# On certain correspondences between Jacobi forms and modular forms, and a non-vanishing result for half-integral weight $L$-functions


By

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Enrolment Number : MATH08200704001

Harish-Chandra Research Institute, Allahabad


A Thesis submitted to the

Board of Studies in Mathematical Sciences

In partial fulfillment of requirements

For the degree of

**DOCTOR OF PHILOSOPHY**

*of*

**HOMI BHABHA NATIONAL INSTITUTE**

**January, 2013**

# Homi Bhabha National Institute

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As members of the Viva Voce Board, we certify that we have read the dissertation prepared by Karam Deo Shankhadhar entitled "On certain correspondences between Jacobi forms and modular forms, and a non-vanishing result for half-integral weight $L$-functions" and recommend that it may be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

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# DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree/diploma at this or any other Institution/University.

Date: **Karam Deo Shankhadhar**

*Dedicated to my mother*

# Acknowledgments

It is indeed my great pleasure to have this opportunity to express my thanks and gratitude to my thesis supervisor Prof. B. Ramakrishnan for his relentless guidance, continuous support and constant encouragement during my Ph.D. work. Again I would like to thank him for interacting with me very extensively and patiently on various topics in number theory, while giving his valuable insights; for encouraging my freedom of expression and for helping me in understanding the subject. I am grateful to him for his patience to rectify my repeated mistakes. This thesis would not have been possible without his careful guidance.

I express my sincere thanks to Prof. S. Böcherer and Prof. M. Manickam for helping me in problems, which cropped up at various stages of this thesis. I am deeply indebted to Prof. W. Kohnen, Prof. K. Murty, Prof. R. Murty, and Prof. N.-P. Skoruppa who helped me enormously with their invaluable suggestions during my Ph.D.. I am specially thankful to Prof. S. D. Adhikari, Dr. S. Gun, Dr. P. Rath and Dr. B. Sahu for providing their advice and suggestions.

I thank the members of my doctoral committee, Dr. D. S. Ramana and Dr. R. Thangadurai for their helpful guidance throughout my stay at HRI. I owe my deepest gratitude to all the faculty members of HRI who have taught me various courses in mathematics, without whom I could not have reached at this stage. I would like to thank the administrative staff of HRI for their cooperation.

I thank all my academic friends specially Archana, Jaban, Kasi and Vijay who shared their valuable thoughts with healthy discussions on and off the subject matter. I am deeply indebted to my school teacher Mr. Sanjay Upadhyay for encouraging me to pursue higher studies in mathematics.

I thank all my HRI friends specially Akhilesh, Bhavin, Jay, Pradip Mishra, Pradeep Rai, Rajarshi, Ram Lal, Ramesh, Senthil, Shailesh and Vikas for being with me and spending memorable time specially in Volleyball game at HRI. I am grateful to all my university friends specially Ambuj, Dharm Vir, Shashank, and Uday Bhan for their support and encouragement. I would like to thank all friends of my life for being with me whenever I needed the most and helping me in various ways.

Last but far from the least I would like to thank all the members of my family specially my parents, sisters, bhabhi, bhaiya Kavit and my wife Indira for their unconditional love and constant support.

# Contents

# Synopsis

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**Title of thesis:** On certain correspondences between Jacobi forms and modular forms, and a non-vanishing result for half-integral weight $L$-functions.

**Name of candidate:** Karam Deo Shankhadhar.

**Name of supervisor:** Prof. B. Ramakrishnan.

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This thesis contains some of my works on Jacobi forms and modular forms during my stay at Harish-Chandra Research Institute as a research scholar.

## 1. Restriction map for Jacobi forms

Let $N, k$ and $m$ be positive integers and $\chi$ be a Dirichlet charecter modulo $N$ with the property that $\chi(-1) = (-1)^k$. We use the notations $M_k(N, \chi)$ and $S_k(N, \chi)$ for the space of modular forms and the space of cusp forms respectively, of weight $k$ and character $\chi$ for the congruence subgroup $\Gamma_0(N)$ (for definition and other details, we refer to Shimura's book [Sh71]). We denote the space of Jacobi forms of weight $k$, index $m$ and character $\chi$ for the Jacobi group $\Gamma_0(N) \ltimes \mathbb{Z}^2$ by $J_{k,m}(N, \chi)$ (for definition and other details, we refer to Eichler-Zagier's monograph [EZ85]). It is well known that any such Jacobi form $\phi(\tau, z)$ can be (uniquely) written as

$$\phi(\tau, z) = \sum_{r=0}^{2m-1} h_{m,r}(\tau)\theta^J_{m,r}(\tau, z),$$

with

$$\begin{aligned}\theta^J_{m,r}(\tau,z) &= \sum_{n\in\mathbb{Z}} e^{2\pi i m((n+\frac{r}{2m})^2\tau+2(n+\frac{r}{2m})z)},\\ h_{m,r}(\tau) &= \sum_{\substack{n\in\mathbb{Z}\\ n\geq r^2/4m}} c_\phi(n,r)e^{2\pi i(n-\frac{r^2}{4m})\tau},\end{aligned}$$

where $c_\phi(n,r)$ denotes the $(n,r)$-th Fourier coefficient of the Jacobi form $\phi$. The (column) vector $\Theta^J(\tau,z) = (\theta^J_{m,r}(\tau,z))_{0\leqslant r<2m}$ satisfies the following transformation property.

$$\Theta^J([\gamma,(0,0),1](\tau,z)) = e^{2\pi i m\frac{cz^2}{c\tau+d}}(c\tau+d)^{\frac{1}{2}}U_m(\gamma)\Theta^J(\tau,z), \tag{1}$$

for all $\gamma = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in SL_2(\mathbb{Z})$. Here, $U_m : SL_2(\mathbb{Z}) \longrightarrow U(2m,\mathbb{C})$ denotes a (projective) representation of $SL_2(\mathbb{Z})$.

The row vector $\mathbf{h} = (h_{m,r})_{0\leqslant r<2m}$ satisfies the transformation property.

$$\mathbf{h}(\gamma\tau) = \chi(d)(c\tau+d)^{k-\frac{1}{2}}\mathbf{h}(\tau)\overline{U_m(\gamma)}^t \tag{2}$$

for all $\gamma = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \Gamma_0(N)$.

Let $\mathcal{D}_0 : J_{k,m}(N,\chi) \longrightarrow M_k(N,\chi)$ be the restriction map given by $\phi(\tau,z) \mapsto \phi(\tau,0)$ and $\mathcal{D}_2$ be the differential operator

$$\mathcal{D}_2 = \left(\frac{k}{2\pi i}\frac{\partial^2}{\partial z^2} - 4\frac{\partial}{\partial\tau}\right)\Big|_{z=0}, \tag{3}$$

which acts on holomorphic functions on $\mathbb{H}\times\mathbb{C}$ and takes Jacobi forms of weight $k$ to cusp forms of weight $k+2$. More generally, one obtains modular forms of weight $k+\nu$ from Jacobi forms of weight $k$ by using certain differential operators $\mathcal{D}_\nu$ as discussed in [EZ85]. Then it is known that $\oplus_{\nu=0}^m \mathcal{D}_{2\nu} : J_{k,m}(N) \longrightarrow M_k(N) \oplus S_{k+2}(N) \oplus ... \oplus S_{k+2m}(N)$ is injective for $k$ even. In this chapter, we study the kernel of $\mathcal{D}_0$ in detail. We denote the kernel of the restriction map $\mathcal{D}_0$ for the space of index $m$ Jacobi forms on $\Gamma_0(N)$ by $J_{k,m}(N,\chi)^0$. When $m=1$, T. Arakawa and S. Böcherer [AB99, Theorem 2] provided two explicit descriptions of $\mathrm{Ker}(\mathcal{D}_0)$: one in terms of modular forms of weight $k-1$ and the other in terms of cusp forms of weight $k+2$ (by applying the differential operator $\mathcal{D}_2$ on $\mathrm{Ker}(\mathcal{D}_0)$). In a subsequent paper [AB03, Theorem 4.3], they proved that $\mathcal{D}_0$ is injective in the case $k = 2$, $m = 1$ and gave some applications. In a private communication, Professor Böcherer informed that one of his students gave a precise description of the image of $\mathcal{D}_0 \oplus \mathcal{D}_2$ in terms of vanishing orders in the cusps ($k$ is an even integer, $m = 1$). Based on this, he conjectured that in the case $k = 2$, one can remove one of the $\mathcal{D}_{2\nu}$ from the direct sum $\oplus_{\nu=0}^{m}\mathcal{D}_{2\nu}$ without affecting the injectivity. In Chapter 1, we give a partial generalization of the work of Arakawa and Böcherer [AB99, AB03], to higher index. In particular, we prove Böcherer's conjecture mentioned above for $m = 2$.

## 1.1. The space of Jacobi forms of index 2

Throughout this section $N = 2$, or an odd square-free positive integer. Define a (projective) representation of $\Gamma_0(2)$ by $\rho_2(\gamma) := (u_{11} + u_{13})^{-1}\overline{U_1}\left(\begin{pmatrix} a & 2b \\ c/2 & d \end{pmatrix}\right)$, for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2)$ with $U_2(\gamma) = (u_{ij})_{0 \leqslant i,j < 4}$. We denote the space of all vector valued modular forms on $\Gamma_0(2N)$ of weight $k$ with character $\chi$ and representation $\rho_2$ by $VM_k(2N, \chi; \rho_2)$. Let $\phi \in J_{k,2}(2N, \chi)^0$, i.e., $\phi$ is in the kernel of the restriction map $\mathcal{D}_0$. This gives us

$$0 = \phi(\tau, 0) = h_{2,0}(\tau)\theta_{2,0}(\tau) + 2h_{2,1}(\tau)\theta_{2,1}(\tau) + h_{2,2}(\tau)\theta_{2,2}(\tau).$$

Define two new functions as

$$\varphi_0(\tau) := \frac{h_{2,0}}{\theta_{2,1}}(\tau), \qquad \varphi_2(\tau) := \frac{h_{2,2}}{\theta_{2,1}}(\tau). \tag{4}$$

Since $\theta_{2,1}$ has no zeros in the upper half plane, $\varphi_0$ and $\varphi_2$ are holomorphic on the upper half plane. We prove that $(\varphi_o, \varphi_2)^t \in VM_{k-1}(2N, \chi; \rho_2)$. Define $\xi_0 := \theta_{2,1}\theta'_{2,0} - \theta_{2,0}\theta'_{2,1}$ and $\xi_2 := \theta_{2,1}\theta'_{2,2} - \theta_{2,2}\theta'_{2,1}$. Proceeding as in [AB99, Proposition 2], we get that $(\xi_0, \xi_2)^t \in VM_3(2, \chi; (\rho_2^{-1})^t)$. Define a subspace of the space $S_k(2N, \chi)$ as follows.

$$S_k^2(2N, \chi)^0 := \{f \in S_k(2N, \chi) \colon f = \varphi_0\xi_0 + \varphi_2\xi_2 \text{ with } (\varphi_0, \varphi_2)^t \in VM_{k-3}(2N, \chi; \rho_2)\}.$$

We prove the following theorem.

**Theorem 1.1.** There is a linear isomorphism

$$\Lambda_2 : J_{k,2}(2N,\chi)^0 \longrightarrow VM_{k-1}(2N,\chi;\rho_2),$$

given by $\phi \mapsto (\varphi_0, \varphi_2)^t$, where $\varphi_0$ and $\varphi_2$ are defined by (4). Further, the map $\mathcal{D}_2 : J_{k,2}(2N,\chi) \longrightarrow S_{k+2}(2N,\chi)$ induces an isomorphism between $J_{k,2}(2N,\chi)^0$ and $S^2_{k+2}(2N,\chi)^0$. More precisely, we have the following commutative diagram of isomorphisms:

$$\begin{array}{ccc} & J_{k,2}(2N,\chi)^0 & \\ \swarrow^{\Lambda_2} & & \searrow^{\mathcal{D}_2} \\ VM_{k-1}(2N,\chi;\rho_2) & \longleftrightarrow & S^2_{k+2}(2N,\chi)^0 \end{array}$$

where the isomorphism in the bottom of the diagram is given by

$$(\varphi_0, \varphi_2) \mapsto 8k(\xi_0\varphi_0 + \xi_2\varphi_2).$$

In [AB03, Corollary 2.3], it was shown that there is no nonzero cusp form of square-free level $N$ and weight $k$ which is divisible by $\eta^{2k-2}(N\tau)$ (under the condition that $k \equiv 4 \pmod{12}$ or $k \equiv 10 \pmod{12}$), where $\eta(\tau)$ is the Dedekind eta-function. Using this result and Atkin-Lehner $W$-operator, we prove the following result as a corollary to Theorem 1.1. This confirms the conjecture made by Böcherer partially in the case $m = 2$ (i.e., we can omit the operator $\mathcal{D}_4$).

**Corollary 1.2.** For $N = 2$ or an odd square-free positive integer, the differential map $\mathcal{D}_0 \oplus \mathcal{D}_2$ is injective on $J_{2,2}(2N)$.

## 1.2. A certain subspace of the space of Jacobi forms of square-free index

Throughout this section m is a square-free positive integer and N is any positive integer. Consider the following subspace of the space of Jacobi forms of index $m$ on $\Gamma_0(mN)$:

$$J^\star_{k,m}(mN,\chi) := \{\phi \in J_{k,m}(mN,\chi) \ : \ h_{m,r} = 0 \ \text{ for all } \ r \neq 0, m\}.$$

Denote the restriction of the space $\mathrm{Ker}(\mathcal{D}_0)$ on this space by $J^\star_{k,m}(mN,\chi)^0$. In this section we study this kernel space. The purpose of considering this restriction is to relate this kernel with index 1 kernel, which was studied in detail by Arakawa and Böcherer [AB99, AB03]. Note that $J^{\star}_{k,1}(N,\chi) = J_{k,1}(N,\chi)$.

Suppose that $\phi \in J^{\star}_{k,m}(mN,\chi)^0$. Then $0 = \phi(\tau,0) = h_{m,0}(\tau)\theta_{m,0}(\tau) + h_{m,m}(\tau)\theta_{m,m}(\tau)$. We define a new function by

$$\varphi(\tau) := \frac{h_{m,0}}{\theta_{m,m}}(\tau) = \frac{-h_{m,m}}{\theta_{m,0}}(\tau). \tag{5}$$

Since $\theta_{m,0}$ and $\theta_{m,m}$ have no zeros in the upper half-plane, $\varphi$ defines a holomorphic function in the upper half-plane. For a given $m$, we define the character $\omega_m\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \det\left(U_1\left(\begin{pmatrix} a & bm \\ c/m & d \end{pmatrix}\right)\right)$, for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(m)$. We simply write $\omega$ (as used in [AB99]) for $\omega_1$. Let $M_k(mN,\chi\overline{\omega}_m)$ denote the space of all modular forms of weight $k$ for $\Gamma_0(mN)$ with character $\chi\overline{\omega}_m$. We prove that the function $\varphi$ belongs to $M_{k-1}(mN,\chi\overline{\omega}_m)$. Define $\xi(\tau) := (\theta_{1,1}\theta'_{1,0} - \theta_{1,0}\theta'_{1,1})(\tau)$ as in [AB99] and $\xi^{\star}_m(\tau) := \xi(m\tau)$. Define the space

$$S^{\star}_k(mN,\chi)^0 := \{f \in S_k(mN,\chi) \ : \ f/\xi^{\star}_m \in M_{k-3}(mN,\chi\overline{\omega}_m)\}.$$

We prove the following theorem.

**Theorem 1.3.** There is a linear isomorphism

$$\Lambda^{\star}_m : J^{\star}_{k,m}(mN,\chi)^0 \longrightarrow M_{k-1}(mN,\chi\overline{\omega}_m)$$

given by $\phi \mapsto \varphi$, where $\varphi$ is defined by (5). The map $\mathcal{D}_2 : J_{k,m}(mN,\chi) \longrightarrow S_{k+2}(mN,\chi)$ induces an isomorphism between $J^{\star}_{k,m}(mN,\chi)^0$ and $S^{\star}_{k+2}(mN,\chi)^0$. Combining these two, we get the following commutative diagram of isomorphisms:

$$\begin{array}{ccc} & J^{\star}_{k,m}(mN,\chi)^0 & \\ \swarrow^{\Lambda^{\star}_m} & & \searrow^{\mathcal{D}_2} \\ M_{k-1}(mN,\chi\overline{\omega}_m) & \longleftrightarrow & S^{\star}_{k+2}(mN,\chi)^0 \end{array}$$

where the isomorphism in the bottom is given by $\varphi \mapsto 4m^2k\xi^{\star}_m\varphi$.

Let $J_{k,m}(mN,\nu)$ denote the space of Jacobi forms of index $m$ on $\Gamma_0(mN)$ with character $\nu$. The isomorphism diagram of Theorem 1.3 gives the following corollary.

**Corollary 1.4.** The two kernel spaces $J_{k,1}(mN,\chi)^0$ and $J^{\star}_{k,m}(mN,\chi\omega_m\overline{\omega})^0$ are isomorphic.

In [AB03, Theorem 4.3], it was shown that $J_{2,1}(N)^0 = \{0\}$ for square-free $N$. Using this theorem with the above corollary, we get

**Corollary 1.5.** Let $N$ be a square-free positive integer and coprime to $m$. Then $J^\star_{2,m}(mN, \omega_m\overline{\omega})^0 = \{0\}$.

We know that the operator $\mathcal{D}_0 \oplus \mathcal{D}_2$ is injective on $J_{k,1}(N)$. Also by [AB03, Theorem 4.3], $\mathcal{D}_0$ is injective on $J_{2,1}(N)$ for square-free $N$. Now we deduce similar kind of results for the space $J^\star_{k,m}(mN)$ in the following two corollaries to Theorem 1.3. Using the fact that $\xi^\star_m$ has no zeroes in the upper half-plane, we have

**Corollary 1.6.** The differential map $\mathcal{D}_0 \oplus \mathcal{D}_2 : J^\star_{k,m}(mN) \longrightarrow M_k(mN) \oplus S_{k+2}(mN)$ is injective.

By using the Atkin-Lehner $W$-operator $W(m)$ and [AB03, Corollary 2.3], we have another corollary.

**Corollary 1.7.** The restriction map $\mathcal{D}_0 : J^\star_{2,m}(mN) \longrightarrow M_2(mN)$ is injective, when $mN$ is square-free, i.e., $J^\star_{2,m}(mN)^0 = \{0\}$.

We make some remarks concerning the subspace studied in this section.

**Remark 1.8.** Note that Theorem 1.3 reduces to [AB99, Theorem 2] in the case of index 1. Moreover, Corollary 1.4 shows that there may exist isomorphic subspaces in the space of Jacobi forms of different index.

**Remark 1.9.** The space $J^\star_{k,m}(mN, \chi)$ as defined above can be quite large for some index. For example, if $k$ is even and $m$ is square-free, using [S05, Theorem 1] we show that dim $J^\star_{k,m}(mN) \geqslant$ dim $J_{k,m}(SL_2(\mathbb{Z}))$, for any positive integer $N$.

**Remark 1.10.** Suppose $N$ is either 2 or an odd square-free positive integer. We show that the space $J^\star_{k,2}(2N, \chi)$ is isomorphic to the full kernel space $J_{k,2}(2N, \chi)^0$. Considering the kernel space $J^\star_{k,2}(2N, \chi)^0$ as a subspace of $J_{k,2}(2N, \chi)^0$, the image of $J^\star_{k,2}(2N, \chi)^0$ under $\Lambda_2$ in the isomorphism diagram of Theorem 1.1, is isomorphic to the space $M_{k-1}(2N, \chi\overline{\omega}_2)$ under the isomorphism $(\varphi_0, \varphi_2) \longmapsto \frac{\varphi_0\theta_{2,1}}{\theta_{2,2}} (= \frac{-\varphi_2\theta_{2,1}}{\theta_{2,0}})$.

## 2. Correspondence between Jacobi cusp forms and elliptic cusp forms

In [SZ88], N. -P. Skoruppa and D. Zagier constructed certain lifting maps (Shimura type correspondence) between the space of holomorphic Jacobi cusp forms of integral weight and a subspace of cusp forms of integral weight, commuting with the action of Hecke operators. Further, in [GKZ], B. Gross, W. Kohnen and D.

Zagier constructed kernel functions for these liftings. By using these correspondences, they obtained deep formulas relating to the height pairings of Heegner points to the Fourier coefficients of holomorphic Jacobi forms. This correspondence for higher levels (for subgroups of the full Jacobi group) was generalized by B. Ramakrishnan in his thesis [BR89]. In [MR], the same results were obtained by showing that the image of the Poincaré series under the Shimura map can be expressed in terms of the holomorphic kernel functions for the periods of cusp forms. In [KB06], K. Bringmann following the method of Gross-Kohnen-Zagier, generalized these liftings to holomorphic Jacobi forms of higher degree (in particular to holomorphic Jacobi forms on $\mathcal{H} \times \mathbb{C}^g$ with matrix index of size $g$, $g \equiv 1 \pmod 8$) on the full Jacobi group. In [Sk88], Skoruppa introduced the concept of skew-holomorphic Jacobi forms, which are holomorphic in $z \in \mathbb{C}$, smooth in $\tau \in \mathcal{H}$ and have slightly modified transformation properties as compared with the holomorphic Jacobi forms. A parallel theory analogous to the holomorphic Jacobi forms can be studied in this case and in [Sk90, Propositions 1 and 2], Skoruppa obtained a correspondence similar to the holomorphic case between the space of skew-holomorphic Jacobi cusp forms (for the full Jacobi group) and elliptic cusp forms of integral weight. In this correspondence, the image of the space of skew-holomorphic Jacobi cusp forms lies in the orthogonal complement of the image of the space of holomorphic Jacobi cusp forms. We notice that in his thesis [M89], M. Manickam also obtainted the same results as mentioned above for the skew-holomorphic case. As an application, he obtained estimates for the Fourier coefficients of the Jacobi cusp forms (both holomorphic and skew-holomorphic) by using the explicit Waldspurger theorem and the estimates for the special values of the $L$-functions of the cusp form of integral weight derived by H. Iwaniec [Iw87]. In Chapter 2, we give a generalization of the correspondence obtained by Bringmann [KB06], Skoruppa [Sk90] and Manickam [M89] to the holomorphic and skew-holomorphic Jacobi forms of higher degree and for congruence subgroups.

First, we recall some basic facts about integral binary quadratic forms, generalized genus character, construction of elliptic cusp forms and Jacobi Poincaré series. Let $N, k, g \in \mathbb{N}$, where $g \equiv 1 \pmod 8$. Let $M$ be a positive definite, symmetric and half-integral $g \times g$ matrix (the last two conditions and $g$ is odd imply that $\frac{1}{2}\det(2M)$ is an integer). Let $\chi$ be a primitive Dirichlet character modulo $N_1$ with $N_1|N$ and $(N_1, \det(2M)) = 1$. For any complex number $z$ and a non-zero real number $c$, we denote by $e_c(z) = e^{2\pi i z/c}$. If $c = 1$, we simply write $e(z)$ instead of $e_1(z)$.

## 2.1. Kernel functions for geodesic cycle integrals

Let $l \in \mathbb{N}$ and $\Delta \in \mathbb{Z}(\geqslant 0)$ be a discriminant. For an integer $\rho \pmod{2l}$ with $\Delta \equiv \rho^2 \pmod{4l}$, we define the following sets of integral binary quadratic forms.

$$\mathcal{Q}_{l,\Delta,\rho} := \{Q(x,y) = ax^2 + bxy + cy^2 | a,b,c \in \mathbb{Z}, b^2 - 4ac = \Delta, a \equiv 0 \pmod{l} \text{ and } b \equiv \rho \pmod{2l}\}$$

and

$$\mathcal{Q}_{l,N,\chi,\Delta,\rho} := \{Q(x,y) \in \mathcal{Q}_{l,\Delta,\rho} | \Delta \equiv 0 \pmod{N_1^2} \text{ and } a \equiv 0 \pmod{lN_1N}\}.$$

Let $D_0$ be a fundamental discriminant dividing $\Delta$, coprime to $N_1$ and suppose that both $D_0$ and $\Delta/D_0$ are squares modulo $4l$. Following [GKZ], we define the generalized genus character $\chi_{D_0} : \mathcal{Q}_{l,\Delta,\rho} \longrightarrow \{0, \pm 1\}$ by

$$\chi_{D_0}(Q) = \begin{cases} \left(\frac{D_0}{t}\right) & \text{if } Q = [al, b, c] \text{ with } (a,b,c,D_0) = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $\left(\frac{D_0}{t}\right)$ denotes the Kronecker symbol. Here t is an integer coprime to $D_0$ represented by the form $[al_1, b, cl_2]$ for some decomposition $l = l_1 l_2$ with $l_1, l_2 > 0$. Then it is known that $\chi_{D_0}$ is well-defined and is $\Gamma_0(l)$ invariant on $\mathcal{Q}_{l,\Delta,\rho}$. For details, we refer to [GKZ, Sect. I]. Also $\chi_{D_0}$ is $\Gamma_0(lN)$ invariant on $\mathcal{Q}_{l,N,\chi,\Delta,\rho}$. Let $S_k(N,\chi)$ denote the space of elliptic cusp forms of weight $k$ with respect to the congruence subgroup $\Gamma_0(N)$ and character $\chi$. For $k \geqslant 2$, define

$$f_{k,l,N,\chi,\Delta,\rho,D_0}(\tau) = \sum_{Q(=[a,b,c]) \in \mathcal{Q}_{l,N,\chi,\Delta,\rho}} \overline{\chi}(c) \chi_{D_0}(Q) Q(\tau,1)^{-k}. \tag{6}$$

Following [GKZ], in [BR89], Ramakrishnan proved that $f_{k,l,N,\chi,\Delta,\rho,D_0}(\tau) \in S_{2k}(lN,\chi^2)$ and obtained its Fourier expansion.

For a cusp form $f \in S_{2k}(lN,\chi^2)$ and $Q = [a,b,c] \in \mathcal{Q}_{l,N,\chi,\Delta,\rho}$ , we define

$$r_{k,l,N,\chi,Q}(f) := \int_{\gamma_Q} f(\tau) Q(-\overline{\tau},1)^{k-1} d\tau, \tag{7}$$

where $\gamma_Q$ is the image in $\Gamma_0(lN)\backslash\mathcal{H}$ of the semicircle $a|\tau|^2+b\mathrm{Re}(\tau)+c=0$, oriented from $\frac{-b-\sqrt{\Delta}}{2a}$ to $\frac{-b+\sqrt{\Delta}}{2a}$ if $a\neq 0$ or if $a=0$ of the vertical line $b\mathrm{Re}(\tau)+c=0$, oriented from $-c/b$ to $i\infty$ if $b>0$ and from $i\infty$ to $-c/b$ if $b<0$. It is known that this integral depends only on the $\Gamma_0(lN)$ equivalence class of $Q$. Further, we define

$$r_{k,l,N,\chi,\Delta,\rho,D_0}(f) := \sum_{Q\in\mathcal{Q}_{l,N,\chi,\Delta,\rho}/\Gamma_0(lN)} \chi(c)\chi_{D_0}(Q)r_{k,l,N,\chi,Q}(f). \tag{8}$$

Then the following holds.

**Proposition 2.1.** For $f\in S_{2k}(lN,\chi^2)$, we have

$$\frac{\pi\binom{2k-2}{k-1}}{i_{lN}2^{2k-2}\Delta^{k-\frac{1}{2}}}\ r_{k,l,N,\chi,\Delta,\rho,D_0}(f) = \langle f, f_{k,l,N,\chi,\Delta,\rho,D_0}\rangle, \tag{9}$$

where $i_{lN}$ denotes the index of $\Gamma_0(lN)$ in $SL_2(\mathbb{Z})$ and $\langle\cdot,\cdot\rangle$ denotes the usual Petersson scalar product for the space of cusp forms of integral weight on $\Gamma_0(lN)$.

For more details of the above discussion, we refer to [St75, KZ84, K85, GKZ, BR89, KB04].

## 2.2. Holomorphic Jacobi cusp forms on $\mathcal{H}\times\mathbb{C}^{(g,1)}$

Let us denote by $J^{\mathrm{cusp}}_{k,M}(N,\chi)$, the vector space of holomorphic Jacobi cusp forms of weight $k$, index $M$ with respect to the generalized Jacobi group $\Gamma^J_g(N) := \Gamma_0(N)\ltimes(\mathbb{Z}^{(g,1)}\times\mathbb{Z}^{(g,1)})$ and character $\chi$ (for the definition of holomorphic Jacobi cusp forms, we refer to [EZ85, Z89]).

**Poincaré Series :** For $n\in\mathbb{Z}, r\in\mathbb{Z}^{(1,g)}$ with $4n>M^{-1}[r^t]$ and $k>g+2$, define a Poincaré series of exponential type by

$$P_{k,M,N,\chi;(n,r)}(\tau,z) := \sum_{X\in\Gamma^J_{g,\infty}\backslash\Gamma^J_g(N)} \overline{\chi}(d)\ e^{(n,r)}|_{k,M}X(\tau,z) \qquad (\tau\in\mathcal{H}, z\in\mathbb{C}^{(g,1)}), \tag{10}$$

where $e^{(n,r)}(\tau,z)=e(n\tau+rz)$ and $\Gamma^J_{g,\infty} := \left\{\left(\begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix},(0,\mu)\right) | m\in\mathbb{Z},\mu\in\mathbb{Z}^{(g,1)}\right\}$.

For $N=1$, Böcherer and Kohnen [BK93] studied this Poincaré series and for level $N$ with trivial character, it was studied by Bringmann [KB04]. One has $P_{k,M,N,\chi;(n,r)}\in J^{\mathrm{cusp}}_{k,M}(N,\chi)$ and the Petersson scalar product on $J^{\mathrm{cusp}}_{k,M}(N,\chi)$ is characterized by the following (see [BK93, KB04]). Let

$$\phi(\tau,z)=\sum_{\substack{m\in\mathbb{Z},r\in\mathbb{Z}^{(1,g)}\\ 4m>M^{-1}[r^t]}} c_\phi(m,r)e(m\tau+rz)\ \in J^{\mathrm{cusp}}_{k,M}(N,\chi).$$

Then, we have

$$\langle \phi, P_{k,M,N,\chi;(n,r)}\rangle = \lambda_{k,M,D,N}\ \ c_\phi(n,r), \tag{11}$$

with

$$\lambda_{k,M,D,N}=\frac{2^{(g-1)(k-g/2-1)-g}\Gamma(k-g/2-1)}{\pi^{k-g/2-1}[\Gamma^J_g(1):\Gamma^J_g(N)]}(\det M)^{k-(g+3)/2}|D|^{-k+g/2+1}, \tag{12}$$

where $D=2^{g-1}\det(M)(M^{-1}[r^t]-4n)$.

For the sake of simplicity, we write $P_{(n,r)}$ for $P_{k,M,N,\chi;(n,r)}$.

## 2.3. Skew-holomorphic Jacobi cusp forms on $\mathcal{H}\times\mathbb{C}^{(g,1)}$

First, let us recall the definition of a skew-holomorphic Jacobi cusp form ([Sk88, A93]). For any pair $X=(\gamma,(\lambda,\mu))\in\Gamma^J_g(1)$ and any function $\phi$ on $\mathcal{H}\times\mathbb{C}^{(g,1)}$, define

$$\phi|^*_{k,M}X(\tau,z)=e\left(M[\lambda]\tau+2\lambda^tMz-M[z+\lambda\tau+\mu]\frac{c}{c\tau+d}\right)|c\tau+d|^{-g}\overline{(c\tau+d)}^{-k+g} \times\phi\left(\frac{a\tau+b}{c\tau+d},\frac{z+\lambda\tau+\mu}{c\tau+d}\right), \tag{13}$$

where $\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{Z})$.

**Definition 2.2.** A function $\phi$ on $\mathcal{H}\times\mathbb{C}^{(g,1)}$ is said to be a skew-holomorphic Jacobi form of weight $k$ and index $M$ with respect to the Jacobi group $\Gamma^J_g(N)$ and character $\chi$, if it satisfies the following conditions.

1. $\phi(\tau,z)$ is a smooth function in $\tau\in\mathcal{H}$ and holomorphic in $z\in\mathbb{C}^{(g,1)}$,
2. $\phi|^*_{k,M}X(\tau,z)=\chi(d)\phi(\tau,z)$ for all $X\left(=\left(\begin{pmatrix}a&b\\c&d\end{pmatrix},(\lambda,\mu)\right)\right)\in\Gamma^J_g(N)$,

3. For each $\gamma \in SL_2(\mathbb{Z})$, $\phi|^*_{k,M}(\gamma,(0,0))$ has a Fourier development of the form

$$\sum_{\substack{n\in\mathbb{Z}, r\in\mathbb{Z}^{(1,g)} \\ \frac{4n}{N}\leq M^{-1}[r^t]}} c_\phi(n,r)e(\frac{n}{N}\overline{\tau}+\frac{i}{2}M^{-1}[r^t]v+rz),$$

where $\tau = u+iv$.

The vector space of all such functions is denoted by $J^*_{k,M}(N,\chi)$. For all $\gamma \in SL_2(\mathbb{Z})$, if $\phi$ satisfies the stronger condition $c_\phi(n,r)=0$ unless $4n/N < M^{-1}[r^t]$, then it is called a skew-holomorphic Jacobi cusp form. We denote by $J^{*,\mathrm{cusp}}_{k,M}(N,\chi)$, the vector subspace of all such cusp forms.

**Definition 2.3.** For skew-holomorphic Jacobi cusp forms $\phi$ and $\psi$ on $\Gamma^J_g(N)$, we define the Petersson scalar product as follows.

$$\langle\phi,\psi\rangle := \frac{1}{[\Gamma^J_g(1):\Gamma^J_g(N)]}\int_{\Gamma^J_g(N)\backslash\mathcal{H}\times\mathbb{C}^{(g,1)}}\phi(\tau,z)\overline{\psi(\tau,z)}e^{-4\pi M[y]\cdot v^{-1}}v^{k-g-2}dudvdxdy, \tag{14}$$

where $\tau = u+iv$ and $z = x+iy$.

**Poincaré Series:** For $n\in\mathbb{Z}, r\in\mathbb{Z}^{(1,g)}$ with $4n < M^{-1}[r^t]$ and $k > g+2$, we define the Poincaré series of exponential type by

$$P^*_{k,M,N,\chi;(n,r)}(\tau,z) := \sum_{X\in\Gamma^J_{g,\infty}\backslash\Gamma^J_g(N)}\overline{\chi}(d)e^{(n,r)}_*|^*_{k,M}X(\tau,z) \qquad (\tau\in\mathcal{H}, z\in\mathbb{C}^{(g,1)}), \tag{15}$$

where $e^{(n,r)}_*(\tau,z) = e(n\overline{\tau}+\frac{i}{2}M^{-1}[r^t]v+rz)$, $\tau = u+iv$. For any pair $(n,r)$ with the property that $4n < M^{-1}[r^t]$, it follows that $P^*_{k,M,N,\chi;(n,r)} \in J^{*,\mathrm{cusp}}_{k,M}(N,\chi)$. Proceeding as in [BK93], we obtain the Fourier expansion of the skew-holomorphic Poincaré series $P^*_{k,M,N,\chi;(n,r)}$ and prove that

$$\langle\phi, P^*_{k,M,N,\chi;(n,r)}\rangle = \lambda_{k,M,D,N} \;\; c_\phi(n,r), \tag{16}$$

where $\phi(\tau,z) = \sum_{\substack{n\in\mathbb{Z}, r\in\mathbb{Z}^{(1,g)} \\ 4n<M^{-1}[r^t]}} c_\phi(n,r)\; e(n\overline{\tau}+\frac{i}{2}M^{-1}[r^t]v+rz) \;\in J^{*,cusp}_{k,M}(N,\chi)$ and $\lambda_{k,M,D,N}$ is the same constant as defined in the holomorphic case.

For the sake of simplicity, we write $P^*_{(n,r)}$ for $P^*_{k,M,N,\chi;(n,r)}$.

Now onwards we assume that $k > \frac{g+3}{2}$.

Let $n_0 \in \mathbb{N}$, $r_0 \in \mathbb{Z}^{(1,g)}$ and $D_0 := -\det \begin{pmatrix} 2n_0 & r_0 \\ r_0^t & 2M \end{pmatrix}$ such that $D_0$ is a fundamental discriminant which is a square modulo $\frac{1}{2}\det(2M)$ and coprime to $N_1$. We further assume that if $p$ divides both $\det(2M)$ and $D_0$, $p^2$ must not divide $\det(2M)$ if $p \neq 2$, $p^3$ must not divide $\det(2M)$ if $p = 2$ and $D_0/4$ is odd, $p^4$ must not divide $\det(2M)$ if $p = 2$ and $D_0/4$ is even. Moreover, if $p \neq 2$, $\prod_{i, i\neq j} m_i$ is assumed to be a square modulo $p$, where the $m_i$ are chosen such that there exists $U \in GL_g(\mathbb{Z}/p\mathbb{Z})$ with $(2M)[U] \equiv \operatorname{diag}[m_1, m_2, ... m_g] \pmod{p}, p|m_j$. Before we proceed to the statement of results, we first give the definition of the lifting maps between the space of Jacobi cusp forms of integral weight with matrix index and the space of cusp forms of integral weight.

**Definition 2.4.** Let $\phi \in J^{\mathrm{cusp}}_{k+\frac{g+1}{2},M}(N,\chi)$(*resp.* $J^{*,\mathrm{cusp}}_{k+\frac{g+1}{2},M}(N,\chi)$) be a holomorphic (resp. skew-holomorphic) Jacobi cusp form of weight $k+\frac{g+1}{2}$, index $M$, level $N$ and character $\chi$ with Fourier expansion

$$\phi(\tau,z) = \sum_{\substack{m\in\mathbb{Z}, r\in\mathbb{Z}^{(1,g)} \\ 4m>M^{-1}[r^t]}} c_\phi(m,r)e(m\tau+rz) \left(\text{resp.} \sum_{\substack{m\in\mathbb{Z}, r\in\mathbb{Z}^{(1,g)} \\ 4m<M^{-1}[r^t]}} c_\phi(m,r)e(m\overline{\tau}+\frac{i}{2}M^{-1}[r^t]v+rz)\right).$$

For $D_0 < 0$ (resp. $D_0 > 0$), we define the $(D_0, r_0)$-th Shimura map on $J^{\mathrm{cusp}}_{k+\frac{g+1}{2},M}(N,\chi)$ (resp. $J^{*,\mathrm{cusp}}_{k+\frac{g+1}{2},M}(N,\chi)$) as follows.

$$S_{D_0,r_0}(\phi)(w) := 2^{\frac{1-g}{2}} \sum_{m\geqslant 1} \left( \sum_{\substack{d|m \\ (d,\frac{N}{N_1})=1}} \chi(d) \left(\frac{D_0}{d}\right) d^{k-1} c_\phi\left(\frac{m^2}{d^2}n_0, \frac{m}{d}r_0\right)\right) e(mw) \ \ (w \in \mathcal{H}). \tag{17}$$

**Definition 2.5.** Let $f \in S_{2k}(\frac{1}{2}\det(2M)N, \chi^2)$ be an elliptic cusp form. For $D_0 < 0$, we define

$$S^*_{D_0,r_0}(f)(\tau,z) := \alpha_{M,D_0,N,\chi} \sum_{\substack{m\in\mathbb{Z}, r\in\mathbb{Z}^{(1,g)} \\ 4m>M^{-1}[r^t]}} \sum_{t|N/N_1} \mu(t)\overline{\chi}(t)\left(\frac{D_0}{t}\right) t^{-k-1} \times r_{k,\frac{1}{2}\det(2M),Nt,\chi,D_0DN_1^2t^2,\rho,D_0}(f)e(m\tau+rz) \tag{18}$$

and for $D_0 > 0$, we define

$$S^*_{D_0,r_0}(f)(\tau,z) := \alpha_{M,D_0,N,\chi} \sum_{\substack{m\in\mathbb{Z}, r\in\mathbb{Z}^{(1,g)}\\ 4m<M^{-1}[r^t]}} \sum_{t|N/N_1} \mu(t)\overline{\chi}(t)\left(\frac{D_0}{t}\right) t^{-k-1} \times r_{k,\frac{1}{2}\det(2M),Nt,\chi,D_0DN_1^2t^2,\rho,D_0}(f) e(m\overline{\tau} + \frac{i}{2}M^{-1}[r^t]v + rz), \tag{19}$$

where $\alpha_{M,D_0,N,\chi} := \left(\frac{i}{\det(2M)}\right)^{k-1} N_1^{-k+\frac{1}{2}} [\Gamma^J_g(1) : \Gamma^J_g(N)] i^{-1}_{(\frac{1}{2}\det(2M))N} \chi(-1) R_2(\chi, D_0)$ and recall that $i_l$ denotes the index of $\Gamma_0(l)$ in $SL_2(\mathbb{Z})$.

We prove the following theorem.

**Theorem 2.6.** Let $P_{(n,r)}$ (resp. $P^*_{(n,r)}$) denote the $(n,r)$-th holomorphic (resp. skew-holomorphic) Poincaré series of weight $k + \frac{g+1}{2}$, index $M$, level $N$ and character $\chi$. For $D_0 < 0$, we have

$$S_{D_0,r_0}(P_{(n,r)})(w) = \beta_{k,D_0,N_1,\chi} \sum_{t|\frac{N}{N_1}} \mu(t)\chi(t)\left(\frac{D_0}{t}\right) t^{k-1} f_{k,\frac{1}{2}\det(2M),\frac{N}{t},\chi,\Delta,\rho,D_0}(tw) \tag{20}$$

and for $D_0 > 0$, we have

$$S_{D_0,r_0}(P^*_{(n,r)})(w) = \beta_{k,D_0,N_1,\chi} \sum_{t|\frac{N}{N_1}} \mu(t)\chi(t)\left(\frac{D_0}{t}\right) t^{k-1} f_{k,\frac{1}{2}\det(2M),\frac{N}{t},\chi,\Delta,-\rho,D_0}(tw), \tag{21}$$

where $\beta_{k,D_0,N_1,\chi} = \frac{\chi(-1)R_2(\chi,D_0)(N_1|D_0|)^{k-\frac{1}{2}}(k-1)!}{(\mathrm{sign}(D_0))^{1/2}2^{\frac{g-1}{2}}(-2\pi i)^k}$, $\Delta = D_0DN_1^2$ and $\rho = -(r_0(2M)^*r^t)N_1$.

As a consequence of the above theorem, by using Proposition 2.1, (11) and (16), we get the following theorem.

**Theorem 2.7.** Let

$$\phi \in \begin{cases} J^{\mathrm{cusp}}_{k+\frac{g+1}{2},M}(N,\chi) & \text{if } D_0 < 0, \\ J^{*,\mathrm{cusp}}_{k+\frac{g+1}{2},M}(N,\chi) & \text{if } D_0 > 0. \end{cases}$$

Then, the function $S_{D_0,r_0}(\phi)(w)$ is an element of $S_{2k}(\frac{1}{2}\det(2M)N,\chi^2)$. If $f$ is a cusp form in $S_{2k}(\frac{1}{2}\det(2M)N,\chi^2)$, then

$$S^*_{D_0,r_0}(f) \in \begin{cases} J^{\mathrm{cusp}}_{k+\frac{g+1}{2},M}(N,\chi) & \text{if } D_0<0,\\ J^{*,\mathrm{cusp}}_{k+\frac{g+1}{2},M}(N,\chi) & \text{if } D_0>0.\end{cases}$$

Moreover, the maps $S_{D_0,r_0}$ and $S^*_{D_0,r_0}$ are adjoint maps with respect to the Petersson scalar products, i.e., for all $f\in S_{2k}(\frac{1}{2}\det(2M)N,\chi^2)$ and for all $\phi$ as above, we have

$$\langle S_{D_0,r_0}(\phi), f\rangle = \langle \phi, S^*_{D_0,r_0}(f)\rangle.$$

**Remark 2.8.** Put $\ell=\frac{1}{2}\det(2M)$ and assume that $(\ell,N)=1$. Let $W_\ell = \begin{pmatrix}\ell\alpha & \beta\\ \ell N\gamma & \ell\delta\end{pmatrix}, \ell\alpha\delta - N\beta\gamma = 1, \alpha\equiv 1 \pmod N, \beta\equiv 1 \pmod \ell$ be the Atkin-Lehner $W$-operator on $S_{2k}(\ell N,\chi^2)$. Then $W_\ell$ preserves the space $S_{2k}(\ell N,\chi^2)$ and the operator $\chi^2(\ell)W_\ell^2$ acts as identity on $S_{2k}(\ell N,\chi^2)$. For details, we refer to [AL78]. Define a subspace of $S_{2k}(\ell N,\chi^2)$ as follows.

$$S_{2k}^{(\pm,\ell)}(\ell N,\chi^2) = \{f\in S_{2k}(\ell N,\chi^2)\ :\ f|W_\ell = \pm(-1)^k\chi(-1)\overline{\chi}(\ell)f\}.$$

When $N=1$, the subspace $S_{2k}^{(\pm,\ell)}(\ell N,\chi^2)$ coincides with the subspace $S^{\pm}_{2k}(\ell)$, which was considered in [SZ88, GKZ, Sk88, Sk90, M89]. It is easy to see that

$$f_{k,\ell,N,\chi,\Delta,\rho,D_0}|W_\ell = \overline{\chi}(\ell) f_{k,\ell,N,\chi,\Delta,-\rho,D_0}.$$

Since

$$f_{k,\ell,N,\chi,\Delta,-\rho,D_0} = \mathrm{sign}(D_0)(-1)^k\chi(-1) f_{k,\ell,N,\chi,\Delta,\rho,D_0},$$

we see that $f_{k,\ell,N,\chi,\Delta,\rho,D_0}\in S_{2k}^{(\mathrm{sign}(D_0),\ell)}(\ell N,\chi^2)$. Therefore, the correspondence given in Theorem 2.7 for $D_0<0$ (resp. $D_0>0$) is actually between the spaces $J^{\mathrm{cusp}}_{k+\frac{g+1}{2},M}(N,\chi)$ and $S_{2k}^{(-,\ell)}(\ell N,\chi^2)$ (resp. $J^{*,\mathrm{cusp}}_{k+\frac{g+1}{2},M}(N,\chi)$ and $S_{2k}^{(+,\ell)}(\ell N,\chi^2)$).

**Remark 2.9.** When $N=1$ and $D_0<0$, the conclusion of Remark 2.8 reduces to [KB06, Theorem 3]. When $N=1$, $g=1$, $D_0>0$, our results (as mentioned in Remark 2.8 reduces to [Sk90, Propositions 1 and 2] and [M89, Theorem 3.3.1].

It is interesting to note that an application of Theorem 2.7 (for the general case $g>1$, $g\equiv 1 \pmod 8$ and $g=1$) gives rise to a correspondence between the space of holomorphic (resp. skew-holomorphic) Jacobi cusp forms of matrix index and the space of holomorphic (resp. skew-holomorphic) Jacobi cusp forms of integer index.

**Theorem 2.10.** There exists a correspondence between the spaces $J^{\mathrm{cusp}}_{k+\frac{g+1}{2},M}(N,\chi)$ (resp. $J^{*,\mathrm{cusp}}_{k+\frac{g+1}{2},M}(N,\chi)$) and $J^{\mathrm{cusp}}_{k+1,\frac{1}{2}\det(2M)}(N,\chi)$ (resp. $J^{*,\mathrm{cusp}}_{k+1,\frac{1}{2}\det(2M)}(N,\chi)$). Moreover, the mappings are adjoint to each other with respect to the Petersson scalar products.

## 3. Non-vanishing of Half-integral weight $L$ - functions

In [K97], Kohnen showed that a certain average of $L$-functions over a basis of orthogonal Hecke eigenforms of integral weight $k$ on $\mathrm{SL}_2(\mathbb{Z})$ does not vanish in large parts of the critical strip. As a consequence, he proved that for sufficiently large $k$ and any point $s$ inside the critical strip not lying on the line $\mathrm{Re}(s) = k/2$, there exists a Hecke eigen cusp form of integral weight $k$ on $SL_2(\mathbb{Z})$, such that the corresponding $L$-function value at $s$ is non-zero. In [Ra05], A. Raghuram generalised Kohnen's method for the average of $L$-functions over a basis of newforms (of integral weight) of level $N$ with primitive character modulo $N$. In Chapter 3, we extend Kohnen's method to the forms of half-integral weight.

Let $N \geq 1$, $k \geq 3$ be integers and $\psi$ be an even Dirichlet character modulo $4N$. Let $S_{k+1/2}(4N,\psi)$ be the space of cusp forms of weight $k+1/2$ with character $\psi$ ([Ko97], [Sh73]). Let $L(f,s)$ be the $L$-function associated to the cusp form $f \in S_{k+1/2}(4N,\psi)$ defined by $L(f,s) = \sum_{n\geq 1} a_f(n)n^{-s}$, where $a_f(n)$ denotes the $n$-th Fourier coefficient of $f$. Then by [MMR, Proposition 1], the completed $L$-function defined by $L^*(f,s) := (2\pi)^{-s}(\sqrt{4N})^s\Gamma(s)L(f,s)$ has the following functional equation

$$L^*(f|H_{4N}, k+1/2-s) = L^*(f,s), \tag{22}$$

where $H_{4N}$ is the Fricke involution on $S_{k+1/2}(4N,\psi)$ defined by

$$f|H_{4N}(\tau) = i^{k+1/2}(4N)^{-k/2-1/4}\tau^{-k-1/2}f(-1/4N\tau).$$

Let $K$ be the operator defined by $f|K(\tau) = \overline{f(-\overline{\tau})}$. Since $KH_{4N} = H_{4N}K$ on $S_{k+1/2}(4N,\psi)$, we have $f|(KH_{4N})^2 = f$. For $f,g \in S_{k+1/2}(4N,\psi)$, let $\langle f,g\rangle$ denote the Petersson scalar product of $f$ and $g$. It is known that the space $S_{k+1/2}(4N,\psi)$ has an orthogonal basis $\{f_1, f_2, \ldots, f_d\}$ of Hecke eigenforms with respect to all Hecke operators $T(p^2)$, $p \nmid 2N$ such that $f_i|KH_{4N} = \lambda_{f_i} f_i$ for all

$i = 1, \ldots, d$, where $d$ is the dimension of the space $S_{k+1/2}(4N, \psi)$ and $\lambda_{f_i} = \pm 1$. Following the proof given by Kohnen [K97], we prove the following theorems.

**Theorem 3.1.** Let $N \geq 1$ be a fixed integer. Let $\{f_1, f_2, \ldots, f_d\}$ be an orthogonal basis as above. Let $r_0 \in \mathbb{R}$ and $\epsilon > 0$. Then there exists a constant $C = C(r_0, \epsilon)$ depending only on $r_0$ and $\epsilon$ such that for $k > C$, the function

$$\sum_{j=1}^{d} \frac{L^*(f_j, s)}{\langle f_j, f_j \rangle} \lambda_{f_j} a_{f_j}(1)$$

doesn't vanish for any point $s = \sigma + ir_0$ with $k/2 - 1/4 < \sigma < k/2 + 1/4 - \epsilon$ or $k/2 + 1/4 + \epsilon < \sigma < k/2 + 3/4$.

**Theorem 3.2.** Let $k \geq 3$ be a fixed integer. Let $\{f_1, f_2, \ldots, f_d\}$ be an orthogonal basis as above. Let $r_0 \in \mathbb{R}$ and $\epsilon > 0$. Then there exists a constant $C' = C'(r_0, \epsilon)$ depending only on $r_0$ and $\epsilon$ such that for $N > C'$, the function

$$\sum_{j=1}^{d} \frac{L^*(f_j, s)}{\langle f_j, f_j \rangle} \lambda_{f_j} a_{f_j}(1)$$

doesn't vanish for any point $s = \sigma + ir_0$ with $k/2 - 1/4 < \sigma < k/2 + 1/4 - \epsilon$ or $k/2 + 1/4 + \epsilon < \sigma < k/2 + 3/4$.

The following corollary is an easy consequence of the above two theorems.

**Corollary 3.3.** Let $s_0$ be a point inside the critical strip $k/2 - 1/4 < \mathrm{Re}(s_0) < k/2 + 3/4$ but not on the line $\mathrm{Re}(s_0) = k/2 + 1/4$. If either $k$ or $N$ is suitably large, then there exists a Hecke eigenform $f$ belonging to $S_{k+1/2}(4N, \psi)$ such that $L(f, s_0) \neq 0$ and $a_f(1) \neq 0$.

Hence, we show that for any given point $s$ inside the critical strip not lying on the line $\mathrm{Re}(s) = k/2 + 1/4$, there exists a Hecke eigen cusp form $f$ of half-integral weight $k + 1/2$ on $\Gamma_0(4N)$ with character $\psi$ such that the corresponding $L$-function value at $s$ is non-zero, and the first Fourier coefficient of $f$ is non-zero. It should be noted that the nonrmalisation of Fourier coefficients of forms of half-integral weight is still an open question. Our results are obtained for $N$ sufficiently large if $k$ is fixed and vice versa. In particular when $N = 1$, we observe that for sufficiently large $k$, either $f$ is a newform in the full space $S_{k+1/2}(4)$ or $f$ is a Hecke eigenform in the Kohnen plus space $S^+_{k+1/2}(4) := \{f \in S_{k+1/2}(4) : a_f(n) = 0 \text{ unless } (-1)^k n \equiv 0, 1 \pmod 4\}$.

# 4. Publications/Preprints

1. B. Ramakrishnan and Karam Deo Shankhadhar, *On the restriction map for Jacobi forms* (submitted for publication, 2011).

2. B. Ramakrishnan and Karam Deo Shankhadhar, *On a correspondence between Jacobi cusp forms and elliptic cusp forms* (Accepted for publication in International Journal of Number Theory, 2012).

3. B. Ramakrishnan and Karam Deo Shankhadhar, *Non-vanishing of L-functions associated to cusp forms of half-integral weight inside the critical strip* (submitted for publication, 2012).

# Chapter 1

# Preliminaries

In this chapter we give some basic definitions and results that will be used in the thesis.

## 1.1 Notations

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ be the set of positive integers, integers, rational numbers, real numbers and complex numbers respectively. For $a, b \in \mathbb{Z}$, we write $a|b$ when $b$ is divisible by $a$ and $a \pmod b$ means that $a$ varies over a complete set of residue classes modulo $b$. For $z \in \mathbb{C}$, Re $z$ denotes the real part of $z$ and Im $z$ denotes the imaginary part of $z$. For any complex number $z$ and a non-zero real number $c$, we denote by $e_c(z) = e^{2\pi i z/c}$. If $c = 1$, we simply write $e(z)$ instead of $e_1(z)$. Let $\mathcal{H} = \{\tau \in \mathbb{C} : \text{Im } \tau > 0\}$ be the complex upper half-plane. We denote by $q = e^{2\pi i \tau}, i = \sqrt{-1}$ for $\tau \in \mathbb{H}$. For a commutative ring $R$, we denote the set of all $n \times n$ matrices with entries in $R$ by $M_n(R)$. The full modular group $SL_2(\mathbb{Z})$ is defined by

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : ad - bc = 1 \right\}.$$

For a positive integer $N$, we define the congruence subgroup $\Gamma_0(N)$ of $SL_2(\mathbb{Z})$ as follows.

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod N \right\}.$$

We denote by $i_N$, the index of $\Gamma_0(N)$ in $SL_2(\mathbb{Z})$.

Let $k$ denote a non-negative integer and $N$ be a positive integer.

## 1.2 Modular Forms of integral weight

The group

$$GL_2^+(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) : ad - bc > 0 \right\}$$

acts on $\mathcal{H}$ by

$$\gamma\tau := \frac{a\tau + b}{c\tau + d}, \text{ where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R}) \text{ and } \tau \in \mathcal{H}.$$

Let $f : \mathcal{H} \to \mathbb{C}$ be a holomorphic function. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$ and a non-negative integer $k$, define

$$f|_k\gamma(\tau) := (\det \gamma)^{k/2}(c\tau + d)^{-k} f(\gamma\tau).$$

We omit the subscript $k$ whenever there is no ambiguity. Let $f|_k\gamma(\tau) = f(\tau)$, for all $\gamma \in \Gamma_0(N)$. Then $f$ is said to be holomorphic at all the cusps of $\Gamma_0(N)$ if for each $\gamma \in SL_2(\mathbb{Z})$, there exists a positive integer $w_\gamma$ such that the function $f|_k\gamma$ has a Fourier expansion of the form

$$f|_k\gamma(\tau) = \sum_{n=0}^{\infty} a_\gamma(n) q^{n/w_\gamma}.$$

If in addition, the constant terms $a_\gamma(0)$ are zero for all $\gamma \in SL_2(\mathbb{Z})$, then we say that $f$ vanishes at all the cusps. If $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then $w_\gamma = 1$ and we write the Fourier expansion at the cusp infinity by, $f(\tau) = \sum_{n=0}^{\infty} a_f(n) q^n$. We call $a_f(n)$, the $n$-th Fourier coefficient of $f$.

**Definition 1.2.1** (Modular form and cusp form of integral weight) Let $\chi$ be a Dirichlet character modulo $N$. A modular form of weight $k$, level $N$ and character $\chi$ is a holomorphic function $f : \mathcal{H} \longrightarrow \mathbb{C}$ such that

(i) $f|_k\gamma(\tau) = \chi(d) f(\tau)$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$,

(ii) $f$ is holomorphic at all the cusps of $\Gamma_0(N)$.

If in addition, $f$ vanishes at all the cusps, then we say that $f$ is a cusp form.

The set of all modular forms (resp. cusp forms) of weight $k$, level $N$ and character $\chi$ on $\Gamma_0(N)$ form a $\mathbb{C}$-vector space and we denote it by $M_k(N,\chi)$ (resp. $S_k(N,\chi)$). If $\chi$ is a trivial character, then we write the space as $M_k(N)$ (resp. $S_k(N)$).

**Definition 1.2.2** (Petersson inner product) Let $f, g \in S_k(N,\chi)$. The Petersson inner product of $f$ and $g$ is defined by

$$\langle f, g\rangle = \frac{1}{i_N}\int_{\mathcal{F}_N} f(\tau)\overline{g(\tau)}v^{k-2}dudv,$$

where $\mathcal{F}_N$ is a fundamental domain for the action of $\Gamma_0(N)$ on $\mathcal{H}$ and $\tau = u+iv$.

**Poincaré series:** Let $k > 2$. For $n \in \mathbb{N}$, define the $n$-th Poincaré series in $S_k(N,\chi)$ as follows.

$$P_{k,N,\chi;n}(\tau) = \frac{1}{2}\sum_{\substack{(c,d)\in\mathbb{Z}^2\\(c,d)=1,N|c}} \overline{\chi}(d)(c\tau+d)^{-k}e\left(n\frac{a_0\tau+b_0}{c\tau+d}\right), \tag{1.1}$$

where in the summation above, for each coprime pair $(c,d)$ with $N|c$, we make a fixed choice of $(a_0,b_0)\in\mathbb{Z}^2$ with $a_0d-b_0c=1$. We have the following characterization of the Poincaré series.

$$\langle f, P_{k,N,\chi;n}\rangle = \frac{\Gamma(k-1)}{i_N(4\pi n)^{k-1}}a_f(n), \tag{1.2}$$

for any cusp form $f\in S_k(N,\chi)$ with Fourier expansion $f(\tau)=\sum_{n\geqslant 1}a_f(n)q^n$.

**Definition 1.2.3** (Hecke operator) For any function $f \in M_k(N,\chi)$, the $n^{th}$ Hecke operator $T(n)$ on $f$ is defined by

$$f|T(n)(\tau) = \frac{1}{n}\sum_{ad=n}\chi(a)a^k\sum_{0\le b<d} f\left(\frac{a\tau+b}{d}\right).$$

The Hecke operator $T(n)$ maps the space $M_k(N,\chi)$ into itself. It also preserves the space $S_k(N,\chi)$. Moreover, if $(n,N)=1$, then $T(n)$ is a hermitian operator with respect to the Petersson inner product.

# 1.3 Modular forms of half-integral weight

For nonzero complex numbers $z$ and $x$, let $z^x = e^{x \log z}$, $\log z = \log|z| + i \arg z$, $-\pi < \arg z \leq \pi$. Let $\zeta$ be a fourth root of unity. Let $G$ denote the four-sheeted covering of $GL_2^+(\mathbb{Q})$ defined as the set of all ordered pairs $(\alpha, \phi(\tau))$, where $\alpha \left( = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \in GL_2^+(\mathbb{Q})$ and $\phi(\tau)$ is a holomorphic function on $\mathcal{H}$ such that $\phi^2(\tau) = \zeta^2 \frac{c\tau+d}{\sqrt{\det \alpha}}$. Then $G$ is a group with the following multiplicative rule.

$$(\alpha, \phi(\tau))(\beta, \psi(\tau)) = (\alpha\beta, \phi(\beta\tau)\psi(\tau)).$$

For a complex valued function $f$ defined on the upper half-plane $\mathcal{H}$ and an element $(\alpha, \phi(\tau)) \in G$, define the stroke operator by

$$f|_{k+1/2}(\alpha, \phi(\tau))(\tau) = \phi(\tau)^{-2k-1} f(\alpha\tau).$$

We omit the subscript $k + 1/2$ whenever there is no ambiguity. For $\Gamma_0(4)$ and its subgroups, we take the lifting $\Gamma_0(4) \to G$ as the collection $\{(\alpha, j(\alpha, \tau))\}$, where $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$, and $j(\alpha, \tau) = \left(\frac{c}{d}\right) \left(\frac{-4}{d}\right)^{-1/2} (c\tau + d)^{1/2}$. Here $\left(\frac{c}{d}\right)$ denotes the generalized quadratic residue symbol and $\left(\frac{-4}{d}\right)^{1/2}$ is equal to 1 or $i$ according as $d$ is 1 or 3 modulo 4 respectively.

**Definition 1.3.1** (Modular form and cusp form of half-integral weight) Let $N$ be any natural number and $\chi$ be any even Dirichlet character modulo $4N$. A holomorphic function $f : \mathcal{H} \longrightarrow \mathbb{C}$ is called a modular form of weight $k + 1/2$ for $\Gamma_0(4N)$ with character $\chi$ if

$$f|_{k+1/2}(\gamma, j(\gamma, \tau))(\tau) = \chi(d) f(\tau), \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N)$$

and $f$ is holomorphic at all the cusps of $\Gamma_0(4N)$. If further, $f$ vanishes at all the cusps, then it is called a cusp form.

The set of modular forms (resp. cusp forms) defined as above form a complex vector space and we denote it by $M_{k+1/2}(4N, \chi)$ (resp. $S_{k+1/2}(4N, \chi)$). If $\chi$ is a trivial character, then the space is denoted by $M_{k+1/2}(4N)$ (resp. $S_{k+1/2}(4N)$).

**Definition 1.3.2** (Petersson inner product) The Petersson inner product of any two cusp forms $f, g \in S_{k+1/2}(4N, \chi)$ is defined by

$$\langle f, g \rangle = \frac{1}{i_{4N}} \int_{\mathcal{F}_{4N}} f(\tau)\overline{g(\tau)} v^{k-3/2} du dv,$$

where $\mathcal{F}_{4N}$ is a fundamental domain for the action of $\Gamma_0(4N)$ on $\mathcal{H}$ and $\tau = u+iv$.

**Poincaré series:** Let $k > 2$. We define the $n$-th Poincaré series in $S_{k+1/2}(4N, \chi)$ as follows.

$$P_{k+1/2,4N,\chi;n}(\tau) = \frac{1}{2} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)=1, 4N|c}} \overline{\chi}(d) \left(\frac{c}{d}\right) \left(\frac{-4}{d}\right)^{k+1/2} (c\tau+d)^{-(k+1/2)} e\left(n \frac{a_0\tau + b_0}{c\tau + d}\right), \tag{1.3}$$

where in the summation above, for each coprime pair $(c, d)$ with $4N|c$, we make a fixed choice of $(a_0, b_0) \in \mathbb{Z}^2$ with $a_0 d - b_0 c = 1$. We have the following characterization of the Poincaré series.

$$\langle f, P_{k+1/2,4N,\chi;n} \rangle = \frac{\Gamma(k - 1/2)}{i_{4N}(4\pi n)^{k-1/2}} a_f(n), \tag{1.4}$$

for any cusp form $f \in S_{k+1/2}(4N, \chi)$ with Fourier expansion $f(\tau) = \sum_{n\geqslant 1} a_f(n) q^n$.

**Definition 1.3.3** (Hecke Operator) For any $f(= \sum_{n\geqslant 0} a_f(n)q^n) \in M_{k+1/2}(4N, \chi)$ and a prime $p \nmid 2N$, we define the action of Hecke operator $T(p^2)$ on $f$ by

$$f|T(p^2)(\tau) = \sum_{n\geq 0} \{a_f(np^2) + \chi(p)\left(\frac{(-1)^k n}{p}\right)p^{k-1} a_f(n) + \chi(p^2) p^{2k-1} a_f(n/p^2)\} q^n.$$

Using the properties

$$T(p^{2(n+1)}) = T(p^2)T(p^{2n}) - p^{2k-1}T(p^{2(n-1)}), \quad (n \geq 1)$$

$$\text{and} \quad T(n^2m^2) = T(n^2)T(m^2), \quad (n, m) = (n, 2N) = (m, 2N) = 1,$$

one can extend the definition of $T(n^2)$ for any positive integer $n$ which is coprime to $2N$.

## 1.4 Holomorphic Jacobi forms on $\mathcal{H} \times \mathbb{C}^{(g,1)}$

Let $g$ be any positive integer and $\Gamma_g^J(N) := \Gamma_0(N) \ltimes (\mathbb{Z}^{(g,1)} \times \mathbb{Z}^{(g,1)})$ denote the generalized Jacobi group. Let $M$ be a positive definite, symmetric, half-integral matrix of size $g \times g$. For any pair $X = (\gamma, (\lambda, \mu)) \in \Gamma_g^J(1)$ and any function $\phi$ on $\mathcal{H} \times \mathbb{C}^{(g,1)}$, define

$$\phi|_{k,M}X(\tau, z) = e\left(M[\lambda]\tau + 2\lambda^t Mz - M[z + \lambda\tau + \mu]\frac{c}{c\tau + d}\right)(c\tau + d)^{-k} \times \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}\right), \tag{1.5}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $M[\lambda] := \lambda^t M \lambda$ ($t$ refers to transpose of vectors and matrices).

**Definition 1.4.1** (Holomorphic Jacobi form) Let $\chi$ be any Dirichlet character modulo $N$. A function $\phi$ on $\mathcal{H} \times \mathbb{C}^{(g,1)}$ is said to be a holomorphic Jacobi form of weight $k$ and index $M$ with respect to the Jacobi group $\Gamma_g^J(N)$ and character $\chi$, if it satisfies the following conditions.

1. $\phi(\tau, z)$ is a holomorphic function,
2. $\phi|_{k,M}X(\tau, z) = \chi(d)\phi(\tau, z)$ for all $X \left( = \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu) \right) \right) \in \Gamma_g^J(N)$,
3. For each $\gamma \in SL_2(\mathbb{Z})$, there exist a positive integer $w_\gamma$ such that $\phi|_{k,M}(\gamma, (0, 0))$ has a Fourier development of the form
$$\sum_{\substack{n \in \mathbb{Z}, r \in \mathbb{Z}^{(1,g)} \\ \frac{4n}{w_\gamma} \geq M^{-1}[r^t]}} c_\gamma(n, r) e\left(\frac{n}{w_\gamma}\tau + rz\right).$$

If $\phi$ satisfies the stronger condition $c_\gamma(n, r) = 0$ unless $4n/w_\gamma > M^{-1}[r^t]$, for all $\gamma \in SL_2(\mathbb{Z})$, then it is called a holomorphic Jacobi cusp form.

**Remark 1.4.2** 1. If $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then $w_\gamma = 1$ and we write the Fourier development, $\phi(\tau, z) = \sum_{\substack{n \in \mathbb{Z}, r \in \mathbb{Z}^{(1,g)} \\ 4n \geq M^{-1}[r^t]}} c_\phi(n, r) e(n\tau + rz)$. We call $c_\phi(n, r)$, the $(n, r)$-th Fourier coefficient of the holomorphic Jacobi form $\phi$.

2. Sometimes we write simply Jacobi forms for holomorphic Jacobi forms if there is no confusion. For $g$ and $N = 1$, the above defines Jacobi forms and Jacobi cusp forms of integer index as considered in [EZ85].

The set of all holomorphic Jacobi forms as defined above form a $\mathbb{C}$-vector space and we denote it by $J_{k,M}(N, \chi)$. We denote by $J^{\text{cusp}}_{k,M}(N, \chi)$, the vector subspace of all holomorphic Jacobi cusp forms as defined above. If $\chi$ is a trivial character, then we write these spaces as $J_{k,M}(N)$ and $J^{\text{cusp}}_{k,M}(N)$ respectively. For more details about holomorphic Jacobi forms, we refer to [EZ85, Z89].

**Definition 1.4.3** (Petersson inner product) For holomorphic Jacobi cusp forms $\phi$ and $\psi$ on $\Gamma_g^J(N)$, we define the Petersson scalar product of them as follows.

$$\langle \phi, \psi \rangle := \frac{1}{[\Gamma_g^J(1) : \Gamma_g^J(N)]} \int_{\Gamma_g^J(N) \backslash \mathcal{H} \times \mathbb{C}^{(g,1)}} \phi(\tau, z) \overline{\psi(\tau, z)} e^{-4\pi M[y] v^{-1}} v^{k-g-2} du dv dx dy, \tag{1.6}$$

where $\tau = u + iv$ and $z = x + iy$.

**Poincaré series :** For $n \in \mathbb{Z}, r \in \mathbb{Z}^{(1,g)}$ with $4n > M^{-1}[r^t]$ and $k > g + 2$, we define the $(n, r)$-th holomorphic Jacobi Poincaré series of exponential type by

$$P_{k,M,N,\chi;(n,r)}(\tau, z) := \sum_{X \in \Gamma^J_{g,\infty} \backslash \Gamma^J_g(N)} \overline{\chi}(d) \; e^{(n,r)}|_{k,M} X(\tau, z) \qquad (\tau \in \mathcal{H}, z \in \mathbb{C}^{(g,1)}), \tag{1.7}$$

where $X = \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu) \right)$, $\Gamma^J_{g,\infty} = \left\{ \left( \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, (0, \mu) \right) | m \in \mathbb{Z}, \mu \in \mathbb{Z}^{(g,1)} \right\}$ and $e^{(n,r)}(\tau, z) = e(n\tau + rz)$. For $N = 1$, S. Böcherer and W. Kohnen [BK93] studied this Poincaré series and for level $N$ with trivial character, it was studied by K. Bringmann [KB04]. Proceeding on the same lines as in the proof of [BK93, Lemma 1], we have $P_{k,M,N,\chi;(n,r)} \in J^{\text{cusp}}_{k,M}(N, \chi)$ and the following Fourier expansion.

$$P_{k,M,N,\chi;(n,r)}(\tau, z) = \sum_{\substack{n' \in \mathbb{Z}, r' \in \mathbb{Z}^{(1,g)} \\ 4n' > M^{-1}[r'^t]}} g^{\pm}_{k,M,N,\chi;(n,r)}(n', r') \; e(n'\tau + r'z), \tag{1.8}$$

where

$$g^{\pm}_{k,M,N,\chi;(n,r)}(n',r') = g_{k,M,N,\chi;(n,r)}(n',r') + \chi(-1)(-1)^k g_{k,M,N,\chi;(n,r)}(n',-r')$$

and

$$g_{k,M,N,\chi;(n,r)}(n',r') = \delta_M(n,r;n',r') + 2\pi i^{-k}(\det 2M)^{-\frac{1}{2}}\,(D'/D)^{(k-g/2-1)/2} \times \sum_{c\geqslant 1, N|c} H_{M,c,\chi}(n,r;n',r') J_{k-\frac{g}{2}-1}\left(\frac{2\pi\sqrt{DD'}}{\det(2M)c}\right). \tag{1.9}$$

Here $D := -\ \det\begin{pmatrix}2n & r\\ r^t & 2M\end{pmatrix}$, $D' := -\ \det\begin{pmatrix}2n' & r'\\ r'^t & 2M\end{pmatrix}$,

$$\delta_M(n,r;n',r') = \begin{cases} 1 & \text{if } D'=D \text{ and } r'\equiv r(\bmod 2M\cdot\mathbb{Z}^{(1,g)}),\\ 0 & \text{otherwise}\end{cases} \tag{1.10}$$

and

$$H_{M,c,\chi}(n,r;n',r') = c^{-\frac{g}{2}-1} e_{2c}(-r'M^{-1}r^t) \sum_{\substack{x (\mathrm{mod}\ c)\\ y(\mathrm{mod}\ c)^*}} \overline{\chi}(y) e_c((M[x]+rx+n)\overline{y}+n'y-r'x). \tag{1.11}$$

In the above sum, $x$ and $y$ run over a complete set of representatives for $\mathbb{Z}^{(g,1)}/c\mathbb{Z}^{(g,1)}$ and $(\mathbb{Z}/c\mathbb{Z})^*$ respectively and $\overline{y}$ denotes the inverse of $y$ (mod $c$). Also, $J_r(x)$ is the Bessel function of order $r$.

For the sake of simplicity, we write $P_{(n,r)}$ for $P_{k,M,N,\chi;(n,r)}$.

The Petersson scalar product on $J^{\mathrm{cusp}}_{k,M}(N,\chi)$ is characterized by the following (see [BK93, KB04]). Let

$$\phi(\tau,z) = \sum_{\substack{m\in\mathbb{Z}, r\in\mathbb{Z}^{(1,g)}\\ 4m>M^{-1}[r^t]}} c_\phi(m,r)e(m\tau+rz) \ \in J^{\mathrm{cusp}}_{k,M}(N,\chi).$$

Then, we have

$$\langle\phi, P_{(n,r)}\rangle = \lambda_{k,M,D,N} c_\phi(n,r), \tag{1.12}$$

with

$$\lambda_{k,M,D,N} = \frac{2^{(g-1)(k-g/2-1)-g}\Gamma(k-g/2-1)}{\pi^{k-g/2-1}[\Gamma^J_g(1):\Gamma^J_g(N)]}(\det M)^{k-(g+3)/2}|D|^{-k+g/2+1}. \tag{1.13}$$

## 1.5 Skew-holomorphic Jacobi forms on $\mathcal{H} \times \mathbb{C}^{(g,1)}$

First, let us recall the definition of a skew-holomorphic Jacobi form ([Sk88, A93]). For any pair $X = (\gamma, (\lambda, \mu)) \in \Gamma_g^J(1)$ and any function $\phi$ on $\mathcal{H} \times \mathbb{C}^{(g,1)}$, define

$$\phi|_{k,M}^* X(\tau, z) = e\left(M[\lambda]\tau + 2\lambda^t M z - M[z + \lambda\tau + \mu]\frac{c}{c\tau + d}\right) |c\tau + d|^{-g}\overline{(c\tau + d)}^{g-k} \times \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}\right), \tag{1.14}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

**Definition 1.5.1** (Skew-holomorphic Jacobi form) Let $\chi$ be any Dirichlet character modulo $N$. A function $\phi$ on $\mathcal{H} \times \mathbb{C}^{(g,1)}$ is said to be a skew-holomorphic Jacobi form of weight $k$ and index $M$ with respect to the Jacobi group $\Gamma_g^J(N)$ and character $\chi$, if it satisfies the following conditions.

1. $\phi(\tau, z)$ is a smooth function in $\tau \in \mathcal{H}$ and holomorphic in $z \in \mathbb{C}^{(g,1)}$,
2. $\phi|_{k,M}^* X(\tau, z) = \chi(d)\phi(\tau, z)$ for all $X \left( = \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu) \right) \right) \in \Gamma_g^J(N)$,
3. For each $\gamma \in SL_2(\mathbb{Z})$, there exist a positive integer $w_\gamma$ such that $\phi|_{k,M}^*(\gamma, (0, 0))$ has a Fourier development of the form

$$\sum_{\substack{n \in \mathbb{Z}, r \in \mathbb{Z}^{(1,g)} \\ \frac{4n}{w_\gamma} \leq M^{-1}[r^t]}} c_\gamma(n, r) e\left(\frac{n}{w_\gamma}\overline{\tau} + \frac{i}{2}M^{-1}[r^t] \operatorname{Im} \tau + rz\right).$$

If $\phi$ satisfies the stronger condition $c_\gamma(n, r) = 0$ unless $4n/w_\gamma < M^{-1}[r^t]$, for all $\gamma \in SL_2(\mathbb{Z})$, then it is called a skew-holomorphic Jacobi cusp form.

**Remark 1.5.2** 1. If $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then $w_\gamma = 1$ and we write the Fourier development, $\phi(\tau, z) = \sum_{\substack{n \in \mathbb{Z}, r \in \mathbb{Z}^{(1,g)} \\ 4n \leq M^{-1}[r^t]}} c_\phi(n, r) e(n\overline{\tau} + \frac{i}{2}M^{-1}[r^t] \operatorname{Im} \tau + rz)$. We call $c_\phi(n, r)$, the $(n, r)$-th Fourier coefficient of the skew- holomorphic Jacobi form $\phi$.

2. For $g$ and $N = 1$, the above defines skew-holomorphic Jacobi forms and Jacobi cusp forms of integer index as considered in [Sk88] and [Sk90].

The set of all skew-holomorphic Jacobi forms as defined above form a $\mathbb{C}$-vector space and we denote it by $J^*_{k,M}(N,\chi)$. We denote the vector subspace of all skew-holomorphic Jacobi cusp forms by $J^{*,\mathrm{cusp}}_{k,M}(N,\chi)$. If $\chi$ is a trivial character, then we write these spaces as $J^*_{k,M}(N)$ and $J^{*,\mathrm{cusp}}_{k,M}(N)$ respectively.

**Definition 1.5.3** (Petersson inner product) For skew-holomorphic Jacobi cusp forms $\phi$ and $\psi$ on $\Gamma^J_g(N)$, we define the Petersson scalar product of them similar to the holomrphic case as follows.

$$\langle \phi, \psi \rangle := \frac{1}{[\Gamma^J_g(1) : \Gamma^J_g(N)]} \int_{\Gamma^J_g(N)\backslash \mathcal{H}\times\mathbb{C}^{(g,1)}} \phi(\tau, z)\overline{\psi(\tau, z)} e^{-4\pi M[y]v^{-1}} v^{k-g-2} dudvdxdy, \tag{1.15}$$

where $\tau = u + iv$ and $z = x + iy$.

**Poincaré series:** For $n \in \mathbb{Z}, r \in \mathbb{Z}^{(1,g)}$ with $4n < M^{-1}[r^t]$ and $k > g + 2$, define the $(n,r)$-th skew-holomorphic Jacobi Poincaré series of exponential type by

$$P^*_{k,M,N,\chi;(n,r)}(\tau, z) := \sum_{X \in \Gamma^J_{g,\infty}\backslash\Gamma^J_g(N)} \overline{\chi}(d) e^{(n,r)}_* |^*_{k,M} X(\tau, z) \qquad (\tau \in \mathcal{H}, z \in \mathbb{C}^{(g,1)}), \tag{1.16}$$

where $X = \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu) \right)$, $\Gamma^J_{g,\infty} = \left\{ \left( \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, (0, \mu) \right) | m \in \mathbb{Z}, \mu \in \mathbb{Z}^{(g,1)} \right\}$ and $e^{(n,r)}_*(\tau, z) = e(n\overline{\tau} + \frac{i}{2} M^{-1}[r^t] \text{ Im } \tau + rz)$. Using the definition and the absolute convergence of the series $P^*_{k,M,n,\chi;(n,r)}$, we get the transformation formula. Following [KB04, Theorem 3.27], one can compute the Fourier expansion of $P^*_{k,M,n,\chi;(n,r)}|_{k,M}(\gamma, (0,0))$ for all $\gamma \in SL_2(\mathbb{Z})$, from which it follows that $P^*_{k,M,N,\chi;(n,r)} \in J^{*,\mathrm{cusp}}_{k,M}(N,\chi)$. We prove the following lemma, which will be used in chapter 3.

**Lemma 1.5.4** *(i) The Poincaré series $P^*_{k,M,N,\chi;(n,r)}$ has the following Fourier expansion.*

$$P^*_{k,M,N,\chi;(n,r)}(\tau, z) = \sum_{\substack{n' \in \mathbb{Z}, r' \in \mathbb{Z}^{(1,g)} \\ 4n' < M^{-1}[r'^t]}} g^{*,\pm}_{k,M,N,\chi;(n,r)}(n', r') e(n'\overline{\tau} + \frac{i}{2} M^{-1}[r'^t] \text{ Im } \tau + r'z), \tag{1.17}$$

*where*

$$g^{*,\pm}_{k,M,N,\chi;(n,r)}(n',r') = g^{*}_{k,M,N,\chi;(n,r)}(n',r')+\chi(-1)(-1)^{k-g}g^{*}_{k,M,N,\chi;(n,r)}(n',-r').$$

*Here*

$$g^{*}_{k,M,N,\chi;(n,r)}(n',r') = \delta_M(n,r;n',r') + 2\pi i^{k-g}(\det 2M)^{-\frac{1}{2}}\,(D'/D)^{(k-g/2-1)/2} \times \sum_{c\geqslant 1, N|c} H_{M,c,\chi}(n,r;n',r')J_{k-\frac{g}{2}-1}\left(\frac{2\pi\sqrt{DD'}}{\det(2M)c}\right), \tag{1.18}$$

*where* $D, D', \delta_M(n,r;n',r')$ *and* $H_{M,c,\chi}(n,r;n',r')$ *are defined in the holomorphic case* (1.10).

*(ii) Let* $\phi(\tau,z) = \sum_{\substack{n\in\mathbb{Z}, r\in\mathbb{Z}^{(1,g)}\\ 4n<M^{-1}[r^t]}} c_\phi(n,r)\; e(n\overline{\tau}+\frac{i}{2}M^{-1}[r^t]\,\mathrm{Im}\,\tau+rz) \;\in J^{*,cusp}_{k,M}(N,\chi)$*.*
*Then, we have*

$$\langle \phi, P^{*}_{k,M,N,\chi;(n,r)}\rangle = \lambda_{k,M,D,N}c_\phi(n,r), \tag{1.19}$$

*where* $\lambda_{k,M,D,N}$ *is the same same constant as defined in the holomorphic case* (1.13)*.*

**Proof.** To prove the above lemma, we proceed as in [BK93, Lemma 1].

**Proof of** $(i)$. Let $\tau = u+iv$. As a set of representatives of $\Gamma^J_{g,\infty}\backslash\Gamma^J_g(N)$, we take the elements

$$\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}, (a\lambda, b\lambda)\right) = \left(\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}, (\lambda,0)\right)\cdot\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}, (0,0)\right),$$

where $c,d\in\mathbb{Z}$ with $(c,d)=1$ and $N|c, \lambda\in\mathbb{Z}^{(g,1)}$ and for each pair $(c,d)$ we have chosen $a,b\in\mathbb{Z}$ such that $ad-bc=1$. This gives us

$$P^{*}_{k,M,N,\chi;(n,r)}(\tau,z) = \sum_{\substack{(c,d)=1, N|c\\ \lambda\in\mathbb{Z}^{(g,1)}}} \overline{\chi}(d)|c\tau+d|^{-g}\overline{(c\tau+d)}^{g-k} e\left(-M[z]\frac{c}{c\tau+d} + M[\lambda]\frac{a\tau+b}{c\tau+d} + 2\lambda^t Mz\frac{1}{c\tau+d} + n\frac{a\overline{\tau}+b}{c\overline{\tau}+d} + rz\frac{1}{c\tau+d} + r\lambda\frac{a\tau+b}{c\tau+d} + \frac{i}{2}M^{-1}[r^t]\frac{v}{|c\tau+d|^2}\right).$$

We now split the sum into the terms with $c = 0$ and the terms with $c \neq 0$. If $c = 0$, then $a = d = \pm 1$, so these terms give

$$\sum_{\lambda \in \mathbb{Z}^{(g,1)}} e((M[\lambda]+r\lambda)\tau+n\overline{\tau}+\frac{i}{2}M^{-1}[r^t]v)[e((r+2\lambda^t M)z)+\chi(-1)(-1)^{k-g}e((-r-2\lambda^t M)z)]$$

$$= \sum_{\substack{n' \in \mathbb{Z}, r' \in \mathbb{Z}^{(g,1)} \\ 4n' < M^{-1}[r'^t]}} (\delta_M(n,r;n',r')e(n'\overline{\tau} + \frac{i}{2}M^{-1}[r'^t]v + r'z)$$
$$+ \chi(-1)(-1)^{k-g}\delta_M(n,r;n',-r')e(n'\overline{\tau} + \frac{i}{2}M^{-1}[r'^t]v - r'z)).$$

The terms for $c < 0$ are obtained from those with $c > 0$ by multiplying with $\chi(-1)(-1)^{k-g}$ and replacing $z$ by $-z$. Thus it suffices to consider the terms with $c > 0$. Using the identities

$$\frac{a\tau+b}{c\tau+d} = \frac{a}{c} - \frac{1}{c(c\tau+d)},$$

$$\frac{1}{c\tau+d}z + \frac{a\tau+b}{c\tau+d}\lambda = \frac{1}{c\tau+d}\left(z - \frac{1}{c}\lambda\right) + \frac{a}{c}\lambda,$$

$$\frac{a\tau+b}{c\tau+d}M[\lambda] + \frac{2}{c\tau+d}\lambda^t Mz - \frac{c}{c\tau+d}M[z] = -\frac{c}{c\tau+d}M\left[z - \frac{1}{c}\lambda\right] + \frac{a}{c}M[\lambda]$$

and replacing $(d, \lambda)$ by $(d + \alpha c, \lambda + \beta c)$, with the new $d$ running $(\mathrm{mod\ c})^*$ and $\lambda(\mathrm{mod\ c}); \alpha \in \mathbb{Z}, \beta \in \mathbb{Z}^{(\mathrm{g},1)}$, we obtain for the terms with $c > 0$ the contribution

$$\sum_{c>0, N|c} c^{-k} \sum_{d(c)^*, \lambda(c)} \overline{\chi}(d)e_c((M[\lambda]+r\lambda+n)\overline{d})\mathcal{F}_{k,M,c;(n,r)}(\tau + \frac{d}{c}, z - \frac{1}{c}\lambda),$$

where

$$\mathcal{F}_{k,M,c;(n,r)}(\tau,z) = \sum_{\alpha \in \mathbb{Z}, \beta \in \mathbb{Z}^{(g,1)}} (\overline{\tau}+\alpha)^{g-k}|\tau+\alpha|^{-g} e\left(-\frac{1}{\tau+\alpha}M[z-\beta] - \frac{n}{c^2(\tau+\alpha)}\right.$$
$$\left. + \frac{1}{c(\tau+\alpha)}r(z-\beta) + \frac{i}{2}\frac{M^{-1}[r^t]}{c^2}\left(\frac{1}{\overline{\tau}+\alpha} - \frac{1}{\tau+\alpha}\right)\right).$$

The function $\mathcal{F}_{k,M,c;(n,r)}(\tau, z)$ has a Fourier expansion of the form

$$\sum_{n' \in \mathbb{Z}, r' \in \mathbb{Z}^{(1,g)}} \gamma(n',r')e(n'\overline{\tau} + \frac{i}{2}M^{-1}[r'^t]v + r'z),$$

where $\gamma(n',r') = \int_{iC_1-\infty}^{iC_1+\infty} \overline{\tau}^{g-k}|\tau|^{-g} e(-n'\overline{\tau} - \frac{1}{4}M^{-1}[r'^t](\tau - \overline{\tau})) \int_{iC_2-\infty}^{iC_2+\infty} \cdots \int_{iC_2-\infty}^{iC_2+\infty}$

$$e\left(\frac{-1}{\tau}M[z] + \frac{1}{c\tau}rz - \frac{n}{c^2\overline{\tau}} + \frac{1}{4}M^{-1}[r^t]\left(\frac{1}{\overline{\tau}} - \frac{1}{\tau}\right) - r'z\right) dz d\tau$$

$$(C_1 > 0, C_2 \in \mathbb{R}).$$

Using

$$D = 2^{-1-g}(\det M)^{-1}\left(\frac{1}{4}M^{-1}[r^t] - n\right)$$

and

$$D' = 2^{-1-g}(\det M)^{-1}\left(\frac{1}{4}M^{-1}[r'^t] - n'\right),$$

we get

$$\begin{aligned}\gamma(n',r') = \int_{iC_1-\infty}^{iC_1+\infty} \overline{\tau}^{g-k}|\tau|^{-g} e\Bigg(&-\frac{1}{4}M^{-1}[r'^t]\tau + 2^{-1-g}(\det M)^{-1}D'\overline{\tau} \\ &+2^{-1-g}(\det M)^{-1}\frac{D}{c^2}\left(\frac{1}{\overline{\tau}} - \frac{1}{\tau}\right)\Bigg) \\ \times \int_{iC_2-\infty}^{iC_2+\infty} \cdots \int_{iC_2-\infty}^{iC_2+\infty} e\left(\frac{-1}{\tau}M[z] - \frac{n}{c^2\tau} + \frac{1}{c\tau}rz - r'z\right) dz d\tau.\end{aligned}$$

The inner multiple integral has been evaluated in the proof of [BK93, Lemma 1 (ii)], which is equal to

$$(\det M)^{-1/2}\left(\frac{\tau}{2i}\right)^{g/2} e_{2c}(-r'M^{-1}r^t) e\left(n'\tau + 2^{-1-g}(\det M)^{-1}\left(D'\tau + \frac{D}{c^2\tau}\right)\right).$$

Therefore

$$\begin{aligned}\gamma(n',r') =& (\det 2M)^{-1/2} e_{2c}(-r'M^{-1}r^t) \\ &\times \int_{iC_1-\infty}^{iC_1+\infty} \left(\frac{\tau}{2i}\right)^{g/2} \overline{\tau}^{g-k}|\tau|^{-g} e\left(-2^{-1-g}(\det M)^{-1}\left(D'\overline{\tau} + \frac{D}{c^2\overline{\tau}}\right)\right) d\tau.\end{aligned}$$

Substituting $\tau \mapsto -\overline{\tau}$, we get

$$\begin{aligned}\gamma(n',r') =& (-1)^{k-g/2}(\det 2M)^{-1/2} e_{2c}(-r'M^{-1}r^t) \\ &\times \int_{iC_1-\infty}^{iC_1+\infty} \left(\frac{\tau}{i}\right)^{g/2} \tau^{-k} e\left(-2^{-1-g}(\det M)^{-1}\left(-D'\tau + \frac{-D}{c^2\tau}\right)\right) d\tau.\end{aligned}$$

The integral in the right hand side of the above equation has been evaluated completely in the proof of [KB04, Lemma 1 (ii)]. Using this evaluation, we have

$$\gamma(n', r') = 2\pi i^{k-g} (\det 2M)^{-1/2} (D'/D)^{k/2-g/4-1/2} c^{k-g/2-1} e_{2c}(-r' M^{-1} r^t) \times J_{k_g/2-1}\left( \frac{\pi}{2^{g-1}(\det M)c} (DD')^{1/2} \right).$$

Thus we obtain for the terms with $c > 0$, the contribution

$$2\pi i^{k-g} (\det 2M)^{-\frac{1}{2}} (D'/D)^{(k-g/2-1)/2} \sum_{c \geqslant 1, N|c} H_{M,c,\chi}(n, r; n', r') J_{k-\frac{g}{2}-1} \left( \frac{2\pi\sqrt{DD'}}{\det(2M)c} \right)$$

and finally this proves $(i)$.

**Proof of $(ii)$.** Let $\tau = u + iv$. Using the standard unfolding argument, we have

$$\langle \phi, P^*_{k,M,N,\chi;(n,r)} \rangle = \frac{1}{[\Gamma^J_g(1) : \Gamma^J_g(N)]} \int_{\Gamma^J_{g,\infty} \backslash \mathcal{H} \times \mathbb{C}^{(g,1)}} \phi(\tau, z) \overline{e^{(n,r)}_*(\tau, z)} v^{k-g-2} e^{-4\pi M[y] v^{-1}} du dv dx dy.$$

A fundamental domain for the action of $\Gamma^J_{g,\infty}$ on $\mathcal{H} \times \mathbb{C}^{(g,1)}$ can be taken as

$$\{(\tau, z) \in \mathcal{H} \times \mathbb{C}^{(g,1)} : \tau = u + iv, 0 \leqslant u \leqslant 1, v > 0; z = x + iy, 0 \leqslant x_\nu \leqslant 1 \text{ for } \nu = 1, \ldots, g, y \in \mathbb{R}^g\},$$

here we have written $x^t = (x_1, \ldots, x_g)$. Inserting the Fourier expansion of $\phi$ in the above integral, using standard orthogonality relations and

$$\int_{\mathbb{R}^g} e^{-4\pi(ry + M[y]v^{-1})} dy = 2^{-g} (\det M)^{-1/2} v^{g/2} e^{\pi v M^{-1}[r^t]}$$

(see [BK93, Lemma 1 (i)]), we get

$$\langle \phi, P^*_{k,M,N,\chi;(n,r)} \rangle = \frac{1}{[\Gamma^J_g(1) : \Gamma^J_g(N)]} (\det 2M)^{-1/2} c_\phi(n, r) \times \int_0^\infty v^{k-g/2-2} e^{-2^{1-g}(\det M)^{-1}\pi D v} dv.$$

Using the definition of the gamma function, we get $(ii)$. □

For the sake of simplicity, we write $P^*_{(n,r)}$ for $P^*_{k,M,N,\chi;(n,r)}$.

# Chapter 2

# Restriction map for Jacobi forms

## 2.1 Introduction

In this chapter, all Jacobi forms are holomorphic Jacobi forms and so we will simply write Jacobi forms. Let $m$ be any positive integer and $\mathcal{D}_0$ be the restriction map from the space of Jacobi forms of weight $k$, index $m$ on the congruence subgroup $\Gamma_0(N)$ to that of elliptic modular forms of the same weight on $\Gamma_0(N)$, given by $\phi(\tau, z) \mapsto \phi(\tau, 0)$. More generally, one obtains modular forms of weight $k+\nu$ from Jacobi forms of weight $k$, by using certain differential operators $\mathcal{D}_\nu$ (for $N = 1$, we refer to [EZ85]). Then it is a fact that $\oplus_{\nu=0}^{m}\mathcal{D}_{2\nu} : J_{k,m}(N) \longrightarrow M_k(N) \oplus S_{k+2}(N) \oplus \ldots \oplus S_{k+2m}(N)$ is injective for $k$ even but when $k \geqslant 4$, it is well-known that the map $\mathcal{D}_0$ is far from being injective. However, J. Kramer [JK86], T. Arakawa and S. Böcherer [AB99, AB03] observed that when $k = 2$ the situation may be different. In fact, when $m = 1$, Arakawa and Böcherer [AB99] provided two explicit descriptions of $\mathrm{Ker}(\mathcal{D}_0)$: one in terms of modular forms of weight $k-1$ and the other in terms of cusp forms of weight $k+2$ (by applying the differential operator $\mathcal{D}_2$ on $\mathrm{Ker}(\mathcal{D}_0)$). In a subsequent paper [AB03], for square-free level $N$ they proved that $\mathcal{D}_0$ is injective in the case $k = 2$, $m = 1$ and gave some applications. In a private communication to us, Professor Böcherer informed that one of his students gave a precise description of the image of $\mathcal{D}_0 \oplus \mathcal{D}_2$ in terms of vanishing orders in the cusps ($k$ arbitrary, $m = 1$). Based on this, he conjectured that in the case $k = 2$ and $N$ square-free, one can remove one of the $\mathcal{D}_{2\nu}$ from the direct sum $\oplus_{\nu=0}^{m}\mathcal{D}_{2\nu}$ without affecting the injectivity property, for any index $m$.

In this chapter, we generalize the results of [AB99] to higher index. In §2.3.1

and §2.3.2, we consider the case $m = 2$ and show that $\mathrm{Ker}(\mathcal{D}_0)$ is isomorphic to the space of vector-valued modular forms of weight $k-1$ and $\mathcal{D}_2(\mathrm{Ker}(\mathcal{D}_0))$ is a certain subspace of cusp forms of weight $k+2$ and these two spaces are related with each other by a simple isomorphism. In §2.3.3, we obtain the injectivity of $\mathcal{D}_0 \oplus \mathcal{D}_2$ on $J_{2,2}(2N)$, where $N = 2$ or an odd square-free positive integer. This confirms the conjecture made by Professor Böcherer partially in the index 2 case (i.e., we can omit the operator $\mathcal{D}_4$ from the direct sum $\oplus_{\nu=0}^{2}\mathcal{D}_{2\nu}$). In §2.4, we consider a subspace of $J_{k,m}(mN,\chi)$, where index $m$ is square-free and $N$ is any positive integer. We obtain results similar to [AB99] and prove the injectivity of $\mathcal{D}_0$ on this subspace when $k = 2$, $mN$ square-free. In §2.5, we make several remarks concerning the subspace studied in §2.4. The results of this chapter are contained in [RS11].

## 2.2 Preliminaries

Let $k$, $m$ and $N$ be positive integers and $\chi$ be a Dirichlet character modulo $N$. Denote the matrices $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ by $I_2$, $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ by $T$ and $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ by $S$. It is well-known that any Jacobi form $\phi \in J_{k,m}(N,\chi)$ can be (uniquely) written as

$$\phi(\tau, z) = \sum_{r=0}^{2m-1} h_{m,r}(\tau)\theta^J_{m,r}(\tau, z).$$

Here

$$\theta^J_{m,r}(\tau, z) = \sum_{n\in\mathbb{Z}} e^{2\pi i m((n+\frac{r}{2m})^2\tau + 2(n+\frac{r}{2m})z)},$$

$$h_{m,r}(\tau) = \sum_{\substack{n\in\mathbb{Z}\\ n\geq r^2/4m}} c_\phi(n,r) e^{2\pi i(n-\frac{r^2}{4m})\tau},$$

where $c_\phi(n,r)$ denotes the $(n,r)$-th Fourier coefficient of the Jacobi form $\phi$ and $\theta^J_{m,r}(\tau,z)$ is the Jacobi theta series.

The (column) vector $\Theta^J(\tau,z) = (\theta^J_{m,r}(\tau,z))_{0\leqslant r<2m}$ satisfies the following transformation rule.

$$\Theta^J\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = e^{2\pi i m\frac{cz^2}{c\tau+d}}(c\tau+d)^{\frac{1}{2}}U_m(\gamma)\Theta^J(\tau,z), \qquad (2.1)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Here, $U_m : SL_2(\mathbb{Z}) \longrightarrow U(2m, \mathbb{C})$ denotes a (projective) representation of $SL_2(\mathbb{Z})$. In the cases $m = 1$ and 2, it is given by

$$U_1(T) = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \; U_1(s) = \frac{e^{-\pi i/4}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix};$$

$$U_2(T) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{i} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \sqrt{i} \end{pmatrix}, U_2(S) = \frac{e^{-\pi i/4}}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}.$$

Using the transformation property of the Jacobi form $\phi$ with (2.1), we get the following transformation property for the row vector $\mathbf{h} = (h_{m,r})_{0 \leqslant r < 2m}$.

$$\mathbf{h}(\gamma\tau) = \chi(d)(c\tau + d)^{k-\frac{1}{2}} \mathbf{h}(\tau) \overline{U_m}(\gamma)^t \tag{2.2}$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(m)$, let $\gamma_m$ denote the $SL_2(\mathbb{Z})$ matrix $\begin{pmatrix} a & bm \\ \frac{c}{m} & d \end{pmatrix}$. Let the matrices $(u_{ij})_{0 \leqslant i,j < 2m}$ and $(u_{ij}^m)_{0 \leqslant i,j \leqslant 1}$ represent $U_m(\gamma)$ and $U_1(\gamma_m)$ respectively. We have the following relations.

$$\theta^J_{m,0}(\tau, z) = \theta^J_{1,0}(m\tau, mz) \;\text{ and }\; \theta^J_{m,m}(\tau, z) = \theta^J_{1,1}(m\tau, mz). \tag{2.3}$$

Using the above relation, for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(m)$, we get the following transformation property.

$$\begin{aligned} \begin{pmatrix} \theta^J_{m,0} \\ \theta^J_{m,m} \end{pmatrix} \left( \frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) &= \begin{pmatrix} \theta^J_{1,0} \\ \theta^J_{1,1} \end{pmatrix} \left( \frac{a(m\tau) + mb}{c/m(m\tau) + d}, \frac{mz}{c/m(m\tau) + d} \right) \\ &= e^{2\pi i m \frac{cz^2}{c\tau + d}} (c\tau + d)^{\frac{1}{2}} U_1(\gamma_m) \begin{pmatrix} \theta^J_{m,0} \\ \theta^J_{m,m} \end{pmatrix} (\tau, z). \end{aligned} \tag{2.4}$$

Comparing the transformation properties given by (2.1) and (2.4) for the action of $\gamma \in \Gamma_0(m)$, we get two linear equations in $(\theta^J_{m,r})_{0 \leqslant r < 2m}$ as follows.

$$(u_{00}-u_{00}^m)\theta^J_{m,0}+(u_{0m}-u_{01}^m)\theta^J_{m,m}+\sum_{j\neq 0,m}u_{0j}\theta^J_{m,j} \;=\; 0; \tag{2.5}$$

$$(u_{m0}-u_{10}^m)\theta^J_{m,0}+(u_{mm}-u_{11}^m)\theta^J_{m,m}+\sum_{j\neq 0,m}u_{mj}\theta^J_{m,j} \;=\; 0. \tag{2.6}$$

Since the set $\{\theta^J_{m,r}\}_{0\leqslant r<2m}$ is linearly independent over $\mathbb{C}$, we have

$$u_{00}=u_{00}^m, u_{0m}=u_{01}^m, u_{0j}=0 \;\text{ for all }\; j\neq 0,m;$$
$$u_{m0}=u_{10}^m, u_{mm}=u_{11}^m, u_{mj}=0 \;\text{ for all }\; j\neq 0,m.$$

Now, for any $\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma_0(mN)$, the above observation for $U_m(\gamma)$ with (2.2), give us the following transformation property.

$$\begin{pmatrix}h_{m,0}\\ h_{m,m}\end{pmatrix}(\gamma\tau) \;=\; \chi(d)(c\tau+d)^{k-\frac{1}{2}}\overline{U_1}(\gamma_m)\begin{pmatrix}h_{m,0}\\ h_{m,m}\end{pmatrix}. \tag{2.7}$$

We will denote $\theta^J_{m,r}(\tau,0)$ by $\theta_{m,r}(\tau)$. Let $\eta(\tau):=e^{2\pi i\tau/24}\prod_{n\geq 1}(1-e^{2\pi in\tau})$ is the Dedekind eta function. One has,

$$\eta^3(2\tau)=\frac{1}{2}\theta_{1,0}(\tau)\theta_{1,1}(\tau)\left(\sum_{n\in\mathbb{Z}}(-1)^ne^{2\pi in^2\tau}\right)$$

(see [R73, (78.4) and (78.6)]). The above identity implies that $\theta_{1,0}$ and $\theta_{1,1}$ have no zero in the upper half-plane. Using (2.3), $\theta_{m,0}(\tau)=\theta_{1,0}(m\tau)$ and $\theta_{m,m}(\tau)=\theta_{1,1}(m\tau)$ for any $m$ and hence $\theta_{m,0}$ and $\theta_{m,m}$ also do not have any zero in the upper half-plane. For a given $m$, we define the character $\omega_m$ on $\Gamma_0(m)$ by $\omega_m(\gamma):=\det(U_1(\gamma_m)), \gamma\in\Gamma_0(m)$. We will simply write $\omega$ (as used in [AB99]) for $\omega_1$. Define,

$$\xi(\tau)=(\theta_{1,1}\theta'_{1,0}-\theta_{1,0}\theta'_{1,1})(\tau). \tag{2.8}$$

It is known that the function $\xi$ is a cusp form of weight 3 for $SL_2(\mathbb{Z})$ with character $\omega$ and it is equal to $-\pi i\eta^6$ ([AB99, Proposition 2]). Since the Dedekind eta function $\eta$ does not have any zero in the upper half-plane, the same is true for $\xi$. Define,

$$\xi^\star_m(\tau):=(\theta_{m,m}\theta'_{m,0}-\theta_{m,0}\theta'_{m,m})(\tau). \tag{2.9}$$

Then $\xi_m^\star(\tau) = \xi(m\tau)$ and hence it is a cusp form of weight 3 for $\Gamma_0(m)$ with character $\omega_m$ and it has no zero in the upper half-plane. We will simply write $\xi$ for $\xi_1^\star$.

Let $\mathcal{D}_0 : J_{k,m}(N,\chi) \longrightarrow M_k(N,\chi)$ be the restriction map given by

$$\phi(\tau,z) \mapsto \phi(\tau,0)$$

and $\mathcal{D}_2$ be the differential operator

$$\mathcal{D}_2 = \left(\frac{k}{2\pi i}\frac{\partial^2}{\partial z^2} - 4\frac{\partial}{\partial \tau}\right)\Big|_{z=0}, \tag{2.10}$$

which acts on holomorphic functions on $\mathcal{H} \times \mathbb{C}$. We denote the kernel of the restriction map $\mathcal{D}_0 : J_{k,m}(N,\chi) \longrightarrow M_k(N,\chi)$ by $J_{k,m}(N,\chi)^0$ and call it kernel space. Let us assume that $\chi(-1) = (-1)^k$ (otherwise $M_k(N,\chi) = \{0\}$). Then one has $h_{m,r} = h_{m,2m-r}$ for all $r = 1,\dots,m$.

**Definition 2.2.1** (vector valued modular form) Let $\rho : \Gamma_0(N) \longrightarrow GL(2,\mathbb{C})$ be any representation. A vector valued modular form on $\Gamma_0(N)$ of weight $k$ with character $\chi$ and representation $\rho$ is a column vector $(f_1,f_2)^t$ of holomorphic functions on the upper half-plane $\mathcal{H}$ satisfying the following conditions.

1. $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}(\gamma\tau) = \chi(d)(c\tau+d)^k\rho(\gamma)\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}(\tau)$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$,

2. $f_1$ and $f_2$ both are holomorphic at all the cusps of $\Gamma_0(N)$.

If further, $f_1$ and $f_2$ both vanishes at all the cusps, then $(f_1,f_2)^t$ is called a vector valued cusp form.

## 2.3 The space of Jacobi forms of index 2

Throughout this section we assume that $N = 2$, or an odd square-free positive integer. In this section, we study the kernel of the restriction map $\mathcal{D}_0$ for the space of index 2 Jacobi forms on $\Gamma_0(2N)$ in detail and deduce the injectivity of $\mathcal{D}_0 \oplus \mathcal{D}_2$ in the weight 2 case.

The action of $U_2$ on the generators of $\Gamma_0(2)$ is given as follows.

$$U_2(-I_2) = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad U_2(T) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{i} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \sqrt{i} \end{pmatrix}$$

and

$$U_2(ST^2S) = \frac{1}{2i}\begin{pmatrix} 1+i & 0 & 1-i & 0 \\ 0 & 1-i & 0 & 1+i \\ 1-i & 0 & 1+i & 0 \\ 0 & 1+i & 0 & 1-i \end{pmatrix}.$$

Therefore, for any $\gamma \in \Gamma_0(2)$, the matrix $U_2(\gamma)$ will always look like $(u_{ij})_{0\leqslant i,j<4}$ with $u_{ij} = 0$ unless $i + j$ is even. Using this observation with (2.1), One has $u_{11} + u_{13} = u_{31} + u_{33}$, since $\theta_{2,1} = \theta_{2,3}$ and $\theta_{21}$ has no zero in the upper half-plane as $\theta_{2,1}(\tau) = \frac{1}{2}\theta_{1,1}(\tau/2)$. This gives us a (projective) representation of $\Gamma_0(2)$ defined by

$$\rho_2(\gamma) := (u_{11} + u_{13})^{-1}\overline{U}_1(\gamma_2), \tag{2.11}$$

where $\gamma_2 = \begin{pmatrix} a & 2b \\ c/2 & d \end{pmatrix} \in SL_2(\mathbb{Z})$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2)$ (as in §2.2).

### 2.3.1 Connection to the space of vector valued modular forms

We start now from a Jacobi form $\phi \in J_{k,2}(2N, \chi)^0$, i.e., $\phi$ is in the kernel of the restriction map $\mathcal{D}_0$. This gives us,

$$0 = \phi(\tau, 0) = h_{2,0}(\tau)\theta_{2,0}(\tau) + 2h_{2,1}(\tau)\theta_{2,1}(\tau) + h_{2,2}(\tau)\theta_{2,2}(\tau). \tag{2.12}$$

Define two new functions as follows.

$$\varphi_0(\tau) := \frac{h_{2,0}}{\theta_{2,1}}(\tau), \qquad \varphi_2(\tau) := \frac{h_{2,2}}{\theta_{2,1}}(\tau). \tag{2.13}$$

Since $\theta_{2,1}$ has no zero in the upper half-plane, $\varphi_0$ and $\varphi_2$ define holomorphic functions on the upper half-plane.

**Proposition 2.3.1** *With $\varphi_0$ and $\varphi_2$ defined as above, $(\varphi_0, \varphi_2)^t$ is a vector valued modular form on $\Gamma_0(2N)$ of weight $(k-1)$ with character $\chi$ and representation $\rho_2$. (We denote the space of all such vector valued modular forms by $VM_{k-1}(2N, \chi; \rho_2)$.)*

**Proof**. We have to check the transformation property and holomorphy condition at the cusps. First we check the transformation property. For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2N)$, using (2.7) for $m = 2$, we have

$$(h_{2,0}, h_{2,2})(\gamma\tau) = \chi(d)(c\tau + d)^{k-\frac{1}{2}}(h_{2,0}, h_{2,2})(\tau)\overline{U_1}(\gamma_2)^t,$$

where $\gamma_2 = \begin{pmatrix} a & 2b \\ c/2 & d \end{pmatrix}$ (as in §2.2). For $(\varphi_0, \varphi_2)$, this means

$$\theta_{2,1}(\gamma\tau)(\varphi_0, \varphi_2)(\gamma\tau) = \chi(d)(c\tau + d)^{k-\frac{1}{2}}\theta_{2,1}(\tau)(\varphi_0, \varphi_2)(\tau)\overline{U}_1(\gamma_2)^t.$$

Using $\theta_{2,1}(\gamma\tau) = (u_{11} + u_{13})(c\tau + d)^{\frac{1}{2}}\theta_{2,1}(\tau)$ and $\theta_{2,1}(\tau) \neq 0$ for $\tau \in \mathcal{H}$, we obtain the following transformation property:

$$(\varphi_0, \varphi_2)(\gamma\tau) = \chi(d)(c\tau + d)^{k-1}(\varphi_0, \varphi_2)(\tau)\rho_2(\gamma)^t. \tag{2.14}$$

It remains to investigate the behaviour of $\varphi_0$ and $\varphi_2$ at each cusp of $\Gamma_0(2N)$. A complete set of cusps of $\Gamma_0(2N)$ is given by the numbers $\frac{a}{c}$ where $c$ runs over positive divisors of $2N$ and for a given $c$, $a$ runs through integers with $1 \leqslant a \leqslant 2N$, $\gcd(a, 2N) = 1$ that are inequivalent modulo $\gcd(c, \frac{2N}{c})$. Since $N$ is either 2 or an odd square-free positive integer, we can assume that all the cusps of $\Gamma_0(2N)$ are $\infty$, 0 and $\frac{1}{c}$ with $c|2N, c \neq 1, 2N$. Choose any cusp $s$ from the above list.

<u>Case 1:</u> Suppose $s = \infty$. If $\phi \in \mathrm{Ker}(\mathcal{D}_0)$ then we have the following.

$$0 = \phi(\tau, 0) = \sum_{\substack{n,r \in \mathbb{Z} \\ r^2 \leqslant 8n}} c_\phi(n, r)q^n = \sum_{n \geqslant 0} \left( \sum_{r \in \mathbb{Z}, r^2 \leqslant 8n} c_\phi(n, r) \right) q^n.$$

Therefore for every integer $n \geqslant 0$, $\sum_{r \in \mathbb{Z}, r^2 \leqslant 8n} c_\phi(n, r) = 0$. In particular taking $n = 0$ we get $c_\phi(0, 0) = 0$. Therefore, $h_{2,0}(\tau) = \sum_{n \geqslant 0} c_\phi(n, 0)q^n = q \sum_{n \geqslant 1} c_\phi(n, 0)q^{n-1}$.

Also we have $h_{2,2}(\tau) = \sum_{n\geqslant\frac{1}{2}} c_\phi(n,2)q^{n-\frac{1}{2}} = q^{\frac{1}{2}}\sum_{n\geqslant 1} c_\phi(n,2)q^{n-1}$ and $\theta_{2,1}(\tau) = \sum_{n\in\mathbb{Z}} q^{2(n+\frac{1}{4})^2} = q^{\frac{1}{8}}\sum_{n\in\mathbb{Z}} q^{(2n^2+n)}$. The $q$-expansion of $\theta_{2,1}$ starts with $q^{1/8}$ whereas the $q$-expansions of $h_{2,2}$ and $h_{2,0}$ start with $q^{1/2}$ and $q$ respectively. Therefore the $q$ expansions of the functions $\varphi_0 = \frac{h_{2,0}}{\theta_{2,1}}$ and $\varphi_2 = \frac{h_{2,2}}{\theta_{2,1}}$ contains only non-negative powers of $q$ and hence they are holomprphic at the cusp $\infty$.

Case 2: Suppose $s \neq \infty$. For $s = 0$ and $\frac{1}{c}$ choose the $SL_2(\mathbb{Z})$ matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as $S$ and $ST^{-c}S$ respectively, which takes $\infty$ to $s$. Using (2.12) and (2.13), we have

$$\begin{aligned}(c\tau+d)^{-k+\frac{1}{2}}(-2h_{2,1}(g\tau)) &= (c\tau+d)^{-k+\frac{1}{2}}(\varphi_0(g\tau),\varphi_2(g\tau))(\theta_{2,0}(g\tau),\theta_{2,2}(g\tau))^t \\ &= (c\tau+d)^{-k+1}(\varphi_0(g\tau),\varphi_2(g\tau)) \\ &\qquad \times (c\tau+d)^{-\frac{1}{2}}(\theta_{2,0}(g\tau),\theta_{2,2}(g\tau))^t.\end{aligned}$$

For the choice of $g$, $U_2(g)$ will have non-zero entries at the $(0,0)$-th place and the $(2,0)$-th place since either $g$ is $S$ or $2\|c$ or $c$ is odd. Using (2.1), we see that $(c\tau+d)^{-\frac{1}{2}}(\theta_{2,0}(g\tau),\theta_{2,2}(g\tau))^t$ tends to a column vector having each component non-zero as Im $\tau$ tends to $\infty$. This shows that $(c\tau+d)^{-k+1}(\varphi_0(g\tau),\varphi_2(g\tau))$ tends to a finite limit as Im $\tau$ goes to $\infty$, since $h_{2,1}$ is holomorphic at all the cusps. □

Conversely, let $(\varphi_0,\varphi_2)^t$ be a vector valued modular form in $VM_{k-1}(2N,\chi;\rho_2)$. Then, we obtain

$$\begin{aligned}\phi(\tau,z) = \varphi_0(\tau)\theta_{2,1}(\tau)\theta^J_{2,0}(\tau,z) &- \frac{1}{2}(\varphi_0\theta_{2,0}+\varphi_2\theta_{2,2})(\tau)(\theta^J_{2,1}+\theta^J_{2,3})(\tau,z) \\ &+ \varphi_2(\tau)\theta_{2,1}(\tau)\theta^J_{2,2}(\tau,z).\end{aligned} \tag{2.15}$$

Using the transformation properties of $(\varphi_0,\varphi_2)$ and $\theta$ functions for $\Gamma_0(2N)$, we see that $\phi \in J_{k,2}(2N,\chi)^0$.

Thus, we have obtained the following theorem.

**Theorem 2.3.2** *There is a linear isomorphism*

$$\Lambda_2 : J_{k,2}(2N,\chi)^0 \longrightarrow VM_{k-1}(2N,\chi;\rho_2),$$

*given by $\phi \mapsto (\varphi_0,\varphi_2)^t$, where $\varphi_0$ and $\varphi_2$ are defined by* (2.13).

### 2.3.2 Connection to the space of cusp forms

Let $\mathcal{D}_2$ be the differential operator as defined in (2.10) and $\phi \in \operatorname{Ker}(\mathcal{D}_0)$ be of the form given by (2.15). Then, using the differential equations

$$\frac{\partial^2}{\partial z^2}\theta^J_{m,r} = 4m(2\pi i)\frac{\partial}{\partial \tau}\theta^J_{m,r} \qquad \text{for } \; r \in \{0, 1, ....2m-1\}, \tag{2.16}$$

we obtain, $\quad \mathcal{D}_2(\phi) = 8k\left(\varphi_0(\theta_{2,1}\theta'_{2,0} - \theta_{2,0}\theta'_{2,1}) + \varphi_2(\theta_{2,1}\theta'_{2,2} - \theta_{2,2}\theta'_{2,1})\right).$

Define

$$\xi_0 := \theta_{2,1}\theta'_{2,0} - \theta_{2,0}\theta'_{2,1} \;\text{ and }\; \xi_2 := \theta_{2,1}\theta'_{2,2} - \theta_{2,2}\theta'_{2,1}. \tag{2.17}$$

Then, we have

$$\mathcal{D}_2(\phi) = 8k(\varphi_0, \varphi_2)(\xi_0, \xi_2)^t. \tag{2.18}$$

Proceeding as in [AB99, Proposition 2], for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2)$, we have

$$(\xi_0, \xi_2)(\gamma\tau) = (c\tau + d)^3(\xi_0, \xi_2)(\tau)(\rho_2(\gamma))^{-1}. \tag{2.19}$$

Analysing the behaviour of $\xi_0$ and $\xi_2$ at the cusps of $\Gamma_0(2)$, we find that the column vector $(\xi_0, \xi_2)^t$ is a vector valued cusp form. Define the space

$$S^2_{k+2}(2N, \chi)^0 := \{f \in S_{k+2}(2N, \chi) \colon f = \varphi_0\xi_0 + \varphi_2\xi_2 \text{ with } (\varphi_0, \varphi_2) \in VM_{k-1}(2N, \chi; \rho_2)\}.$$

We summarize the results of §2.3.1 and §2.3.2 in the following theorem.

**Theorem 2.3.3** *The map $\mathcal{D}_2 : J_{k,2}(2N, \chi) \longrightarrow S_{k+2}(2N, \chi)$ induces an isomorphism between $J_{k,2}(2N, \chi)^0$ and $S^2_{k+2}(2N, \chi)^0$. More precisely, we have the following commutative diagram of isomorphisms:*

$$\begin{array}{ccc} & J_{k,2}(2N,\chi)^0 & \\ {\scriptstyle \Lambda_2}\swarrow & & \searrow{\scriptstyle \mathcal{D}_2} \\ VM_{k-1}(2N,\chi;\rho_2) & \longleftrightarrow & S^2_{k+2}(2N,\chi)^0, \end{array}$$

*where the isomorphism in the bottom of the diagram is given by*

$$(\varphi_0, \varphi_2) \mapsto 8k(\xi_0\varphi_0 + \xi_2\varphi_2).$$

### 2.3.3 Injectivity of $\mathcal{D}_0 \oplus \mathcal{D}_2$

In this section, we shall prove the injectivity of the operator $\mathcal{D}_0 \oplus \mathcal{D}_2$ in the weight 2 case, that is, on the Jacobi form space $J_{2,2}(2N)$.
Let $\phi \in J_{k,2}(2N, \chi)$ such that $\mathcal{D}_0 \oplus \mathcal{D}_2(\phi) = 0$. Since $\phi \in \mathrm{Ker}(\mathcal{D}_0)$, so by (2.18) we get

$$\mathcal{D}_2(\phi) = 8k(\varphi_0\xi_0 + \varphi_2\xi_2),$$

where $\varphi_0$ and $\varphi_2$ are defined by (2.13). Using the assumption that $\phi \in \mathrm{Ker}(\mathcal{D}_2)$, we obtain $\mathcal{D}_2(\phi) = 8k(\varphi_0\xi_0 + \varphi_2\xi_2) = 0$. Define a new function,

$$\psi(\tau) = \frac{\varphi_0}{\xi_2}(\tau) = \frac{-\varphi_2}{\xi_0}(\tau). \tag{2.20}$$

If for any $\tau \in \mathcal{H}$, $\xi_0(\tau) = \xi_2(\tau) = 0$, then the definitions of $\xi_0$ and $\xi_2$ implies that $\theta_{2,1}\xi_2^\star(\tau) = 0$ ($\xi_2^\star$ is defined by (2.9)), which is not true. Therefore, the function $\psi$ is holomorphic in the upper half-plane.
Let $\gamma \in \Gamma_0(2N)$. Using (2.14) and (2.20), we have the following.

$$\psi(\gamma\tau)(\xi_2, -\xi_0)(\gamma\tau) = \chi(d)(c\tau + d)^{k-1}\psi(\tau)(\xi_2, -\xi_0)(\tau)\rho_2(\gamma)^t.$$

Using (2.19) and the non-vanishing of $\xi_0$ and $\xi_2$ at any point in the upper half-plane, we have the following transformation property.

$$\psi(\gamma\tau) = \chi(d)\frac{\overline{\omega}_2(\gamma)}{(u_{11} + u_{13})^2}(c\tau + d)^{k-4}\psi(\tau). \tag{2.21}$$

We prove the following theorem.

**Theorem 2.3.4** *For $N = 2$ or an odd square-free positive integer, the differential map $\mathcal{D}_0 \oplus \mathcal{D}_2$ is injective on $J_{2,2}(2N)$.*

**Proof.** Let $\phi \in \mathrm{Ker}(\mathcal{D}_0 \oplus \mathcal{D}_2)$. Let $\psi$ be the function defined by (2.20) associated to the given weight 2 Jacobi form $\phi$ in the kernel of $\mathcal{D}_0 \oplus \mathcal{D}_2$. Using (2.17), (2.20) and the holomorphicity of $\varphi_0, \varphi_2$ at the cusps of $\Gamma_0(2N)$, we get the holomorphicity of the function $\psi\xi$ at any cusp of $\Gamma_0(2N)$.
When $N = 2$, consider the function $\psi\xi\xi_2^\star$, which is a weight 4 cusp form on $\Gamma_0(4)$. Since $S_4(4) = \{0\}$, we get $\psi = 0$.
When $N$ is an odd square-free positive integer, consider the function $\psi\xi(\xi_2^\star)^3$, which is a weight 10 cusp form for $\Gamma_0(2N)$. Since $\xi_2^\star(\tau) = -\pi i\eta^6(2\tau)$, the cusp

form $\psi\xi(\xi_2^\star)^3$ is divisible by $\eta^{18}(2\tau)$. Since $(2, N) = 1$, the Atkin-Lehner $W$-operator $W(2)$ is an isomorphism on the space $S_{10}(2N)$ (see [AL78, Proposition 1.1]). Applying $W(2)$ on the function $\psi\xi(\xi_2^\star)^3$, it follows that this function is divisible by $\eta^{18}(\tau)$ (since $\eta^{18}(2\tau)|W(2)$ is a constant multiple of $\eta^{18}(\tau)$). In [AB03, Corollary 2.3], it was shown that there is no non-zero cusp form of square-free level $N$ and weight $k, k \equiv 10 \pmod{12}$, which is divisible by $\eta^{2k-2}(N\tau)$. After applying the Fricke involution $W(N)$ this corollary says that there is no non-zero cusp form of square-free level $N$ and weight $k, k \equiv 10 \pmod{12}$, which is divisible by $\eta^{2k-2}(\tau)$. Using this result for $k = 10$ we have $\psi\xi(\xi_2^\star)^3 = 0$, since $2N$ is square-free. Therefore, $\psi = 0$. Using (2.20), we get that $\varphi_0 = \varphi_2 = 0$ and hence $\phi = 0$. □

## 2.4 A certain subspace of the space of Jacobi forms of square-free index

Throughout this section $m$ is a square-free positive integer and $N$ is any positive integer. Consider the following subspace of the space of Jacobi forms of index $m$ on $\Gamma_0(mN)$:

$$J^\star_{k,m}(mN, \chi) := \{\phi \in J_{k,m}(mN, \chi) \ : \ h_{m,r} = 0 \ \text{ for all } \ r \neq 0, m\}. \qquad (2.22)$$

For index 1, the full space of Jacobi forms $J_{k,1}(N, \chi)$ is the same as $J^\star_{k,1}(N, \chi)$. In the case of index 2, we will relate this subspace with the kernel space $J_{k,2}(2N, \chi)^0$ (as studied in §2.3) in §2.5. Denote the restriction of the space $\mathrm{Ker}(\mathcal{D}_0)$ on this subspace by $J^\star_{k,m}(mN, \chi)^0$. In this section we study this kernel space. The purpose of considering this restriction is to relate this kernel with index 1 kernel $J_{k,1}(N, \chi)^0$, which was studied in detail by Arakawa and Böcherer [AB99, AB03].

### 2.4.1 Connection to the space of modular forms

Suppose $\phi \in J^\star_{k,m}(mN, \chi)^0$. Then, we have

$$0 = \phi(\tau, 0) = h_{m,0}(\tau)\theta_{m,0}(\tau) + h_{m,m}(\tau)\theta_{m,m}(\tau).$$

Define a new function by

$$\varphi := \frac{h_{m,0}}{\theta_{m,m}} = \frac{-h_{m,m}}{\theta_{m,0}}. \tag{2.23}$$

Since $\theta_{m,0}$ and $\theta_{m,m}$ have no zero in the upper half-plane, $\varphi$ defines a holomorphic function on the upper half-plane $\mathcal{H}$.

For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(mN)$, let $\gamma_m = \begin{pmatrix} a & mb \\ c/m & d \end{pmatrix}$ (as in §2.2). Taking transpose of the both sides in (2.7), we get the following transformation property.

$$(h_{m,0}(\gamma\tau), h_{m,m}(\gamma\tau)) = \chi(d)(c\tau+d)^{k-\frac{1}{2}}(h_{m,0}(\tau), h_{m,m}(\tau))\overline{U_1}(\gamma_m)^t.$$

Proceeding as in the proof of [AB99, Proposition 1] and using (2.4), the above equation gives the following transformation property:

$$\varphi(\gamma\tau) = \chi(d)\overline{\omega}_m(\gamma)(c\tau+d)^{k-1}\varphi(\tau). \tag{2.24}$$

We now study the behaviour of $\varphi$ at the cusps of $\Gamma_0(mN)$. We can assume that all the cusps of $\Gamma_0(mN)$ are of the form $\frac{a}{c}$, $\gcd(a,c) = 1$ and $c$ varies over positive divisors of $mN$. For such a cusp $s = \frac{a}{c}$, let $\alpha$ be the gcd of $m$ and $c$. Since $m$ is square-free, we have $\gcd(a\frac{m}{\alpha}, c) = 1$. Consider the $SL_2(\mathbb{Z})$ element $g = \begin{pmatrix} a & b \\ c & \frac{m}{\alpha}d' \end{pmatrix}$, where $b$ and $d'$ are integers such that $a\frac{m}{\alpha}d' - bc = 1$. One has $g\infty = \frac{a}{c}$. Let us denote $\frac{m}{\alpha}d'$ by $d$. Now, we have

$$\begin{aligned}(c\tau+d)^{-k+\frac{1}{2}}(h_{m,0}(g\tau), h_{m,m}(g\tau)) &= (c\tau+d)^{-k+\frac{1}{2}}\varphi(g\tau)(\theta_{m,m}(g\tau), -\theta_{m,0}(g\tau)) \\ &= (c\tau+d)^{-k+\frac{1}{2}}\varphi(g\tau) \\ &\qquad \times(\theta_{\alpha,0}(\frac{m}{\alpha}g\tau), \theta_{\alpha,\alpha}(\frac{m}{\alpha}g\tau))\ S.\end{aligned}$$

Since $\begin{pmatrix} \frac{m}{\alpha} & 0 \\ 0 & 1 \end{pmatrix} g\tau = g'\tau'$, where $g' = \begin{pmatrix} a\frac{m}{\alpha} & b \\ c & d' \end{pmatrix} \in \Gamma_0(\alpha)$ and $\tau' = \frac{\alpha}{m}\tau$. Using (2.4), we get

$$\begin{aligned}(c\tau+d)^{-k+\frac{1}{2}}(h_{m,0}(g\tau), h_{m,m}(g\tau)) &= (\alpha/m)^{1/2}(c\tau+d)^{-k+1}\varphi(g\tau) \\ &\quad \times (\theta_{\alpha,0}(\frac{\alpha}{m}\tau), \theta_{\alpha,\alpha}(\frac{\alpha}{m}\tau))U_1(g'_\alpha)^t\ S,\end{aligned}$$

where $g'_\alpha = \begin{pmatrix} am/\alpha & b\alpha \\ c/\alpha & d' \end{pmatrix}$. The left hand side of the above equation remains bounded for Im $\tau \to \infty$, because $h_{m,0}$ and $h_{m,m}$ are modular forms. On the other hand, the term $(\theta_{\alpha,0}(\frac{\alpha}{m}\tau), \theta_{\alpha,\alpha}(\frac{\alpha}{m}\tau))U_1(g'_\alpha)^t\ S$ in the right hand side goes to a row vector, which has atleast one non-zero entry as Im $\tau \to \infty$. Therefore $(c\tau+d)^{-k+1}\varphi(g\tau)$ remains bounded for Im $\tau \to \infty$ and hence $\varphi$ is holomorphic at the cusp $s = \frac{a}{c}$. This shows that the function $\varphi$ is a modular form of weight $k-1$ for $\Gamma_0(mN)$ with character $\chi\overline{\omega}_m$ (we denote the space of all such modular forms by $M_{k-1}(mN, \chi\overline{\omega}_m)$).

Conversely, starting with a modular form $\varphi \in M_{k-1}(mN, \chi\overline{\omega}_m)$, we obtain a Jacobi form

$$\phi(\tau, z) = \varphi(\tau)(\theta_{m,m}(\tau)\theta^J_{m,0}(\tau, z) - \theta_{m,0}(\tau)\theta^J_{m,m}(\tau, z)), \tag{2.25}$$

which belongs to $J^\star_{k,m}(mN, \chi)^0$.
We summarize the results of this subsection in the following theorem.

**Theorem 2.4.1** *There is a linear isomorphism*

$$\Lambda^\star_m : J^\star_{k,m}(mN, \chi)^0 \longrightarrow M_{k-1}(mN, \chi\overline{\omega}_m)$$

*given by $\phi \mapsto \varphi$, where $\varphi$ is defined by (2.23).*

### 2.4.2 Connection to the space of cusp forms

Let $\mathcal{D}_2$ be the differential operator as defined in (2.10). Let $\phi \in J^\star_{k,2}(mN, \chi)^0$ be of the form given by (2.25). Applying $\mathcal{D}_2$ and using the differential equations (2.16), we have

$$\mathcal{D}_2(\phi)(\tau) = 4mk\varphi(\tau)(\theta_{m,m}\theta'_{m,0} - \theta_{m,0}\theta'_{m,m})(\tau) = 4m^2k\varphi(\tau)\xi^\star_m(\tau), \tag{2.26}$$

where $\xi^\star_m$ is defined by (2.9). Define the space

$$S^\star_{k+2}(mN, \chi)^0 := \{f \in S_{k+2}(mN, \chi)\ :\ f/\xi^\star_m \in M_{k-1}(mN, \chi\overline{\omega}_m)\}.$$

We summarize the results of §2.4.1 and §2.4.2 in the following theorem.

**Theorem 2.4.2** *The map $\mathcal{D}_2 : J_{k,m}(mN,\chi) \longrightarrow S_{k+2}(mN,\chi)$ induces an isomorphism between $J^{\star}_{k,m}(mN,\chi)^0$ and $S^{\star}_{k+2}(mN,\chi)^0$. Combining this result with Theorem 2.4.1, we get the following commutative diagram of isomorphisms:*

$$\begin{array}{ccc} & J^{\star}_{k,m}(mN,\chi)^0 & \\ {}^{\Lambda^{\star}_m}\swarrow & & \searrow^{\mathcal{D}_2} \\ M_{k-1}(mN,\chi\overline{\omega}_m) & \longleftrightarrow & S^{\star}_{k+2}(mN,\chi)^0, \end{array}$$

*where the isomorphism in the bottom is given by $\varphi \mapsto 4m^2k\xi^{\star}_m\varphi$.*

### 2.4.3 Connection to the kernel space for index 1 Jacobi forms

We know that the operator $\mathcal{D}_0 \oplus \mathcal{D}_2$ is injective on $J_{k,1}(N)$. Also by [AB03, Theorem 4.3], $\mathcal{D}_0$ is injective on $J_{2,1}(N)$ for square-free $N$. Now we deduce similar kind of results for the space $J^{\star}_{k,m}(mN)$ in the following two corollaries.

**Corollary 2.4.3** *The differential map $\mathcal{D}_0 \oplus \mathcal{D}_2 : J^{\star}_{k,m}(mN) \longrightarrow M_k(mN) \oplus S_{k+2}(mN)$ is injective.*

**Proof.** This follows by using (2.26) and the fact that $\xi^{\star}_m$ has no zero in the upper half-plane. □

**Corollary 2.4.4** *The restriction map $\mathcal{D}_0 : J^{\star}_{2,m}(mN) \longrightarrow M_2(mN)$ is injective, when $mN$ is square-free, i.e., the kernel space $J^{\star}_{2,m}(mN)^0 = \{0\}$.*

**Proof.** By Theorem 2.4.2, the spaces $J^{\star}_{2,m}(mN)^0$ and $S^{\star}_4(mN)$ are isomorphic, where $S^{\star}_4(mN)$ is the subspace of $S_4(mN)$, whose functions are divisible by $\xi^{\star}_m$, in other words, divisible by $\eta^6(m\tau)$. Similar to the proof of Theorem 2.3.4, here we use the Atkin-Lehner $W$-operator $W(m)$ to deduce the corollary. Since $\eta^6(m\tau)|W(m)$ is a constant multiple of $\eta^6(\tau)$, applying the operator $W(m)$ the subspace $S^{\star}_4(mN)$ is isomorphic to the subspace of $S_4(mN)$ whose functions are divisible by $\eta^6(\tau)$. In [AB03, Corollary 2.3, Eq.(2.4)], it was shown that the space $\{f \in S_4(N) | f \text{ divisible by } \eta^6(\tau)\}$ is trivial for square-free $N$. Therefore, $S^{\star}_4(mN) = \{0\}$, since $mN$ is square-free. □

Let $J_{k,m}(mN,\nu)$ denote the space of Jacobi forms of index $m$ on $\Gamma_0(mN)$ with character $\nu$. Using the isomorphism diagram of Theorem 2.4.2, we get the following corollary.

**Corollary 2.4.5** *The two kernel spaces* $J_{k,1}(mN,\chi)^0$ *and* $J^{\star}_{k,m}(mN,\chi\omega_m\overline{\omega})^0$ *are isomorphic.*

Using [AB03, Theorem 4.3] with the above corollary, we have

**Corollary 2.4.6** *Let* $N$ *be a square-free positive integer coprime to* $m$*. Then* $J^{\star}_{2,m}(mN,\omega_m\overline{\omega})^0 = \{0\}$.

## 2.5 Concluding remarks

**Remark 2.5.1** Note that Theorem 2.4.2 reduces to [AB99, Theorem 2] and Corollary 2.4.4 reduces to [AB03, Theorem 4.3] in the case of index 1. Moreover, Corollary 2.4.5 shows that there may exist isomorphic subspaces in the space of Jacobi forms of different index.

**Remark 2.5.2** The space $J^{\star}_{k,m}(mN,\chi)$ as defined in (2.22) can be quite large for some index. For example, if $k$ is even and $m$ is square-free, we verify that $\dim J^{\star}_{k,m}(mN) \geqslant \dim J_{k,m}(= J_{k,m}(1))$. Take any Jacobi form $\phi = \sum_{j=0}^{2m-1} h_{m,j}\theta^J_{m,j}$ of even weight $k$ and square-free index $m$ for the full Jacobi group $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$. Using (2.4) and (2.7), the function defined by $\psi_{0,m} := h_{m,0}\theta^J_{m,0} + h_{m,m}\theta^J_{m,m}$ is a Jacobi form of weight $k$ and index $m$ for the Jacobi group $\Gamma_0(m) \ltimes \mathbb{Z}^2$. Define a mapping $J_{k,m} \longrightarrow J^{\star}_{k,m}(m)$ by $\phi(\tau,z) \longmapsto \psi_{0,m}(\tau,z)$. In [S05, Theorem 1], it was proved that any Jacobi form of square-free index $m$ and even weight $k$ for the full Jacobi group is entirely determined by any one of the associated $h_{m,j}(\tau), j = 1,\dots,2m-1$. This implies the injection of the above map. Hence, for any positive integer $N$, $\dim J^{\star}_{k,m}(mN) \geqslant \dim J^{\star}_{k,m}(m) \geqslant \dim J_{k,m}$.

**Remark 2.5.3** Suppose $N$ is either 2 or an odd square-free positive integer. Using (2.7), the space $J^{\star}_{k,2}(2N,\chi)$ is isomorphic to the space of vector valued modular forms $VM_{k-\frac{1}{2}}(2N,\chi;U^{\star}_2)$, where $U^{\star}_2$ is a (projective) representation defined on $\Gamma_0(2)$ by $U^{\star}_2(\gamma) = \overline{U_1}(\gamma_2)$. Also, the space $VM_{k-\frac{1}{2}}(2N,\chi;U^{\star}_2)$ is isomorphic to the space $VM_{k-1}(2N,\chi;\rho_2)$ by the isomorphism $(\varphi_0,\varphi_2) \longmapsto (\frac{\varphi_0}{\theta_{2,1}}, \frac{\varphi_2}{\theta_{2,1}})$. Combining this observation with Theorem 2.3.2, the full kernel space $J_{k,2}(2N,\chi)^0$ and the space $J^{\star}_{k,2}(2N,\chi)$ are isomorphic.

**Remark 2.5.4** Suppose $N$ is either 2 or an odd square-free positive integer. Consider the space $J^{\star}_{k,2}(2N,\chi)^0$ as a subspace of the full kernel space $J_{k,2}(2N,\chi)^0$. In Theorem 2.3.2, any $\phi \in J^{\star}_{k,2}(2N,\chi)^0$ will correspond to the vector valued modular form $(\varphi_0,\varphi_2)^t$ such that $\varphi_0\theta_{2,0}+\varphi_2\theta_{2,2}=0$. Also, the image of the space $J^{\star}_{k,2}(2N,\chi)^0$ under $\Lambda_2$, is isomorphic, to the space $M_{k-1}(2N,\chi\overline{\omega}_2)$ and the isomorphism is given by $(\varphi_0,\varphi_2)\longmapsto \frac{\varphi_0\theta_{2,1}}{\theta_{2,2}}(=\frac{-\varphi_2\theta_{2,1}}{\theta_{2,0}})$.

# Chapter 3

# Correspondence between Jacobi cusp forms and elliptic cusp forms

## 3.1 Introduction

In [SZ88], N.-P. Skoruppa and D. Zagier constructed certain lifting maps (Shimura type correspondence) between the space of holomorphic Jacobi cusp forms of integral index (for the full Jacobi group) and a subspace of cusp forms of integral weight, commuting with the action of Hecke operators. Further, in [GKZ], B. Gross, W. Kohnen and D. Zagier constructed kernel functions for these lifting maps. By using these correspondences, they obtained deep formulas relating the height pairings of Heegner points to the Fourier coefficients of holomorphic Jacobi forms. This correspondence for higher levels (for subgroups of the full Jacobi group) was generalized by B. Ramakrishnan in his thesis [BR89]. In [MR], M. Manickam and B. Ramakrishnan obtained the same results by showing that the image of the holomorphic Jacobi Poincaré series under the Shimura map can be expressed in terms of the holomorphic kernel functions for the periods of cusp forms. Following the method of Gross-Kohnen-Zagier, in [KB04, KB06], Bringmann generalized these lifting maps to holomorphic Jacobi cusp forms of higher degree (in particular to holomorphic Jacobi cusp forms on $\mathcal{H} \times \mathbb{C}^g$ with matrix index of size $g$, $g \equiv 1 \pmod 8$) on the full Jacobi group.

In [Sk88], Skoruppa introduced the concept of skew-holomorphic Jacobi forms.

Further, in [Sk90, Propositions 1 and 2], he obtained a correspondence similar to the holomorphic case between the space of skew-holomorphic Jacobi cusp forms (for the full Jacobi group) and elliptic cusp forms of integral weight. In this correspondence, the image of the space of skew-holomorphic Jacobi cusp forms lies in the orthogonal complement of the image of the space of holomorphic Jacobi cusp forms. In his thesis [M89], Manickam also obtained the same results as mentioned above for the skew-holomorphic case. As an application, he obtained estimates for the Fourier coefficients of the Jacobi cusp forms (both holomorphic and skew-holomorphic) by using the explicit Waldspurger theorem and the estimates for the special values of the $L$-functions of the cusp form of integral weight derived by H. Iwaniec [Iw87].

In this chapter, we generalize the results of Bringmann, for the congruence subgroups of the full Jacobi group by adopting the techniques of [MR, §3]. We also give a generalization of the correspondence obtained by Skoruppa and Manickam to skew-holomorphic Jacobi cusp forms of higher degree and for congruence subgroups. In fact, our theorem (Theorem 3.3.4) combines both the holomorphic and skew-holomorphic cases. As an application of our results, we obtain a correspondence between holomorphic (resp. skew-holomorphic) Jacobi cusp forms of matrix index and holomorhic (resp. skew-holomorphic) Jacobi cusp forms of integer index, and the mappings are adjoint with respect to the Petersson scalar products (Theorem 3.3.7). The results of this chapter are contained in [RS].

## 3.2 Preliminaries

Let $k, g, N \in \mathbb{N}$, where $g \equiv 1 \pmod 8$. Let $M$ be a positive definite, symmetric and half-integral $g \times g$ matrix (the last two conditions and $g$ is odd imply that $\frac{1}{2}\det(2M)$ is an integer). Let $(2M)^*$ denote the adjoint of the matrix $2M$. Let $\chi$ be a primitive Dirichlet character modulo $N_1$ with $N_1|N$ and $(N_1, \det(2M)) = 1$.

In this section, we recall some basic facts about integral binary quadratic forms, generalized genus character and construction of certain elliptic cusp forms. Let $l \in \mathbb{N}$ and $\Delta \in \mathbb{Z}(\geqslant 0)$ be a discriminant. For an integer $\rho \pmod{2l}$ with $\Delta \equiv \rho^2 \pmod{4l}$, we define the following sets of integral binary quadratic forms.

$$\mathcal{Q}_{l,\Delta,\rho} := \{Q(x,y) = ax^2 + bxy + cy^2 | a,b,c \in \mathbb{Z}, b^2 - 4ac = \Delta, a \equiv 0 \pmod{l} \text{ and } b \equiv \rho \pmod{2l}\}$$

and $$\mathcal{Q}_{l,N,\chi,\Delta,\rho} := \{Q(x,y) \in \mathcal{Q}_{l,\Delta,\rho} | \Delta \equiv 0 \pmod{N_1^2} \text{ and } a \equiv 0 \pmod{lN_1N}\}.$$

Any matrix $\gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in SL_2(\mathbb{Z})$ acts on $Q(x,y)$ in the usual sence by $Q \circ \gamma'(x,y) = Q(a'x + b'y, c'x + d'y)$. For $a,b,c \in \mathbb{Z}$, sometimes we denote $Q(x,y) = ax^2 + bxy + cy^2$ by $[a,b,c]$ also. Let $D_0$ be a fundamental discriminant dividing $\Delta$, coprime to $N_1$ and suppose that both $D_0$ and $\Delta/D_0$ are squares modulo $4l$. Following [GKZ], we define the generalized genus character $\chi_{D_0} : \mathcal{Q}_{l,\Delta,\rho} \longrightarrow \{0, \pm 1\}$ by

$$\chi_{D_0}(Q) = \begin{cases} \left(\frac{D_0}{t}\right) & \text{if } \; Q = [al,b,c] \text{ with } (a,b,c,D_0) = 1, \\ 0 & \quad \text{otherwise}, \end{cases} \tag{3.1}$$

where $\left(\frac{D_0}{t}\right)$ denotes the Kronecker symbol. Here t is an integer coprime to $D_0$ represented by the form $[al_1, b, cl_2]$ for some decomposition $l = l_1 l_2$ with $l_1, l_2 > 0$. Then it is known that $\chi_{D_0}$ is well-defined and is $\Gamma_0(l)$ invariant on $\mathcal{Q}_{l,\Delta,\rho}$. For details, we refer to [GKZ, Sect. I]. Also $\chi_{D_0}$ is $\Gamma_0(lN)$ invariant on $\mathcal{Q}_{l,N,\chi,\Delta,\rho}$. For $k \geqslant 2$, define

$$f_{k,l,N,\chi,\Delta,\rho,D_0}(\tau) = \sum_{Q(=[a,b,c]) \in \mathcal{Q}_{l,N,\chi,\Delta,\rho}} \overline{\chi}(c) \chi_{D_0}(Q) Q(\tau,1)^{-k}. \tag{3.2}$$

Following [GKZ], in [BR89], Ramakrishnan obtained the following proposition.

**Proposition 3.2.1** *The function $f_{k,l,N,\chi,\Delta,\rho,D_0}$ defined by (3.2) is an element of $S_{2k}(lN, \chi^2)$. Further, its Fourier expansion is given by*

$$f_{k,l,N,\chi,\Delta,\rho,D_0}(\tau) = \sum_{m \geqslant 1} c^{\pm}_{k,l,N,\chi}(m, \Delta, \rho, D_0) \; e(m\tau), \tag{3.3}$$

*where*

$$c^{\pm}_{k,l,N,\chi}(m,\Delta,\rho,D_0) = c_{k,l,N,\chi}(m,\Delta,\rho,D_0) + \chi(-1)(-1)^k \mathrm{sign}(D_0) c_{k,l,N,\chi}(m,\Delta,-\rho,D_0),$$

$$\begin{aligned} c_{k,l,N,\chi}(m,\Delta,\rho,D_0) \;=\; & i^{-k} \mathrm{sign}(D_0)^{1/2} \frac{(2\pi)^k}{(k-1)!} \left(m^2/\Delta\right)^{\frac{k-1}{2}} \Big[ (N_1|D_0|)^{-\frac{1}{2}} R_2(\overline{\chi}, D_0) \\ & \times \mathcal{E}_{l,\chi}(m,\Delta,\rho,D_0) + i^{-k} \; \mathrm{sign}(D_0)^{-1/2} \pi \sqrt{2} \left(m^2/\Delta\right)^{\frac{1}{4}} \\ & \times \sum_{a \geqslant 1, N_1 N | a} (la)^{-\frac{1}{2}} S_{la,\overline{\chi}}(m,\Delta,\rho,D_0) J_{k-\frac{1}{2}} \left( \frac{\pi m \sqrt{\Delta}}{la} \right) \Big]. \end{aligned} \tag{3.4}$$

Here $R_2(\overline{\chi}, D_0) = \operatorname{sign}(D_0)^{-1/2} \frac{1}{\sqrt{N_1|D_0|}} \sum_{t(N_1|D_0|)} \overline{\chi}(t) \left(\frac{D_0}{t}\right) e_{N_1|D_0|}(t),$

$$\mathcal{E}_{l,\chi}(m, \Delta, \rho, D_0) = \begin{cases} \chi(\frac{m}{f}) \left(\frac{D_0}{m/f}\right) & \text{if } \Delta = N_1^2 D_0^2 f^2 (f > 0, f|m) \\ & \text{and } N_1|D_0|f \equiv \rho \pmod{2l}, \\ 0 & \text{otherwise}, \end{cases}$$

$$S_{la,\overline{\chi}}(m, \Delta, \rho, D_0) = \sum_{\substack{b(2la) \\ b \equiv \rho(2l) \\ b^2 \equiv \Delta(4la)}} \overline{\chi}\left(\frac{b^2 - \Delta}{4la}\right) \chi_{D_0}\left(\left[al, b, \frac{b^2-\Delta}{4la}\right]\right) e_{2la}(mb) \tag{3.5}$$

*and $J_r(x)$ is the Bessel function of order $r$. (Notation: Here, we have written $b(a)$ in place of $b \pmod a$ for describing the range of summation. Now onwards, we will continue this.)*

For a cusp form $f \in S_{2k}(lN, \chi^2)$ and $Q = [a, b, c] \in \mathcal{Q}_{l,N,\chi,\Delta,\rho}$ , we define

$$r_{k,l,N,\chi,Q}(f) := \int_{\gamma_Q} f(\tau) Q(-\overline{\tau}, 1)^{k-1} d\tau, \tag{3.6}$$

where $\gamma_Q$ is the image in $\Gamma_0(lN)\backslash\mathcal{H}$ of the semicircle $a|\tau|^2 + b\mathrm{Re}\,\tau + c = 0$, oriented from $\frac{-b-\sqrt{\Delta}}{2a}$ to $\frac{-b+\sqrt{\Delta}}{2a}$ if $a \neq 0$ or if $a = 0$ of the vertical line $b\mathrm{Re}\,\tau + c = 0$, oriented from $-c/b$ to $i\infty$ if $b > 0$ and from $i\infty$ to $-c/b$ if $b < 0$. It is known that this integral depends only on the $\Gamma_0(lN)$ equivalence class of $Q$. Further, we define

$$r_{k,l,N,\chi,\Delta,\rho,D_0}(f) := \sum_{Q \in \mathcal{Q}_{l,N,\chi,\Delta,\rho}/\Gamma_0(lN)} \chi(c)\chi_{D_0}(Q) r_{k,l,N,\chi,Q}(f). \tag{3.7}$$

Then the following holds.

**Proposition 3.2.2** *For $f \in S_{2k}(lN, \chi^2)$, we have*

$$\frac{\pi\binom{2k-2}{k-1}}{i_{lN} 2^{2k-2} \Delta^{k-\frac{1}{2}}} r_{k,l,N,\chi,\Delta,\rho,D_0}(f) = \langle f, f_{k,l,N,\chi,\Delta,\rho,D_0} \rangle, \tag{3.8}$$

*where $i_{lN}$ denotes the index of $\Gamma_0(lN)$ in $SL_2(\mathbb{Z})$ and $\langle \cdot, \cdot \rangle$ denotes the usual Petersson scalar product for the space of cusp forms of integral weight on $\Gamma_0(lN)$.*

For more details of the above discussion, we refer to [St75, KZ84, K85, GKZ, BR89, KB04].

## 3.3 Main Results

Throughout this section we assume that $k > \frac{g+3}{2}$.

Let $n_0 \in \mathbb{N}$, $r_0 \in \mathbb{Z}^{(1,g)}$ and $D_0 := -\det\begin{pmatrix} 2n_0 & r_0 \\ r_0^t & 2M \end{pmatrix}$ such that $D_0$ is a fundamental discriminant which is a square modulo $\frac{1}{2}\det(2M)$ and coprime to $N_1$. We further assume that if $p$ divides both $\det(2M)$ and $D_0$, $p^2$ must not divide $\det(2M)$ if $p \neq 2$, $p^3$ must not divide $\det(2M)$ if $p = 2$ and $D_0/4$ is odd, $p^4$ must not divide $\det(2M)$ if $p = 2$ and $D_0/4$ is even. Moreover, if $p \neq 2$, $\prod_{i, i\neq j} m_i$ is assumed to be a square modulo $p$, where the $m_i$ are chosen such that there exists $U \in GL_g(\mathbb{Z}/p\mathbb{Z})$ with $(2M)[U] \equiv \text{diag } [m_1, m_2, ...m_g] \pmod{p}, p|m_j$.

### 3.3.1 Lifting maps

Before we proceed to the statement of results, we first give the definition of the lifting maps between the space of holomorphic and skew-holomorphic Jacobi cusp forms of integral weight with matrix index and the space of cusp forms of integral weight.

**Definition 3.3.1** Let $\phi \in J^{\text{cusp}}_{k+\frac{g+1}{2},M}(N,\chi)(resp.\ \ J^{*,\text{cusp}}_{k+\frac{g+1}{2},M}(N,\chi))$ be a holomorphic (resp. skew-holomorphic) Jacobi cusp form of weight $k+\frac{g+1}{2}$, index $M$, level $N$, (Dirichlet) character $\chi$ and with Fourier expansion

$$\phi(\tau, z) = \sum_{\substack{m\in\mathbb{Z}, r\in\mathbb{Z}^{(1,g)} \\ 4m > M^{-1}[r^t]}} c_\phi(m,r) e(m\tau + rz)$$

$$\left( \text{resp.} \quad \sum_{\substack{m\in\mathbb{Z}, r\in\mathbb{Z}^{(1,g)} \\ 4m < M^{-1}[r^t]}} c_\phi(m,r) e(m\overline{\tau} + \frac{i}{2} M^{-1}[r^t] \text{ Im } \tau + rz) \right).$$

For $D_0 < 0$ (resp. $D_0 > 0$), we define the $(D_0, r_0)$-th Shimura map on $J^{\text{cusp}}_{k+\frac{g+1}{2},M}(N,\chi)$ (resp. $J^{*,\text{cusp}}_{k+\frac{g+1}{2},M}(N,\chi)$) as follows.

$$S_{D_0,r_0}(\phi)(w) := 2^{\frac{1-g}{2}} \sum_{m\geqslant 1} \left( \sum_{\substack{d|m \\ (d,\frac{N}{N_1})=1}} \chi(d) \left(\frac{D_0}{d}\right) d^{k-1} c_\phi\left(\frac{m^2}{d^2} n_0, \frac{m}{d} r_0\right) \right) e(mw) \ \ (w \in \mathcal{H}). \tag{3.9}$$

**Definition 3.3.2** Let $f \in S_{2k}(\frac{1}{2}\det(2M)N, \chi^2)$ be an elliptic cusp form. For $D_0 < 0$, we define the Shintani map $S^*_{D_0,r_0}$ as follows.

$$S^*_{D_0,r_0}(f)(\tau,z) := \alpha_{M,D_0,N,\chi} \sum_{\substack{m\in\mathbb{Z}, r\in\mathbb{Z}^{(1,g)} \\ 4m>M^{-1}[r^t]}} \sum_{t|N/N_1} \mu(t)\overline{\chi}(t)\left(\frac{D_0}{t}\right) t^{-k-1} \times r_{k,\frac{1}{2}\det(2M),Nt,\chi,D_0DN_1^2t^2,\rho,D_0}(f)e(m\tau+rz) \tag{3.10}$$

and for $D_0 > 0$, the Shintani map $S^*_{D_0,r_0}$ is defined as follows.

$$S^*_{D_0,r_0}(f)(\tau,z) := \alpha_{M,D_0,N,\chi} \sum_{\substack{m\in\mathbb{Z}, r\in\mathbb{Z}^{(1,g)} \\ 4m<M^{-1}[r^t]}} \sum_{t|N/N_1} \mu(t)\overline{\chi}(t)\left(\frac{D_0}{t}\right) t^{-k-1} \times r_{k,\frac{1}{2}\det(2M),Nt,\chi,D_0DN_1^2t^2,\rho,D_0}(f)e(m\overline{\tau}+\frac{i}{2}M^{-1}[r^t]\ \mathrm{Im}\ \tau+rz), \tag{3.11}$$

where $\alpha_{M,D_0,N,\chi} := \left(\frac{i}{\det(2M)}\right)^{k-1} N_1^{-k+\frac{1}{2}}[\Gamma^J_g(1):\Gamma^J_g(N)]i^{-1}_{(\frac{1}{2}\det(2M))N}\chi(-1)R_2(\chi,D_0)$ and recall that $i_l$ denotes the index of $\Gamma_0(l)$ in $SL_2(\mathbb{Z})$.

### 3.3.2 Statement of main results

In this subsection, we state the main results of this chapter.

**Theorem 3.3.3** *Let $P_{(n,r)}$ (resp. $P^*_{(n,r)}$) denote the $(n,r)$-th holomorphic (resp. skew-holomorphic) Poincaré series of weight $k+\frac{g+1}{2}$, index $M$, level $N$ and character $\chi$. For $D_0 < 0$, we have*

$$S_{D_0,r_0}(P_{(n,r)})(w) = \beta_{k,D_0,N_1,\chi}\sum_{t|\frac{N}{N_1}} \mu(t)\chi(t)\left(\frac{D_0}{t}\right)t^{k-1}f_{k,\frac{1}{2}\det(2M),\frac{N}{t},\chi,\Delta,\rho,D_0}(tw) \tag{3.12}$$

*and for $D_0 > 0$, we have*

$$S_{D_0,r_0}(P^*_{(n,r)})(w) = \beta_{k,D_0,N_1,\chi}\sum_{t|\frac{N}{N_1}} \mu(t)\chi(t)\left(\frac{D_0}{t}\right)t^{k-1}f_{k,\frac{1}{2}\det(2M),\frac{N}{t},\chi,\Delta,-\rho,D_0}(tw), \tag{3.13}$$

*where $\beta_{k,D_0,N_1,\chi} = \frac{\chi(-1)R_2(\chi,D_0)(N_1|D_0|)^{k-\frac{1}{2}}(k-1)!}{(\mathrm{sign}(D_0))^{1/2}2^{\frac{g-1}{2}}(-2\pi i)^k}$, $\Delta = D_0DN_1^2$ and $\rho = -(r_0(2M)^*r^t)N_1$.*

As a consequence of the above theorem, by using Proposition 3.2.2, (1.12) and (1.19), we get the following theorem.

**Theorem 3.3.4** *Let*

$$\phi \in \begin{cases} J^{\text{cusp}}_{k+\frac{g+1}{2},M}(N,\chi) & \text{if } D_0 < 0, \\ J^{*,\text{cusp}}_{k+\frac{g+1}{2},M}(N,\chi) & \text{if } D_0 > 0. \end{cases}$$

*Then, the function* $S_{D_0,r_0}(\phi)(w)$ *is an element of* $S_{2k}(\frac{1}{2}\det(2M)N,\chi^2)$. *If* $f$ *is a cusp form in* $S_{2k}(\frac{1}{2}\det(2M)N,\chi^2)$, *then*

$$S^*_{D_0,r_0}(f) \in \begin{cases} J^{\text{cusp}}_{k+\frac{g+1}{2},M}(N,\chi) & \text{if } D_0 < 0, \\ J^{*,\text{cusp}}_{k+\frac{g+1}{2},M}(N,\chi) & \text{if } D_0 > 0. \end{cases}$$

*Moreover, the maps* $S_{D_0,r_0}$ *and* $S^*_{D_0,r_0}$ *are adjoint maps with respect to the Petersson scalar products, i.e., for all* $f \in S_{2k}(\frac{1}{2}\det(2M)N,\chi^2)$ *and for all* $\phi$ *as above, we have*

$$\langle S_{D_0,r_0}(\phi), f\rangle = \langle \phi, S^*_{D_0,r_0}(f)\rangle. \tag{3.14}$$

**Remark 3.3.5** Put $\ell = \frac{1}{2}\det(2M)$ and assume that $(\ell, N) = 1$. Let $W_\ell = \begin{pmatrix} \ell\alpha & \beta \\ \ell N\gamma & \ell\delta \end{pmatrix}$, $\ell\alpha\delta - N\beta\gamma = 1, \alpha \equiv 1 \pmod N, \beta \equiv 1 \pmod \ell$ be the Atkin-Lehner $W$-operator on $S_{2k}(\ell N,\chi^2)$. Then $W_\ell$ preserves the space $S_{2k}(\ell N,\chi^2)$ and the operator $\chi^2(\ell)W_\ell^2$ acts as identity on $S_{2k}(\ell N,\chi^2)$. For more details, we refer to [AL78]. Define a subspace of $S_{2k}(\ell N,\chi^2)$ as follows.

$$S^{(\pm,\ell)}_{2k}(\ell N,\chi^2) = \{f \in S_{2k}(\ell N,\chi^2) \ : \ f|W_\ell = \pm(-1)^k\chi(-1)\overline{\chi}(\ell)f\}.$$

When $N = 1$, the subspace $S^{(\pm,\ell)}_{2k}(\ell N,\chi^2)$ coincides with the subspace $S^{\pm}_{2k}(\ell)$, which was considered in [SZ88, GKZ, Sk88, Sk90, M89]. It is easy to see that

$$f_{k,\ell,N,\chi,\Delta,\rho,D_0}|W_l = \overline{\chi}(\ell)f_{k,\ell,N,\chi,\Delta,-\rho,D_0}.$$

Since

$$f_{k,\ell,N,\chi,\Delta,-\rho,D_0} = \text{sign}(D_0)(-1)^k\chi(-1)f_{k,\ell,N,\chi,\Delta,\rho,D_0},$$

we see that $f_{k,\ell,N,\chi,\Delta,\rho,D_0} \in S^{(\text{sign}(D_0),\ell)}_{2k}(\ell N,\chi^2)$. Therefore, the correspondence given in Theorem 3.3.4 for $D_0 < 0$ (resp. $D_0 > 0$) is actually between the spaces

$J^{\text{cusp}}_{k+\frac{g+1}{2},M}(N,\chi)$ and $S^{(-,\ell)}_{2k}(\ell N,\chi^2)$ (resp. $J^{*,\text{cusp}}_{k+\frac{g+1}{2},M}(N,\chi)$ and $S^{(+,\ell)}_{2k}(\ell N,\chi^2)$).

**Remark 3.3.6** When $N=1$ and $D_0<0$, the conclusion of Remark 3.3.5 reduces to [KB06, Theorem 3]. When $N=1$, $g=1$, $D_0>0$, our results (as mentioned in Remark 3.3.5) reduces to [Sk90, Propositions 1 and 2] and [M89, Theorem 3.3.1].

### 3.3.3 Interlink among Jacobi forms

It is interesting to note that an application of Theorem 3.3.4 (for the general case $g>1$, $g\equiv 1 \pmod 8$ and $g=1$) gives rise to a correspondence between the space of holomorphic (resp. skew-holomorphic) Jacobi cusp forms of matrix index and the space of holomorphic (resp. skew-holomorphic) Jacobi cusp forms of integer index.

**Theorem 3.3.7** *There exists a correspondence between the spaces $J^{\text{cusp}}_{k+\frac{g+1}{2},M}(N,\chi)$ (resp. $J^{*,\text{cusp}}_{k+\frac{g+1}{2},M}(N,\chi)$) and $J^{\text{cusp}}_{k+1,\frac{1}{2}\det(2M)}(N,\chi)$ (resp. $J^{*,\text{cusp}}_{k+1,\frac{1}{2}\det(2M)}(N,\chi)$). Moreover, the mappings are adjoint to each other with respect to the Petersson scalar products.*

## 3.4 Proofs

First, let us state a well-known lemma on quadratic Gauss sum, which will be used later.

**Lemma 3.4.1** *Let $p$ be any odd prime integer, $v\in\mathbb{N}$ and $a$ be any integer coprime to $p$. Then*

$$\sum_{t \pmod{p^v}} e_{p^v}(at^2) = \left(\frac{-4}{p^v}\right)^{1/2}\left(\frac{a}{p^v}\right)p^{v/2}. \tag{3.15}$$

In this section, we prove Theorem 3.3.3. Before we prove the theorem we have to show that the right hand side of (3.12) and (3.13) in Theorem 3.3.3 make sence, i.e., $\Delta = D_0DN_1^2$ is a non negative discriminant, $D_0$ and $D$ both are squares modulo $2\det(2M)$ and $D_0D\equiv (r_0(2M)^*r^t)^2 \pmod{2\det(2M)}$. For a proof of the above claims, we refer to [KB04, Lemma 4.20] and [KB06]. We now prove two lemmas, which will be used in proving Theorem 3.3.3.

**Lemma 3.4.2** *For any $m \geqslant 1$, we have*

$$\sum_{\substack{d|m \\ (d,\frac{N}{N_1})=1}} \chi(d) \left(\frac{D_0}{d}\right) d^{k-1} \delta_M \left(n, r; \frac{m^2}{d^2} n_0, \frac{m}{d} r_0\right)$$
$$= (m^2 D_0/D)^{\frac{k-1}{2}} \sum_{\substack{t|m \\ t|\frac{N}{N_1}}} \mu(t)\chi(t) \left(\frac{D_0}{t}\right)$$
$$\times \mathcal{E}_{\frac{1}{2}\det(2M),\chi} \left(\frac{m}{t}, D_0 D N_1^2, \operatorname{sign}(D_0)(r_0(2M)^* r^t) N_1, D_0\right). \tag{3.16}$$

**Proof**. The right hand side of (3.16) is zero unless $D = D_0 f^2$ and $N_1|D_0|f \equiv \operatorname{sign}(D_0)(r_0(2M)^* r^t) N_1 \pmod{\det(2M)}$ for some $f \in \mathbb{N}$ dividing $\frac{m}{t}$, for some divisor $t$ of $(m, N/N_1)$. In this case the right hand side is equal to $\chi(\frac{m}{f}) \left(\frac{D_0}{m/f}\right) (m/f)^{k-1}$ if $(\frac{m}{f}, \frac{N}{N_1}) = 1$ and $0$ otherwise. The left hand side of (3.16) is zero unless $D = \frac{m^2}{d^2} D_0$ and $r \equiv \frac{m}{d} r_0 \pmod{\mathbb{Z}^{(1,g)} \cdot 2M}$, for some divisor $d$ of $m$, which is coprime to $N/N_1$. In this case the left hand side is equal to $\chi(d) \left(\frac{D_0}{d}\right) d^{k-1}$. Setting $f = \frac{m}{d}$, we get the lemma, if we show that the following two congruences are equivalent.

$$D_0 f \equiv r_0 (2M)^* r^t \pmod{\det(2M)} \tag{3.17}$$

and

$$r - r_0 f \equiv 0 \pmod{\mathbb{Z}^{(1,g)} \cdot 2M}. \tag{3.18}$$

The equivalence of above two congruences follows by using [KB04, Lemma 4.27] and the fact that $D_0 = r_0(2M)^* r_0^t - 2n_0 \det(2M)$. □

**Lemma 3.4.3** *Let $\ell = \frac{1}{2}\det(2M)$. Then, for any $m \geqslant 1$, we have*

$$\sum_{\substack{d|m \\ (d,\frac{N}{N_1})=1}} \chi(d) \left(\frac{D_0}{d}\right) d^{-\frac{1}{2}} \sum_{c \geqslant 1} H_{M,Nc,\chi} \left(n, r; \frac{m^2}{d^2} n_0, \frac{m}{d} r_0\right) J_{k-\frac{1}{2}} \left(\frac{\pi m \sqrt{D_0 D}}{\ell N c d}\right)$$
$$= \chi(\operatorname{sign}(D_0)) R_2(\chi, D_0) \sum_{\substack{t|m \\ t|\frac{N}{N_1}}} \mu(t)\chi(t) \left(\frac{D_0}{t}\right)$$
$$\times \sum_{a \geqslant 1} (N_1 N a)^{-\frac{1}{2}} S_{\ell N_1 \frac{N}{t} a, \overline{\chi}} \left(\frac{m}{t}, D_0 D N_1^2, -(r_0(2M)^* r^t) N_1, D_0\right) J_{k-\frac{1}{2}} \left(\frac{\pi m \sqrt{D_0 D}}{\ell N a}\right). \tag{3.19}$$

**Proof.** We put $cd = a$ in the left hand side of (3.19) and sum over $a$. Thus, for every $m, a \geqslant 1$ we have to show that

$$\sum_{\substack{d|(m,a) \\ (d,\frac{N}{N_1})=1}} \chi(d) \left(\frac{D_0}{d}\right) d^{-\frac{1}{2}} H_{M,N\frac{a}{d},\chi}\left(n, r; \frac{m^2}{d^2}n_0, \frac{m}{d}r_0\right)$$
$$= (N_1 Na)^{-\frac{1}{2}} \chi(\mathrm{sign}(D_0)) R_2(\chi, D_0)$$
$$\times \sum_{\substack{t|m \\ t|\frac{N}{N_1}}} \mu(t)\chi(t) \left(\frac{D_0}{t}\right) S_{\ell N_1 \frac{N}{t} a, \overline{\chi}}\left(\frac{m}{t}, D_0 D N_1^2, -(r_0(2M)^* r^t)N_1, D_0\right). \tag{3.20}$$

Applying the Möbius inversion to the following lemma, we get the above claim and hence Lemma 3.4.3. Therefore, it is sufficient to prove the foolowing lemma.

**Lemma 3.4.4** *Let $\Delta = D_0 D N_1^2$ and $\rho = -(r_0(2M)^* r^t)N_1$. For every integer $m, a \geqslant 1$ with $N_1|a$, we have*

$$S_{\frac{1}{2}\det(2M)N_1 a, \overline{\chi}}(m, \Delta, \rho, D_0) = N_1^{\frac{1}{2}} \chi(\mathrm{sign}(D_0)) \overline{R_2(\chi, D_0)} \sum_{d|(m,a)} \chi(d) \left(\frac{D_0}{d}\right) (a/d)^{\frac{1}{2}}$$
$$\times H_{M,\frac{a}{d},\chi}\left(n, r; \frac{m^2}{d^2}n_0, \frac{m}{d}r_0\right), \tag{3.21}$$

**Proof.** Insert the definitions of $S_{la,\chi}$ and $H_{M,c,\chi}$ from (3.5) and (1.11) in (3.21). The left hand side of (3.21) becomes

$$\sum_{\substack{b(\det(2M)N_1 a) \\ b \equiv \rho \equiv -(r_0(2M)^* r^t)N_1(\det(2M)) \\ b^2 \equiv \Delta \equiv N_1^2 D_0 D (2\det(2M)N_1 a)}} \overline{\chi}\left(\frac{b^2 - \Delta}{2\det(2M)N_1 a}\right) \chi_{D_0}\left(\left[\frac{1}{2}\det(2M)N_1 a, b, \frac{b^2-\Delta}{2\det(2M)N_1 a}\right]\right)$$
$$\times e_a\left(\frac{mb}{\det(2M)N_1}\right)$$
$$= \sum_{\substack{b(\det(2M)a) \\ b \equiv -r_0(2M)^* r^t(\det(2M)) \\ b^2 \equiv D_0 D\left(2\det(2M)\frac{a}{N_1}\right)}} \overline{\chi}\left(\frac{b^2 - D_0 D}{2\det(2M)\frac{a}{N_1}}\right) \chi_{D_0}\left(\left[\frac{1}{2}\det(2M)\frac{a}{N_1}, b, \frac{b^2 - D_0 D}{2\det(2M)\frac{a}{N_1}}\right]\right)$$
$$\times e_a\left(\frac{mb}{\det(2M)}\right).$$

Now multiplying both sides of (3.21) by $e_{2a}(m(r_0M^{-1}r^t))$, we see that it remains to show that

$$\sum_{\substack{b(\det(2M)a) \\ b\equiv -r_0(2M)^*r^t(\det(2M)) \\ b^2\equiv D_0D\left(2\det(2M)\frac{a}{N_1}\right)}} \overline{\chi}\left(\frac{b^2-D_0D}{2\det(2M)\frac{a}{N_1}}\right)\chi_{D_0}\left(\left[\frac{1}{2}\det(2M)\frac{a}{N_1}, b, \frac{b^2-D_0D}{2\det(2M)\frac{a}{N_1}}\right]\right) \times e_a\left(\frac{b+r_0(2M)^*r^t}{\det(2M)}m\right)$$
$$= N_1^{\frac{1}{2}}\chi(\mathrm{sign}(D_0))\overline{R_2(\chi,D_0)}\sum_{d|(m,a)}\chi(d)\left(\frac{D_0}{d}\right)(a/d)^{-\frac{g+1}{2}} \times \sum_{\substack{x(a/d)\\ y(a/d)*}}\overline{\chi}(y)e_{a/d}\left(\overline{y}(M[x]+rx+n)+\frac{m^2}{d^2}n_0y-\frac{m}{d}r_0x\right),$$

where $\overline{y}$ denotes the inverse of $y$ modulo $a/d$. Now, both the sides are periodic in $m$ with period $a$. Therefore, it is sufficient to show that their Fourier transforms are equal, i.e., we have to show that for every $h'\in\mathbb{Z}/a\mathbb{Z}$ we have

$$\frac{1}{a}\sum_{\substack{b(\det(2M)a) \\ b\equiv -r_0(2M)^*r^t(\det(2M)) \\ b^2\equiv D_0D\left(2\det(2M)\frac{a}{N_1}\right)}}\sum_{m(a)}\overline{\chi}\left(\frac{b^2-D_0D}{2\det(2M)\frac{a}{N_1}}\right)\chi_{D_0}\left(\left[\frac{1}{2}\det(2M)\frac{a}{N_1}, b, \frac{b^2-D_0D}{2\det(2M)\frac{a}{N_1}}\right]\right) \times e_a\left(\left(\frac{b+r_0(2M)^*r^t}{\det(2M)}-h'\right)m\right)$$
$$= \frac{N_1^{\frac{1}{2}}}{a}\chi(\mathrm{sign}(D_0))\overline{R_2(\chi,D_0)}\sum_{m(a)}\sum_{d|(m,a)}\chi(d)\left(\frac{D_0}{d}\right)(a/d)^{-\frac{g+1}{2}} \times \sum_{\substack{x(a/d)\\ y(a/d)*}}\overline{\chi}(y)e_{a/d}\left(\overline{y}(M[x]+rx+n)+\frac{m^2}{d^2}n_0y-\frac{m}{d}r_0x-h'\frac{m}{d}\right). \tag{3.22}$$

The left hand side of (3.22) is equal to

$$\frac{1}{a}\sum_{\substack{b(\det(2M)a) \\ b\equiv -r_0(2M)^*r^t(\det(2M)) \\ b^2\equiv D_0D\left(2\det(2M)\frac{a}{N_1}\right)}}\overline{\chi}\left(\frac{b^2-D_0D}{2\det(2M)\frac{a}{N_1}}\right)\chi_{D_0}\left(\left[\frac{1}{2}\det(2M)\frac{a}{N_1}, b, \frac{b^2-D_0D}{2\det(2M)\frac{a}{N_1}}\right]\right) \times\sum_{m(a)}e_a\left(\frac{b-h}{\det(2M)}m\right),$$

where $h = \det(2M)h' - r_0(2M)^* r^t$. The above is non-zero only when $b \equiv h \pmod{(\det 2M)a}$. Therefore the left hand side of (3.22) is equal to

$$\begin{cases} \overline{\chi}\left(\frac{h^2-D_0D}{2\det(2M)\frac{a}{N_1}}\right)\chi_{D_0}\left(\left[\frac{1}{2}\det(2M)\frac{a}{N_1}, h, \frac{h^2-D_0D}{2\det(2M)\frac{a}{N_1}}\right]\right) & \\ \qquad\qquad \text{if } h^2 \equiv D_0D \pmod{2\det(2M)\frac{a}{N_1}}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.23}$$

Now, replacing $m$ by $md$ and then $m$ by $-m\overline{y}$ in the right hand side of (3.22), we get

$$\frac{N_1^{\frac{1}{2}}}{a}\chi(\mathrm{sign}(D_0))\overline{R_2(\chi, D_0)}\sum_{d|a}\chi(d)\left(\frac{D_0}{d}\right)(a/d)^{-\frac{g+1}{2}} \times \sum_{\substack{m,x(a/d)\\ y(a/d)*}}\chi(y)e_{a/d}\left(y(M[x]+rx+n+m^2n_0+mr_0x+h'm)\right). \tag{3.24}$$

Thus, for completing the proof of Lemma 3.4.4, it is enough to prove the following lemma.

**Lemma 3.4.5** *Suppose that* $b \equiv -r_0(2M)^* r^t \pmod{\det(2M)}$ *and* $s = \frac{b}{\det(2M)} + r_0(2M)^{-1}r^t$. *Let*

$$F(x,m) := M[x] + r_0xm + n_0m^2 + rx + sm + n; \qquad x \in \mathbb{Z}^{(g,1)}, m \in \mathbb{Z}.$$

*Let*

$$F_{c,\chi}(M, n_0, r_0, n, r, s) := F_{c,\chi} := c^{-\frac{g+1}{2}} \sum_{\substack{m,x(c)\\ y(c)^*}} \chi(y)e_c\left(yF(x,m)\right).$$

*Then for any* $a \geqslant 1$ *with* $N_1|a$, *we have*

$$\frac{N_1^{\frac{1}{2}}}{a}\chi(\mathrm{sign}(D_0)\overline{R_2(\chi, D_0)}\sum_{d|a}\chi(d)\left(\frac{D_0}{d}\right)F_{a/d,\chi} = \begin{cases} \overline{\chi}\left(\frac{b^2-D_0D}{2\det(2M)\frac{a}{N_1}}\right)\chi_{D_0}\left(\left[\frac{1}{2}\det(2M)\frac{a}{N_1}, b, \frac{b^2-D_0D}{2\det(2M)\frac{a}{N_1}}\right]\right) & \text{if } \frac{a}{N_1}\Big|\frac{b^2-D_0D}{2\det(2M)}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.25}$$

**Proof**. Let $N_1' = \prod_{\substack{p^v \| a \\ p | N_1}} p^v$ and $a = N_1' a_1$. Note that $N_1 | a$ and therefore $N_1 | N_1'$. Since $\chi$ is a primitive character modulo $N_1$, the sum in (3.25) varies only over divisors of $a_1$. For any divisor $d$ of $a_1$, decompose $y(a/d)^*$ and $m, x(a/d)$ as follows.

$$y = N_1' y_1 + (a_1/d) y_2 \text{ with } y_1(a_1/d)^* \text{ and } y_2(N_1')^*;$$

$$m = m_1 + m_2 \text{ with } m_1(a_1/d), N_1' | m_1 \text{ and } m_2(N_1'), \frac{a_1}{d} | m_2;$$

$$x = x_1 + x_2 \text{ with } x_1(a_1/d), N_1' | x_1 \text{ and } x_2(N_1'), \frac{a_1}{d} | x_2.$$

Using the decomposition of $y, m$ and $x$, we have for any $d | a_1$,

$$F_{\frac{a}{d},\chi} = \chi\left(\frac{a_1}{d}\right) F_{N_1',\chi} \cdot F_{a_1/d}, \tag{3.26}$$

where

$$F_{N_1',\chi} = N_1'^{-\frac{g+1}{2}} \sum_{\substack{m_2, x_2 (N_1') \\ y_2 (N_1')^*}} \chi(y_2) e_{N_1'}(y_2(M[x_2] + r_0 x_2 m_2 + n_0 m_2^2 + r x_2 + s m_2 + n)) \tag{3.27}$$

and

$$F_{a_1/d} = \left(\frac{a_1}{d}\right)^{-\frac{g+1}{2}} \sum_{\substack{m_1, x_1 \left(\frac{a_1}{d}\right) \\ y_1 \left(\frac{a_1}{d}\right)^*}} e_{a_1/d}(y_1(M[x_1] + r_0 x_1 m_1 + n_0 m_1^2 + r x_1 + s m_1 + n)). \tag{3.28}$$

Therefore,

$$\frac{1}{a} \sum_{d | a_1} \chi(d) \left(\frac{D_0}{d}\right) F_{\frac{a}{d},\chi} = \chi(a_1) \left(\frac{1}{N_1'} F_{N_1',\chi}\right) \left(\frac{1}{a_1} \sum_{d | a_1} \left(\frac{D_0}{d}\right) F_{a_1/d}\right). \tag{3.29}$$

Using [KB06, Lemma 7], we have

$$\frac{1}{a_1} \sum_{d | a_1} \left(\frac{D_0}{d}\right) F_{a_1/d} = \begin{cases} \chi_{D_0}\left(\left[\frac{a_1}{2} \det(2M), b, \frac{b^2 - D_0 D}{2\det(2M) a_1}\right]\right) & \text{if } a_1 | \frac{b^2 - D_0 D}{2\det(2M)}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.30}$$

Now, to complete the proof of the lemma we evaluate $\overline{R_2(\chi, D_0)}F_{N_1',\chi}$. Write $N_1 = \prod_{p^\alpha \| N_1} p^\alpha$ and $\chi = \prod_{p^\alpha \| N_1} \chi_{p^\alpha}$, where $\chi_{p^\alpha}$ is primitive modulo $p^\alpha$. By definition

$$R_2(\chi, D_0) = (\mathrm{sign}(D_0))^{-1/2}(N_1|D_0|)^{-1/2} \sum_{t(N_1|D_0|)} \chi(t) \left(\frac{D_0}{t}\right) e_{N_1|D_0|}(t).$$

First writing $t = N_1 t_1 + |D_0| t_2$ with $t_1 \pmod{|D_0|}$ and $t_2 \pmod{N_1}$ and then using the decomposition $t_2 = \sum_{p^\alpha \| N_1} (N_1/p^\alpha) t_p$ with $t_p \pmod{p^\alpha}$, we get the following.

$$R_2(\chi, D_0) = \chi(|D_0|) \left(\frac{D_0}{N_1}\right) N_1^{-\frac{1}{2}} \left( \prod_{p^\alpha \| N_1} \chi_{p^\alpha}(N_1/p^\alpha) R_{\chi_{p^\alpha}} \right), \tag{3.31}$$

where $R_{\chi_{p^\alpha}} = \sum_{t(p^\alpha)} \chi_{p^\alpha}(t) e_{p^\alpha}(t)$. For $\lambda, y, x$ modulo $N_1'$, $(\lambda, N_1') = 1$, write

$$\begin{aligned} \lambda &= \sum_{p^v \| N_1'} (N_1'/p^v)\lambda_p \ ; \quad \lambda_p (p^v)^*, \\ x &= \sum_{p^v \| N_1'} x_p \ ; \quad (N_1'/p^v) | x_p \text{ and } x_p (p^v), \\ y &= \sum_{p^v \| N_1'} y_p \ ; \quad (N_1'/p^v) | y_p \text{ and } y_p (p^v). \end{aligned}$$

Using the above decomposition, we have

$$\begin{aligned} F_{N_1',\chi} &= N_1'^{-\frac{g+1}{2}} \sum_{\substack{y,x(N_1') \\ \lambda(N_1')^*}} \chi(\lambda) e_{N_1'}(\lambda(M[x] + r_0 xy + n_0 y^2 + rx + sy + n)) \\ &= \prod_{\substack{p^v \| N_1' \\ p^\alpha \| N_1}} \chi_{p^\alpha}(N_1'/p^v) F_{p^v, \chi_{p^\alpha}}, \end{aligned} \tag{3.32}$$

where

$$F_{p^v,\chi_{p^\alpha}} = p^{-v\frac{g+1}{2}} \sum_{\substack{y,x(p^v) \\ \lambda(p^v)^*}} \chi_{p^\alpha}(\lambda) e_{p^v}(\lambda(M[x] + r_0 xy + n_0 y^2 + rx + sy + n)). \tag{3.33}$$

We now evaluate $F_{p^v,\chi_{p^\alpha}}$ for $p|N_1$. Since by assumption $N_1$ is odd, $p$ is an odd prime. Following the argument as in [KB04, KB06], we may assume that the matrix $M$ is a diagonal matrix with entries $m_1, m_2, ....m_g$ satisfying $(m_i, p) = 1$ for $1 \leqslant i \leqslant g$. Let us abbreviate $r := (r_1, r_2, ....r_g)$ and $r_0 := (r'_1, r'_2, ....r'_g)$. Moreover, we have

$$\begin{aligned} D &= \frac{\det(2M)}{2}\left(\sum_{i=1}^{g}\frac{r_i^2}{m_i} - 4n\right), \\ D_0 &= \frac{\det(2M)}{2}\left(\sum_{i=1}^{g}\frac{r_i'^2}{m_i} - 4n_0\right), \\ b &= -\frac{\det(2M)}{2}\left(\sum_{i=1}^{g}\frac{r_i r_i'}{m_i} - 2s\right). \end{aligned} \tag{3.34}$$

To evaluate $F_{p^v,\chi_{p^\alpha}}$, let us first consider the sum over $x(p^v)$.

$$\begin{aligned} &\sum_{x(p^v)} e_{p^v}\left(\lambda[M[x] + r_0 xy + rx]\right) \\ &= \sum_{x(=(x_i)_{1\leqslant i\leqslant g})(p^v)} e_{p^v}\left(\lambda\left[\sum_{i=1}^{g} m_i x_i^2 + \left(\sum_{i=1}^{g} r_i' x_i\right) y + \sum_{i=1}^{g} r_i x_i\right]\right) \\ &= \prod_{i=1}^{g}\sum_{x_i(p^v)} e_{p^v}(\lambda[m_i x_i^2 + (r_i + r_i' y)x_i]) \\ &= \prod_{i=1}^{g}\sum_{x_i(p^v)} e_{p^v}(\lambda(4m_i)^{-1}(4m_i^2 x_i^2 + 4m_i(r_i + r_i' y)x_i)) \\ &= \prod_{i=1}^{g} e_{p^v}(-\lambda(4m_i)^{-1}(r_i + r_i' y)^2) \\ &\qquad\qquad \times \sum_{x_i(p^v)} e_{p^v}(\lambda(4m_i)^{-1}(2m_i x_i + (r_i + r_i' y))^2). \end{aligned}$$

Using (3.15) (Lemma 3.4.1), we have

$$\sum_{x(p^v)} e_{p^v}\left(\lambda[M[x] + r_0 xy + rx]\right) = \prod_{i=1}^{g}\left(\frac{-4}{p^v}\right)^{\frac{1}{2}}\left(\frac{(4m_i)\lambda}{p^v}\right) p^{\frac{v}{2}} e_{p^v}(-\lambda(4m_i)^{-1}(r_i + r_i' y)^2). \tag{3.35}$$

Next, we consider the sum over $y$:

$$\begin{aligned}
&\sum_{y(p^v)} e_{p^v}(\lambda(n_0y^2+sy))\left(\prod_{i=1}^{g} e_{p^v}(-\lambda(4m_i)^{-1}(r_i+r_i'y)^2)\right)\\
&=\sum_{y(p^v)} e_{p^v}\left(-\lambda\left[\left(\sum_{i=1}^{g}\frac{r_i'^2}{4m_i}-n_0\right)y^2+\left(\sum_{i=1}^{g}\frac{r_ir_i'}{2m_i}-s\right)y+\sum_{i=1}^{g}\frac{r_i^2}{4m_i}\right]\right)\\
&=e_{p^v}\left(-\lambda\sum_{i=1}^{g}\frac{r_i^2}{4m_i}\right)\sum_{y(p^v)} e_{p^v}\left(-\frac{\lambda}{4}\left(\frac{D_0}{\det(2M)/2}y^2-\frac{2b}{\det(2M)/2}y\right)\right)\\
&=e_{p^v}\left(-\lambda\sum_{i=1}^{g}\frac{r_i^2}{4m_i}\right)\sum_{y(p^v)} e_{p^v}\left(-\frac{1}{2}\det(2M)^{-1}D_0^{-1}\lambda\left((D_0y-b)^2-b^2\right)\right).
\end{aligned}$$

Again, using (3.15), the above sum over $y$ is equal to

$$\left(\frac{-4}{p^v}\right)^{\frac{1}{2}}\left(\frac{-2\det(2M)D_0\lambda}{p^v}\right)p^{\frac{v}{2}}e_{p^v}\left(\lambda\left(\frac{\det(2M)^{-1}D_0^{-1}b^2}{2}-\sum_{i=1}^{g}\frac{r_i^2}{4m_i}\right)\right). \tag{3.36}$$

Combining (3.35) and (3.36) with (3.33), for any $p|N_1$, we have

$$\begin{aligned}
F_{p^v,\chi_{p^\alpha}}&=\left(\frac{D_0}{p^v}\right)\sum_{\lambda(p^v)^*}\chi_{p^\alpha}(\lambda)e_{p^v}\left(\lambda(b^2-DD_0)\frac{1}{2}\det(2M)^{-1}D_0^{-1}\right)\\
&=\left(\frac{D_0}{p^v}\right)\sum_{\lambda(p^v)^*}\chi_{p^\alpha}(\lambda)e_{p^\alpha}\left(\frac{b^2-DD_0}{2\det(2M)p^{v-\alpha}}\lambda D_0^{-1}\right)\\
&=\left(\frac{D_0}{p^v}\right)\overline{\chi}_{p^\alpha}(D_0^{-1})\sum_{\lambda(p^v)^*}\chi_{p^\alpha}(\lambda)e_{p^\alpha}\left(\lambda\left(\frac{b^2-D_0D}{2\det(2M)p^{v-\alpha}}\right)\right).
\end{aligned} \tag{3.37}$$

Let us denote $\frac{b^2-DD_0}{2\det(2M)}$ by $C$. The sum in the last line of (3.37) is 0 unless $p^{v-\alpha}(2\det(2M))|(b^2-D_0D)$. So, let $p^{v-\alpha}(2\det(2M))|(b^2-D_0D)$, then we have

$$\begin{aligned}
F_{p^v,\chi_{p^\alpha}}&=\left(\frac{D_0}{p^v}\right)\overline{\chi}_{p^\alpha}(D_0^{-1})p^{v-\alpha}\sum_{\lambda(p^\alpha)^*}\chi_{p^\alpha}(\lambda)e_{p^\alpha}\left(\lambda\frac{C}{p^{v-\alpha}}\right)\\
&=p^{v-\alpha}\left(\frac{D_0}{p^v}\right)\overline{\chi}_{p^\alpha}\left(\frac{-C}{p^{v-\alpha}}D_0^{-1}\right)\sum_{\lambda(p^\alpha)^*}\chi_{p^\alpha}(\lambda)e_{p^\alpha}(-\lambda)\\
&=p^{v-\alpha}\left(\frac{D_0}{p^v}\right)\overline{\chi}_{p^\alpha}\left(\frac{C}{p^{v-\alpha}}D_0^{-1}\right)R_{\chi_{p^\alpha}}.
\end{aligned} \tag{3.38}$$

Combining (3.38) with (3.32) and (3.31), we get the following.

$$\overline{R_2(\chi, D_0)} F_{N_1',\chi} = \begin{cases} \chi(\mathrm{sign}(D_0)) N_1' N_1^{-\frac{1}{2}} \left(\frac{D_0}{N_1'/N_1}\right) \overline{\chi}\left(\frac{b^2-DD_0}{2\det(2M)N_1'/N_1}\right) & \text{if } \frac{N_1'}{N_1} \big| \frac{b^2-D_0D}{2\det(2M)}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.39}$$

Using (3.39) and (3.30) with (3.29) and $a = N_1' a_1$, we get the following.

$$\frac{N_1^{\frac{1}{2}}}{a} \chi(\mathrm{sign}(D_0) \overline{R_2(\chi, D_0)} \sum_{d|a} \chi(d) \left(\frac{D_0}{d}\right) F_{a/d,\chi}$$
$$= \begin{cases} \overline{\chi}\left(\frac{b^2-D_0D}{2\det(2M)\frac{a}{N_1}}\right) \left(\frac{D_0}{N_1'/N_1}\right) \chi_{D_0}\left(\left[\frac{a_1}{2}\det(2M), b, \frac{b^2-D_0D}{2\det(2M)a_1}\right]\right) & \text{if } \frac{a}{N_1} \big| \frac{b^2-D_0D}{2\det(2M)}, \\ 0 & \text{otherwise.} \end{cases}$$

Using the definition of $\chi_{D_0}$, we get $\left(\frac{D_0}{N_1'/N_1}\right) \chi_{D_0}\left(\left[\frac{a_1}{2}\det(2M), b, \frac{b^2-D_0D}{2\det(2M)a_1}\right]\right) = \chi_{D_0}\left(\left[\frac{1}{2}\det(2M)\frac{a}{N_1}, b, \frac{b^2-D_0D}{2\det(2M)\frac{a}{N_1}}\right]\right)$ and this proves the lemma. □

Now, we prove Theorem 3.3.3. First, let us assume that $D_0 < 0$ and prove (3.12). We plugin the Fourier coefficients of $f_{k,l,N,\chi,\Delta,\rho,D_0}$ and $P_{(n,r)}$ (using equations (3.3), (3.4), (3.5), (1.8), (1.9), (1.10), (1.11)) in (3.12) and then we compare the $m$-th Fourier coefficients on both the sides. So, the theorem follows if we prove that for any $m \geq 1$,

$$\sum_{\substack{d|m \\ (d,\frac{N}{N_1})=1}} \chi(d) \left(\frac{D_0}{d}\right) d^{k-1} \Bigg\{ \delta_M^{\pm}\left(n, r; \frac{m^2}{d^2}n_0, \frac{m}{d}r_0\right) + 2\pi(-i)^{k+1}(\det 2M)^{-\frac{1}{2}}$$
$$\times \left(\frac{m^2}{d^2}\cdot\frac{D_0}{D}\right)^{\frac{k}{2}-\frac{1}{4}} \sum_{c\geqslant 1, N|c} H^{\pm}_{M,c,\chi}\left(n, r; \frac{m^2}{d^2}n_0, \frac{m}{d}r_0\right) J_{k-\frac{1}{2}}\left(\frac{2\pi m\sqrt{D_0D}}{\det(2M)cd}\right)\Bigg\}$$
$$= (N_1|D_0|)^{k-\frac{1}{2}} \chi(-1) R_2(\chi, D_0) \sum_{\substack{t|m \\ t|\frac{N}{N_1}}} \mu(t)\chi(t)\left(\frac{D_0}{t}\right) t^{k-1} \left(m^2/t^2\Delta\right)^{\frac{k-1}{2}} \Bigg[(N_1|D_0|)^{-\frac{1}{2}}$$
$$\times R_2(\overline{\chi}, D_0) \mathcal{E}^{\pm}_{\frac{1}{2}\det(2M),\chi}\left(\frac{m}{t}, \Delta, \rho, D_0\right) + \sqrt{2}\pi i^{-k} \mathrm{sign}(D_0)^{-1/2} \left(m^2/t^2\Delta\right)^{\frac{1}{4}}$$
$$\times \sum_{\substack{a\geqslant 1 \\ N_1\frac{N}{t}|a}} (\frac{1}{2}\det(2M)a)^{-\frac{1}{2}} S^{\pm}_{\frac{1}{2}\det(2M)a,\overline{\chi}}\left(\frac{m}{t}, \Delta, \rho, D_0\right) J_{k-\frac{1}{2}}\left(\frac{\pi m\sqrt{\Delta}}{\frac{1}{2}\det(2M)at}\right)\Bigg]. \tag{3.40}$$

Comparing the "+" part of first terms on both the sides of (3.40), we need to prove that

$$\sum_{\substack{d|m \\ (d,\frac{N}{N_1})=1}} \chi(d) \left(\frac{D_0}{d}\right) d^{k-1} \delta_M \left(n, r; \frac{m^2}{d^2} n_0, \frac{m}{d} r_0\right)$$
$$= \left(\frac{m^2}{D} D_0\right)^{\frac{k-1}{2}} \sum_{\substack{t|m \\ t|\frac{N}{N_1}}} \mu(t)\chi(t) \left(\frac{D_0}{t}\right) \mathcal{E}_{\frac{1}{2}\det(2M),\chi} \left(\frac{m}{t}, \Delta, \rho, D_0\right). \tag{3.41}$$

Comparing the "+" part of second terms on both the sides of (3.40), we need to prove that

$$\sum_{\substack{d|m \\ (d,\frac{N}{N_1})=1}} \chi(d) \left(\frac{D_0}{d}\right) d^{-\frac{1}{2}} \sum_{c\geqslant 1} H_{M,Nc,\chi} \left(n, r; \frac{m^2}{d^2} n_0, \frac{m}{d} r_0\right) J_{k-\frac{1}{2}} \left(\frac{2\pi m\sqrt{D_0 D}}{\det(2M)Ncd}\right)$$
$$= \chi(-1) R_2(\chi, D_0) \sum_{\substack{t|m \\ t|\frac{N}{N_1}}} \mu(t)\chi(t) \left(\frac{D_0}{t}\right) \sum_{a\geqslant 1} (N_1 Na)^{-\frac{1}{2}}$$
$$\times S_{(\frac{1}{2}\det(2M))N_1 \frac{N}{t} a, \overline{\chi}} \left(\frac{m}{t}, \Delta, \rho, D_0\right) J_{k-\frac{1}{2}} \left(\frac{2\pi m\sqrt{D_0 D}}{\det(2M)Na}\right). \tag{3.42}$$

Since $\rho = -(r_0(2M)^* r^t)N_1$ and $D_0 < 0$, (3.41) and (3.42) follow from Lemma 3.4.2 and Lemma 3.4.3 respectively.

Now, let us assume that $D_0 > 0$ and prove (3.13). In this case, we follow the same lines as in the case of $D_0 < 0$ after replacing $\rho$ by $-\rho$. For the second term equivalence, we compare the "+" part of the left hand side to the "−" part of the right hand side. Hence, we get (3.13) of Theorem 3.3.3. □

# Chapter 4

# Non-vanishing of half-integral weight $L$-functions

## 4.1 Introduction

In [K97], W. Kohnen proved that, given a real number $t_0$ and a positive real number $\epsilon$, for all $k$ large enough the sum of the functions $L^*(f, s)$ with $f$ running over a basis of (properly normalized) Hecke eigenforms of weight $k$ does not vanish on the line segment $\text{Im } s = t_0, (k-1)/2 < \text{ Re } s < k/2 - \epsilon, k/2 + \epsilon < \text{Re } s < (k+1)/2$. As a consequence, he proved that for any such point $s$, for $k$ large enough there exists a Hecke eigenform of weight $k$ on $SL_2(\mathbb{Z})$, such that the corresponding $L$-function value at $s$ is non-zero. Using similar methods, in [Ra05], A. Raghuram generalised Kohnen's result for the average of $L$-functions over a basis of newforms (of integral weight) of level $N$ with primitive character modulo $N$.

In this chapter, we extend Kohnen's method to forms of half-integral weight. As a consequence, we show that for any given point $s$ inside the critical strip, there exists a Hecke eigen cusp form $f$ of half-integral weight $k + 1/2$ on $\Gamma_0(4N)$ with character $\psi$ such that the corresponding $L$-function value at $s$ is non-zero, and the first Fourier coefficient of $f$ is non-zero. It should be noted that the normalisation of Fourier coefficients of forms of half-integral weight is still an open question. Our results are obtained for $N$ sufficiently large if $k$ is fixed and vice versa. In particular when $N = 1$, for sufficiently large $k$, either $f$ is a newform in the full space or $f$ is a Hecke eigenform in the Kohnen plus space. The results of this chapter are contained in [RS12].

## 4.2 Preliminaries

Let $N \geq 1$, $k \geq 3$ be integers and $\psi$ be an even Dirichlet character modulo $4N$. Let $L(f,s)$ be the $L$-function associated to the cusp form $f \in S_{k+1/2}(4N,\psi)$ defined by $L(f,s) := \sum_{n\geq 1} a_f(n)n^{-s}$, where $a_f(n)$ denotes the $n$-th Fourier coefficient of $f$. Then by [MMR, Proposition 1], the completed $L$-function defined by

$$L^*(f,s) := (2\pi)^{-s}(\sqrt{4N})^s\Gamma(s)L(f,s) \tag{4.1}$$

has the following functional equation

$$L^*(f|H_{4N}, k+1/2-s) = L^*(f,s), \tag{4.2}$$

where $H_{4N}$ is the Fricke involution on $S_{k+1/2}(4N,\psi)$ defined by

$$f|H_{4N}(\tau) = i^{k+1/2}(4N)^{-k/2-1/4}\tau^{-k-1/2}f(-1/4N\tau).$$

A cusp form $f \in S_{k+1/2}(4N,\psi)$ is called Hecke eigenform if it is an eigen function of all the Hecke operators $T(p^2), p \not| 2N$. It is known that the space $S_{k+1/2}(4N,\psi)$ has an orthogonal basis (with respect to Petersson scalar product) of Hecke eigenforms. Let $\{f_1, f_2, \ldots, f_d\}$ be such an orthogonal basis of Hecke eigen forms, where $d$ is the dimension of the space $S_{k+1/2}(4N,\psi)$ (see for example [Sh73]). Let $K$ be the operator defined by $f|K(\tau) = \overline{f(-\overline{\tau})}$. Since $KH_{4N} = H_{4N}K$ on $S_{k+1/2}(4N,\psi)$, we have $f|(KH_{4N})^2 = f$. Also, the operators $K$ and $H_{4N}$ commute with the Hecke operators $T(p^2), p \nmid 2N$. Therefore, for the basis elements $f_j, 1 \leqslant j \leqslant d$, we can assume that $f_j|KH_{4N} = \lambda_{f_j} f_j$, where $\lambda_{f_j} = \pm 1$.

## 4.3 Main Results

We state the main results of this chapter.

**Theorem 4.3.1** *Let $N \geq 1$ be a fixed integer. Let $\{f_1, f_2, \ldots, f_d\}$ be an orthogonal basis of Hecke eigenforms as above. Let $r_0 \in \mathbb{R}$. Then there exists a constant $C = C(r_0)$ depending only on $r_0$ such that for $k > C$, the function*

$$\sum_{j=1}^{d} \frac{L^*(f_j,s)}{\langle f_j, f_j\rangle}\lambda_{f_j} a_{f_j}(1)$$

*doesn't vanish for any point $s = \sigma + ir_0$ with $k/2 - 1/4 < \sigma < k/2 + 3/4$.*

**Theorem 4.3.2** *Let $k \geq 3$ be a fixed integer. Let $\{f_1, f_2, \ldots, f_d\}$ be an orthogonal basis of Hecke eigenforms as above. Let $r_0 \in \mathbb{R}$. Then there exists a constant $C' = C'(r_0)$ depending only on $r_0$ such that for $N > C'$, the function*

$$\sum_{j=1}^{d} \frac{L^*(f_j, s)}{\langle f_j, f_j \rangle} \lambda_{f_j} a_{f_j}(1)$$

*doesn't vanish for any point $s = \sigma + ir_0$ with $k/2 - 1/4 < \sigma < k/2 + 3/4$.*

The following corollary is an easy consequence of the above two theorems.

**Corollary 4.3.3** *Let $s_0$ be a point inside the critical strip $k/2 - 1/4 < \mathrm{Re}\ s_0 < k/2 + 3/4$. If either $k$ or $N$ is suitably large, then there exists a Hecke eigenform $f$ belonging to $S_{k+1/2}(4N, \psi)$ such that $L(f, s_0) \neq 0$ and $a_f(1) \neq 0$.*

**Remark 4.3.4** Though we consider Hecke eigenforms of half-integral weight, the $L$-function corresponding to such a Hecke eigenform does not have an Euler product.

**Remark 4.3.5** Let $\psi$ be an even primitive Dirichlet character modulo $4N$. Then it is known from the work of Serre and Stark [SS76] that the space $S_{k+1/2}(4N, \psi)$ is the space of newforms. Hence, the orthogonal basis consists of newforms. In this case, the Hecke eigenform $f$ in Corollary 4.3.3 will be a newform of level $4N$.

**Remark 4.3.6** Let us consider the case $N = 1$. Let $\{f_1, f_2, \ldots, f_{d_1}\}$ be an orthogonal basis of $S^{new}_{k+1/2}(4)$ which are newforms (see [MRV]). Let $\{g_1, g_2, \ldots, g_{d_2}\}$ be an orthogonal basis of $S^{+}_{k+1/2}(4)$ (see [K80]), which are Hecke eigenforms such that the set $\{g_1 \pm g_1|W(4), \ldots, g_{d_2} \pm g_{d_2}|W(4)\}$ forms an orthogonal basis of $S^{old}_{k+1/2}(4)$. Here, $d_1 + 2d_2 = d$ is the dimension of the space $S_{k+1/2}(4)$ and $W(4)$ is the Atkin-Lehner $W$-operator for the prime $p = 2$ on $S_{k+1/2}(4)$. Thus, an orthogonal basis of Hecke eigenforms for the space $S_{k+1/2}(4)$ is given as follows:

$$\{f_1, f_2, \ldots, f_{d_1}, g_1 \pm g_1|W(4), g_2 \pm g_2|W(4), \ldots, g_{d_2} \pm g_{d_2}|W(4)\}.$$

In this case, we get the following result as a consequence of Theorem 4.3.1. Let $s_0$ be a point inside the critical strip $k/2 - 1/4 < \mathrm{Re}\ s_0 < k/2 + 3/4$. If $k$ is suitably large, then there exists a $j$, with $1 \leq j \leq d_1$ or $1 \leq j \leq d_2$ such that

$$L(f_j, s_0) \neq 0, a_{f_j}(1) \neq 0 \quad \text{or} \quad L(g_j \pm g_j|W(4), s_0) \neq 0, a_{g_j}(1) \pm 2^{-k} a_{g_j}(4) \neq 0. \tag{4.3}$$

For the last assertion in the above equation (4.3), we use the fact that $W(4) = 2^{-k}U(4)$ on $S^+_{k+1/2}(4)$, where $U(4)$ is the Hecke operator for $p = 2$ on $S_{k+1/2}(4)$. Note that $W(4) = H_4$ on $S_{k+1/2}(4)$ and therefore, in the second case of (4.3), for a $j$ with $1 \leq j \leq d_2$, it follows from the functional equation that either $L(g_j, s_0) \neq 0$ or $L(g_j, k + 1/2 - s_0) \neq 0$. Hence, for any given point $s$ inside the critical strip, our theorem gives (for sufficiently large $k$) the existence of a newform $f$ in $S^{new}_{k+1/2}(4)$ such that $L(f, s) \neq 0$ or a Hecke eigenform $g$ in the plus space $S^+_{k+1/2}(4)$ such that $L(g, s) \neq 0$ or $L(g, k + 1/2 - s) \neq 0$. Correspondingly, we also get the non-vanishing of the first Fourier coefficient (if it is a newform in $S^{new}_{k+1/2}(4)$) or the first or the 4-th Fourier coefficient (if it is a Hecke eigenform in $S^+_{k+1/2}(4)$).

## 4.4 Proofs

The proof is on the same lines as that of Kohnen [K97].

First, let us define the kernel function for special values of the $L$-function associated to a cusp form of half-integral weight. A similar function for forms of integral weight was considered by Kohnen [K97]. Let $\tau \in \mathcal{H}$ and $s \in \mathbb{C}$ with $1 < \sigma < k - 1/2$, $\sigma = \text{Re } s$. Define

$$R_{s;k,N,\psi}(\tau) = \gamma_k(s) \sum \overline{\psi}(d) \left(\frac{c}{d}\right) \left(\frac{-4}{d}\right)^{k+1/2} (c\tau + d)^{-(k+1/2)} \left(\frac{a\tau + b}{c\tau + d}\right)^{-s}, \tag{4.4}$$

where

$$\gamma_k(s) = \frac{1}{2}\, e^{\pi i s/2}\, \Gamma(s)\Gamma(k + 1/2 - s),$$

and the above sum varies over all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N)$. The condition $1 < \sigma < k - 1/2$ ensures that the above series converges absolutely and uniformly on compact subsets of $\mathcal{H}$ and hence it represents an analytic function on $\mathcal{H}$. The function $R_{s;k,N,\psi}(\tau) \in S_{k+1/2}(4N, \psi)$. For the sake of simplicity, we write $R_{s,\psi}$ for $R_{s;k,N,\psi}$.

For a given $c, d \in \mathbb{Z}$ with $\gcd(c,d) = 1$ and $4N|c$, we choose $a_0, b_0$ such that $a_0 d - b_0 c = 1$. Then any other solution $a, b$ of $ad - bc = 1$ is given by $a = a_0 + nc$ and $b = b_0 + nd$ for $n \in \mathbb{Z}$. Hence,

$$R_{s,\psi}(\tau) = \gamma_k(s) \sum_{\substack{(c,d)=1\\4N|c}} \sum_{n\in\mathbb{Z}} \overline{\psi}(d) \left(\frac{c}{d}\right) \left(\frac{-4}{d}\right)^{k+1/2} (c\tau+d)^{-k-1/2} \left(\frac{a_0\tau + b_0}{c\tau + d} + n\right)^{-s}. \tag{4.5}$$

Using Lipschitz's formula

$$\sum_{n=-\infty}^{\infty} (\tau+n)^{-s} = \frac{e^{-\pi i s/2}(2\pi)^s}{\Gamma(s)} \sum_{n\geq 1} n^{s-1} e(n\tau) \qquad (\tau \in \mathcal{H}, \sigma > 1), \tag{4.6}$$

we get

$$\begin{aligned} R_{s,\psi}(\tau) = \gamma_k(s) \sum_{\substack{(c,d)=1\\4N|c}} \overline{\psi}(d) \left(\frac{c}{d}\right) \left(\frac{-4}{d}\right)^{k+1/2} (c\tau+d)^{-k-1/2} \frac{e^{-\pi i s/2}(2\pi)^s}{\Gamma(s)} \\ \times \sum_{n\geq 1} n^{s-1} e\left(n \frac{a_0\tau+b_0}{c\tau+d}\right) \end{aligned} \tag{4.7}$$

$$\begin{aligned} = (2\pi)^s \Gamma(k+1/2-s) \sum_{n\geq 1} n^{s-1} \\ \times \frac{1}{2} \sum_{\substack{(c,d)=1\\4N|c}} \overline{\psi}(d) \left(\frac{c}{d}\right) \left(\frac{-4}{d}\right)^{k+1/2} (c\tau+d)^{-k-1/2} e\left(n\frac{a_0\tau+b_0}{c\tau+d}\right). \end{aligned}$$

Here, we have used the absolute convergence of the above sum in the region $1 < \sigma < k - \beta - 1/2$ and so the interchange of summations is allowed, where $\beta = k/2 - 1/28$ is the exponent of the estimate for the Fourier coefficients of a cusp form of weight $k + 1/2$ on $\Gamma_0(4N)$ obtained by H. Iwaniec [Iw87]. Thus, for $1 < \sigma < k - \beta - 1/2$,

$$R_{s,\psi}(\tau) = (2\pi)^s \Gamma(k+1/2-s) \sum_{n\geq 1} n^{s-1} P_{n,k+1/2,4N,\psi}(\tau), \tag{4.8}$$

where $P_{n,k+1/2,4N,\psi}$ is the $n$-th Poincaré series defined by (1.3). Using equation (1.4) and (4.1) with the last equation, we get for $1 < \sigma < k - \beta - 1/2$,

$$\langle f, R_{\bar{s},\psi}\rangle = \frac{\pi\Gamma(k-1/2)}{i_{4N} 2^{k-3/2}(4N)^{k/2+1/4-s/2}} L^*(f, k+1/2-s),$$

for all $f \in S_{k+1/2}(4N, \psi)$. Using this, we have

$$R_{s,\psi} = \frac{2^{-2k+1+s}\pi\Gamma(k-1/2)}{i_{4N}N^{k/2+1/4-s/2}} \sum_{j=1}^{d} \frac{L^*(f_j|K, k+1/2-s)}{\langle f_j, f_j \rangle} f_j, \tag{4.9}$$

where the sum varies over the orthogonal basis $\{f_j\}$. Using $f_j|KH_{4N} = \lambda_{f_j} f_j$ together with the functional equation (4.2), we get for $1 < \sigma < k - \beta - 1/2$,

$$R_{s,\psi} = \frac{2^{-2k+1+s}\pi\Gamma(k-1/2)}{i_{4N}N^{k/2+1/4-s/2}} \sum_{j=1}^{d} \frac{L^*(f_j, s)}{\langle f_j, f_j \rangle} \lambda_{f_j} f_j. \tag{4.10}$$

This equality has been established for $1 < \sigma < k - \beta - 1/2$. Since the right hand side is an entire function of $s$, this gives an analytic continuation of the kernel function $R_{s,\psi}$ for all $s \in \mathbb{C}$.
Next, we need the Fourier expansion of the function $R_{s,\psi}$. In an earlier version of [MMR], the Fourier expansion of the function $R_{s,\psi}(\tau)$ was derived, which we present here.

**Lemma 4.4.1** *Let $R_{s,\psi}(\tau) = \sum_{n\geq 1} a_{s,\psi}(n) e^{2\pi i n\tau}$ be the Fourier expansion of $R_{s,\psi}$. The Fourier coefficients $a_{s,\psi}(n)$ are given by*

$$\begin{aligned} a_{s,\psi}(n) =&(2\pi)^s\Gamma(k+1/2-s)n^{s-1} + e^{\pi is/2}(-2\pi i)^{k+1/2}n^{k-1/2} \\ &\times \sum_{\substack{(a,c)\in\mathbb{Z}^2, ac\neq 0\\ \gcd(a,c)=1, 4N|c}} \psi(a)\left(\frac{c}{a}\right)\left(\frac{-4}{a}\right)^{k+1/2} c^{s-k-1/2}a^{-s}e^{2\pi i n a'/c}{}_1f_1(s, k+1/2; -2\pi i n/ac), \end{aligned} \tag{4.11}$$

*where $a'$ is an integer which is the inverse of $a$ modulo $c$ and*

$${}_1f_1(\alpha, \beta; z) = \frac{\Gamma(\alpha)\Gamma(\beta-\alpha)}{\Gamma(\beta)} {}_1F_1(\alpha, \beta; z). \tag{4.12}$$

*Here ${}_1F_1(\alpha, \beta; z)$ is the Kummer's degenerate hypergeometric function, which is defined by the following equation (see [Er54, 5.4, (8)]). For $\alpha, \beta, \mu, \nu, p \in \mathbb{C}$ such that Re $\alpha \geqslant 0$, Re $\beta \geqslant 0$, Re$(\nu-\mu) > 0$ and for any positive real number $T$,*

$$\frac{1}{2\pi i}\int_{T-i\infty}^{T+i\infty} (x+\alpha)^{\mu}(x+\beta)^{-\nu}e^{px}dx = \frac{1}{\Gamma(\nu-\mu)}p^{\nu-\mu-1}e^{\alpha p}\ {}_1F_1(\nu, \nu-\mu; (\beta-\alpha)p). \tag{4.13}$$

**Proof**. Before proceeding to the proof of the lemma, we state the following sublemma which is used to obtain the required Fourier expansion.

**Sublemma:** *For $\alpha, \beta, \mu, \nu \in \mathbb{C}$ with Im $\alpha \geqslant 0$, Im $\beta \geqslant 0$, Re $(\nu - \mu) > 0$ and for $\tau \in \mathcal{H}$, we have*

$$\sum_{m\in\mathbb{Z}} (\tau + \alpha + m)^{\mu} (\tau + \beta + m)^{-\nu} = \sum_{n\geq 1} \gamma_{\alpha,\beta;\mu,\nu}(n) e^{2\pi i n\tau}, \tag{4.14}$$

*where the Fourier coefficients $\gamma_{\alpha,\beta;\mu,\nu}(n)$ are given by*

$$\frac{(-2\pi i)^{\nu-\mu}}{\Gamma(\nu - \mu)} e^{-2\pi i n\alpha} n^{\nu-\mu-1} \ {}_1F_1(\nu, \nu - \mu; -2\pi i n(\beta - \alpha)). \tag{4.15}$$

Here ${}_1F_1(a, b; z)$ is the Kummer's hypergeometric function defined by (4.13).

We assume the sublemma and complete the proof of Lemma 4.4.1. By definition

$$R_{s,\psi}(\tau) = \gamma_k(s) \sum \overline{\psi}(d) \left(\frac{c}{d}\right) \left(\frac{-4}{d}\right)^{k+1/2} (c\tau + d)^{-k-1/2} \left(\frac{a\tau + b}{c\tau + d}\right)^{-s}, \tag{4.16}$$

where the sum varies over all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N)$. We break up the sum into $c = 0$ and $c \neq 0$. The first sum with $c = 0$ gives the following Fourier expansion (by using the Lipschitz's formula (4.6)).

$$2\gamma_k(s) \sum_{n\in\mathbb{Z}} (\tau + n)^{-s} = (2\pi)^s \Gamma(k + 1/2 - s) \sum_{n\geq 1} n^{s-1} e^{2\pi i n\tau}. \tag{4.17}$$

We now consider the case $c \neq 0$. In this case, $a \neq 0$ and we sum over the matrices as follows. For a given coprime pair $(a, c) \in \mathbb{Z}^2$ such that $4N|c$, we choose integers $b_0, d_0$ such that $ad_0 - b_0c = 1$. Then any other solution $b, d$ of $ad - bc = 1$ is given by $b = b_0 + am$, $d = d_0 + cm$, $m \in \mathbb{Z}$. Therefore, the sum with $c \neq 0$ becomes

$$\gamma_k(s) \sum_{\substack{(a,c)\in\mathbb{Z}^2, ac\neq 0 \\ \gcd(a,c)=1, 4N|c}} \psi(a) \left(\frac{c}{a}\right) \left(\frac{-4}{a}\right)^{k+1/2} \sum_{m\in\mathbb{Z}} (c(\tau + m) + d_0)^{-k-1/2} \left(\frac{a(\tau + m) + b_0}{c(\tau + m) + d_0}\right)^{-s}. \tag{4.18}$$

Now, consider the inner sum over $m \in \mathbb{Z}$.

$$\sum_{m\in\mathbb{Z}}(c(\tau+m)+d_0)^{-k-1/2}\left(\frac{a(\tau+m)+b_0}{c(\tau+m)+d_0}\right)^{-s} = c^{-k-1/2+s}a^{-s}\sum_{m\in\mathbb{Z}}(\tau+\frac{d_0}{c}+m)^{-k-\frac{1}{2}+s} \times (\tau+\frac{b_0}{a}+m)^{-s}.$$

Since $k+1/2 = \mathrm{Re}\,(s-(s-k-1/2)) > 3$, using the sublemma (with $\alpha = d_0/c$, $\beta = b_0/a$, $\mu = s-k-1/2$, $\nu = s$), the right-hand side sum has a Fourier expansion with $n$-th Fourier coefficient equal to:

$$\frac{(-2\pi i)^{k+1/2}}{\Gamma(k+1/2)}e^{-2\pi i n d_0/c}n^{k-1/2}{}_1F_1(s,k+1/2;2\pi in/ac).$$

Using this Fourier expansion in (4.18), we see that the $n$-th Fourier coefficient in the case $c \neq 0$ is equal to (interchanging the sum is allowed due to absolute convergence):

$$\gamma_k(s)\frac{(-2\pi i)^{k+1/2}}{\Gamma(k+1/2)}n^{k-1/2}\sum_{\substack{(a,c)\in\mathbb{Z}^2, ac\neq 0\\ \gcd(a,c)=1, 4N|c}}\psi(a)\left(\frac{c}{a}\right)\left(\frac{-4}{a}\right)^{k+1/2}c^{s-k-1/2}a^{-s}e^{-2\pi i n a'/c} \times {}_1F_1(s,k+1/2;2\pi in/ac),$$

where $a'$ is an integer which is the inverse of $a$ modulo $c$. This completes the proof of Lemma 4.4.1.

**Proof of sublemma.** As the sum on the left hand side of (4.14) is periodic in $\tau$ with period 1, its Fourier coefficient is given by

$$\gamma_{\alpha,\beta;\mu,\nu}(n) = \int_{iC-\infty}^{iC+\infty}(\tau+\alpha)^{\mu}(\tau+\beta)^{-\nu}e^{-2\pi in\tau}\,d\tau \quad (C>0). \tag{4.19}$$

Making the substitution $\tau \mapsto it$, the above integral becomes

$$(2\pi)i^{\mu-\nu}\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}(t+\alpha/i)^{\mu}(t+\beta/i)^{-\nu}e^{2\pi nt}\,dt.$$

The above integral is the integral expression for the Kummer's degenerate hypergeometric function with $\alpha,\beta,p$ replaced by $\alpha/i,\beta/i,2\pi n$ respectively. Therefore, we get

$$\gamma_{\alpha,\beta;\mu,\nu}(n) = \frac{(-2\pi i)^{\nu-\mu}}{\Gamma(\nu-\mu)} e^{-2\pi i n\alpha} n^{\nu-\mu-1} \; {}_1F_1(\nu, \nu-\mu; -2\pi i n(\beta-\alpha)).$$

□

We now give the proof of our theorems. Assume that $\sum_j \frac{L^*(f_j,s)}{\langle f_j, f_j\rangle} \lambda_{f_j} a_{f_j}(1) = 0$, for $s$ as in the theorems. Comparing the first Fourier coefficient in (4.10), we get that (for the values of $s$) the first Fourier coefficient of $R_{s,\psi}$ is zero. Therefore, putting $n=1$ in (4.11) and dividing by $(2\pi)^s\Gamma(k+1/2-s)$, we obtain

$$1 + e^{\pi i s/2} \frac{(2\pi)^{k+1/2-s}}{i^{k+1/2}\Gamma(k+1/2-s)} \sum_{\substack{a,c\in\mathbb{Z}, ac\neq 0\\ (a,c)=1, 4N|c}} \psi(a) \left(\frac{c}{a}\right)\left(\frac{-4}{a}\right)^{k+1/2} \frac{c^{s-k-1/2}}{a^s} \times e^{2\pi i a'/c} {}_1f_1(s, k+1/2; -2\pi i/ac) = 0. \tag{4.20}$$

In particular, let $s = k/2 + 1/4 - \delta + ir_0$, where $0 \le \delta < 1/2$. Now, one has

$$|{}_1f_1(s, k+1/2; -2\pi i/ac)| \le 1$$

(see [K97]). Taking the absolute value in (4.20) and using the above estimate, we get

$$\begin{aligned} 1 &\le A(r_0) \frac{\pi^{k/2+1/4+\delta}}{|\Gamma(k/2+1/4+\delta-ir_0)|} \frac{1}{(2N)^{k/2+1/4+\delta}} \left( \sum_{\substack{a,c\in\mathbb{Z}, ac\neq 0,\\ (a,2Nc)=1}} \frac{1}{a^{k/2+1/4-\delta} \cdot c^{k/2+1/4+\delta}} \right) \\ &\le A(r_0)\; B \; \frac{\pi^{k/2+1/4+\delta}}{|\Gamma(k/2+1/4+\delta-ir_0)|} \frac{1}{(2N)^{k/2+1/4+\delta}}, \end{aligned}$$

where $A(r_0)$ is a constant depending only on $r_0$ and $B > 0$ is an absolute constant. To prove Theorem 4.3.1, we fix $N$ and allow $k$ tend to infinity and for the proof of Theorem 4.3.2, we fix $k$ and allow $N$ tend to infinity. In either case, the right-hand side goes to zero (for fixed $N$ one should use the Stirling's approximation), a contradiction. This completes the proof. □

**Remark 4.4.2** In [K97], the non-vanishing result was obtained for $s$ inside the critical strip with the condition that Re $s \neq k/2$, the center of the critical strip. However, since the level of the modular forms considered in this paper is greater than 1, we need not assume this condition and our results are valid for all $s$ inside the critical strip. We also remark that the same is true in [Ra05], since the level $M$ is greater than 1. The reason is as follows. When $M = 1$ (i.e., when one considers the case of forms of integral weight on $SL_2(\mathbb{Z})$), while deriving the Fourier expansion of the function $R_s$, the term corresponding to $ac = 0$ has two contributions ($c = 0$ and $a = 0$). Therefore, in the estimation of the first Fourier coefficient, there is an extra term on the right-hand side (see [K97, p. 189, Eq.(10)]). Due to the appearane of this extra term, in order to get a contradiction, the central values have to be omitted. Since the case $a = 0$ doesn't arise for the levels $M > 1$, we do not get the extra term on the right-hand side in the estimation. Therefore, this gives the advantage of considering all the values of $s$ inside the critical strip. In particular, one obtains non-vanishing results for forms at the center of the critical strip when the level is greater than 1.

**Remark 4.4.3** The average sum in Theorem 4.3.1 (and also in Theorem 4.3.2) contains an extra factor $\lambda_f$ (and of course the first Fourier coefficient), which does not appear in Kohnen's result. In the case of level 1, we have the functional equation $L^*(f, k-s) = (-1)^{k/2}L^*(f, s)$, whereas when the level $M$ is greater than 1, we have a different functional equation in the sense that on the one side we have $L^*(f, s)$ and on the other side we have $L^*(f|H_M, k-s)$ and therefore, in the final form of the functional equation, the root number will depend on the function, especially the eigenvalue of $f$ under the Fricke involution $H_M$. (In the case of half-integral weight, the fricke involution is $H_M = H_{4N}$ in our notation.) Therefore, after using functional equation in (4.9), we will have an extra factor in (4.10), which we call $\lambda_f$. Note that the extra factor corresponding to the eigenvalues under $H_M$ also appears in Raghuram's results (see [Ra05]). Since normalization of Fourier coefficients is not known in the case of half-integral weight, we also have the first Fourier coefficients appearing in the average sum.

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