ON THE TRANSCENDENCE OF CERTAIN NUMBERS AND SOME IDENTITIES

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DECLARATION

I hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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List of Publications arising from the thesis

1. " Some infinite sums identities" Jaban Meher and Sneh bala sinha, *Czechoslovak Mathematical Journal, 2015.*

2. " A generalization of an Identity of Lehmer", Sanoli Gun, Ekata Saha and Sneh Bala Sinha, *Accepted, Acta Arithmetica.*

Sneh Bala Sinha

Dedicated to my parents

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Contents

Synopsis

The thesis consists of two themes, Euler's constant and Identities involving multiple zeta values. In the first part, we consider a generalization of the classical Euler's constant. A special case of these constants was first introduced by Briggs and the notion was later discussed by Lehmer again after which they are known as the Briggs-Lehmer constants. We link these constants to generalized Briggs-Lehmer constants which is also known as generalized Euler-Lehmer constants. We discuss the question of transcendence of these generalized numbers. In the second part, we generalize identities involving Hurwitz zeta function, multiple zeta values proved by I. Mező and show that these are transcendental numbers.

0.1 Euler's constant gamma and generalizations

Euler introduced a famous constant γ, defined as the following limit

$$\gamma := \lim_{x\to\infty} \Big(\sum_{k=1}^{x} \frac{1}{k} - \log x \Big).$$

In 1961, W. Briggs introduced the notion of an Euler's constant associated to an arithmetic progression of integers. This notion was studied in detail by D. H. Lehmer [10] in 1975.

Fix any natural number q; then for every integer a with $0 < a \leq q$, the limit

$$\gamma(a,q) := \lim_{x\to\infty} \Bigg(\sum_{\substack{n\leq x \\ n\equiv a \bmod q}} \frac{1}{n} - \frac{\log x}{q} \Bigg) \tag{1}$$

exists. These numbers $\gamma(a,q)$ are called Euler-Lehmer constants.

While studying the Riemann hypothesis in 2007, Diamond and Ford [5] introduced another generalization of Euler's constant arising from integer sieved by finite sets of primes. For a fixed finite set of primes Ω, define

$$\mathrm{P}_\Omega := \begin{cases} \prod_{p\in\Omega} p & \text{if } \ \Omega \neq \phi, \\ 1 & \text{otherwise,} \end{cases} \tag{2}$$

$$1_\Omega(n) := \begin{cases} 1 & \text{if } (n, \mathrm{P}_\Omega) = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\begin{aligned} \delta_\Omega &:= \lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} 1_\Omega(n) \\ &= \prod_{p\in\Omega} \Big(1 - \frac{1}{p}\Big). \end{aligned}$$

Then the constant $\gamma(\Omega)$ is defined as the limit

$$\gamma(\Omega) := \lim_{x\to\infty} \Big(\sum_{n\leq x} \frac{1_\Omega(n)}{n} - \delta_\Omega \log x \Big).$$

Note that $\gamma = \gamma(\Omega)$ for $\Omega = \emptyset$.

In the second chapter of the thesis we take up a work done with Sanoli Gun and Ekata Saha [7], where we give another proof of a result obtained in [6]. More precisely, this is about the existence of $\gamma(\Omega, a, q)$ and its nature.

Let us first define $\gamma(\Omega, a, q)$.

Definition 0.1.1 For natural numbers a, q and for a finite set of primes Ω not containing any prime factors of q, we define the generalized Euler-Lehmer constant by

$$\gamma(\Omega, a, q) := \lim_{x\to\infty} \Big(\sum_{\substack{n\le x \\ n\equiv a \bmod q}} \frac{1_\Omega(n)}{n} - \delta_\Omega \frac{\log x}{q} \Big).$$

0.1.1 Transcendence of generalized Euler-Lehmer constant

Definition 0.1.2 A complex number α is said to be an **algebraic number** if there is a non-zero polynomial $f(x) \in \mathbb{Q}[x]$ such that $f(\alpha) = 0$; otherwise it is called **transcendental**.

An algebraic number α is said to be of **degree** n if its minimal polynomial $P(x)$ has degree n. Equivalently, $\mathbb{Q}(\alpha)$ is a finite extension of $\mathbb{Q}$ of degree n.

In 2010, **Ram Murty and Saradha** [15] discussed the transcendence nature of $\{\gamma(a, q)\}$ and in 2013 **Ram Murty and Zaytseva** [17] discussed the same for $\{\gamma(\Omega)\}$.

In this context, one would like to ask the following question:

Question 1. Does $\gamma(\Omega, a, q)$ exist? If so, then can we say anything about its nature?

An affirmative answer to the first question is obtained by us in [6] and we have also discussed the transcendence nature of these numbers.

We have thereafter furnished new proofs of existence of these generalized constants in a much simpler way. Here is the prototypical theorem in this context.

Theorem 0.1.3 *For any finite set of primes* Ω *and a natural number* $q \geq 1$ *with* $(q,\ \mathrm{P}_\Omega) = 1$ *where* P_Ω *is defined in equation (2), one has*

$$\gamma(\Omega, a, q) - \delta_\Omega \frac{\gamma}{q} \;=\; \frac{\delta_\Omega}{q} \sum_{p \in \Omega} \frac{\log p}{p-1} - \sum_{\Omega' \subseteq \Omega} \frac{(-1)^{\mathrm{Card}(\Omega')}}{q \mathrm{P}_{\Omega'}} \sum_{\substack{\zeta_q \in \mu_q \\ \zeta_q \neq 1}} \zeta_q^{-a} \log(1 - \zeta_q^{\mathrm{P}_{\Omega'}}), \quad (3)$$

where $\mathrm{Card}(\Omega')$ *denotes the cardinality of the set* Ω'.

Observe that setting $\Omega = \emptyset$ will give the Lehmer's identity (Lehmer [10]):

$$q\gamma(a, q) - \gamma = - \sum_{\substack{\zeta_q \in \mu_q \\ \zeta_q \neq 1}} \zeta_q^{-a} \log(1 - \zeta_q), \qquad (4)$$

where $a, q > 1$ and μ_q is the group of q-th roots of unity in $\bar{\mathbb{Q}}$.

Hence we also derive a different proof of Lehmer's original identity.

To prove the above theorem, we use the following lemmas which can be proved by interchanging relevant sums and and appealing to certain results in cyclotomic fields.

Lemma 0.1.4 *For natural numbers* $a, r > 1, q \geq 1$ *with* $(q, r) = 1$, *we have*

$$\lim_{x \to \infty} \Bigg(\sum_{\substack{n \leq x \\ n \equiv a \bmod q \\ n \equiv 0 \bmod r}} \frac{1}{n} - \frac{1}{qr} \sum_{n \leq x} \frac{1}{n} \Bigg) = \frac{-1}{qr} \sum_{\zeta_q \in \mu_q} \sum_{\substack{\zeta_r \in \mu_r \\ \zeta_q \zeta_r \neq 1}} \zeta_q^{-a} \log(1 - \zeta_q \zeta_r),$$

where μ_r *is the group of* r*-th roots of unity in* $\bar{\mathbb{Q}}$.

Lemma 0.1.5 *For natural numbers a, q and a finite set of primes Ω, we have*

$$\sum_{\substack{n \leq x \\ n \equiv a \bmod q \\ (n, \mathrm{P}_\Omega)=1}} \frac{1}{n} = \sum_{\Omega' \subseteq \Omega} (-1)^{\mathrm{Card}(\Omega')} \sum_{\substack{n \leq x \\ n \equiv a \bmod q \\ n \equiv 0 \bmod \mathrm{P}_{\Omega'}}} \frac{1}{n}.$$

To address transcendental issues, we need to rely on the following theorem of Baker on linear form of logarithms.

Theorem 0.1.6 (Baker) *If $\alpha_1, \ldots, \alpha_n$ are non-zero algebraic numbers such that $\log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over $\mathbb{Q}$, then $1, \log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over $\overline{\mathbb{Q}}$.*

In particular, one has the following Theorem.

Theorem 0.1.7 *If $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ are algebraic numbers with α_i's non-zero, then*

$$\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$$

is either zero or transcendental.

Thus using Baker's theorem along with Theorem 2.4.1, one can prove the following corollaries.

Corollary 0.1.8 *For any natural numbers $a, q > 1$ with $(a, q) = 1$ and for any finite set of primes Ω, the number*

$$\gamma(\Omega, a, q) - \delta_\Omega \frac{\gamma}{q}$$

is transcendental.

In fact, we can prove a stronger corollary.

Corollary 0.1.9 *Let* $\mathrm{U} := \{\Omega_i\}_{i\in\mathbb{N}}$ *be a sequence of finite subsets of primes and* $\mathrm{S} := \{q_j > 1\}_{j\in\mathbb{N}}$ *be a sequence of mutually co-prime natural numbers. Also let* Ω_i*'s do not contain any prime divisors of* q_j*'s for all* i, j *and* a *be a natural number with* $(a, q_j) = 1$ *for all* j*. Then the set*

$$T := \{\ \gamma(\Omega_i, a, q_j)\ \mid\ \Omega_i \in \mathrm{U},\ q_j \in \mathrm{S}\}$$

has at most one algebraic element.

0.2 Some infinite sum identities

In the last chapter we discuss a work with **Jaban Meher** [19] .

For $s > 1$, the Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The values of $\zeta(s)$ at odd positive integers are quite mysterious. The only result in this direction is about the irrationality of $\zeta(3)$. However at even integers the values of $\zeta(s)$ are well known. The values are in terms of powers of π and Bernoulli numbers. This shows that $\zeta(2i)$ is transcendental for any positive integer i. The nth Bernoulli number B_n is a rational number defined by the power series expansion

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1}.$$

The Hurwitz zeta function which is a generalization of the Riemann zeta function is defined as

$$\zeta(s; a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \qquad (s > 1; \quad a \in \mathbb{R}\backslash\mathbb{Z}_-),$$

where $\mathbb{R}$ and $\mathbb{Z}_-$ are the sets of real numbers and non-positive integers respectively. The nth generalized harmonic number of order r is defined as

$$H_{n,r} = \sum_{i=1}^{n} \frac{1}{i^r}.$$

For $n \in \mathbb{N}$, we define

$$H'_{2n-1,r} = \sum_{i=1}^{n} \frac{1}{(2i-1)^r}.$$

Using generalized harmonic number and generalized Riemann zeta function, Istv́an Mező [18] in 2013 proved some identities which we generalize for $2k$ for any positive integer k.

We prove the following results from which one can conclude that the left hand sides of these identities are transcendental numbers.

Proposition 0.2.1 *For any positive integer k,*

$$\sum_{i=1}^{\infty} \frac{\zeta(2k;i)}{i^{2k}} = \left\{ \left(\frac{B_{2k}}{2(2k)!} \right)^2 - \frac{B_{4k}}{2(4k)!} \right\} \frac{(2\pi)^{4k}}{2}.$$

Proposition 0.2.2 *For any positve integer k, the sum*

$$\sum_{i=1}^{\infty} \frac{H_{i,2k}}{i^{2k}} \zeta(2k;i)$$

is equal to

$$\left\{ (-1)^{3k-1} \frac{B_{6k}}{(6k)!} + (-1)^{3k-3} \frac{(B_{2k})^3}{8(2k)!^3} + (-1)^{3k-2} \frac{3B_{2k}B_{4k}}{4(2k)!(4k)!} \right\} \frac{(2\pi)^{6k}}{6}.$$

Proposition 0.2.3 *For any positive integer k, we have*

$$\sum_{i=1}^{\infty} \frac{\zeta(2k; i-1/2)}{(2i-1)^{2k}} = 2^{6k-2}\pi^{4k} \left[\frac{1}{2}\left(1-1/2^{2k}\right)^2 \frac{B_{2k}^2}{((2k)!)^2} - \left(1-1/2^{4k}\right)\frac{B_{4k}}{(4k)!}\right].$$

Proposition 0.2.4 *For any positive integer k, we have*

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{H'_{2n-1,2k}}{(2n-1)^{2k}}\zeta(2k; n-1/2) = 2^{8k}\pi^{6k} \Bigg[& \frac{(-1)^k}{8}\left(1-1/2^{2k}\right)\left(1-1/2^{4k}\right)\frac{B_{2k}B_{4k}}{(2k)!(4k)!} \\ & + \frac{(-1)^{3k-3}}{48}\left(1-1/2^{2k}\right)^3 \frac{B_{2k}^3}{(2k)!^3}\frac{(-1)^{3k-1}}{6} \\ & + \frac{(-1)^{3k-1}}{6}\left(1-1/2^{6k}\right)\frac{B_{6k}}{(6k)!}\Bigg].\end{aligned}$$

CHAPTER 1

Introduction to Euler's constant

In 1731, Euler introduced a remarkable constant, denoted by γ and is given by the following limit:

$$\gamma := \lim_{x\to\infty} (\sum_{n\leq x} \frac{1}{n} - \log x).$$

It is now referred to as Euler's constant. Since Mascheroni had calculated its value up to 32 decimal places in 1790 while solving some problems raised by Euler, it is also known as Euler-Mascheroni constant.
The true nature of Euler's constant is not yet known. One of the major open problem related to γ is the following conjecture.

Conjecture 1.0.5 γ is irrational.

Euler's constant γ appears in several places. We shall give some of these instances in the following sections. Throughout the work empty sums are assumed to be zero.

1.1 Relation to the gamma function and the digamma function

The gamma function, denoted by Γ, is an extension of the factorial function. If n is a positive integer, then

$$\Gamma(n) = (n-1)!.$$

In 1729, Euler introduced the gamma function defined for **Re** $(s) > 0$ by the improper integral

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx, \tag{1.1}$$

while seeking a generalization of the factorial function for non-integral values of n.

Definition 1.1.1 Suppose f is an analytic function defined on a non-empty open subset U of the complex plane $\mathbb{C}$. If V is an open subset of $\mathbb{C}$ containing U and F is an analytic function defined on V such that

$$F(z) = f(z) \quad \forall z \in U,$$

then F is called an analytic continuation of f.

The gamma function $\Gamma(s)$ defined above can be extended analytically to whole complex plane $\mathbb{C}$ except at integers less than or equal to zero.
Since

$$\lim_{n\to\infty} \left(1 - \frac{t}{n}\right)^n = e^{-t},$$

we get the following Gauss representation of gamma function

$$\Gamma(s) = \lim_{n\to\infty} \left(\frac{n^s}{s} \prod_{i=1}^{n} \frac{i}{s+i}\right) = \lim_{n\to\infty} \left(\frac{n^s}{s} \prod_{i=1}^{n} (1 + \frac{s}{i})^{-1}\right). \tag{1.2}$$

Definition 1.1.2 (Digamma function) The digamma function $\psi(z)$ is defined as the meromorphic function given by the logarithmic derivative of the gamma function

$$\psi(z) = \frac{d}{dz} \ln \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

Weierstrass expression for the reciprocal of the gamma function which is an entire function can be derived from (1.2) as

$$\begin{aligned}\frac{1}{\Gamma(z)} &= \lim_{n\to\infty} \left(n^{-z} z \prod_{i=1}^{n} (1 + \frac{z}{i})\right) \\ &= \lim_{n\to\infty} \left(n^{-z} z \prod_{i=1}^{n} e^{\frac{z}{n}} \prod_{i=1}^{n} (1 + \frac{z}{i}) e^{\frac{-z}{n}}\right)\end{aligned}$$

$$= \lim_{n\to\infty} \left(n^{-z} z e^{\gamma z} n^{z} \prod_{i=1}^{n} (1+\frac{z}{i}) e^{\frac{-z}{n}} \right).$$

Hence

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-\frac{z}{n}}.$$

On differentiating the logarithm of the above expression, we obtain

$$-\psi(z) = -\frac{\Gamma'(z)}{\Gamma(z)} = \gamma + \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{n+z} - \frac{1}{n}\right), \tag{1.3}$$

$$\frac{\Gamma'(z+1)}{\Gamma(z+1)} = -\gamma + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+z}\right). \tag{1.4}$$

Substituting $z = 0$ in (1.4), we get $\Gamma'(1) = -\gamma$. Thus γ is related to the classical gamma function.

1.2 Relation to the Riemann zeta function

For $\mathrm{Re}(s) > 1$, the Riemann zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

In 1731, Euler expressed γ in terms of the following convergent series

$$\gamma = \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n}. \tag{1.5}$$

In 1859, Riemann showed that $\zeta(s)$ extends as a meromorphic function to the whole complex plane with its only singularity being a simple pole at $s = 1$. The power series expression for $\zeta(s)$ is given by,

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n (s-1)^n,$$

where the coefficients γ_n for $n \geq 0$ are given by:

$$\gamma_n := \lim_{m\to\infty} \Big(\sum_{k=1}^{m} \frac{(\log k)^n}{k} - \frac{(\log m)^{n+1}}{n+1}\Big),$$

with $\gamma_0 = \gamma$.

1.3 Harmonic series and Euler's constant

In 1729, Euler mentioned that the harmonic numbers are defined as

$$H_n = \sum_{i=1}^{n} \frac{1}{i},$$

which can also be obtained by the integral

$$H_n = \int_0^1 \frac{1-x^n}{1-x} dx.$$

Since the integral $\int_0^1 \frac{1-x^z}{1-x} dx$ is valid for $\mathrm{Re}(z) > 0$, we can define

$$H_z = \int_0^1 \frac{1-x^z}{1-x} dx, \qquad \text{for } \mathrm{Re}(z) > 0.$$

On substituting $z = 1/2$, we get

$$H_{\frac{1}{2}} = 2 - 2\log 2.$$

In 1765, Euler also proved the following result in which γ is related to the harmonic numbers.

Theorem 1.3.1 *For all integers $n \geq 1$,*

$$\psi(n) = -\gamma + H_{n-1}$$

where $H_0 = 0$. Also for integers $n \geq 0$,

$$\psi(n + \frac{1}{2}) = -\gamma - 2\log 2 + 2H_{2n-1} - H_{n-1}$$

where $H_{-1} = 0$.

For the above theorem, see [9, page 29].

1.4 Integral form of Euler's constant

Sometimes the asymptotic value of a partial sum may be calculated very easily by comparing it with an integral. The Euler summation formula estimates the error in this approximation. If f has a continuous derivative f' on the interval $[y, x]$, where $0 < y < x$, then the Euler summation formula is given by,

$$\sum_{y<n\leq x} f(n) = \int_y^x f(t)dt + \int_y^x (t-[t])f'(t)dt + f(x)([x]-x) - f(y)([y]-y). \quad (1.6)$$

Next we give two different integral representations for γ.

- Let us calculate the sum $\sum_{n\leq x} \frac{1}{n}$.

  Taking $f(t) = \frac{1}{t}$ in (1.6), we obtain

$$\begin{aligned}\sum_{n\leq x} \frac{1}{n} &= \int_1^x \frac{dt}{t} - \int_1^x \frac{t-[t]}{t^2}dt + 1 - \frac{x-[x]}{x} \\ &= \log x - \int_1^x \frac{t-[t]}{t^2}dt + 1 + O(\frac{1}{x}) \\ &= \log x - \int_1^\infty \frac{t-[t]}{t^2}dt + \int_x^\infty \frac{t-[t]}{t^2}dt + 1 + O(\frac{1}{x}).\end{aligned}$$

  Since

$$\int_1^\infty \frac{t-[t]}{t^2}dt \leq \int_1^\infty \frac{1}{t^2}dt,$$

  the improper integral is convergent. Also,

$$0 \leq \int_x^\infty \frac{t-[t]}{t^2}dt \leq \int_x^\infty \frac{1}{t^2}dt = \frac{1}{x}.$$

  Thus we obtain

$$\sum_{n\leq x} \frac{1}{n} = \log x - \int_1^\infty \frac{t-[t]}{t^2}dt + 1 + O(\frac{1}{x}). \quad (1.7)$$

Writing $C = 1 - \int_1^\infty \frac{t-[t]}{t^2} dt$ in the above equation and taking $x \to \infty$ in (1.7), we get

$$\lim_{x\to\infty} (\sum_{n\leq x} \frac{1}{n} - \log x) = C.$$

Thus

$$\gamma = 1 - \int_1^\infty \frac{t-[t]}{t^2} dt.$$

- As earlier we mentioned in (1.1)

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

On differentiating the equation we get

$$\Gamma'(x) = \int_0^\infty t^{x-1} e^{-t} \log t dt.$$

Substituting $x = 1$ in the later identity, we get

$$\Gamma'(1) = \int_0^\infty e^{-t} \log t dt.$$

This implies

$$-\gamma = \int_0^\infty e^{-t} \log t dt,$$

which gives an integral form of the Euler's constant.
We end this section by introducing the notion of periods.

Definition 1.4.1 A period, as defined by Kontsevich and Zagier, is a complex number whose real and imaginary parts are values of absolutely convergent integrals of the form

$$\int_\triangle \frac{f(x_1, \ldots, x_n)}{g(x_1, \ldots, x_n)} dx_1 \ldots dx_n,$$

where f and g are polynomials with coefficients in $\mathbb{Q}$, and the domain of integration $\triangle \subset \mathbb{R}^n$ is given by some polynomial inequalities with rational coefficients.

For the above definition, see [9, page 2] and [8].

Example 1.4.1
1. $\log n = \int_1^n \frac{dx}{x}$.
2. $\pi = \int_{x^2+y^2\leq 1} dxdy$.

3. All algebraic numbers. For example $\sqrt{2} = \int_{2x^2 \leq 1} dx$.

Conjecture 1.4.2 Euler's constant is not a period. In particular, Euler's constant is transcendental.

In the following sections, we highlight some of the notions and results from algebraic, analytic as well as transcendental number theory that are relevant to our work.

1.5 Some results from Algebraic Number Theory

From now on all fields are assumed to be characteristic zero.

Definition 1.5.1 A field K is said to be a field extension of a field F, denoted by $K|F$, if F is a subfield of K.

For example, the complex numbers are an extension field of the real numbers, and the real numbers are an extension field of the rational numbers.

Definition 1.5.2 A field $K|F$ is said to be a finite field extension of F of degree n if K forms an n dimensional vector space over F. If F is $\mathbb{Q}$, then K is called an algebraic number field of degree n.

For example $\mathbb{Q}(\sqrt{2})$ is a number field of degree 2.

Definition 1.5.3 A complex number α is said to be **algebraic** if α is a root of a non-zero polynomial $a_nX^n + \cdots + a_0$ in $\mathbb{Q}[X]$.

Definition 1.5.4 Let α_1 and α_2 be two algebraic numbers with the same minimal polynomial in $\mathbb{Q}[X]$. Then α_1, α_2 are said to be **conjugates** of each other over $\mathbb{Q}$.

Remarks 1.5.5 Let K be an algebraic number field of degree n. Then

(1) $K = \mathbb{Q}(\theta)$ for some θ whose minimal polynomial f in $\mathbb{Q}[X]$ is of degree n that is f is the monic polynomial of least degree n such that $f(\theta) = 0$. Let $\theta_1 = \theta, \ldots, \theta_n$ be the distinct roots of f. Then there exists, precisely n distinct embeddings $\sigma_1, \ldots, \sigma_n$ of K into $\mathbb{C}$ which are identity on $\mathbb{Q}$. These are given by $\sigma_i(\theta_1) = \theta_i, i = 1, \ldots, n$. Since $\theta_i \neq \theta_j$ for $i \neq j$, the embeddings are all distinct.

(2) For the same σ_i's defined above, we denote the image $\sigma_i(K)$ of K by $K^{(i)}$ and, for $\alpha \in K, \sigma_i(\alpha)$ by $\alpha^{(i)}$. It gives $K^{(1)}, \ldots, K^{(n)}$ are algebraic number fields of degree n. These are called conjugates of K. If $K^{(i)} \subset \mathbb{R}$, we call it a real conjugate of K, otherwise a complex conjugate of K.

Definition 1.5.6 A complex number α is said to be an **algebraic integer** if α is a root of a monic polynomial in $\mathbb{Z}[X]$.

Proposition 1.5.7 *The following statements are equivalent.*

(i) *α is an algebraic integer.*

(ii) *The minimal polynomial of α is a monic polynomial in $\mathbb{Z}[X]$.*

(iii) *$\mathbb{Z}[\alpha]$ is a finitely generated $\mathbb{Z}$-module.*

(iv) *There exists a finitely generated $\mathbb{Z}$-submodule $M \neq 0$ of $\mathbb{C}$ such that $\alpha M \subset M$.*

For a proof of above proposition, see [24, page 31, Proposition 2.1].
Let $\mathcal{O}_K$ be the set of algebraic integers in K. This forms a ring and is known as the ring of algebraic integers of K.

Theorem 1.5.8 *Let K be an algebraic number field of degree n. Then there exists a $\mathbb{Q}$-base $\{w_1, \ldots, w_n\}$ of K such that $w_i \in \mathcal{O}_K$ and $\mathcal{O}_K = \mathbb{Z}w_1 + \cdots + \mathbb{Z}w_n$.*

For a proof of above theorem, see [24, page 37, Theorem 2.1].

1.5.1 Norm and Trace

Let $L|K$ be a field extension of degree n.

Definition 1.5.9 The **trace** $Tr_{L|K}(\alpha)$ and **norm** $N_{L|K}(\alpha)$ of an element $\alpha \in L$ relative to K are defined to be the trace and determinant, respectively, of the endomorphism

$$\begin{aligned} T_\alpha : & L \mapsto L, \\ & x \mapsto \alpha x \end{aligned}$$

of the K-vector space L :

$$Tr_{L|K}(\alpha) = Tr(T_\alpha), \;\; N_{L|K}(\alpha) = \det(T_\alpha).$$

If the characteristic polynomial of T_α is given by

$$f_\alpha(t) = \det(t\,I - T_\alpha) = t^n - a_1 t^{n-1} + \cdots + (-1)^n a_n \in K[t],$$

then the trace and the norm are

$$a_1 = Tr_{L|K}(\alpha) \;\text{ and }\; a_n = N_{L|K}(\alpha).$$

Proposition 1.5.10 *If $L|K$ is a field extension and $\sigma : L \mapsto \overline{K}$ varies over the different K-embeddings of L into an algebraic closure $\bar{K}$ of K, then we have*

(i) $f_\alpha(t) = \prod_\sigma (t - \sigma\alpha)$,

(ii) $Tr_{L|K}(\alpha) = \sum_\sigma \sigma\alpha$,

(iii) $N_{L|K}(\alpha) = \prod_\sigma \sigma\alpha$.

Proposition 1.5.11 *In a tower of finite field extensions $K \subseteq L \subseteq M$, one has*

$$Tr_{L|K} \circ Tr_{M|L} = Tr_{M|K}, \;\; N_{L|K} \circ N_{M|L} = N_{M|K}.$$

Definition 1.5.12 The **discriminant** of a basis $\{\alpha_1, \cdots, \alpha_n\}$ of a field extension $L|K$ is defined by

$$d(\alpha_1, \cdots, \alpha_n) = [\det(\sigma_i(\alpha_j))]^2,$$

where $\sigma_i, i = 1, \cdots, n$, varies over the K embeddings $L \mapsto \overline{K}$.

For the definitions and propositions of subsection 1.5.1, see [21, page 8-11].

1.5.2 The group of units

Let K be an algebraic number field.

Definition 1.5.13 A non-zero element α in $\mathcal{O}_K$ is called a **unit** of K if $\alpha^{-1} \in \mathcal{O}_K$.

Clearly the units of K form a subgroup of K^*. Observe that $\alpha \in \mathcal{O}_K$ is a unit if and only if $N_{K/\mathbb{Q}}(\alpha) = \pm 1$.

Definition 1.5.14 A complex number α is called a **root of unity** if $\alpha^m = 1$ for some $m \in \mathbb{Z}$, $m \neq 0$. If ρ is a root of unity in K, then its conjugates satisfy $|\rho^{(i)}| = 1$ for $i = 1, 2, \ldots, n$.

Proposition 1.5.15 *The number of roots of unity in K is finite.*

Proposition 1.5.16 *The roots of unity in K^* form a finite cyclic group.*

Proposition 1.5.17 *If α is an algebraic integer all of whose conjugates have absolute value 1, then α is a root of unity.*

For a proof of the above proposition, see [26, Lemma 1.6]

Theorem 1.5.18 *Let r_1 be the number of real conjugates of K, $2r_2$ the number of complex conjugates, and let $r = r_1 + r_2 - 1$. Then there exist $\epsilon_1, \ldots, \epsilon_r$ and a root of unity ζ in K such that any unit $\epsilon \in \mathcal{O}_K$ can be written in the form*

$$\epsilon = \zeta^k \epsilon_1^{k_1} \cdots \epsilon_r^{k_r}, \quad k, k_1, \ldots, k_r \in \mathbb{Z}.$$

The k_i, $i \geq 1$ are uniquely determined, and k is uniquely determined modulo n where n is the order of the group G of roots of unity in K.

For a proof of the above theorem, see [21, Theorem 7.4, Chapter 1].

1.5.3 Cyclotomic Units

We shall now consider the units in cyclotomic field. Fix $\zeta_n = e^{2\pi i/n}$. Let $n \not\equiv 2 \mod 4$ and let V_n be the multiplicative group generated by

$$\{\pm\zeta_n, 1 - \zeta_n^a \mid \quad 1 \leq a \leq n-1\}.$$

Let E_n be the group of units of $\mathbb{Q}(\zeta_n)$ and define

$$C_n = V_n \cap E_n.$$

C_n is called the group of cyclotomic units of $\mathbb{Q}(\zeta_n)$.

For a finite abelian extension K of $\mathbb{Q}$, the celebrated Kronecker-Weber theorem asserts that there exists a minimal n such that $K \subseteq \mathbb{Q}(\zeta_n)$. Let the group of units of K be ${\mathcal{O}_K}^*$, then the group of cyclotomic units of K is

$$C = C_n \cap {\mathcal{O}_K}^*.$$

Theorem 1.5.19 *Let p be prime and $m \geq 1$. Then*

(i) The cyclotomic units of the maximal real subfield $\mathbb{Q}(\zeta_{p^m})^+$ are generated by -1 and the units

$$\xi_a = \zeta_{p^m}^{(1-a)/2} \frac{1-\zeta_{p^m}^a}{1-\zeta_{p^m}}, \ 1 < a < \frac{1}{2}p^m, (a,p) = 1.$$

(ii) The cyclotomic units of $\mathbb{Q}(\zeta_{p^m})$ are generated by ζ_{p^m} and the cyclotomic units of $\mathbb{Q}(\zeta_{p^m})^+$.

Theorem 1.5.20 *Let $n \not\equiv 2 \mod 4$ and $n = \prod_{i=1}^{s} p_i^{e_i}$ be its prime factorisation. Let I run through all subsets of $\{1, \dots, s\}$, except $\{1, \dots, s\}$, and let $n_I = \prod_{i \in I} p_i^{e_i}$. For $1 < a < \frac{n}{2}, (a,n) = 1$, define*

$$\xi_a = \zeta_n^{d_a} \prod_I \frac{1-\zeta_n^{an_I}}{1-\zeta_n^{an_I}}, \quad d_a = \frac{1}{2}(1-a)\sum_I n_I.$$

Then $\{\xi_a\}$ forms a set of multiplicatively independent units for $\mathbb{Q}(\zeta_n)^+$.

For a proof of above theorems, see [26, Theorem 8.3, Lemma 8.1].

1.6 Some results from Analytic Number Theory

Let $\mathbb{N}$ denote the set of natural numbers. An **arithmetic function** f is a complex valued function defined on the natural numbers $\mathbb{N}$. Such an f is called an **additive function** if

$$f(mn) = f(m) + f(n) \tag{1.8}$$

whenever m and n are co-prime. If it holds for all m, n, then f is called **completely additive**. A **multiplicative function** is an arithmetic function f

satisfying $f(1) = 1$ and

$$f(mn) = f(m)f(n) \tag{1.9}$$

whenever m and n are co-prime. If (1.9) holds for all m, n, then f is called **completely multiplicative**. Here are some examples of arithmetic functions. Let $\nu(n)$ denote the number of distinct prime divisors of n while $\Omega(n)$ be the number of prime divisors of n counted with multiplicity. Let $s \in \mathbb{C}$ and consider the divisor functions

$$\sigma_s(n) = \sum_{d|n} d^s,$$

where the summation is over the sth powers of positive divisors of n. The special case $s = 0$ gives the number of divisors of n, usually denoted by $d(n)$.
Following are some more examples of arithmetic functions.

Examples 1.6.1 *(1) The **Möbius function** is a multiplicative function defined by*

$$\mu(n) = \begin{cases} (-1)^{\nu(n)} & \text{if } n \text{ is square free,} \\ 0 & \text{otherwise} \end{cases}$$

and $\mu(1) = 1$. Here $\nu(n)$ denotes the number of distinct prime factors of n.

*(2) The **Euler totient function** given by*

$$\phi(n) = n \prod_{p|n} (1 - \frac{1}{p})$$

is a multiplicative function.

*(3) The **von Mangoldt function,** defined by*

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^\alpha \text{ for some } \alpha \geq 1, \text{ and } p \text{ prime} \\ 0 & \text{otherwise,} \end{cases}$$

is neither multiplicative nor additive.

One has the following identities;

(1)
$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(2) (Möbius inversion formula)
$$f(n) = \sum_{d|n} g(d) \qquad \forall n \in \mathbb{N} \qquad \text{if and only if} \qquad g(n) = \sum_{d|n} \mu(d) f(n/d) \qquad \forall n \in \mathbb{N}.$$

(3) $\sum_{d|n} \phi(d) = n$.

(4) $\dfrac{\phi(n)}{n} = \sum_{d|n} \dfrac{\mu(d)}{d}$.

(5) Let f be multiplicative. Suppose that
$$n = \prod_{p^\alpha || n} p^\alpha$$
is the unique factorization of n into powers of distinct primes. Then
$$\sum_{d|n} f(d) = \prod_{p^\alpha || n} (1 + f(p) + f(p^2) + \cdots + f(p^\alpha)).$$
The function $g(n) = \sum_{d|n} f(d)$ is also multiplicative. The notation $p^\alpha || n$ denotes the exact power of p dividing n.

(6) $\sum_{d|n} \Lambda(d) = \log n$ and hence $\Lambda(n) = -\sum_{d|n} \mu(d) \log d$.

1.6.1 Dirichlet Characters mod q

Consider the group $(\mathbb{Z}/q\mathbb{Z})^*$ of co-prime residue classes mod q. A homomorphism
$$\chi : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$$
is called a **Dirichlet character** (mod q). Since $(\mathbb{Z}/q\mathbb{Z})^*$ has order $\phi(q)$, by Euler's theorem we have
$$a^{\phi(q)} \equiv 1 (\text{mod} q),$$
and so we must have $\chi(a)^{\phi(q)} = 1$ for all $a \in (\mathbb{Z}/q\mathbb{Z})^*$. Thus $\chi(a)$ must be a $\phi(q)$-th root of unity.

We extend the definition of χ to all natural numbers by setting

$$\chi(n) = \begin{cases} \chi(n(\bmod q)) & \text{if } (n,q) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The character $\chi_0 : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$ satisfying $\chi_0(a) = 1$ for all $(a,q) = 1$ is called the **trivial** character $\bmod q$. If χ and ψ are characters modulo q for the same q, so is $\chi\psi$, as well as $\overline{\chi}$, where $\overline{\chi}$ is defined by

$$\overline{\chi}(a) = \overline{\chi(a)}.$$

Proposition 1.6.2 *(1) $(\mathbb{Z}/p\mathbb{Z})^*$ is a cyclic group if p is a prime.*

(2) The group of Dirichlet characters mod q has order $\phi(q)$.

(3) If $\chi \neq \chi_0$, then

$$\sum_{a \bmod q} \chi(a) = 0.$$

(4) Fix $(a,q) = 1$, then

$$\sum_{\chi(\bmod q)} \overline{\chi(a)}\chi(n) = \begin{cases} \phi(q) & \textit{if } n \equiv a(\bmod q), \\ 0 & \textit{otherwise.} \end{cases}$$

For the above section 1.6 , see [12].

1.7 Some results from Transcendental Number Theory

Definition 1.7.1 A complex number which is not algebraic is called a transcendental number.

A classical argument of Cantor shows that the set of algebraic numbers is countable which implies that transcendental numbers exist and that the set of transcendental numbers in $[0,1]$ has Lebesgue measure 1. Nevertheless, it is difficult to check the transcendence of any particular number.

Example 1.7.1 e, π, e^{π} are examples of transcendental numbers.

In 1853, Liouville proved a fundamental theorem concerning approximations of algebraic numbers by rational numbers. This theorem enabled him to construct the first explicit examples of transcendental numbers.

Theorem 1.7.2 (Liouville) *Given a real algebraic number α of degree $n > 1$, there is a positive constant $c = c(\alpha)$ such that for all rational numbers p/q with $(p, q) = 1,\ q > 0$, we have*

$$\left|\alpha - \frac{p}{q}\right| > \frac{c(\alpha)}{q^n}.$$

For a proof of the above theorem, see [13, Theorem 1].
In 1873, Hermite proved that e is transcendental while the transcendence of π was established by Lindemann in 1882. The following theorem unifies these results.

Theorem 1.7.3 (Hermite-Lindemann) *Let α be a non-zero complex number. Then one at least of the two numbers α and e^{α} is transcendental.*

Therefore one can conclude the following:

- If $\alpha \neq 0$ is an algebraic number, then e^{α} is transcendental.
- e is transcendental by taking $\alpha = 1$.
- If $\alpha \neq 0, 1$ is an algebraic number, then $\log \alpha$ is transcendental.
- π is transcendental by taking $\alpha = i\pi$.

Further in 1885, Lindemann and Weierstrass gave a theorem which generalizes the theorem of Hermite and Lindemann.

Theorem 1.7.4 *If $\alpha_1, \ldots, \alpha_s$ are pairwise distinct algebraic numbers, then $e^{\alpha_1}, \ldots, e^{\alpha_s}$ are linearly independent over $\bar{\mathbb{Q}}$.*

For a proof of the above theorem and Theorem 1.7.6 below, see [13, Theorem 5, Theorem 8].

Definition 1.7.5 (Algebraically Independent Set) A set of complex numbers $\alpha_1, \ldots, \alpha_s$ is algebraically independent over $\mathbb{Q}$ if there is no non-zero polynomial $P(x_1, \ldots, x_s)$ in $\mathbb{Z}[x_1, \ldots, x_s]$ such that

$$P(\alpha_1, \ldots, \alpha_s) = 0.$$

Theorem 1.7.4 is equivalent to the following in terms of algebraic independence.

Theorem 1.7.6 (Lindemann-Weierstrass) *If $\alpha_1, \ldots, \alpha_s$ are algebraic numbers linearly independent over $\mathbb{Q}$, then*

$$e^{\alpha_1}, \ldots, e^{\alpha_s}$$

are algebraically independent.

Let $\mathcal{L}$ be the set of logarithms of non-zero algebraic numbers, that is the inverse image of the multiplicative group $\overline{\mathbb{Q}^*}$ by the exponential map:

$$\mathcal{L} = \exp^{-1}(\overline{\mathbb{Q}^*}) = \{\lambda \in \mathbb{C} : e^{\lambda} \in \overline{\mathbb{Q}^*}\}\big(= \{\log \alpha : \alpha \in \overline{\mathbb{Q}^*}\}\big).$$

Note that $\mathcal{L}$ forms $\mathbb{Q}$-vector subspace of $\mathbb{C}$. The Hermite-Lindemann Theorem 1.7.3 can be written as $\mathcal{L} \cap \overline{\mathbb{Q}} = \{0\}$.
In 1900, Hilbert proposed the following question.
Question (Hilbert)
Let $\alpha = e^{\log \alpha}$ with non-zero $\log \alpha$, be an algebraic number and β be an irrational algebraic number. Then

$$\alpha^{\beta} = e^{\beta \log \alpha} \tag{1.10}$$

is transcendental.
In 1934, A. O. Gel'fond and T. Schneider, independently, proved the following theorem.

Theorem 1.7.7 (Gel'fond-Schneider) *If λ_1, λ_2 are $\mathbb{Q}$-linearly independent elements of $\mathcal{L}$, then they are $\bar{\mathbb{Q}}$-linearly independent.*

Observe that the above theorem solves the above question by working with $\log \alpha$ and $\log \alpha^{\beta}$. This shows that $\mathcal{L}$ is not a $\bar{\mathbb{Q}}$-vector space.

For a proof of this, see [13, Corollary 9.3].

In 1966, A. Baker proved the following fundamental result on transcendental numbers.

Theorem 1.7.8 (A. Baker) *If* $\alpha_1, \ldots, \alpha_n$ *are non-zero algebraic numbers such that* $\log \alpha_1, \ldots, \log \alpha_n$ *are linearly independent over* $\mathbb{Q}$*, then*

$$\log \alpha_1, \ldots, \log \alpha_n$$

are linearly independent over $\bar{\mathbb{Q}}$*.*

Baker extended the above theorem to a non-homogeneous case.

Theorem 1.7.9 (A. Baker. Non-homogeneous case) *If* $\alpha_1, \ldots, \alpha_n$ *are non-zero algebraic numbers such that* $\log \alpha_1, \ldots, \log \alpha_n$ *are linearly independent over* $\mathbb{Q}$*, then*

$$1, \log \alpha_1, \ldots, \log \alpha_n$$

are linearly independent over $\bar{\mathbb{Q}}$*.*

For a proof of the above theorem, see [13, Theorem 65].
Observe that the case $n = 1$ is a consequence of the Lindemann-Weierstrass theorem and $n = 2$ implies the Gel' fond-Schneider theorem.
There are some important corollaries to Baker's theorem and for a proof of the following, see [13, Corollaries 20.1 and 20.2].

Corollary 1.7.10 *If* $\alpha_1, \ldots, \alpha_n$ *and* $\beta_1, \ldots, \beta_n$ *are algebraic numbers with* α_i*'s are non-zero, then*

$$\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$$

is either zero or transcendental.

The following corollary generalizes the Gel'fond-Schneider theorem.

Corollary 1.7.11 *If* $\alpha_1, \ldots, \alpha_n$ *and* $\beta_0, \beta_1, \ldots, \beta_n$ *are non-zero algebraic numbers, then*

$$e^{\beta_0} \alpha_1^{\beta_1} \ldots \alpha_n^{\beta_n}$$

is transcendental.

Definition 1.7.12 Let $E|F$ be a field extension. A maximal algebraically independent subset $S \subset E$ over F with respect to the inclusion ordering is called a transcendence base for $E|F$. Any two such transcendence bases have the same cardinality.

Definition 1.7.13 Transcendence degree of $E|F$ is the cardinality of any transcendence base. It is denoted by $tr.deg.(E|F)$.

Example 1.7.2

1. $tr.deg.(\mathbb{Q}(\sqrt{2})|\mathbb{Q}) = 0$.
2. For variables $x_1, \ldots, x_n$,

$$tr.deg.(F(x_1, \ldots, x_n)|F) = n$$

since $x_1, \ldots, x_n$ are algebraically independent.

A. O. Gel'fond in 1948 and Th. Schneider in 1957, suggested the following conjecture.

Conjecture 1.7.14 If $\alpha \neq 0, 1$ is an algebraic number and if β is an irrational algebraic number of degree d, then the $d-1$ numbers $\alpha^{\beta}, \alpha^{\beta^2}, \ldots, \alpha^{\beta^{d-1}}$ are algebraically independent. Here α^{β} is defined as in (1.10).

In 1949, A. O. Gel'fond proved the above statement for a cubic number β.
Further, G. Diaz proved the following statement which is the best result in this direction.
If β is an algebraic number of degree $d \geq 3$, then for any $\lambda \in \mathcal{L} \setminus \{0\}$, at least $[(d+1)/2]$ of the $d-1$ numbers $\alpha^{\beta}, \alpha^{\beta^2}, \ldots, \alpha^{\beta^{d-1}}$ are algebraically independent.
For above history, see [13, page 140].
One of the main open problems in transcendental number theory is Schanuel's conjecture which was stated in 1960's.

Conjecture 1.7.15 (Schanuel's Conjecture) If $\alpha_1, \ldots, \alpha_n$ are any complex numbers linearly independent over $\mathbb{Q}$, then the transcendence degree of

$$\mathbb{Q}(\alpha_1, \ldots, \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n})$$

over $\mathbb{Q}$ is at least n.

One can observe that the following are consequences of the above conjecture.

- For $n = 1$, we get the Hermite-Lindemann Theorem 1.7.3.
- If $\alpha_1, \ldots, \alpha_n$ are non-zero algebraic numbers, then this is the content of Lindemann-Weierstrass Theorem 1.7.4.
- Take $\alpha_i = \beta^{i-1} \log \alpha$, where β is an irrational algebraic number of degree d, we get the d numbers

$$\log \alpha, \alpha^{\beta}, \alpha^{\beta^2}, \ldots, \alpha^{\beta^{d-1}}$$

  are algebraically independent over $\mathbb{Q}$ i.e. we have a stronger version of the Conjecture 1.7.14.

Conjecture 1.7.16 (Algebraic Independence of Logarithms) Let $\lambda_1, \ldots, \lambda_n$ be elements of $\mathcal{L}$ which are linearly independent over $\mathbb{Q}$. Then these numbers are algebraically independent.

The above conjecture is not even known for $n = 2$.

CHAPTER 2

Briggs-Lehmer constants

2.1 Introduction

In 1961, W. E. Briggs introduced the notion of Euler's constant associated to an arithmetic progression of integers

$$a, a+q, a+2q, \ldots (0 < a \leq q).$$

Since D. H. Lehmer discussed the properties of such numbers in 1975, introduced by Briggs, it is known as Briggs-Lehmer constant.

In this chapter, we discuss the existence of Briggs-Lehmer constants, give the proof of Lehmer's identity (2.3) as done by Lehmer and finally furnish a new proof of it.

2.2 Definition and existence of Briggs-Lehmer constant

Fix a natural number $q \geq 1$. For each a satisfying $0 \leq a < q$, the limit

$$\lim_{x\to\infty} \Big(\sum_{\substack{0<n\leq x \\ n\equiv a (\bmod q)}} \frac{1}{n} - \frac{1}{q}\log x \Big), \tag{2.1}$$

exists and is denoted by $\gamma(a,q)$ by Lehmer. We call it as Briggs-Lehmer constants associated to the arithmetic progression $n \equiv a (\mathrm{mod} q)$.

Denote by $H(x,a,q)$ the general harmonic sum associated with the arithmetic progression $a, a+q, a+2q, \ldots,$
that is,

$$H(x,a,q) = \sum_{\substack{0<n\leq x \\ n\equiv a(\bmod q)}} \frac{1}{n}.$$

Note that

$$H(x, a\pm q, q) = H(x,a,q).$$

To show the existence of $\gamma(a,q)$, we divide it into two cases:

- The case $a=0$:

  Since

  $$H(x,0,q) = \frac{1}{q}H(\frac{x}{q},0,1),$$

  we have

  $$\begin{aligned}\gamma(0,q) &= \frac{1}{q}\lim_{x\to\infty}\Big(H(\frac{x}{q},0,1) - \log\frac{x}{q}\Big) - \frac{\log q}{q}.\\ &= (\gamma - \log q)/q.\end{aligned}$$

  Hence the limit of the (2.1) exists.

- The case $a \not\equiv 0(\text{mod} q)$:

  Define

  $$U_n = \frac{1}{a+nq} - \frac{1}{q}\log\big(\frac{a+(n+1)q}{a+nq}\big).$$

  We have

  $$U_n = \int_0^1 \frac{qt}{(a+nq)(a+nq+tq)}dt = O(n^{-2}).$$

  Thus we can construct a convergent series $\sum_{n=0}^{\infty} U_n$ which gives

  $$\begin{aligned}\sum_{n=0}^{\infty} U_n &= \lim_{x\to\infty}\sum_{n=0}^{[x/q]} U_n,\\ &= \lim_{x\to\infty}\Big(H(x,a,q) - \frac{1}{q}\log x\Big).\end{aligned}$$

  This implies that the limit of the (2.1) exists.

From the (2.1) one can deduce the following properties

$$\gamma(0,1) = \gamma,$$
$$\gamma(a \pm q, q) = \gamma(a,q),$$

which show that $\gamma(a,q)$ is a periodic function of a of period q, and

$$\sum_{a=0}^{q-1} \gamma(a,q) = \gamma. \tag{2.2}$$

2.3 Lehmer's Identity

For $q > 1$, one has

$$q\gamma(a,q) = \gamma - \sum_{j=1}^{q-1} e^{-2\pi iaj/q} \log(1 - e^{2\pi ij/q}). \tag{2.3}$$

Proof.— For $a = 0$,

$$\begin{aligned} q\gamma(0,q) &= q\left(\frac{\gamma}{q} - \frac{\log q}{q}\right), \\ &= \gamma - \log q. \end{aligned}$$

Since

$$\begin{aligned} \gamma - \sum_{j=1}^{q-1} \log(1 - e^{2\pi ij/q}) &= \gamma - \log \prod_{j=1}^{q-1}(1 - e^{2\pi ij/q}) \\ &= \gamma - \log q \qquad \text{as} \quad \prod_{j=1}^{q-1}(x - e^{2\pi ij/q}) = \frac{x^q - 1}{x - 1}. \end{aligned}$$

Now suppose that $a \not\equiv 0 (\mathrm{mod} q)$. For simplicity write $\zeta_q = e^{2\pi i/q}$. Consider the finite Fourier series generated by $\gamma(a,q)$, namely

$$\sigma_j = \sum_{\lambda=0}^{q-1} \gamma(a,q)\zeta_q^{\lambda j} \tag{2.4}$$

$$\sigma_o = \sum_{\lambda=0}^{q-1} \gamma(a,q) = \gamma.$$

When $j \neq 0$,

$$\sigma_j = \lim_{x\to\infty}\left(\sum_{\lambda=0}^{q-1}\{H(x,\lambda,q) - \frac{1}{q}\log x\}\zeta_q^{\lambda j}\right)$$

$$\sigma_j = \lim_{x\to\infty}\sum_{\lambda=0}^{q-1}\{H(x,\lambda,q)\}\zeta_q^{\lambda j}, \quad \text{as} \quad \sum_{j=0}^{q-1}\zeta_q^{\lambda j} = 0,$$

$$= \sum_{n=1}^{\infty}\frac{\zeta_q^{nj}}{n},$$

$$= -\log(1-\zeta_q^j). \tag{2.5}$$

Multiplying both sides of (2.4) by ζ_q^{-ja} and summing over j gives us

$$\sum_{j=0}^{q-1}\sigma_j\zeta_q^{-ja} = \sum_{\lambda=0}^{q-1}\gamma(a,q)\sum_{j=0}^{q-1}\zeta_q^{(\lambda-a)j}$$

$$= q\gamma(a,q).$$

Using (2.5) and the value of σ_0, we have the required identity (2.3).

□

2.4 A new proof of Lehmer's identity

We now generalize Lehmer's identity. In order to state the results, we need to introduce few definitions and notations. Throughout the section, we will denote the set of all prime numbers by P, an arbitrary prime number by p. For any finite subset Ω of primes, we define

$$\mathrm{P}_\Omega := \begin{cases} \prod_{p\in\Omega} p & \text{if } \Omega \neq \phi, \\ 1 & \text{otherwise} \end{cases}$$

and

$$\delta_\Omega := \begin{cases} \prod_{p\in\Omega}(1-\frac{1}{p}) & \text{if } \Omega \neq \phi, \\ 1 & \text{otherwise.} \end{cases}$$

For natural numbers $a, q \geq 1$ and for a finite set of primes Ω not containing any prime factors of q, the generalized Briggs-Euler-Lehmer constant $\gamma(\Omega, a, q)$

is defined as

$$\gamma(\Omega, a, q) := \lim_{x\to\infty}\Bigg(\sum_{\substack{n\le x,\\ (n,\mathrm{P}_\Omega)=1,\\ n\equiv a \bmod q}} \frac{1}{n} - \frac{\delta_\Omega \log x}{q} \Bigg).$$

When $q = 1$, one gets back the so-called generalized Euler's constant

$$\gamma(\Omega) := \lim_{x\to\infty}\Big(\sum_{\substack{n\le x,\\ (n,\mathrm{P}_\Omega)=1}} \frac{1}{n} - \delta_\Omega \log x \Big),$$

introduced by Diamond and Ford [5] in 2008. Note that $\gamma(\emptyset, 1, 1) = \gamma(\emptyset) = \gamma(1, 1) = \gamma$. In this context, we have the following theorem which shows the existence of the generalized Briggs-Euler-Lehmer constant $\gamma(\Omega, a, q)$.

Theorem 2.4.1 *For any finite set of primes Ω and a natural number $q \ge 1$ with $(q,\ \mathrm{P}_\Omega) = 1$, one has*

$$\gamma(\Omega, a, q) - \delta_\Omega \frac{\gamma}{q} = \frac{\delta_\Omega}{q}\sum_{p\in\Omega}\frac{\log p}{p-1} - \sum_{\Omega'\subseteq\Omega}\frac{(-1)^{\mathrm{Card}(\Omega')}}{q\mathrm{P}_{\Omega'}}\sum_{\substack{\zeta_q\in\mu_q\\ \zeta_q\ne 1}}\zeta_q^{-a}\log(1-\zeta_q^{\mathrm{P}_{\Omega'}}), \quad (2.6)$$

where $\mathrm{Card}(\Omega')$ *denotes the cardinality of the set* Ω'*.*

Note that when we put $\Omega = \phi$ in (2.6), we recover the identity of Lehmer (2.3). The techniques involved in our proof is different from that of Lehmer and hence gives another proof of Lehmer's original identity.

2.5 Preliminaries

As before, for any finite set Ω of primes, we set

$$\mathrm{P}_\Omega := \begin{cases} \prod_{p\in\Omega} p & \text{if } \ \Omega \ne \phi, \\ 1 & \text{otherwise.} \end{cases}$$

With these notations, one can now easily deduce that

$$\delta_\Omega = \sum_{d\mid \mathrm{P}_\Omega}\frac{\mu(d)}{d} = \sum_{\Omega'\subseteq\Omega}\frac{(-1)^{\mathrm{Card}(\Omega')}}{\mathrm{P}_{\Omega'}} \quad (2.7)$$

$$-\sum_{d|\mathrm{P}_\Omega}\frac{\mu(d)\log d}{d} \;=\; \delta_\Omega\sum_{p\in\Omega}\frac{\log p}{p-1}.$$

For natural numbers $q, r \geq 1$, we have

$$\prod_{\substack{\zeta_q\in\mu_q\\ \zeta_q\neq 1}}(1-\zeta_q) = q \quad \text{and} \quad \prod_{\zeta_q\in\mu_q}(1-\zeta_q\zeta_r) = 1-\zeta_r^q, \tag{2.8}$$

where ζ_r is any fixed r-th root of unity. The above identities follow by substituting $X = 1$ in

$$X^{q-1}+X^{q-2}+\cdots+1 = \prod_{\substack{\zeta_q\in\mu_q,\\ \zeta_q\neq 1}}(X-\zeta_q)$$

and $X = 1/\zeta_r$ in

$$X^q - 1 = \prod_{\zeta_q\in\mu_q}(X-\zeta_q).$$

In order to prove our main theorem, we need the following lemmas.

Lemma 2.5.1 *For natural numbers $a, r > 1, q \geq 1$ with $(q, r) = 1$, we have*

$$\lim_{x\to\infty}\left(\sum_{\substack{n\leq x\\ n\equiv a \bmod q\\ n\equiv 0 \bmod r}}\frac{1}{n} - \frac{1}{qr}\sum_{n\leq x}\frac{1}{n}\right) = \frac{-1}{qr}\sum_{\zeta_q\in\mu_q}\sum_{\substack{\zeta_r\in\mu_r\\ \zeta_q\zeta_r\neq 1}}\zeta_q^{-a}\log(1-\zeta_q\zeta_r).$$

Proof.— Since $(q, r) = 1$, we have $\zeta_q\zeta_r = 1$ if and only if $\zeta_q = \zeta_r = 1$. Hence

$$\begin{aligned}
&\frac{-1}{qr}\sum_{\zeta_q\in\mu_q}\sum_{\substack{\zeta_r\in\mu_r\\ \zeta_q\zeta_r\neq 1}}\zeta_q^{-a}\log(1-\zeta_q\zeta_r)\\
=\; &\frac{-1}{qr}\sum_{\substack{\zeta_r\in\mu_r\\ \zeta_r\neq 1}}\log(1-\zeta_r) - \frac{1}{qr}\sum_{\substack{\zeta_q\in\mu_q\\ \zeta_q\neq 1}}\sum_{\zeta_r\in\mu_r}\zeta_q^{-a}\log(1-\zeta_q\zeta_r)\\
=\; &\frac{1}{qr}\sum_{\substack{\zeta_r\in\mu_r\\ \zeta_r\neq 1}}\sum_{n=1}^{\infty}\frac{\zeta_r^n}{n} + \frac{1}{qr}\sum_{\substack{\zeta_q\in\mu_q\\ \zeta_q\neq 1}}\sum_{\zeta_r\in\mu_r}\sum_{n=1}^{\infty}\frac{\zeta_q^{n-a}\zeta_r^n}{n}\\
=\; &\frac{1}{qr}\lim_{x\to\infty}\left(\sum_{n\leq x}\frac{1}{n}\sum_{\substack{\zeta_r\in\mu_r\\ \zeta_r\neq 1}}\zeta_r^n + \sum_{n\leq x}\frac{1}{n}\sum_{\zeta_r\in\mu_r}\zeta_r^n\sum_{\substack{\zeta_q\in\mu_q\\ \zeta_q\neq 1}}\zeta_q^{n-a}\right)
\end{aligned}$$

$$= \frac{1}{qr}\lim_{x\to\infty}\Big(\sum_{n\le x}\frac{1}{n}\sum_{\zeta_r\in\mu_r}\zeta_r^n\sum_{\zeta_q\in\mu_q}\zeta_q^{n-a}-\sum_{n\le x}\frac{1}{n}\Big)$$

$$= \lim_{x\to\infty}\Big(\sum_{\substack{n\le x\\ n\equiv a \bmod q\\ n\equiv 0 \bmod r}}\frac{1}{n}-\frac{1}{qr}\sum_{n\le x}\frac{1}{n}\Big)$$

since

$$\sum_{\zeta_q\in\mu_q}\zeta_q^n=\begin{cases} q & \text{if } n\equiv 0 \bmod q,\\ 0 & \text{otherwise.}\end{cases}$$

□

Lemma 2.5.2 *For natural numbers a, q and a finite set of primes Ω, we have*

$$\sum_{\substack{n\le x\\ n\equiv a \bmod q\\ (n,\mathrm{P}_\Omega)=1}}\frac{1}{n}=\sum_{\Omega'\subseteq\Omega}(-1)^{\mathrm{Card}(\Omega')}\sum_{\substack{n\le x\\ n\equiv a \bmod q\\ n\equiv 0 \bmod \mathrm{P}_{\Omega'}}}\frac{1}{n}.$$

Proof.— We have

$$\sum_{\substack{n\le x\\ n\equiv a \bmod q\\ (n,\mathrm{P}_\Omega)=1}}\frac{1}{n}=\sum_{\substack{n\le x\\ n\equiv a \bmod q}}\frac{1}{n}\sum_{d|(n,\mathrm{P}_\Omega)}\mu(d) = \sum_{d|\mathrm{P}_\Omega}\mu(d)\sum_{\substack{n\le x\\ n\equiv a \bmod q\\ d|n}}\frac{1}{n}$$

$$= \sum_{d|\mathrm{P}_\Omega}(-1)^{\mathrm{Card}(\Omega_d)}\sum_{\substack{n\le x\\ n\equiv a \bmod q\\ d|n}}\frac{1}{n},$$

where Ω_d is the set of prime divisors of d.

Hence we have

$$\sum_{\substack{n\le x\\ n\equiv a \bmod q\\ (n,\mathrm{P}_\Omega)=1}}\frac{1}{n}=\sum_{\Omega'\subseteq\Omega}(-1)^{\mathrm{Card}(\Omega')}\sum_{\substack{n\le x\\ n\equiv a \bmod q\\ n\equiv 0 \bmod \mathrm{P}_{\Omega'}}}\frac{1}{n}.$$

□

We now give the proof of our main theorem which is a generalization of the identity of Lehmer (2.6).

Proof of 2.4.1.— Using Lemma 2.5.2 and (2.7), we can write

$$\gamma(\Omega,a,q)-\delta_\Omega\frac{\gamma}{q}=\lim_{x\to\infty}\Big(\sum_{\substack{n\le x\\ n\equiv a \bmod q\\ (n,\mathrm{P}_\Omega)=1}}\frac{1}{n}-\frac{\delta_\Omega}{q}\sum_{n\le x}\frac{1}{n}\Big),$$

$$=\lim_{x\to\infty}\left(\sum_{\Omega'\subseteq\Omega}(-1)^{\mathrm{Card}(\Omega')}\sum_{\substack{n\le x\\ n\equiv a \bmod q\\ n\equiv 0 \bmod \mathrm{P}_{\Omega'}}}\frac{1}{n}-\frac{1}{q}\sum_{\Omega'\subseteq\Omega}\frac{(-1)^{\mathrm{Card}(\Omega')}}{\mathrm{P}_{\Omega'}}\sum_{n\le x}\frac{1}{n}\right).$$

Thus

$$\begin{aligned}
&\gamma(\Omega,a,q)-\delta_\Omega\frac{\gamma}{q}\\
&=\lim_{x\to\infty}\left(\sum_{\Omega'\subseteq\Omega}(-1)^{\mathrm{Card}(\Omega')}\Big(\sum_{\substack{n\le x\\ n\equiv a \bmod q\\ n\equiv 0 \bmod \mathrm{P}_{\Omega'}}}\frac{1}{n}-\frac{1}{q\mathrm{P}_{\Omega'}}\sum_{n\le x}\frac{1}{n}\Big)\right)\\
&=\sum_{\Omega'\subseteq\Omega}\frac{(-1)^{\mathrm{Card}(\Omega')+1}}{q\mathrm{P}_{\Omega'}}\sum_{\zeta_q\in\mu_q}\sum_{\substack{\zeta_{\mathrm{P}_{\Omega'}}\in\mu_{\mathrm{P}_{\Omega'}}\\ \zeta_q\zeta_{\mathrm{P}_{\Omega'}}\neq1}}\zeta_q^{-a}\log(1-\zeta_q\zeta_{\mathrm{P}_{\Omega'}}),\quad\text{using Lemma 2.5.1.}\\
&=\sum_{\Omega'\subseteq\Omega}\frac{(-1)^{\mathrm{Card}(\Omega')+1}}{q\mathrm{P}_{\Omega'}}\left(\sum_{\substack{\zeta_{\mathrm{P}_{\Omega'}}\in\mu_{\mathrm{P}_{\Omega'}}\\ \zeta_{\mathrm{P}_{\Omega'}}\neq1}}\log(1-\zeta_{\mathrm{P}_{\Omega'}})+\sum_{\substack{\zeta_q\in\mu_q\\ \zeta_q\neq1}}\sum_{\zeta_{\mathrm{P}_{\Omega'}}\in\mu_{\mathrm{P}_{\Omega'}}}\zeta_q^{-a}\log(1-\zeta_q\zeta_{\mathrm{P}_{\Omega'}})\right)\\
&=\sum_{\Omega'\subseteq\Omega}\frac{(-1)^{\mathrm{Card}(\Omega')+1}}{q\mathrm{P}_{\Omega'}}\left(\log\prod_{\substack{\zeta_{\mathrm{P}_{\Omega'}}\in\mu_{\mathrm{P}_{\Omega'}}\\ \zeta_{\mathrm{P}_{\Omega'}}\neq1}}(1-\zeta_{\mathrm{P}_{\Omega'}})-\sum_{\substack{\zeta_q\in\mu_q\\ \zeta_q\neq1}}\zeta_q^{-a}\sum_{\zeta_{\mathrm{P}_{\Omega'}}\in\mu_{\mathrm{P}_{\Omega'}}}\sum_{m\ge1}\frac{\zeta_q^m\zeta_{\mathrm{P}_{\Omega'}}^m}{m}\right)\\
&=\sum_{\Omega'\subseteq\Omega}\frac{(-1)^{\mathrm{Card}(\Omega')+1}}{q\mathrm{P}_{\Omega'}}\left(\log\mathrm{P}_{\Omega'}-\sum_{\substack{\zeta_q\in\mu_q\\ \zeta_q\neq1}}\zeta_q^{-a}\sum_{\substack{m\ge1\\ m\equiv0 \bmod \mathrm{P}_{\Omega'}}}\frac{\zeta_q^m\mathrm{P}_{\Omega'}}{m}\right)\\
&=\sum_{\Omega'\subseteq\Omega}\frac{(-1)^{\mathrm{Card}(\Omega')+1}}{q\mathrm{P}_{\Omega'}}\left(\log\mathrm{P}_{\Omega'}-\sum_{\substack{\zeta_q\in\mu_q\\ \zeta_q\neq1}}\zeta_q^{-a}\sum_{k=1}^{\infty}\frac{\zeta_q^{k\mathrm{P}_{\Omega'}}}{k}\right)\\
&=\sum_{\Omega'\subseteq\Omega}\frac{(-1)^{\mathrm{Card}(\Omega')+1}}{q\mathrm{P}_{\Omega'}}\left(\log\mathrm{P}_{\Omega'}+\sum_{\substack{\zeta_q\in\mu_q\\ \zeta_q\neq1}}\zeta_q^{-a}\log(1-\zeta_q^{\mathrm{P}_{\Omega'}})\right),\quad\text{using equation (2.8)}\\
&=\sum_{d|\mathrm{P}}\frac{-\mu(d)\log d}{qd}+\sum_{\Omega'\subseteq\Omega}\frac{(-1)^{\mathrm{Card}(\Omega')+1}}{q\mathrm{P}_{\Omega'}}\sum_{\substack{\zeta_q\in\mu_q\\ \zeta_q\neq1}}\zeta_q^{-a}\log(1-\zeta_q^{\mathrm{P}_{\Omega'}})
\end{aligned}$$

$$= \quad \frac{\delta_\Omega}{q} \sum_{p \in \Omega} \frac{\log p}{p-1} - \sum_{\Omega' \subseteq \Omega} \frac{(-1)^{\mathrm{Card}(\Omega')}}{q \mathrm{P}_{\Omega'}} \sum_{\substack{\zeta_q \in \mu_q \\ \zeta_q \neq 1}} \zeta_q^{-a} \log(1 - \zeta_q^{\mathrm{P}_{\Omega'}}), \quad \text{using equation (2.7).}$$

This completes the proof of the theorem.

□

CHAPTER 3

Transcendence of Briggs-Lehmer Constants

In the preceding chapter, we discussed the existence of generalized Briggs-Lehmer constants. Here we consider them from the point of view of transcendence. In 2010 Murty and Saradha [15] proved the following result.

Theorem 3.0.3 *In the infinite list of numbers $\{\gamma(a,q)\}$ with $1 \le a < q, q \ge 2$, at most one number is algebraic. Further, if γ is algebraic, then the only number from the above list which is algebraic is $\gamma(2,4) = \gamma/4$.*

In 2013 Murty and Zaytseva [17] complemented Theorem 3.0.3 by obtaining the following result.

Theorem 3.0.4 *Let S be the set of numbers $\{\gamma(\Omega)\}$, as Ω ranges over all finite sets of distinct primes. Then all numbers of S are transcendental with at most one exception.*

In this chapter we extend these results to the family of numbers $\gamma(\Omega, a, q)$ by proving Theorems 3.2.1 and 3.2.2 in Section 3.2. In both the above theorems, the one possible exception is due to the intractable γ.
We are also interested in the same question for $\gamma(\Omega, a, q)$.

3.1 Preliminaries

To answer such questions for $\gamma(\Omega, a, q)$, we need the following results.

Proposition 3.1.1 *For $n \geq 2$, let $\alpha_1, \ldots, \alpha_n$ be non-zero algebraic numbers such that the number $\log \alpha_1$ belongs to the $\bar{\mathbb{Q}}$-vector space $\bar{\mathbb{Q}} < \log \alpha_2, \ldots, \log \alpha_n >$ generated by the numbers $\log \alpha_2, \ldots, \log \alpha_n$, then the number $\log \alpha_1$ belongs to the $\mathbb{Q}$-vector-space $\mathbb{Q} < \log \alpha_2, \ldots, \log \alpha_n >$ generated by the numbers $\log \alpha_2, \ldots, \log \alpha_n$.*

Proof.— Let E be a maximal $\overline{\mathbb{Q}}$-linearly independent subset of $\{\log \alpha_2, \ldots, \log \alpha_n\}$. If the conclusion of the proposition does not hold then $\{\log \alpha_1\} \cup E$ is linearly independent over $\mathbb{Q}$ hence over $\overline{\mathbb{Q}}$ as well by Baker's Theorem. This contradicts the assumption that $\log \alpha_1$ belongs to the $\overline{\mathbb{Q}}$-vector space generated by E. □

Lemma 3.1.2 *Let $\zeta_q (\neq 1)$ be a q-th root of unity, where $q = p^n$, $n \geq 1$ and p be a prime. Then the norm of $(1 - \zeta_q)$ is p.*

Proof.— Any q-th root of unity distinct from 1 is a primitive r-th root of unity for some r dividing q. We may therefore suppose that ζ_q is a primitive q-th root of unity. On account of the identity

$$\frac{X^{p^n} - 1}{X^{p^{n-1}} - 1} = 1 + X^{p^{n-1}} + \ldots + X^{(p-1)p^{n-1}} \tag{3.1}$$

we see that every primitive q-th root of 1 is a zero of the polynomial on the right hand side of (3.1). There are $\phi(q) = (p-1)p^{n-1}$ primitive q-th roots of unity, which is the same as the degree of this polynomial. Moreover, any two of these primitive roots are Galois conjugates over $\mathbb{Q}$. It follows that

$$\prod_{\theta \text{ conjugate of } \zeta_q} (X - \theta) = 1 + X^{p^{n-1}} + \ldots + X^{(p-1)p^{n-1}}. \tag{3.2}$$

Putting $X = 1$ in (3.2) gives the conclusion of the lemma. □

Lemma 3.1.3 *Let ζ_q be a primitive q-th root of unity, where $q \geq 1$ has at least two prime factors. Then $(1 - \zeta_q)$ is a unit.*

Proof.— See page 12 of Washington [26]. □

3.2 Transcendence of the Numbers $\gamma(\Omega, a, q)$

The above preliminaries together with the celebrated theorem of Baker on linear form of logarithms Theorem 1.7.9 and Corollary 1.7.10 allows us to prove the following theorems.

Theorem 3.2.1 *For any natural numbers $a, q > 1$ with $(a, q) = 1$ and for any finite set of primes Ω, the number*

$$\gamma(\Omega, a, q) - \delta_\Omega \frac{\gamma}{q}$$

is transcendental.

Proof.— We know from Theorem 2.4.1 that

$$\gamma(\Omega, a, q) - \delta_\Omega \frac{\gamma}{q} = \frac{\delta_\Omega}{q} \sum_{p \in \Omega} \frac{\log p}{p - 1} - \sum_{\Omega' \subseteq \Omega} \frac{(-1)^{\mathrm{Card}(\Omega')}}{q \mathrm{P}_{\Omega'}} \sum_{\substack{\zeta_q \in \mu_q \\ \zeta_q \neq 1}} \zeta_q^{-a} \log(1 - \zeta_q^{\mathrm{P}_{\Omega'}}).$$

Now by Corollary 1.7.10, $\gamma(\Omega, a, q) - \delta_\Omega \frac{\gamma}{q}$ is either 0 or transcendental. Assume that it is zero.

Case I. Suppose that Ω is non-empty and $p_0 \in \Omega$. Consider the set

$$\mathrm{E} := \{\log p : p \in \Omega, p \neq p_0\} \cup \{\log(1 - \zeta_q) : \zeta_q \in \mu_q,\ \zeta_q \neq 1\}.$$

Since $\delta_\Omega \neq 0$, we get $\log p_0 \in \bar{\mathbb{Q}}(\mathrm{E})$. Then by Proposition 3.1.1, there are integers $a_0 (\neq 0), a_p, a_{\zeta_q}$ such that

$$a_0 \log p_0 = \sum_{\substack{p \in \Omega \\ p \neq p_0}} a_p \log p \ + \sum_{\substack{\zeta_q \in \mu_q \\ \zeta_q \neq 1}} a_{\zeta_q} \log(1 - \zeta_q)$$

which implies that

$$p_0^{a_0} = \prod_{\substack{p \in \Omega \\ p \neq p_0}} p^{a_p} \prod_{\substack{\zeta_q \in \mu_q \\ \zeta_q \neq 1}} (1 - \zeta_q)^{a_{\zeta_q}}.$$

Since $(q, p_0) = 1$, by taking norms of both sides and applying Lemma 3.1.2 together with Lemma 3.1.3, we get a contradiction.

Case II. Suppose that $\Omega = \phi$. Then we have by Theorem 2.4.1 that

$$\gamma(a,q) - \frac{\gamma}{q} = \frac{-1}{q} \sum_{\substack{\zeta_q \in \mu_q \\ \zeta_q \neq 1}} \zeta_q^{-a} \log(1-\zeta_q) = \frac{-1}{q} \sum_{b=1}^{q-1} \eta_q^{-ab} \log(1-\eta_q^b), \tag{3.3}$$

where η_q is a primitive q-th root of unity. Let E $:= \{\log\alpha_1, \ldots, \log\alpha_t\}$ be a maximal $\mathbb{Q}$-linearly independent subset of $\{\log(1-\eta_q^b) \mid 1 \leq b \leq q-1\}$. Write

$$\log(1-\eta_q^b) = \sum_{c=1}^{t} u_{b,c} \log\alpha_c$$

where $u_{b,c} \in \mathbb{Q}$. Set $\gamma_a := \gamma(a,q) - \frac{\gamma}{q}$. Then by (3.3) we have

$$\gamma_a = \frac{-1}{q} \sum_{b=1}^{q-1} \eta_q^{-ab} \sum_{c=1}^{t} u_{b,c} \log\alpha_c = \frac{-1}{q} \sum_{c=1}^{t} u_c \log\alpha_c \ ,$$

where

$$u_c = \sum_{b=1}^{q-1} \eta_q^{-ab} u_{b,c} \in \mathbb{Q}(\eta_q).$$

Without loss of generality we assume that $\gamma_a = 0$ as by Corollary 1.7.10 the above quantity is either zero or transcendental. Then by our assumption on the set E it follows that $u_c = 0$ for all c. Let σ_ℓ be an element of the Galois group of $\mathbb{Q}(\eta_q)$ over $\mathbb{Q}$ which sends η_q to η_q^ℓ. Thus

$$\begin{aligned}
\gamma_{a\ell} &= \frac{-1}{q} \sum_{b=1}^{q-1} \eta_q^{-ab\ell} \sum_{c=1}^{t} u_{b,c} \log\alpha_c \\
&= \frac{-1}{q} \sum_{b=1}^{q-1} \sigma_\ell(\eta_q^{-ab}) \sum_{c=1}^{t} u_{b,c} \log\alpha_c \\
&= \frac{-1}{q} \sum_{c=1}^{t} \sigma_\ell(u_c) \log\alpha_c.
\end{aligned}$$

Hence $\gamma_{a\ell} = 0$ for all ℓ with $(\ell, q) = 1$. Thus

$$0 = \sum_{\substack{1 \leq r < q \\ (r,q)=1}} \gamma_r \ = \ \lim_{x\to\infty} \sum_{\substack{1 \leq r < q \\ (r,q)=1}} \left(\sum_{\substack{n \leq x \\ n \equiv r \bmod q}} \frac{1}{n} \ - \ \frac{1}{q} \sum_{n \leq x} \frac{1}{n} \right)$$

$$= \lim_{x\to\infty}\Big(\sum_{\substack{n\le x\\(n,q)=1}} \frac{1}{n} - \frac{\varphi(q)}{q}\sum_{n\le x}\frac{1}{n}\Big)$$
$$= \gamma(\Omega_q) - \delta_{\Omega_q}\gamma,$$

where Ω_q is the set of all prime divisors of q. Substituting Ω_q in place of Ω and 1 in place q in Theorem 2.4.1, we get

$$\gamma(\Omega_q) - \delta_{\Omega_q}\gamma = \delta_{\Omega_q}\sum_{p\in\Omega_q}\frac{\log p}{p-1},$$

a contradiction since the set $\{\log p : p \in \Omega_q\}$ is linearly independent over $\mathbb{Q}$. This completes the proof of Theorem 3.2.1.

□

In fact, we can prove a stronger result.

Theorem 3.2.2 *Let* $\mathrm{U} := \{\Omega_i\}_{i\in\mathbb{N}}$ *be a sequence of finite subsets of primes and* $\mathrm{S} := \{q_j > 1\}_{j\in\mathbb{N}}$ *be a sequence of mutually co-prime natural numbers. Also let* Ω_i*'s do not contain any prime divisors of* q_j*'s for all* i, j *and* a *be a natural number with* $(a, q_j) = 1$ *for all* j*. Then the set*

$$T := \{\ \gamma(\Omega_i, a, q_j)\ \mid\ \Omega_i \in \mathrm{U},\ q_j \in \mathrm{S}\}$$

has at most one algebraic element.

Proof.— Suppose that $\gamma(\Omega_1, a, q_1), \gamma(\Omega_2, a, q_2) \in \overline{\mathbb{Q}}$. Then

$$\begin{aligned}
&\frac{\delta_{\Omega_2}}{q_2}\gamma(\Omega_1,a,q_1) - \frac{\delta_{\Omega_1}}{q_1}\gamma(\Omega_2,a,q_2) \qquad (3.4)\\
= {}& \frac{\delta_{\Omega_1}\delta_{\Omega_2}}{q_1q_2}\Big(\sum_{p\in\Omega_1}\frac{\log p}{p-1} - \sum_{p\in\Omega_2}\frac{\log p}{p-1}\Big) - \delta_{\Omega_2}\sum_{\Omega_1'\subseteq\Omega_1}\frac{(-1)^{\mathrm{Card}(\Omega_1')}}{q_1q_2\mathrm{P}_{\Omega_1'}}\\
&\sum_{b=1}^{q_1-1}\eta_{q_1}^{-ab}\log(1-\eta_{q_1}^{b\mathrm{P}_{\Omega_1'}}) \quad + \quad \delta_{\Omega_1}\sum_{\Omega_2'\subseteq\Omega_2}\frac{(-1)^{\mathrm{Card}(\Omega_2')}}{q_1q_2\mathrm{P}_{\Omega_2'}}\\
&\sum_{c=1}^{q_2-1}\eta_{q_2}^{-ac}\log(1-\eta_{q_2}^{c\mathrm{P}_{\Omega_2'}}) \in \overline{\mathbb{Q}}\ ,
\end{aligned}$$

where η_{q_1} and η_{q_2} are primitive q_1-th and q_2-th roots of unity respectively. Hence by Corollary 1.7.10, we know that $\frac{\delta_{\Omega_2}}{q_2}\gamma(\Omega_1, a, q_1) - \frac{\delta_{\Omega_1}}{q_1}\gamma(\Omega_2, a, q_2) = 0$.

Case I. Suppose that $\Omega_1 \neq \Omega_2$. Choose p_0 either from $(\Omega_1 \setminus \Omega_2)$ or from $(\Omega_2 \setminus \Omega_1)$. Then arguing as in Case I of Theorem 3.2.1, and using Lemma 2.5.1, we get the theorem.

Case II. Suppose that $\Omega_1 = \Omega_2 = \Omega$, say. Set $\gamma_a := \frac{1}{q_1}\gamma(\Omega, a, q_2) - \frac{1}{q_2}\gamma(\Omega, a, q_1)$. Then from Theorem 2.4.1, we see that

$$\gamma_a = \sum_{\Omega' \subseteq \Omega} \frac{(-1)^{\mathrm{Card}(\Omega')}}{q_1 q_2 \mathrm{P}_{\Omega'}} \left(\sum_{b=1}^{q_1-1} \eta_{q_1}^{-ab} \log(1 - \eta_{q_1}^{bP'}) \; - \; \sum_{c=1}^{q_2-1} \eta_{q_2}^{-ac} \log(1 - \eta_{q_2}^{cP'}) \right),$$

where η_{q_1} and η_{q_2} are primitive q_1-th and q_2-th roots of unity. Let $\{\log \alpha_1, \ldots, \log \alpha_t\}$ be a maximal $\mathbb{Q}$-linearly independent subset of

$$\{\log(1 - \eta_{q_1}^b), \; \log(1 - \eta_{q_2}^c) \mid 1 \leq b \leq q_1 - 1, 1 \leq c \leq q_2 - 1\}.$$

If we write $\log(1 - \eta_{q_1}^b) = \sum_{r=1}^t d_{b,r} \log \alpha_r$ and $\log(1 - \eta_{q_2}^c) = \sum_{r=1}^t e_{c,r} \log \alpha_r$ where $d_{b,r}, e_{c,r}$ are in $\mathbb{Q}$, then we get $\gamma_a = \sum_{r=1}^t \beta_r \log \alpha_r$, where

$$\beta_r := \sum_{\Omega' \subseteq \Omega} \frac{(-1)^{\mathrm{Card}(\Omega')}}{q_1 q_2 \mathrm{P}_{\Omega'}} (\sum_{b=1}^{q_1-1} d_{b,r} \eta_{q_1}^{-ab} - \sum_{c=1}^{q_2-1} e_{c,r} \eta_{q_2}^{-ac}).$$

Hence by Corollary 1.7.10, we have $\beta_r = 0$ for all r as by assumption $\gamma_a = 0$. Arguing as in Case II, Theorem 3.2.1 and by applying Galois elements of $\mathbb{Q}(\eta_{q_1 q_2})$ over $\mathbb{Q}$, we get that $\gamma_a = 0$ for all $(a, q_1 q_2) = 1$. Hence

$$\sum_{\substack{1 \leq a < q_1 q_2 \\ (a, q_1 q_2) = 1}} \gamma_a = 0.$$

Note that by orthogonality of characters we have

$$\begin{aligned} & \frac{1}{q_1} \sum_{\substack{1 \leq a < q_1 q_2 \\ (a, \; q_1 q_2) = 1}} \sum_{\substack{k \leq x, \\ (k, \mathrm{P}_\Omega) = 1 \\ k \equiv a \bmod q_2}} \frac{1}{k} \\ = \; & \frac{1}{q_1 \phi(q_2)} \sum_{\substack{1 \leq a < q_1 q_2 \\ (a, q_1 q_2) = 1}} \sum_{\substack{k \leq x \\ (k, \mathrm{P}_\Omega) = 1}} \frac{1}{k} \sum_{\chi \bmod q_2} \chi(k) \bar{\chi}(a) \\ = \; & \frac{\phi(q_1)}{q_1} \sum_{\substack{k \leq x, \\ (k, \; q_2 \mathrm{P}_\Omega) = 1}} \frac{1}{k} \end{aligned} \tag{3.5}$$

$$= \delta_{\Omega_{q_1}} \sum_{\substack{k \le x, \\ (k,\ P_{\Omega \cup \Omega_{q_2}})=1}} \frac{1}{k},$$

where Ω_{q_1} is the set of all prime divisors of q_1. Thus using (3.5), we get

$$\begin{aligned} 0 = \sum_{\substack{1 \le a < \ q_1 q_2 \\ (a, q_1 q_2)=1}} \gamma_a &= \lim_{x \to \infty} \sum_{\substack{1 \le a < q_1 q_2 \\ (a, q_1 q_2)=1}} \left(\frac{1}{q_1} \sum_{\substack{k \le x \\ (k, P_\Omega)=1 \\ k \equiv a \bmod q_2}} \frac{1}{k} - \frac{1}{q_2} \sum_{\substack{k \le x \\ (k, P_\Omega)=1 \\ k \equiv a \bmod q_1}} \frac{1}{k} \right) \\ &= \lim_{x \to \infty} \left(\delta_{\Omega_{q_1}} \sum_{\substack{k \le x \\ (k, P_{\Omega \cup \Omega_{q_2}})=1}} \frac{1}{k} - \delta_{\Omega_{q_2}} \sum_{\substack{k \le x \\ (k, P_{\Omega \cup \Omega_{q_1}})=1}} \frac{1}{k} \right) \\ &= \delta_{\Omega_{q_1}} \gamma(\Omega \cup \Omega_{q_2}) - \delta_{\Omega_{q_2}} \gamma(\Omega \cup \Omega_{q_1}). \end{aligned}$$

Here $\Omega_{q_1}, \Omega_{q_2}$ denote respectively the sets of prime divisors of q_1 and q_2 respectively. Now using Theorem 2.4.1, we have that

$$\delta_{\Omega_{q_1}} \gamma(\Omega \cup \Omega_{q_2}) - \delta_{\Omega_{q_2}} \gamma(\Omega \cup \Omega_{q_1}) = \delta_{\Omega \cup \Omega_{q_1} \cup \Omega_{q_2}} \left(\sum_{p \in \Omega_{q_2}} \frac{\log p}{p-1} - \sum_{p \in \Omega_{q_1}} \frac{\log p}{p-1} \right).$$

Since $(q_1, q_2) = 1$, the last expression is transcendental by Corollary 1.7.10.

□

CHAPTER 4

Functions $\zeta(s)$ and $\zeta(s;a)$

4.1 Hurwitz zeta function and its relation to Euler constant

In this chapter, we introduce the Hurwitz zeta function and illustrate some of its properties relevant to our work in the last chapter. For the sake of completion, we furnish proofs of these properties.

Hurwitz zeta function $\zeta(s;a)$ is defined for $s = \sigma + it, \sigma > 1$ by the series

$$\zeta(s;a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \tag{4.1}$$

where a is initially a real number, $0 < a \leq 1$.

Setting $a = 1$,

$$\zeta(s;a) = \sum_{n=0}^{\infty} \frac{1}{(n+1)^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s).$$

Now,

$$\begin{aligned}
\zeta(s;a) - \zeta(s) &= \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} - \sum_{n=1}^{\infty} \frac{1}{n^s} \\
&= \frac{1}{a^s} + \sum_{n=1}^{\infty} \Big(\frac{1}{(n+a)^s} - \frac{1}{n^s} \Big) \\
\lim_{s \to 1} \big(\zeta(s;a) - \zeta(s) \big) &= \frac{1}{a} + \sum_{n=1}^{\infty} \Big(\frac{1}{(n+a)} - \frac{1}{n} \Big)
\end{aligned}$$

$$= -\psi(a) - \gamma \text{ by } (1.3).$$

4.2 L-series and its relation to Hurwitz zeta function

The Dirichlet L-function associated to a Dirichlet character χ is defined as

$$L(s, \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

for $s \in \mathbb{C}$ and $\Re(s) > 1$. In 1837, Dirichlet introduced $L(s, \chi)$ in his celebrated paper to prove that there are infinitely many primes in arithmetic progression $a + nb$, $n = 1, 2, \ldots$, with $\gcd(a, b) = 1$.

One can get Hurwitz zeta function from L-function associated to a Dirichlet character χ mod k as follows,

$$\begin{aligned} L(s, \chi) &= \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \\ &= \sum_{r=1}^{k} \sum_{q=1}^{\infty} \frac{\chi(qk + r)}{(qk + r)^s} \\ &= \frac{1}{k^s} \sum_{r=1}^{k} \chi(r) \sum_{q=1}^{\infty} \frac{1}{(q + \frac{r}{k})^s} \\ &= \frac{1}{k^s} \sum_{r=1}^{k} \chi(r) \zeta(s; \frac{r}{k}). \end{aligned}$$

This representation of $L(s, \chi)$ as a linear combination of Hurwitz zeta function shows that the properties of L-function depend on those of $\zeta(s; x)$.

4.3 Integral representation of Hurwitz zeta function

The following theorem is about the convergence of Hurwitz zeta function.

Theorem 4.3.1 *The series for $\zeta(s;a)$ converges absolutely for $\sigma > 1$. The convergence is uniform in compact sets in every half plane $\sigma \geq 1+\delta, \delta > 0$, so $\zeta(s;a)$ is an analytic function of s in the half plane $\sigma > 1$.*

Proof.— The theorem follows since

$$|\zeta(s;a)| \leq \sum_{n=1}^{\infty} \left| \frac{1}{(n+a)^s} \right| = \sum_{n=1}^{\infty} \frac{1}{(n+a)^{\sigma}} \leq \sum_{n=1}^{\infty} \frac{1}{(n+a)^{1+\delta}} \quad \text{for} \quad \delta > 0.$$

□

Theorem 4.3.2 *For $\sigma > 1$, we have integral representation*

$$\Gamma(s)\zeta(s;a) = \int_0^{\infty} \frac{x^{s-1}e^{-ax}}{1-e^{-x}}dx,$$

where Γ is the gamma function.

4.4 A contour integral representation of Hurwitz zeta function

To extend $\zeta(s;a)$ beyond the line $\sigma = 1$, we derive another representation in terms of a contour integral.

Theorem 4.4.1 *If $0 < a \leq 1$, the function defined by the contour integral*

$$I(s,a) = \int_C \frac{z^{s-1}e^{az}}{1-e^z}dz, \tag{4.2}$$

is an entire function of s. Moreover, we have

$$\zeta(s;a) = \Gamma(1-s)I(s,a) \quad \text{if} \quad \sigma > 1. \tag{4.3}$$

In the above theorem, contour C is a loop around the negative real axis. The loop is composed of three parts C_1, C_2, C_3. C_2 is a positively oriented circle of radius $c < 2\pi$ about the origin and C_1 , C_3 are the lower and upper edges of a "cut" in the z-plane along the negative real axis.

Proof.— Here

$$z^s = \begin{cases} r^s e^{-\pi i s}, & \text{on} \quad C_1 \\ r^s e^{\pi i s}, & \text{on} \quad C_3. \end{cases}$$

To show that $I(s,a)$ is an entire function of s, we will consider an arbitrary compact disk $|s| \leq M$ and prove that the integrals along C_1, C_3 converge uniformly on every such disk.
Along C_1 we have, for $r \geq 1$,

$$|z^{s-1}| = r^{\sigma-1}|e^{-\pi i(\sigma-1+it)}| = r^{\sigma-1}e^{\pi t} \leq r^{M-1}e^{\pi M} \quad \text{since} \quad |s| \leq M.$$

Similarly along C_3 we have, for $r \geq 1$,

$$|z^{s-1}| = r^{\sigma-1}|e^{\pi i(\sigma-1+it)}| = r^{\sigma-1}e^{-\pi t} \leq r^{M-1}e^{\pi M} \quad \text{since} \quad |s| \leq M.$$

Further since $z = -r$ on either C_1 or C_3 we have, for $r \geq 1$,

$$\left|\frac{z^{s-1}e^{az}}{1-e^z}\right| \leq \frac{r^{M-1}e^{\pi M}e^{-ar}}{1-e^{-r}} = \frac{r^{M-1}e^{\pi M}e^{(1-a)r}}{e^r - 1}.$$

But for $r > \log 2,\ e^r - 1 > e^r/2$ so

$$\left|\frac{z^{s-1}e^{az}}{1-e^z}\right| \leq \frac{r^{M-1}e^{\pi M}e^{-ar}}{1-e^{-r}} < Ar^{M-1}e^{-ar}, \quad \text{where} \quad A = 2e^{\pi M}.$$

Now, $\int_c^\infty r^{M-1}e^{-ar}dr$ converges if $c > 0$. This will imply that the integrals along C_1, C_3 converge uniformly on every compact disk $|s| \leq M$. Hence $I(s,a)$ is an entire function of s.
Next, we want to show equation (4.3). We have

$$2\pi i I(s,a) = \left(\int_{C_1} + \int_{C_2} + \int_{C_3}\right)\frac{z^{s-1}e^{az}}{1-e^z},$$

where on C_1 and C_3 we have $z = -r$, and on C_2 we write $z = ce^{i\theta}$, for $-\pi \leq \theta \leq \pi$.
This gives us the following

$$2\pi i I(s,a) = \int_{-\infty}^{c} r^{s-1}e^{-\pi i s}g(-r)dr + \int_{-\pi}^{\pi} c^{s-1}e^{i(s-1)\theta}g(ce^{i\theta})dr \quad +$$

$$\int_c^\infty r^{s-1}e^{\pi is}g(-r)dr, \quad \text{where} \quad g(z) = \frac{e^{az}}{1-e^z}.$$

Therefore, we have

$$2\pi i I(s,a) = 2i\sin(\pi s)\int_c^\infty r^{s-1}e^{\pi is}g(-r)dr + ic^s\int_{-\pi}^{\pi} e^{is\theta}g(ce^{i\theta})d\theta.$$

On dividing by $2i$ we will have

$$\pi I(s,a) = \sin(\pi s)I_1(s,c) + I_2(s,c),$$

and

$$\lim_{c\to 0} I_1(s,c) = \int_0^\infty \frac{r^{s-1}e^{-ar}}{1-e^{-r}}dr = \Gamma(s)\zeta(s;a) \quad \text{for} \quad \sigma > 1.$$

Our next claim is

$$\lim_{c\to 0} I_2(s,c) = 0.$$

Note that $g(z)$ is analytic inside $|z| < 2\pi$ except for a first order pole at $z = 0$. This will imply that $zg(z)$ is analytic every where inside $|z| < 2\pi$ hence bounded there, that is, $|zg(z)| \le A$ where $|z| = c < 2\pi$ and A is a constant. Hence

$$|I_2(s,c)| \le \frac{c^\sigma}{2}\int_{-\pi}^{\pi} e^{-t\theta}\frac{A}{c}d\theta \le Ae^{\pi t}c^{\sigma-1}.$$

As $\sigma > 1$ and $c \to 0$ we find $I_2(s,a) \to 0$.
Thus

$$\pi I(s,c) = \sin(\pi s)\Gamma(s)\zeta(s;a),$$

and using identity $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$ we will have the required result.

□

$\Gamma(1-s)$ and $I(s,a)$ are well defined for complex numbers with $\sigma < 1$. So we can define $\zeta(s;a)$ for $\sigma < 1$ too. This provides the meromorphic continuation of $\zeta(s;a)$ to the whole complex plane.

Theorem 4.4.2 *The function $\zeta(s;a)$ so defined is analytic for all s except for a simple pole at $s = 1$ with residue 1.*

PROOF.— Since $I(s,a)$ is an entire function, the possible singularities of $\zeta(s;a)$ are the poles of $\Gamma(1-s)$ at $s = 1, 2, 3, \ldots$. Since $\zeta(s;a)$ is well defined for $\sigma > 1$,

$s=1$ is the only possible pole of $\zeta(s;a)$.
Further, for $s=n$ in $\mathbb{Z}$,

$$\int_{C_1}\frac{z^{n-1}e^{az}}{1-e^z}dz=-\int_{C_3}\frac{z^{n-1}e^{az}}{1-e^z}dz$$

and

$$\begin{aligned} I(n,a)&=\frac{1}{2\pi i}\Big(\int_{C_1}+\int_{C_2}+\int_{C_3}\Big)\frac{z^{n-1}e^{az}}{1-e^z}dz\\ &=\frac{1}{2\pi i}\int_{C_2}\frac{z^{n-1}e^{az}}{1-e^z}dz\\ &=res\Big(\frac{z^{n-1}e^{az}}{1-e^z};0\Big). \end{aligned}$$

In particular for $n=1$, we have

$$I(1,a)=res\Big(\frac{e^{az}}{1-e^z};0\Big)=\lim_{z\to 0}\frac{ze^{az}}{1-e^z}=\lim_{z\to 0}\frac{z}{1-e^z}=-1.$$

Thus to find residue of $\zeta(s;a)$ at $s=1$,

$$\begin{aligned} \lim_{s\to 1}(s-1)\zeta(s;a)&=\lim_{s\to 1}(s-1)\Gamma(1-s)I(s,a)\\ &=\lim_{s\to 1}\Gamma(2-s)\\ &=\Gamma(1)=1. \end{aligned}$$

□

4.5 Analytic continuation of $\zeta(s)$ and $L(s,\chi)$

One can express $\zeta(s)$ and $L(s,\chi)$ as

$$\zeta(s)=\zeta(s;1)\quad\text{for}\quad\sigma>1$$

and

$$L(s,\chi)=k^{-s}\sum_{r=1}^{k}\chi(r)\zeta(s;\frac{r}{k}),$$

where χ is any Dirichlet character mod k. Thus the identities give analytic continuation of $\zeta(s)$ and $L(s,\chi)$ beyond the line $\sigma = 1$.

The value of $\zeta(-n;a)$ can be calculated explicitly if n is a non negative integer.

4.6 Evaluation of $\zeta(-n;a)$

As we know

$$\zeta(-n;a) = \Gamma(1+n)I(-n,a)$$

and

$$I(-n,a) = res\Big(\frac{z^{-n-1}e^{az}}{1-e^z};0\Big),$$

so while calculating the above residues, a new class of functions called *Bernoulli polynomials* enter naturally into this setting.

Definition 4.6.1 For any complex x we define the function $B_n(x)$ as

$$\frac{ze^{zx}}{e^z-1} = \sum_{n=0}^{\infty}\frac{B_n(x)}{n!}z^n \quad \text{for} \quad |z| < 2\pi. \tag{4.4}$$

The numbers $B_n(0)$ are called Bernoulli numbers and are denoted by B_n,

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty}\frac{B_n}{n!}z^n \quad \text{for} \quad |z| < 2\pi. \tag{4.5}$$

Theorem 4.6.2 *The functions* $B_n(x)$ *are polynomials in* x *given by*

$$B_n(x) = \sum_{k=0}^{n}\binom{n}{k}B_k x^{n-k}.$$

Proof.—

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{B_n(x)}{n!}z^n &= \frac{z}{e^z-1}.e^{xz}\\ &= \Big(\sum_{n=0}^{\infty}\frac{B_n}{n!}z^n\Big)\Big(\sum_{n=0}^{\infty}\frac{x^n}{n!}z^n\Big)\end{aligned}$$

equating coefficient of z^n we find

$$\frac{B_n(x)}{n!} = \sum_{k=0}^{n} \frac{B_k}{k!} \frac{x^{n-k}}{(n-k)!} = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}.$$

□

Thus one has the following theorem:

Theorem 4.6.3 *For every integer* $n \geq 0$

$$\zeta(-n;a) = -\frac{B_{(n+1)}(a)}{n+1}. \tag{4.6}$$

PROOF.— As we know

$$\begin{aligned} I(-n,a) &= res\Big(\frac{z^{-n-1}e^{az}}{1-e^z}\Big) \\ &= -res\Big(\frac{z^{-n-2}ze^{az}}{e^z-1}\Big) \\ &= -res\Big(z^{-n-2}\sum_{m=0}^{\infty}\frac{B_m(a)}{m!}z^m\Big) \\ &= -\frac{B_{n+1}(a)}{(n+1)!}. \end{aligned}$$

Since

$$\zeta(-n;a) = n! I(-n,a),$$

the theorem follows.

□

CHAPTER 5

Some infinite sum identities

5.1 Series arising from multiple zeta values

In this chapter, we generalize the results of a recent work of Istevan Mezo. This is the content of our work [19].

For $s_1 > 1$ and $s_i \geq 1$ for $2 \leq i \leq r$ with s_i integral, the multiple zeta values or the multiple zeta functions are defined as

$$\zeta(s_1, s_2, \dots, s_r) = \sum_{n_1 > n_2 > \cdots > n_r \geq 1} \frac{1}{{n_1}^{s_1} \dots {n_r}^{s_r}}.$$

As seen earlier, the Hurwitz zeta function defined in (4.1) can be extended analytically for all $s \in \mathbb{C}$ except $s = 1$, where it has a simple pole with residue 1. The analytic continuation of $\zeta(s; a)$ can also be further enlarged to complex values of a in the whole complex plane $\mathbb{C} \setminus (-\infty, 0]$.
For $a \in \mathbb{C} \setminus (-\infty, 0]$,

$$\zeta(1-k; a) = -\frac{B_k(a)}{k}$$

is valid for any natural number $k \geq 2$ [16, page 2]. Here $B_k(a)$ is the Bernoulli polynomial.

The nth generalized harmonic number of order r is defined as

$$H_{n,r} = \sum_{i=1}^{n} \frac{1}{i^r}.$$

In 2013, Istevan Mezo in [18] proved the following proposition.

Proposition 5.1.1

$$\sum_{i=1}^{\infty} \frac{\zeta(2;i)}{i^2} = \frac{7\pi^4}{360}.$$

In this work, we generalize the above identity for $2k$.

Proposition 5.1.2 *For any positive integer k,*

$$\sum_{i=1}^{\infty} \frac{\zeta(2k;i)}{i^{2k}} = \left\{ \left(\frac{B_{2k}}{2(2k)!} \right)^2 - \frac{B_{4k}}{2(4k)!} \right\} \frac{(2\pi)^{4k}}{2}. \tag{5.1}$$

Proof.— To prove the above identity, we need the following identity from multiple zeta values.

$$\zeta(s_1)\zeta(s_2) = \zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2). \tag{5.2}$$

Proof of (5.2) is not difficult. The left hand side product is a double sum

$$\sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \frac{1}{{n_1}^{s_1}} \frac{1}{{n_2}^{s_2}},$$

which can be written as

$$\sum_{n_1,n_2=1;n_1>n_2}^{\infty} \frac{1}{{n_1}^{s_1}} \frac{1}{{n_2}^{s_2}} + \sum_{n_1,n_2=1;n_2>n_1}^{\infty} \frac{1}{{n_1}^{s_1}} \frac{1}{{n_2}^{s_2}} + \sum_{n_1,n_2=1;n_1=n_2}^{\infty} \frac{1}{{n_1}^{s_1}} \frac{1}{{n_2}^{s_2}}.$$

This proves (5.2). Now, if $s_1 = s_2 = 2k$, then from (5.2), we get

$$\zeta(2k, 2k) = \frac{\zeta(2k)^2 - \zeta(4k)}{2}. \tag{5.3}$$

Next, we recall the famous theorem due to Euler on the values of Riemann zeta function at even numbers. We have

$$2\zeta(2k) = (-1)^{k-1}(2\pi)^{2k} \frac{B_{2k}}{(2k)!}. \tag{5.4}$$

Now

$$\zeta(2k, 2k) = \sum_{i=1}^{\infty} \frac{1}{i^{2k}} \sum_{j=i+1}^{\infty} \frac{1}{j^{2k}}$$

$$= \sum_{i=1}^{\infty} \frac{1}{i^{2k}} (\zeta(2k) - H_{i,2k}) = \zeta(2k)^2 - \sum_{i=1}^{\infty} \frac{H_{i,2k}}{i^{2k}}. \qquad (5.5)$$

Next consider

$$\sum_{i=1}^{\infty} \frac{H_{i,2k}}{i^{2k}} = \sum_{i=1}^{\infty} \frac{1}{i^{2k}} \sum_{j=1}^{i} \frac{1}{j^{2k}}$$
$$= \sum_{j=1}^{\infty} \frac{1}{j^{2k}} \sum_{i=j}^{\infty} \frac{1}{i^{2k}}$$

(interchanging the summation by using Weierstrass M-test)

$$= \sum_{j=1}^{\infty} \frac{\zeta(2k;j)}{j^{2k}}$$
$$= \sum_{i=1}^{\infty} \frac{\zeta(2k;i)}{i^{2k}}.$$

Thus we have

$$\sum_{i=1}^{\infty} \frac{H_{i,2k}}{i^{2k}} = \sum_{i=1}^{\infty} \frac{\zeta(2k;i)}{i^{2k}}.$$

Thus from the above identity, (5.3) and (5.5), we deduce

$$\sum_{i=1}^{\infty} \frac{\zeta(2k;i)}{i^{2k}} = \zeta(2k)^2 - \frac{\zeta(2k)^2 - \zeta(4k)}{2} = \frac{\zeta(2k)^2 + \zeta(4k)}{2}.$$

Applying 5.4 to the above equality, we finally get

$$\sum_{i=1}^{\infty} \frac{\zeta(2k;i)}{i^{2k}} = \left\{ \left(\frac{B_{2k}}{2(2k)!} \right)^2 - \frac{B_{4k}}{2(4k)!} \right\} \frac{(2\pi)^{4k}}{2}.$$

This proves Proposition (5.1.2).

□

The following identities were derived in [18] (see identities 1 and 8):

Proposition 5.1.3

$$\sum_{i=1}^{\infty} \frac{H_{i,2}}{i^2} \zeta(2;i) = \frac{31\pi^6}{15120}.$$

Proposition 5.1.4

$$\sum_{i=1}^{\infty}\frac{H_{i,4}}{i^4}\zeta(4;i)=\frac{4009\pi^{12}}{3405402000}.$$

For any positive integer k we have the following generalization:

Proposition 5.1.5 *For any positive integer k, we have*

$$\sum_{i=1}^{\infty}\frac{H_{i,2k}}{i^{2k}}\zeta(2k;i)=\left[(-1)^{3k-1}\frac{B_{6k}}{(6k)!}+(-1)^{3k-3}\frac{(B_{2k})^3}{8(2k)!^3}+(-1)^{3k-2}\frac{3B_{2k}B_{4k}}{4(2k)!(4k)!}\right]\frac{(2\pi)^{6k}}{6}. \tag{5.6}$$

PROOF.— As in (5.2), it is not difficult to see that

$$\zeta(s_1,s_2,s_3)+\zeta(s_1,s_2+s_3)+\zeta(s_1,s_3,s_2)+\zeta(s_1+s_3,s_2)+\zeta(s_3,s_1,s_2)=\zeta(s_1,s_2)\zeta(s_3) \tag{5.7}$$

Substituting $s_1=s_2=s_3=2k$ in the above equation together with (5.2), we obtain

$$\zeta(2k,2k,2k)=\frac{\zeta(2k,2k)\zeta(2k)-\zeta(2k)\zeta(4k)+\zeta(6k)}{3}. \tag{5.8}$$

Now

$$\begin{aligned}\zeta(2k,2k,2k)&=\sum_{i=1}^{\infty}\frac{1}{i^{2k}}\sum_{j=i+1}^{\infty}\frac{1}{j^{2k}}\sum_{t=j+1}^{\infty}\frac{1}{t^{2k}}\\&=\sum_{i=1}^{\infty}\frac{1}{i^{2k}}\sum_{j=i+1}^{\infty}\frac{1}{j^{2k}}\left(\zeta(2k)-H_{j,2k}\right)\\&=\sum_{i=1}^{\infty}\frac{1}{i^{2k}}\left(\sum_{j=i+1}^{\infty}\frac{\zeta(2k)}{j^{2k}}-\sum_{j=i+1}^{\infty}\frac{H_{j,2k}}{j^{2k}}\right)\\&=\sum_{i=1}^{\infty}\frac{1}{i^{2k}}\left[\zeta(2k)(\zeta(2k)-H_{i,2k})-\left(\sum_{j=1}^{\infty}\frac{H_{j,2k}}{j^{2k}}-\sum_{j=1}^{i}\frac{H_{j,2k}}{j^{2k}}\right)\right].\end{aligned}$$

Thus we have

$$\sum_{i=1}^{\infty}\frac{1}{i^{2k}}\sum_{j=1}^{i}\frac{H_{j,2k}}{j^{2k}}=\zeta(2k,2k,2k)-\sum_{i=1}^{\infty}\frac{1}{i^{2k}}\left(\zeta(2k)^2-\zeta(2k)H_{i,2k}\right)$$

$$+\sum_{i=1}^{\infty}\frac{1}{i^{2k}}\sum_{j=1}^{\infty}\frac{H_{j,2k}}{j^{2k}}. \tag{5.9}$$

Also we have

$$\sum_{i=1}^{\infty}\frac{1}{i^{2k}}\left(\zeta(2k)^2-\zeta(2k)H_{i,2k}\right)=\zeta(2k)\sum_{i=1}^{\infty}\frac{1}{i^{2k}}\sum_{j=i+1}^{\infty}\frac{1}{j^{2k}}=\zeta(2k)\zeta(2k,2k),$$

and

$$\sum_{i=1}^{\infty}\frac{1}{i^{2k}}\sum_{j=1}^{\infty}\frac{H_{j,2k}}{j^{2k}}=\sum_{i=1}^{\infty}\frac{1}{i^{2k}}\{\zeta(2k)^2-\zeta(2k,2k)\}=\zeta(2k)^3-\zeta(2k)\zeta(2k,2k).$$

From (5.9), we also have

$$\begin{aligned}\sum_{i=1}^{\infty}\frac{1}{i^{2k}}\sum_{j=1}^{i}\frac{H_{j,2k}}{j^{2k}}&=\zeta(2k,2k,2k)-\zeta(2k)\zeta(2k,2k)+\zeta(2k)^3-\zeta(2k)\zeta(2k,2k)\\&=\zeta(2k,2k,2k)-2\zeta(2k)\zeta(2k,2k)+\zeta(2k)^3.\end{aligned}$$

Using (5.3) and (5.8) in the above identity, we get

$$\sum_{i=1}^{\infty}\frac{1}{i^{2k}}\sum_{j=1}^{i}\frac{H_{j,2k}}{j^{2k}}=\frac{1}{6}\{2\zeta(6k)+\zeta(2k)^3+3\zeta(2k)\zeta(4k)\}. \tag{5.10}$$

Using (5.4) in the above equation, we obtain

$$\begin{aligned}\sum_{i=1}^{\infty}\frac{1}{i^{2k}}\sum_{j=1}^{i}\frac{H_{j,2k}}{j^{2k}}=\frac{(2\pi)^{6k}}{6}\Bigg[&(-1)^{3k-1}\frac{B_{6k}}{(6k)!}+\frac{1}{8}(-1)^{3k-3}\frac{B_{2k}^3}{((2k)!)^3}\\&+\frac{3}{4}(-1)^{3k-2}\frac{B_{2k}B_{4k}}{(2k)!(4k)!}\Bigg].\end{aligned} \tag{5.11}$$

Next consider

$$\begin{aligned}\sum_{i=1}^{\infty}\frac{1}{i^{2k}}\sum_{j=1}^{i}\frac{H_{j,2k}}{j^{2k}}&=\sum_{i=1}^{\infty}\frac{1}{i^{2k}}\sum_{j=1}^{i}\frac{1}{j^{2k}}\sum_{t=1}^{j}\frac{1}{t^{2k}}\\&=\sum_{j=1}^{\infty}\frac{1}{j^{2k}}\sum_{i=j}^{\infty}\frac{1}{i^{2k}}\sum_{t=1}^{j}\frac{1}{t^{2k}}\qquad\text{(interchanging summation)}\end{aligned}$$

$$= \sum_{j=1}^{\infty} \frac{1}{j^{2k}} \sum_{t=1}^{j} \frac{1}{t^{2k}} \sum_{i=j}^{\infty} \frac{1}{i^{2k}} \qquad \text{(interchanging summation)}$$
$$= \sum_{i=1}^{\infty} \frac{H_{i,2k}}{i^{2k}} \zeta(2k; i).$$

Thus we see that

$$\sum_{i=1}^{\infty} \frac{1}{i^{2k}} \sum_{j=1}^{i} \frac{H_{j,2k}}{j^{2k}} = \sum_{i=1}^{\infty} \frac{H_{i,2k}}{i^{2k}} \zeta(2k; i).$$

Now using (5.11) and the above identity, we finally obtain

$$\sum_{i=1}^{\infty} \frac{H_{i,2k}}{i^{2k}} \zeta(2k; i) = \frac{(2\pi)^{6k}}{6} \Bigg[(-1)^{3k-1} \frac{B_{6k}}{(6k)!} + \frac{1}{8} (-1)^{3k-3} \frac{B_{2k}^3}{((2k)!)^3} + \frac{3}{4} (-1)^{3k-2} \frac{B_{2k} B_{4k}}{(2k)!(4k)!} \Bigg].$$

□

5.2 Series arising from multiple Hurwitz zeta values

For $s_1 > 1, s_1 + s_2 > 1, \dots s_1 + s_2 + \cdots + s_r > r$, and $\alpha_1, \alpha_2, \dots \alpha_r \in \mathbb{R} \backslash \mathbb{Z}$, the multiple Hurwitz zeta function is defined by

$$\zeta(s_1, s_2, \dots s_r; \alpha_1, \alpha_2, \dots \alpha_r) = \sum_{n_1 > n_2 \cdots > n_r \geq 1} \frac{1}{(n_1 + \alpha_1)^{s_1} \dots (n_r + \alpha_r)^{s_r}}.$$

For more details on multiple Hurwitz zeta function, see [16]. For $n \in \mathbb{N}$, we define

$$H'_{2n-1,r} = \sum_{i=1}^{n} \frac{1}{(2i-1)^r}.$$

The following identity was established in [18].

Proposition 5.2.1

$$\sum_{i=1}^{\infty} \frac{\zeta(2; i - 1/2)}{(2i-1)^2} = \frac{5\pi^4}{96}.$$

We generalize this as follows.

Proposition 5.2.2 *For any positive integer k, we have*

$$\sum_{i=1}^{\infty} \frac{\zeta(2k; i-1/2)}{(2i-1)^{2k}} = 2^{6k-2}\pi^{4k}\left[\frac{1}{2}\left(1-1/2^{2k}\right)^2 \frac{B_{2k}^2}{((2k)!)^2} - \left(1-1/2^{4k}\right)\frac{B_{4k}}{(4k)!}\right].$$

PROOF.— To prove the above identity, we need the following identity from multiple Hurwitz zeta values.

$$\begin{aligned}\zeta(s_1;\alpha)\zeta(s_2;\alpha) &= \zeta(s_1,s_2;\alpha,\alpha)+\zeta(s_2,s_1;\alpha,\alpha)+\zeta(s_1+s_2;\alpha)+\tfrac{1}{\alpha^{s_1}}\zeta(s_2;\alpha)\\ &\quad+\tfrac{1}{\alpha^{s_2}}\zeta(s_1;\alpha)-\tfrac{2}{\alpha^{s_1+s_2}}. \end{aligned} \tag{5.12}$$

Proof of (5.12) is similar to the proof of (5.2), hence we omit the proof. Substituting $s_1 = s_2 = 2k, \alpha = -1/2$ in (5.12), we obtain

$$2\zeta(2k,2k;-1/2,-1/2) = \zeta(2k;-1/2)^2 - \zeta(4k;-1/2) - 2^{2k+1}\zeta(2k;-1/2) + 2^{4k+1}. \tag{5.13}$$

Now

$$\begin{aligned}\zeta(2k,2k;-1/2,-1/2) &= \sum_{i=1}^{\infty}\frac{1}{(i-1/2)^{2k}}\sum_{j=i+1}^{\infty}\frac{1}{(j-1/2)^{2k}}\\ &= 2^{4k}\sum_{i=1}^{\infty}\frac{1}{(2i-1)^{2k}}\sum_{j=i+1}^{\infty}\frac{1}{(2j-1)^{2k}}\\ &= 2^{4k}\sum_{i=1}^{\infty}\frac{1}{(2i-1)^{2k}}\left\{\left(1-\frac{1}{2^{2k}}\right)\zeta(2k) - H'_{2i-1,2k}\right\}\\ &= 2^{4k}\left\{\left(1-\frac{1}{2^{2k}}\right)\zeta(2k)\right\}^2 - 2^{4k}\sum_{i=1}^{\infty}\frac{H'_{2i-1,2k}}{(2i-1)^{2k}}.\end{aligned}$$

Thus from the above identity, we obtain

$$2^{4k}\sum_{i=1}^{\infty}\frac{H'_{2i-1,2k}}{(2i-1)^{2k}} = 2^{4k}\left\{\left(1-\frac{1}{2^{2k}}\right)\zeta(2k)\right\}^2 - \zeta(2k,2k;-1/2,-1/2). \tag{5.14}$$

Now consider

$$\sum_{i=1}^{\infty}\frac{H'_{2i-1,2k}}{(2i-1)^{2k}} = \sum_{i=1}^{\infty}\frac{1}{(2i-1)^{2k}}\sum_{j=1}^{i}\frac{1}{(2j-1)^{2k}}$$

$$= \sum_{j=1}^{\infty} \frac{1}{(2j-1)^{2k}} \sum_{i=j}^{\infty} \frac{1}{(2i-1)^{2k}} \qquad \text{(interchanging summation)}$$
$$= \frac{1}{2^{2k}} \sum_{j=1}^{\infty} \frac{1}{(2j-1)^{2k}} \sum_{j=0}^{\infty} \frac{1}{(j+i-1/2)^{2k}}$$
$$= \frac{1}{2^{2k}} \sum_{i=1}^{\infty} \frac{\zeta(2k; i-1/2)}{(2i-1)^{2k}}.$$

So we see that

$$2^{2k} \sum_{i=1}^{\infty} \frac{H'_{2i-1,2k}}{(2i-1)^{2k}} = \sum_{i=1}^{\infty} \frac{\zeta(2k; i-1/2)}{(2i-1)^{2k}}. \tag{5.15}$$

Using (5.15) in (5.14), we get

$$2^{2k} \sum_{i=1}^{\infty} \frac{\zeta(2k; i-1/2)}{(2i-1)^{2k}} = 2^{4k} \left\{ \left(1 - \frac{1}{2^{2k}}\right) \zeta(2k) \right\}^2 - \zeta(2k, 2k; -1/2, -1/2).$$

Now using (5.13), the identity

$$2^{-2k} \zeta(2k; -1/2) = \zeta(2k) - \frac{\zeta(2k)}{2^{2k}} + 1,$$

and (5.4) in the above equality, we finally obtain

$$\sum_{i=1}^{\infty} \frac{\zeta(2k; i-1/2)}{(2i-1)^{2k}} = 2^{6k-2} (\pi)^{4k} \left[\frac{1}{2} \left(1 - 1/2^{2k}\right)^2 \frac{B_{2k}^2}{((2k)!)^2} - \left(1 - 1/2^{4k}\right) \frac{B_{2k}}{(4k)!} \right].$$

□

Proposition 5.2.3

$$\sum_{n=1}^{\infty} \frac{H'_{2n-1,2}}{(2n-1)^2} \zeta(2; n-1/2) = \frac{61\pi^6}{11520}.$$

Again, the above identity was established in [18]. The generalization of it is as follows.

Proposition 5.2.4 *For any positive integer k, we have*

$$\sum_{n=1}^{\infty} \frac{H'_{2n-1,2k}}{(2n-1)^{2k}} \zeta(2k; n-1/2) = 2^{8k} \pi^{6k} \left[\frac{(-1)^k}{8} \left(1 - 1/2^{2k}\right) \left(1 - 1/2^{4k}\right) \right.$$

$$\frac{B_{2k}B_{4k}}{(2k)!(4k)!} + \frac{(-1)^{3k-3}}{48}\left(1-1/2^{2k}\right)^3\frac{B_{2k}^3}{(2k)!^3} + \frac{(-1)^{3k-1}}{6}\left(1-1/2^{6k}\right)\frac{B_{6k}}{(6k)!}\Bigg].$$

Proof.— To prove the above identity, we need the following identity from multiple Hurwitz zeta values, proof of which is similar to the proof of (5.12).

$$\zeta(s_1, s_2;\alpha,\alpha)\zeta(s_3;\alpha) = \frac{1}{\alpha^{s_3}}\zeta(s_1, s_2;\alpha,\alpha) + \zeta(s_1, s_2, s_3;\alpha,\alpha,\alpha) + \zeta(s_1, s_3, s_2;\alpha, \alpha,\alpha) + \zeta(s_3, s_1, s_2;\alpha,\alpha,\alpha) + \zeta(s_1+s_3, s_2;\alpha,\alpha) + \zeta(s_1, s_2+s_3;\alpha,\alpha).$$

Substituting $s_1 = s_2 = s_3 = 2k$ and $\alpha = -1/2$ in the above equality and using (5.12), we obtain

$$\zeta(2k, 2k, 2k; -1/2, -1/2, -1/2) = \frac{2^{6k}}{3}\left[\frac{\zeta(2k)^3}{2}\left(1-1/2^{2k}\right)^3 - \frac{3}{2}\zeta(2k)\zeta(4k)\left(1-1/2^{2k}\right)\left(1-1/2^{4k}\right) + \left(1-1/2^{6k}\right)\zeta(6k)\right]. \tag{5.16}$$

Now

$$\begin{aligned}\zeta(2k, 2k, 2k; &-1/2, -1/2, -1/2)\\ &= \sum_{i=1}^{\infty}\frac{1}{(i-1/2)^{2k}}\sum_{j=i+1}^{\infty}\frac{1}{(j-1/2)^{2k}}\sum_{t=j+1}^{\infty}\frac{1}{(t-1/2)^{2k}},\\ &= 2^{6k}\Bigg[\left(1-1/2^{2k}\right)^3\zeta(2k)^3 - \left(1-1/2^{2k}\right)\zeta(2k)\sum_{i=1}^{\infty}\frac{H'_{2i-1,2k}}{(2i-1)^{2k}}\\ &\quad-\sum_{i=1}^{\infty}\frac{1}{(2i-1)^{2k}}\sum_{j=1}^{\infty}\frac{H'_{2j-1,2k}}{(2j-1)^{2k}}\Bigg] + 2^{6k}\sum_{i=1}^{\infty}\frac{1}{(2i-1)^{2k}}\sum_{j=1}^{i}\frac{H'_{2j-1,2k}}{(2j-1)^{2k}}.\end{aligned}$$

Now substituting the values of $\sum_{j=1}^{\infty}\frac{H'_{2j-1,2k}}{(2j-1)^{2k}}$ and $\zeta(2k, 2k, 2k; -1/2, -1/2, -1/2)$ from (5.14) and (5.16) respectively in the above identity, we deduce that

$$\sum_{i=1}^{\infty}\frac{1}{(2i-1)^{2k}}\sum_{j=1}^{i}\frac{H'_{2j-1,2k}}{(2j-1)^{2k}} = \frac{1}{2}\zeta(2k)\zeta(4k)\left(1-1/2^{2k}\right)\left(1-1/2^{4k}\right) + \frac{1}{3}\left[\frac{1}{2}\left(1-1/2^{2k}\right)^3\zeta(2k)^3 + \zeta(6k)\left(1-1/2^{6k}\right)\right]. \tag{5.17}$$

Consider

$$\begin{aligned}\sum_{i=1}^{\infty}\frac{1}{(2i-1)^{2k}}\sum_{j=1}^{i}\frac{H'_{2j-1,2k}}{(2j-1)^{2k}} &= \sum_{i=1}^{\infty}\frac{1}{(2i-1)^{2k}}\sum_{j=1}^{i}\frac{1}{(2j-1)^{2k}}\sum_{t=1}^{j}\frac{1}{(2t-1)^{2k}}\\ &= \sum_{j=1}^{\infty}\frac{1}{(2j-1)^{2k}}\sum_{i=j}^{\infty}\frac{1}{(2i-1)^{2k}}\sum_{t=1}^{j}\frac{1}{(2t-1)^{2k}}\\ &\quad \text{(interchanging summation)}\\ &= \sum_{j=1}^{\infty}\frac{1}{(2j-1)^{2k}}\sum_{t=1}^{j}\frac{1}{(2t-1)^{2k}}\sum_{i=j}^{\infty}\frac{1}{(2i-1)^{2k}}\\ &\quad \text{(interchanging summation)}\\ &= 2^{-2k}\sum_{i=1}^{\infty}\frac{H'_{2i-1,2k}}{(2i-1)^{2k}}\zeta(2k; i-1/2).\end{aligned}$$

Thus we have

$$\sum_{i=1}^{\infty}\frac{1}{(2i-1)^{2k}}\sum_{j=1}^{i}\frac{H'_{2j-1,2k}}{(2j-1)^{2k}} = 2^{-2k}\sum_{i=1}^{\infty}\frac{H'_{2i-1,2k}}{(2i-1)^{2k}}\zeta(2k; i-1/2).$$

Now using the above identity and (5.4) in (5.17), we finally obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H'_{2n-1,2k}}{(2n-1)^{2k}}\zeta(2k; n-1/2) &= 2^{8k}\pi^{6k}\Bigg[\frac{(-1)^k}{8}\left(1-1/2^{2k}\right)\left(1-1/2^{4k}\right)\\ &\quad \frac{B_{2k}B_{4k}}{(2k)!(4k)!} + \frac{(-1)^{3k-3}}{48}\left(1-1/2^{2k}\right)^3\frac{B_{2k}^3}{(2k)!^3}\\ &\quad + \frac{(-1)^{3k-1}}{6}\left(1-1/2^{6k}\right)\frac{B_{6k}}{(6k)!}\Bigg].\end{aligned}$$

□

This describes the content of our work [19].

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