A STUDY OF SOME LAMBERT SERIES

By

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DECLARATION

I hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Bibekananda Maji

List of Publications arising from the thesis

Journal

1. "Modular-type relations associated to Rankin-Selberg L-function", Kalyan Chakraborty, Shigeru Kanemitsu, Bibekananda Maji, *Ramanujan J.*, **2017**, *Vol.* 42, 285–299.

Others

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BIBEKANANDA MAJI

Dedicated to my Family

Take up one idea,

make that one idea your life-

think of it, dream of it,

live on that idea.

Let the brain, muscles, nerves,

every part of your boby,

be full of that idea,

and just leave other idea alone.

This is the way to success.

- Swami Vivekananda

If you want to shine like Sun,

first you have to burn like it.

- Dr A. P. J. Abdul Kalam.

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Synopsis

0.1 Introduction

This thesis is about a study of some Lambert series. This works was done during my stay at Harish-Chandra Research Institute as a research scholar. The thesis can be divided into two main chapters. The theme of the first chapter is modular-type relation associated to Rankin-Selberg L-function. Mainly, we will obtain an asymptotic expansion of one interesting Lambert series. The summary of this part is given in Section 0.2. In the second chapter we discuss secant zeta function and its generalization as a Lambert series. The summary of this chapter is given in Section 0.3. Now we begin with some basic definitions.

Definition 0.1.1 (Lambert series) A Lambert series is a series of the form

$$S(q) := \sum_{n=1}^{\infty} a(n) \frac{q^n}{1-q^n},$$

where $a(n)$ is any arithmetic function and $q \in \mathbb{C}$ with $|q| < 1$.

Then expanding naturally we have

$$S(q) = \sum_{n=1}^{\infty} a(n) \sum_{k=1}^{\infty} q^{nk} = \sum_{n=1}^{\infty} b(n) q^n,$$

where $b(n) = \sum_{d|n} a(d)$. If we choose $q = \exp(-z)$, where z is a positive real number, then the Lambert series will be of the form

$$S(z) = \sum_{n=1}^{\infty} b(n) \exp(-nz).$$

Various Lambert series have been studied by many mathematicians. In the following section we will discuss the asymptotic expansion of an interesting Lambert series.

0.2 Modular-type relation associated to Rankin-Selberg L-function

Let $\mathbb{H}$ denote the upper half plane. The Ramanujan cusp form is defined as

$$\Delta(h) := q \prod_{n=1}^{\infty} (1-q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n,$$

where $q = \exp(2\pi i h)$ with $h \in \mathbb{H}$. Ramanujan observed many interesting properties of Ramanujan tau function $\tau(n)$.

In 1981, Zagier [40] conjectured that the Lambert series $\sum_{n=1}^{\infty} \tau^2(n) \exp(-nz)$ should have an asymptotic expansion when $z \to 0^+$, and it can be expressed in terms of the non-trivial zeros of $\zeta(s)$. Hafner and Stopple [17] verified this conjecture. More importantly, Zagier mentioned that the asymptotic expansion of the above series can actually be used to evaluate the non-trivial zeros of $\zeta(s)$, using only the values of the Ramanujan tau function. Since $\tau(n)$ is the nth Fourier coefficient of the Ramanujan cusp form of weight 12, one would naturally like to ask the following question:

Question 0.2.1 *Does the Lambert series $\sum_{n=1}^{\infty} c^2(n) \exp(-nz)$ also have an asymptotic expansion in terms of the non-trivial zeros of $\zeta(s)$ when $z \to 0^+$, where $c(n)$ is the nth Fourier coefficient of any cusp form f over $\Gamma = SL(2, \mathbb{Z})$.*

An affirmative answer to the above question has been obtained.

Let $S_k(\Gamma)$ denote the space of cusp forms of weight k for the full modular group Γ. Let $f \in S_k(\Gamma)$ be a normalized Hecke eigenform with Fourier series expansion

$$f(h) = \sum_{n=1}^{\infty} c(n) e^{2\pi i n h}, \tag{1}$$

where $h \in \mathbb{H}$. Then the associated L-function has an Euler product,

$$L(s,f) := \sum_{n=1}^{\infty} \frac{c(n)}{n^s} = \prod_{p\in\mathbb{P}} \left(1 - \frac{\alpha_p}{p^s}\right)^{-1} \left(1 - \frac{\beta_p}{p^s}\right)^{-1} \quad \text{for} \quad \Re(s) > \frac{k+1}{2},$$

where α_p and β_p are complex numbers satisfying $\alpha_p + \beta_p = c(p)$ and $\alpha_p\beta_p = p^{k-1}$.

Definition 0.2.2 (Symmetric square L-function) Let $f \in S_k(\Gamma)$ be a normalized Hecke eigen form. Then the symmetric square L-function associated to f is defined as follows:

$$\begin{aligned} D(s) &:= L(s, \mathrm{Sym}^2 f, \psi) \\ &= \prod_{p\in\mathbb{P}} \left(1 - \psi(p)\alpha_p^2 p^{-s}\right)^{-1} \left(1 - \psi(p)\alpha_p\beta_p p^{-s}\right)^{-1} \left(1 - \psi(p)\beta_p^2 p^{-s}\right)^{-1}, \end{aligned}$$

where ψ is a Dirichlet character.

Definition 0.2.3 (Rankin-Selberg L-function) The Rankin-Selberg L-function associated to $f \in S_k(\Gamma)$ is defined by the Dirichlet series

$$L(s, f \otimes f) := \sum_{n=1}^{\infty} \frac{|c^2(n)|}{n^s},$$

where $c(n)$ is the nth Fourier coefficients of f.

Shimura [35] has studied symmetric square L-function and proved the important relationship between symmetric square L-function and Rankin-Selberg L-function. These two L-functions can be analytically continued to the whole complex plane except for some poles. Moreover, they satisfy nice functional equations.

Definition 0.2.4 (Confluent hypergeometric function) The following second order differential equation (Kummer's equation)

$$z\frac{d^2w}{dz^2} + (b-z)\frac{dw}{dz} - aw = 0$$

has two linearly independent solutions $M(a, b, z)$ and $U(a, b, z)$. These solutions are known as confluent hypergeometric functions of first and second kind respectively. Confluent hypergeometric function of second kind has an integral representation

$$U(a, b, z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-zt} t^{a-1} (1+t)^{b-a-1} \mathrm{dt},$$

where z is a positive real number, a and b are complex variables.

We now state the main result.

Theorem 0.2.5 *Let $f \in S_k(\Gamma)$ with $f(h) = \sum_{n=1}^{\infty} c(n) e^{2\pi i n h}$. Assume that all the non-trivial zeros of $\zeta(s)$ are simple. Then for positive real z,*

$$\sum_{n=1}^{\infty} |c^2(n)| e^{-4\pi^2 n z} = \frac{\Gamma(k) D(k)}{(4\pi^2 z)^k \zeta(2)} + \mathrm{P}(z) + z^{1-2k} \sum_{n=1}^{\infty} \beta_n e^{-\frac{\mu_n}{z}} U\left(-\frac{1}{2}, k, \frac{\mu_n}{z}\right),$$

where

$$\mathrm{P}(z) = \sum_{\rho} \frac{\Gamma(\frac{\rho}{2} + k - 1)\zeta(\frac{\rho}{2}) D(\frac{\rho}{2} + k - 1)}{\zeta'(\rho)(4\pi^2 z)^{\frac{\rho}{2}+k-1}}$$

with $\rho = x + iy$ running over the non-trivial zeros of $\zeta(s)$. Moreover, the sum over ρ involves bracketing the terms so that the terms for which

$$|y - y'| < \exp\left(-\frac{Ay}{\log y}\right) + \exp\left(-\frac{Ay'}{\log y'}\right)$$

holds are included in the same bracket, where A is a suitable positive constant and $\mu_n = 4\pi^2 n$, $\beta_n = \frac{1}{\sqrt{\pi}} c^2(n)$.

To prove Theorem 0.2.5, we have used the following two important results.

1. The functional equation of Rankin-Selberg L-functional due to Rankin [29].

2. The relation between Rankin-Selberg L-function and symmetric square L-function given by Shimura [35].

The proof of Theorem 0.2.5 was obtained in [8]. This is a joint work with S. Kanemitsu and K. Chakrabory. The first Chapter of the thesis contains details of the proof of this theorem.

0.3 Cotangent and secant zeta function

Lerch [22] in 1904 introduced the cotangent zeta function for algebraic irrationals z and odd positive integers s as follows:

$$\xi(z,s) := \sum_{n=1}^{\infty} \frac{\cot(n\pi z)}{n^s}.$$

In 1973, Berndt [4] considered the cotangent zeta function for $s \in \mathbb{C}$, where his main motivation was to study the generalized Dedekind sums. He found many interesting explicit formulas for $\xi(z,s)$ for a quadratic irrational z and an odd integer $s \geq 3$. Berndt's work implies that

$$\sqrt{j}\xi(\sqrt{j},s)\pi^{-s} \in \mathbb{Q},$$

where j is any positive integer and $s \geq 3$ an odd integer.
Recently, Lalín et al. [21] considered the secant zeta function

$$\psi(z,s) := \sum_{n=1}^{\infty} \frac{\sec(n\pi z)}{n^s} \tag{2}$$

and found its special values at some particular quadratic irrational arguments. They conjectured that, when j is any positive integer and s is an even positive integer,

$$\psi(\sqrt{j},s)\pi^{-s} \in \mathbb{Q}.$$

The main result of Lalín et al. [21] is

Theorem 0.3.1 (Lalín, Rodrigue, Rogers) [21, Theorem 3] *For any algebraic irrational α and $l \in 2\mathbb{N}$, the difference between the following special values* *may be explicitly expressed:*

$$\begin{aligned}&(\alpha+1)^{l-1}\psi\left(\frac{\alpha}{\alpha+1},l\right)-(-\alpha+1)^{l-1}\psi\left(\frac{\alpha}{-\alpha+1},l\right)\\ =&\frac{(\pi i)^l}{l!}\sum_{m=0}^{l}(2^{m-1}-1)B_mE_{l-m}\binom{l}{m}\left[(1+\alpha)^{m-1}-(1-\alpha)^{m-1}\right],\end{aligned}\tag{3}$$

where B_m and E_m indicate the Bernoulli and Euler numbers respectively.

We will introduce a generalization of the secant zeta function as a Lambert series. Using the theory of generalized Dedekind eta-funtion due to Lewittes [23], Berndt [3] and Arakawa [1], we shall give a generalization of the Theorem 0.3.1.

0.3.1 Work of Lewittes, Berndt and Arakawa

Lewittes [23] defined a generalization of the Dedekind eta-function as a Lambert series

$$A(z,s,r_1,r_2):=\sum_{m>-r_1}\sum_{k=1}^{\infty}k^{s-1}e[kr_2+k(m+r_1)z],\tag{4}$$

where $(r_1,r_2)\in\mathbb{R}^2$, $z\in\mathbb{H}$ and $s\in\mathbb{C}$. He introduced its associate as

$$H(z,s,r_1,r_2):=A(z,s,r_1,r_2)+e\left[\frac{s}{2}\right]A(z,s,-r_1,-r_2).\tag{5}$$

Berndt [3] has studied the modular transformation formula of H-function, which will be discussed in the thesis. Arakawa [1] introduced a generalized eta-function as follows

$$\eta(\alpha,s,p,q):=\sum_{n=1}^{\infty}n^{s-1}\frac{e[n(p\alpha+q)]}{1-e[n\alpha]}\qquad\text{with}\quad\Re(\mathrm{s})<0,\tag{6}$$

where $(p,q)\in\mathbb{R}^2$ and α is an algebraic irrational number.

Using the celebrated Thue-Siegel-Roth theorem, Arakawa proved that the series in (6) is absolutely convergent for $\Re(s)<0$.

0.3.2 Generalization of secant zeta function

We introduce two Lambert series corresponding to (6) and (4). Let α be any algebraic irrational number and (p, q) a pair of real numbers. Then we define the series η^* by

$$\eta^*(\alpha, s, p, q) := \sum_{n=1}^{\infty} n^{s-1} \frac{e[n(p\alpha + q)]}{1 + e[n\alpha]} \quad \text{for } \Re(\text{s}) < 0, \tag{7}$$

and another infinite series A^* by

$$A^*(z, s, r_1, r_2) := \sum_{m > -r_1} (-1)^m \sum_{k=1}^{\infty} k^{s-1} e[kr_2 + k(m + r_1)z], \tag{8}$$

for a pair $(r_1, r_2) \in \mathbb{R}^2$, $z \in \mathbb{H}$ and $s \in \mathbb{C}$.

Applying the Thue-Siegel-Roth theorem one can show that the series in (7) is absolutely convergent, when $\Re(s) < 0$ and α is an algebraic irrational number. In particular, if we consider $(r_1, r_2) = (1/2, 0)$ we will get the secant zeta function

$$A^*\left(z, s, \frac{1}{2}, 0\right) = \frac{1}{2} \sum_{k=1}^{\infty} \frac{\sec \pi k z}{k^{1-s}} = \frac{1}{2} \psi(z, 1 - s).$$

Definition 0.3.2 (Hurwitz zeta function) For a positive number a, the Hurwitz zeta function is defined as follows

$$\zeta(s, a) := \sum_{n=0}^{\infty} (n + a)^{-s}, \quad \Re(s) > 1. \tag{9}$$

Let us fix the following matrices:

$$V_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, V_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \text{and} \quad V_2 = V_0^2 V_1^{-1} = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \tag{10}$$

and consider the difference

$$D^*(V_i) := D^*\left(V_i, z, s, \frac{1}{2}, 0\right) := \beta^{-s} A^*\left(V_i z, s, \frac{1}{2}, 0\right) - A^*\left(z, s, \frac{1}{2}, 0\right) \tag{11}$$

for each V_i from (10), where $\beta = cz + d$ and (c, d) is the second row of V_i.

We prove the following result.

Theorem 0.3.3 *For a real algebraic irrational α and a complex variable s with $\Re(s) < 0$, we have*

$$\begin{aligned} D^*(V_0) &= \alpha^{-s}A^*\left(\frac{-1}{\alpha}, s, \frac{1}{2}, 0\right) - A^*\left(\alpha, s, \frac{1}{2}, 0\right) \\ &= -\frac{(2\pi)^{-s}e\left[-\frac{s}{4}\right]}{1-e\left[\frac{s}{2}\right]}\int_{I(\lambda,\infty)} t^{s-1}\frac{\exp(-(\frac{1}{2}+\alpha)t)}{(1+\exp(-t))(1-\exp(-\alpha t))}\,\mathrm{d}t \\ &+ 2^{-2s}\pi^{-s}e\left[-\frac{s}{4}\right]\Gamma(s)\left(\zeta\left(s,\frac{1}{4}\right)-\zeta\left(s,\frac{3}{4}\right)\right) \\ &- 2^{1-s}\sum_{n=1}^{\infty} n^{s-1}\frac{e\left[n\left(\frac{1}{2}\alpha+\frac{1}{4}\right)\right]\left(e\left[\frac{1}{2}n\alpha\right]+1\right)}{1-e[n\alpha]} \\ &+ 2^{2-s}\sum_{n=1}^{\infty}(2n)^{s-1}\frac{e\left[\frac{3}{2}\cdot n\alpha\right]}{1-e[2n\alpha]}. \end{aligned} \tag{12}$$

$$\begin{aligned} D^*(V_1) &= (\alpha+1)^{-s}A^*\left(\frac{\alpha}{\alpha+1}, s, \frac{1}{2}, 0\right) - A^*\left(\alpha, s, \frac{1}{2}, 0\right) \\ &= -\frac{(2\pi)^{-s}e\left[-\frac{s}{4}\right]}{1-e\left[\frac{s}{2}\right]}\int_{I(\lambda,\infty)} t^{s-1}\frac{\exp(-\frac{1}{2}t)}{(1+\exp(-t))}\frac{\exp(-\frac{1}{2}(\alpha+1)t)}{(1-\exp(-(\alpha+1)t))}\,\mathrm{d}t \\ &+ 2^{-s}\sum_{n=1}^{\infty} n^{s-1}\frac{(-1)^{n-1}}{\cos\frac{\pi n}{2}\alpha}. \end{aligned} \tag{13}$$

$$\begin{aligned} D^*(V_2) &= (\alpha-1)^{-s}A^*\left(\frac{-\alpha}{\alpha-1}, s, \frac{1}{2}, 0\right) - A^*\left(\alpha, s, \frac{1}{2}, 0\right) \\ &= -\frac{(2\pi)^{-s}e\left[-\frac{s}{4}\right]}{1-e\left[\frac{s}{2}\right]}\int_{I(\lambda,\infty)} t^{s-1}\frac{\exp(-\frac{1}{2}t)}{(1+\exp(-t))}\frac{\exp(-\frac{1}{2}(\alpha-1)t)}{(1-\exp(-(\alpha-1)t))}\,\mathrm{d}t \\ &- 2^{-s}\sum_{n=1}^{\infty} n^{s-1}\frac{1}{\cos\frac{\pi n}{2}\alpha}, \end{aligned} \tag{14}$$

where $I(\lambda,\infty)$ (for any positive number λ) denotes the integral path consisting of the oriented line segment $(+\infty,\lambda)$, together with the positively-oriented circle of radius λ with center at the origin, and the oriented line segment $(\lambda,+\infty)$.

Adding (13) and (14) we get the main result of Laĺin et al. Theorem 0.3.1. We note it as a corollary.

Corollary 0.3.4

$$\begin{aligned}
&(\alpha+1)^{-s}A^*\left(\frac{\alpha}{\alpha+1},s,\frac{1}{2},0\right)+(\alpha-1)^{-s}A^*\left(\frac{-\alpha}{\alpha-1},s,\frac{1}{2},0\right)\\
&=-\frac{(2\pi)^{-s}e\left[-\frac{s}{4}\right]}{1-e\left[\frac{s}{2}\right]}\int_{I(\lambda,\infty)}t^{s-1}\sum_{m=0}^{\infty}2^{-m-1}E_m\frac{t^m}{m!}\sum_{n=0}^{\infty}(2^{1-n}-1)B_n\\
&\times\frac{\{(\alpha+1)^{n-1}+(\alpha-1)^{n-1}\}t^{n-1}}{n!}\,\mathrm{d}t.
\end{aligned}$$

The following conjecture seems to be plausible:

Conjecture 0.3.5 Let $(p,q)\in\mathbb{R}^2$. If $V_1=\begin{pmatrix}a_1 & b_1\\ c_1 & d_1\end{pmatrix}$ and $V_2=\begin{pmatrix}a_2 & b_2\\ c_2 & d_2\end{pmatrix}$ are inverses of each other in $\mathrm{PSL}_2(\mathbb{Z})$, then

$$(c_1\alpha+d_1)^{-s}A^*\left(V_1\alpha,s,p,q\right)+(c_2\alpha+d_2)^{-s}A^*\left(V_2\alpha,s,p,q\right),$$

can be expressible in terms of rational combination of special values of some zeta functions and L-functions.

The proof of Theorem 0.3.3 is obtained in [19]. This is a joint work with S. Kanemitsu and T. Kuzumaki. We shall see the detailed proof of these results in the second chapter of the thesis.

CHAPTER 0

Basic Notation and definitions

0.1 Basic notation

In this chapter, we shall define various basic notations, definitions and some important results, which will be used throughout the thesis. Let $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ denote the set of natural numbers, integers, rational numbers, real and complex numbers respectively. The set of prime numbers is denoted by $\mathbb{P}$.

A complex number z is said to be an *algebraic* number if there a polynomial $f(x) \in \mathbb{Q}[x]$ such that $f(z) = 0$. A complex number z is said to be *transcendental* if it is not an algebraic number. We let e be the Euler's number and π be the ratio of the circumference of a circle to its diameter. Both e and π are transcendental numbers.

For $x \in \mathbb{R}$, let $\lfloor x \rfloor$ denote the largest integer less than or equal to x and the fractional part of x is defined by $\{x\} := x - \lfloor x \rfloor$.

For $s \in \mathbb{C}$, real and imaginary part of s is denoted by $\Re(s)$ and $\Im(s)$ respectively. The exponential function is defined on the whole complex plane as follows:

$$\exp(s) := e^s = \sum_{n=0}^{\infty} \frac{s^n}{n!}.$$

To make the calculation simpler we use $e[s] := \exp(2\pi i s)$.

Definition 0.1.1 (Bernoulli polynomial) The generating function for Bernoulli polynomial $B_n(x)$ is defined as

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}.$$

For $x = 0$, $B_n(0) = B_n$ is defined as the nth Bernoulli number. Bernoulli numbers have many interesting properties and one can easily prove that $B_n = 0$ for all odd integers $n > 1$.

Definition 0.1.2 (Euler polynomial) The generating function for Euler polynomial $E_n(x)$ is defined as follows:

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}.$$

Corresponding to $x = 1/2$, the nth Euler number is defined as $E_n := 2^n E_n(1/2)$. Again, one can show that $E_n = 0$ for all odd integers $n \geq 1$.

0.2 Some arithmetic functions

Definition 0.2.1 (Arithmetic function) A complex valued function f defined on $\mathbb{N}$ is called an arithmetic function. An arithmetic function f is said to be additive if it satisfies

$$f(mn) = f(m) + f(n),$$

for m, n relatively prime. If this property holds for all m and n, then f is said to be completely additive. For example, $f(n) = \log n$ is completely additive. Similarly, an arithmetic function g is multiplicative if it satisfies

$$f(mn) = f(m)f(n),$$

for m, n relatively prime. If this property holds for all m and n, then f is said to be completely multiplicative. For example, $f(n) = n^{-s}$ with $s \in \mathbb{C}$, is completely multiplicative.

Let $\omega(n)$ denotes the number of distinct prime divisors of n and $d(n)$, known as divisor function, denotes the number of positive divisors of n.

We will write $f(n) = O(g(n))$, for two arithmetic functions f and g, if there is a constant K such that $|f(n)| \leq Kg(n)$ for all $n \in \mathbb{N}$. Sometimes we also use the notation $\ll$ and write $f(n) \ll g(n)$ to indicate the same thing. The following result is well known.

Theorem 0.2.2 *For every $\epsilon > 0$, we have*

$$d(n) = O(n^{\epsilon}). \tag{1}$$

One can find the proof of this theorem in [25, p. 9].

Definition 0.2.3 (Möbius function) The Möbius function is defined as follows:

$$\mu(n) := \begin{cases} (-1)^{\nu(n)}, & \text{if n is square-free,} \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

We can check that this is a multiplicative function.

Definition 0.2.4 (Dirichlet characters) Let $N \in \mathbb{N}$. A Dirichlet character $\chi \pmod N$ is a homomorphism

$$\chi : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*.$$

We extend the definition of χ to all natural numbers by setting

$$\chi(n) = \begin{cases} \chi \quad (n \pmod N), & \text{if} \quad \mathrm{gcd(n, N)} = 1, \\ 0, & \text{otherwise.} \end{cases}$$

If $\chi(n) = 1$ for all $n \in \mathbb{N}$, then we call χ as a trivial character. Dirichlet characters are completely multiplicative function.

0.3 Properties of Riemann zeta function

Definition 0.3.1 (Riemann Zeta function) Let $s \in \mathbb{C}$. The Riemann Zeta function is defined as follows:

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{for} \quad \Re(s) > 1.$$

For $\Re(s) > 1$, this series is absolutely convergent and has the Euler product representation

$$\zeta(s) = \prod_{p\in\mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1}. \tag{3}$$

We can easily see that

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$

Definition 0.3.2 (Dirichlet L-function) A Dirichlet L-series is a function of the form

$$L(s,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

where χ is a Dirichlet character and $s \in \mathbb{C}$ with $\Re(s) > 1$.

If χ is the trivial character, then the associated Dirichlet L-function is the Riemann zeta function. Since Dirichlet characters are completely multiplicative, $L(s,\chi)$ will have an Euler product representation as follows:

$$L(s,\chi) = \prod_{p\in\mathbb{P}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}, \quad \text{for} \quad \Re(\mathrm{s}) > 1.$$

Definition 0.3.3 (Gamma function) Let $s \in \mathbb{C}$ with $\Re(s) > 0$. The classical Gamma function is defined by

$$\Gamma(s) := \int_0^{\infty} x^{s-1} e^{-x} \mathrm{d}x.$$

It satisfies the functional equation $\Gamma(s+1) = s\Gamma(s)$ and can be analytically continued to a meromorphic function on the complex plane with poles at non-positive integers.

Theorem 0.3.4 *The Riemann Zeta function $\zeta(s)$ can be analytically continued to the whole complex plane except for a simple pole at $s = 1$, and it satisfies the functional equation*

$$\xi(s) = \xi(1-s), \tag{4}$$

where

$$\xi(s) := \frac{s(s-1)}{2} \pi^{-s/2} \Gamma(s/2) \zeta(s).$$

One can find the proof of this celebrated result in [37]. The derivative of $\zeta(s)$ will be $\zeta'(s) = \sum_{n=1}^{\infty} \frac{-\log n}{n^s}$ for $\Re(s) > 1$. It can be easily seen that $\zeta(s)$ has trivial zeros at $s = -2n$ for $n \in \mathbb{N}$, which arise due to the poles of $\Gamma(s/2)$. From the Euler product of $\zeta(s)$ we can prove that

Theorem 0.3.5 *The function $\zeta(s)$ has no zeros with $\Re(s) \geq 1$.*

One of the most important conjectures in mathematics is Riemann Hypothesis, which is about non-trivial zeros of $\zeta(s)$.

Conjecture 0.3.6 (Riemann Hypothesis (RH)) All the non-trivial zeros of $\zeta(s)$ lie on the line $\Re(s) = 1/2$.

The following conjecture will also be used in the results.

Conjecture 0.3.7 (Grand Simplicity Hypothesis (GSH)) The (positive) imaginary parts of non-trivial zeros of $L(s, \chi)$ with χ running over all primitive Dirichlet characters are linearly independent over $\mathbb{Q}$ (see Rubinstien and Sarnak [33]).

In particular, GSH implies that all the non-trivial zeros $\zeta(s)$ are simple.

0.4 Modular forms

In this section we recall some basic notions related to classical modular forms. Let $\mathbb{H}$ denote the upper half plane. Let k be an even positive integer. Denote $\mathrm{SL}(2, \mathbb{Z})$ by the

$$\mathrm{SL}(2, \mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}.$$

We know that $\mathrm{SL}(2, \mathbb{Z})$ acts on the upper half plane $\mathbb{H}$ by linear fractional transformation, as follows:

$$\begin{aligned} \mathrm{SL}(2, \mathbb{Z}) : \mathbb{H} &\longrightarrow \mathbb{H} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : h &\longmapsto \frac{ah + b}{ch + d}. \end{aligned} \tag{5}$$

Definition 0.4.1 (Modular form) A complex-valued function f on $\mathbb{H}$ is said to be a modular form of weight k for $\mathrm{SL}(2,\mathbb{Z})$ if it satisfies the following conditions:

1. The function f is a holomorphic on $\mathbb{H}$.
2. For $h \in \mathbb{H}$, $f\left(\frac{ah+b}{ch+d}\right) = (ch+d)^k f(h), \quad \forall\, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$.
3. The function f must be holomorphic as $h \to i\infty$.

One can see that modular forms have a Fourier series expansion of the form

$$f(h) = \sum_{n=0}^{\infty} c(n) q^n,$$

where $q = e^{2\pi i h}$ and $h \in \mathbb{H}$. The coefficients $c(n)$ are knowns as the Fourier coefficients of f. This is an arithmetic function and it has many interesting properties.

Definition 0.4.2 (Cusp form) A modular form f of weight k for $\mathrm{SL}(2,\mathbb{Z})$, is said to be a cusp form if it vanishes at the cusp $i\infty$ i.e., if $c(0) = 0$. Then the Fourier series expansion of f is

$$f(h) = \sum_{n=1}^{\infty} c(n) q^n.$$

Definition 0.4.3 (Ramanujan's cusp form) Ramanujan's delta function $\Delta(h)$ is a cusp form of weight 12 and is defined by

$$\Delta(h) := q \prod_{n=1}^{\infty} (1-q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n.$$

Here the arithmetic function $\tau(n)$ is known as Ramanujan tau function. In 1916, Ramanujan [27] studied this function and stated the following conjectures:

1. $\tau(n)$ is a multiplicative function
2. $\tau(p^{r+1}) = \tau(p)\tau(p^r) - p^{11}\tau(p^{r-1})$
3. $|\tau(p)| \leq p^{11/2}$.

The first two properties were proved by Mordell [24] in 1917 and the third one was proved by Deligne [11] in 1974.

Definition 0.4.4 (Dedekind eta-function) It is defined by

$$\eta(h) := \Delta(h)^{1/24} = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \tag{6}$$

Definition 0.4.5 The L-function associated to a cusp form f of weight k for $\mathrm{SL}(2, \mathbb{Z})$ is defined as follows:

$$L(s, f) := \sum_{n=1}^{\infty} \frac{c(n)}{n^s}.$$

It is absolutely convergent for $\Re(s) > \frac{k+1}{2}$. Hecke proved that the arithmetic function $c(n)$ is a multiplicative function. We denote $S_k(\Gamma)$ by the space of cusp forms of weight k for the full modular group $\Gamma = \mathrm{SL}(2, \mathbb{Z})$.

Theorem 0.4.6 (Hecke) *Let $f \in S_k(\Gamma)$. The function $L(s, f)$ can be analytically continued to an entire function and satisfies the functional equation*

$$(2\pi)^{-s}\Gamma(s)L(s, f) = i^k(2\pi)^{-(k-s)}\Gamma(k - s)L(k - s, f). \tag{7}$$

The proof of this result can be found in [26, Theorem 5.3.7, p. 66].

Theorem 0.4.7 [11, **Deligne's bound**] *Let $c(n)$ be the nth Fourier coefficient of the cusp form $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$. Then*

$$|c(n)| \leq n^{\frac{k-1}{2}} d(n), \tag{8}$$

where $d(n)$ is the divisor function.

Using Theorem 0.2.2, we get

$$|c(n)| \leq n^{\frac{k-1}{2}+\epsilon}, \quad \text{for any} \quad \epsilon > 0.$$

Now we shall state one important and useful theorem in Diophantine equations.

Theorem 0.4.8 (Thue-Siegel-Roth) *Let α be an algebraic irrational number. For every $\epsilon > 0$,*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\epsilon}}$$

can have only finitely many solutions in co-prime integers p and q.

We can see the detail accounts of this theorem in [34].

CHAPTER 1

Modular-type relation associated to Rankin-Selberg L-function

In this chapter, we discuss the asymptotic expansion of a particular Lambert series. Hafner and Stopple proved a conjecture of Zagier, related to the asymptotic expansion of a Lambert series associated to Ramanujan tau function. We generalize their result and prove a similar result for any cusp form for the full modular group. We have mainly studied the Rankin-Selberg L-function and the symmetric square L-function associated to a cusp form and their corresponding functional equations.

1.1 Introduction and Motivation

Ramanujan's work has influenced many areas of number theory. During his stay at Cambridge, he showed the following identity to Hardy and Littlewood.
For any positive real number r,

$$\sum_{n=1}^{\infty} \frac{\mu(n)e^{-r/n^2}}{n} = \sqrt{\frac{\pi}{r}} \sum_{n=1}^{\infty} \frac{\mu(n)e^{-\pi^2/n^2 r}}{n}. \tag{1.1}$$

This formula was missing the contribution of non-trivial zeros of the Riemann zeta function $\zeta(s)$. The corrected version of this formula was given by Hardy and Littelwood and it is as follows:

Theorem 1.1.1 (Ramanujan, Hardy, Littlewood) [18, p. 156, Section 2.5]

Let α and β be two positive real numbers such that $\alpha\beta = \pi$. Assume that the series $\sum_\rho \frac{\Gamma\left(\frac{1-\rho}{2}\right)}{\zeta'(\rho)}\beta^\rho$ converges, where ρ runs through all non-trivial zeros of $\zeta(s)$, and non-trivial zeros of $\zeta(s)$ are simple. Then

$$\sqrt{\alpha}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-\alpha^2/n^2} - \sqrt{\beta}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-\beta^2/n^2} = -\frac{1}{2\sqrt{\beta}}\sum_{\rho}\frac{\Gamma\left(\frac{1-\rho}{2}\right)}{\zeta'(\rho)}. \tag{1.2}$$

For more work related to this identity one can look into Berndt [6, p. 470] and Titchmarsh [37, p. 219]. Dixit [12] obtained a character analogue of the Ramanujan-Hardy-Littlewoood identity. Dixit et al. [14] gave one variable generalization of the above identity and analogues of these identities to Hecke forms. More importantly, in (1.2) one does not actually need to assume the convergence of the infinite series on the right hand side. This series is convergent if the terms ρ are in the same bracket for which

$$|\Im(\rho) - \Im(\rho')| < \exp\left(-A\frac{\rho}{\log\Im(\rho)}\right) + \exp\left(-A\frac{\rho'}{\log(\Im\rho')}\right),$$

where A is a positive constant (see [18, p. 158] and [37, p. 220]). We still do not know whether this series is convergent without the condition of bracketing the terms but mathematicians believe that the series will converge in the usual sense also.
Now we shall give few definitions.

Definition 1.1.2 (Lambert Series) A Lambert series is a series of the form

$$S(q) = \sum_{n=1}^{\infty} a(n)\frac{q^n}{1-q^n},$$

where $a(n)$ is any arithmetic function and $q \in \mathbb{C}$ with $|q| < 1$.

Then expanding naturally, we have

$$S(q) = \sum_{n=1}^{\infty} a(n)\sum_{k=1}^{\infty} q^{nk} = \sum_{n=1}^{\infty} b(n)q^n,$$

where $b(n) = \sum_{d|n} a(d)$. If we choose $q = \exp(-z)$, where z is a positive real number, then the Lambert series becomes

$$S(z) = \sum_{n=1}^{\infty} b(n) \exp(-nz).$$

Various Lambert series have been studied by many mathematicians. In this chapter we will see the asymptotic expansion of one interesting Lambert series.

Definition 1.1.3 (Ramanujan's delta function) The well known Ramanujan delta function is a cusp form of weight 12 over the full modular group $\mathrm{SL}(2, \mathbb{Z})$ and is defined as follows

$$\Delta(h) := q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n,$$

where $q = \exp(2\pi i h)$ with $h \in \mathbb{H}$.

In the introduction, we have seen three important conjectures of Ramanujan related to tau function. Ramanujan also proved many interesting congruence relations related to this tau function. Many mathematicians have intensively studied this function.

In 1981, Zagier [40] conjectured that the Lambert series

$$\sum_{n=1}^{\infty} \tau^2(n) \exp(-nz)$$

should have an asymptotic expansion when $z \to 0^+$ and this can be expressed in terms of the non-trivial zeros of the Riemann zeta function $\zeta(s)$. Hafner and Stopple [17] verified this conjecture. More importantly, Zagier mentioned that the asymptotic expansion of the above series can actually be used to evaluate the non-trivial zeros of $\zeta(s)$ using only the values of the Ramanujan tau function.

For us "*Modular-type relation*" of some L-function would mean some kind of relation which can be obtained using the functional equation of the corresponding L-function. In this thesis, we shall use the functional equation of the Rankin-Selberg L-function to prove that the Lambert series

$$\sum_{n=1}^{\infty} |c^2(n)| \exp(-nz)$$

also admits an asymptotic expansion when $z \to 0^+$, which can be expressed in terms of the non-trivial zeros of $\zeta(s)$, where $c(n)$ is the nth Fourier coefficient of any cusp form f over $\Gamma = \mathrm{SL}(2, \mathbb{Z})$.

1.2 L-function associated to cusp form

Let $S_k(\Gamma)$ denote the space of cusp forms of weight k for the full modular group Γ. Let $f \in S_k(\Gamma)$ be a normalized Hecke eigenform with Fourier series expansion

$$f(h) = \sum_{n=1}^{\infty} c(n) e^{2\pi i n h},$$

where $h \in \mathbb{H}$. Then the associated L-function has an Euler product,

$$L(s, f) := \sum_{n=1}^{\infty} \frac{c(n)}{n^s} = \prod_p \left(1 - \frac{\alpha_p}{p^s}\right)^{-1} \left(1 - \frac{\beta_p}{p^s}\right)^{-1} \quad \text{for } \Re(s) > \frac{k+1}{2},$$

where α_p and β_p are complex numbers satisfying $\alpha_p + \beta_p = c(p)$ and $\alpha_p \beta_p = p^{k-1}$. Shimura [35] has defined the symmetric square L-function as follows:

Definition 1.2.1 (Symmetric square L-function) Let $f \in S_k(\Gamma)$ be a normalized Hecke eigenform. Then the symmetric square L-function associated to f is

$$\begin{aligned} D(s) &:= L(s, \mathrm{Sym}^2 f, \psi) \\ &:= \prod_p \left(1 - \psi(p)\alpha_p^2 p^{-s}\right)^{-1} \left(1 - \psi(p)\alpha_p \beta_p p^{-s}\right)^{-1} \left(1 - \psi(p)\beta_p^2 p^{-s}\right)^{-1}, \end{aligned}$$

where ψ is a primitive Dirichlet character.

Shimura gave the analytic continuation and functional equation of the symmetric square L-function.

Theorem 1.2.2 (Shimura) [35, **Theorem 1**] *Let*

$$R(s) := \pi^{\frac{-3s}{2}} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{s-k+2-\lambda_0}{2}\right) D(s),$$

where λ_0 is 0 or 1 according as $\psi(-1) = 1$ or -1. Then $R(s)$ can be analytically continued to a meromorphic function with possible simple poles at $s = k$ and $s = k - 1$. It satisfies the functional equation

$$R(s) = R(2k - s - 1). \tag{1.3}$$

One can actually show that $R(s)$ is often an entire function. If ψ is the trivial character, then $R(s)$ is an entire function. More general conditions under which $R(s)$ is holomorphic at $s = k$ or $s = k - 1$ is given in Shimura [35, Theorem 2, p. 94].

Definition 1.2.3 (Rankin-Selberg L-function) The Rankin-Selberg L-function associated to f is defined as

$$L(s, f \otimes f) := \sum_{n=1}^{\infty} \frac{|c^2(n)|}{n^s}. \tag{1.4}$$

The following results about Rankin-Selberg L-function were proved by Rankin:

Theorem 1.2.4 (Rankin) [29, Theorem 3] *The Rankin-Selberg L-function $L(s, f \otimes f)$ has the following properties:*

1. *It is absolutely convergent for $\Re(s) > k$.*

2. *It can be continued analytically to the whole complex plane with simple poles at $s = k$, and at $s = k - 1$.*

3. *It satisfies the functional equation*

$$R^*(s) = R^*(2k - 1 - s), \tag{1.5}$$

where

$$R^*(s) = (2\pi)^{-2s}\Gamma(s)\Gamma(s - k + 1)\zeta(2s - 2k + 2)L(s, f \otimes f). \tag{1.6}$$

We follow a more general treatment of Shimura [35, Eqn. (0.4)] which relates Rankin-Selberg L-function and symmetric square L-function by

$$L(s - k + 1, \chi\psi)D(s) = L(2(s - k + 1), \chi^2\psi^2)L(s, f \otimes f, \psi). \tag{1.7}$$

Now with χ and ψ as trivial characters, we get

$$L(s, f \otimes f) = \frac{\zeta(s-k+1)}{\zeta(2(s-k+1))} D(s). \tag{1.8}$$

1.3 Some special functions

We recall the following special functions which will be used in the sequel.

Definition 1.3.1 (Meijer G-function) Meijer G-function is defined by the following line integral

$$G_{p,q}^{m,n}\left(z \middle| \begin{matrix} a_1, .., a_p \\ b_1, .., b_q \end{matrix}\right) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s) z^s}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)} \mathrm{d}s,$$

where m, n, p, q are integers with $0 \leq m \leq q$, $0 \leq n \leq p$ and $a_i - b_j \notin \mathbb{N}$ for $1 \leq i \leq p$, $1 \leq j \leq q$.

Definition 1.3.2 (Confluent hypergeometric function) The following second order differential equation (Kummer's equation)

$$z\frac{d^2w}{dz^2} + (b - z)\frac{dw}{dz} - aw = 0$$

has two linearly independent solutions $M(a, b, z)$ and $U(a, b, z)$. These solutions are known as confluent hypergeometric function of first and second kinds, respectively. Confluent hypergeometric function of second kind has an integral representation

$$U(a, b, z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-zt} t^{a-1} (1 + t)^{b-a-1} \mathrm{d}t,$$

where z is a positive real number, a and b are complex variables.

We refer [20] for more information on these functions. These two functions have many nice relations between them. We will make use of the following well-known formula [20, p. 58] relating G and U functions:

$$G_{1,2}^{2,0}\left(z \middle| \begin{matrix} a \\ b, c \end{matrix}\right) = e^{-z} z^b U\big(a - c, b - c + 1, z\big). \tag{1.9}$$

We now state the main result.

1.4 Main Theorem

Theorem 1.4.1 *Let $f \in S_k(\Gamma)$ with $f(h) = \sum_{n=1}^{\infty} c(n)e^{2\pi inh}$. Assume that the non-trivial zeros of $\zeta(s)$ are simple. Then for positive real z,*

$$\begin{aligned}\sum_{n=1}^{\infty} |c^2(n)|e^{-4\pi^2 nz} &= \frac{\Gamma(k)D(k)}{(4\pi^2 z)^k \zeta(2)} + \mathrm{P}(z) \\ &\quad + z^{1-2k} \sum_{n=1}^{\infty} \beta_n e^{-\frac{\mu_n}{z}} U\left(-\frac{1}{2}, k, \frac{\mu_n}{z}\right),\end{aligned} \tag{1.10}$$

where

$$\mathrm{P}(z) = \sum_{\rho} \frac{\Gamma(\frac{\rho}{2}+k-1)\zeta(\frac{\rho}{2})D(\frac{\rho}{2}+k-1)}{\zeta'(\rho)(4\pi^2 z)^{\frac{\rho}{2}+k-1}}$$

with $\rho = x + iy$ running over the non-trivial zeros of $\zeta(s)$. Moreover, the sum over ρ involves bracketing the terms so that the terms for which

$$|\Im(\rho) - \Im(\rho')| < \exp\left(-A\frac{\rho}{\log(\Im(\rho))}\right) + \exp\left(-A\frac{\rho'}{\log(\Im(\rho'))}\right)$$

holds are included in the same bracket, where A is a suitable positive constant and $\mu_n = 4\pi^2 n$, $\beta_n = \frac{1}{\sqrt{\pi}}|c^2(n)|$.

As in equation (1.2), in the above theorem too the infinite series $P(z)$ converges under the condition of bracketing the terms.

1.5 Work of Roy, Zaharescu and Zaki

In a recent work Roy et al. [32] considered

$$F(s, \chi_1, \chi_2, \chi_3) := \frac{L(s,\chi_3)}{L(s,\chi_1)L(s,\chi_2)} = \sum_{n=1}^{\infty} \frac{g(n)}{n^s} \tag{1.11}$$

for $\sigma = \Re(s) > 1$ and with χ_i's $(i = 1, 2, 3)$ being primitive Dirichlet characters. Restricting χ_i $(i = 1, 2)$ to be trivial and $\chi_3 = \chi$ a primitive character, the above series (1.11) reduces to

$$F(s,\chi)=\frac{L(s,\chi)}{\zeta^2(s)}=\sum_{n=1}^{\infty}\frac{b(n)}{n^s},$$

where $b(n)=(\chi*\mu*\mu)(n)$. Analogously we obtain $\bar{b}(n)=(\bar{\chi}*\mu*\mu)(n)$ while considering $F(s,\bar{\chi})$ (where $\bar{\chi}$ denotes the conjugate character of χ and $*$ denotes convolution). Then $F(s,\chi)$ and $F(s,\bar{\chi})$ satisfy the functional equation

$$\Gamma\left(\frac{1-s}{2}\right)F(s,\chi)=\sqrt{\pi}\epsilon_\chi(\pi q)^{-s}\Gamma\left(\frac{s}{2}\right)F(1-s,\bar{\chi}),$$

where ϵ_χ denotes the Gauss Sum with $|\epsilon_\chi|=\sqrt{q}$.
The authors [32] went further to obtain a modular-type relation with an infinite sum of residues. As can be expected, the main result is the modular-type relation between the two Lambert series, one of the form

$$\sum_{n=1}^{\infty}\frac{b(n)}{n}e^{-\frac{\beta^2}{n^2}}$$

and the other with b and β replaced by $\bar{b}$ and α respectively, where $\alpha>0$ and $\beta>0$ are related by $\alpha\beta=q\pi$. A remarkable feature is that they are related via the infinite sum over the non-trivial zeros of $\zeta(s)$. Their result is:

Theorem 1.5.1 (Roy, Zaharescu and Zaki) [32, Theorem 1] *Let χ be an even primitive character mod q. Assume that the zeros of $\zeta(s)$ are simple and distinct from the zeros of $L(s,\chi)$. Then*

$$\sqrt{\frac{\alpha}{\epsilon_{\bar{\chi}}}}\sum_{n=1}^{\infty}\frac{\bar{b}(n)}{n}e^{-\frac{\alpha^2}{n^2}}-\sqrt{\frac{\beta}{\epsilon_\chi}}\sum_{n=1}^{\infty}\frac{b(n)}{n}e^{-\frac{\beta^2}{n^2}}$$
$$=-\frac{1}{2\sqrt{\beta\epsilon_\chi}}\sum_\rho S_\rho,$$

where $\rho=x+iy$ runs through the non-trivial zeros of $\zeta(s)$ and S_ρ is given by

$$S_\rho=\frac{\beta^\rho L(\rho,\chi)\Gamma\left(\frac{1-\rho}{2}\right)}{(\zeta'(\rho))^2}\left(\log\beta-\frac{\zeta''(\rho)}{\zeta'(\rho)}+\frac{L'(\rho,\chi)}{L(\rho,\chi)}-\frac{1}{2}\psi\left(\frac{1-\rho}{2}\right)\right),$$

where ψ denotes the digamma function. The sum over ρ involves bracketing the *terms so that the terms for which*

$$|y - y'| < \exp\left(-\frac{Ay}{\log y}\right) + \exp\left(-\frac{Ay'}{\log y'}\right),$$

where A is a suitable positive constant, are included in the same bracket.

The main step in their proof deals with estimating the following integral [32, Equn (3.7)]:

$$\frac{1}{2\pi i}\int_{\lambda-iT}^{\lambda+iT} \frac{\Gamma\left(\frac{1-2s}{2}\right)}{F(2s,\chi)^{-1}}\left(\frac{q\pi}{\alpha}\right)^{2s} \mathrm{d}s - \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{\Gamma\left(\frac{1-2s}{2}\right)}{F(2s,\chi)^{-1}}\left(\frac{q\pi}{\alpha}\right)^{2s} \mathrm{d}s$$
$$= \sum_{\rho, -T<\frac{\operatorname{Im}\rho}{2}<T} \operatorname*{Res}_{\rho} + I_2 + I_3.$$

The convergence of the infinite sum is proved as a consequence of the convergence (vanishing) of the two horizontal integrals $I_i (i = 2, 3)$ and the convergence of the vertical integrals. The authors needed a delicate analysis of the distribution of zeros of $\zeta(s)$ for the proof of vanishing of the horizontal integrals.

1.6 Work of Hafner and Stopple

Hafner and Stopple [17] considered the L-function associated to the Ramanujan delta function

$$\begin{aligned} L(s,\Delta) &= \sum_{n=1}^{\infty} \tau(n) n^{-s} \\ &= \prod_p \left(1-\alpha_p p^{-s}\right)^{-1}\left(1-\beta_p p^{-s}\right)^{-1} \quad \text{for } \Re(s) > \frac{13}{2}. \end{aligned}$$

The associated symmetric square L-function is

$$\begin{aligned} D(s) &= L(s, \mathrm{Sym}^2\Delta) \\ &= \prod_p \left(1-\alpha_p^2 p^{-s}\right)^{-1}\left(1-\alpha_p\beta_p p^{-s}\right)^{-1}\left(1-\beta_p^2 p^{-s}\right)^{-1}. \end{aligned}$$

The application of inverse Mellin transform would give

$$\sum_{n=1}^{\infty} \tau^2(n)e^{-zn} = \frac{1}{2\pi i}\int_{(c)} \Gamma(s)\frac{\zeta(s-11)}{\zeta(2(s-11))}D(s)z^{-s}\,\mathrm{d}s. \tag{1.12}$$

Even though a more general function $g(x)$ is considered (which is differentiable and of order $x^{-\alpha}$, $\alpha > 12$) in [17], but the concrete example is given with $g(x) = e^{-xz}$, which leads to the Bochner modular relation with the residual function containing infinitely many terms. They proved:

Theorem 1.6.1 (Hafner, Stopple) [17, Corr. 2.3] *Assume all the non-trivial zeros of $\zeta(s)$ are simple. For $z \to 0^+$,*

$$\begin{aligned}\sum_{n=1}^{\infty} \tau^2(n)e^{-zn} &= 12\Gamma(11)z^{-12} \\ &+ z^{-11-1/4}\sum_{\rho} z^{1/4-\rho/2}\Gamma\left(\frac{\rho}{2}+11\right)\frac{\zeta(\frac{\rho}{2})}{\zeta'(\rho)}D\left(\frac{\rho}{2}+11\right) \\ &+ O(z^{-11+1/2})\end{aligned} \tag{1.13}$$

with ρ running over the non-trivial zeros of $\zeta(s)$.

Clearly, if one assumes the Riemann hypothesis, then Zagier's conjecture follows from the above result [17, Remark]. Here we generalize their work to arbitrary cusp forms.

1.7 Proof of Theorem 1.4.1

We begin with a few lemmas which will be useful in what follows.

Lemma 1.7.1 (Stirling's formula for gamma function) *For $s = \sigma + iT$ in a vertical strip $\alpha \le \sigma \le \beta$,*

$$|\Gamma(\sigma+iT)| = \sqrt{2\pi}|T|^{\sigma-\frac{1}{2}}e^{-\frac{1}{2}\pi|T|}\left(1+O\left(\frac{1}{|T|}\right)\right) \tag{1.14}$$

as $|T| \to \infty$

We can see the proof of this lemma in [26, p. 92].

Lemma 1.7.2 *Let T be a sequence with arbitrarily large absolute value such that $|T-\gamma| > \exp(-A_1\gamma/\log\gamma)$, for every ordinate γ of a zero of $\zeta(s)$ where A_1 is some suitable positive constant. Then,*

$$\frac{1}{|\zeta(\sigma+iT)|} < e^{A_2 T} \tag{1.15}$$

for some suitable constant A_2.

Proof. From [37, p. 218, Equation (9.7.3)], we have

$$\log|\zeta(\rho+iT)| \geq \sum_{|T-\gamma|\leq 1} \log|T-\gamma| + O(\log T).$$

Now choose a sequence of positive numbers T tending to infinity such that $|T-\gamma| > \exp(-A_1\frac{\gamma}{\log\gamma})$ for every ordinate γ of a zero of $\zeta(s)$, where A_1 is a suitable positive constant. Then

$$\log|\zeta(\sigma+iT)| \geq -\sum_{|T-\gamma|\leq 1} A_1\frac{\gamma}{\log\gamma} + O(\log T) > -A_2 T$$

where $A_2 < \pi/4$ if A_1 is small enough. Hence the results follows.

Lemma 1.7.3 *For $s=\sigma+iT$ with $|T|\geq 1$, in a vertical strip $\alpha\leq\sigma\leq\beta$,*

$$|D(s)| \ll |T|^{A(\sigma)+\epsilon}, \tag{1.16}$$

for any $\epsilon>0$, with $A(\sigma)$ being a constant dependent on σ.

Proof. The proof involves utilizing the functional equation (1.3) of $D(s)$ and the Stirling formula for Gamma function.

Lemma 1.7.4 [37, p. 95] *For all $\sigma\geq\sigma_0$ there exist a constant $A(\sigma_0)$, such that*

$$|\zeta(\sigma+iT)| \ll |T|^{A(\sigma_0)} \tag{1.17}$$

as $T\to\infty$.

Proof. It readily follows from the functional equation (4) of Riemann zeta function.

1.7.1 Proof of the main Theorem 1.4.1

Proof. The functional equation (1.5) of the Rankin-Selberg L-function associated to the cusp form f gives

$$\begin{aligned}&(2\pi)^{-2s}\Gamma(s)L(s,f\otimes f)\\&=(2\pi)^{-2(2k-1-s)}\frac{\Gamma(2k-1-s)\Gamma(k-s)\zeta(2(k-s))}{\Gamma(s-k+1)\zeta(2(s-k+1))}L(2k-1-s,f\otimes f).\end{aligned}$$

This would imply that

$$\begin{aligned}&\frac{\Gamma(s)L(s,f\otimes f)}{(4\pi^2)^s}\\&=\frac{1}{\sqrt{\pi}}\frac{\Gamma(2k-1-s)\Gamma(k-s)}{(4\pi^2)^{(2k-1-s)}\Gamma\left(\frac{2k-1}{2}-s\right)}\frac{\zeta(2(k-s))}{\zeta(2k-1-2s)}L(2k-1-s,f\otimes f).\end{aligned} \tag{1.18}$$

Let us assume

$$\begin{aligned}\lambda_n&=\mu_n=4\pi^2 n,\\ \alpha_n&=c^2(n),\\ \beta_n&=\frac{1}{\sqrt{\pi}}c^2(n),\end{aligned}$$

to make (1.18) more symmetric and denote

$$\varphi(s):=\sum_{n=1}^{\infty}\frac{\alpha_n}{\lambda_n^s},\quad\text{and}\quad\psi(s):=\sum_{n=1}^{\infty}\frac{\beta_n}{\mu_n^s}. \tag{1.19}$$

Then with the above notations,

$$\begin{aligned}\varphi(s)&=(4\pi^2)^{-s}L(s,f\otimes f)\\&=(4\pi^2)^{-s}\frac{\zeta(s-k+1)}{\zeta(2(s-k+1))}D(s),\quad\text{(using (1.8))}\end{aligned} \tag{1.20}$$

has a unique simple pole at $s=k$ and infinitely many poles in the critical strip $k-1<\sigma<k-1/2$. In terms of ϕ and ψ the functional equation (1.18) is,

$$\Gamma(s)\varphi(s)=\frac{\Gamma(2k-1-s)\Gamma(k-s)}{\Gamma\left(\frac{2k-1}{2}-s\right)}\frac{\zeta(2(k-s))}{\zeta(2k-1-2s)}\psi(2k-1-s). \tag{1.21}$$

The inverse Mellin transform for the Γ-function is

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)x^{-s}\mathrm{ds}=\begin{cases}e^{-x} & \text{if } \mathrm{c}>0,\\ e^{-x}-1 & \text{if } -1<\mathrm{c}<0.\end{cases}$$

Now applying the inverse Mellin transform, we have (for $\mu > k$)

$$\begin{aligned}\sum_{n=1}^{\infty}c^2(n)e^{-z\lambda_n} &= \frac{1}{2\pi i}\int_{(\mu)}\Gamma(s)\sum_{n=1}^{\infty}\frac{c^2(n)}{\lambda_n^s}z^{-s}\mathrm{ds}\\ &= \frac{1}{2\pi i}\int_{(\mu)}\Gamma(s)\varphi(s)z^{-s}\,\mathrm{d}s.\end{aligned}\tag{1.22}$$

The contour C for the above integration is defined as follows:

C is determined by the line segments (for large real positive T) $[\mu-iT,\mu+iT],[\mu+iT,\lambda+iT],[\lambda+iT,\lambda-iT]$ and $[\lambda-iT,\mu-iT]$ where $\mu>k$ and $\lambda=k-3/2$. Then by applying the residue theorem we get,

$$\frac{1}{2\pi i}\int_C\Gamma(s)\varphi(s)z^{-s}ds=\operatorname*{Res}_{s=k}\Gamma(s)\varphi(s)z^{-s}+\mathrm{P}(z),\tag{1.23}$$

where $\mathrm{P}(z)$ denotes the residual function consisting of infinitely many terms contributed by the non-trivial zeros of $\zeta(2(s-k+1))$.

Firstly, we prove that the horizontal integrals

$$H_1=\frac{1}{2\pi i}\int_{\mu+iT}^{\lambda+iT}\Gamma(s)\phi(s)z^{-s}ds$$

and

$$H_2=\frac{1}{2\pi i}\int_{\lambda-iT}^{\mu-iT}\Gamma(s)\phi(s)z^{-s}ds$$

vanish as $T\to\infty$. We have

$$\begin{aligned}H_1&=\frac{1}{2\pi i}\int_{\mu+iT}^{\lambda+iT}\frac{\Gamma(s)\zeta(s-k+1)D(s)}{\zeta(2(s-k+1))(4\pi^2z)^s}ds\\ &=\frac{1}{2\pi i}\int_{\mu}^{\lambda}\frac{\Gamma(\sigma+iT)\zeta(\sigma-k+1+iT)D(\sigma+iT)}{\zeta(2(\sigma-k+1+iT))(4\pi^2z)^{\sigma+iT}}\mathrm{d}\sigma.\end{aligned}$$

Thus

$$|H_1| \ll \int_{\mu}^{\lambda} \frac{|\Gamma(\sigma+iT)|\,|\zeta(\sigma-k+1+iT)|\,|D(\sigma+iT)|}{|\zeta(2(\sigma-k+1+iT))|(4\pi^2 z)^{\sigma}} \mathrm{d}\sigma. \tag{1.24}$$

Now using (1.14), (1.15), (1.16) and (1.17) in (1.24) it is easily seen that,

$$|H_1| \ll |T|^A \exp\left(A_2 T - \frac{1}{2}\pi|T|\right).$$

Thus $H_1 \to 0$ as $T \to \infty$. Similarly one shows that $H_2 \to 0$ as $T \to \infty$. Therefore (1.23) gives,

$$\sum_{n=1}^{\infty} c^2(n) e^{-z\lambda_n} = \operatorname*{Res}_{s=k} \Gamma(s)\varphi(s)z^{-s} + \mathrm{P}(z) + I,$$

where

$$I = \frac{1}{2\pi i}\int_{(k-3/2)} \frac{\Gamma(2k-1-s)\Gamma(k-s)}{\Gamma\left(\frac{2k-1}{2}-s\right)} \frac{\zeta(2(k-s))}{\zeta(2k-1-2s)} \psi(2k-1-s) z^{-s}\,\mathrm{d}s. \tag{1.25}$$

The first residual term is (by using (1.20))

$$\operatorname*{Res}_{s=k} \Gamma(s)\varphi(s)z^{-s} = \Gamma(k)\frac{(4\pi^2)^{-k}}{\zeta(2)} D(k) z^{-k}. \tag{1.26}$$

We concentrate on the residual function $\mathrm{P}(z)$. It can be written as

$$\begin{aligned}
\mathrm{P}(z) &= \sum_{\rho} \operatorname{Res} \frac{\Gamma(s+k-1)\varphi(s+k-1)}{z^{s+k-1}} \quad (s \to s+k-1) \\
&= \sum_{\rho} \operatorname{Res} \frac{\Gamma(s+k-1)\zeta(s)D(s+k-1)}{(4\pi^2 z)^{s+k-1}\zeta(2s)},
\end{aligned}$$

where ρ runs over all the non-trivial zeros of $\zeta(s)$. As we have assumed that all the zeros of $\zeta(s)$ are simple (Grand simplicity hypothesis 0.3.7), the residue becomes

$$\operatorname{Res}_{\rho/2} = \lim_{s\to\rho/2} \frac{(s-\rho/2)\Gamma(s+k-1)\zeta(s)D(s+k-1)}{(4\pi^2 z)^{s+k-1}\zeta(2s)}$$

$$= \frac{\Gamma(\rho/2+k-1)\zeta(\rho/2)D(\rho/2+k-1)}{(4\pi^2 z)^{\rho/2+k-1}\zeta'(\rho)}.$$

Hence,

$$\mathrm{P}(z) = \frac{1}{z^{k-3/4}} \sum_{\rho} \frac{\Gamma(\frac{\rho}{2}+k-1)\zeta(\frac{\rho}{2})D(\frac{\rho}{2}+k-1)}{z^{\frac{\rho}{2}-1/4}\zeta'(\rho)(4\pi^2)^{\frac{\rho}{2}+k-1}}.$$

Now, if we assume Riemann hypothesis 0.3.6 i.e., $\rho = 1/2 + it$ for some real t, then

$$z^{\rho/2-1/4} = z^{it},$$

which is purely oscillatory. This corroborates with Hafner-Stopple's result.

Let us now look at I in (1.25). It may be re-written as

$$I = z^{1-2k} \frac{1}{2\pi i} \int_{(k+1/2)} \frac{\Gamma(w)\Gamma(w-k+1)}{\Gamma\left(w+\frac{1}{2}-k\right)} \frac{\zeta(2w-2k+2)}{\zeta(2w-2k+1)} \psi(w) \left(\frac{1}{z}\right)^{-w} (-\mathrm{d}w), \tag{1.27}$$

where $w = 2k-1-s$.

We look at the functional equation (1.21) once more. The second factor on the right hand side of (1.21) can be expanded into a Dirichlet series which is absolutely convergent for $\sigma = k - 3/2 < k - 1$ and so we dwell on the first gamma factors. Then the integral on the right of (1.27) is a Meijer G-function $G^{2,0}_{1,2}(z\,|\ldots)$.

Now we will make use of the relation (1.9) of G and U functions. The one which is of our interest is

$$G^{2,0}_{1,2}\left(\frac{1}{z} \,\middle|\, \begin{matrix} \frac{1}{2}-k \\ 0, 1-k \end{matrix}\right) = e^{-z^{-1}} U\left(-\frac{1}{2}, k, \frac{1}{z}\right). \tag{1.28}$$

Thus I becomes (using (1.28))

$$I = z^{1-2k} \frac{1}{2\pi i} \int_{(k+1/2)} \frac{\Gamma(w)\Gamma(w-k+1)}{\Gamma\left(w+\frac{1}{2}-k\right)} \sum_{n=1}^{\infty} \frac{\beta_n}{{\mu_n}^w} \left(\frac{1}{z}\right)^{-w} (-\mathrm{d}w)$$

(as $\frac{\zeta(2w-2k+2)}{\zeta(2w+2k+1)}$ is bounded on the line $\Re(w) = k + 1/2$.)

$$= z^{1-2k} \frac{1}{2\pi i} \sum_{n=1}^{\infty} \beta_n \int_{(k+1/2)} \frac{\Gamma(w)\Gamma(w-k+1)}{\Gamma\left(w+\frac{1}{2}-k\right)} \left(\frac{\mu_n}{z}\right)^{-w} (-\mathrm{d}w)$$

$$= z^{1-2k} \sum_{n=1}^{\infty} \beta_n G^{2,0}_{1,2}\left(\frac{\mu_n}{z} \,\middle|\, \begin{matrix} \frac{1}{2} - k \\ 0, 1-k \end{matrix} \right)$$

$$= z^{1-2k} \sum_{n=1}^{\infty} \beta_n e^{-\frac{\mu_n}{z}} U\left(-\frac{1}{2}, k, \frac{\mu_n}{z}\right). \tag{1.29}$$

This completes the proof of Theorem 1.4.1.

We further refine the result by using first approximation of the U function.

Corollary 1.7.5 *Assume all the conditions of Theorem 1.4.1. Then for $z \to 0^+$, we have*

$$\sum_{n=1}^{\infty} |c^2(n)| e^{-4\pi^2 n z} = c' z^{-k} + \mathrm{P}(z) + \sum_{i=0}^{k} E_i + O(|z|^{3/2-k}), \tag{1.30}$$

where

$$\begin{aligned} c' &= \frac{\Gamma(k) D(k)}{(4\pi^2)^k \zeta(2)}, \\ E_i &= z^{1-2k} \sum_{n=1}^{\infty} \beta_n e^{-\frac{\mu_n}{z}} c_i \left(\frac{\mu_n}{z}\right)^{1/2-i} \quad \text{and} \\ c_i &= (-1)^i \frac{(-1/2)_i(-1/2-k+1)_i}{i!}. \end{aligned}$$

Proof. The main term $c' z^{-k}$ has already been calculated in (1.26). We use the well-known asymptotic formula of U [16, p.278] to get the third term. The approximation is

$$U(a, c, z) = \sum_{n=0}^{N} (-1)^n \frac{(a)_n (a-c+1)_n}{n!} z^{-a-n} + O(|z|^{-a-N-1}), \tag{1.31}$$

where N is any non-negative integer and $-\frac{3}{2}\pi < \arg z < \frac{3}{2}\pi$ with $z \to \infty$. We also use the symbol

$$(a)_n = \frac{a(a+1)\cdots(a+n-1)}{n!}.$$

We use (1.31) in (1.29) and get

$$I = z^{1-2k} \sum_{n=1}^{\infty} \beta_n e^{-\frac{\mu_n}{z}} \left(\sum_{i=0}^{k} c_i \left(\frac{\mu_n}{z}\right)^{1/2-i} + O\left(\left|\frac{\mu_n}{z}\right|^{1/2-k-1} \right) \right),$$

where

$$c_i = (-1)^i \frac{(-1/2)_i(-1/2-k+1)_i}{i!}.$$

Let us split

$$I = \sum_{i=0}^{k} E_i + E$$

with

$$\begin{aligned} E_i &= z^{1-2k} \sum_{n=1}^{\infty} \beta_n e^{-\frac{\mu_n}{z}} c_i \left(\frac{\mu_n}{z}\right)^{1/2-i} \\ E &= z^{1-2k} \sum_{n=1}^{\infty} \beta_n e^{-\frac{\mu_n}{z}} O\left(\left|\frac{\mu_n}{z}\right|^{1/2-k-1} \right). \end{aligned}$$

We want $E = O\left(|z|^{3/2-k}\right)$ and for that we need to check the convergence of $\sum_{n=1}^{\infty} \frac{\beta_n e^{-\frac{\mu_n}{z}}}{{\mu_n}^{k+1/2}}$. For this purpose, we use Deligne's well-known bound for the growth of the Fourier coefficients of a cusp form, i.e.,

$$|c(n)| \le n^{(k-1)/2} d(n),$$

where $d(n)$ is the number of divisors of n and we know that $d(n) = O(n^{\epsilon})$ for any $\epsilon > 0$. Now

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\beta_n e^{-\frac{\mu_n}{z}}}{{\mu_n}^{k+1/2}} &\le \sum_{n=1}^{\infty} \frac{c^2(n)}{n^{k+1/2}} \\ &\le \sum_{n=1}^{\infty} \frac{n^{k-1+\epsilon}}{n^{k+1/2}} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^{3/2-\epsilon}}. \end{aligned}$$

The last series in the above sequel is convergent for any $0 < \epsilon < 1/2$. We end this section with a couple of useful remarks.

Remark 1.7.6 One can easily show that, $E_i \sim k_i\, z^{1-k}$ for each $0 \le i \le k$, where k_i's are some constants. We calculate E_0, and the estimation of the other E_i's follow similarly.

$$\begin{aligned}
E_0 &= z^{1-2k} \sum_{n=1}^{\infty} \beta_n \sqrt{\frac{\mu_n}{z}} e^{-\frac{\mu_n}{z}} \\
&= z^{1/2-2k} \frac{1}{2\pi i} \int_{(c)} \Gamma(s) \psi\left(s - \frac{1}{2}\right) \left(\frac{1}{z}\right)^{-s} \mathrm{d}s \\
&\sim z^{1/2-2k} \operatorname*{Res}_{s=k+1/2} \Gamma\left(s\right) \psi\left(s - \frac{1}{2}\right) \left(\frac{1}{z}\right)^{-s} \\
&\sim k_0 z^{1-k}.
\end{aligned}$$

The constant k_0 is easily calculable and depends on the weight of the particular cusp form in consideration.

$$\begin{aligned}
k_0 &= \operatorname*{Res}_{w=k} \Gamma\left(w + \frac{1}{2}\right) \psi(w) \\
&= \operatorname*{Res}_{w=k} \Gamma\left(w + \frac{1}{2}\right) \frac{\Gamma(w-k+\frac{1}{2})}{\Gamma(w-k+1)} \frac{\zeta(2(w-k)+1)}{\zeta(2(w-k+1))} \Gamma\left(2k-1-w\right) \varphi(2k-1-w) \\
&= \operatorname*{Res}_{w=k} \zeta(2(w-k)+1) \Gamma(k-1) \Gamma(k+\frac{1}{2}) \frac{6\sqrt{\pi}}{\pi^2} \varphi(k-1) \\
&= 3\Gamma(k-1)\Gamma(k+\frac{1}{2}) \frac{D(k-1)}{\pi^{3/2}(4\pi^2)^{k-1}}.
\end{aligned}$$

Remark 1.7.7 We must choose $\frac{z}{4\pi^2}$ in place of z in (1.30), as our sequence is $\lambda_n = 4\pi^2 n$, and with this choice

$$c' = \frac{\Gamma(k)D(k)}{\zeta(2)}.$$

Thus (1.13) can be recovered up to the treatment of the sum of residues, which is of the order z^{-k+1} (according to [17]). A closer analysis of a more general case has been done in [32].

1.8 Conclusion

Hafner and Stopple [17] studied a general case of the function g with the concrete example of the exponential function as is stated after (1.12). There are many other interesting cases of such modular-type relation, each of which will lead to some intriguing results. In our case, this exhibits an interesting phenomenon involving the G-function $G^{2,0}_{1,3}$. In a particular degenerate case $G^{2,0}_{1,3}$ [20, p.245] can be written as,

$$\begin{aligned} &G^{2,0}_{1,3}\left(z \,\middle|\, \begin{matrix} c \\ a,b,c \end{matrix}\right) \\ &= z^{\frac{1}{2}(a+b)}\left\{-\sin((c-b)\pi)\,Y_{a-b}\big(2\sqrt{z}\,\big)+\cos((c-b)\pi)\,J_{a-b}\big(2\sqrt{z}\,\big)\right\} \end{aligned} \tag{1.32}$$

involving J_n and Y_n which are Bessel functions of the first and second kind respectively.

Here we concentrated on studying the integral

$$\frac{1}{2\pi i}\int_{(c)}\Gamma(s)\varphi(s)z^{-s}\mathrm{d}s.$$

Instead, if we had

$$\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s)}{\Gamma(s+\rho+1)}\varphi(s)z^{-s}\,\mathrm{d}s,$$

then we would have got

$$\begin{aligned} &\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s)}{\Gamma(s+\rho+1)}\varphi(s)z^{-s}\,\mathrm{d}s=\operatorname*{Res}_{s=k}\frac{\Gamma(s)}{\Gamma(s+\rho+1)}\varphi(s)z^{-s}+\mathrm{P}(z)+ \\ &\frac{1}{2\pi i}\int_{(k-3/2)}\frac{\Gamma(2k-1-s)\Gamma(k-s)}{\Gamma\left(\frac{2k-1}{2}-s\right)\Gamma(s+\rho+1)}\frac{\zeta(2(k-s))}{\zeta(2k-1-2s)}\psi(2k-1-s)z^{-s}\,\mathrm{d}s. \end{aligned} \tag{1.33}$$

The last integral on the right-hand side of (1.33) becomes

$$\begin{aligned} &z^{1-2k} \\ &\cdot\frac{1}{2\pi i}\int_{(k+1/2)}\frac{\Gamma(w)\Gamma(w-k+1)}{\Gamma\left(w+\frac{1}{2}-k\right)\Gamma(2k+\rho-w)}\frac{\zeta(2w-2k+2)}{\zeta(2w+2k+1)}\psi(w)\left(\frac{1}{z}\right)^{-w}\mathrm{d}w \end{aligned}$$

and corresponds to (1.27). In terms of the G-function, this becomes

$$G^{2,0}_{1,3}\left(\frac{1}{z} \,\middle|\, \begin{matrix} \frac{1}{2}-k \\ 0,1-k,1-(2k+\rho) \end{matrix}\right). \tag{1.34}$$

This corresponds to our previous relation (1.9). Interestingly, in the special case $\rho = -k + \frac{1}{2}$, the G-function $G^{2,0}_{1,3}\big(\frac{1}{z}|...\big)$ as in (1.34), can be explicitly calculated by (1.32).

1.9 Future Work

Our ongoing work is to try to extend the results in the thesis to any cusp form for congruence subgroup. In that case the asymptotic expansion of the Lambert series

$$\sum_{n=1}^{\infty} |c^2(n)| \exp(-nz), \quad z \to 0^+,$$

can be expressed in terms of non-trivial zeros of $L(s, \chi)$, where $c(n)$ is the nth Fourier coefficient of a cusp form f for congruence subgroup.

It would also be interesting to study the problem for different classes of modular forms.

CHAPTER 2

Limiting values of Lambert series and the secant zeta function

In this chapter, we discuss the cotangent and secant zeta functions and their generalizations. Recently, Lalín, Rodrigue and Rogers have studied the secant zeta function and its convergence. They found many interesting values of the secant zeta function at some particular quadratic irrational numbers. They also gave modular transformation properties of the secant zeta function. We have tried to generalize the secant zeta function as a Lambert series and proved a generalized result for Lambert series, from which the main result of Lalín et al. follows as a corollary, using the theory of generalized Dedekind eta-function, developed by Lewittes, Berndt and Arakawa.

2.1 Introduction

The Dedekind eta-function and its limiting values have been considered by several authors starting from Riemann's posthumous fragment [31], Wintner [39] and later by Reyna [30] and Wang [38]. There are many generalizations of the Dedekind eta-function as a Lambert series including that of Lewittes [23], Berndt [3] and Arakawa [1, 2]. In particular cases, they reduce to the cotangent or the cosecant zeta function.

The values of $\zeta(s)$ at positive even integers is given by:

Theorem 2.1.1 (Euler) *For any positive integer k,*

$$\zeta(2k) = \frac{(-1)^{k+1} B_{2k} \pi^{2k}}{2(2k)!},$$

where B_{2k} is a Bernoulli number.

Thus $\zeta(2k) \in \pi^{2k}\mathbb{Q}$, as B_{2n} is a rational number. But, unfortunately for $\zeta(2n+1)$ we do not have such a simple result. Still, $\zeta(2n+1)$ does satisfy some other nice relations and it can be found in Ramanujan's second notebook [5, pp. 275–276]. Immediately, after seeing the above theorem, one can ask

Question 2.1.2 *Is it true that*

$$\zeta(2k+1) \in \pi^{2k+1}\mathbb{Q},$$

for any positive integer k ?

This is an unsolved problem till now. It is even very difficult to show that a particular value of $\zeta(2n+1)$ is an irrational number.

2.2 Cotangent zeta function

Lerch [22] in 1904 introduced the cotangent zeta function for an algebraic irrational number z and an odd positive integer s as

$$\xi(z, s) := \sum_{n=1}^{\infty} \frac{\cot(n\pi z)}{n^s}.$$

He stated the following functional equation for the cotangent zeta function, but without proof.

Theorem 2.2.1 (Lerch, [22]) *For any algebraic irrational number z and sufficiently large positive integer $k = k(z)$, we have*

$$\xi(z, 2k+1) + z^{2k}\xi\left(\frac{1}{z}, 2k+1\right) = (2\pi)^{2k+1}\phi(z, 2k+1), \tag{2.1}$$

where

$$\phi(z,n) := \sum_{i=0}^{n+1} \frac{B_i B_{n+1-j}}{j!(n+1-j)!} z^{j-1},$$

where B_i is the i-th Bernoulli number.

Berndt [4], in 1973, focused on the cotangent zeta function for general $s \in \mathbb{C}$, and proved the Lerch's functional equation for cotangent zeta function. He found many interesting explicit formulas for $\xi(z,s)$ when z is a quadratic irrational and $s \geq 3$ an odd integer. One such pleasing formula is:

$$\xi\left(\frac{1+\sqrt{5}}{2}, 3\right) = -\frac{\pi^3}{45\sqrt{5}}.$$

In fact Berndt's work implies that $\sqrt{j}\xi(\sqrt{j},s)\pi^{-s} \in \mathbb{Q}$, where j is any positive integer and $s \geq 3$ an odd integer. In 1988, Arakawa [2] also studied the analytic continuation and a functional equation of the cotangent zeta function.

2.3 Secant zeta function

Recently, Lalín et al. [21] considered the secant zeta function

$$\psi(z,s) := \sum_{n=1}^{\infty} \frac{\sec(n\pi z)}{n^s} \tag{2.2}$$

and found its special values at some particular quadratic irrational arguments. They proved the following results.

Theorem 2.3.1 (Lalín, Rodrigue, Rogers) [32, **Theorem 1**] *The series* (2.2) *is absolutely convergent in the following cases:*

1. *When $z = \frac{p}{q}$ a rational number with q odd and $s > 1$.*

2. *When z is an algebraic irrational number, and $s \geq 2$.*

To prove this theorem, they have used the celebrated Thue-Siegel-Roth theorem.

Theorem 2.3.2 (Lalín, Rodrigue, Rogers) [32, **Theorem 3**] *Let E_m denote the Euler numbers, and let B_m denote the Bernoulli numbers. Suppose that*

l is an even positive integer. Then for appropriate values of α:

$$\begin{aligned}(\alpha+1)^{l-1}\psi\left(\frac{\alpha}{\alpha+1},l\right)-(-\alpha+1)^{l-1}\psi\left(\frac{\alpha}{-\alpha+1},l\right)\\ =\frac{(\pi i)^k}{k!}\sum_{n=0}^{l}(2^{n-1}-1)B_nE_{k-n}\binom{l}{n}[(1+\alpha)^{n-1}-(1-\alpha)^{n-1}].\end{aligned} \tag{2.3}$$

They found the values of the secant zeta function at some quadratic irrational numbers. For $j\in\mathbb{Z}$,

$$\begin{aligned}\psi(\sqrt{2j(2j+1)},2)&=(3j+1)\frac{\pi^2}{6},\\ \psi(\sqrt{8j(2j+1)},2)&=\frac{\pi^2}{6},\\ \psi(\sqrt{2j(2j+1)},4)&=\frac{75j^2+46j+6}{8j+3}\frac{\pi^4}{180}.\end{aligned} \tag{2.4}$$

After observing these values, they conjectured that

Conjecture 2.3.3 (Lalı̂n, Rodrigue, Rogers) [21, **Cojecture 1**] If j is any positive integer and s is an even positive integer, then

$$\psi(\sqrt{j},s)\pi^{-s}\in\mathbb{Q}.$$

By a clever use of residue theorem, Berndt and Straub [7] proved the above functional equation (2.3) and from it they derived

$$\psi(\sqrt{r},s)\pi^{-s}\in\mathbb{Q},\quad (r\in\mathbb{Q}^+,s\in 2\mathbb{N}).$$

Furthermore, they connected the secant Dirichlet series with Eichler integrals of Eisenstein series and checked uni-modularity of period polynomials. On the other hand, Charollis and Greenberg [10] related the secant Dirichlet series $\psi(\alpha,s)$ to the generalized eta-function which was studied by Arakawa [1]. They proved that for $s\in 2\mathbb{N}$,

$$\psi(\alpha,s)\pi^{-s}\in\mathbb{Q}(\alpha)$$

for all real quadratic irrationals α. They used Arakawa's result to give an explicit formula for $\psi(\alpha,s)$ for real quadratic irrational numbers α.

In this chapter, we will introduce a generalization of the secant zeta function as a Lambert series. Using the theory of generalized Dedekind eta-function due to Lewittes [23], Berndt [3] and Arakawa [1], we shall give a generalization of the Theorem 2.3.2.

We begin by briefly describing the theory of Generalized Dedekind eta-function, developed by Lewittes [23], Berndt [3] and Arakawa[1], which is a main tool in our study.

2.4 Work of Lewittes and Berndt

Lewittes and Berndt treat the case of the upper half-plane $\mathbb{H}$ while Arakawa treats the case of upper half plane limiting to an algebraic irrational number. Hereafter we use the following notations:

$$\begin{aligned} e[w] &:= \exp(2\pi i w), \quad w \in \mathbb{C}, \\ \langle x \rangle \in \mathbb{R}, &\quad 0 < \langle x \rangle \leqq 1, \quad x - \langle x \rangle \in \mathbb{Z}, \\ \{x\} \in \mathbb{R}, &\quad 0 \leqq \{x\} < 1, \quad x - \{x\} \in \mathbb{Z}. \end{aligned}$$

Lewittes [23] defined the generalization of the Dedekind eta-function as a Lambert series. For a pair (r_1, r_2) of real numbers, $z \in \mathbb{H}$ and arbitrary $s \in \mathbb{C}$, he considered the series

$$A(z, s, r_1, r_2) := \sum_{m > -r_1} \sum_{k=1}^{\infty} k^{s-1} e[kr_2 + k(m + r_1)z], \tag{2.5}$$

where the first summation is over all integers m with $m > -r_1$. He also introduced its associate as

$$H(z, s, r_1, r_2) := A(z, s, r_1, r_2) + e\left[\frac{s}{2}\right] A(z, s, -r_1, -r_2). \tag{2.6}$$

Let $s = r_1 = r_2 = 0$. Put $A(z, 0, 0, 0) = A(z)$, then $H(z, 0, 0, 0) = 2A(z)$. Using the product definition (0.4.4) of $\eta(z)$, it is easy to show that

$$\log(\eta(z)) = \frac{\pi i}{12} - A(z).$$

Let us see a couple of examples.

Example 2.4.1 For special choices of parameters r_1 and r_2, the A and H-functions reduce to the cosecant and cotangent zeta functions.

$$\begin{aligned}
\frac{1}{\left(1+e\left[\frac{s}{2}\right]\right)} H\left(z,s,\left(\frac{1}{2},0\right)\right) &= A\left(z,s,\frac{1}{2},0\right) \\
&= \sum_{m>-\frac{1}{2}} \sum_{k=1}^{\infty} k^{s-1} e\left[k\left(m+\frac{1}{2}\right)z\right] \\
&= \sum_{k=1}^{\infty} k^{s-1} \frac{e\left[\frac{1}{2}kz\right]}{1-e[kz]} \\
&= \frac{i}{2}\sum_{k=1}^{\infty} \frac{\operatorname{cosec}(\pi kz)}{k^{1-s}}.
\end{aligned}$$

Also,

$$\begin{aligned}
\frac{1}{\left(1+e\left[\frac{s}{2}\right]\right)} H(z,s,(1,0)) &= A(z,s,1,0) \\
&= \sum_{m>-1} \sum_{k=1}^{\infty} k^{s-1} e[k(m+1)z] \\
&= \sum_{k=1}^{\infty} k^{s-1} \frac{e[kz]}{1-e[kz]} \\
&= \frac{1}{2}\sum_{k=1}^{\infty} k^{s-1}\left\{\frac{1+e[kz]}{1-e[kz]}-1\right\} \\
&= \frac{i}{2}\sum_{k=1}^{\infty} \frac{\cot \pi kz}{k^{1-s}} - \frac{1}{2}\zeta(1-s).
\end{aligned}$$

Some more definitions will be required.

Definition 2.4.1 (Hurwitz zeta function) For a positive number a, the Hurwitz zeta function

$$\zeta(s,a) := \sum_{n=0}^{\infty}(n+a)^{-s}, \quad \Re(s) > 1. \tag{2.7}$$

Definition 2.4.2 Let Ω denote the characteristic function of integers, i.e.

$$\Omega(a) := \begin{cases} 1, & \text{if } \ a \in \mathbb{Z}, \\ 0, & \text{if } \ a \notin \mathbb{Z}. \end{cases} \tag{2.8}$$

For any positive number λ, let $I(\lambda, \infty)$ denote the integration path consisting of the oriented line segment $(+\infty, \lambda)$, the positively-oriented circle of radius λ with center at the origin, and the oriented line segment $(\lambda, +\infty)$.
Let

$$G_2(z, (\omega_1, \omega_2);\ t) := \frac{\exp(-zt)}{(1 - \exp(-\omega_1 t))(1 - \exp(-\omega_2 t))} \tag{2.9}$$

for any pair (ω_1, ω_2) of positive numbers and for $z, t \in \mathbb{C}$.
Berndt [3] proved the following transformation formula.

Theorem 2.4.3 (Berndt) [3, **Theorem 2**] *Let* $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ *with* $c > 0$. *For any pair* (r_1, r_2) *of real numbers, set* $R_1 = r_1 a + r_2 c, R_2 = r_1 b + r_2 d, \rho = \{R_2\}c - \{R_1\}d$. *For* $z \in \mathbb{H}$ *with* $c\Re(z) + d > 0$ *let* $\beta = cz + d$. *Then for arbitrary* $s \in \mathbb{C}$, *we have*

$$\begin{aligned}
&\beta^{-s} H(Vz, s, r_1, r_2) - H(z, s, R_1, R_2) \\
&\quad = -\Omega(r_1)(2\pi)^{-s} e\left[\frac{s}{4}\right] \beta^{-s}\Gamma(s) \left(\zeta(s, \langle r_2 \rangle) + e\left[\frac{s}{2}\right] \zeta(s, \langle -r_2 \rangle)\right) \\
&\qquad + \Omega(R_1)(2\pi)^{-s} e\left[-\frac{s}{4}\right] \Gamma(s) \left(\zeta(s, \langle -R_2 \rangle) + e\left[\frac{s}{2}\right] \zeta(s, \langle R_2 \rangle)\right) \\
&\qquad + (2\pi)^{-s} e\left[-\frac{s}{4}\right] L(z, s, R_1, R_2, c, d),
\end{aligned}$$

where

$$\begin{aligned}
&L(z, s, R_1, R_2, c, d) \\
&= -\sum_{j=1}^{c} \int_{I(\lambda,\infty)} t^{s-1} \frac{\exp(-(1 - \{\frac{(jd+\varrho)}{c}\} + \frac{(cz+d)(j - \{R_1\})}{c})t)}{(1 - \exp(-t))(1 - \exp(-(cz+d)t))} \mathrm{d}t, \\
&\left(0 < \lambda < 2\pi, \frac{2\pi}{|\beta|}\right).
\end{aligned} \tag{2.10}$$

Here, $\log t$ *is understood to be real-valued on the upper segment* $(+\infty, \lambda)$ *of* $I(\lambda, \infty)$.

2.5 Work of Arakawa

Arakawa studied certain Lambert series associated to a complex variable s and an irrational real algebraic number α. Those Lambert series are defined as limiting (boundary) values of the generalized Dedekind eta-functions studied by Berndt [3]. Arakawa obtained transformation formulae under the action of $\mathrm{SL}(2,\mathbb{Z})$ on those α.

For an irrational real algebraic number α and a pair (p,q) of real numbers, Arakawa [1] introduced a generalized eta-function defined as

$$\eta(\alpha,s,p,q) := \sum_{n=1}^{\infty} n^{s-1}\frac{e[n(p\alpha+q)]}{1-e[n\alpha]}, \quad s\in\mathbb{C}, \tag{2.11}$$

and its associate by

$$H(\alpha,s,(p,q)) := \eta(\alpha,s,\langle p\rangle,q) + e\left[\frac{s}{2}\right]\eta(\alpha,s,\langle -p\rangle,-q). \tag{2.12}$$

Example 2.5.1 Again, if we consider $(p,q)=(1/2,0)$ and $(p,q)=(1,0)$, then also we will get the cosecant and cotangent zeta function.

$$\begin{aligned}\frac{1}{\left(1+e\left[\frac{s}{2}\right]\right)}H\left(\alpha,s,\left(\frac{1}{2},0\right)\right) &= \eta\left(\alpha,s,\frac{1}{2},0\right)\\ &= \sum_{k=1}^{\infty}k^{s-1}\frac{e\left[\frac{1}{2}k\alpha\right]}{1-e[k\alpha]}\\ &= \frac{i}{2}\sum_{k=1}^{\infty}\frac{\operatorname{cosec}(\pi k\alpha)}{k^{1-s}},\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\left(1+e\left[\frac{s}{2}\right]\right)}H(\alpha,s,(1,0)) &= \eta(\alpha,s,1,0)\\ &= \sum_{k=1}^{\infty}k^{s-1}\frac{e[kz]}{1-e[kz]}\\ &= \frac{i}{2}\sum_{k=1}^{\infty}\frac{\cot\pi kz}{k^{1-s}} - \frac{1}{2}\zeta(1-s),\end{aligned}$$

where $s \in \mathbb{C}$ with $\Re(s) < 0$.

Theorem 2.5.1 (Arakawa) [1, **Lemma 1 and Theorem 2**] *Suppose $\alpha \in \mathbb{R} \cap \bar{\mathbb{Q}}$ and $\alpha \notin \mathbb{Q}$. Then the infinite series $\eta(\alpha, s, p, q)$ is absolutely convergent if $\Re(s) < 0$. If, in addition, $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 2$ and $(p, q) \in \mathbb{Q}^2$ then $H(\alpha, s, p, q)$ has analytic continuation to $\mathbb{C} - \{0\}$, and the singularity at $s = 0$ is at worst a simple pole.*

Arakawa proved the absolute convergence of $\eta(\alpha, s, p, q)$ for $\Re(s) < 0$, by using the Thue-Siegel-Roth theorem.
Consider the generalized eta-function

$$\eta(z, s, p, q) = \sum_{n=1}^{\infty} n^{s-1} \frac{e[n(pz + q)]}{1 - e[nz]}, \qquad s \in \mathbb{C}, \tag{2.13}$$

corresponding to (2.11), for $z \in \mathbb{H}$ and a pair $(p, q) \in \mathbb{R}^2$ with $p > 0$. Then one can see that this series is absolutely convergent for arbitrary $s \in \mathbb{C}$. It can be easily checked that there is a link between the infinite series $A(z, s, r_1, r_2)$ and $\eta(z, s, r_1, r_2)$.

Lemma 2.5.2 *For any pair $(r_1, r_2) \in \mathbb{R}^2$ and $z \in \mathbb{H}$, we have*

$$A(z, s, r_1, r_2) = \eta(z, s, \langle r_1 \rangle, r_2), \quad s \in \mathbb{C}.$$

Now from the definition of H-function (2.6), we have

$$H(z, s, r_1, r_2) = A(z, s, r_1, r_2) + e\left[\frac{s}{2}\right] A(z, s, -r_1, -r_2).$$

Hence using Lemma 2.5.2, we get

$$H(z, s, r_1, r_2) = \eta(z, s, \langle r_1 \rangle, r_2) + e\left[\frac{s}{2}\right] \eta(z, s, \langle -r_1 \rangle, -r_2).$$

Similarity, we have

Lemma 2.5.3 *For any algebraic irrational number α and a pair $(p, q) \in \mathbb{R}^2$,*

$$A(\alpha, s, p, q) = \eta(\alpha, s, \langle p \rangle, q) \quad \text{with} \quad \Re(\mathrm{s}) < 0.$$

Again by the definition of H-function (2.12)(due to Arakawa), we have

$$H(\alpha, s, p, q) = \eta(\alpha, s, \langle p\rangle, q) + e\left[\frac{s}{2}\right]\eta(\alpha, s, \langle -p\rangle, q).$$

Therefore, by the Lemma 2.5.3, we get

$$H(\alpha, s, p, q) = A(\alpha, s, p, q) + e\left[\frac{s}{2}\right]A(\alpha, s, -p, q).$$

Proposition 2.5.4 (Arakawa) [1, **Proposition 1**] *Let* $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ *and* α *be an irrational real algebraic number and* $(p, q) \in \mathbb{R}^2$ *with* $p > 0$. *Let* $z = \alpha + iy$ *with* $y > 0$. *Set* $z^* = Vz$ *and* $\beta = V\alpha = (a\alpha + b)(c\alpha + d)^{-1}$. *If* $\Re(s) < -3$, *then*

$$\lim_{y\to 0^+} \eta(z^*, s, p, q) = \eta(\beta, s, p, q).$$

Arakawa obtained the following transformation formulae for $H(\alpha, s, (p, q))$, by virtue of the Theorem 2.4.3 of Berndt and Proposition 2.5.4.

Theorem 2.5.5 (Arakawa) [1, **Theorem 1**] *Let* α *be any real algebraic irrational, and let* $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ *with* $c > 0$ *such that* $\beta := c\alpha + d > 0$. *For any pair* (p, q) *of real numbers, set* $p' = pa + qc$, $q' = pb + qd$ *and* $\rho = \{q'\}c - \{p'\}d$. *Then for* $\Re(s) < 0$,

$$\begin{aligned} D_1(V, \alpha, s, (p, q)) :=& \beta^{-s}H(V\alpha, s, (p, q)) - H(\alpha, s, (p, q)V) \\ =& \beta^{-s}H(V\alpha, s, (p, q)) - H(\alpha, s, (p', q')) \\ =& -\Omega(p)(2\pi)^{-s}e\left[\frac{s}{4}\right]\beta^{-s}\Gamma(s)\left(\zeta(s, \langle q\rangle) + e\left[\frac{s}{2}\right]\zeta(s, \langle -q\rangle)\right) \\ &+ \Omega(p')(2\pi)^{-s}e\left[-\frac{s}{4}\right]\Gamma(s)\left(\zeta(s, \langle -q'\rangle) + e\left[\frac{s}{2}\right]\zeta(s, \langle q'\rangle)\right) \\ &+ (2\pi)^{-s}e\left[-\frac{s}{4}\right]L(\alpha, s, (p', q'), c, d), \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} &L(\alpha, s, p', q', c, d) \\ &= -\sum_{j=1}^{c}\int_{I(\lambda,\infty)} t^{s-1}G_2\left(1 - \left\{\frac{(jd+\rho)}{c}\right\} + \frac{(j - \{p'\})\beta}{c},\ (1,\ \beta);\ t\right)\mathrm{d}t \end{aligned} \tag{2.15}$$

$$\left(0 < \lambda < 2\pi, \frac{2\pi}{\beta}\right).$$

Berndt [3, p. 499] found the special values of $L(\alpha, s, (p', q'), c, d)$ at non-negative integral arguments $s = -m$,

$$L(\alpha, -m, (p', q'), c, d) = \frac{2\pi i}{(m+2)!}\sum_{j=1}^{c}\sum_{k=0}^{m+2}\binom{m+2}{k}B_k\left(\frac{j-\{p'\}}{c}\right)\bar{B}_{m+2-k}\left(\frac{jd+\rho}{c}\right)(-\beta)^{k-1},$$

where $B_n(x)$ denotes the nth Bernoulli polynomial and $\bar{B}_n(x) = B_n(\{x\})$.

Lemma 2.5.6 (Arakawa) [1, **Lemma 4**] *Let α be an irrational number in a real quadratic field $\mathbb{Q}(\Delta)$ and let (p, q) be a pair of rational numbers. Then there exist a totally positive unit β of $\mathbb{Q}(\Delta)$ and an element $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $SL(2, \mathbb{Z})$ which satisfy the conditions:*

(i) $c > 0$,

(ii) $(p, q)V \equiv (p, q) \mod 1$,

(iii) $\beta \begin{pmatrix} \alpha \\ 1 \end{pmatrix} = V \begin{pmatrix} \alpha \\ 1 \end{pmatrix}$.

We choose such a $\beta \in \mathbb{Q}(\Delta)$ and $V \in SL(2, \mathbb{Z})$ i.e., which satisfy the conditions of the Lemma 2.5.6. Then using condition (ii), we have

$$H(\alpha, s, (p, q)) = H(\alpha, s, (p, q)V).$$

Since $V\alpha = \alpha$ and $c > 0$, we can see easily from Theorem 2.5.5 that,

$$\begin{aligned} H(\alpha, s, (p, q)) = &- \Omega(p)(2\pi)^{-s} e\left[\frac{s}{4}\right]\Gamma(s)\zeta(s, \langle q\rangle) \\ &+ \Omega(p)(2\pi)^{-s} e\left[-\frac{s}{4}\right]\Gamma(s)\zeta(s, \langle q\rangle)\frac{1 - e[s]\beta^{-s}}{\beta^{-s} - 1} \\ &+ \frac{(2\pi)^{-s} e\left[-\frac{s}{4}\right]}{\beta^{-s} - 1} L(\alpha, s, (p, q), c, d). \end{aligned} \tag{2.16}$$

Example 2.5.2 Let α, β and V as in Lemma 2.5.6, and with $(p, q) = (1, 0)$ and

$(p,q)=(1/2,0)$. Then

$$\begin{aligned}
H(\alpha,s,(1,0)) =&(2\pi)^{-s}\left(-e\left[\frac{s}{4}\right]+e\left[-\frac{s}{4}\right]\frac{1-e\left[s\right]\beta^{-s}}{\beta^{-s}-1}\right)\Gamma(s)\zeta(s)\\
&+\frac{(2\pi)^{-s}e\left[-\frac{s}{4}\right]}{\beta^{-s}-1}L(\alpha,s,(1,0),c,d),\\
H\left(\alpha,s,\left(\frac{1}{2},0\right)\right) =&\frac{(2\pi)^{-s}e\left[-\frac{s}{4}\right]}{\beta^{-s}-1}L\left(\alpha,s,\left(\frac{1}{2},0\right),c,d\right).
\end{aligned}$$

2.5.1 Values at some particular matrices

We fix the following matrices

$$V_0=\begin{pmatrix}0&-1\\1&0\end{pmatrix},V_1=\begin{pmatrix}1&0\\1&1\end{pmatrix}\quad\text{and}\quad V_2=V_0^2V_1^{-1}=\begin{pmatrix}-1&0\\1&-1\end{pmatrix}. \tag{2.17}$$

Example 2.5.3 Theorem 2.5.5 gives the following:

$$\begin{aligned}
D_1(V_0,\alpha,s,(p,q))&=\alpha^{-s}H\left(\frac{-1}{\alpha},s,(p,q)\right)-H(\alpha,s,(q,-p))\\
&=-\Omega(p)(2\pi)^{-s}e\left[\frac{s}{4}\right]\alpha^{-s}\Gamma(s)\left(\zeta(s,\langle q\rangle)+e\left[\frac{s}{2}\right]\zeta(s,\langle -q\rangle)\right)\\
&\quad+\Omega(q)(2\pi)^{-s}e\left[-\frac{s}{4}\right]\Gamma(s)\left(\zeta(s,\langle p\rangle)+e\left[\frac{s}{2}\right]\zeta(s,\langle -p\rangle)\right)\\
&\quad+(2\pi)^{-s}e\left[-\frac{s}{4}\right]L(\alpha,s,(q,-p),1,0)\ ,
\end{aligned} \tag{2.18}$$

$$\begin{aligned}
D_1(V_1,\alpha,s,(p,q))&=(\alpha+1)^{-s}H\left(\frac{\alpha}{\alpha+1},s,(p,q)\right)-H(\alpha,s,(p+q,q))\\
&=-\Omega(p)(2\pi)^{-s}e\left[\frac{s}{4}\right](\alpha+1)^{-s}\Gamma(s)\left(\zeta(s,\langle q\rangle)+e\left[\frac{s}{2}\right]\zeta(s,\langle -q\rangle)\right)\\
&\quad+\Omega(p+q)(2\pi)^{-s}e\left[-\frac{s}{4}\right]\Gamma(s)\left(\zeta(s,\langle -q\rangle)+e\left[\frac{s}{2}\right]\zeta(s,\langle q\rangle)\right)\\
&\quad+(2\pi)^{-s}e\left[-\frac{s}{4}\right]L(\alpha,s,(p+q,q),1,1)\ ,
\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}D_1(V_2,\alpha,s,(p,q)) &= (\alpha-1)^{-s}H\left(\frac{-\alpha}{\alpha-1},s,(p,q)\right) - H(\alpha,s,(-p+q,-q))\\
&= -\Omega(p)(2\pi)^{-s}e\left[\frac{s}{4}\right](\alpha-1)^{-s}\Gamma(s)\left(\zeta(s,\langle q\rangle)+e\left[\frac{s}{2}\right]\zeta(s,\langle -q\rangle)\right)\\
&\quad + \Omega(-p+q)(2\pi)^{-s}e\left[-\frac{s}{4}\right]\Gamma(s)\left(\zeta(s,\langle q\rangle)+e\left[\frac{s}{2}\right]\zeta(s,\langle -q\rangle)\right)\\
&\quad + (2\pi)^{-s}e\left[-\frac{s}{4}\right]L(\alpha,s,(-p+q,-q),1,-1)\ .\end{aligned}\tag{2.20}$$

In particular, when $(p,q)=(1,0)$, we have

$$\begin{aligned}D_1\left(V_0,\alpha,s,(1,0)\right) =&(2\pi)^{-s}e\left[\frac{s}{4}\right]\left\{e\left[-\frac{s}{2}\right]-\alpha^{-s}\right\}\Gamma(s)\left(1+e\left[\frac{s}{2}\right]\right)\zeta(s)\\
&+ (2\pi)^{-s}e\left[-\frac{s}{4}\right]L(\alpha,s,(0,-1),1,0),\end{aligned}$$

$$\begin{aligned}D_1\left(V_1,\alpha,s,(1,0)\right) &= (2\pi)^{-s}e\left[\frac{s}{4}\right]\left\{e\left[-\frac{s}{2}\right]-(\alpha+1)^{-s}\right\}\Gamma(s)\left(1+e\left[\frac{s}{2}\right]\right)\zeta(s)\\
&\quad + (2\pi)^{-s}e\left[-\frac{s}{4}\right]L(\alpha,s,(1,0),1,1),\end{aligned}$$

$$\begin{aligned}D_1\left(V_2,\alpha,s,(1,0)\right) &= (2\pi)^{-s}e\left[\frac{s}{4}\right]\left\{e\left[-\frac{s}{2}\right]-(\alpha-1)^{-s}\right\}\Gamma(s)\left(1+e\left[\frac{s}{2}\right]\right)\zeta(s)\\
&\quad + (2\pi)^{-s}e\left[-\frac{s}{4}\right]L(\alpha,s,(-1,0),1,-1).\end{aligned}$$

If we choose $(p,q)=(1/2,0)$, we get

$$\begin{aligned}D_1\left(V_0,\alpha,s,\left(\frac{1}{2},0\right)\right) &= \alpha^{-s}H\left(\frac{-1}{\alpha},s,\left(\frac{1}{2},0\right)\right) - H\left(\alpha,s,\left(0.-\frac{1}{2}\right)\right)\\
&= (2\pi)^{-s}\left(e\left[\frac{s}{4}\right]+e\left[-\frac{s}{4}\right]\right)\Gamma(s)\zeta\left(s,\frac{1}{2}\right)\\
&\quad + (2\pi)^{-s}e\left[-\frac{s}{4}\right]L\left(\alpha,s,\left(0,-\frac{1}{2}\right),1,0\right),\\
D_1\left(V_1,\alpha,s,\left(\frac{1}{2},0\right)\right) &= (\alpha+1)^{-s}H\left(\frac{\alpha}{\alpha+1},s,\left(\frac{1}{2},0\right)\right) - H\left(\alpha,s,\left(\frac{1}{2},0\right)\right)\\
&= (2\pi)^{-s}e\left[-\frac{s}{4}\right]L\left(\alpha,s,\left(\frac{1}{2},0\right),1,1\right),\end{aligned}$$

$$\begin{aligned}D_1\left(V_2,\alpha,s,\left(\frac{1}{2},0\right)\right) &= (\alpha-1)^{-s}H\left(\frac{-\alpha}{\alpha-1},s,\left(\frac{1}{2},0\right)\right)-H\left(\alpha,s,\left(-\frac{1}{2},0\right)\right)\\ &= (2\pi)^{-s}e\left[-\frac{s}{4}\right]L\left(\alpha,s,\left(-\frac{1}{2},0\right),1,-1\right).\end{aligned}$$

Note that for non-negative integers m, we have the following explicit formulae for V_j, where $j=0,1,2$,

$$\begin{aligned}&L(\alpha,-m,(1,0)V_j,c,d)\\ &=\frac{2\pi i}{(m+2)!}\sum_{k=0}^{m+2}\binom{m+2}{k}B_k(1)\bar{B}_{m+2-k}(1)(-\beta)^{k-1},\\ &L\left(\alpha,-m,\left(\frac{1}{2},0\right)V_j,c,d\right)\\ &=\frac{2\pi i}{(m+2)!}\sum_{k=0}^{m+2}\binom{m+2}{k}B_k\left(\frac{1}{2}\right)\bar{B}_{m+2-k}\left(\frac{1}{2}\right)(-\beta)^{k-1}.\end{aligned}$$

2.6 Generalization of the secant zeta function

We introduce two Lambert series corresponding to (2.11) and (2.5). These include the generalizations of secant and tangent zeta functions as will be shown in Example 2.6.1 below. Let α be any algebraic irrational number and (p,q) a pair of real numbers. Then we define the series η^* by

$$\eta^*(\alpha,s,p,q):=\sum_{n=1}^{\infty}n^{s-1}\frac{e[n(p\alpha+q)]}{1+e[n\alpha]},\qquad \Re(s)<0, \tag{2.21}$$

and another infinite series A^* by

$$A^*(z,s,r_1,r_2):=\sum_{m>-r_1}(-1)^m\sum_{k=1}^{\infty}k^{s-1}e[kr_2+k(m+r_1)z] \tag{2.22}$$

for a pair $(r_1,r_2)\in\mathbb{R}^2$, $z\in\mathbb{H}$ and $s\in\mathbb{C}$.

Example 2.6.1 If we take $(r_1,r_2)=(1,0)$ and $(\frac{1}{2},0)$, then (2.22) becomes

$$\begin{aligned}A^*(\alpha,s,1,0) &= \eta^*(\alpha,s,1,0)\\ &=\sum_{k=1}^{\infty}k^{s-1}\frac{e[k\alpha]}{1+e[k\alpha]}\end{aligned}$$

$$= \frac{1}{2}\sum_{k=1}^{\infty} k^{s-1}\left(\frac{e[k\alpha]-1}{1+e[k\alpha]}+1\right)$$
$$= \frac{i}{2}\sum_{k=1}^{\infty} k^{s-1}\tan\pi k\alpha + \frac{1}{2}\zeta(1-s),$$

and

$$A^*\left(\alpha, s, \frac{1}{2}, 0\right) = \eta\left(\alpha, s, \frac{1}{2}, 0\right) \tag{2.23}$$
$$= \frac{1}{2}\sum_{k=1}^{\infty} k^{s-1}\frac{1}{\cos\pi k\alpha}$$
$$= \frac{1}{2}\psi(\alpha, 1-s)$$

respectively.

By virtue of the results of Lewittes, Berndt and Arakawa, we have the following results.

Lemma 2.6.1 *Let α be an algebraic irrational number and (p, q) be a pair of real numbers. The series $\eta^*(\alpha, s, p, q)$ is absolutely convergent, if $s \in \mathbb{C}$ with $\Re(s) < 0$.*

Proof. One can prove this result applying the Thue-Siegel-Roth theorem, in a similar manner to Arakawa's procedure for proving the absolute convergence of the series $\eta(\alpha, s, p, q)$.

Lemma 2.6.2 *If $z \in \mathbb{H}$ and a pair $(p, q) \in \mathbb{R}^2$ with $p > 0$, then the series $\eta^*(z, s, p, q)$ is absolutely convergent for any $s \in \mathbb{C}$.*

Proof. Since $z \in \mathbb{H}$, assume $z = x + iy$ with $y > 0$. We have

$$\eta^*(z, s, p, q) = \sum_{n=1}^{\infty} n^{s-1}\frac{e[n(pz+q)]}{1+e[nz]}$$
$$= \sum_{n=1}^{\infty} n^{s-1}\sum_{m=0}^{\infty}(-1)^m e[nq + n(m+p)z].$$

Thus,

$$\begin{aligned}\eta^*(z,s,p,q)| &\leq \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} n^{s-1}\exp(-2\pi n(m+p)y)\\ &\leq \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} n^{s-1}\frac{K!}{(2\pi n(m+p)y)^K}\\ &\leq \sum_{n=1}^{\infty} n^{s-1}\frac{C}{n^K} < \infty,\end{aligned}$$

where C is some constant and we can choose a large enough positive integer K such that $K > \Re(s)$ for given any $s \in \mathbb{C}$.

Lemma 2.6.3 *Let $z \in \mathbb{H}$ and α be an irrational algebraic number. Then for any pair of real numbers (r_1, r_2), we have*

$$A^*(z,s,r_1,r_2) = (-1)^{-r_1+\langle r_1\rangle}\eta^*(z,s,\langle r_1\rangle,r_2) \quad \text{for} \quad \mathrm{s} \in \mathbb{C},$$

and

$$A^*(\alpha,s,r_1,r_2) = (-1)^{-r_1+\langle r_1\rangle}\eta^*(\alpha,s,\langle r_1\rangle,r_2) \quad \text{for} \quad \Re(\mathrm{s}) < 0.$$

Proof. If $r_1 \in \mathbb{Z}$, then $m > -r_1$ implies $m = -r_1 + r$ for $r = 1, 2, 3, \cdots$. By the definition of $A^*(z,s,r_1,r_2)$, we know

$$\begin{aligned}A^*(z,s,r_1,r_2) &= \sum_{m>-r_1}(-1)^m\sum_{k=1}^{\infty}k^{s-1}e[kr_2+k(m+r_1)z]\\ &= \sum_{k=1}^{\infty}k^{s-1}\sum_{r=1}^{\infty}(-1)^{-r_1+r}e[kr_2+krz]\\ &= (-1)^{-r_1+1}\sum_{k=1}^{\infty}k^{s-1}e[kr_2+kz]\sum_{r=0}^{\infty}(-1)^r e[krz]\\ &= (-1)^{-r_1+1}\sum_{k=1}^{\infty}k^{s-1}\frac{e[kr_2+kz]}{1+e[kz]}\\ &= (-1)^{-r_1+\langle r_1\rangle}\eta^*(z,s,\langle r_1\rangle,r_2), \quad \text{since} \quad \langle \mathrm{r}_1\rangle = 1.\end{aligned}$$

Again, if $r_1 \notin \mathbb{Z}$, then $m > -r_1$ implies $m = -\lfloor r_1\rfloor + r$ for $r = 0, 1, 2, \cdots$. So we will have

$$\begin{aligned}
A^*(z,s,r_1,r_2) &= \sum_{m>-r_1}(-1)^m\sum_{k=1}^{\infty}k^{s-1}e[kr_2+k(m+r_1)z]\\
&= \sum_{k=1}^{\infty}k^{s-1}\sum_{r=0}^{\infty}(-1)^{-\lfloor r_1\rfloor+r}e[kr_2+k(\langle r_1\rangle+r)z]\\
&= (-1)^{-\lfloor r_1\rfloor}\sum_{k=1}^{\infty}k^{s-1}e[kr_2+k\langle r_1\rangle z]\sum_{r=0}^{\infty}(-1)^r e[krz]\\
&= (-1)^{-\lfloor r_1\rfloor}\sum_{k=1}^{\infty}k^{s-1}\frac{e[kr_2+k\langle r_1\rangle z]}{1+e[kz]}\\
&= (-1)^{-r_1+\langle r_1\rangle}\eta^*(z,s,\langle r_1\rangle,r_2).
\end{aligned}$$

Similarly we can see that

$$A^*(\alpha,s,r_1,r_2)=(-1)^{-r_1+\langle r_1\rangle}\eta^*(\alpha,s,\langle r_1\rangle,r_2)\quad\text{for}\quad \Re(\mathrm{s})<0.$$

Lemma 2.6.4 *If $z\in\mathbb{H}$ then $A^*(z,s,r_1,r_2)$ is absolutely convergent for any $s\in\mathbb{C}$.*

Proof. Using the Lemma 2.6.2 and Lemma 2.6.3, we can show that $A^*(z,s,r_1,r_2)$ is absolutely convergent for $s\in\mathbb{C}$.

2.7 Main Results

Consider the difference

$$D^*(V):=D^*\left(V,\alpha,s,\frac{1}{2},0\right):=\beta^{-s}A^*\left(V\alpha,s,\frac{1}{2},0\right)-A^*\left(\alpha,s,\frac{1}{2},0\right)\qquad(2.24)$$

for each V from (2.17). Now the second term in the above expression is the secant zeta function in view of (2.23). This difference is quite natural in the sense that it expresses the surplus after the modular transformation is applied. We interpret the main result of Lalín et al. Theorem 2.3.2 in this setting as a special case of

$$(\alpha+1)^{-s}A^*\left(V_1\alpha, s, \frac{1}{2}, 0\right) + (\alpha-1)^{-s}A^*\left(V_2\alpha, s, \frac{1}{2}, 0\right) \tag{2.25}$$

for $\Re(s) < 0$ and locate it in a natural way as we will see in Corollary 2.7.2. Our main theorem is the following.

Theorem 2.7.1 *For a real algebraic irrational α and a complex variable s with $\Re(s) < 0$, we have*

$$\begin{aligned}
D^*(V_0) &= \alpha^{-s}A^*\left(\frac{-1}{\alpha}, s, \frac{1}{2}, 0\right) - A^*\left(\alpha, s, \frac{1}{2}, 0\right) \qquad (2.26)\\
&= 2^{1-2s}\pi^{-s}e\left[-\frac{s}{4}\right]\left(\Phi_0 + \Gamma(s)\Omega_0\right) + 2^{1-s}\Psi_0\\
&= -\frac{(2\pi)^{-s}e\left[-\frac{s}{4}\right]}{1-e\left[\frac{s}{2}\right]}\int_{I(\lambda,\infty)} t^{s-1}\frac{\exp(-(\frac{1}{2}+\alpha)t)}{(1+\exp(-t))(1-\exp(-\alpha t))}\,\mathrm{d}t\\
&\quad + 2^{-2s}\pi^{-s}e\left[-\frac{s}{4}\right]\Gamma(s)\left(\zeta\left(s,\frac{1}{4}\right) - \zeta\left(s,\frac{3}{4}\right)\right)\\
&\quad - 2^{1-s}\sum_{n=1}^{\infty} n^{s-1}\frac{e\left[n\left(\frac{1}{2}\alpha+\frac{1}{4}\right)\right]\left(e\left[\frac{1}{2}n\alpha\right]+1\right)}{1-e[n\alpha]}\\
&\quad + 2^{2-s}\sum_{n=1}^{\infty}(2n)^{s-1}\frac{e\left[\frac{3}{2}\cdot n\alpha\right]}{1-e[2n\alpha]}.
\end{aligned}$$

$$\begin{aligned}
D^*(V_1) &= (\alpha+1)^{-s}A^*\left(\frac{\alpha}{\alpha+1}, s, \frac{1}{2}, 0\right) - A^*\left(\alpha, s, \frac{1}{2}, 0\right) \qquad (2.27)\\
&= 2^{1-2s}\pi^{-s}e\left[-\frac{s}{4}\right]\Phi_1 + 2^{1-s}\Psi_1\\
&= -\frac{(2\pi)^{-s}e\left[-\frac{s}{4}\right]}{1-e\left[\frac{s}{2}\right]}\int_{I(\lambda,\infty)} t^{s-1}\frac{\exp(-\frac{1}{2}t)}{(1+\exp(-t))}\frac{\exp(-\frac{1}{2}(\alpha+1)t)}{(1-\exp(-(\alpha+1)t))}\,\mathrm{d}t\\
&\quad + 2^{-s}\sum_{n=1}^{\infty} n^{s-1}\frac{(-1)^{n-1}}{\cos\frac{\pi n}{2}\alpha}.
\end{aligned}$$

Also,

$$\begin{aligned}
D^*(V_2) &= (\alpha-1)^{-s}A^*\left(\frac{-\alpha}{\alpha-1}, s, \frac{1}{2}, 0\right) - A^*\left(\alpha, s, \frac{1}{2}, 0\right) \qquad (2.28)\\
&= 2^{1-2s}\pi^{-s}e\left[-\frac{s}{4}\right]\Phi_2 + 2^{1-s}\Psi_2
\end{aligned}$$

$$= -\frac{(2\pi)^{-s} e\left[-\frac{s}{4}\right]}{1-e\left[\frac{s}{2}\right]} \int_{I(\lambda,\infty)} t^{s-1} \frac{\exp(-\frac{1}{2}t)}{(1+\exp(-t))} \frac{\exp(-\frac{1}{2}(\alpha-1)t)}{(1-\exp(-(\alpha-1)t))} \,\mathrm{d}t$$
$$- 2^{-s} \sum_{n=1}^{\infty} n^{s-1} \frac{1}{\cos \frac{\pi n}{2}\alpha},$$

where Φ_k and Ψ_k, $k = 0,1,2$ are defined later. They indicate the block of L-integrals and the block of H-functions, corresponding to the matrix V_k, respectively. Also, Ω_0 is defined in (2.48).

We recover the main result of Lalín et al. [21, Theorem 3] i.e., Theorem 2.3.2 by adding the equations (2.27) and (2.28). We note it as a corollary.

Corollary 2.7.2

$$(\alpha+1)^{-s} A^*\left(\frac{\alpha}{\alpha+1}, s, \frac{1}{2}, 0\right) + (\alpha-1)^{-s} A^*\left(\frac{-\alpha}{\alpha-1}, s, \frac{1}{2}, 0\right) \tag{2.29}$$
$$= -\frac{(2\pi)^{-s} e\left[-\frac{s}{4}\right]}{1-e\left[\frac{s}{2}\right]} \int_{I(\lambda,\infty)} t^{s-1} \sum_{m=0}^{\infty} 2^{-m-1} E_m \frac{t^m}{m!} \sum_{n=0}^{\infty} (2^{1-n}-1) B_n$$
$$\times \frac{\{(\alpha+1)^{n-1} + (\alpha-1)^{n-1}\} t^{n-1}}{n!} \,\mathrm{d}t.$$

The genesis of the transformation formula of Lalín et al. [21, Theorem 3] for the secant zeta function is given by the sum of $D^*(V_1)$ and $D^*(V_2)$, which we have seen in the above Corollary 2.7.2. We will see in the proof of corollary 2.7.2 that the term $2A^*\left(\alpha, s, \frac{1}{2}, 0\right)$ on the left side and the secant zeta function on the right hand side naturally cancel each other. As this occurs only in such a pairing, this elucidates the hidden structure of the paired transformation formula from a more general standpoint.

2.7.1 Deduction of the main theorem of Lalín et al.

Firstly we deduce Theorem 2.3.2 from Corollary 2.7.2. To do that, let $l = 2k$ be an even positive integer and $s = 1 - l$. Then (2.29) amounts to

$$(\alpha+1)^{2k-1} A^*\left(\frac{\alpha}{\alpha+1}, -2k+1, \frac{1}{2}, 0\right) + (\alpha-1)^{2k-1} A^*\left(\frac{-\alpha}{\alpha-1}, -2k+1, \frac{1}{2}, 0\right)$$
$$= -\frac{(2\pi)^{2k-1} e\left[-\frac{-2k+1}{4}\right]}{1-e\left[\frac{-2k+1}{2}\right]}$$

$$\times \int_{I(\lambda,\infty)} t^{-2k} \sum_{m=0}^{\infty} 2^{-m-1} E_m \frac{t^m}{m!} \sum_{n=0}^{\infty} (2^{1-n}-1) B_n \frac{\{(\alpha+1)^{n-1}+(\alpha-1)^{n-1}\}t^{n-1}}{n!}\,\mathrm{d}t$$
$$= -\frac{2^{2k-1}\pi^{2k}(-1)^k}{2\pi i}$$
$$\times \int_{I(\lambda,\infty)} t^{-2k} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} 2^{-m-1}(2^{1-n}-1)E_m B_n\{(\alpha+1)^{n-1}+(\alpha-1)^{n-1}\}\frac{t^{m+n-1}}{m!n!}\,\mathrm{d}t$$
$$= -2^{2k-1}\pi^{2k}(-1)^k$$
$$\times \sum_{n=0}^{2k} \frac{1}{(2k-n)!n!} 2^{-2k+n-1}(2^{1-n}-1)E_{2k-n}B_n\{(\alpha+1)^{n-1}+(\alpha-1)^{n-1}\}$$
$$= \frac{1}{2}\pi^{2k}(-1)^k \sum_{n=0}^{2k} \frac{1}{(2k-n)!n!}(2^{n-1}-1)E_{2k-n}B_n\{(\alpha+1)^{n-1}+(\alpha-1)^{n-1}\}.$$

This proves Theorem 2.3.2.

The following conjecture seems to be plausible.

Conjecture 2.7.3 Let $W_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ and $W_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ be two matrices in $\mathrm{PSL}_2(\mathbb{Z})$ which are inverses to each other. Then for a pair $(p,q) \in \mathbb{R}^2$,

$$(c_1\alpha+d_1)^{-s}A^*(W_1\alpha,s,p,q) + (c_2\alpha+d_2)^{-s}A^*(W_2\alpha,s,p,q) \tag{2.30}$$

can be expressible in terms of special values of the zeta and L-functions as we have seen for the sum of two *explicit* expressions for

$$(c_1\alpha+d_1)^{-s}A^*\left(V_j\alpha,s,\left(\frac{1}{2},0\right)\right) - A^*\left(\alpha,s,\left(\frac{1}{2},0\right)\right), j=1,2.$$

2.8 A* in terms of A and H-functions

Before proving our main theorem we need to express A^* in terms of A and H. We know that given a sum $S = \sum_n a_n$ with its even and odd parts S_e and S_o, where the even part is over all even integer values and odd part over odd integer values, the sum $2S_e - S$ is the alternating sum $\sum_n (-1)^n a_n$. Using this observation, we have the following result.

Lemma 2.8.1 $A^*(z,s,r_1,r_2) = 2A\left(2z,s,\frac{r_1}{2},r_2\right) - A(z,s,r_1,r_2)$.

Proof. By the definition of $A^*(z,s,r_1,r_2)$, we have

$$\begin{aligned}
A^*(z,s,r_1,r_2) &= \sum_{m>-r_1} (-1)^m \sum_{k=1}^{\infty} k^{s-1} e[kr_2 + k(m+r_1)z] \\
&= 2 \sum_{\substack{m>-r_1 \\ m:even}} \sum_{k=1}^{\infty} k^{s-1} e[kr_2 + k(m+r_1)z] \\
&\quad - \sum_{m>-r_1} \sum_{k=1}^{\infty} k^{s-1} e[kr_2 + k(m+r_1)z] \\
&= 2 \sum_{2m>-r_1} \sum_{k=1}^{\infty} k^{s-1} e[kr_2 + k(2m+r_1)z] \\
&\quad - \sum_{m>-r_1} \sum_{k=1}^{\infty} k^{s-1} e[kr_2 + k(m+r_1)z] \\
&= 2A\left(2z, s, \frac{r_1}{2}, r_2\right) - A(z,s,r_1,r_2).
\end{aligned}$$

There is a duplication formula for $A(z,s,r_1,r_2)$ which is as follows:

Lemma 2.8.2 $A(z,s,r_1,r_2) + A\left(z,s,r_1,r_2+\frac{1}{2}\right) = 2^s A(2z,s,r_1,2r_2)$

Proof. From the definition (2.5) of $A(z,s,r_1,r_2)$, we have

$$\begin{aligned}
&A(z,s,r_1,r_2) + A\left(z,s,r_1,r_2+\frac{1}{2}\right) \\
&= \sum_{m>-r_1} \sum_{k=1}^{\infty} k^{s-1} e\left[kr_2 + k(m+r_1)z\right] \\
&\quad + \sum_{m>-r_1} \sum_{k=1}^{\infty} k^{s-1} e\left[k\left(r_2+\frac{1}{2}\right) + k(m+r_1)z\right] \\
&= \sum_{m>-r_1} \sum_{k=1}^{\infty} k^{s-1} e\left[kr_2 + k(m+r_1)z\right] \left(1 + e\left[\frac{1}{2}k\right]\right) \\
&= 2 \sum_{m>-r_1} \sum_{k=1}^{\infty} (2k)^{s-1} e\left[2kr_2 + 2k(m+r_1)z\right] \\
&= 2^s \sum_{m>-r_1} \sum_{k=1}^{\infty} k^{s-1} e\left[k(2r_2) + k(m+r_1)(2z)\right] \\
&= \; 2^s A(2z,s,r_1,2r_2).
\end{aligned}$$

Using the duplication formula i.e., Lemma 2.8.2 in Lemma 2.8.1, we get

Lemma 2.8.3

$$A^*(z,s,r_1,r_2) = 2^{1-s}A\left(z,s,\frac{r_1}{2},\frac{r_2}{2}\right) + 2^{1-s}A\left(z,s,\frac{r_1}{2},\frac{r_2}{2}+\frac{1}{2}\right) - A(z,s,r_1,r_2). \tag{2.31}$$

On the other hand,

$$H(z,s,r_1,r_2) = A(z,s,r_1,r_2) + e\left[\frac{s}{2}\right]A(z,s,-r_1,-r_2),$$

and

$$H(z,s,-r_1,-r_2) = A(z,s,-r_1,-r_2) + e\left[\frac{s}{2}\right]A(z,s,r_1,r_2).$$

Therefore,

$$A(z,s,r_1,r_2) = \frac{1}{1-e[s]}\left\{H(z,s,r_1,r_2) - e\left[\frac{s}{2}\right]H(z,s,-r_1,-r_2)\right\}. \tag{2.32}$$

Substituting (2.32) in the Lemma 2.8.3, we deduce the following proposition.

Proposition 2.8.4 *For a real algebraic irrational α, a pair (p,q) of real numbers with $p>0$, and a complex variable s with $\Re(s)<0$, we have*

$$\begin{aligned}(1-e[s])A^*(\alpha,s,p,q) &= 2^{1-s}\left\{H\left(\alpha,s,\left(\frac{p}{2},\frac{q}{2}\right)\right) - e\left[\frac{s}{2}\right]H\left(\alpha,s,\left(-\frac{p}{2},-\frac{q}{2}\right)\right)\right\}\\ &+ 2^{1-s}\left\{H\left(\alpha,s,\left(\frac{p}{2},\frac{q}{2}+\frac{1}{2}\right)\right) - e\left[\frac{s}{2}\right]H\left(\alpha,s,\left(-\frac{p}{2},-\frac{q}{2}-\frac{1}{2}\right)\right)\right\}\\ &- (1-e[s])A(\alpha,s,p,q),\end{aligned}$$

where

$$(1-e[s])A(\alpha,s,p,q) = \left\{H(\alpha,s,(p,q)) - e\left[\frac{s}{2}\right]H(\alpha,s,(-p,-q))\right\}. \tag{2.33}$$

as in (2.32).

Example 2.8.1 If we consider $(p,q)=(1,0)$ and $(1/2,0)$, then we get

$A^*(\alpha,s,1,0)$

$$= \frac{1}{1+e\left[\frac{s}{2}\right]}\left\{2^{1-s}H\left(\alpha,s,\left(\frac{1}{2},0\right)\right)+2^{1-s}H\left(\alpha,s,\left(\frac{1}{2},\frac{1}{2}\right)\right)-H(\alpha,s,(1,0))\right\}$$

and

$$\begin{aligned} A^*\left(\alpha,s,\frac{1}{2},0\right) = &\frac{2^{1-s}}{1-e[s]}H\left(\alpha,s,\left(\frac{1}{4},0\right)\right)-\frac{2^{1-s}e\left[\frac{s}{2}\right]}{1-e[s]}H\left(\alpha,s,\left(-\frac{1}{4},0\right)\right) \\ &+\frac{2^{1-s}}{1-e[s]}H\left(\alpha,s,\left(\frac{1}{4},\frac{1}{2}\right)\right)-\frac{2^{1-s}e\left[\frac{s}{2}\right]}{1-e[s]}H\left(\alpha,s,\left(-\frac{1}{4},-\frac{1}{2}\right)\right) \\ &-\frac{1-e\left[\frac{s}{2}\right]}{1-e[s]}H\left(\alpha,s,\left(\frac{1}{2},0\right)\right). \end{aligned} \tag{2.34}$$

For the last term, with s an even integer, we use either

$$H\left(\alpha,s,\left(\frac{1}{2},0\right)\right)=\frac{(2\pi)^{-s}e\left[-\frac{s}{4}\right]}{\beta^{-s}-1}L\left(\alpha,s,\left(\frac{1}{2},0\right),c,d\right) \tag{2.35}$$

or

$$\frac{1}{1+e[\frac{s}{2}]}H\left(\alpha,s,\left(\frac{1}{2},0\right)\right)=\frac{i}{2}\sum_{k=1}^{\infty}\frac{1}{k^{1-s}}\frac{1}{\sin\pi k\alpha} \tag{2.36}$$

which follows from Example 2.4.1 and Example 2.5.2, respectively.

2.9 General Procedure

The general procedure is to transform

$$D^*(V) := D^*(V,\alpha,s,p,q) :=\beta^{-s}A^*\left(V\alpha,s,p,q\right)-A^*\left(\alpha,s,p,q\right). \tag{2.37}$$

We recall the following notations

$$D_1(V,\alpha,s,(p,q))=\beta^{-s}H(V\alpha,s,(p,q))-H(\alpha,s,(p,q)V) \tag{2.38}$$

$$D_0^*(V,\alpha,s,p,q)=\beta^{-s}A\left(V\alpha,s,p,q\right)-A\left(\alpha,s,p,q\right) \tag{2.39}$$

$$A(z,s,r_1,r_2) = \frac{1}{1-e[s]}\left\{H(z,s,r_1,r_2) - e\left[\frac{s}{2}\right]H(z,s,-r_1,-r_2)\right\}. \qquad (2.40)$$

Now using Proposition 2.8.4, we can write

$$\begin{aligned}
&D^*(V,\alpha,s,p,q) + D_0^*(V,\alpha,s,p,q) \qquad (2.41)\\
&= \frac{2^{1-s}}{1-e[s]}\left(\beta^{-s}H\left(V\alpha,s,\left(\frac{p}{2},\frac{q}{2}\right)\right) - H\left(\alpha,s,\left(\frac{p}{2},\frac{q}{2}\right)\right)\right)\\
&- \frac{2^{1-s}e\left[\frac{s}{2}\right]}{1-e[s]}\left(\beta^{-s}H\left(V\alpha,s,\left(-\frac{p}{2},-\frac{q}{2}\right)\right) - H\left(\alpha,s,\left(-\frac{p}{2},-\frac{q}{2}\right)\right)\right)\\
&+ \frac{2^{1-s}}{1-e[s]}\left(\beta^{-s}H\left(V\alpha,s,\left(\frac{p}{2},\frac{q}{2}+\frac{1}{2}\right)\right) - H\left(\alpha,s,\left(\frac{p}{2},\frac{q}{2}+\frac{1}{2}\right)\right)\right)\\
&- \frac{2^{1-s}e\left[\frac{s}{2}\right]}{1-e[s]}\left(\beta^{-s}H\left(V\alpha,s,\left(-\frac{p}{2},-\frac{q}{2}-\frac{1}{2}\right)\right) - H\left(\alpha,s,\left(-\frac{p}{2},-\frac{q}{2}-\frac{1}{2}\right)\right)\right).
\end{aligned}$$

For $(p,q) = (1/2,0)$, we have

$$D_0^*\left(V,\alpha,s,\frac{1}{2},0\right) = \frac{1}{1+e\left[\frac{s}{2}\right]}\left(\beta^{-s}H\left(V\alpha,s,\left(\frac{1}{2},0\right)\right) - H\left(\alpha,s,\left(\frac{1}{2},0\right)\right)\right). \qquad (2.42)$$

We now transform (2.41) by using (2.38),

$$\begin{aligned}
&D^*(V,\alpha,s,p,q) + D_0^*(V,\alpha,s,p,q) \qquad (2.43)\\
&= \frac{2^{1-s}}{1-e[s]}\left(D_1\left(V,\alpha,s,\left(\frac{p}{2},\frac{q}{2}\right)\right) + H\left(\alpha,s,\left(\frac{p}{2},\frac{q}{2}\right)V\right) - H\left(\alpha,s,\left(\frac{p}{2},\frac{q}{2}\right)\right)\right)\\
&- \frac{2^{1-s}e\left[\frac{s}{2}\right]}{1-e[s]}\Bigg(D_1\left(V,\alpha,s,\left(-\frac{p}{2},-\frac{q}{2}\right)\right)\\
&+ H\left(\alpha,s,\left(-\frac{p}{2},-\frac{q}{2}\right)V\right) - H\left(\alpha,s,\left(-\frac{p}{2},-\frac{q}{2}\right)\right)\Bigg)\\
&+ \frac{2^{1-s}}{1-e[s]}\Bigg(D_1\left(V,\alpha,s,\left(\frac{p}{2},\frac{q}{2}+\frac{1}{2}\right)\right)\\
&+ H\left(\alpha,s,\left(\frac{p}{2},\frac{q}{2}+\frac{1}{2}\right)V\right) - H\left(\alpha,s,\left(\frac{p}{2},\frac{q}{2}+\frac{1}{2}\right)\right)\Bigg)\\
&- \frac{2^{1-s}e\left[\frac{s}{2}\right]}{1-e[s]}\Bigg(D_1\left(V,\alpha,s,\left(-\frac{p}{2},-\frac{q}{2}-\frac{1}{2}\right)\right)
\end{aligned}$$

$$+ H\left(\alpha, s, \left(-\frac{p}{2}, -\frac{q}{2} - \frac{1}{2}\right) V\right) - H\left(\alpha, s, \left(-\frac{p}{2}, -\frac{q}{2} - \frac{1}{2}\right)\right)\Bigg),$$

where

$$\begin{aligned} &D_0^*(V, \alpha, s, p, q) \qquad (2.44) \\ &= \frac{1}{1 - e[s]} \left(D_1(V, \alpha, s, (p, q)) + H(\alpha, s, (p, q)V) - H(\alpha, s, (p, q))\right) \\ &\quad - \frac{e\left[\frac{s}{2}\right]}{1 - e[s]} \left(D_1(V, \alpha, s, (-p, -q)) + H(\alpha, s, (-p, -q)V) - H(\alpha, s, (-p, -q))\right) \end{aligned}$$

in case of (2.39), while

$$\begin{aligned} D_0^*\left(V, \alpha, s, \frac{1}{2}, 0\right) &= \frac{1}{1 + e\left[\frac{s}{2}\right]} \Bigg(D_1\left(V, \alpha, s, \left(\frac{1}{2}, 0\right)\right) \qquad (2.45) \\ &+ H\left(\alpha, s, \left(\frac{1}{2}, 0\right) V\right) - H\left(\alpha, s, \left(\frac{1}{2}, 0\right)\right)\Bigg) \end{aligned}$$

in case of (2.42). Hence

$$\begin{aligned} &D^*(V, \alpha, s, p, q) + D_0^*(V, \alpha, s, p, q) \qquad (2.46) \\ &= \frac{2^{1-s}}{1 - e[s]} D_1\left(V, \alpha, s, \left(\frac{p}{2}, \frac{q}{2}\right)\right) - \frac{2^{1-s} e\left[\frac{s}{2}\right]}{1 - e[s]} D_1\left(V, \alpha, s, \left(-\frac{p}{2}, -\frac{q}{2}\right)\right) \\ &+ \frac{2^{1-s}}{1 - e[s]} D_1\left(V, \alpha, s, \left(\frac{p}{2}, \frac{q}{2} + \frac{1}{2}\right)\right) - \frac{2^{1-s} e\left[\frac{s}{2}\right]}{1 - e[s]} D_1\left(V, \alpha, s, \left(-\frac{p}{2}, -\frac{q}{2} - \frac{1}{2}\right)\right) \\ &+ \frac{2^{1-s}}{1 - e[s]} \left(H\left(\alpha, s, \left(\frac{p}{2}, \frac{q}{2}\right) V\right) - H\left(\alpha, s, \left(\frac{p}{2}, \frac{q}{2}\right)\right)\right) \\ &- \frac{2^{1-s} e\left[\frac{s}{2}\right]}{1 - e[s]} \left(H\left(\alpha, s, \left(-\frac{p}{2}, -\frac{q}{2}\right) V\right) - H\left(\alpha, s, \left(-\frac{p}{2}, -\frac{q}{2}\right)\right)\right) \\ &+ \frac{2^{1-s}}{1 - e[s]} \left(H\left(\alpha, s, \left(\frac{p}{2}, \frac{q}{2} + \frac{1}{2}\right) V\right) - H\left(\alpha, s, \left(\frac{p}{2}, \frac{q}{2} + \frac{1}{2}\right)\right)\right) \\ &- \frac{2^{1-s} e\left[\frac{s}{2}\right]}{1 - e[s]} \left(H\left(\alpha, s, \left(-\frac{p}{2}, -\frac{q}{2} - \frac{1}{2}\right) V\right) - H\left(\alpha, s, \left(-\frac{p}{2}, -\frac{q}{2} - \frac{1}{2}\right)\right)\right), \end{aligned}$$

where the last term is either (2.39) or (2.45).

2.10 Proof of Theorem [2.7.1, (2.26)]

The three identities in Theorem 2.7.1 are proved on similar lines. We begin by using (2.43) and (2.45).

$$\begin{aligned}
D^*(V_0) &= D^*\left(V_0, \alpha, s, \frac{1}{2}, 0\right) = \alpha^{-s} A^*\left(\frac{-1}{\alpha}, s, \frac{1}{2}, 0\right) - A^*\left(\alpha, s, \frac{1}{2}, 0\right) \\
&= \frac{2^{1-s}}{1-e[s]} D_1\left(V_0, \alpha, s, \left(\frac{1}{4}, 0\right)\right) + \frac{2^{1-s} e\left[\frac{s}{2}\right]}{1-e[s]} D_1\left(V_0, \alpha, s, \left(-\frac{1}{4}, 0\right)\right) \\
&+ \frac{2^{1-s}}{1-e[s]} D_1\left(V_0, \alpha, s, \left(\frac{1}{4}, \frac{1}{2}\right)\right) - \frac{2^{1-s} e\left[\frac{s}{2}\right]}{1-e[s]} D_1\left(V_0, \alpha, s, \left(-\frac{1}{4}, -\frac{1}{2}\right)\right) \\
&- \frac{1-e\left[\frac{s}{2}\right]}{1-e[s]} D_1\left(V_0, \alpha, s, \left(\frac{1}{2}, 0\right)\right) \\
&+ \frac{2^{1-s}}{1-e[s]} \left(H\left(\alpha, s, \left(\frac{1}{4}, 0\right) V_0\right) - H\left(\alpha, s, \left(\frac{1}{4}, 0\right)\right)\right) \\
&- \frac{2^{1-s}}{1-e[s]} \left\{H\left(\alpha, s, \left(\frac{1}{4}, 0\right) V_0\right) - H\left(\alpha, s, \left(-\frac{1}{4}, 0\right)\right)\right\} \\
&+ \frac{2^{1-s}}{1-e[s]} \left\{H\left(\alpha, s, \left(\frac{1}{4}, \frac{1}{2}\right) V_0\right) - H\left(\alpha, s, \left(\frac{1}{4}, \frac{1}{2}\right)\right)\right\} \\
&- \frac{2^{1-s} e\left[\frac{s}{2}\right]}{1-e[s]} \left\{H\left(\alpha, s, \left(-\frac{1}{4}, -\frac{1}{2}\right) V_0\right) - H\left(\alpha, s, \left(-\frac{1}{4}, -\frac{1}{2}\right)\right)\right\} \\
&- \frac{1-e\left[\frac{s}{2}\right]}{1-e[s]} \left\{H\left(\alpha, s, \left(\frac{1}{2}, 0\right) V_0\right) - H\left(\alpha, s, \left(\frac{1}{2}, 0\right)\right)\right\}.
\end{aligned}$$

Then applying (2.18), we deduce that

$$\begin{aligned}
D^*(V_0) = &\frac{2^{1-s}}{1-e[s]} (2\pi)^{-s} \Big\{ e\left[-\frac{s}{4}\right] L\left(\alpha, s, \left(0, -\frac{1}{4}\right), 1, 0\right) \\
&- e\left[\frac{s}{4}\right] L\left(\alpha, s, \left(0, \frac{1}{4}\right), 1, 0\right) \Big\} \\
&+ \frac{2^{1-s}}{1-e[s]} (2\pi)^{-s} \Big\{ e\left[-\frac{s}{4}\right] L\left(\alpha, s, \left(\frac{1}{2}, -\frac{1}{4}\right), 1, 0\right) \\
&- e\left[\frac{s}{4}\right] L\left(\alpha, s, \left(-\frac{1}{2}, \frac{1}{4}\right), 1, 0\right) \Big\} \\
&- \frac{1-e\left[\frac{s}{2}\right]}{1-e[s]} (2\pi)^{-s} e\left[-\frac{s}{4}\right] L\left(\alpha, s, \left(0, -\frac{1}{2}\right), 1, 0\right) \\
&- \frac{2^{1-s} e\left[\frac{s}{2}\right]}{1-e[s]} (2\pi)^{-s} e\left[-\frac{s}{4}\right] \Gamma(s) \left(\zeta\left(s, \frac{3}{4}\right) + e\left[\frac{s}{2}\right] \zeta\left(s, \frac{1}{4}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2^{1-s}}{1-e[s]}(2\pi)^{-s}e\left[-\frac{s}{4}\right]\Gamma(s)\left(\zeta\left(s,\frac{1}{4}\right)+e\left[\frac{s}{2}\right]\zeta\left(s,\frac{3}{4}\right)\right)\\
&-\frac{1-e\left[\frac{s}{2}\right]}{1-e[s]}(2\pi)^{-s}e\left[-\frac{s}{4}\right]\Gamma(s)\left(\zeta\left(s,\frac{1}{2}\right)+e\left[\frac{s}{2}\right]\zeta\left(s,\frac{1}{2}\right)\right)\\
&-\frac{1-e\left[\frac{s}{2}\right]}{1-e[s]}\left\{H\left(\alpha,s,\left(0,-\frac{1}{2}\right)\right)-H\left(\alpha,s,\left(\frac{1}{2},0\right)\right)\right\}\\
&+\frac{2^{1-s}}{1-e[s]}\left\{H\left(\alpha,s,\left(0,-\frac{1}{4}\right)\right)-H\left(\alpha,s,(\frac{1}{4},0)\right)\right\}\\
&-\frac{2^{1-s}e\left[\frac{s}{2}\right]}{1-e[s]}\left\{H\left(\alpha,s,\left(0,\frac{1}{4}\right)\right)-H\left(\alpha,s,\left(-\frac{1}{4},0\right)\right)\right\}\\
&+\frac{2^{1-s}}{1-e[s]}\left\{H\left(\alpha,s,\left(\frac{1}{2},-\frac{1}{4}\right)\right)-H\left(\alpha,s,\left(\frac{1}{4},\frac{1}{2}\right)\right)\right\}\\
&-\frac{2^{1-s}e\left[\frac{s}{2}\right]}{1-e[s]}\left\{H\left(\alpha,s,\left(-\frac{1}{2},\frac{1}{4}\right)\right)-H\left(\alpha,s,\left(-\frac{1}{4},-\frac{1}{2}\right)\right)\right\}.
\end{aligned}
$$

Let

$$
\begin{aligned}
(1-e[s])\Phi_0=&L\left(\alpha,s,\left(0,-\frac{1}{4}\right),1,0\right)+L\left(\alpha,s,\left(\frac{1}{2},-\frac{1}{4}\right),1,0\right)\\
&-e\left[\frac{s}{2}\right]L\left(\alpha,s,\left(0,\frac{1}{4}\right),1,0\right)-e\left[\frac{s}{2}\right]L\left(\alpha,s,\left(-\frac{1}{2},\frac{1}{4}\right),1,0\right)\\
&-\left(1-e\left[\frac{s}{2}\right]\right)2^{s-1}L\left(\alpha,s,\left(0,-\frac{1}{2}\right),1,0\right),
\end{aligned}
\tag{2.47}
$$

$$
\begin{aligned}
(1-e[s])\Omega_0=&\zeta\left(s,\frac{1}{4}\right)-e[s]\zeta\left(s,\frac{1}{4}\right)\\
&-2^{s-1}\left(1-e\left[\frac{s}{2}\right]\right)\left(\zeta\left(s,\frac{1}{2}\right)+e\left[\frac{s}{2}\right]\zeta\left(s,\frac{1}{2}\right)\right)
\end{aligned}
\tag{2.48}
$$

and

$$
\begin{aligned}
&(1-e[s])\Psi_0\\
=&H\left(\alpha,s,\left(0,-\frac{1}{4}\right)\right)-H\left(\alpha,s,\left(\frac{1}{4},0\right)\right)-e\left[\frac{s}{2}\right]H\left(\alpha,s,\left(0,\frac{1}{4}\right)\right)\\
&+e\left[\frac{s}{2}\right]H\left(\alpha,s,\left(-\frac{1}{4},0\right)\right)+H\left(\alpha,s,\left(\frac{1}{2},-\frac{1}{4}\right)\right)-H\left(\alpha,s,\left(\frac{1}{4},\frac{1}{2}\right)\right)\\
&-e\left[\frac{s}{2}\right]H\left(\alpha,s,\left(-\frac{1}{2},\frac{1}{4}\right)\right)+e\left[\frac{s}{2}\right]H\left(\alpha,s,\left(-\frac{1}{4},-\frac{1}{2}\right)\right)
\end{aligned}
\tag{2.49}
$$

$$- 2^{s-1}\left(1 - e\left[\frac{s}{2}\right]\right)\left\{H\left(\alpha, s, \left(0, -\frac{1}{2}\right)\right) - H\left(\alpha, s, \left(\frac{1}{2}, 0\right)\right)\right\}.$$

Now we can express the difference $D^*(V_0)$ as

$$D^*(V_0) = 2^{1-2s}\pi^{-s}e\left[-\frac{s}{4}\right](\Phi_0 + \Gamma(s)\Omega_0) + 2^{1-s}\Psi_0. \tag{2.50}$$

Using the integral representation (2.15) of $L(\alpha, s, (p', q'), c, d)$, we calculate Φ_0. Therefore,

$$\begin{aligned}(1 - e[s])\Phi_0 = &- \int_{I(\lambda,\infty)} t^{s-1}\frac{\exp(-(\frac{1}{4}+\alpha)t) + \exp(-(\frac{1}{4}+\frac{1}{2}\alpha)t)}{(1-\exp(-t))(1-\exp(-\alpha t))}\,\mathrm{d}t \\ &+ e\left[\frac{s}{2}\right]\int_{I(\lambda,\infty)} t^{s-1}\frac{\exp(-(\frac{3}{4}+\alpha)t) + \exp(-(\frac{3}{4}+\frac{1}{2}\alpha)t)}{(1-\exp(-t))(1-\exp(-\alpha t))}\,\mathrm{d}t \\ &+ 2^{s-1}\left(1 - e\left[\frac{s}{2}\right]\right)\int_{I(\lambda,\infty)} t^{s-1}\frac{\exp(-(\frac{1}{2}+\alpha)t)}{(1-\exp(-t))(1-\exp(-\alpha t))}\,\mathrm{d}t.\end{aligned}$$

Combining the first two integrals, we have

$$\begin{aligned}&(1 - e[s])\Phi_0 \\ &\quad = I_0 + 2^{s-1}\left(1 - e\left[\frac{s}{2}\right]\right)\int_{I(\lambda,\infty)} t^{s-1}\frac{\exp(-(\frac{1}{2}+\alpha)t)}{(1-\exp(-t))(1-\exp(-\alpha t))}\,\mathrm{d}t,\end{aligned}$$

where

$$I_0 = \int_{I(\lambda,\infty)} t^{s-1}\frac{-\exp(-\frac{1}{4}t)\left(1 - \exp\left(\pi i s - \frac{1}{2}t\right)\right)\left(\exp(-\alpha t) + \exp(-\frac{1}{2}\alpha t)\right)}{(1-\exp(-t))(1-\exp(-\alpha t))}\,\mathrm{d}t.$$

Now making the change of variable $t \leftrightarrow 2t$, we get

$$I_0 = 2^s\int_{I(\lambda,\infty)} t^{s-1}\frac{-\exp(-\frac{1}{2}t)\,(1 - \exp(\pi i s - t))\,(\exp(-2\alpha t) + \exp(-\alpha t))}{(1+\exp(-t))(1-\exp(-t))(1+\exp(-\alpha t)(1-\exp(-\alpha t))}\,\mathrm{d}t.$$

Hence after eliminating the common factor, we arrive at

$$(1 - e[s])\Phi_0 = 2^s\int_{I(\lambda,\infty)} t^{s-1}\frac{\exp(-\frac{1}{2}t)\,(-1 + \exp(\pi i s - t))\exp(-\alpha t)}{(1+\exp(-t))(1-\exp(-t))(1-\exp(-\alpha t))}\,\mathrm{d}t$$

$$+ 2^{s-1}\left(1 - e\left[\frac{s}{2}\right]\right)\int_{I(\lambda,\infty)} t^{s-1}\frac{\exp(-(\frac{1}{2}+\alpha)t)}{(1-\exp(-t))(1-\exp(-\alpha t))}\,\mathrm{d}t.$$

Therefore,

$$\Phi_0 \; = -\frac{2^{s-1}}{1-e\left[\frac{s}{2}\right]}\int_{I(\lambda,\infty)} t^{s-1}\frac{\exp(-(\frac{1}{2}+\alpha)t)}{(1+\exp(-t))(1-\exp(-\alpha t))}\,\mathrm{d}t. \tag{2.51}$$

Our next target is to calculate Ψ_0. Using (2.12), we have

$$\begin{aligned}\Psi_0 =& \eta\left(\alpha, s, 1, -\frac{1}{4}\right) - \eta\left(\alpha, s, \frac{1}{4}, 0\right) + \eta\left(\alpha, s, \frac{1}{2}, -\frac{1}{4}\right) \\ & - \eta\left(\alpha, s, \frac{1}{4}, \frac{1}{2}\right) - 2^s\left\{\eta\left(\alpha, s, 1, \frac{1}{2}\right) - \eta\left(\alpha, s, \frac{1}{2}, 0\right)\right\}.\end{aligned} \tag{2.52}$$

Now using the definition of the η-function, we get

$$\begin{aligned}\Psi_0 =& \sum_{n=1}^{\infty} n^{s-1}\frac{e\left[n\left(\alpha-\frac{1}{4}\right)\right]}{1-e[n\alpha]} + \sum_{n=1}^{\infty} n^{s-1}\frac{e\left[n\left(\frac{1}{2}\alpha-\frac{1}{4}\right)\right]}{1-e[n\alpha]} \\ & - \sum_{n=1}^{\infty} n^{s-1}\frac{e\left[\frac{1}{4}n\alpha\right]}{1-e[n\alpha]} - \sum_{n=1}^{\infty} n^{s-1}\frac{e\left[n\left(\frac{1}{4}\alpha+\frac{1}{2}\right)\right]}{1-e[n\alpha]} \\ & - 2^s\sum_{n=1}^{\infty} n^{s-1}\frac{e\left[n\left(\alpha+\frac{1}{2}\right)\right]}{1-e[n\alpha]} + 2^s\sum_{n=1}^{\infty} n^{s-1}\frac{e\left[\frac{1}{2}n\alpha\right]}{1-e[n\alpha]}\end{aligned}$$

$$\begin{aligned}&= \sum_{n=1}^{\infty} n^{s-1}\frac{e\left[n\left(\frac{1}{2}\alpha-\frac{1}{4}\right)\right]\left(e\left[\frac{1}{2}n\alpha\right]+1\right)}{1-e[n\alpha]} \\ &- 2\sum_{n=1}^{\infty}(2n)^{s-1}\frac{e\left[2n\left(\frac{1}{2}\alpha-\frac{1}{4}\right)\right](1+e[n\alpha])}{1-e[2n\alpha]} \\ &- \sum_{n=1}^{\infty} n^{s-1}\frac{e\left[\frac{1}{4}n\alpha\right](1+(-1)^n)}{1-e[n\alpha]} + 2\sum_{n=1}^{\infty}(2n)^{s-1}\frac{e\left[\frac{1}{4}\cdot 2n\alpha\right](1+e[n\alpha])}{1-e[2n\alpha]}. \\ &= -\sum_{n=1}^{\infty} n^{s-1}\frac{e\left[n\left(\frac{1}{2}\alpha+\frac{1}{4}\right)\right]\left(e\left[\frac{1}{2}n\alpha\right]+1\right)}{1-e[n\alpha]} \\ &+ 2\sum_{n=1}^{\infty}(2n)^{s-1}\frac{e\left[\frac{3}{2}\cdot n\alpha\right]}{1-e[2n\alpha]}.\end{aligned}$$

To calculate Ω_0, we use $2^s\zeta\left(s,\frac{1}{2}\right)=\zeta\left(s,\frac{1}{4}\right)+\zeta\left(s,\frac{3}{4}\right)$, and we get

$$\begin{aligned}\Omega_0 &= \left(\zeta\left(s,\frac{1}{4}\right)-2^{s-1}\zeta\left(s,\frac{1}{2}\right)\right) \\ &= 2^{-1}\left(\zeta\left(s,\frac{1}{4}\right)-\zeta\left(s,\frac{3}{4}\right)\right).\end{aligned}\tag{2.53}$$

Finally, combining the expressions for Φ_0, Ψ_0, and Ω_0 we deduce Theorem [2.7.1, (2.26)].

2.11 Proof of Theorem [2.7.1, (2.27)]

By using Proposition 2.8.4 and from (2.19), we have

$$\begin{aligned}
&D^*(V_1)=D^*\left(V_1,\alpha,s,\frac{1}{2},0\right) \\
&=(\alpha+1)^{-s}A^*\left(\frac{\alpha}{\alpha+1},s,\frac{1}{2},0\right)-A^*\left(\alpha,s,\frac{1}{2},0\right) \\
&=\frac{2^{1-s}}{1-e[s]}(2\pi)^{-s} \\
&\quad\times\left\{e\left[-\frac{s}{4}\right]L\left(\alpha,s,\left(\frac{1}{4},0\right),1,1\right)-e\left[\frac{s}{4}\right]L\left(\alpha,s,\left(-\frac{1}{4},0\right),1,1\right)\right\} \\
&+\frac{2^{1-s}}{1-e[s]}(2\pi)^{-s} \\
&\quad\times\left\{e\left[-\frac{s}{4}\right]L\left(\alpha,s,\left(\frac{3}{4},\frac{1}{2}\right),1,1\right)-e\left[\frac{s}{4}\right]L\left(\alpha,s,\left(-\frac{3}{4},-\frac{1}{2}\right),1,1\right)\right\} \\
&+\frac{2^{1-s}}{1-e[s]}\left\{H\left(\alpha,s,\left(\frac{3}{4},\frac{1}{2}\right)\right)-H\left(\alpha,s,\left(\frac{1}{4},\frac{1}{2}\right)\right)\right\} \\
&-\frac{2^{1-s}e\left[\frac{s}{2}\right]}{1-e[s]}\left\{H\left(\alpha,s,\left(-\frac{3}{4},-\frac{1}{2}\right)\right)-H\left(\alpha,s,\left(-\frac{1}{4},-\frac{1}{2}\right)\right)\right\} \\
&-\frac{1-e\left[\frac{s}{2}\right]}{1-e[s]}(2\pi)^{-s}e\left[-\frac{s}{4}\right]L\left(\alpha,s,\left(\frac{1}{2},0\right),1,1\right).
\end{aligned}\tag{2.54}$$

Let

$$\begin{aligned}
(1-e[s])\Phi_1 &= L\left(\alpha,s,\left(\frac{1}{4},0\right),1,1\right)+L\left(\alpha,s,\left(\frac{3}{4},\frac{1}{2}\right),1,1\right) \\
&-e\left[\frac{s}{2}\right]L\left(\alpha,s,\left(-\frac{1}{4},0\right),1,1\right)-e\left[\frac{s}{2}\right]L\left(\alpha,s,\left(-\frac{3}{4},-\frac{1}{2}\right),1,1\right)
\end{aligned}\tag{2.55}$$

$$-\left(1-e\left[\frac{s}{2}\right]\right)2^{s-1}L\left(\alpha,s,\left(\frac{1}{2},0\right),1,1\right).$$

and

$$(1-e[s])\Psi_1 = H\left(\alpha,s,\left(\frac{3}{4},\frac{1}{2}\right)\right)-H\left(\alpha,s,\left(\frac{1}{4},\frac{1}{2}\right)\right) \tag{2.56}$$
$$-e\left[\frac{s}{2}\right]H\left(\alpha,s,\left(-\frac{3}{4},-\frac{1}{2}\right)\right)+e\left[\frac{s}{2}\right]H\left(\alpha,s,\left(-\frac{1}{4},-\frac{1}{2}\right)\right).$$

We now express (2.54) as

$$D^*(V_1)=2^{1-2s}\pi^{-s}e\left[-\frac{s}{4}\right]\Phi_1+2^{1-s}\Psi_1.$$

Now utilizing the integral representation (2.15) of $L(\alpha,s,(p',q'),c,d)$, we have

$$\begin{aligned}
&(1-e[s])\Phi_1\\
&=-\int_{I(\lambda,\infty)}t^{s-1}\frac{\exp(-(\frac{1}{4}+\frac{3}{4}(\alpha+1))t)+\exp(-(\frac{1}{4}+\frac{1}{4}(\alpha+1))t)}{(1-\exp(-t))(1-\exp(-(\alpha+1)t))}\,\mathrm{d}t\\
&+e\left[\frac{s}{2}\right]\int_{I(\lambda,\infty)}t^{s-1}\frac{\exp(-(\frac{3}{4}+\frac{1}{4}(\alpha+1))t)+\exp(-(\frac{3}{4}+\frac{3}{4}(\alpha+1))t)}{(1-\exp(-t))(1-\exp(-(\alpha+1)t))}\,\mathrm{d}t\\
&+2^{s-1}\left(1-e\left[\frac{s}{2}\right]\right)\int_{I(\lambda,\infty)}t^{s-1}\frac{\exp(-(\frac{1}{2}+\frac{1}{2}(\alpha+1))t)}{(1-\exp(-t))(1-\exp(-(\alpha+1)t))}\,\mathrm{d}t.
\end{aligned}$$

Again, we write the left hand side of the above equation as

$$(1-e[s])\Phi_1=I_1 \tag{2.57}$$
$$+2^{s-1}\left(1-e\left[\frac{s}{2}\right]\right)\int_{I(\lambda,\infty)}t^{s-1}\frac{\exp(-(\frac{1}{2}+\frac{1}{2}(\alpha+1))t)}{(1-\exp(-t))(1-\exp(-(\alpha+1)t))}\,\mathrm{d}t,$$

where

$$I_1=\int_{I(\lambda,\infty)}t^{s-1}$$
$$\frac{\left(-\exp(-\frac{1}{4}t)+\exp(\pi is-\frac{3}{4}t)\right)\left(\exp(-\frac{1}{4}(\alpha+1)t)+\exp(-\frac{3}{4}(\alpha+1)t)\right)}{(1-\exp(-t))(1-\exp(-(\alpha+1)t))}\,\mathrm{d}t.$$

Now, by change of variable $t \leftrightarrow 2t$ followed by the elimination of the common factor $1 + \exp(-(\alpha+1)t)$, we get

$$I_1 = 2^s \int_{I(\lambda,\infty)} t^{s-1} \frac{\exp(-\frac{1}{2}t)\,(-1+\exp(\pi i s - t))\exp(-\frac{1}{2}(\alpha+1)t)}{(1+\exp(-t))(1-\exp(-t))(1-\exp(-(\alpha+1)t))}\,\mathrm{d}t.$$

Thus, substituting I_1 in (2.57), we see that

$$\Phi_1 = -\frac{2^{s-1}}{1-e\left[\frac{s}{2}\right]} \int_{I(\lambda,\infty)} t^{s-1} \frac{\exp(-(\frac{1}{2}+\frac{1}{2}(\alpha+1))t)}{(1+\exp(-t))(1-\exp(-(\alpha+1)t))}\,\mathrm{d}t. \tag{2.58}$$

Using the definition of H-function, from (2.56), we have

$$\begin{aligned}\Psi_1 &= \eta\left(\alpha, s, \frac{3}{4}, \frac{1}{2}\right) - \eta\left(\alpha, s, \frac{1}{4}, \frac{1}{2}\right)\\ &= \sum_{n=1}^{\infty} n^{s-1} \frac{e\left[n\left(\frac{3}{4}\alpha+\frac{1}{2}\right)\right]}{1-e[n\alpha]} - \sum_{n=1}^{\infty} n^{s-1} \frac{e\left[n\left(\frac{1}{4}\alpha+\frac{1}{2}\right)\right]}{1-e[n\alpha]}.\end{aligned}$$

The nth summand is

$$n^{s-1} \frac{e\left[n\left(\frac{1}{4}\alpha+\frac{1}{2}\right)\right]\left(e\left[\frac{1}{2}n\alpha\right]-1\right)}{1-e[n\alpha]}$$

from which we may eliminate the common factor $e\left[\frac{1}{2}n\alpha\right]-1$. Therefore,

$$\Psi_1 = \sum_{n=1}^{\infty} n^{s-1} \frac{(-1)^{n-1} e\left[\frac{1}{4}n\alpha\right]}{e\left[\frac{1}{2}n\alpha\right]+1} = \frac{1}{2}\sum_{n=1}^{\infty} n^{s-1} \frac{(-1)^{n-1}}{\cos\frac{\pi n}{2}\alpha}. \tag{2.59}$$

Now we substitute (2.58) and (2.59) in (2.54), and finally get

$$\begin{aligned}&(\alpha+1)^{-s} A^*\left(\frac{\alpha}{\alpha+1}, s, \frac{1}{2}, 0\right) - A^*\left(\alpha, s, \frac{1}{2}, 0\right)\\ &= -\frac{(2\pi)^{-s} e\left[-\frac{s}{4}\right]}{\left(1-e\left[\frac{s}{2}\right]\right)} \int_{I(\lambda,\infty)} t^{s-1} \frac{\exp(-\frac{1}{2}t)}{(1+\exp(-t))} \frac{\exp(-\frac{1}{2}(\alpha+1)t)}{(1-\exp(-(\alpha+1)t))}\,\mathrm{d}t\\ &\quad + 2^{-s} \sum_{n=1}^{\infty} n^{s-1} \frac{(-1)^{n-1}}{\cos\frac{\pi n}{2}\alpha}.\end{aligned} \tag{2.60}$$

This completes the proof of Theorem [2.7.1, (2.27)].

2.12 Proof of Theorem [2.7.1,(2.28)]

We follow the same route, first we use Proposition 2.8.4 and then using (2.20), we obtain

$$
\begin{aligned}
&D^*(V_2) = D^*\left(V_2, \alpha, s, \frac{1}{2}, 0\right)\\
&= (\alpha-1)^{-s} A^*\left(\frac{-\alpha}{\alpha-1}, s, \frac{1}{2}, 0\right) - A^*\left(\alpha, s, \frac{1}{2}, 0\right)\\
&= \frac{2^{1-s}}{1-e[s]}(2\pi)^{-s}\left\{e\left[-\frac{s}{4}\right] L\left(\alpha, s, \left(-\frac{1}{4}, 0\right), 1, -1\right) - e\left[\frac{s}{4}\right] L(\alpha, s, (\frac{1}{4}, 0), 1, -1)\right\}\\
&+ \frac{2^{1-s}}{1-e[s]}(2\pi)^{-s}\left\{e\left[-\frac{s}{4}\right] L\left(\alpha, s, \left(\frac{1}{4}, -\frac{1}{2}\right), 1, -1\right) - e\left[\frac{s}{4}\right] L\left(\alpha, s, \left(-\frac{1}{4}, \frac{1}{2}\right), 1, -1\right)\right\}\\
&- \frac{1-e\left[\frac{s}{2}\right]}{1-e[s]}(2\pi)^{-s} e\left[-\frac{s}{4}\right] L\left(\alpha, s, \left(-\frac{1}{2}, 0\right), 1, -1\right)\\
&+ \frac{2^{1-s}}{1-e[s]}\left\{H\left(\alpha, s, \left(-\frac{1}{4}, 0\right)\right) - H\left(\alpha, s, \left(\frac{1}{4}, 0\right)\right)\right\}\\
&- \frac{2^{1-s} e\left[\frac{s}{2}\right]}{1-e[s]}\left\{H\left(\alpha, s, \left(\frac{1}{4}, 0\right)\right) - H\left(\alpha, s, \left(-\frac{1}{4}, 0\right)\right)\right\}\\
&= 2^{1-2s}\pi^{-s} e\left[-\frac{s}{4}\right]\Phi_2 + 2^{1-s}\Psi_2,
\end{aligned}
$$

where

$$
\begin{aligned}
(1-e[s])\Phi_2 = {}& L\left(\alpha, s, \left(-\frac{1}{4}, 0\right), 1, -1\right) - e\left[\frac{s}{2}\right] L\left(\alpha, s, \left(\frac{1}{4}, 0\right), 1, -1\right)\\
&+ L\left(\alpha, s, \left(\frac{1}{4}, -\frac{1}{2}\right), 1, -1\right) - e\left[\frac{s}{2}\right] L\left(\alpha, s, \left(-\frac{1}{4}, \frac{1}{2}\right), 1, -1\right)\\
&- \left(1 - e\left[\frac{s}{2}\right]\right) 2^{s-1} L\left(\alpha, s, \left(-\frac{1}{2}, 0\right), 1, -1\right),
\end{aligned}
\tag{2.61}
$$

and

$$
\begin{aligned}
(1-e[s])\Psi_2 = {}& H\left(\alpha, s, \left(-\frac{1}{4}, 0\right)\right) - H\left(\alpha, s, \left(\frac{1}{4}, 0\right)\right)\\
&- e\left[\frac{s}{2}\right] H\left(\alpha, s, \left(\frac{1}{4}, 0\right)\right) + e\left[\frac{s}{2}\right] H\left(\alpha, s, \left(-\frac{1}{4}, 0\right)\right)\\
&= \left(1 + e\left[\frac{s}{2}\right]\right)\left(H\left(\alpha, s, \left(-\frac{1}{4}, 0\right)\right) - H\left(\alpha, s, \left(\frac{1}{4}, 0\right)\right)\right).
\end{aligned}
\tag{2.62}
$$

To simplify Φ_2, we make use of the integral representation (2.15) of $L(\alpha, s, (p', q'), c, d)$. So we have

$$(1 - e[s])\Phi_2 = I_2$$
$$+ 2^{s-1}\left(1 - e\left[\frac{s}{2}\right]\right)\int_{I(\lambda,\infty)} t^{s-1}\frac{\exp(-(\frac{1}{2}+\frac{1}{2}(\alpha-1))t)}{(1-\exp(-t))(1-\exp(-(\alpha-1)t))}\,\mathrm{d}t,$$

where

$$I_2 = \int_{I(\lambda,\infty)} t^{s-1}$$
$$\frac{\left(-\exp(-\frac{1}{4}t) + \exp(\pi i s - \frac{3}{4}t)\right)\left(\exp(-\frac{1}{4}(\alpha-1)t) + \exp(-\frac{3}{4}(\alpha-1)t)\right)}{(1-\exp(-t))(1-\exp(-(\alpha-1)t))}\,\mathrm{d}t.$$

As before, by eliminating the common factor $1 - \exp(-t)$, we obtain

$$I_2 = 2^s\int_{I(\lambda,\infty)} t^{s-1}\frac{\exp(-\frac{1}{2}t)\exp(-\frac{1}{2}(\alpha-1)t)(-1+\exp(\pi i s - t))}{(1+\exp(-t))(1-\exp(-t))(1-\exp(-(\alpha-1)t)}\,\mathrm{d}t.$$

Whence it follows that

$$\Phi_2 = -\frac{2^{s-1}}{1-e\left[\frac{s}{2}\right]}\int_{I(\lambda,\infty)} t^{s-1}\frac{\exp(-(\frac{1}{2}+\frac{1}{2}(\alpha-1))t)}{(1+\exp(-t))(1-\exp(-(\alpha-1)t))}\,\mathrm{d}t. \tag{2.63}$$

While handling (2.62), we decompose it as

$$\Psi_2 = \eta\left(\alpha, s, \frac{3}{4}, 0\right) - \eta\left(\alpha, s, \frac{1}{4}, 0\right).$$

In the series expression of Ψ_2 we factor out $e\left[\frac{1}{4}n\alpha\right]$ as before and eliminate the common factor $\left(e\left[\frac{1}{2}n\alpha\right] - 1\right)$ to obtain

$$\Psi_2 = -\sum_{n=1}^{\infty} n^{s-1}\frac{e\left[\frac{1}{4}n\alpha\right]}{e\left[\frac{1}{2}n\alpha\right]+1} = -\frac{1}{2}\sum_{n=1}^{\infty} n^{s-1}\frac{1}{\cos\frac{\pi n}{2}\alpha}. \tag{2.64}$$

Finally substituting the expressions for Φ_2 and Ψ_2, we have

$$(\alpha-1)^{-s}A^*\left(\frac{-\alpha}{\alpha-1}, s, \frac{1}{2}, 0\right) - A^*\left(\alpha, s, \frac{1}{2}, 0\right) \tag{2.65}$$
$$= -\frac{(2\pi)^{-s}e\left[-\frac{s}{4}\right]}{1-e\left[\frac{s}{2}\right]}\int_{I(\lambda,\infty)} t^{s-1}\frac{\exp(-\frac{1}{2}t)}{(1+\exp(-t))}\frac{\exp(-\frac{1}{2}(\alpha-1)t)}{(1-\exp(-(\alpha-1)t)}\,\mathrm{d}t$$

$$- 2^{-s} \sum_{n=1}^{\infty} n^{s-1} \frac{1}{\cos \frac{\pi n}{2} \alpha}.$$

This finishes the proof of Theorem [2.7.1, (2.28)].

2.13 Proof of Corollary 2.7.2

We conclude this chapter by finally proving Corollary 2.7.2. We add (2.60) and (2.65) and derive that

$$\begin{aligned}
&(\alpha+1)^{-s} A^* \left(\frac{\alpha}{\alpha+1}, s, \frac{1}{2}, 0 \right) + (\alpha-1)^{-s} A^* \left(\frac{-\alpha}{\alpha-1}, s, \frac{1}{2}, 0 \right) \\
&- 2A^* \left(\alpha, s, \frac{1}{2}, 0 \right) \\
&= - \frac{(2\pi)^{-s} e\left[-\frac{s}{4}\right]}{1 - e\left[\frac{s}{2}\right]} \int_{I(\lambda,\infty)} t^{s-1} \frac{\exp(-\frac{1}{2}t)}{(1+\exp(-t))} \frac{\exp(-\frac{1}{2}(\alpha+1)t)}{(1-\exp(-(\alpha+1)t))} \,\mathrm{d}t \\
&- \frac{(2\pi)^{-s} e\left[-\frac{s}{4}\right]}{1 - e\left[\frac{s}{2}\right]} \int_{I(\lambda,\infty)} t^{s-1} \frac{\exp(-\frac{1}{2}t)}{(1+\exp(-t))} \frac{\exp(-\frac{1}{2}(\alpha-1)t)}{(1-\exp(-(\alpha-1)t))} \,\mathrm{d}t \\
&+ 2^{-s} \sum_{n=1}^{\infty} n^{s-1} \frac{(-1)^{n-1}}{\cos \frac{\pi n}{2} \alpha} - 2^{-s} \sum_{n=1}^{\infty} n^{s-1} \frac{1}{\cos \frac{\pi n}{2} \alpha} \\
&= - \frac{(2\pi)^{-s} e\left[-\frac{s}{4}\right]}{1 - e\left[\frac{s}{2}\right]} \int_{I(\lambda,\infty)} t^{s-1} \frac{\exp(-\frac{1}{2}t)}{1+\exp(-t)} \left\{ \frac{\exp(-\frac{1}{2}(\alpha+1)t)}{1-\exp(-(\alpha+1)t)} \right. \\
&\left. + \frac{\exp(-\frac{1}{2}(\alpha-1)t)}{1-\exp(-(\alpha-1)t)} \right\} \mathrm{d}t - 2 \cdot 2^{-s} \sum_{n=1}^{\infty} (2n)^{s-1} \frac{1}{\cos \frac{\pi 2n}{2} \alpha}.
\end{aligned}$$

Now in the above expression $2A^*\left(\alpha, s, \frac{1}{2}, 0\right)$ on the left hand side and secant zeta function on the right hand side will cancel each other, as they are the same (from (2.23)). Therefore, we have

$$\begin{aligned}
&(\alpha+1)^{-s} A^* \left(\frac{\alpha}{\alpha+1}, s, \frac{1}{2}, 0 \right) + (\alpha-1)^{-s} A^* \left(\frac{-\alpha}{\alpha-1}, s, \frac{1}{2}, 0 \right) \\
&= - \frac{(2\pi)^{-s} e\left[-\frac{s}{4}\right]}{1 - e\left[\frac{s}{2}\right]} \int_{I(\lambda,\infty)} t^{s-1} \frac{\exp(-\frac{1}{2}t)}{1+\exp(-t)} \left\{ \frac{\exp(-\frac{1}{2}(\alpha+1)t)}{1-\exp(-(\alpha+1)t)} \right. \\
&\left. + \frac{\exp(-\frac{1}{2}(\alpha-1)t)}{1-\exp(-(\alpha-1)t)} \right\} \mathrm{d}t
\end{aligned} \tag{2.66}$$

$$= -\frac{(2\pi)^{-s} e\left[-\frac{s}{4}\right]}{1 - e\left[\frac{s}{2}\right]} \int_{I(\lambda,\infty)} t^{s-1} \sum_{m=0}^{\infty} E_m\left(\frac{1}{2}\right) \frac{t^m}{2m!}$$
$$\times \left\{ \sum_{n=0}^{\infty} B_n\left(\frac{1}{2}\right) \frac{\{(\alpha+1)^{n-1} + (\alpha-1)^{n-1}\} t^{n-1}}{n!} \right\} \mathrm{d}t,$$

and thus the Corollary 2.7.2 follows.

2.14 Future work

Our ongoing project is to derive the general modular transformation formula for $A^*(\alpha, s, p, q)$ for all $(p, q) \in \mathbb{R}^2$, and from which we would like to see the truth of our conjecture.

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