On The Schur multiplier of groups

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STATEMENT BY AUTHOR

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DECLARATION

I hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Sumana Hatui

List of Publications arising from the thesis

Journal

1. "Characterization of finite p-groups by their Schur multiplier", Sumana Hatui, *Proc. Indian Acad. Sci. Math. Sci.*, **2018**, *128*, 128:49.

2. "Finite p-groups having Schur multiplier of maximum order", Sumana Hatui, *J. Algebra*, **2017**, *492*, 490–497.

3. "The Schur multiplier of central product of groups", Sumana Hatui, L. R. Vermani, Manoj K. Yadav, *J. Pure Appl. Algebra*, **2018**, *222*, 3293–3302.

4. "The Schur multipliers of groups of order p^5", Sumana Hatui, Vipul Kakkar, Manoj K. Yadav, *Journal of Group Theory* (Accepted for publication).

Conferences

1. "**Ischia group Theory conference 2018**", held at Ischia, Naples, Italy, March 19-23, 2018.

2. "**83rd Annual Conference of Indian Mathematical Society-an International Meet**", S V University, Tirupati, December 12-15, 2017.

3. "**Groups St Andrews Conference**", held at University of Birmingham, Birmingham, UK, August 05-13, 2017.

Sumana Hatui

Dedicated to

my beloved baba and maa

Sri Saktipada Hatui and Smt. Tanusri Hatui

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Contents

Synopsis

0.1 Introduction

This thesis deals with the Schur multiplier of groups. The thesis contains four chapters. The first chapter consists of basic definitions and preliminaries relevant to the thesis. The second chapter is about characterization of finite p-groups by the order of their Schur multiplier. In the third chapter we study the Schur multiplier of central product of groups. The final part of the thesis consists of a chapter on determining the Schur multiplier, non-abelian tensor square, exterior square and capability of groups of order p^5.

The Schur multiplier was introduced by I. Schur on the study of projective representation of groups [42]. It is the second homology group with integral coefficients. We denote the Schur multiplier of a group G by $M(G)$. By $\mathbb{Z}_p^{(k)}$ we denote $\mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p(k$ times). G' denotes the commutator subgroup of G. $\gamma_i(G)$ denotes the i-th term of the lower central series of a group G and G^{ab} denotes the quotient group $G/\gamma_2(G)$. James [27] classified all p-groups of order p^n $(n \leq 6)$ for odd prime p, upto isoclinism (see Section 1.1 for definition). These isoclinism classes are denoted by Φ_k (see Section 1.1 for details). We use these notations throughout.

0.2 Characterization of finite p-groups by the order of their Schur mulltiplier

In 1956, Green proved the following result:

Theorem 0.2.1 ([15]) *If G is a p-group of order p^n, then $|M(G)| \leq p^{\frac{1}{2}n(n-1)}$.*

So for any finite p-group of order p^n, there is an integer $t(G) \geq 0$ such that $|M(G)| = p^{\frac{1}{2}n(n-1)-t(G)}$. This integer $t(G)$ is called the corank of G. It is an interesting problem to classify the structure of all non-abelian p-groups G by the order of the Schur multiplier $M(G)$, i.e., when $t(G)$ is known. Several authors have studied this problem for various values of $t(G)$.

First Berkovich [1] and Zhou [50] classified all p-groups G for $t(G) = 0, 1, 2$. Ellis [9] also classified groups G for $t(G) = 0, 1, 2, 3$ by a different method. After that several authors classified the groups G of order p^n for $t(G) = 4, 5, 6$ in [34, 35, 26].

In 2009 Niroomand improved the Green's bound by proving the following result:

Theorem 0.2.2 (Main Theorem of [33]) *Let G be a non-abelian finite p-group of order p^n. If $|G'| = p^k$, then*

$$|M(G)| \leq p^{\frac{1}{2}(n+k-2)(n-k-1)+1}.$$

In particular,

$$|M(G)| \leq p^{\frac{1}{2}(n-1)(n-2)+1},$$

and the equality holds in this last bound if and only if $G = H \times Z$, where H is an extra special p-group of order p^3 and exponent p, and Z is an elementary abelian p-group.

This says that for non abelian p-groups G of order p^n, $|M(G)| = p^{\frac{1}{2}(n-1)(n-2)+1-s(G)}$, for some $s(G) \geq 0$. This integer $s(G)$ is called the generalized corank of

G. The structure of non-abelian p-groups for $s(G) = 0, 1, 2$ has been determined in [36, 37] which is the same as to classify groups G for $t(G) = \log_p(|G|) - 2, \log_p(|G|) - 1, \log_p(|G|)$ respectively.

0.2.1 Groups with $t(G) = \log_p(|G|) + 1$

We take the above line of investigation and classify all non-abelian finite p-groups G for which $t(G) = \log_p(|G|) + 1$, which is same as classifying G for $s(G) = 3$, i.e., $|M(G)| = p^{\frac{1}{2}(n-1)(n-2)+1-3} = p^{\frac{1}{2}n(n-3)-1}$. Our main theorem is the following:

Theorem 0.2.3 ([18]) *Let G be a non-abelian finite p-group of order p^n with $t(G) = \log_p(|G|) + 1$. Then for odd prime p, G is isomorphic to one of the following groups:*
$\Phi_2(22), \Phi_3(211)a, \Phi_3(211)b_r, \Phi_2(2111)c, \Phi_2(2111)d, \Phi_3(1^5), \Phi_7(1^5), \Phi_{11}(1^6), \Phi_{12}(1^6),$
$\Phi_{13}(1^6), \Phi_{15}(1^6), (\mathbb{Z}_p^{(4)} \rtimes \mathbb{Z}_p) \times \mathbb{Z}_p^{(2)}.$
Moreover for $p = 2$, G is isomorphic to one of the following groups:
$\mathbb{Z}_2^{(4)} \rtimes \mathbb{Z}_2, G_1 \times \mathbb{Z}_2, G_2, \mathbb{Z}_4 \rtimes \mathbb{Z}_4, D_{16},$
where, $G_1 = \langle a^4 = b^2 = c^2 = 1, [a, c] = b, [a, b] = [b, c] = 1\rangle$,
$G_2 = \langle a^4 = b^4 = c^2 = 1, [a, b] = 1, [a, c] = a^2, [b, c] = b^2\rangle$, and D_{16} is the Dihedral group of order 16.

0.2.2 Groups having Schur multiplier of maximum order

In Theorem 0.2.2, Niroomand classified groups G such that $|M(G)| = p^{\frac{1}{2}(n+k-2)(n-k-1)+1}$ with $k = 1$. We say that $|M(G)|$ attains the bound if $|M(G)| = p^{\frac{1}{2}(n+k-2)(n-k-1)+1}$.

Rai [40] classified finite p-groups G of class 2 such that $|M(G)|$ attains the bound.

Theorem 0.2.4 (Theorem 1.1 of [40]) *Let G be a finite p-group of order p^n and nilpotency class 2 with $|G'| = p^k$. Then $|M(G)| = p^{\frac{1}{2}(n+k-2)(n-k-1)+1}$ if and* *only if G is one of the following groups.*

(i) $G_1 = ES_p(p^3) \times \mathbb{Z}_p^{(n-3)}$*, where p is an odd prime.*

(ii) $G_2 = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_i, \alpha] = \beta_i, [\alpha_1, \alpha_2] = 1, \alpha^p = \alpha_i^p = \beta_i^p = 1\ (i = 1, 2) \rangle$*, where p is an odd prime.*

(iii) $G_3 = \langle \alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \beta_1,\ [\alpha_3, \alpha_1] = \beta_2, \alpha_i^p = \beta_i^p = 1\ (i = 1, 2, 3) \rangle$*, where p is an odd prime.*

Now a natural question arises here as follows: What will happen for groups of nilpotency class ≥ 3? The next theorem gives the answer to this question.

Theorem 0.2.5 ([19]) *There is no non-abelian p-group G of order p^n, $p \neq 3$, having nilpotency class $c \geq 3$ with $|G'| = p^k$ and $|M(G)| = p^{\frac{1}{2}(n+k-2)(n-k-1)+1}$. In particular, $|M(G)| \leq p^{\frac{1}{2}(n+k-2)(n-k-1)}$ for p-groups G of nilpotency class $c \geq 3$ and $p \neq 3$.*

Now one may ask: Is the above statement true for $p = 3$?

The answer to this question is no as shown by the following example. Consider the group

$G = \langle \alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3, \gamma \mid [\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \beta_1, [\alpha_3, \alpha_1] = \beta_2, [\beta_3, \alpha_3] = [\beta_2, \alpha_2] = [\beta_1, \alpha_1] = \gamma, \alpha_i^3 = \beta_i^3 = \gamma^3 = 1\ (i = 1, 2, 3) \rangle$

of order 3^7. Using HAP [12] of GAP [14] we see that $|M(G)| = p^{\frac{1}{2}(n+k-2)(n-k-1)+1} = p^{10}$.

We say that $|M(G)|$ attains the new bound if $|M(G)| = p^{\frac{1}{2}(n+k-2)(n-k-1)}$. So a natural question which arises here is the following:

Question: Do there exist finite p-groups of arbitrary nilpotency class for which the new bound is attained?

The answer to this question is yes for nilpotency classes 3 and 4, as shown by the following examples.

Example 1: Consider the group $G = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha, \alpha_1] = \alpha_2, [\alpha_2, \alpha] =$

$\alpha_3, [\alpha_2, \alpha_1] = \alpha_4, \alpha^p = \alpha_i^p = 1\ (i = 1, 2, 3, 4)\rangle$ from [27]. This is a group of order p^5 with $|G'| = p^3$. The nilpotency class of G is 3. For $p = 5, 7, 11, 13, 17$ using HAP of GAP we obtain $M(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$.

Example 2: Consider the group $G = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_1^p = \alpha_{i+1}^p = 1\ (i = 1, 2, 3)\rangle$ from [27]. This is a group of order p^5 with $|G'| = p^3$. The nilpotency class of G is 4. For $p = 5, 7, 11, 13, 17$ using HAP of GAP we obtain $M(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$.

0.3 The Schur multiplier of Central Product of groups

We say that G is an internal central product of its two normal subgroups H and K amalgamating A if $G = HK$ with $A = H \cap K$ and $[H, K] = 1$. Let H, K be two groups with isomorphic subgroups $A \leq Z(H)$, $B \leq Z(K)$ under an isomorphism $\phi : A \to B$. Consider the normal subgroup $U = \{(a, \phi(a)^{-1}) \mid a \in A\}$. Then the group $G := (H \times K)/U$ is called the external central product of H and K amalgamating A and B via ϕ. The external central product G can be viewed as an internal central product of the images of $H \times 1$ and $1 \times K$ in G. For this reason, we do not differentiate between external and internal central products, and consider only internal ones.

Wiegold [49] proved that if G is a finite group and is the central product of subgroups H and K amalgamating A, then $M(G)$ contains a subgroup isomorphic with $H/A \otimes K/A$. A generalization of this result for arbitrary central quotient of direct product of two arbitrary groups was considered in [8] and it was proved that $H/A \otimes K/A$ is a quotient of $M(G)$.

Here $\mathrm{H}^2(G, D)$ denotes the second cohomology group of a group G with coefficients in D, where D is a divisible abelian group regarded as a trivial G-

module. Unless said otherwise explicitly, here G is always a central product of its normal subgroups H and K with $A = H \cap K$. We study $\mathrm{H}^2(G, D)$, in terms of the second cohomology groups of certain quotients of H and K with coefficients in D. Set $Z = H' \cap K'$. The following result provides a reduction to the case when $Z = 1$.

Theorem 0.3.1 ([20]) *Let B be a subgroup of G such that $B \leq Z$. Then $\mathrm{H}^2(G, D) \cong \mathrm{H}^2(G/B, D)/N$, where $N \cong \mathrm{Hom}(B, D)$.*

Our main theorem is the following.

Theorem 0.3.2 ([20]) *Let $L \cong \mathrm{Hom}\big((A \cap H')/Z, D\big)$, $M \cong \mathrm{Hom}\big((A \cap K')/Z, D\big)$ and $N \cong \mathrm{Hom}(Z, D)$. Then the following statements hold true:*

(i) $\big(\mathrm{H}^2(H/A, D)/L \oplus \mathrm{H}^2(K/A, D)/M\big)/N \oplus \mathrm{Hom}(H/A \otimes K/A, D)$ embeds in $\mathrm{H}^2(G, D)$.

(ii) $\mathrm{H}^2(G, D)$ embeds in $\big(\mathrm{H}^2(H/Z, D) \oplus \mathrm{H}^2(K/Z, D)\big)/N \oplus \mathrm{Hom}(H \otimes K, D)$.

In particular, for $D = \mathbb{C}^\times$, assertion (i) of Theorem 0.3.2 provides a refinement of results from [8] and [49].

Now we present some examples (all of them are finite p-groups) to show that various situations of Theorem 0.3.2 can indeed occur. The following example shows that neither of the two embeddings of Theorem B is necessarily an isomorphism:

Example 1. Let H be the extraspecial p-groups of order p^3 and exponent p and $K = \mathbb{Z}_p^{(n+1)}$, where $n \geq 1$. Let G be a central product of H and K amalgamated at $A \cong H' \cong \mathbb{Z}_p$. Note that $G = H \times \mathbb{Z}_p^{(n)}$. It is easy to see that $M(G) \cong \mathbb{Z}_p^{\left(\frac{1}{2}n(n+3)+2\right)}$. Note that $Z = H' \cap K' = 1$. Then $M(H/A)/\mathrm{Hom}(A \cap H', \mathbb{C}^\times) \oplus M(K/A)/\mathrm{Hom}(A \cap K', \mathbb{C}^\times) \oplus \mathrm{Hom}(H/A \otimes K/A, \mathbb{C}^\times) \cong \mathbb{Z}_p^{\left(\frac{1}{2}n(n+3)\right)}$, which is strictly contained in $M(G)$. Since $M(H) \oplus M(K) \oplus \mathrm{Hom}(H \otimes K, \mathbb{C}^\times) \cong \mathbb{Z}_p^{(\frac{1}{2}(n+1)(n+4)+2)}$, it properly contains $M(G)$.

The following example show that the first embedding in Theorem 0.3.2 can very well be an isomorphism, but the second one can still be strict (i.e., not an isomorphism):

Example 2. $G = \langle \alpha, \alpha_1, \alpha_2, \gamma \mid [\alpha_1, \alpha] = \gamma^{p^2} = \alpha_2, \alpha^p = \alpha_1^p = \alpha_2^p = 1\rangle$. Take $H = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2, \alpha^p = \alpha_1^p = \alpha_2^p = 1\rangle$ and $K = \langle \gamma \rangle \cong \mathbb{Z}_{p^3}$. It can be easily seen that G is a central product of H and K amalgamated at $A \cong \langle \alpha_2 \rangle \cong \langle \gamma^{p^2} \rangle$. Note that $Z = 1$ and $M(H/A)/\operatorname{Hom}(A \cap H', \mathbb{C}^\times) \oplus M(K/A)/\operatorname{Hom}(A \cap K', \mathbb{C}^\times) \oplus \operatorname{Hom}(H/A \otimes K/A, \mathbb{C}^\times) \cong \mathbb{Z}_p^{(2)}$. We have $M(G) \cong \mathbb{Z}_p^{(2)}$. Therefore the first embedding in Theorem 0.3.2 is an isomorphism. It is easy to see that $M(H) \oplus M(K) \oplus \operatorname{Hom}(H \otimes K, \mathbb{C}^\times) \cong \mathbb{Z}_p^{(4)}$, which shows that the second embedding is strict.

We finally present an example which shows that both the embeddings in Theorem B can be isomorphisms.

Example 3. Let H be the extraspecial p-groups of order p^3 and exponent p^2 and $K \cong \mathbb{Z}_{p^{n+1}}$, the cyclic group of order p^{n+1}, where $n \geq 1$. Let G be a central product of H and K amalgamated at $A \cong H' \cong \mathbb{Z}_p$. Note that $G = H \times \mathbb{Z}_{p^n}$. It is easy to see that $M(G) \cong \mathbb{Z}_p^{(2)}$. Note that $Z = H' \cap K' = 1$. Then $M(H/A)/\operatorname{Hom}(A \cap H', \mathbb{C}^\times) \oplus M(K/A)/\operatorname{Hom}(A \cap K', \mathbb{C}^\times) \oplus \operatorname{Hom}(H/A \otimes K/A, \mathbb{C}^\times)$ is isomorphic to $\mathbb{Z}_p^{(2)}$. Also $M(H) \oplus M(K) \oplus \operatorname{Hom}(H \otimes K, \mathbb{C}^\times) \cong \mathbb{Z}_p^{(2)}$. Hence both the embeddings are isomorphisms.

We finally prove

Theorem 0.3.3 *If the second embedding in Theorem 0.3.2 is an isomorphism, then so is the first.*

0.4 The Schur multiplier of groups of order p^5

The non-abelian tensor square $G \otimes G$ of a group G is generated by the symbols $g \otimes h, g, h \in G$, subject to the relations

$$gg' \otimes h = (g^{g'} \otimes h^{g'})(g' \otimes h) \text{ and } g \otimes hh' = (g \otimes h')(g^{h'} \otimes h^{h'})$$

for all $g, g', h, h' \in G$, with $h^g = g^{-1}hg$. The non-abelian exterior square $G \wedge G$ is the quotient of $G \otimes G$ by $\nabla(G)$, where $\nabla(G)$ is the normal subgroup generated by the elements $g \otimes g$, for all $g \in G$. This definition says that there is an epimorphism $f : G \otimes G \to G'$, defined on the generators by $f(g \otimes h) = [g, h]$ where $[g, h] = g^{-1}h^{-1}gh$. This map f induces an epimorphism $f' : G \wedge G \to G'$. The kernel of this map is isomorphic to the Schur multiplier $M(G)$ of G [7]. A group G is called *capable* if there exists a group H such that $G \cong H/Z(H)$, where $Z(H)$ denotes the center of H. We denote the *epicenter* of a group G by $Z^*(G)$, which is the smallest central subgroup of G such that $G/Z^*(G)$ is capable. Here Γ denotes Whitehead's quadratic functor, defined from the category of abelian groups to itself. For an abelian group A, ΓA is the abelian group generated by the symbols $w(a)$, $a \in A$ such that the following relations hold:

(*i*) $w(a) = w(a^{-1})$,

(*ii*) $w(abc)w(a)w(b)w(c) = w(ab)w(bc)w(ca)$,

for all $a, b, c \in A$. For finitely generated groups we have the following:

(*i*) $\Gamma(G \times H) \cong \Gamma(G) \times \Gamma(H) \times (G \otimes H)$.

(*ii*) $\Gamma(\mathbb{Z}_n) = \begin{cases} \mathbb{Z}_n, & \text{n odd} \\ \mathbb{Z}_{2n}, & \text{n even.} \end{cases}$

(*iii*) $\Gamma(\mathbb{Z}) \cong \mathbb{Z}$.

We compute the Schur multiplier, non-abelian tensor square and exterior square of groups of order p^5. As an application of this, we determine the capability of non-abelian p-groups of order p^5. Our main result [21] is presented in the following table (for $p > 3$):

G	$\Gamma(G^{ab})$	$M(G)$	$G \wedge G$	$G \otimes G$	Capability
$\Phi_2(311)a$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p \times \mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(8)}$	Not Capable
$\Phi_2(221)a$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2}^{(2)} \times \mathbb{Z}_p^{(7)}$	Capable
$\Phi_2(221)b$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p \times \mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(8)}$	Not Capable
$\Phi_2(2111)a$	$\mathbb{Z}_p^{(10)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(16)}$	Not Capable
$\Phi_2(2111)b$	$\mathbb{Z}_p^{(10)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(16)}$	Not Capable
$\Phi_2(2111)c$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(10)}$	Not Capable
$\Phi_2(2111)d$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(10)}$	Not Capable
$\Phi_2(1^5)$	$\mathbb{Z}_p^{(10)}$	$\mathbb{Z}_p^{(7)}$	$\mathbb{Z}_p^{(8)}$	$\mathbb{Z}_p^{(18)}$	Capable
$\Phi_2(41)$	$\mathbb{Z}_{p^3} \times \mathbb{Z}_p^{(2)}$	$\{1\}$	$\mathbb{Z}_p$	$\mathbb{Z}_{p^3} \times \mathbb{Z}_p^{(3)}$	Not Capable
$\Phi_2(32)a_1$	$\mathbb{Z}_{p^2}^{(3)}$	$\mathbb{Z}_p$	$\mathbb{Z}_{p^2}$	$\mathbb{Z}_{p^2}^{(4)}$	Not Capable
$\Phi_2(32)a_2$	$\mathbb{Z}_{p^3} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p$	$\mathbb{Z}_{p^2}$	$\mathbb{Z}_{p^3} \times \mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	Not Capable
$\Phi_2(311)b$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p \times \mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(8)}$	Not Capable
$\Phi_2(311)c$	$\mathbb{Z}_{p^3} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p \times \mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_{p^3} \times \mathbb{Z}_p^{(5)}$	Not Capable
$\Phi_2(221)c$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(5)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2}^{(2)} \times \mathbb{Z}_p^{(7)}$	Capable
$\Phi_2(221)d$	$\mathbb{Z}_{p^2}^{(3)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2}^{(4)} \times \mathbb{Z}_p^{(2)}$	Capable
$\Phi_3(2111)a$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(11)}$	Not Capable
$\Phi_3(2111)b_r$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(11)}$	Not Capable
$\Phi_3(1^5)$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(12)}$	Capable
$\Phi_3(311)a$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(5)}$	Not Capable
$\Phi_3(311)b_r$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(5)}$	Not Capable
$\Phi_3(221)a$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(5)}$	Not Capable
$\Phi_3(221)b_r$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p \times \mathbb{Z}_p$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2}^{(2)} \times \mathbb{Z}_p^{(4)}$	Capable
$\Phi_3(2111)c$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(11)}$	Not Capable
$\Phi_3(2111)d$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(6)}$	Not Capable
$\Phi_3(2111)e$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(6)}$	Not Capable

G	$\Gamma(G^{ab})$	$M(G)$	$G \wedge G$	$G \otimes G$	Capability
$\Phi_4(221)a$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(9)}$	Not Capable
$\Phi_4(221)b$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p \times \mathbb{Z}_p$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(8)}$	Capable
$\Phi_4(221)c$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(9)}$	Not Capable
$\Phi_4(221)d_{r,\, r\neq\frac{1}{2}(p-1)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(9)}$	Not Capable
$\Phi_4(221)d_{\frac{1}{2}(p-1)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_{p^2}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(8)}$	Capable
$\Phi_4(221)e$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(9)}$	Not Capable
$\Phi_4(221)f_0$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_{p^2}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(8)}$	Capable
$\Phi_4(221)f_r$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(9)}$	Not Capable
$\Phi_4(2111)a$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(11)}$	Not Capable
$\Phi_4(2111)b$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(11)}$	Not Capable
$\Phi_4(2111)c$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(11)}$	Not Capable
$\Phi_4(1^5)$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(8)}$	$\mathbb{Z}_p^{(14)}$	Capable
$\Phi_5(2111)$	$\mathbb{Z}_p^{(10)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(16)}$	Not Capable
$\Phi_5(1^5)$	$\mathbb{Z}_p^{(10)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(16)}$	Not Capable
$\Phi_6(221)a$	$\mathbb{Z}_p^{(3)}$	$\{1\}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	Not Capable
$\Phi_6(221)b_{r,\, r\neq\frac{1}{2}(p-1)}$	$\mathbb{Z}_p^{(3)}$	$\{1\}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	Not Capable
$\Phi_6(221)b_{\frac{1}{2}(p-1)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(5)}$	Capable
$\Phi_6(221)c_r$	$\mathbb{Z}_p^{(3)}$	$\{1\}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	Not Capable
$\Phi_6(221)d_0$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(5)}$	Capable
$\Phi_6(221)d_r$	$\mathbb{Z}_p^{(3)}$	$\{1\}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	Not Capable
$\Phi_6(2111)a$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(7)}$	Not Capable
$\Phi_6(2111)b_r$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(7)}$	Not Capable
$\Phi_6(1^5)$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(9)}$	Capable
$\Phi_7(2111)a$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(11)}$	Not Capable
$\Phi_7(2111)b_r$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(11)}$	Not Capable

G	$\Gamma(G^{ab})$	$M(G)$	$G \wedge G$	$G \otimes G$	Capability
$\Phi_7(2111)c$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(11)}$	Not Capable
$\Phi_7(1^5)$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(12)}$	Capable
$\Phi_8(32)$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\{1\}$	$\mathbb{Z}_{p^2}$	$\mathbb{Z}_{p^2}^{(2)} \times \mathbb{Z}_p^{(2)}$	Not Capable
$\Phi_9(2111)a$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(7)}$	Not Capable
$\Phi_9(2111)b_r$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(7)}$	Not Capable
$\Phi_9(1^5)$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(3)}$	$\Phi_2(111) \times \mathbb{Z}_p^{(3)}$	$\Phi_2(111) \times \mathbb{Z}_p^{(6)}$	Capable
$\Phi_{10}(2111)a$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(7)}$	Not Capable
$\Phi_{10}(2111)b_r$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(7)}$	Not Capable
$\Phi_{10}(1^5)$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(3)}$	$\Phi_2(111) \times \mathbb{Z}_p^{(3)}$	$\Phi_2(111) \times \mathbb{Z}_p^{(6)}$	Capable

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Notations

G	a group.
$H \leq G$	H is a subgroup of the group G.
$[x, y]$	commutator $x^{-1}y^{-1}xy$ for $x, y \in G$.
G'	commutator subgroup of the group G.
$\Phi(G)$	Frattini subgroup of the group G.
$\gamma_i(G)$	i-th term of the lower central series of G.
G^{ab}	the quotient group $G/\gamma_2(G)$.
$d(G)$	cardinality of a minimal generating set of a finitely generated group G.
$\mathrm{Z}(G)$	center of G.
$\mathbb{Z}_n$	cyclic group of order n.
$G^{(n)}$	direct product of n-copies of G, $n \geq 1$.
$ES(p^{2m+1})$	extraspecial p-group of order p^{2m+1}, $m \geq 1$.
$ES_{p^k}(p^{2m+1})$	extraspecial p-group of order p^{2m+1}, $m \geq 1$ having exponent p^k, $k = 1, 2$.
$[H, K]$	subgroup generated by all commutators $[h, k], h \in H, k \in K$, for two subgroups H and K of G.
$\mathrm{Hom}(H, K)$	groups of all homomorphisms from a group H to a group K.
Γ	Whitehead's quadratic functor from the category of abelian groups to itself.

CHAPTER 1

Background and Preliminaries

In this chapter we provide some basic definitions and results which will be used in succeeding chapters. We present an overview of the Schur multiplier, non-abelian tensor square and exterior square of groups, which are relevant to this thesis.

1.1 Non-abelian p-groups of order upto p^6, p odd

Hall introduced the concept of isoclinism of groups in [17] while studying the classification of prime power order groups. For a given group G, define a map $a_G : G/\,\mathrm{Z}(G) \times G/\,\mathrm{Z}(G) \to G'$ by $a_G(g\,\mathrm{Z}(G), g'\,\mathrm{Z}(G)) = [g, g']$ for $g, g' \in G$. This is a well defined map, called commutator map.

Two groups G and H are said to be *isoclinic* if there are isomorphisms ϕ from $G/\,\mathrm{Z}(G)$ onto $H/\,\mathrm{Z}(H)$ and θ from G' onto H' such that the following diagram commutes.

$$\begin{array}{ccc} G/\,\mathrm{Z}(G) \times G/\,\mathrm{Z}(G) & \xrightarrow{a_G} & G' \\ \downarrow{\scriptstyle \phi\times\phi} & & \downarrow{\scriptstyle \theta} \\ H/\,\mathrm{Z}(H) \times H/\,\mathrm{Z}(H) & \xrightarrow{a_H} & H'. \end{array}$$

The pair (ϕ, θ) is called an isoclinism of G onto H. This is an equivalence relation on groups. In [17], Hall proved that in every family of isoclinic groups there exists a group G such that $Z(G) \leq G'$, which is called a *stem group*.

There are 10 isoclinism classes $\Phi_i, 1 \leq i \leq 10$, of groups of order $p^5, p > 2$ as per the classification by James [27]. The groups in Φ_1 are abelian groups. We recall some notations from [27]. By ν we denote the smallest positive integer which is a non-quadratic residue (mod p), and by ζ we denote the smallest positive integer which is a primitive root (mod p). Relations of the form $[\alpha, \beta] = 1$ for generators α and β are omitted in the presentations of the groups. For an element α_{i+1} of a finite p-group G, by $\alpha_{i+1}^{(p)}$, we mean $\alpha_{i+1}^{p}\alpha_{i+2}^{\binom{p}{2}}\cdots\alpha_{i+k}^{\binom{p}{k}}\cdots\alpha_{i+p}$, where $\alpha_{i+2}, ..., \alpha_{i+p}$ are suitably defined elements of G. Observe that, for the groups of order upto p^5 listed in [27], we have $\alpha_i^{(p)} = \alpha_i^p$ when $p > 3$. Hence for $p > 3$, the presentations of these groups are uniform, but for $p = 2, 3$ groups depend on the prime p. Therefore we assume $p > 3$ for our investigations. On the other hand, for $p = 2, 3$ we use HAP [12] of GAP [14] for our calculations.

From [27], we include presentations of non-abelian groups of order p^n $(n \leq 5)$ and some groups of order p^6, for odd prime p.

1.1.1 Groups of order p^3

The isoclinism family Φ_2 consists of the following groups

(i) $\Phi_2(21) = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2, \alpha^p = \alpha_2, \alpha_1^p = \alpha_2^p = 1\rangle$,

(ii) $\Phi_2(111) = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2, \alpha^p = \alpha_1^p = \alpha_2^p = 1\rangle$.

1.1.2 Groups of order p^4

The isoclinism family Φ_2 consists of the following groups

(i) $\Phi_2(211)a = \Phi_2(21) \times \mathbb{Z}_p$,

(ii) $\Phi_2(1^4) = \Phi_2(111) \times \mathbb{Z}_p$,

(iii) $\Phi_2(31) = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha^{p^2} = \alpha_2, \alpha_1^p = \alpha_2^p = 1 \rangle$,

(iv) $\Phi_2(22) = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha^p = \alpha_2, \alpha_1^{p^2} = \alpha_2^p = 1 \rangle$,

(v) $\Phi_2(211)b = \langle \alpha, \alpha_1, \alpha_2, \gamma \mid [\alpha_1, \alpha] = \gamma^p = \alpha_2, \alpha^p = \alpha_1^p = \alpha_2^p = 1 \rangle$,

(vi) $\Phi_2(211)c = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2, \alpha^{p^2} = \alpha_1^p = \alpha_2^p = 1 \rangle$.

The isoclinism family Φ_3 consists of the following groups

(i) $\Phi_3(211)a = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha^p = \alpha_3, \alpha_1^{(p)} = \alpha_2^p = \alpha_3^p = 1 \rangle$,

(ii) $\Phi_3(211)b_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha]^r = \alpha_1^{(p)} = \alpha_3^r, \alpha^p = \alpha_2^p = \alpha_3^p = 1 \rangle$ for $r = 1$ or ν,

(iii) $\Phi_3(1^4) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_i^{(p)} = \alpha_3^p = 1, (i = 1, 2) \rangle$.

1.1.3 Groups of order p^5

The groups of nilpotency class 2 falls in the isoclinism families Φ_2, Φ_4 and Φ_5.

The isoclinism family Φ_2 consists of the following groups, in which the cyclic direct factor is generated by α_3.

(i) $\Phi_2(311)a = \Phi_2(31) \times \mathbb{Z}_p$,

(ii) $\Phi_2(221)a = \Phi_2(22) \times \mathbb{Z}_p$,

(iii) $\Phi_2(221)b = \Phi_2(21) \times \mathbb{Z}_{p^2}$,

(iv) $\Phi_2(2111)a = \Phi_2(211)a \times \mathbb{Z}_p$,

(v) $\Phi_2(2111)b = \Phi_2(211)b \times \mathbb{Z}_p$,

(vi) $\Phi_2(2111)c = \Phi_2(211)c \times \mathbb{Z}_p$,

(vii) $\Phi_2(2111)d = \Phi_2(111) \times \mathbb{Z}_{p^2}$,

(viii) $\Phi_2(1^5) = \Phi_2(1^4) \times \mathbb{Z}_p$,

(ix) $\Phi_2\,(41) = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha^{p^3} = \alpha_2, \alpha_1^p = \alpha_2^p = 1\rangle$,

(x) $\Phi_2\,(32)\,a_1 = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha^{p^2} = \alpha_2, \alpha_1^{p^2} = \alpha_2^p = 1\rangle$,

(xi) $\Phi_2\,(32)\,a_2 = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_1^p = \alpha_2, \alpha^{p^3} = \alpha_2^p = 1\rangle$,

(xii) $\Phi_2\,(311)\,b = \langle \alpha, \alpha_1, \alpha_2, \gamma \mid [\alpha_1, \alpha] = \gamma^{p^2} = \alpha_2, \alpha^p = \alpha_1^p = \alpha_2^p = 1\rangle$,

(xiii) $\Phi_2\,(311)\,c = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2, \alpha^{p^3} = \alpha_1^p = \alpha_2^p = 1\rangle$,

(xiv) $\Phi_2\,(221)\,c = \langle \alpha, \alpha_1, \alpha_2, \gamma \mid [\alpha_1, \alpha] = \gamma^p = \alpha_2, \alpha^{p^2} = \alpha_1^p = \alpha_2^p = 1\rangle$,

(xv) $\Phi_2\,(221)\,d = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2, \alpha^{p^2} = \alpha_1^{p^2} = \alpha_2^p = 1\rangle$.

Isoclinism family Φ_4 consists of the following groups

(i) $\Phi_4\,(221)\,a = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_i, \alpha] = \beta_i, \alpha^p = \beta_2, \alpha_1^p = \beta_1, \alpha_2^p = \beta_i^p = 1\ (i = 1, 2)\rangle$,

(ii) $\Phi_4\,(221)\,b = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_i, \alpha] = \beta_i, \alpha^p = \beta_2, \alpha_2^p = \beta_1, \alpha_1^p = \beta_i^p = 1\ (i = 1, 2)\rangle$,

(iii) $\Phi_4\,(221)\,c = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_i, \alpha] = \beta_i = \alpha_i^p, \alpha^p = \beta_i^p = 1\ (i = 1, 2)\rangle$,

(iv) $\Phi_4\,(221)\,d_r = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_i, \alpha] = \beta_i, \alpha_1^p = \beta_1^k, \alpha_2^p = \beta_2, \alpha^p = \beta_i^p = 1\ (i = 1, 2)\rangle$, where $k = \zeta^r, r = 1, 2, \ldots, \frac{1}{2}(p-1)$,

(v) $\Phi_4\,(221)\,e = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_i, \alpha] = \beta_i, \alpha_1^p = \beta_2^{-1/4}, \alpha_2^p = \beta_1\beta_2, \alpha^p = \beta_i^p = 1\ (i = 1, 2)\rangle$,

(vi) $\Phi_4\,(221)\,f_0 = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_i, \alpha] = \beta_i, \alpha_1^p = \beta_2, \alpha_2^p = \beta_1^\nu, \alpha^p = \beta_i^p = 1\ (i = 1, 2)\rangle$,

(vii) $\Phi_4\,(221)\,f_r = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_i, \alpha] = \beta_i, \alpha_1^p = \beta_2^k, \alpha_2^p = \beta_1\beta_2, \alpha^p = \beta_i^p = 1\ (i = 1, 2)\rangle$, where $4k = \zeta^{2r+1} - 1$ for $r = 1, 2, \ldots, \frac{1}{2}(p-1)$,

(viii) $\Phi_4\,(2111)\,a = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_i, \alpha] = \beta_i, \alpha^p = \beta_2, \alpha_i^p = \beta_i^p = 1\ (i = 1, 2)\rangle$,

(ix) $\Phi_4\,(2111)\,b = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_i, \alpha] = \beta_i, \alpha_1^p = \beta_1, \alpha^p = \alpha_2^p = \beta_i^p = 1\ (i = 1, 2)\rangle$,

(x) $\Phi_4\,(2111)\,c = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_i, \alpha] = \beta_i, \alpha_2^p = \beta_1, \alpha^p = \alpha_1^p = \beta_i^p = 1\ (i = 1, 2)\rangle$,

(xi) $\Phi_4\,(1^5) = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_i, \alpha] = \beta_i, \alpha^p = \alpha_i^p = \beta_i^p = 1\ (i = 1, 2)\rangle$.

The isoclinism family Φ_5 consists of the following two groups

(i) $\Phi_5(2111) = \langle \alpha_i, \alpha_2, \alpha_3, \alpha_4, \beta \mid [\alpha_1, \alpha_2] = [\alpha_3, \alpha_4] = \alpha_1^p = \beta, \alpha_2^p = \alpha_3^p = \alpha_4^p = \beta^p = 1\rangle$,

(ii) $\Phi_5(1^5) = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta \mid [\alpha_1, \alpha_2] = [\alpha_3, \alpha_4] = \beta, \alpha_1^p = \alpha_2^p = \alpha_3^p = \alpha_4^p = \beta^p = 1\rangle$.

The groups of nilpotency class 3 fall in the isoclinism families Φ_3, Φ_6, Φ_7 and Φ_8.

The class Φ_3 consists of the following groups.

(i) $\Phi_3(2111)a = \Phi_3(211)a \times \mathbb{Z}_p$,

(ii) $\Phi_3(2111)b_r = \Phi_3(211)b_r \times \mathbb{Z}_p$ for $r = 1$ or ν,

(iii) $\Phi_3(1^5) = \Phi_3(1^4) \times \mathbb{Z}_p$,

(iv) $\Phi_3\,(311)\,a = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha^{p^2} = \alpha_3, \alpha_1^{(p)} = \alpha_2^p = \alpha_3^p = 1\rangle$,

(v) $\Phi_3\,(311)\,b_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha]^r = \alpha_1^{p^2} = \alpha_3, \alpha^p = \alpha_2^p = \alpha_3^p = 1\rangle$ for $r = 1$ or ν,

(vi) $\Phi_3\,(221)\,a = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha^p = \alpha_3, \alpha_1^{p^2} = \alpha_2^p = \alpha_3^p = 1\rangle$,

(vii) $\Phi_3\,(221)\,b_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha]^r = \alpha_1^{(p)} = \alpha_3^r, \alpha^{p^2} = \alpha_2^p = \alpha_3^p = 1\rangle$ for $r = 1$ or ν,

(viii) $\Phi_3\,(2111)\,c = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \gamma \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \gamma^p = \alpha_3, \alpha^p = \alpha_i^{(p)} = 1\ (i = 1, 2, 3)\rangle$,

(ix) $\Phi_3\,(2111)\,d = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^{p^2} = \alpha_i^{(p)} = \alpha_3^p = 1\ (i = 1, 2)\rangle$,

(x) $\Phi_3\,(2111)\,e = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_1^{p^2} = \alpha_{i+1}^p = 1\ (i = 1, 2)\rangle$.

The class Φ_6 consists of the following groups

(i) $\Phi_6(221)a = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_i] = \beta_i = \alpha_i^p, \beta^p = \beta_i^p = 1\ (i = 1, 2)\rangle$,

(ii) $\Phi_6(221)b_r = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_i] = \beta_i, \alpha_1^p = \beta_1^k, \alpha_2^p = \beta_2, \beta^p = \beta_i^p = 1\ (i = 1, 2)\rangle$, where $k = \zeta^r, r = 1, 2, \ldots, \frac{1}{2}(p-1)$,

(iii) $\Phi_6(221)c_r = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_i] = \beta_i, \alpha_1^p = \beta_2^{-\frac{1}{4}r}, \alpha_2^p = \beta_1^r\beta_2^r, \beta^p = \beta_i^p = 1\ (i = 1, 2)\rangle$, where $r = 1$ or ν,

(iv) $\Phi_6(221)d_0 = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_i] = \beta_i, \alpha_1^p = \beta_2, \alpha_2^p = \beta_1^\nu, \beta^p = \beta_i^p = 1\ (i = 1, 2)\rangle$,

(v) $\Phi_6(221)d_r = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_i] = \beta_i, \alpha_1^p = \beta_2^k, \alpha_2^p = \beta_1\beta_2, \beta^p = \beta_i^p = 1\ (i = 1, 2)\rangle$ where $4k = \zeta^{2r+1} - 1, r = 1, 2, \ldots, \frac{1}{2}(p-1)$,

(vi) $\Phi_6(2111)a = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_i] = \beta_i, \alpha_1^p = \beta_1, \alpha_2^p = \beta^p = \beta_i^p = 1\ (i = 1, 2)\rangle$ for $p > 3$,

(vii) $\Phi_6(2111)b_r = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_i] = \beta_i, \alpha_2^p = \beta_1^r, \alpha_1^p = \beta^p = \beta_i^p = 1 \; (i = 1, 2)\rangle$ for $r = 1$ or ν and $p > 3$,

(viii) $\Phi_6(1^5) = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_i] = \beta_i, \alpha_i^p = \beta^p = \beta_i^p = 1 \; (i = 1, 2)\rangle$.

The class Φ_7 consists of the following groups

(i) $\Phi_7\,(2111)\,a = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3 = \alpha^p, \alpha_1^{(p)} = \alpha_{i+1}^p = \beta^p = 1 \; (i = 1, 2)\rangle$,

(ii) $\Phi_7\,(2111)\,b_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta]^r = \alpha_3^r = \alpha_1^{(p)}, \alpha^p = \alpha_{i+1}^p = \beta^p = 1 \; (i = 1, 2)\rangle$ for $r = 1$ or ν,

(iii) $\Phi_7\,(2111)\,c = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3 = \beta^p, \alpha^p = \alpha_1^{(p)} = \alpha_{i+1}^p = 1 \; (i = 1, 2)\rangle$,

(iv) $\Phi_7\,(1^5) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \alpha^p = \alpha_1^{(p)} = \alpha_{i+1}^p = \beta^p = 1 \; (i = 1, 2)\rangle$.

The class Φ_8 consists of only one group

(i) $\Phi_8(32) = \langle \alpha_1, \alpha_2, \beta \mid [\alpha_1, \alpha_2] = \beta = \alpha_1^p, \beta^{p^2} = \alpha_2^{p^2} = 1\rangle$

Groups of nilpotency class 4 fall in the isoclinism classes Φ_9 and Φ_{10}. The class Φ_9 consists of the following groups

(i) $\Phi_9\,(2111)\,a = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha_4 = \alpha^p, \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = 1 \; (i = 1, 2, 3)\rangle$,

(ii) $\Phi_9\,(2111)\,b_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha_4^k = \alpha_1^{(p)}, \alpha^p = \alpha_{i+1}^{(p)} = 1 \; (i = 1, 2, 3)\rangle$, where $k = \zeta^r$ for $r + 1 = 1, 2, \ldots, (p-1, 3)$

(iii) $\Phi_9\,(1^5) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = 1 \ (i = 1, 2, 3)\rangle$.

The class Φ_{10} consists of the following groups

(i) $\Phi_{10}\,(2111)\,a_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2]^k = \alpha_4^k = \alpha^p, \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = 1 \ (i = 1, 2, 3)\rangle$, where $k = \zeta^r$ for $r + 1 = 1, 2, \ldots, (p-1, 4)$

(ii) $\Phi_{10}\,(2111)\,b_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2]^k = \alpha_4^k = \alpha_1^{(p)}, \alpha^p = \alpha_{i+1}^{(p)} = 1 \ (i = 1, 2, 3)\rangle$, where $k = \zeta^r$ for $r + 1 = 1, 2, \ldots, (p-1, 3)$ and $p > 3$,

(iii) $\Phi_{10}\,(1^5) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4, \alpha^p = \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = 1 \ (i = 1, 2, 3)\rangle$.

1.1.4 Certain groups of order p^6

We present some groups of order p^6, for odd prime p, from [27], which are needed for our work.

(i) $\Phi_{11}(1^6) = \langle \alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \beta_1, [\alpha_3, \alpha_1] = \beta_2, \alpha_i^{(p)} = \beta_i^p = 1 (i = 1, 2, 3)\rangle$,

(ii) $\Phi_{12}(1^6) = ES_p(p^3) \times ES_p(p^3)$,

(iii) $\Phi_{13}(1^6) = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2 \mid [\alpha_i, \alpha_{i+1}] = \beta_i, [\alpha_2, \alpha_4] = \beta_2, \alpha_i^p = \alpha_3^p = \alpha_4^p = \beta_i^p = 1 (i = 1, 2)\rangle$,

(iv) $\Phi_{15}(1^6) = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2 \mid [\alpha_i, \alpha_{i+1}] = \beta_i, [\alpha_3, \alpha_4] = \beta_1, [\alpha_2, \alpha_4] = \beta_2^\zeta, \alpha_i^p = \alpha_3^p = \alpha_4^p = \beta_i^p = 1 (i = 1, 2)\rangle$.

Remark: The above notations for the groups of order p^n, p odd $(n \leq 6)$, will be used throughout the thesis without further reference.

1.2 Cohomology groups and the Schur multiplier

Let G denote a multiplicative group and D denote a divisible abelian additive group, which is a G-module. A function $f : G^{(n)} \to D$ is called an n-cochain of G in D. Define $C^n(G, D)$ the set of all n-cochains, which form an abelian group under addition. Define a map $d_n : C^n(G, D) \to C^{n+1}(G, D)$ as follows:

$$\begin{aligned}(d_n f)(g_1, g_2, \ldots, g_{n+1}) &= g_1 f(g_2, g_3, \ldots, g_{n+1}) \\ &\quad + \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{n+1}) \\ &\quad + (-1)^{n+1} f(g_1, g_2, \ldots, g_n),\end{aligned}$$

where $f \in C^n(G, D)$ and $g_i \in G$ for $1 \leq i \leq n+1$. The map d_n is a homomorphism. Set $Z^n(G, D) = \ker(d_n)$ and $B^n(G, D) = \mathrm{Im}(d_{n-1})$. We call the elements of $Z^n(G, D)$ as n-cocycles and the elements of $B^n(G, D)$ as n-coboundaries. The group

$$\mathrm{H}^n(G, D) = Z^n(G, D)/B^n(G, D)$$

is called the *n-th cohomology group* of G with coefficients in D. In particular, the second cohomology group of G with coefficients in D is

$$\mathrm{H}^2(G, D) = Z^2(G, D)/B^2(G, D),$$

where $Z^2(G, D) = \{f : G \times G \to D \mid f(xy, z) + f(x, y) = xf(y, z) + f(x, yz),$ for all $x, y, z \in G\}$, the set of 2-cocycles and $B^2(G, D) = \{g : G \times G \to D \mid g(x, y) = xt(y) - t(xy) + t(x)$ for some $t : G \to D\}$, the set of 2-coboundaries.

From now on we consider $H^2(G, D)$, the second cohomology group with trivial action of G on D.

The *Schur multiplier* $\mathrm{M}(G)$ of a group G was introduced by I. Schur in [42] and [43] as an obstruction for a projective representation to become linear representation, and is defined as a second integral homology group $\mathrm{H}_2(G,\mathbb{Z})$, where $\mathbb{Z}$ is a trivial G-module.

From [47, Theorem 11.9.2], it follows that the second cohomology group $\mathrm{H}^2(G,\mathbb{C}^\times)$ is isomorphic to $\mathrm{Hom}\left(\mathrm{H}_2(G,\mathbb{Z}),\mathbb{C}^\times\right)$, where $\mathbb{C}^\times$ is a trivial G-module. For a finite group G, the group $\mathrm{M}(G)$ is isomorphic to the second cohomology group $\mathrm{H}^2(G,\mathbb{C}^\times)$. Let G be a group given by a free presentation F/R. By [24],

$$\mathrm{M}(G) \cong (F' \cap R)/[F,R],$$

which is known as Hopf formula.

Let N be a subgroup of a group G. For a homomorphism $f : G \to D$, consider its restriction on N, say $f|_N$, which defines the *restriction homomorphism* res_N^G : $\mathrm{Hom}(G,D) \to \mathrm{Hom}(N,D)$.

Let $f \in Z^2(G,D)$. Consider the restriction of f on $N \times N$. Then it determines a 2-cocycle $f' : N \times N \to D$. The *restriction homomorphism* res_N^G from $\mathrm{H}^2(G,D)$ to $\mathrm{H}^2(N,D)$ is defined by $\bar{f} \mapsto \bar{f}'$. When the meaning is clear from the context, we write res for res_N^G.

For an arbitrary group G and its subgroup N, we define *inflation homomorphism* $\inf : \mathrm{H}^2(G/N,D) \to \mathrm{H}^2(G,D)$ as follows: for a 2-cocycle $f \in Z^2(G/N,D)$ define $\inf(\bar{f}) = \bar{f}'$, where $f' : G \times G \to D$ given by $f'(g_1,g_2) = f(g_1N,g_2N)$ for $g_1,g_2 \in G$.

Let $1 \to N \to G \xrightarrow{g} G/N \to 1$ be a central extension and $\mu : G/N \to G$ a section. We define *transgression homomorphism* $\mathrm{tra} : \mathrm{Hom}(N,D) \to \mathrm{H}^2(G/N,D)$ as follows: for $\beta \in \mathrm{Hom}(N,D)$, define $\mathrm{tra}(\beta) = \xi \in \mathrm{H}^2(G/N,D)$, where the element ξ is represented by a 2-cocycle f given by $f(\bar{x},\bar{y}) = \beta(\mu(\bar{x})\mu(\bar{y})\mu(\bar{x}\bar{y})^{-1})$, where $\bar{x} = xN \in G/N$ and $\bar{y} = yN \in G/N$.

Theorem 1.2.1 (Hochschild-Serre exact sequence, Theorem 1.5.1 in [30])

Let N be a central subgroup of a group G and D be a trivial G-module. Then for a natural exact sequence $1 \to N \to G \to G/N$, the induced sequence

$$1 \to \mathrm{Hom}(G/N, D) \xrightarrow{\mathrm{inf}} \mathrm{Hom}(G, D) \xrightarrow{\mathrm{res}} \mathrm{Hom}(N, D) \xrightarrow{\mathrm{tra}} \mathrm{H}^2(G/N, D) \xrightarrow{\mathrm{inf}} \mathrm{H}^2(G, D)$$

is exact.

Theorem 1.2.2 ([43]) *Let $G \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$, where $n_{i+1}|n_i$ for all $i \in \{1, \ldots, k-1\}$ and $k \geq 2$. Then*

$$\mathrm{M}(G) \cong \mathbb{Z}_{n_2} \times \mathbb{Z}_{n_3}^{(2)} \times \cdots \times \mathbb{Z}_{n_k}^{(k-1)}.$$

The following result was proved by Schur.

Theorem 1.2.3 ([43]) *For two groups H and K,*

$$\mathrm{M}(H \times K) \cong \mathrm{M}(H) \times \mathrm{M}(K) \times (H/H' \otimes K/K').$$

The Schur multiplier of semi-direct product of groups was studied by Tahara [45], which is the following.

Theorem 1.2.4 ([45]) *If a group G is semi-direct product of a normal subgroup N and a subgroup T, and M is a G-module with trivial G-action, then the following sequence is exact*

$$1 \to \mathrm{H}^1(T, \mathrm{Hom}(N, M)) \to \mathrm{H}^2(G, M)_2 \to \mathrm{H}^2(N, M)^T \to \mathrm{H}^2(T, \mathrm{Hom}(N, M)),$$

where

$$\mathrm{H}^2(G, M)_2 = \mathrm{Ker}(\mathrm{res}^G_T : \mathrm{H}^2(G, M) \to \mathrm{H}^2(T, M))$$

and $\mathrm{H}^2(N, M)^T$ is T-stable subgroup of $\mathrm{H}^2(N, M)$.

The next three theorems give certain upper and lower bounds for $\mathrm{M}(G)$, which were proved by Jones.

Theorem 1.2.5 (Theorem 3.1 of [28]) *Let G be a finite group and K any normal subgroup of G. Set $H = G/K$. Then*

(i) $|\mathrm{M}(H)|$ *divides* $|\mathrm{M}(G)||G' \cap K|$.

(ii) $d(\mathrm{M}(H)) \leq d(\mathrm{M}(G)) + d(G' \cap K)$.

Theorem 1.2.6 (Theorem 4.1 of [28]) *Let G be a finite group and K a central subgroup of G. Set $A = G/K$. Then*

(i) $|\mathrm{M}(G)||G' \cap K|$ *divides* $|\mathrm{M}(A)||\mathrm{M}(K)||A^{ab} \otimes K|$.

(ii) $d(\mathrm{M}(G)) \leq d(\mathrm{M}(A)) + d(\mathrm{M}(K)) + d(A^{ab} \otimes K)$.

Theorem 1.2.7 (Theorem 3.1 of [29]) *Let G be a finite group and N any normal subgroup such that G/N is cyclic. Then*

(i) $|\mathrm{M}(G)|$ *divides* $|\mathrm{M}(N)||N/N'|$.

(ii) $d(\mathrm{M}(G)) \leq d(\mathrm{M}(N)) + d(N/N')$.

Suppose G has a free presentation F/R. Let $Z = S/R$ be a central subgroup of G. Then the map from $F/F'R \times S/R$ to $\mathrm{M}(G)$ defined by $(xF'R, sR) \mapsto [x, s][F, R]$ is a well-defined bilinear map and induces a homomorphism $\lambda : G/G' \otimes Z \to \mathrm{M}(G)$, which is called *Ganea map*. The following theorem was proved by Ganea, which gives the relationship between the groups $\mathrm{M}(G)$ and $\mathrm{M}(G/Z)$, for a central subgroup Z of G.

Theorem 1.2.8 ([13]) *Let Z be a central subgroup of a finite group G. Then the following sequence is exact*

$$G/G' \otimes Z \xrightarrow{\lambda} \mathrm{M}(G) \xrightarrow{\mu} \mathrm{M}(G/Z) \to G' \cap Z \to 1,$$

where λ is the Ganea map.

A group G is called *capable* if there exists a group H such that $G \cong H/\operatorname{Z}(H)$. The *epicenter*, denoted by $Z^*(G)$, of a group G is defined to be the smallest central subgroup K of G such that G/K is capable. Next result is by Beyl et al., which gives the information about the epicenter of a group.

Theorem 1.2.9 (Theorem 4.2 of [2]) *Let Z be a central subgroup of a finite group G. Consider the Ganea map $\lambda : G/G' \otimes Z \to \operatorname{M}(G)$. Then $Z \subseteq Z^*(G)$ if and only if $\operatorname{Ker}\lambda = G/G' \otimes Z$.*

The following theorem gives the upper bound of $|\operatorname{M}(G)|$ for a p-group G, which was proved by Green.

Theorem 1.2.10 ([15]) *If G is a p-group of order p^n, then $|\operatorname{M}(G)| \le p^{\frac{1}{2}n(n-1)}$.*

Later Niroomand improved the Green's bound and proved the following result.

Theorem 1.2.11 (Main Theorem of [33]) *Let G be a non-abelian finite p-group of order p^n. If $|G'| = p^k$, then we have*

$$|\operatorname{M}(G)| \le p^{\frac{1}{2}(n+k-2)(n-k-1)+1}.$$

In particular,

$$|\operatorname{M}(G)| \le p^{\frac{1}{2}(n-1)(n-2)+1},$$

and the equality holds in this latter bound if and only if $G = ES_p(p^3) \times \mathbb{Z}_p^{(n-3)}$.

Lemma 1.2.12 (Lemma 2.2 of [33]) *Let G be an abelian p-group of order p^n which is not elementary abelian. Then*

$$|\operatorname{M}(G)| \le p^{\frac{1}{2}(n-1)(n-2)}.$$

1.2.1 Groups of nilpotency class 2 with G/G' elementary abelian

Now we explain a method by Blackburn and Evens [3] for computing the Schur multiplier of p-groups G of class 2 with G/G' elementary abelian. In this case it follows that G' is elementary abelian. We consider G/G' and G' as vector spaces over $\mathbb{F}_p$, which we denote by V, W respectively. The bilinear map $(-,-)$: $V \times V \to W$ is defined by

$$(v_1, v_2) = [g_1, g_2]$$

for $v_1, v_2 \in V$, where $v_i = g_iG', i \in \{1, 2\}$. The following construction is from [3]. Let X_1 be the subspace of $V \otimes W$ spanned by all

$$v_1 \otimes (v_2, v_3) + v_2 \otimes (v_3, v_1) + v_3 \otimes (v_1, v_2).$$

Consider a map $f : V \to W$ given by $f(gG') = g^p$ for $g \in G$. We denote by X_2, the subspace spanned by all $v \otimes f(v)$, $v \in V$, and take

$$X := X_1 + X_2.$$

Now consider a homomorpism $\sigma : V \wedge V \to (V \otimes W)/X$ given by

$$\sigma(v_1 \wedge v_2) = \big(v_1 \otimes f(v_2) + \tbinom{p}{2} v_2 \otimes (v_1, v_2)\big) + X.$$

Then there exists an abelian group M^* admitting a subgroup N, isomorphic to $(V \otimes W)/X$, such that

$$1 \to N \to M^* \xrightarrow{\xi} V \wedge V \to 1$$

is exact. Now we consider a homomorphism $\rho : V \wedge V \to W$ given by

$$\rho(v_1 \wedge v_2) = (v_1, v_2)$$

for all $v_1, v_2 \in V$. Notice that ρ is an epimorphism. Denote by M, the subgroup of M^* containing N such that $M/N \cong \operatorname{Ker} \rho$. Then it follows that $|M/N| = |V \wedge V|/|W|$, where $N \cong (V \otimes W)/X$. This result will be used for calculating $|M|$, in the second and fourth chapters without further reference.

With the above setting, we have

Theorem 1.2.13 (Theorem 3.1 of [3]) $\mathrm{M}(G) \cong M$.

1.2.2 Groups of nilpotency class $c, c \geq 3$

Here we explain a method given by Ellis and Wiegold in [10] and [11]. Let G be a finite p-group of nilpotency class c and $\gamma_i(G)$ denotes the i-th term of the lower central series of G. Set $\bar{G} = G/\mathrm{Z}(G)$. Define a homomorphism

$$\psi_2 : \bar{G}^{ab} \otimes \bar{G}^{ab} \otimes \bar{G}^{ab} \to \frac{\gamma_2(G)}{\gamma_3(G)} \otimes \bar{G}^{ab}$$

$$\text{by} \quad \psi_2(\bar{x}_1 \otimes \bar{x}_2 \otimes \bar{x}_3) = \overline{[x_1, x_2]} \otimes \bar{x}_3 + \overline{[x_2, x_3]} \otimes \bar{x}_1 + \overline{[x_3, x_1]} \otimes \bar{x}_2.$$

Now define homomorphisms ψ_i for $3 \leq i \leq c$,

$$\psi_i : \bar{G}^{ab} \otimes \bar{G}^{ab} \otimes \cdots \otimes \bar{G}^{ab} \ (i+1 \text{ times}) \to \frac{\gamma_i(G)}{\gamma_{i+1}(G)} \otimes \bar{G}^{ab}$$

by

$$\psi_i(\bar{x}_1 \otimes \bar{x}_2 \otimes \cdots \otimes \bar{x}_{i+1}) \quad = \quad \overline{[x_1, x_2, \ldots, x_i]_l} \otimes \bar{x}_{i+1} + \overline{[x_{i+1}, [x_1, x_2, \ldots, x_{i-1}]_l]} \otimes \bar{x}_i$$

$$+\overline{[[x_i, x_{i+1}]_r, [x_1, x_2, \ldots, x_{i-2}]_l]} \otimes \bar{x}_{i-1}$$
$$+\overline{[[x_{i-1}, x_i, x_{i+1}]_r, [x_1, x_2, \ldots, x_{i-3}]_l]} \otimes \bar{x}_{i-2} + \cdots$$
$$+\overline{[x_2, x_3, \ldots x_{i+1}]_r} \otimes \bar{x}_1,$$

where

$$[x_1, x_2, \ldots, x_i]_l = [[[x_1, x_2], x_3] \ldots, x_i],$$
$$[x_1, x_2, \ldots, x_i]_r = [x_1, \ldots [x_{i-2}, [x_{i-1}, x_i]]],$$

$\bar{x}$ denotes the image in $\bar{G}$ of the element $x \in G$ and $\overline{[x, y]}$ denotes the image in $\frac{\gamma_i(G)}{\gamma_{i+1}(G)}$ of the commutator $[x, y] \in G$.

Proposition 1.2.14 ([10], [11]) *Let G be a finite p-group of nilpotency class c. With the preceding notations, we have*

$$|\mathrm{M}(G)|\,|\gamma_2(G)| \prod_{i=2}^{c} |\mathrm{Im}(\psi_i)| \leq \left|\mathrm{M}(G^{ab})\right| \prod_{i=2}^{c} \left| \frac{\gamma_i(G)}{\gamma_{i+1}(G)} \otimes \bar{G}^{ab} \right|.$$

We use the notations ψ_2, ψ_3 in Chapter 2 without further reference.

1.3 The non-abelian tensor square and exterior square of groups

The notion of *non-abelian tensor product* $G \otimes H$ of two groups G and H, acting on each other and satisfying certain compatibility conditions, was introduced by Brown and Loday [7] as a generalization of abelian tensor product. In particular, when a group G acts on itself by conjugation then $G \otimes G$ is called *non-abelian tensor square* which is defined as follows.

Let G acts on itself by conjugation, i.e., $h^g = g^{-1}hg$ for all $h, g \in G$. Then the non-abelian tensor square $G \otimes G$ of G is the group generated by the symbols $g \otimes h$ for all $g, h \in G$, subject to the relations

$$gg' \otimes h = (g^{g'} \otimes h^{g'})(g' \otimes h)$$

and

$$g \otimes hh' = (g \otimes h')(g^{h'} \otimes h^{h'})$$

for all $g, g', h, h' \in G$.

The *non-abelian exterior square* of G, denoted by $G \wedge G$, is the quotient group of $G \otimes G$ by $\nabla(G)$, where $\nabla(G)$ is the normal subgroup of $G \otimes G$ generated by the elements $g \otimes g$ for all $g \in G$. It follows from the definition that the map $f : G \otimes G \to G'$, defined on the generators by $f(g \otimes h) = [g, h]$, is an epimorphism. The epimorphism f then induce an epimorphism $f' : G \wedge G \to G'$. It follows from [7] that the kernel of f' is isomorphic to the Schur multiplier $\mathrm{M}(G)$ of G.

There is a different description of $G \otimes G$, introduced in [41], which, sometimes, comes more handy for evaluating tensor square of a group G. Let G and G^{ϕ} be the isomorphic groups via the isomorphism $\phi : G \to G^{\phi}$ with $\phi(g) = g^{\phi},\ g \in G$. From now onwards g^{ϕ} denotes the image of the element $g \in G$ in G^{ϕ} via the isomorphism ϕ. Consider the group

$$\nu(G) := \langle G, G^{\phi} \mid \Re, \Re^{\phi}, [g_1, g_2^{\phi}]^g = [g_1^g, (g_2^g)^{\phi}] = [g_1, g_2^{\phi}]^{g^{\phi}} \text{ for all } g, g_1, g_2 \in G \rangle$$

in which $\Re, \Re^{\phi}$ are the defining relations of G and G^{ϕ} respectively. Recall that the commutator subgroup of G and G^{ϕ} in $\nu(G)$ is defined as $[G, G^{\phi}] = \langle [g, h^{\phi}] \mid g, h \in G \rangle$.

Proposition 1.3.1 (Proposition 2.6 of [41]) *The map* $\Phi : G \otimes G \to [G, G^{\phi}]$, *defined by*

$$\Phi(g \otimes h) = [g, h^{\phi}], \quad g, h \in G$$

is an isomorphism.

Let $\pi : [G, G^{\phi}] \to [G, G^{\phi}]/\langle [g, g^{\phi}] \mid g \in G \rangle$ be the natural projection. Then we have an isomorphism between $G \wedge G$ and $[G, G^{\phi}]/\langle [g, g^{\phi}] \mid g \in G \rangle$.

We are now ready to quote some results required for our subsequent investigations.

Lemma 1.3.2 (Lemma 9 of [32]) *If a group G has nilpotency class ≤ 5, then*

$$[x^n, y] = [x, y]^n [x, y, x]^{\binom{n}{2}} [x, y, x, x]^{\binom{n}{3}} [x, y, x, x, x]^{\binom{n}{4}} [x, y, x, [x, y]]^{\sigma(n)}$$

for $x, y \in G$ and any positive integer n, where $\sigma(n) = n(n-1)(2n-1)/6$.

Lemma 1.3.3 ([4] and [41]) *For a group G, the following properties hold in $\nu(G)$.*

(i) *If G is nilpotent of class c, then $\nu(G)$ is nilpotent of class at most $c + 1$.*

(ii) *If G is a p-group, then $\nu(G)$ is a p-group.*

(iii) $[g_1^{\phi}, g_2, g_3] = [g_1, g_2^{\phi}, g_3] = [g_1, g_2, g_3^{\phi}] = [g_1^{\phi}, g_2^{\phi}, g_3] = [g_1^{\phi}, g_2, g_3^{\phi}] = [g_1, g_2^{\phi}, g_3^{\phi}]$ *for all $g_1, g_2, g_3 \in G$.*

(iv) *If either $g \in G'$ or $h \in G'$, then $[g, h^{\phi}] = [h, g^{\phi}]^{-1}$.*

(v) $[g, g^{\phi}] = 1$ *for all $g \in G'$.*

(vi) $[[g_1, g_2^{\phi}], [h_1, h_2^{\phi}]] = [[g_1, g_2], [h_1, h_2]^{\phi}]$ *for all $g_1, g_2, h_1, h_2 \in G$.*

(vii) $[[g_1, g_2^{\phi}], [g_2, g_1^{\phi}]] = 1$ *for all $g_1, g_2 \in G$.*

(viii) *If $g, g_1, g_2 \in G$ such that $[g, g_1] = 1 = [g, g_2]$, then $[g_1, g_2, g^{\phi}] = 1$.*

(ix) $[g, g^{\phi}]$ *is central in $\nu(G)$ for all $g \in G$.*

The following result follows from the proof of [41, Lemma 2.1(iv)]:

Lemma 1.3.4 *For all* $g_1, g_2 \in G$, $[g_1, g_2^\phi] = [g_2, g_1^\phi]^{-1}$ *in* $G \wedge G$, *i.e., modulo* $\nabla(G)$.

The next proposition provides a generating set for $G \wedge G$ when G has a polycyclic generating sequence. This information will be used several times in Chapter 4, as the generating sets for the groups G, given in [27], form polycyclic generating sequences.

Proposition 1.3.5 (Proposition 20 of [4]) *Let* G *be a polycyclic group with a polycyclic generating sequence* $g_1, \dots, g_k$. *Then* $G \wedge G$ *is generated by* $\{[g_i, g_j^\phi], i > j\}$.

By items (vi) and ($viii$) of Lemma 1.3.3, we get the following result.

Lemma 1.3.6 *If* G *is of nilpotency class* 2, *then* $G \otimes G$ *is abelian.*

Let Γ denote Whitehead's quadratic functor which is defined from the category of abelian groups to itself ([48]). For an abelian group A, ΓA is the abelian group generated by the symbols $w(a)$, $a \in A$ such that the following relations hold:

(i) $w(a) = w(a^{-1})$,

(ii) $w(abc)w(a)w(b)w(c) = w(ab)w(bc)w(ca)$,

for all $a, b, c \in A$. For finitely generated groups we have the following (for more details see [48]):

$$(i) \quad \Gamma(G \times H) \cong \Gamma(G) \times \Gamma(H) \times (G \otimes H).$$

$$(ii) \qquad \Gamma(\mathbb{Z}_n) = \begin{cases} \mathbb{Z}_n & \text{n odd} \\ \mathbb{Z}_{2n} & \text{n even .} \end{cases}$$

$$(iii) \qquad \Gamma(\mathbb{Z}) \cong \mathbb{Z}.$$

Theorem 1.3.7 (Theorem 1.3 of [5]) *Let G be a group such that G^{ab} is finitely generated with no elements of order 2. Then $G \otimes G \cong \Gamma(G^{ab}) \times (G \wedge G)$. In particular, if G is a finite p-group, p odd, then $G \otimes G \cong \Gamma(G^{ab}) \times (G \wedge G)$.*

Proposition 1.3.8 (Proposition 11 of [6]) *For the groups G and H, we have*

$$(G \times H) \otimes (G \times H) = (G \otimes G) \times (G \otimes H) \times (H \otimes G) \times (H \otimes H).$$

1.4 Non-abelian tensor square of groups of order p^3 and p^4

In the following result, we present the structures of the Schur multiplier, non-abelian tensor square and exterior square of groups of order p^3 and p^4, $p \geq 5$. For groups of order p^4, results are given in [16]. We mainly work to find the generators of the exterior square.

Theorem 1.4.1 ([21]) *We have the following table for non-abelian groups of order p^3 and p^4, $p \geq 5$.*

G	G^{ab}	$\Gamma(G^{ab})$	$\mathrm{M}(G)$	$G \wedge G$	$G \otimes G$	Generators of $G \wedge G$
$\Phi_2(21)$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\{1\}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(4)}$	$[\alpha_1, \alpha^\phi]$
$\Phi_2(111)$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$[\alpha_1, \alpha^\phi], [\alpha_2, \alpha^\phi], [\alpha_2, \alpha_1^\phi]$
$\Phi_2(211)a$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(9)}$	$[\alpha_1, \alpha^\phi], [\alpha_3, \alpha^\phi], [\alpha_3, \alpha_1^\phi]$
$\Phi_2(1^4)$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(11)}$	$[\alpha_1, \alpha^\phi], [\alpha_2, \alpha^\phi], [\alpha_2, \alpha_1^\phi],$
						$[\alpha_3, \alpha^\phi], [\alpha_3, \alpha_1^\phi]$
$\Phi_2(31)$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\{1\}$	$\mathbb{Z}_p$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(3)}$	$[\alpha_1, \alpha^\phi]$
$\Phi_2(22)$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p$	$\mathbb{Z}_{p^2}$	$\mathbb{Z}_{p^2}^{(2)} \times \mathbb{Z}_p^{(2)}$	$[\alpha_1, \alpha^\phi]$
$\Phi_2(211)b$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(9)}$	$[\alpha_1, \alpha^\phi], [\gamma, \alpha^\phi], [\gamma, \alpha_1^\phi]$
$\Phi_2(211)c$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(5)}$	$[\alpha_1, \alpha^\phi], [\alpha_2, \alpha^\phi], [\alpha_2, \alpha_1^\phi]$
$\Phi_3(211)a$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$[\alpha_1, \alpha^\phi], [\alpha_2, \alpha^\phi], [\alpha_2, \alpha_1^\phi]$
$\Phi_3(211)b_r$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$[\alpha_1, \alpha^\phi], [\alpha_2, \alpha^\phi], [\alpha_2, \alpha_1^\phi]$
$\Phi_3(1^4)$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(7)}$	$[\alpha_1, \alpha^\phi], [\alpha_2, \alpha^\phi], [\alpha_2, \alpha_1^\phi],$
						$[\alpha_3, \alpha^\phi]$

Table 1.1: Groups of order p^3, p^4, $p \geq 5$

Proof. Schur multipliers of groups of order p^4 are taken from [39] for $|G'| = p$ and from [9, page. 4177] for $|G'| = p^2$. The Schur multipliers of groups of order p^3 follows from [30, Theorem 3.3.6]. So we mainly work for computing the exterior squares. Tensor squares will then follow easily by Theorem 1.3.7.

Consider the group $G := \Phi_2(21)$. Since $|\mathrm{M}(G)| = 1$ and $|\gamma_2(G)| = p$, it follows that $G \wedge G \cong \mathbb{Z}_p$. By Lemma 1.3.2, the following identities hold in $G \wedge G$:

$$[\alpha_2, \alpha^\phi] = [\alpha^p, \alpha^\phi] = [\alpha, \alpha^\phi]^p = 1 = [\alpha_2^p, \alpha_1^\phi] = [\alpha_2, \alpha_1^\phi]^p$$

$$[\alpha_2, \alpha_1^\phi] = [\alpha^p, \alpha_1^\phi] = [\alpha, \alpha_1^\phi]^p = [\alpha, (\alpha_1^p)^\phi] = 1.$$

Hence we have

$$G \wedge G \cong \langle [\alpha_1, \alpha^{\phi}] \rangle \cong \mathbb{Z}_p.$$

Consider the group $G := \Phi_2(111)$. Since $|\,\mathrm{M}(G)| = p^2$ and $|\gamma_2(G)| = p$, it follows that $|G \wedge G| = p^3$. By Lemma 1.3.2, the following identities hold in $G \wedge G$:

$$[\alpha_2, \alpha^{\phi}]^p = [\alpha_2^p, \alpha^{\phi}] = 1 = [\alpha_2^p, \alpha_1^{\phi}] = [\alpha_2, \alpha_1^{\phi}]^p$$
$$[\alpha_1, \alpha^{\phi}]^p = [\alpha_1^p, \alpha^{\phi}] = 1.$$

Hence we have

$$G \wedge G \cong [\alpha_1, \alpha^{\phi}], [\alpha_2, \alpha_1^{\phi}], [\alpha_2, \alpha^{\phi}]\rangle \cong \mathbb{Z}_p^{(3)}.$$

Let G be one of the groups $\Phi_2(211)a$ or $\Phi_2(1^4)$, which are direct product of groups. Then the conclusion for $G \otimes G$ follows from Proposition 1.3.8 and for $G \wedge G$ follows from Theorem 1.3.7.

Consider the group $G := \Phi_2(31)$. Since $|\,\mathrm{M}(G)| = 1$ and $|\gamma_2(G)| = p$, it follows that $G \wedge G \cong \mathbb{Z}_p$.

By Lemma 1.3.2, the following identities hold in $G \wedge G$:

$$[\alpha_2, \alpha^{\phi}] = [\alpha^{p^2}, \alpha^{\phi}] = [\alpha, \alpha^{\phi}]^{p^2} = 1 = [\alpha_2^p, \alpha_1^{\phi}] = [\alpha_2, \alpha_1^{\phi}]^p$$
$$[\alpha_2, \alpha_1^{\phi}] = [\alpha^{p^2}, \alpha_1^{\phi}] = [\alpha, \alpha_1^{\phi}]^{p^2} = [\alpha, (\alpha_1^{p^2})^{\phi}] = 1.$$

Hence we have

$$G \wedge G \cong \langle [\alpha_1, \alpha^{\phi}] \rangle \cong \mathbb{Z}_p.$$

Consider the group $G := \Phi_2(22)$. Since $|\,\mathrm{M}(G)| = p$ and $|\gamma_2(G)| = p$, it follows that $|G \wedge G| = p^2$.

By Lemma 1.3.2, the following identities hold in $G \wedge G$:

$$[\alpha_2, \alpha^\phi] = [\alpha^p, \alpha^\phi] = [\alpha, \alpha^\phi]^p = 1 = [\alpha_2^p, \alpha_1^\phi] = [\alpha_2, \alpha_1^\phi]^p$$
$$[\alpha_2, \alpha_1^\phi] = [\alpha^p, \alpha_1^\phi] = [\alpha, \alpha_1^\phi]^p.$$

These identities, along with Proposition 1.3.5, give

$$G \wedge G \cong \langle [\alpha_1, \alpha^\phi] \rangle \cong \mathbb{Z}_{p^2}.$$

Consider the group $G := \Phi_2(211)b$. Since $|\,\mathrm{M}(G)| = p^2$ and $|\gamma_2(G)| = p$, it follows that $|G \wedge G| = p^3$. By Lemma 1.3.3$(viii)$, we have

$$[\alpha_2, \gamma^\phi] = [\alpha_1, \alpha, \gamma^\phi] = 1, \text{ as } \gamma \in \mathrm{Z}(G).$$

By Lemma 1.3.2 the following identities hold:

$$[\alpha_2, \alpha^\phi] = [\gamma^p, \alpha^\phi] = [\gamma, \alpha^\phi]^p = [\gamma, (\alpha^p)^\phi] = 1,$$
$$[\alpha_2, \alpha_1^\phi] = [\gamma^p, \alpha_1^\phi] = [\gamma, \alpha_1^\phi]^p = [\gamma, (\alpha_1^p)^\phi] = 1,$$
$$[\alpha_1, \alpha^\phi]^p = [\alpha_1^p, \alpha^\phi] = 1.$$

These identities, along with Proposition 1.3.5 and Lemma 1.3.6, imply that $G \wedge G$ is generated by $[\alpha_1, \alpha^\phi], [\gamma, \alpha^\phi], [\gamma, \alpha_1^\phi]$, all of which are of order p. Hence

$$G \wedge G \cong \mathbb{Z}_p^{(3)}.$$

Consider the group $G := \Phi_2(211)c$. Since $|\,\mathrm{M}(G)| = p^2$ and $|\gamma_2(G)| = p$, it follows that $|G \wedge G| = p^3$. By Lemma 1.3.2 the following identities hold:

$$[\alpha_2, \alpha^\phi]^p = [\alpha_2^p, \alpha^\phi] = [\alpha_2^p, \alpha_1^\phi] = [\alpha_2, \alpha_1^\phi]^p,$$

$$[\alpha_1, \alpha^{\phi}]^p = [\alpha_1^p, \alpha^{\phi}] = 1.$$

These identities, along with Proposition 1.3.5 and Lemma 1.3.6, imply that

$$G \wedge G \cong \langle [\alpha_1, \alpha^{\phi}], [\alpha_2, \alpha_1^{\phi}], [\alpha_2, \alpha^{\phi}] \rangle \cong \mathbb{Z}_p^{(3)}.$$

For the groups G of order p^4, belonging to the class Φ_3, $G/\operatorname{Z}(G) \cong \Phi_2(111)$, consider the natural epimorphism $[G, G^{\phi}] \to [G/\operatorname{Z}(G), (G/\operatorname{Z}(G))^{\phi}]$ which shows that $[\alpha_1, \alpha^{\phi}], [\alpha_2, \alpha^{\phi}], [\alpha_2, \alpha_1^{\phi}]$ are non-trivial and independent generators of $G \wedge G$.

Now consider the group $G = \Phi_3(211)a$. Since $|\operatorname{M}(G)| = p$, $|G \wedge G| = p^3$. Hence, in view of Lemma 1.3.3(vi),

$$G \wedge G \cong \langle [\alpha_1, \alpha^{\phi}], [\alpha_2, \alpha^{\phi}], [\alpha_2, \alpha_1^{\phi}] \rangle \cong \mathbb{Z}_p^{(3)}.$$

Now consider the group $G = \Phi_3(211)b_r$. Since $|\operatorname{M}(G)| = p$, $|G \wedge G| = p^3$. Hence, in view of Lemma 1.3.3(vi),

$$G \wedge G \cong \langle [\alpha_1, \alpha^{\phi}], [\alpha_2, \alpha^{\phi}], [\alpha_2, \alpha_1^{\phi}] \rangle \cong \mathbb{Z}_p^{(3)}.$$

Now we work out the exterior square of the group $G = \Phi_3(1^4)$. Since $|\operatorname{M}(G)| = p^2$, $|G \wedge G| = p^4$. By Proposition 1.3.5, $G \wedge G$ is generated by the set

$$\{[\alpha_1, \alpha^{\phi}], [\alpha_2, \alpha^{\phi}], [\alpha_2, \alpha_1^{\phi}], [\alpha_3, \alpha^{\phi}], [\alpha_3, \alpha_1^{\phi}], [\alpha_2, \alpha_3^{\phi}]\}.$$

By Lemma 1.3.3$(viii)$, we have

$$[\alpha_2, \alpha_3^{\phi}] = [\alpha_1, \alpha, \alpha_3^{\phi}] = 1 \text{ as } \alpha_3 \in \operatorname{Z}(G).$$

Hall-Witt identity yields

$$\begin{aligned} 1 &= [\alpha_2, \alpha, \alpha_1^{\phi}]^{\alpha^{-1}}[\alpha^{-1}, \alpha_1^{-1}, \alpha_2^{\phi}]^{\alpha_1} \\ &= [\alpha_3, \alpha_1^{\phi}][\alpha_1\alpha\alpha_2^{-1}\alpha^{-1}\alpha_1^{-1}, \alpha_2^{\phi}]^{\alpha_1} \\ &= [\alpha_3, \alpha_1^{\phi}][\alpha\alpha_2^{-1}\alpha^{-1}, \alpha_2^{\phi}] \\ &= [\alpha_3, \alpha_1^{\phi}][\alpha_3\alpha_2^{-1}, \alpha_2^{\phi}] \\ &= [\alpha_3, \alpha_1^{\phi}][\alpha_3, \alpha_2^{\phi}][\alpha_2, \alpha_2^{\phi}]^{-1} \\ &= [\alpha_3, \alpha_1^{\phi}][\alpha_2, \alpha_2^{\phi}]^{-1}. \end{aligned}$$

Consequently, $[\alpha_3, \alpha_1^{\phi}] = [\alpha_2, \alpha_2^{\phi}] = 1$ in $G \wedge G$. Also, by Lemma 1.3.2, we get

$$[\alpha_3, \alpha^{\phi}]^p = [\alpha_3^p, \alpha^{\phi}] = 1 = [\alpha_2^p, \alpha_1^{\phi}] = [\alpha_2, \alpha_1^{\phi}]^p$$

and

$$[\alpha_2, \alpha^{\phi}]^p = [\alpha_2^p, \alpha^{\phi}] = 1 = [\alpha_1^p, \alpha^{\phi}] = [\alpha_1, \alpha^{\phi}]^p.$$

Hence, in view of Lemma 1.3.3(vi),

$$G \wedge G \cong \langle [\alpha_1, \alpha^{\phi}], [\alpha_2, \alpha^{\phi}], [\alpha_2, \alpha_1^{\phi}], [\alpha_3, \alpha^{\phi}] \rangle \cong \mathbb{Z}_p^{(4)}.$$

This completes the proof. □

Remark 1.4.2 *Schur multipliers of groups of order p^3 follows from [30, Theorem 3.3.6] and Schur multipliers of groups of order p^4 follows from [39], [9, page. 4177]. Observe that when G is a non-abelian group of order p^3 or p^4 with $p = 3$, $\mathrm{M}(G)$ is exactly same as presented in Table 1.1.*

CHAPTER 2

Characterization of finite p-groups by the order of their Schur multiplier

In 1956, Green gave an upper bound $p^{\frac{1}{2}n(n-1)}$ on the order of the Schur multiplier $\mathrm{M}(G)$ for non-abelian p-groups G of order p^n (see Theorem 1.2.10). So, for a group G of order p^n, there is an integer $t(G) \geq 0$ such that $|\mathrm{M}(G)| = p^{\frac{1}{2}n(n-1)-t(G)}$. This integer $t(G)$, introduced in [11], is called corank of G. In 2009, Niroomand improved Green's bound by proving that $|\mathrm{M}(G)| \leq p^{\frac{1}{2}(n+k-2)(n-k-1)+1}$, for non-abelian p-groups G of order p^n with $|G'| = p^k$ (see Theorem 1.2.11). In this chapter we classify non-abelian finite p-groups G for which $t(G) = \log_p(|G|) + 1$. We also study non-abelian p-groups G of order p^n such that $|\mathrm{M}(G)| = p^{\frac{1}{2}(n+k-2)(n-k-1)+1}$, where $|G'| = p^k$.

2.1 Overview

It is an interesting problem to determine the structure of all non-abelian p-groups G by the order of their Schur multiplier $\mathrm{M}(G)$, i.e., when $t(G)$ is known. Several authors have studied this problem for various values of $t(G)$.

First Berkovich [1] and Zhou [50] classified all groups G for $t(G) = 0, 1, 2$. Ellis [9] also classified groups G for $t(G) = 0, 1, 2, 3$ by a different method. After that several authors have classified the groups G of order p^n for $t(G) = 4, 5, 6$ in [34, 35, 26].

Later Niroomand improved the Green's bound and showed that for non abelian p-groups G of order p^n, $|\,\mathrm{M}(G)| = p^{\frac{1}{2}(n-1)(n-2)+1-s(G)}$, for some $s(G) \geq 0$, see Theorem 1.2.11. This integer $s(G)$ is called *generalized corank of G* as defined in [38]. The structure of non-abelian p-groups G for $s(G) = 0, 1, 2$ has been determined in [36, 37], which is the same as to classify group G for $t(G) = \log_p(|G|) - 2, \log_p(|G|) - 1, \log_p(|G|)$ respectively.

The following result gives the classification of G for $s(G) = 1$.

Theorem 2.1.1 (Theorem 21 of [36]) *Let G be a p-group of order p^n. Then $t(G) = \log_p(|G|) - 1$ if and only if G is isomorphic to one of the following groups.*

(i) $G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(n-2)}$,

(ii) $G \cong D_8 \times \mathbb{Z}_2^{(n-3)}$,

(iii) $G \cong \mathbb{Z}_p^{(4)} \rtimes \mathbb{Z}_p$ $(p \neq 2)$.

The following result gives the classification of G for $s(G) = 2$.

Theorem 2.1.2 (Theorem 11 of [37]) *Let G be a group of order p^n. Then $t(G) = \log_p(|G|)$ if and only if G is isomorphic to one of the following groups.*

(i) $E(2) \times \mathbb{Z}_p^{(n-2m-2)}$,

(ii) $ES_{p^2}(p^3) \times \mathbb{Z}_p^{(n-3)}$,

(iii) $Q_8 \times \mathbb{Z}_2^{(n-3)}$

(iv) $ES(p^{2m+1}) \times \mathbb{Z}_p^{(n-2m-1)}$ $(m \geq 2)$,

(v) $\langle a, b \mid a^4 = b^4 = 1, [a, b, a] = [a, b, b] = 1, [a, b] = a^2b^2 \rangle$,

(vi) $\langle a, b, c \mid a^2 = b^2 = c^2 = 1, abc = bca = cab \rangle$,

(vii) $\langle a, b \mid a^p = b^p = 1, [a, b, a] = [a, b, b] = 1 \rangle$,

(viii) $\mathbb{Z}_p \times (\mathbb{Z}_p^{(4)} \rtimes \mathbb{Z}_p)$ $(p \neq 2)$,

(ix) $\langle a, b \mid a^9 = b^3 = 1, [a, b, a] = 1, [a, b, b] = a^6, [a, b, b, b] = 1 \rangle$,

(x) $\langle a, b \mid a^p = 1, b^p = 1, [a, b, a] = [a, b, b, a] = [a, b, b, b] = 1 \rangle$ $(p \neq 3)$,

where $E(2)$ *denotes a central product of an extra special* p*-group of order* p^{2m+1} *and a cyclic group of order* p^2*.*

The following result follows from [26].

Theorem 2.1.3 (Theorem A of [26]) *Let* G *be a group of order* p^5*,* p *odd. Then* $t(G) = 6$ *if and only if* G *is isomorphic to* $\Phi_2(2111)c, \Phi_2(2111)d, \Phi_3(1^5)$ *or* $\Phi_7(1^5)$*.*

2.2 Groups G with $t(G) = \log_p(|G|) + 1$

In this section we characterize all non-abelian finite p-groups G for which $t(G) = \log_p(|G|) + 1$, which is same as classifying G for $s(G) = 3$, i.e., $|\mathrm{M}(G)| = p^{\frac{1}{2}n(n-3)-1}$. We start with the following lemma which establishes the result for groups of order p^n for $n \leq 5$.

Lemma 2.2.1 *Let G be a non-abelian p-group of order p^n ($n \leq 5$) with $t(G) = \log_p(|G|) + 1$. Then for an odd prime p, G is isomorphic to $\Phi_2(22), \Phi_3(211)a$, $\Phi_3(211)b_r, \Phi_2(2111)c, \Phi_2(2111)d, \Phi_3(1^5)$ or $\Phi_7(1^5)$, and for $p = 2$, G is isomorphic to $\mathbb{Z}_2^{(4)} \rtimes \mathbb{Z}_2, G_1 \times \mathbb{Z}_2, G_2, D_{16}$ or $\mathbb{Z}_4 \rtimes \mathbb{Z}_4$, where $G_1 = \langle a, b, c \mid a^4 = b^2 = c^2 = 1, [a, c] = b, [a, b] = [b, c] = 1\rangle$ and $G_2 = \langle a, b, c \mid a^4 = b^4 = c^2 = 1, [a, b] = 1, [a, c] = a^2, [b, c] = b^2\rangle$.*

Proof. Let G be a finite p-group with $t(G) = \log_p(|G|)+1$. Observe that there is no group G of order p^3 satisfying this property. If G is of order p^4, then by Theorem 1.4.1 and Remark 1.4.2, $G \cong \Phi_2(22), \Phi_3(211)a$ or $\Phi_3(211)b_r$. If G is of order p^5, then it follows from Theorem 2.1.3 that $G \cong \Phi_2(2111)c, \Phi_2(2111)d, \Phi_3(1^5)$ or $\Phi_7(1^5)$. For $p = 2$, a simple computation with HAP [12] package of GAP [14] establishes the result. □

Lemma 2.2.2 *There is no non-abelian p-group G of order p^n ($n \geq 6$) with $|G'| \geq p^4$ and $t(G) = \log_p(|G|) + 1$.*

Proof. By Theorem 1.2.11, it follows that, for $|G'| \geq p^4$,

$$|\operatorname{M}(G)| \leq p^{\frac{1}{2}(n+4-2)(n-4-1)+1} = p^{\frac{1}{2}n(n-3)-4},$$

which is a contradiction for $n \geq 6$. □

Lemma 2.2.3 *Let G be a non-abelian p-group of order p^n ($n \geq 6$) with $t(G) \leq \log_p(|G|) + 1$. Then G^{ab} is an elementary abelian p-group.*

Proof. Let $|G'| = p^k$. Suppose that G^{ab} is not elementary abelian and $\bar{G} := G/\operatorname{Z}(G)$ is a δ-generator group. Then $\delta \leq (n - k - 1)$ and $|\operatorname{M}(G^{ab})| \leq p^{\frac{1}{2}(n-k-1)(n-k-2)}$ by Lemma 1.2.12. Note that

$$\left|\frac{\gamma_2(G)}{\gamma_3(G)} \otimes \bar{G}^{ab}\right| \left|\frac{\gamma_3(G)}{\gamma_4(G)} \otimes \bar{G}^{ab}\right| \cdots \left|\gamma_c(G) \otimes \bar{G}^{ab}\right|$$

$$= \left|\left(\frac{\gamma_2(G)}{\gamma_3(G)} \oplus \frac{\gamma_3(G)}{\gamma_4(G)} \oplus \cdots \gamma_c(G)\right) \otimes \bar{G}^{ab}\right| \leq p^{k\delta}$$

Now consider the homomorphism ψ_2 defined in Section 1.2.2. Let $\{\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_\delta\}$ be a generating set of $\bar{G}$ such that $\overline{[y_1, y_2]}$ is non-trivial element of $\gamma_2(G)/\gamma_3(G)$. Then $\psi_2(y_1 \otimes y_2 \otimes y_k), 3 \leq k \leq \delta$ gives $\delta - 2$ linearly independent elements in $\frac{\gamma_2(G)}{\gamma_3(G)} \otimes \bar{G}^{ab}$. Hence $|\operatorname{Im}(\psi_2)| \geq p^{\delta-2}$. Now it follows from Proposition 1.2.14 that

$$|\operatorname{M}(G)| \leq p^{\frac{1}{2}(n-k-1)(n-k-2)+(k-1)\delta-(k-2)},$$

which gives $|\operatorname{M}(G)| \leq p^{\frac{1}{2}n(n-3)-\frac{1}{2}(k^2-k)-n+4}$, a contradiction for $n \geq 6$. □

Lemma 2.2.4 *Let G be a non-abelian p-group of order p^n $(n \geq 6)$ and $|G'| = p, p^2$ or p^3 with $t(G) \leq \log_p(|G|) + 1$. Then $\operatorname{Z}(G)$ is of exponent at most p^2, p or p respectively.*

Proof. Suppose that $|G'| = p$. Let the exponent of $\operatorname{Z}(G)$ be p^k $(k \geq 3)$ and K be a cyclic central subgroup of order p^k.

If $|G' \cap K| = p$, then $G' \leq K$, and therefore by Lemma 2.2.3, G/K is elementary abelian and, using Theorem 1.2.6, we have the following:

$$|\operatorname{M}(G)| \leq p^{-1}|\operatorname{M}(G/K)||(G/K)^{ab} \otimes K| \leq p^{-1}p^{\frac{1}{2}(n-k)(n-k-1)}p^{(n-k)} \leq p^{\frac{1}{2}(n-1)(n-4)},$$

which gives a contradiction on the order of $\operatorname{M}(G)$. If $G' \cap K = 1$, then G/K is non-abelian. Now using Theorem 1.2.6 and Theorem 1.2.11, we have

$$\begin{aligned} |\operatorname{M}(G)| \leq |\operatorname{M}(G/K)||(G/K)^{ab} \otimes K| \quad &\leq p^{\frac{1}{2}(n-k-1)(n-k-2)+1}p^{(n-k-1)} \\ &\leq p^{\frac{1}{2}n(n-3)-1-2(n-4)}, \end{aligned}$$

which gives a contradiction on the order of $\operatorname{M}(G)$.

Now suppose that $|G'| = p^2$. By Lemma 2.2.3, G/G' is elementary abelian.

Observe that G' can not be cyclic, otherwise if $G' = \langle t \rangle$ and $[x, y] = t$ for some $x, y \in G$, then the subgroup $H = \langle x, y, G' \rangle$ is of order p^4 with $H' \cong \mathbb{Z}_{p^2}$, which is not possible, see the classification of groups of order p^4 in Section 1.1.2. Hence $G' \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Let the exponent of $\mathrm{Z}(G)$ be p^k $(k \geq 2)$ and K be a cyclic central subgroup of order p^k.

If $|G' \cap K| = p$, then G/K is non-abelian and, using Theorem 1.2.6 and Theorem 1.2.11, we have the following

$$
\begin{aligned}
|\mathrm{M}(G)| \leq p^{-1}|\mathrm{M}(G/K)||(G/K)^{ab} \otimes K| \quad & \leq p^{-1} p^{\frac{1}{2}(n-k-1)(n-k-2)+1} p^{(n-k)} \\
& \leq p^{\frac{1}{2}n(n-3)-(n-3)},
\end{aligned}
$$

which gives a contradiction. If $G' \cap K = 1$, then using Theorem 1.2.6 and Theorem 1.2.11, we have

$$
|\mathrm{M}(G)| \leq |\mathrm{M}(G/K)||(G/K)^{ab} \otimes K| \leq p^{\frac{1}{2}(n-k)(n-k-3)+1} p^{(n-k-2)} \leq p^{\frac{1}{2}n(n-3)-(n-2)},
$$

which is a contradiction.

Similarly for $|G'| = p^3$, we observe that G' can not be cyclic. Let the exponent of $\mathrm{Z}(G)$ be p^k $(k \geq 2)$ and K be a cyclic central subgroup of order p^k. So either $|G' \cap K| = p^2, p$ or 1.

If $|G' \cap K| = p^2$, then by Theorem 1.2.6 and Theorem 1.2.11, we have

$$
\begin{aligned}
|\mathrm{M}(G)| \leq p^{-2}|\mathrm{M}(G/K)||(G/K)^{ab} \otimes K| \quad & \leq p^{\frac{1}{2}(n-k-1)(n-k-2)} p^{(n-k-2)} \\
& \leq p^{\frac{1}{2}n(n-3)-(n-2)},
\end{aligned}
$$

which gives a contradiction. If $|G' \cap K| = p$, then by Theorem 1.2.6 and Theorem

1.2.11, we have the following

$$
\begin{aligned}
|\operatorname{M}(G)| \leq p^{-1}|\operatorname{M}(G/K)||(G/K)^{ab} \otimes K| \quad &\leq p^{\frac{1}{2}(n-k)(n-k-3)} p^{(n-k-2)} \\
&\leq p^{\frac{1}{2}n(n-3)-(n-1)},
\end{aligned}
$$

which is a contradiction. Finally if $G' \cap K = 1$, then by Theorem 1.2.6 and Theorem 1.2.11, we have the following

$$
\begin{aligned}
|\operatorname{M}(G)| \leq |\operatorname{M}(G/K)||(G/K)^{ab} \otimes K| \quad &\leq p^{\frac{1}{2}(n-k+1)(n-k-4)} p^{(n-k-2)} \\
&\leq p^{\frac{1}{2}n(n-3)-(n+1)},
\end{aligned}
$$

which again gives a contradiction, and the proof is complete. □

2.2.1 Groups G of order p^n, $n \geq 6$, with $|G'| = p,\ p^2$

First we consider the groups G such that $|G'| = p$.

Lemma 2.2.5 *There is no non-abelian p-group G of order p^n with $|G'| = p$ and $t(G) = \log_p(|G|) + 1$.*

Proof. Note that G' is a central subgroup of G. By Theorem 1.2.5, we have $|\operatorname{M}(G/G')| \leq |\operatorname{M}(G)||G'|$. By Lemma 2.2.3, it follows that G/G' is an elementary abelian p-group. Hence by Theorem 1.2.2, $|\operatorname{M}(G/G')| = p^{\frac{1}{2}(n-1)(n-2)}$. Therefore $|\operatorname{M}(G)| \geq p^{\frac{1}{2}(n-1)(n-2)-1} = p^{\frac{1}{2}n(n-3)}$, which is a contradiction. □

Now we consider groups G such that $|G'| = p^2$.

Lemma 2.2.6 *Let G be a p-group of order p^n $(n \geq 6)$ with $|G'| = p^2$ and $t(G) \leq \log_p(|G|)+1$. If K is a cyclic subgroup of order p in $G' \cap \operatorname{Z}(G)$, then G/K is isomorphic to one of the following groups: $ES(p^3) \times \mathbb{Z}_p^{(n-4)}$, $E(2) \times \mathbb{Z}_p^{(n-2m-3)}$, $ES(p^{2m+1}) \times \mathbb{Z}_p^{(n-2m-2)}$ $(m \geq 2)$, $D_8 \times \mathbb{Z}_2^{(n-4)}$, $Q_8 \times \mathbb{Z}_2^{(n-4)}$, where $E(2)$ denotes a central product of $ES(p^{2m+1})$ and $\mathbb{Z}_{p^2}$.*

Proof. Suppose that G is a p-group of order p^n with $|G'| = p^2$. Consider a cyclic central subgroup K of order p in $G' \cap \mathrm{Z}(G)$. Then by Theorem 1.2.6 and Theorem 1.2.11, we have

$$|\mathrm{M}(G)| \leq |\mathrm{M}(G/K)|p^{n-3} \leq p^{\frac{1}{2}(n-2)(n-3)+1}p^{n-3} = p^{\frac{1}{2}n(n-3)+1}.$$

Now Theorem 1.2.11, Theorem 2.1.1 and Theorem 2.1.2 provide the structure of G/K, which is precisely as per our assertion. □

Proposition 2.2.7 *There is no non-abelian p-group G of order p^n $(n \geq 6)$ with $|G'| = p^2, |\mathrm{Z}(G)| = p$ and $t(G) \leq \log_p(|G|) + 1$.*

Proof. By Lemma 2.2.3, G/G' is an elementary abelian group of order p^{n-2}. Thus G is an $(n-2)$-generator group. We can choose generators $x, y, \beta_1, \beta_2, \ldots, \beta_{n-4}$ of G such that $[x, y] = z \notin \mathrm{Z}(G)$.

If $[z, x]$ is non-trivial in $\mathrm{Z}(G)$, then $\psi_3(x \otimes y \otimes x \otimes \beta_i)$ for $i = 1, \ldots, n-4$ gives $n-4$ linearly independent elements of $\gamma_3(G) \otimes \bar{G}^{ab}$. By symmetry, if $[z, y]$ is non-trivial in $\mathrm{Z}(G)$, then we have a similar conclusion. So $|\mathrm{Im}(\psi_3)| \geq p^{n-4}$. Note that $|\mathrm{Im}(\psi_2)| \geq p^{n-4}$. So by Proposition 1.2.14, we have

$$p^2|\mathrm{M}(G)||\mathrm{Im}(\psi_2)||\mathrm{Im}(\psi_3)| \leq p^{\frac{1}{2}(n-2)(n-3)}p^{2(n-2)}.$$

It follows that $|\mathrm{M}(G)| \leq p^{\frac{1}{2}n(n-3)-n+5}$, which is not possible for $n \geq 7$.

On the other hand if $[z, x] = [z, y] = 1$, then $[z, \beta_k]$ is non-trivial in $Z(G)$ for some β_k and $\psi_3(x \otimes y \otimes \beta_k \otimes \beta_i)(i \neq k)$ give $n-5$ linearly independent elements of $\gamma_3(G) \otimes \bar{G}^{ab}$. Hence $|\mathrm{Im}(\psi_3)| \geq p^{n-5}$. Note that $|\mathrm{Im}(\psi_2)| \geq p^{n-4}$. So by Proposition 1.2.14 we have

$$p^2|\mathrm{M}(G)||\mathrm{Im}(\psi_2)||\mathrm{Im}(\psi_3)| \leq p^{\frac{1}{2}(n-2)(n-3)}p^{2(n-2)}.$$

It follows that $|\,\mathrm{M}(G)| \leq p^{\frac{1}{2}n(n-3)-n+6}$, which is not possible for $n \geq 8$.

Now if either $\psi_3(x \otimes y \otimes \beta_k \otimes x)$ or $\psi_3(x \otimes y \otimes y \otimes \beta_k)$ is non-trivial then $|\mathrm{Im}(\psi_3)| \geq p^{n-4}$ and

$$p^2|\,\mathrm{M}(G)||\,\mathrm{Im}(\psi_2)||\,\mathrm{Im}(\psi_3)| \leq p^{\frac{1}{2}(n-2)(n-3)}p^{2(n-2)}.$$

It follows that $|\,\mathrm{M}(G)| \leq p^{\frac{1}{2}n(n-3)-n+5}$, which is not possible for $n \geq 7$. Otherwise suppose $\psi_3(x\otimes y\otimes \beta_k \otimes x) = \psi_3(x\otimes y\otimes y\otimes \beta_k) = 1$, then $[x, y, \beta_k] = [\beta_k, x, y] = [y, \beta_k, x]$ and $p = 3$. By HAP [12] of GAP [14] there is no group G of order 3^7 with $|G'| = 3^2, |Z(G)| = 3$ and $|\,\mathrm{M}(G)| = 3^{13}$.

For $|G| = p^6$ $(p \neq 2)$, by [27] it follows that G belongs to the isoclinism class Φ_{22}. In this case $|\,\mathrm{Im}(\psi_2)| \geq p^2$ and $|\,\mathrm{Im}(\psi_3)| \geq p^3$. Hence it follows from Proposition 1.2.14 that $|\,\mathrm{M}(G)| \leq p^7$, which is not our case.

For $p = 2$, that there is no group G of order 2^6 which satisfies the given hypothesis, follows from computation with HAP [12] of GAP [14]. $\square$

Lemma 2.2.8 *Let G be a non-abelian p-group of order p^n $(n \geq 6)$ with $t(G) = \log_p(|G|) + 1$ and $|G'| = p^2$. If there exists a central subgroup K of order p such that $K \cap G' = 1$, then G/K is isomorphic to either $\mathbb{Z}_p^{(4)} \rtimes \mathbb{Z}_p$ or $(\mathbb{Z}_p^{(4)} \rtimes \mathbb{Z}_p) \times \mathbb{Z}_p$ and p is odd.*

Proof. By Theorem 1.2.6 and Theorem 1.2.11, we have

$$|\,\mathrm{M}(G)| \leq |\,\mathrm{M}(G/K)|p^{(n-3)} \leq p^{\frac{1}{2}(n-1)(n-4)+1+n-3} = p^{\frac{1}{2}n(n-3)}.$$

Since $|\,\mathrm{M}(G)| = p^{\frac{1}{2}n(n-3)-1}$, so $|\,\mathrm{M}(G/K)|$ is either $p^{\frac{1}{2}(n-1)(n-4)+1}$ or $p^{\frac{1}{2}(n-1)(n-4)}$. Note that $|G/K| \geq p^5$ with $(G/K)' = p^2$. Now by Theorem 2.1.1, we have $|\,\mathrm{M}(G/K)| = p^{\frac{1}{2}(n-1)(n-4)+1}$ if and only if $G/K \cong \mathbb{Z}_p^{(4)} \rtimes \mathbb{Z}_p$ $(p \neq 2)$. By Theorem 2.1.2, $|\,\mathrm{M}(G/K)| = p^{\frac{1}{2}(n-1)(n-4)}$ if and only if $G/K \cong (\mathbb{Z}_p^{(4)} \rtimes \mathbb{Z}_p) \times \mathbb{Z}_p$ $(p \neq 2)$.

$\square$

Proposition 2.2.9 *There is no non-abelian p-group of order p^7 with $G' = Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$ and $t(G) = \log_p(|G|) + 1$.*

Proof. Note that if $Z^*(G)$ contains any central subgroup Z of G, then using Theorem 1.2.9 in Theorem 1.2.8 we have $|\mathrm{M}(G)| = \frac{|\mathrm{M}(G/Z)|}{|G' \cap Z|}$. Since $|G/Z|$ is of order p^5 or p^6, so $|\mathrm{M}(G)| < p^{13}$, which is not our case. So we have $Z^*(G) = 1$, i.e, G is capable.

First we consider groups G of order p^7 of exponent p^2 for odd p. Note that G/G' is elementary abelian of order p^5 by Lemma 2.2.3. So we take generating sets $\{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5\}$ of G and $\{\eta, \gamma\}$ of G'. It then follows that either $|G^p| = p$ or $G^p = G'$. We claim that $|X| \geq p^8$ (where X, X_1, X_2 are defined in Section 1.2.1).

Let $|G^p| = p$. Without loss of generality assume that η is p-th power of some β_k, say β_1 and $[\beta_i, \beta_j] \notin \langle \eta \rangle$ for some i, j and, all β_k's $(k > 1)$ are of order p. Then $\langle \beta_i \otimes \beta_1^p, i \in \{1,2,3,4,5\} \rangle$ is a subspace of X_2 and $\langle \psi_2(\beta_k \otimes \beta_i \otimes \beta_j), k \in \{1,2,3,4,5\}, k \neq i, j \rangle$ is a subspace of X_1. For $G^p = G'$, without loss of generality, assume that η, γ are p-th power of some β_{k_1}, β_{k_2}, say β_1 and β_2 respectively and all other β_i's are of order p. Then $\langle \beta_i \otimes \beta_1^p, \beta_j \otimes \beta_2^p, i \in \{1,3,4,5\}, j \in \{2,3,4,5\} \rangle$ is a subspace of X_2.

Hence we observe that for non-abelian group G of order p^7 and of exponent p^2, $|X| \geq p^8$ and, by Theorem 1.2.13, $|\mathrm{M}(G)| < p^{13}$, a contradiction.

Now consider groups G of order p^7 and of exponent p. Here G is a special p-group of rank 2, i.e., $G' = Z(G) = \Phi(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where $\Phi(G)$ denotes Frattini subgroup of G. By [23] it follows that there is only one capable special p-group G of rank 2 upto isomorphism which is the following:

$$G = \langle x_1, \cdots, x_5, c_1, c_2 \mid [x_2, x_1] = [x_5, x_3] = c_1, [x_3, x_1] = [x_5, x_4] = c_2 \rangle.$$

By Theorem 1.2.13 we have $|\operatorname{M}(G)| = p^9$ as $|X| = |X_1| = p^9$ which is not our case. □

The following result weaves the next thread in the proof of main theorem.

Theorem 2.2.10 *Let G be a non-abelian p-group of order p^n $(n \geq 6)$ with $|G'| = p^2$, $|\operatorname{Z}(G)| \geq p^2$ and $t(G) = \log_p(|G|) + 1$. Then G is isomorphic to $\Phi_{12}(1^6), \Phi_{13}(1^6)$, $\Phi_{15}(1^6)$ or $(\mathbb{Z}_p^{(4)} \rtimes \mathbb{Z}_p) \times \mathbb{Z}_p^{(2)}$. Moreover, p is always odd.*

Proof. By Lemma 2.2.4, $\operatorname{Z}(G)$ is of exponent p. We consider two cases here.

Case 1: Let $G' = \operatorname{Z}(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p \cong \langle z \rangle \times \langle w \rangle$. The isomorphism type of G/K is as in Lemma 2.2.6. It follows from these structures that there are $n-2$ generators $\{x, y, \alpha_i,\ 1 \leq i \leq n-4\}$ of G such that $[x, y] \in \langle z \rangle$ and $[\alpha_k, x] \in \langle w \rangle$, for some k. Hence $\psi_2(x \otimes y \otimes \alpha_i), i = 1, 2, \ldots, n-4$ and $\psi_2(\alpha_k \otimes x \otimes \alpha_j), j = 1, 2, \ldots, n-4, j \neq k$ gives $(2n-9)$ linearly independent elements in $G' \otimes G/G'$, which implies $|X| \geq p^{2n-9}$. Now by Theorem 1.2.13, we have $|\operatorname{M}(G)| \leq p^{\frac{1}{2}n(n-3)-n+6}$, which is possible only when $n \leq 7$. Thus it only remains to consider groups of order p^6 and p^7.

For groups of order p^7, the result follows from Proposition 2.2.9.

Now consider groups of order p^6 for odd p. Then G belongs to the isoclinism classes Φ_{12}, Φ_{13} or Φ_{15} (see Table 4.1 of [27] for details on the structure of these groups). If G is of exponent p^2, then it is easy to see that $|X| \geq p^5$ and $|\operatorname{M}(G)| \leq p^7$ by Theorem 1.2.13. For $\Phi_{12}(1^6), \Phi_{13}(1^6), \Phi_{15}(1^6)$, we have $|X| = |X_1| = p^4$ and using Theorem 1.2.13 we see that all of these groups have Schur multiplier of order p^8.

A simple GAP check shows that there is no such group for $p = 2$.

Case 2: Consider the complement of Case 1. In these cases we can choose a central subgroup K of order p such that $K \cap G' = 1$. Then by Lemma 2.2.8, it follows that $G/K \cong \mathbb{Z}_p^{(4)} \rtimes \mathbb{Z}_p$ or $(\mathbb{Z}_p^{(4)} \rtimes \mathbb{Z}_p) \times \mathbb{Z}_p$. Hence either G is of order p^6 with $|\operatorname{M}(G)| = p^8$ or G is of order p^7 with $|\operatorname{M}(G)| = p^{13}$. Here

$|\mathrm{Z}(G)/K| = |\mathrm{Z}(G/K)|$. Since G/K and G/G' are of exponent p, so for $g \in G$, $g^p \in K \cap G' = 1$, which implies that G is of exponent p. Then it easily follows that G is isomorphic to $(\mathbb{Z}_p^{(4)} \rtimes \mathbb{Z}_p) \times \mathbb{Z}_p$ or $(\mathbb{Z}_p^{(4)} \rtimes \mathbb{Z}_p) \times \mathbb{Z}_p^{(2)}$ $(p \neq 2)$. Now the consideration of order $\mathrm{M}(G)$ shows that $G \cong (\mathbb{Z}_p^{(4)} \rtimes \mathbb{Z}_p) \times \mathbb{Z}_p^{(2)}$ $(p \neq 2)$. □

2.2.2 Groups G of order p^n, $n \geq 6$ with $|G'| = p^3$

Finally we consider groups G such that $|G'| = p^3$.

Lemma 2.2.11 *Let G be a non-abelian p-group of order p^n with $t(G) = \log_p(|G|)+1$ and $|G'| = p^3$. Then for any subgroup $K \subseteq \mathrm{Z}(G) \cap G'$ of order p, $G/K \cong \mathbb{Z}_p^{(4)} \rtimes \mathbb{Z}_p$ $(p \neq 2)$. In particular, $|G| = p^6$.*

Proof. Consider p odd. By Lemma 2.2.3, G/G' is elementary abelian and, by Theorem 1.2.6 and Theorem 1.2.11, we have

$$|\mathrm{M}(G)| \leq p^{-1}|\mathrm{M}(G/K)||G/G' \otimes K| \leq p^{\frac{1}{2}(n-1)(n-4)+(n-3)} = p^{\frac{1}{2}n(n-3)-1},$$

which implies $|\mathrm{M}(G)| = p^{\frac{1}{2}(n-1)(n-4)+1}$. Now using Theorem 2.1.1, we get $G/K \cong \mathbb{Z}_p^{(4)} \rtimes \mathbb{Z}_p$.

For $p = 2$, $|\mathrm{M}(G)| < p^{\frac{1}{2}n(n-3)-1}$, which is not our case. □

Lemma 2.2.12 *There is no non-abelian finite p-group G with $|G'| = p^3$, $|\mathrm{Z}(G)| = p$ and $t(G) = \log_p(|G|) + 1$.*

Proof. By the preceding lemma we have, $G/\mathrm{Z}(G) \cong \mathbb{Z}_p^{(4)} \rtimes \mathbb{Z}_p \cong \Phi_4(1^5)$ and $|G| = p^6$. Now it follows that (see Table 4.1 of [27] for details) G belongs to one of the isoclinism classes $\Phi_{31}, \Phi_{32}, \Phi_{33}$. Observe that for these groups $|\mathrm{Im}(\psi_2)| \geq p$ and $|\mathrm{Im}(\psi_3)| \geq p$. Now using Proposition 1.2.14 we get $|\mathrm{M}(G)| \leq p^7$, which is not our case. □

Theorem 2.2.13 *Let G be a non-abelian p-group of order p^n with $|G'| = p^3$ and $t(G) = \log_p(|G|) + 1$. Then $G \cong \Phi_{11}(1^6)$.*

Proof. We claim that $\mathrm{Z}(G) \subseteq G'$. Contrarily assume that $\mathrm{Z}(G) \nsubseteq G'$. Then there is a central subgroup K of order p such that $G' \cap K = 1$. Now by Theorem 1.2.6 and Theorem 1.2.11 we have

$$|\,\mathrm{M}(G)| \leq |\,\mathrm{M}(G/K)||G/G'K| \leq p^{\frac{1}{2}n(n-5)+1+(n-4)} = p^{\frac{1}{2}n(n-3)-3},$$

which is a contradiction. Hence $\mathrm{Z}(G) \subseteq G'$.

Note that $|\,\mathrm{Z}(G)| \geq p^2$ by preceding lemma. We can now choose two distinct central subgroups K_i $(i = 1, 2)$ of order p. Then by Lemma 2.2.11 we have $G/K_i \cong \mathbb{Z}_p^{(4)} \rtimes \mathbb{Z}_p$ $(i = 1, 2)$, which are of exponent p. So G is of exponent p. Hence from [27] it follows that G is isomorphic to one of the following groups of order p^6 and of exponent p with $|\,\mathrm{Z}(G)| \geq p^2$, $|G'| = p^3$: $\Phi_6(1^6), \Phi_9(1^6), \Phi_{10}(1^6), \Phi_{11}(1^6), \Phi_{16}(1^6), \Phi_{17}(1^6), \Phi_{18}(1^6), \Phi_{19}(1^6), \Phi_{20}(1^6), \Phi_{21}(1^6)$.

For the groups $G \cong \Phi_6(1^6), \Phi_9(1^6), \Phi_{10}(1^6)$, by a routine check we can show that $|\,\mathrm{M}(G)| \leq p^6$ using Theorem 1.2.3.

Now consider the group $G = \Phi_{11}(1^6)$. Then G is of nilpotency class two with G/G' elementary abelian. Hence by Theorem 1.2.13, it follows that $\Phi_{11}(1^6)$ has Schur multiplier of order p^8 as $|X| = |X_1| = p$.

For other groups G, observe that $|\,\mathrm{Im}(\psi_2)| \geq p$, $|\,\mathrm{Im}(\psi_3)| \geq p$ and hence it follows from Proposition 1.2.14 that $|\,\mathrm{M}(G)| \leq p^7$. □

2.2.3 Main result

Our main theorem is the following:

Theorem 2.2.14 ([18]) *Let G be a finite non-abelian p-group of order p^n with $t(G) = \log_p(|G|) + 1$. Then for odd prime p, G is isomorphic to one of the* *following groups:*

(i) $\Phi_2(22) = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha^p = \alpha_2, \alpha_1^{p^2} = \alpha_2^p = 1 \rangle$,

(ii) $\Phi_3(211)a = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha^p = \alpha_3, \alpha_1^{(p)} = \alpha_2^p = \alpha_3^p = 1 \rangle$,

(iii) $\Phi_3(211)b_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha^p = \alpha_3, \alpha_1^{(p)} = \alpha_2^p = \alpha_3^p = 1 \rangle$,

(iv) $\Phi_2(2111)c = \Phi_2(211)c \times \mathbb{Z}_p$, *where* $\Phi_2(211)c = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2, \alpha^{p^2} = \alpha_1^p = \alpha_2^p = 1 \rangle$,

(v) $\Phi_2(2111)d = ES_p(p^3) \times \mathbb{Z}_{p^2}$,

(vi) $\Phi_3(1^5) = \Phi_3(1^4) \times \mathbb{Z}_p$, *where* $\Phi_3(1^4) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_i^{(p)} = \alpha_3^p = 1 (i = 1, 2) \rangle$,

(vii) $\Phi_7(1^5) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \alpha^p = \alpha_1^{(p)} = \alpha_{i+1}^p = \beta^p = 1 (i = 1, 2) \rangle$,

(viii) $\Phi_{11}(1^6) = \langle \alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \beta_1, [\alpha_3, \alpha_1] = \beta_2, \alpha_i^{(p)} = \beta_i^p = 1 (i = 1, 2, 3) \rangle$,

(ix) $\Phi_{12}(1^6) = ES_p(p^3) \times ES_p(p^3)$,

(x) $\Phi_{13}(1^6) = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2 \mid [\alpha_i, \alpha_{i+1}] = \beta_i, [\alpha_2, \alpha_4] = \beta_2, \alpha_i^p = \alpha_3^p = \alpha_4^p = \beta_i^p = 1 (i = 1, 2) \rangle$,

(xi) $\Phi_{15}(1^6) = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2 \mid [\alpha_i, \alpha_{i+1}] = \beta_i, [\alpha_3, \alpha_4] = \beta_1, [\alpha_2, \alpha_4] = \beta_2^\zeta, \alpha_i^p = \alpha_3^p = \alpha_4^p = \beta_i^p = 1 (i = 1, 2) \rangle$,

(xii) $(\mathbb{Z}_p^{(4)} \rtimes \mathbb{Z}_p) \times \mathbb{Z}_p^{(2)}$.

Moreover for $p = 2$*,* G *is isomorphic to one of the following groups:*

(xiii) $\mathbb{Z}_2^{(4)} \rtimes \mathbb{Z}_2$,

(xiv) $G_1 \times \mathbb{Z}_2$, *where* $G_1 = \langle a, b, c \mid a^4 = b^2 = c^2 = 1, [a, c] = b, [a, b] = [b, c] = 1\rangle$,

(xv) $G_2 = \langle a, b, c \mid a^4 = b^4 = c^2 = 1, [a, b] = 1, [a, c] = a^2, [b, c] = b^2\rangle$,

(xvi) $\mathbb{Z}_4 \rtimes \mathbb{Z}_4$,

(xvii) D_{16}, *the Dihedral group of order* 16.

Proof. Let G be a non-abelian finite p-group of order p^n with $t(G) = \log_p(|G|) + 1$. For $n \leq 5$, result follows from Lemma 2.2.1. Now consider $n \geq 6$. By Lemma 2.2.5, there is no group G with $|G'| = p$. If G' is of order p^2, then the result follows from Proposition 2.2.7 and Theorem 2.2.10. If G' is of order p^3, then the result follows from Theorem 2.2.13. □

2.3 Finite p-groups having Schur multiplier of maximum order

In this section we study non-abelian finite p-groups G of order p^n such that $|\operatorname{M}(G)|$ attains Niroomand's bound, i.e., $|\operatorname{M}(G)| = p^{\frac{1}{2}(n+k-2)(n-k-1)+1}$, where $|G'| = p^k$. For our convenience, instead of writing $|\operatorname{M}(G)| = p^{\frac{1}{2}(n+k-2)(n-k-1)+1}$ we shall write $|\operatorname{M}(G)|$ attains the bound, throughout this section.

In the following result, Rai classified finite p-groups G of class 2 such that $|\operatorname{M}(G)|$ attains the bound.

Theorem 2.3.1 (Theorem 1.1 of [40]) *Let G be a finite p-group of order p^n and nilpotency class 2 with $|G'| = p^k$. Then $|\operatorname{M}(G)| = p^{\frac{1}{2}(n+k-2)(n-k-1)+1}$ if and only if G is one of the following groups.*

(i) $G_1 = ES_p(p^3) \times \mathbb{Z}_p^{(n-3)}$, *for an odd prime* p.

(ii) $G_2 = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_i, \alpha] = \beta_i, [\alpha_1, \alpha_2] = 1, \alpha^p = \alpha_i^p = \beta_i^p = 1 (i = 1, 2)\rangle$, *for an odd prime* p.

(iii) $G_3 = \langle \alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \beta_1, [\alpha_3, \alpha_1] = \beta_2, \alpha_i^p = \beta_i^p = 1 (i = 1, 2, 3)\rangle$, *for an odd prime* p.

We continue this line of investigation and look into the classification of arbitrary finite p-groups attaining this bound. Surprisingly, it turns out that for $p \neq 3$ there is no finite p-group G of nilpotency class $c \geq 3$ such that $|\operatorname{M}(G)|$ attains the bound. Hence for p-groups G of class ≥ 3 and $p \neq 3$ we improve the bound, and in this case $|\operatorname{M}(G)| \leq p^{\frac{1}{2}(n+k-2)(n-k-1)}$, where $|G'| = p^k$.

One can ask what happens for $p = 3$? Is there any group G for $p = 3$ such that $|\operatorname{M}(G)|$ attains the bound? We construct an example in the proof of Lemma 2.3.7, which gives the answer of this question affirmatively.

So the natural question which arises here is the following:

Question: Does there exist finite p-groups of arbitrary nilpotency class for which the improved bound is attained?

The answer to this question is yes for p-groups of nilpotency class 3 and 4; see Section 2.3.1, Example 1 and Example 2.

Now we state some results which will be used to prove Theorem 2.3.8. The following result is a consequence of Theorem 2.2.14.

Theorem 2.3.2 *There is no group G of order p^n and of class $c \geq 3$ such that $|\operatorname{M}(G)| = p^{\frac{1}{2}n(n-1)-(n+1)}$, where $n \geq 6$, p is odd.*

The following result immediately follows from Theorem 2.1.1.

Theorem 2.3.3 *There is no group G of order p^n and of class $c \geq 3$ such that $|\operatorname{M}(G)| = p^{\frac{1}{2}n(n-1)-(n-1)}$.*

Lemma 2.3.4 *Let G be a non-abelian p-group of order p^n with $|G'| = p^k$ and $|\,\mathrm{M}(G)|$ attains the bound. Then the following hold:*

(i) G^{ab} *is an elementary abelian p-group.*

(ii) $\mathrm{Z}(G)$ *is an elementary abelian p-group.*

(iii) $\mathrm{Z}(G) \subseteq G'$ *(except* $G \cong ES_p(p^3) \times \mathbb{Z}_p^{(n-3)}$*).*

Proof. (i) Let K be a central subgroup of order p such that $K \subseteq G'$. Now $|G/K| = p^{n-1}$ and $|(G/K)'| = p^{k-1}$. By Theorem 1.2.11 we get

$$|\,\mathrm{M}(G/K)| \leq p^{\frac{1}{2}(n-1+k-1-2)(n-1-k+1-1)+1} = p^{\frac{1}{2}(n+k-4)(n-k-1)+1}.$$

By Theorem 1.2.6, $|\,\mathrm{M}(G)|p \leq |\,\mathrm{M}(G/K)||G^{ab} \otimes K|$. Hence we have

$$|\,\mathrm{M}(G)| \leq p^{\frac{1}{2}(n+k-4)(n-k-1)+1}p^{(n-k-1)} = p^{\frac{1}{2}(n+k-2)(n-k-1)+1}.$$

Thus $|\,\mathrm{M}(G/K)| = p^{\frac{1}{2}(n+k-4)(n-k-1)+1}$ and $G/G' \cong \mathbb{Z}_p^{(n-k)}$. It follows that $|\,\mathrm{M}(G/K)|$ attains the bound and G/G' is an elementary abelian p-group.

(ii) Suppose that the exponent of $\mathrm{Z}(G) > p$. Consider a cyclic central subgroup K of order p^2. Either $K \subset G'$, $K \cap G' = 1$ or $K \cap G' = \mathbb{Z}_p$.

For the first case $K \subset G'$, $|G/K| = p^{n-2}$ and $|(G/K)'| = p^{k-2}$. Hence by Theorem 1.2.11, we have

$$|\,\mathrm{M}(G/K)| \leq p^{\frac{1}{2}(n-2+k-2-2)(n-2-k+2-1)+1} = p^{\frac{1}{2}(n+k-6)(n-k-1)+1}.$$

Therefore, by Theorem 1.2.6, we have

$$|\,\mathrm{M}(G)| \leq p^{\frac{1}{2}(n+k-6)(n-k-1)+1}p^{(n-k-2)} = p^{\frac{1}{2}(n+k-2)(n-k-1)+1-(n-k)},$$

which is a contradiction.

For the second case $K \cap G' = 1$, $|G/K| = p^{n-2}$ and $|(G/K)'| = p^k$. Therefore from Theorem 1.2.6 and Theorem 1.2.11, we have

$$|\operatorname{M}(G)| \leq p^{\frac{1}{2}(n+k-4)(n-k-3)+1} p^{(n-k-2)} = p^{\frac{1}{2}(n+k-2)(n-k-1)+1-(n+k-3)},$$

which is a contradiction.

For the last case $K \cap G' = \mathbb{Z}_p$, $|G/K| = p^{n-2}$ and $|(G/K)'| = p^{k-1}$. Hence by Theorem 1.2.6 and Theorem 1.2.11, we have

$$|\operatorname{M}(G)| \leq p^{\frac{1}{2}(n+k-5)(n-k-2)} p^{(n-k-1)} = p^{\frac{1}{2}(n+k-2)(n-k-1)+1-(n-2)},$$

which is a contradiction.

(iii) Suppose $\operatorname{Z}(G) \nsubseteq G'$, consider a central subgroup K of order p such that $K \cap G' = 1$. Since $|G/K| = p^{n-1}$ and $|(G/K)'| = p^k$. So by Theorem 1.2.11,

$$|\operatorname{M}(G/K)| \leq p^{\frac{1}{2}(n+k-3)(n-k-2)+1}.$$

Hence by Theorem 1.2.6, we have

$$|\operatorname{M}(G)| \leq p^{\frac{1}{2}(n+k-3)(n-k-2)+1} p^{(n-k-1)} = p^{\frac{1}{2}(n+k-2)(n-k-1)+1-(k-1)},$$

which is a contradiction for $k > 1$. For $k = 1$, it follows from Theorem 1.2.11 that $|\operatorname{M}(G)|$ attains the bound if and only if $G \cong ES_p(p^3) \times \mathbb{Z}_p^{(n-3)}$. $\square$

Lemma 2.3.5 *If G is a p-group of order p^n such that $|\operatorname{M}(G)|$ attains the bound, then for every central subgroup K of order p, $|\operatorname{M}(G/K)|$ also attains the bound.*

Proof. Let K be a cyclic central subgroup of order p. By Lemma 2.3.4(iii), $K \subseteq G'$. Now the result follows from the proof of Lemma 2.3.4(i). $\square$

Lemma 2.3.6 *There is no group G of order p^n $(n \geq 4)$ having maximal nilpotency class such that $|\operatorname{M}(G)|$ attains the bound.*

Proof. First we prove that $|\operatorname{M}(G)| \leq p^{n-2}$ for p-groups G of maximal class. We use induction argument on n to prove this.

Let $n = 4$. Then for $p = 2$, using HAP[12] of GAP[14] and for odd p, by Theorem 1.4.1, it follows that $|\operatorname{M}(G)| \leq p^2 = p^{n-2}$.

Now consider the groups G of order p^n $(n > 4)$ of maximal class. Note that $G/\operatorname{Z}(G)$ is also of maximal class. So by induction hypothesis $|\operatorname{M}(G/\operatorname{Z}(G))| \leq p^{n-3}$. Hence it follows from Theorem 1.2.6 that

$$|\operatorname{M}(G)|p \leq |\operatorname{M}(G/\operatorname{Z}(G))||G^{ab}| \leq p^{n-1}.$$

Hence $|\operatorname{M}(G)| \leq p^{n-2}$. Now if $|\operatorname{M}(G)|$ attains the bound for p-groups G of maximal class, then $|\operatorname{M}(G)| = p^{\frac{1}{2}(n+n-2-2)(n-n+2-1)+1} = p^{n-1}$, which is a contradiction.

□

In view of Lemma 2.3.5 we observe that it is sufficient to consider groups G/K such that $|\operatorname{M}(G/K)|$ attains the bound for every central subgroup K of order p. This observation is going to be key ingredient in the proof of the main theorem. The following lemma refutes the existence of finite p-groups G such that G/K is of nilpotency class 2 and $|\operatorname{M}(G)|$ attains the bound.

Lemma 2.3.7 *There is no non-abelian p-group G of order p^n, $p \neq 3$, having nilpotency class ≥ 3 such that G/K is of nilpotency class 2 for some central subgroup K of order p and $|\operatorname{M}(G)|$ attains the bound. For $p = 3$, there is a group G such that $|\operatorname{M}(G)|$ attains the bound.*

Proof. Suppose that G is a group of order p^n and $|G'| = p^k$ such that $|\operatorname{M}(G)|$ attains the bound. Let G/K be of class 2 for a central subgroup K of order p.

By Lemma 2.3.5, $|\,\mathrm{M}(G/K)|$ also attains the bound. Hence by Theorem 2.3.1, it follows that G/K is isomorphic to G_1, G_2 or G_3. Now we consider the cases depending on the structure of G/K.

If $G/K \cong G_1 = ES_p(p^3)\times\mathbb{Z}_p^{(n-3)}$, then $k=2$ and $|\,\mathrm{M}(G)| = p^{\frac{1}{2}(n+k-2)(n-k-1)+1}$ $= p^{\frac{1}{2}n(n-1)-(n-1)}$, which contradicts Theorem 2.3.3. If $G/K \cong G_2$, then $k=3$ and $|\,\mathrm{M}(G)| = p^8 = p^{\frac{1}{2}6(6-1)-(6+1)}$, which contradicts Theorem 2.3.2.

If $G/K \cong G_3 = \langle \alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3 \mid [\alpha_1,\alpha_2] = \beta_3, [\alpha_2,\alpha_3] = \beta_1, [\alpha_3,\alpha_1] = \beta_2, \alpha_i^p = \beta_i^p = 1\ (i=1,2,3)\rangle$, then $k=4$ and $|\,\mathrm{M}(G)| = p^{\frac{1}{2}(n+k-2)(n-k-1)+1} = p^{10}$. Observe that if $|\,\mathrm{Im}(\psi_3)| \geq p$, then by Proposition 1.2.14, we have $|\,\mathrm{M}(G)| \leq p^9$, which is a contradiction.

Suppose $[\beta_i, \alpha_j]$ is non-trivial in $\gamma_3(G)$ for some $j \in \{1,2,3\}$. Without loss of generality assume that $i=1$. Now if $j=2$ or 3, then $\psi_3(\alpha_2 \otimes \alpha_3 \otimes \alpha_j \otimes \alpha_1)$ is non-trivial element. Hence, $|\,\mathrm{Im}(\psi_3)| \geq p$, which is not possible. If $i=2$ and $j=1$ or 3, then also $|\,\mathrm{Im}(\psi_3)| \geq p$, as described above. We have similar conclusion if $i=3$ and $j=1$ or 2.

Therefore, we consider the case $[\beta_i, \alpha_i] \in \gamma_3(G)$ is non-trivial and $[\beta_i, \alpha_j] = 1$ for $i, j \in \{1,2,3\}, i \neq j$. Now let $|\,\mathrm{Z}(G)| > p$. Without loss of generality assume that $\beta_2 = [\alpha_3, \alpha_1] \in \mathrm{Z}(G)$ and there is a non-trivial element $[\beta_1, \alpha_1] \in \gamma_3(G)$. Then $\psi_3(\alpha_2 \otimes \alpha_3 \otimes \alpha_1 \otimes \alpha_3)$ is non-trivial element. So $|\,\mathrm{Im}(\psi_3)| \geq p$, which is not possible. The remaining case is $\mathrm{Z}(G) = \langle \gamma \rangle \cong \mathbb{Z}_p$. Suppose $\mathrm{Im}(\psi_3) = \{1\}$. Then $\psi_3(\alpha_1 \otimes \alpha_2 \otimes \alpha_3 \otimes \alpha_1) = \psi_3(\alpha_3 \otimes \alpha_1 \otimes \alpha_2 \otimes \alpha_3) = 1$ imply that

$$\big([\alpha_2,\beta_2][\beta_3,\alpha_3]\big) \otimes \bar{\alpha_1} = \big([\beta_2,\alpha_2][\alpha_1,\beta_1]\big) \otimes \bar{\alpha_3} = 1,$$

which forces to have $[\beta_3, \alpha_3] = [\beta_2, \alpha_2] = [\beta_1, \alpha_1]$ and by Hall-Witt identity we have $p = 3$. Using these relations we construct a group

$$G \;=\; \langle \alpha_i, \beta_i, \gamma \mid [\alpha_1,\alpha_2] = \beta_3, [\alpha_2,\alpha_3] = \beta_1, [\alpha_3,\alpha_1] = \beta_2,$$

$$[\beta_3, \alpha_3] = [\beta_2, \alpha_2] = [\beta_1, \alpha_1] = \gamma, \alpha_i^3 = \beta_i^3 = \gamma^3 = 1,\ 1 \leq i \leq 3\rangle,$$

which is of order 3^7. Using HAP of GAP we see that $|\mathrm{M}(G)| = p^{\frac{1}{2}(n+k-2)(n-k-1)+1} = p^{10}$. □

We finally prove the main result, which is the following.

Theorem 2.3.8 ([19]) *There is no non-abelian p-group G of order p^n, $p \neq 3$, having nilpotency class $c \geq 3$ with $|G'| = p^k$ and $|\mathrm{M}(G)| = p^{\frac{1}{2}(n+k-2)(n-k-1)+1}$. In particular, $|\mathrm{M}(G)| \leq p^{\frac{1}{2}(n+k-2)(n-k-1)}$ for p-groups G of nilpotency class $c \geq 3$ and $p \neq 3$.*

Proof. First we prove that there is no group G of order p^n, $p \neq 3$ of class $c = 3$ such that $|\mathrm{M}(G)|$ attains the bound. For a central subgroup K of order p in G, G/K is of class 2 or 3. If G/K is of class 2, then the result follows from Lemma 2.3.7. Now if G/K is of nilpotency class 3, for every central subgroup K of order p, then we use induction on n to prove that there is no such G such that $|\mathrm{M}(G)|$ attains the bound. For $n = 5$, G/K is of maximal class, so our result for $n = 5$ follows from Lemma 2.3.6 and Lemma 2.3.5. Let G be a group of class 3 and $n > 5$. If there is a central subgroup of G/K of order p such that the factor group is of class 2, then the result follows from Lemma 2.3.7 and Lemma 2.3.5. Hence consider that for every central subgroup H/K of G/K of order p, the factor group of G/K by H/K is of class 3 again. Since $|G/K| = p^{n-1}$, by induction hypothesis on n there is no such G/K such that $|\mathrm{M}(G/K)|$ attains the bound. Hence by Lemma 2.3.5, result follows for class $c = 3$.

We have proved our result for $c = 3$. Now we use induction argument to complete the proof for $c > 3$. Let G be a p-group of order p^n and of class $c > 3$. If nilpotency class of G/K is smaller than c, then by induction hypothesis on c, there is no G/K such that $|\mathrm{M}(G/K)|$ attains the bound. Hence the result follows by Lemma 2.3.5. If nilpotency class of G/K is c for every central subgroup K

of order p, then we use induction on n to prove our result. For $n = c + 2$, G/K is of maximal class. So our result is true for $n = c + 2$, by Lemma 2.3.6 and Lemma 2.3.5. Let G be a p-group of order p^n of class c with $n > c + 2$. If there is a central subgroup of G/K of order p such that the factor group is of class smaller than c then the result follows by induction on c and by Lemma 2.3.5. So for every central subgroup of G/K of order p, the factor group of G/K is of class c again. As $|G/K| = p^{n-1}$, by induction hypothesis on n there is no such G/K such that $|\,\mathrm{M}(G/K)|$ attains the bound. Hence our result follows by Lemma 2.3.5. This completes the proof. □

2.3.1 Examples

We conclude by providing some examples of groups G of order p^n with $|G'| = p^k$ such that $|\,\mathrm{M}(G)| = p^{\frac{1}{2}(n+k-2)(n-k-1)}$, i.e., $|\,\mathrm{M}(G)|$ attains the improved bound.

Example 1: Consider the group from [27]

$$G = \langle \alpha, \alpha_i \mid [\alpha, \alpha_1] = \alpha_2, [\alpha_2, \alpha] = \alpha_3, [\alpha_2, \alpha_1] = \alpha_4, \alpha^p = \alpha_i^p = 1,\ 1 \leq i \leq 4\rangle.$$

This is group of order p^5 with $|G'| = p^3$. The nilpotency class of G is 3. For $p = 5, 7, 11, 13, 17$, using HAP of GAP we obtain

$$\mathrm{M}(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p.$$

Note that $|\,\mathrm{M}(G)| = p^{\frac{1}{2}(5+3-2)(5-3-1)} = p^3$, i.e., $|\,\mathrm{M}(G)|$ attains the improved bound.

Example 2: Consider the group from [27]

$$G = \langle \alpha, \alpha_i, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_1^{(p)} = \alpha_{i+1}^{(p)} = 1,\ 1 \leq i \leq 3\rangle.$$

This is group of order p^5 with $|G'| = p^3$. The nilpotency class of G is 4. For $p = 5, 7, 11, 13, 17$ using HAP of GAP we obtain

$$\mathrm{M}(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p.$$

Note that $|\,\mathrm{M}(G)| = p^{\frac{1}{2}(5+3-2)(5-3-1)} = p^3$, i.e., $|\,\mathrm{M}(G)|$ attains the improved bound.

CHAPTER 3

The Schur multiplier of central product of groups

Let G be a central product of two groups H and K. In this chapter we study second cohomology group of G, having coefficients in a divisible abelian group D with trivial G-action, in terms of the second cohomology groups of certain quotients of H and K.

3.1 Motivation

We start with the definitions of internal and external central product of groups. A group G is said to be an *internal central product* of its two normal subgroups H and K amalgamating A if $G = HK$ with $A = H \cap K$ and $[H, K] = 1$. Let H, K be two groups with isomorphic subgroups $A \leq \mathrm{Z}(H)$, $B \leq \mathrm{Z}(K)$ under an isomorphism $\phi : A \to B$. Consider the normal subgroup $U = \{(a, \phi(a)^{-1}) \mid a \in A\}$. Then the group $G := (H \times K)/U$ is called the *external central product* of H and K amalgamating A and B via ϕ. The external central product G can be viewed as an internal central product of the images of $H \times 1$ and $1 \times K$ in

G. For this reason, we do not differentiate between external and internal central products, and consider only internal ones in this chapter.

Let a finite group G be the direct product of two groups H and K. Then the formulation of the Schur multiplier of G in terms of the Schur multipliers of H and K was given by Schur himself, see Theorem 1.2.3. Such a formulation, when G is a semi-direct product of groups H and K, was given by Tahara, see Theorem 1.2.4.

Let G be a internal central product of H and K. Here we study $\mathrm{H}^2(G, D)$, in terms of the second cohomology groups of certain quotients of H and K with coefficients in D, where D is regarded as a trivial G-module, . In this chapter, by the tensor product $G_1 \otimes G_2$ of two groups G_1 and G_2, we always mean the abelian tensor product, i.e., $G_1/G_1' \otimes G_2/G_2'$.

The following result was proved by Wiegold.

Theorem 3.1.1 **([49])** *Let H, K be finite groups, let U, V be isomorphic central subgroups of H, K respectively, and let ϕ be an isomorphism from U onto V. Then the multiplicator of the central product G of H and K amalgamating U with V according to ϕ contains a subgroup isomorphic with $H/U \otimes K/V$.*

A generalization of this result for arbitrary central quotient of direct product of two arbitrary groups was considered in the following result which was proved by Eckmann, Hilton and Stammbach.

Theorem 3.1.2 **([8])** *Let W be central in $A = H \times K$ with quotient G. Let U and V be the images of W under the projection of A onto H and K respectively. Then $H/U \otimes K/V$ is a quotient of $H_2(G, \mathbb{Z})$.*

3.2 Second cohomology of central product of groups

Throughout this section, unless said otherwise explicitly, G is always a internal central product of its normal subgroups H and K with $A = H \cap K$. Set $Z = H' \cap K'$.

Let F/R be a free presentation of X and N be a normal subgroup of X. Let S/R be the induced free presentation of N for some subgroup S of F. The following crucial result then follows from [46, Corollary 3.5].

Lemma 3.2.1 *The inflation homomorphism* $\inf : \mathrm{H}^2(X/N, D) \to \mathrm{H}^2(X, D)$ *is surjective if and only if* $[F, R] = R \cap [F, S]$.

Theorem 3.2.2 *For any central subgroup* B *of* G *contained in* $H' \cap K'$ $(= Z)$, *the inflation homomorphism* $\inf : \mathrm{H}^2(G/B, D) \to \mathrm{H}^2(G, D)$ *is surjective.*

Proof. Let F/R be a free presentation of G. Then the normal subgroups H, K and B can be freely presented as S_1/R, S_2/R and S/R respectively, where S_1, S_2 and S are normal subgroups of F. Further, $Z \cong (S_1' \cap S_2')R/R$. Notice that $F = S_1S_2$, $S \subseteq (S_1' \cap S_2')R$ and $[S_1, S_2] \subseteq R$.

By Lemma 3.2.1, it is enough to prove that $[F, R] = R \cap [F, S]$. Since $[S_1, S, S_2] \subseteq [S_2, S]$, we have

$$\begin{aligned} R \cap [F, S] &= [F, S] = [S_1S_2, S] = [S_1, S][S_1, S, S_2][S_2, S] \\ &= [S_1, S][S_2, S]. \end{aligned}$$

Observe that $[S_1, S] \subseteq [S_1, S_2'R] = [S_1, R][S_1, S_2'][S_1, S_2', R]$. Since both $[S_1, S_2, S_2]$ and $[S_2, S_1, S_2]$ are contained in $[F, R]$, by three subgroup lemma $[S_1, S_2'] \subseteq [F, R]$. Hence $[S_1, S] \subseteq [F, R]$. Now $[S_2, S] \subseteq [S_2, S_1'R] = [S_2, R][S_2, S_1'][S_2, S_1', R]$. Again by Three subgroup lemma $[S_1', S_2] \subseteq [F, R]$. Hence $[S_2, S] \subseteq [F, R]$.

Therefore $R \cap [F,S] \subseteq [F,R]$. Since $[F,R] \subseteq R \cap [F,S]$, it follows that $[F,R] = R \cap [F,S]$, and the proof is complete. $\square$

Now the following observation provides a reduction to the case when $Z = 1$.

Theorem 3.2.3 ([20]) *Let B be a subgroup of G such that $B \leq Z$. Then*

$$\mathrm{H}^2(G,D) \cong \mathrm{H}^2(G/B,D)/N,$$

where $N \cong \mathrm{Hom}(B,D)$.

Proof. It follows from Theorem 3.2.2 that the inf homomorphism, in the following exact sequence, is surjective.

$$0 \to \mathrm{Hom}(B,D) \overset{\mathrm{tra}}{\to} \mathrm{H}^2(G/B,D) \overset{\mathrm{inf}}{\to} \mathrm{H}^2(G,D).$$

Since $\mathrm{Hom}(B,D) \cong \mathrm{Im}(\mathrm{tra}) = \mathrm{Ker}(\mathrm{inf})$, the proof is complete. $\square$

This result is very useful for computational purposes when G is finite and $\mathrm{M}(G/B)$ is known. Just to elaborate, we immediately get the following result for finite extraspecial p-groups proved in [3, Corollary 3.2].

Corollary 3.2.4 *Let G be an extraspecial p-group of order p^{2n+1}, $n \geq 2$. Then $\mathrm{M}(G)$ is an elementary abelian p-group of order p^{2n^2-n-1}.*

Let us consider the following central exact sequence for an arbitrary group X and its central subgroup N:

$$1 \to N \to X \to X/N \to 1.$$

Then we get the exact sequence

$$0 \to \mathrm{Hom}(N \cap X', D) \overset{\mathrm{tra}}{\to} \mathrm{H}^2(X/N,D) \overset{\mathrm{inf}}{\to} \mathrm{H}^2(X,D) \overset{\chi}{\to} \mathrm{H}^2(N,D) \oplus \mathrm{Hom}(X \otimes N, D),$$

where $\chi = (\mathrm{res}, \psi)$ is defined by Iwahori and Matsumoto [25]. To be more precise, $\mathrm{res} : \mathrm{H}^2(X, D) \to \mathrm{H}^2(N, D)$ is the restriction homomorphism and $\psi : \mathrm{H}^2(X, D) \to \mathrm{Hom}(X \otimes N, D)$ is defined as $\psi(\xi)(\bar{x}, n) = f(x, n) - f(n, x)$ for all $\bar{x} = xX' \in X/X'$ and $n \in N$, where $\xi \in \mathrm{H}^2(X, D)$ and f is a 2-cocycle representative of ξ.

Define a map

$$\theta' : \mathrm{H}^2(G, D) \to \mathrm{H}^2(H, D) \oplus \mathrm{H}^2(K, D) \oplus \mathrm{Hom}(H \otimes K, D) \tag{3.2.1}$$

by $\theta' = (\mathrm{res}^G_H, \mathrm{res}^G_K, \nu)$, where $\nu : \mathrm{H}^2(G, D) \to \mathrm{Hom}(H \otimes K, D)$ is a homomorphism defined as follows. If $\xi \in \mathrm{H}^2(G, D)$ is represented by a 2-cocycle f, then $\nu(\xi)$ is the homomorphism $\bar{f} \in \mathrm{Hom}(H \otimes K, D)$ defined by $\bar{f}(\bar{h} \otimes \bar{k}) = f(h, k) - f(k, h)$, where $\bar{h} = hH'$ and $\bar{k} = kK'$. It is now not difficult to see that θ' is indeed a homomorphism.

Consider the natural homomorphisms $\alpha : AH'/H' \otimes K \to H \otimes K$, $\beta : H \otimes AK'/K' \to H \otimes K$ and $\lambda : H \otimes K \to H/A \otimes K/A$. We now get the following exact sequence:

$$(AH'/H' \otimes K) \oplus (H \otimes AK'/K') \xrightarrow{\mu_1} H \otimes K \xrightarrow{\lambda} H/A \otimes K/A \to 0,$$

where $\mu_1(x, y) = \alpha(x) + \beta(y)$.

We have natural epimorphisms $f : H \otimes A \to H \otimes AK'/K'$ and $g : A \otimes K \to AH'/H' \otimes K$. Consider the isomorphism $\eta : K \otimes A \to A \otimes K$, which, on the generators, is defined by $\eta(k \otimes a) = -(a \otimes k)$. Using this, we have an epimorphism $(f, g \circ \eta) : (H \otimes A) \oplus (K \otimes A) \to (H \otimes AK'/K') \oplus (AH'/H' \otimes K)$. Let $\mu = \mu_1 \circ (f, g \circ \eta)$. Then $\mathrm{Im}(\mu_1) = \mathrm{Im}(\mu)$ and the above exact sequence leads to the exact sequence:

$$(H \otimes A) \oplus (K \otimes A) \xrightarrow{\mu} H \otimes K \xrightarrow{\lambda} H/A \otimes K/A \to 0.$$

This exact sequence then gives the exact sequence

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathrm{Hom}(H/A \otimes K/A, D) & \xrightarrow{\lambda^*} & \mathrm{Hom}(H \otimes K, D) \\ & & & & \downarrow{\scriptstyle \mu^*} \\ & & & & \mathrm{Hom}(H \otimes A, D) \oplus \mathrm{Hom}(K \otimes A, D), \end{array}$$

where the homomorphisms μ^* and λ^* are induced by μ and λ respectively.

Let $\alpha : H/H' \oplus K/K' \to G/G'$ be the homomorphism induced by the inclusion maps $H \hookrightarrow G, K \hookrightarrow G$ which is clearly onto. Then α induces an epimorphism $(H/H' \oplus K/K') \otimes A \to G \otimes A$, which in turn induces a monomorphism $\alpha^* : \mathrm{Hom}(G \otimes A, D) \to \mathrm{Hom}(H \otimes A, D) \oplus \mathrm{Hom}(K \otimes A, D)$. Let $\triangle : H^2(A, D) \to H^2(A, D) \oplus H^2(A, D)$ be defined by $\triangle(\xi) = (\xi, \xi)$ for $\xi \in H^2(A, D)$. Set $\bar{G} = G/A$, $\bar{H} = H/A$ and $\bar{K} = K/A$. Let $\xi \in \mathrm{H}^2(\bar{G}, D)$ and f be a 2-cocycle representing ξ. Recall that

$$\theta : \mathrm{H}^2(\bar{G}, D) \to \mathrm{H}^2(\bar{H}, D) \oplus \mathrm{H}^2(\bar{K}, D) \oplus \mathrm{Hom}(\bar{H} \otimes \bar{K}, D)$$

is an isomorphism defined by

$$\theta(\xi) = (\mathrm{res}^{\bar{G}}_{\bar{H}}, \mathrm{res}^{\bar{G}}_{\bar{K}}, \nu_1),$$

where $\nu_1 : \mathrm{H}^2(G, D) \to \mathrm{Hom}(H \otimes K, D)$ is a homomorphism given by $\nu_1(\xi)(\tilde{h} \otimes \tilde{k}) = f(h, k) - f(k, h)$, with $\tilde{h} = h\bar{H}'$ and $\tilde{k} = k\bar{K}'$. Take $X_1 = \mathrm{H}^2(A, D) \oplus \mathrm{Hom}(H \otimes A, D)$, $X_2 = \mathrm{H}^2(A, D) \oplus \mathrm{Hom}(K \otimes A, D)$, $X_3 = \mathrm{Hom}(H \otimes A, D) \oplus$

$\mathrm{Hom}(K \otimes A, D)$ and $Y = \mathrm{H}^2(A, D) \oplus \mathrm{H}^2(A, D)$. We now get the following diagram with exact columns, in which, for want of space, we suppress the use of D, i.e., we write $\mathrm{Hom}(X, D)$ as $\mathrm{Hom}(X)$ and $\mathrm{H}^2(X, D)$ as $\mathrm{H}^2(X)$ for a given group X.

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathrm{Hom}(A \cap G') \arrow[r, "(\mathrm{res},\mathrm{res})"] \arrow[d, "\mathrm{tra}"'] & \mathrm{Hom}(A \cap H') \oplus \mathrm{Hom}(A \cap K') \arrow[d, "(\mathrm{tra},\mathrm{tra},0)"] \\
\mathrm{H}^2(G/A) \arrow[r, "\theta"] \arrow[d, "\mathrm{inf}"'] & \mathrm{H}^2(H/A) \oplus \mathrm{H}^2(K/A) \oplus \mathrm{Hom}(H/A \otimes K/A) \arrow[d, "(\mathrm{inf},\mathrm{inf},\lambda^*)"] \\
\mathrm{H}^2(G) \arrow[r, "\theta'"] \arrow[d, "(\mathrm{res},\psi)"'] & \mathrm{H}^2(H) \oplus \mathrm{H}^2(K) \oplus \mathrm{Hom}(H \otimes K) \arrow[d, "\big((\mathrm{res},\psi),(\mathrm{res},\psi),\mu^*\big)"] \\
\mathrm{H}^2(A) \oplus \mathrm{Hom}(G \otimes A) \arrow[dr, "(\triangle,\alpha^*,\alpha^*)"'] & X_1 \oplus X_2 \oplus X_3 \arrow[d, "\cong"] \\
 & Y \oplus X_3 \oplus X_3.
\end{tikzcd}
$$

Diagram 1

Lemma 3.2.5 *Diagram 1 is commutative.*

Proof. It is a routine check to see that the topmost and middle rectangles are commutative. Observe that $\mathrm{res}_A^H \circ \mathrm{res}_H^G = \mathrm{res}_A^G$ and $\mathrm{res}_A^K \circ \mathrm{res}_K^G = \mathrm{res}_A^G$. It is also clear from the definitions that $(\alpha^*, \alpha^*) \circ \psi = (\psi, \psi, \mu^*) \circ \theta'$. Thus it follows that the bottom part of the diagram is also commutative. $\square$

We now mainly concentrate on the homomorphism θ' defined in (3.2.1). We start with the following result about the kernel of θ'.

Lemma 3.2.6 $\mathrm{Ker}(\theta') = \{\mathrm{inf}(\eta) \mid \eta \in \theta^{-1}\big(\mathrm{Im}(\mathrm{tra}, \mathrm{tra}, 0)\big)\}$.

Proof. Let $\xi \in \ker(\theta')$. By the commutativity of the bottommost part of Diagram 1, it follows that $(\triangle, \alpha^*, \alpha^*)(\mathrm{res}^G_A, \psi)(\xi) = 0$. Since α^* is a monomorphism and $\triangle = (\mathrm{Id}, \mathrm{Id})$, it follows that $(\mathrm{res}^G_A, \psi)(\xi) = 0$. Now the existence of $\eta \in \mathrm{H}^2(G/A, D)$ such that $\xi = \inf(\eta)$ is guaranteed by the exactness of the left column in Diagram 1. Thus $\mathrm{Ker}(\theta') \subseteq \mathrm{Im}\big(\inf : \mathrm{H}^2(G/A, D) \to \mathrm{H}^2(G, D)\big)$.

By the commutativity of the middle rectangle of Diagram 1, it follows that

$$0 = \theta'(\xi) = \theta'(\inf(\eta)) = \theta' \circ \inf(\eta) = (\inf, \inf, \lambda^*) \circ \theta(\eta).$$

Again invoking Diagram 1, we get $\theta(\eta) \in \mathrm{Im}(\mathrm{tra}, \mathrm{tra}, 0)$. Hence $\eta \in \theta^{-1}(\mathrm{Im}(\mathrm{tra}, \mathrm{tra}, 0))$. That $\theta'(\inf(\eta)) = 0$ for $\eta \in \theta^{-1}\big(\mathrm{Im}(\mathrm{tra}, \mathrm{tra}, 0)\big)$ follows from the commutativity of Diagram 1 with the right column exact. This completes the proof. $\square$

We have an exact sequence

$$0 \to H' \cap K' \xrightarrow{\alpha_1} (A \cap H') \oplus (A \cap K') \xrightarrow{\alpha_2} A \cap G' \to 0,$$

where $\alpha_1(z) = (z, z^{-1})$ and $\alpha_2(z_1, z_2) = z_1 z_2$ for $z \in H' \cap K', z_1 \in A \cap H'$ and $z_2 \in A \cap K'$. This sequence induces the following exact sequence

$$0 \to \mathrm{Hom}(A \cap G', D) \xrightarrow{\alpha_2^*} \mathrm{Hom}(A \cap H', D) \oplus \mathrm{Hom}(A \cap K', D) \xrightarrow{\alpha_1^*} \mathrm{Hom}(Z, D) \to 0,$$

in which α_2^* is the homomorphism $(\mathrm{res}, \mathrm{res})$.

The homomorphism α_1^* being surjective, for any $f \in \mathrm{Hom}(Z, D)$, there exists $g \in \mathrm{Hom}(A \cap H', D) \oplus \mathrm{Hom}(A \cap K', D)$ such that $f = \alpha_1^*(g)$. If $g_1 \in \mathrm{Hom}(A \cap H', D) \oplus \mathrm{Hom}(A \cap K', D)$ is another element such that $f = \alpha_1^*(g_1)$, then there exists $\nu \in \mathrm{Hom}(A \cap G', D)$ such that $g - g_1 = \alpha_2^*(\nu)$. For the convenience of writing, set $\zeta = \inf \circ \theta^{-1}$ (recall that θ is an isomorphism). Now, using the commutativity of the topmost rectangle of Diagram 1, we get

$$\begin{aligned}\zeta \circ (\mathrm{tra}, \mathrm{tra}, 0)(g) &= \zeta \circ (\mathrm{tra}, \mathrm{tra}, 0)(g_1) + \zeta \circ (\mathrm{tra}, \mathrm{tra}, 0)(\alpha_2^*(\nu)) \\ &= \zeta \circ (\mathrm{tra}, \mathrm{tra}, 0)(g_1) + \zeta \circ \theta \circ \mathrm{tra}(\nu) \\ &= \zeta \circ (\mathrm{tra}, \mathrm{tra}, 0)(g_1).\end{aligned}$$

Hence $\zeta \circ (\mathrm{tra}, \mathrm{tra}, 0)(g)$ is independent of the choice of $g \in \mathrm{Hom}(A \cap H', D) \oplus \mathrm{Hom}(A \cap K', D)$ with $\alpha_1^*(g) = f$. Setting $\chi(f) = \zeta \circ (\mathrm{tra}, \mathrm{tra}, 0)(g)$, we get a well defined map $\chi : \mathrm{Hom}(Z, D) \to \mathrm{H}^2(G, D)$. It is now clear that χ is a homomorphism.

Theorem 3.2.7 *The following sequence is exact:*

$$0 \to \mathrm{Hom}(Z, D) \xrightarrow{\chi} \mathrm{H}^2(G, D) \xrightarrow{\theta'} \mathrm{H}^2(H, D) \oplus \mathrm{H}^2(K, D) \oplus \mathrm{Hom}(H \otimes K, D).$$

Proof. Suppose that $f \in \mathrm{Hom}(Z, D)$ and $\chi(f) = 0$. Then $\inf \circ \theta^{-1} \circ (\mathrm{tra}, \mathrm{tra}, 0)(g) = 0$ for some $g \in \mathrm{Hom}(A \cap H', D) \oplus \mathrm{Hom}(A \cap K', D)$ such that $f = \alpha_1^*(g)$. Thus there exists $\eta \in \mathrm{Hom}(A \cap G', D)$ such that $\theta^{-1} \circ (\mathrm{tra}, \mathrm{tra}, 0)(g) = \mathrm{tra}(\eta)$ by the commutativity of Diagram 1. Then $(\mathrm{tra}, \mathrm{tra}, 0)(g) = \theta \circ \mathrm{tra}(\eta) = (\mathrm{tra}, \mathrm{tra}, 0) \circ (\mathrm{res}, \mathrm{res})(\eta) = (\mathrm{tra}, \mathrm{tra}, 0) \circ \alpha_2^*(\eta)$. Since $(\mathrm{tra}, \mathrm{tra}, 0)$ is a monomorphism, we have $g = \alpha_2^*(\eta)$. Thus $f = \alpha_1^* \circ \alpha_2^*(\eta) = 0$, which, f being an arbitrary element, proves that χ is a monomorphism. That $\mathrm{Im}(\chi) = \mathrm{Ker}(\theta')$ is now clear from Lemma 3.2.6, and the proof is complete. □

The following is an immediate consequence of the preceding theorem.

Corollary 3.2.8 *If $Z = 1$, then*

$$\theta' : \mathrm{H}^2(G, D) \to \mathrm{H}^2(H, D) \oplus \mathrm{H}^2(K, D) \oplus \mathrm{Hom}(H \otimes K, D)$$

is a monomorphism.

Using commutativity of the middle rectangle of Diagram 1, note that

$$\inf\big(\theta^{-1}(\mathrm{Hom}(H/A \otimes K/A))\big) \cap \ker(\theta') = \{0\}.$$

Hence by Theorem 3.2.7, we get

Corollary 3.2.9 $\mathrm{Hom}(Z, D) \oplus \mathrm{Hom}(H/A \otimes K/A)$ *embeds in* $\mathrm{H}^2(G, D)$.

As we know by Theorem 3.2.3 that $\mathrm{Hom}(Z, D)$ embeds in $\mathrm{H}^2(G/Z, D)$. We now prove a much stronger result in the following

Theorem 3.2.10 $\mathrm{Hom}(Z, D)$ *embeds in* $\mathrm{H}^2(H/A, D)/L \oplus \mathrm{H}^2(K/A, D)/M$*, where* $L \cong \mathrm{Hom}\big((A \cap H')/Z, D\big)$ *and* $M \cong \mathrm{Hom}\big((A \cap K')/Z, D\big)$.

Proof. Let $\alpha : \mathrm{Hom}(A \cap G', D) \to \mathrm{Hom}(Z, D)$ be the epimorphism induced by the inclusion $Z \hookrightarrow A \cap G'$. Set $Y_1 = \mathrm{Im}\big(\inf : \mathrm{H}^2(G/A, D) \to \mathrm{H}^2(G/Z, D)\big)$. Since G/Z is a central product of H/Z and K/Z with $(H/Z)' \cap (K/Z)' = 1$, it follows that Y_1 is isomorphic to $\mathrm{H}^2(H/A, D)/L \oplus \mathrm{H}^2(K/A, D)/M \oplus \mathrm{Hom}(H/A \otimes K/A, D)$, where $L \cong \mathrm{Hom}\big((A \cap H')/Z, D\big)$ and $M \cong \mathrm{Hom}\big((A \cap K')/Z, D\big)$.

Consider the following commutative diagram (with rows not necessarily exact):

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}(A \cap G', D) & \xrightarrow{\mathrm{tra}} & \mathrm{H}^2(G/A, D) & \xrightarrow{\theta} & X \\
 & & \Big\downarrow \alpha & & \Big\downarrow \inf & & \Big\downarrow (p_1, p_2, 1) \\
0 & \longrightarrow & \mathrm{Hom}(Z, D) & \xrightarrow[\mathrm{tra}]{} & Y_1 & \xrightarrow[\bar{\theta}]{} & Y,
\end{array}$$

where

$$X = \mathrm{H}^2(H/A, D) \oplus \mathrm{H}^2(K/A, D) \oplus \mathrm{Hom}(H/A \otimes K/A, D),$$

$$Y = \mathrm{H}^2(H/A, D)/L \oplus \mathrm{H}^2(K/A, D)/M \oplus \mathrm{Hom}(H/A \otimes K/A, D),$$

$\bar{\theta}$ is an isomorphism and p_i, $i = 1, 2$, are natural projections.

Let $\beta \in \mathrm{Hom}(Z, D)$. Then there exists $\bar{\beta} \in \mathrm{Hom}(A \cap G', D)$ such that $\beta = \alpha(\bar{\beta})$. Let $\mathrm{tra}(\bar{\beta}) = \xi \in \mathrm{H}^2(G/A, D)$. The element ξ is represented by a 2-cocycle f given by

$$f(\bar{x}, \bar{y}) = \bar{\beta}(\mu(\bar{x})\mu(\bar{y})\mu(\bar{x}\bar{y})^{-1}), \ \ \bar{x} = xA \in G/A \ \text{ and } \ \bar{y} = yA \in G/A,$$

where μ represents the section $\mu : G/A \to G$ in the exact sequence $1 \to A \to G \to G/A \to 1$.

Recall that $\theta = (\mathrm{res}, \mathrm{res}, \nu)$, where $\nu(\xi)$ for $\bar{h} = hA \in H/A$ and $\bar{k} = kA \in K/A$ is given by

$$\nu(\xi)(\bar{h}, \bar{k}) = f(\bar{h}, \bar{k}) - f(\bar{k}, \bar{h}).$$

Plugging in the value of f we have

$$\begin{aligned}
\nu(\xi)(\bar{h}, \bar{k}) &= f(\bar{h}, \bar{k}) - f(\bar{k}, \bar{h}) \\
&= \bar{\beta}\big(\mu(\bar{h})\mu(\bar{k})\mu(\bar{h}\bar{k})^{-1}\big) - \bar{\beta}\big(\mu(\bar{k})\mu(\bar{h})\mu(\bar{k}\bar{h})^{-1}\big) \\
&= \bar{\beta}\big(\mu(\bar{h})\mu(\bar{k})\mu(\bar{h}\bar{k})^{-1}\mu(\bar{k}\bar{h})\mu(\bar{h})^{-1}\mu(\bar{k})^{-1}\big) \\
&= \bar{\beta}\big(\mu(\bar{h})\mu(\bar{k})\mu(\bar{h}\bar{k})^{-1}\mu(\bar{h}\bar{k})\mu(\bar{k})^{-1}\mu(\bar{h})^{-1}\big) \\
&= 0.
\end{aligned}$$

Hence $\theta(\mathrm{tra}(\bar{\beta})) \in \mathrm{H}^2(H/A, D) \oplus \mathrm{H}^2(K/A, D)$. That $\bar{\theta}(\mathrm{tra}(\beta)) \in \mathrm{H}^2(H/A, D)/L \oplus \mathrm{H}^2(K/A, D)/M$ now follows by the commutativity of the above diagram, which completes the proof. $\square$

Using an argument similar to one as in the preceding proof, we can also prove

Theorem 3.2.11 $\mathrm{Hom}(Z, D)$ *embeds in* $\mathrm{H}^2(H/Z, D) \oplus \mathrm{H}^2(K/Z, D)$.

The following is now an immediate consequence of Theorem 3.2.3 and the preceding theorem.

Corollary 3.2.12 *If $A = Z$, then*

$$\mathrm{H}^2(G, D) \cong \big(\mathrm{H}^2(H/Z, D) \oplus \mathrm{H}^2(K/Z, D)\big)/\operatorname{Hom}(Z, D) \oplus \operatorname{Hom}(H/Z \otimes K/Z, D).$$

Our next result is the following:

Theorem 3.2.13 ([20]) *Let $L \cong \operatorname{Hom}\big((A \cap H')/Z, D\big)$, $M \cong \operatorname{Hom}\big((A \cap K')/Z, D\big)$ and $N \cong Hom(Z, D)$. Then the following statements hold true:*

(i) $\big(\mathrm{H}^2(H/A, D)/L \oplus \mathrm{H}^2(K/A, D)/M\big)/N \oplus \operatorname{Hom}(H/A \otimes K/A, D)$ embeds in $\mathrm{H}^2(G, D)$.

(ii) $\mathrm{H}^2(G, D)$ embeds in $\big(\mathrm{H}^2(H/Z, D) \oplus \mathrm{H}^2(K/Z, D)\big)/N \oplus \operatorname{Hom}(H \otimes K, D)$.

Proof. We already observed that $\operatorname{Im}(\inf : \mathrm{H}^2(G/A, D) \to \mathrm{H}^2(G/Z, D))$ is isomorphic to

$$\mathrm{H}^2(H/A, D)/L \oplus \mathrm{H}^2(K/A, D)/M \oplus \operatorname{Hom}(H/A \otimes K/A, D).$$

The first assertion now follows from Theorem 3.2.3 using Theorem 3.2.10.

By Corollary 3.2.8,

$$\theta' : \mathrm{H}^2(G/Z, D) \to \mathrm{H}^2(H/Z, D) \oplus \mathrm{H}^2(K/Z, D) \oplus \operatorname{Hom}(H \otimes K, D)$$

is a monomorphism. Now the second assertion follows from Theorem 3.2.3 using Theorem 3.2.11. □

In particular, for $D = \mathbb{C}^\times$, assertion (i) of Theorem 3.2.13 provides a refinement of results of Theorem 3.1.2 and Theorem 3.1.1.

Remark 3.2.14

$$\mathrm{H}^2(G, D) \cong (\mathrm{H}^2(H/A, D)/L \oplus \mathrm{H}^2(K/A, D)/M)/N \oplus \mathrm{Hom}(H/A \otimes K/A, D)$$

if and only if $\inf : \mathrm{H}^2(G/A, D) \to \mathrm{H}^2(G/Z, D)$ *is an epimorphism, where* L, M *and* N *are as defined above.*

Proposition 3.2.15 *Let* $\xi \in \mathrm{H}^2(\bar{G}, D)$ *such that* $\theta'(\xi) = (\xi_1, \xi_2, t)$*, where* $\bar{G} = G/Z$*. Further, let either* $\mathrm{res}_{\bar{A}}^{\bar{H}}(\xi_1) = 0$ *or* $\mathrm{res}_{\bar{A}}^{\bar{K}}(\xi_2) = 0$*. Then the following statements are equivalent:*

(i) $\xi \in \mathrm{Im}\big(\inf : \mathrm{H}^2(\bar{G}/\bar{A}, D) \to \mathrm{H}^2(\bar{G}, D)\big)$.

(ii) $\mu^*(t) = 0$.

(iii) $\psi(\xi_1) = \psi(\xi_2) = 0$.

Proof. Consider Diagram 1 for the group $\bar{G} := G/Z$, which is a central product of $\bar{H}$ and $\bar{K}$. Observe that $\xi \in \mathrm{Im}\big(\inf : \mathrm{H}^2(\bar{G}/\bar{A}, D) \to \mathrm{H}^2(\bar{G}, D)\big)$ if and only if $(\mathrm{res}_{\bar{A}}^{\bar{G}}, \psi)(\xi) = 0$. Note that $(\psi(\xi_1), \psi(\xi_2)) = \mu^*(t)$. As $\mathrm{res}_{\bar{A}}^{\bar{H}}(\xi_1) = 0$ or $\mathrm{res}_{\bar{A}}^{\bar{K}}(\xi_2) = 0$, it follows by the commutativity of the bottommost part of Diagram 1, that $(\mathrm{res}_{\bar{A}}^{\bar{G}}, \psi)(\xi) = 0$ if and only if $\psi(\xi_1) = \psi(\xi_2) = 0$. Hence the result follows. $\square$

Corollary 3.2.16 *If* $\inf : \mathrm{H}^2(H/A, D) \to \mathrm{H}^2(H/Z, D)$ *and* $\inf : \mathrm{H}^2(K/A, D) \to \mathrm{H}^2(K/Z, D)$ *are epimorphisms, then*

$$H^2(G, D) \cong \big(H^2(H/Z, D) \oplus H^2(K/Z, D)\big)/\mathrm{Hom}(Z, D) \oplus \mathrm{Hom}(H/A \otimes K/A, D).$$

More precisely, the first embedding in Theorem 3.2.13 is an isomorphism.

In view of Proposition 3.2.15, we have the following result:

Theorem 3.2.17 *If the second embedding in Theorem 3.2.13 is an isomorphism, then so is the first.*

Proof. Since the isomorphism

$$\mathrm{H}^2(G,D) \cong \big(\mathrm{H}^2(H/Z,D) \oplus \mathrm{H}^2(K/Z,D)\big)/\operatorname{Hom}(Z,D) \oplus \operatorname{Hom}(H/Z \otimes K/Z, D)$$

is induced by the monomorphism θ' as defined in (3.2.1) with G replaced by G/Z, it follows from the commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] \\
\operatorname{Hom}(Z,D) \arrow[r, "(\mathrm{Id},\mathrm{Id})"] \arrow[d, "\mathrm{tra}"'] & & \operatorname{Hom}(Z,D) \oplus \operatorname{Hom}(Z,D) \arrow[d, "(\mathrm{tra},\mathrm{tra},0)"] \\
\mathrm{H}^2(G/Z,D) \arrow[r, "\theta'"] & & \mathrm{H}^2(H/Z,D) \oplus \mathrm{H}^2(K/Z,D) \oplus \operatorname{Hom}(H/Z \otimes K/Z, D).
\end{tikzcd}$$

that θ' is an isomorphism.

Let $t \in \operatorname{Hom}(H/Z \otimes K/Z, D)$. Then there exists $\xi \in \mathrm{H}^2(G/Z,D)$ such that $\theta'(\xi) = (0,0,t)$. It then follows from Diagram 1 (for G/Z in place of G) that $(\mathrm{res},\psi)(\xi) = 0$. By Proposition 3.2.15 we then have $\mu^*(t) = 0$, which shows that $\lambda^* : \operatorname{Hom}(H/A \otimes K/A, D) \to \operatorname{Hom}(H/Z \otimes K/Z, D)$ is an epimorphism, and hence, an isomorphism.

Let $\xi_1 \in \mathrm{H}^2(H/Z,D)$. Then there exists $\xi \in \mathrm{H}^2(G/Z,D)$ such that $\theta'(\xi) = (\xi_1, 0, 0)$. Replacing G by G/Z in Diagram 1, by Proposition 3.2.15 it follows that $(\mathrm{res},\psi)(\xi_1) = 0$. Hence $\inf : \mathrm{H}^2(H/A,D) \to \mathrm{H}^2(H/Z,D)$ is an epimorphism, and therefore $\mathrm{H}^2(H/A,D)/\operatorname{Hom}((A \cap H')/Z, D) \cong \mathrm{H}^2(H/Z,D)$.

Similarly, considering an element $\xi_2 \in \mathrm{H}^2(K/Z,D)$ the above argument also shows that $\inf : \mathrm{H}^2(K/A,D) \to \mathrm{H}^2(K/Z,D)$ is an epimorphism. Hence $\mathrm{H}^2(K/A,D)/\operatorname{Hom}((A \cap K')/Z, D) \cong \mathrm{H}^2(K/Z,D)$. It now follows that the first embedding in Theorem 3.2.13 is an isomorphism. □

We conclude this section with the following remark made by J. Wiegold while reviewing [8] for AMS (see MR0349854 (50 #2347)). Let G be the direct product $G = H \oplus K$ of its normal subgroups H and K, and U be an arbitrary central subgroup of G. Then G/U can be viewed as a central product of HU/U and KU/U. Thus all the above results make sense for $\mathrm{H}^2(G/U, D)$.

3.3 Examples

In this section we provide several examples (all of them are finite p-groups) exhibiting various situations of Theorem 3.2.13 can occur i.e., whether or not any embedding in Theorem 3.2.13 actually become isomorphism.

We start with the following example which shows that neither of the two embeddings of Theorem 3.2.13 is necessarily an isomorphism.

Example 1. Let H be the extraspecial p-groups of order p^3 and exponent p and $K = \mathbb{Z}_p^{(n+1)}$, where $n \geq 1$. Let G be a central product of H and K amalgamated at $A \cong H' \cong \mathbb{Z}_p$. Note that $G = H \times \mathbb{Z}_p^{(n)}$. It is easy to see by Theorem 1.2.3 and Theorem 1.4.1 that

$$\mathrm{M}(G) \cong \mathbb{Z}_p^{\left(\frac{1}{2}n(n+3)+2\right)}.$$

Note that $Z = H' \cap K' = 1$ and

$$\mathrm{M}(H/A)/\operatorname{Hom}(A\cap H', \mathbb{C}^\times) \oplus \mathrm{M}(K/A)/\operatorname{Hom}(A\cap K', \mathbb{C}^\times) \oplus \operatorname{Hom}(H/A \otimes K/A, \mathbb{C}^\times)$$

is isomorphic to $\mathbb{Z}_p^{\left(\frac{1}{2}n(n+3)\right)}$, which is strictly contained in $\mathrm{M}(G)$. Since

$$\mathrm{M}(H) \oplus \mathrm{M}(K) \oplus \operatorname{Hom}(H \otimes K, \mathbb{C}^\times) \cong \mathbb{Z}_p^{(\frac{1}{2}(n+1)(n+4)+2)},$$

it properly contains $\mathrm{M}(G)$.

The following two examples show that the first embedding in Theorem 3.2.13 can very well be an isomorphism, but the second one can still be strict (i.e., not an isomorphism).

Example 2. Consider the group G presented as

$$G = \langle \alpha, \alpha_1, \alpha_2, \gamma \mid [\alpha_1, \alpha] = \gamma^{p^2} = \alpha_2, \alpha^p = \alpha_1^p = \alpha_2^p = 1 \rangle.$$

Take $H = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2, \alpha^p = \alpha_1^p = \alpha_2^p = 1 \rangle$ and $K = \langle \gamma \rangle \cong \mathbb{Z}_{p^3}$. It can be easily seen that G is a central product of H and K amalgamated at $A \cong \langle \alpha_2 \rangle \cong \left\langle \gamma^{p^2} \right\rangle$. Note that $Z = 1$ and

$$\mathrm{M}(H/A)/\operatorname{Hom}(A \cap H', \mathbb{C}^\times) \oplus \mathrm{M}(K/A)/\operatorname{Hom}(A \cap K', \mathbb{C}^\times) \oplus \operatorname{Hom}(H/A \otimes K/A, \mathbb{C}^\times)$$

is isomorphic to $\mathbb{Z}_p^{(2)}$. By Theorem 1.2.6, we have $|\,\mathrm{M}(G)| \leq p^2$. Hence $\mathrm{M}(G) \cong \mathbb{Z}_p^{(2)}$, and therefore the first embedding in Theorem 3.2.13 is an isomorphism. It is easy to see that

$$\mathrm{M}(H) \oplus \mathrm{M}(K) \oplus \operatorname{Hom}(H \otimes K, \mathbb{C}^\times) \cong \mathbb{Z}_p^{(4)},$$

which shows that the second embedding is strict.

Example 3. For $p \geq 5$, consider the group G presented as

$$G = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \gamma \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \gamma^p = \alpha_3, \alpha^p = \alpha_i^{(p)} = 1, i = 1, 2, 3 \rangle.$$

Take $H = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha_3, \alpha^p = \alpha_i^{(p)} = 1, i = 1, 2, 3 \rangle$ and $K = \langle \gamma \rangle \cong \mathbb{Z}_{p^2}$. Then G is a central product of H and K amalgamated at

$A \cong \langle \alpha_3 \rangle \cong \langle \gamma^p \rangle$ and $Z = 1$. Note that

$$M(H/A)/\operatorname{Hom}(A \cap H', \mathbb{C}^\times) \oplus M(K/A)/\operatorname{Hom}(A \cap K', \mathbb{C}^\times) \oplus \operatorname{Hom}(H/A \otimes K/A, \mathbb{C}^\times)$$

is isomorphic to $\mathbb{Z}_p^{(3)}$, which embeds in $\mathrm{M}(G)$. Again by Theorem 1.2.6, we have $|\,\mathrm{M}(G)| \leq p^3$. Hence the first embedding is an isomorphism. That the second one is not, can be easily seen as in Example 2.

We finally present an example which shows that both the embeddings in Theorem 3.2.13 can be isomorphisms.

Example 4. Let H be the extraspecial p-groups of order p^3 and exponent p^2 and $K \cong \mathbb{Z}_{p^{n+1}}$, the cyclic group of order p^{n+1}, where $n \geq 1$. Let G be a central product of H and K amalgamated at $A \cong H' \cong \mathbb{Z}_p$. Note that $G = H \times \mathbb{Z}_{p^n}$. By Theorem 1.2.3 and Theorem 1.4.1, $\mathrm{M}(G) \cong \mathbb{Z}_p^{(2)}$. Note that $Z = H' \cap K' = 1$ and

$$\mathrm{M}(H/A)/\operatorname{Hom}(A \cap H', \mathbb{C}^\times) \oplus \mathrm{M}(K/A)/\operatorname{Hom}(A \cap K', \mathbb{C}^\times) \oplus \operatorname{Hom}(H/A \otimes K/A, \mathbb{C}^\times)$$

is isomorphic to $\mathbb{Z}_p^{(2)}$. Also

$$\mathrm{M}(H) \oplus \mathrm{M}(K) \oplus \operatorname{Hom}(H \otimes K, \mathbb{C}^\times) \cong \mathbb{Z}_p^{(2)}.$$

Hence both the embeddings are isomorphisms. $\square$

CHAPTER 4

The Schur multipliers of p-groups of order p^5

In this chapter we compute the Schur multiplier, non-abelian tensor square and non-abelian exterior square of non-abelian p-groups of order p^5. As an application we determine the capability of these groups.

4.1 The Schur multiplier and tensor square

In this section the Schur multiplier , non-abelian exterior square and non-abelian tensor square of groups G of order p^5, $p \geq 5$ are computed. Recall the classification of these groups from Section 1.1.3. Throughout this section, $p \geq 5$, unless stated otherwise and we make calculations in the subgroup $[G, G^{\phi}]$ of $\nu(G)$ modulo $\nabla(G)$, i.e., we work in $G \wedge G$. For commutator and power calculations, we use Lemma 1.3.2.

In Section 1.3 we had seen that $\mathrm{M}(G) \cong \mathrm{Ker}(G \wedge G \to G')$. Hence $|G \wedge G| = |\mathrm{M}(G)||G'|$, which will be used several times throughout this section without any further reference.

4.1.1 Classes Φ_4, Φ_5: Groups of class 2 with G/G' elementary abelian

This section deals with special p-groups of order p^5. First we consider extraspecial p-groups of order p^5, for which we have the following result.

Lemma 4.1.1 *If G is one of the groups $\Phi_5(2111)$ or $\Phi_5(1^5)$, then $\mathrm{M}(G)$ is isomorphic to $\mathbb{Z}_p^{(5)}$ and $G \wedge G$ is isomorphic to $\mathbb{Z}_p^{(6)}$.*

Proof. The groups G are extra-special groups. So it follows from [30, Theorem 3.3.6(i)] that $\mathrm{M}(G)$ is an elementary abelian p-group of order p^5. By [31, Corollary 2.3], we have

$$G \otimes G \cong \mathbb{Z}_p^{(16)}.$$

Now by Theorem 1.3.7,

$$G \wedge G \cong \mathbb{Z}_p^{(6)},$$

which completes the proof. □

The following remark will be used to describe the structure of $G \wedge G$ for groups G in the isoclinism class Φ_4.

Remark 4.1.2 *Let G be any group in the isoclinism class Φ_4. Consider the natural epimorphism*

$$[G, G^\phi] \to [G/\,\mathrm{Z}(G), (G/\,\mathrm{Z}(G))^\phi].$$

Since $G/\,\mathrm{Z}(G) \wedge G/\,\mathrm{Z}(G)$ is elementary abelian of order p^3, it follows that the elements $[\alpha_1, \alpha^\phi], [\alpha_2, \alpha^\phi], [\alpha_2, \alpha_1^\phi]$ are non-trivial and independent in $G \wedge G$. Furthermore, by Lemma 1.3.6, $G \otimes G$ is abelian.

Throughout this section, we use the notations and the method described in Section 1.2.1 to compute $\mathrm{M}(G)$ for groups G in the isoclinism class Φ_4.

Lemma 4.1.3 *If G is one of the groups $\Phi_4(221)a$, $\Phi_4(221)b, \Phi_4\,(221)\,c$, $\Phi_4(221)d_{\frac{1}{2}(p-1)}$, $\Phi_4(221)d_r$ $\big(r \neq \frac{1}{2}(p-1)\big)$, $\Phi_4(221)e$, $\Phi_4(221)f_0$ or $\Phi_4(221)f_r$, then $\mathrm{M}(G)$ is isomorphic to $\mathbb{Z}_p, \mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_p$, $\mathbb{Z}_{p^2}$, $\mathbb{Z}_p$, $\mathbb{Z}_p, \mathbb{Z}_{p^2}$ or $\mathbb{Z}_p$ respectively, and $G \wedge G$ is isomorphic to $\mathbb{Z}_p^{(3)}, \mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}, \mathbb{Z}_p^{(3)}$, $\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$, $\mathbb{Z}_p^{(3)}$, $\mathbb{Z}_p^{(3)}, \mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$ or $\mathbb{Z}_p^{(3)}$ respectively.*

Proof. For the group $G = \Phi_4(221)a$, notice that X_1 is spanned by

$$(\alpha_1 G' \otimes \alpha^p - \alpha_2 G' \otimes \alpha_1^p)$$

and $\dim X_1 = 1$. Observe that $\alpha G' \otimes \alpha^p, \alpha_1 G' \otimes \alpha_1^p \in X_2$ and $(\alpha G' + uG') \otimes \alpha^p, (\alpha_1 G' + uG') \otimes \alpha_1^p \in X_2$ for $uG' \in \operatorname{Ker} f$. So, we have $uG' \otimes \alpha^p, uG' \otimes \alpha_1^p \in X_2$ for all $uG' \in \operatorname{Ker} f$. This implies $\alpha_2 G' \otimes \alpha^p, \alpha_2 G' \otimes \alpha_1^p \in X_2$. Now a general element of X_2 is of the form

$$\begin{aligned}(p_1 \alpha G' + p_2 \alpha_1 G' + p_3 \alpha_2 G') \otimes (p_1 \alpha^p + p_2 \alpha_1^p) \;&=\; p_1^2 \alpha G' \otimes \alpha^p + p_2^2 \alpha_1 G' \otimes \alpha_1^p \\ &\quad + p_1 p_2 (\alpha G' \otimes \alpha_1^p + \alpha_1 G' \otimes \alpha^p) \\ &\quad + p_3 p_1 (\alpha_2 G' \otimes \alpha^p) \\ &\quad + p_3 p_2 (\alpha_2 G' \otimes \alpha_1^p).\end{aligned}$$

This shows that, X_2 is spanned by the set

$$\{\alpha G' \otimes \alpha^p, \alpha_1 G' \otimes \alpha_1^p, \alpha_2 G' \otimes \alpha^p, \alpha_2 G' \otimes \alpha_1^p, (\alpha G' \otimes \alpha_1^p + \alpha_1 G' \otimes \alpha^p)\}.$$

Hence, $\dim X_2 = 5$. Observe that $(\alpha_1 G' \otimes \alpha^p - \alpha_2 G' \otimes \alpha_1^p)$ is not contained in X_2. Thus, $\dim X = 6$, and consequently, $|N| = 1, |M| = p$. Now by Theorem 1.2.13, we have

$$\mathrm{M}(G) \cong \mathbb{Z}_p,$$

which gives

$$|G \wedge G| = p^3.$$

Hence, by Remark 4.1.2,

$$G \wedge G = \langle [\alpha_1, \alpha^\phi], [\alpha_2, \alpha^\phi], [\alpha_1, \alpha_2^\phi] \rangle \cong \mathbb{Z}_p^{(3)}.$$

For the group $G = \Phi_4(221)b$, we see that X_1 is spanned by

$$(\alpha_1 G' \otimes \alpha^p - \alpha_2 G' \otimes \alpha_2^p)$$

and $\dim X_1 = 1$. As described in the preceding case, the subspace X_2 is spanned by the set

$$\{\alpha G' \otimes \alpha^p, \alpha_2 G' \otimes \alpha_2^p, \alpha_1 G' \otimes \alpha^p, \alpha_1 G' \otimes \alpha_2^p, (\alpha G' \otimes \alpha_2^p + \alpha_2 G' \otimes \alpha^p)\},$$

and $\dim X_2 = 5$. Observe that $X_1 \subset X_2$, so $\dim X = 5$ and $|N| = p$. By Theorem 1.2.13 $|M| = |\,\mathrm{M}(G)| = p^2$. Hence, $|G \wedge G| = p^4$. By Lemma 1.3.3$(viii)$,

$$[\beta_1, \beta_2^\phi] = [\alpha_1, \alpha, \beta_2^\phi] = 1.$$

For $i \in \{1, 2\}, x \in \{\alpha, \alpha_1, \alpha_2\}$, by Lemma 1.3.2, we have the following identities:

$$\begin{aligned}
&[\beta_i, x^\phi]^p = [\beta_i^p, x] = 1, \\
&[\beta_2, x^\phi] = [\alpha^p, x^\phi] = [\alpha, x^\phi]^p, \\
&[\beta_1, x^\phi] = [\alpha_2^p, x^\phi] = [\alpha_2, x^\phi]^p, \\
&[\alpha, \alpha_1^\phi]^p = [\alpha, (\alpha_1^p)^\phi] = 1 = [\alpha_2, (\alpha_1^p)^\phi] = [\alpha_2, \alpha_1^\phi]^p, \\
&[\alpha_2, \alpha^\phi]^{p^2} = [\alpha_2^{p^2}, \alpha^\phi] = 1.
\end{aligned}$$

Hence, by Remark 4.1.2, it follows that

$$G \wedge G = \langle [\alpha_2, \alpha^{\phi}], [\alpha_1, \alpha^{\phi}], [\alpha_1, \alpha_2^{\phi}] \rangle \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p$$

and

$$\mathrm{M}(G) \cong \langle [\alpha_2, \alpha^{\phi}]^p, [\alpha_1, \alpha_2^{\phi}] \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p.$$

For the group $G = \Phi_4(221)c$, we see that X_1 is spanned by

$$(\alpha_1 G' \otimes \alpha_2^p - \alpha_2 G' \otimes \alpha_1^p),$$

and $\dim X_1 = 1$. It follows that X_2 is spanned by the set

$$\{\alpha_1 G' \otimes \alpha_1^p, \alpha_2 G' \otimes \alpha_2^p, \alpha G' \otimes \alpha_1^p, \alpha G' \otimes \alpha_2^p, (\alpha_1 G' \otimes \alpha_2^p + \alpha_2 G' \otimes \alpha_1^p)\}$$

and $\dim X_2 = 5$. So $\dim X = 6$ and $|N| = 1, |M| = p$. Hence by Theorem 1.2.13,

$$\mathrm{M}(G) \cong \mathbb{Z}_p$$

and by Remark 4.1.2,

$$G \wedge G \cong \langle [\alpha_1, \alpha^{\phi}], [\alpha_2, \alpha^{\phi}], [\alpha_1, \alpha_2^{\phi}] \rangle \cong \mathbb{Z}_p^{(3)}.$$

For the group $G = \Phi_4(221)d_r$, X_1 is spanned by

$$(\alpha_1 G' \otimes \alpha_2^p - \alpha_2 G' \otimes \beta_1)$$

and $\dim X_1 = 1$. The space X_2 is spanned by the set

$$\{\alpha_1 G' \otimes \beta_1^k, \alpha_2 G' \otimes \alpha_2^p, \alpha G' \otimes \beta_1^k, \alpha G' \otimes \alpha_2^p, (\alpha_1 G' \otimes \alpha_2^p + \alpha_2 G' \otimes \beta_1^k)\}$$

and $\dim X_2 = 5$.

For $r = \frac{1}{2}(p-1)$, it follows that in the presentation of G $k \equiv -1 \pmod p$. So we have $X_1 \subset X_2$. Hence $\dim X = 5$, and therefore $|N| = p$. By Theorem 1.2.13, $|\,\mathrm{M}(G)| = |M| = p^2$ and so $|G \wedge G| = p^4$. By Lemma 1.3.3($viii$),

$$[\beta_1, \beta_2^{\phi}] = [\alpha_1, \alpha, \beta_2^{\phi}] = 1.$$

For $i \in \{1, 2\}, x \in \{\alpha, \alpha_1, \alpha_2\}$, by Lemma 1.3.2, we have

$$\begin{aligned}
&[\beta_i, x^{\phi}]^p = [\beta_i^p, x] = 1,\\
&[\beta_2, x^{\phi}] = [\alpha_2^p, x^{\phi}] = [\alpha_2, x^{\phi}]^p,\\
&[\beta_1, x^{\phi}] = [\alpha_1^{-p}, x^{\phi}] = [\alpha_1^{-1}, x^{\phi}]^p = [\alpha_1, x^{\phi}]^{-p},\\
&[\alpha, \alpha_1^{\phi}]^p = [\alpha^p, \alpha_1^{\phi}] = 1 = [\alpha_2, (\alpha^p)^{\phi}] = [\alpha_2, \alpha^{\phi}]^p,\\
&[\alpha_1, \alpha_2^{\phi}]^{p^2} = [\alpha_1^{p^2}, \alpha_2^{\phi}] = 1.
\end{aligned}$$

Hence, by Remark 4.1.2, it follows that

$$G \wedge G = \langle [\alpha_1, \alpha_2^{\phi}], [\alpha_1, \alpha^{\phi}], [\alpha_2, \alpha^{\phi}] \rangle \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p$$

and, since $[\alpha_1, \alpha_2] = 1$,

$$\mathrm{M}(G) \cong \langle [\alpha_1, \alpha_2^{\phi}] \rangle \cong \mathbb{Z}_{p^2}.$$

For $r \neq \frac{1}{2}(p-1)$, $X_1 \cap X_2 = \emptyset$, so $\dim X = 6$ and therefore $|N| = 1$. Hence

$$\mathrm{M}(G) \cong \mathbb{Z}_p$$

and

$$G \wedge G = \langle [\alpha_1, \alpha^{\phi}], [\alpha_2, \alpha^{\phi}], [\alpha_1, \alpha_2^{\phi}] \rangle \cong \mathbb{Z}_p^{(3)}.$$

For the group $G = \Phi_4(221)e$, X_1 is spanned by

$$(\alpha_1 G' \otimes \beta_2 - \alpha_2 G' \otimes \beta_1)$$

and $\dim X_1 = 1$. The subspace X_2 is spanned by the set

$$\{\alpha_1 G' \otimes \beta_2^{-\frac{1}{4}}, (\alpha_2 G' \otimes \beta_1 + \alpha_2 G' \otimes \beta_2), \alpha G' \otimes \beta_2^{-\frac{1}{4}}, (\alpha G' \otimes \beta_1 + \alpha G' \otimes \beta_2),$$
$$(\alpha_1 G' \otimes \beta_1 + \alpha_1 G' \otimes \beta_2 + \alpha_2 G' \otimes \beta_2^{-\frac{1}{4}})\}$$

and $\dim X_2 = 5$. So $\dim X = 6$. Therefore $|N| = 1$ and by Theorem 1.2.13, $|\,\mathrm{M}(G)| = |M| = p$. Hence

$$\mathrm{M}(G) \cong \mathbb{Z}_p$$

and by Remark 4.1.2,

$$G \wedge G = \langle [\alpha_1, \alpha^\phi], [\alpha_2, \alpha^\phi], [\alpha_1, \alpha_2^\phi] \rangle \cong \mathbb{Z}_p^{(3)}.$$

For the group $G = \Phi_4(221)f_0$, X_1 is spanned by

$$(\alpha_1 G' \otimes \beta_2 - \alpha_2 G' \otimes \beta_1)$$

and $\dim X_1 = 1$. The subspace X_2 is spanned by

$$\{\alpha_1 G' \otimes \beta_2, \alpha_2 G' \otimes \beta_1^\nu, \alpha G' \otimes \beta_2, \alpha G' \otimes \beta_1^\nu, (\alpha_1 G' \otimes \beta_1^\nu + \alpha_2 G' \otimes \beta_2)\}$$

and $\dim X_2 = 5$. Observe that $X_1 \subset X_2$, so $\dim X = 5$. Therefore $|N| = p$ and by Theorem 1.2.13, $|\,\mathrm{M}(G)| = p^2$. By Lemma 1.3.3(viii),

$$[\beta_1, \beta_2^\phi] = [\alpha_1, \alpha, \beta_2^\phi] = 1.$$

For $i \in \{1,2\}, x \in \{\alpha, \alpha_1, \alpha_2\}$, by Lemma 1.3.2, we have

$$[\beta_i, x^\phi]^p = [\beta_i^p, x^\phi] = 1,$$
$$[\beta_2, x^\phi] = [\alpha_1^p, x^\phi] = [\alpha_1, x^\phi]^p,$$
$$[\beta_1, x^\phi] = [\alpha_2^{p\nu^{-1}}, x^\phi] = [\alpha_2, x^\phi]^{p\nu^{-1}},$$
$$[\alpha, \alpha_1^\phi]^p = [\alpha^p, \alpha_1^\phi] = 1 = [\alpha_2, (\alpha^p)^\phi] = [\alpha_2, \alpha^\phi]^p.$$

Hence, by Remark 4.1.2,

$$G \wedge G = \langle [\alpha_1, \alpha_2^\phi], [\alpha_2, \alpha^\phi], [\alpha_1, \alpha^\phi] \rangle \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p$$

and, since $[\alpha_1, \alpha_2] = 1$,

$$\mathrm{M}(G) \cong \langle [\alpha_1, \alpha_2^\phi] \rangle \cong \mathbb{Z}_{p^2}.$$

For the group $G = \Phi_4(221)f_r$, X_1 is spanned by

$$(\alpha_1 G' \otimes \beta_2 - \alpha_2 G' \otimes \beta_1)$$

and $\dim X_1 = 1$. The subspace X_2 is spanned by the set

$$\{\alpha_1 G' \otimes \beta_2^k, (\alpha_2 G' \otimes \beta_1 + \alpha_2 G' \otimes \beta_2), \alpha G' \otimes \beta_2^k, (\alpha G' \otimes \beta_1 + \alpha G' \otimes \beta_2),$$
$$(\alpha_1 G' \otimes \beta_1 + \alpha_1 G' \otimes \beta_2 + \alpha_2 G' \otimes \beta_2^k\}$$

and $\dim X_2 = 5$. So $\dim X = 6$. Therefore $|N| = 1$ and by Theorem 1.2.13, $|\mathrm{M}(G)| = |M| = p$. Hence

$$\mathrm{M}(G) \cong \mathbb{Z}_p$$

and by Remark 4.1.2,

$$G \wedge G = \langle [\alpha_1, \alpha^\phi], [\alpha_2, \alpha^\phi], [\alpha_1, \alpha_2^\phi] \rangle \cong \mathbb{Z}_p^{(3)}.$$

This completes the proof of the lemma. □

Lemma 4.1.4 *If G is one of the groups $\Phi_4(2111)a, \Phi_4(2111)b, \Phi_4(2111)c$ or $\Phi_4(1^5)$, then $\mathrm{M}(G)$ is isomorphic to $\mathbb{Z}_p^{(3)}, \mathbb{Z}_p^{(3)}, \mathbb{Z}_p^{(3)}$ or $\mathbb{Z}_p^{(6)}$ respectively, and $G \wedge G$ is isomorphic to $\mathbb{Z}_p^{(5)}, \mathbb{Z}_p^{(5)}, \mathbb{Z}_p^{(5)}$ or $\mathbb{Z}_p^{(8)}$ respectively.*

Proof. For the group $G = \Phi_4(2111)a$, X_1 is spanned by the element

$$(\alpha_1 G' \otimes \alpha^p - \alpha_2 G' \otimes \beta_1)$$

and $\dim X_1 = 1$. The space X_2 is spanned by the set

$$\{\alpha G' \otimes \alpha^p, \alpha_1 G' \otimes \alpha^p, \alpha_2 G' \otimes \alpha^p\}$$

and $\dim X_2 = 3$. So $\dim X = 4$. Therefore $|N| = p^2$ and by Theorem 1.2.13, $|\mathrm{M}(G)| = |M| = p^3$. So $|G \wedge G| = p^5$. By Lemma 1.3.3$(viii)$,

$$[\beta_1, \beta_2^\phi] = [\alpha_1, \alpha, \beta_2^\phi] = 1.$$

For $i \in \{1, 2\}, x \in \{\alpha, \alpha_1, \alpha_2\}$, by Lemma 1.3.2, we have

$$[\beta_i, x^\phi]^p = [\beta_i^p, x] = 1,$$
$$[\alpha_2, x^\phi]^p = [\alpha_2^p, x^\phi] = 1 = [\alpha_1^p, \alpha^\phi] = [\alpha_1, \alpha^\phi]^p.$$

Therefore every generator of $G \wedge G$ is of order at most p. Hence

$$G \wedge G \cong \mathbb{Z}_p^{(5)}$$

and

$$\mathrm{M}(G) \cong \mathbb{Z}_p^{(3)}.$$

For the group $G = \Phi_4(2111)b$, X_1 is spanned by the element

$$(\alpha_1 G' \otimes \beta_2 - \alpha_2 G' \otimes \alpha_1^p)$$

and $\dim X_1 = 1$. The space X_2 is spanned by the set

$$\{\alpha_1 G' \otimes \alpha_1^p, \alpha G' \otimes \alpha_1^p, \alpha_2 G' \otimes \alpha_1^p\}$$

and $\dim X_2 = 3$. So $\dim X = 4$. Therefore $|N| = p^2$, $|\mathrm{M}(G)| = |M| = p^3$ and $|G \wedge G| = p^5$. By Lemma 1.3.3$(viii)$,

$$[\beta_1, \beta_2^\phi] = [\alpha_1, \alpha, \beta_2^\phi] = 1.$$

For $i \in \{1,2\}, x \in \{\alpha, \alpha_1, \alpha_2\}$, by Lemma 1.3.2, we have

$$[\beta_i, x^\phi]^p = [\beta_i^p, x] = 1,$$
$$[\alpha_2, x^\phi]^p = [\alpha_2^p, x^\phi] = 1 = [\alpha^p, \alpha_1^\phi] = [\alpha, \alpha_1^\phi]^p.$$

Therefore every generator of $G \wedge G$ is of order at most p. Hence

$$G \wedge G \cong \mathbb{Z}_p^{(5)}$$

and

$$\mathrm{M}(G) \cong \mathbb{Z}_p^{(3)}.$$

For the group $G = \Phi_4(2111)c$, X_1 is spanned by the element

$$(\alpha_1 G' \otimes \beta_2 - \alpha_2 G' \otimes \alpha_2^p)$$

and $\dim X_1 = 1$. The space X_2 is spanned by the set

$$\{\alpha_2 G' \otimes \alpha_2^p, \alpha G' \otimes \alpha_2^p, \alpha_1 G' \otimes \alpha_2^p\}$$

and $\dim X_2 = 3$. So $\dim X = 4$. Therefore $|N| = p^2$, $|\,\mathrm{M}(G)| = |M| = p^3$ and $|G \wedge G| = p^5$. By Lemma 1.3.3$(viii)$,

$$[\beta_1, \beta_2^\phi] = [\alpha_1, \alpha, \beta_2^\phi] = 1.$$

For $i \in \{1, 2\}, x \in \{\alpha, \alpha_1, \alpha_2\}$, by Lemma 1.3.2, we have

$$[\beta_i, x^\phi]^p = [\beta_i^p, x] = 1,$$
$$[\alpha_1, x^\phi]^p = [\alpha_1^p, x^\phi] = 1 = [\alpha^p, \alpha_2^\phi] = [\alpha, \alpha_2^\phi]^p.$$

Therefore every generator of $G \wedge G$ is of order at most p. Hence

$$G \wedge G \cong \mathbb{Z}_p^{(5)}$$

and

$$\mathrm{M}(G) \cong \mathbb{Z}_p^{(3)}.$$

For the group $G = \Phi_4(1^5)$, X_1 is spanned by the element

$$(\alpha_1 G' \otimes \beta_2 - \alpha_2 G' \otimes \beta_1)$$

and $\dim X_1 = 1$. Observe that in this case $\dim X = 1$. Therefore $|\,\mathrm{M}(G)| = |M| = p^6$ and $|G \wedge G| = p^8$. By Lemma 1.3.3(*viii*),

$$[\beta_1, \beta_2^{\phi}] = [\alpha_1, \alpha, \beta_2^{\phi}] = 1.$$

For $i \in \{1, 2\}, x \in \{\alpha, \alpha_1, \alpha_2\}$, by Lemma 1.3.2, we have

$$[\beta_i, x^{\phi}]^p = [\beta_i^p, x] = 1,$$
$$[\alpha_i, x^{\phi}]^p = [\alpha_i^p, x^{\phi}] = 1.$$

Therefore every generator of $G \wedge G$ is of order at most p. Hence

$$G \wedge G \cong \mathbb{Z}_p^{(8)}$$

and

$$\mathrm{M}(G) \cong \mathbb{Z}_p^{(6)}.$$

□

4.1.2 Classes Φ_9, Φ_{10}: Groups of maximal class

In this section we consider groups of maximal class i.e., the groups belonging in the isoclinism classes Φ_9 and Φ_{10}.

Lemma 4.1.5 *If G is one of the groups $\Phi_9(2111)a, \Phi_9(2111)b_r, \Phi_{10}(2111)a_r$ or $\Phi_{10}(2111)b_r$, then $\mathrm{M}(G)$ is isomorphic to $\mathbb{Z}_p$ and $G \wedge G$ is isomorphic to $\mathbb{Z}_p^{(4)}$.*

Proof. For the groups G under consideration, taking $K = \mathrm{Z}(G)$, in Theorem 1.2.5(i), we get $p \leq \mathrm{M}(G)$. Thus, $|G \wedge G| \geq p^4$.

If G is either $\Phi_9(2111)a$ or $\Phi_9(2111)b_r$, then by Lemma 1.3.3$(viii)$,

$$[\alpha_2, \alpha_4^\phi] = [\alpha_1, \alpha, \alpha_4^\phi] = 1 = [\alpha_2, \alpha, \alpha_4^\phi] = [\alpha_3, \alpha_4^\phi],$$

as $\alpha_4 \in \mathrm{Z}(G)$. Now we have

$$\begin{aligned}
[\alpha^{-1}, \alpha_1^{-1}, \alpha_2^\phi]^{\alpha_1} &= [\alpha_1\alpha\alpha_2^{-1}\alpha^{-1}\alpha_1^{-1}, \alpha_2^\phi]^{\alpha_1} = [\alpha\alpha_2^{-1}\alpha^{-1}, \alpha_2^\phi] \\
&= [\alpha\alpha_3\alpha^{-1}\alpha_2^{-1}, \alpha_2^\phi] = [\alpha_3\alpha_4^{-1}\alpha_2^{-1}, \alpha_2^\phi] \\
&= [\alpha_3, \alpha_2^\phi][\alpha_4, \alpha_2^\phi]^{-1} = [\alpha_3, \alpha_2^\phi],
\end{aligned}$$

$$\begin{aligned}
[\alpha^{-1}, \alpha_1^{-1}, \alpha_3^\phi]^{\alpha_1} &= [\alpha^{-1}, \alpha_1^{-1}, \alpha_3^\phi]^{\alpha_1} = [\alpha_1\alpha\alpha_2^{-1}\alpha^{-1}\alpha_1^{-1}, \alpha_3^\phi]^{\alpha_1} \\
&= [\alpha\alpha_2^{-1}\alpha^{-1}, \alpha_3^\phi] = [\alpha_3\alpha_4^{-1}\alpha_2^{-1}, \alpha_3^\phi] \\
&= [\alpha_4, \alpha_3^\phi]^{-1}[\alpha_2, \alpha_3^\phi]^{-1} = [\alpha_2, \alpha_3^\phi]^{-1}.
\end{aligned}$$

By Hall-Witt identity, we have

$$\begin{aligned}
1 &= [\alpha_2, \alpha, \alpha_1^\phi]^{\alpha^{-1}}[\alpha^{-1}, \alpha_1^{-1}, \alpha_2^\phi]^{\alpha_1}[\alpha_1, \alpha_2^{-1}, (\alpha^{-1})^\phi]^{\alpha_2} \\
&= [\alpha_3, \alpha_1^\phi][\alpha_3, \alpha_2^\phi], \\
1 &= [\alpha_3, \alpha, \alpha_1^\phi]^{\alpha^{-1}}[\alpha^{-1}, \alpha_1^{-1}, \alpha_3^\phi]^{\alpha_1}[\alpha_1, \alpha_3^{-1}, (\alpha^{-1})^\phi]^{\alpha_3} \\
&= [\alpha_4, \alpha_1^\phi][\alpha_2, \alpha_3^\phi]^{-1}.
\end{aligned}$$

This implies that $[\alpha_3, \alpha_1^\phi] = [\alpha_2, \alpha_3^\phi] = [\alpha_4, \alpha_1^\phi]$ holds in $G \wedge G$ for $G \cong \Phi_9(2111)a$ or $\Phi_9(2111)b_r$.

Now consider the group $G \cong \Phi_9(2111)a$. By Lemma 1.3.2, we have the following identities:

$$
\begin{aligned}
&[\alpha_4, \alpha^\phi]^p = [\alpha_4^p, \alpha^\phi] = 1 = [\alpha_4^p, \alpha_1^\phi] = [\alpha_4, \alpha_1^\phi]^p,\\
&[\alpha_3, \alpha^\phi]^p = [\alpha_3^p, \alpha^\phi] = 1 = [\alpha_3^p, \alpha_1^\phi] = [\alpha_3, \alpha_1^\phi]^p,\\
&[\alpha_2, \alpha^\phi]^p = [\alpha_2^p, \alpha^\phi] = 1 = [\alpha_2^p, \alpha_1^\phi] = [\alpha_2, \alpha_1^\phi]^p,\\
&[\alpha_4, \alpha^\phi] = [\alpha^p, \alpha^\phi] = [\alpha, \alpha^\phi]^p = 1,\\
&[\alpha_4, \alpha_1^\phi] = [\alpha^p, \alpha_1^\phi] = [\alpha, \alpha_1^\phi]^p = [\alpha, (\alpha_1^p)^\phi] = 1.
\end{aligned}
$$

Thus, by Proposition 1.3.5, $G \wedge G$ is generated by

$$\{[\alpha_1, \alpha^\phi], [\alpha_2, \alpha^\phi], [\alpha_3, \alpha^\phi], [\alpha_2, \alpha_1^\phi]\}.$$

Since, by Lemma 1.3.3(vi), $[[\alpha_2, \alpha^\phi], [\alpha_1, \alpha^\phi]] = [\alpha_3, \alpha_2^\phi] = 1$, it follows that $G \wedge G$ is elementary abelian of order p^4. Hence

$$G \wedge G \cong \mathbb{Z}_p^{(4)},$$

and consequently

$$\mathrm{M}(G) \cong \langle [\alpha_2, \alpha_1^\phi] \rangle \cong \mathbb{Z}_p.$$

Now consider the group $G \cong \Phi_9(2111)b_r$. By Lemma 1.3.2, we have the following identities:

$$
\begin{aligned}
&[\alpha_4, \alpha^\phi]^p = [\alpha_4^p, \alpha^\phi] = 1 = [\alpha_4^p, \alpha_1^\phi] = [\alpha_4, \alpha_1^\phi]^p,\\
&[\alpha_3, \alpha^\phi]^p = [\alpha_3^p, \alpha^\phi] = 1 = [\alpha_3^p, \alpha_1^\phi] = [\alpha_3, \alpha_1^\phi]^p,\\
&[\alpha_2, \alpha^\phi]^p = [\alpha_2^p, \alpha^\phi] = 1 = [\alpha_2^p, \alpha_1^\phi] = [\alpha_2, \alpha_1^\phi]^p,\\
&[\alpha_4, \alpha^\phi] = [\alpha_1^{pk^{-1}}, \alpha^\phi] = [\alpha_1, \alpha^\phi]^{pk^{-1}} = [\alpha_1, (\alpha^{pk^{-1}})^\phi] = 1,\\
&[\alpha_4, \alpha_1^\phi] = [\alpha_1^{pk^{-1}}, \alpha_1^\phi] = [\alpha_1, \alpha_1^\phi]^{pk^{-1}} = 1,
\end{aligned}
$$

$$[\alpha_1, \alpha^{\phi}]^p = [\alpha_1, (\alpha^p)^{\phi}] = 1.$$

By Proposition 1.3.5, $G \wedge G$ is generated by

$$\{[\alpha_1, \alpha^{\phi}], [\alpha_2, \alpha^{\phi}], [\alpha_3, \alpha^{\phi}], [\alpha_2, \alpha_1^{\phi}]\}.$$

Since, by Lemma 1.3.3(vi), $[[\alpha_2, \alpha^{\phi}], [\alpha_1, \alpha^{\phi}]] = [\alpha_3, \alpha_2^{\phi}] = 1$, it follows that $G \wedge G$ is elementary abelian of order p^4. Hence

$$G \wedge G \cong \mathbb{Z}_p^{(4)},$$

and consequently

$$\mathrm{M}(G) \cong \langle[\alpha_2, \alpha_1^{\phi}]\rangle \cong \mathbb{Z}_p.$$

If G is either $\Phi_{10}(2111)a_r$ or $\Phi_{10}(2111)b_r$, then by Lemma 1.3.3$(viii)$,

$$[\alpha_2, \alpha_4^{\phi}] = [\alpha_1, \alpha, \alpha_4^{\phi}] = 1 = [\alpha_2, \alpha, \alpha_4^{\phi}] = [\alpha_3, \alpha_4^{\phi}],$$

$$\begin{aligned}
[\alpha^{-1}, \alpha_1^{-1}, \alpha_2^{\phi}]^{\alpha_1} &= [\alpha_1\alpha\alpha_2^{-1}\alpha^{-1}\alpha_1^{-1}, \alpha_2^{\phi}]^{\alpha_1} = [\alpha\alpha_2^{-1}\alpha^{-1}, (\alpha_1^{-1}\alpha_2\alpha_1)^{\phi}] \\
&= [\alpha\alpha_3\alpha^{-1}\alpha_2^{-1}, (\alpha_2\alpha_4^{-1})^{\phi}] = [\alpha_3\alpha_4^{-1}\alpha_2^{-1}, (\alpha_2\alpha_4^{-1})^{\phi}] \\
&= [\alpha_3, \alpha_2^{\phi}][\alpha_4, \alpha_2^{\phi}]^{-1} = [\alpha_3, \alpha_2^{\phi}]
\end{aligned}$$

and

$$\begin{aligned}
[\alpha^{-1}, \alpha_1^{-1}, \alpha_3^{\phi}]^{\alpha_1} &= [\alpha^{-1}, \alpha_1^{-1}, \alpha_3^{\phi}]^{\alpha_1} = [\alpha_1\alpha\alpha_2^{-1}\alpha^{-1}\alpha_1^{-1}, \alpha_3^{\phi}]^{\alpha_1} \\
&= [\alpha\alpha_2^{-1}\alpha^{-1}, \alpha_3^{\phi}] = [\alpha\alpha_3\alpha^{-1}\alpha_2^{-1}, \alpha_3^{\phi}] \\
&= [\alpha_3\alpha_4^{-1}\alpha_2^{-1}, \alpha_3^{\phi}] = [\alpha_4^{-1}, \alpha_3^{\phi}][\alpha_2, \alpha_3^{\phi}]^{-1}
\end{aligned}$$

$$= [\alpha_2, \alpha_3^{\phi}]^{-1}.$$

By Hall-Witt identity, we have

$$\begin{aligned}
1 &= [\alpha_2, \alpha, \alpha_1^{\phi}]^{\alpha^{-1}}[\alpha^{-1}, \alpha_1^{-1}, \alpha_2^{\phi}]^{\alpha_1}[\alpha_1, \alpha_2^{-1}, (\alpha^{-1})^{\phi}]^{\alpha_2} \\
&= [\alpha_3, \alpha_1^{\phi}][\alpha_3, \alpha_2^{\phi}][\alpha_4^{-1}, (\alpha^{-1})^{\phi}]^{\alpha_2} \\
&= [\alpha_3, \alpha_1^{\phi}][\alpha_3, \alpha_2^{\phi}][\alpha_4, \alpha^{\phi}], \\
1 &= [\alpha_3, \alpha, \alpha_1^{\phi}]^{\alpha^{-1}}[\alpha^{-1}, \alpha_1^{-1}, \alpha_3^{\phi}]^{\alpha_1}[\alpha_1, \alpha_3^{-1}, (\alpha^{-1})^{\phi}]^{\alpha_3} \\
&= [\alpha_4, \alpha_1^{\phi}][\alpha_2, \alpha_3^{\phi}]^{-1}.
\end{aligned}$$

This implies that $[\alpha_2, \alpha_3^{\phi}] = [\alpha_4, \alpha_1^{\phi}] = [\alpha_4, \alpha^{\phi}][\alpha_3, \alpha_1^{\phi}]$ holds in $G \wedge G$ for $G \cong \Phi_{10}(2111)a_r$ or $\Phi_{10}(2111)b_r$.

Consider the group $G \cong \Phi_{10}(2111)a_r$. By Lemma 1.3.2, we have the following identities:

$$\begin{aligned}
&[\alpha_4, \alpha^{\phi}]^p = [\alpha_4^p, \alpha^{\phi}] = 1 = [\alpha_4^p, \alpha_1^{\phi}] = [\alpha_4, \alpha_1^{\phi}]^p, \\
&[\alpha_3, \alpha^{\phi}]^p = [\alpha_3^p, \alpha^{\phi}] = 1 = [\alpha_3^p, \alpha_1^{\phi}] = [\alpha_3, \alpha_1^{\phi}]^p, \\
&[\alpha_2, \alpha^{\phi}]^p = [\alpha_2^p, \alpha^{\phi}] = 1 = [\alpha_2^p, \alpha_1^{\phi}] = [\alpha_2, \alpha_1^{\phi}]^p, \\
&[\alpha_4, \alpha^{\phi}] = [\alpha^{pk^{-1}}, \alpha^{\phi}] = [\alpha, \alpha^{\phi}]^{pk^{-1}} = 1, \\
&[\alpha_4, \alpha_1^{\phi}] = [\alpha^{pk^{-1}}, \alpha_1^{\phi}] = [\alpha, \alpha_1^{\phi}]^{pk^{-1}} = [\alpha, (\alpha_1^{pk^{-1}})^{\phi}] = 1, \\
&[\alpha_1, \alpha^{\phi}]^p = [\alpha_1^p, \alpha^{\phi}] = 1.
\end{aligned}$$

Thus, by Proposition 1.3.5, $G \wedge G$ is generated by

$$\{[\alpha_1, \alpha^{\phi}], [\alpha_2, \alpha^{\phi}], [\alpha_3, \alpha^{\phi}], [\alpha_2, \alpha_1^{\phi}]\}.$$

By Lemma 1.3.3(vi), $G \wedge G$ is elementary abelian of order p^4. Hence

$$G \wedge G \cong \mathbb{Z}_p^{(4)},$$

and consequently

$$\mathrm{M}(G) \cong \langle [\alpha_2, \alpha_1^{\phi}][\alpha_3, \alpha^{\phi}] \rangle \cong \mathbb{Z}_p.$$

Consider the group $G \cong \Phi_{10}(2111)b_r$. By Lemma 1.3.2, we have the following identities:

$$[\alpha_4, \alpha^{\phi}]^p = [\alpha_4^p, \alpha^{\phi}] = 1 = [\alpha_4^p, \alpha_1^{\phi}] = [\alpha_4, \alpha_1^{\phi}]^p,$$
$$[\alpha_3, \alpha^{\phi}]^p = [\alpha_3^p, \alpha^{\phi}] = 1 = [\alpha_3^p, \alpha_1^{\phi}] = [\alpha_3, \alpha_1^{\phi}]^p,$$
$$[\alpha_2, \alpha^{\phi}]^p = [\alpha_2^p, \alpha^{\phi}] = 1 = [\alpha_2^p, \alpha_1^{\phi}] = [\alpha_2, \alpha_1^{\phi}]^p,$$
$$[\alpha_4, \alpha^{\phi}] = [\alpha_1^{pk^{-1}}, \alpha^{\phi}] = [\alpha_1, \alpha^{\phi}]^{pk^{-1}} = [\alpha_1, (\alpha^{pk^{-1}})^{\phi}] = 1,$$
$$[\alpha_4, \alpha_1^{\phi}] = [\alpha_1^{pk^{-1}}, \alpha_1^{\phi}] = [\alpha_1, \alpha_1^{\phi}]^{pk^{-1}} = 1,$$
$$[\alpha_1, \alpha^{\phi}]^p = [\alpha_1, (\alpha^p)^{\phi}] = 1.$$

Thus, by Proposition 1.3.5, $G \wedge G$ is generated by

$$\{[\alpha_1, \alpha^{\phi}], [\alpha_2, \alpha^{\phi}], [\alpha_3, \alpha^{\phi}], [\alpha_2, \alpha_1^{\phi}]\}.$$

By Lemma 1.3.3(vi), $G \wedge G$ is elementary abelian of order p^4. Hence

$$G \wedge G \cong \mathbb{Z}_p^{(4)},$$

and consequently

$$\mathrm{M}(G) \cong \langle [\alpha_2, \alpha_1^{\phi}][\alpha_3, \alpha^{\phi}] \rangle \cong \mathbb{Z}_p.$$

This completes the proof. $\square$

Lemma 4.1.6 *If G is one of the groups $\Phi_9(1^5)$ or $\Phi_{10}(1^5)$, then* $\mathrm{M}(G)$ *is isomorphic to $\mathbb{Z}_p^{(3)}$, and $G \wedge G$ is isomorphic to $\Phi_2(111) \times \mathbb{Z}_p^{(3)}$.*

Proof. Consider the group $G \cong \Phi_9(1^5)$. Let F be the free group generated by $\{\alpha, \alpha_1\}$. Define $\alpha_{i+1} = [\alpha_i, \alpha]$, $1 \leq i \leq 3$. Set $\beta_1 = [\alpha_1, \alpha_2]$, $\beta_2 = [\alpha_4, \alpha]$ and $\beta_3 = [\alpha_1, \alpha_4]$. Then, modulo $\gamma_6(F)$, we have

$$\begin{aligned}
[\alpha^{-1}, \alpha_1^{-1}, \alpha_3]^{\alpha_1} &= [\alpha^{-1}, \alpha_1^{-1}, \alpha_3]^{\alpha_1} = [\alpha_1 \alpha \alpha_2^{-1} \alpha^{-1} \alpha_1^{-1}, \alpha_3]^{\alpha_1} \\
&= [\alpha \alpha_2^{-1} \alpha^{-1}, \alpha_3[\alpha_3, \alpha_1]] = [\alpha \alpha_3 \alpha^{-1} \alpha_2^{-1}, \alpha_3[\alpha_3, \alpha_1]] \\
&= [\alpha_3 \alpha \alpha_4^{-1} \alpha^{-1} \alpha_2^{-1}, \alpha_3[\alpha_3, \alpha_1]] = [\alpha_3 \alpha \beta_2 \alpha^{-1} \alpha_4^{-1} \alpha_2^{-1}, \alpha_3[\alpha_3, \alpha_1]] \\
&= [\alpha_3 \beta_2 \alpha_4^{-1} \alpha_2^{-1}, \alpha_3[\alpha_3, \alpha_1]] = [\alpha_2^{-1}, \alpha_3] \\
&= [\alpha_2, \alpha_3]^{-1},
\end{aligned}$$

$$\begin{aligned}
[\alpha^{-1}, \alpha_1^{-1}, \alpha_2]^{\alpha_1} &= [\alpha^{-1}, \alpha_1^{-1}, \alpha_2]^{\alpha_1} = [\alpha_1 \alpha \alpha_2^{-1} \alpha^{-1} \alpha_1^{-1}, \alpha_2]^{\alpha_1} \\
&= [\alpha_3 \beta_2 \alpha_4^{-1} \alpha_2^{-1}, \alpha_2 \beta_1^{-1}] = [\alpha_3, \alpha_2 \beta_1^{-1}][\alpha_2^{-1}, \alpha_2 \beta_1^{-1}] \\
&= [\alpha_3, \alpha_2][\alpha_2, \beta_1].
\end{aligned}$$

By Hall-Witt identity, we have the following identities modulo $\gamma_6(F)$:

$$\begin{aligned}
1 &= [\alpha_3, \alpha, \alpha_1]^{\alpha^{-1}} [\alpha^{-1}, \alpha_1^{-1}, \alpha_3]^{\alpha_1} [\alpha_1, \alpha_3^{-1}, \alpha^{-1}]^{\alpha_3} \\
&= [\alpha_4, \alpha_1]^{\alpha^{-1}} [\alpha_2, \alpha_3]^{-1} [\alpha_3, \alpha_1, \alpha_3^{-1} \alpha^{-1} \alpha_3] \\
&= [\alpha_4, \alpha_1][\alpha_2, \alpha_3]^{-1} [\alpha_3, \alpha_1, \alpha_4 \alpha^{-1}] \\
&= [\alpha_4, \alpha_1][\alpha_2, \alpha_3]^{-1} [\alpha_3, \alpha_1, \alpha^{-1}], \qquad (4.1.1)
\end{aligned}$$

$$\begin{aligned}
1 &= [\alpha_2, \alpha, \alpha_1]^{\alpha^{-1}} [\alpha^{-1}, \alpha_1^{-1}, \alpha_2]^{\alpha_1} [\alpha_1, \alpha_2^{-1}, \alpha^{-1}]^{\alpha_2} \\
&= [\alpha_3, \alpha_1]^{\alpha^{-1}} [\alpha_3, \alpha_2][\alpha_2, \beta_1][\beta_1^{-1}, \alpha_2^{-1} \alpha^{-1} \alpha_2]
\end{aligned}$$

$$\begin{aligned}
&= [\alpha_3, \alpha_1][\alpha_3, \alpha_1, \alpha^{-1}][\alpha_3, \alpha_2][\alpha_2, \beta_1][\beta_1^{-1}, \alpha_3\alpha^{-1}] \\
&= [\alpha_3, \alpha_1][\alpha_3, \alpha_1, \alpha]^{-1}[\alpha_3, \alpha_2][\alpha_2, \beta_1][\beta_1^{-1}, \alpha^{-1}] \\
&= [\alpha_3, \alpha_1][\alpha_3, \alpha_1, \alpha]^{-1}[\alpha_3, \alpha_2][\alpha_2, \beta_1][\beta_1, \alpha]. \qquad (4.1.2)
\end{aligned}$$

Now consider

$$H_1 = F/\langle \gamma_6(F), F^p, \beta_1, \beta_3, [\alpha_3, \alpha_1, \alpha], [\alpha_3, \alpha_1, \alpha_1]\rangle.$$

Using (4.1.1) and (4.1.2) we have $[\alpha_4, \alpha_1] = [\alpha_2, \alpha_3] = [\alpha_3, \alpha_1] = 1$ in H_1. So

$$H_1 \cong \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_2 \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_4, \alpha] = \beta_2, \alpha^p = \alpha_1^p = \alpha_{i+1}^p = \beta_2^p = 1 \quad (i = 1, 2, 3)\rangle,$$

which is the group $\Phi_{35}(1^6)$ of order p^6 in [27]. Now consider

$$H_2 = F/\langle \gamma_6(F), F^p, \beta_1, [\alpha_3, \alpha_1, \alpha], [\alpha_3, \alpha_1, \alpha_1]\rangle.$$

Using (4.1.1) and (4.1.2), we have $[\alpha_4, \alpha_1] = [\alpha_2, \alpha_3] = [\alpha_3, \alpha_1]$ in H_2. Observe that

$$\begin{aligned}
H_2 \cong \langle \alpha, \alpha_i, \alpha_4, \beta_2, \beta_3 \mid & [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_4, \alpha] = \beta_2, [\alpha_1, \alpha_4] = [\alpha_1, \alpha_3] = \\
& [\alpha_3, \alpha_2] = \beta_3, \alpha^p = \alpha_i^p = \alpha_4^p = \beta_2^p = \beta_3^p = 1,\ 1 \le i \le 3\rangle.
\end{aligned}$$

Since $H_2/\langle \beta_3\rangle \cong H_1$, it follows that H_2 is of order p^7. Now consider

$$H = F/\langle \gamma_6(F), F^p, [\beta_1, \alpha], [\beta_1, \alpha_1], [\alpha_3, \alpha_1, \alpha], [\alpha_3, \alpha_1, \alpha_1]\rangle.$$

Observe that

$$\begin{aligned} H \quad \cong \quad & \langle \alpha, \alpha_j, \beta_i \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \beta_1, [\alpha_4, \alpha] = \beta_2, [\alpha_1, \alpha_4] = [\alpha_1, \alpha_3] =, \\ & [\alpha_3, \alpha_2] = \beta_3 \alpha^p = \alpha_j^p = \beta_i^p = 1,\ 1 \leq i \leq 3,\ 1 \leq j \leq 4 \rangle. \end{aligned}$$

Then H is a group of order p^8 and

$$H/\langle \beta_1, \beta_2, \beta_3 \rangle \cong G.$$

Take $Z = \langle \beta_1, \beta_2, \beta_3 \rangle$. Now the image of $\mathrm{tra} : \mathrm{Hom}(Z, \mathbb{C}^*) \to \mathrm{M}(H/Z)$ is

$$H' \cap Z \cong Z \cong \mathbb{Z}_p^{(3)}.$$

Hence $\mathbb{Z}_p^{(3)}$ is contained in $\mathrm{M}(G)$. By Theorem 1.2.6, $|\,\mathrm{M}(G)| \leq p^3$. Hence

$$\mathrm{M}(G) \cong \mathbb{Z}_p^{(3)}.$$

As $|G'| = p^3$, we get

$$|G \wedge G| = p^6.$$

Since $\alpha_4 \in \mathrm{Z}(G)$, by Lemma 1.3.3$(viii)$

$$[\alpha_2, \alpha_4^{\phi}] = [\alpha_3, \alpha_4^{\phi}] = 1.$$

Thus, by Proposition 1.3.2, $G \wedge G$ is generated by the set

$$\{[\alpha_1, \alpha^{\phi}], [\alpha_2, \alpha^{\phi}], [\alpha_3, \alpha^{\phi}], [\alpha_4, \alpha^{\phi}], [\alpha_2, \alpha_1^{\phi}], [\alpha_3, \alpha_1^{\phi}], [\alpha_4, \alpha_1^{\phi}], [\alpha_3, \alpha_2^{\phi}]\}.$$

Now we have

$$\begin{aligned}
[\alpha^{-1}, \alpha_1^{-1}, \alpha_2^{\phi}]^{\alpha_1} &= [\alpha_1 \alpha \alpha_2^{-1} \alpha^{-1} \alpha_1^{-1}, \alpha_2^{\phi}]^{\alpha_1} = [\alpha \alpha_2^{-1} \alpha^{-1}, \alpha_2^{\phi}] \\
&= [\alpha \alpha_3 \alpha^{-1} \alpha_2^{-1}, \alpha_2^{\phi}] = [\alpha_3 \alpha_4^{-1} \alpha_2^{-1}, \alpha_2^{\phi}] \\
&= [\alpha_3, \alpha_2^{\phi}][\alpha_4, \alpha_2^{\phi}]^{-1} = [\alpha_3, \alpha_2^{\phi}],
\end{aligned}$$

$$\begin{aligned}
[\alpha^{-1}, \alpha_1^{-1}, \alpha_3^{\phi}]^{\alpha_1} &= [\alpha^{-1}, \alpha_1^{-1}, \alpha_3^{\phi}]^{\alpha_1} = [\alpha_1 \alpha \alpha_2^{-1} \alpha^{-1} \alpha_1^{-1}, \alpha_3^{\phi}]^{\alpha_1} \\
&= [\alpha \alpha_2^{-1} \alpha^{-1}, \alpha_3^{\phi}] = [\alpha \alpha_3 \alpha^{-1} \alpha_2^{-1}, \alpha_3^{\phi}] \\
&= [\alpha_3 \alpha_4^{-1} \alpha_2^{-1}, \alpha_3^{\phi}] = [\alpha_2, \alpha_3^{\phi}]^{-1}.
\end{aligned}$$

By Hall-Witt identity, we have

$$\begin{aligned}
1 &= [\alpha_2, \alpha, \alpha_1^{\phi}]^{\alpha^{-1}} [\alpha^{-1}, \alpha_1^{-1}, \alpha_2^{\phi}]^{\alpha_1} [\alpha_1, \alpha_2^{-1}, (\alpha^{-1})^{\phi}]^{\alpha_2} \\
&= [\alpha_3, \alpha_1^{\phi}][\alpha_3, \alpha_2^{\phi}], \\
1 &= [\alpha_3, \alpha, \alpha_1^{\phi}]^{\alpha^{-1}} [\alpha^{-1}, \alpha_1^{-1}, \alpha_3^{\phi}]^{\alpha_1} [\alpha_1, \alpha_3^{-1}, (\alpha^{-1})^{\phi}]^{\alpha_3} \\
&= [\alpha_4, \alpha_1^{\phi}][\alpha_2, \alpha_3^{\phi}]^{-1}.
\end{aligned}$$

This implies that $[\alpha_4, \alpha_1^{\phi}] = [\alpha_2, \alpha_3^{\phi}] = [\alpha_3, \alpha_1^{\phi}]$. By Lemma 1.3.3$(vi)$,

$$[[\alpha_1, \alpha^{\phi}], [\alpha_2, \alpha^{\phi}]] = [\alpha_2, \alpha_3^{\phi}].$$

By Lemma 1.3.2, we have

$$\begin{aligned}
&[\alpha_4, \alpha^{\phi}]^p = [\alpha_4^p, \alpha^{\phi}] = 1 = [\alpha_3^p, \alpha^{\phi}] = [\alpha_3, \alpha^{\phi}]^p, \\
&[\alpha_3, \alpha_2^{\phi}]^p = [\alpha_3^p, \alpha_2^{\phi}] = 1 = [\alpha_2^p, \alpha_1^{\phi}] = [\alpha_2, \alpha_1^{\phi}]^p, \\
&[\alpha_2, \alpha^{\phi}]^p = [\alpha_2^p, \alpha^{\phi}] = 1 = [\alpha_1^p, \alpha^{\phi}] = [\alpha_1, \alpha^{\phi}]^p.
\end{aligned}$$

Hence

$$\begin{aligned} G\wedge G &\cong \langle [\alpha_1,\alpha^{\phi}],[\alpha_2,\alpha^{\phi}],[\alpha_2,\alpha_3^{\phi}]\rangle\times\langle[\alpha_2,\alpha_1^{\phi}]\rangle\times\langle[\alpha_3,\alpha^{\phi}]\rangle\times\langle[\alpha_4,\alpha^{\phi}]\rangle\\ &\cong \Phi_2(111)\times\mathbb{Z}_p^{(3)}. \end{aligned}$$

Now consider the group $G\cong\Phi_{10}(1^5)$. Let F be the free group generated by $\{\alpha,\alpha_1\}$. Define $\alpha_{i+1}=[\alpha_i,\alpha]$, $i=1,2,3$. Set $\beta=[\alpha_1,\alpha_2]$, $\beta_1=[\alpha_4,\alpha]$, $\beta_2=[\alpha_1,\alpha_4]$ and $\beta_3=\alpha_4^{-1}[\alpha_1,\alpha_2]$. Reading Modulo $\gamma_6(F)$, we have

$$\begin{aligned} [\alpha^{-1},\alpha_1^{-1},\alpha_3]^{\alpha_1} &= [\alpha^{-1},\alpha_1^{-1},\alpha_3]^{\alpha_1}=[\alpha_1\alpha\alpha_2^{-1}\alpha^{-1}\alpha_1^{-1},\alpha_3]^{\alpha_1}\\ &= [\alpha\alpha_2^{-1}\alpha^{-1},\alpha_3[\alpha_3,\alpha_1]]=[\alpha\alpha_3\alpha^{-1}\alpha_2^{-1},\alpha_3[\alpha_3,\alpha_1]]\\ &= [\alpha_3\alpha\alpha_4^{-1}\alpha^{-1}\alpha_2^{-1},\alpha_3[\alpha_3,\alpha_1]]=[\alpha_3\alpha\beta_1\alpha^{-1}\alpha_4^{-1}\alpha_2^{-1},\alpha_3[\alpha_3,\alpha_1]]\\ &= [\alpha_3\beta_1\alpha_4^{-1}\alpha_2^{-1},\alpha_3[\alpha_3,\alpha_1]]=[\alpha_2^{-1},\alpha_3]\\ &= [\alpha_2,\alpha_3]^{-1}, \end{aligned}$$

$$\begin{aligned} [\alpha^{-1},\alpha_1^{-1},\alpha_2]^{\alpha_1} &= [\alpha^{-1},\alpha_1^{-1},\alpha_2]^{\alpha_1}=[\alpha_1\alpha\alpha_2^{-1}\alpha^{-1}\alpha_1^{-1},\alpha_2]^{\alpha_1}\\ &= [\alpha_3\beta_1\alpha_4^{-1}\alpha_2^{-1},\alpha_2\beta_3^{-1}\alpha_4^{-1}]=[\alpha_3,\alpha_2\beta_3^{-1}\alpha_4^{-1}][\alpha_2^{-1},\alpha_2\beta_3^{-1}\alpha_4^{-1}]\\ &= [\alpha_3,\alpha_2][\alpha_2,\beta_3]. \end{aligned}$$

By Hall-Witt identity, we have the following identities in F modulo $\gamma_6(F)$:

$$\begin{aligned} 1 &= [\alpha_3,\alpha,\alpha_1]^{\alpha^{-1}}[\alpha^{-1},\alpha_1^{-1},\alpha_3]^{\alpha_1}[\alpha_1,\alpha_3^{-1},\alpha^{-1}]^{\alpha_3}\\ &= [\alpha_4,\alpha_1]^{\alpha^{-1}}[\alpha_2,\alpha_3]^{-1}[\alpha_3,\alpha_1,\alpha_3^{-1}\alpha^{-1}\alpha_3]\\ &= [\alpha_4,\alpha_1][\alpha_2,\alpha_3]^{-1}[\alpha_3,\alpha_1,\alpha_4\alpha^{-1}]\\ &= [\alpha_4,\alpha_1][\alpha_2,\alpha_3]^{-1}[\alpha_3,\alpha_1,\alpha^{-1}], \qquad (4.1.3) \end{aligned}$$

$$\begin{aligned}1 &= [\alpha_2, \alpha, \alpha_1]^{\alpha^{-1}}[\alpha^{-1}, \alpha_1^{-1}, \alpha_2]^{\alpha_1}[\alpha_1, \alpha_2^{-1}, \alpha^{-1}]^{\alpha_2} \\ &= [\alpha_3, \alpha_1]^{\alpha^{-1}}[\alpha_3, \alpha_2][\alpha_2, \beta_3][\beta_3^{-1}\alpha_4^{-1}, \alpha_2^{-1}\alpha^{-1}\alpha_2] \\ &= [\alpha_3, \alpha_1][\alpha_3, \alpha_1, \alpha^{-1}][\alpha_3, \alpha_2][\alpha_2, \beta_3][\beta_3^{-1}\alpha_4^{-1}, \alpha_3\alpha^{-1}] \\ &= [\alpha_3, \alpha_1][\alpha_3, \alpha_1, \alpha]^{-1}[\alpha_3, \alpha_2][\alpha_2, \beta_3][\beta_3, \alpha][\alpha_4, \alpha]. \qquad (4.1.4)\end{aligned}$$

Consider

$$H_1 = F/\langle \gamma_6(F), F^p, \beta_1, \beta_3, [\alpha_3, \alpha_1, \alpha], [\alpha_3, \alpha_1, \alpha_1]\rangle.$$

By (4.1.3) and (4.1.4), respectively, we have $[\alpha_4, \alpha_1] = [\alpha_2, \alpha_3]$ and $[\alpha_1, \alpha_3] = [\alpha_4, \alpha][\alpha_3, \alpha_2]$ in H_1. Observe that

$$\begin{aligned}H_1 \cong \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_2 \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4, [\alpha_1, \alpha_4] = [\alpha_3, \alpha_2] = \\ [\alpha_1, \alpha_3] = \beta_2, \alpha^p = \alpha_1^p = \alpha_{i+1}^p = \beta_2^p = 1 \ (i = 1, 2, 3)\rangle.\end{aligned}$$

Then H_1 is the group $\Phi_{39}(1^6)$ of order p^6 in [27]. Now define

$$H_2 = F/\langle \gamma_6(F), F^p, \beta_3, [\alpha_3, \alpha_1, \alpha], [\alpha_3, \alpha_1, \alpha_1]\rangle.$$

Again, by (4.1.3) and (4.1.4), we have $[\alpha_4, \alpha_1] = [\alpha_2, \alpha_3]$ and $[\alpha_1, \alpha_3] = [\alpha_4, \alpha][\alpha_3, \alpha_2]$ in H_2; hence

$$\begin{aligned}H_2 \cong \langle \alpha, \alpha_i, \alpha_4, \beta_1, \beta_2 \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4, [\alpha_4, \alpha] = \beta_1, [\alpha_1, \alpha_4] = \\ [\alpha_3, \alpha_2] = \beta_2, [\alpha_1, \alpha_3] = \beta_1\beta_2, \alpha^p = \alpha_{i+1}^p = \beta_1^p = \beta_2^p = 1, \ 1 \leq i \leq 3\rangle.\end{aligned}$$

Notice that $|H_2| = p^7$.

Finally consider

$$H = F/\langle \gamma_6(F), F^p, [\alpha_3, \alpha_1, \alpha], [\alpha_3, \alpha_1, \alpha_1], [\beta_3, \alpha], [\beta_3, \alpha_1]\rangle.$$

It follows that

$$\begin{aligned} H \;\cong\; & \langle \alpha, \alpha_j, \beta_i \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4\beta_3, [\alpha_4, \alpha] = \beta_1, [\alpha_1, \alpha_4] = \\ & [\alpha_3, \alpha_2] = \beta_2, [\alpha_1, \alpha_3] = \beta_1\beta_2, \alpha^p = \alpha_{i+1}^p = \beta_i^p = 1,\ 1 \leq i \leq 3,\ 1 \leq j \leq 4\rangle. \end{aligned}$$

Notice that $|H| = p^8$ and $H/\langle \beta_1, \beta_2, \beta_3\rangle \cong G$. Then, as in the preceding case, we have

$$\mathrm{M}(G) \cong \mathbb{Z}_p^{(3)}.$$

As $|G'| = p^3$, we get

$$|G \wedge G| = p^6.$$

Since $\alpha_4 \in \mathrm{Z}(G)$, by Lemma 1.3.3$(viii)$,

$$[\alpha_2, \alpha_4^\phi] = [\alpha_1, \alpha, \alpha_4^\phi] = 1 = [\alpha_2, \alpha, \alpha_4^\phi] = [\alpha_3, \alpha_4^\phi].$$

Hence, by Proposition 1.3.5, $G \wedge G$ is generated by the set

$$\{[\alpha_1, \alpha^\phi], [\alpha_2, \alpha^\phi], [\alpha_3, \alpha^\phi], [\alpha_4, \alpha^\phi], [\alpha_2, \alpha_1^\phi], [\alpha_3, \alpha_1^\phi], [\alpha_4, \alpha_1^\phi], [\alpha_3, \alpha_2^\phi]\}.$$

We have

$$\begin{aligned} [\alpha^{-1}, \alpha_1^{-1}, \alpha_2^\phi]^{\alpha_1} &= [\alpha_1\alpha\alpha_2^{-1}\alpha^{-1}\alpha_1^{-1}, \alpha_2^\phi]^{\alpha_1} = [\alpha\alpha_2^{-1}\alpha^{-1}, (\alpha_1^{-1}\alpha_2\alpha_1)^\phi] \\ &= [\alpha\alpha_3\alpha^{-1}\alpha_2^{-1}, (\alpha_2\alpha_4)^\phi] = [\alpha_3\alpha_4^{-1}\alpha_2^{-1}, (\alpha_2\alpha_4)^\phi] \\ &= [\alpha_4, \alpha_2^\phi]^{-1}[\alpha_3, \alpha_2^\phi] = [\alpha_3, \alpha_2^\phi], \end{aligned}$$

$$\begin{aligned} [\alpha^{-1}, \alpha_1^{-1}, \alpha_3^\phi]^{\alpha_1} &= [\alpha^{-1}, \alpha_1^{-1}, \alpha_3^\phi]^{\alpha_1} = [\alpha_1\alpha\alpha_2^{-1}\alpha^{-1}\alpha_1^{-1}, \alpha_3^\phi]^{\alpha_1} \\ &= [\alpha\alpha_2^{-1}\alpha^{-1}, \alpha_3^\phi] = [\alpha\alpha_3\alpha^{-1}\alpha_2^{-1}, \alpha_3^\phi] \end{aligned}$$

$$= [\alpha_3\alpha_4\alpha_2^{-1}, \alpha_3^{\phi}] = [\alpha_2, \alpha_3^{\phi}]^{-1}.$$

By Hall-Witt identity, we have

$$\begin{aligned}
1 &= [\alpha_2, \alpha, \alpha_1^{\phi}]^{\alpha^{-1}}[\alpha^{-1}, \alpha_1^{-1}, \alpha_2^{\phi}]^{\alpha_1}[\alpha_1, \alpha_2^{-1}, (\alpha^{-1})^{\phi}]^{\alpha_2} \\
&= [\alpha_3, \alpha_1^{\phi}][\alpha_3, \alpha_2^{\phi}][\alpha_4^{-1}, (\alpha^{-1})^{\phi}]^{\alpha_2} \\
&= [\alpha_3, \alpha_1^{\phi}][\alpha_3, \alpha_2^{\phi}][\alpha_4, \alpha^{\phi}], \\
1 &= [\alpha_3, \alpha, \alpha_1^{\phi}]^{\alpha^{-1}}[\alpha^{-1}, \alpha_1^{-1}, \alpha_3^{\phi}]^{\alpha_1}[\alpha_1, \alpha_3^{-1}, (\alpha^{-1})^{\phi}]^{\alpha_3} \\
&= [\alpha_4, \alpha_1^{\phi}][\alpha_2, \alpha_3^{\phi}]^{-1}.
\end{aligned}$$

This implies that $[\alpha_4, \alpha_1^{\phi}] = [\alpha_2, \alpha_3^{\phi}] = [\alpha_4, \alpha^{\phi}][\alpha_3, \alpha_1^{\phi}]$. By Lemma 1.3.3$(vi)$, $[[\alpha_1, \alpha^{\phi}], [\alpha_2, \alpha^{\phi}]] = [\alpha_2, \alpha_3^{\phi}]$. By Lemma 1.3.2, we have the following identities

$$\begin{aligned}
[\alpha_4, \alpha^{\phi}]^p = [\alpha_4^p, \alpha^{\phi}] = 1 = [\alpha_3^p, \alpha^{\phi}] = [\alpha_3, \alpha^{\phi}]^p, \\
[\alpha_3, \alpha_2^{\phi}]^p = [\alpha_3^p, \alpha_2^{\phi}] = 1 = [\alpha_2^p, \alpha_1^{\phi}] = [\alpha_2, \alpha_1^{\phi}]^p, \\
[\alpha_2, \alpha^{\phi}]^p = [\alpha_2^p, \alpha^{\phi}] = 1 = [\alpha_1^p, \alpha^{\phi}] = [\alpha_1, \alpha^{\phi}]^p.
\end{aligned}$$

Hence

$$\begin{aligned}
G \wedge G &\cong \langle [\alpha_1, \alpha^{\phi}], [\alpha_2, \alpha^{\phi}], [\alpha_2, \alpha_3^{\phi}] \rangle \times \langle [\alpha_2, \alpha_1^{\phi}] \rangle \times \langle [\alpha_3, \alpha^{\phi}] \rangle \times \langle [\alpha_4, \alpha^{\phi}] \rangle \\
&\cong \Phi_2(111) \times \mathbb{Z}_p^{(3)}.
\end{aligned}$$

The proof is now complete. $\square$

4.1.3 Classes $\Phi_2, \Phi_3, \Phi_6, \Phi_7, \Phi_8$: Remaining Groups of order p^5

We start with the groups which occur as direct product of groups of smaller orders. For such groups we compute the Schur multiplier in the following lemma, whose proof follows from Theorem 1.2.3 and Theorem 1.4.1.

Lemma 4.1.7 *The following assertions hold:*

(i) $\mathrm{M}(\Phi_2(311)a) \cong \mathrm{M}(\Phi_2(221)b) \cong \mathbb{Z}_p \times \mathbb{Z}_p$,

(ii) $\mathrm{M}(\Phi_2(221)a) \cong \mathbb{Z}_p^{(3)}$,

(iii) $\mathrm{M}(\Phi_2(2111)a) \cong \mathrm{M}(\Phi_2(2111)b) \cong \mathbb{Z}_p^{(5)}$,

(iv) $\mathrm{M}(\Phi_2(2111)c) \cong \mathrm{M}(\Phi_2(2111)d) \cong \mathbb{Z}_p^{(4)}$,

(v) $\mathrm{M}(\Phi_2(1^5)) \cong \mathbb{Z}_p^{(7)}$,

(vi) $\mathrm{M}(\Phi_3(2111)a) \cong \mathrm{M}(\Phi_3(2111)b_r) \cong \mathbb{Z}_p^{(3)}$,

(vii) $\mathrm{M}(\Phi_3(1^5)) \cong \mathbb{Z}_p^{(4)}$.

The proof of the following lemma is a direct consequence of Proposition 1.3.8 and Theorem 1.4.1.

Lemma 4.1.8 *The following assertions hold:*

(i) $\Phi_2(311)a \otimes \Phi_2(311)a \cong \Phi_2(221)b \otimes \Phi_2(221)b \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(8)}$,

(ii) $\Phi_2(221)a \otimes \Phi_2(221)a \cong \mathbb{Z}_{p^2}^{(2)} \times \mathbb{Z}_p^{(7)}$,

(iii) $\Phi_2(2111)a \otimes \Phi_2(2111)a \cong \Phi_2(2111)b \otimes \Phi_2(2111)b \cong \mathbb{Z}_p^{(16)}$,

(iv) $\Phi_2(2111)c \otimes \Phi_2(2111)c \cong \Phi_2(2111)d \otimes \Phi_2(2111)d \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(10)}$,

(v) $\Phi_2(1^5) \otimes \Phi_2(1^5) \cong \mathbb{Z}_p^{(18)}$,

(vi) $\Phi_3(2111)a \otimes \Phi_3(2111)a \cong \Phi_3(2111)b_r \otimes \Phi_3(2111)b_r \cong \mathbb{Z}_p^{(11)}$,

(vii) $\Phi_3(1^5) \otimes \Phi_3(1^5) \cong \mathbb{Z}_p^{(12)}$.

Lemma 4.1.9 *If G is one of the groups $\Phi_2(41), \Phi_2(32)a_1$, $\Phi_2(32)a_2$ or $\Phi_8(32)$, then $\mathrm{M}(G)$ is isomorphic to $\{1\}, \mathbb{Z}_p$, $\mathbb{Z}_p$ or $\{1\}$ respectively, and $G \wedge G$ is isomorphic to $\mathbb{Z}_p, \mathbb{Z}_{p^2}, \mathbb{Z}_{p^2}$ or $\mathbb{Z}_{p^2}$ respectively.*

Proof. Since these groups are metacyclic, the assertion about the Schur multipliers follows from [30, Theorem 2.11.3]. Now we compute the exterior square. Since the Schur multiplier of the groups G isomorphic to $\Phi_2\,(41)$ or $\Phi_8\,(32)$ is trivial, we have $G \wedge G \cong G'$. Hence

$$\Phi_2\,(41) \wedge \Phi_2\,(41) \cong \mathbb{Z}_p$$

and

$$\Phi_8\,(32) \wedge \Phi_8\,(32) \cong \mathbb{Z}_{p^2}.$$

Now we consider $G = \Phi_2\,(32)\,a_1$. By Lemma 1.3.2, we have the following identities

$$\begin{aligned}
&[\alpha_2, \alpha^\phi] = [\alpha^{p^2}, \alpha^\phi] = [\alpha, \alpha^\phi]^{p^2} = 1 = [\alpha_2^p, \alpha_1^\phi] = [\alpha_2, \alpha_1^\phi]^p,\\
&[\alpha_2, \alpha_1^\phi] = [\alpha^{p^2}, \alpha_1^\phi] = [\alpha, \alpha_1^\phi]^{p^2} = [\alpha, (\alpha_1^{p^2})^\phi] = 1.
\end{aligned}$$

Hence, by Proposition 1.3.5, $G \wedge G$ is a cyclic group generated by $[\alpha_1, \alpha^\phi]$. Since both $\mathrm{M}(G)$ and $\gamma_2(G)$ are of order p, it follows that

$$G \wedge G \cong \langle [\alpha_1, \alpha^\phi] \rangle \cong \mathbb{Z}_{p^2}.$$

Similarly for $G = \Phi_2\,(32)\,a_2$, we have

$$[\alpha_2, \alpha_1^{\phi}] = [\alpha_1^p, \alpha_1^{\phi}] = [\alpha_1, \alpha_1^{\phi}]^p = 1,$$
$$[\alpha_2, \alpha^{\phi}] = [\alpha_1^p, \alpha^{\phi}] = [\alpha_1, \alpha^{\phi}]^p.$$

Hence $G \wedge G$ is generated by $[\alpha_1, \alpha^{\phi}]$. As above,

$$G \wedge G \cong \langle [\alpha_1, \alpha^{\phi}] \rangle \cong \mathbb{Z}_{p^2},$$

and the proof is complete. $\square$

Lemma 4.1.10 *If G is one of the groups $\Phi_2(311)b$ or $\Phi_2(311)c$, then $\mathrm{M}(G)$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$, and $G \wedge G$ is isomorphic to $\mathbb{Z}_p^{(3)}$.*

Proof. For the group $G = \Phi_2(311)b$, taking $K = \mathrm{Z}(G)$ in Theorem 1.2.6, we have $|\,\mathrm{M}(G)| \leq p^2$. On the other hand, taking $K = \Phi(G)$ in Theorem 1.2.5(ii), we get $d(\mathrm{M}(G)) \geq 2$. Hence,

$$\mathrm{M}(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p.$$

By Proposition 1.3.5, the group $G \wedge G$ is generated by the set

$$\{[\alpha_1, \alpha^{\phi}], [\alpha_2, \alpha^{\phi}], [\alpha_2, \alpha_1^{\phi}], [\gamma, \alpha^{\phi}], [\gamma, \alpha_1^{\phi}], [\gamma, \alpha_2^{\phi}]\}.$$

By Lemma 1.3.3($viii$),

$$[\alpha_2, \gamma^{\phi}] = [\alpha_1, \alpha, \gamma^{\phi}] = 1.$$

By Lemma 1.3.2, we get the following set of identities:

$$[\gamma, \alpha^{\phi}]^p = [\gamma, (\alpha^p)^{\phi}] = 1 = [\gamma, (\alpha_1^p)^{\phi}] = [\gamma, \alpha_1^{\phi}]^p,$$
$$[\alpha_2, \alpha^{\phi}] = [\gamma^{p^2}, \alpha^{\phi}] = [\gamma, \alpha^{\phi}]^{p^2} = 1,$$

$$[\alpha_2, \alpha_1^{\phi}] = [\gamma^{p^2}, \alpha_1^{\phi}] = [\gamma, \alpha_1^{\phi}]^{p^2} = 1,$$
$$[\alpha, \alpha_1^{\phi}]^p = [\alpha^p, \alpha_1^{\phi}] = 1.$$

Hence, by Proposition 1.3.5, $G \wedge G$ is generated by $\{[\alpha_1, \alpha^{\phi}], [\gamma, \alpha^{\phi}], [\gamma, \alpha_1^{\phi}]\}$. Since the nilpotency class of G is 2, by Lemma 1.3.6, $G \wedge G$ is abelian. Hence,

$$G \wedge G \cong \mathbb{Z}_p^{(3)}.$$

Now consider the group $G = \Phi_2(311)c$. Again using Theorem 1.2.6 with $K = G'$, we get $|\,\mathrm{M}(G)| \leq p^2$. By Theorem 1.2.5$(ii)$ with $K = \langle \alpha^p \rangle$, we have $d(\mathrm{M}(G)) \geq 2$. Hence

$$\mathrm{M}(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p.$$

Since $|G'| = p$, it follows that $G \wedge G$ is of order p^3.

By Lemma 1.3.2, we have the following:

$$[\alpha_2, \alpha_1^{\phi}]^p = [\alpha_2^p, \alpha_1^{\phi}] = 1,$$
$$[\alpha_2, \alpha^{\phi}]^p = [\alpha_2^p, \alpha^{\phi}] = 1,$$
$$[\alpha_1, \alpha^{\phi}]^p = [\alpha_1^p, \alpha^{\phi}] = 1.$$

Consequently, by Proposition 1.3.5, $G \wedge G$ is generated by $\{[\alpha_1, \alpha^{\phi}], [\alpha_2, \alpha^{\phi}], [\alpha_2, \alpha_1^{\phi}]\}$, which shows that

$$G \wedge G \cong \mathbb{Z}_p^{(3)}.$$

The proof is now complete. □

Lemma 4.1.11 *If G is one of the groups $\Phi_3(311)a, \Phi_3(311)b_r$ or $\Phi_2(221)d$, then $\mathrm{M}(G)$ is isomorphic to $\mathbb{Z}_p, \mathbb{Z}_p$ or $\mathbb{Z}_p^{(3)}$ respectively, and $G \wedge G$ is isomorphic to $\mathbb{Z}_p^{(3)}, \mathbb{Z}_p^{(3)}$ or $\mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p$ respectively.*

Proof. Let G be one of the groups $\Phi_3(311)a$ or $\Phi_3(311)b_r$. Then taking $K = \mathrm{Z}(G)$ in Theorem 1.2.5(i), $p \leq |\,\mathrm{M}(G)|$. Since $|G'| = p^2$, it follows that $|G \wedge G| \geq p^3$.

For the group $G = \Phi_3\,(311)\,a$, by Lemma 1.3.3$(viii)$, $[\alpha_2, \alpha_3^{\phi}] = [\alpha_1, \alpha, \alpha_3^{\phi}] = 1$. It now follows, by Proposition 1.3.5, that $G \wedge G$ is generated by the set

$$\{[\alpha_1, \alpha^{\phi}], [\alpha_2, \alpha^{\phi}], [\alpha_2, \alpha_1^{\phi}], [\alpha_3, \alpha^{\phi}], [\alpha_3, \alpha_1^{\phi}]\}.$$

By Lemma 1.3.2, The following identities hold:

$$\begin{aligned}
&[\alpha_3, \alpha^{\phi}] = [\alpha^{p^2}, \alpha^{\phi}] = [\alpha, \alpha^{\phi}]^{p^2} = 1,\\
&[\alpha_3, \alpha_1^{\phi}]^p = [\alpha_3^p, \alpha_1^{\phi}] = 1,\\
&[\alpha_2, \alpha^{\phi}]^p = [\alpha_2^p, \alpha^{\phi}] = 1 = [\alpha_2^p, \alpha_1^{\phi}] = [\alpha_2, \alpha_1^{\phi}]^p,\\
&[\alpha_1, \alpha^{\phi}]^p = [\alpha_1^p, \alpha^{\phi}] = 1,\\
&[\alpha_3, \alpha_1^{\phi}] = [\alpha^{p^2}, \alpha_1^{\phi}] = [\alpha, \alpha_1^{\phi}]^{p^2} = 1.
\end{aligned}$$

Hence $G \wedge G$ is generated by $\{[\alpha_1, \alpha^{\phi}], [\alpha_2, \alpha^{\phi}], [\alpha_2, \alpha_1^{\phi}]\}$. Since, in view of Lemma 1.3.3(vi), $G \wedge G$ is abelian, we have

$$G \wedge G \cong \mathbb{Z}_p^{(3)}.$$

Consequently,

$$\mathrm{M}(G) \cong [\alpha_1, \alpha_2^{\phi}]\rangle \cong \mathbb{Z}_p.$$

Consider the group $G = \Phi_3(311)b_r$, for $r = 1, \nu$. By Lemma 1.3.3$(viii)$, $[\alpha_2, \alpha_3^{\phi}] = [\alpha_1, \alpha, \alpha_3^{\phi}] = 1$. It now follows, by Proposition 1.3.5, that $G \wedge G$ is generated by the set

$$\{[\alpha_1, \alpha^{\phi}], [\alpha_2, \alpha^{\phi}], [\alpha_2, \alpha_1^{\phi}], [\alpha_3, \alpha^{\phi}], [\alpha_3, \alpha_1^{\phi}]\}.$$

By Lemma 1.3.2, The following identities hold:

$$[\alpha_3,\alpha_1^{\phi}]=[\alpha_1^{p^2},\alpha_1^{\phi}]=[\alpha_1,\alpha_1^{\phi}]^{p^2}=1,$$
$$[\alpha_3,\alpha^{\phi}]^p=[\alpha_3^p,\alpha^{\phi}]=1,$$
$$[\alpha_2,\alpha^{\phi}]^p=[\alpha_2^p,\alpha^{\phi}]=1=[\alpha_2^p,\alpha_1^{\phi}]=[\alpha_2,\alpha_1^{\phi}]^p,$$
$$[\alpha_1,\alpha^{\phi}]^p=[\alpha_1,(\alpha^p)^{\phi}]=1,$$
$$[\alpha_3,\alpha^{\phi}]=[\alpha_1^{p^2},\alpha^{\phi}]=[\alpha_1,\alpha^{\phi}]^{p^2}=1.$$

Hence $G\wedge G$ is generated by $\{[\alpha_1,\alpha^{\phi}],[\alpha_2,\alpha^{\phi}],[\alpha_2,\alpha_1^{\phi}]\}$. Since, in view of Lemma 1.3.3(vi), $G\wedge G$ is abelian, we have

$$G\wedge G\cong\mathbb{Z}_p^{(3)}.$$

Consequently,

$$\mathrm{M}(G)\cong[\alpha_1,\alpha_2^{\phi}]\rangle\cong\mathbb{Z}_p.$$

Finally we consider the group $G=\Phi_2(221)d$. By Proposition 1.3.5, the group $G\wedge G$ is generated by the set

$$\{[\alpha_1,\alpha^{\phi}],[\alpha_2,\alpha^{\phi}],[\alpha_1,\alpha_2^{\phi}]\}.$$

By Lemma 1.3.6, $G\wedge G$ is abelian. We have the identities, by Lemma 1.3.2:

$$[\alpha_2,\alpha_1^{\phi}]^p=[\alpha_2^p,\alpha_1^{\phi}]=1=[\alpha_2^p,\alpha^{\phi}]=[\alpha_2,\alpha^{\phi}]^p,$$
$$[\alpha_1,\alpha^{\phi}]^{p^2}=[\alpha_1^{p^2},\alpha^{\phi}]=1.$$

Now the natural epimorphism $[G,G^{\phi}]\to[G^{ab},(G^{ab})^{\phi}]$ implies that the order of $[\alpha_1,\alpha^{\phi}]$ is p^2 in $G\wedge G$. Notice that $G/\langle\alpha_1^p\rangle\cong\Phi_2(211)c$. Consider the natural epimorphism

$$[G, G^{\phi}] \to [G/\langle\alpha_1^p\rangle, (G/\langle\alpha_1^p\rangle)^{\phi}] \cong [\Phi_2(211)c, (\Phi_2(211)c)^{\phi}],$$

which induces an epimorphism $G \wedge G \to (\Phi_2(211)c) \wedge (\Phi_2(211)c)$. By Theorem 1.4.1, we know that $\Phi_2(211)c \wedge \Phi_2(211)c \cong \mathbb{Z}_p^{(3)}$. Hence the generators $[\alpha_1, \alpha^{\phi}], [\alpha_2, \alpha^{\phi}], [\alpha_1, \alpha_2^{\phi}]$ of $G \wedge G$ are non-trivial, independent and $[\alpha_2, \alpha^{\phi}], [\alpha_1, \alpha_2^{\phi}]$ have order p in $G \wedge G$. As a consequence, we get

$$G \wedge G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p,$$

which, by observing the fact that $G' = \langle[\alpha_1, \alpha]\rangle$, gives

$$\mathrm{M}(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p.$$

This completes the proof. □

Lemma 4.1.12 *If G is one of the groups $\Phi_3(221)a$ or $\Phi_2(221)c$, then $\mathrm{M}(G)$ is isomorphic to $\mathbb{Z}_p$ or $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$ respectively, and $G \wedge G$ is isomorphic to $\mathbb{Z}_p^{(3)}$ or $\mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p$ respectively.*

Proof. The group $G = \Phi_3(221)a$ is a semidirect product of its subgroups

$$N = \langle\alpha_2, \alpha, \alpha_3\rangle \cong \Phi_2(21)$$

and

$$T = \langle\alpha_1\rangle \cong \mathbb{Z}_{p^2},$$

where N is normal.

By Theorem 1.4.1, we know that $\mathrm{M}(N) = 1$. Now using Theorem 1.2.4, we get the following exact sequence

$$1 \to \mathrm{H}^1(T, \mathrm{Hom}(N, \mathbb{C}^*)) \to \mathrm{H}^2(G, \mathbb{C}^*) \to 1.$$

Hence,

$$\mathrm{M}(G) \cong \mathrm{H}^1(T, \mathrm{Hom}(N, \mathbb{C}^*)).$$

Notice that $\mathrm{Hom}(N, \mathbb{C}^*) \cong N/N' \cong \langle \alpha N', \alpha_2 N' \rangle$. Let ζ be a primitive p-th root of unity and $\mathrm{Hom}(N, \mathbb{C}^*) \cong \langle \phi_1, \phi_2 \rangle$, where $\phi_i : N \to \mathbb{C}^*$, $i = 1, 2$, are defined by setting

$$\phi_1(\alpha) = \zeta, \quad \phi_1(\alpha_2) = 1$$

and

$$\phi_2(\alpha) = 1, \quad \phi_2(\alpha_2) = \zeta^{-1}.$$

Recall that T acts on $\mathrm{Hom}(N, \mathbb{C}^*)$ as follows. For $\phi_1 \in \mathrm{Hom}(N, \mathbb{C}^*)$, we set

$$^{\alpha_1}\phi_1(\alpha) = \phi_1(\alpha_1^{-1}\alpha\alpha_1) = \phi_1(\alpha)$$

and

$$^{\alpha_1}\phi_1(\alpha_2) = \phi_1(\alpha_1^{-1}\alpha_2\alpha_1) = \phi_1(\alpha_2).$$

So, $^{\alpha_1}\phi_1 = \phi_1$. Similarly the action of α_1 on ϕ_2 is given by $^{\alpha_1}\phi_2 = \phi_1\phi_2$.

Define the map $Norm : \mathrm{Hom}(N, \mathbb{C}^*) \to \mathrm{Hom}(N, \mathbb{C}^*)$ given by

$$Norm(\phi) = \, ^{(1+\alpha_1+\alpha_1^2+\cdots+\alpha_1^{p-1})}\phi.$$

It is easy to check that $\mathrm{Ker}(Norm) = \mathrm{Hom}(N, \mathbb{C}^*)$. Define $\beta : \mathrm{Hom}(N, \mathbb{C}^*) \to \mathrm{Hom}(N, \mathbb{C}^*)$ by

$$\beta(\phi) = \, ^{(\alpha_1 - 1)}\phi$$

Then $\mathrm{Im}(\beta) = \langle \phi_1 \rangle$. It is a general fact (see Step 3 in the proof of Theorem 5.4 of [22]) that

$$\mathrm{H}^1(T, \mathrm{Hom}(N, \mathbb{C}^*)) \cong \frac{\mathrm{Ker}(Norm)}{\mathrm{Im}(\beta)} \cong \mathbb{Z}_p.$$

Hence,

$$\mathrm{M}(G) \cong \mathbb{Z}_p.$$

By Lemma 1.3.3$(viii)$, we have

$$[\alpha_2, \alpha_3^{\phi}] = [\alpha_1, \alpha, \alpha_3^{\phi}] = 1.$$

By Lemma 1.3.2, we have the following identities:

$$[\alpha_3, \alpha^{\phi}] = [\alpha^p, \alpha^{\phi}] = [\alpha, \alpha^{\phi}]^p = 1,$$
$$[\alpha_2, \alpha^{\phi}]^p = [\alpha_2^p, \alpha^{\phi}] = 1 = [\alpha_2^p, \alpha_1^{\phi}] = [\alpha_2, \alpha_1^{\phi}]^p,$$
$$[\alpha_3, \alpha_1^{\phi}] = [\alpha^p, \alpha_1^{\phi}] = [\alpha, \alpha_1^{\phi}]^p.$$

Now we get

$$\begin{aligned}
[\alpha^{-1}, \alpha_1^{-1}, \alpha_2^{\phi}]^{\alpha_1} &= [\alpha_1 \alpha \alpha_2^{-1} \alpha^{-1} \alpha_1^{-1}, \alpha_2^{\phi}]^{\alpha_1} = [\alpha \alpha_2^{-1} \alpha^{-1}, \alpha_2^{\phi}] \\
&= [\alpha_3 \alpha_2^{-1}, \alpha_2^{\phi}] = [\alpha_3, \alpha_2^{\phi}][\alpha_2, \alpha_2^{\phi}]^{-1} \\
&= 1.
\end{aligned}$$

By Hall-Witt identity,

$$\begin{aligned}
1 &= [\alpha_2, \alpha, \alpha_1^{\phi}]^{\alpha^{-1}} [\alpha^{-1}, \alpha_1^{-1}, \alpha_2^{\phi}]^{\alpha_1} \\
&= [\alpha_3, \alpha_1^{\phi}].
\end{aligned}$$

Hence, $G \wedge G$ is generated by the set

$$\{[\alpha_1, \alpha^{\phi}], [\alpha_2, \alpha^{\phi}], [\alpha_2, \alpha_1^{\phi}]\}.$$

By Lemma 1.3.3(vi) $G \wedge G$ is abelian, and therefore

$$G \wedge G \cong \mathbb{Z}_p^{(3)}.$$

The group $G = \Phi_2(221)c$ is a semi-direct product of its normal subgroup $N = \langle \alpha_1, \gamma \rangle$ and $T = \langle \alpha \rangle$. Now using Theorem 1.2.4, we get the following exact sequence

$$1 \to \mathrm{H}^1(T, \mathrm{Hom}(N, \mathbb{C}^*)) \to \mathrm{H}^2(G, \mathbb{C}^*) \to \mathrm{H}^2(N, \mathbb{C}^*)^T.$$

As above, $\mathrm{H}^1(T, \mathrm{Hom}(N, \mathbb{C}^*)) \cong \mathbb{Z}_{p^2}$, which embeds in $\mathrm{M}(G)$. Now by Theorem 1.2.5(ii), taking $K = \langle \gamma^p \rangle$, we have $2 \leq d(\mathrm{M}(G))$. By Theorem 1.2.6, taking $K = \langle \gamma \rangle$, we have $|\,\mathrm{M}(G)| \leq p^3$. Hence

$$\mathrm{M}(G) \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p.$$

We have the following identities:

$$[\alpha_2, \gamma^{\phi}] = [\alpha_1, \alpha, \gamma^{\phi}] = 1, \text{ by Lemma 1.3.3}(viii),$$

$$[\alpha_2, \alpha_1^{\phi}] = [\gamma^p, \alpha_1^{\phi}] = [\gamma, \alpha_1^{\phi}]^p = [\gamma, (\alpha_1^p)^{\phi}] = 1,$$

$$[\alpha_2, \alpha^{\phi}] = [\gamma^p, \alpha^{\phi}] = [\gamma, \alpha^{\phi}]^p,$$

$$[\alpha_1, \alpha^{\phi}]^p = [\alpha_1^p, \alpha^{\phi}] = 1.$$

Hence $G \wedge G$ is generated by the set

$$\{[\alpha_1, \alpha^\phi], [\gamma, \alpha^\phi], [\gamma, \alpha_1^\phi]\}.$$

Since the nilpotency class of G is 2, by Lemma 1.3.6, $G \wedge G$ is abelian; hence

$$G \wedge G \cong \langle [\gamma, \alpha^\phi] \rangle \times \langle [\gamma, \alpha_1^\phi] \rangle \times \langle [\alpha_1, \alpha^\phi] \rangle \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p.$$

The proof is now complete. $\square$

Lemma 4.1.13 *If G is one of the groups $\Phi_3(2111)c, \Phi_3(2111)d$ or $\Phi_3(2111)e$, then $\mathrm{M}(G)$ is isomorphic to $\mathbb{Z}_p^{(3)}$, $\mathbb{Z}_p \times \mathbb{Z}_p$ or $\mathbb{Z}_p \times \mathbb{Z}_p$ respectively, and $G \wedge G$ is isomorphic to $\mathbb{Z}_p^{(5)}, \mathbb{Z}_p^{(4)}$ or $\mathbb{Z}_p^{(4)}$ respectively.*

Proof. For the group $G = \Phi_3(2111)c$, by Theorem 1.2.6, taking $K = \mathrm{Z}(G)$, we get $|\,\mathrm{M}(G)| \leq p^3$, and by Theorem 1.2.5($i$), taking $K = \langle \gamma^p \rangle$, it follows that p^3 divides $|\,\mathrm{M}(G)|$; hence $|\,\mathrm{M}(G)| = p^3$.

By Lemma 1.3.2, we have the following identities:

$$[\alpha_2, \gamma^\phi] = [\alpha_1, \alpha, \gamma^\phi] = 1 = [\alpha_2, \alpha, \gamma^\phi] = [\alpha_3, \gamma^\phi], \text{ by Lemma 1.3.3}(viii).$$

$$[\alpha_3, \alpha^\phi] = [\gamma^p, \alpha^\phi] = [\gamma, \alpha^\phi]^p = [\gamma, (\alpha^p)^\phi] = 1,$$

$$[\alpha_3, \alpha_i^\phi] = [\gamma^p, \alpha_i^\phi] = [\gamma, \alpha_i^\phi]^p = [\gamma, (\alpha_i^p)^\phi] = 1 \text{ for } i = 1, 2,$$

$$[\alpha_2, \alpha_1^\phi]^p = [\alpha_2^p, \alpha_1^\phi] = 1 = [\alpha_2^p, \alpha^\phi] = [\alpha_2, \alpha^\phi]^p,$$

$$[\alpha_1, \alpha^\phi]^p = [\alpha_1^p, \alpha^\phi] = 1.$$

The group $G \wedge G$ is generated by

$$\{[\alpha_1, \alpha^\phi], [\alpha_2, \alpha^\phi], [\alpha_2, \alpha_1^\phi], [\gamma, \alpha^\phi], [\gamma, \alpha_1^\phi]\},$$

and every generator has order at most p. By Lemma 1.3.3(vi), $G \wedge G$ is abelian.

Hence

$$\mathrm{M}(G) \cong \mathbb{Z}_p^{(3)}.$$

Since $|G'| = p^2$, we get

$$G \wedge G \cong \mathbb{Z}_p^{(5)}.$$

Consider the group $G = \Phi_3(2111)d$. By Theorem 1.2.5(i), taking $K = \langle \alpha^p \rangle$, $p^2 \leq \mathrm{M}(G)$. Since $|G'| = p^2$, we have

$$|G \wedge G| \geq p^4.$$

By Lemma 1.3.3($viii$),

$$[\alpha_2, \alpha_3^\phi] = [\alpha_1, \alpha, \alpha_3^\phi] = 1.$$

We have

$$\begin{aligned}
[\alpha^{-1}, \alpha_1^{-1}, \alpha_2^\phi]^{\alpha_1} &= [\alpha_1 \alpha \alpha_2^{-1} \alpha^{-1} \alpha_1^{-1}, \alpha_2^\phi]^{\alpha_1} = [\alpha \alpha_2^{-1} \alpha^{-1}, \alpha_2^\phi] \\
&= [\alpha_3 \alpha_2^{-1}, \alpha_2^\phi] = [\alpha_3, \alpha_2^\phi][\alpha_2, \alpha_2^\phi]^{-1} = [\alpha_2, \alpha_2^\phi]^{-1} \\
&= 1.
\end{aligned}$$

By Hall-Witt identity,

$$\begin{aligned}
1 &= [\alpha_2, \alpha, \alpha_1^\phi]^{\alpha^{-1}} [\alpha^{-1}, \alpha_1^{-1}, \alpha_2^\phi]^{\alpha_1} \\
&= [\alpha_3, \alpha_1^\phi].
\end{aligned}$$

By Lemma 1.3.2, we have the following identities:

$$\begin{aligned}
&[\alpha_3, \alpha^\phi]^p = [\alpha_3^p, \alpha^\phi] = 1 = [\alpha_2^p, \alpha_1^\phi] = [\alpha_2, \alpha_1^\phi]^p, \\
&[\alpha_2, \alpha^\phi]^p = [\alpha_2^p, \alpha^\phi] = 1 = [\alpha_1^p, \alpha^\phi] = [\alpha_1, \alpha^\phi]^p.
\end{aligned}$$

So, by Proposition 1.3.5, $G \wedge G$ is generated by the set

$$\{[\alpha_1, \alpha^{\phi}], [\alpha_2, \alpha^{\phi}], [\alpha_2, \alpha_1^{\phi}], [\alpha_3, \alpha^{\phi}]\}.$$

By Lemma 1.3.3(vi), it follows that $G \wedge G$ is elementary abelian p-group, and hence $|G \wedge G| \leq p^4$. Thus,

$$G \wedge G \cong \mathbb{Z}_p^{(4)}$$

and

$$\mathrm{M}(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p.$$

For the group $G = \Phi_3(2111)e$, by Theorem 1.2.5(i), taking $K = \langle \alpha_1^p \rangle$, we get $p^2 \leq |\,\mathrm{M}(G)|$. Since $|G'| = p^2$, we have $|G \wedge G| \geq p^4$. As in the preceding case, $[\alpha_2, \alpha_3^{\phi}] = 1$ and by Hall-Witt identity, $[\alpha_3, \alpha_1^{\phi}] = 1$. By Lemma 1.3.2, we have the following identities:

$$[\alpha_3, \alpha^{\phi}]^p = [\alpha_3^p, \alpha^{\phi}] = 1 = [\alpha_2^p, \alpha_1^{\phi}] = [\alpha_2, \alpha_1^{\phi}]^p,$$
$$[\alpha_2, \alpha^{\phi}]^p = [\alpha_2^p, \alpha^{\phi}] = 1 = [\alpha_1, (\alpha^p)^{\phi}] = [\alpha_1, \alpha^{\phi}]^p.$$

So, by Proposition 1.3.5, $G \wedge G$ is generated by the set

$$\{[\alpha_1, \alpha^{\phi}], [\alpha_2, \alpha^{\phi}], [\alpha_2, \alpha_1^{\phi}], [\alpha_3, \alpha^{\phi}]\}.$$

By Lemma 1.3.3(vi), it follows that $G \wedge G$ is elementary abelian p-group, and hence $|G \wedge G| \leq p^4$. Thus,

$$G \wedge G \cong \mathbb{Z}_p^{(4)}$$

and

$$\mathrm{M}(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p.$$

The proof is now complete. $\square$

Lemma 4.1.14 *If G is one of the groups $\Phi_3(221)b_r$, $\Phi_6(221)b_{\frac{1}{2}(p-1)}$ or $\Phi_6(221)d_0$, then $\mathrm{M}(G)$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_p$ or $\mathbb{Z}_p$ respectively, and $G \wedge G$ is isomorphic to $\mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p$ in all the cases.*

Proof. From [27, Table 4.1], it follows that for the groups E of order p^6 in the isoclinism classes Φ_{25}, Φ_{26}, Φ_{42} or Φ_{43}, $E/\mathrm{Z}(E)$ is isomorphic to $\Phi_3\,(221)\,b_1$, $\Phi_3\,(221)\,b_\nu$, $\Phi_6(221)b_{\frac{1}{2}(p-1)}$ or $\Phi_6(221)d_0$ respectively. Thus, all the groups G, under consideration, are capable; hence $Z^*(G) = 1$.

For the group $G = \Phi_3\,(221)\,b_r$, by Theorem 1.2.8, taking $Z = \langle \alpha^p \rangle$, we have the following exact sequence

$$G/G' \otimes Z \xrightarrow{\lambda} \mathrm{M}(G) \to \mathbb{Z}_p \to 1.$$

By Theorem 1.2.9, it follows that $\alpha_1 \otimes \alpha^p \notin \mathrm{Ker}\,\lambda$, as $\alpha \otimes \alpha^p \in \mathrm{Ker}\,\lambda$. So $|\,\mathrm{Im}(\lambda)| = p$, and $|\,\mathrm{M}(G)| = p^2$. Since $|G'| = p^2$, $|G \wedge G| = p^4$. By Lemma 1.3.3$(viii)$,

$$[\alpha_2, \alpha_3^\phi] = [\alpha_1, \alpha, \alpha_3^\phi] = 1.$$

By Lemma 1.3.2, the following identities hold:

$$\begin{aligned}
&[\alpha_3, \alpha^\phi]^p = [\alpha_3^p, \alpha^\phi] = 1 = [\alpha_2^p, \alpha_1^\phi] = [\alpha_2, \alpha_1^\phi]^p,\\
&[\alpha_2, \alpha^\phi]^p = [\alpha_2^p, \alpha^\phi] = 1,\\
&[\alpha_3, \alpha_1^\phi] = [\alpha_1^{pr^{-1}}, \alpha_1^\phi] = [\alpha_1, \alpha_1^\phi]^{pr^{-1}} = 1,\\
&[\alpha_3, \alpha^\phi] = [\alpha_1^{pr^{-1}}, \alpha^\phi] = [\alpha_1, \alpha^\phi]^{pr^{-1}},\\
&[\alpha_1, \alpha^\phi]^{p^2} = [\alpha_1^{p^2}, \alpha^\phi] = 1.
\end{aligned}$$

Observe that $G \wedge G$ is generated by the set $\{[\alpha_1, \alpha^\phi], [\alpha_2, \alpha^\phi], [\alpha_1, \alpha_2^\phi]\}$, by Proposition 1.3.5. By Lemma 1.3.3(vi), $G \wedge G$ is abelian, and therefore

$$G \wedge G \cong \langle [\alpha_1, \alpha^{\phi}] \rangle \times \langle [\alpha_2, \alpha^{\phi}] \rangle \times \langle [\alpha_2, \alpha_1^{\phi}] \rangle \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p.$$

Hence

$$\mathrm{M}(G) \cong \langle [\alpha_1, \alpha^{\phi}]^p \rangle \times \langle [\alpha_2, \alpha_1^{\phi}] \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p.$$

For the group $G \cong \Phi_6(221)d_0$, taking $Z = \langle \beta_2 \rangle$, notice that $G/Z \cong \Phi_3(211)b_{\nu}$. By Theorem 1.4.1, $\mathrm{M}(G/Z) \cong \mathbb{Z}_p$. Then by Theorem 1.2.8, we have the exact sequence

$$G/G' \otimes Z \xrightarrow{\lambda} M(G) \xrightarrow{\mu} \mathbb{Z}_p \to \mathbb{Z}_p \to 1.$$

By Theorem 1.2.9, it follows that $\alpha_2 G' \otimes \alpha_1^p \notin \operatorname{Ker} \lambda$, as $\alpha_1 G' \otimes \alpha_1^p \in \operatorname{Ker} \lambda$. Hence $|\operatorname{Im}(\lambda)| = p$, and therefore

$$\mathrm{M}(G) \cong \mathbb{Z}_p.$$

Since $|G'| = p^3$, we have

$$|G \wedge G| = p^4.$$

By Lemma 1.3.3$(viii)$, $[\beta, \beta_1^{\phi}] = [\alpha_1, \alpha_2, \beta_1^{\phi}] = 1$. Similarly, $[\beta, \beta_2^{\phi}] = [\beta_2, \beta_1^{\phi}] = 1$.

By Lemma 1.3.2, we have the following identities:

$$\begin{aligned}
&[x, y^{\phi}]^p = [x^p, y^{\phi}] = 1 \; for \; x \in \{\beta, \beta_1, \beta_2\}, y \in \{\alpha_1, \alpha_2\},\\
&[\beta_1, \alpha_1^{\phi}] = [\alpha_2^{p\nu^{-1}}, \alpha_1^{\phi}] = [\alpha_2, \alpha_1^{\phi}]^{p\nu^{-1}},\\
&[\beta_1, \alpha_2^{\phi}] = [\alpha_2^{p\nu^{-1}}, \alpha_2^{\phi}] = [\alpha_2, \alpha_2^{\phi}]^{p\nu^{-1}} = 1,\\
&[\beta_2, \alpha_1^{\phi}] = [\alpha_1^p, \alpha_1^{\phi}] = [\alpha_1, \alpha_1^{\phi}]^p = 1,\\
&[\beta_2, \alpha_2^{\phi}] = [\alpha_1^p, \alpha_2^{\phi}] = [\alpha_1, \alpha_2^{\phi}]^p,\\
&[\alpha_1, \alpha_2^{\phi}]^{p^2} = [\beta_2, \alpha_2^{\phi}]^p = 1.
\end{aligned}$$

Hence, by Proposition 1.3.5, $G \wedge G$ is generated by $\{[\alpha_1, \alpha_2^\phi], [\beta, \alpha_1^\phi], [\beta, \alpha_2^\phi]\}$ and

$$G \wedge G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p.$$

Finally we consider the group $G = \Phi_6(221)b_{\frac{1}{2}(p-1)}$. As in the preceding case, taking $Z = \langle \beta_2 \rangle$, it follows that

$$\mathrm{M}(G) \cong \mathbb{Z}_p.$$

Recall that ζ is the smallest positive integer which is a primitive root (mod p). In this case, $\zeta^{\frac{1}{2}(p-1)} \equiv -1 \pmod p$. By Lemma 1.3.3(*viii*), $[\beta, \beta_1^\phi] = [\alpha_1, \alpha_2, \beta_1^\phi] = 1$. Similarly $[\beta, \beta_2^\phi] = [\beta_2, \beta_1^\phi] = 1$.

By Lemma 1.3.2, we have the following identities:

$$\begin{aligned}
&[x, y^\phi]^p = [x^p, y^\phi] = 1 \text{ for } x \in \{\beta, \beta_1, \beta_2\}, y \in \{\alpha_1, \alpha_2\},\\
&[\beta_1, \alpha_1^\phi] = [\alpha_1^{-p}, \alpha_1^\phi] = [\alpha_1^{-1}, \alpha_1^\phi]^p = 1,\\
&[\beta_1, \alpha_2^\phi] = [\alpha_1^{-p}, \alpha_2^\phi] = [\alpha_1^{-1}, \alpha_2^\phi]^p = [\alpha_1, \alpha_2^\phi]^{-p},\\
&[\beta_2, \alpha_1^\phi] = [\alpha_2^p, \alpha_1^\phi] = [\alpha_2, \alpha_1^\phi]^p,\\
&[\beta_2, \alpha_2^\phi] = [\alpha_2^p, \alpha_2^\phi] = [\alpha_2, \alpha_2^\phi]^p = 1\\
&[\alpha_2, \alpha_1^\phi]^{p^2} = [\beta_2, \alpha_1^\phi]^p = 1.
\end{aligned}$$

Hence, $G \wedge G$ is generated by the set $\{[\alpha_1, \alpha_2^\phi], [\beta, \alpha_1^\phi], [\beta, \alpha_2^\phi]\}$, and therefore

$$G \wedge G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p,$$

which completes the proof. □

Lemma 4.1.15 *If G is one of the groups $\Phi_6(221)a, \Phi_6(221)b_r$ $\left(r \neq \frac{1}{2}(p-1)\right)$, $\Phi_6(221)c_r$ or $\Phi_6(221)d_r$, then $\mathrm{M}(G)$ is trivial and $G \wedge G \cong \mathbb{Z}_p^{(3)}$.*

Proof. It follows from [44, Theorem 2.2] that, for the groups G under consideration,

$$\mathrm{M}(G) = 1.$$

Hence, consequently,

$$G \wedge G \cong G' \cong \mathbb{Z}_p^{(3)}.$$

□

Lemma 4.1.16 *If G is one of the groups $\Phi_6(2111)a, \Phi_6(2111)b_r$ or $\Phi_6(1^5)$, then $\mathrm{M}(G)$ is isomorphic to $\mathbb{Z}_p, \mathbb{Z}_p$, or $\mathbb{Z}_p^{(3)}$ respectively, and $G \wedge G \cong \mathbb{Z}_p^{(4)}, \mathbb{Z}_p^{(4)}$ or $\mathbb{Z}_p^{(6)}$ respectively.*

Proof. For the group $G \cong \Phi_6(2111)a$, by Theorem 1.2.5(i), taking $K = \langle \beta_1 \rangle$, we have $p \leq |\mathrm{M}(G)|$, and so, since $|G'| = p^3$, it follows that $p^4 \leq |G \wedge G|$. By Lemma 1.3.3($viii$),

$$[\beta, \beta_1^\phi] = [\beta, \beta_2^\phi] = [\beta_2, \beta_1^\phi] = 1.$$

By Lemma 1.3.2, the following hold:

$$\begin{aligned}
&[x, y^\phi]^p = [x^p, y^\phi] = 1 \text{ for } x \in \{\beta, \beta_1, \beta_2\}, y \in \{\alpha_1, \alpha_2\},\\
&[\beta_1, \alpha_1^\phi] = [\alpha_1^p, \alpha_1^\phi] = [\alpha_1, \alpha_1^\phi]^p = 1,\\
&[\beta_1, \alpha_2^\phi] = [\alpha_1^p, \alpha_2^\phi] = [\alpha_1, \alpha_2^\phi]^p = [\alpha_1, (\alpha_2^p)^\phi] = 1.
\end{aligned}$$

Now we have

$$\begin{aligned}
[\alpha_2^{-1}, \alpha_1^{-1}, \beta^\phi]^{\alpha_1} &= [\alpha_2\alpha_1\beta^{-1}\alpha_1^{-1}\alpha_2^{-1}, \beta^\phi]^{\alpha_1} = [\alpha_2\alpha_1\beta_1\alpha_1^{-1}\beta^{-1}\alpha_2^{-1}, \beta^\phi]^{\alpha_1}\\
&= [\alpha_2\beta_1\beta^{-1}\alpha_2^{-1}, \beta^\phi]^{\alpha_1} = [\alpha_2\beta_1\beta_2\alpha_2^{-1}\beta^{-1}, \beta^\phi]^{\alpha_1}\\
&= [\beta_1\beta_2\beta^{-1}, \beta^\phi]^{\alpha_1} = [\beta, \beta^\phi]^{-1} = 1.
\end{aligned}$$

By Hall-Witt identity, we get

$$\begin{aligned} 1 &= [\beta, \alpha_2, \alpha_1^{\phi}]^{\alpha_2^{-1}}[\alpha_2^{-1}, \alpha_1^{-1}, \beta^{\phi}]^{\alpha_1}[\alpha_1, \beta^{-1}, \alpha_2^{-\phi}]^{\beta} \\ &= [\beta_2, \alpha_1^{\phi}]^{\alpha_2^{-1}}[\beta\beta_1\beta^{-1}, \alpha_2^{-\phi}]^{\beta} \\ &= [\beta_2, \alpha_1^{\phi}][\beta_1, \alpha_2^{\phi}]^{-1}. \end{aligned}$$

Hence, $[\beta_2, \alpha_1^{\phi}] = [\beta_1, \alpha_2^{\phi}] = 1$. Consequently, $G \wedge G$ is generated by the set

$$\{[\alpha_1, \alpha_2^{\phi}], [\beta, \alpha_1^{\phi}], [\beta, \alpha_2^{\phi}], [\beta_2, \alpha_2^{\phi}]\},$$

and is elementary abelian by Lemma 1.3.3(vi). Hence

$$G \wedge G \cong \mathbb{Z}_p^{(4)},$$

and therefore

$$\mathrm{M}(G) \cong \mathbb{Z}_p.$$

For the group $G \cong \Phi_6(2111)b_r$, taking $K = \langle \beta_1 \rangle$, we have $p \leq |\,\mathrm{M}(G)|$, and so, since $|G'| = p^3$, it follows that $p^4 \leq |G \wedge G|$. By Lemma 1.3.3$(viii)$,

$$[\beta, \beta_1^{\phi}] = [\beta, \beta_2^{\phi}] = [\beta_2, \beta_1^{\phi}] = 1.$$

By Lemma 1.3.2, the following hold:

$$\begin{aligned} &[x, y^{\phi}]^p = [x^p, y^{\phi}] = 1 \text{ for } x \in \{\beta, \beta_1, \beta_2\}, y \in \{\alpha_1, \alpha_2\}, \\ &[\beta_1, \alpha_2^{\phi}] = [\alpha_2^{pr^{-1}}, \alpha_2^{\phi}] = [\alpha_2, \alpha_2^{\phi}]^{pr^{-1}} = 1, \\ &[\beta_1, \alpha_1^{\phi}] = [\alpha_2^{pr^{-1}}, \alpha_1^{\phi}] = [\alpha_2, \alpha_1^{\phi}]^{pr^{-1}} = [\alpha_2, (\alpha_1^{pr^{-1}})^{\phi}] = 1. \end{aligned}$$

By Hall-Witt identity, we get

$$[\beta_2, \alpha_1^{\phi}] = [\beta_1, \alpha_2^{\phi}] = 1.$$

Consequently, $G \wedge G$ is generated by the set

$$\{[\alpha_1, \alpha_2^{\phi}], [\beta, \alpha_1^{\phi}], [\beta, \alpha_2^{\phi}], [\beta_2, \alpha_2^{\phi}]\},$$

and is elementary abelian by Lemma 1.3.3(vi). Hence

$$G \wedge G \cong \mathbb{Z}_p^{(4)},$$

and therefore

$$\mathrm{M}(G) \cong \mathbb{Z}_p.$$

Now we consider the group $G = \Phi_6(1^5)$. By Theorem 1.2.6, taking $K = \langle \beta_1 \rangle$, we have $|\mathrm{M}(G)| \leq p^3$. Since $|G'| = p^3$, we get $|G \wedge G| \leq p^6$. As described above,

$$[\beta, \beta_1^{\phi}] = [\beta, \beta_2^{\phi}] = [\beta_2, \beta_1^{\phi}] = 1$$

and, by Hall-Witt identity, we get

$$[\beta_2, \alpha_1^{\phi}] = [\beta_1, \alpha_2^{\phi}].$$

Thus, by Proposition 1.3.5, $G \wedge G$ is generated by the set

$$\{[\alpha_1, \alpha_2^{\phi}], [\beta, \alpha_1^{\phi}], [\beta, \alpha_2^{\phi}], [\beta_1, \alpha_1^{\phi}], [\beta_1, \alpha_2^{\phi}], [\beta_2, \alpha_2^{\phi}]\}.$$

It follows from Lemma 1.3.3(vi) that $G \wedge G$ is abelian. A straightforward calculation (as above) shows that each generator of $G \wedge G$ is of order at most

p.

Consider the natural epimorphism

$$[G, G^\phi] \to [G/\langle\beta_1\rangle, (G/\langle\beta_1\rangle)^\phi] \cong [\Phi_3(1^4), \Phi_3(1^4)^\phi],$$

which shows that the elements $[\alpha_1, \alpha_2^\phi], [\beta, \alpha_1^\phi], [\beta, \alpha_2^\phi], [\beta_2, \alpha_2^\phi]$ are non-trivial and independent in $G \wedge G$. Now, consider the natural epimorphism

$$[G, G^\phi] \to [G/\langle\beta_2\rangle, (G/\langle\beta_2\rangle)^\phi] \cong [\Phi_3(1^4), \Phi_3(1^4)^\phi],$$

which shows that the elements $[\alpha_1, \alpha_2^\phi], [\beta, \alpha_1^\phi], [\beta, \alpha_2^\phi], [\beta_1, \alpha_1^\phi]$ are non-trivial and independent in $G \wedge G$. We take the quotient group

$$\begin{aligned}
G/\langle\beta_1\beta_2\rangle &= \langle\alpha_1, \alpha_2, \beta, \beta_1 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = [\alpha_2, \beta] = \beta_1, \alpha_1^p = \alpha_2^p = \beta^p = \beta_1^p = 1\rangle \\
&\cong \langle\alpha_1\alpha_2, \alpha_1, \beta, \beta_1 \mid [\alpha_1, \alpha_1\alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\beta, \alpha_1\alpha_2] = 1, \alpha_1^p = (\alpha_1\alpha_2)^p \\
&\qquad\qquad = \beta^p = \beta_1^p = 1\rangle \\
&\cong \Phi_3(1^4).
\end{aligned}$$

In $[G/\langle\beta_1\beta_2\rangle, (G/\langle\beta_1\beta_2\rangle)^\phi]$, by Hall-Witt identity we have

$$\begin{aligned}
1 &= [\beta, \alpha_1, \alpha_2^\phi]^{\alpha_1^{-1}}[\alpha_1^{-1}, \alpha_2^{-1}, \beta^\phi]^{\alpha_2}[\alpha_2, \beta^{-1}, \alpha_1^{-\phi}]^\beta \\
&= [\beta_1, \alpha_2^\phi]^{\alpha_1^{-1}}[\alpha_1\alpha_2\beta\alpha_2^{-1}\alpha_1^{-1}, \beta^\phi]^{\alpha_2}[\beta\beta_1^{-1}\beta^{-1}, \alpha_1^{-\phi}]^\beta \\
&= [\beta_1, \alpha_2^\phi][\alpha_1\beta\beta_1\alpha_1^{-1}, \beta^\phi]^{\alpha_2}[\beta_1^{-1}, \alpha_1^{-\phi}]^\beta \\
&= [\beta_1, \alpha_2^\phi][\alpha_1\beta\alpha_1^{-1}\beta_1, \beta^\phi]^{\alpha_2}[\beta_1, \alpha_1^\phi] \\
&= [\beta_1, \alpha_2^\phi][\beta\alpha_1\beta_1^{-1}\alpha_1^{-1}\beta_1, \beta^\phi]^{\alpha_2}[\beta_1, \alpha_1^\phi] \\
&= [\beta_1, \alpha_2^\phi][\beta, \beta^\phi][\beta_1, \alpha_1^\phi] \\
&= [\beta_1, \alpha_2^\phi][\beta_1, \alpha_1^\phi].
\end{aligned}$$

It follows from Theorem 1.4.1 that

$$G/\langle\beta_1\beta_2\rangle \wedge G/\langle\beta_1\beta_2\rangle \cong \langle[\alpha_1\alpha_2,\alpha_1^{\phi}],[\beta^{-1},\alpha_1^{\phi}],[\beta^{-1},(\alpha_1\alpha_2)^{\phi}],[\beta_1^{-1},\alpha_1^{\phi}]\rangle.$$

A straightforward computation now shows that

$$G/\langle\beta_1\beta_2\rangle \wedge G/\langle\beta_1\beta_2\rangle \cong \langle[\alpha_1,\alpha_2^{\phi}],[\beta,\alpha_1^{\phi}],[\beta,\alpha_2^{\phi}],[\beta_1,\alpha_2^{\phi}]\rangle \cong \mathbb{Z}_p^{(4)}.$$

Now consider the natural epimorphism

$$[G,G^{\phi}] \to [G/\langle\beta_1\beta_2\rangle,(G/\langle\beta_1\beta_2\rangle)^{\phi}] \cong [\Phi_3(1^4),\Phi_3(1^4)^{\phi}],$$

which shows that the generators $[\alpha_1,\alpha_2^{\phi}],[\beta,\alpha_1^{\phi}],[\beta,\alpha_2^{\phi}],[\beta_1,\alpha_2^{\phi}]$ of $G\wedge G$ are independent. Thus, it follows that all the six generators of $G\wedge G$ are non-trivial and independent, and so, we have $|G\wedge G|\geq p^6$. Hence,

$$G\wedge G\cong \mathbb{Z}_p^{(6)}$$

and

$$\mathrm{M}(G)\cong \mathbb{Z}_p^{(3)}.$$

The proof is now complete. $\square$

Lemma 4.1.17 *If G is one of the groups $\Phi_7(2111)a, \Phi_7(2111)b_r, \Phi_7(2111)c$ or $\Phi_7(1^5)$, then $\mathrm{M}(G)$ is isomorphic to $\mathbb{Z}_p^{(3)}, \mathbb{Z}_p^{(3)}, \mathbb{Z}_p^{(3)}$ or $\mathbb{Z}_p^{(4)}$ respectively, and $G\wedge G$ is isomorphic to $\mathbb{Z}_p^{(5)}, \mathbb{Z}_p^{(5)}, \mathbb{Z}_p^{(5)}$ or $\mathbb{Z}_p^{(6)}$ respectively.*

Proof. For the groups G belonging to the isoclinism class Φ_7, it follows from Theorem 1.2.5(ii), taking $K=\mathrm{Z}(G)$, that $d(\mathrm{M}(G))\geq 3$.

For the group $G = \Phi_7(2111)a$, consider the normal subgroup

$$N = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \rangle \cong \Phi_3(211)a$$

of G of order p^4. By Theorem 1.4.1, $\mathrm{M}(N) \cong \mathbb{Z}_p$. Then, by Theorem 1.2.7, $|\,\mathrm{M}(G)|$ divides $|\,\mathrm{M}(N)||N/N'|$. Since $|N/N'| = p^2$, it follows that $|\,\mathrm{M}(G)| = p^3$. Hence $G \wedge G$ is of order p^5. By Proposition 1.3.5, $G \wedge G$ is generated by

$$\{[\alpha_1, \alpha^\phi], [\alpha_2, \alpha^\phi], [\alpha_3, \alpha^\phi], [\beta, \alpha^\phi], [\alpha_2, \alpha_1^\phi], [\alpha_3, \alpha_1^\phi], [\beta, \alpha_1^\phi], [\alpha_3, \alpha_2^\phi], [\beta, \alpha_2^\phi], [\beta, \alpha_3^\phi]\}.$$

By Lemma 1.3.3$(viii)$, we have

$$[\alpha_2, \alpha_3^\phi] = [\alpha_1, \alpha, \alpha_3^\phi] = 1.$$

For $x \in \{\alpha, \alpha_1, \alpha_2, \alpha_3, \beta\}$, by Lemma 1.3.2, the following identities hold:

$$[\alpha_3, x^\phi]^p = [\alpha_3^p, x^\phi] = 1 = [\beta^p, x^\phi] = [\beta, x^\phi]^p,$$
$$[\alpha_2, x^\phi]^p = [\alpha_2^p, x^\phi] = 1 = [\alpha_1^p, \alpha^\phi] = [\alpha_1, \alpha^\phi]^p.$$

Hence all the generators of $G \wedge G$ is of order at most p. By Lemma 1.3.3(vi), it follows that $G \wedge G$ is abelian, and consequently

$$G \wedge G \cong \mathbb{Z}_p^{(5)}$$

and

$$\mathrm{M}(G) \cong \mathbb{Z}_p^{(3)}.$$

For the group $G = \Phi_7(2111)b_r$, $(r = 1$ or $\nu)$, consider the normal subgroup

$$N = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \rangle \cong \Phi_3(211)b_r$$

of G of order p^4. By Theorem 1.4.1, $\mathrm{M}(N) \cong \mathbb{Z}_p$. Then, using Theorem 1.2.7, $|\,\mathrm{M}(G)|$ divides p^3. Hence $|\,\mathrm{M}(G)| = p^3$ and so, $|G \wedge G| = p^5$. By Proposition 1.3.5, $G \wedge G$ is generated by

$$\{[\alpha_1, \alpha^\phi], [\alpha_2, \alpha^\phi], [\alpha_3, \alpha^\phi], [\beta, \alpha^\phi], [\alpha_2, \alpha_1^\phi], [\alpha_3, \alpha_1^\phi], [\beta, \alpha_1^\phi], [\alpha_3, \alpha_2^\phi], [\beta, \alpha_2^\phi], [\beta, \alpha_3^\phi]\}.$$

By Lemma 1.3.3($viii$), we have

$$[\alpha_2, \alpha_3^\phi] = [\alpha_1, \alpha, \alpha_3^\phi] = 1.$$

For $x \in \{\alpha, \alpha_1, \alpha_2, \alpha_3, \beta\}$, by Lemma 1.3.2, the following identities hold:

$$\begin{aligned} &[\alpha_3, x^\phi]^p = [\alpha_3^p, x^\phi] = 1 = [\beta^p, x^\phi] = [\beta, x^\phi]^p, \\ &[\alpha_2, x^\phi]^p = [\alpha_2^p, x^\phi] = 1 = [\alpha_1, (\alpha^p)^\phi] = [\alpha_1, \alpha^\phi]^p. \end{aligned}$$

Hence all the generators of $G \wedge G$ is of order at most p. By Lemma 1.3.3(vi), it follows that $G \wedge G$ is abelian, and consequently

$$G \wedge G \cong \mathbb{Z}_p^{(5)}$$

and

$$\mathrm{M}(G) \cong \mathbb{Z}_p^{(3)}.$$

For the group $G = \Phi_7(2111)c$, consider the normal subgroup

$$N = \langle \alpha_1, \alpha\beta, \alpha_2, \alpha_3 \rangle \cong \Phi_3(211)a$$

of G of order p^4. By Theorem 1.4.1, $\mathrm{M}(N) \cong \mathbb{Z}_p$. Then, using Theorem 1.2.7, $|\,\mathrm{M}(G)|$ divides p^3. Hence $|\,\mathrm{M}(G)| = p^3$ and so, $|G \wedge G| = p^5$. By Proposition

1.3.5, $G \wedge G$ is generated by

$$\{[\alpha_1, \alpha^\phi], [\alpha_2, \alpha^\phi], [\alpha_3, \alpha^\phi], [\beta, \alpha^\phi], [\alpha_2, \alpha_1^\phi], [\alpha_3, \alpha_1^\phi], [\beta, \alpha_1^\phi], [\alpha_3, \alpha_2^\phi], [\beta, \alpha_2^\phi], [\beta, \alpha_3^\phi]\}.$$

By Lemma 1.3.3($viii$), we have

$$[\alpha_2, \alpha_3^\phi] = [\alpha_1, \alpha, \alpha_3^\phi] = 1.$$

For $x \in \{\alpha, \alpha_1, \alpha_2, \alpha_3, \beta\}$, by Lemma 1.3.2, the following identities hold:

$$[\alpha_3, x^\phi]^p = [\alpha_3^p, x^\phi] = 1 = [\alpha_2^p, x^\phi] = [\alpha_2, x^\phi]^p,$$
$$[\alpha_1, x^\phi]^p = [\alpha_1^p, x^\phi] = 1 = [\beta, (\alpha^p)^\phi] = [\beta, \alpha^\phi]^p.$$

Hence all the generators of $G \wedge G$ is of order at most p. By Lemma 1.3.3(vi), it follows that $G \wedge G$ is abelian, and consequently

$$G \wedge G \cong \mathbb{Z}_p^{(5)}$$

and

$$\mathrm{M}(G) \cong \mathbb{Z}_p^{(3)}.$$

Finally consider the group $G = \Phi_7(1^5)$. It follows from [18, Main Theorem] that

$$\mathrm{M}(G) \cong \mathbb{Z}_p^{(4)}.$$

So $|G \wedge G| = p^6$. For $x \in \{\alpha, \alpha_1, \alpha_2, \alpha_3, \beta\}$, by Lemma 1.3.2, the following identities hold:

$$[\alpha_3, x^\phi]^p = [\alpha_3^p, x^\phi] = 1 = [\alpha_2^p, x^\phi] = [\alpha_2, x^\phi]^p,$$
$$[\alpha_1, x^\phi]^p = [\alpha_1^p, x^\phi] = 1 = [\beta^p, x^\phi] = [\beta, x^\phi]^p.$$

Hence all the generators of $G \wedge G$ is of order at most p. By Lemma 1.3.3(vi), it follows that $G \wedge G$ is abelian, and consequently

$$G \wedge G \cong \mathbb{Z}_p^{(6)},$$

which completes the proof. □

4.1.4 Main result

Theorem 1.3.7 and all lemmas in Sections 4.1.1, 4.1.2 and 4.1.3 yield our main result, which we present in the following table:

G	G^{ab}	$\Gamma(G^{ab})$	$\mathrm{M}(G)$	$G \wedge G$	$G \otimes G$
$\Phi_2(311)a$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p \times \mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(8)}$
$\Phi_2(221)a$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2}^{(2)} \times \mathbb{Z}_p^{(7)}$
$\Phi_2(221)b$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p \times \mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(8)}$
$\Phi_2(2111)a$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(10)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(16)}$
$\Phi_2(2111)b$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(10)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(16)}$
$\Phi_2(2111)c$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(10)}$
$\Phi_2(2111)d$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(10)}$
$\Phi_2(1^5)$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(10)}$	$\mathbb{Z}_p^{(7)}$	$\mathbb{Z}_p^{(8)}$	$\mathbb{Z}_p^{(18)}$
$\Phi_2(41)$	$\mathbb{Z}_{p^3} \times \mathbb{Z}_p$	$\mathbb{Z}_{p^3} \times \mathbb{Z}_p^{(2)}$	$\{1\}$	$\mathbb{Z}_p$	$\mathbb{Z}_{p^3} \times \mathbb{Z}_p^{(3)}$
$\Phi_2(32)a_1$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$	$\mathbb{Z}_{p^2}^{(3)}$	$\mathbb{Z}_p$	$\mathbb{Z}_{p^2}$	$\mathbb{Z}_{p^2}^{(4)}$
$\Phi_2(32)a_2$	$\mathbb{Z}_{p^3} \times \mathbb{Z}_p$	$\mathbb{Z}_{p^3} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p$	$\mathbb{Z}_{p^2}$	$\mathbb{Z}_{p^3} \times \mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$
$\Phi_2(311)b$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p \times \mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(8)}$
$\Phi_2(311)c$	$\mathbb{Z}_{p^3} \times \mathbb{Z}_p$	$\mathbb{Z}_{p^3} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p \times \mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_{p^3} \times \mathbb{Z}_p^{(5)}$
$\Phi_2(221)c$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(5)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2}^{(2)} \times \mathbb{Z}_p^{(7)}$
$\Phi_2(221)d$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$	$\mathbb{Z}_{p^2}^{(3)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2}^{(4)} \times \mathbb{Z}_p^{(2)}$

G	G^{ab}	$\Gamma(G^{ab})$	$\mathrm{M}(G)$	$G \wedge G$	$G \otimes G$
$\Phi_3(2111)a$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(11)}$
$\Phi_3(2111)b_r$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(11)}$
$\Phi_3(1^5)$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(12)}$
$\Phi_3(311)a$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(5)}$
$\Phi_3(311)b_r$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(5)}$
$\Phi_3(221)a$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(5)}$
$\Phi_3(221)b_r$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p \times \mathbb{Z}_p$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2}^{(2)} \times \mathbb{Z}_p^{(4)}$
$\Phi_3(2111)c$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(11)}$
$\Phi_3(2111)d$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(6)}$
$\Phi_3(2111)e$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(6)}$
$\Phi_4(221)a$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(9)}$
$\Phi_4(221)b$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p \times \mathbb{Z}_p$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(8)}$
$\Phi_4(221)c$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(9)}$
$\Phi_4(221)d_{r,\, r\neq\frac{1}{2}(p-1)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(9)}$
$\Phi_4(221)d_{\frac{1}{2}(p-1)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_{p^2}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(8)}$
$\Phi_4(221)e$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(9)}$
$\Phi_4(221)f_0$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_{p^2}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(8)}$
$\Phi_4(221)f_r$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(9)}$
$\Phi_4(2111)a$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(11)}$
$\Phi_4(2111)b$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(11)}$
$\Phi_4(2111)c$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(11)}$
$\Phi_4(1^5)$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(8)}$	$\mathbb{Z}_p^{(14)}$
$\Phi_5(2111)$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(10)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(16)}$
$\Phi_5(1^5)$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(10)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(16)}$
$\Phi_6(221)a$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\{1\}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$
$\Phi_6(221)b_{r,\, r\neq\frac{1}{2}(p-1)}$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\{1\}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$
$\Phi_6(221)b_{\frac{1}{2}(p-1)}$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(5)}$

G	G^{ab}	$\Gamma(G^{ab})$	$\mathrm{M}(G)$	$G \wedge G$	$G \otimes G$
$\Phi_6(221)c_r$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\{1\}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$
$\Phi_6(221)d_0$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(5)}$
$\Phi_6(221)d_r$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\{1\}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$
$\Phi_6(2111)a$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(7)}$
$\Phi_6(2111)b_r$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(7)}$
$\Phi_6(1^5)$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(9)}$
$\Phi_7(2111)a$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(11)}$
$\Phi_7(2111)b_r$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(11)}$
$\Phi_7(2111)c$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(11)}$
$\Phi_7(1^5)$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(12)}$
$\Phi_8(32)$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\{1\}$	$\mathbb{Z}_{p^2}$	$\mathbb{Z}_{p^2}^{(2)} \times \mathbb{Z}_p^{(2)}$
$\Phi_9(2111)a$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(7)}$
$\Phi_9(2111)b_r$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(7)}$
$\Phi_9(1^5)$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(3)}$	$\Phi_2(111) \times \mathbb{Z}_p^{(3)}$	$\Phi_2(111) \times \mathbb{Z}_p^{(6)}$
$\Phi_{10}(2111)a_r$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(7)}$
$\Phi_{10}(2111)b_r$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(7)}$
$\Phi_{10}(1^5)$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(3)}$	$\Phi_2(111) \times \mathbb{Z}_p^{(3)}$	$\Phi_2(111) \times \mathbb{Z}_p^{(6)}$

Table 4.1: Groups of order p^5, $p \geq 5$

4.2 Capability of groups of order $p^5, p \geq 5$

In this section we determine the capability of non-abelian p-groups of order p^5, $p \geq 5$. For the convenience of reader we work out the details for some groups. For all other groups proof goes on the same lines.

Consider the group $G = \Phi_2(311)a$. By Theorem 1.2.8, the following sequence is exact

$$G/G' \otimes Z \xrightarrow{\lambda} \mathrm{M}(G) \to \mathrm{M}(G/Z) \to G' \cap Z \to 1.$$

For the central subgroup $Z = \langle \alpha^p \rangle$, $G/Z \cong \mathbb{Z}_p^{(3)}$ and $G' \cap Z \cong \mathbb{Z}_p$. Since $\mathrm{M}(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$, so by the above exact sequence $\ker \lambda = G/G' \otimes Z$. Hence, by Theorem 1.2.9, $\langle \alpha^p \rangle \subseteq Z^*(G)$. Similarly we observe that $\langle \alpha_3 \rangle$ is not in $Z^*(G)$. Hence G is not capable and $Z^*(G) = \langle \alpha^p \rangle$.

Consider groups G isomorphic to $\Phi_4(221)b$, $\Phi_4(221)d_{\frac{1}{2}(p-1)}$ or $\Phi_4(221)f_0$. Since $|\,\mathrm{M}(G)| = p^2$, by the exact sequence in Theorem 1.2.8, it follows that $\ker \lambda = G/G' \otimes Z$ if and only if either $|\,\mathrm{M}(G/Z)| = p^3$ with $|G' \cap Z| = p$ or $|\,\mathrm{M}(G/Z)| = p^4$ with $|G' \cap Z| = p^2$. It is easy to observe that this will not happen for any central subgroup Z of G. Hence, by Theorem 1.2.9, $Z^*(G) = 1$.

In the following table the description of epicenter $Z^*(G)$ is given for each group G under consideration.

G	Capability	Epicenter	G	Capability	Epicenter
$\Phi_2(311)a$	Not Capable	$\langle \alpha^p \rangle$	$\Phi_4(221)e$	Not capable	$Z(G)$
$\Phi_2(221)a$	Capable	$\langle 1 \rangle$	$\Phi_4(221)f_0$	Capable	$\langle 1 \rangle$
$\Phi_2(221)b$	Not Capable	$\gamma_2(G) \times \langle \alpha_3^p \rangle$	$\Phi_4(221)f_r$	Not capable	$Z(G)$
$\Phi_2(2111)a$	Not Capable	$\gamma_2(G)$	$\Phi_4(2111)a$	Not capable	$\langle \beta_2 \rangle$
$\Phi_2(2111)b$	Not Capable	$\gamma_2(G)$	$\Phi_4(2111)b$	Not capable	$\langle \beta_1 \rangle$
$\Phi_2(2111)c$	Not Capable	$\langle \alpha^p \rangle$	$\Phi_4(2111)c$	Not Capable	$\langle \beta_1 \rangle$
$\Phi_2(2111)d$	Not Capable	$\langle \alpha_3^p \rangle$	$\Phi_4(1^5)$	Capable	$\langle 1 \rangle$
$\Phi_2(1^5)$	Capable	$\langle 1 \rangle$	$\Phi_5(2111)$	Not capable	$Z(G)$
$\Phi_2(41)$	Not Capable	$Z(G)$	$\Phi_5(1^5)$	Not capable	$Z(G)$
$\Phi_2(32)a_1$	Not Capable	$\gamma_2(G)$	$\Phi_6(221)a$	Not capable	$Z(G)$
$\Phi_2(32)a_2$	Not capable	$\langle \alpha^{p^2} \rangle$	$\Phi_6(221)b_{r,\, r \neq \frac{1}{2}(p-1)}$	Not capable	$Z(G)$
$\Phi_2(311)b$	Not capable	$\langle \gamma^p \rangle$	$\Phi_6(221)b_{\frac{1}{2}(p-1)}$	Capable	$\langle 1 \rangle$
$\Phi_2(311)c$	Not capable	$\langle \alpha^p \rangle$	$\Phi_6(221)c_r$	Not capable	$Z(G)$
$\Phi_2(221)c$	Capable	$\langle 1 \rangle$	$\Phi_6(221)d_0$	Capable	$\langle 1 \rangle$
$\Phi_2(221)d$	Capable	$\langle 1 \rangle$	$\Phi_6(221)d_r$	Not capable	$Z(G)$
$\Phi_3(2111)a$	Not capable	$\langle \alpha_3 \rangle$	$\Phi_6(2111)a$	Not capable	$\langle \beta_1 \rangle$
$\Phi_3(2111)b_r$	Not capable	$\langle \alpha_3 \rangle$	$\Phi_6(2111)b_r$	Not capable	$\langle \beta_1 \rangle$
$\Phi_3(1^5)$	Capable	$\langle 1 \rangle$	$\Phi_6(1^5)$	Capable	$\langle 1 \rangle$
$\Phi_3(311)a$	Not capable	$Z(G)$	$\Phi_7(2111)a$	Not capable	$Z(G)$
$\Phi_3(311)b_r$	Not capable	$Z(G)$	$\Phi_7(2111)b_r$	Not capable	$Z(G)$
$\Phi_3(221)a$	Not capable	$Z(G)$	$\Phi_7(2111)c$	Not capable	$Z(G)$
$\Phi_3(221)b_r$	Capable	$\langle 1 \rangle$	$\Phi_7(1^5)$	Capable	$\langle 1 \rangle$
$\Phi_3(2111)c$	Not capable	$\langle \gamma^p \rangle$	$\Phi_8(32)$	Not capable	$Z(G)$
$\Phi_3(2111)d$	Not capable	$\langle \alpha^p \rangle$	$\Phi_9(2111)a$	Not capable	$Z(G)$
$\Phi_3(2111)e$	Not capable	$\langle \alpha_1^p \rangle$	$\Phi_9(2111)b_r$	Not capable	$Z(G)$
$\Phi_4(221)a$	Not capable	$Z(G)$	$\Phi_9(1^5)$	Capable	$\langle 1 \rangle$
$\Phi_4(221)b$	Capable	$\langle 1 \rangle$	$\Phi_{10}(2111)a_r$	Not capable	$Z(G)$
$\Phi_4(221)c$	Not capable	$Z(G)$	$\Phi_{10}(2111)b_r$	Not capable	$Z(G)$
$\Phi_4(221)d_{r,\, r \neq \frac{1}{2}(p-1)}$	Not capable	$Z(G)$	$\Phi_{10}(1^5)$	Capable	$\langle 1 \rangle$
$\Phi_4(221)d_{\frac{1}{2}(p-1)}$	Capable	$\langle 1 \rangle$			

Table 4.2: Capability of groups of order p^5, $p \geq 5$

4.3 Non-abelian groups of order 2^5 and 3^5

The following table takes care of groups of order 2^5. These results are obtained by GAP [14].

Group ID	M(G)	$G \wedge G$	$G \otimes G$	Capability	Epicenter
2	$\mathbb{Z}_2^{(3)}$	$\mathbb{Z}_4 \times \mathbb{Z}_2^{(2)}$	$\mathbb{Z}_4^{(4)} \times \mathbb{Z}_2^{(2)}$	Capable	$\{1\}$
4	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_8 \times \mathbb{Z}_4^{(3)}$	Not Capable	$\mathbb{Z}_2$
5	$\mathbb{Z}_2^{(2)}$	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2^{(3)}$	Not Capable	$\mathbb{Z}_2$
6	$\mathbb{Z}_2^{(2)}$	$\mathbb{Z}_4 \times \mathbb{Z}_2^{(2)}$	$\mathbb{Z}_4^{(2)} \times \mathbb{Z}_2^{(4)}$	Capable	$\{1\}$
7	$\mathbb{Z}_2$	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2^{(3)}$	Not Capable	$\mathbb{Z}_2$
8	$\mathbb{Z}_2$	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2^{(3)}$	Not Capable	$\mathbb{Z}_2$
9	$\mathbb{Z}_2^{(2)}$	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2^{(3)}$	Capable	$\{1\}$
10	$\mathbb{Z}_2$	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2^{(3)}$	Not Capable	$\mathbb{Z}_2$
11	$\mathbb{Z}_2$	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_4^{(2)} \times \mathbb{Z}_2^{(3)}$	Not Capable	$\mathbb{Z}_2$
12	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_8 \times \mathbb{Z}_4^{(2)} \times \mathbb{Z}_2$	Not Capable	$\mathbb{Z}_4$
13	$\mathbb{Z}_2$	$\mathbb{Z}_8$	$\mathbb{Z}_8 \times \mathbb{Z}_4^{(2)} \times \mathbb{Z}_2$	Capable	$\{1\}$
14	$\mathbb{Z}_2$	$\mathbb{Z}_8$	$\mathbb{Z}_8 \times \mathbb{Z}_4^{(2)} \times \mathbb{Z}_2$	Not Capable	$\mathbb{Z}_2$
15	$\{1\}$	$\mathbb{Z}_4$	$\mathbb{Z}_8 \times \mathbb{Z}_4^{(2)} \times \mathbb{Z}_2$	Not Capable	$\mathbb{Z}_4$
17	$\{1\}$	$\mathbb{Z}_2$	$\mathbb{Z}_{16} \times \mathbb{Z}_2^{(3)}$	Not Capable	$\mathbb{Z}_8$
18	$\mathbb{Z}_2$	$\mathbb{Z}_{16}$	$\mathbb{Z}_{16} \times \mathbb{Z}_2^{(3)}$	Capable	$\{1\}$
19	$\{1\}$	$\mathbb{Z}_8$	$\mathbb{Z}_{16} \times \mathbb{Z}_2^{(3)}$	Not Capable	$\mathbb{Z}_2$
20	$\{1\}$	$\mathbb{Z}_8$	$\mathbb{Z}_{16} \times \mathbb{Z}_2^{(3)}$	Not Capable	$\mathbb{Z}_2$
22	$\mathbb{Z}_2^{(4)}$	$\mathbb{Z}_4 \times \mathbb{Z}_2^{(3)}$	$\mathbb{Z}_4^{(2)} \times \mathbb{Z}_2^{(8)}$	Capable	$\{1\}$
23	$\mathbb{Z}_2^{(3)}$	$\mathbb{Z}_4 \times \mathbb{Z}_2^{(2)}$	$\mathbb{Z}_4^{(3)} \times \mathbb{Z}_2^{(6)}$	Not Capable	$\mathbb{Z}_2$
24	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_4 \times \mathbb{Z}_2^{(2)}$	$\mathbb{Z}_4^{(2)} \times \mathbb{Z}_2^{(7)}$	Capable	$\{1\}$

Group ID	M(G)	$G \wedge G$	$G \otimes G$	Capability	Epicenter
25	$\mathbb{Z}_2^{(3)}$	$\mathbb{Z}_4 \times \mathbb{Z}_2^{(2)}$	$\mathbb{Z}_4^{(2)} \times \mathbb{Z}_2^{(7)}$	Not Capable	$\mathbb{Z}_2$
26	$\mathbb{Z}_2^{(2)}$	$\mathbb{Z}_2^{(3)}$	$\mathbb{Z}_4^{(3)} \times \mathbb{Z}_2^{(6)}$	Not Capable	$\mathbb{Z}_2 \times \mathbb{Z}_2$
27	$\mathbb{Z}_2^{(4)}$	$\mathbb{Z}_4^{(2)} \times \mathbb{Z}_2^{(2)}$	$\mathbb{Z}_4^{(2)} \times \mathbb{Z}_2^{(8)}$	Capable	$\{1\}$
28	$\mathbb{Z}_2^{(3)}$	$\mathbb{Z}_4^{(2)} \times \mathbb{Z}_2$	$\mathbb{Z}_4^{(2)} \times \mathbb{Z}_2^{(7)}$	Capable	$\{1\}$
29	$\mathbb{Z}_2^{(2)}$	$\mathbb{Z}_4 \times \mathbb{Z}_2^{(2)}$	$\mathbb{Z}_4^{(3)} \times \mathbb{Z}_2^{(6)}$	Not Capable	$\mathbb{Z}_2$
30	$\mathbb{Z}_2^{(2)}$	$\mathbb{Z}_4 \times \mathbb{Z}_2^{(2)}$	$\mathbb{Z}_4^{(2)} \times \mathbb{Z}_2^{(7)}$	Not Capable	$\mathbb{Z}_2$
31	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_4^{(2)} \times \mathbb{Z}_2$	$\mathbb{Z}_4^{(3)} \times \mathbb{Z}_2^{(6)}$	Capable	$\{1\}$
32	$\mathbb{Z}_2$	$\mathbb{Z}_2^{(3)}$	$\mathbb{Z}_4^{(2)} \times \mathbb{Z}_2^{(7)}$	Not Capable	$\mathbb{Z}_2 \times \mathbb{Z}_2$
33	$\mathbb{Z}_2$	$\mathbb{Z}_2^{(3)}$	$\mathbb{Z}_4^{(2)} \times \mathbb{Z}_2^{(7)}$	Not Capable	$\mathbb{Z}_2 \times \mathbb{Z}_2$
34	$\mathbb{Z}_2^{(2)} \times \mathbb{Z}_4$	$\mathbb{Z}_4^{(3)}$	$\mathbb{Z}_4^{(3)} \times \mathbb{Z}_2^{(6)}$	Capable	$\{1\}$
35	$\mathbb{Z}_2^{(2)}$	$\mathbb{Z}_4 \times \mathbb{Z}_2^{(2)}$	$\mathbb{Z}_4^{(4)} \times \mathbb{Z}_2^{(5)}$	Not Capable	$\mathbb{Z}_2$
37	$\mathbb{Z}_2^{(2)}$	$\mathbb{Z}_2^{(3)}$	$\mathbb{Z}_8 \times \mathbb{Z}_2^{(8)}$	Not Capable	$\mathbb{Z}_4$
38	$\mathbb{Z}_2^{(2)}$	$\mathbb{Z}_2^{(3)}$	$\mathbb{Z}_4 \times \mathbb{Z}_2^{(8)}$	Not Capable	$\mathbb{Z}_4$
39	$\mathbb{Z}_2^{(3)}$	$\mathbb{Z}_8 \times \mathbb{Z}_2^{(2)}$	$\mathbb{Z}_8 \times \mathbb{Z}_2^{(8)}$	Capable	$\{1\}$
40	$\mathbb{Z}_2^{(2)}$	$\mathbb{Z}_4 \times \mathbb{Z}_2^{(2)}$	$\mathbb{Z}_8 \times \mathbb{Z}_2^{(8)}$	Not Capable	$\mathbb{Z}_2$
41	$\mathbb{Z}_2^{(2)}$	$\mathbb{Z}_4 \times \mathbb{Z}_2^{(2)}$	$\mathbb{Z}_8 \times \mathbb{Z}_2^{(8)}$	Not Capable	$\mathbb{Z}_2$
42	$\mathbb{Z}_2^{(2)}$	$\mathbb{Z}_4 \times \mathbb{Z}_2^{(2)}$	$\mathbb{Z}_4 \times \mathbb{Z}_2^{(8)}$	Not Capable	$\mathbb{Z}_2$
43	$\mathbb{Z}_2^{(2)}$	$\mathbb{Z}_4 \times \mathbb{Z}_2^{(2)}$	$\mathbb{Z}_4 \times \mathbb{Z}_2^{(8)}$	Not Capable	$\mathbb{Z}_2$
44	$\mathbb{Z}_2^{(2)}$	$\mathbb{Z}_4 \times \mathbb{Z}_2^{(2)}$	$\mathbb{Z}_4 \times \mathbb{Z}_2^{(8)}$	Not Capable	$\mathbb{Z}_2$
46	$\mathbb{Z}_2^{(6)}$	$\mathbb{Z}_4 \times \mathbb{Z}_2^{(5)}$	$\mathbb{Z}_4 \times \mathbb{Z}_2^{(15)}$	Capable	$\{1\}$
47	$\mathbb{Z}_2^{(5)}$	$\mathbb{Z}_2^{(6)}$	$\mathbb{Z}_4^{(2)} \times \mathbb{Z}_2^{(14)}$	Not Capable	$\mathbb{Z}_2$
48	$\mathbb{Z}_2^{(5)}$	$\mathbb{Z}_2^{(6)}$	$\mathbb{Z}_2^{(16)}$	Not Capable	$\mathbb{Z}_2$
49	$\mathbb{Z}_2^{(5)}$	$\mathbb{Z}_2^{(6)}$	$\mathbb{Z}_2^{(16)}$	Not Capable	$\mathbb{Z}_2$
50	$\mathbb{Z}_2^{(5)}$	$\mathbb{Z}_2^{(6)}$	$\mathbb{Z}_2^{(16)}$	Not Capable	$\mathbb{Z}_2$

Table 4.3: Groups of order 2^5

The following table takes care of groups of order 3^5. These results are obtained by GAP [14].

Group ID	M(G)	$G \wedge G$	$G \otimes G$	Capability	Epicenter
2	$\mathbb{Z}_3^{(3)}$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(2)}$	$\mathbb{Z}_9^{(4)} \times \mathbb{Z}_3^{(2)}$	Capable	$\{1\}$
3	$\mathbb{Z}_3^{(2)}$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(3)}$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(6)}$	Capable	$\{1\}$
4	$\mathbb{Z}_3$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(2)}$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(5)}$	Capable	$\{1\}$
5	$\{1\}$	$\mathbb{Z}_3^{(3)}$	$\mathbb{Z}_3^{(6)}$	Not Capable	$\mathbb{Z}_3^{(2)}$
6	$\mathbb{Z}_3$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(2)}$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(5)}$	Not Capable	$\mathbb{Z}_3$
7	$\{1\}$	$\mathbb{Z}_3^{(3)}$	$\mathbb{Z}_3^{(6)}$	Not Capable	$\mathbb{Z}_3^{(2)}$
8	$\mathbb{Z}_3$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(2)}$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(5)}$	Not Capable	$\mathbb{Z}_3$
9	$\mathbb{Z}_3$	$\mathbb{Z}_3^{(4)}$	$\mathbb{Z}_3^{(7)}$	Capable	$\{1\}$
11	$\mathbb{Z}_3$	$\mathbb{Z}_9$	$\mathbb{Z}_9^{(4)}$	Not Capable	$\mathbb{Z}_3$
12	$\mathbb{Z}_3^{(2)}$	$\mathbb{Z}_3^{(3)}$	$\mathbb{Z}_{27} \times \mathbb{Z}_3^{(5)}$	Not Capable	$\mathbb{Z}_9$
13	$\mathbb{Z}_3^{(2)}$	$\mathbb{Z}_3^{(4)}$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(6)}$	Capable	$\{1\}$
14	$\mathbb{Z}_3^{(2)}$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(2)}$	$\mathbb{Z}_9^{(2)} \times \mathbb{Z}_3^{(4)}$	Capable	$\{1\}$
15	$\mathbb{Z}_3^{(2)}$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(2)}$	$\mathbb{Z}_9^{(2)} \times \mathbb{Z}_3^{(4)}$	Not Capable	$\mathbb{Z}_3$
16	$\mathbb{Z}_3$	$\mathbb{Z}_3^{(3)}$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(5)}$	Not Capable	$\mathbb{Z}_9$
17	$\mathbb{Z}_3^{(2)}$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(2)}$	$\mathbb{Z}_9^{(2)} \times \mathbb{Z}_3^{(4)}$	Not Capable	$\mathbb{Z}_3$
18	$\mathbb{Z}_3$	$\mathbb{Z}_3^{(3)}$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(5)}$	Not Capable	$\mathbb{Z}_3 \times \mathbb{Z}_3$
19	$\mathbb{Z}_3$	$\mathbb{Z}_3^{(3)}$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(5)}$	Not Capable	$\mathbb{Z}_9$
20	$\mathbb{Z}_3$	$\mathbb{Z}_3^{(3)}$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(5)}$	Not Capable	$\mathbb{Z}_9$
21	$\mathbb{Z}_3$	$\mathbb{Z}_9$	$\mathbb{Z}_{27} \times \mathbb{Z}_9 \times \mathbb{Z}_3^{(2)}$	Not Capable	$\mathbb{Z}_3$
22	$\{1\}$	$\mathbb{Z}_9$	$\mathbb{Z}_9^{(2)} \times \mathbb{Z}_3^{(2)}$	Not Capable	$\mathbb{Z}_3$
24	$\{1\}$	$\mathbb{Z}_3$	$\mathbb{Z}_{27} \times \mathbb{Z}_3^{(3)}$	Not Capable	$\mathbb{Z}_{27}$
25	$\mathbb{Z}_3$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(2)}$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(5)}$	Not Capable	$\mathbb{Z}_3$

Group ID	M(G)	$G \wedge G$	$G \otimes G$	Capability	Epicenter
26	$\mathbb{Z}_9 \times \mathbb{Z}_3$	X	$X \times \mathbb{Z}_3^{(3)}$	Capable	$\{1\}$
27	$\mathbb{Z}_3$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(2)}$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(5)}$	Not Capable	$\mathbb{Z}_3$
28	$\mathbb{Z}_9$	Y	$Y \times \mathbb{Z}_3^{(3)}$	Capable	$\{1\}$
29	$\mathbb{Z}_3$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(2)}$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(5)}$	Not Capable	$\mathbb{Z}_3$
30	$\mathbb{Z}_3$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(2)}$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(5)}$	Not Capable	$\mathbb{Z}_3$
32	$\mathbb{Z}_3^{(4)}$	$\mathbb{Z}_3^{(5)}$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(10)}$	Not Capable	$\mathbb{Z}_3$
33	$\mathbb{Z}_3^{(3)}$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(2)}$	$\mathbb{Z}_9^{(2)} \times \mathbb{Z}_3^{(7)}$	Capable	$\{1\}$
34	$\mathbb{Z}_9 \times \mathbb{Z}_3$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(2)}$	$\mathbb{Z}_9^{(2)} \times \mathbb{Z}_3^{(7)}$	Capable	$\{1\}$
35	$\mathbb{Z}_3^{(4)}$	$\mathbb{Z}_3^{(5)}$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(10)}$	Not Capable	$\mathbb{Z}_3$
36	$\mathbb{Z}_3^{(2)}$	$\mathbb{Z}_3^{(3)}$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(8)}$	Not Capable	$\mathbb{Z}_3 \times \mathbb{Z}_3$
37	$\mathbb{Z}_3^{(6)}$	$\mathbb{Z}_3^{(8)}$	$\mathbb{Z}_3^{(14)}$	Capable	$\{1\}$
38	$\mathbb{Z}_3^{(3)}$	$\mathbb{Z}_3^{(5)}$	$\mathbb{Z}_3^{(11)}$	Not Capable	$\mathbb{Z}_3$
39	$\mathbb{Z}_3^{(3)}$	$\mathbb{Z}_3^{(5)}$	$\mathbb{Z}_3^{(11)}$	Not Capable	$\mathbb{Z}_3$
40	$\mathbb{Z}_3^{(3)}$	$\mathbb{Z}_3^{(5)}$	$\mathbb{Z}_3^{(11)}$	Not Capable	$\mathbb{Z}_3$
41	$\mathbb{Z}_3$	$\mathbb{Z}_3^{(3)}$	$\mathbb{Z}_3^{(9)}$	Not Capable	$\mathbb{Z}_3 \times \mathbb{Z}_3$
42	$\mathbb{Z}_3^{(2)}$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(2)}$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(8)}$	Capable	$\{1\}$
43	$\mathbb{Z}_9$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(2)}$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(8)}$	Capable	$\{1\}$
44	$\mathbb{Z}_3$	$\mathbb{Z}_3^{(3)}$	$\mathbb{Z}_3^{(9)}$	Not Capable	$\mathbb{Z}_3 \times \mathbb{Z}_3$
45	$\mathbb{Z}_9$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(2)}$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(8)}$	Capable	$\{1\}$
46	$\mathbb{Z}_3$	$\mathbb{Z}_3^{(3)}$	$\mathbb{Z}_3^{(9)}$	Not Capable	$\mathbb{Z}_3 \times \mathbb{Z}_3$
47	$\mathbb{Z}_3$	$\mathbb{Z}_3^{(3)}$	$\mathbb{Z}_3^{(9)}$	Not Capable	$\mathbb{Z}_3 \times \mathbb{Z}_3$
49	$\mathbb{Z}_3^{(2)}$	$\mathbb{Z}_3^{(3)}$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(8)}$	Not Capable	$\mathbb{Z}_9$
50	$\mathbb{Z}_3^{(2)}$	$\mathbb{Z}_3^{(3)}$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(8)}$	Not Capable	$\mathbb{Z}_9$
51	$\mathbb{Z}_3^{(3)}$	$\mathbb{Z}_3^{(5)}$	$\mathbb{Z}_3^{(11)}$	Not Capable	$\mathbb{Z}_3$
52	$\mathbb{Z}_3^{(3)}$	$\mathbb{Z}_3^{(5)}$	$\mathbb{Z}_3^{(11)}$	Not Capable	$\mathbb{Z}_3$

Group ID	M(G)	$G \wedge G$	$G \otimes G$	Capability	Epicenter
53	$\mathbb{Z}_3^{(4)}$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(4)}$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(10)}$	Capable	$\{1\}$
54	$\mathbb{Z}_3^{(3)}$	$\mathbb{Z}_3^{(5)}$	$\mathbb{Z}_3^{(11)}$	Not Capable	$\mathbb{Z}_3$
55	$\mathbb{Z}_3^{(3)}$	$\mathbb{Z}_3^{(5)}$	$\mathbb{Z}_3^{(11)}$	Not Capable	$\mathbb{Z}_3$
56	$\mathbb{Z}_3^{(3)}$	$\mathbb{Z}_3^{(5)}$	$\mathbb{Z}_3^{(11)}$	Not Capable	$\mathbb{Z}_3$
57	$\mathbb{Z}_3^{(3)}$	$\mathbb{Z}_3^{(5)}$	$\mathbb{Z}_3^{(11)}$	Not Capable	$\mathbb{Z}_3$
58	$\mathbb{Z}_3^{(4)}$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(4)}$	$\mathbb{Z}_9 \times \mathbb{Z}_3^{(10)}$	Capable	$\{1\}$
59	$\mathbb{Z}_3^{(3)}$	$\mathbb{Z}_3^{(5)}$	$\mathbb{Z}_3^{(11)}$	Not Capable	$\mathbb{Z}_3$
60	$\mathbb{Z}_3^{(3)}$	$\mathbb{Z}_3^{(5)}$	$\mathbb{Z}_3^{(11)}$	Not Capable	$\mathbb{Z}_3$
62	$\mathbb{Z}_3^{(7)}$	$\mathbb{Z}_3^{(8)}$	$\mathbb{Z}_3^{(18)}$	Capable	$\{1\}$
63	$\mathbb{Z}_3^{(5)}$	$\mathbb{Z}_3^{(6)}$	$\mathbb{Z}_3^{(16)}$	Not Capable	$\mathbb{Z}_3$
64	$\mathbb{Z}_3^{(5)}$	$\mathbb{Z}_3^{(6)}$	$\mathbb{Z}_3^{(16)}$	Not Capable	$\mathbb{Z}_3$
65	$\mathbb{Z}_3^{(5)}$	$\mathbb{Z}_3^{(6)}$	$\mathbb{Z}_3^{(16)}$	Not Capable	$\mathbb{Z}_3$
66	$\mathbb{Z}_3^{(5)}$	$\mathbb{Z}_3^{(6)}$	$\mathbb{Z}_3^{(16)}$	Not Capable	$\mathbb{Z}_3$

Table 4.4: Groups of order 3^5

The groups X and Y in Table 4.4 are given by

$$X = \langle a, b, c \mid [b, a] = c^3, [a, c] = [b, c] = 1, a^9 = b^9 = c^9 = 1\rangle$$

and

$$Y = \langle a, b, c \mid [b, a] = c^3, [a, c] = [b, c] = 1, a^9 = b^3 = c^9 = 1\rangle.$$

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