PROBLEMS IN THE THEORY OF MODULI SPACES OF REPRESENTATIONS OF QUIVERS

By
S. Manikandan
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Harish-Chandra Research Institute, Allahabad

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S. Manikandan

DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

S. Manikandan

List of Publications arising from the thesis

Journal

1. "Holomorphic aspects of moduli of representations of quivers", Pradeep Das, S. Manikandan and N. Raghavendra, *Indian J. Pure Appl. Math.*, **2019**, *50*, 549-595.

Others

1. "Holomorphic sectional curvature of the moduli of representations of quivers", S. Manikandan, Preprint.

S. Manikandan

To

my father Subramanian
and my mother Rajeshwari

The acute perception of the presence of something strong, very real and at the same time very delicate is what one one may mean by "beauty", in its thousand-fold aspects. That someone is ambitious doesn't mean that one cannot also feel the presence of beauty in them; but it is not the attribute of ambition which evokes this feeling.... —Excerpts from **"Crops and Seeds: Reflections and Testimony about my Past as a Mathematician"** by ***Alexander Grothendieck***

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S. Manikandan

Contents

Main Conventions and Notations

We will denote by $\mathbb{C}$ the field of complex numbers. The letter Q or $Q = (Q_0, Q_1, \mathrm{s}, \mathrm{t})$ will always denote a quiver. The elements of Q_0 are called the *vertices of* Q, and those of Q_1 are called the *arrows of* Q. For any arrow α of Q, the vertex $\mathrm{s}(\alpha)$ is called the *source* of α, and the vertex $\mathrm{t}(\alpha)$ is called the *target* of α.

$\mathbf{Rep}_k(Q)$ denote an abelian category whose objects are representations of Q over k, and whose morphisms are defined in 2.3.1 and for any two representations (V,ρ) and (W,σ), $\mathrm{Hom}((V,\rho),(W,\sigma))$ is the set of all morphisms in the category $\mathbf{Rep}_k(Q)$.

An element of the $\mathbb{R}$-vector space $\mathbb{R}^{Q_0}$ is called a weight of Q which mostly denoted by θ. For every non zero representation (V,ρ), $\dim(V,\rho)$, $\mathrm{rk}(V,\rho)$ and $\mu_\theta(V,\rho)$ are respectively denote the dimension vector, rank and θ-slope of (V,ρ) as in 2.3.2.

We denote the set of all skew-Hermitian endomorphisms of (V,ρ) with respect to a Hermitian metric h by $\mathrm{End}(V,\rho,h)$. Some notation regarding families of representations over Q will be introduced in Section 2.4.

The main notations in section 3.1.2 and 3.2.3 is the following: Q is a non-empty finite quiver, $d = (d_a)_{a\in Q_0}$ a non-zero element of $\mathbb{N}^{Q_0}$, and $V = (V_a)_{a\in Q_0}$ a family of $\mathbb{C}$-vector spaces, such that $\dim_{\mathbb{C}}(V_a) = d_a$ for all $a \in Q_0$. Fix a family $h = (h_a)_{a\in Q_0}$ of Hermitian inner products $h_a : V_a \times V_a \to \mathbb{C}$. In addition, we also fix now a rational weight $\theta \in \mathbf{Q}^{Q_0}$ of Q.

We will denote by $\mathscr{A}$ the finite-dimensional $\mathbb{C}$-vector space $\bigoplus_{\alpha\in Q_1} \mathrm{Hom}_{\mathbb{C}}(V_{\mathrm{s}(\alpha)}, V_{\mathrm{t}(\alpha)})$. For any subset X of $\mathscr{A}$, let X_{schur} (respectively, X_{s}) denote the set of all points ρ in X, such that the representation (V,ρ) of Q is Schur (respectively, θ-stable). Also, denote by X_{eh} (respectively, X_{irr}) the set of all $\rho \in X$, such that the Hermitian metric h on (V,ρ) is Einstein-Hermitian with respect to θ (respectively, irreducible).

We will denote by G be the complex Lie group $\prod_{a\in Q_0} \mathrm{Aut}_{\mathbb{C}}(V_a)$, H the central complex Lie subgroup of G consisting of all elements of the form ce, as c runs over $\mathbb{C}^{\times}$, where $e = (\mathbf{1}_{V_a})_{a \in Q_0}$ is the identity element of G and K the compact subgroup $\prod_{a \in Q_0} \mathrm{Aut}(V_a, h_a)$, where, for each $a \in Q_0$, $\mathrm{Aut}(V_a, h_a)$ is the subgroup of $\mathrm{Aut}_{\mathbb{C}}(V_a)$ consisting of $\mathbb{C}$-automorphisms of V_a which preserve the Hermitian inner product h_a on V_a.

We will define $\mathscr{B} = \mathscr{A}_{\mathrm{schur}}$. We will denote by M the moduli space $\mathscr{B}/G$ of Schur representations of Q with dimension vector d and M_{s} the moduli space of θ-stable representations of Q with dimension vector d.

The letters g, h, etc., will denote either a group element or a metric according to the context.

SUMMARY

This thesis is about holomorphic and differential-geometric aspect of moduli spaces of complex finite-dimensional representations of a finite quiver. The methods involved in this study are elementary in nature. They are based on Kähler geometry, and do not use any results from geometric invariant theory.

This thesis is divided into two parts. In the first part, we construct a structure of complex premanifold, that is, a non-Hausdorff complex manifold, on the moduli spaces of complex Schur representations of a finite quiver. We construct a structure of complex manifold and a natural Kähler metric on the moduli spaces of the finite-dimensional complex representations of a finite quiver, which are stable with respect to a fixed rational weight. We then exhibit a Hermitian holomorphic line bundle on this moduli space, whose Chern form is essentially an integral multiple of the Kähler form of this metric. In particular, when the moduli space of stable representations are compact, by the Kodaira embeding theorem, the results as stated above give an analytic proof of the projectivity of these moduli spaces. In the second part, we calculate the holomorphic sectional curvature of the moduli space of stable representations, and be shown that is non-negative.

Chapter 1

Introduction

A quiver is simply a finite oriented graph, and a quiver representation is obtained by interpreting the vertices as vector spaces, and the edges as corresponding linear maps. Moduli spaces of representations of quivers are of interest because of their relations with the moduli spaces of representations of algebras [20], and with the moduli spaces of sheaves on projective schemes [2]. A general survey about the moduli spaces of representations of quivers is [31]. These moduli spaces can be constructed as algebraic quotients using geometric invariant theory (for bundles, the construction was given by Mumford, Newstead and Seshadri [29, 30, 34], and for quivers, this construction was given by King [20]).

In this thesis we study holomorphic and differential geometric aspects of the moduli space of finite-dimensional complex representations of a finite quiver, which are stable with respect to a fixed rational weight. The methods involved in this study are elementary in nature. They are based on Kähler geometry, and do not use any results from geometric invariant theory.

1.1 Structure of the thesis

This thesis is organized as follows.

Chapter 2 starts by introducing the theory of quiver representations towards the study of analytic geometric aspects of moduli space of complex representations of a quiver. We view the stability of representations of quivers as a special case of Rudakov's theory of stability structures on an abelian category. Accordingly, we begin by recalling this theory in Section 2.1. Any finite positive family of additive functions on an abelian category, and a corresponding family of real numbers, called a *weight*, define an stability structure on the category. There is a natural hyperplane arrangement on the space of weights, and the stability condition remains constant within every facet of this hyperplane arrangement. We describe this idea in Section 2.2. It includes, as a special case, the stability of representations of a finite quiver with respect to a given weight. We recall some basic notions about quivers and their representations in Section 2.3. We also describe a theorem of King, which relates stability of a representation of a quiver to the existence of a certain kind of inner product on the representation, which we call an *Einstein-Hermitian metric*, because of its similarity to Einstein-Hermitian metrics on vector bundles. We discuss families of representations in Section 2.4, and explain a criterion for two representations in a family to be separated from each other.

In Chapter 3, we describe the construction of a natural Kähler metric on the moduli space of stable representations. We construct the moduli space of Schur representations in Section 3.1. It is, in general, a non-Hausdorff complex manifold. Its open subset of stable representations is Hausdorff, and has a natural Kähler metric, as we explain in Section 3.2. The main results of this section are Theorem 3.1.22 and Theorem 3.2.31.

In Chapter 4, we give a description of a natural Hermitian holomorphic line bundle on the moduli space of stable representations and show that its curvature is essentially an integral multiple of the Kähler form on the moduli space. The main result of this section is Theorem 4.3.1.

In Chapter 5, we calculate the holomorphic sectional curvature of the moduli space of stable representations, to be non-negative. The main result of this section is Corollary 5.4.4

Chapter 2

Representation of quivers and stability structures

This chapter provides the background to the results in the thesis. it consists of the following four sections:

1. Stability structures
2. Stability with respect to a weight
3. Representations of quivers
4. Families of representations

2.1 Stability structures

The stability of representations of a quiver is a special case of the notion of a stability structure that was defined by Rudakov [5, Definition 1.1]. The subsections of the current section are devoted to a brief discussion of *some* of the properties of stability structures. For instance, definition of stability structures, the Schur lemma about the endomorphisms of stable objects, Jordan-Hölder and Harder Narasimhan filtrations, and the notion of S-equivalence of semistable objects.

2.1.1 Semistable objects of an abelian category

Let $\mathscr{A}$ be an abelian category, and $\preceq$ a total preorder on the set of non-zero objects of $\mathscr{A}$. For any two non-zero objects M and N of $\mathscr{A}$, write $M \succeq N$ if $N \preceq M$, and define

$$\begin{aligned} M \prec N &\quad \text{if} \quad M \preceq N \text{ and } M \not\succeq N, \\ M \asymp N &\quad \text{if} \quad M \preceq N \text{ and } M \succeq N, \\ M \succ N &\quad \text{if} \quad M \succeq N \text{ and } M \not\preceq N. \end{aligned}$$

Then, $\succeq$ is a total preorder, $\prec$ and $\succ$ irreflexive transitive relations, and $\asymp$ an equivalence relation, on the set of non-zero objects of $\mathscr{A}$. Moreover, for any two non-zero objects M and N of $\mathscr{A}$, exactly one of the three statements

$$M \prec N, \quad M \asymp N, \quad M \succ N$$

holds. We say that $\preceq$ has the *seesaw property* if for every short exact sequence

$$0 \to M' \xrightarrow{f'} M \xrightarrow{f} M'' \to 0$$

of non-zero objects of $\mathscr{A}$, exactly one of the three statements

$$M' \prec M \prec M'', \quad M' \asymp M \asymp M'', \quad M' \succ M \succ M''$$

is true.

Definition 2.1.1 A *stability structure* on $\mathscr{A}$ is a total preorder on the set of non-zero objects of $\mathscr{A}$, which has the seesaw property.

We fix a stability structure $\preceq$ on an abelian category $\mathscr{A}$. The seesaw property implies that if M and M' are two isomorphic non-zero objects of $\mathscr{A}$, then $M \asymp M'$. It follows that if $M \cong M'$ and $N \cong N'$ are isomorphisms of objects in $\mathscr{A}$, then $M \preceq N$ (respectively, $M \prec N$, $M \asymp N$) if and only if $M' \preceq N'$ (respectively, $M' \prec N'$, $M' \asymp N'$). In particular, if $i : N \to M$ and $i' : N' \to M$ are two equivalent non-zero subobjects of M, we have $N \preceq M$ (respectively, $N \prec M$, $N \asymp M$) if and only if $N' \preceq M$ (respectively, $N' \prec M$, $N' \asymp M$).

Let $f : M \to M'$ be an isomorphism, and let $i : M \to M \oplus M'$, $i' : M' \to M \oplus M'$, $p : M \oplus M' \to M$, and $p' : M \oplus M' \to M'$, be the canonical morphisms. Since the relation $\asymp$ is reflexive, we have $M' \asymp M'$, hence the exact sequence

$$0 \to M' \xrightarrow{i'} M \oplus M' \xrightarrow{f \circ p} M' \to 0$$

implies that $M' \asymp M \oplus M'$. Similarly, the exact sequence

$$0 \to M \xrightarrow{i} M \oplus M' \xrightarrow{f^{-1} \circ p'} M \to 0$$

implies that $M \asymp M \oplus M'$. Therefore, as $\asymp$ is symmetric and transitive, we get the relation $M \asymp M'$.

Now, suppose $M \cong M'$ and $N \cong N'$. Then, by the above paragraph, $M \asymp M'$ and $N \asymp N'$. If $M \preceq N$, then $M' \preceq M \preceq N \preceq N'$, hence $M' \preceq N'$. If $M \prec N$, then $M \preceq N$ and $N \not\preceq M$, hence, by the previous statement, $M' \preceq N'$, and $N' \not\preceq M'$, so $M' \prec N'$. Lastly, if $M \asymp N$, then, by the symmetry and transitivity of $\asymp$, we get $M' \asymp N'$.

Definition 2.1.2 An object M of $\mathscr{A}$ is called *semistable* (respectively, *stable*) if it is *nonzero*, and satisfies

$$N \preceq M \quad (\text{respectively, } N \prec M)$$

for every non-zero proper subobject N of M.

Definition 2.1.3 We say that an object of $\mathscr{A}$ is *polystable* if it is semistable, and is isomorphic to a finite family of stable objects of $\mathscr{A}$.

It is obvious that stable $\Rightarrow$ polystable $\Rightarrow$ semistable, and that all three properties are preserved by isomorphisms in $\mathscr{A}$. It is also easy to verify the following statements about semistable objects.

Proposition 2.1.4 *Let $\preceq$ be a stability structure on an abelian category $\mathscr{A}$. Let S be an $\asymp$-equivalence class in the set of non-zero objects of $\mathscr{A}$, and let $\mathscr{A}(S)$ be the the full subcategory of $\mathscr{A}$, whose objects are either zero objects of $\mathscr{A}$, or semistable objects of $\mathscr{A}$ which belong to S. Then:*

1. *A non-zero object M of $\mathscr{A}$ is semistable (respectively, stable) if and only if for every non-zero epimorphism $f : M \to N$ which is not an isomorphism, we have*

$$M \preceq N \quad (\text{respectively, } M \prec N).$$

2. *Let*

$$0 \to M' \xrightarrow{f'} M \xrightarrow{f} M'' \to 0$$

be a short exact sequence of non-zero objects of $\mathscr{A}$. Suppose that

$$M' \asymp M \asymp M''.$$

Then, M is semistable if and only if both M' and M'' are semistable.

3. *Let M and N be two non-zero objects of $\mathscr{A}$. Then, the object $M \oplus N$ is semistable if and only if both M and N are semistable, and $M \asymp N$. In that case,*

$$(M \oplus N) \asymp M \asymp N.$$

4. *Let M and N be two semistable objects of $\mathscr{A}$, and let $f : M \to N$ be a morphism. Suppose that $M \asymp N$. Then, each of the objects* $\mathrm{Ker}(f)$, $\mathrm{Im}(f)$, $\mathrm{Coimg}(f)$, *and* $\mathrm{Coker}(f)$, *is either zero, or is semistable and $\asymp$-related to M.*

5. *The category $\mathscr{A}(S)$ is an abelian subcategory of $\mathscr{A}$.*

Proof.

(1) Let $f : M \to N$ be a non-zero epimorphism which is not an isomorphism, and let K be its kernel. Then, K is a non-zero proper subobject of M, and we have an exact sequence

$$0 \to K \to M \xrightarrow{f} N \to 0.$$

Therefore, by the seesaw property, we have $K \preceq M$ if and only if $M \preceq N$, and $K \prec M$ if and only if $M \prec N$. On the other hand, if X is a non-zero proper subobject of M, and if Y is the quotient object M/X, then the canonical morphism $\pi : M \to Y$ is a non-zero epimorphism which is not an isomorphism, and its kernel equals X. The stated criteria follow from these observations.

(2) Suppose that M is semistable. If $g' : N' \to M'$ is a non-zero monomorphism, then $f' \circ g' : N' \to M$ is also a non-zero monomorphism, hence, as M is semistable, we get $N' \preceq M$. On the other hand, as $M \asymp M'$, we have $M \preceq M'$. It follows that $N' \leq M'$. Therefore, M' is semistable. On the other hand, if $g'' : M'' \to N''$ is a

non-zero epimorphism, then $g'' \circ f : M \to N''$ is also a non-zero epimorphism, hence, as M is semistable, we get $M \preceq N''$. As $M \asymp M''$, this implies that $M'' \preceq N''$. It follows that M'' also is semistable.

Conversely, suppose that M' and M'' are semistable. Let N be a non-zero subobject of M. We have to prove that $N \preceq M$. Let $N' = f'^{-1}(N)$, that is, the kernel of $\pi \circ f'$, where $\pi : a \to M/N$ is the canonical morphism. Let $N'' = f(N)$, that is the image of $f \circ i : N \to M''$, where $i : N \to M$ is the canonical morphism. We then have an exact sequence

$$0 \to N' \xrightarrow{g'} N \xrightarrow{g} N'' \to 0,$$

where g' and g are induced by f' and f, respectively. If $N' = 0$, then g is an isomorphism, hence $N \asymp N''$, and, as M'' is semistable and $M \asymp M''$, we get $N \preceq N'' \preceq M'' \preceq M$, so $N \preceq M$. Similarly, if $N'' = 0$, then g' is an isomorphism, hence $N \asymp N'$, and, as M' is semistable and $M \asymp M'$, we get $N \preceq N' \preceq M' \preceq M$, so $N \preceq M$. Lastly, if both N and N' are non-zero, then, by the seesaw property, either $N' \preceq N \preceq N''$, or $N' \succ N \succ N''$. In the first case, since M'' is semistable and $M \asymp M''$, we get $N \preceq N'' \preceq M'' \preceq M$, hence $N \preceq M$. In the other case, since M' is semistable and $M \asymp M'$, we get $N \prec N' \preceq M' \preceq M$, hence we again have $N \preceq M$. It follows that M is semistable.

(3) Let $X = M \oplus N$. We then have a short exact sequence

$$0 \to M \xrightarrow{i} X \xrightarrow{p} N \to 0$$

of non-zero objects of $\mathscr{A}$. Suppose M is semistable. Then, since $N \neq 0$, the canonical morphism from N to X is a non-zero monomorphism, hence $N \preceq X$. On the other hand, again since $N \neq 0$, the morphism p in the above exact sequence is a non-zero epimorphism, hence $X \succeq N$. Therefore, $N \asymp X$. Now, the seesaw property and the above exact sequence imply that $M \asymp X$. Thus, $X \asymp M \asymp N$, and, by (2), both M and N are semistable. Conversely, suppose both M and N are semistable, and $M \asymp N$. Then, by the seesaw property, $M \asymp X \asymp N$, hence, by (2), X is semistable.

(4) (a) Suppose $\mathrm{Ker}(f) \neq 0$. If $\mathrm{Im}(f) = 0$, then $f = 0$, hence $\mathrm{Ker}(f) = M$. Therefore, we can assume that $\mathrm{Im}(f) \neq 0$. We then have a canonical short exact sequence

$$0 \to \mathrm{Ker}(f) \xrightarrow{i} M \xrightarrow{p} \mathrm{Im}(f) \to 0$$

of non-zero objects of $\mathscr{A}$. Since $\mathrm{Im}(f) \neq 0$, the morphism p in the above exact sequence is a non-zero epimorphism, hence, by (1), we have $M \preceq \mathrm{Im}(f)$. On the other hand, $\mathrm{Im}(f)$ is a non-zero subobject of N, hence $\mathrm{Im}(f) \preceq N \asymp M$. It follows that $M \asymp \mathrm{Im}(f)$. Therefore, by the seesaw property, we get $\mathrm{Ker}(f) \asymp M \asymp \mathrm{Im}(f)$. Now, by (2), both $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$ are semistable.

(b) Suppose $\mathrm{Im}(f) \neq 0$. If $\mathrm{Ker}(f) = 0$, then f induces an isomorphism from M to $\mathrm{Im}(f)$, hence $\mathrm{Im}(f)$ is semistable and $\mathrm{Im}(f) \asymp M$. So, we can assume that $\mathrm{Ker}(f) \neq 0$. Then, it was shown in (a) that $\mathrm{Im}(f)$ is semistable and $\mathrm{Im}(f) \asymp M$.

(c) Suppose $\mathrm{Coimg}(f) \neq 0$. If $\mathrm{Ker}(f) = 0$, then the canonical morphism from M to $\mathrm{Coimg}(f)$ is an isomorphism, hence $\mathrm{Coimg}(f)$ is semistable and $\mathrm{Coimg}(f) \asymp M$. So, we can assume that $\mathrm{Ker}(f)$ is non-zero. We then have an exact sequence

$$0 \to \mathrm{Ker}(f) \to M \to \mathrm{Coimg}(f) \to 0$$

of non-zero objects of $\mathscr{A}$. Moreover, by (a), $\mathrm{Ker}(f) \asymp M$. Therefore, by the seesaw property, $\mathrm{Ker}(f) \asymp M \asymp \mathrm{Coimg}(f)$. Now, as M is semistable, by (2), so is $\mathrm{Coimg}(f)$.

(d) Suppose $\mathrm{Coker}(f) \neq 0$. If $\mathrm{Im}(f) = 0$, then the canonical morphism from N to $\mathrm{Coker}(f)$ is an isomorphism, hence $\mathrm{Coker}(f)$ is semistable and $\mathrm{Coker}(f) \asymp N$. So, we can assume that $\mathrm{Im}(f)$ is non-zero. We then have an exact sequence

$$0 \to \mathrm{Im}(f) \to N \to \mathrm{Coker}(f) \to 0$$

of non-zero objects of $\mathscr{A}$. Moreover, by (b), $\mathrm{Im}(f) \asymp N$. Therefore, by the seesaw property, $\mathrm{Im}(f) \asymp N \asymp \mathrm{Coker}(f)$. Now, as N is semistable, by (2), so is $\mathrm{Coker}(f)$.

(5) Recall that an *abelian subcategory* of an abelian category $\mathscr{C}$ is a subcategory $\mathscr{C}'$ of $\mathscr{C}$, such that $\mathscr{C}'$ is also abelian, and every exact sequence in $\mathscr{C}'$ is an exact sequence in $\mathscr{C}$ [39, § 1.2, p. 7]. Thus, we have to check that $\mathscr{A}(S)$ is an abelian category, and that every exact sequence in $\mathscr{A}(S)$ is exact in $\mathscr{A}$. Since $\mathscr{A}(S)$ is a full subcategory of $\mathscr{A}$, it suffices to check the following [13, Theorem 3.41]:

(a) There exists a zero object of $\mathscr{A}$, which is an object of $\mathscr{A}(S)$.

(b) Every pair (M,N) of objects of $\mathscr{A}(S)$ has a coproduct (X,i,j) in $\mathscr{A}$, such that X is an object $\mathscr{A}(S)$.

(c) For any two objects M and N of $\mathscr{A}(S)$, and for every morphism $f : M \to N$ in $\mathscr{A}$, there exist a kernel $i : K \to M$ in $\mathscr{A}$, and a cokernel $p : N \to C$ in $\mathscr{A}$, such that K and C are objects of $\mathscr{A}(S)$.

By definition, every zero object is an object of $\mathscr{A}(S)$, so the first condition is satisfied. The other two conditions follow from (3) and (4). $\square$

2.1.2 The Schur lemma

Recall that an object M of an additive category $\mathscr{A}$ is called *simple* if it is non-zero, and has no non-zero proper subobject.

Definition 2.1.5 We say that M is a *Schur* object, or a *brick*, if the ring $\mathrm{End}(M)$ is a division ring.

Remark 2.1.6 It is easy to see that every simple object of $\mathscr{A}$ is Schur. If M is simple, then it is Schur. Since M is non-zero, the ring $\mathrm{End}(M)$ is non-zero. Let f be a non-zero element of $\mathrm{End}(M)$, and let $i : I \to M$ be an image of f. Then, I is a non-zero object of $\mathscr{C}$, and i is a monomorphism. Therefore, as M is simple, i is an isomorphism. Let $f' : M \to I$ be the unique morphism such that $i \circ f' = f$. Then, f' is an epimorphism. It follows that f is an epimorphism. On the other hand, let $k : K \to M$ be a kernel of f. Then, k is a monomorphism; but it is not an isomorphism, since $f \circ k = 0$, and $f \neq 0$. Therefore, as M is simple, K must be a zero object. This implies that f is a monomorphism. Thus, f is both a monomorphism and an epimorphism. Since $\mathscr{C}$ is abelian, this implies that f is an isomorphism, and is hence a unit in the ring $\mathrm{End}(M)$. It follows that $\mathrm{End}(M)$ is a division ring.

The following Proposition follows directly from [32, Theorem 1].

Proposition 2.1.7 *Let $\mathscr{A}$, $\preceq$, S, and $\mathscr{A}(S)$, be as in Proposition 2.1.4.*

1. *Let M and N be two semistable objects of $\mathscr{A}$, and let $f : M \to N$ be a non-zero morphism. Suppose that $M \succeq N$. Then:*

   (a) $M \asymp N$.

   (b) If M is stable, then f is a monomorphism.

   (c) If N is stable, then f is an epimorphism.

(d) *If both M and N are stable, then f is an isomorphism.*

2. *An element M of S is a simple object of the abelian category $\mathscr{A}(S)$ if and only if it is stable.*

3. *(Schur Lemma) Every stable object of $\mathscr{A}$ is a Schur object of $\mathscr{A}$.*

4. *Suppose that $\mathscr{A}$ is a K-linear abelian category, where K is an algebraically closed field, and let M be an object of $\mathscr{A}$. Suppose also that the K-vector space* $\mathrm{End}(M)$ *is finite-dimensional. Then, M is a Schur object of $\mathscr{A}$ if and only if for every endomorphism f of M in $\mathscr{A}$, there exists a unique element $\lambda \in K$, such that $f = \lambda \mathbf{1}_M$.*

Proof.

(1) This is [32, Theorem 1].

(2) Let $M \in S$. Suppose M is a simple object of $\mathscr{A}(S)$. Then, M is semistable, hence, by Proposition 2.1.4(2), every non-zero subobject N of M, such that $N \asymp M$, is semistable. Thus, the inclusion morphism $i : N \to M$ is a non-zero monomorphism in $\mathscr{A}(S)$. Since M is simple in $\mathscr{A}(S)$, i must be an isomorphism, hence $N = M$ as subobjects of M. This implies that for every proper non-zero subobject N of M, we have $N \prec M$, hence M is stable.

Conversely, suppose M is stable. Then, M is semistable, and is hence an object of $\mathscr{A}(S)$. Let $f : N \to M$ be a non-zero monomorphism in $\mathscr{A}(S)$. Then, by (1), f is an epimorphism. Thus f is both a monomorphism and an epimorphism in $\mathscr{A}$, and is hence an isomorphism. It follows that M is simple in $\mathscr{A}(S)$.

(3) Let M be a stable object of $\mathscr{A}$, and let $A = \mathrm{End}(M)$. Then, M is non-zero, so its identity morphism $\mathbf{1}_M$ is a non-zero element of A, hence the ring A is non-zero. Moreover, if f is a non-zero element of A, then, by (1d), f is an automorphism of M, and is hence a unit in the ring A. Thus, every non-zero element of A is a unit in A.

(4) If M satisfies the stated condition, then the map $\lambda \mapsto \lambda \mathbf{1}_M$ is an isomorphism of rings from K to $\mathrm{End}(M)$. Therefore, $\mathrm{End}(M)$ is a field, and hence a division ring, so M is Schur. Conversely, if M is Schur, then, by definition, the K-algebra $\mathrm{End}(M)$ is a division ring.

2.1.3 Jordan-Hölder filtrations

A sequence $(M_n)_{n\in\mathbb{N}}$ of subobjects of an object M of $\mathscr{A}$ is called *stationary* if there exists $n_0 \in \mathbb{N}$, such that $M_n = M_{n+1}$ for all $n \geq n_0$.

Definition 2.1.8 We say that an object M of $\mathscr{A}$ is

1. *Noetherian* (respectively, *Artinian*) if every sequence $(M_n)_{n\in\mathbb{N}}$ of subobjects of M, such that $M_n \subset M_{n+1}$ (respectively, $M_n \supset M_{n+1}$) for all $n \in \mathbb{N}$, is stationary.
2. *quasi-Noetherian* with respect to $\preceq$ if every sequence $(M_n)_{n\in\mathbb{N}}$ of subobjects of M, such that $M_n \subset M_{n+1}$, and $M_n \preceq M_{n+1}$ for all $n \in \mathbb{N}$, is stationary.
3. *weakly Artinian* with respect to $\preceq$ if every sequence $(M_n)_{n\in\mathbb{N}}$ of subobjects of M, such that $M_n \supset M_{n+1}$, and $M_n \preceq M_{n+1}$ for all $n \in \mathbb{N}$, is stationary.
4. *weakly Noetherian* with respect to $\preceq$ if it is quasi-Noetherian with respect to $\preceq$, and if every sequence $(M_n)_{n\in\mathbb{N}}$ of subobjects of M, such that $M_n \subset M_{n+1}$, and $M_n \succeq M_{n+1}$ for all $n \in \mathbb{N}$, is stationary.

The category $\mathscr{A}$ is called *Noetherian* (respectively, *Artinian*) if every object in it is Noetherian (respectively, Artinian). It is called *quasi-Noetherian* (respectively, *weakly Artinian*, *weakly Noetherian*) with respect to $\preceq$ if every object in it is quasi-Noetherian (respectively, weakly Artinian, weakly Noetherian) with respect to $\preceq$.

Definition 2.1.9 If M is a semistable object of $\mathscr{A}$, then a *Jordan-Hölder filtration* of M with respect to $\preceq$ is a sequence $(M_i)_{i=0}^n$ of subobjects of M, such that $n \geq 1$, $M_0 = M$, $M_n = 0$, and $M_i \subset M_{i-1}$, M_{i-1}/M_i is stable, and $M_{i-1} \asymp M$, for every $i = 1, \ldots, n$.

Definition 2.1.10 If M is an arbitrary object of $\mathscr{A}$, then a *Harder-Narasimhan filtration* of M with respect to $\preceq$ is a sequence $(M_i)_{i=0}^n$ of subobjects of M, such that $n \in \mathbb{N}$, $M_0 = M$, $M_n = 0$, $M_i \subset M_{i-1}$ and $G_i = M_{i-1}/M_i$ is semistable for every $i = 1, \ldots, n$, and $G_{i-1} \prec G_i$ for every $i = 2, \ldots, n$.

The statements in the following Proposition are proved in [32, Theorems 2 and 3].

Proposition 2.1.11 *Let $\preceq$ be a stability structure on an abelian category $\mathscr{A}$.*

1. *Suppose that $\mathscr{A}$ is quasi-Noetherian, and weakly Artinian, with respect to $\preceq$. Then, every semistable object M of $\mathscr{A}$ has a Jordan-Hölder filtration with respect to $\preceq$. Moreover, if $(M_i)_{i=0}^n$ and $(N_j)_{j=0}^m$ are two Jordan-Hölder filtrations of M with respect to $\preceq$, then $n = m$, and there exists a permutation $\pi \in \mathrm{S}_n$ such that M_{i-1}/M_i is isomorphic to $N_{\pi(i)-1}/N_{\pi(i)}$ for every $i = 1, \ldots, n$.*

2. *Suppose that $\mathscr{A}$ is weakly Noetherian, and weakly Artinian, with respect to $\preceq$. Then, every object of $\mathscr{A}$ has a unique Harder-Narasimhan filtration with respect to $\preceq$.*

Proof.

(1) Let M be a semistable object of $\mathscr{A}$, and S the $\asymp$-equivalence class of M. Since $\mathscr{A}$ is quasi-Noetherian and weakly Artinian, the abelian category $\mathscr{A}(S)$ is Noetherian and Artinian. Therefore, the existence of a Jordan-Hölder filtration of the object M of $\mathscr{A}(S)$ follows from [36, Chapter III, Proposition 3.5, and Chapter IV, Proposition 5.3]. Its uniqueness follows from [36, p. 92]. See also [32, Theorem 3].

(2) This is [32, Theorem 2]. □

Let M and N be two semistable objects of $\mathscr{A}$. Let $(M_i)_{i=0}^n$ and $(N_j)_{j=0}^m$ be Jordan-Hölder filtrations of M and N, respectively, with respect to $\preceq$, which exist by Proposition 2.1.11(1). Then, we say that M is *S-equivalent* to N with respect to $\preceq$, if $n = m$, and there exists a permutation $\pi \in \mathrm{S}_n$, such that M_{i-1}/M_i is isomorphic $N_{\pi(i)-1}/N_{\pi(i)}$ for every $i = 1, \ldots, n$. By the above Proposition, this is independent of the choices of the Jordan-Hölder filtrations, and defines an equivalence relation on the set of all semistable objects of $\mathscr{A}$.

2.2 Stability with respect to a weight

In subsection 2.2.1, we consider a special kind of the stability structures defined in section 2.1, namely stability structures defined by a finite family of positive additive functions on an abelian category, and a corresponding family of real numbers called weights, which form a finite-dimensional real vector space, the weight space. Moreover, in subsection 2.2.3, we will see that fixing the values of the additive functions defines a hyperplane arrangement on the weight space and we describe how semistability and other related notions behave with respect to this hyperplane arrangement.

2.2.1 Semistablity with respect to a weight

Recall that a function φ from the set $\mathrm{Ob}(\mathscr{A})$ of objects of $\mathscr{A}$ to an abelian group G is called *additive* if for every short exact sequence

$$0 \to M' \xrightarrow{f'} M \xrightarrow{f} M'' \to 0$$

in $\mathscr{A}$, we have

$$\varphi(M) = \varphi(M') + \varphi(M'')$$

in G. This condition implies that $\varphi(0) = 0$ for every zero object 0 of $\mathscr{A}$, $\varphi(M \oplus N) = \varphi(M) + \varphi(N)$ for any two objects M and N of $\mathscr{A}$, and $\varphi(M) = \varphi(N)$ if M is isomorphic to N. The set $\mathbf{Add}(\mathscr{A}, G)$ of all additive functions from $\mathscr{A}$ to G has a natural structure of an abelian group. If A is a ring, the abelian group $\mathbf{Add}(\mathscr{A}, A)$ has a natural structure of an A-module.

Let $\cong$ be the isomorphism relation on the set $\mathrm{Ob}(\mathscr{A})$, and $\mathrm{Ob}(\mathscr{A})/\cong$ its quotient set. We denote the isomorphism class of any object M of $\mathscr{A}$ by $[M]$. Let $F(\mathscr{A})$ be the free abelian group on the set $\mathrm{Ob}(\mathscr{A})/\cong$, and identify that set with its canonical image in $F(\mathscr{A})$. The *Grothendieck group* of $\mathscr{A}$ is the quotient $K_0(\mathscr{A})$ of $F(\mathscr{A})$ by the subgroup generated by elements of the form

$$[M] - [M'] - [M''],$$

as

$$0 \to M' \xrightarrow{f'} M \xrightarrow{f} M'' \to 0$$

runs over all the short exact sequences in $\mathscr{A}$. The composite of the canonical functions

$$\mathrm{Ob}(\mathscr{A}) \to \mathrm{Ob}(\mathscr{A})/\cong \to F(\mathscr{A}) \to K_0(\mathscr{A})$$

is an additive function $\pi : \mathscr{A} \to K_0(\mathscr{A})$. For any abelian group G, the function $\psi \mapsto \psi \circ \pi$ is an isomorphism of abelian groups from $\mathrm{Hom}_{\mathbb{Z}}(K_0(\mathscr{A}), G)$ onto $\mathbf{Add}(\mathscr{A}, G)$. For every additive function φ from $\mathscr{A}$ to G, the unique homomorphism of groups $\psi : K_0(\mathscr{A}) \to G$, such that $\varphi = \psi \circ \pi$, is called the homomorphism *induced* by φ.

If φ is a positive additive function from $\mathscr{A}$ to G, then $\varphi(X) \geq 0$ for every object X of $\mathscr{A}$, hence the exactness of the obvious sequence

$$0 \to M' \to M \to M/M' \to 0$$

implies that $\varphi(M') = \varphi(M) - \varphi(M/M') \leq \varphi(M)$. Moreover, $\varphi(M') = \varphi(M)$ if and only if $\varphi(M/M') = 0$. As φ is positive, $\varphi(X) > 0$ for every non-zero object X of $\mathscr{A}$, hence this implies that $M/M' = 0$, that is, $M' = M$.

Here is a verification that the category $\mathscr{A}$ is Noetherian and Artinian if there exists a positive additive function φ from $\mathscr{A}$ to $\mathbb{Z}$.

Let M be an object of $\mathscr{A}$, and $(M_n)_{n\in\mathbb{N}}$ a sequence of subobjects of M such that $M_n \subset M_{n+1}$ for all $n \in \mathbb{N}$. Then, as φ is a positive additive function from $\mathscr{A}$ to $\mathbb{Z}$, $(\varphi(M_n))_{n\in\mathbb{N}}$ is an increasing sequence of integers in the compact interval $[0, \varphi(M)]$ in $\mathbb{R}$. Therefore, there exists $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$, we have $\varphi(M_n) = \varphi(M_{n+1})$, so $M_n = M_{n+1}$. Therefore, the sequence (M_n) is stationary, hence M is a Noetherian object of $\mathscr{A}$.

Next, let M be an object of $\mathscr{A}$, and $(M_n)_{n\in\mathbb{N}}$ a sequence of subobjects of M such that $M_n \supset M_{n+1}$ for all $n \in \mathbb{N}$. Then, as φ is a positive additive function from $\mathscr{A}$ to $\mathbb{Z}$, $(\varphi(M_n))_{n\in\mathbb{N}}$ is a decreasing sequence of integers in the compact interval $[0, \varphi(M)]$ in $\mathbb{R}$. Therefore, there exists $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$, we have $\varphi(M_n) = \varphi(M_{n+1})$, so $M_n = M_{n+1}$. Therefore, the sequence (M_n) is stationary, hence M is an Artinian object of $\mathscr{A}$.

Definition 2.2.1 Let $\mathscr{A}$ be an abelian category. We say that a family $(\varphi_i)_{i\in I}$ of additive functions from the set of objects of $\mathscr{A}$ to an ordered abelian group G is *positive* if $\varphi_i(M) \geq 0$ for every object M of $\mathscr{A}$ and for every $i \in I$, and if for every non-zero object M of $\mathscr{A}$, there exists an $i \in I$, such that $\varphi_i(M) > 0$. In particular, we say that an additive function φ from $\mathscr{A}$ to G is *positive* if the singleton family (φ) is positive.

Remark 2.2.2 If φ is a positive additive function from $\mathscr{A}$ to G, then an object M of $\mathscr{A}$ is zero if and only if $\varphi(M) = 0$, hence, if M' is a subobject of an object M, then $\varphi(M') \leq \varphi(M)$, and equality holds if and only if $M' = M$. The category $\mathscr{A}$ is Noetherian and Artinian if there exists a positive additive function from $\mathscr{A}$ to $\mathbb{Z}$.

We now fix a non-empty finite positive family $(\varphi_i)_{i\in I}$ of additive functions from $\mathscr{A}$ to $\mathbb{Z}$.

Definition 2.2.3

1. The *dimension vector* of any object M of $\mathscr{A}$ is the element $\varphi(M)$ of $\mathbb{N}^I$ that is defined by

$$\varphi(M) = (\varphi_i(M))_{i\in I}.$$

2. The *rank* of M is the natural number $\mathrm{rk}(M)$ defined by

$$\mathrm{rk}(M) = \sum_{i\in I} \varphi_i(M).$$

   Since $(\varphi_i)_{i\in I}$ is a positive family of additive functions from $\mathscr{A}$ to $\mathbb{Z}$, the function rk is a positive additive function from $\mathscr{A}$ to $\mathbb{Z}$.

3. An element of the $\mathbb{R}$-vector space $\mathbb{R}^I$ is called a *weight* of $\mathscr{A}$. We say that a weight is *rational* (respectively, *integral*) if it belongs to the subset $\mathbf{Q}^I$ (respectively, $\mathbb{Z}^I$) of $\mathbb{R}^I$. We fix a weight θ of $\mathscr{A}$.

4. We define the θ-*degree* of any object M of $\mathscr{A}$ to be the real number $\deg_\theta(M)$ given by

$$\deg_\theta(M) = \sum_{i \in I} \theta_i \varphi_i(M).$$

5. If $M \neq 0$, we define another real number $\mu_\theta(M)$ by

$$\mu_\theta(M) = \frac{\deg_\theta(M)}{\mathrm{rk}(M)},$$

and call it the θ-*slope* of M.

Proposition 2.2.4 *Let θ be any weight of $\mathscr{A}$. Define a relation $\preceq_\theta$ on the set of non-zero objects of $\mathscr{A}$, by setting $M \preceq_\theta N$ if $\mu_\theta(M) \leq \mu_\theta(N)$. Then, $\preceq_\theta$ is a stability structure on $\mathscr{A}$.*

Proof. Let $c(M) = \deg_\theta(M)$, and $r(M) = \mathrm{rk}(M)$, for every object M of $\mathscr{A}$. Then, c is an additive function from $\mathscr{A}$ to the ordered abelian group $\mathbb{R}$, and r is a positive additive function from $\mathscr{A}$ to $\mathbb{Z}$. Moreover, in the notation of [32, Definition 3.1], the function μ_θ is the $(c:r)$ slope, and the relation $\preceq_\theta$ the $(c:r)$ preorder, on the set of non-zero objects of $\mathscr{A}$. Therefore, it follows from [32, Lemma 3.2 and Remark] that $\preceq_\theta$ is a stability structure on $\mathscr{A}$. □Let θ be a weight of $\mathscr{A}$.

Definition 2.2.5 An object of $\mathscr{A}$ is called θ-*semistable* (respectively, θ-*stable*, θ-*polystable*) if it is semistable (respectively, stable, polystable) with respect to the stability structure $\preceq_\theta$ on $\mathscr{A}$.

Remark 2.2.6

1. If ζ is a strictly positive real number, and $\omega = \zeta\theta$, then an object of $\mathscr{A}$ is θ-semistable (respectively, θ-stable, θ-polystable) if and only if it is ω-semistable (respectively, ω-stable, ω-polystable).

2. There are obvious special versions of the statements in Propositions 2.1.4–2.1.7,

with semistable (respectively, stable) objects replaced by θ-semistable (respectively, θ-stable) objects of $\mathscr{A}$. Moreover, since rk is a positive additive function from $\mathscr{A}$ to $\mathbb{Z}$, the category $\mathscr{A}$ is Noetherian and Artinian. In particular, by Proposition 2.1.11, every θ-semistable object of $\mathscr{A}$ has a Jordan-Hölder filtration, and every object of $\mathscr{A}$ has a unique Harder-Narasimhan filtration, with respect to $\preceq_\theta$. We say that two θ-semistable objects of $\mathscr{A}$ are S_θ*-equivalent* if they are S-equivalent with respect to $\preceq_\theta$.

There is a well-known definition of the stability of objects of $\mathscr{A}$ that is defined by King in [20, p. 516]. Let λ be an additive function from $\mathscr{A}$ to $\mathbb{R}$. Then, King defines an object M of $\mathscr{A}$ to be λ-semistable if it is non-zero, if $\lambda(M) = 0$, and if $\lambda(N) \geq 0$ (respectively, $\lambda(N) > 0$) for every non-zero proper subobject N of M. The following Proposition shows that the notion of θ-semistability (respectively, θ-stability) defined above is a special case of this definition of λ-semistability (respectively, λ-stability).

Proposition 2.2.7 *Let θ be a weight of $\mathscr{A}$, μ a real number, and c a strictly positive real number. Define an additive function λ from $\mathscr{A}$ to $\mathbb{R}$, by putting $\lambda(M) = c(\mu \operatorname{rk}(M) - \deg_\theta(M))$ for every object M of $\mathscr{A}$. Let $O^{\mathrm{ss}}(\theta,\mu)$ (respectively, $O^{\mathrm{s}}(\theta,\mu)$) be the set of all θ-semistable (respectively, θ-stable) objects M of $\mathscr{A}$, such that $\mu_\theta(M) = \mu$. Let $K^{\mathrm{ss}}(\lambda)$ (respectively, $K^{\mathrm{s}}(\lambda)$) be the set of all λ-semistable (respectively, λ-stable) objects of $\mathscr{A}$, in the sense of King. Then, $O^{\mathrm{ss}}(\theta,\mu) = K^{\mathrm{ss}}(\lambda)$, and $O^{\mathrm{s}}(\theta,\mu) = K^{\mathrm{s}}(\lambda)$.*

Proof. For any non-zero object M of $\mathscr{A}$, we have

$$\lambda(M) = c \operatorname{rk}(M)\,(\mu - \mu_\theta(M)).$$

In particular, since both c and $\operatorname{rk}(M)$ are non-zero, we have $\lambda(M) = 0$ if and only if $\mu_\theta(M) = \mu$. Suppose that this is indeed the case. Then, for every non-zero proper subobject N of M, since c and $\mathrm{rk}(N)$ are strictly positive, the above equation implies that

$$\lambda(N) \geq 0 \;\Leftrightarrow\; \mu \geq \mu_\theta(N) \;\Leftrightarrow\; \mu_\theta(N) \leq \mu_\theta(M).$$

Moreover, all the three inequalities are strict if any one of them is. It follows that M belongs to $K^{\mathrm{ss}}(\lambda)$ (respectively, $K^{\mathrm{s}}(\lambda)$) if and only if it belongs to $O^{\mathrm{ss}}(\theta,\mu)$ (respectively, $O^{\mathrm{s}}(\theta,\mu)$). □

2.2.2 Facets with respect to a hyperplane arrangement

Recall that an *affine space* modeled after a finite-dimensional $\mathbb{R}$-vector space T is a non-empty set E, together with a free and transitive right action

$$(e,t) \mapsto e+t : E \times T \to E$$

of the additive group of T on E.

A subset F of E is called an *affine subspace* of E if it is non-empty, and if there exists a subspace L of T, such that $F = e+L$ for all $e \in F$. In that case, L is unique, and its dimension is called the dimension of F. Moreover, if F is a subset of E for which there exist an element e_0 of F, and a subspace L of T, such that $F = e_0+L$, then $F = e+L$ for all $e \in F$, hence F is an affine subspace of E.

A function $u : E \to \mathbb{R}$ is called affine if there exists a linear function $u' : T \to \mathbb{R}$, such that $u(e+t) = u(e)+u'(t)$ for all $e \in E$ and $t \in T$. In that case, for any element e of the non-empty set E, we have $u'(t) = u(e+t)-u(e)$ for all $t \in T$, hence u' is unique. Moreover, if $u : E \to \mathbb{R}$ is a function for which there exists an element e_0 of E, and a linear function $u' : T \to \mathbb{R}$, such that $u(e_0+t) = u(e_0)+u'(t)$ for all $t \in \mathbb{R}$, then $u(e+t) = u(e)+u'(t)$ for all $e \in E$ and $t \in E$, hence u is affine.

Let E be an affine space modeled after a finite-dimensional $\mathbb{R}$-vector space T. We will give T its usual topology, that is, the unique topology which makes it a Hausdorff topological vector space. We also give E the unique topology which makes the translation $t \mapsto e+t : T \to E$ a homeomorphism for every $e \in E$.

Definition 2.2.8 A *hyperplane* in E is an affine subspace H of E, such that $\dim(H) = \dim(E) - 1$.

Remark 2.2.9 1. It is very useful description, or, a equivalent definition of hyperplane when we describe, in subsection (2.2.3), how semistability and other related notations behave with respect to this hyperplane.

A subset H of E is a hyperplane in E if and only if there exists a non-constant affine function $u : E \to \mathbb{R}$, such that $H = u^{-1}(0)$. Such a function is called a *defining function* of H. For this, Let H be a hyperplane in E. Then, there exists an element e_0 of H, and a hyperplane L in T, such that $H = e_0 + L$. Let $u' : T \to \mathbb{R}$ be a linear function, such that $L = \mathrm{Ker}(u')$, and let $u : E \to \mathbb{R}$ be the unique function such that $u(e_0+t) = u'(t)$ for all $t \in \mathbb{R}$. Then, u is affine, and

$$u^{-1}(0) = e_0 + \mathrm{Ker}(u') = e_0 + L = H,$$

hence u is a defining function of H.

2. If u and v are two defining functions of H, then there exists a unique non-zero real number a, such that $u(e) = av(e)$ for all $e \in E$. For this, if $v : E \to \mathbb{R}$ is another defining function of H, and if $v' : E \to \mathbb{R}$ is the unique linear function such that $v(e+t) = v(e) + v'(t)$ for all $e \in E$ and $t \in T$, then, as $H = v^{-1}(0)$, we have $v(e_0) = 0$, hence $v'(t) = v(e_0+t)$ for all $t \in T$. Therefore, $L = \mathrm{Ker}(v')$. It follows that there exists a unique non-zero real number a, such that $u' = av'$, or, equivalently, $u = av$.

Let H be a hyperplane in E. Then, $E \setminus H$ has two connected components, which are called the *open half-spaces* bounded by H. Their closures in E are called the *closed half-spaces* bounded by H. We say that two points x and y in E are *strictly on the same side* of H if they belong to the same open half-space bounded by H. They are said to be on *opposite sides* of H if they belong to distinct open half-spaces bounded by H.

If u is a defining function of H, then the sets $u^{-1}((0,+\infty))$ and $u^{-1}((-\infty,0))$ are the open half-spaces bounded by H, and the sets $u^{-1}([0,+\infty))$ and $u^{-1}((-\infty,0])$ are the closed half-spaces bounded by H. In particular, all the half-spaces bounded by H are convex subsets of E. Two points x and y of E are strictly on the same side of H if and only if and only if $u(x)u(y) > 0$. They are on opposite sides of H if and only if $u(x)u(y) < 0$.

Definition 2.2.10 A *hyperplane arrangement* in E is a locally finite set of hyperplanes in E.

We fix a hyperplane arrangement $\mathscr{H}$ in E. For any subset X of E, we define

$$\mathscr{H}(X) = \{H \in \mathscr{H} \,|\, H \cap X \neq \emptyset\}.$$

If X is a singleton $\{x\}$, we write $\mathscr{H}(x)$ for $\mathscr{H}(X)$.

For every hyperplane H in E, we define a relation $\sim_H$ on E by setting $x \sim_H y$ if x and y belong to H, or if x and y are strictly on the same side of H. It is clear that $\sim_H$ is an equivalence relation on E. We now define the relation $\sim$ on E to be the intersection of the relations $\sim_H$ as H runs over $\mathscr{H}$. Thus, $x \sim y$ if and only if $x \sim_H y$ for all $H \in \mathscr{H}$. Being the intersection of a family of equivalence relations on E, the relation $\sim$ also is an equivalence relation on E. The $\sim$-equivalence class of an element a of E is called the *facet* of E through a with respect to $\mathscr{H}$.

Definition 2.2.11 A subset of E is called a *facet* of E with respect to $\mathscr{H}$ if it equals the facet of E through some element of E.

Let F be a facet of E. Then, for every point a in F, and for any element H of $\mathscr{H}(a)$, we have $F \subset H$, hence

$$\mathscr{H}(F) = \mathscr{H}(a) = \{H \in \mathscr{H} \,|\, F \subset H\}.$$

In particular, since $\mathscr{H}$ is locally finite, the set $\mathscr{H}(F)$ is finite. The intersection $\mathrm{Supp}(F)$ of all the elements of $\mathscr{H}(F)$ is an affine subspace of E, and is called the *support* of F. The dimension of the affine space $\mathrm{Supp}(F)$ is called the *dimension* of F, and is denoted by $\dim(F)$.

Let F be a facet of E, and L its support. Then, the closure $\overline{F}$ of F in E is a subset of L. For, F is contained in every element H of $\mathscr{H}(F)$, and hence in the intersection L of all such H. Now, as L is an affine subspace of E, it is closed in E. It follows that $\overline{F} \subset L$. Thus, the closure $\overline{F}$ of F in E is a subset of $\mathrm{Supp}(F)$. The following statement is proved in [4, Chapter V, § 1, no. 2, Proposition 3].

Proposition 2.2.12 *Let F be a facet of E, and L its support. Then, F equals the interior of $\overline{F}$ in L. In particular, F is open in L.*

Proposition 2.2.13 *A subset F of E is a facet of E if and only if it is a maximal element of the set of all connected subsets X of E, such that for every hyperplane $H \in \mathscr{H}$, X is a subset, either of H, or of $E \setminus H$.*

Proof. Let $\mathscr{X}$ denote the set of all connected subsets X of E, such that for every hyperplane $H \in \mathscr{H}$, X is a subset, either of H, or of $E \setminus H$.

Suppose that F is a facet of E. Then, it is a convex subset of E, and is hence path-connected. If $H \in \mathscr{H}$, and F is not a subset of H, then $H \notin \mathscr{H}(F)$, so $H \subset E \setminus F$. Therefore, $F \in \mathscr{X}$. Let X be any element of $\mathscr{X}$, such that $F \subset X$. Let a be an element of the non-empty set F. To show that $X \subset F$, it suffices to check that $x \sim_H a$ for all $H \in \mathscr{H}$. If $X \subset H$, then $x \in X \subset H$ and $a \in F \subset X \subset H$, hence $x \sim_H a$. If $X \not\subset H$, then $X \subset E \setminus H$. As X is connected, this implies that it is contained in an open half-space D bounded by H. Thus, $x \in X \subset D$ and $a \in F \subset X \subset D$, hence x and a are strictly on the same side of H, so $x \sim_H a$. This proves that $X \subset F$, hence $X = F$. Thus, F is a maximal element of $\mathscr{X}$.

Suppose that F is a maximal element of $\mathscr{X}$. We first claim that F is non-empty. Suppose $F = \emptyset$. Since E is non-empty, it has a point e. Thus, $F \subset X = \{e\}$. Clearly, $X \in \mathscr{X}$. Therefore, by the maximality of F, we get $F = X$, a contradiction, as $X \neq \emptyset$. This proves that F is non-empty. Let a be an element of F, and F' the facet of E through a. We claim that $F \subset F'$. To see this, let $x \in F$. Since $F \in \mathscr{X}$, for any $H \in \mathscr{H}$, either $F \subset H$ or $F \subset X \setminus H$. If $F \subset H$, then x and a belong to H, hence $x \sim_H a$. If $F \subset E \setminus H$, then, as F is connected, it is contained in an open half-space D bounded by H. Thus, x and a belong to D, hence they are strictly on the same side of H, so $x \sim_H a$ again. It follows that $x \sim a$, hence x belongs to the facet F' through a. The claim that $F \subset F'$ is verified. Now, by the previous paragraph, the facet F' is an element of $\mathscr{X}$. Therefore, by the maximality of F implies that $F = F'$. It follows that F is a facet of E. □

In some references, for instance, [15, Definition 4.1], a subset F of E satisfying the latter condition is called a "chamber" of E, rather than a "facet" of E as here. We are following the terminology in [4, Chapter V, § 1], where the term "chamber" is reserved for a facet that does not meet any element of $\mathscr{H}$.

2.2.3 The hyperplane arrangement on the weight space

Let $\mathscr{A}$ be an abelian category, $(\varphi_i)_{i\in I}$ a non-empty finite positive family of additive functions from $\mathscr{A}$ to $\mathbb{Z}$, and rk a positive additive function from $\mathscr{A}$ to $\mathbb{Z}$, as in Section

2.2.1. The $\mathbb{R}$-vector space $\mathbb{R}^I$ is called the *weight space* of $\mathscr{A}$. We give it the usual, that is, the product topology.

For all elements $\theta = (\theta_i)_{i\in I}$ of $\mathbb{R}^I$ and $d = (d_i)_{i\in I}$ of $\mathbb{N}^I$, we define a real number $\deg_\theta(d)$, and a natural number $\mathrm{rk}(d)$, by

$$\deg_\theta(d) = \sum_{i\in I} \theta_i d_i, \quad \mathrm{rk}(d) = \sum_{i\in I} d_i.$$

If d is a non-zero element of $\mathbb{N}^I$, then $\mathrm{rk}(d) > 0$, so for each $\theta \in \mathbb{R}^I$, we have a real number $\mu_\theta(d)$, which is defined by

$$\mu_\theta(d) = \frac{\deg_\theta(d)}{\mathrm{rk}(d)}.$$

For any two non-zero elements d and e of $\mathbb{N}^I$, we define an $\mathbb{R}$-linear function $f(d,e) : \mathbb{R}^I \to \mathbb{R}$ by

$$f(d,e)(\theta) = \mu_\theta(d) - \mu_\theta(e).$$

Remark 2.2.14 $f(d,e) = 0$ if and only if $e \in \mathbf{Q}d$. For this, Let

$$f_i = \frac{d_i}{\mathrm{rk}(d)} - \frac{e_i}{\mathrm{rk}(e)}$$

for each $i \in I$. Then,

$$f(d,e)(\theta) = \sum_{i\in I} \theta_i f_i$$

for all $\theta \in \mathbb{R}^I$. Therefore, $f(d,e)$ is $\mathbb{R}$-linear. Moreover, $f(d,e) = 0$ if and only if $f_i = 0$ for all $i \in I$. Therefore, if $f(d,e) = 0$, then $e = \frac{\mathrm{rk}(e)}{\mathrm{rk}(d)} d$, hence $e \in \mathbf{Q}d$. Conversely, suppose $e = \lambda d$ for some $\lambda \in \mathbf{Q}$. Then, as e is non-zero, λ is also non-zero, and

$$\mathrm{rk}(e) = \sum_{i \in I} e_i = \lambda \sum_{i \in I} d_i = \lambda \mathrm{rk}(d).$$

Therefore,

$$f_i = \frac{d_i}{\mathrm{rk}(d)} - \frac{e_i}{\mathrm{rk}(e)} = \frac{d_i}{\mathrm{rk}(d)} - \frac{\lambda d_i}{\lambda \mathrm{rk}(e)} = 0$$

for all $i \in I$. It follows that $f(d,e) = 0$.

Remark 2.2.15

Remark 2.2.16 Let ψ be an additive function from $\mathscr{A}$ to an ordered abelian group G, such that $\psi(M) \geq 0$ for every object M of $\mathscr{A}$. Then, $\psi(N) \leq \psi(M)$ for every object M of $\mathscr{A}$, and for every subobject N of M. For, the canonical exact sequence

$$0 \to N \to M \to M/N \to 0$$

implies that

$$\psi(M) = \psi(N) + \psi(M/N) \geq \psi(N),$$

since $\psi(M/N) \geq 0$.

Fix a non-zero element d of $\mathbb{N}^I$. Let S_d denote the set of all elements e of $\mathbb{N}^I \setminus \mathbf{Q}d$, for which there exist an object M of $\mathscr{A}$, and a subobject N of M, such that $\varphi(M) = d$ and $\varphi(N) = e$. Since the family $(\varphi_i)_{i \in I}$ of additive functions on $\mathscr{A}$ is positive, from the 2.2.16, it is clear that the set S_d is contained in $\prod_{i \in I}(\mathbb{N} \cap [0, d_i])$, and is hence finite. For each $e \in S_d$, let

$$H(d,e) = \mathrm{Ker}(f(d,e)) = \{\theta \in \mathbb{R}^I \mid \mu_\theta(e) = \mu_\theta(d)\}.$$

Since $e \notin \mathbf{Q}d$, the function $f(d,e)$ is non-zero, hence $H(d,e)$ is a hyperplane in $\mathbb{R}^I$. We thus get a finite hyperplane arrangement

$$\mathscr{H}(d) = \{H(d,e) \,|\, e \in S_d\}$$

in the affine space $\mathbb{R}^I$.

We define sgn to be the function from $\mathbb{R}$ to the subset $\{-1,0,1\}$ of $\mathbb{R}$, which is -1 at every negative real number, vanishes at 0, and is 1 at every positive real number.

Proposition 2.2.17 *Let F be a facet of $\mathbb{R}^I$ with respect to the hyperplane arrangement $\mathscr{H}(d)$, and let θ and ω be two elements of F. Let M be an object of $\mathscr{A}$, such that $\varphi(M) = d$, and let N be a non-zero subobject of M. Then,*

$$\operatorname{sgn}(\mu_\theta(M) - \mu_\theta(N)) = \operatorname{sgn}(\mu_\omega(M) - \mu_\omega(N)).$$

Proof. Let $e = \varphi(N)$, and let $f = f(d,e) : \mathbb{R}^I \to \mathbb{R}$. Then, for each weight λ of $\mathscr{A}$, we have

$$\mu_\lambda(M) - \mu_\lambda(N) = \mu_\lambda(d) - \mu_\lambda(e) = f(\lambda).$$

Therefore, we have to prove that $\operatorname{sgn}(f(\theta)) = \operatorname{sgn}(f(\omega))$. If $f(\theta) = f(\omega) = 0$, then the equality to be proved is obvious. Suppose that either $f(\theta)$ or $f(\omega)$ is non-zero. By interchanging θ and ω, we can assume that $f(\theta) \neq 0$. Then, $e \notin \mathbf{Q}d$. As $\varphi(M) = d$ and $\varphi(N) = e$, this implies that $e \in S_d$. Thus, the hyperplane $H = H(d,e)$ is an element of $\mathscr{H}(d)$. Now, as $H = \operatorname{Ker}(f)$, f is a defining function of H. Moreover, $\theta \notin H$, since $f(\theta) \neq 0$. As θ and ω both belong to the same facet F, we have $\theta \sim_H \omega$, so θ and ω are strictly on the same side of H. Therefore, $f(\theta)f(\omega) > 0$. It follows that $\operatorname{sgn}(f(\theta)) = \operatorname{sgn}(f(\omega))$. □

Proposition 2.2.18 *Let d be a non-zero element of N^I, F a facet of $\mathbb{R}^I$ with respect* *to the hyperplane arrangement $\mathscr{H}(d)$, and θ and ω two elements of F. Let M be an object of $\mathscr{A}$, such that $\varphi(M) = d$. Then, M is θ-semistable (respectively, θ-stable) if and only if it is ω-semistable (respectively, ω-stable).*

Proof. As $\varphi(M) = d$ is non-zero, and the family $(\varphi_i)_{i \in I}$ is positive, M is non-zero. For each weight λ of $\mathscr{A}$, and for each non-zero subobject N of M, define

$$g_\lambda(N) = \mu_\lambda(M) - \mu_\lambda(N).$$

Then, M is λ-semistable if and only if $g_\lambda(N) \geq 0$, or equivalently, $\operatorname{sgn}(g_\lambda(N))$ belongs to $\{0, 1\}$, for every non-zero subobject N of M. Similarly, M is λ-stable if and only if $\operatorname{sgn}(g_\lambda(N)) = 1$ for every non-zero proper subobject N of M. Therefore, it suffices to check that $\operatorname{sgn}(g_\theta(N)) = \operatorname{sgn}(g_\omega(N))$ for every non-zero proper subobject N of M. As θ and ω belong to the same facet F, this is a consequence of Proposition 2.2.17. □

Proposition 2.2.19 *Let d be a non-zero element of N^I, F a facet of $\mathbb{R}^I$ with respect to the hyperplane arrangement $\mathscr{H}(d)$, and θ and ω two elements of F. Let M be an object of $\mathscr{A}$, such that $\varphi(M) = d$. Suppose that M is θ-semistable, and let $(M_i)_{i=0}^n$ be a Jordan-Hölder filtration of M with respect to θ. Then, M is ω-semistable, and $(M_i)_{i=0}^n$ is a Jordan-Hölder filtration of M with respect to ω also.*

Proof. The fact that M is ω-semistable has already been proved in Proposition 2.2.18. For every $i = 1, \dots, n$, we have $\mu_\theta(M_{i-1}) = \mu_\theta(M)$, hence, by Proposition 2.2.17,

$$\operatorname{sgn}(\mu_\omega(M) - \mu_\omega(M_{i-1})) = \operatorname{sgn}(\mu_\theta(M) - \mu_\theta(M_{i-1})) = 0,$$

so $\mu_\omega(M_{i-1}) = \mu_\omega(M)$. It remains to prove that the quotient object $N_i = M_{i-1}/M_i$ is ω-stable for every $i = 1, \dots, n$.

We will first verify that $\mu_\omega(N_i) = \mu_\omega(M)$ for all $i = 1, \dots, n$. If $i = n$, this follows from the above paragraph, since $N_n = M_{n-1}$. Suppose $1 \leq i \leq n-1$. Then, both i and $i+1$ belong to $\{1,\ldots,n\}$, hence, by the above paragraph,

$$\mu_\omega(M_i) = \mu_\omega(M_{i-1}) = \mu_\omega(M).$$

We also have a short exact sequence

$$0 \to M_i \to M_{i-1} \to N_i \to 0,$$

of non-zero objects of $\mathscr{A}$. Therefore, by the seesaw property of $\preceq_\omega$, we get

$$\mu_\omega(N_i) = \mu_\omega(M_{i-1}) = \mu_\omega(M).$$

This proves that $\mu_\omega(N_i) = \mu_\omega(M)$ for all $i = 1,\ldots,n$.

Let $i \in \{1,\ldots,n\}$. In view of the previous paragraph, to show that N_i is ω-stable, it suffices to show that $\mu_\omega(X) < \mu_\omega(M)$ for every proper non-zero subobject X of N_i. To begin with, since N_i is θ-stable, we have

$$\mu_\theta(X) < \mu_\theta(N_i) = \mu_\theta(M).$$

Suppose first that $i = n$. Then, $N_i = M_{n-1}$, and X is a non-zero subobject of M, hence, by Proposition 2.2.17 and the above inequality,

$$\operatorname{sgn}(\mu_\omega(M) - \mu_\omega(X)) = \operatorname{sgn}(\mu_\theta(M) - \mu_\theta(X)) = 1.$$

It follows that $\mu_\omega(X) < \mu_\omega(M)$. Suppose next that $1 \leq i \leq n-1$. Let $\pi : M_{i-1} \to N_i$

be the canonical projection, and let $Y = \pi^{-1}(X)$, that is, the kernel of the composite

$$M_{i-1} \xrightarrow{\pi} N_i \to N_i/X.$$

Thus, Y is a non-zero subobject of M_{i-1}, and we have a short exact sequence

$$0 \to M_i \to Y \to X \to 0$$

of non-zero objects of $\mathscr{A}$. By the above paragraphs,

$$\mu_\theta(M_i) = \mu_\theta(M) = \mu_\theta(N_i) > \mu_\theta(X).$$

Therefore, by the seesaw property of $\preceq_\theta$,

$$\mu_\theta(M) = \mu_\theta(M_i) > \mu_\theta(Y),$$

hence, by Proposition 2.2.17,

$$\operatorname{sgn}(\mu_\omega(M) - \mu_\omega(Y)) = \operatorname{sgn}(\mu_\theta(M) - \mu_\theta(Y)) = 1,$$

so

$$\mu_\omega(M_i) = \mu_\omega(M) > \mu_\omega(Y).$$

Again, by the seesaw property of $\preceq_\omega$, we get $\mu_\omega(M_i) > \mu_\omega(X)$, hence $\mu_\omega(M) > \mu_\omega(X)$. This proves that N_i is ω-stable. □

Proposition 2.2.20 *Let d be a non-zero element of $\mathbb{N}^I$, F a facet of $\mathbb{R}^I$ with respect to the hyperplane arrangement $\mathscr{H}(d)$, and θ and ω two elements of F. Let M be an object of $\mathscr{A}$, such that $\varphi(M) = d$. Then, M is θ-polystable if and only if it is*

ω-polystable.

Proof. Suppose M is θ-polystable. Then, M is θ-semistable, and there exists a sequence $(M_i)_{i=1}^n$ of θ-stable objects of $\mathscr{A}$, such that $n \in \mathbb{N}$, M_i is θ-stable for each $i = 1, \dots, n$, and M is isomorphic to $\bigoplus_{i=1}^n M_i$. As M is non-zero, we in fact have $n \geq 1$. Also, by Proposition 2.2.18, M is ω-semistable. Let $N = \bigoplus_{i=1}^n M_i$. Then, since N is isomorphic to M, it is both θ-semistable and ω-semistable, and $\varphi(M) = d$. For each $i = 0, \dots, n$, define $N_i = \bigoplus_{j=i+1}^n M_i$. Then, $(N_i)_{i=0}^n$ is a decreasing sequence of subobjects of N, $N_0 = N$, $N^n = 0$, and for each $i = 1, \dots, n$, N_{i-1}/N_i is isomorphic to M_i, and is hence θ-stable. Moreover, since N is θ-semistable, by Proposition 2.1.4(3), for every $i = 1, \dots, n$, we get

$$\mu_\theta(N) = \mu_\theta(M_i) = \mu_\theta(N_{i-1}).$$

Therefore, $(N_i)_{i=0}^n$ is a Jordan-Hölder filtration of N with respect to θ. By Proposition 2.2.19, it is a Jordan-Hölder filtration of N with respect to ω also. In particular, for each $i = 1, \dots, n$, N_{i-1}/N_i is ω-stable, hence M_i is ω-stable. As M is ω-semistable, and isomorphic to $\bigoplus_{i=1}^n M_i$, it follows that M is ω-polystable. □

Proposition 2.2.21 *Let d be a non-zero element of N^I, F a facet of $\mathbb{R}^I$ with respect to the hyperplane arrangement $\mathscr{H}(d)$, and θ and ω two elements of F. Let M and N be two objects of $\mathscr{A}$, such that $\varphi(M) = \varphi(N) = d$. Suppose that M and N are θ-semistable, and that M is S_θ-equivalent to N. Then, M and N are ω-semistable, and M is S_ω-equivalent to N.*

Proof. Let $(M_i)_{i=0}^n$ and $(N_j)_{j=0}^m$ be Jordan-Hölder filtrations of M and N, respectively, with respect to θ. Then, by Proposition 2.2.19, M and N are ω-semistable, and $(M_i)_{i=0}^n$ and $(N_j)_{j=0}^m$ are Jordan-Hölder filtrations of M and N, respectively, with respect to ω. Now, because M is S_θ-equivalent to N, $n = m$, and there exists a permutation $\pi \in \mathrm{S}_n$, such that M_{i-1}/M_i is isomorphic $N_{\pi(i)-1}/N_{\pi(i)}$ for every $i = 1,\dots,n$. As $(M_i)_{i=0}^n$ and $(N_j)_{j=0}^m$ are Jordan-Hölder filtrations of M and N, respectively, with respect to ω, it follows that M is S_ω-equivalent to N. □

Lemma 2.2.22 *Let V be a finite-dimensional $\mathbb{R}$-vector space, and V' a $\mathbf{Q}$-structure on V. Let $(f_i)_{i\in I}$ be a family of $\mathbb{R}$-linear functions from V to $\mathbb{R}$, which are rational over $\mathbf{Q}$, for the $\mathbf{Q}$-structure $\mathbf{Q}$ on $\mathbb{R}$. Let $L = \bigcap_{i\in I} \operatorname{Ker}(f_i)$. Then, with respect to the usual topology on V, the closure of $V' \cap L$ in V equals L.*

Proof. For each $i \in I$, the $\mathbb{R}$-subspace $L_i = \operatorname{Ker}(f_i)$ of V is rational over $\mathbf{Q}$, since f_i is rational over $\mathbf{Q}$ [3, Chapter II, § 8, no. 3, Corollary 2 to Proposition 3]. Therefore, the intersection L of the L_i is also rational over $\mathbf{Q}$ [3, Chapter II, § 8, no. 1, Corollary 1 to Proposition 2]. This implies that $V' \cap L$ generates L as an $\mathbb{R}$-vector space. Now, let $x \in L$. Then, there exist a finite family $(v_i)_{i\in I}$ of elements of $V' \cap L$, and real numbers a_i ($i \in I$), such that $x = \sum_{i\in I} a_i v_i$. For each $i \in I$, let $(q_{in})_{n\in\mathbb{N}}$ be a sequence of rational numbers converging to a_i in $\mathbb{R}$. Define $x_n = \sum_{i\in I} q_{in} v_i$ for each $n \in \mathbb{N}$. Then, as V' is a $\mathbf{Q}$-subspace of V, $x_n \in V' \cap L$ for all $n \in \mathbb{N}$. Moreover,

$$\lim_{n\to+\infty} x_n = \sum_{i\in I} (\lim_{n\to+\infty} q_{in}) v_i = \sum_{i\in I} a_i v_i = x.$$

It follows that x belongs to the closure $\overline{V' \cap L}$ of $V' \cap L$ in V. Thus, $L \subset \overline{V' \cap L}$. Conversely, since every $\mathbb{R}$-subspace of V is closed in V, L is closed in V, hence $\overline{V' \cap L} \subset L$. It follows that $\overline{V' \cap L} = L$. □

Proposition 2.2.23 *Every facet of $\mathbb{R}^I$ with respect to the hyperplane arrangement $\mathscr{H}(d)$ contains an integral weight, that is, an element of $\mathbb{Z}^I$.*

Proof. Give $\mathbb{R}^I$ the $\mathbf{Q}$-structure $\mathbf{Q}^I$. Every element e of S_d is an element of $\mathbb{N}^I$, hence $f(d,e)(\mathbf{Q}^I) \subset \mathbf{Q}$, so $f(d,e)$ is rational over $\mathbf{Q}$. Let F be a facet of $\mathbb{R}^I$ with respect to

$\mathscr{H}(d)$, and let $L = \mathrm{Supp}(F)$. Let K denote the set of all elements e of S_d such that $F \subset H(d,e)$. Then,

$$L = \bigcap_{e \in K} H(d,e) = \mathrm{Ker}(f(d,e)),$$

hence, by Lemma 2.2.22, the closure of $\mathbf{Q}^I \cap L$ in $\mathbb{R}^I$ equals L. Now, by Proposition 2.2.12, F is open in L. Therefore, there exists an open subset U of $\mathbb{R}^I$, such that $F = U \cap L$. Let θ be an element of the non-empty set F. Then, θ belongs to the closure L of $\mathbf{Q}^I \cap L$ in $\mathbb{R}^I$, and U is an open neighbourhood of θ in $\mathbb{R}^I$, hence there exists an element ξ in $(\mathbf{Q}^I \cap L) \cap U = \mathbf{Q}^I \cap F$. Let n be a strictly positive integer such that $\omega = n\xi$ belongs to $\mathbb{Z}^I$. We claim that $\omega \in F$. To see this, let $e \in S_d$. Then, $f(d,e)$ is $\mathbb{R}$-linear, hence $f(d,e)(\omega) = nf(d,e)(\xi)$. As $n > 0$, this implies that $\mathrm{sgn}(f(d,e)(\omega)) = \mathrm{sgn}(f(d,e)(\xi))$. Since $f(d,e)$ is a defining function of $H(d,e)$, it follows that $\omega \sim_{H(d,e)} \xi$. As this is true for all $e \in S_d$, $\omega \sim \xi$, hence ω belongs to the facet F of $\mathbb{R}^I$ through ξ. Thus, ω is an element of $\mathbb{Z}^I \cap F$. □

Proposition 2.2.24 *Let d be a non-zero element of $\mathbb{N}^I$, and let $\theta \in \mathbb{R}^I$. Then, there exists an integral weight ω of $\mathscr{A}$, with the following properties:*

1. *Any object M of $\mathscr{A}$, such that $\varphi(M) = d$, is θ-semistable (respectively, θ-stable, θ-polystable) if and only if it is ω-semistable (respectively, ω-stable, ω-polystable).*

2. *If M is a θ-semistable object of $\mathscr{A}$ such that $\varphi(M) = d$, then every Jordan-Hölder filtration of M with respect to θ is a Jordan-Hölder filtration of M with respect to ω, and conversely.*

3. *Two θ-semistable objects M and N of $\mathscr{A}$, such that $\varphi(M) = \varphi(N) = d$, are S_θ-equivalent if and only if they are S_ω-equivalent.*

Proof. This Proposition follows immediately from Propositions 2.2.18–2.2.21 and

2.2.23. □

2.3 Representations of quivers

Here, we will specialise the constructs of the previous sections to the specific abelian category of the representations of a quiver over a field. To do so, the followings

- Quivers and their representations
- Notion of the semistability of a representation of a quiver with respect to a weight, as an instance of the general theory described in section 2.2.1.
- Hermitian metrics on complex representations of a quiver

are discussed in respective subsections.

2.3.1 The category of representations

Definition 2.3.1 1. A *quiver* Q is a quadruple $(Q_0, Q_1, \mathrm{s}, \mathrm{t})$, where Q_0 and Q_1 are sets, and $\mathrm{s} : Q_1 \to Q_0$, and $\mathrm{t} : Q_1 \to Q_0$ are functions. The elements of Q_0 are called the *vertices of* Q, and those of Q_1 are called the *arrows of* Q. For any arrow α of Q, the vertex $\mathrm{s}(\alpha)$ is called the *source* of α, and the vertex $\mathrm{t}(\alpha)$ is called the *target* of α. If $\mathrm{s}(\alpha) = a$ and $\mathrm{t}(\alpha) = b$, then we say that α is an arrow *from* a *to* b, and write $\alpha : a \to b$.

2. We say that Q is *vertex-finite* if the set Q_0 is finite, *arrow-finite* if the set Q_1 is finite, and *finite* if it is both vertex-finite and arrow-finite.

3. The quiver $(\emptyset, \emptyset, \mathrm{s}, \mathrm{t})$, where s and t are the empty functions, is called the *empty* quiver. We say that a quiver Q is *non-empty* if it is not equal to the empty quiver, or equivalently, if the set Q_0 of its vertices is non-empty.

Definition 2.3.2 Let k be a field. A *representation of Q over k* is a pair (V,ρ), where $V=(V_a)_{a\in Q_0}$ is a family of finite-dimensional k-vector spaces, and $\rho=(\rho_\alpha)_{\alpha\in Q_1}$ is a family of k-linear maps $\rho_\alpha : V_{\mathrm{s}(\alpha)}\to V_{\mathrm{t}(\alpha)}$. We will often drop the base field k from the terminology. If (V,ρ) and (W,σ) are two representations of Q, then a *morphism from* (V,ρ) *to* (W,σ) is a family $f=(f_a)_{a\in Q_0}$ of k-linear maps $f_a : V_a\to W_a$, such that for every $\alpha\in Q_1$, the diagram

$$\begin{array}{ccc} V_{\mathrm{s}(\alpha)} & \xrightarrow{\rho_\alpha} & V_{\mathrm{t}(\alpha)} \\ {\scriptstyle f_{\mathrm{s}(\alpha)}}\downarrow & & \downarrow{\scriptstyle f_{\mathrm{t}(\alpha)}} \\ W_{\mathrm{s}(\alpha)} & \xrightarrow[\sigma_\alpha]{} & W_{\mathrm{t}(\alpha)} \end{array}$$

commutes.

If (V,ρ), (W,σ), and (X,τ) are three representations of Q, f a morphism from (V,ρ) to (W,σ), and g a morphism from (W,σ) to (X,τ), then the *composite* of f and g is the family $g\circ f$ defined by $g\circ f=(g_a\circ f_a)_{a\in Q_0}$. It is easy to verify that $g\circ f$ is a morphism from (V,ρ) to (X,τ). We thus get a category $\mathbf{Rep}_k(Q)$, whose objects are representations of Q over k, and whose morphisms are defined as above.

For any two representations (V,ρ) and (W,σ) of Q, the set $\mathrm{Hom}((V,\rho),(W,\sigma))$ is a k-subspace of the k-vector space $\bigoplus_{a\in Q_0}\mathrm{Hom}_k(V_a,W_a)$. If (X,τ) is another representation of Q, then the composition operator

$$\mathrm{Hom}((W,\sigma),(X,\tau))\times\mathrm{Hom}((V,\rho),(W,\sigma))\to\mathrm{Hom}((V,\rho),(X,\tau))$$

is k-bilinear. Any representation (V,ρ) such that the k-vector space V_a is zero for all $a\in Q_0$ is a zero object in this category. For every finite family $(V_i,\rho_i)_{i\in I}$ of representations of Q, the pair (V,ρ), which is defined by

$$V_a=\bigoplus_{i\in I}V_{i,a},\quad \rho_\alpha=\bigoplus_{i\in I}\rho_{i,\alpha},$$

for all $a \in Q_0$ and $\alpha \in Q_1$, is a coproduct of $(V_i, \rho_i)_{i\in I}$ in $\mathbf{Rep}_k(Q)$. Thus, $\mathbf{Rep}_k(Q)$ is a k-linear additive category.

If $f : (V,\rho) \to (W,\sigma)$ is a morphism of representations of Q, then the pair (V',ρ'), where $V'_a = \mathrm{Ker}(f_a)$ for all $a \in Q_0$, and $\rho'_\alpha : V'_{\mathrm{s}(\alpha)} \to V'_{\mathrm{t}(\alpha)}$ is the restriction of ρ_α for each $\alpha \in Q_1$, is a representation of Q, and there is an obvious morphism $i : (V',\rho') \to (V,\rho)$, which is given by the inclusion maps $i_a : V'_a \to V_a$ for all $a \in Q_0$. The representation (V',ρ'), together with the morphism i, is a kernel of f in $\mathbf{Rep}_k(Q)$. Similarly, the pair (W',σ'), where $W'_a = \mathrm{Coker}(f_a)$ for all $a \in Q_0$, and $\sigma'_\alpha : W'_{\mathrm{s}(\alpha)} \to W'_{\mathrm{t}(\alpha)}$ is the k-linear map induced by σ_α for each $\alpha \in Q_1$, is a representation of Q, and there is a morphism $\pi : (W,\sigma) \to (W',\sigma')$, which is given by the canonical projections $\pi_a : W_a \to W'_a$ for all $a \in Q_0$. The representation (W',σ'), together with the morphism π, is a cokernel of f in $\mathbf{Rep}_k(Q)$. Thus, every morphism in this additive category has a kernel and a cokernel. It is obvious that the canonical morphism from $\mathrm{Coker}(i)$ to $\mathrm{Ker}(\pi)$ is an isomorphism. It follows that $\mathbf{Rep}_k(Q)$ is a k-linear abelian category.

Definition 2.3.3 A *subrepresentation* of a representation (V,ρ) of Q is a subobject of (V,ρ) in the category $\mathbf{Rep}_k(Q)$.

Every family $(V_i,\rho_i)_{i\in I}$ of subrepresentations of (V,ρ) has a meet $\bigcap_{i\in I}(V_i,\rho_i)$, and a join $\sum_{i\in I}(V_i,\rho_i)$. Thus, the category $\mathbf{Rep}_k(Q)$ has all meets and joins of subobjects.

Remark 2.3.4 For every representation (V,ρ) of Q, and for any element c of k, we have a representation $(V, c\rho)$ of Q, where $c\rho = (c\rho_\alpha)_{\alpha\in Q_1}$. If (W,σ) is a subrepresentation of (V,ρ), then $(W, c\sigma)$ is a subrepresentation of $(V, c\rho)$. It is also obvious that if $(V_i,\rho_i)_{i\in I}$ is a finite family of representations of Q, then

$$(\bigoplus V_i, c(\bigoplus_{i\in I} \rho_i)) = (\bigoplus V_i, \bigoplus_{i\in I}(c\rho_i)).$$

Lastly, if $f : (V,\rho) \to (W,\sigma)$ is a morphism of representations of Q, then f is also a morphism of representations from $(V, c\rho)$ to $(W, c\sigma)$.

2.3.2 Semistability and stability of representations

Fix a non-empty vertex-finite quiver Q, and a field k. For every representation (V,ρ) of Q over k, and $a \in Q_0$, let

$$\dim_a(V,\rho) = \dim_k(V_a).$$

Then, $(\dim_a(V,\rho))_{a\in Q_0}$ is a non-empty finite positive family of additive functions from the abelian category $\mathbf{Rep}_k(Q)$ to $\mathbb{Z}$. Therefore, the statements of Section 2.2 are applicable here. The following are the versions for representations of some of the notions defined there.

For every representation (V,ρ), the element

$$\dim(V,\rho) = (\dim_k(V_a))_{a\in Q_0}.$$

of $\mathbb{N}^{Q_0}$ is called the *dimension vector* of (V,ρ), and the natural number

$$\mathrm{rk}(V,\rho) = \sum_{a\in Q_0} \dim_k(V_a).$$

is called the *rank* of (V,ρ).

An element of the $\mathbb{R}$-vector space $\mathbb{R}^{Q_0}$ is called a weight of Q. We say that a weight is rational (respectively, integral) if it belongs to the subset $\mathbf{Q}^{Q_0}$ (respectively, $\mathbb{Z}^{Q_0}$) of $\mathbb{R}^{Q_0}$. We fix a weight θ of Q.

For any representation (V,ρ) of Q, the θ-degree of (V,ρ) is the real number

$$\deg_\theta(V,\rho) = \sum_{a\in Q_0} \theta_a \dim_k(V_a).$$

If $(V,\rho) \neq 0$, the real number

$$\mu_\theta(V,\rho) = \frac{\deg_\theta(V,\rho)}{\mathrm{rk}(V,\rho)},$$

is called the θ-slope of (V,ρ).

Definition 2.3.5 A representation (V,ρ) of Q is called θ-semistable (respectively, θ-stable) if it is non-zero, and if

$$\mu_\theta(W,\sigma) \leq \mu_\theta(V,\rho) \quad \text{(respectively, } \mu_\theta(W,\sigma) < \mu_\theta(V,\rho))$$

for every non-zero proper subrepresentation (W,σ) of (V,ρ).

Definition 2.3.6 We say that a representation of Q is θ-polystable if it is θ-semistable, and is isomorphic to a finite family of θ-stable representations of Q.

There are obvious versions for representations of all the results of Section 2.2.

Remark 2.3.7 For any non-zero element c of k, the θ-semistability (respectively, θ-stability, θ-polystability) of a representation (V,ρ) of Q over k is equivalent to the θ-semistability (respectively, θ-stability, θ-polystability) of $(V,c\rho)$. Also, if ζ is a strictly positive real number, and $\omega = \zeta\theta$, then a representation of Q is θ-semistable (respectively, θ-stable, θ-polystable) if and only if it is ω-semistable (respectively, ω-stable, ω-polystable).

2.3.3 Einstein-Hermitian metrics on complex representations

Let Q be a non-empty finite quiver, and fix a weight θ of Q. All the representations of Q considered in this subsection will be over $\mathbb{C}$.

Definition 2.3.8 A *Hermitian metric* on a representation (V,ρ) of Q is a family $h = (h_a)_{a\in Q_0}$ of Hermitian inner products $h_a : V_a \times V_a \to \mathbb{C}$.

Given a Hermitian metric h on (V,ρ), for every vertex a of Q_0, we have an endomorphism $K_\theta(V,\rho)_a$ of the $\mathbb{C}$-vector space V_a, which is defined by

$$K_\theta(V,\rho)_a = \theta_a \mathbf{1}_{V_a} + \sum_{\alpha\in \mathrm{t}^{-1}(a)} \rho_\alpha \circ \rho_\alpha^* - \sum_{\alpha\in \mathrm{s}^{-1}(a)} \rho_\alpha^* \circ \rho_\alpha,$$

where, for each $\alpha \in Q_1$, $\rho_\alpha^* : V_{\mathrm{t}(\alpha)} \to V_{\mathrm{s}(\alpha)}$ is the adjoint of $\rho_\alpha : V_{\mathrm{s}(\alpha)} \to V_{\mathrm{t}(\alpha)}$ with respect to the Hermitian inner products $h_{\mathrm{s}(\alpha)}$ and $h_{\mathrm{t}(\alpha)}$ on $V_{\mathrm{s}(\alpha)}$ and $V_{\mathrm{t}(\alpha)}$, respectively.

Definition 2.3.9 We say that the metric h is *Einstein-Hermitian* with respect to θ if there exists a constant $c \in \mathbb{C}$, such that

$$K_\theta(V,\rho)_a = c\mathbf{1}_{V_a}$$

for all $a \in Q_0$.

If this is the case, and if (V,ρ) is non-zero, then it is easy to see that $c = \mu_\theta(V,\rho)$, hence

$$\sum_{\alpha\in \mathrm{t}^{-1}(a)} \rho_\alpha \circ \rho_\alpha^* - \sum_{\alpha\in \mathrm{s}^{-1}(a)} \rho_\alpha^* \circ \rho_\alpha = (\mu_\theta(V,\rho) - \theta_a)\, \mathbf{1}_{V_a}$$

for all $a \in Q_0$. For this, let h be a Hermitian metric on a non-zero representation (V,ρ) of Q, and suppose that c is a complex number such that $K_\theta(V,\rho)_a = c\mathbf{1}_{V_a}$ for all $a \in Q_0$. Thus,

$$\theta_a \mathbf{1}_{V_a} + \sum_{\alpha\in \mathrm{t}^{-1}(a)} \rho_\alpha \circ \rho_\alpha^* - \sum_{\alpha\in \mathrm{s}^{-1}(a)} \rho_\alpha^* \circ \rho_\alpha = c\mathbf{1}_{V_a},$$

hence

$$\theta_a \dim_k(V_a) + \sum_{\alpha \in \mathrm{t}^{-1}(a)} \mathrm{Tr}(\rho_\alpha \circ \rho_\alpha^*) - \sum_{\alpha \in \mathrm{s}^{-1}(a)} \mathrm{Tr}(\rho_\alpha^* \circ \rho_\alpha) = c \dim_k(V_a)$$

for all $a \in Q_0$. Since the families $(\mathrm{s}^{-1}(a))_{a \in Q_0}$ and $(\mathrm{t}^{-1}(a))_{a \in Q_0}$ are partitions of the set Q_1, adding over $a \in Q_0$ on both sides of the above equation gives

$$\deg_\theta(V,\rho) + \sum_{\alpha \in Q_1} \mathrm{Tr}(\rho_\alpha \circ \rho_\alpha^*) - \sum_{\alpha \in Q_1} \mathrm{Tr}(\rho_\alpha^* \circ \rho_\alpha) = c \dim(V,\rho).$$

As $\mathrm{Tr}(\rho_\alpha \circ \rho_\alpha^*) = \mathrm{Tr}(\rho_\alpha^* \circ \rho_\alpha)$ for all $\alpha \in Q_1$, and (V,ρ) is non-zero, it follows that $c = \mu_\theta(V,\rho)$.

If we are considering more than one Hermitian metric on (V,ρ), and want to indicate the dependence of $K_\theta(V,\rho)$ on the metric, we will write $K_\theta(V,\rho,h)$ instead of $K_\theta(V,\rho)$.

The following Proposition is a consequence of [20, Proposition 6.5]. The restriction to rational weights here is due to the fact that the cited result is proved in that reference only for integral weights.

Proposition 2.3.10 *Let θ a rational weight of Q, and (V,ρ) a non-zero representation of Q. Then, (V,ρ) has an Einstein-Hermitian metric with respect to θ if and only if it is θ-polystable. Moreover, if h_1 and h_2 are two Einstein-Hermitian metrics on (V,ρ) with respect to θ, then there exists an automorphism f of (V,ρ), such that*

$$h_{1,a}(v,w) = h_{2,a}(f_a(v), f_a(w))$$

for all $a \in Q_0$ and $v,w \in V_a$.

Proof. Let $\mu = \mu_\theta(V,\rho)$. Since θ is a rational weight, μ is a rational number. Therefore, there exists an integer $n > 0$, such that $n\theta \in \mathbb{Z}^{Q_0}$ and $n\mu \in \mathbb{Z}$. Define $\omega \in \mathbb{Z}^{Q_0}$ by putting $\omega_a = n(\mu - \theta_a)$ for all $a \in Q_0$. Then, $\deg_\omega = n(\mu \mathrm{rk} - \deg_\theta)$. Therefore, by putting $\deg_\omega$ in the place of λ in Proposition 2.2.7, and in the notation used there, we have

$$O^{\mathrm{ss}}(\theta,\mu) = K^{\mathrm{ss}}(\deg_\omega), \quad O^{\mathrm{s}}(\theta,\mu) = K^{\mathrm{s}}(\deg_\omega).$$

Suppose (V,ρ) is θ-polystable. Then, (V,ρ) is θ-semistable, and there exist a finite family $(V_i,\rho_i)_{i\in I}$ of θ-stable representations of Q, such that (V,ρ) us isomorphic to $\bigoplus_{i\in I}(V_i,\rho_i)$. Let $(W,\sigma) = (V,\sqrt{n}\rho)$, and $(W_i,\sigma_i) = (V_i,\sqrt{n}\rho_i)$. Then, as $n \neq 0$, by Remark 2.3.7, (W,σ) is θ-semistable, and (W_i,σ_i) is θ-stable for every $i \in I$. Clearly, $\mu_\theta(W,\sigma) = \mu_\theta(V,\rho) = \mu$, and, by Proposition 2.1.4(3), $\mu_\theta(W_i,\sigma_i) = \mu_\theta(V_i,\rho_i) = \mu$ for all $i \in I$. Therefore, (W,σ) belongs to $K^{\mathrm{ss}}(\deg_\omega)$, and (W_i,σ_i) belongs to $K^{\mathrm{s}}(\deg_\omega)$ for every $i \in I$. Moreover, by Remark 2.3.4, (W,σ) is isomorphic to $\bigoplus_{i\in I}(W_i,\sigma_i)$. Therefore, by [20, Proposition 6.5], (W,σ) has an Einstein-Hermitian metric h with respect to ω. Thus, for every $a \in Q_0$, we have

$$\sum_{\alpha\in h^{-1}(a)} \sigma_\alpha \circ \sigma_\alpha^* - \sum_{\alpha\in t^{-1}(a)} \sigma_\alpha^* \circ \sigma_\alpha = \omega_a \, \mathbf{1}_{V_a},$$

where the adjoints are taken with respect to h, hence, as $\sigma_\alpha = \sqrt{n}\rho_\alpha$ and $\sigma_\alpha^* = \sqrt{n}\rho_\alpha^*$, we get

$$\sum_{\alpha\in h^{-1}(a)} \rho_\alpha \circ \rho_\alpha^* - \sum_{\alpha\in t^{-1}(a)} \rho_\alpha^* \circ \rho_\alpha = (\mu_\theta(V,\rho) - \theta_a)\, \mathbf{1}_{V_a}.$$

It follows that h is an Einstein-Hermitian metric on (V,ρ) with respect to θ. Moreover, if h_1 and h_2 are two Einstein-Hermitian metrics on (V,ρ), then, by [20, Proposition 6.5], there exists an automorphism f of (W,σ) such that

$$h_{1,a}(v,w) = h_{2,a}(f(v),f(w))$$

for all $a \in Q_0$ and $v, w \in W_a$. By Remark 2.3.4, f is an automorphism of (V,ρ) also.

Conversely, suppose that (V,ρ) has an Einstein-Hermitian metric h with respect to θ. Then, by the above relations, h is an Einstein-Hermitian metric on $(W,\sigma) = (V, \sqrt{n}\rho)$ with respect to ω. Therefore, by [20, Proposition 6.5], there exist a finite family $(W_i, \sigma_i)_{i \in I}$ of elements of $K^{\mathrm{s}}(\deg_\omega)$, such that (W,σ) is isomorphic to $\bigoplus_{i \in I}(W_i, \sigma_i)$. Thus, (W_i, σ_i) belongs to $O^{\mathrm{s}}(\theta, \mu)$ for every $i \in I$. It follows from Proposition 2.1.4(3) that (W,σ) is θ-semistable. Let $(V_i, \rho_i) = (W_i, (\sqrt{n})^{-1}\sigma_i)$ for every $i \in I$. Then, by Remarks 2.3.4–2.3.7, (V,ρ) is θ-semistable, (V_i, ρ_i) is θ-stable for every $i \in I$, and (V,ρ) is isomorphic to $\bigoplus_{i \in I}(V_i, \rho_i)$. Thus, (V,ρ) is θ-polystable. □

Definition 2.3.11 Given a Hermitian metric h on a representation (V,ρ) of Q, we say that two subrepresentations (V_1, ρ_1) and (V_2, ρ_2) of (V,ρ) are *orthogonal* with respect to h if for every $a \in Q_0$, the subspaces $V_{1,a}$ and $V_{2,a}$ of V_a are orthogonal with respect to the Hermitian inner product h_a on V_a.

Corollary 2.3.12 *Let θ be a rational weight of Q, (V,ρ) a non-zero representation of Q, and h an Einstein-Hermitian metric on (V,ρ) with respect to θ. Then, there exists a finite family $(V_i, \rho_i)_{i \in I}$ of θ-stable subrepresentations of Q, such that $(V,\rho) = \bigoplus_{i \in I}(V_i, \rho_i)$, and such that, for all $i, j \in I$ with $i \neq j$, (V_i, ρ_i) and (V_j, ρ_j) are orthogonal with respect to h.*

Proof. By Proposition 2.3.10, (V,ρ) is θ-polystable, hence there exists a finite family $(W_i, \sigma_i)_{i \in I}$ of θ-stable subrepresentations of Q, such that $(V,\rho) = \bigoplus_{i \in I}(W_i, \sigma_i)$. Again, by Proposition 2.3.10, for each $i \in I$, there exists an Einstein-Hermitian metric h_i on (W_i, σ_i). Let h' denote the Hermitian metric $\oplus_{i \in I} h_i$ on (V,ρ). Thus,

$$h'_a\Big(\sum_{i \in I} v_i, \sum_{i \in I} w_i\Big) = \sum_{i \in I} h_i(v_i, w_i)$$

for all $a \in Q_0$ and $v_i, w_i \in W_{i,a}$ $(i \in I)$. Then, for any $\alpha \in Q_1$, the adjoint $\rho_\alpha^{*'}$ of ρ_α with respect to h' equals $\oplus_{i \in I} \sigma_{i,\alpha}^{*_i}$, where $\sigma_{i,\alpha}^{*_i}$ is the adjoint of $\sigma_{i,\alpha}$ with respect to h_i $(i \in I)$. By Proposition 2.1.4(3), $\mu_\theta(W_i, \sigma_i) = \mu_\theta(V, \rho)$ for all $i \in I$, hence this implies that h' is an Einstein-Hermitian metric on (V, ρ). Therefore, by Proposition 2.3.10, there exists an automorphism f of (V, ρ), such that $h'_a(v, w) = h_a(f_a(v), f_a(w))$ for all $a \in Q_0$ and $v, w \in V_a$. For each $i \in I$, let $(V_i, \rho_i) = f(W_i, \sigma_i)$. Then, f induces an isomorphism from (W_i, σ_i) to (V_i, ρ_i), hence (V_i, ρ_i) is θ-stable. It is obvious that $(V, \rho) = \bigoplus_{i \in I}(V_i, \rho_i)$. For this, Since f is an automorphism of (V, ρ), we have

$$\sum_{i \in I}(V_i, \rho_i) = \sum_{i \in I} f(W_i, \sigma_i) = f\Big(\sum_{i \in I}(W_i, \sigma_i)\Big) = f(V, \rho) = (V, \rho).$$

Also, for all $i \in I$, we have

$$\begin{aligned}(V_i, \rho_i) \cap \sum_{j \in I \setminus \{i\}} (V_j, \rho_j) &= f(W_i, \sigma_i) \cap \sum_{j \in I \setminus \{i\}} f(W_j, \sigma_j) \\ &= f((W_i, \sigma_i) \cap \sum_{j \in I \setminus \{i\}} (W_j, \sigma_j)) = f(0) = 0.\end{aligned}$$

Therefore, $(V, \rho) = \bigoplus_{i \in I}(V_i, \rho_i)$.

If $i, j \in I$ and $i \neq j$, then, for all $a \in Q_0$, $v_i \in V_{i,a}$, and $v_j \in V_{j,a}$, there exist $w_i \in W_{i,a}$ and $w_j \in W_{j,a}$, such that $f_a(w_i) = v_i$ and $f_a(w_j) = v_j$, hence

$$h_a(v_i, v_j) = h_a(f_a(w_i), f_a(w_j)) = h'_a(w_i, w_j) = 0,$$

since (W_i, σ_i) and (W_j, σ_j) are orthogonal with respect to h'. Thus, (V_i, ρ_i) and (V_j, ρ_j) are orthogonal with respect to h. □

Definition 2.3.13 Let h be a Hermitian metric on (V, ρ). We say that an endomorphism f of (V, ρ) is *skew-Hermitian* with respect to h if for every $a \in Q_0$, the endomorphism

f_a of V_a is skew-Hermitian with respect to h_a, that is,

$$h_a(f_a(v),w)+h_a(v,f_a(w))=0$$

for all $v,w \in V_a$.

We denote the set of all skew-Hermitian endomorphisms of (V,ρ) with respect to h by $\mathrm{End}(V,\rho,h)$. It is an $\mathbb{R}$-subspace of the $\mathbb{C}$-vector space $\mathrm{End}(V,\rho)$.

Definition 2.3.14 We say that h is *irreducible* if for every endomorphism f of (V,ρ) that is skew-Hermitian with respect to h, there exists $\lambda \in \mathbb{C}$, such that $f=\lambda \mathbf{1}_{(V,\rho)}$. The complex number λ is then purely imaginary, hence h is irreducible if and only if

$$\mathrm{End}(V,\rho,h)=\sqrt{-1}\,\mathbb{R}\,\mathbf{1}_{(V,\rho)}.$$

Proposition 2.3.15 *Let (V,ρ) be a non-zero representation of Q, and h a Hermitian metric on (V,ρ). Then, the following are equivalent:*

1. *h is irreducible.*
2. *If $(V_i,\rho_i)_{i\in I}$ is a finite family of subrepresentations of (V,ρ),*

$$(V,\rho)=\bigoplus_{i\in I}(V_i,\rho_i),$$

and (V_i,ρ_i) and (V_j,ρ_j) are orthogonal with respect to h for all $i,j\in I$ with $i\neq j$, then there exists $i\in I$, such that $(V_i,\rho_i)=(V,\rho)$.

Proof. (1)⇒(2) : Suppose h is irreducible. Let (V_i,ρ_i) be a finite family of subrepresentations of (V,ρ), such that $(V,\rho)=\bigoplus_{i\in I}(V_i,\rho_i)$, and such that for all $i,j\in I$ with $i\neq j$, (V_i,ρ_i) and (V_j,ρ_j) are orthogonal with respect to h. We have to prove that there exists $i\in I$, such that $(V_i,\rho_i)=(V,\rho)$. Let J be the set of all $i\in I$ such that (V_i,ρ_i) is non-zero.

Then, $(V,\rho) = \bigoplus_{i\in J}(V_i,\rho_i)$. Let $\lambda : J \to \mathbb{C}$ be an injective function such that $\lambda(i)$ is purely imaginary for every $i \in J$, and let $f = \oplus_{i\in J} f_i$, where $f_i = \lambda(i)\mathbf{1}_{(V_i,\rho_i)}$ for all $i \in J$. Then, for all $i \in J$, $f_i \in \mathrm{End}(V_i,\rho_i,h_i)$, where h_i is the Hermitian metric on (V_i,ρ_i) induced by h. Moreover, as the (V_i,ρ_i) are pairwise orthogonal, $f^* = \oplus_{i\in J} f_i^{*_i}$, where f^* is the adjoint of f with respect to h, and $f_i^{*_i}$ is the adjoint of f_i with respect to h_i ($i \in J$). Therefore, $f \in \mathrm{End}(V,\rho,h)$. Since f is irreducible, there exists a complex number ζ, such that $f = \zeta\mathbf{1}_{(V,\rho)}$. This implies that $f_i = \zeta\mathbf{1}_{(V_i,\rho_i)}$; thus, $\lambda(i)\mathbf{1}_{(V_i,\rho_i)} = \zeta\mathbf{1}_{(V_i,\rho_i)}$; as (V_i,ρ_i) is non-zero, $\mathbf{1}_{(V_i,\rho_i)}$ is a non-zero element of the $\mathbb{C}$-vector space $\mathrm{End}(V_i,\rho_i)$, hence we get $\lambda(i) = \zeta$ for all $i \in J$. As λ is injective, it follows that $\mathrm{card}(J) \leq 1$. As (V,ρ) is non-zero, $J \neq \emptyset$, hence $\mathrm{card}(J) = 1$. Thus, there exists a unique element $j \in J$, and

$$(V,\rho) = (V_j,\rho_j) \oplus (\oplus_{i\in I\setminus J}(V_i,\rho_i)) = (V_j,\rho_j),$$

since (V_i,ρ_i) is zero for all $i \in I \setminus J$.

(2)⇒(1) : Suppose (V,ρ) satisfies (2). Let f be a skew-Hermitian endomorphism of (V,ρ). For each $\lambda \in \mathbb{C}$, let $(V_\lambda,\rho_\lambda) = \mathrm{Ker}(f - \lambda\mathbf{1}_{(V,\rho)})$. Since (V,ρ) is non-zero, there exists a vertex a of Q, such that V_a is non-zero. The characteristic polynomial p_a of the $\mathbb{C}$-endomorphism f_a of V_a has degree $\dim_{\mathbb{C}}(V_a) \geq 1$, and hence has a root ζ in $\mathbb{C}$. Thus, $\det(f_a - \zeta\mathbf{1}_{V_a}) = p_a(\zeta) = 0$, hence $\mathrm{Ker}(f_a - \zeta\mathbf{1}_{V_a}) \neq 0$. It follows that (V_ζ,ρ_ζ) is non-zero.

Now, for every vertex b of Q, the $\mathbb{C}$-endomorphism f_b of V_b is skew-Hermitian, and hence normal, with respect to the Hermitian inner product h_b on V_b. Therefore, V_b has a basis that is orthonormal with respect to h_b, each of whose elements is an eigenvector of f_b. This implies that

$$V_b = \oplus_{\lambda\in\mathbb{C}}\mathrm{Ker}(f_b - \lambda\mathbf{1}_{V_b}) = \mathrm{Ker}(f_b - \zeta\mathbf{1}_{V_b}) \oplus W_b = V_{\zeta,b} \oplus W_b,$$

where

$$W_b = \sum_{\lambda \in \mathbb{C} \setminus \{\zeta\}} \mathrm{Ker}(f_b - \lambda \mathbf{1}_{V_b}).$$

Thus,

$$(V, \rho) = (V_\zeta, \rho_\zeta) \oplus (W, \sigma),$$

where

$$(W, \sigma) = \sum_{\lambda \in \mathbb{C} \setminus \{\zeta\}} (V_\lambda, \rho_\lambda).$$

Moreover, for all $b \in Q_0$, $\lambda, \mu \in \mathbb{C}$ with $\lambda \neq \mu$, and $v \in V_{\lambda,b}$ and $w \in V_{\mu,b}$ with $v \neq 0$ and $w \neq 0$, we have

$$\lambda h_b(v,w) = h_b(f(v),w) = -h_b(v, f(w)) = -h_b(v, \mu w) = -\overline{\mu} h_b(v,w) = \mu h_b(v,w),$$

since μ is purely imaginary, hence

$$(\lambda - \mu) h_b(v,w) = 0.$$

As $\lambda \neq \mu$, this implies that $h_b(v,w) = 0$. Thus, the family $(V_{\lambda,b})_{\lambda \in \mathbb{C}}$ of subspaces of V_b is pairwise orthogonal with respect to h_b. Therefore, $V_{\zeta,b}$ and W_b are orthogonal with respect to h_b, for all $b \in Q_0$. In other words, (V_ζ, ρ_ζ) and (W, σ) are orthogonal with respect to h. The hypothesis now implies that one of them must be equal to (V, ρ). If $(W, \sigma) = (V, \rho)$, then (V_ζ, ρ_ζ) is zero, a contradiction. Therefore, $(V_\zeta, \rho_\zeta) = (V, \rho)$. It follows that $f = \zeta \mathbf{1}_{V,\rho}$. This proves that (V, ρ) is irreducible. □

Proposition 2.3.16 *Let θ be a rational weight of Q, (V, ρ) a non-zero representation of Q, and h an Einstein-Hermitian metric on (V, ρ) with respect to θ. Then, the following are equivalent:*

1. *h is irreducible.*

2. (V,ρ) *is* θ*-stable.*

3. (V,ρ) *is Schur.*

Proof. (1)⇒(2): Suppose h is irreducible. Since h is Einstein-Hermitian with respect to θ, by Corollary 2.3.12, there exists a finite family $(V_i,\rho_i)_{i\in I}$ of θ-stable subrepresentations of Q, such that $(V,\rho) = \bigoplus_{i\in I}(V_i,\rho_i)$, and such that, for all $i,j\in I$ with $i\neq j$, (V_i,ρ_i) and (V_j,ρ_j) are orthogonal with respect to h. As h is irreducible, by Proposition 2.3.15, there exists $i\in I$ such that $(V,\rho)=(V_i,\rho_i)$. Therefore, (V,ρ) is θ-stable.

(2)⇒(3): Follows from Proposition 2.1.7(3).

(3)⇒(1): Suppose f is a skew-Hermitian endomorphism of (V,ρ). Then, by Proposition 2.1.7(3)–(4), there exists $\lambda\in\mathbb{C}$ such that $f=\lambda\mathbf{1}_{(V,\rho)}$. Therefore, h is irreducible.

□

2.4 Families of representations

Subsection 2.4.4, deals with the definition of families of complex representations parametrised by a complex space, analytic subset of a complex space, and state Propositions 2.4.21, 2.4.23. While, in Subsection 2.4.5, we explain a criterion for two representations in a family to be separated from each other.

2.4.1 Preliminaries

First, we recall some basic definitions and facts which will needed in this section and further more.

Recall that a *ringed space* is a pair $(X,\mathscr{O}_X)$ where X is a topological space, and $\mathscr{O}_X$ is a sheaf of rings on X.

Let $(X,\mathscr{O}_X)$ and $(Y,\mathscr{O}_Y)$ be two ringed spaces. A *morphism of ringed spaces* from $(X,\mathscr{O}_X)$ to $(Y,\mathscr{O}_Y)$ is a pair $(f,\tilde{f})$, where

1. $f\colon X\to Y$ is a continuous map.
2. $\tilde{f}$ is an assignment which attaches to each open subset V of Y , a homomorphism of rings $\tilde{f}_V\colon \mathscr{O}_Y(V)\to\mathscr{O}_X(f^{-1}(V))$, such that for every pair (V,V') of open subsets of Y with $V\supset V'$, the diagram

$$\begin{array}{ccc} \mathscr{O}_Y(V) & \xrightarrow{\tilde{f}_V} & \mathscr{O}_X(f^{-1}(V)) \\ {\scriptstyle \rho^V_{V'}}\downarrow & & \downarrow{\scriptstyle \rho^{f^{-1}(V)}_{f^{-1}(V')}} \\ \mathscr{O}_Y(V') & \xrightarrow[\tilde{f}_{V'}]{} & \mathscr{O}_X(f^{-1}(V')) \end{array}$$

   commutes.

Let $(X,\mathscr{O}_X)$ and $(Y,\mathscr{O}_Y)$ be ringed spaces, and $(u,\tilde{u})$ a morphism from $(X,\mathscr{O}_X)$ to $(Y,\mathscr{O}_Y)$. Then for every $x\in X$, we have a canonical homomorphism of rings,

$$u_x\colon \mathscr{O}_{Y,u(x)}\to\mathscr{O}_{X,x}$$

defined as follows: Let $\theta\in\mathscr{O}_{Y,u(x)}$. Then, there exist an open neighborhood U of $u(x)$ in Y , and $s\in\mathscr{O}_Y(U)$ such that, $(s)_{u(x)}=\theta$. Since $s\in\mathscr{O}_Y(U)$, $\tilde{u}_U(s)$ is a section of $\mathscr{O}_X$ on the open neighborhood $u^{-1}(U)$ of x. Therefore, $(\tilde{u}_U(s))_x$ is an element of $\mathscr{O}_{X,x}$. Define $u_x(\theta)=(\tilde{u}_U(s))_x$. It follows from this definition that, u_x is independent of the choice of a pair (U,s).

A ringed space $(X,\mathscr{O}_X)$ is called a *locally ringed space* if for every point $x\in X$, the stalk $\mathscr{O}_{X,x}$ is a local ring. In that case, the maximal ideal in $\mathscr{O}_{X,x}$ is denoted by $\mathfrak{m}_{X,x}$ or $\mathfrak{m}_x$. The residue field $\mathscr{O}_{X,x}/\mathfrak{m}_{X,x}$ of $\mathscr{O}_{X,x}$ is denoted by $k(x)$.

Let $(X,\mathscr{O}_X)$ and $(Y,\mathscr{O}_Y)$ be locally ringed spaces. A *morphism of locally ringed spaces* from $(X,\mathscr{O}_X)$ to $(Y,\mathscr{O}_Y)$ is a morphism of ringed spaces $(u,\tilde{u})\colon (X,\mathscr{O}_X)\to$

$(Y, \mathscr{O}_Y)$such that, for all $x \in X$, the canonical homomorphism

$$u_x \colon \mathscr{O}_{Y,u(x)} \to \mathscr{O}_{X,x}$$

is a local homomorphism, that is, $u_x(\mathfrak{m}_{Y,u(x)}) \subset \mathfrak{m}_{X,x}$.

An $\mathscr{O}_X$-module $\mathscr{F}$ is said to be *finitely generated* if for every point $x \in X$, there exists an open neighborhood U of x such that $\mathscr{F}|_U$ is generated by a finite family of sections of $\mathscr{F}$ on U. That is, there exist a natural number n, and a surjective morphism $\varphi \colon (\mathscr{O}_X|_U)^n \to \mathscr{F}|_U$ of $\mathscr{O}_X|_U$-modules.

Let $(X, \mathscr{O}_X)$ be a ringed space. We say that an $(X, \mathscr{O}_X)$-module $\mathscr{F}$ is *coherent* if it satisfies the following conditions:

1. For every point $x \in X$, there exists an open neighborhood U of x such that $\mathscr{F}|_U$ is generated by a finite family of sections of $\mathscr{F}$ on U.

2. For every open subset U of X, for every integer $p \in \mathbf{N}$, and for every morphism of $(X, \mathscr{O}_X)$-modules
$$\varphi \colon (\mathscr{O}_X|_U)^p \to \mathscr{F}|_U,$$
the $\mathscr{O}_X|_U$-submodule $\mathrm{Ker}(\varphi)$ of $(\mathscr{O}_X|_U)^p$ satisfies condition (1) above.

Remark 2.4.1 1. If $\mathscr{F}$ and $\mathscr{G}$ are two coherent $\mathscr{O}_X$-modules, and if $\varphi \colon \mathscr{F} \to \mathscr{G}$ is a morphism of $\mathscr{O}_X$-modules, then *Image*(φ), $\mathrm{Ker}(\varphi)$, and $\mathrm{Coker}(\varphi)$ are coherent $\mathscr{O}_X$-modules.

2. If $\mathscr{F}$ and $\mathscr{G}$ are two coherent $\mathscr{O}_X$-modules, then so is $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$.

Let $(X, \mathscr{O}_x)$ be a ringed space. We say that an $\mathscr{O}_X$-module $\mathscr{F}$ is *locally free* if for every point $x \in X$, there exist an open neighborhood U of x, and a set I, such that $F|_U \cong (\mathscr{O}_X)^I|_U$ as an $\mathscr{O}_X|_U$-module.

A sheaf of $\mathbb{C}$-algebras $\mathscr{A}$ is called a *sheaf of local* $\mathbb{C}$-*algebras* if every stalk $\mathscr{A}_x$ is a local ring with (unique) maximal ideal $\mathfrak{m}(\mathscr{A}_x)$ so that the quotient epimorphism $\mathscr{A}_x \to \mathscr{A}_x/\mathfrak{m}(\mathscr{A}_x)$ always induces an isomorphism $\mathbb{C}\tilde{\to}\mathfrak{m}(\mathscr{A}_x)$. One identifies $\mathscr{A}_x/\mathfrak{m}(\mathscr{A}_x)$ with $\mathbb{C}$ and thus has a canonical direct sum $\mathscr{A}_x = \mathbb{C} \oplus \mathfrak{m}(\mathscr{A}_x)$ as $\mathbb{C}$-vector space.

A ringed space $(X, \mathscr{A})$ is called a $\mathbb{C}$-*ringed space* if $\mathscr{A}$ is a sheaf of local $\mathbb{C}$-algebras.

We next recall the notion of a complex model space. Let D be an open subset in $\mathbb{C}^n$ and let $\mathscr{J}$ be an ideal sheaf in $\mathscr{O}_D$, which is of "finite type" on D, i.e. to every point $z \in D$ there exists an open neighborhood $U \subset D$ of z and functions $f_i \in \mathscr{O}(U)$ $(1 \leq i \leq k)$ such that the sheaf $\mathscr{J}$ is generated over U by f_i's. i.e.

$$\mathscr{J}(V) = \mathscr{O}(V) f_1|_V + \cdots + \mathscr{O}(V) f_k|_V,$$

for every open subset V of U. The quotient sheaf $\mathscr{O}_D/\mathscr{J}$ is a sheaf of rings on D. We consider its support $Y := Supp(\mathscr{O}_D/\mathscr{J})$, that is the set of all points $z \in D$, where $\mathscr{J}_z \neq \mathscr{O}_z$. So locally Y is the zero set of finitely many holomorphic functions. The restriction

$$\mathscr{O}_Y := (\mathscr{O}_D/\mathscr{J})|_Y$$

of $\mathscr{O}_D/\mathscr{J}$ is a sheaf of rings on Y. The ringed space $(Y, \mathscr{O}_Y)$ is called a *complex model space* in D.

Definition 2.4.2 Let $(X, \mathscr{O}_X)$ be a $\mathbb{C}$-ringed space such that X is a Hausdorff space. We call $(X, \mathscr{O}_X)$ a *Complex Space* if every point of X has an open neighborhood U such that the open $\mathbb{C}$-ringed subspace $(U, \mathscr{O}_U)$ of $(X, \mathscr{O}_X)$ is isomorphic to a complex model space.

2.4.2 Families parametrised by arbitrary ringed spaces

For any ringed space T, we will denote by $\mathscr{O}_T$ the structure sheaf of T. If (f,u) is a morphism from a ringed space T to another ringed space T', we will, as usual, just write f instead of (f,u), and will denote by $\tilde{f}$ the morphism $u : \mathscr{O}_{T'} \to f_*(\mathscr{O}_T)$ of sheaves of rings on T'. By a locally free $\mathscr{O}_T$-module, we mean a locally finitely generated and locally free $\mathscr{O}_T$-module. If E is any $\mathscr{O}_T$-module, then for any $t \in T$, we denote by E_t the $\mathscr{O}_{T,t}$-module which is the stalk of E at t. If U is an open neighbourhood of t, and $s \in E(U)$, we denote the *germ* of s at t, that is, the canonical image of s in E_t, by s_t. If $f : E \to F$ is a morphism of $\mathscr{O}_T$-modules, then for each $t \in T$, we have a canonical $\mathscr{O}_{T,t}$-linear map $f_t : E_t \to F_t$.

Let Q be a non-empty finite quiver.

Definition 2.4.3 A *family of representations* of Q parametrised by a ringed space T is a pair (V,ρ), where $V = (V_a)_{a\in Q_0}$ is a family of locally free $\mathscr{O}_T$-modules, and $\rho = (\rho_\alpha)_{\alpha\in Q_1}$ is a family of morphisms $\rho_\alpha : V_{t(\alpha)} \to V_{h(\alpha)}$ of $\mathscr{O}_T$-modules.

Definition 2.4.4 If (V,ρ) and (W,σ) are two families of representations of Q parametrised by T, then a *morphism* from (V,ρ) to (W,σ) is a family $(f_a)_{a\in Q_0}$ of morphisms $f_a : V_a \to W_a$ of $\mathscr{O}_T$-modules, such that for every $\alpha \in Q_1$, the diagram

$$\begin{array}{ccc} V_{\mathrm{s}(\alpha)} & \xrightarrow{\rho_\alpha} & V_{\mathrm{t}(\alpha)} \\ {\scriptstyle f_{\mathrm{s}(\alpha)}}\downarrow & & \downarrow{\scriptstyle f_{\mathrm{t}(\alpha)}} \\ W_{\mathrm{s}(\alpha)} & \xrightarrow[\sigma_\alpha]{} & W_{\mathrm{t}(\alpha)} \end{array}$$

commutes.

If (V,ρ), (W,σ), and (X,τ) are three families of representations of Q parametrised by T, f a morphism from (V,ρ) to (W,σ), and g a morphism from (W,σ) to (X,τ). Then, the *composite* of f and g is the family $g\circ f$ defined by $g\circ f = (g_a \circ f_a)_{a\in Q_0}$. It is easy to verify that $g \circ f$ is a morphism from (V,ρ) to (X,τ). We thus get category $\mathbf{Rep}_T(Q)$, whose objects are representations of Q parametrised by T, and whose morphisms are the morphisms defined above.

For any two families of representations (V,ρ) and (W,σ) of Q parametrised by T, the set $\mathrm{Hom}((V,\rho),(W,\sigma))$ is an $\mathscr{O}_T(T)$-submodule of the $\mathscr{O}_T(T)$-module $\oplus_{a\in Q_0}\mathrm{Hom}(V_a,W_a)$.

For any family (V,ρ) of representations of Q parametrised by T, and for every open subspace U of T, we get a family $(V,\rho)|_U = (V|_U,\rho|_U)$ of representations of Q parametrised by U, where $V|_U = (V_a|_U)_{a\in Q_0}$, and $\rho|_U = (\rho_\alpha|_U)_{\alpha\in Q_1}$. We call $(V,\rho)|_U$ the *restriction* of (V,ρ) to U. If $f:(V,\rho)\to(W,\sigma)$ is a morphism of families of representations of Q parametrised by T, we have a morphism

$$f|_U : (V,\rho)|_U \to (W,\sigma)|_U$$

of families of representations of Q parametrised by U, which is defined by

$$f|_U = (f_a|_U)_{a\in Q_0}.$$

Suppose (V,ρ) and (W,σ) are two families of representations of Q parametrised by T. We then have a sheaf $\mathscr{H}om((V,\rho),(W,\sigma))$ on T, which is defined by

$$\mathscr{H}om((V,\rho),(W,\sigma))(U) = \mathrm{Hom}((V,\rho)|_U,(W,\sigma)|_U)$$

for every open subset U of T, and $\rho_{VU}(f) = f|_V$ for all open subsets U and V of T such that $U \supset V$, and for all f in $\mathscr{H}om((V,\rho),(W,\sigma))(U)$. The sheaf

$$\mathscr{H}om((V,\rho),(W,\sigma))$$

is an $\mathscr{O}_T$-submodule of the $\mathscr{O}_T$-module $\oplus_{a\in Q_0}\mathscr{H}om_{\mathscr{O}_T}(V_a,W_a)$.

Remark 2.4.5 Given two families of representations (V,ρ) and (W,σ) of Q parametrised by T, we define two $\mathscr{O}_T$-modules E and F by

$$E = \oplus_{a\in Q_0}\mathscr{H}om_{\mathscr{O}_T}(V_a,W_a), \quad F = \oplus_{\alpha\in Q_1}\mathscr{H}om_{\mathscr{O}_T}(V_{t(\alpha)},W_{h(\alpha)}),$$

and a morphism $u : E \to F$ of $\mathscr{O}_T$-modules by

$$u_U(f) = (f_{\mathrm{t}(\alpha)} \circ \rho_\alpha|_U - \sigma_\alpha|_U \circ f_{\mathrm{s}(\alpha)})_{\alpha\in Q_1},$$

for every open subset U of T, and for every $f = (f_a)_{a\in Q_0}$ in

$$E(U) = \oplus_{a\in Q_0}\mathrm{Hom}(V_a|_U,W_a|_U).$$

Note that the right hand side of the above equation is indeed an element of

$$F(U) = \oplus_{\alpha\in Q_1}\mathrm{Hom}(V_{\mathrm{s}(\alpha)}|_U,W_{\mathrm{t}(\alpha)}|_U).$$

The assumption that Q is finite, and the fact that V_a is locally free for every $a \in Q_0$, imply that E and F are locally free. In particular, E and F are coherent $\mathscr{O}_T$-modules. By the definition of u, we have $(\mathrm{Ker}(u))(U) = \mathrm{Hom}((V,\rho)|_U,(W,\sigma)|_U)$ for every open subset U of T, hence $\mathscr{H}om((V,\rho),(W,\sigma)) = \mathrm{Ker}(u)$. Since the kernel of any morphism between two coherent $\mathscr{O}_T$-modules is coherent, it follows that the $\mathscr{O}_T$-module $\mathscr{H}om((V,\rho),(W,\sigma))$ is coherent.

Remark 2.4.6 A reference for the coherence of the kernel of a morphism between two coherent $\mathscr{O}_T$-modules is [19, Chapter 0, Proposition 5.3.2].

Let T and T' be two ringed spaces, $f : T' \to T$ a morphism of ringed spaces, and (V,ρ) a family of representations of Q parametrised by T. For each $a \in Q_0$, define an

$\mathscr{O}_{T'}$-module M_a by $M_a = V_a{}^*(f)$. Then, for each $\alpha \in Q_1$, we have a morphism

$$\varphi_\alpha = \rho_\alpha{}^*(f) : M_{s(\alpha)} \to M_{t(\alpha)}$$

of $\mathscr{O}_{T'}$-modules. We thus, get a family (M, φ) of representations of Q parametrised by T'. We call it the *pullback* of (V, ρ) by f, and will denote it by $V, \rho^*(f)$. In particular, if U is an open ringed subspace of T, and if $f : U \to T$ is the canonical morphism of ringed spaces, then $f^*(V, \rho)$ is canonically isomorphic to the restriction $(V, \rho)|_U$ defined above. If A is any ringed subspace of T, and $f : A \to T$ the canonical morphism, we will call $f^*(V, \rho)$ the *restriction* of (V, ρ) to A, and will denote it by $(V, \rho)|_A$. Note that if $f' : T'' \to T'$ is another morphism of ringed spaces, then we have a canonical isomorphism $(f \circ f')^*(V, \rho) \cong f'^*(f^*(V, \rho))$ of families of representations parametrised by T.

2.4.3 Families parametrised by locally ringed spaces

Let T be a locally ringed space. For any point t of T, we will denote the maximal ideal in the local ring $\mathscr{O}_{T,t}$ by $\mathfrak{m}_t$, and the residue field $\mathscr{O}_{T,t}/\mathfrak{m}_t$ by $k(t)$. For any $\mathscr{O}_T$-module E and $t \in T$, we will denote the fibre of E at t, that is, the $k(t)$-vector space $E_t/\mathfrak{m}_t E_t$ by $E(t)$. It is canonically identified with the $k(t)$-vector space $k(t) \otimes_{\mathscr{O}_{T,t}} E_t$. For any element γ of E_t, we denote the *value* of γ at t, that is, the canonical image of γ in $E(t)$, by $\gamma(t)$. If U is an open neighbourhood of t, and $s \in E(U)$, we denote the *value* of s at t, that is, the element $s_t(t)$ of $E(t)$, by $s(t)$. If $f : E \to F$ is a morphism of $\mathscr{O}_T$-modules, then for each $t \in T$, we have a canonical $k(t)$-linear map $f(t) : E(t) \to F(t)$.

Remark 2.4.7 The assignment $\gamma + \mathfrak{m}_t E_t \mapsto 1 \otimes \gamma$ is a well-defined isomorphism from $E(t)$ onto the $k(t)$-vector space $k(t) \otimes_{\mathscr{O}_{T,t}} E_t$, where we consider $k(t)$ an $\mathscr{O}_{T,t}$-algebra through the canonical projection $\mathscr{O}_{T,t} \to k(t)$. The inverse of this isomorphism is the unique $\mathscr{O}_{T,t}$-linear map which takes $\lambda \otimes \gamma$ to $\lambda \cdot \gamma(t)$. If U is an open neighbourhood of t, and $s \in E(U)$, then under this isomorphism, $s(t)$ corresponds to $1 \otimes s_t$.

Definition 2.4.8 We say that a subset A of T is *analytic* if for every point $t_0 \in T$, there exist an open neighbourhood U of t_0 in T, and a finite subset E of $\mathscr{O}_T(U)$, such that

$$U \cap A = \{t \in U \mid f(t) = 0 \text{ for all } f \in E\} = \{t \in U \mid f_t \in \mathfrak{m}_t \text{ for all } f \in E\}.$$

Lemma 2.4.9 *Let T be a locally ringed space. Then:*

1. *Every analytic subset of T is closed in T.*
2. *A subset A of T is analytic if and only if every point t_0 of T has an open neighbourhood U in T, such that $U \cap A$ is an analytic subset of the locally ringed subspace U of T. In other words, the analyticity of a subset of T is a local property.*
3. *Every open and closed subset of T is analytic.*
4. *The union and the intersection of a finite family of analytic subsets of T are analytic.*

Proof. (1) For any open subset V of T, and for every element h of $\mathscr{O}_T(V)$, let

$$V_h = \{t \in V \mid h(t) \neq 0\} = \{t \in V \mid h_t \notin \mathfrak{m}_t\}.$$

Then, V_h is open in T, and $h|_{V_h}$ is a unit in the ring $\mathscr{O}_T(V_h)$ [19, Chapter 0, Proposition 5.5.1].

Now, suppose A is an analytic subset of T. Then, there exist an open cover $(U_i)_{i\in I}$ of T, and, for each $i \in I$, a finite subset E_i of $\mathscr{O}_T(U_i)$, such that

$$U_i \cap A = \{t \in U \mid f(t) = 0 \text{ for all } f \in E_i\} = \bigcap_{f \in E_i} (U_i \setminus (U_i)_f),$$

that is,

$$U_i \setminus A = \bigcup_{f \in E} (U_i)_f.$$

Thus, $U_i \setminus A$ is open in T for each $i \in I$. Since

$$T \setminus A = \bigcup_{i \in I} (U_i \setminus A),$$

it follows that $T \setminus A$ is open in T, hence A is closed in T.

(2) Immediate from the definition of analyticity.

(3) Let A be an open and closed subset of T. If t_0 is a point in A, then $U = A$ is an open neighbourhood of t_0 in T, and the subset $E = \{0\}$ of the ring $\mathscr{O}_T(U)$ has the property that

$$U \cap A = U = \{t \in U \mid f(t) = 0 \text{ for all } f \in E\}.$$

On the other hand, if $t_0 \in T \setminus A$, then $U = T \setminus A$ is an open neighbourhood of t_0 in T, and the subset $E = \{1\}$ of the ring $\mathscr{O}_T(U)$ has the property that

$$U \cap A = \emptyset = \{t \in U \mid f(t) = 0 \text{ for all } f \in E\},$$

since $\mathfrak{m}_t$ is a proper ideal in $\mathscr{O}_{T,t}$ for all $t \in T$. It follows that A is analytic.

(4) Let $(A_i)_{i\in I}$ be a finite family of analytic subsets of T, $A = \bigcup_{i\in I} A_i$, and $B = \bigcap_{i\in I} A_i$. Let $t_0 \in T$. Then, for each $i \in I$, there exist an open neighbourhood U_i of t_0 in

T, and a finite subset E_i of $\mathscr{O}_T(U_i)$, such that

$$U_i \cap A_i = \{t \in U_i \,|\, f(t) = 0 \text{ for all } f \in E_i\}.$$

Let $U = \bigcap_{i \in I} U_i$,

$$E = \{\prod_{i \in I} (f_i|_U) \,|\, f_i \in E_i \text{ for every } i \in I\},$$

and

$$F = \{f \in \mathscr{O}_T(U) \,|\, \text{ there exist } i \in I \text{ and } g \in E_i \text{ such that} f = g|_U\}.$$

Since I is finite, U is an open neighbourhood of t_0 in T, and E and F are finite subsets of $\mathscr{O}_T(U)$. Therefore, it suffices to show that

$$U \cap A = \{t \in U \,|\, f(t) = 0 \text{ for all } f \in E\},$$

and

$$U \cap B = \{t \in U \,|\, f(t) = 0 \text{ for all } f \in F\}.$$

If $t \in U \cap A$ and $f \in E$, then there exist $i \in I$ and $f_j \in E_j$ ($j \in I$), such that $t \in A_i$ and $f = \prod_{j \in I}(f_j|_U)$; as $t \in U \cap A_i \subset U_i \cap A_i$, we have $f_i(t) = 0$, hence $f(t) = \prod_{j \in I}(f_j(t)) = 0$. Therefore,

$$U \cap A \subset \{t \in U \,|\, f(t) = 0 \text{ for all } f \in E\}.$$

Conversely, suppose t belongs to the right hand side of the above inclusion. Suppose $t \notin A$. Then, for each $i \in I$, $t \notin A_i$; as $t \in U \subset U_i$, this implies that there exists $f_i \in E_i$, such that $f_i(t) \neq 0$. Let $f = \prod_{i \in I}(f_i|_U)$. Then, $f(t) = \prod_{i \in I}(f_i(t))$. As $\mathfrak{m}_t$ is a prime ideal in $\mathscr{O}_{T,t}$, and $f_i(t) \neq 0$ for all $i \in I$, we get $f(t) \neq 0$, a contradiction since t belongs to the right hand side of the above inclusion. Therefore $t \in A$, and

$$U \cap A = \{t \in U \mid f(t) = 0 \text{ for all } f \in E\}.$$

On the other hand, if $t \in U \cap B$ and $f \in F$, then there exist $i \in I$ and $g \in E_i$, such that $g|_U = f$; as $t \in U \cap B \subset U_i \cap A_i$, we get $f(t) = g(t) = 0$. Therefore,

$$U \cap B \subset \{t \in U \mid f(t) = 0 \text{ for all } f \in F\}.$$

Conversely, if t belongs to the right hand side of the above inclusion, then, for each $i \in I$ and $g \in E_i$, we have $t \in U \subset U_i$ and $f = g|_U \in F$, hence $g(t) = f(t) = 0$, so $t \in U_i \cap A_i \subset B$. Therefore $t \in B$, and

$$U \cap B = \{t \in U \mid f(t) = 0 \text{ for all } f \in F\}.$$

□

Remark 2.4.10 Let A be a local ring, M an A-module, $n \in \mathbb{N}$, and $(e_i)_{i=1}^n$ a family of elements of M. Let $\mathfrak{m}(A)$ denote the maximal ideal of A, $\mathbf{k}(A)$ the residue field $A/\mathfrak{m}(A)$ of A, and V the $\mathbf{k}(A)$-vector space $M/\mathfrak{m}(A)M$. Let $a \mapsto \overline{a} : A \to \mathbf{k}(A)$ and $x \mapsto \overline{x} : M \to V$ be the canonical projections, and $(\overline{e_i})_{i=1}^n$. Then, $(e_i)_{i=1}^n$ is an A-basis of M if and only if M is free, its rank equals n, and $(\overline{e_i})_{i=1}^n$ is a $\mathbf{k}(A)$-basis of V.

The map $\overline{x} \mapsto 1 \otimes x$ is an isomorphism from V onto the $\mathbf{k}(A)$-vector space $\mathbf{k}(A) \otimes_A M$, where we consider $\mathbf{k}(A)$ an A-module through the canonical projection $A \to \mathbf{k}(A)$. The inverse of this isomorphism is the unique A-linear map which takes $\lambda \otimes x$ to $\overline{\lambda x}$.

Suppose $(e_i)_{i=1}^n$ is a basis of M. Then, $(1 \otimes e_i)_{i=1}^n$ is a $\mathbf{k}(A)$-basis of $\mathbf{k}(A) \otimes_A M$. Therefore, by the above observation, $(\overline{e_i})_{i=1}^n$ is a $\mathbf{k}(A)$-basis of V. Conversely, suppose M is free, its rank equals n, and $(\overline{e_i})_{i=1}^n$ is a $\mathbf{k}(A)$-basis of V. Then, by Nakayama's lemma, $(e_i)_{i=1}^n$ generates the A-module M [12, Corollary 4.8]. As M is free, and its rank equals n, $(e_i)_{i=1}^n$ is a basis of M [12, Corollary 4.4(b)].

Lemma 2.4.11 *Let T be a locally ringed space, E an $\mathscr{O}_T$-module, $n \in \mathbb{N}$, and $(e_i)_{i=1}^n$ a family of elements of $E(T)$. Then, the following are equivalent:*

1. *The morphism $\varphi : \mathscr{O}_T^n \to E$ of $\mathscr{O}_T$-modules associated with the family $(e_i)_{i=1}^n$ is an isomorphism. (Recall that φ is defined by $\varphi_U(a) = \sum_{i=1}^n a_i \cdot e_i|_U$ for every open subset U of T, and for every $a = (a_1, \dots, a_n) \in \mathscr{O}_T^n(U)$.)*
2. *For every $t \in T$, $(e_{i,t})_{i=1}^n$ is an $\mathscr{O}_{T,t}$-basis of E_t.*
3. *For every $t \in T$, the $\mathscr{O}_{T,t}$-module E_t is free, n equals its rank, and $(e_i(t))_{i=1}^n$ is a $k(t)$-basis of $E(t)$.*

Proof. Let $(\xi_i)_{i=1}^n$ be the canonical $\mathscr{O}_T(T)$-basis of $\mathscr{O}_T^n(T)$. Then, for every $t \in T$, $(\xi_{i,t})_{i=1}^n$ is an $\mathscr{O}_{T,t}$-basis of $\mathscr{O}_{T,t}^n$, and the homomorphism $\varphi_t : \mathscr{O}_{T,t}^n \to E_t$ maps $\xi_{i,t}$ to $e_{i,t}$ $(1 \leq i \leq n)$. Since φ is an isomorphism of $\mathscr{O}_T$-modules if and only if φ_t is an isomorphism of $\mathscr{O}_{T,t}$-modules, we get the equivalence of (1) and (2). The equivalence of (2) and (3) is immediate from Remark 2.4.10. □

Remark 2.4.12 Let K be a field, E and F two finite-dimensional K-vector spaces, and $u : E \to F$ a K-linear map. Let $(e_j)_{j=1}^n$ be a basis of E, $(f_i)_{i=1}^m$ a basis of E, and $a = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ the matrix of u with respect to these bases. Then, the rank of u is the maximum of the set of integers r, such that $0 \leq r \leq \min(m,n)$, and a has a non-zero minor of order r [3, Chapter III, § 8, no. 7, Corollary to Proposition 15].

Lemma 2.4.13 *Let T be a locally ringed space, E and F two locally free $\mathscr{O}_T$-modules, and $u : E \to F$ a morphism of $\mathscr{O}_T$-modules. Then, for each $d \in \mathbb{N}$, the subset*

$$\{t \in T \mid \dim_{k(t)}(\mathrm{Ker}(u(t))) \geq d\}$$

is analytic. In particular, the function

$$t \mapsto \dim_{k(t)}(\mathrm{Ker}(u(t))) : T \to \mathbb{N}$$

is upper semi-continuous.

Proof. Let α denote the given function, $d \in \mathbb{N}$, and

$$A = \{t \in T \,|\, f(t) \geq d\}.$$

We have to prove that A is an analytic subset of T. By Lemma 2.4.9(2), this is a local property on T. Therefore, we can assume that there exist $n, m \in \mathbb{N}$, and isomorphisms $\varphi : \mathscr{O}_T^n \to E$ and $\psi : \mathscr{O}_T^m \to F$ of $\mathscr{O}_T$-modules. Define a function $\beta : T \to \mathbb{N}$ by

$$\beta(t) = \mathrm{rk}_{k(t)}(u(t)) = \dim_{k(t)}(\mathrm{Im}(u(t))).$$

Then, for all $t \in T$, we have $\dim_{k(t)}(E(t)) = n$, hence

$$\mathrm{rk}_{k(t)}(u(t)) = \dim_{k(t)}(\mathrm{Im}(u(t))) = \dim_{k(t)}(E(t)/\mathrm{Ker}(u(t))) = n - \alpha(t).$$

Therefore,

$$A = \{t \in T \,|\, \mathrm{rk}_{k(t)}(u(t)) \leq n - d\}. \tag{2.1}$$

Let $(\xi_j)_{j=1}^n$ be the canonical $\mathscr{O}_T(T)$-basis of $\mathscr{O}_T^n$, and e_j the element $\varphi_T(\xi_j)$ of $E(T)$ ($1 \leq j \leq n$). Similarly, let $(\eta_i)_{i=1}^m$ be the canonical $\mathscr{O}_T(T)$-basis of $\mathscr{O}_T^n$, and f_i the element $\psi_T(\eta_i)$ of $F(T)$ ($1 \leq i \leq m$). Then, since φ_T is an isomorphism of $\mathscr{O}_T(T)$-modules, $(e_j)_{j=1}^n$ is an $\mathscr{O}_T(T)$-basis of $E(T)$; similarly, $(f_i)_{i=1}^m$ is an $\mathscr{O}_T(T)$-basis of $F(T)$. Moreover, by Lemma 2.4.11, for every $t \in T$, $(e_{j,t})_{j=1}^n$ is an $\mathscr{O}_{T,t}$-basis of E_t, and $(e_j(t))_{j=1}^n$ is a $k(t)$-basis of $E(t)$; similarly, $(f_{i,t})_{i=1}^m$ is an $\mathscr{O}_{T,t}$-basis of F_t, and

$(f_i(t))_{i=1}^m$ is a $k(t)$-basis of $F(t)$. Let $a = (a_{ij})_{1\leq i\leq m, 1\leq j\leq n}$ be the matrix of the $\mathscr{O}_T(T)$-linear map $u_T : E(T) \to F(T)$, with respect to the bases $(e_j)_{j=1}^n$ and $(f_i)_{i=1}^m$. Thus, $a_{ij} \in \mathscr{O}_T(T)$ for all $i = 1,\ldots,m$ and $j = 1,\ldots,n$, and

$$u_T(e_j) = \sum_{i=1}^{n} a_{ij} f_i$$

for all $j = 1,\ldots,n$. Then, for every $t \in T$, we have

$$u_t(e_{j,t}) = (u_T(e_j))_t = \sum_{i=1}^{n} a_{ij,t} f_{i,t}$$

for all $j = 1,\ldots,n$, hence $a_t = (a_{ij,t})_{1\leq i\leq m, 1\leq j\leq n}$ is the matrix of the $\mathscr{O}_{T,t}$-linear map $u_t : E_t \to F_t$, with respect to the bases $(e_{j,t})_{j=1}^n$ and $(f_{i,t})_{i=1}^m$; similarly,

$$(u(t))(e_j(t)) = (u_t(e_{j,t}))(t) = \sum_{i=1}^{n} a_{ij}(t) f_i(t)$$

for all $j = 1,\ldots,n$, hence $a(t) = (a_{ij}(t))_{1\leq i\leq m, 1\leq j\leq n}$ is the matrix of the $k(t)$-linear map $u(t) : E(t) \to F(t)$, with respect to the bases $(e_j(t))_{j=1}^n$ and $(f_i(t))_{i=1}^m$.

Let Λ be the set of all pairs (I,J) of finite sequences of integers $I = (i_1,\ldots,i_l)$ and $J = (j_1,\ldots,j_l)$, such that $n-d+1 \leq l \leq \min(m,n)$, $1 \leq i_1 < \cdots i_l \leq m$, and $1 \leq j_1 < \cdots j_l \leq n$. For each such pair (I,J), let $b_{IJ} = (b_{pq})_{1\leq p,q\leq l}$, and $b_{pq} = a_{i_p,j_q}$ $(1 \leq p,q \leq l)$. Let h_{IJ} denote the element $\det(b_{IJ})$ of $\mathscr{O}_T(T)$. Then, by (2.1) and Remark 2.4.12,

$$A = \{t \in T \mid h_{IJ}(t) = 0 \text{ for all } (I,J) \in \Lambda\}.$$

Therefore, A is an analytic subset of T. In particular, by Lemma 2.4.9(1), A is closed in T. The upper semi-continuity of the given function follows. □

Proposition 2.4.14 *Let T be a locally ringed space, (V,ρ) and (W,σ) two families of representations of Q parametrised by T, and $u : E \to F$ the morphism of $\mathscr{O}_T$-modules* *defined in Remark 2.4.5. Then, for every $t \in T$, there is a canonical isomorphism*

$$\mathrm{Hom}((V(t),\rho(t)),(W(t),\sigma(t))) \cong \mathrm{Ker}(u(t))$$

of $k(t)$-vector spaces. In particular, for every $n \in \mathbb{N}$, the subset

$$\{t \in T \,|\, \dim_{k(t)}(\mathrm{Hom}((V(t),\rho(t)),(W(t),\sigma(t)))) \geq n\}$$

of T is analytic, and the function

$$t \mapsto \dim_{k(t)}(\mathrm{Hom}((V(t),\rho(t)),(W(t),\sigma(t)))) : T \to \mathbb{N}$$

is upper semi-continuous.

Proof. Let $t \in T$. We first note that the canonical $\mathscr{O}_{T,t}$-homomorphism

$$E_t \to \oplus_{a \in Q_0}(\mathscr{H}om_{\mathscr{O}_T}(V_a,W_a)_t)$$

is an isomorphism. Also, for every $a \in Q_0$, the canonical $\mathscr{O}_{T,t}$-homomorphism

$$\mathscr{H}om_{\mathscr{O}_T}(V_a,W_a)_t \to \mathrm{Hom}_{\mathscr{O}_{T,t}}(V_{a,t},W_{a,t})$$

is an isomorphism, since the $\mathscr{O}_T$-module V_a is locally finitely presented [16, Proposition 4.1.1]. Therefore, the $\mathscr{O}_{T,t}$-homomorphism $u_t : E_t \to F_t$ can be identified with the map

$$f = (f_a)_{a \in Q_0} \mapsto (f_{h(\alpha)} \circ \rho_{\alpha,t} - \sigma_{\alpha,t} \circ f_{t(\alpha)})_{\alpha \in Q_1} :$$
$$\oplus_{a \in Q_0} \mathrm{Hom}_{\mathscr{O}_{T,t}}(V_{a,t},W_{a,t}) \to \oplus_{\alpha \in Q_1} \mathrm{Hom}_{\mathscr{O}_{T,t}}(V_{t(\alpha),t},W_{h(\alpha),t}).$$

We also note that the canonical $k(t)$-homomorphism

$$k(t) \otimes_{\mathscr{O}_{T,t}} (\oplus_{a \in Q_0} \mathrm{Hom}_{\mathscr{O}_{T,t}}(V_{a,t}, W_{a,t})) \to \oplus_{a \in Q_0} (k(t) \otimes_{\mathscr{O}_{T,t}} \mathrm{Hom}_{\mathscr{O}_{T,t}}(V_{a,t}, W_{a,t}))$$

is an isomorphism, since the tensor product commutes with direct sums. Also, the canonical $k(t)$-homomorphism

$$k(t) \otimes_{\mathscr{O}_{T,t}} \mathrm{Hom}_{\mathscr{O}_{T,t}}(V_{a,t}, W_{a,t}) \to \mathrm{Hom}_{\mathscr{O}_{T,t}}(k(t) \otimes_{\mathscr{O}_{T,t}} V_{a,t},\ k(t) \otimes_{\mathscr{O}_{T,t}} W_{a,t})$$

is an isomorphism, since the $k(t)$-module $V_{a,t}$ is finitely generated and projective [3, Chapter II, § 5, no. 4, Proposition 7]. Through these isomorphisms, the $k(t)$-homomorphism $u(t) = \mathbf{1}_{k(t)} \otimes u_t$ is identified with the map

$$f = (f_a)_{a \in Q_0} \mapsto (f_{h(\alpha)} \circ \rho_\alpha(t) - \sigma_\alpha(t) \circ f_{t(\alpha)})_{\alpha \in Q_1} :$$
$$\oplus_{a \in Q_0} \mathrm{Hom}_{k(t)}(V_a(t), W_a(t)) \to \oplus_{\alpha \in Q_1} \mathrm{Hom}_{k(t)}(V_{t(\alpha)}(t), W_{h(\alpha)}(t)).$$

We thus get a canonical $k(t)$-isomorphism

$$\mathrm{Ker}(u(t)) \cong \mathrm{Hom}((V(t), \rho(t)), (W(t), \sigma(t))).$$

The rest of the proposition now follows from Lemma 2.4.13. □

Corollary 2.4.15 *Let T be a locally ringed space, and (V, ρ) a family of representations of Q parametrised by T. Then, the subset*

$$\{t \in T \mid \dim_{k(t)}(\mathrm{End}(V(t), \rho(t))) \neq 1\}.$$

of T is analytic.

Proof. Let

$$A = \{t \in T \,|\, \dim_{k(t)}(\mathrm{End}(V(t),\rho(t))) \geq 2\},$$

and

$$B = \{t \in T \,|\, \mathrm{End}(V(t),\rho(t)) = 0\}.$$

Then, the given set equals $A \cup B$. Now, by Proposition 2.4.14, A is an analytic subset of T. On the other hand, for any $t \in T$, we have $\mathrm{End}(V(t),\rho(t)) = 0$ if and only if $(V(t),\rho(t)) = 0$ if and only if $V_a(t) = 0$ for all $a \in Q_0$, hence $B = \bigcap_{a \in Q_0} B_a$, where

$$B_a = \{t \in T \,|\, V_a(t) = 0\},$$

for each $a \in Q_0$. As the $\mathscr{O}_T$-module V_a is locally free, it is follows from Lemma 2.4.11(3), its rank function

$$t \mapsto \dim_{k(t)}(V_a(t)) : T \to \mathbb{N}$$

is locally constant. Hence B_a is open and closed in T for every $a \in Q_0$. As any intersection of open and closed sets is open and closed, B is also open and closed in T. That B and $A \cup B$ are analytic follows from Lemma 2.4.9(3)–(4). □

For every open subset U of T, let

$$\begin{aligned}\mathscr{Z}_T(U) &= \{f \in \mathscr{O}_T(U) \,|\, f(t) = 0 \text{ for all } t \in U\} \\ &= \{f \in \mathscr{O}_T(U) \,|\, f(t) \in \mathfrak{m}_t \text{ for all } t \in U\}.\end{aligned}$$

Then, for every open subset U of T, for every open cover $(U_i)_{i \in I}$ of U, and for every $f \in \mathscr{O}_T(U)$, we have $f \in \mathscr{Z}_T(U)$ if and only if $f|_{U_i} \in \mathscr{Z}_T(U_i)$ for all $i \in I$. It follows that for all open subsets U and V of T such that $U \supset V$, the restriction map $\mathscr{O}_T(U) \to \mathscr{O}_T(V)$ induces a map $\mathscr{Z}_T(U) \to \mathscr{Z}_T(V)$. We thus get a subsheaf $\mathscr{Z}_T$ of $\mathscr{O}_T$. This subsheaf is obviously an ideal in $\mathscr{O}_T$. We will call it the *vanishing ideal* of T. If T is a scheme, then $\mathscr{Z}_T$ equals the nilradical $\mathscr{N}_T$ of $\mathscr{O}_T$. Similarly, if T is a complex space, then also, by the Rückert Nullstellensatz, we have $\mathscr{Z}_T = \mathscr{N}_T$.

Remark 2.4.16 A reference for the above application of the Rückert Nullstellensatz is [14, Chapter 3, § 2, no. 2, Corollary]..

Lemma 2.4.17 *Let T be a locally ringed space, and E a locally finitely generated $\mathscr{O}_T$-module. Suppose that the vanishing ideal $\mathscr{Z}_T$ of T is zero, and that its rank function*

$$t \mapsto \dim_{k(t)}(E(t)) : T \to \mathbb{N}$$

is locally constant. Then, E is locally free.

Proof. (See [9, Chapter III, Lemma 1.6] and [14, Chapter 4, § 4, no. 2, Criterion 2].) Let $t_0 \in T$, $n = \dim_{k(t_0)}(E(t_0))$, and $(e_i)_{i=1}^n$ a $k(t_0)$-basis of $E(t_0)$. Then, there exist an open neighbourhood U of t_0 in T, and elements $s_1, \dots, s_n$ of $E(U)$, such that $s_i(t_0) = e_i$ for all $i = 1, \dots, n$. Since the rank function of E is locally constant, we can assume that $\dim_{k(t)}(E(t)) = n$ for all $t \in U$. Now, since $(s_i(t_0))_{i=1}^n$ generates the $k(t_0)$-vector space $E(t_0)$, by Nakayama's lemma, $(s_{i,t_0})_{i=1}^n$ generates the $\mathscr{O}_{T,t_0}$-module E_{t_0} [12, Corollary 4.8]. As E is locally finitely generated, this implies that there exists an open neighbourhood V of t_0 in U, such that for each $t \in V$, $(s_{i,t})_{i=1}^n$ generates the $\mathscr{O}_{T,t}$-module E_t [19, Chapter 0, Proposition 5.2.2(i)]. Thus, for each $t \in V$, $(s_i(t))_{i=1}^n$ generates the $k(t)$-vector space $E(t)$. Since $\dim(E(t)) = n$, $(s_i(t))_{i=1}^n$ is a $k(t)$-basis of $E(t)$ for all $t \in V$.

We claim that $(s_{i,t})_{i=1}^n$ is an $\mathscr{O}_{T,t}$-basis of E_t for every $t \in V$. Let $t \in V$, and suppose $\gamma_1, \dots, \gamma_n$ are elements of $\mathscr{O}_{T,t}$, such that $\sum_{i=1}^n \gamma_i s_{i,t} = 0$. Choose an open neighbourhood W of t in V, and elements $a_1, \dots, a_n$ of $\mathscr{O}_T(W)$, such that $\gamma_i = a_{i,t}$. Then, there exists an open neighbourhood X of t in W, such that $\sum_{i=1}^n a_i|_X \cdot s_i|_X = 0$. In particular, $\sum_{i=1}^{n} a_i(x)s_i(x) = 0$ for all $x \in X$. Since $(s_i(x))_{i=1}^{n}$ is a $k(t)$-basis of $E(x)$, we get $a_i(x) = 0$ for all $i = 1,\ldots,n$ and $x \in X$. Thus, for every $i = 1,\ldots,n$, $a_i|_X$ belongs to $\mathscr{Z}_T(X)$; since $\mathscr{Z}_T$ is zero, this implies that $a_i|_X = 0$, hence $\gamma_i = s_{i,t} = 0$. Therefore, $(s_{i,t})_{i=1}^{n}$ is $\mathscr{O}_{T,t}$-linearly independent. Since it generates the $\mathscr{O}_{T,t}$-module E_t, it is an $\mathscr{O}_{T,t}$-basis of E_t, as claimed.

Now, by Lemma 2.4.11, the morphism $\varphi : \mathscr{O}_V^n \to E_V$ of $\mathscr{O}_V$-modules associated with the family $(s_i)_{i=1}^{n}$ is an isomorphism, where $\mathscr{O}_V = \mathscr{O}_T|_V$ and $E_V = E|_V$. This proves that E is locally free. □

Lemma 2.4.18 *Let T be a locally ringed space, E and F two locally free $\mathscr{O}_T$-modules, and $u : E \to F$ a morphism of $\mathscr{O}_T$-modules. Suppose that the vanishing ideal $\mathscr{Z}_T$ of T is zero, and that the function*

$$t \mapsto \dim_{k(t)}(\mathrm{Ker}(u(t))) : T \to \mathbb{N}$$

is locally constant. Then, the $\mathscr{O}_T$-module $\mathrm{Ker}(u)$ *is locally free. Moreover, for every $t \in T$, there is a canonical $k(t)$-isomorphism*

$$(\mathrm{Ker}(u))(t) \cong \mathrm{Ker}(u(t)).$$

Proof. Let K, I, and C, denote the $\mathscr{O}_T$-modules $\mathrm{Ker}(u)$, $\mathrm{Im}(u)$, and $\mathrm{Coker}(u)$, respectively, and let $\alpha : K \to E$, $\beta : I \to F$, and $\pi : F \to C$ be the canonical morphisms of $\mathscr{O}_T$-modules. Then, since E and F are locally free, they are coherent, hence so are K, I, and C [19, Chapter 0, Corollary 5.3.4]; in particular, they are locally finitely generated. Moreover, there exists a unique morphism $u' : E \to I$ of $\mathscr{O}_T$-modules, such that $\beta \circ u' = u$, and we have exact sequences

$$E \xrightarrow{u} F \xrightarrow{\pi} C \to 0, \tag{2.2}$$

$$0 \to I \xrightarrow{\beta} F \xrightarrow{\pi} C \to 0, \tag{2.3}$$

and

$$0 \to K \xrightarrow{\alpha} E \xrightarrow{u'} I \to 0, \tag{2.4}$$

of $\mathscr{O}_T$-modules.

Since taking stalks is an exact functor, and since the tensor product is a right exact functor, (2.2) induces an exact sequence

$$E(t) \xrightarrow{u(t)} F(t) \xrightarrow{\pi(t)} C(t) \to 0$$

of $k(t)$-vector spaces for all $t \in T$. Thus,

$$\begin{aligned}
\dim_{k(t)}(C(t)) &= \dim_{k(t)}(F(t)) - \dim_{k(t)}(\mathrm{Ker}(\pi(t))) \\
&= \dim_{k(t)}(F(t)) - \dim_{k(t)}(\mathrm{Im}(u(t))) \\
&= \dim_{k(t)}(F(t)) - \dim_{k(t)}(E(t)) + \dim_{k(t)}(\mathrm{Ker}(u(t))).
\end{aligned}$$

Since E and F are locally free, their rank functions are locally constant, by Lemma 2.4.11. Therefore, the hypothesis on u, and the above equation implies that the rank function of C is also locally constant. As C is locally finitely generated, and the vanishing ideal of T is zero, it follows from Lemma 2.4.17 that C is locally free.

Now, for each $t \in T$, (2.3) induces an exact sequence

$$\mathrm{Tor}_1^{\mathscr{O}_{T,t}}(k(t), C_t) \to I(t) \xrightarrow{\beta(t)} F(t) \xrightarrow{\pi(t)} C(t) \to 0$$

of $\mathscr{O}_{T,t}$-modules. As C is locally free, for every $t \in T$, the $\mathscr{O}_{T,t}$-module C_t is free, and hence flat. Therefore, $\mathrm{Tor}_1^{\mathscr{O}_{T,t}}(k(t), C_t) = 0$, and the sequence

$$0 \to I(t) \xrightarrow{\beta(t)} F(t) \xrightarrow{\pi(t)} C(t) \to 0 \tag{2.5}$$

of $k(t)$-vector spaces is exact. Thus,

$$\dim_{k(t)}(I(t)) = \dim_{k(t)}(F(t)) - \dim_{k(t)}(C(t)).$$

Since F and C are locally free, their rank functions are locally constant by Lemma 2.4.11, so the above relation implies that the rank function of I also is locally constant. Again, by Lemma 2.4.17, it follows that I is locally free.

Next, for each $t \in T$, (2.4) induces an exact sequence

$$\mathrm{Tor}_1^{\mathscr{O}_{T,t}}(k(t), I_t) \to K(t) \xrightarrow{\alpha(t)} E(t) \xrightarrow{u'(t)} I(t) \to 0$$

of $\mathscr{O}_{T,t}$-modules. As I is locally free, for every $t \in T$, the $\mathscr{O}_{T,t}$-module I_t is free, and hence flat. Therefore, $\mathrm{Tor}_1^{\mathscr{O}_{T,t}}(k(t), I_t) = 0$, and the sequence

$$0 \to K(t) \xrightarrow{\alpha(t)} E(t) \xrightarrow{u'(t)} C(t) \to 0 \tag{2.6}$$

of $k(t)$-vector spaces is exact. Thus,

$$\dim_{k(t)}(K(t)) = \dim_{k(t)}(E(t)) - \dim_{k(t)}(I(t)).$$

Since E and I are locally free, their rank functions are locally constant by Lemma 2.4.11, so the above relation implies that the rank function of K also is locally constant. Again, by Lemma 2.4.17, it follows that K is locally free. This proves the first part of the proposition.

Lastly, since $u = \beta \circ u'$, we have $u(t) = \beta(t) \circ u'(t)$. By (2.5), $\beta(t)$ is injective. Therefore, $\mathrm{Ker}(u(t)) = \mathrm{Ker}(u'(t))$. Now, by (2.6), $\alpha(t)$ is injective, and

$$\mathrm{Im}(\alpha(t)) = \mathrm{Ker}(u'(t)) = \mathrm{Ker}(u(t)).$$

Therefore, $\alpha(t)$ induces a $k(t)$-isomorphism from $K(t)$ onto $\mathrm{Ker}(u(t))$. □

Proposition 2.4.19 *Let T be a locally ringed space, and (V,ρ) and (W,σ) two families of representations of Q parametrised by T. Suppose that the vanishing ideal $\mathscr{Z}_T$ of T is zero, and that the function*

$$t \mapsto \dim_{k(t)}(\mathrm{Hom}((V(t),\rho(t)),(W(t),\sigma(t)))) : T \to \mathbb{N}$$

is locally constant. Then, the $\mathscr{O}_T$-module $\mathscr{H}om((V,\rho),(W,\sigma))$ is locally free. Moreover, for every $t \in T$, there is a canonical $k(t)$-isomorphism

$$(\mathscr{H}om((V,\rho),(W,\sigma)))(t) \cong \mathrm{Hom}((V(t),\rho(t)),(W(t),\sigma(t))).$$

Proof. Let $u : E \to F$ be the morphism of $\mathscr{O}_T$-modules defined in Remark 2.4.5. Then,

$$\mathscr{H}om((V,\rho),(W,\sigma)) = \mathrm{Ker}(u).$$

Moreover, by Proposition 2.4.14, for every $t \in T$, we have a canonical isomorphism

$$\mathrm{Hom}((V(t),\rho(t)),(W(t),\sigma(t))) \cong \mathrm{Ker}(u(t))$$

of $k(t)$-vector spaces. The proposition now follows from Lemma 2.4.18. □

Suppose (V,ρ) is a family of representations of Q parametrised by T. Then, for each point $t \in T$, we then get a representation $(V(t),\rho(t))$ of Q over the residue field $k(t)$, which is defined by $V(t) = (V_a(t))_{a\in Q_0}$ and $\rho(t) = (\rho_\alpha(t))_{\alpha\in Q_1}$. If P is any property of representations of Q over an arbitrary field, we say that (V,ρ) *has the property* P if for every $t \in T$, the representation $(V(t),\rho(t))$ of Q over $k(t)$ has the property P. We can thus speak of a *family of non-zero representations* of Q parametrised by T, a *family of Schur representations* of Q parametrised by T, etc. If θ is a weight in $\mathbb{R}^{Q_0}$, we can speak of a *family of θ-stable representations* of Q parametrised by T, a *family of θ-semistable representations* of Q parametrised by T, etc.

2.4.4 Families parametrised by complex spaces

Fix a non-empty finite quiver Q. We will consider only complex representations of Q in this subsection. Let T be a complex space. Then, there exists a unique topology on T, whose closed sets are precisely the analytic subsets of T. We will call it the *Zariski topology* on T. We will call the given topology on T the *strong topology*. In this context, if we say "open", "continuous", etc., without any qualifiers, we mean "open with respect to the strong topology", "continuous with respect to the strong topology", etc. As observed in Section 2.4.3, the Zariski topology on T is coarser than the strong topology on T.

Remark 2.4.20 A proof of the fact that there exists a unique topology on T, whose closed sets are precisely the analytic subsets of T can be found in [14, Chapter 4, § 1, no. 1, and Chapter 5, § 6, no. 1, Corollary to Proposition]. That the Zariski topology on T is coarser than the strong topology on T follows from Lemma 2.4.9(1).

Every complex space T is a ringed space over $\mathbb{C}$, that is, its structure sheaf $\mathscr{O}_T$ is a sheaf of $\mathbb{C}$-algebras. It is also a locally ringed space. Moreover, for every point $t \in T$, the canonical homomorphism from $\mathbb{C}$ to the residue field $k(t) = \mathscr{O}_{T,t}/\mathfrak{m}_t$ is an isomorphism [17, Proposition 2.3]. Equivalently, $\mathscr{O}_{T,t} = \mathbb{C}1_t + \mathfrak{m}_t$, where 1_t is the identity element of $\mathscr{O}_{T,t}$. We thus get an internal direct sum decomposition $\mathscr{O}_{T,t} = \mathbb{C}1_t \oplus \mathfrak{m}_t$ of the $\mathbb{C}$-vector space $\mathscr{O}_{T,t}$. We will identify $k(t)$ with $\mathbb{C}$ through the above isomorphism. Thus, for every point $t \in T$, there is a canonical identification of the residue field $k(t)$ with $\mathbb{C}$. If E is an $\mathscr{O}_T$-module, then through this identification, the fibre $E(t)$ becomes a $\mathbb{C}$-vector space. In particular, if (V,ρ) is a representation of Q parametrised by T, we get a representation $(V(t),\rho(t))$ of Q over $\mathbb{C}$.

Suppose $f : T \to T'$ is a morphism of complex spaces, that is, a morphism of ringed spaces over $\mathbb{C}$, which means that for every open subset V of T', the homomorphism of rings $\tilde{f}_V : \mathscr{O}_{T'}(V) \to \mathscr{O}_T(f^{-1}(V))$ is $\mathbb{C}$-linear. Then, f is a morphism of locally ringed spaces, that is, the homomorphism $f_t^{\sharp} : \mathscr{O}_{T',f(t)} \to \mathscr{O}_{T,t}$ of local rings induced by f is a local homomorphism [17, Lemma 2.6].

Proposition 2.4.21 *Let (V,ρ) a family of representations of Q parametrised by a complex analytic space T. Then, the subset of T, consisting of all the points $t \in T$ such that the representation $(V(t),\rho(t))$ of Q over $\mathbb{C}$ is Schur, is open with respect to the Zariski topology on T.*

Proof. Let T^0 denote the said subset of T. By Proposition 2.1.7(4), T^0 equals the set of all the points $t \in T$, such that $\dim_{\mathbb{C}}(\mathrm{End}(V(t),\rho(t))) = 1$. Therefore, by Corollary 2.4.15, $T \setminus T^0$ is an analytic subset of T, and is hence Zariski closed. □

Remark 2.4.22 Let S and T be two complex spaces, $f : S \to T$ a morphism of complex analytic spaces, and (V,ρ) a family of representations of Q parametrised by T. Let $(M,\varphi) = V,\rho^*(f)$, the pullback of (V,ρ) by f. It is a family of representations of Q

parametrised by S. For every $a \in Q_0$ and $s \in S$, we have canonical $\mathbb{C}$-isomorphisms

$$\begin{aligned} M_a(s) &\cong \mathbb{C} \otimes_{\mathscr{O}_{S,s}} M_{a,s} \cong \mathbb{C} \otimes_{\mathscr{O}_{S,s}} \mathscr{O}_{S,s} \otimes_{\mathscr{O}_{T_1,f(s)}} V_{a,f(s)} \\ &\cong \mathbb{C} \otimes_{\mathscr{O}_{T,f(s)}} V_{a,f(s)} \cong V_a(f(s)). \end{aligned}$$

Through these isomorphisms, for any $\alpha \in Q_1$, the $\mathbb{C}$-homomorphism

$$\varphi_\alpha(s) : M_{s(\alpha)}(s) \to M_{t(\alpha)}(s)$$

is identified with the map

$$\rho_\alpha(f(s)) : V_{s(\alpha)}(f(s)) \to V_{t(\alpha)}(f(s)).$$

Thus, for every $s \in S$, we have a canonical isomorphism

$$(M(s), \varphi(s)) \cong (V(f(s)), \rho(f(s)))$$

of representations of Q over $\mathbb{C}$.

Proposition 2.4.23 *Let $f_1 : S \to T_1$ and $f_2 : S \to T_2$ be morphisms of complex analytic spaces, (V, ρ) a family of representations of Q parametrised by T_1, and (W, σ) a family of representations of Q parametrised by T_2. Then, the function*

$$s \mapsto \dim_{\mathbb{C}}(\mathrm{Hom}((V(f_1(s)), \rho(f_1(s))), (W(f_2(s)), \sigma(f_2(s))))) : S \to \mathbb{N}$$

is upper semi-continuous with respect to the Zariski topology on S.

Proof. Let $(M,\varphi) = {f_1}^*(V,\rho)$ and $(N,\psi) = {f_2}^*(W,\sigma)$. Then, it is follows from remark 2.4.22 for every $s \in S$, we have canonical $\mathbb{C}$-isomorphisms

$$(M(s),\varphi(s)) \cong (V(f_1(s)),\rho(f_1(s))), \quad (N(s),\psi(s)) \cong (W(f_2(s)),\sigma(f_2(s)))$$

of representations of Q over $\mathbb{C}$. We thus get a canonical $\mathbb{C}$-isomorphism

$$\mathrm{Hom}((M(s),\varphi(s)),(N(s),\psi(s))) \cong \mathrm{Hom}((V(f_1(s)),\rho(f_1(s))),(W(f_2(s)),\sigma(f_2(s))))$$

for every $s \in S$. Now, by Proposition 2.4.14, for every $n \in \mathbb{N}$, the subset

$$\{s \in S \,|\, \dim_{\mathbb{C}}(\mathrm{Hom}((M(s),\varphi(s)),(N(s),\psi(s)))) \geq n\}$$

of S is analytic, that is, closed in the Zariski topology on S. Therefore, the given function is upper semi-continuous with respect to the Zariski topology on S. □

Remark 2.4.24 Let T be a complex analytic space, and A an analytic subset of T. Then, A has a canonical structure of a reduced closed complex analytic subspace of T [14, Chapter 4, § 3, no. 1, Proposition].

Proposition 2.4.25 *Let S, T_1, and T_2 be three complex spaces, $f_1 : S \to T_1$ and $f_2 : S \to T_2$ two morphisms of complex spaces, $\theta \in \mathbb{R}^{Q_0}$ a weight of Q, (V,ρ) a family of θ-stable representations of Q parametrised by T_1, and (W,σ) a family of θ-stable representations of Q parametrised by T_2. Suppose that*

$$\dim(V(f_1(s)),\rho(f_1(s))) = \dim(W(f_2(s)),\sigma(f_2(s)))$$

for all $s \in S$. Then, the subset

$$A = \{s \in S \,|\, (V(f_1(s)), \rho(f_1(s))) \cong (W(f_2(s)), \sigma(f_2(s)))\}$$

of S is analytic. Moreover, if we give A its canonical structure of a reduced closed complex analytic subspace of S, and let $g_1 = f_1|_A : A \to T_1$ and $g_2 = f_2|_A : A \to T_2$, then the $\mathscr{O}_A$-module

$$L = \mathscr{H}om({g_1}^*(V,\rho), {g_2}^*(W,\sigma)),$$

is invertible, and for every $s \in A$, there is a canonical $\mathbb{C}$-isomorphism

$$L(s) \cong \mathrm{Hom}((V(f_1(s)), \rho(f_1(s))), (W(f_2(s)), \sigma(f_2(s)))).$$

Proof. By hypothesis,

$$\dim(V(f_1(s)), \rho(f_1(s))) = \dim(W(f_2(s)), \sigma(f_2(s)))$$

for all $s \in S$. Therefore,

$$\mu_\theta(V(f_1(s)), \rho(f_1(s))) = \mu_\theta(W(f_2(s)), \sigma(f_2(s)))$$

for all $s \in S$. It follows from Proposition 2.1.7(1d) that

$$A = \{s \in S \,|\, \dim_{\mathbb{C}}(\mathrm{Hom}((V(f_1(s)), \rho(f_1(s))), (W(f_2(s)), \sigma(f_2(s))))) \geq 1\}.$$

Therefore, by Proposition 2.4.23, A is an analytic subset of S.

Give A its canonical structure of a reduced closed complex subspace of S. Let

$$(M,\varphi) = {g_1}^*(V,\rho), \quad (N,\psi) = {g_2}^*(W,\sigma).$$

Then, for all $s \in A$, we have $g_1(s) = f_1(s)$ and $g_2(s) = f_2(s)$, hence

$$(M(s), \varphi(s)) \cong (V(f_1(s)), \rho(f_1(s))), \quad (N(s), \psi(s)) \cong (W(f_2(s)), \rho(f_2(s))),$$

so $(M(s), \varphi(s))$ and $(N(s), \psi(s))$ are θ-stable; moreover, $(M(s), \varphi(s)) \cong (N(s), \psi(s))$, hence we have $\mathbb{C}$-isomorphisms

$$\mathrm{Hom}((M(s), \varphi(s)), (N(s), \psi(s))) \cong \mathrm{Hom}((M(s), \varphi(s)), (M(s), \varphi(s))).$$

Therefore, by Proposition 2.1.7(3)–(4),

$$\dim_{\mathbb{C}}(\mathrm{Hom}((M(s), \varphi(s)), (N(s), \psi(s)))) = 1$$

for all $s \in A$. Now, since A is reduced, by the Rückert Nullstellensatz, the vanishing ideal $\mathscr{X}_A$ of A is zero. Therefore, by Proposition 2.4.19, the $\mathscr{O}_A$-module

$$L = \mathscr{H}om((M, \varphi), (N, \psi))$$

is locally free, and we have a canonical $\mathbb{C}$-isomorphism

$$L(s) \cong \mathrm{Hom}((M(s), \varphi(s)), (N(s), \psi(s)))$$

for all $s \in A$. Thus, $\dim_{\mathbb{C}}(L(s)) = 1$ for all $s \in A$, hence, it follows that from the Lemma 2.4.11 that the rank of L is 1, so L is an invertible $\mathscr{O}_A$-module. □

Proposition 2.4.26 *Let $f_1 : S \to T_1$ and $f_2 : S \to T_2$ be morphisms of complex analytic spaces, $\theta \in \mathbb{R}^{Q_0}$ a weight of Q, (V, ρ) a family of θ-stable representations of Q parametrised by T_1, and (W, σ) a family of θ-stable representations of Q parametrised by T_2. Suppose that for each $a \in Q_0$, we have $V_a = \mathscr{O}_{T_1} \otimes_{\mathbb{C}} D_a$ and $W_a = \mathscr{O}_{T_2} \otimes_{\mathbb{C}} D_a$,* *where D_a is a finite-dimensional $\mathbb{C}$-vector space. Give the analytic subset*

$$A = \{s \in S \,|\, (V(f_1(s)), \rho(f_1(s))) \cong (W(f_2(s)), \sigma(f_2(s)))\},$$

of S its canonical structure of a reduced closed complex subspace of S (Proposition 2.4.25). Then, for every point $s_0 \in A$, there exist an open neighbourhood U of s_0 in A, and a family $h = (h_a)_{a\in Q_0}$ of elements $h_a \in \mathscr{O}_A(U) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(D_a)$, such that for every point $s \in U$, the family $h(s) = (h_a(s))_{a\in Q_0}$ is an isomorphism of representations of Q over $\mathbb{C}$ from $(D, \rho(f_1(s)))$ to $(D, \sigma(f_2(s)))$, where $D = (D_a)_{a\in Q_0}$.

Proof. For all $s \in S$ and $a \in Q_0$, we have $V_a(f_1(s)) = W_a(f_2(s)) = D_a$, hence $\dim(V(f_1(s)), \rho(f_1(s))) = \dim(W(f_2(s)), \sigma(f_2(s)))$. Therefore, if we let $g_1 = f_1|_A : A \to T_1$, $g_2 = f_2|_A : A \to T_2$, $(M, \varphi) = V, \rho^*(g_1)$ and $(N, \psi) = W, \sigma^*(g_2)$, then, by Proposition 2.4.25, the $\mathscr{O}_A$-module $L = \mathscr{H}om((M, \varphi), (N, \psi))$ is invertible. Thus, for every $s_0 \in A$, there exist an open neighbourhood U of s_0 in A, and an isomorphism $F : \mathscr{O}_U \to L_U$ of $\mathscr{O}_U$-modules, where $\mathscr{O}_U = \mathscr{O}_A|_U$ and $L_U = L|_U$.

The isomorphism F of $\mathscr{O}_U$-modules defines an isomorphism $F_U : \mathscr{O}_A(U) \to L(U)$ of $\mathscr{O}_A(U)$-modules. Let u denote the element $F_U(1)$ of $L(U)$, where 1 is the identity element of the ring $\mathscr{O}_A(U)$. Then, $u \in \mathrm{Hom}((M, \varphi)|_U, (N, \psi)|_U)$. Therefore, u is a family $(u_a)_{a\in Q_0}$ of homomorphisms $u_a : M_a|_U \to N_a|_U$ of $\mathscr{O}_U$-modules satisfying certain conditions. Now, since $V_a = \mathscr{O}_{T_1} \otimes D_a$, $M_a|_U = \mathscr{O}_U \otimes D_a$, and, similarly, $N_a|_U = \mathscr{O}_U \otimes D_a$. Therefore, u_a defines an element h_a of $\mathscr{O}_A(U) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(D_a)$. Moreover, $h_a(s) = u_a(s) : D_a \to D_a$ for all $s \in U$. Since $u(s)$ is a morphism of representations of Q over $\mathbb{C}$ from $(D, \varphi(s))$ to $(D, \psi(s))$, so is the family $h(s) = (h_a(s))_{a\in Q_0}$. Now, for all $s \in A$, we have $g_1(s) = f_1(s)$ and $g_2(s) = f_2(s)$, hence $\varphi(s) = \rho(f_1(s))$ and $\psi(s) = \sigma(f_2(s))$. Thus, $h(s)$ is a morphism of representations of Q from $(D, \rho(f_1(s)))$ to $(D, \sigma(f_2(s)))$.

It remains to show that for every $s \in U$, $h(s)$ is an isomorphism of representations of representations of Q. Let $s \in U$. Since $F : \mathscr{O}_U \to L_U$ is an isomorphism of $\mathscr{O}_U$-modules, $F(s) : \mathbb{C} \to L(s)$ is an isomorphism. Therefore, $u(s) = (F_U(1))(s) = (F(s))(1)$ is a non-zero element of $L(s)$. Thus, $h(s)$ is a non-zero morphism from $(D, \rho(f_1(s)))$ to $(D, \sigma(f_2(s)))$. Clearly,

$$\mu_\theta(D, \rho(f_1(s))) = \mu_\theta(D, \sigma(f_2(s))) = \mu_\theta(d),$$

where $d = (\dim_{\mathbb{C}}(D_a))_{a \in Q_0}$. As $D = V(f_1(s)) = W(f_2(s))$, $(D, \rho(f_1(s)))$ and $(D, \sigma(f_2(s)))$ are θ-stable, hence by Proposition 2.1.7(1d), $h(s)$ is an isomorphism of representations of Q. □

Remark 2.4.27 The proof of the above Proposition is similar to that of [26, Lemma 8.3.3, p. 132]. It uses the following basic facts about trivial vector bundles.

Let T be a ringed space over a field k, and D a finite-dimensional k-vector space. Then, we define an $\mathscr{O}_T$-module $\mathscr{O}_T \otimes_k D$ by setting

$$(\mathscr{O}_T \otimes_k D)(U) = \mathscr{O}_T(U) \otimes_k D$$

for all open subsets U of T, and

$$\rho_{VU} = \rho_{VU} \otimes \mathbf{1}_D : \mathscr{O}_T(U) \otimes_k D \to \mathscr{O}_T(V) \otimes_k D$$

for all open subsets U and V of T such that $U \supset V$. It is easy to see that these assignments define a sheaf, and hence an $\mathscr{O}_T$-module.

1. For every point $t \in T$, there is a canonical isomorphism

$$(\mathscr{O}_T \otimes_k D)_t \cong \mathscr{O}_{T,t} \otimes_k D$$

of $\mathscr{O}_{T,t}$-modules. If U is an open neighbourhood of t, $s \in \mathscr{O}_T(U)$, and $v \in D$, then under the above identification, we have $(s \otimes v)_t = s_t \otimes v$.

2. If $(e_i)_{i=1}^n$ is a k-basis of D, then the morphism of $\mathscr{O}_T$-modules $\varphi : \mathscr{O}_T^n \to \mathscr{O}_T \otimes_k D$ associated with the family $(1 \otimes e_i)_{i=1}^n$ of elements of $(\mathscr{O}_T \otimes_k D)(T)$ (Lemma 2.4.11) is an isomorphism. Therefore, $\mathscr{O}_T \otimes_k D$ is a finitely generated and free $\mathscr{O}_T$-module.

3. If $f : S \to T$ is a morphism of ringed spaces over k, then there is a canonical isomorphism
$$f^{-1}(\mathscr{O}_T \otimes_k D) \cong f^{-1}(\mathscr{O}_T) \otimes_k D$$
of $f^{-1}(\mathscr{O}_T)$-modules on the ringed space $(S, f^{-1}(\mathscr{O}_T))$ over k. We thus get canonical isomorphisms
$$\mathscr{O}_T \otimes_k D^*(f) \cong \mathscr{O}_S \otimes_{f^{-1}(\mathscr{O}_T)} f^{-1}(\mathscr{O}_T \otimes_k D) \cong \mathscr{O}_S \otimes_k D$$
of $\mathscr{O}_S$-modules.

4. If D' is another finite-dimensional k-vector space, and if $u : \mathscr{O}_T \otimes_k D \to \mathscr{O}_T \otimes_k D'$ is a morphism of $\mathscr{O}_T$-modules, then the homomorphism
$$u_T : \mathscr{O}_T(T) \otimes_k D \to \mathscr{O}_T(T) \otimes_k D'$$
of $\mathscr{O}_T$-modules gives an element
$$h \in (\mathscr{O}_T \otimes_k \operatorname{Hom}_k(D, D'))(T) = \mathscr{O}_T(T) \otimes_k \operatorname{Hom}_k(D, D')$$

through the isomorphism

$$\mathrm{Hom}_{\mathscr{O}_T(T)}(\mathscr{O}_T(T)\otimes_k D, \mathscr{O}_T(T)\otimes_k D') \cong \mathscr{O}_T(T)\otimes_k \mathrm{Hom}_k(D,D')$$

of $\mathscr{O}_T(T)$-modules, which exists since D is a finitely generated projective k-module [3, Chapter II, § 5, no. 3, Proposition 7]. If $(e_j)_{j=1}^n$ is a k-basis of D, and $(e_i')_{i=1}^m$ a k-basis of D', then there exist a unique matrix $(u_{ij})_{1\le i\le m, 1\le j\le n}$ of elements of $\mathscr{O}_T(T)$, such that

$$u_T(1\otimes e_j) = \sum_{i=1}^m u_{ij}\otimes e_i'$$

for all $j=1,\dots,n$. We have

$$h = \sum_{i=1}^n \sum_{j=1}^n u_{ij}\otimes f_{ij},$$

where, for $i=1,\dots,m$ and $j=1,\dots,n$, $f_{ij}: D\to D'$ is the unique k-linear map such that $f_{ij}(e_{j'}) = \delta_{jj'}e_i$ $(1\le j'\le n)$, where δ is the Kronecker delta. In particular, by (1), for all $t\in T$, we have

$$h_t = \sum_{i=1}^n \sum_{j=1}^n u_{ij,t}\otimes f_{ij}$$

in $(\mathscr{O}_T\otimes_k \mathrm{Hom}_k(D,D'))_t = \mathscr{O}_{T,t}\otimes_k \mathrm{Hom}_k(D,D')$.

5. Suppose that T is a complex analytic space, and $k=\mathbb{C}$. Then, by (1), for every $t\in T$, we have canonical $\mathbb{C}$-isomorphisms

$$(\mathscr{O}_T\otimes_k D)(t) \cong \mathbb{C}\otimes_{\mathscr{O}_{T,t}} (\mathscr{O}_T\otimes_{\mathbb{C}} D)_t \cong \mathbb{C}\otimes_{\mathscr{O}_{T,t}} (\mathscr{O}_{T,t}\otimes_{\mathbb{C}} D) \cong D,$$

since $k(t)=\mathbb{C}$. If U is an open neighbourhood of t, $s\in\mathscr{O}_T(U)$, and $v\in D$, then under the above identification, we have

$$(s \otimes v)(t) = s(t)v, \tag{2.7}$$

where we note that $s(t) \in k(t) = \mathbb{C}$. Now, in the situation of (4), for each $t \in T$, we have a $\mathbb{C}$-linear map

$$u(t) : (\mathscr{O}_T \otimes_k D)(t) = D \to (\mathscr{O}_T \otimes_{\mathbb{C}} D')(t) = D',$$

and an element

$$h(t) \in (\mathscr{O}_T \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}}(D, D'))(t) = \mathrm{Hom}_{\mathbb{C}}(D, D').$$

In fact,

$$u(t) = h(t) : D \to D'. \tag{2.8}$$

For, if $1 \leq j \leq n$, then by (2.7),

$$\begin{aligned} u(t)(e_j) &= u(t)((1 \otimes e_j)(t)) = (u_T(1 \otimes e_j))(t) \\ &= \Big(\sum_{i=1}^{n} u_{ij} \otimes e'_i\Big)(t) = \sum_{i=1}^{n} u_{ij}(t) e'_i, \end{aligned}$$

while,

$$\begin{aligned} h(t)(e_j) &= \Big(\sum_{i=1}^{n} \sum_{j'=1}^{n} u_{ij'} \otimes f_{ij'}\Big)(t)(e_j) \\ &= \Big(\sum_{i=1}^{n} \sum_{j'=1}^{n} u_{ij'}(t) f_{ij'}\Big)(e_j) = \sum_{i=1}^{n} u_{ij}(t) e'_i. \end{aligned}$$

This verifies (2.8).

2.4.5 The Hausdorff property

Let Q be a non-empty finite quiver. If T is a topological space, and R an equivalence relation on T.

Definition 2.4.28 We say that two points t_1 and t_2 in T are *separated* with respect to R if there exist an open neighbourhood U_1 of t_1, and an open neighbourhood U_2 of t_2, in T, such that $U_1 \cap U_2 = \emptyset$, and both U_1 and U_2 are saturated with respect to R. This is equivalent to the condition that there exist an open neighbourhood U_1' of $\pi(t_1)$, and an open neighbourhood U_2' of $\pi(t_2)$, in T', such that $U_1' \cap U_2' = \emptyset$, where $T' = T/R$ is the quotient topological space of T by R, $\pi : T \to T'$ the canonical projection.

Definition 2.4.29 We say that R is *open* if the saturation with respect to R of any open subset of T is open in T. This is equivalent to the condition that $\pi : T \to T'$ is an open map.

Proposition 2.4.30 *Let T be a topological space, and R an open equivalence relation on T. Then, the folloeing statements are true:*

1. *The closure of R in $T \times T$ equals the set of all points (t_1, t_2) in $T \times T$ such that t_1 and t_2 are not separated with respect to R.*
2. *The quotient topological space $T' = T/R$ is Hausdorff if and only if R is closed in $T \times T$.*

Proof.

(1) Suppose t_1 and t_2 are not separated with respect to R. We have to show that for every open neighbourhood U_1 of t_1 in T, and for every open neighbourhood U_2 of t_2 in T, the set $R \cap (U_1 \times U_2)$ is non-empty. Since R is open, the saturation W_i of U_i with respect to R is a saturated open neighbourhood of t_i in T. Since t_1 and t_2

are not separated with respect to R, there exists a point s_0 in $V_1 \cap V_2$. As V_1 is the saturation of U_1 with respect to R, there exists a point $s_1 \in U_1$, such that $s_0 R s_1$. Similarly, there exists a point $s_2 \in U_2$, such that $s_0 R s_2$. Since R is an equivalence relation, we get $s_1 R s_2$, hence $(s_1, s_2) \in R$. Thus, $(s_1, s_2) \in R \cap (U_1 \times U_2)$.

Conversely, suppose (t_1, t_2) belongs to F. For $i = 1, 2$, let U_i be any saturated open neighbourhood of t_i in T. Then, there exists a point (s_1, s_2) in $R \cap (U_1 \times U_2)$. Since $(s_1, s_2) \in R$, we have $s_1 R s_2$. As U_1 is saturated, and $s_2 \in U_2$, this implies that $s_1 \in U_2$. Thus, $s_1 \in U_1 \cap U_2$, hence $U_1 \cap U_2 \neq \emptyset$. Therefore, every saturated open neighbourhood of t_1 meets every saturated open neighbourhood of t_2. In other words, t_1 and t_2 are not separated with respect to R.

(2) (See [5, Chapter I, § 8, no. 3, Proposition 8].) Let T' denote the quotient topological space T/R, and $\pi : T \to T'$ the canonical projection. Assume that T' is Hausdorff. Let $(t_1, t_2) \in \overline{R}$. Then, by (1), t_1 and t_2 are not separated with respect to R. Therefore, every open neighbourhood of $\pi(t_1)$ meets every open neighbourhood of $\pi(t_2)$, in T'. Since T' is Hausdorff, this implies that $\pi(t_1) = \pi(t_2)$, hence $(t_1, t_2) \in R$. Therefore, R is closed in $T \times T$.

Conversely, suppose R is closed in $T \times T$, and let x_1 and x_2 be two distinct points of T'. Choose $t_1, t_2 \in T$, such that $\pi(t_1) = x_1$ and $\pi(t_2) = x_2$. Then, $(t_1, t_2) \notin R$. As R is closed in $T \times T$, this implies that $(t_1, t_2) \notin \overline{R}$. By (1), t_1 and t_2 are separated with respect to R. Therefore, there exist an open neighbourhood U_1' of $\pi(t_1)$, and an open neighbourhood U_2' of $\pi(t_2)$, in T', such that $U_1' \cap U_2' = \emptyset$. It follows that T' is Hausdorff. □

Definition 2.4.31 Let T be a complex analytic space, and (V, ρ) a family of representations of Q parametrised by T. Define a relation R on T by setting $t_1 R t_2$ if the representations $(V(t_1), \rho(t_1))$ and $(V(t_2), \rho(t_2))$ of Q over $\mathbb{C}$ are isomorphic. This is an equivalence relation on T. We will call it the equivalence relation on T *induced* by (V,ρ).

Remark 2.4.32 Let T_1 and T_2 be two complex spaces. Then, there exists a product P of T_1 and T_2 in the category of complex spaces; moreover, the underlying topological space of P is a product of the underlying topological spaces of T_1 and T_2 in the category of topological spaces ([18, Theorem 2.1] and [14, Chapter 1, § 3, no.4, Theorem]). Therefore, we will identify the underlying topological space of P with the product topological space $T_1 \times T_2$, and the underlying continuous maps of the canonical projections from P to T_1 and T_2, with the canonical projections from the topological space $T_1 \times T_2$ to T_1 and T_2.

Proposition 2.4.33 *Let T be a complex space, (V,ρ) a family of non-zero representations of Q parametrised by T, and R the equivalence relation on T induced by (V,ρ). Let Z denote the Zariski closure of R in the product complex space $T \times T$. Then, Z is contained in the set of all points (t_1,t_2) in $T \times T$, for which there exist non-zero morphisms*

$$f : (V(t_1),\rho(t_1)) \to (V(t_2),\rho(t_2)), \quad g : (V(t_2),\rho(t_2)) \to (V(t_1),\rho(t_1))$$

of representations of Q over $\mathbb{C}$.

Proof. Let

$$A_1 = \{(t_1,t_2) \in T \times T \,|\, \mathrm{Hom}((V(t_1),\rho(t_1)),(V(t_2),\rho(t_2))) \neq 0\},$$

and

$$A_2 = \{(t_1,t_2) \in T \times T \,|\, \mathrm{Hom}((V(t_2),\rho(t_2)),(V(t_1),\rho(t_1))) \neq 0\}.$$

Then, it is obvious that R is a subset of $A_1 \cap A_2$. On the other hand, by Proposition 2.4.23, A_1 and A_2 are closed in the Zariski topology on $T \times T$, and hence so is $A_1 \cap A_2$. It follows that $Z \subset A_1 \cap A_2$. □

Proposition 2.4.34 *Let T be a complex space, (V,ρ) a family of non-zero representations of Q parametrised by T, and t_1 and t_2 two points of T. Suppose that the equivalence relation on T induced by (V,ρ) is open, and that t_1 and t_2 are not separated with respect to (V,ρ). Then, there exist non-zero morphisms*

$$f : (V(t_1),\rho(t_1)) \to (V(t_2),\rho(t_2)), \quad g : (V(t_2),\rho(t_2)) \to (V(t_1),\rho(t_1))$$

of representations of Q over $\mathbb{C}$.

Proof. By Remark 2.4.30(1), (t_1,t_2) belongs to the closure F of R with respect to the product topology $\mathscr{P}$ induced by the strong topology on T. On the other hand, $\mathscr{P}$ equals the strong topology on $T \times T$, and is hence finer than the Zariski topology on $T \times T$. It follows that F is contained in the Zariski closure Z of R in $T \times T$. Therefore, $(t_1,t_2) \in Z$. The proposition now follows from Proposition 2.4.33. □

Remark 2.4.35 The above proof follows that of [27, Proposition 2.9].

Proposition 2.4.36 *Let T be a complex space, $\theta \in \mathbb{R}^{Q_0}$, and (V,ρ) a family of θ-stable representations of Q parametrised by T. Suppose that the equivalence relation R on T induced by (V,ρ) is open. Then, the quotient topological space T/R is Hausdorff.*

Proof. By Remark 2.4.30(2), it suffices to prove that R is closed in $T \times T$ with respect to the product topology $\mathscr{P}$ induced by the strong topology on T. Let F denote the closure of R with respect to $\mathscr{P}$, and let $(t_1,t_2) \in F$. Now, $\mathscr{P}$ equals the strong topology on $T \times T$, and is hence finer than the Zariski topology on $T \times T$. It follows that F is contained in the Zariski closure Z of R in $T \times T$. Therefore, $(t_1,t_2) \in Z$, so, by

Proposition 2.4.33, there exists a non-zero morphism

$$f : (V(t_1), \rho(t_1)) \to (V(t_2), \rho(t_2))$$

of representations of Q over $\mathbb{C}$. Now, for every $a \in Q_0$, the $\mathscr{O}_T$-module V_a is locally free, hence its rank function $t \mapsto \dim_{\mathbb{C}}(V_a(t)) : T \to \mathbb{N}$ is locally constant. Therefore the function $t \mapsto \dim(V(t), \rho(t)) : T \to \mathbb{N}^{Q_0}$ is continuous with respect to the discrete topology on $\mathbb{N}^{Q_0}$. Thus, as $\mathbb{N}^{Q_0}$ is Hausdorff, the set

$$G = \{(t,t') \in T \times T \,|\, \dim(V(t), \rho(t)) = \dim(V(t'), \rho(t'))$$

is closed in $T \times T$. Clearly, $R \subset G$. Therefore, $F \subset G$, hence

$$\dim(V(t_1), \rho(t_1)) = \dim(V(t_2), \rho(t_2))\}.$$

This implies that

$$\mu_\theta(V(t_1), \rho(t_1)) = \mu_\theta(V(t_2), \rho(t_2)).$$

Since $(V(t), \rho(t))$ is θ-stable for all $t \in T$, by Proposition 2.1.7(1d), we see that f is an isomorphism. Therefore, $t_1 R t_2$, hence $(t_1, t_2) \in R$. This proves that F is closed with respect to $\mathscr{P}$. □

Proposition 2.4.37 *Let T be a complex space, $\theta : T \to \mathbb{R}^{Q_0}$ a continuous function, and (V, ρ) a family of representations of Q parametrised by T. Suppose that the equivalence relation R on T induced by (V, ρ) is open, that the representation $(V(t), \rho(t))$ over $\mathbb{C}$ is $\theta(t)$-stable for every $t \in T$, and that the function θ is R-invariant. Then, the quotient topological space T/R is Hausdorff.*

Proof. By Remark 2.4.30(2), it suffices to prove that R is closed in $T \times T$ with respect to the product topology $\mathscr{P}$ induced by the strong topology on T. Let F denote the closure of R with respect to $\mathscr{P}$, and let $(t_1,t_2) \in F$. Now, $\mathscr{P}$ equals the strong topology on $T \times T$, and is hence finer than the Zariski topology on $T \times T$. It follows that F is contained in the Zariski closure of R in $T \times T$. Therefore, by Proposition 2.4.33, there exists a non-zero morphism

$$f : (V(t_1), \rho(t_1)) \to (V(t_2), \rho(t_2))$$

of representations of Q over $\mathbb{C}$. Now, for every $a \in Q_0$, the $\mathscr{O}_T$-module V_a is locally free, hence its rank function $t \mapsto \dim_{\mathbb{C}}(V_a(t)) : T \to \mathbb{N}$ is locally constant. Therefore, as θ is continuous, the function $\varphi : T \to \mathbb{N}^{Q_0} \times \mathbb{R}^{Q_0}$, which is defined by

$$\varphi(t) = (\dim(V(t), \rho(t)), \theta(t))$$

is continuous, if we give $\mathbb{N}^{Q_0}$ the discrete topology, and $\mathbb{R}^{Q_0}$ the usual topology. Thus, as $\mathbb{N}^{Q_0} \times \mathbb{R}^{Q_0}$ is Hausdorff, the set

$$G = \{(t,t') \in T \times T \mid \varphi(t) = \varphi(t')\}$$

is closed in $T \times T$. As θ is R-invariant, $R \subset G$. Therefore, $F \subset G$, hence $\varphi(t_1) = \varphi(t_2)$. This implies that

$$\mu_\theta(V(t_1), \rho(t_1)) = \mu_\theta(V(t_2), \rho(t_2)).$$

Since $(V(t), \rho(t))$ is θ-stable for all $t \in T$, by Proposition 2.1.7(1d), we see that f is an isomorphism. Therefore, $t_1 R t_2$, hence $(t_1,t_2) \in R$. This proves that F is closed with respect to $\mathscr{P}$. □

Chapter 3

Kahler structures on the moduli of stable representations

3.1 The moduli space of Schur representations

In subsection 3.1.1, we give a detailed construction of complex premanifolds in the context of free actions of complex Lie groups and Corollary 3.1.18. In subsection 3.1.2, we apply the Corollary 3.1.18 to quiver representations over the field of complex numbers to prove the first main Theorem 3.1.22 and draw an important conclusion that the moduli of Schur representations gets the structure of complex premanifold.

3.1.1 Quotient premanifolds

Definition 3.1.1 By a *complex premanifold*, we mean a complex manifold without any separation or countability conditions, that is, a topological space with a maximal holomorphic atlas.

Definition 3.1.2 We use the term *complex manifold* for a complex premanifold whose underlying topological space is *Hausdorff*. Note that, we are not assuming second countablity condition in the definition of complex manifold.

Alternatively, a complex manifold is a ringed space $(X, \mathscr{O}_X)$ over $\mathbb{C}$ with the property that for every point $x \in X$, there exist an open neighbourhood U of x in X, $n \in \mathbb{N}$, an open subset V of $\mathbb{C}^n$, and an isomorphism $\varphi : (U, \mathscr{O}_U) \to (V, \mathscr{O}_V)$ of ringed spaces over $\mathbb{C}$, where $\mathscr{O}_U = \mathscr{O}_X|_U$, and $\mathscr{O}_V$ is the sheaf of holomorphic functions on V.

We denote the holomorphic tangent space of a complex premanifold X at a point x in X by $\mathrm{T}_x(X)$, and the holomorphic tangent bundle of X by $\mathrm{T}(X)$.

Definition 3.1.3

1. A holomorphic map $f : X \to Y$ of complex premanifolds is called an *immersion* at a point $x \in X$ if the linear map $\mathrm{T}_x(f) : \mathrm{T}_x(X) \to \mathrm{T}_{f(x)}(Y)$ is injective. It is called an *immersion* if it is immersion at every point of X.

2. A holomorphic map $f : X \to Y$ is called an *embedding* if it is an immersion, and the map $f' : X \to f(X)$ induced by f is a homeomorphism.

3. A holomorphic map $f : X \to Y$ is called a *submersion* at a point $x \in X$ if the linear map $\mathrm{T}_x(f) : \mathrm{T}_x(X) \to \mathrm{T}_{f(x)}(Y)$ is surjective. It is called an *immersion* if it is submersion at every point of X.

The following remark is an useful result, in general and from view points of its utility on several occasion in this work, in particular.

Remark 3.1.4 A holomorphic map $p : X \to Y$ of complex premanifolds is a submersion at a point $a \in X$ if and only if there exist an open neighbourhood V of $p(a)$ in Y, and a holomorphic section $s : V \to X$ of p, such that $s(p(a)) = a$ [8, 5.9.1].

Let R be an equivalence relation on a complex premanifold X, Y the quotient topological space X/R, and $p : X \to Y$ the canonical projection. It is a theorem of Godement that the following statements are equivalent:

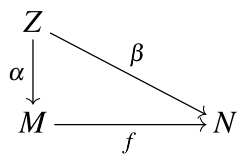
1. There exists a structure of a complex premanifold on Y with the property that p is a holomorphic submersion.

2. The relation R is a subpremanifold of $X \times X$, and the restricted projection $\mathrm{pr}_1 : R \to X$ is a submersion.

Moreover, in that case, such a complex premanifold structure on Y is unique [33, Part II, Chapter III, § 12, Theorems 1–2].

Remark 3.1.5 To see the uniqueness part in the fact above, we give the following facts:

1. Let M, N, Z be any three complex manifolds, and let the following be a commutative diagram of maps between sets:



Assume that α is a surjective holomorphic submersion, and that β is holomorphic. Then f is holomorphic. If in addition β is a submersion, then so is f. Further, surjectivity of β implies that of f.

For this, let $m \in M$. Then $m = \alpha(z)$ for some $z \in Z$. Since α is submersion at z, we see that there exists an open neighborhood U of m in M and a holomorphic section $s\colon U \to Z$ of α on U such that $s(m) = z$. It follows that $f = f \circ \alpha \circ s = \beta \circ s$ on U. Hence, f is holomorphic on U.

2. Let $p\colon M \to N$ be a surjective map between sets. Assume that M has a structure of complex manifolds. Then, N has at most one structure of complex manifold for which p is a holomorphic submersion.

To see this, for $j = 1,2$, let N_j be N equipped with a structure of complex manifold such that $p\colon M \to N_j$ is a holomorphic submersion. Let $I\colon N_1 \to N_2$ be the identity map. Then, by the above fact I is holomorphic. Similarly, the inverse of I is holomorphic, hence I is a biholomorphism.

Remark 3.1.6 Another reference for Godement's theorem is [8, 5.9.5]. It is also mentioned there that a map g from Y to a complex premanifold Z is holomorphic if and only if $g \circ p : X \to Z$ is holomorphic. Equivalently, the map $p : X \to Y$ is a coequaliser of the restricted projections $\mathrm{pr}_1 : R \to X$ and $\mathrm{pr}_2 : R \to X$ in the category of complex premanifolds. In fact, if $f : M \to N$ is a surjective holomorphic submersion of complex premanifolds, and if R_f is the equivalence relation on M defined by f, then f is a coequaliser of the in the category of complex premanifolds restricted projections $\mathrm{pr}_1 : R_f \to M$ and $\mathrm{pr}_2 : R_f \to M$ [33, Part II, Chapter III, § 11, Lemma 2]. Yet another reference for Godement's theorem is [38, § 5.6].

We will use the above theorem in the context of group actions. Let X be a topological space, and G a topological group. Suppose that we are given a continuous right action of G on X. Let R denote the equivalence relation on X defined by this action, and $\tau : X \times G \to R$ the map $(x,g) \mapsto (x,xg)$.

For any two subsets A and B of X, let

$$P_G(A,B) = \{g \in G \,|\, Ag \cap B \neq \emptyset\}.$$

If the action of G on X is free, then for every $(x,y) \in R$, there exists a unique element $\varphi(x,y)$ of G, such that $y = x\varphi(x,y)$; we thus get a map $\varphi : R \to G$, which is called the *translation* map of the given action.

Definition 3.1.7 We say that the action of G on X is *principal* if it is free, and its translation map is continuous.

Proposition 3.1.8 *The following assertions are true:*

1. *The action of G on X is free if and only if the map τ is injective.*

2. *The following statements are equivalent:*

   (a) *The action of G on X is principal.*

   (b) *The action of G on X is free, and its translation map is continuous at (x,x) for all $x \in X$.*

   (c) *The action of G on X is free, and for every point $x \in X$, and for every neighbourhood V of the identity element e of G, there exists a neighbourhood U of x in X, such that $P_G(U,U) \subset V$.*

   (d) *The map τ is a homeomorphism.*

Proof. (1) For all $(x,g) \in X \times G$, we have $\tau^{-1}(\tau(\mathbf{x},\mathbf{g})) = \{\mathbf{x}\} \times \mathbf{G_x g}$. Thus, if the action of G on X is free, then $G_x = \{e\}$, and $\tau^{-1}(\tau(x,g)) = \{(x,g)\}$ for all $(x,g) \in X \times G$, hence τ is injective. Conversely, if τ is injective, then, for all $x \in X$, we have $\{x\} \times G_x = \tau^{-1}(\tau(x,e)) = \{(x,e)\}$, hence $G_x = \{e\}$, so the action of G on X is free.

(2) (2a)⇔(2b): If the action of G on X is principal, then, by definition, it is free, and its translation map φ is continuous, and hence continuous at (x,x) for all $x \in X$. Conversely, suppose that the action of G on X is free, and that its translation map φ is continuous at (x,x) for all $x \in X$. Let $(x,y) \in R$, and let $\varphi(x,y) = g$. Then, every neighbourhood of g in G is of the form Wg, where W is a neighbourhood of e in G. Since φ is continuous at (x,x) and $\varphi(x,x) = e$, there exist open neighbourhoods U and V of x in X, such that $\varphi((U \times V) \cap R) \subset W$. Now, $U \times Vg$ is a neighbourhood of (x,y) in $X \times X$, so, to show that φ is continuous at (x,y), it suffices to prove that $\varphi((U \times Vg) \cap R) \subset Wg$. Let $(a,b) \in (U \times Vg) \cap R$. Then, there exists $c \in V$ such that $b = cg$. Thus, $a\varphi(a,b) = b = cg$, hence $c = a\varphi(a,b)g^{-1}$. Therefore, $(a,c) \in$

$(U \times V) \cap R$, and $\varphi(a,c) = \varphi(a,b)g^{-1}$. As $\varphi(a,c) \in W$, we get $\varphi(a,b) \in Wg$. This proves that $\varphi((U \times Vg) \cap R) \subset Wg$, hence φ is continuous at (x,y). Therefore, φ is continuous, that is, the action of G on X is principal.

(2b)⇔(2c): Suppose that the action of G on X is free, and let φ denote its translation map. Then, $\mathbf{P_G(A,B) = \varphi((A \times B) \cap R)}$ for all subsets A and B of X. To see this, let $g \in P_G(A,B)$. Then, there exist $a \in A$ and $b \in B$, such that $b = ag$. Thus, $(a,b) \in (A \times B) \cap R$, and $g = \varphi(a,b)$ belongs to $\varphi((A \times B) \cap R)$. Conversely, let $g \in \varphi((A \times B) \cap R)$. Then, there exist $a \in A$ and $b \in B$, such that $(a,b) \in R$, and $g = \varphi(a,b)$. Thus, $b = a\varphi(a,b) = ag$ belongs to $Ag \cap B$, hence $g \in P_G(A,B)$. This proves that $P_G(A,B) = \varphi((A \times B) \cap R)$ for all subsets A and B of X.

Now, suppose φ is continuous at (x,x) for every $x \in X$. Let $x \in X$, and let V be a neighbourhood of e in G. Then, there exists an open neighbourhood T of (x,x) in R, such that $\varphi(T) \subset V$. Let U be an open neighbourhood of x in X, such that $(U \times U) \cap R \subset T$. Then, $P_G(U,U) = \varphi((U \times U) \cap R) \subset \varphi(T) \subset V$. Conversely, suppose that for every point $x \in X$, and for every neighbourhood V of the identity element e of G, there exists a neighbourhood U of x in X, such that $P_G(U,U) \subset V$. Let $x \in X$, and let V be an open neighbourhood of $e = \varphi(x,x)$ in G. Then, by the hypothesis, there exists an open neighbourhood U of x in X, such that $P_G(U,U) \subset V$. The set $T = (U \times U) \cap R$ is an open neighbourhood of (x,x) in R, and $\varphi(T) = P_G(U,U) \subset V$. Therefore, φ is continuous at (x,x).

(2a)⇔(2d): By (1), the action of G on X is free if and only if τ is injective. Suppose that this is the case, and let φ be the translation map of the action. Then, since the action of G on X is continuous, and $R = \tau(X \times G)$, τ is a continuous bijection, and $\boldsymbol{\tau^{-1}(x,y) = (x, \varphi(x,y))}$ for all $(x,y) \in R$. Therefore, τ^{-1} is continuous if and only if φ is continuous. It follows that τ is a homeomorphism if and only if the action of G on X is principal.

Lemma 3.1.9 *Let G be a complex Lie group acting holomorphically on the right of a complex premanifold X, and let $x \in X$. Then, the stabiliser G_x of x is a complex Lie subgroup of G, and $\mathrm{Lie}(G_x)$ equals the set of all $\xi \in \mathrm{Lie}(G)$ such that $\xi^\sharp(x) = 0$. In particular, if $G_x = \{e\}$, $\xi \in \mathrm{Lie}(G)$, and $\xi^\sharp(x) = 0$, then $\xi = 0$.*

Proof. Let $\mu_x : G \to X$ denote the orbit map of x. Then, $\mu_x \circ \rho_g = \rho_g \circ \mu_x$, hence $\mathrm{rk}(\mathrm{T}_g(\mu_x)) = \mathrm{rk}(\mathrm{T}_e(\mu_x))$ for all $g \in G$. Thus, μ_x has constant rank. This implies that $\mu_x^{-1}(x) = G_x$ is a subpremanifold of G, and $\mathrm{T}_e(G_x) = \mathrm{Ker}(\mathrm{T}_e(\mu_x))$ ([8, 5.10.5 and 5.10.6] and [38, Proposition 5.39]). The multiplication and the inversion laws of G_x are induced by those of G, and hence are holomorphic. Therefore, G_x is a complex Lie subgroup of G, and $\mathrm{Lie}(G_x) = \mathrm{Ker}(\mathrm{T}_e(\mu_x))$. Now,

$$\mathrm{T}_e(\mu_x)(\xi) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} x\exp(t\xi) = \xi^\sharp(x)$$

for all $\xi \in \mathrm{Lie}(G)$. This proves the first assertion of the lemma. In particular, if $G_x = \{e\}$, then, since the holomorphic tangent space of a discrete complex premanifold at any point is zero, we have $\mathrm{Lie}(G_x) = 0$, hence the second assertion follows. □

Lemma 3.1.10 *Let X be a complex premanifold, and G a complex Lie group. Suppose that we are given a principal holomorphic right action of G on X. Let R be the equivalence relation on X defined by the action of G, and $\tau : X \times G \to R$ the map $(x,g) \mapsto (x,xg)$. Then, R is a complex subpremanifold of $X \times X$, and τ is a biholomorphism.*

Proof. Let $\sigma : X \times G \to X \times X$ be the map $(x,g) \mapsto (x,xg)$. Thus, $\sigma(X \times G) = R$, and $\tau : X \times G \to R$ is the map induced by σ. The map σ is obviously holomorphic. It is an immersion, since the action of G on X is free. Indeed, for any complex Lie group G, we define $\mathrm{Lie}(G)$ to be the holomorphic tangent space $\mathrm{T}_e(G)$ to G at e. It is canonically identified with the complex Lie subalgebra of $\mathrm{H}^0(G,\mathrm{T}(G))$ consisting of left invariant global sections of the holomorphic tangent bundle $\mathrm{T}(G)$ of G. For any $g \in G$, the diagram

$$\begin{array}{ccc} X \times G & \xrightarrow{\sigma} & X \times X \\ {\scriptstyle \mathbf{1}_X \times \rho_g}\downarrow & & \downarrow{\scriptstyle \mathbf{1}_X \times \rho_g} \\ X \times G & \xrightarrow[\sigma]{} & X \times X \end{array}$$

commutes. That is, for all $g \in G$, we have $\sigma \circ (\mathbf{1}_X \times \rho_g) = (\mathbf{1}_X \times \rho_g) \circ \sigma$, where ρ_g denotes the translation by g on every right G-space. Therefore, by the chain rule, for every $x \in X$, the diagram

$$\begin{array}{ccc} \mathrm{T}_x(X) \oplus \mathrm{Lie}(G) & \xrightarrow{\mathrm{T}_{(x,e)}(\sigma)} & \mathrm{T}_x(X) \oplus \mathrm{T}_x(X) \\ {\scriptstyle \mathbf{1}_{\mathrm{T}_x(X)} \oplus \mathrm{T}_e(\rho_g)}\downarrow & & \downarrow{\scriptstyle \mathbf{1}_{\mathrm{T}_x(X)} \oplus \mathrm{T}_x(\rho_g)} \\ \mathrm{T}_x(X) \oplus \mathrm{T}_g(G) & \xrightarrow[\mathrm{T}_{(x,g)}(\sigma)]{} & \mathrm{T}_x(X) \oplus \mathrm{T}_{xg}(X) \end{array}$$

commutes. Since both the vertical arrows are isomorphisms, the bottom arrow is injective if and only if the top arrow is injective. Therefore, to check that σ is an immersion at (x,g), it suffices to check that it is an immersion at (x,e). For that, we need to compute the differential $\mathrm{T}_{(x,e)}(\sigma)$ of σ at the point (x,e). Let $c : I \to X$ be a holomorphic map such that I is an open neighbourhood of 0 in $\mathbb{C}$, $c(0) = x$, and $\dot{c}(0) = v$. Then,

$$\begin{aligned} \mathrm{T}_{(x,e)}(\sigma)(v,\sigma) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \sigma(c(t), \exp(t\xi)) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (c(t), c(t)\exp(t\xi)) \\ &= \left(\dot{c}(0), \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mu(c(t), \exp(t\xi))\right) = (v, \mathrm{T}_{(x,e)}(\mu)(v,\xi)). \end{aligned}$$

Thus, For all $v \in \mathrm{T}_x(X)$ and $\xi \in \mathrm{Lie}(G)$, we have $\mathrm{T}_{(\mathbf{x},\mathbf{e})}(\sigma)(\mathbf{v},\xi) = (\mathbf{v}, \mathrm{T}_{(\mathbf{x},\mathbf{e})}(\mu)(\mathbf{v},\xi))$, where $\mu : X \times G \to X$ is the action. Therefore, if (v,ξ) belongs to the kernel of $\mathrm{T}_{(x,e)}(\sigma)$, then $v = 0$, and $\mathrm{T}_{(x,e)}(\mu)(0,\xi) = 0$. But, note that for the constant map

$c : \mathbb{C} \to X$ with value x, we have $\dot{c}(0) = 0$, hence

$$\mathrm{T}_{(x,e)}(\mu)(0,\xi) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mu(c(t), \exp(t\xi)) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} x\exp(t\xi).$$

On the other hand, the holomorphic vector field $\xi^\sharp$ on X induced by ξ is defined by

$$\xi^\sharp(y) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} y\exp(t\xi)$$

for all $y \in X$. It follows that $\mathrm{T}_{(\mathbf{x},\mathbf{e})}(\mu)(\mathbf{0},\xi) = \xi^\sharp(\mathbf{x})$. Therefore, $0 = \mathrm{T}_{(x,e)}(\mu)(0,\xi) = \xi^\sharp(x)$. Thus, $\xi^\sharp(x) = 0$. Since the action of G on X is free, from the Lemma 3.1.9, it follows that $\xi = 0$, and, hence $\mathrm{T}_{(x,e)}(\sigma)$ is injective. This proves that σ is an immersion.

Now, since the action of G on X is principal, by Remark 3.1.8, the map τ is a homeomorphism. By the previous paragraph, σ is an immersion. Therefore, σ is a holomorphic embedding, its image R is a complex subpremanifold of $X \times X$(from [8, 5.8.3] and [38, Theorem 5.34]), and τ is a biholomorphism. □

Definition 3.1.11 Let B, F be complex premanifolds. A *fiber bundle* over a premanifold B (the base premanifold) with fiber modeled on F is a complex premanifold M together with a holomorphic submersion $p\colon M \to B$ such that for every $a \in B$ there exists an open neighborhood U of a in B and a biholomorphism $\tau\colon p^{-1}(U) \to U \times F$ (trivialization) such that the following diagram commutes:

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\ \tau\ } & U \times F \\ & {\scriptstyle p}\searrow & \downarrow {\scriptstyle Pr_1} \\ & & U \end{array}$$

In particular, this means that every fiber $p^{-1}(x)$, for $x \in B$ is biholomorphic to the complex premanifold F.

Definition 3.1.12 Let G be a Lie group. A *principal fiber bundle* with structure group G over B, or a *principle-G bundle* is a fiber bundle $p\colon M \to B$ (with fiber modeled on G) together with a holomorphic right action of G on M such that for every $a \in B$ there exists an open neighborhood U of a in B and a trivialization $\tau\colon p^{-1}(U) \to U \times G$ in which the action becomes trivialized in the sense that the following diagram commutes, for all $g \in G$:

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\tau} & U \times G \\ {\scriptstyle\rho_g}\downarrow & & \downarrow{\scriptstyle \mathbf{1}_U \times r_g} \\ p^{-1}(U) & \xrightarrow[\tau]{} & U \times G \end{array}$$

Here ρ_g denotes the right action $M \to M$, $(x \mapsto xg)$ and r_g denotes the right multiplication $G \to G$, $(h \mapsto hg)$.

Lemma 3.1.13 *Let $p : X \to Y$ be a surjective holomorphic submersion of complex premanifolds, and G a complex Lie group. Suppose that we are given a principal holomorphic right action of G on X, such that $p^{-1}(p(x)) = xG$ for all $x \in X$. Then, this action makes p a holomorphic principal G-bundle.*

Proof. Let R be the equivalence relation on X defined by the action of G, and τ : $X \times G \to R$ the map $(x,g) \mapsto (x,xg)$. Then, by Lemma 3.1.10, R is a complex subpremanifold of $X \times X$, and τ is a biholomorphism. Let $y \in Y$. As p is surjective, there exists a point $a \in p^{-1}(y)$. Since p is a submersion at a, there exist an open neighbourhood V of y in Y, and a holomorphic section $s : V \to X$ of p, such that $s(y) = a$. The hypothesis on the fibres of p implies that the map $(c,g) \mapsto s(c)g$ is a G-equivariant holomorphic bijection u from $V \times G$ onto $p^{-1}(V)$. Its inverse is the composite

$$p^{-1}(V) \xrightarrow{\alpha} (p^{-1}(V) \times p^{-1}(V)) \cap R \xrightarrow{\beta} p^{-1}(V) \times G \xrightarrow{\gamma} V \times G,$$

where

$$\alpha(x) = (x, s(p(x))), \quad \beta(y,z) = \tau^{-1}(y,z), \quad \gamma(x,g) = (p(x), g)$$

for all $x \in p^{-1}(V)$, $(y,z) \in (p^{-1}(V) \times p^{-1}(V)) \cap R$, and $g \in G$. Since τ^{-1} is holomorphic, u^{-1} is also holomorphic. By definition, $p(u(c,g)) = c$ for all $c \in V$ and $g \in G$. Thus, u is a local trivialisation of p at y. It follows that p is a holomorphic principal G-bundle. □

Remark 3.1.14 The proof of Lemma 3.1.13 also works to show that if $p : X \to Y$ is a surjective smooth submersion of smooth premanifolds, and G a real Lie group, and if we are given a principal smooth right action of G on X, such that $p^{-1}(p(x)) = xG$ for all $x \in X$, then this action makes p a smooth principal G-bundle.

Proposition 3.1.15 *Let X be a complex premanifold, and G a complex Lie group. Suppose that we are given a principal holomorphic right action of G on X. Let Y be the quotient topological space X/G, and $p : X \to Y$ the canonical projection. Then, there exists a unique structure of a complex premanifold on Y such that p is a holomorphic submersion. This structure makes p a holomorphic principal G-bundle.*

Proof. Let R be the equivalence relation on X defined by the action of G, and $\tau : X \times G \to R$ the map $(x,g) \mapsto (x,xg)$. Then, by Lemma 3.1.10, R is a complex subpremanifold of $X \times X$, and τ is a biholomorphism. Since $\mathrm{pr}_1 \circ \tau = \mathrm{pr}_1$, and $\mathrm{pr}_1 : X \times G \to X$ is clearly a submersion, it follows that $\mathrm{pr}_1 : R \to X$ is a submersion. Therefore, by Godement's theorem, there exists a unique structure of a complex premanifold on Y, such that p is a holomorphic submersion. It is obvious that p is surjective, and that $p^{-1}(p(x)) = xG$ for all $x \in X$. Therefore, by Lemma 3.1.13, p is a holomorphic principal G-bundle. □

There is a partial converse for Proposition 3.1.15, as follows.

Proposition 3.1.16 *Suppose that G is second countable, X is Hausdorff, the action of G on X is free, and there exists a structure of a complex premanifold on Y such that p is a holomorphic submersion. Then, the action of G on X is principal.*

Proof. Let R be the equivalence relation on X defined by the action of G, $\sigma : X \times G \to X \times X$ the map $(x,g) \mapsto (x,xg)$, and $\sigma : X \times G \to R$ the map induced by σ. By Godement's theorem, R is a complex presubmanifold of $X \times X$. Since the action of G on X is free, by Remark 3.1.8, and the first paragraph in the proof of Proposition 3.1.15, σ is an injective immersion.

Now, as p is a submersion, the pair (p,p) is transversal, hence its fiber product R is a subpremanifold of $X \times X$, and, for all $(x,y) \in R$, $\mathrm{T}_{(x,y)}(R)$ is the fibre product of $\mathrm{T}_x(X)$ and $\mathrm{T}_y(X)$ over $\mathrm{T}_{p(x)}(Y)$ ([8, 5.11.2] and [38, Theorem 5.47]); moreover, if g is an element of G such that $y = xg$, then the translation $\rho_g : X \to X$ by g takes x to y, hence $\dim_x(X) = \dim_y(Y)$, and $\dim_{(x,y)}(R) = 2\dim_x(X) - \dim_{p(x)}(Y)$.

Next, since p is a holomorphic submersion, for every $x \in X$, the orbit $xG = p^{-1}(p(x))$ is a subpremanifold of X, and $\mathrm{T}_x(xG) = \mathrm{Ker}(\mathrm{T}_x(p))$ ([8, 5.10.5] and [38, Corollary 5.40]). Every premanifold is a Baire space, so xG is a Baire space. Since X is Hausdorff, xG is also Hausdorff. As G is a second-countable manifold, it is σ-compact, that is a locally compact space which has a countable compact cover. Now, by [7, Chapter IX, § 5, no. 3, Proposition 6], if a σ-compact topological group H acts continuously and transitively on the right of a Hausdorff Baire space Z, then the orbit map $h \mapsto zh : G \to Z$ of every point z of Z is an open map. Therefore, the orbit map $\mu_x : G \to X$ induces a homeomorphism from G onto xG. Also, since $G_x = \{e\}$, μ_x is a holomorphic immersion ([8, 5.12.5]). Therefore, it is a holomorphic embedding, and hence induces a biholomorphism from G onto xG. We thus get an exact sequence

$$0 \to \mathrm{Lie}(G) \xrightarrow{\mathrm{T}_e(\mu_x)} \mathrm{T}_x(X) \xrightarrow{\mathrm{T}_x(p)} \mathrm{T}_{p(x)}(Y) \to 0$$

of finite-dimensional $\mathbb{C}$-vector spaces. This implies that

$$\dim_x(X) = \dim_{p(x)}(Y) + \dim(G).$$

Therefore,

$$\dim_{\tau(x,g)}(R) = 2\dim_x(X) - \dim_{p(x)}(Y) = \dim_x(X) + \dim(G) = \dim_{(x,g)}(X \times G)$$

for all $(x,g) \in X \times G$. As σ is an injective holomorphic immersion whose image equals R, τ is a bijective holomorphic immersion. The above equality implies that it is a local biholomorphism. It follows that τ is a biholomorphism. In particular, it is a homeomorphism. Therefore, by Remark 3.1.8, the action of G on X is principal. □

Let G be a complex Lie group acting holomorphically on the right of a complex premanifold X, Y the quotient topological space X/G, and $p : X \to Y$ the canonical projection. Let H be a normal complex Lie subgroup of G, $\overline{G}$ the complex Lie group $H\backslash G$, and $\pi : G \to \overline{G}$ the canonical projection. If the stabiliser G_x of any point $x \in X$ equals H, then there is an induced holomorphic right action of $\overline{G}$ on X which is defined as follows: $xu = xg$ for all $x \in X$ and $u \in \overline{G}$, where g is any element of G such that $u = \pi(g)$. Since $xh = x$ for all $x \in X$ and $h \in H$, this action is well-defined.

The quotient topological group $\overline{G}$ has a unique structure of a complex premanifold such that the canonical projection from G to $\overline{G}$ is a holomorphic submersion. With this structure, $\overline{G}$ is a complex Lie group, π is a homomorphism of complex Lie groups, and the kernel of $\mathrm{T}_e(\pi) : \mathrm{Lie}(G) \to \mathrm{Lie}(\overline{G})$ equals $\mathrm{Lie}(H)$ ([33, Part II, Chapter IV, § 5, Remark 2, p. 108] and [6, Chapter III, § 1, no. 6, Proposition 11]). The induced right action of $\overline{G}$ on X is holomorphic by [8, 5.9.6], or [6, Chapter III, § 1, no. 6, Proposition 13]. Alternatively, we have a commutative diagram

$$\begin{array}{ccc} X \times G & & \\ {\scriptstyle \mathbf{1}_X \times \pi}\downarrow & \overset{\mu}{\searrow} & \\ X \times \overline{G} & \xrightarrow[\overline{\mu}]{} & X \end{array}$$

where μ and $\overline{\mu}$ are the action maps. Since $\mathbf{1}_X \times \pi$ is a surjective submersion, and μ is holomorphic, this implies that μ' is also holomorphic.

In the following Corollary, by an H-*invariant* subset of G, we mean a subset of G that is invariant under the canonical left action of H on G, that is, a subset V of G, such that $HV = V$, where HV is the set of all elements of G of the form hx, with $h \in H$ and $x \in V$. Note that if $HV \subset V$, then $HV = V$, since $e \in H$.

Remark 3.1.17 For all subsets A and B of X, we have

$$P_{\overline{G}}(A,B) = \pi(P_G(A,B)).$$

To see this, let $u \in P_{\overline{G}}(A,B)$. Then, there exists $a \in A$, $b \in B$ such that $b = au$. But, since $u \in \overline{G}$, there is a $g \in G$ such that $\pi(g) = u$. Now, we have $b = au = a\pi(g) = ag$. It follows that $u = \pi(g) \in \pi(P_G(A,B))$.

Corollary 3.1.18 *Suppose that the stabiliser G_x of any point $x \in X$ equals H, and that for each $x \in X$, and H-invariant neighbourhood V of e in G, there exists a neighbourhood U of x in X, such that $P_G(U,U) \subset V$. Then, the action of $\overline{G}$ on X is principal, and there exists a unique structure of a complex premanifold on Y, such that p is a holomorphic submersion. Moreover, with the induced action of $\overline{G}$ on X, p is a holomorphic principal $\overline{G}$-bundle.*

Proof. The induced action of $\overline{G}$ on X is free, since $G_x = H$ for all $x \in X$. Let $x \in X$, and let W be a neighbourhood of e in $\overline{G}$. Then, $V = \pi^{-1}(W)$ is an H-invariant neighbourhood of e in G. Therefore, by hypothesis, there exists an open neighbourhood U of x in X, such that $P_G(U,U) \subset V$. Now, $U = U \cap X$ is an open neighbourhood of x in X, and $P_{\overline{G}}(U,U) \subset \pi(P_G(U,U)) \subset \pi(P_G(U,U)) \subset \pi(V) \subset W$. Therefore, by Remark 3.1.8, the action of $\overline{G}$ on X is principal. It is obvious that the induced action of $\overline{G}$ on X is holomorphic, that $X/\overline{G} = X/G = Y$, and that the canonical projection from X to $X/\overline{G}$ equals p. The Corollary now follows from Proposition 3.1.15. □

3.1.2 The complex premanifold of Schur representations

Let Q be a non-empty finite quiver. We will consider only complex representations of Q in this subsection. Let $d = (d_a)_{a\in Q_0}$ be a non-zero element of $\mathbb{N}^{Q_0}$, and fix a family $V = (V_a)_{a\in Q_0}$ of $\mathbb{C}$-vector spaces, such that $\dim_{\mathbb{C}}(V_a) = d_a$ for all $a \in Q_0$.

Denote by $\mathscr{A}$ the finite-dimensional $\mathbb{C}$-vector space $\bigoplus_{\alpha\in Q_1} \mathrm{Hom}_{\mathbb{C}}(V_{\mathrm{s}(\alpha)}, V_{\mathrm{t}(\alpha)})$. For every element ρ of $\mathscr{A}$, we have a representation (V,ρ) of Q. Moreover, for every representation (W,σ) of Q, such that $\dim(W,\sigma) = d$, there exists an element ρ of $\mathscr{A}$, such that the representations (V,ρ) and (W,σ) are isomorphic.

We give the vector space $\mathscr{A}$ the usual topology, and the usual structure of a complex manifold. That is, the usual structure of a complex manifold on the $\mathbb{C}$-vector space $\mathscr{A}$ is the unique structure of a complex manifold on $\mathscr{A}$ which makes all $\mathbb{C}$-linear functions on $\mathscr{A}$ holomorphic.

For each $a \in Q_0$, denote by E_a the trivial holomorphic vector bundle $\mathscr{A} \times V_a$ on $\mathscr{A}$. Then, for every $\alpha \in Q_1$, we have a morphism $\theta_\alpha : E_{\mathrm{s}(\alpha)} \to E_{\mathrm{t}(\alpha)}$ of holomorphic vector bundles, which is defined by $\theta_\alpha(\rho, v) = (\rho, \rho_\alpha(v))$ for all (ρ, v) in $E_{\mathrm{s}(\alpha)}$. We thus get a family (E,θ) of representations of Q parametrised by $\mathscr{A}$, where $E = (E_a)_{a\in Q_0}$ and $\theta = (\theta_\alpha)_{\alpha\in Q_1}$. By definition, for each point $\rho \in \mathscr{A}$, the fibre representation $E(\rho)$ is precisely (V,ρ).

Let G be the complex Lie group $\prod_{a\in Q_0}\mathrm{Aut}_{\mathbb{C}}(V_a)$. There is a canonical holomorphic linear right action $(\rho,g)\mapsto \rho g$ of G on $\mathscr{A}$, which is defined by

$$(\rho\mathbf{g})_\alpha = \mathbf{g}_{\mathrm{t}(\alpha)}^{-\mathbf{1}}\circ\rho_\alpha\circ\mathbf{g}_{\mathrm{s}(\alpha)}$$

for all $\rho\in\mathscr{A}$, $g\in G$, and $\alpha\in Q_1$. For all $\rho,\sigma\in\mathscr{A}$ and $g\in G$, we have $\sigma=\rho g$ if and only if g is an isomorphism of representations of Q, from (V,σ) to (V,ρ). In other words, two points ρ and σ of $\mathscr{A}$ lie on the same orbit of G if and only if the representations (V,ρ) and (V,σ) of Q are isomorphic. Thus, the map which takes every point ρ of $\mathscr{A}$ to the representation (V,ρ) induces a bijection from the quotient set $\mathscr{A}/\mathbf{G}$ onto the set of **isomorphism classes of representations** $(\mathbf{W},\sigma)$ of Q, such that $\dim(\mathbf{W},\sigma)=\mathbf{d}$.

Denote by H the central complex Lie subgroup of G consisting of all elements of the form ce, as c runs over $\mathbb{C}^\times$, where $e=(\mathbf{1}_{V_a})_{a\in Q_0}$ is the identity element of G. Let $\overline{G}$ denote the complex Lie group $H\backslash G$, $\pi: G\to\overline{G}$ the canonical projection. Define $\mathscr{B}$ to be the set of all points ρ of $\mathscr{A}$, such that the representation (V,ρ) of Q is Schur. It is a G-invariant subset of $\mathscr{A}$. By Proposition 2.1.7(4), a point ρ of $\mathscr{A}$ lies in $\mathscr{B}$ if and only if its stabiliser G_ρ equals H. Proposition 2.4.21, applied to the family (E,θ) of representations of Q parametrised by $\mathscr{A}$, implies that $\mathscr{B}$ is Zariski open in $\mathscr{A}$, and is hence an open complex submanifold of $\mathscr{A}$. Let M denote the quotient topological space $\mathscr{B}/G$, and $p:\mathscr{B}\to M$ the canonical projection. By the above observation, there is a canonical bijection from M onto the set of isomorphism classes of Schur representations (W,σ) of Q, such that $\dim(W,\sigma)=d$. We will call M the *moduli space* of Schur representations of Q with dimension vector d. Note that the action of G on $\mathscr{A}$ induces a holomorphic right action of $\overline{G}$ on $\mathscr{B}$.

The Lie algebra $\mathrm{Lie}(G)$ of G is the direct sum Lie algebra $\bigoplus_{a\in Q_0}\mathrm{End}_{\mathbb{C}}(V_a)$, where, for each $a \in Q_0$, the associative $\mathbb{C}$-algebra $\mathrm{End}_{\mathbb{C}}(V_a)$ is given its usual Lie algebra structure. Note that $\mathrm{Lie}(G)$ has a canonical structure of an associative unital $\mathbb{C}$-algebra, and that G is the group of units of the underlying ring of $\mathrm{Lie}(G)$, and is open in $\mathrm{Lie}(G)$. The Lie algebra of H is the Lie subalgebra of $\mathrm{Lie}(G)$ consisting of all elements of the form ce, as c runs over $\mathbb{C}$. To see this, Let $f : \mathbb{C} \to \mathrm{Lie}(G)$ be the $\mathbb{C}$-linear map $t \mapsto te$. Then, as d is a non-zero element of $\mathbb{N}^{Q_0}$, we have $e \neq 0$, hence f is injective. Therefore, it is an isomorphism onto the $\mathbb{C}$-subspace $\mathbb{C}e$ of $\mathrm{Lie}(G)$. Since $\mathbb{C}^\times$ and H are open subsets of $\mathbb{C}$ and $\mathbb{C}e$, respectively, f induces a biholomorphism $g : \mathbb{C}^\times \to H$. Therefore,

$$\mathrm{Lie}(H) = \mathrm{T}_e(H) = \mathrm{T}_{g(1)}(H) = \mathrm{T}_1(g)(\mathbb{C}).$$

But, as f is $\mathbb{C}$-linear, $\mathrm{T}_1(g) = f : \mathbb{C} \to \mathrm{Lie}(G)$, hence $\mathrm{Lie}(H) = f(\mathbb{C}) = \mathbb{C}e$.

Let $\mathrm{Tr} : \mathrm{Lie}(G) \to \mathbb{C}$ be the $\mathbb{C}$-linear function defined by $\mathrm{Tr}(\xi) = \sum_{a \in Q_0} \mathrm{Tr}(\xi_a)$ for all elements $\xi = (\xi_a)_{a \in Q_0}$ of $\mathrm{Lie}(G)$, and let $\mathrm{Lie}(G)^0$ denote its kernel. Then, as $d \neq 0$, $\mathrm{Tr}(e) = \mathrm{rk}(d) = \sum_{a \in Q_0} d_a$ is a non-zero natural number, and we have a decomposition $\mathrm{Lie}(G) = \mathrm{Lie}(H) \oplus \mathrm{Lie}(G)^0$. To see this, the $\mathbb{C}$-linearity of Tr implies that

$$\mathrm{Tr}(\xi^0) = \mathrm{Tr}(\xi) - c(\xi)\mathrm{Tr}(e) = \mathrm{Tr}(\xi) - c(\xi)\mathrm{rk}(d) = \mathrm{Tr}(\xi) - \mathrm{Tr}(\xi) = 0,$$

hence $\xi^0 \in \mathrm{Lie}(G)^0$. Thus, $\mathrm{Lie}(G) = \mathrm{Lie}(H) + \mathrm{Lie}(G)^0$. If $c \in \mathbb{C}$ and $ce \in \mathrm{Lie}(G)^0$, then

$$c\mathrm{rk}(d) = c\mathrm{Tr}(e) = \mathrm{Tr}(ce) = 0,$$

hence, as $\mathrm{rk}(d) \neq 0$, we have $c = 0$. Thus, $\mathrm{Lie}(H) \cap \mathrm{Lie}(G)^0 = 0$. It follows that $\mathrm{Lie}(G) = \mathrm{Lie}(H) \oplus \mathrm{Lie}(G)^0$.

As d is a non-zero element of $\mathbb{N}^{Q_0}$, we have $e \neq 0$, hence the map $t \mapsto te : \mathbb{C} \to \mathrm{Lie}(H)$ is a $\mathbb{C}$-isomorphism. For any element ξ of $\mathrm{Lie}(G)$, we define $(c(\xi), \xi^0)$ to be the unique element of $\mathbb{C} \times \mathrm{Lie}(G)^0$, such that $\xi = c(\xi)e + \xi^0$. Then, $c(\xi) = \frac{\mathrm{Tr}(\xi)}{\mathrm{rk}(d)}$, and

$\xi^0 = \xi - c(\xi)e$ for all $\xi \in \mathrm{Lie}(G)$.

For every element ρ of $\mathscr{A}$, we denote the *orbit map* $g \mapsto \rho g : G \to \mathscr{A}$ by μ_ρ, and by D_ρ the $\mathbb{C}$-linear map $\mathrm{T}_e(\mu_\rho) : \mathrm{Lie}(G) \to \mathscr{A}$. Thus,

$$D_\rho(\xi) = (\rho_\alpha \circ \xi_{\mathrm{s}(\alpha)} - \xi_{\mathrm{t}(\alpha)} \circ \rho_\alpha)_{\alpha \in Q_1}$$

for all $\xi \in \mathrm{Lie}(G)$. To see this, Let $\xi \in \mathrm{Lie}(G)$. Then,

$$\begin{aligned} D_\rho(\xi) = \mathrm{T}_e(\mu_\rho)(\xi) &= \frac{\mathrm{d}}{\mathrm{d}h}\Big|_{h=0} \mu_\rho(\exp(h\xi)) \\ &= \left(\frac{\mathrm{d}}{\mathrm{d}h}\Big|_{h=0} \big(\exp(-h\xi_{\mathrm{t}(\alpha)}) \circ \rho_\alpha \circ \exp(h\xi_{\mathrm{s}(\alpha)})\big)\right)_{\alpha \in Q_1} \\ &= \big(-\xi_{\mathrm{t}(\alpha)} \circ \rho_\alpha \circ \mathbf{1}_{V_{\mathrm{s}(\alpha)}} + \mathbf{1}_{V_{\mathrm{t}(\alpha)}} \circ \rho_\alpha \circ \xi_{\mathrm{s}(\alpha)}\big)_{\alpha \in Q_1} \\ &= \big(\rho_\alpha \circ \xi_{\mathrm{s}(\alpha)} - \xi_{\mathrm{t}(\alpha)} \circ \rho_\alpha\big)_{\alpha \in Q_1}. \end{aligned}$$

So, $\mathrm{Ker}(D_\rho) = \mathrm{End}(V, \rho)$. In particular, $\mathrm{Ker}(D_\rho) = \mathrm{Lie}(H)$ if $\rho \in \mathscr{B}$.

It will be convenient to fix a family $h = (h_a)_{a \in Q_0}$ of Hermitian inner products $h_a : V_a \times V_a \to \mathbb{C}$. Thus, for every point $\rho \in \mathscr{A}$, h is a Hermitian metric on the representation (V, ρ) of Q.

We need some linear algebra concepts which we use in the proof of the Theorem 3.1.22 below. For any two finite-dimensional Hermitian inner product spaces V and W, we have a Hermitian inner product $\langle \cdot, \cdot \rangle$ on the $\mathbb{C}$-vector space $\mathrm{Hom}_{\mathbb{C}}(V, W)$, which is defined by $\langle u, v \rangle = \mathrm{Tr}(u \circ v^*)$ for all $u, v \in \mathrm{Hom}_{\mathbb{C}}(V, W)$, where, $v^* : W \to V$ is the adjoint of u. We denote the norm associated to this Hermitian inner product by $\|\cdot\|$.

Proposition 3.1.19 *1.* $\|u(x)\| \leq \|u\|\|x\|$ *for all* $u \in \mathrm{Hom}_{\mathbb{C}}(V, W)$ *and* $x \in V$

2. $\|u^*\| = \|u\|$, *and* $\|\mathbf{1}_V\| = \sqrt{\dim_{\mathbb{C}}(V)}$

3. For all finite-dimensional Hermitian inner product spaces V, W, *and* X, *and for* *all* $u \in \mathrm{Hom}_{\mathbb{C}}(V,W)$ *and* $v \in \mathrm{Hom}_{\mathbb{C}}(W,X)$*, we have* $\|v \circ u\| \leq \|v\|\|u\|$.

Proof. (1) Let V and W be two finite-dimensional Hermitian inner product spaces. Then, we have an inner product $\langle \cdot,\cdot \rangle$ on $\mathrm{Hom}_{\mathbb{C}}(V,W)$, which is defined by $\langle u,v \rangle = \mathrm{Tr}(uv^*)$ for all $u,v \in \mathrm{Hom}_{\mathbb{C}}(V,W)$. If $B = (e_j)_{j=1}^n$ and $C = (f_i)_{i=1}^m$ are orthonormal $\mathbb{C}$-bases of V and W, respectively, then $\langle u,v \rangle = \sum_{i=1}^m \sum_{j=1}^n a_{ij}\overline{b_{ij}}$, where $(a_{ij})_{1\leq i\leq m, 1\leq j\leq n}$ and $(b_{ij})_{1\leq i\leq m, 1\leq j\leq n}$ are, respectively, the matrices of u and v with respect to B and C. Therefore, $\|u\|^2 = \sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2$. Thus, if $x = \sum_{j=1}^n x_j e_j$ is any element of V, then, by the Cauchy-Schwartz inequality, for every $i = 1,\dots,m$, we have

$$|\sum_{j=1}^n a_{ij}x_j|^2 \leq \sum_{j=1}^n |a_{ij}^2| \sum_{j=1}^n |x_j|^2 = \|x\|^2 \sum_{j=1}^n |a_{ij}^2|,$$

hence

$$\begin{aligned}\|u(x)\|^2 &= \|\sum_{j=1}^n x_j u(e_j)\|^2 = \|\sum_{i=1}^m \sum_{j=1}^n x_j a_{ij} f_i\|^2 = \|\sum_{i=1}^m \Big(\sum_{j=1}^n a_{ij}x_j\Big) f_i\|^2 \\ &= \sum_{i=1}^m |\sum_{j=1}^n a_{ij}x_j|^2 \leq \sum_{i=1}^m \Big(\|x\|^2 \sum_{j=1}^n |a_{ij}|^2\Big) = \|x\|^2 \sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2 \\ &= \|x\|^2 \|u\|^2.\end{aligned}$$

Therefore, $\|u(x)\| \leq \|u\|\|x\|$.

(2) The matrix $(a_{ji}^*)_{1\leq j\leq n, 1\leq i\leq m}$ of $u^* : W \to V$, with respect to the above bases of V and W, is given by $a_{ji}^* = \overline{a_{ij}}$. Therefore,

$$\|u^*\| = \sum_{j=1}^n \sum_{i=1}^m |a_{ji}^*|^2 = \sum_{j=1}^n \sum_{i=1}^m |\overline{a_{ij}}|^2 = \sum_{j=1}^n \sum_{i=1}^m |a_{ij}|^2 = \sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2,$$

hence $\|u^*\| = \|u\|$.

If $W = V$, then

$$\|\mathbf{1}_V\|^2 = \sum_{i=1}^n \sum_{j=1}^n \delta_{ij}^2 = n = \dim_{\mathbb{C}}(V),$$

where δ is the Kronecker delta, hence $\|\mathbf{1}_V\| = \sqrt{\dim_{\mathbb{C}}(V)}$.

3 Suppose $u : V \to W$ and $v : W \to X$ are two $\mathbb{C}$-linear maps of finite-dimensional Hermitian vector spaces. Let $B = (e_j)_{j=1}^n$, $C = (f_i)_{i=1}^m$, and $D = (g_h)_{h=1}^l$, be orthonormal $\mathbb{C}$-bases of V and W, respectively, and let $(a_{ij})_{1 \le i \le m, 1 \le j \le n}$ $(b_{hi})_{1 \le h \le l, 1 \le i \le m}$, $(c_{hj})_{1 \le h \le l, 1 \le j \le n}$, respectively, be the matrices of u, v, and $v \circ u$, with respect to these bases. Then, by the Cauchy-Schwartz inequality, for all $h = 1, \ldots, l$ and $j = 1, \ldots, n$, we have

$$|c_{hj}|^2 = |\sum_{i=1}^m b_{hi} a_{ij}|^2 \le \Big(\sum_{i=1}^m |b_{hi}|^2\Big)\Big(\sum_{i=1}^m |a_{ij}|^2\Big).$$

Therefore,

$$\begin{aligned} \|v \circ u\|^2 &= \sum_{h=1}^l \sum_{j=1}^n |c_{hj}|^2 \le \sum_{h=1}^l \sum_{j=1}^n \Big(\sum_{i=1}^m |b_{hi}|^2 \sum_{i=1}^m |a_{ij}|^2\Big) \\ &= \Big(\sum_{h=1}^l \sum_{i=1}^m |b_{hi}|^2\Big)\Big(\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2\Big) = \|v\|^2 \|u\|^2. \end{aligned}$$

It follows that $\|v \circ u\| \le \|v\| \|u\|$.

Remark 3.1.20 Using the above facts, it is easy to verify that for every $u \in \mathrm{Hom}_{\mathbb{C}}(V, W)$, there exists a real number $\theta > 0$, such that $\theta \|x\| \le \|u(x)\|$ for all $x \in \mathrm{Ker}(u)^\perp$, where $X^\perp$ denotes the orthogonal complement of any subset X of a finite-dimensional Hermitian inner product space.

Lastly, here is a proof of Remark 3.1.20.

Lemma 3.1.21 *Let $u : V \to W$ be a $\mathbb{C}$-linear map of finite-dimensional Hermitian inner product spaces. Then, there exists a real number $\theta > 0$, such that $\theta \|x\| \le \|u(x)\|$ for all $x \in \mathrm{Ker}(u)^\perp$.*

Proof. Let $H = \mathrm{Ker}(u)$, $I = u(V)$, and $u' : H^\perp \to I$ the $\mathbb{C}$-linear map induced by u. If $x \in \mathrm{Ker}(u')$, then x belongs to the domain $H^\perp$ of u', and $u(x) = u'(x) = 0$, hence

$x \in \operatorname{Ker}(u) = H$ also, so $x = 0$. Therefore, u' is injective. For any $y \in I$, since $u(V) = I$, there exists $x \in V$, such that $u(x) = y$. As $V = H + H^{\perp}$, we have $x = x_1 + x_2$, where $x_1 \in H$ and $x_2 \in H^{\perp}$. Thus, $y = u(x) = u(x_2) = u'(x_2)$, since $x_1 \in H = \operatorname{Ker}(u)$. It follows that u' is surjective, and is hence a $\mathbb{C}$-isomorphism.

Let $v' : I \to H^{\perp}$ be the inverse of u', that is, $v' \circ u' = \mathbf{1}_{H^{\perp}}$ and $u' \circ v' = \mathbf{1}_I$. If we give $H^{\perp}$ and I the Hermitian inner products induced from V and W, respectively, then, by one of the above paragraphs, we have $\|v'(y)\| \leq \|v'\|\|y\|$ for all $y \in I$. For all $x \in H^{\perp}$, we have $x = v'(u'(x)) = v'(u(x))$; since $u(x) \in I$, by the previous inequality, we get $\|x\| = \|v'(u(x))\| \leq \|v'\|\|u(x)\|$. Now, since $\lim_{t\to 0, t\in\mathbb{R}}(t\|v'\|) = 0$, there exists a real number θ, such that $\theta > 0$, and $\theta\|v'\| \leq 1$. The previous inequality implies that $\theta\|x\| \leq \|u(x)\|$ for all $x \in H^{\perp}$. □

In particular, the family h induces a Hermitian inner product $\langle\cdot,\cdot\rangle$ on the $\mathbb{C}$-vector space $\operatorname{Hom}_{\mathbb{C}}(V_a, V_b)$ for all $a, b \in Q_0$. We give $\operatorname{Lie}(G)$ the Hermitian inner product $\langle\cdot,\cdot\rangle$ which is the direct sum of the Hermitian inner products $\langle\cdot,\cdot\rangle$ on $\operatorname{End}_{\mathbb{C}}(V_a)$ as a runs over Q_0. Note that $\|e\| = \sqrt{\operatorname{rk}(d)}$ with respect to this Hermitian inner product, and that $\operatorname{Lie}(H)^{\perp} = \operatorname{Lie}(G)^0$. To see this, for all $\xi \in \operatorname{Lie}(G)^0$ and $c \in \mathbb{C}$, we have

$$\langle \xi, ce\rangle = \overline{c}\langle \xi, e\rangle = \overline{c}\sum_{a\in Q_0}\langle \xi_a, \mathbf{1}_{V_a}\rangle = \overline{c}\sum_{a\in Q_0}\operatorname{Tr}(\xi_a) = \overline{c}\operatorname{Tr}(\xi) = 0.$$

Therefore, $\operatorname{Lie}(G)^0 \subset \operatorname{Lie}(H)^{\perp}$. Conversely, let $\xi \in \operatorname{Lie}(H)^{\perp}$. Then, $\xi^0 \in \operatorname{Lie}(G)^0 \subset \operatorname{Lie}(H)^{\perp}$, so $c(\xi)e = \xi - \xi^0$ belongs to $\operatorname{Lie}(H) \cap \operatorname{Lie}(H)^{\perp}$, and is, hence, zero. It follows that $\xi = \xi^0 \in \operatorname{Lie}(G)^0$, that is, $\operatorname{Lie}(H)^{\perp} \subset \operatorname{Lie}(G)^0$. This proves that $\operatorname{Lie}(G)^0 = \operatorname{Lie}(H)^{\perp}$. Let $u : \operatorname{Lie}(G) \to \operatorname{Lie}(H)$ be the corresponding orthogonal projection.

Similarly, we give $\mathscr{A}$ the Hermitian inner product $\langle\cdot,\cdot\rangle$ which is the direct sum of the Hermitian inner products $\langle\cdot,\cdot\rangle$ on $\operatorname{Hom}_{\mathbb{C}}(V_{\mathrm{s}(\alpha)}, V_{\mathrm{t}(\alpha)})$ as α runs over Q_1. For every $\rho \in \mathscr{A}$, we have the adjoint $D_\rho^* : \mathscr{A} \to \operatorname{Lie}(G)$ of the $\mathbb{C}$-linear map $D_\rho : \operatorname{Lie}(G) \to \mathscr{A}$ which was defined above.

Theorem 3.1.22 *The action of $\overline{G}$ on $\mathscr{B}$ is principal. In particular, there exists a unique structure of a complex premanifold on the moduli space M of complex Schur representations of Q with dimension vector d, such that $p : \mathscr{B} \to M$ is a holomorphic submersion. Moreover, the topological space M is second-countable, and this complex premanifold structure makes p a holomorphic principal $\overline{G}$-bundle.*

Proof. Let ρ be an arbitrary point of $\mathscr{B}$. Then, as observed above, $\mathscr{B}$ is a G-invariant open complex submanifold of $\mathscr{A}$, so there is an induced holomorphic right action of G on $\mathscr{B}$. Also, $G_\rho = H$ for all $\rho \in \mathscr{B}$. Therefore, by Corollary 3.1.18, it suffices to prove that for every H-invariant neighbourhood V of e in G, there exists an open neighbourhood U of ρ in $\mathscr{B}$, such that $P_G(U,U) \subset V$.

Let $D_\rho : \mathrm{Lie}(G) \to \mathscr{A}$ be the $\mathbb{C}$-linear map defined earlier. Then, as noted above, $\mathrm{Ker}(D_\rho) = \mathrm{Lie}(H)$. Therefore, $\mathrm{Ker}(D_\rho)^\perp = \mathrm{Lie}(H)^\perp = \mathrm{Lie}(G)^0$, hence, by Remark 3.1.20, there exists a real number $\theta > 0$, such that $\theta\|f\| \leq \|D_\rho(f)\|$ for all $f \in \mathrm{Lie}(G)^0$. Let $q_1 = \mathrm{card}(Q_1)$. Then, the continuity of the norm function on $\mathscr{A}$ implies that the set X of all $(\sigma,\tau) \in \mathscr{A} \times \mathscr{A}$, such that $q_1(\|\sigma-\rho\| + \|\tau-\rho\|) < \theta$, is an open neighbourhood of (ρ,ρ) in $\mathscr{A} \times \mathscr{A}$.

Consider a point $(\sigma,\tau) \in X$, let $g \in G$, and suppose $\tau = \sigma g$. Then, by the above paragraph,

$$\theta\|g^0\| \leq \|D_\rho(g^0)\|.$$

Let $\sigma' = \sigma - \rho$ and $\tau' = \tau - \rho$. The relation $\tau = \sigma g$ implies that

$$D_\rho(g) = \left(g_{\mathrm{t}(\alpha)} \circ \tau'_\alpha - \sigma'_\alpha \circ g_{\mathrm{s}(\alpha)}\right)_{\alpha \in Q_1}.$$

Now, we will claim that

$$\|D_\rho(g)\| \leq \sum_{\alpha \in Q_1} \|g_{\mathrm{t}(\alpha)} \circ \tau'_\alpha - \sigma'_\alpha \circ g_{\mathrm{s}(\alpha)}\|$$

For this, let $\rho, \sigma, \tau \in \mathscr{A}$ and $g \in G$, and suppose $\tau = \sigma g$. Let $\sigma' = \sigma - \rho$ and $\tau' = \tau - \rho$. Then,

$$D_\rho(g) = \big(\rho_\alpha \circ g_{\mathrm{s}(\alpha)} - g_{\mathrm{t}(\alpha)} \circ \rho_\alpha\big)_{\alpha \in Q_1}.$$

Now, since $\tau = \sigma g$, for each $\alpha \in Q_1$, we have $\tau_\alpha = g_{\mathrm{t}(\alpha)}^{-1} \circ \sigma_\alpha \circ g_{\mathrm{s}(\alpha)}$, hence $g_{\mathrm{t}(\alpha)} \circ \tau_\alpha = \sigma_\alpha \circ g_{\mathrm{s}(\alpha)}$, that is, $g_{\mathrm{t}(\alpha)} \circ (\tau'_\alpha + \rho_\alpha) = (\sigma'_\alpha + \rho_\alpha) \circ g_{\mathrm{s}(\alpha)}$, hence

$$\rho_\alpha \circ g_{\mathrm{s}(\alpha)} - g_{\mathrm{t}(\alpha)} \circ \rho_\alpha = g_{\mathrm{t}(\alpha)} \circ \tau'_\alpha - \sigma'_\alpha \circ g_{\mathrm{s}(\alpha)}.$$

Therefore,

$$D_\rho(g) = \big(g_{\mathrm{t}(\alpha)} \circ \tau'_\alpha - \sigma'_\alpha \circ g_{\mathrm{s}(\alpha)}\big)_{\alpha \in Q_1}.$$

In view of the above relation, and the fact that the Hermitian inner product on $\mathscr{A}$ is the direct sum of the Hermitian inner products on $\mathrm{Hom}_{\mathbb{C}}(V_{\mathrm{s}(\alpha)}, V_{\mathrm{t}(\alpha)})$ as α runs over Q_1, we have

$$\|D_\rho(g)\| = \Big(\sum_{\alpha \in Q_1} \|g_{\mathrm{t}(\alpha)} \circ \tau'_\alpha - \sigma'_\alpha \circ g_{\mathrm{s}(\alpha)}\|^2\Big)^{\frac{1}{2}}.$$

Now, if $(a_i)_{i \in I}$ is a finite family of real numbers ≥ 0, then

$$\Big(\sum_{i \in I} a_i^2\Big)^{\frac{1}{2}} \leq \sum_{i \in I} a_i,$$

since

$$\Big(\sum_{i \in I} a_i\Big)^2 = \sum_{i \in I} a_i^2 + \sum_{\substack{(i,j) \in I \times I \\ i \neq j}} a_i a_j \geq \sum_{i \in I} a_i^2.$$

Therefore,

$$\|D_\rho(g)\| \leq \sum_{\alpha \in Q_1} \|g_{\mathrm{t}(\alpha)} \circ \tau'_\alpha - \sigma'_\alpha \circ g_{\mathrm{s}(\alpha)}\|,$$

as stated.

Therefore, as $c(g)e \in \mathrm{Ker}(D_\rho)$, we have

$$\begin{aligned}\|D_\rho(g^0)\| = \|D_\rho(g)\| &\le \sum_{\alpha\in Q_1} \|g_{\mathrm{t}(\alpha)}\circ\tau'_\alpha - \sigma'_\alpha\circ g_{\mathrm{s}(\alpha)}\| \\ &\le \sum_{\alpha\in Q_1}\left(\|g_{\mathrm{t}(\alpha)}\circ\tau'_\alpha\| + \|\sigma'_\alpha\circ g_{\mathrm{s}(\alpha)}\|\right) \le \sum_{\alpha\in Q_1}\left(\|g_{\mathrm{t}(\alpha)}\|\|\tau'_\alpha\| + \|\sigma'_\alpha\|\|g_{\mathrm{s}(\alpha)}\|\right) \\ &\le q_1\left(\|g\|\|\tau'\| + \|\sigma'\|\|g\|\right) = q_1\|g\|(\|\tau'\|+\|\sigma'\|) \\ &\le q_1\left(|c(g)|\|e\| + \|g^0\|\right)\left(\|\tau'\|+\|\sigma'\|\right) \\ &= q_1\left(|c(g)|\sqrt{\mathrm{rk}(d)} + \|g^0\|\right)\left(\|\tau'\|+\|\sigma'\|\right).\end{aligned}$$

Thus,

$$\theta\|g^0\| \le q_1\left(|c(g)|\sqrt{\mathrm{rk}(d)} + \|g^0\|\right)\left(\|\sigma'\|+\|\tau'\|\right),$$

hence

$$\|g^0\|(\theta - q_1(\|\sigma'\|+\|\tau'\|)) \le q_1|c(g)|\sqrt{\mathrm{rk}(d)}(\|\sigma'\|+\|\tau'\|).$$

As $(\sigma,\tau)\in X$, we have $q_1(\|\sigma'\|+\|\tau'\|) < \theta$, hence this implies that

$$\|g^0\| \le \frac{q_1|c(g)|\sqrt{\mathrm{rk}(d)}(\|\sigma'\|+\|\tau'\|)}{\theta - q_1(\|\sigma'\|+\|\tau'\|)}.$$

In particular, $c(g)\neq 0$; for, if $c(g)=0$, then, by the above inequality, we get $g^0=0$, hence $g = c(g)e + g^0 = 0$; therefore, as $g_a \in \mathrm{Aut}_{\mathbb{C}}(V_a)$, we have $V_a = 0$ for every $a\in Q_0$, a contradiction, since d is a non-zero element of $\mathbb{N}^{Q_0}$. It follows that

$$\left\|\frac{1}{c(g)}g - e\right\| = \left\|\frac{1}{c(g)}g^0\right\| = \frac{1}{|c(g)|}\|g^0\| \le \frac{q_1\sqrt{\mathrm{rk}(d)}(\|\sigma'\|+\|\tau'\|)}{\theta - q_1(\|\sigma'\|+\|\tau'\|)}.$$

We have thus shown that for all $(\sigma,\tau)\in X$ and $g\in G$, such that $\tau = \sigma g$, we have $c(g)\neq 0$, and

$$\left\|\frac{1}{c(g)}g - e\right\| \le \delta(\sigma,\tau),$$

where $\delta : X \to [0,+\infty)$ is the function defined by

$$\delta(\sigma,\tau) = \frac{q_1\sqrt{\operatorname{rk}(d)}(\|\sigma-\rho\|+\|\tau-\rho\|)}{\theta - q_1(\|\sigma-\rho\|+\|\tau-\rho\|)}.$$

Now, let V be an H-invariant open neighbourhood of e in G. Then, as G is open in $\operatorname{Lie}(G)$, there exists $\varepsilon > 0$, such that the open ball $B(e,\varepsilon)$ in $\operatorname{Lie}(G)$, with radius ε and centre e, is contained in V. As X is an open neighbourhood of (ρ,ρ) in $\mathscr{A}\times\mathscr{A}$, and the function δ is continuous, there exists an open neighbourhood U of ρ in $\mathscr{B}$, such that $U\times U \subset X$, and $\delta(\sigma,\tau) < \varepsilon$ for all $(\sigma,\tau)\in U\times U$. We claim that $P_G(U,U)\subset V$. Let $g\in P_G(U,U)$. Then, there exists a point (σ,τ) of $U\times U$, such that $\tau = \sigma g$. As $(\sigma,\tau)\in X$, by the above paragraph, $c(g)\neq 0$, and $\|\frac{1}{c(g)}g - e\| \le \delta(\sigma,\tau) < \varepsilon$, hence $\frac{1}{c(g)}g \in B(e,\varepsilon)\subset V$. As V is H-invariant, $g = \big(c(g)e\big)\big(\frac{1}{c(g)}g\big)\in HV\subset V$. Thus, $P_G(U,U)\subset V$.

Lastly, since $\mathscr{A}$, being a finite-dimensional complex vector space, is second countable, so is its open subset $\mathscr{B}$. As the map $p:\mathscr{B}\to M$ is surjective, continuous, and open, it follows that the topological space M is also second-countable. □

3.2 The Kähler metric on moduli of stable representations

In subsection 3.2.1, we define moment map for a lie group action on a symplectic manifold, list some properties of these moment maps, and give an explicit description, in Lemma 3.2.17, for moment maps for linear action on a finite dimensional vector spaces. In subsection 3.2.2, we give general theory for quotient Kähler manifold arising holomorphic action of a complex lie group on a Kähler manifolds, and prove the proposition 3.2.23. In subsection 3.2.3, we apply this theory to quiver representations and prove one of our main contributory Theorem 3.2.31. More precisely, in the Theorem 3.2.31, we prove that the moduli of stable representations is Hausdorff and gets the canonical structure of kähler manifold.

3.2.1 Moment maps

Definition 3.2.1 A *symplectic manifold* is a pair (X,Ω), where X is a smooth manifold, and Ω is a non-degenerate closed smooth 2-form on X.

Example 3.2.1 $(\mathbf{R}^{2n},\omega_0)$ is a symplectic manifold, where $\mathbf{R}^{2n}$ is the euclidean space of dimension $2n$, and $\omega_0 := \sum_{i=1}^{n} dx_i \wedge dy_i$, for the global coordinates $(x_1,...,x_n,y_1,...,y_n)$ of the smooth manifold $\mathbf{R}^{2n}$.

Example 3.2.2 Cotangent bundle T^*X of any smooth manifold X carries a canonical symplectic form. The idea comes from Hamilton's formulation of mechanics where points of the manifold represent position and cotangent vectors represent momentum.

Given a smooth manifold X, its cotangent bundle $M := T^*X$ has a canonical 1-form λ_{can} (sometimes called the tautological 1-form) defined as follows: let $\pi\colon T^*X \to X$ be the projection map, then for a given point $(x,y) \in T^*X$ (x is a point in the manifold X and y is a covector at x), $\lambda_{can},(x,y) := \pi^* y$ (where on the RHS we view y as a 1-form on T_xX).

Then we define the canonical symplectic form on T^*X by $\omega_{can} := -d\lambda_{can}$. Thus, (T^*X,ω_{can}) is an exact symplectic manifold by construction.

In coordinates: given a point $x \in X$, choose local coordinates $(x_1,...,x_n)$ on X (in the spirit of Hamilton one should think of them as the position of a system of particles). Given this choice of coordinates, T_x^*X has a canonical basis $dx_1,...,dx_n$. Given any $y \in T_x^*X$, write it as

$$y := \sum_{i=1}^{n} y_i dx_i$$

and define the canonical symplectic form on T^*X by

$$\omega_{can} := \sum_{i=1}^{n} dx_i \wedge dy_i$$

Example 3.2.3 The spheres S^{2n}, for $n \geq 2$ are not symplectic, because its second cohomology $H^2(S^{2n}) = 0$. To see this: assume (M,ω) is symplectic. Then, M is orientable (ω^n is a volume form). If M is closed then $[\omega] \in H^2_{dR}(M)$ is non zero. Indeed, since ω is closed, it defines an element $[\omega] \in H^2_{dR}(M)$. And, since ω is nondegenerate $[\omega^n] = [\omega]^n$ defines a volume form, i.e., a non zero element of $H^2_{dR}(M) \cong \mathbb{R}$.

Kähler manifolds are examples of symplectic manifolds. Kähler manifold is a manifold with three mutually compatible structures: a complex structure, a Riemannian structure, and a symplectic structure. To define these manifolds we need some linear algebra concepts (taken from [40]) which we recall now.

Let E be a complex vector space of complex dimension n. Let E' be the real dual space to the underlying real vector space of E, and let $F = E' \otimes_{\mathbf{R}} \mathbf{C}$ be the complex vector space of complex-valued real-linear mappings of E to $\mathbf{C}$. Then F has complex dimension $2n$, and we let

$$\Lambda = \sum_{p=0}^{2n} \Lambda^p F$$

be the $\mathbf{C}$-linear exterior algebra of F . We will refer to an $\omega \in \Lambda^p F$ as a p-form or as a p-covector (on E). Now, ΛF is equipped with a natural conjugation obtained by setting, if $\omega \in \Lambda^p F$,

$$\overline{\omega}(v_1,\dots,v_n) = \overline{\omega(v_1,\dots,v_n)},\ v_j \in E.$$

We say that $\omega \in \Lambda^p F$ is real if $\omega = \overline{\omega}$, and we will let $\Lambda^p_{\mathbf{R}} F$ denote the real elements of $\Lambda^p F$ (noting that $\Lambda^p_{\mathbf{R}} F \cong \Lambda^p E'$).

Let $\Lambda^{1,0}F$ be the subspace of $\Lambda^1 F$ consisting of complex-linear 1-forms on E, and let $\Lambda^{0,1}F$ be the subspace of conjugate-linear 1-forms on E. Then, we see that $\overline{\Lambda^{1,0}F} = \Lambda^{0,1}F$ and moreover, the following relation holds

$$\Lambda^1 F = \Lambda^{1,0}F \oplus \Lambda^{1,0}F$$

and this induces a bigrading on ΛF,

$$\Lambda F = \sum_{r=0}^{2n} \sum_{p+q=r} \Lambda^{p,q}F,$$

and we see that if $\omega \in \Lambda^{p,q}F$, then $\overline{\omega} \in \Lambda^{q,p}F$.

A Hermitian inner product on a complex vector space E is a map

$$\langle \cdot, \cdot \rangle : E \times E \to \mathbf{C}$$

such that $\langle \cdot, \cdot \rangle$ is

1. sesquilinear (i.e. complex linear in the first variabe and conjugate linear in tha second variable)

2. Hermitian symmetric (i.e. $\langle u, v \rangle = \overline{\langle v, u \rangle}$, for all u, v are in E)

3. Positive definite (i.e. $\langle u, u \rangle > 0$, for all nonzero u in E).

Now we suppose that our complex vector space is equipped with a Hermitian inner product $\langle \cdot, \cdot \rangle$. This inner product can be represented in the following manner.

Let $\{e_1, \ldots, e_n\}$ be a complex basis for E, and $\{z_1, \ldots, z_n\}$ be its dual basis for $\Lambda^{1,0}F = \mathrm{Hom}(E, \mathbf{C})$, and $\{\bar{z}_1, \ldots, \bar{z}_n\}$ be a basis for $\Lambda^{0,1}F$. For u, v are in E, let

$h(u,v) := \langle u,v\rangle$, we have

$$\begin{aligned} h(u,v) &= \sum_{\mu,\nu=1}^{n} h(e_\mu,e_\nu) z_\mu(u)\cdot \bar{z}_\nu(v) \\ &= \sum_{\mu,\nu=1}^{n} h(e_\mu,e_\nu) z_\mu \otimes \bar{z}_\nu(u,v) \\ &= \sum_{\mu,\nu=1}^{n} h_{\mu\nu}\, z_\mu \otimes \bar{z}_\nu(u,v), \end{aligned}$$

where $h_{\mu\nu} = \langle e_\mu, e_\nu\rangle$. Then $(h_{\mu\nu})$ is a positive definite Hermition symmetric matrix.

Now h is a complex-valued sesquilinear form acting on $E\times E$, and we can write

$$h = S + \sqrt{-1}A,$$

where S and A are real bilinear forms acting on E. One finds that S is a symmetric positive definite bilinear form, which represents the Euclidean inner product induced on the underlying real vector space of E by the Hermitian metric on E. Moreover, one can calculate easily that

$$\begin{aligned} A &= \frac{1}{2\sqrt{-1}} \sum_{\mu,\nu=1}^{n} h_{\mu\nu}\, (z_\mu \otimes \bar{z}_\nu - \bar{z}_\nu \otimes z_\mu) \\ &= -\sqrt{-1} \sum_{\mu,\nu=1}^{n} h_{\mu\nu}\, z_\mu \wedge \bar{z}_\nu \end{aligned}$$

Let us define

$$\Omega = \frac{\sqrt{-1}}{2} \sum_{\mu,\nu=1}^{n} h_{\mu\nu}\, z_\mu \wedge \bar{z}_\nu, \tag{3.1}$$

the fundamental 2-form associated to the Hermitian metric h. One sees immediately that

$$\Omega == -\frac{1}{2}A = -\frac{1}{2}Im(h),$$

and thus $h = S - 2\sqrt{-1}\Omega$.

Moreover, Ω is a real 2-form of type $(1,1)$. We can always choose a basis $\{z_\mu\}$ of $\Lambda^{1,0}F$ so that h has the form

$$h = \sum_{\mu=1}^{n} z_\mu \wedge \bar{z}_\mu .$$

Thus from equation 3.1, with respect to this basis,

$$\Omega = \frac{\sqrt{-1}}{2} \sum_{\mu=1}^{n} z_\mu \wedge \bar{z}_\mu .$$

An almost complex structure on a manifold X is a smooth field of complex structures on the tangent spaces: $x \mapsto J_x \colon \mathrm{T}_x(X) \to \mathrm{T}_x(X)$ linear, and ${J_x}^2 = Id$. The pair (X, J) is then called an *almost complex manifold*.

Let (X, Ω) be a symplectic manifold. An almost complex structure J on X is called compatible (with Ω or Ω-compatible) if the assignment

$$x \mapsto g_x \colon \mathrm{T}_x(X) \times \mathrm{T}_x(X) \to \mathbf{R}$$
$$g_x(u, v) := \Omega_x(u, J_x v)$$

is a positive real inner product on $\mathrm{T}_x(X)$.

Let X be a Hermitian complex manifold with Hermitian metric h. Then, there is associated to X and h a fundamental form Ω , which at each point $x \in X$ is the form of type $(1,1)$, which is the fundamental form associated as in equation 3.1 with the Hermitian bilinear form

$$h_x \colon \mathrm{T}_x(X) \times \mathrm{T}_x(X) \to \mathbf{C},$$

given by the Hermitian metric.

Definition 3.2.2 A Hermitian metric h on X is called a Kähler metric if the fundamental form Ω associated with h is closed; i.e., $d\Omega = 0$.

Definition 3.2.3 A complex manifold equipped with a Kähler metric is called a Kähler manifold: or, equivalently, A Kähler manifold is a symplectic manifold (M,Ω) equipped with an integrable Ω-compatible almost complex structure. The symplectic form Ω is then called a Kähler form.

Example 3.2.4 Let $X=\mathbf{C}^n$ and let $h=\sum_{\mu=1}^n z_\mu\wedge\bar{z}_\mu$. Then

$$\Omega=\frac{\sqrt{-1}}{2}\sum_{\mu=1}^n z_\mu\wedge\bar{z}_\mu=\sum_{\mu=1}^n x_\mu\wedge\bar{y}_\mu$$

where $z_\mu=x_\mu+\sqrt{-1}y_\mu$, $\mu=1,...,n$, is the usual notation for real and imaginary coordinates. Then, clearly, $d\Omega=0$, since Ω has constant coefficients, and hence h is a Kähler metric on $\mathbf{C}^n$.

Example 3.2.5 Every complex manifold X of complex dimension 1 (a Riemann surface) is of Khler type. To see this, let h be an arbitrary Hermitian metric on X. Then it suffices to show that this metric is indeed a Kähler metric. But this is trivial, since the associated fundamental form Ω is of type $(1,1)$ and therefore of total degree 2 on X. Since X has two real dimensions, it follows that $d\Omega=0$, since there are no forms of higher degree.

Definition 3.2.4 Let M be a complex manifold of complex dimension n. A function $\rho\in C^\infty(M;R)$ is *strictly plurisubharmonic* (s.p.s.h.) if, on each local complex chart $(U,z_1,...,z_n)$ the matrix $(\frac{\partial^2\rho}{\partial z_\mu\partial\bar{z}_\nu}(p))$ is positive-definite at all $p\in U$.

Proposition 3.2.5 *[1] Let M be a complex manifold and let $\rho\in C^\infty(M;R)$ be s.p.s.h.. Then*

$$\Omega=\frac{\sqrt{-1}}{2}\,\partial\bar{\partial}(\rho)$$

is Kähler. A function ρ as in the proposition is called a (global) Kähler potential.

Proof. A trivial observation is that

$$\partial\Omega = \frac{\sqrt{-1}}{2}\,\partial^2\overline{\partial}\rho = 0,$$
$$\overline{\partial}\Omega = \frac{\sqrt{-1}}{2}\,\partial\overline{\partial}^2\rho = 0.$$

$$d\Omega = \partial\Omega + \overline{\partial}\Omega = 0 \Rightarrow \Omega \text{ is closed.}$$
$$\overline{\Omega} = -\frac{\sqrt{-1}}{2}\,\overline{\partial}\partial\rho = \frac{\sqrt{-1}}{2}\,\partial\overline{\partial}\rho = \Omega \Rightarrow \Omega \text{ is real.}$$
$$\Omega \textit{is of}\,(1,1)\textit{ type} \Rightarrow J^*\Omega = \Omega \Rightarrow \Omega(.,J.) \textit{ is symmetric.}$$

As $\partial\rho = \sum \frac{\partial\rho}{\partial z_j}\,dz_j$, and $\overline{\partial}\rho = \sum \frac{\partial\rho}{\partial \bar{z}_j}\,d\bar{z}_j$, we have

$$\begin{aligned}
\Omega &= \frac{\sqrt{-1}}{2}\,\partial\overline{\partial}\rho \\
&= \frac{\sqrt{-1}}{2}\sum \frac{\partial}{\partial z_j}\Big(\frac{\partial}{\partial \bar{z}_k}\Big)\,dz_j\wedge d\bar{z}_k \\
&= \frac{\sqrt{-1}}{2}\sum \frac{\partial^2}{\partial z_j\partial \bar{z}_k}\,dz_j\wedge d\bar{z}_k \\
&= \frac{\sqrt{-1}}{2}\sum h_{jk}\,dz_j\wedge d\bar{z}_k,
\end{aligned}$$

where $h_{jk} = \frac{\partial^2}{\partial z_j\partial \bar{z}_k}$. Thus, ρ is s.p.s.h $\Rightarrow$ The matrix (h_{jk}) is positive definite $\Rightarrow \Omega(.,J.)$ is positive. In particular, Ω is nondegenerate. □

Example 3.2.6 Any complex submanifold of a Kähler manifold is also Kähler.

Example 3.2.7 Complex vector space $(\mathbf{C}^n,\Omega_0)$ is Kähler where $\Omega_0 = \frac{\sqrt{-1}}{2}\sum_{j,k=1}^n dz_j\wedge d\bar{z}_k$. Every complex submanifold of $\mathbf{C}^n$ is Kähler. To see this, As $\mathbf{C}^n \cong \mathbf{R}^{2n}$, with complex coordinates $(z_1,...,z_n)$ and corresponding real coordinates $(x_1,y_1,...,x_n,y_n)$ via

$z_j = x_j + \sqrt{-1}y_j$. Let

$$\rho(x_1, y_1, ..., x_n, y_n) := \sum_{j=1}^{n} x_j^2 + y_j^2 = \sum_{j=1}^{n} |z_j|^2 = \sum_{j=1}^{n} z_j \bar{z}_j.$$

Then $\frac{\partial}{\partial z_j}(\frac{\partial}{\partial \bar{z}_k}\rho) = \frac{\partial}{\partial z_j}(z_k) = \delta_{jk}$. So

$$(h_{jk}) = (\frac{\partial^2}{\partial z_j \partial \bar{z}_k}) = (Id)$$

is a positive definite matrix. By using the proposition 3.2.5, we have ρ is s.p.s.h. The corresponding Kähler form

$$\begin{aligned}\Omega &= \frac{\sqrt{-1}}{2}\,\partial\bar{\partial}\rho \\ &= \frac{\sqrt{-1}}{2}\sum_{j,k=1}^{n} \delta_{jk}\, dz_j \wedge d\bar{z}_k \\ &= \frac{\sqrt{-1}}{2}\sum_{j=1}^{n} dz_j \wedge d\bar{z}_j\end{aligned}$$

is the standard form.

The purpose of the following example is to describe the natural Kähler structure on complex projective space, $\mathbb{C}P^n$.

Example 3.2.8 [1]

1. Note that the function

$$z = (z_0, z_1 ..., z_n) \mapsto log\ (1 + |z|^2) \colon \mathbb{C}^{n+1} \to \mathbb{R}$$

is strictly plurisubharmonic, where $|z|^2 = |z_0|^2 + ... + \|z_n|^2$. So, by proposition

3.2.5 it follows that the 2-form

$$\Omega_{FS} = \frac{\sqrt{-1}}{2}\, \partial\overline{\partial} log\ (1+|z|^2)$$

is a Kähler form on $\mathbb{C}^{n+1}$.(It is usually called the *Fubini-Study* form on $\mathbb{C}^{n+1}$.)

2. Let U be the open subset of $\mathbb{C}^{n+1}$ defined by the inequality $z_0 \neq 0$, and let $\varphi\colon U \to U$ be the map

$$\varphi(z_0, ..., z_n) = \frac{1}{z_0}(1, z_1, ..., z_n).$$

Then φ maps U biholomorphically onto U and that

$$\begin{aligned}\varphi^* log\ (1+|z|^2) &= log\ (1+|z|^2) + log\ \frac{1}{|z_0|^2}\\ &= log\ (1+|z|^2) - log\ z_0 - log\ \overline{z_0}.\end{aligned}$$

As, φ^* commute with ∂ and $\overline{\partial}$, we have $\varphi^*(\Omega_{FS}) = \Omega_{FS}$.

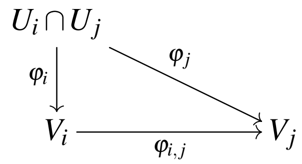
3. Recall that $\mathbb{C}P^n$ is obtained from $\mathbb{C}^{n+1} \setminus 0$ by making the identifications $(z_0, ..., z_n) \sim (\lambda z_0, ..., \lambda z_n)$ for all $\in \mathbb{C} \setminus 0$; We denote by $[z_0, ..., z_n]$ is the equivalence class of $(z_0, ..., z_n)$. For $i = 0, 1, ..., n$, let

$$U_i = \{[z_0, ..., z_n] \in \mathbb{C}P^n : z_i \neq 0\},$$

and $\varphi_i\colon U_i \to \mathbb{C}^n$ defined by

$$\varphi_i([z_0, ..., z_n]) = (\frac{z_0}{z_i}, ... \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, ..., \frac{z_n}{z_i}).$$

Then, we have a following commutative diagram



,where $V_i = \{(z_1, ..., z_n) \in \mathbb{C}^n : z_i \neq 0\}$, $V_j = \{(z_1, ..., z_n) \in \mathbb{C}^n : z_j \neq 0\}$ and $\varphi_{i,j}(z_1, ..., z_n) = (\frac{z_1}{z_i}, ..., \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, ..., \frac{z_n}{z_i})$. Now the set U is equal to the sets V_i and V_j, and the map φ coincides with $\varphi_{i,j}$. As $\varphi_i^*(\Omega_{FS}) = \varphi_j^*(\Omega_{FS})$ on $U_i \cap U_j$, the Kähler forms $\varphi_i^*(\Omega_{FS})$ glue together to define a Kähler structure on $\mathbb{C}P^n$. This is called the *Fubini-Study* form on complex projective space.

Definition 3.2.6 Recall that a smooth vector field ξ on X is called *symplectic vector field* if $L_\xi(\Omega) = 0$, where L_ξ is the Lie derivative with respect to ξ. We denote by $V(X, \Omega)$ the set of symplectic vector fields on X.

By [21, Chapter I, Corollary 3.7], a smooth vector field ξ on X is symplectic if and only if, for every point $x \in X$, and for some (and hence every) local flow $\varphi : I \times U \to X$ of ξ at x, and for all $t \in I$, we have $\varphi_t^*(\Omega) = \Omega|_U$, where $\varphi_t = y \mapsto \varphi(t, y) : U \to X$.

Suppose (X, Ω) be a smooth symplectic manifold. Then, the non-degenerate smooth 2-form Ω induces an isomorphism of smooth real vector bundles $\tilde{\Omega} : \mathrm{T}(X) \to \mathrm{T}^*(X)$, where $\mathrm{T}(X)$ denotes the tangent bundle of X, and $\mathrm{T}^*(X)$ its dual. Thus, for all $x \in X$, and $v, w \in \mathrm{T}_x(X)$, we have

$$\tilde{\Omega}(v)(w) = \Omega(x)(v, w)$$

The $S(X)$-linear map $\tilde{\Omega} : V(X) \to \mathrm{A}^1(X)$ induced by $\tilde{\Omega}$ is given by

$$\tilde{\Omega}(\xi) = i_\xi(\Omega),$$

for all smooth vector fields ξ on X. The differential $\mathrm{d}f$ of any smooth real function f on X is a smooth global section of $\mathrm{T}^*(X)$, so there exists a unique smooth vector field $H(f)$ on X, such that $\tilde{\Omega}(H(f)) = \mathrm{d}f$, that is, $\Omega(x)(H(f),w) = (\mathrm{d}f)(x)(w) = w(f)$ for all $x \in X$ and $w \in \mathrm{T}_x(X)$. Thus, we have the following definition;

Definition 3.2.7 Let (X,Ω) be a smooth symplectic manifold. For any smooth real function f on X, the unique smooth vector field $H(f)$ on X, such that $\Omega(x)(H(f)(x),w) = w(f)$ for all $x \in X$ and $w \in \mathrm{T}_x(X)$ (i.e.$i_\xi(\Omega) = dH(f)$), where $\mathrm{T}_x(X)$ denotes the tangent space of X at x, is called the *Hamiltonian vector field* with *Hamiltonian function* f.

It is easy to see from the definition that for a symplectic manifold (X,Ω),

- ξ is symplectic vector field on $X \Leftrightarrow i_\xi(\Omega)$ is a closed one form on X.
- ξ is Hamiltonian vector field on $X \Leftrightarrow i_\xi(\Omega)$ is an exact one form on X.

Remark 3.2.8

1. Locally, on every contractible open set, every symplectic vector field is Hamiltonian. If $H^1_{dR}(X) = 0$, then globally every symplectic vector field is Hamiltonian. In general, $H^1_{dR}(X)$ measures the obstruction for symplectic vector fields to be Hamiltonian.

2. for any smooth real function f on X, the Hamiltonian vector field $H(f)$ of f is a symplectic vector field, because

$$L_{H(f)}(\Omega) = \mathrm{d}\, i_{H(f)}(\Omega) + i_{H(f)}(\mathrm{d}\Omega) = \mathrm{d}\, i_{H(f)}(\Omega) = \mathrm{d}(\tilde{\Omega}(H(f))) = \mathrm{d}\,\mathrm{d}f = 0,$$

since Ω is closed.

3. The set $V(X,\Omega)$ of symplectic vector fields on X is a Lie subalgebra of the real Lie algebra $V(X)$ of smooth vector fields on X. For this, for every $p \in \mathbb{Z}$, let

$\mathrm{A}^p(X)$ denote the $\mathbb{R}$-vector space of smooth p-forms on X, and let $\mathrm{A}^{\cdot}(X) = \bigoplus_{p\in\mathbb{Z}} \mathrm{A}^p(X)$ be the graded $\mathbb{R}$-algebra of all smooth forms on X. Given any smooth vector field ξ on X, we define the *interior multiplication* by ξ to be the unique $S(X)$-linear map $i_\xi : \mathrm{A}^p(X) \to \mathrm{A}^{p-1}(X)$, such that

$$i_\xi(\omega)(\xi_1,\dots,\xi_{p-1}) = \omega(\xi,\xi_1,\dots,\xi_{p-1})$$

for all $\omega \in \mathrm{A}^p(X)$, and $\xi_1,\dots,\xi_{p-1} \in \mathbf{Vec}(k)X$. On the other hand, we have the *Lie derivative* $L_\xi : \mathrm{A}^p(X) \to \mathrm{A}^p(X)$. As $\mathbb{R}$-endomorphisms of $\mathrm{A}^{\cdot}(X)$, these two operators satisfy the conditions

$$L_\xi = \mathrm{d}\circ i_\xi + i_\xi \circ \mathrm{d}, \quad [L_\xi, i_\eta] = i_{[\xi,\eta]},$$

for all smooth vector fields ξ and η on X ([8, 8.4.7 and 8.5.7], [21, Chapter I, Proposition 3.10], and [37, Proposition 2.25]). Thus, if ξ and ν are two symplectic vector fields on X, then, since Ω is closed,

$$L_{[\xi,\eta]}(\Omega) = \mathrm{d}\, i_{[\xi,\eta]}(\Omega) + i_{[\xi,\eta]}(\mathrm{d}\Omega) = \mathrm{d}\, i_{[\xi,\eta]}(\Omega).$$

On the other hand, since $L_\xi(\Omega) = 0$, we have

$$i_{[\xi,\eta]}(\Omega) = [L_\xi, i_\eta](\Omega) = L_\xi(i_\eta(\Omega)) = \mathrm{d}\, i_\xi(i_\eta(\Omega)) + i_\xi(\mathrm{d}\, i_\eta(\Omega)).$$

But,

$$\mathrm{d}\, i_\eta(\Omega) = L_\eta(\Omega) - i_\eta(\mathrm{d}\Omega) = 0,$$

since $L_\eta(\Omega) = 0$, and Ω is closed. Thus,

$$i_{[\xi,\eta]}(\Omega) = \mathrm{d}(\Omega(\eta,\xi)).$$

Therefore,

$$L_{[\xi,\eta]}(\Omega) = \mathrm{d}\,\mathrm{d}\,\Omega(\eta,\xi) = 0,$$

hence $[\xi,\eta]$ is also a symplectic vector field. This verifies that $V(X,\Omega)$ is a Lie subalgebra of $V(X)$.

4. We define the Poisson bracket of any two smooth real functions f and g on X, by $\{f,g\} = \Omega(H(g),H(f))$. This makes the $\mathbb{R}$-vector space $S(X)$ of smooth real functions on X a Lie algebra, and the map $f \mapsto H(f)$ is a homomorphism of real Lie algebras $H : S(X) \to V(X,\Omega)$. To see this, it is obvious that the Poisson bracket is an $\mathbb{R}$-bilinear map from $S(X) \times S(X)$ to $S(X)$. As Ω is alternating,

$$\{f,g\} = \Omega(H(g),H(f)) = -\Omega(H(f),H(g)) = -\{g,f\}$$

for all $f,g \in S(X)$, hence the Poisson bracket is alternating. As Ω is closed, for all smooth vector fields a,b,c on X, we have, by [8, 8.5.7] or [37, Proposition 2.25],

$$\begin{aligned}
0 &= \mathrm{d}\Omega(a,b,c) \\
&= a\Omega(b,c) - b\Omega(a,c) + c\Omega(a,b) - \Omega([a,b],c) + \Omega([a,c],b) - \Omega([b,c],a) \\
&= a\Omega(b,c) + b\Omega(c,a) + c\Omega(a,b) - \Omega([a,b],c) - \Omega([b,c],a) - \Omega([c,a],b).
\end{aligned}$$

Therefore, for all smooth real functions f, g, h,

$$\begin{aligned}&H(f)\Omega(H(g),H(h))+H(g)\Omega(H(h),H(f))+H(h)\Omega(H(f),H(g))\\&\quad-\Omega([H(f),H(g)],H(h))-\Omega([H(g),H(h)],H(f))-\Omega([H(h),H(f)],H(g))\\&\quad=0.\end{aligned}$$

Now, if a and b are two symplectic vector fields, then

$$\begin{aligned}i_{[a,b]}(\Omega)&=[L_a,i_b](\Omega)=L_a(i_b(\Omega)))=(\mathrm{d}\,i_a+i_a\,\mathrm{d})i_b(\Omega)=\mathrm{d}\,i_ai_b(\Omega)+i_a\,\mathrm{d}\,i_b(\Omega)\\&=\mathrm{d}\,i_ai_b(\Omega)+i_a(L_b-i_b\,\mathrm{d})(\Omega)=\mathrm{d}\,i_ai_b(\Omega)=\mathrm{d}((i_b(\Omega))(a))=\mathrm{d}(\Omega(b,a)),\end{aligned}$$

since $L_a(\Omega)=L_b(\Omega)=\mathrm{d}\,\Omega=0$; therefore, for any smooth vector field c on X, we have

$$-\Omega([a,b],c)=-i_{[a,b]}(c)=-\,\mathrm{d}(\Omega(b,a))(c)=-c\Omega(b,a)=c\Omega(a,b).$$

In particular,

$$\begin{aligned}-\Omega([H(f),H(g)],H(h))&=H(h)\Omega(H(f),H(g)),\\-\Omega([H(g),H(h)],H(f))&=H(f)\Omega(H(g),H(h)),\\-\Omega([H(h),H(f)],H(g))&=H(g)\Omega(H(h),H(f)).\end{aligned}$$

Thus,

$$2\big(H(f)\Omega(H(g),H(h))+H(g)\Omega(H(h),H(f))+H(h)\Omega(H(f),H(g))\big)=0.$$

Now, for all smooth real functions φ and ψ on X,

$$H(\varphi)\psi = \Omega(H(\psi), H(\varphi)) = \{\varphi, \psi\}.$$

Therefore,

$$H(f)\Omega(H(g), H(h)) = -H(f)\Omega(H(h), H(g)) = -\{f, \Omega(H(h), H(g))\}.$$

Thus,

$$\begin{aligned} H(f)\Omega(H(g), H(h)) &= -\{f, \{g, h\}\} \\ H(g)\Omega(H(h), H(f)) &= -\{g, \{h, f\}\} \\ H(h)\Omega(H(f), H(g)) &= -\{h, \{f, g\}\}. \end{aligned}$$

It follows that

$$-2\big(\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\}\big) = 0.$$

Thus, the Poisson bracket satisfies the Jacobi identity, and hence makes $S(X)$ a real Lie algebra.

The map $H = f \mapsto H(f) : S(X) \to V(X, \Omega)$ is obviously $\mathbb{R}$-linear. Moreover, by the above observation, for any two symplectic vector fields a and b on X, we have

$$\tilde{\Omega}([a, b]) = i_{[a,b]}(\Omega) = \mathrm{d}(\Omega(b, a)) = \tilde{\Omega}(H(\Omega(b, a))),$$

hence

$$[a, b] = H(\Omega(b, a)).$$

In particular,

$$[H(f),H(g)] = H(\Omega(H(g),H(f))) = H(\{f,g\}).$$

Therefore, H is a homomorphism of real Lie algebras.

Let K be a real Lie group. Suppose we are given a smooth right action of K on X, which is symplectic, by which we mean that it preserves the symplectic form Ω on X. Thus, Ω is K-invariant, that is, $\rho_g^*(\Omega) = \Omega$ for all $g \in K$, where ρ_g denotes the translation by g on every right K-space.

Definition 3.2.9 For every element ξ of the Lie algebra $\mathrm{Lie}(K)$ of K, the induced vector field $\xi^\sharp$ on X which is defined by

$$\xi^\sharp(x) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(x\exp(t\xi))$$

for all $x \in X$, where $\exp\colon \mathrm{Lie}(K) \to K$ is the exponential mapping of the Lie group K.

Proposition 3.2.10 *Notations as in above;*

1. *For every element ξ of $\mathrm{Lie}(K)$, the induced vector field $\xi^\sharp$ on X is symplectic.*
2. *The map $\xi \mapsto \xi^\sharp$ is a homomorphism of real Lie algebras from $\mathrm{Lie}(K)$ to $V(X,\Omega)$, which is K-equivariant for the adjoint action of K on $\mathrm{Lie}(K)$, and the canonical action of K on $V(X,\Omega)$.*

Proof. (1) : The flow of $\xi^\sharp$ is the smooth map $\varphi : \mathbb{R} \times X \to X$, which is given by $\varphi(t,x) = \varphi_t(x) = x\exp(t\xi)$ for all $(t,x) \in \mathbb{R} \times X$. Thus, $\varphi_t = \rho_{\exp(t\xi)}$ for all $t \in \mathbb{R}$. By definition, for all $x \in X$, we have

$$L_{\xi^\sharp}(\Omega)(x) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big(\varphi_t^*(\Omega)(x)\big) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big(\rho_{\exp(t\xi)}^*(\Omega)(x)\big).$$

As Ω is K-invariant, $\rho^*_{\exp(t\xi)}(\Omega) = \Omega$ for all $t \in \mathbb{R}$, hence

$$L_{\xi^\sharp}(\Omega)(x) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big(\Omega(x)\big) = 0.$$

Therefore, $\xi^\sharp$ is symplectic.

(2) : We now check that the map $\xi \mapsto \xi^\sharp$ is a homomorphism of real Lie algebras from $\mathrm{Lie}(K)$ to $V(X,\Omega)$. It is obviously $\mathbb{R}$-linear. Recall that if $f : X \to X'$ is a smooth map of smooth manifolds, and if ξ and ξ' are two smooth vector fields on X and X', respectively, then ξ is said to be *f-related* to ξ', if $\mathrm{T}_x(f)(\xi(x)) = \xi'(f(x))$ for all $x \in X$. It is easy to see that if ξ and η are two smooth vector fields on X, ξ' and η' two smooth vector fields on X', ξ f-related to ξ', and η f-related to η', then $[\xi,\eta]$ is f-related to $[\xi',\eta']$ ([8, 8.5.6] and [37, Proposition 1.55]). Now, let $\xi \in \mathrm{Lie}(K)$, $x \in X$, and $\nu_x : K \to X$ the orbit map of x. Then, $\xi^\sharp(\mathbf{x}) = \mathrm{T}_{\mathbf{e}}(\nu_{\mathbf{x}})(\xi)$. Let $\overline{\xi}$ denote the left-invariant vector field on K defined by ξ. We claim that $\overline{\xi}$ is ν_x-related to $\xi^\sharp$. Let $g \in K$. Then, $\overline{\xi}(g) = \mathrm{T}_e(\lambda_g)(\xi)$, where λ_g denotes the left translation by g on every left K-space. Thus,

$$\begin{aligned}\mathrm{T}_g(\nu_x)(\overline{\xi}(g)) &= \mathrm{T}_g(\nu_x)(\mathrm{T}_e(\lambda_g)(\xi)) = \mathrm{T}_e(\nu_x \circ \lambda_g)(\xi) \\ &= \mathrm{T}_e(\nu_{xg})(\xi) = \xi^\sharp(xg) = \xi^\sharp(\nu_x(g)),\end{aligned}$$

since $\nu_x \circ \lambda_g = \nu_{xg} : K \to X$. This verifies that $\overline{\xi}$ is ν_x-related to $\xi^\sharp$. Therefore, if ξ and η are two elements of $\mathrm{Lie}(K)$, then, by the above fact, $[\overline{\xi},\overline{\eta}]$ is ν_x-related to $[\xi^\sharp,\eta^\sharp]$. In particular,

$$\mathrm{T}_e(\nu_x)([\overline{\xi},\overline{\eta}](e)) = [\xi^\sharp,\eta^\sharp](\nu_x(e)) = [\xi^\sharp,\eta^\sharp](x).$$

But, by the definition of the bracket in $\mathrm{Lie}(K)$, we have $[\xi,\eta] = [\overline{\xi},\overline{\eta}](\mathbf{e})$, hence

$$[\xi,\eta]^\sharp(x) = \mathrm{T}_e(\nu_x)([\xi,\eta]) = \mathrm{T}_e(\nu_x)([\overline{\xi},\overline{\eta}](e)) = [\xi^\sharp,\eta^\sharp](x).$$

It follows that $[\xi,\eta]^\sharp = [\xi^\sharp,\eta^\sharp]$, hence the said map is a homomorphism of real Lie algebras.

The above homomorphism of Lie algebras is K-equivariant, for the adjoint action of K on $\mathrm{Lie}(K)$, and the canonical action of K on $V(X,\Omega)$. Recall that the adjoint action of K on $\mathrm{Lie}(K)$ is the left action given by $(g,\xi)\mapsto \mathrm{Ad}(g)(\xi)$, where, for each $g\in K$, $\mathrm{Ad}(g)$ is the homomorphism of real Lie algebras $\mathrm{T}_e(\mathrm{Int}(g)) : \mathrm{Lie}(K)\to\mathrm{Lie}(K)$, and $\mathrm{Int}(g)$ is the inner automorphism $h\mapsto ghg^{-1}$ of K. The canonical action of K on $V(X)$ is the left action given by $(g\xi)(x) = \mathrm{T}_x(\rho_g)^{-1}(\xi(xg)) = \mathrm{T}_{xg}(\rho_{g^{-1}})(\xi(xg))$ for all $g\in K$, $\xi\in V(X)$, and $x\in X$. Thus,

$$\mathrm{T}_x(\rho_g)((g\xi)(x)) = \mathrm{T}_x(\rho_g)(\mathrm{T}_x(\rho_g)^{-1}(\xi(xg))) = \xi(xg).$$

Therefore, $g\xi$ is ρ_g-related to ξ. Now, if $f : X\to X'$ is a smooth map of smooth manifolds, and ξ a vector field on X that is f-related to a smooth vector field ξ' on X', then $L_\xi(f^*(\omega')) = f^*(L_{\xi'}(\omega'))$ for every smooth differential form ω' on X' [8, 8.4.9]. Therefore, $L_{g\xi}(\rho_g^*(\omega)) = \rho_g^*(L_\xi(\omega))$ for every smooth differential form ω on X. In particular, if ξ is a symplectic vector field on X, then

$$L_{g\xi}(\Omega) = L_{g\xi}(\rho_g^*(\Omega)) = \rho_g^*(L_\xi(\Omega)) = \rho_g^*(0) = 0,$$

hence $g\xi$ is symplectic too. Thus, the subset $V(X,\Omega)$ of $V(X)$ is K-invariant. Moreover, for all $\xi\in\mathrm{Lie}(G)$, we have

$$(g\xi^\sharp)(x) = \mathrm{T}_{xg}(\rho_{g^{-1}})(\xi^\sharp(xg)) = \mathrm{T}_{xg}(\rho_{g^{-1}})(\mathrm{T}_e(\nu_{xg})(\xi)) = \mathrm{T}_e(\rho_{g^{-1}}\circ\nu_{xg})(\xi).$$

But, $\rho_{g^{-1}}\circ\nu_{xg} = \nu_x\circ(\mathrm{Int}(g))$, hence

$$(g\xi^\sharp)(x) = \mathrm{T}_e(\nu_x\circ(\mathrm{Int}(g)))(\xi) = \mathrm{T}_e(\nu_x)(\mathrm{T}_e(\mathrm{Int}(g))(\xi)) = \mathrm{T}_e(\nu_x)(\mathrm{Ad}(g)\xi).$$

Thus, $(g\xi^\sharp)(x) = (\mathrm{Ad}(g)\xi)^\sharp(x)$, that is, $g\xi^\sharp = (\mathrm{Ad}(g)\xi)^\sharp$. This proves that the map $\xi \mapsto \xi^\sharp : \mathrm{Lie}(K) \to V(X,\Omega)$ is K-equivariant.

Lastly, note that the map $H = f \mapsto H(f) : S(X) \to V(X,\Omega)$ is also K-invariant. The canonical action of K on $S(X)$ is the left action given by $(f,g) \mapsto gf$, where $(gf)(x) = f(xg)$ for all $g \in K$, $f \in S(X)$, and $x \in X$. Now, for all $x \in X$ and $w \in \mathrm{T}_x(X)$, we have

$$\begin{aligned}
\Omega(x)((gH(f))(x),w) &= \Omega(x)(\mathrm{T}_{xg}(\rho_{g^{-1}})(H(f)(xg)),\mathrm{T}_{xg}(\rho_{g^{-1}})(\mathrm{T}_x(\rho_g)(w))) \\
&= \rho^*_{g^{-1}}(\Omega)(xg)(H(f)(xg),\mathrm{T}_x(\rho_g)(w)) \\
&= \Omega(xg)(H(f)(xg),\mathrm{T}_x(\rho_g)(w)) \\
&= \mathrm{T}_x(\rho_g)(w)(f).
\end{aligned}$$

On the other hand,

$$\Omega(x)(H(gf)(x),w) = w(gf) = w(f \circ \rho_g) = \mathrm{T}_x(\rho_g)(w)(f).$$

It follows that

$$\Omega(x)((gH(f))(x),w) = \Omega(x)(H(gf)(x),w)$$

for all $w \in \mathrm{T}_x(X)$, hence $gH(f)(x) = H(gf)(x)$. Thus, $gH(f) = H(gf)$, that is, H is K-equivariant. □

Definition 3.2.11 A *moment map* for the action of K on X is a smooth map $\Phi : X \to \mathrm{Lie}(K)^*$, which is K-invariant for the coadjoint action of K on $\mathrm{Lie}(K)^*$, and has the property that $H(\Phi^\xi) = \xi^\sharp$ for all $\xi \in \mathrm{Lie}(K)$, where Φ^ξ is the smooth real function $x \mapsto \Phi(x)(\xi)$ on X.

The smooth function $\Phi^\xi : X \to \mathrm{Lie}(K)^*$ in the above definition is called the component of Φ along ξ.

Thus, if $\Phi : X \to \mathrm{Lie}(K)^*$ is a moment map for an action K on X, then for every element $\xi \in \mathrm{Lie}(K)$, the symplectic vector field $\xi^\sharp$ is a Hamiltonian vector field with Hamiltonian function Φ^ξ, where Φ^ξ is the component of Φ along ξ.

Definition 3.2.12 An action $\Psi\colon K \times X \to X$ is said to be Hamiltonian action if there exist a moment map for Ψ.

Example 3.2.9 For a symplectic manifold (X,Ω), the map $\Psi \mapsto \xi_x := \frac{d\Psi_t(x)}{dt}$ $(x \in X)$ induces a bijective correspondence:

$$\{\textit{symplectic actions of } \mathbb{R} \textit{ on } X\} \underset{\longleftrightarrow}{1-1} \{\textit{complete symplectic vector fields on } X\}$$

with inverse $\xi \mapsto \Psi := exp\, t\xi$.

Example 3.2.10 Let (X,Ω) be a symplectic manifold with a symplectic action $\Psi\colon \mathbb{R} \times X \to X$. Then, Ψ is Hamiltonian action $\Leftrightarrow$ there exists a smooth function $H\colon X \to \mathbb{R}$ such that $dH = i_\xi(\Omega)$, where ξ is the vector field on X generated by Ψ.

Example 3.2.11 Let (X,Ω) be a symplectic manifold with a symplectic action $\Psi\colon \mathbb{S}^1 \times X \to X$. Then Ψ is an action of $\mathbb{R}$ on X which is 2π-periodic(i.e. $\Psi_{2\pi} = \Psi_0$). Then, Ψ is Hamiltonian action $\Leftrightarrow$ the underlying action $\mathbb{R}$ on X is Hamiltonian.

Now, we will see some properties of moment maps. For this, we need some linear algebra facts which we will give now:

Lemma 3.2.13 *Let V be a finite-dimensional vector space over a field k, B a non-degenerate k-bilinear form on V, and W a k-subspace of V. Then, $W^{\perp(\Omega)\perp(\Omega)} = W$, where $S^{\perp(\Omega)}$ denotes the set of all elements of V that are B-orthogonal to any subset S of V.*

Proof. Let $f : V \to W^*$ be the k-homomorphism defined by $f(v)(w) = B(v,w)$ for all $v \in V$ and $w \in W$. For any $\beta \in W^*$, let α be an element of V^*, such that $\alpha|_W = \beta$. As B is non-degenerate, there exists $v \in V$, such that $\alpha(y) = B(v,y)$ for all $y \in V$. Thus, $f(v)(w) = B(v,w) = \alpha(w) = \beta(w)$ for all $v \in V$ and $w \in W$, hence $f(v) = \beta$. Therefore, f is surjective. Since $\mathrm{Ker}(f) = W^{\perp(\Omega)}$, this implies that $\dim_k(V) - \dim_k(W^{\perp(\Omega)}) = \dim_k(W)$, hence $\dim_k(W) + \dim_k(W^{\perp(\Omega)}) = \dim_k(V)$. Putting $W^{\perp(\Omega)}$ in the place of W, we also get $\dim_k(W^{\perp(\Omega)}) + \dim_k(W^{\perp(\Omega)\perp(\Omega)}) = \dim_k(V)$. Subtracting the second of these two equations from the first gives $\dim_k(W) = \dim_k(W^{\perp(\Omega)\perp(\Omega)})$. As $W \subset W^{\perp(\Omega)\perp(\Omega)}$, it follows that $W^{\perp(\Omega)\perp(\Omega)} = W$. □

Lemma 3.2.14 *Let k be a field, and $f : V \to W$ a k-linear map of k-vector spaces. Then,* $\mathrm{Im}(f^*) = \mathrm{Ann}(\mathrm{Ker}(f))$.

Proof. We have an exact sequence of k-vector spaces

$$\mathrm{Ker}(f) \xrightarrow{i} V \xrightarrow{f} W,$$

where $i : \mathrm{Ker}(f) \to V$ is the inclusion map. Since dualising is an exact functor on the category of k-vector spaces, this induces an exact sequence

$$W^* \xrightarrow{f^*} V^* \xrightarrow{i^*} \mathrm{Ker}(f)^*.$$

Therefore, $\mathrm{Im}(f^*) = \mathrm{Ker}(i^*)$. But, since $i^*(\alpha) = \alpha \circ i = \alpha|_{\mathrm{Ker}(f)}$ for all $\alpha \in V^*$, we have $\mathrm{Ker}(i^*) = \mathrm{Ann}(\mathrm{Ker}(f))$. □

Proposition 3.2.15 1. *If Φ is a moment map, then, for every $x \in X$, the $\mathbb{R}$-linear map $\mathrm{T}_x(\Phi) : \mathrm{T}_x(X) \to \mathrm{Lie}(K)^*$ is given by*

$$\mathrm{T}_x(\Phi)(v)(\xi) = \Omega(x)(\xi^\sharp(x), v)$$

for all $v \in \mathrm{T}_x(X)$ *and* $\xi \in \mathrm{Lie}(K)$.

2. $\mathrm{Ker}(\mathrm{T}_x(\Phi)) = \mathrm{Im}(\mathrm{T}_e(\nu_x))^{\perp(\Omega)}$, *where* $\nu_x : K \to X$ *is the orbit map of* x, $\mathrm{T}_e(\nu_x) : \mathrm{Lie}(K) \to \mathrm{T}_x(X)$ *is the induced* $\mathbb{R}$*-linear map, and* $S^{\perp(\Omega)}$ *denotes the set of elements of* $\mathrm{T}_x(X)$ *that are* $\Omega(x)$*-orthogonal to any subset* S *of* $\mathrm{T}_x(X)$.

3. $\mathrm{Ker}(\mathrm{T}_x(\Phi))^{\perp(\Omega)} = \mathrm{Im}(\mathrm{T}_e(\nu_x))$

4. $\mathrm{Im}(\mathrm{T}_x(\Phi)) = \mathrm{Ann}(\mathrm{Lie}(K_x))$, *where* K_x *is the stabiliser of* x *in* K, *and* $\mathrm{Ann}(M)$ *denotes the annihilator in* $\mathrm{Lie}(K)^*$ *of any subset* M *of* $\mathrm{Lie}(K)$. *In particular,* Φ *is a submersion at* x *if and only if the subgroup* K_x *of* K *is discrete.*

Proof. (1) : Let $x \in X$, $w \in \mathrm{T}_x(X)$, and $\xi \in \mathrm{Lie}(K)$, and let $c : I \to X$ be a smooth map, such that I is an open neighbourhood of 0 in $\mathbb{R}$, $c(0) = x$, and $\dot{c}(0) = w$. Then,

$$\mathrm{T}_x(\Phi)(w) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Phi(c(t)).$$

Therefore,

$$\begin{aligned}\mathrm{T}_x(\Phi)(w)(\xi) &= \Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Phi(c(t))\Big)(\xi) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \big(\Phi(c(t))(\xi)\big) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Phi^{\xi}(c(t)) = \mathrm{T}_x(\Phi^{\xi})(w) = w(\Phi^{\xi}) \\ &= \Omega(x)(H(\Phi^{\xi})(x), w) = \Omega(x)(\xi^{\sharp}(x), w).\end{aligned}$$

This proves (1).

(2) : From (1) , we can write

$$\mathrm{Ker}(\mathrm{T}_x(\Phi)) = \{w \in \mathrm{T}_x(X) \,|\, \Omega(x)(\xi^{\sharp}(x), w) = 0 \text{ for all } \xi \in \mathrm{Lie}(K)\} = S_x^{\perp(\Omega)},$$

where

$$S_x = \{\xi^{\sharp}(x) \,|\, \xi \in \mathrm{Lie}(K)\}.$$

But,

$$\mathrm{T}_e(\nu_x)(\xi) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \nu_x(\exp(t\xi)) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (x\exp(t\xi)) = \xi^\sharp(x)$$

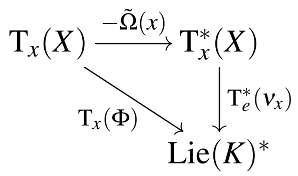
for all $\xi \in \mathrm{Lie}(K)$, hence $S_x = \mathrm{Im}(\mathrm{T}_e(\nu_x))$. Therefore, $\mathrm{Ker}(\mathrm{T}_x(\Phi)) = \mathrm{Im}(\mathrm{T}_e(\nu_x))^{\perp(\Omega)}$.

(3) : From (2) and the Lemma 3.2.13 implies that $\mathrm{Ker}(\mathrm{T}_x(\Phi))^{\perp(\Omega)} = \mathrm{Im}(\mathrm{T}_e(\nu_x))$.

(4) : For any $w \in \mathrm{T}_x(X)$ and $\xi \in \mathrm{Lie}(K)$, we have

$$\begin{aligned} \mathrm{T}_x(\Phi)(w)(\xi) &= \Omega(x)(\xi^\sharp(x), w) = -\Omega(x)(w, \xi^\sharp(x)) \\ &= -\tilde{\Omega}(x)(w)(\xi^\sharp(x)) = -\tilde{\Omega}(x)(w)(\mathrm{T}_e(\nu_x)(\xi)) \\ &= -\mathrm{T}_e^*(\nu_x)(\tilde{\Omega}(x)(w))(\xi), \end{aligned}$$

where $\tilde{\Omega}(x) : \mathrm{T}_x(X) \to \mathrm{T}_x^*(X)$ is the $\mathbb{R}$-isomorphism $a \mapsto \Omega(x)(a, \cdot)$. Thus, the diagram



It follows that

$$\mathrm{Im}(\mathrm{T}_x(\Phi)) = \mathrm{Im}(\mathrm{T}_e^*(\nu_x)).$$

Therefore, by the Lemma 3.2.14,

$$\mathrm{Im}(\mathrm{T}_x(\Phi)) = \mathrm{Ann}(\mathrm{Ker}(\mathrm{T}_e(\nu_x))).$$

But, since $\nu_x : K \to X$ has constant rank, $K_x = \nu_x^{-1}(x)$ is a real Lie subgroup of G, and $\mathrm{Lie}(K_x) = \mathrm{T}_e(K_x) = \mathrm{Ker}(\mathrm{T}_e(\nu_x))$ ([8, 5.10.5 and 5.12.5] and [38, Proposition 5.39 and Corollary 6.10]). Therefore,

$$\mathrm{Im}(\mathrm{T}_x(\Phi)) = \mathrm{Ann}(\mathrm{Lie}(K_x)).$$

The map $\mathrm{T}_x(\Phi)$ is surjective if and only if $\mathrm{Ann}(\mathrm{Lie}(K_x)) = \mathrm{Lie}(K)^*$. For any field k, k-vector space V, and k-subspace W of V, we have $W = 0$ if and only if $\mathrm{Ann}(W) = V^*$. Therefore, $\mathrm{Ann}(\mathrm{Lie}(K_x)) = \mathrm{Lie}(K)^*$ if and only if $\mathrm{Lie}(K_x) = 0$. But, as $\mathrm{Lie}(K_x) = \mathrm{T}_e(K_x)$, we have $\mathrm{Lie}(K_x) = 0$ if and only if $\dim_e(K_x) = 0$. A premanifold M has dimension 0 at a point m if and only if m is an isolated point of M (by definition, this means that $\{m\}$ is open in M). Also, a topological group is discrete if and only if its identity element is an isolated point. It follows that Φ is a submersion at x if and only if the subgroup K_x of K is discrete. □

Remark 3.2.16 Let $\Phi : X \to \mathrm{Lie}(K)^*$ be a moment map for the action of a Lie group K on a symplectic manifold (X, Ω). Let U be a K-invariant open submanifold of X. Then, $\Phi|_U : U \to \mathrm{Lie}(K)^*$ is a moment map for the action of K on $(U, \Omega|_U)$. To see this, for all $x \in U$, $w \in \mathrm{T}_x(X)$, and $\xi \in \mathrm{Lie}(K)$, we have

$$\begin{aligned} w(\Phi|_U^{\xi}) &= \mathrm{T}_x(\Phi|_U^{\xi})(w) \\ &= \mathrm{T}_x(\Phi^{\xi})(w) \\ &= w(\Phi^{\xi}), \end{aligned}$$

and also, we have

$$\begin{aligned} \Omega(\xi|_U^{\sharp}(x), w) &= \Omega(\mathrm{T}_e(\nu|_U)(x), w) \\ &= \Omega(\mathrm{T}_e(\nu)(x), w) \\ &= \Omega(\xi^{\sharp}(x), w). \end{aligned}$$

Thus, for all $x \in U$, $w \in \mathrm{T}_x(X)$, and $\xi \in \mathrm{Lie}(K)$, we have

$$w(\Phi|_U^{\xi}) = \Omega(\xi|_U^{\sharp}(x), w),$$

and hence $\Phi|_U : U \to \mathrm{Lie}(K)^*$ is a moment map for the action of K on $(U, \Omega|_U)$.

For later use, we state here a simple fact about moment maps for linear actions. Let V be a finite-dimensional $\mathbb{R}$-vector space, and Ω a symplectic form on V. Since the tangent space of V at any point is canonically isomorphic to V itself, Ω defines a smooth 2-form on V. By abuse of notation, we will denote this smooth 2-form also by Ω. In any linear coordinate system on V, we can express Ω as a form with constant coefficients, so $\mathrm{d}\Omega = 0$. Therefore, (V, Ω) is a symplectic manifold.

Let K be a real Lie group, and suppose that we are given a smooth linear right action of K on V, which preserves the symplectic form Ω on V. For any element ξ of $\mathrm{Lie}(K)$, let $\xi^\sharp$ be the vector field on V defined by ξ; then $\xi^\sharp$ is an $\mathbb{R}$-endomorphism of V, and $\Omega(\xi^\sharp(x), y) + \Omega(x, \xi^\sharp(y)) = 0$ for all $x, y \in V$. For each element α of $\mathrm{Lie}(K)^*$, define a map $\Phi_\alpha : V \to \mathrm{Lie}(K)^*$ by

$$\blacksquare_\alpha(\mathbf{x})(\xi) = \frac{\mathbf{1}}{\mathbf{2}}\blacksquare(\xi^\sharp(\mathbf{x}), \mathbf{x}) + \alpha(\xi)$$

for all $x \in V$ and $\xi \in \mathrm{Lie}(K)$.

Lemma 3.2.17 *The map $\alpha \mapsto \Phi_\alpha$ is a bijection from the set of K-invariant elements of* $\mathrm{Lie}(K)^*$ *onto the set of moment maps for the action of K on V.*

Proof. We first check that Φ_0 is a moment map for the action of K on V. It is obviously smooth. For all $\xi \in \mathrm{Lie}(K)$, $x \in V$, and $g \in K$, we have

$$\begin{aligned}\xi^\sharp(xg) &= \mathrm{T}_e(\nu_{xg})(\xi) = \mathrm{T}_e(\rho_g \circ \nu_x \circ \mathrm{Int}(g))(\xi) = \mathrm{T}_x(\rho_g)(\mathrm{T}_e(\nu_x)(\mathrm{Ad}(g)\xi)) \\ &= \mathrm{T}_x(\rho_g)((\mathrm{Ad}(g)\xi)^\sharp(x)) = \rho_g((\mathrm{Ad}(g)\xi)^\sharp(x)) = (\mathrm{Ad}(g)\xi)^\sharp(x)g.\end{aligned}$$

Since Ω is K-invariant, this implies that Φ_0 is K-equivariant. For all $\xi \in \mathrm{Lie}(K)$, $x \in V$, and $w \in \mathrm{T}_x(V) = V$, we have

$$w(\Phi_0^\xi) = \mathrm{T}_x(\Phi_0^\xi)(w) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\Omega(\xi^\sharp(x+tw), x+tw)$$
$$= \frac{1}{2}(\Omega(\xi^\sharp(x), w) + \Omega(\xi^\sharp(w), x))$$
$$= \frac{1}{2}(\Omega(\xi^\sharp(x), w) - \Omega(w, \xi^\sharp(x))) = \Omega(\xi^\sharp(x), w).$$

Therefore, $H(\Phi_0^\xi) = \xi^\sharp$, and Φ_0 is a moment map for the action of K on V.

Let $\alpha \in \mathrm{Lie}(K)^*$. Then, for every $\xi \in \mathrm{Lie}(K)$, we have $\Phi_\alpha^\xi = \Phi_0^\xi + \alpha(\xi)$, and hence $\Phi_\alpha = \Phi_0 + \alpha$. As $\alpha(\xi)$ is a constant real function on V, we get $\mathrm{T}_x(\Phi_\alpha^\xi) = \mathrm{T}_x(\Phi_0^\xi)$, hence $H(\Phi_\alpha^\xi) = H(\Phi_0^\xi) = \xi^\sharp$. On the other hand, as Φ_0 is K-equivariant, for all $x \in V$, $g \in K$, and $\xi \in \mathrm{Lie}(K)$, we have

$$\Phi_\alpha(xg)(\xi) = \Phi_0(xg)(\xi) + \alpha(\xi) = \Phi_0(x)(\mathrm{Ad}(g)\xi) + \alpha(\xi)$$
$$= \Phi_\alpha(x)(\mathrm{Ad}(g)\xi) - (\alpha(\mathrm{Ad}(g)\xi) - \alpha(\xi)).$$

Thus, Φ_α is K-equivariant if and only if $\alpha(\mathrm{Ad}(g)\xi) = \alpha(\xi)$ for all $g \in G$ and $\xi \in \mathrm{Lie}(K)$, that is, if and only if α is K-invariant.

Now, the map $\alpha \mapsto \Phi_\alpha$ is clearly injective. Suppose $\Psi : V \to \mathrm{Lie}(K)^*$ is a moment map for the action of K on V. Then, for all $\xi \in \mathrm{Lie}(K)$, we have $H(\Psi^\xi) = H(\Phi_0^\xi) = \xi^\sharp$. By the definition of Hamiltonian vector field , we have $\mathrm{T}_x(\Psi^\xi) = \mathrm{T}_x(\Phi_0^\xi)$ for all $x \in X$; as V is connected, this implies that $\Psi^\xi - \Phi_0^\xi = \alpha(\xi)$ for some $\alpha(\xi) \in \mathbb{R}$. The map $\alpha : \xi \mapsto \alpha(\xi)$ is $\mathbb{R}$-linear, since $\alpha = \Psi(0) - \Phi_0(0)$, and $\Psi(x)$ and $\Phi_0(x)$ are $\mathbb{R}$-linear functions on $\mathrm{Lie}(K)$, for every $x \in V$. It follows that $\alpha \mapsto \Phi_\alpha$ is surjective. □

Lemma 3.2.18 *If K is a connected real Lie group, then the set of K-invariant elements of* $\mathrm{Lie}(K)^*$ *equals* $\mathrm{Ann}([\mathrm{Lie}(K), \mathrm{Lie}(K)])$*, the annihilator in* $\mathrm{Lie}(K)^*$ *of the derived* *algebra of* Lie(K).

Proof. For each $\xi \in \mathrm{Lie}(K)$, define a function $\psi_\xi : K \to \mathbb{R}$ by setting

$$\psi_\xi(g) = \alpha(\mathrm{Ad}(g)\xi)$$

for all $g \in K$. Then, for all $\eta \in \mathrm{Lie}(K)$, we have

$$\begin{aligned}\mathrm{T}_e(\psi_\xi)(\eta) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\psi_\xi(\exp(t\eta)) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\alpha(\mathrm{Ad}(\exp(t\eta))\xi)\\ &= \alpha\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big(\mathrm{Ad}(\exp(t\eta))\xi\big)\Big) = \alpha(\mathrm{ad}(\eta)\xi) = \alpha([\eta,\xi]).\end{aligned}$$

Now, if g is an arbitrary element of K, we have $\psi_\xi \circ \rho_g = \psi_{\mathrm{Ad}(g)\xi}$, where $\rho_g : K \to K$ is the right translation by g. Therefore, $\mathrm{T}_g(\psi_\xi) \circ \mathrm{T}_e(\rho_g) = \mathrm{T}_e(\psi_{\mathrm{Ad}(g)\xi})$. Thus, $\mathrm{T}_g(\psi_\xi) = 0$ for all $\xi \in \mathrm{Lie}(K)$ if and only if $\mathrm{T}_e(\psi_\xi) = 0$. Now, as $\mathrm{T}_e(\rho_g) : \mathrm{Lie}(K) \to \mathrm{T}_g(K)$ and $\mathrm{Ad}(g) : \mathrm{Lie}(K) \to \mathrm{Lie}(K)$ are isomorphisms for all $g \in K$, and since K is connected, the following statements are equivalent:

1. α is K-invariant.
2. ψ_ξ is constant for all $\xi \in \mathrm{Lie}(K)$.
3. $\mathrm{T}_e(\psi_\xi) = 0$ for all $\xi \in \mathrm{Lie}(K)$.
4. $\alpha([\eta,\xi]) = 0$ for all $\xi, \eta \in \mathrm{Lie}(K)$.
5. $\alpha \in \mathrm{Ann}([\mathrm{Lie}(K), \mathrm{Lie}(K)])$.

□

We now state one of the most significant results of symplectic geometry, namely the famous *Marsden-Weinstein reduction theorem*. Let (X,Ω) be a symplectic manifold and G be a real Lie group acting symplectically on X. Assume that there is a moment map $\Phi\colon X \to \mathrm{Lie}(G)^*$ satisfying the following conditions:

1. $0 \in \mathrm{Lie}(G)^*$ is a regular value of Φ. (If $d\Phi_x \colon \mathrm{T}_x(X) \to \mathrm{Lie}(G)^*$ is surjective for every $x \in \Phi^{-1}(0)$, then the implicit function theorem guarantees 1.)

2. The action G on $\Phi^{-1}(0)$ is free and that each point $x \in \Phi^{-1}(0)$ there exist a submanifold S_x of $\Phi^{-1}(0)$ through x which is transversal to the orbit $G(x)$ in the sense that
$$\mathrm{T}_x(\Phi^{-1}(0)) = \mathrm{T}_x(S_x) + \mathrm{T}_x(G(x)).$$

Then there exist a unique symplectic form ω on $\Phi^{-1}(0)/G$ such that
$$\pi^*(\omega) = i^*(\Omega)$$
on $\Phi^{-1}(0)$; where $i\colon \Phi^{-1}(0) \to X$ is the canonical inclusion, and $\pi\colon \Phi^{-1}(0) \to \Phi^{-1}(0)/G$ is the canonical projection.

Example 3.2.12

$$\textit{Let } \Omega = \frac{\sqrt{-1}}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j = \sum_{j=1}^{n} dx_j \wedge dy_j$$
$$= \sum_{j=1}^{n} r_j dr_j \wedge d\theta_j$$

be the standard symplectic form on $\mathbb{C}^n$. Consider the following smooth action $\Psi\colon \mathbb{S}^1 \times \mathbb{C}^n \to \mathbb{C}^n$ defined by
$$(t,z) \mapsto t.z$$

The action Ψ is Hamiltonian with moment map $\Phi\colon \mathbb{C}^n \to \mathbb{R}$ is given by
$$\Phi(z) = -\frac{1}{2}\,|z|^2 + \textit{Constant},$$

for all $z = (z_1, ..., z_n) \in \mathbb{C}^n$. In polar coordinates, this is given by

$$\Phi(re^{\sqrt{-1}\theta}) = -\frac{1}{2}\, r^2.$$

For this, one can calculate the following:

- $d\Phi = -\frac{1}{2}d(\sum_{j=1}^{n}\; {r_j}^2)$

- The vector field $\xi^\sharp$ generated by the action Ψ is given by

$$\xi^\sharp = \frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\theta_2} + ... + \frac{\partial}{\partial\theta_n}$$

- $i_{\xi^\sharp}(\Omega) = -\frac{1}{2}\sum_{j=1}^{n}\; d{r_j}^2.$

The above calculation shows that Φ is a moment map for Ψ. If we choose the constant to be $\frac{1}{2}$, then $\Phi^{-1}(0) = \mathbb{S}^{2n-1}$ is the unit sphere. The orbit space of the zero level of the moment map is

$$\Phi^{-1}(0)/\mathbb{S}^1 = \mathbb{S}^{2n-1}/\mathbb{S}^1 \cong \mathbb{P}^{n-1}.$$

Thus the reduced space is isomorphic to $\mathbb{P}^{n-1}$.

3.2.2 Kähler quotients

First, we recall some basic facts and fix some notations which we need further.

Let V be a finite-dimensional $\mathbb{R}$-vector space, and V^{c} its complexification, the $\mathbb{C}$-vector space $\mathbb{C} \otimes_{\mathbb{R}} V$. The map $v \mapsto 1 \otimes v$ is an $\mathbb{R}$-monomorphism from V to V^{c}. We will identify V with an $\mathbb{R}$-subspace of V^{c} through this map. Then, we have an $\mathbb{R}$-vector space decomposition

$$V^{\mathrm{c}} = V \oplus \sqrt{-1}V.$$

This decomposition induces an anti-endomorphism $\bar{\cdot}$ of the $\mathbb{C}$-vector space V^{c}, which is defined by

$$\overline{v+\sqrt{-1}w} = v - \sqrt{-1}w$$

for all $v, w \in V$.

Let J be a complex structure on V, that is, an $\mathbb{R}$-endomorphism of V, such that $J^2 = -\mathbf{1}_V$. Denote by J^{c} the $\mathbb{C}$-endomorphism $\mathbf{1}_{\mathbb{C}} \otimes J$ of V^{c}. Then, we have a $\mathbb{C}$-vector space decomposition

$$V^{\mathrm{c}} = V' \oplus V'',$$

where V' is the $\sqrt{-1}$-eigenspace of J^{c}, and V'' the $(-\sqrt{-1})$-eigenspace of J^{c}. Let $\pi' : V^{\mathrm{c}} \to V'$ and $\pi'' : V^{\mathrm{c}} \to V''$ be the projections defined by this decomposition. We have

$$\pi'(v) = \frac{1}{2}(v - \sqrt{-1}J^{\mathrm{c}}(v)), \quad \pi''(v) = \frac{1}{2}(x + \sqrt{-1}J^{\mathrm{c}}(v))$$

for all $v \in V^{\mathrm{c}}$, and

$$V'' = \overline{V'}, \quad V' = \overline{V''},$$

where $\overline{M}$ denotes the image of any subset M of V^{c} under the anti-endomorphism $\bar{\cdot}$ of V^{c}. Moreover, π' restricts to $\mathbb{R}$-isomorphisms $p' : V \to V'$ and $p'' : V \to V''$. We will identify V with V', as $\mathbb{R}$-vector spaces, through p'. If we want to specify V in the notation for any of these maps, we will write them as π'_V, p'_V, etc.

Let V and W be two finite-dimensional vector spaces, with complex structures J_V and J_W, respectively. Suppose $f : V \to W$ is an $\mathbb{R}$-homomorphism, and let $f^{\mathrm{c}} : V^{\mathrm{c}} \to W^{\mathrm{c}}$. Then, the following statements are equivalent:

1. $J_W \circ f = f \circ J_V$.

2. $f^{\mathrm{c}}(V') \subset W'$.

3. $f^{\mathrm{c}}(V'') \subset W''$.

If these conditions hold, the diagrams

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ {\scriptstyle p'_V}\downarrow & & \downarrow{\scriptstyle p'_W} \\ V' & \xrightarrow[f']{} & W' \end{array} \qquad \begin{array}{ccc} V & \xrightarrow{f} & W \\ {\scriptstyle p''_V}\downarrow & & \downarrow{\scriptstyle p''_W} \\ V'' & \xrightarrow[f'']{} & W'' \end{array}$$

commute, where f' and f'' are the restrictions of f^{c}. In particular, under the above $\mathbb{R}$-identification of V with V' through p'_V, and the analogous identification for W, the maps $f : V \to W$ and $f' : V' \to W'$ are identified as $\mathbb{R}$-homomorphisms.

Now, we will use the above linear algebra concepts to the theory of manifolds which we are going to recall here. let X be a complex premanifold. Let $\mathrm{T}(X)$ denote the tangent bundle of the underlying smooth manifold of X, and $\mathrm{T}^{\mathrm{c}}(X)$ its complexification, the smooth $\mathbb{C}$-vector bundle $\mathbb{C} \otimes_{\mathbb{R}} \mathrm{T}(X)$ on X. For each point $x \in X$, the map $v \mapsto 1 \otimes v$ is an $\mathbb{R}$-monomorphism from the fibre $\mathrm{T}_x(X)$ of $\mathrm{T}(X)$ at x, to the fibre $\mathrm{T}^{\mathrm{c}}_x(X)$ of $\mathrm{T}^{\mathrm{c}}(X)$ at x. This gives a monomorphism $\mathrm{T}(X) \to \mathrm{T}^{\mathrm{c}}(X)$ of smooth $\mathbb{R}$-vector bundles on X. We will identify $\mathrm{T}(X)$ with a smooth $\mathbb{R}$-subbundle of $\mathrm{T}^{\mathrm{c}}(X)$ through this map. Then, we have a smooth $\mathbb{R}$-vector bundle decomposition

$$\mathrm{T}^{\mathrm{c}}(X) = \mathrm{T}(X) \oplus \sqrt{-1}\,\mathrm{T}(X)$$

on X. This decomposition induces an anti-endomorphism $\bar{\cdot}$ of the smooth $\mathbb{C}$-vector bundle $\mathrm{T}^{\mathrm{c}}(X)$ on X, which is defined by

$$\overline{v + \sqrt{-1}w} = v - \sqrt{-1}w$$

for all $x \in X$, and $v, w \in \mathrm{T}_x(X)$.

For any point $x \in X$, let $S_x(X)$ denote the $\mathbb{R}$-algebra of germs of smooth real functions at x, and $S^{\mathrm{c}}_x(X)$ the $\mathbb{C}$-algebra of germs of smooth complex functions at x. Then,

$S_x(X)$ is an $\mathbb{R}$-subalgebra of $S_x^{\mathrm{c}}(X)$, and we have an $\mathbb{R}$-vector space decomposition

$$S_x^{\mathrm{c}}(X) = S_x(X) \oplus \sqrt{-1}S_x(X).$$

This decomposition induces an anti-endomorphism $\bar{\cdot}$ of the $\mathbb{C}$-algebra $S_x^{\mathrm{c}}(V)$, which is defined by

$$\overline{\gamma + \sqrt{-1}\eta} = \gamma - \sqrt{-1}\eta$$

for all $\gamma, \eta \in S_x(X)$.

The $\mathbb{R}$-vector space $\mathrm{T}_x(X)$ is canonically identified with the $\mathbb{R}$-vector space $\mathrm{D}_{\mathbb{R}}(S_x(X), \mathbb{R})$ of $\mathbb{R}$-derivations of $S_x(X)$. This identification extends to a $\mathbb{C}$-isomorphism from $\mathrm{T}_x^{\mathrm{c}}(X)$ onto the $\mathbb{C}$-vector space $\mathrm{D}_{\mathbb{C}}(S_x^{\mathrm{c}}(X), \mathbb{C})$ of $\mathbb{C}$-derivations of $S_x^{\mathrm{c}}(X)$. We will identify the $\mathbb{C}$-vector spaces $\mathrm{T}_x^{\mathrm{c}}(X)$ and $\mathrm{D}_{\mathbb{C}}(S_x^{\mathrm{c}}(X), \mathbb{C})$ through this $\mathbb{C}$-isomorphism. Under this identification, we have

$$(v + \sqrt{-1}w)(\gamma + \sqrt{-1}\eta) = (v(\gamma) - w(\eta)) + \sqrt{-1}(v(\eta) + w(\gamma))$$

for all $v, w \in \mathrm{T}_x(X)$ and $\gamma, \eta \in S_x(X)$. We have

$$\bar{v}(\gamma) = \overline{v(\bar{\gamma})}$$

for all $v \in \mathrm{T}_x^{\mathrm{c}}(X)$ and $\gamma \in S_x^{\mathrm{c}}(X)$. The $\mathbb{R}$-subspace $\mathrm{T}_x(X)$ of $\mathrm{T}_x^{\mathrm{c}}(X)$ consists of all $v \in \mathrm{T}_x^{\mathrm{c}}(X)$ such that $v(S_x(X)) \subset \mathbb{R}$.

The complex manifold structure on X induces a smooth almost complex structure on X, that is, an endomorphism of the smooth $\mathbb{R}$-vector bundle $\mathrm{T}(X)$ on X, such that $J^2 = -\mathbf{1}_{\mathrm{T}(X)}$. Denote by J^{c} the endomorphism $\mathbf{1}_{\mathbb{C}} \otimes J$ of the smooth $\mathbb{C}$-vector bundle

$\mathrm{T}^{\mathrm{c}}(X)$ on X. Then, we have a smooth $\mathbb{C}$-vector bundle decomposition

$$\mathrm{T}^{\mathrm{c}}(X) = \mathrm{T}'(X) \oplus \mathrm{T}''(X)$$

on X, such that, for each $x \in X$, the fibre $\mathrm{T}'_x(X)$ of $\mathrm{T}'(X)$ is the $\sqrt{-1}$-eigenspace of $J^{\mathrm{c}}(x)$, and the fibre $\mathrm{T}''_x(X)$ of $\mathrm{T}''(X)$ is the $(-\sqrt{-1})$-eigenspace of $J^{\mathrm{c}}(x)$. Let $\pi' : \mathrm{T}^{\mathrm{c}}(X) \to \mathrm{T}'(X)$ and $\pi'' : \mathrm{T}^{\mathrm{c}}(X) \to \mathrm{T}''(X)$ be the projections defined by this decomposition. We have

$$\pi'(v) = \frac{1}{2}(v - \sqrt{-1}J^{\mathrm{c}}(x)(v)), \quad \pi''(v) = \frac{1}{2}(v + \sqrt{-1}J^{\mathrm{c}}(x)(v))$$

for all $x \in X$ and $v \in \mathrm{T}^{\mathrm{c}}_x(X)$. Moreover, π' restricts to isomorphisms $p' : \mathrm{T}(X) \to \mathrm{T}'(X)$ and $p'' : \mathrm{T}(X) \to \mathrm{T}''(X)$ of smooth $\mathbb{R}$-vector bundles on X. We will identify $\mathrm{T}(X)$ with $\mathrm{T}'(X)$, as smooth $\mathbb{R}$-vector bundles on X, through p'. If we want to specify X in the notation for any of these maps, we will write them as π'_X, p'_X, etc.

Let $O_x(X)$ denote the $\mathbb{C}$-subalgebra of $S^{\mathrm{c}}_x(X)$ consisting of the germs of holomorphic functions at x. Then, the map $v \mapsto v|_{O_x(X)}$ is a $\mathbb{C}$-isomorphism from $\mathrm{T}'_x(X)$ onto the $\mathbb{C}$-vector space $\mathrm{D}_{\mathbb{C}}(O_x(X), \mathbb{C})$ of $\mathbb{C}$-derivations of $O_x(X)$. We will identify the $\mathbb{C}$-vector spaces $\mathrm{T}'_x(X)$ and $\mathrm{D}_{\mathbb{C}}(O_x(X), \mathbb{C})$ through this isomorphism. Thus, the $\mathbb{C}$-subspace $\mathrm{T}'_x(X)$ of is identified with the holomorphic tangent space of X at x, and the smooth $\mathbb{C}$-subbundle $\mathrm{T}'(X)$ of $\mathrm{T}^{\mathrm{c}}(X)$ with the holomorphic tangent bundle of X.

Let X and Y be two complex manifolds, with almost complex structures J_X and J_Y, respectively. Suppose $f : X \to Y$ is a smooth map. For every point $x \in X$, let $\mathrm{T}^{\mathrm{c}}_x(f)$ denote the $\mathbb{C}$-homomorphism $\mathrm{T}_x(f)^{\mathrm{c}} : \mathrm{T}^{\mathrm{c}}_x(X) \to \mathrm{T}^{\mathrm{c}}_{f(x)}(Y)$. Then, the following statements are equivalent:

1. f is holomorphic.

2. For all $x \in X$, we have $J_Y(f(x)) \circ \mathrm{T}_x(f) = \mathrm{T}_x(f) \circ J_X(x)$.

3. For all $x \in X$, we have $\mathrm{T}^{\mathrm{c}}_x(f)(\mathrm{T}'_x(X)) \subset \mathrm{T}'_{f(x)}(Y)$.

4. For all $x \in X$, we have $\mathrm{T}^{\mathrm{c}}_x(f)(\mathrm{T}''_x(X)) \subset \mathrm{T}''_{f(x)}(Y)$.

If these conditions hold, the diagrams

$$
\begin{array}{ccc}
\mathrm{T}_x(X) & \xrightarrow{\mathrm{T}_x(f)} & \mathrm{T}_{f(x)}(Y) \\
{\scriptstyle p'_X(x)}\downarrow & & \downarrow{\scriptstyle p'_Y(f(x))} \\
\mathrm{T}'_x(X) & \xrightarrow[\mathrm{T}'_x(f)]{} & \mathrm{T}'_{f(x)}(Y)
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{T}_x(X) & \xrightarrow{\mathrm{T}_x(f)} & \mathrm{T}_{f(x)}(Y) \\
{\scriptstyle p''_X(x)}\downarrow & & \downarrow{\scriptstyle p''_Y(f(x))} \\
\mathrm{T}''_x(X) & \xrightarrow[\mathrm{T}''_x(f)]{} & \mathrm{T}''_{f(x)}(Y)
\end{array}
$$

commute, where $\mathrm{T}'_x(f)$ and $\mathrm{T}''_x(f)$ are the restrictions of $\mathrm{T}^{\mathrm{c}}_x(f)$. In particular, under the above $\mathbb{R}$-identification of $\mathrm{T}_x(X)$ with $\mathrm{T}'_x(X)$ through $p'_X(x)$, and the analogous identification for $\mathrm{T}_{f(x)}(Y)$, the maps $\mathrm{T}_x(f) : \mathrm{T}_x(X) \to \mathrm{T}_{f(x)}(Y)$ and $\mathrm{T}'_x(f) : \mathrm{T}'_x(X) \to \mathrm{T}'_{f(x)}(Y)$ are identified as $\mathbb{R}$-homomorphisms.

Having identified $\mathrm{T}_x(X)$ and $\mathrm{T}'_x(X)$ as $\mathbb{R}$-vector spaces, we denote both of them by the same symbol $\mathrm{T}_x(X)$. Similarly, for any holomorphic map $f : X \to Y$ between complex premanifolds, we write $\mathrm{T}_x(f)$ for both $\mathrm{T}_x(f) : \mathrm{T}_x(X) \to \mathrm{T}_{f(x)}(Y)$ and $\mathrm{T}'_x(f) : \mathrm{T}'_x(X) \to \mathrm{T}'_{f(x)}(Y)$, which are identified as $\mathbb{R}$-homomorphisms.

For any complex premanifold X and $x \in X$, we will identify the tangent space at x of the underlying smooth manifold of X, with the holomorphic tangent space $\mathrm{T}_x(X)$ of X at x, using the canonical $\mathbb{R}$-isomorphism between them. With this identification, for any holomorphic map $f : X \to Y$ of complex premanifolds, and $x \in X$, the real differential of f at x is equal to the $\mathbb{C}$-linear map $\mathrm{T}_x(f) : \mathrm{T}_x(X) \to \mathrm{T}_{f(x)}(Y)$, considered as an $\mathbb{R}$-linear map.

Definition 3.2.19 Let X be a complex manifold, g a Kähler metric on X, and Ω its Kähler form, that is, the closed real 2-form on X defined by $\Omega(x)(v,w) = -2\Im(g(x)(v,w))$ for all $x \in X$, and $v, w \in \mathrm{T}_x(X)$, where $\Im(t)$ denotes the imaginary part of a complex number t.

Definition 3.2.20 We say that a smooth real 2-form ω on X is *positive* if for every $x \in X$, we have $\omega(\sqrt{-1}v, \sqrt{-1}w) = \omega(v,w)$ for all $v,w \in \mathrm{T}_x(X)$, and $\omega(x)(v,\sqrt{-1}v) > 0$ for all non-zero $v \in \mathrm{T}_x(X)$.

For any $x \in X$ and $v,w \in \mathrm{T}_x(X)$, we have $g(x)(\sqrt{-1}v, \sqrt{-1}w) = g(x)(v,w)$, and $\Omega(x)(v,\sqrt{-1}v) = 2g(x)(v,v)$, hence Ω is positive. A positive 2-form is clearly non-degenerate. As Ω is closed, it follows that (X,Ω) is a smooth symplectic manifold.

Lemma 3.2.21 *Let B be the smooth Riemannian metric on X defined by $B(x)(v,w) = 2\Re(g(x)(v,w))$, where $\Re(t)$ denotes the real part of a complex number t. Then, for every point $x \in X$, and $\mathbb{R}$-subspace W of $\mathrm{T}_x(X)$, we have $W^{\perp(\Omega)} = (\sqrt{-1}W)^{\perp(B)}$, where $S^{\perp(\Omega)}$ (respectively, $S^{\perp(B)}$) is the set of all elements of $\mathrm{T}_x(X)$ that are $\Omega(x)$-orthogonal (respectively, $B(x)$-orthogonal) to any subset S of $\mathrm{T}_x(X)$. In particular, we have an $\mathbb{R}$-vector space decomposition $\mathrm{T}_x(X) = W^{\perp(\Omega)} \oplus (\sqrt{-1}W)$.*

Proof. For all $v,w \in \mathrm{T}_x(X)$, we have

$$\begin{aligned} B(x)(v,\sqrt{-1}w) &= 2\Re(g(x)(v,\sqrt{-1}w)) = 2\Re(-\sqrt{-1}g(x)(v,w)) \\ &= 2\Im(g(x)(v,w)) = -\Omega(x)(v,w). \end{aligned}$$

Therefore, an element v of $\mathrm{T}_x(X)$ is $B(x)$-orthogonal to $\sqrt{-1}W$ if and only if it is $\Omega(x)$-orthogonal to W. It follows that $(\sqrt{-1}W)^{\perp(B)} = W^{\perp(\Omega)}$. As $B(x)$ is a real inner product on $\mathrm{T}_x(X)$, we get $W^{\perp(\Omega)\perp(B)} = (\sqrt{-1}W)^{\perp(B)\perp(B)} = \sqrt{-1}W$, and $\mathrm{T}_x(X) = W^{\perp(\Omega)} \oplus W^{\perp(\Omega)\perp(B)} = W^{\perp(\Omega)} \oplus \sqrt{-1}W$. □

Let G be a complex Lie group, and K a compact subgroup of G; in particular, K is a real Lie subgroup of G. Suppose that we are given a holomorphic right action of G on X, such that the induced action of K on X preserves the Kähler metric g on X. Then, the Kähler form Ω on X is K-invariant, that is, the action of K on X is symplectic.

Let $\Phi : X \to \mathrm{Lie}(K)^*$ be a moment map for the action of K on X, X_{m} the closed subset $\Phi^{-1}(0)$ of X, and $X_{\mathrm{ms}} = X_{\mathrm{m}}G$. Then, X_{ms} is G-invariant, and, since Φ is K-equivariant, X_{m} is a K-invariant subset of X_{ms}. Denote by Y the quotient topological space X/G, and let $p : X \to Y$ be the canonical projection. Let $Y_{\mathrm{ms}} = p(X_{\mathrm{ms}})$, $p_{\mathrm{ms}} : X_{\mathrm{ms}} \to Y_{\mathrm{ms}}$ the map induced by p, and $p_{\mathrm{m}} = p_{\mathrm{ms}}|_{X_{\mathrm{m}}} : X_{\mathrm{m}} \to Y_{\mathrm{ms}}$.

Remark 3.2.22 The statement that K is a real Lie subgroup of G follows from the fact that every closed subgroup of a real Lie group G is a real Lie subgroup of G [6, Chapter III, § 8, no. 2, Theorem 2].

Proposition 3.2.23 *Suppose that the action of G on X is principal, and that*

$$P_G(X_{\mathrm{m}}, X_{\mathrm{m}}) \subset K.$$

Then:

1. *The set X_{m} is a closed smooth submanifold of X, X_{ms} is open in X, Y_{ms} is open in Y, the action of K on X_{m} is principal, and $p_{\mathrm{m}} : X_{\mathrm{m}} \to Y_{\mathrm{ms}}$ is a smooth principal K-bundle.*

2. *The action of G on X_{ms} is proper, Y_{ms} is a Hausdorff open subspace of Y, the action of G on X_{ms} is principal, and $p_{\mathrm{ms}} : X_{\mathrm{ms}} \to Y_{\mathrm{ms}}$ is a holomorphic principal G-bundle.*

3. *There exists a unique Kähler metric h_{ms} on Y_{ms}, such that $p_{\mathrm{m}}^*(\Theta_{\mathrm{ms}}) = \Omega_{\mathrm{m}}$, where Θ_{ms} is the Kähler form of h_{ms}, $\Omega_{\mathrm{m}} = i_{\mathrm{m}}^*(\Omega)$, and $i_{\mathrm{m}} : X_{\mathrm{m}} \to X$ is the inclusion map.*

Proof. (1) We will use the the remarks and the notation in Section 3.2.1. Since the action of G on X is free, we have $K_x = \{e\}$ for all $x \in X$, hence the moment map $\Phi : X \to \mathrm{Lie}(K)^*$ is a submersion. Therefore, $X_{\mathrm{m}} = \Phi^{-1}(0)$ is a closed submanifold of X, and, for all $x \in X_{\mathrm{m}}$, we have $\mathrm{T}_x(X_{\mathrm{m}}) = \mathrm{Ker}(\mathrm{T}_x(\Phi)) = \mathrm{Im}(\mathrm{T}_e(\nu_x))^{\perp(\Omega)}$.

We will next check that X_{ms} is open in X. Let $\mu_{\mathrm{m}} : X_{\mathrm{m}} \times G \to X$ be the restriction of the action map $\mu : X \times G \to X$. We claim that the smooth map μ_{m} is a submersion. For all $g \in G$, we have $\mu_{\mathrm{m}} \circ (\mathbf{1}_{X_{\mathrm{m}}} \times \rho_g) = \rho_g \circ \mu_{\mathrm{m}}$, where ρ_g denotes the translation by g on any right G-space. Therefore, it suffices to check that the $\mathbb{R}$-linear map

$$\mathrm{T}_{(x,e)}(\mu_{\mathrm{m}}) : \mathrm{T}_x(X_{\mathrm{m}}) \oplus \mathrm{Lie}(G) \to \mathrm{T}_x(X)$$

is surjective for every $x \in X_{\mathrm{m}}$. For all $w \in \mathrm{T}_x(X_{\mathrm{m}})$ and $\xi \in \mathrm{Lie}(G)$, we have

$$\mathrm{T}_{(x,e)}(\mu_{\mathrm{m}})(w,\xi) = \mathrm{T}_{(x,e)}(\mu)(w,\xi) = w + \xi^{\sharp}(x) = w + \mathrm{T}_e(\mu_x)(\xi),$$

where $\mu_x : G \to X$ is the orbit map of X. Now, putting $W = \mathrm{Im}(\mathrm{T}_e(\nu_x))$ in Lemma 3.2.21, we get

$$\mathrm{T}_x(X) = \mathrm{Im}(\mathrm{T}_e(\nu_x))^{\perp(\Omega)} \oplus (\sqrt{-1}\mathrm{Im}(\mathrm{T}_e(\nu_x))) = \mathrm{T}_x(X_{\mathrm{m}}) \oplus (\sqrt{-1}\mathrm{Im}(\mathrm{T}_e(\nu_x))).$$

Therefore, for each $u \in \mathrm{T}_x(X)$, there exist $w \in \mathrm{T}_x(X_{\mathrm{m}})$ and $\eta \in \mathrm{Lie}(K)$, such that $u = w + \sqrt{-1}\mathrm{T}_e(\nu_x)(\eta)$. But, since $\nu_x : K \to X$ is the restriction of $\mu_x : G \to X$, we have $\mathrm{T}_e(\nu_x)(\eta) = \mathrm{T}_e(\mu_x)(\eta)$. Also, since μ_x is holomorphic, the map $\mathrm{T}_e(\mu_x) : \mathrm{Lie}(G) \to \mathrm{T}_x(X)$ is $\mathbb{C}$-linear. Therefore,

$$u = w + \sqrt{-1}\mathrm{T}_e(\mu_x)(\eta) = w + \mathrm{T}_e(\mu_x)(\sqrt{-1}\eta) = \mathrm{T}_{(x,e)}(\mu_{\mathrm{m}})(w, \sqrt{-1}\eta).$$

This proves that $\mathrm{T}_{(x,e)}(\mu_{\mathrm{m}})$ is surjective for all $x \in X_{\mathrm{m}}$, hence μ_{m} is a submersion. Therefore, it is an open map. In particular, $X_{\mathrm{ms}} = X_{\mathrm{m}}G = \mu_{\mathrm{m}}(X_{\mathrm{m}} \times G)$ is open in X.

The map $p : X \to Y$ is the quotient map for a continuous action of a topological group, and is hence an open map. Therefore, as X_{ms} is open in X, $Y_{\mathrm{ms}} = p(X_{\mathrm{ms}})$ is open in Y. Moreover, as X_{ms} is G-invariant, we have $X_{\mathrm{ms}} = p^{-1}(p(X_{\mathrm{ms}})) = p^{-1}(Y_{\mathrm{ms}})$.

Now, by Proposition 3.1.15, there exists a unique structure of a complex premanifold on Y, such that p is a holomorphic submersion; moreover, this structure makes p a holomorphic principal G-bundle. Therefore, p_{ms} is also a holomorphic principal G-bundle.

The map $p_{\mathrm{m}} : X_{\mathrm{m}} \to Y_{\mathrm{ms}}$ is obviously smooth. It is surjective, because $Y_{\mathrm{ms}} = p(X_{\mathrm{ms}}) = p(X_{\mathrm{m}}G) = p(X_{\mathrm{m}}) = p_{\mathrm{m}}(X_{\mathrm{m}})$. We will now check that it is a submersion. Let $x \in X_{\mathrm{m}}$, and $w \in \mathrm{T}_{p(x)}(Y)$. Then, since $p : X \to Y$ is a holomorphic submersion, there exists $v \in \mathrm{T}_x(X)$, such that $\mathrm{T}_x(p)(v) = w$. As $\mathrm{T}_x(X) = \mathrm{T}_x(X_{\mathrm{m}}) \oplus (\sqrt{-1}\mathrm{Im}(\mathrm{T}_e(\nu_x)))$, there exist $v_{\mathrm{m}} \in \mathrm{T}_x(X_{\mathrm{m}})$ and $\xi \in \mathrm{Lie}(K)$, such that $v = v_{\mathrm{m}} + \sqrt{-1}\mathrm{T}_e(\nu_x)(\xi)$. Now, $\mathrm{T}_e(\nu_x)(\xi) = \mathrm{T}_e(\mu_x)(\xi)$ belongs to the $\mathbb{C}$-subspace $\mathrm{Im}(\mathrm{T}_e(\mu_x)) = \mathrm{T}_x(xG) = \mathrm{Ker}(\mathrm{T}_x(p))$ of $\mathrm{T}_x(X)$, hence $\mathrm{T}_x(p)(\sqrt{-1}\mathrm{T}_e(\nu_x)(\xi)) = 0$. Therefore, $w = \mathrm{T}_x(p)(v_{\mathrm{m}}) = \mathrm{T}_x(p_{\mathrm{m}})(v_{\mathrm{m}})$. This proves that p_{m} is a submersion. The condition $P_G(X_{\mathrm{m}}, X_{\mathrm{m}}) \subset K$, and the K-invariance of p_{m}, imply that $p_{\mathrm{m}}^{-1}(p_{\mathrm{m}}(x)) = xK$ for all $x \in X_{\mathrm{m}}$. Lastly, if R is the graph of the action of G on X, R_{m} that of the action of K on X_{m}, and $\varphi : R \to G$ and $\varphi_{\mathrm{m}} : R_{\mathrm{m}} \to K$ the translation maps, then $R_{\mathrm{m}} \subset R$, and φ_{m} is induced by φ. As the action of G on X is principal, φ is continuous, hence so is φ_{m}, so the action of K on X_{m} is also principal. Now, by Remark 3.1.14, p_{m} is a smooth principal K-bundle.

(2) As $p_{\mathrm{m}} : X_{\mathrm{m}} \to Y_{\mathrm{ms}}$ is a smooth principal G-bundle, it is an open map. Therefore, $p_{\mathrm{m}} \times p_{\mathrm{m}} : X_{\mathrm{m}} \times X_{\mathrm{m}} \to Y_{\mathrm{ms}} \times Y_{\mathrm{ms}}$ is also an open map. Since it is also a continuous surjection, it is a quotient map. Now, $R_{\mathrm{m}} = (p_{\mathrm{m}} \times p_{\mathrm{m}})^{-1}(\Delta_{\mathrm{ms}})$, where R_{m} is the graph of the action of K on X_{m}, and Δ_{ms} is the diagonal of Y_{ms}. Since K is compact and X_{m} is Hausdorff, the action of K on X_{m} is proper, hence R_{m} is closed in $X_{\mathrm{m}} \times X_{\mathrm{m}}$. Therefore, Δ_{ms} is closed in $Y_{\mathrm{ms}} \times Y_{\mathrm{ms}}$, so Y_{ms} is Hausdorff. The graph R_{ms} of the action of G on X_{ms} equals $(p_{\mathrm{ms}} \times p_{\mathrm{ms}})^{-1}(\Delta_{\mathrm{ms}})$, and is hence closed in $X_{\mathrm{ms}} \times X_{\mathrm{ms}}$. Moreover, R_{ms} is contained in the graph R of the action of G on X, and the translation map $\varphi_{\mathrm{ms}} : R_{\mathrm{ms}} \to G$ is the restriction of the translation map $\varphi : R \to G$. As the action of

G on X is principal, this implies that the action of G on X_{ms} is also principal. Let $\sigma_{\mathrm{ms}} : X_{\mathrm{ms}} \times G \to X_{\mathrm{ms}} \times X_{\mathrm{ms}}$ be the map $(x,g) \mapsto (x,xg)$, and let $\tau_{\mathrm{ms}} : X_{\mathrm{ms}} \times G \to R_{\mathrm{ms}}$ be the map induced by σ_{ms}. Then, by Remark (3.1.8), τ_{ms} is a homeomorphism. Thus, we can write $\sigma_{\mathrm{ms}} : X_{\mathrm{ms}} \times G \to X_{\mathrm{ms}} \times X_{\mathrm{ms}}$ is the composition of the homeomorphism $\tau_{\mathrm{ms}} : X_{\mathrm{ms}} \times G \to R_{\mathrm{ms}}$ followed by the inclusion map $j\colon R_{\mathrm{ms}} \to X_{\mathrm{ms}} \times X_{\mathrm{ms}}$.

Since R_{ms} is closed in $X_{\mathrm{ms}} \times X_{\mathrm{ms}}$, it follows that the map σ_{ms} is proper. In other words, the action of G on X_{ms} is proper. It has been proved above that Y_{ms} is open in Y, and p_{ms} is a holomorphic principal G-bundle.

(3) By hypothesis, the Kähler metric g on X is K-invariant, hence its Kähler form Ω is K-invariant. Therefore, its restriction Ω_{m} to the K-invariant smooth submanifold X_{m} of X is also K-invariant. Let $x \in X_{\mathrm{m}}$, $v \in \mathrm{T}_x(X_{\mathrm{m}})$, and $\xi \in \mathrm{Lie}(K)$. Then, since $\mathrm{T}_x(X_{\mathrm{m}}) = \mathrm{Im}(\mathrm{T}_e(\nu_x))^{\perp(\Omega)}$, we have $\Omega_{\mathrm{m}}(x)(v, \mathrm{T}_e(\nu_x)(\xi)) = \Omega(x)(v, \mathrm{T}_e(\nu_x)(\xi)) = 0$. Therefore, $\Omega(x)(v,w) = 0$ if either v or w is a vertical tangent vector at x for the principal K-bundle $p_{\mathrm{m}} : X_{\mathrm{m}} \to Y_{\mathrm{ms}}$. It follows that there exists a unique smooth 2-form Θ_{ms} on X_{ms}, such that $p_{\mathrm{m}}^*(\Theta_{\mathrm{ms}}) = \Omega_{\mathrm{m}}$. As Ω is closed, so is Ω_{m}, and hence so is Θ_{ms}.

We claim that Θ_{ms} is positive. Let $y \in Y_{\mathrm{ms}}$, and $w, w' \in \mathrm{T}_y(Y)$. Let $x \in p_{\mathrm{m}}^{-1}(y)$. Then, there exists $v, v' \in \mathrm{T}_x(X_{\mathrm{m}})$, such that $w = \mathrm{T}_x(p_{\mathrm{m}})(v)$ and $w' = \mathrm{T}_x(p_{\mathrm{m}})(v')$. Since $\mathrm{T}_x(X) = \mathrm{T}_x(X_{\mathrm{m}}) \oplus (\sqrt{-1}\mathrm{Im}(\mathrm{T}_e(\nu_x)))$, there exist $a, a' \in \mathrm{T}_x(X_{\mathrm{m}})$ and $\xi, \xi' \in \mathrm{Lie}(K)$, such that $\sqrt{-1}v = a + \sqrt{-1}\mathrm{T}_e(\nu_x)(\xi)$ and $\sqrt{-1}v' = a' + \sqrt{-1}\mathrm{T}_e(\nu_x)(\xi')$. As $\mathrm{T}_x(p)$ is $\mathbb{C}$-linear,

$$
\begin{aligned}
\sqrt{-1}w &= \sqrt{-1}\mathrm{T}_x(p_{\mathrm{m}})(v) = \sqrt{-1}\mathrm{T}_x(p)(v) = \mathrm{T}_x(p)(\sqrt{-1}v) \\
&= \mathrm{T}_x(p)(a + \sqrt{-1}\mathrm{T}_e(\nu_x)(\xi)) = \mathrm{T}_x(p)(a) + \sqrt{-1}\mathrm{T}_x(p)(\mathrm{T}_e(\nu_x)(\xi)) \\
&= \mathrm{T}_x(p_{\mathrm{m}})(a) + \sqrt{-1}\mathrm{T}_x(p)(\mathrm{T}_e(\mu_x)(\xi)) = \mathrm{T}_x(p_{\mathrm{m}})(a).
\end{aligned}
$$

Similarly, $\sqrt{-1}w' = \mathrm{T}_x(p_{\mathrm{m}})(a')$. Therefore, as $\Omega(x)$ vanishes on vertical tangent vectors for p_{m},

$$
\begin{aligned}
\Theta_{\mathrm{ms}}(y)(\sqrt{-1}w, \sqrt{-1}w') &= \Theta_{\mathrm{ms}}(y)(\mathrm{T}_x(p_{\mathrm{m}})(a), \mathrm{T}_x(p_{\mathrm{m}})(a')) = \Omega_{\mathrm{m}}(x)(a,a') = \Omega(x)(a,a') \\
&= \Omega(x)(\sqrt{-1}(v - \mathrm{T}_e(\nu_x)(\xi)), \sqrt{-1}(v' - \mathrm{T}_e(\nu_x)(\xi'))) \\
&= \Omega(x)(v - \mathrm{T}_e(\nu_x)(\xi), v' - \mathrm{T}_e(\nu_x)(\xi')) \\
&= \Omega_{\mathrm{m}}(x)(v - \mathrm{T}_e(\nu_x)(\xi), v' - \mathrm{T}_e(\nu_x)(\xi')) \\
&= \Omega_{\mathrm{m}}(x)(v,v') = \Theta_{\mathrm{ms}}(y)(w,w').
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\Theta_{\mathrm{ms}}(y)(w, \sqrt{-1}w) &= \Theta_{\mathrm{ms}}(y)(\mathrm{T}_x(p_{\mathrm{m}})(v), \mathrm{T}_x(p_{\mathrm{m}})(a)) = \Omega_{\mathrm{m}}(x)(v,a) \\
&= \Omega_{\mathrm{m}}(x)(v,a) - \Omega_{\mathrm{m}}(x)(\mathrm{T}_e(\nu_x)(\xi), a) \\
&= \Omega_{\mathrm{m}}(x)(v - \mathrm{T}_e(\nu_x)(\xi), a) = \Omega(x)(v - \mathrm{T}_e(\nu_x)(\xi), a) \\
&= \Omega(x)(v - \mathrm{T}_e(\nu_x)(\xi), \sqrt{-1}(v - \mathrm{T}_e(\nu_x)(\xi))).
\end{aligned}
$$

Now, if $w = \mathrm{T}_x(p_{\mathrm{m}})(v)$ is non-zero, then $v \neq \mathrm{T}_e(\nu_x)(\xi)$, hence, as $\Omega(x)$ is positive, $\Omega(x)(v - \mathrm{T}_e(\nu_x)(\xi), \sqrt{-1}(v - \mathrm{T}_e(\nu_x)(\xi))) > 0$. Therefore, $\Theta_{\mathrm{ms}}(y)(w, \sqrt{-1}w) > 0$. It follows that Θ_{ms} is positive. Thus, the rule

$$
h_{\mathrm{ms}}(y)(w,w') = \frac{1}{2}\big(\Theta_{\mathrm{ms}}(y)(w, \sqrt{-1}w') - \sqrt{-1}\Theta_{\mathrm{ms}}(y)(w,w')\big),
$$

for all $y \in Y_{\mathrm{ms}}$ and $w, w' \in \mathrm{T}_y(Y)$, defines a Kähler metric on Y_{ms}, whose Kähler form equals Θ_{ms}. If h is another Kähler metric on Y_{ms}, whose Kähler form Θ satisfies the condition $p_{\mathrm{m}}^*(\Theta) = \Omega_{\mathrm{m}}$, then, since Θ_{ms} is the unique smooth 2-form on Y_{ms} such that $p_{\mathrm{m}}^*(\Theta_{\mathrm{ms}}) = \Omega_{\mathrm{m}}$, we get $\Theta = \Theta_{\mathrm{ms}}$. Therefore,

$$
h(y)(w,w') = \frac{1}{2}\big(\Theta(y)(w, \sqrt{-1}w') - \sqrt{-1}\Theta(y)(w,w')\big) = h_{\mathrm{ms}}(y)(w,w')
$$

for all $y \in Y_{\rm ms}$ and $w, w' \in {\rm T}_y(Y)$. This establishes the uniqueness of $h_{\rm ms}$. □

Let X be a complex manifold, g a Kähler metric on X, and Ω its Kähler form. Let G be a complex Lie group, and K a compact subgroup of G. Suppose that we are given a holomorphic right action of G on X, such that the induced action of K on X preserves the Kähler metric g. Let $\Phi : X \to {\rm Lie}(K)^*$ be a moment map for the action of K on X, $X_{\rm m}$ the closed subset $\Phi^{-1}(0)$ of X, and $X_{\rm ms} = X_{\rm m}G$. Denote by Y the quotient topological space X/G, and let $p : X \to Y$ be the canonical projection. Let $Y_{\rm ms} = p(X_{\rm ms})$, $p_{\rm ms} : X_{\rm ms} \to Y_{\rm ms}$ the map induced by p, and $p_{\rm m} = p_{\rm ms}|_{X_{\rm m}} : X_{\rm m} \to Y_{\rm ms}$.

Let H be a normal complex Lie subgroup of G, $\overline{G}$ the complex Lie group $H\backslash G$, and $\pi : G \to \overline{G}$ the canonical projection. Let $\overline{K}$ be the compact subgroup $\pi(K)$ of $\overline{G}$, and $\pi_K : K \to \overline{K}$ the homomorphism of real Lie groups induced by π. The subset $H \cap K$ of G is a real Lie subgroup of G, and ${\rm Lie}(H \cap K)$ equals the real Lie subalgebra ${\rm Lie}(H) \cap {\rm Lie}(K)$ of ${\rm Lie}(G)$. The map ${\rm T}_e(\pi) : {\rm Lie}(G) \to {\rm Lie}(\overline{G})$ is a surjective homomorphism of complex Lie algebras with kernel ${\rm Lie}(H)$, and ${\rm T}_e(\pi_K) : {\rm Lie}(K) \to {\rm Lie}(\overline{K})$ is a surjective homomorphism of real Lie algebras with kernel ${\rm Lie}(H \cap K)$.

Remark 3.2.24 If G is a real Lie group, and if H and K are two real Lie subgroups of G, then $H \cap K$ is also a Lie subgroup of G, and its Lie algebra equals the Lie subalgebra ${\rm Lie}(H) \cap {\rm Lie}(K)$ of ${\rm Lie}(G)$ [6, Chapter III, § 6, no. 2, Corollary 2].

Corollary 3.2.25 *Suppose that $G_x = H$ for all $x \in X$, that the induced action of $\overline{G}$ on X is principal, and that*

$$\Phi(X) \subset {\rm Ann}({\rm Lie}(H \cap K)), \quad P_G(X_{\rm m}, X_{\rm m}) \subset HK.$$

Then:

1. *The set $X_{\rm m}$ is a closed smooth submanifold of X, $X_{\rm ms}$ is open in X, $Y_{\rm ms}$ is open in Y, the action of $\overline{K}$ on $X_{\rm m}$ is principal, and $p_{\rm m} : X_{\rm m} \to Y_{\rm ms}$ is a smooth principal*

$\overline{K}$-bundle.

2. *The action of $\overline{G}$ on X_{ms} is proper, Y_{ms} is a Hausdorff open subspace of Y, the action of $\overline{G}$ on X_{ms} is principal, and $p_{\mathrm{ms}} : X_{\mathrm{ms}} \to Y_{\mathrm{ms}}$ is a holomorphic principal $\overline{G}$-bundle.*

3. *There exists a unique Kähler metric h_{ms} on Y_{ms}, such that $p_{\mathrm{m}}^*(\Theta_{\mathrm{ms}}) = \Omega_{\mathrm{m}}$, where Θ_{ms} is the Kähler form of h_{ms}, $\Omega_{\mathrm{m}} = i_{\mathrm{m}}^*(\Omega)$, and $i_{\mathrm{m}} : X_{\mathrm{m}} \to X$ is the inclusion map.*

Proof. The action of $\overline{K}$ on X induced by that of $\overline{G}$ on X preserves the Kähler metric g on X. Now, Consider a real subspace $\mathscr{W}$ of $\mathrm{Lie}(K)^*$ which is defined by

$$\mathscr{W} := \{f \in \mathrm{Lie}(K)^* : f \in \mathrm{Ann}(\mathrm{Lie}(H \cap K))\}.$$

Since $\mathrm{T}_e(\pi_K)\colon \mathrm{Lie}(K) \to \mathrm{Lie}(\overline{K})$ is surjective, its real dual $\mathrm{T}_e(\pi_K)^*\colon \mathrm{Lie}(\overline{K})^* \to \mathrm{Lie}(K)^*$ is injective. Moreover, since $\mathrm{Ker}(\mathrm{T}_e(\pi_K)) = \mathrm{Lie}(H \cap K)$, we have $\mathit{Image}(\mathrm{T}_e(\pi_K)^*) = \mathrm{Ker}(\mathrm{T}_e(\pi_K))^\perp = \mathrm{Lie}(H \cap K)^\perp = \mathscr{W}$. Thus, we can identify $\mathrm{Lie}(\overline{K})^*$ with $\mathscr{W} = \mathit{Image}(\mathrm{T}_e(\pi_K)^*)$ as a subspace of $\mathrm{Lie}(K)^*$.

Since $\Phi(X) \subset \mathrm{Ann}(\mathrm{Lie}(H \cap K))$, we have $\Phi(X) \subset \mathscr{W}$, and hence we get a unique induced smooth map $\overline{\Phi}\colon X \to \mathrm{Lie}(\overline{K})^* = \mathscr{W}$ such that $i \circ \overline{\Phi} = \Phi$, where $i\colon \mathscr{W} \to \mathrm{Lie}(K)^*$ is the inclusion map and $\Phi\colon X \to \mathrm{Lie}(K)^*$ is the given moment map for the action K on X. For any $x \in X$, if we denote $\nu_x\colon K \to X$ and $\overline{\nu}_x\colon \overline{K} \to X$ are the orbit maps for the actions K and $\overline{K}$ on X, then $\overline{\nu}_x \circ \pi_K = \nu_x$ and $\mathrm{T}_e(\nu_x) = \mathrm{T}_{\overline{e}}(\overline{\nu}_x) \circ \mathrm{T}_e(\pi_K)$. So, for any $\xi \in \mathrm{Lie}(K)$ and, $x \in X$, we have $\xi^\sharp(x) = \mathrm{T}_e(\nu_x)(\xi) = \mathrm{T}_{\overline{e}}(\overline{\nu}_x)(\mathrm{T}_e(\pi_K)(\xi)) = (\overline{\xi})^\sharp(x)$, and $\overline{\Phi}^{\overline{\xi}}(x) = \overline{\Phi}(x)(\overline{\xi}) = \overline{\Phi}(x) \circ (\mathrm{T}_e(\pi_K)(\xi)) = \Phi(x)(\xi) = \Phi^\xi(x)$.

Therefore, for all $x \in X$ and, $w \in \mathrm{T}_x(X)$, we have $w(\overline{\Phi}^{\overline{\xi}}) = \mathrm{T}_x(\overline{\Phi}^{\overline{\xi}}) = \mathrm{T}_x(\Phi^\xi) = w(\Phi^\xi)$, and $\Omega(\overline{\xi}^\sharp(x), w) = \Omega(\xi^\sharp(x), w)$, where $\overline{\xi} = \mathrm{T}_e(\pi_K)(\xi)$. This Proves $\overline{\Phi}\colon X \to \mathrm{Lie}(\overline{K})^*$ is a moment map for the action of $\overline{K}$ on X.

We need to verify $\overline{\Phi}^{-1}(0) = \Phi^{-1}(0)$. For this, let $x \in \Phi^{-1}(0)$. Then we have $0 \equiv \Phi(x) = \overline{\Phi}(x) \circ \mathrm{T}_e(\pi_K)$, and $\overline{\Phi}(x)(X) = \{0\}$ (since $\mathrm{T}_e(\pi_K)$ is surjective), implies that $\overline{\Phi}(x) \equiv 0$, and hence $x \in \overline{\Phi}^{-1}(0)$. Conversely, let $x \in \overline{\Phi}^{-1}(0)$. Then, we have $\Phi(x) = \overline{\Phi}(x) \circ \mathrm{T}_e(\pi_K) = 0 \circ \mathrm{T}_e(\pi_K) \equiv 0$, and hence $x \in \Phi^{-1}(0)$. Thus, $\overline{\Phi}^{-1}(0) = \Phi^{-1}(0) = X_{\mathrm{m}}$.

Moreover, $\overline{\Phi}^{-1}(0)\overline{G} = X_{\mathrm{m}}G = X_{\mathrm{ms}}$, and $P_{\overline{G}}(X_{\mathrm{m}}, X_{\mathrm{m}}) \subset \pi(P_G(X_{\mathrm{m}}, X_{\mathrm{m}})) \subset \pi(HK) \subset \pi(K) = \overline{K}$ (by using remark (3.1.17)). It is obvious that $X/\overline{G} = X/G = Y$, and the canonical projection from X to $X/\overline{G}$ equals p. Therefore, the Corollary follows from Proposition 3.2.23. □

3.2.3 The Kähler metric on the moduli of stable representations

We will follow the notation of Section 3.1.2. Thus, Q is a non-empty finite quiver, $d = (d_a)_{a \in Q_0}$ a non-zero element of $\mathbb{N}^{Q_0}$, and $V = (V_a)_{a \in Q_0}$ a family of $\mathbb{C}$-vector spaces, such that $\dim_{\mathbb{C}}(V_a) = d_a$ for all $a \in Q_0$. Fix a family $h = (h_a)_{a \in Q_0}$ of Hermitian inner products $h_a : V_a \times V_a \to \mathbb{C}$. In additon, we also fix now a rational weight $\theta \in \mathbf{Q}^{Q_0}$ of Q.

Denote by $\mathscr{A}$ the finite-dimensional $\mathbb{C}$-vector space $\bigoplus_{\alpha \in Q_1} \mathrm{Hom}_{\mathbb{C}}(V_{\mathrm{s}(\alpha)}, V_{\mathrm{t}(\alpha)})$. For any subset X of $\mathscr{A}$, let X_{schur} (respectively, X_{s}) denote the set of all points ρ in X, such that the representation (V, ρ) of Q is Schur (respectively, θ-stable). Also, denote by X_{eh} (respectively, X_{irr}) the set of all $\rho \in X$, such that the Hermitian metric h on (V, ρ) is Einstein-Hermitian with respect to θ (respectively, irreducible).

Recall that G is the complex Lie group $\prod_{a \in Q_0} \mathrm{Aut}_{\mathbb{C}}(V_a)$, with its canonical canonical holomorphic linear right action on $\mathscr{A}$. Denote by H the central complex Lie subgroup $\mathbb{C}^{\times}e$ of G, $\overline{G}$ the complex Lie group $H\backslash G$, and $\pi : G \to \overline{G}$ the canonical projection. Let K denote the compact subgroup $\prod_{a \in Q_0} \mathrm{Aut}(V_a, h_a)$, where, for each $a \in Q_0$, $\mathrm{Aut}(V_a, h_a)$ is the subgroup of $\mathrm{Aut}_{\mathbb{C}}(V_a)$ consisting of $\mathbb{C}$-automorphisms of V_a which preserve the Hermitian inner product h_a on V_a. Let $\overline{K}$ be the compact subgroup

$\pi(K)$ of $\overline{G}$, and $\pi_K : K \to \overline{K}$ the homomorphism of real Lie groups induced by π.

Let $\mathscr{B} = \mathscr{A}_{\mathrm{schur}}$. Then, by Proposition 2.1.7(3), $\mathscr{B}_{\mathrm{s}} = \mathscr{A}_{\mathrm{s}}$. On the other hand, by Proposition 2.3.16, $\mathscr{B}_{\mathrm{eh}} = \mathscr{A}_{\mathrm{eh}} \cap \mathscr{A}_{\mathrm{irr}} = \mathscr{A}_{\mathrm{eh}} \cap \mathscr{A}_{\mathrm{s}} = \mathscr{A}_{\mathrm{eh}} \cap \mathscr{B}_{\mathrm{s}}$. As noted in Section 3.1.2, $\mathscr{B}$ is a G-invariant open complex submanifold of $\mathscr{A}$, and, by Proposition 2.1.7(4), $G_\rho = H$ for all $\rho \in \mathscr{B}$. Let M denote the moduli space $\mathscr{B}/G$ of Schur representations of Q with dimension vector d, and $p : \mathscr{B} \to M$ the canonical projection. It was proved in Theorem 3.1.22 that the action of $\overline{G}$ on $\mathscr{B}$ is principal, that there exists a unique structure of a complex premanifold on M such that p is a holomorphic submersion, and that this structure in fact makes p a holomorphic principal $\overline{G}$-bundle. Let $M_{\mathrm{s}} = p(\mathscr{B}_{\mathrm{s}})$, $p_{\mathrm{s}} : \mathscr{B}_{\mathrm{s}} \to M_{\mathrm{s}}$ the map induced by p, and $p_{\mathrm{eh}} = p_{\mathrm{s}}|_{\mathscr{B}_{\mathrm{eh}}} : \mathscr{B}_{\mathrm{eh}} \to M_{\mathrm{s}}$. Recall that for any two subsets A and B of $\mathscr{A}$, $P_G(A,B)$ denotes the set of all $g \in G$, such that $Ag \cap B \neq \emptyset$.

Now, we are going to verify the hypothesis, as in the corollary (3.2.25), for the quiver representations settings. We will write these below as few lemmas.

Lemma 3.2.26 *We have $\mathscr{B}_{\mathrm{eh}}G = \mathscr{B}_{\mathrm{s}}$ and $P_G(\mathscr{B}_{\mathrm{eh}}, \mathscr{B}_{\mathrm{eh}}) \subset HK$.*

Proof. It is obvious that the subset $\mathscr{B}_{\mathrm{s}}$ of $\mathscr{B}$ is G-invariant. By the above paragraph, $\mathscr{B}_{\mathrm{eh}} \subset \mathscr{B}_{\mathrm{s}}$. Therefore, $\mathscr{B}_{\mathrm{eh}}G \subset \mathscr{B}_{\mathrm{s}}$. Conversely, if $\sigma \in \mathscr{B}_{\mathrm{s}}$, then, by Proposition 2.3.10, (V,σ) has an Einstein-Hermitian metric k with respect to θ. For each $a \in Q_0$, h_a and k_a are two Hermitian inner products on V_a, hence there exists a $\mathbb{C}$-automorphism g_a of V_a, such that $h_a(g_a(x), g_a(y)) = k_a(x,y)$ for all $x, y \in V_a$. We thus get an element $g = (g_a)_{a \in Q_0}$ of G.

Let $\rho = \sigma g^{-1}$. Then, for all $\alpha \in Q_1$, we have $\rho_\alpha^{*(h)} = g_{\mathrm{s}(\alpha)} \circ \sigma_\alpha^{*(k)} \circ g_{\mathrm{t}(\alpha)}^{-1}$, where $\rho_\alpha^{*(h)}$ is the adjoint of ρ_α with respect to $h_{\mathrm{s}(\alpha)}$ and $h_{\mathrm{t}(\alpha)}$, and $\rho_\alpha^{*(k)}$ the adjoint of ρ_α

with respect to $k_{\mathrm{s}(\alpha)}$ and $k_{\mathrm{t}(\alpha)}$. To see this, for all $v \in V_{\mathrm{s}(\alpha)}$ and $w \in V_{\mathrm{t}(\alpha)}$, we have

$$\begin{aligned} h_{\mathrm{s}(\alpha)}(v, \rho_\alpha^{*(h)}(w)) &= h_{\mathrm{t}(\alpha)}(\rho_\alpha(v), w) \\ &= h_{\mathrm{t}(\alpha)}(g_{\mathrm{t}(\alpha)}(\sigma_\alpha(g_{\mathrm{s}(\alpha)}^{-1}(v))), w) \\ &= k_{\mathrm{t}(\alpha)}(\sigma_\alpha(g_{\mathrm{s}(\alpha)}^{-1}(v)), g_{\mathrm{t}(\alpha)}^{-1}(w)) \\ &= k_{\mathrm{s}(\alpha)}(g_{\mathrm{s}(\alpha)}^{-1}(v), \sigma_\alpha^{*(k)}(g_{\mathrm{t}(\alpha)}^{-1}(w))) \\ &= h_{\mathrm{s}(\alpha)}(v, g_{\mathrm{s}(\alpha)}(\sigma_\alpha^{*(k)}(g_{\mathrm{t}(\alpha)}^{-1}(w)))). \end{aligned}$$

It follows that

$$\rho_\alpha^{*(h)} = g_{\mathrm{s}(\alpha)} \circ \sigma_\alpha^{*(k)} \circ g_{\mathrm{t}(\alpha)}^{-1}.$$

Now, we will calculate $K_\theta(V, \rho, h)$. For this,

$$\begin{aligned} K_\theta(V,\rho,h)_a &= \theta_a \mathbf{1}_{V_a} + \sum_{\alpha \in \mathrm{t}^{-1}(a)} \rho_\alpha \circ \rho_\alpha^{*(h)} - \sum_{\alpha \in \mathrm{s}^{-1}(a)} \rho_\alpha^{*(h)} \circ \rho_\alpha \\ &= \theta_a \mathbf{1}_{V_a} + \sum_{\alpha \in \mathrm{t}^{-1}(a)} g_{\mathrm{t}(\alpha)} \circ \sigma_\alpha \circ \sigma_\alpha^{*(k)} \circ g_{\mathrm{t}(\alpha)}^{-1} - \sum_{\alpha \in \mathrm{s}^{-1}(a)} g_{\mathrm{s}(\alpha)} \circ \sigma_\alpha^{*(k)} \circ \sigma_\alpha \circ g_{\mathrm{s}(\alpha)}^{-1} \\ &= \theta_a \mathbf{1}_{V_a} + g_a \circ \Big(\sum_{\alpha \in \mathrm{t}^{-1}(a)} \sigma_\alpha \circ \sigma_\alpha^{*(k)} - \sum_{\alpha \in \mathrm{s}^{-1}(a)} \sigma_\alpha^{*(k)} \circ \sigma_\alpha \Big) \circ g_a^{-1} \\ &= g_a \circ K_\theta(V,\sigma,k) \circ g_a^{-1} = g_a \circ (\mu_\theta(V,\sigma)\mathbf{1}_{V_a}) \circ g_a^{-1} = \mu_\theta(d)\mathbf{1}_{V_a}. \end{aligned}$$

Therefore, for every $a \in Q_0$, we get

$$K_\theta(V,\rho,h)_a = g_a \circ K_\theta(V,\sigma,k) \circ g_a^{-1} = \mu_\theta(d)\mathbf{1}_{V_a}.$$

Thus, the Hermitian metric h on (V, ρ) is Einstein-Hermitian, so $\rho \in \mathscr{B}_{\mathrm{eh}}$, and $\sigma = \rho g$ belongs to $\mathscr{B}_{\mathrm{eh}} G$. This proves that $\mathscr{B}_{\mathrm{eh}} G = \mathscr{B}_{\mathrm{s}}$.

Next, let $g \in P_G(\mathscr{B}_{\mathrm{eh}}, \mathscr{B}_{\mathrm{eh}})$. Then, there exist $\rho, \sigma \in \mathscr{B}_{\mathrm{eh}}$, such that $\sigma = \rho g$. Then, g is an isomorphism from (V, σ) to (V, ρ). For every $a \in Q_0$, define a Hermitian inner product k_a on V_a by $k_a(x,y) = h_a(g_a(x), g_a(y))$ for all $x, y \in V_a$. Then, as observed above, since $\sigma \in \mathscr{B}_{\mathrm{eh}}$, we have

$$K_\theta(V,\rho,k)_a = g_a^{-1} \circ K_\theta(V,\sigma,h) \circ g_a = \mu_\theta(d)\mathbf{1}_{V_a}$$

for all $a \in Q_0$, hence the Hermitian metric $k = (k_a)_{a\in Q_0}$ on (V,ρ) is Einstein-Hermitian with respect to θ. As $\rho \in \mathscr{B}_{\mathrm{eh}}$, the Hermitian metric h on (V,ρ) is also Einstein-Hermitian with respect to θ. Therefore, by Proposition 2.3.10, there exists an automorphism f of (V,ρ), such that $k_a(x,y) = h_a(f_a(x), f_a(y))$ for all $a \in Q_0$, and $x, y \in V_a$. Now, since $\rho \in \mathscr{B}$, $f = ce$ for some $c \in \mathbb{C}$. As the dimension vector d is non-zero, we have $c \neq 0$, and

$$h_a\Big(\frac{1}{c}g_a(x), \frac{1}{c}g_a(y)\Big) = k_a\Big(\frac{1}{c}x, \frac{1}{c}y\Big) = h_a\Big(\frac{1}{c}f_a(x), \frac{1}{c}f_a(y)\Big) = h_a(x,y)$$

for all $a \in Q_0$, and $x, y \in V_a$. Therefore, $\frac{1}{c}g_a \in \mathrm{Aut}(V_a, h_a)$ for all $a \in Q_0$, so $\frac{1}{c}g \in K$. Thus, $g = (ce)\big(\frac{1}{c}g\big)$ belongs to HK. It follows that $P_G(\mathscr{B}_{\mathrm{eh}}, \mathscr{B}_{\mathrm{eh}}) \subset HK$. □

The family h induces a Hermitian inner product $\langle\cdot,\cdot\rangle$ on the $\mathbb{C}$-vector space $\mathscr{A}$, and the complex Lie algebra $\mathrm{Lie}(G)$. For every point ρ of $\mathscr{A}$, the $\mathbb{C}$-vector space $\mathrm{T}_\rho(\mathscr{A})$ is canonically isomorphic to $\mathscr{A}$. Therefore, the Hermitian inner product $\langle\cdot,\cdot\rangle$ on $\mathscr{A}$ defines a Hermitian metric g on the complex manifold $\mathscr{A}$, namely $g(\rho)(\sigma,\tau) = \langle\sigma,\tau\rangle$ for all $\rho,\sigma,\tau \in \mathscr{A}$. The fundamental 2-form Ω of g is given by $\Omega(\rho)(\sigma,\tau) = -2\Im(\langle\sigma,\tau\rangle)$ for all $\rho,\sigma,\tau \in \mathscr{A}$. Since $\Omega(\rho)$ is independent of ρ, we have $\mathrm{d}\Omega = 0$, hence the Hermitian metric g on $\mathscr{A}$ is Kähler.

Remark 3.2.27

- The action of K on $\mathscr{A}$ induced by that of G preserves the Hermitian inner product $\langle\cdot,\cdot\rangle$, and hence the Kähler metric g on $\mathscr{A}$. To see this, let

$\rho = (\rho_\alpha)_{\alpha \in Q_1}$, $\sigma = (\sigma_\alpha)_{\alpha \in Q_1}$ are in $\mathscr{A}$, and $g = (g_a)_{a \in Q_0}$ in K. Then

$$\begin{aligned}
\langle \rho g, \sigma g \rangle &= \sum_{\alpha \in Q_1} \mathrm{Tr}({(\sigma g)_\alpha}^* \circ (\rho g)_\alpha) \\
&= \sum_{\alpha \in Q_1} \mathrm{Tr}(g_{s(\alpha)}^* \circ \sigma_\alpha^* \circ {g_{t(\alpha)}^{-1}}^* \circ g_{t(\alpha)}^{-1} \circ \rho_\alpha \circ g_{s(\alpha)}) \\
&= \sum_{\alpha \in Q_1} \mathrm{Tr}(g_{s(\alpha)}^* \circ \sigma_\alpha^* \circ Id \circ \rho_\alpha \circ g_{s(\alpha)}) \\
&= \sum_{\alpha \in Q_1} \mathrm{Tr}(g_{s(\alpha)}^* \circ \sigma_\alpha^* \circ Id \circ \rho_\alpha \circ g_{s(\alpha)}) \\
&= \sum_{\alpha \in Q_1} \mathrm{Tr}(g_{s(\alpha)}^* \circ \sigma_\alpha^* \circ \rho_\alpha \circ g_{s(\alpha)}) \\
&= \sum_{\alpha \in Q_1} \mathrm{Tr}(g_{s(\alpha)}^* \circ g_{s(\alpha)} \circ \sigma_\alpha^* \circ \rho_\alpha) \\
&= \sum_{\alpha \in Q_1} \mathrm{Tr}(\sigma_\alpha^* \circ \rho_\alpha) \\
&= \langle \rho, \sigma \rangle .
\end{aligned}$$

- Similarly, the action of K on $\mathrm{Lie}(G)$ induced by that of G preserves the Hermitian inner product on $\mathrm{Lie}(G)$.

Definition 3.2.28 For each $a \in Q_0$, let $\mathrm{End}(V_a, h_a)$ denote the real Lie subalgebra of $\mathrm{End}_{\mathbb{C}}(V_a)$ consisting of $\mathbb{C}$-endomorphisms u of V_a that are skew-Hermitian with respect to h_a, that is,

$$h_a(u(x), y) + h_a(x, u(y)) = 0$$

for all $x, y \in V_a$.

Then, $\mathrm{Lie}(K)$ equals the real Lie subalgebra $\bigoplus_{a \in Q_0} \mathrm{End}(V_a, h_a)$ of $\mathrm{Lie}(G)$. The Hermitian inner product $\langle \cdot, \cdot \rangle$ on $\mathrm{Lie}(G)$ restricts to a real inner product on $\mathrm{Lie}(K)$, which is given by

$$\begin{aligned}\langle \xi, \eta \rangle &= \sum_{a \in Q_0} \mathrm{Tr}(\xi_a \circ \eta_a^*) \\ &= \sum_{a \in Q_0} \mathrm{Tr}(\xi_a \circ (-\eta_a)) \\ &= -\sum_{a \in Q_0} \mathrm{Tr}(\xi_a \circ \eta_a)\end{aligned}$$

for all $\xi, \eta \in \mathrm{Lie}(K)$.

Now, we are going to define a moment map for the action K on $\mathscr{A}$.

For any point ρ in $\mathscr{A}$, let $\nu_\rho : K \to \mathscr{A}$ be the orbit map of ρ, and denote by D_ρ the $\mathbb{R}$-linear map $\mathrm{T}_e(\nu_\rho) : \mathrm{Lie}(K) \to \mathscr{A}$. Then, as in Section 3.1.2, we have

$$D_\rho(\xi) = (\rho_\alpha \circ \xi_{\mathrm{s}(\alpha)} - \xi_{\mathrm{t}(\alpha)} \circ \rho_\alpha)_{\alpha \in Q_1}.$$

For every element ξ of $\mathrm{Lie}(K)$, the vector field $\xi^\sharp$ on $\mathscr{A}$ induced by ξ is the $\mathbb{C}$-endomorphism of $\mathscr{A}$ given by $\xi^\sharp(\rho) = \mathrm{T}_e(\nu_\rho)(\xi) = D_\rho(\xi)$ for all $\rho \in \mathscr{A}$.

Recall the notation

$$\deg_\theta(d) = \sum_{a \in Q_0} \theta_a d_a, \quad \mathrm{rk}(d) = \sum_{a \in Q_0} d_a, \quad \mu_\theta(d) = \frac{\deg_\theta(d)}{\mathrm{rk}(d)},$$

where $\theta \in \mathbb{R}^{Q_0}$ is the rational weight of Q that we have fixed. If $a, b \in Q_0$, and $f \in \mathrm{Hom}_{\mathbb{C}}(V_a, V_b)$, let $f^* \in \mathrm{Hom}_{\mathbb{C}}(V_b, V_a)$ be the adjoint of f with respect to the Hermitian inner products h_a and h_b on V_a and V_b, respectively. For every point ρ of $\mathscr{A}$, and $a \in Q_0$, define an element $L_\theta(\rho)_a$ of $\mathrm{End}(V_a, h_a)$ by

$$L_\theta(\rho)_a = \sqrt{-1}\Big((\theta_a - \mu_\theta(d))\mathbf{1}_{V_a} + \sum_{\alpha \in \mathrm{t}^{-1}(a)} \rho_\alpha \circ \rho_\alpha^* - \sum_{\alpha \in \mathrm{s}^{-1}(a)} \rho_\alpha^* \circ \rho_\alpha\Big),$$

and let $L_\theta(\rho)$ be the element $(L_\theta(\rho)_a)_{a\in Q_0}$ of $\mathrm{Lie}(K)$. Define a map $\Phi_\theta : \mathscr{A} \to \mathrm{Lie}(K)^*$ by

$$\Phi_\theta(\rho)(\xi) = \langle \xi, L_\theta(\rho) \rangle$$

for all $\rho \in \mathscr{A}$ and $\xi \in \mathrm{Lie}(K)$, where $\langle \cdot, \cdot \rangle$ is the real inner product on $\mathrm{Lie}(K)$.

Next, we need to check that Φ_θ is a moment map for the action K on $\mathscr{A}$. For this, we will use the criterion which we had done in the Lemma 3.2.17, in subsection 3.2.1.

Lemma 3.2.29 *Let η denote the element $\left(\sqrt{-1}(\theta_a - \mu_\theta(d))\mathbf{1}_{V_a}\right)_{a\in Q_0}$ of $\mathrm{Lie}(K)$, and α the element of $\mathrm{Lie}(K)^*$, which is defined by $\alpha(\xi) = \langle \xi, \eta \rangle$ for all $\xi \in \mathrm{Lie}(K)$. Then,*

$$\Phi_\theta(\rho)(\xi) = \frac{1}{2}\Omega(\xi^\sharp(\rho), \rho) + \alpha(\xi)$$

for all $\rho \in \mathscr{A}$ and $\xi \in \mathrm{Lie}(K)$. In particular, Φ_θ is a moment map for the action of K on $\mathscr{A}$.

Proof. Let $\rho \in \mathscr{A}$ and $\xi \in \mathrm{Lie}(K)$. For every $a \in Q_0$, define an element $A(\rho)_a$ of $\mathrm{End}(V_a, h_a)$, by

$$A(\rho)_a = \sqrt{-1}\Big(\sum_{\alpha\in \mathrm{t}^{-1}(a)} \rho_\alpha \circ \rho_\alpha^* - \sum_{\alpha\in \mathrm{s}^{-1}(a)} \rho_\alpha^* \circ \rho_\alpha \Big),$$

and let $A(\rho)$ denote the element $(A(\rho)_a)_{a\in Q_0}$ of $\mathrm{Lie}(K)$. Then,

$$L_\theta(\rho) = A(\rho) + \eta, \quad \Phi_\theta(\rho)(\xi) = \langle \xi, A(\rho) \rangle + \alpha(\xi).$$

We claim that

$$\langle \xi, A(\rho) \rangle = \frac{1}{2}\Omega(\xi^\sharp(\rho), \rho).$$

By the definition of Ω,

$$\frac{1}{2}\Omega(\xi^\sharp(\rho),\rho) = -\Im(\langle \xi^\sharp(\rho),\rho\rangle) = \frac{\sqrt{-1}}{2}(\langle \xi^\sharp(\rho),\rho\rangle - \langle \rho, \xi^\sharp(\rho)\rangle).$$

Since K preserves the Hermitian inner product on $\mathscr{A}$, the $\mathbb{C}$-endomorphism $\xi^\sharp$ of $\mathscr{A}$ is skew-Hermitian, that is,

$$\langle \xi^\sharp(\rho),\rho\rangle + \langle \rho, \xi^\sharp(\rho)\rangle = 0.$$

Therefore,

$$\frac{1}{2}\Omega(\xi^\sharp(\rho),\rho) = \sqrt{-1}\langle \xi^\sharp(\rho),\rho\rangle = \sqrt{-1}\langle D_\rho(\xi),\rho\rangle.$$

It is easy to see that

$$\langle D_\rho(\xi),\rho\rangle = \sqrt{-1}\sum_{a\in Q_0} \mathrm{Tr}(\xi_a \circ A(\rho)_a).$$

To see this,

$$\begin{aligned}
\langle D_\rho(\xi),\rho\rangle &= \sum_{\alpha\in Q_1}\mathrm{Tr}(D_\rho(\xi)_\alpha\circ\rho_\alpha^*) = \sum_{\alpha\in Q_1}\mathrm{Tr}((\rho_\alpha\circ\xi_{\mathrm{s}(\alpha)}-\xi_{\mathrm{t}(\alpha)}\circ\rho_\alpha)\circ\rho_\alpha^*)\\
&= \sum_{\alpha\in Q_1}\mathrm{Tr}(\rho_\alpha\circ(\xi_{\mathrm{s}(\alpha)}\circ\rho_\alpha^*)) - \sum_{\alpha\in Q_1}\mathrm{Tr}(\xi_{\mathrm{t}(\alpha)}\circ\rho_\alpha\circ\rho_\alpha^*)\\
&= \sum_{\alpha\in Q_1}\mathrm{Tr}(\xi_{\mathrm{s}(\alpha)}\circ\rho_\alpha^*\circ\rho_\alpha) - \sum_{\alpha\in Q_1}\mathrm{Tr}(\xi_{\mathrm{t}(\alpha)}\circ\rho_\alpha\circ\rho_\alpha^*)\\
&= \mathrm{Tr}\Big(\sum_{\alpha\in Q_1}\xi_{\mathrm{s}(\alpha)}\circ\rho_\alpha^*\circ\rho_\alpha - \sum_{\alpha\in Q_1}\xi_{\mathrm{t}(\alpha)}\circ\rho_\alpha\circ\rho_\alpha^*\Big)\\
&= \mathrm{Tr}\Big(\sum_{a\in Q_0}\sum_{\alpha\in\mathrm{s}^{-1}(a)}\xi_{\mathrm{s}(\alpha)}\circ\rho_\alpha^*\circ\rho_\alpha - \sum_{a\in Q_0}\sum_{\alpha\in\mathrm{t}^{-1}(a)}\xi_{\mathrm{t}(\alpha)}\circ\rho_\alpha\circ\rho_\alpha^*\Big)\\
&= \mathrm{Tr}\Big(\sum_{a\in Q_0}\sum_{\alpha\in\mathrm{s}^{-1}(a)}\xi_a\circ\rho_\alpha^*\circ\rho_\alpha - \sum_{a\in Q_0}\sum_{\alpha\in\mathrm{t}^{-1}(a)}\xi_a\circ\rho_\alpha\circ\rho_\alpha^*\Big)\\
&= \mathrm{Tr}\Big(\sum_{a\in Q_0}\xi_a\circ\Big(\sum_{\alpha\in\mathrm{s}^{-1}(a)}\rho_\alpha^*\circ\rho_\alpha - \sum_{\alpha\in\mathrm{t}^{-1}(a)}\rho_\alpha\circ\rho_\alpha^*\Big)\Big)\\
&= \mathrm{Tr}\Big(\sum_{a\in Q_0}\xi_a\circ(\sqrt{-1}A(\rho)_a)\Big)\\
&= \sqrt{-1}\sum_{a\in Q_0}\mathrm{Tr}(\xi_a\circ A(\rho)_a).
\end{aligned}$$

Thus,

$$\frac{1}{2}\Omega(\xi^\sharp(\rho),\rho) = -\sum_{a\in Q_0}\mathrm{Tr}(\xi_a\circ A(\rho)_a) = \langle\xi,A(\rho)\rangle,$$

which proves the above claim, and gives the relation

$$\Phi_\theta(\rho)(\xi) = \frac{1}{2}\Omega(\xi^\sharp(\rho),\rho)+\alpha(\xi)$$

for all $\rho\in\mathscr{A}$ and $\xi\in\mathrm{Lie}(K)$. Now, the inner product on $\mathrm{Lie}(K)$ is K-invariant, and for all $g\in K$ and $\xi\in\mathrm{Lie}(K)$, we have $\mathrm{Ad}(g)^{-1}\eta = (g_a^{-1}\circ\eta_a\circ g_a)_{a\in Q_0} = \eta$, hence

$$\alpha(\mathrm{Ad}(g)\xi) = \langle\mathrm{Ad}(g)\xi,\eta\rangle = \langle\xi,\mathrm{Ad}(g)^{-1}\eta\rangle = \langle\xi,\eta\rangle = \alpha(\xi).$$

Therefore, the element α of $\mathrm{Lie}(K)^*$ is K-invariant. It follows from Lemma 3.2.17 that Φ_θ is a moment map for the action of K on $\mathscr{A}$. □

Lemma 3.2.30 *We have* $\Phi_\theta(\mathscr{A}) \subset \mathrm{Ann}(\mathrm{Lie}(H \cap K))$, *and* $\Phi_\theta^{-1}(0) = \mathscr{A}_{\mathrm{eh}}$.

Proof. Let $\rho \in \mathscr{A}$, and $\xi \in \mathrm{Lie}(H \cap K) = \mathrm{Lie}(H) \cap \mathrm{Lie}(K)$. Then, there exists a real number c, such that $\xi = \sqrt{-1}ce$, where $e = (\mathbf{1}_{V_a})_{a \in Q_0}$ is the identity element of $G \subset \mathrm{Lie}(G)$. Therefore,

$$\Phi_\theta(\rho)(\xi) = \langle \xi, L_\theta(\rho) \rangle = -\sum_{a \in Q_0} \mathrm{Tr}(\xi_a \circ L_\theta(\rho)_a) = -\sqrt{-1}c \sum_{a \in Q_0} \mathrm{Tr}(L_\theta(\rho)_a).$$

But, with $A(\rho)$ as in the proof of Lemma 3.2.29, we have, by definition,

$$L_\theta(\rho)_a = A(\rho)_a + \sqrt{-1}(\theta_a - \mu_\theta(d))\mathbf{1}_{V_a}$$

for all $a \in Q_0$. Since $\mathrm{Tr}(\mathbf{1}_{V_a}) = \dim_{\mathbb{C}}(V_a) = d_a$ for all $a \in Q_0$, and since the families

$(\mathrm{s}^{-1}(a))_{a\in Q_0}$ and $(\mathrm{t}^{-1}(a))_{a\in Q_0}$ are partitions of the set Q_1, we have

$$\begin{aligned}
\sum_{a\in Q_0}\mathrm{Tr}(L_\theta(\rho)_a) &= \sum_{a\in Q_0}\big(\mathrm{Tr}(A(\rho)_a)+\sqrt{-1}(\theta_a-\mu_\theta(d))d_a\big)\\
&= \sum_{a\in Q_0}\big(\mathrm{Tr}(A(\rho)_a)+\sqrt{-1}(\theta_a-\mu_\theta(d))d_a\big)\\
&= \sum_{a\in Q_0}\mathrm{Tr}(A(\rho)_a)+\sqrt{-1}\Big(\sum_{a\in Q_0}\theta_a d_a-\mu_\theta(d)\sum_{a\in Q_0}d_a\Big)\\
&= \sum_{a\in Q_0}\mathrm{Tr}(A(\rho)_a)+\sqrt{-1}(\deg_\theta(d)-\mu_\theta(d)\mathrm{rk}(d))\\
&= \sum_{a\in Q_0}\mathrm{Tr}(A(\rho)_a)+\sqrt{-1}(\deg_\theta(d)-\deg_\theta(d))\\
&= \sum_{a\in Q_0}\mathrm{Tr}(A(\rho)_a)=\mathrm{Tr}\Big(\sum_{a\in Q_0}A(\rho)_a\Big)\\
&= \sqrt{-1}\mathrm{Tr}\Big(\sum_{a\in Q_0}\Big(\sum_{\alpha\in \mathrm{t}^{-1}(a)}\rho_\alpha\circ\rho_\alpha^*-\sum_{\alpha\in \mathrm{s}^{-1}(a)}\rho_\alpha^*\circ\rho_\alpha\Big)\Big)\\
&= \sqrt{-1}\mathrm{Tr}\Big(\sum_{\alpha\in Q_1}\rho_\alpha\circ\rho_\alpha^*-\sum_{\alpha\in Q_1}\rho_\alpha^*\circ\rho_\alpha\Big)\\
&= \sqrt{-1}\sum_{\alpha\in Q_1}\big(\mathrm{Tr}(\rho_\alpha\circ\rho_\alpha^*)-\mathrm{Tr}(\rho_\alpha^*\circ\rho_\alpha)\big)\\
&= 0.
\end{aligned}$$

Therefore, $\Phi_\theta(\rho)(\xi)=0$, hence $\Phi_\theta(\mathscr{A})\subset \mathrm{Ann}(\mathrm{Lie}(H\cap K))$.

Lastly, in the notation of Section 2.3.3, we have $L_\theta(\rho)=\sqrt{-1}(K_\theta(V,\rho)-\mu_\theta(d)e)$ for all $\rho\in\mathscr{A}$. As $L_\theta(\rho)\in\mathrm{Lie}(K)$, and $\langle\cdot,\cdot\rangle$ is an inner product on $\mathrm{Lie}(K)$, we have

$$\Phi_\theta(\rho)=0\Leftrightarrow L_\theta(\rho)=0\Leftrightarrow K_\theta(V,\rho)=\mu_\theta(d)e.$$

Therefore, $\Phi^{-1}(0)=\mathscr{A}_{\mathrm{eh}}$. □

Theorem 3.2.31 *With notation as above, the following statements are true:*

1. *The set $\mathscr{B}_{\mathrm{eh}}$ is a closed smooth submanifold of $\mathscr{B}$, $\mathscr{B}_{\mathrm{s}}$ is open in $\mathscr{B}$, M_{s} is open in*

M, the action of $\overline{K}$ on $\mathscr{B}_{\mathrm{eh}}$ is principal, and $p_{\mathrm{eh}} : \mathscr{B}_{\mathrm{eh}} \to M_{\mathrm{s}}$ is a smooth principal $\overline{K}$-bundle.

2. *The action of $\overline{G}$ on $\mathscr{B}_{\mathrm{s}}$ is proper, M_{s} is a Hausdorff open subspace of M, the action of $\overline{G}$ on $\mathscr{B}_{\mathrm{s}}$ is principal, and $p_{\mathrm{s}} : \mathscr{B}_{\mathrm{s}} \to M_{\mathrm{s}}$ is a holomorphic principal $\overline{G}$-bundle.*

3. *There exists a unique Kähler metric h_{s} on M_{s}, such that $p_{\mathrm{eh}}^*(\Theta_{\mathrm{s}}) = \Omega_{\mathrm{eh}}$, where Θ_{s} is the Kähler form of h_{s}, $\Omega_{\mathrm{eh}} = i_{\mathrm{eh}}^*(\Omega)$, $i_{\mathrm{eh}} : \mathscr{B}_{\mathrm{eh}} \to \mathscr{B}$ is the inclusion map, and Ω is the Kähler form on $\mathscr{B}$.*

Proof. The stabiliser G_ρ of any point ρ of $\mathscr{B}$ equals H, and, by Theorem 3.1.22, the induced action of $\overline{G}$ on $\mathscr{B}$ is principal. Let $\Psi_\theta : \mathscr{B} \to \mathrm{Lie}(K)$ be the restriction of Φ_θ. By Lemma 3.2.29, Φ_θ is a moment map for the action of K on $\mathscr{A}$. Since the restriction of a moment map to an open submanifold is a moment map (remark 3.2.16), Ψ_θ is a moment map for the action of K on $\mathscr{B}$. Moreover, $\Psi_\theta(\mathscr{B}) \subset \Phi_\theta(\mathscr{A}) \subset \mathrm{Ann}(\mathrm{Lie}(H \cap K))$, and $\mathscr{B}_{\mathrm{m}} := \Psi_\theta^{-1}(0) = \Phi_\theta^{-1}(0) \cap \mathscr{B} = \mathscr{A}_{\mathrm{eh}} \cap \mathscr{B} = \mathscr{B}_{\mathrm{eh}}$. Finally, by Lemma 3.2.26, we have $\mathscr{B}_{\mathrm{ms}} := \mathscr{B}_{\mathrm{m}} G = \mathscr{B}_{\mathrm{eh}} G = \mathscr{B}_{\mathrm{s}}$, and $P_G(\mathscr{B}_{\mathrm{m}}, \mathscr{B}_{\mathrm{m}}) = P_G(\mathscr{B}_{\mathrm{eh}}, \mathscr{B}_{\mathrm{eh}}) \subset HK$. Therefore, the Theorem follows from Corollary 3.2.25. □

Chapter 4

Holomorphic Hermitian line bundle on the moduli space of stable representations

In the section 4.1, we will recall some concepts like connection, curvature, etc., which are important for the sequel. Section 4.2, is devoted to the study of descent of the trivial line bundle over a vector space, on which a complex Lie group acts linearly, to the quotient of the vector space and a decent of a Hermitian metric, connection of these line bundles to the quotient of the vector space. The same is formalized in the proposition 4.2.2 and as an immediate consequence of it, we prove one of the main Theorem 4.3.1 for quiver representations.

4.1 Preliminaries

We recall some basic concepts which are needed in the sequel (taken from [40]).

Definition 4.1.1 A smooth (resp, holomorphic) complex *vector bundle* of rank r over a smooth (resp, complex) manifold X is a pair (E,π), where, E is a smooth (resp, complex) manifold and $\pi\colon E\to X$ is a smooth (resp, holomorphic) map such that

- $E_p := \pi^{-1}(p)$, for $p\in X$, is a complex vector space of dimension r (E_p is called the fibre over p).
- For every $p\in X$, there is an open neighborhood U of p, and a diffeomorphism (resp, bi-holomorphism) $h\colon \pi^{-1}(U)\to U\times\mathbb{C}^r$ such that $h(E_p)\subset\{p\}\times\mathbb{C}^r$, and $h^p\colon E_p\to\mathbb{C}^r$, defined by the composition $E_p \underrightarrow{h} \{p\}\times\mathbb{C}^r \underrightarrow{proj} \mathbb{C}^r$, is a isomorphism of complex vector space. The pair (U,h) as above is called a local trivialization.

Definition 4.1.2 Let $E\to X$ be a smooth vector bundle. A *Hermitian metric* h on E is an assignment of a Hermitian inner product $h_x\colon E_x\times E_x\to\mathbb{C}$ ($x\in X$) such that for any open set $U\subset X$ and smooth sections $\zeta,\eta\in\Gamma(U,E)$ the function

$$\langle\zeta,\eta\rangle\colon U\to\mathbb{C},$$

$x\mapsto h_x(\zeta(x),\eta(x))$ is smooth.

A smooth vector bundle E equipped with a Hermitian metric h is called a *Hermitian vector bundle*.

Suppose that (E,h) is a Hermitian vector bundle of rank r. Then a (local) representative for the Hermitian metric h with respect to a local frame $f=(e_1,...,e_r)$ over some open set U is defined by $h(f)=[h(f)_{\rho\sigma}]$ of $r\times r$ matrix of the smooth functions, where $h(f)_{\rho\sigma}=\langle e_\rho,e_\sigma\rangle$. Thus, $h(f)$ is a positive definite Hermitian symmetric matrix.

Definition 4.1.3 A smooth *connection* D on a smooth vector bundle $E\to X$ is a $\mathbb{C}$-

linear mapping

$$D\colon \Gamma(X,E) \to A^1(X,E),$$

which satisfies $D(\varphi\xi) = d\varphi \otimes \xi + \varphi D\xi$, where $\varphi \in S(X)$ (smooth function on X) and $\xi \in \Gamma(X,E)$(smooth section on X). Note that we use the identification $A^1(X,E) \cong A^1(X) \otimes \Gamma(X,E)$ in the above definition.

Now, we recall a local description for a connection as well. We define the connection matrix $\theta(D, f)$ associated with the connections D and the frame $f = (e_1, ..., e_r)$ over some open set U of X, by setting, $\theta(D, f) = \theta(f) := [\theta_{\rho\sigma}(D, f)]$, where $\theta_{\rho\sigma}(D, f) \in A^1(U)$ is given by

$$De_\sigma = \sum_{\rho=1}^{r} \theta_{\rho\sigma}(D, f) \otimes e_\rho.$$

Remark 4.1.4 Given a connection D on a smooth vector bundle $E \to X$, we define the *covariant derivative* with respect to a vector field V of X by

$$D_V : \Gamma(X,E) \to \Gamma(X,E),$$

$s \mapsto \langle Ds, V\rangle$ and this map satisfies the following properties:

- $D_V(s)$ is linear over $C^\infty(X)$ in V, i.e. for all $f, g \in C^\infty(X)$,

$$D_{fV_1+gV_2}(s) = fD_{V_1}(s) + gD_{V_2}(s).$$

- $D_V(s)$ is linear over $\mathbb{R}$ in s, i.e. for all $a, b \in \mathbb{R}$,

$$D_V(as+bt) = aD_V(s) + bD_V(t).$$

- $D_V(fs) = fD_V(s) + V(f)\, s$, for $f \in C^\infty(X)$.

Definition 4.1.5 Let $E \to X$ be a smooth vector bundle with a connection D and let $\theta(f)$ be the associated connection matrix for a local frame f on an open set U on X. We define an $r \times r$ matrix of 2-forms

$$\Theta(D,f) := d\theta(f) + \theta(f) \wedge \theta(f),$$

where $\Theta_{\rho\sigma}(D,f) = d\theta_{\rho\sigma} + \sum_{k=1}^{r}\ \theta_{\rho k} \wedge \theta_{k\sigma}$. We call $\Theta(D,f)$ the curvature matrix associated with the connection matrix $\theta(f)$. Thus, the unique $\mathbb{C}$-linear mapping

$$\Theta\colon \Gamma(X,E) \to A^2(X,E)$$

such that for all local frame f, has the representation as

$$\Theta(f) = d\theta(f) + \theta(f) \wedge \theta(f)$$

is called the *curvature* associated to the connection D on X and is denoted by $\Theta_E(D)$.

Remark 4.1.6 Let $E \to X$ be a smooth vector bundle with a connection D. Then, for any vector fields V,W of X, the curvature $\Theta_E(D)$ induces a map

$$\Theta(V,W)\colon \Gamma(X,E) \to \Gamma(X,E),$$

such that $\Theta(V,W) = D_V D_W - D_W D_V - D_{[V,W]}$, is called the curvature transform determined by V,W.

Let D be the connection of a complex line bundle $E \to X$. Then, there exists $\omega \in \Omega^2(X)$ such that for any vector fields V, W of X, we have

$$\omega(V,W) = D_V D_W - D_W D_V - D_{[V,W]}.$$

The form ω is determined by this equation, is closed and is known as the *curvature* of D.

Definition 4.1.7 Let $E \to X$ be a smooth line bundle equipped with a connection D. Then, the first Chern form of E relative to the connection D is defined to be

$$c_1(E,D) := \frac{\sqrt{-1}}{2\pi}\Theta_E(D),$$

and the first Chern class of E denoted by $c_1(E)$, is the cohomology class of $c_1(E,D)$ in the de Rham group $H^2(X,\mathbb{C})$.

Suppose now that $E \to X$ is a holomorphic vector bundle over a complex manifold X. If E, as a differentiable bundle, is equipped with a smooth Hermitian metric h, we call to it as a Hermitian holomorphic vector bundle. Suppose that

$$D\colon \Gamma(X,E) \to A^1(X,E) = A^{1,0}(X,E) \oplus A^{0,1}(X,E)$$

is a connection on E. Then, D splits naturally into $D = D' + D''$, where

$$D'\colon \Gamma(X,E) \to A^{1,0}(X,E)$$
$$D''\colon \Gamma(X,E) \to A^{0,1}(X,E)$$

Some properties of holomorphic vector bundles are recalled for their usage in the sequel.

- If (E,h) is a Hermitian holomorphic vector bundle over a complex manifold X, then h induces canonically the connection, $D(h)$, on E which satisfies, for an open set W of X,

  1. For $\xi,\eta \in \Gamma(W,E)$, $d\langle \xi,\eta\rangle = \langle D\xi,\eta\rangle + \langle \xi, D\eta\rangle$ i.e., D is compatible with the metric h.
  2. If $\xi \in \mathscr{O}(W,E)$, i.e., is a holomorphic section of E, then $D''\xi = 0$.

In local coordinates: For a holomorphic frame f, we have

$$\begin{aligned} \theta(f) &= h(f)^{-1}\partial h(f), \\ D' &= \partial + \theta(f), \\ D'' &= \overline{\partial}. \end{aligned}$$

- Let D be the canonical connection of a Hermitian holomorphic vector bundle $E \to X$, with Hermitian metric h. Let $\theta(f)$ and $\Theta(f)$ be the connection and curvature matrices defined by D with respect to a holomorphic frame f. Then

  1. $\theta(f)$ is of type $(1,0)$.
  2. $\Theta(f) = \overline{\partial}\theta(f)$ and hence $\Theta(f)$ is of type $(1,1)$.

With restriction to holomorphic line bundles, the following is true.

Let $\widetilde{H}^2(X,\mathbb{Z})$ denote the image of $H^2(X,\mathbb{Z})$ in $H^2(X,\mathbb{R})$ under the natural homomorphism induced by the inclusion of the constant sheaves $\mathbb{Z} \subset \mathbb{R}$.

- Let $E \to X$ be a complex line bundle. Then $c_1(E) \in \widetilde{H}^2(X,\mathbb{Z})$.

We want to give one principal example (see [40]) concerning the computation of connections and curvatures and also this example gives some understanding for the theory involved in this chapter.

Example 4.1.1 [40] Consider the complex manifold $\mathbb{P}_{n-1}(\mathbb{C}) = \frac{\mathbb{C}^n/\{0\}}{\sim}$, where $\sim$ is the obvious relation. Let

$$U_{1,n} := \{(p,v) : p \in \mathbb{P}_{n-1}, v \in [p]\} \subset \mathbb{P}_{n-1} \times \mathbb{C}^n.$$

Then, for $\alpha = 0,\dots,n-1$, the biholomorphism $f_\alpha : U_\alpha \to \mathbb{C}^n$ by

$$f_\alpha([z_0,\dots,z_{n-1}]) = (\frac{z_0}{z_\alpha},\dots,\frac{z_{n-1}}{z_\alpha})$$

induce a holomorphic map

$$g_{\alpha\beta} := f_\alpha \circ f_\beta{}^{-1} : U_\alpha \cap U_\beta \to GL(n,\mathbb{C});\ \ (\alpha,\beta = 0,\dots,n-1)$$

gives the holomorphic line bundle structure on $U_{1,n}$. Moreover, we can think each $f_\alpha : U_\alpha \to \mathbb{C}^n$ is a local section (i.e. an element of $\Gamma(U_\alpha, U_{1,n})$) which is also a holomorphic local frame of the line bundle $U_{1,n}$ on U_α.

We define a metric on $U_{1,n}$ by letting

$$h(f_\alpha) := \overline{f_\alpha}^t f_\alpha,$$

for each $\alpha \in \{0,\dots,n-1\}$. This metric is the restricting of the standard Hermitian metric on $\mathbb{C}^n$ to the fibres of $U_{1,n} \to \mathbb{P}_{n-1}$. First we note that $h(f_\alpha)$ is positive definite since, $\overline{z}^t h(f_\alpha) z = \overline{f_\alpha(z)^t} f_\alpha(z) = |f_\alpha(z)|^2 > 0$ if $z \neq 0$ in U_α. Thus we get a well defined Hermitian metric h on $U_{1,n}$.

We can now compute the canonical connection and curvature for $U_{1,n}$ with respect to this natural metric. Note that from above discussion that (letting $f = f_\alpha$, $U = U_\alpha$, e.t.c), we have

$$\theta(f) = h(f)^{-1}\partial h(f),$$
$$\Theta(f) = \overline{\partial}\theta(f).$$

Since $\overline{\partial}h(f)^{-1} = -h(f)^{-1}\,\overline{\partial}h(f)\,h(f)^{-1}$ and $df = \partial f$, we have

$$\begin{aligned}\Theta(f) &= \overline{\partial}\theta(f) \\ &= \overline{\partial}\{h(f)^{-1}\partial h(f)\} \\ &= h^{-1}\cdot\overline{df}^t\wedge df - h^{-1}\cdot\overline{df}^t\cdot f\cdot h^{-1}\wedge\overline{f}^t\cdot df \\ &= -\frac{\langle f,f\rangle\langle df,df\rangle - \langle df,f\rangle\wedge\langle f,df\rangle}{\langle f,f\rangle^2}.\end{aligned}$$

If we write $f = (\xi_1,...,\xi_n)$ (as a column vector), where $\xi_j \in \mathscr{O}(U)$, then

$$\begin{aligned}df &= (d\xi_1,...,d\xi_n), \textit{(as a column vector)}, \\ \overline{df}^t &= (d\overline{\xi}_1,...,d\overline{\xi}_n),\end{aligned}$$

and we obtain

$$\Theta(f) = -\frac{|f|^2\,\sum_{i=1}^n d\xi_i\wedge d\overline{\xi}_i - \sum_{i,j=1}^n \overline{\xi}_i\xi_j d\xi_i\wedge d\overline{\xi}_j}{|f|^4},$$

where $|f|^2 = \sum_{i=1}^n |\xi_i|^2 \neq 0$. As $\xi_1,...,\xi_n$ are homogeneous coordinates for $\mathbb{P}_{n-1}$, and by the homogeneity of the equation of $\Theta(f)$ above, we see that the local expression of Θ induces a well-defined 2-form on all of $\mathbb{P}_{n-1}$. On the other hand, the Kähler form of $\mathbb{P}_{n-1}$ associated to the Kähler metric h with respect to those homogeneous coordinates

$\xi_1,\ldots,\xi_n$ is given by

$$\Omega(f)=\frac{\sqrt{-1}}{2}\,\frac{|f|^2\ \sum_{i=1}^n d\xi_i\wedge d\overline{\xi}_i-\sum_{i,j=1}^n\overline{\xi}_i\xi_j d\xi_i\wedge d\overline{\xi}_j}{|f|^4}.$$

4.2 Line bundles on quotients of vector spaces

Let V be a finite-dimensional $\mathbb{C}$-vector space, $\langle\cdot,\cdot\rangle$ a Hermitian inner product on V, and $\Omega=-2\Im(\langle\cdot,\cdot\rangle)$ its fundamental 2-form. We will consider V to be a Kähler manifold in the usual way. Let G be a complex Lie group, and K a real Lie subgroup of G. Suppose that we are given a holomorphic linear right action of G, and that the induced action of K on V preserves the Hermitian inner product $\langle\cdot,\cdot\rangle$ on V.

Let $\chi : G\to\mathbb{C}^\times$ be a character of G, and suppose that $\chi(K)\subset \mathrm{U}(1)$. Then, $\mathrm{T}_e(\chi)(\mathrm{Lie}(K))$ is contained in the $\mathbb{R}$-subspace $\mathrm{Lie}(\mathrm{U}(1))=\sqrt{-1}\mathbb{R}$ of $\mathrm{Lie}(\mathbb{C}^\times)=\mathbb{C}$. Fix a non-zero real number λ. Let α be the element of $\mathrm{Lie}(K)^*$ defined by $\alpha(\xi)=-\frac{\sqrt{-1}}{\lambda}\mathrm{T}_e(\chi)(\xi)$ for all $\xi\in\mathrm{Lie}(K)$. Since $\chi(gag^{-1})=\chi(a)$ for all $a,g\in G$, α is K-invariant. Therefore, by Lemma 3.2.17, the map $\Phi_\alpha : V\to\mathrm{Lie}(K)^*$, which is defined by

$$\Phi_\alpha(x)(\xi)=\frac{1}{2}\Omega(\xi^\sharp(x),x)+\alpha(\xi)$$

for all $x\in V$ and $\xi\in\mathrm{Lie}(K)$, is a moment map for the action of K on V.

Let E denote the trivial holomorphic line bundle $V\times\mathbb{C}$ on V. Given a character $\chi : G\to\mathbb{C}^\times$ of G with $\chi(K)\subset\mathrm{U}(1)$, we lift the G-action to the trivial line bundle E as follows. Define a right action of G on E by setting $(x,a)g=(xg,\chi(g)^{-1}a)$ for all $(x,a)\in E$ and $g\in G$. Let $\Gamma(E)$ denote the $\mathbb{C}$-vector space of smooth sections of E on V. For each $\xi\in\mathrm{Lie}(G)$ and $s\in\Gamma(E)$, define another section $\xi s\in\Gamma(E)$ by

$$(\xi s)(x)=\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big(s(x\exp(t\xi))\exp(-t\xi)\big)$$

for all $x \in V$. For every $x \in V$, define a Hermitian inner product $h(x) : E(x) \times E(x) \to \mathbb{C}$ by

$$h(x)((x,a),(x,b)) = \exp(\lambda \|x\|^2) a\bar{b}$$

for all $a, b \in \mathbb{C}$, where $\|x\|^2 = \langle x, x \rangle$. Define a map $s_0 : V \to E$ by $s_0(x) = (x, 1)$ for all $x \in V$. It is a global holomorphic frame of E on V. Now, for any holomorphic sections $s, t \in \Gamma(V, E)$, we can write $s = f.s_0$ and $t = g.s_0$ for some holomorphic functions $f, g \colon V \to \mathbb{C}$. Then, for all $x \in V$,

$$\langle s,t \rangle(x) = h(x)(f(x)(x,1), g(x)(x,1)) = f(x)\bar{g}(x) \exp(\lambda \|x\|^2),$$

and hence $\langle s, t \rangle \colon V \to \mathbb{C}$ is a smooth function, h is smooth metric. Thus, (E, h) is a holomorphic Hermitian line bundle on V.

This gives a smooth Hermitian metric h on E.

Lemma 4.2.1 *Let ∇ be the canonical connection of the Hermitian holomorphic line bundle (E,h) on V. Then:*

1. *For all $\xi \in \mathrm{Lie}(K)$ and $s \in \Gamma(E)$, we have $\nabla_{\xi^\sharp}(s) = \xi s - \lambda \sqrt{-1} \Phi_\alpha^\xi s$.*
2. *The first Chern form $c_1(E,h)$ of ∇ equals $-\frac{\lambda}{2\pi}\Omega$.*

Proof. Let $\xi \in \mathrm{Lie}(K)$. Define a map $s_0 : V \to E$ by $s_0(x) = (x, 1)$ for all $x \in V$. It is a holomorphic frame of E on V. We have

$$\nabla_v(s_0) = \lambda \langle v, x \rangle s_0(x)$$

for all $x \in V$ and $v \in \mathrm{T}_x(V) = V$. To see this , Let ω be the connection form of ∇ with respect to the holomorphic frame s_0 of E on V. Then, ω is a smooth $(1,0)$-form on V, and

$$\nabla(s_0) = \omega \otimes s_0.$$

If $H : V \to \mathbb{R}$ is the smooth positive function defined by

$$H(x) = h(x)(s_0(x), s_0(x)) = h(x)((x,1),(x,1)) = \exp(\lambda N(x)),$$

then

$$\omega = (\partial H)H^{-1} = \partial(\log H) = \partial(\lambda N),$$

where $N : V \to \mathbb{R}$ is the smooth function $x \mapsto \|x\|^2$. Let $(e_i)_{i=1}^n$ be an orthonormal basis of V, and $(z_i)_{i=1}^n$ the dual basis of V^*. Then,

$$N = \sum_{i=1}^n z_i \overline{z_i}.$$

Therefore,

$$\omega = \lambda \partial N = \lambda \sum_{i=1}^n \overline{z_i} \partial z_i = \lambda \sum_{i=1}^n \overline{z_i} \, \mathrm{d} z_i,$$

since $\partial \overline{z_i} = 0$, $\overline{\partial} z_i = 0$, and $\mathrm{d} z_i = \partial z_i$, as $\overline{z_i}$ is anti-holomorphic, and z_i is holomorphic, for all $i = 1, \dots, n$. Thus, if $x = \sum_{i=1}^n x_i e_i$ is a point in V, then

$$\omega(x) = \lambda \sum_{i=1}^n \overline{z_i}(x) \, \mathrm{d} z_i(x) = \lambda \sum_{i=1}^n \overline{x_i} z_i,$$

hence, for every element $v = \sum_{i=1}^n v_i e_i$ of $\mathrm{T}_x(V) = V$, we have

$$\omega(x)(v) = \lambda \Big(\sum_{i=1}^n \overline{x_i} z_i \Big) \Big(\sum_{j=1}^n v_j e_j \Big) = \lambda \sum_{i,j=1}^n \overline{x_i} v_j \delta_{ij} = \lambda \sum_{i=1}^n v_i \overline{x_i} = \lambda \langle v, x \rangle.$$

It follows that

$$\nabla_v(s_0) = (\omega(x) \otimes s_0(x))(v) = \omega(x)(v) s_0(x) = \lambda \langle v, x \rangle s_0(x).$$

Now, let $\xi \in \mathrm{Lie}(K)$ and $x \in V$. For every complex number t, we have $-2\Im(t) =$

$\sqrt{-1}(t-\bar{t})$. Also,

$$\overline{\langle \xi^\sharp(x), x\rangle} = \langle x, \xi^\sharp(x)\rangle.$$

Therefore,

$$\Omega(\xi^\sharp(x), x) = -2\Im(\langle \xi^\sharp(x), x\rangle) = \sqrt{-1}(\langle \xi^\sharp(x), x\rangle - \langle x, \xi^\sharp(x)\rangle).$$

Since the $\mathbb{C}$-endomorphism $\xi^\sharp$ of V is skew-Hermitian, we have

$$\langle \xi^\sharp(x), x\rangle - \langle x, \xi^\sharp(x)\rangle = 0.$$

Thus, we get

$$\Omega(\xi^\sharp(x), x) = 2\sqrt{-1}\langle \xi^\sharp(x), x\rangle,$$

hence

$$\langle \xi^\sharp(x), x\rangle = -\frac{\sqrt{-1}}{2}\Omega(\xi^\sharp(x), x). \tag{4.1}$$

Therefore,

$$\begin{aligned}\nabla_{\xi^\sharp}(s_0)(x) &= \lambda\langle \xi^\sharp(x), x\rangle s_0(x)\\ &= -\frac{\lambda\sqrt{-1}}{2}\Omega(\xi^\sharp(x), x)s_0(x). \ (\textit{by equation } 4.1)\end{aligned}$$

On the other hand,

$$\begin{aligned}(\xi s_0)(x) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big(s_0(\exp(t\xi))\exp(-t\xi)\big) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big(x, \chi(\exp(-t\xi))^{-1}\big)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big(\chi(\exp(t\xi))s_0(x)\big) = \Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\chi(\exp(t\xi))\Big)s_0(x)\\ &= \mathrm{T}_e(\chi)(\xi)s_0(x)\\ &= \lambda\sqrt{-1}\alpha(\xi)s_0(x).\end{aligned}$$

Thus,

$$(\xi s_0)(x)-\nabla_{\xi^\sharp}(s_0)(x)=\lambda\sqrt{-1}\big(\alpha(\xi)+\frac{1}{2}\Omega(\xi^\sharp(x),x)\big)s_0(x)=\lambda\sqrt{-1}\Phi_\alpha(x)(\xi)s_0(x).$$

It follows that

$$\xi s_0-\nabla_{\xi^\sharp}(s_0)=\lambda\sqrt{-1}\Phi_\alpha^\xi s_0.$$

Now, let s be an arbitrary element of $\Gamma(E)$. Then, there exists a smooth complex function f on V, such that $s=fs_0$. Then,

$$\begin{aligned}(\xi(fs))(x)&=\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\Big(f(x\exp(t\xi))s(x\exp(t\xi))\exp(-t\xi)\Big)\\&=\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}f(x\exp(t\xi))\Big)s(x)+f(x)\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}s(x\exp(t\xi))\exp(-t\xi)\Big)\\&=\mathrm{T}_x(f)(\xi^\sharp(x))s(x)+f(x)(\xi s)(x)\\&=\xi^\sharp(f)(x)s(x)+f(x)(\xi s)(x).\end{aligned}$$

Thus,

$$\xi(fs_0)=\xi^\sharp(f)s_0+f(\xi s_0).$$

Therefore,

$$\begin{aligned}\xi s-\nabla_{\xi^\sharp}(s)&=\big(\xi^\sharp(f)s_0+f(\xi s_0)\big)-\big(\xi^\sharp(f)s_0+f\nabla_{\xi^\sharp}(s_0)\big)\\&=f(\xi s_0-\nabla_{\xi^\sharp}(s_0))=f\lambda\sqrt{-1}\Phi_\alpha^\xi s_0=\lambda\sqrt{-1}\Phi_\alpha^\xi s.\end{aligned}$$

This proves (1).

Let ω be the connection form of ∇ with respect to the holomorphic frame s_0 of E on V, and R the curvature form of ∇. Then, $\omega=\lambda\partial N$, where $N:V\to\mathbb{R}$ is the smooth function $x\mapsto\|x\|^2$.

Lastly, the curvature form R of ∇ is the $(1,1)$-form on V given by

$$R = \overline{\partial}\omega = \lambda\overline{\partial}\partial N = -\lambda\partial\overline{\partial}N.$$

We claim that

$$\sqrt{-1}\partial\overline{\partial}N = \Omega.$$

To see this, first recall that $N = \sum_{i=1}^{n} z_i\overline{z_i}$. Therefore, $\overline{\partial}N = \sum_{i=1}^{n} z_i\overline{\partial}\overline{z_i}$, and

$$\partial\overline{\partial}N = \sum_{i=1}^{n} \partial z_i \wedge \overline{\partial}\overline{z_i} + \sum_{i=1}^{n} z_i \wedge \partial\overline{\partial}\overline{z_i} = \sum_{i=1}^{n} \partial z_i \wedge \overline{\partial}\overline{z_i} = \sum_{i=1}^{n} \mathrm{d}z_i \wedge \mathrm{d}\overline{z_i},$$

since $\overline{\partial}z_i = 0$, $\partial\overline{z_i} = 0$, $\mathrm{d}z_i = \partial z_i$, $\mathrm{d}\overline{z_i} = \overline{\partial}\overline{z_i}$, and $\partial\overline{\partial}\overline{z_i} = -\overline{\partial}\partial\overline{z_i} = 0$, as z_i is holomorphic, and $\overline{z_i}$ anti-holomorphic, for all $i = 1,\dots,n$. Let $x \in V$, $v,w \in \mathrm{T}_x(V) = V$. Write $v = \sum_{i=1}^{n} v_i e_i$ and $w = \sum_{i=1}^{n} w_i e_i$, where $v_i, w_i \in \mathbb{C}$ for all $i = 1,\dots,n$. Then

$$\begin{aligned}(\partial\overline{\partial}N)(x)(v,w) &= \Big(\sum_{k=1}^{n} \mathrm{d}z_k(x) \wedge \mathrm{d}\overline{z_k}(x)\Big)(v,w) = \Big(\sum_{k=1}^{n} z_k \wedge \overline{z_k}\Big)(v,w)\\ &= \sum_{k=1}^{n} (z_k(v) \wedge \overline{z_k}(w) - z_k(w) \wedge \overline{z_k}(v)) = \sum_{k=1}^{n} (v_k\overline{w_k} - w_k\overline{v_k})\\ &= \sum_{k=1}^{n} (v_k\overline{w_k} - \overline{v_k\overline{w_k}}) = 2\sqrt{-1}\Im\Big(\sum_{k=1}^{n} v_k\overline{w_k}\Big) = 2\sqrt{-1}\Im(\langle v,w\rangle).\end{aligned}$$

Therefore,

$$\sqrt{-1}(\partial\overline{\partial}N)(x)(v,w) = -2\Im(\langle v,w\rangle) = \Omega(v,w) = \Omega(x)(v,w).$$

Thus, $\sqrt{-1}\partial\overline{\partial}N = \Omega$.

It follows that

$$R = -\lambda\partial\overline{\partial}N = \lambda\sqrt{-1}\Omega.$$

Thus, the first Chern form of (E,h) is given by

$$c_1(E,h) = \frac{\sqrt{-1}}{2\pi}R = -\frac{\lambda}{2\pi}\Omega,$$

as stated in (2). □

Let H be a normal complex Lie subgroup of G, $\overline{G}$ the complex Lie group $H\backslash G$, and $\pi : G \to \overline{G}$ the canonical projection. Let $\overline{K}$ be the compact subgroup $\pi(K)$ of $\overline{G}$, and $\pi_K : K \to \overline{K}$ the homomorphism of real Lie groups induced by π.

Let X be a G-invariant open subset of V, X_{m} the closed subset $\Phi_\alpha^{-1}(0) \cap X$ of X, and $X_{\mathrm{ms}} = X_{\mathrm{m}}G$. Denote by Y the quotient topological space X/G, and let $p : X \to Y$ be the canonical projection. Let $Y_{\mathrm{ms}} = p(X_{\mathrm{ms}})$, $p_{\mathrm{ms}} : X_{\mathrm{ms}} \to Y_{\mathrm{ms}}$ the map induced by p, and $p_{\mathrm{m}} = p_{\mathrm{ms}}|_{X_{\mathrm{m}}} : X_{\mathrm{m}} \to Y_{\mathrm{ms}}$.

The subset $E_X = X \times \mathbb{C}$ is a G-invariant open subset of E. Let F denote the quotient topological space E_X/G, and $q : E_X \to F$ the canonical projection. There is a canonical continuous surjection from F to Y, and every fibre of this map has a canonical structure of a 1-dimensional $\mathbb{C}$-vector space. Thus, F is a family of 1-dimensional $\mathbb{C}$-vector spaces on Y. Let F_{m} (respectively, F_{ms}) denote the restriction of this family to the subspace Y_{m} (respectively, Y_{ms}) of Y. For every $x \in X$, the map $q : E \to F$ restricts to a $\mathbb{C}$-isomorphism $q(x) : E(x) \to F(p(x))$.

Note that if H is contained in the kernel of the character $\chi : G \to \mathbb{C}^\times$, then we have an induced action of $\overline{G}$ on E, and hence on E_X. If, moreover, the action of $\overline{G}$ on X is principal, then so is its action on E_X. Thus, in that case, there is a unique structure of a complex premanifold on F, such that q is a holomorphic submersion. With this structure, the family F of 1-dimensional $\mathbb{C}$-vector spaces is a holomorphic line bundle on X. For every holomorphic (respectively, smooth) section t of F on any open subset V of Y, there exists a unique holomorphic (respectively, smooth) section s of E_X on $p^{-1}(V)$, which is $\overline{G}$-invariant (that is, $s(xa) = s(x)a$ for all $x \in p^{-1}(V)$ and $a \in \overline{G}$), such that $q(s(x)) = t(p(x))$ for all $x \in p^{-1}(V)$.

Proposition 4.2.2 *Consider the context of Corollary 3.2.25. Suppose that $G_x = H$ for all $x \in X$, the induced action of $\overline{G}$ on X is principal, $H \subset \mathrm{Ker}(\chi)$, and*

$$\Phi_\alpha(X) \subset \mathrm{Ann}(\mathrm{Lie}(H \cap K)), \quad P_G(X_{\mathrm{m}}, X_{\mathrm{m}}) \subset HK.$$

Then, there exists a unique smooth Hermitian metric k_{ms} on the holomorphic line bundle F_{ms} on X_{ms}, such that $c_1(F_{\mathrm{ms}}, k_{\mathrm{ms}}) = -\frac{\lambda}{2\pi}\Theta_{\mathrm{ms}}$, where Θ_{ms} is the Kähler form on the open complex submanifold X_{ms} of X.

Proof. For every point $y \in Y_{\mathrm{ms}}$, define $k_{\mathrm{ms}}(y) : F(y) \times F(y) \to \mathbb{C}$ by $k_{\mathrm{ms}}(y)(a,b) = h(x)(a',b')$, where x is any point of $p_{\mathrm{m}}^{-1}(y)$ and $a', b' \in E(x)$ are such that $q(a') = a$ and $q(b') = b$. Then, since $p_{\mathrm{m}} : X_{\mathrm{m}} \to Y_{\mathrm{ms}}$ is a smooth principal $\overline{K}$-bundle, and the metric h is K-invariant, the above rule gives a well-defined smooth Hermitian metric k_{ms} on F_{ms}.

Suppose t is a smooth section of F_{ms} on an open subset V of Y_{ms}, $y \in Y_{\mathrm{ms}}$, and $w \in \mathrm{T}_y(Y)$. We will define an element $\nabla'_w(t)$ of $F(y)$ as follows. Let $x \in p_{\mathrm{m}}^{-1}(y)$, and choose $v \in \mathrm{T}_x(X_{\mathrm{m}})$, such that $\mathrm{T}_x(p_{\mathrm{m}})(v) = w$. Let s be the unique K-invariant section of E on $p_{\mathrm{m}}^{-1}(V)$ which projects to t. Define $\nabla'_w(t) = q(\nabla_v(s))$. If $x' \in p_{\mathrm{m}}^{-1}(y)$ and $v' \in \mathrm{T}_{x'}(X_{\mathrm{m}})$ are two other choices, such that $\mathrm{T}_{x'}(p_{\mathrm{m}})(v') = w$, then there exists a unique $g \in K$, such that $x' = xg$. Now, $v' - \mathrm{T}_x(\rho_g)(v)$ belongs to $\mathrm{Ker}(\mathrm{T}_{x'}(p_{\mathrm{m}}))$, and is hence of the form $\xi^\sharp(x')$ for some $\xi \in \mathrm{Lie}(K)$. Thus, by Lemma 4.2.1,

$$\nabla_{v'}(s) = \nabla_{\xi^\sharp(x')}(s) + \nabla_{\mathrm{T}_x(\rho_g)(v)} = \big((\xi s)(x') - \lambda\sqrt{-1}\Phi^\xi_\alpha(x')s(x')\big) + \big(\nabla_v(s)\big)g,$$

since the action of K preserves the metric h on E, and hence its canonical connection ∇ also. Now, since s is K-invariant, we have $\xi s = 0$, and since $x' \in X_{\mathrm{m}}$, we have

$\Phi_\alpha^\xi(x') = 0$. Therefore, $\nabla_{v'}(s) = \big(\nabla_v(s)\big)g$, hence $q(\nabla_{v'}(s)) = q(\big(\nabla_v(s)\big))$. It follows that $\nabla'_w(t)$ is well-defined. Since q_{m} is a smooth principal $\overline{G}$-bundle, this rule defines a smooth connection ∇' on F_{ms}.

We claim that ∇' is the canonical connection of the Hermitian holomorphic line bundle $(F_{\mathrm{ms}}, k_{\mathrm{ms}})$ on Y_{ms}. As ∇ is compatible with the metric h on E, and K preserves h, ∇' is compatible with the metric k_{ms} on F_{ms}. Therefore, we only need to check that ∇' is compatible with the holomorphic structure on F_{ms}. Let t be a holomorphic section of F_{ms} on an open subset V of Y_{ms}, $y \in V$, and $w \in \mathrm{T}_y(Y)$. We have to check that $\nabla'_{\sqrt{-1}w}(t) = \sqrt{-1}\nabla'_{\sqrt{-1}w}(t)$. Let s be the G-invariant holomorphic section of E on $p_{\mathrm{ms}}^{-1}(V)$ corresponding to t. Let $x \in p_{\mathrm{m}}^{-1}(y)$, and choose $v \in \mathrm{T}_x(X_{\mathrm{m}})$, such that $\mathrm{T}_x(p_{\mathrm{m}})(v) = w$. Then, by Lemma 3.2.21, $\sqrt{-1}v = v' + \sqrt{-1}\xi^\sharp(x)$, where $v' \in \mathrm{T}_x(X_{\mathrm{m}})$ and $\xi \in \mathrm{Lie}(K)$. By definition, $\nabla'_w(t) = \nabla_v(s)$. Similarly, since $\mathrm{T}_x(p_{\mathrm{m}})(v') = \mathrm{T}_x(p)(\sqrt{-1}(v - \xi^\sharp(x))) = \sqrt{-1}w$, we have $\nabla'_{\sqrt{-1}w}(t) = \nabla_{v'}(s)$. Now, since ∇ is compatible with the holomorphic structure on E, we get

$$\nabla_{v'}(s) = \nabla_{\sqrt{-1}(v-\xi^\sharp(x))}(s) = \sqrt{-1}(\nabla_v(s) - \nabla_{\xi^\sharp(x)}(s)) =$$

But, as we saw above, $\nabla_{\xi^\sharp(x)}(s) = 0$. It follows that $\nabla'_{\sqrt{-1}w}(t) = \sqrt{-1}\nabla'_w(t)$. This proves the above claim.

Thus, the canonical connection ∇' on $(F_{\mathrm{ms}}, k_{\mathrm{ms}})$ is the descent of ∇ through $p_{\mathrm{m}} : X_{\mathrm{m}} \to Y_{\mathrm{ms}}$. Therefore,

$$p_{\mathrm{m}}^* c_1(F_{\mathrm{ms}}, k_{\mathrm{ms}}) = i_{\mathrm{m}}^* c_1(E,h),$$

where $i_{\mathrm{m}} : X_{\mathrm{m}} \to X$ is the inclusion. But, by Lemma 4.2.1, $c_1(E,h) = -\frac{\lambda}{2\pi}\Omega$, hence

$$p_{\mathrm{m}}^* c_1(F_{\mathrm{ms}}, k_{\mathrm{ms}}) = i_{\mathrm{m}}^* c_1(E,h) = -\frac{\lambda}{2\pi} i_{\mathrm{m}}^*(\Omega) - \frac{\lambda}{2\pi} p_{\mathrm{m}}^* \Theta_{\mathrm{ms}}.$$

As p_{m} is a smooth submersion, it follows that $c_1(F_{\mathrm{ms}}, k_{\mathrm{ms}}) = -\frac{\lambda}{2\pi}\Theta_{\mathrm{ms}}$. □

4.3 The line bundle on the moduli space

We will follow the notation of Section 3.2.3. Recall that θ is a rational weight of Q. Let n be an integer > 0, such that $n(\theta_a - \mu_\theta(d)) \in \mathbb{Z}$ for all $a \in Q_0$. Let $\lambda = -n$. Let $\chi : G \to \mathbb{C}^\times$ be the character

$$\chi(g) = \prod_{a \in Q_0} \det(g_a)^{n(\mu_\theta(d) - \theta_a)}.$$

Then, $\chi(K) \subset \mathrm{U}(1)$, and $H \subset \mathrm{Ker}(\chi)$, since $\sum_{a \in Q_0} (\mu_\theta(d) - \theta_a) d_a = 0$. Let $\alpha = -\frac{\sqrt{-1}}{\lambda} \mathrm{T}_e(\chi)$. Then,

$$\alpha(\xi) = \frac{\sqrt{-1}}{n} \mathrm{T}_e(\chi)(\xi) = \frac{\sqrt{-1}}{n} \sum_{a \in Q_0} n(\mu_\theta(d) - \theta_a) \mathrm{Tr}(\xi_a) = -\sqrt{-1} \sum_{a \in Q_0} (\theta_a - \mu_\theta(d)) \mathrm{Tr}(\xi_a) = \langle \xi, \eta \rangle,$$

where $\eta = \left(\sqrt{-1}(\theta_a - \mu_\theta(d))\right)_{a \in Q_0}$. Thus,

$$\Phi_\alpha(\rho)(\xi) = \frac{1}{2} \Omega(\xi^\sharp(x), x) + \alpha(\xi) = \frac{1}{2} \Omega(\xi^\sharp(x), x) + \langle \xi, \eta \rangle = \Phi_\theta(\rho)(\xi)$$

for all $\rho \in \mathscr{A}$ and $\xi \in \mathrm{Lie}(K)$.

Let E be the trivial line bundle on $\mathscr{A}$ with the action of G defined by χ as above. Let F_s be its quotient by G on M_s. As above, F_s is a holomorphic line bundle on M_s. Now, the following result is an immediate consequence of Proposition 4.2.2.

Theorem 4.3.1 *Let n be any positive integer, such that $n(\theta_a - \mu_\theta(d)) \in \mathbb{Z}$ for all $a \in Q_0$. There exists a unique smooth Hermitian k_s on the holomorphic line bundle F_s on M_s, such that $c_1(F_\mathrm{s}, k_\mathrm{s}) = \frac{n}{2\pi} \Theta_\mathrm{s}$, where Θ_s is the Kähler form on M_s.* □

A compact complex manifold X which admits an embedding into $\mathbb{P}_n(\mathbb{C})$ (for some n) is called a *projective algebraic manifold*.

If X is a compact complex manifold, then a d-closed differential form φ on X is said to be integral if its cohomology class in the de Rham group, $[\varphi] \in H^*(X,\mathbb{C})$, is in the image of the natural mapping $H^*(X,\mathbb{Z}) \to H^*(X,\mathbb{C})$.

Let h be a Kähler metric on a complex manifold of Kähler type and let Ω be the associated fundamental form. If Ω is an integral differential form, then Ω is called a Hodge form on X, and h is called a Hodge metric. A manifold of Kähler type is called a *Hodge manifold* if it admits a Hodge metric.

Kodaira's Projective Embedding Theorem: Let X be a compact Hodge manifold. Then, X is a projective algebraic manifold.

We can adjust the metric on the line bundle over $\mathscr{A}$ so that we can prove that the kähler form Θ_{s} on M_{s} is an integral form (from (4.3.1)). Hence, $(M_{\mathrm{s}},\Theta_{\mathrm{s}})$ is a Hodge manifold if M_{s} is compact. Thus, by Kodaira's Embedding Theorem, we have

Corollary 4.3.2 *if M_{s} is compact manifold then the moduli space M_{s} of stable representations is a projective algebraic manifold.*

Chapter 5

Holomorphic sectional curvature of the moduli space of stable representations

In Section 5.1, we recall some definitions which will be used in the sequel. In Section 5.2, we state a lemma which relates the holomorphic sectional curvatures of almost Hermitian manifolds in an almost Hermitian submersion. Finally, we conclude this chapter by a contributory result in corollary 5.4.4 that the holomorphic sectional curvature of the moduli space of stable representations is non-negative.

5.1 Preliminaries

Let M be a smooth manifold with a smooth metric g. Let $R(X,Y)$ denote the *curvature transformation* of $\chi(M)$, the $\mathbb{R}$-Lie algebra of smooth vector fields of M), determined by $X,Y \in \chi(M)$, i.e,

$$R(X,Y)\colon \chi(M) \to \chi(M),$$

defined by

$$R(X,Y)(Z) := \nabla_X \nabla_Y(Z) - \nabla_Y \nabla_X(Z) - \nabla_{[X,Y]}(Z),$$

for all $Z \in \chi(M)$, where ∇ is the Levi-Civita connection of M. The *Riemannian curvature tensor* of (M,g), denoted by R, is defined by

$$R\colon \chi(M) \times \chi(M) \times \chi(M) \times \chi(M) \to C^\infty(M),$$

$$R(X,Y,Z,W) = g(R(Z,W)Y,X),$$

for all X,Y,Z,W in $\chi(M)$. By [22, Chapter V, proposition 2.1],

Proposition 5.1.1 *The Riemannian curvature tensor of M, considered as real quadrilinear mapping $R_x\colon \mathrm{T}_x(M) \times \mathrm{T}_x(M) \times \mathrm{T}_x(M) \times \mathrm{T}_x(M) \to \mathbb{R}$ ($x \in M$) possess the following properties:*

a) $R(X_1,X_2,X_3,X_4) = -R(X_2,X_1,X_3,X_4)$,

b) $R(X_1,X_2,X_3,X_4) = -R(X_1,X_2,X_4,X_3)$,

c) $R(X_1,X_2,X_3,X_4) + R(X_1,X_3,X_4,X_2) + R(X_1,X_4,X_2,X_3) = 0$*, and hence*

d) $R(X_1,X_2,X_3,X_4) = R(X_3,X_4,X_1,X_2)$,

for all $X_i \in \mathrm{T}_x(M), 1 \le i \le 4$.

Definition 5.1.2 Let (M,g) be a Riemannian manifold. For each 2-dimensional $\mathbb{R}$-subspaces P of $\mathrm{T}_x(M)$, the *sectional curvature* $k(P)$ along P is defined by

$$k(P) := R(X_1,X_2,X_1,X_2),$$

where $\{X_1,X_2\}$ is any orthonormal basis for P with respect to g_x.

Note that the definition of $k(P)$ is independent of the choice of an orthonormal basis of P.

Definition 5.1.3 Let M be a Kähler manifold with an Hermitian metric h, and $J\colon TM \to TM$ be its complex structure (i.e., $J^2 = -Id$). Let $g = 2\Re(h)$. Then, (M,g) is a Riemannian manifold. Let x be a point in M. For each 2- dimensional $\mathbb{R}$-subspace of the $\mathbb{C}$- vector space $\mathrm{T}_x(M)$ with $J_x(P) \subset P$, the *holomorphic sectional curvature* along the holomorphic plane P is defined by

$$k(P) := R(X,JX,X,JX),$$

where X is a unit vector in P with respect to g_x.

Definition 5.1.4 If $k(P)$ is constant for all holomorphic planes P in $\mathrm{T}_x(M)$ and for all $x \in M$, then M is called a space of constant holomorphic curvature.

Example 5.1.1 [23, Chapter IX, Theorem 7.8]

- For any positive real number c, the complex projective space $\mathbb{P}_n(\mathbb{C})$ carries a Kähler metric of constant holomorphic sectional curvature c.
- For any negative real number c, the open unit ball $\mathbb{D}^n$ in $\mathbb{C}^n$ carries a Kähler metric of constant holomorphic sectional curvature c.
- The Euclidean space $\mathbb{C}^n$ with usual Kähler metric is an example of space of constant holomorphic sectional curvature 0.

Definition 5.1.5 Let (M,g) and (N,h) be two Riemannian manifolds and $f\colon M \to N$ a smooth submersion. Then, f is said to be a *Riemannian submersion* if the $\mathbb{R}$-linear isomorphism

$$\mathrm{T}_x(f)\colon \mathrm{Ker}(\mathrm{T}_x(f))^{\perp} \to \mathrm{T}_{f(x)}(N)$$

is an isometry with respect to g_x and $h_{f(x)}$, where $\mathrm{Ker}(\mathrm{T}_x(f))^{\perp}$ is the orthogonal complement of $\mathrm{Ker}(\mathrm{T}_x(f))$ in $\mathrm{T}_x(M)$ with respect to g_x.

An example of a Riemannian submersion arises when a Lie group G acts isometrically, freely and properly on a Riemannian manifold (M,g). The projection $\pi\colon M \to N$ to the quotient space $N = M/G$ equipped with the quotient metric is a Riemannian submersion.

Definition 5.1.6 Let (M,g,I) and (N,h,J) be two almost Hermitian manifolds and $f\colon M \to N$ a smooth submersion. Then, f is said to be an *almost Hermitian submersion* if

1. The map f is a Riemannian submersion. That is, the $\mathbb{R}$-linear isomorphism

$$\mathrm{T}_x(f)\colon \mathrm{Ker}(\mathrm{T}_x(f))^{\perp} \to \mathrm{T}_{f(x)}(N)$$

   is an isometry with respect to g_x and $h_{f(x)}$, where $\mathrm{Ker}(\mathrm{T}_x(f))^{\perp}$ is the orthogonal complement of $\mathrm{Ker}(\mathrm{T}_x(f))$ in $\mathrm{T}_x(M)$ with respect to g_x.

2. The map f is an almost complex mapping. That is, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Ker}(\mathrm{T}_p(\pi))^{\perp} & \xrightarrow{\mathrm{T}_p(\pi)} & \mathrm{T}_{\pi(p)}(W) \\ {\scriptstyle J(p)}\downarrow & & \downarrow{\scriptstyle J_W(\pi(p))} \\ \mathrm{Ker}(\mathrm{T}_p(\pi))^{\perp} & \xrightarrow{\mathrm{T}_p(\pi)} & \mathrm{T}_{\pi(p)}(W). \end{array}$$

5.2 General theory of holomorphic sectional curvatures

Let V be a Kähler manifold with Kähler metric h, and complex structure J. Let $g = 2\Re(h)$ be the smooth Riemannian metric of V. Let N be a CR sub-manifold of V and TN its tangent bundle.

Set $T^hN = TN \cap J(TN)$. Since N is a CR sub-manifold of V, so T^hN is a smooth complex sub-bundle of $TV|_N$. Let T^vN be the orthogonal complement of T^hN in TN with respect to g. Thus, we have an orthogonal direct sum of smooth bundle as

$$TV|_N = T^hN \oplus T^vN \oplus T^\perp N,$$

where $T^\perp N$ is the orthogonal complement of TN in $TV|_N$ with respect to g.

We make following assumptions :

a) J interchange T^vN and $T^\perp N$;

b) There is a submersion $\pi : N \to W$ of N onto an almost Hermitian manifold (W, h_W, J_W) such that

   i) $T^v_pN = \mathrm{Ker}(\mathrm{T}_p(\pi))$, for all $p \in N$.

   ii) The $\mathbb{R}$-isomorphism $\mathrm{T}_p(\pi) : T^h_pN \to T_{\pi(p)}W$ is a complex isometry for every $p \in N$, i.e., the diagram

$$\begin{array}{ccc} \mathrm{Ker}(\mathrm{T}_p(\pi))^\perp & \xrightarrow{\mathrm{T}_p(\pi)} & \mathrm{T}_{\pi(p)}(N) \\ {\scriptstyle J(p)}\downarrow & & \downarrow{\scriptstyle J_W(\pi(p))} \\ \mathrm{Ker}(\mathrm{T}_p(\pi))^\perp & \xrightarrow{\mathrm{T}_p(\pi)} & \mathrm{T}_{\pi(p)}(N) \end{array}$$

   commutes.

Lemma 5.2.1 *([25, Theorem 1.3]) Under preceding assumptions, W is a Kähler manifold. Further, if H^V and H^W denote the respective holomorphic sectional curvature of V and W, then for all $p \in N$ and for any horizontal unit vector $v \in T^h_pN$, we have*

$$H^V(v) = H^W(\mathrm{T}_p(\pi)(v)) - 4\,|C(v,v)|^2,$$

where $C\colon TN\times TN\to T^{\perp}N$ is the second fundamental form of N in V.

5.3 Quiver setup

Fix a non-empty finite quiver $Q=(Q_0,Q_1,s,t)$, $d=(d_a)_{a\in Q_0}$ a non-zero element of $\mathbb{N}^{Q_0}$, and $V=(V_a)_{a\in Q_0}$ a family of $\mathbb{C}$-vector spaces, such that $\dim_{\mathbb{C}}(V_a)=d_a$, for all $a\in Q_0$. We also fix a family $h=(h_a)_{a\in Q_0}$ of Hermitian inner products $h_a:V_a\times V_a\to\mathbb{C}$. In addition, we also fix a rational weight $\theta\in\mathbf{Q}^{Q_0}$ of Q.

Denote by $\mathscr{A}$ the finite-dimensional $\mathbb{C}$-vector space $\bigoplus_{\alpha\in Q_1}\mathrm{Hom}_{\mathbb{C}}(V_{\mathrm{s}(\alpha)},V_{\mathrm{t}(\alpha)})$. We give usual topology and usual structure of a complex manifold on the vector space $\mathscr{A}$.

The family h induces a Hermitian inner product $\langle\cdot,\cdot\rangle$ on the $\mathbb{C}$-vector space $\mathscr{A}$. For every point ρ of $\mathscr{A}$, the $\mathbb{C}$-vector space $\mathrm{T}_\rho(\mathscr{A})$ is canonically isomorphic to $\mathscr{A}$. Therefore, the Hermitian inner product $\langle\cdot,\cdot\rangle$ on $\mathscr{A}$ defines a Hermitian metric g on the complex manifold $\mathscr{A}$, namely $g(\rho)(\sigma,\tau)=\langle\sigma,\tau\rangle$ for all $\rho,\sigma,\tau\in\mathscr{A}$. The fundamental 2-form Ω of g is given by $\Omega(\rho)(\sigma,\tau)=-2\Im(\langle\sigma,\tau\rangle)$ for all $\rho,\sigma,\tau\in\mathscr{A}$, where $\Im(\langle\sigma,\tau\rangle)$ is the imaginary part of the complex number $\Omega(\rho)(\sigma,\tau)$. Since $\Omega(\rho)$ is independent of ρ, we have $\mathrm{d}\Omega=0$, hence the Hermitian metric g on $\mathscr{A}$ is Kähler. Let B be the smooth Riemannian metric on the underlying smooth manifold of $\mathscr{A}$ defined by $B=2\Re(g)$. Let J be the complex structure on $\mathscr{A}$ defined by $J(\rho)(\sigma)=\sqrt{-1}\sigma$, for every $\rho,\sigma\in\mathscr{A}$.

Define $\mathscr{B}$ to be the set of all points ρ of $\mathscr{A}$, such that the representation (V,ρ) of Q is Schur. Then, $\mathscr{B}$ is an open complex sub-manifold of the Kähler manifold $\mathscr{A}$, and hence $\mathscr{B}$ gets an induced Kähler metric which we also denote by g.

Let G be the complex Lie group $\prod_{a\in Q_0}\mathrm{Aut}_{\mathbb{C}}(V_a)$, with its canonical holomorphic linear right action on $\mathscr{A}$. Denote by H the central complex Lie subgroup $\mathbb{C}^{\times}e$ of G, $\overline{G}$ the complex Lie group $H\backslash G$, and $\pi:G\to\overline{G}$ the canonical projection. Let K denote the compact subgroup $\prod_{a\in Q_0} \mathrm{Aut}(V_a, h_a)$, where, for each $a \in Q_0$, $\mathrm{Aut}(V_a, h_a)$ is the subgroup of $\mathrm{Aut}_{\mathbb{C}}(V_a)$ consisting of $\mathbb{C}$-automorphisms of V_a which preserve the Hermitian inner product h_a on V_a. Let $\overline{K}$ be the compact subgroup $\pi(K)$ of $\overline{G}$. Moreover, $\mathscr{B}$ is invariant under the action of G on $\mathscr{A}$, and the induced action of K on $\mathscr{A}$ preserves the Kähler metric g.

Let M denote the quotient topological space $\mathscr{B}/G$, and $p : \mathscr{B} \to M$ the canonical projection. Then, we have the following (Chapter 3, Section 3.1, Theorem 3.1.22):

Theorem 5.3.1 *The action of $\overline{G}$ on $\mathscr{B}$ is principal. In particular, there exists a unique structure of a complex premanifold on the moduli space M of complex Schur representations of Q with dimension vector d, such that $p : \mathscr{B} \to M$ is a holomorphic submersion. Moreover, this structure makes the map p a holomorphic principal $\overline{G}$-bundle.*

For any complex premanifold X and $x \in X$, we will identify the tangent space at x of the underlying smooth manifold of X, with the holomorphic tangent space $\mathrm{T}_x(X)$ of X at x, using the canonical $\mathbb{R}$-isomorphism between them. With this identification, for any holomorphic map $f : X \to Y$ of complex premanifolds, and $x \in X$, the real differential of f at x is equal to the $\mathbb{C}$-linear map $\mathrm{T}_x(f) : \mathrm{T}_x(X) \to \mathrm{T}_{f(x)}(Y)$, considered as an $\mathbb{R}$-linear map.

For any point ρ in $\mathscr{A}$, let $\nu_\rho : K \to \mathscr{A}$ be the orbit map of ρ, and denote by D_ρ the $\mathbb{R}$-linear map $\mathrm{T}_e(\nu_\rho) : \mathrm{Lie}(K) \to \mathscr{A}$. Then, for every element ξ of $\mathrm{Lie}(K)$, the vector field $\xi^\sharp$ on $\mathscr{A}$ induced by ξ is the $\mathbb{C}$-endomorphism of $\mathscr{A}$ given by $\xi^\sharp(\rho) = \mathrm{T}_e(\nu_\rho)(\xi)$, for all $\rho \in \mathscr{A}$.

Let η denote the element $\left(\sqrt{-1}(\theta_a - \mu_\theta(d))\mathbf{1}_{V_a}\right)_{a\in Q_0}$ of $\mathrm{Lie}(K)$, and α the element of $\mathrm{Lie}(K)^*$, where $\alpha(\xi) = \langle \xi, \eta \rangle$, for all $\xi \in \mathrm{Lie}(K)$. Then, setting

$$\Phi_\theta(\rho)(\xi) := \frac{1}{2}\Omega(\xi^\sharp(\rho), \rho) + \alpha(\xi),$$

for all $\rho \in \mathscr{A}$ and $\xi \in \mathrm{Lie}(K)$, the map $\Phi_\theta : \mathscr{A} \to \mathrm{Lie}(K)^*$ is a moment map for the action of K on $\mathscr{A}$ (Chapter 3, Section 3.2.1, Lemma 3.2.17).

Let $\Psi_\theta : \mathscr{B} \to \mathrm{Lie}(K)^*$ be the restriction of Φ_θ on $\mathscr{B}$. Then, Ψ_θ becomes a moment map for the action of K on $\mathscr{B}$. (Chapter 3, Section 3.2.1, Remark 3.2.16)

let $\mathscr{B}_{\mathrm{s}}$ denote the set of all points ρ in $\mathscr{B}$, such that the representation (V,ρ) of Q is θ-stable, and $\mathscr{B}_{\mathrm{eh}}$ the set of all $\rho \in \mathscr{B}$, such that the Hermitian metric h on (V,ρ) is Einstein-Hermitian with respect to θ. Then, we have (Chapter 3, Section 3.2.3, Lemma 3.2.26, 3.2.30)

i) $\mathscr{B}_{\mathrm{eh}} = \Psi_\theta^{-1}(0)$,

ii) $\mathscr{B}_{\mathrm{eh}} G = \mathscr{B}_{\mathrm{s}}$.

Recall that M denotes the moduli space $\mathscr{B}/G$ of Schur representations of Q, and $p : \mathscr{B} \to M$ the canonical projection. Let $M_{\mathrm{s}} = p(\mathscr{B}_{\mathrm{s}})$ be the moduli space of θ-stable representations of Q, $p_{\mathrm{s}} : \mathscr{B}_{\mathrm{s}} \to M_{\mathrm{s}}$ the map induced by p, and $p_{\mathrm{eh}} = p_{\mathrm{s}}|_{\mathscr{B}_{\mathrm{eh}}} : \mathscr{B}_{\mathrm{eh}} \to M_{\mathrm{s}}$. Then, we have the following (Chapter 3, Section 3.2.3, Theorem 3.2.31):

Theorem 5.3.2 *With notations as above, the following statements are true:*

1. *The set $\mathscr{B}_{\mathrm{eh}}$ is a closed smooth submanifold of $\mathscr{B}$, $\mathscr{B}_{\mathrm{s}}$ is open in $\mathscr{B}$, M_{s} is open in M, the action of $\overline{K}$ on $\mathscr{B}_{\mathrm{eh}}$ is principal, and $p_{\mathrm{eh}} : \mathscr{B}_{\mathrm{eh}} \to M_{\mathrm{s}}$ is a smooth principal $\overline{K}$-bundle.*

2. *The action of $\overline{G}$ on $\mathscr{B}_{\mathrm{s}}$ is proper, M_{s} is a Hausdorff open subspace of M, the action of $\overline{G}$ on $\mathscr{B}_{\mathrm{s}}$ is principal, and $p_{\mathrm{s}} : \mathscr{B}_{\mathrm{s}} \to M_{\mathrm{s}}$ is a holomorphic principal $\overline{G}$-bundle.*

3. *There exists a unique Kähler metric h_{s} on M_{s}, such that $p_{\mathrm{eh}}^*(\Theta_{\mathrm{s}}) = \Omega_{\mathrm{eh}}$, where Θ_{s} is the Kähler form of h_{s}, $\Omega_{\mathrm{eh}} = i_{\mathrm{eh}}^*(\Omega)$, $i_{\mathrm{eh}} : \mathscr{B}_{\mathrm{eh}} \to \mathscr{B}$ is the inclusion map, and Ω is the Kähler form on $\mathscr{B}$.*

5.4 Calculating the holomorphic sectional curvature of the moduli space

We set V to be the Kähler manifold $\mathscr{B}$ of Schur representations of Q with the Kähler metric g, the Kähler form $\Omega = -2\Im(g)$, the smooth Riemannian metric $B = 2\Re(g)$, and the complex structure J which were defined earlier.

Let N be the closed smooth submanifold $\mathscr{B}_{\mathrm{eh}}$ of $\mathscr{B}$ of irreducible Einstein Hermitian representations of Q.

Further, let W be the moduli space M_{s} of θ-stable representations of Q which is a Kähler manifold with Kähler metric h_{s}, Kähler form Θ_{s}, smooth Riemannian metric B_{s}, and complex structure J_{s} are given by (3.2.31) ; $\pi : N \to W$ be the smooth map $p_{\mathrm{eh}} = p_{\mathrm{s}}|_{\mathscr{B}_{\mathrm{eh}}} : \mathscr{B}_{\mathrm{eh}} \to M_{\mathrm{s}}$ which is the restriction of the holomorphic map $p : \mathscr{B} \to M$.

Let $\rho \in \mathscr{B}_{\mathrm{eh}}$ be fixed. Denote by U the $\mathbb{R}$-vector subspace $\mathrm{Im}(\mathrm{T}_e(\nu_\rho))$ of the $\mathbb{C}$-vector space $\mathscr{A}$, where $\nu_\rho : K \to \mathscr{A}$ is the orbit map of ρ.

Let $\mathrm{T}_\rho(\mathscr{B}_{\mathrm{eh}})$ be the tangent space of $\mathscr{B}_{\mathrm{eh}}$ at ρ, and $\mathrm{T}_\rho(\mathscr{B}_{\mathrm{eh}})^\perp$ the orthogonal complement of $\mathrm{T}_\rho(\mathscr{B}_{\mathrm{eh}})$ in $\mathrm{T}_\rho(\mathscr{B}) = \mathscr{A}$ with respect to $B(\rho)$.

Let us set $\mathrm{T}^h_\rho\mathscr{B}_{\mathrm{eh}} = \mathrm{T}_\rho(\mathscr{B}_{\mathrm{eh}}) \cap J(\mathrm{T}_\rho(\mathscr{B}_{\mathrm{eh}}))$, and $\mathrm{T}^v_\rho\mathscr{B}_{\mathrm{eh}}$ to be the orthogonal complement of $\mathrm{T}^h_\rho\mathscr{B}_{\mathrm{eh}}$ in $\mathrm{T}_\rho(\mathscr{B}_{\mathrm{eh}})$ with respect to $B(\rho)$.

Proposition 5.4.1 *With notations as above, the following statements are true:*

1. $\mathrm{T}_\rho(\mathscr{B}_{\mathrm{eh}}) = U^{\perp(\Omega)}$
2. $U^{\perp(\Omega)} = (\sqrt{-1}U)^{\perp(B)}$
3. $\mathrm{T}^\perp_\rho\mathscr{B}_{\mathrm{eh}} = \sqrt{-1}U$
4. $J(\mathrm{T}_\rho(\mathscr{B}_{\mathrm{eh}})) = U^{\perp(B)}$
5. $\mathrm{T}^h_\rho\mathscr{B}_{\mathrm{eh}} = U^{\perp(g)}$

6. $\mathrm{T}^{v}_{\rho}\mathscr{B}_{\mathrm{eh}} = U$

7. $J(\mathrm{T}^{h}_{\rho}\mathscr{B}_{\mathrm{eh}}) = \mathrm{T}^{h}_{\rho}\mathscr{B}_{\mathrm{eh}}$

8. $J(\mathrm{T}^{v}_{\rho}\mathscr{B}_{\mathrm{eh}} \oplus \mathrm{T}^{\perp}_{\rho}\mathscr{B}_{\mathrm{eh}}) = \mathrm{T}^{v}_{\rho}\mathscr{B}_{\mathrm{eh}} \oplus \mathrm{T}^{\perp}_{\rho}\mathscr{B}_{\mathrm{eh}}$.

Proof.

- The differential of the moment map for the action of K on $\mathscr{A}$ at ρ is the $\mathbb{R}$-linear map $\mathrm{T}_{\rho}(\Phi_{\theta}) : \mathscr{A} \to \mathrm{Lie}(K)^{*}$ given by $\mathrm{T}_{\rho}(\Phi_{\theta})(\sigma)(\xi) = \Omega(\xi^{\sharp}(\rho), \sigma)$ for all $\sigma \in \mathscr{A}$ and $\xi \in \mathrm{Lie}(K)$(by using proposition 3.2.15 (1)). Thus (by using proposition 3.2.15 (2)), we have $\mathrm{T}_{\rho}(\mathscr{A}_{\mathrm{eh}}) = \mathrm{Ker}(\mathrm{T}_{\rho}(\Phi_{\theta})) = \mathrm{Im}(\mathrm{T}_{e}(\nu_{\rho}))^{\perp(\Omega)} = U^{\perp(\Omega)}$. Since $\mathscr{B}_{\mathrm{eh}} = \mathscr{A}_{\mathrm{eh}} \cap \mathscr{B}$ is open in $\mathscr{A}_{\mathrm{eh}}$, we get $\mathrm{T}_{\rho}(\mathscr{B}_{\mathrm{eh}}) = \mathrm{T}_{\rho}(\mathscr{A}_{\mathrm{eh}}) = U^{\perp(\Omega)}$.

- For all $\sigma, \tau \in \mathscr{A}$, we have

$$\begin{aligned} B(\sigma, \sqrt{-1}\tau) &= 2\Re(g(\sigma, \sqrt{-1}\tau)) = 2\Re(g(\sqrt{-1}\sigma, -\tau)) \\ &= 2\Re(-\sqrt{-1}g(\sigma,\tau)) = 2\Im(g(\sigma,\tau)) = -\Omega(\sigma,\tau). \end{aligned}$$

  Therefore, an element σ of $\mathscr{A}$ is B-orthogonal to $\sqrt{-1}U$ if and only if it is Ω-orthogonal to U. Its follows that $U^{\perp(\Omega)} = (\sqrt{-1}U)^{\perp(B)}$.

- As B is a real inner product on $\mathscr{A}$, we get

$$\begin{aligned} \mathrm{T}^{\perp}_{\rho}\mathscr{B}_{\mathrm{eh}} &= \mathrm{T}_{\rho}(\mathscr{B}_{\mathrm{eh}})^{\perp(B)} = U^{\perp(\Omega)\perp(B)} && (by (1)) \\ &= (\sqrt{-1}U)^{\perp(B)\perp(B)} = \sqrt{-1}U. && (by (2)) \end{aligned}$$

  Moreover, we have

$$\mathrm{T}_{\rho}(\mathscr{B}) = \mathscr{A} = U^{\perp(\Omega)} \oplus U^{\perp(\Omega)\perp(B)} = U^{\perp(\Omega)} \oplus \sqrt{-1}U. \tag{5.1}$$

- Let $\sigma \in \mathrm{T}_\rho(\mathscr{B}_{\mathrm{eh}})$. Then, for all $\tau \in U$, we have

$$\begin{aligned} B(\sqrt{-1}\sigma,\tau) &= B(-\sigma,\sqrt{-1}\tau) = -B(\sigma,\sqrt{-1}\tau) \\ &= 0 \qquad (\textit{since } \sigma \in \mathrm{T}_\rho(\mathscr{B}_{\mathrm{eh}}) = (\sqrt{-1}U)^{\perp(B)}). \end{aligned}$$

It follows that $\sqrt{-1}\sigma \in U^{\perp(B)}$. This proves that $J(\mathrm{T}_\rho(\mathscr{B}_{\mathrm{eh}})) \subset U^{\perp(B)}$. Conversely, let $\sigma \in U^{\perp(B)}$. Write $\sigma = \sqrt{-1}(-\sqrt{-1}\sigma) = \sqrt{-1}\tau$, where $\tau = -\sqrt{-1}\sigma$. Then we claim that $\tau \in \mathrm{T}_\rho(\mathscr{B}_{\mathrm{eh}}) = (\sqrt{-1}U)^{\perp(B)}$. To see it, note that for all $\delta \in U$, we have

$$\begin{aligned} B(\tau,\sqrt{-1}\delta) &= B(-\sqrt{-1}\sigma,\sqrt{-1}\delta) = -B(\sigma,\delta) \\ &= 0. \qquad (\because \sigma \in U^{\perp(B)}, \delta \in U) \end{aligned}$$

This implies that $\tau \in \mathrm{T}_\rho(\mathscr{B}_{\mathrm{eh}})$, and this proves the claim. Thus, $\sigma = \sqrt{-1}(-\sqrt{-1}\sigma) = \sqrt{-1}\tau \in J(\mathrm{T}_\rho(\mathscr{B}_{\mathrm{eh}}))$. It follows that $J(\mathrm{T}_\rho(\mathscr{B}_{\mathrm{eh}})) \supset U^{\perp(B)}$.

- Since for any $\sigma,\tau \in \mathscr{A}$, $g(\sigma,\tau) = \frac{1}{2}(B(\sigma,\tau) - \sqrt{-1}\Omega(\sigma,\tau))$, it follows that $\sigma \in U^{\perp(g)}$ if and only if $\sigma \in U^{\perp(\Omega)}$ and $\sigma \in U^{\perp(B)}$. Therefore, $\sigma \in U^{\perp(g)}$ if and only if $\sigma \in \mathrm{T}_\rho(\mathscr{B}_{\mathrm{eh}})$ and $\sigma \in J(\mathrm{T}_\rho(\mathscr{B}_{\mathrm{eh}}))$. Thus, $U^{\perp(g)} = \mathrm{T}_\rho(\mathscr{B}_{\mathrm{eh}}) \cap J(\mathrm{T}_\rho(\mathscr{B}_{\mathrm{eh}})) = \mathrm{T}^h_\rho \mathscr{B}_{\mathrm{eh}}$.

- Now,

$$\begin{aligned} \mathrm{T}^v_\rho \mathscr{B}_{\mathrm{eh}} &= \{\sigma \in \mathrm{T}_\rho(\mathscr{B}_{\mathrm{eh}}) : B(\sigma,\tau) = 0, \textit{for all } \tau \in \mathrm{T}^h_\rho \mathscr{B}_{\mathrm{eh}}\} \\ &= \{\sigma \in U^{\perp(\Omega)} : B(\sigma,\tau) = 0, \textit{for all } \tau \in U^{\perp(g)}\} \end{aligned}$$

First, we claim that $U \subset U^{\perp(\Omega)}$ (i.e, orbits are isotropic submanifolds). For this, let $\sigma \in U$. Then, we can write as $\sigma = \mathrm{T}_e(\nu_\rho)(\eta) = \eta^\sharp(\rho)$, for some $\eta \in \mathrm{Lie}(K)$.

For any $\tau \in U$, write $\tau = \mathrm{T}_e(\nu_\rho)(\xi) = \xi^\sharp(\rho)$, for some $\xi \in \mathrm{Lie}(K)$. now,

$$\begin{aligned}
\Omega(\rho)(\sigma,\tau) &= \Omega(\rho)(\eta^\sharp(\rho),\xi^\sharp(\rho)) \\
&= \Omega(H(\Psi^\eta_\theta)(\rho),\xi^\sharp(\rho)) \\
&= \xi^\sharp(\rho)(\Psi^\eta_\theta) \\
&= \mathrm{T}_\rho(\Psi^\eta_\theta)(\xi^\sharp(\rho)) \\
&= \mathrm{T}_\rho(\Psi^\eta_\theta)(\mathrm{T}_e(\nu_\rho)(\xi)) \\
&= \mathrm{T}_e(\Psi^\eta_\theta \circ \nu_\rho)(\xi) \\
&= \mathrm{T}_e(0\colon \mathrm{Lie}(k) \to \mathbb{R})(\xi) \\
&= 0.
\end{aligned}$$

Thus, $\sigma \in U^{\perp(\Omega)}$, and hence $U \subset U^{\perp(\Omega)}$.

Since $\sigma \in U$, we have $g(\tau,\sigma) = 0$, for all $\tau \in \mathrm{T}^h_\rho\mathscr{B}_{\mathrm{eh}} = U^{\perp(g)}$. So, $B(\sigma,\tau) = B(\tau,\sigma) = 0$, for all $\tau \in \mathrm{T}^h_\rho\mathscr{B}_{\mathrm{eh}}$. As $U \subset U^{\perp(\Omega)} = \mathrm{T}_\rho(\mathscr{B}_{\mathrm{eh}})$, we have

$$U \subset \mathrm{T}^v_\rho\mathscr{B}_{\mathrm{eh}}$$

Now,

$$\begin{aligned}
\mathscr{A} &= \mathrm{T}_\rho(\mathscr{B}_{\mathrm{eh}}) \oplus \mathrm{T}^\perp_\rho\mathscr{B}_{\mathrm{eh}} \\
&= \mathrm{T}^h_\rho\mathscr{B}_{\mathrm{eh}} \oplus \mathrm{T}^v_\rho\mathscr{B}_{\mathrm{eh}} \oplus \mathrm{T}^\perp_\rho\mathscr{B}_{\mathrm{eh}} \\
&= U^{\perp(g)} \oplus \mathrm{T}^v_\rho\mathscr{B}_{\mathrm{eh}} \oplus \sqrt{-1}U.
\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathscr{A} &= U^C \oplus (U^C)^{\perp(g)} \\ &= U^C \oplus U^{\perp(g)} \\ &= U \oplus \sqrt{-1}U \oplus U^{\perp(g)}.\end{aligned}$$

It follows that $\dim_{\mathbb{R}}(U) = \dim_{\mathbb{R}}(\mathrm{T}^v_\rho \mathscr{B}_{\mathrm{eh}})$. But we have proved that $U \subset \mathrm{T}^v_\rho \mathscr{B}_{\mathrm{eh}}$. Hence $U = \mathrm{T}^v_\rho \mathscr{B}_{\mathrm{eh}}$.

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$$\begin{aligned}J(\mathrm{T}^h_\rho \mathscr{B}_{\mathrm{eh}}) &= J(J(\mathrm{T}_\rho(\mathscr{B}_{\mathrm{eh}})) \cap \mathrm{T}_\rho(\mathscr{B}_{\mathrm{eh}})) = J^2(\mathrm{T}_\rho(\mathscr{B}_{\mathrm{eh}})) \cap J(\mathrm{T}_\rho(\mathscr{B}_{\mathrm{eh}})) \\ &= \sqrt{-1}^2 \mathrm{T}_\rho(\mathscr{B}_{\mathrm{eh}})) \cap J(\mathrm{T}_\rho(\mathscr{B}_{\mathrm{eh}})) = -(\mathrm{T}_\rho(\mathscr{B}_{\mathrm{eh}})) \cap J(\mathrm{T}_\rho(\mathscr{B}_{\mathrm{eh}})) \\ &= \mathrm{T}^h_\rho \mathscr{B}_{\mathrm{eh}}.\end{aligned}$$

In particular, the $\mathbb{R}$-vector subspace $\mathrm{T}^h_\rho \mathscr{B}_{\mathrm{eh}}$ of the $\mathbb{C}$-vector space $\mathscr{A}$ is a $\mathbb{C}$-subspace of $\mathscr{A}$.

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$$\begin{aligned}J(\mathrm{T}^v_\rho \mathscr{B}_{\mathrm{eh}} \oplus \mathrm{T}^\perp_\rho \mathscr{B}_{\mathrm{eh}}) &= \sqrt{-1}(U \oplus (\sqrt{-1}U)) = \sqrt{-1}U \oplus -(U) = \sqrt{-1}U \oplus U \\ &= \mathrm{T}^\perp_\rho \mathscr{B}_{\mathrm{eh}} \oplus \mathrm{T}^v_\rho \mathscr{B}_{\mathrm{eh}} = \mathrm{T}^v_\rho \mathscr{B}_{\mathrm{eh}} \oplus \mathrm{T}^\perp_\rho \mathscr{B}_{\mathrm{eh}}.\end{aligned}$$

□

From the above proposition (5.4.1), we have an orthogonal direct sum of $\mathbb{R}$-vector spaces,

$$\mathrm{T}_\rho(\mathscr{B}) = \mathscr{A} = \mathrm{T}^h_\rho \mathscr{B}_{\mathrm{eh}} \oplus \mathrm{T}^v_\rho \mathscr{B}_{\mathrm{eh}} \oplus \mathrm{T}^\perp_\rho \mathscr{B}_{\mathrm{eh}}.$$

Now, we are going to prove two facts which are used in the proof of (6) in proposition (5.4.1).

1. Proof that $U^{\perp(g)} = (U^C)^{\perp(g)}$: Since $U^C \supset U$, we have $(U^C)^{\perp(g)} \subset U^{\perp(g)}$. conversely, let $\sigma \in U^{\perp(g)}$. Then for every $\tau \in U$, we have $g(\sigma,\tau) = 0$. Now, for all $\tau \in U$, we also have $g(\sigma, \sqrt{-1}\tau) = (-\sqrt{-1})g(\sigma,\tau) = 0$. Therefore, for every $\delta \in U^C$, we get $g(\sigma,\delta) = 0$. It follows that $\sigma \in (U^C)^{\perp(g)}$, and hence $U^{\perp(g)} \subset (U^C)^{\perp(g)}$.

2. Proof that $(U^C)^{\perp(g)} = (U^C)^{\perp(B)}$: Clearly, $(U^C)^{\perp(g)} \subset (U^C)^{\perp(B)}$. Conversely, let $\sigma \in (U^C)^{\perp(B)}$. Then, for every $\tau \in U^C$, we have $B(\sigma,\tau) = 0$. Since U^C is a $\mathbb{C}$-subspace of $\mathscr{A}$, for every $\tau \in U^C$, we have $\sqrt{-1}\tau \in U^C$. This implies that for every $\tau \in U^C$, we have

$$\begin{aligned}\Omega(\sigma,\tau) &= -2\Im(g(\sigma,\tau)) = -2\Im(\sqrt{-1}g(\sigma,\sqrt{-1}\tau)) = -2\Re(g(\sigma,\sqrt{-1}\tau))\\ &= -B(\sigma,\sqrt{-1}\tau)) = 0.\end{aligned}$$

Thus, $B(\sigma,\tau) = \Omega(\sigma,\tau) = 0$, for all $\tau \in U^C$, and hence $g(\sigma,\tau) = 0$, for all $\tau \in U^C$. It follows that $\sigma \in (U^C)^{\perp(g)}$. This proves that $(U^C)^{\perp(B)} \subset (U^C)^{\perp(g)}$.

□

Recall that M denotes the moduli space $\mathscr{B}/G$ of Schur representations of Q, and $p : \mathscr{B} \to M$ the canonical projection. Let $M_{\mathrm{s}} = p(\mathscr{B}_{\mathrm{s}})$ be the moduli of θ-stable representations of Q, $p_{\mathrm{s}} : \mathscr{B}_{\mathrm{s}} \to M_{\mathrm{s}}$ the map induced by p, and $p_{\mathrm{eh}} = p_{\mathrm{s}}|_{\mathscr{B}_{\mathrm{eh}}} : \mathscr{B}_{\mathrm{eh}} \to M_{\mathrm{s}}$. For any two subsets A and B of $\mathscr{A}$, $P_G(A,B)$ denotes the set of all $g \in G$, such that $Ag \cap B \neq \emptyset$. Further, we had proved the following fact (Section 3.2.3, Lemma 3.2.26)

$$P_G(\mathscr{B}_{\mathrm{eh}}, \mathscr{B}_{\mathrm{eh}}) \subset HK.$$

Let $\rho \in \mathscr{B}_{\text{eh}}$ be fixed, and let $\rho' = p_{\text{eh}}(\rho)$. In this setting of notations, we prove the following.

Proposition 5.4.2 *The following statements are true:*

1. *The smooth map $p_{\text{eh}} : \mathscr{B}_{\text{eh}} \to M_{\text{s}}$ is a submersion.*

2. $p_{\text{eh}}^{-1}(p_{\text{eh}}(\rho)) = \rho \overline{K}$.

3. $\operatorname{Ker}(\mathrm{T}_\rho(p_{\text{eh}})) = \mathrm{T}^v_\rho \mathscr{B}_{\text{eh}}$.

4. $\mathrm{T}_\rho(p_{\text{eh}})|_{\mathrm{T}^h_\rho \mathscr{B}_{\text{eh}}} : \mathrm{T}^h_\rho \mathscr{B}_{\text{eh}} \to \mathrm{T}_{\rho'}(M_{\text{s}})$ *is a complex isometry, i.e., the diagram*

$$\begin{array}{ccc} \mathrm{T}^h_\rho \mathscr{B}_{\text{eh}} & \xrightarrow{\mathrm{T}_\rho(p_{\text{eh}})} & \mathrm{T}_{\rho'}(M_{\text{s}}) \\ {\scriptstyle J(\rho)=\sqrt{-1}}\downarrow & & \downarrow{\scriptstyle J_s(\rho')} \\ \mathrm{T}^h_\rho \mathscr{B}_{\text{eh}} & \xrightarrow{\mathrm{T}_\rho(\pi)} & \mathrm{T}_{\rho'}(M_{\text{s}}) \end{array} \tag{5.2}$$

commutes.

Proof. (1) The smooth submersion of the map $p_{\text{eh}} : \mathscr{B}_{\text{eh}} \to M_{\text{s}}$ follows from (3.2.31).

(2) Let $\sigma \in p_{\text{eh}}^{-1}(p_{\text{eh}}(\rho))$. Then, $p_{\text{eh}}(\sigma) = p_{\text{eh}}(\rho)$, and hence there exists $g \in G$ such that $\sigma = \rho g$. Therefore $g \in P_G(\mathscr{B}_{\text{eh}}, \mathscr{B}_{\text{eh}})$. Since $P_G(\mathscr{B}_{\text{eh}}, \mathscr{B}_{\text{eh}}) \subset HK$, there exist $h_0 \in H$ and $k_0 \in K$ such that $g = h_0 k_0$. Now, $\sigma = \rho g = (\rho h_0) k_0 = \rho k_0 = \rho \pi_K(k_0)$. It follows that $\sigma \in \rho \overline{K}$, and hence $p_{\text{eh}}^{-1}(p_{\text{eh}}(\rho)) \subset \rho \overline{K}$. Conversely, since p_{eh} is $\overline{K}$-invariant, we have $\rho \overline{K} \subset p_{\text{eh}}^{-1}(p_{\text{eh}}(\rho))$.

(3)

$$\begin{aligned} \operatorname{Ker}(\mathrm{T}_\rho(p_{\text{eh}})) &= \mathrm{T}_\rho(p_{\text{eh}}^{-1}(p_{\text{eh}}(\rho))) = \mathrm{T}_\rho(\rho \overline{K}) = \mathrm{T}_\rho(\rho K) \\ &= Im(\mathrm{T}_e(\nu_\rho)) = U = \mathrm{T}^v_\rho \mathscr{B}_{\text{eh}}. \end{aligned}$$

(4) Since $\mathrm{Ker}(\mathrm{T}_\rho(p_{\mathrm{eh}})) = \mathrm{T}^v_\rho\mathscr{B}_{\mathrm{eh}}$, and $\mathrm{T}_\rho(\mathscr{B}_{\mathrm{eh}}) = \mathrm{T}^h_\rho\mathscr{B}_{\mathrm{eh}} \oplus \mathrm{T}^v_\rho\mathscr{B}_{\mathrm{eh}}$, it follows that $\mathrm{T}_\rho(p_{\mathrm{eh}})|_{\mathrm{T}^h_\rho\mathscr{B}_{\mathrm{eh}}} : \mathrm{T}^h_\rho\mathscr{B}_{\mathrm{eh}} \to \mathrm{T}_{\rho'}(M_{\mathrm{s}})$ is a $\mathbb{R}$-linear isomorphism.

Let J and J_s be the complex structures of $\mathscr{B}$, and M_{s} respectively. Under the canonical $\mathbb{C}$-isomorphism of $\mathrm{T}_\rho(\mathscr{B})$ and $\mathscr{A}$, we have the diagram of $\mathbb{C}$-linear maps

$$\begin{array}{ccc} \mathscr{A} = \mathrm{T}_\rho(\mathscr{B}) & \xrightarrow{\mathrm{T}_\rho(p)} & \mathrm{T}_{\rho'}(M) \\ {\scriptstyle J(\rho)=\sqrt{-1}}\downarrow & & \downarrow{\scriptstyle J_S(\rho')} \\ \mathscr{A} = \mathrm{T}_\rho(\mathscr{B}) & \xrightarrow[\mathrm{T}_\rho(p)]{} & \mathrm{T}_{\rho'}(M) \end{array} \quad . \tag{5.3}$$

Since $p : \mathscr{B} \to M$ is holomorphic, the above diagram commutes.

Since $J(\mathrm{T}^h_\rho\mathscr{B}_{\mathrm{eh}}) = \mathrm{T}^h_\rho\mathscr{B}_{\mathrm{eh}}$ (by (7)), and the $\mathbb{R}$-linear map $\mathrm{T}_\rho(p_{\mathrm{eh}})$ is the restriction of the $\mathbb{C}$-linear map $\mathrm{T}_\rho(p)$, we have the commutative diagram

$$\begin{array}{ccc} \mathrm{T}^h_\rho\mathscr{B}_{\mathrm{eh}} & \xrightarrow{\mathrm{T}_\rho(p_{\mathrm{eh}})} & \mathrm{T}_{\rho'}(M_{\mathrm{s}}) \\ {\scriptstyle J(\rho)=\sqrt{-1}}\downarrow & & \downarrow{\scriptstyle J_s(\rho')} \\ \mathrm{T}^h_\rho\mathscr{B}_{\mathrm{eh}} & \xrightarrow[\mathrm{T}_\rho(p_{\mathrm{eh}})]{} & \mathrm{T}_{\rho'}(M_{\mathrm{s}}) \end{array} \quad . \tag{5.4}$$

We recall the Kähler form Θ_{s} on M_{s} described in (3.2.31): Let σ', $\tau' \in \mathrm{T}_{\rho'}(M_{\mathrm{s}})$. If for any σ, $\tau \in \mathrm{T}_\rho(\mathscr{B}_{\mathrm{eh}})$, $\mathrm{T}_\rho(p_{\mathrm{eh}})(\sigma) = \sigma'$ and $\mathrm{T}_\rho(p_{\mathrm{eh}})(\tau) = \tau'$, then $\Theta_{\mathrm{s}}(\sigma', \tau')$ is defined as the value $\Omega_{\mathrm{eh}}(\sigma, \tau)$, where Ω_{eh} the Kähler form on $\mathscr{B}_{\mathrm{eh}}$.

If we write $\sigma = \sigma^h + \sigma^v$, where $\sigma^h \in \mathrm{T}^h_\rho\mathscr{B}_{\mathrm{eh}}$, and $\sigma^v \in \mathrm{T}^v_\rho\mathscr{B}_{\mathrm{eh}}$, then we have $\sigma' = \mathrm{T}_\rho(p_{\mathrm{eh}})(\sigma^h)$, because of (3). Therefore, we can assume $\Theta_{\mathrm{s}}(\sigma', \tau') = \Omega_{\mathrm{eh}}(\sigma, \tau)$, where σ, $\tau \in \mathrm{T}^h_\rho\mathscr{B}_{\mathrm{eh}}$. By using the commutative diagram 5.4, we see that $\mathrm{T}_\rho(p_{\mathrm{eh}})(\sqrt{-1}\sigma) = J_s(\rho')(\sigma')$ for σ, $\tau \in \mathrm{T}^h_\rho\mathscr{B}_{\mathrm{eh}}$.

Let B_s be the smooth Riemannian metric on M_{s}. Then we have

$$B_s(\sigma', \tau') = -\Theta_{\mathrm{s}}(J_s(\rho')(\sigma'), \tau') = -\Omega_{\mathrm{eh}}(\sqrt{-1}\sigma, \tau)$$
$$= B(\sigma, \tau)$$

Thus, we have proved that $\mathrm{T}_\rho(p_{\mathrm{eh}})|_{\mathrm{T}^h_\rho \mathscr{B}_{\mathrm{eh}}} : \mathrm{T}^h_\rho \mathscr{B}_{\mathrm{eh}} \to \mathrm{T}_{\rho'}(M_{\mathrm{s}})$ is a complex isometry.

□

We have now shown that all the assumptions of Lemma 5.2.1 hold in the quiver setup described in Section 5.3. We, therefore, have the following result.

Theorem 5.4.3 *Let $H^{\mathscr{B}}$, H^S denote the holomorphic sectional curvature of $\mathscr{B}$, and M_{s} respectively. Then, for all $\rho \in \mathscr{B}_{\mathrm{eh}}$, for any horizontal unit vector $\sigma \in \mathrm{T}^h_\rho \mathscr{B}_{\mathrm{eh}}$ of $\mathscr{B}_{\mathrm{eh}}$, we have*

$$H^{\mathscr{B}}(\sigma) = H^S(\mathrm{T}_\rho(p_{\mathrm{eh}})(\sigma)) - 4\,|C(\sigma, \sigma)|^2,$$

where $C : \mathrm{T}_\rho(\mathscr{B}_{\mathrm{eh}}) \times \mathrm{T}_\rho(\mathscr{B}_{\mathrm{eh}}) \to \mathrm{T}^\perp_\rho \mathscr{B}_{\mathrm{eh}}$ denote the second fundamental form of $\mathscr{B}_{\mathrm{eh}}$ in $\mathscr{B}$.

Corollary 5.4.4 *The holomorphic sectional curvature of the moduli space M_{s} of θ-stable complex representations of Q with dimension vector d is non negative.*

Proof. Since the Kähler metric g on $\mathscr{B}$ is flat, its holomorphic sectional curvature $H^{\mathscr{B}}$ vanishes. The corollary follows from Theorem (5.4.3).

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