FOURIER INTEGRAL OPERATORS, WAVE EQUATION AND MAXIMAL OPERATORS

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DEDICATIONS

To my loving parents

SMT. DIPALI MANNA & SHREE DILIP MANNA.

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Synopsis

This Ph. D. dissertation entitled "Fourier integral operators, Wave equation and Maximal operators" carries out mainly the study of local smoothing of Fourier integral operators in the plane with application to the wave equation and Bourgain's circular maximal operator. Apart from this, the L^p- boundedness of the maximal function along hypersurfaces is also studied. The approach uses the concepts and techniques from real variable method in Fourier analysis and micro-local analysis, in particular the Littlewood-Paley theory, multipliers, oscillatory integrals, Fourier integral operators etc.

The content of the present dissertation is divided into six chapters. Chapter 1 is the introduction, where we briefly discuss the Fourier integral operators, wave equation, maximal operators and discusses some basic question concerning it. In particular, we discuss the regularity theory of Fourier integral operators and some of its basic properties. Naturally, the geometry and singularities of phase functions reflect in the L^p- boundedness of the Fourier integral operators. The analysis of Fourier integral operators has developed into a field of active research, with extensions in many different directions. For example, a Fourier integral operator with phase function $\phi(x,t,\xi) = x \cdot \xi + t|\xi|$ arises in the study of problems involving wave equations, see Hörmander [18]. They are a natural generalization of pseudo-differential operators for which $\phi(x,\xi) = x \cdot \xi$.

As in the case of pseudo-differential operators, a Fourier integral operator with symbol of order zero is bounded on L^2_{loc} under some non-degeneracy condition on the phase function, as proved by Hörmander [18]. It is well known that pseudo-differential operators with symbol of order zero are bounded on L^p for $1 < p < \infty$, see [37]. This follows from the fact that they are operators of the Calderon-Zygmund type. But for Fourier integral operators which are not pseudo-differential, this property does not hold. Under some non-degeneracy condition on the phase function, Seeger et al. [31] showed that the corresponding Fourier integral operator is bounded on $L^p_{loc}(\mathbb{R}^n)$ for $1 < p < \infty$ provided that the symbol of the operator belongs to S^m, the symbol class of order $m \leq -(n-1)|\frac{1}{p} - \frac{1}{2}|$. The simplest example of such an operator corresponds to the phase functions $\phi(x,t,\xi) = x \cdot \xi + t|\xi|$ arising in the solution to the wave equation.

The basic concepts and results of oscillatory integrals that is relevant in the study of regularity property of Fourier integral operators, constitute the content of chapter 2. In this chapter, we explicitly construct a partition of unity based on angular decomposition, which will be needed in the later chapters. In fact, the decomposition into angular components was made to control the L^1 norm of the kernels of Fourier integral operators.

In chapter 3, we study the local smoothing estimates for Fourier integral operators of the form

$$\mathcal{F}f(x,t) = \rho_1(t) \int_{\mathbb{R}^2} e^{ix\cdot\xi} e^{it|\xi|}\, a(\xi)\, \hat{f}(\xi)\, d\xi. \tag{1}$$

Here $a \in S^m(\mathbb{R}^2)$, the symbol class of order $m \leq 0$ and $\rho_1 \in C_c^\infty((1,2))$. Fourier integral operators of the form (1) arise in wave equation and also in the study of spherical maximal operators.

Consider the Cauchy problem for the wave equation on $\mathbb{R}^2$:

$$\begin{cases} (\partial_t^2 - \Delta)\, u(x,t) = 0, \\ u(x,0) = f(x), \partial_t u(x,0) = g(x). \end{cases} \tag{2}$$

For $f, g \in \mathscr{S}(\mathbb{R}^2)$, the solution can be easily obtained via Fourier transform:

$$u(x,t) = \mathcal{F}_t f(x) + \mathcal{G}_t f(x),$$

where

$$\mathcal{F}_t f(x) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{ix\cdot\xi} \cos(t|\xi|)\, \hat{f}(\xi)\, d\xi$$

and

$$\mathcal{G}_t g(x) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{ix\cdot\xi} \frac{\sin(t|\xi|)}{|\xi|}\, \hat{g}(\xi)\, d\xi.$$

These multipliers can be expressed in terms of the oscillatory integrals of the form

$$\mathcal{F}h(x,t) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{ix\cdot\xi} e^{it|\xi|}\, a(\xi)\, \hat{h}(\xi)\, d\xi, \tag{3}$$

where $a(\xi) \equiv 1$ or $1/|\xi|$.

The regularity property of the solution operators $\mathcal{F}_t$ and $\mathcal{G}_t$ has been studied by Peral [30] and Miyachi [27] on $\mathbb{R}^n$, $n \geq 2$. For $\alpha \geq 0$, consider $L^p_\alpha = (-\Delta + I)^{-\alpha/2} L^p$, the Sobolev space of L^p functions on $\mathbb{R}^n$ with α derivatives in L^p, see [37]. Note that L^p_α is a Banach space with norm $\|f\|_{L^p_\alpha} = \|(-\Delta + I)^{\alpha/2} f\|_{L^p}$. When $\alpha < 0$, these are spaces of tempered distributions. It has been shown by Peral and Miyachi that the following L^p Sobolev inequalities

$$\|\mathcal{F}_t f\|_{L^p_\alpha(dx)} \leq C_p \|f\|_{L^p_1(dx)}, \quad \|\mathcal{G}_t g\|_{L^p_\alpha(dx)} \leq C_p \|g\|_{L^p(dx)}$$

hold if and only if $\alpha - 1 = -\left|\frac{1}{2} - \frac{1}{p}\right|$, $1 < p < \infty$.

In 1991, Sogge [34] made an interesting observation: if one averages over $t \in [1, 2]$, there is a gain of regularity in L^p for $2 < p < \infty$. Let $\mathcal{F}f$ be the Fourier integral operator given by (3) with the amplitude function $a(\xi) \equiv 1$. He showed that there is an $\epsilon(p) > 0$ such that the following estimate

$$\left(\int_{t=1}^{2} \int_{\mathbb{R}^2} \left|(I - \Delta)^{\frac{\sigma}{2}}(\mathcal{F}f)(x,t)\right|^p \, dxdt\right)^{\frac{1}{p}} \leq c_{\sigma,p} ||f||_{L^p(dx)},$$

holds for all $\sigma < \left(\frac{1}{p} - \frac{1}{2}\right) + \epsilon(p)$ and for each $p \in (2, \infty)$.

Comparing with the estimates of Peral and Miyachi (with $\sigma = \alpha - 1$), the above estimate shows that there is a gain in regularity by $\epsilon(p)$. The above estimate is called the local smoothing estimate of order $\epsilon(p)$. Borrowing a term from a similar situation involving the Schrödinger equation, Sogge called this phenomenon as local smoothing. In a latter joint work with B. Mockenhaupt and A. Seeger [28], they made a further improvement in dimension two, by showing that $\epsilon(p) < \frac{1}{2p}$, for $p \geq 4$ and $\epsilon(p) < \frac{1}{2}(\frac{1}{2} - \frac{1}{p})$ for $2 < p \leq 4$.

This motivates us to initiate the study of local smoothing problem of Fourier integral operators. In this chapter, we prove the local smoothing estimate for the Fourier integral operator, given by (1) with the amplitude function $a \in S^0(\mathbb{R}^2)$. Proof of the refined estimate of this type requires frequency localization with suitable smooth cut off functions which form a partition of unity, and a delicate machinery from Littlewood-Paley theory and micro-local analysis.

As an application of the above local smoothing estimate, we give a proof of the local smoothing estimate for the initial value problem (2) for the wave equation and this part constitute the content of chapter 4. This requires consideration of a slightly more general class of Fourier integral operators with symbol $a \in S^m(\mathbb{R}^2)$, the symbol class of order $m \leq 0$ to handle the initial velocity in the problem of wave equation.

In chapter 5, as an application of the local smoothing estimate, we give an alternative proof of the L^p boundedness of the circular maximal operator on $L^p(\mathbb{R}^2)$ for $p > 2$. Given a function f, continuous and compactly supported, consider the averaging operator $S_t f(x) = \int_{\mathbb{S}^{n-1}} f(x - ty)\, d\sigma(y)$, for each $x \in \mathbb{R}^n$ and $t > 0$. Here, $d\sigma$ denotes the normalized Lebesgue measure over the unit sphere $\mathbb{S}^{n-1}$. Stein [36] introduced the maximal function

$$\mathcal{M}f(x) = \sup_{t>0} |S_t f(x)|,$$

and showed that $\mathcal{M} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ for the optimal range of $p's$, that is, $\frac{n}{n-1} < p \leq \infty$, provided $n \geq 3$. The case $n = 2$ was settled by Bourgain [1]. Bourgain's proof of the circular maximal theorem relies more directly on the geometry involved.

The main step of our proof of the circular maximal theorem is to decompose the averaging operator into dyadic operators, and then express each dyadic operator in terms of Fourier integral operator by using the stationary phase method. In fact, we reduce our problem to the estimates where the supremum is only taken over $t \in [1, 2]$. This reduction follows from the arguments, given in [28, 33]. To complete the proof, we will use the local smoothing estimates of Fourier integral operators, obtained in chapter 3.

Finally, in chapter 6, we study the L^p-boundedness of maximal operators along a class of hypersurfaces in $\mathbb{R}^{n+1}$ given by the graph of a function. Let $S = \Phi^{-1}(0)$ be a hypersurface in $\mathbb{R}^{n+1}$ given by a function $\Phi \in C^1(\mathbb{R}^{n+1})$ with $\nabla\Phi \neq 0$. We write the elements in $\mathbb{R}^{n+1}$ as $(x, x_0) \in \mathbb{R}^n \times \mathbb{R}$ and for simplicity, assume that Φ is of the form $\Phi(x, x_0) = h(x) - x_0$, where h is a non-negative C^1 function defined on $\mathbb{R}^n$. In this case, S is the graph of h, and we assume that $h(0) = 0$, and $\nabla h(x) \neq 0$ for all $x \in S_r$, where $S_r = \{x \in \mathbb{R}^n : h(x) = r\}$ is the level set of the function h, at height $r > 0$ and each S_r is a compact C^1 hypersurface in $\mathbb{R}^n$. We also assume that h satisfies the following α-homogeneity condition:

$$h(r^\alpha x) = rh(x), \;\; \alpha > 0 \tag{4}$$

for each $r > 0$ and $x \in \mathbb{R}^n$. For $r \geq 0$, let $\Sigma_r = \{\tilde{x} = (x, x_0) \in S : 0 \leq x_0 \leq r\}$ and for $f \in \mathscr{S}(\mathbb{R}^{n+1})$, consider the average

$$A_r f(\tilde{x}) = \frac{1}{\mu(\Sigma_r)} \int\limits_{\Sigma_r} |f(\tilde{x} - \tilde{y})| \, d\mu(\tilde{y}), \tag{5}$$

where μ denotes the surface measure on S induced by the Lebesgue measure on $\mathbb{R}^{n+1}$. Define the corresponding maximal operator by

$$Mf(\tilde{x}) = \sup_{r>0} A_r f(\tilde{x}). \tag{6}$$

In the last several years, considerable attention has been given to the study of maximal operators along surfaces and curves, (see, [10, 21, 38, 39] and references therein). Note that our maximal operator is closer in spirit to the maximal operators studied in [10] which answers a problem posed by Stein and Wainger in [39] and refers to hypersurfaces in $\mathbb{R}^3$ given by the graph of $\phi(s,t) = |s|^\alpha\,|t|^\beta$ on $\mathbb{R}^2$. This has been extended to more general hypersurface that arise as a graph of certain function of the form $\Gamma(|x|,|y|)$ on $\mathbb{R}^n \times \mathbb{R}^m$ in [21]. These maximal operators are called the multi-parameter maximal functions, as they correspond to averages over images of rectangles on the surface.

The maximal operator (6) that we consider here on the other hand is obtained from averages over the portion of the hypersurface upto 'height' r i.e., Σ_r in (5). We proved that, the maximal operator given by (6) is bounded on $L^p(\mathbb{R}^{n+1})$ for $p > \frac{k+1}{k}$, in case of hypersurfaces that arise as the graph of a function $h \in C^1(\mathbb{R}^n \setminus \{0\})$, which is α-homogeneous, in the sense of (4), for which the lower dimensional surface $S_1 = \{x \in \mathbb{R}^n : h(x) = 1\}$ has at least k, $1 \leq k \leq n-1$, non vanishing principal curvature everywhere on S_1. Our approach is via a factorization technique, which considerably simplify the proof. The key idea is to factorise the maximal operator along hypersurface into a generalized one dimensional Hardy-Littlewood maximal operator, and a dilated maximal operator associated with a compact hypersurface in $\mathbb{R}^n$. The main step in this reduction is a factorisation of the surface measure.

Chapter 0

Notation and Definition

The purpose of this chapter is to establish basic notations and some function spaces that will be used throughout this dissertation.

Symbols

- $\mathbb{N}$ will denote the set of positive integers, $\mathbb{Z}$ the set of integers, $\mathbb{R}$ the set of real numbers, $\mathbb{C}$ the set of complex numbers. We will be working with $\mathbb{N}^n$, $\mathbb{Z}^n$, $\mathbb{R}^n$ and n will always denote the dimension.

- The notation $A \approx B$ means $c^{-1} A \leq B \leq c\, A$, for some $c \geq 1$.

- $[x]$ will denote the largest integer less than or equal to x.

- If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, we set

$$x \cdot y = \sum_{i=1}^{n} x_i\, y_i, \;\; |x| = \sqrt{x \cdot x}.$$

- For the partial derivatives, we set

$$\partial_{x_i} = \frac{\partial}{\partial x_i},$$

  and for higher-order derivative we use multi-index notation.

- A multi-index is an ordered n-tuple of non-negative integers.

If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index. We set

$$|\alpha| = \sum_{i=1}^{n} \alpha_i, \ \partial_x^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

- The most important partial differential operator is the Laplacian

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_j^2}.$$

Function Spaces and Definitions

We will consider certain well known function spaces and some definition that we present now.

- $C(\mathbb{R}^n)$ will denote the space of continuous functions on $\mathbb{R}^n$, $C_c^\infty(\mathbb{R}^n)$ the space of smooth functions on $\mathbb{R}^n$ with compact support.

- The set of Schwartz-class functions $\mathscr{S}(\mathbb{R}^n)$, consists of all $\phi \in C^\infty(\mathbb{R}^n)$ satisfying

$$\sup_{x\in\mathbb{R}^n} \left|x^\beta\, \partial^\alpha \phi(x)\right| < \infty, \tag{1}$$

for all multi-indices α, β. Note that $\mathscr{S}(\mathbb{R}^n)$ is a Fréchet space with the topology defined by the semi-norms (1). In fact, the set of all compactly supported C^∞ functions $C_c^\infty(\mathbb{R}^n)$ is contained in $\mathscr{S}(\mathbb{R}^n)$. Moreover, we can define linear and continuous functional on the Schwartz space $\mathscr{S}(\mathbb{R}^n)$, the so-called tempered distributions, and the space of tempered distributions will be denoted by $\mathscr{S}'(\mathbb{R}^n)$. For the details, see [35], and [13, p. 293].

- $L^p(\mathbb{R}^n)$ will denote the usual Lebesgue space of measurable functions $f(x)$ for which the following norm

$$\|f\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^p\, dx\right)^{1/p}, \ (1 \le p < \infty),$$

is finite. L^p_{loc} will denote the space of measurable function $f : \mathbb{R}^n \to \mathbb{C}$ (with respect to Lebesgue measure) if $\left(\int_K |f(x)|^p\, dx\right)^{1/p}$ is finite for every bounded measurable set $K \subset \mathbb{R}^n$.

- $L^{\infty}(\mathbb{R}^n)$ norm is given by

$$\|f\|_{L^{\infty}(\mathbb{R}^n)} = ess \cdot sup_{x\in\mathbb{R}^n}|f(x)|.$$

- $l^p(\mathbb{Z}^n)$ will denote the spaces of sequences on $\mathbb{Z}^n$ for which the following norm

$$\|a\|_{l^p} = \left(\sum_{m\in\mathbb{Z}^n} |a_m|^p\right)^{1/p}$$

is finite.

- The Fourier transform $\widehat{\cdot}: \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx,\ \xi \in \mathbb{R}^n. \tag{2}$$

Then, Fourier transform $f \longrightarrow \widehat{f}$ is an isomorphism of $\mathscr{S}(\mathbb{R}^n)$ into $\mathscr{S}(\mathbb{R}^n)$ and the inverse Fourier transform is given by

$$f^{\vee}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)\, d\xi,\ x \in \mathbb{R}^n. \tag{3}$$

Also, this Fourier transform can be uniquely extended to $\mathbb{F} : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$. For details, see [35, 13].

- For $s \in \mathbb{C}$, we define operators $(I-\Delta)^{s/2} : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ by

$$(I-\Delta)^{s/2} f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} (1+|\xi|^2)^{s/2} \hat{f}(\xi)\, d\xi.$$

Finally, L^p_s will denote the Sobolev space of all $f \in \mathscr{S}'(\mathbb{R}^n)$ for which $(I-\Delta)^{s/2} f$ is a function and

$$\|u\|_{L^p_s(\mathbb{R}^n)} = \| (I-\Delta)^{s/2} f\|_{L^p(\mathbb{R}^n)},\ \ (1 \le p < \infty,\ \text{and } s \in \mathbb{R}),$$

is finite.

Chapter 1

Introduction and Preliminaries

The aim of the first section of this chapter is to introduce the Fourier integral operators and gather some basic properties of these operators. In section 1.2, we introduce the wave equation and discuss some basic questions concerning it. In the last section, we briefly describe the spherical maximal operator and the maximal function along hypersurfaces.

1.1 Fourier Integral Operators

The theory of Fourier integral operators was developed by Hörmander in [18]. In this work, we study the smoothing property of a certain class of Fourier integral operators, which take functions on $\mathbb{R}^2$ to functions on $\mathbb{R}^3$. Fourier integral operators are important for many different applications, in particular in a variety of problems arising in partial differential equations. They are a natural generalization of pseudo-differential operators which are important for the applications to the theory of elliptic differential equations. On the other hand, Fourier integral operators have been used in the solution to problems involving hyperbolic partial differential equations. In the last several years, Fourier integral operators have become an important tool in the study of dilated maximal operators. It is hoped thus to convey an idea of how the classical theory of Fourier integral operators fits into contemporary developments in the smoothing estimates for the solutions of the wave equation and the L^p-boundedness of the circular maximal operator.

In the following section, we give precise definitions of symbol class, the wave front set of a distribution and Fourier integral operators. We also discuss the regularity theory of Fourier integral operators and some of its basic properties.

1.1.1 The Symbol Class

In the following definition, we consider complex valued functions a defined in $X \times \mathbb{R}^N$, where X is an open subset of $\mathbb{R}^n$ with n possibly different from N. The following definition is from [18, Definition 1.1.1].

Definition 1.1.1 Let m be a fixed real number. We denote by $S^m(X \times \mathbb{R}^N)$, the symbol class of order m, the set of all $a \in C^\infty(X \times \mathbb{R}^N)$ such that for every compact set $K \subset X$ and all multi-indices α and β, the estimate

$$\left|\partial_x^\beta \, \partial_\xi^\alpha a(x,\xi)\right| \leq C_{\alpha,\beta,K}\,(1+|\xi|)^{m-|\alpha|}\,,\ x \in K,\ \xi \in \mathbb{R}^N \tag{1.1}$$

holds for some constant $C_{\alpha,\beta,K}$.

The elements of $S^m(X \times \mathbb{R}^N)$ are called the symbols of order m. We also use the notation $S^m(\mathbb{R}^N)$ when the symbols are independent of x. We sometimes write only S^m and talk about symbols of order m. S^m is non empty for each m, as $a(\xi) = (1+|\xi|^2)^{m/2} \in S^m$, the symbol class of order m. Note that S^m is a Fréchet space with the topology defined by taking as seminorms the best constant $C_{\alpha,\beta,K}$, which can be used in (1.1). We refer the reader to [18] for more details about the symbols.

1.1.2 Oscillatory Integrals

In this section, we shall discuss oscillatory integrals of the form

$$I_\phi(au) = \int_X \int_{\mathbb{R}^N} e^{i\phi(x,\xi)}\, a(x,\xi)\, u(x)\, dx\, d\xi, \quad u \in C_c^\infty(X),\ X \subset \mathbb{R}^n \tag{1.2}$$

where $a(x,\xi) \in S^m(X \times \mathbb{R}^N)$, the symbol class of order $m \in \mathbb{R}$, and the phase function ϕ is a real valued positively homogeneous of degree 1 with respect to the ξ variable and that $\phi \in C^\infty(X \times \mathbb{R}^N \setminus \{0\})$. We shall need the following definition to make sense of the integral (1.2) with arbitrary $a \in S^m$.

Definition 1.1.2 We say that a phase function ϕ defined in a neighborhood of a point (x_0, ξ_0) has (x_0, ξ_0) as a critical point if

$$(\nabla_\xi \phi)(x_0, \xi_0) = \left(\frac{\partial \phi}{\partial \xi_1}, \dots, \frac{\partial \phi}{\partial \xi_n} \right) \Bigg|_{(x,\xi)=(x_0,\xi_0)} = 0.$$

Since $a \in S^m(X \times \mathbb{R}^N)$ and $u \in C_c^\infty(X)$, the integral (1.2) is absolutely convergent for every $a \in S^m(X \times \mathbb{R}^N)$ provided that $m + N < 0$. In particular, it is well defined if $a(x,\xi) \in C^\infty(X \times \mathbb{R}^N)$ and $a(x,\xi) = 0$ for $|\xi| > 1$. We will extend the definition of (1.2) to arbitrary $a \in S^m(X \times \mathbb{R}^N)$, for all m (see Proposition 1.1.4). We will see that the definition of (1.2) is always possible if ϕ has no critical point with $\xi \neq 0$. This depends on an integration by parts argument in (1.2). The following lemma (see, [18, Lemma 1.1.3]) is crucial to apply integration by parts formula.

Lemma 1.1.3 *If ϕ has no critical point (x,ξ) with $\xi \neq 0$, then one can find a first order differential operator*

$$L = \sum a_j \frac{\partial}{\partial \xi_j} + \sum b_j \frac{\partial}{\partial x_j} + c$$

with $a_j \in S^0(X \times \mathbb{R}^N)$ and $b_j, c \in S^{-1}(X \times \mathbb{R}^N)$ such that $L^\dagger e^{i\phi} = e^{i\phi}$ where $L^\dagger$ is the adjoint of L.

The proof of Lemma 1.1.3 follows from [18, Lemma 1.2.1]. Now, if $a(x,\xi) = 0$ for large $|\xi|$, we can integrate by parts in (1.2) after replacing $e^{i\phi}$ by $L^\dagger e^{i\phi}$. This gives

$$I_\phi(au) = \int \int e^{i\phi(x,\xi)} \, L \left[a(x,\xi) \, u(x) \right] \, dx \, d\xi,$$

and hence after iteration, we get

$$I_\phi(au) = \int \int e^{i\phi(x,\xi)} \, L^k \left[a(x,\xi) \, u(x) \right] \, dx \, d\xi, \;\; k = 0, 1, 2, \dots \tag{1.3}$$

Now, L is a continuous map of $S^m(X \times \mathbb{R}^N)$ into $S^{m-1}(X \times \mathbb{R}^N)$ for the topology induced by $S^m(X \times \mathbb{R}^N)$. Hence, L^k maps $S^m(X \times \mathbb{R}^N)$ continuously into $S^{m-k}(X \times \mathbb{R}^N)$. Thus, if $m - k < -N$, the integral (1.3) is absolutely convergent on all of $S^m(X \times \mathbb{R}^N)$. The following Proposition is a restatement of [18, Proposition 1.2.2], which says that the integral (1.2) is convergent for $a \in S^m(X \times \mathbb{R}^N)$, for all m if ϕ has no critical point with $\xi \neq 0$.

Proposition 1.1.4 *If ϕ has no critical points, then the definition of the integral (1.2) can be extended in one and only one way to arbitrary $a \in S^m(X \times \mathbb{R}^N)$, for all m and $u \in C_c^\infty(\mathbb{R}^n)$ so that $I_\phi(au)$ is continuous function of $a \in S^m$ for every fixed m. The linear form $A : u \longrightarrow I_\phi(au)$ defines a distribution of order $\leq k$ if $a \in S^m$ and $m - k < -N$.*

The interested reader may find a proof of above Proposition in the aforementioned paper [18, Proposition 1.2.2].

1.1.3 The wave front set of a distribution

In this section, we give the definition of the wave front set of a distribution and compute the wave front sets of some distribution. If $u \in D'(\mathbb{R}^n)$ (compactly supported distribution) then sing supp u, the singular support of u, is the set of all $x \in \mathbb{R}^n$ such that x has no open neighborhood on which the restriction of u is C^∞. The wave front set of a distribution is a refinement of the notion of singular support of a distribution, which also carries the information on the direction along which the singularity exists. The following definition is from [9, Definition 1.3.1].

Definition 1.1.5 Let $u \in D'(X)$, X open in $\mathbb{R}^n$, then the wave front set of u, denoted by $WF(u)$ is defined as the complement of largest open cone in $X \times (\mathbb{R}^n \setminus \{0\})$ of the collection of all $(x_0, \xi_0) \in X \times (\mathbb{R}^n \setminus \{0\})$ such that the Fourier transform inequality

$$|\widehat{\varphi u}(\xi)| \leq C_N \left(1 + |\xi|\right)^{-N} \tag{1.4}$$

holds for all $\varphi \in C_c^\infty$ in a neighborhood U of x_0 and for all ξ in a conic neighborhood V of ξ_0 for all $N \in \mathbb{N}$.

We refer the reader to [9] for more on the wave front set of a distribution. Let us give some simple example of wave front set of a distribution.

1.1.4 Example

1. In $\mathbb{R}$, let $u(x) = \delta_0(x)$, the Dirac delta distribution. The singular support of $\delta_0(x)$ is $\{0\}$ and as a distribution $\widehat{\phi\delta_0}(\xi) = \phi(0)$, where ϕ is a smooth

function supported in some neighborhood of the origin. Thus, $\widehat{\phi\delta_0}(\xi) = \phi(0)$ is not decreasing and this proves that

$$WF(u) = \{(0, \xi) : \xi \in \mathbb{R} \setminus \{0\}\}.$$

2. In $\mathbb{R}$, let $u(x) = (x + i0)^{-1} = \lim_{\epsilon \longrightarrow 0^+}(x + i\epsilon)^{-1} = p.v.\ \frac{1}{x} - \pi i \delta_0(x)$. Then, u has Fourier transform equal to $-2\pi i \chi_{[0,\infty)}$. Thus,

$$\widehat{\phi u}(\xi) = \hat{\phi} * \hat{u}(\xi) = -\int_{-\infty}^{\xi} \hat{\phi}(\eta)\, d\eta,$$

where $\phi \in C_c^\infty(\mathbb{R})$, the support of ϕ is in some neighborhood of the origin, $\phi(0) \neq 0$. Then $\widehat{\phi u}(\xi)$ tends to $-2\pi i \phi(0) \neq 0$ for $\xi \to \infty$.

Let $F(\xi) = -\int_{-\infty}^{\xi} \hat{\phi}(\eta)\, d\eta$. It is well known that the Fourier transform of a test function ϕ is rapid decreasing i.e., for any integer $\mathbb{N}$, there is a constant C_N for which $|\hat{\phi}(\eta)| \leq C_N(1 + |\eta|)^{-N}$. Thus, for $\xi < 0$,

$$\begin{aligned} |F(\xi)| &= \int_{-\infty}^{\xi} C_N(1 + |\eta|)^{-N}\, d\eta \\ &= \int_{-\xi}^{\infty} C_N(1 + \eta)^{-N}\, d\eta = \frac{C_N}{N-1}(1 - \xi)^{(1-N)} \end{aligned}$$

is rapidly decreasing and for any $\xi > 0$,

$$|F(\xi)| \leq \int_{-\infty}^{\infty} C_N(1 + |\eta|)^{-N}\, d\eta = \frac{2C_N}{N-1}.$$

This proves that

$$WF(u) = \{(0, \xi) : \xi > 0\}.$$

1.1.5 Definition of Fourier Integral Operators

Given a function $a(x, \xi) \in S^m(X \times \mathbb{R}^n)$, the symbol class of order $m \in \mathbb{R}$, and a function $\phi(x, \xi) \in C^\infty(X \times \mathbb{R}^n \setminus \{0\})$, real valued positively homogeneous of degree one in the ξ variable, we consider a Fourier integral operator F_a^ϕ acting a-priori on Schwartz class functions f, by setting

$$F_a^\phi f(x) = \int_{\mathbb{R}^n} e^{i\phi(x,\xi)}\, a(x, \xi)\, \hat{f}(\xi)\, d\xi, \;\; X \subset \mathbb{R}^n.$$

The function $a : X \times \mathbb{R}^n \to \mathbb{C}$ is called the amplitude of F_a^ϕ and $\phi : X \times \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}$ is called the phase function. We refer the reader to [18] for more details about Fourier integral operators.

We always assume that the phase function ϕ satisfies the following crucial non-degeneracy condition: for $\xi \neq 0$,

$$det\left(\frac{\partial^2\phi}{\partial x_i \partial \xi_j}\right) \neq 0$$

on the support of the amplitude function a. We are interested in the regularity property of such operators. It turns out that the treatment of such operator requires two complementary analysis. The first is in terms of their kernel K, which are given by

$$K(x,y) = \int_{\mathbb{R}^n} e^{i[\phi(x,\xi)-y\cdot\xi]}\, a(x,\xi)\, d\xi.$$

A key aspect of Fourier integral operators is that these operators have kernel whose singular support is not limited to the diagonal unlike pseudo-differential operators. We observe that for fixed x, the singular support of the kernel $K(x,y)$ is contained in $\Sigma_x = \{y : y = \nabla_\xi \Phi(x,\xi)\}$ for some ξ, that is, where the phase function $\xi \to \phi(x,\xi) - y\cdot\xi$ has a critical point (see, [18, Theorem 1.4.1], [38]). One of the simplest examples is given by the phase functions $\phi_\pm(x,t,\xi) = x\cdot\xi \pm t|\xi|$, arising in the solution to the wave equation.

The second way of looking at the operator comes from decomposing the frequency space, that is the ξ - space. In particular, dyadic decomposition is important in our analysis of Fourier integral operators, see chapter 3. We also need a further decomposition in the ξ- space in terms of angular variable, namely, each dyadic shell $2^{j-1} \leq |\xi| \leq 2^{j+1}$, $j \geq 0$, is split into approximately $2^{j(n-1)/2}$ thin sectors, corresponding to truncated cones of aperture $\approx 2^{-j/2}$, see Section 2.1.2, Chapter 2, for such decomposition in dimension $n = 2$. In chapter 3, we exploit both points of view to proof our main result (Theorem 3.1.1, Chapter 3).

1.1.6 Known Regularity Results

In this section, we discuss the known regularity results of Fourier integral operators which send functions of n variables to functions of $n+1$ variables. Such operators are of the form

$$Ff(x,t) = \int_{\mathbb{R}^n} e^{i\phi(x,t,\xi)}\, a(x,t,\xi)\, \hat{f}(\xi)\, d\xi,\ \ f \in \mathscr{S}(\mathbb{R}^n)$$

where $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, the amplitude function $a \in C^\infty$ is in a suitable symbol class, and the phase function $\phi \in C^\infty(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \setminus \{0\})$ is real-valued and homogeneous of degree one in the ξ variable.

A natural question related to such operators is regarding the regularity: How does the smoothness of Ff compare to that f. The answer to this question depends on how we treat the time variable t. First, if we fix t and measures the smoothness of Ff only in the x variable, then the study of the regularity of Fourier integral operators in L^2 spaces goes back, to the pioneering work of Hörmander [18]. He proved an L^2_{loc} estimate when the amplitude $a \in S^0$, the symbol class of order 0. For $1 < p < \infty$, the optimal L^p_{loc} boundedness of Fourier integral operator Ff is due to A. Seeger et al. [31]. They proved that the operator Ff is bounded on $L^p_{loc}(\mathbb{R}^n)$ for $1 < p < \infty$ provided that the symbol of the operator belongs to $S^m(\mathbb{R}^n \times \mathbb{R}^n)$, the symbol class of order $m \le -(n-1)|\frac{1}{p} - \frac{1}{2}|$. The sharpness of the order $-(n-1)|\frac{1}{p} - \frac{1}{2}|$ was shown by Miyachi [27] and Peral [30], (see also [31]).

On the other hand, if one treats t as a variable, Ff may exhibit additional smoothing. Bourgain proved such an estimate for the circular averaging operator; in that setting any additional smoothing is enough to imply $L^p \to L^p$ bounds on the circular maximal operator. Improved smoothing estimates for the Fourier integral operators were first proved by Sogge [34]. He made an interesting observation: If one integrate the operator Ff over the space variables, then there is some smoothing, while integrating over both space and time variables leads to increased smoothing. This phenomenon was proved for the solution operators to the wave equation (see, [34, 28]). This motivates us to initiate the study of the smoothing problem of Fourier integral operators.

In this work, we begin to examine what smoothing estimates are possible for the Fourier integral operators with phase function $\phi(x,t,\xi) = x \cdot \xi + t|\xi|$ and amplitude function $a \in S^m$, the symbol class of order $m \le 0$. Here, we restrict ourselves to dimension $n = 2$. Investigating and answering these questions is, precisely, the topic of interest, and constitute the main part of this dissertation, and we will return to these issues in chapter 3.

1.2 The Wave Equation

Consider the Cauchy problem for the wave equation on $\mathbb{R}^n$:

$$\begin{cases} (\partial_t^2 - \Delta)\, u(x,t) = 0, \\ u(x,0) = f(x), \partial_t u(x,0) = g(x), \end{cases} \tag{1.5}$$

where $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ and $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ with f and g are complex valued function on $\mathbb{R}^n$.

For $f, g \in \mathscr{S}(\mathbb{R}^n)$; taking the Fourier transform with respect to space variable x in (1.5), we obtain

$$\begin{cases} \partial_t^2\, \hat{u}(\xi,t) = -|\xi|^2\, \hat{u}(\xi,t), \\ \hat{u}(\xi,0) = \hat{f}(\xi) \text{ and } \partial_t\, \hat{u}(\xi,0) = \hat{g}(\xi). \end{cases}$$

Hence, the solution of this family of ordinary differential equations, with parameter ξ, can be written as

$$\hat{u}(\xi,t) = \left[\frac{\hat{f}(\xi)}{2} + \frac{\hat{g}(\xi)}{2i|\xi|}\right] e^{it|\xi|} + \left[\frac{\hat{f}(\xi)}{2} - \frac{\hat{g}(\xi)}{2i|\xi|}\right] e^{-it|\xi|},$$

and then taking inverse Fourier transform, we write the solution $u(x,t)$ of (1.5) as $u(x,t) = \mathcal{F}_t f(x) + \mathcal{G}_t g(x)$, where

$$\mathcal{F}_t f(x) = (2\pi)^{-2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \cos(t|\xi|) \hat{f}(\xi) d\xi$$

and

$$\mathcal{G}_t g(x) = (2\pi)^{-2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{\sin(t|\xi|)}{|\xi|} \hat{g}(\xi) d\xi. \tag{1.6}$$

These multipliers can be expressed in terms of the oscillatory integrals of the form

$$\mathcal{F}h(x,t) = (2\pi)^{-2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{it|\xi|}\, a(\xi)\, \hat{h}(\xi)\, d\xi, \tag{1.7}$$

with amplitude function $a(\xi) = 1$ or $1/|\xi|$.

Therefore, a natural question arises at this point: To what extent is the following true: (assume $g = 0$ here)

$$\|u(x,t)\|_{L^p(\mathbb{R}^n)} \leq B\|f\|_{L^p(\mathbb{R}^n)}, \ f \in \mathscr{S}(\mathbb{R}^n)?$$

The case $p = 2$ is related to the well known "law of conservation of energy". The regularity property of the solution operators $\mathcal{F}_t$ and $\mathcal{G}_t$ has been studied by J. Peral and A. Miyachi on $\mathbb{R}^n$, $n \geq 2$. The result of Peral and Miyachi gives the optimal regularity for the solution $u(x,t)$ of (1.5) for each fixed t, (see, [30, 27]). On the other hand, we investigate the smoothing property of solutions to the Cauchy problem for the wave equation in dimension $n = 2$, and we will return to these issues in chapter 4.

1.3 A Brief Overview of Some Maximal Operators

In this section, we briefly discuss the spherical maximal operator, the maximal function along hypersurfaces and discusses some of its L^p- mapping property.

1.3.1 Spherical Maximal Operator

Given a function f, continuous and compactly supported, the spherical mean

$$A_t f(x) = \int_{\mathbb{S}^{n-1}} f(x - ty)\, d\sigma(y), \ x \in \mathbb{R}^n \text{ and } t > 0,$$

is well-defined for every $r > 0$, where $d\sigma$ denotes the normalized Lebesgue measure over the unit sphere $\mathbb{S}^{n-1}$. The operator $A_t f(x)$ is the mean value of f over the sphere of radius t centered at x and it defines a bounded operator on $L^p(\mathbb{R}^n)$ for $1 \leq p \leq \infty$. Consider now the spherical maximal operator given by

$$\mathcal{M}f(x) = \sup_{t>0} |A_t f(x)|. \tag{1.8}$$

Then, spherical maximal operator $\mathcal{M}$ defines a bounded operator on $L^p(\mathbb{R}^n)$ if and only if $p > \frac{n}{n-1}$ with $n > 1$.

This result was first proved by Stein [36], for $n \geq 3$. Stein's proof of the spherical maximal theorem for $n \geq 3$ exploit the curvature via the decay of the Fourier transform of the surface measure on the sphere and a g- function argument. In the case of the sphere, the Fourier transform decays like $|\xi|^{-\frac{n-1}{2}}$ at infinity (see, [22]). The decay estimates are weaker for surfaces with flat directions, which is reflected in the range of exponents p in maximal and averaging estimates. For the original proofs of the above result, we refer the reader to the aforementioned paper. The result was extended to $L^p(\mathbb{R}^n)$ by Stein and Wainger [39]. A consequence of this is that if $f \in L^p(\mathbb{R}^n)$, where $\frac{n}{n-1} < p \leq \infty$, then the spherical means $A_t f(x)$ tends to $f(x)$ almost everywhere, as t tends to 0.

The 2-dimensional version of the spherical maximal operator was proved by Bourgain [1]. He proved that $\|\mathcal{M}f\|_{L^p(\mathbb{R}^2)} \leq \|f\|_{L^p(\mathbb{R}^2)}$, for $2 < p \leq \infty$. An alternative approach to the result for circular maximal operators has been devised by Mockenhaupt et al. [28]. Since the averaging operators $A_t f$ can be expressed as Fourier integral operators, the authors of [28] develop a more sophisticated theory of local smoothing estimates of Fourier integral operators, so that they can improve many estimates for maximal operators. These local smoothing estimates for Fourier integral operators have also other applications, one of which is to deal with a special class which contains the solution operator for the Cauchy problem associated to the wave equation and we will return to these issues in chapter 4 and chapter 5.

1.3.2 Maximal Function along Hypersurfaces

In this section, we study slightly different maximal operator on $\mathbb{R}^{n+1}$ compared to the dilated maximal operator. We investigate the L^p boundedness for maximal operators along a class of hypersurfaces given by the graph of a function. Let $S = \Phi^{-1}(0)$ be a hypersurfaces in $\mathbb{R}^{n+1}$ given by a function $\Phi \in C^1(\mathbb{R}^{n+1})$ with $\nabla\Phi \neq 0$. We write the elements in $\mathbb{R}^{n+1}$ as $(x, x_0) \in \mathbb{R}^n \times \mathbb{R}$ and for simplicity, we may assume that Φ is of the form

$$\Phi(x, x_0) = h(x) - x_0,$$

where h is a non negative C^1 function on $\mathbb{R}^n$. In this case, S is the graph of h, and we assume that $h(0) = 0$, and $\nabla h(x) \neq 0$ for all $x \in S_r$, where

$$S_r = \{x \in \mathbb{R}^n : h(x) = r\} \tag{1.9}$$

is the level set of the function h, at height $r > 0$ and each S_r is a compact C^1 hypersurfaces in $\mathbb{R}^n$. For $r \geq 0$, let $\Sigma_r = \{\tilde{x} = (x, x_0) \in S : 0 \leq x_0 \leq r\}$ and for $f \in \mathscr{S}(\mathbb{R}^{n+1})$, consider the maximal operator

$$Mf(\tilde{x}) = \sup_{r>0} \frac{1}{\mu(\Sigma_r)} \int_{\Sigma_r} |f(\tilde{x} - \tilde{y})| \, d\mu(\tilde{y}), \tag{1.10}$$

where μ denotes the surface measure on S induced by the Lebesgue measure on $\mathbb{R}^{n+1}$.

In the last several years, considerable attention has been given to the study of maximal operators along surfaces and curves, (see, [10, 21, 38, 39] and references therein). Note that our maximal operator is closer in spirit to the maximal operators studied in [3, 10] which answers a problem posed by Stein and Wainger in [39] and refers to hypersurfaces in $\mathbb{R}^3$ given by the graph of $\phi(s,t) = |s|^\alpha \, |t|^\beta$ on $\mathbb{R}^2$. This has been extended to general quadratic surfaces in [4] and to more general hypersurface that arise as a graph of certain function of the form $\Gamma(|x|, |y|)$ on $\mathbb{R}^n \times \mathbb{R}^m$ in [21]. These maximal operators are called the multi-parameter maximal functions, as they correspond to averages over images of rectangles on the surface. This also leads to a similar problem for maximal operators by averaging over images of balls.

The maximal operator (1.10) that we consider here on the other hand, is obtained from averages over the portion of the hypersurface upto 'height' r i.e., Σ_r as in (1.10). In fact, for the hypersurfaces with spherically symmetric 'cross section' S_r given by (1.9), we answer the above mentioned problem for maximal operators along hypersurfaces in $\mathbb{R}^{n+1}$, obtained by averaging over images of balls in $\mathbb{R}^n$. We prove the L^p boundedness of the maximal operator (1.10) for a wide class of hypersurfaces that arise as the graph of a $\mathbb{C}^1$ function and we will return to these problems in chapter 6.

Chapter 2

Some Oscillatory Integral Estimates Related to Fourier Integral Operators

The aim of this chapter is to introduce some basic oscillatory integral estimates. We also introduce the angular decomposition of the frequency space and gather some basic properties of the kernels of the Fourier integral operators which will be needed in the later chapters. In the last section, we explicitly construct a partition of unity based on angular decomposition.

2.1 Introduction

We start with examining the behavior of the kernels of the Fourier integral operators of the form

$$\mathcal{F}f(x,t) = \rho_1(t) \int_{\mathbb{R}^2} e^{i(x\cdot\xi + t|\xi|)}\, a(\xi)\, \hat{f}(\xi)\, d\xi,\ \ f \in \mathscr{S}(\mathbb{R}^2), \tag{2.1}$$

where $\rho_1 \in C_c^\infty((1,2))$ and the amplitude function $a \in S^0(\mathbb{R}^2)$, the symbol class of order 0. Let K^a be the kernel of $\mathcal{F}f$, given by

$$K^a(x,t) = \rho_1(t) \int_{\mathbb{R}^2} e^{i(x\cdot\xi + t|\xi|)}\, a(\xi)\, d\xi.$$

The proof of local smoothing estimates (Theorem 3.1.1, Chapter 3) involved explicit expressions for the kernels of such operators (2.1), which provided an estimate for boundedness on $L^4(\mathbb{R}^2)$ and $L^\infty(\mathbb{R}^2)$. Thus, in this chapter, we concentrate on the point-wise estimates of the kernels that we require in the next chapter.

We first discuss the dyadic decomposition of the dual space (the ξ-space) that is needed to prove the various oscillatory integral estimates.

2.1.1 The Dyadic Decomposition

The proof of our main result makes use of the division of the dual (frequency) space into dyadic shells. The idea of dyadic decomposition was originated in the work of Littlewood and Paley. We will now describe this decomposition as follows: Let ρ be a non negative radial function in $C_c^\infty(\mathbb{R}^2)$ supported in $\{\frac{1}{2} \leq |\xi| \leq 2\}$ such that

$$\sum_{j=-\infty}^{\infty} \rho(2^{-j}\xi) = 1 \text{ for } \xi \neq 0.$$

For example, we shall take,

$$\Phi(\xi) = \begin{cases} 1, & \text{if } |\xi| \leq \frac{1}{2} \\ 0, & \text{if } |\xi| \geq 1. \end{cases}$$

and

$$\rho(\xi) \ = \ \Phi(\frac{\xi}{2}) - \Phi(\xi).$$

Then, one can easily see that $\sum\limits_{j\in\mathbb{Z}} \rho(2^{-j}\xi) \ = \ 1$ for $\xi \neq 0$ (see [11]). Setting $\phi_0 = \sum_{j\leq 0} \rho(2^{-j}|\xi|)$, we can write $1 = \phi_0 + \sum_{j\in\mathbb{N}} \rho(2^{-j}|\xi|)$, where ϕ_0 is compactly supported. Thus, for each $j \in \mathbb{N}$, we set

$$K_j^a(x,t) = \rho_1(t) \int_{\mathbb{R}^2} e^{i(x\cdot\xi + t|\xi|)}\, a(\xi)\, \rho(2^{-j}|\xi|)\, d\xi.$$

Now we prove some auxiliary estimates required in the proof of the L^p boundedness of Fourier integral operator, see chapter 3. We need to do a further decomposition of the dual space with respect to the angular variable. Such decomposition has been used in [31]. Here we will be a bit more specific to our application and also get some relevant estimates easily by a more geometric approach.

2.1.2 Angular Decomposition

In this section, we discuss the second dyadic decomposition of the ξ- space with respect to the angle. First, for fixed integer $j \geq 1$, let $N = N(j) = 4\,[2^{j/2}]$ where $[x]$ denotes the largest integer less than or equal to x. We now choose N equally spaced points $\xi_0, \xi_1, \ldots, \xi_{N-1}$ on the unit circle $S^1 = \{\xi \in \mathbb{R}^2 : |\xi| = 1\}$ with $\xi_0 = e_1$. In fact, we take $\xi_\nu = O\xi_0$ for $1 \leq \nu \leq N-1$, where O is the rotation in counter clockwise direction by an angle $2\pi\nu/N$. Notice that the distance between two consecutive points ξ_ν and $\xi_{\nu'}$, is given by

$$|\xi_\nu - \xi_{\nu'}| \approx 2\sin(\pi/N) \approx 2^{-\frac{j}{2}}, \tag{2.2}$$

as $N = N(j) \approx 2^{\frac{j}{2}}$.

Let $\mathcal{C}_0$ be the arc in S^1 given by $|\theta| \leq 2\pi/N$. Let ψ be a smooth function on the unit circle S^1 with support $= \mathcal{C}_0$, such that $0 \leq \psi(\xi) \leq 1$ and $\psi(\xi_0) = 1$. Note that ψ defines a homogeneous function on $\mathbb{R}^2 \setminus \{0\}$ supported on the sector given by $|\theta| \leq 2\pi/N$.

With $N = N(j)$ as above, we can construct a homogeneous partition of unity $\{\chi_\nu\}_{\nu=0}^{N-1}$ on $\mathbb{R}^2 \setminus \{0\}$ with the following properties:

$$\chi_0(\xi) = \psi(\theta),\ \chi_\nu(\xi) = \chi_0(O^{-1}\xi),\ 1 \leq \nu \leq N-1, \tag{2.3}$$

where $\xi/|\xi| = (\cos\theta, \sin\theta)$ and O is a rotation by the angle $2\pi\nu/N$ and

$$|\partial_{\xi_1}^k \chi_0(\xi)| \leq C_k,\ |\partial_{\xi_2}^k \chi_0(\xi)| \leq C_k N^k \approx C_k\, 2^{\frac{jk}{2}} \text{ for } |\xi| = 1, \tag{2.4}$$

for all $k \in \mathbb{N}$, with constant C_k is independent of N (hence independent of j). An explicit construction of such a partition of unity is carried out in Section 3.

Note that χ_ν as a homogeneous function on $\mathbb{R}^2$ is supported on an angular sector given by $|\theta - \theta_\nu| \leq 2\pi/N$, where $\theta_\nu = 2\pi\nu/N$. Using the homogeneous partitions of unity $\{\chi_\nu\}_\nu$, we set

$$K_{j,\nu}^a(x,t) = \rho_1(t) \int_\xi e^{i(x\cdot\xi + t|\xi|)}\, \rho(2^{-j}|\xi|)\, a(\xi)\, \chi_\nu(\xi)\ d\xi, \tag{2.5}$$

for $j \geq 1$ and $0 \leq \nu \leq N-1$.

2.2 Some Kernel Estimates

The following kernel estimate is a crucial ingredient for various L^p estimates for $\mathcal{F}f$, see chapter 3. This result is essentially in the spirit of the result in Section 4.5, Ch. 9 in [38], for $\phi(x,t,\xi) = x \cdot \xi + t|\xi|$.

Lemma 2.2.1 *Let $K^a_{j,\nu}$ be as in (2.5) and ξ_ν be as in (2.2). Then $K^a_{j,\nu}$ satisfies the inequality*

$$|K^a_{j,\nu}(x,t)| \le C\, 2^{3j/2} \rho_1(t)\, \Psi_j(Tx + te_1),$$

where $\Psi_j(x) = \left[1 + 2^{2j}|x_1|^2\right]^{-k} \left[1 + 2^j|x_2|^2\right]^{-k}$, $k \in \mathbb{N}$ with a constant $C = C_k$ independent of a, j and ν, and $T \in SO(2)$ is such that $T\xi_\nu = e_1, 0 \le \nu \le N-1$.

Proof. We first consider the case $\xi_\nu = \xi_0 = e_1$ and estimate $K^a_{j,0}(x,t)$ by oscillatory integral techniques as in [31]. By (2.5),

$$K^a_{j,0}(x,t) = \rho_1(t) \int_\xi e^{i(x\cdot\xi + t|\xi|)}\, \rho(2^{-j}|\xi|)\, a(\xi)\, \chi_0(\xi)\, d\xi. \tag{2.6}$$

Let L_j be the differential operator $\left(I - 2^{2j}\partial^2_{\xi_1}\right)\left(I - 2^j \partial^2_{\xi_2}\right)$, so that

$$L^k_j\, e^{i(x+te_1)\cdot\xi} = \left[1 + 2^{2j}|x_1 + t|^2\right]^k \left[1 + 2^j|x_2|^2\right]^k e^{i(x+te_1).\xi}, \;\; k \in \mathrm{N}\,.$$

Re writing $e^{i(x\cdot\xi+t|\xi|)}$ as $e^{it(|\xi|-\xi_1)}\, e^{i(x+te_1)\cdot\xi}$ and using the above formula, we get

$$e^{i(x\cdot\xi+t|\xi|)} = \left[1 + 2^{2j}|x_1 + t|^2\right]^{-k} \left[1 + 2^j|x_2|^2\right]^{-k} e^{it(|\xi|-\xi_1)}\; L^k_j e^{i(x+te_1)\cdot\xi}.$$

Using this formula in (2.6) and an integration by parts shows that

$$K^a_{j,0}(x,t) = \Psi_j(x + te_1)\, A^a_{j,0}(x,t) \tag{2.7}$$

where $\Psi_j(x) = \left[1 + 2^{2j}|x_1|^2\right]^{-k} \left[1 + 2^j|x_2|^2\right]^{-k}$ and

$$A^a_{j,0}(x,t) = \int_\xi e^{i[x\cdot\xi + t\xi_1]}\; L^k_j \left[e^{it(|\xi|-\xi_1)}\, \rho(2^{-j}|\xi|)\, a(\xi)\, \chi_0(\xi)\,\right] d\xi. \tag{2.8}$$

Note that the integrand in (2.8) is supported on the set

$$E = \operatorname{supp} \chi_0 \cap \{\xi : 2^{j-1} \le |\xi| \le 2^{j+1}\}.$$

To complete the proof for $\nu = 0$, we need to show that $|A^a_{j,0}(x,t)| \leq 2^{3j/2}$. This will follow from (2.8) once we verify the following;

- The measure of E is bounded by a constant times $2^{3j/2}$
- $L^k_j \left[e^{it(|\xi|-\xi_1)} \rho(2^{-j}|\xi|)\, a(\xi)\, \chi_0(\xi) \right]$ is bounded uniformly in j.

The first part is clear, since $|\xi_2| \leq \xi_1 \sin(\pi/N) \leq 2\pi 2^{j/2}$ and $\xi_1 \leq 2^{j+1}$ on E.

For the second part, we observe that L^k_j is a linear combination of various derivatives $(2^j \partial_{\xi_1})^{k_1}(2^{j/2}\partial_{\xi_2})^{k_2}$ with $k_1 + k_2 \leq 4k$. Note that each of the above derivative of the functions $\rho(2^{-j}|\xi|)$ and $a(\xi)$ are uniformly bounded in j. Also in view of (2.4), and the fact that $\chi_0(\xi)$ is homogeneous of degree zero, the above derivatives of $\chi_0(\xi)$ are also uniformly bounded in $j \in \mathbb{N}$. Since the integration is actually on the sector E, all the above derivatives applied to $e^{it(|\xi|-\xi_1)}$ also give functions bounded uniformly in j, by Lemma 2.2.3 below.

To estimate $A^a_{j,\nu}$ for general ν, first note that $\chi_\nu(\xi) = \chi_0(O^{-1}\xi)$ by (2.3) where $O \in SO(2)$ is such that $\xi_\nu = Oe_1$. Thus using the change of variable $\xi \to O\xi$ in (2.5), we see that $K^a_{j,\nu}(x,t) = K^{a^\nu}_{j,0}(O^{-1}x,t)$ where $a^\nu(\xi) = a(O\xi)$. Notice that the estimate for $|A^{a^\nu}_{j,0}(x,t)|$ depends on the derivatives of $a^\nu = a \circ O$ which has the same bound as a. Hence the proof follows with $T = O^{-1}$. $\square$

Remark 2.2.2 In the above Lemma, we only considered the critical case $m = 0$. In fact when $a \in S^m(\mathbb{R}^2)$ then we have the estimate

$$|K^a_{j,\nu}(x,t)| \leq C\, 2^{jm}\, 2^{3j/2} \rho_1(t)\, \Psi_j(Tx + te_1)$$

and this follows since $|\partial^\alpha_\xi a| \leq C|\xi|^{m-|\alpha|}$, $|\alpha| \geq 0$ as $a \in S^m(\mathbb{R}^2)$. Since $|\xi| \approx 2^j$ on the support of $\rho_0(2^{-j}|\xi|)$, the arguments in the proof of the above lemma leads to the improved estimate in the case $m < 0$.

Lemma 2.2.3 *Let $h(\xi) = |\xi| - \xi_1$ for $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, then we have*

$$|\partial^k_{\xi_1} h(\xi)| \leq A_k\, 2^{-kj}, |\partial^k_{\xi_2} h(\xi)| \leq B_k\, 2^{-\frac{kj}{2}}, \text{ for } k \geq 1,$$

on the set $E = supp\, \chi_0 \cap \{\xi = (\xi_1,\xi_2) : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$.

Proof. Writing $\xi = |\xi|(\cos\theta, \sin\theta)$, we see that $\partial_{\xi_1} h(\xi) = \frac{\xi_1}{|\xi|} - 1 = -2\sin^2(\theta/2)$.

Since $|\sin\theta| \le |\theta|$ and $|\theta| \le 2\pi/N$ on the support of χ_0, we have

$$|\partial_{\xi_1} h(\xi)| \le \frac{2\pi^2}{N^2} \le 2\pi^2 2^{-j},\ \xi \in E$$

as $N \ge 2^{j/2}$, which proves the case $k=1$. To deal with the case $k>1$, we write $\partial_{\xi_1}^k h(\xi) = \partial_{\xi_1}^{k-1} g(\xi)$, where $g = \partial_{\xi_1} h$ which is a function homogeneous of degree zero on $\mathbb{R}^2$, hence $\partial_{\xi_1}^{k-1} g$ is homogeneous of degree $1-k$. it follows that

$$\partial_{\xi_1}^k h(\xi) = |\xi|^{1-k} (\partial_{\xi_1}^{k-1} g)(\xi/|\xi|). \tag{2.9}$$

Now, note that $g(\xi) = -2\sin^2(\theta/2) := \tilde{g}(\theta)$ as computed above and also $\partial_{\xi_1} = -\sin\theta\,\partial_\theta$ on homogeneous functions, see (2.16) in Section 3. An easy induction argument shows that $(-\sin\theta\,\partial_\theta)^{k-1}\tilde{g}(\theta) = P_k(\cos\theta)\sin^2\theta$ where P_k is a polynomial of degree $k-1$. Now for $\xi = (r\cos\theta, r\sin\theta) \in E$, we have $|\theta| \le 2\pi/N$, and hence $|P_k(\cos\theta)\sin^2\theta| \le c_k|\sin^2\theta| \le 4c_k\pi^2 N^{-2} \approx C_k 2^{-j}$ for some constant C_k independent of j. It follows from (2.9) that

$$|\partial_{\xi_1}^k h(\xi)| \le C_k 2^{-kj}$$

since $|\xi| \approx 2^j$ on E. Hence the case $k>1$.

Since $|\partial_{\xi_2} h(\xi)| = \frac{\xi_2}{|\xi|}$, the required inequality is clear on E, for $k=1$. For $k \ge 2$, note that $\partial_{\xi_2}^k h(\xi) = \partial_{\xi_2}^k |\xi|$. Since the function $g_1(\xi) = |\xi|$ is homogeneous of degree 1, these derivatives are homogeneous functions of degree $1-k$. It follows that $|\partial_{\xi_2}^k h(\xi)| \le C_k|\xi|^{1-k} \le C_k|\xi|^{-k/2}$ on E, for $k \ge 2$ and hence the required inequality holds on E. $\square$

Remark 2.2.4 We note that the kernel $K_{j,\nu}^a$ also satisfy the following point-wise estimate, which is useful in the proof of Proposition 3.6.2, Chapter 3.

$$|K_{j,\nu}^a(x,t)| \le C_k 2^{3j/2}\left[1 + 2^{2j}|\langle x, \xi_\nu\rangle + t|^2\right]^{-k}\left[1 + 2^j|\langle x, \xi_\nu^\perp\rangle|^2\right]^{-k} \tag{2.10}$$

where $\xi_\nu^\perp$ is a unit vector perpendicular to ξ_ν.

In fact, the function $\Psi_j(x)$ defining $K_{j,\nu}^a$ in Lemma 2.2.1 is a function of $2^j|x_1|$ and $2^{j/2}|x_2|$. Now for $T \in SO(2)$, $(Tx)_1 := \langle Tx, e_1\rangle = \langle x, \xi_\nu\rangle$ if $T\xi_\nu = e_1$. Similarly $(Tx)_2 = \langle x, \xi_\nu^\perp\rangle$. It follows that

$$\Psi_j(Tx + te_1) = \left[1 + 2^{2j}|\langle x, \xi_\nu\rangle + t|^2\right]^{-k}\left[1 + 2^j|\langle x, \xi_\nu^\perp\rangle|^2\right]^{-k}.$$

Hence (2.10) follows from Lemma 2.2.1 and the fact that $|\rho_1(t)| \leq 1$.

2.3 Homogeneous Partition of Unity

In this section, we construct an explicit partition of unity $\{\chi_\nu\}_{\nu=0}^{N-1}$ on $\mathbb{R}^2$ for any $N \in \mathbb{N}$, with special properties given by (2.3) and (2.4). We do this by constructing a partition of unity on $\mathbb{R}$ and transfer it to S^1 via the covering map $x \to (\cos x, \sin x), x \in \mathbb{R}$.

For $N \in \mathbb{N}$, let $\{\xi_0, \ldots, \xi_{N-1}\}$ be a set of equally spaced points on S^1. In fact, we can take $\xi_\nu = \left(\cos(\frac{2\pi\nu}{N}), \sin(\frac{2\pi\nu}{N})\right),\ \nu = 0, 1, \cdots, N-1$ as mentioned before. Notice that the Euclidean distance between the consecutive points ξ_ν and $\xi_{\nu+1}$ is $|\xi_\nu - \xi_{\nu+1}| \approx 2\sin(\pi/N),\ \nu = 0, \ldots, N-1$ with $\xi_N = \xi_0$.

Let Φ be a smooth, non negative even periodic function on $\mathbb{R}$ with period 4π, which is strictly increasing on $(-2\pi, 0)$ with $\Phi(0) = 1$ and $\Phi(2\pi) = 0$. For instance, the function $\phi \in C_c^\infty(|x| \leq 2\pi)$ given by $\phi(x) = e^{-\frac{x^2}{4\pi^2 - x^2}}$ for $|x| \leq 2\pi$, can be periodised to get one such.

Then $x \to \Phi(Nx)$ is a periodic function on $\mathbb{R}$ with period $4\pi/N$. Consider the family $\{\psi_\nu\}_{\nu \in \mathbb{Z}}$ of smooth compactly supported functions on $\mathbb{R}$ given by

$$\psi_\nu(x) = \frac{\Phi(Nx - 2\nu\pi)}{\Phi(Nx - 2\nu\pi) + \Phi(Nx - 2(\nu+1)\pi)}, \text{ for } |x - 2\nu\pi/N| \leq 2\pi/N \quad (2.11)$$

and for $\psi_\nu = 0$ for $|x - 2\nu\pi/N| > 2\pi/N$. Note that ψ_ν is a translate by $2\nu\pi/N$ of the function $\psi_0 \in C_c^\infty(|x| \leq 2\pi/N)$.

Note that $\psi_\nu \geq 0$, $\psi_\nu \in C_c^\infty\left(\left[\frac{2(\nu-1)\pi}{N}, \frac{2(\nu+1)\pi}{N}\right]\right)$ and satisfies $\psi_\nu(2\nu\pi/N) = 1$ by choice of Φ. More over $\psi_\nu + \psi_{\nu+1} \equiv 1$ on the common intervals where the two functions are defined. This follows from the 4π periodicity of Φ. Thus the family $\{\psi_\nu\}_{\nu \in \mathbb{Z}}$ defines a smooth partition of unity on $\mathbb{R}$ with supports of exactly two consecutive functions ψ_ν and $\psi_{\nu\pm1}$, intersect on a set of positive measure. Through the covering map $\theta \to (\cos\theta, \sin\theta)$ of the circle, the family $\{\psi_\nu(\theta)\}_{\nu=0}^{N-1}$ gives a smooth partition of unity on the circle such that

$$|\partial_\theta^k \psi_\nu(\theta)| \leq N^k, \text{ for } 0 \leq \nu \leq N-1,\ k \in \mathbb{N}, \quad (2.12)$$

since $\psi_\nu(x)$ given by (2.11) is a function of Nx.

The functions $\psi_\nu, 0 \leq \nu \leq N-1$ defines a homogeneous partition of unity

$\{\chi_\nu\}$ on $\mathbb{R}^2 \setminus \{0\}$ by setting

$$\chi_\nu(\xi) = \psi_\nu(\theta), \text{ for } \xi = |\xi|(\cos\theta, \sin\theta), \tag{2.13}$$

which is non zero only for $\theta \in \left[\frac{2(\nu-1)\pi}{N}, \frac{2(\nu+1)\pi}{N}\right]$. Note that $\chi_\nu \in C^\infty(\mathbb{R}^2 \setminus \{0\})$. The estimates (2.12) for $\nu = 0$ give the following derivative estimates for χ_0:

$$|\partial_{\xi_1}^k \chi_0| \leq 1, \ |\partial_{\xi_2}^k \chi_0| \leq N^k \tag{2.14}$$

for all $k \in \mathbb{N}$. This can be seen as follows.

Taking powers of the tangential derivative ∂_θ in (2.13) on the unit circle and using (2.12), we see that for all $k \in \mathbb{N}$,

$$\left|(\xi^\perp \cdot \nabla)^k \chi_\nu(\xi)\right| \ \leq \ CN^k \text{ for } |\xi| = 1. \tag{2.15}$$

Note that for $\xi = (\cos\theta, \sin\theta)$

$$\xi^\perp \cdot \nabla\chi_\nu(\xi) = -\sin\theta\, \partial_{\xi_1}\chi_\nu(\xi) + \cos\theta\, \partial_{\xi_2}\chi_\nu(\xi) \tag{2.16}$$

Similarly, differentiation in (2.13) in the radial direction gives $(\xi\cdot\nabla)\chi_\nu(\xi)=0$. At the point $\xi=(\cos\theta, \sin\theta)$, this gives the relation $\cos\theta\, \partial_{\xi_1}\chi_\nu(\xi)=-\sin\theta\, \partial_{\xi_2}\chi_\nu(\xi)$. Using this relation in (2.16), we see that

$$\xi^\perp \cdot \nabla = -\frac{1}{\sin\theta}\partial_{\xi_1} = \frac{1}{\cos(\theta)}\partial_{\xi_2}. \tag{2.17}$$

Since $\sin\theta = \xi_2/|\xi|$ and $\cos\theta = \xi_1/|\xi|$, we see that $\partial_{\xi_1}(1/\sin\theta) = \xi_1/\xi_2$, and $\partial_{\xi_2}(1/\cos\theta) = 2\xi_2/\xi_1$ on $|\xi| = 1$. A simple calculation using this, shows that the estimates (2.15) translate to the following inequalities

$$|\partial_{\xi_1}^k \chi_\nu(\xi)| \leq |N\sin\theta|^k \ \text{ and } |\partial_{\xi_2}^k \chi_\nu(\xi)| \leq |N\cos\theta|^k \tag{2.18}$$

for $\theta \in \left[\frac{2(\nu-1)\pi}{N}, \frac{2(\nu+1)\pi}{N}\right]$, the sector in $\mathbb{R}^2$ where χ_ν is supported. In particular, when $\nu = 0$ we have $|\sin\theta| \leq |\theta| \leq 2\pi/N$, and $|\cos\theta| \leq 1$. This gives the estimates (2.14).

Chapter 3

Local Smoothing Estimate of Fourier Integral Operator

In this chapter, we prove the local smoothing estimate for the Fourier integral operators with amplitude function $a \in S^0$, the symbol class of order 0 and the phase function of the form $\phi(x,t,\xi) = x \cdot \xi + t|\xi|$. In section 3.1, we give an overview of the local smoothing results which have been proven to date, and then we discuss the techniques which lead to the proof of the local smoothing estimate. In section 3.2.1, we study the smoothing property of the Fourier integral operators with compactly supported amplitude function. Sections 3.3–3.7 are devoted to the proof of our main result.

3.1 Introduction

In this section, we study the local smoothing property of the Fourier integral operators of the form

$$\mathcal{F}f(x,t) = \rho_1(t) \int_{\mathbb{R}^2} e^{ix\cdot\xi} e^{it|\xi|}\, a(\xi)\, \hat{f}(\xi)\, d\xi,\ \ f \in \mathscr{S}(\mathbb{R}^2), (x,t) \in \mathbb{R}^2 \times \mathbb{R}. \quad (3.1)$$

Here $a \in S^0(\mathbb{R}^2)$, the symbol class of order 0 and $\rho_1 \in C_c^\infty(\mathbb{R})$.

We assume without loss of generality that supp $(\rho_1) = [1,2]$, by composing ρ_1 with an affine transformation on $\mathbb{R}$ if necessary. Operators of the form (3.1) arise in wave equation and also in the study of spherical maximal operators. For fixed t, the regularity property of Fourier integral operators has been extensively studied by Seeger et al. [31]. The result of Seeger et al. says that the operator

$\mathcal{F}f$ is bounded on $L^p_{loc}(\mathbb{R}^2)$ for $1 < p < \infty$ provided that the amplitude function $a \in S^m(\mathbb{R}^2)$ for $m \le -|1/p - 1/2|$.

On the other hand, if one averages over $t \in [1,2]$, then there is a gain of regularity in L^p for $2 < p < \infty$. In fact, this phenomenon for the Fourier integral operator $\mathcal{F}f$ with the amplitude function $a(\xi) \equiv 1$ has been studied by C. D. Sogge [34]. He proved that there is an $\epsilon(p) > 0$ such that the following estimate

$$\left(\int_{t=1}^{2} \int_{\mathbb{R}^2} \left| (I - \Delta)^{\frac{\sigma}{2}} \mathcal{F}f(x,t) \right|^p \, dxdt \right)^{\frac{1}{p}} \le c_{\sigma,p} \|f\|_{L^p(dx)}, \tag{3.2}$$

holds for all σ with $\mathrm{Re}(\sigma) < \left(\frac{1}{p} - \frac{1}{2}\right) + \epsilon(p)$ for each $p \in (2, \infty)$. Comparing with the estimates of Seeger et al. (with $\sigma = m$), this amounts to a gain in regularity for $\mathcal{F}f$ by $\epsilon(p)$.

The estimate (3.2) is called the local smoothing estimate of order $\epsilon(p)$. Borrowing a term from a similar situation involving the Schrödinger equation [32], Sogge called this phenomenon as *local smoothing*. In a latter joint work with B. Mockenhaupt and A. Seeger [28], they made a further improvement in dimension two, by showing that $\epsilon(p) < \frac{1}{2p}$, for $p \ge 4$ and $\epsilon(p) < \frac{1}{2}(\frac{1}{2} - \frac{1}{p})$ for $2 < p \le 4$.

In the following theorem, we prove the local smoothing results for the Fourier integral operator of the form (3.1) with amplitude function $a \in S^0(\mathbb{R}^2)$.

Theorem 3.1.1 *Let $\mathcal{F}f$ be as in (3.1) with amplitude function $a \in S^0(\mathbb{R}^2)$. Then, the inequality*

$$\|(I - \Delta)^{\frac{\sigma}{2}} \mathcal{F}f\|_{L^p(\mathbb{R}^2 \times \mathbb{R})} \le C_{\sigma,p} \|f\|_{L^p(\mathbb{R}^2)} \tag{3.3}$$

holds for all $f \in L^p(\mathbb{R}^2)$, for

$$\begin{cases} Re(\sigma) < 1/2(1/p - 1/2), & \text{if } 2 < p \le 4, \\ Re(\sigma) < \frac{3}{2p} - \frac{1}{2}, & \text{if } 4 \le p < \infty. \end{cases}$$

Remark 3.1.2 In fact, we will consider a slightly more general class of Fourier integral operators of the form

$$\mathcal{F}f(x,t) = \int_{\mathbb{R}^2} e^{i(x\cdot\xi + t|\xi|)} \, a_1(t,\xi) \, \hat{f}(\xi) \, d\xi, \;\; f \in \mathscr{S}(\mathbb{R}^2)$$

with amplitude function $a_1(t,\xi) = \rho_1(t)a(\xi)$, where $\rho_1 \in C_c^\infty((0,\infty))$ and $a \in S^m(\mathbb{R}^2)$, $m \le 0$. We will see that an improved smoothing estimate holds when $a \in S^m(\mathbb{R}^2)$ for $m < 0$, and is useful to handle the initial velocity in the wave equation. So we state and prove the case $m < 0$ as a separate Theorem in the next chapter.

In fact, the estimate (3.3) for $2 < p \le 4$ and $\mathrm{Re}(\sigma) < 1/2(1/p - 1/2)$, follows by analytic interpolation (see, [39]), once we have the following two estimates:

$$\begin{aligned}
\|(I-\Delta)^{\frac{\sigma}{2}}\mathcal{F}f\|_{L^2(\mathbb{R}^2\times\mathbb{R})} &\le C_\sigma\|f\|_{L^2(\mathbb{R}^2)},\ Re(\sigma) = 0\\
\|(I-\Delta)^{\frac{\sigma}{2}}\mathcal{F}f\|_{L^4(\mathbb{R}^2\times\mathbb{R})} &\le C_\sigma\|f\|_{L^4(\mathbb{R}^2)},\ Re(\sigma) < -1/8.
\end{aligned}$$

The above L^2 estimate will follow immediately from the Plancheral theorem for $\mathrm{Re}(\sigma) \le 0$. Thus to prove Theorem 3.1.1, enough to prove (3.3) for $4 \le p < \infty$.

We first decompose the operator $\mathcal{F}$ into a family of operators $\{\mathcal{F}_j\}_{j\ge 0}$. We can actually obtain the regularity estimate for $\mathcal{F}$ from the corresponding L^p estimate for each of the operators $\mathcal{F}_j, j \ge 0$. In fact, each $\mathcal{F}_j$ is an infinitely smoothing operator, as it corresponds to a smooth kernel. Hence the actual regularity of $\mathcal{F}$ with respect to the L^p Sobolev space rely on the exact growth of $\|\mathcal{F}_j f\|_{L^p(\mathbb{R}^3)}$ as $j \to \infty$, see Section 3.7. We use wave front set analysis as in [28] to single out the region where the Fourier transform has rapid decay. The precise estimate for $\mathcal{F}_j f$ requires careful analysis with frequency decomposition and delicate machinery from Littlewood - Paley theory as employed in the previous works [28], [29] and [31] in this direction.

In contrast to the above works, our approach for estimating $\mathcal{F}_j f$ relies on a duality argument, and the use of a square function based on angular decomposition. The L^4 boundedness of such a square functions is established by Cordoba in [7]. Using this, we show that the norms of the operators $\mathcal{F}_j : L^4(\mathbb{R}^2) \to L^4(\mathbb{R}^3)$ has growth $2^{j/8}$, see Proposition 3.6.5. Our approach also yields the same smoothing as obtained in [28].

3.2 Decomposition of Fourier Integral Operator

In this section, we will discuss the dyadic decomposition of Fourier integral operator and give an outline of the proof of Theorem 3.1.1. Dyadic decomposition is an important tool to analyze such operators.

We first express the Fourier integral operator $\mathcal{F}$ as an infinite sum of Fourier integral operators $\{\mathcal{F}_j\}_{j\geq 0}$ as follows: Choose $\rho_0 \in C_c^\infty([\frac{1}{2}, 2])$ such that $1 = \sum_{j\in\mathbb{Z}} \rho_0(2^{-j}|\xi|)$. See section 2.1.1 Chapter 2, for the construction of such a $\rho_0 \geq 0$. For technical reasons, we take ρ_0 to be of the form $\rho_0 = \rho^2$ with $\rho \in C_c^\infty([\frac{1}{2}, 2])$. Setting $\phi_0 = \sum_{j\leq 0} \rho_0(2^{-j}|\xi|)$, we can write $1 = \phi_0 + \sum_{j\in\mathbb{N}} \rho_0(2^{-j}|\xi|)$, where ϕ_0 is a smooth function supported in the ball $|\xi| \leq 2$. Since $\rho_0(2^{-j}|\xi|)$ is supported on the annulus $2^{j-1} \leq |\xi| \leq 2^{j+1}$, the functions $a_j(t,\xi) = \rho_1(t)\,\rho_0(2^{-j}|\xi|)\,a(\xi)$ are vanishing on $|\xi| < 1$, for all $j \in \mathbb{N}$. Thus for each $j \in \mathbb{N}$, we set

$$\mathcal{F}_j f(x,t) = \int_{\mathbb{R}^2} e^{i(x\cdot\xi + t|\xi|)}\, a_j(t,\xi)\, \hat{f}(\xi)\, d\xi \tag{3.4}$$

so that

$$\mathcal{F} f(x,t) = \mathcal{F}_0 f(x,t) + \sum_{j\in\mathbb{N}} \mathcal{F}_j f(x,t) \tag{3.5}$$

as a tempered distribution. Here $\mathcal{F}_0 f$ is the Fourier integral operator with compactly supported amplitude function $a_0(t,\xi) := \rho_1(t)\, a(\xi)\, \phi_0(\xi)$. It turns out that $\mathcal{F}_0 f$ is a smoothing operator, see Corollary 3.2.3.

The L^p estimate for $\mathcal{F}_j f$ for $4 \leq p \leq \infty$, follows by Riesz-Thorin interpolation theorem, once we have the following estimates for all $j \in \mathbb{N}$ and for fixed $\epsilon > 0$,

$$\|\mathcal{F}_j f\|_{L^4(\mathbb{R}^2\times\mathbb{R})} \leq C\, j^{7/4+b}\, 2^{j(1+3\epsilon)/8}\, \|f\|_{L^4(\mathbb{R}^2)}, \text{ for some } b > 0 \tag{3.6}$$

$$\|\mathcal{F}_j f\|_{L^\infty(\mathbb{R}^2\times\mathbb{R})} \leq C\, 2^{j/2}\, \|f\|_{L^\infty(\mathbb{R}^2)}, \tag{3.7}$$

with a constant C independent of j. The estimate (3.7) follows by standard arguments expressing $\mathcal{F}_j$ as a convolution operator with an L^1 kernel, see Proposition 3.3.1. The estimate (3.6) is subtle and requires more sophisticated arguments. The regularity estimate for $\mathcal{F}_j f$ is then deduced from these L^p estimates using a Sobolev estimate given by Lemma 3.7.1. The precise regularity estimates for $\mathcal{F} f$ follows from the regularity estimate for $\mathcal{F}_j f$, via summability.

The main technical difficulties are in proving the estimate (3.6) for $j \geq 1$. First, by a wave front set analysis, we can identify the region where $\mathcal{F}_j f$ has rapid decay.

Remark 3.2.1 Taking Fourier transform in both the x and t variables in (3.1), we get

$$\widehat{\mathcal{F}f}(\xi,\tau) = a(\xi)\,\hat{f}(\xi)\int_t e^{-it(\tau-|\xi|)}\rho_1(t)\,dt.$$

We see that $\hat{\rho}_1(\tau-|\xi|)$ has no decay along the cone $\tau=|\xi|$. In fact, the wave front set of the distribution (see for definition, Section 1.1.3, Chapter 1) on $\mathbb{R}^3$ given by the Fourier integral operator (3.1) is contained in the set $\{(x,t,\xi,|\xi|) : |x|=t, \xi\in\mathbb{R}^2\smallsetminus\{0\}\}$ provided $a\equiv 0$ near the zero section $\xi=0$. This follows by Proposition 2.5.7 in [18]. Thus if $a\equiv 0$ near the zero section, then the Fourier transform of $\mathcal{F}f(x,t)=\mathcal{F}_t f(x)$ has rapidly decay away from the light cone $(\xi,|\xi|)$.

In view of Remark 3.2.1, we see that the wave front set of the distribution given by the Fourier integral operator (3.4) is actually contained in the conic set

$$C=\{(x,t,\xi,|\xi|) : |x|=t,\ \xi\in\mathbb{R}^2\setminus\{0\}\}.$$

Note that each $\mathcal{F}_j, j\geq 1$ is a Fourier integral operator with a distribution kernel having singularities "along the direction" $\tau=|\xi|$ in the frequency domain. So it is natural to split the kernel localising around the wave front set and away from it and analyze separately. This leads to the two operators Q_δ and R_δ defined as follows.

Choose an even function $\psi\in C_c^\infty(-2,2)$ such that $\psi=1$ on $[-1,1]$. For $\delta>0$, this defines a cut of function ψ^δ supported near the cone $|\xi|=\tau$ in $\mathbb{R}^3$ by

$$\psi^\delta(\xi,\tau)=\psi\left(\frac{|\xi|-\tau}{\delta}\right),\ \ (\xi,\tau)\in\mathbb{R}^2\times\mathbb{R}. \tag{3.8}$$

Let Q_δ and $\mathcal{R}_\delta$ denote the multiplier operators on R^3 with multipliers ψ^δ and $1-\psi^\delta$ respectively:

$$\begin{aligned}\widehat{Q_\delta(\mathcal{F}_j f)}(\xi,\tau) &= \psi^\delta(\xi,\tau)\,\widehat{\mathcal{F}_j f}(\xi,\tau),\\ \widehat{\mathcal{R}_\delta(\mathcal{F}_j f)}(\xi,\tau) &= [1-\psi^\delta(\xi,\tau)]\,\widehat{\mathcal{F}_j f}(\xi,\tau).\end{aligned} \tag{3.9}$$

Since $\mathcal{F}_j f = Q_\delta(\mathcal{F}_j f)+\mathcal{R}_\delta(\mathcal{F}_j f)$, the L^p estimate for $\mathcal{F}_j f$ follows from the corresponding estimates for $Q_\delta(\mathcal{F}_j f)$ and $\mathcal{R}_\delta(\mathcal{F}_j f)$. The estimate for $\mathcal{R}_\delta(\mathcal{F}_j f)$ easy and follows via standard kernel estimate as in [28]. Our estimate for $Q_\delta(\mathcal{F}_j f)$ is more direct, via a duality argument which leads to the proof of Theorem 3.1.1.

3.2.1 Fourier Integral Operator with Compactly Supported Amplitude Function

In this section, we discuss the L^p mapping properties of the Fourier integral operator with amplitude function $\rho_1(t)\, a(\xi)\, \phi_0(\xi)$ which is of compact support. For $a \in C_c^\infty(\mathbb{R}^2)$ set

$$\mathcal{F}_0 f(x,t) = \rho_1(t) \int_{\mathbb{R}^2} e^{i(x\cdot\xi + t|\xi|)}\, a(\xi)\, \hat{f}(\xi)\, d\xi. \tag{3.10}$$

In the following proposition, we prove that the operator $\mathcal{F}_0 f$ is a smoothing operator.

Proposition 3.2.2 *Let $\mathcal{F}_0 f(x,t)$ be as in (3.10), with supp $a \subset \{\xi : |\xi| \leq 2\}$. Then for each $t \in [1,2]$, the operator $\mathcal{F}_t$ given by $\mathcal{F}_t f(x) = \mathcal{F}_0 f(x,t)$ is a smoothing operator and satisfy the estimate $\|\partial_x^\alpha \mathcal{F}_t f\|_{L^p dx} \leq C_\alpha \|f\|_p$ for all $1 < p < \infty$ and for all α, with C_α, independent of $t \in [1,2]$.*

Proof. From (3.10), we see that

$$\mathcal{F}_t f(x) := \mathcal{F}_0 f(x,t) = \rho_1(t)\, K_t * f(x),$$

where

$$K_t(x) = \int e^{ix.\xi}\, e^{it|\xi|} a(\xi)\, d\xi.$$

Clearly $K_t \in C^\infty(\mathbb{R}^2)$ because K_t being Fourier transform of a compactly supported function and hence $\mathcal{F}_t$ is a smoothing operator for $f \in \mathscr{S}(\mathbb{R}^2)$. Now, we show that the operator $T_{t,\alpha} : f(x) \to \partial_x^\alpha \mathcal{F}_t f(x)$ is bounded from $L^p(\mathbb{R}^2) \to L^p(\mathbb{R}^2)$, with norm bound C_α, independent of t, for all multi-index α with $|\alpha| \geq 0$. Note that, for each $t \in [1,2]$, $T_{t,\alpha}$ is the multiplier transformation whose multiplier is $m_{t,\alpha}(\xi)$, with

$$m_{t,\alpha}(\xi) = (i\xi)^\alpha\, e^{it|\xi|}\, a(\xi).$$

Since, supp $a \subset \{\xi : |\xi| \leq 2\}$, $m_{t,\alpha}$ is compactly supported with supp $m_{t,\alpha} \subset$

$\{\xi : |\xi| \leq 2\}$. Moreover, we also see easily that

$$|m_{t,\alpha}(\xi)| \leq B, \quad \left| \left(\frac{\partial}{\partial \xi}\right)^{\alpha} m_{t,\alpha}(\xi) \right| \leq \frac{B}{|\xi|^{|\alpha|}},$$

for all $|\alpha| \leq 2$, where B is independent of t. Thus, by the Hörmander's multiplier theorem (see, [37, Theorem 3]), we get

$$\|T_{t,\alpha} f\|_{L^p(\mathbb{R}^2)} \leq C_\alpha \|f\|_{L^p(\mathbb{R}^2)},$$

for $f \in L^2 \cap L^p,\ 1 < p < \infty$ and with C_α independent of t. Hence the proof. □

Corollary 3.2.3 *Let $\mathcal{F}_0$ be as in (3.10) with a smooth amplitude function supported in $\{\xi : |\xi| \leq 2\}$. Then the following estimate holds for all $\alpha \in \mathbb{C}$.*

$$\|(I - \Delta)^\alpha \mathcal{F}_0 f\|_{L^p(\mathbb{R}^2 \times \mathbb{R})} \leq C_{\alpha,p} \|f\|_{L^p(\mathbb{R}^2)}, \quad 1 < p < \infty. \tag{3.11}$$

Proof. Note that $(I - \Delta)^\alpha \mathcal{F}_0$ is a Fourier integral operator with amplitude function $(1 + |\xi|^2)^\alpha\, a(\xi) \in C_c^\infty(|\xi| \leq 2)$. The proof follows from Proposition 3.2.2 applied to the Fourier integral operator $(I - \Delta)^\alpha \mathcal{F}_0$, followed by a t-integration over $[1, 2]$, as the constant C_α in Proposition 3.2.2, is independent of t. □

3.3 L^∞ Estimate for $\mathcal{F}_j$

In this section, we prove the L^∞ estimate (3.7) for $\mathcal{F}_j$. For that, we also need a further decomposition of the operators $\mathcal{F}_j$ in terms of the angular variable, as discussed in Section 2.1.2, Chapter 2. Recall that χ_ν as a homogeneous function on $\mathbb{R}^2$ is supported on an angular sector given by $|\theta - \theta_\nu| \leq 2\pi/N$, where $\theta_\nu = 2\pi\nu/N$. Using the homogeneous partitions of unity $\{\chi_\nu\}_\nu$, we define the operators

$$\mathcal{F}_{j,\nu} f(x,t) = \int_{\mathbb{R}^2} e^{i(x\cdot\xi + t|\xi|)}\, a_j(t,\xi)\, \chi_\nu(\xi)\, \hat{f}(\xi)\, d\xi \tag{3.12}$$

for $j \geq 1,\ 0 \leq \nu \leq N-1$, where $a_j(t,\xi) = \rho_1(t)\, \rho_0(2^{-j}|\xi|)\, a(\xi)$. Note that $\mathcal{F}_j f = \sum_{\nu=0}^{N-1} \mathcal{F}_{j,\nu} f$.

Recall that by choice $\rho_0 = \rho^2$. We also need to consider the Fourier integral operators $\tilde{\mathcal{F}}_{j,\nu}$ given by

$$\tilde{\mathcal{F}}_{j,\nu} f(x,t) = \int_{\mathbb{R}^2} e^{i(x\cdot\xi + t|\xi|)}\, \tilde{a}_j(t,\xi)\, \chi_\nu(\xi)\, \hat{f}(\xi)\, d\xi \tag{3.13}$$

with amplitudes $\tilde{a}_j(t,\xi) = \rho_1(t)\,\rho(2^{-j}|\xi|)\,a(\xi)$. We have

$$\tilde{\mathcal{F}}_{j,\nu} f(x,t) = \int_{y\in\mathbb{R}^2} K^a_{j,\nu}(x-y,t)\, f(y)\, dy,$$

with kernel $K^a_{j,\nu}$, as in (2.5), Chapter 2. Note that replacing ρ in (2.5) by ρ^2 gives the kernel $k_{j,\nu}$ for the Fourier integral operator $\mathcal{F}_{j,\nu}$.

Now, in the following proposition, we prove the L^∞ estimate (3.7) mentioned in Section 3.2.

Proposition 3.3.1 *Let $\mathcal{F}_j$ be the operator given by (3.4) for $j \in \mathbb{N}$. Then $\mathcal{F}_j$ satisfies the following inequality*

$$\|\mathcal{F}_j f\|_{L^\infty(\mathbb{R}^2\times\mathbb{R})} \le C 2^{j/2} \|f\|_{L^\infty(\mathbb{R}^2)}$$

with a constant C independent of j.

Proof. We have $\mathcal{F}_j = \sum_\nu \mathcal{F}_{j,\nu}$ where $\mathcal{F}_{j,\nu}$ is given by (3.12) with kernel

$$k_{j,\nu}(x,t) = \rho_1(t) \int_{\mathbb{R}^2} e^{i(x\cdot\xi + t|\xi|)}\, \rho^2(2^{-j}|\xi|)\, a(\xi)\, \chi_\nu(\xi)\, d\xi.$$

Note that $k_{j,\nu}$ differs from $K^a_{j,\nu}$ in (2.5), only in the power of ρ. Hence by the same arguments as in Lemma 2.2.1, Chapter 2, we get the following uniform L^1 estimate

$$\|k_{j,\nu}(\cdot,t)\|_{L^1 dx} \le C_N |\rho_1(t)|\, 2^{3j/2} \|\Psi_j(x)\|_{L^1 dx} = C_N \rho_1(t)| \|\Psi_0(x)\|_{L^1 dx}$$

by a change of variable. It follows that

$$\|\mathcal{F}_{j,\nu} f(\cdot,t)\|_{L^\infty(\mathbb{R}^2)} \le C_N \rho_1(t)| \|\Psi_0(x)\|_{L^1 dx} \|f\|_{L^\infty dx},$$

for $0 \le \nu \le N-1$. Summing over ν, this gives the required estimate after a t integration, observing that there are $N = N(j) \approx 2^{j/2}$ terms in the sum. $\square$

3.4 Estimates of Square Function

In this section, we study the square function based on angular decomposition, which is crucial in the L^4 estimate for $\mathcal{F}_j f$ in Section 3.6. We first consider the equally spaced decomposition in τ and ξ variables respectively. For this, let $\phi \in C_c^\infty([-1,1])$ be such that $\sum_{n\in\mathbb{Z}} \phi(\tau - n) = 1$.

For $n \in \mathbb{Z}$ and $m = (m_1, m_2) \in \mathbb{Z}^2$, we define the multiplier operators P^n and P_m by

$$\widehat{(P^n g)}(\xi,\tau) = \phi(2^{-j/2}\tau - n)\,\hat{g}(\xi,\tau) \tag{3.14}$$

$$\widehat{(P_m g)}(\xi,\tau) = \prod_{i=1}^{2} \phi(2^{-j/2}\xi_i - m_i)\,\hat{g}(\xi,\tau) \tag{3.15}$$

for $g \in \mathcal{S}(\mathbb{R}^3)$. Note that $\sum_n P^n = I$ and $\sum_m P_m = I$.
By support property of ϕ, we see that $\widehat{P^n g}\widehat{P^{n'} h} \not\equiv 0$ only if $|n - n'| \le 2$. Thus for a given n, there are atmost 5 n's for which $\widehat{P^n g}\widehat{P^{n'} h}$ is non trivial. This fact is used in the next lemma, which is crucial for the proof of the estimate for the square function Sg in Proposition 3.4.5. This result is essentially in the spirit of the result in [28, Lemma 1.2].

Lemma 3.4.1 *Let $\{g_n(x,t)\}_{n=1}^N$ be a sequence of functions in $\mathscr{S}(\mathbb{R}^2 \times \mathbb{R})$ and P^n be as in (3.14). Then for $2 \le p \le \infty$, we have*

$$\left\| \sum_{n=1}^N P^n g_n \right\|_{L^p(\mathbb{R}^3)} \le C\, N^{1/2-1/p} \left\| \left(\sum_{n=1}^N |P^n g_n|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^3)}.$$

Proof. The proof follows by Riesz-Thorin interpolation between the cases $p = 2$ and $p = \infty$. We first consider the case $p = 2$.

Using Plancheral theorem, expanding the square and in view of the above observation on $\widehat{P^n g_n}\widehat{P^{n'} g_{n'}}$, we see that

$$\begin{aligned} \left\| \sum_{n=1}^N P^n g_n \right\|_{L^2(\mathbb{R}^3)}^2 &= \int_{\mathbb{R}^3} \left| \sum_n \widehat{P^n g_n} \right|^2 d\xi d\tau \\ &\le 5 \int_{\mathbb{R}^3} \sum_n \left| \widehat{P^n g_n} \right|^2 d\xi d\tau \end{aligned}$$

$$= \quad 5 \left\| \left(\sum_{n=1}^{N} |P^n g_n|^2 \right)^{1/2} \right\|_{L^2(\mathbb{R}^3)}^2$$

which settles the case $p = 2$.

For $p = \infty$, using Cauchy-Schwarz inequality with respect to the sum, we get

$$\left\| \sum_{n=1}^{N} P^n g_n \right\|_{L^\infty(\mathbb{R}^3)} \leq \left\| N^{1/2} \left(\sum_{n=1}^{N} |P^n g_n|^2 \right)^{1/2} \right\|_{L^\infty(\mathbb{R}^3)}$$

$$= \quad N^{1/2} \left\| \left(\sum_{n=1}^{N} |P^n g_n|^2 \right)^{1/2} \right\|_{L^\infty(\mathbb{R}^3)} .$$

Thus for $2 \leq p \leq \infty$, interpolation yields

$$\left\| \sum_{n=1}^{N} P^n g_n \right\|_{L^p(\mathbb{R}^3)} \leq C\, N^{1/2-1/p} \left\| \left(\sum_{n=1}^{N} |P^n g_n|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^3)} .$$

This completes the proof. □

Next we introduce a square function based on angular decomposition, which is crucial in the L^4 estimate for $Q^\delta \mathcal{F}_j f$ in Section 3.6. This is based on the multiplier operator $T^\delta_{\nu,j}$ defined as follows:

Let $\tilde{\chi}_\nu$ be a homogeneous function on $\mathbb{R}^2$, which is smooth as a function on S^1, satisfying $\tilde{\chi}_\nu \chi_\nu = \chi_\nu$, where χ_ν is as in (2.3), Chapter 2. In fact, we can define $\tilde{\chi}_\nu(\xi) = \tilde{\psi}_\nu(\theta)$ as in (2.13) Chapter 2, where $\tilde{\psi}_\nu$ is defined exactly as in (2.11) with $\Phi(x)$ replaced by $\Phi(x/2)$.

For $0 \leq \nu \leq N-1, j \in \mathbb{N}$ and $\delta > 0$, let $\{T^\delta_{\nu,j}\}_{\nu=0}^{N-1}$ be a family of operators given by

$$\widehat{T^\delta_{\nu,j} g}(\xi,\tau) = \tilde{\chi}_\nu(\xi)\, \rho(2^{-j}|\xi|)\, \psi\left(\frac{|\xi| - \tau}{\delta} \right) \hat{g}(\xi,\tau),\ \ g \in \mathscr{S}(\mathbb{R}^3). \tag{3.16}$$

Note that on the support of $\widehat{P^n T^\delta_{\nu,j} g}$, we have

$$(n-1)2^{j/2} \leq \tau \leq (n+1)2^{j/2},\ \ \tau - 2\delta \leq |\xi| \leq \tau + 2\delta \tag{3.17}$$

which gives,

$$\left(n-1-2\delta 2^{-j/2}\right) 2^{j/2} \leq |\xi| \leq \left(n+1+2\delta 2^{-j/2}\right) 2^{j/2}. \tag{3.18}$$

On the other hand, on the support of $\widehat{P_m g}$ for $m=(m_1,m_2)\in\mathbb{Z}^2$, we have, $(m_i-1)2^{j/2}\leq \xi_i \leq (m_i+1)2^{j/2}$, for $i=1,2$. and hence

$$(|m|-2\sqrt{2})2^{j/2} \leq |\xi| \leq (|m|+\sqrt{2})2^{j/2}. \tag{3.19}$$

In view of (3.18) and (3.19), we see that $P^n\widehat{T^\delta_{s,j}P_m}g$ is non zero only for $||m|-n|\leq 7$, for $\delta<2^{j/2}$.

Hence, writing $g=\sum_m P_m g$, we see that

$$P^n T^\delta_{\nu,j} g(x,t) = P^n T^\delta_{\nu,j} g_n(x,t) \tag{3.20}$$

where $g_n=\sum_{\{m\in\mathbb{Z}^2:||m|-n|\leq 7\}} P_m g$. For $n\in\mathbb{N}$ and $0\leq\nu\leq N-1$, set $I_{n,\nu}=\{m\in\mathbb{Z}^2\cap \operatorname{supp}[\tilde{\chi}_\nu\,\rho(2^{-j}\cdot)]:||m|-n|\leq 7\}$ and define

$$g_{n,\nu}=\sum_{m\in I_{n,\nu}} P_m g. \tag{3.21}$$

Note that, $T^\delta_{\nu,j}g_n(x,t)=T^\delta_{\nu,j}g_{n,\nu}(x,t)$. Now, we prove the following proposition, which will be used in the proof of the main square function estimate given in Proposition 3.4.5. This result is essentially in the spirit of the result in [28, pp. 213].

Proposition 3.4.2 *Let $T^\delta_{\nu,j}g$ be as in (3.16) and g_n as in (3.20). Then the following square function estimate holds*

$$\left\|\left(\sum_{n,\nu}|T^\delta_{\nu,j}g_n|^2\right)^{\frac{1}{2}}\right\|_{L^4(\mathbb{R}^3)} \leq C\,\delta^{1/4}\,j^{3/4}\,\|g\|_{L^4(\mathbb{R}^3)} \tag{3.22}$$

for all $g\in\mathcal{S}(\mathbb{R}^3)$ with a constant C independent of j.

Proof. We write

$$T^\delta_{\nu,j}g_n(x,t)=\int_{\mathbb{R}^3}\tilde{k}^\delta_{j,\nu}(x-y,t-s)\,g_n(y,s)\,dy\,ds, \tag{3.23}$$

where

$$\begin{aligned}\tilde{k}^{\delta}_{j,\nu}(x,t) &= \int_{\mathbb{R}^3} e^{i[x\cdot\xi + t\tau]} \left[\tilde{\chi}_\nu(\xi)\,\rho(2^{-j}|\xi|)\,\psi\left(\frac{|\xi|-\tau}{\delta}\right)\right] d\xi\, d\tau \\ &= \delta\psi^{\vee}(\delta t)\, K_{j,\nu}(x),\end{aligned}$$

with $K_{j,\nu}(x,t) = \int_{\xi\in\mathbb{R}^2} e^{i(x\cdot\xi+t|\xi|)}\,\rho(2^{-j}|\xi|)\,\chi_\nu(\xi)\,d\xi$. Note that $K_{j,\nu}$ is same as $K^a_{j,\nu}$ in (2.5), Chapter 2, with $a \equiv 1$. Hence by the same arguments as in Lemma 2.2.1, Chapter 2, we have

$$\int_{x,t} |\tilde{k}^{\delta}_{j,\nu}(x,t)| dxdt \leq C \tag{3.24}$$

with a constant independent of j and δ. Since $T^{\delta}_{\nu,j} g_n(x,t) = T^{\delta}_{\nu,j} g_{n,\nu}(x,t)$, an application of Cauchy-Schwarz inequality in (3.23) yields

$$\begin{aligned}|T^{\delta}_{\nu,j} g_n(x,t)|^2 &\leq \left(\int_{\mathbb{R}^3} |g_{n,\nu}(y,s)|^2 |\tilde{k}^{\delta}_{j,\nu}(x-y,t-s)|\, dy\, ds\right) \\ &\quad\times \left(\int_{\mathbb{R}^3} |\tilde{k}_{j,\nu}(x-y,t-s)|\, dy\, ds\right) \\ &\leq C \int_{\mathbb{R}^3} |g_{n,\nu}(y,s)|^2 |\tilde{k}^{\delta}_{j,\nu}(x-y,t-s)| dy\, ds,\end{aligned}$$

in view of (3.24), where $g_{n,\nu}$ is as in (3.21). Summing over n and ν, squaring and integrating, this leads to the inequality

$$\left\|\left(\sum_{n,\nu} |T^{\delta}_{\nu,j} g_n|^2\right)^{\frac{1}{2}}\right\|^2_{L^4(\mathbb{R}^3)} \tag{3.25}$$

$$\begin{aligned}&\leq C \left[\int_{x,t}\left[\int_{y,s}\sum_{n,\nu} |g_{n,\nu}(y,s)|^2 |\tilde{k}^{\delta}_{j,\nu}(x-y,t-s)| dy\, ds\right]^2 dxdt\right]^{\frac{1}{2}} \\ &= C \sup_{\|h\|_{L^2}=1} \left|\int_{x,t}\left[\int_{y,s}\sum_{n,\nu} |g_{n,\nu}(y,s)|^2 |\tilde{k}^{\delta}_{j,\nu}(x-y,t-s)|\, dy\, ds\right] h(x,t)\, dxdt\right|\end{aligned}$$

where we used duality to express the $L^2(dxdt)$ norm. Using the change of variable $t \to t+s$, and setting $h_s(x,t) = \delta\psi^{\vee}(\delta t) h(x,t+s)$, we see that

$$\int_{x,t} |\tilde{k}^{\delta}_{j,\nu}(x-y,t-s)|\, h(x,t)\, dx\, dt$$

$$\leq \sup_{\nu} \int_{x,t} |K_{j,\nu}(x-y,t)|\,|h_s(x,t)|\,dx\,dt$$

by writing $\tilde{k}^{\delta}_{j,\nu}(x,t) = K_{j,\nu}(x)\,\delta\,\psi^{\vee}(\delta t)$. Thus an interchange of integrals in (3.25) shows that the right hand side of (3.25) is atmost

$$\sup_{\|h\|_{L^2}=1} \left| \int_{y,s} \sum_{n,\nu} |g_{n,\nu}(y,s)|^2 \left[\sup_{\nu} \int_{x,t} |K_{j,\nu}(x-y,t)|\,|h_s(x,t)|\,dxdt \right] dy\,ds \right|.$$

By Cauchy Schwarz inequality in the variables (y,s), the term inside the modulus sign is at most

$$C \left[\int_{y,s} \left(\sum_{n,\nu} |g_{n,\nu}(y,s)|^2 \right)^2 dy\,ds \right]^{1/2} \times \tag{3.26}$$
$$\left[\int_{y,s} \sup_{\nu} \left| \int_{x,t} |K_{j,\nu}(x-y,t)|\,|h_s(x,t)|\,dxdt \right|^2 dy\,ds \right]^{\frac{1}{2}}.$$

We first consider the second term. Since the kernel $K_{j,\nu}$ satisfies the point-wise estimate (2.10) in chapter 2, appealing to Lemma 1.4 in [28], we get,

$$\begin{aligned} \int_y \sup_{\nu} \left| \int_{x,t} |K_{j,\nu}(x-y,t)|\,|h_s(x,t)|\,dxdt \right|^2 dy &\leq C|\log(2^{-j})|^{3/2} \|h_s\|^2_{L^2(\mathbb{R}^3)} \\ &\leq Cj^3 \|h_s\|^2_{L^2(\mathbb{R}^3)}, \end{aligned}$$

with C independent of j. Since $h_s(x,t) = \delta\psi^{\vee}(\delta t)h(x,t+s)$, integrating with respect to the s-variable on both sides gives

$$\begin{aligned} &\int_{y,s} \sup_{\nu} \left| \int_{x,t} |K_{j,\nu}(x-y,t)|\,|h_s(x,t)|\,dxdt \right|^2 dy\,ds \\ &\quad\leq C\,j^3 \int_s \int_{x,t} |\delta\psi^{\vee}(\delta t)h(x,t+s)|^2\,dxdt\,ds \\ &\quad= C\,j^3\,\delta \int_t |\psi^{\vee}(t)|^2 \left[\int_{x,s} |h(x,t+s)|^2\,dxds \right] dt \\ &\quad\leq C\,\delta\,j^3\,\|h\|^2_{L^2(\mathbb{R}^3)}. \end{aligned}$$

It follows that the the second term in (3.26) is bounded by $C\,\delta^{1/2}\,j^{3/2}\|h\|_{L^2(\mathbb{R}^3)}$.

To complete the proof, we need to show that the first term of (3.26) is bounded by $C\,\|g\|^2_{L^4(\mathbb{R}^3)}$, with C independent of j, which follows from Lemma 3.4.3 below. Using these estimates in (3.25) and taking the square root, the proof follows. □

Lemma 3.4.3 *Let $g_{n,\nu}$ be as in (3.21). Then the following estimate holds*

$$\left\| \left(\sum_{n,\nu} |g_{n,\nu}|^2 \right)^{1/2} \right\|_{L^4(\mathbb{R}^3)} \leq C_p\,\|g\|_{L^4(\mathbb{R}^3)}$$

for all $g \in L^4(\mathbb{R}^3)$.

Proof. Form (3.21), $g_{n,\nu} = \sum\limits_{m\in I_{n,\nu}} P_m g$, where

$$I_{n,\nu} = \{m \in \mathbb{Z}^2 \cap \operatorname{supp}[\tilde{\chi}_\nu\,\rho(2^{-j}\cdot)] : ||m| - n| \leq 7\}.$$

Let κ denotes the cardinality of $I_{n,\nu}$. Then by Cauchy-Schwarz inequality, we have

$$|g_{n,\nu}|^2 = |\sum_{m\in I_{n,\nu}} P_m g\,|^2 \leq \kappa \sum_{m\in I_{n,\nu}} |P_m g|^2.$$

Note that the cardinality κ is uniformly bounded in n and j. In fact the annular region $n - 7 \leq |m| \leq n + 7$ has area $28\pi n$, and hence the number of lattice points in this annulus grows like n as n increases. But from (3.17), we see that $n \leq 7|\xi|2^{-j/2} \leq C\,2^{j/2}$ since $|\xi| \approx 2^j$ on the support of $\rho(2^{-j}|\xi|)$. Note that $I_{n,\nu}$ is the set of integer lattice points in the intersection of the above annulus with a sector given by $|\theta| \leq \frac{4\pi}{N}$, hence this intersection has area $4 \cdot \frac{28n\pi}{N} \leq C\,\frac{28\pi 2^{j/2}}{N}$. This is uniformly bounded in j as $N \approx 2^{j/2}$.

Now summing over (n,ν) and observing that $m \in I_{n,\nu}$ if and only if $(n,\nu) \in J_m := \{(n,\nu) : ||m| - n| \leq 7, m \in \operatorname{supp}[\tilde{\chi}_\nu\,\rho(2^{-j}\cdot)\}$, we get

$$\begin{aligned}
\sum_{n\in\mathbb{N}} \sum_{\nu=0}^{N-1} |g_{n,\nu}|^2 &\leq \kappa \sum_{n,\nu} \sum_{m\in I_{n,\nu}} |P_m g|^2 \leq \kappa \sum_{m\in\mathbb{Z}^2} \sum_{(n,\nu)\in J_m} |P_m g|^2 \\
&\leq 60\kappa \sum_{m\in\mathbb{Z}^2} |P_m g|^2
\end{aligned}$$

since the cardinality of J_m is at most 60: There are at most 15 integers n such that $||m|-n| \le 7$ and any given $m \neq 0$ is in the support of at most four $\tilde{\chi}_\nu$s. It follows that

$$\begin{aligned}\left\| \left(\sum_{n,\nu} |g_{n,\nu}(\cdot,t)|^2 \right)^{1/2} \right\|_{L^4(\mathbb{R}^2,dx)} &\le C \left\| \left(\sum_{m\in\mathbb{Z}^2} |P_m g(\cdot,t)|^2 \right)^{1/2} \right\|_{L^4(\mathbb{R}^2)} \\ &\le C_p \, \|g(\cdot,t)\|_{L^4(\mathbb{R}^2,dx)}\end{aligned}$$

for some constant C_p independent of j, by the Littlewood-Paley estimate corresponding to equally spaced decomposition in $\mathbb{R}^2$ given in [14, Theorem 2.16, p. 489] applied to the function $g_t(x) = g(x,t)$ for each t. The required estimate follows by a further t integration. $\square$

Now, we shall need the following overlap lemma [28, Lemma 1.3], which is crucial in the L^4 estimate of the square function Sg, given in Proposition 3.4.5. For each fixed $n \in \mathbb{Z}$ and for a given fixed $\epsilon > 0$, consider the set:

$$\begin{aligned} V_\nu^n = \{(\xi,\tau) \in \mathbb{R}^3 : & \\ & |2^{-j/2}\tau - n| \le 1,\ \xi \in \text{ supp } [\rho(2^{-j}\cdot)\,\tilde{\chi}_\nu] \text{ and } \left|\frac{|\xi|-\tau}{2}\right| \le 2^{\epsilon(j-1)}\}, \end{aligned} \tag{3.27}$$

for $0 \le \nu \le N-1$.

Lemma 3.4.4 *Let V_ν^n be as in (3.27) and $V_\nu^n + V_{\nu'}^{n'}$ denote the algebraic sum of two such sets. Then, for fixed values of $n,\ n' \in \mathbb{Z}$ and for fixed $\epsilon > 0$, we have*

$$\sum_{\nu,\nu'} \chi_{V_\nu^n + V_{\nu'}^{n'}}(\xi,\tau) \le C\, j\, 2^{\epsilon j/2},$$

with a constant C, independent of $n,\ n'$ and j.

Proof. On the support of $\rho(2^{-j}|\xi|)$, we have $2^{j-1} \le |\xi| \le 2^{j+1}$ and hence, $|\frac{|\xi|-\tau}{2}| \le 2^{\epsilon(j-1)} \le |\xi|^\epsilon$. It follows that $V_\nu^n \subset U_\nu^n$, where U_ν^n is the set given by

$$U_\nu^n = \{(\xi,\tau) \in \mathbb{R}^3 : |2^{-j/2}\tau - n| \le 1,\ \xi \in \text{ supp } [\tilde{\chi}_\nu] \text{ and } \left|\frac{|\xi|-\tau}{2}\right| \le |\xi|^\epsilon\},$$

for each n and ν. Hence, $V_\nu^n + V_{\nu'}^{n'} \subset U_\nu^n + U_{\nu'}^{n'}$ and the proof follows by volume packing arguments as in [28, Lemma 1.3]. $\square$

We end this section with the following square function estimate: With $T^{\delta}_{\nu,j}$ as in (3.16), we define the square function

$$Sg(x,t) = \left(\sum_{\nu=0}^{N-1} |T^{\delta}_{\nu,j} g(x,t)|^2 \right)^{\frac{1}{2}}. \tag{3.28}$$

Proposition 3.4.5 *Let Sg be as in (3.28) with $\delta = 2^{\epsilon(j-1)}$, $\epsilon \in (0, \frac{1}{2})$ and $j \in \mathbb{N}$. Then there exists a constant C such that the inequality*

$$\|Sg\|_{L^p(\mathbb{R}^3)} \le C\, j\, 2^{j/8}\, \delta^{3/8}\, \|g\|_{L^p(\mathbb{R}^3)},$$

holds for all $g \in \mathscr{S}(\mathbb{R}^3)$ and for $4/3 \le p \le 4$.

Proof. We use the Rademacher function argument as in Stein [37], page 106 to reduce the square function estimate to a multiplier problem. Recall that the Rademacher functions $\{r_k\}_{k\ge 0}$ are functions on $\mathbb{R}$ defined as follows. Let r_0 be the periodic function on $\mathbb{R}$ with period 1 defined by

$$r_0(s) = \chi_{[0,1/2]}(s) - \chi_{(1/2,1)}(s), \text{ for } 0 \le s < 1.$$

For $k \in \mathbb{N}$, define $r_k(s) = r_0(2^k s)$, $k \ge 1$.

The Rademacher functions have the following interesting property: if $F(s) = \sum_\nu a_\nu r_\nu(s) \in L^2([0,1])$ then $F \in L^p([0,1])$ for all $p \in (1,\infty)$. In fact, we have

$$c_1 \|F\|_p \le \|F\|_2 \le c_2 \|F\|_p, \tag{3.29}$$

for positive constants c_1, c_2 depending only on p (and not on the particular function F), see [37], page 277.

For each $s \in [0,1)$, set

$$P(s,x,t) = \sum_{\nu=0}^{N-1} r_\nu(s)\, T^{\delta}_{\nu,j} g(x,t). \tag{3.30}$$

By the orthonormality of the collection $\{r_\nu\}$ and the property (3.29), we see that for each $(x,t) \in \mathbb{R}^3$

$$|Sg(x,t)| = \left(\int_{[0,1)} |P(s,x,t)|^2 ds \right)^{1/2} \le C_p\, \|P(\cdot,x,t)\|_{L^p(ds)}$$

for $1 < p < \infty$, with a constant C_p independent of (x,t).

It follows that

$$\int_{\mathbb{R}^3} |Sg(x,t)|^p dx\, dt \le C_p^p \int_{\mathbb{R}^3} \int_{[0,1)} |P(s,x,t)|^p ds\, dxdt. \tag{3.31}$$

Let $T_j^{\delta,s}$ denote the multiplier operator corresponding to the multiplier

$$\tilde{m}_{s,j}(\xi) = \sum_{\nu=0}^{N-1} r_\nu(s)\, \tilde{\chi}_\nu(\xi)\, \rho(2^{-j}|\xi|)\, \psi\left(\frac{|\xi| - \tau}{\delta}\right)$$

so that $P(s,x,t) = T_j^{\delta,s} g(x,t)$ in view of (3.30) and (3.16).

Thus (3.31) reads as

$$\int_{\mathbb{R}^3} |Sg(x,t)|^p dx\, dt \le c_2^p \int_{[0,1)} \int_{\mathbb{R}^3} |T_j^{\delta,s} g(x,t)|^p dx\, dt\, ds. \tag{3.32}$$

Thus to prove L^p boundedness of S enough to show the L^p boundedness of $T_j^{\delta,s}$ with norm bound independent of s. We first estimate the L^4 norm of $T_j^{\delta,s} g$. Following an idea of Fefferman [12] we reduce it to an L^2 estimate for a bilinear expression. This L^2 norm can be dominated by the L^4 norm of the associated square function.

We now come to the details. In view of (3.20) and using the identity $\sum_n P^n = I$, we get $T_j^{\delta,s} g = \sum\limits_{n\in\mathbb{Z}} P^n T_j^{\delta,s} g_n$. But from (3.17), we see that $n \approx |\xi| 2^{-j/2} \approx 2^{j/2}$ since $|\xi| \approx 2^j$ on the support of $\rho(2^{-j}|\xi|)$.

Following as in [28], in view of Lemma 3.4.1 with $N \approx 2^{j/2}$, we see that,

$$\begin{aligned}
\left\| T_j^{\delta,s} g \right\|_{L^4(\mathbb{R}^3)} &= \left\| \sum_n P^n T_j^{\delta,s} g_n \right\|_{L^4(\mathbb{R}^3)} \\
&\le C\, 2^{j/8} \left\| \left(\sum_n |P^n T_j^{\delta,s} g_n|^2 \right)^{1/2} \right\|_{L^4(\mathbb{R}^3)} \\
&= C\, 2^{j/8} \left\| \left(\sum_n |P^n \sum_\nu r_\nu(s) T_{\nu,j}^{\delta} g_n|^2 \right)^{1/2} \right\|_{L^4(\mathbb{R}^3)}
\end{aligned}$$

since $T_j^{\delta,s} g = \sum_\nu r_\nu(s)\, T_{\nu,j}^{\delta} g$.

We have

$$\left(\sum_n |P^n \sum_\nu r_\nu(s) T^\delta_{\nu,j} g_n|^2\right)^2 = \sum_{n,n'} \left| P^n \sum_\nu r_\nu T^\delta_{\nu,j} g_n \cdot P^{n'} \sum_{\nu'} r_{\nu'} T^\delta_{\nu',j} g_{n'} \right|^2$$
$$= \sum_{n,n'} \left| \sum_{\nu,\nu'} r_\nu(s)\, r_{\nu'}(s)\, P^n T^\delta_{\nu,j} g_n \cdot P^{n'} T^\delta_{\nu',j} g_{n'} \right|^2 .$$

Integrating with respect to x and t variables, using Plancherel's theorem, followed by the Cauchy-Schwarz inequality and using the fact that

$$\text{supp}\left[\widehat{(P^n) T^\delta_{\nu,j}} g_n * \widehat{(p^{n'}) T^\delta_{\nu',j}} g_{n'} \right] \subset V^n_\nu + V^{n'}_{\nu'},$$

where V^n_ν is as in (3.27) with $\delta = 2^{\epsilon(j-1)}$, we get

$$\left\| \left(\sum_n |P^n \sum_\nu r_\nu(s) T^\delta_{\nu,j} g_n|^2 \right)^{1/2} \right\|^4_{L^4(\mathbb{R}^3)}$$
$$= \int_{\mathbb{R}^3} \sum_{n,n'} \left| \sum_{\nu,\nu'} r_\nu(s)\, r_{\nu'}(s)\, \widehat{(P^n) T^\delta_{\nu,j}} g_n * \widehat{(p^{n'}) T^\delta_{\nu',j}} g_{n'} \right|^2 d\xi d\tau$$
$$\leq \int_{\mathbb{R}^3} \sum_{n,n'} \left| \sum_{\nu,\nu'} r_\nu(s)\, r_{\nu'}(s)\, \chi_{V^n_\nu + V^{n'}_{\nu'}}(\xi,\tau)\, \widehat{(P^n) T^\delta_{\nu,j}} g_n * \widehat{(P^{n'}) T^\delta_{\nu',j}} g_{n'} \right|^2 d\xi d\tau$$
$$\leq \int_{\mathbb{R}^3} \sum_{n,n'} \left(\sum_{\nu,\nu'} \chi_{V^n_\nu + V^{n'}_{\nu'}}(\xi,\tau) \right) \sum_{\nu,\nu'} \left| \widehat{P^n T^\delta_{\nu,j}} g_n * \widehat{P^{n'} T^\delta_{\nu',j}} g_{n'} \right|^2 d\xi d\tau.$$

In view of Lemma 3.4.4, with $\delta = 2^{\epsilon(j-1)}$, the above is atmost

$$C\, j\, 2^{\epsilon/2}\, \delta^{1/2} \int_{\mathbb{R}^3} \sum_{n,n'} \sum_{\nu,\nu'} \left| P^n T^\delta_{\nu,j} g_n \cdot P^{n'} T^\delta_{\nu',j} g_{n'} \right|^2 dx dt$$
$$= C\, j\, 2^{\epsilon/2}\, \delta^{1/2} \left\| \left(\sum_{n,\nu} |P^n T^\delta_{\nu,j} g_n|^2 \right)^{1/2} \right\|^4_{L^4(\mathbb{R}^3)} .$$

Now we show that

$$\left\| \left(\sum_{n,\nu} |P^n T^\delta_{\nu,j} g_n|^2 \right)^{1/2} \right\|_{L^4(\mathbb{R}^3)} \leq C \left\| \left(\sum_{n,\nu} |T^\delta_{\nu,j} g_n|^2 \right)^{1/2} \right\|_{L^4(\mathbb{R}^3)} \tag{3.33}$$

from which, the result will follow in view of Proposition 3.4.2.

To see (3.33), note that P^n is a multiplier in the t variable, and hence

$$P^n T^\delta_{\nu,j} g_n(x,t) = \int_{\mathbb{R}} T^\delta_{\nu,j} g_n(x, t-r)\, k(r)\, dr,$$

where $k(r) = 2^{j/2} e^{i2^{j/2}nr} \phi^\vee(2^{j/2} r)$. An application of the Minkowski's inequality with respect sum over n and ν, yields

$$\left(\sum_{n,\nu} |P^n T^\delta_{\nu,j} g_n(x,t)|^2 \right)^{1/2} \leq \int_{\mathbb{R}} \left(\sum_{n,\nu} |T^\delta_{\nu,j} g_n(x, t-r)|^2 \right)^{1/2} |k(r)|\, dr.$$

By Minkowski's inequality for integrals, (3.33) follows from this, since $\int_{\mathbb{R}} |k| = \int_{\mathbb{R}} |\phi^\vee|$ is independent of j. This gives the required L^4 estimate for $T^\delta_{s,j} g$.

Since the multiplier operator $T^\delta_{s,j}$ is bounded on $L^4(\mathbb{R}^3)$ with norm bound independent of s, we also have $T^\delta_{s,j}$ is bounded on $L^{4/3}$ with same bound as $4/3$ is the index dual to 4. Thus by Riesz-Thorin interpolation theorem [see, [39]], we see that $T^\delta_{s,j}$ is bounded on $L^p(\mathbb{R}^3)$ for $4/3 \leq p \leq 4$, with norm independent of s. Hence the required square function estimate follows from (3.32). $\square$

Remark 3.4.6 The L^4 estimate for Sg is essentially in the spirit of the result in [28]. Using Rademacher function argument we are able to extend this to $4/3 \leq p \leq 4$ which is new in our argument of local smoothing estimate.

3.5 L^p estimates for $\mathcal{R}_\delta(\mathcal{F}_j f)$

Recall that $\mathcal{R}_\delta$ was defined in (3.9) as a multiplier operator on $\mathbb{R}^3$ with multiplier $1 - \psi^\delta(\xi, \tau)$. For notational convenience, we write $R^\delta_j f$ for $\mathcal{R}_\delta(\mathcal{F}_j f)$.

The estimate for $R^\delta_j f$ relies on the rapid decay of the Fourier transform. The following lemma is the key ingredient for the same.

Lemma 3.5.1 *For $j \in \mathbb{N},\ 0 < \delta < 2^j$, consider the set*

$$A^\delta_j = \{(\xi, \tau) \in \mathbb{R}^2 \times \mathbb{R} : 2^{j-1} \leq |\xi| \leq 2^{j+1}, |\tau - |\xi|| > \delta\}.$$

Then, for each $0 < \epsilon < \frac{1}{2}$, *there exists* $C_{j,\epsilon,\delta} > 0$ *such that the estimate*

$$|\tau - |\xi|| > C_{j,\epsilon,\delta}(|\tau| + |\xi|)^{\epsilon}$$

holds in A_j^δ *with* $C_{j,\epsilon,\delta} = 6^{-\epsilon}\,\delta 2^{-j\epsilon}$. *In particular* $C_{j,\epsilon,\delta} := C_\epsilon = 12^{-\epsilon}$ *when* $\delta = 2^{\epsilon(j-1)}$.

Proof. The required inequality clearly holds when $\tau \leq 0$, with $C_{j,\epsilon,\delta} = 1$. So we assume $\tau > 0$. We write $A_j^\delta = B_1 \cup B_2$ where

$$B_1 = \{(\xi,\tau) \in A_j^\delta : \tau > 2|\xi|\},\ B_2 = \{(\xi,\tau) \in A_j^\delta : \tau \leq 2|\xi|\}.$$

We show that $\inf_{(\xi,\tau)\in B_i} \frac{|\tau-|\xi||}{(\tau+|\xi|)^\epsilon} \geq C_{j,\epsilon,\delta}$ for $i = 1, 2$ for some $C_{j,\epsilon,\delta} > 0$. Since $\tau > 2|\xi|$ on B_1, we have

$$\frac{|\tau - |\xi||}{(\tau + |\xi|)^\epsilon} = \frac{\tau - |\xi|}{(\tau + |\xi|)^\epsilon} = \tau^{1-\epsilon}\frac{1-\theta}{(1+\theta)^\epsilon}$$

where $\theta = \frac{|\xi|}{\tau} < \frac{1}{2}$ on B_1. Hence $1 - \theta > \frac{1}{2}$ and $1 + \theta < \frac{3}{2}$. It follows that for $(\xi,\tau) \in B_1$,

$$\frac{|\tau - |\xi||}{(\tau + |\xi|)^\epsilon} > \frac{1}{2}\left(\frac{2}{3}\right)^\epsilon 2^{j(1-\epsilon)}.$$

On the other hand on B_2, we have

$$\frac{|\tau - |\xi||}{(\tau + |\xi|)^\epsilon} > \frac{\delta}{(\tau + |\xi|)^\epsilon} \geq \frac{\delta}{(3|\xi|)^\epsilon} \geq \frac{\delta}{2^{j\epsilon}}6^{-\epsilon}. \tag{3.34}$$

Clearly, if we choose $\delta = 2^{\epsilon(j-1)}$, we get $\inf_{(\xi,\tau)\in B_2} \frac{|\tau-|\xi||}{(\tau+|\xi|)^\epsilon} \geq 12^{-\epsilon}$. □

Proposition 3.5.2 *Let* R_j^δ *be as in* (3.9) *with* $\delta = 2^{\epsilon(j-1)}, \epsilon \in (0, \frac{1}{2})$ *and* $j \in \mathbb{N}$. *Then,* $R_j^\delta f$ *satisfies the inequality*

$$\|R_j^\delta f\|_{L^p(\mathbb{R}^2\times\mathbb{R})} \leq C_2\|f\|_{L^p},\ 1 \leq p \leq \infty \tag{3.35}$$

for all $f \in L^p(\mathbb{R}^2)$ *with constant* C_2, *independent of* δ *and* j.

Proof. Since $f \to R_j^\delta f$ is a linear map, it is enough to estimate (3.35) for $f \in \mathscr{S}(\mathbb{R}^2)$. We have

$$\widehat{R_j^\delta(f)}(\xi,\tau) = [1 - \psi^\delta(\xi,\tau)]\,\hat{f}(\xi)\,a(\xi)\,\rho_0(2^{-j}|\xi|)\,\hat{\rho}_1(\tau - |\xi|) \tag{3.36}$$

Using (3.36) and expanding $\hat{f}$, we get

$$\begin{aligned} R_j^\delta f(x,t) &= \int_{(\xi,\tau)\in\mathbb{R}^2\times\mathbb{R}} e^{i[x.\xi+t\tau]}\, \widehat{R_j^\delta(f)}(\xi,\tau)d\xi d\tau \\ &= \int f(y)\mathcal{K}_j^\delta(x-y,t)dy \end{aligned} \tag{3.37}$$

where,

$$\mathcal{K}_j^\delta(x,t) = \int_{\xi.\tau} e^{ix.\xi}e^{it.\tau}[1-\psi^\delta(\xi,\tau)]a(\xi)\rho_0(2^{-j}|\xi|)\hat{\rho}_1(\tau-|\xi|)\, d\xi d\tau. \tag{3.38}$$

Now observe that for all $N\in\mathbb{N}$,

$$(1+|x|^2)^N(1+|t|^2)^N\, e^{i[x.\xi+t.\tau]} = (I-\Delta_\xi)^N(I-\partial_\tau^2)^N\, e^{i[x.\xi+t.\tau]}.$$

Hence an integration by parts shows that

$$\begin{aligned} &(1+|x|^2)^N\,(1+|t|^2)^N\,\mathcal{K}_j^\delta(x,t) \\ &\quad= \int_{\xi,\tau} e^{i[x.\xi+t.\tau]}\,(I-\Delta_\xi)^N(I-\partial_\tau^2)^N\, b_j(\xi,\tau)\, d\xi d\tau, \end{aligned} \tag{3.39}$$

where $b_j(\xi,\tau) = [1-\psi^\delta(\xi,\tau)]a(\xi)\rho_0(2^{-j}|\xi|)\hat{\rho}_1(\tau-|\xi|)$.

Note that $(I-\Delta_\xi)^N(I-\partial_\tau^2)^N\, b_j(\xi,\tau)$ is a sum of terms that involves various partial derivatives of order up to $4N$, of the functions $\psi^\delta(\xi,\tau), a(\xi), \rho_0(2^{-j}|\xi|)$ and $\hat{\rho}_1(\tau-|\xi|)$. Each derivative on ψ^δ brings in a negative power of δ and since $\delta = 2^{\epsilon(j-1)}$, all these derivatives are bounded uniformly in ϵ and j. Same is the case with a and ρ_0.

Since $\hat{\rho}_1$ is a Schwartz class function, for each $M,N\in\mathbb{N}$, there is a constant $C_{M,N}$ such that $\hat{\rho}_1(y)$ and all its derivatives are bounded by a constant times $(1+|y|)^{-M}$. It follows that for each $N,M\in\mathbb{N}$, there is a constant $C_{M,N}$, independent of ϵ and j such that

$$\begin{aligned} |(I-\Delta_\xi)^N(I-\partial_\tau^2)^N\, b_j(\xi,\tau)| &\le C_{M,N}(1+|\tau-|\xi||)^{-M} \\ &\le C_{M,N}(1+C_\epsilon(|\tau|+|\xi|)^\epsilon)^{-M} \end{aligned} \tag{3.40}$$

for $|\tau-|\xi||>\delta$, by Lemma 3.5.1. Note that the integral in (3.38) and hence in (3.39) is actually over the set $|\tau-|\xi||>\delta$, as $\psi^\delta(\xi,\tau)=1$ on $|\tau-|\xi||\le\delta$, hence Lemma 3.5.1 is applicable.

Using (3.40) in (3.39), the right hand side on (3.39) is bounded by

$$\begin{aligned}
&C_{M,N}\int_{\mathbb{R}^2\times\mathbb{R}}(1+C_\epsilon(|\tau|+|\xi|)^\epsilon)^{-M}\,d\xi d\tau\\
&\qquad\le C_{M,N}\int_{|(\xi,\tau)|\le 1}d\xi d\tau + C_{M,N}\,(C_\epsilon)^{-M}\int_{|(\xi,\tau)|>1}(|\tau|+|\xi|)^{-\epsilon M}\,d\xi d\tau\\
&\qquad\le C_{M,N}\left(1+(C_\epsilon)^{-M}\int_{|(\xi,\tau)|>1}(|\tau|^2+|\xi|^2)^{-\epsilon M/2}\,d\xi d\tau\right)\\
&\qquad\le C_{M,N}\left(1+(C_\epsilon)^{-M}\int_{r>1}(r)^{-\epsilon M}\,r^2\,dr\right)\le C_{M,N}(1+(C_\epsilon)^{-M})
\end{aligned}$$

The last step follows by choosing M such that $M>3/\epsilon$ for a given fixed $\epsilon>0$. Hence, (3.39) translates to the inequality,

$$\mathcal{K}_j^\delta(x,t)\le\frac{C_{M,N}(1+(C_\epsilon)^{-M})}{(1+|x|^2)^N(1+|t|^2)^N} \tag{3.41}$$

It follows that $\mathcal{K}_j\in L^1(\mathbb{R}^2)$ for $N>1$ and

$$\|\mathcal{K}_j\|_{L^1(dx)}\lesssim\frac{(1+(C_\epsilon)^{-M})}{(1+|t|^2)^N}\le\frac{(1+12^4)}{(1+|t|^2)^N},$$

as $C_\epsilon=12^{-\epsilon}$ and choosing M such that $3<\epsilon M\le 4$. Using (3.41) in (3.37), we get

$$\left|R_j^\delta f(x,t)\right|\le\frac{C_{M,N}(1+(C_\epsilon)^{-M})}{(1+|t|^2)^N}\cdot\left(f*\frac{1}{(1+|x|^2)^N}\right).$$

Hence by Young's inequality, we get for $1\le p\le\infty$

$$\|R_j^\delta f\|_{L^p(dx)}^p\lesssim\left[\frac{(1+12^4)}{(1+|t|^2)^N}\right]^p\cdot\|f\|_{L^p(dx)}^p.$$

Thus, a t-integration gives the required estimate and hence the proof. □

3.6 L^4 estimates for Q_j^δ and $\mathcal{F}_j f$

We use the duality argument to estimate $Q_\delta(\mathcal{F}_j f)$, combined with a Littlewood-Paley type argument, but using a square function based on angular decomposition discussed in Section 3.4. In the following proposition, we first estimate

$Q_\delta(\mathcal{F}_j f)$ in terms of a square-function.

Proposition 3.6.1 *Let $Q_\delta(\mathcal{F}_j f)$ be as in (3.9), with $\delta = 2^{\epsilon(j-1)}, \epsilon \in (0, \frac{1}{2})$ and $j \in \mathbb{N}$ and $\tilde{\mathcal{F}}_{j,\nu} f$ be as in (3.13). Then the inequality*

$$\|Q_\delta(\mathcal{F}_j f)\|_{L^4(\mathbb{R}^3)} \leq C\, j\, 2^{j/8}\, \delta^{3/8} \left\|(\sum_\nu |\tilde{\mathcal{F}}_{j,\nu} f|^2)^{\frac{1}{2}}\right\|_{L^4(\mathbb{R}^3)}$$

holds for all $f \in \mathscr{S}(\mathbb{R}^2)$ with a constant C independent of j and δ.

Proof. Writing $Q_\delta(\mathcal{F}_j f) = \sum_\nu Q_\delta(\mathcal{F}_{j,\nu} f)$, and by duality we have

$$||Q_\delta(\mathcal{F}_j f)||_{L^4} = \sup_{\|H\|_{4/3} \leq 1} \int_{\mathbb{R}^3} \sum_\nu Q_\delta(\mathcal{F}_{j,\nu} f)(x,t)\, \overline{H}(x,t)\, dx dt.$$

By Parseval's theorem for the Fourier transform, we have

$$\begin{aligned} \int_{\mathbb{R}^3} Q_\delta(\mathcal{F}_{j,\nu} f)(x,t)\, \overline{H}(x,t)\, dx dt \quad &= \int_{\mathbb{R}^3} \widehat{Q_\delta(\mathcal{F}_{j,\nu}} f)(\xi,\tau)\, \overline{\widehat{H}}(\xi,\tau)\, d\xi d\tau \\ &= \int_{\mathbb{R}^3} (\tilde{\mathcal{F}}_{j,\nu} f)(x,t)\, T^\delta_{\nu,j} H(x,t)\, dx dt \qquad (3.42) \end{aligned}$$

in view of (3.9), where $\tilde{\mathcal{F}}_{j,\nu}$ is as in (3.13), and $T^\delta_{\nu,j}$ is the multiplier operator given by

$$\widehat{T^\delta_{\nu,j} H}\,(\xi,\tau) = \tilde{\chi_\nu}(\xi)\, \rho_0(2^{-j}|\xi|)\, \psi\left(\frac{|\xi| - \tau}{\delta}\right)\, \hat{H}(\xi,\tau), \;\; H \in \mathscr{S}(\mathbb{R}^3) \qquad (3.43)$$

with $\tilde{\chi}_\nu$ is as in (3.16), such that $\tilde{\chi}_\nu \chi_\nu = \chi_\nu$.

Now summing over ν in (3.42) and using Cauchy-Schwarz inequality with respect to ν and an application of Hölder's inequality yields

$$\begin{aligned} \langle Q_\delta(\mathcal{F}_j f), H\rangle &= \int_{\mathbb{R}^3} \sum_\nu (\tilde{\mathcal{F}}_{j,\nu} f)(x,t)\, T^\delta_{\nu,j} H(x,t)\, dx dt \\ &\leq \left\| \left(\sum_\nu |(\tilde{\mathcal{F}}_{j,\nu} f)|^2 \right)^{\frac{1}{2}} \right\|_{L^4(\mathbb{R}^3)} \left\| \left(\sum_\nu \left|T^\delta_{\nu,j} H\right|^2 \right)^{\frac{1}{2}} \right\|_{L^{4/3}(\mathbb{R}^3)}. \qquad (3.44) \end{aligned}$$

Note that $T^\delta_{\nu,j}$ defined by (3.43) is a multiplier operator on $\mathbb{R}^3$, with multiplier $\tilde{\chi_\nu}(\xi)\, \rho_0(2^{-j}|\xi|)\, \psi\left(\frac{|\xi|-\tau}{\delta}\right)$. Hence by Proposition 3.4.5, the second term on the right hand side of (3.44) is bounded by $C\, j\, 2^{j/8}\, \delta^{3/8}\, \|H\|_{4/3}$. Taking supremum over $\|H\|_{4/3} \le 1$, this yields

$$\|Q_\delta(\mathcal{F}_j f)\|_4 \le C\, j\, 2^{j/8}\, \delta^{3/8} \left\| \left(\sum_\nu |\tilde{\mathcal{F}}_{j,\nu} f|^2 \right)^{\frac{1}{2}} \right\|_{L^4(\mathbb{R}^3)} .$$

Hence the proof. □

Next we prove the following $L^4(\mathbb{R}^2) \to L^4(\mathbb{R}^3)$ square function estimate. This result is essentially in the spirit of the result in [28, pp. 213]. With χ_ν as in (2.3), we first define the multiplier operator f_ν by

$$\widehat{f_\nu}(\xi) = \tilde{\tilde{\chi}}_\nu(\xi)\, \hat{f}(\xi), \tag{3.45}$$

where $\tilde{\tilde{\chi}}_\nu$ denote the characteristic function of the support of χ_ν.

Proposition 3.6.2 *Let $\tilde{\mathcal{F}}_{j,\nu} f$ and f_ν be as in (3.13) and (3.45) respectively. Then the following square function estimate holds*

$$\left\| \left(\sum_\nu |\tilde{\mathcal{F}}_{j,\nu} f_\nu|^2 \right)^{\frac{1}{2}} \right\|_{L^4(\mathbb{R}^3)} \le C\, j^{3/4+b}\, \|f\|_{L^4(\mathbb{R}^2)} \tag{3.46}$$

for all $f \in \mathscr{S}(\mathbb{R}^2)$ with a constant C independent of j and for some $b > 0$.

Proof. Note that $\tilde{\mathcal{F}}_{j,\nu} f = \tilde{\mathcal{F}}_{j,\nu} f_\nu$ in view of (3.45) as $\chi_\nu = \chi_\nu \tilde{\tilde{\chi}}_\nu$. Thus

$$\tilde{\mathcal{F}}_{j,\nu} f_\nu(x,t) = \int_{\mathbb{R}^2} K^a_{j,\nu}(x-y,t)\, f_\nu(y)\, dy, \tag{3.47}$$

with $K^a_{j,\nu}$ as in (2.5).
Using Cauchy-Schwarz inequality in (3.47) and summing over ν, we get

$$\begin{aligned} \sum_\nu |\tilde{\mathcal{F}}_{j,\nu} f_\nu|^2 &\le \sum_\nu \left(\int_{\mathbb{R}^2} |f_\nu|^2 |K^a_{j,\nu}(x-y,t)|\, dy \right) \left(\int_{\mathbb{R}^2} |K^a_{j,\nu}(x-y,t)|\, dy \right) \\ &\le C \int_{y\in\mathbb{R}^2} \sum_\nu |f_\nu|^2 |K^a_{j,\nu}(x-y,t)| dy \end{aligned} \tag{3.48}$$

since $\left\| K^a_{j,\nu}(\cdot,t) \right\|_{L^1 dx} \le C$ for some constant C independent of t and j, by Lemma

2.2.1.

Squaring and integrating, this leads to the inequality

$$\left\| \left(\sum_\nu |\tilde{\mathcal{F}}_{j,\nu} f_\nu|^2 \right)^{\frac{1}{2}} \right\|^2_{L^4(\mathbb{R}^3)} \tag{3.49}$$

$$\begin{aligned}
&\leq C \left[\int_{x,t} \left[\int_y \sum_\nu |f_\nu(y)|^2 |K^a_{j,\nu}(x-y,t)| dy \right]^2 dxdt \right]^{\frac{1}{2}} \\
&= C \sup_{\|g\|_{L^2}=1} \left| \int_{x,t} \left[\int_y \sum_\nu |f_\nu(y)|^2 |K^a_{j,\nu}(x-y,t)|\, dy \right] g(x,t)\, dxdt \right| \\
&\leq C \sup_{\|g\|_{L^2}=1} \left| \int_y \sum_\nu |f_\nu(y)|^2 \left[\int_{x,t} |K^a_{j,\nu}(x-y,t)|\, |g(x,t)|\, dxdt \right] dy \right|
\end{aligned}$$

where we used duality in the above equality for the $L^2(dxdt)$ norm, and Fubini's theorem in the last step.

By Cauchy Schwarz inequality in y variable, the term inside the modulus sign is at most

$$C \left[\int_y \left(\sum_\nu |f_\nu(y)|^2 \right)^2 dy \right]^{1/2} \times$$
$$\left[\int_y \sup_\nu \left| \int_{x,t} |K^a_{j,\nu}(x-y,t)|\, |g(x,t)|\, dxdt \right|^2 dy \right]^{\frac{1}{2}}.$$

Note that the first term above is $\left\| (\sum_\nu |f_\nu(y)|^2)^{1/2} \right\|^2_{L^4}$. But, Cordoba [7] proved that there exist a constants C independent of N, so that the following inequality holds

$$\| \left(\sum_{\nu=1}^N |f_\nu|^2 \right)^{1/2} \|_{L^4(\mathbb{R}^2)} \leq C[\log N]^b \, \|f\|_{L^4(\mathbb{R}^2)} \tag{3.50}$$

for some $b > 0$. Hence by (3.50) with $N \approx 2^{j/2}$, we see that the first term above is at most $Cj^{2b} \|f\|^2_{L^4}$, with C independent of j.

Also since the kernel $K^a_{j,\nu}$ satisfies the point wise estimate (2.10), appealing to Lemma 1.4 in [28], we conclude that the second term above satisfy the estimate

$$\left[\int_y \sup_\nu \left|\int_{x,t} |K^a_{j,\nu}(x-y,t)|\,|g(x,t)|\,dxdt\right|^2 dy\right]^{\frac{1}{2}} \le C|\log(2^{-j})|^{3/2}\|g\|_{L^2(\mathbb{R}^3)}$$
$$= Cj^{3/2}\|g\|_{L^2(\mathbb{R}^3)}$$

with C independent of j. Using these estimates in (3.49) and taking the square root, the proof follows. □

Remark 3.6.3 Proposition 3.6.2 relies on the result of Cordoba [7], which brings the restriction in dimension $n = 2$. Hence, the local smoothing estimate proved in this dissertation is valid only for $n = 2$.

Proposition 3.6.4 *Let $Q_\delta \mathcal{F}_j f$ be as in (3.9), with $\delta = 2^{\epsilon(j-1)}, \epsilon \in (0, \frac{1}{2})$ and $j \in \mathbb{N}$. Then we have the following estimate*

$$\|Q_\delta \mathcal{F}_j f\|_4 \le C\, j^{7/4+b}\, 2^{j/8}\, \delta^{3/8}\, \|f\|_{L^4(\mathbb{R}^2)},$$

for all $f \in L^4(\mathbb{R}^2)$, with a constant C independent of j and for some $b > 0$.

Proof. The proof follows from Proposition 3.6.1 and Proposition 3.6.2, by the density of $\mathscr{S}(\mathbb{R}^2)$ in $L^4(\mathbb{R}^2)$. □

Proposition 3.6.5 *Let $\mathcal{F}_j f$ be as in (3.12). Then, for a given fixed $\epsilon \in (0, \frac{1}{2})$, we have the following estimate*

$$\|\mathcal{F}_j f\|_4 \le C\, j^{7/4+b}\, 2^{j(1+3\epsilon)/8}\, \|f\|_{L^4(\mathbb{R}^2)},$$

for all $f \in L^4(\mathbb{R}^2)$, with a constant C independent of j and for some $b > 0$.

Proof. Writing $\mathcal{F}_j f = Q_\delta(\mathcal{F}_j f) + \mathcal{R}_\delta(\mathcal{F}_j f)$, with $\delta = 2^{\epsilon(j-1)}$ and then the proof follows from Proposition 3.5.2 and Proposition 3.6.4, by the density of $\mathscr{S}(\mathbb{R}^2)$ in $L^4(\mathbb{R}^2)$. □

3.7 L^p Regularity Estimates

In this section, we give the proof of Theorem 3.1.1. We will show that the local smoothing estimate (3.3) will follow from the regularity estimates for $\mathcal{F}_j f$ by a summability argument. We first deduce the regularity estimate for $\mathcal{F}_j f$ from the L^p estimates. A key step in this reduction is the following Lemma.

Lemma 3.7.1 *For $f \in \mathscr{S}(\mathbb{R}^2)$, let $f_{\sigma,j}$ be given by*

$$\widehat{f_{\sigma,j}}(\xi) = \hat{f}(\xi)\,\rho(2^{-j}|\xi|)\,(1+|\xi|^2)^{\sigma/2}, \;\; Re(\sigma) < 0.$$

Then for $1 \leq p \leq \infty$, we have

$$\|f_{\sigma,j}\|_{L^p(\mathbb{R}^2)} \leq C\,2^{j\,Re(\sigma)}\,\|f\|_{L^p(\mathbb{R}^2)}.$$

Proof. We have,

$$f_{\sigma,j}(x) = f * k(x),$$

where k is the inverse Fourier transform of $\rho(2^{-j}|\xi|)\,(1+|\xi|^2)^{\sigma/2}$:

$$\begin{aligned} k(x) &= (2\pi)^{-2}\int_\xi e^{ix\cdot\xi}\rho(2^{-j}|\xi|)\,(1+|\xi|^2)^{\sigma/2}\,d\xi \\ &= \frac{2^{2j}}{(2\pi)^2}\int_{\frac{1}{2}\leq|\xi|\leq 2} e^{i2^jx\cdot\xi}\rho(|\xi|)\,(1+2^{2j}|\xi|^2)^{\sigma/2}\,d\xi. \end{aligned}$$

Thus the proof follows by Young's inequality if we show that $\|k\|_1 \leq C2^{j\mathrm{Re}(\sigma)}$.

Since $(1+|2^jx|^2)^N\,e^{i2^jx\cdot\xi} = (I-\Delta_\xi)^N e^{i2^jx\cdot\xi}$ for $N \in \mathbb{N}$, an integration by parts shows that

$$k(x) = (2\pi)^{-2}\,\frac{2^{2j}}{(1+|2^jx|^2)^N}\int_{\frac{1}{2}\leq|\xi|\leq 2} e^{i2^jx\cdot\xi}\,(I-\Delta)^N\left[\rho(|\xi|)\,(1+2^{2j}|\xi|^2)^{\sigma/2}\right]\,d\xi.$$

We show that $\|k\|_1 \leq C2^{j\mathrm{Re}(\sigma)}$, by estimating the above integral. Observe that on the support of ρ we have

$$|\partial^\alpha(1+2^{2j}|\xi|^2)^{\sigma/2}| \leq C_N(1+2^{2j}|\xi|^2)^{\mathrm{Re}(\sigma)/2}$$

for $|\alpha| \leq N$, for some constant C_N independent of j. Thus expanding $(1-\Delta)^N$ and using the fact that ρ and all its partial derivatives are bounded, we see that for $\mathrm{Re}(\sigma) < 0$

$$|k(x)| \;\; \leq \;\; C_N'\frac{2^{2j}}{(1+|2^jx|^2)^N}\int_{\frac{1}{2}\leq|\xi|\leq 2}(1+2^{2j}|\xi|^2)^{\mathrm{Re}(\sigma)/2}\,d\xi$$

$$
\begin{aligned}
&\leq C_N' \frac{2^{2j}}{(1+|2^j x|^2)^N} \int_{\frac{1}{2}\leq|\xi|\leq 2} C\,(2^{2j}|\xi|^2)^{\mathrm{Re}(\sigma)/2}\,d\xi \\
&\leq C_N'' \frac{2^{2j}}{(1+|2^j x|^2)^N}\, 2^{j\mathrm{Re}(\sigma)}
\end{aligned}
$$

for all $N \in \mathbb{N}$. Taking $N = 2$ we get the required estimate and this ends the proof. □

We recall the Fourier integral operator $\tilde{\mathcal{F}}_j$ introduced in (3.13)

$$
\tilde{\mathcal{F}}_j f(x,t) = \rho_1(t) \int_{\mathbb{R}^2} e^{i(x\cdot\xi + t|\xi|)}\, \rho(2^{-j}|\xi|)\, a(\xi)\, \hat{f}(\xi)\, d\xi \tag{3.51}
$$

which was used in estimating the L^4 norm of $\mathcal{F}_j f$ through Proposition 3.6.1 and Proposition 3.6.2. Note that the ρ_0 dependence on norm bounds in the Proposition 3.6.1 and Proposition 3.6.2 are through the bound of ρ_0 and its derivatives, which in turn depend only on the bound for ρ and its derivatives, as $\rho_0 = \rho^2$.

In fact, the Fourier integral operator $\tilde{\mathcal{F}}_j f$ also satisfies the same norm estimates as in Proposition 3.6.5 and Proposition 3.3.1, with the constant C depending only on ρ and its derivatives:

$$
\|\tilde{\mathcal{F}}_j f\|_{L^4(\mathbb{R}^2\times\mathbb{R})} \leq C\, j^{7/4+b}\, 2^{j(1+3\epsilon)/8}\, \|f\|_{L^4(\mathbb{R}^2)}, \tag{3.52}
$$
$$
\|\tilde{\mathcal{F}}_j f\|_{L^\infty(\mathbb{R}^2\times\mathbb{R})} \leq C\, 2^{j/2}\, \|f\|_{L^\infty(\mathbb{R}^2)}. \tag{3.53}
$$

Proposition 3.7.2 *Let $\mathcal{F}_j f$ be as in (3.4) with $a \in S^0(\mathbb{R}^2)$ and $Re(\sigma) < 0$. Then for each $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ independent of j such that the following estimate holds*

$$
\|(I-\Delta_x)^{\sigma/2}\mathcal{F}_j f\|_{L^p(\mathbb{R}^2\times\mathbb{R})} \leq C_\epsilon\, 2^{\theta j}\, \|f\|_{L^p(\mathbb{R}^2)}
$$

for $4 \leq p \leq \infty$, with $\theta = \theta_\epsilon = (17/2+4b)\epsilon/p + Re(\sigma) + (1/2 - 3/2p)$.

Proof. Set $\mathcal{L} = (I-\Delta_x)^{1/2}$. Then we have $\mathcal{L}^\sigma(\mathcal{F}_j f) = \tilde{\mathcal{F}}_j(f_{\sigma,j})$ where $\tilde{\mathcal{F}}_j$ and $f_{\sigma,j}$ are as in (3.13) and Lemma 3.7.1 respectively. This can be seen by taking Fourier transform in the x-variable and the fact that $\rho_0 = \rho\cdot\rho$.

Now by Riesz-Thorin interpolation, (3.52) and (3.53) yields

$$
\|\tilde{\mathcal{F}}_j f\|_{L^p(\mathbb{R}^2\times\mathbb{R})} \leq C\, 2^{j(1+3\epsilon)(1-t)/8}\, j^{(7/4+b)(1-t)}\, 2^{tj/2}\, \|f\|_{L^p(\mathbb{R}^2)}, \tag{3.54}
$$

for $4 \leq p \leq \infty$, where $\frac{1}{p} = \frac{1-t}{4}$.

Using (3.54) with $f_{\sigma,j}$, $\mathrm{Re}(\sigma) < 0$, and in view of the estimate for $f_{\sigma,j}$, given by Lemma 3.7.1, we see that

$$\|\mathcal{L}^\sigma(\mathcal{F}_j f)\|_{L^p(\mathbb{R}^2\times\mathbb{R})} \leq C\, j^{(7+4b)/p}\, 2^{(1/2-3/2p)j}\, 2^{3\epsilon j/2p}\, 2^{j\mathrm{Re}(\sigma)}\, \|f\|_{L^p(\mathbb{R}^2)}. \quad (3.55)$$

Since $j \leq C_\epsilon 2^{\epsilon j}$ for any $\epsilon > 0$, we have $j^{(7+4b)/p}\, 2^{3\epsilon j/2p}\, 2^{(1/2-3/2p)j}\, 2^{j\mathrm{Re}(\sigma)} \leq C_\epsilon 2^{\theta j}$, where $\theta = \theta_\epsilon = (17/2+4b)\epsilon/p + (1/2-3/2p) + \mathrm{Re}(\sigma)$. Since $\epsilon > 0$ is arbitrary, this completes the proof. □

Remark 3.7.3 Note that the Proposition 3.7.2 also holds for $\mathcal{L}^s\tilde{\mathcal{F}}_j$ for $s \leq 0$. In fact, the composition of $\mathcal{L}^s$ with the Fourier integral operator $\tilde{\mathcal{F}}_j$ has the effect of multiplying the amplitude function by $(1+|\xi|^2)^{s/2}$ whose derivatives are all bounded when $s \leq 0$.

Now we give the proof of Theorem 3.1.1.

Proof. (of Theorem 3.1.1) In view of Proposition 3.7.2, we have

$$\|\mathcal{L}^\sigma(\mathcal{F}_j f)\|_{L^p(\mathbb{R}^2\times\mathbb{R})} \leq C_\epsilon\, 2^{\theta j}\, \|f\|_{L^p(\mathbb{R}^2)},\ \ 4 \leq p \leq \infty$$

where $\theta = \theta_\epsilon = (17/2+4b)\epsilon/p + (1/2-3/2p) + \mathrm{Re}(\sigma)$. Note that $\theta < 0$ whenever $\mathrm{Re}(\sigma) < \frac{3}{2p} - \frac{1}{2} - (17/2+4b)\epsilon/p$ and hence $\sum_{j=0}^{\infty} \mathcal{L}^\sigma\mathcal{F}_j f$ is absolutely summable in $L^p(\mathbb{R}^3)$. It follows that

$$\|\mathcal{L}^\sigma\mathcal{F}f\|_{L^p(\mathbb{R}^2\times\mathbb{R})} \leq \sum_{j=0}^{\infty} \|\mathcal{L}^\sigma\mathcal{F}_j f\|_{L^p(\mathbb{R}^2\times\mathbb{R})} \leq C_{\epsilon,\sigma}\|f\|_{L^p(\mathbb{R}^2)}, \quad (3.56)$$

for $\mathrm{Re}(\sigma) < \sigma_\epsilon = \frac{3}{2p} - \frac{1}{2} - (17/2+4b)\epsilon/p$ with $C_{\epsilon,\sigma} = C_\epsilon \sum_{j=0}^{\infty} 2^{\theta j} < \infty$. Note that $\epsilon > 0$ is arbitrary, and $\sigma_\epsilon \to \frac{3}{2p} - \frac{1}{2}$ as $\epsilon \to 0$. Thus for any given σ with $\mathrm{Re}(\sigma) < \frac{3}{2p} - \frac{1}{2}$, we have $\mathrm{Re}(\sigma) < \sigma_\epsilon$ for some small $\epsilon > 0$. Hence it follows that for $4 \leq p \leq \infty$ and $\mathrm{Re}(\sigma) < \frac{3}{2p} - \frac{1}{2}$, there exists a C_σ such that the estimate (3.56) holds. Hence the proof. □

Chapter 4

An Application to the Wave Equation

In this brief chapter, we extend the local smoothing estimate obtained in chapter 3 to Fourier integral operators with amplitude function $a \in S^m(\mathbb{R}^2)$, $m < 0$. Using this, we obtain a local smoothing estimate for the wave equation in the plane with prescribed initial profile and velocity.

Consider the Cauchy problem for the wave equation in the plane:

$$\begin{cases} (\partial_t^2 - \Delta)\, u(x,t) = 0, \\ u(x,0) = f(x), \partial_t u(x,0) = g(x). \end{cases} \tag{4.1}$$

Here $x \in \mathbb{R}^2,\ t \in \mathbb{R}$ and Δu means $\sum_{i=1}^{2} \frac{\partial^2 u(x,t)}{\partial x_i^2}$. In this chapter, we prove the local smoothing estimate of the solution $u(x,t)$ of (4.1). We write the solution $u(x,t)$ of (4.1) as

$$u(x,t) = (\mathcal{F}_t f)(x) + (\mathcal{G}_t g)(x), \tag{4.2}$$

where $\mathcal{F}_t$ and $\mathcal{G}_t$ are as in (1.6), Chapter 1 with $n = 2$. For fixed t, the study of the regularity property of the solution operator $u(x,t)$ on $\mathbb{R}^n, n \geq 2$ goes back to the work of Peral [30] and Miyachi [27]. The result of Peral and Miyachi says that

$$\|\mathcal{F}_t f\|_{L^p} \leq b_p(t) \|f\|_{L^p_\alpha} \text{ if and only if } |1/p - 1/2| \leq \alpha, \text{ and}$$

$$\|\mathcal{G}_t g\|_{L^p} \le c_p(t)\|g\|_{L^p_\alpha} \text{ if and only if } |1/p-1/2| \le \alpha+1,$$

where the constants $b_p(t)$ and $c_p(t)$ are computed explicitly in [27]. In this case $b_p(t) = B_p(1+t)^{|1/p-1/2|}$ and $c_p(t)$ has the following form : when $\alpha \ge 0$, $c_p(t) = C_p\, t\,(1+t)^{a(p)}$, where $a(p) = \max\{|1/p-1/2|-1, 0\}$; and when $-1 \le \alpha < 0$,

$$c_p(t) = \begin{cases} C_p\, t, & t \ge 1, \\ C_p t^{1+\alpha}, & 0 < t < 1, \end{cases}$$

where the constants B_p and C_p are independent of t.

The result of Peral and Miyachi gives the regularity of the solution for each fixed t and the estimate is the best possible valid for all t. However, when t is treated as a variable, $u(x,t)$ may exhibit additional smoothing. Therefore, a natural question arises at this point: To what extent is the following true: (assume $f = 0$ here)

$$\|u(x,t)\|_{L^p(\mathbb{R}^2\times[1,2])} \le B\|g\|_{L^p(\mathbb{R}^2)},\ g \in \mathscr{S}(\mathbb{R}^2)?$$

To answer this question, we need to consider a slightly more general class of Fourier integral operators of the form

$$\mathcal{F}f(x,t) = \int_{\mathbb{R}^2} e^{i(x\cdot\xi+t|\xi|)}\, a_1(t,\xi)\, \hat{f}(\xi)\, d\xi,\ f \in \mathscr{S}(\mathbb{R}^2) \tag{4.3}$$

with amplitude function $a_1(t,\xi) = \rho_1(t)a(\xi)$, where $\rho_1 \in C_c^\infty((1,2))$ and $a \in S^m(\mathbb{R}^2)$, $m \le 0$. Note that, the case $m = 0$ has been studied in chapter 3. The case $m < 0$ can be deduced from the case $m = 0$ as shown in Theorem 4.0.4 below.

Theorem 4.0.4 *Let $\mathcal{F}f$ be as in (4.3) with amplitude function $a \in S^m(\mathbb{R}^2)$, the symbol class of order $m < 0$. Then, the inequality*

$$\|(I-\Delta)^{\frac{\sigma-m}{2}}\mathcal{F}f\|_{L^p(\mathbb{R}^2\times\mathbb{R})} \le C_{\sigma,m,p}\|f\|_{L^p(\mathbb{R}^2)} \tag{4.4}$$

holds for all $f \in L^p(\mathbb{R}^2)$, for

$$\begin{cases} Re(\sigma) < 1/2(1/p-1/2), & \text{if } 2 < p \le 4, \\ Re(\sigma) < \frac{3}{2p} - \frac{1}{2}, & \text{if } 4 \le p < \infty. \end{cases}$$

Proof. Let $\mathcal{F}_j$ and $\tilde{\mathcal{F}}_j$ be the Fourier integral operators, as in (3.4) and (3.51) respectively in chapter 3 with the amplitude function $a \in S^m(\mathbb{R}^2)$, $m < 0$. Set $\mathcal{L} = (I - \Delta)^{1/2}$. We write $\mathcal{L}^{\sigma-m}(\mathcal{F}_j f) = [\mathcal{L}^{-m}\tilde{\mathcal{F}}_j](f_{\sigma,j})$, where $f_{\sigma,j}$ is as in Lemma 3.7.1, Chapter 3, given by

$$\widehat{f_{\sigma,j}}(\xi) = \hat{f}(\xi)\,\rho(2^{-j}|\xi|)\,(1 + |\xi|^2)^{\sigma/2},\ \mathrm{Re}(\sigma) < 0.$$

This can be seen by taking Fourier transform in the x-variable and the fact that $\rho_0 = \rho \cdot \rho$.

The key observation is that $\mathcal{L}^{-m}\tilde{\mathcal{F}}_j$ is a Fourier integral operator with symbol in $S^0(\mathbb{R}^2)$ if $a \in S^m(\mathbb{R}^2)$. This follows since $(1 + |\xi|^2)^{-m/2} \in S^{-m}(\mathbb{R}^2)$ and the fact that by Leibniz's rule, the order of the product of two symbols is the sum of their orders.

Thus using the estimates in Proposition 3.7.2, Chapter 3, for the Fourier integral operator $\mathcal{L}^{-m}\tilde{\mathcal{F}}_j$, which is valid in view of Remark 3.7.3, Chapter 3, we get

$$\|\mathcal{L}^{\sigma-m}(\mathcal{F}_j f)\|_{L^p(\mathbb{R}^2\times\mathbb{R})} \leq C_{\epsilon,m}\, 2^{\theta j}\, \|f\|_{L^p(\mathbb{R}^2)},\ 4 \leq p \leq \infty$$

where the constant $C_{\epsilon,m}$ is independent of j for each $\epsilon > 0$ and $\theta = (17/2 + 4b)\epsilon/p + \mathrm{Re}(\sigma) + (1/2 - 3/2p)$ for some $b > 0$. Thus by the summability and the limiting arguments as in the proof of Theorem 3.1.1, Chapter 3, we get the estimate

$$\|\mathcal{L}^{\sigma-m}\mathcal{F}f\|_{L^p(\mathbb{R}^2\times\mathbb{R})} \leq C_{\sigma,m}\|f\|_{L^p(\mathbb{R}^2)} \tag{4.5}$$

valid for $\mathrm{Re}(\sigma) < \frac{3}{2p} - \frac{1}{2}$, $4 \leq p \leq \infty$.

The L^2 estimate for the Fourier integral operator $\mathcal{L}^{\sigma-m}(\mathcal{F}f)$ will follow immediately from Plancheral theorem for $\mathrm{Re}(\sigma) \leq 0$. Hence, the required estimate (4.4) for $2 < p \leq 4$ and $\mathrm{Re}(\sigma) < 1/2(1/p - 1/2)$, follows by analytic interpolation (see, [40]) between the above L^2 and L^4 estimates for the Fourier integral operator $\mathcal{L}^{\sigma-m}(\mathcal{F}f)$. This completes the proof. □

Let Ff be the Fourier integral operator as in (4.3) with amplitude function $a \in S^0(\mathbb{R}^2)$, the symbol class of order 0. Then we have $\mathcal{L}^\sigma(Ff) = F(\mathcal{L}^\sigma f)$, where $\mathcal{L} = (I - \Delta)^{1/2}$. This can be seen by taking Fourier transform in the x-

variable. Now, we can re write the estimate in Theorem 3.1.1, Chapter 3, as

$$\|Ff\|_{L^p(\mathbb{R}^2\times I)} \le C_\sigma \|\mathcal{L}^{-\sigma} f\|_p := C_\sigma \|f\|_{L^p_{-\sigma}}, \tag{4.6}$$

with $\mathrm{Re}(\sigma) < \frac{3}{2p} - \frac{1}{2}$ for $4 \le p \le \infty$ and $\mathrm{Re}(\sigma) < 1/2(1/p - 1/2)$ for $2 < p < 4$. Since only real valued σ is relevant here, we can assume that σ is real.

Using Theorem 4.0.4, we obtain the following local smoothing estimate for the initial value problem (4.1) for the wave equation in terms of $L^p_\alpha(\mathbb{R}^2)$. In fact, we get the same smoothing as obtained in [28].

Theorem 4.0.5 *Let $u(x,t)$ denote the solution to the Cauchy problem (4.1). Set $\sigma_p = 1/2(1/p - 1/2)$, for $2 < p \le 4$ and $\sigma_p = \frac{3}{2p} - \frac{1}{2}$, for $4 \le p < \infty$. Then u satisfies the inequality*

$$\|u(x,t)\|_{L^p(\mathbb{R}^2\times I)} \le C_I \left(\|f\|_{L^p_{-\sigma}(\mathbb{R}^2)} + \|g\|_{L^p_{-(\sigma+1)}(\mathbb{R}^2)} \right) \tag{4.7}$$

for $\sigma < \sigma_p$, for any compact time interval $I \subset (0,\infty)$. The constant C_I depends on p, σ and I.

Proof. The solution $u(x,t)$ to the wave equation (4.1) is given by (4.2) in terms of the operators $\mathcal{F}_t$ and $\mathcal{G}_t$. Writing $\cos(t|\xi|) = (e^{it|\xi|} + e^{-it|\xi|})/2$ and $\sin(t|\xi|) = (e^{it|\xi|} - e^{-it|\xi|})/2i$, we see that $\mathcal{F}_t$ is the average of the operators $\mathcal{F}^\pm$ given by

$$\mathcal{F}^\pm f(x,t) = \int_{\mathbb{R}^2} e^{i(x\cdot\xi \pm t|\xi|)} \hat{f}(\xi) d\xi.$$

The operators $\mathcal{F}^\pm f$ are Fourier integral operators as in (3.1) Chapter 3, with the amplitude function $a(\xi) \equiv 1$. Hence the estimate (4.6) holds for both the operators $\mathcal{F}^\pm f$ which yields

$$\|\mathcal{F}_t f(x)\|_{L^p(\mathbb{R}^2\times I)} \le C_\sigma \|f\|_{L^p_{-\sigma}(\mathbb{R}^2)}, \tag{4.8}$$

for $\sigma < \frac{3}{2p} - \frac{1}{2}$ for $4 \le p \le \infty$ and $\sigma < 1/2(1/p - 1/2), 2 < p < 4$.

Now $\mathcal{G}_t$ is the difference of the operators

$$\mathcal{G}^\pm g(x,t) = \frac{1}{2i} \int_{\mathbb{R}^2} e^{i(x\cdot\xi \pm t|\xi|)} \frac{\hat{g}(\xi)}{|\xi|} d\xi.$$

Note that $\mathcal{G}^\pm$ is not a Fourier integral operator of the form considered in (3.1)

Chapter 3, hence we cannot apply Theorem 3.1.1 to estimate it, *a priori*. So we split $\mathcal{G}^{\pm} = \mathcal{G}_0^{\pm} + \mathcal{G}_1^{\pm}$ using a cut off function $\varphi \in C_c^{\infty}(\mathbb{R}^2)$ supported in $|\xi| \leq 1$, so that $\mathcal{G}_0^{\pm}$ will have compactly supported amplitude function supported near the origin. And the amplitude function of $\mathcal{G}_1^{\pm}$ vanishes near origin. Hence $\mathcal{G}_1^{\pm}$ is a Fourier integral operator of the form (4.3) with amplitude function in $S^{-1}(\mathbb{R}^2)$ and appealing to Theorem 4.0.4 with $m = -1$, we get $\|\mathcal{L}^{\sigma+1}(\mathcal{G}_1^{\pm} g)\|_p \leq C_\sigma \|g\|_p$, for $\sigma < \frac{3}{2p} - \frac{1}{2}$ for $4 \leq p \leq \infty$, and $\sigma < 1/2(1/p - 1/2)$ for $2 < p < 4$. Again since $\mathcal{L}^{\sigma+1}(\mathcal{G}_1^{\pm} g) = \mathcal{G}_1^{\pm}(\mathcal{L}^{\sigma+1} g)$, the above estimate may be re written as

$$\|\mathcal{G}_1^{\pm} g\|_{L^p(\mathbb{R}^2 \times I)} \leq C_\sigma \|\mathcal{L}^{-(\sigma+1)} g\|_{L^p(\mathbb{R}^2)}, \tag{4.9}$$

for $\sigma < \frac{3}{2p} - \frac{1}{2}$ for $4 \leq p \leq \infty$, and $\sigma < 1/2(1/p - 1/2)$ for $2 < p < 4$.

Note that, $\mathcal{G}_0 = \mathcal{G}_0^{+} - \mathcal{G}_0^{-}$ is a Fourier integral operator with compactly supported amplitude function. We will see that the estimate for $\mathcal{G}_0$ can be deduced from the estimate for $\mathcal{F}_0$ considered in chapter 3. Since

$$\mathcal{G}_0^{\pm} g(x,t) = \frac{1}{2i} \int_{\mathbb{R}^2} e^{i(x\cdot\xi \pm t|\xi|)}\, \varphi(\xi)\, \frac{\hat{g}(\xi)}{|\xi|}\, d\xi,$$

we have $\partial_t \mathcal{G}_0^{\pm} g(x,t) = \mathcal{F}_0^{\pm} g(x,t)$. Again since $\mathcal{L}^{\sigma+1}(\partial_t \mathcal{G}_0^{\pm} g) = \partial_t \mathcal{G}_0^{\pm}(\mathcal{L}^{\sigma+1} g)$, we can re-write the estimate in Corollary 3.2.3, Chapter 3, as

$$\|\partial_t \mathcal{G}_0^{\pm} g\|_{L^p(\mathbb{R}^2 \times I)} \leq C_\sigma \|\mathcal{L}^{-(\sigma+1)} g\|_{L^p(\mathbb{R}^2)}, \tag{4.10}$$

for $\sigma < \frac{3}{2p} - \frac{1}{2}$ for $4 \leq p \leq \infty$, and $\sigma < 1/2(1/p - 1/2)$ for $2 < p < 4$.

Now, we shall use a Sobolev embedding Theorem (see, [37, 35]) to replace

$$\int_{t=1}^{2} |\partial_t \mathcal{G}_0 g(x,t)|^p \, dt, \ p > 1,$$

with $\sup_{1 \leq t \leq 2} |\mathcal{G}_0 g(x,t)|^p$.

Using this for each x, on the left hand side of the inequality (4.10) yields the inequality $\| \sup_{t \in I} |\mathcal{G}_0 g(\cdot,t)| \|_{L^p(\mathbb{R}^2)} \leq C_\sigma \|\mathcal{L}^{-(\sigma+1)} g\|_{L^p(\mathbb{R}^2)}$. This leads to the uniform estimate

$$\|\mathcal{G}_0 g(\cdot,t)\|_{L^p(\mathbb{R}^2)}^p \leq C_\sigma^p \|\mathcal{L}^{-(\sigma+1)} g\|_{L^p(\mathbb{R}^2)}^p$$

with C_α independent of t. Thus a t-integration over $I = [1,2]$ gives

$$\|\mathcal{G}_0 g\|_{L^p(\mathbb{R}^2\times I)} \leq C_\sigma \|\mathcal{L}^{-(\sigma+1)} g\|_{L^p(\mathbb{R}^2)}. \tag{4.11}$$

The estimates (4.9) and (4.11) together yields the estimate

$$\|\mathcal{G}_t g(x)\|_{L^p(\mathbb{R}^2\times I)} \leq C_\sigma \|\mathcal{L}^{-(\sigma+1)} g\|_{L^p(\mathbb{R}^2)} := C_\sigma \|g\|_{L^p_{-(\sigma+1)}(\mathbb{R}^2)}, \tag{4.12}$$

for $\sigma < \frac{3}{2p} - \frac{1}{2}$ for $4 \leq p \leq \infty$, and $\sigma < 1/2(1/p - 1/2)$ for $2 < p < 4$. Hence the required estimate for $u(x,t)$ follows from (4.8) and (4.12).

Note that Theorem 3.1.1, Chapter 3, Theorem 4.0.4 and also the estimates (4.8) and (4.12) are valid with the t interval $[1,2]$ replaced by any compact interval $I \subset (0,\infty)$. This involves composing ρ_1 with an affine transformation, and the constant in the estimate will depend on the length of the interval I in this case. This ends the proof. $\square$

Chapter 5

An Application to the Circular Maximal Operator

In this chapter, we give an alternative proof of the L^p-boundedness of the circular maximal operator on $L^p(\mathbb{R}^2)$ for $p > 2$. In section 5.1, we briefly recall the circular maximal operator and some known results concerning it. Section 5.2 is devoted to the proof of our main result and in section 5.2.1, we discuss the Littlewood-Paley square function arguments, which is one of the key steps to the proof of the circular maximal theorem.

5.1 Introduction

In this section, we study the L^p- mapping property of the circular maximal operator, given by

$$\mathcal{M}f(x) = \sup_{t>0} |A_t f(x)|, \tag{5.1}$$

where, $A_t f$ is the averaging operator given by

$$A_t f(x) = \int_{\mathbb{S}^1} f(x - ty) d\sigma(y), \; f \in \mathscr{S}(\mathbb{R}^2),$$

for each $x \in \mathbb{R}^2$ and $t > 0$, where $d\sigma$ denotes the normalized Lebesgue measure over the unit circle $\mathbb{S}^1$. Note that, $A_t f(x)$ is the mean value of f over the circle of radius t centered at x and it defines a bounded operator on $L^p(\mathbb{R}^2)$ for $1 \leq p \leq \infty$.

The circular maximal operator $\mathcal{M}$ defines a bounded operator on $L^p(\mathbb{R}^2)$ if and only if $p > 2$. This was first proved by Bourgain (see, [1]). The analogous result of the spherical maximal operator was proved by Stein [36, 39], for $n \geq 3$ and also showed that the associated maximal operator is bounded on $L^p(\mathbb{R}^n)$ only if $p > \frac{n}{n-1}$, $n \geq 2$. Bourgain's proof of the circular maximal theorem relies more directly on the geometry involved (for further details, see [1]). Other proof for $n = 2$ is due to Mockenhaupt et al. [28] and proof of this result is based on their local smoothing estimates. In this chapter, we give an alternative proof of the L^p-boundedness of Bourgain's circular maximal operator by using a stationary phase method and the local smoothing estimates of Fourier integral operator, obtained in chapter 3.

Now, in the following theorem, we state our main result of this chapter.

Theorem 5.1.1 *Let f be a bounded measurable function on $\mathbb{R}^2$. Then, the maximal operator $\mathcal{M}f$, given by* (5.1) *satisfies the inequality*

$$\|\mathcal{M}f\|_{L^p(\mathbb{R}^2)} \leq C_p \|f\|_{L^p(\mathbb{R}^2)}$$

for $p > 2$.

The proof of Theorem 5.1.1 will consist of three main steps. First, we shall decompose each averaging operator A_t into dyadic operators, and then express each dyadic operator in terms of the Fourier integral operator by using the stationary phase method. In fact, we reduce our problem to the estimates where the supremum is only taken over $t \in [1, 2]$. This reduction follows from the Littlewood-Paley square function argument, see section 5.2.1. To complete the proof, we shall then use the results on the local smoothing estimates of Fourier integral operators, obtained in chapter 3. We now detail the dyadic decomposition of the dual space that is needed to prove the Theorem 5.1.1.

5.1.1 The Dyadic Maximal Operator

In this section, we express the averaging operator $A_t f$ as an infinite sum of dyadic operators $\{A_t^j\}_{j\geq 0}$ as follows: Let ψ be a non-negative radial function in $C_c^\infty(\mathbb{R}^2)$ supported in $\{\frac{1}{2} \leq |\xi| \leq 2\}$ such that $\sum_{j=0}^\infty \psi(2^{-j}\xi) = 1$ for $|\xi| \geq 1$. Define $\psi_j(\xi) = \psi(2^{-j}\xi)$ for $j \geq 0$, and $\phi(\xi) = 1 - \sum_{j=0}^\infty \psi_j(\xi)$, see Section 2.1.1, Chapter 2, for the construction of such a $\psi \geq 0$.

Denote by σ^j, μ the C^∞ functions given by

$$\widehat{(\sigma^j)}(\xi) = \widehat{(d\sigma)}(\xi)\,\psi_j(\xi) \quad \text{and} \quad \widehat{\mu}(\xi) = \widehat{(d\sigma)}(\xi)\,\phi(\xi). \tag{5.2}$$

Let A_t^j and B_t be the operators defined by

$$\widehat{A_t^j f}(\xi) = \hat{f}(\xi)\,\widehat{(\sigma^j)}(t\,\xi) \text{ and } \widehat{B_t f}(\xi) = \hat{f}(\xi)\,\widehat{\mu}(t\,\xi). \tag{5.3}$$

Notice that $A_t f(x) = \sum_{j=0}^{\infty} A_t^j f(x) + B_t f(x)$.

In fact, B_t is point-wise majorized by a constant times the Hardy-Littlewood maximal operator M, since B_t is a convolution operator with a smooth kernel of uniformly bounded L^1- norm. Thus, we see that

$$\mathcal{M} f(x) \leq \sum_{j=0}^{\infty} \sup_{t>0} |A_t^j f(x)| + CMf(x). \tag{5.4}$$

We shall use C as a constant independent of j, in several times without mentioning it.

5.2 Proof of Theorem 5.1.1

In the proof of the Theorem 5.1.1, we shall use a scaling argument and a technical lemma (see, Lemma 5.2.1) in order to reduce our problem to the local smoothing estimates of Fourier integral operators, discussed in chapter 3.

In view of (5.4), it is enough to prove the following: There exists a constant $\epsilon(p) > 0$ such that for $p > 2$, the inequality

$$\int_{\mathbb{R}^2} \sup_{t>0} |A_t^j f(x)|^p \, dx \,\leq\, C_p \, 2^{-j\,p\,\epsilon(p)} \int_{\mathbb{R}^2} |f(x)|^p \, dx, \tag{5.5}$$

holds with constant C_p, independent of j. In fact, it is enough to take supremum over $t \in [1,2]$ (see, section 5.2.1). Next, we claim that there exists a constant $\epsilon(p) > 0$ such that for $p > 2$, the inequality

$$\int_{\mathbb{R}^2} \sup_{1\leq t\leq 2} |A_t^j f(x)|^p \, dx \,\leq\, C_p \, 2^{-j\,p\,\epsilon(p)} \int_{\mathbb{R}^2} |f(x)|^p \, dx, \tag{5.6}$$

holds with a constant C_p, independent of j. To prove (5.6), we shall use the following lemma (see, [35, Lemma 2.4.2]).

Lemma 5.2.1 *Suppose that $F \in C^1(\mathbb{R})$. Then if $p > 1$ and $1/p + 1/p' = 1$, we have*

$$\sup_{\lambda} |F(\lambda)|^p \leq |F(0)|^p + p\left(\int |F(\lambda)|^p \, d\lambda\right)^{1/p'} \cdot \left(\int |F'(\lambda)|^p \, d\lambda\right)^{1/p}.$$

Proof. Using the fundamental theorem of calculus, we write,

$$|F(\lambda)|^p = |F(0)|^p + \int_0^\lambda \frac{d}{ds}|F(s)|^p ds = |F(0)|^p + p\int_0^\lambda |F(s)|^{p-1} \cdot |F'(s)| \, ds.$$

Now, if we use Hölder's inequality, we get the desired result. □

Next, we choose a cut off function $\tilde{\rho} \in C_c^\infty(\mathbb{R})$ supported in $[\frac{1}{2}, 4]$ such that $\tilde{\rho}(t) = 1$ if $1 \leq t \leq 2$. Using Lemma 5.2.1 and the fact that $\tilde{\rho}(0) = 0$, we get

$$\begin{aligned}
&\sup_{t\in\mathbb{R}} \left|\tilde{\rho}(t)\, A_t^j f(x)\right|^p \qquad (5.7)\\
\leq\ & p\left(\int_{-\infty}^{\infty} \left|\tilde{\rho}(t)\, A_t^j f(x)\right|^p \, dt\right)^{1/p'} \left(\int_{-\infty}^{\infty} \left|\partial_t\left[\tilde{\rho}(t)\, A_t^j f(x)\right]\right|^p \, dt\right)^{\frac{1}{p}}\\
\leq\ & p\left(\int_{\frac{1}{2}}^{4} \left|A_t^j f(x)\right|^p dt\right)^{1/p'} \left(\int_{\frac{1}{2}}^{4} \left|\partial_t\left[A_t^j f(x)\right]\right|^p dt\right)^{\frac{1}{p}} + Cp\int_{\frac{1}{2}}^{4} \left|A_t^j f(x)\right|^p dt,
\end{aligned}$$

with constant $C = \|\tilde{\rho}'(t)\|_{L^\infty(\mathbb{R})}$.

In the last step we have used the fact that $\tilde{\rho}$ is supported in $[1/2, 4]$ and $\tilde{\rho}$, $\tilde{\rho}'$ are uniformly bounded. Integrating (5.7) with respect to the x variable and by Hölder's inequality, we get

$$\begin{aligned}
&\|\sup_{1\leq t\leq 2} A_t^j f(x)\|_{L^p(\mathbb{R}^2)}^p \qquad (5.8)\\
\leq\ & p\left(\int_{\frac{1}{2}}^{4}\int_{x\in\mathbb{R}^2} |A_t^j f(x)|^p \, dx\, dt\right)^{1/p'} \left(\int_{\frac{1}{2}}^{4}\int_{x\in\mathbb{R}^2} |\partial_t A_t^j f(x)|^p \, dx\, dt\right)^{\frac{1}{p}}\\
&+C\, p\int_{\frac{1}{2}}^{4}\int_{x\in\mathbb{R}^2} |A_t^j f(x)|^p \, dx\, dt.
\end{aligned}$$

Now, we will estimate each term in the right hand side of (5.8) separately. To estimate the norm in (5.8), we use the following Proposition from [35].

Proposition 5.2.2 *The Fourier transform of the surface measure $d\sigma$ of the unit circle $\mathbb{S}^1$ can be written as:*

$$\widehat{d\sigma}(\xi) = e^{2i|\xi|}\omega_+(|\xi|) + e^{-2i|\xi|}\omega_-(|\xi|), \tag{5.9}$$

with smooth functions $\omega_\pm$ on $\mathbb{R}_+$ satisfying $|\frac{\partial^k}{\partial s^k}\omega_\pm(s)| \leq C_k(1+s)^{-\frac{1}{2}-k}$, $k = 0, 1, 2, \ldots$ i.e., $\omega_\pm \in S^{-1/2}$, the symbol class of order $-1/2$.

In view of the Proposition 5.2.2 and by Fourier inversion formula in (5.3), we have

$$\begin{aligned} A_t^j f(x) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{i\xi\cdot x}\, \widehat{(\sigma^j)}(t\xi)\, \hat{f}(\xi)\, d\xi &= (2\pi)^{-2} \int_{\mathbb{R}^2} e^{i\xi\cdot x}\, \widehat{(d\sigma)}(t\xi)\, \psi_j(t\xi)\, \hat{f}(\xi)\, d\xi \\ = \quad (2\pi)^{-2} \int_{\mathbb{R}^2} e^{i\xi\cdot x}\, e^{it|\xi|}\, \omega_+(t|\xi|)\, \psi(t|\xi|2^{-j})\, \hat{f}(\xi)\, d\xi \\ + \quad (2\pi)^{-2} \int_{\mathbb{R}^2} e^{i\xi\cdot x}\, e^{-it|\xi|}\, \omega_-(t|\xi|)\, \psi(t|\xi|2^{-j})\, \hat{f}(\xi)\, d\xi, \end{aligned} \tag{5.10}$$

which is a sum of two Fourier integral operators $\mathcal{F}_j^\pm f(x,t)$ similar to the one considered in chapter 3 :

$$\mathcal{F}_j^\pm f(x,t) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{i[x\cdot\xi \pm t|\xi|]}\, a_{j,t}^\pm(\xi)\, \hat{f}(\xi)\, d\xi, \;\; (x,t) \in \mathbb{R}^2 \times [1/2, 4] \tag{5.11}$$

for $j \geq 0$, where $a_{j,t}^\pm(\xi) = \omega_\pm(t|\xi|)\, \psi(t|\xi|2^{-j}) \in S^{-1/2}$, by Proposition 5.2.2.

Remark 5.2.3 Note that in chapter 3, we consider the Fourier integral operator with amplitude function $\tilde{a}(\xi,t)$ of the form $\rho_1(t)\, a(\xi)\, \psi(2^{-j}|\xi|)$ and the estimate for the Fourier integral operator then involved various derivatives of ψ. Thus for $t \in [1/2, 4]$, the same estimate holds with $\psi(2^{-j}|\xi|)$ replaced by $\psi(t2^{-j}|\xi|)$ and the same regularity estimate (see, Proposition 3.7.2, Chapter 3) holds for the Fourier integral operator $\mathcal{F}_j^\pm f$ as well.

Setting $\mathcal{L} = (I - \Delta)^{1/2}$. Then, we write $\mathcal{F}_j^\pm f = \mathcal{L}^{-1/2}[\mathcal{L}^{1/2}\mathcal{F}_j^\pm]f$. The key observation is that $\mathcal{L}^{1/2}\mathcal{F}_j^\pm$ is a Fourier integral operator with symbol in S^0, the symbol class of order 0. This follows since $(1+|\xi|^2)^{1/4} \in S^{1/2}$ and the fact that the order of the product of two symbols is the sum of their orders.

Thus, in view of the Remark 5.2.3 and by Proposition 3.7.2 in chapter 3 with

$\sigma = -1/2$, we get

$$\|\mathcal{F}_j^{\pm} f(x,t)\|_{L^p(\mathbb{R}^2\times[1/2,4])} \leq C_\epsilon\, 2^{\epsilon'(p)j}\, \|f\|_{L^p(\mathbb{R}^2)}, \tag{5.12}$$

for $4 \leq p \leq \infty$, with constant C_ϵ independent of j for each $\epsilon > 0$, where $\epsilon'(p) = (17/2+4b)\epsilon/p - 1/2 + (1/2 - 3/2p)$ for some $b > 0$.

Now, differentiating $\mathcal{F}_j^{\pm} f$ in (5.11) with respect to t variable, we get

$$\partial_t \left(\mathcal{F}_j^{\pm} f\right)(x,t) = (2\pi)^{-2} \int_\xi e^{i[\xi\cdot x \pm t|\xi|]} P_j^{\pm}(\xi,t)\, \hat{f}(\xi)\, d\xi, \tag{5.13}$$

where

$$\begin{aligned} P_j^{\pm}(\xi,t) &= i\,(\pm|\xi|)\,\omega_\pm(t|\xi|)\,\psi(t|\xi|2^{-j}) + \partial_t[\omega_\pm(t|\xi|)]\,\psi(t|\xi|2^{-j}) \\ &+ \omega_\pm(t|\xi|)\,(|\xi|2^{-j})\,\psi'(t|\xi|2^{-j}) \\ &= b_{j,t}(\xi)\,\omega_\pm(t|\xi|) + c_{j,t}(\xi)\,\omega'_\pm(t|\xi|), \end{aligned} \tag{5.14}$$

where, $b_{j,t}(\xi) = i(\pm|\xi|)\psi(t|\xi|2^{-j}) + (2^{-j}|\xi|)\psi'(t|\xi|2^{-j})$, $\quad c_{j,t}(\xi) = |\xi|\,\psi(t|\xi|2^{-j})$ and $\partial_t[\omega_\pm(t|\xi|)] = \omega'_\pm(t|\xi|)\cdot|\xi|$.

Now, we see that for each $t \in [1/2,4]$, $b_{j,t}$ and $c_{j,t}$ are smooth functions with $\|b_{j,t}(\xi)\|_{L^\infty(\mathbb{R}^2)} \leq 2^{j+2}$ and $\|c_{j,t}(\xi)\|_{L^\infty(\mathbb{R}^2)} \leq 2^{j+2}$, since $|\xi| \in [2^{j-3}, 2^{j+2}]$ on the support of $\psi(t|\xi|2^{-j})$. Note that, $\omega'_\pm \in S^{-3/2}$, the symbol class of order $-3/2$. This follows since $\omega_\pm \in S^{-1/2}$, the symbol class of order $-1/2$ by Proposition 5.2.2. Thus, in view of (5.13) and (5.14), we have

$$\begin{aligned} \partial_t(\mathcal{F}_j^{\pm} f)(x,t) &= (2\pi)^{-2} \int_\xi e^{i[\xi\cdot x\pm t|\xi|]}\, b_{j,t}(\xi)\,\omega_\pm(t|\xi|)\,\hat{f}(\xi)\,d\xi \\ &+ (2\pi)^{-2} \int_\xi e^{i[\xi\cdot x\pm t|\xi|]}\, c_{j,t}(\xi)\,\omega'_\pm(t|\xi|)\,\hat{f}(\xi)\,d\xi. \end{aligned} \tag{5.15}$$

Remark 5.2.4 We notice that the Fourier integral operators $\partial_t(\mathcal{F}_j^{\pm} f)$ in (5.15) involve smooth functions $b_{j,t}(\xi)$ and $c_{j,t}(\xi)$ for $t \in [1/2,4]$, instead of $\psi(2^{-j}|\xi|)$, which was considered in chapter 3. In fact, for $|\xi| \in [2^{j-3}, 2^{j+2}]$, we have

$$\|(2^j\partial_\xi)^\alpha[b_{j,t}(\xi)]\|_{L^\infty(\mathbb{R}^2)} \leq C_\alpha\, 2^j\, \left[\|\partial_\xi^{\alpha_1}[\psi]\|_{L^\infty(\mathbb{R}^2)} + \|\partial_\xi^{\alpha_2}[\psi']\|_{L^\infty(\mathbb{R}^2)}\right],$$

and

$$\|(2^j\partial_\xi)^\alpha[c_{j,t}(\xi)]\|_{L^\infty(\mathbb{R}^2)} \leq C_\alpha\, 2^j\, \|\partial_\xi^{\alpha_1}[\psi]\|_{L^\infty(\mathbb{R}^2)},$$

for all multi-indices $\alpha,\ \alpha_1,\ \alpha_2$ such that $|\alpha| \geq |\alpha_1| \geq 0$ and $|\alpha| \geq |\alpha_2| \geq 0$ with a constant C_α, independent of j.

Note that, the kernel estimate (see, Lemma 2.2.1) for the Fourier integral operator in chapter 2 involved various derivatives of ψ. In fact, in view of the above observation and the same arguments as in Lemma 2.2.1, Chapter 2, the kernels of $\partial_t(\mathcal{F}_j^{\pm} f)$ satisfy the same estimate as in Lemma 2.2.1 with a constant $C\, 2^j$. And this translates to the extra 2^j factor in the regularity estimate (see, Proposition 3.7.2 Chapter 3) for the Fourier integral operators $\partial_t(\mathcal{F}_j^{\pm} f)$.

Hence, in view of above Remark 5.2.4, and by Proposition 3.7.2 in chapter 3, we get

$$\|\partial_t(\mathcal{F}_j^{\pm} f)(x,t)\|_{L^p(\mathbb{R}^2\times[1/2,4])} \leq C\, 2^{\epsilon'(p)j}\, 2^j\, \|f\|_{L^p(\mathbb{R}^2)} \tag{5.16}$$

for $4 \leq p \leq \infty$ with constant C independent of j and $\epsilon'(p)$ as in (5.12).

Using (5.12) and (5.16) in (5.8), we get for $4 \leq p \leq \infty$

$$\begin{aligned} \|\sup_{1\leq t\leq 2} A_t^j f(x)\|_{L^p(\mathbb{R}^2)}^p &\leq C\, p\, 2^{[p\,\epsilon'(p)+1]j}\, \|f\|_{L^p(\mathbb{R}^2)}^p + C\, p\, 2^{p\,\epsilon'(p)\, j}\, \|f\|_{L^p(\mathbb{R}^2)}^p \\ &\leq 2C\, p\, 2^{[p\,\epsilon'(p)+1]j}\, \|f\|_{L^p(\mathbb{R}^2)}^p. \end{aligned}$$

Since, $p\,\epsilon'(p)+1 = p\ [(17/2+4b)\epsilon/p\ -1/2\ +\ (1/2-3/2p)]+1 = (17/2+4b)\epsilon - 1/2 < 0$, as $\epsilon > 0$ is arbitrary small, the inequality (5.6) holds for $4 \leq p < \infty$. Hence, this completes the proof of (5.6) for $4 \leq p < \infty$. To deal with the case $2 < p < 4$, we first observe that

$$\|e^{\pm it|\xi|}\, \omega_{\pm}(t|\xi|)\, \psi(t|\xi|2^{-j})\|_{L^\infty(\mathbb{R}^2)} \leq C\, 2^{-j/2},\ 1/2 \leq t \leq 4,$$

by Proposition 5.2.2. Therefore, using Plancheral theorem, we get

$$\|\mathcal{F}_j^{\pm} f(x,t)\|_{L^2(\mathbb{R}^2\times[1/2,4])} \leq C\, 2^{-j/2}\, \|f\|_{L^2(\mathbb{R}^2)}. \tag{5.17}$$

By Riesz-Thorin interpolation between (5.12) with $p = 4$ and (5.17), we get

$$\|\mathcal{F}_j^{\pm} f(x,t)\|_{L^p(\mathbb{R}^2\times[1/2,4])} \leq C\, 2^{\epsilon''(p)j}\, \|f\|_{L^p(\mathbb{R}^2)}, \tag{5.18}$$

for $2 \leq p \leq 4$, with $\epsilon^{''}(p) = \tilde{\epsilon} - 1/4 - 1/2p$, where $\tilde{\epsilon} = (17/2 + 4b)\,(1/2 - 1/p)\,\epsilon$.

Similarly, we have, $\|e^{\pm it|\xi|} A_j^{\pm}(\xi, t)\|_{L^\infty(\mathbb{R}^2)} \leq C\, 2^{j/2},\ 1/2 \leq t \leq 4$ by Proposition 5.2.2. Therefore, using Plancheral theorem, we get

$$\|\partial_t(\mathcal{F}_j^{\pm} f)(x, t)\|_{L^2(\mathbb{R}^2 \times [1/2,4])} \leq C\, 2^{j/2}\, \|f\|_{L^2(\mathbb{R}^2)}. \tag{5.19}$$

By Riesz-Thorin interpolation between (5.16) with $p = 4$ and (5.19), we get

$$\|\partial_t(\mathcal{F}_j^{\pm} f)(x, t)\|_{L^p(\mathbb{R}^2 \times [1/2,4])} \leq C\, 2^{\epsilon^{''}(p) j}\, 2^j\, \|f\|_{L^p(\mathbb{R}^2)}, \tag{5.20}$$

for $2 \leq p \leq 4$ and $\epsilon^{''}(p)$ as in (5.18).

Finally, using (5.18) and (5.20) in (5.8), we get for $2 < p < 4$

$$\begin{aligned} \| \sup_{1 \leq t \leq 2} A_t^j f(x)\|^p_{L^p(\mathbb{R}^2)} &\leq C\, p\, 2^{[p\,\epsilon''(p)+1]j}\, \|f\|^p_{L^p(\mathbb{R}^2)} + C\, p\, 2^{p\,\epsilon''(p)\, j}\, \|f\|^p_{L^p(\mathbb{R}^2)} \\ &\leq 2C\, p\, 2^{[p\,\epsilon''(p)+1]j}\, \|f\|^p_{L^p(\mathbb{R}^2)}. \end{aligned}$$

Thus, the inequality (5.6) for $2 < p < 4$ follows from the fact that $p\,\epsilon''(p) + 1 = p\,[\tilde{\epsilon} - 1/4 - 1/2p] + 1 = p\,\tilde{\epsilon} - p/4 + 1/2 < 0$, as $\tilde{\epsilon} = (17/2 + 4b)\,(1/2 - 1/p)\,\epsilon$ is arbitrary small for $p > 2$. Hence, this completes the proof of the claim (5.6) for $2 < p < 4$.

To complete the proof of Theorem 5.1.1, we need to use the Littlewood-Paley square function arguments, which we will discuss in the next section.

5.2.1 Littlewood-Paley Square Function Arguments

In this section, we shall discuss how to obtain the estimate for supremum over $t > 0$ in (5.5) from the estimate for (5.6) with supremum over $t \in [1, 2]$. This follows from the arguments given in Mockenhaupt et al. [28]. We discuss it briefly here for completeness. First, consider the Littlewood-Paley operators $L_k,\ k \in \mathbb{Z}$, defined by $\widehat{(L_k f)}(\xi) = \psi(2^{-k}|\xi|)\, \hat{f}(\xi)$, where ψ is as in (5.2). From (5.3), we get

$$\widehat{A_t^j f}(\xi) = \hat{f}(\xi)\, \widehat{\sigma^j}(t\xi) \;=\; \hat{f}(\xi)\, \widehat{(d\sigma)}(t\xi)\, \psi_j(t\xi).$$

Note that for $l \in \mathbb{Z}$ and $t \in [2^l, 2^{l+1}]$, the support of $\widehat{A_t^j f}$ is contained in

$$\{\xi : 2^{j-1}/t \leq |\xi| \leq 2^{j+1}/t\} \subset \{\xi : 2^{j-l-2} \leq |\xi| \leq 2^{j-l+2}\}.$$

In fact, when $t \in [2^l, 2^{l+1}]$, by the above observation, we have

$$A_t^j f(x) \;=\; A_t^j \left(\sum_{|k+l-j|\le 3} L_k f \right)(x). \tag{5.21}$$

This follows from the following fact: we write, $f = \sum_{k\in\mathbb{Z}} L_k f$ and for each k, we have

$$\widehat{A_t^j(L_k f)}(\xi) = \widehat{L_k f}(\xi)\,\widehat{(d\sigma)}(t\xi)\,\psi_j(t\xi) \;=\; \psi(2^{-k}|\xi|)\hat{f}(\xi)\,\widehat{(d\sigma)}(t\xi)\,\psi(t\,2^{-j}|\xi|).$$

Note that, for a given j and l, we have $\widehat{A_t^j(L_k f)} \equiv 0$ when $k > j-l+3$ or $k < j-l-3$. This follows from the support properties of $\psi(t\,2^{-j}|\xi|)$ and $\psi(2^{-k}|\xi|)$ for $t \in [2^l, 2^{l+1}]$. Thus, for a given j and l, $A_t^j(L_k f)$ is non-trivial only for $k \in [j-l-3, j-l+3]$. It follows that for each j and l, there are 7 non-trivial functions $A_t^j(L_k f)$.

Hence, for a given l and j, there are only seven k's that matters in the above summation (5.21). Now, using Hölder inequality for summation, we get

$$\left| \sum_{\{k:|k+l-j|\le 3\}} L_k f \right|^p \le 7^{p-1} \sum_{\{k:|k+l-j|\le 3\}} |L_k f|^p\,,\ p \ge 1. \tag{5.22}$$

Next, by rescaling, we also see that the inequality (5.6) will be true for $\sup\limits_{2^l\le t\le 2^{l+1}}$ with the same constant. Thus, in view of (5.21), (5.22) and the above observation, we have

$$\begin{aligned}
\int \sup_{t>0} |A_t^j f(x)|^p\,dx \;&\le\; \sum_{l=-\infty}^{\infty} \int \sup_{t\in[2^l,2^{l+1}]} \left|A_t^j f(x)\right|^p\,dx \\
&=\; \sum_{l=-\infty}^{\infty} \int \sup_{t\in[2^l,2^{l+1}]} \left| A_t^j \left(\sum_{\{k:|k+l-j|\le 3\}} L_k f \right)(x) \right|^p dx \\
&\le\; C_p\,2^{-j\epsilon_p p} \sum_{l=-\infty}^{\infty} \int \left| \sum_{\{k:|k+l-j|\le 3\}} L_k f(x) \right|^p dx \\
&\le\; C_p\,7^{p-1}\,2^{-j\epsilon_p p} \sum_{l=-\infty}^{\infty} \int \sum_{\{k:|k+l-j|\le 3\}} |L_k f(x)|^p\,dx
\end{aligned}$$

$$
\begin{aligned}
&\leq \; C_p \, 7^p \, 2^{-j\epsilon_p \, p} \int \sum_{k=-\infty}^{\infty} |L_k f(x)|^p \; dx \\
&\leq \; C_p \, 7^p \, 2^{-j\epsilon_p \, p} \int \left(\sum_{k=-\infty}^{\infty} |L_k f(x)|^2 \right)^{(\frac{p}{2})} \; dx.
\end{aligned}
$$

In the last step, we have used the fact that $p > 2$.

If we now use the L^p- boundedness of Littlewood-Paley square function for $p > 2$, we get

$$\int \sup_{t>0} |A_t^j f(x)|^p \, dx \leq C_p \, 7^p \, 2^{-j\epsilon_p \, p} \int_x |f(x)|^p \, dx.$$

This completes the proof of the inequality (5.5) and hence the Theorem 5.1.1 follows. □

Chapter 6

Maximal Functions along Hypersurfaces

In this chapter, we study the L^p-boundedness for maximal operators along a class of hypersurfaces in $\mathbb{R}^{n+1}$ given by the graph of a function. Section 6.1 is the introduction, where we briefly discuss the Hardy-Littlewood maximal operator, maximal function along hypersurfaces, and then we state our main result. In section 6.2, we discuss a factorisation of the surface measure, which is analogous to the polar decomposition of the Lebesgue measure on $\mathbb{R}^n$. In section 6.3, we study the L^p- mapping property of some auxiliary maximal operators. Section 6.4 is devoted to the proof of our main result.

6.1 Introduction

Let f be a locally integrable function on $\mathbb{R}^n$. Then, the Hardy-Littlewood maximal operator Mf is given by

$$Mf(x) = \sup_{r>0} \frac{1}{m(B)} \int_B |f(x - ry)| \, dy, \tag{6.1}$$

where $m(B)$ denotes the Lebesgue measure of the unit ball $B = B(0, 1)$ centered at the origin. One fundamental fact about Hardy Littlewood maximal operator Mf that attracts our interest is the L^p- inequality

$$\|Mf\|_{L^p(\mathbb{R}^n)} \leq A_p \|f\|_{L^p(\mathbb{R}^n)}, \quad f \in L^p(\mathbb{R}^n) \tag{6.2}$$

for all $1 < p \leq \infty$, and is of weak type $(1,1)$.

The case $n = 1$ of the inequality (6.2) was first studied by G. H. Hardy and J. E. Littlewood [17], while the higher dimensional case was due to J. Marcinkiewicz and A. Zygmund [26] and N. Wiener [42]. However, the maximal operators associated with averages with respect to singular measures, like surface carried measures are not bounded in $L^p(\mathbb{R}^n)$ for all $p > 1$. A classic example is the L^p-boundedness of spherical maximal operator on $L^p(\mathbb{R}^n)$ for $p > n/(n-1)$, $n \geq 2$.

The spherical maximal theorem has been extended to the maximal operator associated to the dilates of more general compact hypersurfaces in $\mathbb{R}^n$, by Greenleaf [15], and Sogge [33] with the range of p depending on the curvature of the surface. In fact, E. M. Stein and S. Wainger had already remarked in [39], the role of curvature in the boundedness of the maximal operator. The work of Greenleaf and Sogge explicitly showed this connection in terms of the principal curvatures, see Theorem 6.3.3.

The above maximal operators are 'dilated maximal operators' associated with singular measures on $\mathbb{R}^n$. In this chapter, we consider a slightly different maximal operator on $\mathbb{R}^{n+1}$ obtained by averaging along hypersurfaces in $\mathbb{R}^{n+1}$, see (6.5) for the definition, which we also call maximal function along hypersurfaces. What is interesting is that these are maximal operators associated with singular averages, but with better L^p mapping properties than the dilated maximal operators considered by Greenleaf and Sogge.

We now discuss the maximal operator along hypersurface S given by $S = \{(x, x_0) : h(x) = x_0\}$, where $h \in C^1(\mathbb{R}^n \setminus \{0\})$. We assume that $h(0) = 0$ and $\nabla h(x) \neq 0$ for all $x \in S_r$, where

$$S_r = \{x \in \mathbb{R}^n : h(x) = r\} \tag{6.3}$$

is the level set of the function h, at height $r > 0$ and each S_r is a compact C^1 hypersurface in $\mathbb{R}^n$.

For $r \geq 0$, let $\Sigma_r = \{(x, x_0) \in S : 0 \leq x_0 \leq r\}$ and for $f \in \mathscr{S}(\mathbb{R}^{n+1})$, consider the average

$$A_r f(x, x_0) = \frac{1}{\mu(\Sigma_r)} \int\limits_{\Sigma_r} |f(x - y', x_0 - y_0)| \, d\mu(y), \quad y = (y', y_0) \tag{6.4}$$

where μ denotes the surface measure on S induced by the Lebesgue measure on

$\mathbb{R}^{n+1}$. Define the corresponding maximal operator by

$$Mf(x, x_0) = \sup_{r>0} A_r f(x, x_0). \tag{6.5}$$

Our main result is the following maximal theorem for hypersurfaces given by graph of functions that are α homogeneous; i.e., satisfying

$$h(r^\alpha x) = rh(x), \ \alpha > 0$$

for each $r > 0$ and $x \in \mathbb{R}^n$.

Theorem 6.1.1 *Let S be a hypersurface in $\mathbb{R}^{n+1}$ given by the graph of an α-homogeneous function h. Suppose that the level set $S_1 = \{x \in \mathbb{R}^n : h(x) = 1\}$ has at least k, $1 \leq k \leq n-1$, non-vanishing principal curvatures everywhere on S_1. Then, the maximal operator M, given by (6.5) satisfies the inequality*

$$\|Mf\|_{L^p(\mathbb{R}^{n+1})} \leq C_p \|f\|_{L^p(\mathbb{R}^{n+1})}$$

for $p > \frac{k+1}{k}$.

The novelty of our approach is a factorisation technique, which gives a very simple proof of Theorem 6.1.1 via geometric arguments. The key idea is to factorise the maximal operator along hypersurface into a generalized one-dimensional Hardy-Littlewood maximal operator, and a dilated maximal operator associated with a compact hypersurface in $\mathbb{R}^n$. This reduction is based on a factorisation of the surface measure discussed in the next section.

6.2 A decomposition of the surface measure

One of the key ideas in our proof as mentioned above is a decomposition of the surface measure on S in terms of the surface measure $d\sigma_r$ on the slice $\tilde{S}_r = \{(x, r) : x \in S_r\}$ and the Lebesgue measure dr on $[0, \infty)$. Note that for each $r > 0$, the slice $\tilde{S}_r$ is the "vertical translate" of S_r given by (6.3), and $S = \cup_{r=0}^{\infty} \tilde{S}_r$. This gives rise to a decomposition of the surface measure on S in terms of the surface measure $d\sigma_r$ on S_r, and the Lebesgue measure on $(0, \infty)$:

$$d\mu(y) \quad = \quad \tilde{W}(x)\, d\sigma_r(x)\, dr, \ \ y = (x, r),$$

with $x \in S_r,\ r > 0$, and $\tilde{W}$ is a weight function defined on $\mathbb{R}^n$. This is analogous to the polar decomposition on $\mathbb{R}^n$. In fact, in Proposition (6.2.1), we also give a class of surfaces for which such a decomposition holds.

We assume that h satisfies the invariance property

$$h(G_r(x)) \;=\; r\,h(x), \text{ for } \; x \in \mathbb{R}^n, \tag{6.6}$$

where $\{G_r\}_{r>0}$ is a one-parameter group of C^1 diffeomorphism of $\mathbb{R}^n$ to itself, with $G_r \circ G_t = G_{rt}$ for all $r, t > 0$ and with the identity element G_1. Clearly, the above invariance property shows that G_r maps the surface S_1 onto S_r for $r > 0$. Note that the invariance property is a generalisation of the homogeneity property. We also assume that the map $G : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}^n$ is C^1.

Differentiating (6.6) with respect to r, we get the useful identity

$$\left\langle \frac{\partial}{\partial r} G_r(x), \nabla h(G_r(x)) \right\rangle = h(x),\ x \in \mathbb{R}^n. \tag{6.7}$$

Now, we prove the following decomposition result for the surface measure on the hypersurface S.

Proposition 6.2.1 *Let S be a hypersurface in $\mathbb{R}^{n+1}$ given by the graph of a function $h \in C^1(\mathbb{R}^n)$ with $\nabla h(x) \neq 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$. We assume that $h(x) = 0$, only for $x = 0$, Range$(h) = [0, \infty)$ and that h satisfies the invariance property (6.6), for some one-parameter group $\{G_r\}_{r>0}$ of C^1 diffeomorphisms on $\mathbb{R}^n$. Then the surface measure $d\mu$ on S has the decomposition*

$$d\mu(y) = \tilde{W}(x)\, d\sigma_t(x)\, dt,\ y = (x, t) \in S_t \times \mathbb{R}_+$$

where $d\sigma_t$ is the surface measure on the level set $S_t = \{x \in \mathbb{R}^n : h(x) = t\}$, for each $t > 0$. Moreover, the weight $\tilde{W}(x)$ can be expressed in terms of $|\nabla h|$.

Proof. We have $S = \{(x, x_0) \in \mathbb{R}^n \times [0, \infty) : x_0 = h(x)\}$. Since, $\nabla h(x) \neq 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$, each of the level sets $S_r = \{x : h(x) = r\}$ of h are $n - 1$ dimensional C^1 hypersurfaces in $\mathbb{R}^n$, for $r > 0$.

The unit normal vector field on S is given by the gradient of the function $\Phi(x, x_0) = h(x) - x_0,\ (x, x_0) \in \mathbb{R}^n \times \mathbb{R}$:

$$N(x, x_0) = \frac{\nabla\Phi(x, x_0)}{|\nabla\Phi(x, x_0)|} = \frac{(N_1(x), \ldots, N_n(x), -1)}{\sqrt{1 + |\nabla h(x)|^2}} \tag{6.8}$$

where $N_i(x) = \frac{\partial h}{\partial_{x_i}}(x)$, for $(x, x_0) \in S$, $i = 1, 2, \ldots, n$. The surface S has the natural parametrization $x \to (x, h(x))$, being the graph of h. However it is convenient to use another parametrisation, defined in terms of the parametrisations for S_r. For each $r > 0$, let $\phi^r : Q \to S_r$ be a C^1 parametrisation of S_r, where $Q \subset \mathbb{R}^{n-1}$ is the parameter domain. The parametrisation ϕ^r is assumed to be regular, in the sense that the tangent vectors $\frac{\partial \phi^r}{\partial \theta_i}(\theta)$, $i = 1, 2, \ldots, n-1$, based at the point $\phi^r(\theta) \in S_r$ are linearly independent, for all $\theta = (\theta_1, \ldots, \theta_{n-1}) \in Q$.

Note that we can choose the same parameter domain for all ϕ^r. In fact, choosing a regular parametrisation $\phi : Q \to S_1$ for S_1, we can set $\phi^r = G_r \circ \phi$. Note that ϕ^r defines a parametrisation for S_r, since G_r maps S_1 to S_r. In fact, ϕ^r is regular, since the derivative $DG_r(x)$ is a nonsingular matrix. Let

$$\phi^r(\theta) = (\phi^r_1(\theta), \ldots, \phi^r_n(\theta)) \in S_r \tag{6.9}$$

for $\theta \in Q$. Note that, by choosing ϕ such that $\phi(Q)$ covers all of S_1 except possibly a set of measure zero in S_1, we can practically work with a single parametrisation for each S_r. This is possible, for instance when the level sets S_r are diffeomorphic to an $(n-1)$-dimensional sphere or torus.

Since S_r is the level set of h, the unit normal vector field ν on S_r is given by the gradient of h:

$$\nu(x) = \frac{\nabla h(x)}{|\nabla h(x)|} \tag{6.10}$$

for $x = \phi^r(\theta) \in S_r$. The surface measure is given by the determinant of the raw vectors consisting of the normal vector field ν and the tangent vector fields $\frac{\partial \phi^r}{\partial \theta_i}$, $i = 1, 2, \ldots, n-1$, on S_r, (see, Thorpe [41]):

$$d\sigma_r(\theta) = det \begin{pmatrix} \nu(r, \theta) \\ \frac{\partial \phi^r}{\partial \theta_1}(\theta) \\ \vdots \\ \frac{\partial \phi^r}{\partial \theta_{n-1}}(\theta) \end{pmatrix} d\theta. \tag{6.11}$$

Now the parametrisation ϕ^r of S_r gives rise to a parametrisation of the hypersurface S (excluding origin): Define $\varphi : (0, \infty) \times Q \to S \subset \mathbb{R}^{n+1}$ by :

$$\varphi(r, \theta) = (\phi_1^r(\theta), \ldots, \phi_n^r(\theta), r), \tag{6.12}$$

where $\phi_i^r(\theta), 1 \leq i \leq n$ are as in (6.9). With respect to this parametrisation, the n-dimensional volume of the hypersurface $S \subset \mathbb{R}^{n+1}$ is given by

$$d\mu(r,\theta) = \frac{1}{\sqrt{1 + |\nabla h(\phi^r(\theta))|^2}} \begin{vmatrix} N_1(r,\theta) & N_2(r,\theta) & \ldots & N_n(r,\theta) & -1 \\ \frac{\partial \phi_1^r(\theta)}{\partial \theta_1} & \frac{\partial \phi_2^r(\theta)}{\partial \theta_1} & \ldots & \frac{\partial \phi_n^r(\theta)}{\partial \theta_1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial \phi_1^r(\theta)}{\partial \theta_{n-1}} & \frac{\partial \phi_2^r(\theta)}{\partial \theta_{n-1}} & \ldots & \frac{\partial \phi_n^r(\theta)}{\partial \theta_{n-1}} & 0 \\ \frac{\partial \phi_1^r(\theta)}{\partial r} & \frac{\partial \phi_2^r(\theta)}{\partial r} & \ldots & \frac{\partial \phi_n^r(\theta)}{\partial r} & 1 \end{vmatrix} dr\, d\theta,$$

where the first raw contains the components of the normal vector to S at $\varphi(r, \theta)$, given by (6.8) and the remaining n raw vectors are the tangent vectors to S at $\varphi(r, \theta)$. By adding the first raw to the last, we can simplify the above determinant to get

$$d\mu(r,\theta) = \frac{1}{\sqrt{1 + |\nabla h|^2}}\, det \begin{pmatrix} w(r,\theta) \\ \frac{\partial \phi^r}{\partial \theta_1}(r,\theta) \\ \vdots \\ \frac{\partial \phi^r}{\partial \theta_{n-1}}(r,\theta) \end{pmatrix} dr\, d\theta.$$

where $w(r, \theta)$ is the vector field on $S_r \subset \mathbb{R}^n$, given by

$$w(r,\theta) = \left(\left[\frac{\partial \phi_1^r}{\partial r}(\theta) + N_1(r,\theta) \right], \ldots, \left[\frac{\partial \phi_n^r}{\partial r}(\theta) + N_n(r,\theta) \right] \right) \tag{6.13}$$

and the components $N_i(r, \theta)$ are as in (6.8): $N_i(r, \theta) = \frac{\partial h}{\partial_{x_i}}(\phi^r(\theta)), 1 \leq i \leq n$. Now using the decomposition

$$w(r,\theta) = \langle w, \nu \rangle\, \nu(r,\theta) + \langle w, v \rangle\, v(r,\theta) \tag{6.14}$$

in terms of the unit normal vector $\nu(r, \theta) = \frac{\nabla h(\phi^r(\theta))}{|\nabla h(\phi^r(\theta))|}$, and a unit tangent vector v to S_r at $\phi^r(\theta)$, we see that the above determinant is a sum of two determinants, one with w replaced by $\langle w, \nu \rangle\, \nu(r, \theta)$ and the other with w replaced by

$\langle w, v \rangle\, v(r,\theta)$. But the latter determinant is zero: v is a linear combination of other raw vectors which span the tangent space $T_{\phi^r(\theta)}S_r$. Hence we see that

$$\begin{aligned} d\mu(r,\theta) &= \frac{\langle w,\nu\rangle}{\sqrt{1+|\nabla h|^2}}\, det \begin{pmatrix} \nu(r,\theta) \\ \frac{\partial \phi^r}{\partial \theta_1}(\theta) \\ \vdots \\ \frac{\partial \phi^r}{\partial \theta_{n-1}}(\theta) \end{pmatrix} d\theta\, dr \\ &= \frac{\langle w,\nu\rangle}{\sqrt{1+|\nabla h|^2}}\, d\sigma_r(\theta)\, dr, \end{aligned} \tag{6.15}$$

by (6.11). It is also easy to see that the function $\langle w, \nu\rangle$ arising in (6.15) is positive on all surfaces $S_r, r > 0$. In fact from (6.13), using (6.8) and (6.10) we see that

$$\begin{aligned} \langle w,\nu\rangle &= \frac{1}{|\nabla h|}\left\langle \frac{\partial \phi^r(\theta)}{\partial r}, \nabla h(\phi^r(\theta))\right\rangle + |\nabla h| \\ &= \frac{1}{|\nabla h|} + |\nabla h| = \frac{1+|\nabla h|^2}{|\nabla h|} \end{aligned} \tag{6.16}$$

where we used the fact that $\left\langle \frac{\partial \phi^r(\theta)}{\partial r}, \nabla h(\phi^r(\theta))\right\rangle = 1$, which follows from (6.7) since $\phi^r(\theta) = G_r(\phi(\theta))$ and $\phi(\theta) \in S_1$. Thus setting

$$\tilde{W}(x) = \frac{\sqrt{1+|\nabla h(x)|^2}}{|\nabla h(x)|} = W(r,\theta) \text{ for } x = \phi^r(\theta), \tag{6.17}$$

we obtain the decomposition,

$$d\mu(r,\theta) = W(r,\theta)\, d\sigma_r(\theta) dr \tag{6.18}$$

in view of (6.15) and (6.16), as asserted. □

By using Cauchy-Schwarz inequality in (6.7), we also see that

$$\frac{1}{|\nabla h(G_r(x))|} \le \left|\frac{\partial G_r(x)}{\partial r}\right|, \text{ for } x \in S_1 .$$

Since $G \in C^1(\mathbb{R}^n \times (0,\infty))$, $\frac{\partial G_r}{\partial r}$ is continuous and hence $sup_{x\in S_r}\left|\frac{\partial G_r(x)}{\partial r}\right| < \infty$ by compactness of S_r, for each $r > 0$. Hence $W(r,\theta) < \infty$ for each $r > 0$.

Theorem 6.1.1 of this paper concerns the one parameter group of diffeomorphisms of the form $G_r(x) = \lambda(r)x,\ x \in \mathbb{R}^n$ for some non-negative function λ on $(0, \infty)$. The group property $G_{r_1} \circ G_{r_2} = G_{r_1 r_2}$ shows that $\lambda(r) = r^\alpha$ for some $\alpha > 0$. We denote this dilation by G_r^α:

$$G_r^\alpha(x) = r^\alpha x. \tag{6.19}$$

The invariance property $h(G_r^\alpha(x)) = rh(x)$ in this case is the α-homogeneity

$$h(r^\alpha x) = rh(x),\ x \in \mathbb{R}^n. \tag{6.20}$$

There are plenty of smooth functions h satisfying such property: For instance for each positive integer k, and $\alpha > 0$, consider the function h_k defined by

$$h_k(x) = (x_1^{2k} + \cdots + x_n^{2k})^{1/2k\alpha}, x \in \mathbb{R}^n. \tag{6.21}$$

Note that $h_k \in C^1(\mathbb{R}^n \setminus \{0\})$ and satisfies the invariance property (6.20). Also $|\nabla h_k(x)| \neq 0$ for $x \neq 0$ and hence the level sets $h_k(x) = r$ are smooth surfaces for $r > 0$.

The next lemma is crucial in our approach.

Lemma 6.2.2 *Let S be a hypersurface in $\mathbb{R}^{n+1}$ given by the graph of a function h as above, satisfying the homogeneity property (6.20), with $\alpha > 0$. Then the surface measure $d\mu$ has the behaviour*

$$d\mu(r, \theta) \approx (1 + r^{\alpha-1})\, d\sigma_r(\theta) dr \tag{6.22}$$

where $d\sigma_r$ is the surface measure on S_r for $r > 0$.

Proof. In view of (6.18), it is enough to show that there are positive constants C_1 and C_2 such that

$$C_1(1 + r^{\alpha-1}) \leq W(r, \theta) \leq C_2(1 + r^{\alpha-1}). \tag{6.23}$$

By (6.17), we have $W(r, \theta) = \sqrt{1 + \frac{1}{|\nabla h(x)|^2}}$, for $x = \phi^r(\theta) = r^\alpha \phi(\theta) \in S_r$, where ϕ denotes the parametrisation for S_1 as in Proposition 6.2.1.

Partial differentiation in x variable in (6.20) shows that

$$|\nabla h(r^\alpha x)| = r^{1-\alpha} |\nabla h(x)|. \tag{6.24}$$

There exists positive constants $\tilde{C_1}$ and $\tilde{C_2}$ such that $\tilde{C_1} \leq |\nabla h(x)| \leq \tilde{C_2}$ for $x \in S_1$, as ∇h is continuous and non-vanishing on S_1. Thus for all $x \in S_1$, we have

$$C_1(1 + r^{2(\alpha-1)}) \leq 1 + \frac{1}{|\nabla h(r^\alpha x)|^2} \leq C_2(1 + r^{2(\alpha-1)})$$

for constants C_1 and C_2. Since $\sqrt{1 + r^{2(\alpha-1)}} \approx 1 + r^{\alpha-1}$, the proof follows. □

Lemma 6.2.3 *Let S_1 be a smooth surface in $\mathbb{R}^n$ and set $S_t := G_t^\alpha S_1 = t^\alpha S_1$. If $d\sigma_t$ denotes the surface measure on S_t, then we have $d\sigma_t = t^{(n-1)\alpha} d\sigma_1$.*

Proof. The proof follows from the explicit expression for $d\sigma_t$ given by (6.11). In view of (6.24), it is clear from (6.10) that $\nu(t, \theta) = \nu(1, \theta)$, for $t > 0$. Also since $\phi^t(\theta) = t^\alpha \phi(\theta)$, we have $\frac{\partial \phi^t}{\partial \theta_i} = t^\alpha \frac{\partial \phi}{\partial \theta_i}$ for $i = 1, 2, \ldots, n-1$. Thus the conclusion follows since the determinant is a multi-linear function of its raw vectors. □

6.3 Some auxiliary maximal theorems

In this section, we introduce a general maximal operator on $\mathbb{R}^n$, in the spirit of the Hardy-Littlewood maximal operator and prove an L^p-boundedness result. Let ν be a measure on $\mathbb{R}^n$, which is locally absolutely continuous in the sense that $d\nu(x) = \varphi(x)\, dx$ with density $\varphi \in L^1_{loc}(\mathbb{R}^n)$. Consider the maximal operator

$$M_\nu f(x) = \sup_{r>0} \frac{1}{\nu(\{|y| < r\})} \int_{|y|<r} |f(x-y)|\, d\nu(y). \tag{6.25}$$

Note that $\nu(\{|y| < r\}) < \infty$ for each $r > 0$, since $\varphi \in L^1_{loc}(\mathbb{R}^n)$.

We only need the L^p-boundedness of the following one-dimensional maximal operator M_1 defined by

$$M_1 f(x) = \sup_{r>0} \frac{1}{\nu(0, r)} \int_0^r |f(x-y)|\, d\nu(y), \ x \in \mathbb{R}. \tag{6.26}$$

The maximal operator M_ν is interesting in its own right, which prompts us to give an L^p-boundedness result for the n-dimensional case. However, here we give a proof only in the special case, where the density satisfies certain averaging condition, which holds in our case. The required one-dimensional version can also be easily deduced from the following n-dimensional result.

Theorem 6.3.1 *Let M_ν be the maximal operator given by (6.25), where ν is a locally absolutely continuous measure on $\mathbb{R}^n$, with density $\varphi \geq 0$. Suppose also that φ satisfies the averaging condition*

$$\varphi(y) \leq \frac{C}{r^n} \int_{|y|<r} \varphi(y)\, dy, \text{ for } |y| < r, \text{ and } r > 0. \tag{6.27}$$

Then M_ν is bounded on $L^p(R^n, dx)$ for all $p > 1$, and is of weak type $(1,1)$.

Proof. The proof follows by a simple reduction to the Hardy-Littlewood maximal operator. In fact, since ν is locally absolutely continuous with density φ, we have $\nu(\{|y| \leq r\})) = \int_{|y|<r} \varphi(y)dy$. Hence, the averaging condition (6.27) says that

$$\frac{\varphi(y)}{\nu(\{|y| < r\})} \leq \frac{C}{r^n}$$

for $|y| < r$. Using this in (6.25), we immediately see that

$$M_\nu f(x) \leq \sup_{r>0} \frac{\tilde{C}}{|\{|y| < r\}|} \int_{|y|<r} |f(x-y)|\, dy$$

which is a constant times the Hardy-Littlewood maximal operator on $\mathbb{R}^n$. Hence, the proof follows from the L^p-boundedness of the Hardy-Littlewood maximal operator. $\square$

Remark 6.3.2 By taking $\varphi \in L^1_{loc}(\mathbb{R})$ supported in $[0, \infty)$, the one-dimensional case of the above theorem gives the boundedness of the maximal operator M_1.

The averaging condition (6.27) is a growth condition on φ. In fact, the function $\varphi_s(y) = |y|^s$, $s \geq 0$ and hence any finite linear combinations of such φ_{s_i}'s with positive coefficients satisfy the above averaging condition, and so is the function $\varphi(y) = log(1 + |y|)$. But the exponential function $\varphi(y) = e^{|y|}$ fails to satisfy the averaging condition.

We also need the L^p-boundedness of the 'dilated maximal operators' associated with compact surfaces in $\mathbb{R}^n$. Let S be a compact hypersurface in $\mathbb{R}^n$ with surface measure $d\sigma$ and let $d\tilde{\sigma}$ denotes the normalised surface measure on S which defines a Borel probability measure on S. Then consider the dilated maximal operator $\mathcal{M}$ given by

$$\mathcal{M}f(x) = \sup_{t>0} \int_S |f(x-ty)| d\tilde{\sigma}(y), \tag{6.28}$$

which generalizes the Stein's spherical maximal operator considered in [36].

Allan Greenleaf has studied the L^p-boundedness of such maximal operators in $\mathbb{R}^n$. Greenleaf has shown in [15], that if S has k non-vanishing principal curvatures on S for $2 \le k \le n-1$, then $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^n)$ for $p > \frac{k+1}{k}$. In fact, he considers maximal operators associated with non-isotropic dilations, but the proof goes along with the method of Stein and Wainger [39], by showing that the required decay property of the Fourier transform of the surface carried measure $d\tilde{\sigma}$ holds under the non-vanishing curvature assumption.

C. D. Sogge in [33], improved the result of Greenleaf by showing that the L^p-boundedness holds for dilated maximal operators, even with one non vanishing principal curvature everywhere on the surface, for $2 < p \le \infty$. In the following theorem, we combine the results of Sogge and Greenleaf, in the special case when the hypersurface is compact:

Theorem 6.3.3 *Let $S = \{x \in \mathbb{R}^n : \Phi(x) = 1\}$ be a compact $\mathbb{C}^\infty$ hypersurface in $\mathbb{R}^n$, $n \ge 2$, given by a function $\Phi \in \mathbb{C}^\infty$ with $\nabla\Phi \ne 0$ on S. Suppose that at every point on S, there are k non-vanishing principal curvatures, $1 \le k \le n-1$. Then the maximal operator $\mathcal{M}$ given by (6.28) satisfies the inequality*

$$\|\mathcal{M}f\|_{L^p} \le C_p \|f\|_{L^p},$$

for all $f \in L^p(\mathbb{R}^n)$, for $p > \frac{k+1}{k}$, for some constant C_p.

6.4 Proof of main theorem

Now we come back to the study of maximal function along a hypersurface in $\mathbb{R}^{n+1}$. Recall that $S = \{(x, x_0) : h(x) = x_0\}$, where $h \in C^1(\mathbb{R}^n \setminus \{0\})$ and satisfies the invariant property (6.6) with $G_r(x) = r^\alpha x$ for some $\alpha > 0$: $h(r^\alpha x) = rh(x), x \in S_1$. We also assume that $h(0) = 0$ and $\nabla h \ne 0$ on $\mathbb{R}^n \setminus \{0\}$.

Note that a function satisfying the above α-homogeneity property is completely determined by its values on the level set $S_1 = \{x \in \mathbb{R}^n : h(x) = 1\}$. This also means that one can pick up an arbitrary compact C^1 surface $S_1 \subset \mathbb{R}^n$, that encloses the origin, and set $h = 1$ on S_1 and extend it to $\mathbb{R}^n$ by the homogeneity condition: $h(y) = h(r^\alpha x) = rh(x) = r$. Note that any $0 \neq y \in \mathbb{R}^n$ is of the form $r^\alpha x$ for some $x \in S_1$. Such an extension defines a C^1 function on $\mathbb{R}^n \setminus \{0\}$ for $\alpha > 0$. We can get a variety of hypersurfaces S given by graph of such functions. For instance, the functions given in (6.21) give surfaces that are C^1 away from the origin for $\alpha > 0$.

We now proceed to prove Theorem 6.1.1. We have a crucial reduction for the proof, for hypersurfaces given by the graph of "α-homogeneous" functions, which we highlight as:

Remark 6.4.1 In view of Lemma 6.2.2, for the purpose of L^p-boundedness, we can assume without loss of generality that $d\mu$ is of the form $d\mu(r,\theta) = \lambda(r)d\sigma_r(\theta)dr$, where $\lambda(r) = (1 + r^{\alpha-1})$, with $\alpha > 0$.

By the above remark, we write the averages given by (6.4) as

$$\begin{aligned} A_r f(x, x_0) &= \frac{1}{\mu(\Sigma_r)} \int\limits_{t=0}^{r} \left[\int_{y \in S_t} |f(x-y, x_0 - t)|\, d\sigma_t(y) \right] (1 + t^{\alpha-1})\, dt \\ &= \frac{1}{\mu(\Sigma_r)} \int\limits_{t=0}^{r} \left[\int_{S_t} |f(x-y, x_0 - t)|\, d\tilde{\sigma}_t(y) \right] \psi(t)\, dt. \end{aligned} \tag{6.29}$$

Here $\psi(t) = (1+t^{\alpha-1})\,\mathrm{vol}(S_t)$ and $d\tilde{\sigma}_t = \frac{d\sigma_t(y)}{\mathrm{vol}(S_t)}$ is the normalised surface measure on S_t, which defines a one-parameter family of Borel probability measures on $\mathbb{R}^n$, for $t > 0$. Also by Lemma 6.2.3,

$$\psi(t) = \mathrm{vol}(S_1)(1 + t^{\alpha-1})t^{(n-1)\alpha} \tag{6.30}$$

and $\frac{1}{\mu(\Sigma_r)} \int_0^r \psi(t)dt = 1$ by the assumption in Remark 6.4.1.

Proof of Theorem 6.1.1. In view of Remark 6.4.1, we can assume that the averages in (6.4) are given by normalised measures of the form $d\mu_r(x) = \lambda(r)d\sigma_r(\theta)dr$. In the equation (6.29) for the average $A_r(f)(x)$, the expression inside the square bracket satisfies the obvious inequality

$$\frac{1}{\mathrm{vol}(S_t)} \int\limits_{y \in S_t} |f(x-y, x_0 - t)|\, d\sigma_t(y) \leq \mathcal{M} f_{x_0 - t}(x) \tag{6.31}$$

where $f_s(x) = f(x,s)$ and

$$\mathcal{M}f_s(x) = \sup_{r>0} \frac{1}{\mathrm{vol}[r^\alpha S_t]} \int\limits_{y \in r^\alpha S_t} |f_s(x-y)| \, d\sigma_{rt}(y).$$

Also since $S_t = G_t^\alpha S_1 = t^\alpha S_1$ by (6.19), we have $r^\alpha S_t = S_{rt}$, and hence

$$\begin{aligned} \mathcal{M}f_s(x) &= \sup_{r>0} \frac{1}{\mathrm{vol}(S_{rt})} \int\limits_{y \in S_{rt}} |f_s(x-y)| \, d\sigma_{rt}(y) \\ &= \sup_{r>0} \frac{1}{\mathrm{vol}(S_r)} \int\limits_{y \in S_r} |f_s(x-y)| \, d\sigma_r(y). \end{aligned} \tag{6.32}$$

For $s = x_0 - t$, this is the same as the maximal operator in the x variable obtained by taking supremum of averages over all the $n-1$ dimensional surfaces

$$S_r \times \{x_0 - t\} = \{(y, x_0 - t) : y \in S_r\} \subset \mathbb{R}^n \times \{x_0 - t\},$$

lying in the hyperplane in $\mathbb{R}^{n+1}$ with last co-ordinate $x_0 - t$ fixed. Thus from (6.29), we see that

$$\begin{aligned} A_r f(x, x_0) &\leq \frac{1}{\mu(\Sigma_r)} \int\limits_{t=0}^{r} \mathcal{M}f_{x_0 - t}(x) \, \psi(t) \, dt \\ &\leq M_1[\mathcal{M}f_s(x)](x_0). \end{aligned} \tag{6.33}$$

where M_1 is the maximal operator with respect to the s-variable, given by

$$M_1 g(x_0) = \sup_{r>0} \frac{1}{\mu(\Sigma_r)} \int\limits_{t=0}^{r} |g(x_0 - t)| \, \psi(t) \, dt. \tag{6.34}$$

Note that this M_1 is the same as the maximal operator on $\mathbb{R}$, considered in (6.26) with $d\nu(t) = \psi(t)dt$, with density $\psi(t) = \mathrm{vol}(S_1)(1 + t^{\alpha-1})t^{(n-1)\alpha}$ given in (6.30). Since $\psi(t)$ satisfies the averaging condition (6.27), it follows that M_1 is bounded on $L^p(\mathbb{R})$ for $p > 1$, in view of Theorem 6.3.1 and Remark 6.3.2.

Taking supremum over all $r > 0$ in (6.33), we see that

$$Mf(x, x_0) \leq M_1[\mathcal{M}f_s(x)](x_0). \tag{6.35}$$

We first observe that the map $s \to \mathcal{M}f_s(x)$ is in $L^p(\mathbb{R})$, for almost all $x \in \mathbb{R}^n$. In fact, by applying Theorem 6.3.3 for the dilated maximal operator given by (6.32), for each fixed $s \in \mathbb{R}$, we get

$$\int_x |\mathcal{M}f_s(x)|^p\, dx \;\le\; C_2^p \int_x |f_s(x)|^p\, dx \tag{6.36}$$

for $p > \frac{k+1}{k}, 1 \le k \le n-1$ with C_2 independent of s. Since $f_s(x) = f(x,s)$, a further integration in (6.36) with respect to the s variable yields

$$\int_s \int_x |\mathcal{M}f_s(x)|^p\, dxds \le C_2^p \int_s \int_x |f(x,s)|^p\, dxds < \infty. \tag{6.37}$$

Hence by Fubini's theorem, $\int_s |\mathcal{M}f(x,s)|^p\, ds < \infty$, for almost all $x \in \mathbb{R}^n$.

Thus, in view of (6.35) and the L^p mapping result for M_1, we see that,

$$\begin{aligned} \|Mf\|^p_{L^p(\mathbb{R}^{n+1})} \;&\le\; \int_{x\in\mathbb{R}^n}\int_{x_0\in\mathbb{R}} |M_1[\mathcal{M}f_s(x)](x_0)|^p\, dx_0\, dx \\ &\le\; \int_x C_1^p \int_s |\mathcal{M}f_s(x)|^p\, ds\, dx \\ &\le\; C_1^p C_2^p\, \|f\|^p_{L^p(\mathbb{R}^{n+1})} \end{aligned} \tag{6.38}$$

for $p > \frac{k+1}{k}$ by (6.36), and as $f_s(x) = f(x,s)$. This completes the proof. □

6.4.1 Final Remarks

We have saved this concluding section to briefly outline of few research directions for future which could be based upon the work presented here.

1. It would be interesting to know if the above maximal operator is bounded on $L^p(\mathbb{R}^{n+1})$ for the full range $1 < p < \infty$. The restriction on p in our theorem came from the use of the maximal theorem of Greenleaf and Sogge for the dilated maximal operators, which involves the curvature.

   In fact the curvature condition in Theorem 6.1.1 could be relaxed, in dimension $n = 4$, if we use Theorem 1.2 in [20](valid for dimension $n = 3$) instead of using Theorem 6.3.3 in our proof. The maximal theorem proved in [20] gives the L^p boundedness of the dilated maximal operator, using certain height function defined in terms of Newton polyhedra, see [20], page 164. However, use of this theorem gives L^p boundedness for a smaller range of p: a subset of $(2, \infty]$.

2. Although we apply Greenleaf's result [15], which is valid for more general non-isotropic dilations, our approach via factorisation does not seem to yield results in the case of one-parameter group of non-isotropic dilations of the form $G_t^\alpha(x) = (t^{\alpha_1} x_1, \ldots, t^{\alpha_n} x_n)$. It would be interesting to know whether the result holds in that case as well.

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