Hermitian Jacobi group over the ring of integers of the imaginary quadratic field $\mathbb{Q}(i)$ (see Section 2.2) to construct modular forms for $SL(2, \mathbb{Z})$. This is the analogue of the heat operator defined and studied for classical Jacobi forms by M. Eichler and D. Zagier in [16] in the sense that we also use the Taylor expansion of such a Hermitian Jacobi form around the origin for the construction. However, here there are three variables, and so we restrict to the "diagonal part" of such a form to define the differential operator (see 2.2.1 for details).

Our differential operator is different from the corresponding object defined by Y. Choie et. al in [12].

Remark 2.1.1. For convenience of notation, in this chapter we use the notation D_ν for the operators constructed from the Taylor expansion of Hermitian Jacobi forms instead of a more suggestive $D_\nu(\mathcal{O}_K)$, unless there is any confusion with the corresponding operators occuring in the theory of classical Jacobi forms.

As we have the notion of Petersson inner product on the space of Hermitian Jacobi cusp forms (as defined in Chapter 1), this makes it into a Hilbert space. From generalities, the Poincaré series span the space and so it is enough to define the effect of D_ν on a Poincaré series. We compute the Fourier expansion of the adjoint of D_ν in Section 2.3. This is done by expressing the image of a Hermitian Poincaré series (defined in Section 2.3.2) under D_ν as an infinite sum of elliptic Poincaré series for $SL(2,\mathbb{Z})$. The calculations follow those of Y. Tokuno in [42]. We quote one of the main results below, which allows us to construct Hermitian Jacobi forms from the Fourier development of elliptic modular forms in an explicit manner:

Let $f \in S_{k+2\nu}$ and $(,)$ be the Petersson inner product on $S_{k+2\nu}$. Let $\langle , \rangle$ be the Petersson inner product on $J_{k,m}^{cusp}(\mathcal{O}_K)$ and $D_\nu^* : S_{k+2\nu} \longrightarrow J_{k,m}^{cusp}(\mathcal{O}_K)$ be the adjoint of D_ν with respect to the above inner products.

Theorem 2.1.1. *With the above notations the Fourier development of $D_\nu^* f$ is given by*

$$D_\nu^* f\,(\tau, z_1, z_2) = \sum_{n=0}^{\infty} \sum_{\substack{r \in \mathcal{O}_K^\sharp \\ nm \geq N(r)}} c_{D_\nu^* f}(n,r) e^{2\pi i (n\tau + r z_1 + \bar{r} z_2)} \qquad \text{where}$$

$$\begin{aligned} c_{D_\nu^* f}(n,r) = & \frac{\nu! (-1)^\nu (4\pi)^{2\nu-1} \Gamma(k+2\nu-1) m^{\nu-k+3} \left(nm - N(r)\right)^{k-2}}{\Gamma(k-2)(k-1)^{(\nu)}} \\ & \times \sum_{\lambda \in \mathfrak{O}_K} \frac{a\left(mN(\lambda) + r\lambda + \bar{r}\bar{\lambda} + n, f\right)}{\left(mN(\lambda) + r\lambda + \bar{r}\bar{\lambda} + n\right)^{k+\nu-1}} \\ & \times \sum_{j=0}^{\nu} \frac{(-1)^j (k-1)^{(2\nu-j)}}{(\nu-j)!^2 j!} \left(\frac{N(m\lambda + \bar{r})}{m\left(mN(\lambda) + r\lambda + \bar{r}\bar{\lambda} + n\right)} \right)^{\nu-j}, \end{aligned} \tag{2.1.1}$$

where $f(\tau) = \sum_{n=1}^{\infty} a\,(n,f)\, e^{2\pi i n \tau}$.

We remark that the operators D_ν are not Hecke equivariant, and so the above does not give a homomorphism of the corresponding spaces as a module over the corresponding Hecke algebras.

In [38], N. P. Skoruppa proved that $J_{1,m} = \{0\}\,, \forall\, m \geq 1$. The proof uses the Theta correspondence for classical Jacobi forms and a result on Serre and Stark on modular forms of weight 1/2, since the Theta components are half-integral weight modular forms for certain congruence subgroups of $SL(2,\mathbb{Z})$. We prove the analogous result in the case of Hermitian Jacobi forms, viz., $J_{1,m}(\mathcal{O}_K)$ for all $m \geq 1$ in Lemma 2.3.1, which follows from the corresponding result of N. P. Skoruppa mentioned above.

Further, we define a map from the tensor product of two copies of classical Jacobi forms (denoted $J_{k,m}$) to $J_{k,m}(\mathcal{O}_K)$ and construct Hermitian Jacobi forms from those of smaller weights and indices using partial derivaties with respect to the variables z_1 and z_2. We illustrate the results in the propositions cited below. The proofs are given in section 2.3.4. Fix $k_1, k_2 \in \mathbb{N}$. Let $\phi_j \in J_{k_j,m}$, $j \in \{1,2\}$. Define

$$H\,(\phi_1,\phi_2)\,(\tau,z_1,z_2) = \sum_{\epsilon\in\mathcal{O}_K^\times} \phi_1\left(\tau, \frac{1}{2}(z_1+z_2)\right)\phi_2\left(\tau, \frac{i}{2}(z_1-z_2)\right) |_k\, \epsilon I.$$

Corollary 2.1.2. *Then* $H\,(\phi_1,\phi_2) \in J_{k_1+k_2,m}(\mathcal{O}_K)$. *If* ϕ_i *are cusp forms, then so is* $H\,(\phi_1,\phi_2)$.

Proposition 2.1.3. Let ϕ and ψ *be Hermitian Jacobi forms of weights* k_1 *and* k_2 *and index* m_1 *and* m_2 *respectively. Then*

(*i*) $m_1\phi\psi_{(2)} - m_2\psi\phi_{(2)}$ *is a Hermitian Jacobi form of weight* k_1+k_2+1 and index m_1+m_2.

(*ii*) $$\left(m_1\phi\psi_{(1)} - m_2\psi\phi_{(1)}\right)^2 + m_1\phi^2\left(\psi_{(1)}^2 - \psi\psi_{(1,1)}\right) + m_2\psi^2\left(\phi_{(1)}^2 - \phi\phi_{(1,1)}\right)$$

is a Hermitian Jacobi form of weight $2(k_1+k_2+1)$ *and index* $2(m_1+m_2)$.

(In the above, $\phi_{(j)} := \frac{\partial}{\partial z_j}\phi$ for $j = 1, 2$ and $\phi_{(r,s)} = \frac{\partial^2}{\partial z_s \partial z_r}\phi$ for $r, s = 1, 2$)

Also, in analogy with Jacobi forms, D_ν commutes with V_l ($l \in \mathbb{N}$) operators in a certain sense (see Section 2.4). This is useful in computing the first few Taylor coefficients of $V_l\phi$ from those of ϕ. In Section 2.5, using the Theta Correspondence between $J_{k,m}(\mathcal{O}_K)$ and modular forms on congruence subgroups of $SL(2, \mathbb{Z})$, we compute the number of Fourier coefficients that determine $J_{k,m}(\mathcal{O}_K)$ (see Proposition 2.5.3 in section 2.5). This is a Sturm type result for Hermitian Jacobi forms.

Proposition 2.1.4. In the Fourier expansion (1.2.7) of a Hermitian Jacobi form ϕ, *suppose that* $c_\phi(n, r) = 0$ for $0 \leq n \leq \kappa(k, m)$. *Then* $\phi \equiv 0$; *i.e.,* ϕ *"is determined" by the first* $R(4m\,\kappa(k, m))$ *of it's Fourier coefficients, where for* $4m|l$

$$R(l) := R_m(l) = \sum_{0 \leq n \leq \frac{l}{4m}} \sum_{0 \leq d \leq 4mn} r(d),$$

$$\kappa(k, m) = \left[\frac{4m^2(k-1)}{3} \prod_{p|4m} \left(1 - \frac{1}{p^2}\right) + \frac{m}{2} \right].$$

Using this, an embedding of a certain subspace $J^{Spez}_{k,m}(\mathcal{O}_K)$ (see Definition 2.5.4) of $J_{k,m}(\mathcal{O}_K)$ into a finite direct sum of spaces of modular forms for $SL(2, \mathbb{Z})$ using the D_ν maps is obtained (following [16]). An analogous embedding of $J_{k,m}(\mathcal{O}_K)$ is desirable, but one way to do that would be to prove the Hermitian Theta-Wronskian to be nowhere vanishing on the upper half plane. Such an embedding will of course play an important role as in the case of classical Jacobi forms. To the knowledge of the author, unfortunately we do not have this at present.

2.2 A non-holomorphic differential operator on $J_{k,m}(\mathcal{O}_K)$

Definition 2.2.1. *Let* $\phi_0 := \sum_{\nu \geq 0} \chi_{\nu,\nu}(\tau)\,(z_1 z_2)^\nu$ *be the 'diagonal part' of* $\phi \in J_{k,m}(\mathcal{O}_K)$. *We denote the vector space of 'diagonal parts' arising from* $J_{k,m}(\mathcal{O}_K)$ *by* $J^0_{k,m}(\mathcal{O}_K)$, *i.e.,*

$J^0_{k,m}(\mathcal{O}_K) := \{\phi_0 \mid \phi \in J_{k,m}(\mathcal{O}_K)\}$. *We define the operator*

$$L_{k,m} := 8\pi im\frac{\partial}{\partial\tau} - \frac{(2k-2)}{z_1}\frac{\partial}{\partial z_2} - \frac{(2k-2)}{z_2}\frac{\partial}{\partial z_1} - 4\frac{\partial^2}{\partial z_1\partial z_2}.$$

Lemma 2.2.2. *Let ϕ be a holomorphic function on $\mathcal{H}\times\mathbb{C}^2$. Then,*

$$L_{k,m}(\phi|_{k,m}M) = (L_{k,m}\phi)|_{k+2,m}M, \text{ where } M \in SL(2,\mathbb{R}). \tag{2.2.1}$$

Proof. The proof is by direct verification. We include the details for completeness. Let $M = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in SL(2,\mathbb{Z})$. For convenience, let $L := L_{k,m}$ and $\xi := \left(M\tau, \frac{z_1}{c\tau+d}, \frac{z_2}{c\tau+d}\right)$. Further, let $\phi_\tau := \frac{\partial\phi}{\partial\tau}$, $\phi_1 := \frac{\partial\phi}{\partial z_1}$, $\phi_2 := \frac{\partial\phi}{\partial z_2}$, $\phi_{1,2} := (\phi_1)_2$. Then we have

$$\begin{aligned}(L\phi)|_{k+2,m}M &= (c\tau+d)^{-(k+2)}e\left(\frac{-2\pi imcz_1z_2}{c\tau+d}\right)L\phi(\xi)\\ &= (c\tau+d)^{-(k+2)}e\left(\frac{-2\pi imcz_1z_2}{c\tau+d}\right)\\ &\quad\times\left(8\pi im\phi_\tau(\xi) - \frac{(2k-2)}{(c\tau+d)z_1}\phi_2(\xi) - \frac{(2k-2)}{(c\tau+d)z_2}\phi_1(\xi) - 4\phi_{1,2}(\xi)\right).\end{aligned} \tag{2.2.2}$$

On the other hand, after some calculation we find:

$$\begin{aligned}(\phi\ |_{k,m}\ M)_\tau &= (c\tau+d)^{-(k+2)}e\left(\frac{-2\pi imcz_1z_2}{c\tau+d}\right) \quad (2.2.3)\\ &\quad\times\left(\phi_\tau(\xi) - cz_1\phi_1(\xi) - cz_2\phi_2(\xi) - kc(c\tau+d)\phi(\xi) + 2\pi imc^2z_1z_2\phi(\xi)\right).\\ (\phi|_{k+2,m}M)_1 &= (c\tau+d)^{-(k+1)}e\left(\frac{-2\pi imcz_1z_2}{c\tau+d}\right)\left(\phi_1(\xi) - 2\pi imcz_2\phi(\xi)\right). \quad (2.2.4)\\ (\phi|_{k+2,m}M)_2 &= (c\tau+d)^{-(k+1)}e\left(\frac{-2\pi imcz_1z_2}{c\tau+d}\right)\left(\phi_2(\xi) - 2\pi imcz_1\phi(\xi)\right). \quad (2.2.5)\\ (\phi|_{k+2,m}M)_{1,2} &= (c\tau+d)^{-(k+2)}e\left(\frac{-2\pi imcz_1z_2}{c\tau+d}\right)\left(\phi_{1,2}(\xi) - 2\pi imcz_1\phi_1(\xi) - \right. \quad (2.2.6)\\ &\quad\left. - 2\pi imcz_2(\phi_2(\xi) - 2\pi imcz_1\phi(\xi)) - 2\pi imc(c\tau+d)\phi(\xi)\right).\end{aligned}$$

Combining equations (2.2.3), (2.2.4), (2.2.5) and (2.2.6), we calculate $L(\phi\ |_{k,m}\ \phi)$. It is easily seen that the result is the same as the expression of $(L\phi)|_{k+2,m}M$ in equation (2.2.2). This proves the lemma. ☐

Lemma 2.2.3. *Consider the power series expansion of $\phi \in J_{k,m}(\mathcal{O}_K)$ as in (1.2.23). Then the following are equivalent:*

$$(i) \quad \phi|_{k,m}M = \phi$$

$$(ii) \quad \chi_{\alpha,\beta}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{k+\alpha+\beta}\sum_{\alpha\geq\nu,\beta\geq\nu}\frac{1}{\nu!}\left(\frac{2\pi imc}{c\tau+d}\right)^{\nu}\chi_{\alpha-\nu,\beta-\nu}(\tau),$$

where $M = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in SL(2,\mathbb{Z})$.

Proof. $\phi|_{k,m}M = \phi$

$$\Leftrightarrow (c\tau+d)^{-k}e^{\frac{-2\pi imcz_1z_2}{c\tau+d}}\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z_1}{c\tau+d}, \frac{z_2}{c\tau+d}\right) = \phi(\tau, z_1, z_2)$$

$$\Leftrightarrow \sum_{\alpha\geq 0,\beta\geq 0}\chi_{\alpha,\beta}\left(\frac{a\tau+b}{c\tau+d}\right)z_1^{\alpha}z_2^{\beta} = (c\tau+d)^{k+\alpha+\beta}\left(\sum_{\nu\geq 0}\frac{1}{\nu!}\left(\frac{2\pi imc}{c\tau+d}\right)^{\nu}(z_1z_2)^{\nu}\right) \times \left(\sum_{\alpha,\beta}\chi_{\alpha,\beta}(\tau)z_1^{\alpha}z_2^{\beta}\right)$$

$$\Leftrightarrow \chi_{\alpha,\beta}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{k+\alpha+\beta}\sum_{\alpha\geq\nu,\beta\geq\nu}\frac{1}{\nu!}\left(\frac{2\pi imc}{c\tau+d}\right)^{\nu}\chi_{\alpha-\nu,\beta-\nu}(\tau).$$

□

Remark 2.2.1. The diagonal part $\phi_0 := \sum_{\nu\geq 0,\nu\geq 0}\chi_{\nu,\nu}(\tau)(z_1z_2)^{\nu}$ satisfies $\phi_0|_{k,m}M = \phi_0$, for all $\phi \in J_{k,m}(\mathcal{O}_K)$. This follows from the above Proposition by replacing ϕ by it's diagonal part ϕ_0 and retracing the proof from the last line.

We denote the space of holomorphic functions on $\mathcal{H}\times\mathbb{C}^2$ satisfying the conditions of Lemma 2.2.3 (i.e., invariant under the action of $SL(2,\mathbb{Z})$) by $J^0_{k,m}$, so $J^0_{k,m}(\mathcal{O}_K) \subset J^0_{k,m}$ from the above Remark. From the transformation (1.2.5) we get that $\chi_{\alpha,\beta} = \epsilon^{k-\alpha+\beta}\chi_{\alpha,\beta}$ ($\epsilon \in \mathcal{O}_K^{\times}$). Hence $\chi_{\alpha,\alpha} \neq 0$ only when $k \equiv 0 \pmod 4$. So from now on we assume $k \equiv 0 \pmod 4$. We define the non-holomorphic differential operators next (for $k \equiv 0 \pmod 4$), but they can be defined for other congruence classes of k as well. See Remark 2.2.3 at the end of this section.

Definition 2.2.4. *For each $\nu \geq 0$, we denote by $\tilde{D}_\nu$ the composite map :*

$$\tilde{D}_\nu : J^0_{k,m} \xrightarrow{L_{k,m}} \cdots \xrightarrow{L_{k+2\nu-2,m}} J^0_{k+2\nu,m}.$$

Clearly $\left(\tilde{D}_\nu \phi\right)|_{k+2\nu,m} M = \tilde{D}_\nu\left(\phi|_{k,m} M\right) = \tilde{D}_\nu(\phi)$ for all $\phi \in J^0_{k,m}$ and $M \in SL(2,\mathbb{Z})$. Let $\pi_{k,m} : J_{k,m}(\mathcal{O}_K) \longrightarrow J^0_{k,m}$ be the projection $\phi \mapsto \phi_0$.

Proposition 2.2.5. *The composite map $D_\nu \phi := \tilde{D}_\nu \circ \pi_{k,m}\phi\,(\tau, z_1, z_2)\,|_{z_1=0,z_2=0}$ defines a linear map from $J_{k,m}(\mathcal{O}_K)$ to M_k for $\nu = 0$ and to $S_{k+2\nu}$ for $\nu \geq 1$.*

Proof. By Remark 2.2.1, $\chi_{0,0}(\tau)$ is a modular form for $SL(2,\mathbb{Z})$. Since $\tilde{D}_\nu$ are invariant under the action of $SL(2,\mathbb{Z})$, we get the proposition. The assertion about cusp forms when $\nu \geq 1$ is trivial. □

Proposition 2.2.6. *With the notation of definition (2.2.4), we have the expansion :*

$$\tilde{D}_\nu \phi = \sum_{\alpha=0}^{\infty} \sum_{\mu=0}^{\nu} (-4)^{\nu-\mu} (8\pi i m)^\mu \binom{\nu}{\mu} \frac{(\alpha+\nu-\mu)!}{\alpha!} \frac{(\alpha+k+2\nu-\mu-2)!}{(\alpha+k+2\nu-2)!} \times$$
$$\times\, \chi^{(\mu)}_{\alpha+\nu-\mu,\alpha+\nu-\mu}(\tau)\,(z_1 z_2)^\alpha, \qquad (2.2.7)$$

where $g^{(\nu)}(\tau) = \left(\frac{\partial}{\partial\tau}\right)^\nu g(\tau)$.

Proof. This is easily checked by induction. For $\nu = 0, 1$ it is obvious. For $\nu > 1$, we have by definition,

$$\begin{aligned} L_{k,m}\phi &= \sum_{\alpha\geq 0} \left(8\pi i m \frac{\partial}{\partial\tau}\chi_{\alpha,\alpha} - 4(k-1)(\alpha+1)\chi_{\alpha+1,\alpha+1} - 4(\alpha+1)^2\chi_{\alpha+1,\alpha+1}\right)(z_1 z_2)^\alpha \\ &= \sum_{\alpha\geq 0} \left(8\pi i m \frac{\partial}{\partial\tau}\chi_{\alpha,\alpha} - 4(\alpha+1)(k+\alpha)\chi_{\alpha+1,\alpha+1}\right)(z_1 z_2)^\alpha. \end{aligned}$$

Using the fact that, $\tilde{D}_{\nu+1} = L_{k+2\nu} \circ \tilde{D}_\nu$, the $(\alpha,\alpha)^{th}$ coefficient of $\tilde{D}_{\nu+1}\phi$

$$
\begin{aligned}
&= 8\pi i m \frac{\partial}{\partial \tau}\tilde{\chi}_{\alpha,\alpha} - 4(\alpha+1)(k+2\nu+\alpha)\tilde{\chi}_{\alpha+1,\alpha+1} \left(\text{where} \tilde{\chi}_{\alpha,\alpha} = (\alpha,\alpha)^{th}\text{coeff. of} \tilde{D}_\nu \phi\right) \\
&= \sum_{\mu=0}^{\nu} (-4)^{\nu-\mu} (8\pi i m)^{\mu} \binom{\nu}{\mu} \frac{(\alpha+\nu-\mu)!}{\alpha!} \frac{(k+\alpha+2\nu-\mu-2)!}{(\alpha+k+\nu-2)!} \chi^{(\mu+1)}_{\alpha+\nu-\mu} - 4(k+2\nu+\alpha) \\
&\times (\alpha+1) \left(\sum_{\mu=0}^{\nu} (-4)^{\nu-\mu} (8\pi i m)^{\mu} \binom{\nu}{\mu} \frac{(\alpha+\nu+1-\mu)!}{(\alpha+1)!} \frac{(k+\alpha+2\nu-\mu-1)!}{(\alpha+k+\nu-2)!} \chi^{(\mu)}_{\alpha+\nu-\mu+1} \right) \\
&= (8\pi i m)^{\mu} \chi^{(\mu+1)}_{\alpha,\alpha} - (\alpha+1)(k+2\nu+\alpha)(-4)^{\nu+1} \frac{(\alpha+\nu+1)!}{(\alpha+1)!} \frac{(k+\alpha+2\nu-1)!}{(\alpha+k+\nu-2)!} \chi^{(\mu)}_{\alpha+1+\nu} \\
&+ \sum_{\mu=1}^{\nu} (8\pi i m)^{\mu} (-4)^{\nu-\mu+1} \left\{ \binom{\nu}{\mu-1} \frac{(\alpha+\nu-\mu+1)!}{\alpha!} \frac{(k+\alpha+2\nu-\mu-1)!}{(\alpha+k+\nu-2)!} \chi^{(\mu)}_{\alpha+\nu-\mu+1} \right. \\
&\left. + (k+2\nu+\alpha) \binom{\nu}{\mu} \frac{(\alpha+\nu-\mu+1)!}{\alpha!} \frac{(k+\alpha+2\nu-\mu-1)!}{(\alpha+k+\nu-1)!} \chi^{(\mu)}_{\alpha+\nu-\mu+1} \right\} \\
&= \sum_{\mu=0}^{\nu+1} (-4)^{\nu+1-\mu} (8\pi i m)^{\mu} \binom{\nu+1}{\mu} \frac{(\alpha+\nu-\mu+1)!}{\alpha!} \frac{(\alpha+k+2\nu-\mu)!}{(\alpha+k+\nu-1)!} \chi^{(\mu)}_{\alpha+\nu-\mu+1}(\tau),
\end{aligned}
$$

after simplification. For convenience of notation we have let $\chi_\alpha := \chi_{\alpha,\alpha}$. This proves the proposition. □

Corollary 2.2.7. *For $\phi \in J_{k,m}(\mathcal{O}_K)$,*

$$
D_\nu \phi = \nu! \left(\sum_{\mu=0}^{\nu} (-4)^{\nu-\mu} (8\pi i m)^{\mu} \frac{(k+2\nu-\mu-2)!}{\mu!\,(k+\nu-2)!} \chi^{(\mu)}_{\nu-\mu,\nu-\mu}(\tau) \right). \tag{2.2.8}
$$

Proof. This follows by considering the $(0,0)^{th}$ coefficients in the above Proposition. □

Remark 2.2.2. Inverting the formula in Corollary 2.2.7, we get (letting $\xi_\nu = D_\nu \phi$)

$$
\chi_{\nu,\nu}(\tau) = \frac{1}{\nu! 4^\nu} \left(\sum_{\mu=0}^{\nu} (-1)^{\nu-\mu} (8\pi i m)^{\mu} \binom{\nu}{\mu} \frac{(k+2\nu-2\mu-1)\,(k+\nu-\mu-2)!}{(k+2\nu-\mu-1)!} \xi^{(\mu)}_{\nu-\mu}(\tau) \right). \tag{2.2.9}
$$

Remark 2.2.3. We can also define the D_ν maps on the subseries of ϕ:

$$
\text{Let} \quad \phi_n := \sum_{\nu \geq 0} \chi_{\nu+n,\nu}(\tau)(z_1 z_2)^\nu \quad \text{and} \quad \phi^n := \sum_{\nu \geq 0} \chi_{\nu,\nu+n}(\tau)(z_1 z_2)^\nu.
$$

It then follows from Remark 2.2.1 in the same way as in the case of ϕ_0 that ϕ_n (resp. ϕ^n) is invariant under the action of $SL(2,\mathbb{Z})$. So we can define D_ν operators on them. The formula for D_ν in this case is the same as in the case of ϕ_0 except that k is replaced by $k+n$ and $\chi_{\alpha,\alpha}$ by $\chi_{\alpha+n,\alpha}$ (resp. by $\chi_{\alpha+n,\alpha}$). For example, if $k \equiv 1,2,3 \pmod 4$, one can define the D_ν operators on ϕ_n ($n \equiv 1,2,3 \pmod 4$) or on ϕ^n (resp. $n \equiv 3,2,1 \pmod 4$) and composing with the projection from $J_{k,m}(\mathcal{O}_K)$.

2.3 Construction of Hermitian Jacobi forms

We need to consider $J_{k,m}(\mathcal{O}_K)$ only for $k > 1$, because of the following lemma:

Lemma 2.3.1. $J_{1,m}(\mathcal{O}_K) = \{0\}$ *for all* $m \geq 1$.

Proof. The proof is a simple application of the corresponding result for classical Jacobi forms, proved by N.P. Skoruppa([38]). We use the U_ρ operator on Hermitian Jacobi forms defined as

$$U_\rho \colon J_{1,m}(\mathcal{O}_K) \to J_{1,N(\rho)m}(\mathcal{O}_K), \quad (U_\rho \phi)(\tau, z_1, z_2) = \phi(\tau, \rho z_1, \bar{\rho} z_2) \quad (\rho \in \mathcal{O}_K).$$

Then we apply the π operator (restriction) to get down to classical Jacobi forms :

$$\pi \colon J_{1,m}(\mathcal{O}_K) \to J_{1,m}, \quad (\pi\phi)(\tau, z) = \phi(\tau, z, z).$$

Let $\phi \in J_{1,m}(\mathcal{O}_K)$. Since $J_{1,m} = \{0\}$, we have the following for each $\rho \in \mathcal{O}_K$:

$$(\pi \circ U_\rho)\,\phi = 0 \tag{2.3.1}$$

Considering the power series expansion of ϕ as in (1.2.23) we get

$$\sum \chi_{\alpha,\beta} z_1^\alpha z_2^\beta \overset{U_\rho}{\longmapsto} \sum \chi_{\alpha,\beta} \rho^\alpha \bar{\rho}^\beta z_1^\alpha z_2^\beta \overset{\pi}{\longmapsto} \sum \chi_{\alpha,\beta} \rho^\alpha \bar{\rho}^\beta z^{\alpha+\beta} = 0$$

This clearly implies that $\chi_{0,0} \equiv 0$. For each $n \geq 1$ we get the following equation

$$\sum_{\alpha=0}^{n} \left(\frac{\rho}{\bar{\rho}}\right)^\alpha \chi_{\alpha,n-\alpha} = 0. \tag{2.3.2}$$

We choose $\{\rho_0, \rho_1, \cdots \rho_n\} \in \mathcal{O}_K$ such that each $\rho_i \in \mathcal{O}_K \backslash \mathbb{Z}$ and for each pair $(i,j) \in \{0, 1, \cdots n\}$ with $i \neq j$, $\rho_i \bar{\rho}_j \neq \rho_j \bar{\rho}_i$; i.e., $\rho_i \bar{\rho}_j \in \mathcal{O}_K \backslash \mathbb{Z}$. (For two sets A, B we have used the notation $A \backslash B := \{x \in A \mid x \notin B\}$.)

With these choices of ρ in equation (2.3.2), we get a system of equations for each $n \geq 1$

$$M \cdot \Xi_n = 0, \text{ where } M_{\gamma,\alpha} = \left(\frac{\rho_\gamma}{\bar{\rho}_\gamma}\right)^\alpha \text{ and } \Xi_n = (\chi_{\alpha,n-\alpha})_{0 \leq \alpha \leq n}.$$

Clearly $M = V\left(\frac{\rho_0}{\bar{\rho}_0}, \cdots \frac{\rho_n}{\bar{\rho}_n}\right)$, and for complex numbers $a_0, a_2, \cdots a_l$, $V(a_0, a_2, \cdots a_l)$ is the Vandermonde determinant which is non-zero when $a_i \neq a_j \; \forall i \neq j$. With our choice of ρ_i's we conclude that $\Xi_n \equiv 0$. Since this happens for every n, we conclude that $\chi_{\alpha,\beta} \equiv 0$ for all α, β and so $\phi \equiv 0$. □

2.3.1 Fourier expansion of the adjoint of D_ν

We recall Theorem 2.1.1 below. Let $f \in S_{k+2\nu}$ and $(,)$ be the Petersson inner product on $S_{k+2\nu}$. Let $\langle , \rangle$ be the Petersson inner product on $J_{k,m}^{cusp}(\mathcal{O}_K)$ and $D_\nu^* : S_{k+2\nu} \longrightarrow J_{k,m}^{cusp}(\mathcal{O}_K)$ be the adjoint of D_ν with respect to the above inner products.

Theorem 2.1.1. *With the above notations the Fourier development of $D_\nu^* f$ is given by*

$$D_\nu^* f\,(\tau, z_1, z_2) = \sum_{n=0}^{\infty} \sum_{\substack{r \in \mathcal{O}_K^\sharp \\ nm \geq N(r)}} c_{D_\nu^* f}(n,r) e^{2\pi i(n\tau + rz_1 + \bar{r}z_2)} \quad \text{where}$$

$$\begin{aligned} c_{D_\nu^* f}(n,r) = & \frac{\nu!(-1)^\nu (4\pi)^{2\nu-1} \Gamma(k+2\nu-1) m^{\nu-k+3} \left(nm - N(r)\right)^{k-2}}{\Gamma(k-2)(k-1)^{(\nu)}} \\ & \times \sum_{\lambda \in \mathfrak{O}_K} \frac{a\left(mN(\lambda) + r\lambda + \bar{r}\bar{\lambda} + n, f\right)}{\left(mN(\lambda) + r\lambda + \bar{r}\bar{\lambda} + n\right)^{k+\nu-1}} \\ & \times \sum_{j=0}^{\nu} \frac{(-1)^j (k-1)^{(2\nu-j)}}{(\nu-j)!^2 j!} \left(\frac{N(m\lambda + \bar{r})}{m\left(mN(\lambda) + r\lambda + \bar{r}\bar{\lambda} + n\right)}\right)^{\nu-j}, \end{aligned} \tag{2.3.3}$$

where $\quad f(\tau) = \sum_{n=1}^{\infty} a\,(n, f)\, e^{2\pi i n\tau}$.

We defer the proof until a later section (2.3.3), after some preliminaries on Hermitian Jacobi Poincaré series.

2.3.2 Poincaré series for Hermitian Jacobi forms

For the proof we make use of the Hermitian Jacobi Poincaré series. Let $n \in \mathbb{Z}, r \in \mathcal{O}_K^\sharp$, and write Γ^J for $\Gamma^J(\mathcal{O}_K)$. Let $\Gamma_\infty^J := \left\{ \left(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, (0,\mu) \right) \mid n \in \mathbb{Z}, \mu \in \mathcal{O}_K \right\} \subset \Gamma^J$ be the stabilizer group of the function $e^{n,r} := e^{2\pi i(n\tau + rz_1 + \bar{r}z_2)}$ under the action of Γ^J. Let $P_{n,r}^{k,m} \left(n \in \mathbb{Z}, r \in \mathcal{O}_K^\sharp \right)$ be the (n,r)-th Hermitian Jacobi Poincaré series of weight $k > 4$ and index m defined by

$$P_{n,r}^{k,m}(\tau, z_1, z_2) = \sum_{\gamma \in \Gamma_\infty^J \backslash \Gamma^J} e\left(n\tau + rz_1 + \bar{r}z_2\right) |_{k,m} \gamma(\tau, z_1, z_2) \tag{2.3.4}$$

We have defined the notion of an inner product $\langle \cdot, \cdot \rangle_H$ for Hermitian Jacobi forms in Chapter 1. Like in the case of classical Jacobi forms, we have :

Lemma 2.3.2.

$$\langle \phi, P_{n,r}^{k,m} \rangle_H = \lambda_{n,r}^{k,m} c_\phi(n,r) \quad \forall\, \phi \in J_{k,m}(\mathcal{O}_K),$$

where $\lambda_{n,r}^{k,m} = \frac{m^{k-4}}{(4\pi)^{k-3}} \frac{\Gamma(k-2)}{(mn - N(r))^{k-3}}$.

Proof. We know that $dV^J = v^{-4} du dv dx_1 dy_1 dx_2 dy_2$ is the invariant volume element on $\mathcal{H} \times \mathbb{C}^2$ for Γ^J. We have, by the usual un-folding argument,

$$\begin{aligned} \langle \phi, P_{n,r}^{k,m} \rangle &= \int_{\Gamma^J \backslash \mathcal{H} \times \mathbb{C}^2} \phi(\tau, z_1, z_2) \sum_{\gamma \in \Gamma_\infty^J \backslash \Gamma^J} \overline{e^{n,r}(\tau, z_1, z_2)|_{k,m}\gamma} \quad e^{\frac{-\pi m}{v}|z_1 - \bar{z}_2|^2} v^k dV^J \\ &= \int_{\Gamma_\infty^J \backslash \mathcal{H} \times \mathbb{C}^2} \phi(\tau, z_1, z_2) \overline{e^{n,r}(\tau, z_1, z_2)} \quad e^{\frac{-\pi m}{v}|z_1 - \bar{z}_2|^2} v^k dV^J, \end{aligned}$$

where $\tau = u + iv$, $z_i = x_j + iy_j$, $j = 1, 2$. As a fundamental domain for the action of Γ_∞^J on $\mathcal{H} \times \mathbb{C}^2$ we take

$$\Gamma_\infty^J \backslash \mathcal{H} \times \mathbb{C}^2 = \{0 \le u \le 1, 0 < v, 0 \le x_1 \le 1, 0 \le y_1 \le 1\}.$$

We make the substitution $\bar{z}_2 - z_1 = z'$. Noting that $\mathcal{O}_K^\sharp = \frac{i}{2}\mathcal{O}_K$,

$$
\begin{aligned}
\langle \phi, P_{n,r}^{k,m} \rangle_H &= \sum_{\substack{l \geq 1 \\ nm > N(s)}} \sum_{s \in \mathcal{O}_K^\sharp} c_\phi(l,s) \int_{\Gamma_\infty^J \backslash \mathcal{H} \times \mathbb{C}^2} e^{-2\pi v(l+n)} e^{2\pi i (l-n)u} e^{4\pi i Re((s-r)z_1) + \bar{s}\bar{z}' - rz'} e^{\frac{-\pi m}{v}|z'|^2} v^k dV'^J \\
&= c_\phi(n,r) \int_0^\infty v^{k-4} e^{-4\pi n v} \left\{ \int_{\mathcal{U}C} e^{\frac{-\pi m}{v}|z'|^2 + 4\pi Im(rz')} dz' \right\} dv \\
&= \frac{c_\phi(n,r)}{m} \int_0^\infty v^{k-3} e^{-\frac{4\pi v}{m}(mn - N(r))} dv \\
&= c_\phi(n,r) \frac{m^{k-4} \quad \Gamma(k-2)}{(4\pi)^{k-3} (mn - N(r))^{k-3}}.
\end{aligned}
$$

□

Lemma 2.3.3. *We have*

$$
\frac{\partial^\alpha}{\partial {z_1}^\alpha} \frac{\partial^\alpha}{\partial {z_2}^\alpha} \left(\exp\left(az_1 + bz_2 + cz_1 z_2\right) \right)|_{z_1 = z_2 = 0} = \sum_{h=0}^{\alpha} (ab)^h c^{\alpha - h} \binom{\alpha}{h}^2 (\alpha - h)!
$$

Proof.

$$
\begin{aligned}
&\frac{\partial^\alpha}{\partial {z_1}^\alpha} \frac{\partial^\alpha}{\partial {z_2}^\alpha} \left(\exp\left(az_1 + bz_2 + cz_1 z_2\right) \right) = \frac{\partial^\alpha}{\partial {z_1}^\alpha} \exp\left(az_1\right) \frac{\partial^\alpha}{\partial {z_2}^\alpha} \left(\exp\left(b + cz_1\right) z_2 \right) \\
&= \frac{\partial^\alpha}{\partial {z_1}^\alpha} \exp\left(az_1\right) \left[(b + cz_1)^\alpha \exp\left(bz_2 + cz_1 z_2\right) \right] = \exp\left(bz_2\right) \frac{\partial^\alpha}{\partial {z_1}^\alpha} \left[(b + cz_1)^\alpha \exp\left((a + cz_2) z_1\right) \right] \\
&= \exp\left(bz_2\right) \sum_{h=0}^{\alpha} \binom{\alpha}{h} \frac{\alpha!}{(\alpha - h)!} c^h (b + cz_1)^{\alpha - h} (a + cz_2)^{\alpha - h} \exp\left(az_1 + cz_1 z_2\right),
\end{aligned}
$$

from which the lemma easily follows upon changing $h \mapsto \alpha - h$. □

2.3.3 Proof of Theorem 2.1.1

Proof. Since the proof is quite similar to that in [42, Theorem 1.1], we will only include the results of our computation. From Lemma (2.3.2) we can compute the (n, r)-th Fourier coefficient of $D_\nu^* f$ as

$$
\langle D_\nu^* f, P_{n,r} \rangle = \frac{c_{D_\nu^* f}(n,r) m^{k-4} \Gamma(k-2)}{(4\pi)^{k-2} (nm - N(r))^{k-3}} = \left(f, D_\nu(P_{n,r}) \right). \tag{2.3.5}
$$

Next, we compute $D_\nu(P_{n,r})$ as an infinite linear combination of elliptic Poincaré series (see equation (2.3.11)). We start with the definition of Hermitian Poincaré series :

$$P_{n,r}(\tau,z_1,z_2)=\sum_{\lambda\in\mathcal{O}_K}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}(c\tau+d)^{-k}e\left(-\frac{mcz_1z_2}{c\tau+d}+(mN(\lambda)+r\lambda+\bar{r}\bar{\lambda}+n)\gamma\tau\right.$$
$$\left.+\frac{(m\bar{\lambda}+r)z_1+(m\lambda+r)z_2}{c\tau+d}\right)\quad(2.3.6)$$

By Lemma 2.3.3 with $a=\frac{2\pi i(m\bar{\lambda}+r)}{c\tau+d}, b=\frac{2\pi i(m\lambda+\bar{r})}{c\tau+d}, c=\frac{-2\pi imc}{c\tau+d}$

we obtain a formula for $\tilde{\chi}_{\alpha,\alpha}$, where the power series expansion of $P_{n,r}$ is $\sum_{\alpha,\beta\geq 0}\tilde{\chi}_{\alpha,\beta}\,z_1^\alpha z_2^\beta$.

$$\begin{aligned}\tilde{\chi}_{\alpha,\alpha}=&\sum_{h=0}^{\alpha}\frac{(2\pi i)^{\alpha+h}}{(h!)^2(\alpha-h)!}\times\\ &\times\sum_{\lambda\in\mathcal{O}_K}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}(-mc)^{\alpha-h}(c\tau+d)^{-(k+h+\alpha)}N(m\lambda+\bar{r})^h e\left(mN(\lambda)+r\lambda+\bar{r}\bar{\lambda}+n\right).\end{aligned}\quad(2.3.7)$$

For convenience of notation, we let $T:=mN(\lambda)+r\lambda+\bar{r}\bar{\lambda}+n$.

Put $\tilde{P}(\tau)=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}(-mc)^{\alpha-h}(c\tau+d)^{-(k+h+\alpha)}e(T)$. Then we have the following formula ([42, p.30]) :

$$\begin{aligned}\tilde{P}^{(\mu)}(\tau)=&\sum_{j=0}^{\mu}\binom{\mu}{j}(-1)^{\mu-j}(2\pi i)^j(k+h+\alpha+j)^{(\mu-j)}\times\\ &\times\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}(c\tau+d)^{-(k+h+\alpha+\mu+j)}c^{\mu-j}(-mc)^{\alpha-h}T^je(T).\end{aligned}\quad(2.3.8)$$

where for non-negative integers a,b , $a^{(b)}:=\frac{(a+b-1)!}{(b-1)!}$.

From equation (2.3.7) and (2.3.8) we get the following expression for $\tilde{\chi}_{\alpha,\alpha}$:

$$\begin{aligned}\tilde{\chi}_{\alpha,\alpha}=&\sum_{h=0}^{\alpha}\frac{(2\pi i)^{\alpha+h}}{(h!)^2(\alpha-h)!}\sum_{\lambda\in\mathcal{O}_K}N(m\lambda+\bar{r})^h\sum_{j=0}^{\mu}\binom{\mu}{j}(-1)^{\mu-j}(2\pi i)^j\\ &\times(k+h+\alpha+j)^{(\mu-j)}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}(c\tau+d)^{-(k+h+\alpha+\mu+j)}\,c^{\mu-j}(-mc)^{\alpha-h}\,T^je(T).\end{aligned}\quad(2.3.9)$$

Finally taking $\alpha=\nu-\mu$ in equation (2.3.9) we arrive at the following expression for

$D_\nu(P_{n,r})(\tau)$:

$$D_\nu(P_{n,r})(\tau) = \sum_{\lambda\in\mathcal{O}_K}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\sum_{\mu=0}^{\nu}\sum_{h=0}^{\nu-\mu}\sum_{j=0}^{\mu}(-1)^{\mu+h+j}\binom{\mu}{j}(2\pi i)^{\nu+h+j}\frac{4^\nu \nu!}{\mu!(h!)^2(\nu-\mu-h)!}$$
$$\times (k+h+\nu-\mu+j)^{(\mu-j)}\,\frac{(k+2\nu-\mu-2)!}{(k+\nu-2)!}m^{\nu-h}N(m\lambda+\bar{r})^h \tag{2.3.10}$$
$$\times (c\tau+d)^{-(k+h+\nu+j)}\,c^{\nu-h-j}\,T^j e(T).$$

Using the identities as in [42, p.31] we get the expansion of $D_\nu(P_{n,r})(\tau)$ in terms of the Poincaré series:

$$\sum_{j=0}^{\nu}\frac{(-1)^j(k+2\nu-j-2)!\,\nu!\,(4\pi)^{2\nu}}{j!(\nu-j)!^2(k+\nu-2)!}\sum_{\lambda\in\mathcal{O}_K}N(m\lambda+\bar{r})^{\nu-j}(mT)^j\,P_T^{k+2\nu}(\tau), \tag{2.3.11}$$

where $P_T^{k+2\nu}$ denotes the T-th Poincaré series of weight $k+2\nu$ for $SL(2,\mathbb{Z})$

Using Lemma 2.3.2 and equation (2.3.5) and the fact that $\left(f, P_n^k\right) = \frac{a(n,f)\Gamma(k-1)}{(4\pi n)^{k-1}}$ for $f=\sum_{n=1}^{\infty}a(n,f)q^n \in S_k$ and P_n^k the n-th Poincaré series of weight k, we get the desired formula (2.3.3) in Theorem 2.3.1. □

2.3.4 Construction of Hermitian Jacobi forms using classical Jacobi Forms

In this section we define a map (which is essentially a change of variables) from 2 copies of Jacobi forms to Jacobi forms for the group $SL(2,\mathbb{Z})\ltimes\mathcal{O}_K^2$, denoted by $J_{k,m}^1(\mathcal{O}_K)$ (see [33]); the transformation properties for these Jacobi forms being the same as in (1.2.5) and (1.2.6) except that we take $\epsilon=1$. Obviously $J_{k,m}(\mathcal{O}_K)\subset J_{k,m}^1(\mathcal{O}_K)$. We then average over the units in $\mathcal{O}_K$ to get Hermitian Jacobi forms.

Proposition 2.3.4. *Fix $k_1,k_2\in\mathbb{N}$. Let $\phi_j\in J_{k_j,m}$, $j\in\{1,2\}$. Define*

$$H(\phi_1,\phi_2)(\tau,z_1,z_2) = \sum_{\epsilon\in\mathcal{O}_K^\times}\phi_1\left(\tau,\frac{1}{2}(z_1+z_2)\right)\phi_2\left(\tau,\frac{i}{2}(z_1-z_2)\right).$$

Then $H(\phi_1,\phi_2)\in J_{k_1+k_2,m}^1(\mathcal{O}_K)$. If ϕ_i are cusp forms, then so is $H(\phi_1,\phi_2)$.

Proof. (i) Invariance under $|_{k_1+k_2,m}M$ where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$.

$$\begin{aligned}
H(\phi_1,\phi_2)|_{k_1+k_2,m}M &= (c\tau+d)^{-(k_1+k_2)} e\left(\frac{-mcz_1z_2}{c\tau+d}\right) H(\phi_1,\phi_2)\left(M\tau, \frac{z_1}{c\tau+d}, \frac{z_2}{c\tau+d}\right) \\
&= (c\tau+d)^{-(k_1+k_2)} e\left(\frac{-mcz_1z_2}{c\tau+d}\right)(c\tau+d)^{k_1} e\left(\frac{mc\left(\frac{z_1+z_2}{2}\right)^2}{c\tau+d}\right)\phi_1\left(\tau, \frac{1}{2}(z_1+z_2)\right) \times \\
&\times (c\tau+d)^{k_2} e\left(\frac{-mc\left(\frac{z_1-z_2}{2}\right)^2}{c\tau+d}\right)\phi_2\left(\tau, \frac{i}{2}(z_1-z_2)\right) \\
&= (c\tau+d)^{-(k_1+k_2)} e\left(\frac{-mcz_1z_2}{c\tau+d}\right)(c\tau+d)^{k_1+k_2} e\left(\frac{mc\left(\frac{z_1+z_2}{2}\right)^2 - mc\left(\frac{z_1-z_2}{2}\right)^2}{c\tau+d}\right) \\
&\qquad \times \phi_1\left(\tau, \frac{1}{2}(z_1+z_2)\right)\phi_2\left(\tau, \frac{i}{2}(z_1-z_2)\right) \\
&= H(\phi_1,\phi_2).
\end{aligned}$$

(ii) Invariance under $|_m[\lambda,\mu]$, where $\lambda,\mu \in \mathcal{O}_K$. We set $\lambda = \lambda_1 + i\lambda_2$ and $\mu = \mu_1 + i\mu_2$, so that $\lambda_j, \mu_j \in \mathbb{Z}$. Also let $e^m(z) := e(mz)$.

$$\begin{aligned}
H(\phi_1,\phi_2)|_m[\lambda,\mu] &= e^m\left(N(\lambda)\tau + \bar{\lambda}z_1 + \lambda z_2\right)\phi_1\left(\tau, \frac{z_1+z_2}{2} + \frac{\lambda+\bar{\lambda}}{2}\tau + \frac{\mu+\bar{\mu}}{2}\right) \\
&\qquad \times \phi_2\left(\tau, i\left(\frac{z_1-z_2}{2} + \frac{\lambda-\bar{\lambda}}{2}\tau + \frac{\mu-\bar{\mu}}{2}\right)\right) \\
&= e^m\left(N(\lambda)\tau + \bar{\lambda}z_1 + \lambda z_2\right)\phi_1\left(\tau, \frac{z_1+z_2}{2} + \lambda_1\tau + \mu_1\right)\phi_2\left(\tau, \frac{i}{2}(z_1-z_2) - \lambda_2\tau - \mu_2\right) \\
&= e^m\left(N(\lambda)\tau + \bar{\lambda}z_1 + \lambda z_2\right) e^m\left\{-\left(\lambda_1^2\tau + \lambda_1(z_1+z_2)\right) - \left(\lambda_2^2\tau - i\lambda_2(z_1-z_2)\right)\right\} \\
&\qquad \times \phi_1\left(\tau, \frac{1}{2}(z_1+z_2)\right)\phi_2\left(\tau, \frac{i}{2}(z_1-z_2)\right) \\
&= e^m\left(N(\lambda)\tau + \bar{\lambda}z_1 + \lambda z_2\right) e^m\left\{-N(\lambda)\tau - (\lambda_1 - i\lambda_2)z_1 - (\lambda_1 + i\lambda_2)z_2\right\} \\
&\qquad \times \phi_1\left(\tau, \frac{1}{2}(z_1+z_2)\right)\phi_2\left(\tau, \frac{i}{2}(z_1-z_2)\right) \\
&= H(\phi_1,\phi_2).
\end{aligned}$$

The assertion about cusp forms is easily checked by writing the Fourier expansion of $H(\phi_1,\phi_2)$ from those of ϕ_1 and ϕ_2. ☐

Corollary 2.3.5. *H is a bilinear map and hence by the above proposition, defines a unique linear map, which we still denote by H. Further averaging over the units of* $\mathcal{O}_K$*,* *i.e., considering the map* $\Lambda\colon J^1_{k,m}(\mathcal{O}_K) \to J_{k,m}(\mathcal{O}_K)$ *given by* $\phi \mapsto \sum_{\epsilon \in \mathcal{O}_K^\times} \phi \mid_k \epsilon I$ *we have the following map into Hermitian Jacobi forms:*

$$J_{k_1,m} \otimes J_{k_2,m} \xrightarrow{H} J^1_{k_1+k_2,m}(\mathcal{O}_K) \xrightarrow{\Lambda} J_{k_1+k_2,m}(\mathcal{O}_K) \tag{2.3.12}$$

preserving cusp forms.

Remark 2.3.6. *If* $\phi_1 = \sum_{\mu \pmod{2m}} h_\mu \theta_{m,\mu}(\tau, z) \in J_{k_1,m}$ *and* $\phi_2 = \sum_{\mu \pmod{2m}} g_\mu \theta_{m,\mu}(\tau, z) \in J_{k_2,m}$ *be their Theta decompositions (see [16]).Then the Theta decomposition (see Section 2.5 for definition) of* $\phi_1 \otimes \phi_2 \in J_{k_1+k_2,m}(\mathcal{O}_K)$ *is given by (clearly the construction of the Theta decomposition for* $J^1_{k_1+k_2,m}(\mathcal{O}_K)$ *is exactly the same as for* $J_{k_1+k_2,m}(\mathcal{O}_K)$*)*

$$\phi_1 \otimes \phi_2(\tau, z_1, z_2) = \sum_{\epsilon \in \mathcal{O}_K^\times} \epsilon^{-k_1-k_2} \sum_{s \in \mathcal{O}_K^\sharp / m\mathcal{O}_K} h_{\mathrm{Re}(s)}(\tau)\, g_{\mathrm{Im}(s)}(\tau) \cdot \theta^H_{m,\epsilon s}(\tau, z_1, z_2).$$

This follows easily from the fact that $H(\theta_{m,\mu}, \theta_{m,\nu}) = \theta^H_{m,\frac{\mu}{2}+i\frac{\nu}{2}}$ and that $\theta^H_{m,s} \mid_{1,m} \epsilon I = \bar{\epsilon}\theta^H_{m,\epsilon s}$. It would be interesting to study this map and to find how large the image is.

2.3.5 Construction by differentiation

Finally we construct Hermitian Jacobi forms from smaller weights and indices using differentiation of the variables z_1, z_2. This is the analogue of the corresponding construction for the classical Jacobi forms [16, Theorem 9.5]. For a function $\phi\colon \mathcal{H} \times \mathbb{C}^2 \to \mathbb{C}$ we let $\phi_{(j)} := \frac{\partial}{\partial z_j}\phi$ for $j = 1, 2$ and $\phi_{(r,s)} = \frac{\partial^2}{\partial z_s \partial z_r}\phi$ for $r, s = 1, 2$.

Proposition 2.3.7. *Let* $\phi \in J_{k_1,m_1}(\mathcal{O}_K)$ *and* $\psi \in J_{k_2,m_2}(\mathcal{O}_K)$*. Then*

(i) $m_1\phi\psi_{(2)} - m_2\psi\phi_{(2)} \in J_{k_1+k_2,m_1+m_2}(\mathcal{O}_K)$.

(ii) $\Big(m_1\phi\psi_{(1)} - m_2\psi\phi_{(1)}\Big)^2 + m_1\phi^2\Big(\psi^2_{(1)} - \psi\psi_{(1,1)}\Big) + m_2\psi^2\Big(\phi^2_{(1)} - \phi\phi_{(1,1)}\Big) \in J_{k',m'}(\mathcal{O}_K)$,

where $k' = k_1 + k_2$ *and* $m' = m_1 + m_2$.

Proof. (*i*) A meromorphic Hermitian Jacobi form ϕ of weight k and index 0 is a meromorphic function $\phi\colon \mathcal{H}\times\mathbb{C}^2\to\mathbb{C}$ satisfying

$$\phi\left(M\tau, \frac{\epsilon z_1}{c\tau+d}, \frac{\bar{\epsilon} z_2}{c\tau+d}\right) = \epsilon^k (c\tau+d)^k \phi(\tau, z_1, z_2),\ \forall \epsilon\in\mathcal{O}_K^{\times}, M\in SL(2,\mathbb{Z}) \quad \text{and}$$

$$\phi(\tau, z_1+\lambda\tau+\mu, z_2+\bar{\lambda}\tau+\bar{\mu}) = \phi(\tau, z_1, z_2) \text{ for all } \lambda,\mu \text{ in } \mathcal{O}_K.$$

Clearly $\phi_{(2)}$ (resp. $\phi_{(1,1)}$) is a meromorphic Hermitian Jacobi form of weight $k+1$ (resp. $k+2$) and index 0 (resp. 0) since in our case $K=\mathbb{Q}(i)$ the unit group is isomorphic to $\mathbb{Z}/4\mathbb{Z}$. Therefore, given $\phi\in J_{k_1,m_1}(\mathcal{O}_K)$ and $\phi\in J_{k_2,m_2}(\mathcal{O}_K)$, we consider the quotient $\Phi := \frac{\phi^{m_2}}{\psi^{m_1}}$ which is a meromorphic Hermitian Jacobi form of weight $k_1m_2-k_2m_1$ and index 0. Then

$$\Phi_{(2)} = \frac{\phi^{m_2-1}}{\psi^{m_1+1}}(m_2\psi\phi_{(2)} - m_1\phi\psi_{(2)}).$$

This proves that $m_1\phi\psi_{(2)} - m_2\psi\phi_{(2)}$ is a meromorphic Hermitian Jacobi form of weight k_1+k_2+1 and index m_1+m_2. Holomorphicity at the cusps is easy to see by writing the Fourier expansions of ϕ and ψ. In case $k_1m_2-k_2m_1<0$, we consider $\Phi=\frac{\psi^{m_1}}{\phi^{m_2}}$ to get the same result.

(*ii*) We calculate $\Phi_{(1,1)}$

$$\begin{aligned}\Phi_{(1,1)} &= \frac{\partial}{\partial z_1}\Big(\frac{\phi^{m_2-1}}{\psi^{m_1+1}}(m_2\psi\phi_{(2)} - m_1\phi\psi_{(2)})\Big)\\ &= \frac{\phi^{m_2-2}}{\psi^{m_1+2}}\left\{\Big(m_1\phi\psi_{(1)} - m_2\psi\phi_{(1)}\Big)^2 + m_1\phi^2\Big(\psi_{(1)}^2 - \psi\psi_{(1,1)}\Big) + m_2\psi^2\Big(\phi_{(1)}^2 - \phi\phi_{(1,1)}\Big)\right\}.\end{aligned}$$

The same arguments as in the proof of (*i*) completes the proof. □

2.4 Commutation with Hecke Operators

Definition 2.4.1. *For $l\in\mathbb{N}$ and $\phi\colon\mathcal{H}\times\mathbb{C}^2\to\mathbb{C}$ let*

$$\phi|_{k,m}V_l(\tau,z_1,z_2) := l^{k-1}\sum_{\substack{\gamma\in SL(2,\mathbb{Z})\backslash M(2,\mathcal{U}Z)\\ \det\gamma=l}} (c\tau+d)^{-k} e\left(-\frac{mlc\,z_1z_2}{c\tau+d}\right)\phi\left(\gamma\tau, \frac{lz_1}{c\tau+d}, \frac{lz_2}{c\tau+d}\right). \tag{2.4.1}$$

Let $\phi \in J_{k,m}(\mathcal{O}_K)$. Then it was shown by K. Haverkamp in [20] that $\phi|_{k,m}V_l \in J_{k,ml}(\mathcal{O}_K)$ (see [16] for classical Jacobi forms). We consider the Fourier development of the action of $\phi|_{k,m}V_l$ in the next Lemma :

Lemma 2.4.2.

$$\phi|_{k,m}V_l(\tau, z_1, z_2) = \sum_{n=1}^{\infty} \sum_{\substack{t\in\mathcal{O}_K^\sharp \\ nm\geq N(t)}} \left(\sum_{\substack{a|(n,l) \\ t/a\in\mathcal{O}^\sharp}} a^{k-1} c\left(\frac{nl}{a^2}, \frac{t}{a}\right) \right) e\left(n\tau + tz_1 + \bar{t}z_2\right). \tag{2.4.2}$$

Proof. The proof is standard. We choose as the set of representatives in the above sum (2.4.1)

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \quad a, d > 0, \quad b \pmod d, \quad ad = l.$$

$$\begin{aligned} \phi|_{k,m}V_l(\tau, z_1, z_2) &:= l^{k-1} \sum_{ad=l} \sum_{b(mod\, d)} d^{-k} \phi\left(\frac{a\tau + b}{d}, az_1, az_2\right) \\ &= l^{k-1} \sum_{ad=l} d^{-k} \sum_{b(mod\, d)} \sum_{n,r} c_\phi(n,r) e\left(\frac{an\tau}{d}\right) \cdot e\left(arz_1 + a\bar{r}z_2\right) \cdot e\left(\frac{bn}{d}\right). \end{aligned}$$

Interchanging the second and third summations, we can rewrite the above as

$$\begin{aligned} & l^{k-1} \sum_{ad=l} d^{1-k} \sum_{\substack{n,r \\ n\equiv 0(mod\, d)}} c_\phi(n,r) e\left(\frac{an\tau}{d}\right) \cdot e\left(arz_1 + a\bar{r}z_2\right) \\ &= \sum_{ad=l} a^{k-1} \sum_{n,r} c_\phi(\frac{ln}{a}, r) e\left(an\tau\right) \cdot e\left(arz_1 + a\bar{r}z_2\right) \\ &= \sum_{n=1}^{\infty} \sum_{\substack{t\in\mathcal{O}_K^\sharp \\ lnm\geq N(t)}} \left(\sum_{\substack{a|(n,l) \\ t/a\in\mathcal{O}^\sharp}} a^{k-1} c\left(\frac{nl}{a^2}, \frac{t}{a}\right) \right) e\left(n\tau + tz_1 + \bar{t}z_2\right). \end{aligned}$$

□

Proposition 2.4.3. *Let* $\nu \geq 0$, $\phi \in J_{k,m}(\mathcal{O}_K)$, $l \in \mathbb{N}$, V_l *as in Definition (2.4.1). Then*

$$D_\nu\left(\phi|_{k,m}V_l\right) = \left(D_\nu\phi\right)|_{k+2\nu}T_l, \tag{2.4.3}$$

where T_l is the usual Hecke operator on elliptic modular forms.

Proof. From the definition of D_ν operators in (2.2.5) it is enough to prove that the following diagrams are commutative for all k, m :

$$\begin{array}{ccc} J^0_{k,m}(\mathcal{O}_K) & \xrightarrow{L_{k,m}} & J^0_{k+2,m}(\mathcal{O}_K) \\ \big\downarrow V_l & & \big\downarrow V_l \\ J^0_{k,ml}(\mathcal{O}_K) & \xrightarrow{L_{k,m}} & J^0_{k+2,ml}(\mathcal{O}_K) \end{array} \quad , \quad \begin{array}{ccc} J^0_{k,m}(\mathcal{O}_K) & \xrightarrow{z_1=z_2=0} & M_k \\ \big\downarrow V_l & & \big\downarrow T_l \\ J^0_{k,ml}(\mathcal{O}_K) & \xrightarrow{z_1=z_2=0} & M_k \end{array}$$

The first diagram is commutative since V_l maps ϕ_0 (the diagonal part of ϕ) to $l^{\frac{k}{2}-1}\sum_M (\phi_0|M)\left(\tau, \sqrt{l}z_1, \sqrt{l}z_2\right)$ and $L_{k,m}$ commutes with $|_{k,m}M$ (2.2.2). That the second diagram is commutative follows from (2.4.1) and the definition of T_l. $\square$

2.5 Number of Fourier coefficients that determine a Hermitian Jacobi form

We recall the Theta correspondence between Hermitian Jacobi forms and vector-valued modular forms studies by K. Haverkamp in [20], [21] : Let $\phi \in J_{k,m}(\mathcal{O}_K)$ with Fourier expansion (1.2.7)

$$\phi = \sum_{n=0}^{\infty} \sum_{\substack{r \in \mathcal{O}_K^\sharp \\ nm \geq N(r)}} c_\phi(n,r) e^{2\pi i(n\tau + rz_1 + \bar{r}z_2)}.$$

Let $N\colon K \to \mathbb{Q}$ be the norm map. It is known ([21], [20]) that $c_\phi(n,r)$ depends only on $r \pmod{m\mathcal{O}_K}$ and $D(n,r) = nm - N(r)$. Therefore if we define

$$c_s(L) := \begin{cases} c_\phi(n,r) & \text{if } r \equiv s \pmod{m\mathcal{O}_K} \text{ and } L = 4D(n,r) \\ 0 & \text{otherwise} \end{cases}$$

where $s \in \mathcal{O}_K^\sharp / m\mathcal{O}_K$ and $L \in \mathbb{Z}$, we can rewrite the Fourier expansion of ϕ as the following, known as the `Theta decomposition` for Hermitian Jacobi forms:

$$\phi(\tau, z_1, z_2) = \sum_{s \in \mathcal{O}_K^\sharp / m\mathcal{O}_K} h_s(\tau) \cdot \theta^H_{m,s}(\tau, z_1, z_2), \tag{2.5.1}$$

where

$$h_s(\tau) := \sum_{\substack{L=0 \\ N(s)+L/4\in m\mathbb{Z}}}^{\infty} c_s(L) e^{\frac{2\pi i L\tau}{4m}}, \text{ and} \tag{2.5.2}$$

$$\theta^H_{m,s}(\tau, z_1, z_2) := \sum_{r\equiv s (mod\, m\mathcal{O}_K)} e\left(\frac{N(r)}{m}\tau + r z_1 + \bar{r} z_2\right).$$

Definition 2.5.1. *For a positive integer m, define*

$$\kappa(k,m) = \left[\frac{4m^2(k-1)}{3}\prod_{p|4m}\left(1-\frac{1}{p^2}\right)+\frac{m}{2}\right], \tag{2.5.3}$$

$[\cdot]$ *being the greatest-integer function. Note that $\kappa(k,m)$ also equals $\left[\frac{\omega}{m}\right]$, where*

$$\omega := [SL(2,\mathbb{Z})\colon \Gamma(4m)]\cdot \frac{k-1}{48} + \frac{m^2}{2}. \tag{2.5.4}$$

Let $r(n)$ denote the number of integral solutions of $x^2+y^2=n$. It is well known that $r(n) = 4\delta(n)$, where $\delta(n) = \sum_{d|n}\left(\frac{-4}{d}\right)$, $\left(\frac{-4}{\cdot}\right)$ being the unique primitive Dirichlet character modulo 4 (see [19] for instance).

Definition 2.5.2. *For $4m|l$, let $R(l) := R_m(l) = \sum_{0\le n\le \frac{l}{4m}} \sum_{0\le d\le 4mn} r(d)$.*

Proposition 2.5.3. *In the Fourier expansion (1.2.7) of a Hermitian Jacobi form ϕ, suppose that $c_\phi(n,r)=0$ for $0\le n\le \kappa(k,m)$. Then $\phi\equiv 0$; i.e., ϕ "is determined" by the first $R(4m\,\kappa(k,m))$ of it's Fourier coefficients.*

Proof. The proof will use the Theta decomposition (1.2.4). By Remark 1.2.1, only finitely many coefficients determine each h_s. Namely, if the Fourier coefficients $c_s(L)$ of h_s vanish for all $0\le L\le \kappa(k,m)$, then h_s itself vanish (see [37] p.120).

Let $1 \le L \le [SL(2,\mathbb{Z})\colon \Gamma(4m)]\cdot \frac{k-1}{12}, s\in \mathcal{O}_K^\sharp/m\mathcal{O}_K$. From the definition of $c_s(L)$ above, $c_s(L)=0$ unless $N(2s)+L\equiv 0 \pmod{4m\mathbb{Z}}$. As a set of representatives $\mathcal{S}$ of $\mathcal{O}_K^\sharp$ in $\mathcal{O}_K^\sharp/m\mathcal{O}_K$ we take

$$\mathcal{S}\colon \left\{\frac{p}{2}+i\frac{q}{2}\right\}, \text{ where } (p,q)\in[-m,m-1]\times[-m,m-1]. \tag{2.5.5}$$

With the above choice, note that $\underset{s\in\mathcal{S}}{max}(N(s)) = \frac{m^2}{2}$. Therefore, when $N(2s) + L = 4mn \in 4m\mathbb{Z}$, the bound on L implies that $0 \leq n \leq \kappa(k,m)$. Since there are $\sum\limits_{0\leq d\leq 4mn} r(d)$ coefficients $c_\phi(n,r)$ for each $n \geq 0$, this proves the proposition. $\square$

Definition 2.5.4 ([21],[20]). *We define the following subspace of $J_{k,m}(\mathcal{O}_K)$:*

$$J^{Spez}_{k,m}(\mathcal{O}_K) := \{\phi \in J_{k,m}(\mathcal{O}_K) |\, c_\phi(n,r) \text{ depends only on } nm - N(r)\} \tag{2.5.6}$$

Proposition 2.5.5. *Suppose that in the power series decomposition (1.2.23) of $\phi \in J^{Spez}_{k,m}(\mathcal{O}_K)$, $\chi_{\nu,\nu} = 0$ for all $0 \leq \nu \leq R(4m\,\kappa(k,m))$. Then $\phi \equiv 0$.*

Proof. Since each $\chi_{\nu,\nu}$ is periodic in τ with period 1, (this follows by taking $M = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ in the identity (2.2.3)) and is holomorphic on $\mathcal{H}$, it has a unique Fourier expansion

$$\nu!^2\chi_{\nu,\nu} = \sum_n \left(\sum_r (-4\pi^2 N(r))^\nu c(n,r) \right) e(n\tau).$$

Define $\tilde{r}(n) := Card.\, \{0 \leq d \leq 4mn,\, d = \square\}$, where $d = \square$ means d is a sum of two squares. Noticing that $c(n,r) = c(n,r')$ if $N(r) = N(r')$, we let $c(n,d) := c(n,r)$, if $d = N(2r)$. Also $\chi_{0,0} \equiv 0$ implies $c(0,0) = 0$. From the hypothesis we get for each $1 \leq n \leq R(4m\,\kappa(k,m))$

$$\sum_{\substack{1\leq d\leq 4mn \\ N(2r)=d}} r(d) d^\nu c(n,d) = 0.$$

For a fixed n as above, considering this equation for $1 \leq \nu \leq \tilde{r}(n)$ we get a $\tilde{r}(n) \times \tilde{r}(n)$ matrix $M(n)$ such that (since $\tilde{r}(n) < R(4m\,\kappa(k,m))$).

$$M(n) \cdot C(n) = 0, \text{ where } M(n)_{\nu,d} = r(d)d^\nu \quad \text{and} \quad C(n) = (c(n,d))_{1\leq d\leq 4mn, d=\square}.$$

Now $\det M(n) = c \cdot \det V\,(1, 2, \cdots \tilde{r}(n)) \neq 0$, where c is a non-zero constant, and for integers $a_1, a_2, \cdots a_l$, $V(a_1, a_2, \cdots a_l)$ is the Vandermonde determinant which is non-zero when $a_i \neq a_j\, \forall i \neq j$. Therefore, $C(n) \equiv 0$.

Doing this for each $0 \leq n \leq R(4m\,\kappa(k,m))$, we get $c(n,r) = 0$, for $0 \leq n \leq R(4m\,\kappa(k,m))$, so $\phi \equiv 0$ from the previous Proposition. $\square$

Theorem 2.5.6. *The map* $\mathcal{D}\colon J_{k,m}^{Spez}(\mathcal{O}_K) \longrightarrow M_k \bigoplus_{1 \le \nu \le R(4m\,\kappa(k,m))} S_{k+2\nu}$ *defined by*

$$\mathcal{D}(\phi) = (D_\nu \phi)_{0 \le \nu \le R(4m\,\kappa(k,m))}\,, \tag{2.5.7}$$

is injective.

Proof. If $\phi \in \ker \mathcal{D}$, by definition of $\mathcal{D}$ we have $\xi_\nu := D_\nu \phi \equiv 0$ for all ν as in the Theorem. By Remark (2.2.2) we obtain $\chi_{\nu,\nu} \equiv 0$ for $0 \le \nu \le R(4m\,\kappa(k,m))$. Therefore $\phi \equiv 0$ by the previous Proposition. □

Therefore we have embedded $J_{k,m}^{Spez}(\mathcal{O}_K)$ into finitely many copies of elliptic modular forms. The natural question is whether the full space $J_{k,m}(\mathcal{O}_K)$ can be so embedded. As mentioned in the Introduction, this is possible if one can prove the non-vanishing of the Hermitian Theta-Wronskian on the upper half plane.

Chapter 3

Hermitian Jacobi forms of index 1 and 2

3.1 Introduction

Hermitian Jacobi forms of integer weight and index are defined for the Jacobi group over the ring of integers $\mathcal{O}_K$ of an imaginary quadratic field K. Such a form $\phi(\tau, z_1, z_2)$ gives rise to a classical Jacobi form for the Jacobi group defined over $\mathbb{Z}$ by the restriction $\pi_\rho \colon J_{k,m}(\mathcal{O}_K) \to J_{k,N(\rho)m}$ defined by $\pi_\rho\phi(\tau, z_1, z_2) = \phi(\tau, \rho z, \bar{\rho} z)$ ($\rho \in \mathcal{O}_K$, see chapter 1 or [21]). In the previous chapter differential operators were constructed from the Taylor expansion of Hermitian Jacobi forms in analogy to that of classical Jacobi forms studied in [16]. Further, a certain subspace of Hermitian Jacobi forms was realized as a subspace of a direct product of finitely many copies of elliptic modular forms for the full modular group.

In this chapter, we treat classical Jacobi forms as an intermediate space between Hermitian Jacobi forms and elliptic modular forms. As a Corollary of the Theorems we also get a description of the kernels of the restriction maps, using the description of the kernel of the restriction map D_0 from classical Jacobi forms of index 1 to elliptic modular forms that has been studied in [3] and [4]. We use the Theta decomposition of Hermitian Jacobi forms and several differential operators as the main tool throughout this Chapter. In the cases of index $1, 2$ the restriction maps give sufficient information to obtain relations with simpler or known spaces of modular forms.

Preliminaries on Hermitian Jacobi forms have been presented in Chapters 1 and 2. In Section 3.2 we compare the spaces of classical Jacobi forms of index $1, 2$ with Hermitian Jacobi forms of index 1 via the restriction maps. For $k \equiv 0 \pmod 4$, we show that the restriction map π_{1+i} is an isomorphism, whereas for $k \equiv 2 \pmod 4$, the restriction map π_1 is injective (refer to Theorem 3.2.1 and Corollary 3.2.7). Recently R. Sasaki in [36] has described the structure of Hermitian Jacobi forms of index 1. We recover one of his results in Corollary 3.2.3.

In section 3.3 we consider Hermitain Jacobi forms of index 2. The main results in this case are Proposition 3.3.3, 3.3.8 and Theorems 3.3.7, 3.1.1. When $k \equiv 1, 3 \pmod 4$, the restriction maps π_{1+i} are isomorphisms, whereas in each of the cases $k \equiv 0, 2 \pmod 4$, $\pi_1 \times \pi_{1+i}$ gives an embedding $J_{k,2}(\mathcal{O}_K) \hookrightarrow J_{k,2} \times J_{k,4}$. Further, there exists an exact sequence of vector spaces connecting Hermitian, classical Jacobi forms with elliptic modular forms via the restriction maps and known differential operators on Jacobi forms. In the proof of Theorem 3.1.1 we explicitly determine a basis of $J_{4,2}(\mathcal{O}_K)$ in terms of their Theta decompositions; as the Eisenstein and Poincaré series are defined only for weights > 4. We state one of the main theorems (whose proof will be given in section 3.3), which deals with index 2 forms of weight $k \equiv 0 \pmod 4$.

Theorem 3.1.1. *Let* $k \equiv 0 \pmod 4$. *We have the following exact sequence of vector spaces*

$$0 \to J_{k,2}(\mathcal{O}_K) \xrightarrow{\pi_1 \times \pi_{1+i}} J_{k,2} \times J_{k,4} \xrightarrow{\Lambda(2) - \Lambda(4)} M_k \times S_{k+2} \to 0 \tag{3.1.1}$$

where $\Lambda(m) := D_0 + \frac{2}{m} D_2 \colon J_{k,m} \to M_k \times S_{k+2}$; D_0 and D_2 *are well known differential operators on the classical Jacobi forms defined in Chapter* 1.

From the exact sequences or from the isomorphisms with classical Jacobi forms, one can clearly embed Hermitian Jacobi forms of index $1,2$ into elliptic modular forms at the level of vector spaces. In subsection 3.3.5 we use the embedding of $J_{k,m}(\mathcal{O}_K)$, $m=1,2$ into classical Jacobi forms to give upper bounds on the order of vanishing of a Hermitian Jacobi form at the origin.

We also compute the rank of index m forms of weight a multiple of 2 and 4 (denoted as $J_{n*,m}(\mathcal{O}_K)$, $n=2,4$) as a module over the algebra of elliptic modular forms. Unlike the classical Jacobi forms, the number of homogeneous products of degree m of the index 1 generators is less than the rank. Following the argument as in [16, p.97], we easily see that $J_{*,*}(\mathcal{O}_K)$ is free over M_*, and $J_{n*,m}(\mathcal{O}_K)$ is of finite rank $R_n(m)$ over M_*. We have the following Proposition which is proved in § 3.4.

Proposition 3.1.2. *(i)* $R_4(m)=m^2+2$, *(ii)* $R_2(m)=2(m^2+1)$.

3.2 Comparision of $J_{k,1}$ and $J_{k,1}(\mathcal{O}_K)$

We consider the Jacobi forms of index 1 arising from the restriction map π_1 of Hermitian Jacobi forms of index 1, where $\pi_\rho\phi(\tau,z)=\phi(\tau,\rho z,\bar{\rho}z)$ $(\phi\in J_{k,1}(\mathcal{O}_K),\rho\in\mathcal{O}_K)$. It is sufficient to consider the restriction maps π_ϵ $(\epsilon\in\mathcal{O}_K^\times)$ for $\epsilon=1$, since by the first transformation rule (1.2.5) we have $\phi(\tau,\epsilon z,\bar{\epsilon}z)=\epsilon^k\phi(\tau,z,z)$.

As a set of representatives of $\mathcal{O}_K^\sharp$ in $\mathcal{O}_K^\sharp/\mathcal{O}_K$ $(\cong\frac{\mathbb{Z}}{2\mathbb{Z}}\times\frac{\mathbb{Z}}{2\mathbb{Z}})$ we take $\mathcal{S}_1:=\left\{0,\frac{i}{2},\frac{1}{2},\frac{1+i}{2}\right\}$. In this section we denote the corresponding Theta components by $h_{r,s}$ and the Hermitian Theta functions of index 1 by $\theta^H_{r,s}$, where $\{r,s\}\in\{0,1\}$. We denote the Jacobi Theta functions of index 1 by $\theta_{1,0}(\tau,z)$, $\theta_{1,1}(\tau,z)$. Further we let

$$\vartheta_0(\tau)=\sum_{r\in\mathbb{Z}}e\left(r^2\tau\right),\qquad \vartheta_1(\tau)=\sum_{r\equiv 1\pmod 2}e\left(\frac{r^2}{4}\tau\right)\quad(\tau\in\mathcal{H}).$$

3.2.1 The case $k \equiv 2 \pmod 4$

Theorem 3.2.1. *1. Let $k \equiv 2 \pmod 4$. Then there is an exact sequence of vector spaces*

$$0 \longrightarrow J_{k,1}(\mathcal{O}_K) \xrightarrow{\pi_1} J_{k,1} \xrightarrow{D_0} M_k \longrightarrow 0, \tag{3.2.1}$$

where D_0 denotes the restriction to modular forms $\phi(\tau, z) \mapsto \phi(\tau, 0)$.

2. Let $k \equiv 2 \pmod 4$. Then π_{1+i} is the zero map.

Proof. 1. Let $\phi \in J_{k,1}(\mathcal{O}_K)$. From the Theta decomposition (1.2.4) and the fact that when $k \equiv 2 \pmod 4$, $h_{0,0} = h_{1,1} = 0$ and $h_{0,1} = -h_{1,0}$ (follows from equation (1.2.13)) we get that

$$\pi_1\phi = h_{0,1}\left(\vartheta_1\theta_{1,0} - \vartheta_0\theta_{1,1}\right).$$

Since $\vartheta_1\theta_{1,0} - \vartheta_0\theta_{1,1} \not\equiv 0$ (see [3]), we clearly have that π_1 is injective and $\mathrm{Im}(\pi_1) \subseteq \ker D_0$.

Let $\phi \in \ker D_0$. From [3, Theorem 1] we see that $\phi(\tau, z) = \varphi(\tau)\left(\vartheta_1\theta_{1,0} - \vartheta_0\theta_{1,1}\right)$, where $\varphi \in M_{k-1}(SL(2,\mathbb{Z}), \bar{\omega})$ which consists of holomorphic functions $f\colon \mathcal{H} \to \mathbb{C}$ bounded at infinity and satisfying $f\mid_{k-1} S = \bar{\omega}(S)f$, $f\mid_{k-1} T = \bar{\omega}(T)f$ (also see [3]). Here ω is the linear character of $SL(2,\mathbb{Z})$ defined by $\omega(T) = i$, $\omega(S) = i$.

So, we only need to check that if $\phi \in J_{k,1}(\mathcal{O}_K)$, then $h_{0,1} \in M_{k-1}(SL(2,\mathbb{Z}), \bar{\omega})$. We already know $h_{0,1}$ is in $M_{k-1}((\Gamma(4))$, so it suffices to check it has the right transformation properties under $SL(2,\mathbb{Z})$. From (1.2.11), (1.2.12) we have

$$h_{0,1}\mid_{k-1} T = e^{-2\pi i N(i/2)} h_{0,1}, \quad h_{0,1}\mid_{k-1} S = \frac{i}{2}\sum_{s \in \mathcal{O}_K^\sharp/\mathcal{O}_K} e^{-4\pi i \mathrm{Re}(-is/2)} h_s$$

which give $h_{0,1}\mid_{k-1} T = -ih_{0,1}(\tau)$ and $h_{0,1}\mid_{k-1} S = -ih_{0,1}(\tau)$, as desired.

Finally D_0 is surjective. This follows in view of the isomorphism $D_0 + D_2\colon J_{k,1} \to M_k \oplus S_{k+2}$, where $D_2 = \left(\frac{k}{2\pi i}\frac{\partial^2}{\partial z^2} - 2\frac{\partial}{\partial \tau}\right)_{z=0}$ (see also §1.1.3.2). Another way to see this using Hermitian Jacobi forms is as follows. Let $V := \mathrm{Im}(D_0)$. Then we have the exact sequence

$$0 \longrightarrow J_{k,1}(\mathcal{O}_K) \xrightarrow{\pi_1} J_{k,1} \xrightarrow{D_0} V \longrightarrow 0$$

Therefore $\dim V = \dim J_{k,1} - \dim J_{k,1}(\mathcal{O}_K)$, which equals $\dim M_k$. The last equality can be seen as follows. First let $k > 4$. Then from [20, p.25] we easily compute when $k \equiv 2 \pmod 4$ that

$$\dim J_{k,1}^{Eis}(\mathcal{O}_K) = 0 \quad \text{and from [20, p.93] we have} \quad \dim J_{k,1}^{cusp}(\mathcal{O}_K) = \left[\frac{k+2}{12}\right].$$

Since $J_{k,1}(\mathcal{O}_K) = J_{k,1}^{Eis}(\mathcal{O}_K) \oplus J_{k,1}^{cusp}(\mathcal{O}_K)$ (see [20]), we get the desired equality of dimensions. When $k = 2$, $J_{k,1} = 0$ and hence so is it's subspace $J_{k,1}(\mathcal{O}_K)$. This shows $V = M_k$ and completes the exactness of the sequence (3.2.1).

2. Let $\phi \in J_{k,1}(\mathcal{O}_K)$ have the Theta decomposition (1.2.4). From Lemma 3.3.9, we write down the Theta decomposition of $\pi_{1+i}\phi$:

$$\begin{aligned}\pi_{1+i}\phi = (h_{0,0}a_0 + h_{1,1}a_2)\theta_{2,0} + (h_{1,0}a_1 + h_{0,1}a_3)\theta_{2,1} \\ + (h_{0,0}a_2 + h_{1,1}a_0)\theta_{2,2} + (h_{0,1}a_1 + h_{1,0}a_3)\theta_{2,3},\end{aligned}$$

where $\theta_{2,\mu} := \theta_{2,\mu}(\tau, z)$, $(\mu \in \mathbb{Z}/4\mathbb{Z})$ are the Jacobi Theta functions of index 2 and $a_\mu := \theta_{2,\mu}(\tau, 0)$ (note that $a_1 = a_3$). Since $h_{0,0} = h_{1,1} = 0$ and $h_{0,1} + h_{1,0} = 0$, when $k \equiv 2 \pmod 4$ the Theorem follows. □

From the above Theorem and the results of [3] we get an isomorphism of $J_{k,1}(\mathcal{O}_K)$ with S_{k+2}, which was also obtained by Sasaki in [36].

Corollary 3.2.2. *Let $k \equiv 2 \pmod 4$. Then $J_{k,1}(\mathcal{O}_K) \cong M_{k-1}(SL(2,\mathbb{Z}), \bar{\omega})$.*

Proof. Let $\phi \in J_{k,1}(\mathcal{O}_K)$. It follows from the proof of the above Theorem that the map sending ϕ to $h_{0,1}$ gives the desired isomorphism. □

Corollary 3.2.3. *Let $k \equiv 2 \pmod 4$. Then the composite*

$$J_{k,1}(\mathcal{O}_K) \overset{\pi_1}{\hookrightarrow} J_{k,1} \overset{D_2}{\to} S_{k+2} \tag{3.2.2}$$

gives an isomorphism from $J_{k,1}(\mathcal{O}_K)$ with S_{k+2} where D_2 is defined as in the proof of the above theorem.

Proof. The result follows from [3, Theorem 2], which in the case $N = 1$ says that $D_2\colon J_{k,1} \to S_{k+2}$ gives an isomorphism of $\ker D_0$ with

$$S^{\circ}_{k+2} := \left\{ f \in S_{k+2} \mid \varphi := \frac{f}{\xi} \in M_{k-1}(SL(2,\mathbb{Z}), \bar{\omega}) \right\},$$

where ω is defined as in the proof of the above Theorem and $\xi = \vartheta_1\vartheta_0' - \vartheta_0\vartheta_1'$. But $S^{\circ}_{k+2} = S_{k+2}$ when $N = 1$, since by [3, Proposition 2], $\xi \in S_3(SL(2,\mathbb{Z}), \omega)$. From equation (3.2.1) we have $\mathrm{Im}(\pi_1) = \ker D_0$. Therefore the Corollary follows. □

Corollary 3.2.4. *Let $k \equiv 2 \pmod 4$. Then multiplication by ξ gives an isomorphism $M_{k-1}(SL(2,\mathbb{Z}), \bar{\omega}) \xrightarrow{\cong} S_{k+2}$.*

Proof. Follows from the previous two Corollaries. □

We define $J_{k,1}(\mathcal{O}_K, N)$ to be the space of Hermitian Jacobi forms for the congruence subgroup $\Gamma_0(N)$ in the usual way. It is immediate that the same proof as in Theorem 3.2.1 applies to this case when $k \equiv 2 \pmod 4$ (see also [3] where the case of classical Jacobi forms is done) and we have an exact sequence of vector spaces

$$0 \longrightarrow J_{k,1}(\mathcal{O}_K, N) \xrightarrow{\pi_1} J_{k,1}(N) \xrightarrow{D_0} M_k(N). \tag{3.2.3}$$

Corollary 3.2.5. *Let $N > 1$. Then $J_{2,1}(\mathcal{O}_K, N) = 0$.*

Proof. A result of Arakawa and Böcherer [4] says that D_0 in (3.2.3) is injective when $k = 2$ and $N > 1$. Therefore the Corollary follows. □

3.2.2 The case $k \equiv 0 \pmod 4$

We now treat the case $k \equiv 0 \pmod 4$. We recall that

$$D_0 + D_2\colon J_{k,1} \xrightarrow{\cong} M_k + S_{k+2}; \tag{3.2.4}$$

where D_0 and D_2 are differential operators constructed from the Taylor expansion of a Jacobi form around the origin, as discussed in Chapter 1.

We recall from Chapter 1 that from the Taylor expansion of Hermitian Jacobi forms, one can define the $D_\nu(\mathcal{O}_K)$ operators in the same way as for the case of Jacobi forms (see [13], [36]). Let $\phi(\tau, z_1, z_2) = \sum_{\alpha,\beta \geq 0} \chi_{\alpha,\beta}(\tau) z_1^\alpha z_2^\beta \in J_{k,1}(\mathcal{O}_K)$ be the Taylor expansion of ϕ around $z_1 = z_2 = 0$. Let $\xi_{1,1} := D_1(\mathcal{O}_K)\phi$, $\xi_{2,2} := D_2(\mathcal{O}_K)\phi$. Then

$$\xi_{1,1} := \chi_{1,1} - \frac{2\pi i}{k}\chi'_{0,0}, \quad \xi_{2,2} := \chi_{2,2} - \frac{2\pi i}{k+2}\chi'_{1,1} + \frac{(2\pi i)^2}{2(k+1)(k+2)}\chi''_{0,0} \tag{3.2.5}$$

define linear maps from $J_{k,1}(\mathcal{O}_K)$ to S_{k+2} and from $J_{k,1}(\mathcal{O}_K)$ to S_{k+4} respectively.

In [36], Sasaki proved that when $k \equiv 0 \pmod 4$

$$\xi \colon J_{k,1}(\mathcal{O}_K) \to M_k \oplus S_{k+2} \oplus S_{k+4}, \qquad \phi \mapsto \chi_{0,0} + \xi_{1,1} + \xi_{2,2} - 6(\chi_{4,0} + \chi_{0,4}) \tag{3.2.6}$$

is an isomorphism.

Remark 3.2.1. We remark here that from the Fourier expansion of a Hermitian Jacobi form ϕ of index 1 (1.2.7), we get $\phi(\tau, z_1, z_2) = \phi(\tau, z_2, z_1)$ if $k \equiv 0 \pmod 4$ and hence in it's Taylor expansion we have $\chi_{\alpha,\beta} = \chi_{\beta,\alpha} \,\forall\, \alpha, \beta \geq 0$. Hence the isomorphism is also given by $\phi \mapsto \chi_{0,0} + \xi_{1,1} + \xi_{2,2} - 12(\chi_{0,4})$. Hence the 4 Taylor coefficients $\chi_{0,0}, \chi_{0,4}, \chi_{1,1}, \chi_{2,2}$ determine ϕ, as expected in analogy with classical Jacobi forms.

Theorem 3.2.6. *1. Let $k \equiv 0 \pmod 4$. Then there is an exact sequence of vector spaces*

$$0 \longrightarrow S_{k+4} \overset{\xi^{-1}|_{S_{k+4}}}{\longrightarrow} J_{k,1}(\mathcal{O}_K) \overset{\pi_1}{\longrightarrow} J_{k,1} \longrightarrow 0 \tag{3.2.7}$$

where $\xi \colon J_{k,1}(\mathcal{O}_K) \to M_k \oplus S_{k+2} \oplus S_{k+4}$ is the isomorphism given in [36].

2. Let $k \equiv 0 \pmod 4$. Then π_{1+i} induces an isomorphism between $J_{k,1}(\mathcal{O}_K)$ and $J_{k,2}$.

Proof. 1. Follows directly from Lemma 3.2.8 given below.

2. When $k \equiv 0 \pmod 4$, in the Theta decomposition of $\phi \in J_{k,1}(\mathcal{O}_K)$ we have $h_{0,1} = h_{1,0}$. Let $\phi \in \ker \pi_{1+i}$. From the Theta decomposition of $\pi_{1+i}\phi$ (see [14]) we easily deduce that $h_{0,1} = h_{1,0} = 0$, $(a_0^2 - a_2^2)h_{0,0} = (a_0^2 - a_2^2)h_{1,1} = 0$. But $a_0^2 \not\equiv a_2^2$ since the Wronskian Wr_2 doesnot vanish on $\mathcal{H}$ (see the proof of *Step 1* of Theorem 3.1.1).

Hence, the kernel is trivial. Moreover, from Corollary 3.2.3, considering the dimensions, we conclude that π_{1+i} is an isomorphism. □

Corollary 3.2.7. *Let* $k \equiv 0 \pmod 4$. *Then*

$$J_{k,2} \xrightarrow{D_0+D_2+D_4} M_k \oplus S_{k+2} \oplus S_{k+4} \xrightarrow{\xi^{-1}} J_{k,1}(\mathcal{O}_K)$$

is an isomorphism.

Proof. In fact, each map is an isomorphism. The first map is injective as proved by Eichler and Zagier [16] and dimension count shows that it is an isomorphism. □

Remark 3.2.2. In the above Theorem, it is clear that if $f \in S_{k+4}$,

$$\xi^{-1} f = \{\phi \in J_{k,1}(\mathcal{O}_K) \mid \chi_{2,2} - 12\,\chi_{0,4} = f\}\,.$$

Lemma 3.2.8. *The following diagram is commutative*

$$\begin{array}{ccc} J_{k,1}(\mathcal{O}_K) & \xrightarrow{\pi_1} & J_{k,1} \\ \cong \Big\downarrow \xi & & \cong \Big\downarrow D_0 + \frac{(2\pi i)^2}{2k} D_2 \\ M_k \oplus S_{k+2} \oplus S_{k+4} & \xrightarrow{pr.} & M_k \oplus S_{k+2} \end{array}$$

Proof. The proof is immediate from definitions. We compute $(pr. \circ \xi)\phi = \chi_{0,0} - \frac{2\pi i}{k}\chi'_{0,0} + \chi_{1,1}$. On the other hand, $(D_0 + \frac{(2\pi i)^2}{2k} D_2) \circ \pi_1 \phi = \chi_{0,0} + (\chi_{0,2} + \chi_{2,0} + \chi_{1,1}) - \frac{2\pi i}{k}\chi'_{0,0} = \chi_{0,0} - \frac{2\pi i}{k}\chi'_{0,0} + \chi_{1,1}$, since $\chi_{0,2} = \chi_{2,0} = 0 = \chi_{1,0} = \chi_{0,1}$ when $k \equiv 0 \pmod 4$. In fact, $\chi_{\alpha,\beta} = 0$ unless $\alpha - \beta \equiv k \pmod 4$ follows from first transformation rule for Hermitian Jacobi forms (1.2.5). □

3.3 Hermitian Jacobi forms of index 2

In this section we consider Hermitian Jacobi forms of index 2 by relating them to classical Jacobi forms and elliptic modular forms via several restriction maps. Let $\mathcal{D} := 2i\mathcal{O}_K$, the Different of K. We use a representation of the group defined for a positive integer m :

$$G_m := \{\mu \in \mathcal{O}_K/m\mathcal{D} \mid N(\mu) \equiv 1 \pmod{4m}\}\,.$$

For $m = 2$, we consider the representation of $G_2 = U_K \cong \mathbb{Z}/4\mathbb{Z}$ defined in [20] :

$$\rho_2 \colon G_2 \longrightarrow Aut\left(J_{k,2}(\mathcal{O}_K)\right), \quad \mu \mapsto W_\mu,$$

where W_μ is defined by

$$W_\mu\left(\Theta_2^t \cdot h\right) = \Theta_2^t \cdot h^{(\mu)}, \quad h^{(\mu)} := \left(h_{\mu s}\right)_{s \in \mathcal{O}_K^\sharp/2\mathcal{O}_K}.$$

Accordingly we have a decomposition of $J_{k,2}(\mathcal{O}_K)$:

$$J_{k,2}(\mathcal{O}_K) = \bigoplus_{\eta \in G_2^*} J_{k,2}^\eta(\mathcal{O}_K), \tag{3.3.1}$$

where G_2^* is the group of characters of G and

$$J_{k,2}^\eta(\mathcal{O}_K) := \{\phi \in J_{k,2}(\mathcal{O}_K) \mid W_\mu \phi = \eta(\mu)\phi \quad \forall \mu \in G_2\}. \tag{3.3.2}$$

$G_2 \cong \mathbb{Z}/4\mathbb{Z}$ via $i \mapsto 1$, $-1 \mapsto 2$, $-i \mapsto 3$, $1 \mapsto 0$. Also,

$$G_2^* = \left\{\eta_\alpha := \left(x \mapsto e^{\frac{2\pi i \alpha x}{4}}\right); \, x, \alpha \in \mathbb{Z}/4\mathbb{Z}\right\}.$$

We take as a set of representatives of $\mathcal{O}_K^\sharp$ in $\mathcal{O}_K^\sharp/m\mathcal{O}_K$ as the set

$$\mathcal{S}_m := \left\{\frac{a}{2} + i\frac{b}{2} \mid a, b \in \mathbb{Z}/2m\mathbb{Z}\right\}.$$

We denote the corresponding Theta components of $\phi \in J_{k,2}(\mathcal{O}_K)$ by $h_{a,b}$ and the Hermitian Theta functions of weight 1 and index m by $\theta^H_{m;a,b}$ (or by $\theta^H_{m;s}$, $s = \frac{a+ib}{2} \in \mathcal{S}_m$) in this section, but we drop the index unless it is necessary. Also we denote by $\theta_{m,\mu}(\tau, z)$ (μ (mod $2m$)) the classical Theta functions. The following Lemmas give the Theta decomposition of the images of Hermitian Jacobi forms of index 2 under the restriction maps. We define for convenience of notation $a_\mu := \theta_{2,\mu}(\tau, 0)$ ($\mu \in \mathbb{Z}/4\mathbb{Z}$) and $b_\mu := \theta_{4,\mu}(\tau, 0)$ ($\mu \in \mathbb{Z}/8\mathbb{Z}$).

Lemma 3.3.1. *Let* $\pi_1 \colon J_{k,2}(\mathcal{O}_K) \to J_{k,2}$ *be given by* $\phi(\tau, z_1, z_2) \mapsto \phi(\tau, z, z)$ *and* $\pi_1\phi = \sum_{\mu \in \mathbb{Z}/4\mathbb{Z}} H_\mu(\tau) \cdot \theta_{2,\mu}(\tau, z)$ *be it's Theta decomposition, where* $H_\mu = (-1)^k H_{-\mu}$ ($\mu \in \mathbb{Z}/4\mathbb{Z}$).

Then,

$$H_0 = h_{0,0}a_0 + h_{0,1}a_1 + h_{0,2}a_2 + h_{0,3}a_3, \tag{3.3.3}$$

$$H_1 = h_{1,0}a_0 + h_{1,1}a_1 + h_{1,2}a_2 + h_{1,3}a_3, \tag{3.3.4}$$

$$H_2 = h_{2,0}a_0 + h_{2,1}a_1 + h_{2,2}a_2 + h_{2,3}a_3. \tag{3.3.5}$$

Proof. Let $s \in \mathcal{S}_2$. The effect of π_1 on $\theta^H_{2;s}$ is given below.

$$\begin{aligned}
\pi_1\theta^H_{2;s} &= \sum_{\substack{r\equiv s \pmod{2\mathcal{O}_K} \\ r\in\mathcal{O}_K^\sharp}} e\left(\frac{N(r)}{2}\tau + 2\mathrm{Re}(r)\cdot z\right) \\
&= \sum_{\substack{\mathrm{Re}(2r)\equiv\mathrm{Re}(2s) \pmod{4\mathbb{Z}} \\ \mathrm{Re}(2r)\in\mathbb{Z}}} e\left(\frac{(\mathrm{Re}(2r))^2}{8}\tau + \mathrm{Re}(2r)\cdot z\right) \\
&\times \sum_{\substack{\mathrm{Im}(2r)\equiv\mathrm{Im}(2s) \pmod{4\mathbb{Z}} \\ \mathrm{Im}(2r)\in\mathbb{Z}}} e\left(\frac{(\mathrm{Im}(2r))^2}{8}\tau\right) \\
&= \theta_{2,\mathrm{Re}(2s)}(\tau,z)\cdot a_{\mathrm{Im}(2s)}.
\end{aligned}$$

This shows that $\pi_1\phi = \sum\limits_{\mu\in\mathbb{Z}/4\mathbb{Z}} \left(\sum\limits_{\substack{s\in\mathcal{S}_2 \\ \mathrm{Re}(2s)=\mu}} h_s \cdot a_{\mathrm{Im}(2s)} \right) \theta_{2,\mu}(\tau,z)$, which proves the Lemma. □

Lemma 3.3.2. *Let* $\pi_{1+i}\colon J_{k,2}(\mathcal{O}_K) \to J_{k,4}$ *be given by* $\phi(\tau,z_1,z_2) \mapsto \phi(\tau,(1+i)z,(1-i)z)$ *and* $\pi_{1+i}\phi = \sum\limits_{\mu\in\mathbb{Z}/8\mathbb{Z}} \bar{H}_\mu(\tau)\cdot\theta_{4,\mu}(\tau,z)$ *be it's Theta decomposition, where* $\bar{H}_\mu = (-1)^k\bar{H}_{-\mu}$ *(*$\mu\in\mathbb{Z}/8\mathbb{Z}$*). Then,*

$$\bar{H}_0 = h_{0,0}b_0 + h_{1,1}b_2 + h_{2,2}b_4 + h_{3,3}b_6, \tag{3.3.6}$$

$$\bar{H}_1 = h_{1,0}b_1 + h_{2,1}b_3 + h_{3,2}b_5 + h_{0,3}b_7, \tag{3.3.7}$$

$$\bar{H}_2 = h_{2,0}b_2 + h_{3,1}b_4 + h_{0,2}b_6 + h_{1,3}b_0, \tag{3.3.8}$$

$$\bar{H}_3 = h_{3,0}b_3 + h_{0,1}b_5 + h_{1,2}b_7 + h_{2,3}b_1, \tag{3.3.9}$$

$$\bar{H}_4 = h_{0,0}b_4 + h_{1,1}b_6 + h_{2,2}b_0 + h_{3,3}b_2. \tag{3.3.10}$$

Proof. We note that $2(1+i)\mathcal{O}_K = 4\mathcal{O}_K \cup 2(1+i)+4\mathcal{O}_K$ (disjoint union) as abelian groups. Let $s = \frac{\mu}{2} + i\frac{\lambda}{2} \in \mathcal{S}_2$. We have

$$\begin{aligned}
U_{1+i}\theta^H_{2,s}(\tau, z_1, z_2) &= \sum_{r\equiv s \pmod{2\mathcal{O}_K}} e\left(\frac{N(r)}{2}\tau + (1+i)rz_1 + (1-i)\bar{r}z_2\right) \\
&= \sum_{r'\equiv(1+i)s \pmod{2(1+i)\mathcal{O}_K}} e\left(\frac{N(r')}{4}\tau + r'z_1 + \bar{r}'z_2\right) \\
&= \sum_{r'\equiv\frac{\mu-\lambda}{2}+i\frac{\mu+\lambda}{2} \pmod{4\mathcal{O}_K}} e\left(\frac{N(r')}{4}\tau + r'z_1 + \bar{r}'z_2\right) \\
&\quad + \sum_{r'\equiv\frac{\mu-\lambda+4}{2}+i\frac{\mu+\lambda+4}{2} \pmod{4\mathcal{O}_K}} e\left(\frac{N(r')}{4}\tau + r'z_1 + \bar{r}'z_2\right),
\end{aligned}$$

from which the Lemma follows easily. □

From the transformation $h_s \mid_{k-1} \epsilon I = \epsilon h_{\epsilon s}$ $(\epsilon \in \mathcal{O}_K^\times)$, we conclude that

$$h_{a,b} = i^k h_{-b,a}, \quad h_{a,b} = (-1)^k h_{-a,-b}. \tag{3.3.11}$$

From the direct-sum decomposition (3.3.1) or from the above equation (3.3.11) we see that $J_{k,2}(\mathcal{O}_K) = J^{\eta_\alpha}_{k,2}(\mathcal{O}_K)$ for $k + \alpha \equiv 0 \pmod 4$.

3.3.1 $\eta = \eta_1$

In this case $k \equiv 3 \pmod 4$. It is easy to see that $h_{0,0} = h_{2,2} = h_{0,2} = h_{2,0} = 0$, and after some calculation, we get

$$h_{0,3} = -h_{0,1}, \quad h_{1,0} = -ih_{0,1}, \quad h_{1,3} = -ih_{1,1}, \quad h_{2,1} = ih_{1,2}, \quad h_{2,3} = -ih_{1,2} \tag{3.3.12}$$

$$h_{3,0} = ih_{0,1}, \quad h_{3,1} = ih_{1,1}, \quad h_{3,2} = -h_{1,2}, \quad h_{3,3} = -h_{1,1}. \tag{3.3.13}$$

We consider the map $\pi_{1+i}\colon J^{\eta_1}_{k,2}(\mathcal{O}_K) \to J_{k,4}$. Using Lemma 3.3.2 we have

$$\pi_{1+i}\phi(\tau, z) = \sum_{\mu \pmod 8} \bar{H}_\mu \theta_{4,\mu}(\tau, z) \quad \text{where } \bar{H}_0 = \bar{H}_4 = 0, \tag{3.3.14}$$

$$\bar{H}_1 = -(1+i)h_{0,1}b_1 - (1-i)h_{1,2}b_3, \bar{H}_2 = ih_{1,1}(b_4 - b_0), \tag{3.3.15}$$

$$\bar{H}_3 = (1+i)h_{0,1}b_3 + (1-i)h_{1,2}b_1, \tag{3.3.16}$$

from which we conclude that π_{1+i} is injective. But for $k > 4$, from [20, Satz 2.5] (or Chapter 1) we get $\dim J_{k,2}^{Eis}(\mathcal{O}_K) = 0$. Also for $k > 4$, using the Trace formula (see [21, Theorem 3], [20, Korollar 2.5, p.92], or Chapter 1) we get $\dim J_{k,2}(\mathcal{O}_K) = \frac{k-3}{4} = \dim J_{k,2}$, where the last equality follows from [16, Cor. Theorem 9.2, p.105]). When $k = 3$, $J_{3,4} = 0$ and therefore so is $J_{3,2}(\mathcal{O}_K)$. Therefore,

Proposition 3.3.3. *Let $k \equiv 3 \pmod 4$. Then π_{1+i} induces an isomorphism between $J_{k,2}(\mathcal{O}_K)$ and $J_{k,4}$.*

3.3.2 $\eta = \eta_2$

In this case $k \equiv 2 \pmod 4$ and using the equations (3.3.11) we find that $h_{0,0} = h_{2,2} = 0$ and every other Theta component h_s of $\phi \in J_{k,2}^{\eta_2}(\mathcal{O}_K)$ is an unit times $h_{0,1}, h_{0,2}, h_{1,1}, h_{1,2}$:

$$h_{0,3} = h_{0,1}, \quad h_{1,0} = -h_{0,1}, \quad h_{1,3} = -h_{1,1}, \quad h_{2,1} = -h_{1,2}, \quad h_{2,3} = -h_{1,2} \tag{3.3.17}$$

$$h_{3,0} = -h_{0,1}, \quad h_{3,1} = -h_{1,1}, \quad h_{3,2} = h_{1,2}, \quad h_{3,3} = h_{1,1}. \tag{3.3.18}$$

Further, we calculate the transformation of $h_{0,1}, h_{0,2}, h_{1,1}, h_{1,2}$ under S from equation (1.2.12):

$$h_{0,1} \mid_{k-1} S = \frac{i}{2}(h_{0,1} + h_{0,2} + h_{1,2}) \tag{3.3.19}$$

$$h_{0,2} \mid_{k-1} S = i(h_{0,1} - h_{1,2}) \tag{3.3.20}$$

$$h_{1,1} \mid_{k-1} S = -ih_{1,1} \tag{3.3.21}$$

$$h_{1,2} \mid_{k-1} S = \frac{i}{2}(h_{0,1} - h_{0,2} + h_{1,2}) \tag{3.3.22}$$

Also, the formula $h_s \mid_{k-1} T = e^{-\pi i N(s)} h_s$ (from (1.2.11) when $m = 2$), gives

$$h_{1,1}(\tau + 1) = -ih_{1,1}(\tau), \quad h_{0,1}(\tau + 1) = \frac{1-i}{\sqrt{2}} h_{0,1}(\tau), \tag{3.3.23}$$

$$h_{0,2}(\tau + 1) = -h_{0,2}(\tau), \quad h_{1,2}(\tau + 1) = -\frac{1-i}{\sqrt{2}} h_{1,2}(\tau). \tag{3.3.24}$$

We note the above observations in the following Lemmas :

Lemma 3.3.4. *Let $k \equiv 2 \pmod 4$. Then in the Theta decomposition (1.2.4) of $\phi \in J_{k,2}(\mathcal{O}_K)$, $h_{1,1} \in M_{k-1}(SL(2,\mathbb{Z}),\bar{\omega})$, where ω is the linear character of $SL(2,\mathbb{Z})$ defined by $\omega(T) = \omega(S) = i$.*

Proof. From the above we get $h_{1,1} \in M_{k-1}(SL(2,\mathbb{Z}),\bar{\omega})$; since $h_{1,1}$ is already a modular form for $\Gamma(8)$, the holomorphicity at infinity is automatic. □

Lemma 3.3.5. $k \equiv 2 \pmod 4$. *Then $J^{Spez}_{k,2}(\mathcal{O}_K) = 0$.*

Proof. By [20, Proposition 5.6] the homomorphism $\iota\colon J^{Spez}_{k,2}(\mathcal{O}_K) \to M^*_{k-1}\left(\Gamma_0(8), (\frac{-4}{\cdot})\right)$ is injective, where

$$M^*_{k-1}\left(\Gamma_0(8), \left(\frac{-4}{\cdot}\right)\right) = \{f(\tau) = \sum_n a(n)e(n\tau) \in M_{k-1}\left(\Gamma_0(8), \left(\frac{-4}{\cdot}\right)\right) \mid \tag{3.3.25}$$

$$a(n) \neq 0 \Rightarrow \exists \lambda \in \mathcal{O}_K\colon n \equiv -N(\lambda) \pmod 8\} \tag{3.3.26}$$

and $\iota(\phi)(\tau) = \sum_{s \in \in \mathcal{O}_K^\sharp/\mathcal{O}_K} h_s(8\tau)$, which turns out to be 0 in this case from the equations (3.3.17) and (3.3.18). □

Lemma 3.3.6. *Let $p, q \in \mathbb{Z}/4\mathbb{Z}$ so that $\frac{p}{2} + i\frac{q}{2} \in \mathcal{S}_2$. Then*

$$\frac{\partial^6}{\partial z_1^6}\left(\theta^H_{p,q}(\tau,z_1,z_2) - \theta^H_{q,p}(\tau,z_1,z_2)\right)_{z_1=z_2=0} = 2\,(16\pi i)^3\left(\left(a'''_p a_q - a'''_q a_p\right) + 15\left(a''_q a'_p - a''_p a'_q\right)\right)$$

where x, y, z are the "Theta constants" and a_μ are as defined at the beginning of this section.

Proof. L.H.S. =

$$= (2\pi i)^6 \sum_{\substack{x \equiv p \pmod 4\\ y \equiv q \pmod 4}} (x+iy)^6 e\left(\frac{x^2+y^2}{8}\tau\right) - (2\pi i)^6 \sum_{\substack{y \equiv p \pmod 4\\ x \equiv q \pmod 4}} (x+iy)^6 e\left(\frac{x^2+y^2}{8}\tau\right)$$

$$= (2\pi i)^6 \sum_{\substack{x \equiv p \pmod 4\\ y \equiv q \pmod 4}} (x+iy)^6 e\left(\frac{x^2+y^2}{8}\tau\right) + (2\pi i)^6 \sum_{\substack{x \equiv p \pmod 4\\ y \equiv q \pmod 4}} (x-iy)^6 e\left(\frac{x^2+y^2}{8}\tau\right)$$

$$= 2\,(2\pi i)^6 \sum_{\substack{x \equiv p \pmod 4\\ y \equiv q \pmod 4}} (x^6 - 15x^4y^2 + 15x^2y^4 - y^6) e\left(\frac{x^2+y^2}{8}\tau\right)$$

$$= 2\,(16\pi i)^3\left(\left(a'''_p a_q - a'''_q a_p\right) + 15\left(a''_q a'_p - a''_p a'_q\right)\right) = R.H.S.$$

□

Theorem 3.3.7. *Let* $k \equiv 2 \pmod 4$. *We have the following exact sequence of vector spaces*

$$0 \longrightarrow S_{k+2} \times S_{k+6} \xrightarrow{\sigma} J_{k,2}(\mathcal{O}_K) \xrightarrow{\pi_1} J_{k,2} \xrightarrow{D_0} M_k \longrightarrow 0. \tag{3.3.27}$$

— *Note* : The map σ is defined as follows. We will prove that $\ker \pi_1 \cong M_{k-1}(SL(2,\mathbb{Z}),\bar{\omega}) \times S_{k+6}$; $\phi \mapsto \Big(h_{1,1},\, D_0(6)(\phi - h_{1,1}\left(\theta^H_{1,1} - \theta^H_{1,3} - \theta^H_{3,1} + \theta^H_{3,3}\right))\Big)$ where $D_0(6)\phi = \chi_{6,0}$, the coefficient of z_1^6 in the Taylor expansion of ϕ around $z_1 = z_2 = 0$ (see Chapter 2 for the definition of Differential operators D_ν, $\nu \in \mathbb{Z}_{\geq 0}$). σ will be the inverse of this isomorphism composed with the isomorphism from S_{k+2} to $M_{k-1}(SL(2,\mathbb{Z}),\bar{\omega})$ (see Corollary 3.2.4).

Proof. We divide the proof into 3 steps.

Step 1. Consider the restriction map $\pi_1\colon J_{k,2}(\mathcal{O}_K) \to J_{k,2}$. Let $\phi \in \ker \pi_1$. We obtain the Theta decomposition of $\pi_1\phi$ from Lemma 3.3.1. Keeping the notation of the Lemma,

$$H_0 = h_{0,0}a_0 + h_{0,1}a_1 + h_{0,2}a_2 + h_{0,3}a_3 = 2h_{0,1}a_1 + h_{0,2}a_2 \tag{3.3.28}$$

$$H_1 = h_{1,0}a_0 + h_{1,1}a_1 + h_{1,2}a_2 + h_{1,3}a_3 = -h_{0,1}a_0 + h_{1,2}a_2 \tag{3.3.29}$$

$$H_2 = h_{2,0}a_0 + h_{2,1}a_1 + h_{2,2}a_2 + h_{2,3}a_3 = -h_{0,2}a_0 - 2h_{1,2}a_1, \tag{3.3.30}$$

upon using equations (3.3.17), (3.3.18); and $H_1 = H_3$. Since $\phi \in \ker \pi_1$ we get

$$\frac{h_{0,1}}{a_2} = \frac{-h_{0,2}}{2a_1} = \frac{h_{1,2}}{a_0} := \psi. \tag{3.3.31}$$

ψ is well defined since it is well known that a_μ ($\mu \in \mathbb{Z}/4\mathbb{Z}$) never vanishes on $\mathcal{H}$. Therefore

$$\phi = \psi\Big(a_2\left(\theta^H_{0,1} + \theta^H_{0,3} - \theta^H_{1,0} - \theta^H_{3,0}\right) - 2a_1\left(\theta^H_{0,2} - \theta^H_{2,0}\right) + a_0\left(\theta^H_{1,2} - \theta^H_{2,1} - \theta^H_{2,3} + \theta^H_{3,2}\right)\Big) \tag{3.3.32}$$

$$+ h_{1,1}\left(\theta^H_{1,1} - \theta^H_{1,3} - \theta^H_{3,1} + \theta^H_{3,3}\right).$$

Furthermore from the definition of ψ above and using the transformation formulas (3.3.19), (3.3.20), (3.3.22) for $(h_{0,1}, h_{0,2}, h_{1,2})$ we get the following transformation formulas for ψ :

$$\psi\left(-\frac{1}{\tau}\right) = \frac{1-i}{\sqrt{2}}\tau^{k-3/2}\psi(\tau), \quad \psi(\tau+1) = -\frac{1-i}{\sqrt{2}}\psi(\tau). \tag{3.3.33}$$

Further, from equations (3.3.19), (3.3.20), (3.3.22), (3.3.4) and from Proposition 1.2.4 (since the transformations of $(h_{0,1}, h_{0,2}, h_{1,2})$ and $h_{1,1}$ under S, T are independent of each other) we conclude that $h_{1,1}\left(\theta^H_{1,1} - \theta^H_{1,3} - \theta^H_{3,1} + \theta^H_{3,3}\right) \in J_{k,2}(\mathcal{O}_K)$ and hence so is $\phi - h_{1,1}\left(\theta^H_{1,1} - \theta^H_{1,3} - \theta^H_{3,1} + \theta^H_{3,3}\right)$.

We define $\ker \pi_1{}^\circ := \{\phi \in \ker \pi_1 \mid h_{1,1} = 0\}$. By the same reasoning as in the above paragraph,

$$\ker \pi_1 \cong M_{k-1}(SL(2,\mathbb{Z}), \bar{\omega}) \times \ker \pi_1{}^\circ \text{ via } \phi \mapsto \left(h_{1,1},\ \phi - h_{1,1}(\theta^H_{1,1} - \theta^H_{1,3} - \theta^H_{3,1} + \theta^H_{3,3})\right),$$

using Lemma 3.3.4 and that $\theta^H_{1,1} - \theta^H_{1,3} - \theta^H_{3,1} + \theta^H_{3,3} \not\equiv 0$. The latter fact follows from (cf. [20])

$$\int_{P_\tau} \theta^H_{m;s}(\tau, z_1, z_2) \cdot \overline{\theta^H_{m;t}(\tau, z_1, z_2)} e^{-\pi m N(z_1 - \bar{z}_2)/v} dz_1 dz_2 = \frac{\delta_{s,t}(m\mathcal{O}_K)v}{m}$$

where $\tau = u + iv \in \mathcal{H}$,

$$\delta_{s,t}(m\mathcal{O}_K) := \begin{cases} 1 & \text{if } s \equiv t \pmod{m\mathcal{O}_K} \\ 0 & \text{otherwise} \end{cases},$$

and the parallelotope

$$P_\tau := \{(\alpha + \beta i + \gamma\tau + \delta i\tau), (\alpha - \beta i + \gamma\tau - \delta i\tau);\ 0 \le \alpha, \beta, \gamma, \delta \le 1\} \subset \mathbb{C}^2.$$

We now prove that $D_0(6)\colon \ker \pi_1{}^\circ \to S_{k+6}$ is an isomorphism.

Let $\phi \in \ker \pi_1{}^\circ$. From equation (3.3.32) and Lemma 3.3.6 we get

$$\begin{aligned}
D_0(6)\phi &= 2c\psi a_2\Big((a_0'''a_1 - a_1'''a_0) + 15\,(a_1''a_0' - a_0''a_1') \Big) \\
&\quad - 2c\psi a_1\Big((a_0'''a_2 - a_2'''a_0) + 15\,(a_2''a_0' - a_0''a_2') \Big) \\
&\quad + 2c\psi a_0\Big((a_1'''a_2 - a_2'''a_1) + 15\,(a_2''a_1' - a_1''a_2') \Big) \\
&= 30c\psi\Big(a_0\,(a_2''a_1' - a_1''a_2') - a_1\,(a_2''a_0' - a_0''a_2') + a_2\,(a_1''a_0' - a_0''a_1') \Big) \\
&= 15c\psi \cdot Wr_2(\tau) = 15c'\psi\eta^{15}(\tau),
\end{aligned}$$

where $c = 2\,(16\pi i)^3$, $c' = c\left(\frac{\pi i}{4}\right)^3 2!4!$ and $Wr_m(\tau) = 2^{m-1}\det\left(\theta^{(\nu)}_{m,\mu}(\tau,0)_{0\leq\nu,\mu\leq m}\right)$ is the Jacobi-Theta Wronskian of order 2. The equality $Wr_2(\tau) = \frac{\pi i}{4}^3 2!4!\,\eta^{15}(\tau)$ ($\eta(\tau)$ being the Dedekind's η - function) follows from the work of Kramer [28].

Clearly $D_0(6)\mid_{\ker\pi_1}$ is injective. To show it's surjectivity it suffices to check that given $f \in S_{k+6}$, if we define φ by $\varphi\cdot\eta^{15} := f$, then φ has the transformation properties (3.3.33). It is then easy to check that the equation (3.3.31) defining the vector-valued modular form $(h_{0,1}, h_{0,2}, h_{1,2})$ gives it's transformation formulas from those of φ and (a_0, a_1, a_2) (see [16, p.59]), the conditions at infinity being trivially true. Since we already know that

$$\varphi\Big(a_2\left(\theta^H_{0,1} + \theta^H_{0,3} - \theta^H_{1,0} - \theta^H_{3,0}\right) - 2a_1\left(\theta^H_{0,2} - \theta^H_{2,0}\right) + a_0\left(\theta^H_{1,2} - \theta^H_{2,1} - \theta^H_{2,3} + \theta^H_{3,2}\right)\Big) \in \ker\pi_1{}^\circ$$

from the argument after equation (3.3.33), the assertion about sufficiency is true.

It remains to check the sufficiency. We know $\eta^{15}\left(-\frac{1}{\tau}\right) = \left(\frac{\tau}{i}\right)^{15}\eta(\tau)$, $\eta^{15}(\tau+1) = e^{\frac{5\pi i}{4}}\eta(\tau)$. From the definition of φ, we have

$$\varphi\left(-\frac{1}{\tau}\right)\eta^{15}\left(-\frac{1}{\tau}\right) = \tau^{k+6}\varphi(\tau)\eta^{15}(\tau), \quad \varphi(\tau+1)\eta^{15}(\tau+1) = \varphi(\tau)\eta^{15}(\tau)$$

which clearly gives the right transformation properties for ψ. From the definition of σ (after the statement of Theorem 3.3.7) we see that σ induces an isomorphism between $S_{k+2}\times S_{k+6}$ and $\ker\pi_1$.

Step 2. The case $k = 2$. We claim $J_{2,2}(\mathcal{O}_K) = 0$. Indeed, considering the restriction map π_1 we find that $\dim J_{2,2}(\mathcal{O}_K) = \dim\ker\pi_1 + \dim\mathrm{Im}(\pi_1) = \dim S_4 + \dim S_8 = 0$ since

$J_{2,2} = 0$ and using *Step 1* for the description of $\ker \pi_1$. The Theorem is trivially true in this case.

Step 3. $\ker D_0 = \mathrm{Im}(\pi_1)$ for $k > 4$. From equations (3.3.28), (3.3.29), (3.3.30) it follows that

$$\begin{aligned} D_0 \circ \pi_1 &= a_0 H_0 + 2a_1 H_1 + a_2 H_2 \\ &= a_0(2h_{0,1}a_1 + h_{0,2}a_2) - 2a_1(h_{0,1}a_0 + h_{1,2}a_2) - a_2(h_{0,2}a_0 - 2h_{0,2}a_1) = 0 \end{aligned}$$

This could also be seen from the fact that $(D_0 \circ \pi_1)\phi = D_0(\mathcal{O}_K)\phi = \chi_{0,0} = 0$ since $\chi_{\alpha,\beta} = 0$ unless $\alpha - \beta \equiv k \pmod 4$. So $\mathrm{Im}(\pi_1) \subseteq \ker D_0$. But a direct check (considering $k \equiv 2, 6, 10 \pmod{12}$) shows

$$\begin{aligned} \dim \mathrm{Im}(\pi_1) &= \dim J_{k,2}(\mathcal{O}_K) - \dim \ker \pi_1 = \left[\frac{k-1}{3}\right] - \dim S_{k+2} - \dim S_{k+6} \\ &= \dim J_{k,2} - \dim M_k = \dim \ker D_0, \end{aligned}$$

since D_0 is surjective. The dimension formula for $J_{k,2}^{cusp}(\mathcal{O}_K)$ $(k > 4)$ follows from [20, Korollar 8.11, p.92] or [21, Theorem 3] (see also Chapter 1) and the fact that $\dim J_{k,2}^{Eis}(\mathcal{O}_K) = 0$ for $k \equiv 2 \pmod 4$ (see [20, Satz 2.5, p.25]) gives $\dim J_{k,2}(\mathcal{O}_K) = \left[\frac{k-1}{3}\right]$. This completes the proof of the Theorem. □

3.3.3 $\eta = \eta_3$

In this case $k \equiv 1 \pmod 4$. It is easy to see that $h_{0,0} = h_{2,2} = h_{0,2} = h_{2,0} = 0$, and after a calculation,

$$h_{0,3} = -h_{0,1}, \quad h_{1,0} = ih_{0,1}, \quad h_{1,3} = ih_{1,1}, \quad h_{2,1} = -ih_{1,2}, \quad h_{2,3} = ih_{1,2} \tag{3.3.34}$$

$$h_{3,0} = -ih_{0,1}, \quad h_{3,1} = -ih_{1,1}, \quad h_{3,2} = -h_{1,2}, \quad h_{3,3} = -h_{1,1}. \tag{3.3.35}$$

Exactly the same argument as in the case $\eta = \eta_1$ works here, i.e., we consider the map $\pi_{1+i} \colon J^{\eta_3}_{k,2}(\mathcal{O}_K) \to J_{k,4}$. Using Lemma 3.3.2 we have

$$\pi_{1+i}\phi(\tau, z) = \sum_{\mu \pmod 8} \bar{H}_\mu \theta_{4,\mu}(\tau, z) \quad \text{where } \bar{H}_0 = \bar{H}_4 = 0, \tag{3.3.36}$$

$$\bar{H}_1 = -(1-i)h_{0,1}b_1 - (1+i)h_{1,2}b_3, \bar{H}_2 = ih_{1,1}(b_0 - b_4), \tag{3.3.37}$$

$$\bar{H}_3 = (1-i)h_{0,1}b_3 + (1+i)h_{1,2}b_1, \tag{3.3.38}$$

from which we conclude that π_{1+i} is injective. Also from the dimension formula for $k > 4$ we get $\dim J_{k,2}(\mathcal{O}_K) = \frac{k-5}{4}$ ($= \dim J_{k,4}$ from [16]), whereas $J_{1,2}(\mathcal{O}_K) \hookrightarrow J_{1,4} = 0$. Hence we have the following Proposition :

Proposition 3.3.8. *Let $k \equiv 1 \pmod 4$. Then π_{1+i} induces an isomorphism between $J_{k,2}(\mathcal{O}_K)$ and $J_{k,4}$.*

3.3.4 $\eta = \eta_0$

In this case $k \equiv 0 \pmod 4$, and

$$h_{0,1} = h_{0,3} = h_{1,0} = h_{3,0}, \quad h_{0,2} = h_{2,0}, \tag{3.3.39}$$

$$h_{1,2} = h_{2,1} = h_{2,3} = h_{3,2}, \quad h_{1,1} = h_{1,3} = h_{3,1} = h_{3,3}. \tag{3.3.40}$$

For $k > 4$ from [20, Satz 2.5, p.25] or Chapter 1, we find $\dim J^{Eis}_{k,2}(\mathcal{O}_K) = 2$, and from [20, Korollar 8.1, p.92] or [21, Theorem 3] that $\dim J^{cusp}_{k,2}(\mathcal{O}_K) = \frac{k-4}{2}$ via the Trace formula for Hecke Operators. Therefore $\dim J_{k,2}(\mathcal{O}_K) = \frac{k}{2}$.

We prove a Lemma which will be used in the proof of the next Theorem.

Lemma 3.3.9. *Let $\phi \in J_{k,1}(\mathcal{O}_K)$ with Theta decomposition $\phi = h_0\theta^H_{1;0} + h_{\frac{1}{2}}\theta^H_{1;\frac{1}{2}} + h_{\frac{i}{2}}\theta^H_{1;\frac{1}{2}} + h_{\frac{1+i}{2}}\theta^H_{1;\frac{1}{2}}$. Then the Theta decomposition of $U_{1+i}\phi$ is given by:*

$$U_{1+i}\phi = h_0\theta^H_{0,0} + h_0\theta^H_{2,2} + h_{\frac{1}{2}}\theta^H_{1,1} + h_{\frac{1}{2}}\theta^H_{3,3} + h_{\frac{i}{2}}\theta^H_{3,1} + h_{\frac{i}{2}}\theta^H_{1,3} + h_{\frac{1+i}{2}}\theta^H_{0,2} + h_{\frac{1+i}{2}}\theta^H_{2,0} \tag{3.3.41}$$

Proof. First we note $(1+i)\mathcal{O}_K = 2\mathcal{O}_K \cup (1+i) + 2\mathcal{O}_K$ (disjoint union) as abelian groups. Let $s = \frac{x}{2} + i\frac{y}{2} \in \mathcal{S}_2$. We have

$$\begin{aligned} U_{1+i}\theta^H_{1,s}(\tau, z_1, z_2) &= \sum_{r \equiv s \pmod{\mathcal{O}_K}} e\left(N(r)\tau + (1+i)rz_1 + (1-i)\bar{r}z_2\right) \\ &= \sum_{r' \equiv \frac{x-y}{2} + i\frac{x+y}{2} \pmod{(1+i)\mathcal{O}_K}} e\left(\frac{N(r')}{2}\tau + r'z_1 + \bar{r}'z_2\right). \end{aligned}$$

Using the above formula and that $(1+i)\mathcal{O}_K = 2\mathcal{O}_K \cup (1+i) + 2\mathcal{O}_K$, we see that

$$U_{1+i}\theta^H_{1;0} = \theta^H_{0,0} + \theta^H_{2,2}; \qquad U_{1+i}\theta^H_{1;\frac{1}{2}} = \theta^H_{1,1} + \theta^H_{3,3}; \tag{3.3.42}$$

$$U_{1+i}\theta^H_{1;\frac{i}{2}} = \theta^H_{3,1} + \theta^H_{1,3}; \qquad U_{1+i}\theta^H_{1;\frac{1+i}{2}} = \theta^H_{0,2} + \theta^H_{2,0}. \tag{3.3.43}$$

The lemma now follows at once. □

Lemma 3.3.10. *Let $\phi_{4,1}$ be the basis element of $J_{4,1}(\mathcal{O}_K)$ given in [36]. Then, $\{U_{1+i}\phi_{4,1},\ \phi_{4,1} \mid V_2\}$ is a basis of $J_{4,2}(\mathcal{O}_K)$.*

Proof. The Taylor expansion of $\phi_{4,1}$ around $z_1 = z_2 = 0$ is $\phi_{4,1}(\tau, z_1, z_2) = 2E_4 + \pi i E_4' z_1 z_2 + \cdots$, from which the proof follows easily by writing down the corresponding Taylor expansions of $U_{1+i}\phi_{4,1}$ and $\phi_{4,1} \mid V_2$. □

Now we state and prove Theorem 3.1.1 mentioned in the Introduction.

Theorem 3.1.1. *Let $k \equiv 0 \pmod 4$. We have the following exact sequence of vector spaces*

$$0 \to J_{k,2}(\mathcal{O}_K) \xrightarrow{\pi_1 \times \pi_{1+i}} J_{k,2} \times J_{k,4} \xrightarrow{\Lambda(2) - \Lambda(4)} M_k \times S_{k+2} \to 0, \tag{3.3.44}$$

where $\Lambda(m) := D_0 + \frac{2}{m}D_2 \colon J_{k,m} \to M_k \times S_{k+2}$; D_0 and D_2 are well known differential operators on Jacobi forms given by, $D_0\phi := \phi\,|_{z=0}$ and $D_2\phi := \left(\frac{k}{2\pi i}\frac{\partial^2}{\partial z^2}\phi - 2\frac{\partial}{\partial \tau}\phi\right)_{z=0}$.

Proof. We divide the proof into two steps.

Step 1. Let $\phi \in \ker(\pi_1 \times \pi_{1+i})$. We invoke Lemmas 3.3.1 and 3.3.2. Keeping the same notation as those in the Lemmas, we get $\pi_{1+i}\phi = \sum_{\mu \in \mathbb{Z}/8\mathbb{Z}} \bar{H}_\mu(\tau) \cdot \theta_{4,\mu}(\tau, z)$, where

$\bar{H}_\mu = \bar{H}_{-\mu}$ ($\mu \in \mathbb{Z}/8\mathbb{Z}$) and

$$\bar{H}_0 = h_{0,0}b_0 + 2h_{1,1}b_2 + h_{2,2}b_4 = 0, \tag{3.3.45}$$

$$\bar{H}_1 = 2h_{0,1}b_1 + 2h_{1,2}b_3 = 0, \tag{3.3.46}$$

$$\bar{H}_2 = 2h_{0,2}b_2 + 2h_{1,1}b_4 = 0, \tag{3.3.47}$$

$$\bar{H}_3 = 2h_{0,1}b_3 + 2h_{1,2}b_1 = 0, \tag{3.3.48}$$

$$\bar{H}_4 = h_{0,0}b_4 + 2h_{1,1}b_2 + h_{2,2}b_0 = 0, \tag{3.3.49}$$

since $b_\mu = b_{-\mu}$. Further,

$$H_0 = h_{0,0}a_0 + 2h_{0,1}a_3 + h_{0,2}a_2 = 0, \tag{3.3.50}$$

$$H_1 = h_{0,1}a_0 + 2h_{1,1}a_1 + h_{1,2}a_2 = 0, \tag{3.3.51}$$

$$H_2 = h_{0,2}a_0 + 2h_{1,2}a_1 + h_{2,2}a_2 = 0. \tag{3.3.52}$$

From (3.3.46) and (3.3.48) we get $(b_1^2 - b_3^2)h_{0,1} = (b_1^2 - b_3^2)h_{1,2} = 0$.

We claim that $\theta_{m,\mu}(\tau, 0) \neq \theta_{m,\nu}(\tau, 0)$ for $\mu \neq \nu$ ($0 \leq \mu, \nu \leq m$), $\tau \in \mathcal{H}$. Suppose not. Then the Wronskian Wr_m of $\theta_{m,\mu}$ ($0 \leq \mu \leq m$), would be identically zero on $\mathcal{H}$, contradicting the fact that it is a non-zero multiple of Dedekind's η- function [28].

Therefore $b_1^2(\tau) \neq b_3^2(\tau)$ for all $\tau \in \mathcal{H}$, which implies that $h_{0,1} = h_{1,2} = 0$ (only $b_1^2 \not\equiv b_3^2$ would have sufficed to get this conclusion). Finally (3.3.51) and (3.3.47) together imply that $h_{1,1} = h_{0,2} = 0$. From (3.3.45) and (3.3.49) we get $(b_0^2 - b_4^2)h_{0,0} = (b_0^2 - b_4^2)h_{2,2} = 0$. By the above, we get $h_{0,0} = h_{2,2} = 0$. Hence $\phi = 0$.

Step 2. $\mathrm{Im}\,(\pi_1 \times \pi_{1+i}) \subseteq \ker\,(\Lambda(2) - \Lambda(4))$. We use the Taylor expansions of the Jacobi forms involved. Let $\phi(\tau, z_1, z_2) = \sum_{\alpha,\beta \geq 0} \chi_{\alpha,\beta}(\tau) z_1^\alpha z_2^\beta \in J_{k,2}(\mathcal{O}_K)$ be the Taylor expansion of ϕ around $z_1 = z_2 = 0$. Then the Taylor developments of $\pi_1\phi$ and $\pi_{1+i}\phi$ are

$$\pi_1\phi = \chi_{0,0} + \chi_{1,1}\, z^2 + (\chi_{0,4} + \chi_{2,2} + \chi_{4,0})\, z^4 + \cdots, \tag{3.3.53}$$

$$\pi_{1+i}\phi = \chi_{0,0} + 2\chi_{1,1}\, z^2 - 4\,(\chi_{0,4} - \chi_{2,2} + \chi_{4,0})\, z^4 + \cdots, \tag{3.3.54}$$

from which it easily follows $\Lambda(2)\pi_1\phi = \Lambda(4)\pi_{1+i}\phi$.

Clearly $\Lambda(2) - \Lambda(4)$ is surjective, since $\Lambda(2)$ is surjective (recall that $D_0 + D_2 + D_4 \colon J_{k,2} \to M_k \times S_{k+2} \times S_{k+4}$ is an isomorphism).

$\mathrm{Im}\,(\pi_1 \times \pi_{1+i}) = \ker\,(\Lambda(2) - \Lambda(4))$. We show that they have the same dimension (for $k \geq 4$). First of all we have,

$$\dim \mathrm{Im}\,(\pi_1 \times \pi_{1+i}) = \dim J_{k,2}(\mathcal{O}_K) = \frac{k}{2}$$

(for $k > 4$, use Haverkamp's dimension formula, see Lemma 3.3.10 for $k = 4$). Whereas,

$$\dim \ker\,(\Lambda(2) - \Lambda(4)) = \dim J_{k,2} + \dim J_{k,4} - \dim M_k - \dim S_{k+2}.$$

From part 2 of Theorem 3.2.7 and the fact $\dim J_{k,1} = \frac{k}{4}$ (for $k \equiv 0 \pmod 4$), see [36, Theorem 1]) or computing directly we get $\dim J_{k,2} = \frac{k}{4}$. A direct check now shows that

$$\dim J_{k,4} - \dim M_k - \dim S_{k+2} = \dim S_{k+4} + \dim S_{k+6} + \dim S_{k+8} = \frac{k}{4}.$$

(Recall that $D_0 + D_2 + D_4 + D_6 + D_8 \colon J_{k,4} \to M_k \times S_{k+2} \times S_{k+4} \times S_{k+6} \times S_{k+8}$ is an isomorphism.)

This completes the proof of Theorem 3.1.1. □

Next, we give the explicit Theta decompositions of two particular basis elements of $J_{4,2}(\mathcal{O}_K)$. One could perhaps write down the Theta decomposition of $\phi_{4,1} \mid V_2$ from the corresponding decomposition of $\phi_{4,1}$, but we use a different method.

Proposition 3.3.11. *The space $J_{4,2}(\mathcal{O}_K)$ is spanned by the two linearly independent elements $\Phi_{4,2}$ and $\tilde{\Phi}_{4,2}$ given by*

$$\Phi_{4,2} = (x^6 + y^6)(\theta^H_{0,0} + \theta^H_{2,2}) + z^6(\theta^H_{1,1} + \theta^H_{3,3} + \theta^H_{1,3} + \theta^H_{3,1}) + (x^6 - y^6)(\theta^H_{0,2} + \theta^H_{2,0}), \quad \textit{and} \tag{3.3.55}$$

$$\tilde{\Phi}_{4,2} = 2x^3y^3(\theta^H_{0,0} - \theta^H_{2,2}) + z^3(x^3 - y^3)(\theta^H_{0,1} + \theta^H_{1,0} + \theta^H_{0,3} + \theta^H_{3,0}) + \\ + z^3(x^3 + y^3)(\theta^H_{1,2} + \theta^H_{2,1} + \theta^H_{2,3} + \theta^H_{3,2}). \tag{3.3.56}$$

Proof. Let $\Phi_{4,1}$ be a basis element of $J_{4,1}(\mathcal{O}_K)$ explicitly given in [36]:

$$\Phi_{4,1} = (x^6+y^6)\theta^H_{1,0} + z^6(\theta^H_{1,\frac{1}{2}} + \theta^H_{1,\frac{i}{2}}) + (x^6-y^6)\theta^H_{1,\frac{1+i}{2}}, \tag{3.3.57}$$

where $x = \sum_{n\in\mathbb{Z}} e\left(\frac{n^2\tau}{2}\right)$, $y = \sum_{n\in\mathbb{Z}} (-1)^n e\left(\frac{n^2\tau}{2}\right)$, $z = \sum_{t\in\frac{1}{2}+\mathbb{Z}} e\left(\frac{t^2\tau}{2}\right)$ are the so called "Theta constants".

Let $\Phi_{4,2} := U_{1+i}\Phi_{4,1}$. We will produce another element of $J_{4,2}(\mathcal{O}_K)$ linearly independent of $\Phi_{4,2}$. To this end, we compute the Theta decomposition of $\Phi_{4,2}$ using Lemma 3.3.9 and the fact that $h_{\frac{1}{2}} = h_{\frac{i}{2}}$ (h_s being Theta components of an element in $J_{k,1}(\mathcal{O}_K)$, $k \equiv 0 \pmod 4$) :

$$\Phi_{4,2} = (x^6+y^6)(\theta^H_{0,0}+\theta^H_{2,2}) + z^6(\theta^H_{1,1}+\theta^H_{3,3}+\theta^H_{1,3}+\theta^H_{3,1}) + (x^6-y^6)(\theta^H_{0,2}+\theta^H_{2,0}). \tag{3.3.58}$$

Here we consider the restriction π_1. Since $\dim J_{4,2} = 1$, and π_1 is non-zero, we also have π_1 is surjective. Since $\Phi_{4,2} \notin \ker\pi_1$, $\dim\ker\pi_1 = 1$. We will determine $\tilde{\Phi}_{4,2} \neq 0$ by the condition $\tilde{\Phi}_{4,2} - \Phi_{4,2} \in \ker\pi_1$, which will prove the Proposition.

Let $\phi \in J_{4,2}(\mathcal{O}_K)$. The transformation formulas for it's Theta components under S are as follows:

$$h_{0,0} \mid_3 S = \frac{i}{4}(h_{0,0}+h_{2,2}+2h_{0,2}+4h_{1,1}+4h_{0,1}+4h_{1,2}) \tag{3.3.59}$$

$$h_{2,2} \mid_3 S = \frac{i}{4}(h_{0,0}+h_{2,2}+2h_{0,2}+4h_{1,1}-4h_{0,1}-4h_{1,2}) \tag{3.3.60}$$

$$h_{0,1} \mid_3 S = \frac{i}{4}(h_{0,0}-h_{2,2}+2h_{0,1}-2h_{1,2}) \tag{3.3.61}$$

$$h_{1,2} \mid_3 S = \frac{i}{4}(h_{0,0}-h_{2,2}-2h_{0,1}+2h_{1,2}) \tag{3.3.62}$$

$$h_{0,2} \mid_3 S = \frac{i}{4}(h_{0,0}+h_{2,2}+2h_{0,2}-4h_{1,1}) \tag{3.3.63}$$

$$h_{1,1} \mid_3 S = \frac{i}{4}(h_{0,0}+h_{2,2}-2h_{0,2}) \tag{3.3.64}$$

Let us denote the Theta components of $\Phi_{4,2}$ by $\hat{h}_s$ ($s \in \mathcal{S}_2$). From (3.3.58) we see that $\hat{h}_{0,1} = \hat{h}_{1,2} = 0$ which implies by equations (3.3.59) to (3.3.64) that $\hat{h}_{0,0} = \hat{h}_{2,2}$ and the following transformation formulas under S:

$$\hat{h}_{0,0} \mid_3 S = \frac{i}{2}(\hat{h}_{0,0} + \hat{h}_{0,2} + 2\hat{h}_{1,1}) \tag{3.3.65}$$

$$\hat{h}_{0,2} \mid_3 S = \frac{i}{2}(\hat{h}_{0,0} + \hat{h}_{0,2} - 2\hat{h}_{1,1}) \tag{3.3.66}$$

$$\hat{h}_{1,1} \mid_3 S = \frac{i}{2}(\hat{h}_{0,0} - \hat{h}_{0,2}) \tag{3.3.67}$$

From the above formulas (3.3.59) to (3.3.64) we note that if we assume $\tilde{h}_{1,1} = \tilde{h}_{0,2} = 0$ in a Hermitian Jacobi form $\tilde{\phi}$ of weight 4 and index 2, (conditions complementary to that in $\Phi_{4,2}$), we get $\tilde{h}_{0,0} + \tilde{h}_{2,2} = 0$ and a "honest" vector-valued modular form of weight 3:

$$\tilde{h}_{0,0} \mid_3 S = \frac{i}{2}(\tilde{h}_{0,1} + \tilde{h}_{1,2}) \tag{3.3.68}$$

$$\tilde{h}_{0,1} \mid_3 S = \frac{i}{2}(\tilde{h}_{0,0} + \tilde{h}_{0,1} - \tilde{h}_{1,2}) \tag{3.3.69}$$

$$\tilde{h}_{1,2} \mid_3 S = \frac{i}{2}(\tilde{h}_{0,0} - \tilde{h}_{0,1} + \tilde{h}_{1,2}), \tag{3.3.70}$$

the transformation formulas under T remaining the same. Therefore Theorem 1.2.4 will give a Hermitian Jacobi form of weight 4 index 2 with the above Theta components, which we denote by $\tilde{\Phi}_{4,2}$. If $\tilde{\Phi}_{4,2}$ exists and is non-zero, we are done, since by construction it is linearly independent of $\Phi_{4,2}$.

We will determine $\tilde{\Phi}_{4,2}$ by imposing the condition that $\tilde{\Phi}_{4,2} - \Phi_{4,2} \in \ker \pi_1$, i.e., $\pi_1\tilde{\Phi}_{4,2} = \pi_1\Phi_{4,2}$. Upon using Lemma 3.3.1 and equation (3.3.58), the Theta components $\tilde{h}_s$ of $\tilde{\Phi}_{4,2}$ satisfy the following system of equations:

$$\tilde{h}_{0,0}a_0 + 2\tilde{h}_{0,1}a_1 = (x^6 + y^6)a_0 + (x^6 - y^6)a_2 \tag{3.3.71}$$

$$\tilde{h}_{0,1}a_0 + \tilde{h}_{1,2}a_2 = 2z^6a_1 \tag{3.3.72}$$

$$-\tilde{h}_{0,0}a_2 + 2\tilde{h}_{1,2}a_1 = (x^6 - y^6)a_0 + (x^6 + y^6)a_2. \tag{3.3.73}$$

Using the relations $x = a_0 + a_2$, $y = a_0 - a_2$, $z = 2a_1$, we find after a calculation,

$$\tilde{h}_{0,0} = 2x^3y^3, \quad \tilde{h}_{0,1} = z^3(x^3 - y^3), \quad \tilde{h}_{1,2} = z^3(x^3 + y^3). \tag{3.3.74}$$

It is now easy to see (using the formulas in [16, p.59]) that $\tilde{h}_{0,0}, \tilde{h}_{0,1}, \tilde{h}_{1,2}$ given by equation (3.3.74) satisfy the right transformation formulas (3.3.68), (3.3.69), (3.3.70) under S. Whence, $\tilde{\Phi}_{4,2}$ is non-zero and has the Theta decomposition:

$$\tilde{\Phi}_{4,2} = 2x^3y^3(\theta^H_{0,0} - \theta^H_{2,2}) + z^3(x^3 - y^3)(\theta^H_{0,1} + \theta^H_{1,0} + \theta^H_{0,3} + \theta^H_{3,0}) + \\ + z^3(x^3 + y^3)(\theta^H_{1,2} + \theta^H_{2,1} + \theta^H_{2,3} + \theta^H_{3,2}).$$

As noted earlier, this completes the proof of the Proposition. □

3.3.5 Order of vanishing at the origin

For $\phi \in J_{k,m}(\mathcal{O}_K)$, let $\phi(\tau, z_1, z_2) = \sum_{\alpha,\beta\geq 0} \chi_{\alpha,\beta}(\tau) z_1^\alpha z_2^\beta$ be the Taylor expansion around $z_1 = z_2 = 0$. Define a non-negative integer $\varrho_{k,m}\phi$ by

$$\varrho_{k,m}\phi = \min\{\alpha + \beta \mid \chi_{\alpha,\beta}(\tau) \not\equiv 0\} \qquad \text{if } \phi \not\equiv 0 \tag{3.3.75}$$

$$= \infty \qquad \text{otherwise} \tag{3.3.76}$$

i.e., $\varrho_{k,m}\phi$ can be interpeted as the order of vanishing of ϕ at the origin. From the relation with Jacobi forms, we can give upper bounds on $\varrho_{k,m}\phi$ for any $\phi \in J_{k,m}(\mathcal{O}_K)$ $(m = 1, 2)$.

Proposition 3.3.12. *(i) Let $\phi \in J_{k,1}(\mathcal{O}_K)$ be non zero. Then*

$$0 \leq \varrho_{k,1}\phi \leq 2 \ \textit{ if } k \equiv 2 \pmod 4; \quad 0 \leq \varrho_{k,1}\phi \leq 4 \ \textit{ if } k \equiv 0 \pmod 4 \tag{3.3.77}$$

(ii) Let $\phi \in J_{k,2}(\mathcal{O}_K)$. Then

$$0 \leq \varrho_{k,2}\phi \leq 5 \ \textit{ if } k \equiv 1,3 \pmod 4; \quad 0 \leq \varrho_{k,2}\phi \leq 8 \ \textit{ if } k \equiv 0,2 \pmod 4 \tag{3.3.78}$$

Proof. All of these assertions except the case $k \equiv 2 \pmod 4$, $m = 2$ follow easily from Propositions 3.3.8, 3.3.3 and Theorem **??** and that the first $2m$ of the Taylor coefficients of the Taylor development of a Jacobi form of index m around the origin determine it (see Chapter 1). In the case $k \equiv 2 \pmod 4$, $m = 2$ we have $J_{k,2}(\mathcal{O}_K) \overset{\pi_1 \times \pi_{1+i}}{\hookrightarrow} J_{k,2} \times J_{k,4}$ (as in the case $k \equiv 0 \pmod 4$). This follows from Lemmas 3.3.1 and 3.3.2 and equations (3.3.17) and (3.3.18). For convenience, we give the proof. Let $\phi \in J_{k,2}(\mathcal{O}_K)$, with Theta components $h_{a,b}$ $(a, b \in \mathcal{S}_2)$. From $\pi_{1+i}\phi = 0$, we get $h_{1,1} = 0$ and from $\pi_1\phi = 0$ that (using $h_{0,0} = h_{2,2} = 0$)

$$\begin{pmatrix} 2a_1 & a_2 & 0 \\ a_0 & 0 & a_2 \\ 0 & a_0 & 2a_1 \end{pmatrix} \begin{pmatrix} h_{0,1} \\ h_{0,2} \\ h_{1,2} \end{pmatrix} = 0$$

Since $\det \left(\begin{smallmatrix} 2a_1 & a_2 & 0 \\ a_0 & 0 & a_2 \\ 0 & a_0 & 2a_1 \end{smallmatrix}\right) = -4a_0a_1a_2$, we get the required injectivity and hence the Proposition. □

3.4 Rank of $J_{n*,m}(\mathcal{O}_K)$ over M_* and algebraic independence of $\phi_{4,1}$, $\phi_{8,1}$, $\phi_{12,1}$

We refer to Chapter 1, Definition 1.2.10 for the definition of the index 1 forms $\phi_{4,1}$, $\phi_{8,1}$, $\phi_{12,1}$ which form a basis for $J_{4*,1}(\mathcal{O}_K) := \bigoplus_{k\geq 0} J_{4k,1}(\mathcal{O}_K)$ as a module over M_*.

Remark 3.4.1. *For $m \geq 1$, $J_{n*,m}(\mathcal{O}_K)$ $(n = 2, 4)$ are modules over M_* via the algebra isomorphism*

$$M_* = \mathbb{C}[E_4, E_6] \xrightarrow{E_4 \mapsto E_4, E_6 \mapsto E_6^2} \mathbb{C}[E_4, E_6^2], \tag{3.4.1}$$

because $E_6 \cdot J_{n*,m}(\mathcal{O}_K) \notin J_{n*,m}(\mathcal{O}_K)$. From the argument in [16, p.97], we easily see that $J_{*,*}(\mathcal{O}_K)$ is free over M_*, and $J_{n*,m}(\mathcal{O}_K)$ is of finite rank $r_n(m)$ over M_*.

Now we state and prove Proposition 3.1.2 mentioned in the Introduction.

Proposition 3.1.2. (i) $R_4(m) = m^2 + 2$, $\quad (ii)$ $R_2(m) = 2(m^2 + 1)$.

Proof. The proof is immediate from the dimension formula of $J_{k,m}(\mathcal{O}_K)$ in Theorem 1.2.14 (see also [21, Theorem 3]). We find that $\dim J_{k,m}(\mathcal{O}_K) = (m^2+2)\dim M_k + f(m) + O(1)$, (resp. $= m^2 \dim M_k + g(m) + O(1)$) where $f(m), g(m)$ are functions depending only on

m when $k \equiv 0 \pmod 4$ (resp. $k \equiv 2 \pmod 4$). Letting $k \to \infty$, we get (i). Since there can be no linear relation between generators of weights $0 \pmod 4$ and $2 \pmod 4$ by Remark 3.4.1, (ii) follows. □

Proposition 3.4.2. *The Hermitian Jacobi forms $\phi_{4,1}$, $\phi_{8,1}$, $\phi_{12,1}$ are algebraically independent over M_*.*

Proof. It is enough to prove the algebraic independence of $\psi_{8,1}, \tilde{\psi}_{16,1}, \psi_{16,1}$, where $\psi_{8,1} = E_4\phi_{4,1} - \phi_{8,1}$, $\psi_{12,1} = E_4\phi_{8,1} - \phi_{12,1}$, $\tilde{\psi}_{16,1} = 5E_4\psi_{12,1} - 3\psi_{16,1}$; $\psi_{8,1}, \psi_{12,1}, \psi_{16,1}$ being the generators of $J^{cusp}_{4*,1}(\mathcal{O}_K)$ over M_* (see [36] or Chapter 1 for their definition).

Let $f(X,Y,Z) = \sum_{a+b+c=m} Q_{a,b,c} X^a Y^b Z^c$ be a homogeneous polynomial over M_* of least degree m such that $f(\psi_{8,1}, \tilde{\psi}_{16,1}, \psi_{16,1}) = 0$. Applying the map π_1 in the above relation we get

$$\sum_{b+c=m} Q_{0,b,c} \, (\pi_1\tilde{\psi}_{16,1})^b \, (\pi_1\psi_{16,1})^c = 0, \qquad \text{since } \pi_1\psi_{8,1} = 0.$$

From Lemma 3.4.3 $\pi_1\tilde{\psi}_{16,1} \neq 0$, $D_0\,\pi_1\tilde{\psi}_{16,1} = 0$, $D_0\,\pi_1\psi_{16,1} \neq 0$. Hence, the argument in [16, p.90] for classical Jacobi forms applies, showing $Q_{0,b,c} = 0$ for all b, c such that $b + c = m$. Hence we have

$$\sum_{a+b+c=m,\, a\geq 1} Q_{a,b,c} \, \psi_{8,1}^a \, \tilde{\psi}_{16,1}^b \, \psi_{16,1}^c = 0,$$

giving an equation of lower degree. Hence the Proposition is proved. □

Lemma 3.4.3. *(i)* $\psi_{16,1} = -2^8\Delta\phi_{4,1} + 2E_4^2\phi_{8,1} - E_4\phi_{12,1}$.

(ii) $D_0\psi_{16,1} = -5\cdot 2^8\Delta E_4, \quad D_0\psi_{12,1} = -3\cdot 2^8\Delta$.

(iii) $\pi_1\tilde{\psi}_{16,1} = (\frac{2}{9}E_4^3 + 3\cdot 2^9\Delta)\,E_{4,1} + \frac{8}{9}E_4E_6\,E_{6,1}$*; $E_{4,1}, E_{6,1}$ being the normalised Jacobi Eisenstein series, which are a basis of $J_{2*,1}$ over M_* and $\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24}$ is the Discriminant cusp form of weight* 12*.*

Proof. The calculations follow from the Theta decompositions of $\phi_{4j,1}$ $(j = 1,2,3)$ given in [36] and using the Theta relations (see [31]) : Let $\theta^H_s(\tau) := \theta^H_{0,0}(\tau, 0, 0)$ $(s \in \mathcal{S}_1)$. Then we have

$$\theta^H_{0,0}(\tau) = \frac{1}{2}(x^2 + y^2),\ \theta^H_{0,1}(\tau) = \theta^H_{1,0}(\tau) = \frac{1}{2}z^2,\ \theta^H_{1,1}(\tau) = \frac{1}{2}(x^2 - y^2);$$

where

$$x = \sum_{n\in\mathbb{Z}} e\left(\frac{n^2\tau}{2}\right), \quad y = \sum_{n\in\mathbb{Z}} (-1)^n e\left(\frac{n^2\tau}{2}\right), \quad z = \sum_{t\in\frac{1}{2}+\mathbb{Z}} e\left(\frac{t^2\tau}{2}\right)$$

are the "Theta constants". The rest of the calculation is straightforward. □

3.5 Concluding remarks

1. The restriction maps that we use in this Chapter do not commute with Hecke operators. Nevertheless it is expected that the following should be true. There should exist finitely many algebraic integers $\rho_j \in \mathcal{O}_K$, $1 \leq j \leq n$ (where n depends only on the index m) such that we have an embedding/isomorphism :

$$\pi_{\rho_1} \times \cdots \times \pi_{\rho_n} : J_{k,m}(\mathcal{O}_K) \hookrightarrow J_{k,mN(\rho_1)} \times \cdots \times J_{k,mN(\rho_n)}$$

   where $N : K \to \mathbb{Q}$ is the norm map. From the results of this Chapter (see Theorem 3.3.7, Proposition 3.3.12) this is true for $m = 1, 2$ and these cases suggest that ρ_j and n above should be related to the decomposition of m in $\mathcal{O}_K$.

2. We know that for $k \equiv 0 \pmod 4$, $\dim J_{k,2}(\mathcal{O}_K) = \frac{k}{2} = 2(\dim M_{k-4} + \dim M_{k-8} + \dim M_{k-12})$. This suggests what the minimal weights of the 6 generators of $J_{4*,2}(\mathcal{O}_K)$ over M_* should be, but the calculations seem to be much more than that in the case of classical Jacobi forms.

Chapter 4

Non-vanishing of Jacobi Poincaré series

4.1 Introduction

The theory of Poincaré series is old and is based on a simple idea going back to Petersson. Let $\mathcal{H}$ be the upper half plane. For each $k \in \mathbb{Z}$, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$ acts on functions $f\colon \mathcal{H} \to \mathbb{C}$ by $f \mid_k \gamma := (c\tau + d)^{-k} f(\gamma\tau)$. One considers the "average over $SL(2,\mathbb{Z})$, modulo the stabilizer G of f under the previous action ", i.e., define for $\tau \in \mathcal{H}$ the following.

Definition 4.1.1. $\mathtt{P}(f)(\tau) := \sum\limits_{\gamma \in G\backslash SL(2,\mathbb{Z})} f \mid_k \gamma = \sum\limits_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G\backslash SL(2,\mathbb{Z})} (c\tau + d)^{-k} f(\frac{a\tau+b}{c\tau+d})$.

If $\mathtt{P}(f)$ converges to a holomorphic function on $\mathcal{H}$ and has a nice Fourier development, it clearly defines a modular form. If we take $f(\tau) = e(m\tau)$, (where $e(z) := \exp(2\pi i z)$, $m \in \mathbb{Z}$), we obtain the Poincaré series which have been the objects of extensive research in the theory of modular forms. The space of elliptic cusp forms of weight k, denoted by S_k, are equipped with a positive definite inner product $(\ ,\ )$ (the Petersson inner product), which gives a Hilbert space structure on S_k. Moreover, it is classical that the collection of Poincaré series P_m^k $(m \in \mathbb{Z})$ span S_k. Further we have

$$(f, P_m^k) = \frac{a(m,f)\Gamma(k-1)}{(4\pi m)^{k-1}}, \text{ where } f(\tau) = \sum_{n\geq 1} a(n,f)e(n\tau).$$

The next question is then, when is $P_m^k \neq 0$? Reformulating the same question, we can ask: Is there a cusp form f_m $(m \geq 1)$ such that it's m-th Fourier coefficient is non-zero? The precise conjecture is stated after the next theorem.

In [34] R. A. Rankin has proved that the *m-th* Poincaré series P_m^k of weight k, where k, m are positive integers, for the full modular group $SL(2,\mathbb{Z})$ is not identically zero for sufficiently large k and finitely many m depending on k. C. J. Mozzochi extended Rankin's result to integral weight modular forms for congruence subgroups in [30]. More precisely,

Theorem 4.1.1. *(Rankin, [34]) There exist positive constants k_0 and B, where $B > 4\log 2$, such that, for all even $k \geq k_0$ and all positive integers $m \leq k^2 \exp\{-\frac{B\log k}{\log\log k}\}$, the Poincaré series P_m^k doesnot vanish identically.*

Rankin uses very precise estimates of Bessel functions and the estimates of one dimensional Kloosterman sums from Esterman's work ([17]) on the estimates of exponential sums. This remains the best result in the direction of the following conjecture for elliptic modular forms (also see [35]):

Conjecture 4.1.2. $P_m^k \neq 0$ *for all* $m \geq 1$, *whenever* $\dim S_k \neq 0$.

This is a hard problem, for example, when $k = 12$, this is easily seen to be equivalent to Lehmer's conjecture on non-vanishing of the Ramanujan's τ function. In [35], an algebraic criterion for the Conjecture 4.1.2 is given in terms of the principal terms in the Fourier expansion of certain meromorphic modular forms on $SL(2,\mathbb{Z})$. It would be nice, if such a criterion can be found for Jacobi forms as well. Finally we mention that the analytic approaches do not seem to give a complete answer to the above question.

In this chapter we prove similar results for higher degree Jacobi Poincaré series defined for the full Jacobi group $\Gamma_g^J = SL(2,\mathbb{Z}) \ltimes (\mathbb{Z}^g \times \mathbb{Z}^g)$, where g is a positive integer and is referred to as the degree of the Jacobi group. The Jacobi group operates on $\mathcal{H} \times \mathbb{C}^g$ and also on functions $\phi\colon \mathcal{H} \times \mathbb{C}^g \to \mathbb{C}$. We denote the latter action by $|_{k,m}$. (See Chapter 1 for the definitions.)

Let $k, g \in \mathbb{Z}$ and m be a symmetric, positive-definite, half-integral $(g \times g)$ matrix. The vector space of Jacobi cusp forms of weight k, index m and degree g, denoted by $J^{cusp}_{k,m,g}$, is defined to be the space of holomorphic functions $\phi\colon \mathcal{H} \times \mathbb{C}^g \to \mathbb{C}$ satisfying $\phi|_{k,m}\gamma = \phi$ (where $\gamma \in \Gamma^J_g$) and having a Fourier expansion

$$\phi(\tau, z) = \sum_{n \in \mathbb{N}, r \in \mathbb{Z}^g, 4n > m^{-1}[r^t]} c_\phi(n, r) e(n\tau + rz).$$

If $g = 1$, we denote $J^{cusp}_{k,m,1}$ by $J^{cusp}_{k,m}$.

For $n \in \mathbb{N}$, $r \in \mathbb{Z}^g$ with $4n > m^{-1}[r^t]$, let $P^{k,m}_{n,r}$ be the (n, r)-*th* Poincaré series of weight k and index m (of exponential type) defined for $k > g + 2$. (see Chapter 1 for definition). It is well-known that the Poincaré series $P^{k,m}_{n,r}$ $(n \in \mathbb{Z}, r \in \mathbb{Z}^g)$ span $J^{cusp}_{k,m,g}$. It is then natural to ask when such a Poincaré series vanishes identically or when it does not. We prove the following theorems, which give a partial answer to the above question by using analytic methods; more precisely, using compatible estimates of Bessel functions and Kloosterman sums.

Let $D = \det \left(\begin{smallmatrix} 2n & r \\ r^t & 2m \end{smallmatrix}\right)$ and define $k' := k - g/2 - 1$.

Theorem 4.1.3. *Let k be even when $2r \equiv 0 \pmod{\mathbb{Z}^g \cdot 2m}$. Then there exist an integer k_0 and a constant $B > 3\log 2$ such that for all $k \geq k_0$ (depending only on g), the Jacobi Poincaré series $P^{k,m}_{n,r}$ does not vanish identically for*

$$k' \leq \frac{\pi D}{\det(2m)} \leq k'^{1+\alpha(g)} \exp\left\{-\frac{B \log k'}{\log\log k'}\right\},$$

where $\alpha(g) = \begin{cases} \frac{2}{3(g+2)} & \textit{if} \quad 1 \leq g \leq 4, \\ \frac{2}{3g} & \textit{if} \quad g \geq 5. \end{cases}$

We mention here another theorem by Petersson [32]. To the knowledge of the author, he was the first to give examples when certain Poincaré series does not vanish identically.

Theorem 4.1.4 (Petersson, [32]). *Let $n \geq 1$ and let $d := \dim S_k \neq 0$. Then $P_1^k, \ldots, P_d^k$ are a basis for S_k. In particular they are non-zero.*

We construct a basis of $J_{k,m}^{cusp}$ (when this space is non-zero) consisting of the 'first' $\dim J_{k,m}^{cusp}$ Poincaré series (see Theorem 4.3.1 in section 4.3). One has to make the precise notion of 'first' in this situation. We explain that for k even. The other case is analogous. First we note that the (n,r)-th Poincaré series $P_{n,r}^{k,m}$ can also be denoted as $P_{D,r}^{k,m}$; where $D := 4mn - r^2$. This follows from the fact that $P_{n,r}^{k,m}$ depends only on $r \pmod{4m}$ and $4mn - r^2$ (> 0) (see [16]). So, in the next theorem we index the relevant Poincaré series by the above formalism by choosing the least positive integer $D_\mu \equiv -\mu^2 \pmod{4m}$ for each $\mu = 0, 1, \ldots m$.

Theorem 4.1.5. 1. *Let $k \geq m + 12$. If k is even, the collection $\left\{ P_{D_\mu + 4m\lambda_\mu, \mu}^{k,m} \right\}$, $\mu = 0, 1, \ldots, m$; $\lambda_\mu = 0, 1, \ldots, \dim S_{k+2\mu} - \left[\frac{\mu^2}{4m}\right] - 1$, where $D_\mu := 4m\left(\left[\frac{\mu^2}{4m}\right] + 1\right) - \mu^2$ forms a basis for $J_{k,m}^{cusp}$.*

A theorem analogous to Theorem 4.1.1 is easily seen to hold in the half-integral weight case also. Using this and the fact that the Eichler-Zagier map from classical Jacobi forms of index 1 to half-integral weight modular forms maps Poincaré series to the corresponding Poincaré series, we get a better result in the case of Jacobi forms of index 1. This is the content of Section 4.3.2.

We also give conditions for non-vanishing of Poincaré series independent of the weight for classical Jacobi forms ($g = 1$). The technique is the same as that in the proof of Theorem 4.1.3, but different estimates of Bessel functions are used as the classical estimates do not give the desired result.

Define $M(x) := \exp\left\{\frac{B_1 \log x}{\log \log 2x}\right\}$ $(x \geq 2,\ B_1 > \log 2)$ as in [34].

Theorem 4.1.6. *Let $g = 1$. For $D > \frac{m}{\pi}$, we have $P_{D,r}^{k,m} \not\equiv 0$ for*

$$M\left(\frac{\pi D}{m}\right) \sigma_0(D)\, D < \frac{m^{\frac{8}{7}}}{\lambda},$$

where $\lambda = (2\sqrt{2}\pi^{\frac{5}{3}}A)^{\frac{3}{2}}$, $A = \frac{1}{\pi}\left(\frac{2}{6^{\frac{2}{3}}} + \frac{54}{2^{\frac{5}{6}}} + \frac{16}{2^{\frac{3}{4}}}\right)$ *and* $\sigma_0(D) = \sum_{d|D} 1$.

Finally following Rankin ([34]), we give conditional statements on the non-vanishing of Jacobi Poincaré series, based on the relation of g-dimensional Kloosterman sums with corresponding 1-dimensional sums and identities involving them.

Theorem 4.1.7. *Let p be an odd prime, $\mu \in \mathbb{N}$. Suppose that* $p \mid m, p \mid r, p \nmid n$. *If* $P^{k,p^\mu m}_{p^\mu n, p^\mu r} \not\equiv 0$, *then*

$$\text{either } P^{k,mp^{\mu-1}}_{np^{\mu-1},rp^{\mu-1}} \not\equiv 0 \quad \text{or} \quad P^{k,p^{2\mu}m}_{np^{2\mu},rp^{2\mu}} \not\equiv 0 \quad \text{and} \quad P^{k,p^{2\mu}m}_{n,rp^{\mu}} \not\equiv 0. \tag{4.1.1}$$

(Here $p \mid m$ means p divides every entry of m; since $2m$ is a $(g \times g)$ matrix with integer entries and p is odd, this makes sense.)

The proofs of the above theorems are presented in section 4.2. We make the following remarks.

Remark 4.1.1.

1. In Section 4.2 we first prove the trivial case where the Poincaré series $P^{k,m}_{n,r}$ does not vanish when the ratio $\frac{\pi D}{\det(2m)}$, $\left(\frac{D}{\det(2m)} = 2n - 2m^{-1}[\frac{1}{2}r^t]\right)$ by which we measure the non-vanishing of Jacobi Poincaré series, is $O(k)$, but with explicit range of the weight k where this is valid. This follows from Proposition 4.2.1 for arbitrary g and also from Theorem 4.1.5 in the case $g = 1$ (recall that $\dim J^{cusp}_{k,m,1} \sim O\left(\frac{k(m+1)}{12}\right)$).

2. Theorem 4.1.3 therefore improves the trivial case mentioned in the previous remark. However, achieving the "order of $k^{2-\epsilon}$ $(\epsilon > 0)$" as in [34] in the case of Jacobi Poincaré series using Rankin's methods seems difficult mainly because of the presence of the factor (c, D) instead of $(c, D)^{\frac{1}{2}}$ in the estimate of Kloosterman sums of degree g (even for small g, see section 4.2).

3. The condition that k be even when $2r \equiv 0 \pmod{\mathbb{Z}^g \cdot 2m}$ in Theorem 4.1.3 is

necessary, as the (n,r)-*th* Poincaré series vanish when k is odd and $2r \equiv 0 \pmod{\mathbb{Z}^g \cdot 2m}$. The restriction $k' \leq \frac{\pi D}{\det(2m)}$ in Theorem 4.1.3 is natural since we know the result in the complement (see Proposition 4.2.1). Same is true for the condition $D > \frac{m}{\pi}$ in Theorem 4.1.6.

4.2 Proofs for arbitrary degree g

4.2.1 Proof of Proposition 4.2.1

From the Fourier expansion of $P^{k,m}_{n,r}$ and from the inner product formula, we see that in order to prove that it is non-zero, it is enough to prove that $|S(n,r)| < \frac{1}{2\pi}$ (noting that $2m$ is a positive-definite matrix with integer entries, hence $\det(2m) \geq 1$), where

$$S(n,r) := \det(2m)^{-1/2} \sum_{c \geq 1} H^{\pm}_{m,c}(n,r,n,r) J_{k-g/2-1}\left(\frac{2\pi D}{det(2m) \cdot c}\right). \tag{4.2.1}$$

We will need the following estimates. (See [41],[43] and [7] respectively for details):

$$\text{(i)} \qquad |J_\nu(x)| \leq \min\left\{1, \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu\right\} \text{ for } x > 0 \text{ and } \nu \geq 2. \tag{4.2.2}$$

$$\text{(ii)} \qquad |H_{m,c}(n,r,n,\pm r)| \leq 2^{\omega(c)} c^{g/2-1}(D,c), \tag{4.2.3}$$

where $\omega(c)$ is the number of distinct prime divisors of c, $(D,c) = gcd(D,c)$.

First, we first obtain the following proposition which follows easily from trivial estimates of Bessel functions.

Proposition 4.2.1. *1. Let k be even when $2r \equiv 0 \pmod{\mathbb{Z}^g \cdot 2m}$. Then there exists an integer k_0 such that the (n,r)-th Poincaré series $P^{k,m}_{n,r}$ does not vanish identically*

$$when \ k \geq k_0 \quad and \quad D \leq \frac{k'}{\pi e} \cdot \det(2m).$$

If $k > g+3$, then one can take $k_0 = max\left(g+4, [\frac{g}{2}] + 69\right)$.

2. *For all* $D < \frac{1}{\pi}\det(2m)$, *the Poincaré series* $P^{k,m}_{n,r}$ *does not vanish identically, whenever the condition is non-void and* $k > \begin{cases} g+3 & \text{if } g \geq 2, \\ 5 & \text{if } g = 1. \end{cases}$

Lemma 4.2.2. *The condition in Proposition 4.2.1(2) is non-void for* $n < \frac{1}{6} + \frac{(2m-3)^2}{144m}$ *when* $g = 1$.

Proof. Here $D = 4mn - r^2$. Suppose that $D < \frac{1}{\pi}(2m) < \frac{1}{3}(2m)$. Then we have $2m(2n - \frac{1}{3}) < r^2 < 4mn$. Noticing that there is a square in the interval $[x, y]$, $(x, y \in \mathbb{R}^+)$ when $2\sqrt{x} + 1 < y - x$, we need to have,

$$2\sqrt{2m(2n - \frac{1}{3})} + 1 < \frac{2m}{3}, \quad \text{or} \quad n < \frac{1}{6} + \frac{(2m-3)^2}{144m}.$$

So, in the case $g = 1$, the Poincaré series $P^{n,r}_{k,m}$ does not vanish identically when $k > 4$ and n satisfies the condition of the lemma. □

Proof of Proposition 4.2.1. In a straightforward manner, using estimates (4.2.2) and (4.2.3), we get $|S(n,r)| \leq \frac{2}{\Gamma(k'+1)} \left(\frac{S}{2}\right)^{k'} \sum_c \frac{2^{\omega(c)}}{c^{k-g-1}}$, where $S := \frac{2\pi D}{det(2m)}$. The series $\sum_c \frac{2^{\omega(c)}}{c^{k-g-1}}$ converges for $k > g + 2$ (using the fact that for any $\delta > 0$, we have that $\omega(c) \sim o(c^\delta), c \mapsto \infty$). Using this, and the Stirling's formula, the second part of 1 is immediate. Part 2 of Proposition 4.2.1 follows from Corollary 4.2.3 given below. □

Corollary 4.2.3. *If* $k > g + 3$, *then one can take* $k_0 = max\left(g + 4, [\frac{g}{2}] + 69\right)$ *in Proposition 4.2.1.*

Proof. Examining the proof of Proposition 4.2.1, we see that when $k \geq g + 4$ the series

$$\sum_c \frac{2^{\omega(c)}}{c^{k-g-1}} < \zeta(2) = \frac{\pi^2}{6},$$

using the trivial bound $2^{\omega(c)} \leq c$. The rest follows by using Stirling's formula :

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\lambda_n}, \quad \text{where} \quad \frac{1}{12n+1} < \lambda_n < \frac{1}{12n}, \quad \text{for } n \in \mathbb{N}.$$

□

4.2.2 Poincaré series of small weights

For $\mathrm{Re}(s) > \frac{1}{2}(\frac{g}{2} - k + 2)$, using the 'Hecke trick', the Jacobi Poincaré series is defined by K. Bringmann in [9],

$$P^{k,m}_{n,r;s}(\tau, z) = \sum_{\gamma \in \Gamma^J_{g,\infty} \backslash \Gamma^J_g} \left(\frac{v}{|c\tau + d|^2} \right)^s e(n\tau + rz)|_{k,m}\gamma(\tau, z),$$

where, $\tau = u + iv \in \mathcal{H}$, $z \in \mathbb{C}^g$, $s \in \mathbb{C}$. Then for $k > \frac{g}{2} + 2$, $\; P^{k,m}_{n,r;0} \in J^{cusp}_{k,m}$ and has the same Fourier properties as $P^{k,m}_{n,r}$. We also consider conditions for it's non-vanishing in the following Proposition.

Proposition 4.2.4. *Under the hypotheses of Proposition 4.2.1 there exists an integer $C(m)$, depending on m such that the Poincaré series $P^{k,m}_{n,r;0}$ doesnot vanish identically*

$$\forall k \in \left[\max \left\{ C(m), \frac{g+7}{2} \right\}, \infty \right) \text{ and } (n, r) \in \mathbb{N} \times \mathbb{Z}^g \text{ with } D \leq \frac{k'}{\pi e} \cdot \det(2m).$$

Remark 4.2.5. *Though Proposition 4.2.1 is applicable here, the above theorem accounts for (possibly) smaller values of k.*

Proof. This theorem again follows from the arguments of the proof of Proposition 4.2.1. Here we use the following estimate for the Kloosterman sums of degree g (see [7, p.508,512]):

$$|H_{m,c}(n, r, n \pm r)| \leq 2^{\omega(c)} c^{-1/2} (D, c), \; \forall c \geq C(m),$$

where $C(m)$ is a constant depending on m. With the notation of Proposition 4.2.1, we have for some positive constant $C_1(m)$,

$$\begin{aligned} |S(n,r)| &\leq \sum_{1 \leq c \leq C(m)} \frac{2^{\omega(c)+1} c^{g/2-1} (D,c)}{\Gamma(k'+1)} \left(\frac{S}{2c} \right)^{k'} + \sum_{c > C(m)} \frac{2^{\omega(c)+1} c^{-1/2} (D,c)}{\Gamma(k'+1)} \left(\frac{S}{2c} \right)^{k'} \\ &\leq C(m)^{(g-1)/2} \sum_{1 \leq c \leq C(m)} \frac{2^{\omega(c)+1} c^{-1/2} (D,c)}{\Gamma(k'+1)} \left(\frac{S}{2c} \right)^{k'} + \sum_{c > C(m)} \frac{2^{\omega(c)+1} c^{-1/2} (D,c)}{\Gamma(k'+1)} \left(\frac{S}{2c} \right)^{k'} \\ &\leq \frac{2C_1(m)}{\Gamma(k'+1)} \left(\frac{S}{2} \right)^{k'} \sum_c \frac{2^{\omega(c)}}{c^{k-(g+3)/2}}. \end{aligned}$$

The condition $k > \frac{g+7}{2}$ precisely guarantees convergence of the series above. The rest of the proof is identical to that of Proposition 4.2.1. □

4.2.3 Proof of Theorem 4.1.3

We now come to the main result of this section. Let $D = \det\left(\begin{smallmatrix} 2n & r \\ r^t & 2m \end{smallmatrix}\right)$ and define $k' := k - g/2 - 1$.

Theorem 4.1.3. *Let k be even when $2r \equiv 0 \pmod{\mathbb{Z}^g \cdot 2m}$. Then there exist an integer k_0 and a constant $B > 3\log 2$ such that for all $k \geq k_0$ (depending only on g), the Jacobi Poincaré series $P_{n,r}^{k,m}$ does not vanish identically for*

$$k' \leq \frac{\pi D}{\det(2m)} \leq {k'}^{1+\alpha(g)} \exp\left\{-\frac{B\log k'}{\log\log k'}\right\},$$

where $\alpha(g) = \begin{cases} \frac{2}{3(g+2)} & \textit{if} \quad 1 \leq g \leq 4, \\ \frac{2}{3g} & \textit{if} \quad g \geq 5. \end{cases}$

Proof. We use Rankin's method as in [34]. With $S(n,r)$ as in equation (4.2.1), we need to prove $|S(n,r)| < \frac{1}{2\pi}$. Define

$$\sigma = {k'}^{-1/6}, \quad Q^* = \frac{2\pi D}{k' \det(2m)}, \quad M(D) = \exp\left\{\frac{B_1 \log D}{\log\log 2D}\right\}.$$

We have $|S(n,r)| \leq \det(2m)^{-1/2}|S_1(n,r)| + \det(2m)^{-1/2}|S_2(n,r)|$, where

$$|S_1(n,r)| = \sum_{1\leq c\leq Q^*} |H^{\pm}_{m,c}(n,r,n,r)||J_{k'}\left(\frac{k'Q^*}{c}\right)|,$$
$$|S_2(n,r)| = \sum_{c>Q^*} |H^{\pm}_{m,c}(n,r,n,r)||J_{k'}\left(\frac{k'Q^*}{c}\right)|.$$

Let $d = (c, D)$, $c = dr$, $r < \frac{Q^*}{d} < D$. Proceeding as in [34] (see also [30]) we get

$$\begin{aligned}
|S_1(n,r)| &\leq \sum_{d|D} 2^{\omega(d)} d \sum_{r<Q^*/d} 2^{\omega(r)} (dr)^{g/2-1} |J_{k'}\left(\frac{k'Q^*}{c}\right)| \\
&\leq M(D) Q^{*g/2-1} \sum_{d|D} 2^{\omega(d)} d \sum_{r<Q^*/d} \left(\frac{Q^*}{rd}\right)^{1-g/2} |J_{k'}\left(\frac{k'Q^*}{c}\right)| \\
&\leq M(D) Q^{*g/2-1} \sum_{d|D} 2^{\omega(d)} d \sum_{r<Q^*/d} \left(\frac{Q^*}{rd}\right)^{1/2} |J_{k'}\left(\frac{k'Q^*}{c}\right)| \\
&\leq A_1 M(D) Q^{*g/2-1} \sum_{d|D, d<Q^*} 2^{\omega(d)} d \left\{\frac{Q^*\sigma^3}{d} + 3\sigma^2\right\} \qquad (4.2.4) \\
&\leq A_2 M(D)^3 \frac{Q^{*g/2}}{k'^{1/2}} + A_3 M(D)^3 \frac{Q^{*g/2}}{k'^{1/3}} \\
&\leq A_4 M(D)^3 \frac{\left(\frac{\pi D}{\det(2m)}\right)^{g/2}}{k'^{g/2+1/2}} + A_5 M(D)^3 \frac{\left(\frac{\pi D}{\det(2m)}\right)^{g/2}}{k'^{g/2+1/3}}, \qquad (4.2.5)
\end{aligned}$$

where in inequality (4.2.4), we have used the estimate of Bessel functions given in [34, Lemma 4.4] and the implied constants are absolute.

However, the other sum $S_2(n, r)$ needs to be handled differently. We have

$$\begin{aligned}
|S_2(n,r)| \leq &\sum_{Q^*<c\leq k'Q^*} 2^{\omega(c)+1} c^{g/2-1} (D,c) |J_{k'}\left(\frac{2\pi D}{c\cdot\det(2m)}\right)| \qquad (4.2.6) \\
&+ \sum_{c>k'Q^*} 2^{\omega(c)+1} c^{g/2-1} (D,c) |J_{k'}\left(\frac{2\pi D}{c\cdot\det(2m)}\right)| \\
&\leq 2M(D) \frac{\left(\frac{\pi D}{\det(2m)}\right)^{k'}}{\Gamma(k'+1)} \sum_{Q^*<c\leq k'Q^*} \frac{1}{c^{k'-g/2}} + 2 \sum_{c>k'Q^*} c^{g/2+1} |J_{k'}\left(\frac{2\pi D}{c\cdot\det(2m)}\right)| \\
&\leq \frac{2M(D)}{Q^{*k'-g/2-1-\epsilon}} \frac{\left(\frac{\pi D}{\det(2m)}\right)^{k'}}{\Gamma(k'+1)} \sum_{Q^*<c\leq k'Q^*} \frac{1}{c^{1+\epsilon}} + \frac{2\left(\frac{\pi D}{\det(2m)}\right)^{g/2+2+\delta}}{\Gamma(k'+1)} \sum_{c>k'Q^*} \frac{1}{c^{1+\delta}} \qquad (4.2.7) \\
&\leq \frac{a_0 M(D)}{k'^{g/2+3/2+\epsilon}} \left(\frac{\pi D}{\det(2m)}\right)^{g/2+1+\epsilon} + a_1 \frac{\left(\frac{\pi D}{\det(2m)}\right)^{g/2+2+\delta}}{k'^{k'+1/2}}, \qquad (4.2.8)
\end{aligned}$$

where a_i, A_j are absolute constants, and $0 < \epsilon, \delta < 1$. Now for any $g \geq 1$, and $\alpha(g) = \frac{2}{3g+2}$; we choose $0 < \epsilon, \delta < \frac{1}{2}$ and find that $S_1(n,r)$ and $S_2(n,r)$ are small if we choose k large. If $g \geq 5$, then we find that a better bound $\alpha(g) = \frac{2}{3g}$ works. This completes the proof. □

4.3 Explicit basis for $J_{k,m}^{cusp}$ and proof of Theorem 4.1.5

4.3.1 Enumeration of the basis

H. Petersson proved that the first d_k $(= \dim S_k)$ Poincaré series $P_1^k, \cdots, P_{d_k}^k$ is a basis for the space of cusp forms S_k for $SL(2,\mathbb{Z})$. We prove the corresponding result for Jacobi forms. The proof is based on the dimension formula given in [16].

Theorem 4.3.1. *Let $k \geq m + 12$. Then we have the following basis for $J_{k,m}^{cusp}$:*

1. *If k is even,* $\left\{P^{k,m}_{D_\mu+4m\lambda_\mu,\mu}\right\}$ *,* $\mu = 0, 1, \ldots, m$ *;* $\lambda_\mu = 0, 1, \ldots, \dim S_{k+2\mu} - \left[\frac{\mu^2}{4m}\right] - 1$ *where* $D_\mu := 4m\left(\left[\frac{\mu^2}{4m}\right] + 1\right) - \mu^2$.
2. *If k is odd,* $\left\{P^{k,m}_{D_\mu+4m\lambda_\mu,\mu}\right\}$ *,* $\mu = 1, \ldots, m-1$ *;* $\lambda_\mu = 0, 1, \ldots, \dim S_{k+2\mu-1} - \left[\frac{\mu^2}{4m}\right] - 1.$

Proof. We prove the Theorem for k even, the other case is analogous. The condition $k \geq m + 12$ ensures that $\dim S_{k+2\mu} \geq \left[\frac{\mu^2}{4m}\right] + 1$ (see [16, p.103]). The proof follows Petersson's argument in the elliptic case (see [32], [39]). Let $d = \dim J_{k,m}^{cusp}$ and $\phi_1, \ldots, \phi_d$ be an orthonormal basis. We write

$$\phi_j(\tau, z) = \sum_{\substack{r \in \mathbb{Z} \\ D'>0, D' \equiv -r^2 \pmod{4m}}} c_j(D', r) e\left(\frac{D' + r^2}{4m}\tau + rz\right).$$

Then it is easy to verify using the orthonormality of the ϕ_j's that

$$P^{k,m}_{D_\mu+4m\lambda_\mu,\mu} = \lambda^{-1}_{k,m,D_\mu+4m\lambda_\mu} \sum_{j=1}^{d} c_j(D_\mu + 4m\lambda_\mu, \mu)\phi_j,$$

where μ and λ_μ varies as in the statement of the Theorem. We get a $d \times d$ matrix indexed by pairs $(D_\mu + 4m\lambda_\mu, \lambda_\mu)$ and j. It suffices to prove the matrix is invertible. If not, let there be a linear relation

$$\sum_{j=1}^{d} \xi_j\, c_j(D_\mu + 4m\lambda_\mu, \mu) = 0,\ (\xi_1, \ldots, \xi_d) \neq (0, \ldots, 0),\ \text{for all}\,(D_\mu + 4m\lambda_\mu, \mu).$$

Considering the non-zero Jacobi form $\Phi := \sum_{j=1}^{d} \xi_j \phi_j$, we see that the Fourier coefficients $c_\Phi(D_\mu + 4m\lambda_\mu, \mu)$ (μ and λ_μ as in the Theorem) are zero.

`Claim` : This implies that $D_{2\mu}\Phi = 0$ for $\mu = 0, \ldots, m$, (see chapter 1 for the definition of operators $D_{2\mu}$).

Accepting the claim, we infer that $\Phi = 0$, since we have an injection

$$\bigoplus_{0\le\nu\le m} D_{2\nu} \colon J^{cusp}_{k,m} \hookrightarrow \bigoplus_{0\le\nu\le m} S_{k+2\nu}.$$

This is a contradiction and hence completes the proof of the theorem.

`Proof of claim` : Let $\Phi \in J^{cusp}_{k,m}$. Then we have the following Fourier expansion of the modular form $D_{2\nu}\Phi$ of weight $k + 2\nu$, (cf. [16, p. 32], k even $, \nu = 0, \ldots, m$)

$$D_{2\nu}\Phi = A_{k,\nu} \sum_{n\ge0} \left(\sum_{r\colon r^2<4mn} \left(\sum_{0\le\mu\le\nu} \frac{(k+2\nu-\mu-2)!}{(k+2\nu-2)!} \frac{(-mn)^\mu\, r^{2\nu-2\mu}}{\mu!\,(2\nu-2\mu)!} \right) c_\Phi(n, r) \right) q^n, \tag{4.3.1}$$

where, $A_{k,\nu} := (2\pi i)^{-\nu} \frac{(k+2\nu-2)!\,(2\nu)!}{(k+\nu-2)!}$ and $q := e(\tau)$.

Let ℓ be an a positive even integer. Let $d_\ell := \dim S_\ell$. Since an elliptic cusp form $f = \sum_{n=1}^{\infty} a(n, f) q^n$ of weight ℓ is determined by the first d_ℓ of it's Fourier coefficients $a(1, f), \ldots, a(d_\ell, f)$, looking at equation (4.3.1) we need to prove that $c_\Phi(n_\nu, r_\nu) = 0$ for all r_ν such that $r_\nu^2 < 4mn_\nu\,, 0 \le r_\nu \le m$ and all n_ν such that $[\frac{r_\nu^2}{4m}] + 1 \le n_\nu \le d_{k+2\nu}$ ($\nu = 0, \ldots, m$). From now on let ℓ denote one of $k + 2\nu$, ($\nu = 0, \ldots, m$) and for convenience, we drop the suffix ν. To see this, first, if $|r| > 2m$ in equation (4.3.1), we can consider $-m \le r' = r - 2mx \le m$ for a suitable integer x and an $n' \ge 1$ such that $4mn' - r'^2 = 4mn - r^2$ and use the fact that $c_\Phi(n', r') = c_\Phi(n, r)$ and that $n \ge n' \ge 1$, so n' also satisfies the same upper bound as that of n (namely, $[\frac{r'^2}{4m}] + 1 \le n' \le d_\ell$). We can finally reduce to the case $0 \le r \le m$ since $c_\Phi(n, r) = c_\Phi(n, -r)$ as ℓ is even.

But any such n_ν can be written as $n_\nu = [\frac{\nu^2}{4m}] + 1 + \lambda_\nu = \frac{D_\nu + 4m\lambda_\nu + \nu^2}{4m}$ with $0 \le \nu \le m$ and D_ν, λ_ν as in the statement of the theorem. This proves the claim. $\square$

4.3.2 Non-vanishing of classical Poincaré series of index 1

Let $M^+_{k-1/2} := M^+_{k-1/2}(\Gamma_0(4))$ be the Kohnen's plus space for $\Gamma_0(4)$. See Chapter 1 for it's definition. The Eichler-Zagier map for Jacobi forms of integral weight and index 1 denoted by, $Z_1 \colon J_{k,1} \to M^+_{k-1/2}$ is defined by

$$Z_1 \colon \sum_{\substack{D>0, r\in\mathbb{Z} \\ D\equiv -r^2 \pmod 4}} c(D) e\left(\frac{D+r^2}{4}\tau + rz\right) \mapsto \sum_{0<D\in\mathbb{Z}} c(D) e(D\tau)$$

where the Fourier coefficient $c(D)$ does not depend on r.

Let k be even. Following the notation in [26], let $P_{k-1,4,D}$ ($D \equiv 0, -1 \pmod 4$) be the Poincaré series in $M^+_{k-1/2}$. From now on fix $D \in \mathbb{N}$ and $r \in \mathbb{Z}$.

Proposition 4.3.2. *Z_1 maps $P_{D,r} \in J^{cusp}_{k,1}$ to $3\, P_{k-1,4,D} \in M^+_{k-1/2}$.*

The proposition follows by direct calculation. We write down the Fourier expansion of $P_{D,r}$ (for convenience, as it can be read off from Proposition 1.1.8 for $g = 1$ and $m = 1$) and show that under Z_1, the (D', r')-th coefficient of $P_{D,r}$ maps to $3\cdot$ the D'-th Fourier coefficient of $P_{k-1,4,D}$.

— `Fourier development of` $P_{D,r}$:

Proposition 4.3.3 (Böcherer-Kohnen, [7])**.**

$$P_{D,r}(\tau, z) = \sum_{n'\in\mathbb{N}, r'\in\mathbb{Z}, 4n'>r^2} c_{n,r}(n', r') e(n'\tau + r'z), \quad \textit{where}$$

$$c_{n,r}(n',r') = \delta_1(n,r,n',r') + (-1)^k \delta_1(n,r,n',-r') + 2\pi i^k\, 2^{-1/2} \cdot (D'/D)^{k/2-3/2}$$
$$\times \sum_{c\geq 1} \left(H_{1,c}(n,r,n',r') + (-1)^k H_{1,c}(n,r,n',-r')\right) J_{k-3/2}\left(\frac{2\pi\sqrt{DD'}}{2c}\right) \quad (4.3.2)$$

where $D = 4n - r^2$, $D' = 4n' - r'^2$, $\delta_1(n,r,n',r') := \begin{cases} 1 & \text{if } D = D', \\ 0 & \text{otherwise} , \end{cases}$
and $H_{1,c}(n,r,n',r') := c^{-3/2} \sum_{x(c),y(c)^*} e_c\left((x^2+rx+n)\bar{y} + n'y + r'x\right) e_{2c}(rr')$,
where in the summation x *(resp.* y*) run over a complete set of representatives for* $\mathbb{Z}/c\mathbb{Z}$ *(resp.* $(\mathbb{Z}/c\mathbb{Z})^*$*),* $\bar{y}$ *denotes an inverse of* $y \pmod c$*, and* $J_{k-3/2}$ *denotes the Bessel function of order* $k-3/2$*.*

— `Fourier development of` $P_{k-1,4,D}$:

Proposition 4.3.4 (Kohnen, [26])**.**

$$P_{k-1,4,D}(\tau) = \sum_{t\geq 1, t\equiv 0,3(4)} g_D(t) e(t\tau), \quad \textit{with}$$

$$g_D(t) = \frac{2}{3}\left[\delta_{D,t} + (-1)^{k/2}\pi\sqrt{2}(t/D)^{k/2-3/4} \sum_{c\geq 1} H_c(t,D) J_{k-3/2}\left(\frac{\pi}{c}\sqrt{tD}\right)\right]. \quad (4.3.3)$$

Here $\delta_{t,D}$ *is the Kronecker delta, and,*

$$H_c(t,D) = (1-(-1)^{k-1}i)\left(1+\left(\frac{4}{c}\right)\right)\frac{1}{4c}\sum_{\delta(4c)^*}\left(\frac{4c}{\delta}\right)\left(\frac{-4}{\delta}\right)^{k-1/2} e_{4c}(t\delta + D\delta^{-1}).$$
(4.3.4)

Definition 4.3.5. *(i) Let* w *be an integer,* $c \geq 1$ *be even and* $u, v \equiv 0, (-1)^w \pmod 4$*. Let* $\alpha \in \{1,2\}$*. We define the following exponential sum,*

$$\mathbb{H}_{\alpha c}(u,v) := (1-(-1)^w i)\frac{1}{4c}\sum_{1\leq \delta \leq \alpha c - 1,\, (\delta,4c)=1}\left(\frac{4c}{\delta}\right)\left(\frac{-4}{\delta}\right)^{w+1/2} e_{4c}(u\delta + v\delta^{-1}), \quad (4.3.5)$$

where $\delta\delta^{-1} \equiv 1 \pmod{4c}$*.*

Lemma 4.3.6. *Let w be an integer, $c \geq 1$ be even and $u, v \equiv 0, (-1)^w \pmod 4$.*

1. $H_c(u,v) = (1+(-1)^{u+v+c/2})\,\mathbb{H}_{2c}(u,v)$ *Therefore, when $c \equiv 2 \pmod 4$, $H_c(u,v)$ vanishes unless u, v have different pairity.*

2. *Let $c \equiv 0 \pmod 4$. Then $H_c(u,v) = (1+(-1)^{u+v})(1+e_4(u-v))\,\mathbb{H}_c(u,v)$.*

Remark 4.3.1. In the sequel, we will use Definition 4.3.5 and the above lemma with $w = k-1$, with k even.

Proof. 1. This is easily seen by splitting the exponential sum as follows:

$$
\begin{aligned}
&\sum_{1\leq\delta\leq 4c-1\,,(\delta,4c)=1} \left(\frac{4c}{\delta}\right)\left(\frac{-4}{\delta}\right)^{w+1/2} e_{4c}(u\delta+v\delta^{-1}) \\
&= \sum_{1\leq\delta_1\leq 2c-1\,,(\delta_1,4c)=1} \left(\frac{4c}{\delta_1}\right)\left(\frac{-4}{\delta_1}\right)^{w+1/2} e_{4c}(u\delta_1+v{\delta_1}^{-1}) \\
&+ \sum_{1\leq\delta_1\leq 2c-1\,,(\delta_1,4c)=1} \left(\frac{4c}{\delta_1+2c}\right)\left(\frac{-4}{\delta_1+2c}\right)^{w+1/2} e_{4c}\left(u(\delta_1+2c)+v({\delta_1}^{-1}+2c)\right) \\
&= (1+(-1)^{u+v+c/2}) \sum_{1\leq\delta_1\leq 2c-1\,,(\delta_1,4c)=1} \left(\frac{4c}{\delta_1}\right)\left(\frac{-4}{\delta_1}\right)^{w+1/2} e_{4c}(u\delta_1+v{\delta_1}^{-1}), \qquad (4.3.6)
\end{aligned}
$$

where clearly $\left(\frac{-4}{\delta_1+2c}\right) = \left(\frac{-4}{\delta_1}\right)$ for c even, and,

$$
\left(\frac{4c}{\delta_1+2c}\right) = \left(\frac{2}{\delta_1+2c}\right)\cdot\left(\frac{c/2}{\delta_1+2c}\right) = (-1)^{c/2}\left(\frac{2}{\delta_1}\right)\cdot\left(\frac{c/2}{\delta_1}\right) = (-1)^{c/2}\left(\frac{4c}{\delta_1}\right).
$$

This completes the proof of 1.

2. Since 1. holds for any even c, we split the exponential sum in $\mathbb{H}_{2c}(u,v)$ and use that

with $c \equiv 0 \pmod 4$. We have:

$$\begin{aligned}
&\sum_{1\le\delta\le 2c-1\,,(\delta,4c)=1} \left(\frac{4c}{\delta}\right)\left(\frac{-4}{\delta}\right)^{w+1/2} e_{4c}(u\delta+v\delta^{-1})\\
&=\sum_{1\le\delta_1\le c-1\,,(\delta_1,4c)=1} \left(\frac{4c}{\delta_1}\right)\left(\frac{-4}{\delta_1}\right)^{w+1/2} e_{4c}(u\delta_1+v{\delta_1}^{-1})\\
&+\sum_{1\le\delta_1\le c-1\,,(\delta_1,4c)=1} \left(\frac{4c}{\delta_1+c}\right)\left(\frac{-4}{\delta_1+c}\right)^{w+1/2} e_{4c}\left(u(\delta_1+c)+v({\delta_1}^{-1}-c)\right) \qquad (4.3.7)\\
&\text{where } (\delta_1+c)({\delta_1}^{-1}-c)\equiv 1 \pmod{4c}\,,\ \text{since } \delta_1\equiv{\delta_1}^{-1} \pmod 4\\
&=(1+e_4(u-v))\sum_{1\le\delta_1\le c-1\,,(\delta_1,4c)=1} \left(\frac{4c}{\delta_1}\right)\left(\frac{-4}{\delta_1}\right)^{w+1/2} e_{4c}(u\delta_1+v{\delta_1}^{-1}) \qquad (4.3.8)
\end{aligned}$$

where clearly $\left(\frac{-4}{\delta_1+c}\right)=\left(\frac{-4}{\delta_1}\right)$ for $c \equiv 0 \pmod 4$, and

$$\left(\frac{4c}{\delta_1+c}\right)=\left(\frac{4}{\delta_1+c}\right)\cdot\left(\frac{c/4}{\delta_1+c}\right)=\left(\frac{4}{\delta_1}\right)\cdot\left(\frac{c/4}{\delta_1}\right)=\left(\frac{4c}{\delta_1}\right).$$

This completes the proof of 2. □

Definition 4.3.7. *We denote by $G(a,b,c)$ the quadratic Gauss sum defined by*

$$G(a,b,c) := \sum_{n \pmod c} e_c(an^2+bn) \quad \textit{where} \quad a,b,c\in\mathbb{Z}. \qquad (4.3.9)$$

The values of the Gauss sums are well known and so are their properties; like multiplicity properties, reciprocity law etc. Below we summarise the facts that will be needed in the sequel. See [2], [6] etc. for details.

Proposition 4.3.8. *We have the following:*

1. *$G(a,b,c)$ only depends on the residue classes of a,b modulo c.*
2. *$G(a,b,c_1c_2)=G(c_2a,b,c_1)\,G(c_1a,b,c_2)$, where $(c_1,c_2)=1$.*

3. *Let* $(a,c)=1$.

$$G(a,b,c)=\begin{cases}\epsilon_c\sqrt{c}\left(\frac{a}{c}\right)e_c\left(-\psi(a)b^2\right) & \text{if } c\equiv 1 \pmod 2, 4a\psi(a)\equiv 1 \pmod c\\ 2\,G(2a,b,\frac{c}{2}) & \text{if } c\equiv 2 \pmod 4, b\equiv 1 \pmod 2\\ 0 & \text{if } c\equiv 2 \pmod 4, b=0\\ (1+i){\epsilon_a}^{-1}\sqrt{c}\left(\frac{c}{a}\right) & \text{if } c\equiv 0 \pmod 4, b=0\\ 0 & \text{if } c\equiv 0 \pmod 4, b\equiv 1 \pmod 2.\end{cases} \tag{4.3.10}$$

To prove Proposition 4.3.2, we need to compare the Kloosterman type sums occuring in the Fourier development of index 1 Jacobi-Poincaré series with those in the Fourier development of half-integral weight Poincaré series. The next proposition addresses that.

We need some more definitions relating to Kloosterman type sums from [26]. Note that the factor $\frac{1}{Nc}$ is missing in the definition of $\mathscr{H}'_{Nc}(u,v)$ on [26, p. 256, equation (33)] and the factor $4^{-2} \pmod c$ is missing in [26, p. 256, equation (36)]; taking these into account,

Definition 4.3.9. *Let* $u,v,w\in\mathbb{Z}$, $c\geq 1$. *Define*

$$(i)\quad \mathscr{H}'_c(u,v):=\frac{1}{c}\left(\frac{4}{-c}\right)\left(\frac{-4}{c}\right)^{-w-1/2}\sum_{\delta(c)^*}\left(\frac{\delta}{c}\right)e_c\left(u\delta+v\delta^{-1}\right), \tag{4.3.11}$$

$$(ii)\quad \mathscr{H}_{4c}(u,v):=\left(1+\left(\frac{4}{c}\right)\right)\frac{1}{4c}\sum_{\delta(4c)^*}\left(\frac{4c}{\delta}\right)\left(\frac{-4}{\delta}\right)^{w+1/2}e_{4c}(t\delta+D\delta^{-1}). \tag{4.3.12}$$

Proposition 4.3.10 (Kohnen, [26])**.** *Let* $(4c_1,c_2)=1$.

$$\mathscr{H}_{4c_1c_2}(u,v)=\mathscr{H}_{4c_1}(u,v\,{c_2}^{-2})\,\mathscr{H}'_{c_2}(u,v\,4^{-2}\,{c_1}^{-2}), \tag{4.3.13}$$

where ${c_1}^{-1}c_1\equiv 1 \pmod{c_2}$, ${c_2}^{-1}c_2\equiv 1 \pmod{4c_1}$.

Proposition 4.3.11 (Kohnen, [26])**.** *Let* $u,v\equiv 0,(-1)^w \pmod 4$. *Then*

(i) $\mathscr{H}_4(u,v)=\frac{1}{4}(-1)^{uv}(1+(-1)^w)$.

(ii) $\mathscr{H}_8(u,v)=\frac{1}{2\sqrt{2}}\left(\frac{u+v}{2}\right)(1+(-1)^w)$.

Remark 4.3.2. (i) In the sequel, we will use the above definitions and Proposition 4.3.11 with $w = k - 1$, with k even.

(ii) We have $H_c(u, v) = (1 - (-1)^w i)\, \mathscr{H}_{4c}(u, v)$.

(iii) In the sequel we will use Proposition 4.3.10 with $c_1 = 1, 2$.

Proposition 4.3.12. *Let $c \geq 1$ and k even. Then* $H_{1,c}(n, r, n', \pm r') = H_c(D', D)$.

Proof. We distinguish 3 cases, for c odd and $c \equiv 0, 2 \pmod 4$. We recall that $\epsilon_\delta = \begin{cases} 1 & \text{if } \delta \equiv 1 \pmod 4, \\ i & \text{if } \delta \equiv 3 \pmod 4. \end{cases}$

1. $c \equiv 1 \pmod 2$

We use the values of Gauss sums from table (4.3.10) in Proposition 4.3.8.

$$
\begin{aligned}
H_{1,c}(n, r, n', r') &= c^{-3/2} \sum_{x(c), y(c)^*} e_c\left((x^2 + rx + n)\bar{y} + n'y + r'x\right) e_{2c}(rr') \\
&= c^{-3/2} \sum_{y(c)^*} G(\bar{y}, r\bar{y} + r', c)\, e_c(n\bar{y} + n'y) e_{2c}(rr') \qquad (4.3.14) \\
&= c^{-3/2} \epsilon_c \sqrt{c} \sum_{y(c)^*} \left(\frac{\bar{y}}{c}\right) e_c\left(-4^{-1} y (r\bar{y} + r')^2 + n\bar{y} + n'y\right) e_{2c}(rr') \\
&= \frac{\epsilon_c}{c} \sum_{y(c)^*} \left(\frac{\bar{y}}{c}\right) e_c\left(D'y + 4^{-2}\bar{y}D\right) e_2(DD') \\
&= \epsilon_c \left(\frac{4}{c}\right) \left(\frac{-4}{c}\right)^{k-1/2} \epsilon_c{}^{-1} (-1)^{DD'} 4 (-1)^{-DD'} (1 - i)^{-1} \mathscr{H}_{4c}(D', 4^{-2}D), \\
&= 2(1 + i) \mathscr{H}_{4c}(D', 4^{-2}D) = 2H_c(D', D), \text{ after simplification}, \qquad (4.3.15)
\end{aligned}
$$

where in the above $44^{-1} \equiv 1 \pmod c$, y varies over a reduced residue system modulo c and $y\bar{y} \equiv 1 \pmod c$. The equalities in the last two lines of the above calculation follow from Proposition 4.3.11 and Proposition 4.3.10 with $c_1 = 1$, $c_2 = c$.

2. $c \equiv 2 \pmod 4$

Let $c = 2c'$, with c' odd. From table (4.3.10), and Lemma 4.3.6, we see that $H_{1,c}(n, r, n', r') = 0 = H_c(D', D)$ if r and r' or equivalently D and D' have the same pairity. When they have opposite pairity, using the multiplicative property of Gauss sum in Proposition 4.3.8 and applying the formula from 4.3.10 we have:

$$\begin{aligned} H_{1,c}(n,r,n',r') &= c^{-3/2} \sum_{x(c),y(c)^*} e_c\left((x^2+rx+n)\bar{y}+n'y+r'x\right) e_{2c}(rr') \\ &= 2c^{-3/2} \sum_{y(2c')^*} G(2\bar{y}, r\bar{y}+r', c')\, e_{2c'}(n\bar{y}+n'y) e_{4c'}(rr') \\ &= 2c^{-3/2}\epsilon_{c'}\sqrt{c'} \sum_{y(2c')^*} \left(\frac{2\bar{y}}{c'}\right) e_{c'}\left(-8^{-1}y(r\bar{y}+r')^2\right) e_{2c'}\left(n\bar{y}+n'y\right) e_{4c'}(rr'). \end{aligned} \tag{4.3.16}$$

We make a change of variable $y \mapsto 2y+c'$ and find after simplification that the above sum, (in which y now varies over a reduced residue system modulo c' and $(2y+c')(2\cdot 4^{-1}\bar{y}+c') \equiv 1 \pmod{2c'}$, $44^{-1} \equiv 1 \pmod{c'}$) :

$$\begin{aligned} &= \frac{\epsilon_{c'}}{\sqrt{2c'}} \sum_{y(c')^*} \left(\frac{\bar{y}}{c'}\right) e_{c'}\left(D'y+4^{-3}\bar{y}D\right) e_2(n+n'+rr'/2) \\ &= \frac{1}{\sqrt{2}}\left(\frac{D'+D}{2}\right)\cdot \frac{\epsilon_{c'}}{c'} \sum_{y(c')^*} \left(\frac{\bar{y}}{c'}\right) e_{c'}\left(D'y+4^{-3}\bar{y}D\right) \\ &= (1+i)\mathscr{H}_8(D',D)\, \mathscr{H}'_{c'}(D',4^{-3}D) = (1+i)\mathscr{H}_{8c'}(D',D) = H_c(D',D), \end{aligned} \tag{4.3.17}$$

where the equalities in the last line follows from Proposition 4.3.11 and Proposition 4.3.10 with $c_1 = 2$, $c_2 = c'$.

3. *$c \equiv 0 \pmod{4}$*

From table (4.3.10), and Lemma 4.3.6, we see that $H_{1,c}(n,r,n',r') = 0 = H_c(D',D)$ if r and r' or equivalently D and D' have opposite pairity. When they have the same pairity, again applying the formula from 4.3.10, we have the following:

$$\begin{aligned} H_{1,c}(n,r,n',r') &= c^{-3/2}\sum_{y(c)^*} G(\bar{y}, r\bar{y}+r', c)\, e_c(n\bar{y}+n'y)e_{2c}(rr') \\ &= c^{-3/2}\sum_{y(c)^*} G(\bar{y}, 0, c)\, e_{4c}(D'y+D\bar{y}) \\ &= (1+i)c^{-3/2}\sum_{y(c)^*} {\epsilon_{\bar{y}}}^{-1}\sqrt{c}\left(\frac{c}{\bar{y}}\right)\, e_{4c}(D'y+D\bar{y}) \\ &= 4\,\mathbb{H}_c(D',D) = H_c(D',D)\,, \text{ since in this case } D\equiv D' \pmod 4, \end{aligned} \tag{4.3.18}$$

where the equality in the last line follows from Lemma 4.3.6(2) with $w = k-1$. ☐

Proof of Proposition 4.3.2. First trivially we have, $\delta_1(n,r,n',\pm r') = \delta_{D,D'}$. Therefore comparing the two Fourier developments and noting that k is even, we see that it is sufficient to prove for all $c \geq 1$ that $H_{1,c}(n,r,n',\pm r') = const.\cdot H_c(D',D)$. Combining 1, 2 and 3 from Proposition 4.3.12, we finally arrive at the conclusion that when k is even,

$$c_{n,r}(n',r') = 3\,g_D(D') \text{ for all } n,r,n',r', \tag{4.3.19}$$

where $c_{n,r}(n',r')$ and $g_D(D')$ are the coefficients on the Fourier expansions of the relevant Poincaré series defined above. This completes the proof of Proposition 4.3.2. ☐

Proposition 4.3.13. *There exist positive constants k_0 and B, where $B > 4\log 2$, such that, for all even $k \geq k_0$ and all positive integers $D \leq k^2\exp\{-B\log k/\log\log k\}$, the Poincaré series $P_{k-1,4,D}$ and hence also the Poincaré series $P^{k,1}_{D,r}$ does not vanish identically.*

Proof. From the Fourier expansion of $P_{k-1,4,D}$ given in [26], we see that the proof is the same as in the case of integral weight Poincaré series for congruence subgroups of $SL(2,\mathbb{Z})$ given in [30]; so we omit it. The next part of the Proposition follows from Proposition 4.3.2. Therefore in the case of Jacobi forms of index 1, we get a result in tune with Rankin's, and of course better than those in the other theorems in this chapter. ☐

4.3.3 Proof of Theorem 4.1.6

Define $M(x) := \exp\left\{\frac{B_1 \log x}{\log\log 2x}\right\}$ $(x \geq 2,\ B_1 > \log 2)$ as in [34].

Theorem 4.1.6. *Let $g = 1$. For $D > \frac{m}{\pi}$, we have $P^{k,m}_{D,r} \not\equiv 0$ for*

$$M\left(\frac{\pi D}{m}\right)\sigma_0(D)\,D < \frac{m^{\frac{8}{7}}}{\lambda},$$

where $\lambda = (2\sqrt{2}\pi^{\frac{5}{3}}A)^{\frac{3}{2}}$, $A = \frac{1}{\pi}\left(\frac{2}{6^{\frac{2}{3}}} + \frac{54}{2^{\frac{5}{6}}} + \frac{16}{2^{\frac{3}{4}}}\right)$ and $\sigma_0(D) = \sum_{d|D} 1$.

Proof. We write $S(n,r) = S_1(n,r) + S_2(n,r)$, where

$$S_1(n,r) = i^k \pi\sqrt{2} m^{-1/2} \sum_{1 \leq c \leq \frac{\pi D}{m}} H^{\pm}_{m,c}(n,r,n,r) J_{k'}\left(\frac{\pi D}{mc}\right),$$
$$S_2(n,r) = i^k \pi\sqrt{2} m^{-1/2} \sum_{c > \frac{\pi D}{m}} H^{\pm}_{m,c}(n,r,n,r) J_{k'}\left(\frac{\pi D}{mc}\right).$$

We use the following estimate of Bessel functions to estimate $S_1(n,r)$:

$$|J_\nu(r)| \leq A r^{-1/3}, \quad \text{where} \quad \nu \geq 0,\ r \geq 1 \tag{4.3.20}$$

(cf. [18, Lemma 3.4], the constant C appearing in the Lemma can be computed to be the constant A in Theorem 4.1.6 using [40, p. 333].)

$$\begin{aligned}
|S_1(n,r)| &\leq \frac{2\sqrt{2}\pi}{m^{1/2}} \sum_{1 \leq c \leq \frac{\pi D}{m}} \frac{2^{\omega(c)}(D,c)}{c^{1/2}} |J_{k'}\left(\frac{\pi D}{mc}\right)| \\
&\leq \frac{2\sqrt{2} m^{1/3}\pi^{2/3}}{D^{1/3} m^{1/2}} M\left(\frac{\pi D}{m}\right) \sum_{1 \leq c \leq \frac{\pi D}{m}} \frac{(D,c)}{c^{1/6}} \\
&\leq \frac{2\sqrt{2}\pi^{2/3}}{D^{1/3} m^{1/6}} M\left(\frac{\pi D}{m}\right) \sum_{d|D, d < \frac{\pi D}{m}} d \\
&\leq \frac{2\sqrt{2} D^{2/3}\pi^{5/3}}{m^{7/6}} M\left(\frac{\pi D}{m}\right) \sigma_0(D).
\end{aligned} \tag{4.3.21}$$

$$
\begin{aligned}
|S_2(n,r)| &\le \frac{2\sqrt{2}\pi}{m^{1/2}} \sum_{c>\frac{\pi D}{m}} c^{3/2} |J_{k'}\left(\frac{\pi D}{mc}\right)| \\
&\le \frac{2\sqrt{2}\pi}{\Gamma(k'+1)m^{1/2}} \sum_{c>\frac{\pi D}{m}} c^{3/2}\left(\frac{\pi D}{mc}\right)^{3/2+2} \\
&\le \frac{2\sqrt{2}\pi^{9/2}D^{7/2}}{\Gamma(k'+1)m^4} \sum_{c>\frac{\pi D}{m}} \frac{1}{c^2} \le \frac{2\sqrt{2}\pi^{13/2}D^{7/2}}{6\,\Gamma(k'+1)m^4}.
\end{aligned} \tag{4.3.22}
$$

From the bound given in Theorem 4.1.6, it follows from estimates (4.3.21) and (4.3.22) that S_1 and S_2 are both less than $\frac{1}{2}$ in absolute value. Finally, from the expression of the (n,r)-*th* Fourier coefficient of $P^{k,m}_{n,r}$ given in Proposition 1.1.8, we get the Theorem. □

4.4 Further results

Recall the one dimensional Kloosterman sum for a positive integer c:

$$
S(r,m;c) = \sum_{\substack{h=1 \\ (h,c)=1}}^{c} e_c(rh+mh'), \text{ where } hh' \equiv 1 \pmod{c}. \tag{4.4.1}
$$

It is well known that (see [34, §3] for example) the following relation holds for an odd prime p :

$$
S(rp^\rho, mp^\mu; c) = S(r, mp^{\rho+\mu}; c) + pS(rp^{\rho-1}, mp^{\mu-1}; c/p), \text{ where } p|c, p\nmid r, p\nmid m\,(\rho,\mu\ge 1). \tag{4.4.2}
$$

Definition 4.4.1. *We let*

$$
K_{m,c}(n,r,n',r') = \sum_{x(c),y(c)^*} e_c\left((m[x]+rx+n)\bar{y}+n'y+r'x\right) \tag{4.4.3}
$$

$$
= c^{g/2+1}\, e_{2c}\left(-r'm^{-1}r^t\right) H_{m,c}(n,r,n',r') \tag{4.4.4}
$$

where in the above sum, $x \in \mathbb{Z}^g/c\mathbb{Z}^g, r \in \mathbb{Z}^g$.

Lemma 4.4.2. *Let p be a odd prime such that* $p|(c,r,r')$, $p \mid 2m$, $p \nmid n, p \nmid n'$. *Then the following identity holds :*

$$K_{mp^{\mu},c}(p^{\mu}n, p^{\mu}r, p^{\rho}n', r') = K_{mp^{\rho+\mu},c}(p^{\rho+\mu}n, p^{\rho+\mu}r, n', r')+ \\ + p^2 K_{mp^{\mu-1},c/p}(p^{\mu-1}n, p^{\mu-1}r, p^{\rho-1}n', r'/p) \tag{4.4.5}$$

Proof. The proof follows by noting that,

$$K_{m,c}(n,r,n',r') = \sum_{x \pmod c} e_c(r'x)\, S(n', m[x]+rx+n; c)\,, \tag{4.4.6}$$

from which the *L.H.S.* and the first term of the *R.H.S.* in (4.4.5) are taken care of by summing both sides of the equation (4.4.2) with appropriate arguments over $x \pmod c$. For the last term, we split the summation in (4.4.6) (after replacing $(m,n,r,n';c)$ by $(p^{\mu-1}m, p^{\mu-1}n, p^{\mu-1}r, p^{\rho-1}n'; \frac{c}{p})$ respectively) as $x = \frac{c}{p}x_1 + x_2$, where x_1 (resp.) x_2 range over $\mathbb{Z}^g/p\mathbb{Z}^g$ (resp.) $\mathbb{Z}^g/\frac{c}{p}\mathbb{Z}^g$. We have

$$\begin{aligned} &\sum_{x \pmod c} e_c(r'x)\, S\left(p^{\rho-1}n', p^{\mu-1}(m[x]+rx+n); c/p\right) \\ &= \sum_{x_1,x_2} e_c\left(r'(c/p\, x_1 + x_2)\right) S\left(p^{\rho-1}n', p^{\mu-1}\left((c/p\, x_1+x_2)^t m(c/p\, x_1 + x_2)\right.\right. \\ &\left.\left. + r(c/p\, x_1 + x_2) + n\right); c/p\right) \\ &= \sum_{x_1} e_p\left(r'c/p\, x_1\right) \sum_{x_2} e_{c/p}\left(r'/p\, x_2\right) S\left(p^{\rho-1}n', p^{\mu-1}(m[x_2]+rx_2+n); c/p\right) \\ &= pK_{mp^{\mu-1},c/p}(p^{\mu-1}n, p^{\mu-1}r, p^{\rho-1}n', r'/p). \end{aligned}$$

Therefore using (4.4.2) the lemma follows. □

4.4.1 Proof of Theorem 4.1.7

Now we come to the main result of this section, mentioned in the inroduction of this chapter.

Theorem 4.1.7. *Let p be an odd prime, $\mu \in \mathbb{N}$. Suppose that $p \mid m, p \mid r, p \nmid n$. If $P^{k,p^\mu m}_{p^\mu n, p^\mu r} \not\equiv 0$, then*

$$\text{either } P^{k,mp^{\mu-1}}_{np^{\mu-1}, rp^{\mu-1}} \not\equiv 0 \quad \text{or} \quad P^{k,p^{2\mu}m}_{np^{2\mu}, rp^{2\mu}} \not\equiv 0 \quad \text{and} \quad P^{k,p^{2\mu}m}_{n, rp^{\mu}} \not\equiv 0. \tag{4.4.7}$$

(Here $p \mid m$ means p divides every entry of m; since $2m$ is a $(g \times g)$ matrix with integer entries and p is odd, this makes sense.)

Proof. From Lemma 4.4.2 we easily deduce that under the conditions of the lemma,

$$\begin{aligned} H_{mp^\mu, c}\left(p^\mu n, p^\mu r, p^\rho n', r'\right) &= H_{mp^{\rho+\mu}, c}\left(p^{\rho+\mu} n, p^{\rho+\mu} r, n', r'\right) \\ &\quad + p^{-\frac{g}{2}+1} H_{mp^{\mu-1}, \frac{c}{p}}\left(p^{\mu-1} n, p^{\mu-1} r, p^{\rho-1} n', \frac{r'}{p}\right). \end{aligned} \tag{4.4.8}$$

In the case $p \nmid c$, we note that we have the equality from the definition,

$$H_{mp^\mu, c}\left(p^\mu n, p^\mu r, p^\rho n', r'\right) = H_{mp^{\rho+\mu}, c}\left(p^{\rho+\mu} n, p^{\rho+\mu} r, n', r'\right). \tag{4.4.9}$$

We sum equation (4.4.8) over $c \geq 1$ such that $p \mid c$, equation (4.4.9) over all $c \geq 1$ and add them. Gathering all of above and noting that $\frac{2\pi\sqrt{D'D}}{det(2m)\cdot c}$ is the same in all the three sums (putting $\rho = \mu$ and $n' = n$, $r' = p^\mu r$), we get positive constants α_1 and α_2, such that

$$c^{k,p^\mu m}\left(p^\mu n, p^\mu r\right) = \alpha_1\, c^{k,p^{2\mu} m}\left(p^{2\mu} n, p^{2\mu} r; n, p^\mu r\right) + \alpha_2\, c^{k,p^{\mu-1} m}\left(p^{\mu-1} n, p^{\mu-1} r\right)$$

(where we have used the notation $c_{P^{k,m}_{n,r}}(n,r) := c^{k,m}(n,r;n,r) = c^{k,m}(n,r)$). This immediately implies (4.4.7) and thus completes the proof of Theorem **??**. □

Remark 4.4.1. The constants α_1, α_2 in the above proof can be determined explicitly and may give a better result in the same vein as Theorem 4.2.4 (see [34, Section 6]).

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