MODULATION SPACES AND NONLINEAR EVOLUTION EQUATIONS

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STATEMENT BY AUTHOR

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DECLARATION

I hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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List of Publications arising from the thesis

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To My

Grandparents

Smt. Dholiben Bhimani & Shree Mohanbhai Bhimani

&

Parents

Smt. Kailasben Bhimani & Shree Ghanshyambhai Bhimani

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Abstract

The study of the local and global well-posedness of nonlinear evolution PDE in spaces of low regularity represents one of the most active research fields, where the deepest machinery of modern harmonic analysis is applied. The principal aim of this PhD dissertation is to study nonlinear Schrödinger, wave and Klein-Gordon equations in the case of modulation $M^{p,q}(\mathbb{R}^d)$ and Wiener amalgam $W^{p,q}(\mathbb{R}^d)$ (time-frequency) spaces.

In the last decade, many mathematicians have used these spaces as a regularity class for the Cauchy problem. In fact, fantastic progress have been done in the last decade from the PDE point of view in these spaces. But some of the fundamental issues were left open by active researchers in this field. For instance: (1) Whether one can take power type nonlinearity $u|u|^{\alpha}$ ($\alpha \in (0,\infty) \setminus 2\mathbb{N}$) in Schrödinger equation to obtain local well-posedness result?(2) The global well-posedness for the NLS with initial data(large) in modulation spaces has not yet clear due to lack of any useful conservation laws in these spaces by which one can guarantee global well-posedness.

To handle these issues we have studied composition operators on modulation and Wiener amalgam spaces. As an application, we point out the standard method for proving the well-posedness results for nonlinear dispersive (Schrödinger/wave/Klein-Gordon) equations cannot be handled for nonlinearity of the form $F(u) = u|u|^{\alpha}, \alpha \in (0,\infty) \setminus 2\mathbb{N}$.

We have obtained some sufficient conditions for nonlinearity $uF(u)$ and $|u|$ to be in $M^{1,1}(\mathbb{R})$ whenever $u \in M^{1,1}(\mathbb{R})$ and F is a contraction on $\mathbb{C}$.

We study the Cauchy problem for Hartree type equations, that is, Schrödinger equation with cubic convolution nonlinearity $F(u) = (K * |u|^2)u$ under a specified condition on the potential K with Cauchy data in modulation spaces. We have established local and global well-posedness results for the Hartree type equations.

In fact, these time-frequency spaces are present in various problems in the analysis, which also involves the study of twisted convolution. Finally, we take an excursion to the study of factorization problems with respect to twisted convolution in the realm of time-frequency and Lebesgue spaces. We have also illustrated its applications to functional analysis.

Contents

Synopsis

0.1 Introduction

This PhD dissertation entitled "Modulation Spaces and Nonlinear Evolution Equations" carries out the study of nonlinear Schrödinger, wave and Klein-Gordon equations and factorization problems in the realm of modulation and Wiener amalgam spaces using the concepts and techniques from harmonic and time-frequency analysis.

The content of the present dissertation is divided into five chapters. The concepts and results of modulation and Wiener amalgam spaces which, playing an important role in the study of nonlinear evolution equations and factorization problems, constitute the content of Chapter 1.

We start with the nonlinear Schrödinger equation (NLS):

$$i\frac{\partial}{\partial t}u(x,t)+\Delta_x u(x,t)=F(u(x,t)),\quad u(x,0)=u_0(x), \tag{1}$$

where $\Delta_x = \sum_{j=1}^{d}\frac{\partial^2}{\partial x_j^2}$ is the Laplacian on $\mathbb{R}^d$, $(x,t)\in\mathbb{R}^d\times\mathbb{R}, i=\sqrt{-1}$, u_0 is a complex valued function on $\mathbb{R}^d$ and the nonlinearity is given by a complex function F on $\mathbb{C}$.

It is well-known that the Schrödinger semi-group $e^{it\Delta}$ is bounded in $L^p(\mathbb{R}^d)$ if and only if $p=2$. Thus we cannot expect to solve linear Schrödinger equation in the usual Lebesgue spaces $L^p(\mathbb{R}^d)$ $(p \neq 2)$; and so the NLS as well. It is therefore very natural to seek function spaces in which we can solve the NLS. In fact, inspired by uniform(in contrast to dyadic) decomposition techniques and in search of obtaining local well-posedness results for some nonlinear evolution equations, in particular the NLS, Wang-Zhao-Guo [66] have constructed the spaces $E^{\lambda}_{p,q}$, and asserts that the Schrödinger semi-group $e^{it\Delta}$ is bounded on these spaces and the space $E^{0}_{2,1}$ is an algebra under pointwise multiplication. And as a consequence, ensured the local well-posedness results (see [66, Theorem 1.1]) for the power type nonlinearity $F(u) = u|u|^{2k}$ $(k \in \mathbb{N})$. Roughly speaking, a Cauchy data in an $E^{\lambda}_{p,q}$ is rougher than any given one in a fractional Bessel potential space (for instance: Sobolev space $H^s(\mathbb{R}^d) \subset E^0_{2,1}(\mathbb{R}^d)$ $(s > d/2)$) and this low-regularity is desirable in many situations. In the subsequent papers [68, 4] it has been recognized that the spaces $E^{\lambda}_{p,q}$ is in fact the well-known modulation spaces.

It may be recalled that in 1983 Feichtinger [21] introduced a class of Banach spaces, which allow a measurement of space variable and Fourier transform variable of a function or distribution f on $\mathbb{R}^d$ simultaneously using short-time Fourier transform (STFT), the so-called modulation spaces. More precisely, the STFT of f with respect to a window function $0 \neq g \in \mathcal{S}(\mathbb{R}^d)$ (Schwartz space) is defined by

$$V_g f(x, w) = \langle f, M_w T_x g\rangle,$$

where $T_x f(t) = f(t - x)$, $M_w f(t) = e^{2\pi i w\cdot t} f(t)$, and $\langle f, g\rangle$ denotes the the action of the tempered distribution f on the Schwartz class function g. And the weighted modulation spaces $M^{p,q}_s(\mathbb{R}^d)$ $(1 \leq p, q \leq \infty, s \in \mathbb{R})$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ for which, the following norm

$$\|f\|_{M_s^{p,q}(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x,w)|^p dx \right)^{q/p} (1+|w|^2)^{sq/2} dw \right)^{1/q}$$

is finite, with the usual modification if p or q is infinite. We put $M_0^{p,q}(\mathbb{R}^d) = M^{p,q}(\mathbb{R}^d)$. By reversing the order of integration we can obtain another family of spaces, the so-called Wiener amalgam spaces $W_s^{p,q}(\mathbb{R}^d)$.

In the last decade, these spaces have turned out to be very fruitful for the nonlinear evolution equations and many mathematicians have found these spaces attractive. In fact, the unimodular Fourier multiplier operator $e^{i|D|^\alpha}$ is not bounded on most of the Lebesgue spaces $L^p(\mathbb{R}^d)$ $(p \neq 2)$ or even Besov spaces; in contrast, it is bounded on $W^{p,q}(\mathbb{R}^d)(1 \leq p,q \leq \infty)$ for $\alpha \in [0,1]$, and on $M^{p,q}(\mathbb{R}^d)(1 \leq p,q \leq \infty)$ for $\alpha \in [0,2]$ (cf. [2, 4, 15]). The cases $\alpha = 1,2$ are of particular interest because they occur in the time evolution of wave and Schrödinger equations respectively. In particular, we mention, in 2009 Bényi-Okoudjou in [2] have used time-frequency (in contrast, to uniform decomposition) techniques to obtain the local well-posedness result (see [2, Theorem 1.1]) in $M^{p,1}(\mathbb{R}^d)$ $(1 \leq p \leq \infty)$ with the nonlinearity of the generic form $F(u) = g(|u|^2)\, u$, for some complex-entire function $g(z)$, and immediately after this, it has been noted by Cordero-Nicola [12] that this non-linearity can be replaced by real entire function.

The proof of the above mentioned local well-posedness results highly depend on the fact that $M_s^{p,1}(\mathbb{R}^n)$ is an algebra under pointwise multiplication:

$$\||u|^{2k}u\|_{M_s^{p,1}(\mathbb{R}^d)} = \|u^{k+1}\bar{u}^k\|_{M_s^{p,1}(\mathbb{R}^d)} \lesssim \|u\|^{2k+1}_{M_s^{p,1}(\mathbb{R}^d)}.$$

Hence, the nonlinearity of the type $F(z) = z|z|^\alpha$, $\alpha \in 2\mathbb{N}$ can be handled in this way. Of course it is very natural to ask, how far can one go, to include more general nonlinear terms in these dispersive equations on modulation spaces? It was in this context Ruzhansky-Sugimoto-Wang [49] raised the open problem:

Does $\||u|^{\alpha}u\|_{M^{p,1}(\mathbb{R}^d)} \leq \|u\|^{\alpha+1}_{M^{p,1}(\mathbb{R}^d)}$ hold for all $\alpha \in (0,\infty) \setminus 2\mathbb{N}$?

This question inspires us to study the nonlinear mapping properties (see Section 0.2 for description) on the modulation and Wiener amalgam spaces and this is precisely the starting point for investigation in this dissertation, which constitute the content of Chapter 2.

The knowledge of nonlinear mapping properties naturally leads us to the study of contraction (see Section 0.3 for description) of functions in $M^{1,1}(\mathbb{R})$, which constitute the content of Chapter 3.

In Chapter 4, we illustrate the method of the contraction mapping theorem to obtain local well-posedness results for NLS, NLW and NLKG equations for the 'real entire' nonlinearities in some weighted modulation spaces $M^{p,q}_s(\mathbb{R}^d)$, and highlights the fundamental importance of our previous results(Chapter 2) by pointing out that the standard method for the evolution of nonlinear dispersive (Schrödinger/wave/Klein-Gordon) equations cannot be considered for nonlinearity of the form $F(u) = u|u|^{\alpha}, \alpha \in (0,\infty) \setminus 2\mathbb{N}$.

After having these local well-posedness results in modulation spaces, of course, it is natural to investigate the global well-posedness results, and in fact some attempts have been made in the literature. In particular, we mention the global well-posedness results for the Schrödinger equation with the power type nonlinearity $F(u) = |u|^{2k}u$ $(k \in \mathbb{N})$ are obtained in [68, 32] with small initial data from $M^{p,1}(\mathbb{R}^d)$ $(1 \leq p \leq 2)$. However, the global well-posedness result for the large initial data (without any restriction to initial data) in modulation space is not yet clear, in fact it is an open question [49, p.280], because one of the main obstacle is a lack of useful conservation laws in modulation spaces by which one can guarantee the global existence result. These considerations inspires us to investigate Schrödinger equation with cubic convolution nonlinearity (Hartree type equation):

$$iu_t + \Delta u = (K * |u|^2)u, \quad u(x, t_0) = u_0(x); \tag{2}$$

where $t_0 \in \mathbb{R}$ and potential K of the following type:

$$K(x) = \frac{\lambda}{|x|^\gamma}, (\lambda \in \mathbb{R}, \gamma > 0, x \in \mathbb{R}^d), \tag{3}$$

and we established (see Section 0.4 for description) local and global well-posedness results, which forms the principal part of Chapter 4.

Finally, we divert our attention slightly from the main line of investigation in the present dissertation. In fact, these spaces are also present in various other current trends (pseudo-differential operators, [14], et al.) of investigation which involves the study of twisted convolution $\natural$. On the other hand, in 1939 Salem [57] proved factorization theorem $L^1(\mathbb{T}) = L^1(\mathbb{T}) * L^1(\mathbb{T})$, since then major mathematicians (for instance: Walter Rudin, Paul Cohen, Edwin Hewitt, et al.) have contributed to factorization problems; and it found strong impact on other parts of harmonic analysis. This motivates us to initiate the study of factorization problems (see Section 0.5 for description) with respect to the twisted convolution $\natural$ in the realm of modulation, Wiener amalgam, and Lebesgue spaces, and this part constitute the content of Chapter 5.

0.2 Composition Operators on $M^{p,q}(\mathbb{R}^d)$ and $W^{p,q}(\mathbb{R}^d)$

Let X and Y be normed spaces of functions. For a given function $F : \mathbb{R}^2 \to \mathbb{C}$, we associate it, with the composition operator $T_F : f \mapsto F(f)$ which maps X to Y, that is, $F(f) \in Y$ whenever $f \in X$; where $F(f)$ is the composition of functions F and f. If $T_F(X) \subset X$, we say the composition operator T_F acts on X. Composition operators are simple examples of nonlinear mappings.

Theorem 0.2.1 (Necessary Condition) *Suppose that T_F is the composition operator associated to a complex function F on $\mathbb{C}$, and $1 \leq p \leq \infty$ and $1 \leq q < 2$.*

1. *If T_F maps $M^{p,1}(\mathbb{R}^d)$ to $M^{p,q}(\mathbb{R}^d)$, then F must be real analytic on $\mathbb{R}^2$. Moreover, $F(0) = 0$ if $p < \infty$.*

2. *If T_F maps $W^{p,1}(\mathbb{R}^d)$ to $W^{p,q}(\mathbb{R}^d)$, then F must be real analytic on $\mathbb{R}^2$. Moreover, $F(0) = 0$ if $p < \infty$.*

Theorem 0.2.2 (Sufficient Condition) *Let F be a real analytic function on $\mathbb{R}^2$ with $F(0) = 0$. Then T_F acts on $M^{1,1}(\mathbb{R}^d)$.*

0.3 Contraction of Functions in $M^{1,1}(\mathbb{R})$

As a consequence of Theorem 0.2.1(1), there exist functions $f \in M^{1,1}(\mathbb{R})$ such that $|f|, f|f|^{2k+1}$ $(k \in \mathbb{N})$ does not belong to $M^{1,1}(\mathbb{R})$. In view of this, one is prompted to ask: given $f \in M^{1,1}(\mathbb{R})$, under which sufficient condition, one can ensure the membership for nonlinearity $|f|$ and $f|f|^{2k+1}$ in $M^{1,1}(\mathbb{R})$?

Definition 0.3.1 A function $F : \mathbb{C} \to \mathbb{C}$ is called a contraction if it satisfies the inequality:$|F(z_1) - F(z_2)| \leq |z_1 - z_2|, (z_1, z_2 \in \mathbb{C})$. If f is a complex valued function, we say the function $F(f)$ a contraction of f.

Definition 0.3.2 We call $\lambda(w)$ a negative definite function if it has the form

$$\lambda(w) = \int_0^\infty \frac{\sin^2 2\pi w\alpha}{\alpha^2} d\mu(\alpha), \ \ (\mu(0) = 0)$$

where $\mu(\alpha)$ is a non-decreasing function such that the integral converges for every real w.

We denote by $A(\mathbb{R}^d)$ the algebra of Fourier transforms. In other words, $f \in A(\mathbb{R}^d)$ if there exists some $\psi \in L^1(\mathbb{R}^d)$ such that $f(w) = \hat{\psi}(w)$ $(w \in \mathbb{R}^d)$, and we define the Beurling algebra $A^*(\mathbb{R}) = \{f = \hat{\psi} \in A(\mathbb{R}) : \sup_{|\xi|>|x|} |\psi(\xi)| \in L^1(\mathbb{R})\}$.

Theorem 0.3.3 *Suppose that $f \in M^{1,1}(\mathbb{R}) \cap A^*(\mathbb{R})$ and $F(f)$ be a contraction of f such that $F(f)$ vanishes at infinity. Then $fF(f) \in M^{1,1}(\mathbb{R})$, and $\|fF(f)\|_{M^{1,1}} \lesssim \|f\|_{M^{1,1}} \|F(f)\|_{A(\mathbb{R})}$.*

Theorem 0.3.4 *Suppose that $f \in M^{1,1}(\mathbb{R})$. If there is a negative definite function $\lambda(w)$ such that $|V_g f|^2 \beta + \beta^{-1} \in L^1(\mathbb{R}^2)$, where $\beta(x,w) = \lambda(w)\gamma(x)$ for some function $\gamma(x)$ $(x, w \in \mathbb{R})$, then $|f| \in M^{1,1}(\mathbb{R})$, and $\|f|f|^{2k+1}\|_{M^{1,1}} \lesssim \|f\|_{M^{1,1}} \||f|\|_{M^{1,1}}^{2k+1}$.*

0.4 Nonlinear Evolution Equations

Theorem 0.4.1 *Assume that $u_0 \in M^{1,1}(\mathbb{R}^d)$ and let K be given by (3) with $\lambda \in \mathbb{R}$, and $0 < \gamma < min\{2, d/2\}, d \in \mathbb{N}$. Then there exists a unique global solution of (2) such that $u \in C(\mathbb{R}, M^{1,1}(\mathbb{R}^d))$.*

Theorem 0.4.2 *Let $K \in A(\mathbb{R}^d), d \in \mathbb{N}$. Then, for any $u_0 \in M^{p,q}(\mathbb{R}^d)$ $(1 \leq q \leq min\{p, p'\}$, where $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{p'} = 1)$, there exists a unique global solution $u(t)$ of (2) such that $u(t) \in C(\mathbb{R}, M^{p,q}(\mathbb{R}^d))$.*

Theorem 0.4.3 *Assume that* $u_0 \in M^{p,1}(\mathbb{R}^d)$ $(1 \leq p \leq \infty)$, *and* $K \in M^{1,\infty}(\mathbb{R}^d), d \in \mathbb{N}$. *Then, there exist* $T^* = T^*(\|u_0\|_{M^{p,1}}) > t_0$ *and* $T_* = T_*(\|u_0\|_{M^{p,1}}) < t_0$ *such that* (2) *has a unique solution* $u \in C([T_*, T^*], M^{p,1}(\mathbb{R}^d))$.

0.5 On Twisted Convolution and Modulation Spaces

Theorem 0.5.1 *Let* $1 \leq p, q < \infty$ *and* E *denote any one of* $L^p(\mathbb{R}^{2d})$ *or* $M^{p,q}(\mathbb{R}^{2d})$ *or* $W^{p,q}(\mathbb{R}^{2d})$. *Then*

1. $E = L^1(\mathbb{R}^{2d}) \natural E$.

2. $M^{2,2}(\mathbb{R}^{2d}) \natural M^{2,2}(\mathbb{R}^{2d}) \subsetneq M^{2,2}(\mathbb{R}^{2d})$.

Theorem 0.5.2 *Let* $1 \leq p, q < \infty$, *and* $E^2 = E * E$.

1. $E^2 \neq E$, *where* $E = M^{p,1}(\mathbb{T}^d)$.

2. $E = L^1(\mathbb{R}^d) * E$, *where* $E = M^{p,q}(\mathbb{R}^d)$ *or* $W^{p,q}(\mathbb{R}^d)$.

Theorem 0.5.3 (Applications) *Let* $1 \leq p, q \leq \infty$.

1. *Let* E *denote any one of* $M^{p,q}(\mathbb{R}^{2d})$ *or* $W^{p,q}(\mathbb{R}^{2d})$. *If* T *is any map from* $L^1(\mathbb{R}^{2d})$ *to* E *such that* $T(f \natural h) = f * T(h)$ *for all* $f, h \in L^1(\mathbb{R}^{2d})$, *then*

$$\|T(f)\|_E \lesssim \|f\|_{L^1(\mathbb{R}^{2d})} \text{ for all } f \in L^1(\mathbb{R}^{2d}).$$

2. *Let* E *denote any one of* $L^p(\mathbb{R}^{2d})$ *or* $M^{p,q}(\mathbb{R}^{2d})$ *or* $W^{p,q}(\mathbb{R}^{2d})$. *If* T *is any map from* $L^1(\mathbb{R}^{2d})$ *to* E *such that* $T(f \natural h) = f \natural T(h)$ *for all* $f, h \in L^1(\mathbb{R}^{2d})$, *then*

$$\|T(f)\|_E \lesssim \|f\|_{L^1(\mathbb{R}^{2d})} \text{ for all } f \in L^1(\mathbb{R}^{2d}).$$

3. *Every positive linear functional on* $(L^1(\mathbb{R}^{2d}), \natural)$ *is continuous.*

4. *Every maximal left ideal in* $(L^1(\mathbb{R}^{2d}), \natural)$ *is closed.*

Chapter 0

Notations and Definitions

The purpose of this chapter is to establish notations and function spaces that will be used throughout this dissertation.

Symbols

- $\mathbb{N}$ will denote the set of positive integers, $\mathbb{Z}$ the set of integers, $\mathbb{R}$ the set of real numbers, $\mathbb{C}$ the set of complex numbers. We will be working with $\mathbb{N}^d$, $\mathbb{Z}^d, \mathbb{R}^d, \mathbb{C}^d$, and d will always denote the dimension.
- The notation $A \lesssim B$ means $A \leq cB$ for a some constant $c > 0$, whereas $A \asymp B$ means $c^{-1}A \leq B \leq cA$, for some $c \geq 1$.
- The symbol $A_1 \hookrightarrow A_2$ denotes the continuous embedding of the topological linear space A_1 into A_2.
- If $x, y \in \mathbb{R}^d$, we set $x \cdot y = \sum_1^d x_j y_j$, $|x| = \sqrt{x \cdot x}$.
- For $s \in \mathbb{R}, w \in \mathbb{R}^d$, we put $\langle w \rangle^s = (1 + |w|^2)^{s/2}$.
- For the partial derivatives, we set

  $$\partial_j = \frac{\partial}{\partial x_j},$$

  and for higher-order derivative we use multi-index notation.
- A multi-index is an ordered d-tuple of non-negative integers. If $\alpha = (\alpha_1, ..., \alpha_d)$ is a multi-index, we set $|\alpha| = \sum_1^d \alpha_j, \partial^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d}$.

Function Spaces and Definitions

We will consider certain well known function spaces and some definitions that we record now.

- $C(\mathbb{R}^d)$ will denote the space of continuous functions on $\mathbb{R}^d$, $C_0(\mathbb{R}^d)$ the space of continuous functions $\mathbb{R}^d$ which vanishes at infinity, and $C_c^\infty(\mathbb{R}^d)$ the space of smooth functions on $\mathbb{R}^d$ with compact support.

- $L^{p,q}(\mathbb{R}^d \times \mathbb{R}^d)$ will denote the spaces of measurable functions $f(x,w)$ for which the following mixed norm

$$\|f\|_{L^{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x,w)|^p dx\right)^{q/p} dw\right)^{1/q} \quad (1 \leq p, q < \infty)$$

  is finite. We note that if $p = q$, we have $L^{p,p}(\mathbb{R}^d \times \mathbb{R}^d) = L^p(\mathbb{R}^{2d})$ the usual Lebesgue spaces. To emphasize the dimension, we shall also use the notation $\|f\|_{L^{p,q}(\mathbb{R}^d \times \mathbb{R}^d)}$ for the above norm.

- $L^\infty(\mathbb{R}^d)$ norm is given by

$$\|f\|_{L^\infty} = \text{ess.sup}_{x\in\mathbb{R}^d} |f(x)|.$$

  We note that the above mixed $L^{p,q}$ can be defined by natural modification if p or q is infinite.

- $\ell^q(\mathbb{Z}^d)$ will denote the spaces of sequences on $\mathbb{Z}^d$ for which the following norm

$$\|a\|_{\ell^q} = \left(\sum_{m\in\mathbb{Z}^d} |a_m|^q\right)^{1/q}$$

  is finite.

- For any non-negative integer N and any multi-index α, we define

$$\|f\|_{(N,\alpha)} = \sup_{x\in\mathbb{R}^d} (1 + |x|)^N |\partial^\alpha f(x)|.$$

  Then the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ can be defined by

$$\mathcal{S}(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d) : \|f\|_{(N,\alpha)} < \infty \text{ for all } N, \alpha\}.$$

We note that $\mathcal{S}(\mathbb{R}^d)$ is a Fréchet space with the topology defined by the norms $\|\cdot\|_{(N,\alpha)}$. Moreover, we can define linear and continuous functionals on the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, the so-called tempered distributions, and the space of tempered distributions will be denoted by $\mathcal{S}'(\mathbb{R}^d)$. For the details, see [24, Proposition 8.2] and [24, p.293].

- The Fourier transform $\mathcal{F}:\mathcal{S}(\mathbb{R}^d)\to\mathcal{S}(\mathbb{R}^d)$ is defined by

$$\mathcal{F}f(w)=\widehat{f}(w)=\int_{\mathbb{R}^d} f(t)e^{-2\pi it\cdot w}dt,\ w\in\mathbb{R}^d. \tag{1}$$

Then $\mathcal{F}$ is a bijection and the inverse Fourier transform is given by

$$\mathcal{F}^{-1}f(x)=f^{\vee}(x)=\int_{\mathbb{R}^d} f(w)\,e^{2\pi ix\cdot w}dw,\quad x\in\mathbb{R}^d, \tag{2}$$

and this Fourier transform can be uniquely extended to $\mathcal{F}:\mathcal{S}'(\mathbb{R}^d)\to\mathcal{S}'(\mathbb{R}^d)$. For details, see [24, Corollary 2.28] and [24, p.296].

- The Fourier algebra on $d-$torus will be denoted by $A(\mathbb{T}^d)$, and, it is the space of functions on the $d-$torus $\mathbb{T}^d$ having absolutely convergent Fourier series:

$$A(\mathbb{T}^d)=\{f:\mathbb{T}^d\to\mathbb{C}:\sum_{m\in\mathbb{Z}^d}|\hat{f}(m)|<\infty\},$$

where $\hat{f}(m)=\int_{\mathbb{T}^d} f(x)e^{-2\pi im\cdot x}dx$, the mth Fourier coefficient of f. The space $A(\mathbb{T}^d)$ is a Banach algebra under pointwise addition and multiplication, with respect to the norm

$$\|f\|_{A(\mathbb{T}^d)}:=\sum_{m\in\mathbb{Z}^d}|\hat{f}(m)|.$$

- The algebra of Fourier transforms will be denoted by $A(\mathbb{R}^d)$. We say $f\in A(\mathbb{R}^d)$ if there exists some $\psi\in L^1(\mathbb{R}^d)$ such that

$$f(w)=\hat{\psi}(w)\ (w\in\mathbb{R}).$$

The space $A(\mathbb{R}^d)$ is a Banach algebra under pointwise addition and multiplication, with respect to the norm:

$$\|f\|_{A(\mathbb{R}^d)} := \|\psi\|_{L^1} \;(f \in A(\mathbb{R}^d)).$$

We note that $A(\mathbb{R}^d)$ is also denoted by $\mathcal{F}L^1(\mathbb{R}^d)$.

- Let $(A, \|\cdot\|_A)$ be a Banach algebra. A Banach space $(L, \|\cdot\|_L)$ is called a left Banach $A-$module if there exists a multiplication operation between elements of A and elements of L, denoted by $\cdot$, such that L is an algebraic left module over A with respect to this multiplication and $\|a \cdot x\|_L \leq C\|a\|_A\|x\|_L$ for all $a \in A,\; x \in L$, and for some constant $C \geq 1$.

- Let $I \subset \mathbb{R}$ be an interval and X be a Banach space. The notation $C(I, X)$ will denote the space of continuous functions $u : I \to X$.

- Let $I \subset \mathbb{R}$ be an interval and X be a Banach space. The notation $L^p(I, X)$ will denote the space of measurable functions $u : I \to X$ such that $\|u\|_{L^p(I)} < \infty$.

Chapter 1

Introduction and Preliminaries

The aim of the first section of this chapter is to introduce the nonlinear Schrödinger equation and raise some basic questions concerning it. In Sections 1.2-1.4, we introduce modulation and Wiener amalgam spaces and gather some basic properties of these spaces which will be needed in the later chapters. In the last section, we revisit some of the questions of the first section to see how it leads to modulation spaces, and investigation of later chapter starts. It is hoped thus to convey an idea of how the classical theory of modulation spaces fits into contemporary developments in the area of partial differential equations.

1.1 The Nonlinear Schrödinger Equation

In the early 1925s Erwin Schrödinger has considered the following equation:

$$i\frac{\partial}{\partial t}u(x,t) + \Delta_x u(x,t) = 0, \quad u(x,0) = u_0(x), \tag{1.1}$$

where $\Delta_x = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$ is the Laplacian on $\mathbb{R}^d$, $(x,t) \in \mathbb{R}^d \times \mathbb{R}, i = \sqrt{-1}$, u_0 is a complex valued function on $\mathbb{R}^d$. Taking the Fourier transform with respect to the space variable x in (1.1), we obtain

$$\begin{cases} \widehat{\partial_t u}(\xi,t) = \partial_t \hat{u}(\xi,t) = \widehat{i\Delta u}(\xi,t) = -4\pi^2 i|\xi|^2 \hat{u}(\xi,t), \\ \hat{u}(\xi,0) = \hat{u_0}(\xi). \end{cases}$$

The solution of this ordinary differential equations in t, with parameter ξ, can be written as, $\hat{u}(\xi,t) = e^{-4\pi^2 it|\xi|^2}\hat{u_0}(\xi)$; and then taking inverse Fourier transform, we have

$$u(x,t) = (e^{-4\pi^2 it|\xi|^2}\hat{u_0}(\xi))^\vee = e^{it\Delta}u_0(x).$$

Now it is worth noting the following well-known facts (for instance, see [41, Proposition 4.2], [41, p.63]):

- For all $t \in \mathbb{R}$, $e^{it\Delta} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is an isometry; which implies $\|e^{it\Delta}f\|_{L^2} = \|f\|_{L^2}$.
- For $0 \neq t \in \mathbb{R}$, $e^{it\Delta}$ is not a bounded operator on $L^p(\mathbb{R}^d)$ if $p \neq 2$, that is, $m(\xi) = e^{-4\pi^2 it|\xi|^2}$ is not a L^p multiplier for $p \neq 2$.

Next we consider the initial value problem (with nonlinear term)

$$(NLS) \quad i\frac{\partial}{\partial t}u(x,t) + \Delta_x u(x,t) = F(u(x,t)), \quad u(x,0) = u_0(x),$$

where the nonlinearity is given by a complex function F on $\mathbb{C}$. This equation is known as the nonlinear Schrödinger equation (NLS for short).

In view of this we may conclude that we cannot expect to solve linear Schrödinger equation in the usual Lebesgue spaces $L^p(\mathbb{R})$ $(p \neq 2)$; and so the NLS as well.

A couple of questions arise at this point quite naturally: (1) For which functions spaces one can expect to solve linear Schrödinger equation? (2) For which function spaces one can expect to solve the NLS? (3) For the NLS with a given initial data, does there exists a solution locally in time? Whether is it unique in the considered function space (local well-posedness)? When a local solution can be extended to a global in time? Is it unique (global well-posedness)? (4) If we can solve the NLS in some specific function space, with which nonlinearity?

Investigating and answering these questions is, precisely, the topic of interest, and the main part of this dissertation, and we will return to some of these issues in Section 1.5 and then in Chapter 4.

1.2 The Short-Time Fourier Transform

We know for mathematicians Fourier transform is a wonderful tool and it is indispensable in many situations, but it involves the whole function at once and sometimes it is not an efficient way to measure the different frequencies entered at different times. One way to handle this is not to consider the Fourier transform of f only, but to consider the Fourier transform of f multiplied by translation of g, which leads to the notion of the Short-time Fourier transform of f. More precisely, the short-time Fourier transform(STFT) of a function f with respect to a window function $g \in \mathcal{S}(\mathbb{R}^d)$ is defined by

$$V_g f(x,w) = \int_{\mathbb{R}^d} f(t)\overline{g(t-x)}\, e^{-2\pi i w\cdot t}\, dt, \;\; (x,w) \in \mathbb{R}^{2d} \tag{1.2}$$

whenever the integral exists.

For $x, w \in \mathbb{R}^d$ the translation operator T_x and the modulation operator M_w are defined by $T_x f(t) = f(t-x)$ and $M_w f(t) = e^{2\pi i w\cdot t} f(t)$. Operators of the form $T_x M_w$ or $M_w T_x$ are called time-frequency shifts. We put $g^*(y) = \overline{g(-y)}$.

The STFT is linear in f and conjugate linear in g. Usually the window g is kept fixed, and $V_g f$ is considered a linear mapping from functions on $\mathbb{R}^d$ to functions on $\mathbb{R}^{2d}$. The next lemma reveals many interesting faces of the STFT.

Lemma 1.2.1 *If $f, g \in L^2(\mathbb{R}^d)$, then $V_g f$ is uniformly continuous on $\mathbb{R}^{2d}$, and*

$$\begin{aligned}
V_g f(x,w) &= (\widehat{f \cdot T_x \bar{g}})(w) && (1.3)\\
&= \langle f, M_w T_x g \rangle && (1.4)\\
&= \langle \hat{f}, T_w M_{-x} \hat{g} \rangle && (1.5)\\
&= e^{-2\pi i x\cdot w} (\widehat{\hat{f} \cdot T_w \bar{\hat{g}}})(-x) && (1.6)\\
&= e^{-2\pi i x\cdot w} V_{\hat{g}} \hat{f}(w, -x) && (1.7)\\
&= e^{-2\pi i x\cdot w} (f * M_w g^*)(x) && (1.8)\\
&= (\hat{f} * M_{-x} \hat{g}^*)(w) && (1.9)\\
&= e^{-\pi i x\cdot w} \int_{\mathbb{R}^d} f(t+\frac{x}{2}) \bar{g}(t-\frac{x}{2}) e^{-2\pi i t\cdot w} dt. && (1.10)
\end{aligned}$$

Proof. The proof can be found in [28, Lemma 3.1.1]. In fact, all the identities follows by the straightforward calculations. To derive (1.5) from (1.4), we may use Parseval formula. The uniform continuity of $V_g f$ follows from the continuity of translation $\{T_x\}$ and modulation operators $\{M_w\}$ on $L^2(\mathbb{R}^d)$. □

Remark 1.2.2 A bit roughly speaking, the formulas (1.3) and (1.6) tells us that the STFT is a (local) Fourier transform of f and $\hat{f}$.

Since Schwartz space is invariant under time-frequency shifts, Lemma 1.2.1 (1.4), suggests us to define the STFT for $f \in \mathcal{S}'(\mathbb{R}^d)$ and $g \in \mathcal{S}(\mathbb{R}^d)$ as follows:

$$V_g f(x,w) = \langle f, M_w T_x g \rangle, \tag{1.11}$$

where $\langle f, g \rangle$ denotes the the action of the tempered distribution f on the Schwartz class function g. Thus $V : (f,g) \to V_g(f)$ extends to a bilinear form on $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ and $V_g(f)$ defines a uniformly continuous function on $\mathbb{R}^d \times \mathbb{R}^d$ whenever $f \in \mathcal{S}'(\mathbb{R}^d)$ and $g \in \mathcal{S}(\mathbb{R}^d)$.

The STFT may be considered as the sesquilinear form $(f,g) \mapsto V_g f$ defined on $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. Let $f \otimes g$ be the (tensor) product $f \otimes g(x,t) = f(x)g(t)$, let $\mathcal{T}_a$ be the asymmetric coordinate transform

$$\mathcal{T}_a F(x,t) = F(t, t-x), \tag{1.12}$$

and let $\mathcal{F}_2$ be the partial Fourier transform

$$\mathcal{F}_2 F(x,w) = \int_{\mathbb{R}^d} F(x,t) e^{-2\pi i t \cdot w} dt. \tag{1.13}$$

A straightforward computation gives the following factorization for the STFT:

Lemma 1.2.3 *If $f, g \in L^2(\mathbb{R}^d)$, then*

$$V_g f = \mathcal{F}_2 \mathcal{T}_a (f \otimes \bar{g}). \tag{1.14}$$

Remark 1.2.4 (1) Note first that both operators $\mathcal{T}_a$ and $\mathcal{F}_2$ are isomorphisms on $\mathcal{S}'(\mathbb{R}^{2d})$. If $f, g \in \mathcal{S}'(\mathbb{R}^d)$, then $f \otimes \bar{g} \in \mathcal{S}'(\mathbb{R}^{2d})$ as well. Thus, $V_g f$ is well defined tempered distribution whenever $f, g \in \mathcal{S}'(\mathbb{R}^d)$. See also [23, Proposition 1.42].
(2) For more detail on previous discussion, see [28, Theorem 11.2.3].
(3) Remembering the Bargmann transform

$$Bf(z) = \int_{\mathbb{R}^d} f(t) e^{2\pi t \cdot z - \pi t^2 - \frac{\pi}{2} z^2} dt \quad (z \in \mathbb{C}^d),$$

and taking the Gaussian window function as $g(x) = e^{-\pi x^2}$, it is easy to see that,

$$V_g f(x, -w) = e^{\pi i x \cdot w} Bf(z) e^{-\pi |z|^2/2} \quad (z = x + iw),$$

so we may say that the STFT is a real variable reformulation of the Bargmann transform $Bf(z)$.

1.3 Modulation and Wiener Amalgam Spaces

In 1983 Feichtinger [21] introduced a class of Banach spaces, which allow a measurement of space variable and Fourier transform variable of a function or distribution f on $\mathbb{R}^d$ simultaneously using the STFT, the so-called modulation spaces.

Definition 1.3.1 (modulation spaces) For $1 \leq p, q \leq \infty, s \geq 0$, and for given a non zero smooth rapidly decreasing function $g \in \mathcal{S}(\mathbb{R}^d)$, the weighted modulation space $M_s^{p,q}(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ for which, the following norm

$$\|f\|_{M_s^{p,q}(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x, w)|^p dx \right)^{q/p} \langle w \rangle^{sq} dw \right)^{1/q}$$

is finite, with the usual modification if p or q is infinite.

Remark 1.3.2 (1) The definition of the modulation space given above, is independent of the choice of the particular window function. In fact if g and g' are any two window functions, then we have the relation

$$\|V_{g'} f\|_{L_s^{p,q}} \lesssim \|V_{g'} g\|_{L_s^{1,1}} \|V_g f\|_{L_s^{p,q}},$$

see [28, Proposition 11.3.2, p.233]. It follows that, the modulation space norms given by g and g' are equivalent.
(2) The modulation spaces can also be defined for exponents $0 < p, q < 1$, see [64, 66, 40].
(3) When $s = 0$, we simply write $M_0^{p,q}(\mathbb{R}^d) = M^{p,q}(\mathbb{R}^d)$.
(4) If there is no confusion, we also use the notation for the norm $\|f\|_{M_s^{p,q}(\mathbb{R}^d)} = \|f\|_{M_s^{p,q}}$.

By reversing the order of integration we define the another family of spaces, so-called Wiener amalgam spaces.

Definition 1.3.3 (Wiener amalgam spaces) For $1 \leq p, q \leq \infty, s \geq 0$, and $0 \neq g \in \mathcal{S}(\mathbb{R}^d)$, the weighted Wiener amalgam space $W_s^{p,q}(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that the norm

$$\|f\|_{W_s^{p,q}(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x,w)|^q \langle w \rangle^{sq} dw \right)^{p/q} dx \right)^{1/p}$$

is finite, with usual modifications if p or $q = \infty$.

Remark 1.3.4 (1) When $s = 0$, we simply write $W_0^{p,q}(\mathbb{R}^d) = W^{p,q}(\mathbb{R}^d)$.
(2) If there is no confusion, we also use the notation for the norm $\|f\|_{W_s^{p,q}(\mathbb{R}^d)} = \|f\|_{W_s^{p,q}}$.

By Lemma 1.4.1(1.7), we have

$$|V_g f(x,w)| = |V_{\hat{g}} \hat{f}(w,-x)|; \tag{1.15}$$

and as a consequence, we have

$$\|f\|_{W^{p,q}} \asymp \|\hat{f}\|_{M^{q,p}}. \tag{1.16}$$

This relation tells us that the fundamental properties of $W_s^{p,q}(\mathbb{R}^d)$ we may derive from $M_s^{p,q}(\mathbb{R}^d)$. For example, the definition of $W_s^{p,q}(\mathbb{R}^d)$ is independent of the choice of the window function $0 \neq g \in \mathcal{S}(\mathbb{R}^d)$, that is, different window functions yield equivalent norms since this is the case for the modulation space $M^{p,q}(\mathbb{R}^d)$. See Remark 1.3.2.

We note that there is another characterization [64, 66, 40] of modulation and Wiener amalgam spaces: let $\phi \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\text{supp}\phi \subset (-1,1)^d$$

and

$$\sum_{m \in \mathbb{Z}^d} \phi(w-m) = 1, \forall w \in \mathbb{R}^d.$$

Then we have the equivalence

$$\|f\|_{M_s^{p,q}} \asymp \|\|\langle m\rangle^s \phi(D-m)f\|_{L^p}\|_{\ell^q},$$

and

$$\|f\|_{W_s^{p,q}} \asymp \|\|\langle m\rangle^s \phi(D-m)f\|_{\ell^q}\|_{L^p},$$

where $\phi(D-m)f = \mathcal{F}^{-1}(\widehat{f} \cdot T_m\phi)$.

Note. In the above definition, we have followed notations as in [53, 50, 5].

Remark 1.3.5 The space $M^{1,1}(\mathbb{R}^d)$ is a Segal algebra. In the literature, it is also known as the Feichtinger algebra and often denoted by $S_0(\mathbb{R}^d)$. See [18].

1.4 Basic Properties of Modulation and Wiener Amalgam Spaces

Now we collect some basic properties of modulation and Wiener amalgam spaces which we shall need later.

Lemma 1.4.1 *Let $p, q, p_i, q_i \in [1, \infty]$ $(i = 1, 2)$.*

1. $\mathcal{S}(\mathbb{R}^d) \hookrightarrow M^{p,q}(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$ *and* $\mathcal{S}(\mathbb{R}^d) \hookrightarrow W^{p,q}(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$.

2. *If $q_1 \leq q_2$ and $p_1 \leq p_2$, then* $W^{p_1,q_1}(\mathbb{R}^d) \hookrightarrow W^{p_2,q_2}(\mathbb{R}^d)$ *and* $M^{p_1,q_1}(\mathbb{R}^d) \hookrightarrow M^{p_2,q_2}(\mathbb{R}^d)$.

3. $M^{p,q_1}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d) \hookrightarrow M^{p,q_2}(\mathbb{R}^d)$ *and* $W^{p,q_1}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d) \hookrightarrow W^{p,q_2}(\mathbb{R}^d)$ *holds for $q_1 \leq min\{p, p'\}$ and $q_2 \geq max\{p, p'\}$ with $\frac{1}{p} + \frac{1}{p'} = 1$.*

4. $M^{p,q}(\mathbb{R}^d) \hookrightarrow W^{p,q}(\mathbb{R}^d)$ *when $q \leq p$ and* $W^{p,q}(\mathbb{R}^d) \hookrightarrow M^{p,q}(\mathbb{R}^d)$ *when $p \leq q$.*

5. $\mathcal{S}(\mathbb{R}^d)$ *is dense in $M^{p,q}(\mathbb{R}^d)$ if p and $q < \infty$.*

6. *The spaces $W^{p,q}(\mathbb{R}^d)$ and $M^{p,q}(\mathbb{R}^d)$ are Banach spaces.*

7. *The spaces $W^{p,q}(\mathbb{R}^d)$ and $M^{p,q}(\mathbb{R}^d)$ are invariant under complex conjugation. In particular, we have the inequality $\|Ref\|_{M_s^{p,q}} \leq \|f\|_{M_s^{p,q}}$ and $\|Imf\|_{M_s^{p,q}} \leq \|f\|_{M_s^{p,q}}$.*

8. *The Fourier transform establish an isomorphism* $\mathcal{F} : W^{p,q}(\mathbb{R}^d) \to M^{q,p}(\mathbb{R}^d)$.

Proof. All these statements are well-known and the interested reader may find a proof in [28, 50, 69, 21].

The proof of statement (1) follows from [28, Theorem 11.2.5]. In fact, by Lemma 1.2.3, we have the factorization $V_g f = \mathcal{F}_2 \mathcal{T}_a (f \otimes \bar{g})$ of the STFT into coordinate transformation $\mathcal{T}_a$ and the partial Fourier transform $\mathcal{F}_2$. If $f, g \in \mathcal{S}(\mathbb{R}^d)$, then $f \otimes \bar{g} \in \mathcal{S}(\mathbb{R}^{2d})$. Since $\mathcal{S}(\mathbb{R}^{2d})$ is invariant under both operators $\mathcal{T}_a$ and $\mathcal{F}_2$, it follows that $V_g f \in \mathcal{S}(\mathbb{R}^{2d})$. Hence, the proof of statement (1) follows. For the proof of statement (2), see [28, Theorem 12.2.2]. For the proof of statement (3), see [59, Proposition 1.7] and [53]. For the proof of statement (4), see [50, Section 5]. For the proof of statement (5), see [28, Proposition 11.3.4]. For the proof of statement (6), see [28, Theorem 11.3.5]. The proof of statement (7) follows by definition. In view of the fundamental identity (1.7) of time-frequency analysis it follows that

$$\|f\|_{W^{p,q}} \asymp \|\hat{f}\|_{M^{q,p}},$$

which established the proof of statement (8). □

Remark 1.4.2 There are several embedding results between Lebesgue, Sobolev, or Besov spaces and modulation spaces, see for example, [43, 54, 59, 27]. In fact, the necessary and sufficient condition for embedding between Besov spaces $B^s_{p,q}$, and modulation spaces $M^{p,q}_s$ for all $s \in \mathbb{R}, 0 < p, q \leq \infty$ has been obtained in [67, 68]. We note, in particular that the L^2 Sobolev space $H^s(\mathbb{R}^d)$ coincides with $M^{2,2}_s(\mathbb{R}^d)$, see [28, Proposition 11.3.1].

Proposition 1.4.3 (Algebra Property) *Let* $p_i, q_i \in [1, \infty]$ $(i = 0, 1, 2)$.

1. $M^{p_1,q_1}(\mathbb{R}^d) * M^{p_2,q_2}(\mathbb{R}^d) \hookrightarrow M^{p_0,q_0}(\mathbb{R}^d)$ *for* $\frac{1}{p_1} + \frac{1}{p_2} = 1 + \frac{1}{p_0}$ *and* $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_0}$ *with norm inequality*

$$\|f * h\|_{M^{p_0,q_0}} \lesssim \|f\|_{M^{p_1,q_1}} \|h\|_{M^{p_2,q_2}}.$$

In particular, $M^{p,q}(\mathbb{R}^d)$ *is a left Banach* $L^1(\mathbb{R}^d)-$*module with respect to convolution.*

2. *$W^{p_1,q_1}(\mathbb{R}^d) * W^{p_2,q_2}(\mathbb{R}^d) \hookrightarrow W^{p_0,q_0}(\mathbb{R}^d)$ for $\frac{1}{p_1}+\frac{1}{p_2}=1+\frac{1}{p_0}$ and $\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{q_0}$ with norm inequality*

$$\|f*h\|_{W^{p_0,q_0}} \lesssim \|f\|_{W^{p_1,q_1}}\|h\|_{W^{p_2,q_2}}.$$

In particular, $W^{p,q}(\mathbb{R}^d)$ is a left Banach $L^1(\mathbb{R}^d)-$module with respect to convolution.

3. *$M^{p_1,q_1}(\mathbb{R}^d)\cdot M^{p_2,q_2}(\mathbb{R}^d) \hookrightarrow M^{p_0,q_0}(\mathbb{R}^d)$ for $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_0}$ and $\frac{1}{q_1}+\frac{1}{q_2}=1+\frac{1}{q_0}$ with norm inequality*

$$\|f\cdot h\|_{M^{p_0,q_0}} \lesssim \|f\|_{M^{p_1,q_1}}\|h\|_{M^{p_2,q_2}}.$$

In particular, $M^{p,q}(\mathbb{R}^d)$ is a left Banach $\mathcal{F}L^1(\mathbb{R}^d)-$module with respect to pointwise multiplication.

4. *$W^{p_1,q_1}(\mathbb{R}^d)\cdot W^{p_2,q_2}(\mathbb{R}^d) \hookrightarrow W^{p_0,q_0}(\mathbb{R}^d)$ for $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_0}$ and $\frac{1}{q_1}+\frac{1}{q_2}=1+\frac{1}{q_0}$ with norm inequality*

$$\|h\cdot f\|_{W^{p_0,q_0}} \lesssim \|h\|_{W^{p_1,q_1}}\|f\|_{W^{p_2,q_2}}.$$

Proof. Since $\mathcal{S}(\mathbb{R}^d)*\mathcal{S}(\mathbb{R}^d)\hookrightarrow\mathcal{S}(\mathbb{R}^d)$, $g:=g_0*g_0\in\mathcal{S}(\mathbb{R}^d)$ for $g_0\in\mathcal{S}(\mathbb{R}^d)$, we recall $g^*(y)=\overline{g(-y)}$ and we note that $M_w(g^*)=M_w(g_0^**g_0^*)=M_wg_0^**M_wg_0^*$, and in view of (1.8), Young's inequality, and Hölder's inequality, we may find

$$\begin{aligned}
\|h*f\|_{M^{p_0,q_0}} &\lesssim \|\|(h*M_wg_0^*)*(f*M_wg_0^*)\|_{L^{p_0}}\|_{L^{q_0}}\\
&\lesssim \|\|h*M_wg_0^*\|_{L^{p_1}}\|f*M_wg_0^*\|_{L^{p_2}}\|_{L^{q_0}}\\
&\lesssim \|\|h*M_wg_0^*\|_{L^{p_1}}\|_{L^{q_1}}\|\|f*M_wg_0^*\|_{L^{p_2}}\|_{L^{q_2}}\\
&\lesssim \|h\|_{M^{p_1,q_1}}\|f\|_{M^{p_2,q_2}}.
\end{aligned}$$

By Lemma 1.4.1(3), we have $L^1(\mathbb{R}^d)\hookrightarrow M^{1,\infty}(\mathbb{R}^d)$. This completes the proof of statement (1). Now we shall see how the statement (4) can be derived from statement (1). By Lemma 1.4.1(8) and statement (1), we have

$$\begin{aligned}
\|hf\|_{W^{p_0,q_0}} &\asymp \|\hat{h}*\hat{f}\|_{M^{q_0,p_0}}\\
&\lesssim \|\hat{h}\|_{M^{q_1,p_1}}\|\hat{f}\|_{M^{q_2,p_2}}\\
&\lesssim \|h\|_{W^{p_1,q_1}}\|f\|_{W^{p_2,q_2}}.
\end{aligned}$$

This completes the proof of statement (4). The proof of statement (3) can be found in [59, Theorem 2.4]. The proof of statement (2) can be derived from (3) via Lemma 1.4.1(8).

In fact, these algebra properties are well-known, and can also be found in [59], [50, Section 5]. □

Next we prove an approximation result on the modulation space $M^{p,q}(\mathbb{R}^d)$ for $1 \leq p, q < \infty$. Let $\phi \in \mathcal{S}(\mathbb{R}^d)$, with $\int_{\mathbb{R}^d} \phi = 1$ and and set $\phi_r(x) := r^{-d}\phi(x/r), r > 0$. Then the family $\{\varphi_r\}_{r>0}$ is called an approximate identity in $M^{p,q}(\mathbb{R}^d)$ in view of the next lemma.

Lemma 1.4.4 (Approximate identity) *Let $\{\phi_r\}_{r>0}$ be as above and $f \in M^{p,q}(\mathbb{R}^d), 1 \leq p, q < \infty$. Then given $\epsilon > 0$, there exists a $\delta > 0$ such that $\|f * \phi_r - f\|_{M^{p,q}} < \epsilon$ whenever $r < \delta$.*

Proof. The proof is straightforward. First we assume that $f \in \mathcal{S}(\mathbb{R}^d)$. Since $\int_{\mathbb{R}^d} \phi = 1$, setting $y = rz$, we see that,

$$\begin{aligned} f * \phi_r(t) - f(t) &= \int_{\mathbb{R}^d} [f(t-y) - f(t)]\phi_r(y)dy \\ &= \int_{\mathbb{R}^d} [f(t-rz) - f(t)]\phi(z)dz \\ &= \int_{\mathbb{R}^d} [T_{rz}f(t) - f(t)]\phi(z)dz. \end{aligned}$$

Put $h_r(t) = f * \phi_r(t) - f(t)$; and take $0 \neq g \in \mathcal{S}(\mathbb{R}^d)$. Then

$$V_g h_r(x, w) = \int_{\mathbb{R}^d} V_g(T_{rz}f - f)(x, w)\, \phi(z)dz.$$

Taking mixed $L^{p,q}$ norm and an application of Minkowski's inequality for integrals, this gives,

$$\|h_r\|_{M^{p,q}} \leq \int_{\mathbb{R}^d} \|T_{rz}f - f\|_{M^{p,q}} |\phi(z)|dz.$$

Now the proof follows from the dominated convergence theorem. Note that $\|T_{rz}f - f\|_{M^{p,q}} \leq 2\|f\|_{M^{p,q}}$ by translation invariance of $M^{p,q}$ norm.

Also since $V_g T_{rz} f(x, w) = M_{(0,-rz)} \left(T_{(rz,0)} V_g f\right)(x, w)$, we have

$$\|T_{rz}f - f\|_{M^{p,q}} = \|V_g T_{rz}f - V_g f\|_{L^{p,q}}$$

$$\begin{aligned} &= \|M_{(0,-rz)}(T_{(rz,0)}V_g f) - M_{(0,-rz)}(V_g f) + M_{(0,-rz)}(V_g f) - V_g f\|_{L^{p,q}} \\ &\leq \|T_{(rz,0)}(V_g f) - V_g f\|_{L^{p,q}} + \|M_{(0,-rz)}(V_g f) - V_g f\|_{L^{p,q}} \end{aligned}$$

each of these tend to 0 as $r \to 0$, again by the continuity of the translation and modulation operators in the mixed L^p space $L^{p,q}(\mathbb{R}^{2d})$, $(1 \leq p, q < \infty)$.

To complete the proof, we note that, if f is a general element in $M^{p,q}(\mathbb{R}^d)$, then by density, we can choose a $g \in \mathcal{S}(\mathbb{R}^d)$ such that $\|f - g\|_{M^{p,q}} < \frac{\epsilon}{4}$. Then

$$\begin{aligned} &\|f * \phi_r - f\|_{M^{p,q}} \\ &\qquad \leq \|(f-g) * \phi_r\|_{M^{p,q}} + \|g * \phi_r - g\|_{M^{p,q}} + \|g - f\|_{M^{p,q}} \\ &\qquad \leq 2\|(f-g)\|_{M^{p,q}} + \|g * \phi_r - g\|_{M^{p,q}} \end{aligned}$$

in view of Proposition 1.4.3(1). Thus the general case follows since $g \in \mathcal{S}(\mathbb{R}^d)$. □

Remark 1.4.5 For future use we record that, if there are finitely many functions $f_1, ..., f_N$, a single δ can be chosen that works for all f_i's, by simply choosing $\delta = \min\{\delta_i : i = 1, 2, ...N\}$.

Some of the weighted modulations spaces $M_s^{p,q}(\mathbb{R}^d)$ are multiplicative algebras. To be more specific, we state the following result. For the proof, see [53, Proposition 3.2], [66], [2, Corollary 2.7].

Proposition 1.4.6 *Let $X = M_s^{p,q}(\mathbb{R}^d), 1 \leq p, q \leq \infty$ and $s > d/q'$, or $X = M_s^{p,1}(\mathbb{R}^d), 1 \leq p \leq \infty, s \geq 0$. Then X is a multiplication algebra, and we have the inequality*

$$\|f \cdot g\|_X \lesssim \|f\|_X \|g\|_X, \tag{1.17}$$

for all $f, g \in X$.

The next proposition gives a sufficient condition for a function to be in $M^{1,1}(\mathbb{R}^d)$. See [28, p.250] for a proof.

Proposition 1.4.7 *Let $L_s^2(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} |f(x)|^2 (1 + |x|)^{2s} < \infty\}$. If both f and $\widehat{f}$ are in $L_s^2(\mathbb{R}^d)$ for some $s > d$, then $f \in M^{1,1}(\mathbb{R}^d)$.*

Finally, in this section, we note that to see how modulation and Wiener amalgam spaces arise in the early eighties and how it has been further studied and generalized to the theory of co-orbit spaces by Feichtinger-Gröchening [19, 20]. And how it fits into the development of contemporary time-frequency analysis, we refer the interested reader to the excellent article (historical development point of view) of Feichtinger [22].

1.5 Known Results for the NLS in Modulation Spaces

For $f \in \mathcal{S}(\mathbb{R}^d)$, we define the Schrödinger propagator $e^{it\Delta}$ for $t \in \mathbb{R}$ as follows:

$$e^{it\Delta} f(x) := \int_{\mathbb{R}^d} e^{i\pi t|\xi|^2}\, \hat{f}(\xi)\, e^{2\pi i\xi\cdot x}\, d\xi = \sigma_t^{\vee} * f(x), (x \in \mathbb{R}^d), \tag{1.18}$$

where $\sigma_t(\xi) := e^{i\pi t|\xi|^2}$, $(\xi \in \mathbb{R}^d)$.

The next proposition shows that the uniform boundedness of the Schrödinger propagator $e^{it\Delta}$ in modulation spaces.

Proposition 1.5.1 ([4]) *Let $t \in \mathbb{R}$, $p, q \in [1, \infty]$. Then*

$$\|e^{it\Delta} f\|_{M^{p,q}} \leq C(t^2+1)^{d/4} \|f\|_{M^{p,q}}, \tag{1.19}$$

where C is some constant depending only on d.

Before the proof of Proposition 1.5.1, we recall the Fourier transform of generalized Gaussian, which enables us to compute the modulation and Wiener amalgam space norm of the multiplier σ_t. We need the following temporary definitions. Let f is a generalized Gaussian of the form

$$f(x) \quad = \quad e^{-\pi x\cdot Ax + 2\pi b\cdot x + c}, \tag{1.20}$$

where $A \in GL(d, \mathbb{C})$ is an invertible $d \times d$ matrix over $\mathbb{C}$ with positive definite real part and $b \in \mathbb{C}^d, c \in \mathbb{C}$.

Definition 1.5.2 Let B be an invertible $d \times d$ matrix over $\mathbb{R}$ and C be a symmetric $d \times d$ matrix over $\mathbb{R}$. Then we define

$$U_B f(x) = |\det\, B|^{1/2} f(Bx)$$

and

$$N_C f(x) = e^{-\pi i x\cdot Cx} f(x)$$

to be the unitary operators of coordinate change and multiplication by the chirp $e^{-\pi i x\cdot Cx}$.

Next lemma gives the explicit form of the Fourier transform of the generalized Gaussian. For the proof, see [28, Lemma 4.4.2, p.70].

Lemma 1.5.3 *Let f be the generalized Gaussian of the form (1.20) and write $A = B + iC$ with B real-valued positive definite and C symmetric. Also write $b = b_1 + ib_2, b_1, b_2 \in \mathbb{R}^d$.*

1. *Then*

$$f = kM_{b_2 - CB^{-1}b_1} T_{B^{-1}b_1} N_C U_{B^{1/2}} \phi_1, \tag{1.21}$$

   where $k \in \mathbb{C}$ and $\phi_1(x) = e^{-\pi x^2}$.

2. *The Fourier transform of f is again Gaussian, specifically,*

$$\hat{f} = (det\ A)^{-1/2} k T_{b_2 - CB^{-1}b_1} M_{-B^{-1}b_1}(e^{-\pi w\cdot A^{-1}w}). \tag{1.22}$$

Proof of Proposition 1.5.1. In view of (1.18), and Proposition 1.4.3(1), we may find

$$\|e^{it\Delta} f\|_{M^{p,q}} \quad \lesssim \quad \|\sigma_t^\vee\|_{M^{1,\infty}} \|f\|_{M^{p,q}};$$

and note that $\|\sigma_t^\vee\|_{M^{1,\infty}} \asymp \|\sigma_t\|_{W^{\infty,1}}$, and by exploiting calculation as in [4, Theorem 14] one can obtain $\|\sigma_t\|_{W^{\infty,1}} = C_d(1+t^2)^{d/4}$.

The explicit computation for the norm $\|\sigma_t^\vee\|_{M^{1,\infty}}$ is delicate, however, to give the flavor, and to illustrate how it can be done, now we will sketch the proof.

We use the Gaussian $g(\xi) = e^{-\pi|\xi|^2}$ as a window for the short-time Fourier transform. Then the STFT $V_g\sigma_t$ can be calculated explicitly by using Gaussian integrals.

Let $x, w \in \mathbb{R}^d, t \in \mathbb{R}$, and by (1.2), we have

$$V_g\sigma_t(x,w) \quad = \quad \int_{\mathbb{R}^d} e^{i\pi t|\xi|^2} e^{-2\pi i\xi\cdot w} e^{-\pi|\xi - x|^2} d\xi$$

$$= \quad e^{-\pi|x|^2}\int_{\mathbb{R}^d} e^{-\pi(1-it)|\xi|^2}e^{2\pi\xi\cdot x}e^{-2\pi i\xi\cdot w}d\xi. \tag{1.23}$$

In (1.23), the integral is the Fourier transform of a generalized Gaussian, say $h(y) = e^{-\pi(1-it)|y|^2}e^{2\pi y\cdot x} = e^{-\pi y\cdot Ay+2\pi b\cdot y}$, where A is the $d\times d$ diagonal matrix, with diagonal entries are $1-it$ and $b = x$. By Lemma 1.5.3, we obtain

$$V_g\sigma_t(x,w) = e^{-\pi|x|^2}(1-it)^{-d/2}e^{\pi(1-it)|x|^2}T_{tx}M_{-x}(e^{-\pi|w|^2/(1-it)})$$

(where the square root $(1-it)^{1/2}$ is taken with positive imaginary part). After taking absolute values and performing some cancellations we arrive at the expression

$$|V_g\sigma_t(x,w)| = (1+t^2)^{-d/4}e^{-\pi|w-tx|^2/(1+t^2)}.$$

Since $\int_{\mathbb{R}^d} e^{-a|x|^2}dx = a^{-d/2}$ (see [24, Proposition 2.53]), we may obtain

$$\begin{aligned}\|\sigma_t\|_{W^{\infty,1}} &= \sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}|V_g\sigma_t(x,w)|dw\\ &= (1+t^2)^{-d/4}\int_{\mathbb{R}^d} e^{-\frac{\pi}{t^2+1}|w|^2}dw\\ &= \pi^{-d/2}(t^2+1)^{d/4}.\end{aligned}$$

□

Now we are in a position to state without proof one of the main known result from the existing literature and we shall return to this later in Chapter 4. In fact, by using Propositions 1.5.1 and 1.4.3(3), we can prove the following well-posedness result, for detail see Theorem 4.2.6.

Theorem 1.5.4 (Local well-posedness) *Assume that* $u_0 \in M^{p,1}(\mathbb{R}^d)$ $(1 \leq p \leq \infty)$ *and the nonlinearity* F *has the form* $F(z) = |z|^{2k}z, k \in \mathbb{N}$. *Then, there exists* $T = T(\|u_0\|_{M^{p,1}}) > 0$ *such that NLS has a unique solution* $u \in C([0,T], M^{p,1}(\mathbb{R}^d))$. *Moreover, if* $T < \infty$ *then* $\limsup_{t\to T}\|u(\cdot,t)\|_{M^{p,1}} = \infty$.

This result deserves some historical remarks: in search of obtaining well-posedness results for the NLS, and inspired from the uniform decomposition techniques in 2006, Wang-Zhao-Guo [66] have constructed the spaces $E^{\lambda}_{p,q}$ (in fact, immediately in the subsequent papers [68, 4] it has been recognized that it is modulation spaces). They asserted that the Schrödinger propagator is bounded on $E^{\lambda}_{p,q}$ spaces independently(in contrast to time-frequency [see Proposition 1.5.1 above] techniques). They also showed that space $E^{0}_{2,1}$ is an algebra under pointwise multiplication. And showed that the NLS is locally well-posed (see [66, Theorem 1.1]) in $E^{0}_{2,1} = M^{2,1}(\mathbb{R}^d)$. Since then many mathematicians have been attracted in this direction, and the modulation and Wiener amalgam spaces have made their own place in partial differential equations(PDEs), see [68, 2, 12, 32]. In particular we mention, in 2009 Bényi-Okoudjou in [2] have used time-frequency techniques to obtain the local well-posedness result (see [2, Theorem 1.1]) in $M^{p,1}(\mathbb{R}^d)$ $(1 \leq p \leq \infty)$ with the non-linearity of the generic form $F(u) = g(|u|^2)\, u$, for some complex-entire function $g(z)$, and immediately after this, it has been noted by Cordero-Nicola [12] that this non-linearity can be replaced by real entire function.

One of the key points in the above local well-posedness results is that, the above nonlinearities map the modulation space to itself. In fact, the proof of the above local well-posedness results crucially relies on the fact that $M^{p,1}_s(\mathbb{R}^n)$ is a function algebra under pointwise multiplication: $\|fg\|_{M^{p,1}_s} \leq C\|f\|_{M^{p,1}_s}\|g\|_{M^{p,1}_s}$ for some constant C. Therefore, if $\alpha = 2k$, $|u|^{\alpha}u = u^{k+1}\bar{u}^k$ $(k \in \mathbb{N})$ and hence

$$\||u|^{2k}u\|_{M^{p,1}_s} \leq C\|u\|^{2k+1}_{M^{p,1}_s}.$$

Hence the nonlinearity of the type $F(z) = z|z|^{\alpha}$, $\alpha \in 2\mathbb{N}$ can be handled in this way. Of course it is very natural to ask, how far can one go, to include more general nonlinear terms in these dispersive equations on modulation spaces? It was in this context Ruzhansky-Sugimoto-Wang [49, p.280] raised the open problem:

$$\text{Does } \||u|^{\alpha}u\|_{M^{p,1}} \leq \|u\|^{\alpha+1}_{M^{p,1}} \text{ hold for all } \alpha \in (0,\infty) \setminus 2\mathbb{N}?$$

This question inspires us to study the mapping properties (see Chapter 2 below) on the modulation and Wiener amalgam spaces and this is precisely the starting point for the investigation of this dissertation.

Finally, in this section, we note that there is also an equivalent definition of modulation spaces using frequency-uniform decomposition techniques (which is quite similar in the spirit of Besov spaces), independently studied by Wang et al. in [66], which has turned out to be very fruitful in PDEs, see [68]. For a brief survey of modulation spaces and nonlinear evolution equations, we refer the interested reader to [49] and for further reading from the PDEs viewpoint we refer to [69] and the references therein.

Chapter 2

Composition Operators on $M^{p,q}(\mathbb{R}^d)$ and $W^{p,q}(\mathbb{R}^d)$

The aim of this chapter is to study composition operators on modulation and Wiener amalgam spaces and as a consequence to answer the question concerning general power type linearity mentioned in Section 1.5.

2.1 Introduction

Let X and Y be normed spaces of functions. For a given function $F : \mathbb{R}^2 \to \mathbb{C}$, we associate with it, the composition operator $T_F : f \mapsto F(f)$ which maps X to Y, that is, $F(f) \in Y$ whenever $f \in X$; where $F(f)$ is the composition of functions F and f. If $T_F : X \to X$, we say the composition operator T_F acts on X.

Can we characterize functions F for which the composition operator T_F maps X to Y?

Of course, the properties of the operator T_F strongly depend on X and Y. The aim of this chapter is to take a small step toward the answer in the case of modulation and Wiener amalgam spaces.

In Section 1.5, we have noted that how much modulation spaces is important from the PDE point of view, in fact, both modulation and Wiener amalgam spaces have turned out to be very fruitful in various applications. In fact, these spaces are nowadays present in investigations that concern problems Fourier multipliers, pseudo differential operators, Fourier integral operators, Strichartz estimates, nonlinear partial differential equations(PDEs), and so on (cf. [2, 4, 13, 15, 30, 31]). For instance: the unimodular Fourier multiplier operator $e^{i|D|^\alpha}$ is not bounded on most of the Lebesgue spaces $L^p(\mathbb{R}^d)$ $(p \neq 2)$ or even Besov spaces [36]; in contrast it is bounded on $W^{p,q}(\mathbb{R}^d)(1 \leq p, q \leq \infty)$ for $\alpha \in [0,1]$, and on $M^{p,q}(\mathbb{R}^d)(1 \leq p, q \leq \infty)$ for $\alpha \in [0,2]$ (cf. [2, 4, 15]). The cases $\alpha = 1, 2$ are of particular interest because they occur in the time evolution of wave and Schrödinger equations respectively. Many mathematicians have been using these spaces as a regularity class of initial data for the Cauchy problem for nonlinear evolution equations [66, 69, 2, 4, 68]), see also Chapter 4 below. In particular, we mention, Cordero-Nicola [12] have used these spaces as underlying working spaces for the nonlinear wave equation, with real entire nonlinearity.

But one of the underneath issue in the nonlinear PDEs in the realm of modulation and Wiener amalgam spaces is to determine, which is the most general nonlinearity one can take. This is not yet completely clear (see Section 1.5 above), and therefore the problem stated in the first paragraph lies at the interface between the time-frequency analysis (modulation/ Wiener amalgam spaces) and nonlinear PDEs, and hopefully the answer will serve the bridge between them.

Inspired from these considerations, and in pursuing our aim, we have obtained the necessary condition (see Theorem 2.2.1(1) below): if T_F maps $M^{p,1}(\mathbb{R}^d)$ to $M^{p,q}(\mathbb{R}^d)$ $(1 \leq p \leq \infty, 1 \leq q < 2)$, then F is real analytic on $\mathbb{R}^2$. The proof relies on the "localized" version of the "time-frequency" spaces, which can be identified with the Fourier algebra on the torus $A(\mathbb{T}^d)$. As a consequence, there exist functions (see Corollary 2.2.5 below) $f \in M^{p,1}(\mathbb{R}^d)$ such that $f|f|^\alpha, \alpha \in (0,\infty) \setminus 2\mathbb{N}$ does not belong to $M^{p,q}(\mathbb{R}^d)$. The analogous necessary condition(see Theorem 2.2.1(2)) holds in the case of Wiener spaces. On the other hand, we show that (see Theorem 2.3.3 below) F is real analytic on $\mathbb{R}^2$, and $F(0) = 0$, then T_F maps $M^{1,1}(\mathbb{R}^d)$ to $M^{1,1}(\mathbb{R}^d)$. And as consequence of these necessary and sufficient conditions, we answer the above problem completely in $M^{1,1}(\mathbb{R}^d)$:

A composition operator T_F acts on $M^{1,1}(\mathbb{R}^d)$ if and only if $F(0) = 0$ and F is real analytic on $\mathbb{R}^2$.

We note that the proof for the sufficient condition relies on the invariant property of the modulation space $M^{1,1}(\mathbb{R}^d)$ under the Fourier transform. This invariance is not available for $M^{p,1}(\mathbb{R}^d)$, when $p > 1$, however, this inspires us to obtain (Theorem 2.3.9 below) a partial converse to Theorem 2.2.1(necessary condition): if we restrict the domain of the T_F to be a subclass of $M^{p,1}(\mathbb{R}^d)$ or $W^{p,1}(\mathbb{R}^d)(1 < p < \infty)$ which is invariant under Fourier transform and vanishing at infinity.

2.2 Necessary Condition

In this section, we prove that if the composition operator T_F maps modulation spaces $M^{p,1}(\mathbb{R}^d)$ to $M^{p,q}(\mathbb{R}^d)$, then F is necessarily real analytic on $\mathbb{R}^2$. A similar necessity condition is also proved for Wiener amalgam spaces.

Theorem 2.2.1 *Suppose that T_F is the composition operator associated to a complex function F on $\mathbb{C} = \mathbb{R}^2$, and $1 \leq p \leq \infty$ and $1 \leq q < 2$.*

1. *If T_F maps $M^{p,1}(\mathbb{R}^d)$ to $M^{p,q}(\mathbb{R}^d)$, then F must be real analytic on $\mathbb{R}^2$. Moreover, $F(0) = 0$ if $p < \infty$.*
2. *If T_F maps $W^{p,1}(\mathbb{R}^d)$ to $W^{p,q}(\mathbb{R}^d)$, then F must be real analytic on $\mathbb{R}^2$. Moreover, $F(0) = 0$ if $p < \infty$.*

We start with following:

Definition 2.2.2 A complex valued function F, defined on an open set E in the plane $\mathbb{R}^2$, is said to be real analytic on E, if to every point $(s_0, t_0) \in E$, there corresponds an expansion of the form

$$F(s,t) = \sum_{m,n=0}^{\infty} a_{mn}\,(s - s_0)^m\,(t - t_0)^n, \quad a_{mn} \in \mathbb{C}$$

which converges absolutely for all (s,t) in some neighbourhood of (s_0, t_0).

If $E = \mathbb{R}^2$ and if the above series converges absolutely for all $(s,t) \in \mathbb{R}^2$, then F is called real entire. In that case F has the power series expansion

$$F(s,t) = \sum_{m,n=0}^{\infty} a_{mn}\, s^m\, t^n \tag{2.1}$$

that converges absolutely for every $(s,t) \in \mathbb{R}^2$.

Remark 2.2.3 If F is real analytic at a point $(s_0, t_0) \in \mathbb{R} \times \mathbb{R}$, then the above power series expansion shows that F has an analytic extension $F(s+is', t+it')$ to an open set in the complex domain $\mathbb{C} \times \mathbb{C}$ containing (s_0, t_0). Also, if F is real analytic in an open set in $\mathbb{R}^2$, then fixing one variable, F is a real analytic function of the other variable.

Remark 2.2.4 Note that F is real analytic everywhere on $\mathbb{R}^2$, does not imply that F is real entire. A standard example is the function $F(x,y) = \frac{1}{(1+x^2)(1+y^2)}$ which is real analytic everywhere on $\mathbb{R}^2$, but the power series expansion around $(0,0)$, converges only in the unit disc $x^2+y^2<1$.

Notation. If F is a real entire function given by (2.1), then we denote by $\tilde{F}$ the function given by the power series expansion

$$\tilde{F}(s,t) = \sum_{m,n=0}^{\infty} |a_{mn}|\, s^m\, t^n. \tag{2.2}$$

Note that $\tilde{F}$ is real entire if F is real entire. Moreover, as a function on $[0,\infty) \times [0,\infty)$, it is monotonically increasing with respect to each of the variables s and t.

Before proving a Theorem 2.2.1, we discuss some interesting consequences of this result. First notice that for $\alpha > 0$, the complex function

$$F(z) = |z|^\alpha z = (x^2+y^2)^{\frac{\alpha}{2}}(x+iy),$$

as a mapping from $\mathbb{R}^2 \to \mathbb{R}^2$ may be written as

$$F(x,y) = \left((x^2+y^2)^{\alpha/2}x, (x^2+y^2)^{\alpha/2}y\right).$$

Note that the functions $(x,y) \mapsto (x^2+y^2)^{\alpha/2}x$ and $(x,y) \mapsto (x^2+y^2)^{\alpha/2}y$ are real analytic at zero only if $\alpha \in 2\mathbb{N}$. Thus the above theorem answers negatively, the open question raised in [49] regarding the validity of an inequality of the form

$$\| |u|^\alpha u\|_{M^{p,1}} \lesssim \|u\|_{M^{p,1}}^{\alpha+1},$$

for all $u \in M^{p,1}(\mathbb{R}^d)$, for $\alpha \in (0,\infty) \setminus 2\mathbb{N}$. In fact, we have the following

Corollary 2.2.5 *There exists $f \in M^{p,1}(\mathbb{R}^d)$ such that $f|f|^\alpha \notin M^{p,1}(\mathbb{R}^d)$, for any $\alpha \in (0,\infty) \setminus 2\mathbb{N}$.*

Proof. If possible, suppose that $F(f) \in M^{p,1}(\mathbb{R}^d)$ for all $f \in M^{p,1}(\mathbb{R}^d)$, where $F : \mathbb{C}(\approx \mathbb{R}^2) \to \mathbb{C}$ given by $F(z) = z|z|^\alpha = x(x^2+y^2)^{\alpha/2} + iy(x^2+y^2)^{\alpha/2}$, for $\alpha \in (0,\infty) \setminus 2\mathbb{N}$. But then by Theorem 2.2.1(1), F must be real analytic on $\mathbb{R}^2$, which is absurd. □

Corollary 2.2.6 *If $f \in M^{p,1}(\mathbb{R})$ then $|f|$ need not be in $M^{p,1}(\mathbb{R})$. Conversely $|f| \in M^{p,1}(\mathbb{R})$ does not imply that $f \in M^{p,1}(\mathbb{R})$.*

Proof. The function $F(z) = |z| = (x^2+y^2)^{1/2}$ is not real analytic on $\mathbb{C} \approx \mathbb{R}^2$, which shows the first part. For the converse, consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1-x & \text{if } 0 \le x < 1, \\ -1-x, & \text{if } -1 \le x < 0, \\ 0, & \text{if } |x| \ge 1. \end{cases}$$

Note that f is discontinuous and hence does not belong to $M^{p,1}(\mathbb{R})$; as $M^{p,1}(\mathbb{R}) \subset C(\mathbb{R})$ (see Corollary 5.3.9). But $|f| = (1-|x|)_+$, which is the triangle function, with Fourier transform $\left(\frac{\sin(\pi w)}{\pi w}\right)^2$. Thus by Proposition 1.4.7, $|f| \in M^{1,1}(\mathbb{R}) \subset M^{p,1}(\mathbb{R})$. □

Corollary 2.2.7 *There exists $f \in W^{p,1}(\mathbb{R}^d)$ such that $f|f|^\alpha \notin W^{p,q}(\mathbb{R}^d)$ $(1 \le p \le \infty, 1 \le q < 2)$, for any $\alpha \in (0,\infty) \setminus 2\mathbb{N}$.*

Proof. The nonlinear mapping $F : \mathbb{R}^2 \to \mathbb{R}^2 : z \mapsto z|z|^\alpha$ is not real analytic on $\mathbb{R}^2$ for $\alpha \in (0,\infty) \setminus 2\mathbb{N}$. □

Now we proceed to prove Theorem 2.2.1(1). Our proof is motivated by a classical result [33, p.156] of Helson, Kahane, Katznelson and Rudin, for abstract Fourier algebras. We let $A^q(\mathbb{T}^d)$ be the class of all complex functions f on the $d-$torus $\mathbb{T}^d$ whose Fourier coefficients

$$\widehat{f}(m) = \int_{\mathbb{T}^d} f(x) e^{-2\pi i m\cdot x} dx, \ (m \in \mathbb{Z}^d)$$

satisfy the condition

$$\|f\|_{A^q(\mathbb{T}^d)} := \|\widehat{f}\|_{\ell^q} < \infty.$$

Now we recall, the classical theorem of Katznelson [33, p.156], see also, [48, Theorem 6.9.2] for $A^1(\mathbb{T})$ which had proved in 1959, and later generalized by Rudin [47] in 1962 for $A^q(G)$, where G is infinite compact abelian group and $1 < q < 2$. We rephrase it here by combining both of it as required in our context.

Theorem 2.2.8 (Katznelson-Rudin) *Suppose that T_F is the composition operator associated to a complex function F on $\mathbb{C}$, and $1 \leq q < 2$. If T_F takes $A^1(\mathbb{T}^d)$ to $A^q(\mathbb{T}^d)$, then F is real analytic on $\mathbb{R}^2$.*

Now we introduce periodic Wiener amalgam and modulation spaces, and for this reason, first we recall some definitions, and introduce temporary notations, as given in [51, 50]. We start by noting that there is a one-to-one corresponding between functions on $\mathbb{R}^d$ that are 1-periodic in each of the coordinate directions and functions on torus $\mathbb{T}^d$; and we may identify $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ with $[0,1)^d$. Let $\mathcal{D}(\mathbb{T}^d)$ be the vector space $C^\infty(\mathbb{T}^d)$ endowed with the usual test function topology, and let $\mathcal{D}'(\mathbb{T}^d)$ be its dual, the space of distributions on $\mathbb{T}^d$. Let $\mathcal{S}(\mathbb{Z}^d)$ denote the space of rapidly decaying functions $\mathbb{Z}^d \to \mathbb{C}$. Let $\mathcal{F}_T : \mathcal{D}(\mathbb{T}^d) \to \mathcal{S}(\mathbb{Z}^d)$ be the toroidal Fourier transform (hence the subscript T) defined by

$$(\mathcal{F}_T f)(\xi) := \hat{f}(\xi) = \int_{\mathbb{T}^d} f(x) e^{-2\pi i \xi \cdot x} dx, \quad (\xi \in \mathbb{Z}^d).$$

Then $\mathcal{F}_T$ is a bijection and the inverse Fourier transform is given by

$$(\mathcal{F}_T^{-1} f)(x) := \sum_{\xi \in \mathbb{Z}^d} \hat{f}(\xi) e^{2\pi i \xi \cdot x}, \quad (x \in \mathbb{T}^d),$$

and this Fourier transform is extended uniquely to $\mathcal{F}_T : \mathcal{D}'(\mathbb{T}^d) \to \mathcal{S}'(\mathbb{Z}^d)$, (see [52, Section 3.1] for detail).

The Wiener amalgam spaces $W^{p,q}(\mathbb{T}^d)$ consists of all $f \in \mathcal{D}'(\mathbb{T}^d)$ such that

$$\|f\|_{W^{p,q}(\mathbb{T}^d)} := \| \|\phi(D_T - k) f\|_{\ell^q} \|_{L^p(\mathbb{T}^d)} < \infty, \tag{2.3}$$

and modulation spaces $M^{p,q}(\mathbb{T}^d)$ consists of all $f \in \mathcal{D}'(\mathbb{T}^d)$ such that

$$\|f\|_{M^{p,q}(\mathbb{T}^d)} := \| \|\phi(D_T - k) f\|_{L^p(\mathbb{T}^d)} \|_{\ell^q} < \infty, \tag{2.4}$$

for some ϕ with compact support in the discrete topology of $\mathbb{Z}^d$, where $\phi(D_T - k)f = \mathcal{F}_T^{-1}(T_k\phi \cdot \mathcal{F}_T f)$.

Note. In the above definition, we have followed notation as in [50, 5, 51].

Proposition 2.2.9 *Let $1 \leq p, q \leq \infty$. Then, we have,*

$$M^{p,q}(\mathbb{T}^d) = W^{p,q}(\mathbb{T}^d) = A^q(\mathbb{T}^d),$$

with norm inequality

$$\|f\|_{M^{p,q}(\mathbb{T}^d)} \asymp \|f\|_{W^{p,q}(\mathbb{T}^d)} \asymp \|f\|_{A^q(\mathbb{T}^d)}.$$

Proof. For the proof we refer to [50, Section 5]. □

We now define the local-in-time versions of the Wiener amalgam and modulation spaces in the following way. Given an interval $I = [0,1)^d$, let $W^{p,q}(I)$ be the restriction of $W^{p,q}(\mathbb{R}^d)$ onto I via

$$\|f\|_{W^{p,q}(I)} := \inf\{\|g\|_{W^{p,q}(\mathbb{R}^d)} : g = f \text{ on } I\}, \tag{2.5}$$

and $M^{p,q}(I)$ be the restriction of $M^{p,q}(\mathbb{R}^d)$ onto I via

$$\|f\|_{M^{p,q}(I)} = \inf\{\|g\|_{M^{p,q}(\mathbb{R}^d)} : g = f \text{ on } I\}. \tag{2.6}$$

We note that Bényi-Oh has proved the "equivalence" of the periodic function spaces ($M^{p,q}(\mathbb{T}^d)$ and $W^{p,q}(\mathbb{T}^d)$) and their local-in-time versions (defined on a bounded interval $I = [0,1)^d$, that is $M^{p,q}(I)$ and $W^{p,q}(I)$) in [5, Appendix B] (see also [5, Remark 3.3]) via establishing the equivalent of norms:

$$\|f\|_{M^{p,q}(\mathbb{T}^d)} \asymp \|f\|_{M^{p,q}(I)} \text{ and } \|f\|_{W^{p,q}(\mathbb{T}^d)} \asymp \|f\|_{W^{p,q}(I)}, \tag{2.7}$$

where $1 \leq p, q \leq \infty$.

We first prove the following result.

Lemma 2.2.10 *Let f be a periodic function on $\mathbb{R}^d$ with absolutely convergent Fourier series. Then f is a tempered distribution on $\mathbb{R}^d$ and the Fourier transform of f is the discrete measure $\mu = \sum_{m\in\mathbb{Z}^d} \hat{f}(m)\,\delta_m$, where $\hat{f}(m)$ denotes the mth Fourier coefficient of f, and δ_m the Dirac mass at $m \in \mathbb{R}^d$.*

Proof. Note that f is continuous on the torus $\mathbb{T}^d$ since the Fourier series is absolutely convergent. Thus f viewed as a periodic function on $\mathbb{R}^d$, is bounded and hence defines a tempered distribution.

We have $f(x) = \sum_{m\in\mathbb{Z}^d} \hat{f}(m)\, e^{2\pi i m\cdot x}$ for all $x \in \mathbb{R}^d$. Thus for $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} f\,\widehat{\varphi} = \sum_{m\in\mathbb{Z}^d} \hat{f}(m) \int_{\mathbb{R}^d} e^{2\pi i m\cdot x}\,\widehat{\varphi}(x)\,dx = \sum_{m\in\mathbb{Z}^d} \hat{f}(m)\,\varphi(m).$$

Writing $\varphi(m) = \delta_m(\varphi)$, this shows that $\langle \widehat{f}, \varphi\rangle = \left\langle \sum_{m\in\mathbb{Z}^d} \hat{f}(m)\,\delta_m, \varphi \right\rangle$ for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Thus the Fourier transform of f as a tempered distribution, is given by $\widehat{f} = \sum_{m\in\mathbb{Z}^d} \hat{f}(m)\,\delta_m$ as asserted. $\square$

Note that the μ defined above is a complex Borel measure on $\mathbb{R}^d$, with total variation norm $\|\mu\| = |\mu|(\mathbb{R}^d) = \sum_{m\in\mathbb{Z}^d} |\hat{f}(m)| < \infty$.

Proposition 2.2.11 *Suppose that T_F is the composition operator associated to a complex function F on $\mathbb{C}, 1 \le p \le \infty$, and $1 \le q < 2$. If T_F maps $M^{p,1}(\mathbb{R}^d)$ to $M^{p,q}(\mathbb{R}^d)$, then T_F maps $A^1(\mathbb{T}^d)$ to $A^q(\mathbb{T}^d)$.*

Proof. Let $f \in A^1(\mathbb{T}^d)$. Then $f^*(x) = f(e^{2\pi i x_1}, ..., e^{2\pi i x_d})$ is a periodic function on $\mathbb{R}^d$ with absolutely convergent Fourier series

$$f^*(x) = \sum_{m\in\mathbb{Z}^d} \widehat{f}(m)\, e^{2\pi i m\cdot x}.$$

Choose $g \in C_c^\infty(\mathbb{R}^d)$ such that $g \equiv 1$ on $Q_d = [0,1)^d$. Then we claim that $gf^* \in M^{1,1}(\mathbb{R}^d) \subset M^{p,1}(\mathbb{R}^d)$. Once the claim is assumed, by hypothesis, we have

$$F(gf^*) \in M^{p,q}(\mathbb{R}^d). \tag{2.8}$$

Note that if $z \in \mathbb{T}^d$, then $z = (e^{2\pi i x_1}, ..., e^{2\pi i x_d})$ for some $x = (x_1, ..., x_d) \in Q_d$, hence

$$F(f(z)) = F(f^*(x)) = F(gf^*(x)), \text{ for } x \in Q_d. \tag{2.9}$$

Now if $\phi \in C_c^\infty(\mathbb{T}^d)$, then $g\phi^*$ is a compactly supported smooth function on $\mathbb{R}^d$. Also $\phi(z) = g(x)\phi^*(x)$ for every $x \in Q_d$, as per the notation above and hence

$$\phi(z)F(f)(z) = g(x)\phi^*(x)F(gf^*)(x), \tag{2.10}$$

for some $x \in Q_d$.

By (2.10), Proposition 2.2.9, (2.7), (2.5), and Proposition 1.4.3(3), we obtain

$$\begin{aligned}\|\phi F(f)\|_{A^q(\mathbb{T}^d)} &= \|g\phi^* F(gf^*)\|_{A^q(\mathbb{T}^d)} \\ &\asymp \|g\phi^* F(gf^*)\|_{M^{p,q}(\mathbb{T}^d)} \\ &\asymp \|g\phi^* F(gf^*)\|_{M^{p,q}(Q_d)} \\ &\lesssim \|g\phi^* F(gf^*)\|_{M^{p,q}} \\ &\lesssim \|g\phi^*\|_{M^{\infty,1}} \|F(gf^*)\|_{M^{p,q}},\end{aligned}$$

which is finite for every smooth cutoff function ϕ supported on Q_d in view of Lemma 1.4.1 (1), and (2.8). Now by compactness of $\mathbb{T}^d$, a partition of unity argument shows that $F(f) \in A^q(\mathbb{T}^d)$.

To complete the proof, we need to prove the claim. Since $M^{1,1}(\mathbb{R}^d)$ is invariant under Fourier transform, enough to show that $\widehat{gf^*} = \widehat{g} * \widehat{f^*} \in M^{1,1}(\mathbb{R}^d)$. By Lemma 2.2.10, applied to f^*, we see that

$$\widehat{f^*} = \mu = \sum_{m \in \mathbb{Z}^d} \hat{f}(m)\, \delta_m.$$

Hence,

$$\widehat{g} * \widehat{f^*} = \sum_{m \in \mathbb{Z}^d} \hat{f}(m)\, \widehat{g} * \delta_m = \sum_{m \in \mathbb{Z}^d} \hat{f}(m)\, T_m \widehat{g}.$$

Since the translation operator T_m is an isometry on $M^{1,1}(\mathbb{R}^d)$, it follows that the above series is absolutely convergent in $M^{1,1}(\mathbb{R}^d)$, and hence $\widehat{gf^*} \in M^{1,1}(\mathbb{R}^d)$ as claimed. □

Proof of Theorem 2.2.1 (1). If T_F takes $M^{p,1}(\mathbb{R}^d)$ to $M^{p,q}(\mathbb{R}^d)$, then T_F takes $A^1(\mathbb{T}^d)$ to $A^q(\mathbb{T}^d)$ by Proposition 2.2.11. Hence the analyticity follows from Theorem 2.2.8.

Note that the zero function $u_0 \equiv 0 \in M^{p,1}(\mathbb{R}^d)$ and $F(u_0)(x) = F(0)$ for all $x \in \mathbb{R}^d$. But the constant functions in $M^{p,q}(\mathbb{R}^d)$ $(1 \leq p < \infty, 1 \leq q < 2)$ is the zero function only. It follows that $F(0) = 0$ if $p < \infty$. □

Proof of Theorem 2.2.1 (2). Exploiting the ideas from the proof of Theorem 2.2.1(2), the proof can be produced; and so we omit the details. □

2.3 Sufficient Conditions

In this section, we obtain sufficient conditions: properties of F, which gives guarantees, the associated composition operator T_F takes the space $M^{p,1}(\mathbb{R}^d)$(or subclass of it) to the space $M^{p,1}(\mathbb{R}^d)$.

We start with following sufficient condition which is easy to obtain.

Theorem 2.3.1 *Suppose that T_F is the composition operator associated to a complex function F on $\mathbb{C}$, and X denotes $M^{p,1}_s(\mathbb{R}^d)$, $1 \leq p \leq \infty, s \geq 0$, or $X = M^{p,q}_s(\mathbb{R}^d)$, $1 \leq p,q \leq \infty, s > d/q'$. If F is a real entire function given by $F(x,y) = \sum_{m,n} a_{mn}x^m y^n$, with $F(0) = 0$, then T_F acts on X, and in particular we have*

$$\|F(f)\|_X \lesssim \tilde{F}\left(\|f_1\|_X, \|f_2\|_X\right), \;\; f = f_1 + if_2 \tag{2.11}$$

for all $f \in X$, where $\tilde{F}(x,y)$ is the real entire function given by $\tilde{F}(x,y) = \sum_{m,n} |a_{mn}|x^m y^n$.

Proof. Let $f \in X$ with $f_1 = \frac{f+\bar{f}}{2}$ and $f_2 = \frac{f-\bar{f}}{2i}$. Then $f_1, f_2 \in X$ and so $f_1^m, f_2^n \in X$ by Proposition 1.4.6. Since the series $\sum_{m,n=0}^{\infty} a_{mn}x^m y^n$, converges absolutely for all (x,y), the series $\sum_{n,m=0}^{\infty} a_{mn}f_1^m f_2^n$ is converges in the norm of X; and its sum is $F(f) = \sum_{n,m=0}^{\infty} a_{mn}f_1^m f_2^m$; and hence

$$\|F(f)\|_X \leq \sum_{m,n=0}^{\infty} |a_{mn}| \cdot \|f_1\|_X^m \|f_2\|_X^n.$$

□

Remark 2.3.2 Corollary 3.3 of [53, p.355] is a particular case of Theorem 2.3.1; as every complex-entire function is real entire as a function on $\mathbb{R}^2$.

Our next theorem says that under a weaker hypothesis on F, the associated composition T_F takes $M^{1,1}(\mathbb{R}^d)$ to $M^{1,1}(\mathbb{R}^d)$.

Theorem 2.3.3 *Let F be a real analytic function on $\mathbb{R}^2$ with $F(0) = 0$. Then $F(f) \in M^{1,1}(\mathbb{R}^d)$ for all $f \in M^{1,1}(\mathbb{R}^d)$.*

For arbitrary real analytic function F, we do not have a favourable estimate like (2.11); and our approach is inspired by the classical Wiener-Lévy [72, 42] sufficient condition: if F is real analytic on $\mathbb{R}^2$, then the composition operator T_F acts on $A^1(\mathbb{T})$.

First we collect some technical results which should be regarded as the tool for proving Theorem 2.3.3.

We start with the following definition.

Definition 2.3.4 Let f be a function defined on $\mathbb{R}^d$, we say that f belongs to $M^{p,1}(\mathbb{R}^d)$ locally at a point $x_0 \in \mathbb{R}^d$ if there is a neighbourhood V of x_0 and a function $g \in M^{p,1}(\mathbb{R})$ such that $f(x) = g(x)$ for every $x \in V$. We say that f belongs to $M^{p,1}(\mathbb{R}^d)$ at ∞, if there is a compact set $K \subset \mathbb{R}^d$ and a function $h \in M^{p,1}(\mathbb{R}^d)$ such that $f(x) = h(x)$ for all $x \in \mathbb{R}^d \setminus K$.

We denote by $M^{p,1}_{loc}(\mathbb{R}^d)$, the space of functions that are locally in $M^{p,1}(\mathbb{R}^d)$ at each point $x_0 \in \mathbb{R}^d$.

Lemma 2.3.5 *Let $1 \leq p \leq \infty$. A function $f \in M^{p,1}_{loc}(\mathbb{R}^d)$, if and only if $\varphi f \in M^{p,1}(\mathbb{R}^d)$ for every $\varphi \in C_c^\infty(\mathbb{R}^d)$.*

A function f belongs to $M^{p,1}(\mathbb{R}^d)$ at ∞, if and only if there exists a $\varphi \in C_c^\infty(\mathbb{R}^d)$ such that $(1-\varphi)f \in M^{p,1}(\mathbb{R}^d)$.

Proof. If $\varphi f \in M^{p,1}(\mathbb{R}^d)$ for all $\varphi \in C_c^\infty(\mathbb{R}^d)$, then f is clearly in $M^{p,1}_{loc}(\mathbb{R}^d)$. In fact for any point $x \in \mathbb{R}^d$, we can choose a smooth function φ with compact support, which has value one in a neighbourhood of x, by smooth version of Urysohn lemma, see [24, p.245]. Then $f \equiv \varphi f$ in that neighbourhood.

Conversely, suppose $f \in M^{p,1}_{loc}(\mathbb{R}^d)$ and $\varphi \in C_c^\infty(\mathbb{R}^d)$ with support K. By hypothesis, for each point $x \in K$, there is an an open ball $B_r(x)$ of radius r and centered at x such that f coincides with a $g \in M^{p,1}(\mathbb{R}^d)$ in that ball. By compactness of K, we can find finitely many points $x_1, x_2, ..x_N$ such that the balls $B_{r_i}(x_i), i = 1, 2, .., N$ cover K. Let $\{\varphi_i : i = 1, 2, ..., N\}$ be a partition of unity subordinate to this cover.

Let $g_i \in M^{p,1}(\mathbb{R}^d)$ be such that $f = g_i$ on $B_{r_i}(x_i)$. Since ϕ_i is supported in $B_{r_i}(x_i)$, we also have $\varphi_i f = \varphi_i g_i$ on $B_{r_i}(x_i)$, and $\varphi_i g_i \in M^{p,1}(\mathbb{R}^d)$ since $\varphi_i \in C_c^\infty(\mathbb{R}^d) \subset M^{p,1}(\mathbb{R}^d)$ and by Proposition 1.4.3 (3). Note that we also have $\varphi\,\varphi_i g_i \in M^{p,1}(\mathbb{R}^d)$, since $\varphi\varphi_i$ is also in $C_c^\infty(\mathbb{R}^d)$. Thus $\varphi\,\varphi_i f \in M^{p,1}(\mathbb{R}^d)$ for each i. But $\sum_{i=1}^N \varphi_i = 1$, implies $\varphi f = \sum_{i=1}^N \varphi\,\varphi_i f \in M^{p,1}(\mathbb{R}^d)$, thus proves the first part of the Lemma.

Again, if $\varphi \in C_c^\infty(\mathbb{R}^d)$ is such that $(1-\varphi)f \in M^{p,1}(\mathbb{R}^d)$, clearly f coincides with a function in $M^{p,1}(\mathbb{R}^d)$ in the compliment of a compact set, namely the function $(1-\varphi)f$. On the other hand, suppose there exists a $g \in M^{p.1}(\mathbb{R}^d)$ such that $f = g$ on the complement of a large ball $B(0,R)$ of radius R, centered at origin. Let φ be a smooth function with support $B(0,R)$. Then $(1-\varphi) \equiv 1$ on $|x| > R$ and hence $(1-\varphi)f = (1-\varphi)g = g - \varphi g \in M^{p,1}(\mathbb{R}^d)$, as both g and φg are in $M^{p,1}(\mathbb{R}^d)$. This completes the proof. □

The following lemma gives a useful test for a function to be in $M^{p,1}(\mathbb{R}^d)$.

Lemma 2.3.6 *If $f \in M^{p,1}_{loc}(\mathbb{R}^d)$ and f belongs to $M^{p,1}(\mathbb{R}^d)$ at infinity, for $1 \leq p \leq \infty$, then $f \in M^{p,1}(\mathbb{R}^d)$.*

Proof. Since f belongs to $M^{p,1}(\mathbb{R}^d)$ at infinity, there exists a $\varphi \in C_c^\infty(\mathbb{R}^d)$ such that $(1-\varphi)f \in M^{p,1}(\mathbb{R}^d)$. Now $f = \varphi f + (1-\varphi)f$, and both φf and $(1-\varphi)f$ are in $M^{p,1}(\mathbb{R}^d)$, by Lemma 2.3.5. Hence, $f \in M^{p,1}(\mathbb{R}^d)$. This completes the proof. □

Now we proceed to prove Theorem 2.3.3. We start with the following technical result.

Proposition 2.3.7 *Let $f \in M^{1,1}(\mathbb{R}^d)$, $x_0 \in \mathbb{R}^d$ and $\epsilon > 0$. Then there exists a $\phi \in C_c^\infty(\mathbb{R}^d)$ such that $\|\phi\,[f - f(x_0)]\,\|_{M^{1,1}} < \epsilon$. The function ϕ can be chosen so that $\phi \equiv 1$ in some neighbourhood of x_0.*

There also exists a $\psi \in C_c^\infty(\mathbb{R}^d)$ such that $\|(1-\psi)f\|_{M^{1,1}} < \epsilon$.

Proof. Let φ be a smooth function supported in the ball $B_2(0)$ such that $\varphi \equiv 1$ on $B_1(0)$ and set $\varphi^\lambda(x) = \varphi(\lambda x)$. To prove the first part, enough to show that the $M^{1,1}$ norm of the function $h^\lambda(x) := \varphi^\lambda(x - x_0)[f(x) - f(x_0)]$ tends to zero as $\lambda \to \infty$.

For notational convenience, we assume $x_0 = 0$. Note that

$$h^\lambda(x) = \varphi(x)\, h^\lambda(x), \tag{2.12}$$

for $\lambda > 2$, as $\varphi \equiv 1$ on the support of φ^λ in this case. Since the Fourier transform is an isometry on $M^{1,1}(\mathbb{R}^d)$, enough to estimate $\widehat{h^\lambda}$. Since $\widehat{\varphi\, h^\lambda} = \widehat{\varphi} * \widehat{h^\lambda}$, in view of (2.12) and Proposition 1.4.3(1), we see that

$$\|\widehat{h^\lambda}\|_{M^{1,1}} \quad = \quad \|\widehat{\varphi\, h^\lambda}\|_{M^{1,1}}$$

$$\leq \|\widehat{\varphi}\|_{M^{1,1}}\|\widehat{h^\lambda}\|_{L^1}.$$

Since $\widehat{h^\lambda} = \widehat{\varphi^\lambda f} - f(0)\widehat{\varphi^\lambda} = \widehat{\varphi^\lambda} * \widehat{f} - f(0)\widehat{\varphi^\lambda}$, writing $f(0) = \int_{\mathbb{R}^d} \widehat{f}(y)dy$, we see that

$$\begin{aligned} \widehat{h^\lambda}(\xi) &= \int_{\mathbb{R}^d} \widehat{f}(y)\left[\widehat{\varphi^\lambda}(\xi - y) - \widehat{\varphi^\lambda}(\xi)\right] dy \\ &= \int_{\mathbb{R}^d} \widehat{f}(y)\frac{1}{\lambda^d}\left[\widehat{\varphi}\left(\frac{\xi - y}{\lambda}\right) - \widehat{\varphi}\left(\frac{\xi}{\lambda}\right)\right] dy. \end{aligned}$$

Taking the L^1 norm on both sides and by the change of variable $\xi \to \lambda\xi$, we see that

$$\begin{aligned} \int_\xi |\widehat{h^\lambda}| d\xi &\leq \int_{\mathbb{R}^d} |\widehat{f}(y)| \int_\xi \left|\widehat{\varphi}\left(\xi - \frac{y}{\lambda}\right) - \widehat{\varphi}\left(\xi\right)\right| d\xi\, dy \\ &\leq \int_{\mathbb{R}^d} |\widehat{f}(y)| \left\|\widehat{\varphi}\left(\cdot - \frac{y}{\lambda}\right) - \widehat{\varphi}\left(\cdot\right)\right\|_{L^1} dy. \end{aligned} \tag{2.13}$$

Now we note that $M^{1,1}(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$ and hence $\widehat{f} \in L^1(\mathbb{R}^d)$. Thus the above tends to zero as $\lambda \to \infty$, by dominated convergence theorem and the continuity of the translation in $L^1(\mathbb{R}^d)$.

For general x_0, we can continue the same proof by taking $\varphi^\lambda(x - x_0)$ and carrying out the proof as above.

To prove the second part, we choose a $\chi \in C_c^\infty(\mathbb{R}^d)$ with $\chi(0) = 1$, and estimate the $M^{1,1}$ norm of $[1-\chi(\lambda x)]f(x)$, for $\lambda > 1$. As before, since $M^{1,1}(\mathbb{R}^d)$ is invariant under the Fourier transform, enough to estimate the Fourier transform of $[1 - \chi(\lambda x)]f(x)$, which is $\widehat{f}(\xi) - \widehat{f} * \varphi_\lambda(\xi)$, with $\varphi = \widehat{\chi}$. This tends to zero in $M^{1,1}(\mathbb{R}^d)$ as $\lambda \to 0$, by Lemma 1.4.4 since $\int \widehat{\varphi} = \varphi(0) = 1$.

Now we can choose for ϕ, any φ^λ for sufficiently small λ. This completes the proof. □

Remark 2.3.8 If there are finitely many functions $f_1, f_2, ...f_N$, then one can choose a single ϕ and ψ that works for all these functions. All we need to do is to dominate the inequality (2.13) with $|\widehat{f}\,|$ replaced by $\sum_1^N |\widehat{f_i}|$, to get a single ϕ valid for all f_i's.

On the other hand, if $\psi_i = \varphi_{\lambda_i}$ for f_i, then if $\lambda = \min\{\lambda_i, i = 1, 2, ...N\}$, then $\psi = \varphi_\lambda$ will work for all f_i, as observed in Remark 1.4.5.

Proof of Theorem 2.3.3. Write $f = f_1 + if_2 \in M^{1,1}(\mathbb{R}^d)$, where f_1 and f_2 are real functions, and with an abuse of notation, we write $F(f) = F(f_1, f_2)$. To show that $F(f)$ is in $M^{1,1}(\mathbb{R}^d)$, enough to show, in view of Lemma 2.3.6 that $F(f) \in M^{1,1}_{loc}(\mathbb{R}^d)$ and $F(f)$ belongs to $M^{1,1}(\mathbb{R}^d)$ at ∞. First we show that $F(f) \in M^{1,1}_{loc}(\mathbb{R}^d)$.

Fix $x_0 \in \mathbb{R}^d$ and put $f(x_0) = s_0 + it_0$. Since F is real analytic at (s_0, t_0), there exists a $\delta > 0$ such that F has the power series expansion

$$F(s,t) = F(s_0,t_0) + \sum_{m,n=0}^{\infty} a_{mn}(s-s_0)^m(t-t_0)^n, \quad (a_{00} = 0) \tag{2.14}$$

which converges absolutely for $|s - s_0| \leq \delta, |t - t_0| \leq \delta$. Then

$$\begin{aligned} F(f_1(x), f_2(x)) &= F(s_0, t_0) \\ &\quad + \sum_{(m,n)\neq(0,0)} a_{mn}[f_1(x) - f_1(x_0)]^m [f_2(x) - f_2(x_0)]^n \end{aligned} \tag{2.15}$$

whenever the series converges.

Note that both f_1 and f_2 are in $M^{1,1}(\mathbb{R}^d)$, being the real and imaginary part of f. Hence by Proposition 2.3.7, and Remark 2.3.8, we can find a $\phi \in C_c^\infty(\mathbb{R}^d)$, such that $\phi \equiv 1$ near x_0 and $\|\phi[f_i - f_i(x_0)]\|_{M^{1,1}} < \delta$, for $i = 1, 2$. Now consider the function G on $\mathbb{R}^d$ defined by

$$\begin{aligned} G(x) &= \phi(x)\, F(s_0, t_0) \\ &\quad + \sum_{(m,n)\neq(0,0)} a_{mn} \left(\phi(x)[f_1(x) - f_1(x_0)]\right)^m \left(\phi(x)[f_2(x) - f_2(x_0)]\right)^n . \end{aligned}$$

Since $\|\varphi[f_i - f_i(x_0)]\|_{M^{1,1}} < \delta$, for $i = 1, 2$ and in view of the algebraic inequality (1.17), we see that the above series is absolutely convergent in $M^{1,1}(\mathbb{R}^d)$. Also since $\phi \equiv 1$ in some neighbourhood of x_0, it follows that $G \equiv F(f)$ in some neighbourhood of x_0. Since x_0 is arbitrary, this shows that $F(f) \in M^{1,1}_{loc}(\mathbb{R}^d)$

To show that $F(f) \in M^{1,1}(\mathbb{R}^d)$ at infinity, we take $(s_0, t_0) = (0, 0)$ in equation (2.14). Since $F(0) = 0$, the expansion (2.15) now becomes

$$F(f_1(x), f_2(x)) = \sum_{(m,n)\neq(0,0)} a_{mn}\, [f_1(x)]^m\, [f_2(x)]^n,$$

whenever the series converges.

By Proposition 2.3.7, we have $\|(1 - \psi)f_i\|_{M^{1,1}} < \delta$, for $i = 1, 2$ for some

$\psi \in C_c^\infty(\mathbb{R}^d)$. Now consider the function H defined by

$$H(x) = \sum_{(m,n)\neq(0,0)} a_{mn} \left[(1-\psi(x))f_1(x)\right]^m \left[(1-\psi(x))f_2(x)\right]^n.$$

The above series is absolutely convergent in $M^{1,1}(\mathbb{R}^d)$, in view of the above norm estimates, hence $H \in M^{1,1}(\mathbb{R}^d)$. Also since ψ is compactly supported, $1-\psi \equiv 1$ in the complement of a large ball centered at the origin, hence $H = F(f)$ in the compliment of a compact set. This shows that $F(f)$ belongs to $M^{1,1}(\mathbb{R}^d)$ at infinity. □

We note that the proof for the sufficient condition(Theorem 2.3.3) relies on the invariant property of the modulation space $M^{1,1}(\mathbb{R}^d)$ under the Fourier transform. This invariance is not available for $M^{p,1}(\mathbb{R}^d)$, when $p>1$.

Now we proceed to obtain a partial converse to Theorem 2.2.1: if we restrict the domain of the T_F to be a subclass of $M^{p,1}(\mathbb{R}^d)$ or $W^{p,1}(\mathbb{R}^d)(1<p<\infty)$ which is invariant under the Fourier transform and vanishing at infinity. More specifically, we have the following:

Theorem 2.3.9 *Let $1<p<\infty$, and suppose that T_F is the composition operator associated to a complex function F on $\mathbb{C}$.*

1. *Let $X = \{f, \hat{f} \in M^{p,1}(\mathbb{R}^d) : f$ vanishes at infinity$\}$. If F is real analytic on $\mathbb{R}^2$ which takes origin to itself, then T_F takes X to $W^{p,1}(\mathbb{R}^d)$.*
2. *Let $X = \{f, \hat{f} \in W^{p,1}(\mathbb{R}^d) : f$ vanishes at infinity$\}$. If F is real analytic on $\mathbb{R}^2$ which takes origin to itself, then T_F takes X to $W^{p,1}(\mathbb{R}^d)$.*

To prove this theorem, first we need some technical lemmas.

Lemma 2.3.10 *Suppose $f \in W^{1,1}(\mathbb{R}^d), \gamma_0 \in \mathbb{R}^d$, and $\delta>0$. Then there exists $h \in W^{1,1}(\mathbb{R}^d)$ such that $\|h\|_{W^{1,1}} < \delta$ and*

$$\widehat{h}(\gamma) = \widehat{f}(\gamma) - \widehat{f}(\gamma_0) \tag{2.16}$$

for all γ in some neighbourhood of γ_0.

Proof. Choose $k \in \mathcal{S}(\mathbb{R}^d)$ with $\widehat{k}=1$ in some neighbourhood of the origin. For $\lambda>0$, put,

$$k_\lambda(x) = e^{2\pi i \gamma_0 \cdot x}\lambda^{-d}k(x/\lambda), (x \in \mathbb{R}^d) \tag{2.17}$$

and define

$$\phi_\lambda(x) = (f * k_\lambda)(x) - \widehat{f}(\gamma_0)k_\lambda(x). \tag{2.18}$$

Again, we choose $\psi \in \mathcal{S}(\mathbb{R}^d)$ such that $\widehat{\psi} = 1$ in some neighbourhood of the γ_0; and define

$$h_\lambda(x) = (\psi * \phi_\lambda)(x), (x \in \mathbb{R}^d). \tag{2.19}$$

Note that $\phi_\lambda \in L^1(\mathbb{R}^d)$ and by Proposition 1.4.3 (1), we have, $h_\lambda \in W^{1,1}(\mathbb{R}^d)$, and since $\widehat{k_\lambda}(\gamma) = 1$ in some neighbourhood V_λ of γ_0, and by virtue of ψ we may assume that $\widehat{\psi}(\gamma) = 1$ in V_λ; therefore it follows that,

$$\widehat{h_\lambda}(\gamma) = \widehat{f}(\gamma) - \widehat{f}(\gamma_0)$$

holds for all γ in some neighbourhood V_λ of γ_0; therefore equality in (2.16) holds for $\gamma \in V_\lambda$ with h_λ in place of h.

Next, we claim that, $\|h_\lambda\|_{W^{1,1}} \to 0$ as $\lambda \to \infty$; and this completes the proof of the lemma.

By Proposition 1.4.3(2), we have, $\|\phi_\lambda * \psi\|_{W^{1,1}} \le \|\psi\|_{W^{1,1}} \cdot \|\phi_\lambda\|_{L^1}$, and $\|\psi\|_{W^{1,1}} < \infty$; it suffices to prove the claim , by showing that $\|\phi_\lambda\|_{L^1} \to 0$ as $\lambda \to \infty$.

Observe that,

$$\begin{aligned}\phi_\lambda(x) &= \int_{\mathbb{R}^d} f(y)[k_\lambda(x-y) - e^{-2\pi i\gamma_0\cdot y}k_\lambda(x)]dy \\ &= \int_{\mathbb{R}^d} f(y)e^{2\pi i\gamma_0\cdot(x-y)}[\lambda^{-d}k(\lambda^{-1}(x-y) - k(\lambda^{-1}x)]dy;\end{aligned}$$

and hence,

$$\|\phi_\lambda\|_{L^1} \le \int_{\mathbb{R}^d} |f(y)| \left(\int_{\mathbb{R}^d} |k(z - \lambda^{-1}y) - k(z)|dz \right) dy; \tag{2.20}$$

by the change of variable $x = \lambda z$. The inner integral in (2.20) is at most $2\|k\|_{L^1}$, and it tends to zero for every $y \in \mathbb{R}^d$, as $\lambda \to \infty$. Hence, $\|\phi_\lambda\|_{L^1} \to 0$ as $\lambda \to \infty$, by the dominated convergence. □

Lemma 2.3.11 *If $f \in W^{p,1}(\mathbb{R}^d)(1 < p < \infty), \gamma_0 \in \mathbb{R}^d$, and $\delta > 0$, then there* *exists* $h \in W^{1,1}(\mathbb{R}^d)$ *such that* $\|h\|_{W^{1,1}} < \delta$, *and*

$$\widehat{h}(\gamma) = \widehat{f}(\gamma) - \widehat{f}(\gamma_0) \tag{2.21}$$

for all γ *in some neighbourhood* V_{γ_0} *of* γ_0.

Proof. Fix $\gamma_0 \in \mathbb{R}^d$, and choose some neighbourhood of γ_0 sufficiently small, say V_{γ_0}, and a compact set K containing it, that is, $V_{\gamma_0} \subset K$, and K is compact in $\mathbb{R}^d$.

By Lemma 1.4.1(3), we have $W^{p,1}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$. Since $L^p(\mathbb{R}^d) \subset L^1_{loc}(\mathbb{R}^d)$, we can choose $g \in L^1(\mathbb{R}^d)$ such that $g(\gamma) = f(\gamma)$ for every $\gamma \in V_{\gamma_0}$ and $g(\gamma) = 0$ outside compact set K, and so support of g is contained in K, that is, $\operatorname{supp} g \subset K$.

We choose, $\phi \in \mathcal{S}(\mathbb{R}^d)$ so that $\widehat{\phi} = 1$ in some neighbourhood of γ_0 and define

$$h_1(x) = (\phi * g)(x) - \phi(x)\widehat{g}(\gamma_0), \ (x \in \mathbb{R}^d).$$

We note that $h_1 \in \mathcal{S}(\mathbb{R}^d) \subset W^{1,1}(\mathbb{R}^d)$; so we can apply Lemma 2.3.10, for h_1 and (2.21) follows. □

Lemma 2.3.12 *If* $f \in M^{p,1}(\mathbb{R}^d)$ $(1 \le p < \infty)$ *and* $\epsilon > 0$. *There exists* $v \in M^{p,1}(\mathbb{R}^d)$ *such that* $\widehat{v}$ *has a compact support and* $\|f - f * v\|_{M^{p,1}} < \epsilon$.

Proof. In Lemma 1.4.4, we choose, $\phi \in \mathcal{S}(\mathbb{R}^d)$ such that $\widehat{\phi} \in C_c^\infty(\mathbb{R}^d)$ and $\widehat{\phi}(0) = 1$, and the proof follows. □

Lemma 2.3.13 *If* $f \in W^{p,1}(\mathbb{R}^d), (1 \le p < \infty)$ *and* $\epsilon > 0$. *There exists* $v \in W^{p,1}(\mathbb{R}^d)$ *such that* $\widehat{v}$ *has a compact support and* $\|f - f * v\|_{W^{p,1}} < \epsilon$.

Proof. By Minkowski inequality for integral, we have, $\|f\|_{W^{p,1}} \le \|f\|_{M^{p,1}}$; and then the proof follows by Lemma 2.3.12. □

Proof of Theorem 2.3.9(2). By Lemma 2.3.6, it is enough to show that $F(f)$ belongs to $W^{p,1}(\mathbb{R}^d)$ locally at every point of $\mathbb{R}^d \cup \{\infty\}$.

Fix $\gamma_0 \in \mathbb{R}^d \cup \{\infty\}$, put $f(\gamma_0) = s_0 + it_0$, and choose $\delta > 0$ such that the series

$$F(s,t) = F(s_0,t_0) + \sum_{m,n=0}^{\infty} a_{mn}(s - s_0)^m (t - t_0)^n, (a_{00} = 0) \tag{2.22}$$

converges absolutely for $|s-s_0| \leq \delta, |t-t_0| \leq \delta$.

Since $f \in X$, we have, $f^\vee \in X$, and if $\gamma_0 \in \mathbb{R}^d$, then Lemma 2.3.11 applies to $f^\vee$; so there exists a function $h \in W^{1,1}(\mathbb{R}^d)$ such that $\|h\|_{W^{1,1}} < \delta$, and

$$\widehat{h}(\gamma) = f(\gamma) - f(\gamma_0) \tag{2.23}$$

in some neighbourhood V_{γ_0} of γ_0. Put $\widehat{h} = \widehat{h_1} + i\widehat{h_2}$ ($\widehat{h_1}, \widehat{h_2}$ real), since $\|h\|_{W^{1,1}} = \|\widehat{h}\|_{W^{1,1}}$, we have, $\|\widehat{h_1}\|_{W^{1,1}} < \delta$ and $\|\widehat{h_2}\|_{W^{1,1}} < \delta$. In view of (2.22), one can conclude that, the series

$$\sum_{m,n=0}^{\infty} a_{mn} \widehat{h_1}^m \widehat{h_2}^n$$

converges, in the norm of $W^{1,1}(\mathbb{R}^d)$, to a function $g \in W^{1,1}(\mathbb{R}^d)$. If we put, $f(\gamma) = f_1(\gamma) + if_2(\gamma)$, ($f_1, f_2$ real), then by (2.23), we have,

$$(\widehat{h_1}(\gamma), \widehat{h_2}(\gamma)) = (f_1(\gamma) - s_0, f_2(\gamma) - t_0); \forall \gamma \in V_{\gamma_0}.$$

But then, for $\gamma \in V_{\gamma_0}$, we have

$$\begin{aligned} F(f(\gamma)) &= F(s_0, t_0) + \sum_{m,n=0}^{\infty} a_{mn} \widehat{h_1}(\gamma)^m \widehat{h_2}(\gamma)^n \\ &= F(s_0, t_0) + g(\gamma). \end{aligned}$$

Next, we can choose $\psi \in C_c^\infty(\mathbb{R}^d)$ so that $\psi(\gamma) = 1$ for all $\gamma \in V_{\gamma_0}$; and therefore, it follows that, $F(s_0, t_0)\psi + g \in W^{1,1}(\mathbb{R}^d)$, and it coincide with $F(f)$ on some neighbourhood of γ_0, that is, $F(s_0, t_0)\psi(\gamma) + g(\gamma) = F(f(\gamma))$ for all $\gamma \in V_{\gamma_0}$; thus $F(f)$ belongs to $W^{1,1}(\mathbb{R}^d)$ locally at γ_0. Since $W^{1,1}(\mathbb{R}^d) \subset W^{p,1}(\mathbb{R}^d)$, $F(f)$ belongs to $W^{p,1}(\mathbb{R}^d)$ locally at γ_0.

For the case, $\gamma_0 = \infty$, we use Lemma 2.3.13 for $f^\vee$, and we get, $h = f^\vee - f^\vee * v$, (where v is as chosen in Lemma 2.3.13), so that, $\|h\|_{W^{p,1}} < \delta$, and

$$f(\gamma) = \widehat{h}(\gamma),$$

for all γ in the complement of some compact subset K of $\mathbb{R}^d$. In this case, we notice that $f(\gamma_0) = 0$, and similar argument as before, it is easy to conclude that, there exists some function g (in fact, the series $\sum_{m,n=0} a_{mn} \widehat{h_1}^m \widehat{h_2}^n$ converges in the $W^{p,1}$ norm, to some function in $W^{p,1}(\mathbb{R}^d)$, say it is g) in $W^{p,1}(\mathbb{R}^d)$ which coincide with $F(f)$ in compliment of a compact set; hence $F(f)$ belongs to $W^{p,1}(\mathbb{R}^d)$ at ∞. $\square$

Lemma 2.3.14 *If $f \in M^{p,1}(\mathbb{R}^d)$ $(1 < p < \infty), \gamma_0 \in \mathbb{R}^d$, and $\delta > 0$, then there exists $h \in M^{1,1}(\mathbb{R}^d)$ such that $\|h\|_{M^{1,1}} < \delta$, and*

$$\widehat{h}(\gamma) = \widehat{f}(\gamma) - \widehat{f}(\gamma_0) \tag{2.24}$$

for all γ in some neighbourhood V_{γ_0} of γ_0.

Proof. By Minkowski's integral inequality, we observe, $\|f\|_{M^{1,1}} \asymp \|f\|_{W^{1,1}}$ and $M^{p,1}(\mathbb{R}^d) \subset W^{p,1}(\mathbb{R}^d)$; the proof follows by Lemma 2.3.11. $\square$

Proof of Theorem 2.3.9(1). Taking Lemmas 2.3.12 and 2.3.14 into our account and exploiting the method of Theorem 2.3.9(1); the proof follows. $\square$

2.4 Concluding Remark

Composition operators are simple examples of nonlinear mappings. In this chapter, we have studied composition operators(for instance, see Theorems 2.2.1 and 2.3.3), and gained the complete understating of composition operators on $M^{1,1}(\mathbb{R}^d)$.

We hope to investigate composition operators on the weighted modulation and Wiener amalgam spaces for the various remaining cases(see the question posed in Section 2.1) in our future work.

Chapter 3

Contraction of Functions in $M^{1,1}(\mathbb{R})$

In this brief chapter, we will obtain some sufficient conditions for nonlinearity $fF(f)$ and $|f|$ to be in $M^{1,1}(\mathbb{R})$ whenever $f \in M^{1,1}(\mathbb{R})$ and F is a contraction on $\mathbb{C}$.

3.1 Introduction

In the last chapter, we have shown that: A composition operator T_F acts on $M^{1,1}(\mathbb{R}^d)$ if and only if $F(0) = 0$ and F is real analytic on $\mathbb{R}^2$. As a consequence, there exist functions $f \in M^{1,1}(\mathbb{R})$ such that $|f|, f|f|^{2k+1}$ $(k \in \mathbb{N})$ does not belong to $M^{1,1}(\mathbb{R})$. In view of this, one is prompted to ask: given $f \in M^{1,1}(\mathbb{R})$, under which sufficient condition, one can ensure the membership for nonlinearity $|f|$ and $f|f|^{2k+1}$ in $M^{1,1}(\mathbb{R})$?

The purpose of this chapter is to investigate this question. We start by recalling formal Fourier series and taking glance at classical results. For $f \in L^1(\mathbb{T})$ its formal Fourier series is given by

$$f(e^{2\pi i\theta}) \sim \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{-2\pi in\theta},$$

where $\hat{f}(n)$ denotes the n^{th} Fourier coefficient of f. We denote by $A(\mathbb{T})$ the class of all functions on the unit circle whose Fourier series is absolutely convergent.

The Wiener-Lévy theorem [73, p.245] asserts that if F is analytic on the range of some $f \in A(\mathbb{T})$, then $F(f) \in A(\mathbb{T})$. Katznelson [38] has established the converse: if F is defined (for instance) on the interval $[-1, 1]$ of the real axis and if $F(f) \in A(\mathbb{T})$ for all $f \in A(\mathbb{T})$ whose range is in $[-1,1]$, then F is analytic on $[-1,1]$. As a consequence, there exist functions $f \in A(\mathbb{T})$ such that $|f|, f|f|^{2k+1}$ $(k \in \mathbb{N})$ does not belong to $A(\mathbb{T})$. On the other hand, Beurling [7] has shown that if $f \in A(\mathbb{T})$ is such that $|\hat{f}(\pm n)| \leq c_n$ $(n \in \mathbb{N} \cup \{0\})$, where c_n is a non-increasing sequence of numbers with a finite sum, then $|f| \in A(\mathbb{T})$. The analogous results have been proved in which, the underlying group $\mathbb{T}$ is replaced by $\mathbb{R}$, that is, the algebra $A(\mathbb{R})$ of Fourier transforms; cf. [33, 7].

One of the interesting subclass of $A(\mathbb{R})$, from the PDEs viewpoint, is the modulation space $M^{1,1}(\mathbb{R})$. In the last decade, modulation spaces have made their own place in PDEs; as they provide a remarkable properties which are known to fail on usual Lebesgue spaces. For instance, $M^{p,1}(\mathbb{R})$ is an algebra under pointwise multiplication; the Schrödinger and wave propagator are not $L^p(\mathbb{R})$ $(p \neq 2)$ bounded but bounded on $M^{p,q}(\mathbb{R})$ $(1 \leq p, q \leq \infty)$. So, the modulation spaces have been used as a regularity class of initial data for the Cauchy problem for non-linear evolution equations, mainly with nonlinearity of form $f^{2k+1}(F(f))^{2k}$, where $F(z) = \bar{z}$ $(z \in \mathbb{C})$; cf. [2, 4, 66, 68]. What about the nonlinearity $f(F(f))^{2k+1}$ when $F(z) = |z|$? This problem is delicate and the answer is still unclear, cf. [53], which shows the importance of the above problem.

And in view of these considerations we are inspired to investigate the above question, and sufficient conditions(Theorems 3.3.1, 3.3.2 below) are obtained in terms of Beurling's algebra $A^*(\mathbb{R}) \subset A(\mathbb{R})$ (Definition 3.2.1 below) and negative definite functions(Definition 3.2.2 below, introduced by Beurling in [7]), which has occurred naturally while investigating it; in fact A^* and negative definite functions are intimately related which we will see in Section 3. The underneath ideas of our main results are to use the contraction properties of $\mathbb{C}$ which we have done using the negative definite functions. We start with the following definition:

Definition 3.1.1 A complex function F on $\mathbb{C}$ is called a contraction if it satisfies the following inequality

$$|F(z_1) - F(z_2)| \leq |z_1 - z_2|, (z_1, z_2 \in \mathbb{C}).$$

If f is a complex valued function, we say the function $F(f)$ a contraction of f.

More precisely, we show (Theorems 3.3.1 below) $fF(f) \in M^{1,1}(\mathbb{R})$ whenever $f \in M^{1,1}(\mathbb{R}) \cap A^*(\mathbb{R})$ and $F(f)$ vanishes at infinity, where $F(f)$ is a contraction of f. Also, we show $|f| \in M^{1,1}(\mathbb{R})$ whenever $f \in M^{1,1}(\mathbb{R})$ and with some suitable condition on Short-time Fourier transform of f. See Theorem 3.3.2 below.

3.2 The Beurling algebra $A^*(\mathbb{R})$ and $M^{1,1}(\mathbb{R})$

We denote by $A(\mathbb{R})$ the algebra of Fourier transforms. In other words, $f \in A(\mathbb{R})$ if there exists some $\psi \in L^1(\mathbb{R})$ such that

$$f(w) = \hat{\psi}(w) \ (w \in \mathbb{R}).$$

The space $A(\mathbb{R})$ is a Banach algebra under pointwise addition and multiplication, with respect to the norm:

$$\|f\|_{A(\mathbb{R})} := \|\psi\|_{L^1} \ (f \in A(\mathbb{R})).$$

Definition 3.2.1 We define the Beurling algebra $A^*(\mathbb{R})$ by functions $f = \hat{\psi}$ in $A(\mathbb{R})$ for which

$$\psi^*(x) := \sup_{|\xi|>|x|} |\psi(\xi)|, (x \in \mathbb{R}) \tag{3.1}$$

belongs to $L^1(\mathbb{R})$:

$$A^*(\mathbb{R}) = \{f \in A(\mathbb{R}) : \psi^* \in L^1(\mathbb{R})\}.$$

The space $A^*(\mathbb{R})$ is normed by the L^1-norm on $\mathbb{R}$:

$$\|f\|_{A^*(\mathbb{R})} := \|\psi^*\|_{L^1} \ (f \in A^*(\mathbb{R})). \tag{3.2}$$

For a further study of the space A^* we refer the reader to [6].

The space $A^*(\mathbb{R})$ was born due to Beurling while studying the contraction of functions in $A(\mathbb{R})$ and asserted that the important tool in order to study of contraction is the following negative definite functions. We make the following definition:

Definition 3.2.2 We call $\lambda(w)$ a negative definite function if it has the form

$$\lambda(w) = \int_0^\infty \frac{\sin^2 2\pi w\alpha}{\alpha^2} d\mu(\alpha), \ \ (\mu(0) = 0)$$

where $\mu(\alpha)$ is a non-decreasing function such that the integral converges for every real w.

For a further study of the above integral form we refer the reader to [65].

Lemma 3.2.3 ([7]) *Let f be a non-increasing and $f \in L^1((0,\infty))$. Then a negative definite function λ exists such that*

$$f(w) \ \ \leq \ \ \frac{1}{\lambda(w)}, \ (w > 0) \tag{3.3}$$

$$\int_0^\infty \frac{dw}{\lambda(w)} \leq 24 \int_0^\infty f(w)dw. \tag{3.4}$$

Proposition 3.2.4 ([7]) *Let $f \in A(\mathbb{R})$ and $F(f)$ be a contraction of f such that $F(f)$ vanishing at infinity. If there is a negative definite function λ such that $|f^\vee|^2\lambda + \lambda^{-1} \in L^1(\mathbb{R})$, then $F(f) \in A(\mathbb{R})$.*

Theorem 3.2.5 (Beurling) *Let $f \in A^*(\mathbb{R})$ and $F(f)$ be a contraction of f such that $F(f)$ vanishes at infinity. Then $F(f) \in A(\mathbb{R})$.*

Proof. In view of (3.1), we note that ψ^* is non-increasing, and now Lemma 3.2.3 and Proposition 3.2.4 give the desired result. □

Proposition 3.2.6 (a) $\mathcal{S}(\mathbb{R}) \subset A^*(\mathbb{R})$. (b) There exists a function in $M^{1,1}(\mathbb{R})$ which does not belong to $A^*(\mathbb{R})$.

Proof. (a) Let $f \in \mathcal{S}(\mathbb{R})$. Since Fourier transform is an isomorphism on $\mathcal{S}(\mathbb{R})$, there exist $\psi \in \mathcal{S}(\mathbb{R})$ such that $\hat{\psi} = f$. Put $\psi^*(x) = \sup_{|\xi|>|x|} |\psi(\xi)|$, and observe that $|\psi^*(x)| \leq \sup_{|\xi|>|x|} \frac{C}{(1+|\xi|)^n}$ for some constant C and $n \in \mathbb{N}$. Taking n large enough, it follows that, $\psi^* \in L^1(\mathbb{R})$. Hence, $f \in A^*(\mathbb{R})$.

(b)If possible, suppose that $M^{1,1}(\mathbb{R}) \subset A^*(\mathbb{R})$. Then by Beurling Theorem 3.2.5, it follows that, $|f| \in A(\mathbb{R})$ for all $f \in M^{1,1}(\mathbb{R})$. Therefore, there exist $\psi \in L^1(\mathbb{R})$ such that $\hat{\psi} = |f|$. But then it follows that, $f|f| \in M^{1,1}(\mathbb{R})$ whenever $f \in M^{1,1}(\mathbb{R})$ by Proposition 1.4.3(1); which is absurd due to Corollary 2.2.5. □

3.3 Sufficient Conditions

In this section we prove our main Theorems 3.3.1 and 3.3.2 but before this it is worth noting the following: in view of $\mathcal{S}(\mathbb{R}) \subset M^{1,1}(\mathbb{R})$, Proposition 3.2.6 has inspired us to impose the hypothesis, $f \in A^*(\mathbb{R}) \cap M^{1,1}(\mathbb{R})$, of Theorem 3.3.1.

Theorem 3.3.1 *Suppose that $f \in M^{1,1}(\mathbb{R}) \cap A^*(\mathbb{R})$ and $F(f)$ be a contraction of f such that $F(f)$ vanishes at infinity. Then $fF(f) \in M^{1,1}(\mathbb{R})$, and $\|fF(f)\|_{M^{1,1}} \lesssim \|f\|_{M^{1,1}} \|F(f)\|_{A(\mathbb{R})}$.*

Proof of Theorem 3.3.1. By Beurling's Theorem 3.2.5, $F(f) \in A(\mathbb{R})$, and so there exists $\psi \in L^1(\mathbb{R})$ such that $\hat{\psi} = F(f)$. Hence, in view of Lemma 1.4.1(8) and Proposition 1.4.3(1), we have,

$$\begin{aligned} \|fF(f)\|_{M^{1,1}} &= \|f^{\vee} * \psi\|_{M^{1,1}} \\ &\lesssim \|f\|_{M^{1,1}} \|\psi\|_{L^1} \\ &= \|f\|_{M^{1,1}} \|F(f)\|_{A(\mathbb{R})}. \end{aligned}$$

□

Theorem 3.3.2 *Suppose that $f \in M^{1,1}(\mathbb{R})$. If there is a negative definite function $\lambda(w)$ such that $|V_g f|^2 \beta + \beta^{-1} \in L^1(\mathbb{R}^2)$, where $\beta(x, w) = \lambda(w)\gamma(x)$ for some function $\gamma(x)$ $(x, w \in \mathbb{R})$, then $|f| \in M^{1,1}(\mathbb{R})$, and $\|f|f|^{2k+1}\|_{M^{1,1}} \lesssim \|f\|_{M^{1,1}} \||f|\|_{M^{1,1}}^{2k+1}$.*

Now to proving Theorem 3.3.2 we need the following technical lemma which has been observed in [7].

Lemma 3.3.3 *Let $h \in L^2(\mathbb{R})$ and $\alpha > 0$. Put $H_n(w) = \int_{-n}^{n} h(t)e^{-2\pi i w t} dt$ $(w \in \mathbb{R}, n \in \mathbb{N})$, and*

$$R_n(w) = \int_{-n}^{n} e^{-2\pi i w t}(h(t+\alpha) - h(t-\alpha))dt - (e^{2\pi i \alpha w} - e^{-2\pi i \alpha w})H_n(w).$$

Then R_n converges to 0 in $L^2(\mathbb{R})$ as n tends to infinity.

Proof. By change of variable,

$$\int_{-n}^{n} h(t)e^{-2\pi i(t-\alpha)w} dt = \int_{-n-\alpha}^{n-\alpha} h(t+\alpha)e^{-2\pi i t w} dt$$

and

$$\int_{-n}^{n} h(t)e^{-2\pi i(\alpha+t)w}dt = \int_{-n+\alpha}^{n+\alpha} h(t-\alpha)e^{-2\pi iwt}dt;$$

and this motivates us to define

$$r_n(t) = \begin{cases} h(t-\alpha) & \text{if } t \in [n, n+\alpha), \\ h(t+\alpha) & \text{if } t \in [n-\alpha, n), \\ -h(t-\alpha) & \text{if } t \in [-n, -n+\alpha), \\ -h(t+\alpha) & \text{if } t \in [-n-\alpha, -n), \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

By (3.5), we may rewrite $R_n(w)$ as, $R_n(w) = \int_{\mathbb{R}} r_n(t)e^{-2\pi iwt}dt$. By Plancherel theorem,

$$\|R_n\|_{L^2}^2 = \|r_n\|_{L^2}^2 = \left(\int_{-n-\alpha}^{-n+\alpha} + \int_{n-\alpha}^{n+\alpha}\right) |h(t)|^2 dt$$

and since $h \in L^2(\mathbb{R})$, we have, $\sum_{n\in\mathbb{Z}} \int_{n-\alpha}^{n+\alpha} |h(t)|^2$ is finite, but this implies, $\int_{n-\alpha}^{n+\alpha} |h(t)|^2 dt$ tends to 0 as $|n| \to \infty$. It follows that, $\|R_n\|_{L^2} \to 0$ as $n \to \infty$. □

Proof of the Theorem 3.3.2. By (1.2),

$$V_g f(x, w) = \widehat{(fT_x\bar{g})}(w) \ (x, w \in \mathbb{R}). \tag{3.6}$$

Fixing a space variable x, and taking the inverse Fourier transform with respect to the frequency variable w in the (3.6), we have, $(fT_x\bar{g})(\xi) = (V_g f)^\vee(x, \xi)$. For $\alpha > 0$, we have,

$$(fT_x\bar{g})(\xi+\alpha) - (fT_x\bar{g})(\xi-\alpha) = (F_x^\alpha)^\vee(\xi),$$

where $F_x^\alpha(t) = V_g f(x,t)(e^{2\pi it\alpha} - e^{-2\pi it\alpha})$. By the Plancherel theorem, we have,

$$\begin{aligned} &\int_{\mathbb{R}} |(fT_x\bar{g})(\xi+\alpha) - (fT_x\bar{g})(\xi-\alpha)|^2 d\xi \\ &\quad = 4\int_{\mathbb{R}} |V_g f(x,w)|^2 \sin^2(2\pi w\alpha) dw. \end{aligned} \tag{3.7}$$

Multiplying both sides in (3.7) by $\alpha^{-2}d\mu(\alpha)$ and integrating over $(0, \infty)$, we get by inverting the order of integration,

$$\int_{\mathbb{R}} \int_0^\infty |fT_x\bar{g}(\xi+\alpha) - fT_x\bar{g}(\xi-\alpha))|^2 \alpha^{-2} d\mu(\alpha) d\xi \\ = 4\int_{\mathbb{R}} |V_g(x,w)|^2 \lambda(w) dw. \tag{3.8}$$

Now taking integration on both sides with respect to x, we obtain

$$\int_{\mathbb{R}^2} \int_0^\infty |(fT_x\bar{g})(\xi+\alpha) - (fT_x\bar{g})(\xi-\alpha))|^2 \alpha^{-2} \gamma(x) d\mu(\alpha) d\xi dx \\ = \int_{\mathbb{R}^2} |V_g f(x,w)|^2 \beta(x,w) dw dx. \tag{3.9}$$

Let $h = |f|$ be a contraction of f and define

$$H_n^x(w) := \int_{-n}^{n} (hT_x\bar{g})(t) e^{-2\pi i w\cdot t} dt, (n \in \mathbb{N}).$$

We put R_n^x (keeping x fixed) as follows:

$$\int_{-n}^{n} e^{-2\pi i t w} \left((hT_x\bar{g})(t+\alpha) - (hT_x\bar{g})(t-\alpha)\right) dt \\ = \left(e^{2\pi i\alpha w} - e^{-2\pi i\alpha w}\right) H_n^x(w) + R_n^x(w),$$

then by Lemma 3.3.3, it follows that, the remainder R_n^x converges to 0 in $L^2(\mathbb{R})$; and $hT_x\bar{g}(t+\alpha) - hT_x\bar{g}(t-\alpha) \in L^2(\mathbb{R})$, and the sequence

$$(e^{2\pi i\alpha\cdot w} - e^{-2\pi i t w}) H_n^x(w)$$

converges in $L^2(\mathbb{R})$ to a certain function which can be written in the form

$$(e^{2\pi i\alpha w} - e^{-2\pi i\alpha w}) H^x(w).$$

It follows that

$$\int_{\mathbb{R}} |(hT_x\bar{g})(\xi+\alpha) - (hT_x\bar{g})(\xi-\alpha)|^2 d\xi = 4\int_{\mathbb{R}} |H^x(w)|^2 \sin^2(2\pi w\alpha) dw.$$

Performing as before, we have,

$$\int_{\mathbb{R}^2}\int_0^\infty |(hT_x\bar{g})(\xi+\alpha)-(hT_x\bar{g})(\xi-\alpha))|^2\alpha^{-2}\gamma(x)d\mu(\alpha)d\xi dx = \int_{\mathbb{R}^2}|H^x(w)|^2\beta(x,w)dwdx. \tag{3.10}$$

Since h is a contraction of f, we have for $g(x)=e^{-\pi|x|^2/2}>0$,

$$\int_{\mathbb{R}^2}\int_0^\infty |(hT_x\bar{g})(\xi+\alpha)-(hT_x\bar{g})(\xi-\alpha))|^2\alpha^{-2}\gamma(x)d\mu(\alpha)d\xi dx \leq \int_{\mathbb{R}^2}\int_0^\infty |(fT_x\bar{g})(\xi+\alpha)-(fT_x\bar{g})(\xi-\alpha))|^2\alpha^{-2}\gamma(x)d\mu(\alpha)d\xi dx. \tag{3.11}$$

By (4.63), (3.9), (4.40), and Schwartz's inequality, it follows that, $H^x(w)\in L^1(\mathbb{R}^2)$. Thus, it follow that $|f|\in M^{1,1}(\mathbb{R})$ and since $M^{1,1}(\mathbb{R})$ is an algebra under pointwise multiplication, we get the desired inequality, $\|f|f|^{2k+1}\|_{M^{1,1}}\lesssim \|f\|_{M^{1,1}}\||f|\|^{2k+1}_{M^{1,1}}$. □

3.4 Concluding Remarks

1. We have proved Theorems 3.3.1 and 3.3.2 for the one dimension; it would be interesting to know whether the analogous results are true or not for the dimension greater than one.

2. Taking Theorem 2.2.1(1) into the account, it would be very natural to investigate whether the analogue of Theorems 3.3.1 and 3.3.2 are true or not for $M^{p,1}(\mathbb{R}),(1<p<\infty)$.

Chapter 4

Nonlinear Evolution Equations

After taking a brief introduction to the nonlinear evolution equations in the first section, in Section 4.2 of this chapter, we illustrate the method of the contraction mapping theorem to obtain local well-posedness results for NLS, NLW and NLKG equations for the 'real entire' nonlinearities in some weighted modulation spaces $M_s^{p,q}(\mathbb{R}^d)$. In Section 4.3 we highlights the fundamental importance of our previous results of Chapter 2.

Section 4.4 is devoted to the Cauchy problem for Schrödinger equation with cubic convolution nonlinearity, in fact, with this nonlinearity we establish local and global well-posedness results.

4.1 Introduction

Nonlinear evolution equations, i.e., partial differential equations with time t as one of the independent variables, arise not only from many fields of mathematics, but also from other branches of science such as physics, mechanics and material science. Just as an example, Navier-stokes arises from heat transforms, nonlinear Schrödinger equations from quantum mechanics, and so on.

The first question to ask in the theoretical study is whether for a nonlinear evolution equation with given initial data, is there a solution, at least locally in time, and whether it is unique in the considered class (local well-posedness). The next step is to investigate when a local solution can be extended to set a global one in time, and whether it is unique(global well-posedness).

Complexity of nonlinear evolution equations and challenges in their theoretical study have attracted a lot of interest from many mathematicians and scientists in nonlinear sciences. In fact, over the last several decades, many authors have contributed on this subject and now the theory of nonlinear evolution equations is vast, and still, the topic is of interest in the current trend of new investigations. We cannot hope to acknowledge here all these who have contributed to the theory of nonlinear evolution equations, however, for a sample of results and a nice introduction to the field, we refer the reader to monographs [10, 60, 41], and for the recent development and the connection between modulation (Wiener amalgam) spaces and nonlinear evolution equations, we recommend the monograph [69], and the references therein.

The aim of next the two section is to focus on the Cauchy problem for the nonlinear Schrödinger equation (NLS), the nonlinear wave equation (NLW), and the nonlinear Klein-Gordon equation (NLKG) in the realm of modulation spaces. In fact, in Subsections 4.2.1-4.2.3, as an application of Theorem 2.3.1, we illustrate how the local well-posedness of the NLS, NLW and NLKG equations for the 'real entire' nonlinearities can be obtained in some weighted modulation spaces $M_s^{p,q}(\mathbb{R}^d)$ using the contraction mapping principle; and in the later section, in view of this and as an aid to our previous results(Chapter 2), we point out the standard method for the evolution of nonlinear evolution (Schrödinger/wave/Klein-Gordon) equations cannot be considered for nonlinearity of the form $u|u|^\alpha, \alpha \in (0,\infty) \setminus 2\mathbb{N}$.

The aim of Section 4.4 is to focus on the Cauchy problem for Schrödinger equation with cubic convolution nonlinearity $F(u) = (K * |u|^2)u$ (see Subsection 4.4.1 below for the motivation) under a specified condition on potential K with Cauchy data in modulation spaces $M^{p,q}(\mathbb{R}^d)$. We establish global well-posedness results in $M^{1,1}(\mathbb{R}^d)$ when $K(x) = \lambda |x|^{-\gamma}$ $(\lambda \in \mathbb{R}, 0 < \gamma < \min\{2, d/2\})$; in $M^{p,q}(\mathbb{R}^d)$ $(1 \leq q \leq \min\{p, p'\}$ where p' is the Hölder conjugate of $p \in [1,2])$ when K is in Fourier algebra $\mathcal{F}L^1(\mathbb{R}^d)$, and local well-posedness result in $M^{p,1}(\mathbb{R}^d)(1 \leq p \leq \infty)$ when $K \in M^{1,\infty}(\mathbb{R}^d)$.

4.2 The Local Well-Posedness of the NLS, NLW and NLKG

In this section, we study initial value problems for the NLS, nonlinear wave equation(NLW), and nonlinear Klein-Gordon equation(NLKG). Specifically, we study

$$(NLS)\ i\frac{\partial u}{\partial t} + \Delta_x u = F(u),\ \ u(x,t_0) = u_0(x), \tag{4.1}$$

$$(NLW)\ \frac{\partial^2 u}{\partial t^2} - \Delta_x u = F(u),\ \ u(x,t_0) = u_0(x), \frac{\partial u}{\partial t}(x,t_0) = u_1(x), \tag{4.2}$$

$$(NLKG)\ \frac{\partial^2 u}{\partial t^2} + (I - \Delta_x)u = F(u),\ \ u(x,t_0) = u_0(x), \frac{\partial u}{\partial t}(x,t_0) = u_1(x), \tag{4.3}$$

where $t_0 \in \mathbb{R}$, u_0, u_1 are complex valued functions on $\mathbb{R}^d$, I is the identity map and F is a real entire function with $F(0) = 0$.

In fact, our Theorem 2.3.1 has inspired us to consider nonlinearites of the form,

$$F(u) = G(u_1, u_2); \tag{4.4}$$

where $u = u_1 + iu_2$ and $G : \mathbb{R}^2 \to \mathbb{C}$ is real entire on $\mathbb{R}^2$ with $G(0) = 0$. This generalizes the nonlinearities previously studied in modulation spaces. With the help of estimate (2.11) for real entire nonlinearities given by Theorem 2.3.1, and well established Fourier multiplier estimates, we prove the local well-posedness results of NLS (4.1), NLW (4.2), and $NLKG$ (4.3) with Cauchy data in X, where X denotes the spaces $M^{p,1}_s(\mathbb{R}^d), (1 \leq p \leq \infty, s \geq 0)$; or $M^{p,q}_s(\mathbb{R}^d), (1 \leq p, q \leq \infty, s > d/q')$, see Theorems 4.2.6, 4.2.7 and Theorem 4.2.8.

We start with the observation that the partial derivatives $\partial_x F(x,y)$ and $\partial_y F(x,y)$ are real entire functions if F is real entire. This can be easily seen from the power series expansion $F(x,y) = \sum_{m,n=0}^{\infty} a_{mn} x^m y^n,\ (x,y) \in \mathbb{R}^2$. In fact we can do term by term differentiation and get

$$\partial_x F(x,y) = \sum_{m\geq 1, n\geq 0} m\, a_{mn}\, x^{m-1}\, y^n. \tag{4.5}$$

This is justified because the above power series is absolutely convergent on $\mathbb{R}^2$: In fact, we have $m \leq 2^{m-1}$ for $m \geq 1$ and hence

$$|m\, x^{m-1}| \leq (2|x|)^{m-1} \leq (1 + 2|x|)^m.$$

Thus $\sum_{m\geq 1, n\geq 0} m\,|a_{mn}|\,|x|^{m-1}\,|y|^n \leq \tilde{F}(1+2|x|,|y|) < \infty$ for all $(x,y)\in\mathbb{R}^2$. (See notation (2.2).) Similarly, we can also show that $\partial_y F$ is real entire and has the expansion

$$\partial_y F(x,y) = \sum_{m\geq 0, n\geq 1} n\,a_{mn}\,x^m\,y^{n-1} \tag{4.6}$$

valid for all $(x,y)\in\mathbb{R}^2$. From (4.5) and (4.6) we also get the inequalities

$$\begin{aligned} |\partial_x F(x,y)| &\leq \widetilde{\partial_x F}(|x|,|y|) := \sum_{m\geq 1, n\geq 0} m|a_{mn}||x|^{m-1}\,|y|^n, \\ |\partial_y F(x,y)| &\leq \widetilde{\partial_y F}(|x|,|y|) := \sum_{m\geq 0, n\geq 1} n|a_{mn}||x|^{m}\,|y|^{n-1}. \end{aligned}$$

Note that we cannot expect a similar inequality by replacing x and y by functions u and v in the Banach algebra X as $\partial_x F(u)$ and $\partial_y F(u)$ need not be in X because of the possible nonzero constant term in the power series expansion. However, we have the following substitute given in the following

Lemma 4.2.1 *Let F be a real entire function on $\mathbb{R}^2$, then the partial derivatives $\partial_x F(x,y)$ and $\partial_y F(x,y)$ are also real entire functions. Moreover, if $u = u_1 + iu_2 \in X$, the modulation space mentioned above, then the following estimates hold*

$$\|w\partial_x F(u_1,u_2)\|_X \lesssim \|w\|_X\,\widetilde{\partial_x F}(\|u_1\|_X,\|u_2\|_X), \tag{4.7}$$

$$\|w\partial_y F(u_1,u_2)\|_X \lesssim \|w\|_X\widetilde{\partial_y F}(\|u_1\|_X,\|u_2\|_X) \tag{4.8}$$

for every $w\in X$.

Proof. We have already observed that $\partial_x F(x,y)$ and $\partial_y F(x,y)$ are real entire functions with absolutely convergent power series expansions (4.5) and (4.6) valid for all $(x,y)\in\mathbb{R}^2$. Now we observe that the series

$$\sum_{m\geq 1, n\geq 0} m\,a_{mn}\,w\,u_1^{m-1}\,u_2^n$$

is absolutely convergent in X for every $w\in X$. In fact, since X is an algebra, we have

$$\|w\,u_1^{m-1}\,u_2^n\|_X \lesssim \|w\|_X\|u_1\|_X^{m-1}\,\|u_2\|_X^n$$

by (1.17). It follows that $w\,\partial_x F(u_1,u_2)\in X$ and

$$\begin{aligned}\|w\,\partial_x F(u_1,u_2)\|_X &\lesssim \sum_{m\geq 1, n\geq 0} m|a_{mn}|\,\|w\|_X\,\|u_1\|_X^{m-1}\,\|u_2\|_X^n \\ &= \|w\|_X\,\widetilde{\partial_x F}(\|u_1\|_X,\|u_2\|_X).\end{aligned} \tag{4.9}$$

Similarly, $w\,\partial_y F(u_1,u_2)\in X$ and

$$\begin{aligned}\|w\,\partial_y F(u_1,u_2)\|_X &\lesssim \sum_{m\geq 0, n\geq 1} n|a_{mn}|\,\|w\|_X\,\|u_1\|_X^{m}\,\|u_2\|_X^{n-1} \\ &= \|w\|_X\,\widetilde{\partial_y F}(\|u_1\|_X,\|u_2\|_X).\end{aligned} \tag{4.10}$$

Hence, the lemma. □

The following proposition gives the essential estimate required to establish the contraction estimate.

Proposition 4.2.2 *Let F be a real entire function on $\mathbb{R}^2$ and X be the modulation space as in Lemma 4.2.1. Then we have*

$$\begin{aligned}&\|F(u_1,u_2)-F(v_1,v_2)\|_X \\ &\qquad\lesssim 2\|u-v\|_X\left[\left(\widetilde{\partial_x F}+\widetilde{\partial_y F}\right)(\|u\|_X+\|v\|_X,\|u\|_X+\|v\|_X)\right]\end{aligned}$$

for every $u,v\in X$.

Proof. Let $u,v\in X$, with $u=u_1+iu_2$ and $v=v_1+iv_2$, where $u_1=\mathrm{Re}(u)$ and $v_1=\mathrm{Re}(v)$. Using the formula

$$F(x,y)-F(x',y')=\int_0^1\frac{d}{ds}\left[F(x'+s(x-x'),y'+s(y-y'))\right]ds$$

for $x,x',y,y'\in\mathbb{R}$, we see that

$$\begin{aligned}&F(u_1,u_2)-F(v_1,v_2) \\ &\quad=\int_{s=0}^1 (u_1-v_1)\,\partial_x F(u_1+s(u_1-v_1),u_2+s(u_2-v_2))\,ds \\ &\quad+\int_{s=0}^1 (u_2-v_2)\,\partial_y F(u_1+s(u_1-v_1),u_2+s(u_2-v_2))\,ds.\end{aligned} \tag{4.11}$$

Taking norm on both sides and applying Minkowski's inequality for integral, and the Lemma 4.2.1, with $w = v_i - u_i,\ i = 1,2$ we get,

$$\begin{aligned} & \|F(u_1,u_2) - F(v_1,v_2)\|_X \qquad (4.12) \\ \lesssim\ & \|u_1 - v_1\|_X \int_{s=0}^{1} \widetilde{\partial_x F}(\|(u_1 + s(u_1 - v_1))\|_X, \|u_2 + s(u_2 - v_2)\|_X) ds \\ +\ & \|u_2 - v_2\|_X \int_{s=0}^{1} \widetilde{\partial_y F}(\|u_1 + s(u_1 - v_1)\|_X, \|(u_2 + s(u_2 - v_2)\|_X) ds. \end{aligned}$$

Note that

$$\|u_i + s(v_i - u_i)\|_X \leq (1-s)\|u_i\|_X + s\|v_i\|_X \leq \|u\|_X + \|v\|_X$$

for $i = 1,2$ in view of Lemma 1.4.1(7). Thus using the monotonicity of $\widetilde{\partial_x F}$ and $\widetilde{\partial_y F}$ on $[0,\infty)$ in each of its variables, the above integrands are dominated by

$$\left(\widetilde{\partial_x F} + \widetilde{\partial_y F}\right)(\|u\|_X + \|v\|_X, \|u\|_X + \|v\|_X).$$

In view of these observations, (4.12) yields the estimate

$$\begin{aligned} \|F(u_1,u_2) \ - \ & F(v_1,v_2)\|_X \lesssim (\|u_1 - v_1\|_X) + \|u_2 - v_2\|_X) \\ & \times \left(\widetilde{\partial_x F} + \widetilde{\partial_y F}\right)(\|u\|_X + \|v\|_X, \|u\|_X + \|v\|_X). \end{aligned}$$

Since $u_1 - v_1 = \mathrm{Re}(u - v)$ and $u_2 - v_2 = \mathrm{Im}(u - v)$, the required inequality follows from this, in view of Lemma 1.4.1(7). □

The estimates for the linear propagators associated to the Schrödinger, the wave and the Klein-Gordon equations are given by the multiplier theorems on modulation spaces $M_s^{p,q}(\mathbb{R}^d)$, for three sets of multipliers listed below.

For a bounded measurable function σ on $\mathbb{R}^d$, let H_σ denote the Fourier multiplier operator given by

$$H_\sigma f(x) = \int_{\mathbb{R}^d} \sigma(\xi)\, \widehat{f}(\xi)\, e^{2\pi i \xi \cdot x}\, d\xi, \qquad (4.13)$$

for $f \in \mathcal{S}(\mathbb{R}^d)$. The function σ is called the multiplier. Here we are concerned with the following families of multipliers defined on $\mathbb{R}^d$:

1. $\sigma(\xi) = e^{-it4\pi^2|\xi|^2}$,

2. $\sigma^1(\xi) = \sin(2\pi t|\xi|)/2\pi|\xi|, \quad \sigma^2(\xi) = \cos(2\pi t|\xi|)$,

3. $\mu_1(\xi) = \sin[t(1 + |2\pi\xi|^2)^{1/2}]/(1 + |2\pi\xi|^2)^{1/2}, \ \mu_2(\xi) = \cos[t(1 + |2\pi\xi|^2)^{1/2}]$.

We use the following results, and we refer to [2, Lemma 2.2], and also [4, Theorem 1, Corollary 18], for the proof of these facts. See also Proposition 1.5.1.

Proposition 4.2.3 *Let σ be as in (1) and H_σ be the Fourier multiplier as in (4.13). Then H_σ extends to a bounded operator on $M_s^{p,q}(\mathbb{R}^d)$ for $1 \leq p, q \leq \infty, s \geq 0$. Moreover, H_σ satisfies the inequality*

$$\|H_\sigma f\|_{M_s^{p,q}} \leq c_d(1 + t^2)^{d/4}\|f\|_{M_s^{p,q}} \tag{4.14}$$

for some constant c_d.

Proposition 4.2.4 *Let σ^1 and σ^2 be as in (2). Then the corresponding Fourier multiplier operators $H_{\sigma^1}, H_{\sigma^2}$ can be extended as a bounded operators on $M_s^{p,q}(\mathbb{R}^d)$ for $1 \leq p, q \leq \infty, s \geq 0$. Moreover, they satisfy the inequalities*

$$\|H_{\sigma^i} f\|_{M_s^{p,q}} \leq c_d(1 + t^2)^{d/4}\|f\|_{M_s^{p,q}}. \tag{4.15}$$

Proposition 4.2.5 *Let μ_1 and μ_2 be as in (3). Then the Fourier multiplier operators $H_{\mu_i}, i = 1, 2$ can be extended as a bounded operators on $M_s^{p,q}(\mathbb{R}^d)$, for $1 \leq p, q \leq \infty, s \geq 0$. Moreover, these operators satisfy the inequalities*

$$\|H_{\mu_i} f\|_{M_s^{p,q}} \leq c_d(1 + t^2)^{d/4}\|f\|_{M_s^{p,q}}. \tag{4.16}$$

Now we proceed to prove the well-posedness results, starting with nonlinear Schrödinger equation.

4.2.1 The Nonlinear Schrödinger Equation

Theorem 4.2.6 *Assume that $u_0 \in X$ and the nonlinearity F has the form (4.4). Then, there exists $T_* = T_*(\|u_0\|_X) < t_0$ and $T^* = T^*(\|u_0\|_X) > t_0$ such that (4.1) has a unique solution $u \in C([T_*, T^*], X)$. Moreover, if $|T^*| < \infty$ then $\limsup_{t\to T^*} \|u(\cdot, t)\|_X = \infty$. Similarly $\limsup_{t\to T_*} \|u(\cdot, t)\|_X = \infty$, if $|T_*| < \infty$.*

Proof. We start by noting that (4.1) can be written in the equivalent form

$$u(\cdot,t) = S(t-t_0)u_0 - i\mathcal{A}F(u) \tag{4.17}$$

where

$$S(t) = e^{it\triangle},\ (\mathcal{A}v)(t,x) = \int_{t_0}^{t} S(t-\tau)\,v(t,x)\,d\tau. \tag{4.18}$$

This equivalence is valid in the space of tempered distributions on $\mathbb{R}^d$. For simplicity, we assume that $t_0 = 0$ and prove the local existence on $[0,T]$. Similar arguments also apply to interval of the form $[-T',0]$ for proving local solutions.

We show that the equivalent integral equation (4.17) has a unique solution, by showing that the mapping $\mathcal{J}$ given by

$$\mathcal{J}(u) = S(t)u_0 - i\int_0^t S(t-\tau)\,[F(u(\cdot,\tau))]\,d\tau \tag{4.19}$$

has a unique fixed point in an appropriate functions space, for small t. For this, we consider the Banach space $X_T = C([0,T],X)$, with norm

$$\|u\|_{X_T} = \sup_{t\in[0,T]} \|u(\cdot,t)\|_X\,,\ (u\in X_T).$$

By the Fourier multiplier estimate (4.14) in Proposition 4.2.3 we see that

$$\|S(t)u_0\|_X \le c_d(1+t^2)^{d/4}\,\|u_0\|_X$$

for $t\in\mathbb{R}$. It follows that, for $0\le t\le T$

$$\|S(t)u_0\|_{X_T} \le C_T\,\|u_0\|_{X_T} \tag{4.20}$$

with $C_T = c_d(1+T^2)^{d/4}$.

Also, note that if $u\in X_T$, then $u(\cdot,t)\in X$ for each $t\in[0,T]$. Hence by the estimate (2.11) of Theorem 2.3.1, $F(u(\cdot,t))\in X$ and we have

$$\begin{aligned}\|F(u(\cdot,t))\|_X &\le \tilde{F}(\|u(\cdot,t)\|_X, \|u(\cdot,t)\|_X) \\ &\le \tilde{F}(\|u\|_{X_T}, \|u\|_{X_T}),\end{aligned} \tag{4.21}$$

where the last inequality follows from the fact that $\tilde{F}$ is monotonically increasing on $[0,\infty)\times[0,\infty)$ with respect to each of its variables.

Now an application of Minkowski's inequality for integrals, the Fourier multiplier estimate (4.14) and the estimate (4.21), yields

$$\begin{aligned} \left\| \int_0^t S(t-\tau)[F(u(\cdot,\tau))]\,d\tau \right\|_X &\leq \int_0^t \|S(t-\tau)[F(u(\cdot,\tau))]\|_X \,d\tau \\ &\leq TC_T\,\tilde{F}(\|u\|_{X_T}, \|u\|_{X_T}) \end{aligned} \tag{4.22}$$

for $0 \leq t \leq T$. Using the estimates (4.20) and (4.47) in (4.19), we see that

$$\begin{aligned} \|\mathcal{J}(u)\|_{X_T} &\leq C_T \left(\|u_0\|_X + T\tilde{F}(\|u\|_{X_T}, \|u\|_{X_T}) \right) \\ &\leq C_T \left(\|u_0\|_X + T\,\|u\|_{X_T} G(\|u\|_{X_T}) \right) \end{aligned} \tag{4.23}$$

where G is a real analytic function on $[0, \infty)$ such that $\tilde{F}(x, x) = x\,G(x)$. This factorisation follows from the fact that the constant term in the power series expansion for $\tilde{F}$ is zero, (i.e., $\tilde{F}(0, 0) = 0$). We also note that G is increasing on $[0, \infty)$.

For $M > 0$, put $X_{T,M} = \{u \in X_T : \|u\|_{X_T} \leq M\}$, which is the closed ball of radius M, and centered at the origin in X_T. We claim that

$$\mathcal{J} : X_{T,M} \to X_{T,M},$$

for suitable choice of M and small $T > 0$. Note that $C_T \leq C_1$ for $0 < T \leq 1$. Hence, putting $M = 2C_1\,\|u_0\|_X$, from (4.23) we see that for $u \in X_{T,M}$ and $T \leq 1$

$$\|\mathcal{J}(u)\|_{X_T} \leq \frac{M}{2} + TC_1\,MG(M) \leq M \tag{4.24}$$

for $T \leq T_1$, where

$$T_1 = \min\left\{1, \frac{1}{2C_1 G(M)}\right\}. \tag{4.25}$$

Thus $\mathcal{J} : X_{T,M} \to X_{T,M}$, for $M = 2C_1\,\|u_0\|_X$, and all $T \leq T_1$, hence the claim.

Now we show that $\mathcal{J}$ satisfies the contraction estimate

$$\|\mathcal{J}(u) - \mathcal{J}(v)\|_{X_T} \leq \frac{1}{2}\|u - v\|_{X_T} \tag{4.26}$$

on $X_{T,M}$ if T sufficiently small.

From (4.19) and the estimate (4.13) in Proposition 4.2.3, we see that

$$\begin{aligned}\|\mathcal{J}(u(\cdot,t)) - \mathcal{J}(v(\cdot,t))\|_X &\leq \int_0^t \|S(t-\tau)\,[F(u(\cdot,\tau)) - F(v(\cdot,\tau))]\|_X\, d\tau. \\ &\leq C_t \int_0^t \|F(u(\cdot,\tau)) - F(v(\cdot,\tau))\|_X\, d\tau, \end{aligned} \tag{4.27}$$

since $C_{t-\tau} \leq C_t$. By Proposition 4.2.2 this is at most

$$2C_t \int_0^t \|u-v\|_X \left[\left(\widetilde{\partial_x F} + \widetilde{\partial_y F}\right)(\|u\|_X + \|v\|_X, \|u\|_X + \|v\|_X)\right] d\tau.$$

Now taking supremum over all $t \in [0,T]$, we see that

$$\begin{aligned}\|\mathcal{J}(u) &- \mathcal{J}(v)\|_{X_T} \\ &\leq 2TC_T\|u-v\|_{X_T}\left(\widetilde{\partial_x F} + \widetilde{\partial_y F}\right)(\|u\|_{X_T} + \|v\|_{X_T}, \|u\|_{X_T} + \|v\|_{X_T}).\end{aligned}$$

Now if u and v are in $X_{T,M}$, the RHS of the above inequality is at most

$$2TC_T\|u-v\|_{X_T}\left(\widetilde{\partial_x F} + \widetilde{\partial_y F}\right)(2M,2M) \leq \frac{\|u-v\|_{X_T}}{2} \tag{4.28}$$

for all $T \leq T_2$, where

$$T_2 = \min\left\{1, \left[4C_1\left(\widetilde{\partial_x F} + \widetilde{\partial_y F}\right)(2M,2M)\right]^{-1}\right\}. \tag{4.29}$$

Thus from (4.28), we see that the estimate (4.26) holds for all $T < T_2$. Now choosing $T^1 = \min\{T_1, T_2\}$ where T_1 is given by (4.25), so that both the inequalities (4.24) and (4.26) are valid for $T < T^1$. Hence for such a choice of T, $\mathcal{J}$ is a contraction on the Banach space $X_{T,M}$ and hence has a unique fixed point in $X_{T,M}$, by the Banach's contraction mapping principle. Thus we conclude that $\mathcal{J}$ has a unique fixed point in $X_{T,M}$ which is a solution of (4.55) on $[0,T]$ for any $T < T^1$. Note that T^1 depends on $\|u_0\|_X$.

The arguments above also give the solution for the initial data corresponding to any given time t_0, on an interval $[t_0, t_0 + T^1]$ where T^1 is given by the same formula with $\|u(0)\|_X$ replaced by $\|u(t_0)\|_X$. In other words, the dependence of the length of the interval of existence on the initial time t_0 is only through the norm $\|u(t_0)\|_X$. Thus if the solution exists on $[0,T']$ and if $\|u(T')\|_X < \infty$, the above arguments can be carried out again for the initial value problem with the new initial data $u(T')$ to extend the solution to the larger interval $[0, T'']$. This procedure can be continued and hence we get a solution on maximal interval $[0, T^*]$ having the following blow up alternative: either $\|u(\cdot, T^*)\|_X = \infty$ or $\lim_{t\to T^*} \|u(\cdot, t)\|_X = \infty$.

Similar arguments can be carried out, to extend the solution to a maximal intervals to the left, of the form $[T_*, 0]$. This gives the blow up alternative.

The uniqueness also follows from the uniqueness of the fixed point for $\mathcal{J}$. This completes the proof. □

By similar arguments, using the multiplier estimates given in Proposition 4.2.4, Proposition 4.2.5, and using the Proposition 4.2.2 to prove contraction estimates, we can establish analogous local well-posedness results, for the initial value problems for the wave equation and the Klein-Gordon equation. Instead of repeating the arguments, we only indicate the equivalent integral equation in terms of the one parameter groups involved, and the relevant estimates, to carry out the proof as above.

4.2.2 The Nonlinear Wave Equation

Theorem 4.2.7 *Assume that $u_0, u_1 \in X$ and the nonlinearity F has the form (4.4). Then, there exists $T^* = T^*(\|u_0\|_X, \|u_1\|_X)$ such that (4.2) has a unique solution $u \in C([0, T^*], X)$. Moreover, if $T^* < \infty$, then $\limsup_{t\to T^*} \|u(\cdot, t)\|_X = \infty$.*

Proof. Equation (4.2) can be written in the equivalent form

$$u(\cdot, t) = \tilde{K}(t)u_0 + K(t)u_1 - \mathcal{B}F(u) \tag{4.30}$$

where

$$K(t) = \frac{\sin(t\sqrt{-\triangle})}{\sqrt{-\triangle}}, \tilde{K}(t) = \cos(t\sqrt{-\triangle}), (\mathcal{B}v)(t, x) = \int_0^t K(t-\tau)v(\tau, x)d\tau.$$

Consider the mapping

$$\mathcal{J}(u) = \tilde{K}(t)u_0 + K(t)u_1 - \mathcal{B}F(u).$$

By using Proposition 4.2.4 for the first two inequalities below, and estimate

(2.11) for the last inequality, we can write,

$$\begin{cases} \|\tilde{K}(t)u_0\|_X \le C_T\|u_0\|_X, \\ \|K(t)u_1\|_X \le C_T\|u_1\|_X, \\ \|\mathcal{B}F(u)\|_X \le TC_T\tilde{F}(\|u\|_X, \|u\|_X), \end{cases} \tag{4.31}$$

where C_T is some constant times $(1+T^2)^{d/4}$, as before. Thus the standard contraction mapping argument can be applied to $\mathcal{J}$ to complete the proof. □

4.2.3 The Nonlinear Klein-Gordon Equation

Theorem 4.2.8 *Assume that $u_0, u_1 \in X$ and the nonlinearity F has the form (4.4). Then, there exists $T^* = T^*(\|u_0\|_X, \|u_1\|_X)$ such that (4.3) has a unique solution $u \in C([0,T^*], X)$. Moreover, if $T^* < \infty$, then* $\limsup_{t\to T^*} \|u(\cdot,t)\|_X = \infty$.

Proof. The equivalent form of equation (4.3) is

$$u(\cdot,t) = \tilde{K}(t)u_0 + K(t)u_1 + \mathcal{C}F(u), \tag{4.32}$$

where now

$$K(t) = \frac{\sin t(I-\triangle)^{1/2}}{(I-\triangle)^{1/2}}, \tilde{K}(t) = \cos t(I-\triangle)^{1/2}, \ (\mathcal{C}v)(t,x) = \int_0^t K(t-\tau)v(\tau,x)d\tau.$$

By using Proposition 4.2.5 and the notations above, we can write

$$\begin{cases} \|\tilde{K}u_0\|_X \le C_T\|u_0\|_X, \\ \|K(t)u_1\|_X \le C_T\|u_1\|_X, \\ \|\mathcal{C}F(u)\|_X \le TC_T\tilde{F}(\|u\|_X, \|u\|_X), \end{cases} \tag{4.33}$$

Now the standard contraction mapping argument applied to $\mathcal{J}$ gives the proof.
□

Remark 4.2.9 We would like to point out that the local wellposedness has already been proved for wave equation with real entire nonlinearity in [12] by essentially the same method.

4.3 Comments on the Preceding Theorems

We would like to point out that Theorem 2.2.1(1) throws light on the limitation of the prevailing method of studying well-posedness in modulation spaces $M^{p,1}_s(\mathbb{R}^d)$ using the algebraic property available in these spaces. Our result (Theorem 2.2.1(1)) shows that this approach using the algebraic property or even the general mapping property of the nonlinearity of the modulation space to itself, can handle only the so-called real analytic nonlinearities on $M^{p,1}(\mathbb{R}^d)$. In particular, the nonlinearities of interest in applications, namely the power type $F(u) = |u|^\alpha u$ for $\alpha \notin 2\mathbb{N}$, and also the exponential type $F(u) = e^{u|u|} - 1$ are ruled out in this approach. This leads to the fact that to deal with local existence for nonlinear Schrödinger equation and other dispersive equations with power type nonlinearity $|u|^\alpha u$ when α is not an even integer, requires some new approach. We would also like to point out that our Theorem 2.3.3, naturally, raise the interesting open question(see Section 4.5 below).

4.4 The Cauchy Problem for the Hartree Type Equation

4.4.1 Motivation

Inspired from the work of Chadam-Glassey [8] in 1980s Ginibre-Velo [25] have studied the Schrödinger equation with cubic convolution nonlinearity due to both their strong physical background and theoretical importance. This kind of nonlinearity appears in quantum theory of boson stars, atomic and nuclear physics, describing superfluids, etc.. This model is known as the Hartree type equation:

$$iu_t + \Delta u = (K * |u|^2)u, \quad u(x, t_0) = u_0(x); \tag{4.34}$$

where $u(x,t)$ is a complex valued function on $\mathbb{R}^d \times \mathbb{R}$, Δ is the Laplacian on $\mathbb{R}^d$, u_0 is a complex valued function on $\mathbb{R}^d$, K is some suitable potential (function) on $\mathbb{R}^d$, time $t_0 \in \mathbb{R}$, and $*$ denotes the convolution in $\mathbb{R}^d$.

In subsequent years the local and global well-posedness, regularity, and scattering theory for Eq. (4.34) have attracted a lot of attention by many mathematicians. Almost exclusively, the techniques developed so far restrict to Cauchy problems with initial data in Sobolev spaces, mainly because of the crucial role played by the Fourier transform in the analysis of partial differential operators. See [9, 25, 10].

We note that over the past ten years there has been increasing interest for many mathematicians to consider Cauchy data in modulation spaces $M^{p,q}(\mathbb{R}^d)$ (Definition 1.3.1) for nonlinear dispersive equations because these spaces are rougher than any given one in a fractional Bessel potential space and this low-regularity is desirable in many situations. For instance, we mention, the local well-posedness result of Schrödinger equation, especially, with power type non-linearity $F(u) = |u|^{2k}u$ ($k \in \mathbb{N}$) are obtained in [66, 2] with Cauchy data from $M^{p,1}(\mathbb{R}^d)$ and a global existence result in [68, 32] with small initial data from $M^{p,1}(\mathbb{R}^d)$ $(1 \leq p \leq 2)$. However, the global well-posedness result for the large initial data (without any restriction to initial data) in modulation space is still unknown, see the open question in [49, p.280], because one of the main obstacle is a lack of useful conservation laws in modulation spaces by which one can guarantee the global existence result.

Taking these considerations into our account, in this chapter, we will investigate Hartree type equation (4.34) with potentials of the following types:

$$K(x) = \frac{\lambda}{|x|^{\gamma}}, (\lambda \in \mathbb{R}, \gamma > 0, x \in \mathbb{R}^d), \tag{4.35}$$

$$K \in \mathcal{F}L^1(\mathbb{R}^d), \tag{4.36}$$

$$K \in M^{1,\infty}(\mathbb{R}^d). \tag{4.37}$$

The homogeneous kernel of the form (4.35) is known as Hartree potential. Now we note that the solutions to (4.34) enjoy (for instance see Proposition 4.4.4 below) the mass conservation law,

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2} \quad (t \in \mathbb{R}),$$

and exploiting this mass conservation law and techniques from time-frequency analysis we prove global existence result (Theorem 5.1.1 below) for Eq. (4.34) in the space $M^{1,1}(\mathbb{R}^d)$ for K of the form (4.35); the proof relies on some suitable decomposition of Fourier transform of Hartree potential into Lebesgue spaces (Eq. (4.39) below). We prove global existence result (Theorem 4.4.8) in the space $M^{p,q}(\mathbb{R}^d)$ when potential $K \in \mathcal{F}L^1(\mathbb{R}^d)$ (Definition (4.52) below) and local existence (Theorem 4.4.10 below) via uniform estimates for the Schrödinger propagator in modulation spaces $M^{p,q}(\mathbb{R}^d)$ and algebraic properties of the space $M^{p,q}(\mathbb{R}^d)$.

4.4.2 Global Well-Posedness in $M^{1,1}$ for the Hartree Potential

In this section, we prove global existence result (Theorem 5.1.1) for (4.34) with the Hartree potential (4.35).

Theorem 4.4.1 *Assume that $u_0 \in M^{1,1}(\mathbb{R}^d)$ and let K be given by (4.35) with $\lambda \in \mathbb{R}$, and $0 < \gamma < min\{2, d/2\}, d \in \mathbb{N}$. Then there exists a unique global solution of (4.34) such that $u \in C(\mathbb{R}, M^{1,1}(\mathbb{R}^d))$.*

We recall the Fourier transform of Hartree potential:

Proposition 4.4.2 *Let $d \geq 1$ and $0 < \gamma < d$. There exists $C = C(d, \gamma)$ such that the Fourier transform of K defined by (4.35) is*

$$\widehat{K}(\xi) = \frac{\lambda C}{|\xi|^{d-\gamma}}. \tag{4.38}$$

Proof. See [3, Proposition 1.29, p.23].

We start with decomposing Fourier transform of Hartree potential into Lebesgue spaces: indeed, in view of Proposition 4.4.2, we have

$$\widehat{K} = k_1 + k_2 \in L^p(\mathbb{R}^d) + L^q(\mathbb{R}^d), \tag{4.39}$$

where $k_1 := \chi_{\{|\xi|\leq 1\}}\widehat{K} \in L^p(\mathbb{R}^d)$ for all $p \in [1, \frac{d}{d-\gamma})$ and $k_2 := \chi_{\{|\xi|>1\}}\widehat{K} \in L^q(\mathbb{R}^d)$ for all $q \in (\frac{d}{d-\gamma}, \infty]$.

Definition 4.4.3 A pair $(p, q) \neq (2, \infty)$ is called a admissible if $p \geq 2, q \geq 2,$

and
$$\frac{2}{p} = d\left(\frac{1}{2} - \frac{1}{q}\right).$$

The next proposition establishes the global well-posedness for (4.34) in $L^2(\mathbb{R}^d)$. For a proof, see [9, Proposition 2.3].

Proposition 4.4.4 ([9]) *Let $d \geq 1$, and K be given by (4.35) with $\lambda \in \mathbb{R}$ and $0 < \gamma < min\{2, d\}$. If $u_0 \in L^2(\mathbb{R}^d)$, then (4.34) has a unique global solution*
$$u \in C(\mathbb{R}, L^2(\mathbb{R}^d)) \cap L^{8/\gamma}_{loc}(\mathbb{R}, L^{4d/(2d-\gamma)}(\mathbb{R}^d)).$$
In addition, its L^2-norm is conserved,
$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \ \forall t \in \mathbb{R},$$
and for all admissible pairs $(p, q), u \in L^p_{loc}(\mathbb{R}, L^q(\mathbb{R}^d))$.

Lemma 4.4.5 (Gronwall inequality, integral form) *Let $A : [t_0, t_1] \to [0, \infty)$ be continuous and non-negative, and suppose that A obeys the integral inequality*
$$A(t) \leq C + \int_{t_0}^{t_1} B(s)A(s)ds, \ \forall t \in [t_0, t_1],$$
where $C \geq 0$ and $B : [t_0, t_1] \to [0, \infty)$ is continuous and nonnegative. Then we have
$$A(t) \leq C \exp\left(\int_{t_0}^{t} B(s)ds\right) \forall t \in [t_0, t].$$

Lemma 4.4.6 *Let $0 < \gamma < d$. For any $f, g \in M^{1,1}(\mathbb{R}^d)$, we have*
$$\|(K*|f|^2)f-(K*|g|^2)g\|_{M^{1,1}} \lesssim (\|f\|^2_{M^{1,1}}+\|f\|_{M^{1,1}}\|g\|_{M^{1,1}}+\|g\|^2_{M^{1,1}})\|f-g\|_{M^{1,1}}.$$

Proof. By Proposition1.4.3(3), (4.39), Hölder's inequality, Lemma 1.4.1(3), Lemma 1.4.1(8), and Lemma 1.4.1(7), we obtain
$$\begin{aligned}
\|(K * |f|^2)(f - g)\|_{M^{1,1}} &\lesssim \|K * |f|^2\|_{\mathcal{F}L^1}\|f - g\|_{M^{1,1}} \\
&\lesssim \left(\|k_1\widehat{|f|^2}\|_{L^1} + \|k_2\widehat{|f|^2}\|_{L^1}\right)\|f - g\|_{M^{1,1}} \\
&\lesssim \left(\|k_1\|_{L^1}\|\widehat{|f|^2}\|_{L^\infty} + \|k_2\|_{L^\infty}\|\widehat{|f|^2}\|_{L^1}\right)\|f - g\|_{M^{1,1}} \\
&\lesssim \left(\||f|^2\|_{L^1} + \|\widehat{|f|^2}\|_{L^1}\right)\|f - g\|_{M^{1,1}}
\end{aligned}$$

$$\lesssim \|f\|^2_{M^{1,1}} \|f-g\|_{M^{1,1}} \tag{4.40}$$

and,

$$\begin{aligned}
\|(K*(|f|^2-|g|^2))g\|_{M^{1,1}} &\lesssim \|K*(|f|^2-|g|^2)\|_{\mathcal{F}L^1}\|g\|_{M^{1,1}} \\
&\lesssim \left(\||f|^2-|g|^2\|_{L^1} + \|\widehat{|f|^2-|g|^2}\|_{L^1}\right)\|g\|_{M^{1,1}} \\
&\lesssim \||f|^2-|g|^2\|_{M^{1,1}}\|g\|_{M^{1,1}} \\
&\lesssim \left(\|f\|_{M^{1,1}}+\|g\|_{M^{1,1}}\right)\|f-g\|_{M^{1,1}}\|g\|_{M^{1,1}}. \qquad (4.41)
\end{aligned}$$

Now taking the identity

$$(K*|f|^2)f-(K*|g|^2)g = (K*|f|^2)(f-g)+(K*(|f|^2-|g|^2))g$$

into our account, (4.40) and (4.41) gives the desired result.

Lemma 4.4.7 *Let K be given by (4.35) with $\lambda \in \mathbb{R}$, and $0<\gamma<d$. Then for any $f \in M^{1,1}(\mathbb{R}^d)$, we have,*

$$\|(K*|f|^2)f\|_{M^{1,1}} \lesssim \|f\|^3_{M^{1,1}}. \tag{4.42}$$

Proof. By Proposition 1.4.3(3), (4.39), Hölder's inequality, Lemma 1.4.1(3), Lemma 1.4.1(8), and Lemma 1.4.1(7), we obtain

$$\begin{aligned}
\|(K*|f|^2)f)\|_{M^{1,1}} &\lesssim \|K*|f|^2\|_{\mathcal{F}L^1}\|f\|_{M^{1,1}} \\
&\lesssim \left(\|k_1\widehat{|f|^2}\|_{L^1}+\|k_2\widehat{|f|^2}\|_{L^1}\right)\|f\|_{M^{1,1}} \\
&\lesssim \left(\|k_1\|_{L^1}\|\widehat{|f|^2}\|_{L^\infty}++\|k_2\|_{L^\infty}\|\widehat{|f|^2}\|_{L^1}\right)\|f\|_{M^{1,1}} \\
&\lesssim \left(\||f|^2\|_{L^1}+\|\widehat{|f|^2}\|_{M^{1,1}}\right)\|f\|_{M^{1,1}} \\
&\lesssim \||f|^2\|_{M^{1,1}}\|f\|_{M^{1,1}} \\
&\lesssim \|f\|^3_{M^{1,1}}. \qquad (4.43)
\end{aligned}$$

□

Proof of Theorem 5.1.1. By Duhamel's formula, we note that (4.34) can be written in the equivalent form

$$u(\cdot,t) = S(t-t_0)u_0 - i\mathcal{A}F(u) \tag{4.44}$$

where S and $\mathcal{A}$ are as in (4.18).

For simplicity, we assume that $t_0 = 0$ and prove the local existence on $[0,T]$. Similar arguments also apply to interval of the form $[-T',0]$ for proving local solutions.

We consider now the mapping

$$\mathcal{J}(u) = S(t)u_0 - i\int_0^t S(t-\tau)\left[(K*|u|^2(\tau))u(\tau)\right]d\tau. \tag{4.45}$$

By Proposition 1.5.1,

$$\|S(t)u_0\|_{M^{1,1}} \le C(1+t^2)^{d/4}\|u_0\|_{M^{1,1}} \tag{4.46}$$

for $t \in \mathbb{R}$, and where C is a universal constant depending only on d.
By Minkowski's inequality for integrals, Proposition 1.5.1, and Lemma 4.4.7, we obtain

$$\begin{aligned}\left\|\int_0^t S(t-\tau)[(K*|u|^2(\tau))u(\tau)]\,d\tau\right\|_{M^{1,1}} &\le \int_0^t \left\|S(t-\tau)[(K*|u|^2(\tau))u(\tau)]\right\|_{M^{1,1}} d\tau\\ &\le TC_T\,\|[(K*|u|^2(t))u(t)]\|_{M^{1,1}}\\ &\le TC_T\|u(t)\|^3_{M^{1,1}}\end{aligned} \tag{4.47}$$

where $C_T = C(1+t^2)^{d/4}$.
By (4.46) and (4.47), we have

$$\|\mathcal{J}u\|_{C([0,T],M^{1,1})} \le C_T\left(\|u_0\|_{M^{1,1}} + cT\|u\|^3_{M^{1,1}}\right), \tag{4.48}$$

for some universal constant c.

For $M > 0$, put $B_{T,M} = \{u \in C([0,T],M^{1,1}(\mathbb{R}^d)) : \|u\|_{C([0,T],M^{1,1})} \le M\}$, which is the closed ball of radius M and centered at the origin in $C([0,T],M^{1,1}(\mathbb{R}^d))$. Next, we show that the mapping $\mathcal{J}$ takes $B_{T,M}$ into itself for suitable choice of M and small $T > 0$. Indeed, if we let, $M = 2C_T\|u_0\|_{M^{1,1}}$ and $u \in B_{T,M}$, from

(4.48) we obtain

$$\|\mathcal{J}u\|_{C([0,T],M^{1,1})} \leq \frac{M}{2} + cC_T T M^3. \tag{4.49}$$

We choose a T such that $cC_T TM^2 \leq 1/2$, that is, $T \leq \tilde{T}(\|u_0\|_{M^{1,1}}, d, \gamma)$ and as a consequence we have

$$\|\mathcal{J}u\|_{C([0,T],M^{1,1})} \leq \frac{M}{2} + \frac{M}{2} = M, \tag{4.50}$$

that is, $\mathcal{J}u \in B_{T,M}$. By Lemma 4.4.6, and the arguments as before, we obtain

$$\|\mathcal{J}u - \mathcal{J}v\|_{C([0,T],M^{1,1})} \leq \frac{1}{2}\|u - v\|_{C([0,T],M^{1,1})}. \tag{4.51}$$

Therefore, using the Banach's contraction mapping principle, we conclude that $\mathcal{J}$ has a fixed point in $B_{T,M}$ which is a solution of (4.44).

Now we shall see that the solution constructed before is global in time. In fact, in view of Proposition 4.4.4, to prove Theorem 5.1.1, it suffices to prove that the modulation space norm of u, that is, $\|u\|_{M^{1,1}}$ cannot become unbounded in finite time.

In view of (4.39) and to use the Hausdorff-Young inequality we let $1 < \frac{d}{d-\gamma} < q \leq 2$, and we obtain

$$\begin{aligned}
&\|u(t)\|_{M^{1,1}}\\
\lesssim &C_T \left(\|u_0\|_{M^{1,1}} + \int_0^t \|(K * |u(\tau)|^2)u(\tau)\|_{M^{1,1}} d\tau \right)\\
\lesssim &C_T \left(\|u_0\|_{M^{1,1}} + \int_0^t \|K * |u(\tau)|^2\|_{\mathcal{F}L^1} \|u(\tau)\|_{M^{1,1}} d\tau \right)\\
\lesssim &C_T \|u_0\|_{M^{1,1}} + C_T \int_0^t \left(\|k_1\|_{L^1} \|u(\tau)\|_{L^2}^2 + \|k_2\|_{L^q} \|\widehat{|u(\tau)|^2}\|_{L^{q'}} \right) \|u(\tau)\|_{M^{1,1}} d\tau\\
\lesssim &C_T \|u_0\|_{M^{1,1}} + C_T \int_0^t \left(\|k_1\|_{L^1} \|u_0\|_{L^2}^2 + \|k_2\|_{L^q} \||u(\tau)|^2\|_{L^q} \right) \|u(\tau)\|_{M^{1,1}} d\tau\\
\lesssim &C_T \|u_0\|_{M^{1,1}} + C_T \int_0^t \|u(\tau)\|_{M^{1,1}} d\tau + C_T \int_0^t \|u(\tau)\|_{L^{2q}}^2 \|u(\tau)\|_{M^{1,1}} d\tau.
\end{aligned}$$

where we have used Proposition 1.4.3(3), Hölder's inequality, and the conservation of the L^2-norm of u.

We note that the requirement on q can be fulfilled if and only if $0 < \gamma < d/2$.

To apply Proposition 4.4.4, we let $\alpha > 1$ and $(2\alpha, 2q)$ is admissible. This is possible provided that $2q < \frac{2d}{d-2}$ when $d \geq 3$: this condition is compatible with the requirement $q > d/(d-\gamma)$ if and only if $\gamma < 2$. Using the Hölder's inequality for the last integral, we obtain

$$\|u(t)\|_{M^{1,1}} \lesssim C_T \|u_0\|_{M^{1,1}} + C_T \int_0^t \|u(\tau)\|_{M^{1,1}} d\tau + C_T \|u\|^2_{L^{2\alpha}([0,T],L^{2q})} \|u\|_{L^{\alpha'}[0,T],M^{1,1})},$$

where α' is the Hölder conjugate exponent of α. Put,

$$h(t) := \sup_{0 \leq \tau \leq t} \|u(\tau)\|_{M^{1,1}}.$$

For a given $T > 0$, h satisfies an estimate of the form,

$$h(t) \lesssim C_T \|u_0\|_{M^{1,1}} + C_T \int_0^t h(\tau) d\tau + C_T C_0(T) \left(\int_0^t h(\tau)^{\alpha'} d\tau \right)^{\frac{1}{\alpha'}},$$

provided that $0 \leq t \leq T$, and where we have used the fact that α' is finite. Using the Hölder's inequality we infer that,

$$h(t) \lesssim C_T \|u_0\|_{M^{1,1}} + C_1(T) \left(\int_0^t h(\tau)^{\alpha'} d\tau \right)^{\frac{1}{\alpha'}}.$$

Raising the above estimate to the power α', we find that

$$h(t)^{\alpha'} \lesssim C_2(T) \left(1 + \int_0^t h(\tau)^{\alpha'} d\tau \right).$$

In view of Gronwall inequality as in Lemma 4.4.5, one may conclude that $h \in L^\infty([0,T])$. Since $T > 0$ is arbitrary, $h \in L^\infty_{loc}(\mathbb{R})$, and the proof of Theorem 5.1.1 follows. □

4.4.3 Global Well-Posedness in $M^{p,q}$ for Potential in $\mathcal{F}L^1$

In this section, we will prove global existence result(Theorem 4.4.8) for (4.34) with the potential in Fourier algebra $\mathcal{F}L^1(\mathbb{R}^d)$.

Theorem 4.4.8 *Let $K \in \mathcal{F}L^1(\mathbb{R}^d), d \in \mathbb{N}$. Then, for any $u_0 \in M^{p,q}(\mathbb{R}^d)$ $(1 \leq q \leq min\{p, p'\}$ where p' is the Hölder conjugate of $p \in [1,2])$, there exists a unique global solution $u(t)$ of (4.34) such that $u(t) \in C(\mathbb{R}, M^{p,q}(\mathbb{R}^d))$.*

We denote by $\mathcal{F}L^1(\mathbb{R}^d)$ the space of all Fourier transforms of $L^1(\mathbb{R}^d)$, that is,

$$\mathcal{F}L^1(\mathbb{R}^d) = \{f \in L^\infty : \hat{f} \in L^1(\mathbb{R}^d)\}. \tag{4.52}$$

The space $\mathcal{F}L^1(\mathbb{R}^d)$ is a Banach algebra under pointwise addition and multiplication, with respect to the norm:

$$\|f\|_{\mathcal{F}L^1} := \|\hat{f}\|_{L^1} \ (f \in \mathcal{F}L^1(\mathbb{R}^d)),$$

and we call $\mathcal{F}L^1(\mathbb{R}^d)$ the Fourier algebra.

Lemma 4.4.9 *Let $K \in \mathcal{F}L^1(\mathbb{R}^d)$. For any $f, g \in M^{p,q}(\mathbb{R}^d)$ $(p \in [1,2],\ 1 \le q \le min\{p, p'\},\ \frac{1}{p} + \frac{1}{p'} = 1)$, we have*

$$\|(K * |f|^2)f - (K * |g|^2)g\|_{M^{p,q}} \lesssim \|K\|_{\mathcal{F}L^1} \left(\|f\|^2_{M^{p,1}} + \|f+g\|_{M^{p,q}}\right) \|f-g\|_{M^{p,q}}.$$

Proof. By Proposition 1.4.3(3), Hölder inequality, Hausdroff-Young inequality, Lemma 1.4.1(3), and in view of identity

$$2(|f|^2 - |g|^2) = (f-g)(\bar{f} + \bar{g}) + (\bar{f} - \bar{g})(f+g),$$

we obtain

$$\begin{aligned} \|(K * |f|^2)(f-g)\|_{M^{p,q}} &\lesssim \|K * |f|^2\|_{\mathcal{F}L^1} \|f-g\|_{M^{p,q}} \\ &\lesssim \|K\|_{\mathcal{F}L^1} \|\widehat{|f|^2}\|_{L^\infty} \|f-g\|_{M^{p,q}} \\ &\lesssim \|K\|_{\mathcal{F}L^1} \|f\|^2_{L^2} \|f-g\|_{M^{p,q}} \\ &\lesssim \|K\|_{\mathcal{F}L^1} \|f\|^2_{M^{p,q}} \|f-g\|_{M^{p,q}}, \end{aligned} \tag{4.53}$$

and

$$\begin{aligned} &\|(K * (|f|^2 - |g|^2))g\|_{M^{p,q}} \\ &\lesssim \|K * (|f|^2 - |g|^2)\|_{\mathcal{F}L^1} \|g\|_{M^{p,q}} \\ &\lesssim \|K\|_{\mathcal{F}L^1} \||f|^2 - |g|^2\|_{L^1} \|g\|_{M^{p,q}} \\ &\lesssim \|K\|_{\mathcal{F}L^1} \left(\|f+g\|_{M^{p,q}}\right) \|f-g\|_{M^{p,q}} \|g\|_{M^{p,q}}. \end{aligned} \tag{4.54}$$

Taking the identity

$$(K * |f|^2)f - (K * |g|^2)g = (K * |f|^2)(f - g) + (K * (|f|^2 - |g|^2))g$$

into our account and combining (4.53) and (4.54), gives the desired inequality. □

Proof of Theorem 4.4.8. We recall that, by Duhamel's formula, (4.34) can be written in the equivalent form

$$u(\cdot, t) = S(t - t_0)u_0 - i\mathcal{A}F(u) \tag{4.55}$$

where

$$S(t) = e^{it\Delta}, \ (\mathcal{A}v)(t, x) = \int_{t_0}^{t} S(t - \tau)\, v(t, x)\, d\tau. \tag{4.56}$$

For simplicity, we assume that $t_0 = 0$ and prove the local existence on $[0, T]$.

We consider now the mapping

$$\mathcal{J}(u) = S(t)u_0 - i \int_0^t S(t - \tau)\, [(K * |u|^2(\tau))u(\tau)]\, d\tau. \tag{4.57}$$

By (1.5.1) and Minkowski's inequality for integrals, we obtain

$$\|\mathcal{J}u\|_{C([0,T],M^{p,q})} \leq C_T \left(\|u_0\|_{M^{p,q}} + cT\|u\|^3_{M^{p,q}}\right), \tag{4.58}$$

where $C_T = C(1 + t^2)^{d/4}$ and c is some universal constant.

For $M > 0$, put $E_{T,M} = \{u \in C([0, T], M^{p,q}(\mathbb{R}^d)) : \|u\|_{C([0,T],M^{p,q})} \leq M\}$, which is the closed ball of radius M, and centered at the origin in $C([0, T], M^{p,q}(\mathbb{R}^d))$.

Next, we show that the mapping $\mathcal{J}$ takes $E_{T,M}$ into itself for suitable choice of M and small $T > 0$. Indeed, if we let, $M = 2C_T\|u_0\|_{M^{p,q}}$ and $u \in E_{T,M}$, from (4.58) we obtain

$$\|\mathcal{J}u\|_{C([0,T],M^{p,q})} \leq \frac{M}{2} + cC_T T M^3. \tag{4.59}$$

We choose a T such that $cC_T TM^2 \leq 1/2$, that is, $T \leq \tilde{T}(\|u_0\|_{M^{p,q}})$ and as a consequence, we obtain

$$\|\mathcal{J}u\|_{C([0,T],M^{p,q})} \leq \frac{M}{2} + \frac{M}{2} = M, \tag{4.60}$$

that is, $\mathcal{J}u \in E_{T,M}$. By Lemma 4.4.9, and the arguments as before, we obtain

$$\|\mathcal{J}u - \mathcal{J}v\|_{C([0,T],M^{p,q})} \leq \frac{1}{2}\|u - v\|_{C([0,T],M^{p,q})}. \tag{4.61}$$

Therefore, using Banach's contraction mapping principle, we conclude that $\mathcal{J}$ has a fixed point in $B_{T,M}$ which is a solution of (4.55).

Indeed, the solution constructed before is global in time: in view of the conservation of L^2 norm, we have

$$\begin{aligned}
\|u((t)\|_{M^{p,q}} &\lesssim C_T \left(\|u_0\|_{M^{p,q}} + \int_0^t \|K * |u(\tau)|^2\|_{\mathcal{F}L^1} \|u(\tau)\|_{M^{p,q}} d\tau \right) \\
&\lesssim C_T \left(\|u_0\|_{M^{p,q}} + \int_0^t \|K\|_{\mathcal{F}L^1} \||u(t)|^2\|_{L^1} \|u(\tau)\|_{M^{p,q}} d\tau \right) \\
&\lesssim C_T \left(\|u_0\|_{M^{p,q}} + \|K\|_{\mathcal{F}L^1} \|u_0\|_{L^2}^2 \int_0^t \|u(\tau)\|_{M^{p,q}} d\tau \right),
\end{aligned} \tag{4.62}$$

and by Gronwall inequality, we conclude that $\|u(t)\|_{M^{p,q}}$ remains bounded on finite time intervals. This completes the proof. □

4.4.4 Local Well-Posedness in $M^{p,1}$ for Potential in $M^{1,\infty}$

In this section, we prove local existence result (Theorem 4.4.10) for (4.34) with the potential in modulation space $M^{1,\infty}(\mathbb{R}^d)$.

Theorem 4.4.10 *Assume that $u_0 \in M^{p,1}(\mathbb{R}^d)$ $(1 \leq p \leq \infty)$, and $K \in M^{1,\infty}(\mathbb{R}^d)$, $d \in \mathbb{N}$. Then, there exist $T^* = T^*(\|u_0\|_{M^{p,1}}) > t_0$ and $T_* = T_*(\|u_0\|_{M^{p,1}}) < t_0$ such that (4.34) has a unique solution $u \in C([T_*, T^*], M^{p,1}(\mathbb{R}^d))$.*

Lemma 4.4.11 *Let $K \in M^{1,\infty}(\mathbb{R}^d)$. For any $f, g \in M^{p,1}(\mathbb{R}^d)$ $(1 \leq p \leq \infty)$, we have,*

$$\begin{aligned}
&\|(K * |f|^2)f - (K * |g|^2)g\|_{M^{p,1}} \\
&\lesssim \left((\|f\|_{M^{p,1}}^2 + \|f\|_{M^{p,1}}\|g\|_{M^{p,1}} + \|g\|_{M^{p,1}}^2) \right) \|f - g\|_{M^{p,1}}.
\end{aligned}$$

Proof. By Proposition 1.4.3(3), Proposition 1.4.3(1), we obtain

$$\begin{aligned}
\|(K * |f|^2)(f - g)\|_{M^{p,1}} &\lesssim \|K * |f|^2\|_{M^{\infty,1}} \|f - g\|_{M^{p,1}} \\
&\lesssim \|K\|_{M^{1,\infty}} \||f|^2\|_{M^{\infty,1}} \|f - g\|_{M^{p,1}}
\end{aligned}$$

$$\lesssim \|K\|_{M^{1,\infty}}\|f\|^2_{M^{p,1}}\|f-g\|_{M^{p,1}}, \tag{4.63}$$

and

$$\begin{aligned}
&\|(K*(|f|^2-|g|^2))g\|_{M^{p,1}}\\
\lesssim&\|K*(|f|^2-|g|^2)\|_{M^{p,1}}\|g\|_{M^{p,1}}\\
\lesssim&\|K\|_{M^{1,\infty}}\||f|^2-|g|^2\|_{M^{p,1}}\|g\|_{M^{p,1}}\\
\lesssim&\|K\|_{M^{1,\infty}}\left(\|f\|_{M^{p,1}}+\|g\|_{M^{p,1}}\right)\|f-g\|_{M^{p,1}}\|g\|_{M^{p,1}}.
\end{aligned} \tag{4.64}$$

Taking the identity

$$(K*|f|^2)f-(K*|g|^2)g=(K*|f|^2)(f-g)+(K*(|f|^2-|g|^2))g$$

into our account and combining (4.63) and (4.64), gives the desired inequality.

Proof of Theorem 4.4.10. Taking Lemma 4.4.11 into our account and exploiting the method from previous results the proof follows. □

4.5 Concluding Remarks

1. The analogue of Theorem 4.4.10 holds for the general nonlinearity $(K*|u|^{2k})u, k\in\mathbb{N}$, that is, for the Schrödinger equation with the nonlinearity $(K*|u|^{2k})u, k\in\mathbb{N}$.

2. In Section 4.2, we have shown the local well-posedness results for real entire nonlinearities on $M^{p,1}(\mathbb{R}^d)$ for $1\leq p\leq\infty$. Since any real analytic function vanishing at origin, maps $M^{1,1}(\mathbb{R}^d)$ to itself, by Theorem 2.3.3. In view of this, it would be interesting to whether the local well-posedness can be proved in $M^{1,1}(\mathbb{R}^d)$, for real analytic nonlinearities.

Chapter 5

On Twisted Convolution and Modulation Spaces

The main purpose of this chapter is to initiate the study of factorization problems with respect to twisted convolution in the realm of modulation, Wiener amalgam and Lebesgue spaces.

In Section 5.1, we briefly recall well established factorization results and its importance in applications; and then we mention our motivation to study factorization problems for twisted convolution and finally we state our main results. Sections 5.2-5.5 are devoted to the proof of our main results and in Section 5.6, we discuss future problems in this direction.

5.1 Introduction

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the circle group and we define the convolution of functions $f, h \in L^1(\mathbb{T})$ on the circle group by

$$(f * h)(x) = \int_{\mathbb{T}} f(y)h(y^{-1}x)dy.$$

Similarly, this definition can be generalized to any locally compact group, and the convolution operation pervades throughout analysis and indispensable in many situations. Under this operation, Lebesgue space $L^1(\mathbb{T})$ forms a complex Banach algebra, however, $L^1(\mathbb{T})$ possesses no identity element relative to convolution. A question with an algebraic flavor arise quite naturally at this point: whether every $f \in L^1(\mathbb{T})$ can be factored into a convolution product $g * h$ with $g, h \in$

$L^1(\mathbb{T})$?

In 1939 using the classical techniques of Fourier series Salem [57] has asserted that $L^1(\mathbb{T})$ factor, that is, $L^1(\mathbb{T}) * L^1(\mathbb{T}) = L^1(\mathbb{T})$. Using a Euclidean Fourier transform and particular functions on $\mathbb{R}$, Walter Rudin, in 1958 [45] proved $L^1(\mathbb{R}) * L^1(\mathbb{R}) = L^1(\mathbb{R})$. In 1959, he [46] also proved $L^1(G) = L^1(G) * L^1(G)$, where G is any locally Euclidean abelian group. Subsequently Paul Cohen [11] observed in 1959 that the essential ingredient in Rudin's argument was the presence of a bounded approximate identity in the algebra $L^1(G)$, and took the most significant step towards the factorization property by asserting that any Banach algebra with bounded left approximate identity factor. In particular, $L^1(G) = L^1(G) * L^1(G)$ for any locally compact group. On the other hand, $L^p(G)$ fail to factor for $p > 1$ if G is infinite; this has been established for compact groups in [35, 34.40], and for non-compact groups in [56]. However, the Cohen's result has been generalized by Edwin Hewitt [34] to the Banach modules over Banach algebras with a bounded left approximate identity, and in particular $L^p(G) = L^1(G) * L^p(G)$ $(1 \leq p < \infty)$ for any locally compact group G.

These factorization results have found immense application in harmonic analysis, for instance: using the Cohen's factorization theorem, Varopoulos [55] has ensured that every positive linear functional on a Banach *-algebra with a bounded approximate identity is continuous; and Green [26] has showed that every maximal left ideal in a Banach algebra with a bounded right approximate identity is closed and so on.

There is an extensive and interesting history for factorization and non-factorization results and its impact on other parts of harmonic analysis, and we cannot hope to acknowledge here all those who made the theory of factorization such a success story; instead, we refer the interested reader to the excellent survey articles(historical development point of view) [44, 39] and monographs [16, 71] and the references therein.

The main purpose of this chapter is to investigate these factorization results with respect to twisted convolution (relatively less familiar operation, and in comparison to the usual convolution here the difference consisting in the exponential modulating factor under the integral—hence the name twisted convolution, see Definition 5.2.3) in the realm of modulation, Wiener amalgam and Lebesgue spaces, and to illustrate its applications, for various reasons, which we shall now describe briefly. Twisted convolution appeared during the work of von-Newman while characterizing (see [23]) irreducible representation of Heisenberg group, in fact, its definition is set up so that Weyl transform ρ takes twisted convolution into composition in the sense that $\rho(f \natural h) = \rho(f)\rho(h)$, see [23, p.26], [61, 70] for detail. The twisted convolution provides an excellent substitute and having interesting properties that are known to fail for the ordinary convolution, for instance, $L^p(\mathbb{R}^{2d})$ $(1 < p \leq 2)$ is a Banach algebra under twisted convolution (see Proposition 5.2.5 (4) below), on the other hand, it fails to be a Banach algebra under ordinary convolution [56].

In the early eighties around 1980-1983, Feichtinger [18, 21] introduced modulation and Wiener amalgam spaces, both are closely related, and the idea of these spaces is to consider decaying property of a function with respect to the space variable and the variable of its Fourier transform simultaneously, see Section 1.3; to handle some problems in time-frequency analysis, see [28].

Since the early nineties, these spaces have been used in the analysis of pseudo-differential operators (See [58, 29, 59, 14, 28]) and twisted convolution has played a vital role in the background. In fact, twisted convolution is intimately connected to the psuedo-differential calculus, in the sense that the Fourier transform of a Weyl product(twisted product) is essentially a twisted convolution, see [28, p.325] and also [14]. To see the further connection between twisted convolution and pseudo-differential calculus, and its importance, we refer the interested reader to see [62, 63].

Thus, these spaces have found their way into different areas of mathematical analysis and applications, and nowadays present in investigations that concern problems on Fourier multipliers, pseudo differential operators, Fourier integral operators, Strichartz estimates, nonlinear PDEs, etc. (cf. [14, 4, 59, 58, 15, 5, 28, 69]). In short, the time-frequency analysis, pseudo differential operators, and twisted convolution are intimately related. See [28, 23].

Keeping all these considerations into our account, we are inspired to investigate the factorization problems in these spaces. We prove that the spaces $L^p(\mathbb{R}^{2d})$, $M^{p,q}(\mathbb{R}^{2d})$ and $W^{p,q}(\mathbb{R}^d)$ factor over $L^1(\mathbb{R}^{2d})$ with respect to twisted convolution $\natural$ (see Theorem 5.1.1 (1)). We prove Theorem 5.1.1 by constructing a bounded approximate identity in $L^1(\mathbb{R}^{2d})$ with respect to $\natural$. In fact, this identity is also a left approximate identity in $M^{p,q}(\mathbb{R}^{2d}), W^{p,q}(\mathbb{R}^{2d})$, and $L^p(\mathbb{R}^{2d})$ $(1 \leq p, q < \infty)$ with respect to $\natural$. We also asserts that we cannot hope to factor $M^{1,1}(\mathbb{R}^{2d})$ over $M^{1,1}(\mathbb{R}^{2d})$ with respect to twisted convolution (see Remark 5.1.2 below), and we prove that a Banach algebra $(L^2(\mathbb{R}^{2d}), \natural)$ fail to factor (see Theorem 5.1.1 (2) below).

Coming to the ordinary convolution, we show that $M^{p,q}(\mathbb{R}^d)$ and $W^{p,q}(\mathbb{R}^d)$ can be factored over $L^1(\mathbb{R}^d)$ with respect to convolution (see Theorem 5.1.3 (2)), and recapture the fact that $M^{p,1}(\mathbb{R}^d) \subset C(\mathbb{R}^d)$ (see Corollary 5.3.9 below). We also show that $M^{p,1}(\mathbb{T}^d)$ fail to factor with respect to convolution.

As a consequence of these and exploiting well established results, we derive some interesting applications (see Theorem 5.1.4 and Remark 5.1.5 below) in other parts of harmonic analysis.

We state our main results.

Theorem 5.1.1 *Let $1 \leq p, q < \infty$ and E denote any one of $L^p(\mathbb{R}^{2d})$ or $M^{p,q}(\mathbb{R}^{2d})$ or $W^{p,q}(\mathbb{R}^{2d})$.*

1. *For any $f \in E$ and $\epsilon > 0$ there exists $g \in L^1(\mathbb{R}^d)$ and $h \in E$ with the following properties:*

   (a) $f = g \natural h$,

   (b) $\|f - h\|_E \leq \epsilon$.

   In particular, $E = L^1(\mathbb{R}^{2d}) \natural E$.

2. *$M^{2,2}(\mathbb{R}^{2d}) \natural M^{2,2}(\mathbb{R}^{2d}) \subsetneq M^{2,2}(\mathbb{R}^{2d})$.*

Remark 5.1.2 In Theorem 5.1.1 (1) when $E = M^{1,1}(\mathbb{R}^{2d})$ it is impossible to replace $L^1(\mathbb{R}^{2d})$ by $M^{1,1}(\mathbb{R}^{2d})$. See Proposition 5.4.3 below.

Theorem 5.1.3 *Let $1 \leq p, q < \infty$, and $E^2 = E * E$.*

1. *$E^2 \subsetneq E$, where $E = M^{p,1}(\mathbb{T}^d)$.*

2. *$E = L^1(\mathbb{R}^d) * E$, where $E = M^{p,q}(\mathbb{R}^d)$ or $W^{p,q}(\mathbb{R}^d)$.*

Theorem 5.1.4 (Applications) *Let $1 \leq p, q \leq \infty$.*

1. *Let E denote any one of $M^{p,q}(\mathbb{R}^{2d})$ or $W^{p,q}(\mathbb{R}^{2d})$. If T is any function from $L^1(\mathbb{R}^{2d})$ to E such that $T(f \natural h) = f * T(h)$ for all $f, h \in L^1(\mathbb{R}^{2d})$, then T is a bounded linear transformation. In particular, we have*

$$\|T(f)\|_E \lesssim \|f\|_{L^1(\mathbb{R}^{2d})} \text{ for all } f \in L^1(\mathbb{R}^{2d}).$$

2. *Let E denote any one of $L^p(\mathbb{R}^{2d})$ or $M^{p,q}(\mathbb{R}^{2d})$ or $W^{p,q}(\mathbb{R}^{2d})$. If T is any function from $L^1(\mathbb{R}^{2d})$ to E such that $T(f\natural h) = f\natural T(h)$ for all $f, h \in L^1(\mathbb{R}^{2d})$, then T is a bounded linear transformation. In particular, we have*
$$\|T(f)\|_E \lesssim \|f\|_{L^1(\mathbb{R}^{2d})} \text{ for all } f \in L^1(\mathbb{R}^{2d}).$$

3. *Every positive linear functional on $(L^1(\mathbb{R}^{2d}), \natural)$ is continuous.*

4. *Every maximal left ideal in $(L^1(\mathbb{R}^{2d}), \natural)$ is closed.*

Remark 5.1.5 The analogue of Theorem 5.1.4 is true if we replace the same function spaces on $\mathbb{R}^d$ and the ordinary convolution instead of twisted convolution.

The sequel contains required background for twisted convolution in Section 5.2, the proof of factorization Theorems 5.1.1(1) and 5.1.3(2) in Section 5.3, the proof of non-factorization Theorems 5.1.1(2) and 5.1.3(1) in Section 5.4,the proof of Theorem 5.1.4 in Section 5.5.

5.2 Twisted Convolution

In this section, we provide relevant information for twisted convolution. Our representation owes to [28] where a more complete overview of the subject is given; and most of the proof can be found in it. See also [23, 61, 70].

For $x, w \in \mathbb{R}^d$, we recall the translation operator T_x and the modulation operator M_w:

$$T_x f(t) = f(t-x), \quad M_w f(t) = e^{2\pi i w\cdot t} f(t); \tag{5.1}$$

and we obtain

$$M_w T_x f(t) = e^{2\pi i w\cdot t} f(t-x), \quad T_x M_w f(t) = e^{-2\pi i w\cdot x} M_w T_x f(t).$$

Thus T_x and M_w commute if and only if $x \cdot w \in \mathbb{Z}$. Operators of the form $T_x M_w$ or $M_w T_x$ are called time-frequency shifts. Taking the composition of time-frequency shifts, we have

$$(T_x M_w)(T_{x'} M_{w'}) = e^{2\pi i x'\cdot w} T_{x+x'} M_{w+w'} \tag{5.2}$$

In many situations, for instance, in estimates with absolute values, the phase factor $e^{2\pi i x'\cdot w}$ does not matter. However, this factor is absolutely essential for a deeper understanding of the mathematical structure of time- frequency shifts, and it is the very reason for the appearance of a non-commutative group in the analysis.

For this we introduce a third coordinate in addition to time and frequency. By (5.2), time-frequency shifts, which are parametrized by $\mathbb{R}^{2d}$, are not closed under composition. As is suggested by (5.2), we "add" the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and look for a group multiplication on $\mathbb{R}^{2d} \times \mathbb{T}$ that is consistent with (5.2). We are lead to following abstract group multiplication on $\mathbb{R}^{2d} \times \mathbb{T}$.

Definition 5.2.1 The reduced Heisenberg group $\mathbb{H}_d^{red}$ is the locally compact space $\mathbb{H}_d^{red} = \mathbb{R}^{2d} \times \mathbb{T}$ under the multiplication

$$(x, w, e^{2\pi i\tau}) \cdot (x', w', e^{2\pi i\tau'}) = (x + x', w + w', e^{2\pi i(\tau+\tau')}e^{\pi i(x'\cdot w - x\cdot w')}).$$

We note that the product $(x, 0, 1) \cdot (0, w, 1) = (x, w, e^{-\pi i x\cdot w})$ corresponds to the time-frequency shifts $T_x M_w$, whereas the product $(0, w, 1) \cdot (x, 0, 1) = (x, w, e^{\pi i x\cdot w})$ corresponds to $M_w T_x \neq T_x M_w$. The group law in $\mathbb{H}_d^{red}$ reflects the non-commutativity of time-frequency shifts. Now we introduce the full Heisenberg group:

Definition 5.2.2 The full Heisenberg group $\mathbb{H}_d$ is the Euclidean space $\mathbb{R}^{2d} \times \mathbb{R}$ under the group multiplication

$$(x, w, \tau) \cdot (x', w', \tau') = (x + x', w + w', \tau + \tau' + \frac{1}{2}(x' \cdot w - x \cdot w')).$$

We take a note that $\mathbb{H}_d$ and $\mathbb{H}_d^{red}$ carries Haar measure that is invariant under group translations, in fact, it is the Lebesgue measure $dh := dxdwd\tau$ on $\mathbb{R}^{2d+1}$ and $\mathbb{R}^{2d} \times \mathbb{T}$; the convolution of $F_1, F_2 \in L^1(\mathbb{H}_d)$ or $L^1(\mathbb{H}_d^{red})$ respectively given by

$$(F_1 * F_2)(h_0) = \int_{\mathbb{H}_d \text{ (or } \mathbb{H}_d^{red})} F_1(h) F_2(h^{-1}h_0) dh;$$

and $L^1(\mathbb{H}_d)$ forms a non-commutative Banach algebra under convolution.

In time-frequency analysis the physical variables are x and w, whereas the auxiliary variable τ is added to create a group structure. It is often necessary to extend a function from the time-frequency plane to a function on the Heisenberg group. Since $\mathbb{T}$ is compact, it is more convenient to extend to the reduced Heisenberg group $\mathbb{H}_d^{red}$. Then it is of interest to understand how such extension is compatible with the convolution on $\mathbb{H}_d^{red}$.

Since the interesting action takes place in the third coordinate, we look for functions on $\mathbb{H}_d^{red}$ of the form

$$F^0(x, w, e^{2\pi i\tau}) = e^{-2\pi i\tau} F(x, w).$$

Then $\|F^0\|_{L^p(\mathbb{H}_d^{red})} = \|F\|_{L^p(\mathbb{R}^{2d})}$, and the convolution of the extensions yields a new operation on $L^1(\mathbb{R}^{2d})$. Let $F, G \in L^1(\mathbb{R}^{2d})$, and F^0, G^0 be as above. Then

$$\begin{aligned}&(F^0 * G^0)(x, w, e^{2\pi i\tau})\\ =&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_0^1 F^0(x', w', e^{2\pi i\tau'})G^0(x - x', w - w', e^{2\pi i(\tau-\tau')}e^{\pi i(x'\cdot w - x\cdot w')})dx'dw'd\tau'\\ =&e^{-2\pi i\tau}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F(x, x')G(x - x', w - w')e^{\pi i(x\cdot w' - x'\cdot w)}dx'dw';\end{aligned}$$

this formula naturally inspires to the following new operation on $L^1(\mathbb{R}^{2d})$:

Definition 5.2.3 For $\lambda \in \mathbb{R}$, the $\lambda-$ twisted convolution of f and h is the function defined by

$$f\natural_\lambda h(x, w) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x', w')h(x - x', w - w')e^{i\lambda\pi(x\cdot w' - x'\cdot w)}dx'dw';$$

for all $(x, w) \in \mathbb{R}^{2d}$ such that the integral exists.

For simplicity , we may identify the time-frequency plane $\mathbb{R}^{2d}$ with $\mathbb{C}^d$ via $(x, w) \mapsto z = x + iw$. Then $[(x, w), (x', w')] = x' \cdot w - x \cdot w' = \text{Im }(z \cdot \bar{z'})$, where $z' = (x', w')$, and the above formula we may rewrite as follows:

$$f\natural_\lambda h(z) = \int_{\mathbb{C}^d} f(z')h(z - z')e^{-i\lambda\pi \text{ Im}(z\cdot\bar{z'})}dz'. \tag{5.3}$$

Remark 5.2.4 When $\lambda = 0$ in Definition 5.2.3, it is just an ordinary convolution on $\mathbb{R}^{2d}$, and for $\lambda = 1$ we simply put, $f\natural_1 h = f\natural h$, and call it twisted convolution of f and h.

Proposition 5.2.5 *Let $1 \leq p, q, r \leq \infty$, $\lambda \in \mathbb{R}$, and assuming that all the integrals in question exist.*

1. *$f\natural_\lambda h = h\natural_{-\lambda} f$. In particular, the $\lambda-$twisted convolution, in general, is non-commutative.*

2. *$(f\natural_\lambda g)\natural_\lambda h = f\natural_\lambda(g\natural_\lambda h)$.*

3. *$L^p(\mathbb{R}^{2d})\natural_\lambda L^q(\mathbb{R}^{2d}) \subset L^r(\mathbb{R}^{2d})$ for $\frac{1}{p}+\frac{1}{q} = 1+\frac{1}{r}$. In particular, $L^p(\mathbb{R}^{2d})$ is a left Banach $L^1(\mathbb{R}^{2d})-$module with respect to $\lambda-$twisted convolution.*

4. *$L^p(\mathbb{R}^{2d})\natural L^p(\mathbb{R}^{2d}) \subset L^p(\mathbb{R}^{2d})$ for $1 \leq p \leq 2$. In particular, $L^p(\mathbb{R}^{2d})$ is a non-commutative Banach algebra with respect to twisted convolution.*

Proof. The proof of statements (1) and (2) is straightforward and follows by definition and performing change of variable; for instance, see [70, Proposition 9.1]. The proof of statement (3) follows by Young's inequality. For the proof of statement (4), see [28, p.326], [62, Proposition 2.1] and see also [23, p.27], [61, p.17]. □

5.3 Factorization in $L^p(\mathbb{R}^n)$, $M^{p,q}(\mathbb{R}^n)$, $W^{p,q}(\mathbb{R}^n)$

In this section, we will prove factorization results (Theorems 5.1.1(1) and 5.1.3(2)), that is, the possibility of factorizations $f = g\natural h$ and $f = g * h$ for f, g, h are in specified function spaces.

We start with constructing an approximate identity in $L^1(\mathbb{R}^{2d})$ with respect to twisted convolution. If ϕ is any function on $\mathbb{R}^{2d}$ and $r > 0$, we set

$$\phi_r(z) := r^{-2d}\phi(r^{-1}z). \tag{5.4}$$

Note. In what follows, the notation for ϕ_r will remain as defined in (5.4).

If $\phi \in L^1(\mathbb{R}^{2d})$, then $\int_{\mathbb{R}^{2d}} \phi_r(z)dz$ is independent of r, in fact, we have

$$\int_{\mathbb{R}^{2d}} \phi_r(z)dz = \int_{\mathbb{R}^{2d}} \phi(z)dz. \tag{5.5}$$

Proposition 5.3.1 *Suppose $\phi \in L^1(\mathbb{R}^{2d})$ with $\hat{\phi}(0) = 1$, and $\lambda \in \mathbb{R}$. If $f \in L^p(\mathbb{R}^{2d})$ $(1 \le p < \infty)$, then $f\natural_\lambda \phi_r \to f$ in the L^p norm as $r \to 0$.*

Proof. Setting $z' = ry$, and in view of (5.3) and (5.5), we have

$$\begin{aligned} f\natural\phi_r(z) - f(z) &= \int_{\mathbb{C}^d} f(z-z')\phi_r(z')e^{i\lambda\pi \mathrm{Im}(z\bar{z'})}dz' - f(z) \\ &= \int_{\mathbb{C}^d} [f(z-ry)e^{i\lambda r \mathrm{Im}(z\bar{y})} - f(z)]\phi(y)dy \\ &= \int_{\mathbb{C}^d} [T_{ry}f(z)e^{i\lambda r\pi \mathrm{Im}(z\bar{y})} - f(z)]\phi(y)dy. \end{aligned} \tag{5.6}$$

Apply Minkowski's inequality for the integrals:

$$\|f\natural\phi_r - f\|_{L^p(\mathbb{R}^{2d})} \le \int_{\mathbb{C}^d} \|T_{ry}fe^{i\lambda r\pi \mathrm{Im}(z\bar{y})} - f\|_{L^p(\mathbb{R}^{2d})}|\phi(y)|dy.$$

We note that $\|e^{i\lambda r\pi \mathrm{Im}(z\bar{y})}T_{ry}f - f\|_{L^p(\mathbb{R}^{2d})}$ is bounded by $2\|f\|_{L^p(\mathbb{R}^{2d})}$ and tends to 0 as $r \to 0$ for each y. Assertion therefore follows from the dominated convergence theorem. □

Proposition 5.3.2 *Let $1 \le p, q \le \infty$. Then*

$$L^1(\mathbb{R}^{2d})\natural M^{p,q}(\mathbb{R}^{2d}) \hookrightarrow M^{p,q}(\mathbb{R}^{2d})$$

with norm inequality

$$\|f\natural h\|_{M^{p,q}(\mathbb{R}^{2d})} \le \|h\|_{L^1(\mathbb{R}^{2d})}\|f\|_{M^{p,q}(\mathbb{R}^{2d})}.$$

In particular, $M^{p,q}(\mathbb{R}^{2d})$ is a left Banach $L^1(\mathbb{R}^{2d})-$module with respect to twisted convolution.

Proof. Putting $z = (x_1, x_2), t = (t_1, t_2) \in \mathbb{R}^{2d}$, and in view of (5.3), we may write,

$$(f\natural h)(z) = \int_{\mathbb{R}^{2d}} f(t)g(z-t)e^{-i\pi \mathrm{Im}(z\cdot\bar{t})}dt. \tag{5.7}$$

Let $z, w \in \mathbb{R}^{2d}$, and put $M^{\#}_{wz}g(t) := e^{2\pi iwt}\bar{g}(-t)e^{-i\pi \mathrm{Im}\ (z\bar{t})}$, and by (5.7) and

(1.2), we have

$$\begin{aligned}(f\natural M^{\#}_{wz}g)(z) &= e^{2\pi i w\cdot z}\int_{\mathbb{R}^{2d}} f(t)\overline{g(t-z)}e^{-2\pi i wt}dt \\ &= e^{2\pi i w\cdot z}V_g f(z,w). \qquad (5.8)\end{aligned}$$

In view of (5.8), Proposition 5.2.5(2) and Young's inequality, we obtain

$$\begin{aligned}\|h\natural f\|_{M^{p,q}(\mathbb{R}^{2d})} &= \|V_g(h\natural f)\|_{L^{p,q}(\mathbb{R}^{2d})} \\ &= \|(h\natural f)\natural M^{\#}_{wz}g\|_{L^{p,q}(\mathbb{R}^{4d})} \\ &\lesssim \|h\natural (f\natural M^{\#}_{wz}g)\|_{L^{p,q}(\mathbb{R}^{4d})} \\ &\leq \|h\|_{L^1(\mathbb{R}^{2d})}\|f\|_{M^{p,q}(\mathbb{R}^{2d})}.\end{aligned}$$

□

Proposition 5.3.3 *Let $1 \leq p,q \leq \infty$. Then $L^1(\mathbb{R}^{2d})\natural W^{p,q}(\mathbb{R}^{2d}) \subset W^{p,q}(\mathbb{R}^{2d})$ with norm inequality*

$$\|f\natural h\|_{W^{p,q}(\mathbb{R}^{2d})} \leq \|h\|_{L^1(\mathbb{R}^{2d})}\|f\|_{W^{p,q}(\mathbb{R}^{2d})}.$$

In particular, $W^{p,q}(\mathbb{R}^{2d})$ is a left Banach $L^1(\mathbb{R}^{2d})-$module with respect to twisted convolution.

Proof. Exploiting the ideas from Proposition 5.3.2, the proof can be produced similarly and so we omit the details. □

The proof of the next proposition goes along lines as in the proof of Lemma 1.4.4.

Proposition 5.3.4 *Let $\phi \in \mathcal{S}(\mathbb{R}^{2d})$ with $\int_{\mathbb{C}^d}\phi(z)dz = 1$. If $f \in M^{p,q}(\mathbb{R}^{2d})$ $(1 \leq p,q < \infty)$, then $\phi_r\natural f \to f$ in the $M^{p,q}$ norm as $r \to 0$.*

Proof. Putting $z = (x_1,x_2), t = (t_1,t_2) \in \mathbb{R}^{2d}$, and in view of (5.3), we may write

$$(f\natural h)(z) = \int_{\mathbb{R}^{2d}} f(t)g(z-t)e^{-i\pi \mathrm{Im}(z\cdot\bar{t})}dt.$$

Setting $t = ry$, we note that

$$(\phi_r\natural f)(z) - f(z) = \int_{\mathbb{R}^{2d}} f(z-t)\phi_r(t)e^{-i\pi\mathrm{Im}(z\cdot\bar{t})}dt - f(z)$$

$$\begin{aligned} &= \int_{\mathbb{R}^{2d}} [e^{-\pi i r \mathrm{Im}(z\cdot\bar{y})} f(z-ry) - f(z)]\phi(y)dy \\ &= \int_{\mathbb{R}^{2d}} [M^*_{ry} T_{ry} f(z) - f(z)]\phi(y)dy; \end{aligned}$$

where $M^*_{ry}T_{ry}f(z) = e^{-i\pi r \mathrm{Im}(z\cdot\bar{y})} f(z-ry)$. We put, $h_r(z) = (\phi_r \natural f)(z) - f(z)$. Let $z = (x_1, x_2) \in \mathbb{R}^{2d}$, $w = (w_1, w_2) \in \mathbb{R}^{2d}$, we have

$$V_g h_r(z,w) = \int_{\mathbb{R}^{2d}} (V_g M^*_{ry} T_{ry} f - V_g f)(z,w)\phi(y)dy. \tag{5.9}$$

Taking mixed $L^{p,q}-$ norm on the both sides of (5.9), and using Minkowski's inequality for the integrals, we may find

$$\|h_r\|_{M^{p,q}(\mathbb{R}^{2d})} \le \int_{\mathbb{R}^{2d}} \|M^*_{ry}T_{ry}f - f\|_{M^{p,q}(\mathbb{R}^{2d})} |\phi(y)|dy \tag{5.10}$$

Notice that,

$$\begin{aligned} V_g(M^*_{ry}T_{ry}f)(z,w) &= e^{-2\pi i w r y} \int_{\mathbb{R}^{2d}} e^{-i\pi r \mathrm{Im}(t\cdot\bar{y})} f(t) \overline{g(t-(z-ry))} e^{-2\pi i w\cdot t} dt \\ &= M_{(0,-ry)}(T_{(ry,0)} V_g M^*_{ry} f)(z,w); \end{aligned} \tag{5.11}$$

where $M^*_{ry}f(t) = e^{-i\pi r \mathrm{Im}(t\cdot\bar{y})} f(t)$. And by (5.11), we may find

$$\begin{aligned} &\|M^*_{ry}T_{ry}f - f\|_{M^{p,q}(\mathbb{R}^{2d})} \\ =&\|M_{(0,-ry)}(T_{(ry,0)}V_g M^*_{ry} f) - V_g f\|_{L^{p,q}(\mathbb{R}^{4d})} \\ =&\|M_{(0,-ry)}(T_{(ry,0)}V_g M^*_{ry} f) - M_{(0,-ry)}V_g f + M_{(0,-ry)}V_g f - V_g f\|_{L^{p,q}(\mathbb{R}^{4d})} \\ \le&\|M_{(0,-ry)}(T_{(ry,0)}V_g M^*_{ry} f) - M_{(0,-ry)}V_g f\|_{L^{p,q}(\mathbb{R}^{4d})} + \|M_{(0,-ry)}V_g f - V_g f\|_{L^{p,q}(\mathbb{R}^{4d})}, \end{aligned}$$

and each of these tends to 0 as $r \to 0$. Now by dominated convergence theorem, it follows that the right hand side of (5.10) tends to as $r \to 0$. Hence, the proof. □

Proposition 5.3.5 *Let $\phi \in \mathcal{S}(\mathbb{R}^{2d})$ with $\hat{\phi}(0) = 1$. If $f \in W^{p,q}(\mathbb{R}^{2d})$ $(1 \le p, q < \infty)$, then $\phi_r \natural f \to f$ in the $W^{p,q}$ norm as $r \to 0$.*

Proof. Exploiting the ideas from Proposition 5.3.4, the proof can be produced similarly and so we omit the details. □

Now we are in a position to apply following well known factorization theorem, to prove Theorems 5.1.1(1) and 5.1.3(2)). We recall

Theorem 5.3.6 (Edwin Hewitt [34]) *Let L be a left Banach $A-$module with the property: for every finite set $\{a_1, a_2, ..., a_m\} \subset A$, every $x \in L$, and every $\epsilon > 0$, there is $a \in A$ such that $\|aa_j - a_j\|_A < \epsilon, j = 1, 2, ..., m$, and $\|a \cdot x - x\|_L < \epsilon$. Then the mapping from $A \times L$ to L is surjective. Furthermore, for any $z \in L$ and $\epsilon > 0$ there exists $x \in A$ and $y \in L$ such that*

(i) $z = x \cdot y$;

(ii) $\|z - y\|_L < \epsilon$.

Remark 5.3.7 We note that Banach algebra is a left Banach module over itself, and we say it has the Cohen's factorization property if it satisfies the above properties (i) and (ii).

Proof of Theorem 5.1.1 (1). In view of Propositions 5.3.3, 5.3.2, and 5.3.5, we notice that, the mapping $(f, h) \mapsto f\natural h$ of $L^1(\mathbb{R}^{2d}) \times E$ into E is a left Banach $L^1(\mathbb{R}^{2d})-$module; and in view of Propositions 5.3.1 and 5.3.4, we may apply Theorem 5.3.6. Hence, the proof. □

Proposition 5.3.8 *Let $\psi \in \mathcal{S}(\mathbb{R}^d)$ with $\hat{\psi}(0) = 1$ and $\psi_t(x) = t^{-d}\psi(xt^{-1}), t > 0, x \in \mathbb{R}^d$, and $E = M^{p,q}(\mathbb{R}^d)$ or $W^{p,q}(\mathbb{R}^d)$ with $1 \leq p, q < \infty$. Then $\|\psi_t * f - f\|_E \to 0$ as $t \to 0$ for $f \in E$.*

Proof. For $E = M^{p,q}(\mathbb{R}^d)$, the proof has been established in Lemma 1.4.4; and the proof can be given similarly for $E = W^{p,q}(\mathbb{R}^d)$. □

Proof of Theorem 5.1.3 (2). Taking Proposition 5.3.8 into our account, and in view of Lemma 1.4.1 (1) and (2), we may apply Theorem 5.3.6. Hence, the proof. □

Corollary 5.3.9 *Let $1 \leq p < \infty$, and $E = M^{p,1}(\mathbb{R}^d)$ or $W^{p,1}(\mathbb{R}^d)$. Then $E \subset C(\mathbb{R}^d)$.*

Proof. In view of Theorem 5.1.3 (2), and Lemmas 1.4.1 (3) and 1.4.1 (2), we may find $E = L^1(\mathbb{R}^d) * E \subset L^1(\mathbb{R}^d) * L^\infty(\mathbb{R}^d) \subset C(\mathbb{R}^d)$. □

5.4 Non-factorization in $L^p(\mathbb{R}^n)$, $M^{p,q}(\mathbb{R}^n)$, $W^{p,q}(\mathbb{R}^n)$

In the last section, we have shown every $f \in L^1(\mathbb{R}^{2d})$ can be factorized as $f = g\natural h$ with $g, h \in L^1(\mathbb{R}^{2d})$ and h is close to f in sense of L^1-norm; in this section we shall show if we replaced $L^1(\mathbb{R}^{2d})$ by $M^{1,1}(\mathbb{R}^{2d})$ or $L^2(\mathbb{R}^{2d})$ this is impossible (Theorem 5.1.1 (2) and Remark (5.1.2)).

Lemma 5.4.1 *There is no bounded approximate identity in $M^{1,1}(\mathbb{R}^{2d})$ with respect to twisted convolution $\natural$.*

Proof. If possible, suppose that $\{e_r\}$ is an approximate identity in $M^{1,1}(\mathbb{R}^{2d})$ bounded by C. By Lemma 1.4.1 (3), and Proposition 5.2.5(1), we have

$$\begin{aligned} \|f\|_{L^1(\mathbb{R}^{2d})} &\lesssim \|f\|_{M^{1,1}(\mathbb{R}^{2d})} \\ &= \lim \|e_r \natural_1 f\|_{M^{1,1}(\mathbb{R}^{2d})} \\ &= \lim \|f \natural_{-1} e_r\|_{M^{1,1}(\mathbb{R}^{2d})} \\ &\leq \limsup \|f\|_{L^1(\mathbb{R}^{2d})} \|e_r\|_{M^{1,1}(\mathbb{R}^{2d})} \\ &\lesssim C\|f\|_{L^1(\mathbb{R}^{2d})}. \end{aligned}$$

But this means that the norms $\|\cdot\|_{L^1}$ and $\|\cdot\|_{M^{1,1}}$ are equivalent on $M^{1,1}(\mathbb{R}^{2d})$ which is contradiction as $M^{1,1}(\mathbb{R}^{2d})$ is a proper dense subset of $L^1(\mathbb{R}^{2d})$. ☐

Theorem 5.4.2 (Altman [1]) *A Banach algebra has a Cohen's factorization property if and only if it has a bounded left approximate identity.*

Proposition 5.4.3 *A Banach algebra $(M^{1,1}(\mathbb{R}^{2d}), \natural)$ does not have Cohen's factorization property.*

Proof. By Lemma 5.4.1, Remark 5.3.7 and Theorem 5.4.2, the proof follows. ☐

We recall that $M^{2,2}(\mathbb{R}^{2d}) = L^2(\mathbb{R}^{2d}) = W^{2,2}(\mathbb{R}^{2d})$ (see Lemma 1.4.1(3)) and proceed to prove that it fails to factor in the sense that $L^2(\mathbb{R}^{2d})\natural L^2(\mathbb{R}^{2d}) \neq L^2(\mathbb{R}^{2d})$.

Proposition 5.4.4 *Let $1 \leq p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then $L^p(\mathbb{R}^{2d})\natural L^q(\mathbb{R}^{2d}) \subset C(\mathbb{R}^{2d})$.*

Proof. Let $y \in \mathbb{C}^d$, and in view of (5.3), we may find

$$T_y(f\natural h)(z) - (f\natural h)(z) = \int_{\mathbb{C}^d} [T_y h(z-z')e^{i\pi \mathrm{Im}(y\bar{z'})} - h(z-z')]f(z')e^{-i\pi \mathrm{Im}(z\bar{z'})}dz'.$$

From this it follows that,

$$\begin{aligned} \|T_y(f\natural h) - (f\natural h)\|_u &= \|T_y(f\natural h) - (f\natural h)\|_{L^\infty(\mathbb{R}^{2d})} \\ &\leq \|e^{i\pi \mathrm{Im}(y(\cdot))}T_y h - h\|_{L^p(\mathbb{R}^{2d})}\|f\|_{L^q(\mathbb{R}^{2d})}; \end{aligned}$$

which is tends to 0 as $y \to 0$. □

Proof of Theorem 5.1.1 (2). If possible, suppose that a Banach algebra $(L^2(\mathbb{R}^{2d}), \natural)$ can be factored; and as a consequence we have $L^2(\mathbb{R}^{2d}) \subset L^2(\mathbb{R}^{2d})\natural L^2(\mathbb{R}^{2d})$. But then by Proposition 5.4.4, we have $L^2(\mathbb{R}^{2d}) \subset C(\mathbb{R}^{2d})$, which is absurd. Thus, there is a function in $L^2(\mathbb{R}^{2d})$ which cannot be factored as twisted convolution of two members in $L^2(\mathbb{R}^{2d})$. □

Remark 5.4.5 (1) We denote by $A(\mathbb{R}^d)$ (Fourier algebra) the space of all Fourier transforms of $L^1(\mathbb{R}^d)$, that is, $A(\mathbb{R}^d) = \{f \in L^\infty : \hat{f} \in L^1(\mathbb{R}^d)\}$. The space $A(\mathbb{R}^d)$ is a Banach algebra under pointwise addition and multiplication, with respect to the norm:

$$\|f\|_{A(\mathbb{R}^d)} := \|\hat{f}\|_{L^1} \ (f \in A(\mathbb{R}^d)),$$

and it is well-known that $L^2(\mathbb{R}^{2d}) * L^2(\mathbb{R}^{2d}) = A(\mathbb{R}^{2d})$.
In contrast, we have $L^2(\mathbb{R}^{2d})\natural L^2(\mathbb{R}^{2d}) \neq A(\mathbb{R}^{2d})$: if $L^2(\mathbb{R}^{2d})\natural L^2(\mathbb{R}^{2d}) = A(\mathbb{R}^{2d})$, then $L^2(\mathbb{R}^{2d}) * L^2(\mathbb{R}^{2d}) \subset L^2(\mathbb{R}^{2d})$, which is absurd. (2) There does not exists bounded approximate identity in $(L^2(\mathbb{R}^{2d}), \natural)$.

Next, we prove non-factorization result for periodic modulation spaces (see (2.4) and Proposition 2.2.9).

Proof of Theorem 5.1.3 (1). The proof is straightforward. Clearly, $A(\mathbb{T}^d) \subset L^2(\mathbb{T}^d)$, and by using Cauchy-Schwartz inequality, we have $A(\mathbb{T}^d) * A(\mathbb{T}^d) \subset A(\mathbb{T}^d)$.

We define

$$f(e^{2\pi it}) := \sum_{n\in\mathbb{Z}^d} a_n e^{2\pi int};$$

where $a_n = \frac{1}{n_1^2 n_2^2 \cdots n_d^2}$ for $n = (n_1, ..., n_d) \in \mathbb{N}^d$ and $a_n = 0$ otherwise. Then we note that $f \in A(\mathbb{T}^d)$ as $\hat{f} \in \ell^1(\mathbb{Z}^d)$ but $\hat{f} \notin \ell^1(\mathbb{Z}^d) \cdot \ell^1(\mathbb{Z}^d) = \{x \cdot y : x, y \in \ell^1(\mathbb{Z}^d)\}$. It follow that, $f \notin A(\mathbb{T}^d) * A(\mathbb{T}^d)$. □

Remark 5.4.6 It is well-known that $A(\mathbb{T}^d) = L^2(\mathbb{T}^d) * L^2(\mathbb{T}^d)$.

5.5 Applications

We collect and state some well known results which has made use of Cohen-Hewitt factorization theorem in it's proofs.

Theorem 5.5.1 (N. Th.Varopoulos [55]) *Let A be a Banach *-algebra with a bounded approximate identity. Then every positive functional p on A is continuous.*

Theorem 5.5.2 (B. E. Johnson [37]) *Let A be a Banach algebra with a bounded left approximate identity and let X be a left Banach A-module. If T is a function from A into X such that $T(ab) = aT(b)$ for all $a, b \in A$, then T is a bounded linear transformation.*

Theorem 5.5.3 ((M. D. Green [26]) *Let A be a Banach algebra with a bounded right approximate identity. Then every maximal left ideal I in A is closed.*

Proof of Theorem 5.1.4. Taking Propositions 5.3.2, 1.4.3(1), into our account, we may apply Theorem 5.5.2, and the assertion in Theorem 5.1.4 (1) and (2) follows. By defining $f^*(x) := \overline{f(x)}$, $f \in L^1(\mathbb{R}^{2d})$, we have

$$\begin{aligned} (f \natural h)^* &= \overline{f \natural_1 h} \\ &= \bar{f} \natural_{-1} \bar{h} \\ &= \bar{h} \natural_1 \bar{f} \\ &= h^* \natural f^*, \end{aligned}$$

and the mapping $* : f \mapsto \bar{f}$ form an involution on Banach algebra $(L^1(\mathbb{R}^{2d}), \natural)$. And a Banach algebra $L^1(\mathbb{R}^{2d})$ equipped with this as an involution forms a Banach *-algebra. Hence, in view of Proposition 5.3.1, we may apply Theorem 5.5.1, and the assertion in Theorem 5.1.4 (3) follows. In view of Proposition 5.3.1, and Theorem 5.5.3, the assertion in Theorem 5.1.4 (4) follows. □

5.6 Final Remarks

1. What information we have gained so far, concerning factorization problems in the realm of Lebesgue, modulation, and Wiener amalgam spaces is very little information, and further research needs to be done to gain a complete understanding of the factorization problems, and this consideration, inspires us to raise the following questions:

   - Taking Proposition 5.2.5 (4), Theorem 5.1.1 (2) and Remark 5.4.5, into our account, it would be interesting to know: what is $L^p(\mathbb{R}^{2d})\natural L^p(\mathbb{R}^{2d})$ exactly for $p \in (1, 2]$?

   - Taking the Proposition 3 into our account, it would be interesting to know: what is the set $L^p(\mathbb{R}^{2d})\natural L^q(\mathbb{R}^{2d})$ exactly for $p, q \in (1, \infty)$, and $\frac{1}{p} + \frac{1}{q} - 1 \geq 0$?

   - Taking Proposition 5.3.2 and Remark 5.1.2 into our account, it would be interesting to know: what $M^{1,1}(\mathbb{R}^{2d})\natural M^{1,1}(\mathbb{R}^{2d})$ exactly?

   - Taking Theorem 5.1.3 (1) into our account, it would be interesting to know: what is the set $E * E$ exactly for $E = M^{p,1}(\mathbb{T}^d)$ or $M^{p,1}(\mathbb{R}^d)$ or $W^{p,1}(\mathbb{R}^d)$ $(1 \leq p \leq \infty)$?

   - Taking Proposition 1.4.3, into our account, it would be interesting to know what are the sets $M^{p_1,q_1}(\mathbb{R}^d) * M^{p_2,q_2}(\mathbb{R}^d)$ and $W^{p_1,q_1}(\mathbb{R}^d) * W^{p_2,q_2}(\mathbb{R}^d)$ exactly for $\frac{1}{p_1} + \frac{1}{p_2} - 1 > 0$ and $\frac{1}{q_1} + \frac{1}{q_2} \geq 0$ $(p_i, q_i \in [1, \infty], i = 1, 2)$?

2. The results of this chapter are yet under investigation for the possible generalization; and we hope to address the above and related issues in future.

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