# Nonlinear Schrödinger equation and the twisted Laplacian

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# DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree/diploma at this or any other Institution/University.

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## List of Publications arising from the thesis

### Journal

1. "Nonlinear Schrödinger equation for the twisted Laplacian", P.K. Ratnakumar, Vijay Kumar Sohani, J. Funct. Anal., **2013**, 265 (1), 1-27.

2. "Strichartz estimates for the Schrödinger propagator for the Laguerre operator", Vijay Kumar Sohani, Proc. Indian Acad. Sci. (Math. Sci.), **2013**, 123 (4), 525-537.

### Others

1. "Nonlinear Schrödinger equation for the twisted Laplacian- global well posedness", P. K. Ratnakumar, Vijay Kumar Sohani, communicated.

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Date: **Vijay Kumar Sohani**

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# Contents

# Synopsis

## 1. Introduction

The free Schrödinger equation on $\mathbb{R}^n$ is the PDE

$$i\partial_t\psi(x,t) + \Delta\psi(x,t) = 0, \qquad x \in \mathbb{R}^n,\ t \in \mathbb{R}$$

which gives the quantum mechanical description of the evolution of a free particle in $\mathbb{R}^n$. If $\psi$ is the solution of the Schrödinger equation, then $|\psi(x,t)|^2$ is interpreted as the probability density for finding the position of the particle in $\mathbb{R}^n$ at a given time $t$. More generally for any self adjoint differential operator $L$ on $\mathbb{R}^n$, we consider the initial value problem for the Schrödinger equation for the operator $L$:

$$\begin{aligned} i\partial_t u(x,t) - Lu(x,t) &= 0,\ x \in \mathbb{R}^n,\ t \in \mathbb{R} \\ u(x,0) &= f(x) \end{aligned}$$

with $L$ now representing the corresponding Hamiltonian of the quantum mechanical system.

The significance of this view point is that, most Hamiltonians of interest, namely the perturbation of the Laplacian with a potential $V$ (of the form $L = -\Delta + V$) or the magnetic Laplacian corresponding to the magnetic potential $(A_1(x), ..., A_n(x))$ (of the form $L = \sum_{j=1}^n \left(i\partial_{x_j} + A_j(x)\right)^2$) on $\mathbb{R}^n$, can be analysed with our approach, in terms of the spectral theory of the Hamiltonian.

In this thesis we consider the twisted Laplacian. The twisted laplacian on $\mathbb{C}^n$ is given by

$$\mathcal{L} = \sum_{j=1}^n \left[\left(i\partial_{x_j} + \frac{y_j}{2}\right)^2 + \left(i\partial_{y_j} - \frac{x_j}{2}\right)^2\right]$$

which is of the form $\sum_{j=1}^{2n} \left[(i\partial_{w_j} - A_j(w))^2\right]$, hence represents a Schrödinger operator on $\mathbb{C}^n$ for the magnetic vector potential $A(z) = \frac{iz}{2}, z \in \mathbb{C}^n$.

The Schrödinger equation for the magnetic potential with magnetic field decaying at infinity has been studied by many authors, see for instance Yajima [39], where author studies the propagator for the linear equation. In contrast, the nonlinear equation in our situation corresponds to a magnetic equation with a constant magnetic field, which has no decay. For more details on general magnetic Schrödinger equation corresponding to magnetic field without decay, see [1]. In [40] Zhang and Zheng proved the local well posedness for the nonlinear Schrödinger equation with twisted Laplacian and polynomial nonlinearity. The well posedness result for nonlinear Schrödinger equation on $\mathbb{R}^n$ has been studied by many others, see Ginibre Velo [12, 13, 14], Cazenave Weissler [6, 7, 8], Tsutsumi [36], Kato [16], Begout [2], Sjögren Torrea [27], to mention only a few.

The magnetic Laplacian naturally arises in the study of system in the presence of a magnetic field, hence there is an active interest and extensive research is going on in the study with magnetic Laplacian. In explicit terms the twisted Laplacian looks like

$$\mathcal{L} = -\Delta + \frac{1}{4}|z|^2 - i\sum_{1}^{n}\left(x_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial x_j}\right).$$

In this thesis we will study the well posedness, i.e., local existence, uniqueness, stability and blowup alternative of the initial value problem (see Section 3)

$$\begin{aligned} i\partial_t u(z,t) - \mathcal{L}u(z,t) &= G(z,u), \qquad\qquad z \in \mathbb{C}^n,\ t \in \mathbb{R} \\ u(z,t_0) &= f(z) \end{aligned}$$

with $f$ in certain first order Sobolev spaces related to the twisted Laplacian and also in $L^2(\mathbb{C}^n)$, see Sections 4, 6, 7, 8. This work is based on [24] (to appear in J. Funct. Anal. 265 (1) (2013) 1-27) and [25, 29].

Twisted Laplacian and Laguerre operator are closely related to each other in the following sense. If $f \in \mathcal{S}(\mathbb{C}^n)$ is radial then $\mathcal{L}f(z) = L_{n-1}f(r)$ where $L_{n-1}$ is 1-dimensional Laguerre operator of type $n-1$ given by (9) and $r = |z|$. More generally we can consider $n$-dimensional Laguerre operator $L_\beta$ on $\mathbb{R}^n_+ = (0,\infty)^n$ of type $\beta \in \left(-\frac{1}{2},\infty\right)^n$ which has singularity at $x_j = 0$, $1 \le j \le n$. By similar analysis we also prove the local well posedness of the initial value problem for Schrödinger equation with the Laguerre operator and initial value in $L^2(\mathbb{R}^n_+, d\nu)$ where $d\nu = \left(\prod_{j=1}^n x_j^{2\beta_j+1}\right) dx$, see Section 9. This work is based on the Strichartz estimates for the Laguerre operator proved in Sohani [28] (to appear in Proc. Math. Sci.).

# 2. Schrödinger propagator for the twisted Laplacian

Now we define the Schrödinger propagator $e^{-it\mathcal{L}}$ through the spectral theory of the twisted Laplacian $\mathcal{L}$. The twisted Laplacian is closely related to the sub Laplacian on the Heisenberg group, hence the spectral theory of this operator is closely connected with the representation theory of the Heisenberg group. Here we give a brief review of the spectral theory of the twisted Laplacian $\mathcal{L}$. The material discussed here is based on the books by Folland [11] and Thangavelu [33, 34].

The eigenfunctions of the operator $\mathcal{L}$ are called the special Hermite functions, which are defined in terms of the Fourier-Wigner transform. For a pair of functions $f, g \in L^2(\mathbb{R}^n)$, the Fourier-Wigner transform is defined to be

$$V(f,g)(z) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f\left(\xi + \frac{y}{2}\right) \overline{g}\left(\xi - \frac{y}{2}\right) d\xi,$$

where $z = x + iy \in \mathbb{C}^n$. For any two multi-indices $\mu, \nu$ the special Hermite functions $\Phi_{\mu\nu}$ are given by

$$\Phi_{\mu\nu}(z) = V(h_\mu, h_\nu)(z)$$

where $h_\mu$ and $h_\nu$ are Hermite functions on $\mathbb{R}^n$. Recall that for each nonnegative integer $k$, the one dimensional Hermite functions $h_k$ are defined by

$$h_k(x) = \frac{(-1)^k}{\sqrt{2^k k! \sqrt{\pi}}} \left(\frac{d^k}{dx^k} e^{-x^2}\right) e^{\frac{x^2}{2}}.$$

Now for each multi index $\nu = (\nu_1, \cdots, \nu_n)$, the n-dimensional Hermite functions are defined by the tensor product :

$$h_\nu(x) = \prod_{i=1}^{n} h_{\nu_i}(x_i), \qquad x = (x_1, \cdots, x_n).$$

$\Phi_{\mu\nu}$ are eigenfunctions of $\mathcal{L}$ with eigenvalue $2|\nu|+n$ and they also form a complete orthonormal system in $L^2(\mathbb{C}^n)$. Thus every $f \in L^2(\mathbb{C}^n)$ has the expansion

$$f = \sum_{\mu,\nu} \langle f, \Phi_{\mu\nu} \rangle \Phi_{\mu\nu}$$

in terms of the eigenfunctions of $\mathcal{L}$. The above expansion may be written as

$$f = \sum_{k=0}^{\infty} P_k f$$

where

$$P_k f = \sum_{\mu, |\nu|=k} \langle f, \Phi_{\mu,\nu} \rangle \Phi_{\mu\nu}$$

is the spectral projection corresponding to the eigenvalue $2k+n$. Now for any $f \in L^2(\mathbb{C}^n)$ such that $\mathcal{L}f \in L^2(\mathbb{C}^n)$, by self adjointness of $\mathcal{L}$, we have $P_k(\mathcal{L}f) = (2k+n)P_k f$. It follows that for $f \in L^2(\mathbb{C}^n)$ with $\mathcal{L}f \in L^2(\mathbb{C}^n)$

$$\mathcal{L}f = \sum_{k=0}^{\infty} (2k+n) P_k f.$$

Thus, we can define Schrödinger propagator $e^{-it\mathcal{L}}$ as

$$e^{-it\mathcal{L}} f = \sum_{k=0}^{\infty} e^{-it(2k+n)} P_k f.$$

Note that $P_k f$ has the compact representation

$$P_k f(z) = (2\pi)^{-n} (f \times \varphi_k)(z)$$

in terms of the Laguerre function $\varphi_k(z) = L_k^{n-1}(\frac{1}{2}|z|^2) e^{-\frac{1}{4}|z|^2}$, see [33]. Hence formally we can express $e^{-it\mathcal{L}}$ as a twisted convolution operator:

$$e^{-it\mathcal{L}} f = f \times K_{it}$$

for $f \in \mathcal{S}(\mathbb{C}^n)$ where $K_{it}(z) = \frac{(4\pi i)^{-n}}{(\sin t)^n} e^{\frac{i(\cot t)|z|^2}{4}}$.

# 3. Nonlinear Schrödinger equation for the twisted Laplacian

We consider the initial value problem for the nonlinear Schrödinger equation for the twisted Laplacian $\mathcal{L}$:

$$i\partial_t u(z,t) - \mathcal{L}u(z,t) = G(z,u), \quad z \in \mathbb{C}^n,\ t \in \mathbb{R} \tag{1}$$

$$u(z,t_0) = f(z). \tag{2}$$

Here we consider the nonlinearity $G$ of the form

$$G(z,w) = \psi(x,y,|w|)\, w,\ (x,y,w) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{C}, \tag{3}$$

where $z = x+iy \in \mathbb{C}^n$, $w \in \mathbb{C}$ and $\psi \in C(\mathbb{R}^n \times \mathbb{R}^n \times [0,\infty)) \cap C^1(\mathbb{R}^n \times \mathbb{R}^n \times (0,\infty))$ satisfy the following inequality

$$|F(x,y,\eta)| \leq \lambda |\eta|^\alpha \tag{4}$$

for $F = \psi, \partial_{x_j}\psi,\ \partial_{y_j}\psi\ (1 \leq j \leq n)$ and $w\partial_w \psi(x,y,w)$, $\alpha \geq 0$ and some constant $\lambda \geq 0$. The class of nonlinearity given by (3), (4) includes in particular, power type nonlinearity of the form $|u|^\alpha u$.

When $G \equiv 0$ and $f \in L^2(\mathbb{C}^n)$ the solution to this initial value problem is given by the Schrödinger propagator

$$u(z,t) = e^{-i(t-t_0)\mathcal{L}} f(z).$$

When $G(z,u) = g(z)$, the solution is given by the Duhamel formula

$$u(z,t) = e^{-i(t-t_0)\mathcal{L}} f(z) - i \int_{t_0}^{t} e^{-i(t-s)\mathcal{L}} g(z)\, ds.$$

Thus in the linear case, the solution is determined once the functions $f$ and $g$ are known.

The basic idea in nonlinear analysis is the following heuristic reasoning based on the above formula. If the solution $u$ is known, then one would expect $u$ to satisfy the above equation with $g(z)$ replaced by $G(z,u(z,s))$:

$$u(z,t) = e^{-i(t-t_0)\mathcal{L}} f(z) - i \textstyle\int_{t_0}^{t} e^{-i(t-s)\mathcal{L}} G(z,u(z,s))\, ds. \tag{5}$$

Indeed one can show that $u$ from a reasonable function space satisfies a PDE of the form (1), (2), if and only if $u$ satisfies an integral equation of the form (5).

This reduces the existence theorem for the solution to the nonlinear Schrödinger equation to a fixed point theorem for the operator

$$\mathcal{H}(u)(z,t) = e^{-i(t-t_0)\mathcal{L}} f(z) - i \int_{t_0}^{t} e^{-i(t-s)\mathcal{L}} G(z, u(z,s)) ds$$

in a suitable subset of the relevant function space.

# 4. Some auxiliary function spaces

## The Sobolev space $\tilde{W}^{1,p}(\mathbb{C}^n)$

Let $L_j$ and $M_j$ be the differential operators defined by

$$L_j = \left( \frac{\partial}{\partial x_j} + i\frac{y_j}{2} \right) \quad \text{and} \quad M_j = \left( \frac{\partial}{\partial y_j} - i\frac{x_j}{2} \right), \quad j = 1, 2, ..., n.$$

We consider the following space

$$\tilde{W}^{1,p}(\mathbb{C}^n) \quad = \quad \{ f \in L^p(\mathbb{C}^n) : L_j f, M_j f \in L^p(\mathbb{C}^n), 1 \leq j \leq n \}.$$

It is easy to see that $\tilde{W}^{1,p}(\mathbb{C}^n)$ is a Banach space with respect to norm $\|f\| = \|f\|_{L^p(\mathbb{C}^n)} + \sum_{j=1}^{n} \left( \|L_j f\|_{L^p(\mathbb{C}^n)} + \|M_j f\|_{L^p(\mathbb{C}^n)} \right)$. The differential operators $L_j$ and $M_j$ are the natural ones adaptable to the power type nonlinearity $G(u) = |u|^\alpha u$ and the generality that we consider here. The natural choice, namely the standard Sobolev space $W^{1,p}_{\mathcal{L}}(\mathbb{C}^n)$ defined using the twisted Laplacian $\mathcal{L}$ (see [35]), is not suitable for treating such nonlinearities.

An interesting relation between the Sobolev space $\tilde{W}^{1,p}(\mathbb{C}^n)$ and the ordinary Sobolev space $W^{1,p}(\mathbb{C}^n)$ is the following: If $u \in \tilde{W}^{1,p}(\mathbb{C}^n)$, then $|u| \in W^{1,p}(\mathbb{C}^n)$. We have the continuous inclusion

$$\tilde{W}^{1,p_1}(\mathbb{C}^n) \hookrightarrow L^{p_2}(\mathbb{C}^n) \quad \begin{array}{ll} \text{for } p_1 \leq p_2 \leq \frac{2np_1}{2n-p_1} & \text{if } p_1 < 2n \\ \text{for } p_1 \leq p_2 < \infty & \text{if } p_1 = 2n \\ \text{for } p_1 \leq p_2 \leq \infty & \text{if } p_1 > 2n. \end{array} \tag{6}$$

The differential operators $L_j$ and $M_j$ $(1 \leq j \leq n)$ have the following commutativity properties

$$\begin{aligned} L_j e^{-it\mathcal{L}} f &= e^{-it\mathcal{L}} L_j f \\ M_j e^{-it\mathcal{L}} f &= e^{-it\mathcal{L}} M_j f \\ L_j \int_{t_0}^{t} e^{-i(t-s)\mathcal{L}} g(z,s)ds &= \int_{t_0}^{t} e^{-i(t-s)\mathcal{L}} L_j g(z,s)ds \\ M_j \int_{t_0}^{t} e^{-i(t-s)\mathcal{L}} g(z,s)ds &= \int_{t_0}^{t} e^{-i(t-s)\mathcal{L}} M_j g(z,s)ds \end{aligned}$$

where $f \in \mathcal{S}'(\mathbb{C}^n)$, $t, t_0 \in \mathbb{R}$, $g \in L^{q'}_{\text{loc}}\left(I, \tilde{W}^{1,p'}(\mathbb{C}^n)\right)$ for some admissible pair $(q,p)$ (see Definition 5.1) and open interval $I$ containing $t, t_0$.

## The Sobolev space $\tilde{W}^{1,p}_{\mathcal{L}}(\mathbb{C}^n)$

The local well posedness of the nonlinear Schrödinger equation for the twisted Laplacian has been studied in [24] for initial value in $\tilde{W}^{1,2}(\mathbb{C}^n)$. However this approach does not conclude energy conservation.

We overcome this difficulty by introducing the Sobolev space $\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$ defined using the operators $Z_j$ and $\overline{Z}_j$, which is the natural one in this context where

$$Z_j = \frac{\partial}{\partial z_j} + \frac{1}{2}\bar{z}_j, \ \overline{Z}_j = -\frac{\partial}{\partial \bar{z}_j} + \frac{1}{2} z_j$$

and $\frac{\partial}{\partial z_j}$, $\frac{\partial}{\partial \bar{z}_j}$ denote the complex derivatives $\frac{\partial}{\partial_{x_j}} \mp i \frac{\partial}{\partial_{y_j}}$ respectively. Though they do not commute with $e^{-it\mathcal{L}}$, they have a reasonable commutative relation, suitable for us. The advantage of working with this Sobolev space is that we get energy conservation in this case. From this we can show that there is no finite time blowup, hence can conclude global existence in the Sobolev space $\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$.

We consider the following Banach space

$$\tilde{W}^{1,p}_{\mathcal{L}}(\mathbb{C}^n) \quad = \quad \{f \in L^p(\mathbb{C}^n) : Z_j f, \overline{Z}_j f \in L^p(\mathbb{C}^n), 1 \leq j \leq n\}$$

with norm $\|f\| = \|f\|_{L^p(\mathbb{C}^n)} + \sum_{j=1}^n \left( \|Z_j f\|_{L^p(\mathbb{C}^n)} + \|\overline{Z}_j f\|_{L^p(\mathbb{C}^n)} \right)$.

The Sobolev space $\tilde{W}^{1,p}_{\mathcal{L}}(\mathbb{C}^n)$ will also satisfy embedding (6) as similar to space $\tilde{W}^{1,p}(\mathbb{C}^n)$. Operators $Z_j$ and $\bar{Z}_j$ $(1 \leq j \leq n)$ have the following quasi commutativity properties

$$
\begin{aligned}
Z_j e^{-it\mathcal{L}} f &= e^{-2it} e^{-it\mathcal{L}} Z_j f \\
\overline{Z}_j e^{-it\mathcal{L}} f &= e^{2it} e^{-it\mathcal{L}} \overline{Z}_j f \\
Z_j \int_{t_0}^{t} e^{-i(t-s)\mathcal{L}} g(z,s) ds &= e^{-2it} \int_{t_0}^{t} e^{-i(t-s)\mathcal{L}} e^{2is} Z_j g(z,s) ds \\
\overline{Z}_j \int_{t_0}^{t} e^{-i(t-s)\mathcal{L}} g(z,s) ds &= e^{2it} \int_{t_0}^{t} e^{-i(t-s)\mathcal{L}} e^{-2is} \overline{Z}_j g(z,s) ds
\end{aligned}
$$

where $f \in \mathcal{S}'(\mathbb{C}^n)$, $t, t_0 \in \mathbb{R}$, $g \in L^{q'}_{\text{loc}}\left(I, \tilde{W}^{1,p'}_{\mathcal{L}}(\mathbb{C}^n)\right)$ for some admissible pair $(q,p)$ and open interval $I$ containing $t, t_0$.

# 5. Strichartz estimates

Strichartz estimate is an important tool in the study of local existence of solutions to dispersive equations, in which no derivatives are present in the nonlinearity. Strichartz estimates were first proved by Strichartz [30] for free Schrödinger and wave equations on $\mathbb{R}^n$. They were generalized to general admissible pairs $(q,p)$ by Ginibre and Velo [14, 15], Lindblad and Sogge [19]. The end point estimates were proved by Keel and Tao [17]. End point estimates were also proved by D'Ancona, Fanelli, Vega and Viscigia [9] for magnetic Schrödinger equation with some conditions on the potential $A$ and $V$.

The Homogeneous Strichartz estimate (7) for twisted Laplacian is proved by Ratnakumar [22]. We begin with the following definition of admissible pair and discuss the Strichartz estimates.

**Definition 5.1** Let $n \geq 1$. We say that a pair $(q,p)$ is *admissible* if

$$
\begin{gathered}
1 \leq q \leq 2, \quad 0 \leq n\left(\frac{1}{2} - \frac{1}{p}\right) < \frac{1}{2} \qquad \text{or} \\
2 < q \leq \infty \quad \text{and } 0 \leq n\left(\frac{1}{2} - \frac{1}{p}\right) \leq \frac{1}{q}.
\end{gathered}
$$

**Remark 5.2** The admissibility condition on $(q,p)$ implies that $2 \leq p < \frac{2n}{n-1}$.

Since Strichartz estimates will be in terms of mixed $L^p$ spaces, we define space

$L^q((a,b), L^p(\mathbb{C}^n))$ by the following

$$L^q((a,b), L^p(\mathbb{C}^n)) = \{\text{g is measurable on } \mathbb{C}^n \times (a,b) \ : \|g\|_{L^q((a,b),L^p(\mathbb{C}^n))} < \infty\}$$

where $\|g\|_{L^q((a,b),L^p(\mathbb{C}^n))} = \left(\int_a^b \|g\|^q_{L^p(\mathbb{C}^n)} dt\right)^{\frac{1}{q}}$.

The main Strichartz type estimates is compiled in the following theorem.

**Theorem 5.3 (Strichartz Estimates)** *Let $(q,p), (q_1,p_1)$ be two admissible pairs, $(a,b)$ be a finite interval with $t_0 \in [a,b]$, $f \in L^2(\mathbb{C}^n)$ and $g \in L^{q_1'}((a,b), L^{p_1'})$ where $q_1'$ and $p_1'$ are conjugate exponents of $q_1$ and $p_1$ respectively. Then the following estimates hold over $\mathbb{C}^n \times (a,b)$:*

$$\|e^{-it\mathcal{L}} f\|_{L^q((a,b),L^p(\mathbb{C}^n))} \leq C\|f\|_2 \tag{7}$$

$$\left\|\int_{t_0}^t e^{-i(t-s)\mathcal{L}} g(z,s)ds\right\|_{L^q((a,b),L^p(\mathbb{C}^n))} \leq C\|g\|_{L^{q_1'}((a,b),L^{p_1'}(\mathbb{C}^n))} \tag{8}$$

*where the constant $C$ depends on admissible pairs but independent of $t_0$. Moreover $e^{-it\mathcal{L}} f \in C(\mathbb{R}, L^2(\mathbb{C}^n))$ and $\int_{t_0}^t e^{-i(t-s)\mathcal{L}} g(z,s)ds \in C([a,b], L^2)$.*

**Remark 5.4** Note that $e^{-it\mathcal{L}} f(z)$ is $2\pi$ periodic in $t$, hence we can not expect the above Strichartz inequalities for unbounded intervals except when $q = \infty$. Since $|\sin t|$ is $\pi$ periodic, constant $C$ in the inequalities (7) and (8) can be chosen independent of interval $(a,b)$ provided $b - a \leq \pi$.

## 6. A Local existence result

We consider initial value $f \in \tilde{W}^{1,2}(\mathbb{C}^n)$. We have proved the local well posedness of initial value problem (1), (2) in this case, see [24]. Now we state the main theorems.

**Theorem 6.1** (Local existence) *Assume that $G$ is as in (3), (4), $\alpha \in [0, \frac{2}{n-1})$ and $u(\cdot, t_0) = f \in \tilde{W}^{1,2}(\mathbb{C}^n)$. Then there exist a number $T = T(\|u(\cdot,t_0)\|)$ such that the initial value problem (1), (2) has a unique solution $u \in C([t_0 - T, t_0 + T]; \tilde{W}^{1,2}(\mathbb{C}^n))$.*

**Theorem 6.2** *Let $u(\cdot,t_0) = f \in \tilde{W}^{1,2}(\mathbb{C}^n)$, $\alpha \in [0, \frac{2}{n-1})$ and $G$ be as in (3), (4). Initial value problem (1), (2) has unique maximal solution $u \in C((T_*, T^*), \tilde{W}^{1,2}) \cap L^{q_1}_{loc}\Big((T_*, T^*), \tilde{W}^{1,p_1}(\mathbb{C}^n)\Big)$, where $t_0 \in (T_*, T^*)$ and $(q_1, p_1)$ be an arbitrary admissible pair. Fix $p = 2 + \alpha$. Moreover the following properties hold:*

**(i)(Uniqueness)** *Solution is unique in* $C((T_*, T^*), \tilde{W}^{1,2}) \cap L^{q_1}_{loc}\left((T_*, T^*), \tilde{W}^{1,p}\right)$ *for every admissible pair* $(q_1, p)$ *with* $q_1 > 2$.

**(ii)(Blowup alternative)** *If* $T^* < \infty$ *(respectively,* $T_* > -\infty$*), then* $\|u(\cdot, t)\|_{\tilde{W}^{1,2}} \to \infty$ *as* $t \to T^*$ *(respectively,* $t \to T_*$*).*

**(iii)(Stability)** *If* $f_j \to f$ *in* $\tilde{W}^{1,2}(\mathbb{C}^n)$*, then* $u_j \to u$ *in* $L^{q_1}(I, \tilde{W}^{1,p_1}(\mathbb{C}^n))$ *for every admissible pair* $(q_1, p_1)$ *and every interval* $I$ *with* $\bar{I} \subset (T_*, T^*)$.

## 7. Global well posedness in $\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$

In this Section we consider initial value $f \in \tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$. As similar to Theorem 6.1 and Theorem 6.2, we have the following Theorem (see [25]).

**Theorem 7.1** (Local well posedness) *Let* $f = u(\cdot, t_0) \in \tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$, $\alpha \in [0, \frac{2}{n-1})$ *and* $G$ *be as in (3) and (4). Then the Initial value problem (1), (2) has unique maximal solution* $u \in C((T_*, T^*), \tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)) \cap L^{q_1}_{loc}\left((T_*, T^*), \tilde{W}^{1,p_1}_{\mathcal{L}}\right)$, *where* $t_0 \in (T_*, T^*)$ *and* $(q_1, p_1)$ *be an arbitrary admissible pair. Fix* $p = 2 + \alpha$. *Moreover the following properties hold:*

**(i)(Uniqueness)** *Solution is unique in* $C((T_*, T^*), \tilde{W}^{1,2}_{\mathcal{L}}) \cap L^{q_1}_{loc}\left((T_*, T^*), \tilde{W}^{1,p}_{\mathcal{L}}\right)$ *for every admissible pair* $(q_1, p)$ *with* $q_1 > 2$.

**(ii)(Blowup alternative)** *If* $T^* < \infty$ *(respectively,* $T_* > -\infty$*), then* $\|u(\cdot, t)\|_{\tilde{W}^{1,2}_{\mathcal{L}}} \to \infty$ *as* $t \to T^*$ *(respectively,* $t \to T_*$*).*

**(iii)(Stability)** *If* $f_j \to f$ *in* $\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$*, then* $u_j \to u$ *in* $\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$ *for each* $t \in (T_*, T^*)$ *and also in* $L^{q_1}\left(I, \tilde{W}^{1,p_1}(\mathbb{C}^n)\right)$ *for every admissible pair* $(q_1, p_1)$ *and every interval* $I$ *with* $\bar{I} \subset (T_*, T^*)$.

Our main result is the following theorem (see [25]).

**Theorem 7.2** (Global well posedness) *Let* $f \in \tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$, $\alpha \in [0, \frac{2}{n-1})$ *and* $\psi : \mathbb{R}^n \times \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ *be real valued as in (3) and (4). Then the solution* $u \in C((T_*, T^*), \tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)) \cap L^{q_1}_{loc}\left((T_*, T^*), \tilde{W}^{1,p_1}_{\mathcal{L}}\right)$ *of the initial value problem (1.0.13), (1.0.14) as obtained in Theorem 7.1 satisfies the following properties:*

**(i)(Conservation of charge)** $\|u(\cdot, t)\|_{L^2(\mathbb{C}^n)} = \|f\|_{L^2(\mathbb{C}^n)}, \quad t \in (T_*, T^*)$.

**(ii)(Conservation of energy)** *$E(u(\cdot,t)) = E(u(\cdot,t_0))$, $t \in (T_*, T^*)$, where*

$$E(u) = \frac{1}{4}\sum_{j=1}^{n}\int_{\mathbb{C}^n}\left(|Z_j u(z,t)|^2 + |\overline{Z}_j u(z,t)|^2\right) dz + \int_{\mathbb{C}^n} \tilde{G}(z,|u|)dz.$$

**(iii)(Global existence)** *If $\psi \geq 0$ is nonnegative, the solution extends to the whole of $\mathbb{R}$. For nonpositive $\psi$, the solution is global if $0 \leq \alpha < \frac{2}{n}$.*

# Critical Case $\alpha = \frac{2}{n-1}$

Now we consider the critical case $\alpha = \frac{2}{n-1}$. In subcritical case $0 \leq \alpha < \frac{2}{n-1}$ for each $\alpha$, we have some $q > 2$ such that $(q, 2+\alpha)$ be an admissible pair, which is not the case when $\alpha = \frac{2}{n-1}$. To treat critical case, we adopt the truncation argument of Cazenave and Weissler [7]. To prove local existence, we truncate the nonlinearity $G$ and obtain solution for the truncated problem. We obtain solution $u$ for nonlinearity $G$ by using Strichartz estimates and by passing to the limit.

Now we state the main theorem, see [29].

**Theorem 7.3** *Let $f \in \tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$ and $G$ be as in (3) and (4) with $\alpha = \frac{2}{n-1}$ and $n \geq 2$. Initial value problem (1), (2) has unique maximal solution $u \in C((T_*,T^*),\tilde{W}^{1,2}_{\mathcal{L}}) \cap L^{q_1}_{loc}\left((T_*,T^*),\tilde{W}^{1,p_1}_{\mathcal{L}}(\mathbb{C}^n)\right)$, where $t_0 \in (T_*,T^*)$ and $(q_1,p_1)$ be an arbitrary admissible pair. Moreover the following properties hold:*

**(i)(Uniqueness)** *Solution is unique in $C((T_*,T^*),\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)) \cap L^{\gamma}((T_*,T^*),\tilde{W}^{1,\rho}_{\mathcal{L}})$ where $\rho = \frac{2n^2}{n^2-n+1}$, $\gamma = \frac{2n}{n-1}$.*

**(ii)(Blowup alternative)** *If $T^* < \infty$ then $\|u\|_{L^q\left((t_0,T^*),\tilde{W}^{1,p}_{\mathcal{L}}\right)} = \infty$ for every admissible pair $(q,p)$ with $2 < p$ and $\frac{1}{q} = n\left(\frac{1}{2} - \frac{1}{p}\right)$. Similar conclusion holds if $T_* > -\infty$.*

**(iii)(Stability)** *If $f_j \to f$ in $\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$ then $\|u - \tilde{u}_j\|_{L^q\left(I,\tilde{W}^{1,p}_{\mathcal{L}}(\mathbb{C}^n)\right)} \to 0$ as $j \to \infty$ for every admissible pair $(q,p)$ and every interval $I$ with $\overline{I} \subset (T_*,T^*)$, where $u, \tilde{u}_j$ are solutions corresponding to $f, f_j$ respectively.*

# 8. Global well posedness in $L^2(\mathbb{C}^n)$

Now we discuss global well posedness in $L^2(\mathbb{C}^n)$ for subcritical case $0 \leq \alpha < \frac{2}{n}$ (see [25]). However in critical case $\alpha = \frac{2}{n}$, we can prove global well posedness in $L^2(\mathbb{C}^n)$ only for sufficiently small intial value, see Remark 8.3.

## Subcritical Case $0 \leq \alpha < \frac{2}{n}$

**Theorem 8.1** *Let $u(\cdot, t_0) = f \in L^2(\mathbb{C}^n)$, $0 \leq \alpha < \frac{2}{n}$ and $G$ be as in (3) and (4). Initial value problem (1), (2) has unique maximal solution $u \in C((T_*, T^*), L^2) \cap L^{q_2}_{loc}((T_*, T^*), L^{p_2}(\mathbb{C}^n))$, where $t_0 \in (T_*, T^*)$ and $(q_2, p_2)$ be an arbitrary admissible pair. Fix $p = 2 + \alpha$. Moreover the following properties hold:*

**(i)(Uniqueness)** *Solution is unique in $C((T_*, T^*), L^2(\mathbb{C}^n)) \cap L^{q_2}_{loc}((T_*, T^*), L^p)$ where $q_2 \in [q_1, q]$, $\frac{1}{q} = n\left(\frac{1}{2} - \frac{1}{p}\right)$ and $q_1 = \frac{2p(p-1)}{2p+2n-np} \geq 1$.*

**(ii)(Blowup alternative)** *If $T^* < \infty$ then $\|u\|_{L^{q_2}((t_0,T^*),L^p(\mathbb{C}^n))} = \infty$ where $q_2 \in [q_1, q]$. Similar conclusion holds if $T_* > -\infty$.*

**(iii)(Stability)** *If $f_j \to f$ in $L^2(\mathbb{C}^n)$, then $u_j \to u$ in $L^{q_2}(I, L^{p_2}(\mathbb{C}^n))$ for every interval $I$ with $\overline{I} \subset (T_*, T^*)$ and for every admissible pair $(q_2, p_2)$, where $u_j$ and $u$ are solutions corresponding to $f_j$ and $f$ respectively.*

**(iv)(Conservation of charge and global existence)** *If $\psi$ is real valued, then we have conservation of charge $\|u(\cdot, t)\|_{L^2(\mathbb{C}^n)} = \|f\|_{L^2(\mathbb{C}^n)}$ for every $t \in (T_*, T^*)$. Moreover solution is global, i.e., $T_* = -\infty$ and $T^* = \infty$.*

## Critical Case $\alpha = \frac{2}{n}$

**Theorem 8.2** *Let $u(\cdot, t_0) = f \in L^2(\mathbb{C}^n)$, $\alpha = \frac{2}{n}$ and $G$ be as in (3) and (4). Initial value problem (1), (2) has unique maximal solution $u \in C((T_*, T^*), L^2(\mathbb{C}^n)) \cap L^{q_1}_{loc}((T_*, T^*), L^{p_1}(\mathbb{C}^n))$, where $t_0 \in (T_*, T^*)$ and $(q_1, p_1)$ be an arbitrary admissible pair. Fix $p = 2 + \alpha$. Moreover the following properties hold:*

**(i)(Uniqueness)** *Solution is unique in $C((T_*, T^*), L^2(\mathbb{C}^n)) \cap L^p((T_*, T^*), L^p(\mathbb{C}^n))$.*

**(ii)(Blowup alternative)** *If $T^* < \infty$ then $\|u\|_{L^p((t_0,T^*),L^p)} = \infty$. Similar conclusion holds if $T_* > -\infty$.*

**(iii)(Stability)** *If $f_j \to f$ in $L^2(\mathbb{C}^n)$, then $u_j \to u$ in $L^{q_1}(I, L^{p_1}(\mathbb{C}^n))$ for every interval $I$ with $\overline{I} \subset (T_*, T^*)$ and for every admissible pair $(q_1, p_1)$, where $u_j$ and $u$ are solutions corresponding to $f_j$ and $f$ respectively.*

**(iv)(Conservation of charge)** *If $\psi : \mathbb{R}^n \times \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ is real valued, then we have conservation of charge $\|u(\cdot, t)\|_{L^2(\mathbb{C}^n)} = \|f\|_{L^2(\mathbb{C}^n)}$ for every $t \in (T_*, T^*)$.*

**Remark 8.3** If $\|f\|_{L^2(\mathbb{C}^n)}$ is sufficiently small, then

$$\|e^{-i(t-t_0)\mathcal{L}} f\|_{L^p(I,L^p)} \leq C\|f\|_{L^2} < \delta$$

where $p = \frac{2(n+1)}{n}$. Since $C$ will not depend on $I$ and $t_0$ as long as $|I| \leq \pi$ and from conservation of charge, we get global solution, i.e., $-T_* = T^* = \infty$ in Theorem 8.2.

# 9. The case of the Laguerre operator

As discussed in Section 1, here we consider the Laguerre case. Laguerre operator $L_\beta$ on $\mathbb{R}^n_+ = (0, \infty)^n$ with $\beta \in (-\frac{1}{2}, \infty)^n$ is given by,

$$L_\beta = -\Delta - \textstyle\sum_{j=1}^n \left(\frac{2\beta_j+1}{x_j} \frac{\partial}{\partial x_j}\right) + \frac{|x|^2}{4}. \tag{9}$$

For each multi index $\mu \in \mathbb{Z}^n_{\geq 0}$ and $\beta \in (-\frac{1}{2}, \infty)^n$, the $n$-dimensional Laguerre functions are defined by the tensor product of 1-dimensional Laguerre functions

$$\psi^\beta_\mu(x) = \prod_{j=1}^n \psi^{\beta_j}_{\mu_j}(x_j), \;\; x \in \mathbb{R}^n_+$$

where $\psi^{\beta_j}_k(y) = \left(\frac{2^{-\beta_j} k!}{\Gamma(k+\beta_j+1)}\right)^{\frac{1}{2}} L^{\beta_j}_k(\frac{y^2}{2}) e^{-\frac{y^2}{4}}$, $y \in \mathbb{R}_+, k \geq 0$ and Laguerre polynomial $L^{\beta_j}_k(y)$ is given by the following

$$L^{\beta_j}_k(y) = \sum_{j=0}^k \frac{\Gamma(k+\beta_j+1)}{\Gamma(k-j+1)\Gamma(j+\beta_j+1)} \frac{(-y)^j}{j!}.$$

Laguerre functions $\psi^\beta_\mu(x)$ form a complete orthonormal family in $L^2(\mathbb{R}^n_+, d\nu)$ where $d\nu(x) = x_1^{2\beta_1+1} \cdots x_n^{2\beta_n+1} dx_1 \cdots dx_n$. Each $\psi^\beta_\mu$ is an eigenfunction of the

Laguerre operator $L_\beta$ with eigenvalue $\left(2|\mu| + \sum_{j=1}^n \beta_j + n\right)$.

## Schrödinger Propagator $e^{-itL_\beta}$

If $f \in C^2 \cap L^2(\mathbb{R}_+^n, d\nu)$ such that $L_\beta f \in L^2(\mathbb{R}_+^n, d\nu)$ then we observe that

$$\left\langle L_\beta f, \psi_\mu^\beta \right\rangle_\nu = \left\langle f, L_\beta \psi_\mu^\beta \right\rangle_\nu = \left(2|\mu| + n + \sum_{j=1}^n \beta_j\right) \left\langle f, \psi_\mu^\beta \right\rangle_\nu .$$

Therefore for $f \in L^2(\mathbb{R}_+^n, d\nu)$, we define $e^{-itL_\beta} f$ as $L^2(\mathbb{R}_+^n, d\nu)$ function by the following

$$e^{-itL_\beta} f = \sum_{k=0}^\infty e^{-it\left(2k+n+\sum_{j=1}^n \beta_j\right)} \sum_{|\mu|=k} \left\langle f, \psi_\mu^\beta \right\rangle_\nu \psi_\mu^\beta.$$

It is easy to see that $e^{-itL_\beta}$ is a unitary operator with adjoint operator $e^{itL_\beta}$ on $L^2(\mathbb{R}_+^n, d\nu)$.

**Remark 9.1** $e^{-itL_\beta} f$ is periodic in $t$ if and only if $\sum_{j=1}^n \beta_j$ is rational whereas $e^{it\sum \beta_j} e^{-itL_\beta} f$ and $|e^{-itL_\beta} f|$ are always periodic in $t$ with period $\leq 2\pi$.

## Strichartz estimates

**Definition 9.2** Let $n \geq 1$ and $\beta \in (-\frac{1}{2}, \infty)^n$. We say that a pair $(q, p)$ is *admissible* in the Laguerre case if

$$1 \leq q \leq 2, \quad 0 \leq \left(n + \sum_{j=0}^n \beta_j\right)\left(1 - \frac{2}{p}\right) < 1 \qquad \text{or}$$

$$2 < q \leq \infty \text{ and } 0 \leq \left(n + \sum_{j=0}^n \beta_j\right)\left(1 - \frac{2}{p}\right) \leq \frac{2}{q}.$$

**Remark 9.3**

**(i)** The admissibility condition on $(q,p)$ implies that

$$0 \le \left(n + \sum_{j=0}^{n} \beta_j\right)\left(1 - \frac{2}{p}\right) < 1.$$

**(ii)** If $1 \le q \le 2, n = 1, 1+\beta < 1$, then $p \in [2,\infty]$.

**(iii)** If $1 \le q \le 2, n = 1, 1+\beta = 1$, then $p \in [2,\infty)$.

**(iv)** If $1 \le q \le 2, \left(n + \sum_{j=0}^{n} \beta_j\right) > 1$, then $p \in \left[2, \frac{2\left(n+\sum_{j=0}^{n}\beta_j\right)}{\left(n+\sum_{j=0}^{n}\beta_j\right)-1}\right)$.

The main Strichartz type estimates is compiled in following theorem (see [28]).

**Theorem 9.4 (Strichartz Estimates)** *Let $(q,p),(q_1,p_1)$ be two admissible pairs according to definition 9.2, $(a,b)$ be a finite interval with $t_0 \in [a,b]$, $f \in L^2(\mathbb{R}^n_+, d\nu)$ and $g \in L^{q_1'}((a,b),\, L^{p_1'}(d\nu))$. Then the following estimates hold over $\mathbb{R}^n_+ \times (a,b)$:*

$$\|e^{-itL_\beta} f\|_{L^q((a,b),L^p(d\nu))} \le C\|f\|_{L^2(d\nu)} \qquad (10)$$

$$\left\|\int_{t_0}^{t} e^{-i(t-s)L_\beta} g(x,s)ds\right\|_{L^q((a,b),L^p(d\nu))} \le C\|g\|_{L^{q_1'}((a,b),\, L^{p_1'}(d\nu))} \qquad (11)$$

*where constant $C$ depends on admissible pairs but independent of $t_0$. Moreover $e^{-itL_\beta} f \in C(\mathbb{R}, L^2(\mathbb{R}^n_+, d\nu))$ and $\int_{t_0}^{t} e^{-i(t-s)L_\beta} g(x,s)ds \in C([a,b], L^2(\mathbb{R}^n_+, d\nu))$.*

**Remark 9.5** As similar to Remark 5.4, we can not expect the above Strichartz inequalities for unbounded intervals except when $q = \infty$. Also Since $|\sin t|$ is $\pi$ periodic, constant $C$ in the inequalities (10) and (11) can be chosen independent of interval $(a,b)$ provided $b - a \le \pi$.

# Local well posedness in $L^2(\mathbb{R}^n_+, d\nu)$

We consider the initial value problem for the nonlinear Schrödinger equation for the Laguerre operator $L_\beta$:

$$i\partial_t u(x,t) - L_\beta u(x,t) = G(x,u), \quad x \in \mathbb{R}^n_+,\ t \in \mathbb{R} \qquad (12)$$

$$u(x,t_0) = f(x) \qquad (13)$$

where nonlinearity $G$ is a function on $\mathbb{R}^n_+ \times \mathbb{C}$ satisfying similar conditions as in (3), (4).

Since $L_\beta$ has no decomposition in terms of first differential operators as the twisted Laplacian $\mathcal{L}$ has, we only consider the initial value in $L^2(\mathbb{R}^n_+, d\nu)$. As similar to the twisted Laplacian case, we can prove the local well posedness of the above IVP.

Now we discuss the local well posedness result for the above IVP for subcritical case $0 \leq \alpha < \frac{2}{n+\sum_{j=1}^n \beta_j}$ and critical case $\alpha = \frac{2}{n+\sum_{j=1}^n \beta_j}$.

## Subcritical case $0 \leq \alpha < \frac{2}{n+\sum_{j=1}^n \beta_j}$

Now we state the main Theorem for the subcritical case $0 \leq \alpha < \frac{2}{n+\sum_{j=1}^n \beta_j}$.

**Theorem 9.6** *Let $u(\cdot, t_0) = f \in L^2(\mathbb{R}^n_+, d\nu)$, $0 \leq \alpha < \frac{2}{n+\sum_{j=1}^n \beta_j}$ and $G$ be a function satisfying similar conditions as in (3), (4). Initial value problem (12), (13) has unique maximal solution $u \in C((T_*, T^*), L^2(\mathbb{R}^n_+, d\nu)) \cap L^{q_2}_{loc}((T_*, T^*), L^{p_2}(d\nu))$ for every admissible pair $(q_2, p_2)$, where $t_0 \in (T_*, T^*)$. Fix $p = 2 + \alpha$. Moreover the following properties hold:*

**(i)(Uniqueness)** *Solution is unique in $C((T_*, T^*), L^2(d\nu)) \cap L^{q_2}_{loc}((T_*, T^*), L^p(d\nu))$ where $q_2 \in [q_1, q]$ and*

$$\frac{1}{q} = \left(n + \sum_{j=1}^n \beta_j\right)\left(\frac{1}{2} - \frac{1}{p}\right), \quad q_1 = \frac{2p(p-1)}{2p - \left(n + \sum_{j=1}^n \beta_j\right)(p-2)} \geq 1.$$

**(ii)(Blowup alternative)** *If $T^* < \infty$ (respectively, $T_* > -\infty$), then $u \notin L^{q_2}((t_0,\, T^*), L^p(\mathbb{R}^n_+, d\nu))$ (respectively, $u \notin L^{q_2}((T_*, t_0), L^p(\mathbb{R}^n_+, d\nu))$) where $q_2 \in [q_1, q]$.*

**(iii)(Stability)** *If $f_j \to f$ in $L^2(\mathbb{R}^n_+, d\nu)$, then $u_j \to u$ in $L^{q_2}\left(I, L^{p_2}(\mathbb{R}^n_+, d\nu)\right)$ for every interval $I$ with $\overline{I} \subset (T_*, T^*)$ and for every admissible pair $(q_2, p_2)$, where $u_j$ and $u$ are solutions corresponding to $f_j$ and $f$ respectively.*

## Critical case $\alpha = \frac{2}{n+\sum_{j=1}^n \beta_j}$

Now we state the main theorem for the critical case $\alpha = \frac{2}{n+\sum_{j=1}^n \beta_j}$.

**Theorem 9.7** *Let $u(\cdot, t_0) = f \in L^2(\mathbb{R}^n_+, d\nu)$, $\alpha = \frac{2}{n+\sum_{j=1}^n \beta_j}$ and $G$ be a function satisfying similar conditions as in (3), (4). Initial value problem (12), (13) has unique maximal solution $u \in C((T_*, T^*), L^2(\mathbb{R}^n_+, d\nu)) \cap L^{q_1}_{loc}\left((T_*, T^*), L^{p_1}(\mathbb{R}^n_+, d\nu)\right)$*

*for every admissible pair* $(q_1, p_1)$*, where* $t_0 \in (T_*, T^*)$*. Fix* $p = 2 + \alpha$*. Moreover the following properties hold:*

**(i)(Uniqueness)** *Solution is unique in* $C((T_*, T^*), L^2(d\nu)) \cap L^p((T_*, T^*), L^p(d\nu))$.

**(ii)(Blowup alternative)** *If* $T^* < \infty$ *then* $\|u\|_{L^p((t_0,T^*),L^p(d\nu))} = \infty$*. Similar conclusion holds if* $T_* > -\infty$.

**(iii)(Stability)** *If* $f_j \to f$ *in* $L^2(\mathbb{C}^n)$*, then* $u_j \to u$ *in* $L^{q_1}\big(I, L^{p_1}(\mathbb{R}^n_+, d\nu)\big)$ *for every interval* $I$ *with* $\overline{I} \subset (T_*, T^*)$ *and for every admissible pair* $(q_1, p_1)$*, where* $u_j$ *and* $u$ *are solutions corresponding to* $f_j$ *and* $f$ *respectively.*

## 10. Publications/Preprints


1. P. K. Ratnakumar, V. K. Sohani, Nonlinear Schrödinger equation for the twisted Laplacian, J. Funct. Anal. 265 (1) (2013) 1-27.

2. V. K. Sohani, Strichartz Estimates for the Schrödinger propagator for the Laguerre Operator (accepted for publication in Proc. Indian Acad. Sci. (Math. Sci.), 2012).

3. P. K. Ratnakumar, V. K. Sohani, Nonlinear Schrödinger equation for the twisted Laplacian-global well posedness, communicated.

4. V. K. Sohani, Nonlinear Schrödinger equation for the twisted Laplacian in the critical case, communicated.

# Chapter 1

# Introduction

In this thesis we will study the well posedness problem for the nonlinear Schrödinger equation for the magnetic Laplacian on $\mathbb{R}^{2n}$, corresponding to constant magnetic field. The magnetic Laplacian in this case corresponds to the so called "twisted Laplacian" on $\mathbb{C}^n$. We establish the well posednes in certain first order Sobolev spaces associated to the twisted Laplacian, and also in $L^2(\mathbb{C}^n)$. In this connection, we also study Schrödinger equation for the ($n$-dimensional) Laguerre differential operator.

Schrödinger equations, arise in quantum mechanics as evolution equations describing the dynamics of the quantum particles. Hence the natural problem to study is the Cauchy problem: Find $u(\cdot,t)$ for any time $t$, for a given initial data $u(t_0) = f$ at time $t = t_0$.

A Cauchy problem is said to be locally well posed in a Banach space $B$, if for any given initial data $f = u(\cdot, t_0) \in B$, for $t = t_0$, there exists an interval $I$ containing $t_0$ and a unique solution $u \in C(I, B)$ to the Cauchy problem which is stable, i.e., depends continuously on the initial data. If $I = \mathbb{R}$, we say that the problem is globally well posed.

The Schrödinger equation is also an example of a dispersive equation, in the sense that the solutions spread out in space as time $t \to \infty$. This feature usually translates into a suitable decay estimate for the solution with respect to time $t$ as $t \to \infty$. For free Schrödinger equation on $\mathbb{R}^n$, this is given by the $L^1 \to L^\infty$ estimate of the form

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq |2t|^{-\frac{n}{2}} \|f\|_{L^1(\mathbb{R}^n)} \tag{1.0.1}$$

where $u(x,t)$ is given by (1.0.4). Such decay estimates are useful in the analysis of dispersive equations, especially in establishing Strichartz estimates, a very crucial tool in modern approach to dispersive equations, see [12].

## The Schrödinger equation

The free Schrödinger equation on $\mathbb{R}^n$ is the PDE

$$i\partial_t \psi(x,t) + \Delta \psi(x,t) = 0, \qquad x \in \mathbb{R}^n,\ t \in \mathbb{R}$$

which gives the quantum mechanical description of the evolution of a free particle in $\mathbb{R}^n$. If $\psi$ is the solution of the Schrödinger equation, then $|\psi(x,t)|^2$ is interpreted as the probability density for finding the position of the particle in $\mathbb{R}^n$ at a given time $t$. Let us consider the initial value problem

$$\begin{aligned} i\partial_t u(x,t) + \Delta u(x,t) &= 0, \qquad x \in \mathbb{R}^n,\ t \in \mathbb{R} && (1.0.2)\\ u(x,0) &= f(x). && (1.0.3) \end{aligned}$$

For $f \in L^2(\mathbb{R}^n)$, the solution is given by the Fourier transform:

$$u(x,t) = \int_{\mathbb{R}^n} e^{-it|\xi|^2}\, \hat{f}(\xi)\, e^{ix\xi}\, d\xi. \tag{1.0.4}$$

This may be written as a convolution operator

$$u(x,t) = (2it)^{-\frac{n}{2}} \left( f * e^{\frac{i|x|^2}{4t}} \right)(x) \tag{1.0.5}$$

which leads to the dispersive estimate mentioned in (1.0.1).

In view of (1.0.4) we write

$$u(x,t) = e^{it\Delta} f(x)$$

interpreting the Fourier inversion formula as the spectral decomposition in terms of the eigenfunctions of the Laplacian, see [30], [31]. Using Plancheral theorem in (1.0.4), we see that

$$\|u(\cdot,t)\|_2 = \|e^{it\Delta} f\|_2 = \|f\|_2 \tag{1.0.6}$$

which physically represents the charge conservation.

Now let us consider the inhomogeneous Schrödinger equation

$$
\begin{aligned}
i\partial_t u(x,t) + \Delta u(x,t) &= g(x,t), \qquad x \in \mathbb{R}^n,\ t \in \mathbb{R} &(1.0.7)\\
u(x,t_0) &= f(x). &(1.0.8)
\end{aligned}
$$

The solution in this case is given by the Duhamel's formua:

$$
u(x,t) = e^{i(t-t_0)\Delta} f - i \int_{t_0}^{t} e^{i(t-s)\Delta} g(x,s)ds. \tag{1.0.9}
$$

This can be seen as follows. Taking Fourier transform in the $x$-variable, we have

$$
\begin{aligned}
i\partial_t \hat{u}(\xi,t) - |\xi|^2 \hat{u}(\xi,t) &= \hat{g}(\xi,t)\\
i\partial_t (e^{it|\xi|^2} \hat{u}(\xi,t)) &= e^{it|\xi|^2} \hat{g}(\xi,t).
\end{aligned}
$$

Now integrate with respect to the $t$-variable on the interval $(t_0, t)$, we have

$$
\begin{aligned}
i(e^{it|\xi|^2} \hat{u}(\xi,t) - e^{it_0|\xi|^2} \hat{u}(\xi,t_0)) &= \int_{t_0}^{t} e^{is|\xi|^2} \hat{g}(\xi,s)ds\\
\hat{u}(\xi,t) = e^{-i(t-t_0)|\xi|^2} \hat{f}(\xi) &- i \int_{t_0}^{t} e^{-i(t-s)|\xi|^2} \hat{g}(\xi,s)ds.
\end{aligned}
$$

By taking inverse Fourier transform in the $\xi$-variable, this yields (1.0.9).

This formal computation suggests that $u$ given by the above equation should be a solution to the initial value problem (1.0.7), (1.0.8). This equivalence is crucial in local existence theory. In fact, we prove such equivalence for the twisted Laplacian $\mathcal{L}$ in Lemma 5.0.21.

More generally for any self adjoint differential operator $L$ on $\mathbb{R}^n$, having the spectral representation $L = \int_E \lambda \, dP_\lambda$, we can associate the Schrödinger propagator $\{e^{-itL} : t \in \mathbb{R}\}$ given by

$$
e^{-itL} f = \int_E e^{-it\lambda} dP_\lambda(f) \tag{1.0.10}
$$

for $f \in L^2(\mathbb{R}^n)$. Here $dP_\lambda$ denote the spectral projection for $L$, i.e., a projection valued measure supported on the spectrum $E$ of $L$, see [26].

In this case, the function $u(x,t) = e^{-itL}f(x)$ solves the initial value problem for the Schrödinger equation for the operator $L$:

$$\begin{aligned} i\partial_t u(x,t) - Lu(x,t) &= 0,\ x \in \mathbb{R}^n,\ t \in \mathbb{R} && (1.0.11) \\ u(x,0) &= f(x) && (1.0.12) \end{aligned}$$

with $L$ now representing the corresponding Hamiltonian of the quantum mechanical system.

The significance of this view point is that, most Hamiltonians of interest, namely the perturbation of the Laplacian with a potential $V$ (of the form $L = -\Delta + V$) or the magnetic Laplacian corresponding to the magnetic potential $(A_1(x), ..., A_n(x))$ (of the form $L = \sum_{j=1}^{n} \left(i\partial_{x_j} + A_j(x)\right)^2$) on $\mathbb{R}^n$, can be analysed, in terms of the spectral theory of the Hamiltonian, see [24] and [20, 21].

In this thesis, we concentrate on Schrödinger equation for an interesting magnetic Laplacian on $\mathbb{C}^n$ of the form $\sum\limits_{j=1}^{2n} \left[(i\partial_{w_j} - A_j(w))^2\right]$, corresponding to the magnetic vector potential $A(z) = \frac{iz}{2}, z \in \mathbb{C}^n$. This happens to be the twisted Laplacian on $\mathbb{C}^n$.

## Twisted Laplacian

The twisted Laplacian $\mathcal{L}$ on $\mathbb{C}^n$ is given by

$$\mathcal{L} = \frac{1}{2} \sum_{j=1}^{n} \left(Z_j \overline{Z}_j + \overline{Z}_j Z_j\right)$$

where $Z_j = \frac{\partial}{\partial z_j} + \frac{1}{2}\bar{z}_j,\ \overline{Z}_j = -\frac{\partial}{\partial \bar{z}_j} + \frac{1}{2} z_j,\ j = 1, 2, \ldots, n$. Here $\frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial \bar{z}_j}$ denote the complex derivatives $\frac{\partial}{\partial_{x_j}} \mp i\frac{\partial}{\partial_{y_j}}$ respectively. The operator $\mathcal{L}$ may be viewed as the complex analogue of the quantum harmonic oscillator Hamiltonian $H = -\Delta + |x|^2$ on $\mathbb{R}^n$, which has the representation

$$H = \frac{1}{2} \sum_{j=1}^{n} \left(A_j A_j^* + A_j^* A_j\right)$$

in terms of the creation operators $A_j = -\frac{\partial}{\partial x_j} + x_j$ and the annihilation operators $A_j^* = \frac{\partial}{\partial x_j} + x_j,\ j = 1, 2, \ldots, n$. The operator $\mathcal{L}$ was introduced by R. S. Strichartz

[31], and called the special Hermite operator and it looks quite similar to the Hermite operator on $\mathbb{C}^n$. In explicit terms the twisted Laplacian looks like

$$\mathcal{L} = -\Delta + \frac{1}{4}|z|^2 - i\sum_1^n \left( x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j} \right).$$

This may be re written as

$$\mathcal{L} = \sum_{j=1}^n \left[ \left( i\partial_{x_j} + \frac{y_j}{2} \right)^2 + \left( i\partial_{y_j} - \frac{x_j}{2} \right)^2 \right]$$

which is of the form $\sum_{j=1}^{2n} \left[ (i\partial_{w_j} - A_j(w))^2 \right]$ hence represents a Schrödinger operator on $\mathbb{C}^n$ for the magnetic vector potential $A(z) = \frac{iz}{2}, z \in \mathbb{C}^n$.

## Nonlinear Schrödinger equation for the twisted Laplacian

We consider the initial value problem for the nonlinear Schrödinger equation for the twisted Laplacian $\mathcal{L}$:

$$i\partial_t u(z,t) - \mathcal{L}u(z,t) = G(z,u), \qquad z \in \mathbb{C}^n,\ t \in \mathbb{R} \tag{1.0.13}$$

$$u(z,t_0) = f(z) \tag{1.0.14}$$

for $f \in L^2(\mathbb{C}^n)$. Here we consider the nonlinearity $G$ of the form

$$G(z,w) = \psi(x,y,|w|)\, w,\ (x,y,w) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{C}, \tag{1.0.15}$$

where $z = x+iy \in \mathbb{C}^n, w \in \mathbb{C}$ and $\psi \in C(\mathbb{R}^n \times \mathbb{R}^n \times [0,\infty)) \cap C^1(\mathbb{R}^n \times \mathbb{R}^n \times (0,\infty))$ satisfy the following inequality

$$|F(x,y,\eta)| \le C|\eta|^\alpha \tag{1.0.16}$$

with $F = \psi, \partial_{x_j}\psi,\ \partial_{y_j}\psi\ (1 \le j \le n)$ and $\eta\partial_\eta\psi(x,y,\eta),\ \alpha \ge 0$ and for some constant $C$. By mean value theorem, we see that

$$|G(z,u) - G(z,v)| \le |u-v|\, \Psi(u,v) \tag{1.0.17}$$

where $\Psi(u,v) = (|\eta\partial_\eta\psi(x,y,\eta)| + |\psi(x,y,\eta)|)\,|_{\eta=\theta|u|+(1-\theta)|v|}$ for some $0<\theta<1$. Notice that in view of the condition (1.0.16) on $\psi$, we have

$$|G(z,u) - G(z,v)| \leq C(|u|^\alpha + |v|^\alpha)|u-v| \tag{1.0.18}$$

for some constant $C$, where $u, v \in \mathbb{C}$ and $z \in \mathbb{C}^n$.

When $G \equiv 0$ and $f \in L^2(\mathbb{C}^n)$ the solution to this initial value problem is given by the Schrödinger propagator

$$u(z,t) = e^{-i(t-t_0)\mathcal{L}} f(z).$$

When $G(z,u) = g(z)$, the solution is given by the Duhamel's formula (see equation (1.0.9))

$$u(z,t) = e^{-i(t-t_0)\mathcal{L}} f(z) - i\int_{t_0}^{t} e^{-i(t-s)\mathcal{L}} g(z)ds. \tag{1.0.19}$$

Thus in the linear case, the solution is determined once the functions $f$ and $g$ are known.

The basic idea in nonlinear analysis is the following heuristic reasoning based on the above formula. If the solution $u$ is known, then one would expect $u$ to satisfy the above equation with $g(z)$ replaced by $G(z, u(z,s))$:

$$u(z,t) = e^{-i(t-t_0)\mathcal{L}} f(z) - i\int_{t_0}^{t} e^{-i(t-s)\mathcal{L}} G(z,u(z,s))ds. \tag{1.0.20}$$

Indeed one can show that $u$ from a reasonable function space satisfies a PDE of the form (1.0.13), (1.0.14), if and only if $u$ satisfies an integral equation of the form (1.0.20), see Lemma 5.0.21.

This reduces the existence theorem for the solution to the nonlinear Schrödinger equation to a fixed point theorem for the operator

$$\mathcal{H}(u)(z,t) = e^{-i(t-t_0)\mathcal{L}} f(z) - i\int_{t_0}^{t} e^{-i(t-s)\mathcal{L}} G(z,u(z,s))ds \tag{1.0.21}$$

in a suitable subset of the relevant function space.

## Conservation laws

Now we discuss the conservation of mass and energy for the magnetic Schrödinger equation (1.0.13). For proving mass conservation, we assume that $\psi$ is real valued. Taking $L^2(\mathbb{C}^n)$ inner product with $u$ on both sides of Schrödinger equation (1.0.13) and taking imaginary part, we get the mass conservation

$$\frac{d}{dt}\|u(\cdot,t)\|_2^2 = 0. \tag{1.0.22}$$

Taking $L^2(\mathbb{C}^n)$ inner product with respect to $\partial_t u$ on both sides and taking real part, we get the energy conservation

$$\frac{d}{dt}\left(\frac{1}{4}\sum_{j=1}^{n}(\|Z_j u(z,t)\|_2^2 + \left\|\overline{Z}_j u(z,t)\right\|_2^2) + \int_{\mathbb{C}^n}\tilde{G}(z,u)dz\right) = 0$$

where $\tilde{G} : \mathbb{C}^n \times [0,\infty) \to \mathbb{C}$ is given by the following

$$\tilde{G}(z,\tau) = \int_0^\tau \psi(z,s)s\,ds. \tag{1.0.23}$$

This leads to the conservation of the energy $E$:

$$E(u(\cdot,t)) = \frac{1}{4}\sum_{j=1}^{n}(\|Z_j u(\cdot,t)\|_2^2 + \left\|\overline{Z}_j u(\cdot,t)\right\|_2^2) + \int_{\mathbb{C}^n}\tilde{G}(z,|u|)dz. \tag{1.0.24}$$

In Theorem 6.0.33 in chapter 6 we prove that these formal identities are valid in the space of existence of the solution. If $G(z,u) = \lambda|u|^\alpha u$, then $\tilde{G}(z,|u|) = \frac{\lambda}{\alpha+2}|u|^{\alpha+2}$. Note that for each $z \in \mathbb{C}^n$, $\tilde{G}(z,\cdot) : [0,\infty) \to \mathbb{R}^2$ is a $C^1$ map and $\frac{\partial \tilde{G}}{\partial\sigma}(z,\sigma) = G(z,\sigma)$. Also note that by mean value theorem

$$\begin{aligned}|\tilde{G}(z,\sigma_1) - \tilde{G}(z,\sigma_2)| &= |\sigma_1 - \sigma_2|\ |G(z,\theta\sigma_1 + (1-\theta)\sigma_2)| \text{ where } \theta \in (0,1) \\ &\leq C|\sigma_1-\sigma_2|(|\sigma_1|^{1+\alpha} + |\sigma_2|^{1+\alpha}). \end{aligned} \tag{1.0.25}$$

In this thesis we will study the well posedness, i.e., local existence, uniqueness, stability and blowup alternative of the initial value problem (1.0.13), (1.0.14) with $f$ in certain first order Sobolev spaces related to the twisted Laplacian and also in $L^2(\mathbb{C}^n)$, see chapters 4, 5, 6, 7. This work is based on [24] (published in

J. Funct. Anal. 265 (1) (2013) 1-27) and [25, 29].

The Schrödinger equation for the magnetic potential with magnetic field decaying at infinity has been studied by many authors, see for instance Yajima [39], where author studies the propagator for the linear equation. In contrast, the nonlinear equation in our situation corresponds to a magnetic equation with a constant magnetic field, which has no decay. For more details on general magnetic Schrödinger equation corresponding to magnetic field without decay, see [1]. In [40] Zhang and Zheng also consider the nonlinear Schrödinger equation for the twisted Laplacian and with polynomial nonlinearity. They obtain product rule for fractional derivatives using Littlewood Paley theory and as a consequence prove the local well posedness result. There is a vast literature available for the well posedness results for nonlinear Schrödinger equation on $\mathbb{R}^n$. See for instance the papers by Ginibre and Velo [12, 13, 14], Kato [16], Cazenave and Weisler [6, 7, 8], Tsutsumi [36], Begout [2], Sjögren Torrea [27], the books by Cazenave [4] and Tao[32] and the extensive references there in. Some of the references that we came across dealing with magnetic Schrödinger equation are [39], [1] and [5] as mentioned before. In fact, the stability result discussed in [5], is actually the stability problem for the nonlinear Schrödinger equation for the twisted Laplacian in the plane.

The class of nonlinearity given by (1.0.15), (1.0.16) includes in particular, power type nonlinearity of the form $|u|^\alpha u$ and is also adaptable to the Schrödinger equation for the twisted Laplacian, for local existence via Kato's method [16]. The main difficulty in this approach is caused by the noncommutativity of $\mathcal{L}$ with $\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}$ and the noncompatibility of $\mathcal{L}$ with the powertype nonlinearty as observed in [5]. We are able to overcome this difficulty by introducing the appropriate set of differential operators $L_j, M_j$ and operators $Z_j, \overline{Z}_j$ $(1 \leq j \leq n)$ and working with suitable Sobolev spaces defined using these operators (see chapter 4 for definition).

We follow Kato's method [16] to prove the local existence in first order Sobolev spaces related to operators $L_j, M_j$ and operators $Z_j, \overline{Z}_j$ $(1 \leq j \leq n)$. Conservation laws have been an important tool for proving the existence of solutions of nonlinear Schrödinger equations, which is available for a large class of nonlinearities, see Ginibre and Velo [12]. In [12] Ginibre and Velo studied the Cauchy problem in the energy space for power type nonlinearities. T. Kato ([16]), introduced a method using Strichartz estimates which was applicable even for those nonlinear problems, where conservation laws are not available. In chapter 4 we observe that the operators

$$L_j = \left(\frac{\partial}{\partial x_j} + i\frac{y_j}{2}\right), \quad \text{and} \quad M_j = \left(\frac{\partial}{\partial y_j} - i\frac{x_j}{2}\right), \quad j = 1, 2, ..., n$$

commute with both the operators $e^{-it\mathcal{L}}$ and $\int_0^t e^{-i(t-s)\mathcal{L}}ds$, for $j = 1, 2, \ldots, n$. These operators are also compatible with the nonlinearity $G$ considered in (1.0.15), (1.0.16). Therefore we consider the following Banach space

$$\tilde{W}^{1,2}(\mathbb{C}^n) \;=\; \{f \in L^2(\mathbb{C}^n) : L_j f, M_j f \in L^2(\mathbb{C}^n), 1 \leq j \leq n\}$$

with norm $\|f\|_{\tilde{W}^{1,2}(\mathbb{C}^n)} = \|f\|_{L^2(\mathbb{C}^n)} + \sum_{j=1}^n \left(\|L_j f\|_{L^2(\mathbb{C}^n)} + \|M_j f\|_{L^2(\mathbb{C}^n)}\right)$. In chapter 5 we prove the local well posedness for initial value $f$ in $\tilde{W}^{1,2}(\mathbb{C}^n)$.

Observe that in view of (1.0.24), $\tilde{W}^{1,2}(\mathbb{C}^n)$ is not the energy space, therefore energy conservation is not possible in the above case. We overcome this situation by introducing the Sobolev space $\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$ (see chapter 4) defined using the operators $Z_j$ and $\overline{Z}_j$, which is the energy space and natural one in this context. Though they do not commute with $e^{-it\mathcal{L}}$, they have a reasonable commutation relation, suitable for our purpose. The advantage of working with this Sobolev space is that we get energy conservation in this case, see Theorem 6.0.33 in chapter 6. From this we can show that there is no finite time blow up in defocussing case (when $\psi$ is nonnegative) and also in focusing case (when $\psi$ is nonpositive) with $0 \leq \alpha < \frac{2}{n}$, hence in Theorem 6.0.33 we conclude the global existence in the Sobolev space $\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$.

In chapter 6 we also consider the critical case $\alpha = \frac{2}{n-1}$. In subcritical case $0 \leq \alpha < \frac{2}{n-1}$ for each $\alpha$, we have some $q > 2$ such that $(q, 2+\alpha)$ be an admissible pair, which may not be the case when $\alpha = \frac{2}{n-1}$. To treat the critical case, we adopt truncation argument of Cazenave and Weissler [7]. To prove local existence, we truncate the nonlinearity $G$ and obtain solution for the truncated problem. We obtain solution $u$ for the nonlinearity $G$ by using Strichartz estimates and by passing to the limit.

In chapter 7 we prove the global well posedness in $L^2(\mathbb{C}^n)$ for subcritical case $0 \leq \alpha < \frac{2}{n}$ using mass conservation. However in critical case $\alpha = \frac{2}{n}$, we can prove global well posedness in $L^2(\mathbb{C}^n)$ for intial value with sufficiently small norm in $L^2(\mathbb{C}^n)$. Our approach is based on Cazenave and Weissler [7].

Twisted Laplacian and Laguerre operator are closely related to each other in the following sense. If $f \in \mathcal{S}(\mathbb{C}^n)$ is radial then $\mathcal{L}f(z) = L_{n-1}f(r)$ where $L_{n-1}$ is 1-dimensional Laguerre operator of type $n-1$ given by (8.0.1) and $r = |z|$. More generally we can consider $n$-dimensional Laguerre operator $L_\beta$ on $\mathbb{R}^n_+ = (0,\infty)^n$ of type $\beta \in \left(-\frac{1}{2},\infty\right)^n$ which has singularity at $x_j = 0$, $1 \leq j \leq n$. Moreover special Hermite functions $\Phi_{\mu+\tilde{\mu},\mu}, \Phi_{\mu,\mu+\tilde{\mu}}$ on $\mathbb{C}^n$ with $\tilde{\mu} \in \mathbb{Z}^n_{\geq 0}$ are related with $n$-dimensional Laguerre functions $\psi^{\tilde{\mu}}_{\mu}$, see Theorem 1.3.4 and Theorem 1.3.5, page 19-20 in [33], where $\psi^{\tilde{\mu}}_{\mu}$ are given by (8.0.2). By similar analysis we also prove the local well posedness of the initial value problem for the Schrödinger equation with the Laguerre operator and initial value in $L^2(\mathbb{R}^n_+, d\nu)$ where $d\nu = \left(\prod_{j=1}^n x_j^{2\beta_j+1}\right) dx$, see chapter 8. This work is based on the Strichartz estimates for the Laguerre operator proved in Sohani [28] (to appear in Proc. Math. Sci.).

# Chapter 2

# Schrödinger propagator for the twisted Laplacian

Now we define the Schrödinger propagator $e^{-it\mathcal{L}}$ through the spectral theory of the twisted Laplacian. The twisted Laplacian is closely related to the sub Laplacian on the Heisenberg group, hence the spectral theory of this operator is closely connected with the representation theory of the Heisenberg group. Here we give a brief review of the spectral theory of the twisted Laplacian $\mathcal{L}$. The materials discussed here is based on the the following books: Folland [11], and Thangavelu [33, 34].

The eigenfunctions of the operator $\mathcal{L}$ are called the special Hermite functions, which are defined in terms of the Fourier-Wigner transform. For a pair of functions $f, g \in L^2(\mathbb{R}^n)$, the Fourier-Wigner transform is defined to be

$$V(f,g)(z) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f\left(\xi + \frac{y}{2}\right) \overline{g}\left(\xi - \frac{y}{2}\right) d\xi,$$

where $z = x + iy \in \mathbb{C}^n$. For any two multi-indices $\mu, \nu$ the special Hermite functions $\Phi_{\mu\nu}$ are given by

$$\Phi_{\mu\nu}(z) = V(h_\mu, h_\nu)(z)$$

where $h_\mu$ and $h_\nu$ are Hermite functions on $\mathbb{R}^n$. Recall that for each nonnegative integer $k$, the one dimensional Hermite functions $h_k$ are defined by

$$h_k(x) = \frac{(-1)^k}{\sqrt{2^k k! \sqrt{\pi}}} \left(\frac{d^k}{dx^k} e^{-x^2}\right) e^{\frac{x^2}{2}}.$$

Now for each multi index $\nu = (\nu_1, \cdots, \nu_n)$, the n-dimensional Hermite functions are defined by the tensor product :

$$h_\nu(x) = \prod_{i=1}^{n} h_{\nu_i}(x_i), \qquad x = (x_1, \cdots, x_n).$$

Since the Hermit functions satisfy the recursion relations

$$\begin{aligned} \left(-\frac{d}{dx} + x\right) h_k(x) &= (2k+2)^{\frac{1}{2}} h_{k+1}(x), \\ \left(\ \frac{d}{dx} + x\right) h_k(x) &= (2k)^{\frac{1}{2}} h_{k-1}(x), \end{aligned}$$

it follows that the special Hermit functions satisfy the relations

$$Z_j \Phi_{\mu,\nu} = i(2\nu_j)^{\frac{1}{2}} \Phi_{\mu,\nu-e_j}, \qquad \overline{Z}_j \Phi_{\mu,\nu} = -i(2\nu_j+2)^{\frac{1}{2}} \Phi_{\mu,\nu+e_j}. \qquad (2.0.1)$$

Since $\mathcal{L} = \frac{1}{2} \sum_{j=1}^{n} \left(Z_j \overline{Z}_j + \overline{Z}_j Z_j\right)$, it follows that $\Phi_{\mu\,\nu}$ are eigenfunctions of $\mathcal{L}$ with eigenvalue $2|\nu| + n$ and moreover, they form a complete orthonormal system in $L^2(\mathbb{C}^n)$. Thus every $f \in L^2(\mathbb{C}^n)$ has the expansion

$$f = \sum_{\mu,\,\nu} \langle f, \Phi_{\mu\,\nu} \rangle \Phi_{\mu\,\nu} \qquad (2.0.2)$$

in terms of the eigenfunctions of $\mathcal{L}$. The above expansion may be written as

$$f = \sum_{k=0}^{\infty} P_k f \qquad (2.0.3)$$

where

$$P_k f = \sum_{\mu,|\nu|=k} \langle f, \Phi_{\mu,\nu} \rangle \Phi_{\mu\nu} \qquad (2.0.4)$$

is the spectral projection corresponding to the eigenvalue $2k + n$. We also have the Plancheral theorem for the special Hermit expansion

$$\|f\|_2^2 = \sum_{k=0}^{\infty} \|P_k f\|_2^2. \qquad (2.0.5)$$

Now for any $f \in L^2(\mathbb{C}^n)$ such that $\mathcal{L}f \in L^2(\mathbb{C}^n)$, by self adjointness of $\mathcal{L}$, we have $P_k(\mathcal{L}f) = (2k+n)P_kf$. It follows that for $f \in L^2(\mathbb{C}^n)$ with $\mathcal{L}f \in L^2(\mathbb{C}^n)$

$$\mathcal{L}f = \sum_{k=0}^{\infty}(2k+n)P_kf. \tag{2.0.6}$$

Thus, we can define $e^{-it\mathcal{L}}$ as

$$e^{-it\mathcal{L}}f = \sum_{k=0}^{\infty} e^{-it(2k+n)}P_kf. \tag{2.0.7}$$

Note that $P_kf$ has the compact representation

$$P_kf(z) = (2\pi)^{-n}(f \times \varphi_k)(z) = (2\pi)^{-n}\int_{\mathbb{C}^n} f(z-w)\varphi_k(w)e^{\frac{i}{2}\operatorname{Im}(z.\overline{w})}dw \tag{2.0.8}$$

in terms of the Laguerre function $\varphi_k(z) = L_k^{n-1}(\frac{1}{2}|z|^2)e^{-\frac{1}{4}|z|^2}$, see [22, 33]. For $f \in L^2(\mathbb{C}^n)$, we have compact form $e^{-(r+it)\mathcal{L}}f = f \times K_{r+it}(z)$ (see [22]) where $r > 0$ and

$$K_{r+it}(z) \quad = \quad (2\pi)^{-n}e^{-n(r+it)}(1-e^{-2(r+it)})^{-n}e^{-\frac{1+e^{-2(r+it)}}{1-e^{-2(r+it)}}\cdot\frac{|z|^2}{4}}.$$

Let us consider sequence $\{r_m\}$ of positive real numbers converging to zero. We observe that $e^{-(r_m+it)\mathcal{L}}f \to e^{-it\mathcal{L}}f$ as $r_m \to 0$ in $L^2(\mathbb{C}^n)$ and therefore upto a subsequence $e^{-(r_m+it)\mathcal{L}}f(z) \to e^{-it\mathcal{L}}f(z)$ for a.e. $z$ as $r_m \to 0$. Since

$$|K_{r_m+it}(z)| \leq 2|\sin t|^{-n},$$

for $f \in L^1 \cap L^2(\mathbb{C}^n)$,

$$f \times K_{r_m+it}(z) \to f \times K_{it}(z) \quad \text{as} \quad r_m \to 0$$

for a.e. $z$. Hence we can express $e^{-it\mathcal{L}}$ as a twisted convolution operator:

$$e^{-it\mathcal{L}}f = f \times K_{it}$$

for $f \in L^1 \cap L^2(\mathbb{C}^n)$ where $K_{it}(z) = \frac{(4\pi i)^{-n}}{(\sin t)^n}e^{\frac{i(\cot t)|z|^2}{4}}$.

# Chapter 3

# Strichartz estimates

Strichartz estimates are useful for establishing existence of solution for semilinear Schrödinger and wave equations, in which no derivatives are present in the nonlinearity. Strichartz estimates were first proved by Strichartz [30] for solutions of Schrödinger and wave equations on $\mathbb{R}^n$. They were generalized by Ginibre and Velo [14, 15], Lindblad and Sogge [19]. In [17] Keel and Tao proved Strichartz estimates including endpoint for the wave and the Schrödinger equations.

Homogeneous Strichartz estimates for twisted Laplacian is proved by Ratnakumar in [22]. We begin with the following definitions of mixed $L^p$ space $L^q((a,b), L^p(\mathbb{C}^n))$, admissible pair and prove the Strichartz estimates. For mixed $L^p$ spaces we would like to refer to section 8.18 in Edwards [10].

**Definition 3.0.1** Let $n \geq 1$ and $1 \leq p, q \leq \infty$. We define $L^q((a,b), L^p(\mathbb{C}^n))$ by the following

$$L^q((a,b), L^p(\mathbb{C}^n)) = \{\text{g is measurable on } \mathbb{C}^n \times (a,b) : \|g\|_{L^q((a,b),L^p(\mathbb{C}^n))} < \infty\}$$

where $\|g\|_{L^q((a,b),L^p(\mathbb{C}^n))} = \left(\int_a^b \|g\|^q_{L^p(\mathbb{C}^n)} dt\right)^{\frac{1}{q}}$.

**Definition 3.0.2** Let $n \geq 1$. We say that a pair $(q,p)$ is *admissible* if

$$1 \leq q \leq 2, \quad 0 \leq n\left(\frac{1}{2} - \frac{1}{p}\right) < \frac{1}{2} \qquad \text{or}$$

$$2 < q \leq \infty \text{ and } 0 \leq n\left(\frac{1}{2} - \frac{1}{p}\right) \leq \frac{1}{q} < \frac{1}{2}.$$

**Remark 3.0.3** The admissibility condition on $(q,p)$ implies that $2 \leq p < \frac{2n}{n-1}$.

Admissible condition is basically coming from the following Lemma 3.0.4 and Remark 3.0.5 which are useful in proving Strichartz estimate (3.0.3). This Lemma was proved in [22] (see Lemma 2, p. 293-294) for compact interval $[-\pi,\pi]$, we state here for arbitrary compact interval $[a,b]$. Same proof will work here, so we skip the proof.

**Lemma 3.0.4** *Let $(a,b)$ be a bounded interval and $T$ be the operator given by*

$$Tf(t) = \int_a^b K(t-s)f(s)ds.$$

*Then the following inequality*

$$\|Tf\|_q \;\leq\; C_K\|f\|_{q'}$$

*holds for $q=\infty$ if $K \in L^\infty(a-b,b-a)$, for $q \in (2,\infty)$ if $K \in$ weak $L^{\frac{q}{2}}(a-b,b-a)$ and also for $1 \leq q \leq 2$ if $K \in L^1(a-b,b-a)$. The constant $C_K$ is independent of $f$ but depends on $K$ and interval $(a,b)$.*

**Remark 3.0.5** Let $p \in [2,\infty], a,b \in \mathbb{R}$ and $a < b$. $|\sin t|^{-2n\left(\frac{1}{2}-\frac{1}{p}\right)} \in$ weak $L^{\frac{q}{2}}(a-b,b-a)$ with $q \in (2,\infty)$ if $2 < q \leq \frac{1}{n(\frac{1}{2}-\frac{1}{p})}$ or $n(\frac{1}{2}-\frac{1}{p}) \leq \frac{1}{q} < \frac{1}{2}$. Also $|\sin t|^{-2n\left(\frac{1}{2}-\frac{1}{p}\right)} \in L^1(a-b,b-a)$ if $2n\left(\frac{1}{2}-\frac{1}{p}\right) < 1$. If we consider $p=2$ then $|\sin t|^{-2n\left(\frac{1}{2}-\frac{1}{p}\right)} = 1 \in L^\infty(a-b,b-a)$.

Now we state a Lemma which is helpful in proving Strichartz estimates (Theorem 3.0.7). For proof we refer to Lemma 3 in [22].

**Lemma 3.0.6** *Let $[a,b]$ be a bounded interval containing $t_0$. Let $h_j(z,t) \in L^{q_j'}((a,b),L^2(\mathbb{C}^n))$, where $q_j'$ is conjugate exponent of $q_j$ with $1 \leq q_j \leq \infty$ for $j=1,2$. Then the functions*

$$e^{-i(t-t_0)\mathcal{L}}h_1(z,t)e^{-i(s-t_0)\mathcal{L}}h_2(z,s), \quad h_1(z,t)e^{i(t-s)\mathcal{L}}h_2(z,s)$$

*belong to $L^1(\mathbb{C}^n \times (a,b) \times (a,b))$.*

The main Strichartz type estimates in this chapter is compiled in the following theorem. Homogeneous Strichartz estimate (3.0.1) is proved in [22]. For completeness, we also give the proof of the estimate (3.0.1). Our approach is similar to the Euclidean case discussed in Cazenave [4].

**Theorem 3.0.7 (Strichartz Estimates)** *Let* $(q,p),(q_1,p_1)$ *be two admissible pairs,* $(a,b)$ *a finite interval with* $t_0 \in [a,b]$, $f \in L^2(\mathbb{C}^n)$ *and* $g \in L^{q_1'}((a,b),\, L^{p_1'}(\mathbb{C}^n))$ *where* $q_1'$ *and* $p_1'$ *are conjugate exponents of* $q_1$ *and* $p_1$ *respectively. Then the following estimates hold over* $\mathbb{C}^n \times (a,b)$*:*

$$\|e^{-it\mathcal{L}}f\|_{L^q((a,b),L^p(\mathbb{C}^n))} \leq C\|f\|_2 \tag{3.0.1}$$

$$\left\|\int_{t_0}^{t} e^{-i(t-s)\mathcal{L}}g(z,s)ds\right\|_{L^q((a,b),L^p(\mathbb{C}^n))} \leq C\|g\|_{L^{q_1'}((a,b),L^{p_1'}(\mathbb{C}^n))} \tag{3.0.2}$$

*where the constant* $C$ *depends on admissible pairs and independent of* $t_0$*. Moreover* $e^{-it\mathcal{L}}f \in C(\mathbb{R}, L^2(\mathbb{C}^n))$ *and* $\int_{t_0}^{t} e^{-i(t-s)\mathcal{L}}g(z,s)ds \in C([a,b], L^2(\mathbb{C}^n))$*.*

**Remark 3.0.8** Note that $e^{-i(t-t_0)\mathcal{L}}f(z)$ is $2\pi$ periodic in $t$, hence we can not expect the above Strichartz inequalities for unbounded intervals except when $q=\infty$. Also Since $|\sin t|$ is $\pi$ periodic, in view of Remark 3.0.5, constant $C$ in the inequalities (3.0.1) and (3.0.2) can be chosen independent of the interval $(a,b)$ provided $b-a \leq \pi$.

**Proof**. We prove the Theorem in the following steps. In step 2 we prove estimate (3.0.1) and $e^{-it\mathcal{L}}f \in C(\mathbb{R}, L^2(\mathbb{C}^n))$, whereas in step 6 we prove estimate (3.0.2).

Step 1: We will prove estimate (3.0.2) when $(q,p)=(q_1,p_1)$. Using Minkowski's inequality for integrals and from Proposition 1 in [22], we get

$$\left\|\int_{t_0}^{t} e^{-i(t-s)\mathcal{L}}g(z,s)ds\right\|_{L^p(\mathbb{C}^n)} \leq C\int_a^b |\sin(t-s)|^{-2n(\frac{1}{2}-\frac{1}{p})}\|g(\cdot,s)\|_{L^{p'}(\mathbb{C}^n)}ds.$$

Now taking $L^q$-norm with respect to the $t$-variable on the interval $(a,b)$ and using Lemma 3.0.4 with Remark 3.0.5 we get the estimate (3.0.2) for $(q,p)=(q_1,p_1)$, i.e.,

$$\left\|\int_{t_0}^{t} e^{-i(t-s)\mathcal{L}}g(z,s)ds\right\|_{L^q((a,b),L^p(C^n))} \leq C\|g\|_{L^{q'}((a,b);\,L^{p'}(\mathbb{C}^n))} \tag{3.0.3}$$

Step 2: To prove estimate (3.0.1), we first prove the following estimate

$$\left\| \int_a^b e^{i(t-t_0)\mathcal{L}} g(z,t)dt \right\|_{L^2(\mathbb{C}^n)} \leq C\|g\|_{L^{q'}((a,b),L^{p'}(\mathbb{C}^n))}. \tag{3.0.4}$$

By density argument it is enough to prove estimate (3.0.4) for $g \in L^{q'}((a,b), L^2 \cap L^{p'}(\mathbb{C}^n))$. Since $e^{-it\mathcal{L}}$ is the adjoint of $e^{it\mathcal{L}}$ on $L^2(\mathbb{C}^n)$, from Lemma 3.0.6, the Hölder's inequality for the mixed $L^p$ spaces and the estimate (3.0.3), we get estimate (3.0.4):

$$\begin{aligned}
&\left\| \int_a^b e^{i(t-t_0)\mathcal{L}} g(\cdot,t)dt \right\|^2_{L^2(\mathbb{C}^n)} \\
&= \left\langle \int_a^b e^{i(t-t_0)\mathcal{L}} g(\cdot,t)dt, \int_a^b e^{i(s-t_0)\mathcal{L}} g(\cdot,s)ds \right\rangle \\
&= \int_a^b \int_a^b \left\langle e^{i(t-t_0)\mathcal{L}} g(\cdot,t), e^{i(s-t_0)\mathcal{L}} g(\cdot,s) \right\rangle dsdt \\
&= \int_a^b \int_a^b \left\langle g(\cdot,t), e^{-i(t-s)\mathcal{L}} g(\cdot,s) \right\rangle dsdt \\
&= \int_a^b \left\langle g(\cdot,t), \int_a^b e^{-i(t-s)\mathcal{L}} g(\cdot,s)ds \right\rangle dt \\
&\leq \|g\|_{L^{q'}((a,b),L^{p'}(\mathbb{C}^n))} \left\| \int_a^b e^{-i(t-s)\mathcal{L}} g(z,s)ds \right\|_{L^q((a,b),L^p(\mathbb{C}^n))} \\
&\leq C\|g\|^2_{L^{q'}((a,b),L^{p'}(\mathbb{C}^n))}.
\end{aligned}$$

Since $e^{-it\mathcal{L}}$ is unitary operator on $L^2(\mathbb{C}^n)$, the estimate (3.0.1) follows if $(q,p) = (\infty, 2)$. For $q < \infty$, estimate (3.0.1) follows from a duality argument, using estimate (3.0.4), Lemma 3.0.6 and the fact that $e^{-it\mathcal{L}}$ is adjoint operator of $e^{it\mathcal{L}}$ on $L^2(\mathbb{C}^n)$. Since $|e^{-it(2k+n)} - 1| \leq 2$ and $\|P_k f\|_{L^2(\mathbb{C}^n)} \in l^2(\mathbb{Z}_{\geq 0})$, $e^{-it\mathcal{L}} f(z) \in C(\mathbb{R}, L^2(\mathbb{C}^n))$ follows from the dominated convergence theorem.

Step 3: Now we will prove estimate (3.0.2) by using a duality argument in the $z$-variable when $(q,p) = (\infty, 2)$. By a density argument it is enough to prove the estimate (3.0.2) for $g \in L^{q_1'}((a,b), L^2 \cap L^{p_1'}(\mathbb{C}^n))$. Let $h \in \mathcal{S}(\mathbb{C}^n)$ with $\|h\|_{L^2(\mathbb{C}^n)} = 1$. By Hölder's inequality, Lemma 3.0.6, estimate (3.0.1) and the fact that $e^{-it\mathcal{L}}$ is the adjoint of $e^{it\mathcal{L}}$ on $L^2(\mathbb{C}^n)$, we get

$$\left| \left\langle \int_{t_0}^t e^{-i(t-s)\mathcal{L}} g(\cdot,s)ds, h \right\rangle \right|$$

$$
\begin{aligned}
&= \left| \int_{t_0}^{t} \left\langle g(\cdot,s), e^{i(t-s)\mathcal{L}} h \right\rangle ds \right| \\
&\leq \int_a^b \left| \left\langle g(\cdot,s), e^{i(t-s)\mathcal{L}} h \right\rangle \right| ds \\
&\leq \|g\|_{L^{q_1'}((a,b),L^{p_1'}(d\nu))} \left\| e^{-is\mathcal{L}}(e^{it\mathcal{L}} h) \right\|_{L^{q_1}((a,b)(ds),L^{p_1}(\mathbb{C}^n))} \\
&\leq C\|g\|_{L^{q_1'}((a,b),L^{p_1'}(\mathbb{C}^n))} \|e^{it\mathcal{L}} h\|_{L^2(\mathbb{C}^n)} \\
&= C\|g\|_{L^{q_1'}((a,b),L^{p_1'}(\mathbb{C}^n))} \|h\|_{L^2(\mathbb{C}^n)}.
\end{aligned}
$$

Taking supremum over all $h$ with $\|h\|_{L^2(\mathbb{C}^n)} = 1$ and then supremum over $t \in (a,b)$, we get the estimate

$$
\left\| \int_{t_0}^{t} e^{-i(t-s)\mathcal{L}} g(z,s) ds \right\|_{L^\infty((a,b),L^2(\mathbb{C}^n))} \leq C \|g\|_{L^{q_1'}((a,b);\, L^{p_1'}(\mathbb{C}^n))}. \tag{3.0.5}
$$

Step 4: Now we will prove estimate (3.0.2) when $(q_1, p_1) = (\infty, 2)$. Estimate (3.0.2) follows from estimate (3.0.5) if $(q,p) = (\infty, 2)$. So we assume that $(q,p) \neq (\infty, 2)$. To prove estimate (3.0.2), take $h \in L^{q'}\left((a,b), L^2 \cap L^{p'}(\mathbb{C}^n)\right)$. Now from Lemma 3.0.6 and the fact that $e^{-it\mathcal{L}}$ is the adjoint of $e^{it\mathcal{L}}$ on $L^2(\mathbb{C}^n)$, we observe the following

$$
\begin{aligned}
& \int_a^b \left\langle \int_{t_0}^{t} e^{-i(t-s)\mathcal{L}} g(\cdot,s) ds, h(\cdot,t) \right\rangle_{\mathbb{C}^n} dt \\
= \; & \left( \int_{t=a}^{t_0} \int_{s=t}^{t_0} + \int_{t=t_0}^{b} \int_{s=t_0}^{t} \right) \left\langle g(\cdot,s), e^{i(t-s)\mathcal{L}} h(\cdot,t) \right\rangle_{\mathbb{C}^n} ds\, dt \\
= \; & \left( \int_{s=a}^{t_0} \int_{t=a}^{s} + \int_{s=t_0}^{b} \int_{t=s}^{b} \right) \left\langle g(\cdot,s), e^{i(t-s)\mathcal{L}} h(\cdot,t) \right\rangle_{\mathbb{C}^n} dt\, ds \\
= \; & \int_a^{t_0} \left\langle g(\cdot,s), \int_a^s e^{-i(s-t)\mathcal{L}} h(\cdot,t) dt \right\rangle_{\mathbb{C}^n} ds \\
& + \int_{t_0}^{b} \left\langle g(\cdot,s), \int_s^b e^{-i(s-t)\mathcal{L}} h(\cdot,t) dt \right\rangle_{\mathbb{C}^n} ds.
\end{aligned}
$$

In view of estimate (3.0.5) and Hölder's inequality, we get the estimate

$$
\left\| \int_{t_0}^{t} e^{-i(t-s)\mathcal{L}} g(z,s) ds \right\|_{L^q((a,b),L^p(\mathbb{C}^n))} \leq C\|g\|_{L^1((a,b),L^2(\mathbb{C}^n))}. \tag{3.0.6}
$$

Step 5: Now we assume that $q, q_1 > 2$ and

$$\frac{1}{q} = n\left(\frac{1}{2} - \frac{1}{p}\right), \quad \frac{1}{q_1} = n\left(\frac{1}{2} - \frac{1}{p_1}\right).$$

In this case estimate (3.0.2) follows from bilinear Riesz-Thorin interpolation theorem and estimates (3.0.3), (3.0.5), (3.0.6) (see step 4 at page no. 36 in Cazenave [4]).

Step 6: To prove estimate (3.0.2), let us define $\tilde{q}, \tilde{q_1}$ by the following

$$\frac{1}{\tilde{q}} = n\left(\frac{1}{2} - \frac{1}{p}\right), \ \frac{1}{\tilde{q_1}} = n\left(\frac{1}{2} - \frac{1}{p_1}\right).$$

Then $1 \leq q \leq \tilde{q}$, $1 \leq q_1 \leq \tilde{q_1}$ and $2 < \tilde{q}, \tilde{q}_1$. By Hölder's inequality in the $t$-variable and step 4 we obtain estimate (3.0.2):

$$\begin{aligned}
\left\|\int_{t_0}^{t} e^{-i(t-s)\mathcal{L}} g ds\right\|_{L^q((a,b),L^p(\mathbb{C}^n))} &\leq (b-a)^{\kappa_1} \left\|\int_{t_0}^{t} e^{-i(t-s)\mathcal{L}} g ds\right\|_{L^{\tilde{q}}((a,b),L^p(\mathbb{C}^n))} \\
&\leq C(b-a)^{\kappa_1} \|g\|_{L^{\tilde{q_1}'}((a,b),L^{p_1'}(\mathbb{C}^n))} \\
&\leq C(b-a)^{\kappa_1+\kappa_2} \|g\|_{L^{q_1'}((a,b),L^{p_1'}(\mathbb{C}^n))} \\
&\leq C \|g\|_{L^{q_1'}((a,b),L^{p_1'}(\mathbb{C}^n))}
\end{aligned}$$

where

$$\kappa_1 = \frac{1}{q} - n\left(\frac{1}{2} - \frac{1}{p}\right), \ \kappa_2 = \frac{1}{q_1} - n\left(\frac{1}{2} - \frac{1}{p_1}\right).$$

Step 7: Now we prove $\int_{t_0}^{t} e^{-i(t-s)\mathcal{L}} g(z,s) ds \in C([a,b], L^2(\mathbb{C}^n))$. Let $t_m \to t$. Consider $h \in \mathcal{S}(\mathbb{C}^n)$ and we see that

$$\begin{aligned}
&\left|\left\langle \int_{t_0}^{t} e^{-i(t_m-s)\mathcal{L}} g(\cdot,s) ds - \int_{t_0}^{t} e^{-i(t-s)\mathcal{L}} g(\cdot,s) ds, h \right\rangle_{(\mathbb{C}^n)}\right| \\
&= \left|\int_{t_0}^{t} \left\langle e^{-i(t_m-s)\mathcal{L}} g(\cdot,s) - e^{-i(t-s)\mathcal{L}} g(\cdot,s), h\right\rangle_{(\mathbb{C}^n)} ds\right| \\
&= \left|\int_{t_0}^{t} \left\langle g(\cdot,s), \left(e^{i(t_m-s)\mathcal{L}} - e^{i(t-s)\mathcal{L}}\right) h\right\rangle_{(\mathbb{C}^n)} ds\right| \\
&\leq \int_a^b \|g(\cdot,s)\|_{L^{p'}(\mathbb{C}^n)} \|e^{-i(s-t_0)\mathcal{L}} \left(e^{i(t_m-t_0)\mathcal{L}} h - e^{i(t-t_0)\mathcal{L}} h\right)\|_{L^p(\mathbb{C}^n)} ds \\
&\leq \|g\|_{L^{q'}((a,b),L^{p'}(\mathbb{C}^n))} \|e^{-i(s-t_0)\mathcal{L}} \left(e^{i(t_m-t_0)\mathcal{L}} h - e^{i(t-t_0)\mathcal{L}} h\right)\|_{L^q((a,b)(ds),L^p(\mathbb{C}^n))} \\
&\leq C\|g\|_{L^{q'}((a,b),L^{p'}(\mathbb{C}^n))} \|\left(e^{i(t_m-t_0)\mathcal{L}} - e^{i(t-t_0)\mathcal{L}}\right) h\|_{L^2(\mathbb{C}^n)}.
\end{aligned}$$

By density of $\mathcal{S}(\mathbb{C}^n)$ in $L^2(\mathbb{C}^n)$, this shows that

$$\int_{t_0}^{t} e^{-i(t_m-s)\mathcal{L}} g(\cdot,s)ds \to \int_{t_0}^{t} e^{-i(t-s)\mathcal{L}} g(\cdot,s)ds$$

weakly in $L^2(\mathbb{C}^n)$. Also note that $L^2(\mathbb{C}^n)$ norm of this sequence is constant and equal to:

$$\begin{aligned}\left\|\int_{t_0}^{t} e^{-i(t_m-s)\mathcal{L}} g(\cdot,s)ds\right\|_{L^2(\mathbb{C}^n)} &= \left\|e^{-i(t_m-t)\mathcal{L}} \int_{t_0}^{t} e^{-i(t-s)\mathcal{L}} g(\cdot,s)ds\right\|_{L^2(\mathbb{C}^n)} \\ &= \left\|\int_{t_0}^{t} e^{-i(t-s)\mathcal{L}} g(\cdot,s)ds\right\|_{L^2(\mathbb{C}^n)}.\end{aligned}$$

Therefore we have the convergence in $L^2(\mathbb{C}^n)$:

$$\int_{t_0}^{t} e^{-i(t_m-s)\mathcal{L}} g(\cdot,s)ds \to \int_{t_0}^{t} e^{-i(t-s)\mathcal{L}} g(\cdot,s)ds.$$

Also since $\left\|\int_{t}^{t_m} e^{-i(t_m-s)\mathcal{L}} g(\cdot,s)ds\right\|_{L^2(\mathbb{C}^n)} \leq C\|g\|_{L^{q'}([t,t_m],L^{p'}(\mathbb{C}^n))} \to 0$ as $t_m \to t$, we conclude that $\int_{t_0}^{t} e^{-i(t-s)\mathcal{L}} g(z,s)ds \in C([a,b],L^2(\mathbb{C}^n))$.

# Chapter 4

# Some auxiliary function spaces

## The Sobolev space $\tilde{W}^{1,p}(\mathbb{C}^n)$

Let $L_j$ and $M_j$ be the differential operators defined by

$$L_j = \left(\frac{\partial}{\partial x_j} + i\frac{y_j}{2}\right) \quad \text{and} \quad M_j = \left(\frac{\partial}{\partial y_j} - i\frac{x_j}{2}\right), \quad j = 1, 2, ..., n. \tag{4.0.1}$$

We consider the space

$$\tilde{W}^{1,p}(\mathbb{C}^n) \;\; = \;\; \{f \in L^p(\mathbb{C}^n) : L_j f, M_j f \in L^p(\mathbb{C}^n), 1 \leq j \leq n\}$$

with norm $\|f\| = \|f\|_{L^p(\mathbb{C}^n)} + \sum_{j=1}^n \left(\|L_j f\|_{L^p(\mathbb{C}^n)} + \|M_j f\|_{L^p(\mathbb{C}^n)}\right)$. If $\{f_k\}$ is a Cauchy sequence in $\tilde{W}^{1,p}(\mathbb{C}^n)$ then there exists $f, g_j, h_j \in L^p(\mathbb{C}^n)$ such that $f_k \to f, L_j f_k \to g_j, M_j f_k \to h_j$ in $L^p(\mathbb{C}^n)$ as $k \to \infty$ for $1 \leq j \leq n$. Since $L_j, M_j$ are skew adjoint operators, it is easy to see that $L_j f = g_j, M_j f = h_j$ in $\mathcal{S}'(\mathbb{C}^n)$ for $1 \leq j \leq n$. This shows that $f \in \tilde{W}^{1,p}(\mathbb{C}^n)$ and $f_k \to f$ in $\tilde{W}^{1,p}(\mathbb{C}^n)$. Hence $\tilde{W}^{1,p}(\mathbb{C}^n)$ is a Banach space.

An interesting relation between the Sobolev space $\tilde{W}^{1,p}(\mathbb{C}^n)$ and the ordinary Sobolev space $W^{1,p}(\mathbb{C}^n)$ is the following: If $u \in \tilde{W}^{1,p}(\mathbb{C}^n)$, then $|u| \in W^{1,p}(\mathbb{C}^n)$.

**Lemma 4.0.9** *[Sobolev Embedding Theorem] We have the continuous inclusion*

$$\tilde{W}^{1,p_1}(\mathbb{C}^n) \hookrightarrow L^{p_2}(\mathbb{C}^n) \quad \begin{array}{ll} \text{for } p_1 \leq p_2 \leq \frac{2np_1}{2n-p_1} & \text{if } p_1 < 2n \\ \text{for } p_1 \leq p_2 < \infty & \text{if } p_1 = 2n \\ \text{for } p_1 \leq p_2 \leq \infty & \text{if } p_1 > 2n \end{array}$$

*where* $1 < p_1 < \infty$.

**Proof**. Let $f \in \tilde{W}^{1,p_1}(\mathbb{C}^n)$ and $\epsilon > 0$. Consider $u_\epsilon = e^{-\epsilon\mathcal{L}} f$. Note that (4.0.6) is also valid for $f \in L^{p_1}(\mathbb{C}^n)$. Since $K_\epsilon$ given by (4.0.7) is in $\mathcal{S}(\mathbb{C}^n)$ and from Lemma 4.0.17 $u_\epsilon = f \times K_\epsilon \in \tilde{W}^{1,p_1}(\mathbb{C}^n) \cap C^\infty(\mathbb{C}^n)$ and we have

$$2|u_\epsilon|\frac{\partial}{\partial x_j}|u_\epsilon| = \frac{\partial}{\partial x_j}(\overline{u_\epsilon}u_\epsilon) = 2\Re\left(\overline{u_\epsilon}\,\frac{\partial}{\partial x_j}u_\epsilon\right) = 2\Re\left(\overline{u_\epsilon}(\frac{\partial}{\partial x_j} + \frac{iy_j}{2})\,u_\epsilon\right).$$

Hence on the set $A_\epsilon = \{z \in \mathbb{C}^n \mid u_\epsilon(z) \neq 0\}$, we have

$$\left|\frac{\partial}{\partial x_j}|u_\epsilon|\right| = \left|\Re\left(\frac{\overline{u_\epsilon}}{|u_\epsilon|}(\frac{\partial}{\partial x_j} + \frac{iy_j}{2})\,u_\epsilon\right)\right| \leq |L_j(u_\epsilon)|\,.$$

Similarly $\left|\frac{\partial}{\partial y_j}|u_\epsilon|\right| \leq |M_j u_\epsilon|$ on $A_\epsilon$. Note that $\|u_\epsilon\|_{L^{p_2}(\mathbb{C}^n)} = \|u_\epsilon\chi_{A_\epsilon}\|_{L^{p_2}(\mathbb{C}^n)}$. By the usual Sobolev embedding on $\mathbb{C}^n$ and above observations, we have inequality $\|u_\epsilon\|_{L^{p_2}(\mathbb{C}^n)} \leq C\||u_\epsilon\chi_{A_\epsilon}|\|_{W^{1,p_1}} \leq C\,\|u_\epsilon\|_{\tilde{W}^{1,p_1}}$. Since $Se^{-\epsilon\mathcal{L}}f = e^{-\epsilon\mathcal{L}}Sf$ for $S = L_j, M_j (1 \leq j \leq n)$ (see Lemma 4.0.10), therfore by Lemma 4.0.17 $u_\epsilon = e^{-\epsilon\mathcal{L}}f \to f$ in $\tilde{W}^{1,p_1}(\mathbb{C}^n)$ and also in $L^{p_2}(\mathbb{C}^n)$ as $\epsilon \to 0$. Therefore we have $\|f\|_{L^{p_2}(\mathbb{C}^n)} \leq C\|f\|_{\tilde{W}^{1,p_1}(\mathbb{C}^n)}$. Hence Lemma is proved.

**Lemma 4.0.10** *Let* $f \in \mathcal{S}'(\mathbb{C}^n)$. *Then for every* $t, t_0 \in \mathbb{R}$, *we have the following equalities in* $\mathcal{S}'(\mathbb{C}^n)$

$$\begin{aligned} L_j e^{-i(t-t_0)\mathcal{L}} f &= e^{-i(t-t_0)\mathcal{L}} L_j f \\ M_j e^{-i(t-t_0)\mathcal{L}} f &= e^{-i(t-t_0)\mathcal{L}} M_j f. \end{aligned}$$

**Proof**. Since $f \in \mathcal{S}'(\mathbb{C}^n)$, $L_j e^{-i(t-t_0)\mathcal{L}} f$, $M_j e^{-i(t-t_0)\mathcal{L}} f \in \mathcal{S}'(\mathbb{C}^n)$. In view of (1.3.17), (1.3.18), (1.3.21) and (1.3.22) in [33], we have

$$\begin{aligned} L_j\Phi_{\mu,\nu} &= \frac{i}{2}\left((2\mu_j)^{\frac{1}{2}}\Phi_{\mu-e_j,\nu} + (2\mu_j+2)^{\frac{1}{2}}\Phi_{\mu+e_j,\nu}\right) \\ M_j\Phi_{\mu,\nu} &= \frac{1}{2}\left((2\mu_j)^{\frac{1}{2}}\Phi_{\mu-e_j,\nu} - (2\mu_j+2)^{\frac{1}{2}}\Phi_{\mu+e_j,\nu}\right). \end{aligned}$$

Since $L_j, M_j$ are skew adjoint operators and finite linear combination of special Hermite functions are dense in $\mathcal{S}(\mathbb{C}^n)$ (Theorem 1.4.4 in [34]), Lemma follows from the following observations

$$\begin{aligned}\left\langle L_j e^{-i(t-t_0)\mathcal{L}} f, \Phi_{\mu,\nu}\right\rangle &= \left\langle e^{-i(t-t_0)\mathcal{L}} L_j f, \Phi_{\mu,\nu}\right\rangle \\ \left\langle M_j e^{-i(t-t_0)\mathcal{L}} f, \Phi_{\mu,\nu}\right\rangle &= \left\langle e^{-i(t-t_0)\mathcal{L}} M_j f, \Phi_{\mu,\nu}\right\rangle\end{aligned}$$

for every $\mu, \nu \in (\mathbb{Z}_{\geq 0})^n$.

**Lemma 4.0.11** *Let $t_0 \in \mathbb{R}$ and $I$ an open interval containing $t_0$. Let $g \in L^{q'}_{loc}\left(I, \tilde{W}^{1,p'}(\mathbb{C}^n)\right)$, where $(q,p)$ be an admissible pair. Then $\int_{t_0}^t e^{-i(t-s)\mathcal{L}} g(z,s)ds \in C(I, \tilde{W}^{1,2}(\mathbb{C}^n))$. Moreover for each $t \in I$, we have the following equalities in $L^2(\mathbb{C}^n)$*

$$\begin{aligned}L_j \int_{t_0}^t e^{-i(t-s)\mathcal{L}} g(z,s)ds &= \int_{t_0}^t e^{-i(t-s)\mathcal{L}} L_j g(z,s)ds \\ M_j \int_{t_0}^t e^{-i(t-s)\mathcal{L}} g(z,s)ds &= \int_{t_0}^t e^{-i(t-s)\mathcal{L}} M_j g(z,s)ds.\end{aligned}$$

**Proof**. Since $g \in L^{q'}_{\text{loc}}\left(I, \tilde{W}^{1,p'}(\mathbb{C}^n)\right)$, by Strichartz estimates (Theorem 3.0.7) $\int_{t_0}^t e^{-i(t-s)\mathcal{L}} Sg(z,s)ds \in C(I, L^2(\mathbb{C}^n))$, where $S = L_j, M_j$, $1 \leq j \leq n$. In view of Theorem 1.4.4 in [34], Lemma follows from the following observations

$$\begin{aligned}\left\langle L_j \int_{t_0}^t e^{-i(t-s)\mathcal{L}} g(z,s)ds, \Phi_{\mu,\nu}\right\rangle &= \left\langle \int_{t_0}^t e^{-i(t-s)\mathcal{L}} L_j g(z,s)ds, \Phi_{\mu,\nu}\right\rangle \\ \left\langle M_j \int_{t_0}^t e^{-i(t-s)\mathcal{L}} g(z,s)ds, \Phi_{\mu,\nu}\right\rangle &= \left\langle \int_{t_0}^t e^{-i(t-s)\mathcal{L}} M_j g(z,s)ds, \Phi_{\mu,\nu}\right\rangle\end{aligned}$$

for every $\mu, \nu \in (\mathbb{Z}_{\geq 0})^n$.

## The Sobolev space $\tilde{W}^{1,p}_{\mathcal{L}}(\mathbb{C}^n)$

The local well posedness of the nonlinear Schrödinger equation for the twisted Laplacian is discussed in chapter 5 for initial values in $\tilde{W}^{1,2}(\mathbb{C}^n)$. However this approach does not give the energy conservation. We overcome this difficulty by introducing the Sobolev space $\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$ defined using the operators $Z_j$ and $\overline{Z}_j$

$$Z_j = \frac{\partial}{\partial z_j} + \frac{1}{2}\bar{z}_j,\ \overline{Z}_j = -\frac{\partial}{\partial \bar{z}_j} + \frac{1}{2} z_j,$$

which is the natural one in this context. Here $\frac{\partial}{\partial z_j}$, $\frac{\partial}{\partial \bar{z}_j}$ denote the complex derivatives $\frac{\partial}{\partial_{x_j}} \mp i\frac{\partial}{\partial_{y_j}}$ respectively.

**Definition 4.0.12** Let $m$ be a nonnegative integer and $1 \leq p < \infty$. We define space $\tilde{W}^{m,p}_{\mathcal{L}}(\mathbb{C}^n)$ by the following

$$\tilde{W}^{m,p}_{\mathcal{L}}(\mathbb{C}^n) = \{f \in L^p(\mathbb{C}^n) : S^\alpha f \in L^p(\mathbb{C}^n), |\alpha| \leq m\}$$

where $S^\alpha$ denotes monomial in $Z_1, \ldots, Z_n, \bar{Z}_1, \ldots, \bar{Z}_n$ of degree $|\alpha| = \alpha_1 + \cdots + \alpha_{2n}$. $\tilde{W}^{m,p}_{\mathcal{L}}(\mathbb{C}^n)$ is a Banach space with norm given by

$$\|f\|_{\tilde{W}^{m,p}_{\mathcal{L}}} = \sum_{|\alpha| \leq m} \|S^\alpha f\|_{L^p}.$$

**Lemma 4.0.13** *Let $f \in \tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$. Then we have*

$$\|f\|_{\tilde{W}^{1,2}_{\mathcal{L}}} \approx \sum_{1 \leq j \leq n} (\|Z_j f\|_{L^2} + \|\bar{Z}_j f\|_{L^2}).$$

**Proof**. Clearly

$$\|f\|_{\tilde{W}^{1,2}_{\mathcal{L}}} \geq \sum_{1 \leq j \leq n} (\|Z_j f\|_{L^2} + \|\bar{Z}_j f\|_{L^2}).$$

Now we show that

$$\|f\|_{\tilde{W}^{1,2}_{\mathcal{L}}} \leq 2 \sum_{1 \leq j \leq n} (\|Z_j f\|_{L^2} + \|\bar{Z}_j f\|_{L^2}).$$

Enough to show that $\|\bar{\mathbb{Z}}_j f\|_2 \geq \|f\|_2,\ 1 \leq j \leq n$. This follows from the Plancheral theorem for the special Hermite expansion

$$\|f\|_2^2 = \sum_{\mu,\nu} |\langle f, \phi_{\mu,\nu}\rangle|^2,$$

for $f \in L^2(\mathbb{C}^n)$. In view of (2.0.1) and $Z_j, \bar{Z}_j$ are adjoint of each other, we have

$$\bar{Z}_j f \quad = \quad \sum_{\mu,\,\nu} \langle \bar{Z}_j f, \Phi_{\mu\,\nu}\rangle \Phi_{\mu\,\nu} = -\sum_{\mu,\,\nu_j \geq 1} i(2\nu_j)^{\frac{1}{2}} \langle f, \Phi_{\mu\,\nu - e_j}\rangle \Phi_{\mu\,\nu}. \qquad (4.0.2)$$

Thus in view of equation (4.0.2), we have

$$\|\bar{Z}_j f\|_2^2 = \sum_{\mu,\,\nu_j \geq 1} 2\nu_j |\langle f, \Phi_{\mu\,\nu - e_j}\rangle|^2 = \sum_{\mu,\nu} (2\nu_j + 2)|\langle f, \Phi_{\mu,\nu}\rangle|^2 \geq \|f\|_2^2,$$

which completes the proof.
Though the operators $Z_j$ and $\bar{Z}_j$ $(1 \leq j \leq n)$ do not commute with $e^{-it\mathcal{L}}$, they have a reasonable commutation relation, suitable for our purpose, see Lemma 4.0.15. The advantage of working with this Sobolev space is that we get energy conservation in this case. Using this we can show that there is no finite time blow up in the defocussing case and also in the focusing case with $0 \leq \alpha < \frac{2}{n}$, which yields the global existence in the Sobolev space $\tilde{W}_{\mathcal{L}}^{1,2}(\mathbb{C}^n)$.

We have the following embedding theorem for the Sobolev space $\tilde{W}_{\mathcal{L}}^{1,p}(\mathbb{C}^n)$.

**Lemma 4.0.14** *( Sobolev Embedding Theorem ) We have the continuous inclusion*

$$\begin{array}{lll} \tilde{W}_{\mathcal{L}}^{1,p_1}(\mathbb{C}^n) \hookrightarrow L^{p_2}(\mathbb{C}^n) & \text{for } p_1 \leq p_2 \leq \frac{2np_1}{2n-p} & \text{if } p_1 < 2n \\ & \text{for } p_1 \leq p_2 < \infty & \text{if } p_1 = 2n \\ & \text{for } p_1 \leq p_2 \leq \infty & \text{if } p_1 > 2n \end{array}$$

*where* $1 < p_1 < \infty$.

**Proof**. Let $f \in \tilde{W}_{\mathcal{L}}^{1,p_1}(\mathbb{C}^n)$ and $\epsilon > 0$. Consider $u_\epsilon = e^{-\epsilon\mathcal{L}} f$. Then $u_\epsilon \in \tilde{W}_{\mathcal{L}}^{1,p_1}(\mathbb{C}^n) \cap C^\infty(\mathbb{C}^n)$ and we have

$$2|u_\epsilon| \frac{\partial}{\partial x_j}|u_\epsilon| = \frac{\partial}{\partial x_j}(\overline{u_\epsilon} u_\epsilon) = 2\Re\left(\overline{u_\epsilon}\frac{\partial}{\partial x_j} u_\epsilon\right) = 2\Re\left(\overline{u_\epsilon}(\frac{\partial}{\partial x_j} - \frac{iy_j}{2})\, u_\epsilon\right).$$

Note that

$$\frac{1}{2}\left(Z_j + \bar{Z}_j\right) = -i\left(\frac{\partial}{\partial y_j} + \frac{ix_j}{2}\right), \quad \frac{1}{2}\left(Z_j - \bar{Z}_j\right) = \left(\frac{\partial}{\partial x_j} - \frac{iy_j}{2}\right). \qquad (4.0.3)$$

Hence on the set $A_\epsilon = \{z \in \mathbb{C}^n \,|\, u_\epsilon(z) \neq 0\}$, we have

$$\left|\frac{\partial}{\partial x_j}|u_\epsilon|\right| = \left|\Re\left(\frac{\overline{u_\epsilon}}{|u_\epsilon|}(\frac{\partial}{\partial x_j} - \frac{iy_j}{2})\, u_\epsilon\right)\right| \leq \frac{1}{2}(|Z_j u_\epsilon| + |\bar{Z}_j u_\epsilon|).$$

Similarly $\left|\frac{\partial}{\partial y_j}|u_\epsilon|\right| \leq \frac{1}{2}(|Z_j u_\epsilon| + |\bar{Z}_j u_\epsilon|)$ on $A_\epsilon$. Note that $\|u_\epsilon\|_{L^{p_2}(\mathbb{C}^n)} = \|u_\epsilon \chi_{A_\epsilon}\|_{L^{p_2}(\mathbb{C}^n)}$. By the usual Sobolev embedding on $\mathbb{C}^n$ and above observations, we have inequality $\|u_\epsilon\|_{L^{p_2}(\mathbb{C}^n)} \leq C\| |u_\epsilon \chi_{A_\epsilon}| \|_{W^{1,p_1}} \leq C\, \|u_\epsilon\|_{\tilde{W}_{\mathcal{L}}^{1,p_1}}$. By Lemma 4.0.17 $u_\epsilon =$

$e^{-\epsilon\mathcal{L}}f \to f$ in $\tilde{W}^{1,p_1}_{\mathcal{L}}(\mathbb{C}^n)$ and also in $L^{p_2}(\mathbb{C}^n)$ as $\epsilon \to 0$. Therefore we have $\|f\|_{L^{p_2}(\mathbb{C}^n)} \leq C\|f\|_{\tilde{W}^{1,p_1}_{\mathcal{L}}(\mathbb{C}^n)}$, where constant $C$ is a generic constant independent of $f$. Hence Lemma is proved.

**Lemma 4.0.15** *(Quasi commutativity) Let $f \in \mathcal{S}'(\mathbb{C}^n)$. Then for every $t, t_0 \in \mathbb{R}$, we have the following equalities in $\mathcal{S}'(\mathbb{C}^n)$*

$$\begin{aligned} Z_j e^{-i(t-t_0)\mathcal{L}} f &= e^{-2i(t-t_0)} e^{-i(t-t_0)\mathcal{L}} Z_j f \\ \overline{Z}_j e^{-i(t-t_0)\mathcal{L}} f &= e^{2i(t-t_0)} e^{-i(t-t_0)\mathcal{L}} \overline{Z}_j f. \end{aligned}$$

**Proof**. Note that both $Z_j e^{-i(t-t_0)\mathcal{L}} f$ and $\bar{Z}_j e^{-i(t-t_0)\mathcal{L}} f$ are in $\mathcal{S}'(\mathbb{C}^n)$ for $f \in \mathcal{S}'(\mathbb{C}^n)$. Since every tempered distribution has a special Hermite expansion, enough to show the identities

$$\begin{aligned} \left\langle Z_j e^{-i(t-t_0)\mathcal{L}} f, \Phi_{\mu,\nu} \right\rangle &= e^{-2i(t-t_0)} \left\langle e^{-i(t-t_0)\mathcal{L}} Z_j f, \Phi_{\mu,\nu} \right\rangle \\ \left\langle \overline{Z}_j e^{-i(t-t_0)\mathcal{L}} f, \Phi_{\mu,\nu} \right\rangle &= e^{2i(t-t_0)} \left\langle e^{-i(t-t_0)\mathcal{L}} \overline{Z}_j f, \Phi_{\mu,\nu} \right\rangle \end{aligned}$$

for every $\mu, \nu \in (\mathbb{Z}_{\geq 0})^n$.

Since $Z_j$ and $\bar{Z}_j$ are adjoint of each other, both identities in the Lemma can be easily verified using the relations

$$e^{-i(t-t_0)\mathcal{L}} \overline{Z}_j \Phi_{\mu,\nu} = e^{2i(t-t_0)}\, \overline{Z}_j e^{-i(t-t_0)\mathcal{L}} \Phi_{\mu,\nu} \tag{4.0.4}$$

$$e^{-i(t-t_0)\mathcal{L}} Z_j \Phi_{\mu,\nu} = e^{-2i(t-t_0)}\, Z_j e^{-i(t-t_0)\mathcal{L}} \Phi_{\mu,\nu} \tag{4.0.5}$$

which follows from the relations (2.0.1) and the fact that $e^{-i\tau\mathcal{L}}\Phi_{\mu,\nu} = e^{-i\tau(2k+n)}\Phi_{\mu,\nu}$.

**Lemma 4.0.16** *(Quasi commutativity) Let $t_0 \in \mathbb{R}$ and $I$ an open interval such that $t_0 \in I$. Let $g \in L^{q'}_{loc}\left(I, \tilde{W}^{1,p'}_{\mathcal{L}}(\mathbb{C}^n)\right)$, where $(q, p)$ be an admissible pair. Then $\int_{t_0}^t e^{-i(t-s)\mathcal{L}} g(z,s)ds \in C(I, \tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n))$. Moreover for each $t \in I$, we have the following equalities in $L^2(\mathbb{C}^n)$*

$$\begin{aligned} Z_j \int_{t_0}^t e^{-i(t-s)\mathcal{L}} g(z,s)ds &= e^{-2it} \int_{t_0}^t e^{-i(t-s)\mathcal{L}} e^{2is} Z_j g(z,s)ds \\ \overline{Z}_j \int_{t_0}^t e^{-i(t-s)\mathcal{L}} g(z,s)ds &= e^{2it} \int_{t_0}^t e^{-i(t-s)\mathcal{L}} e^{-2is} \overline{Z}_j g(z,s)ds. \end{aligned}$$

**Proof**. Since $g \in L^{q'}_{\text{loc}}\left(I, \tilde{W}^{1,p'}_{\mathcal{L}}(\mathbb{C}^n)\right)$, by Strichartz estimates (Theorem 3.0.7) $\int_{t_0}^{t} e^{-i(t-s)\mathcal{L}} Sg(z,s)ds \in C(I, L^2(\mathbb{C}^n))$, where $S = Z_j, \bar{Z}_j$, $1 \le j \le n$. As discussed in Lemma 4.0.15, Lemma follows from the following observations

$$\begin{aligned}\left\langle Z_j \int_{t_0}^{t} e^{-i(t-s)\mathcal{L}} g(z,s)ds, \Phi_{\mu,\nu}\right\rangle &= e^{-2it}\left\langle \int_{t_0}^{t} e^{-i(t-s)\mathcal{L}} e^{2is} L_j g(z,s)ds, \Phi_{\mu,\nu}\right\rangle \\ \left\langle \bar{Z}_j \int_{t_0}^{t} e^{-i(t-s)\mathcal{L}} g(z,s)ds, \Phi_{\mu,\nu}\right\rangle &= e^{2it}\left\langle \int_{t_0}^{t} e^{-i(t-s)\mathcal{L}} e^{-2is} M_j g(z,s)ds, \Phi_{\mu,\nu}\right\rangle\end{aligned}$$

for every $\mu, \nu \in (\mathbb{Z}_{\ge 0})^n$. These identities can be easily verified using the relations (4.0.4), (4.0.5).

Now we discuss the heat operator associated with the twisted Laplacian. For $\epsilon > 0$, consider the heat operator for the twisted Laplacian given by

$$e^{-\epsilon\mathcal{L}} f = \sum_{k=0}^{\infty} e^{-\epsilon(2k+n)} P_k f$$

for $f \in L^2(\mathbb{C}^n)$. By orthogonality of the special Hermit functions $\Phi_{\mu,\nu}$, $P_k f$ are orthogonal projections. Hence it is clear that $e^{-\epsilon\mathcal{L}}$ is contraction on $L^2(\mathbb{C}^n)$.

$$\|e^{-\epsilon\mathcal{L}} f\|_2^2 = \sum_{k=1}^{\infty} e^{-2\epsilon(2k+n)} \|P_k f\|_2^2.$$

The heat operator $e^{-\epsilon\mathcal{L}}$ has the following integral representation as a twisted convolution operator

$$e^{-\epsilon\mathcal{L}} f = f \times K_\epsilon, \tag{4.0.6}$$

where

$$K_\epsilon(z) = (2\pi)^{-n} e^{-n\epsilon}(1 - e^{-2\epsilon})^{-n} e^{-\frac{(1+e^{-2\epsilon})}{(1-e^{-2\epsilon})}\frac{|z|^2}{4}} \tag{4.0.7}$$

see (2.10), (2.11) in [22]. Note that $\|K_\epsilon\|_{L^1(\mathbb{C}^n)} = 2^n e^{-n\epsilon}(1 + e^{-2\epsilon})^{-n} < 1$ and $\lim_{\epsilon\to 0} \|K_\epsilon\|_{L^1(\mathbb{C}^n)} = 1$.

**Lemma 4.0.17** *For $\epsilon > 0$, $e^{-\epsilon\mathcal{L}} : L^p(\mathbb{C}^n) \to \tilde{W}^{m,p}_{\mathcal{L}}(\mathbb{C}^n)$ defines a bounded operator for each nonnegative integer $m$ and $1 \le p \le \infty$. In particular we have the* *following inequalities:*

$$\|e^{-\epsilon\mathcal{L}}f\|_{L^p(\mathbb{C}^n)} \leq C\|f\|_{L^p(\mathbb{C}^n)} \quad (4.0.8)$$

$$\|e^{-\epsilon\mathcal{L}}f\|_{\tilde{W}^{1,p}_{\mathcal{L}}(\mathbb{C}^n)} \leq C\|f\|_{\tilde{W}^{1,p}_{\mathcal{L}}(\mathbb{C}^n)} \quad (4.0.9)$$

$$\|e^{-\epsilon\mathcal{L}}f\|_{\tilde{W}^{m,p}_{\mathcal{L}}(\mathbb{C}^n)} \leq C_\epsilon\|f\|_{L^p(\mathbb{C}^n)} \quad (4.0.10)$$

*with constant* $C$ *in (4.0.8) and (4.0.9) is independent of* $\epsilon \in (0,1]$. *Moreover, for* $f \in \tilde{W}^{m,p}_{\mathcal{L}}(\mathbb{C}^n)$, $e^{-\epsilon\mathcal{L}}f \to f$ *in* $\tilde{W}^{m,p}_{\mathcal{L}}(\mathbb{C}^n)$, $1 < p < \infty$.

**Proof**. In view of (4.0.6) and the fact that $|f \times g| \leq |f| * |g|$, we see that

$$|e^{-\epsilon\mathcal{L}}f| \leq |f| * K_\epsilon, \quad (4.0.11)$$

where $K_\epsilon$ is given by (4.0.7). Since

$$\|K_\epsilon\|_{L^1(\mathbb{C}^n)} = 2^n e^{-n\epsilon}(1 + e^{-2\epsilon})^{-n} \leq 1,$$

estimate (4.0.8) follows from Young's inequality, see Folland [11] with $C = 1$.

As in Lemma 4.0.15 we see that $e^{-\epsilon\mathcal{L}}f, Z_j e^{-\epsilon\mathcal{L}}f, \overline{Z}_j e^{-\epsilon\mathcal{L}}f \in \mathcal{S}'(\mathbb{C}^n)$, for $\epsilon > 0$, for $f \in \mathcal{S}'(\mathbb{C}^n)$, and the following equalities hold:

$$Z_j e^{-\epsilon\mathcal{L}}f = e^{2\epsilon}e^{-\epsilon\mathcal{L}}Z_j f, \quad \overline{Z}_j e^{-\epsilon\mathcal{L}}f = e^{-2\epsilon}e^{-\epsilon\mathcal{L}}\overline{Z}_j f, \quad (4.0.12)$$

hence the estimate (4.0.9) follows from the estimate (4.0.8). To prove (4.0.10), enough to prove

$$\|\tilde{S}e^{-\epsilon\mathcal{L}}f\|_{L^p(\mathbb{C}^n)} \leq C_\epsilon\|f\|_{L^p(\mathbb{C}^n)}$$

for any monomial $\tilde{S}$ in $(Z_1, \ldots, Z_n, \bar{Z}_1, \ldots, \bar{Z}_n)$ of degree at most $m$. In view of (4.0.6) and equation (1.3.10) in Thangavelu [33], we have

$$\tilde{S}e^{-\epsilon\mathcal{L}}f = \tilde{S}(f \times K_\epsilon) = f \times \tilde{S}K_\epsilon.$$

Since $K_\epsilon \in \mathcal{S}(\mathbb{C}^n)$, $\tilde{S}K_\epsilon \in L^1(\mathbb{C}^n)$, hence (4.0.10) follows by Young's inequality.

To prove the convergence in $\tilde{W}^{m,p}_{\mathcal{L}}(\mathbb{C}^n)$, we first observe that for $f \in L^2$, $e^{-\epsilon\mathcal{L}}f \to f$ in $L^2(\mathbb{C}^n)$ as $\epsilon \to 0$. This follows from the identity

$$\|e^{-\epsilon\mathcal{L}}f - f\|^2_{L^2(\mathbb{C}^n)} = \sum_{k=0}^{\infty} |1 - e^{-\epsilon(2k+n)}|^2 \|P_k f\|^2_{L^2(\mathbb{C}^n)}$$

by an application of the dominated convergence theorem applied to the sum.

First we consider the simple case $m = 0$. In view of the uniform estimate (4.0.8), enough to prove the convergence on a dense set in $L^p$. For $2 < p < \infty$, writing $\frac{1}{p} = \frac{\beta}{2} + \frac{1-\beta}{\infty} = \frac{\beta}{2}$ and an application of Hölder's inequality using the estimate (4.0.8), we see that

$$\begin{aligned} \|e^{-\epsilon\mathcal{L}}f - f\|_{L^p(C^n)} &\leq \|e^{-\epsilon\mathcal{L}}f - f\|^{\beta}_{L^2(C^n)} \|e^{-\epsilon\mathcal{L}}f - f\|^{1-\beta}_{L^\infty(C^n)} \\ &\leq \|e^{-\epsilon\mathcal{L}}f - f\|^{\beta}_{L^2(C^n)} (2\|f\|_{L^\infty(C^n)})^{1-\beta} \end{aligned}$$

which goes to zero as $\epsilon \to 0$, for $f \in L^2 \cap L^\infty(\mathbb{C}^n)$. Similarly we can prove convergence in $L^p(\mathbb{C}^n)$ for $1 < p < 2$.

For $m \neq 0$, we need to show $\tilde{S}(e^{-\epsilon\mathcal{L}}f - f) \to 0$ in $L^p(\mathbb{C}^n)$ as $\epsilon \to 0$. But in view of (4.0.12), we have

$$\tilde{S}(Z, \bar{Z})(e^{-\epsilon\mathcal{L}}f) = \tilde{S}(e^{2\epsilon}, e^{-2\epsilon})\; e^{-\epsilon\mathcal{L}}(\tilde{S}(Z, \bar{Z})f).$$

Hence applying the previous argument to the $L^p$ function $g = \tilde{S}(Z, \bar{Z})f$, and observing that $\tilde{S}(e^{2\epsilon}, e^{-2\epsilon}) \to 1$, the result follows.

# Chapter 5

# A local existence result

In this chapter we prove the local well posedness of the initial value problem (1.0.13), (1.0.14):

$$\begin{aligned} i\partial_t u(z,t) - \mathcal{L}u(z,t) &= G(z,u), \qquad\qquad z \in \mathbb{C}^n,\ t \in \mathbb{R} \\ u(z,t_0) &= f(z). \end{aligned}$$

in the first order Sobolev space $\tilde{W}^{1,2}(\mathbb{C}^n)$. The differential operators $L_j$ and $M_j$ are the natural ones adaptable to the power type nonlinearity $G(u) = |u|^\alpha u$ and the generality that we consider here. Moreover we have the embedding theorem (Lemma 4.0.9) and the operators $L_j, M_j$ commute with $e^{-i(t-t_0)\mathcal{L}}$ and $\int_{t_0}^t e^{-i(t-s)\mathcal{L}}$, see Lemma 4.0.10. Theorem 5.0.23 and Theorem 5.0.26 are main results of this chapter, see [24]. Now we prove some auxilliary estimates.

**Lemma 5.0.18** *Let $f \in \tilde{W}^{1,2}(\mathbb{C}^n)$ and $t_0 \in \mathbb{R}$. Then for every bounded interval $I$ and every admissible pair $(q_1,p_1)$, $t \to e^{-i(t-t_0)\mathcal{L}} f \in C(\mathbb{R}, \tilde{W}^{1,2}(\mathbb{C}^n)) \cap L^{q_1}_{loc}(\mathbb{R}, \tilde{W}^{1,p_1}(\mathbb{C}^n))$ and the following estimates hold :*

$$\|e^{-i(t-t_0)\mathcal{L}} f\|_{L^\infty\left(\mathbb{R},\tilde{W}^{1,2}(\mathbb{C}^n)\right)} = \|f\|_{\tilde{W}^{1,2}(\mathbb{C}^n)} \tag{5.0.1}$$

$$\|e^{-i(t-t_0)\mathcal{L}} f\|_{L^{q_1}\left(I,\tilde{W}^{1,p_1}(\mathbb{C}^n)\right)} \leq C\|f\|_{\tilde{W}^{1,2}(\mathbb{C}^n)} \tag{5.0.2}$$

*where the constant $C$ is independent of $f$ and $t_0$.*

**Proof**. Since both $L_j$ and $M_j$ commute with the isometry $e^{-i(t-t_0)\mathcal{L}}$, we have

$$Se^{-i(t-t_0)\mathcal{L}} f = e^{-i(t-t_0)\mathcal{L}} Sf, \quad \|Se^{-i(t-t_0)\mathcal{L}} f\|_{L^2(\mathbb{C}^n)} = \|Sf\|_{L^2(\mathbb{C}^n)}$$

for every $t \in \mathbb{R}$ with $S = L_j$, or $M_j$, $j = 1, 2, ..., n$ from which (5.0.1) follows. Estimate (5.0.2) follows from the Strichartz type estimate (3.0.2) for $e^{-i(t-t_0)\mathcal{L}}$ using the above commutativity.

**Lemma 5.0.19** *Let $I$ be a bounded interval and $(q, p)$ an admissible pair with $p = 2 + \alpha$ and $q > 2$. Let $G$ be as in (1.0.15), (1.0.16), $\alpha \in [0, \frac{2}{n-1})$ and $u, v \in L^\infty(I, \tilde{W}^{1,2}(\mathbb{C}^n))$. Then $u, v \in L^q(I, L^p(\mathbb{C}^n))$ and the following estimate holds*

$$\begin{aligned}\|G(z,u) - G(z,v)\|_{L^{q'}(I,L^{p'}(\mathbb{C}^n))} &\leq C|I|^{\frac{q-q'}{qq'}} \|u - v\|_{L^q(I,L^p(\mathbb{C}^n))} \\ &\times \left(\|u\|^\alpha_{L^\infty(I,\tilde{W}^{1,2})} + \|u\|^\alpha_{L^\infty(I,\tilde{W}^{1,2})}\right). \quad (5.0.3)\end{aligned}$$

**Proof**. Since $I$ is a bounded interval, in view of embedding theorem (Lemma 4.0.9), $u, v \in L^q(I, L^p(\mathbb{C}^n))$. By estimate (1.0.18), $\frac{1}{p'} = \frac{\alpha}{p} + \frac{1}{p}$, Holder's inequality in the $z$-variable and Sobolev embedding theorem (Lemma 4.0.9), we observe that

$$\begin{aligned}\|G(\cdot,u) - G(\cdot,v)\|_{L^{p'}(\mathbb{C}^n)} &\leq C\||u - v|(|u|^\alpha + |v|^\alpha)\|_{L^{p'}(\mathbb{C}^n)} \\ &\leq C\|(u - v)(\cdot,t)\|_{L^p(\mathbb{C}^n)} \left(\|u\|^\alpha_{L^p(\mathbb{C}^n)} + \|v\|^\alpha_{L^p(\mathbb{C}^n)}\right) \quad (5.0.4) \\ &\leq C\|(u - v)(\cdot,t)\|_{L^p(\mathbb{C}^n)} \left(\|u(\cdot,t)\|^\alpha_{\tilde{W}^{1,2}} + \|v(\cdot,t)\|^\alpha_{\tilde{W}^{1,2}}\right) \\ &\leq C\|(u - v)(\cdot,t)\|_{L^p(\mathbb{C}^n)} \left(\|u\|^\alpha_{L^\infty(I,\tilde{W}^{1,2})} + \|v\|^\alpha_{L^\infty(I,\tilde{W}^{1,2})}\right)\end{aligned}$$

where $t \in I$. Now by taking $L^{q'}$ norm in the $t$-variable on the interval $I$ in the above inequality, we get the required estimate (5.0.3).

**Proposition 5.0.20** *Let $t_0 \in \mathbb{R}$ and $I$ an open interval containing $t_0$. Let $G$ be as in (1.0.15), (1.0.16), $\alpha \in [0, \frac{2}{n-1})$ and $(q, p)$ an admissible pair with $p = \alpha + 2$, $q > 2$. If $u \in L^\infty_{loc}(I, \tilde{W}^{1,2}(\mathbb{C}^n)) \cap L^q_{loc}(I, \tilde{W}^{1,p}(\mathbb{C}^n))$, then $G(z, u(z,t)) \in L^{q'}_{loc}(I, \tilde{W}^{1,p'})$ and $\int_{t_0}^t e^{-i(t-s)\mathcal{L}} G(z, u(z,s))ds \in C(I, \tilde{W}^{1,2})$. Moreover, for every bounded interval $J$ with $\overline{J} \subset I$, $t_0 \in \overline{J}$ and every admissible pair $(q_1, p_1)$, the following inequalities hold:*

$$\begin{aligned}\|SG(z,u(z,t))\|_{L^{q'}(J,L^{p'}(\mathbb{C}^n))} &\leq C|J|^{\frac{q-q'}{qq'}} \|u\|^\alpha_{L^\infty(J,\tilde{W}^{1,2}(\mathbb{C}^n))} \quad (5.0.5) \\ &\times (\|u\|_{L^q(J,L^p(\mathbb{C}^n))} + \|Su\|_{L^q(J,L^p(\mathbb{C}^n))}) \\ \|G(z,u(z,t))\|_{L^{q'}(J,\tilde{W}^{1,p'}(\mathbb{C}^n))} &\leq C|J|^{\frac{q-q'}{qq'}} \|u\|^\alpha_{L^\infty(J,\tilde{W}^{1,2}(\mathbb{C}^n))} \quad (5.0.6) \\ &\times \|u\|_{L^q(J,\tilde{W}^{1,p}(\mathbb{C}^n))}\end{aligned}$$

$$\left\| \int_{t_0}^{t} e^{-i(t-s)\mathcal{L}} G(z,u(z,s))ds \right\|_{L^{q_1}(J,\tilde{W}^{1,p_1}(\mathbb{C}^n))} \leq C|J|^{\frac{q-q'}{qq'}} \|u\|^{\alpha}_{L^\infty(J,\tilde{W}^{1,2})} \times \|u\|_{L^q(J,\tilde{W}^{1,p}(\mathbb{C}^n))} \quad (5.0.7)$$

*where* $S = Id, L_j, M_j$ $(1 \leq j \leq n)$ *and the constant* $C$ *is independent of* $u$ *and* $t_0$.

**Proof**. To prove the inequality (5.0.5), we first observe that

$$\begin{aligned} L_j[\psi(x,y,|u|)u] &= \psi(x,y,|u|)\, L_j u + u\,(\partial_{|u|}\psi)(x,y,|u|)\, \Re\left(\frac{\bar{u}}{|u|} L_j u\right) \\ &+ u(\partial_{x_j}\psi)(x,y,|u|) \quad (5.0.8) \\ M_j[\psi(x,y,|u|)u] &= \psi(x,y,|u|)\, M_j u + u\,(\partial_{|u|}\psi)(x,y,|u|)\, \Re\left(\frac{\bar{u}}{|u|} M_j u\right) \\ &+ u(\partial_{y_j}\psi)(x,y,|u|). \quad (5.0.9) \end{aligned}$$

Thus we see that for $S = L_j$ and $M_j$, $|SG|$ satisfies an inequality of the form

$$|SG| \leq |\psi(x,y,|u|)\, Su| + |\tilde{\psi_1}(x,y,|u|)\, Su| + |\tilde{\psi_2}(x,y,|u|)\, u| \quad (5.0.10)$$

where $\tilde{\psi_1}(x,y,|u|) = u\partial_{|u|}\psi$ and $\tilde{\psi_2}(x,y,|u|) = u\partial_{x_j}\psi$ or $u\partial_{y_j}\psi$ depending on $S = L_j$ or $M_j$. Moreover, by assumption (1.0.16) on $\psi$, we have $|\tilde{\psi_i}(x,y,|u|)| \leq C|u|^\alpha,\ i = 1,2$. Therefore

$$|SG| \leq C|u|^\alpha(|u| + |Su|) \quad (5.0.11)$$

for $S = \mathrm{Id}, L_j, M_j; 1 \leq j \leq n$. Since $\frac{q'}{q} + \frac{q-q'}{q} = 1$, an application of the Hölder's inequality in the $t$-variable shows that for $q > 2$

$$\begin{aligned} \|SG(z,u(z,t))\|_{L^{q'}\left(J;L^{p'}(\mathbb{C}^n)\right)} &\leq |J|^{\frac{q-q'}{qq'}} \|SG(z,u(z,t))\|_{L^q(J;L^{p'}(\mathbb{C}^n))} \\ &\leq C|J|^{\frac{q-q'}{qq'}} \||u|^\alpha(|u| + |Su|)\|_{L^q(J;L^{p'})}. \quad (5.0.12) \end{aligned}$$

A further application of Hölder's inequality in the $z$-variable, using $\frac{p'}{p} + \frac{\alpha p'}{p} = 1$ and Lemma 4.0.9, we see that for a.e. $t \in J$

$$\begin{aligned} \||u|^\alpha(|u| + |Su|)\|_{L^{p'}(\mathbb{C}^n)} &\leq \|u(\cdot,t)\|^\alpha_{L^p(\mathbb{C}^n)} (\|u(\cdot,t)\|_{L^p(\mathbb{C}^n)} + \|Su(\cdot,t)\|_{L^p(\mathbb{C}^n)}) \\ &\leq C\|u(\cdot,t)\|^\alpha_{\tilde{W}^{1,2}(\mathbb{C}^n)} (\|u(\cdot,t)\|_{L^p(\mathbb{C}^n)} + \|Su(\cdot,t)\|_{L^p(\mathbb{C}^n)}) \\ &\leq C\|u\|^\alpha_{L^\infty\left(J,\tilde{W}^{1,2}\right)} (\|u(\cdot,t)\|_{L^p} + \|Su(\cdot,t)\|_{L^p}). \quad (5.0.13) \end{aligned}$$

Now taking $L^q$ norm with respect to the $t$-variable on both sides, and substituting in the RHS of inequality (5.0.12) gives the estimate (5.0.5).

Estimate (5.0.6) follows from the estimate (5.0.5). Estimate (5.0.7) follows from Strichartz estimates (Theorem 3.0.7) and the estimate (5.0.6). The fact that $\int_{t_0}^t e^{-i(t-s)\mathcal{L}} G(z,u(z,s))\ ds \in C(I,\tilde{W}^{1,2})$ follows from Lemma 4.0.11 and $G(z,u) \in L^{q'}_{\text{loc}}(I,\tilde{W}^{1,p'})$. Hence we have proved the Proposition.

# Equivalence of IVP and integral equation

In this section we will show the equivalence of the differential equations (1.0.13), (1.0.14) and the integral equation (1.0.20).

**Lemma 5.0.21** *Let $I$ be an open interval containing $t_0$ and $G$ as in (1.0.15), (1.0.16). Let $\alpha \in [0, \frac{2}{n-1}]$ for $n \geq 2$ and $\alpha \in [0,\infty)$ for $n = 1$. Let $u \in X$, where*

$$X = \{u \in C(I, L^2(\mathbb{C}^n)) : G(z,u(z,t)) \in L^{q_1'}_{loc}(I, L^{p_1'}(\mathbb{C}^n))\}$$

*for some admissible pair $(q_1, p_1)$. Then $u$ satisfies the nonlinear Schrödinger equation (1.0.13), with initial data (1.0.14) if and only if $u$ satisfies the integral equation (1.0.20).*

**Proof**. First observe that the following equalities

$$\frac{\partial}{\partial_t}(e^{-i(t-t_0)\mathcal{L}} f) = -i\mathcal{L}e^{-i(t-t_0)\mathcal{L}} f \tag{5.0.14}$$

$$\frac{\partial}{\partial_t}\int_{t_0}^t e^{-i(t-s)\mathcal{L}} G(z,s)ds = -i\mathcal{L}\int_{t_0}^t e^{-i(t-s)\mathcal{L}} G(z,s)ds + G(z,t) \tag{5.0.15}$$

are valid in the distribution sense for $f \in L^2(\mathbb{C}^n)$, $G \in L^{q_1'}_{\text{loc}}(I, L^{p_1'}(\mathbb{C}^n))$. Using these we now show the equivalence of the initial value problem (1.0.13), (1.0.14) and the integral equations (1.0.20).

If $u$ satisfies (1.0.20) then using (5.0.14) and (5.0.15), we conclude that $u$ satisfies (1.0.13) and (1.0.14).

On the otherhand, if $u$ satisfies (1.0.13) and (1.0.14) then the function $v$ given by

$$v(z,t) = u(z,t) - e^{-i(t-t_0)\mathcal{L}} f + i\int_{t_0}^t e^{-i(t-s)\mathcal{L}} G(z,s)ds,$$

satisfies

$$\begin{aligned} i\partial_t v(z,t) - \mathcal{L}v(z,t) = 0, \\ v(z,t_0) = 0. \end{aligned}$$

In view of Theorem 3.0.7 and given hypothesis, we have $v \in C(I, L^2(\mathbb{C}^n))$. Therefore $i\mathcal{L}^{-1}\partial_t v(z,t) - v(z,t) = 0$ and $\mathcal{L}^{-1}\partial_t v(z,t) \in C(I, L^2(\mathbb{C}^n))$. Now for each $\mu, \nu \in (\mathbb{Z}_{\geq 0})^n$, and $\phi \in C_c^\infty(I)$, we observe that

$$\begin{aligned} \left\langle \mathcal{L}^{-1}\partial_t v(\cdot,t), \Phi_{\mu,\nu} \right\rangle &= (2|\nu|+n)^{-1} \langle \partial_t v(\cdot,t), \Phi_{\mu,\nu} \rangle \in C(I) \quad \text{and} \\ \left\langle i\mathcal{L}^{-1}\partial_t v(\cdot,t), \Phi_{\mu,\nu}\phi \right\rangle &= i(2|\nu|+n)^{-1} \left\langle \frac{d}{dt} \langle v(\cdot,t), \Phi_{\mu,\nu} \rangle, \phi \right\rangle. \end{aligned}$$

Therefore we take inner product of the equation $i\mathcal{L}^{-1}\partial_t v(z,t) - v(z,t) = 0$ with $\Phi_{\mu,\nu}$ and observe that

$$\begin{aligned} \left\langle i\mathcal{L}^{-1}\partial_t v(\cdot,t) - v(\cdot,t), \Phi_{\mu,\nu} \right\rangle &= 0 \\ i(2|\nu|+n)^{-1}\frac{d}{dt} \langle v(\cdot,t), \Phi_{\mu,\nu} \rangle - \langle v(\cdot,t), \Phi_{\mu,\nu} \rangle &= 0 \\ \frac{d}{dt}\left(e^{i(2|\nu|+n)t} \langle v(\cdot,t), \Phi_{\mu,\nu} \rangle\right) &= 0. \end{aligned}$$

Since $t \to e^{i(2|\nu|+n)t} \langle v(\cdot,t), \Phi_{\mu,\nu} \rangle$ is continuous and its distributional derivative is zero, this function must be constant. This shows that

$$\langle v(\cdot,t), \Phi_{\mu,\nu} \rangle = e^{-i(2|\nu|+n)(t-t_0)} \langle v(\cdot,t_0), \Phi_{\mu,\nu} \rangle = 0$$

for every $\mu, \nu \in (\mathbb{Z}_{\geq 0})^n$. Therefore $v(\cdot,t) = 0$ in $L^2(\mathbb{C}^n)$ for each $t \in I$ and hence $u$ satisfies (1.0.20).

Now we prove (5.0.14) and (5.0.15). Let $\phi \in C_c^\infty(\mathbb{C}^n \times I)$. Since $I$ is an open interval, supp $\phi \subset A \times B$, for some compact set $A \subset \mathbb{C}^n$ and some compact interval $B \subset I$. Clearly,

$$\frac{\partial}{\partial_t}(e^{-i(t-t_0)\mathcal{L}}\bar{\phi}) = e^{-i(t-t_0)\mathcal{L}}\frac{\partial}{\partial_t}\bar{\phi} - e^{-i(t-t_0)\mathcal{L}} i\mathcal{L}\bar{\phi}.$$

Also since $\phi(z,\cdot)$ has compact support in $I$ for each $z$, $\int_I \frac{\partial}{\partial_t}(e^{-i(t-t_0)\mathcal{L}}\phi)dt = 0$, hence

$$\int_I e^{-i(t-t_0)\mathcal{L}}\ \partial_t\overline{\phi}\ dt = \int_I e^{-i(t-t_0)\mathcal{L}} i\mathcal{L}\overline{\phi}\ dt = i\ \overline{\int_I e^{i(t-t_0)\mathcal{L}}\mathcal{L}\phi\ dt}. \qquad (5.0.16)$$

Using this and the pairing $\langle f, \varphi\rangle = \int f\bar{\varphi}$, we see that

$$\begin{aligned}\int_{\mathbb{C}^n\times I} e^{-i(t-t_0)\mathcal{L}} f(z)\,\frac{\partial}{\partial_t}\overline{\phi(z,t)}\,dzdt &= \left\langle e^{-i(t-t_0)\mathcal{L}} f, \frac{\partial}{\partial_t}\phi\right\rangle = \left\langle f, e^{i(t-t_0)\mathcal{L}}\frac{\partial}{\partial_t}\phi\right\rangle \\ &= \left\langle f, -ie^{i(t-t_0)\mathcal{L}}\mathcal{L}\phi\right\rangle = \left\langle i\mathcal{L}e^{-i(t-t_0)\mathcal{L}} f, \phi\right\rangle.\end{aligned}$$

This proves (5.0.14) in the distribution sense.

To prove (5.0.15), choose a sequence $\{G_m\}$ in $C_c^\infty(A\times B)$ such that $G_m \to G$ in $L^{q_1'}(B, L^{p_1'}(A))$. Note that $G_m, \mathcal{L}G_m \in L^2(A\times B)$ hence,

$$\lim_{h\to 0}\frac{1}{h}\left[e^{-i(t+h-s)\mathcal{L}} - e^{-i(t-s)\mathcal{L}}\right] G_m(z,s) = -i\mathcal{L}\,e^{-i(t-s)\mathcal{L}} G_m(z,s)$$

and $\lim_{s\to t} e^{-i(t-s)\mathcal{L}} G_m(z,s) = G_m(z,t)$ where both the limits are taken in $L^2(\mathbb{C}^n)$ sense. Thus as an $L^2(\mathbb{C}^n)$ valued integral on $I$, we have

$$\begin{aligned}&\frac{\partial}{\partial_t}\int_{t_0}^t e^{-i(t-s)\mathcal{L}} G_m(z,s)ds \\ &= \lim_{h\to 0}\frac{1}{h}\left(\int_{t_0}^{t+h} e^{-i(t+h-s)\mathcal{L}} G_m(z,s)ds - \int_{t_0}^t e^{-i(t-s)\mathcal{L}} G_m(z,s)ds\right) \\ &= \lim_{h\to 0}\frac{1}{h}\int_{t_0}^{t+h}(e^{-i(t+h-s)\mathcal{L}} - e^{-i(t-s)\mathcal{L}}) G_m(z,s)ds \\ &\quad + \lim_{h\to 0}\frac{1}{h}\int_t^{t+h} e^{-i(t-s)\mathcal{L}} G_m(z,s)ds \\ &= -i\mathcal{L}\int_{t_0}^t e^{-i(t-s)\mathcal{L}} G_m(z,s)ds + G_m(z,t). \end{aligned} \qquad (5.0.17)$$

Observe that $\int_{t_0}^t e^{-i(t-s)\mathcal{L}} G_m(z,s)ds \to \int_{t_0}^t e^{-i(t-s)\mathcal{L}} G(z,s)ds$ in $L^{q_1}(B; L^{p_1}(A))$ as $m\to\infty$. This follows from Strichartz estimates (Theorem 3.0.7), since $B$ is a bounded interval. Thus using (5.0.17), we see that

$$\left\langle \int_{t_0}^t e^{-i(t-s)\mathcal{L}} G(z,s)ds, \frac{\partial}{\partial_t}\phi\right\rangle = \lim_{m\to\infty}\left\langle \int_0^t e^{-i(t-s)\mathcal{L}} G_m(z,s)ds, \frac{\partial}{\partial_t}\phi\right\rangle$$

$$
\begin{aligned}
&= \lim_{m\to\infty} \left\langle -\frac{\partial}{\partial_t} \int_0^t e^{-i(t-s)\mathcal{L}} G_m(z,s)ds, \phi \right\rangle \\
&= \lim_{m\to\infty} \left\langle i\mathcal{L} \int_0^t e^{-i(t-s)\mathcal{L}} G_m(z,s)ds - G_m(z,t), \phi \right\rangle \\
&= \lim_{m\to\infty} \left\langle \int_0^t e^{-i(t-s)\mathcal{L}} G_m(z,s)ds, -i\mathcal{L}\phi \right\rangle - \langle G(z,t), \phi \rangle \\
&= -\left\langle \int_0^t e^{-i(t-s)\mathcal{L}} G(z,s)ds, i\mathcal{L}\phi \right\rangle - \langle G(z,t), \phi \rangle.
\end{aligned}
$$

This shows that (5.0.15) holds in the distribution sense.

## Local existence

In this section we prove local existence of solutions in the first order Sobolev space $\tilde{W}^{1,2}(\mathbb{C}^n)$. We follow Kato's approach [16] using Strichartz estimates. The key step is to identify some complete metric space that lie in $L^\infty((t_0-T, t_0+T); \tilde{W}^{1,2}(\mathbb{C}^n))$, for a suitable $T$, where the operator $\mathcal{H}$ given by (1.0.21) is a contraction. We proceed as follows:

For given positive numbers $T$ and $M$, consider the set $E = E_{T,M}$ given by

$$
E = \left\{ u \in L^\infty\left(I_T; \tilde{W}^{1,2}\right) \cap L^q\left(I_T, \tilde{W}^{1,p}\right) \;\middle|\; \begin{array}{l} \|u\|_{L^\infty\left(I_T,\tilde{W}^{1,2}\right)} \le M, \\ \|u\|_{L^q\left(I_T,\tilde{W}^{1,p}\right)} \le M \end{array} \right\}
$$

where $I_T = (t_0 - T, t_0 + T)$. Introduce a metric on $E$, by setting

$$
d(u,v) = \|u-v\|_{L^\infty(I_T,L^2(\mathbb{C}^n))} + \|u-v\|_{L^q(I_T,L^p(\mathbb{C}^n))}.
$$

**Proposition 5.0.22** *$(E,d)$ is a complete metric space.*

**Proof**. Let $\{u_m\}$ be a Cauchy sequence in $(E,d)$. Then $\{u_m\}$ be a Cauchy sequence in $L^\infty(I_T, L^2(\mathbb{C}^n))$ and $L^q(I_T, L^p(\mathbb{C}^n))$. Since these spaces are complete (see section 8.18.1 in [10]), there exists $u \in L^\infty(I_T, L^2(\mathbb{C}^n)) \cap L^q(I_T, L^p(\mathbb{C}^n))$ and $u_m \to u$ in $L^\infty(I_T, L^2(\mathbb{C}^n))$ and also in $L^q(I_T, L^p(\mathbb{C}^n))$.

We need to show that $u \in L^\infty(I_T; \tilde{W}^{1,2}(\mathbb{C}^n)) \cap L^q(I_T; \tilde{W}^{1,p}(\mathbb{C}^n))$ with

$$
\max\{\|u\|_{L^\infty(I_T,\tilde{W}^{1,2}(\mathbb{C}^n))}, \|u\|_{L^q(I_T,\tilde{W}^{1,p}(\mathbb{C}^n))}\} \le M.
$$

Let $S = \mathrm{Id},\ L_j$ or $M_j$ with $1 \le j \le n$ and $\varphi \in C_c^\infty(\mathbb{C}^n \times I_T)$. Then for fixed

$t \in I_T$, using the pairing $\langle , \rangle_z$ in the $z$-variable, we see that

$$\begin{aligned}|\langle u(\cdot,t), S^*\varphi(\cdot,t)\rangle_z| &\leq |\langle (u-u_m)(\cdot,t), S^*\varphi(\cdot,t)\rangle| + |\langle Su_m(\cdot,t), \varphi(\cdot,t)\rangle| \\ &\leq \|u(\cdot,t) - u_m(\cdot,t)\|_{L^p(\mathbb{C}^n,dz)} \|S^*\varphi(\cdot,t)\|_{L^{p'}(\mathbb{C}^n,dz)} \\ &\quad + \|Su_m(\cdot,t)\|_{L^p(\mathbb{C}^n,dz)} \|\varphi(\cdot,t)\|_{L^{p'}(\mathbb{C}^n,dz)}.\end{aligned}$$

Integrating with respect to $t$ and applying the Hölder's inequality in the $t$-variable, this yields

$$|\langle Su, \varphi\rangle_{z,t}| \leq \|u-u_m\|_{L^q(I_T,L^p(\mathbb{C}^n))} \|S^*\varphi\|_{L^{q'}(I_T,L^{p'}(\mathbb{C}^n))} + \|Su_m\|_{L^q(I_T,L^p(\mathbb{C}^n))} \|\varphi\|_{L^{q'}(I_T,L^{p'})}.$$

Since $u_m \to u \in L^q(I_T, L^p(\mathbb{C}^n))$, letting $m \to \infty$, we get

$$|\langle Su, \varphi\rangle_{z,t}| \leq \liminf_{m\to\infty} \|Su_m\|_{L^q(I_T,L^p(\mathbb{C}^n))} \|\varphi\|_{L^{q'}(I_T,L^{p'}(\mathbb{C}^n))}.$$

Taking supremum over all $\varphi \in C_c^\infty(\mathbb{C}^n \times I_T)$ with $\|\varphi\|_{L^{q'}(I_T,L^{p'}(\mathbb{C}^n))} \leq 1$, this gives

$$\|Su\|_{L^q(I_T;L^p)} \leq \liminf_{m\to\infty} \|Su_m\|_{L^q(I_T,L^p(\mathbb{C}^n))}. \tag{5.0.18}$$

Therefore

$$\|u\|_{L^q\left(I_T;\tilde{W}^{1,p}\right)} \leq \liminf_{m\to\infty} \|u_m\|_{L^q(I_T,\tilde{W}^{1,p}(\mathbb{C}^n))} \leq M.$$

To get estimate for the pair $(\infty, 2)$, take $\varphi \in C_c^\infty(\mathbb{C}^n)$, and by the same arguments as before

$$|\langle Su(\cdot,t), \varphi\rangle_z| \leq \liminf_{m\to\infty} \|Su_m(\cdot,t)\|_{L^2(\mathbb{C}^n)} \|\varphi\|_{L^2(\mathbb{C}^n)}$$

for almost every $t \in I_T$. Taking supremum over all $\varphi \in C_c^\infty(\mathbb{C}^n)$ with $\|\varphi\|_{L^2} \leq 1$ this gives

$$\|Su(\cdot,t)\|_{L^2(\mathbb{C}^n)} \leq \liminf_{m\to\infty} \|Su_m\|_{L^\infty(I_T,L^2(\mathbb{C}^n))}. \tag{5.0.19}$$

Taking the essential supremum over $t \in I_T$, we get

$$\|Su\|_{L^\infty(I_T;L^2)} \leq \liminf_{m\to\infty} \|Su_m\|_{L^\infty(I_T,L^2(\mathbb{C}^n))}.$$

Therefore

$$\|u\|_{L^\infty\left(I_T;\tilde{W}^{1,2}\right)} \leq \liminf_{m\to\infty} \|u_m\|_{L^\infty(I_T,\tilde{W}^{1,2}(\mathbb{C}^n))} \leq M.$$

Now we prove the following theorem in which we give the existence of soultion of the initial value problem (1.0.13), (1.0.14).

**Theorem 5.0.23** *(Local existence) Assume that $G$ is as in (1.0.15), (1.0.16) with $\alpha \in [0, \frac{2}{n-1})$ and $u(z,t_0) = f(z) \in \tilde{W}^{1,2}(\mathbb{C}^n)$. Then there exist a number $T = T(\|u_0\|)$ such that the initial value problem (1.0.13), (1.0.14) has a unique solution $u \in C([t_0 - T, t_0 + T]; \tilde{W}^{1,2}(\mathbb{C}^n)) \cap L^q((t_0 - T, t_0 + T), \tilde{W}^{1,p}(\mathbb{C}^n))$, where $(q,p)$ be an admissible pair with $p = 2 + \alpha, q > 2$.*

**Proof**. In view of Lemma 5.0.21, we show existence of solution by showing that operator $\mathcal{H}$ given by (1.0.21) has fixed point in the complete metric space $E$ for suitable $T > 0$ and $M > 0$. Let $u \in E$. In view of equation (1.0.21) and from estimates in Lemma 5.0.18 and Proposition 5.0.20, we see that

$$\begin{aligned}
&\max\{\|\mathcal{H}u\|_{L^\infty\left(I_T,\tilde{W}^{1,2}\right)}, \|\mathcal{H}u\|_{L^q\left(I_T,\tilde{W}^{1,p}\right)}\} \\
&\leq C\,\|f\|_{\tilde{W}^{1,2}} + C\,T^{\frac{q-q'}{qq'}}\,\|u\|^\alpha_{L^\infty(I_T;\tilde{W}^{1,2})}\|u\|_{L^q(I_T,\tilde{W}^{1,p}(\mathbb{C}^n))} \\
&\leq C\,\|f\|_{\tilde{W}^{1,2}} + C\,T^{\frac{q-q'}{qq'}}\,M^{1+\alpha}.
\end{aligned} \tag{5.0.20}$$

For $u, v \in E$, using Strichartz estimate (3.0.2) and Lemma (5.0.19), we get

$$\begin{aligned}
d(\mathcal{H}u, \mathcal{H}v) &= \|\mathcal{H}u - \mathcal{H}v\|_{L^\infty(I_T,L^2)} + \|\mathcal{H}u - \mathcal{H}v\|_{L^q(I_T,L^p)} \\
&\leq C\|G(z,u) - G(z,v)\|_{L^{q'}\left(I_T,L^{p'}\right)} \\
&\leq CT^{\frac{q-q'}{qq'}}(\|u\|^\alpha_{L^\infty\left(I_T,\tilde{W}^{1,2}\right)} + \|v\|^\alpha_{L^\infty\left(I_T,\tilde{W}^{1,2}\right)})\|u-v\|_{L^q(I_T,L^p)} \\
&\leq CT^{\frac{q-q'}{qq'}}M^\alpha d(u,v).
\end{aligned} \tag{5.0.21}$$

Choose

$$M = \begin{cases} 1 & \text{if } f = 0 \\ 2C\|f\|_{\tilde{W}^{1,2}(\mathbb{C}^n)} & \text{if } f \neq 0 \end{cases} \tag{5.0.22}$$

and

$$T = \begin{cases} \min\{\pi, (2C)^{-\frac{qq'}{q-q'}}\} & \text{if } f = 0 \\ \min\{\pi, (2C)^{-(1+\alpha)\frac{qq'}{q-q'}}\|f\|^{-\alpha\frac{qq'}{q-q'}}_{\tilde{W}^{1,2}(\mathbb{C}^n)}\} & \text{if } f \neq 0 \end{cases} \tag{5.0.23}$$

where $C$ is the same constant that appears in the inequalities (5.0.20), (5.0.21) and is independent of $T$, see Remark 3.0.8.

For these choices of $M$ and $T$, operator $\mathcal{H}$ maps $E$ to $E$ and also is a contaction on $E$. Therefore $\mathcal{H}$ has unique fixed point in $E$. From Lemma 5.0.18 and Proposition 5.0.20, we conclude that $u \in C(\bar{I}_T, \tilde{W}^{1,2}(\mathbb{C}^n)) \cap L^{q_1}(I_T, \tilde{W}^{1,p_1}(\mathbb{C}^n))$ for every admissible pair $(q_1, p_1)$. In view of Lemma 5.0.21 and estimate 5.0.6, $u$ is solution of the initial value problem (1.0.13), (1.0.14).

## Blowup alternative, uniqueness and stability

In Theorem 5.0.26, we will prove blowup alternative, uniqueness and stability. We first prove the following two Propositions which are useful in proving Theorem 5.0.26.

**Proposition 5.0.24** *Let $\Phi$ be a continuous complex valued function on $\mathbb{C}$ such that $|\Phi(w)| \leq C|w|^\alpha$ for $0 \leq \alpha < \frac{2}{n-1}$. Let $a, b \in \mathbb{R}$. Suppose $\{u_m\}$ be a sequence in $L^q\left((a,b), \tilde{W}^{1,p}(\mathbb{C}^n)\right) \cap L^\infty((a,b), \tilde{W}^{1,2}(\mathbb{C}^n))$ with $p = 2+\alpha,\ q \geq 2$, such that*

$$\sup_{m\in\mathbb{N}} \|u_m\|_{L^\infty\left((a,b),\tilde{W}^{1,2}(\mathbb{C}^n)\right)} \leq M < \infty.$$

*If $u_m \to u$ in $L^q((a,b), L^p(\mathbb{C}^n))$ and $u \in L^q((a,b), \tilde{W}^{1,p}) \cap L^\infty((a,b), \tilde{W}^{1,2})$ then $[\Phi(u_m) - \Phi(u)]Su \to 0$ in $L^{q'}\left((a,b), L^{p'}(\mathbb{C}^n)\right)$, for $S = Id, L_j, M_j; 1 \leq j \leq n$.*

**Proof**. Since $u_m \to u$ in $L^q((a,b), L^p(\mathbb{C}^n))$, we can extract a subsequence still denoted by $u_k$ such that

$$\|u_{k+1} - u_k\|_{L^q((a,b),L^p(\mathbb{C}^n))} \leq \frac{1}{2^k}$$

for all $k \geq 1$ and $u_k(z,t) \to u(z,t)$ a.e. $(z,t)$, see Theorem 4.9 in Brezis [3]. Hence by continuity of $\Phi$,

$$[\Phi(u_k) - \Phi(u)]Su \to 0 \quad \text{for a.e. } (z,t) \in \mathbb{C}^n \times (a,b). \tag{5.0.24}$$

We establish the norm convergence by appealing to a dominated convergence argument in the $z$ and $t$ variables successively.

Consider the function $H(z,t) = \sum_{k=1}^\infty |u_{k+1}(z,t) - u_k(z,t)|$. Clearly $H \in L^q\left((a,b), L^p(\mathbb{C}^n)\right)$, since the above series converges absolutely in that space. Also for $l > k$, $|(u_l - u_k)(z,t)| \leq |u_l - u_{l-1}| + \cdots + |u_{k+1} - u_k| \leq H(z,t)$ hence

$|u_k - u| \leq H$. This leads to the pointwise almost everywhere inequality

$$|u_k(z,t)| \leq |u(z,t)| + H(z,t) = v(z,t).$$

Hence

$$|\,[\Phi(u_k) - \Phi(u)]\, Su(z,t)|^{p'} \leq |[v^\alpha + |u|^\alpha] Su(z,t)|^{p'}.$$

Since $u, v \in L^q\left((a,b), L^p(\mathbb{C}^n)\right)$ and $p = 2+\alpha$, using Hölder's inequality with $\frac{p'}{p} + \frac{\alpha p'}{p} = 1$, we get

$$\begin{aligned}\int_{\mathbb{C}^n} &|(v^\alpha + |u|^\alpha) Su(z,t)|^{p'} dz \\ &\leq (\|v(\cdot,t)\|^{\alpha p'}_{L^p(\mathbb{C}^n)} + \|u(\cdot,t)\|^{\alpha p'}_{L^p(\mathbb{C}^n)}) \|Su(\cdot,t)\|^{p'}_{L^p(\mathbb{C}^n)}.\end{aligned} \tag{5.0.25}$$

Thus using dominated convergence theorem in the $z$-variable, we see that

$$\|\,[\Phi(u_k) - \Phi(u)]\, Su(\cdot,t)\|_{L^{p'}(\mathbb{C}^n)} \to 0 \tag{5.0.26}$$

as $k \to \infty$, for a.e. $t$.

Again, in view of Lemma 4.0.9 and Hölder's inequality as above, we get

$$\begin{aligned}\|[\Phi(u_k) \;&-\; \Phi(u)]\, Su(\cdot,t)\|_{L^{p'}(\mathbb{C}^n)} \\ &\leq\; C\left(\|u_k\|^\alpha_{L^\infty\left([a,b],\tilde{W}^{1,2}(\mathbb{C}^n)\right)} + \|u\|^\alpha_{L^\infty\left([a,b],\tilde{W}^{1,2}(\mathbb{C}^n)\right)}\right)\|Su(\cdot,t)\|_{L^p} \\ &\leq\; C(M^\alpha + \|u\|^\alpha_{L^\infty\left([a,b],\tilde{W}^{1,2}(\mathbb{C}^n)\right)})\|Su(\cdot,t)\|_{L^p(\mathbb{C}^n)}.\end{aligned}$$

Since $\|Su(\cdot,t)\|_{L^p(\mathbb{C}^n)} \in L^{q'}((a,b))$ and $q \geq 2$, an application of the Hölder's inequality in the $t$-variable shows that

$$\int_a^b \|Su(\cdot,t)\|^{q'}_{L^p(\mathbb{C}^n)} dt \leq [b-a]^{\frac{q-q'}{q}} \|Su(\cdot,t)\|^{q'}_{L^q((a,b),L^p(\mathbb{C}^n))}.$$

Hence a further application of dominated convergence theorem in the $t$-variable shows that $\|\,(\Phi(u_k) - \Phi(u))\, Su\|_{L^{q'}((a,b),L^{p'})} \to 0$, as $k \to \infty$.

Thus we have shown that $[\Phi(u_{m_k}) - \Phi(u)]\, Su \to 0$ in $L^{q'}((a,b), L^{p'}(\mathbb{C}^n))$ whenever $u_m \to u$ in $L^q((a,b), L^p(\mathbb{C}^n))$. But the above arguments are also valid if we had started with any subsequence of $u_m$. It follows that any subsequence of $[\Phi(u_m) - \Phi(u)]\, Su$ has a subsequence that converges to 0 in $L^{q'}((a,b), L^{p'}(\mathbb{C}^n))$. From this we conclude that the original sequence $[\Phi(u_m) - \Phi(u)]\, Su$ converges to zero in $L^{q'}((a,b),L^{p'}(\mathbb{C}^n))$, hence the Proposition.

**Proposition 5.0.25** *Let $\{f_m\}_{m\geq 1}$ be a sequence in $\tilde{W}^{1,2}(\mathbb{C}^n)$ such that $f_m \to f$ in $\tilde{W}^{1,2}(\mathbb{C}^n)$ as $m\to\infty$. Let $u_m$ and $u$ be the solutions corresponding to the initial data $f_m$ and $f$ respectively, at time $t=t_0$. Then there exists $\tau$, depending on $\|f\|_{\tilde{W}^{1,2}}$ such that $\|(u_m-u)(\cdot,t)\|_{\tilde{W}^{1,2}(\mathbb{C}^n)}\to 0$ for each $t\in[t_0-\tau,t_0+\tau]$ and $\|u_m-u\|_{L^{q_1}([t_0-\tau,t_0+\tau],\tilde{W}^{1,p_1}(\mathbb{C}^n))}\to 0$ as $m\to\infty$ for every admissible pair $(q_1,p_1)$. Moreover $\|G(z,u_m(z,t))-G(z,u(z,t))\|_{L^{q'}([t_0-\tau,t_0+\tau],\tilde{W}^{1,p'}(\mathbb{C}^n))}\to 0$ as $m\to\infty$.*

**Proof**. Since $\|f_m\|_{\tilde{W}^{1,2}(\mathbb{C}^n)}\to\|f\|_{\tilde{W}^{1,2}(\mathbb{C}^n)}$, by (5.0.22), (5.0.23) and by taking $m$ large if necessary, we can assume that solutions $u_m$ are defined on $[t_0-\tau,t_0+\tau]$ for $\tau<T$. Setting $G_m(z,t)=G(z,u_m(z,t))$, we have

$$(u_m-u)(z,t)=e^{-i(t-t_0)\mathcal{L}}(f_m-f)(z)-i\int_{t_0}^{t}e^{-i(t-s)\mathcal{L}}(G_m-G)(z,s)ds$$

for all $t\in I_\tau=[t_0-\tau,t_0+\tau]$.

First we consider the case $f\equiv 0$. Since $\mathcal{H}(0)=0$ and the fixed point of $\mathcal{H}$ in $E$ is unique, in this case the solution $u\equiv 0$. Thus by Lemma 5.0.18, Proposition 5.0.20 and Strichartz estimates (Theorem 3.0.7), we see that

$$\|u_m\|_{L^q(I_\tau,\tilde{W}^{1,p})} \leq C\|f_m\|_{\tilde{W}^{1,2}}+C\tau^{\frac{q-q'}{qq'}}\|u_m\|^{\alpha}_{L^\infty(I_\tau,\tilde{W}^{1,2})}\|u_m\|_{L^q(I_\tau,\tilde{W}^{1,p})}. \tag{5.0.27}$$

Note that $\|u_m\|_{L^\infty(I_\tau,\tilde{W}^{1,2})}\leq M_m$. $M_m$ is given by the following

$$M_m=\begin{cases}1 & \text{if } f_m=0\\ 2C\|f_m\|_{\tilde{W}^{1,2}} & \text{if } f_m\neq 0\end{cases}$$

and $\|f_m\|_{\tilde{W}^{1,2}}\to\|f\|_{\tilde{W}^{1,2}}=0$, therefore we have $M_m\leq 1$ for large $m$. Now we choose $\tau$ sufficiently small so that $C\tau^{\frac{q-q'}{qq'}}<\frac{1}{2}$ and from (5.0.27) we see that

$$\|u_m\|_{L^q(I_\tau,\tilde{W}^{1,p})}\leq 2C\|f_m\|_{\tilde{W}^{1,2}}\to 0 \text{ as } m\to\infty. \tag{5.0.28}$$

Therefore by estimate (5.0.6)

$$\|G_m\|_{L^{q'}(I_\tau,\tilde{W}^{1,p'})}\leq C\tau^{\frac{q-q'}{qq'}}\|u_m\|^{\alpha}_{L^\infty(I_\tau,\tilde{W}^{1,2})}\|u_m\|_{L^q(I_\tau,\tilde{W}^{1,p})}\to 0$$

as $m \to \infty$ and by Strichartz estimates

$$\|u_m\|_{L^{q_1}(I_\tau, \tilde{W}^{1,p_1})} \leq C\|f_m\|_{\tilde{W}^{1,2}} + C\|G_m\|_{L^{q'}(I_\tau, \tilde{W}^{1,p'})} \to 0$$

as $m \to \infty$ for every admissible pair $(q_1, p_1)$.

Now we consider the case $f \neq 0$. We choose $m$ sufficiently large such that $\|f_m - f\|_{\tilde{W}^{1,2}} < \|f\|_{\tilde{W}^{1,2}}$. Therefore we have $\|f_m\|_{\tilde{W}^{1,2}} \leq 2\|f\|_{\tilde{W}^{1,2}}$ and $M_m := 2C\|f_m\|_{\tilde{W}^{1,2}} \leq 2M := 4C\|f\|_{\tilde{W}^{1,2}}$. By Lemma 5.0.19, equation (5.0.22), Lemma 5.0.18, Proposition 5.0.20 and the fact that $M_m \leq 2M$ we get

$$\|G_m - G\|_{L^{q'}(I_\tau, L^{p'}(\mathbb{C}^n))} \leq C\tau^{\frac{q-q'}{qq'}}\|f\|^{\alpha}_{\tilde{W}^{1,2}}\|u_m - u\|_{L^q(I_\tau, L^p(\mathbb{C}^n))} \quad (5.0.29)$$

$$\|u_m - u\|_{L^q(I_\tau, L^p(\mathbb{C}^n))} \leq C\|f_m - f\|_{L^2} + C\tau^{\frac{q-q'}{qq'}}\|f\|^{\alpha}_{\tilde{W}^{1,2}}\|u_m - u\|_{L^q(I_\tau, L^p(\mathbb{C}^n))}.$$

Now we choose $\tau$ small so that $C\tau^{\frac{q-q'}{qq'}}\|f\|^{\alpha}_{\tilde{W}^{1,2}} < \frac{1}{2}$, we see that

$$\|u_m - u\|_{L^q(I_\tau, L^p(\mathbb{C}^n))} \leq 2C\|f_m - f\|_{L^2(\mathbb{C}^n)} \to 0 \quad (5.0.30)$$

as $m \to \infty$. From estimate 5.0.29, we see that $\|G_m - G\|_{L^{q'}(I_\tau, L^{p'}(\mathbb{C}^n))} \to 0$ as $m \to \infty$ and from Strichartz estimates $\|u_m - u\|_{L^{q_1}(I_\tau, L^{p_1}(\mathbb{C}^n))} \to 0$ as $m \to \infty$ for every admissible pair $(q_1, p_1)$.

For $S = L_j, M_j$ $(1 \leq j \leq n)$ and by using (5.0.8), (5.0.9) with the notation $\psi_m = \psi(z, |u_m(z,t)|)$, we have

$$\begin{aligned} S(G_m - G) = {} & \psi_m S(u_m - u) + (\psi_m - \psi)Su + (\partial_j \psi_m)(u_m - u) \\ & + (\partial_j \psi_m - \partial_j \psi)u + (\partial_{2n+1}\psi_m)u_m \Re(\frac{\overline{u_m}}{|u_m|}S(u_m - u)) \\ & + (\partial_{2n+1}\psi_m)u_m \Re(\frac{\overline{u_m}}{|u_m|}Su) - (\partial_{2n+1}\psi)u\Re(\frac{\overline{u}}{|u|}Su) \end{aligned} \quad (5.0.31)$$

where $\partial_j = \partial_{x_j}$ for $S = L_j$ and $\partial_j = \partial_{y_j}$ for $S = M_j,\ 1 \leq j \leq n$.

Using the assumption (1.0.16) on $\psi$, Lemma 4.0.9, and the fact that $M_m \leq 2M$, similar computations as in Proposition 5.0.20 shows that

$$\|\psi_m S(u_m - u)\|_{L^{q'}\left(I_\tau, L^{p'}(\mathbb{C}^n)\right)} \leq C\tau^{\frac{q-q'}{qq'}}\|f\|^{\alpha}_{\tilde{W}^{1,2}(\mathbb{C}^n)}\|S(u_m - u)\|_{L^q(I_\tau, L^p(\mathbb{C}^n))}$$

$$\|(\partial_j \psi_m)(u_m - u)\|_{L^{q'}\left(I_\tau, L^{p'}(\mathbb{C}^n)\right)} \leq C\tau^{\frac{q-q'}{qq'}}\|f\|^{\alpha}_{\tilde{W}^{1,2}}\|u_m - u\|_{L^q(I_\tau, L^p(\mathbb{C}^n))}$$

$$\|(\partial_{2n+1}\psi_m)u_m\Re(\frac{\overline{u_m}}{|u_m|}S(u_m-u))\|_{L^{q'}\left(I_\tau,L^{p'}(\mathbb{C}^n)\right)}$$
$$\leq C\tau^{\frac{q-q'}{qq'}}\|f\|^{\alpha}_{\tilde{W}^{1,2}}\|S(u_m-u)\|_{L^q(I_\tau,L^p(\mathbb{C}^n))}.$$

Since $\|u_m-u\|_{L^q(I_\tau,L^p(\mathbb{C}^n))}\to 0$, by second inequality in the above estimates, $(\partial_j\psi_m)(u_m-u)\to 0$ as $m\to\infty$. Now $G$ is $C^1$, so in view of the condition (1.0.16) on $\psi$ and Proposition 5.0.24, the sequences $(\psi_m-\psi)Su,$ , $(\partial_j\psi_m-\partial_j\psi)u$ and $(\partial_4\psi_m)u_m\Re(\frac{\overline{u_m}}{|u_m|}Su)-(\partial_4\psi)u\Re(\frac{\overline{u}}{|u|}Su)$ converges to zero in $L^{q'}(I_\tau,L^{p'}(\mathbb{C}^n))$ as $m\to\infty$. Using these observations and from (5.0.31), we get

$$\begin{aligned}\|S(G_m-G)\|_{L^{q'}(I_\tau,L^{p'})} &\leq C\tau^{\frac{q-q'}{qq'}}\|f\|^{\alpha}_{\tilde{W}^{1,2}}\|S(u_m-u)\|_{L^q(I_\tau,L^p(\mathbb{C}^n))}+a_m\\ \|G_m-G\|_{L^{q'}(I_\tau,\tilde{W}^{1,p'})} &\leq C\tau^{\frac{q-q'}{qq'}}\|f\|^{\alpha}_{\tilde{W}^{1,2}}\|u_m-u\|_{L^q(I_\tau,\tilde{W}^{1,p}(\mathbb{C}^n))}+a_m \quad (5.0.32)\end{aligned}$$

where $a_m\to 0$ as $m\to\infty$ and $S=L_j,M_j$; $1\leq j\leq n$. By Lemma 5.0.18 and estimate (5.0.32), we see that

$$\|u_m-u\|_{L^q(I_\tau,\tilde{W}^{1,p})} \leq C\|f_m-f\|_{\tilde{W}^{1,2}}+C\tau^{\frac{q-q'}{qq'}}\|f\|^{\alpha}_{\tilde{W}^{1,2}}\|u_m-u\|_{L^q(I_\tau,\tilde{W}^{1,p})}+a_m.$$

Now choose $\tau$ sufficiently small so that

$$C\tau^{\frac{q-q'}{qq'}}\|f\|^{\alpha}_{\tilde{W}^{1,2}}\leq\frac{1}{2}$$

and we see that $\|u_m-u\|_{L^q(I_\tau,\tilde{W}^{1,p})}\leq 2C\|f_m-f\|_{\tilde{W}^{1,2}}+2a_m\to 0$ as $m\to\infty$. Now from (5.0.32), $\|G_m-G\|_{L^{q'}(I_\tau,\tilde{W}^{1,p'})}\to 0$ as $m\to\infty$ and from Strichartz estimates $\|u_m-u\|_{L^{q_1}(I_\tau,\tilde{W}^{1,p_1})}\to 0$ as $m\to\infty$ for every admissible pair $(q_1,p_1)$. Since $u_m,u\in C(\overline{I}_\tau,\tilde{W}^{1,2}(\mathbb{C}^n))$ for each $m$, therefore $\|(u_m-u)(\cdot,t)\|_{\tilde{W}^{1,2}}\leq\|u_m-u\|_{L^\infty(I_\tau,\tilde{W}^{1,2})}$ for each $t\in\overline{I}_\tau$. Since $(\infty,2)$ is an admissible pair, therefore $\|(u_m-u)(\cdot,t)\|_{\tilde{W}^{1,2}}\to 0$ as $m\to\infty$ for each $t\in\overline{I}_\tau$. Hence the proposition.

**Theorem 5.0.26** *Let $u(\cdot,t_0)=f\in\tilde{W}^{1,2}(\mathbb{C}^n)$, $\alpha\in[0,\frac{2}{n-1})$ and $G$ be as in (1.0.15), (1.0.16). Then the initial value problem (1.0.13), (1.0.14) has unique maximal solution $u\in C((T_*,T^*),\tilde{W}^{1,2}(\mathbb{C}^n))\cap L^{q_1}_{loc}\Big((T_*,T^*),\tilde{W}^{1,p_1}(\mathbb{C}^n)\Big)$, where $t_0\in(T_*,T^*)$ and $(q_1,p_1)$ be an arbitrary admissible pair. Fix $p=2+\alpha$. Moreover the following properties hold:*

**(i)(Uniqueness)** *Solution is unique in $C((T_*,T^*),\tilde{W}^{1,2})\cap L^{q_1}_{loc}\Big((T_*,T^*),\tilde{W}^{1,p}\Big)$ for every admissible pair $(q_1,p)$ with $q_1>2$.*

**(ii)(Blowup alternative)** *If* $T^* < \infty$ *(respectively,* $T_* > -\infty$*), then* $\|u(\cdot,t)\|_{\tilde{W}^{1,2}} \to \infty$ *as* $t \to T^*$ *(respectively,* $t \to T_*$*).*

**(iii)(Stability)** *If* $f_j \to f$ *in* $\tilde{W}^{1,2}(\mathbb{C}^n)$*, then* $u_j(\cdot,t) \to u(\cdot,t)$ *in* $\tilde{W}^{1,2}(\mathbb{C}^n)$ *for each* $t \in (T_*, T^*)$ *and also* $u_j \to u$ *in* $L^{q_1}(I, \tilde{W}^{1,p_1}(\mathbb{C}^n))$ *for every admissible pair* $(q_1, p_1)$ *and every interval* $I$ *with* $\bar{I} \subset (T_*, T^*)$*, where* $u_j$ *is the solution of equation (1.0.13) with initial value* $u_j(\cdot, t_0) = f_j$*.*

**Proof**. By local existence (Theorem 5.0.23), the solution exists in $C(\overline{I}_T : \tilde{W}^{1,2}(\mathbb{C}^n))$ where $I_T = (t_0 - T, t_0 + T)$. Since $\|u(\cdot, t_0 + T)\|_{\tilde{W}^{1,2}(\mathbb{C}^n)} < \infty$, the argument in the proof of Theorem 5.0.23 can be carried out with $t_0 + T$ as the initial time, to extend the solution to the interval $[t_0 + T, T_1]$. This procedure can be continued and we can get a sequence $\{T_m\}$ such that $t_0 + T < T_1 < T_2 < \cdots < T_m < \cdots$ as long as $\|u(\cdot, T_m)\|_{\tilde{W}^{1,2}(\mathbb{C}^n)} < \infty$. Let $T^* = \sup_m T_m$ so that the solution extends to $[t_0, T^*)$. In the same way we can extend the solution to the left side to the interval $(T_*, t_0]$ to get a solution in $C((T_*, T^*), \tilde{W}^{1,2}(\mathbb{C}^n))$. Now we prove blowup alternative, uniqueness and stability.

**Blowup alternative:** Suppose $T^* < \infty$ and $\sup_{t \in [t_0, T^*)} \|u(z,t)\|_{\tilde{W}^{1,2}} = M_0 < \infty$. If $f = 0$, then $\mathcal{H}(0) = 0$ and since $\mathcal{H}$ has unique fixed point in $E$, $u(\cdot, t) = 0$ for $t \in [t_0 - T, t_0 + T]$ where $T = \min\{\pi, (2C)^{-\frac{qq'}{q-q'}}\}$, see (5.0.23). By considering $t_0 - T$ and $t_0 + T$ as a initial time, by the same reasoning solution $u(\cdot, t) = 0$ for $t \in [t_0 - 2T, t_0 + 2T]$. By continuing this process, solution $u$ is global and $u(\cdot, t) = 0$ for $t \in \mathbb{R}$. This contradicts the fact that $T^* < \infty$. Therefore $f \neq 0$.

Now we choose a sequence $t_j \uparrow T^*$. From local existence (see (5.0.23)) we can choose $T_j = \min\{C_1 \|u(., t_j)\|_{\tilde{W}^{1,2}}^{-\alpha\frac{qq'}{q-q'}}, \pi\}$ such that $u \in C([t_j - T_j, t_j + T_j], \tilde{W}^{1,2})$ where $C_1 = (2C)^{-(1+\alpha)\frac{qq'}{q-q'}}$. Hence by assumption $T_j \geq \min\{C_1 M_0^{-\alpha\frac{qq'}{q-q'}}, \pi\}$, a constant independent of $t_j$, for $q > 2$. Thus we can choose $j$ so large that $t_j + T_j > T^*$, which contradicts maximality of $T^*$. Hence if $T^* < \infty$ then $\lim_{t \to T^*} \|u(z,t)\|_{\tilde{W}^{1,2}} = \infty$. Similarly, we can show that $\lim_{t \to T_*} \|u(.,t)\|_{\tilde{W}^{1,2}} = \infty$, if $T_* > -\infty$.

**Uniqueness:** Suppose that $u, v \in C((T_*, T^*), \tilde{W}^{1,2}) \cap L^q_{\text{loc}}((T_*, T^*), \tilde{W}^{1,p})$ are two solutions of the equations (1.0.13) and (1.0.14) where $(q,p)$ be an admissible pair with $p = 2 + \alpha$ and $q > 2$. Then $u$ and $v$ will satisfy integral equation (1.0.20), see Lemma 5.0.21. Since $u(\cdot, t_0) = v(\cdot, t_0) = f$ and the solution given by the contraction mapping is unique on $[t_0 - T, t_0 + T]$, $u(\cdot, t) = v(\cdot, t)$ for

$t \in [t_0 - T, t_0 + T]$. Let $\tilde{t} \in (T_*, T^*)$ be such that $u(\cdot, \tilde{t}) = v(\cdot, \tilde{t})$. For $\tau \in (\tilde{t}, T^*)$, we have

$$\begin{aligned} u(z,\tau) &= e^{-i(\tau-\tilde{t})\mathcal{L}} u(z,\tilde{t}) - i \int_{\tilde{t}}^{\tau} e^{-i(\tau-s)\mathcal{L}} G(z, u(z,s)) ds, \\ v(z,\tau) &= e^{-i(\tau-\tilde{t})\mathcal{L}} v(z,\tilde{t}) - i \int_{\tilde{t}}^{\tau} e^{-i(\tau-s)\mathcal{L}} G(z, v(z,s)) ds. \end{aligned}$$

By Strichartz estimate (3.0.2) and Lemma 5.0.19, we have

$$\begin{aligned} \|u - v\|_{L^q\left((\tilde{t},\tau), L^p(\mathbb{C}^n)\right)} &= \left\| \int_{\tilde{t}}^{\tau} e^{-i(t-s)\mathcal{L}} \left(G(u) - G(v)\right)(z,s) ds \right\|_{L^q\left((\tilde{t},\tau), L^p(\mathbb{C}^n)\right)} \\ &\leq C|\tau - \tilde{t}|^{\frac{q-q'}{qq'}} M_{\tilde{t},\tau}^{\alpha} \|u - v\|_{L^q\left((\tilde{t},\tau), L^p(\mathbb{C}^n)\right)} \end{aligned}$$

for all $\tau \in (\tilde{t}, T^*)$ where $M_{\tilde{t},\tau} = \max\{\|u\|_{L^\infty\left((\tilde{t},\tau), \tilde{W}^{1,2}\right)}, \|v\|_{L^\infty\left((\tilde{t},\tau), \tilde{W}^{1,2}\right)}\}$, see (5.0.22). Since $u, v \in C\left([t_0, T^*), \tilde{W}^{1,2}\right)$, we have $M_{\tilde{t},\tau} < \infty$. Choose $\tau \in [\tilde{t}, T^*)$ sufficiently close to $\tilde{t}$ such that $C|\tau - \tilde{t}|^{\frac{q-q'}{qq'}} M_{\tilde{t},\tau}^{\alpha} = c < 1$, so that

$$0 \leq (1 - c)\|u - v\|_{L^q((\tilde{t},\tau):L^p(\mathbb{C}^n))} \leq 0.$$

Hence $u = v$ on the larger interval $[\tilde{t}, \tau]$.

Now let $\theta$ =sup$\{\tilde{T} : t_0 < \tilde{T} < T^* : \|u - v\|_{L^q([t_0,\tilde{T}], L^p)} = 0\}$. If $\theta < T^*$, then for sufficiently small $\epsilon > 0$, choose $\tilde{t} = \theta - \epsilon, \tau = \theta + \epsilon$ and by the above observation, $\|u - v\|_{L^q((\theta-\epsilon,\theta+\epsilon), L^p)} = 0$, which contradicts the definition of $\theta$. Thus we conclude that $\theta = T^*$, proving the uniqueness on $[t_0, T^*)$. Similarly one can show uniqueness on $(T_*, t_0]$.

**Stability:** Let $\{f_m\}_{m \geq 1}$ be a sequence in $\tilde{W}^{1,2}(\mathbb{C}^n)$ such that $f_m \to f$ in $\tilde{W}^{1,2}$ as $m \to \infty$. Let $u_m$ and $u$ be the solutions corresponding to the initial values $f_m$ and $f$ respectively. Let $(T_*, T^*)$ and $(T_{*,m}, T_m^*)$ be maximal intervals for the solutions $u$ and $u_m$ respectively and $I \subset (T_*, T^*)$ be a compact interval.

The key idea is to extend the stability result proved in Proposition 5.0.25 to the interval $I$ by covering it with finitely many intervals obtained by successive application of Proposition 5.0.25. This is possible provided $u_m$ is defined on $I$, for all but finitely many $m$. In fact, we prove $I \subset (T_{*,m}, T_m^*)$ for all but finitely many $m$.

We can assume that $t_0 \in I = [a, b]$, and give a proof by the method of contradiction. Suppose there exist infinitely many $T^*_{m_j} \leq b$ and let $c = \liminf T^*_{m_j}$. Then for $\epsilon > 0$, $[t_0, c-\epsilon] \subset [t_0, T^*_{m_j})$ for all $m_j$ sufficiently large and $u_{m_j}$ are defined on $[t_0, c-\epsilon]$.

By compactness, the stability result proved in Proposition 5.0.25 can be extended to the interval $[t_0, c-\epsilon]$ by covering it with finitely many intervals obtained by successive application of Proposition 5.0.25. Hence

$$\|u_{m_j}(\cdot, c-\epsilon)\|_{\tilde{W}^{1,2}} \to \|u(\cdot, c-\epsilon)\|_{\tilde{W}^{1,2}} \quad \text{as} \quad j \to \infty.$$

Also by continuity we have

$$\|u(\cdot, c-\epsilon)\|_{\tilde{W}^{1,2}} \to \|u(\cdot, c)\|_{\tilde{W}^{1,2}} \quad \text{as} \quad \epsilon \to 0.$$

Thus, for any $\delta > 0$, we have

$$\|u_{m_j}(\cdot, c-\epsilon)\|_{\tilde{W}^{1,2}}^{-\alpha\frac{qq'}{q-q'}} > \delta \text{ whenever } \|u(\cdot, c)\|_{\tilde{W}^{1,2}}^{-\alpha\frac{qq'}{q-q'}} > \delta, \tag{5.0.33}$$

for sufficiently small $\epsilon$ and for all $j \geq j_0(\epsilon)$. Therefore by applying the local existence theorem (see equation 5.0.23), with $c-\epsilon$ as the initial time, without loss of generality we can assume that $u_{m_j}$ extends to $[t_0, c-\epsilon+C_1\|u_{m_j}(\cdot, c-\epsilon)\|_{\tilde{W}^{1,2}}^{-\alpha\frac{qq'}{q-q'}}]$ for large $j$ where $C_1 = (2C)^{-(1+\alpha)\frac{qq'}{q-q'}}$. Now choosing $\epsilon < \frac{C_1}{2}\delta$, we have by (5.0.33)

$$c - \epsilon + C_1\|u_{m_j}(\cdot, c-\epsilon)\|_{\tilde{W}^{1,2}}^{-\alpha\frac{qq'}{q-q'}} > c + \frac{C_1}{2}\delta \quad \text{for all} \quad j \geq j_0(\epsilon).$$

It follows that $T^*_{m_j} \geq c + \frac{C_1}{2}\delta$, hence contradicts the fact that $\liminf T^*_{m_j} = c$.

Similarly we can show that $[a, t_0] \subset (T_{*,m}, t_0]$ for all but finitely many $m$ which completes the proof of stability.

# Chapter 6

# Global well posedness in $\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$

In chapter 5, we proved the local well posedness of the initial value problem (1.0.13), (1.0.14) in $\tilde{W}^{1,2}(\mathbb{C}^n)$. The reason for considering this space was that the operators $L_j, M_j$ $(1 \leq j \leq n)$ commute with $e^{-it\mathcal{L}}$ and $\int_{t_0}^{t} e^{-(t-s)\mathcal{L}}$ and also compatible with the nonlinearity $G$. From (1.0.24), we see that $\tilde{W}^{1,2}(\mathbb{C}^n)$ is not the energy space and therefore energy conservation is not possible in this case. Thus this approach does not conclude global existence.

Hence in this chapter we consider initial value in the space $\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$. This space has the advantage of being the energy space, see (1.0.24). For proving mass conservation we assume that $\psi$ is real valued. Using these conservation laws, we can show that there is no finite time blow up in defocussing case (when $\psi$ is nonnegative) with $0 \leq \alpha < \frac{2}{n-1}$ and also in focusing case (when $\psi$ is nonpositive) with $0 \leq \alpha < \frac{2}{n}$, hence in Theorem 6.0.33 we conclude global existence for initial value in the Sobolev space $\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$.

In this chapter we consider both subcritical $0 \leq \alpha < \frac{2}{n-1}$ (see [25]) and critical case $\alpha = \frac{2}{n-1}$ (see [29]). Theorem 6.0.33 and Theorem 6.0.39 are main results of this chapter.

## Subcritical case $0 \leq \alpha < \frac{2}{n-1}$

In this section, first we prove some auxilliary estimates.

**Lemma 6.0.27** *Let $f \in \tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$ and $t_0 \in \mathbb{R}$. Then for every bounded interval $I$ and every admissible pair $(q_1, p_1)$, $t \to e^{-i(t-t_0)\mathcal{L}} f \in C(\mathbb{R}, \tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)) \cap L^{q_1}_{loc}(\mathbb{R}, \tilde{W}^{1,p_1}_{\mathcal{L}}(\mathbb{C}^n))$ and the following estimates hold :*

$$\|e^{-i(t-t_0)\mathcal{L}}f\|_{L^\infty\left(\mathbb{R},\tilde{W}_{\mathcal{L}}^{1,2}(\mathbb{C}^n)\right)} = \|f\|_{\tilde{W}_{\mathcal{L}}^{1,2}(\mathbb{C}^n)}, \tag{6.0.1}$$

$$\|e^{-i(t-t_0)\mathcal{L}}f\|_{L^{q_1}\left(I,\tilde{W}_{\mathcal{L}}^{1,p_1}(\mathbb{C}^n)\right)} \leq C\|f\|_{\tilde{W}_{\mathcal{L}}^{1,2}(\mathbb{C}^n)}, \tag{6.0.2}$$

*where the constant $C$ is independent of $f$ and $t_0$.*

**Proof**. From Lemma 4.0.15 we see that $|Z_j e^{-i(t-t_0)\mathcal{L}}f| = |e^{-i(t-t_0)\mathcal{L}}Z_j f|$ and $|\overline{Z}_j e^{-i(t-t_0)\mathcal{L}}f| = |e^{-i(t-t_0)\mathcal{L}}\overline{Z}_j f|$. Hence the proof follows from Theorem 3.0.7.

**Lemma 6.0.28** *Let $I$ be a finite interval and $(q,p)$ an admissible pair with $p = 2+\alpha$ and $q > 2$. Let $G$ be as in (1.0.15), (1.0.16) with $\alpha \in [0, \frac{2}{n-1})$ and $u, v \in L^\infty(I, \tilde{W}_{\mathcal{L}}^{1,2}(\mathbb{C}^n))$. Then $u, v \in L^q(I, L^p(\mathbb{C}^n))$ and the following estimate holds*

$$\begin{aligned}\|G(z,u) - G(z,v)\|_{L^{q'}(I,L^{p'}(\mathbb{C}^n))} &\leq C|I|^{\frac{q-q'}{qq'}}\|u-v\|_{L^q(I,L^p(\mathbb{C}^n))} \\ &\times \left(\|u\|^\alpha_{L^\infty(I,\tilde{W}_{\mathcal{L}}^{1,2})} + \|u\|^\alpha_{L^\infty(I,\tilde{W}_{\mathcal{L}}^{1,2})}\right).\end{aligned} \tag{6.0.3}$$

**Proof**. Since $I$ is a finite interval, in view of embedding theorem (Lemma 4.0.14), $u, v \in L^q(I, L^p(\mathbb{C}^n))$. By estimate (1.0.18), $\frac{1}{p'} = \frac{\alpha}{p} + \frac{1}{p}$, Holder's inequality in the $z$-variable and Lemma 4.0.14, we observe that

$$\begin{aligned}\|G(\cdot,u) - G(\cdot,v)\|_{L^{p'}(\mathbb{C}^n)} &\leq C\||u-v|(|u|^\alpha + |v|^\alpha)\|_{L^{p'}(\mathbb{C}^n)} \\ &\leq C\|(u-v)(\cdot,t)\|_{L^p(\mathbb{C}^n)}\left(\|u\|^\alpha_{L^p(\mathbb{C}^n)} + \|v\|^\alpha_{L^p(\mathbb{C}^n)}\right) \\ &\leq C\|(u-v)(\cdot,t)\|_{L^p(\mathbb{C}^n)}\left(\|u(\cdot,t)\|^\alpha_{\tilde{W}_{\mathcal{L}}^{1,2}} + \|v(\cdot,t)\|^\alpha_{\tilde{W}_{\mathcal{L}}^{1,2}}\right) \\ &\leq C\|(u-v)(\cdot,t)\|_{L^p(\mathbb{C}^n)}\left(\|u\|^\alpha_{L^\infty(I,\tilde{W}_{\mathcal{L}}^{1,2})} + \|v\|^\alpha_{L^\infty(I,\tilde{W}_{\mathcal{L}}^{1,2})}\right)\end{aligned}$$

where $t \in I$. Now by taking $L^{q'}$ norm in the $t$-variable on the interval $I$ in the above inequality, we get the required estimate (6.0.3).

**Proposition 6.0.29** *Let $t_0 \in \mathbb{R}$ and $I$ an open interval containing $t_0$. Let $G$ be as in (1.0.15), (1.0.16) with $\alpha \in [0, \frac{2}{n-1})$. Let $(q,p)$ be an admissible pair with $p = \alpha + 2$, $q > 2$.*

*If $u \in L^\infty_{loc}(I, \tilde{W}_{\mathcal{L}}^{1,2}(\mathbb{C}^n)) \cap L^q_{loc}(I, \tilde{W}_{\mathcal{L}}^{1,p}(\mathbb{C}^n))$, then for every bounded interval $J$ with $\overline{J} \subset I$, $t_0 \in \overline{J}$ and every admissible pair $(q_1, p_1)$, the following inequalities hold:*

$$\|SG(z,u(z,t))\|_{L^{q'}(J,L^{p'}(\mathbb{C}^n))} \leq C|J|^{\frac{q-q'}{qq'}}\|u\|^{\alpha}_{L^{\infty}(J,\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n))} \times\|u\|_{L^{q}(J,\tilde{W}^{1,p}_{\mathcal{L}}(\mathbb{C}^n))} \quad (6.0.4)$$

$$\|G(z,u(z,t))\|_{L^{q'}(J,\tilde{W}^{1,p'}_{\mathcal{L}}(\mathbb{C}^n))} \leq C|J|^{\frac{q-q'}{qq'}}\|u\|^{\alpha}_{L^{\infty}(J,\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n))} \times\|u\|_{L^{q}(J,\tilde{W}^{1,p}_{\mathcal{L}}(\mathbb{C}^n))} \quad (6.0.5)$$

$$\left\|\int_{t_0}^{t} e^{-i(t-s)\mathcal{L}}G(z,u(z,s))ds\right\|_{L^{q_1}(J,\tilde{W}^{1,p_1}_{\mathcal{L}})} \leq C|J|^{\frac{q-q'}{qq'}}\|u\|^{\alpha}_{L^{\infty}(J,\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n))} \times\|u\|_{L^{q}(J,\tilde{W}^{1,p}_{\mathcal{L}}(\mathbb{C}^n))} \quad (6.0.6)$$

*for* $S = Id, Z_j, \overline{Z}_j$ $(1 \leq j \leq n)$ *and for some constant* $C$ *independent of* $u$ *and* $t_0$. *Moreover* $\int_{t_0}^{t} e^{-i(t-s)\mathcal{L}}G(z,u(z,s))ds \in C(I,\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n))$.

**Proof**. To prove the inequality (6.0.4), we first observe that

$$(\partial_{x_j} - \frac{iy_j}{2})[\psi(x,y,|u|)u] = \psi(x,y,|u|)\,(\partial_{x_j} - \frac{iy_j}{2})u + u(\partial_{x_j}\psi)(x,y,|u|) + u\,(\partial_{|u|}\psi)(x,y,|u|)\,\Re\left(\frac{\bar{u}}{|u|}(\partial_{x_j} - \frac{iy_j}{2}y_j)u\right) \quad (6.0.7)$$

$$(\partial_{y_j} + \frac{ix_j}{2})[\psi(x,y,|u|)u] = \psi(x,y,|u|)\,(\partial_{y_j} + \frac{ix_j}{2})u + u(\partial_{y_j}\psi)(x,y,|u|) + u\,(\partial_{|u|}\psi)(x,y,|u|)\,\Re\left(\frac{\bar{u}}{|u|}(\partial_{y_j} + \frac{ix_j}{2})u\right). \quad (6.0.8)$$

Thus we see that for $S_j = (\partial_{x_j} - \frac{iy_j}{2})$ and $(\partial_{y_j} + \frac{ix_j}{2})$, $|S_jG|$ satisfies an inequality of the form

$$|S_jG| \leq |\psi(x,y,|u|)\,u| + |\tilde{\psi_1}(x,y,|u|)\,u| + |\tilde{\psi_2}(x,y,|u|)\,Su|$$

where $\tilde{\psi_1}(x,y,|u|) = u\partial_{x_j}\psi$ or $u\partial_{y_j}\psi$ depending on $S_j = (\partial_{x_j} - \frac{iy_j}{2})$ or $(\partial_{y_j} + \frac{ix_j}{2})$ and $\tilde{\psi_2}(x,y,|u|) = u\partial_{|u|}\psi$. Moreover, by assumption (1.0.16) on $\psi$, we have $|\tilde{\psi_i}(x,y,|u|)| \leq C|u|^{\alpha},\ i = 1,2$. Therefore

$$|S_jG| \leq C|u|^{\alpha}(|u| + |Su|)$$

for $S_j = \mathrm{Id}, (\partial_{x_j} - \frac{iy_j}{2}), (\partial_{y_j} + \frac{ix_j}{2}); 1 \leq j \leq n$. From the observations $\frac{1}{2}(Z_j - \bar{Z}_j) = (\partial_{x_j} - \frac{i}{2}y_j)$ and $\frac{i}{2}(Z_j + \bar{Z}_j) = (\partial_{y_j} + \frac{i}{2}x_j)$ (see 4.0.3), we get the inequality

$$|SG| \leq C|u|^{\alpha}\left[|u| + |Z_j u| + |\bar{Z}_j u|\right] \tag{6.0.9}$$

for $S = \mathrm{Id}, Z_j, \bar{Z}_j$ $(1 \leq j \leq n)$ and for some constant $C$.

An application of the Hölder's inequality in the $z$-variable, using $\frac{p'}{p} + \frac{\alpha p'}{p} = 1$ and Lemma 4.0.14, we see that for a.e. $t \in J$ and $S = \mathrm{Id}, Z_j, \bar{Z}_j$

$$\begin{aligned}
\|SG(\cdot, u(\cdot,t))\|_{L^{p'}(\mathbb{C}^n)} &\leq C\||u|^{\alpha}(|u| + |Z_j u| + |\bar{Z}_j u|)\|_{L^{p'}(\mathbb{C}^n)} \\
&\leq C\|u\|^{\alpha}_{L^p(\mathbb{C}^n)}\left(\|u\|_{L^p(\mathbb{C}^n)} + \|Z_j u\|_{L^p(\mathbb{C}^n)} + \|\bar{Z}_j u\|_{L^p(\mathbb{C}^n)}\right) \\
&\leq C\|u(\cdot,t)\|^{\alpha}_{\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)}\|u(\cdot,t)\|_{\tilde{W}^{1,p}_{\mathcal{L}}(\mathbb{C}^n)} \\
&\leq C\|u\|^{\alpha}_{L^{\infty}\left(J,\tilde{W}^{1,2}_{\mathcal{L}}\right)}\|u(\cdot,t)\|_{\tilde{W}^{1,p}_{\mathcal{L}}(\mathbb{C}^n)}.
\end{aligned}$$

Now taking $L^{q'}$ norm with respect to the $t$-variable on both sides and an application of the Hölder's inequality in the $t$-variable with $\frac{q'}{q} + \frac{q-q'}{q} = 1$, for $q > 2$ gives the estimate (6.0.4).

Estimate (6.0.5) follows from the estimate (6.0.4). Estimate (6.0.6) follows from Strichartz estimates (Theorem 3.0.7) and the estimate (6.0.5). The fact that $\int_{t_0}^{t} e^{-i(t-s)\mathcal{L}}G(z,u(z,s))\ ds \in C(I, \tilde{W}^{1,2}_{\mathcal{L}})$ follows from Lemma 4.0.16 and $G(z,u) \in L^{q'}_{\text{loc}}(I, \tilde{W}^{1,p'}_{\mathcal{L}})$. Hence the Proposition.

**Proposition 6.0.30** *Let $I$ be an open interval in $\mathbb{R}$, $\tilde{G}$ be as in (1.0.23) with $0 \leq \alpha < \frac{2}{n-1}$. Let $p = 2 + \alpha$ and $\{\epsilon_m\}$ a sequence of nonnegative real numbers converging to* 0. *Then*

$$\lim_{m\to\infty}\int_{\mathbb{C}^n}\tilde{G}(z, |e^{-\epsilon_m\mathcal{L}}v_m(z,t)|)dz = \int_{\mathbb{C}^n}\tilde{G}(z,|v(z,t)|)dz.$$

*whenever $v_m \to v$ in $C \cap L^{\infty}(I, L^p(\mathbb{C}^n))$.*

**Proof**. Since $v, v_m \in C \cap L^{\infty}(I, L^p(\mathbb{C}^n))$, for each $t \in I$,

$$\|(v_m - v)(\cdot,t)\|_{L^p(\mathbb{C}^n)} \leq \|v_m - v\|_{L^{\infty}(I,L^p(\mathbb{C}^n))} \to 0$$

as $m \to \infty$. By adding and subtracting appropriate terms we have

$$\begin{aligned}\int_{\mathbb{C}^n} & \left|\tilde{G}(z,|e^{-\epsilon_m\mathcal{L}}v_m(z,t)|)-\tilde{G}(z,|v(z,t)|)\right|dz\\ &\le \int_{\mathbb{C}^n}\left|\tilde{G}(z,|e^{-\epsilon_m\mathcal{L}}v_m|)-\tilde{G}(z,|e^{-\epsilon_m\mathcal{L}}v|)\right|dz\\ &\qquad +\int_{\mathbb{C}^n}\left|\tilde{G}(z,|e^{-\epsilon_m\mathcal{L}}v|)-\tilde{G}(z,|v|)\right|dz.\end{aligned}$$

In view of (1.0.25) and Lemma 4.0.17, an application of Hölder's inequality with $(1+\alpha)p' = p$ shows that

$$\begin{aligned}&\int_{\mathbb{C}^n}\left|\tilde{G}(z,|e^{-\epsilon_m\mathcal{L}}v_m|)-\tilde{G}(z,|e^{-\epsilon_m\mathcal{L}}v|)\right|dz\\ &\le C\int_{\mathbb{C}^n}|e^{-\epsilon_m\mathcal{L}}(v_m-v)|\ (|e^{-\epsilon_m\mathcal{L}}v_m|^{1+\alpha}+|e^{-\epsilon_m\mathcal{L}}v|^{1+\alpha})dz\\ &\le C\|e^{-\epsilon_m\mathcal{L}}(v_m-v)\|_{L^p(\mathbb{C}^n)}(\|e^{-\epsilon_m\mathcal{L}}v_m\|^{\alpha+1}_{L^p(\mathbb{C}^n)}+\|e^{-\epsilon_m\mathcal{L}}v\|^{\alpha+1}_{L^p(\mathbb{C}^n)})\\ &\le C\|(v_m-v)(\cdot,t)\|_{L^p(\mathbb{C}^n)}(\|v_m(\cdot,t)\|^{\alpha+1}_{L^p(\mathbb{C}^n)}+\|v(\cdot,t)\|^{\alpha+1}_{L^p(\mathbb{C}^n)}).\end{aligned}$$

Since $\{v_m\}$ is a Cauchy sequence in $L^\infty(I, L^p(\mathbb{C}^n))$, $\|v_m(\cdot,t)\|_{L^p(\mathbb{C}^n)}$ is bounded for $t$ fixed. Hence

$$\int_{\mathbb{C}^n}\left|\tilde{G}(z,|e^{-\epsilon_m\mathcal{L}}v_m|)-\tilde{G}(z,|e^{-\epsilon_m\mathcal{L}}v|)\right|dz\to 0$$

as $m\to\infty$.

A similar argument shows that $\int_{\mathbb{C}^n}(\tilde{G}(z,|e^{-\epsilon_m\mathcal{L}}v(z,t)|)-\tilde{G}(z,|v(z,t)|))dz$ tends to zero as $m\to\infty$, hence the Lemma.

# Local wellposednes in $\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$

**Theorem 6.0.31** *(Local well posedness) Let $f = u(\cdot,t_0)\in\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$, and $G$ be as in (1.0.15) and (1.0.16) with $\alpha\in[0,\frac{2}{n-1})$. Then the initial value problem (1.0.13), (1.0.14) has a unique maximal solution $u\in C((T_*,T^*),\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n))\cap L^{q_1}_{loc}\Big((T_*,T^*),\tilde{W}^{1,p_1}_{\mathcal{L}}\Big)$, where $t_0\in(T_*,T^*)$ and $(q_1,p_1)$ be an arbitrary admissible pair. Fix $p=2+\alpha$. Moreover the following properties hold:*

**(i)(Uniqueness)** *Solution is unique in $C((T_*,T^*),\tilde{W}^{1,2}_{\mathcal{L}})\cap L^{q_1}_{loc}\Big((T_*,T^*),\tilde{W}^{1,p}_{\mathcal{L}}\Big)$ for every admissible pair $(q_1,p)$ with $q_1>2$.*

**(ii)(Blowup alternative)** *If $T^* < \infty$ (respectively, $T_* > -\infty$), then $\|u(\cdot,t)\|_{\tilde{W}^{1,2}_{\mathcal{L}}} \to \infty$ as $t \to T^*$ (respectively, $t \to T_*$).*

**(iii)(Stability)** *If $f_j \to f$ in $\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$, then $u_j(\cdot,t) \to u(\cdot,t)$ in $\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$ for each $t \in (T_*, T^*)$ and also $u_j \to u$ in $L^{q_1}\left(I, \tilde{W}^{1,p_1}(\mathbb{C}^n)\right)$ for every admissible pair $(q_1, p_1)$ and every interval $I$ with $\bar{I} \subset (T_*, T^*)$.*

**Proof**. Proof follows by similar arguments as in Theorem 5.0.23 and Theorem 5.0.26. For completeness we give the proof.

**Local existence:** We establish the local existence of solution for the problem (1.0.13), (1.0.14) by establishing the existence of solution in the space $X$ (see Lemma 5.0.21) for the equivalent integeral equation

$$u(z,t) = e^{-i(t-t_0)\mathcal{L}} f(z) - i \int_{t_0}^{t} e^{-i(t-s)\mathcal{L}} G(z, u(z,s)) ds. \tag{6.0.10}$$

For given positive numbers $T$ and $M$, consider the set $E = E_{T,M}$ given by

$$E = \left\{ u \in L^\infty\left(I_T; \tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)\right) \cap L^q\left(I_T, \tilde{W}^{1,p}_{\mathcal{L}}(\mathbb{C}^n)\right) \;\middle|\; \begin{array}{l} \|u\|_{L^\infty\left(I_T, \tilde{W}^{1,2}_{\mathcal{L}}\right)} \leq M, \\ \|u\|_{L^q\left(I_T, \tilde{W}^{1,p}_{\mathcal{L}}\right)} \leq M \end{array} \right\}$$

where $I_T = (t_0 - T, t_0 + T)$ and $(q,p)$ be an admissible pair with $p = 2 + \alpha$ and $q > 2$. Then $E$ with the metric given by

$$d(u,v) = \|u - v\|_{L^\infty(I_T, L^2(\mathbb{C}^n))} + \|u - v\|_{L^q(I_T, L^p(\mathbb{C}^n))}$$

is a complete metric space. This can be verified by similar arguments as in Proposition 5.0.22.

First we verify that $\mathcal{H}$ given by (1.0.21) maps $E_{T,M}$ to $E_{T,M}$ for small $T$. If $u \in E_{T,M}$, using the estimates from Lemma 6.0.27 and Proposition 6.0.29, in (1.0.20), we see that,

$$\begin{aligned} \max&\left\{ \|\mathcal{H}u\|_{L^\infty\left(I_T, \tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)\right)}, \|\mathcal{H}u\|_{L^q\left(I_T, \tilde{W}^{1,p}_{\mathcal{L}}(\mathbb{C}^n)\right)} \right\} \\ &\leq C \|f\|_{\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)} + C\, T^{\frac{q-q'}{qq'}} \|u\|^{\alpha}_{L^\infty(I_T; \tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n))} \|u\|_{L^q(I_T, \tilde{W}^{1,p}_{\mathcal{L}}(\mathbb{C}^n))} \\ &\leq C \|f\|_{\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)} + C\, T^{\frac{q-q'}{qq'}} M^{1+\alpha}. \end{aligned} \tag{6.0.11}$$

This quantity is at most $M$ provided we choose

$$T \leq T_0 = \left( \frac{M - C\|f\|_{\tilde{W}^{1,2}_{\mathcal{L}}}}{CM^{1+\alpha}} \right)^{\frac{qq'}{q-q'}}.$$

Thus for a given $M > C\|f\|_{\tilde{W}^{1,2}_{\mathcal{L}}}$, $\mathcal{H}$ maps $E_{T,M}$ to $E_{T,M}$ for all $T \leq T_0$.

For $u, v \in E$, using Strichartz estimate (3.0.2) and the estimate (6.0.3), we get

$$\begin{aligned} d(\mathcal{H}u, \mathcal{H}v) &= \|\mathcal{H}u - \mathcal{H}v\|_{L^\infty(I_T, L^2(\mathbb{C}^n))} + \|\mathcal{H}u - \mathcal{H}v\|_{L^q(I_T, L^p(\mathbb{C}^n))} \\ &\leq C\|G(z,u) - G(z,v)\|_{L^{q'}\left(I_T, L^{p'}(\mathbb{C}^n)\right)} \\ &\leq CT^{\frac{q-q'}{qq'}} \left[ \|u\|^\alpha_{L^\infty\left(I_T, \tilde{W}^{1,2}_{\mathcal{L}}\right)} + \|v\|^\alpha_{L^\infty\left(I_T, \tilde{W}^{1,2}_{\mathcal{L}}\right)} \right] \|u - v\|_{L^q(I_T, L^p(\mathbb{C}^n))} \\ &\leq CT^{\frac{q-q'}{qq'}} M^\alpha d(u,v) \end{aligned} \tag{6.0.12}$$

Now we choose

$$M = \begin{cases} 1 & \text{if } f = 0 \\ 2C\|f\|_{\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)} & \text{if } f \neq 0 \end{cases} \tag{6.0.13}$$

and

$$T = \begin{cases} \min\{\pi, (2C)^{-\frac{qq'}{q-q'}}\} & \text{if } f = 0 \\ \min\{\pi, (2C)^{-(1+\alpha)\frac{qq'}{q-q'}} \|f\|^{-\alpha\frac{qq'}{q-q'}}_{\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)}\} & \text{if } f \neq 0 \end{cases} \tag{6.0.14}$$

where $C$ is the same constant that appears in the inequalities (6.0.11), (6.0.12) and is independent of $T$. For these choices of $M$ and $T$, the operator $\mathcal{H}$ maps $E$ to $E$ and also is a contaction on $E$. Therefore $\mathcal{H}$ has unique fixed point in $E$. From Lemma 6.0.27 and Proposition 6.0.29, we conclude that $u \in C(I_T, \tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)) \cap L^{q_1}(I_T, \tilde{W}^{1,p_1}_{\mathcal{L}}(\mathbb{C}^n))$ for every admissible pair $(q_1, p_1)$.

Now we consider inital time $t_0 - T$ and $t_0 + T$. By the above argument we get open intervals containing $t_0 - T$ and $t_0 + T$ on which solution exists to the initial value problem (1.0.13), (1.0.14). By continuing this process, we get maximal interval $(T_*, T^*)$ containing $t_0$ and solution $u$ of the initial value problem (1.0.13), (1.0.14) lies in $C((T_*, T^*), \tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)) \cap L^{q_1}_{\text{loc}}((T_*, T^*), \tilde{W}^{1,p_1}_{\mathcal{L}}(\mathbb{C}^n))$ for every admissible pair $(q_1, p_1)$.

**Blowup alternative:** Suppose $T^* < \infty$ and $\sup\limits_{t\in[t_0,T^*)} \|u(z,t)\|_{\tilde{W}^{1,2}_{\mathcal{L}}} = M_0 < \infty$. Clearly $f \neq 0$, see blowup alternative in Theorem 5.0.26. Now we choose a sequence $t_j \uparrow T^*$. From local existence we can choose

$$T_j = \min\{C_1 \|u(.,t_j)\|_{\tilde{W}^{1,2}_{\mathcal{L}}}^{-\alpha\frac{qq'}{q-q'}}, \pi\}$$

such that $u \in C([t_j - T_j, t_j + T_j], \tilde{W}^{1,2}_{\mathcal{L}})$ where $C_1 = (2C)^{-(1+\alpha)\frac{qq'}{q-q'}}$, see (6.0.14). Hence by assumption $T_j \geq \min\{C_1 M_0^{-\alpha\frac{qq'}{q-q'}}, \pi\}$, a constant independent of $t_j$, for $q > 2$. Thus we can choose $j$ so large that $t_j + T_j > T^*$, which contradicts maximality of $T^*$. Hence if $T^* < \infty$ then $\lim\limits_{t\to T^*} \|u(z,t)\|_{\tilde{W}^{1,2}_{\mathcal{L}}} = \infty$. Similarly, we can show that $\lim\limits_{t\to T_*} \|u(.,t)\|_{\tilde{W}^{1,2}_{\mathcal{L}}} = \infty$, if $T_* > -\infty$.

**Uniqueness:** Suppose that $u, v \in C((T_*, T^*), \tilde{W}^{1,2}_{\mathcal{L}}) \cap L^q_{\text{loc}}((T_*, T^*), \tilde{W}^{1,p}_{\mathcal{L}})$ are two solutions of the equations (1.0.13) and (1.0.14) where $(q,p)$ be an admissible pair with $p = 2+\alpha$ and $q > 2$. Then $u$ and $v$ will satisfy integral equation (1.0.20), see Lemma 5.0.21. Since $u(\cdot,t_0) = v(\cdot,t_0) = f$ and the solution given by the contraction mapping is unique on $[t_0 - T, t_0 + T]$, $u(\cdot,t) = v(\cdot,t)$ for $t \in [t_0 - T, t_0 + T]$. Let $\tilde{t} \in (T_*, T^*)$ be such that $u(\cdot,\tilde{t}) = v(\cdot,\tilde{t})$. For $\tau \in (\tilde{t}, T^*)$, we have

$$\begin{aligned} u(z,\tau) &= e^{-i(\tau-\tilde{t})\mathcal{L}} u(z,\tilde{t}) - i\int_{\tilde{t}}^{\tau} e^{-i(\tau-s)\mathcal{L}} G(z,u(z,s))ds, \\ v(z,\tau) &= e^{-i(\tau-\tilde{t})\mathcal{L}} v(z,\tilde{t}) - i\int_{\tilde{t}}^{\tau} e^{-i(\tau-s)\mathcal{L}} G(z,v(z,s))ds. \end{aligned}$$

By Strichartz estimate (3.0.2) and Lemma 6.0.28, we have

$$\begin{aligned} \|u-v\|_{L^q\left((\tilde{t},\tau),L^p(\mathbb{C}^n)\right)} &= \left\| \int_{\tilde{t}}^{\tau} e^{-i(t-s)\mathcal{L}} \left(G(u) - G(v)\right)(z,s)ds \right\|_{L^q\left((\tilde{t},\tau),L^p(\mathbb{C}^n)\right)} \\ &\leq C|\tau - \tilde{t}|^{\frac{q-q'}{qq'}} M^{\alpha}_{\tilde{t},\tau} \|u-v\|_{L^q\left((\tilde{t},\tau),L^p(\mathbb{C}^n)\right)} \end{aligned}$$

for all $\tau \in (\tilde{t}, T^*)$ where $M_{\tilde{t},\tau} = \max\{\|u\|_{L^\infty\left((\tilde{t},\tau),\tilde{W}^{1,2}_{\mathcal{L}}\right)}, \|v\|_{L^\infty\left((\tilde{t},\tau),\tilde{W}^{1,2}_{\mathcal{L}}\right)}\}$, see (6.0.13). Since $u, v \in C\left([t_0, T^*), \tilde{W}^{1,2}_{\mathcal{L}}\right)$, we have $M_{\tilde{t},\tau} < \infty$. Choose $\tau \in [\tilde{t}, T^*)$ sufficiently close to $\tilde{t}$ such that $C|\tau - \tilde{t}|^{\frac{q-q'}{qq'}} M^{\alpha}_{\tilde{t},\tau} = c < 1$, so that

$$0 \leq (1-c)\|u-v\|_{L^q((\tilde{t},\tau):L^p(\mathbb{C}^n))} \leq 0.$$

Hence $u = v$ on the larger interval $[\tilde{t}, \tau]$.

Now let $\theta$ =sup$\{\tilde{T} : t_0 < \tilde{T} < T^* : \|u - v\|_{L^q([t_0,\tilde{T}],L^p)} = 0\}$. If $\theta < T^*$, then for sufficiently small $\epsilon > 0$, choose $\tilde{t} = \theta - \epsilon, \tau = \theta + \epsilon$ and by the above observation, $\|u-v\|_{L^q((\theta-\epsilon,\theta+\epsilon),L^p)} = 0$, which contradicts the definition of $\theta$. Thus we conclude that $\theta = T^*$, proving the uniqueness on $[t_0, T^*)$. Similarly one can show uniqueness on $(T_*, t_0]$.

**Stability:** Stability follows by similar arguments as in Theorem 5.0.26.

# Blowup analysis in $\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$

In this section we show that the maximal solution established in Theorem 6.0.31 is actually a global one. This is established by showing that there is no finite time blow up. We use a blow up analysis as in [12] using the conservation laws, to conclude that there is no finite time blow up.

The mass conservation (1.0.22) formally derived in chapter 1, is valid for $u \in C((T_*, T^*), \tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n))$ but the formal computation for the energy conservation law (1.0.24) given there is valid only for $u(\cdot, t) \in \tilde{W}^{2,2}_{\mathcal{L}}(\mathbb{C}^n)$ for each $t$ in the interval of existence. However, since the Schrödinger equation does not have regularizing property, we can not expect $u$ to be in the second order Sobolev space $\tilde{W}^{2,2}_{\mathcal{L}}(\mathbb{C}^n)$, for the initial data $f \in \tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$. So we need some alternate argument to prove the energy conservation in $\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$.

We deduce the energy conservation for $u \in \tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$ from the equation (1.0.24) valid for $u(\cdot, t) \in \tilde{W}^{2,2}_{\mathcal{L}}(\mathbb{C}^n)$ by an approximation argument, using the stability result obtained for the maximal solution, and a regularization argument on the nonlinearity.

Let $\{f_m\}_{n\in\mathbb{N}}$ be a sequence of functions in $\tilde{W}^{2,2}_{\mathcal{L}}(\mathbb{C}^n)$ such that $f_m \to f$ in $\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$ and for $\epsilon > 0$, set $G_\epsilon(z, u) = e^{-\epsilon\mathcal{L}} G(z, e^{-\epsilon\mathcal{L}} u)$.

In view of estimates (4.0.8), (4.0.9) in Lemma 4.0.17 and estimate (6.0.3) in Lemma 6.0.28, for $v_1, v_2 \in L^\infty(I, \tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n))$, we observe that

$$\begin{aligned}
\|G_\epsilon(z,v_1)-G_\epsilon(z,v_2)\|_{L^{q'}(I,L^{p'})} &\leq C\|G(z,e^{-\epsilon\mathcal{L}}v_1)-G(z,e^{-\epsilon\mathcal{L}}v_2)\|_{L^{q'}(I,L^{p'}(\mathbb{C}^n))}\\
&\leq C|I|^{\frac{q-q'}{qq'}}\|e^{-\epsilon\mathcal{L}}(v_1-v_2)\|_{L^q(I,L^p(\mathbb{C}^n))}\\
&\quad\times\left(\|e^{-\epsilon\mathcal{L}}v_1\|^\alpha_{L^\infty(I,\tilde{W}^{1,2}_{\mathcal{L}})}+\|e^{-\epsilon\mathcal{L}}v_2\|^\alpha_{L^\infty(I,\tilde{W}^{1,2}_{\mathcal{L}})}\right)\\
&\leq C|I|^{\frac{q-q'}{qq'}}\|v_1-v_2\|_{L^q(I,L^p(\mathbb{C}^n))}\\
&\quad\times\left(\|v_1\|^\alpha_{L^\infty(I,\tilde{W}^{1,2}_{\mathcal{L}})}+\|v_2\|^\alpha_{L^\infty(I,\tilde{W}^{1,2}_{\mathcal{L}})}\right) \quad (6.0.15)
\end{aligned}$$

where constant $C$ is independent of $\epsilon$ for $\epsilon\in(0,1]$.

Using estimates (4.0.8), (4.0.9) in Lemma 4.0.17 and estimate (6.0.5) in Proposition 6.0.29, for $v\in L^\infty(I,\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n))\cap L^q(I,\tilde{W}^{1,p}_{\mathcal{L}}(\mathbb{C}^n))$, we observe that

$$\begin{aligned}
\|G_\epsilon(z,v(z,t))\|_{L^{q'}(I,\tilde{W}^{1,p'}_{\mathcal{L}}(\mathbb{C}^n))} &\leq C\|G(z,e^{-\epsilon\mathcal{L}}v(z,t))\|_{L^{q'}(I,\tilde{W}^{1,p'}_{\mathcal{L}}(\mathbb{C}^n))}\\
&\leq C|I|^{\frac{q-q'}{qq'}}\|e^{-\epsilon\mathcal{L}}v\|^\alpha_{L^\infty(I,\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n))}\|e^{-\epsilon\mathcal{L}}v\|_{L^q(I,\tilde{W}^{1,p}_{\mathcal{L}}(\mathbb{C}^n))}\\
&\leq C|I|^{\frac{q-q'}{qq'}}\|v\|^\alpha_{L^\infty(I,\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n))}\|v\|_{L^q(I,\tilde{W}^{1,p}_{\mathcal{L}}(\mathbb{C}^n))} \quad (6.0.16)
\end{aligned}$$

where constant $C$ is independent of $\epsilon$ for $\epsilon\in(0,1]$.

Now we consider the initial value problem

$$i\partial_t u(z,t)-\mathcal{L}u(z,t)=G_\epsilon(z,u), \qquad z\in\mathbb{C}^n,\ t\in\mathbb{R} \qquad (6.0.17)$$

$$u(\cdot,t_0)=f_m. \qquad (6.0.18)$$

Observe that, in view of the estimates (6.0.15) and (6.0.16), the arguments used in the proof of Theorem 6.0.31 is valid for the above problem and we get a unique solution $u^\epsilon_m\in C(I,\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n))$ that satisfies the integral equation (see Lemma 5.0.21)

$$u^\epsilon_m(z,t)=e^{-i(t-t_0)\mathcal{L}}f_m(z)-i\int_{t_0}^t e^{-i(t-s)\mathcal{L}}G_\epsilon(z,u^\epsilon_m(z,s))ds. \qquad (6.0.19)$$

Moreover since $\|f_m\|_{\tilde{W}^{1,2}_{\mathcal{L}}}\to\|f\|_{\tilde{W}^{1,2}_{\mathcal{L}}}$, in view of (6.0.14), we can choose interval $I$ containing $t_0$ small and assume that $u^\epsilon_m$ are defined on $I$ for every $m$ and $\epsilon>0$. For the same reason, in view of (6.0.13), we can also find an $M$ such that

$$\sup_{t\in I}\|u^\epsilon_m\|_{\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)}\leq M \qquad (6.0.20)$$

valid for all $0 < \epsilon \leq 1$ and $m \in \mathbb{N}$.

The following convergence result is crucial in the proof of conservation of energy in $\tilde{W}^{1,2}_{\mathcal{L}}$.

**Proposition 6.0.32** *Let $\epsilon > 0$ and $f \in \tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$. Let $u^\epsilon$ and $u$ be the solutions of the equations (1.0.13), and (6.0.17) respectively with same initial value $u(\cdot, t_0) = u^\epsilon(\cdot, t_0) = f$. Then there exists $T > 0$ such that*

$$u^\epsilon \to u, \quad as \quad \epsilon \to 0$$

*in $L^\infty(I, L^{p_1}(\mathbb{C}^n))$ for all $2 \leq p_1 < \frac{2n}{n-1}$ where $I = (t_0 - T, t_0 + T)$.*

**Proof**. By local existence argument, the solution $u$ and $u^\epsilon$ exists on an interval $(t_0 - T, t_0 + T)$, where $T$ depends only on $\|f\|_{\tilde{W}^{1,2}_{\mathcal{L}}}$, see (6.0.14). In view of (1.0.21) and Lemma 5.0.21, we have

$$u^\epsilon(z,t) - u(z,t) = -i \int_{t_0}^{t} e^{-i(t-s)\mathcal{L}}[G_\epsilon(z,u) - G(z,u)]ds.$$

Hence by estimate (3.0.2), we see that

$$\|u^\epsilon - u\|_{L^{q_1}(I, L^{p_1}(\mathbb{C}^n))} \leq C \|G_\epsilon - G\|_{L^{q'}(I, L^{p'}(\mathbb{C}^n))} \tag{6.0.21}$$

for every admissible pair $(q_1, p_1)$. Since $G_\epsilon(z,u) = e^{-\epsilon\mathcal{L}} G(z, e^{-\epsilon\mathcal{L}} u)$, by adding and subtracting appropriate terms, we see that

$$\begin{aligned} \|G_\epsilon(z,u^\epsilon) \quad - \quad & G(z,u)\|_{L^{q'}(I, L^{p'}(\mathbb{C}^n))} \\ \leq \quad & \|G_\epsilon(z,u^\epsilon) - G_\epsilon(z,u)\|_{L^{q'}(I, L^{p'}(\mathbb{C}^n))} \\ & + \|e^{-\epsilon\mathcal{L}}[G(z, e^{-\epsilon\mathcal{L}}u) - G(z,u)]\|_{L^{q'}(I, L^{p'}(\mathbb{C}^n))} \\ & + \|e^{-\epsilon\mathcal{L}} G(z,u) - G(z,u)\|_{L^{q'}(I, L^{p'}(\mathbb{C}^n))}. \end{aligned} \tag{6.0.22}$$

We first estimate the last two terms. In view of Lemma 4.0.17, we have

$$\begin{aligned} \|e^{-\epsilon\mathcal{L}} G(\cdot, u(\cdot,t)) - G(\cdot, u(\cdot,t))\|_{L^{p'}(\mathbb{C}^n)} &= o(\epsilon) \\ \|e^{-\epsilon\mathcal{L}} u(\cdot,t) - u(\cdot,t)\|_{L^{p}(\mathbb{C}^n)} &= o(\epsilon) \end{aligned}$$

as $\epsilon \to 0$, for each $t \in I$. Hence an application of the dominated convergence theorem in the $t$-variable shows that as $\epsilon \to 0$

$$\|e^{-\epsilon\mathcal{L}}G(z,u) - G(z,u)\|_{L^{q'}(I,L^{p'}(\mathbb{C}^n))} = o(\epsilon) \tag{6.0.23}$$
$$\|e^{-\epsilon\mathcal{L}}u - u\|_{L^{q}(I,L^{p}(\mathbb{C}^n))} = o(\epsilon). \tag{6.0.24}$$

Equation (6.0.23) gives the required estimate for the third term in the RHS of (6.0.22). By estimate (4.0.8) and estimate (6.0.3), we see that

$$\begin{aligned}&\|e^{-\epsilon\mathcal{L}}[G(z,e^{-\epsilon\mathcal{L}}u) - G(z,u)]\|_{L^{q'}(I,L^{p'}(\mathbb{C}^n))} \\ &\leq CT^{\frac{q-q'}{qq'}}\|u\|^{\alpha}_{L^\infty(I,\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n))}\|e^{-\epsilon\mathcal{L}}u - u\|_{L^q(I_T,L^p(\mathbb{C}^n))} = o(\epsilon)\end{aligned} \tag{6.0.25}$$

as $\epsilon \to 0$ by (6.0.24), which gives the estimate for the second term on the RHS of (6.0.22). Again estimate (6.0.15) gives the inequality

$$\|G_\epsilon(z,u^\epsilon) - G_\epsilon(z,u)\|_{L^{q'}(I,L^{p'}(\mathbb{C}^n))} \leq CT^{\frac{q-q'}{qq'}}M^{\alpha}\|u^\epsilon - u\|_{L^q(I,L^p(\mathbb{C}^n))}. \tag{6.0.26}$$

Now from the estimate (6.0.21), (6.0.22) and in view of (6.0.26), (6.0.25) and (6.0.23) we see that

$$\|u^\epsilon - u\|_{L^{q_1}(I,L^{p_1}(\mathbb{C}^n))} \leq CT^{\frac{q-q'}{qq'}}M^{\alpha}\|u^\epsilon - u\|_{L^q(I,L^p(\mathbb{C}^n))} + o(\epsilon) \tag{6.0.27}$$
$$\|u^\epsilon - u\|_{L^{q}(I,L^{p}(\mathbb{C}^n))} \leq CT^{\frac{q-q'}{qq'}}M^{\alpha}\|u^\epsilon - u\|_{L^q(I,L^p(\mathbb{C}^n))} + o(\epsilon). \tag{6.0.28}$$

Let us choose $T$ sufficiently small so that $CT^{\frac{q-q'}{qq'}}M^{\alpha} < \frac{1}{2}$ with constant $C$ in the inequality (6.0.28). This gives

$$\|u^\epsilon - u\|_{L^q(I,L^p(\mathbb{C}^n))} = o(\epsilon)$$

as $\epsilon \to 0$. From estimate (6.0.27) with pair $(\infty, 2)$, we have

$$\|u^\epsilon - u\|_{L^\infty(I,L^2(\mathbb{C}^n))} = o(\epsilon). \tag{6.0.29}$$

Now we prove

$$\|u^\epsilon - u\|_{L^\infty(I,L^{p_1}(\mathbb{C}^n))} = o(\epsilon) \tag{6.0.30}$$

as $\epsilon \to 0$ for all $2 \leq p_1 < \frac{2n}{n-1}$.

Now we choose $r \in (p_1, \frac{2n}{n-1})$ and $\lambda \in (0,1)$ such that $\frac{1}{p_1} = \frac{\lambda}{2} + \frac{1-\lambda}{r}$. Thus by Hölders inequality with indices $\frac{r}{(1-\lambda)p_1}$ and $\frac{2}{\lambda p_1}$, we get

$$\|u^\epsilon - u\|_{L^{p_1}(\mathbb{C}^n)} \leq \|u^\epsilon - u\|^{1-\lambda}_{L^r(\mathbb{C}^n)} \|u^\epsilon - u\|^{\lambda}_{L^2(\mathbb{C}^n)}.$$

Recall that by local existence theorem $u, u^\epsilon \in C(I, \tilde{W}^{1,2}_{\mathcal{L}})$ and there exist $M$ such that

$$\sup_{t\in I} \|u\|_{\tilde{W}^{1,2}_{\mathcal{L}}} \leq M, \quad \sup_{t\in I} \|u^\epsilon\|_{\tilde{W}^{1,2}_{\mathcal{L}}} \leq M$$

by (6.0.20). Thus by Lemma 4.0.14, we have $\|u^\epsilon - u\|_{L^r(\mathbb{C}^n)} \leq \|u - u^\epsilon\|_{\tilde{W}^{1,2}_{\mathcal{L}}} \leq 2M$ for $2 \leq r \leq \frac{2n}{n-1}$. Thus we see that

$$\|u^\epsilon - u\|_{L^\infty(I, L^{p_1}(\mathbb{C}^n))} \leq (2M)^{1-\lambda} \|u^\epsilon - u\|^{\lambda}_{L^\infty(I, L^2(\mathbb{C}^n))}. \qquad (6.0.31)$$

This proves (6.0.30) in view of (6.0.29).

Our main result in this section is the following Theorem.

**Theorem 6.0.33** *(Global well posedness) Let $f \in \tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$ and $\psi$ be real valued function as in (1.0.15) and (1.0.16) with $\alpha \in [0, \frac{2}{n-1})$. Then the solution $u \in C((T_*, T^*), \tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)) \cap L^{q_1}_{loc}\left((T_*, T^*), \tilde{W}^{1,p_1}_{\mathcal{L}}\right)$ of the initial value problem (1.0.13), (1.0.14) as obtained in Theorem 6.0.31 satisfies the following properties:*

**(i)(Conservation of charge)** $\|u(\cdot, t)\|_{L^2(\mathbb{C}^n)} = \|f\|_{L^2(\mathbb{C}^n)}, \; t \in (T_*, T^*)$.

**(ii)(Conservation of energy)** $E(u(\cdot, t)) = E(u(\cdot, t_0)), \; t \in (T_*, T^*)$, *where*

$$E(u) = \frac{1}{4} \sum_{j=1}^{n} \int_{\mathbb{C}^n} \left(|Z_j u(z,t)|^2 + |\overline{Z}_j u(z,t)|^2\right) dz + \int_{\mathbb{C}^n} \tilde{G}(z, |u|) dz. \; (6.0.32)$$

**(iii)(Global existence)** *If $\psi \geq 0$ is nonnegative, the solution extends to the whole of $\mathbb{R}$. For nonpositive $\psi$, the solution is global if $0 \leq \alpha < \frac{2}{n}$.*

**Proof**. The proof of conservation of charge (1.0.22) given in chapter 1 is valid for $u \in ((T_*, T^*), \tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n))$ as observed before. Thus we need to prove energy conservation. Let $u^\epsilon_m(z,t)$ denote the solution to the regularized problem (6.0.17), (6.0.18). Then $u^\epsilon_m \in C(I, \tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)) \cap L^{q_1}(I; \tilde{W}^{1,p_1}_{\mathcal{L}}(\mathbb{C}^n))$ for every admissible pair $(q_1, p_1)$ and is given by the integral equation (6.0.19), where $I =$

$(t_0-T,t_0+T)$. Since $f_m\in\tilde{W}^{2,2}_{\mathcal{L}}(\mathbb{C}^n)$, therefore from equation (4.0.12) and Theorem 3.0.7 $e^{-i(t-t_0)\mathcal{L}}f_m\in C(I,\tilde{W}^{2,2}_{\mathcal{L}}(\mathbb{C}^n))$. From estimate (6.0.16), $G_\epsilon(z,u^\epsilon_m)\in L^{q'}(I,\tilde{W}^{1,p'}_{\mathcal{L}}(\mathbb{C}^n))$ and from estimate (4.0.10) $G_\epsilon(z,u^\epsilon_m)\in L^{q'}(I,\tilde{W}^{2,p'}_{\mathcal{L}}(\mathbb{C}^n))$. From these observations, equation (6.0.19) and equation (4.0.12), we conclude that $u^\epsilon_m\in C(I,\tilde{W}^{2,2}_{\mathcal{L}}(\mathbb{C}^n))$ for each $m\in\mathbb{N}$ and $\epsilon>0$ .

Since $\langle G_\epsilon(z,u^\epsilon_m),\partial_t u^\epsilon_m\rangle=\langle G(z,e^{-\epsilon\mathcal{L}}u^\epsilon_m),e^{-\epsilon\mathcal{L}}\partial_t u^\epsilon_m\rangle$, taking $L^2(\mathbb{C}^n)$ inner product with $\partial_t u^\epsilon_m$ on both sides of the equation (6.0.17) with $u$ replaced by $u^\epsilon_m$ and a computation similar to the one that led to (1.0.24) yields the energy conservation:

$$\frac{1}{4}\sum_{j=1}^{n}(\|Z_ju^\epsilon_m(\cdot,t)\|_2^2+\left\|\overline{Z}_ju^\epsilon_m\right\|_2^2)+\int_{\mathbb{C}^n}\tilde{G}(z,|e^{-\epsilon\mathcal{L}}u^\epsilon_m|)dz=E(f_m). \quad (6.0.33)$$

By stability $u^\epsilon_m(\cdot,t)\to u^\epsilon(\cdot,t)$ in $\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$ for each $t\in I$ and also $u^\epsilon_m\to u^\epsilon$ in $L^{q_1}(I;\tilde{W}^{1,p_1}_{\mathcal{L}}(\mathbb{C}^n))$ for every admissible pair $(q_1,p_1)$ as $m\to\infty$. By Lemma 4.0.14 $C\cap L^\infty(I,L^p(\mathbb{C}^n))\subset C\cap L^\infty(I,\tilde{W}^{1,2}_{\mathcal{L}})$ and $u^\epsilon_m\to u^\epsilon$ in $L^\infty(I,L^p(\mathbb{C}^n))$ as $m\to\infty$. From estimate (4.0.8), $e^{-\epsilon\mathcal{L}}u^\epsilon_m\to e^{-\epsilon\mathcal{L}}u^\epsilon$ in $L^\infty(I,L^p(\mathbb{C}^n))$ as $m\to\infty$.

Thus letting $m\to\infty$ in (6.0.33) and using Proposition 6.0.30, we get the energy conservation for $u^\epsilon$:

$$E(f)=\frac{1}{4}\sum_{j=1}^{n}(\|Z_ju^\epsilon(\cdot,t)\|_2^2+\left\|\overline{Z}_ju^\epsilon(\cdot,t)\right\|_2^2)+\lambda\int_{\mathbb{C}^n}\tilde{G}(z,|e^{-\epsilon\mathcal{L}}u^\epsilon|)dz \quad (6.0.34)$$

for each $\epsilon>0$. From Proposition 6.0.32) $u^\epsilon\to u$ in $L^\infty(I,L^p(\mathbb{C}^n))$ and therefore from Proposition 6.0.30, we see that

$$\lim_{\epsilon\to0}\int_{\mathbb{C}^n}\tilde{G}(z,|e^{-\epsilon\mathcal{L}}u^\epsilon|)dz=\int_{\mathbb{C}^n}\tilde{G}(z,|u|)dz. \quad (6.0.35)$$

From Proposition 6.0.32 $u^\epsilon(\cdot,t)\to u(\cdot,t)$ in $L^2(\mathbb{C}^n)$ for each $t\in I$. Therefore for any sequence $\{\epsilon_m\}$ of positive real numbers converging to 0, we see

$$\begin{aligned}\|Su(\cdot,t)\|_{L^2(\mathbb{C}^n)} &= \sup_{\phi\in\mathcal{S}(\mathbb{C}^n),\|\phi\|_{L^2}\le1}|\langle Su(\cdot,t),\phi\rangle|\\ &= \sup_{\phi\in\mathcal{S}(\mathbb{C}^n),\|\phi\|_{L^2}\le1}|\langle u(\cdot,t),S^*\phi\rangle|\\ &= \sup_{\phi\in\mathcal{S}(\mathbb{C}^n),\|\phi\|_{L^2}\le1}\lim_{m\to\infty}|\langle u^{\epsilon_m}(\cdot,t),S^*\phi\rangle|\end{aligned}$$

$$\begin{aligned}
&= \sup_{\phi\in\mathcal{S}(\mathbb{C}^n),\|\phi\|_{L^2}\le 1} \lim_{m\to\infty} |\langle S u^{\epsilon_m}(\cdot,t),\phi\rangle| \\
&\le \sup_{\phi\in\mathcal{S}(\mathbb{C}^n),\|\phi\|_{L^2}\le 1} \liminf_{m\to\infty} \|S u^{\epsilon_m}(\cdot,t)\|_{L^2}\|\phi(\cdot,t)\|_{L^2} \\
&\le \liminf_{m\to\infty} \|S u^{\epsilon_m}(\cdot,t)\|_{L^2(\mathbb{C}^n)} \qquad (6.0.36)
\end{aligned}$$

for $S = Z_j, \bar{Z}_j$.

Taking limit as $\epsilon_m \to 0$ in (6.0.34), in view of the inequality (6.0.36) and the identity (6.0.35), we see that $E(u(\cdot,t)) \le E(f) = E(u(\cdot,t_0))$ for each $t \in I = [t_0-T, t_0+T]$. This shows that $E(u(\cdot,t))$ has local maximum at $t_0$. This argument can be repeated for any point in $(T_*, T^*)$ instead of $t_0$. Since $t \to E(u(\cdot,t))$ is continuous and it has local maximum at every point, therefore $E(u(\cdot,t))$ is constant on $(T_*, T^*)$. This proves the energy conservation.

**Global existence:** Now we will prove global existence. Let us assume that $\psi$ is nonnegative. Then $\tilde{G} : \mathbb{C}^n \times [0,\infty) \to [0,\infty)$ is also nonnegative and by conservation of energy

$$\begin{aligned}
E(f) &= E(u(z,t)) \\
&= \frac{1}{4}\sum_{j=1}^{n}(\|Z_j u(z,t)\|_2^2 + \left\|\overline{Z}_j u(z,t)\right\|_2^2) + \int_{\mathbb{C}^n} \tilde{G}(z,|u|)dz \qquad (6.0.37) \\
&\ge \frac{1}{4}\sum_{j=1}^{n}(\|Z_j u(z,t)\|_2^2 + \left\|\overline{Z}_j u(z,t)\right\|_2^2) \nrightarrow \infty
\end{aligned}$$

as $t \to T_*$ or $t \to T^*$. By blowup alternative and Lemma 4.0.13, we have global existence, i.e., $-T_* = T^* = \infty$.

To deal with nonpositive $\psi$ with $0 \le \alpha < \frac{2}{n}$, we first get an estimate for $\int_{\mathbb{C}^n} \tilde{G}(z,u(z,t))dz$ comparing with $\|u(\cdot,t)\|_{\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)}$. In view of (1.0.25) and the fact that $\alpha + 2 = p$, we see that

$$\int_{\mathbb{C}^n} \tilde{G}(z,u)dz \le C \int_{\mathbb{C}^n} |u(z,t)|^p dz.$$

Since $p = 2+\alpha$, $\alpha \in [0, \frac{2}{n})$, we have $2 \le p \le \frac{4}{4-p} < \frac{2n}{n-1}$. Since $p \le \frac{4}{4-p}$, we can choose $p_1$ such that $\frac{4}{4-p} < p_1 < \frac{2n}{n-1}$. Let $\theta \in (0,1]$ such that $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{p_1}$. Then $p\theta = \frac{2(p_1-p)}{(p_1-2)}$, $(1-\theta)p = \frac{p_1(p-2)}{(p_1-2)}$. An application of Hölder's inequality with indices

$\frac{2}{p\theta}$ and $\frac{p_1}{(1-\theta)p}$ in the above shows that

$$\begin{aligned}\int_{\mathbb{C}^n} \tilde{G}(z,u)dz &\leq C\int_{\mathbb{C}^n} |u(z,t)|^{\theta p}\, |u(z,t)|^{(1-\theta)p}\, dz \\ &\leq C\left(\int_{\mathbb{C}^n} |u(z,t)|^2 dz\right)^{\frac{p\theta}{2}} \left(\int_{\mathbb{C}^n} |u(z,t)|^{p_1} dz\right)^{\frac{(1-\theta)p}{p_1}} \\ &\leq C\|u(\cdot,t)\|^{p\theta}_{L^2(\mathbb{C}^n)} \|u(\cdot,t)\|^{(1-\theta)p}_{\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)} \\ &\leq C\|f\|^{p\theta}_{L^2(\mathbb{C}^n)} \|u(\cdot,t)\|^{(1-\theta)p}_{\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)}. \end{aligned} \tag{6.0.38}$$

In the third inequality, we use embedding theorem (Lemma 4.0.14) and fourth inequality follows from conservation of charge. Since $\psi$ is nonpositive, from (1.0.23) $\tilde{G}$ is also nonpositive. Hence from (6.0.37), by Lemma 4.0.13 and in view of the estimate (6.0.38), we see that for all $t \in (T_*, T^*)$

$$E(f) \geq C_1 \|u(\cdot,t)\|^2_{\tilde{W}^{1,2}_{\mathcal{L}}} - C\ \|f\|^{p\theta}_{L^2(\mathbb{C}^n)}\ \|u(\cdot,t)\|^{(1-\theta)p}_{\tilde{W}^{1,2}_{\mathcal{L}}}. \tag{6.0.39}$$

Note that for $\alpha < \frac{2}{n}$, $(1-\theta)p = \frac{p_1(p-2)}{(p_1-2)} < 2$. Thus the above inequality shows that $\|u(\cdot,t)\|_{\tilde{W}^{1,2}_{\mathcal{L}}}$, can not blowup as $t \to T_*$ or $t \to T^*$. Hence by blow up alternative, the maximal interval is $\mathbb{R}$ and proves the global existence.

## Critical Case $\alpha = \frac{2}{n-1}$

In Theorem 6.0.31, we proved the local well posedness in $\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$ for subcritical case $\alpha \in [0, \frac{2}{n-1})$. In this section we will consider critical case $\alpha = \frac{2}{n-1}$ with $n \geq 2$. In Theorem 6.0.31, for critical case $\alpha = \frac{2}{n-1}$ the main difficulty is that we don't have any $q > 2$ so that $(q, 2+\frac{2}{n-1})$ becomes an admissible pair. We overcome this difficulty by considering admissible pair $(\gamma, \rho)$ and by using embedding theorem (Lemma 4.0.14), where

$$\rho = \frac{2n^2}{n^2-n+1}, \quad \gamma = \frac{2n}{n-1}.$$

To treat the critical case, we adopt truncation argument of Cazenave and Weissler [7]. To prove local existence, we truncate the nonlinearity $G$ and obtain solution for the truncated problem. We obtain solution $u$ for the nonlinearity $G$ by using Strichartz estimates and by passing to the limit.

For $m \geq 1$, consider $G_m(z,u) = \psi_m(z,|u|)u : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}$, where

$$\psi_m(z,\sigma) = \begin{cases} \psi(z,\sigma) & \text{if } 0 \leq \sigma \leq m \\ m^2\left(\frac{\psi(z,\sigma)}{\sigma^2} - \frac{\psi(z,m)}{\sigma^2} + \frac{\psi(z,m)}{m^2}\right) & \text{if } \sigma \geq m. \end{cases}$$

For $m = 0$, we define $G_0(z,u) = G(z,u)$ and $\psi_0(z,|u|) = \psi(z,|u|)$. Note that $\psi_m$ is differentiable at $\sigma = m$ with respect to $\sigma$ and also note that $G_m$ will satisfy (1.0.15) and (1.0.16) with $\alpha = \frac{2}{n-1}$ as well as $\alpha = 0$. For $m \geq 1$, $G_m(z,\cdot) : \mathbb{C} \to \mathbb{C}$ is globally Lipschitz from mean value theorem and

$$|G_m(z,u) - G_m(z,v)| \leq C_m|u-v| \quad \text{for} \quad m \geq 1 \tag{6.0.40}$$

where constant $C_m$ depends on $m \in \mathbb{N}$ but independent of $z \in \mathbb{C}^n$ and $u, v \in C$. Moreover by mean value theorem we also see that

$$|G_m(z,u) - G_m(z,v)| \leq C(|u| + |v|)^{\frac{2}{n-1}}|u-v| \quad \text{for} \quad m \geq 0 \tag{6.0.41}$$

where constant $C$ is independent of $m \in \mathbb{Z}_{\geq 0}, z \in \mathbb{C}^n$ and $u, v \in C$.

Since $F_0$ satisfies estimate (1.0.16) with $\alpha = \frac{2}{n-1}$, we conclude that

$$|F_m(z,\sigma)| \leq C\sigma^{\frac{2}{n-1}}, \tag{6.0.42}$$

where $F_m = \psi_m, \partial_{x_j}\psi_m, \partial_{y_j}\psi_m, \sigma\partial_\sigma\psi_m(x,y,\sigma)$ with $1 \leq j \leq n$ and constant $C$ is independent of $m$.

In view of Duhamel's formula (see, Lemma 5.0.21) and in order to find solution for given IVP (1.0.13), (1.0.14) with initial value $f \in \tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$ and nonlinearity $G_m$, it is sufficient to find the solution of the following equation

$$u(z,t) = e^{-i(t-t_0)\mathcal{L}}f(z) - i\int_{t_0}^{t} e^{-i(t-s)\mathcal{L}}G(z,u_m(z,s))ds.$$

This reduces the existence theorem for the solution to the nonlinear Schrödinger equation to a fixed point theorem for the operator with $m \geq 0$

$$\mathcal{H}_m(u)(z,t) = e^{-i(t-t_0)\mathcal{L}}f(z) - i\int_{t_0}^{t} e^{-i(t-s)\mathcal{L}}G_m(z,u(z,s))ds. \tag{6.0.43}$$

# Some auxilliary estimates

**Lemma 6.0.34** *Let* $u, v \in L^\gamma\left(I, \tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)$ *for some interval* $I$*, then the following estimate holds for each* $m \in \mathbb{Z}_{\geq 0}$

$$\begin{aligned} \|G_m(z,u) - G_m(z,v)\|_{L^{\gamma'}\left(I, L^{\rho'}(\mathbb{C}^n)\right)} &\leq C\|u-v\|_{L^\gamma(I,L^\rho(\mathbb{C}^n))}\times \\ &\left(\|u\|_{L^\gamma\left(I,\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)} + \|v\|_{L^\gamma\left(I,\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)}\right)^{\frac{2}{n-1}} \end{aligned} \tag{6.0.44}$$

*where the constant* $C$ *is independent of* $u, v, m, t_0$ *and* $I$*.*

**Proof**. Since $\frac{1}{\rho'} = \frac{1}{\rho} + \frac{n-1}{n^2} = \frac{1}{\rho} + \frac{2}{n-1}\cdot\frac{n-1}{n\gamma}$, by using Hölder's inequality in the $z$-variable in (6.0.41) and by embedding theorem (Lemma 4.0.14), we get

$$\begin{aligned} &\|G_m(\cdot, u(\cdot,t)) - G_m(\cdot, v(\cdot,t))\|_{L^{\rho'}(\mathbb{C}^n)} \\ \leq\ & C\|(u-v)(\cdot,t)\|_{L^\rho(\mathbb{C}^n)}\left(\|u(\cdot,t)\|_{L^{\frac{n\gamma}{n-1}}(\mathbb{C}^n)} + \|v(\cdot,t)\|_{L^{\frac{n\gamma}{n-1}}(\mathbb{C}^n)}\right)^{\frac{\gamma}{n}} \\ \leq\ & C\|(u-v)(\cdot,t)\|_{L^\rho(\mathbb{C}^n)}\left(\|u(\cdot,t)\|_{\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)} + \|v(\cdot,t)\|_{\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)}\right)^{\frac{\gamma}{n}} \end{aligned} \tag{6.0.45}$$

for each $t \in I$. Since $\frac{1}{\gamma'} = \frac{1}{\gamma} + \frac{1}{n}$, by taking $L^{\gamma'}$ norm in the $t$-variable in this inequality and then by using the Höder's inequality we get the desired estimate (6.0.44).

**Lemma 6.0.35** *Let* $I$ *be a bounded interval and* $u \in L^\infty(I, \tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)) \cap L^\gamma$ $(I, \tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n))$*, then following estimate holds*

$$\begin{aligned} &\|G_m(z,u(z,t)) - G(z,u(z,t))\|_{L^{\gamma'}\left(I,L^{\rho'}(\mathbb{C}^n)\right)} \\ &\leq C|I|^{\frac{n-1}{2n}} m^{-\frac{1}{n(n-1)}}\|u\|^{\frac{n^2-n+1}{n(n-1)}}_{L^\infty\left(I,\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)\right)}\|u\|^{\frac{2}{n-1}}_{L^\gamma\left(I,\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)} \end{aligned}$$

*for all* $m \geq 1$*, where the constant* $C$ *is independent of* $m, u$ *and* $I$*.*

**Proof**. Note that

$$G_m(z,u(z,t)) - G(z,u(z,t)) = (u\chi_{|u(z,t)|>m}(z,t))(\psi_m(z,|u|) - \psi(z,|u|)).$$

Therefore $|G_m(z,u(z,t)) - G(z,u(z,t))| \leq C|u\chi_{|u(z,t)|>m}(z,t)|\ |u|^{\frac{2}{n-1}}$. By Taking

$L^{\rho'}$-norm in the $z$-variable, we have

$$\begin{aligned}\|G_m(z,u)-G(z,u)\|_{L^{\rho'}(\mathbb{C}^n)} &\leq C\|u\chi_{|u|>m}(\cdot,t)\|_{L^\rho(\mathbb{C}^n)}\|u(\cdot,t)\|^{\frac{\gamma}{n}}_{L^{\frac{n\gamma}{n-1}}(\mathbb{C}^n)} \\ &\leq C\|u\chi_{|u|>m}(\cdot,t)\|_{L^\rho(\mathbb{C}^n)}\|u(\cdot,t)\|^{\frac{\gamma}{n}}_{\tilde{W}^{1,\rho}_{\mathcal{L}}}. \quad (6.0.46)\end{aligned}$$

Now we observe the following

$$\begin{aligned}\|u\ \chi_{|u|>m}(\cdot,t)\|^\rho_{L^\rho(\mathbb{C}^n)} &= \int_{\mathbb{C}^n}|u|^\rho\chi_{|u|>m}(z,t)dz \\ &\leq \int_{\mathbb{C}^n} m^{-\frac{\rho}{n(n-1)}}\ |u|^{\frac{2n}{n-1}}dz \\ &\leq m^{-\frac{\rho}{n(n-1)}}\ \|u\|^{\frac{(n^2-n+1)\rho}{n(n-1)}}_{L^{\frac{2n}{n-1}}(\mathbb{C}^n)} \\ &\leq m^{-\frac{\rho}{n(n-1)}}\ \|u\|^{\frac{(n^2-n+1)\rho}{n(n-1)}}_{\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)} \\ \|u\ \chi_{|u|>m}(\cdot,t)\|_{L^\rho} &\leq m^{-\frac{1}{n(n-1)}}\ \|u\|^{\frac{(n^2-n+1)}{n(n-1)}}_{\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)}.\end{aligned}$$

By taking $L^\gamma$-norm in the $t$-variable we have

$$\|u\chi_{|u|>m}\|_{L^\gamma(I,L^\rho(\mathbb{C}^n))} \leq |I|^{\frac{n-1}{2n}}m^{-\frac{1}{n(n-1)}}\|u\|^{\frac{(n^2-n+1)}{n(n-1)}}_{L^\infty\left(I,\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)\right)}. \quad (6.0.47)$$

By taking $L^{\gamma'}$-norm in the $t$-variable in the estimate (6.0.46) and using Hölder's inequality, we get

$$\|G_m(z,u)-G(z,u)\|_{L^{\gamma'}\left(I,L^{\rho'}\right)} \leq C\|u\chi_{|u|>m}\|_{L^\gamma(I,L^\rho)}\|u\|^{\frac{2}{n-1}}_{L^\gamma\left(I,\tilde{W}^{1,\rho}_{\mathcal{L}}\right)}.$$

By using inequality (6.0.47) in the above inequality, we get the desired estimate.

**Lemma 6.0.36** *Let $u \in L^\gamma\left(I,\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)$ for some interval $I$. Then for each $m \in \mathbb{Z}_{\geq 0}$, $G_m(z,u(z,t)) \in L^{\gamma'}\left(I,\tilde{W}^{1,\rho'}_{\mathcal{L}}(\mathbb{C}^n)\right)$ and the following estimates hold:*

$$\|SG_m(z,u(z,t))\|_{L^{\gamma'}\left(I,L^{\rho'}(\mathbb{C}^n)\right)} \leq C\|u\|^{\frac{n+1}{n-1}}_{L^\gamma\left(I,\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)} \quad (6.0.48)$$

$$\|G_m(z,u(z,t))\|_{L^{\gamma'}\left(I,\tilde{W}^{1,\rho'}_{\mathcal{L}}(\mathbb{C}^n)\right)} \leq C\|u\|^{\frac{n+1}{n-1}}_{L^\gamma\left(I,\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)} \quad (6.0.49)$$

*where $S = Id, Z_j, \overline{Z}_j$, $1 \leq j \leq n$ and the constant $C$ is independent of $u$ and $I$.*

**Proof**. Since $\psi_m, \partial_{x_j}\psi_m, \partial_{y_j}\psi_m, |u|\partial_{|u|}\psi_m$ satisfy estimate (6.0.42), we have

$$|SG_m(z,u)| \le C|u|^{\frac{2}{n-1}}(|u| + |Z_j u| + |\bar{Z}_j u|)$$

where $S = Id, Z_j, \overline{Z}_j$ $(1 \le j \le n)$, see estimate 6.0.9. Now estimate (6.0.48) follows from Hölder's inequality and embedding theorem (Lemma 4.0.14) as we used in the proof of Lemma 6.0.34. Estimate (6.0.49) is a consequence of estimate (6.0.48).

**Proposition 6.0.37** *Let $I$ be a bounded interval such that $t_0 \in \overline{I}$.*

**(i)** *If $u, v \in L^\gamma\left(I, \tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)$, then $\mathcal{H}_m u - \mathcal{H}_m v \in L^{q_1}(I, L^{p_1}(\mathbb{C}^n))$ for every admissible pair $(q_1, p_1)$, for every $m \ge 0$ and the following estimate holds:*

$$\begin{aligned} &\|\mathcal{H}_m u - \mathcal{H}_m v\|_{L^{q_1}(I, L^{p_1}(\mathbb{C}^n))} \qquad (6.0.50)\\ \le\ & C\|u - v\|_{L^\gamma(I, L^\rho(\mathbb{C}^n))}\left(\|u\|_{L^\gamma\left(I, \tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)} + \|v\|_{L^\gamma\left(I, \tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)}\right)^{\frac{2}{n-1}}. \end{aligned}$$

**(ii)** *If $u \in L^\infty\left(I, \tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)\right) \cap L^\gamma\left(I, \tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)$, then $\mathcal{H}_m u - \mathcal{H}u \in L^{q_1}(I, L^{p_1}(\mathbb{C}^n))$ for every admissible pair $(q_1, p_1)$, for every $m \ge 1$ and the following estimate holds*

$$\begin{aligned} &\|\mathcal{H}_m u - \mathcal{H}u\|_{L^{q_1}(I, L^{p_1}(\mathbb{C}^n))} \qquad (6.0.51)\\ \le\ & C|I|^{\frac{n-1}{2n}} m^{-\frac{1}{n(n-1)}} \|u\|^{\frac{n^2-n+1}{n(n-1)}}_{L^\infty\left(I, \tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)\right)} \|u\|^{\frac{2}{n-1}}_{L^\gamma\left(I, \tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)} \end{aligned}$$

*where the constant $C$ is independent of $u, v, m$ and $t_0$.*

**Proof**. Estimate (6.0.50) follows from Strichartz estimates (Theorem 3.0.7) and Lemma 6.0.34, whereas estimate (6.0.51) follows from Theorem 3.0.7 and Lemma 6.0.35.

Now we state the following Proposition, which is useful in proving continuous dependence. Proof is similar to Proposition 5.0.24. But for completeness, we give the proof.

**Proposition 6.0.38** *Let $\Phi$ be a continuous complex valued function on $\mathbb{C}$ such that $|\Phi(w)| \le C|w|^{\frac{2}{n-1}}$ with $n \ge 2$. Let $\{u_m\}$ be a bounded sequence in $L^\gamma\left(I, \tilde{W}^{1,\rho}_{\mathcal{L}}\right)$*

*for some interval I. If* $u_m \to u$ *in* $L^\gamma(I, L^\rho(\mathbb{C}^n))$ *then* $u \in L^\gamma\left(I, \tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)$ *and* $[\Phi(u_m) - \Phi(u)]Su \to 0$ *in* $L^{\gamma'}\left(I, L^{\rho'}(\mathbb{C}^n)\right)$, *for* $S = Id, Z_j, \overline{Z}_j; 1 \leq j \leq n$.

**Proof**. First we will prove $u \in L^\gamma\left(I, \tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)$. By a duality argument (also see Lemma A.2.1 in [12]), we have

$$\begin{aligned}
\|Su\|_{L^\gamma(I,L^\rho(\mathbb{C}^n))} &= \sup_{\phi \in C_c^\infty(\mathbb{C}^n \times I), \|\phi\|_{L^{\gamma'}(I,L^{\rho'}(\mathbb{C}^n))} \leq 1} \left| \langle Su, \phi \rangle_{z,t} \right| \\
&= \sup_{\phi} \left| \langle u, S^*\phi \rangle_{z,t} \right| \\
&= \sup_{\phi} \lim_{m\to\infty} \left| \langle u_m, S^*\phi \rangle_{z,t} \right| \\
&= \sup_{\phi} \lim_{m\to\infty} \left| \langle Su_m, \phi \rangle_{z,t} \right| \\
&\leq \sup_{\phi} \liminf_{m\to\infty} \|Su_m\|_{L^\gamma(I,L^\rho(\mathbb{C}^n))} \|\phi\|_{L^{\gamma'}(I,L^{\rho'}(\mathbb{C}^n))} \\
&\leq \liminf_{m\to\infty} \|Su_m\|_{L^\gamma(I,L^\rho(\mathbb{C}^n))} \qquad (6.0.52)
\end{aligned}$$

for $S = Z_j, \bar{Z}_j$; $1 \leq j \leq n$. Therefore

$$\|u\|_{L^\gamma(I,\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n))} \leq \liminf_{m\to\infty} \|u_m\|_{L^\gamma(I,\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n))} < \infty.$$

Since $u_m \to u$ in $L^\gamma(I, L^\rho(\mathbb{C}^n))$, we can extract a subsequence still denoted by $u_k$ such that

$$\|u_{k+1} - u_k\|_{L^\gamma(I,L^\rho(\mathbb{C}^n))} \leq \frac{1}{2^k}$$

for all $k \geq 1$ and $u_k(z,t) \to u(z,t)$ a.e. Hence by continuity of $\Phi$,

$$[\Phi(u_k) - \Phi(u)]Su \to 0 \quad \text{for a.e } (z,t) \in \mathbb{C}^n \times I. \qquad (6.0.53)$$

We establish the norm convergence by appealing to a dominated convergence argument in $z$ and $t$ variables successively.

Consider the function $H(z,t) = \sum_{k=1}^\infty |u_{k+1}(z,t) - u_k(z,t)|$. Clearly $H \in L^\gamma(I, L^\rho(\mathbb{C}^n))$. Also for $l > k$,

$$|(u_l - u_k)(z,t)| \leq |u_l - u_{l-1}| + \cdots + |u_{k+1} - u_k| \leq H(z,t),$$

hence $|u_k - u| \leq H$. This leads to the pointwise almost everywhere inequality

$$|u_k(z,t)| \leq |u(z,t)| + H(z,t) = v(z,t).$$

Hence

$$|\,[\Phi(u_k) - \Phi(u)]\, Su(z,t)|^{\rho'} \leq C[v^{\frac{2}{n-1}} + |u|^{\frac{2}{n-1}}]^{\rho'} \ \ |Su(z,t)|^{\rho'}. \qquad (6.0.54)$$

Since $u, v \in L^{\gamma}(I, L^{\rho}(\mathbb{C}^n))$, using Hölder's inequality with $\frac{1}{\rho'} = \frac{1}{\rho} + \frac{n-1}{n^2} = \frac{1}{\rho} + \frac{2}{n-1} \cdot \frac{n-1}{n\gamma}$ and Lemma 4.0.14, we get

$$\begin{aligned} &\int_{\mathbb{C}^n} [v^{\frac{2}{n-1}} + |u|^{\frac{2}{n-1}}]^{\rho'} \ \ |Su(z,t)|^{\rho'} dz \qquad (6.0.55)\\ &\leq (\|v(\cdot,t)\|_{L^{\frac{n\gamma}{n-1}}(\mathbb{C}^n)} + \|u(\cdot,t)\|_{L^{\frac{n\gamma}{n-1}}(\mathbb{C}^n)})^{\frac{\rho'\gamma}{n}} \|Su(\cdot,t)\|^{\rho'}_{L^{\rho}(\mathbb{C}^n)}. \\ &\leq (\|v(\cdot,t)\|_{\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)} + \|u(\cdot,t)\|_{\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)})^{\frac{\rho'\gamma}{n}} \|Su(\cdot,t)\|^{\rho'}_{L^{\rho}(\mathbb{C}^n)} < \infty \end{aligned}$$

for a.e. $t \in I$. Thus in view of (6.0.54), (6.0.55) and using dominated convergence theorem in the $z$-variable, we see that

$$\|\,[\Phi(u_k) - \Phi(u)]\, Su(\cdot,t)\|_{L^{\rho'}(\mathbb{C}^n)} \to 0 \qquad (6.0.56)$$

as $k \to \infty$, for a.e. $t$.

Again, in view of (6.0.54) and (6.0.55), we get

$$\begin{aligned} &\|[\Phi(u_k) - \Phi(u)]\, Su(\cdot,t)\|_{L^{\rho'}(\mathbb{C}^n)} \\ &\leq C(\|v(\cdot,t)\|_{\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)} + \|u(\cdot,t)\|_{\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)})^{\frac{\gamma}{n}} \|Su(\cdot,t)\|_{L^{\rho}(\mathbb{C}^n)}. \end{aligned}$$

Since $\frac{1}{\gamma'} = \frac{1}{\gamma} + \frac{1}{n}$, an application of the Hölder's inequality in the $t$-variable shows that

$$\begin{aligned} &\|[\Phi(u_k) - \Phi(u)]\, Su\|_{L^{\gamma'}(I, L^{\rho'}(\mathbb{C}^n))} \\ &\leq C(\|v\|_{L^{\gamma}(I, \tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n))} + \|u\|_{L^{\gamma}(I, \tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n))})^{\frac{\gamma}{n}} \|Su\|_{L^{\gamma}(I, L^{\rho}(\mathbb{C}^n))}. \end{aligned}$$

Hence a further application of dominated convergence theorem with (6.0.56) shows that $\|\,(\Phi(u_k) - \Phi(u))\, Su\|_{L^{\gamma'}(I, L^{\rho'})} \to 0$, as $k \to \infty$.

Thus we have shown that $[\Phi(u_{m_k}) - \Phi(u)]\, Su \to 0$ in $L^{\gamma'}(I, L^{\rho'}(\mathbb{C}^n))$ for some subsequence $u_{m_k}$ whenever $u_m \to u$ in $L^\gamma(I, L^\rho(\mathbb{C}^n))$. But the above arguments are also valid if we had started with any subsequence of $u_m$. It follows that any subsequence of $[\Phi(u_m) - \Phi(u)]\, Su$ has a subsequence that converges to 0 in $L^{\gamma'}(I, L^{\rho'}(\mathbb{C}^n))$. From this we conclude that the original sequence $[\Phi(u_m) - \Phi(u)]\, Su$ converges to zero in $L^{\gamma'}(I, L^{\rho'}(\mathbb{C}^n))$, hence the proposition.

## Local well posedness for critical case $\alpha = \frac{2}{n-1}$

Now we state the main theorem of this section.

**Theorem 6.0.39** *Let $f \in \tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$ and $G$ be as in (1.0.15) and (1.0.16) with $\alpha = \frac{2}{n-1}$ and $n \geq 2$. Initial value problem (1.0.13), (1.0.14) has maximal solution $u \in C((T_*, T^*), \tilde{W}^{1,2}_{\mathcal{L}}) \cap L^{q_1}_{loc}\left((T_*, T^*), \tilde{W}^{1,p_1}_{\mathcal{L}}(\mathbb{C}^n)\right)$, where $t_0 \in (T_*, T^*)$ and $(q_1, p_1)$ be an arbitrary admissible pair. Moreover the following properties hold:*

**(i)(Uniqueness)** *Solution is unique in $C((T_*, T^*), \tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)) \cap L^\gamma((T_*, T^*), \tilde{W}^{1,\rho}_{\mathcal{L}})$.*

**(ii)(Blowup alternative)** *If $T^* < \infty$ then $\|u\|_{L^{\tilde{q}}\left((t_0, T^*), \tilde{W}^{1,\tilde{p}}_{\mathcal{L}}\right)} = \infty$ for every admissible pair $(\tilde{q}, \tilde{p})$ with $2 < \tilde{p}$ and $\frac{1}{\tilde{q}} = n\left(\frac{1}{2} - \frac{1}{\tilde{p}}\right)$. Similar conclusion holds if $T_* > -\infty$.*

**(iii)(Stability)** *If $f_k \to f$ in $\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$ then $\|u - \tilde{u}_k\|_{L^{q_1}\left(I, \tilde{W}^{1,p_1}_{\mathcal{L}}(\mathbb{C}^n)\right)} \to 0$ as $k \to \infty$ for every admissible pair $(q_1, p_1)$ and every interval $I$ with $\overline{I} \subset (T_*, T^*)$, where $u, \tilde{u}_k$ are solutions corresponding to $f, f_k$ respectively.*

**Proof**. **Local existence:** Since $G_m(z, \cdot) : \mathbb{C} \to \mathbb{C}$ is globally Lipschitz for each $m \geq 1$, see estimate (6.0.40), from Theorem 6.0.31, it follows that there exists a unique global solution $u_m \in C(\mathbb{R}, \tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n))$ of the initial value problem

$$i\partial_t v(z,t) - \mathcal{L}v(z,t) = G_m(z, v), \qquad z \in \mathbb{C}^n,\ t \in \mathbb{R} \tag{6.0.57}$$

$$v(\cdot, t_0) = f. \tag{6.0.58}$$

Furthermore $\mathcal{H}_m u_m = u_m$ (see 6.0.43) and $u_m \in L^{q_1}_{\text{loc}}(\mathbb{R}, \tilde{W}^{1,p_1}_{\mathcal{L}}(\mathbb{C}^n))$ for every admissible pair $(q_1, p_1)$. We deduce from Lemma 6.0.36 and Strichartz estimates

(Theorem 3.0.7) that

$$\begin{aligned}
&\|u_m\|_{L^{q_1}\left((t_0,t_0+T),\tilde{W}^{1,p_1}_{\mathcal{L}}(\mathbb{C}^n)\right)} \\
&\leq \|e^{-i(t-t_0)\mathcal{L}}f\|_{L^{q_1}\left((t_0,t_0+T),\tilde{W}^{1,p_1}_{\mathcal{L}}(\mathbb{C}^n)\right)} + C\|u_m\|^{\frac{n+1}{n-1}}_{L^{\gamma}\left((t_0,t_0+T),\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)}.
\end{aligned} \tag{6.0.59}$$

Let $l \geq m$, we see that

$$u_m - u_l = (\mathcal{H}_m(u_m) - \mathcal{H}_m(u_l)) + (\mathcal{H}_m(u_l) - \mathcal{H}(u_l)) + (\mathcal{H}(u_l) - \mathcal{H}_l(u_l)).$$

From Proposition 6.0.37, we deduce that

$$\begin{aligned}
\|u_m - u_l\|_{L^{q_1}((t_0,t_0+T),L^{p_1}(\mathbb{C}^n))} &\leq C\left(\|u_m\|_{L^{\gamma}((t_0,t_0+T),\tilde{W}^{1,\rho}_{\mathcal{L}})} + \|u_l\|_{L^{\gamma}((t_0,t_0+T),\tilde{W}^{1,\rho}_{\mathcal{L}})}\right)^{\frac{2}{n-1}} \times \\
&\left(\|u_m - u_l\|_{L^{\gamma}((t_0,t_0+T),L^{\rho})} + T^{\frac{n-1}{2n}} m^{-\frac{1}{n(n-1)}} \|u_l\|^{\frac{n^2-n+1}{n(n-1)}}_{L^{\infty}((t_0,t_0+T),\tilde{W}^{1,2}_{\mathcal{L}})}\right).
\end{aligned} \tag{6.0.60}$$

We choose $T \leq \pi$, therefore we can take constant $C$ to be independent of $T$. Let $\tilde{C}$ be larger than the constant $C$ that appear in (6.0.59), (6.0.60), (6.0.50), (6.0.51) and in Strichartz estimates (Theorem 3.0.7) for the particular choice of the admissible pairs $(q,p) = (\gamma,\rho)$ and $(q_1,p_1) = (\gamma,\rho)$. Fixed $\delta$ small enough so that

$$\tilde{C}(4\delta)^{\frac{2}{n-1}} < \frac{1}{2}. \tag{6.0.61}$$

We claim that if $0 < T \leq \pi$ is such that

$$\|e^{-i(t-t_0)\mathcal{L}}f\|_{L^{\gamma}\left((t_0,t_0+T),\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)} \leq \delta \tag{6.0.62}$$

then

$$\sup_{m\geq 1} \|u_m\|_{L^{\gamma}\left((t_0,t_0+T),\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)} \leq 2\delta \tag{6.0.63}$$

$$\sup_{m\geq 1} \|u_m\|_{L^{q_1}\left((t_0,t_0+T),\tilde{W}^{1,p_1}_{\mathcal{L}}(\mathbb{C}^n)\right)} < \infty \tag{6.0.64}$$

for every admissible pair $(q_1,p_1)$. Let $\theta_m(t) = \|u_m\|_{L^{\gamma}\left((t_0,t_0+t),\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)}$. From (6.0.59), we see that

$$\theta_m(t) \leq \delta + \tilde{C}\theta_m(t)^{\frac{n+1}{n-1}}.$$

If $\theta_m(t) = 2\delta$ for some $t \in (t_0, t_0 + T]$, then

$$2\delta \leq \delta + \tilde{C}(2\delta)^{\frac{n+1}{n-1}} < 2\delta$$

which is a contradiction. Since $\theta_m$ is a continuous function with $\theta_m(t_0) = 0$, we conclude that $\theta_m(t) < 2\delta$ for all $t \in (t_0, t_0 + T]$, which proves (6.0.63). From (6.0.59), we see that

$$\begin{aligned}\sup_m \|u_m\|_{L^{q_1}\left((t_0,t_0+T),\tilde{W}^{1,p_1}_{\mathcal{L}}(\mathbb{C}^n)\right)} &\leq \|e^{-i(t-t_0)\mathcal{L}} f\|_{L^{q_1}\left((t_0,t_0+T),\tilde{W}^{1,p_1}_{\mathcal{L}}(\mathbb{C}^n)\right)} + C(2\delta)^{\frac{n+1}{n-1}} \\ &\leq C(q_1, p_1, n, \delta, f) < \infty.\end{aligned}$$

This proves (6.0.64). By taking $(q_1, p_1) = (\gamma, \rho)$ in (6.0.60), we see that

$$\begin{aligned}\|u_m - u_l\|_{L^\gamma((t_0,t_0+T),L^\rho(\mathbb{C}^n))} &\leq \frac{1}{2}\left(\|u_m - u_l\|_{L^\gamma((t_0,t_0+T),L^\rho(\mathbb{C}^n))} + CT^{\frac{n-1}{2n}} m^{-\frac{1}{n(n-1)}}\right) \\ &\leq 2CT^{\frac{n-1}{2n}} m^{-\frac{1}{n(n-1)}} \to 0 \quad \text{as} \quad m \to \infty.\end{aligned}$$

This shows that $u_m$ is a Cauchy sequence in $L^\gamma\left((t_0, t_0 + T), L^\rho(\mathbb{C}^n)\right)$ and from (6.0.60) it is also Cauchy sequence in $L^{q_1}\left((t_0, t_0 + T), L^{p_1}(\mathbb{C}^n)\right)$ for every admissible pair $(q_1, p_1)$. Let $u$ be its limit, then $u_m \to u$ in $L^{q_1}\left((t_0, t_0 + T), L^{p_1}(\mathbb{C}^n)\right)$ for every admissible pair $(q_1, p_1)$. By a duality argument (see (6.0.52)) and from estimates (6.0.63), (6.0.64), we have

$$\|u\|_{L^\gamma\left((t_0,t_0+T),\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)} \leq 2\delta \tag{6.0.65}$$

$$\|u\|_{L^{q_1}\left((t_0,t_0+T),\tilde{W}^{1,p_1}_{\mathcal{L}}(\mathbb{C}^n)\right)} < \infty. \tag{6.0.66}$$

From Lemma 6.0.36, $G_m(z, u(z, t)) \in L^{\gamma'}\left((t_0, t_0 + T), \tilde{W}^{1,\rho'}_{\mathcal{L}}(\mathbb{C}^n)\right)$ for each $m \geq 0$. From Strichartz estimates (Theorem 3.0.7) and (6.0.43) with $m = 0$, $\mathcal{H}u \in L^{q_1}((t_0, t_0 + T), \tilde{W}^{1,p_1}_{\mathcal{L}}(\mathbb{C}^n))$ for every admissible pair $(q_1, p_1)$.

From Lemma 6.0.34 $\|G_m(z, u_m) - G_m(z, u)\|_{L^{\gamma'}\left((t_0,t_0+T),L^{\rho'}(\mathbb{C}^n)\right)} \to 0$ and from Lemma 6.0.35, $\|G_m(z, u) - G(z, u)\|_{L^{\gamma'}\left((t_0,t_0+T),L^{\rho'}(\mathbb{C}^n)\right)} \to 0$ as $m \to \infty$. Therefore

$$\|G_m(z, u_m) - G(z, u)\|_{L^{\gamma'}((t_0,t_0+T),L^{\rho'})} \to 0 \quad \text{as} \quad m \to \infty.$$

Since $u_m = \mathcal{H}_m u_m$ for each $m \geq 1$, from Strichartz estimates we deduce that

$$\begin{aligned} \|u_m - \mathcal{H}u\|_{L^{q_1}((t_0,t_0+T),L^{p_1}(\mathbb{C}^n))} &= \|\mathcal{H}_m u_m - \mathcal{H}u\|_{L^{q_1}((t_0,t_0+T),L^{p_1}(\mathbb{C}^n))} \\ &\leq C\|G_m(z,u_m) - G(z,u)\|_{L^{\gamma'}((t_0,t_0+T),L^{\rho'})} \to 0 \end{aligned}$$

as $m \to \infty$. Therefore for $t \in (t_0, t_0+T)$

$$u = \mathcal{H}u = e^{-i(t-t_0)\mathcal{L}} f(z) - i\int_{t_0}^{t} e^{-i(t-s)\mathcal{L}} G(z,u(z,s))ds. \tag{6.0.67}$$

From Strichartz estimates and estimate (6.0.66), $u \in C([t_0, t_0+T], \tilde{W}^{1,2}_{\mathcal{L}}) \cap L^{q_1}((t_0,t_0+T), \tilde{W}^{1,p_1}_{\mathcal{L}}(\mathbb{C}^n))$ for every admissible pair $(q_1,p_1)$. In view of Lemma 5.0.21, $u$ is also a solution to the initial value problem (1.0.13), (1.0.14). Similarly solution exists on the interval $[t_0 - T', t_0]$ for some $T' > 0$. Now we continue this process with initial time $t_0+T$ and $t_0 - T'$. By continuing this process, we get maximal interval $(T_*, T^*)$ and solution $u \in C((T_*,T^*), \tilde{W}^{1,2}_{\mathcal{L}}) \cap L^{q_1}_{\text{loc}}\left((T_*,T^*), \tilde{W}^{1,p_1}_{\mathcal{L}}(\mathbb{C}^n)\right)$ for every admissible pair $(q_1,p_1)$.

**Blowup alternative:** We prove blowup alternative by method of contradiction. Let us assume that $T^* < \infty$ and $u \in L^{\tilde{q}}((t_0,T^*), \tilde{W}^{1,\tilde{p}}_{\mathcal{L}})$ for some admissible pair $(\tilde{q},\tilde{p})$ with $2 < \tilde{p}$ and $\frac{1}{\tilde{q}} = n\left(\frac{1}{2} - \frac{1}{\tilde{p}}\right)$. Since $2 < \tilde{p} < \frac{2n}{n-1}$, $n \geq 2$, $\tilde{p} < 2n$. We choose admissible pair $(q_1,p_1)$ as follows

$$\frac{1}{p_1'} = \frac{1}{p_1} + \frac{2}{n-1}\left(\frac{1}{\tilde{p}} - \frac{1}{2n}\right), \quad \frac{1}{q_1'} = \frac{1}{q_1} + \frac{2}{n-1}\frac{1}{\tilde{q}}.$$

Let us choose $s$ and $t$ such that $t_0 \leq s < t < T^*$. Since $|S_jG(z,u(z,t))| \leq C|u|^{\frac{2}{n-1}}(|u| + |Z_ju| + |\overline{Z}_ju|)$ for $S_j = Id, Z_j, \overline{Z}_j$ $(1 \leq j \leq n)$ (see estimate 6.0.9), by Lemma 4.0.14 and Hölder's inequality we see that

$$\|G(z,u(z,\tau))\|_{L^{q_1'}((s,t),\tilde{W}^{1,p_1'}_{\mathcal{L}})} \leq C\|u\|_{L^{q_1}((s,t),\tilde{W}^{1,p_1}_{\mathcal{L}})}\|u\|^{\frac{2}{n-1}}_{L^{\tilde{q}}((s,t),\tilde{W}^{1,\tilde{p}}_{\mathcal{L}})}. \tag{6.0.68}$$

Since $(t_0,T^*)$ is a bounded interval, so we can choose constant $C$ to be independent of $s$ and $t$, where $t_0 \leq s < t < T^*$. Now we see that

$$u(z,\tau) = e^{-i(\tau-s)\mathcal{L}}u(\cdot,s)(z) - i\int_s^{\tau} e^{-i(\tau-s_1)\mathcal{L}} G(z,s_1,u(z,s_1))ds_1.$$

Therefore from Strichartz estimates (Theorem 3.0.7) and estimate (6.0.68), we deduce that

$$\|u\|_{L^{q_1}\left((s,t),\tilde{W}_{\mathcal{L}}^{1,p_1}\right)} \leq C\|u(\cdot,s)\|_{\tilde{W}_{\mathcal{L}}^{1,2}} + C\|u\|_{L^{q_1}\left((s,t),\tilde{W}_{\mathcal{L}}^{1,p_1}\right)}\|u\|^{\frac{2}{n-1}}_{L^{\tilde{q}}\left((s,t),\tilde{W}_{\mathcal{L}}^{1,\tilde{p}}\right)}$$

where constant $C$ is independent of $s$ and $t$. Since $\tilde{p} \neq 2$, so $\tilde{q} < \infty$ and $u \in L^{\tilde{q}}\left((t_0,T^*),\tilde{W}_{\mathcal{L}}^{1,\tilde{p}}(\mathbb{C}^n)\right)$, we choose $s$ sufficiently close to $T^*$ such that

$$C\|u\|^{\frac{2}{n-1}}_{L^{\tilde{q}}\left((s,T^*),\tilde{W}_{\mathcal{L}}^{1,\tilde{p}}(\mathbb{C}^n)\right)} \leq \frac{1}{2}.$$

Therefore we get

$$\|u\|_{L^{q_1}\left((s,t),\tilde{W}_{\mathcal{L}}^{1,p_1}(\mathbb{C}^n)\right)} \leq 2C\|u(\cdot,s)\|_{\tilde{W}_{\mathcal{L}}^{1,2}}.$$

Since RHS is independent of $t \in (s,T^*)$, we have $u \in L^{q_1}\left((s,T^*),\tilde{W}_{\mathcal{L}}^{1,p_1}(\mathbb{C}^n)\right)$. Therefore $u \in L^{q_1}\left((t_0,T^*),\tilde{W}_{\mathcal{L}}^{1,p_1}\right)$ and $G(z,u(z,\tau)) \in L^{q_1'}\left((t_0,T^*),\tilde{W}_{\mathcal{L}}^{1,p_1'}\right)$ follows from (6.0.68). Now from Strichartz estimates and (6.0.67), $u \in L^{q_2}((t_0,T^*),\tilde{W}_{\mathcal{L}}^{1,p_2}(\mathbb{C}^n)) \cap C([t_0,T^*],\tilde{W}_{\mathcal{L}}^{1,2}(\mathbb{C}^n))$ for every admissible pair $(q_2,p_2)$. Now by considering $T^*$ as a initial time and by local existence argument, we get contradiction to maximality of $T^*$.

**Uniqueness:** Suppose $u,v \in C((T_*,T^*),\tilde{W}_{\mathcal{L}}^{1,2}) \cap L^{\gamma}_{\text{loc}}((T_*,T^*),\tilde{W}_{\mathcal{L}}^{1,\rho})$ are two solutions of the equations (1.0.13) and (1.0.14). Then in view of Lemma 5.0.21 $u$ and $v$ will satisfy integral equation (1.0.20). From estimate 6.0.49 with $m = 0$, $G(z,u) \in L^{\gamma'}_{\text{loc}}((T_*,T^*),\tilde{W}_{\mathcal{L}}^{1,\rho'})$. Since $u(\cdot,t_0) = v(\cdot,t_0) = f$, from estimate (6.0.50) with $m = 0$ and $(q_1,p_1) = (\gamma,\rho)$, there exists sufficiently small $T$ such that $u(\cdot,t) = v(\cdot,t)$ for $t \in [t_0 - T, t_0 + T]$. Let $\tilde{t} \in (T_*,T^*)$ be such that $u(\cdot,\tilde{t}) = v(\cdot,\tilde{t})$. For $\tau \in (\tilde{t},T^*)$, we have

$$\begin{aligned} u(z,\tau) &= e^{-i(\tau-\tilde{t})\mathcal{L}}u(z,\tilde{t}) - i\int_{\tilde{t}}^{\tau} e^{-i(\tau-s)\mathcal{L}}G(z,u(z,s))ds, \\ v(z,\tau) &= e^{-i(\tau-\tilde{t})\mathcal{L}}v(z,\tilde{t}) - i\int_{\tilde{t}}^{\tau} e^{-i(\tau-s)\mathcal{L}}G(z,v(z,s))ds. \end{aligned}$$

By Strichartz estimate (3.0.2) and estimate (6.0.50) with $m = 0$ and $(q_1,p_1) = (\gamma,\rho)$, we have

$$
\begin{aligned}
\|u-v\|_{L^\gamma\left((\tilde{t},\tau),L^\rho(\mathbb{C}^n)\right)} &= \left\| \int_{\tilde{t}}^{\tau} e^{-i(t-s)\mathcal{L}} \left(G(u)-G(v)\right)(z,s)ds \right\|_{L^\gamma\left((\tilde{t},\tau),L^\rho(\mathbb{C}^n)\right)} \\
&\leq C\|u-v\|_{L^\gamma\left((\tilde{t},\tau),L^\rho\right)} \left( \|u\|_{L^\gamma\left((\tilde{t},\tau),\tilde{W}^{1,\rho}_{\mathcal{L}}\right)} + \|v\|_{L^\gamma\left((\tilde{t},\tau),\tilde{W}^{1,\rho}_{\mathcal{L}}\right)} \right)^{\frac{2}{n-1}}
\end{aligned}
$$

for all $\tau \in (\tilde{t}, T^*)$. Choose $\tau \in (\tilde{t}, T^*)$ sufficiently close to $\tilde{t}$ such that

$$
C\left( \|u\|_{L^\gamma\left((\tilde{t},\tau),\tilde{W}^{1,\rho}_{\mathcal{L}}\right)} + \|v\|_{L^\gamma\left((\tilde{t},\tau),\tilde{W}^{1,\rho}_{\mathcal{L}}\right)} \right)^{\frac{2}{n-1}} \leq \frac{1}{2}.
$$

Therefore $\|u-v\|_{L^\gamma\left((\tilde{t},\tau),L^{rho}(\mathbb{C}^n)\right)} \leq \frac{1}{2}\|u-v\|_{L^\gamma\left((\tilde{t},\tau),L^\rho(\mathbb{C}^n)\right)}$. Hence $u=v$ on the larger interval $[\tilde{t},\tau]$.

Now let $\theta$ =sup$\{\tilde{T} : t_0 < \tilde{T} < T^* : \|u-v\|_{L^\gamma([t_0,\tilde{T}],L^\rho)} = 0\}$. If $\theta < T^*$, then for sufficiently small $\epsilon > 0$, choose $\tilde{t} = \theta - \epsilon, \tau = \theta + \epsilon$ and by the above observation, $\|u-v\|_{L^\gamma((\theta-\epsilon,\theta+\epsilon),L^\rho)} = 0$, which contradicts the definition of $\theta$. Thus we conclude that $\theta = T^*$, proving the uniqueness on $[t_0, T^*)$. Similarly one can show uniqueness on $(T_*, t_0]$.

**Stability:** We prove stability in the following two steps.

**Step 1:** Let $f_k \to f$ in $\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$. Then for each $T > 0$,

$$
\|e^{-i(t-t_0)\mathcal{L}}(f-f_k)\|_{L^\gamma\left(I_T,\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)} \leq C\|f-f_k\|_{\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)} \to 0 \ \text{ as } \ k \to \infty
$$

where $I_T = (t_0 - T, t_0 + T)$. Therefore for given $\delta > 0$ in (6.0.61), choose $T(\delta)$ sufficiently small such that

$$
\|e^{-i(t-t_0)\mathcal{L}} f\|_{L^\gamma(I_T,\tilde{W}^{1,\rho}_{\mathcal{L}})} \leq \frac{\delta}{2} \tag{6.0.69}
$$

and choose $k$ sufficiently large so that

$$
\|e^{-i(t-t_0)\mathcal{L}}(f-f_k)\|_{L^\gamma\left(I_T,\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)} \leq C\|f-f_k\|_{\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)} \leq \frac{\delta}{2}.
$$

Therefore choose $k_0(T)$ so large such that

$$
\|e^{-i(t-t_0)\mathcal{L}} f_k\|_{L^\gamma\left(I_T,\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)} \leq \delta \tag{6.0.70}
$$

for $k \geq k_0(T)$.

Let $u$ and $\tilde{u_k}$ are solutions corresponding to initial values $f$ and $f_k$ at time $t_0$ respectively for $k \geq 1$. In view of estimates (6.0.65) and (6.0.66), $u, \tilde{u_k}$ will satisfy following estimates

$$\|u\|_{L^\gamma\left(I_T, \tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)} \leq 2\delta \tag{6.0.71}$$

$$\|u\|_{L^{q_1}\left((t_0,t_0+T), \tilde{W}^{1,p_1}_{\mathcal{L}}(\mathbb{C}^n)\right)} < \infty \tag{6.0.72}$$

$$\sup_{k\geq k_0(T)} \|\tilde{u}_k\|_{L^\gamma\left(I_T, \tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)} \leq 2\delta \tag{6.0.73}$$

$$\sup_{k\geq k_0(T)} \|\tilde{u}_k\|_{L^{q_1}\left(I_T, \tilde{W}^{1,p_1}_{\mathcal{L}}(\mathbb{C}^n)\right)} < \infty \tag{6.0.74}$$

where $(q_1, p_1)$ be any admissible pair. Now from Strichartz estimates and Lemma 6.0.34,

$$\begin{aligned}\|u-\tilde{u_k}\|_{L^\gamma(I_T,L^\rho)} &= \|\mathcal{H}u - \mathcal{H}\tilde{u_k}\|_{L^\gamma(I_T,L^\rho)} \\ &\leq C\|f-f_k\|_{\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)} + C\|G(z,u)-G(z,\tilde{u_k})\|_{L^{\gamma'}(I_T,L^{\rho'})} \\ &\leq C\|f-f_k\|_{\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)} + C\|u-\tilde{u_k}\|_{L^\gamma(I_T,L^\rho(\mathbb{C}^n))} \times \\ &\quad \left(\|u\|_{L^\gamma\left(I_T,\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)} + \|\tilde{u_k}\|_{L^\gamma\left(I_T,\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)}\right)^{\frac{2}{n-1}}.\end{aligned}$$

From (6.0.61) and (6.0.65),

$$C\left(\|u\|_{L^\gamma\left(I_T,\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)} + \|\tilde{u_k}\|_{L^\gamma\left(I_T,\tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)}\right)^{\frac{2}{n-1}} \leq \frac{1}{2}.$$

Therefore $\|u-\tilde{u_k}\|_{L^\gamma(I_T,L^\rho)} \leq 2C\|f-f_k\|_{\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)} \to 0$ as $k\to\infty$. Since $\{\tilde{u_k}\}$ is a bounded sequence in $L^\gamma\left(I_T, \tilde{W}^{1,\rho}_{\mathcal{L}}(\mathbb{C}^n)\right)$, therefore from Lemma 6.0.34 with $m=0$, $\|G(z,u(z,t)) - G(z,\tilde{u_k}(z,t))\|_{L^{\gamma'}\left(I_T,L^{\rho'}(\mathbb{C}^n)\right)} \to 0$ as $j\to\infty$. Since $\mathcal{H}u = u, \mathcal{H}\tilde{u}_k = \tilde{u}_k$, therefore from Theorem 3.0.7

$$\|u-\tilde{u_k}\|_{L^{q_1}(I_T,L^{p_1})} \leq C\|f-f_k\|_{L^2} + C\|G(z,u)-G(z,\tilde{u_k})\|_{L^{\gamma'}\left(I_T,L^{\rho'}(\mathbb{C}^n)\right)} \to 0$$

as $k\to\infty$ for every admissible pair $(q_1,p_1)$. Note that $(\partial_{x_j} - \frac{iy_j}{2}) = \frac{1}{2}(Z_j - \overline{Z}_j)$ and $(\partial_{y_j} + \frac{ix_j}{2}) = \frac{i}{2}(Z_j + \overline{Z}_j)$. For $S = (\partial_{x_j} - \frac{iy_j}{2}), (\partial_{y_j} + \frac{ix_j}{2})$ and using the notation $\psi_{(k)} = \psi\left(z, |\tilde{u}_k(z,t)|\right)$ (see equation (5.0.31)), we have

$$S(G_{(k)}-G)=\psi_{(k)}S(\tilde{u}_k-u)+(\psi_{(k)}-\psi)Su+(\partial_j\psi_{(k)})(\tilde{u}_k-u)$$
$$+(\partial_j\psi_{(k)}-\partial_j\psi)u+(\partial_{2n+1}\psi_{(k)})\tilde{u}_k\Re(\frac{\overline{\tilde{u}_k}}{|\tilde{u}_k|}S(\tilde{u}_k-u)) \qquad (6.0.75)$$
$$+(\partial_{2n+1}\psi_{(k)})\tilde{u}_k\Re(\frac{\overline{\tilde{u}_k}}{|\tilde{u}_k|}Su)-(\partial_{2n+1}\psi)u\Re(\frac{\overline{u}}{|u|}Su)$$

where $\partial_j=\partial_{x_j}$ for $S=(\partial_{x_j}-\frac{iy_j}{2})$ and $\partial_j=\partial_{y_j}$ for $S=(\partial_{y_j}+\frac{ix_j}{2})$, $1\le j\le n$. Using the assumption (1.0.16) on $\psi$, Lemma 4.0.14, and by similar computations as used in Lemma 6.0.34 and Proposition 6.0.38, we have

$$\|\psi_{(k)}S(\tilde{u}_k-u)\|_{L^{\gamma'}\left(I_T,L^{\rho'}\right)}\le C\|S(\tilde{u}_k-u)\|_{L^{\gamma}(I_T,L^{\rho})}\|\tilde{u}_k\|^{\frac{2}{n-1}}_{L^{\gamma}\left(I_T,\tilde{W}^{1,\rho}_{\mathcal{L}}\right)}$$

$$\|(\partial_j\psi_{(k)})(\tilde{u}_k-u)\|_{L^{\gamma'}\left(I_T,L^{\rho'}\right)}\le C\|\tilde{u}_k-u\|_{L^{\gamma}(I_T,L^{\rho})}\|\tilde{u}_k\|^{\frac{2}{n-1}}_{L^{\gamma}\left(I_T,\tilde{W}^{1,\rho}_{\mathcal{L}}\right)}$$

$$\|(\partial_{2n+1}\psi_{(k)})\tilde{u}_k\Re(\frac{\overline{\tilde{u}_k}}{|\tilde{u}_k|}S(\tilde{u}_k-u))\|_{L^{\gamma'}\left(I_T,L^{\rho'}\right)}$$
$$\le C\|S(\tilde{u}_k-u)\|_{L^{\gamma}(I_T,L^{\rho})}\|\tilde{u}_k\|^{\frac{2}{n-1}}_{L^{\gamma}\left(I_T,\tilde{W}^{1,\rho}_{\mathcal{L}}\right)}.$$

Since $\|\tilde{u}_k-u\|_{L^{\gamma}(I_T,L^{\rho}(\mathbb{C}^n))}\to 0$ and $\{\tilde{u}_k\}$ is a bounded sequence in $L^{\gamma}(I_T,\tilde{W}^{1,\rho}_{\mathcal{L}})$, therefore by second inequality in the above estimates, $(\partial_j\psi_{(k)})(\tilde{u}_k-u)\to 0$ as $k\to\infty$ in $L^{\gamma'}\left(I_T,L^{\rho'}(\mathbb{C}^n)\right)$. Since $G$ is $C^1$, so in view of the condition (1.0.16) on $\psi$ and Proposition 6.0.38, the sequences $(\psi_{(k)}-\psi)Su,$, $(\partial_j\psi_{(k)}-\partial_j\psi)u$ and $(\partial_{2n+1}\psi_{(k)})\tilde{u}_k\Re(\frac{\overline{\tilde{u}_k}}{|\tilde{u}_k|}Su)-(\partial_{2n+1}\psi)u\Re(\frac{\overline{u}}{|u|}Su)$ converges to zero in $L^{\gamma'}(I_\tau,L^{\rho'})$ as $k\to\infty$. Using these observations in (6.0.75), we get

$$\|S(G_{(k)}-G)\|_{L^{\gamma'}(I_T,L^{\rho'})}\le C\|\tilde{u}_k\|^{\frac{2}{n-1}}_{L^{\gamma}\left(I_T,\tilde{W}^{1,\rho}_{\mathcal{L}}\right)}\|S(\tilde{u}_k-u)\|_{L^{\gamma}(I_T,L^{\rho}(\mathbb{C}^n))}+a_k$$

where $S=(\partial_{x_j}-\frac{iy_j}{2}),(\partial_{y_j}+\frac{ix_j}{2})$ $(1\le j\le n)$ and $a_k\to 0$ as $k\to\infty$. Since $(\partial_{x_j}-\frac{iy_j}{2})=\frac{1}{2}(Z_j-\overline{Z}_j)$ and $(\partial_{y_j}+\frac{ix_j}{2})=\frac{i}{2}(Z_j+\overline{Z}_j)$, therefore we have

$$\|G_{(k)}-G\|_{L^{\gamma'}(I_T,\tilde{W}^{1,\rho'}_{\mathcal{L}})}\ \le\ C\|\tilde{u}_k\|^{\frac{2}{n-1}}_{L^{\gamma}\left(I_T,\tilde{W}^{1,\rho}_{\mathcal{L}}\right)}\|\tilde{u}_k-u\|_{L^{\gamma}(I_T,\tilde{W}^{1,\rho}_{\mathcal{L}})}+a_k. \qquad (6.0.76)$$

Now from Strichartz estimates and above estimate, we have

$$\|\tilde{u}_k - u\|_{L^\gamma(I_T, \tilde{W}^{1,\rho}_{\mathcal{L}})} \leq C\|f_k - f\|_{\tilde{W}^{1,2}_{\mathcal{L}}} + C\|\tilde{u}_k\|^{\frac{2}{n-1}}_{L^\gamma\left(I_T, \tilde{W}^{1,\rho}_{\mathcal{L}}\right)} \|\tilde{u}_k - u\|_{L^\gamma(I_T, \tilde{W}^{1,\rho}_{\mathcal{L}})} + a_k. \tag{6.0.77}$$

Now we choose $\delta > 0$ sufficiently small such that it satisfies condition (6.0.61) and

$$C(2\delta)^{\frac{2}{n-1}} \leq \frac{1}{2}$$

where constant $C$ is appearing in the inequality (6.0.77). Note that $T$ depends on $\delta$ through (6.0.69). Therefore from estimates (6.0.73) and (6.0.77), we have

$$\|\tilde{u}_k - u\|_{L^q(I_T, \tilde{W}^{1,p}_{\mathcal{L}})} \leq 2C\|f_k - f\|_{\tilde{W}^{1,2}_{\mathcal{L}}} + 2a_k \to 0$$

as $k \to \infty$. Now from estimates (6.0.76) and (6.0.73)

$$\|G_{(k)} - G\|_{L^{\gamma'}(I_T, \tilde{W}^{1,\rho'}_{\mathcal{L}})} \to 0$$

as $k \to \infty$. From Strichartz estimates,

$$\|\tilde{u}_k - u\|_{L^{q_1}(I_T, \tilde{W}^{1,p_1}_{\mathcal{L}})} \leq C\|f_k - f\|_{\tilde{W}^{1,2}_{\mathcal{L}}} + C\|G_{(k)} - G\|_{L^{\gamma'}(I_T, \tilde{W}^{1,\rho'}_{\mathcal{L}})} \to 0$$

as $k \to \infty$ for every admissible pair $(q_1, p_1)$.

**Step 2:** Let $(T_{*,k}, T^*_k)$ be the maximal interval for the solutions $\tilde{u}_k$ and $I \subset (T_*, T^*)$ be a compact interval. As discussed in Theorem 5.0.26, in order to prove stability for interval $I$, it is enough to prove that $\tilde{u}_k$ is defined on $I$, for all but finitely many $k$. In fact, we prove $I \subset (T_{*,k}, T^*_k)$ for all but finitely many $k$.

Without loss of generality, we assume that $t_0 \in I = [a, b]$, and give a proof by the method of contradiction. Suppose there exist infinitely many $T^*_{k_m} \leq b$ and let $c = \liminf T^*_{k_m}$. Then for $\epsilon > 0$, $[t_0, c - \epsilon] \subset [t_0, T^*_{k_m})$ for all $k_m$ sufficiently large and $\tilde{u}_{k_m}$ are defined on $[t_0, c - \epsilon]$.

By compactness and step 1, the stability result proved above can be extended to the interval $[t_0, c - \epsilon]$.

For given $\delta > 0$, choose $\epsilon > 0$ sufficiently small such that

$$\begin{aligned}
\|e^{-i(t-(c-\epsilon))\mathcal{L}}u(\cdot, c-\epsilon) - e^{-i(t-(c-\epsilon))\mathcal{L}}u(\cdot, c)\|_{L^\gamma\left((c-\epsilon,c+\epsilon),\tilde{W}^{1,\rho}_{\mathcal{L}}\right)} \\
\leq C\|u(\cdot, c-\epsilon) - u(\cdot, c)\|_{\tilde{W}^{1,2}_{\mathcal{L}}} \leq \frac{\delta}{6} \\
\|e^{-i(t-(c-\epsilon))\mathcal{L}}u(\cdot, c) - e^{-i(t-c)\mathcal{L}}u(\cdot, c)\|_{L^\gamma\left((c-\epsilon,c+\epsilon),\tilde{W}^{1,\rho}_{\mathcal{L}}\right)} \\
\leq C\|e^{-i\epsilon t\mathcal{L}}u(\cdot, c) - u(\cdot, c)\|_{\tilde{W}^{1,2}_{\mathcal{L}}} \leq \frac{\delta}{6} \\
\|e^{-i(t-c)\mathcal{L}}u(\cdot, c)\|_{L^\gamma\left((c-\epsilon,c+\epsilon),\tilde{W}^{1,\rho}_{\mathcal{L}}\right)} \leq \frac{\delta}{6}.
\end{aligned}$$

Now we choose $k_0(\epsilon)$ such that following estimate holds for all $k \geq k_0$

$$\begin{aligned}
\|e^{-i(t-(c-\epsilon))}\tilde{u}_{k_m}(\cdot, c-\epsilon) - e^{-i(t-(c-\epsilon))}u(\cdot, c-\epsilon)\|_{L^\gamma\left((c-\epsilon,c+\epsilon),\tilde{W}^{1,\rho}_{\mathcal{L}}\right)} \\
\leq C\|\tilde{u}_{k_m}(\cdot, c-\epsilon) - u(\cdot, c-\epsilon)\|_{\tilde{W}^{1,2}_{\mathcal{L}}} \leq \tfrac{\delta}{2}.
\end{aligned}$$

Therefore $\|e^{-i(t-(c-\epsilon))}\tilde{u}_{k_m}(\cdot, c-\epsilon)\|_{L^\gamma\left((c-\epsilon,c+\epsilon),\tilde{W}^{1,\rho}_{\mathcal{L}}\right)} \leq \delta$ for all $k_m \geq k_0$. Now by local existence argument (see (6.0.62)), $\tilde{u}_{k_m}$ is defined on $(t_0, c+\epsilon)$ and therefore $T^*_{k_m} \geq c+\epsilon$ for all $k_m \geq k_0$, hence contradicts the fact that $\liminf T^*_{k_m} = c$.

Similarly we can show that $[a, t_0] \subset (T_{*,k}, t_0]$ for all but finitely many $k$ which completes the proof of stability.

# Chapter 7

# Global well posedness in $L^2(\mathbb{C}^n)$

In this chapter we will prove global well posedness in $L^2(\mathbb{C}^n)$ for the subcritical case $0 \leq \alpha < \frac{2}{n}$. However in the critical case $\alpha = \frac{2}{n}$, we can prove the global well posedness in $L^2(\mathbb{C}^n)$ only for sufficiently small intial value, see Remark 7.0.42. We follow method of Cazenave and Weissler [7]. Theorem 7.0.40 and Theorem 7.0.41 are main results of this chapter.

## Subcritical Case $0 \leq \alpha < \frac{2}{n}$

**Theorem 7.0.40** *Let $u(\cdot, t_0) = f \in L^2(\mathbb{C}^n)$ and $G$ be as in (1.0.15) and (1.0.16) with $0 \leq \alpha < \frac{2}{n}$. Initial value problem (1.0.13), (1.0.14) has unique maximal solution $u \in C((T_*, T^*), L^2(\mathbb{C}^n)) \cap L^{q_2}_{loc}((T_*, T^*), L^{p_2}(\mathbb{C}^n))$, where $t_0 \in (T_*, T^*)$ and $(q_2, p_2)$ be an arbitrary admissible pair. Fix $p = 2+\alpha$. Moreover the following properties hold:*

**(i)(Uniqueness)** *Solution is unique in $C((T_*, T^*), L^2(\mathbb{C}^n)) \cap L^{q_2}_{loc}((T_*, T^*), L^p)$ where $q_2 \in [q_1, q]$, $\frac{1}{q} = n\left(\frac{1}{2} - \frac{1}{p}\right)$ and $q_1 = \frac{2p(p-1)}{2p+2n-np} \geq 1$.*

**(ii)(Blowup alternative)** *If $T^* < \infty$ (respectively, $T_* > -\infty$), then $\|u\|_{L^{q_2}((t_0,T^*),L^p(\mathbb{C}^n))} = \infty$ (respectively, $\|u\|_{L^{q_2}((T_*,t_0),L^p(\mathbb{C}^n))} = \infty$), where $q_2 \in [q_1, q]$.*

**(iii)(Stability)** *If $f_j \to f$ in $L^2(\mathbb{C}^n)$, then $u_j \to u$ in $L^{q_1}(I, L^{p_1}(\mathbb{C}^n))$ for every interval $I$ with $\overline{I} \subset (T_*, T^*)$ and for every admissible pair $(q_1, p_1)$, where $u_j$ and $u$ are solutions corresponding to $f_j$ and $f$ respectively.*

**(iv)(Conservation of charge and global existence)** *If $\psi$ is real valued, then we have conservation of charge $\|u(\cdot,t)\|_{L^2(\mathbb{C}^n)} = \|f\|_{L^2(\mathbb{C}^n)}$ for every $t \in (T_*, T^*)$. Moreover solution is global, i.e., $T_* = -\infty$ and $T^* = \infty$.*

**Proof**. The key point is that we prove local existence without using embedding theorems (Lemma 4.0.9, Lemma 4.0.14). For given positive real numbers $T$ and $M$ with $T \leq \pi$, consider the metric space

$$E_{T,M} = \{u \in L^q(I, L^p(\mathbb{C}^n)) : \|u\|_{L^q(I,L^p(\mathbb{C}^n))} \leq M\}$$

with metric $d(u,v) = \|u - v\|_{L^q(I,L^p(\mathbb{C}^n))}$, where $I = (t_0 - T, t_0 + T)$. We show existence of solution to the initail value problem (1.0.13), (1.0.14) by showing that operator $\mathcal{H}$ given by (1.0.21) is contraction on complete metric spce $E_{T,M}$ for suitable $T$ and $M$.

Let $q_1 = \frac{2p(p-1)}{2p+2n-np}$. Since $0 \leq \alpha < \frac{2}{n}, p = 2 + \alpha$, therefore $1 \leq q_1 < q$ and $\frac{1}{q'} = \frac{\alpha}{q_1} + \frac{1}{q_1}$. Let $u, v \in L^q(I, L^p(\mathbb{C}^n))$. By taking $L^{q'}$ norm with respect to the $t$-variable in the inequality (5.0.4) and using the Hölder's inequality, we get

$$\begin{aligned}
\|G(z,u) - G(z,v)\|_{L^{q'}(I,L^{p'})} &\leq C(\|u\|^{\alpha}_{L^{q_1}(I,L^p)} + \|v\|^{\alpha}_{L^{q_1}(I,L^p)}) \\
&\quad \times \|u - v\|_{L^{q_1}(I,L^p)} \qquad (7.0.1) \\
&\leq CT^{\frac{q-q_1}{qq_1}(1+\alpha)}(\|u\|^{\alpha}_{L^q(I,L^p)} + \|v\|^{\alpha}_{L^q(I,L^p)}) \\
&\quad \times \|u - v\|_{L^q(I,L^p)}. \qquad (7.0.2)
\end{aligned}$$

From Strichartz estimates (Theorem 3.0.7), above estimate and for $u \in E_{T,M}$, we observe that

$$\begin{aligned}
\|\mathcal{H}u\|_{L^q(I,L^p)} &\leq C\|f\|_{L^2(\mathbb{C}^n)} + C\|G(z,u(z,t))\|_{L^{q'}(I,L^{p'})} \\
&\leq C\|f\|_{L^2(\mathbb{C}^n)} + CT^{\frac{q-q_1}{qq_1}(1+\alpha)}\|u\|^{\alpha}_{L^q(I,L^p)}\|u\|_{L^q(I,L^p(\mathbb{C}^n))} \\
&\leq C\|f\|_{L^2(\mathbb{C}^n)} + CT^{\frac{q-q_1}{qq_1}(1+\alpha)}M^{1+\alpha}. \qquad (7.0.3)
\end{aligned}$$

From Theorem 3.0.7, estimate (7.0.2) and for $u, v \in E$, we observe that

$$\begin{aligned}
\|\mathcal{H}u - \mathcal{H}v\|_{L^q(I,L^p(\mathbb{C}^n))} &\leq C\|G(z,u) - G(z,v)\|_{L^{q'}(I,L^{p'})} \\
&\leq CT^{\frac{q-q_1}{qq_1}(1+\alpha)}M^{\alpha}\|u - v\|_{L^q(I,L^p(\mathbb{C}^n))}. \qquad (7.0.4)
\end{aligned}$$

Choose

$$M = \begin{cases} 1 & \text{if } f = 0 \\ 2C\|f\|_{L^2(\mathbb{C}^n)} & \text{if } f \neq 0 \end{cases} \tag{7.0.5}$$

and

$$T = \begin{cases} \min\{\pi, (2C)^{-\frac{qq_1}{(q-q_1)(1+\alpha)}}\} & \text{if } f = 0 \\ \min\{\pi, (2C)^{-\frac{qq_1}{q-q_1}}\|f\|_{L^2(\mathbb{C}^n)}^{-\frac{\alpha qq_1}{(q-q_1)(1+\alpha)}}\} & \text{if } f \neq 0 \end{cases} \tag{7.0.6}$$

where $C$ is the same constant that appears in the inequalities (7.0.3), (7.0.4) and is independent of $T$. For this choice of $M$ and $T$, operator $\mathcal{H}$ is contaction on $E$. Therefore $\mathcal{H}$ has unique fixed point in $E$. From estimate 7.0.2 with $v = 0$, $G(z, u(z,t)) \in L^{q'}(I, L^{p'}(\mathbb{C}^n))$, from Strichartz estimates $u \in C(\overline{I}, L^2(\mathbb{C}^n)) \cap L^{\tilde{q}}(I, L^{\tilde{p}}(\mathbb{C}^n))$ for every admissible pair $(\tilde{q}, \tilde{p})$. In view of Lemma 5.0.21, u is also a solution of the initial value problem (1.0.13), (1.0.14).

Now we consider inital time $t_0 - T$ and $t_0 + T$. Then by the above argument, solution $u$ is defined on the interval $[T_{-1}, T_1]$ for some $T_{-1} < t_0 - T$ and $T_1 > t_0 + T$. By continuing this process, we get maximal interval $(T_*, T^*)$ and solution $u$ is defined on this interval. Moreover $u \in C((T_*, T^*), L^2(\mathbb{C}^n)) \cap L^{\tilde{q}}_{\text{loc}}((T_*, T^*), L^{\tilde{p}}(\mathbb{C}^n))$ for every admissible pair $(\tilde{q}, \tilde{p})$. In view of estimates (7.0.1), (7.0.2), (7.0.4) uniqueness follows by similar arguments as in Theorem 5.0.26.

**Stability:** We prove stability in the following two steps.

Step 1: Let $\{f_m\}_{m\geq 1}$ be a sequence in $L^2(\mathbb{C}^n)$ such that $f_m \to f$ in $L^2(\mathbb{C}^n)$ as $m \to \infty$. Let $u_m$ and $u$ be the solutions corresponding to the initial data $f_m$ and $f$ respectively, at time $t = t_0$.

Since $f_m \to f$ in $L^2(\mathbb{C}^n)$, in view of (7.0.6) we can choose $\tau < T$ sufficiently small so that $u_m$ are defined on $I_\tau = (t_0 - \tau, t_0 + \tau)$ for sufficiently large $m$. Also note that

$$u_m - u = e^{-i(t-t_0)\mathcal{L}}(f_m - f) - i\int_{t_0}^{t} e^{-i(t-s)\mathcal{L}}(G(z, u_m(z,s)) - G(z, u(z,s)))ds.$$

By Theorem 3.0.7 and estimate (7.0.2),

$$\|u_m - u\|_{L^q(I_\tau, L^p)} \leq C\|f_m - f\|_{L^2} + C\tau^{\frac{q-q_1}{qq_1}(1+\alpha)}(M^\alpha + M_m^\alpha)\|u_m - u\|_{L^q(I,L^p)}$$

where $M_m$ is given by (7.0.5) with $f$ replaced by $f_m$. Since $f_m \to f$ in $L^2(\mathbb{C}^n)$, $\{M_m\}$ is a bounded sequence. We choose $\tau$ sufficiently small so that

$$C\tau^{\frac{q-q_1}{qq_1}(1+\alpha)}(M^\alpha + M_m^\alpha) \leq \frac{1}{2}$$

for $m \geq 1$. Thus $\|u_m - u\|_{L^q(I_\tau, L^p)} \leq 2C\|f_m - f\|_{L^2} \to 0$ as $m \to \infty$. From estimate (7.0.2) $\|G(z, u_m) - G(z, u)\|_{L^{q'}(I_\tau, L^{p'})} \to 0$ as $m \to \infty$. From Theorem 3.0.7

$$\|u_m - u\|_{L^{q_1}(I_\tau, L^{p_1})} \leq C\|f_m - f\|_{L^2} + C\|G(z, u_m) - G(z, u)\|_{L^{q'}(I_\tau, L^{p'})} \to 0$$

as $m \to \infty$ for every admissible pair $(q_1, p_1)$.

Step 2: Let $(T_{*,m}, T_m^*)$ be the maximal interval for the solutions $u_m$ and $I \subset (T_*, T^*)$ be a compact interval. As discussed in Theorem 5.0.26, in order to prove stability for interval $I$, it is enough to prove that $u_m$ is defined on $I$, for all but finitely many $m$. In fact, we prove $I \subset (T_{*,m}, T_m^*)$ for all but finitely many $m$.

Without loss of generality, we assume that $t_0 \in I = [a, b]$, and give a proof by the method of contradiction. Suppose there exist infinitely many $T_{m_j}^* \leq b$ and let $c = \liminf T_{m_j}^*$. Then for $\epsilon > 0$, $[t_0, c - \epsilon] \subset [t_0, T_{m_j}^*)$ for all $m_j$ sufficiently large and $u_{m_j}$ are defined on $[t_0, c - \epsilon]$.

By compactness, the stability result proved in step 1 can be extended to the interval $[t_0, c - \epsilon]$. Hence

$$\|u_{m_j}(\cdot, c - \epsilon)\|_{L^2} \to \|u(\cdot, c - \epsilon)\|_{L^2} \text{ as } j \to \infty.$$

Also by continuity we have

$$\|u(\cdot, c - \epsilon)\|_{L^2} \to \|u(\cdot, c)\|_{L^2} \text{ as } \epsilon \to 0.$$

Thus, for any $\delta > 0$, we have

$$\|u_{m_j}(\cdot, c - \epsilon)\|_{L^2}^{-\frac{\alpha q q_1}{(q-q_1)(1+\alpha))}} > \delta \text{ whenever } \|u(\cdot, c)\|_{L^2}^{-\frac{\alpha q q_1}{(q-q_1)(1+\alpha)}} > \delta, \tag{7.0.7}$$

for sufficiently small $\epsilon$ and for all $j \geq j_0(\epsilon)$. Therefore by applying the local existence argument (see equation 7.0.6), with $c - \epsilon$ as the initial time, without loss of generality we can assume that $u_{m_j}$ extends to $[t_0, c - \epsilon + C_1\|u_{m_j}(\cdot, c -$

$\epsilon)\|_{L^2}^{-\frac{\alpha q q_1}{(q-q_1)(1+\alpha)}}]$ for large $j$ where $C_1 = (2C)^{-\frac{qq'}{q-q'}}$. Now choosing $\epsilon < \frac{C_1}{2}\,\delta$, we have by (7.0.7)

$$c - \epsilon + C_1\|u_{m_j}(\cdot, c-\epsilon)\|_{L^2}^{-\frac{\alpha q q_1}{(q-q_1)(1+\alpha)}} > c + \frac{C_1}{2}\delta \quad \text{for all} \ \ j \geq j_0(\epsilon).$$

It follows that $T^*_{m_j} \geq c + \frac{C_1}{2}\delta$, hence contradicts the fact that $\liminf T^*_{m_j} = c$.

Similarly we can show that $[a, t_0] \subset (T_{*,m}, t_0]$ for all but finitely many $m$ which completes the proof of stability.

**Blowup alternative:** We prove blowup alternative by method of contradiction. Suppose $T^* < \infty$ and $u \in L^{\tilde{q}}((t_0, T^*), L^p(\mathbb{C}^n))$ for some $\tilde{q} \in [q_1, q]$. Then by estimates (7.0.1), $G(z, u) \in L^{q'}((t_0, T^*), L^{p'}(\mathbb{C}^n))$ and by Strichartz estimates $u \in C([t_0, T^*], L^2(\mathbb{C}^n)) \cap L^{q_2}((t_0, T^*), L^{p_2}(\mathbb{C}^n))$ for every admissible pair $(q_2, p_2)$. By taking $T^*$ as initial time and by local existence argument, solution exists on the interval $[t_0, T^* + \epsilon)$ for some $\epsilon > 0$, which is a contradiction for maximality of $T^*$.

**Conservation of charge and global existence:** Let $\{f_m\}$ be a sequence in $\tilde{W}^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$ converging to $f$ in $L^2(\mathbb{C}^n)$. By stability $u_m \to u$ in $L^\infty(I, L^2(\mathbb{C}^n))$ for every interval $I$ with $\overline{I} \subset (T_*, T^*)$. By conservation of charge (see Theorem 6.0.33), $\|u_m(\cdot, t)\|_{L^2(\mathbb{C}^n)} = \|f_m\|_{L^2(\mathbb{C}^n)}$ for each $t \in I$. Therefore by taking limit $m \to \infty$, we get $\|u(\cdot, t)\|_{L^2(\mathbb{C}^n)} = \|f\|_{L^2(\mathbb{C}^n)}$ for each $t \in I$. Hence $\|u(\cdot, t)\|_{L^2(\mathbb{C}^n)} = \|f\|_{L^2(\mathbb{C}^n)}$ for each $t \in (T_*, T^*)$. From conservation of charge and local existence argument ((7.0.5), (7.0.6)), we will get global solution, i.e., $-T_* = T^* = \infty$.

## Critical case $\alpha = \frac{2}{n}$

**Theorem 7.0.41** *Let $u(\cdot, t_0) = f \in L^2(\mathbb{C}^n)$, $\alpha = \frac{2}{n}$ and $G$ be as in (1.0.15) and (1.0.16). Initial value problem (1.0.13), (1.0.14) has unique maximal solution $u \in C((T_*, T^*), L^2(\mathbb{C}^n)) \cap L^{q_1}_{loc}((T_*, T^*), L^{p_1}(\mathbb{C}^n))$, where $t_0 \in (T_*, T^*)$ and $(q_1, p_1)$ be an arbitrary admissible pair. Fix $p = 2 + \alpha$. Moreover the following properties hold:*

**(i)(Uniqueness)** *Solution is unique in $C((T_*, T^*), L^2(\mathbb{C}^n)) \cap L^p((T_*, T^*), L^p(\mathbb{C}^n))$.*

**(ii)(Blowup alternative)** *If $T^* < \infty$ (respectively, $T_* > -\infty$), then $\|u\|_{L^p((t_0,T^*),L^p(\mathbb{C}^n))} = \infty$ (respectively, $\|u\|_{L^p((T_*,t_0),L^p(\mathbb{C}^n))} = \infty$).*

**(iii)(Stability)** *If $f_j \to f$ in $L^2(\mathbb{C}^n)$, then $u_j \to u$ in $L^{q_1}(I, L^{p_1}(\mathbb{C}^n))$ for every interval $I$ with $\overline{I} \subset (T_*, T^*)$ and for every admissible pair $(q_1, p_1)$, where $u_j$ and $u$ are solutions corresponding to $f_j$ and $f$ respectively.*

**(iv)(Conservation of charge)** *If $\psi : \mathbb{R}^n \times \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ is real valued, then we have conservation of charge $\|u(\cdot, t)\|_{L^2(\mathbb{C}^n)} = \|f\|_{L^2(\mathbb{C}^n)}$ for every $t \in (T_*, T^*)$.*

**Proof**. First we prove local existence. For given positive real numbers $T$ and $\delta$ with $T \leq \pi$, consider the metric space

$$E_{T,\delta} = \{u \in L^p(I, L^p(\mathbb{C}^n)) : \|u\|_{L^p(I,L^p(\mathbb{C}^n))} \leq \delta\}$$

with metric $d(u, v) = \|u - v\|_{L^p(I,L^p(\mathbb{C}^n))}$, where $I = (t_0 - T, t_0 + T)$ and $p = \frac{2(n+1)}{n}$. Since $\frac{1}{p} = n(\frac{1}{2} - \frac{1}{p})$, $(p, p)$ be an admissible pair. We show existence of solution to the initail value problem (1.0.13), (1.0.14) by showing that operator $\mathcal{H}$ given by (1.0.21) is contraction on complete metric spce $E_{T,\delta}$ for suitable $T$ and $\delta$.

Let $u, v \in L^p(I, L^p(\mathbb{C}^n))$. By taking $L^{p'}$ norm with respect to the $t$-variable in the inequality (5.0.4) and using the Hölder's inequality, we get

$$\begin{aligned} \|G(z,u) - G(z,v)\|_{L^{p'}(I,L^{p'})} &\leq C(\|u\|^{\alpha}_{L^p(I,L^p)} + \|v\|^{\alpha}_{L^p(I,L^p)}) \\ &\quad \times \|u - v\|_{L^p(I,L^p)}. \end{aligned} \tag{7.0.8}$$

From Strichartz estimates (Theorem 3.0.7), above estimate and for $u, v \in E_{T,\delta}$, we observe that

$$\begin{aligned} \|\mathcal{H}u\|_{L^p(I,L^p(\mathbb{C}^n))} &\leq \|e^{-i(t-t_0)\mathcal{L}} f\|_{L^p(I,L^p)} + C\|u\|^{\alpha}_{L^p(I,L^p)}\|u\|_{L^p(I,L^p)} \\ &\leq \|e^{-i(t-t_0)\mathcal{L}} f\|_{L^p(I,L^p)} + C\delta^{1+\alpha} \end{aligned} \tag{7.0.9}$$

$$\begin{aligned} \|\mathcal{H}u - \mathcal{H}v\|_{L^p(I,L^p(\mathbb{C}^n))} &\leq C(\|u\|^{\alpha}_{L^p(I,L^p)} + \|v\|^{\alpha}_{L^p(I,L^p)})\|u - v\|_{L^p(I,L^p(\mathbb{C}^n))} \\ &\leq C\delta^{\alpha}\|u - v\|_{L^p(I,L^p(\mathbb{C}^n))}. \end{aligned} \tag{7.0.10}$$

Choose

$$\delta = (4C)^{-\frac{1}{\alpha}} \tag{7.0.11}$$

and $T \leq \pi$ sufficiently small such that

$$\|e^{-i(t-t_0)\mathcal{L}} f\|_{L^p(I,L^p(\mathbb{C}^n))} \leq \frac{\delta}{2} \tag{7.0.12}$$

where $C$ is the same constant that appears in the inequalities (7.0.8), (7.0.9), (7.0.10) and is independent of $T$. For this choice of $\delta$ and $T$, operator $\mathcal{H}$ is a contaction on $E_{T,\delta}$. Therefore $\mathcal{H}$ has unique fixed point in $E_{T,\delta}$. Since $G(z, u(z,t)) \in L^{p'}(I, L^{p'}(\mathbb{C}^n))$, from Strichartz estimates $u \in C(\overline{I}, L^2(\mathbb{C}^n)) \cap L^{\tilde{q}}(I, L^{\tilde{p}}(\mathbb{C}^n))$ for every admissible pair $(\tilde{q}, \tilde{p})$. In view of Lemma 5.0.21, $u$ is also a solution of the initial value problem (1.0.13), (1.0.14).

By succesive application of local existence argument, solution can be extended to maximal interval $(T_*, T^*)$ and $u \in C((T_*, T^*), L^2(\mathbb{C}^n)) \cap L^{\tilde{q}}_{\text{loc}}((T_*, T^*), L^{\tilde{p}}(\mathbb{C}^n))$ for every admissible pair $(\tilde{q}, \tilde{p})$. In view of (7.0.8), (7.0.10) and (7.0.11), uniqueness follows by similar arguments as in Theorem 5.0.26.

**Stability:** We prove stability in the following two steps.

Step 1: Let $\{f_m\}_{m\geq 1}$ be a sequence in $L^2(\mathbb{C}^n)$ such that $f_m \to f$ in $L^2(\mathbb{C}^n)$ as $m \to \infty$. Let $u_m$ and $u$ be the solutions corresponding to the initial data $f_m$ and $f$ respectively, at time $t = t_0$. Since $f_m \to f$ in $L^2(\mathbb{C}^n)$, by Theorem 3.0.7,

$$\|e^{-i(t-t_0)\mathcal{L}}(f_m - f)\|_{L^p(I,L^p(\mathbb{C}^n))} \leq C\|f_m - f\|_{L^2(C^n)} \to 0$$

as $m \to \infty$. Choose $\tau < T$, then by (7.0.12), $\|e^{-i(t-t_0)\mathcal{L}} f\|_{L^p(I_\tau,L^p(\mathbb{C}^n))} < \frac{\delta}{2}$, $\|e^{-i(t-t_0)\mathcal{L}} f_m\|_{L^p(I_\tau,L^p(\mathbb{C}^n))} < \frac{\delta}{2}$ and $u_m$ are defined on $I_\tau$ for sufficiently large $m$, where $I_\tau = (t_0 - \tau, t_0 + \tau)$. Setting $G_m(z,t) = G(z, u_m(z,t))$, we have

$$(u_m - u)(z,t) = e^{-i(t-t_0)\mathcal{L}}(f_m - f)(z) - i\int_{t_0}^{t} e^{-i(t-s)\mathcal{L}}(G_m - G)(z,s)ds \tag{7.0.13}$$

for all $t \in I_\tau$. From estimate (7.0.8), we see that

$$\begin{aligned} \|G(z,u_m) - G(z,u)\|_{L^{p'}(I_\tau,L^{p'}(\mathbb{C}^n))} &\leq C(\|u\|^\alpha_{L^p(I,L^p)} + \|v\|^\alpha_{L^p(I,L^p)})\|u - v\|_{L^p(I,L^p)} \\ &\leq C\delta^\alpha \|u - v\|_{L^p(I,L^p)}. \end{aligned} \tag{7.0.14}$$

Now from equation (7.0.13), Theorem 3.0.7 and above estimate, we have

$$\|u_m - u\|_{L^p(I_\tau,L^p(\mathbb{C}^n))} \leq C\|f_m - f\|_{L^2} + C\delta^\alpha\|u_m - u\|_{L^p(I_\tau,L^p(\mathbb{C}^n))}.$$

Note that constant $C$ in the second term of RHS of the above inequality is same constant that appears in the inequality (7.0.10). Now from (7.0.11), $C\delta^{\alpha}=\frac{1}{4}$ and $\|u_m-u\|_{L^p(I_\tau,L^p(\mathbb{C}^n))}\leq\frac{4C}{3}\|f_m-f\|_{L^2}\to 0$ as $m\to\infty$. Now from estimate (7.0.14) $\|G(z,u_m)-G(z,u)\|_{L^{p'}(I_\tau,L^{p'}(\mathbb{C}^n))}\to 0$ as $m\to\infty$. Now from equation (7.0.13) and theorem 3.0.7, $u_m\to u$ in $L^{q_1}(I_\tau,L^{p_1}(\mathbb{C}^n))$ for every admissible pair $(q_1,p_1)$. Since $u_m,u\in C(\overline{I}_\tau,L^2(\mathbb{C}^n))$ for every $m$, therefore $\|(u_m-u)(\cdot,t)\|_{L^2(\mathbb{C}^n)}\leq\|u_m-u\|_{L^\infty(I_\tau,L^2(\mathbb{C}^n))}\to 0$ as $m\to\infty$ for each $t\in I_\tau$.

Step 2: Let $(T_{*,m},T_m^*)$ be the maximal interval for the solutions $u_m$ and $I\subset(T_*,T^*)$ be a compact interval. The key idea is to extend the local stability result proved above to the interval $I$ by covering it with finitely many intervals obtained by successive application of the above local stability argument. This is possible provided $u_m$ is defined on $I$, for all but finitely many $m$. In fact, we prove $I\subset(T_{*,m},T_m^*)$ for all but finitely many $m$.

Without loss of generality, we assume that $t_0\in I=[a,b]$, and give a proof by the method of contradiction. Suppose there exist infinitely many $T_{m_j}^*\leq b$ and let $c=\liminf T_{m_j}^*$. Then for $\epsilon>0$, $[t_0,c-\epsilon]\subset[t_0,T_{m_j}^*)$ for all $m_j$ sufficiently large and $u_{m_j}$ are defined on $[t_0,c-\epsilon]$.

By compactness, the stability result proved in step 1 can be extended to the interval $[t_0,c-\epsilon]$. Hence for any interval $J$ with $|J|\leq\pi$, we have

$$\|e^{-i(t-(c-\epsilon))\mathcal{L}}(u_{m_j}-u)(z,c-\epsilon)\|_{L^p(J(dt),L^p(\mathbb{C}^n))}\leq C\|(u_{m_j}-u)(\cdot,c-\epsilon)\|_{L^2}\to 0$$

as $j\to\infty$. Also by continuity we have

$$\|e^{-i(t-(c-\epsilon))\mathcal{L}}(u(z,c-\epsilon)-u(z,c))\|_{L^p(J,L^p(\mathbb{C}^n))}\leq C\|u(\cdot,c-\epsilon)-u(\cdot,c)\|_{L^2}\to 0$$

as $\epsilon\to 0$. We also observe that

$$\|e^{-i(t-c)\mathcal{L}}(e^{-i\epsilon\mathcal{L}}u(z,c)-u(z,c))\|_{L^p(J,L^p(\mathbb{C}^n))} \leq C\|e^{-i\epsilon\mathcal{L}}u(\cdot,c)-u(\cdot,c)\|_{L^2}\to 0$$

as $\epsilon\to 0$. Now choose $\eta>0$ sufficiently small such that

$$\|e^{-i(t-c)\mathcal{L}}u(z,c))\|_{L^p(J_\eta,L^p(\mathbb{C}^n))}<\frac{\delta}{2}$$

where $J_\eta = (c-\eta, c+\eta)$ and $\delta$ is given by (7.0.11). By the above observations, we can choose $\epsilon < \frac{\eta}{2}$ sufficiently small such that

$$\|e^{-i(t-(c-\epsilon))\mathcal{L}} u_{m_j}(z, c-\epsilon)\|_{L^p(J_\eta, L^p(\mathbb{C}^n))} < \frac{\delta}{2}$$

for sufficiently large $j \geq j_0(\epsilon)$. Therefore by applying the local existence argument (see equation 7.0.12), with $c-\epsilon$ as a initial time, $u_{m_j}$ extends to $[t_0, c-\epsilon+\eta]$ for large $j$. It follows that $T^*_{m_j} \geq c-\epsilon+\eta \geq c+\frac{\eta}{2}$, hence contradicts the fact that $\liminf T^*_{m_j} = c$.

Similarly we can show that $[a, t_0] \subset (T_{*,m}, t_0]$ for all but finitely many $m$ which completes the proof of stability.

Conservation of charge follows exactly as in Theorem 7.0.40.

**Blowup alternative:** We prove blowup alternative by method of contradiction. Suppose $T^* < \infty$ and $u \in L^p((t_0, T^*), L^p(\mathbb{C}^n))$. Then by estimates (7.0.8), $G(z, u) \in L^{p'}((t_0, T^*), L^{p'}(\mathbb{C}^n))$ and by Strichartz estimates $u \in C([t_0, T^*], L^2) \cap L^{\tilde{q}}\big((T_*, T^*), L^{\tilde{p}}(\mathbb{C}^n)\big)$ for every admissible pair $(\tilde{q}, \tilde{p})$. By taking $T^*$ as a initial time and by local existence argument, solution exists on the interval $[t_0, T^*+\epsilon)$ for some $\epsilon > 0$, which is a contradiction for maximality of $T^*$.

**Remark 7.0.42** If $\|f\|_{L^2(\mathbb{C}^n)}$ is sufficiently small, then $\|e^{-i(t-t_0)\mathcal{L}} f\|_{L^p(I, L^p)} \leq C\|f\|_{L^2} < \delta$ where $p = \frac{2(n+1)}{n}$. Since $C$ is independent of $t_0$ and interval $I = (t_0 - T, t_0 + T)$ provided $2T \leq \pi$, from conservation of charge and (7.0.12) we get global solution, i.e., $-T_* = T^* = \infty$ in Theorem 7.0.41.

# Chapter 8

# The case of the Laguerre operator

As discussed in chapter 1, in this chapter we consider the Laguerre case. Laguerre operator $L_\beta$ on $\mathbb{R}_+ = (0,\infty)$ with $\beta \in (-1,\infty)$ is given by,

$$L_\beta = -\frac{d^2}{dx^2} - \frac{2\beta+1}{x}\frac{d}{dx} + \frac{x^2}{4}. \tag{8.0.1}$$

The one dimensional Laguerre polynomials $L_k^\beta(x)$ of type $\beta > -1$ are defined by the generating function identity

$$\sum_{k=0}^{\infty} L_k^\beta(x) t^k = (1-t)^{-\beta-1} e^{-\frac{xt}{1-t}}, \quad |t| < 1.$$

Here $x > 0$ and $k \in \mathbb{Z}_{\geq 0}$. Each $L_k^\beta$ is a polynomial of degree $k$ and explicitly given by

$$L_k^\beta(x) = \sum_{j=0}^{k} \frac{\Gamma(k+\beta+1)}{\Gamma(k-j+1)\Gamma(j+\beta+1)} \frac{(-x)^j}{j!}.$$

Laguerre functions $\psi_k^\beta(x) = \left(\frac{2^{-\beta}k!}{\Gamma(k+\beta+1)}\right)^{\frac{1}{2}} L_k^\beta(\frac{x^2}{2}) e^{-\frac{x^2}{4}}$ form a complete orthonormal family in $L^2(\mathbb{R}_+, x^{2\beta+1}dx)$. Each $\psi_k^\beta$ is an eigenfunction of the Laguerre operator $L_\beta$ given by (8.0.1) with eigenvalue $(2k+\beta+1)$, i.e.,

$$L_\beta \psi_k^\beta = (2k+\beta+1)\psi_k^\beta.$$

If $f, g \in L^2(\mathbb{R}_+, x^{2\beta+1}dx)$ with $L_\beta f, L_\beta g \in L^2(\mathbb{R}_+, x^{2\beta+1}dx)$, then $\langle L_\beta f, g\rangle_{x^{2\beta+1}dx} = \langle f, L_\beta g\rangle_{x^{2\beta+1}dx}$ where inner product is with respect to the measure $x^{2\beta+1}dx$. Therefore we say that Laguerre operator $L_\beta$ is self adjoint with repect to the measure $x^{2\beta+1}dx$. Thus for every $f \in L^2(\mathbb{R}_+, x^{2\beta+1}dx)$ has the Laguerre expansion

$$f = \sum_{k=0}^{\infty} \left\langle f, \psi_k^\beta \right\rangle_{x^{2\beta+1}dx} \psi_k^\beta.$$

We call $\left\langle f, \psi_k^\beta \right\rangle_{x^{2\beta+1}dx}$ as the $k$-th Fourier-Laguerre coefficient of $f$. Now for each multi index $\mu = (\mu_1, \cdots, \mu_n) \in (\mathbb{Z}_{\geq 0})^n$ and $\beta = (\beta_1, \cdots, \beta_n) \in (-1, \infty)^n$, the $n$-dimensional Laguerre functions are defined by the tensor product of 1-dimensional Laguerre functions

$$\psi_\mu^\beta(x) = \prod_{j=1}^{n} \psi_{\mu_j}^{\beta_j}(x_j), \ \ x \in \mathbb{R}_+^n = (\mathbb{R}_+)^n. \tag{8.0.2}$$

The $n$-dimensional Laguerre operator $L_\beta$ for $\beta = (\beta_1, \cdots, \beta_n) \in (-1, \infty)^n$, is defined as the sum of 1-dimensional Laguerre operators $L_{\beta_j}$

$$L_\beta = \sum_{j=1}^{n} L_{\beta_j} = -\Delta - \sum_{j=1}^{n} \left( \frac{2\beta_j + 1}{x_j} \frac{\partial}{\partial x_j} \right) + \frac{|x|^2}{4}.$$

Therefore $L_\beta \psi_\mu^\beta = \left(2|\mu| + \sum_{j=1}^{n} \beta_j + n\right) \psi_\mu^\beta$, where $|\mu| = \sum_{j=1}^{n} \mu_j$. Hence, $\psi_\mu^\beta$ are eigenfunctions of $L_\beta$ with eigenvalue $2|\mu| + \sum_{j=1}^{n} \beta_j + n$ and they also form a complete orthonormal system in $L^2(\mathbb{R}_+^n, d\nu(x))$ where

$$d\nu(x) = x_1^{2\beta_1+1} \cdots x_n^{2\beta_n+1} dx_1 \cdots dx_n.$$

Also note that Laguerre operator $L_\beta$ is self adjoint with repect to measure $d\nu$. Thus for every $f \in L^2(\mathbb{R}_+^n, d\nu(x))$ has the Laguerre expansion

$$f = \sum_{\mu} \left\langle f, \psi_\mu^\beta \right\rangle_\nu \psi_\mu^\beta = \sum_{k=0}^{\infty} P_k f,$$

where inner product is with respect to measure $\nu$ and $P_k$ denotes the Laguerre projection operator corresponding to the eigenvalue $2k + \sum_{j=1}^{n} \beta_j + n$ given by

$$P_k f = \sum_{|\mu|=k} \left\langle f, \psi_\mu^\beta \right\rangle_\nu \psi_\mu^\beta.$$

The material discussed here is based on the books by Thangavelu [33] and Lebedev [18].

**Remark 8.0.43** In view of estimate (1) in Watson [38] (see section 3.31, page 49), in this chapter we only consider $\beta \in \left(-\frac{1}{2}, \infty\right)^n$.

**Remark 8.0.44** Note that $L^\infty(\mathbb{R}_+^n, dx) = L^\infty(\mathbb{R}_+^n, d\nu)$ with equality of norms $\|f\|_{L^\infty(\mathbb{R}_+^n, dx)} = \|f\|_{L^\infty(\mathbb{R}_+^n, d\nu)}$, where $dx$ denote the usual Lebesgue measure on $\mathbb{R}_+^n$.

## Schrödinger Propagator $e^{-itL_\beta}$

If $f \in C^2 \cap L^2(\mathbb{R}_+^n, d\nu)$ such that $L_\beta f \in L^2(\mathbb{R}_+^n, d\nu)$ then we observe that

$$\left\langle L_\beta f, \psi_\mu^\beta \right\rangle_\nu = \left\langle f, L_\beta \psi_\mu^\beta \right\rangle_\nu = \left(2|\mu| + n + \sum_{j=1}^n \beta_j\right) \left\langle f, \psi_\mu^\beta \right\rangle_\nu.$$

Therefore for $f \in L^2(\mathbb{R}_+^n, d\nu)$, we define $e^{-itL_\beta} f$ as $L^2(\mathbb{R}_+^n, d\nu)$ function by the following

$$e^{-itL_\beta} f = \sum_{k=0}^\infty e^{-it\left(2k+n+\sum_{j=1}^n \beta_j\right)} \sum_{|\mu|=k} \left\langle f, \psi_\mu^\beta \right\rangle_\nu \psi_\mu^\beta.$$

It is easy to see that $e^{-itL_\beta}$ is unitary operator with adjoint $e^{itL_\beta}$ on $L^2(\mathbb{R}_+^n, d\nu)$.

**Remark 8.0.45** $e^{-itL_\beta} f$ is periodic in $t$ if and only if $\sum_{j=1}^n \beta_j$ is rational whereas $e^{it\sum \beta_j} e^{-itL_\beta} f$ and $|e^{-itL_\beta} f|$ are always periodic in $t$.

Now we state the following Lemma. This Lemma is proved in Sohani [28]. Proof relies on regularization argument introduced in [20] (also see [22], [23]) and Mehler's formula for Laguerre functions (see, (4.17.6) in [18]), so we skip the proof.

**Lemma 8.0.46** *Let $r > 0, \beta \in \left(-\frac{1}{2}, \infty\right)^n$. Then $e^{-(r+it)L_\beta}$ is an integral operator on $L^2(\mathbb{R}^n_+, d\nu)$. Moreover*

$$\begin{aligned} e^{-(r+it)L_\beta} f(x) &= \int_{\mathbb{R}^n_+} f(y) K(x,y,r,t,\beta) d\nu(y) \\ K(x,y,r,t,\beta) &= e^{-nr} e^{-it(n+\sum \beta_j)} \left(1 - e^{-2(r+it)}\right)^{-n} e^{-\left(\frac{|x|^2+|y|^2}{4}\right)\left(\frac{1+e^{-2(r+it)}}{1-e^{-2(r+it)}}\right)} \\ &\quad \times \prod_{j=1}^{n} \left( (x_j y_j)^{-\beta_j} (e^{-2it})^{\frac{-\beta_j}{2}} I_{\beta_j} \left( \frac{x_j y_j e^{-r} (e^{-2it})^{\frac{1}{2}}}{1 - e^{-2(r+it)}} \right) \right) \end{aligned}$$

*where $I_{\beta_j}$ is the modified Bessel function of first kind and $|\arg(e^{-2it})| < \pi$.*

**Lemma 8.0.47** *Let $K(x,y,r,t,\beta)$ be the kernel as in Lemma 8.0.46. Then we have uniform estimate for $K$ in $r \in (0,1]$.*

$$|K(x,y,r,t,\beta)| \leq \frac{C}{|\sin t|^{n+\sum_{j=1}^n \beta_j}} \tag{8.0.3}$$

*where $C$ only depends on $n$ and $\beta$.*

**Proof**. Let $\arg(e^{-2it}) = -2\tilde{t}$ with $|\tilde{t}| < \frac{\pi}{2}$, then $e^{-2it} = e^{-2i\tilde{t}}$, $(e^{-2it})^{\frac{1}{2}} = e^{-i\tilde{t}}$ and $\cos 2t = \cos 2\tilde{t}$. Now we observe the following

$$\begin{aligned} \left|1 - e^{-2(r+it)}\right| &= \left(1 + e^{-4r} - 2e^{-2r}\cos 2t\right)^{\frac{1}{2}} \\ \frac{x_j y_j e^{-(r+i\tilde{t})}}{1 - e^{-2(r+it)}} &= x_j y_j e^{-r} \left( \frac{(1 - e^{-2r})\cos\tilde{t} - i(1 + e^{-2r})\sin\tilde{t}}{1 + e^{-4r} - 2e^{-2r}\cos 2t} \right) \\ \left| \mathrm{Re}\left( \frac{x_j y_j e^{-(r+i\tilde{t})}}{1 - e^{-2(r+it)}} \right) \right| &\leq \frac{x_j y_j e^{-r}\left(1 - e^{-2r}\right)}{1 + e^{-4r} - 2e^{-2r}\cos 2t} \\ \frac{1 + e^{-2(r+it)}}{1 - e^{-2(r+it)}} &= \frac{(1 - e^{-4r}) - 2ie^{-2r}\sin 2t}{1 + e^{-4r} - 2e^{-2r}\cos 2t} \end{aligned}$$

Now we observe that

$$|I_\delta(z)| \leq \frac{|z|^\delta}{2^\delta \Gamma(\delta+1)} \exp(|\mathrm{Re}(z)|), \quad \text{for } \delta > -\frac{1}{2} \tag{8.0.4}$$

which follows from inequality (1) in section 3.31, page 49 in Watson [38] and equalities (5.7.4) and (5.7.6) in Lebedev [18]. We also observe that

$$(1+e^{-2r})\left(|x|^2+|y|^2\right)-4e^{-r}\sum x_jy_j$$
$$=(1-e^{-r})^2\left(|x|^2+|y|^2\right)+2e^{-r}\textstyle\sum_{j=1}^{n}(x_j-y_j)^2\geq(1-e^{-r})^2\left(|x|^2+|y|^2\right).$$

Using the above observations, we see that

$$\begin{aligned}
&|K(x,y,r,t,\beta)|\\
\leq\;& Ce^{-r(n+\sum\beta_j)}\left(1+e^{-4r}-2e^{-2r}\cos 2t\right)^{-\frac{(n+\sum\beta_j)}{2}}e^{-\frac{(1-e^{-2r})(1-e^{-r})^2(|x|^2+|y|^2)}{4\left(1+e^{-4r}-2e^{-2r}\cos 2t\right)}}\\
\leq\;& C\left(1+e^{-4r}-2e^{-2r}\cos 2t\right)^{-\frac{(n+\sum\beta_j)}{2}}. \qquad (8.0.5)
\end{aligned}$$

Now for $r\in(0,1]$ we have

$$1+e^{-4r}-2e^{-2r}\cos 2t=(1-e^{-2r})^2+4e^{-2r}\sin^2 t\geq 4e^{-2}\sin^2 t.$$

Therefore using this estimate in (8.0.5) we get the desired estimate.

**Lemma 8.0.48** *Let $t\notin\frac{\pi}{2}\mathbb{Z}$, $2\leq p\leq\infty$ and $p'=\frac{p}{p-1}$. Then*

$$\|e^{-itL_\beta}f\|_{L^p(d\nu)}\leq C|\sin t|^{-\left(1-\frac{2}{p}\right)(n+\sum\beta_j)}\|f\|_{L^{p'}(d\nu)}$$

*where constant $C$ depends only on $n,p,\beta$.*

**Proof**. For $f\in L^2(\mathbb{R}^n_+,d\nu)$ we have

$$\|e^{-itL_\beta}f\|^2_{L^2(d\nu)}=\sum_{k=0}^{\infty}|e^{-it(2k+n+\sum\beta_j)}|^2.\|P_kf\|^2_{L^2(d\nu)}=\|f\|^2_{L^2(d\nu)}. \qquad (8.0.6)$$

For $f\in L^1\cap L^2(\mathbb{R}^n_+,d\nu)$ we observe from Lemma 8.0.47 and Remark 8.0.44 that

$$\|e^{-(r+it)L_\beta}f\|_{L^\infty(\mathbb{R}^n_+,d\nu)}\leq C|\sin t|^{-(n+\sum\beta_j)}\|e^{-(r+it)L_\beta}f\|_{L^1(\mathbb{R}^n_+,d\nu)}.$$

Since $e^{-(r_m+it)L_\beta}f\to e^{-itL_\beta}f$ in $L^2(d\nu)$ as $r_m\to 0$, $e^{-(r_{m_j}+it)L_\beta}f(x)\to e^{-itL_\beta}f(x)$ a.e. $x$ for some subsequence $\{r_{m_j}\}$. Also observe that

$$\int_{\mathbb{R}^n_+}f(y)K(x,y,r_{m_j},t,\beta)d\nu(y)\to\int_{\mathbb{R}^n_+}f(y)K(x,y,0,t,\beta)d\nu(y)$$

for a.e. $x \in \mathbb{R}^n_+$. Therefore for $f \in L^1 \cap L^2(\mathbb{R}^n_+, d\nu)$ we get

$$e^{-itL_\beta} f(x) = \int_{\mathbb{R}^n_+} f(y) K(x, y, 0, t, \beta) d\nu(y). \tag{8.0.7}$$

From Remark 8.0.44 and Lemma 8.0.47 we observe that

$$\|e^{-itL_\beta} f\|_{L^\infty(\mathbb{R}^n_+, d\nu)} \leq C |\sin t|^{-(n+\sum \beta_j)} \|f\|_{L^1(\mathbb{R}^n_+, d\nu)}. \tag{8.0.8}$$

This inequality can be proved for $f \in L^1(\mathbb{R}^n_+, d\nu)$ by density argument. Using Riesz-Thorin interpolation theorem (see Folland [11]) and in view of (8.0.6), (8.0.8) Lemma follows.

## Strichartz estimates

**Definition 8.0.49** Let $n \geq 1$ and $\beta \in (-\frac{1}{2}, \infty)^n$. We say that a pair $(q, p)$ is *admissible* in the Laguerre case if

$$1 \leq q \leq 2, \quad 0 \leq \left(n + \sum_{j=0}^{n} \beta_j\right)\left(1 - \frac{2}{p}\right) < 1 \qquad \text{or}$$

$$2 < q \leq \infty \text{ and } 0 \leq \left(n + \sum_{j=0}^{n} \beta_j\right)\left(1 - \frac{2}{p}\right) \leq \frac{2}{q} < 1.$$

**Remark 8.0.50** **(i)** The admissibility condition on $(q, p)$ implies that

$$0 \leq \left(n + \sum_{j=0}^{n} \beta_j\right)\left(1 - \frac{2}{p}\right) < 1.$$

**(ii)** If $1 \leq q \leq 2, n = 1, 1 + \beta < 1$, then $p \in [2, \infty]$.

**(iii)** If $1 \leq q \leq 2, n = 1, 1 + \beta = 1$, then $p \in [2, \infty)$.

**(iv)** If $1 \leq q \leq 2$, $\left(n + \sum_{j=0}^{n} \beta_j\right) > 1$, then $p \in \left[2, \frac{2\left(n+\sum_{j=0}^{n} \beta_j\right)}{\left(n+\sum_{j=0}^{n} \beta_j\right)-1}\right)$.

Admissible condition is basically coming from the Lemma 3.0.4 and Remark 8.0.51.

**Remark 8.0.51** Let $p \in [2, \infty], a, b \in \mathbb{R}$. $|\sin t|^{-\left(1-\frac{2}{p}\right)(n+\sum \beta_j)} \in$ weak $L^{\frac{q}{2}}(a-b, b-a)$ with $q \in (2, \infty)$ if $1 < \frac{q}{2} \leq \frac{1}{(n+\sum_{j=0}^{n} \beta_j)(1-\frac{2}{p})}$ or $\left(n + \sum_{j=0}^{n} \beta_j\right)(1-\frac{2}{p}) \leq \frac{2}{q} < 1$. Also $|\sin t|^{-\left(1-\frac{2}{p}\right)(n+\sum \beta_j)} \in L^1(a-b, b-a)$ if $\left(n + \sum_{j=0}^{n} \beta_j\right)\left(1-\frac{2}{p}\right) < 1$. If we consider $p = 2$ then $|\sin t|^{-\left(1-\frac{2}{p}\right)(n+\sum \beta_j)} = 1 \in L^\infty(a-b, b-a)$.

Now we prove a Lemma which is helpful in proving Strichartz estimates.

**Lemma 8.0.52** *Let $[a, b]$ be a bounded interval containing $t_0$. Let $h_j(x, t) \in L^{q_j'}((a, b), L^2(\mathbb{R}_+^n, d\nu(x)))$, where $q_j'$ is conjugate exponent of $q_j$ with $1 \leq q_j \leq \infty$ for $j = 1, 2$. Then the functions*

$$e^{-i(t-t_0)L_\beta} h_1(x, t) e^{-i(s-t_0)L_\beta} h_2(x, s), \quad h_1(x, t) e^{i(t-s)L_\beta} h_2(x, s)$$

*belong to $L^1(\mathbb{R}_+^n \times (a, b) \times (a, b), d\nu(x) \times dt \times ds)$.*

**Proof**. For simplicity we are considering $h_1 = h_2 = h$ and $q_1 = q_2 = q$. Since $h \in L^{q'}((a, b), L^2(d\nu))$, $h(\cdot, t) \in L^2(\mathbb{R}_+^n, d\nu)$ for a.e. $t \in (a, b)$. Therefore $e^{-i(t-t_0)L_\beta} h(\cdot, t) \in L^2(\mathbb{R}_+^n, d\nu)$ for a.e. $t \in (a, b)$. Then by Hölder's inequality $e^{-i(t-t_0)L_\beta} h(\cdot, t) e^{-i(s-t_0)L_\beta} h(\cdot, s) \in L^1(\mathbb{R}_+^n, d\nu)$ for a.e. $t, s \in (a, b)$ and

$$\int_{\mathbb{R}_+^n} \left| e^{-i(t-t_0)L_\beta} h(x, t) e^{-i(s-t_0)L_\beta} h(x, s) \right| d\nu(x) \leq \|h(\cdot, t)\|_{L^2(d\nu)} \|h(\cdot, s)\|_{L^2(d\nu)}.$$

Integrating with respect to $t$ and $s$ over $(a, b) \times (a, b)$ and using Hölder's inequality in the $t$-variable, we get

$$\begin{aligned} &\int_a^b \int_a^b \int_{\mathbb{R}_+^n} |e^{-i(t-t_0)L_\beta} h(x, t) e^{-i(s-t_0)L_\beta} h(x, s)| d\nu(x) dt ds \\ &\leq \left( \int_a^b \|h(\cdot, t)\|_{L^2(d\nu)} dt \right)^2 \leq (b-a)^{\frac{2}{q}} \|h\|^2_{L^{q'}((a,b), L^2(d\nu))}. \end{aligned}$$

Similarly $h_1(x, t) e^{i(t-s)L_\beta} h_2(x, s) \in L^1(\mathbb{R}_+^n \times (a, b) \times (a, b), d\nu(x) \times dt \times ds)$ can be proved.

The main Strichartz type estimates in this chapter is compiled in following theorem which is proved in [28]. Proof follows by similar arguments as in Theorem 3.0.7.

**Theorem 8.0.53 (Strichartz Estimates)** *Let $(q,p),(q_1,p_1)$ be two admissible pairs according to definition 8.0.49. Let $(a,b)$ be a finite interval with $t_0 \in [a,b]$, $f \in L^2(\mathbb{R}^n_+, d\nu)$ and $g \in L^{q_1'}((a,b), L^{p_1'}(\mathbb{R}^n_+, d\nu))$ where $q_1'$ and $p_1'$ are conjugate exponents of $q_1$ and $p_1$ respectively. Then the following estimates hold over $\mathbb{R}^n_+ \times (a,b)$:*

$$\|e^{-itL_\beta} f\|_{L^q((a,b),L^p(d\nu))} \leq C\|f\|_{L^2(d\nu)} \tag{8.0.9}$$

$$\left\| \int_{t_0}^t e^{-i(t-s)L_\beta} g(x,s)ds \right\|_{L^q((a,b),L^p(d\nu))} \leq C\|g\|_{L^{q_1'}((a,b),L^{p_1'}(d\nu))} \tag{8.0.10}$$

*where the constant $C$ depends on admissible pairs but independent of $t_0$. Moreover $e^{-itL_\beta} f \in C(\mathbb{R}, L^2(d\nu))$ and $\int_{t_0}^t e^{-i(t-s)L_\beta} g(x,s)ds \in C([a,b], L^2(d\nu))$.*

**Remark 8.0.54** Note that $e^{-itL_\beta} f(x)$ is $2\pi$ periodic in $t$, hence we can not expect the above Strichartz inequalities for unbounded intervals except when $q = \infty$. Also Since $|\sin t|$ is $\pi$ periodic, in view of Remark 8.0.51 and Remark 3.0.8, constant $C$ in the inequalities (8.0.9) and (8.0.10) can be chosen independent of interval $(a,b)$ provided $b - a \leq \pi$.

## Local well posedness in $L^2(\mathbb{R}^n_+, d\nu)$

We consider the initial value problem for the nonlinear Schrödinger equation for the Laguerre operator $L_\beta$:

$$i\partial_t u(x,t) - L_\beta u(x,t) = G(x,u), \qquad x \in \mathbb{R}^n_+, \; t \in \mathbb{R} \tag{8.0.11}$$

$$u(x,t_0) = f(x) \tag{8.0.12}$$

where $G$ is a function on $\mathbb{R}^n_+ \times \mathbb{C}$ satisfying similar conditions as in (1.0.15), (1.0.16). Here we consider the nonlinearity $G$ of the form

$$G(x,w) = \psi(x,|w|)\, w, \; (x,w) \in \mathbb{R}^n_+ \times \mathbb{C}, \tag{8.0.13}$$

where $\psi \in C(\mathbb{R}^n_+ \times [0,\infty)) \cap C^1(\mathbb{R}^n_+ \times (0,\infty))$ satisfy the following inequality

$$|F(x,\eta)| \leq C|\eta|^\alpha \tag{8.0.14}$$

with $F = \psi, \partial_{x_j}\psi$ $(1 \leq j \leq n)$ and $\eta\partial_\eta\psi(x,\eta)$, $\alpha \geq 0$ and for some constant $C$. By mean value theorem and estimate 8.0.14 on $\psi$, we have

$$|G(x,u) - G(x,v)| \leq C(|u|^\alpha + |v|^\alpha)|u-v| \tag{8.0.15}$$

where $C$ is independent of $u, v \in \mathbb{C}$ and $x \in \mathbb{R}^n_+$.

Since $L_\beta$ has no decomposition in terms of first differential operators as the twisted Laplacian $\mathcal{L}$ has, therefore we only consider the initial value in $L^2(\mathbb{R}^n_+, d\nu)$. As similar to the twisted Laplacian case, we can prove the local well posedness of the initial value problem (8.0.11), (8.0.12).

Now we discuss the local well posedness result for the above IVP for subcritical case $0 \leq \alpha < \frac{2}{n+\sum_{j=1}^n \beta_j}$ and critical case $\alpha = \frac{2}{n+\sum_{j=1}^n \beta_j}$.

## Subcritical case $\mathbf{0} \leq \alpha < \frac{\mathbf{2}}{\mathbf{n}+\sum_{\mathbf{j=1}}^{\mathbf{n}} \beta_{\mathbf{j}}}$

Now we state the following Theorem for the subcritical case $0 \leq \alpha < \frac{2}{n+\sum_{j=1}^n \beta_j}$. Proof follows by similar arguments as in Theorem 7.0.40.

**Theorem 8.0.55** *Let* $u(\cdot, t_0) = f \in L^2(\mathbb{R}^n_+, d\nu)$, $0 \leq \alpha < \frac{2}{n+\sum_{j=1}^n \beta_j}$ *and* $G$ *be as in (8.0.13), (8.0.14). Initial value problem (8.0.11), (8.0.12) has unique maximal solution* $u \in C((T_*, T^*), L^2(\mathbb{R}^n_+, d\nu)) \cap L^{q_2}_{loc}\left((T_*, T^*), L^{p_2}(\mathbb{R}^n_+, d\nu)\right)$, *where* $t_0 \in (T_*, T^*)$ *and* $(q_2, p_2)$ *be an arbitrary admissible pair. Fix* $p = 2+\alpha$. *Moreover the following properties hold:*

**(i)(Uniqueness)** *Solution is unique in* $C((T_*, T^*), L^2(d\nu)) \cap L^{q_2}_{loc}((T_*, T^*), L^p(d\nu))$ *where* $q_2 \in [q_1, q]$ *and*

$$\frac{1}{q} = \left(n + \sum_{j=1}^n \beta_j\right)\left(\frac{1}{2} - \frac{1}{p}\right), \quad q_1 = \frac{2p(p-1)}{2p - \left(n + \sum_{j=1}^n \beta_j\right)(p-2)} \geq 1.$$

**(ii)(Blowup alternative)** *If* $T^* < \infty$ *(respectively,* $T_* > -\infty$*), then* $u \notin L^{q_2}((t_0, T^*), L^p(\mathbb{R}^n_+, d\nu))$ *(respectively,* $u \notin L^{q_2}((T_*, t_0), L^p(\mathbb{R}^n_+, d\nu))$*) where* $q_2 \in [q_1, q]$.

**(iii)(Stability)** *If* $f_j \to f$ *in* $L^2(\mathbb{R}^n_+, d\nu)$, *then* $u_j \to u$ *in* $L^{q_2}\left(I, L^{p_2}(\mathbb{R}^n_+, d\nu)\right)$ *for every interval* $I$ *with* $\overline{I} \subset (T_*, T^*)$ *and for every admissible pair* $(q_2, p_2)$, *where* $u_j$ *and* $u$ *are solutions corresponding to* $f_j$ *and* $f$ *respectively.*

## Critical case $\alpha = \frac{2}{n+\sum_{j=1}^{n} \beta_j}$

Now we state the following theorem for the critical case $\alpha = \frac{2}{n+\sum_{j=1}^{n} \beta_j}$. Proof follows by similar arguments as in Theorem 7.0.41.

**Theorem 8.0.56** *Let $u(\cdot, t_0) = f \in L^2(\mathbb{R}_+^n, d\nu)$, $\alpha = \frac{2}{n+\sum_{j=1}^{n} \beta_j}$ and $G$ be as in (8.0.13), (8.0.14). Initial value problem (8.0.11), (8.0.12) has unique maximal solution $u \in C((T_*, T^*), L^2(\mathbb{R}_+^n, d\nu)) \cap L^{q_1}_{loc}\big((T_*, T^*), L^{p_1}(\mathbb{R}_+^n, d\nu)\big)$, where $t_0 \in (T_*, T^*)$ and $(q_1, p_1)$ be an arbitrary admissible pair. Fix $p = 2 + \alpha$. Moreover the following properties hold:*

**(i)(Uniqueness)** *Solution is unique in $C((T_*, T^*), L^2(d\nu)) \cap L^p((T_*, T^*), L^p(d\nu))$.*

**(ii)(Blowup alternative)** *If $T^* < \infty$ then $\|u\|_{L^p((t_0, T^*), L^p(d\nu))} = \infty$. Similar conclusion holds if $T_* > -\infty$.*

**(iii)(Stability)** *If $f_j \to f$ in $L^2(\mathbb{C}^n)$, then $u_j \to u$ in $L^{q_1}\big(I, L^{p_1}(\mathbb{R}_+^n, d\nu)\big)$ for every interval $I$ with $\overline{I} \subset (T_*, T^*)$ and for every admissible pair $(q_1, p_1)$, where $u_j$ and $u$ are solutions corresponding to $f_j$ and $f$ respectively.*

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