Existence of Darboux chart on some Fréchet manifolds

By

Pradip Kumar

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Pradip Kumar

Publications/Preprints

1. Pradip Kumar, Almost complex structure on path space. International Journal of Geometric Methods in Modern Physics Vol. 10, No. 3 (2013) 1220034.
2. Pradip Kumar, Existence of 'Darboux chart' on loop space,arXiv:1309.2190
3. Pradip Kumar, Darboux chart on projective limit of weak symplectic Banach manifold, arXiv:1309.1693

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HRI.

Synopsis

Title of thesis: Existence of Darboux chart on some Fréchet manifolds.

Name of candidate: Pradip Kumar.

Name of supervisor: Prof. Rukmini Dey.

Affiliation: Harish-Chandra Research Institute, Allahabad.

This thesis contains some of my work on local symplectic geometry of some Fréchet manifolds during my stay at Harish-Chandra Research Institute as a research scholar.

1. Loop space and weak symplectic structure on it

Let (M,ω) be a finite dimensional symplectic manifold. Loop space $LM := C^\infty(S^1, M)$ is a Fréchet manifold. For manifold structure on LM we refer [20],[25]. For $\gamma \in LM$, $T_\gamma LM = \Gamma_{S^1}(\gamma^* TM)$. As M is oriented, we can identify $\Gamma_{S^1}(\gamma^* TM)$ with $L\mathbb{R}^n$.

For $X, Y \in T_\gamma LM = \Gamma_{S^1}(\gamma^* TM)$, $X(t), Y(t) \in \gamma^*(TM)$, define:

$$\Omega^\omega_\gamma(X,Y) = \int_0^1 \omega_{\gamma(t)}(X(t), Y(t))dt$$

This is a weak symplectic structure on LM.

Recall, by a weak symplectic form Ω on a Fréchet manifold $\mathcal{M}$, we mean that the induced map $\Omega^b : T\mathcal{M} \to T^*\mathcal{M}$ is an injective map and Ω is a closed 2 form. By a strong symplectic form Ω on an infinite dimension manifold $\mathcal{M}$ (Banach or Fréchet manifold), we mean that the corresponding map Ω^b : $T\mathcal{M} \to T^*\mathcal{M}$ is a bundle isomorphism.

We mention here that there does not exist any strong symplectic form on LM. For example, in the case of $L\mathbb{R}^n$, where $T_\gamma L\mathbb{R}^n = L\mathbb{R}^n$ but $T^*_\gamma L\mathbb{R}^n$, the dual space of $L\mathbb{R}^n$, is the set of all $\mathbb{R}^n$ valued distribution on circle [31]. Therefore $T_\gamma L\mathbb{R}^n$ can not be topologically isomorphic to $T^*_\gamma L\mathbb{R}^n$. In general if Ω is a symplectic form on a Fréchet manifold, it can not be strong symplectic. However it may be weak symplectic (Ω^b injective) or quasi symplectic (kernel of Ω^b is finite dimensional).

Let $(\mathcal{M}, \sigma)$ be an infinite dimensional Fréchet manifold with a weak symplectic structure σ.

Definition. By a Darboux chart around $p \in \mathcal{M}$, we mean a coordinate chart $\{(\mathfrak{U}, \Phi), \Phi : \mathfrak{U} \to E\}$ around p such that there exists a bounded alternating bilinear map $\mathcal{F}$ on E for which

$$\sigma_q(v_1, v_2) = \mathcal{F}(d\Phi_q(v_1), d\Phi_q(v_2))$$

for every $q \in \mathfrak{U}$ and $v_1, v_2 \in T_q\mathcal{M}$. This chart around p is called a Darboux chart around p.

We say that $(\mathcal{M}, \sigma)$ admits a Darboux chart if for every $p \in \mathcal{M}$ there is a Darboux chart around $p \in \mathcal{M}$. In [PK2], we proved that the weak symplectic manifold $(L\mathbb{R}^{2n}, \Omega^\omega)$ admits a Darboux chart if there is a global Darboux chart for $(\mathbb{R}^{2n}, \omega)$.

For the weak symplectic Fréchet manifold $(\mathcal{M}, \sigma)$, Kriegl and Michor [20] introduced the notion of symplectic cohomology (different from Floer cohomology). They introduced a subspace $C^\infty_\sigma(\mathcal{M}, \mathbb{R}) \subset C^\infty(LM, \mathbb{R})$. We mention a theorem by Kriegl and Michor:

Theorem (§48.9,[20])**.** *If $(\mathcal{M}, \sigma)$ is a smooth weakly symplectic manifold which admits smooth partitions of unity in $C^\infty_\sigma(\mathcal{M}, \mathbb{R})$, and which admits Darboux chart, then the symplectic cohomology equals the De Rham cohomology: $H^k_\sigma(\mathcal{M}) = H^k_{DR}(\mathcal{M})$.*

In [PK2], we proved that $(L\mathbb{R}^{2n}, \Omega^\omega)$ admits smooth partitions of unity in $C^\infty_\sigma(L\mathbb{R}^{2n}, \mathbb{R})$. This will imply that in the case when there is a global Darboux chart for $(\mathbb{R}^{2n}, \omega)$. The inclusion map from the symplectic cohomology (as defined by Kriegl and Michor [20]) of the loop space over $\mathbb{R}^{2n}$ to the De Rham cohomology of $L\mathbb{R}^{2n}$ is an isomorphism.

Let (M, J) be an almost complex manifold, Indranil Biswas and Saikat Chatterjee in [3] defined an almost complex structure $\tilde{J}$ on the path space

$C^\infty([0,1],M)$. Same definition gives an almost complex structure on the loop space LM. This is defined as following.

For $\gamma \in LM$ and $X \in T_\gamma LM$. We can think X a vector field along γ. Define

$$\begin{aligned}
&\tilde{J} : TLM \to TLM;\\
&\tilde{J}_\gamma : T_\gamma LM \to T_\gamma LM \text{ by}\\
&\tilde{J}_\gamma(X)(t) := J(X(\gamma(t)))
\end{aligned}$$

We have $\tilde{J}_\gamma^2 = -Id_{T_\gamma LM}$ because $J^2 = -Id$. We see that $\tilde{J}$ is smooth with a smooth inverse (For smoothness we need to show if $c : \mathbb{R} \to TLM$ is smooth then $\tilde{J} \circ c : \mathbb{R} \to TLM$ should be smooth). Thus $\tilde{J}$ is an almost complex structure on the loop space LM.

Lempert [21] gave the following definition.

Definition. (Weak integrable in local sense): Let $(\mathcal{M}, \tilde{J})$ be an almost complex manifold (Banach or Fréchet). We say $(\mathcal{M}, \tilde{J})$ is weak integrable in local sense, if for any $p \in \mathcal{M}$ and any non zero $v \in T_p\mathcal{M}$, there is a neighborhood U of p and a $\tilde{J}$-*holomorphic* function F on U such that $v(F) \neq 0$.

In [PK1], we showed that $(LM, \tilde{J})$ is weak integrable in local sense.

2. Projective limit of Banach manifolds

We say M be the projective limit of Banach manifolds (PLB-Manifold) modeled on a PLB-space $E = \varprojlim E_i$ if we have the following.

1. There is a projective system of Banach manifolds $\{M_i, \phi_{ji}\}_{i,j\in\mathbb{N}}$ such that $M = \varprojlim M_i$.
2. For each $p \in M$, we have $p = (p_i)$. $p_i \in M_i$, and there is a chart (U_i, ψ_i) of $p_i \in M_i$ such that
   (a) $\phi_{ji}(U_j) \subset U_i$, $j \geq i$.

(b) Let $\{E_i, \rho_{ji}\}_{i,j\in\mathbb{N}}$ be a projective systems of Banach spaces, where each ρ_{ji} is inclusion map and the diagram

$$\begin{array}{ccc} U_j & \xrightarrow{\psi_j} & \psi_j(U_j) \\ \downarrow{\scriptstyle \phi_{ji}} & & \downarrow{\scriptstyle \rho_{ji}} \\ U_i & \xrightarrow{\psi_i} & \psi_i(U_i) \end{array}$$

commutes.

(c) $\varprojlim \psi_i(U_i)$ is open in E and $\varprojlim U_i$ is open in M with the inverse limit topology.

The space M satisfying above properties has a natural Fréchet manifold structure. The differential structure on M is determined by the co-ordinate map $\psi : U = \varprojlim U_i \to \psi(U) = \varprojlim \psi_i(U_i)$. Therefore a smooth structure on these type of manifolds is completely determined by the smooth structure on the sequence. George Galanis in [10, 11, 12, 13] has studied this type of manifolds and smooth structures on them.

G. Galanis, in a series of articles [10, 11, 12, 13] discussed various properties of PLB-manifolds.

Suppose M is endowed with a weak symplectic structure σ and each M_i is endowed with corresponding weak symplectic structures σ^i.

In 1969, for the case of a strong symplectic Banach manifold $(\mathcal{M}, \sigma)$, Weinstein [32] proved that $(\mathcal{M}, \sigma)$ admits Darboux chart. In 1972, Marsden [24] showed that the Darboux theorem fails for a weak symplectic Banach manifold. In 1999 Bambusi [2] gave a necessary and sufficient condition for existence of Darboux charts for a weak symplectic Banach manifold (in the case when the model space is reflexive).

In [PK3], we defined the notion of a compatible weak symplectic structure σ on the PLB manifold with the projective system and we proved a version of a Darboux theorem which is explained below.

Suppose $\{E_i, \phi_{ji}\}_{i,j\in\mathbb{N}}$ and $\{F_i, \rho_{ji}\}_{i,j\in\mathbb{N}}$ be the projective system of Banach spaces and $E = \varprojlim E_i$ and $F = \varprojlim F_i$. E and F are Fréchet space.

Definition. [Projective system of mapping]. We say $\{f_i : E_i \to F_i\}_{i\in\mathbb{N}}$ is a projective system of mapping if the following diagram commutes.

$$\begin{array}{ccc} E_j & \xrightarrow{f_j} & F_j \\ \Big\downarrow{\scriptstyle \phi_{ji}} & & \Big\downarrow{\scriptstyle \rho_{ji}} \\ E_i & \xrightarrow{f_i} & F_i. \end{array}$$

We denote the canonical mapping of $E \to E_i$ by e_i and $F \to F_i$ by e'_i.

Definition. We say $f : E \to F$ is the projective limit of a system $\{f_i : E_i \to F_i\}_{i \in \mathbb{N}}$ if for each i, the following diagram commutes.

$$\begin{array}{ccc} E_i & \xrightarrow{f_i} & F_i \\ \Big\downarrow{\scriptstyle e_i} & & \Big\downarrow{\scriptstyle e'_i} \\ E & \xrightarrow{f} & F. \end{array}$$

We define the map $\varprojlim f_i$ as the following.

If $\{f_i\}$ is the projective system of mappings then we see that for any $x = (x_i) \in \varprojlim E_i$, $(f_i(x_i)) \in \varprojlim F_i$. We define $(\varprojlim f_i)(x) = (f_i(x_i)) \in F$. If f be the projective limit of system $\{f_i\}$ then we have

$$f(x) = (f_i(x_i)).$$

We denote the projective limit of system $\{f_i\}$ by $\varprojlim f_i$. Also $\left(\varprojlim f_i\right)(x) = (f_i(x_i)) = f(x)$.

We will need a notion of Lipsctiz map.

Definition (Projective μ-Lipschitz map)**.** Let $E = \varprojlim E_i$ be a Fréchet space. A mapping $\phi : E \to E$ is called projective μ-Lipschitz (μ, a positive real number) if there are $\phi_i : E_i \to E_i$ such that $\phi = \varprojlim \phi_i$ and for every i, ϕ_i is a μ Lipschitz map on each E_i.

Suppose M and N be PLB manifolds modeled over a PLB space E. We say $\phi : M \to N$ is locally μ-Lipsctiz map if there exists a coordinate chart around p and $f(p)$ such that on that chart ϕ is projective μ-Lipscitz.

In some other co-ordinate chart ϕ may not be locally μ Lipscitz.

Basics about weak symplectic structure

Let M be a PLB manifold and $\{M_i, \phi_{ji}\}_{i,j \in \mathbb{N}}$ be a projective system of Banach manifolds with $M = \varprojlim M_i$. Suppose each M_i is modeled over a reflexive Banach space E_i and each M_i has a weak symplectic structure σ^i.

Let $x \in M$, we have $x = (x_i)$ where $\phi_{ji}(x_j) = x_i$. For each x_i, following [2], we define a norm on $T_{x_i}M_i$, for $X \in T_{x_i}M_i$,

$$\|X\|_{\mathcal{F}_{x_i}} := \sup_{\|Y\|_i=1} |\sigma^i_{x_i}(X, Y)|$$

where $\|.\|_i$ is the norm on the Banach space $T_{x_i}M_i$. Let $\mathcal{F}_{x_i}$ be the completion of $T_{x_i}M_i$ with respect to the $\|.\|_{\mathcal{F}_{x_i}}$ norm. As each $T_{x_i}M_i$ is a reflexive Banach space, we have that the induced map (Lemma 2.8,[2]),

$$(\sigma^i_{x_i})^b : T_{x_i}M_i \to \mathcal{F}^*_{x_i}; \;\; X \to \sigma^i_{x_i}(X, .)$$

is a topological isomorphism.

Define for each $i, j \in \mathbb{N}$ $j \geq i$ and given $x = (x_i) \in \varprojlim M_i = M$,

$$\psi_{ji} : \mathcal{F}^*_{x_j} \to \mathcal{F}^*_{x_i} \text{ by}$$

$$\psi_{ji} = (\sigma^i_{x_i})^b \circ T_{x_j}\phi_{ji} \circ ((\sigma^j_{x_j})^b)^{-1} \tag{0.0.1}$$

where $T_{x_j}\phi_{ji}$ is the differential of $\phi_{ji} : M_j \to M_i$ at x_j. We see that $\{\mathcal{F}^*_{x_i}, \psi_{ji}\}_{i,j\in\mathbb{N}}$ is a projective system of Banach spaces and smooth maps (since for any $k \geq j \geq i$, we have $\psi_{ki} = \psi_{kj} \circ \psi_{ji}$). We see that $\{(\sigma^i_{x_i})^b : T_{x_i}M_i \to \mathcal{F}^*_{x_i}\}$ is a projective system of mappings because

$$(\sigma^j_{x_j})^b \circ \psi_{ji} = T_{x_j}\phi_{ji} \circ (\sigma^i_{x_i})^b.$$

Fix some point $p = (p_i) \in M$. We know that for a fixed $p = (p_i) \in M$, $\{T_{p_j}M_j, T_{p_j}\phi_{ji}\}_{i,j\in\mathbb{N}}$ is a projective system of Banach spaces. In section 1.3.3 we saw that $T_pM \simeq \varprojlim T_{p_j}M_j$. Let h_p be the isomorphism from $T_pM \to \varprojlim T_{p_j}M_j$ as defined in [14].

For some coordinate neighborhood $U = \varprojlim U_i$ around p, let $\sigma_p := \sigma|_{x=p}$ be the constant symplectic structure on U with a natural parallelism $TU \simeq U \times E$ where $E \simeq T_pM$ is a Fréchet space. On each U_i we have a corresponding constant symplectic structure $\sigma^i_{p_i}$ with the natural parallelism.

For $t \in [-1, 1]$ we define $\sigma^t := \sigma + t(\sigma - \sigma_p)$ similarly $(\sigma^i)^t := \sigma^i + t(\sigma^i - \sigma^i_{p_i})$. Suppose for some $q = (q_i) \in M$ and for some $t \in [-1, 1]$, $((\sigma_q)^{tb})^{-1}$ and $(((\sigma^i_{q_i}))^{tb})^{-1}$ exist for each i. Then $\{(((\sigma^i_{q_i}))^{tb})^{-1} : \mathcal{F}^{*t}_{q_i} \to T_{q_i}M_i\}$ is a projective system of map. Here $\mathcal{F}^{*t}_{q_i}$ is defined in the same way as $\mathcal{F}^*_{q_i}$ above. Where $\mathcal{F}^*_{q_i}$ are spaces corresponding to σ^i, $\mathcal{F}^{*t}_{q_i}$ are spaces corresponding to weak symplectic structure $(\sigma^i)^t$. Also for fixed t corresponding to the

ψ_{ji} maps, we have the maps ψ_{ji}^t (for the weak symplectic structure $(\sigma^i)^t$). Therefore we see that for each t, the collections $\{(((\sigma_{q_i}^i))^{tb})^{-1} : \mathcal{F}_{q_i}^{*t} \to T_{q_i} M_i\}$ are a projective system of mappings.

For a weak symplectic structure σ on a PLB manifold M as discussed above, we have for each $p = (p_i)$, σ_p^b is a map on T_pM that is $\sigma_p^b : T_pM \to T_p^*M$. Let $h_p : T_pM \to \varprojlim T_{p_j}M_j$ be an isomorphism [14]. With this identification we can consider σ_p^b as a map defined on $\varprojlim T_{p_j}M_j$.

Now we are in a position to define a compatible symplectic structure.

Compatible symplectic structure

We say that a weak symplectic structure σ on M is compatible with the projective system if the following is satisfied:

1. Suppose there are weak symplectic structure σ^i on M_i such that for every $x \in M$, $\sigma_x^b := \varprojlim (\sigma_{x_i}^i)^b$.
2. If for some $p \in M$, there exists a 1 - form α such that for each $x \in M$, $\alpha_x = (\alpha_{x_i}^i) \in \varprojlim \mathcal{F}_{x_i}^*$, we must have $h_x((\sigma_x)^{tb})^{-1}(\alpha_x) = \big(((\sigma_{x_i}^i)^{tb})^{-1}(\alpha_{x_i}^i)\big)$ whenever defined.
3. For such α as in above, whenever $Y_t^i(x_i) := [\big(x_i, ((\sigma_{x_i}^i)^{tb})^{-1}(\alpha_{x_i}^i)\big)]$ is defined on some open set U_i of M_i, it is defined on whole M_i. Each Y_t^i is locally projective μ-Lipschitz smooth map for some fixed positive real $\mu > 0$.

Main theorem

For a fixed x, we define $H_x := \{\sigma_x(X, .) : X \in E\}$. For compatible σ, as set, we have $\varprojlim \mathcal{F}_{x_i}^* = H_x$. With the projective limit topology, H_x becomes a Fréchet space. We state the theorem for some open neighborhood of $0 \in E$.

Theorem. *Suppose*

1. *There exists a neighborhood $\mathcal{W}$ of $0 \in E$, such that all H_x are identical and $\sigma_x^{tb} : E \to H$ is an isomorphism for each t and for each $x \in \mathcal{W}$.*
2. *There exists a vector field $X = (X_i)$ on E such that on $\mathcal{W}$, $L_X\sigma = \sigma - \sigma_0$.*
3. *For every i and $t \in [-1, 1]$, $X_i(x_i)$ as element on E_i is bounded by $\frac{M}{\|((\sigma_{x_i}^i)^{tb})^{-1}\|_{op}}$ for some positive real M.*

then there exists a coordinate chart $(\mathcal{V}, \Phi)$ *around zero such that* $\Phi^*\sigma = \sigma_0$.

Proof of above theorem uses the Moser trick.

Table of Contents

Chapter 1

Fréchet manifolds

In this chapter we will give some basic definitions and results that will be used in this thesis in subsequent chapters. Our discussion will be more geometric rather function analytic. Most of the definitions and other functional treatment of global analysis in the non-Banach setting, we refer to the interesting monograph [20]. Content of this chapter is influenced by [20], [15], [31], [10] and includes basic calculations of my articles [17], [18] and [19].

1.1 Fréchet space

1.1.1 Definition of Fréchet space and examples

Definition 1.1.1 (Frechet Space)**.** A Fréchet space is a complete Hausdorff metrizable locally convex topological vector space.

The topology on a locally convex space is metrizable if and only if it can be derived from countable semi-norms. Therefore if E is a Fréchet space, we have countable semi-norms $\{\rho_n : n \in \mathbb{N}\}$ which generate topology on E.

Example 1.1.1. Trivial example is a **Banach space**.

Example 1.1.2. **Loop space** $L\mathbb{R} := C^\infty(S^1, \mathbb{R})$. For each $\gamma \in LR$ and $k \in \mathbb{N} \cup \{0\}$, define

$$\|\gamma\|_k := \sum_{i=0}^{k} \sup_{t \in S^1} |\gamma^{(i)}(t)|.$$

where $\gamma^{(i)}(t)$ denotes the i-th derivative of γ at t. Each $\|.\|_k$ is a semi-norm on $L\mathbb{R}$ (in-fact it is a norm). This countable collection gives the locally convex topology on $L\mathbb{R}$.

This topology is metrizable and complete (§1.46, [29]). Therefore $L\mathbb{R}$ is a Fréchet space. It is worth mentioning here that $L\mathbb{R}$ is not normable [29].

Similarly $L\mathbb{R}^n := C^\infty(S^1, \mathbb{R}^n)$ is also a Fréchet space. Identify S^1 with $[0,1]/\sim$ and elements of $I = [0,1]$ by t. In this thesis we will use this identification frequently.

Example 1.1.3. **Projective limit of Banach space**: Suppose $\{E_i, \rho_{ji}\}_{i,j\in\mathbb{N}}$ be the projective system of Banach space. Let $E = \varprojlim E_i$. The projective limit topology on E is given as following.

For each $n \in \mathbb{N}$, define a seminorm on E by $\|(x_i)\|_n := \|x_n\|_n$. This countable collection of seminorms on E makes E a Fréchet space. It is worth mentioning here that the projective limit topology of countable collection is always complete [30].

In fact other way is also true. Suppose E is a Fréchet space. Topology on E can be realized by an increasing countable family of semi-norms [30].

$$\rho_1 \leq \rho_2 \leq ....$$

E can then be realized up to topological isomorphism as the projective limit of projective system $\{E_i; \rho_{ji}\}_{i,j\in\mathbb{N}}$, where E_i is the completion of the quotient $E/ker(\rho_i)$ and connecting morphism $\rho_{ji}(j \geq i)$ are given by

$$\rho_{ji}([x + ker(\rho_j)]_j) = [x + ker(\rho_i)]_i.$$

Here bracket denotes the corresponding equivalence classes.

In the sequel each point x of E will be considered as $(x_i)_{i\in\mathbb{N}}$ of its projections onto the Banach factors of the limit, with respect to the canonical mappings $f_i : E \to E_i$. If norms on each E_i is denoted by $\|.\|_i$. The construction allows us to consider each f_i as an isometry in the sense that

$$\rho_i(x) = \|f_i(x)\|_i \;\; x \in E$$

Remark 1.1.1. Here we fix the symbol i and j for the natural numbers. In the next whenever we will use i and j together we will mean by natural numbers $j \geq i$.

1.1.2 Bornology

Let (E, τ) be a locally convex topological vector space with the locally convex topology τ. Bornology of E is the the collection of all bounded set of E. The topology can vary considerably without changing the bornology. In the next section, we will see that it is not the topology but the bornology on which smoothness of a map depends.

The bornologification [30] E_{born} of a locally convex space (E, τ) is the finest locally convex topology τ^* on E having the same bornology.

A locally convex space is called bornological if it is stable under the bornologification. We have the following equivalent criterion for the bornological locally convex space.

1. A locally convex spaces E is bornological if and only if every convex balanced set V in E which absorbs every bounded set B (that is $B \subset tV$, for some $t > 0$) is a 0 - neighborhood.

2. A locally convex spaces E is bornological if and only if each semi-norm on E that is bounded on bounded sets, is continuous.

Every normed space is bornological since any set which absorbs the unit ball must contain the ball of some positive radius. It is a fact that inductive limit of a family of bornological space is bornological [30]. Thus a Fréchet space topology is bornological.

1.1.3 Dual of a Fréchet space

Let E be a Fréchet space. Dual of E is the set of all bounded linear map from $E \to \mathbb{R}$ with the topology of uniform convergence on bounded sets in E. It is a well known fact that the dual of a Fréchet is the Fréchet space if and only if Fréchet space is the Banach space.

Example 1.1.4. Dual of loop space $L\mathbb{R}^n$ defined as in Example 1.1.2 is the space of $\mathbb{R}^n$-valued distributions on the circle (§4.5 [31]).

1.2 Smooth map on Fréchet space

When one strays outside the realm of Banach spaces there are a lot of way to define the derivative. Even in Fréchet space there are three inequivalent way to define the derivative [20]. Therefore choice of the calculus is required to be fixed.

Let E be a not normable locally convex space (for example the Fréchet space as in example 1.1.2) define

$$\begin{aligned} F : E' \times E &\to \mathbb{R} \\ (f, e) &\to f(e) \end{aligned} \tag{1.2.1}$$

where E' is the strong dual of E as define in section 1.1.3. F is not continuous when $E' \times E$ is given product topology [31]. In most of the applications (at least in our situation) we want that this map to be a smooth map. So the problem is to define the suitable notion of smoothness.

Also the definition the smoothness in Fréchet space or even in a locally convex space should agree with the definition of smooth maps in Banach space situation and fits very well in our application.

Kriegl and Michor in [20] starts by defining the smooth curve in a locally convex space. Following [20], we can define the smoothness of a curve in the same way as in Banach space (§1.2,[20]). In this way, we have the collection of smooth curves on any Fréchet space.

For the locally convex space E, we have the following theorems.

Theorem 1.2.1 (Theorem 2.14(4),[20])**.** *The curve $c : \mathbb{R} \to E$ is smooth if and only if the curves $l \circ c : \mathbb{R} \to \mathbb{R}$ are smooth for all $l \in E^*$, continuous dual of E.*

Theorem 1.2.2 (Corollary 2.11, [20])**.** *A linear map $l : E \to F$ between two locally convex vector spaces is bounded if and only if it maps smooth curves in E to smooth curves in F.*

Now following [20], we give the definition of a smooth map.

Definition 1.2.1. [§3.11,[20]] A function $f : U(\subset E) \to F$ defined on an open subset U of E is smooth if it takes smooth curves in U to smooth curves in F.

Remark 1.2.1. With this definition, with the product topology on $E' \times E$, the evaluation map F in equation 1.2.1 (being linear and bounded) is a smooth function, but it is not continuous in the product topology.

Therefore we see that smoothness of a map does not depend upon the topology of the space but it depends upon the bornology. For the Fréchet space the bornologificaton of Fréchet space is the same as Fréchet space topology. Therefore in the case of Fréchet spaces, with the above definition of a smooth curve and a smooth map, smooth map is always a continuous map. In the case of $E' \times E$, the product topology on $E' \times E$ does not make $E' \times E$ a Fréchet space.

The main benefit of the definition of smooth map as above, is having the crucial tool as following.

Theorem 1.2.3 (Exponential law, Theorem 3.12 [20])**.** *Let $U \subset E$ be a open subset of Fréchet space then $C^\infty(U_1 \times U_2, F) \approx C^\infty(U_1, C^\infty(U_2, F))$.*

The derivative is given explicity by following theorem.

Theorem 1.2.4 (Theorem 3.18,[20]). *Let E and F be Fréchet space and $U \subset E$ be open set then the differential operator*

$$d : C^\infty(U, F) \to C^\infty(U, L(E, F))$$

$$df(x)(v) := \lim_{t\to 0} \frac{f(x + tv) - f(x)}{t}$$

exists and bounded(smooth). Also the chain rule holds

$$d(f \circ g)(x)(v) = df(g(x))dg(x)v$$

Remark 1.2.2. In above theorem space $L(E, F)$ denote the space of all bounded linear mappings from E to F. It is closed linear subspace of $C^\infty(E, F)$. Following (§3.17 ,[20]), a mapping $f : U \to L(E, F)$ is smooth if and only if the composite mapping $U \to L(E, F) \to C^\infty(E, F)$ is smooth.

In the following, we will discuss the smoothness for particular cases of Fréchet spaces.

1.2.1 Smooth maps on loop space $L\mathbb{R}^n$

Suppose $f : \mathcal{U} \subset L\mathbb{R}^n \to E$ be a map from open subset of $L\mathbb{R}^n$ to a Fréchet space.

Suppose we know the collection $C^\infty(\mathbb{R}, L\mathbb{R}^n)$, then by the definition 1.2.1, f is smooth if and only $f \circ c$ is smooth.

Let $c : \mathbb{R} \to L\mathbb{R}^n$ be a smooth curve, define $c^\vee : \mathbb{R} \times S^1 \to \mathbb{R}^n$, by

$$c^\vee(t, s) := c(t)(s)$$

$c^\vee$ is called the adjoint of c. By theorem 1.2.3, we have the following theorem for the loop space which is a particular case of the exponential law.

Theorem 1.2.5 (proposition 3.7, [31]). *A curve $c : \mathbb{R} \to L\mathbb{R}^n$ is smooth if and only if its adjoint $c^\vee : \mathbb{R} \times S^1 \to \mathbb{R}^n$ is smooth.*

This is an easy criterion for checking smooth curve in the loop space and which will help to determine the smoothness of a map.

In chapter 2, we will use above theorem in proving smoothness of some particular maps.

1.2.2 Smooth maps on PLB space

Suppose $\{E_i, \phi_{ji}\}_{i,j\in\mathbb{N}}$ and $\{F_i, \rho_{ji}\}_{i,j\in\mathbb{N}}$ be the projective system of Banach spaces and $E = \varprojlim E_i$ and $F = \varprojlim F_i$. E and F are Fréchet spaces.

Definition 1.2.2. [Projective system of mapping]. We say $\{f_i : E_i \to F_i\}_{i\in\mathbb{N}}$ is a projective system of mapping if the following diagram commutes.

$$\begin{array}{ccc} E_j & \xrightarrow{f_j} & F_j \\ \Big\downarrow{\scriptstyle \phi_{ji}} & & \Big\downarrow{\scriptstyle \rho_{ji}} \\ E_i & \xrightarrow{f_i} & F_i. \end{array}$$

We denote the canonical mapping of $E \to E_i$ by e_i and $F \to F_i$ by e'_i.

Definition 1.2.3. We say $f : E \to F$ is the projective limit of a system $\{f_i : E_i \to F_i\}_{i\in\mathbb{N}}$ if for each i, the following diagram commutes.

$$\begin{array}{ccc} E_i & \xrightarrow{f_i} & F_i \\ \Big\downarrow{\scriptstyle e_i} & & \Big\downarrow{\scriptstyle e'_i} \\ E & \xrightarrow{f} & F. \end{array}$$

We define the map $\varprojlim f_i$ as following.

If $\{f_i\}$ is the projective system of mappings then we see that for any $x = (x_i) \in \varprojlim E_i$, $(f_i(x_i)) \in \varprojlim F_i$. We define $(\varprojlim f_i)(x) = (f_i(x_i)) \in F$. If f be the projective limit of system $\{f_i\}$ then we have

$$f(x) = (f_i(x_i)).$$

We denote the projective limit of system $\{f_i\}$ by $\varprojlim f_i$. Also $\left(\varprojlim f_i\right)(x) = (f_i(x_i)) = f(x)$.

We are interested in knowing the criterion of checking smoothness of the map

$$f : E \to F \text{ such that } f = \varprojlim f_i.$$

G. Galanis has given the following criterion.

Theorem 1.2.6 (Lemma 1.2,[10]). *Suppose $E = \varprojlim E_i$ and $F = \varprojlim F_i$ and $\{f_i : E_i \to F_i\}_{i\in\mathbb{N}}$ be a projective system of smooth mapping then the following holds.*

1. *f is C^∞, in the sense of J. Leslie.[22]*

2. *$df(x) = \varprojlim df_i(x_i)$, $x = (x_i) \in E$.*

3. $df = \varprojlim df_i$.

Following [20], we already defined a smooth map between Fréchet space (definition 1.2.1). In PLB-space if a map $f = \varprojlim f_i$ (as in theorem 1.2.6) is smooth in the sense of J. Leslie then it will be smooth in the sense of Kriegl and Michor too. This can be seen as following.

Let $c : \mathbb{R} \to \varprojlim U_i \subset E$ be a smooth curve and $f := \varprojlim f_i : \varprojlim U_i \to E$ be a map which is smooth in the sense of J. Leslie (i.e f satisfies theorem 1.2.6). We can identify $c(t)$ as $c(t) = (c_i(t))$ where $\phi_{ji}(c_j(t)) = c_i(t)$. c is smooth if and only each c_i is smooth (here we are using the fact that $\pi_i : \varprojlim E_i \to E_i$ is a smooth map and $c_i = \pi_i \circ c$).

Now let $\tilde{c} : \mathbb{R} \to \varprojlim U_i$, defined by $\tilde{c}(t) = (f \circ c)(t)$, we see that

$$\tilde{c}(t) = \varprojlim (f_i \circ c_i)(t)$$

As each $f_i \circ c_i$ is smooth, the derivative of every order exists. Therefore by theorem 1.2.6, $\tilde{c}$ is smooth in the sense of J. Leslie. Recall that smoothness of curves is defined by in the same way by J. Leslie and by Kriegl and Michor. This proves $f \circ c$ is smooth curve for every smooth curve c.

Therefore f defined as in theorem 1.2.6 is smooth in the sense of Kriegl and Michor too.

1.3 Fréchet manifolds

Definition 1.3.1 (Fréchet manifold)**.** A Fréchet manifold is a set $\mathcal{M}$ together with a smooth structure represented by an atlas $(U_\alpha, u_\alpha)_{\alpha \in A}$ such that the canonical topology on $\mathcal{M}$ with respect to this structure is Hausdorff.

As usual, charts are bijections from open subsets of $\mathcal{M}$ to open subsets of a fixed Fréchet space. An atlas is maximal cover of $\mathcal{M}$ by charts, where all transition functions are defined on open subsets and are required to be smooth.

Definition 1.3.2 (Smooth map)**.** A map $f : \mathcal{M} \to \mathcal{N}$ of a Fréchet manifolds is said to be smooth at $p \in \mathcal{M}$ if it is smooth in one and hence all pair(s) of charts around p and $f(p)$. The map is smooth if it is smooth at all points of M.

We have following proposition which helps in checking smooth map.

Proposition 1.3.1. *(§27.2,[20]) f is smooth if and only if $f \circ \gamma$ is smooth for every smooth curve $\gamma : \mathbb{R} \to \mathcal{M}$.*

We will require Fréchet manifolds to be smoothly Hausdorff (smooth functions separate points). One can prove (§16.10, [20]) that each Fréhet manifold is smoothly paracompact, that is, each open cover admits a smooth partition of unity subordinated to it. We will discuss more about smooth partition of unity in chapter 2.

In next sections, the symbol $\mathcal{M}$ is fixed for infinite dimension manifold (Banach or Fréchet).

1.3.1 Loop space as Fréchet manifold

Loop space $L\mathbb{R}^n$ is a Fréchet space and we discussed the space of smooth curve $C^\infty(\mathbb{R}, L\mathbb{R}^n)$ in section 1.2.1. In this section we will give a manifold structure on the loop space LM, where M is a finite dimension oriented manifold.

In this thesis whenever we will discuss about the loop space, we will mean by loop space over a finite dimension smooth manifold.

We start with a proposition.

Proposition 1.3.2. *Any oriented vector bundle over S^1 is trivial.*

The main point is that up to isomorphism, every real vector bundle over the circle is either trivial or the Whitney sum of a trivial bundle with the Mobius bundle. The latter is not orientable.

Definition 1.3.3 (Local addition)**.** A local addition on M consists of a smooth map $\eta : TM \to M$ such that

1. The composition of η with the zero section is the identity on M.
2. There exists an open neighborhood of V of the diagonal of $M \times M$ such that $\pi \times \eta : TM \to M \times M$ is diffeomorphism onto V. Here $\pi : TM \to M$ is the projection map.

For any $[(p, v)] \in TM$, $\pi \times \eta([(p, v)]) = (\pi([(p, v)]), \eta([(p, v)]) = (p, \eta([(p, v)])$. We have following proposition.

Proposition 1.3.3 ([25],[31])**.** *For any finite dimension smooth manifold M, local addition always exits.*

Let $\eta : TM \to M$ be a local addition. Let $V \subset M \times M$ be the image of the map $\pi \times \eta : TM \to M \times M$, where V is an open neighborhood of diagonal in M. By the definition of local addition $\pi \times \eta$ is a diffeomorphism on to V.

Suppose $\alpha \in LM$, we define $(\mathcal{U}_\alpha, \Psi_\alpha)$ as a coordinate chart around α, where $\mathcal{U}_\alpha$ is defined as following:

$$\mathcal{U}_\alpha := \{\beta \in LM : (\alpha, \beta) \in LV\} \subset LM$$

The pre-image of $\{\alpha\} \times \mathcal{U}_\alpha$ under $\pi \times \eta^L$ is naturally identified with $\Gamma_{s^1}(\alpha^* TM)$.

$\Psi_\alpha : \Gamma_{S^1}(\alpha^* TM) \to \mathcal{U}_\alpha$ is defined as following. Let $\beta \in \Gamma_{S^1}(\alpha^* TM)$ and $\tilde{\beta}$ be the corresponding loop in TM, so $\beta(t) = (t, \tilde{\beta}(t))$. Then we have $(\pi \times \eta)^L(\tilde{\beta}) = (\alpha, \eta^L(\tilde{\beta}))$. Now define

$$\Psi_\alpha := \eta^L(\tilde{\beta})$$

This gives a chart for loop space around $\alpha \in LM$.

For the full discussion we refer to various article [20],[25] etc.

Proposition 1.3.4 ([31]). *With the atlas consisting charts as above and with the manifold topology on the loop space LM, LM is Hausdorff, regular, second countable and paracompact.*

Thus the loop space LM is a Fréchet manifold.

1.3.2 PLB manifold

We say M is the projective limit of Banach manifolds (PLB-Manifold) modeled on a PLB-space $E = \varprojlim E_i$ if we have followings.

1. There is a projective system of Banach manifolds $\{M_i, \phi_{ji}\}_{i,j\in\mathbb{N}}$ such that $M = \varprojlim M_i$.

2. For each $p \in M$, we have $p = (p_i)$. $p_i \in M_i$, and there is a chart (U_i, ψ_i) of $p_i \in M_i$ such that

   (a) $\phi_{ji}(U_j) \subset U_i$, $j \geq i$.

   (b) Let $\{E_i, \rho_{ji}\}_{i,j\in\mathbb{N}}$ be a projective systems of Banach spaces, where each ρ_{ji} is inclusion map and the diagram

$$\begin{tikzcd} U_j \arrow[r, "\psi_j"] \arrow[d, "\phi_{ji}"] & \psi_j(U_j) \arrow[d, "\rho_{ji}"] \\ U_i \arrow[r, "\psi_i"] & \psi_i(U_i) \end{tikzcd}$$

   commutes.

(c) $\varprojlim \psi_i(U_i)$ is open in E and $\varprojlim U_i$ is open in M with the inverse limit topology.

The space M satisfying the above properties has a natural Fréchet manifold structure. The differential structure on M is determined by the co-ordinate map $\psi : U = \varprojlim U_i \to \psi(U) = \varprojlim \psi_i(U_i)$. Therefore a smooth structure on these type of manifolds is completely determined by the smooth structure on the sequence. George Galanis in [10, 11, 12, 13] has studied this type of manifolds and smooth structure on it.

G. Galanis, in a series of articles [10, 11, 12, 13] discussed various properties of PLB-manifolds. H.Omri [27] discussed the ILB-manifolds (similar to PLB manifolds) and ILB-normal manifolds (here ILB stands for inverse limit of Banach). A PLB-manifold is the projective limit of Banach manifolds in general not only modeled over ILB space in contrary Omri-strong ILB-manifolds

In view of discussion in section 1.2.2, we see that the calculus on PLB-manifolds agrees with the Kriegl and Michor calculus.

In particular we have the following proposition.

Proposition 1.3.5. *Let E be a Fréchet space and $E = \varprojlim E_i$, E_is are Banach spaces. The co-ordinate map ψ_i as defined above, $\psi_i : U = \varprojlim U_i \to \psi(U) = \varprojlim \psi_i(U_i)$ is smooth in the sense of Kriegl and Michor [20].*

Proof. We have to check that $c : \mathbb{R} \to \varprojlim U_i$ is smooth if and only if $\tilde{c} : \mathbb{R} \to \varprojlim E_i$ defined by

$$\tilde{c}(t) := (\varprojlim \psi_i) \circ c(t) = \varprojlim (\psi_i \circ c_i)(t)$$

is smooth. But this follows by the remark after theorem 1.2.6. □

Summarizing above, PLB manifolds are Fréchet manifolds and fit with Kriegl and Michor calculus.

1.3.3 Tangent bundle

Let $p \in E$ be a point in a Fréchet space E. The kinematic tangent space T_pE of E at p is the set of all pairs (p, X) with $X \in E$. Equivalently, T_pE is the set of equivalence classes of smooth curves through p, where $\gamma_1 \sim \gamma_2$ if both have the same derivative at p. Each tangent vector $X \in T_pE$ yields a continuous (hence bounded) derivation $X : C^\infty(E \supset \{p\}, \mathbb{R}) \to \mathbb{R}$ on the germs of smooth functions at p.

In the general Fréchet space, it is not true that each such derivation comes from a tangent vector. However if E is a nuclear Fréchet reflexive space (§28.7, [20]), T_pE does coincide with the set derivations on the stalk $C^\infty(E \supset \{p\}, \mathbb{R})$.

If Fréchet manifold is not modeled over nuclear Fréchet space then there are two notion of vector which may not agree. We call the space of tangent vector at a point p as *Kinematic tangent space*. We call the set of all derivation at a point as *Operational tangent space*.

Let $\mathcal{M}$ be the Fréchet manifold with a smooth atlas $(\mathcal{M} \supset U_\alpha \to u_\alpha)_{\alpha \in A}$. We define the kinematic tangent bundle $T\mathcal{M}$ of a Fréchet manifold $\mathcal{M}$. to be the quotient of the disjoint union $\bigcup_\alpha \{\alpha\} \times U_\alpha \times E$ by the equivalence relation

$$(\alpha, p, X)\ (\beta, q, Y) \Leftrightarrow p = q,\ d(u_{\alpha\beta})(u_\beta(p))(Y) = X,$$

where the $u_\alpha : M \supset U_\alpha \to E$ denote the charts, and $u_{\alpha\beta} = u_\alpha \circ u_\beta^{-1}$ transition function of the manifold.

We will denote the kinematic tangent bundle of a Fréchet manifold $\mathcal{M}$ by $T\mathcal{M}$. Kriegl and Michor defined the operational tangent bundle $D\mathcal{M}$ in (§28.12, [20]). In our application (see section 1.4.4) we will need only kinematic tangent bundle therefore in next, by tangent bundle we will mean by kinematic tangent bundle.

Tangent bundle of the loop space

$\Gamma_{S^1}(\gamma^*(TM))$ seems trivially the tangent space at γ because chart map at γ identifies it. But this is not the case. There are many possibility of chart maps because of various choices of local addition. Hence we do not have a canonical choice for the tangent space.

Andrew Stacey in [31] showed that the tangent bundle TLM of the loop space LM has the structure of a bundle of $L\mathbb{R}$-modules (§4.1, [31]). Using this fact further he proved in [31] that TLM and LTM are diffeomorphic, covering the identity on LM (§4.1,[31]). This proves that the tangent space (kinematic tangent space) at the point $\gamma \in LM$

$$T_\gamma LM \approx \Gamma_{S^1}(\gamma^*(TM)) \equiv L\mathbb{R}^n.$$

Tangent bundle of a PLB manifold

We will follow [14],[1] for the following discussion on the tangent bundle of a PLB manifold. If $\{M_i, \phi_{ji}\}$ be a projective system of Banach manifolds and $M = \varprojlim M_i$ is the PLB manifold for this projective system. For $p \in M = \varprojlim M_i$, we have $p = (p_i)$. We observe that $\{T_{p_j}M_j, T_{p_j}\phi_{ji}\}$ is a projective system of Banach spaces ($T_{p_j}\phi_{ji}$ is the usual Banach space derivative of ϕ_{ji} at point p_j). The identification $T_pM \simeq \varprojlim T_{p_i}M_i$ is given by the mapping $h := \varprojlim T_p\phi_i$, where ϕ_i are the canonical projection of M. We refer to [14] or [1] for the proof.

Galanis proved in [14] that $\{TM_i, T\phi_{ji}\}$ is a projective system of Banach manifolds and TM is a PLB manifold and

$$TM \simeq \varprojlim TM_i \text{ by } g = \varprojlim T\phi_i.$$

Remark 1.3.1. The strong dual of a Fréchet space need not be metrizable. If we consider the strong dual of T_pM as the cotangent space, we would drop out of the Fréchet space category. In order to avoid this we will consider tensors not as sections of a certain bundle, but simply as smooth, fiberwise multilinear maps $A : TM \times_M ... \times_M TM \to E$ with $\pi \circ A = id$ and $\pi : E \to M$ a vector bundle over M.

1.4 Differential geometry on Fréchet manifold

1.4.1 Vector fields

As we discussed in section 1.3.3, for Fréchet spaces in general there are two types of tangent space, the kinematic tangent space and the operational tangent space. Therefore vector fields on a general Fréchet manifold $\mathcal{M}$ are of two types.

A kinematic vector field X on $\mathcal{M}$ is a smooth section of the kinematic tangent bundle $T\mathcal{M} \to \mathcal{M}$. The space of all kinematic vector fields will be denoted by $\mathfrak{X}(\mathcal{M})$.

By an operational vector field we mean a bounded derivation of the sheaf $C^\infty(., \mathbb{R})$. That is for an open set $U \subset \mathcal{M}$ we are given bounded derivations $X_U : C^\infty(U, \mathbb{R}) \to C^\infty(U, \mathbb{R})$ commuting with the restriction mappings. This can be identified with the smooth sections of the operational tangent bundle (§32.2,[20]). Denote the space of all operational vector fields by $Der(C^\infty(M, \mathbb{R}))$

For a reflexive nuclear Fréchet space both notion of vector field agrees ([20]). But in general we have the following proposition:

Proposition 1.4.1 (Lemma 32.3,[20])**.** *There is a natural embedding of convenient vector spaces*

$$\mathfrak{X}(M) \to Der(C^\infty(M, \mathbb{R}))$$

For our purpose (for example while working on 2 form on a Fréchet manifold) we need only kinematic vector field. For example in section 1.4.4 we will see that for defining the differential form we need only kinematic vector fields.

Vector field (kinematic) on the loop space LM

A vector field X over LM is defined as a smooth section of TLM. We denote the collection of all vector fields on LM by $\mathfrak{X}(LM)$. There are vector field on the loop space which arise from the base manifold and there are other type of vector fields which do not arise in this way.

Following Biswas and Chatterjee [3], we define a special vector field on the loop space LM as following:

Definition 1.4.1. A vector field ξ on LM is said to be the vector field on LM associated to a vector field K on M, if

$$ev_{t_*}(\xi(\gamma)) = (K)_{\gamma(t)}, \forall \gamma \in LM, t \in [0,1] \tag{1.4.1}$$

where for each $t \in [0,1]$, $ev_t : LM \to M$, defined by $ev_t(\gamma) = \gamma(t)$, is a smooth map. Let $\mathfrak{X}'(LM)$ is a collection of all vector fields ξ on LM such that there is a vector field K of M and which satisfies 1.4.1.

Many physicists ([3],[6],[7]) use the above collection $\mathfrak{X}'(LM)$ as a definition of vector field on the loop space. But below we will see that, in the manifold structure on LM given as in section 1.3.1, $\mathfrak{X}'(LM)$ is not same as $\mathfrak{X}(LM)$.

Example 1.4.1. Define $X(\gamma) = (\gamma, \gamma')$ from $L\mathbb{R}^n \to TL\mathbb{R}^n = L\mathbb{R}^n \times L\mathbb{R}^n$. This is a bounded linear map and hence smooth. Thus this is an example of a vector field on $L\mathbb{R}^n$.

For this vector field, it is trivial to see that $X \notin \mathfrak{X}'(L\mathbb{R}^n)$. Therefore $\mathfrak{X}'(LM) \subsetneq \mathfrak{X}(LM)$.

1.4.2 Flow of a vector field

In this section we will discuss the existence of the flow of a kinematic vector field.

Let $c : J \to \mathcal{M}$ be a smooth curve in a manifold $\mathcal{M}$ defined on an interval J. It will be called an integral curve or flow line of a kinematic vector field $X \in \mathfrak{X}(\mathcal{M})$ if $c'(t) = X(c(t))$ holds for all $t \in J$. For a Fréchet manifold, the flow line of a vector field may not exist (page 330, [20]).

Let $X \in \mathfrak{X}(\mathcal{M})$ be a kinematic vector field. A local flow F for X is smooth mapping $F : \mathcal{U} \subset \mathcal{M} \times \mathbb{R} \to \mathcal{M}$ defined on an open neighborhood $\mathcal{U}$ of $\mathcal{M} \times \{0\}$ such that

1. $\mathcal{U} \cap (\{x\} \times \mathbb{R})$ is a connected open interval.

2. If $F(s,x)$ exists then $F(t+s,x)$ exists if and only if $F(t,F(s,x))$ exits and we have the equality.

3. For each $x \in \mathcal{U}$, $\gamma_x(t) := F(t,x)$ is the integral curve of X passing through a point x at $t=0$.

Suppose $Y : U \to E$ be a vector field on an open subset U of a Fréchet space E. For ensuring existence of the flow of a vector field on a Fréchet space we can go either in direction set by H. Omri as in [27] or we can demand for some kind of tame condition as in [15]. For a general Fréchet space, the flow of a vector field may not exist for example see [20].

The situation become simpler while working on some special vector fields on PLB-manifolds. In rest of this section we will use the notation of example 1.1.3.

Definition 1.4.2 (Projective μ-Lipschitz map)**.** Let $E = \varprojlim E_i$ be a Fréchet space. A mapping $\phi : E \to E$ is called projective μ-Lipschitz (μ, a positive real number) if there are $\phi_i : E_i \to E_i$ such that $\phi = \varprojlim \phi_i$ and for every i, ϕ_i is a μ Lipschitz map on each E_i.

We have a theorem below which will be helpful in determining existence of the flow.

Theorem 1.4.2. *Let $E = \varprojlim E_i$ be a Fréchet space and suppose $X : E \to E$ is a projective μ Lipschitz map such that each X_i is smooth map on E_i and $X = \varprojlim X_i$. Suppose for each i,*

$$M := \sup\{\rho_i(X(x)) : i \in \mathbb{N},\ \ x \in E\} < +\infty$$

Then there is a unique C^∞ curve $x(t)$ defined on $\mathbb{R}$ such that

$$x'(t) = X(x(t)),\ \ x(0) = x_0. \tag{1.4.2}$$

Above theorem is a version of theorem proved by G. Galanis (Theorem 3, [13]). The proof below is motivated from [13].

Proof. $\{X_i\}_{i\in\mathbb{N}}$ be a family of smooth map realizing X. Equation 1.4.2 gives a system of ordinary differential equations on the Banach spaces E_i defined by

$$x_i'(t) = X_i(x_i(t)),\ \ x_i(0) = x_0^i \tag{1.4.3}$$

where $x_0 = (x_0^i)$. It is given that each X_i is μ - Lipschiz and $\|X_i(x_i)\|_i < M$. Therefore by (§4.1, [23]), a unique smooth solution can be defined for each equation 1.4.3 on $\mathbb{R}$.

These solution are related. For any $j \geq i$, we have

$$(\rho_{ji} \circ x_j)'(t) = \rho_{ji}(x_j'(t)) = \rho_{ji}(X_j(x_j(t))) = X_i(x_i(t)).$$

We refer to (theorem 3, [13]) for the above calculation.

Both $\rho_{ji} \circ x_j$ and x_i emanate from the same initial point. Therefore they coincide, as a result, the mapping $x = \varprojlim x_i$ is defined on the $\mathbb{R}$. Following (theorem 3, [14]), we see that x is unique desired solution of the given differential equation 1.4.2. Each x_i are smooth curve, therefore $x := \varprojlim x_i$ is a smooth curve (see 1.2.2). $\square$

Theorem 1.4.3. *Let $X : E \to E$ be a smooth vector field on E as in theorem 1.4.2. Then for every $y \in E$ there is a unique integral curve $t \to x^y(t) \in E$ defined on $\mathbb{R}$ such that $x^y(0) = y$. Also the map $F : \mathbb{R} \times E \to E$ defined by*

$$F(t, p) := x_p(t)$$

is a smooth map.

Proof given below is taken from [23], [13] and [5].

Proof. Theorem 1.4.2, implies that for every $y \in E$, there exists a unique curve

$$x^y : \mathbb{R} \to E \text{ such that } x^y(0) = y.$$

In fact $x^y = \varprojlim x_i^{y_i}$ ($\varprojlim x_i^{y_i}$ is well defined as we saw in the theorem 1.4.2, where $x_i^{y_i} : \mathbb{R} \to E_i$ be the integral curves passing through y_i of corresponding vector fields X_i).

For each i define $F_i : \mathbb{R} \times E_i \to E_i$ such that $F_i(t, p_i) = x_i^{p_i}(t)$ and $F : \mathbb{R} \times E \to E$ such that $F(t, p) = x^p(t)$.

$\{F_i\}$ makes projective system of map with projective limit F. That is we have $F = \varprojlim F_i$. Authors in [23] showed that each F_i is smooth map. This proves that F is a smooth map. $\square$

1.4.3 Lie Bracket

Let $X, Y \in \mathfrak{X}(\mathcal{M})$ where $\mathfrak{X}(\mathcal{M})$ is the collection of kinematic vector field. Define a map

$$f \to X(Y(f)) - Y(X(f)).$$

This is a bounded derivation of sheaf $C^\infty(., \mathbb{R})$. We denote it by $[X, Y]$.

In general Fréchet space where two notions of vector fields (kinematic and operational) do not agree, it is not obvious that $[X, Y]$ is a kinematic vector field. Though by definition $[X, Y]$ is an operational vector field.

We mention here a theorem in (§32.8 [20]). Theorem states that the bounded derivation $[X, Y] \in \mathfrak{X}(\mathcal{M})$ whenever $X, Y \in \mathfrak{X}(\mathcal{M})$. We call this map as Lie bracket of X and Y.

1.4.4 Differential form and de Rham cohomology

Space of k-differential forms on $\mathcal{M}$ is the closed linear subspace of $C^\infty(T\mathcal{M} \times_{\mathcal{M}} ... \times_{\mathcal{M}} T\mathcal{M}, \mathbb{R})$ consisting of all fiber-wise k - linear alternating smooth functions in the vector bundle structure $T\mathcal{M} \oplus .. \oplus T\mathcal{M}$. We denote this space by $\Omega^k(\mathcal{M})$.

For example, a 2-form $\omega \in \Omega^2(\mathcal{M})$ is a fiberwise bilinear alternating smooth function in the vector bundle structrure $T\mathcal{M} \oplus T\mathcal{M}$. $T\mathcal{M} \oplus T\mathcal{M}$ has a vector bundle structure (§29.4,[20]) as in the case of finite dimension manifold. Therefore for each $p \in M$, ω_p (as a bilinear map) is a bounded map (being smooth map, we refer section 1.2).

As we discussed that there are two types of tangent bundle, kinematic tangent bundle $(T\mathcal{M})$ and operational tangent bundle $(D\mathcal{M})$. In view of these two types of tangent bundles there are many ways to define differential forms which agree with the usual differential form in the finite dimensional case.

Kriegl and Michor (§33, [20]) showed that there are 12 classes of possible differential form. But out of these there is only one (defined above) which satisfies all the useful identities as in the finite dimensional case.

With this definition of the differential form all the important mappings are defined in usual way and smooth:

$$d : \Omega^k(\mathcal{M}) \to \Omega^{k+1}(\mathcal{M})(\text{ exterior derivative, §33.12 [20]}).$$
$$i : \mathfrak{X}(\mathcal{M}) \times \Omega^k(\mathcal{M}) \to \Omega^{k-1}(\mathcal{M})(\text{ insertion operator, §33.10 [20]}).$$
$$\mathcal{L} : \mathfrak{X}(\mathcal{M}) \times \Omega^k(\mathcal{M}) \to \Omega^k(\mathcal{M})(\text{ Lie derivative, §33.17 [20]}).$$
$$f^* : \Omega^k(\mathcal{M}) \to \Omega^k(\mathcal{N})(\text{ pull back operator, §33.9 [20]}).$$

Following (§34,[20]), for a Fréchet manifold $\mathcal{M}$ consider a graded algebra

$$\Omega(\mathcal{M}) = \oplus_{k=0}^{\infty} \Omega^k(\mathcal{M})$$

$d(\phi \wedge \psi) = d\phi \wedge \psi + (-1)^{deg(\phi)} \phi \wedge d\psi$. Define

$$H^k_{DR}(\mathcal{M}) = \frac{\{\omega \in \Omega^k(\mathcal{M}) : d\omega = 0\}}{\{d\phi : \phi \in \Omega^{k-1}(\mathcal{M})\}} \tag{1.4.4}$$

$H^k_{DR}(\mathcal{M})$ is called the k-th de-Rham cohomology of $\mathcal{M}$.

1.5 Symplectic geometry on a Fréchet manifold

1.5.1 Weak symplectic structure

Definition 1.5.1. A 2 form $\sigma \in \Omega^2(\mathcal{M})$ is called a weak symplectic form on $\mathcal{M}$ if it is closed ($d\sigma = 0$) and if the associated vector bundle homomorphism $\sigma^b : T\mathcal{M} \to T^*\mathcal{M}$ is injective.

Definition 1.5.2. A 2 form σ on a Banach manifold is called a strong symplectic if it is closed ($d\sigma = 0$) and its associated vector bundle homomorphism $\sigma^b : T\mathcal{M} \to T^*\mathcal{M}$ is invertible with smooth inverse.

In the case of strong symplectic Banach manifold, the vector bundle $T\mathcal{M}$ has reflexive fibers $T_x\mathcal{M}$. For a Fréchet manifold with the strong topology $T^*_x\mathcal{M}$ is not topologically isomorphic to $T_x\mathcal{M}$. Hence for our situation that is for the case of the loop space and for the case of PLB manifold, there is only weak symplectic structure. There are no strong symplectic structure on these.

We will discuss a symplectic structure on the loop space in chapter 2. In chapter 3, we will discuss a weak symplectic structure on the PLB manifold.

1.5.2 Symplectic cohomology defined by Kriegl and Michor

By the symplectic cohomology, we mean the definition given by Kriegl and Michor in [20]. We will follow notations and definitions of section 48 of [20]. For sake of completeness, below we will define the required terms.

Let $(\mathcal{M}, \sigma)$ be a weak symplectic Fréchet manifold. Let $T^\sigma_x\mathcal{M}$ denotes the real linear subspace $T^\sigma_x\mathcal{M} = \sigma^b_x(T_x\mathcal{M}) \subset T^*_x(\mathcal{M})$. These vector space fit to form a sub bundle of $T^*\mathcal{M}$ and $\sigma^b : T\mathcal{M} \to T^\sigma\mathcal{M}$ is bundle isomorphism (§48.4,[20]). Define $C^\infty_\sigma(\mathcal{M}, \mathbb{R}) \subset C^\infty(\mathcal{M}, \mathbb{R})$ to be the linear subspace consisting of all smooth functions $f : \mathcal{M} \to \mathbb{R}$ such that $df : \mathcal{M} \to T^*\mathcal{M}$ factors to a smooth mapping $\mathcal{M} \to T^\sigma\mathcal{M}$.

In other words, $f \in C^\infty_\sigma(\mathcal{M}, \mathbb{R})$ if there exists a smooth σ-gradient $grad^\sigma f \in \mathfrak{X}(\mathcal{M})$ such that for given $p \in \mathcal{M}$ and $Y \in T_p\mathcal{M}$ we have $df_p(Y) = \sigma_p(grad^\sigma f|_p, Y)$. Detailed description of these spaces and analysis of weak symplectic manifold is given in (§48,[20]).

Let $C^\infty(L^k_{alt}(T\mathcal{M}, \mathbb{R})^\sigma)$ be the space of smooth sections of a vector bundle with fiber $L^k_{alt}(T_x\mathcal{M}, \mathbb{R})^{\sigma_x}$ consisting of all bounded skew symmetric forms ω with $\omega(., X_2, ..., X_k) \in$

$T_x^\sigma \mathcal{M}$. Let

$$\Omega_\sigma^k(\mathcal{M}) := \{\omega \in C^\infty(L_{alt}^k(T\mathcal{M}, \mathbb{R})^\sigma) : d\omega \in C^\infty(L_{alt}^{k+1}(T\mathcal{M}, \mathbb{R})^\sigma)\}.$$

$d^2 = 0$ and the wedge product of σ dual forms is again a σ- dual form (see page 527 of [20]). We have a graded differential subalgebra $(\Omega_\sigma(\mathcal{M}), d)$, whose cohomology is called the symplectic cohomology and will be denoted by $H_\sigma^k(\mathcal{M})$. We mention that this definition of the symplectic cohomology is not same as the symplectic cohomology defined by Floer.

1.5.3 Darboux chart

Let E be a Fréchet space. We have $TE = E \times E$. Let $\mathcal{F}$ be a bounded, skew symmetric, non singular, bilinear map $\mathcal{F} : E \times E \to \mathbb{R}$. $\mathcal{F}$ defines a 2-form ω on E by the following:

$$\omega_x : T_xE \times T_xE \to \mathbb{R}$$

$$\omega_x(v, w) := \mathcal{F}(\tilde{v}, \tilde{w})$$

where T_xE is identified with E and $\tilde{v}$ and $\tilde{w}$ corresponds to $v, w \in T_xE$.

Example 1.5.1. Let $E = L\mathbb{R}^n$, then we have $TE = L\mathbb{R}^n \times L\mathbb{R}^n$. Define $\mathcal{F} : L\mathbb{R}^n \times L\mathbb{R}^n \to \mathbb{R}$ by

$$\mathcal{F}(X, Y) := \int_0^1 \langle X'(t), Y(t)\rangle dt$$

Then $\mathcal{F}$ is a skew symmetric bilinear map. $\mathcal{F}$ defines a symplectic structure on $L\mathbb{R}^n$ as following: For $\gamma \in L\mathbb{R}^n$ and $X, Y \in T_\gamma L\mathbb{R}^n$, define:

$$\omega_\gamma(X, Y) := \int_0^1 \langle \tilde{X}'(t), \tilde{Y}(t)\rangle dt.$$

Now we will proceed to define a Darboux chart:

For a general infinite dimensional smooth manifold $\mathcal{M}$, we have the following definition:

By a Darboux chart around $p \in \mathcal{M}$, we mean a coordinate chart $\{(\mathfrak{U}, \Phi), \Phi : \mathfrak{U} \to E\}$ around p such that there exists a bounded alternating bilinear map $\mathcal{F}$ on E for which, for $v_1, v_2 \in T_q\mathcal{M}$ and for every $q \in \mathfrak{U}$

$$\sigma_q(v_1, v_2) = \mathcal{F}(d\Phi_q(v_1), d\Phi_q(v_2))$$

This chart around p is called a Darboux chart around p. We say that $(\mathcal{M}, \sigma)$ admits Darboux chart if for every $p \in \mathcal{M}$ there is a Darboux chart around $p \in \mathcal{M}$.

In the case of finite dimensional manifold M, Darboux theorem states that every point in M has a coordinate neighborhood N with coordinate functions $(x_1, .., x_n, y_1, .., y_n)$ such that $\sigma = \sum_{i=1}^{n} dx_i \wedge dy_i$ on N.

Chapter 2

Weak symplectic structure on a loop space over $\mathbb{R}^{2n}$

For a finite dimensional symplectic manifold (M, ω) with a symplectic form ω the corresponding loop space ($LM = C^\infty(S^1, M)$) is a (nuclear) Fréchet manifold modeled on $L\mathbb{R}^{2n}$ (see section 1.3.1). It admits a weak symplectic form Ω^ω. We prove that the loop space over $\mathbb{R}^{2n}$ admits a Darboux chart for the weak symplectic structure Ω^ω if there is a global Darboux chart for $(\mathbb{R}^{2n}, \omega)$.

As a corollary of existence of Darboux chart, we will see that the inclusion map from the symplectic cohomology (as we discussed in section 1.5.2) of the loop space over $\mathbb{R}^{2n}$ to the de Rham cohomology (section 1.4.4) of the loop space is an isomorphism. We will make a remark for the general loop space LM in place of $L\mathbb{R}^{2n}$.

Further we will discuss about an almost complex structure $\tilde{J}$ on the loop space LM which is compatible with the weak symplectic structure Ω^ω. We will see that $\tilde{J}$ is weak integrable in the local sense (a notion defined by L. Lempert [21]).

2.1 Symplectic structure Ω^ω on the loop space

For $\gamma \in LM$, we know that $T_\gamma LM = \Gamma_{S^1}(\gamma^* TM)$. As M is oriented, we can identify $\Gamma_{S^1}(\gamma^* TM)$ with $L\mathbb{R}^n$.

For $X, Y \in T_\gamma LM = \Gamma_{S^1}(\gamma^* TM)$, $X(t), Y(t) \in \gamma^*(TM)$. Define

$$\Omega^\omega_\gamma(X, Y) = \int_0^1 \omega_{\gamma(t)}(X(t), Y(t))dt \tag{2.1.1}$$

This is a weak symplectic form on LM. Dr. Saikat Chatterjee introduced the symplectic structure Ω^ω on the path space $C^\infty([0,1], M)$ as defined in equation 2.1.1 in his work (with Prof. Indranil Biswas and Prof. Rukmini Dey) on pre-quantization of the path space. They proved that Ω^ω is a weak symplectic structure.

2.2 Darboux chart on the loop space $L\mathbb{R}^{2n}$

In this section we will see that for some symplectic structure ω on $\mathbb{R}^{2n}$, the corresponding weak symplectic structure Ω^ω on $L\mathbb{R}^{2n}$ admits a Darboux chart.

2.2.1 Isotopy on loop space

We start by defining isotopy on the loop space $L\mathbb{R}^n$. We will give an example of isotopy which we will use in later section.

Definition 2.2.1. Smooth map $\phi : \mathbb{R} \times L\mathbb{R}^n \to L\mathbb{R}^n$ is called an isotopy if each

$$\phi_s : L\mathbb{R}^n \to L\mathbb{R}^n$$

is a diffeomorphism and $\phi_0 = Id_{L\mathbb{R}^n}$.

We restate the checking criterion of smoothness of a map for the case of loop space as discussed in section 1.2.1. A curve $c : \mathbb{R} \to L\mathbb{R}^n$ is smooth if and only if its adjoint $c^\vee : \mathbb{R} \times S^1 \to \mathbb{R}^n$ is smooth. A map $\phi : \mathcal{U} \subset L\mathbb{R}^n \to L\mathbb{R}^n$ is smooth if it takes a smooth curve to a smooth curve.

Let $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be an isotopy. Define

$$\phi^L : \mathbb{R} \times L\mathbb{R}^n \to L\mathbb{R}^n; \;\; \phi^L(s, \gamma) = (t \to \phi_s(\gamma(t))). \tag{2.2.1}$$

Proposition 2.2.1. *ϕ^L as defined above is an isotopy on $L\mathbb{R}^n$. For $X \in T_\gamma L\mathbb{R}^n \approx C^\infty(S^1, \mathbb{R}^n)$, the derivative of ϕ^L_s at γ is given by $d\phi^L_s(\gamma)(X) = (t \to d\phi_s(\gamma(t))(X(t)))$.*

Proof. Let c be a smooth curve on $\mathbb{R} \times L\mathbb{R}^n$.

$$c : \mathbb{R} \to \mathbb{R} \times L\mathbb{R}^n, c(u) = (c_1(u), c_2(u)).$$

Then

$$\begin{aligned}
\tilde{c} \;\; &: \;\; \mathbb{R} \to L\mathbb{R}^n \quad \text{is defined by:} \\
\tilde{c}(u) \;\; &= \;\; \phi^L \circ c(u) = \phi^L(c_1(u), c_2(u)) \\
\tilde{c}(u) \;\; &= \;\; \big(t \to \phi_{c_1(u)}(c_2(u)(t))\big).
\end{aligned}$$

By the definition of a smooth map in $L\mathbb{R}^n$, $\tilde{c}$ is smooth if and only if $\tilde{c}^\vee : \mathbb{R} \times S^1 \to \mathbb{R}^n$ defined by $\tilde{c}^\vee(u,t) \to \phi(c_1(u), c_2(u)(t))$ is smooth. As ϕ and c_1, c_2 smooth map, this shows that $\tilde{c}^\vee$ is a smooth map and therefore $\tilde{c}$ is smooth.

This shows that for every $c : \mathbb{R} \to \mathbb{R} \times L\mathbb{R}^n$, we have $\phi^L \circ c$ is smooth. This implies that ϕ^L is a smooth map.

For $s \in \mathbb{R}$, ϕ_s is a diffeomorphism. Let ψ_s be the inverse of ϕ_s. ψ_s^L, defined in a similar way as above, will be the smooth inverse of ϕ_s^L. Therefore we see that for each s, ϕ_s^L is a diffeomorphism. We also have $\phi_0^L = Id_{L\mathbb{R}^n}$. This proves that ϕ^L is an isotopy of $L\mathbb{R}^n$.

Now we will calculate the derivative of the map ϕ_s^L for a fixed $s \in \mathbb{R}$.

$$\phi_s^L : L\mathbb{R}^n \to L\mathbb{R}^n; \quad \phi_s^L(\gamma)(t) = \phi_s(\gamma(t)).$$

Let $\gamma \in L\mathbb{R}^n$, then $d\phi_s^L(\gamma) : T_\gamma L\mathbb{R}^n \to T_{\phi_s^L(\gamma)} L\mathbb{R}^n$. Take $X \in T_\gamma L\mathbb{R}^n = \Gamma_{S^1}(\gamma^* T\mathbb{R}^n) \simeq C^\infty(S^1, \mathbb{R}^n)$. $d\phi_s^L(\gamma)(X)$ is a vector field along $\phi_s^L(\gamma)$. The derivative is determined by the directional derivative (see theorem 1.2.4).

$d\phi_s^L(\gamma)(X)$ is an element of $T_{\phi_s^L(\gamma)} L\mathbb{R}^n \approx C^\infty(S^1, \mathbb{R}^n)$ which is the limit of the following net indexed by $u \in \mathbb{R}^+$:

$$\left\{ \frac{\phi_s^L(\gamma + uX) - \phi^L(\gamma)}{u} \right\}.$$

Evaluation at time t is a continuous linear map $L\mathbb{R}^n \to \mathbb{R}$ which takes this net to

$$\begin{aligned} & \left\{ \frac{\phi_s^L(\gamma + uX)(t) - \phi_s^L(\gamma)(t)}{u} \right\} \\ = & \left\{ \frac{\phi_s(\gamma(t) + uX(t)) - \phi_s(\gamma(t))}{u} \right\} \end{aligned}$$

This is the differential quotient which tends to $d\phi_s(\gamma(t))(X(t))$. Since a loop is completely determined by its values at each time, therefore we have

$$d\phi_s^L(\gamma)(X) = (t \to d\phi_s(\gamma(t))(X(t))) \tag{2.2.2}$$

□

We will need the following function in next few sections. Let us define a map (suggested by Dr. Saikat chatterjee) on the loop space corresponding to a map on the base manifold. If $f : \mathbb{R}^n \to \mathbb{R}$ be a C^∞ function then define

$$\begin{aligned} & \tilde{f} : L\mathbb{R}^n \to \mathbb{R} \text{ by} \\ & \tilde{f}(\gamma) = \int_0^1 f(\gamma(t))dt \end{aligned} \tag{2.2.3}$$

Then using the same argument as above, we have for $\gamma \in L\mathbb{R}^n$, $X \in T_\gamma(L\mathbb{R}^n) \approx C^\infty(S^1, \mathbb{R}^n)$,

$$d\tilde{f}_\gamma(X) = \int_0^1 df_{\gamma(t)}(X(t))dt \tag{2.2.4}$$

2.2.2 $(L\mathbb{R}^{2n}, \Omega^\omega)$ admits a Darboux chart.

Let ω_0 be the standard symplectic structure on $\mathbb{R}^{2n}$. With the natural identification of $TL\mathbb{R}^{2n}$ with $L\mathbb{R}^{2n} \times L\mathbb{R}^{2n}$, for $X, Y \in T_\gamma L\mathbb{R}^{2n} \simeq L\mathbb{R}^{2n}$, we have

$$\Omega^{\omega_0}_\gamma(X, Y) := \int_0^1 \omega_0(X(t), Y(t))dt$$

Suppose there is a weak symplectic structure Ω^ω on $L\mathbb{R}^{2n}$ defined as equation 2.1.1. We want to prove that there exists a change of co-ordinate such that Ω^ω changes to Ω^{ω_0}.

Let ω_s be a 1 parameter family of 2 form on $\mathbb{R}^{2n}$. Denote $\Omega^s := \Omega^{\omega_s}$ for the corresponding 1 parameter family of 2 form on $L\mathbb{R}^{2n}$. We have the following theorem.

Theorem 2.2.2. *Suppose there is an isotopy ϕ on $\mathbb{R}^{2n}$ such that $\phi_s^*(\omega_s) = \omega_0$, then with the corresponding isotopy ϕ^L on loop space we have $(\phi_s^L)^*(\Omega^s) = \Omega^0$*

Proof. We have given an isotopy $\phi : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $(\phi^s)^*(\omega^s) = \omega^0$. Define

$$\begin{aligned} & \phi^L : \mathbb{R} \times L\mathbb{R}^{2n} \to L\mathbb{R}^{2n} \\ \text{by} \qquad & \phi^L_s(\gamma) = (t \to \phi^s(\gamma(t))); \end{aligned}$$

we have already seen that by equation 2.2.2, $d\phi^L_s(\gamma)(X) = (t \to d\phi_s(\gamma(t))(X(t)))$. Therefore we have

$$\begin{aligned} (\phi_s^L)^*(\Omega^s)_\gamma(X, Y) &= \Omega^s_{\phi_s^L(\gamma)}(d\phi_s^L(\gamma)(X), d\phi_s^L(\gamma)(Y)) \\ &= \int \omega^s_{\phi_s(\gamma(t))}(d\phi_s(\gamma(t))(X(t)), d\phi_s(\gamma(t))(Y(t)))dt \\ &= \int \omega^0_{\gamma(t)}(X(t), Y(t))dt \\ &= \Omega^0_\gamma(X, Y) \end{aligned}$$

□

Therefore if $(\mathbb{R}^{2n}, \omega)$ admits Darboux charts then above theorem 2.2.2 concludes that corresponding weak symplectic manifold $(L\mathbb{R}^{2n}, \Omega^\omega)$ admits Darboux chart.

2.3 Partition of unity on LM

Let $\mathcal{M}$ be a Fréchet manifold and $S \subset C(\mathcal{M}, \mathbb{R})$ be a subalgebra.

Definition 2.3.1. A S-partition of unity on the space $\mathcal{M}$ is a set $\mathcal{F}$ of functions $f \in S$ which satisfy the following conditions.

1. For all $f \in \mathcal{F}$ and $x \in X$ one has $f(x) \geq 0$.
2. The set $\{supp(f) : f \in \mathcal{F}\}$ of all supports is a locally finite covering of X.
3. The sum $\sum_{f \in \mathcal{F}} f(x)$ equals 1 for all $x \in X$.

The partition of unity is called subordinate to an open covering $\mathcal{U}$ of $\mathcal{M}$ if for every $f \in \mathcal{F}$, there exists an $U \in \mathcal{U}$ with $supp(f) \subset U$.

Definition 2.3.2. We say that $\mathcal{M}$ is S-paracompact if every open cover $\mathcal{U}$ admits a S-partition of unity subordinate to it.

Remark 2.3.1. $L\mathbb{R}^n$ is a Fréchet space and therefore $L\mathbb{R}^n$ admits $C(L\mathbb{R}^n, \mathbb{R})$-partition of unity. LM is smoothly embedded in $L\mathbb{R}^n$ for some n. Therefore LM admits a $C(LM, \mathbb{R})$ partition of unity.

We have the following proposition.

Proposition 2.3.1 (§42.3, [20]). *For finite dimensional second countable manifolds M, N the smooth manifold $C^\infty(M, N)$ is smoothly paracompact.*

The above proposition implies that LM is smoothly paracompact, which means that it admits smooth partition of unity subordinate to any open cover.

For a weak symplectic Fréchet manifold $(\mathcal{M}, \sigma)$, in section 1.4, we defined the space $C^\infty_\sigma(\mathcal{M}, \mathbb{R})$. We are interested in looking at the question whether for $\sigma = \Omega^\omega$, weak symplectic manifold (LM, σ) admits $C^\infty_\sigma(LM, \mathbb{R})$ partition of unity.

2.3.1 $C^\infty_\sigma(LM, \mathbb{R})$ partition of unity on LM for $\sigma = \Omega^\omega$

For the weak symplectic Fréchet manifold (LM, Ω^ω), we have the following proposition.

Proposition 2.3.2. *(LM, σ) admits smooth partition of unity in $C^\infty_\sigma(LM, \mathbb{R})$.*

Proof. As we discussed earlier LM admits a smooth partition of unity in $C^\infty(LM,\mathbb{R})$ but we are required to show that LM admits a smooth partition of unity in $C^\infty_\sigma(LM,\mathbb{R})$.

Let $\{\mathcal{U}_\alpha\}$ be a covering of LM. For each $\mathcal{U}_\alpha$, identify S^1 with $[0,1]/\sim$ and elements of $I=[0,1]$ by t and define:

$$U_\alpha := \{p=\gamma(t) \text{ for some } \gamma\in\mathcal{U}_\alpha \text{ and for some } t\in S^1\}$$

For the loop space LM, $ev_t : LM \to M$ is an open map for any fixed t (see [31]). Therefore U_α is an open subset of M and this collection covers M. Let $\{f_\alpha\}$ be the partition of unity subordinate to the covering $\{U_\alpha\}$. For each α, as in equation 2.2.3, define

$$\hat{f}_\alpha(\gamma) = \int_0^1 f_\alpha(\gamma(t))dt$$

We see that the support of $\hat{f}_\alpha \subset \mathcal{U}_\alpha$ since the support of $f_\alpha \subset U_\alpha$.

For each f_α there is a vector field X_α on M such that $df_\alpha = \omega(X_\alpha, .)$. Let $\hat{X}_\alpha$ be a vector field on LM defined by $\hat{X}_\alpha(\gamma)(t) := X_\alpha(\gamma(t))$. Call this vector field on LM a vector field associated with X_α.

From equation 2.2.4 we have for $X \in T_\gamma LM$,

$$(d\hat{f}_\alpha)_\gamma(X) = \int_0^1 df_{\gamma(t)}(X(t))dt.$$

This gives

$$(d\hat{f}_\alpha)_\gamma(X) = \int_0^1 \omega_{\gamma(t)}(X_\alpha(\gamma(t)), X(t))dt$$

That is

$$(d\hat{f}_\alpha)_\gamma(X) = \Omega^\omega_\gamma(\hat{X}_\alpha(\gamma), X)$$

This proves

$$\hat{f}_\alpha \subset C^\infty_\sigma(LM,\mathbb{R})$$

Also $\sum_\alpha \hat{f}_\alpha = 1$. Hence the collection $\{\hat{f}_\alpha\}$ is the required partition of unity. □

2.4 Symplectic cohomology and de-Rham cohomology of loop space

In section 1.4.4 we defined the de Rham cohomology of a Fréchet manifold. In section 1.5.2 we defined the symplectic cohomology of a weak symplectic Fréchet manifold.

It is worth mentioning that for a strong symplectic manifold, both the cohomology groups will be the same. For a weak symplectic manifold, we have the following theorem by Kriegl and Michor.

Theorem 2.4.1 (§48.9,[20])**.** *If (M, σ) is a smooth weakly symplectic manifold which admits smooth partitions of unity in $C^\infty_\sigma(M, \mathbb{R})$, and which admits Darboux chart, then the symplectic cohomology equals the De Rham cohomology: $H^k_\sigma(M) = H^k_{DR}(M)$.*

We have following theorem.

Theorem 2.4.2. *If there is a global Darboux chart for $(\mathbb{R}^{2n}, \omega)$ then for $(L\mathbb{R}^{2n}, \Omega^\omega)$, we have for every $k \in \{0\} \cup \mathbb{N}$,*

$$H^k_{DR}(L\mathbb{R}^{2n}) \simeq H^k_{\Omega^\omega}(L\mathbb{R}^{2n}).$$

Proof. Combining theorem 2.4.1 with proposition 2.3.2 and theorem 2.2.2 we conclude the result. □

Remark 2.4.1. Proposition 2.3.2 is not only true for $(L\mathbb{R}^{2n}, \Omega^\omega)$ but it is true for any (LM, Ω^ω). But in calculating the derivative of ϕ^L_s in proposition 2.2.2 and proving that $(L\mathbb{R}^{2n}, \Omega^\omega)$ admits a Darboux chart 2.2.2, we used that $\mathbb{R}^{2n}$ admits a global Darboux chart for any symplectic form ω. For general loop space LM in place of $L\mathbb{R}^{2n}$, the proof given in section 2, 4 will not work.

We believe that (LM, Ω^ω) may admit Darboux chart for most symplectic manifolds (M, ω). Some work in this direction is discussed in chapter 3. As there exists a PLB manifold structure on loop space LM ([20], in chapter 3, we studied a general class of Fréchet manifolds, called PLB manifold and necessary conditions for existence of Darboux chart on weak symplectic PLB (projective limit of Banach) manifolds.

2.5 Almost complex structure $\tilde{J}$ on the loop space

Let (M, J) be an almost complex manifold. Indranil Biswas and Saikat Chatterjee in [3] defined an almost complex structure $\tilde{J}$ on the path space $C^\infty([0,1], M)$. the same definition gives an almost complex structure on the loop space LM.

For $\gamma \in LM$ and $X \in T_\gamma LM$. Then X a vector field on M along γ. Define

$$\begin{aligned} &\tilde{J} : TLM \to TLM; \\ &\tilde{J}_\gamma : T_\gamma LM \to T_\gamma LM \text{ by} \\ &\tilde{J}_\gamma(X)(t) := J(X(\gamma(t))) \end{aligned}$$

We have $\tilde{J}_\gamma^2 = -Id_{T_\gamma LM}$ because $J^2 = -Id$. Using the definition of smoothness (if $c : \mathbb{R} \to TLM$ is smooth then $\tilde{J} \circ c : \mathbb{R} \to TLM$ should be smooth) we see that $\tilde{J}$ is smooth with a smooth inverse.

This proves that $\tilde{J}$ is an almost complex structure on the loop space LM.

2.5.1 $\tilde{J}$ is compatible with Ω^ω whenever J is compatible with ω

Here we define a metric (inner product) on the loop space LM of a finite dimensional Riemannian manifold (M, g). The metric $\tilde{g}$ on the tangent space LM is a symmetric fiberwise bilinear map $TLM \times_M TLM \to \mathbb{R}$ with the property that the induced map $\tilde{g}^\vee : TLM \to T^*LM$ satisfies $\tilde{g}^\vee(v)v > 0$ for $v \neq 0$.

There is a lot of literature which deal with metrics on LM. For example we can see [4], [3] etc. Following these articles, we have a metric $\tilde{g}$ on the loop space defined as following.

For $\gamma \in LM$ and $X, Y \in T_\gamma LM$. $X, Y \in \Gamma_{S^1}(\gamma^* TM)$. We define

$$\tilde{g}_\gamma(X, Y) := \int_0^1 g_{\gamma(t)}(X(t), Y(t))dt. \tag{2.5.1}$$

$\tilde{g}$ is a weak metric on LM. By a weak metric, we mean that the induced map by the metric $\tilde{g}$ from the tangent bundle of LM to the cotangent bundle T^*LM is not a topological isomorphism. In fact there does not exist any strong metric on any Fréchet manifold.

For a symplectic form ω on M, we have the corresponding symplectic structure Ω^ω as defined in the equation 2.1.1. Ω^ω is given as following. For $X, Y \in T_\gamma LM \simeq \Gamma_{S^1}(\gamma^*(TM))$ we have

$$\Omega_\gamma^\omega(X, Y) = \int_0^1 \omega_{\gamma(t)}(X(t), Y(t))dt \tag{2.5.2}$$

Also the almost complex structure is given by

$$\tilde{J}_\gamma(X)(t) := J_{\gamma(t)}(X(t)) \tag{2.5.3}$$

where we have $X \in T_\gamma LM$ that is $X : S^1 \to \gamma^*(TM)$.

Suppose for a smooth manifold (M, g), symplectic structure ω on M and almost complex structure J are compatible. This means we have

$$g(u, v) = \omega(u, Jv).$$

then, $\tilde{g}(X, Y) = \Omega^\omega(X, \tilde{J}Y)$.

2.5.2 $\tilde{J}$ is formally integrable

Let $(LM, \tilde{J})$ be the almost complex Fréchet manifold defined as in section 2.5. For given two vector fields X and $Y \in \mathfrak{X}(LM)$, define the Nijenhuis-tensor as

$$N_{\tilde{J}}(X,Y) := [\tilde{J}(X), \tilde{J}(Y)] - [X,Y] - \tilde{J}([X, \tilde{J}Y]) - \tilde{J}([\tilde{J}X, Y]).$$

In [4], Brylinski defined and used this tensor in the case of the loop space. We say that $(LM, \tilde{J})$ is $formally\, integrable$ if the Nijenhuis tensor vanishes identically. Indranil Biswas and Saikat Chaterjee in [3] showed that $(PM, \tilde{J})$ is formally integrable. The same proof shows that $(LM, \tilde{J})$ is formally integrable.

Integrability of $(LM, \tilde{J})$ is not known. J.-L. Brylinski in [4] introduced an almost complex structure on a loop space which contains only immersion (We call it the Brylinski loop space). The almost complex structure on the Brylinski loop space turns out to be formally integrable. But Lempert [21] proved that this almost complex structure on the Brylinski loop space is not integrable.

Lempert [21] defined the notion of weak integrability in local sense for infinite dimensional manifolds. In next section we will discuss about the weak integrability of $(LM, \tilde{J})$.

2.5.3 $\tilde{J}$ is weak integrable in local sense when J is integrable

Definition 2.5.1. (Weak integrable in local sense) Let (M, J) be an almost complex manifold (Banach or Fréchet). We say (M, J) is weak integrable in local sense, if for any $p \in M$ and any non zero $v \in T_pM$, there is a neighborhood U of p and a *J-holomorphic* function F on U such that $v(F) \neq 0$.

We recall the following definition.

Definition 2.5.2. If (M, J) and (M', J') be almost complex manifolds. A map

$$f : (M, J) \to (M', J')$$

is said to to pseudo-holomorphic or simply holomorphic if

$$(f_*)J = J'(f_*)$$

If $f : (M, J) \to \mathbb{C}$, then pseudo-holomorphic functions are sometimes called *J-holomophic*. When J and J' are integrable, then f is pseudo- holomorphic if and only if f is holomorphic. Following we will define the notion of a neighborhood of $\gamma \in LM$ associated to chart (U, ϕ) of M.

Given $\gamma \in LM$ and for some fixed t, let (U, ϕ) be a coordinate chart of M around $\gamma(t) \in M$. Let $\mathfrak{U}$ be a neighborhood of $\gamma \in LM$. Define

$$\mathfrak{U}_\gamma^t := \{\alpha \in \mathfrak{U} : \alpha(t) \in U\}$$

$\gamma \in \mathfrak{U}_\gamma^t$. As for fixed $t \in [0,1]$, $ev_t : LM \to M$ is a continuous map, we have $\mathfrak{U}_\gamma^t = ev_t^{-1}(U) \cap \mathfrak{U} \neq \phi$. Hence $\mathfrak{U}_\gamma^t$ is open set for each t. We say $\mathfrak{U}_\gamma^t$ is an open neighborhood of γ associated with the chart (U, ϕ) at time t.

Now we will give an example of a holomorphic ($\tilde{J}$- holomorphic) function.

Proposition 2.5.1. *If M be a complex manifold, for given $\gamma \in LM$ and fixed t, let $\mathfrak{U}_\gamma^t$ be an open set associated with the chart $(U, \phi = (\phi_1, ...\phi_n))$. For each $t \in [0,1]$, define*

$$F_i^t : \mathfrak{U}_\gamma^t \to \mathbb{C}; \quad \gamma \to \phi_i \circ ev_t(\gamma) \tag{2.5.4}$$

Then F_i^t is $\tilde{J} -$ holomorpic map on some neighborhood of $\gamma \in LM$.

Proof. Corollary 2.3 of [3] proves that ev_t is $\tilde{J}$-*holomorphic* and ϕ_i is a coordinate map of a complex manifold M and hence ϕ_i is holomorphic. This gives F_γ^t is $\tilde{J}$-*holomorphic.* □

Hence for each $\gamma \in LM$ there is a neighborhood of γ at time t, $\mathfrak{U}_\gamma^t$ and a $\tilde{J}$-*holomorphic* function F_i^t defined on this neighborhood. This will work as a possible candidate in proving the weak integrability (local sense) of $(LM, \tilde{J})$.

If M is a complex manifold, we have the following proposition.

Theorem 2.5.2. *$(LM, \tilde{J})$ is weak integrable in local sense.*

Proof. Let $X \neq 0$ and $X \in T_\gamma LM$. We have $X \in C^\infty(S^1, \gamma^*(LM))$ and there is some t_0 such that $X(t_0) \neq 0$. Let $(U, (\phi_1, ..\phi_n))$ be a coordinate neighborhood of $\gamma(t_0)$.

As $X(t_0) \in T_{\gamma(t_0)}M$, suppose locally (in some neighborhood of t_0)

$$X(t) = \sum_{j=1}^{n} X^j(t) \frac{\partial}{\partial z_j}|_{\gamma(t)}.$$

If $X(t_0) \neq 0$ then there exists some j such that $X^j(t_0) \neq 0$. Now take $F_j^{t_0}$ as defined in Proposition 2.5.1. $F_j^{t_0}$ will suffice our purpose.

For fixed t, we have:

$$\begin{aligned} F_{i_*}^t(\gamma)(X) &= d\phi_i(\gamma(t)) \circ ev_{t_*}(\gamma)(X) \\ &= d\phi_i(\gamma(t))(X(t)) = X^i(t) \end{aligned}$$

$\left(d\phi_{i_{\gamma(t)}}(X(t)) = d\phi_{i_{\gamma(t)}}(\sum_{j=1}^{n} X^j(t)\frac{\partial}{\partial z_j}) = X^i(t) \right)$.

That is for $t = t_0$, we have $XF_j^{t_0} = X^j(t_0) \neq 0$

Hence for $X \neq 0, X \in T_\gamma LM$, we have a holomorphic function $F_j^{t_0}$ defined on $\mathfrak{U}_\gamma^{t_0} \subset LM$ such that $XF_j^{t_0} \neq 0$ $\square$

This proves that $(LM, \tilde{J})$ is weak integrable in local sense.

Chapter 3

Weak symplectic structure on PLB manifolds

In chapter 2 we discussed a symplectic structure Ω^ω on the loop space LM. We proved very easily that $(L\mathbb{R}^{2n}, \Omega^\omega)$ admits a Darboux chart. But for the case of (LM, Ω^ω) it is not straight forward to see whether it admits Darboux chart.

We denote for $i \in \mathbb{N} \cup \{0\}$, $L^iM := C^i(S^1, M)$. L^iM is a Banach manifold and $LM = \varprojlim L^iM$. For $\gamma \in LM$, we have $\gamma \in L^iM$ for each i. Fix a local addition η for M and let $\pi \times \eta : TM \to V$ is diffeomorphism on to a open neighborhood V of the diagonal of $M \times M$. Define

$$U_\gamma^i := \{\alpha \in L^iM : (\gamma, \alpha) \in L^iV\}$$

and ψ_i is an injective map from $U_\gamma^i \to L\mathbb{R}^n$ as defined in the section 1.3.1. We write the following trivial facts.

1. $\{L^iM, \phi_{ji} = I(\text{inclusion map})\}$ is a projective system of Banach manifolds.

2. $\{L^i\mathbb{R}^n, \rho_{ji} = I(\text{inclusion map})\}$ is a projective system of Banach spaces.

3. We see that for $\gamma \in LM$, $\phi_{ji}(U_\gamma^j) \subset U_\gamma^i$ for every $j \geq i$.

4. We have $\rho_{ji} \circ \psi_j = \psi_i \phi_{ji}$.

5. $\varprojlim \psi_i(U_\gamma^i) = U_\gamma$.

This makes LM a PLB manifold.

In this chapter we will define a weak symplectic structure on a PLB manifold $M = \varprojlim M_i$ such that each M_i is modelled on a reflexive Banach space. Further we will prove that if certain conditions holds then PLB manifold admits a Darboux chart. We will start by discussing the case of strong symplectic Banach manifolds. Then we will discuss weak symplectic Banach manifolds and in the end we will discuss our main object of study, namely PLB manifold with a particular weak symplectic structure.

Our analysis of PLB manifold does not completely hold for the case of loop space. This is because, we could find the condition for existence of Darboux chart for the case of PLB manifold $M := \varprojlim M_i$ when each M_i is modeled over a reflexive Banach space E_i. In the case of loop space LM, each L^iM is not reflexive space. Therefore a general discussion on loop space is still lacking.

3.1 Strong symplectic Banach manifolds

Let (M, Ω) be a strong symplectic Banach manifold modeled on a Banach space B. For ensuring existence of Darboux chart, it is enough to know the existence of Darboux chart on some open neighborhood of zero in B.

Let Ω be a symplectic structure on a neighborhood of zero in a Banach space B. In 1969, Alan Weinstein proved the following theorem.

Theorem 3.1.1 ([32]). *Let Ω_1 be the strong symplectic structure on B which is constant with respect to the natural parallelism on B and equal to Ω at 0. Then there are neighborhoods U and V of 0 and a diffeomorphism $f : U \to V$ such that $f(0) = 0$, $f_*(0)$ is the identity, and $f^*(\Omega_1) = \Omega$.*

Ω_1 is said to be constant with respect to natural parallelism ($TB \simeq B \times B$) on B, if there exists a non singular, skew symmetric, bounded bilinear map $\mathcal{F}$ on B such that $\Omega_1(x) := \mathcal{F}$ for every x. Theorem demands existence of Ω_1 such that $\Omega_1(0) = \Omega(0)$. This theorem proves that the local classification of strong symplectic structures on a manifold modeled on the Banach space B is thus reduced to the classification of non singular, skew symmetric, bounded bilinear forms on B.

Further if B is the Hilbert space, every such form is equal to $\sum_{i \in I} \zeta_i \wedge \eta_i$ for some basis $\{\zeta_i\} \cup \{\eta_i\}$ of B^*. Proof of this theorem is completely based on the Moser trick.

This theorem completely solves the problem of existence of Darboux chart on strong symplectic Banach manifold.

3.2 Weak symplectic structure on Banach manifolds

If Ω is not strong symplectic but only it is a weak symplectic, then Darboux type theorem fails. In 1972, J. Marsden, has given an example which shows that the weak symplectic structure on a Hilbert manifold may not admit Darboux chart [24].

3.2.1 Sufficient condition for the existence of Darboux chart on weak symplectic Banach manifolds

Dario Bambusi in [2] has given sufficient conditions for existence of Darboux chart on weak symplectic Banach manifold. In this section we will give a brief review of Bambusi theorem.

Suppose (M, Ω) is a Banach manifold modeled over a reflexive Banach space E. Since the Darboux theorem is a local result it is enough to consider the case where M is an open set U of a Banach space E. Without loss of generality we can assume $0 \in E$.

We have $\Omega_x : E \times E \to \mathbb{R}$. Using Ω_x define a norm on E

$$\|X\|_{\mathcal{F}_x} := \sup_{\|Y\|_E = 1} |\Omega_x(X, Y)|$$

and consider the completion $\mathcal{F}$ of E in such a norm. It is clear that Ω_x can be extended to a continuous bilinear form on $E \times \mathcal{F}_x$.

Bambusi proved the following theorem:

Theorem 3.2.1. *[Theorem 2.1 [2]] Assume that there exists a neighborhood $\mathcal{W}$ of 0 such that, for all $x \in \mathcal{W}$ the spaces $\mathcal{F}_x$ coincide and moreover that the map $x \to \Omega_x$ is differentiable as an application from $\mathcal{W}$ to the continuous bilinear forms on $\mathcal{F} \times E$, where $\mathcal{F} := \mathcal{F}_x$; then there exists a neighborhood $\mathcal{V}$ of 0 and a change of coordinates ψ defined on $\mathcal{V}$ which reduces Ω to the constant two form $\Omega_0 := \Omega_x|_{x=0}$.*

A condition given in the theorem assumes that there exists a neighborhood $\mathcal{W}$ of 0 such that, for all $x \in \mathcal{W}$ the spaces $\mathcal{F}_x$. This means for every $x, y \in \mathcal{W}$, Banach spaces $\mathcal{F}_x$ and $\mathcal{F}_y$ is topologically isomorphic. This condition is a necessary condition for the existence of Darboux chart. Bambusi proved the following:

Theorem 3.2.2 (Proposition 2.6 [2])**.** *The existence of Darboux chart about 0 in which the symplectic form is constant implies that there exists a neighborhood of 0 such that for each x in such a neighborhood there exists an isomorphism between $\mathcal{F}_x$ and $\mathcal{F}_0$ which restricts to an isomorphism of Banach space E with itself.*

Therefore first condition on the theorem 3.2.1 is a necessary condition.

As Ω is a weak symplectic structure, we have given that as a map $\Omega : \mathcal{U} \to L(E \times E, \mathbb{R})$ is smooth. Second condition of the theorem demands that

$$\Omega : \mathcal{U} \to L(E \times \mathcal{F}, \mathbb{R})$$

should be smooth. Bambusi pointed out that this condition is not automatic.

3.3 Weak symplectic structure on PLB manifold

In chapter 1, we defined a PLB manifold. Suppose (M, σ) be a PLB manifold with a weak symplectic form σ. In this section, we will define the compatibility of σ with the projective system. Then we will prove a version of Darboux theorem for the PLB manifold with a weak symplectic structure compatible with the projective system.

Our idea is similar to the idea of Weinstein [32] and Bambusi [2]. We extended the main ideas of Bambusi [2] in the context of PLB manifold.

We recall that for a Fréchet manifold, there does not exist any strong symplectic structure because the induced map $\sigma_p^b : T_pM \to T_p^*M$ is never a topological isomorphism. T_p^*M is not even a Fréchet space with strong dual topology. But there are weak symplectic structures on Fréchet manifolds.

3.3.1 Basics about weak symplectic structures

Let M be a PLB manifold and $\{M_i, \phi_{ji}\}_{i,j\in\mathbb{N}}$ be a projective system of Banach manifolds with $M = \varprojlim M_i$. Suppose each M_i is modeled over a reflexive Banach space E_i and each M_i has a weak symplectic structure σ^i.

Let $x \in M$, we have $x = (x_i)$ where $\phi_{ji}(x_j) = x_i$. For each x_i, following [2], we define a norm on $T_{x_i}M_i$, for $X \in T_{x_i}M_i$,

$$\|X\|_{\mathcal{F}_{x_i}} := \sup_{\|Y\|_i=1} |\sigma^i_{x_i}(X, Y)|$$

where $\|.\|_i$ is the norm on the Banach space $T_{x_i}M_i$. Let $\mathcal{F}_{x_i}$ be the completion of $T_{x_i}M_i$ with respect to the $\|.\|_{\mathcal{F}_{x_i}}$ norm. As each $T_{x_i}M_i$ is a reflexive Banach space, we have that the induced map (Lemma 2.8,[2]),

$$(\sigma^i_{x_i})^b : T_{x_i}M_i \to \mathcal{F}^*_{x_i}; \quad X \to \sigma^i_{x_i}(X, .)$$

is a topological isomorphism.

Define for each $i, j \in \mathbb{N}$ $j \geq i$ and given $x = (x_i) \in \varprojlim M_i = M$,

$$\psi_{ji} : \mathcal{F}^*_{x_j} \to \mathcal{F}^*_{x_i} \text{ by}$$

$$\psi_{ji} = (\sigma^i_{x_i})^b \circ T_{x_j}\phi_{ji} \circ ((\sigma^j_{x_j})^b)^{-1} \tag{3.3.1}$$

where $T_{x_j}\phi_{ji}$ is the differential of $\phi_{ji} : M_j \to M_i$ at x_j. We see that $\{\mathcal{F}^*_{x_i}, \psi_{ji}\}_{i,j\in\mathbb{N}}$ is a projective system of Banach spaces and smooth maps (since for any $k \geq j \geq i$, we have $\psi_{ki} = \psi_{kj} \circ \psi_{ji}$). We see that $\{(\sigma^i_{x_i})^b : T_{x_i}M_i \to \mathcal{F}^*_{x_i}\}$ is a projective system of mappings because

$$(\sigma^j_{x_j})^b \circ \psi_{ji} = T_{x_j}\phi_{ji} \circ (\sigma^i_{x_i})^b.$$

Fix some point $p = (p_i) \in M$. We know that for a fixed $p = (p_i) \in M$, $\{T_{p_j}M_j, T_{p_j}\phi_{ji}\}_{i,j\in\mathbb{N}}$ is a projective system of Banach spaces. In section 1.3.3 we saw that $T_pM \simeq \varprojlim T_{p_j}M_j$. Let h_p be the isomorphism from $T_pM \to \varprojlim T_{p_j}M_j$ as defined in [14].

For some coordinate neighborhood $U = \varprojlim U_i$ around p, let $\sigma_p := \sigma|_{x=p}$ be the constant symplectic structure on U with a natural parallelism $TU \simeq U \times E$ where $E \simeq T_pM$ is a Fréchet space. On each U_i we have a corresponding constant symplectic structure $\sigma^i_{p_i}$ with the natural parallelism.

For $t \in [-1, 1]$ we define $\sigma^t := \sigma + t(\sigma - \sigma_p)$ similarly $(\sigma^i)^t := \sigma^i + t(\sigma^i - \sigma^i_{p_i})$. Suppose for some $q = (q_i) \in M$ and for some $t \in [-1, 1]$, $((\sigma_q)^{tb})^{-1}$ and $(((\sigma^i_{q_i}))^{tb})^{-1}$ exist for each i. Then $\{(((\sigma^i_{q_i}))^{tb})^{-1} : \mathcal{F}^{*t}_{q_i} \to T_{q_i}M_i\}$ is a projective system of map. Here $\mathcal{F}^{*t}_{q_i}$ is defined in the same way as $\mathcal{F}^*_{q_i}$ above. Where $\mathcal{F}^*_{q_i}$ are spaces corresponding to σ^i, $\mathcal{F}^{*t}_{q_i}$ are spaces corresponding to weak symplectic structure $(\sigma^i)^t$. Also for fixed t corresponding to the ψ_{ji} maps, we have the maps ψ^t_{ji} (for the weak symplectic structure $(\sigma^i)^t$). Therefore we see that for each t, the collections $\{(((\sigma^i_{q_i}))^{tb})^{-1} : \mathcal{F}^{*t}_{q_i} \to T_{q_i}M_i\}$ are a projective system of mappings.

For a weak symplectic structure σ on a PLB manifold M as discussed above, we have for each $p = (p_i)$, σ^b_p is a map on T_pM that is $\sigma^b_p : T_pM \to T^*_pM$. Let $h_p : T_pM \to \varprojlim T_{p_j}M_j$ be an isomorphism [14]. With this identification we can consider σ^b_p as a map defined on $\varprojlim T_{p_j}M_j$.

Now we are in a position to define a compatible symplectic structure.

3.3.2 Compatible symplectic structure

We say that a weak symplectic structure σ on an open subset $U = \varprojlim U_i$ of a PLB space $E = \varprojlim E_i$ is compatible with the projective system (U_i, σ_i) (σ_i are the weak symplectic structure on U_i) if the following is satisfied:

1. For every $x \in U$, $\sigma^b_x := \varprojlim(\sigma^i_{x_i})^b$.

2. If for some $p \in U$, there exists a 1 - form α such that for each $x \in U$, $\alpha_x = (\alpha^i_{x_i}) \in \varprojlim \mathcal{F}^*_{x_i}$, we must have $h_x((\sigma_x)^{tb})^{-1}(\alpha_x) = \big(((\sigma^i_{x_i})^{tb})^{-1}(\alpha^i_{x_i})\big)$ whenever defined.

3. For such α as in above, whenever $Y_t^i(x_i) := ((\sigma^i_{x_i})^{tb})^{-1}(\alpha^i_{x_i})$ is defined on some open set W_i of U_i, it is defined on whole E_i. Each Y_t^i is projective μ-Lipschitz smooth map for some fixed positive real $\mu > 0$.

Remark 3.3.1. In condition 1, $\varprojlim(\sigma^i_{x_i})^b$ make sense because $\{(\sigma^i_{x_i})^b\}$ makes projective system of mapping. σ^b_x as a map $\sigma^b_x : \varprojlim T_{x_i}M_i \to \varprojlim \mathcal{F}^*_{x_i}$ is the projective limit of the maps $(\sigma^i_{x_i})^b$.

Remark 3.3.2. For every $j \geq i$, condition 2 demands

$$h_x((\sigma_x)^{tb})^{-1}(\alpha_x) = \big(((\sigma^i_{x_i})^{tb})^{-1}(\alpha^i_{x_i})\big) \in \varprojlim T_{x_i}M_i.$$

This means that we have

$$T_{x_j}\phi_{ji}(((\sigma^i_{x_j})^{tb})^{-1}(\alpha^j_{x_j})) = ((\sigma^i_{x_i})^{tb})^{-1}(\alpha^i_{x_i}).$$

Definition 3.3.1 (Compatible symplectic structure)**.** Suppose (M, σ) be a PLB manifold and $M = \varprojlim M_i$. Let σ_i be the weak symplectic structure on M_i. We say σ is compatible with the projective system if for every point $p = (p_i) \in M$ there is a co-ordinate system $(\varprojlim U_i, \psi := \varprojlim \psi_i)$ such that each $\psi_i : U_i(\subset E_i) \to M_i$ is a co-ordinate map around each p_i and weak symplectic structure $(U, \psi^*\sigma)$ is compatible with the projective system $(U_i, \psi_i^*\sigma_i)$.

The definition of compatibility of σ arise while exploring the possibility of existence of Darboux chart for the case of the loop space (LM, Ω^ω) discussed in chapter 2. In the introduction of this chapter we saw that $LM := \varprojlim L^iM$ is a PLB manifold. We denote for $i \geq 0$, $L^iM := C^i(S^1, M)$. L^iM is a Banach manifold and there exists a PLB manifold structure on $LM := \varprojlim L^iM$. For a symplectic structure Ω^ω defined in chapter 2, if we define Ω^ω_i symplectic structure on L^iM by the exactly same formulae as of Ω^ω. Then above condition arise while analyzing the relation of Ω^ω_i and Ω^ω.

3.3.3 A Fréchet space used in the theorem

Since the Darboux theorem is a local result, we will work on some open subset $\mathcal{U}$ (containing zero) of a Fréchet space E which is the projective limit of Banach spaces. We have $\{E_i, \rho_{ji}\}$ is the inverse system of Banach spaces (manifolds) E_i and $E = \varprojlim E_i$. As E is a PLB manifold, each ρ_{ji} is the inclusion map. We can assume that $\mathcal{U} = \varprojlim \mathcal{U}_i$, where $\mathcal{U}_i$ are open subsets of Banach spaces E_i. Let σ be a compatible symplectic structure on the PLB manifold $\mathcal{U}$.

As we discussed earlier, for a fixed $x = (x_i) \in E$, $\{T_{x_j}E_j = E_j, T_{x_j}\phi_{ji} = \phi_{ji}\}_{i,j\in\mathbb{N}}$ is a projective system of Banach spaces. In section 1.3.3, we saw that $E = T_xE = \varprojlim T_{x_j}E_j$.

If the topology on E is generated by the collection of semi-norms $\{\rho_k : k \in \mathbb{N}\}$ then for each $x \in E$ and $k \in \mathbb{N}$, define norms on E as the following. For $X \in E$,

$$\rho_k^x(X) := \sup_{\rho_k(Y)=1} |\sigma_x(X, Y)|.$$

All ρ_k^x are norms on E and collection $\{\rho_k^x : k \in \mathbb{N}\}$ generate a topology on E. Let completion of E with respect to this collection be denoted by $\mathcal{F}_x$. Then $\mathcal{F}_x$ is a Fréchet space.

For a fixed $x \in \mathcal{U}$, let $H_x := \{\sigma_x(X, .) : X \in E\}$. We can extend σ as a continuous bilinear map $E \times \mathcal{F}_x \to \mathbb{R}$.

For $x = (x_i)$, in the section 3.3.1 we defined $\mathcal{F}_{x_i}^*$. If σ on E is compatible with the inductive maps, for $X \in T_xE = E$

$$\sigma_x(X, .) = (\sigma_{x_i}^i(X_i, .)) \in \varprojlim \mathcal{F}_{x_i}^*$$

Therefore $H_x \subset \varprojlim \mathcal{F}_{x_i}^*$ as a set.

We know that each $(\sigma_{x_i}^i)^b : T_{x_i}E_i(= E_i) \to \mathcal{F}_{x_i}^*$ is a topological isomorphism. Therefore a typical element of $\mathcal{F}_{x_i}^*$ will be given by $\sigma_{x_i}^i(X_i, .)$ for some $X_i \in E_i$. Hence typical element of $\varprojlim \mathcal{F}_{x_i}^*$ will be given by $(\sigma_{x_i}^i(X_i, .))$ where we must have $\psi_{ji}(\sigma_{x_j}^j(X_j, .)) = \sigma_{x_i}^i(X_i, .)$. This will happen if and only if $T_{x_j}\rho_{ji}(X_j) = \rho_{ji}(X_j) = X_i$. (since we can identify $(\sigma_{x_i}^i(X_i, .)) = \sigma_x(X, .)$). This shows as set, $\varprojlim \mathcal{F}_{x_i}^* \subset H_x$.

Therefore we have, as set, $\varprojlim \mathcal{F}_{x_i}^* = H_x$.

On H_x, we have two possible topology:

1. Projective limit topology when we identify H_x as $\varprojlim \mathcal{F}_{x_i}^*$. Projective limit topology on H_x given as follows: For $i \in \mathbb{N}$, we have $\|\sigma_x(X, .)\|_i := \|\sigma_{x_i}^i(X_i, .)\|_{op}$. Here on the right hand side of expression $\|.\|_{op}$ is the operator norm of $\sigma_{x_i}^i(X_i, .)$ as an element of $\mathcal{F}_{x_i}^*$.

2. The induced topology when we consider H_x as subset of E^*.

We fix the notation $\|.\|_i$ for norm on Banach space $T_{x_i}E_i = E_i$ and we recall that $\|.\|_{\mathcal{F}_{x_i}}$ is the norm on $\mathcal{F}_{x_i}$ as defined in section 3.3.1.

We have for $X, Y \in T_{x_i}E_i = E_i$,

$$\|X\|_i\|Y\|_{\mathcal{F}_{x_i}} \geq |\sigma_{x_i}^i(X, Y)|. \tag{3.3.2}$$

Fix $\epsilon > 0$, for $Y \in \mathcal{F}_{x_i}$ we have a sequence $(Y_n) \in E_i$ such that $\lim_{n\to\infty} Y_n = Y$ in $\mathcal{F}_{x_i}$. This means that there exists a natural number $N^Y \in \mathbb{N}$ such that

$$\|Y_n\|_{\mathcal{F}_{x_i}} - \epsilon < \|Y\|_{\mathcal{F}_{x_i}} < \|Y_n\|_{\mathcal{F}_{x_i}} + \epsilon;\ \ \forall n \geq N^Y. \tag{3.3.3}$$

Let $A = \{Y \in \mathcal{F}_{x_i} : \|Y\|_{\mathcal{F}_{x_i}} = 1\}$. For each $Y \in A$, there exists a sequence (Y_n) in E_i such that equation 3.3.3 is satisfied for some N^Y. We have a collection call A^Y_ϵ and B^Y_ϵ as follows: For each $Y \in A$, fix a sequence $(Y_n) \subset E_i$ such that $\lim_{n\to\infty} Y_n = Y$.

$$A^Y_\epsilon := \{Y_n : n \geq N^Y\} \ \text{ and } B^Y_\epsilon := \{y \in E_i : \|y\|_{\mathcal{F}_{x_i}} - \epsilon < \|Y\|_{\mathcal{F}_{x_i}}\}.$$

Let $A^\epsilon := \bigcup_{Y\in A} A^Y_\epsilon$ and $B^\epsilon = \bigcup_{Y\in A} B^Y_\epsilon$.

If $X \in E_i$ is fixed and f is defined on E_i such that $f(Y) = |\sigma^i_{x_i}(X,Y)|$ and we have a continuous extension of f to $\mathcal{F}_{x_i}$. Then we have

$$\sup_{Y\in A} f(Y) \leq \sup_{Y\in A^\epsilon} f(Y)$$

Also $A^\epsilon \subset B^\epsilon$, we have

$$\sup_{Y\in A} f(Y) \leq \sup_{Y\in B^\epsilon} f(Y).$$

On B^ϵ, that is for $X \in E_i$ and $Y \in B^\epsilon$, we have $|\sigma^i_{x_i}(X,Y)| \leq (1+\epsilon)|\|X\|_i$. Therefore we have

$$\begin{aligned}\|\sigma^i_{x_i}(X,.)\|_{op} &= \sup_{Y\in A} |\sigma^i_{x_i}(X,Y)| \\ &\leq \sup_{Y\in B^\epsilon} |\sigma^i_{x_i}(X,Y)| \\ &\leq (1+\epsilon)\|X\|_i.\end{aligned}$$

Since this is true for every ϵ, we have for $X \in T_{x_i}E_i = E_i$,

$$\|\sigma^i_{x_i}(X,.)\|_{op} \leq \|X\|_i. \tag{3.3.4}$$

Remark 3.3.3. In above discussion by $\|\sigma^i_{x_i}(X,.)\|_{op}$, we mean the operator norm of $\sigma^i_{x_i}(X,.)$ as an element of $\mathcal{F}^*_{x_i}$.

We have the following proposition.

Proposition 3.3.1. *$\sigma^b_x : E \to H_x$ is an isomorphism.*

Proof. As discussed earlier, for $x = (x_i)$, $H_x = \varprojlim \mathcal{F}^*_{x_i}$. For each x_i, we know $(\sigma^i_{x_i})^b : E_i(= T_{x_i}E_i) \to F^*_{x_i}$ is an isomorphism (lemma 2.8,[2]). Let the inverse of $(\sigma^i_{x_i})^b$ is denoted by $J^i_{x_i}$. Define

$$\sigma^b_x : E = \varprojlim T_{x_i}E_i \to \varprojlim \mathcal{F}^*_{x_i}; \quad \sigma^b_x(X) = \big((\sigma^i_{x_i})^b(X_i)\big) \text{ and}$$
$$J_x : H_x(= \varprojlim \mathcal{F}^*_{x_i}) \to \varprojlim T_{x_i}E_i = E \text{ is defined as follows.}$$

If $\alpha_x \in H_x$, then J_x can be defined using $J^i_{x_i} = ((\sigma^i_{x_i})^b)^{-1}$.

$$J_x(\alpha_x) := (J^i_{x_i}(\alpha_{x_i}))$$

As σ is compatible with the inductive limit $(J^i_{x_i}(\alpha_{x_i})) \in \varinjlim T_{x_i}E_i = E$.

J_x is the inverse of σ^b_x and this gives an isomorphism. □

Remark 3.3.4. Form above proposition, it is clear that H_x and H_y are topologically isomorphic.

3.3.4 Condition for the existence of a Darboux chart on weak symplectic PLB manifolds

We state the theorem for some open neighborhood of $0 \in E$.

Theorem 3.3.2. *Suppose*

1. *There exists a neighborhood $\mathcal{W}$ of $0 \in E$, such that all H_x are identical and $\sigma^{tb}_x : E \to H$ is an isomorphism for each t and for each $x \in \mathcal{W}$.*

2. *There exists a vector field $X = (X_i)$ on E such that on $\mathcal{W}$, $L_X\sigma = \sigma - \sigma_0$.*

3. *For every i and $t \in [-1, 1]$, $\|X_i(x_i)\|_i.\|((\sigma^i_{x_i})^{tb})^{-1}\|_{op}$ is bounded by M for some positive real M.*

then there exists a coordinate chart $(\mathcal{V}, \Phi)$ around zero such that $\Phi^*\sigma = \sigma_0$.

For the rest of the discussion we denote $\overline{\sigma} = \sigma - \sigma_0$. $\overline{\sigma}$ is defined on $\mathcal{W}$, an open neighborhood of 0.

Remark 3.3.5. Suppose σ is a weak symplectic structure as defined earlier (compatible with projective system) and there exists a vector field X on $\mathcal{W}$ such that $L_X\sigma = \overline{\sigma}$. Then we have $d(i_X\sigma) = \overline{\sigma}$. Denote $\alpha = i_X\sigma = \sigma^b(X)$. We define

$$\alpha^i_{x_i} := (\sigma^i_{x_i})^b(X_i(x_i))$$

We can extend these $\alpha^i_{x_i}$ as $\alpha^i_{x_i} \in \mathcal{F}^*_{x_i}$. We see that $\psi_{ji}(\alpha^j_{x_j}) = \alpha^i_{x_i}$.

Therefore $\varinjlim \alpha^i_{x_i}$ exists and as σ is compatible with inductive maps, we have

$$\alpha_x = \varinjlim(\alpha^i_{x_i})$$

Remark 3.3.6. For any $x_i \in E_i$, $\alpha^i_{x_i} \in E_i^*$. As an element of E_i^*, we have

$$\begin{aligned}\|\alpha^i_{x_i}\|_{op} &= \sup_{\|Y\|_i=1} |\alpha^i_{x_i}(Y)| \\ &= \sup_{\|Y\|_i=1} |\sigma^i_{x_i}(X_i(x_i), Y)| \\ &= \|X_i(x_i)\|_{\mathcal{F}_{x_i}}\end{aligned}$$

But as an an element of $\mathcal{F}^*_{x_i}$, by equation 3.3.4, we have

$$\|\alpha^i_{x_i}\|_{op} \leq \|X_i(x_i)\|_i$$

3.3.5 Proof of the theorem

Proof. There is an open neighborhood of $0 \in E$, $\mathcal{W} = \varprojlim \mathcal{W}_i$. $0 \in \varprojlim \mathcal{W}_i$ is identified with $0 = (0, 0, ..)$. On $\varprojlim \mathcal{W}_i$, define $\overline{\sigma} = \sigma_0 - \sigma$ and $\sigma^t = \sigma + t\overline{\sigma}$ for $t \in [-1, 1,]$.

By remark 3.3.5, we have on $\varprojlim \mathcal{W}_i$, $\alpha_x = (\alpha^i_{x_i})$ and

$$d(\alpha) = \overline{\sigma}.$$

As $\alpha = i_X\sigma$, we have $\alpha \in H$. We want to solve for $Y_t : \mathcal{W} \to E$ such that

$$i_{Y_t}\sigma^t = -\alpha$$

Consider

$$(\sigma^t_x)^b : E \to H.$$

For $x \in \varprojlim \mathcal{W}_i$, $(\sigma^t_x)^b$ is isomorphism for all t. Hence for $x \in \varprojlim \mathcal{W}_i$,

$$Y_t(x) = ((\sigma^t_x)^b)^{-1}(\alpha_x) \quad \text{is well defined.}$$

We define $Y^i_t : \mathcal{U}_i \to E_i$ such that

$$Y^i_t(x_i) := \left(((\sigma^i_{x_i})^t)^b\right)^{-1}(\alpha^i_{x_i})$$

As σ is compatible with inductive limits, we have $Y_t(x) = (Y^i_t(x_i))$, that is to say, $Y_t = \varprojlim Y^i_t$ for each t. Y^i_t defined on $\mathcal{W}_i$ is a smooth map and therefore Y_t is a smooth map.

By the definition of compatibility of σ with the inductive maps, we have that each Y^i_t is μ-Lipscitz map defined on E_i. Therefore Y_t is defined on E.

We want to make sure that the isotopy of the time dependent vector field Y_t exists. For this we will use theorems 1.4.2 and 1.4.3. Comparing with the notation of the theorem 1.4.2, $\rho_i(Y_t(x)) := \|Y^i_t(x_i)\|_i$.

Here $\|.\|_i$ denotes norm on each E_i and we know that topology of $E = \varprojlim E_i$ is generated by countable norms $\{\|.\|_i : i \in \mathbb{N}\}$.

$$\begin{aligned}
\rho_i(Y_t(x)) \quad &= \|Y_t^i(x)\|_i \\
&= \|(((\sigma_{x_i}^i)^t)^b)^{-1}(\alpha_{x_i})\|_i \\
&\leq \|((\sigma_{x_i}^i)^{tb})^{-1}\|_{op}.\|\alpha_{x_i}^i\|_{op} \\
&\leq \|((\sigma_{x_i}^i)^{tb})^{-1}\|_{op}.\|X_i(x_i)\|_i \\
&\leq M.
\end{aligned}$$

We define $\tilde{Y}(x,t) := (Y_t(x), \frac{d}{dt})$ a vector field on $E \times \mathbb{R}$. Vector field $\tilde{Y}$ satisfies the condition of the theorem 1.4.3. Therefore, flow of time dependent vector field Y_t exists.

Each flow is defined for all $t \in [-1,1]$ and there exists an isotopy ϕ_t for the time dependent vector field Y_t.

We have:

$$\begin{aligned}
\frac{d}{dt}\phi_t^*\sigma_t &= \phi_t^*(L_{Y_t}\sigma_t) + \phi_t^*\frac{d}{dt}\sigma_t \\
&= \phi_t^*(-d\alpha + \overline{\sigma}) = 0
\end{aligned}$$

Hence we have $\phi_1^*\sigma_1 = \phi_0^*\sigma_0$.

This gives

$$\phi_1^*\sigma = \sigma_0$$

$\square$

This proves existence of Darboux chart.

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