SOME RESULTS RELATED TO RIESZ SUM AND K-FREE INTEGERS

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DECLARATION

I hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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List of Publications arising from the thesis

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1. "Linearized Product of two Riemann zeta functions", Debika Banerjee and Jay Mehta, *Proceedings of The Japan Academy, Series A, Mathematical Sciences*, **2014**, *Vol. 90, Ser. A, No 8*, 19-24.

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Dedicated to my Parents

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Abstract

The thesis is divided into three chapters, a brief description of each chapter is given below. The first two chapters deal with the results related to Riesz sum and last chapter comprises of results concerning k-free integers.

The first chapter is based on a result related to Riesz Sum. In a joint work with Jay Mehta, we have given another proof of well-known Wilton's formula to obtain the linearized product of two Riemann zeta functions.

In the second chapter, a brief review of Dirichlet divisor problem starting with its definition, history and present development are given. In this chapter, we studied a new kind of the divisor problem related to the nth co-efficient of the square of the derived zeta function. In a joint work with Makoto Minamide, we have derived the asymptotic formula for Reisz sum of order one for this new kind of the divisor function. As a result, we have deduced an asymptotic formula for the continuous mean value of error term. Also we have investigated the difference of continuous and discrete mean value of error term of order one and two respectively using Furuya's result.

In the last chapter, the definition of square-free as well as k-free integers, their distribution, density and their application are introduced. Then a few results related to square free integers are described which is a joint work with Makoto Minamade. The last part of the chapter, deals with problems related to non k-free integers.

Contents

Synopsis

0.1 Introduction

One may define analytic number theory as the branch of mathematics that deals with the analytic methods to solve number-theoritic problem. That is why it becomes constant source of inspiration for many number theorists. By analytic methods, we mean a clever manipulation of sums, series and integrals, error analysis, Fourier series and transform, contour integration and residues. Some analytic number theorists also uses advanced tools such as modular form and Laplacian spectral theory. There are many methods adopted by them to solve problems such as Sieve Method, Circle method and L-functions (study of their properties). In present days, methods developed in analytic number theory have helped to solve many important problems in other fields. There are various problems in number theory: the very first and main theorem which is the main motivation of studying this subject is 'The Prime Number Theorem' which describes the asymptotic distribution of prime numbers, Goldbach Conjecture-the most oldest unsolved problem till now which states that even integers greater than two can be expressed as sum of two primes, Waring problem, Riemann hypothesis and many more. Proceeding towards the history, the credit goes to

Leonhard Euler who first used analytical techniques for studying properties of integers. Euler's used divergence of the zeta function to prove the existence of infinite number of primes. That was the first step which initiated the study of analytic number theory. P.G.L Dirichlet had tremendous contribution in this regard whose theory of L-functions for characters resulted the proof of infinity of prime numbers in arithmetic progression.

However, in the last few decades there has been efforts and various tools have been developed to enhance our understanding of analytic number theory. This thesis mainly focuses on particular classes of problems in analytic number theory. In this short introduction, we will try to give a glimpse of work done in this thesis.

The theme of the first chapter is the linear representations of the product of two zeta functions. A brief description is given in 0.3. The second chapter is based on the average behaviour of a new kind of divisor function. 0.4 contains a short summary of this part. The third chapter is devoted to some problems concerning k-free integers. 0.5 gives a brief view of this part.

0.2 A brief introduction to Riesz sum

Riesz sum was first introduced by M. Riesz which has been used extensively in connection with summability of Dirichlet series. Let $\{a_n\}$ be a sequence of complex numbers and $\{\lambda_k\}$ be a sequence of real numbers, then Riesz sum of order κ is defined as the following:

$$\begin{aligned} A_\lambda^\kappa(x) &= {\sum_{\lambda_k \leq x}}' (x-\lambda_k)^\kappa a_k \\ &= \kappa \int_0^x (x-t)^{\kappa-1} A_\lambda(t) dt \end{aligned}$$

where $A_\lambda(x) = \sum_{\lambda_k \le x} a_k$ and the prime on the summation sign means that the corresponding term is to be halved when $\lambda_k = x$.

0.3 Linearized product of two Riemann zeta functions

In 1929, Wilton [W2] gave an approximate functional equation for the product of two zeta functions in the critical regions which is analogous to the approximate functional equation for $\zeta^2(s)$ obtained by Hardy and Littlewood [HL2]. In course of proving the above, he obtained the following

Theorem 0.3.1 *For $Re(u) > -1$, $Re(v) > -1$, $Re(u+v) > 0$ and $u+v \neq 2$, we have*

$$\begin{aligned} &\zeta(u)\zeta(v) \\ &= \zeta(u+v-1)\left(\frac{1}{u-1}+\frac{1}{v-1}\right) \\ &\quad + 2(2\pi)^{u-1}\sum_{n=1}^{\infty}\sigma_{1-u-v}(n)n^{u-1}u\int_{2\pi n}^{\infty}x^{-u-1}\sin x\,\mathrm{d}x \\ &\quad + 2(2\pi)^{v-1}\sum_{n=1}^{\infty}\sigma_{1-u-v}(n)n^{v-1}v\int_{2\pi n}^{\infty}x^{-v-1}\sin x\,\mathrm{d}x \end{aligned} \tag{1}$$

Wilton obtained the above by combining a result in [HL1] with a result in [W1]. In 2003, M. Nakajima [N] proved this well-known Wilton's formula with the help of Atkinson dissection. In a joint work with Jay Mehta in [BM4], we gave an alternative proof by considering Riesz sum of the order one and using a property of incomplete gamma function. In the first chapter, the proof of the above theorem is given elaborately.

0.4 The average behaviour of the error term in a new kind of the divisor problem

A long standing unsolved problem in number theory is the Dirichlet Divisor Problem. We first begin with the definition of arithmetical function $d(n)$ which counts the number of divisors of n. Also another way of defining it is the nth coefficient of the square of zeta function. The asymptotic formula for $\sum_{n\leq x} d(n)$ is

$$D(x) = \sum_{n\leq x} d(n) = x\log x + (2\gamma - 1)x + \Delta(x), \tag{2}$$

where $\Delta(x)$ denotes the error term. From number theoretic point of view it is of great importance to estimate the error term and in literature estimating the error term $\Delta(x)$ is known as *Dirichlet Divisor problem*. First attempt was made by Dirichlet himself who proved that $\Delta(x) = O(x^{\frac{1}{2}})$ and later it was conjectured that $\Delta(x) \ll x^{\frac{1}{4}+\varepsilon}$ (for any $\varepsilon > 0$) based on mean value considerations. Since then many mathematicians made an attempt to improve the bound. In 1904, G. Voronoi [V] made the first attempt and showed that error term can be improved to $O\left(x^{1/3}\log x\right)$. The proof was based on interpolation and is nearly forty pages long. As a result in 1917, it was considered to be one of the deepest in the analytic theory of numbers by Hardy and Ramanujan [HR1]. In 1922, J.G.Van der Corput [Co1] showed that the error term is of order $O(x^{\frac{33}{100}})$. His proof required estimates for exponential sums. After six years in 1928, he [Co2] improved it to $O(x^{\frac{27}{82}})$. Later in 1969, Kolesnik [K] improved it to $O\left(x^{12/37}\right)$. These results lead to the study of continuous and discrete mean value of $\Delta(x)$. Voronoi is the first one to work in this direction. Further Dixon and Ferrar [DF] made a remarkable contribution in this regard. First they estimated the asymptotic formula for weighted divisor sum that is Riesz sum of the type $\sum_{n\leq x}(x-n)^{\alpha-1}d(n)$ for $\alpha > 2$ and the established the formula for $\alpha = 2$ by analytic continuation argument. Using this they derived an asymptotic formual for $\int_1^X \Delta(x)dx$. Also continuous and discrete mean values have been studied for higher powers of $\Delta(x)$. Hardy [Har] obtained some results in this direction. Moreover, Tong [To], Preissmann [P], Lau and Tsang [LT] have several contributions in this regard. The higher-power moments of $\Delta(x)$ was also studied by Zhai [Z1], [Z2], Ivić and Sargos [IS].

Inspired by above, a new kind of divisor problem has been studied by Makoto Minamide in [M]. He investigated the nth coefficient of square of derived zeta function and it turns out to be $\sum_{d|n}(-\log d)(-\log \frac{n}{d})$ which he denoted by $D_{(1)}(n)$. In his paper, he obtained an asymptotic formula $\sum_{n\leq x} D_{(1)}(n)$. Just like Dirichlet divisor problem he used the notation $\Delta_{(1)}(x)$ to denote the error term in the asymptotic formula. He also studied $\int_1^X \Delta_{(1)}^2(x)dx$. So naturally one can ask the following two questions

Question 1. Is it possible to calculate $\int_1^X \Delta_{(1)}(x)dx$?

The answer is affirmative. The calculation is given elaborately in chapter 1.

Question 2. Do $\int_1^X \Delta_{(1)}(x)dx$ and $\sum_{n\leq X} \Delta_{(1)}(n)$ behave differently for large values of X ?

An affirmative answer to the above question is obtained by our first result given below. Similar is the case with $\int_1^X \Delta_{(1)}^2(x)dx$ and $\sum_{n\leq X} \Delta_{(1)}^2(x)dx$. Thus in a joint work with Makoto Minamide in [BM3], we established the relationship between the discrete and continuous mean values of the error term.

Proceeding as in the line of Dixon and Ferrar in [DF], we obtained the following result:

Theorem 0.4.1 *Keeping the notations as above, let ε be a small positive con-*

stant. Then we have

$$\int_1^X \Delta_{(1)}(x)\mathrm{dx} = \frac{\pi X^2}{2}\sum_{n=1}^{\infty} d(n)\lambda_2(4\pi\sqrt{nX})\log^2 nX + 2\pi X^2\sum_{n=1}^{\infty} d_{(0,1)}(n)\lambda_2(4\pi\sqrt{nX})\log nX + 2\pi X^2\sum_{n=1}^{\infty} D_{(1)}(n)\lambda_2(4\pi\sqrt{nX}) + O\left(X^{\frac{1}{2}+\varepsilon}\right), \tag{3}$$

The proof is similar to the proof given by Dixon Ferrar in case of $\Delta(x)$ which begins with considering Riesz sum of order greater than 2 but is more technical and dicussed in chapter 2. Further, using the asymptotic formula for λ_2 we get

Corollary 0.4.2

$$\int_1^X \Delta_{(1)}(x)dx = \frac{X^{\frac{3}{4}}\log^2 X}{8\sqrt{2}\pi^2}\sum_{n=1}^{\infty}\frac{d(n)}{n^{\frac{5}{4}}}\sin\left(4\pi\sqrt{nX}-\frac{\pi}{4}\right) + \frac{X^{\frac{3}{4}}}{\sqrt{2}\pi^2}\sum_{n=1}^{\infty}\frac{\frac{3}{8}d(n)\log^2 n - \frac{1}{2}l(n)}{n^{\frac{5}{4}}}\sin\left(4\pi\sqrt{nX}-\frac{\pi}{4}\right) + O\left(X^{\frac{1}{2}+\varepsilon}\right), \tag{4}$$

where $l(n) = \sum_{d|n}\log^2 d$.

Proceeding further with the help of Furuya's result which is a generalization of Segal's result we obtained the following

Theorem 0.4.3

$$\sum_{n\le x}\Delta_{(1)}(n) = \int_1^x \Delta_{(1)}(u)du + \frac{xP_{(3)}(\log x)}{2} + O\left(x^{\frac{1}{3}+\epsilon}\right), \tag{5}$$

$$\sum_{n\le x}\Delta_{(1)}(n) = \frac{x}{2}P_{(3)}(\log x) + O\left(x^{\frac{3}{4}}\log^2 x\right) \tag{6}$$

$$= \frac{1}{12}x\log^3 x - \frac{1}{4}x\log^2 x + \frac{1-2\gamma_1}{2}x\log x - \left(2\gamma_2-\gamma_1+\frac{1}{2}\right)x + O\left(x^{\frac{3}{4}}\log^2 x\right).$$

The first assertion of the above theorem shows that the difference is very large.

That is the means in continuous and discrete case behave differently for large values of x. In case of square we have the following

Theorem 0.4.4

$$\sum_{n\leq x}\Delta^2_{(1)}(n)=\int_1^x\Delta^2_{(1)}(u)du+\frac{1}{216}x\log^6 x-\frac{1}{36}x\log^5 x+\frac{1}{36}(5-4\gamma_2)\,x\log^4 x$$
$$+a_3x\log^3 x+a_2x\log^2 x+a_1x\log x+a_0x+O\left(x^{\frac{3}{4}}\log^5 x\right),$$

where $a_3=(4\gamma_1-2\gamma_2-5)/9$, $a_2=(2\gamma_1^2-4\gamma_1+2\gamma_2+5)/3$, $a_1=-(4\gamma_1^2-8\gamma_1\gamma_2-8\gamma_1+4\gamma_2+10)/3$, $a_0=(4\gamma_1^2-8\gamma_1\gamma_2+8\gamma_2^2-8\gamma_1+4\gamma_2+10)/3$.

The proofs of the above theorems are discussed elaborately in chapter 2. Now we shall move to next part of my thesis where we will discuss problems related to k-free integers

0.5 Problems related to k-free integers

To analyse many number theoritic algorithms we need to look on those integers which have small prime factors. In literature, this types of numbers are called "smooth numbers". This term seems to have been first introduced by Leonard Adleman and this has developed a vast research area. Similarly study of integers with large prime factors has became a topic of deep research for many number theorists. Dickman [D], N.G.de Bruijn [Br1] and [Br2], Buchstab [Bu2], Friedlander [FR1], [Fr2], [Fr3], [Fr4] and [Fr5], Granville [Gra], Hilderbrand [H] and [HT], Pomerance [IP], Tenenbaum [IT] and Ivić [I1] and [I2], K. Ramachandra [R1], [R2] and [R3] are the renowned mathematicians to work in this field. There are many more. In 1930, Dickman [D] obtained a famous result which is the following:

Let $\Psi(x,y)=\sum_{\substack{n\le x\\ p|n\Rightarrow p\le y}} 1$. That is the number of integers up to x all of whose prime factors are less than or equal to y. Dickman showed that $\Psi(x,y)\sim x\rho(u)$ as $x\to\infty$ where $x=y^u$. Here $\rho(u)$ is called Dickman-de Bruijn $\rho-$ function and which is defined by

$$\rho(u)=\begin{cases}1 & \text{for} \quad 0\le u\le 1,\\ \frac{1}{u}\int_{u-1}^{u}\rho(t)dt & \text{for} \qquad u>1\end{cases}$$

Infact we have the following result:

$$\Psi(x,y)=x\rho(u)+\left(\frac{x}{\log y}\right)$$

uniformly for $x\ge y\ge 2$. Similarly, if we count by $\Phi(x,y)=\sum_{\substack{n\le x\\ p|n\Rightarrow p>y}} 1$, that is the number of integers upto x all of whose prime factors are greater than y, then one can obtain

$$\Phi(x,y)=\frac{x\omega(u)-y}{\log y}+\left(\frac{x}{\log^2 y}\right)$$

uniformly for $x\ge y\ge 2$. Here $u\omega(u)=1$ for $1\le u\le 2$ and $1+\int_1^{u-1}\omega(v)dv$ for $u>2$ (see, e.g., Tenenbaum [T, p. 368, p. 400]).

There are many other results in this regard. Instead of going to further details, we shall focus on studying $\Psi(x,y,f)=\sum_{\substack{n\le x\\ p|n\Rightarrow p\le y}} f(n)$ and $\Phi(x,y,f)=\sum_{\substack{n\le x\\ p|n\Rightarrow p\le y}} f(n)$ where f is a multiplicative function. We take f to be μ, μ^2 and μ/N, where μ is the Möbius function and $N(n)=n$. In a joint work with Makoto Minamide in [BM1], we have studied them.

For studying the above we have prepared the following analogues of Buchstab's identity:

Let us assume some restrictions on f. We assume all the functions satisfy the following.:

$$\begin{cases} (A) \quad f \text{ is multiplicative,} \\ (B) \quad f(p^m) = 0 \text{ for any prime and positive integer } m \geq 2. \end{cases}$$

Under these assumptions, we obtained analogues of Buchstab's identity (see, e.g., Tenenbaum [T, p. 365, p. 398]).

Theorem 0.5.1 *Keeping the notations as above and for $x \geq z \geq y \geq 2$,*

$$\psi(x;y;f) = 1 + \sum_{p<y} f(p)\psi\left(\frac{x}{p}, p; f\right),$$
$$\psi(x,y;f) = \psi(x,z;f) - \sum_{y\leq p<z} f(p)\psi\left(\frac{x}{p}, p; f\right),$$
$$\Phi(x,y;f) = 1 + \sum_{y<p\leq x} f(p)\Phi\left(\frac{x}{p}, p; f\right),$$
$$\Phi(x,y;f) = \Phi(x,z;f) + \sum_{y<p\leq z} f(p)\Phi\left(\frac{x}{p}, p; f\right).$$

As an application of the above we obtained the following:

Theorem 0.5.2 *For $x^{\varepsilon} < y \leq x$, then*

$$\Phi\left(x, y; \frac{\mu}{N}\right) = \rho(u) + O\left(\frac{1}{\log y}\right),$$

where $u = \log x/\log y$ and $\rho(u)$ is the Dickman function.

As another application of the above analogues of Buchstab's identity, we shall define

$$\mathcal{Q}(x,y) = \Phi(x,y;\mu^2) = \sum_{\substack{n\leq x \\ p|n\Rightarrow p>y}} \mu^2(n),$$

$$\mathcal{R}(x,y) = \Phi(x,y;\mu) = \sum_{\substack{n\le x\\ p|n\Rightarrow p>y}} \mu(n).$$

which will lead us to

Theorem 0.5.3 *For $x^{\varepsilon} < y \le x$, by the prime number theorem of the form ($\pi(x) = \sum_{p\le x} 1 = x/\log x + O(x/\log^2 x)$), we have*

$$\mathcal{Q}(x,y) = \frac{x\omega(u) - y}{\log y} + O\left(\frac{x}{\log^2 y}\right),$$
$$\mathcal{R}(x,y) = \frac{x\rho'(u) + y}{\log y} + O\left(\frac{x}{\log^2 y}\right),$$

where $u = \log x/\log y$, $\omega(u)$ is the Buchstab's function and $\rho'(u)$ is the derivative of $\rho(u)$. Also we have

Theorem 0.5.4 *Uniformly for $x \ge y \ge 2$, we have*

$$\mathcal{Q}(x,y) = \frac{x\omega(u) - y}{\log y} + O\left(\frac{x}{\log^2 y}\right),$$
$$\mathcal{R}(x,y) = \frac{x\rho'(u) + y}{\log y} + O\left(\frac{x}{\log^2 y}\right),$$

where the notation is same as the above.

Definition 0.5.5 Let m be a positive square-free integer and $\mathcal{N}(m)$ the number of prime factors of m. For $x \ge y \ge 1$ we define the following counting functions:

$$\mathcal{Q}_{\text{even}}(x,y) := \sum_{\substack{m\le x,\ m \text{ square-free},\ \mathcal{N}(m):\ \text{even}\\ p|m\Rightarrow p>y}} 1 = \sum_{\substack{n\le x\\ p|n\Rightarrow p>y}} \frac{\mu^2(n)+\mu(n)}{2},$$
$$\mathcal{Q}_{\text{odd}}(x,y) := \sum_{\substack{m\le x,\ m \text{ square-free},\ \mathcal{N}(m):\ \text{odd}\\ p|m\Rightarrow p>y}} 1 = \sum_{\substack{n\le x\\ p|n\Rightarrow p>y}} \frac{\mu^2(n)-\mu(n)}{2},$$

For large y, we get some interesting results.

Theorem 0.5.6 *Uniformly for $x \geq y \geq 2$ we have*

$$\mathcal{Q}_{\text{even}}(x,y) = \frac{x}{\log y}\left(\frac{\omega(u)+\rho'(u)}{2}\right) + O\left(\frac{x}{\log^2 y}\right),$$
$$\mathcal{Q}_{\text{odd}}(x,y) = \frac{x}{\log y}\left(\frac{\omega(u)-\rho'(u)}{2}\right) - \frac{y}{\log y} + O\left(\frac{x}{\log^2 y}\right).$$

We shall provide all the details in chapter 3.

0.5.1 On non k-free integers

We say that an integer n is k-free ($k \geq 2$) if for every prime p the valuation $v_p(n) < k$. If $Q_k(x)$ denotes the the number of positive integers such that n is k-free. Then it is easy to prove that $Q_k(x) = \frac{x}{\zeta(k)} + \left(x^{\frac{1}{k}}\right)$. Here our main aim is to study

$$\Phi_k(x,y) := \sum_{\substack{n\leq x \\ p|n \Rightarrow p>y}} q_k(n),$$

where

$$q_k(n) := \begin{cases} 1 & n \text{ is } k\text{-free}, \\ 0 & \text{otherwise}. \end{cases}$$

In a joint work with Makoto Minamide in [BM2] for $\Phi_k(x,y)$ we have the following theorem

Theorem 0.5.7 *For $x^{\varepsilon} < y \leq x$ (any $\varepsilon > 0$) and $u = \log x/\log y$, by the PNT ($\pi(x) := \sum_{p\leq x} 1 = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right)$) we have*

$$\Phi_k(x,y) = \frac{x\omega(u) - y}{\log y} + O\left(\frac{x}{\log^2 y}\right),$$

where ω is the Buchstab function .

In [BM2], we studied $D_k(x,y) := \Phi(x,y) - \Phi_k(x,y)$ which is the number of non k-free integers $\leq x$ whose prime factors $> y$. Using the prime number theorem of the form $\pi(x) = \sum_{p\leq x} 1 = x/\log x + O(x/\log^2 x)$ (as $x \to \infty$) and Buchstabstab's identity' we showed the following upper bounds

$$D_k(x,y) = \begin{cases} 0, & x^{\frac{1}{k}} \leq y \leq x, \\ \frac{kx^{\frac{1}{k}}}{\log x} - \frac{y}{\log y} + O\left(\frac{x^{\frac{1}{k}}}{\log^2 y}\right), & x^{\frac{1}{k+1}} \leq y < x^{\frac{1}{k}}, \\ O\left(\frac{xy^{1-k}}{\log^2 y}\right), & \substack{x^{\frac{1}{k+l+1}} \leq y < x^{\frac{1}{k+l}}, \\ l=1,2,\ldots.} \end{cases}$$

Also, one can have

Theorem 0.5.8 *Using the PNT of the form $\pi(x) = \int_2^x \frac{dt}{\log t} + O\left(x \exp\left(-c\sqrt{\log x}\right)\right)$, we have*

$$D_k(x,y) = O\left(\frac{xy^{1-k}}{\log^2 y}\right)$$

uniformly for $x^{1/(k+1)} \geq y \geq 2$.

The proof of all the theorems is the content of the third and final part of the thesis.

CHAPTER 1

Linearized product of two Riemann zeta functions

In this chapter, we elucidate the well-known Wilton's formula for the product of two Riemann zeta functions. A proof of Wilton's expression for product of two zeta functions was given by M. Nakajima using Atkinson dissection. On the similar line we derive Wilton's formula using Riesz sum of the order $\kappa = 1$.

1.1 Introduction

In [N], M. Nakajima derived an expression for the product of two Dirichlet series. Using the derived expression he proved the well known Wilton's formula for the product of two Riemann zeta function. The main tool exploited was the Atkinson dissection. The main aim of this chapter is to provide an alternative proof using Riesz sum.

By the Atkinson dissection method, we mean, splitting of the double $\sum_{m,\ n=1}^{\infty}$ as

$$\sum_{m,n=1}^{\infty} = \sum_{m=1}^{\infty}\sum_{n<m} + \sum_{m=n=1}^{\infty} + \sum_{n=1}^{\infty}\sum_{m<n}.$$

Nakajima divided the double sum by the following trick

$$\sum_{m,n} = \sum_{m=1}^{\infty}\sum_{n\le m}{}' + \sum_{n=1}^{\infty}\sum_{m\le n}{}',$$

where $\sum{}'$ means that the corresponding term in the summation is to be halved when $m = n$.

On the other hand, the Riesz sum, introduced by M. Riesz, have been studied in connection with summability of Fourier series and of Dirichlet series [C] and [HR2]. For a given increasing sequence $\{\lambda_k\}$ of reals and a sequence $\{\alpha_k\}$ of complex numbers, the Riesz sum of order κ is defined by

$$A^{\kappa}(x) = A_{\lambda}^{\kappa}(x) = \sum_{\lambda_k \le x}{}' (x - \lambda_k)^{\kappa} \alpha_k$$
$$= \kappa \int_0^x (x - t)^{\kappa - 1} A_{\lambda}(t) \mathrm{d}t,$$

where $A_{\lambda}(x) = A_{\lambda}^0(x) = \sum'_{\lambda_k \le x} \alpha_k$, and the prime on the summation sign means that when $\lambda_k = x$, the correponding term is to be halved.

Consider the Dirichlet series $\varphi(s)$ and $\Phi(s)$ defined as

$$\varphi(s) = \sum_{n=1}^{\infty} \frac{\alpha_n}{\lambda_n^s}, \ \sigma > \sigma_{\varphi} \text{ and } \Phi(s) = \sum_{n=1}^{\infty} \frac{a_n}{\gamma_n^s}, \ \sigma > \sigma_{\Phi},$$

where $\{\lambda_n\}$ and $\{\gamma_n\}$ are increasing sequences of real numbers and α_n and a_n are complex numbers. Assume that they can be analytically continued to the whole complex plane (they may have poles) and saitsfy the following growth condition

$$\varphi(\sigma + it) << (|t| + 1)^{s_{\varphi}(\sigma)}, \ \Phi(\sigma + it) << (|t| + 1)^{s_{\Phi}(\sigma)},$$

in the strip $-b < \sigma < c$. In particular, for Riemann zeta function, $s_{\zeta}(-b) = \frac{1}{2} + b$.

Let us consider an integral of the following form for Dirichlet series ϕ and φ, (for $c > 0$ and $\kappa \ge 0$),

$$\mathcal{F}_c(\varphi(u), \Phi(v); x) = \mathcal{F}_c^{\kappa}(\varphi(u), \Phi(v); x)$$
$$= \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(w)}{\Gamma(w + \kappa + 1)} \varphi(u + w) \Phi(v - w) x^{w + \kappa} \, \mathrm{d}w,$$

where $\int_{(c)}$ means $\int_{-c-i\infty}^{-c+i\infty}$. The above integral has its counterpart $\mathcal{F}_c(\Phi(u), \varphi(v); x)$ under the condition

$$Re(u) > \sigma_{\varphi} + c, \quad Re(v) > \sigma_{\Phi} + c.$$

Subsequently we shall assume ϕ and φ to be the Riemann zeta functions.

The Atkinson dissection is the special case of the Riesz sum $A^\kappa(x)$ with $\kappa = 0$ in the sense that

$$\begin{aligned} A^\kappa(x) &= \mathcal{F}^\kappa_{(c)}(\varphi(u), \Phi(v); x) + \mathcal{F}^\kappa_{(c)}(\Phi(u), \varphi(v); x) \\ &= \frac{1}{\Gamma(\kappa+1)} \sum_{m=1}^\infty a_m \gamma_m^{-v-\kappa} \sideset{}{'}\sum_{\lambda_n \le \gamma_m x} \alpha_n \lambda_n^{-u} (\gamma_m x - \lambda_n)^\kappa \\ &\quad + \frac{1}{\Gamma(\kappa+1)} \sum_{n=1}^\infty \alpha_n \lambda_n^{-u-\kappa} \sideset{}{'}\sum_{\gamma_m \le \lambda_n x} a_m \gamma_m^{-v} (\lambda_n x - \gamma_m)^\kappa, \end{aligned} \tag{1.1}$$

implies

$$\begin{aligned} \sum_{m,n=1}^\infty \alpha_m \lambda_m^{-u} a_n \gamma_n^{-v} &= \sum_{m=1}^\infty \sum_{n<m} \alpha_m \lambda_m^{-u} a_n \gamma_n^{-v} \\ &\quad + \sum_{m=n}^\infty \alpha_n \lambda_n^{-u} a_n \gamma_n^{-v} + \sum_{n=1}^\infty \sum_{m<n} \alpha_m \lambda_m^{-u} a_n \gamma_n^{-v}. \end{aligned} \tag{1.2}$$

In this chapter, we prove the well-known Wilton's formula by taking $\kappa = 1$ and making use of a formula related to incomplete gamma function. The Wilton's formula can be stated as

Theorem 1.1.1 *For $Re(u) > -1$, $Re(v) > -1$, $Re(u+v) > 0$ and $u+v \neq 2$, we have*

$$\begin{aligned} &\zeta(u)\zeta(v) \\ &= \zeta(u+v-1)\left(\frac{1}{u-1} + \frac{1}{v-1}\right) \\ &\quad + 2(2\pi)^{u-1} \sum_{n=1}^\infty \sigma_{1-u-v}(n) n^{u-1} u \int_{2\pi n}^\infty x^{-u-1} \sin x \, dx \\ &\quad + 2(2\pi)^{v-1} \sum_{n=1}^\infty \sigma_{1-u-v}(n) n^{v-1} v \int_{2\pi n}^\infty x^{-v-1} \sin x \, dx. \end{aligned} \tag{1.3}$$

In the next section, first we will prove the above formula for the region $Re(u) > 1$, $Re(v) > 1$. We claim that (1.3) is valid in the region stated above by analytic continuation since the integrals and the summations in the right hand side of (1.3) are absolutely and uniformly convergent for $Re(u) > -1$, $Re(v) > -1$ and $Re(u+v) > 0$. In particular, if $u = 1$ or $v = 1$ but $u+v \neq 2$ then left hand side of the above expression coincides with the right hand side due to the presence of $(u-1)^{-1}$ and $(v-1)^{-1}$ factors in the right hand side of the above expression.

1.2 Proof of the Theorem

Let us start the proof by considering first the integral:

$$\mathcal{F}^{\kappa}_{(c)}(\zeta(u),\zeta(v);x) = \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(w)}{\Gamma(w+\kappa+1)}\zeta(u+w)\zeta(v-w)x^{w+\kappa}\mathrm{d}w, \quad (1.4)$$

for $Re(u) > 1+c'$ and $Re(v) > 1+c$, with $c, c' > 0$ and $\kappa \geq 1$. These conditions together with the fact $\Gamma(s) \sim e^{-\frac{\pi}{2}|t|}|t|^{\sigma-\frac{1}{2}}$ for σ in any fixed interval $[a, b]$ and $|t| \longrightarrow \infty$ implies the absolute convergence of the above integral.
Hence by Cauchy's residue theorem

$$\begin{aligned}\mathcal{F}^{\kappa}_{(c)}(\zeta(u),\zeta(v);x) = \mathcal{F}^{\kappa}_{(-b)}(\zeta(u),\zeta(v);x) + \frac{x^{\kappa}\zeta(u)\zeta(v)}{\Gamma(\kappa+1)} \\ - \frac{x^{\kappa-1}\zeta(u-1)\zeta(v+1)}{\Gamma(\kappa)} + \frac{\zeta(u+v-1)x^{\kappa+1-u}}{\Gamma(2+\kappa-u)},\end{aligned} \quad (1.5)$$

where

$$\mathcal{F}^{\kappa}_{(-b)}(\zeta(u),\zeta(v);x) = \frac{1}{2\pi i}\int_{(-b)} \frac{\Gamma(w)}{\Gamma(w+\kappa+1)}\zeta(u+w)\zeta(v-w)x^{w+\kappa}\mathrm{d}w, \quad (1.6)$$

and we take $0 < Re(u) - 1 < b < \frac{3}{2}$ and $0 < Re(v) - 1 < b < \frac{3}{2}$. The integrand in (1.4) has poles at $w = -1, 0, 1-u$ each of order one, so we get the above expression (1.5).

Let us first estimate the integral (1.6). First of all, let us recall the following bound for the zeta function

$$\zeta(s) \ll \begin{cases} |t|^{\frac{1}{2}-\sigma+\varepsilon}, & \sigma \leq 0, \\ |t|^{\frac{1}{2}(1-\sigma)+\varepsilon}, & 0 \leq \sigma \leq 1, \\ |t|^{\varepsilon}, & \sigma \geq 1, \end{cases} \quad (1.7)$$

Hence the integral (1.6) is absolutely convergent. We know that the functional equation for the Riemann zeta function $\zeta(s) = 2(2\pi)^{s-1}\sin\left(\frac{s\pi}{2}\right)\Gamma(1-$

$s)\zeta(1-s)$. Now applying functional equation for $\zeta(u+w)$ we have

$$\begin{aligned}&\mathcal{F}^{\kappa}_{(-b)}(\zeta(u),\zeta(v);x)\\&=2\frac{1}{2\pi i}\int_{(-b)}S_{\kappa}(w)f(w)(2\pi)^{u+w-1}\sin(\frac{\pi}{2}(u+w))\ \Gamma(1-u-w)\mathrm{d}w,\end{aligned}$$

where

$$\frac{\Gamma(w)}{\Gamma(w+\kappa+1)}x^{w+\kappa}=S_{\kappa}(w),$$

and

$$\zeta(1-u-w)\zeta(v-w)=f(w).$$

By change of variable and assuming $Re(u)<b$, we have

$$\begin{aligned}&\frac{1}{\pi i}\int_{(b)}S_{\kappa}(-z)f(-z)(2\pi)^{u-z-1}\sin(\frac{\pi}{2}(u-z))\ \Gamma(1-u+z)\mathrm{d}z\\&=\frac{1}{\pi i}\int_{(b)}S_{\kappa}(-z)(2\pi)^{u-z-1}\sin(\frac{\pi}{2}(u-z))\ \sum_{n=1}^{\infty}\sigma_{v+u-1}(n)n^{-v-z}\Gamma(1-u+z)\mathrm{d}z\\&=\frac{1}{\pi i}\int_{(b)}S_{\kappa}(-z)(2\pi)^{u-z-1}\sin(\frac{\pi}{2}(u-z))\ \sum_{n=1}^{\infty}\sigma_{1-u-v}(n)n^{u-z-1}\Gamma(1-u+z)\mathrm{d}z\\&=\frac{1}{\pi i}\sum_{n=1}^{\infty}\sigma_{1-u-v}(n)(2n\pi)^{u-1}\int_{(b)}S_{\kappa}(-z)\sin(\frac{\pi}{2}(u-z))\ (2n\pi)^{-z}\Gamma(1-u+z)\mathrm{d}z,\end{aligned}$$

where the second step follows due to the fact $\zeta(s)\zeta(s-\alpha)=\sum_{n=1}^{\infty}\frac{\sigma_{\alpha}(n)}{n^s}$ with $\sigma>$max$(1,1+Re(\alpha))$ and in our case $\alpha=u+v-1$ and and in subsequent step we have used the relation $\sigma_{\alpha}(n)=n^{\alpha}\sigma_{-\alpha}(n)$. As we are only interested in the case of $\kappa=1$, so putting $\kappa=1$ we have

$$\begin{aligned}&\mathcal{F}^{1}_{(-b)}(\zeta(u),\zeta(v);x)\\&=\frac{1}{\pi i}\sum_{n=1}^{\infty}\sigma_{1-u-v}(n)(2n\pi)^{u-1}\int_{(b)}S_{1}(-z)\sin(\frac{\pi}{2}(u-z))\ (2n\pi)^{-z}\Gamma(1-u+z)\mathrm{d}z\\&=\frac{1}{\pi i}\sum_{n=1}^{\infty}\sigma_{1-u-v}(n)(2n\pi)^{u-1}\int_{(b)}S_{1}(-z)(2n\pi)^{-z}\ \left\{\frac{e^{i\frac{\pi}{2}(u-z)}-e^{-i\frac{\pi}{2}(u-z)}}{2i}\right\}\Gamma(1-u+z)\mathrm{d}z.\end{aligned}$$

Let

$$h_b(u,v;x)=\frac{1}{2\pi i}\int_{(b)}\frac{x^{-z+1}(2n\pi)^{-z}}{z(z-1)}e^{i\frac{\pi}{2}(u-z-1)}\Gamma(1-u+z)\mathrm{d}z.$$

Similarly, let

$$g_b(u,v;x) = \frac{1}{2\pi i}\int_{(b)} \frac{x^{-z+1}(2n\pi)^{-z}}{z(z-1)} e^{-i\frac{\pi}{2}(u-z-1)}\Gamma(1-u+z)\mathrm{d}z.$$

Differentiation of the above integral $h_b(u,v;x)$ with respect to x, will lead us to

$$\begin{aligned} h_b(u,v;x)' = &-\frac{1}{2\pi i}\int_{(b)} \frac{x^{-z}(2n\pi)^{-z}}{z} e^{i\frac{\pi}{2}(u-z-1)}\Gamma(1-u+z)\mathrm{d}z \\ = &-\frac{1}{2\pi i} e^{i\frac{\pi}{2}(u-1)}\int_{(b)} \frac{(2\pi n x e^{i\frac{\pi}{2}})^{-z}}{z}\Gamma(1-u+z)\mathrm{d}z, \end{aligned}$$

Then shifting the path of integration to the left, we get

$$h_b(u,v;x)' = -\frac{1}{2\pi i} e^{i\frac{\pi}{2}(u-1)}\int_{(b_1)} \frac{(2\pi n x e^{i\frac{\pi}{2}})^{-z}}{z}\Gamma(1-u+z)\mathrm{d}z,$$

where $\frac{1}{2} + b_1 < \Re(u),\ 0 < \Re(u-1) < b_1$. Similarly, we have

$$g_b(u,v;x)' = -\frac{1}{2\pi i} e^{i\frac{\pi}{2}(1-u)}\int_{(b_1)} \frac{(2\pi n x e^{-i\frac{\pi}{2}})^{-z}}{z}\Gamma(1-u+z)\mathrm{d}z.$$

Now here we will make use of a formula for incomplete gamma function, which is given as

$$\frac{1}{2\pi i}\int_{(c)} x^{-s}\Gamma(s+\alpha)\frac{1}{s}\mathrm{d}s = \Gamma(\alpha,x) \qquad (c>0, \Re x > 0),$$

$$\frac{1}{2\pi i}\int_{(c)} (ix)^{-s}\Gamma(s+\alpha)\frac{\mathrm{d}s}{s} = \Gamma(\alpha, ix),$$

$$(c>0, \Re\alpha < \frac{1}{2} - c,\ x \in R),$$

where

$$\Gamma(\alpha,x) = \int_x^\infty t^{\alpha-1}e^{-t}\mathrm{d}t \qquad (|\arg\alpha| < \pi)$$

is the incomplete gamma function.

As a result we have,

$$\begin{aligned} h_b(u,v;x)' + g_b(u,v;x)' = &-e^{-i\frac{\pi}{2}(1-u)}\Gamma(1-u, 2\pi n x e^{\frac{i\pi}{2}}) \\ &- e^{i\frac{\pi}{2}(1-u)}\Gamma(1-u, -2\pi n x e^{\frac{i\pi}{2}}) \end{aligned}$$

$$= -2\int_{2\pi nx}^{\infty} t^{-u}\cos t dt \qquad (\Re u > 0),$$

where we have used the following formula

$$\int_{u}^{\infty} x^{\alpha-1}\cos x dx = \frac{1}{2}e^{-i\frac{\pi}{2}\alpha}\Gamma(\alpha, iu) + \frac{1}{2}e^{i\frac{\pi}{2}\alpha}\Gamma(\alpha, -iu).$$

Hence, we obtain

$$\left(\mathcal{F}^1_{(-b)}(\zeta(u), \zeta(v); x)\right)' = -\sum_{n=1}^{\infty}\sigma_{1-u-v}(n)(2\pi n)^{u-1}2\int_{2\pi nx}^{\infty} t^{-u}\cos t dt. \qquad (1.8)$$

We note that the differentiated series is absolutely convergent.
On the other hand if we put $\kappa = 1$ in (1.5), then we obtain

$$\mathcal{F}^1_{(c)}(\zeta(u), \zeta(v); x) = \mathcal{F}^1_{(-b)}(\zeta(u), \zeta(v); x) + x\zeta(u)\zeta(v) - \zeta(u-1)\zeta(v+1) + \frac{\zeta(u+v-1)x^{2-u}}{(u-2)(u-1)}. \qquad (1.9)$$

Now we want to use the result obtained in (1.8), for that we need to differentiate the expression in (1.9)

$$\left(\mathcal{F}^1_{(c)}(\zeta(u), \zeta(v); x)\right)' = \left(\mathcal{F}^1_{(-b)}(\zeta(u), \zeta(v); x)\right)' + \zeta(u)\zeta(v) - \frac{\zeta(u+v-1)x^{1-u}}{(u-1)}. \qquad (1.10)$$

Also, we know that

$$\left(\mathcal{F}^1_{(c)}(\zeta(u), \zeta(v); x)\right)' = \sum_{m=1}^{\infty} m^{-v} \sideset{}{'}\sum_{n\le mx} n^{-u}. \qquad (1.11)$$

Lastly we put (1.8) and (1.11) in (1.10)and take $x = 1$ which will give rise to

$$\sum_{m=1}^{\infty} m^{-v} \sideset{}{'}\sum_{n\le m} n^{-u} = -2\sum_{n=1}^{\infty}\sigma_{1-u-v}(n)(2\pi n)^{u-1}u\int_{2\pi n}^{\infty} t^{-u-1}\sin t dt + \zeta(u)\zeta(v) - \frac{\zeta(u+v-1)}{(u-1)}. \qquad (1.12)$$

Similarly, interchanging the role of u and v in (1.4) we have

$$\sum_{n=1}^{\infty} n^{-u} \sum_{m\leq n}{}' m^{-v}$$
$$= -2\sum_{n=1}^{\infty}\sigma_{1-u-v}(n)(2\pi n)^{v-1}v\int_{2\pi n}^{\infty} t^{-v-1}\sin t dt + \zeta(u)\zeta(v) - \frac{\zeta(u+v-1)}{(v-1)}. \tag{1.13}$$

Adding the equations (1.12) and (1.13), we get

$$\begin{aligned}\zeta(u)\zeta(v) = & -2\sum_{n=1}^{\infty}\sigma_{1-u-v}(n)(2\pi n)^{u-1}u\int_{2\pi n}^{\infty} t^{-u-1}\sin t dt \\ & -2\sum_{n=1}^{\infty}\sigma_{1-u-v}(n)(2\pi n)^{v-1}v\int_{2\pi n}^{\infty} t^{-v-1}\sin t dt \\ & + 2\zeta(u)\zeta(v) - \zeta(u+v-1)\left\{\frac{1}{v-1}+\frac{1}{u-1}\right\},\end{aligned}$$

and hence

$$\begin{aligned}\zeta(u)\zeta(v) = \zeta(u+v-1)\left\{\frac{1}{v-1}+\frac{1}{u-1}\right\} \\ + 2(2\pi)^{u-1}\sum_{n=1}^{\infty}\sigma_{1-u-v}(n)n^{u-1}u\int_{2\pi n}^{\infty} t^{-u-1}\sin t dt \\ + 2(2\pi)^{v-1}\sum_{n=1}^{\infty}\sigma_{1-u-v}(n)n^{v-1}v\int_{2\pi n}^{\infty} t^{-u-1}\sin t dt.\end{aligned}$$

This proves the theorem.

CHAPTER 2

The average behaviour of the error term in a new kind of divisor problem

In this chapter, we define a new kind of divisor function $D_{(1)}(n)$ by the nth coefficient of the Dirichlet series $(\zeta'(s))^2$ and denote by $\Delta_{(1)}(x)$ the error term in the asymptotic formula for $\sum_{n\leq x} D_{(1)}(n)$. Then we compute the first moment of $\Delta_{(1)}(x)$ that is $\int_1^X \Delta_{(1)}(x)\mathrm{d}x$ and consider 'discrete mean values' $\sum_{n\leq x} \Delta_{(1)}^k(n)$ $(k=1,2)$ and deduce asymptotic formulas which are analogous to the results obtained by Voronoï, Hardy and Furuya.

2.1 Introduction

We begin with the asymptotic formula

$$D(x) = \sum_{n\leq x} d(n) = x\log x + (2\gamma - 1)x + \Delta(x),$$

where $d(n)$ stands for divisor function which is due to Dirichlet and $\Delta(x)$ denotes the error term. From number theoretic point of view it is of great importance to estimate the error term and in literature estimating the error term $\Delta(x)$ is known as Dirichlet divisor problem. First attempt was made by Dirichlet himself who proved that $\Delta(x) = O(x^{\frac{1}{2}})$ and later it was conjectured that $\Delta(x) \ll x^{\frac{1}{4}+\varepsilon}$ (for any $\varepsilon > 0$) based on mean value considerations. Since then it has been studied by many number theorists. For more details see Titchmarsh [Ti, Ch. 12] and Ivić [I3, Ch. 3]. In 2003, Huxley [Hux] showed that $\Delta(x) = O\left(x^{\frac{131}{416}}(\log x)^{\frac{26947}{8320}}\right)$ which is possibly the best known improvement in our present knowledge. However the conjecture is still an open question.

In [V], Voronoï studied the first moment of $\Delta(x)$, that is, $\int_1^x \Delta(u)\mathrm{d}u$ and obtained the formula (2.1) below. On the other hand, Cramér obtained the formula (2.2) for $\int_1^x \Delta^2(u)\mathrm{d}u$ in [Cr].

$$\int_1^x \Delta(u)\mathrm{d}u = \frac{1}{4}x + \frac{1}{2\sqrt{2}\pi^2}x^{\frac{3}{4}}\sum_{n=1}^{\infty} d(n)n^{-\frac{5}{4}}\sin\left(4\pi\sqrt{nx} - \frac{\pi}{4}\right) + O\left(x^{\frac{1}{4}}\right), \tag{2.1}$$

$$\int_1^x \Delta^2(u)\mathrm{d}u = \frac{1}{6\pi^2}\sum_{n=1}^{\infty}\frac{d^2(n)}{n^{\frac{3}{2}}}x^{\frac{3}{2}} + O\left(x^{\frac{5}{4}+\varepsilon}\right), \quad \text{(for any } \varepsilon > 0). \tag{2.2}$$

The error term $O(x^{5/4+\varepsilon})$ in (2.2) was improved to $O\left(x\log^5 x\right)$ by Tong [To]. Later Lau and Tsang [LT] showed the bound to be $O\left(x(\log x)^3 \log\log x\right)$. Further the higher-power moments of $\Delta(x)$ was initiated by Tsang [Ts], Zhai [Z1], [Z2], Ivić and Sargos [IS]. In literature these integrals $\int_1^x \Delta^k(u)\mathrm{d}u$ (where k is a positive integer) are called 'continuous mean values.' Voronoï considered the discrete counter part and showed that

$$\sum_{n\le x}\Delta(n) = \frac{1}{2}x\log x + \left(\gamma - \frac{1}{4}\right)x + O\left(x^{\frac{3}{4}}\right)$$

for $x \ge 1$, where γ is the Euler constant. The sums $\sum_{n\le x}\Delta^k(n)$ $(k = 1, 2, \ldots)$ are called 'discrete mean values.' Further, Hardy [Har] considered the difference between the continuous and the discrete mean values of $\mathcal{D}^2(x)$, where $\mathcal{D}(x)$ is a function similar to $\Delta(x)$. From his result we can obtain

$$\sum_{n\le x}\Delta^2(n) = \int_1^x \Delta^2(u)\mathrm{d}u + O\left(x^{1+\varepsilon}\right), \tag{2.3}$$

where $\varepsilon > 0$ is any small constant. See the appendix in [CFTZ]. Recently, Furuya [F] started the investigation of error term in (2.3) in further details. He improved (2.3) as follows:

$$\sum_{n\le x}\Delta^2(n)=\int_1^x\Delta^2(u)\mathrm{d}u+\frac{1}{6}x\log^2 x+c_1x\log x+c_2x+O\left(x^{\frac{3}{4}}\log x\right),\quad (2.4)$$

where $c_1=(8\gamma-1)/12$ and $c_2=(8\gamma^2-2\gamma+1)/12$. Moreover he deduced the formula:

$$\sum_{n\le x}\Delta^3(n)=\int_1^x\Delta^3(u)\mathrm{d}u+c_3x^{\frac{3}{2}}\log x+c_4x^{\frac{3}{2}}+O\left(x\log^5 x\right)\quad (2.5)$$

with $c_3=3C/2$ and $c_4=C(3\gamma-1)$, where C denotes the coefficient of the main term in (2.2). Lately, Cao, Furuya, Tanigawa, and Zhai developed the formula similar to (2.5) up to 10th power-moments [CFTZ].

This chapter, takes its origin from the above consideration on $\Delta(x)$ and a new kind of divisor problem which was studied recently by Makoto Minamide [M]. First of all, we shall set

$$\Delta_{(1)}(x):=\sum_{n\le x}D_{(1)}(n)-xP_{(3)}(\log x)\quad (2.6)$$

where

$$P_{(3)}(u)=\frac{u^3}{3!}-\frac{u^2}{2!}+\frac{1-2\gamma_1}{1!}u+2\gamma_1-4\gamma_2-1,\quad (2.7)$$

and the constants γ_1 and γ_2 are defined as follows:

$$\zeta(s)=\frac{1}{s-1}+\gamma+\gamma_1(s-1)+\gamma_2(s-1)^2+\cdots.\quad (2.8)$$

In [M, p. 330, (9)], it is shown that $xP_{(3)}(\log x)=Res_{s=1}\left(\zeta'(s)\right)^2x^s/s$ and an arithmetical function $D_{(1)}(n)$ is defined by

$$D_{(1)}(n):=\sum_{d|n}(-\log d)\left(-\log\frac{n}{d}\right).$$

The definition of $D_{(1)}(n)$ comes from the square of the derivative of the Riemann zeta function,

$$(\zeta'(s))^2 = \sum_{n=1}^{\infty} \frac{D_{(1)}(n)}{n^s} \quad (\mathrm{Re}\, s > 1). \tag{2.9}$$

Similarly, we shall define $d_{(0,1)}(n)$ by the following identity,

$$\zeta(s)\zeta'(s) = \sum_{n=1}^{\infty} \frac{d_{(0,1)}(n)}{n^s} \quad (\mathrm{Re}\, s > 1). \tag{2.10}$$

It turns out to be $d_{(0,1)}(n) = \sum_{d|n}(-\log d)$. Arithmetical functions like $D_{(1)}(n)$ and $d_{(0,1)}(n)$ were appeared in Gonek [Gon], Hall [H], [FMT1], and [FMT2]. By the definitions (2.9) and (2.10) we observe that $D_{(1)}(n) \ll d(n)\log^2 n$ and $d_{(0,1)}(n) = -(1/2)d(n)\log n$. A result of [M] established the mean square formula for $\Delta_{(1)}(x)$:

$$\int_1^x \Delta_{(1)}^2(u)\mathrm{d}u = \frac{1}{96\pi^2}\sum_{n=1}^{\infty}\frac{d^2(n)}{n^{\frac{3}{2}}} x^{\frac{3}{2}}\log^4 x + O\left(x^{\frac{3}{2}}\log^3 x\right).$$

For details of this formula and the truncated Voronoï formula for $\Delta_{(1)}(x)$ see [M]. The main object of this chapter, is to study the asymptotic formula for $\int_1^x \Delta_{(1)}(u)\mathrm{d}u$ by employing Dixon and Ferrar's method [DF]. Then next goal is to investigate the bounds for the difference between the discrete mean value and the continuous one for $\Delta_{(1)}^k(x)$ in case of $k = 1$ and 2 with the help of an identity by Furuya [F] which is a generalization of Segal's identity [S].

A necessary ingredient for this chapter is an identity analogous to a famous identity in [DF] for the Riesz sum $\sum_{n\le x} d(n)(x-n)^{q-1}$ where q is a real number and $q > 2$ which is the following:

$$\frac{1}{\Gamma(q)}\sum_{n\le x} d(n)(x-n)^{q-1} - \frac{x^{q-1}}{4\Gamma(q)} - \frac{x^q}{\Gamma(q+1)}\{\gamma + \log x - \psi(1+q)\}$$
$$= 2\pi x^q \sum_{n=1}^{\infty} d(n)\lambda_q(4\pi\sqrt{(nx)}),$$

where λ_q is defined below. Then Dixon and Ferrar established the identity for $q = 2$ by analytic continuation. Here λ_q is the generalized Bessel function and the original definition for $\lambda_q(z)$ is

$$\lambda_q(z) := \frac{\pi}{2} \times \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \frac{(z/2)^{2s-2}}{\Gamma(s)\Gamma(s+q)\cos^2(\pi s/2)} \mathrm{d}s, \tag{2.11}$$

where $\Gamma(z)$ is the Gamma function, q is a complex variable, and α real satisfying

$$\mathrm{Re}\, q > 0, \ -1 < \alpha < 1, \ \mathrm{Re}\,(q) + 2\alpha - 2 > 0.$$

This has the expression:

$$\begin{aligned} \lambda_q(z) = -\frac{2}{\pi} \sum_{m=0}^{\infty} \frac{(z/2)^{4m}}{\Gamma(2m+1)\Gamma(2m+1+q)} \times \\ \times \left(2\log(z/2) - \psi(2m+1) - \psi(2m+1+q)\right), \end{aligned}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$. If we take $q = 1$, we observe that

$$-\frac{2z}{\pi}\lambda_1(2z) = K_1(2z) + \frac{\pi}{2} Y_1(2z),$$

where K_1 and Y_1 are Bessel functions in the usual sense. For details of these Bessel functions, see [DF, Section 3], [I3, Chapter 3] and Jutila [J, Chapter 1].

To be short for a statement of a result of this chapter, we shall modify the definition (2.6) as follows.

$$\Delta_{(1)}(x) := \sum_{n \le x} D_{(1)}(n) - Res_{s=1} \left(\zeta'(s)\right)^2 \frac{x^s}{s} - Res_{s=0} \left(\zeta'(s)\right)^2 \frac{x^s}{s}. \tag{2.12}$$

2.2 The main results

Theorem 2.2.1 *Keeping the notations as above let ε be a small positive constant. Then we have*

$$\begin{aligned} \int_1^X \Delta_{(1)}(x)\mathrm{d}x = & \frac{\pi X^2}{2} \sum_{n=1}^{\infty} d(n)\lambda_2(4\pi\sqrt{nX}) \log^2 nX \\ & + 2\pi X^2 \sum_{n=1}^{\infty} d_{(0,1)}(n)\lambda_2(4\pi\sqrt{nX}) \log nX \\ & + 2\pi X^2 \sum_{n=1}^{\infty} D_{(1)}(n)\lambda_2(4\pi\sqrt{nX}) + O\left(X^{\frac{1}{2}+\varepsilon}\right). \end{aligned} \tag{2.13}$$

Using the asymptotic formula for λ_2 (see [DF, p. 41], [I3, p. 91]) we get

Corollary 2.2.2

$$\int_1^X \Delta_{(1)}(x)\mathrm{d}x = \frac{X^{\frac{3}{4}}\log^2 X}{8\sqrt{2}\pi^2}\sum_{n=1}^{\infty}\frac{d(n)}{n^{\frac{5}{4}}}\sin\left(4\pi\sqrt{nX}-\frac{\pi}{4}\right)$$
$$+\frac{X^{\frac{3}{4}}}{\sqrt{2}\pi^2}\sum_{n=1}^{\infty}\frac{\frac{1}{8}d(n)\log^2 n-\frac{1}{2}l(n)}{n^{\frac{5}{4}}}\sin\left(4\pi\sqrt{nX}-\frac{\pi}{4}\right)+O\left(X^{\frac{1}{2}+\varepsilon}\right), \tag{2.14}$$

where

$$l(n)=\sum_{d|n}\log^2 d. \tag{2.15}$$

Here note that the term of $\log X$ *does not appear.*

By Corollary 2.2.2 we have $\int_1^x \Delta_{(1)}(u)\mathrm{d}u \ll x^{3/4}\log^2 x$. From this estimate and Lemma 1 of [F, p. 6] we find formulas for the differences $\sum_{n\le x}\Delta_{(1)}^k(n) - \int_1^x \Delta_{(1)}^k(u)\mathrm{d}u$ $(k=1,2)$. Thus we have

Theorem 2.2.3

$$\sum_{n\le x}\Delta_{(1)}(n)=\int_1^x \Delta_{(1)}(u)\mathrm{d}u+\frac{xP_{(3)}(\log x)}{2}+O\left(x^{\frac{1}{3}+\varepsilon}\right), \tag{2.16}$$

$$\sum_{n\le x}\Delta_{(1)}(n)=\frac{x}{2}P_{(3)}(\log x)+O\left(x^{\frac{3}{4}}\log^2 x\right) \tag{2.17}$$
$$=\frac{1}{12}x\log^3 x-\frac{1}{4}x\log^2 x+\frac{1-2\gamma_1}{2}x\log x-\left(2\gamma_2-\gamma_1+\frac{1}{2}\right)x+O\left(x^{\frac{3}{4}}\log^2 x\right).$$

The first assertion of the above theorem shows that the difference is very large. That is the means in continuous and discrete case behave differently for large values of x.

Theorem 2.2.4

$$\sum_{n\le x}\Delta_{(1)}^2(n)=\int_1^x \Delta_{(1)}^2(u)\mathrm{d}u+\frac{1}{216}x\log^6 x-\frac{1}{36}x\log^5 x+\frac{1}{36}(5-4\gamma_1)x\log^4 x$$

$$+ a_3 x \log^3 x + a_2 x \log^2 x + a_1 x \log x + a_0 x + O\left(x^{\frac{3}{4}} \log^5 x\right),$$

where $a_3 = (4\gamma_1 - 2\gamma_2 - 5)/9$, $a_2 = (2\gamma_1^2 - 4\gamma_1 + 2\gamma_2 + 5)/3$, $a_1 = -(4\gamma_1^2 - 8\gamma_1\gamma_2 - 8\gamma_1 + 4\gamma_2 + 10)/3$, $a_0 = (4\gamma_1^2 - 8\gamma_1\gamma_2 + 8\gamma_2^2 - 8\gamma_1 + 4\gamma_2 + 10)/3$.

Theorems 2.2.3 and 2.2.4 are analogues of Voronoï's result (2.1) and Hardy and Furuya's results (2.3) and (2.4) respectively.

Remark 2.2.5 Furuya, Minamide, and Tanigawa [FMT1] generalized $\Delta_{(1)}(x)$ to $\Delta_{(k,l)}(x)$ (k and l are arbitrarily nonnegative integers) which arises from the product of the kth and the lth derivatives of the Riemann zeta function. The authors proved a 'Chowla-Walum formula' for $\Delta_{(k,l)}(x)$, and $\Delta_{(k,l)} = O\left(x^{229/696} \log^{k+l-1} x\right)$. On the other hand, [FMT2] is an attempt to a 'new circle problem.' The authors investigated a 'truncated Voronoï formula' for $\zeta'(s)L'(s,\chi_4)$, where $L(s,\chi_4)$ is the Dirichlet L-function with the Dirichlet character mod 4.

2.3 Preliminary Theorem

Following the method of [DF] we shall start the proof of Theorem 2.2.1. Let $s = \sigma + it$ (σ, $t \in \mathbb{R}$), a real variable $q > 2$, (for a while, later q will be a complex variable) and

$$G_q(s) := \frac{\Gamma(s)}{\Gamma(s+q)},$$

where $\Gamma(s)$ is the Gamma function. Here we shall remark that

$$G_q(s) \ll \frac{1}{|t|^q} \tag{2.18}$$

for σ in any fixed interval $[a,b]$ and $|t| > 2$ because of the bound $\Gamma(s) = O\left(e^{-\frac{\pi}{2}|t|}|t|^{\sigma-\frac{1}{2}}\right)$ for σ in any fixed interval $[a,b]$ and $|t| \longrightarrow \infty$.

We shall consider the following integral:

$$\frac{1}{2\pi i}\int_{(c_0)} \left(\zeta'(s)\right)^2 x^{s+q-1} G_q(s) \mathrm{d}s,$$

where x is a large parameter, $c_0 > 1$ is a constant, and $q > 2$. We recall the definition (2.9) and estimates of $\zeta'(s)$ ([Gon, p. 127, (20)]).

$$\zeta'(s) \ll \begin{cases} |t|^{\frac{1}{2}-\sigma+\varepsilon}, & \sigma \leq 0, \\ |t|^{\frac{1}{2}(1-\sigma)+\varepsilon}, & 0 \leq \sigma \leq 1, \\ |t|^{\varepsilon}, & \sigma \geq 1, \end{cases} \tag{2.19}$$

where ε is arbitrary and $|t| \geq 1$. Now we know that

$$\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{\Gamma(s)}{\Gamma(s+q)}\left(\frac{x}{n}\right)^s \mathrm{d}s = \begin{cases} 0, & 0 \leq x \leq n \\ \frac{1}{\Gamma(q)}(1-\frac{n}{x})^{q-1}, & x > n \end{cases} \tag{2.20}$$

given that $b > 0$ and $q > 1$. Now we are familiar with the fact $D_{(1)}(n) = O\left(n^{\varepsilon}\right)$ (for any $\varepsilon > 0$) so the series $\sum_{n=1}^{\infty} D_{(1)}(n)n^{-s}$ converges absolutely for $Re(s) > 1$. Using (2.20) and taking $c_0 > 1$ we get

$$\begin{aligned}
\frac{x^{q-1}}{\Gamma(q)}\sum_{n\leq x} D_{(1)}(n)\left(1-\frac{n}{x}\right)^{q-1} &= x^{q-1}\sum_{n\leq x} D_{(1)}(n)\frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty}\frac{\Gamma(s)}{\Gamma(s+q)}\left(\frac{x}{n}\right)^s \mathrm{d}s \\
&+ x^{q-1}\sum_{n\geq x} D_{(1)}(n)\frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty}\frac{\Gamma(s)}{\Gamma(s+q)}\left(\frac{x}{n}\right)^s \mathrm{d}s \\
&= \frac{1}{2\pi i}\int_{(c_0)}\sum_{n=1}^{\infty}\frac{D_{(1)}(n)}{n^s}x^{s+q-1}G_q(s)\mathrm{d}s.
\end{aligned}$$

We have interchanged in the last step summation and integration (The integrals converge absolutely as $\Gamma(s) = O\left(e^{-\frac{\pi}{2}|t|}|t|^{\sigma-\frac{1}{2}}\right)$ for σ in any fixed interval $[a,b]$ and $|t| \longrightarrow \infty$). Now we shall fix a constant c such that $0 < c < \min(1/2, q/2-1)$. By the residue theorem, (2.19) and (2.18) we get

$$\begin{aligned}
D_{q-1}^{(1)}(x) &:= \frac{x^{q-1}}{\Gamma(q)}\sum_{n\leq x} D_{(1)}(n)\left(1-\frac{n}{x}\right)^{q-1} \\
&= \frac{1}{2\pi i}\int_{(c_0)}\left(\zeta'(s)\right)^2 x^{s+q-1}G_q(s)\mathrm{d}s \\
&= \frac{1}{2\pi i}\int_{(-c)}\left(\zeta'(s)\right)^2 x^{s+q-1}G_q(s)\mathrm{d}s + \left(Res_{s=0}+Res_{s=1}\right)\zeta'(s)^2 x^{s+q-1}G_q(s) \\
&= I + \left(Res_{s=0}+Res_{s=1}\right)\zeta'(s)^2 x^{s+q-1}G_q(s),
\end{aligned} \tag{2.21}$$

where

$$I := \frac{1}{2\pi i}\int_{(-c)} \zeta'(s)^2 x^{s+q-1} G_q(s)\mathrm{d}s.$$

Also we have

$$\begin{aligned} Res_{s=0}\left(\zeta'(s)\right)^2 x^{s+q-1}G_q(s) &= \lim_{s\longrightarrow 0} s\left(\zeta'(s)\right)^2 x^{s+q-1}G_q(s) \\ &= \left(\zeta'(0)\right)^2 \frac{x^{q-1}}{\Gamma(q)} = \frac{(\log^2 2\pi)\, x^{q-1}}{4\Gamma(q)}, \end{aligned} \tag{2.22}$$

$$\begin{aligned} Res_{s=1}\left(\zeta'(s)\right)^2 x^{s+q-1}G_q(s) &= \lim_{s\longrightarrow 1} \frac{1}{3!}\frac{d^3}{\mathrm{d}s^3}\left\{(s-1)^4\left(\zeta'(s)\right)^2 x^{s+q-1}G_q(s)\right\} \\ &= \frac{x^q}{\Gamma(1+q)}\left(\frac{\log^3 x}{3!} - \frac{2\gamma_1 \log x}{1!} - 4\gamma_2\right) \\ &+ x^q g_q(1)\left(\frac{\log^2 x}{2!} - 2\gamma_1\right) + x^q g_q(2)\frac{\log x}{1!} + x^q g_q(3), \end{aligned} \tag{2.23}$$

where γ_i is defined in (2.8) and $g_q(i)$ $(i = 1, 2, 3)$ is defined as follows:

$$\frac{\Gamma(s)}{\Gamma(s+q)} = \frac{1}{\Gamma(1+q)} + g_q(1)(s-1) + g_q(2)(s-1)^2 + g_q(3)(s-1)^3 + \cdots$$

Using the functional equation for $\zeta(s)$, that is, $\zeta(s) = \chi(s)\zeta(1-s)$ (where $\chi(s) = (2\pi)^s/2\Gamma(s)\cos(s\pi/2)$), we observe that

$$\zeta'(s)^2 = \chi'(s)^2\zeta^2(1-s) - 2\chi(s)\chi'(s)\zeta(1-s)\zeta'(1-s) + \chi^2(s)\zeta'(1-s)^2. \tag{2.24}$$

Now we use (2.24) in (2.21) to get the following

$$\begin{aligned} I = &\frac{1}{2\pi i}\int_{(-c)} \chi'(s)^2\zeta^2(1-s)x^{s+q-1}G_q(s)\mathrm{d}s \\ &- 2\frac{1}{2\pi i}\int_{(-c)} \chi(s)\chi'(s)\zeta(1-s)\zeta'(1-s)x^{s+q-1}G_q(s)\mathrm{d}s \\ &+ \frac{1}{2\pi i}\int_{(-c)} \chi^2(s)\zeta'(1-s)^2 x^{s+q-1}G_q(s)\mathrm{d}s. \end{aligned}$$

Then we interchange the integration and summation to obtain the expression (2.25) below. Since using the formula $\chi(s) \sim \left(\frac{|t|}{2\pi}\right)^{1/2-\sigma}$ as $|t| \to \infty$ and $\chi'(s) \sim$

$\left(-\log\frac{|t|}{2\pi}\right)\left(\frac{|t|}{2\pi}\right)^{1/2-\sigma}$ as $|t|\to\infty$ (see (2.32) below) one can easily show that the integrals are absolutely convergent.

$$\begin{aligned} I =& \sum_{n=1}^{\infty}\frac{d(n)}{n^q}\frac{1}{2\pi i}\int_{(-c)}\left(\chi'(s)\right)^2(nx)^{s+q-1}G_q(s)\mathrm{d}s \\ &-\sum_{n=1}^{\infty}\frac{d_{(0,1)}(n)}{n^q}\frac{1}{2\pi i}\int_{(-c)}2\chi(s)\chi'(s)(nx)^{s+q-1}G_q(s)\mathrm{d}s \\ &+\sum_{n=1}^{\infty}\frac{D_{(1)}(n)}{n^q}\frac{1}{2\pi i}\int_{(-c)}\chi(s)^2(nx)^{s+q-1}G_q(s)\mathrm{d}s. \end{aligned} \tag{2.25}$$

Now in order to calculate I, we shall define

$$\begin{aligned} J_1(Y) :=& \frac{1}{2\pi i}\int_{(-c)}\left(\chi'(s)\right)^2 Y^{s+q-1}G_q(s)\mathrm{d}s, \\ J_2(Y) :=& \frac{1}{2\pi i}\int_{(-c)}2\chi(s)\chi'(s)Y^{s+q-1}G_q(s)\mathrm{d}s, \\ J_3(Y) :=& \frac{1}{2\pi i}\int_{(-c)}\chi^2(s)Y^{s+q-1}G_q(s)\mathrm{d}s. \end{aligned}$$

Thus we can rewrite the right hand side of (2.25) as follows.

$$I=\sum_{n=1}^{\infty}\frac{d(n)}{n^q}J_1(nx)-\sum_{n=1}^{\infty}\frac{d_{(0,1)}(n)}{n^q}J_2(nx)+\sum_{n=1}^{\infty}\frac{D_{(1)}(n)}{n^q}J_3(nx). \tag{2.26}$$

We shall try to express $J_i(Y)$ $(i=1,2,3)$ by the generalized Bessel function $\lambda_q\left(4\pi\sqrt{Y}\right)$. We have in Section 1 (2.11) and also [I3, Ch. 3]. By (2.11) ([DF, p. 39, (3.21)]) we have

$$J_3(Y)=2\pi Y^q\lambda_q\left(4\pi\sqrt{Y}\right). \tag{2.27}$$

Our next purpose is to express $J_1(Y)$ and $J_2(Y)$ in tems of $J_3(Y)$ as we already have information about it. Hence we have

Lemma 2.3.1

$$J_1(Y) = \frac{\log^2 Y}{4}J_3(Y)+O\left(\left(1+\frac{1}{|q-2c-1|^3}\right)Y^{q-1-c}\log^2 Y\right), \tag{2.28}$$

$$J_2(Y) = -(\log Y)J_3(Y)+O\left(\left(1+\frac{1}{|q-2c-1|}\right)Y^{q-1-c}\right). \tag{2.29}$$

To show Lemma 2.3.1, we shall recall the following estimates on $\chi(s)$.

Lemma 2.3.2 *For $s = \sigma + it$, $|t| \geq 2$, and $a \leq \sigma \leq b$ (a, b are arbitrary fixed real number) we have*

$$\chi(\sigma + it) = \left(\frac{|t|}{2\pi}\right)^{\frac{1}{2}-\sigma-it} e^{i\left(t \pm \frac{\pi}{4}\right)} \left(1 + O\left(\frac{1}{|t|}\right)\right), \tag{2.30}$$

$$\chi^2(\sigma + it) = \left(\frac{|t|}{2\pi}\right)^{1-2\sigma} e^{iE(t)} \left(1 + O\left(\frac{1}{|t|}\right)\right), \tag{2.31}$$

where

$$E(t) = \begin{cases} -2t\log\frac{|t|}{2\pi} + 2t + \frac{\pi}{2} & (t > 0), \\ -2t\log\frac{|t|}{2\pi} + 2t - \frac{\pi}{2} & (t < 0). \end{cases}$$

Moreover, we assume that $|(\sigma-1)/t| < 1$. For the kth derivative of $\chi(s)$, we have

$$\chi^{(k)}(\sigma + it) = \chi(\sigma + it)\left(-\log\frac{|t|}{2\pi}\right)^k + O\left(|t|^{-\frac{1}{2}-\sigma}(\log|t|)^{k-1}\right). \tag{2.32}$$

Proof. For the formula (2.30) see [Ti, p. 78, (4.12.3)]. For the formula (2.32) refer Gonek [Gon, p. 133, Lemma 6].

Proof of Lemma 2.3.1. By integration by parts, we have

$$\begin{aligned} J_2(Y) =& \frac{1}{2\pi i}\int_{(-c)} \left(\chi^2(s)\right)' Y^{s+q-1} G_q(s) \mathrm{d}s \\ =& -\frac{\log Y}{2\pi i}\int_{(-c)} \chi^2(s) Y^{s+q-1} G_q(s) \mathrm{d}s \\ & -\frac{1}{2\pi i}\int_{(-c)} \chi^2(s) Y^{s+q-1} G_q(s)(\psi(s) - \psi(s+q)) \mathrm{d}s, \end{aligned}$$

where $\psi(s) = \Gamma'(s)/\Gamma(s)$ is the digamma function. For the second integral, we divide the integral line $(-c)$ as

$$\int_{(-c)} = \int_{-c-i\infty}^{-c-2i} + \int_{-c-2i}^{-c+2i} + \int_{-c+2i}^{-c+i\infty}. \tag{2.33}$$

Trivially we have

$$\int_{-c-2i}^{-c+2i} \chi^2(s) Y^{s+q-1} G_q(s) \mathrm{d}s \ll Y^{q-1-c}.$$

Since $\chi^2(-c+it) \ll |t|^{1+2c}$ (for $|t| \geq 2$, by (2.31)) and $\psi(-c+it) - \psi(-c+q+it) \ll_q 1/|t|$ (q is bounded and $|t| \geq 2$) we see that

$$\left(\int_{-c-i\infty}^{-c-2i} + \int_{-c+2i}^{-c+i\infty} \right) \chi^2(s) Y^{s+q-1} G_q(s) \left(\psi(s) - \psi(s+q) \right) \mathrm{d}s$$
$$\ll Y^{q-1-c} \int_2^\infty t^{2c-q} \mathrm{d}t \ll \frac{Y^{q-1-c}}{|q-2c-1|}.$$

Therefore we get (2.29).

Before showing (2.28) we shall set a notation. Since through out this chapter, we will often use the division (2.33) of the integral line $(-c)$, to avoid complicated integral expressions we shall write

$$\fint_{(-c)} := \int_{-c-i\infty}^{-c-2i} + \int_{-c+2i}^{-c+i\infty} \quad \text{and} \quad \fint_{-\infty}^{\infty} := \int_{-\infty}^{-2} + \int_2^\infty .$$

Now we proceed to show (2.28). Since $(\chi'(s))^2 = (\chi(s)\chi'(s))' - \chi(s)\chi''(s)$ we have

$$\begin{aligned} J_1(Y) =& \frac{1}{2\pi i} \int_{(-c)} (\chi(s)\chi'(s))' \, Y^{s+q-1} G_q(s) \mathrm{d}s \\ & - \frac{1}{2\pi i} \int_{(-c)} \chi(s)\chi''(s) Y^{s+q-1} G_q(s) \mathrm{d}s \\ =& - \frac{\log Y}{2} J_2(Y) - \frac{1}{2\pi i} \fint_{(-c)} \chi(s)\chi''(s) Y^{s+q-1} G_q(s) \mathrm{d}s \\ & + O\left(Y^{q-1-c} \fint_{-\infty}^{\infty} |t|^{2c-q} \log|t| \mathrm{d}t \right) + O\left(Y^{q-1-c}\right) \\ =& - \frac{\log Y}{2} J_2(Y) + O\left(\left(1 + \frac{1}{|q-2c-1|} + \frac{1}{|q-2c-1|^2} \right) Y^{q-1-c} \right) \\ & - \frac{1}{2\pi i} \fint_{(-c)} \chi(s)\chi''(s) Y^{s+q-1} G_q(s) \mathrm{d}s. \end{aligned}$$

Here we shall make use of Lemma 2.3.2 and (2.18).

$$J_1' = - \frac{1}{2\pi i} \fint_{(-c)} \chi(s)\chi''(s) Y^{s+q-1} G_q(s) \mathrm{d}s$$

$$= -\frac{1}{2\pi i}\fint_{-\infty}^{\infty}\left(\frac{|t|}{2\pi}\right)^{1+2c}\left(-\log\frac{|t|}{2\pi}\right)^{2}e^{iE(t)}Y^{-1-c+q+it}G_q(-c+it)i\mathrm{d}t$$
$$+O\left(Y^{q-1-c}\fint_{-\infty}^{\infty}|t|^{2c-q}\log^2|t|\mathrm{d}t\right).$$

We have by integration by parts, using $\frac{d}{\mathrm{d}t}\left(\frac{e^{iE(t)}}{2i}\right)=\left(-\log\frac{|t|}{2\pi}\right)e^{iE(t)}$,

$$-\frac{1}{2\pi i}\fint_{-\infty}^{\infty}\left(\frac{|t|}{2\pi}\right)^{1+2c}\left(-\log\frac{|t|}{2\pi}\right)^{2}e^{iE(t)}Y^{-1-c+q+it}G_q(-c+it)i\mathrm{d}t$$
$$=\frac{\log Y}{2}\frac{1}{2\pi i}\fint_{-\infty}^{\infty}\left(\frac{|t|}{2\pi}\right)^{1+2c}\left(-\log\frac{|t|}{2\pi}\right)e^{iE(t)}Y^{-1-c+q+it}G_q(-c+it)i\mathrm{d}t$$
$$+O\left(\left(1+\frac{1}{|q-2c-1|}+\frac{1}{|q-2c-1|^2}+\frac{1}{|q-2c-1|^3}\right)Y^{q-1-2c}\right).$$

Once again by integration by parts we have

$$J_1' = -\frac{(\log Y)^2}{4}\fint_{-\infty}^{\infty}\left(\frac{|t|}{2\pi}\right)^{1+2c}e^{iE(t)}Y^{-1-c+q+it}G_q(-c+it)i\mathrm{d}t$$
$$+O\left(\left(1+\frac{1}{|q-2c-1|^3}\right)Y^{q-1-c}\log Y\right).$$

Using Lemma 2.3.2 (2.31) we obtain

$$\frac{1}{2\pi i}\fint_{-\infty}^{\infty}\left(\frac{|t|}{2\pi}\right)^{1+2c}e^{iE(t)}Y^{-1-c+q+it}G_q(-c+it)i\mathrm{d}t$$
$$=J_3(Y)+O\left(\left(1+\frac{1}{|q-2c-1|}\right)Y^{q-1-c}\right).$$

Hence

$$J_1' = -\frac{(\log Y)^2}{4}J_3(Y)+O\left(\left(1+\frac{1}{|q-2c-1|^3}\right)Y^{q-2c-1}\log^2 Y\right).$$

Hence we obtain the assertion of (2.28).

Now we take $Y=nx$ for (2.27), (2.28), and (2.29), then by (2.26) we get the following representation for the integral in (2.25).

Theorem 2.3.3 *Let $q>2$ and $0<c<\min(1/2,(q/2)-1)$. For the integral I which is defined in (2.25) we have the following expression involving the* *generalized Bessel function* λ_q.

$$\begin{aligned} I =&\frac{\pi}{2}x^q \sum_{n=1}^{\infty} d(n)(\log nx)^2 \lambda_q\left(4\pi\sqrt{nx}\right) \\ &+ 2\pi x^q \sum_{n=1}^{\infty} d_{(0,1)}(n)(\log nx)\lambda_q\left(4\pi\sqrt{nx}\right) \\ &+ 2\pi x^q \sum_{n=1}^{\infty} D_{(1)}(n)\lambda_q\left(4\pi\sqrt{nx}\right) \\ &+ O\left(\left(1+\frac{1}{|q-2c-1|^3}\right) x^{q-1-c}\log^2 x\right). \end{aligned} \tag{2.34}$$

Remark 2.3.4 Since $\lambda_q\left(4\pi\sqrt{nx}\right) \ll (nx)^{-\frac{q}{2}-\frac{1}{4}}$ (for detail see [DF, p. 42, (4.12)]) the above infinite series are absolutely and uniformly convergent for bounded $q\,(>3/2)$ with respect to x in any closed x-interval excluding origin.

Using this theorem and the asymptotic formula for $\lambda_q(z)$ we shall prove Theorem 2.2.1 in the next section.

2.4 Proof of Theorem 2.2.1 and Corollary 2.2.2

Here, we shall prove Theorem 2.2.1. By (2.21), (2.25) and Theorem 2.3.3 we have

$$\begin{aligned} &\frac{1}{\Gamma(q)} \sum_{n\le x} D_{(1)}(n)(x-n)^{q-1} \\ =&\frac{\pi x^q}{2} \sum_{n=1}^{\infty} d(n)(\log nx)^2 \lambda_q(4\pi\sqrt{nx}) \\ &+ 2\pi x^q \sum_{n=1}^{\infty} d_{(0,1)}(n)(\log nx)\lambda_q(4\pi\sqrt{nx}) \\ &+ 2\pi x^q \sum_{n=1}^{\infty} D_{(1)}(n)\lambda_q(4\pi\sqrt{nx}) \\ &+ Res_{s=0}\left(\zeta'(s)\right)^2 x^{s+q-1}G_q(s) + Res_{s=1}\left(\zeta'(s)\right)^2 x^{s+q-1}G_q(s) + O\left(x^{q-1-c}\right) \\ &+ O\left(\left(1+\frac{1}{|q-2c-1|^3}\right) x^{q-1-c}\log^2 x\right) \end{aligned} \tag{2.35}$$

for $q>2$ and $0<c<\min\left(1/2, (q/2)-1\right)$.

Here we shall consider the asymptotic formula ([DF, p. 42 (4.12)]) for $\lambda_q(4\pi\sqrt{nx})$:

$$\begin{aligned}\lambda_q(4\pi\sqrt{nx}) =& \frac{1}{(nx)^{\frac{q}{2}+\frac{1}{4}}}\left(A_q \sin\left(4\pi\sqrt{nx}-\frac{\pi}{4}-\frac{q\pi}{2}\right)+B_q e^{-4\pi\sqrt{nx}}\right)\\ &+O\left(\frac{1}{(nx)^{\frac{q}{2}+\frac{5}{4}}}\right)+O\left(\frac{1}{(nx)^2}\right)\end{aligned} \tag{2.36}$$

and we note that A_q and B_q are bounded for all q, n and x. Moreover, the above O-constant does not depend on q over any finite part of q-plane. However, for a fixed x the convergence of the series in (2.35) are uniform with respect to any q in any finite part of $Re(q) > \frac{3}{2}$. Both the sides of (2.35) are analytic functions of q in the region $Re(q) > \frac{3}{2}$. By the grace of the above formula (2.36) and continuity we observe that (2.35) is valid for $q = 2$.

If $q = 2$, by (2.22) and (2.23) we observe that

$$\begin{aligned}&Res_{s=0}\left(\zeta'(s)\right)^2 x^{s+1}\frac{\Gamma(s)}{\Gamma(s+2)} = \frac{(\log 2\pi)^2}{4}x,\\ &Res_{s=1}\left(\zeta'(s)\right)^2 x^{s+1}\frac{\Gamma(s)}{\Gamma(s+2)}\\ &= x^2\left(\frac{\log^3 x}{12}-\frac{3\log^2 x}{8}+\left(\frac{7}{8}-\gamma_1\right)\log x+\left(-2\gamma_2+\frac{3}{2}\gamma_1-\frac{15}{16}\right)\right).\end{aligned}$$

By the definition (2.12) of $\Delta_{(1)}(x)$ we see that

$$\begin{aligned}\int_1^X \Delta_{(1)}(x)\mathrm{d}x =& \int_1^X\left(\sum_{n\le x}D_{(1)}(n)-Res_{s=1}\left(\zeta'(s)\right)^2\frac{x^s}{s}-\left(\zeta'(0)\right)^2\right)\mathrm{d}x\\ =& \sum_{n\le X}D_{(1)}(X-n)-\int_1^X Res_{s=1}\left(\zeta'(s)\right)^2\frac{x^s}{s}\mathrm{d}x-\left(\zeta'(0)\right)^2 X+O(1).\end{aligned} \tag{2.37}$$

We shall consider $\int_1^X Res_{s=1}\left(\zeta'(s)\right)^2 \frac{x^s}{s}\mathrm{d}x$. Here we shall note that in (2.35) if we put $q = 2$, then we have $Res_{s=1}\left(\zeta'(s)\right)^2 x^{s+1}G_2(s) = Res_{s=1}\left(\zeta'(s)\right)^2 x^{s+1}/(s(s+1))$. Moreover we observe that

$$Res_{s=1}\left(\zeta'(s)\right)^2\frac{x^s}{s} = \frac{d}{\mathrm{d}x}Res_{s=1}\left(\zeta'(s)\right)^2\frac{x^{s+1}}{s(s+1)}.$$

This implies that

$$\int_1^X Res_{s=1} \left(\zeta'(s)\right)^2 \frac{x^s}{s} \mathrm{d}x = Res_{s=1} \left(\zeta'(s)\right)^2 \frac{X^{s+1}}{s(s+1)} + O(1). \tag{2.38}$$

By (2.37) and (2.38) we get

$$\sum_{n \leq X} D_{(1)}(n)(X-n) = \int_1^X \Delta_{(1)}(x)\mathrm{d}x + Res_{s=1} \left(\zeta'(s)\right)^2 \frac{X^{s+1}}{s(s+1)} + \left(\zeta'(0)\right)^2 X + O(1). \tag{2.39}$$

Therefore substituting (2.39) into (2.35) with $q = 2$, and choosing $c = 1/2 - \delta$ (a small $\delta > 0$) we obtain the assertion of Theorem 2.2.1.

Next we shall show Corollary 2.2.2. Using [I3, p. 85, p. 91]

$$\lambda_q(z) \sim \left(\frac{2}{z}\right)^q \left(-Y_q(z) + \frac{2e^{q\pi i}}{\pi} K_q(z)\right) - \frac{2}{\pi} \sum_{r=1}^{\infty} \frac{\Gamma(2r)}{\Gamma(q-2r+1)} \left(\frac{2}{z}\right)^{4r},$$

$$Y_q(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} \left\{ \sin\left(z - \frac{\pi q}{2} - \frac{\pi}{4}\right) \sum_{m=0}^{\infty} \frac{(-1)^m (q,2m)}{(2z)^{2m}} + \cos\left(z - \frac{\pi q}{2} - \frac{\pi}{4}\right) \sum_{m=0}^{\infty} \frac{(-1)^m (q,2m+1)}{(2z)^{2m+1}} \right\},$$

$$K_q(z) \sim \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \sum_{m=0}^{\infty} \frac{(q,m)}{(2z)^m},$$

where $(q,m) = \Gamma(q+m+1/2)/(m!\Gamma(q-m+1/2))$, we obtain

$$\lambda_2(4\pi\sqrt{nx}) = \frac{1}{4\sqrt{2}\pi^3 (nx)^{5/4}} \sin\left(4\pi\sqrt{nx} - \frac{\pi}{4}\right) + O\left(\frac{1}{(nx)^{7/4}}\right). \tag{2.40}$$

We substitute (2.40) into (2.13), then

$$\begin{aligned} &\int_1^X \Delta_{(1)}(x)\mathrm{d}x \\ &= \frac{X^{\frac{3}{4}} \log^2 X}{8\sqrt{2}\pi^2} \sum_{n=1}^{\infty} \frac{d(n)}{n^{5/4}} \sin\left(4\pi\sqrt{nX} - \frac{\pi}{4}\right) \\ &+ \frac{X^{\frac{3}{4}} \log X}{4\sqrt{2}\pi^2} \sum_{n=1}^{\infty} \frac{\left(d(n)\log n + 2d_{(0,1)}(n)\right)}{n^{5/4}} \sin\left(4\pi\sqrt{nX} - \frac{\pi}{4}\right) \end{aligned}$$

$$+\frac{X^{\frac{3}{4}}}{\sqrt{2}\pi^2}\sum_{n=1}^{\infty}\frac{\left(\frac{1}{8}d(n)\log^2 n+\frac{1}{2}d_{(0,1)}(n)\log n+\frac{1}{2}D_{(1)}(n)\right)}{n^{5/4}}\sin\left(4\pi\sqrt{nX}-\frac{\pi}{4}\right)$$
$$+O\left(X^{\frac{1}{2}+\varepsilon}\right).$$

We remark the relation in ([Hal, p. 297])

$$d_{(0,1)}(n)=-\frac{1}{2}d(n)\log n,\quad D_{(1)}(n)=\frac{1}{2}d(n)\log^2 n-l(n)$$

(for the definition of $l(s)$ see (2.15)) to obtain the assertion of Corollary 2.2.2.

2.5 Proof of Theorems 2.2.3 and 2.2.4

We give proofs of Theorems 2.2.3 and Theorem 2.2.4. To this end, we apply Lemma 1 of [F, p. 6] which is a generalization of a Lemma of Segal [S, p. 279, p. 765].

Lemma 2.5.1 (Furuya [F, p. 6]) *Let $f(n)$ be an arithmetical function saisfying*

$$\sum_{n\le x}f(n)=g(x)+E(x),$$

where $g(x)$ is the main term and $E(x)$ is the error term. We assume that $g(x)$ is continuously differentiable for $x\ge 1$ and fix a natural number k. Then we have

$$\sum_{n\le x}E^k(n)=\left(\frac{1}{2}-\psi(x)\right)E^k(x)+\int_1^x E^k(u)\mathrm{d}u$$
$$+k\int_1^x\left(\frac{1}{2}-\psi(u)\right)g'(u)E^{k-1}(u)\mathrm{d}u,$$

where $\psi(x)=x-[x]-1/2$.

Proof of Theorem 2.2.3. We use this lemma for $k=1$, $g(x)=xP_{(3)}(\log x)+(\zeta'(0))^2$ and $E(x)=\Delta_{(1)}(x)$ (see (2.7) and (2.12)). Also throughout this section, we will use the notation $B_1(x)$ to denote the Bernoulli polynomial $\psi(x)=x-[x]-1/2$ in order to avoid any discrepancy (notation for digamma function is also $\psi(u)$ which we have used already). Then

$$\sum_{n\le x}\Delta_{(1)}(n) = \left(\frac{1}{2}-B_1(x)\right)\Delta_{(1)}(x) + \int_1^x \Delta_{(1)}(u)\mathrm{d}u + \frac{1}{2}(g(x)-g(1)) - \int_1^x B_1(u)g'(u)\mathrm{d}u.$$

Since $\Delta_{(1)}(x) \ll x^{1/3+\varepsilon}$ which is proved in ([M]) and $\int_1^x B_1(u)g'(u)\mathrm{d}u \ll \log^3 x$ (see (2.41) and (2.42)), immediately we get the first assertion (2.16) of Theorem 2.2.3. Moreover, $\int_1^x \Delta_{(1)}(u)\mathrm{d}u \ll x^{3/4}\log^2 x$ so we obtain the second assertion of (2.17).

Let us prove $\int_1^x B_1(u)g'(u)\mathrm{d}u \ll \log^3 x$. We take $N=[x]$. Also we will make use of the following facts

$$g'(u) = \frac{\log^3 u}{3!} - 2\gamma_1\log u - 4\gamma_2, \tag{2.41}$$

$B_2'(t) = B_1(t)\ \forall t\in(0,1)$, both the functions are periodic with period 1 and $B_2(1)=B_2(0)=\frac{1}{6}$ to estimate the following:

$$\begin{aligned}
&\int_1^x B_1(u)g'(u)\mathrm{d}u\\
&= \int_1^N B_1(u)g'(u)\mathrm{d}u + O\left(\log^3 N\right)\\
&= \sum_{n=1}^{N-1}\int_n^{n+1} B_1(u)g'(u)\mathrm{d}u + O\left(\log^3 N\right)\\
&= \sum_{n=1}^{N-1}\int_0^1 B_1(t)g'(n+t)\mathrm{d}t + O\left(\log^3 N\right)\\
&= \sum_{n=1}^{N-1}\int_0^1 B_2'(t)g'(n+t)\mathrm{d}t + O\left(\log^3 N\right)\\
&= \sum_{n=1}^{N-1}\left(B_2(t)g'(n+t)|_0^1 - \int_0^1 B_2(t)g''(n+t)\mathrm{d}t\right) + O\left(\log^3 N\right)\\
&= \frac{1}{6}\sum_{n=1}^{N-1}(g'(n+1)-g'(n)) + O\left(\sum_{n=1}^{N-1}\int_0^1\frac{\log^2(n+t)}{(n+t)}\mathrm{d}t\right) + O\left(\log^3 N\right)\\
&= O\left(\log^3 N\right) + O\left(\log^2 N\sum_{n=1}^{N-1}\frac{1}{n}\right)
\end{aligned}$$

$$= O\left(\log^3 N\right) \tag{2.42}$$

Proof of Theorem 2.2.4. We shall apply Lemma 2.5.1 for $k = 2$, and get

$$\begin{aligned}\sum_{n\le x}\Delta_{(1)}^2(n) = \int_1^x \Delta_{(1)}^2(u)\mathrm{d}u + O\left(x^{\frac{2}{3}+\varepsilon}\right) \\ + 2\int_1^x\left(\frac{1}{2} - B_1(u)\right)g'(u)\Delta_{(1)}(u)\mathrm{d}u,\end{aligned} \tag{2.43}$$

where $\varepsilon > 0$ is a small constant. Since we have (2.41) so we need to estimate the following integrals one by one

$$\int_1^x\left(\frac{1}{2} - B_1(u)\right)\Delta_{(1)}(u)\log^j u\mathrm{d}u \quad (\text{for } j = 3, 1, 0). \tag{2.44}$$

We have denoted $[x]$ by N, so we have

$$\begin{aligned}\int_1^x\left(\frac{1}{2} - B_1(u)\right)\Delta_{(1)}(u)\log^j u\mathrm{d}u = \int_1^N\left(\frac{1}{2} - B_1(u)\right)\Delta_{(1)}(u)\log^j u\mathrm{d}u \\ + O\left(x^{\frac{2}{3}+\varepsilon}\right).\end{aligned} \tag{2.45}$$

We remark that

$$\begin{aligned}&\int_1^N\left(\frac{1}{2} - B_1(u)\right)\Delta_{(1)}(u)\log^j u\mathrm{d}u \\ &= \sum_{n=1}^{N-1}\int_n^{n+1}(1 - \{u\})\Delta_{(1)}(u)\log^j u\mathrm{d}u \\ &= \int_0^1(1-t)\sum_{n=1}^{N-1}\left(\Delta_{(1)}(n+t)\log^j(n+t) - \Delta_{(1)}(n)\log^j n\right)\mathrm{d}t \\ &\quad + \left(\int_0^1(1-t)\mathrm{d}t\right)\sum_{n=1}^{N-1}\Delta_{(1)}(n)\log^j n \quad (j = 3, 1, 0).\end{aligned} \tag{2.46}$$

To calculate the above we prepare the following formulas.

Lemma 2.5.2 *Let j denote non-negative integers, $T \ge 1$ a real number and $N \ge 1$ an integer. We have*

$$\int_1^T \log^j t\mathrm{d}t = T\sum_{k=0}^{j}(-1)^k\frac{j!}{(j-k)!}\log^{j-k}T - (-1)^j j!, \tag{2.47}$$

$$\sum_{n=1}^{N} \log^j n = \int_1^N \log^j t \mathrm{d}t + O\left(\log^j N\right). \tag{2.48}$$

Proof. Using integration by parts we get (2.47) and by the Euler-Maclaurin formula we find the second assertion (2.48). Moreover, we need the following lemmas

Lemma 2.5.3

$$\begin{aligned}\sum_{n=1}^{N-1} \Delta_{(1)}(n) \log^3 n =& \frac{N \log^6 N}{12} - \frac{N \log^5 N}{2} + \left(\frac{5}{2} - \gamma_1\right) N \log^4 N \\ &- 2c_1 N \log^3 N + 6c_1 N \log^2 N - 12c_1 N \log N + 12c_1 N \\ &+ O\left(N^{\frac{3}{4}} \log^3 N\right)\end{aligned} \tag{2.49}$$

$$\begin{aligned}\sum_{n=1}^{N-1} \Delta_{(1)}(n) \log n =& \frac{N \log^3 N}{12} - \frac{1}{3} N \log^3 N + (1-\gamma_1) N \log^2 N - c_2 N \log N \\ &+ 2c_2 N + O\left(N^{\frac{3}{4}} \log^4 N\right),\end{aligned} \tag{2.50}$$

where $c_1 = \gamma_2 - 2\gamma_1 + 5$ *and* $c_2 = \gamma_2 - \gamma_1 + 1$.

Proof. Using the result of Thorem 2.2.3 and partial summation, we have

$$\begin{aligned}&\sum_{n=1}^{N-1} \Delta_{(1)}(n) \log^3 n \\ &= \frac{N \log^6 N}{12} - \frac{N \log^4 N}{4} + \frac{(1-2\gamma_1) N \log^4 N}{2} - \left(2\gamma_2 - \gamma_1 + \frac{1}{2}\right) N \log^3 N \\ &- \frac{1}{4} \int_1^{N-1} \log^5 t \mathrm{d}t + \frac{3}{4} \int_1^{N-1} \log^4 t \mathrm{d}t - \frac{3(1-2\gamma_1)}{2} \int_1^{N-1} \log^3 t \mathrm{d}t \\ &+ 3\left(2\gamma_2 - \gamma_1 + \frac{1}{2}\right) + O\left(N^{\frac{3}{4}} \log^3 N\right).\end{aligned}$$

By (2.47) of Lemma 2.5.2 we have the first assertion (2.49). By the same way, we get the second assertion (2.50). We shall now consider the case $j = 3$ in (2.46).

$$\int_1^N \left(\frac{1}{2} - B_1(u)\right) (\log u)^3 \Delta_{(1)}(u) \mathrm{d}u$$

$$
\begin{aligned}
&= \int_0^1 (1-t) \sum_{n=1}^{N-1} \Delta_{(1)}(n+t) \left(\log^3(n+t) - \log^3 n\right) \mathrm{d}t \\
&\quad + \int_0^1 (1-t) \sum_{n=1}^{N-1} \left(\Delta_{(1)}(n+t) - \Delta_{(1)}(n)\right) \log^3 n \mathrm{d}t \\
&\quad + \frac{1}{2} \sum_{n=1}^{N-1} \Delta_{(1)}(n) \log^3 n \\
&=: K_1 + K_2 + K_3 \text{ (say)}.
\end{aligned} \tag{2.51}
$$

As $\log^3(n+t) - \log^3 n = O\left(n^{-1}\log^2 n\right)$, we have by ([M]) $\Delta_{(1)}(n+t) \ll n^{1/3+\varepsilon}$. Therefore we get $K_1 = x^{1/3+\varepsilon}$. For K_2, since $(n+t)\log^3(n+t) - n\log^3 n = t\log^3 n + 3t\log^2 n + O\left(n^{-1}\log^2 n\right)$, $(n+t)\log^2(n+t) - n\log^2(n) = t\log^2 n + 2t\log n + O\left(n^{-1}\log n\right)$, and $(n+t)\log(n+t) - n\log n = t\log n + t + O\left(n^{-1}\right)$ we find that

$$
\begin{aligned}
\Delta_{(1)}(n+t) - \Delta_{(1)}(n) &= -(g(n+t) - g(n)) \\
&= -\left(\frac{t\log^3 n}{6} - 2\gamma_1 t\log n - 4\gamma_2 t + O\left(\frac{\log^2 n}{n}\right)\right).
\end{aligned} \tag{2.52}
$$

By (2.52) and Lemma 2.5.2 we have

$$
\begin{aligned}
K_2 &= -\frac{1}{6}\int_0^1 (1-t)t\mathrm{d}t \sum_{n=1}^{N-1} \log^6 n + 2\gamma_1 \int_0^1 (1-t)t\mathrm{d}t \sum_{n=1}^{N-1} \log^4 n \\
&\quad + 4\gamma_2 \int_0^1 (1-t)t\mathrm{d}t \sum_{n=1}^{N-1} \log^3 n + O\left(\int_0^1 (1-t)\mathrm{d}t \sum_{n=1}^{N-1} \frac{\log^5 n}{n}\right) \\
&= -\frac{1}{36} N\log^6 N + \frac{1}{6} N\log^5 N + \left(\frac{\gamma_1}{3} - \frac{5}{6}\right) N\log^4 N \\
&\quad + \frac{2}{3}\left(\gamma_2 - 2\gamma_1 + 5\right) N\log^3 N - 2\left(\gamma_2 - 2\gamma_1 + 5\right) N\log^2 N \\
&\quad + 4(\gamma_2 - 2\gamma_1 + 5) N\log N - 4(\gamma_2 - 2\gamma_2 + 5)N + O\left(\log^6 N\right).
\end{aligned}
$$

For K_3 we use (2.49) of Lemma 2.5.3. Hence we get

$$\begin{aligned}&\int_1^N \left(\frac{1}{2}-B_1(u)\right)\Delta_{(1)}(u)\log^3 u du\\&=\frac{1}{72}N\log^6 N-\frac{1}{12}N\log^5 N+\left(\frac{5}{12}-\frac{\gamma_1}{6}\right)N\log^4 N-\frac{c_1}{3}N\log^3 N\\&\quad+c_1N\log^2 N-2c_1N\log N+2c_1N+O\left(N^{\frac{3}{4}}\log^5 N\right),\quad (c_1=\gamma_2-2\gamma_1+5).\end{aligned} \tag{2.53}$$

By calculations (using Lemma 2.5.2) similar to the above we obtain

$$\begin{aligned}&\int_1^N \left(\frac{1}{2}-B_1(u)\right)\Delta_{(1)}(u)\log u du\\&=\frac{1}{72}N\log^4 N-\frac{1}{18}N\log^3 N+\left(\frac{1}{6}-\frac{\gamma_1}{6}\right)N\log^2 N-\frac{c_2}{3}N\log N\\&\quad+\frac{c_2}{3}N\log N+\frac{c_2}{3}N+O\left(N^{\frac{3}{4}}\log^4 N\right),\quad (c_2=\gamma_2-\gamma_1+1).\end{aligned} \tag{2.54}$$

Moreover by the same argument as above and using Theorem 2.2.3 we get

$$\begin{aligned}&\int_1^N \left(\frac{1}{2}-B_1(u)\right)\Delta_{(1)}(u)du\\&=\frac{1}{72}N\log^3 N-\frac{1}{24}N\log^2 N+\left(\frac{1}{12}-\frac{\gamma_1}{6}\right)N\log N\\&\quad+\left(\frac{2\gamma_2}{3}+\frac{\gamma_1}{6}-\frac{1}{12}\right)+O\left(N^{\frac{3}{4}}\log^2 N\right).\end{aligned} \tag{2.55}$$

From the above results (2.43)–(2.46), (2.51), and (2.53)–(2.55) we obtain the assertion of Theorem 2.2.4.

2.6 A concluding remark

In this chapter, we studied averages of $\Delta_{(1)}(x)$ as a new direction from the classical divisor problem and a further development of [M]. Here we shall compare Voronoï's result (2.1) and our result (2.14). These results are similar. But the error term in (2.14) is bigger than that of (2.1). The cause is the employment of Gonek's Lemma (2.32) in calculations for I_1 and I_2 of (2.25). It rises the error terms in (2.35). To overcome this difficuty, we have to analyze the integrals

$$\frac{1}{2\pi i}\int_{(-c)} \chi(s)\chi'(s)(nx)^{s+q-1}G_q(s)\mathrm{d}s,$$
$$\frac{1}{2\pi i}\int_{(-c)} \left(\chi'(s)\right)^2 (nx)^{s+q-1}G_q(s)\mathrm{d}s,$$

directly, and find formula which are connected to the generalaized Bessel function $\lambda_q(z)$. This is a problem for future consideration. We expect that the error terms $O\left(X^{\frac{1}{2}+\varepsilon}\right)$ in (2.13) and (2.14)to be smaller. We conjecture :

Conjecture. *The error terms $O\left(X^{1/2+\varepsilon}\right)$ in (2.13) and (2.14) can be improved to $O\left(X^{\frac{1}{4}+\varepsilon}\right)$.*

CHAPTER 3

Some problems on k-free integers

In this chapter, we consider sums of the type $\Phi(x,y;f) = \sum_{n\le x, p|n \Rightarrow p>y} f(n)$ and $\psi(x,y;f) = \sum_{n\le x, p|n\Rightarrow p<y} f(n)$. Through out this chapter, p will denote a prime, f certain kind of arithmetical functions and prove some identities for Φ and ψ which are analogous to the 'so-called' Buchstab's identity. As an application, we will prove some formulas for square-free integers and also we will deduce a formula for the number of non k-free integers $\le x$ whose prime factors $> y$.

3.1 Introduction

An integer with all its prime factors $\le y$ is called y-smooth number. An important problem is to study the distributions of all the y-smooth numbers up to x and $\Psi(x,y)$ denotes the number of integers up to x whose prime factors are all $\le y$ that is:

$$\Psi(x,y) = \sum_{\substack{n\le x \\ p|n\Rightarrow p\le y}} 1.$$

An important identity related to it is the following

$$\Psi(x,y) = \Psi(x,z) - \sum_{y<p\le z} \Psi\left(\frac{x}{p}, p\right), \tag{3.1}$$

where x, y and z are positive real numbers such that $2 \le y \le z \le x$. The above identity is called Buchstab's identity ([CM]). By using this identity (3.1), Chebycheff's estimate

$$\pi(x) = \sum_{p \le x} 1 = O\left(\frac{x}{\log x}\right), \tag{3.2}$$

and Mertens' formula

$$\sum_{p \le x} \frac{1}{p} = \log\log x + A + E_1(x), \quad E_1(x) = O\left(\frac{1}{\log x}\right), \tag{3.3}$$

we obtain for any $\varepsilon > 0$ and $x^{\varepsilon} < y \le x$,

$$\Psi(x, y) = x\rho(u) + O\left(\frac{x}{\log y}\right),$$

where $u = \log x / \log y$ and the function $\rho(u)$ is defined by

$$\rho(u) = \begin{cases} 1 & (0 \le u \le 1), \\ 1 - \int_1^u \frac{\rho(v-1)}{v} dv & (u \ge 1). \end{cases} \tag{3.4}$$

This function $\rho(u)$ is called Dickman's function [D]. Similarly, one can consider the following analogue of Buchstab's identity by defining $\Phi(x, y)$ to be the number of integers $n \le x$ all of whose prime factors are greater than y:

$$\Phi(x, y) = \sum_{\substack{n \le x \\ p | n \Rightarrow p > y}} 1.$$

For $x \ge z \ge y \ge 2$, we have

$$\Phi(x, y) = \Phi(x, z) + \sum_{y < p \le x} \Phi\left(\frac{x}{p}, p\right) + O\left(\frac{x}{y}\right).$$

This identity helps one to derive an asymptotic formula for $\Phi(x, y)$. For any $\varepsilon > 0$ and $x^{\varepsilon} < y \le x$, using the prime number theorem of the form

$$\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right) \tag{3.5}$$

one can get

$$\Phi(x,y) = \frac{x\omega(u) - y}{\log y} + O\left(\frac{x}{\log^2 y}\right). \tag{3.6}$$

Here $\omega(u)$ is defined recursively as:

$$\omega(u) = \begin{cases} \frac{1}{u} & (1 \le u \le 2), \\ \frac{1}{u} + \frac{1}{u}\int_1^{u-1} \omega(v)dv & (u \ge 2). \end{cases} \tag{3.7}$$

Some analogues of $\Psi(x,y)$ and $\Phi(x,y)$ are considered by Alladi [Al1], [Al2] and Ivić [I2]. Being motivated by these studies we shall consider analogues of Buchstab's identity and deduce some results concerning square free integers.

Now we shall define three summatory functions concerned with f as follows:

Definition 3.1.1 Let $x \ge y \ge 2$ and for an arithmetical function f, we define

$$M(x;f) = \sum_{n \le x} f(n),$$
$$\psi(x,y;f) = \sum_{\substack{n \le x \\ p \mid n \Rightarrow p < y}} f(n),$$
$$\Phi(x,y;f) = \sum_{\substack{n \le x \\ p \mid n \Rightarrow p > y}} f(n).$$

Remark 3.1.2 If $y \ge x$, then clearly,

$$\psi(x,y;f) = M(x;f) + O(|f(x)|) \quad \text{and} \quad \Phi(x,y;f) = 1.$$

Now we add two restrictions on f:

$$\begin{cases} (A) & f \text{ is multiplicative}, \\ (B) & f(p^m) = 0 \text{ for any prime and positive integer } m \ge 2. \end{cases} \tag{3.8}$$

Under these assumptions, we obtain analogues of Buchstab's identity (see, e.g., Tenenbaum [T, p. 365, p. 398]).

Theorem 3.1.3 *Keeping the notations as above and for $x \geq z \geq y \geq 2$,*

$$\psi(x;y;f) = 1 + \sum_{p<y} f(p)\psi\left(\frac{x}{p}, p; f\right), \tag{3.9}$$

$$\psi(x,y;f) = \psi(x,z;f) - \sum_{y\leq p<z} f(p)\psi\left(\frac{x}{p}, p; f\right), \tag{3.10}$$

$$\Phi(x,y;f) = 1 + \sum_{y<p\leq x} f(p)\Phi\left(\frac{x}{p}, p; f\right), \tag{3.11}$$

$$\Phi(x,y;f) = \Phi(x,z;f) + \sum_{y<p\leq z} f(p)\Phi\left(\frac{x}{p}, p; f\right). \tag{3.12}$$

We shall apply the above formulas (3.11) and (3.12) to the arithmetical functions μ, μ^2 and μ/N, where μ is the Möbius function and $N(n) = n$. These three functions satisfy the required conditions (3.8).

For example we have

Theorem 3.1.4 *For $x^\varepsilon < y \leq x$, then*

$$\Phi\left(x, y; \frac{\mu}{N}\right) = \rho(u) + O\left(\frac{1}{\log y}\right), \tag{3.13}$$

where $u = \log x/\log y$ and $\rho(u)$ is the Dickman function.

Corollary 3.1.5 *For any $\alpha > 1$*

$$\lim_{x\to\infty} \Phi\left(x, x^{1/\alpha}; \frac{\mu}{N}\right) = \rho(\alpha).$$

As another application of Theorem 3.1.3, we shall define

$$\Phi_2(x,y) = \Phi(x,y;\mu^2) = \sum_{\substack{n\leq x \\ p|n\Rightarrow p>y}} \mu^2(n), \tag{3.14}$$

$$\mathcal{R}(x,y) = \Phi(x,y;\mu) = \sum_{\substack{n\leq x \\ p|n\Rightarrow p>y}} \mu(n). \tag{3.15}$$

By formulas (3.11) and (3.12) we have

Theorem 3.1.6 *For $x^\varepsilon < y \leq x$, by the prime number theorem of the form*

(3.5), we have

$$\Phi_2(x,y) = \frac{x\omega(u) - y}{\log y} + O\left(\frac{x}{\log^2 y}\right), \tag{3.16}$$

$$\mathcal{R}(x,y) = \frac{x\rho'(u) + y}{\log y} + O\left(\frac{x}{\log^2 y}\right), \tag{3.17}$$

where $u = \log x/\log y$, $\omega(u)$ is the Buchstab function (see (3.7)) and $\rho'(u)$ is the derivative of $\rho(u)$.

Trivially, when $y \geq x \geq 1$ we see $\Phi_2(x,y) = \mathcal{R}(x,y) = 1$.

Remark 3.1.7 Alladi [Al1, p. 87, Theorem 1] studied the asymptotic formula for $\mathcal{R}(x,y)$, by using (3.5). His result showed the error term of (3.17) is $O(x \cdot u^2/\log^2 y)$ uniformly for $x \geq y \geq 2$. We will consider the above theorem by observing the prime number theorem of the form

$$\pi(x) = li(x) + O\left(x \exp\left(-c\sqrt{\log x}\right)\right), \tag{3.18}$$

(where $li(x) = \int_2^x \frac{dt}{\log t}$, $c > 0$ is a constant),

3.2 Proof of Theorem 3.1.3 and an application

First of all we shall prove Theorem 3.1.3. Let f be an arithmetical function satisfying (3.8). By Definition 3.1.1 we have the assertion (3.9) as follows

$$\psi(x,y;f) = 1 + \sum_{p<y} \sum_{\substack{pm \leq x,\, p\nmid m \\ q|m \Rightarrow q<p}} f(pm) = 1 + \sum_{p<y} f(p) \sum_{\substack{pm \leq x \\ q|m \Rightarrow q<p}} f(m).$$

The second assertion (3.10) follows from (3.9) easily.

By an argument similar to the above, we have the formula (3.11)

$$\Phi(x,y;f) = 1 + \sum_{y<p\leq x} \sum_{\substack{pm \leq x,\, p\nmid m \\ q|m \Rightarrow q>p}} f(pm) = 1 + \sum_{y<p\leq x} f(p) \sum_{\substack{m \leq x/p \\ q|m \Rightarrow q>p}} f(m).$$

Form (3.11), we can obtain the identity (3.12) easily.

Let $u = \log x/\log y$. As an application of Theorem 3.1.3, we shall prove Theorem 3.1.4.

Proof of Theorem 3.1.4 Let us assume $u \in (1,2]$, then by Eratosthenes' sieve and (3.3) we observe that

$$\Phi\left(x, y; \frac{\mu}{N}\right) = 1 - \sum_{y<p\le x} \frac{1}{p} = 1 - \log u + O\left(\frac{1}{\log y}\right).$$

Now we assume that the formula (3.13) is true for $u \in (1,2], (2,3], \ldots, (K-1, K]$. In the case of $u \in (K, K+1]$, we put in (3.12), $y = x^{1/u}$ and $z = x^{1/K}$ then

$$\Phi\left(x, x^{1/u}; \frac{\mu}{N}\right) = \rho(K) + O\left(\frac{1}{\log y}\right) - \sum_{x^{1/u}<p\le x^{1/K}} \frac{1}{p}\Phi\left(\frac{x}{p}, p; \frac{\mu}{N}\right).$$

In the above sum, since $\frac{\log \frac{x}{p}}{\log p} \le K$ we shall apply our assumption to get

$$\begin{aligned}
&\sum_{x^{1/u}<p\le x^{1/K}} \left\{\frac{1}{p}\rho\left(\frac{\log x}{\log p} - 1\right) + O\left(\frac{1}{p\log p}\right)\right\} \\
&= \sum_{x^{1/u}<p\le x^{1/K}} \frac{1}{p}\rho\left(\frac{\log x}{\log p} - 1\right) + O\left(\frac{1}{\log y}\right) \\
&= \int_{x^{1/u}}^{x^{1/K}} \rho\left(\frac{\log x}{\log w} - 1\right) d\log\log w + \int_{x^{1/u}}^{x^{1/K}} \rho\left(\frac{\log x}{\log w} - 1\right) dE_1(w) \\
&\quad + O\left(\frac{1}{\log y}\right) \\
&:= A + B + O\left(\frac{1}{\log y}\right) \quad \text{(say)},
\end{aligned}$$

where $E_1(\cdot)$ is the same error term in (3.3). Putting $v = \log x/\log w$ we have $A = \int_K^u \rho(v-1)v^{-1}dv$. Moreover since ρ, ρ' are bounded ([T, p. 366]) we see $B = O(1/\log y)$. Therefore, for $u \in (K, K+1]$ we obtain

$$\begin{aligned}
\Phi\left(x, x^{1/u}; \frac{\mu}{N}\right) &= \rho(K) - \int_K^u \frac{\rho(v-1)}{v} dv + O\left(\frac{1}{\log y}\right) \\
&= \rho(u) + O\left(\frac{1}{\log y}\right).
\end{aligned}$$

From this, we observe that the assertion (3.13) is valid for $x^{\varepsilon} < y \le x$.

3.3 On square-free integers

In this section, we shall consider applications of (3.11) and (3.12) on square-free numbers. So we shall prove Theorem 3.1.6.

Proof of Theorem 3.1.6: We will prove the formula (3.17) only. The other formula (3.16) follows by a similar method. First we shall notice that

$$\rho'(u) = \begin{cases} -\frac{1}{u} & (1 \le u \le 2), \\ -\frac{1}{u} - \frac{1}{u}\int_2^u \rho'(v-1)dv & (u \ge 2). \end{cases} \tag{3.19}$$

By (3.11), Eratosthenes' sieve, the prime number theorem (3.5), and (3.19) we have

$$\begin{aligned} \mathcal{R}(x,y) &= 1 + \sum_{y<p\le x} \mu(p) = 1 - \pi(x) + \pi(y) \\ &= \frac{x\rho'(u) + y}{\log y} + O\left(\frac{x}{\log^2 y}\right) \quad \textit{for } u \in (1,2] \ (\textit{or } \sqrt{x} \le y < x). \end{aligned} \tag{3.20}$$

For $u \in (2,3]$, using (3.12) with $f = \mu$, $y = x^{1/u}$, and $z = x^{1/2}$ we have

$$\mathcal{R}(x,y) = \mathcal{R}(x,x^{1/2}) - \sum_{x^{1/3}<p\le x^{1/2}} \mathcal{R}\left(\frac{x}{p},p\right) + O\left(\frac{x}{\log^2 y}\right).$$

Since $(\log x/p)/\log p = \log x/\log p - 1 \le 2$, using (3.20) we can show that (3.17) is valid for $u \in (2,3]$ (the method is similar to the generalized argument just below).

Here we assume the formula (3.17) is true for $u \in (3,4], (4,5], \ldots, (N-1,N]$ $(N \ge 3$). We shall consider it for $u \in (N, N+1]$ and take $f = \mu$, $y = x^{1/u}$ and $z = x^{1/N}$ in (3.12), then we have

$$\begin{aligned} \mathcal{R}(x,y) = &\frac{x\rho'(N) + x^{1/N}}{\log x^{1/N}} + \frac{y}{\log y} - \frac{y}{\log y} \\ &- \sum_{x^{1/u}<p\le x^{1/N}} \mathcal{R}\left(\frac{x}{p},p\right) + O\left(\frac{x}{\log^2 y}\right). \end{aligned}$$

Since $\frac{\log\frac{x}{p}}{\log p} = \frac{\log x}{\log p} - 1 \le N$ we can get

$$\sum_{x^{1/u}<p\le x^{1/N}} \mathcal{R}\left(\frac{x}{p},p\right) = x \sum_{x^{1/u}<p\le x^{1/N}} \frac{\rho'\left(\frac{\log x}{\log p}-1\right)}{p\log p} + \sum_{x^{1/u}<p\le x^{1/N}} \frac{p}{\log p}$$
$$+ O\left(x \sum_{x^{1/u}<p\le x^{1/N}} \frac{1}{p\log^2 p}\right)$$
$$=: xA + B + C \quad \text{(say)}.$$

Using (3.5) and (3.3) we have B, $C \ll x/\log^2 y$ respectively. Also by (3.3) we see

$$A = \int_{x^{1/u}}^{1/N} \frac{\rho'\left(\frac{\log x}{\log w}-1\right)}{\log w} d\log\log w + \int_{x^{1/u}}^{x^{1/N}} \frac{\rho'\left(\frac{\log x}{\log w}-1\right)}{\log w} dE_1(w).$$

By putting $v = \log x/\log w$, the former integral is

$$-\frac{1}{\log x}\int_u^N \rho'(v-1)dv,$$

and the latter integral is

$$\left[\frac{\rho'\left(\frac{\log x}{\log w}-1\right)}{\log w}E_1(w)\right]_{x^{1/u}}^{x^{1/N}} \tag{3.21}$$
$$+ \log x \int_{x^{1/u}}^{x^{1/N}} \frac{\rho''\left(\frac{\log x}{\log w}-1\right)}{w\log^3 w}E_1(w)dw + \int_{x^{1/u}}^{x^{1/N}} \frac{\rho'\left(\frac{\log x}{\log w}-1\right)}{w\log^2 w}E_1(w)dw.$$

Since ρ' is bounded and $E_1(w) = O(1/\log w)$ the first part of (3.21) is estimated as $O(1/\log^2 y)$. Moreover, since $1/\log y = O(N/\log x)$ and $\log((N+1)/N) = O(1/N)$ we can estimate the middle and last parts of (3.21) as $O(1/\log^2 y)$ respectively. Hence for $u \in (N, N+1]$ we obtain

$$\mathcal{R}(x,y) = \frac{x\rho'(N)}{\log x^{1/N}} + \frac{x}{\log x}\int_u^N \rho'(v-1)dv + \frac{y}{\log y} + O\left(\frac{x}{\log^2 y}\right)$$
$$= \frac{x}{\log y}\left(\frac{(\log y)\rho'(N)}{\log x^{1/N}} + \frac{\log y}{\log x}\int_u^N \rho'(v-1)dv\right) + \frac{y}{\log y} + O\left(\frac{x}{\log^2 y}\right)$$

$$= \frac{x}{\log y}\left(\frac{N}{u}\left(-\frac{1}{N}-\frac{1}{N}\int_2^N \rho'(v-1)dv\right)-\frac{1}{u}\int_N^u \rho'(v-1)dv\right)$$
$$+\frac{y}{\log y}+O\left(\frac{x}{\log^2 y}\right)$$
$$= \frac{x\rho'(u)+y}{\log y}+O\left(\frac{x}{\log^2 y}\right).$$

This shows that the formula (3.17) is valid for $x^\varepsilon < y \le x$. We will consider some applications of Theorem 3.1.6 on square-free integers.

Definition 3.3.1 Let m be a positive square-free integer and $\mathcal{N}(m)$ the number of prime factors of m. For $x \ge y \ge 1$ we define the following counting functions:

$$\mathcal{Q}_{\text{even}}(x,y) := \sum_{\substack{m\le x,\ m \text{ square-free},\ \mathcal{N}(m):\ \text{even} \\ p|m \Rightarrow p>y}} 1 = \sum_{\substack{n\le x \\ p|n\Rightarrow p>y}} \frac{\mu^2(n)+\mu(n)}{2},$$
$$\mathcal{Q}_{\text{odd}}(x,y) := \sum_{\substack{m\le x,\ m \text{ square-free},\ \mathcal{N}(m):\ \text{odd} \\ p|m \Rightarrow p>y}} 1 = \sum_{\substack{n\le x \\ p|n\Rightarrow p>y}} \frac{\mu^2(n)-\mu(n)}{2},$$

where we regard $\mathcal{N}(1) = 1$ as even.

If we use $M(x;\mu) = o(x)$ (which is equivalent to the prime number theorem in the form $\pi(x) \sim x/\log x$) and $M(x;\mu^2) = \frac{6}{\pi^2}x + O(\sqrt{x})$, then we have easily

$$\mathcal{Q}_{\text{even}}(x,1) = \frac{3}{\pi^2}x + o(x) \text{ and } \mathcal{Q}_{\text{odd}}(x,1) = \frac{3}{\pi^2}x + o(x).$$

However if y is large by Theorem 3.1.6 we get the following corollary.

Corollary 3.3.2 *For $x^\varepsilon < y \le x$ and $u = \frac{\log x}{\log y}$,*

$$\mathcal{Q}_{\text{even}}(x,y) = \frac{x}{\log y}\left(\frac{\omega(u)+\rho'(u)}{2}\right) + O\left(\frac{x}{\log^2 y}\right),$$
$$\mathcal{Q}_{\text{odd}}(x,y) = \frac{x}{\log y}\left(\frac{\omega(u)-\rho'(u)}{2}\right) - \frac{y}{\log y} + O\left(\frac{x}{\log^2 y}\right).$$

3.4 Remarks

In this final section, following [T, p. 400, Theorem 3] we shall attempt to extend the range $x^\varepsilon < y \le x$ in Theorem 3.1.6. By the prime number theorem of the form (3.18) we have

$$\sum_{p\le x}\frac{1}{p}=\log\log x+A+O\left(\exp\left(-B\sqrt{\log x}\right)\right), \tag{3.22}$$

where A and B are some positive constants. With the help of (3.18) and (3.22) we obtain the following.

Theorem 3.4.1 *Uniformly for $x\ge y\ge 2$, we have*

$$\Phi_2(x,y)=\frac{x\omega(u)-y}{\log y}+O\left(\frac{x}{\log^2 y}\right), \tag{3.23}$$

$$\mathcal{R}(x,y)=\frac{x\rho'(u)+y}{\log y}+O\left(\frac{x}{\log^2 y}\right), \tag{3.24}$$

where the notation is same as the above.

Proof. Since trivially $\Phi_2(x,y)$ and $\mathcal{R}(x,y)=O(x)$, so if y is bounded then (3.23) and (3.24) are obviously true. So we assume that $y\ge y_0$, where y_0 is a sufficiently large constant. We may also assume that $u>3$ in fact we have already proved the result for $1\le u\le 3$ in theorem 3.1.6. Let $\Delta(x,y)$ be the function implicitly defined by the formula

$$\mathcal{R}(x,y)=\frac{x}{\log y}\left(\rho'(u)+\frac{\Delta(x,y)}{\log y}\right), \tag{3.25}$$

We shall establish by induction on integers $k\ge 3$, that the quantity

$$\Delta_k:=\sup\left\{|\Delta(x,y)|\,|\,y\ge y_0,\ 2<u\le k\right\}.$$

is finite and bounded independently of k. By Theorem 3.1.6 we see that $\Delta_3<+\infty$. Let $k\ge 3$ be such that $\Delta_k<+\infty$. We shall consider the case $y\ge y_0$ and $2<u\le k+1$. By the identity (3.12) with $f=\mu$ and $z=\sqrt{x}$ and (3.25) we observe that

$$\mathcal{R}(x,y)=\mathcal{R}\left(x,\sqrt{x}\right)-\sum_{y<p\le\sqrt{x}}\frac{x}{p\log p}\left\{\rho'\left(\frac{\log x}{\log p}-1\right)+\frac{\theta_p\Delta_k}{\log p}\right\}$$

with $\theta_p = \theta_p(x) \in [-1, 1]$. By (3.5) we have

$$\mathcal{R}\left(x, \sqrt{x}\right) = -\frac{x}{\log x} + O\left(\frac{x}{\log^2 y}\right),$$

By (3.22) for any sufficiently large $y \geq y_0$ we have

$$\sum_{p>y} \frac{1}{p \log^2 p} = \frac{\frac{1}{2} + O\left(\exp\left(-B\sqrt{\log x}\right)\right)}{\log^2 y} \leq \frac{3}{4 \log^2 y},$$

$$H(v) = \sum_{x^{1/v} < p \leq \sqrt{x}} \frac{1}{p} = \log \frac{v}{2} + O\left(\exp\left(-B\sqrt{\log x^{1/v}}\right)\right). \tag{3.26}$$

By the Stieltjes integral with (3.26) we see that

$$\sum_{y<p\leq\sqrt{x}} \frac{\rho'\left(\frac{\log x}{\log p} - 1\right)}{p \log p} = \frac{1}{\log x} \int_2^u \rho'(v-1)dv + O\left(\frac{u \exp\left(-B\sqrt{\log y}\right)}{\log x}\right)$$
$$= \frac{-u\rho'(u) + 1}{\log x} + O\left(\frac{u \exp\left(-B\sqrt{\log y}\right)}{\log x}\right).$$

Collecting the above calculations we have

$$\mathcal{R}(x, y) = \frac{x}{\log y}\left(\rho'(u) + O\left(\exp\left(-B\sqrt{\log y}\right)\right)\right) + \frac{x\left(\theta\Delta_k + O(1)\right)}{\log^2 y}. \tag{3.27}$$

where $|\theta| \leq \frac{3}{4}$. By (3.27) we see that $\Delta_{k+1} \leq max(\Delta_k, \frac{3}{4}\Delta_k + C)$ with a constant $C > 0$. Since $\Delta_k \leq \Delta_{k+1}$, this completes the proof of (3.24). By a similar argument we may prove (3.23). We have also

Corollary 3.4.2 *Uniformly for $x \geq y \geq 2$ we have*

$$\mathcal{Q}_{\text{even}}(x, y) = \frac{x}{\log y}\left(\frac{\omega(u) + \rho'(u)}{2}\right) + O\left(\frac{x}{\log^2 y}\right),$$
$$\mathcal{Q}_{\text{odd}}(x, y) = \frac{x}{\log y}\left(\frac{\omega(u) - \rho'(u)}{2}\right) - \frac{y}{\log y} + O\left(\frac{x}{\log^2 y}\right).$$

The proofs are similar.

Now we will look into some further topics related to non-square free integers and widening the range to non k-free integers.

3.5 Results on non k-free integers

From (3.6) and (3.16), we have

$$D_2(x,y) := \sum_{\substack{n\le x\\ p|n\Rightarrow p>y}} \left(1-\mu^2(n)\right) = \Phi(x,y) - \Phi_2(x,y) = O\left(\frac{x}{\log^2 y}\right)$$

for $(x^\varepsilon < y \le x)$. This quantity $D_2(x,y)$ denotes the number of positive non square-free integers up to x whose all prime factors is greater than y, where a non square-free integer n means that n is divided by a square of a prime p. It may be interesting to improve the above O-term. We shall now generalize this problem. Fix $k \ge 2$ an integer, and define the function

$$\Phi_k(x,y) := \sum_{\substack{n\le x\\ p|n\Rightarrow p>y}} q_k(n),$$

where

$$q_k(n) := \begin{cases} 1 & n \text{ is } k\text{-free},\\ 0 & \text{otherwise}. \end{cases}$$

As in the two cases of $\Phi(x,y)$ and $\Phi_2(x,y)$, applying a 'Buchstab's identity' for $\Phi_k(x,y)$ (see Lemma 3.6.1 below) we shall prove the following theorem.

Theorem 3.5.1 *If $x = y$, then $\Phi_k(x,y) = 1$. For $x^\varepsilon < y \le x$ (any $\varepsilon > 0$) and $u = \log x/\log y$, by the PNT (3.5) we have*

$$\Phi_k(x,y) = \frac{x\omega(u) - y}{\log y} + O\left(\frac{x}{\log^2 y}\right), \tag{3.28}$$

where ω is the Buchstab function (3.7).

From (3.6) and (3.28) we immediately observe that

$$D_k(x,y) := \Phi(x,y) - \Phi_k(x,y) = O\left(\frac{x}{\log^2 y}\right) \quad \text{for } x^\varepsilon < y \le x. \tag{3.29}$$

Here $D_k(x,y)$ denotes that the number of positive 'non k-free' integers $\le x$ whose prime factors $> y$. If an integer n has a prime power p^α ($\alpha \ge k$), then we shall call n as non k-free integer. In this present section, our aim is to improve the above estimate (3.29). It is not difficult to see that the number of non k-free integers $\le x$ is $(1-1/\zeta(k))x + O\left(x^{1/k}\right)$, where $\zeta(k)$ is the value of the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ $(\mathrm{Re}\, s > 1)$ at $s = k$. However, we shall investigate an influence of the condition 'prime factors of $n \le x$ which are greater than y.'

Theorem 3.5.2 *Under the same condition and notation, we have*

$$D_k(x,y) = \begin{cases} 0, & x^{\frac{1}{k}} \le y \le x, \\ \frac{kx^{\frac{1}{k}}}{\log x} - \frac{y}{\log y} + O\left(\frac{x^{\frac{1}{k}}}{\log^2 y}\right), & x^{\frac{1}{k+1}} \le y < x^{\frac{1}{k}}, \\ O\left(\frac{xy^{1-k}}{\log^2 y}\right), & x^{\frac{1}{k+l+1}} \le y < x^{\frac{1}{k+l}},\ l=1,2,\ldots. \end{cases} \tag{3.30}$$

To prove this, we will introduce an analogue of Buchstab's identity for $D_k(x,y)$ given in the next section. By a direct approach with $\pi(t) \sim t/\log t$ and $p_j \sim j \log j$ (p_j denotes jth prime) we get

$$D_k(x,y) \le \sum_{y<p\le x^{1/k}} \frac{x}{p^k} = x \sum_{\pi(y)<j\le\pi\left(x^{1/k}\right)} \frac{1}{p_j^k} \ll \frac{xy^{1-k}}{\log y}.$$

The third assertion of Theorem 3.5.2 is better by a factor $1/\log y$ from the above. In the proof of the theorem, it is important to estimate the difference $\Phi(x,y) - \Phi_k(x,y)$ carefully.

In the above, we need the PNT (3.5). In addition, if we use the prime number theorem in the form

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O\left(x \exp\left(-c\sqrt{\log x}\right)\right), \tag{3.31}$$

(where $c > 0$ is a constant) we obtain the following theorem.

Theorem 3.5.3 *Using the PNT (3.31), we have*

$$D_k(x,y) = O\left(\frac{xy^{1-k}}{\log^2 y}\right) \tag{3.32}$$

uniformly for $x^{1/(k+1)} \ge y \ge 2$.

3.6 Buchstab's identity for k-free integers

To prove Theorem 3.5.1 we shall deduce an analogue of Buchstab's identity for k-free integers.

Lemma 3.6.1 *Keeping the notation in Section 1. For $x \geq z \geq y \geq 1$ we have*

$$\Phi_k(x,y) = 1 + \sum_{y<p\leq x} \sum_{j=1}^{k-1} \Phi_k\left(\frac{x}{p_j}, p\right), \tag{3.33}$$

$$\Phi_k(x,y) = \Phi_k(x,z) + \sum_{y<p\leq z} \sum_{j=1}^{k-1} \Phi_k\left(\frac{x}{p^j}, p\right). \tag{3.34}$$

Proof. Denote by q a prime. By the multiplicative property of $q_k(n)$, and the definition of k-free integers, we obtain the formula (3.33) by the following way.

$$\begin{aligned}
&\Phi_k(x,y) \\
&= 1 + \sum_{y<p\leq x}\left(\sum_{\substack{pm\leq x\\ q|m\Rightarrow q>p}} q_k(pm) + \sum_{\substack{p^2m\leq x\\ q|m\Rightarrow q>p}} q_k\left(p^2m\right) + \cdots + \sum_{\substack{p^{k-1}m\leq x\\ q|m\Rightarrow q>p}} q_k\left(p^{k-1}m\right)\right) \\
&= 1 + \sum_{y<p\leq x}\left(\sum_{\substack{m\leq x/p\\ q|m\Rightarrow q>p}} q_k(m) + \cdots + \sum_{\substack{m\leq x/p^{k-1}\\ q|m\Rightarrow q>p}} q_k(m)\right) \\
&= 1 + \sum_{y<p\leq x} \sum_{j=1}^{k-1} \Phi_k\left(\frac{x}{p^j}, p\right).
\end{aligned}$$

The formula (3.34) is an immediate consequence of (3.33).

We shall recall the following Buchstab identity on $\Phi(x,y)$ (see, e.g. [CM, p. 78]):

$$\Phi(x,y) = \Phi(x,z) + \sum_{y<p\leq z} \sum_{j=1}^{\infty} \Phi\left(\frac{x}{p^j}, p\right) \quad (x \geq z \geq 1). \tag{3.35}$$

Subtracting (3.34) from (3.35) we obtain the Buchstab identity for $D_k(x,y)$ which will help us to deduce Theorem 3.5.2.

Lemma 3.6.2 *For $x \geq z \geq y \geq 1$, we have*

$$D_k(x,y) = D_k(x,z) + \sum_{y<p\leq z} \sum_{j=1}^{k-1} D_k\left(\frac{x}{p^j}, p\right) + \sum_{y<p\leq z} \sum_{j=k}^{\infty} \Phi\left(\frac{x}{p^j}, p\right). \qquad (3.36)$$

Remark 3.6.3 In a proof of the formula (3.6), we use (3.35) and estimate the sum

$$\sum_{y<p\leq z} \sum_{j\geq 2} \Phi\left(\frac{x}{p^j}, p\right) = O\left(\frac{x}{y}\right).$$

However, in our proof of Theorem 3.5.2 we treat

$$\sum_{y<p\leq z} \sum_{j\geq k} \Phi\left(\frac{x}{p^j}, y\right)$$

carefully as it plays an important role in the study. We learn that it is important for the study of the distribution of non k-free integers. This is a new aspect of the Buchstab identity.

3.7 Proof of theorem 3.5.1

First of all, we shall prove Theorem 3.5.1. This is a natural generalization of (3.16). Let k be fixed an integer ≥ 2 and $u = \log x/\log y$, where x and y are large parameters. We will use the PNT (3.5). When $x = y$ we have $\Phi_k(x,x) = 1$, trivially. Next we shall consider $\Phi_k(x,y)$ in the case $x^{1/2} \leq y < x$. Since $x/p < p$ we see $\Phi_k(x/p,p) = 1$. Moreover, for $j \geq 2$ since $x/p^j < 1$ we observe that $\Phi_k(x/p^j,p) = 0$ $(2 \leq j \leq k-1)$. Hence, by (3.33) in Lemma 3.6.1 and (3.5) we obtain

$$\begin{aligned}\Phi_k(x,y) &= 1 + \sum_{y<p\leq x} 1 = \pi(x) - \pi(y) + 1 \\ &= \frac{x}{\log x} - \frac{y}{\log y} + O\left(\frac{x}{\log^2 y}\right) \qquad (3.37)\\ &= \frac{x}{\log y}\frac{1}{u} - \frac{y}{\log y} + O\left(\frac{x}{\log^2 y}\right).\end{aligned}$$

We see that Theorem 3.5.1 is true for $x^{1/2} \le y \le x$. Now we shall consider the case $x^{1/3} \le y < x^{1/2}$. In formula (3.34) of Lemma 3.6.1, take $z = x^{1/2}$ then we have by (3.37)

$$\begin{aligned}\Phi_k(x,y) &= \Phi_k\left(x, x^{\frac{1}{2}}\right) + \sum_{y<p\le x^{\frac{1}{2}}} \sum_{j=1}^{k-1} \Phi_k\left(\frac{x}{p^j}, p\right) \\ &= \frac{x}{\log x^{\frac{1}{2}}}\frac{1}{2} - \frac{x^{\frac{1}{2}}}{\log x^{\frac{1}{2}}} + \sum_{y<p\le x^{\frac{1}{2}}} \Phi_k\left(\frac{x}{p}, p\right) \\ &\quad + \sum_{y<p\le x^{1/2}} \sum_{j=2}^{k-1} \Phi_k\left(\frac{x}{p^j}, p\right) + O\left(\frac{x}{\log^2 y}\right).\end{aligned} \tag{3.38}$$

By using the trivial bound $\Phi_k(x,y) \ll x$ we have

$$\sum_{y<p\le x^{1/2}} \sum_{j=2}^{k-1} \Phi_k\left(\frac{x}{p^j}, p\right) \ll \sum_{y<p\le x^{1/2}} \frac{x}{p^2} \ll x^{\frac{2}{3}}.$$

Here we shall remark that the primes p $(y < p \le x^{1/2})$ satisfy $(x/p)^{1/2} < p \le x/p$. Then by (3.37) we have

$$\begin{aligned}&\sum_{y<p\le x^{\frac{1}{2}}} \Phi_k\left(\frac{x}{p}, p\right) \\ &= \sum_{y<p\le x^{\frac{1}{2}}} \frac{x}{p\log p}\frac{1}{\frac{\log x}{\log p} - 1} - \sum_{y<p\le x^{\frac{1}{2}}} \frac{p}{\log p} + O\left(\sum_{y<p\le x^{\frac{1}{2}}} \frac{x}{p\log^2 p}\right).\end{aligned} \tag{3.39}$$

In the right hand side of (3.39), the second sum is estimated by Chebycheff's estimate $\pi(x) \ll x/\log x$ as follows:

$$\sum_{y<p\le x^{\frac{1}{2}}} \frac{p}{\log p} \ll \frac{x}{\log^2 y}. \tag{3.40}$$

In O-term of the right hand side of (3.39), the sum is estimated by the Mertens formula

$$\sum_{p\le t} \frac{1}{p} = \log\log t + c + R(t), \quad R(t) = O\left(\frac{1}{\log t}\right), \tag{3.41}$$

as follows:

$$\sum_{y<p\leq x^{\frac{1}{2}}} \frac{x}{p\log^2 p} \ll \frac{x}{\log^2 y} \sum_{y<p\leq x^{\frac{1}{2}}} \frac{1}{p} \ll \frac{x}{\log^2 y}. \tag{3.42}$$

For the first sum in the right hand side of (3.39) we shall express it by the Stieltjes integral and by the formula (3.41) we get

$$\begin{aligned}
&\sum_{y<p\leq x^{\frac{1}{2}}} \frac{1}{p\log p} \frac{1}{\frac{\log x}{\log p}-1} \\
&= \int_y^{x^{\frac{1}{2}}} \frac{1}{\left(\frac{\log x}{\log w}-1\right)\log w} d\left(\sum_{p\leq w}\frac{1}{p}\right) \\
&= \int_y^{x^{\frac{1}{2}}} \frac{1}{\left(\frac{\log x}{\log w}-1\right)\log w} d\log\log w + \int_y^{x^{\frac{1}{2}}} \frac{1}{\left(\frac{\log x}{\log w}-1\right)\log w} dR(w) \\
&=: A + B \quad \text{(say)}.
\end{aligned} \tag{3.43}$$

For the integral A we write $v = \log x/\log w$, then

$$\frac{d\log\log w}{dw} = \frac{1}{w\log w} \quad \textit{and} \quad \frac{dv}{dw} = -\frac{v}{w\log w}.$$

Hence we have

$$\begin{aligned}
A &= \int_u^2 \frac{1}{(v-1)\frac{\log x}{v}}\left(-\frac{1}{v}\right)dv = \frac{1}{\log x}\int_2^u \frac{1}{v-1}dv = \frac{1}{\log y}\frac{\log y}{\log x}\int_1^{u-1}\frac{dv}{v} \\
&= \frac{1}{\log y}\cdot\frac{1}{u}\int_1^{u-1}\frac{dv}{v}.
\end{aligned} \tag{3.44}$$

For the integral B, integral by parts gives

$$\begin{aligned}
B &= \left[\frac{1}{\left(\frac{\log x}{\log w}-1\right)\log w}R(w)\right]_y^{x^{\frac{1}{2}}} - \int_y^{x^{\frac{1}{2}}} \frac{1}{w\left(\frac{\log x}{\log w}-1\right)^2\log^2 w}R(w)dw \\
&\ll \frac{1}{\log^2 y} + \int_y^{x^{\frac{1}{2}}} \frac{1}{w\log^3 w}dw \ll \frac{1}{\log^2 y}.
\end{aligned} \tag{3.45}$$

Collecting (3.38), (3.39), (3.40), (3.42), (3.43), (3.44) and (3.45) we observe that

$$\begin{aligned}\Phi_k(x,y) &= \frac{x}{\log x^{\frac{1}{2}}}\frac{1}{2} - \frac{x^{\frac{1}{2}}}{\log x^{\frac{1}{2}}} + \frac{x}{\log y}\frac{1}{u}\int_1^{u-1}\frac{dv}{v} + O\left(\frac{x}{\log^2 y}\right)\\ &= \frac{x}{\log y}\left(\frac{1}{u} + \int_1^{u-1}\frac{dv}{v}\right) - \frac{y}{\log y} + \frac{y\log x^{\frac{1}{2}} - x^{\frac{1}{2}}\log y}{(\log y)(\log x^{\frac{1}{2}})} + O\left(\frac{x}{\log^2 y}\right)\\ &= \frac{x\omega(u) - y}{\log y} + O\left(\frac{x}{\log^2 y}\right) \quad \textit{for} \quad x^{\frac{1}{3}} \le y < x^{\frac{1}{2}}.\end{aligned}$$

So with the previous result (3.37) we obtain Theorem 3.5.1 in the case of $x^{1/3} \le y \le x^{1/2}$. Here we shall assume that Theorem 3.5.1 is true for $x^{1/(m+1)} \le y \le x^{1/m}$ $(m = 1, 2, \ldots, N-1)$. We shall consider the case $x^{1/(N+1)} \le y < x^{1/N}$. In the formula (3.34) we chose $z = x^{1/N}$, and use the assumption. The primes p in $y < p \le x^{1/N}$ satisfy $(x/p)^{1/N} \le p \le (x/p)^{1/(N-1)}$, then we can use the assumption. Then

$$\begin{aligned}\Phi_k(x,y) = \Phi_k\left(x, x^{\frac{1}{N}}\right) + \sum_{y<p\le x^{\frac{1}{N}}}\frac{\omega\left(\frac{\log x}{\log p} - 1\right)}{p\log p} - \sum_{y<p\le x^{\frac{1}{N}}}\frac{p}{\log p}\\ + O\left(\sum_{y<p\le x^{\frac{1}{N}}}\frac{x}{p\log^2 p}\right) + O\left(x^{1-\frac{1}{N+1}}\right).\end{aligned} \tag{3.46}$$

The assumption leads us to

$$\begin{aligned}\Phi_k\left(x, x^{\frac{1}{N}}\right) &= \frac{x}{\log x^{\frac{1}{N}}}\omega(N) - \frac{x^{\frac{1}{N}}}{\log x^{\frac{1}{N}}} + O\left(\frac{x}{\log^2 y}\right)\\ &= \frac{x}{\log y}\frac{\log y}{\log x}N\left(\frac{1}{N} + \frac{1}{N}\int_1^{N-1}\omega(v)dv\right)\\ &\quad - \frac{y}{\log y} + \left(\frac{y}{\log y} - \frac{x^{\frac{1}{N}}}{\log x^{\frac{1}{N}}}\right) + O\left(\frac{x}{\log^2 y}\right)\\ &= \frac{x}{\log y}\left(\frac{1}{u} + \frac{1}{u}\int_1^{N-1}\omega(v)dv\right) - \frac{y}{\log y} + O\left(\frac{x}{\log^2 y}\right).\end{aligned}$$

The second sum in (3.46) is by Chebycheff's estimate which is estimated as

$$\sum_{y<p\le x^{\frac{1}{N}}}\frac{p}{\log p} \ll \frac{x^{\frac{1}{N}}}{\log y}\sum_{y<p\le x^{\frac{1}{N}}}1 \ll \frac{x^{\frac{2}{N}}}{\log^2 y},$$

and the third sum in (3.46) is estimated as

$$\sum_{y<p\le x^{\frac{1}{N}}} \frac{x}{p\log^2 p} \ll \frac{x}{\log^2 y} \sum_{y<p\le x^{\frac{1}{N}}} \frac{1}{p} \ll \frac{x}{\log^2 y},$$

by the Merten's formula (3.41). For the first sum in (3.46), by (3.41) we shall write

$$\begin{aligned}
&\sum_{y<p\le x^{1/N}} \frac{\omega\left(\frac{\log x}{\log p}-1\right)}{p\log p} = \int_y^{x^{1/N}} \frac{\omega\left(\frac{\log x}{\log w}-1\right)}{\log w} d\left(\sum_{p\le w}\frac{1}{p}\right)\\
&= \int_y^{x^{\frac{1}{N}}} \frac{\omega\left(\frac{\log x}{\log w}-1\right)}{\log w} d\log\log w + \int_y^{x^{\frac{1}{N}}} \frac{\omega\left(\frac{\log x}{\log w}-1\right)}{\log w} dR(w)\\
&=: C + D \quad \text{(say)}.
\end{aligned}$$

We put $v = \log x/\log w$ in C, then we get

$$C = \frac{1}{\log y}\frac{1}{u}\int_{N-1}^{u-1} \omega(v)dv.$$

For D, by integration by parts, we write

$$\begin{aligned}
D &= \left[\frac{\omega\left(\frac{\log x}{\log w}-1\right)}{\log w} R(w)\right]_y^{x^{1/N}} + \int_y^{x^{\frac{1}{N}}} \frac{\omega'\left(\frac{\log x}{\log w}-1\right)\log x}{w\log^3 w} R(w)dw\\
&+ \int_y^{x^{\frac{1}{N}}} \frac{\omega\left(\frac{\log x}{\log w}-1\right)}{w\log^2 w} R(w)dt\\
&=: D_1 + D_2 + D_3 \quad \text{(say)}.
\end{aligned}$$

Since ω is bounded we have $D_1 = O(1/\log^2 w)$, and

$$D_3 \ll \frac{1}{\log^3 y}\int_y^{x^{\frac{1}{N}}} \frac{1}{w}dw \ll \frac{1}{\log^2 y}.$$

For D_2 we shall remark that ω' is also bounded and

$$D_2 \ll \frac{\log x}{\log^3 y}\int_y^{x^{1/N}} \frac{1}{w\log w}dw \ll \frac{1}{\log^2 y}.$$

By the above argument, we have

$$\Phi_k(x,y) = \frac{x}{\log y}\left(\frac{1}{u} + \frac{1}{u}\int_1^{N-1}\omega(v)dv\right) - \frac{y}{\log y} + \frac{x}{\log y}\frac{1}{u}\int_{N-1}^{u-1}\omega(v)dv + O\left(\frac{x}{\log^2 y}\right),$$

where the above O-constant does not depend on k. Thus the completion of the proof.

3.8 Proof of theorem 3.5.2

We shall prove Theorem 3.5.2. The first assertion of (3.30) is trivial. In fact, if there is a non k-free integer $n \le x$ whose prime factor $p > y$ with the condition $x^{1/k} < y \le x$, then $n \ge p^k > x$. This is a contradiction. Therefore the assertion holds for $x^{1/k} < y \le x$.

Next we shall try to prove the second assertion of (3.30). For $x^{1/(k+1)} \le y < x^{1/k}$, we choose $z = x^{1/k}$ in the Buchstab identity (3.36) of Lemma 3.6.2 to get

$$D_k(x,y) = \sum_{y<p\le x^{\frac{1}{k}}}\sum_{j=1}^{k-1} D_k\left(\frac{x}{p^j}, p\right) + \sum_{y<p\le x^{1/k}}\sum_{j=k}^{\infty}\Phi\left(\frac{x}{p^j}, p\right).$$

In this situation, we shall remark that the primes p ($y < p \le x^{1/k}$) satisfy

$$\left(\frac{x}{p^j}\right)^{\frac{1}{k-j+1}} \le p \le \left(\frac{x}{p^j}\right)^{\frac{1}{k-j}} \qquad (j = 1, 2, \ldots, k-1).$$

By the previous result we know $D_k(x/p^j, p) = 0$ for $j = 1, 2, \ldots, k-1$, so we have

$$D_k(x,y) = \sum_{y<p\le x^{\frac{1}{k}}}\sum_{j=k}^{\infty}\Phi\left(\frac{x}{p^j}, p\right).$$

Here if $j \ge k+1$ we observe that $x/p^j < x/y^{k+1} \le 1$ and $\Phi(x/p^j, p) = 0$. If $j = k$ we observe that $1 \le x/p^k < p$. Then we see that

$$\Phi\left(\frac{x}{p^k}, p\right) = \sum_{\substack{n \le x/p^k \\ q|n \Rightarrow q>p}} 1 = 1.$$

Therefore by (3.5) we obtain the second assertion in (3.30), that is,

$$D_k(x,y) = 1 + \sum_{y<p\le x^{\frac{1}{k}}} 1 = \frac{kx^{\frac{1}{k}}}{\log x} - \frac{y}{\log y} + O\left(\frac{x^{\frac{1}{k}}}{\log^2 y}\right).$$

To show the third assertion in (3.30), first we shall consider $D_k(x,y)$ for $x^{1/(k+2)} \le y < x^{1/(k+1)}$ (the case $l=1$). Taking $z = x^{1/(k+1)}$ in (3.36) and using the second assertion in (3.30) we get

$$D_k(x,y) = \sum_{y<p\le x^{\frac{1}{k+1}}} \sum_{j=1}^{k-1} D_k\left(\frac{x}{p^j}, p\right) + \sum_{y<p\le x^{\frac{1}{k+1}}} \sum_{j=k}^{\infty} \Phi\left(\frac{x}{p^j}, p\right) + O\left(\frac{x^{\frac{1}{k}}}{\log y}\right).$$

We shall consider the second sum. If $j \ge k+2$, then $\sum_{j\ge k+2} \Phi(x/p^j, p) = 0$, since $x/p^j < x/y^{k+2} \le 1$. In the case of $j = k+1$, we observe that $1 \le x/p^{k+1} < p$ and $\Phi(x/p^{k+1}, p) = 1$. By the PNT (3.5) we have

$$\sum_{y<p\le x^{\frac{1}{k+1}}} \Phi\left(\frac{x}{p^{k+1}}, p\right) \ll \frac{x^{\frac{1}{k+1}}}{\log y}.$$

For $\sum_{y<p\le x^{1/(k+1)}} \Phi(x/p^{k+1}, p)$, we shall remark that $(x/p^k)^{1/2} \le p \le x/p^k$. Then by the formula (3.6) we get

$$\sum_{y<p\le x^{\frac{1}{k+1}}} \Phi\left(\frac{x}{p^k}, p\right) \ll \sum_{y<p\le x^{\frac{1}{k+1}}} \frac{x}{p^k \log p} = x \int_y^{x^{\frac{1}{k+1}}} \frac{d\pi(t)}{t^k \log t}.$$

Using the PNT (3.5) and $y^{1-k} > x^{\frac{1-k}{k+1}}$ we have an upper bound of the right hand side.

$$\int_y^{x^{\frac{1}{k+1}}} \frac{d\pi(t)}{t^k \log t} = \left[\frac{\pi(t)}{t^k \log t}\right]_y^{x^{\frac{1}{k+1}}} + \int_y^{x^{\frac{1}{k+1}}} \frac{kt^{-k-1}}{\log t}\pi(t)dt + \int_y^{x^{\frac{1}{k+1}}} \frac{t^{-k-1}}{\log^2 t}\pi(t)dt$$
$$\ll \frac{y^{1-k}}{\log^2 y}.$$

Then we see that

$$D_k(x,y) = \sum_{y<p\le x^{\frac{1}{k+1}}} \sum_{j=1}^{k-1} D_k\left(\frac{x}{p^j}, p\right) + O\left(\frac{xy^{1-k}}{\log^2 y}\right) + O\left(\frac{x^{\frac{1}{k}}}{\log y}\right).$$

Here we shall remark that

$$xy^{1-k} > x\cdot x^{\frac{1-k}{k+1}} = x^{\frac{1}{k+1}} \quad \textit{and} \quad \frac{2}{k+1} > \frac{1}{k} \quad (k\ge 2).$$

Then

$$D_k(x,y) = \sum_{y<p\le x^{\frac{1}{k+1}}} \sum_{j=1}^{k-1} D_k\left(\frac{x}{p^j}, p\right) + O\left(\frac{xy^{1-k}}{\log^2 y}\right) \quad \textit{for } x^{\frac{1}{k+2}} \le y < x^{\frac{1}{k+1}}.$$

For the remainder sum we note that the primes $p \in (y, x^{1/(k+1)}]$ satisfy the following inequalities.

$$\left(\frac{x}{p^j}\right)^{\frac{1}{k+2-j}} \le p \le \left(\frac{x}{p^j}\right)^{\frac{1}{k+1-j}} \qquad (j = 1, 2, \ldots, k-1).$$

In the cases of $j = 2, 3, \ldots, k-1$, by the first result in (3.30) we learn that

$$\sum_{y<p\le x^{\frac{1}{k+1}}} \sum_{j=2}^{k-1} D_k\left(\frac{x}{p^j}, p\right) = 0.$$

In the case of $j = 1$, by the second result in (3.30) we get

$$\sum_{y<p\le x^{\frac{1}{k+1}}} D_k\left(\frac{x}{p}, p\right) \ll \sum_{y<p\le x^{\frac{1}{k+1}}} \left(\frac{x}{p}\right)^{\frac{1}{k}} \frac{1}{\log p} = x^{\frac{1}{k}} \int_y^{x^{\frac{1}{k+1}}} \frac{d\pi(t)}{t^{\frac{1}{k}}\log t}.$$

By the PNT (3.5) we have

$$\int_y^{x^{\frac{1}{k+1}}} \frac{d\pi(t)}{t^{\frac{1}{k}}\log t} \ll \frac{x^{\frac{k-1}{k(k+1)}}}{\log^2 y},$$

therefore since $y^{1-k} > x^{(1-k)/(k+1)}$

$$\sum_{y<p\leq x^{\frac{1}{k+1}}} D_k\left(\frac{x}{p},p\right) \ll \frac{x^{\frac{2}{k+1}}}{\log^2 y} = \frac{x^{\frac{(k+1)+2-(k+1)}{k+1}}}{\log^2 y} = \frac{x^{1+\frac{1-k}{k+1}}}{\log^2 y} \ll \frac{xy^{1-k}}{\log^2 y}$$

for $x^{1/(k+2)} \leq y < x^{1/(k+1)}$.

Collecting these estimates we see that the third assertion of (3.30) is true for $l = 1$.

Next we shall assume that the assertion is true for $l = 1, 2, \ldots, m$, and consider an estimate on $D_k(x,y)$ in the case $l = m+1$ i.e., $x^{1/(k+m+2)} \leq y < x^{1/(k+m+1)}$. In (3.36) we take $z = x^{1/(k+m+1)}$, and use the assumption, then

$$D_k(x,y) = \sum_{y<p\leq x^{\frac{1}{m+k+1}}} \sum_{j=1}^{k-1} D_k\left(\frac{x}{p^j},p\right) + \sum_{y<p\leq x^{\frac{1}{m+k+1}}} \sum_{j=k}^{\infty} \Phi\left(\frac{x}{p^j},p\right) + O\left(\frac{xy^{1-k}}{\log^2 y}\right).$$

On the second sum, when $j \geq k+m+2$, since $x/p^j < x/y^{k+m+2} \leq 1$ then $\sum_{j\geq k+m+2} \Phi(x/p^j,p) = 0$. For the case $j = k+m+1$, since $1 \leq x/p^{k+m+1} < p$ we get by the PNT (3.5) and $y^{1-k} > x^{(1-k)/(k+m+1)}$

$$\sum_{y<p\leq x^{\frac{1}{k+m+1}}} \Phi\left(\frac{x}{p^{k+m+1}},p\right) \ll \frac{x^{\frac{1}{k+m+1}}}{\log y} \ll \frac{xy^{1-k}}{\log^2 y}.$$

When $k \leq j \leq k+m$, we shall remark that

$$\left(\frac{x}{p^j}\right)^{\frac{1}{k+m+2-j}} \leq p \leq \left(\frac{x}{p^j}\right)^{\frac{1}{k+m+1-j}}.$$

We shall use the formula (3.6) and the PNT (3.5),

$$\sum_{y<p\leq x^{\frac{1}{k+m+1}}} \sum_{j=k}^{k+m} \Phi\left(\frac{x}{p^j},p\right)$$

$$\sum_{y<p\leq x^{\frac{1}{k+m+1}}} \left(\frac{x}{p^k\log p} + \frac{x}{p^{k+1}\log p} + \cdots + \frac{x}{p^{k+m}\log p}\right)$$

$$\begin{aligned}
&\ll x \sum_{y<p\le x^{\frac{1}{k+m+1}}} \frac{1}{p^k \log p} = x \int_y^{x^{\frac{1}{k+m+1}}} \frac{d\pi(t)}{t^k \log t} \\
&\ll \frac{xy^{1-k}}{\log^2 y}.
\end{aligned}$$

That is, we get

$$\sum_{y<p\le x^{\frac{1}{k+m+1}}} \sum_{j=k}^{\infty} \Phi\left(\frac{x}{p^j}, p\right) \ll \frac{xy^{1-k}}{\log^2 y} \quad \textit{for} \quad x^{\frac{1}{k+m+2}} \le y < x^{\frac{1}{k+m+1}}.$$

On $\sum_{y<p\le x^{1/(k+m+1)}} \sum_{j=1}^{k-1} D_k(x/p^j, p)$, we remark that

$$\left(\frac{x}{p^j}\right)^{\frac{1}{k+m+2-j}} \le p \le \left(\frac{x}{p^j}\right)^{\frac{1}{k+m+1-j}} \qquad (1 \le j \le k-1).$$

Hence we can apply the assumption of induction, and by the formula (3.41) we obtain

$$\begin{aligned}
&\sum_{y<p\le x^{\frac{1}{k+m+2}}} \sum_{j=1}^{k-1} D_k\left(\frac{x}{p^j}, p\right) \\
&\ll \sum_{y<p\le x^{\frac{1}{k+m+1}}} \left(\frac{xp^{1-k}}{p\log^2 p} + \frac{xp^{1-k}}{p^2\log^2 p} + \cdots + \frac{xp^{1-k}}{p^{k-1}\log^2 p}\right) \\
&\ll \frac{xy^{1-k}}{\log^2 y} \sum_{y<p\le x^{\frac{1}{k+m+1}}} \frac{1}{p} \\
&= \frac{xy^{1-k}}{\log^2 y}\left(\log\left(\frac{1}{k+m+1}\frac{\log x}{\log y}\right) + O\left(\frac{1}{\log y}\right)\right) \\
&\ll \frac{xy^{1-k}}{\log^2 y}.
\end{aligned}$$

From the above arguments we get

$$D_k(x,y) \ll \frac{xy^{1-k}}{\log^2 y} \quad \textit{for} \quad x^{\frac{1}{k+m+2}} \le y < x^{\frac{1}{k+m+1}}.$$

It completes the proof of Theorem 3.5.2.

Remark 3.8.1 We shall deduce a simple lower bound for $D_k(x,y)$. Let x and y be satisfied $x^{1/(k+l+1)} \le y < x^{1/(k+l)}$ where $l = 1, 2, \ldots$. We take $z = x^{1/(k+l)}$ in (3.36), then by PNT (3.5) we have

$$\begin{aligned} D_k(x,y) &\ge \sum_{y<p\le x^{\frac{1}{k+l}}} \sum_{j=k}^{\infty} \Phi\left(\frac{x}{p^j}, p\right) = \sum_{y<p\le x^{\frac{1}{k+l}}} \sum_{j=k}^{k+l} \Phi\left(\frac{x}{p^j}, p\right) \\ &\ge (l+1) \sum_{y<p\le x^{\frac{1}{k+l}}} 1 \\ &= \frac{(l+1)(k+l)x^{\frac{1}{k+l}}}{\log x} - \frac{(l+1)y}{\log y} + O\left(\frac{(l+1)x^{\frac{1}{k+l}}}{\log^2 y}\right). \end{aligned}$$

3.9 Proof of theorem 3.5.3

Finally, we shall prove Theorem 3.5.3. To prove this, we shall use the PNT of the form (3.31). When y is finite, the assertion (3.32) is trivial. From here let $y_0 \ge 2$ be sufficiently large and $y \ge y_0$. We shall define two functions $\Delta(x,y)$ and $\Delta_m(x,y)$ by

$$\begin{aligned} D_k(x,y) &= \frac{xy^{1-k}}{\log^2 y}\Delta(x,y), \\ \Delta_m &:= \sup\{|\Delta(x,y)| \mid y \ge y_0,\ k+1 < u \le m\}, \end{aligned}$$

where $u = \log x / \log y$ and $x^{1/(k+1)} \ge y \ge y_0$.

We shall try to show $\Delta_m(x,y) < +\infty$ for any integer $m \ge k+2$. For $m = k+2$ we observe that Δ_{k+2} is finite by Theorem 3.5.2. Now assume that Δ_m is finite for $m = k+2, k+3, \ldots, M$. We shall consider Δ_{M+1} (i.e. $k+1 < u \le M+1$). In Lemma 3.6.2 we shall take $z = x^{1/(k+1)}$, then

$$D_k(x,y) = \sum_{y<p\le x^{\frac{1}{k+1}}} \sum_{j=1}^{k-1} D_k\left(\frac{x}{p^j}, p\right) + \sum_{y<p\le x^{\frac{1}{k+1}}} \sum_{j\ge k} \Phi\left(\frac{x}{p^j}, p\right) + O\left(\frac{xy^{1-k}}{\log^2 y}\right).$$

Since $\Phi(x/p^j, p) \ll x/p^j$ we have

$$\sum_{y<p\le x^{\frac{1}{k+1}}} \sum_{j\ge k+1} \Phi\left(\frac{x}{p^j}, p\right) \ll x \sum_{y<p\le x^{\frac{1}{k+1}}} \frac{1}{p^{k+1}} \ll \frac{x}{y^k} \ll \frac{xy^{1-k}}{\log^2 y}.$$

On the other hand, by the PNT (3.31) since $\Phi(x/p^k, p) \ll x/(p^k \log p)$, hence we get

$$\sum_{y<p\le x^{\frac{1}{k+1}}} \sum_{j\ge k} \Phi\left(\frac{x}{p^j}, p\right) \ll \frac{xy^{1-k}}{\log^2 y}.$$

On $\sum_{y<p\le x^{1/(k+1)}} \sum_{j=1}^{k-1} D_k\left(\frac{x}{p^j}, p\right)$, we observe that

$$\frac{\log(x/p^{k-1})}{\log p} \le \cdots \le \frac{\log(x/p^2)}{\log p} \le \frac{\log(x/p)}{\log p} < u-1 \le M.$$

Then, by the assumption of induction we see that

$$\sum_{y<p\le x^{\frac{1}{k+1}}} \sum_{j=2}^{k-1} D_k\left(\frac{x}{p^j}, p\right) \le \frac{xy^{1-k}}{\log^2 y} \sum_{y<p\le x^{\frac{1}{k+1}}} \frac{1}{p^2} \ll \frac{xy^{1-k}}{\log^2 y},$$

where the O-constant is independent on k, and by also (3.41)

$$\sum_{y<p\le x^{\frac{1}{k+1}}} D_k\left(\frac{x}{p}, p\right) \le xy^{1-k}\Delta_M \sum_{y<p\le x^{\frac{1}{k+1}}} \frac{1}{p\log^2 p} \le \frac{3}{4}\frac{xy^{1-k}}{\log^2 y}\Delta_M.$$

From the above arguments we get $\Delta_{M+1} \le \frac{3}{4}\Delta_M + C$ ($C > 0$ is a constant). This implies that $\Delta_{M+1} \le 4C$. It completes the proof of Theorem 3.5.3.

By Theorems 3.5.1 and 3.5.3 we have a corollary.

Corollary 3.9.1 *By the PNT (3.31) we have*

$$\Phi_k(x,y) = \frac{x\omega(u) - y}{\log y} + O\left(\frac{x}{\log^2 y}\right)$$

for $x \ge y \ge 2$ uniformly.

Proof. By the grace of the PNT (3.31), it is known that the formula (3.6) is valid for $x \ge y \ge 2$ uniformly. Since $\Phi_k(x,y) = \Phi(x,y) + D_k(x,y)$, we get the assertion for $x \ge y \ge 2$.

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