ALGEBRAIC AND COMBINATORIAL METHODS IN ZERO-SUM PROBLEMS

By

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree/diploma at this or any other Institution/University.

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Others

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To my

Maa and Baba

The more we come out and do
good to others, the more our hearts
will be purified, and God will be in them.

Swami Vivekananda

Arise, awake and stop not
till the goal is reached.

Swami Vivekananda

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Eshita Mazumdar

Contents

SYNOPSIS

In this thesis some problems on combinatorial number theory, specially on "zero-sum problems", have been studied.

We start by defining some zero-sum constants.

By a sequence over a group G we mean a finite sequence of terms from G which is unordered and repetition of terms is allowed. We view sequences over group G as elements of the free abelian monoid $\mathcal{F}(G)$ and use multiplicative notation (so our notation is consistent with [29], [32], [37]).

For a sequence $S = g_1 \cdot \ldots \cdot g_l$, l is the length of the sequence. The sequence S is called a *zero-sum sequence* if $g_1 + \cdots + g_l = 0$, where 0 is the identity element of the group.

Now we come to the definitions of two fundamental combinatorial invariants, one is known as the Davenport constant and another the EGZ constant.

Davenport Constant: For a finite abelian group G (written additively) of order $|G|$, the Davenport constant $\mathsf{D}(G)$ is defined to be the smallest integer l such that every sequence $g_1 \cdot \ldots \cdot g_l$ has some non-empty sum of g_i being 0, the identity element of G.

For an abelian group $G = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$ with $n_1|n_2|\cdots|n_r$, one can show very easily that

$$1 + \sum_{i=1}^{r}(n_i - 1) \leq \mathsf{D}(G) \leq |G|.$$

In 1961 Erdős-Ginzburg-Ziv proved the following :

Theorem 0.0.0.1. *For a positive integer n, any sequence $a_1, \cdot \ldots \cdot, a_{2n-1}$ of $2n-1$ integers has a subsequence of n elements whose sum is 0 modulo n.*

This theorem is known as the EGZ theorem.

Based on the above theorem, the following two invariants for a finite abelian group will come into the picture:

EGZ constant: For a finite abelian group G of exponent n, EGZ constant $\mathsf{s}(G)$ is defined to be the least integer k such that any sequence $g_1 \cdot \ldots \cdot g_k$ has 0 as the sum of a subsequence of length n.

Another constant is the Gao constant $\mathsf{E}(G)$, which is defined as follows: The constant $\mathsf{E}(G)$ for a finite abelian group of order $|G|$ is defined to be the least integer k such that any sequence $g_1 \cdot \ldots \cdot g_k$ over G has 0 as a $|G|$-sum.

It has been seen in [28] that for any finite abelian group G,

$$\mathsf{E}(G) = n + \mathsf{D}(G) - 1.$$

One can notice also that for simplest finite abelian group of order n, i.e. $G = \mathbb{Z}_n$, $E(G) = \mathsf{s}(G)$.

This thesis has been divided into three parts.

In the first Chapter titled **"Modification of Griffiths Theorem"** we consider the following:

For a different set of weights A one can define the weighted *EGZ constant*, denoted by $\mathsf{s}_A(G)$ as follows. If G is a finite abelian group with $\exp(G) = n$, then for a non-empty subset A of $[1, n-1]$, one defines $\mathsf{s}_A(G)$ to be the least integer k such that any sequence S with length k of elements in G has an *A-weighted zero-sum subsequence* of length $\exp(G) = n$, i.e. for any sequence $x_1 \cdot \ldots \cdot x_k$ with $x_i \in G$, there exists a subset

$I \subset [1,k]$ with $|I| = n$ and, for each $i \in I$, some element $a_i \in A$ such that

$$\sum_{i \in I} a_i x_i = 0.$$

Clearly, for $A = \{1\}$, $\mathsf{s}_A(G) = \mathsf{s}(G)$.

We have seen that Griffiths proved in [35] the following result for odd integer n.

Theorem 0.0.0.2. (Griffiths theorem): *Let $n = p_1^{a_1} \cdots p_k^{a_k}$ be an odd integer and let $a = \sum_s a_s$. For each s, let $A_s \subset \mathbb{Z}_{p_s^{a_s}}$ be a subset with its size $|A_s| > p_s^{a_s}/2$, and let $A = A_1 \times \cdots \times A_k$. Then for $m > a$, every sequence $x_1 \cdot \ldots \cdot x_{m+a}$ over $\mathbb{Z}_n$ has 0 as an A-weighted m-sum.*

He also proved a similar kind of result for even n.

Modifying the method he used to prove the above result, we obtained in [10] the following result, which clearly shows that we are trying to generalize the condition on weight set A for a group $G = \mathbb{Z}_n$:

Theorem 0.0.0.3. *Let $n = p_1^{a_1} \cdots p_k^{a_k}$ be an odd integer and let $a = \sum_s a_s$. For each s, let $A_s \subset \mathbb{Z}_{p_s^{a_s}}$ be a subset with $|A_s| > (4/9)p_s^{a_s}$, and let $A = A_1 \times \cdots \times A_k$. Then for $m > 2a$, every sequence $x_1 \cdot \ldots \cdot x_{m+2a}$ over $\mathbb{Z}_n$ has 0 as an A-weighted m-sum.*

This result improves on a number of other results. We discuss these in Chapter I.

In the second Chapter **"Modification of a polynomial method of Rónyai"** we discuss the following:

For a finite abelian group G with $\exp(G) = n$, $\mathsf{s}_{mn}(G)$ is defined to be the least integer k such that any sequence S with length k of elements in G has a zero-sum subsequence of length mn. Putting $m = 1$, one observes that the constant $\mathsf{s}(G)$ is the same as $\mathsf{s}_n(G)$.

As before, for a non-empty subset A of $[1, n-1]$, one defines $\mathsf{s}_{mn,A}(G)$ to be the least integer k such that any sequence S with length k of elements in G has an A-weighted zero-sum subsequence of length mn.

We know from EGZ theorem that $\mathsf{s}_n(\mathbb{Z}_n^d)$ for $d = 1$ is $2n-1$. So one can ask similar question in higher dimension i.e. what is the value of $\mathsf{s}_n(\mathbb{Z}_n^d)$ for $d > 1$. For the case $d = 2$ Kemnitz conjecture was that $\mathsf{s}_n(\mathbb{Z}_n^2) = 4n - 3$, which has been proved by Reiher in [50].

In order to prove Kemnitz conjecture, it is enough to prove that $\mathsf{s}_p(\mathbb{Z}_p^2) = 4p - 3$ for all primes p, which was done by Reiher. Now one can be immediately ask for the value of $\mathsf{s}_n(\mathbb{Z}_n^d)$ for $d = 3$ or $d = 4$ also. However, it does not appear to be easy to evaluate the value of $\mathsf{s}_{n,A}(\mathbb{Z}_n^3)$ for $A = \{1\}$.

Keeping these things in mind, we fix our length to be either $2p$ or $3p$. Then we get some lower as well as upper bound for $\mathsf{s}_{3p,A}(\mathbb{Z}_p^3)$ and $\mathsf{s}_{2p,A}(\mathbb{Z}_p^3)$ for $A = \{1, -1\}$. Also we have bounds for $\mathsf{s}_{3p,A}(\mathbb{Z}_p^{2k})$ for $k \geq 3$ for different A. We discuss these in details in Chapter II. Here we give the theorems ([10]) which we prove in Chapter II rigorously:

Theorem 0.0.0.4. *For $A = \{\pm 1\}$, and an odd prime p, we have*

$$2p + 3\lfloor \log_2 p \rfloor \leq \mathsf{s}_{2p,A}(\mathbb{Z}_p^3) \leq \frac{(7p-3)}{2}.$$

Theorem 0.0.0.5. *For $A = \{\pm 1\}$, and an odd prime p, we have*

$$3p + 3\lfloor \log_2 p \rfloor \leq \mathsf{s}_{3p,A}(\mathbb{Z}_p^3) \leq \frac{(9p-3)}{2}.$$

To prove the theorem we use the polynomial method of Rónyai from [52].
By slight modification of the polynomial method of Rónyai we get in [11] a generalized result for $d = 2k$ i.e. for d even, which is the following:

Theorem 0.0.0.6. *Let p be an odd prime and $k \geq 3$ a divisor of $p-1$, θ an element of order k in $\mathbb{Z}_p^*$ and A the subgroup generated by θ. Then, we have*

$$\mathsf{s}_{3p,A}(\mathbb{Z}_p^{2k}) \leq 5p - 2.$$

In the third Chapter **"Relation between $\mathsf{s}_{\pm}(G)$ and $\eta_{\pm}(G)$"** we talk about a recently introduced invariant $\eta_A(G)$ for a finite abelian group G, which is defined to be the smallest positive integer t such that any sequence of length t of elements of G contains a non-empty A-weighted zero-sum subsequence of length at most $exp(G)$, this generalizes the constant $\eta(G)$, which correspond to the case $A = \{1\}$.

Similar to the relation between $\mathsf{E}(G)$ and $\mathsf{D}(G)$ discussed before we can talk about relation between the constants $\eta(G)$ and $\mathsf{s}(G)$. The conjecture of Gao et al. [30], that $\mathsf{s}(G) = \eta(G) + \exp(G) - 1$ holds for an abelian group G, was established in the case of rank at most two by Geroldinger and Halter-Koch [32].

Regarding the weighted analogue

$$\mathsf{s}_A(G) = \eta_A(G) + \exp(G) - 1,$$

for a finite cyclic group $G = \mathbb{Z}_n$, it coincides with a result established by Grynkiewicz et al. [38].

In Chapter III, we consider this analogue in the case of the weight $A = \{1, -1\}$ and try to see that whether equality holds or not. It has been observed by Moriya [46] that

$$\mathsf{s}_{\pm}(\mathbb{Z}_n \oplus \mathbb{Z}_n) > \eta_{\pm}(\mathbb{Z}_n \oplus \mathbb{Z}_n) + \exp(\mathbb{Z}_n \oplus \mathbb{Z}_n) - 1,$$

for an odd integer $n > 7$. However, he also proved that $\mathsf{s}_{\pm}(G) = \eta_{\pm}(G) + \exp(G) - 1$, for any abelian group G of order 8 and 16.

We have proved in our paper [12] that the result is also true for any abelian group of order 32. Then we make the following conjecture:

Conjecture 1. *The relation*

$$\mathsf{s}_{\pm}(G) = \eta_{\pm}(G) + \exp(G) - 1,$$

holds for any finite abelian 2-group G.

So in our Chapter III we discuss proof of the relation for the group of order 32 thoroughly. Going through all those things we shall see that $\mathsf{s}_{\pm}(\mathbb{Z}_2 \oplus \mathbb{Z}_{2n}) \leq 2n + \lceil \log_2 2n \rceil + 1$, and we will get a good lower bound for $\eta_{\pm}(\mathbb{Z}_2 \oplus \mathbb{Z}_{2n})$ i.e. we get the following result:

Lemma 0.0.0.7. *For positive integers r and n, we have*

$$\eta_{\pm}(\mathbb{Z}_2^r \oplus \mathbb{Z}_{2n}) \geq \max\left\{\lfloor \log_2 2n \rfloor + r + \left\lfloor \frac{r}{2n-1} \right\rfloor, r + A(r,n)\right\} + 1,$$

where

$$A(r,n) = \begin{cases} 1 & \text{if } r \leq n, \\ \lfloor \frac{r}{n} \rfloor & \text{if } r > n. \end{cases}$$

Chapter 0

Introduction

For a finite abelian group G (written additively), $\mathcal{F}(G)$ will denote a free abelian multiplicative monoid with basis G, and $\exp(G)$ will denote the exponent of G. A finite sequence S over G, is an element of $\mathcal{F}(G)$, i.e $S = (x_1, ..., x_t) = x_1 \cdot \ldots \cdot x_t$, where the repetition of elements of G is allowed and their order is disregarded (so our notation is consistent with [29], [32], [37]).

For $S \in \mathcal{F}(G)$, if

$$S = x_1 x_2 \cdot \ldots \cdot x_t = \prod_{g \in G} g^{\mathsf{v}_g(S)},$$

where $\mathsf{v}_g(S) \geq 0$ is the *multiplicity* of g in S,

$$|S| = t = \sum_{g \in G} \mathsf{v}_g(S)$$

is the *length* of S. The sequence S *contains* some $g \in G$ if $\mathsf{v}_g(S) \geq 1$.

If S and T are sequences over G, then T is said to be a *subsequence* of S if $\mathsf{v}_g(T) \leq \mathsf{v}_g(S)$ for every $g \in G$. If T is a subsequence of S, we write $T|S$ and ST^{-1} denotes the sequence obtained by deleting the terms of T from the sequence S.

The sequence S is called a zero-sum sequence if $x_1 + ... + x_t = 0$, the identity element of G. A typical direct zero-sum problem studies the conditions which ensure that given sequences have non-empty zero-sum subsequences with prescribed properties. The associated inverse zero-sum problem studies the structure of extremal sequences which have no such zero-sum subsequences.

Throughout this thesis for integers $m < n$, we shall use the notation $[m, n]$ to denote the set $\{m, m+1, \ldots, n\}$ and for a finite set A, we denote its size by $|A|$, which is the number of elements of A.

In this thesis, we mainly focus on some invariants associated to zero-sum problem for finite abelian groups. In general, sometimes it seems difficult to find out these invariants precisely. Under such circumstances we try to bound these constants or establish some identities so that we can guess the nature of the constants.

This thesis contains the detailed discussion about different types of zero-sum constants. The natural question arises why one should bother about these constants. First of all, we try to give an answer to this question.

0.1 Introduction of the Davenport Constant:

For a finite abelian group G, the *Davenport constant* $D(G)$ is defined to be the smallest natural number k such that any sequence of length k over G has a non-empty zero-sum subsequence. It should be remarked that K. Rogers [51] was the first one to work on this constant and his work was missed out by most of the authors in this area.

The motivation behind the introduction of the constant $D(G)$ by Roger and Davenport is as follows;

Let K be a finite field extension over $\mathbb{Q}$ and $\mathcal{O}_K$ be its ring of integers. Let $x \in \mathcal{O}_K$

be an irreducible element. Then, it is known that the principal ideal

$$x\mathcal{O}_K = \mathcal{P}_1 \cdots \mathcal{P}_\ell$$

can be factored into product of prime ideals in $\mathcal{O}_K$ where $\mathcal{P}_i$ is a prime ideal in $\mathcal{O}_K$. There arises a very natural question: *What is the upper bound for ℓ? That is, in the factorization of the principal ideal $x\mathcal{O}_K$, how many prime ideals occur?*

Let $\mathcal{C}_K$ denote the class group of $\mathcal{O}_K$. Since $\mathcal{C}_K$ is an abelian group one can find out the Davenport constant for $\mathcal{C}_K$. The length ℓ of the decomposition of $x\mathcal{O}_K$ into prime ideals is bounded above by $D(\mathcal{C}_K)$. For otherwise, suppose $\ell > D(\mathcal{C}_K)$. It is well-known result that every ideal class contains a prime ideal representation. Hence, the prime ideals occuring in the factorization of $x\mathcal{O}_K$ are the elements of group $\mathcal{C}_K$. Since principal ideal $x\mathcal{O}_K$ represents the zero element of the class group $\mathcal{C}_K$, the prime ideals product $\mathcal{P}_1 \cdots \mathcal{P}_\ell$ is the zero element of $\mathcal{C}_K$. Since $\ell \geq D(\mathcal{C}_K) + 1$, we can consider $\mathcal{P}_1, \ldots, \mathcal{P}_{D(\mathcal{C}_K)}$. By the definition of $D(\mathcal{C}_K)$, there is a subsequence $\mathcal{P}_{i_1}, \ldots, \mathcal{P}_{i_r}$ whose product is zero in $\mathcal{O}_K$. That means, there exists $y \in \mathcal{O}_K$ which is not a unit in $\mathcal{O}_K$ such that

$$\mathcal{P}_{i_1} \cdots \mathcal{P}_{i_r} = y\mathcal{O}_K.$$

Since

$$x\mathcal{O}_K = \mathcal{P}_1 \cdots \mathcal{P}_\ell,$$

we conclude that there exists $z \in \mathcal{O}_K$ such that

$$x\mathcal{O}_K = y\mathcal{O}_K z\mathcal{O}_K = (yz)\mathcal{O}_K.$$

This means that y divides x and y is not a unit, a contradiction to the irreducibility of x.

Hence $\ell \leq D(\mathcal{C}_K)$.

This is just the initial motivation for the Davenport constant. There were other applications as well.

Given a finite abelian group $G = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_r}$, with $n_1|n_2|\cdots|n_r$, we define $D^*(G) = 1 + \sum_{i=1}^{r}(n_i - 1)$. It is trivial to see that $D^*(G) \leq D(G) \leq |G|$. Olson [48] [49] proved that $D(G) = D^*(G)$ when G is of rank 2 or it is a p-group. It is also known that $D(G) > D^*(G)$ for infinitely many finite abelian groups of rank $d > 3$ (see [34], for instance). The following general upper bound for $D(G)$ was obtained by Emde Boas and Kruyswijk [21] and Meshulam [45]; Alford, Granville and Pomerance [14] gave another proof which involves a generalization of Olson's proof of $D(G) = D^*(G)$ for finite abelian p-groups.

Theorem 0.1.0.1. *We have*

$$D(G) \leq n\left(1 + \log\frac{|G|}{n}\right),$$

where $n = \exp(G)$.

The precise value of this constant in terms of the group invariants of G is still unknown. But it is very interesting to see that in a similar manner one can define various combinatorial invariants, which are very useful and sometimes it is easy to find them.

0.2 Constant E(G):

A significant part in the study of zero-sum problem includes Erdős-Ginzburg-Ziv (i.e EGZ) theorem which is the following:

Theorem 0.2.0.1. (EGZ Theorem) *For a positive integer n, any sequence $a_1, \cdot \ldots \cdot, a_{2n-1}$ of $2n-1$ integers has a subsequence of n elements whose sum is 0 modulo n.*

There is an another combinatorial constant associated to a finite abelian group defined in the following way:

The constant $\mathsf{E}(G)$ for a finite abelian group G of order $|G|$ is defined to be the least integer k such that any sequence $g_1 \cdot \ldots \cdot g_k$ over G has 0 the identity element, as a sum of a subsequence of length $|G|$. Therefore the Erdős-Ginzburg-Ziv theorem (1961) shows that $\mathsf{E}(\mathbb{Z}_n) \leq 2n-1$. One can notice that sequence $0^{(n-1)}1^{(n-1)}$ in $\mathbb{Z}_n$ does not have a subsequence of length n, whose sum will be 0. So, $\mathsf{E}(\mathbb{Z}_n) \geq 2n-1$. Therefore, $\mathsf{E}(\mathbb{Z}_n) = 2n-1$.

It is very easy to see that $\mathsf{D}(\mathbb{Z}_n) = n$. So one can conclude that $\mathsf{E}(\mathbb{Z}_n) = n+\mathsf{D}(\mathbb{Z}_n)-1$. Remarkably the same is true for any finite abelian group G, namely, $\mathsf{E}(G) = |G|+\mathsf{D}(G)-1$. It is trivial that $\mathsf{E}(G) \geq |G| + \mathsf{D}(G) - 1$, because given a sequence $g_1 \cdot \ldots \cdot g_{D-1}$ with no non-empty sum being 0, we adjoin $|G|-1$ zeros to this sequence and obtain a sequence of length $|G|+D-2$ without 0 as an $|G|$-sum (i.e as a sum of a subsequence of length $|G|$). The reverse inequality has been proved by Gao [28]. Therefore the problem of finding out $\mathsf{D}(G)$ is reduced to that of finding $\mathsf{E}(G)$.

0.2.1 EGZ Constant:

For a finite abelian group G of exponent n, the EGZ constant $\mathsf{s}(G)$ is defined to be the least positive integer k such that any sequence $g_1 \cdot \ldots \cdot g_k \in \mathcal{F}(G)$ has 0 as an n-sum. For the simplest finite abelian group $\mathbb{Z}_n$ the value of $\mathsf{s}(G)$ and $\mathsf{E}(G)$ is same trivially.

One can further generalize the constant $\mathsf{s}(G)$ to $\mathsf{s}_A(G)$ for different weight sets $A \subseteq \{1, 2, \cdots, \exp(G)\}$. $\mathsf{s}_A(G)$ defined as follows: If G is a finite abelian group with $\exp(G) = n$, then for a non-empty subset A of $[1, n-1]$, $\mathsf{s}_A(G)$ is the least integer k such that any sequence S with length k of elements in G has an A-*weighted zero-sum subsequence* of length n, that is, for any sequence $x_1 \cdot \ldots \cdot x_k$ with $x_i \in G$, there exists a subset $I \subset [1, k]$ with $|I| = n$ and, for each $i \in I$, some element $a_i \in A$ such that

$$\sum_{i \in I} a_i x_i = 0.$$

For $A = \{1\}$, $\mathsf{s}_A(G) = \mathsf{s}(G)$, and we call a sequence to be a zero-sum sequence if the sum of their elements will be the identity element of the group.

The above weighted version and some other invariants with weights were initiated by Adhikari, Chen, Friedlander, Konyagin and Pappalardi [6], Adhikari and Chen [5] and Adhikari, Balasubramanian, Pappalardi and Rath [3]. People are very interested in finding out the precise value of the constant $\mathsf{s}_A(G)$ for different groups G and weight set A. But it seems very difficult to find out these constants precisely. Therefore, it is natural to look for good bounds to $\mathsf{s}_A(G)$. For the developments on the bounds of $\mathsf{s}_A(G)$ in the case of abelian groups with higher rank and related references, one may look into the recent paper of Adhikari, Grynkiewicz and Sun [8].

0.3 Modification of Griffiths Theorem:

In the Chapter 1 of the thesis, we mainly concern about the value of $\mathsf{s}_A(G)$ keeping the group G fixed and varying the weight set A. For the group $G = \mathbb{Z}_n$ and $A = \mathbb{Z}_n^* = \{a \in [1, n-1] | (a, n) = 1\}$, the set of units of $\mathbb{Z}_n$, Luca [43] and Griffiths [35] independently proved the following result which had been conjectured in [6]:

$$\mathsf{s}_A(\mathbb{Z}_n) \leq n + \Omega(n),$$

where $\Omega(n)$ denotes the number of prime factors of n, counted with multiplicity.

0.3.1 Griffiths Theorem

Griffiths in his paper [35] proved the folllowing result for an odd integer n.

Theorem 0.3.1.1. *Let $n = p_1^{a_1} \cdots p_k^{a_k}$ be an odd integer and let $a = \sum_s a_s$. For each s, let $A_s \subset \mathbb{Z}_{p_s^{a_s}}$ be a subset with its size $|A_s| > p_s^{a_s}/2$, and let $A = A_1 \times \cdots \times A_k$. Then for $m > a$, every sequence $x_1 \cdot \ldots \cdot x_{m+a}$ over $\mathbb{Z}_n$ has 0 as an A-weighted m-sum, i.e every sequence $x_1 \cdot \ldots \cdot x_{m+a}$ over $\mathbb{Z}_n$ has a subsequence of length m whose A-weighted sum will be* 0.

He also proved a similar result for n even, which is the following:

Theorem 0.3.1.2. *Let $n = 2^{a_1} \cdots p_k^{a_k}$ be an even integer and let $a = \sum_s a_s$. Let $A_1 \in \mathbb{Z}_{2_1^a}$ be such that either $|A_1| > 2^{a_1-1}$ or $|A_1| > 2^{a_1-2}$ and $A_1 \subset \mathbb{Z}^*_{2^{a_1}}$. For each $s \geq 2$, let $A_s \subset \mathbb{Z}_{p_s^{a_s}}$ be a subset with its size $|A_s| > p_s^{a_s}/2$, and let $A = A_1 \times \cdots \times A_k$. Then for even $m > a$, every sequence $x_1 \cdot \ldots \cdot x_{m+a}$ over $\mathbb{Z}_n$ has 0 as an A-weighted m-sum.*

The following three lemmas and the observation have been used as tools to prove Griffiths theorem for odd n in [35].

Lemma 0.3.1.3. *Let p^a be an odd prime power and $A \subset \mathbb{Z}_{p^a}$ be a subset such that $|A| > p^a/2$. If $x, y \in \mathbb{Z}^*_{p^a}$, the group of units in $\mathbb{Z}_{p^a}$, then given any $t \in \mathbb{Z}_{p^a}$, there exist $\alpha, \beta \in A$ such that*

$$\alpha x + \beta y = t.$$

Lemma 0.3.1.4. *Let p^a be an odd prime power and let $A \subset \mathbb{Z}_{p^a}$ be such that $|A| > p^a/2$. Let $x_1 \cdot \ldots \cdot x_m$ be a sequence over $\mathbb{Z}_{p^a}$ such that for each $b \in [1, a]$, $|T_b| \neq 1$ where $T_b = \{i | x_i \neq 0 \pmod{p^b}\}$. Then $x_1 \cdot \ldots \cdot x_m$ is an A-weighted zero-sum sequence.*

Lemma 0.3.1.5. *Given subsets $X_1, \cdots, X_a$ of the set $V = [1, m+a]$, where $m > a$, there exists a set $I \subset [1, m+a]$ with $|I| = m$ and $|I \cap X_s|$ is not equal to 1, for all $s = 1, \cdots, a$.*

Observation: If $n = p_1^{a_1} \cdots p_k^{a_k}$ and $A = A_1 \times A_2 \times \cdots \times A_k$ is a subset of $\mathbb{Z}_n$, where $A_s \subset \mathbb{Z}_{p_s^{a_s}}$ for each $s \in [1, k]$, then a sequence of $x_1 \cdot \ldots \cdot x_m$ over $\mathbb{Z}_n$ is an A-weighted zero-sum sequence in $\mathbb{Z}_n$ if and only if for each $s \in [1, k]$, the sequence $x_1^{(s)} \cdot \ldots \cdot x_m^{(s)}$ is an A_s-weighted zero-sum sequence in $\mathbb{Z}_{p_s^{a_s}}$.

In our Chapter 1 we completely focus on Griffiths theorem 0.3.1.1. We observe that this result of Griffiths can be modified by closely following his method, and the modified version is better in some sense that we discuss that in Chapter 1 elaborately.

Following is the modified version [10] of Griffiths Theorem:

Theorem 0.3.1.6. *Let $n = p_1^{a_1} \cdots p_k^{a_k}$ be an odd integer and let $a = \sum_s a_s$. For each s, let $A_s \subset \mathbb{Z}_{p_s^{a_s}}$ be a subset with $|A_s| > (4/9)p_s^{a_s}$, and let $A = A_1 \times \cdots \times A_k$. Then for $m > 2a$, every sequence $x_1 \cdot \ldots \cdot x_{m+2a}$ over $\mathbb{Z}_n$ has 0 as an A-weighted m-sum.*

0.4 Relation between two combinatorial constants:

In Chapter 3 we introduce a new invariant $\eta_A(G)$ and discuss the relation between $\eta_A(G)$ and $\mathsf{s}_A(G)$. The invariant $\eta_A(G)$ is defined to be the smallest positive integer t such that any sequence of length t of elements of G contains a non-empty A-weighted zero-sum subsequence of length at most $\exp(G) = n$. These generalizes the constant $\eta(G)$ which corresponds to the case $A = \{1\}$.

For the group $G = \mathbb{Z}_n$, $\eta(G) = D(G)$. We have seen earlier in Section 0.2 the relation between $\mathsf{E}(G)$ and $\mathsf{D}(G)$. The immediate question arises whether there exists a similar relationship between $\eta(G)$ and $\mathsf{s}(G)$. The conjecture of Gao et al. [30] that

$\mathsf{s}(G) = \eta(G) + \exp(G) - 1$ holds for a finite abelian group G was established for the groups of rank at most two by Geroldinger and Halter-Koch [32].

One can consider the following weighted analogue of the above mentioned conjecture.

$$\mathsf{s}_A(G) = \eta_A(G) + \exp(G) - 1,$$

For a finite cyclic group $G = \mathbb{Z}_n$, it coincides with a result established by Grynkiewicz et al. [38].

In Chapter 3, we consider this analogue for the weight $A = \{\pm 1\}$ and try to see whether equality holds or not. In the case of the weight $A = \{1, -1\}$ or $\{\pm 1\}$ we write $\eta_\pm(G)$ for $\eta_{\{1,-1\}}(G)$ and $\mathsf{s}_\pm(G)$ for $\mathsf{s}_{\{1,-1\}}(G)$ respectively. It has been observed by Moriya [46] that

$$\mathsf{s}_\pm(\mathbb{Z}_n \oplus \mathbb{Z}_n) > \eta_\pm(\mathbb{Z}_n \oplus \mathbb{Z}_n) + \exp(\mathbb{Z}_n \oplus \mathbb{Z}_n) - 1,$$

for an odd integer $n > 7$. However he also proved that $\mathsf{s}_\pm(G) = \eta_\pm(G) + \exp(G) - 1$ for any abelian group G of order 8 and 16. In [12] we prove that the same relation is true for any abelian group of order 32 also. Then we make the following conjecture:

Conjecture 2. *The relation*

$$\mathsf{s}_\pm(G) = \eta_\pm(G) + \exp(G) - 1,$$

holds for any finite abelian 2*-group* G*.*

In Chapter 3, we provide a detailed proof of the theorem. While going through all those things, we noticed that $\mathsf{s}_\pm(\mathbb{Z}_2 \oplus \mathbb{Z}_{2n}) \leq 2n + \lceil \log_2 2n \rceil + 1$, and we get a good lower bound for $\eta_\pm(\mathbb{Z}_2 \oplus \mathbb{Z}_{2n})$ i.e we get the following result:

Lemma 0.4.0.1. *For positive integers r and n, we have*

$$\eta_{\pm}(\mathbb{Z}_2^r \oplus \mathbb{Z}_{2n}) \geq \max\left\{\lfloor \log_2 2n \rfloor + r + \left\lfloor \frac{r}{2n-1} \right\rfloor, r + A(r,n)\right\} + 1,$$

where

$$A(r,n) = \begin{cases} 1 & \text{if } r \leq n, \\ \lfloor \frac{r}{n} \rfloor & \text{if } r > n. \end{cases}$$

0.5 Higher dimensional analogue

Keeping in mind the previous definitions which one now we will introduce a new invariant. For a finite abelian group G with $\exp(G) = n$, $\mathsf{s}_{mn}(G)$ is defined to be the least integer k such that any sequence S with length k of elements in G has a zero-sum subsequence of length mn. One observes that the constant $\mathsf{s}(G)$ is the same as $\mathsf{s}_n(G)$ for $m = 1$.

As before, for a non-empty subset A of $[1, n-1]$, one defines $\mathsf{s}_{mn,A}(G)$ to be the least integer k such that any sequence S with length k of elements in G has an A-weighted zero-sum subsequence of length mn. Substituting $m = 1$ and $A = \{1\}$, we will have $\mathsf{s}(G) = \mathsf{s}_{n,\{1\}}(G)$.

Clearly, these $\mathsf{s}_{mn}(G)$ and $\mathsf{s}_{mn,A}(G)$ are generalised version of $\mathsf{s}(G)$. Now as we know the EGZ theorem can be derived from many different ways. One of them is the following theorem:

Theorem 0.5.0.1 (Alon)**.** *Let $A = [a_{ij}]$ be an $n \times n$ matrix over a field F such that* per $A \neq 0$. *Then for any vector $c = (c_1, \ldots, c_n) \in F^n$, and any family of sets $S_1, \ldots, S_n$ of F, each of cardinality 2, there is $s = (s_1, \ldots, s_n) \in S_1 \times S_2 \times \cdots \times S_n$ such that for every i, the ith coordinate of As differs from c_i.*

where, given an n by n matrix $A = (a_{ij})$ over a field, its *permanent*, denoted by per A, is defined by

$$\text{per } A = \sum a_{1\ \alpha(1)} a_{2\ \alpha(2)} \cdots a_{n\ \alpha(n)},$$

where the summation is taken over all permutations α of $[1, n]$. This derivation is one of the five proofs presented by Alon and Dubiner in [17]. A slight modification of this application of Alon's Theorem was later employed in [9] to obtain a result on weighted versions of some zero-sum constants.

We know from EGZ theorem that $\mathsf{s}_n(\mathbb{Z}_n) = 2n - 1$. A higher dimensional analogue to EGZ theorem is $\mathsf{s}_n(\mathbb{Z}_n^d) =?$ for $d > 1$. Since the number of elements of $\mathbb{Z}_n^d$ having coordinates 0 or 1 is 2^d, considering a sequence where each of these elements are repeated $(n-1)$ times, one observes that $1 + 2^d(n-1) \leq \mathsf{s}_n(\mathbb{Z}_n^d)$. Again, observing that in any sequence of $1 + n^d(n-1)$ elements of $\mathbb{Z}_n^d$ will have at least one vector appearing at least n times, so we have

$$1 + 2^d(n-1) \leq \mathsf{s}_n(\mathbb{Z}_n^d) \leq 1 + n^d(n-1).$$

For $d = 2$, Kemnitz conjecture [41] tells that $\mathsf{s}_n(\mathbb{Z}_n^2) = 4n - 3$, which has been proved by Reiher in [50].

In view of proving Kemnitz conjecture, it is enough to prove that $\mathsf{s}_p(\mathbb{Z}_p^2) = 4p-3$, for all primes p and that is what Reiher did. Now one can immediately get interested in finding out the value of $\mathsf{s}_n(\mathbb{Z}_n^d)$ for $d > 2$. But finding out this value seems very challenging.

Keeping those things in mind we are able to give some bound to $\mathsf{s}_{3p,A}(\mathbb{Z}_p^3)$ for $A = \{1, -1\}$, which is the following:

Theorem 0.5.0.2. *For $A = \{\pm 1\}$, and an odd prime p, we have*

$$3p + 3\lfloor \log_2 p \rfloor \leq \mathsf{s}_{3p,A}(\mathbb{Z}_p^3) \leq \frac{(9p-3)}{2}.$$

Details of the result has been discussed in Chapter 2. To prove the theorem we use the polynomial method of Rónyai in [52]. Slightly modifying the polynomial method of Rónyai, we get the following generalized result for $d = 2k$ i.e for d even.

Theorem 0.5.0.3. *Let p be an odd prime and $k \geq 3$, be a divisor of $p-1$, θ be an element of order k in $\mathbb{Z}_p^*$ and A be the subgroup generated by θ. Then, we have*

$$3p + 2k \leq \mathsf{s}_{3p,A}(\mathbb{Z}_p^{2k}) \leq 5p - 2.$$

We have lot of observations related to these results all of those we will discuss in the Chapter 2 in detail.

Chapter 1

Modification of Griffiths Theorem

1.1 Introduction:

The principal goal of this chapter is to modify a beautiful result of Griffiths and also to see an improvement of the theorem. In the theory of zero-sum problem, the main object of study are the sequences of elements from an abelian group G. However, we will generally not need that our sequences be ordered, only that they allow multiple repetition of elements. Thus we adopt the algebraic viewpoint of considering sequences over G as elements of the free abelian monoid $\mathcal{F}(G)$.

One of the most fundamental and useful theorems in Inverse Additive Theory is Kneser's theorem. This theorem plays a vital role in proving the main theorem of this chapter. In this theorem we are interested in determining the structure of A and B with small sumset. Let us state the theorem.

We need the following definitions to state the theorem. For a non-empty subset A of an abelian group G, the stabilizer of A, denoted by $Stab(A)$, is defined as:

$$Stab(A) = \{x \in G : x + A = A\}.$$

One can easily see that $0 \in Stab(A)$ and $Stab(A)$ is a subgroup of G. Moreover, $Stab(A)$ is a largest subgroup of G such that $Stab(A)+A = A$. In particular, $Stab(A) = G$ if and only if $A = G$.

Following is known as Kneser Theorem.

Theorem 1.1.0.1. *Let G be an abelian group, $G \neq \{0\}$, and let A and B be non-empty finite subsets of G. If $|A| + |B| \leq |G|$ then $|A + B| \geq |A + H| + |B + H| - |H|$, where $H = Stab(A + B)$ is the stabilizer of $A + B$.*

We know that the simplest finite abelian group of exponent n is $G = \mathbb{Z}_n$. In 1961 it was proved by Erdős, Ginzburg and Ziv that $\mathsf{s}(\mathbb{Z}_n) = 2n - 1$. So, the immediate question that arises, what will be the value of $\mathsf{s}_A(G)$ for an abelian group G with $\exp(G) = n$ and weight set $A \subseteq [1, n-1]$. For $G = \mathbb{Z}_n$ we have the following results. For $A = \mathbb{Z}_n \setminus \{0\}$, $\mathsf{s}_A(\mathbb{Z}_n) = n + 1$ (see [6]), and for $A = \mathbb{Z}_n^* = \{a \in [1, n-1] | (a, n) = 1\}$, Luca [43] and Griffiths [35] independently proved the following result which had been conjectured in [6]:

$$\mathsf{s}_A(\mathbb{Z}_n) \leq n + \Omega(n), \tag{1.1}$$

where $\Omega(n)$ denotes the number of prime factors of n, counted with multiplicity.

One can notice very easily that for $A = \mathbb{Z}_n^*$ if n is the product of a (not necessarily distinct) primes $n = q_1 \cdot \ldots \cdot q_a$ (so that $\Omega(n) = a$) then the sequence $1, q_1, q_1q_2, \cdot \ldots \cdot, q_1 .. q_{a-1}$ has no non-empty subset with a weighted sum to 0. Thus adjoining 0's $(n-1)$ times we have a sequence of length $n+\Omega(n)-1$ which does not have subsequence of length n whose A-weighted sum will be 0. Therefore

$$\mathsf{s}_A(\mathbb{Z}_n) \geq n + \Omega(n)$$

so that we can conclude that $\mathsf{s}_A(\mathbb{Z}_n) = n + \Omega(n)$.

When $n = p$, a prime, and A is the set of quadratic residues modulo p, Adhikari and Rath [13] proved that

$$\mathsf{s}_A(\mathbb{Z}_p) = p + 2.$$

For a general n, considering the set A of squares in the group of units in the cyclic group $\mathbb{Z}_n$, it was proved by Adhikari, Chantal David and Urroz [7] that if n is a square-free integer, coprime to 6, then

$$\mathsf{s}_A(\mathbb{Z}_n) = n + 2\Omega(n). \tag{1.2}$$

Later, removing the requirement that n is square-free Chintamani and Moriya [22] showed that if n is a power of 3 or coprime to $30 = 2 \times 3 \times 5$, then the result (1.2) holds, where A is again the set of squares in the group of units in $\mathbb{Z}_n$.

For the weight set $A = \{\pm 1\}$, Adhikari, Chen, Friedlander, Konyagin and Pappalardi [6] proved that $\mathsf{s}_A(\mathbb{Z}_n) = n + \lfloor \log_2 n \rfloor$ for any positive integer n. For the same weight set and $G = \mathbb{Z}_n^2$ Adhikari, Balasubramanian, Pappalardi and Rath [3] proved that $\mathsf{s}_A(G) = 2n - 1$ for n to be odd.

Later, it was proved by Adhikari, Grynkiewicz and Zhi-Wei Sun [8] that for any finite abelian group G of rank r and even exponent there exists a constant k_r, depending only on r, such that

$$\mathsf{s}_{\{\pm 1\}}(G) \leq \exp(G) + \log_2 |G| + k_r \log_2 \log_2 |G|.$$

Now we state the following result of Griffiths [35] which generalizes the result (1.1) for an odd integer n:

Theorem 1.1.0.2. *Let $n = p_1^{a_1} \cdots p_k^{a_k}$ be an odd integer and let $a = \sum_s a_s$. For each s, let $A_s \subset \mathbb{Z}_{p_s^{a_s}}$ be a subset with its size $|A_s| > p_s^{a_s}/2$, and let $A = A_1 \times \cdots \times A_k$. Then for $m > a$, every sequence $x_1 \cdot \ldots \cdot x_{m+a}$ over $\mathbb{Z}_n$ has 0 as an A-weighted m-sum.*

Griffiths [35] also had a similar result when n is even; we only need to mention the case when n is odd.

With suitable modifications of the method used in [35], we establish the following result:

Theorem 1.1.0.3. *Let $n = p_1^{a_1} \cdots p_k^{a_k}$ be an odd integer and let $a = \sum_s a_s$. For each s, let $A_s \subset \mathbb{Z}_{p_s^{a_s}}$ be a subset with $|A_s| > (4/9)p_s^{a_s}$, and let $A = A_1 \times \cdots \times A_k$. Then for $m > 2a$, every sequence $x_1 \cdot \ldots \cdot x_{m+2a}$ over $\mathbb{Z}_n$ has 0 as an A-weighted m-sum.*

In this whole chapter we will discuss about this main theorem and its proof.

1.2 Comparison with some earlier results:

Before going to the proof the theorem 1.1.0.3 we give a comparison of this result with earlier ones. In 1961 Erdős-Ginzburg-Ziv supply a details of proof of the fact that if $a_1, a_2, \ldots, a_{2n-1}$ are integers, then there exists a set $I \subseteq \{1, 2, \ldots, 2n-1\}$ with $|I| = n$ such that

$$\sum_{i \in I} a_i \equiv 0 \pmod{n}.$$

Then weighted version of this problem was initiated by Adhikari, Chen, Friedlander, Konyagin and Papapalardi in [6].

Modifying in 2007 Luca [43] proved that if $a_1, \ldots, a_{n+\Omega(n)}$ are integers, then there exists a subset $M \subseteq \{1, \ldots, n + \Omega(n)\}$ with $|M| = n$ such that the equation

$$\sum_{i \in M} a_i x_i \equiv 0 \pmod{n}.$$

admits a solution $x_i \in U(\mathbb{Z}_n)$, where $U(\mathbb{Z}_n)$ stands for the multiplicative group modulo n and $\Omega(n) = |U(\mathbb{Z}_n)|$.

In 2008 for any odd n Griffiths [35] generalizes the result (1.1) by using some graphical method.

Modifying the Griffiths theorem we show that for $n = p_1^{a_1} \cdots p_k^{a_k}$ to be an odd integer and $a = \sum_s a_s$, if A is same as in the statement of Theorem 1.1.0.3 then by using the theorem 1.1.0.3, since $n \geq 3^a > 2a$, it follows that any sequence of length $n + 2a$ of elements of $\mathbb{Z}_n$ has 0 as an A-weighted n-sum. In other words, $\mathsf{s}_A(\mathbb{Z}_n) \leq n + 2\Omega(n)$.

Clearly, Theorem 1.1.0.3 covers many subsets $A = A_1 \times \cdots \times A_k$ with $A_s \subset \mathbb{Z}_{p_s^{a_s}}$, which were not covered by the result of Griffiths. We proceed to give one such example where it determines the exact value of $\mathsf{s}_A(\mathbb{Z}_n)$.

When $n = p$, a prime, and A is the set of quadratic residues modulo p, it attains the upper bound i.e

$$\mathsf{s}_A(\mathbb{Z}_p) = p + 2. \tag{1.3}$$

For general n, considering the set A of squares in the group of units in the cyclic group $\mathbb{Z}_n$, that if n is a square-free integer, coprime to 6, then

$$\mathsf{s}_A(\mathbb{Z}_n) = n + 2\Omega(n). \tag{1.4}$$

Later, removing the requirement that n is a square-free, Chintamani and Moriya [22] showed that if n is a power of 3 or n is coprime to $30 = 2\times3\times5$, then the result (1.2) holds, where A is again the set of squares in the group of units in $\mathbb{Z}_n$. However, Chintamani and Moriya [22] had only to prove that $\mathsf{s}_A(\mathbb{Z}_n) \leq n+2\Omega(n)$ as the corresponding inequality in the other direction for odd n (and so for n coprime to 30) had already been established by Adhikari, Chantal David and Urroz [7]. We mention that a lower bound for $\mathsf{s}_A(\mathbb{Z}_n)$ when n is even has been given by Grundman and Owens [36].

Considering an odd integer $n = p_1^{a_1} \cdots p_k^{a_k}$, the set A of squares in the group of units in $\mathbb{Z}_n$ is $A = A_1 \times \cdots \times A_k$, where A_s is the set of squares in the group of units in $\mathbb{Z}_{p_s^{a_s}}$ and it satisfies $|A_s| = \frac{p_s^{a_s}}{2}\left(1 - \frac{1}{p_s}\right)$. Observing that $\frac{1}{2}\left(1 - \frac{1}{p_s}\right) > (4/9)$, if $p_s \geq 11$, Theorem 1.1.0.3 gives the required upper bound in the above mentioned result of Chintamani and Moriya [22] when n is coprime to $2 \times 3 \times 5 \times 7$.

1.3 Proof of Theorem 1.1.0.3

For the proof of Theorem 1.1.0.3, we shall closely follow the method of Griffiths [35]. Though we shall be able to use many of the ideas in [35] with straight forward modifications, some modifications need some work and some new observations have to be made to make things work.

We start by proving couple of lemmas which we shall use in the proof of the main theorem.

Lemma 1.3.0.1. *Let p^a be an odd prime power and $A \subset \mathbb{Z}_{p^a}$ be a subset such that $|A| > \frac{4}{9}p^a$. If $x, y, z \in \mathbb{Z}_{p^a}^*$, the group of units in $\mathbb{Z}_{p^a}$, then given any $t \in \mathbb{Z}_{p^a}$, there exist $\alpha, \beta, \gamma \in A$ such that*

$$\alpha x + \beta y + \gamma z = t.$$

Proof: Considering the sets

$$A_1 = \{\alpha x : \alpha \in A\},\ B_1 = \{\beta y : \beta \in A\},\ C_1 = \{\gamma z : \gamma \in A\},$$

and observing that $|A_1| = |B_1| = |C_1| = |A|$, by Kneser's theorem ([42], may also see Chapter 4 of [47]) we have

$$|A_1 + B_1| \geq |A_1| + |B_1| - |H| > \frac{8p^a}{9} - |H|, \tag{1.5}$$

where $H = H(A_1 + B_1)$ is the stabilizer of $A_1 + B_1$.

Now, $H = \mathbb{Z}_{p^a}$ would imply $A_1 + B_1 = \mathbb{Z}_{p^a}$, which in turn would imply that $A_1 + B_1 + C_1 = \mathbb{Z}_{p^a}$ and we are through.

Otherwise, $|H|$ being a power of an odd prime $p \geq 3$, we have

$$|H| \leq p^{a-1} = \frac{p^a}{p} \leq \frac{p^a}{3}$$

and hence from (1.5),

$$|A_1 + B_1| > \frac{8p^a}{9} - \frac{p^a}{3} = \frac{5p^a}{9}.$$

Therefore, we have

$$|A_1 + B_1| + |t - C_1| > \frac{5p^a}{9} + \frac{4p^a}{9} = p^a,$$

which implies that the sets $A_1 + B_1$ and $t - C_1$ intersect and we are through. $\square$

Lemma 1.3.0.2. *Let p^a be an odd prime power and let $A \subset \mathbb{Z}_{p^a}$ be such that $|A| > (4/9)p^a$. Let $x_1 \cdot \ldots \cdot x_m$ be a sequence over $\mathbb{Z}_{p^a}$ such that for each $b \in [1, a]$, writing $T_b = \{i | x_i \neq 0 \pmod{p^b}\}$, its cardinality $|T_b| \notin \{1, 2\}$. Then $x_1 \cdot \ldots \cdot x_m$ is an A-weighted zero-sum sequence.*

Proof: Let c be minimal such that $\{i | x_i \neq 0 \pmod{p^c}\}$ is non-empty. If no such c exists then $T_b = \emptyset$ for all b and we are done. Therefore, $\{i | x_i \neq 0 \pmod{p^c}\}$ has at least three elements; without loss of generality let $x_1, x_2, x_3 \neq 0 \pmod{p^c}$.

Set

$$x_i' = x_i / p^{c-1} \in \mathbb{Z}_{p^{a-(c-1)}},$$

for $i \in [1, m]$. If elements of A meets less than $(4/9)p^{a-(c-1)}$ congruence classes modulo $p^{a-(c-1)}$, then $|A| < (4/9)p^{a-(c-1)} \times p^{(c-1)} = (4/9)p^a$, which is a contradiction to our assumption. Therefore, the elements of A must meet more than $(4/9)p^{a-(c-1)}$ congruence classes modulo $p^{a-(c-1)}$.

Picking up arbitrarily $\alpha_4, \alpha_5, \cdots, \alpha_m \in A$, by Lemma 1.3.0.1, there exist $\alpha_1, \alpha_2, \alpha_3 \in A$ satisfying

$$\alpha_1 x_1' + \alpha_2 x_2' + \alpha_3 x_3' = -\alpha_4 x_4' - \cdots - \alpha_m x_m'$$

in $\mathbb{Z}_{p^{a-(c-1)}}$, and hence

$$\alpha_1 x_1 + \cdots + \alpha_m x_m = 0$$

in $\mathbb{Z}_{p^a}$. $\square$

Let $n = p_1^{a_1} \cdots p_k^{a_k}$. Then, $\mathbb{Z}_n$ is isomorphic to $\mathbb{Z}_{p_1^{a_1}} \times \cdots \times \mathbb{Z}_{p_k^{a_k}}$ and an element $x \in \mathbb{Z}_n$ can be written as $x = (x^{(1)}, \ldots, x^{(k)})$, where $x^{(s)} \equiv x \pmod{p_s^{a_s}}$ for each s. As has been observed in [35], it is not difficult to see that if $A = A_1 \times A_2 \times \cdots \times A_k$ is a subset of $\mathbb{Z}_n$, where $A_s \subset \mathbb{Z}_{p_s^{a_s}}$ for each $s \in [1, k]$, then a sequence of $x_1 \cdot \ldots \cdot x_m$ over $\mathbb{Z}_n$ is an A-weighted zero-sum sequence in $\mathbb{Z}_n$ if and only if for each $s \in [1, k]$, the sequence $x_1^{(s)} \cdot \ldots \cdot x_m^{(s)}$ is an A_s-weighted zero-sum sequence in $\mathbb{Z}_{p_s^{a_s}}$.

We shall need the following definitions.

Given subsets $X_1, \cdots, X_a$ of the set $V = [1, m + 2a]$, a *path* is a sequence of distinct vertices $v_1, \cdots, v_l$ and distinct sets $X_{i_1}, \cdots, X_{i_{l+1}}$ such that $v_1 \in X_{i_1} \cap X_{i_2}, \cdots, v_l \in X_{i_l} \cap X_{i_{l+1}}$. A *cycle* is a sequence of distinct vertices $v_1, \cdots, v_l$ and distinct sets $X_{i_1}, \cdots, X_{i_l}$ such that $v_1 \in X_{i_1} \cap X_{i_2}, \cdots, v_l \in X_{i_l} \cap X_{i_1}$.

Lemma 1.3.0.3. *Given subsets $X_1, \cdots, X_a$ of the set $V = [1, m + 2a]$, where $m > 2a$, there exists a set $I \subset [1, m + 2a]$ with $|I| = m$ and $|I \cap X_s| \notin \{1, 2\}$, for all $s = 1, \cdots, a$.*

Proof: Given a ground set $V = [1, m + 2a]$ and subsets $X_1, \cdots, X_a$ of V, for $I \subset V$, we define $S(I)$ to be the set $\{s : |I \cap X_s| \geq 3\}$ and I will be called *valid* if $|I \cap X_s| \notin \{1, 2\}$, for all $s \in [1, a]$.

We proceed by induction on a. In the case $a = 1$, we have $V = [1, m + 2]$ where $m > 2$. If $0 \leq |X_1| \leq 2$, then we can take $I \subset V \setminus X_1$, such that $|I| = m$ and we have $|I \cap X_1| = 0 \notin \{1, 2\}$.

Now, let $|X_1| > 2$. If $|X_1| \geq m$, we take $I \subset X_1$ such that $|I| = m$, so that $|I \cap X_1| = m > 2$. If $|X_1| < m$, we choose I with $|I| = m$ and $X_1 \subset I \subset V$ so that $|I \cap X_1| = |X_1| > 2$.

Now, assume that $a > 1$ and the statement is true when the number of subsets is not more than $a - 1$.

If one of the sets, say X_a, has no more than two elements, then without loss of generality, let $X_a \subset \{m+2a, m+2a-1\}$ and consider the sets $X_i' = X_i \cap [1, m+2(a-1)]$, for $i \in [1, a-1]$. Since $m > 2a > 2(a-1)$, by the induction hypothesis there exists $I \subset [1, m + 2(a-1)]$ with $|I| = m$ and $|I \cap X_i'| \notin \{1, 2\}$, for $i \in [1, a-1]$. Clearly, $|I \cap X_i| \notin \{1, 2\}$, for $i \in [1, a-1]$ and $|I \cap X_a| = 0$. So, we are through. Hence we assume that

$$|X_s| \geq 3, \quad \text{for all } s.$$

If there exists a non-empty valid set $J \subset V = [1, m + 2a]$ such that $2|S(J)| \geq |J|$, then considering the ground set $V \setminus J$ and the subsets $\{X_s : s \notin S(J)\}$, observing that $|V \setminus J| = m + 2a - |J| \geq m + 2(a - |S(J)|)$, by the induction hypothesis there is a set $J' \subset V \setminus J$ with $|J'| = m - |J|$ such that $|J' \cap X_s| \notin \{1, 2\}$, for all $s \notin S(J)$.

Since $J \subset V$ is valid, $J \cap X_s$ is empty for any X_s with $s \notin S(J)$. Therefore, it is clear that $I = J \cup J'$ is valid for the ground set $V = [1, m + 2a]$ and subsets $X_1, \cdots, X_a$. Since $|I| = m$, we are through.

Now, let $J (\neq \emptyset)$ be a subset of V such that $2|S(J)| \geq |J|$. If J is not valid, then there exists X_s such that $|J \cap X_s| \in \{1, 2\}$.

If $|J \cap X_s| = 1$, then since $|X_s| \geq 3$, we can choose $i, j \in X_s \setminus J$ and consider the set $K = J \cup \{i,j\}$. Then, $|K| = |J| + 2$ and $|S(K)| \geq |S(J)| + 1$ so that $2|S(K)| \geq 2|S(J)| + 2 \geq |J| + 2 = |K|$.

Similarly, if $|J \cap X_s| = 2$, we can choose $i \in X_s \setminus J$ and consider the set $K = J \cup \{i\}$. We have $|K| = |J|+1$ and $|S(K)| \geq |S(J)|+1$ so that $2|S(K)| \geq 2|S(J)|+2 \geq |J|+2 > |K|$.

Therefore, iterating this process we arrive at a valid set L with $2|S(L)| \geq |L|$ and by our previous argument L can be extended to a valid set I with $|I| = m$. So, we assume that for all non-empty $J \subset [1, m + 2a]$ we have

$$2|S(J)| < |J|.$$

If there are X_u, X_v, $u \neq v$, such that $i, j \in X_u \cap X_v$, then taking $k \in X_u \setminus \{i,j\}$ and $l \in X_v \setminus \{i,j\}$ and considering $I = \{i,j,k,l\}$, we have $2|S(I)| \geq 4 \geq |I|$, contradicting the above assumption. So, we assume that for every pair X_u, X_v for $u \neq v$, we have

$$|X_u \cap X_v| \leq 1.$$

If there is a cycle, consisting of distinct vertices $v_1, \cdots, v_l$ and distinct sets $X_{i_1}, \cdots, X_{i_l}$ such that $v_1 \in X_{i_1} \cap X_{i_2}, \cdots, v_l \in X_{i_l} \cap X_{i_1}$, then considering the set $K = \{v_1, \cdots, v_l\}$ and observing that $|X_s| \geq 3$ for all s, we can choose $t_j \in X_{i_j}$ for $j \in [1,l]$ so that taking $J = K \cup \{t_1, \cdots, t_l\}$, $|X_{i_j} \cap J| \geq 3$ for all $j \in [1,l]$. Then $2|S(J)| \geq 2l \geq |J|$, which is a contradiction to our assumption. Therefore, it is assumed that there is no cycle.

Define a *leaf* to be a set X_s such that $|X_s \cap (\cup_{t \neq s} X_t)| \leq 1$. We claim that there must be at least two leaves.

If the sets X_s are pairwise disjoint, then for any s, $|X_s \cap (\cup_{t \neq s} X_t)| = 0$; since $a > 1$, we have two leaves. So we assume that there are two sets which meet. Without loss of generality, let $X_1 \cap X_2 \neq \emptyset$.

Now we consider a path of maximum length involving X_1; by the assumption above, its length is at least 2. Let $X_{i_1}, \cdots, X_{i_l}$ be the distinct sets corresponding to this path, where X_{i_1}, X_{i_l} are end sets and $v_1 \in X_{i_1} \cap X_{i_2}, \cdots, v_{i_{l-1}} \in X_{i_{l-1}} \cap X_{i_l}$. By the maximality condition, $X_{i_1} \setminus \{v_1\}$ and $X_{i_l} \setminus \{v_{l-1}\}$ cannot intersect with the sets not on the path and since there are no cycles, they cannot intersect with the sets on the path as well. Therefore, $|X_{i_1} \cap (\cup_{t \neq i_1} X_t)| = 1$ and similarly, $|X_{i_l} \cap (\cup_{t \neq i_l} X_t)| = 1$. This establishes the claim that there are at least two leaves.

Consider the case $a = 2$ so that $m \geq 2a + 1 = 5$. If either $X_{a-1} \cap X_a \neq \emptyset$, or $X_{a-1} \cap X_a = \emptyset$ and $m \geq 6$, in both these cases, one can easily find $I \subset V$, such that $|I| = m \geq 5$ and $|I \cap X_i| \geq 3$ for $i = 1, 2$. If $X_{a-1} \cap X_a = \emptyset$ and $m = 5$, then $m + 2a = 9$ and at least one of the sets X_{a-1}, X_a, say X_a, has no more than 4 elements. Therefore there is $I \subset V \setminus X_a$ with $|I| = 5 = m$ such that $|I \cap X_{a-1}| \geq 3$, $|I \cap X_a| = 0$ and we are through. So, henceforth we assume that $a > 2$.

We call a point $t \in X_i$ a *free vertex* if $t \notin \cup_{j \neq i} X_j$.

First we consider the case when there are two sets, say X_{a-1}, X_a, each having at least four free vertices. Let $m + 2a, m + 2a - 1, m + 2a - 2, m + 2a - 3$ be free vertices in X_a and $m + 2a - 4, m + 2a - 5, m + 2a - 6, m + 2a - 7$ be free vertices in X_{a-1}. Considering the set $W = [1, m + 2a - 8]$, by the induction hypothesis there is a set $J \subset W$ such that $|J| = m - 4$ and $|J \cap X_i| \notin \{1, 2\}$, for $i \in [1, a - 2]$. If J intersects both X_{a-1} and X_a, we take $I = J \cup \{m + 2a, m + 2a - 1, m + 2a - 4, m + 2a - 5\}$. If J does not intersect at least one of them, say X_a, we take $I = J \cup \{m + 2a - 4, m + 2a - 5, m + 2a - 6, m + 2a - 7\}$. Clearly, I is a valid set with $|I| = m$.

Next, suppose there is exactly one set, say X_a, which has more than three free vertices. Let $m + 2a, m + 2a - 1, m + 2a - 2, m + 2a - 3$ be free vertices in X_a. As there are two leaves, there must be one leaf among the other sets; let X_{a-1} be a leaf, without loss of generality. Now, $|X_{a-1}| \geq 3$ and $|X_{a-1} \cap (\cup_{t \neq a-1} X_t)| \leq 1$. Since by our assumption X_{a-1} does not have more than three free vertices, $|X_{a-1}| \in \{3, 4\}$.

If X_{a-1} has three elements, say $m+2a-4, m+2a-5, m+2a-6$, by the induction hypothesis there exists $J \subset [1, m+2a-7]$ such that $|J| = m-3 > 2(a-2)$ and $|J \cap X_i| \notin \{1, 2\}$ for $i \in [1, a-2]$. Since J does not intersect X_{a-1}, taking $I = J \cup \{m + 2a, m+2a-1, m+2a-2\}$, I is a valid set with $|I| = m$.

If X_{a-1} has four elements, say $m+2a-4, m+2a-5, m+2a-6, m+2a-7$, by the induction hypothesis there exists $J \subset [1, m+2a-8]$ such that $|J| = m-4 > 2(a-2)$ and $|J \cap X_i| \notin \{1, 2\}$ for $i \in [1, a-2]$. Since J does not intersect X_{a-1}, taking $I = J \cup \{m+2a, m+2a-1, m+2a-2, m+2a-3\}$, I is a valid set with $|I| = m$.

Now we assume that no set X_s has more than three free vertices. We claim that for $a > 1$,

$$| \cup_s X_s| \leq 4a - 1.$$

We proceed by induction. For $a = 2$, since no set has more than three free vertices and $|X_1 \cap X_2| \leq 1$, $|X_1 \cup X_2| \leq 7 = 4a - 1$.

Now, assume $a > 2$. By the induction hypothesis, $|X_1 \cup \cdots \cup X_{a-1}| \leq 4(a-1) - 1 = 4a-5$. Since no set has more than three free vertices, $|\cup_s X_s| \leq 4a-5+3 = 4a-2 \leq 4a-1$. Hence the claim is established.

Since, $m > 2a$, for $a > 1$, we have $4a - 1 < 4a < m + 2a$ and hence there are two vertices, say $m+2a, m+2a-1$, which are not in any of the sets $X_1, \cdots, X_a$.

Let X_a be one of the leaves. As had been observed earlier, by our assumptions, $|X_a| \in \{3, 4\}$. If $|X_a| = 3$, then X_a has two free vertices, say $m+2a-2, m+2a-3$. By the induction hypothesis there exists $J \subset [1, m+2a-4]$ such that $|J| = m-2 > 2(a-1)$ and $|J \cap X_i| \notin \{1, 2\}$ for $i \in [1, a-1]$. If J meets X_a, we take $I = J \cup \{m+2a-2, m+2a-3\}$ and if J does not meet X_a, we take $I = J \cup \{m+2a, m+2a-1\}$ and in either case we obtain a valid set I with $|I| = m$.

Now, let $|X_a| = 4$ so that X_a has three free vertices; let $m+2a-2, m+2a-3, m+2a-4$ be free vertices in X_a. As we have at least two leaves, let the other leaf be X_{a-1}. By the above argument we are done except when $|X_{a-1}| = 4$ in which case X_{a-1} has three free vertices; and let $m+2a-5, m+2a-6, m+2a-7$ be free vertices in X_{a-1}.

By the induction hypothesis there exists $J \subset [1, m+2a-8]$ such that $|J| = m-4 > 2(a-2)$ and $|J \cap X_i| \notin \{1,2\}$ for $i \in [1, a-2]$. If J meets both X_a, X_{a-1}, we take $I = J \cup \{m+2a-2, m+2a-3, m+2a-5, m+2a-6\}$. If J does not meet one of these two sets, say X_{a-1}, we take $I = J \cup \{m+2a-2, m+2a-3, m+2a-4, m+2a\}$. In either case we obtain a valid set I with $|I| = m$.

Proof of Theorem 1.1.0.3. Given a sequence $x_1 \cdot \ldots \cdot x_{m+2a}$ over $\mathbb{Z}_n$, we define $X_b^{(s)} \subset [1, m+2a]$ for $s \in [1, k]$ and $b \in [1, a_s]$ by

$$X_b^{(s)} = \{i : x_i \neq 0 \pmod{p_s^b}\}.$$

By Lemma 1.3.0.3, there exists $I \subset [1, m+2a]$ with $|I| = m$ and $|I \cap X_b^{(s)}| \notin \{1,2\}$ for all s, b. Let $I = \{i_1, \cdots, i_m\}$. Then by Lemma 1.3.0.2 and the observation made after the proof of Lemma 1.3.0.2, it follows that $x_{i_1}, \cdots, x_{i_m}$ is an A-weighted zero-sum sequence. $\square$

Chapter 2

Modification of a polynomial method of Rónyai

2.1 Introduction:

One of the most potent methods for solving additive questions is the polynomial method. There are numerous variations to this method but the core idea is to use polynomials, generally over a field, in order to solve, or help solve, additive problems; Alon's Combinatorial Nullstellensatz [16] unifies several of them. We see applications of some versions of the polynomial method in the study of some zero-sum problems; for more information on them as well important applications of the polynomial method to other additive problems, one may look into [47], [37] and [16], for instance. Here we present only the two most commonly used ones: The Combinatorial Nullstellensatz and the Chevalley-Warning Theorem. In its purest form, the method is unparalleled for tackling problems over a field that do not involve structural concerns, perhaps only asking for a lower bound of some sort.

2.1.1 Preliminaries and few known facts:

We start with the result of Erdős, Ginzburg and Ziv [26]. EGZ theorem is a prototype of zero-sum theorems and it plays a vital role in the development of zero-sum results in additive combinatorics. This theorem has several proofs; some among the most interesting proofs involve elementary algebraic techniques.

Alon [15] (see also [17]) proved the EGZ theorem using the following result.

Theorem 2.1.1.1. (Chevalley-Warning) *For $i = 1, \ldots, r$, let $f_i(x_1, x_2, \ldots, x_n)$ be a polynomial of degree d_i over the finite field of q elements and of characteristic p. If $\sum_{i=1}^{r} d_i < n$, then the number N of common zeros of $f_1, f_2, \ldots, f_r$ in that particular field is divisible by p.*

For a proof of the above theorem, one may look into [1] or [40], for instance.

Given an n by n matrix $A = (a_{ij})$ over a field F, its *permanent*, denoted by per A, is defined by

$$\text{per } A = \sum a_{1\ \alpha(1)} a_{2\ \alpha(2)} \cdots a_{n\ \alpha(n)},$$

where the summation is taken over all permutations α of $[1, n]$.

The following is known as *the permanent lemma*; the statement here is as in [16]; this is a slightly extended version of a lemma first proved in [20] (see also [19]).

Theorem 2.1.1.2. (Alon) *Let $A = [a_{ij}]$ be an $n \times n$ matrix over a field F such that* per $A \neq 0$. *Then for any vector $c = (c_1, \ldots, c_n) \in F^n$, and any family of sets $S_1, \ldots, S_n$ of F, each of cardinality 2, there is $s = (s_1, \ldots, s_n) \in S_1 \times S_2 \times \cdots \times S_n$ such that for every i, the ith coordinate of As differs from c_i.*

The permanent lemma is an immediate corollary (see [16]) of the following.

Theorem 2.1.1.3. (Alon) *Let $f(x_1, x_2, \dots, x_n) \in F[x_1, x_2, \dots, x_n]$, where F is a field. If the degree of f is $\sum_{i=1}^n t_i$, where $t_i \geq 0$ are integers, and the coefficient of $\prod_{i=1}^n x_i^{t_i}$ in f is non-zero, then given subsets $S_1, \dots, S_n$ of F with $|S_i| > t_i$, there are $s_i \in S_i$ for $i = 1, \dots, n$, such that*

$$f(s_1, \dots, s_n) \neq 0.$$

Theorem 2.1.1.3 in turn follows (see [16]) from the following.

Theorem 2.1.1.4. (Alon) *If S_i's for $1 \leq i \leq n$, are nonempty subsets of a field F and $f(x_1, x_2, \dots, x_n) \in F[x_1, x_2, \dots, x_n]$ is such that $f(s_1, \dots, s_n) = 0$, for all $s_i \in S_i$ for $i = 1, \dots, n$ (this is equivalent to saying that f vanishes over all the common zeros of $g_i(x_i) = \prod_{s \in S_i}(x_i - s)$, for $i = 1, \dots, n$).*

Then there are polynomials $h_1, \dots, h_n \in F[x_1, x_2, \dots, x_n]$ satisfying $\deg(h_i) \leq \deg(f) - |S_i| = \deg(f) - \deg(g_i)$ so that

$$f = \sum_{i=1}^{n} h_i g_i.$$

Theorems 2.1.1.4 and 2.1.1.3 are known as Alon's Combinatorial Nullstellensatz [16].

In the previous chapter we have explained the definition of *the Erdős-Ginzburg-Ziv constant* $\mathsf{s}(G)$ for a finite abelian group G. It has been observed that for any integer $d \geq 1$,

$$1 + 2^d(n-1) \leq \mathsf{s}(\mathbb{Z}_n^d) \leq 1 + n^d(n-1),$$

From EGZ theorem we have $\mathsf{s}(\mathbb{Z}_n) = 2n - 1$. Regarding the corresponding question in dimension two, the Kemnitz Conjecture [41] $\mathsf{s}(\mathbb{Z}_n^2) = 4n - 3$ has now been settled by Reiher [50].

In higher dimensional cases the general upper bound of Alon and Dubiner [18] says that there is an absolute constant $c > 0$ so that

$$\mathsf{s}(\mathbb{Z}_n^d) \leq (cd \log_2 d)^d n, \text{ for all } \ n,$$

which shows that the growth of $\mathsf{s}(\mathbb{Z}_n^d)$ is linear in n. However, this bound is far from the expected one; it has been conjectured [18] that there is an absolute constant c such that

$$\mathsf{s}(\mathbb{Z}_n^d) \leq c^d n, \text{ for all } n \text{ and } d.$$

There are some lower bounds also for $\mathsf{s}(\mathbb{Z}_n^d)$. It has been proved by Elsholtz [25], that for an odd integer $n \geq 3$,

$$\mathsf{s}(\mathbb{Z}_n^d) \geq (1.125)^{[\frac{d}{3}]} (n-1)2^d + 1.$$

Hence, in particular, for $d \geq 3$ and an odd integer $n \geq 3$,

$$1 + 2^d(n-1) < \mathsf{s}(\mathbb{Z}_n^d).$$

Improvements in the above lower bounds have been obtained in [24]. In the Chapter 0 we have seen that for a finite abelian group G, Gao [28] proved the following relation between constant $E(G)$ and the Davenport constant $D(G)$.

$$E(G) = D(G) + |G| - 1. \tag{2.1}$$

For more knowledge about these constants one may look into the expository article of Gao and Geroldinger [29] and Section 4.2 in the survey of Geroldinger [31].

2.2 Generalization of the Erdős-Ginzburg-Ziv constant:

For a finite abelian group G with $\exp(G) = n$ and a finite non-empty subset A of $[1, n-1]$, we have discussed about *Erdős-Ginzburg-Ziv constant* and *A-weighted zero-sum subsequence* in previous chapter. That was one particular direction of generalizing of *Erdős-Ginzburg-Ziv constant*.

The constants $\mathsf{s}_{mn}(G)$ and $\mathsf{s}_{mn,A}(G)$ defined in Chapter 0 are further generalizations of the constant $\mathsf{s}(G)$. In this chapter we mainly concern about the generalized version of $\mathsf{s}(G)$ i.e $\mathsf{s}_{mn,A}(G)$.

2.2.1 Main Results:

In 1961 after getting the value of $\mathsf{s}(\mathbb{Z}_p)$, p being a prime it will become a multi-dimensional problem from this line of research is to determine the value of $\mathsf{s}(\mathbb{Z}_p^d)$ for $d > 1$.

In this chapter, we will consider the non-empty set $A = \{\pm 1\}$ and our main focus will be to talk about the length of the sequence from which we can easily get a $\{\pm 1\}$-weighted subsequence of length $2p$ or $3p$. Through this chapter, we will talk about for $d = 3$ or $2k$ for $k \geq 3$.

Recently, the problem for the rank 3 case was taken up in [10] and we observed that a suitable modification of a polynomial method used by Rónyai ([52]) yields the following results for dimension 3.

Theorem 2.2.1.1. *For $A = \{\pm 1\}$, and an odd prime p, we have*

$$2p + 3\lfloor \log_2 p \rfloor \leq \mathsf{s}_{2p,A}(\mathbb{Z}_p^3) \leq \frac{(7p-3)}{2}.$$

Theorem 2.2.1.2. *For $A = \{\pm 1\}$, and an odd prime p, we have*

$$3p + 3\lfloor \log_2 p \rfloor \leq \mathsf{s}_{3p,A}(\mathbb{Z}_p^3) \leq \frac{(9p-3)}{2}.$$

It was observed in Theorem 2.2.1.1 and Theorem 2.2.1.2 there is a big gap between the upper and lower bounds of $\mathsf{s}_{2p,A}(\mathbb{Z}_p^3)$ as well as $\mathsf{s}_{2p,A}(\mathbb{Z}_p^3)$.

A slight modification of Rónyai ([52]) yeilds the following result also:

Theorem 2.2.1.3. *Let p be an odd prime and $k \geq 3$ a divisor of $p-1$, θ an element of order k in $\mathbb{Z}_p^*$ and A the subgroup generated by θ. Then, we have*

$$3p + 2k \leq \mathsf{s}_{3p,A}(\mathbb{Z}_p^{2k}) \leq 5p - 2.$$

2.2.2 Few Observations and Lemmas:

We start with some observartions and lemmas that help us in proving Theorem 2.2.1.3, Theorem 2.2.1.1 and Theorem 2.2.1.2. Let (e_1, e_2, e_3) be a basis of $\mathbb{Z}_p^3$ and let $e_0 = e_1 + e_2 + e_3$.

Observation 1. The sequence

$$S = \prod_{\nu=0}^{3} e_\nu^{p-1}$$

has no plus-minus zero-sum subsequence of length p since obtaining $(0,0,0)$ happens either by adding an element with its additive inverse (if the inverse is not in the sequence, but the element repeats, it can be obtained by multiplying with (-1)) or by adding the sum of the elements e_1, e_2, e_3 with the additive inverse of e_0. Each involves an even number of elements in the sequence and p is an odd prime. Thus,

$$\mathsf{s}_{p,\{\pm 1\}}(\mathbb{Z}_p^3) \geq 4p - 3. \tag{2.2}$$

Observation 2. We consider the sequence

$$T = e_0^3 \prod_{\nu=1}^{3} e_\nu^{p-1}.$$

Since T is a subsequence of S, it does not have a plus-minus zero-sum subsequence of length p. However, multiplying one of the e_0's by (-1) and then adding with the remaining elements, it follows that the sequence is a plus-minus zero-sum sequence of length $3p$. However, observing that the sequence

$$S = \mathbf{0}^{3p-1} \prod_{\nu=0}^{r}(2^\nu e_1) \prod_{\nu=0}^{r}(2^\nu e_2) \prod_{\nu=0}^{r}(2^\nu e_3)$$

where r is defined by $2^{r+1} \leq p < 2^{r+2}$, does not have any plus-minus zero-sum subsequence of length $3p$, we obtain:

$$\mathsf{s}_{3p,\{\pm 1\}}(\mathbb{Z}_p^3) \geq 3p + 3\lfloor \log_2 p \rfloor. \tag{2.3}$$

Similarly, one can observe that

$$\mathsf{s}_{2p,A}(\mathbb{Z}_p^3) \geq 2p + 3\lfloor \log_2 p \rfloor. \tag{2.4}$$

So, one way inequality of Theorem 2.2.1.1 and Theorem 2.2.1.2 follows.

Observation 3. Let $(e_1, e_2, \ldots, e_{2k})$ be a basis of $\mathbb{Z}_p^{2k}$. Observing that the sequence

$$S = \mathbf{0}^{3p-1} \prod_{i=1}^{2k} e_i$$

does not have any A-weighted zero-sum subsequence of length $3p$, we obtain:

$$\mathsf{s}_{3p,A}(\mathbb{Z}_p^{2k}) \geq 3p + 2k. \tag{2.5}$$

Therefore one way inequality also holds of the Theorem 2.2.1.3.

Observation 4. Given a sequence $\prod_{i=1}^{t} w_i$ over $\mathbb{Z}_p^d$, where $d = 2k$, $t = 3p - 2$ and $w_i = (a_{i1}, \ldots, a_{id})$ with $a_{i1}, \ldots, a_{id}$ in $\mathbb{Z}_p$, we consider the following system of equations over field $\mathbb{Z}_p$.

$$\sum_{i=1}^{t} a_{i1} x_i^{\frac{p-1}{k}} = 0, \ \sum_{i=1}^{t} a_{i2} x_i^{\frac{p-1}{k}} = 0, \quad \ldots, \quad \sum_{i=1}^{t} a_{id} x_i^{\frac{p-1}{k}} = 0; \ \sum_{i=1}^{t} x_i^{p-1} = 0.$$

The sum of the degrees of each of the polynomials for the above system of equqtions is $\frac{d(p-1)}{k} + (p-1) = 3(p-1) < 3p - 2 = t$ and $x_1 = x_2 = \cdots = x_t = 0$ is a solution, and therefore by Chevalley-Warning theorem there exists a nontrivial solution $(y_1, \ldots, y_t)$ of the above system. Writing $I = \{i : y_i \neq 0\}$, from the first d equations it follows that $\sum_{i \in I} \epsilon_i (a_{i1}, \ldots, a_{id}) = (0, \ldots, 0)$, where $\epsilon_i \in \{\theta, \ldots, \theta^k = 1\} = A$. By Fermat's little theorem, from the last equation we have $|I| = p$ or $|I| = 2p$.

Hence, *a sequence of $3p - 2$ elements of $\mathbb{Z}_p^{2k}$ must have an A-weighted zero-sum subsequence of length p or $2p$.*

From this, it is easy to see that *a sequence of $4p - 2$ elements of $\mathbb{Z}_p^{2k}$ must have an A-weighted zero-sum subsequence of length $2p$.*

Now, if we consider *a sequence of $5p-2$ elements of $\mathbb{Z}_p^{2k}$ such that it has an A-weighted zero-sum subsequence of length p, then from the above discussion it follows that the sequence has an A-weighted zero-sum subsequence of length $3p$; Theorem 2.2.1.2 says that this holds unconditionally.*

Observation 5. Considering a sequence $(a_1, b_1, c_1) \cdot \ldots \cdot (a_t, b_t, c_t)$ over $\mathbb{Z}_p^3$, where $t = \frac{(7p-3)}{2}$, by following the same argument as in observation 4, we can say that it has a $\{\pm 1\}$-weighted zero-sum subsequence of length $2p$. Therefore, we have

$$\mathsf{s}_{2p,A}(\mathbb{Z}_p^3) \leq \frac{(7p-3)}{2}. \tag{2.6}$$

Thus if the sequence of length $\frac{(9p-3)}{2}$ has a plus-minus zero-sum subsequence of length p then it must have a plus-minus zero-sum subsequence of length $3p$.

Remarks. From the above observations we conclude that Theorem 2.2.1.1 holds. From (2.5) we have seen that if $k = p-1$, then the lower bound is $3p + 2(p-1) = 5p - 2$ and hence the upper bound in Theorem 2.2.1.3 is tight. For the constant $\mathsf{s}_{p,A}(\mathbb{Z}_p^3)$ obtaining any reasonable upper bound would be rather difficult.

Lemma 2.2.2.1. *With p, k, θ, and A stated as in Theorem 2.2.1.3, for a positive integer m, the monomials $\prod_{1\leq i\leq m} x_i^{r_i}$, $r_i \in [0, k]$ constitute a basis of the $\mathbb{Z}_p$-linear space of all functions from $D = \{0, \theta, \theta^2, \ldots, \theta^k = 1\}^m$ to $\mathbb{Z}_p$.*

Proof. It is easy to observe that the dimension of the space spanned by the monomials $\prod_{1\leq i\leq m} x_i^{r_i}, r_i \in \{0, 1, \ldots, k\}$ over $\mathbb{Z}_p$ is $(k+1)^m$ which is the same as that of the $\mathbb{Z}_p$-linear space of all functions from $D = \{0, \theta, \ldots, \theta^k = 1\}^m$ to $\mathbb{Z}_p$.

For a point $(x_1, x_2, \ldots, x_m)$ in D, and a subset W of $[1, m]$, considering the function $f_{0,W}(x_1, x_2, \ldots, x_m) := \prod_{j\in W}(1 - x_j^k)$, we observe that

$$f_{0,W}(x_1, x_2, \ldots, x_m) = \begin{cases} 1, & \text{if } x_j = 0 \text{ for all } j \in W, \\ 0, & \text{otherwise.} \end{cases}$$

If we define $f_{k,W}(x_1,x_2,\ldots,x_m) := \prod_{j\in W} x_j k^{-1}(1+x_j+\cdots+x_j^{k-1})$, then

$$f_{k,W}(x_1,x_2,\ldots,x_m) = \begin{cases} 1, & \text{if } x_j = 1 \text{ for all } j \in W, \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, for $t \in [1,k-1]$, defining

$$f_{t,W}(x_1,x_2,\ldots,x_m) := \prod_{j\in W} \frac{\prod_{i\neq t}(x_j-\theta^i)}{\prod_{i\neq t}(\theta^t-\theta^i)},$$

we observe that

$$f_{t,W}(x_1,x_2,\ldots,x_m) = \begin{cases} 1, & \text{if } x_j = \theta^t \text{ for all } j \in W, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, if $W_0, W_1, \ldots, W_k$ are disjoint subsets of $[1,m]$ such that their union is $[1,m]$, then the function

$$f_{W_0,W_1,\ldots,W_k}(x_1,x_2,\ldots,x_m) := \prod_{t\in[0,k]} f_{t,W_t}(x_1,x_2,\ldots,x_m)$$

takes the value 1 precisely at the point $(x_1,x_2,\ldots,x_m)$ of D where $x_j = 0$ for $j \in W_0$, and $x_j = \theta^t$ for $j \in W_t$ for $t \in [1,k]$.

Since the functions $f_{W_0,W_1,\ldots,W_k}$ clearly span the linear space of functions from D to $\mathbb{Z}_p$, we are through. $\square$

Going through the similar argument as in the previous lemma one can have the following result:

Lemma 2.2.2.2. *Let F be a field which is not of characteristic 2 and m be a positive integer. Then the monomials $\prod_{1\leq i\leq m} x_i^{r_i}, r_i \in \{0,1,2\}$ constitute a basis of the F-linear* *space of all functions from $D = \{0, 1, -1\}^m$ to F.*

2.2.3 Proof of Theorem 2.2.1.2 and 2.2.1.3.

Proof of Theorem 2.2.1.3.

Let p, k, θ and A be as in the statement of the theorem. We write $d = 2k$.

Let $S = \prod_{i=1}^{m} w_i$ be a sequence over $\mathbb{Z}_p^d$, where $m = 5p - 2$ and $w_i = (a_{i1}, \ldots, a_{id})$ with $a_{i1}, \ldots, a_{id}$ in $\mathbb{Z}_p$. We proceed to show that it must have a A-weighted zero-sum subsequence of length $3p$. If possible, let there be no such subsequence. So, by the last statement in Observation 4, S has no A-weighted zero-sum subsequence of length p.

Let

$$\sigma(x_1, x_2, \ldots, x_m) := \sum_{I \subset [1,m], |I| = p} \prod_{i \in I} x_i^k,$$

the p-th elementary symmetric polynomial of the variables $x_1^k, x_2^k, \ldots, x_m^k$.

Next we consider the following polynomials in $\mathbf{F}_p[x_1, x_2, \ldots, x_m]$:

$P_1(x_1, x_2, \ldots, x_m) :=$

$$\left(\left(\sum_{i=1}^{m} a_{i1} x_i\right)^{p-1} - 1\right)\left(\left(\sum_{i=1}^{m} a_{i2} x_i\right)^{p-1} - 1\right) \cdots \left(\left(\sum_{i=1}^{m} a_{id} x_i\right)^{p-1} - 1\right),$$

$$P_2(x_1, x_2, \ldots, x_m) := \left(\left(\sum_{i=1}^{m} x_i^k\right)^{p-1} - 1\right),$$

$$P_3(x_1, x_2, \ldots, x_m) := (\sigma(x_1, x_2, \ldots, x_m) - 4)(\sigma(x_1, x_2, \ldots, x_m) - 2)$$

and

$$P(x_1, x_2, \ldots, x_m) := P_1(x_1, x_2, \ldots, x_m) \cdot P_2(x_1, x_2, \ldots, x_m) \cdot P_3(x_1, x_2, \ldots, x_m).$$

Given $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ in $\{0, \theta, \theta^2, \ldots, \theta^k = 1\}^m$, if the number of non-zero entries of α is $2p$, then $\sigma(\alpha) = \binom{2p}{p} = 2 \in \mathbf{F}_p$ and therefore the second factor in P_3 vanishes for $(x_1, x_2, \ldots, x_m) = \alpha$.

Similarly, if the number of non-zero entries of α is $4p$, then $\sigma(\alpha) = \binom{4p}{p} = 4 \in \mathbf{F}_p$ and therefore the first factor in P_3 vanishes for $(x_1, x_2, \ldots, x_m) = \alpha$ in this case.

If the number of non-zero entries of α is p or $3p$, then by our assumption, P_1 vanishes for $(x_1, x_2, \ldots, x_m) = \alpha$ in this case.

Finally, P_2 vanishes unless the number of non-zero entries of α is divisible by p.

Therefore, P vanishes on all vectors in $\{0, \theta, \ldots, \theta^k\}^m$ except at $\mathbf{0}$ and $P(\mathbf{0}) = -8$ as $d = 2k$. Thus, P and $-8f_{\emptyset,\ldots,\emptyset,[m]}$ ($f_{\emptyset,\ldots,\emptyset,[m]}$ as defined in the proof of Lemma 2.2.2.1) are the same as functions on $\{0, \theta, \ldots, \theta^k\}^m$. We observe that $\deg P \leq d(p-1) + k(p-1) + 2kp = 5kp - 3k$.

We now reduce P into a linear combination of monomials of the form $\prod_{1 \leq i \leq m} x_i^{r_i}, r_i \in [0, k]$ by replacing each x_i^{tk+r}, $t \geq 1, r \in [1, k]$ by x_i^r and let Q denote the resulting expression.

We note that as functions on $\{0, \theta, \theta^2, \ldots, \theta^k\}^m$, P and Q are the same. Therefore, as a function on $\{0, \theta, \theta^2, \ldots, \theta^k\}^m$,

$$Q = -8f_{\emptyset,\ldots,\emptyset,[m]}.$$

Also, since reduction can not increase the degree, we have $\deg Q \leq 5kp - 3k$. But, because of the uniqueness part in view of Lemma 2.2.2.1, Q has to be identical with $-8(1 - x_1^k)(1 - x_2^k) \cdots (1 - x_m^k)$. This leads to a contradiction since the later has degree $km = 5kp - 2k$.

$\square$

Proof of Theorem 2.2.1.2.

The line of proof of the Theorem 2.2.1.2 is similar to that of Theorem 2.2.1.3.

Consider $S = (a_1, b_1, c_1) \cdot \ldots \cdot (a_m, b_m, c_m)$ a sequence over $\mathbb{Z}_p^3$ where $m = \frac{(9p-3)}{2}$. We have to show that it must have a plus-minus zero-sum subsequence of length $3p$.

If possible, let there be no such subsequence. By observation 5, there is no plus-minus zero-sum subsequence of length p.

The key steps in proving are to consider the polynomials,

$$\sigma(x_1, x_2, \cdots, x_m) := \sum_{I \subset [1,m], |I|=p} \prod_{i \in I} x_i^2,$$

the p-th elementary symmetric polynomial of the variables x_1^2, x_2^2, $\cdots, x_m^2$, and

$$\begin{aligned} &P(x_1, x_2, \cdots, x_m) \\ := &\left(\left(\sum_{i=1}^m a_i x_i\right)^{p-1} - 1\right)\left(\left(\sum_{i=1}^m b_i x_i\right)^{p-1} - 1\right)\left(\left(\sum_{i=1}^m c_i x_i\right)^{p-1} - 1\right) \\ &\left(\left(\sum_{i=1}^m x_i^2\right)^{p-1} - 1\right)(\sigma(x_1, x_2, \cdots, x_m) - 4)(\sigma(x_1, x_2, \cdots, x_m) - 2). \end{aligned}$$

in $\mathbf{F}_p[x_1, x_2, \ldots, x_m]$ After that the argument goes along the same lines as in the proof of the Theorem 2.2.1.3. $\square$

Chapter 3

Relation between $\mathsf{s}_{\pm}(G)$ and $\eta_{\pm}(G)$

3.1 Introduction:

Consider G to be a finite abelian group (written additively) with $\exp(G) = n$. In this chapter our main aim is to discuss about two particular types of zero-sum invariants i.e $\eta_A(G)$ and $\mathsf{s}_A(G)$ for $A \subseteq [1, n-1]$. Both of these invariants are the generalizations of two classical invariants $\eta(G)$ and $\mathsf{s}(G)$ in zero-sum theory defined in Chapter 0

To continue our discussion more efficiently we shall need a recent result on a weighted analogue of the Harborth constant of a class of finite abelian groups. We will discuss the result in Section 3.2 but before that let us define this constant first. *The Harborth constant* $\mathsf{g}(G)$ of a finite abelian group G is the smallest positive integer l such that any subset of G of cardinality l has a subset of cardinality equal to $\exp(G)$ whose elements sum to the identity element. For the plus-minus weighted analogue $\mathsf{g}_{\pm}(G)$ of $\mathsf{g}(G)$, one requires a subset of cardinality $\exp(G)$, a $\{\pm 1\}$-weighted sum of whose elements is equal to the identity element.

3.2 Few known facts:

It was conjectured by W. Gao et al. [30] that $\mathsf{s}(G) = \eta(G) + \exp(G) - 1$ holds for any finite abelian group G. This relation was established for finite abelian groups of rank at most 2 by Geroldinger and Halter-Koch [32]. It is very natural to ask whether the weighted analogue of the above conjecture, i.e.

$$\mathsf{s}_A(G) = \eta_A(G) + \exp(G) - 1, \tag{3.1}$$

holds. For $G = \mathbb{Z}_n$ the relation (3.1) holds and it was established by Grynkiewicz et al. [38]. Since by definition of $\eta_A(G)$, there is a sequence of length $\eta_A(G) - 1$ which does not have any non-empty A-weighted zero-sum subsequence of length not exceeding $\exp(G)$, appending a sequence of 0's of length $\exp(G) - 1$, we observe that

$$\mathsf{s}_A(G) \geq \eta_A(G) + \exp(G) - 1. \tag{3.2}$$

Therefore the oneway inequality holds in (3.1).

When G is an elementary 2-group, then $\mathsf{s}_{\pm}(G) = \mathsf{s}(G)$ and $\eta_{\pm}(G) = \eta(G)$.

One has the following trivial bounds for the problem of Harborth [39]

$$1 + 2^d(n-1) \leq \mathsf{s}(\mathbb{Z}_n^d) \leq 1 + n^d(n-1). \tag{3.3}$$

When $n = 2$, from (3.3), we have

$$\mathsf{s}_{\pm}(\mathbb{Z}_2^d) = \mathsf{s}(\mathbb{Z}_2^d) = 1 + 2^d.$$

Since a sequence of length $2^d - 1$ of all distinct non-zero elements of $\mathbb{Z}_2^d$ does not have any zero-sum subsequence of length ≤ 2, $\eta_{\pm}(\mathbb{Z}_2^d) = \eta(\mathbb{Z}_2^d) \geq 2^d$.

Again, if a sequence of length 2^d of elements of $\mathbb{Z}_2^d$, does not contain the additive identity $\mathbf{0}$ of $\mathbb{Z}_2^d$, then at least one element gets repeated, and one has a zero-sum subsequence of length 2 and hence, $\eta_{\pm}(\mathbb{Z}_2^d) = \eta(\mathbb{Z}_2^d) \leq 2^d$.

Thus,

$$\mathsf{s}_{\pm}(\mathbb{Z}_2^d) = 1 + 2^d = \eta_{\pm}(\mathbb{Z}_2^d) + 1.$$

Recently, it has been shown in [46] that for the weight set $A = \{1, -1\}$, the relation (3.1) does not hold for $G = \mathbb{Z}_n \oplus \mathbb{Z}_n$ for an odd integer $n > 7$. However, in the same paper [46], it has been shown that the relation holds with $A = \{1, -1\}$ for any abelian group G of order 8 and 16.

In this chapter, we shall observe that in the case of the weight set $A = \{1, -1\}$, the relation (3.1) holds for the groups $\mathbb{Z}_2 \oplus \mathbb{Z}_{2n}$, when n is a power of 2. Also we shall show that the relation (3.1) holds for any abelian group G of order 32, and shall also make some related observations.

Regarding the relation (3.1) with $A = \{1, -1\}$, we are tempted to make the following conjecture.

Conjecture 3. *The relation*

$$\mathsf{s}_{\pm}(G) = \eta_{\pm}(G) + \exp(G) - 1,$$

holds for any finite abelian 2-group G.

To continue the discussion, we shall also require the following results:

Theorem 3.2.0.1. *(Marchan et al. [44]) Let $n \in \mathbb{N}$. For $n \geq 3$ we have*

$$\mathsf{g}_{\pm}(\mathbb{Z}_2 \oplus \mathbb{Z}_{2n}) = 2n + 2.$$

Moreover, $\mathsf{g}_{\pm}(\mathbb{Z}_2 \oplus \mathbb{Z}_2) = \mathsf{g}_{\pm}(\mathbb{Z}_2 \oplus \mathbb{Z}_4) = 5.$

and

Theorem 3.2.0.2. *(see [8]) Let G be a finite and nontrivial abelian group and let $S \in \mathcal{F}(G)$ be a sequence.*

1. *If $|S| \geq \log_2 |G| + 1$ and G is not an elementary 2-group, then S contains a proper nontrivial $\{\pm 1\}$-weighted zero-sum subsequence.*

2. *If $|S| \geq \log_2 |G| + 2$ and G is not an elementary 2-group of even rank, then S contains a proper nontrivial $\{\pm 1\}$-weighted zero-sum subsequence of even length.*

3. *If $|S| > \log_2 |G|$, then S contains a nontrivial $\{\pm 1\}$-weighted zero-sum subsequence, and if $|S| > \log_2 |G| + 1$, then such a subsequence may be found with even length.*

3.3 Results and Lemmas:

Consider the set $A = \{1, -1\}$ and the group to be $\mathbb{Z}_2 \oplus \mathbb{Z}_{2n}$. We have the following result:

Theorem 3.3.0.1. $\mathsf{s}_{\pm}(\mathbb{Z}_2 \oplus \mathbb{Z}_{2n}) \leq 2n + \lceil \log_2 2n \rceil + 1.$

Proof. Let S be a sequence over $G = \mathbb{Z}_2 \oplus \mathbb{Z}_{2n}$ of length $|S| = 2n + \lceil \log_2 2n \rceil + 1$. We proceed to show that there exists a $\{\pm 1\}$-weighted zero-sum subsequence of S of length $\exp(G) = 2n$.

We write the sequence S of the form $S = T^2 U$, where T and U are subsequences of S and U is square-free.

If $2|T| \leq \lceil \log_2 2n \rceil - 1$, then $|U| = |S| - 2|T| \geq 2n + 2$ and by Theorem 3.2.0.1, U has a $\{\pm 1\}$-weighted zero-sum subsequence of length $\exp(G) = 2n$ and we are through.

So we assume that

$$2|T| \geq \lceil \log_2 2n \rceil.$$

Case I. ($\lceil \log_2 2n \rceil$ is even)

Because of our assumption on $|T|$, in this case we may write, $S = V^2 W$ where $2|V| = \lceil \log_2 2n \rceil$, W being the remaining part of the sequence S. We have, $|W| = |S| - \lceil \log_2 2n \rceil = 2n + 1$.

Since by Part (2) of Theorem 3.2.0.2, any sequence of length at least $\lceil \log_2 2n \rceil + 3$ has a proper non-trivial $\{\pm 1\}$-weighted zero-sum subsequence of even length, we get pairwise disjoint $\{\pm 1\}$ zero-sum subsequences $A_1, \ldots, A_l$ of W, each of even length, such that

$$\begin{aligned} 2n + 1 - \sum_{i=1}^{l} |A_i| &\leq \lceil \log_2 2n \rceil + 2, \\ \Rightarrow 2n - 1 - \lceil \log_2 2n \rceil &\leq \sum_{i=1}^{l} |A_i|, \\ &\leq 2n + 1 \qquad \text{(Since } |W| = 2n + 1\text{)}. \end{aligned}$$

Since each $|A_i|$ is even,

$$2n - \lceil \log_2 2n \rceil \leq \sum_{i=1}^{l} |A_i| \leq 2n.$$

Since $2|V| = \lceil \log_2 2n \rceil$, there exists a subsequence V_1 of V such that $V_1^2 \prod_{i=1}^{l} A_i$ is a $\{\pm 1\}$-weighted zero-sum subsequence of length $2n$.

Case II. ($\lceil \log_2 2n \rceil$ is odd)

In this case also we may write $S = V^2 W$ where $2|V| = \lceil \log_2 2n \rceil + 1$ and W is the remaining part of the sequence S. So, $|W| = |S| - \lceil \log_2 2n \rceil - 1 = 2n$. Proceeding as in

Case I, we will get subsequence $\prod_{i=1}^{l} A_i$ of W such that

$$|W| - \sum_{i=1}^{l} |A_i| \leq \lceil \log_2 2n \rceil + 2.$$

Therefore, since $\lceil \log_2 2n \rceil$ is odd and each $|A_i|$ is even and $|W| = 2n$, we have

$$2n - \lceil \log_2 2n \rceil - 1 \leq \sum_{i=1}^{l} |A_i| \leq 2n.$$

Since $2|V| = \lceil \log_2 2n \rceil + 1$, there exists a subsequence V_1 of V such that $V_1^2 \prod_{i=1}^{l} A_i$ is a desired subsequence of S. $\square$

Now we have the following corollary of the above theorem:

Corollary 3.3.0.2. *If n is a power of 2 then,*

$$\mathsf{s}_{\pm}(\mathbb{Z}_2 \oplus \mathbb{Z}_{2n}) = 2n + \lceil \log_2 2n \rceil + 1 = \eta_{\pm}(\mathbb{Z}_2 \oplus \mathbb{Z}_{2n}) + 2n - 1.$$

Proof.

$$\begin{aligned}
2n + \lceil \log_2 2n \rceil + 1 &\geq \mathsf{s}_{\pm}(\mathbb{Z}_2 \oplus \mathbb{Z}_{2n}) \quad \text{(by Theorem 3.3.0.1)} \\
&\geq \eta_{\pm}(\mathbb{Z}_2 \oplus \mathbb{Z}_{2n}) + 2n - 1 \quad \text{(by (3.2))} \\
&\geq \lfloor \log_2 2n \rfloor + 2n + 1.
\end{aligned}$$

The last inequality follows by considering the sequence $(1,0)(0,1)(0,2) \cdot \ldots \cdot (0,2^r)$, where r is defined by $2^{r+1} \leq 2n < 2^{r+2}$, so that

$$\eta_{\pm}(\mathbb{Z}_2 \oplus \mathbb{Z}_{2n}) \geq \lfloor \log_2 2n \rfloor + 2.$$

If n is a power of 2, $\lceil \log_2 2n \rceil = \lfloor \log_2 2n \rfloor$ and hence the corollary follows. $\square$

Next we establish a lower bound on $\eta_{\pm}(G)$ for a class of finite abelian groups G; these bounds will be useful in the next section.

Lemma 3.3.0.3. *For positive integers r and n, we have*

$$\eta_{\pm}(\mathbb{Z}_2^r \oplus \mathbb{Z}_{2n}) \geq \max\left\{ \lfloor \log_2 2n \rfloor + r + \left\lfloor \frac{r}{2n-1} \right\rfloor, r + A(r,n) \right\} + 1,$$

where

$$A(r,n) = \begin{cases} 1 & \text{if } r \leq n, \\ \left\lfloor \frac{r}{n} \right\rfloor & \text{if } r > n. \end{cases}$$

Proof. For $n = 1$, as observed in Section 3.2, $\eta_{\pm}(\mathbb{Z}_2^{r+1}) = 2^{r+1}$, and the lower bound in the lemma holds.

Now, we assume $n > 1$ and consider the sequence

$$S = \prod_{i=1}^{r} e_i \prod_{t=0}^{s} f_t \prod_{j=1}^{k} g_j,$$

where $s = \lfloor \log_2 2n \rfloor - 1$, $k = \left\lfloor \frac{r}{2n-1} \right\rfloor$ and e_i, f_t, and g_j are defined as follows:

$$e_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0),$$

having 1 at the i-th position, for $1 \leq i \leq r$,

$$f_t = (0, 0, \ldots, 0, 2^t), \quad \text{for } 0 \leq t \leq s,$$

and

$$g_{j+1} = (0, 0, \ldots 0, 1, 1, \ldots, 1, 0, \ldots, 0, 1),$$

having 1 at the $(r+1)$-th position and positions $(2n-1)j+1, (2n-1)j+2, \ldots, (2n-1)j+2n-1$, for $0 \leq j \leq k-1$.

Now $\prod_{i=1}^{r} e_i \prod_{t=1}^{s} f_t$ is a zero-sum free sequence with respect to weight set $\{1,-1\}$, and therefore any $\{\pm 1\}$-weighted zero-sum subsequence of S must involve one or more g_i's. However, any $\{\pm 1\}$-weighted zero-sum subsequence containing one of the g_i's, must have at least $2n-1$ elements among the e_i's and an element from f_i's. Thus the length of any $\{\pm\}$ weighted zero-sum subsequence of S will be at least $2n+1$, thereby implying that

$$\eta_{\pm}(\mathbb{Z}_2^r \oplus \mathbb{Z}_{2n}) \geq r + \lfloor \log_2 2n \rfloor + \left\lfloor \frac{r}{2n-1} \right\rfloor + 1. \tag{3.4}$$

We proceed to observe that

$$\eta_{\pm}(\mathbb{Z}_2^r \oplus \mathbb{Z}_{2n}) \geq r + A(r,n) + 1. \tag{3.5}$$

If $r \leq n$, then since the sequence $S_1 = f_1 \prod_{i=1}^{r} e_i$ has no $\{\pm\}$ weighted zero-sum subsequence, and we are done in this case.

If $r > n$, we consider the sequence $S = \prod_{i=1}^{r} e_i \prod_{j=1}^{u} h_j$, where $u = \left\lfloor \frac{r}{n} \right\rfloor$ and define

$$h_{j+1} = (0, 0, \ldots 0, 1, 1, \ldots, 1, 0, \ldots, 0, 1),$$

having 1 at $(r+1)$-th position and the positions $nj+1, nj+2, \ldots, nj+n$, for $0 \leq j \leq u-1$.

Observing that $\prod_{i=1}^{r} e_i$ is a zero-sum free sequence with respect to weight $\{1,-1\}$, and therefore any $\{\pm 1\}$-weighted zero-sum subsequence of S must involve one or more h_i's. However, any $\{\pm 1\}$-weighted zero-sum subsequence containing one of the h_i's, must have at least two h_i's considering the last position and hence has to be at least of length $2n+2$ and hence we have (3.5).

From (3.4) and (3.5), the lemma follows. $\square$

3.4 The case of an abelian group of order 32

First we have some lemmas dealing with different cases of abelian groups of order 32.

Lemma 3.4.0.1. *We have*

$$\mathsf{s}_{\pm}(\mathbb{Z}_2^3 \oplus \mathbb{Z}_4) = 10 \text{ and } \eta_{\pm}(\mathbb{Z}_2^3 \oplus \mathbb{Z}_4) = 7.$$

Proof. Let $S = a_1 \cdot \ldots \cdot a_{10} \in \mathcal{F}(\mathbb{Z}_2^3 \oplus \mathbb{Z}_4)$. Suppose that S is square-free, that is, the elements in S are distinct. Since there are $\binom{10}{2} = 45 > 32$ subsequences of S of length 2, we shall have $a_i + a_j = a_k + a_l$, for some $i, j, k, l \in \{1, 2, \ldots, 10\}$ with $\{i, j\} \neq \{k, l\}$. The assumption that S is square-free forces that $\{i, j\} \cap \{k, l\} = \emptyset$ and hence we obtain a $\{\pm 1\}$-weighted zero-sum subsequence of length 4.

Without loss of generality, we now assume that $a_1 = a_2$ so that a_1a_2 is a $\{\pm 1\}$-weighted zero-sum subsequence of length 2 and write $T = a_3a_4 \cdot \ldots \cdot a_{10}$. We can assume that the elements in T are distinct, otherwise, S will trivially have a $\{\pm 1\}$-weighted zero-sum subsequence of length 4.

To complete the proof, we proceed to show that T has a $\{\pm 1\}$-weighted zero-sum subsequence of length 2 or 4.

If T has at most 1 element of order 4, so that we have at least 7 elements, say $a_3, a_4, a_5, a_6, a_7, a_8, a_9$ of order at most 2, then observing that the number of two length subsequences of $a_3a_4 \cdot \ldots \cdot a_9$ is $\binom{7}{2} = 21 > 15$, while the number of elements of order 2 in the group $\mathbb{Z}_2^3 \oplus \mathbb{Z}_4$ is 15, there are two distinct subsequences with the same sum, thus giving us a $\{\pm 1\}$-weighted zero-sum subsequence of length 4.

Therfore we assume that T has at least 2 elements of order 4 and consider the collection of weighted sums:

$$
\begin{aligned}
D &= \{a_i \pm a_j, 3 \leq i < j \leq 10, \quad a_i, a_j \text{ are of order } 4\} \\
&\quad \cup \{a_r + a_s, 3 \leq r < s \leq 10, \quad a_r, a_s \text{ are of order } 2\}.
\end{aligned}
$$

Since $2 \cdot \binom{c}{2} + \binom{d}{2}$, where c and d are the numbers of a_i's with $3 \leq i < j \leq 10$ of order 4 and of order 2 respectively (so that $c + d = 8$), is not less than 16 as c varies from 2 to 8, two weighted sums in D must be equal.

If one of the $\{\pm 1\}$-weighted sums $a_i \pm a_j$, where a_i, a_j are of order 4 is equal to some $a_r + a_s$, where a_r, a_s are of order 2, we get a $\{\pm 1\}$-weighted zero-sum subsequence of length 4.

Since a_i's are distinct, if two distinct sums $a_r + a_s$, $a_u + a_v$ are equal where a_r, a_s, a_u, a_v of order 2, one must have $\{r, s\} \cap \{u, v\} = \emptyset$, so that one has a $\{\pm 1\}$-weighted zero-sum subsequence of length 4.

Finally, consider the following case; the argument here is marked as an observation so that it can be quoted later on.

Observation I. This is the case when some $a_i \pm a_j$ is equal to some $a_p \pm a_q$, where a_i, a_j, a_p, a_q are of order 4.

If $\{i, j\} \cap \{p, q\} = \emptyset$, then we get a $\{\pm 1\}$-weighted zero-sum subsequence of length 4.

Since a_j is of order 4, $a_i + a_j \neq a_i - a_j$. Hence, the other possibilities are

$$a_i + \epsilon a_j = a_p + \delta a_q,$$

for some $i, j, p, q,\ i < j,\ p < q,\ |\{i, j\} \cap \{p, q\}| = 1$ and $\epsilon, \delta \in \{1, -1\}$.

If $i = p$, one gets a $\{\pm 1\}$-weighted zero-sum subsequence of length 2. If $i = q$ (by symmetry, the case $p = j$ is similar), and $\delta = 1$, then once again we get a $\{\pm 1\}$-weighted zero-sum subsequence of length 2. If $i = q$, and $\delta = -1$, we shall get an expression of the form

$$a_s + \lambda a_t = 2a_u, \text{ where } \{s, t, u\} = \{i, j\} \cup \{p, q\}, \ \eta \in \{1, -1\}. \tag{3.6}$$

If $j = q$, then since $i \neq p$ and hence $a_i \neq a_p$ by our assumption, it is forced that $\epsilon \neq \delta$ and once again we get an expression of the form (3.6).

Now, (3.6) implies that $a_s + (\lambda + 2)a_t = 2(a_u + a_t) = 0$, since a sum of two order 4 elements here is of order 2.

Since $(\lambda + 2) \in \{1, 3\}$ and $3a_t = -a_t$ we get $a_s a_t$ to be a $\{\pm 1\}$-weighted zero-sum subsequence of length 2.

Hence, in this case, we always get a $\{\pm 1\}$-weighted zero-sum subsequence of length 4 or 2.

Therefore, we have proved that $\mathsf{s}_{\pm}(\mathbb{Z}_2^3 \oplus \mathbb{Z}_4) \leq 10$.

Now, by Lemma 3.3.0.3,

$$\eta_{\pm}(\mathbb{Z}_2^3 \oplus \mathbb{Z}_4) \geq \lfloor \log_2 4 \rfloor + 3 + \left\lfloor \frac{3}{4-1} \right\rfloor + 1 = 7.$$

From the above and (3.2) we have

$$10 \geq \mathsf{s}_{\pm}(\mathbb{Z}_2^3 \oplus \mathbb{Z}_4) \geq \eta_{\pm}(\mathbb{Z}_2^3 \oplus \mathbb{Z}_4) + 3 \geq 10.$$

and we are done. □

Lemma 3.4.0.2. *We have*

$$\mathsf{s}_{\pm}(\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4) = 9 \text{ and } \eta_{\pm}(\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4) = 6.$$

Proof. Let $S = a_1 \cdot \ldots \cdot a_9 \in \mathcal{F}(\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4)$. If S is square-free, then since there are $\binom{9}{2} = 36 > 32$ subsequences of S of length 2, as in the proof of the previous lemma, we obtain a $\{\pm 1\}$-weighted zero-sum subsequence of length 4.

Without loss of generality, we now assume that $a_1 = a_2$ so that $S_1 = a_1 a_2$ is a $\{\pm 1\}$-weighted zero-sum subsequence of length 2 and write $T = a_3 a_4 \cdot \ldots \cdot a_9$ and assume that the elements in T are distinct, otherwise, we shall have a $\{\pm 1\}$-weighted zero-sum subsequence of T of length 2 and we shall be through.

We shall now show that T has a $\{\pm 1\}$-weighted zero-sum subsequence of length 2 or 4. Noting that the group $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4$ consists of the identity element, 7 elements of order 2 and 24 elements of order 4, we shall proceed to take care of various cases depending on the number of elements of order 4 in the sequence T.

If $(x, y, z) \in \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4$ is an element of order 4, depending on whether only y or only z is of order 4 in $\mathbb{Z}_4$, we call it respectively of Type 1 and Type 2; if both y and z are of order 4 in $\mathbb{Z}_4$, we call it of Type 3. One observes that sum of two order 4 elements of the same type is of order 2, sum of two order 4 elements of different types is of order 4 and the sum of three elements of order 4, one from each type, is of order 2. From the fact that sum of two order 4 elements of different types is of order 4, we have the following.

Observation II. If a_i, a_j, a_k are order 4 elements of distinct types, then the four sums $a_i \pm a_j \pm a_k$ are distinct.

Case (i). If T does not have more than 2 elements of order 4, so that it has at least 5 elements, say a_3, a_4, a_5, a_6, a_7, of order at most 2, then observing that the number of two length subsequences of $a_3a_4 \cdot \ldots \cdot a_7$ is $\binom{5}{2} = 10 > 8$, there are two distinct subsequences with the same sum, thus giving us a $\{\pm 1\}$-weighted zero-sum subsequence of length 4.

Case (ii). Suppose T has exactly three elements, say a_7, a_8, a_9, of order 4, so that a_3, a_4, a_5, a_6 are of order 2.

The number of subsequences of length two of $a_3a_4a_5a_6$ is $\binom{4}{2} = 6$ and corresponding to each such subsequence a_ia_j, $3 \leq i < j \leq 6$, we have an order 2 element $a_i + a_j$.

Now, if among a_7, a_8, a_9, at least two elements, say a_7, a_8 are of the same type, then consider the elements $a_7 \pm a_8$. Since a_8 is of order 4, $a_7 + a_8 \neq a_7 - a_8$.

Therefore, if none of the elements $a_i + a_j$, $3 \leq i < j \leq 6$ and $a_7 \pm a_8$ is 0, then either two among the distinct sums $a_i + a_j$, $3 \leq i < j \leq 6$ will be equal or one of the sums $a_i + a_j$, $3 \leq i < j \leq 6$ will be equal to one of the sums $a_7 \pm a_8$. Thus in any case we shall get a $\{\pm 1\}$-weighted zero-sum subsequence S_2 of length 4 or 2. If $|S_2| = 2$, then S_1S_2 is a $\{\pm 1\}$-weighted zero-sum subsequence of length 4.

If the three elements a_7, a_8, a_9 are of three distinct types, then by Observation II above, the four sums $a_7 \pm a_8 \pm a_9$ are distinct elements of the subgroup $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. If one of the elements a_3, a_4, a_5, a_6 is equal to one of the sums $a_7 \pm a_8 \pm a_9$, it gives us a $\{\pm 1\}$-weighted zero-sum subsequence of length 4. Otherwise, the elements a_3, a_4, a_5, a_6 together with the four sums $a_7 \pm a_8 \pm a_9$ will be the all distinct elements of the subgroup $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, and since, for $k > 1$, the sum of the all distinct elements of $\mathbb{Z}_2^k$ is zero, observing that $4a_7 = 0$ (a_7 being an element in a group of exponent 4), here we have $a_3 + a_4 + a_5 + a_6 = 0$, and we are through.

Case (iii). Suppose T has four elements, say a_6, a_7, a_8, a_9, of order 4, so that a_3, a_4, a_5 are of order 2.

Now, if among a_6, a_7, a_8, a_9, at least three elements, say a_6, a_7, a_8 are of the same type, then consider the elements $a_6 \pm a_7, a_6 \pm a_8, a_7 \pm a_8$ along with three sums $a_i + a_j$,

$3 \le i < j \le 5$, two of them must be equal, and hence by Observation I in the proof of Lemma 3.4.0.1, providing us with a $\{\pm 1\}$-weighted zero-sum subsequence of length 4.

If it happens that there are two elements, say a_6, a_7 of a particular type and a_8, a_9 of another, we consider the elements $a_6 \pm a_7, a_8 \pm a_9$, along with three sums $a_i + a_j$, $3 \le i < j \le 5$. If any two of these are equal or any one of them is zero, then we are through. If it is not the case, then being all the non-zero elements of order 2, as was observed in the previous case, their sum is 0; however, since the sum is $2(a_6 + a_8)$ and $(a_6 + a_8)$ is of order 4, it is not possible.

Finally, if two elements, say a_6, a_7 are of a particular type and among the remaining elements a_8, a_9, one element each is in the remaining types, consider the collection

$$a_5 + a_6 \pm a_8 \pm a_9, a_6 + a_7, a_6 + a_7 + a_3 + a_5, a_6 + a_7 + a_4 + a_5.$$

Once again, if any two of these are equal or any one of them is zero, then we are through. Otherwise, their sum $3a_6 + 3a_7 + a_3 + a_4 = -a_6 - a_7 + a_3 + a_4$ is zero, providing us with a $\{\pm 1\}$-weighted zero-sum subsequence of length 4.

Case (iv). Suppose there are at least six elements, say $a_4, a_5, a_6, a_7, a_8, a_9$, which are of order 4.

If there are four elements, say a_4, a_5, a_6, a_7, which are of the same type, then consider

$$a_4 \pm a_j, \quad j \in \{5, 6, 7\}, \text{ and } a_5 \pm a_6.$$

If all of them are distinct, then one of them is 0, thus providing us with a $\{\pm 1\}$-weighted zero-sum subsequence of length 2. If two of them are equal, then once again we are through by the argument in Observation I made during the proof of Lemma 3.4.0.1.

If there are not more than three elements of a particular type, then we have the following possibilities.

If it happens that there are three elements of a particular type so that at least two of another, without loss of generality, let a_4, a_5, a_6 be of one type and a_7, a_8 of another and we are through by considering the elements $a_4 \pm a_5, a_4 \pm a_6, a_5 \pm a_6, a_7 \pm a_8$.

Otherwise, there are two elements of order 4 of each type. Let a_4, a_5 are of type 1, a_6, a_7 are of type 2, and a_8, a_9 of type 3.

By Observation II, the elements $a_4 \pm a_6 \pm a_8$ are 4 distinct elements and similarly, $a_4 \pm a_7 \pm a_9$ are 4 distinct elements.

If one among the first group is equal to one of the second group, we get a $\{\pm 1\}$-weighted zero-sum subsequence of length 4. Otherwise, it gives the complete list of 8 distinct elements of order 2. If a_3 is of order 2, then it is equal to one of these and once again, we get a $\{\pm 1\}$-weighted zero-sum subsequence of length 4. If a_3 is of order 4, then there are three elements of a particular type and two of another, a case which has been already covered.

Case (v). The last case to deal with is the one where T has five elements, say a_5, a_6, a_7, a_8, a_9, of order 4, so that a_3, a_4 are of order 2.

Among the elements of order 4, if there are four elements of the same type, or there are three elements of a particular type and two of another, it has been taken care of while dealing with Case (iv).

If none of these happen, then there are order 4 elements of all the three types. In fact, the following two situations will arise.

If there is one element, say a_5, of a particular type, two elements, say a_6, a_7 of another type and a_8, a_9 are of the remaining type. In this situation, if one among the four distinct elements $a_5 \pm a_6 \pm a_8$ is equal to one among the distinct elements $a_5 \pm a_7 \pm a_9$, we get a

$\{\pm 1\}$-weighted zero-sum subsequence of length 4. If all these eight elements are distinct, a_3 must be equal to one of these, thus giving us a $\{\pm 1\}$-weighted zero-sum subsequence of length 4.

In the second possible situation, there will be three elements, say a_5, a_6, a_7 of a particular type and from the remaining elements a_8, a_9, one element each in the remaining types.

Consider the elements

$$a_5 \pm a_8 \pm a_9, a_5 \pm a_6 + a_3, a_5 \pm a_7 + a_3,$$

where, as seen before, equality of two of these will make us through.

If they are distinct, as argued before, a_4 is equal to one of these and we get a $\{\pm 1\}$-weighted zero-sum subsequence of length 4.

Therefore,

$$\mathsf{s}_{\pm}(\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4) \leq 9. \tag{3.7}$$

Since the sequence $g_1 \cdot \ldots \cdot g_5$, where g_i's are defined by

$$g_1 = (1, 0, 0), g_2 = (0, 1, 0), g_3 = (0, 2, 0), g_4 = (0, 0, 1), g_5 = (0, 0, 2)$$

does not have a non-empty $\{\pm 1\}$-weighted zero-sum subsequence, we have

$$\eta_{\pm}(\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4) \geq 6. \tag{3.8}$$

From (3.7), (3.8) and (3.2) we have

$$9 \geq \mathsf{s}_{\pm}(\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4) \geq \eta_A(\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4) + \exp(\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4) - 1 \geq 6 + 4 - 1 = 9$$

and hence the lemma. $\square$

Lemma 3.4.0.3. *If G be an abelian group of order 32 with $\exp(G) = 8$, then*

$$\mathsf{s}_{\pm}(G) = 13 \text{ and } \eta_{\pm}(G) = 6.$$

Proof. Let $S = a_1 \cdot \ldots \cdot a_{13} \in \mathcal{F}(G)$. We proceed to show that S has a $\{\pm 1\}$-weighted zero-sum subsequence of length 8.

Case (A). If S is square-free, then observing that $\binom{13}{2} = 78 > 32$, we shall have $a_i + a_j = a_k + a_l$, where $\{i,j\} \cap \{k,l\} = \emptyset$ and we obtain a $\{\pm 1\}$-weighted zero-sum subsequence $S_1 = a_i a_j a_k a_l$ of length 4. Since $\binom{9}{2} = 36 > 32$, the sequence SS_1^{-1} being of length 9 will have another $\{\pm 1\}$-weighted zero-sum subsequence S_2 of length 4. This shows that S has a $\{\pm 1\}$-weighted zero-sum subsequence $S_1 S_2$ of length 8, in this case.

Case (B). If S is not square-free, let $a_1 = a_2$ so that $T = a_1 a_2$ is a $\{\pm 1\}$-weighted zero-sum subsequence of length 2.

Subcase (B-1). If ST^{-1} is square-free, then observing that $\binom{11}{2} > 32$, we have a $\{\pm 1\}$-weighted zero-sum subsequence T_1 of ST^{-1} of length 4.

Since $|ST^{-1}T_1^{-1}| = 7$, by Part (3) of Theorem 3.2.0.2, $ST^{-1}T_1^{-1}$ has a $\{\pm 1\}$-weighted zero-sum subsequence T_2 with $|T_2| \in \{2, 4, 6\}$.

If $|T_2| = 2$, then TT_1T_2 is a $\{\pm 1\}$-weighted zero-sum subsequence of length 8. Considering, T_1T_2 when $|T_2| = 4$ and TT_2 when $|T_2| = 6$, we get the required $\{\pm 1\}$-weighted zero-sum subsequence of length 8.

Subcase (B-2). If ST^{-1} is not square-free, let $a_3 = a_4$ so that $U_1 = a_3 a_4$ is a $\{\pm 1\}$-weighted zero-sum subsequence of length 2. Since $|ST^{-1}U_1^{-1}| = 9$, and $\binom{9}{2} > 32$, $ST^{-1}U_1^{-1}$ will have a $\{\pm 1\}$-weighted zero-sum subsequence U_2 of length 2 or 4. If $|U_2| = 4$, we are through; if $|U_2| = 2$, so that $|ST^{-1}U_1^{-1}U_2^{-1}| = 7$, we invoke Part (3) of Theorem 3.2.0.2, as in the above subcase and we are through.

Thus we have established that

$$\mathsf{s}_{\pm}(G) \leq 13. \tag{3.9}$$

Now, G can be one of the following groups

$$\mathbb{Z}_4 \oplus \mathbb{Z}_8, \quad \mathbb{Z}_2^2 \oplus \mathbb{Z}_8.$$

If $G = \mathbb{Z}_4 \oplus \mathbb{Z}_8$, then the sequence $b_1 \cdot \ldots \cdot b_5$, where $b_i \in G$ are defined by

$$b_1 = (0,1), b_2 = (0,2), b_3 = (0,4), b_4 = (1,0), b_5 = (2,0)$$

does not have a non-empty $\{\pm 1\}$-weighted zero-sum subsequence and if $G = \mathbb{Z}_2^2 \oplus \mathbb{Z}_8$, then the sequence $c_1 \cdot \ldots \cdot c_5$, where $c_i \in G$ are defined by

$$c_1 = (0,0,1), c_2 = (0,0,2), c_3 = (0,0,4), c_4 = (0,1,0), c_5 = (1,0,0)$$

does not have a non-empty $\{\pm 1\}$-weighted zero-sum subsequence. Therefore,

$$\eta_{\pm}(G) \geq 6. \tag{3.10}$$

From (3.9), (3.10) and (3.2), we have

$$13 \geq \mathsf{s}_{\pm}(G) \geq \eta_{\pm}(G) + \exp(G) - 1 \geq 6 + 8 - 1 = 13$$

and hence the lemma. $\square$

Theorem 3.4.0.4. *If G is an abelian group of order 32, the following relation holds:*

$$s_{\pm}(G) = \eta_{\pm}(G) + \exp(G) - 1.$$

Proof. When G is any cyclic group $\mathbb{Z}_n$, the relation in the theorem holds by a result established in [6]; as has been mentioned in Section 3.2, in the case of a finite cyclic group, even the corresponding result for general weights coincides with a result established by Grynkiewicz et al. [38].

As mentioned in Section 3.2 the relation stated in the theorem holds for elementary 2-groups.

If $G = \mathbb{Z}_2 \oplus \mathbb{Z}_{16}$, the theorem follows from Corollary 3.3.0.2 in and if $\exp(G) = 8$, the theorem follows from Lemma 3.4.0.3.

Finally, Lemmas 3.4.0.1, 3.4.0.2 take care of the remaining case $\exp(G) = 4$. $\square$

Bibliography

[1] Sukumar Das Adhikari,*Aspects of combinatorics and combinatorial number theory*, (Narosa, New Delhi, 2002).

[2] Sukumar Das Adhikari, An extremal problem in combinatorial number theory, *Proceedings of an International Conference in Celebration of the 125th Anniversary of Ramanujan's Birth; University of Delhi, 17-22 December 2012*, Eds. Bruce C. Berndt and Dipendra Prasad, (Ramanujan Mathematical Society Lecture Notes Series, Number 20, 2013) pp. 1–11.

[3] S. D. Adhikari, R. Balasubramanian, F. Pappalardi and P. Rath, *Some zero-sum constants with weights*, Proc. Indian Acad. Sci. (Math. Sci.), **118** (2008), 183–188.

[4] S. D. Adhikari, R. Balasubramanian and P. Rath, Some combinatorial group invariants and their generalizations with weights, in *Additive combinatorics*, Eds. Granville, Nathanson, Solymosi, (CRM Proc. Lecture Notes, 43, Amer. Math. Soc., Providence, RI, 2007) pp. 327–335.

[5] S. D. Adhikari and Y. G. Chen, *Davenport constant with weights and some related questions - II*, J. Combinatorial Theory, Ser. A, **115** (2008), 178–184.

[6] S. D. Adhikari, Y. G. Chen, J. B. Friedlander, S. V. Konyagin and F. Pappalardi, *Contributions to zero–sum problems*, Discrete Math. **306** (2006), 1–10.

[7] S. D. Adhikari, Chantal David and Jorge Jiménez Urroz, *Generalizations of some zero-sum theorems*, Integers, **8** (2008), paper A 52.

[8] S. D. Adhikari, David J. Grynkiewicz and Zhi-Wei Sun,*On weighted zero-sum sequences*, Advances in Applied Mathematics, **48** (2012), 506–527.

[9] S. D. Adhikari, Sanoli Gun and Purusottam Rath, Remarks on some zero-sum theorems, *Proc. Indian Acad. Sci. (Math. Sci.)*, **119**, No. 3, (2009) 275–281.

[10] Sukumar Das Adhikari, Eshita Mazumdar, *Modification of some methods in the study of zero-sum constant*, Integers **14** (2014), paper A 25.

[11] Sukumar Das Adhikari, Eshita Mazumdar, *The polynomial method in the study of zero-sum theorems*, Int. J. Number Theory 11 (2015), no. 5, 1451-1461.

[12] Sukumar Das Adhikari, Eshita Mazumdar, B. K. Moriya, *Relation between two weighted zero-sum constants*, (submitted)

[13] S. D. Adhikari and P. Rath,*Davenport constant with weights and some related questions*, Integers, **6** (2006), paper A 30.

[14] W. R. Alford, A. Granville and C. Pomerance, There are infinitely many Carmichael numbers, *Annals of Math.*, **139** (2), no. 3, (1994) 703–722.

[15] N. Alon, Tools from higher algebra, *Handbook in Combinatorics, Vol. II*, Eds. Graham, Grötschel and Lovász, (North Holland, 1995) 1749–1783.

[16] N. Alon, Combinatorial Nullstellensatz, in *Recent trends in combinatorics (Mátraháza, 1995), Combin. Probab. Comput.*, **8** (1999) 7–29.

[17] N. Alon and M. Dubiner, Zero-sum sets of prescribed size, in *Combinatorics, Paul Erdős is eighty (Volume 1), Keszthely (Hungary)*, (1993) 33–50.

[18] N. Alon and M. Dubiner, A lattice point problem and additive number theory, *Combinatorica*, **15** (1995) 301–309.

[19] N. Alon, N. Linial and R. Meshulam, Additive bases of vector spaces over prime fields, *J. Combinatorial Theory, Ser A* **57** (1991) 203–210.

[20] N. Alon and M. Tarsi, A nowhere-zero point in linear mappings, *Combinatorica* **9** (1989) 393-395.

[21] P. van Emde Boas and D. Kruswjik, A combinatorial problem on finite abelian group III, *Z. W.1969-008* (Math. Centrum, Amsterdam).

[22] M. N. Chintamani and Bhavin K. Moriya, *Generalizations of some zero sum theorems*, Proc. Indian Acad. Sci. (Math. Sci.), **122**, No. 1 (2012), 15–21.

[23] M. N. Chintamani, B. K. Moriya, W. D. Gao, P. Paul and R. Thangadurai, New upper bounds for the Davenport and for the Erdős-Ginzburg-Ziv constants, *Arch. Math.* **98** (2012) 133-142.

[24] Y. Edel, C. Elsholtz, A. Geroldinger, S. Kubertin and L. Rackham, Zero-sum problems in finite abelian groups and affine caps, *Quart. J. Math.* **58** (2007) 159–186.

[25] C. Elsholtz, *Lower bounds for multidimensional zero sums*, Combinatorica, **24**, no. 3, 351–358 (2004).

[26] P. Erdős, A. Ginzberg, A. Ziv, *Theorem in the additive number theory*, Bull. Res. Council Israel (F), **10** (1961) 41-43.

[27] Yushuang Fan, W. Gao and Qinghai Zhong, On the Erdős-Ginzburg-Ziv constant of finite abelian groups of high rank, *J. Number Theory*, **131** (2011) 1864–1874.

[28] W.D.Gao, A Combinatorial problem on finite Abelian groups, J. Number Theory 58 (1996)100-103.

[29] W. D. Gao and A. Geroldinger, *Zero-sum problems in finite abelian groups: a survey*, Expo. Math. **24** (2006), 337–369.

[30] W. D. Gao, Q. H. Hou, W. A. Schmid, R. Thangadurai, *On short zero-sum subsequences II.* Integers **7** (2007), A21.

[31] A. Geroldinger, Additive group theory and non-unique factorizations, *in Combinatorial number theory and additive group theory, in: Adv. Courses Math. CRM Barcelona*, Eds. A. Geroldinger, I. Ruzsa, (Birkhäuser Verlag, Basel, 2009).

[32] A. Geroldinger and F. Halter-Koch, *Non-unique factorizations: Algebraic, combinatorial and analytic theory*, Pure and Applied Mathematics (Boca Raton), 278, (Chapman & Hall/CRC, Boca Raton, FL, 2006).

[33] A. Geroldinger, Imre Z. Ruzsa, *Combinatorial Number Theory and Additive Group Theory*, Advanced Courses in Mathematics CRM Barcelona, Birkhaűser, 2008.

[34] A. Geroldinger and R. Schneider, On Davenport's constant, *J. Combin. Theory, Ser. A*, **61**, no. 1, (1992) 147-152.

[35] S. Griffiths, *The Erdős-Ginzburg-Ziv theorem with units*, Discrete Math. **308**, No. 23 (2008), 5473–5484.

[36] H. G. Grundman and C. S. Owens, *On bounds for two Davenport-type constants*, Integers, **13**, (2013) paper A 7.

[37] D. J. Grynkiewicz, *Structural Additive Theory*, Developments in Mathematics, Springer, 2013.

[38] D. J. Grynkiewicz, L. E. Marchan, O. Ordaz, *A Weighted Generalization of Two Theorems of Gao.* Ramanujan J. **28** (2012), no. 3, 323-340.

[39] H. Harborth, *Ein Extremalproblem für Gitterpunkte*, J. Reine Angew. Math. **262/263**, 356–360 (1973).

[40] Kenneth Ireland and Michael Rosen, *A Classical Introduction to Modern Number Theory*, 2nd edition, (Springer, 1990).

[41] A. Kemnitz, *On a lattice point problem Ars Combin.*, 16b (1983), pp. 151160

[42] M. Kneser, *Abschätzung der asymptotischen Dichte von Summenmengen*, Math. Z. **58** (1953), 459–484.

[43] F. Luca, *A generalization of a classical zero-sum problem*, Discrete Math. **307**, No. 13 (2007), 1672–1678.

[44] L. E. Marchan, O. Ordaz, D. Ramos and W. A. Schmid, *Some exact values of the Harborth constant and its plus-minus weighted analogue.* Arch. Math. (Basel) **101** (2013), no. 6, 501–512.

[45] R. Meshulam, An uncertainty inequality and zero subsums,*Discrete Math.*, **84**, no. 2, (1990) 197-200.

[46] B. K. Moriya, *On weighted zero sum subsequences of short length*, Integers **14** (2014), Paper No. A21.

[47] Melvyn B. Nathanson, *Additive Number Theory: Inverse Problems and the Geometry of Sumsets*, Springer, 1996.

[48] J. E. Olson, A combinatorial problem in finite abelian groups, I, *J. Number Theory*, **1** (1969) 8–10.

[49] J. E. Olson, A combinatorial problem in finite abelian groups, II, *J. Number Theory*, **1** (1969) 195–199.

[50] C. Reiher, *On Kemnitz conjecture concerning lattice-points in the plane*, Ramanujan J. **13**, no. 1-3 (2007), 333-337.

[51] K. Rogers, A Combinatorial problem in Abelian groups, *Proc. Cambridge Phil. Soc.*, **59**, (1963) 559–562.

[52] L. Rónyai, *On a conjecture of Kemnitz*, Combinatorica, **20** (4) (2000), 569–573.

[53] S. Savchev and Fang Chen, *Kemnitz conjecture revisited*, Discrete Math. **297** (2005), 196–201.

[54] W. A. Schmid and J. J. Zhuang, On short zero-sum subsequences over p-groups, *Ars Comb.* **95** (2010) 343–352.

[55] T. Tao, Van H. Yu, *Additive Combinatorics*, Cambridge, 2006.