SOME TOPICS ON DIRICHLET L-FUNCTIONS

By

MITHUN KUMAR DAS

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Harish-Chandra Research Institute, Prayagraj

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D. S. R

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(Guide)

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Mithun Kumar Das

Mithun Kumar Das

DECLARATION

I, Mithun Kumar Das, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree or diploma at this or any other Institution or University.

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List of publications/preprints arising from the thesis

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2. "Higher dimensional Dedekind sums and twisted mean value of Dirichlet L-series", Mithun Kumar Das, Abhishak Juyal, *Journal of the Ramanujan Mathematical Society*, **2020**, *35*, no. 4, 327–340.

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Mithun Kumar Das

To

My family, for their inexhaustible support.

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Chapter 1

Background

This chapter is a collage of various facts from analytic number theory, mainly pertaining to the Riemann zeta function and Dirichlet L-functions, that we will find useful later. The material of Sections 1.2 to 1.5 is entirely standard; the reader may refer, for example, to the relevant sections of H. Cohen's textbooks [16] and [17] for detailed coverage. We begin by recalling the unifying notion of the Selberg class.

1.1 The Selberg class

Let $s \in \mathbb{C}$ and $(a_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers. Then a series of the form

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

is called a Dirichlet series. In 1989, Selberg [70] introduced a class $\mathbb{S}$ of functions on $\mathbb{C}$ defined by a Dirichlet series, now called the Selberg class, and formulated some conjectures on this class. An element F of $\mathbb{S}$ satisfies the following axioms:

(i) The function F is given by an absolutely convergent Dirichlet series $\sum_{n\geq 1} \frac{a_n}{n^s}$ in the half-plane $\mathrm{Re}(s) = \sigma > 1$.

(ii) Analyticity : F has a meromorphic extension to $\mathbb{C}$ which has no poles on $\mathbb{C}$ except possibly at $s = 1$. If $m \geq 0$ is the order of the pole $s = 1$ of F, then the function $s \mapsto (s-1)^m F(s)$, which is thus an entire function, is of finite order.

(iii) Functional equation: there exist an integer $k \geq 1$, real numbers Q, $w_1, w_2, \ldots, w_k$ and complex numbers ω, $\mu_1, \mu_2, \ldots, \mu_k$ satisfying

$$|\omega| = 1, Q > 0 \text{ and } w_j > 0, \mathrm{Re}\,(\mu_j) \geq 0$$

for all j, $1 \leq j \leq k$, such that

$$\Phi(s) = Q^s \prod_{j=1}^{k} \Gamma(w_j s + \mu_j) F(s)$$

satisfies the relation

$$\Phi(s) = \omega \overline{\Phi(1-\overline{s})}, \tag{1.1}$$

for all s in $\mathbb{C}$.

(iv) Ramanujan hypothesis: for any given $\delta > 0$ we have $a_n \ll_\delta n^\delta$ for all $n \geq 1$.

(v) Euler product : we have $a_1 = 1$ and that there exists a sequence of complex numbers $(b_n)_{n\in\mathbb{N}}$ with $\sum_n \frac{b_n}{n^s}$ an absolutely convergent Dirichlet series for $\mathrm{Re}\,(s) > 1$ such that in this half-plane we have

$$\log F(s) = \sum_{n\geq 1} \frac{b_n}{n^s}. \tag{1.2}$$

Moreover, $b_n = 0$ except when n is a prime power and $b_n \ll_\theta n^\theta$ for some $\theta < 1/2$.

In (v) above and elsewhere $\log F$ means, of course, a function G such that $\exp(G) = F$ on the domain in question, for instance, the half-plane $\mathrm{Re}\,(s) > 1$ in (v).

An $F \in \mathbb{S}$ is said to be primitive if the relation $F = F_1 F_2$ with F_1 and F_2 in $\mathbb{S}$ implies either $F_1 = F$ or $F_2 = F$. Selberg [70] made, among others, the following *pair of conjectures*.

1. For each F in the class $\mathbb{S}$ there is an integer n_F, which is 1 whenever F is primitive, so that for all $X \geq 1$ we have

$$\sum_{p \leq X} \frac{|a_p|^2}{p} = n_F \log\log X + O(1) \, . \tag{1.3}$$

2. For any distinct primitive elements F, F' in the class $\mathbb{S}$ we have

$$\sum_{p \leq X} \frac{a_p a'_p}{p} \ll 1 \, . \tag{1.4}$$

for all $X \geq 1$.

In these statements, of course, the a_n and the a'_n are the coefficients of the Dirichlet series defining F and F' respectively and the sums (1.3) and (1.4) are over prime numbers p.

A problem of great interest is to classify members in $\mathbb{S}$. To this end, one defines the degree d_F of an $F \in \mathbb{S}$ by the relation

$$d_F = 2 \sum_{j=1}^{k} w_j.$$

We have from [18] that the only function of degree 0 in the Selberg class is the constant function $F = 1$ and that there are no elements with degree d satisfying $0 < d < 1$. To describe the degree 1 elements of the Selberg class, we now review the basic facts on the Riemann zeta function and the Dirichlet L-functions.

1.2 The Riemann zeta function

For any $s = \sigma + it \in \mathbb{C}$ with $\sigma > 1$, we set

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{1.5}$$

on noting that the right-hand side is an absolutely convergent Dirichlet series for such s. Thus the function $s \mapsto \zeta(s)$ is a holomorphic function on the half-plane $\sigma > 1$ and, as we explain in more detail later, extends as a meromorphic function on $\mathbb{C}$. This meromorphic function is denoted by ζ and is called the Riemann zeta function.

The study on this series began about 1730, by Swiss mathematician, Leonhard Euler (1707-1783). He studied the sum

$$1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\cdots$$

for integer $s>1$, which is precisely $\zeta(s)$. Euler discovered a formula relating $\zeta(2k)$, $k\geq 1$, to the Bernoulli numbers as

$$\zeta(2k)=\frac{(-1)^{k-1}B_{2k}(2\pi)^{2k}}{2(2k)!},$$

where B_{2k} is the $2k$-th Bernoulli number. This formula yields $\zeta(2)=\frac{\pi^2}{6}$, $\zeta(4)=\frac{\pi^4}{90}$ and so on.

The function ζ is intimately connected with the distribution of primes and was originally studied by Riemann in his famous memoir published in 1859. Fundamental to this connection is the fact that $\zeta(s)$ can be written as an infinite product involving only the primes numbers, i.e.,

$$\zeta(s)=\prod_p\left(1-\frac{1}{p^s}\right)^{-1}. \tag{1.6}$$

This relation, which was first discovered by Euler for real values $s>1$, remains valid for all complex s in the half-plane $\sigma>1$, with the product being absolutely and uniformly convergent on each compact subset of this half-plane. It follows that ζ is in fact a non-vanishing holomorphic function in the half-plane $\sigma>1$. Further, (1.6) can be written as

$$\log\zeta(s)=\sum_{n\geq 2}\frac{\Lambda(n)}{n^s\log n} \tag{1.7}$$

for $\sigma>1$, where Λ is the classical Von Mangoldt function. This version of (1.6) is in the form (1.2) with $b_1 = 0$ and $b_n = \frac{\Lambda(n)}{\log n}$ for $n \geq 2$.

1.3 Dirichlet characters

We recall that a character of a group is a homomorphism from the group to the multiplicative group of non-zero complex numbers $\mathbb{C} \setminus \{0\}$. A function χ from $\mathbb{Z}$ to $\mathbb{C}$ is called a Dirichlet character modulo q or to the modulus q, where $q \geq 1$ is an integer, if there exists a character g of the group $G(q)$ of invertible residue classes modulo q such that for each integer n we have

$$\chi(n) = \begin{cases} g(\hat{n}) & \text{if } (n,q) = 1, \\ 0 & \text{if } (n,q) > 1, \end{cases} \tag{1.8}$$

where $\hat{n}$ is the residue class of n modulo q.

A Dirichlet character modulo q, for an integer $q \geq 1$, is therefore a periodic completely multiplicative arithmetic function with period q that takes non-zero complex values at the integers coprime to q and is 0 for all other integers. Indeed, since $G(q)$ is a finite group of order $\phi(q)$, the values of each of its characters lie in the subgroup of $\phi(q)$-th roots of unity in $\mathbb{C}$. Thus for any Dirichlet character modulo q and any integer n we in fact have that $\chi(n)$ is a $\phi(q)$-th root of unity if $(n,q) = 1$ and hence $|\chi(n)| = 1$ in that case and, of course, $\chi(n) = 0$ otherwise. Further, there are $\phi(q)$ Dirichlet characters modulo q, since their number is the same as the number of characters of the group $G(q)$, which is $\phi(q)$ as well.

For any integer $q \geq 1$, the Dirichlet character χ_0, defined by $\chi_0(n) = 1$ for $(q,n) = 1$ and $\chi_0(n) = 0$ otherwise is called the principal character to modulus q. It arises from the trivial character on $G(q)$ by (1.8). All other Dirichlet characters modulo q are called non-principal characters. Note that when $q = 1$ there is a unique Dirichlet character χ modulo q and it is the principal character modulo 1, which satisfies $\chi(n) = 1$ for *all* integers n.

Let χ be a Dirichlet character modulo an integer $q \geq 1$ and let d be a divisor of q. Then the Dirichlet character χ is said to be defined modulo d if there is a Dirchlet character ψ modulo d such that $\chi(n) = \psi(n)$ for all integers n satisfying $(n, q) = 1$. Plainly, for a Dirichlet character ψ modulo d to exist such that this relation holds a necessary condition is that $\chi(n) = 1$ for all integers n satisfying $(n, q) = 1$ and $n \equiv 1 \pmod{d}$, since $\psi(n) = 1$ for such n. It is known that this condition is also sufficient.

The conductor of χ of a Dirichlet character modulo q is the smallest of the divisors d of q such that χ is defined modulo d. A Dirichlet character χ modulo q is said to be primitive if its conductor is q. For example, all the non-principal characters to modulus p, a prime number, are primitive. On the other hand, the conductor of the principal Dirichlet character modulo any integer $q \geq 1$ is 1. If d is the conductor of a Dirichlet character χ modulo q then there is a primitive Dirichlet character ψ modulo d such that

$$\chi(n) = \chi_0(n)\psi(n) \text{ for all integers } n, \tag{1.9}$$

where χ_0 is the principal character modulo q.

A Dirichlet character χ is said to be an even character, and respectively an odd character, if $\chi(-1) = 1$, respectively if $\chi(-1) = -1$. Thus if for any Dirichlet character χ we set

$$\mathfrak{a} = \mathfrak{a}(\chi) = \frac{1 - \chi(-1)}{2}. \tag{1.10}$$

then χ is even or odd, respectively, if $\mathfrak{a} = 0$ or $\mathfrak{a} = 1$.

For any Dirichlet character χ to a modulus q we have the following :

$$\sum_{m \bmod q} \chi(m) = \begin{cases} \varphi(q) & \text{if } \chi \text{ is the principal character,} \\ 0 & \text{else.} \end{cases} \tag{1.11}$$

Also, one has

$$\sum_{\chi \bmod q} \chi(m) = \begin{cases} \varphi(q) & \text{if } m \equiv 1 \mod q, \\ 0 & \text{else,} \end{cases} \tag{1.12}$$

where the sum is now over all Dirichlet characters χ modulo q. The relation (1.12) yields a fundamental orthogonality relation for Dirichlet characters modulo q, which is that for any integers m and a with $(a, q) = 1$ we have

$$\frac{1}{\varphi(q)} \sum_{\chi \bmod q} \overline{\chi}(a)\chi(m) = \begin{cases} 1 & \text{if } m \equiv a \bmod q, \\ 0 & \text{else.} \end{cases} \tag{1.13}$$

Next, we recall an important sum $G(n, \chi)$, called the Gauss sum associated to a Dirichlet character χ modulo an integer $q \geq 1$. It is defined by

$$G(n, \chi) = \sum_{m \bmod q} \chi(m) e\left(\frac{nm}{q}\right), \tag{1.14}$$

for any integer n and Dirichlet character χ modulo q. Note that if $(n,q) = 1$, then nm varies over the complete set of residue classes modulo q when m varies over the same. From this fact and the properties of χ recalled above we get

$$G(n, \chi) = \overline{\chi}(n) \sum_{m \bmod q} \chi(mn) e\left(\frac{mn}{q}\right) = \overline{\chi}(n) G(1, \chi),$$

for such integers n. If, moreover, χ is a primitive character modulo q, then one can show that $G(n, \chi) = 0$ when $(n, q) \neq 1$ and therefore for such χ the above identity holds for all integers n.

We define $\tau(\chi)$ to be $G(1, \chi)$ for any Dirichlet character χ modulo q. Then one can prove the following basic lemma starting from the identity above :

Lemma 1.3.1 *Let χ be a primitive character modulo q. Then we have that*

(1) $|\tau(\chi)| = \sqrt{q}$,

(2) $\chi(n) = \frac{\tau(\chi)}{q} \sum_{m \bmod q} \overline{\chi}(m) e\left(-\frac{mn}{q}\right)$.

1.4 Dirichlet L-functions

Given a Dirichlet character χ and an $s = \sigma + it$ with $\sigma > 1$ we set

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}. \tag{1.15}$$

Indeed, we have $|\chi(n)| \leq 1$ for all $n \geq 1$ and by comparison with the series on the right-hand side of (1.5) it follows that the Dirichlet series on the right-hand side of (1.15) is also absolutely convergent in the half-plane $\sigma > 1$. Thus the function $s \mapsto L(s,\chi)$ is a holomorphic function in this half-plane. As we review in Section 1.5 this function, like ζ, can be extended as a meromorphic function to $\mathbb{C}$. This meromorphic function is called the Dirichlet L-function associated with χ. Note that the Dirichlet L-function associated with the unique Dirichlet character modulo 1 is the same as the Riemann zeta function since the series on the right-hand sides of (1.15) and (1.5) are identical in that case. Thus Dirichlet L-functions are a generalization of the Riemann zeta function.

Since a Dirichlet character is a completely multiplicative function we obtain the following analogue of the product formula (1.6) for Dirichlet L-functions :

$$L(s,\chi) = \prod_{p} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}, \tag{1.16}$$

The product on the right-hand side of (1.16) is also absolutely and uniformly convergent on each compact subset in the half-plane $\sigma > 1$ and hence $s \mapsto L(s,\chi)$ is, in fact, a non-vanishing holomorphic function in this half-plane. As with the ζ function, one may rewrite (1.16) as

$$\log L(s,\chi) = \sum_{n\geq 2} \frac{\Lambda(n)\chi(n)}{n^s \log n} \tag{1.17}$$

for $\sigma > 1$, which is again in the form (1.2) with $b_1 = 0$ and $b_n = \frac{\Lambda(n)\chi(n)}{\log n}$ for $n \geq 2$.

Suppose now that χ is a Dirichlet character to the modulus q and let the conductor of χ be the divisor d of q. If then ψ is the Dirichlet character modulo d such that (1.9) holds, we have using (1.16) that

$$L(s,\chi) = L(s,\psi)\prod_{p|q}\left(1-\frac{\psi(p)}{p^s}\right). \tag{1.18}$$

Since the product on the right-hand side of (1.18) is a finite product, which is thus easily understood in various situations, we shall often restrict ourselves to the case when χ is a primitive character modulo q when describing properties of Dirichlet L-functions.

Dirichlet introduced his L-functions in 1837 to prove the celebrated theorem that there are infinitely many prime numbers in any arithmetic progression

$$\{qn+a : n \in \mathbb{N}\cup\{0\}\}$$

with $(a,q) = 1$. To do this, Dirichlet used (1.13) together with (1.17) to obtain the relations

$$\sum_{p\equiv a \bmod q} \frac{1}{p^s} = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \overline{\chi}(a) \sum_p \frac{\chi(p)}{p^s} = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \overline{\chi}(a) \log L(s,\chi) + \mathrm{O}(1),$$

for real $s > 1$. Then on noting that $\log L(s,\chi_0) \to \infty$ as $s \to 1$, Dirichlet reduced the proof of his theorem to showing that $L(1,\chi) \neq 0$ for $\chi \neq \chi_0$ modulo q, which he obtained using his class number formula.

1.5 The functional equation

Riemann in his memoir of 1859 showed that the function ζ can be extended as a meromorphic function in the complex plane and that it has a unique pole, a simple pole at $s=1$ with residue 1. Further, he established the following functional equation for ζ: for all s in $\mathbb{C}$ we have

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s), \tag{1.19}$$

where Γ is the gamma function. We recall that this function is defined on the half-plane $\mathrm{Re}(s)=\sigma>0$ by

$$\Gamma(s)=\int_0^\infty x^{s-1}e^{-x}dx. \tag{1.20}$$

It is easily seen from this definition that Γ is a holomorphic function on the aforementioned half-plane. Moreover, that this function can be extended as a meromorphic function on $\mathbb{C}$ with its only poles being simple poles at $s=-n$ for any integer $n\geq 0$, where it has residue $\frac{(-1)^n}{n!}$. Further, for all s in $\mathbb{C}$ we have that Γ satisfies the relations

$$\Gamma(s)=(s-1)\Gamma(s-1) \quad \text{and} \quad \Gamma(s)\Gamma(1-s)=\frac{\pi}{\sin\pi s} \tag{1.21}$$

and

$$\Gamma(s)=\pi^{-1/2}2^{s-1}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right). \tag{1.22}$$

The first relation in (1.21) is called the functional equation for the gamma function while the second relation in (1.21) is called the reflection formula for the gamma function. The relation (1.22) is called the duplication formula for the gamma function.

The function Γ is a non-vanishing function on $\mathbb{C}$. Indeed, the reflection formula above tells us that any zero of Γ is necessarily an integer. However, as we have already remarked, the integers $n \leq 0$ are poles of Γ while $\Gamma(n) = (n-1)!$ for integers $n \geq 1$, by the functional equation in (1.21) and $\Gamma(1) = 1$.

We now give a brief sketch of a proof of meromorphic extension and the functional equation (1.19) for the function ζ. Thus suppose that $s = \sigma + it$ with $\sigma > 1$. Then from the definition of the gamma function (1.20) we can write

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)n^{-s} = \int_0^\infty t^{\frac{1}{2}s-1}e^{-\pi n^2 t}dt.$$

Now, summing over all integers $n \neq 0$ and interchanging summation and integration which is easily justified, we obtain

$$2\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty t^{\frac{s}{2}-1}(\psi(t) - \psi(0))dt, \tag{1.23}$$

where for any t in $(0, \infty)$ we have set

$$\psi(t) = \sum_n e^{-\pi n^2 t}$$

with the summation ranging over all integers n. Note, of course, that $\psi(0) = 1$. The function $\psi : t \mapsto \psi(t)$ is essentially Jacobi's theta function (see Chapter 10 of [71]) and it is known that ψ satisfies the functional equation

$$\psi(t) = \frac{1}{\sqrt{t}}\psi(\frac{1}{t}) \tag{1.24}$$

for all t in $(0, \infty)$. On splitting the range of integration in the integral on the right-hand side of (1.23) into $(0, 1)$ and $[1, \infty)$ and using the the relation (1.24) for ψ in the integral over $(0,\, 1)$ we obtain the first equality in

$$\begin{aligned} 2\pi^{-\frac{s}{2}}\Gamma\Big(\frac{s}{2}\Big)\zeta(s) &= \int_1^\infty t^{\frac{s}{2}-1}(\psi(t)-\psi(0))dt + \int_0^1 \left(\frac{1}{\sqrt{t}}\psi\Big(\frac{1}{t}\Big)-\psi(0)\right)t^{\frac{s}{2}-1}dt \\ &= \int_1^\infty \frac{\psi(t)-\psi(0)}{t}\left(t^{\frac{s}{2}}+t^{\frac{1-s}{2}}\right)dt + 2\psi(0)\left(\frac{1}{s-1}-\frac{1}{s}\right). \end{aligned} \tag{1.25}$$

The second equality above results on making the change of variables $t \mapsto \frac{1}{t}$ in the integral over $(0, 1)$ on the right-hand side of the first equality, rearranging the terms and evaluating the integrals of $t^{-\frac{s}{2}-1}$ and $t^{\frac{1-s}{2}-1}$ over $(1,\, \infty)$. The function $(\psi(t)-\psi(0))/t$ tends to 0 exponentially as $t \to \infty$. Hence the integral on the right-hand side of the second equality above converges absolutely and uniformly on any compact subset of $\mathbb{C}$ and thus defines an entire function of s.

Let us now set

$$\Lambda(s) = \pi^{-\frac{s}{2}}\Gamma\Big(\frac{s}{2}\Big)\zeta(s). \tag{1.26}$$

Then the conclusions of the preceding paragraph and the relation (1.25) tell us that the function $\Lambda : s \mapsto \Lambda(s)$, defined as a holomorphic function on the half-plane $\sigma > 1$, extends as a meromorphic function on $\mathbb{C}$ with two poles, one at $s = 1$ with residue $\psi(0) = 1$ and the other at $s = 0$ with residue $-\psi(0) = -1$. Since $\pi^{-\frac{s}{2}}$ is an entire function of s and since $\Gamma(\frac{s}{2})$ has a simple pole at $s = 0$ with residue -1, we see that ζ extends as meromorphic function on $\mathbb{C}$ with a unique pole, a simple pole at $s = 1$ with residue 1 (since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$). In particular, $(s-1)\zeta(s)$ is an entire function. It can be shown to be of order 1; we refer to Theorem 3.20, page 245 of [77], which gives a stronger conclusion. Furthermore, since the right-hand side of (1.25) is invariant under $s \mapsto 1-s$ we obtain that $\Lambda(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ satisfies for all s in $\mathbb{C}$ the relation

$$\Lambda(s) = \Lambda(1-s), \tag{1.27}$$

which is the functional equation of Riemann as stated in (1.19). For all s in $\mathbb{C}$, the function Λ satisfies $\Lambda(s) = \overline{\Lambda(\overline{s})}$ as can be seen by analytic continuation from the half-plane $\mathrm{Re}(s) = \sigma > 0$. Consequently, (1.27) can be written as

$$\Lambda(s) = \overline{\Lambda(1-\overline{s})},$$

for all s in $\mathbb{C}$, which is in the form (1.1). With these remarks and those of Section 1.2, we obtain that ζ is a degree 1 function of the Selberg class.

The arguments outlined above can be extended to cover the general case of a Dirichlet L-function associated with a *primitive* Dirichlet character χ of modulus q. In light of (1.18), this is a reasonable assumption on χ. The details omitted in our sketch below can be found on pages 173 to 175 of [17]. Here the role of $\psi(t)$ is played by

$$\psi(t,\chi) = \sum_n n^{\mathfrak{a}} \chi(n) e^{-\frac{\pi n^2 t}{q}} \tag{1.28}$$

with t as before in $(0,\infty)$ and $\mathfrak{a} = \mathfrak{a}(\chi)$, defined in (1.10). Note that $\psi(0,\chi) = \chi(0) = 0$ for any χ of modulus $q \geq 2$. It can be shown that the function $t \mapsto \psi(t,\chi)$ on $(0,\infty)$ satisfies the functional equation

$$\psi(t,\chi) = \mathfrak{w}(\chi) t^{-\frac{2\mathfrak{a}+1}{2}} \psi\left(\frac{1}{t}, \overline{\chi}\right), \tag{1.29}$$

where $\overline{\chi}$ is defined by $\overline{\chi}(n) = \overline{\chi(n)}$ for all integers n and

$$\mathfrak{w}(\chi) = \frac{\tau(\chi)}{i^{\mathfrak{a}} q^{\frac{1}{2}}}. \tag{1.30}$$

Then extending (1.26) we define

$$\Lambda(s,\chi) = \left(\frac{\pi}{q}\right)^{-\frac{s}{2}} \Gamma\left(\frac{s+\mathfrak{a}}{2}\right) L(s,\chi) \tag{1.31}$$

and obtain as a generalization of (1.23) the relation

$$2\Lambda(s,\chi) = \left(\frac{\pi}{q}\right)^{\frac{\mathfrak{a}}{2}} \int_0^\infty t^{\frac{s+\mathfrak{a}}{2}-1}(\psi(t,\chi)-\psi(0,\chi))dt, \tag{1.32}$$

from which we eventually deduce using same principles as explained above for ζ that the function $s \mapsto \Lambda(s,\chi)$ extends as a meromorphic function on $\mathbb{C}$ with two poles, one at $s=1$ with residue $\left(\frac{\pi}{q}\right)^{\frac{\mathfrak{a}}{2}} \psi(0,\chi)$ and the other at $s=0$ with residue $-\left(\frac{\pi}{q}\right)^{\frac{\mathfrak{a}}{2}} \psi(0,\chi)$. Since, however, $\psi(0,\chi)=\chi(0)=0$ unless χ is the trivial character to modulus $q=1$, in which case $\chi(n)=1$ for all integers n and hence $L(s,\chi)=\zeta(s)$, it follows that the function $s \mapsto L(s,\chi)$ extends as an entire function when χ is any primitive character to a modulus $q \geq 2$. Using (1.18) we then infer that for any Dirichlet character χ, not necessarily primitive, with conductor $q \geq 2$, the function $s \mapsto L(s,\chi)$ extends as an entire function on $\mathbb{C}$ and also that when the conductor $q=1$, it extends as a meromorphic function to $\mathbb{C}$ with a unique pole, a simple pole at $s=1$. Furthermore, when χ is primitive, for all s in $\mathbb{C}$ we have the functional equation

$$\Lambda(s,\chi) = \mathfrak{w}(\chi)\Lambda(1-s,\overline{\chi}) = \mathfrak{w}(\chi)\overline{\Lambda(1-\overline{s},\chi)}, \tag{1.33}$$

generalizing (1.27). Also, from (1) of Lemma 1.3.1 we see that $|\mathfrak{w}(\chi)|=1$. Thus with $Q=\sqrt{\frac{q}{\pi}}$ we conclude that (1.33) is in the form (1.1). Moreover, it can be shown that $s \mapsto L(s,\chi)$ is of order 1 as an entire function, for any Dirichlet character χ of modulus $q \geq 2$; we refer to Theorem 8.24, page 385 of [77] for a stronger conclusion. With these remarks and those of Section 1.2 we obtain that the function $s \mapsto L(s,\chi)$ for any primitive Dirichlet character χ of modulus $q \geq 2$ is a degree 1 function of the Selberg class.

If F is an element of the Selberg class and α is any real number, then the imaginary translate of F by α, defined to be the function $s \mapsto F(s+i\alpha)$, satisfies all axioms describing the Selberg class given in Section 1.1 except possibly the axiom (ii). Indeed, if F is entire then so is its imaginary translate by any α but if F has a pole at $s=1$ then $s \mapsto F(s+i\alpha)$

has a pole at $1-i\alpha$, thus violating the axiom (ii) of Section 1.1 when $\alpha \neq 0$.

In conclusion, the family of imaginary translates of Dirichlet L-functions associated to non-trivial primitive characters, that is, the functions $s \mapsto L(s+i\alpha,\chi)$, where L is the Dirichlet L-function associated to a primitive Dirichlet character χ to a modulus $q \geq 2$ and α is a real number, are all degree 1 elements of the Selberg class, while $s \mapsto \zeta(s+i\alpha)$ belongs to the Selberg class only when $\alpha = 0$ and is of degree 1 in that case. A remarkable theorem of Kaczorowski and Perelli [42] tells us that these are the only elements of degree 1 in the Selberg class. A simple proof of the aforementioned theorem is given by K. Soundararajan in [72].

1.6 Zeros of degree one L-functions

We have remarked in Section 1.2 that ζ has no zeros in the half-plane $\mathrm{Re}(s) = \sigma > 1$ on account of the Euler product (1.27). Further, the functional equation (1.19) gives

$$\zeta(s) = \pi^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \zeta(1-s). \tag{1.34}$$

Since Γ has no zeros in $\mathbb{C}$, it now follows from that (1.34) that the zeros of ζ in the half-plane $\sigma < 0$ are the same as the zeros of $s \mapsto \Gamma\left(\frac{s}{2}\right)^{-1}$ in this half-plane and have the same orders. Hence ζ has a simple zero at $s = -2n$ for each integer $n \geq 1$ and these are the only zeros of the function ζ in the union of the half-planes $\sigma > 1$ and $\sigma < 0$. These zeros are called the trivial zeros of the function ζ. A fundamental result on the zeros of ζ is that there are no zeros of this function on the line $\sigma = 1$; see Theorem 3.13, page 239 of [77], for instance. Using the (1.34) and that fact ζ has a simple pole at $s = 1$ and Γ at $s = 0$, we conclude that ζ has no zeros on the line $\sigma = 0$ as well. Thus all the zeros of zeta function in the complex plane, except for the trivial zeros, lie in the strip $0 < \sigma < 1$. This strip is called the critical strip. The zeros of ζ in this strip are called the non-trivial zeros of ζ. The function ζ has no non-trivial real zeros. This follows, for instance, from the relation

$$1-\frac{1}{2^{\sigma}} \le \sum_{n\ge 1}\frac{(-1)^{n+1}}{n^{\sigma}} = \left(1-\frac{2}{2^{\sigma}}\right)\zeta(\sigma), \tag{1.35}$$

valid for all σ in the interval $(0,1)$.

The famous Riemann Hypothesis is the assertion that all the non-trivial zeros of ζ lie on the line $\sigma=\frac{1}{2}$, that is, that they are of the form $\frac{1}{2}+it$ for real t. The line $\sigma=\frac{1}{2}$ is called the critical line.

The function ζ satisfies $\zeta(s)=\overline{\zeta(\bar{s})}$ for all s in $\mathbb{C}$ and hence the zeros of ζ are symmetrical about the real line, that is, if $\sigma+it$ is a zero of ζ then so is $\sigma-it$. For any real $T>0$, we now let $N(T)$ be the number of zeros $\sigma+it$ of ζ with $\{0<\sigma<1, 0\le t\le T\}$. Then we have the formula for $N(T)$ given by the theorem below, which was stated by Riemann with a outline of his proof and fully established by Von Mangoldt.

Theorem 1.6.1 *Let $T\ge 2$ be a real number. Then*

$$N(T)=\frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi}+\mathrm{O}(\log T). \tag{1.36}$$

This result is proved by considering the entire function ξ defined by

$$\xi(s)=s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)=s(s-1)\Lambda(s).$$

Plainly, $\xi(s)=\xi(1-s)$, from (1.19), and also $\xi(s)=\overline{\xi(\bar{s})}$ for all s in $\mathbb{C}$. Applying the argument principle to the function ξ on the positively oriented rectangle with vertices $2\pm iT$, $-1\pm iT$ and taking account of the symmetries of this function as expressed by the preceding relations we see that

$$N(T)=\frac{1}{\pi}\mathrm{var}(\xi(s),\mathscr{C}). \tag{1.37}$$

Here and below we use $\mathrm{var}(f(s),\mathscr{C})$ to denote the variation of the argument of a function $s\mapsto f(s)$ along the rectilinear path $\mathscr{C}$ joining $2, 2+iT, 1/2+iT$ in that order. Since the variation of argument is additive over the factors defining the function ξ, and since $s\Gamma(\frac{s}{2}) = 2\Gamma(\frac{s}{2}+1)$ by the functional equation in (1.21), we have

$$\operatorname{var}(\xi(s),\mathscr{C}) = \operatorname{var}(s-1,\mathscr{C}) + \operatorname{var}(\pi^{-\frac{s}{2}},\mathscr{C}) + \operatorname{var}(\Gamma(\frac{s}{2}+1),\mathscr{C}) + \operatorname{var}(\zeta(s),\mathscr{C}). \quad (1.38)$$

The first two terms on the right-hand side of the above relation are easily computed to be $\frac{\pi}{2} + O(\frac{1}{T})$ and $-\frac{T}{2}\log\pi$ respectively. For the third term, we use Stirling's formula in its complex form to get

$$\operatorname{var}(\Gamma(\frac{s}{2}+1),\mathscr{C}) = \frac{T}{2}\log\frac{T}{2} - \frac{T}{2} + \frac{3\pi}{8} + \mathrm{O}\left(\frac{1}{T}\right). \quad (1.39)$$

The last term on the right-hand side of (1.38) is traditionally denoted by $S(T)$, so that, on combining (1.37), (1.38) and (1.39) we may write

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O(\frac{1}{T}). \quad (1.40)$$

$S(T)$ is an oscillatory function of T that can be shown to satisfy $S(T) \ll \log T$. Using this (1.40) we obtain Theorem 1.6.1; for the details missing from our account we refer to page 244 of [77]. The essential fact is that, very remarkably, the main contribution to the count for the zeros of the zeta function arises from the gamma factor in the definition of ξ via (1.39).

Much of what we have described above for ζ extends in a natural manner to the L-function $s \mapsto L(s,\chi)$, where χ is a primitive Dirichlet character to a modulus $q \geq 2$. The new features are the dependence of the results on the character χ and the conductor q. We begin by recalling from Section 1.4 that $L(s,\chi)$ is non-vanishing function of s on the half-plane $\sigma > 0$ on account of (1.16). Further, as with ζ, it is a fundamental theorem that $s \mapsto L(s,\chi)$ does not have a zero on the line $\sigma = 1$; this follows from Theorems 8.20 and 8.21 on page 378 of [77]. Thus on writing the functional equation (1.33) in the form

$$L(s,\chi) = \mathfrak{w}(\chi)\left(\frac{\pi}{q}\right)^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{1-s+\mathfrak{a}}{2}\right)}{\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)} L(1-s,\overline{\chi}) \tag{1.41}$$

for all s in $\mathbb{C}$, noting that $s \mapsto L(1-s,\overline{\chi})$ is an entire function and, as before, that Γ has no zeros in $\mathbb{C}$, we see that the zeros of $s \mapsto L(s,\chi)$ in the half-plane $\sigma \le 0$ are the same as the zeros of $s \mapsto \Gamma\left(\frac{s+\mathfrak{a}}{2}\right)^{-1}$ in this half-plane and have the same orders. Hence $s \mapsto L(s,\chi)$ has a simple zero at $s = -2n - \mathfrak{a}(\chi)$ for each integer $n \ge 0$ and these are the only zeros of this function in the union of the half-planes $\sigma \ge 1$ and $\sigma \le 0$ when χ is a primitive Dirichlet character of conductor $q \ge 2$. These zeros are called the trivial zeros of the function $s \mapsto L(s,\chi)$. Thus all other zeros of this function, called the non-trivial zeros, lie in the strip $0 < \sigma < 1$, again called the critical strip.

By analogy with the Riemann Hypothesis for the function ζ we have the Generalized Riemann Hypothesis which is the assertion that for any Dirichlet character χ, all the zeros of $s \mapsto L(s,\chi)$ in the strip $0 < \sigma < 1$ lie on the line $\sigma = \frac{1}{2}$, that is, that they are of the form $\frac{1}{2} + it$ for real t. The line $\sigma = \frac{1}{2}$ is again called the critical line in this general context.

Finally, we have the following extension of Theorem 1.6.1, which is proved by a similar argument. Here for a primitive character χ modulo q we have set $N(T,\chi)$ to be the number of zeros $L(s,\chi)$ in the rectangular region $\{0 < \sigma < 1, |t| \le T\}$.

Theorem 1.6.2 *For $T \ge 2$ we have*

$$\frac{1}{2}N(T,\chi) = \frac{T}{2\pi}\log\frac{qT}{2\pi} - \frac{T}{2\pi} + \mathrm{O}(\log qT).$$

1.7 Hardy's Z-function

The Riemann Hypothesis that all non-trivial zeros of the Riemann zeta function lie on the line $\sigma = \frac{1}{2}$, the critical line, has neither been proved nor disproved so far and is one of the most famous unsolved problems in Mathematics. Already in 1914, however, Hardy [34] obtained the remarkable result that the Riemann zeta function has infinitely many zeros on the critical line.

To prove this result, Hardy introduced his Z-function in the following manner. For any s in $\mathbb{C}$ we set

$$\rho(s) = \pi^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \tag{1.42}$$

so that the functional equation for the ζ from (1.42) now reads $\zeta(s) = \rho(s)\zeta(1-s)$. Plainly, the function $s \mapsto \rho(s)$ is a non-vanishing holomorphic function in the critical strip $0 < \sigma < 1$. Let $\rho^{-\frac{1}{2}}$ be a chosen holomorphic branch of the square root of ρ^{-1} in this strip. Then for all real t we set

$$Z(t) = \zeta\left(\frac{1}{2} + it\right) \rho\left(\frac{1}{2} + it\right)^{-\frac{1}{2}}. \tag{1.43}$$

It is not difficult to see that $Z(t)$ takes real values for all real t; we shall in fact review a more general argument in the context of Dirichlet L-functions below. Further, it is clear from (1.43) that the zeros of the function Z on the real line are in bijection with the zeros of ζ on the critical line by the map $t \to \frac{1}{2} + it$.

Hardy showed that the function Z changes sign infinitely often on the real line and concluded that the zeta function has infinitely many zeros on the critical line. In 1942, Atle Selberg proved that the number of sign change of $Z(t)$ in the interval $[0, T]$ is greater than $cT \log T$, for a small but effectively computable real number $c > 0$. As a result one obtains that

$$N_0(T) \gg T \log T,$$

where $N_0(T)$ denotes the number of zeros $s = \frac{1}{2} + it$ of the Riemann zeta function on the critical line with $0 < t < T$. By comparing with the formula (1.36) for $N(T)$, the number of zeros of ζ in the critical strip $0 < \sigma < 1$ up to height T, we may conclude that a positive, effectively computable proportion, though small, of the zeros of ζ lie on the critical line. A much improved result in this direction was subsequently proved by N. Levinson [51], who showed that

$$N_0(T) \geq \frac{1}{3}N(T), \text{ for large enough } T. \tag{1.44}$$

After the efforts of many mathematicians over a long period of time, it is currently known [19, 62] that

$$N_0(T) \geq .4149\, N(T),$$

for sufficiently large T. The proof of all of these positive proportion results for the zeros of the ζ function depends on the idea of mollification that we discuss in Section 1.10.

Let us now consider a primitive Dirichlet character χ to a modulus $q \geq 1$. One can then extend the notion of Hardy's Z-function to the Dirichlet L-function $L(s,\chi)$ as follows. We now set

$$\rho(s,\chi) = \mathfrak{w}(\chi)\left(\frac{\pi}{q}\right)^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{1-s+\mathfrak{a}}{2}\right)}{\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)}. \tag{1.45}$$

Then $\rho : s \mapsto \rho(s,\chi)$ is a non-vanishing holomorphic function in the strip $0 < \sigma < 1$. Thus ρ has a holomorphic square root on this strip. We choose one such square root and momentarily denote it by $s \mapsto a(s,\chi)$ so that we have $a(s,\chi)^2 = \rho(s,\chi)$ for s in the strip $0 < \sigma < 1$. Further, using (1.45) again and recalling that $|\mathfrak{w}(\chi)| = 1$ and $\overline{\Gamma(s)} = \Gamma(\overline{s})$ for all complex s, we have that $|\rho(s,\chi)| = 1$ for s on the critical line $\sigma = \frac{1}{2}$, since $1-s = \overline{s}$ for such s. Consequently, $a(s,\chi)\overline{a(s,\chi)} = |a(s,\chi)^2| = 1$ for such s. Now the functional equation (1.41) for $L(s,\chi)$ gives

$$L(s,\chi) = \rho(s,\chi)L(1-s,\overline{\chi}) = a(s,\chi)^2\overline{L(s,\chi)}$$

for s on the critical line, since $L(\overline{s},\overline{\chi}) = \overline{L(s,\chi)}$ for all s. It then follows that for all s on this line we have

$$\frac{L(s,\chi)}{a(s,\chi)} = a(s,\chi)\overline{a(s,\chi)}\overline{\left(\frac{L(s,\chi)}{a(s,\chi)}\right)} = \overline{\left(\frac{L(s,\chi)}{a(s,\chi)}\right)}. \tag{1.46}$$

Thus, if for any real t we define

$$Z(t,\chi) = L\left(\frac{1}{2}+it,\chi\right) a\left(\frac{1}{2}+it,\chi\right)^{-1}, \tag{1.47}$$

then (1.46) tells us that $Z(t,\chi)$ is real for all real t. Moreover, it is plain from the above definition that the zeros of $Z(t,\chi)$ on the real line are in bijection with the zeros $L(s,\chi)$ on the critical line $\sigma = \frac{1}{2}$ by the map $t \mapsto \frac{1}{2}+it$. Thus $t \mapsto Z(t,\chi)$ extends Hardy's Z-function to the Dirichlet L-function associated to a primitive Dirichlet character χ.

We shall hereafter write $\rho(s,\chi)^{-\frac{1}{2}}$ to denote $a(s,\chi)^{-1}$. For the sake of definiteness, we may fix $\varepsilon(\chi)$ to be a square root of $\mathfrak{w}(\chi) = \frac{\tau(\chi)}{i^{\mathfrak{a}} q^{\frac{1}{2}}}$, f to be the branch of the square root of the non-vanishing holomorphic function $s \mapsto \dfrac{\Gamma((1-s+\mathfrak{a})/2)}{\Gamma((s+\mathfrak{a})/2)}$ on the strip $0<\sigma<1$ that takes the value $+1$ at $s=1$ and set

$$\rho(s,\chi)^{-\frac{1}{2}} = \varepsilon(\chi)^{-1}\left(\frac{\pi}{q}\right)^{\frac{1}{4}-\frac{s}{2}} f(s)^{-1} \tag{1.48}$$

for all s in the aforementioned strip.

For any primitive Dirichlet character χ and real number $T>0$, we set $N_0(T,\chi)$ to be the number of zeros of $s \mapsto L(s,\chi)$ with $s = \frac{1}{2}+it$ and $|t| \le T$. In 1976, T. Hilano extended Levinson's $\frac{1}{3}$ result (1.44) to this L-function by proving that $N_0(T,\chi) > \frac{1}{3}N(T,\chi)$, where $N(T,\chi)$ is as in Theorem 1.6.2. Later, Bauer [5] improved this result to the following:

$$\limsup_{T\to\infty} \frac{N_0(T,\chi)}{N(T,\chi)} > 0.365815.$$

1.8 Approximate functional equation

An important device in the theory of L-functions is the approximate functional equation, which provides an approximation to an L-function at a point $s = \sigma + it$ within the critical strip by Dirichlet polynomials of length $\ll \sqrt{t}$. The original example of an approximate functional equation was obtained by Hardy and Littlewood for the Riemann zeta function. It is given by the following theorem, for a proof of which we refer to Theorem 4.15 in [78].

Theorem 1.8.1 *Let $x > c > 0$, $y > c > 0$ and $t > c > 0$ be such that $2\pi xy = t$ with c a fixed positive real number. Then uniformly in $0 \leq \sigma \leq 1$, we have*

$$\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} + \rho(s) \sum_{n \leq y} \frac{1}{n^{1-s}} + \mathrm{O}\left(x^{-\sigma}\right) + \mathrm{O}\left(t^{\frac{1}{2}-\sigma} y^{\sigma-1}\right).$$

Many authors have studied the approximate functional equation for zeta and L-functions in the Selberg class and more general class of functions [74, 76, 52, 14]. In 1968, A. F. Lavrik [49] proved an approximate functional equation for $L(s,\chi)$, given by the following and stated in Theorem 1 and Corollary 1, page 93 of [49]:

Theorem 1.8.2 *Let χ be a primitive Dirichlet character to the modulus q and let $x > c > 0$, $y > c > 0$ and $t > c > 0$ be such that $2\pi xy = qt$ with c a fixed positive real number. Then uniformly in $0 < \sigma < 1$, we have*

$$L(s,\chi) = \sum_{n \leq x} \frac{\chi(n)}{n^s} + \rho(s,\chi) \sum_{n \leq y} \frac{\overline{\chi}(n)}{n^{1-s}} + R_{x,y}(s),$$

where

$$R_{x,y}(s) \ll \sqrt{q}\left(y^{-\sigma} + x^{\sigma-1}(qt)^{\frac{1}{2}-\sigma}\right)\log 2t.$$

Note that the condition $2\pi xy = qt$ is expressed as (2) on page 92 of [49]. The above functional equation is important for the reason that the modulus q of χ appears explicitly in the error term. From [47, p. 79] we have another simple and useful approximation to $L(s,\chi)$:

$$L(s,\chi) = \sum_{n\leq qx} \frac{\chi(n)}{n^s} + \mathrm{O}(q^{1-\sigma}x^{-\sigma}), \tag{1.49}$$

where χ is a primitive Dirichlet character modulo q, $0 < \sigma_0 < \sigma \leq 2$, $\pi x \geq |t| \geq 2\pi$ and O-constant depends on σ_0. Approximate functional equations play a crucial role in determining the mean values and size of L-functions. In the next section we recall some results on mean values of L-functions of interest to us.

1.9 Results on mean values

The basic mean values of an L-function $F(s)$ are :

$$\int_0^T |F(\sigma+it)|^2 dt \quad \text{or} \quad \int_0^T F(\sigma+it)dt.$$

Here one can take the range of integration in the above to be $[T, 2T]$ or the short interval $[T, T+H]$ in place of $[0, T]$, where $1 \leq H \leq T$.

1.9.1 k-th moment of $\zeta(s)$ and $L(s,\chi)$

For positive k, if we set

$$I_k(\sigma,T) = \int_0^T |(\zeta(\sigma+it))^k|^2 dt = \int_0^T |\zeta(\sigma+it)|^{2k} dt$$

then by using Theorem 1.8.1, or even a weaker version given by Theorem 4.11 of [78], one can establish the following, proved as Theorem 7.2 in [78] : for $k=1$ and $\sigma > 1/2$,

$$I_1(\sigma, T) = \int_0^T |\zeta(\sigma+it)|^2 dt \sim \zeta(2\sigma)T \quad \text{as } T \to \infty.$$

The above result implies that for $\sigma > 1/2$, the values of zeta function are of constant size in an average sense. In 1918, Hardy and Littlewood proved the following mean value result at $\sigma = 1/2$:

$$I_1(1/2, T) \sim T \log T \quad \text{as } T \to \infty.$$

In the article [39] Ingham established

$$I_2(1/2, T) \sim \frac{T}{2\pi^2} \log^4 T \quad \text{as } T \to \infty.$$

For $k > 2$, the moments $I_k(1/2, T)$ are still conjectures. For a Dirichlet L-function $L(s, \chi)$, the mean square depends on the conductor q, and if $\sigma > 1/2$ then it is, in the interval $[0, T]$, asymptotic to $c(q, \sigma)T$, where $c(q, \sigma)$ as a positive constant depends on q and σ. At $\sigma = 1/2$ the mean square is the following:

Theorem 1.9.1 *[59] Let χ be a primitive Dirichlet character modulo a positive integer q and $T \geq 1$. Then we have*

$$\int_0^T \left| L\left(\frac{1}{2}+it, \chi\right) \right|^2 dt = \frac{\varphi(q)}{q} \left(T \log \frac{qT}{2\pi} + 2\gamma - 1 + 2\sum_{p|q} \frac{\log p}{p-1} \right) + E(T, q),$$

where $E(T, q)$ is the error term given by

$$E(T, q) \ll \left(q^{7/12} T^{5/12} + q\right) \log^2(qT) + \left(q^{3/2} \log(qT) + q^{19/12}\right) T^{5/12} + q^{3/4} T^{1/4} (\log T)^{3/2} + q^2 \log(qT).$$

The equality above is from Theorem 1 of [59] incorporating the correction given in [60], while the bound for $E(T,q)$ is from Theorem 2 of [59].

1.10 Mollification

A mollifier $M(s)$ for a Dirichlet series or in particular for $\zeta(s)$ is an entire function which "pretends" to behave like $1/\zeta(s)$. A natural choice of $M(s)$ is of the following kind:

$$M(s) = \sum_{n\leq X} \frac{\mu(n)P(n)}{n^s},$$

where P is a smooth function that ensures the convergence of the sum. The mollification of $\zeta(s)$ was successfully used by A. Selberg [69] in 1942 to prove one of his breakthrough results, the positive zero density estimate of critical zeros of zeta function. His choice of the mollifier is the following: For $\sigma > 1$, write

$$\frac{1}{\sqrt{\zeta(s)}} = \sum_{n=1}^{\infty} \frac{\alpha(n)}{n^s}.$$

Using coefficients, he defined $\beta(n) := \alpha(n)\left(1 - \frac{\log n}{\log X}\right)$ and the Dirichlet polynomial

$$\eta(s) := \sum_{n\leq X} \frac{\beta(n)}{n^s},$$

then he used $\eta^2(s)$ as a mollifier for $\zeta(s)$. In fact, he proved one of the key results of this article in [70, Lemma 15]. We write this result as a lemma with a modification following [28, Lemma 3]:

Lemma 1.10.1 *For $0 \leq k_1, k_2 \leq (\log T)^{-1/2}$ and $U = T^a$ with $1/2 < a \leq 3/5$, we have*

$$\int_T^{T+U} Z(t+k_1)Z(t+k_2)|\eta_{k_1}(t)\eta_{k_2}(t)|^2 dt = 2U\mathbf{K}(k_1-k_2) + \mathrm{O}(\sqrt{T}X^7),$$

where $\eta_{k_1}(t) := \eta(1/2+i(t+k_1), \eta_{k_2}(t) := \eta(1/2+i(t+k_2)$ *and*

$$\mathbf{K}(u) := \mathrm{Re}\left(\tau^{iu} \sum_{\substack{\nu_i<X \\ i=1,2,3,4}} \frac{\beta(\nu_1)\beta(\nu_2)\beta(\nu_3)\beta(\nu_4)}{\nu_1\nu_2\nu_3\nu_4} \frac{\kappa^{1+iu}}{(\nu_2\nu_3)^{iu}} \sum_{n<\frac{\tau\kappa}{\nu_2\nu_4}} \frac{1}{n^{1+iu}}\right),$$

with $\tau = \sqrt{T/2\pi}$*,* $\kappa := (\nu_1\nu_3, \nu_2\nu_4)$ *and* $0 < u < (\log T)^{-1/2}$.

Selberg showed that in $0 < u < (\log T)^{-1/2}$, the function $\mathbf{K}(u) = \mathrm{O}(1)$, but by continuity of $\mathbf{K}$ it can be extends up to $u = 0$ (see [28], page 196, the paragraph below (2.8)). Thus, one can conclude for $X = T^{\theta}, 0 < \theta < 1/100$ and $U = T^{a}$, $1/2 < a \leq 1$, the following mean value:

$$\int_T^{T+U} Z^2(t) \left|\eta\left(\frac{1}{2}+it\right)\right|^4 dt \ll U. \tag{1.50}$$

Since $|Z(t)| = |\zeta(1/2+it)|$, one can compare the above result with $I_1(1/2, T)$ and conclude that the mollification replaces the log factor by an absolute constant in the mean square. In 1985, R. Balasubramanian, J. B. Conrey, D. R. Heath-Brown [4] took another mollifier and proved an asymptotic result for the mean square. Namely,

$$\int_0^T Z^2(t) \left|M_{T^{\theta}}\left(\frac{1}{2}+it\right)\right|^2 dt \sim \left(1+\frac{1}{\theta}\right) T \quad \text{as } T \to \infty, \tag{1.51}$$

where $0 < \theta < 9/17$ and

$$M_{T^{\theta}}(s) := \sum_{n \leq T^{\theta}} \frac{\mu(n)}{n^s}\left(1 - \frac{\log n}{\log X}\right).$$

Note that the Möbius function $\mu(n)$ is the n-th coefficient of the Dirichlet series of $(\zeta(s))^{-1}$ for $\sigma > 1$. In the next chapter we will establish the mean value result for $Z(t, \chi)$ similar to (1.50). In that case the mollifier is

$$\Psi_{T^{\theta}}(s, \chi) = \sum_{n \leq T^{\theta}} \frac{\alpha(n)\chi(n)}{n^s}\left(1 - \frac{\log n}{\log X}\right),$$

where $0 < \theta < 1/40$.

1.11 Analytic tools

In this section, we collect together important analytic tools, mostly without proof, which we will use in the next chapters.

1.11.1 Exponential integrals

We begin with some results on exponential integrals.

Lemma 1.11.1 *[78, Lemma 4.5] Suppose $f : [a, b] \to \mathbb{R}$ is a real valued and twice differentiable function satisfying $f''(x) \geq r > 0$ or $f''(x) \leq -r < 0$ in $[a, b]$. Further, let g be a real valued function such that $|g(x)| \leq M$ and $\frac{g}{f'}(x)$ is monotonic. Then*

$$\left| \int_a^b g(x) e^{if(x)} dx \right| \leq \frac{8M}{\sqrt{r}}.$$

Lemma 1.11.2 *[69, Lemma 2] Let $T > 4$, $\sqrt{T} \leq U \leq T^{3/5}$ and $\xi > 0$ be a real number. Then*

$$\int_T^{T+U} (t/e\xi)^{it} dt = \begin{cases} \mathrm{O}\left(\left(\log \frac{T+\sqrt{T}}{\xi}\right)^{-1}\right) & \text{if } \xi \leq T, \\ (2\pi\xi)^{\frac{1}{2}} e^{i(\frac{\pi}{4}-\xi)} + \mathrm{O}\left(\frac{1}{\log \frac{\xi}{T-\sqrt{T}}} + \frac{1}{\log \frac{T+U+\sqrt{T}}{\xi}}\right) & \text{if } T \leq \xi \leq T+U, \\ \mathrm{O}\left(\left(\log \frac{\xi}{T+U-\sqrt{T}}\right)^{-1}\right) & \text{if } \xi \geq T+U. \end{cases}$$

1.11.2 Summation formulas

In this subsection, we state some basic lemmas on summation formulas that we will need in the sequel.

Lemma 1.11.3 (Euler's summation formula) *Let f be a continuously differentiable* *function in* $[A,B]$ *with* $0<A<B$*, then*

$$\sum_{A<n\le B} f(n)=\int_A^B f(x)dx+\int_A^B (x-[x])f'(x)dx+f(B)([B]-B)-f(A)([A]-A).$$

Lemma 1.11.4 (Abel's partial summation formula) *Let* a_n *be a sequence of complex numbers and let*

$$A(x)=\sum_{n\le x} a_n$$

with $A(x)=0$ *if* $x<1$*. Let* f *be a continuously differentiable function in* $[y,x]$ *with* $0<y<x$*. Then we have*

$$\sum_{y<n\le x} a_n f(n)=A(x)f(x)-A(y)f(y)-\int_y^x A(t)f'(t)dt.$$

1.11.3 Mean value theorem of Dirichlet polynomials

A Dirichlet polynomial is a Dirichlet series with finitely many non-zero coefficients. The mean value of Dirichlet polynomials is very useful tool in analytic number theory. There are two variants of this mean value: one is an integral form and other is a discrete form. The integral variant is important for our application and is given below:

Theorem 1.11.5 *[40, Theorem 5.2] Let* $a_1,a_2,\ldots,a_X$ *be a sequence of complex numbers. Then we get*

$$\int_0^T \left|\sum_{n\le X} a_n n^{it}\right|^2 dt = T\sum_{n\le X}|a_n|^2+\mathrm{O}\left(\sum_{n\le X} n\,|a_n|^2\right).$$

Note that this theorem is also valid as $X\to\infty$ if $\sum_{n=1}^{\infty} n\,|a_n|^2$ converges. The first step towards the proof of the above theorem is to expand the square, separate the diagonal and non-diagonal terms and integrate them. Thus, we get

$$\int_0^T \left|\sum_{n\le X} a_n n^{it}\right|^2 dt = T\sum_{n\le X}|a_n|^2 + \sum_{\substack{m,n\le X\\ m\neq n}} \frac{a_m\bar{a}_n}{\log(m/n)}\left((m/n)^{iT}-1\right). \tag{1.52}$$

It remains to show

$$\sum_{\substack{m,n\le X\\ m\neq n}} \left|\frac{a_m\bar{a}_n}{\log(m/n)}\right| \ll \sum_{n\le X} n|a_n|^2,$$

which has been proved in [40, Section 5.2]. From the equation (1.52), one can show that this result also holds for short interval $[T, T+H]$ with $H\le T$.

1.12 Selberg's central limit theorem

Understanding the distribution of values of Riemann zeta function and other L-functions is an important problem in the theory of these functions. For example, one may ask for the distribution of $\zeta(\sigma+it)$ when σ is fixed and t varies in $[T, 2T]$, with large T. For $\sigma>\frac{1}{2}$ this was studied in depth in the classical works of Bohr and Jessen [10, 12]. The case $\sigma=\frac{1}{2}$ was first taken up by Selberg [69], who established the remarkable result that the quantity $\zeta(\frac{1}{2}+it)/\sqrt{\frac{1}{2}\log\log t}$ for $t\in[T, 2T]$, behaves "approximately" like a standard complex normal random variable. Later, in [70], Selberg stated this result for any element of the Selberg Class and indicated a proof.

For our purposes, it will be convenient to state Selberg's general result in the form given by Hejhal in [33] for a class of functions satisfying conditions that are stated only slightly different from the axioms defining the Selberg class given in Section 1.1. In addition Hejhal requires that an analogue of Selberg's Density theorem be satisfied by the functions in the class. The precise form of this requirement is given by inequality (D) on page 553 of [33]. This theorem is known to hold for Dirichlet L-functions associated with primitive characters. A general method for deriving such a result, following Selberg, is explained in Section 4, page 610 of [67]; see also [58]. Since of course our interest here is eventually in Dirichlet L-functions, we shall hereafter not mention the condition (D), taking it as known for the functions being considered.

Let $L_1(s), \ldots, L_m(s)$ be m L-functions. We restate here the hypotheses (i)–(v) given on pages 551 and 552 of [33].

(i) Each $L_j(s)$ can be expressed in the form

$$L_j(s) = \sum_{n=1}^{\infty} \frac{a_j(n)}{n^s} = \prod_p \left(1 - \alpha_{1pj} p^{-s}\right)^{-1} \cdots \left(1 - \alpha_{dpj} p^{-s}\right)^{-1}$$

for $\mathrm{Re}\,(s) > 1$.

(ii) Each $L_j(s)$ admits an analytic continuation on $\mathbb{C}$ as a meromorphic function of finite order and has a finite number of poles along the line $\mathrm{Re}\,(s) = 1$.

(iii) For each j, the aforementioned analytic continuation satisfies

$$e^{i\omega_j} G(s) L_j(s) = e^{-i\omega_j} \overline{G(1-\bar{s}) L_j(1-\bar{s})}$$

with certain $\omega_j \in \mathbb{R}$ and a common gamma factor G which is given by

$$G(s) = Q^s \prod_{i=1}^{h} \Gamma(\lambda_i s + \mu_i),$$

where $Q > 0, \lambda_i > 0$ and $\mathrm{Re}(\mu_i) \geq 0$.

(iv) The coefficients of the L-functions $L_1(s), \ldots, L_m(s)$ satisfy

$$\sum_{p \leq x} \frac{a_j(p)\overline{a_k(p)}}{p} = \delta_{jk} n_j \log\log x + c_{jk} + \mathrm{O}\left(\frac{1}{\log x}\right),$$

for some constants $n_j > 0$, $c_{jk} \in \mathbb{C}$ and $x \geq 2$.

(v) The "roots" α_{kpj} are tempered in the sense that $\alpha_{kpj} \leq 1$.

Thus let L be a function defined for $\mathrm{Re}(s) = \sigma > 1$ by

$$L(s) = \sum_{n\geq 1} \frac{a(n)}{n^s} \tag{1.53}$$

and satisfying the conditions (i) to (v) given above with $m = 1$. Note that, condition (iv) takes the form

$$\sum_{p\leq X} \frac{|a(p)|^2}{p} = n \log\log X + C_1 + O(\frac{1}{\log X}) \tag{1.54}$$

for certain constants $n > 0$ and C_1. Further, following equations (3.1), page 555 and the line below equation (4.20), *ibid.*, page 563, for any $\sigma \geq \frac{1}{2}$ and any real $T \geq 2$ let us set

$$\psi = \psi(\sigma, T) = \sum_{p\leq T} \frac{|a(p)|^2}{p^{2\sigma}}\,. \tag{1.55}$$

Then by equation (4.21), *ibid.*, page 563 we have the following result.

Theorem 1.12.1 *For any real numbers $a < b$, let $\mathfrak{U}_{a,b}$ be the characteristic function of the interval (a, b). Then for any $\sigma \geq \frac{1}{2}$ we have the relation*

$$\frac{1}{T}\int_T^{2T} \mathfrak{U}_{a,b}(\log|L(\sigma+it)|)\,dt = \int_{a/\sqrt{\pi\psi}}^{b/\sqrt{\pi\psi}} e^{-\pi u^2} du + \mathrm{O}\left(\frac{(\log\psi)^2}{\sqrt{\psi}}\right) \tag{1.56}$$

for all real $T \geq 2$.

Suppose now that L_1 and L_2 are functions satisfying the conditions (i) to (v) of [33] with $m = 2$. Note that if $\{a_1(n)\}_{n\geq 1}$ and $\{a_2(n)\}_{n\geq 1}$ are the coefficients of the Dirichlet series expansions of L_1 and L_2, then (iv) of the conditions in [33] now requires that in addition to each L_1, L_2 satisfying (1.54) we must also have the orthogonality condition

$$\sum_{p\leq X} \frac{a_1(p)\overline{a_2(p)}}{p} = C_2 + O(\frac{1}{\log X}). \tag{1.57}$$

As remarked by Hejhal in the fourth line on page 564 of [33], then there is a natural analogue of Theorem 1.12.1 for $x_1 \log|L_1(\sigma+it)| + x_2 \log|L_2(\sigma+it)|$, where x_1, x_2 are any real numbers, at least one distinct from 0. Indeed, we have the following result on writing $\mathscr{L}_{x_1,x_2}(\sigma+it)$ for the preceding expression and taking account of (4.14) and (4.1) of [33] .

Theorem 1.12.2 *With $a, b, \mathfrak{U}_{a,b}$ and σ as in Theorem 1.12.1 we have*

$$\frac{1}{T}\int_T^{2T} \mathfrak{U}_{a,b}\left(\mathscr{L}_{x_1,x_2}(\sigma+it)\right)dt = \int_{a/\sqrt{\pi\psi}}^{b/\sqrt{\pi\psi}} e^{-\pi u^2}du + \mathrm{O}\left(\frac{(\log\psi)^2}{\sqrt{\psi}}\right) \tag{1.58}$$

for all real $T \geq 2$ where now

$$\psi = \psi(\sigma, T) = \sum_{p\leq T} \frac{|x_1a_1(p)+x_2a_2(p)|^2}{p^{2\sigma}}. \tag{1.59}$$

Finally, we set $L(s) = L(s,\chi)$, $L_1(s) = L(s,\chi_1)$ and $L_2(s) = L(s,\chi_2)$, where χ, χ_1 and χ_2 are all primitive Dirichlet characters, with χ_1 and χ_2 distinct. Then they satisfy (1.54) with $n = 1$ and also (1.57). This can be verfied using the orthogonality relations (1.13). Also, note that this implies that L-functions associated to primitive Dirichlet characters satisfy Selberg's conjectures of Section 1.1. In fact, this last assertion is true in much greater generality; see [54]. Our remarks now imply the following corollaries to Theorems 1.12.1 and 1.12.2.

Corollary 1.12.3 *For any primitive Dirichlet character χ and all real $T \geq 4$ we have*

$$\frac{1}{T}\int_T^{2T} \mathfrak{U}_{a,b}\left(\frac{\log|L(\frac{1}{2}+it,\chi)|}{\sqrt{\pi\log\log t}}\right)dt = \int_a^b e^{-\pi u^2}du + \mathrm{O}\left(\frac{(\log\log\log T)^2}{\sqrt{\log\log T}}\right). \tag{1.60}$$

Proof. It suffices to put $\sigma = \frac{1}{2}$ and $a\sqrt{\pi\log\log T}$ and $b\sqrt{\pi\log\log T}$ in place of a and b respectively in (1.56) where ψ is given by (1.55) with $a(p) = \chi(p)$ and take into account (1.54) with $n = 1$.

Corollary 1.12.4 *Let χ_1, χ_2 be two distinct primitive Dirichlet characters. Then for any real a, b with $a < b$ we have*

$$\frac{1}{T}\int_T^{2T} \mathfrak{U}_{a,b}\left(\frac{\log|L(\frac{1}{2}+it,\chi_1)|-\log|L(\frac{1}{2}+it,\chi_2)|}{\sqrt{2\pi\log\log t}}\right)dt$$
$$= \int_a^b e^{-\pi u^2}du + \mathrm{O}\left(\frac{(\log\log\log T)^2}{\sqrt{\log\log T}}\right). \tag{1.61}$$

Proof. We set $\sigma=\frac{1}{2}$ but $a\sqrt{2\pi\log\log T}$ and $b\sqrt{2\pi\log\log T}$ in place of a and b in (1.58) where now ψ is given by (1.59) with $x_1=1$, $a_1(p)=\chi_1(p)$ and $x_2=-1$, $a_2(p)=\chi_2(p)$. Note that

$$\sum_{p\le T}\frac{|\chi_1(p)-\chi_2(p)|^2}{p} = 2\log\log T + C_3 + O(\frac{1}{\log T}),$$

for a constant C_3, which can be verified by opening the square in the sum and using (1.54) and (1.57).

From Theorem 1.12.2 we find that $\mathscr{L}_{x_1,x_2}(\sigma+it)$ is approximately normally distributed with mean 0 and variance $\sum_{p\le T}\frac{|x_1a_1(p)+x_2a_2(p)|^2}{p^{2\sigma}}$, for any real numbers x_1,x_2. Moreover, by using hypothesis (iv) given at the beginning of this section we get

$$\sum_{p\le T}\frac{|a_1(p)+a_2(p)|^2}{p^{2\sigma}} = (n_1+n_2)\log\log T + \mathrm{O}(1)$$
$$= \sum_{p\le T}\frac{|a_1(p)|^2}{p^{2\sigma}} + \sum_{p\le T}\frac{|a_2(p)|^2}{p^{2\sigma}} + \mathrm{O}(1). \tag{1.62}$$

It is a standard fact in the theory of probability that a bivariate random variable (X,Y) is jointly normally distributed if and only if every linear combination of X,Y is normally distributed (see [26, Theorem 5.5.32]). For two random variables X,Y, the covariance $\mathrm{Cov}(X,Y)$ is defined by $\mathrm{Cov}(X,Y) := \mathbb{E}[(X-\mathbb{E}(\mathbb{X}))(Y-\mathbb{E}(\mathbb{Y}))]$, where $\mathbb{E}(X)$ is the mean or expectation of X. Let $\mathrm{Var}(X)$ denote the variance of X, then the correlation of X,Y, denoted by $\rho(X,Y)$, is defined by

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}.$$

We recall that X and Y are uncorrelated if $\rho(X,Y)=0$ and we know that X and Y are uncorrelated if and only if $\operatorname{Var}(X+Y)=\operatorname{Var}(X)+\operatorname{Var}(Y)$ (see [26, Theorem 5.3.18]). A bivariate random variable (X,Y) is associated with the mean vector $(\mathbb{E}(X),\mathbb{E}(Y))$ and the covariance matrix

$$\begin{pmatrix} \operatorname{Var}(X) & \rho\operatorname{Var}(X)\operatorname{Var}(Y) \\ \rho\operatorname{Var}(X)\operatorname{Var}(Y) & \operatorname{Var}(Y) \end{pmatrix}.$$

Thus, by Theorem 1.12.2 and (1.62), we deduce that

$$\Big(\log L_1(\sigma+it),\ \log L_2(\sigma+it)\Big)$$

is approximately jointly bivariate normal with the mean vector $(0,0)$ and the covariance matrix

$$\begin{pmatrix} \sum_{p\le T}\frac{|a_1(p)|^2}{p^{2\sigma}} & 0 \\ 0 & \sum_{p\le T}\frac{|a_2(p)|^2}{p^{2\sigma}} \end{pmatrix} = \begin{pmatrix} n_1\log\log T & 0 \\ 0 & n_2\log\log T \end{pmatrix}.$$

Moreover, we know that two random variables are jointly normally distributed and uncorrelated if and only if they are independent (see [26, Theorem 5.3.25]). Thus, we conclude that $\log L_1(\sigma+it)$ and $\log L_2(\sigma+it)$ are independent.

Chapter 2

Value Distribution of Karatsuba's and Generalized Davenport-Heilbronn Z-function

2.1 Introduction

We recall from Section 1.7 that Hardy's Z-function is a real-valued function on the real line defined by

$$Z(t)=\zeta\left(\frac{1}{2}+it\right)\rho\left(\frac{1}{2}+it\right)^{-\frac{1}{2}},$$

where $\rho(s)=2^s\pi^{s-1}\sin(\pi s/2)\Gamma(1-s)$ and $\rho(s)^{\frac{1}{2}}$ is the branch of the square root of ρ satisfying $\rho(\frac{1}{2})^{\frac{1}{2}}=+1$. The zeros of the function Z on the real line are in bijection with the zeros of ζ on the critical line by $t\rightarrow\frac{1}{2}+it$. Further, by a theorem of Selberg, it is known that $Z(t)$ has $\gg T\log T$ zeros in an interval of length T. Thus, it is a highly oscillatory function. From now onwards we denote the Lebesgue measure of a subset S of the real line by $\operatorname{meas}(S)$. From Selberg's central limit theorem (Theorem 1.12.1) one can obtain the following:

$$\text{meas}\left(\{t \in [T, 2T] : \log|Z(t)| > 0\}\right) \sim \frac{T}{2} \quad \text{as } T \to \infty.$$

A similar result holds for the case when $\log|Z(t)| < 0$. However, from these results, we cannot immediately conclude anything about the distribution of the signs of Z-function. In 2010, A. Ivić asked the following question:

Problem 2.1.1 [41, Prob. 11.7] Do there exist constants $a_+ > 0$, $a_- > 0$ such that

$$\text{meas}\left(\{t \in [T, 2T] : Z(t) > 0\}\right) = (a_+ + \mathrm{o}(1))T \quad \text{as } T \to \infty,$$
$$\text{meas}\left(\{t \in [T, 2T] : Z(t) < 0\}\right) = (a_- + \mathrm{o}(1))T \quad \text{as } T \to \infty?$$

In 2016, Gonek and Ivić [28] have given the theoretical evidence towards an affirmative answer to this question. More precisely, they proved the following:

Theorem 2.1.2 *[28, Theorem 1] For sufficiently large T, we have*

$$\text{meas}\left(\{t \in [T, 2T] : Z(t) > 0\}\right) \gg T, \quad \text{meas}\left(\{t \in [T, 2T] : Z(t) < 0\}\right) \gg T.$$

Also, using numerical computations they conjectured:

$$\text{meas}\left(\{t \in [T, 2T] : Z(t) > 0\}\right) \sim \frac{T}{2}, \quad \text{meas}\left(\{t \in [T, 2T] : Z(t) < 0\}\right) \sim \frac{T}{2} \quad \text{as } T \to \infty.$$

Analogous to $Z(t)$, one can define the Z-function $Z(t, \chi)$ associated with a Dirichlet L-function for a primitive Dirichlet character χ modulo q, i.e.,

$$Z(t,\chi) = L\left(\frac{1}{2} + it, \chi\right) \rho\left(\frac{1}{2} + it, \chi\right)^{-\frac{1}{2}}, \tag{2.1}$$

where $\rho(s, \chi)$ is the analogue of $\rho(s)$ arising from the functional equation (1.33) of $L(s, \chi)$. Recently, R. Mawia [65, 66] extended the result of Gonek and Ivić for $Z(t, \chi)$.

Indeed, he proved that for sufficiently large q and T,

$$\operatorname{meas}(\{t \in [T, T+H] : Z(t, \chi) > 0\}) \gg \frac{\varphi^2(q)}{q^2 4^{\omega(q)}} H, \tag{2.2}$$

where $q < T^\nu$, $H = T^\theta$ with $\nu > 0$, $\theta > 0$ satisfying $\frac{1}{2} + \frac{\nu}{2} < \theta \leq 1$. In 1991, Karatsuba [45] considered the following: let $\chi_1, \chi_2, \ldots, \chi_r$ be arbitrary primitive Dirichlet characters modulo $q_1, q_2, \ldots, q_r$ respectively, with the q_i all having the same parity and let $a_1, a_2, \ldots, a_r$ be arbitrary real numbers. Define

$$\Lambda(s) = \sum_{j=1}^{r} a_j (\rho(s, \chi_j))^{-\frac{1}{2}} L(s, \chi_j)$$

and set

$$\Omega(t) = \Lambda\left(\frac{1}{2} + it\right) = \sum_{j=1}^{r} a_j Z(t, \chi_j). \tag{2.3}$$

One can observe that $\Omega(t)$ is an analogue of $Z(t, \chi)$. In particular, $\Omega(t)$ is real for all real values of t. The function Ω is called Karatsuba's Z-function. Indeed, for all large T, Karatsuba proved that $\Omega(t)$ has at least $H(\log T)^{\frac{2}{\varphi(K)} - \varepsilon}$ odd order zeros in the interval $[T, T+H]$, where $K = \operatorname{lcm}(q_1, q_2, \ldots, q_r)$, $H = T^{\frac{27}{82} + \varepsilon'}$ and $0 < \varepsilon < \frac{1}{100}$, $0 < \varepsilon' < \frac{1}{100}$ are fixed positive real numbers. According to the web page of Selberg in the Institute for Advanced Study Archives, early in 1998, Selberg obtained his celebrated result on the existence of a positive proportion of zeros along the critical line for linear combinations of L-functions. For the function Ω, his result says that the number of zeros of $\Omega(t)$ in $[T, 2T]$ is $\geq \frac{c}{r \log r} T \log T$, $\quad T > T_0(\Omega)$, $r \geq 2$, where c is a positive constant and $T_0(\Omega)$ is a constant that depends on Ω. Thus, Ω also fluctuates considerably. Naturally, it is reasonable to ask if we can extend Theorem 2.1.2 to $Z(t, \chi)$ and more generally to $\Omega(t)$.

The first part of this chapter gives an affirmative answer to the aforementioned question. The second part of this chapter, comprising Sections 2.5 to 2.7, deals with a generalized Davenport-Heilbronn function. Section 2.5 gives an introduction to the contents of this part of the chapter.

2.2 Results on Karatsuba's Z-function

We prove the following results with the notations as above:

Theorem 2.2.1 *Let $\varepsilon > 0$ and let T be a sufficiently large real number. Further, suppose that $1 \leq q_j \leq T^{\frac{1}{5}-\varepsilon}, 1 \leq j \leq r$, where r may grow with T but not faster than $\mathrm{O}(e^{(1-\varepsilon)(\log\log T)^{\frac{1}{4}}})$. Then we have*

$$\operatorname{meas}(\{T < t \leq 2T : \Omega(t) > 0\}) \gg \frac{T}{r^2} \text{ and } \operatorname{meas}(\{T < t \leq 2T : \Omega(t) < 0\}) \gg \frac{T}{r^2},$$

where the implied constants are independent of r, the a_j's and the q_j's.

For $r = 1$, we obtain the following corollary.

Corollary 2.2.2 *Let $\varepsilon > 0$ and $1 \leq q \leq T^{\frac{1}{5}-\varepsilon}$. Then we have*

$\operatorname{meas}(\{T < t \leq 2T : Z(t,\chi) > 0\}) \gg T$, $\operatorname{meas}(\{T < t \leq 2T : Z(t,\chi) < 0\}) \gg T$,

where χ is a primitive Dirichlet character modulo q and the implied constants are independent of q.

Let $\omega(q)$ be the number of distinct prime factors of q. When q varies over the integers with $\omega(q)$ bounded by a constant, then the result of R. Mawia [65, 66], at $H = T$, holds for $q < T$, but Corollary 2.2.2 holds for $q \leq T^{\frac{1}{5}-\varepsilon}$, $\varepsilon > 0$. Thus, in this case, Corollary 2.2.2 is weaker than the result of R. Mawia.

In 1917, Hardy and Ramanujan [36] proved that for almost all integers n, $\omega(n)$ satisfies the following asymptotic formula:

$$\omega(n) \sim \log\log n.$$

Thus, when q varies with T, for almost all $q \leq T^{\frac{1}{5}-\varepsilon}$ $\varepsilon > 0$, Corollary 2.2.2 is stronger than the result of R. Mawia [65, 66]. Because, in this case, Corollary 2.2.2 implies that both the measures, mentioned in Corollary 2.2.2, are at least cT for some positive absolute constant c. But R. Mawia's result implies that both the measures are at most $cT(\log q)^{-1}$ for some positive absolute constant c. Note that, in general, at least one of the sets mentioned in Corollary 2.2.2 has measure cT for some positive absolute constant c.

The idea of the proof of R. Mawia and ours is the same. But, the upper bound estimate for the mean square in (2.5) is different from [66, Lemma 9]. There are two reasons behind this. The first reason is that Lemma 2.3.3 is the same as Lemma 4 and Lemma 5 of [66]. But, we give different arguments while proving Lemma 2.3.3 and consequently get sildly different estimates compare to the lemmas of [66]. Secondly, the expression $A_1 + A_2$ in Lemma 2.3.5, is analogous to the expression $S(0)$, which is defined in [66, p. 241]. We have calculated $A_1 + A_2$ more carefully and a bit differently. Thus, here we get $A_1 + A_2 = \mathrm{O}(1)$ (see Lemma 2.3.5) while R. Mawia gets $S(0) = \mathrm{O}(4^{\omega(q)} q^2/\varphi(q)^2)$ (see [66, p. 241–243]).

The next theorem is a short interval version of Corollary 2.2.2 and we can compare this result in the same way as above with the result of R. Mawia [66].

Theorem 2.2.3 *Let $\varepsilon > 0$ and χ be a primitive Dirichlet character modulo q. Also, let $q < T^{\nu}, \nu < \frac{1}{5}$ and $T^{\frac{3+\nu}{4}+\varepsilon} < H \leq T$. Then we have*

$$\mathrm{meas}(\{T < t \leq T+H : Z(t,\chi) > 0\}) \gg H, \quad \mathrm{meas}(\{T < t \leq T+H : Z(t,\chi) < 0\}) \gg H,$$

where the implied constants are independent of q.

The basic idea of the proof of Theorem 2.2.1 is the following: for any measurable function Ψ, we have the inequality

$$\mathrm{meas}\left(I_\Omega^{\pm}(T,H)\right) \geq \frac{1}{4}\frac{\left(\int_T^{T+H}|\Omega\Psi|dt \pm \int_T^{T+H}\Omega|\Psi|dt\right)^2}{\int_T^{T+H}|\Omega\Psi|^2 dt}, \tag{2.4}$$

where $I_\Omega^{+}(T,H) = \{T < t \leq T+H : \Omega(t) > 0\}$ and $I_\Omega^{-}(T,H) = \{T < t \leq T+H : \Omega(t) < 0\}$ and we denote them together by $I_\Omega^{\pm}(T,H)$. We apply this inequality with Ψ, which is chosen to be a mollifier for one of the Dirichlet L-functions in Ω. The mollifier Ψ is the square of a Dirichlet polynomial ψ_j, and ψ_j looks quite similar to a truncated Dirichlet series obtained from a branch of the square root of $L(s,\chi_j)$, for a $j \in \{1,2,\ldots,r\}$. The precise definition of ψ_j has been given at the beginning of the next s Section (see (2.12)). To obtain Theorem 2.2.3, we again use the inequality (2.4) with $\chi_j = \chi$, $\Psi = \psi^2$ and $\Omega(t) = Z(t,\chi)$ and then we use the following proposition.

Proposition 2.2.4 *(a) For $X = T^\theta$ and $1 \le q \le T^{\frac{1}{5}-8\theta-\varepsilon}$ with $0 < \theta < \frac{1}{40} - \frac{\varepsilon}{8}$, we have*

$$\int_T^{T+V} Z(t,\chi)^2 \left|\psi\left(\frac{1}{2}+it\right)\right|^4 dt \ll V, \tag{2.5}$$

where $V = T^b$, $3/5 \le b \le 1$ and whenever q is fixed we have $1/2 < b \le 1$.

(b) For $2 \le H \le T$,

$$\int_T^{T+H} Z(t,\chi)\left|\psi\left(\frac{1}{2}+it\right)\right|^2 dt \ll q^{\frac{1}{4}} T^{\frac{3}{4}} X (\log T)^2,$$
$$\int_T^{T+H} |Z(t,\chi)|\left|\psi\left(\frac{1}{2}+it\right)\right|^2 dt \ge H + \mathrm{O}\left(\sqrt{qT} X \log X\right).$$

In the proof of Proposition 2.2.4, the major work is to obtain the mean square result in (2.5). The idea of the proof of this result is taken from the work of A. Selberg on the positive density estimate of the number of the zeros of the Riemann zeta function on the critical line (see [69]).

To prove (b), we use an approximate functional equation, some exponential integral bounds, and contour integration. The methodology of this is analogous to the recent work by Gonek-Ivić [28]. By using the three integrals from Proposition 2.2.4 in (2.4), we obtain Theorem 2.2.3.

In the following proposition, we obtain the three integrals for Ω which are mentioned in (2.4).

Proposition 2.2.5 *Let $X \le T$ and j be that index in $\{1,\ldots,r\}$ for which we defined the* *function* Ψ. *Then we have*

$$\int_T^{2T} \Omega(t)^2 \left|\psi_j\left(\frac{1}{2}+it\right)\right|^4 dt \ll_q r\sum_{i=1}^{r} a_i^2 T(\log T)^3, \tag{2.6}$$

$$\int_T^{2T} |\Omega(t)| \left|\psi_j\left(\frac{1}{2}+it\right)\right|^2 dt \geq |\sum_{i=i}^{r} a_i \mathfrak{w}(\chi_i)^{-\frac{1}{2}}| \Big(H + \mathrm{O}(\sqrt{qT}X\log X)\Big), \tag{2.7}$$

$$\int_T^{2T} \Omega(t) \left|\psi_j\left(\frac{1}{2}+it\right)\right|^2 dt \ll \Big(\sum_{i=1}^{r} a_i\Big) q^{\frac{1}{4}} T^{\frac{3}{4}} X(\log T)^2. \tag{2.8}$$

The proof of (2.7) and (2.8) in this proposition follow from part (b) of Proposition 2.2.4. To prove (2.6) we need an upper bound of

$$\int_T^{2T} Z(t,\chi_i)^2 \left|\psi_j\left(\frac{1}{2}+it\right)\right|^4 dt,$$

where $i \neq j$. Following the idea of the proof of the mean square result in (2.5), it is possible to prove

$$\int_T^{2T} Z(t,\chi_i)^2 \left|\psi_j\left(\frac{1}{2}+it\right)\right|^4 dt \ll_q T(\log T)^3,$$

whenever $i \neq j$. Thus, by using Proposition 2.2.5 in (2.4), we can show that

$$\mathrm{meas}(I_\Omega^{\pm}) \gg_{q,r} T(\log T)^{-3}.$$

Clearly, this result is very weak because at least one of the sets I_Ω^+ and I_Ω^- has measure $\gg T$. To get a stronger result, we have to obtain a strong bound in (2.6), that is, we need to show the upper bound without the factor $(\log T)^{-3}$. Note that instead of the mollifier ψ_j^2 if we use Karatsuba's mollifier from [45], which is also defined as g_j in Section 2.7 after (2.95), then we get the upper bound in (2.6) as $\ll_{K,r} T(\log T)^{1-\frac{2}{\varphi(K)}}$ (see [45, eq: 74-75]), where $K \geq 3$ is the lcm of the conductors q_j, $j = 1,\ldots,r$. If we use the above bound in (2.4), we still get a weaker result.

To complete the proof of Theorem 2.2.1, we have to suitably estimate the three integrals of (2.4) with this choice of Ψ. This is done as follows:

Proposition 2.2.6 *Let $X = T^{\theta}$ and $1 \le q_j \le T^{\frac{1}{5}-8\theta-\varepsilon}$ with $0 < \theta < \frac{1}{40} - \frac{\varepsilon}{8}$, where ε is a small positive real number and let j be the index in $\{1, \ldots, r\}$ using which we defined Ψ. Then we have*

$$\int_{S_j} \Omega(t)^2 \left|\psi_j\left(\frac{1}{2}+it\right)\right|^4 dt \ll r a_j^2 T, \tag{2.9}$$

$$\int_{S_j} |\Omega(t)| \left|\psi_j\left(\frac{1}{2}+it\right)\right|^2 dt \ge |a_j| \mu(S_j)(1+\mathrm{o}(1)) \quad \text{as } T \to \infty, \tag{2.10}$$

$$\int_{S_j} \Omega(t) \left|\psi_j\left(\frac{1}{2}+it\right)\right|^2 dt = o(T), \text{ as } T \to \infty, \tag{2.11}$$

where S_j is an appropriate subset of $[T, 2T]$ of measure $\ge \frac{T}{r} - \mathrm{O}\left(T(\log\log T)^{-\frac{1}{4}}\right)$ and r may grow with T but not faster than $\mathrm{O}(e^{(1-\varepsilon)(\log\log T)^{\frac{1}{4}}})$.

The set $S_j \subset [T, 2T]$ is obtained from Selberg's central limit theorem stated in Corollary 1.12.4 and has the property that for $t \in S_j$, we have $\log|L(1/2+it, \chi_j)| > \log|L(1/2+it, \chi_l)| + (\log\log T)^{\frac{1}{4}}$, for all $l \ne j$, $1 \le l \le r$.

Note that Proposition 2.2.6 is established when $S_j \subset [T, 2T]$. The natural question is whether we can extend this Proposition when $S_j \subset [T, T+H]$ for any $0 < H < T$. From the methods used to prove Proposition 2.2.6, we find that this extension is possible if we can extend the results of Selberg in Lemma 2.3.6 to the short interval $[T, T+H]$. To prove (a), (b) and (c) of Lemma 2.3.6 we need an extension of Corollary 1.12.3. This seems to be highly non-trivial.

For $r = 1$, we obtain Corollary 2.2.2 from Theorem 2.2.1.

2.3 Preliminaries

First of all, we choose a mollifier for Ω which mollifies exactly one Dirichlet L-function in Ω, say $L(s,\chi_j)$. Let $(\alpha(n))_n$ be a sequence of real numbers such that

$$\prod_p \left(1-\frac{1}{p^s}\right)^{\frac{1}{2}} = \sum_{n=1}^{\infty} \frac{\alpha(n)}{n^s}, \qquad \mathrm{Re}(s) > 1,$$

where p runs through the primes. For a fixed $j,\ 1 \le j \le r$ and $\theta \in \left(0, \frac{1}{2}\right)$, write

$$\beta_j(n) = \begin{cases} \alpha(n)\chi_j(n)\left(1-\frac{\log n}{\log X}\right), & 1 \le n < X = T^\theta, \\ 0, & n \ge X. \end{cases}$$

Then we consider the following Dirichlet polynomial corresponding to $L(s,\chi_j)$:

$$\psi_j(s) := \sum_{\substack{n \le X \\ (q_j,n)=1}} \frac{\beta_j(n)}{n^s}. \tag{2.12}$$

It is clear that $|\alpha(n)| \le 1$ and $|\beta_j(n)| \le 1$ for all $n \in \mathbb{N}$. We use ψ_j^2 to mollify Ω, or in particular, $L(s,\chi_j)$. We want to compute the integrals of the functions $\Omega(t)\left|\psi_j\left(\frac{1}{2}+it\right)\right|^2$, $|\Omega(t)|\left|\psi_j\left(\frac{1}{2}+it\right)\right|^2$ and $\Omega^2(t)\left|\psi_j\left(\frac{1}{2}+it\right)\right|^4$ in the interval [T, T+H]. The following lemmas are useful to estimate the above integrals.

Lemma 2.3.1 *Let $s = \sigma + it$ with $-\frac{1}{2} \le \sigma \le \frac{3}{2}$ and $t \ge t_0$, where t_0 is a positive constant. Then we have the following upper bound:*

$$L(s,\chi) \ll (qt)^{(1-\sigma)/2} \log t. \tag{2.13}$$

Proof. In [63, Theorem 3], H. Rademacher proved that for $-\frac{1}{2} \le \eta \le \sigma \le 1+\eta \le \frac{3}{2}$, for all moduli $q \ge 1$ and for all primitive characters χ modulo q,

$$|L(s,\chi)| \le \left(\frac{q|1+s|}{2\pi}\right)^{\frac{1+\eta-\sigma}{2}} \zeta(1+\eta).$$

Taking $\eta = \dfrac{1}{\log t}$ with $t \geq t_0 \geq e^2$ and using the result $\zeta\left(1+\dfrac{1}{\log t}\right) \ll \log t$ in the above inequality we conclude this lemma.

The following lemma is an approximation of $Z(t,\chi)$, which is defined in (2.1).

Lemma 2.3.2 *For $\varepsilon > 0$, let $U = T^a$, $\frac{1}{2} < a < \frac{3}{4} - \varepsilon$ and $T \leq t \leq T+U$, with $\tau = \sqrt{\frac{qT}{2\pi}}$. Then for any character χ modulo q, we have*

$$Z(t,\chi) = \Delta(t,T;\chi) + \overline{\Delta}(t,T;\chi) + \mathrm{O}\left(q^{\frac{1}{4}} T^{a-\frac{3}{4}}\right), \tag{2.14}$$

where $\Delta(t,T;\chi) = \rho\left(\frac{1}{2}+it,\chi\right)^{-\frac{1}{2}} \sum_{n\leq\tau} \dfrac{\chi(n)}{n^{\frac{1}{2}+it}}$.

Proof. Taking $s = \frac{1}{2}+it$ and $x = y$ in Theorem 1.8.2, we have

$$L\left(\frac{1}{2}+it,\chi\right) = \sum_{n\leq\sqrt{\frac{qt}{2\pi}}} \frac{\chi(n)}{n^{\frac{1}{2}+it}} + \rho\left(\frac{1}{2}+it,\chi\right) \sum_{n\leq\sqrt{\frac{qt}{2\pi}}} \frac{\overline{\chi}(n)}{n^{\frac{1}{2}-it}} + \mathrm{O}\left(\left(\frac{q}{t}\right)^{\frac{1}{4}} \log 2t\right). \tag{2.15}$$

Let us denote

$$S := \sum_{\tau<n\leq\sqrt{\frac{qt}{2\pi}}} \frac{\chi(n)}{n^{\frac{1}{2}+it}}.$$

Then we estimate S trivially as

$$S \leq \sum_{\tau<n\leq\sqrt{\frac{qt}{2\pi}}} \frac{1}{n^{\frac{1}{2}}} \leq \frac{1}{\sqrt{\tau}}\left(\sqrt{\frac{q(T+U)}{2\pi}} - \tau\right) \ll \sqrt{\tau}\left(\sqrt{1+\frac{U}{T}} - 1\right) \ll \sqrt{\tau}\frac{U}{T} \ll q^{\frac{1}{4}} T^{a-\frac{3}{4}}.$$

So, we obtain

$$L\left(\frac{1}{2}+it,\chi\right) = \sum_{n\leq\tau} \frac{\chi(n)}{n^{\frac{1}{2}+it}} + \rho\left(\frac{1}{2}+it,\chi\right) \sum_{n\leq\tau} \frac{\overline{\chi}(n)}{n^{\frac{1}{2}-it}} + \mathrm{O}\left(q^{\frac{1}{4}} T^{a-\frac{3}{4}}\right). \tag{2.16}$$

Hence, by using (2.16) in the definition of $Z(t,\chi)$ we get the expression (2.14).

The next lemma gives two summation formula estimates that we derived from Selberg's work [69, Lemma 12] on the Riemann zeta function.

Lemma 2.3.3 *Let χ be a Dirichlet character modulo q, $1 \leq d \leq X$ and let ρ be a positive integer. For $r = 1, 2$ we have*

$$\sum_{\substack{n \leq X/d \\ (n,\rho)=1}} \frac{\alpha(n)|\chi(n)|}{n} \left(\log \frac{X}{dn}\right)^r = \mathrm{O}\left(\left(\frac{q}{\varphi(q)}\right)^{\frac{1}{2}} \prod_{p|\rho}\left(1+\frac{1}{p^{\frac{3}{4}}}\right)(\log X)^{r-\frac{1}{2}}\right). \quad (2.17)$$

Also, we obtain

$$\sum_{\substack{n \leq X/d \\ (n,\rho)=1}} \frac{\alpha(n)|\chi(n)|}{n} \log \frac{X}{dn} (\log(nd)) = \mathrm{O}\left(\left(\frac{q}{\varphi(q)}\right)^{\frac{1}{2}} \prod_{p|\rho}\left(1+\frac{1}{p^{\frac{3}{4}}}\right)(\log X)^{\frac{3}{2}}\right), \quad (2.18)$$

where $\alpha(n)$ is the n-th coefficient of $\zeta(s)^{-\frac{1}{2}}$ (see the beginning of Section 2.3).

Proof. The proof of this lemma is analogous to the proof of [69, Lemma 12] at $\gamma = 0$. If we define the function

$$g(s) := \prod_{p|q}\left(1-\frac{1}{p^{1+s}}\right)^{-\frac{1}{2}} \left(\prod_{p|\rho}\left(1-\frac{1}{p^{1+s}}\right) s\zeta(1+s)\right)^{-\frac{1}{2}},$$

where ρ is an integer, then in our case, the function $g(s)$ would play the role of the function $f(s)$, which was defined in [69, Lemma 11]. The estimates $f(0)$, $f'(0)$, $f''(0)$ and the remainder term $R(s)$ of the Taylor series expansion for $f(s)$ at zero are used to obtain the result in [69, Lemma 12]. Let us denote

$$h(s) := \prod_{p|q}\left(1-\frac{1}{p^{1+s}}\right)^{-\frac{1}{2}}.$$

Since $\prod_{p|q}\left(1-\frac{1}{p}\right) = \frac{\varphi(q)}{q}$, by logarithmic differentiation of $h(s)$ at $s=0$, we have

$$h^{(k)}(0) \ll \left(\frac{q}{\varphi(q)}\right)^{\frac{1}{2}} \sum_{p|q} \frac{(\log p)^k}{p}, \quad \text{for } k = 0, 1, 2.$$

By using [51, Lemma 3.9 and Lemma 3.10] we get

$$h^{(k)}(0) \ll \left(\frac{q}{\varphi(q)}\right)^{\frac{1}{2}} (\log\log q)^k, \text{ for } k = 0,\, 1,\, 2.$$

Now, by using Leibniz formula and [69, Lemma 11] we obtain

$$g^{(j)}(0) \ll \prod_{p|\rho}\left(1+p^{-\frac{3}{4}}\right)\left(\frac{q}{\varphi(q)}\right)^{\frac{1}{2}} (\log\log q)^j \text{ for } j = 0,\, 1,\, 2.$$

The Taylor series expansion of $g(s)$ at $s=0$, for $r = 1, 2$ gives

$$g(s) = \sum_{j=0}^{r}\frac{s^j}{j!}g^{(j)}(0) + s^{r+1}R_1(s). \tag{2.19}$$

Following [69, Lemma 11], for $s = it$ and $-2 \le t \le 2$, we get

$$R_1(it) \ll \prod_{p|\rho}\left(1+p^{-\frac{3}{4}}\right)\left(\frac{q}{\varphi(q)}\right)^{\frac{1}{2}}.$$

Now, following the first half of the proof of [69, Lemma 12], we obtain that

$$\begin{aligned}\sum_{\substack{n\le X/d\\(n,\rho)=1}}\frac{\alpha(n)|\chi(n)|}{n}\left(\log\frac{X}{dn}\right)^r =& \frac{r!}{2\pi i}\int_{-2i}^{2i}\frac{(X/d)^s}{s^{r+\frac{1}{2}}}g(s)ds\\ &+\mathrm{O}\left(\prod_{p|\rho}\left(1+p^{-\frac{3}{4}}\right)\left(\frac{q}{\varphi(q)}\right)^{\frac{1}{2}}(\log\log q)^2\right).\end{aligned} \tag{2.20}$$

Replacing $g(s)$ from (2.19) into (2.20), and we see that the above is

$$\begin{aligned}\sum_{\substack{n\le X/d\\(n,\rho)=1}}\frac{\alpha(n)|\chi(n)|}{n}\left(\log\frac{X}{dn}\right)^r =& \frac{r!}{2\pi i}\sum_{j=0}^{r}\frac{g^{(j)}(0)}{j!}\int_{-2i}^{2i}\frac{\left(\frac{X}{d}\right)^s\sqrt{s}}{s^{r+1-j}}ds + \frac{r!}{2\pi i}\int_{-2i}^{2i}\left(\frac{X}{d}\right)^s\sqrt{s}R_1(s)ds\\ &+\mathrm{O}\left(\prod_{p|\rho}\left(1+p^{-\frac{3}{4}}\right)\left(\frac{q}{\varphi(q)}\right)^{\frac{1}{2}}(\log\log q)^2\right).\end{aligned}$$

Now, by applying [69, Lemma 9] in the above equality, we obtain

$$\sum_{\substack{n\le X/d\\(n,\rho)=1}}\frac{\alpha(n)|\chi(n)|}{n}\left(\log\frac{X}{dn}\right)^{r}=\frac{r!}{2\pi}\sum_{j=0}^{r}\frac{|g^{(j)}(0)|}{j!}\mathrm{O}\left(\left(\log\frac{X}{d}\right)^{r-j-\frac{1}{2}}\right)$$
$$+\mathrm{O}\left(\prod_{p|\rho}\left(1+p^{-\frac{3}{4}}\right)\left(\frac{q}{\varphi(q)}\right)^{\frac{1}{2}}(\log\log q)^{2}\right).$$

Thus, the above equality shows that the upper bound estimate is

$$\sum_{\substack{n\le X/d\\(n,\rho)=1}}\frac{\alpha(n)|\chi(n)|}{n}\left(\log\frac{X}{dn}\right)^{r}=\mathrm{O}\left(g(0)\left(\log X\right)^{r-\frac{1}{2}}\right).$$

This completes the proof of the first part of this lemma.

For the second part we begin by multiplying the formula (2.17) by $\log X$ for $r=1$, and then subtracting the formula (2.17) for $r=2$, to get

$$\sum_{\substack{n\le X/d\\(n,\rho)=1}}\frac{\alpha(n)|\chi(n)|}{n}\left(\log X\log\frac{X}{dn}-\left(\log\frac{X}{dn}\right)^{2}\right)=\mathrm{O}\left(\left(\frac{q}{\varphi(q)}\right)^{\frac{1}{2}}\prod_{p|\rho}\left(1+\frac{1}{p^{\frac{3}{4}}}\right)(\log X)^{\frac{3}{2}}\right).$$

Now using the fact that

$$\left(\log X\log\frac{X}{dn}-\left(\log\frac{X}{dn}\right)^{2}\right)=\log\frac{X}{dn}\log(nd),$$

we obtain the required result. This completes the proof.

The following lemma is an estimate of an arithmetic sum which we shall use to prove Lemma 2.3.5.

Lemma 2.3.4 *For $Y>0$ and a positive integer q we have*

$$\sum_{\substack{m\le Y\\(q,m)=1}}\frac{1}{m}=\frac{\varphi(q)}{q}\left(\log Y+\gamma+A_q\right)+\mathrm{O}\left(\frac{d(q)}{Y}\right),$$

where γ is the Euler's constant and $A_q := \sum_{p|q} \frac{\log p}{p-1} \ll \log\log q$.

Proof. By using the property of Möbius function we can write

$$\sum_{\substack{m\leq Y\\(q,m)=1}} \frac{1}{m} = \sum_{m\leq Y} \frac{1}{m} \sum_{d|m, d|q} \mu(d) = \sum_{d|q} \frac{\mu(d)}{d} \sum_{n\leq \frac{Y}{d}} \frac{1}{d}.$$

Using Lemma 1.11.3 one can show $\sum_{n\leq x} \frac{1}{n} = \log x + \gamma + \mathrm{O}(x^{-1})$. Thus, we get

$$\begin{aligned}\sum_{\substack{m\leq Y\\(q,m)=1}} \frac{1}{m} &= \sum_{d|q} \frac{\mu(d)}{d} \left(\log \frac{Y}{d} + \gamma + \mathrm{O}\left(\frac{d}{Y}\right)\right)\\ &= \sum_{d|q} \frac{\mu(d)}{d} \left(\log Y + \gamma - \log d\right) + \mathrm{O}\left(\frac{d(q)}{Y}\right).\end{aligned}$$

Now, for squarefree d we write $\log d = \sum_{p|d} \log p$, for prime p. Then we have

$$\sum_{d|q} \frac{\mu(d)}{d} \log d = \sum_{d|q} \frac{\mu(d)}{d} \sum_{pr=d} \log p = -\sum_{p|q} \frac{\log p}{p} \sum_{r|\frac{q}{p}} \frac{\mu(r)}{r}.$$

Note that $\sum_{d|q} \frac{\mu(d)}{d} = \frac{\varphi(q)}{q}$. Thus, we have $\sum_{d|q} \frac{\mu(d)}{d} \log d = -\frac{\varphi(q)}{q} A_q$, and hence

$$\sum_{\substack{m\leq Y\\(q,m)=1}} \frac{1}{m} = \frac{\varphi(q)}{q} \left(\log Y + \gamma + A_q\right) + \mathrm{O}\left(\frac{d(q)}{Y}\right).$$

Now, it remains to bound A_q. By power series expansion of $(1-\frac{1}{p})^{-1}$, we can write

$$A_q = \sum_{p|q} \frac{\log p}{p} + \mathrm{O}(1).$$

Hence, the upper bound of A_q follows from [51, Lemma 3.9, p. 400]. This completes the proof.

Now recall ψ_j from (2.12) and we put ψ for a character χ modulo q. We can write

$$\psi(s)^2 = \sum_{n_1,n_2 \le X} \frac{\beta(n_1)\beta(n_2)}{(n_1 n_2)^s} = \sum_{n \le X^2} \frac{b(n)}{n^s}, \tag{2.21}$$

where $b(n) := \sum_{d|n} \beta(d)\beta(\frac{n}{d})$, and d, $\frac{n}{d} \le X$. Note that $|b(n)| \le d(n)$, where $d(n)$ is the usual divisor function and $b(1) = 1$. Using the above definition of $b(n)$ we have the following lemma:

Lemma 2.3.5 *Let $X = T^\theta$ with $0 < \theta < \dfrac{1}{4}$ and $\tau = \sqrt{qT/(2\pi)}$ with $q \le T$. We define A_1, A_2 by*

$$A_1 := \sum_{\substack{m,n \le X^2 \\ n<m}} \frac{b(m)\overline{b(n)}}{mn} \overline{\chi}(m)\chi(n)(m,n) \sum_{\frac{\tau(m,n)}{m} \le r \le \frac{\tau(m,n)}{n}} \frac{|\chi(r)|}{r},$$

$$A_2 := \sum_{\substack{k,l \le \tau; m,n \le X^2 \\ kn=lm}} \frac{\chi(l)\overline{\chi}(k)b(m)\overline{b(n)}}{\sqrt{klmn}}.$$

Then we have

$$A_1 + A_2 = \mathrm{O}(1),$$

where the implied constant is independent of q.

Proof. From the definition of $b(n)$, we observe that the non-zero contribution to A_2 comes from the term with the condition $(mn,q) = 1$. For such m, n, let $d = (m,n)$. Then $m = dm_1, n = dn_1$ where $(m_1,n_1) = 1$. From the above conditions, we have $|\chi(d)| = 1$, and hence

$$\chi(n)\overline{\chi}(m) = \chi(n_1)\overline{\chi}(m_1)|\chi(d)|^2 = \chi(n_1)\overline{\chi}(m_1). \tag{2.22}$$

Also, the equation $kn = lm$ can be written as $kn_1 = lm_1$, and hence $k = vm_1, l = vn_1$ for some v, $1 \leq v \leq \min\{\tau/m_1,\, \tau/n_1\}$. Thus we get

$$\chi(l)\overline{\chi}(k) = \chi(vn_1)\,\overline{\chi}(vm_1) = \chi(n_1)\,\overline{\chi}(m_1)\,|\chi(v)|^2 = \chi(n_1)\,\overline{\chi}(m_1)\,|\chi(v)|.$$

Denoting $\min\{\tau/m_1,\, \tau/n_1\}$ by α and using the above identity, we obtain

$$A_2 = \sum_{m,n \leq X^2} \frac{b(m)\overline{b(n)}}{\sqrt{mn}} \sum_{\substack{l,k \leq \tau \\ km = ln}} \frac{\chi(l)\overline{\chi}(k)}{\sqrt{kl}} = \sum_{m,n \leq X^2} \frac{b(m)\overline{b(n)}}{\sqrt{mn}} \sum_{1 \leq v \leq \alpha} \frac{\chi(n_1)\,\overline{\chi}(m_1)\,|\chi(v)|}{\sqrt{m_1 n_1}\, v}.$$

Note that $\sqrt{m_1 n_1} = \sqrt{mn}/d$ and $\alpha = \min\{\tau d/m,\, \tau d/n\} = \tau d/\max\{m, n\}$. Now using the fact $\chi(n)\overline{\chi}(m) = \chi(n_1)\,\overline{\chi}(m_1)$, we see that

$$A_2 = \sum_{m,n \leq X^2} \frac{b(m)\overline{b(n)}}{mn} \chi(n)\overline{\chi}(m)(m,n) \sum_{v \leq \frac{(m,n)\tau}{\max(m,n)}} \frac{|\chi(v)|}{v}. \tag{2.23}$$

Hence, we obtain

$$A_1 + A_2 = \sum_{m,n \leq X^2} \frac{b(m)\overline{b(n)}}{mn} \chi(n)\overline{\chi}(m)(m,n) \sum_{v \leq \frac{(m,n)\tau}{n}} \frac{|\chi(v)|}{v}. \tag{2.24}$$

For simplicity, we write

$$B(X) := \sum_{m,n \leq X^2} \frac{b(m)\overline{b(n)}}{mn} \chi(n)\overline{\chi}(m)(m,n). \tag{2.25}$$

Note that $B(X)$ can be written as

$$B(X) = \sum_{\substack{d \leq X^2 \\ (d,q)=1}} \varphi(d) \left(\sum_{\substack{m \leq X^2 \\ d|m}} \frac{b(m)\overline{\chi(m)}}{m} \right) \left(\sum_{\substack{n \leq X^2 \\ d|n}} \frac{\overline{b(n)}\chi(n)}{n} \right), \tag{2.26}$$

where φ is the Euler's function. Now, from (2.21) we write

$$\sum_{\substack{m\le X^2\\ d|m}} \frac{b(m)\overline{\chi(m)}}{m} = \frac{1}{(\log X)^2} \sum_{\substack{m_1,m_2\le X\\ d|m_1m_2}} \frac{\alpha(m_1)\alpha(m_2)|\chi(m_1)||\chi(m_2)|}{m_1m_2} \log\frac{X}{m_1}\log\frac{X}{m_2}. \tag{2.27}$$

Since $d|m_1m_2$, there exist $d_i\in\mathbf{N}(\,i=1,\,2)$ such that $m_i=d_in_i$, $d|d_1d_2$ and $(d,\,n_i)=1$, that is, $d_i=\prod\limits_{p|d,\,p^{\alpha}||m_i} p^{\alpha}$. Now (2.27) can be written as follows:

$$\begin{aligned}\sum_{\substack{m\le X^2\\ d|m}} \frac{b(m)\overline{\chi(m)}}{m} &= \frac{1}{(\log X)^2} \sum_{\substack{d|d_1d_2\\ d_1,d_2\le X\\ d_1d_2|d^\infty}} \frac{\alpha(d_1)\alpha(d_2)|\chi(d_1)||\chi(d_2)|}{d_1d_2}\\ &\times\left(\sum_{\substack{n_1\le\frac{X}{d_1}\\ (n_1,d)=1}} \frac{\alpha(n_1)|\chi(n_1)|}{n_1}\log\frac{X}{n_1d_1}\right)\left(\sum_{\substack{n_2\le\frac{X}{d_2}\\ (n_2,d)=1}} \frac{\alpha(n_2)|\chi(n_2)|}{n_2}\log\frac{X}{n_2d_2}\right).\end{aligned} \tag{2.28}$$

Using Lemma 2.3.3 in the two inner sums that appear on the right-hand side of (2.28), we get

$$\begin{aligned}\sum_{\substack{m\le X^2\\ d|m}} \frac{b(m)\overline{\chi(m)}}{m} &\ll \frac{q}{\varphi(q)}\frac{1}{\log X}\prod_{p|d}(1+p^{-3/4})^2 \sum_{\substack{d|d_1d_2\\ d_1,d_2\le X\\ d_1d_2|d^\infty}} \frac{|\alpha(d_1)\alpha(d_2)|.|\chi(d_1)\chi(d_2)|}{d_1d_2}\\ &\ll \frac{q}{\varphi(q)}\frac{1}{\log X}\prod_{p|d}(1+p^{-3/4})^2 \sum_{d|n,n|d^\infty} \frac{|\chi(n)|}{n} \sum_{d_1d_2=n} |\alpha(d_1)\alpha(d_2)|.\end{aligned}$$

From [47, Ch.VI, Section 3, Lemma 1], the last inner sum $\sum_{d_1d_2=n}|\alpha(d_1)\alpha(d_2)|$ is dominated by 1. So, we obtain

$$\begin{aligned}\sum_{\substack{m\le X^2\\ d|m}} \frac{b(m)\overline{\chi(m)}}{m} &\ll \frac{q}{\varphi(q)}\frac{1}{\log X}\prod_{p|d}(1+p^{-3/4})^2 \sum_{d|n,n|d^\infty}\frac{|\chi(n)|}{n}\\ &\ll \frac{q}{\varphi(q)}\frac{1}{d\log X}\prod_{p|d}(1+p^{-3/4})^2\prod_{p|d}\left(1-\frac{1}{p}\right)^{-1}\\ &\ll \frac{q}{\varphi(q)\varphi(d)\log X}\prod_{p|d}(1+p^{-3/4})^2. \end{aligned}\tag{2.29}$$

Now using (2.29) in (2.26), we get

$$B(X) = \mathrm{O}\left(\frac{q^2}{\varphi^2(q)(\log X)^2}\sum_{\substack{d\le X^2\\ (q,d)=1}}\frac{1}{\varphi(d)}\prod_{p|d}(1+p^{-3/4})^4\right). \tag{2.30}$$

Since

$$(1+x)^4 = 1+x(4+6x+4x^2+x^3) < 1+10x \text{ for } x \le p^{-3/4} \le 2^{-3/4},$$

$$\prod_{p|d}(1+p^{-3/4})^4 < \prod_{p|d}(1+10p^{-3/4}) = \sum_{\delta|d}\frac{10^{\omega(\delta)}}{\delta^{3/4}}.$$

Hence the inner sum of (2.30) can be estimated as

$$\sum_{\substack{d\le X^2\\ (q,d)=1}}\frac{1}{\varphi(d)}\prod_{p|d}(1+p^{-3/4})^4 \ll \sum_{\substack{d\le X^2\\ (q,d)=1}}\frac{1}{\varphi(d)}\sum_{\delta|d}\frac{10^{\omega(\delta)}}{\delta^{\frac34}} \ll \sum_{\substack{\delta\le X^2\\ (q,\delta)=1}}\frac{10^{\omega(\delta)}}{\varphi(\delta)\delta^{\frac34}}\sum_{\substack{m\le X^2/\delta\\ (q,m)=1}}\frac{1}{\varphi(m)}. \tag{2.31}$$

The last inequality follows from the fact that $\varphi(m\delta)\ge\varphi(m)\varphi(\delta)$. Note that the series

$$\sum_{\delta\ge1}\frac{10^{\omega(\delta)}}{\varphi(\delta)\delta^{\frac34}}$$

is convergent. So, it is enough to estimate the last inner sum of (2.31). By using the identity

$$\frac{1}{\varphi(m)} = \frac{1}{m}\sum_{d|m}\frac{\mu^2(d)}{\varphi(d)},$$

we write

$$\sum_{\substack{m\le X^2\\(q,m)=1}}\frac{1}{\varphi(m)} = \sum_{\substack{m\le X^2\\(q,m)=1}}\frac{1}{m}\sum_{d|m}\frac{\mu^2(d)}{\varphi(d)} = \sum_{\substack{d\le X^2\\(q,d)=1}}\frac{\mu^2(d)}{d\varphi(d)}\sum_{\substack{m\le X^2/d\\(q,m)=1}}\frac{1}{m}. \tag{2.32}$$

We know that the series $\sum_{d\ge 1}\frac{\mu^2(d)}{d\varphi(d)}$ is convergent, and thus we can use Lemma 2.3.4 to estimate the inner sum in the right-hand side of (2.32). Hence, we obtain

$$\sum_{\substack{m\le X^2\\(q,m)=1}}\frac{1}{\varphi(m)} \ll \frac{\varphi(q)}{q}(\log X+\log\log q)+d(q)\frac{(\log X)^2}{X}.$$

So, we have

$$\sum_{\substack{d\le X^2\\(q,d)=1}}\frac{1}{\varphi(d)}\prod_{p|d}(1+p^{-3/4})^4 \ll \frac{\varphi(q)}{q}(\log X+\log\log q)+d(q)\frac{(\log X)^2}{X}. \tag{2.33}$$

Further, using (2.33) in (2.30) for $q\le T$, we find

$$B(X) = \mathrm{O}\left(\frac{q}{\varphi(q)\log X}\right). \tag{2.34}$$

Let

$$C(X) := \sum_{m,n\le X^2}\frac{b(m)\overline{b(n)}}{mn}\chi(n)\overline{\chi}(m)(m,n)\log\,(m,n) \tag{2.35}$$

and for $n\ge 1$, we denote

$$\varphi'(n) := n\sum_{d|n}\frac{\mu(d)}{d}\log\frac{n}{d}.$$

Then we have

$$C(X)=\sum_{\substack{d\le X^2\\(q,d)=1}}\varphi'(d)\left(\sum_{\substack{m\le X^2\\d|m}}\frac{b(m)\overline{\chi(m)}}{m}\right)\left(\sum_{\substack{n\le X^2\\d|n}}\frac{\overline{b(n)}\chi(n)}{n}\right).$$

Using the fact that $\varphi'(n)\le 2\varphi(n)\log n$ (see [69, 5.40]) and (2.29), we get

$$C(X)=\mathrm{O}\left(\frac{q^2}{(\varphi(q)\log X)^2}\sum_{\substack{d\le X^2\\(q,d)=1}}\frac{\log d}{\varphi(d)}\prod_{p|d}(1+p^{-3/4})^4\right).$$

Now, to proceed as in the case of (2.31), we need an upper bound for the sum

$$\sum_{\substack{m\le X^2\\(q,m)=1}}\frac{\log m}{\varphi(m)}$$

in the place of (2.32). The estimate of the above sum is

$$\ll\frac{\varphi(q)}{q}(\log X+\log\log q)\log X+d(q)\frac{(\log X)^3}{X}.$$

Since $q\le T$, we obtain the estimate

$$C(X)=\mathrm{O}\left(\frac{q}{\varphi(q)}\right). \tag{2.36}$$

Let

$$D(X)=\sum_{m,n\le X^2}\frac{b(m)\overline{b(n)}}{mn}\chi(n)\overline{\chi}(m)(m,n)\log m,$$

which can also be written as:

$$D(X)=\sum_{d\le X^2}\varphi(d)\left(\sum_{\substack{m\le X^2\\d|m}}\frac{b(m)\overline{\chi}(m)\log m}{m}\right)\left(\sum_{\substack{n\le X^2\\d|n}}\frac{\overline{b(n)}\chi(n)}{n}\right). \tag{2.37}$$

Simplifying as in (2.27), we get

$$\sum_{\substack{m\le X^2\\ d|m}}\frac{b(m)\overline{\chi(m)}\log m}{m}=\frac{1}{(\log X)^2}\sum_{\substack{m_1,m_2\le X\\ d|m_1m_2}}\frac{\alpha(m_1)\alpha(m_2)|\chi(m_1)||\chi(m_2)|}{m_1m_2}$$
$$\times\log\frac{X}{m_1}\log\frac{X}{m_2}(\log m_1+\log m_2). \tag{2.38}$$

Proceeding similar to (2.28), we see that the above is

$$\sum_{\substack{m\le X^2\\ d|m}}\frac{b(m)\overline{\chi(m)}\log m}{m}=\frac{2}{(\log X)^2}\sum_{\substack{d|d_1d_2\\ d_1,d_2\le X\\ d_1d_2|d^\infty}}\frac{\alpha(d_1)\alpha(d_2)|\chi(d_1)||\chi(d_2)|}{d_1d_2}$$
$$\times\left(\sum_{\substack{n_1\le\frac{X}{d_1}\\ (n_1,d)=1}}\frac{\alpha(n_1)|\chi(n_1)|\log(n_1d_1)}{n_1}\log\frac{X}{n_1d_1}\right)\left(\sum_{\substack{n_2\le\frac{X}{d_2}\\ (n_2,d)=1}}\frac{\alpha(n_2)|\chi(n_2)|}{n_2}\log\frac{X}{n_2d_2}\right). \tag{2.39}$$

Using (2.18) in the first inner sum and (2.17) in the second inner sum on the right-hand side of (2.39), and then proceeding in a similar manner to (2.29), we get

$$\sum_{\substack{m\le X^2\\ d|m}}\frac{b(m)\overline{\chi}(m)\log m}{m}\ll\frac{q}{\varphi(q)\varphi(d)}\prod_{p|d}(1+p^{-3/4})^2. \tag{2.40}$$

Now, using (2.40) and (2.29) in (2.37), we get

$$\begin{aligned}D(X)&=\mathrm{O}\left(\frac{q^2}{\varphi^2(q)\log X}\sum_{d\le X^2}\frac{1}{d}\prod_{p|d}(1+p^{-3/4})^4\right)\\ &=\mathrm{O}\left(\frac{q}{\varphi(q)}\right).\end{aligned} \tag{2.41}$$

Hence, by using (2.3.4) in the inner sum of (2.24), $A_1 + A_2$ can be written as;

$$A_1 + A_2 = \frac{\varphi(q)}{q}\big((\log\tau + \gamma + A_q)B(X) + C(X) - D(X)\big) + \mathrm{O}\left(\frac{d(q)X^2(\log X)^4}{\tau}\right). \tag{2.42}$$

Finally using (2.34), (2.36) and (2.41) in (2.42) and replacing τ by $\sqrt{\frac{qT}{2\pi}}$, X by T^{θ}, for $q \le T$ we conclude that

$$A_1 + A_2 = \mathrm{O}(1).$$

In the next lemma, we recall some results from the unpublished work of Selberg [68]. Some parts of the following lemma can be deduced from central limit theorem due to Selberg.

Lemma 2.3.6 (Selberg [68]) *(a) For a fixed j with $1 \le j, l \le r$ and $j \ne l$ we have*

$$|\log|L(\frac{1}{2}+it,\chi_j)| - \log|L(\frac{1}{2}+it,\chi_l)|| > (\log\log T)^{\frac{1}{4}}$$

in the interval $[T, 2T]$, except for a set $S \subset [T, 2T]$ of measure $\mathrm{O}\left(T(\log\log T)^{-\frac{1}{4}}\right)$.

(b) We can divide the set $[T, 2T] \setminus S$ into r subsets S_j, $1 \le j \le r$ such that

$$\mathrm{meas}(S_j) \ge \frac{T}{r} - \mathrm{O}\left(T(\log\log T)^{-\frac{1}{4}}\right)$$

and for $t \in S_j$, we have

$$\log|L(\frac{1}{2}+it,\chi_j)| > \log|L(\frac{1}{2}+it,\chi_l)| + (\log\log T)^{\frac{1}{4}} \text{ for all } l \ne j.$$

(c) Let $\frac{1}{\log T} < h < \frac{\log\log T}{\log T}$ and $t \in S_j$. Then

$$\log|L(\frac{1}{2}+it',\chi_j)| > \log|L(\frac{1}{2}+it',\chi_l)| + \frac{1}{2}(\log\log T)^{\frac{1}{4}}$$

for all $l \neq j$ *and* $t \leq t' \leq t+h$ *except in a subset of* $(t, t+h)$ *of measure* $\mathrm{O}\left(h(\log\log T)^{-\frac{1}{5}}\right)$.

(d) *Recall the definition of* ψ_j *from* (2.12) *and* $Z(u, \chi_j)$ *from* (2.1). *Put*

$$M_{\chi_j}(t,h) := \int_t^{t+h} \left(L\left(\frac{1}{2}+iu, \chi_j\right) \psi_j\left(\frac{1}{2}+iu\right)^2 - 1 \right) du,$$

and

$$I_{\chi_j}(t,h) := \int_t^{t+h} Z\left(u, \chi_j\right) \left|\psi_j\left(\frac{1}{2}+iu\right)\right|^2 du.$$

Then

$$\int_T^{2T} |M_{\chi_j}(t,h)|^2 dt = \mathrm{O}\left(T\frac{h^{\frac{3}{2}}}{\sqrt{\log T}}\right) \quad \text{and} \quad \int_T^{2T} |I_{\chi_j}(t,h)|^2 dt = \mathrm{O}\left(T\frac{h^{\frac{3}{2}}}{\sqrt{\log T}}\right).$$

For the statements $(a), (b), (c)$ see page no. $8-9$ and 11, and for (d) see page no. 6 of [68], respectively. Similar results have been proved for degree two L-functions in [67]. The statement (a) follows from Corollary 1.12.4 and arguments for this is as follows. Let $\wp$ be a subset of $[T, 2T]$, defined by

$$\wp = \left\{ t \in [T, 2T] : \left| \log\left|L\left(\frac{1}{2}+it, \chi\right)\right| - \log\left|L\left(\frac{1}{2}+it, \chi^{'}\right)\right| \right| \leq (\log\log T)^{\frac{1}{4}} \right\}.$$

Also, we define

$$\omega(t) := \frac{\log\left|L\left(\frac{1}{2}+it, \chi\right)\right| - \log\left|L\left(\frac{1}{2}+it, \chi^{'}\right)\right|}{\sqrt{2\pi\log\log t}}.$$

Since we can write $\int_T^{2T} = \int_{\wp} + \int_{\wp^c}$, then from Corollary 1.12.4 we have

$$\int_{\wp} \mathfrak{U}_{a,b}(\omega(t))dt + \int_{\wp^c} \mathfrak{U}_{a,b}(\omega(t))dt = \int_a^b e^{-\pi u^2} du + \mathrm{O}\left(\frac{(\log\log\log T)^2}{\sqrt{\log\log T}}\right).$$

By choosing $a = \frac{-1}{\sqrt{2\pi}(\log\log T)^{1/4}}$ and $b = \frac{1}{\sqrt{2\pi}(\log\log T)^{1/4}}$ we get that

$$\mathfrak{U}_{a,b}(\omega(t)) = \begin{cases} 1 & \text{if } t \in \wp, \\ 0 & \text{if } t \in \wp^c. \end{cases}$$

Thus, we get

$$\mathrm{meas}(\wp) = T\int_{\frac{-1}{\sqrt{2\pi}(\log\log T)^{1/4}}}^{\frac{1}{\sqrt{2\pi}(\log\log T)^{1/4}}} e^{-\pi u^2}du + \mathrm{O}\left(T\frac{(\log\log\log T)^2}{\sqrt{\log\log T}}\right).$$

Since the function $e^{-\pi u^2} \leq 1$, we obtain

$$\mathrm{meas}(\wp) \ll \frac{T}{(\log\log T)^{1/4}} + T\frac{(\log\log\log T)^2}{\sqrt{\log\log T}} \ll \frac{T}{(\log\log T)^{1/4}}.$$

This completes the proof of (a).

Selberg concluded (b) by using the fact that $\log|L(s,\chi_j)|,\ j = 1,\dots,r$ are 'statistically independent'. Thus, each $\log L(1/2+it,\chi_j)$ dominates all others with equal probability.

Proof of (c) is indicated by Selberg in [68, page 10–11]; see also [67, page 6-7] for the degree 2 setting.

A result similar to (d) for the Riemann zeta function has been obtained by Selberg [69, Lemma 15, Lemma 16], following which one may prove (d) above. Although, in this case, we have to be careful about the conductor. But, Selberg pointed out in [68, P. 6], that the implied constants in the result (d) are independent of characters.

The following lemma is the core of the proof of Proposition 2.3.8, which is the main proposition of this chapter.

Lemma 2.3.7 *Let $X = T^\theta$ and $1 \leq q \leq T^{\frac{1}{5}-8\theta-\varepsilon}$ with $0 < \theta < \frac{1}{40} - \frac{\varepsilon}{8}$, where ε is a small positive real number and χ is a character modulo q for which we defined the mollifier.*

Then we have

$$I(T;U) := \int_T^{T+U} Z(t,\chi)^2 \left|\psi\left(\frac{1}{2}+it\right)\right|^4 dt \ll U, \tag{2.43}$$

where $U = T^{\frac{3}{5}}$ and the $\ll$-constant is independent of q. Moreover, for a fixed q, we have $I(T;T^a) \ll T^a$, where $\frac{1}{2}+4\theta \le a \le \frac{3}{5}$ and $0 < \theta \le \frac{3}{80}-\varepsilon$.

Proof. Squaring (2.14), we get

$$\begin{aligned} Z(t,\chi)^2 \left|\psi\left(\frac{1}{2}+it\right)\right|^4 &= 2\,\mathrm{Re}\,\Delta(t,T;\chi)^2 \left|\psi\left(\frac{1}{2}+it\right)\right|^4 + 2|\Delta(t,T;\chi)|^2 \left|\psi\left(\frac{1}{2}+it\right)\right|^4 \\ &+ \mathrm{O}\left(q^{\frac{1}{4}} T^{a-\frac{3}{4}} |\Delta(t,T;\chi)| \left|\psi\left(\frac{1}{2}+it\right)\right|^4\right) + \mathrm{O}\left(q^{\frac{1}{2}} T^{2a-\frac{3}{2}} \left|\psi\left(\frac{1}{2}+it\right)\right|^4\right) \\ &= 2\,\mathrm{Re}\,(P_1) + 2P_2 + R_1 + R_2, \end{aligned}$$

where $P_1 = \Delta(t,T;\chi)^2 \left|\psi\left(\frac{1}{2}+it\right)\right|^4$, $P_2 = |\Delta(t,T;\chi)|^2 \left|\psi\left(\frac{1}{2}+it\right)\right|^4$ and $R_1,\ R_2$ are, respectively, the third and fourth terms of the above equation. Hence,

$$I(T;U) = 2\,\mathrm{Re} \int_T^{T+U} P_1 dt + 2\int_T^{T+U} P_2 dt + \int_T^{T+U} (R_1+R_2) dt. \tag{2.44}$$

One gets the estimates

$$\left|\psi\left(\frac{1}{2}+it\right)\right| \le \sum_{n\le X} \frac{1}{\sqrt{n}} \ll \sqrt{X} \tag{2.45}$$

and

$$\int_T^{T+U} |\Delta(t,T;\chi)|^2 dt \ll U \sum_{n\le\tau} n^{-1} + \sum_{\substack{m,n\le\tau \\ m\ne n}} (mn)^{-\frac{1}{2}} \left|\log\frac{m}{n}\right|^{-1}.$$

From [69, Lemma 1], we obtain $\int_T^{T+U} |\Delta(t,T;\chi)|^2 dt \ll U\log T$ and by the Cauchy-Schwarz inequality we have $\int_T^{T+U} |\Delta(t,T;\chi)| dt \ll U\sqrt{\log T}$. Therefore, we get

$$\int_T^{T+U} R_1 dt = \mathrm{O}\left(q^{\frac{1}{4}} T^{a-\frac{3}{4}} X^2 \int_T^{T+U} |\Delta(t,T;\chi)|\right) = \mathrm{O}\left(q^{\frac{1}{4}} T^{a-\frac{3}{4}} X^2 U\sqrt{\log T}\right).$$

We also get $\int_T^{T+U} R_2 dt = \mathrm{O}(q^{\frac{1}{2}} T^{2a-\frac{3}{2}} U X^2)$. To evaluate the integral $\int_T^{T+U} P_1 dt$, we follow the method of Selberg. From (2.21) we can write

$$\Delta(t,T;\chi)^2\left|\psi\left(\frac{1}{2}+it\right)\right|^4=\sum_{m,n\le X^2}\frac{b(m)\overline{b(n)}}{\sqrt{mn}}\Delta(t,T;\chi)^2\left(\frac{n}{m}\right)^{it}.$$

Thus, by interchanging summation and integration we can write

$$\int_T^{T+U}P_1\,dt=\sum_{m,n\le X^2}\frac{b(m)\overline{b(n)}}{\sqrt{mn}}\int_T^{T+U}\Delta(t,T;\chi)^2\left(\frac{n}{m}\right)^{it}dt. \tag{2.46}$$

By Stirling's formula (see [77, corollary II.0.13]), we get

$$\rho\left(\frac{1}{2}+it,\chi\right)^{-1}=\overline{\mathfrak{w}}(\chi)\left(\frac{2\pi}{qt}\right)^{-it}e^{-it-\pi i(1-2\mathfrak{a})/4}\left\{1+\mathrm{O}\left(\frac{1}{t}\right)\right\}.$$

Now, we consider coprime integers μ_1, μ_2 smaller than X^2. In this case, by using the above formula for the ratio of the values of the gamma function, we write

$$\int_T^{T+U}\Delta(t,T;\chi)^2\left(\frac{\mu_2}{\mu_1}\right)^{it}dt=\overline{\mathfrak{m}(\chi)}e^{-\frac{\pi i}{4}(1-2\mathfrak{a})}\sum_{u,v\le\tau}\frac{\chi(uv)}{\sqrt{uv}}\int_T^{T+U}\left(\frac{q\mu_2 t}{2\pi e\mu_1 uv}\right)^{it}dt$$
$$+\mathrm{O}\Big(\sum_{u,v\le\tau}\frac{1}{\sqrt{uv}}\int_T^{T+U}\frac{dt}{t}\Big). \tag{2.47}$$

Note that $\int_T^{T+U}\frac{dt}{t}=\mathrm{O}(U/T)$ and hence the error term would be $\mathrm{O}(U\tau/T)$, which is at most of order $\sqrt{qT}$. We see that the main term in the right-hand side of (2.47) is the same as in Lemma 1.11.2 with $\gamma=0$, $\xi=\frac{2\pi\mu_1 uv}{q\mu_2}$. If we put $\tau=\sqrt{qT/(2\pi)}$, $\tau_1=\sqrt{q(T+U)/(2\pi)}$ and take $U=T^a$ for $\frac{1}{2}<a\le\frac{3}{5}$ then by following [69, eq: 4.9–4.13] we get

$$\int_T^{T+U}\Delta(t,T;\chi)^2\left(\frac{\mu_2}{\mu_1}\right)^{it}dt=2\pi\overline{\mathfrak{m}(\chi)}e^{\frac{\pi i\mathfrak{a}}{2}}\sqrt{\frac{\mu_1}{q\mu_2}}\sum_{u,v\le\tau}{}'\chi(uv)e\Big(-\frac{uv\mu_1}{q\mu_2}\Big)+\mathrm{O}(\sqrt{qT}X^2),$$

where $\mu_2<\mu_1$, $e(x):=e^{2\pi ix}$ and $\sum'$ means the summation over $\frac{\mu_2}{\mu_1}\tau^2\le uv\le\frac{\mu_2}{\mu_1}\tau_1^2$. We rewrite the sum $\sum'$ as

$$\sum_{u,v\leq\tau}{}' \chi(uv)e(-\frac{uv\mu_1}{q\mu_2}) = \sum_{\frac{\mu_2}{\mu_1}\tau\leq v\leq\tau} \chi(v) \sum_{\frac{\tau^2\mu_2}{v\mu_1}\leq u\leq\min\{\tau,\,\frac{\tau_1^2\mu_2}{v\mu_1}\}} \chi(u)e\left(-\frac{uv\mu_1}{q\mu_2}\right).$$

If we replace the upper bound of u, i.e., $\min\left\{\tau, \frac{\tau_1^2\mu_2}{v\mu_1}\right\}$ by $\frac{\tau_1^2\mu_2}{v\mu_1}$ then the error term is $\mathrm{O}\left(\sqrt{\frac{\mu_2}{\mu_1}\frac{qU^2}{T}}\right)$. This can be evaluated in a similar way as in [69, page-104]. Thus, we get

$$\sum_{u,v\leq\tau}{}' \chi(uv)e(-\frac{uv\mu_1}{q\mu_2}) = \sum_{\frac{\mu_2}{\mu_1}\tau\leq v\leq\tau} \chi(v) \sum_{\frac{\tau^2\mu_2}{v\mu_1}\leq u\leq\frac{\tau_1^2\mu_2}{v\mu_1}} \chi(u)e\left(-\frac{uv\mu_1}{q\mu_2}\right) + \mathrm{O}\left(\sqrt{\frac{\mu_2}{\mu_1}\frac{qU^2}{T}}\right). \tag{2.48}$$

Now, we evaluate the inner sum in the right-hand side of (2.48) in two cases according as μ_2 divides v or not. If $\mu_2 \nmid v$, then by using the identity $\chi(u) = \tau(\overline{\chi})^{-1} \sum_{a \bmod q} \overline{\chi}(a)e\left(\frac{au}{q}\right)$ in the inner sum of the right-hand side of (2.48) we get

$$\sum_{u,v\leq\tau}{}' \chi(uv)e(-\frac{uv\mu_1}{q\mu_2}) = \frac{1}{\tau(\overline{\chi})} \sum_{\frac{\mu_2}{\mu_1}\tau\leq v\leq\tau} \chi(v) \sum_{a \bmod q} \overline{\chi}(a) \sum_{\frac{\tau^2\mu_2}{v\mu_1}\leq u\leq\frac{\tau_1^2\mu_2}{v\mu_1}} e\left(\frac{u(a\mu_2 - v\mu_1)}{q\mu_2}\right).$$

But the inner most sum on the right-hand side of the above equation is bounded by

$$\left\| \frac{(a\mu_2 - v\mu_1)}{q\mu_2} \right\|^{-1}.$$

So, first we have to estimate the sum

$$S_q := \sum_{a=1}^{q-1} \left\| \frac{a\mu_2 - v\mu_1}{q\mu_2} \right\|^{-1}.$$

Note that $a\mu_2 \neq v\mu_1$. Let $x_a = \frac{a}{q} - \frac{v\mu_1}{q\mu_2}$ for $1 \leq a \leq q-1$. Then for any two elements $x_a, x_{a'}$ we get $\| x_a - x_{a'} \| \geq q^{-1}$. Thus there is at most one rational x_a in the intervals of the form $\left(\frac{k-1}{q}, \frac{k}{q}\right)$ for $1 \le k \le q$. Further, when $x_a \in \left(\frac{k-1}{q}, \frac{k}{q}\right)$ for $k \ge 2$, we have

$$\| x_a \|^{-1} \ge \left(\min\left\{\frac{k}{q}, \frac{k}{q-k}\right\}\right)^{-1}$$

and when $x_a \in \left(0, \frac{k}{q}\right)$ we have $\| x_a \|^{-1} \ge q\mu_2$.

Hence,

$$S_q \ll \sum_{a=1}^{q/2} \frac{q}{a} + q\mu_2 \ll q(\log(eq) + \mu_2).$$

So, for the case $\mu_2 \nmid v$ the upper bound of (2.48) is $\mathrm{O}(\sqrt{T}q(\log(eq)+\mu_2))$. The remaining case is $\mu_2 \mid v$ and in this case if we write $\mu_2 r = v$, then the right-hand side of (2.48) is

$$\chi(\mu_2) \sum_{\frac{\tau}{\mu_1} \le r \le \frac{\tau}{\mu_2}} \chi(r) \sum_{\frac{\tau^2}{r\mu_1} \le u \le \frac{\tau_1^2}{r\mu_1}} \chi(u) e\left(-\frac{ur\mu_1}{q}\right) + \mathrm{O}\left(\sqrt{\frac{\mu_2}{\mu_1}} \frac{qU^2}{T}\right).$$

Let us write $u = ql + t$. Then the inner sum can be written as

$$\sum_{\frac{\tau^2}{r\mu_1} \le u \le \frac{\tau_1^2}{r\mu_1}} \chi(u) e\left(-\frac{ur\mu_1}{q}\right) = \sum_{t \bmod q} \chi(t) e\left(-\frac{tr\mu_1}{q}\right)\left(\frac{\tau_1^2 - \tau^2}{\mu_1 rq} + \mathrm{O}(1)\right).$$

Since $(r\mu_1, q) > 1$ gives no contribution in $\int_T^{T+U} P_1 dt$, we can take $(r\mu_1, q) = 1$, and then a change of variable $-tr\mu_1 = z$ gives

$$\begin{aligned} \sum_{\frac{\tau^2}{r\mu_1} \le u \le \frac{\tau_1^2}{r\mu_1}} \chi(u) e\left(-\frac{ur\mu_1}{q}\right) &= \frac{U}{2\pi\mu_1 r}\overline{\chi}(r\mu_1) \sum_{z \bmod q} \chi(-z) e\left(\frac{z}{q}\right) + \mathrm{O}(q) \\ &= \frac{U}{2\pi\mu_1 r}\overline{\chi}(r\mu_1)\sqrt{q}\mathfrak{m}(\chi) i^{-\mathfrak{a}} + \mathrm{O}(q). \end{aligned} \tag{2.49}$$

Using (2.49) in (2.48) we obtain

$$\int_T^{T+U} \Delta(t,T;\chi)^2 \left(\frac{\mu_2}{\mu_1}\right)^{it} dt = U\frac{\overline{\chi}(\mu_1)\chi(\mu_2)}{\sqrt{\mu_1\mu_2}} \sum_{\frac{\tau}{\mu_1}\le r\le\frac{\tau}{\mu_2}} \frac{|\chi(r)|}{r} + \mathrm{O}(\sqrt{qT}X^2)$$
$$+\mathrm{O}\left(\sqrt{\frac{qT\mu_1}{\mu_2}}\log(eq) + \sqrt{qT\mu_1\mu_2}\right). \tag{2.50}$$

Thus, inserting (2.50) in (2.46) with the choice $\mu_1 = m/(m,n)$, $\mu_2 = n/(m,n)$ and by using (2.22), we get

$$\begin{aligned}\int_T^{T+U} P_1 dt &= UA_1 + \mathrm{O}\left(\sqrt{qT}X^2\left(\sum_{l\le X}\frac{1}{\sqrt{l}}\right)^4\right)\\ &\quad+\mathrm{O}\left(\sqrt{qT}\left(\log(eq)\sum_{1\le n<m\le X^2}\frac{d(m)d(n)}{n} + \sum_{1\le n<m\le X^2}\frac{d(m)d(n)}{(m,n)}\right)\right)\\ &= UA_1 + \mathrm{O}\left(\sqrt{qT}X^4(\log T)^2\right),\end{aligned} \tag{2.51}$$

where A_1 is defined in Lemma 2.3.5. Next, we have to evaluate the integral of P_2. We can write

$$\begin{aligned}\int_T^{T+U} P_2 dt &= \sum_{\substack{k,l\le\tau\\ m,n\le X^2}} \frac{\chi(l)\overline{\chi}(k)b(m)\overline{b(n)}}{\sqrt{klmn}} \int_T^{T+U}\left(\frac{kn}{lm}\right)^{it} dt\\ &= UA_2 + \sum_{\substack{k,l\le\tau;m,n\le X^2\\ kn\ne lm}} \frac{\chi(l)\overline{\chi}(k)b(m)\overline{b(n)}}{i\sqrt{klmn}\log(kn/lm)}\left((kn/lm)^{i(T+U)} - (kn/lm)^{iT}\right),\end{aligned} \tag{2.52}$$

where A_2 is defined in Lemma 2.3.5. First, we estimate the second sum of (2.52). If $d_k(n)$

denotes the kth generalized divisor function, the second sum is

$$\ll \sum_{\substack{u,v\le\tau X^2\\ u\neq v}} \frac{d_3(u)d_3(v)}{\sqrt{uv}}\left|\log\frac{u}{v}\right|^{-1} \ll \sum_{\substack{u,v\le\tau X^2\\ u\neq v}} \frac{d_3^2(u)+d_3^2(v)}{\sqrt{uv}}\left|\log\frac{u}{v}\right|^{-1}$$
$$\ll \sum_{\substack{u,v\le\tau X^2\\ u\neq v}} \frac{d_3^2(u)}{\sqrt{uv}}\left|\log\frac{u}{v}\right|^{-1}.$$

For any fixed u we break the sum as

$$\sum_{\substack{v\le\tau X^2\\ u\neq v}} \frac{d_3^2(u)}{\sqrt{uv}}\left|\log\frac{u}{v}\right|^{-1} = \left(\sum_{v<\frac{u}{2}} + \sum_{\substack{\frac{u}{2}\le v\le\frac{3u}{2}\\ u\neq v}} + \sum_{v>\frac{3u}{2}}\right)\left(\frac{d_3^2(u)}{\sqrt{uv}}\left|\log\frac{u}{v}\right|^{-1}\right),$$

and it is not hard to show that the first and the third sums are bounded by $\ll X\sqrt{\tau}$. For the second sum we write $v=u+k$, such that $|k|\le\frac{u}{2}$. So,

$$\left|\log\frac{u}{v}\right|^{-1} = \mathrm{O}\left(\frac{u}{|k|}\right)$$

and hence

$$\sum_{\substack{\frac{u}{2}\le v\le\frac{3u}{2}\\ u\neq v}} \frac{1}{\sqrt{v}}\left|\log\frac{u}{v}\right|^{-1} \ll \sqrt{u}\log u.$$

By using all the above estimates, the second sum of (2.52) is

$$\ll \sum_{u,v\le\tau X^2} \frac{d_3^2(u)}{\sqrt{u}}(\sqrt{u}\log(u)+X\sqrt{\tau}) \ll \tau X^2(\log T)^9 \ll \sqrt{qT}X^2(\log T)^9.$$

Taking into account all the integrals from (2.44) and taking $U=T^a$ for $\frac{1}{2}<a\le\frac{3}{5}$, we get

$$\begin{aligned} I(T;U) =& 2U\,\mathrm{Re}\,(A_1+A_2) + \mathrm{O}\left(q^{\frac{1}{2}}T^{2a-\frac{3}{2}}UX^2 + q^{\frac{1}{4}}T^{a-\frac{3}{4}}X^2U\sqrt{\log T}\right)\\ &+\mathrm{O}\left(\sqrt{qT}X^4(\log T)^2+\sqrt{qT}X^2(\log T)^9\right). \end{aligned} \tag{2.53}$$

Clearly, the error term in the right-hand side of the above expression will be $\mathrm{o}(U)$ if

$$q^{\frac{1}{2}}T^{2a-\frac{3}{2}+2\theta}+q^{\frac{1}{4}}T^{a-\frac{3}{4}+2\theta}(\log T)^{\frac{1}{2}}+q^{\frac{1}{2}}T^{\frac{1}{2}-a+4\theta}(\log T)^{2}+q^{\frac{1}{2}}T^{\frac{1}{2}-a+2\theta}(\log T)^{9}=\mathrm{o}(1), \tag{2.54}$$

as $T\to\infty$ and for any q which may be fixed or depends on T. For fixed q, the left-hand side of (2.54) is indeed $\mathrm{o}(1)$ as $T\to\infty$ if $\frac{1}{2}+4\theta<a\leq\frac{3}{5}$ and $0<\theta\leq\frac{3}{80}-\varepsilon$ with a small $\varepsilon>0$. If q varies with T, then we want to choose $a\in\left(\frac{1}{2},\frac{3}{5}\right]$ and $\theta<\frac{1}{4}$ in such a way that q attains its maximum value and the above error term will be $\mathrm{o}(1)$ as $T\to\infty$. Therefore, choosing $a=\frac{3}{5}$ and small $\varepsilon>0$, we get $0<\theta<\frac{1}{40}-\frac{\varepsilon}{8}$, $1\leq q\leq T^{\frac{1}{5}-8\theta-\varepsilon}$ and the above error term is $\mathrm{o}(1)$. So, we conclude from Lemma 2.3.5 that $I(T;U)\ll U$. This completes the proof of Lemma 2.3.7.

Now, we estimate the integrals which are given in Proposition 2.2.5 that help us to prove Theorem 2.2.1 and Theorem 2.6.2. We prove them in three different propositions in three subsections. Recall that $\Omega(t)$ and ψ_j are as in (2.3) and (2.12) respectively.

2.3.1 Upper bound estimate for $\int\Omega^2|\psi_j|^4$ over S_j

Proposition 2.3.8 *Suppose $X=T^{\theta}$ and $1\leq q_j\leq T^{\frac{1}{5}-8\theta-\varepsilon}$ with $0<\theta<\frac{1}{40}-\frac{\varepsilon}{8}$, where ε is a small positive real number and χ_j is a character modulo q_j for which we defined the mollifier. Then we have*

$$J(T):=\int_{S_j}\Omega(t)^2\left|\psi_j\left(\frac{1}{2}+it\right)\right|^4dt\ll ra_j^2T,$$

where $S_j\subset[T,2T]\setminus S$ is as defined in (b) of Lemma 2.3.6 and r may grow with T but not faster than $\mathrm{O}(e^{(1-\varepsilon)(\log\log T)^{\frac{1}{4}}})$.

Proof. First, we prove that

$$\int_{S_j}Z(t,\chi_j)^2\left|\psi_j\left(\frac{1}{2}+it\right)\right|^4dt\ll T. \tag{2.55}$$

Now, we split the interval $[T, 2T]$ into sub-intervals of length U. Let any such sub-interval be $[T+mU, T+(m+1)U]$, where m is a non-negative integer. By using Lemma 2.3.7 we can show that $I(T+mU;U) \ll U$. Hence,

$$I(T;T) = \sum_{m} I(T+mU;U) \ll T. \tag{2.56}$$

If we replace $[T, T+U]$ by $E_j(U) := [T, T+U] \cap S_j$ in $I(T;U)$ and write $I(E_j(U))$ in place of $I(T;U)$, we get $I(E_j(U)) \ll U$. As a result we get

$$\int_{S_j} Z(t, \chi_j)^2 \left|\psi_j\left(\frac{1}{2}+it\right)\right|^4 dt \ll T.$$

Since $\left|L\left(\frac{1}{2}+it, \chi_j\right)\right| = |Z(t, \chi_j)|$, we get

$$\int_{T}^{2T} \left|L\left(\frac{1}{2}+it, \chi_j\right)\right|^2 \left|\psi_j\left(\frac{1}{2}+it\right)\right|^4 dt \ll T. \tag{2.57}$$

Now, by the Cauchy-Schwarz inequality, we have

$$\Omega(t)^2 \leq r \sum_{l=1}^{r} a_l^2 Z(t, \chi_l)^2.$$

So, for $J(T)$, which is defined in Proposition 2.3.8, we get

$$J(T) \leq r a_j^2 \int_{S_j} Z(t, \chi_j)^2 \left|\psi_j\left(\frac{1}{2}+it\right)\right|^4 dt + r \sum_{\substack{l=1 \\ l \neq j}}^{r} a_l^2 \int_{S_j} Z(t, \chi_l)^2 \left|\psi_j\left(\frac{1}{2}+it\right)\right|^4 dt. \tag{2.58}$$

Whenever $t \in S_j$ and $l \neq j$, from part (b) of Lemma 2.3.6 we have

$$\log|L(s, \chi_j)| > \log|L(s, \chi_l)| + (\log\log T)^{\frac{1}{4}}.$$

This implies

$$|L(s, \chi_l)| < |L(s, \chi_j)| e^{-(\log\log T)^{\frac{1}{4}}}. \tag{2.59}$$

Now using (2.55), (2.59) and (2.57) in the right-hand side of (2.58), we get

$$J(T) \ll ra_j^2 T + r \sum_{\substack{l=1 \\ l \neq j}}^{r} a_l^2 T e^{-2(\log\log T)^{\frac{1}{4}}}.$$

Thus, by hypothesis we get

$$J(T) \ll ra_j^2 T.$$

In the next proposition we estimate the lower bound of the first moment.

2.3.2 Lower bound estimate for $\int |\Omega \psi_j^2|$ over S_j

Proposition 2.3.9 *Let $X = T^{\theta}$ and $1 \leq q_j \leq T^{\frac{1}{5}-8\theta-\varepsilon}$ with $0 < \theta < \frac{1}{40} - \frac{\varepsilon}{8}$, where ε is a small positive real number and q_j is the same as in Proposition 2.3.8. Then we have*

$$\int_{S_j} |\Omega(t)| \left| \psi_j \left(\frac{1}{2} + it \right) \right|^2 dt \geq |a_j| \operatorname{meas}(S_j)(1 + \mathrm{o}(1)) \quad \text{as } T \to \infty, \tag{2.60}$$

where $S_j \subset [T, 2T] \setminus S$ is as defined in Lemma 2.3.6 and r may grow with T but not faster than $\mathrm{O}(e^{(1-\varepsilon)(\log\log T)^{\frac{1}{4}}})$.

Proof. Since all the characters have same parity, the value of $\mathfrak{a}$ in the expression (1.45) of $\rho(s, \chi_j)$ remains the same for $l = 1, 2, \ldots, r$. So we have

$$\rho(s, \chi_l) := \mathfrak{w}(\chi_l) \left(\frac{\pi}{q} \right)^{s-\frac{1}{2}} \frac{\Gamma((1-s+\mathfrak{a})/2)}{\Gamma((s+\mathfrak{a})/2)}.$$

This gives

$$\Omega(t) = \left(\left(\frac{\pi}{q} \right)^{it} \frac{\Gamma((\frac{1}{2} - it + \mathfrak{a})/2)}{\Gamma((\frac{1}{2} + it + \mathfrak{a})/2)} \right)^{-\frac{1}{2}} \sum_{l=1}^{r} a_l \mathfrak{w}(\chi_l)^{-\frac{1}{2}} L\left(\frac{1}{2} + it, \chi_l \right).$$

So, we can write

$$\int_{S_j} |\Omega(t)| \left|\psi_j\left(\frac{1}{2}+it\right)\right|^2 dt = \int_{S_j} \left|\sum_{l=1}^{r} a_l \mathfrak{w}(\chi_l)^{-\frac{1}{2}} L\left(\frac{1}{2}+it,\chi_l\right)\psi_j\left(\frac{1}{2}+it\right)^2\right| dt \quad (2.61)$$
$$\geq \left|a_j \mathfrak{w}(\chi_j)^{-\frac{1}{2}} \int_{S_j} L\left(\frac{1}{2}+it,\chi_j\right)\psi_j\left(\frac{1}{2}+it\right)^2 dt\right|$$
$$- \left|\sum_{\substack{l=1\\ l\neq j}}^{r} a_l \mathfrak{w}(\chi_l)^{-\frac{1}{2}} \int_{S_j} L\left(\frac{1}{2}+it,\chi_l\right)\psi_j\left(\frac{1}{2}+it\right)^2 dt\right|.$$

Using the Cauchy-Schwarz inequality and then Lemma 2.3.6 (b) and (2.57) in the last part of the right-hand side of (2.61), we get

$$\int_{S_j} |\Omega(t)| \left|\psi_j\left(\frac{1}{2}+it\right)\right|^2 dt \geq \left|a_j \mathfrak{w}(\chi_j)^{-\frac{1}{2}} \int_{S_j} L\left(\frac{1}{2}+it,\chi_j\right)\psi_j\left(\frac{1}{2}+it\right)^2 dt\right|$$
$$+\mathrm{O}\left(Te^{-(\log\log T)^{\frac{1}{4}}}\right). \quad (2.62)$$

Now, setting

$$f(t) := L\left(\frac{1}{2}+it,\chi_j\right)\psi_j\left(\frac{1}{2}+it\right)^2 - 1,$$

we have

$$\int_{S_j} L\left(\frac{1}{2}+it,\chi_j\right)\psi_j\left(\frac{1}{2}+it\right)^2 dt = \int_{S_j}(1+f(t))dt = \mathrm{meas}(S_j) + \int_{S_j} f(t)dt. \quad (2.63)$$

We fix a δ, $0<\delta<1$, and define

$$h := \frac{(\log\log T)^{\delta}}{\log T}. \quad (2.64)$$

Next, we recall the definition of $M_{\chi_j}(t;h)$ from part (d) of Lemma 2.3.6 and let V_j be a subset of $[T,2T]$ such that for $t\in V_j$ we have

$$|M_{\chi_j}(t;h)| > h(\log\log T)^{-\frac{\delta}{8}}.$$

Then we get

$$\operatorname{meas}(V_j) \leq \int_{V_j} \left| \frac{M_{\chi_j}(t,h)}{h(\log\log T)^{-\frac{\delta}{8}}} \right|^2 dt \leq \frac{(\log\log T)^{\frac{\delta}{4}}}{h^2} \int_T^{2T} |M_{\chi_j}(t,h)|^2 dt. \tag{2.65}$$

Using the first result of Lemma 2.3.6 (d) in the right-hand side of (2.65), we get $\operatorname{meas}(V_j) = \mathrm{O}\left(T(\log\log T)^{-\frac{\delta}{4}}\right)$. From (2.57) we can write

$$\int_t^{t+h} \left| L\left(\frac{1}{2}+iu, \chi_j\right) \psi_j\left(\frac{1}{2}+iu\right)^2 \right|^2 du < h(\log\log T)^{\frac{1}{6}},$$

except in a subset Q_j of $[T, 2T]$ of measure $\mathrm{O}(T(\log\log T)^{-\frac{1}{6}})$. Then we split S_j as $A_j \sqcup B_j$ (say) where

$$A_j = S_j \setminus (Q_j \cup V_j), \quad B_j = S_j \cap (Q_j \cup V_j).$$

Thus, we get

$$\int_{S_j} f(t)dt = \left(\int_{A_j} + \int_{B_j} \right) f(t)dt. \tag{2.66}$$

Clearly, we see that

$$\begin{aligned} \operatorname{meas}(B_j) \leq \operatorname{meas}(V_j) + \operatorname{meas}(Q_j) &\ll T(\log\log T)^{-\frac{\delta}{4}} + T(\log\log T)^{-\frac{1}{6}} \\ &\ll T(\log\log T)^{-\gamma}, \end{aligned}$$

where $\gamma = \min\left(\frac{\delta}{4}, \frac{1}{6}\right)$. Hence, by the Cauchy-Schwarz inequality we get

$$\int_{B_j} f(t)dt \leq \sqrt{\mu(B_j)} \left(\int_T^{2T} |f(t)|^2 dt \right)^{\frac{1}{2}}.$$

But by the Cauchy-Schwarz inequality and (2.56), we find that

$$\int_T^{2T} |f(t)|^2 dt \leq 2I(T;T) + 2T \ll T,$$

where $I(T;T)$ is defined as in (2.43). As a result we get that

$$\int_{B_j} f(t)dt \ll T(\log\log T)^{-\gamma/2}. \tag{2.67}$$

To estimate the integral over A_j in (2.66), we cover the set A_j by segments of the form $e_\nu = [t_\nu, t_\nu + h]$, $\nu = 1, 2, 3, \ldots, n$. We construct the segments in the following way; Let t_1 be the smallest element of the set A_j. Then we set $e_1 := [t_1, t_1 + h]$. Let t_2 be the smallest element of the set $(t_1 + h, 2T] \cap A_j$, then we define $e_2 = [t_2, t_2 + h]$. Similarly, we choose the ν-th interval $e_\nu := [t_\nu, t_\nu + h]$, where t_ν is the smallest element of the set $(t_{\nu-1} + h, 2T] \cap A_j$. Let n be the smallest positive integer such that $A_j \subseteq \cup_{\nu \leq n} e_\nu$. From our construction, we have $n \leq \frac{T}{h} + 1$. Thus we get

$$\begin{aligned}\int_{A_j} f(t)dt = \sum_{\nu=1}^{n} \int_{A_j \cap e_\nu} f(t)dt &= \sum_{\nu=1}^{n} \left(\int_{t_\nu}^{t_\nu + h} - \int_{B_j \cap e_\nu} \right) f(t)dt \\ &= \sum_{\nu=1}^{n} (M_{\chi_j}(t_\nu; h) - M'_{\chi_j}(t_\nu; h)),\end{aligned}$$

where $M'_{\chi_j}(t_\nu; h))$ is defined as follows:

$$M'_{\chi_j}(t_\nu; h)) = \int_{B_j \cap e_\nu} f(t)dt = \int_{B_j \cap e_\nu} \left(L\left(\frac{1}{2} + it, \chi_j\right) \psi_j \left(\frac{1}{2} + it\right)^2 - 1 \right) dt.$$

Since $t_\nu \notin V_j$, from the definition of V_j we get

$$|M_{\chi_j}(t_\nu; h)| \leq h(\log\log T)^{-\frac{\delta}{8}} \quad \text{for all } \nu = 1, 2, \ldots, n.$$

Hence, the first sum is

$$\sum_{\nu=1}^{n} M_{\chi_j}(t_\nu;h) \ll \frac{nh}{(\log\log T)^{\frac{\delta}{8}}} \ll \left(\frac{T}{h}+1\right)\frac{h}{(\log\log T)^{\frac{\delta}{8}}} \ll \frac{T}{(\log\log T)^{\frac{\delta}{8}}}.$$

Further, for any $t\in B_j\cap e_\nu = B_j\cap[t_\nu, t_\nu+h]$ we have $t\notin S_j$. Since $t_\nu\in S_j$, by Lemma 2.3.6 (c), the measure of the set $B_j\cap e_\nu$ in the integral $M'_{\chi_j}(t_\nu;h)$ is

$$\ll h(\log\log T)^{-\frac{1}{5}}.$$

Therefore, by the Cauchy-Schwarz inequality we get

$$|M'_{\chi_j}(t_\nu;h)| \le (\mu(B_j\cap e_\nu))^{\frac{1}{2}}\left(\int_{t_\nu}^{t_\nu+h}|f(t)|^2\right)^{\frac{1}{2}}.$$

Using Minkowski inequality on the last integral and then applying Lemma 2.3.6 (e), we obtain

$$\left(\int_{t_\nu}^{t_\nu+h}|f(t)|^2\right)^{\frac{1}{2}} \ll \sqrt{h(\log\log T)^{\frac{1}{6}}}.$$

Thus,

$$\sum_{\nu=1}^{n}|M'_{\chi_j}(t_\nu;h)| \ll n\sqrt{h(\log\log T)^{-\frac{1}{5}}.h(\log\log T)^{\frac{1}{6}}} \ll T(\log\log T)^{-\frac{1}{60}}.$$

As a result we get

$$\int_{A_j} f(t)dt \ll T(\log\log T)^{-\min\left(\frac{1}{60},\frac{\delta}{8}\right)}. \tag{2.68}$$

Using (2.68) and (2.67) in the right-hand side of (2.66) and then using (2.66) in (2.63), we obtain

$$\int_{S_j} L\left(\frac{1}{2}+it,\chi_j\right)\psi_j\left(\frac{1}{2}+it\right)^2 dt = \mu(S_j) + \mathrm{O}\left(T(\log\log T)^{-\min\left(\frac{1}{60},\frac{\delta}{8}\right)}\right).$$

Hence, substituting the last result in the right-hand side of (2.62), we conclude the proposition.

2.3.3 Upper bound estimate for $\int \Omega |\psi_j|^2$ over S_j

Proposition 2.3.10 *Let $X = T^{\theta}$ and $1 \le q_j \le T^{\frac{1}{5}-8\theta-\varepsilon}$ with $0 < \theta < \frac{1}{40} - \frac{\varepsilon}{8}$, where ε is a small positive real number and q_j is the same as in Proposition 2.3.8. Then*

$$I(T) := \int_{S_j} \Omega(t) \left| \psi_j \left(\frac{1}{2} + it \right) \right|^2 dt = o(T), \text{ as } T \to \infty,$$

where $S_j \subset [T, 2T] \setminus S$ is as defined in Lemma 2.3.6 and r may grow with T but not faster than $\mathrm{O}(e^{(1-\varepsilon)(\log\log T)^{\frac{1}{4}}})$.

Proof. Note that $I(T)$ can be written as follows;

$$I(T) = a_j \int_{S_j} Z(t, \chi_j) \left| \psi_j \left(\frac{1}{2} + it \right) \right|^2 dt + \sum_{\substack{l=1 \\ l \ne j}}^{r} a_l \int_{S_j} Z(t, \chi_l) \left| \psi_j \left(\frac{1}{2} + it \right) \right|^2 dt. \tag{2.69}$$

Applying the Cauchy-Schwarz inequality and then Lemma 2.3.6 (b) and (2.57) in the second integral that appears on the right-hand side of (2.69), we can see that it is of order $\mathrm{O}\left(Te^{-(\log\log T)^{\frac{1}{4}}} \right)$. It remains to show that

$$L_j := \int_{S_j} Z(t, \chi_j) \left| \psi_j \left(\frac{1}{2} + it \right) \right|^2 dt = \mathrm{o}(T) \quad \text{as } T \to \infty. \tag{2.70}$$

The method of proof of (2.70) is very similar to the method used to estimate the integral of $f(t)$ given in (2.63). Choose $h = \log T (\log\log T)^{-\delta},\ 0 < \delta < 1$. If V_j' is the subset of $[T, 2T]$ such that

$$|I_{\chi_j}(t; h)| > h(\log\log T)^{-\frac{\delta}{8}},$$

then following (2.65) and the second result in Lemma 2.3.6 (d), we get $\mathrm{meas}(V_j') =$

$\mathrm{O}\left(T(\log\log T)^{-\frac{\delta}{4}}\right)$. Now we split S_j as $A_j^{'} \sqcup B_j^{'}$ (say) where

$$A_j^{'} = S_j \setminus (Q_j \cup V_j^{'}), \quad B_j^{'} = S_j \cap (Q_j \cup V_j^{'}).$$

Here Q_j's are given as in the proof of Proposition 2.3.9. Thus, we write

$$\int_{S_j} Z(t,\chi_j)\left|\psi_j\left(\frac{1}{2}+it\right)\right|^2 dt = \left(\int_{A_j^{'}} + \int_{B_j^{'}}\right) Z(t,\chi_j)\left|\psi_j\left(\frac{1}{2}+it\right)\right|^2 dt. \tag{2.71}$$

Naturally,

$$\mathrm{meas}(B_j^{'}) \le \mathrm{meas}(V_j^{'}) + \mathrm{meas}(Q_j) \ll T(\log\log T)^{-\frac{\delta}{4}} + T(\log\log T)^{-\frac{1}{6}} \ll T(\log\log T)^{-\gamma},$$

where $\gamma = \min\left(\frac{\delta}{4}, \frac{1}{6}\right)$. Now, by the Cauchy-Schwarz inequality and (2.56) we get

$$\left|\int_{B_j^{'}} Z(t,\chi_j)\left|\psi_j\left(\frac{1}{2}+it\right)\right|^2 dt\right| \le \sqrt{\mathrm{meas}(B_j^{'})}\sqrt{I(T;T)} \ll T(\log\log T)^{-\gamma/2}. \tag{2.72}$$

Our next task is to estimate the integral over $A_j^{'}$ given in (2.71). For this we cover the set $A_j^{'}$ by segments of the form $e_\nu = [t_\nu, t_\nu + h]$, $\nu = 1, 2, 3, \ldots, n$. The construction of these segments is similar to what we did for A_j in Proposition 2.3.9 (see the paragraph following (2.67)). Thus, we have $n \le \frac{T}{h} + 1$, and we write

$$\begin{aligned}\int_{A_j^{'}} Z(t,\chi_j)\left|\psi_j\left(\frac{1}{2}+it\right)\right|^2 dt &= \sum_{\nu=1}^{n}\left(\int_{t_\nu}^{t_\nu+h} - \int_{B_j^{'}\cap e_\nu}\right) Z(t,\chi_j)\left|\psi_j\left(\frac{1}{2}+it\right)\right|^2 dt \\ &= \sum_{\nu=1}^{n}\left(I_{\chi_j}(t_\nu;h) - I_{\chi_j}^{'}(t_\nu;h)\right).\end{aligned}$$

From the definition of $V_j^{'}$, if $t_\nu \notin V_j^{'}$ we get

$$|I_{\chi_j}(t_\nu;h)| \le h(\log\log T)^{-\frac{\delta}{8}} \quad \text{for any } \nu = 1, 2, \ldots, n.$$

So, for the first sum we have

$$\Big|\sum_{\nu=1}^{n} I_{\chi_j}(t_\nu;h)\Big| \ll \frac{nh}{(\log\log T)^{\frac{\delta}{8}}} \ll \left(\frac{T}{h}+1\right)\frac{h}{(\log\log T)^{\frac{\delta}{8}}} \ll \frac{T}{(\log\log T)^{\frac{\delta}{8}}}.$$

Next, for any $t \in B_j^{'} \cap e_\nu = B_j^{'} \cap [t_\nu, t_\nu + h]$ we have $t \notin S_j$. Since $t_\nu \in S_j$ by Lemma 2.3.6 (c), the measure of the set $B_j^{'} \cap e_\nu$ in the integral $I_{\chi_j}^{'}(t_\nu;h)$ is

$$\ll h(\log\log T)^{-\frac{1}{5}}.$$

Note that $|Z(t,\chi_j)| = |L(t,\chi_j)|$. So by the Cauchy-Schwarz inequality, we get

$$|I_{\chi_j}^{'}(t_\nu;h)| \leq (\text{meas}(B_j^{'}\cap e_\nu))^{\frac{1}{2}} \left(\int_{t_\nu}^{t_\nu+h} \left|Z(t,\chi_j)\psi_j\left(\frac{1}{2}+it\right)^2\right|^2 dt\right)^{\frac{1}{2}}.$$

By applying Lemma 2.3.6 (e) to the above integral, we obtain

$$\sum_{\nu=1}^{n} |I_{\chi_j}^{'}(t_\nu;h)| \ll n\sqrt{h(\log\log T)^{-\frac{1}{5}} h(\log\log T)^{\frac{1}{6}}} \ll T(\log\log T)^{-\frac{1}{60}}.$$

As a result we get

$$\int_{A_j^{'}} Z(t,\chi_j)\left|\psi_j\left(\frac{1}{2}+it\right)\right|^2 dt \ll T(\log\log T)^{-\min\left(\frac{1}{60},\frac{\delta}{8}\right)}. \tag{2.73}$$

Using (2.73) and (2.72) in the right-hand side of (2.71), we obtain (2.70), namely,

$$L_j \ll T(\log\log T)^{-\min\left(\frac{1}{60},\frac{\delta}{8}\right)}.$$

2.4 Proof of theorems on Karatsuba's Z-function

Proof of Theorem 2.2.1. For $2 \leq H \leq T$, let us define

$$\begin{aligned} E_j^+ &:= \{T < t \leq T+H : t \in S_j \text{ and } \Omega(t) > 0\}, \\ E_j^- &:= \{T < t \leq T+H : t \in S_j, \text{ and } \Omega(t) < 0\}, \end{aligned}$$

where S_j is as defined in Lemma 2.3.6 and Ω is as given in (2.3).

Now, recall ψ_j from (2.12). One can write

$$\int_{S_j} \Omega(t) \left|\psi_j\left(\frac{1}{2}+it\right)\right|^2 dt = \left(\int_{E_j^+} + \int_{E_j^-}\right) \Omega(t) \left|\psi_j\left(\frac{1}{2}+it\right)\right|^2 dt \tag{2.74}$$

and

$$\int_{S_j} |\Omega(t)| \left|\psi_j\left(\frac{1}{2}+it\right)\right|^2 dt = \left(\int_{E_j^+} - \int_{E_j^-}\right) \Omega(t) \left|\psi_j\left(\frac{1}{2}+it\right)\right|^2 dt. \tag{2.75}$$

Adding (2.74) and (2.75) we get

$$\begin{aligned} \int_{E_j^+} \Omega(t) \left|\psi_j\left(\frac{1}{2}+it\right)\right|^2 dt = \frac{1}{2}\int_{S_j} \Omega(t) \left|\psi_j\left(\frac{1}{2}+it\right)\right|^2 dt \\ + \frac{1}{2}\int_{S_j} |\Omega(t)| \left|\psi_j\left(\frac{1}{2}+it\right)\right|^2 dt. \end{aligned} \tag{2.76}$$

Subtracting (2.74) from (2.75), we get

$$\begin{aligned} -\int_{E_j^-} \Omega(t) \left|\psi_j\left(\frac{1}{2}+it\right)\right|^2 dt = \frac{1}{2}\int_{S_j} |\Omega(t)| \left|\psi_j\left(\frac{1}{2}+it\right)\right|^2 dt \\ - \frac{1}{2}\int_{S_j} \Omega(t) \left|\psi_j\left(\frac{1}{2}+it\right)\right|^2 dt. \end{aligned} \tag{2.77}$$

By applying Proposition 2.3.9 and 2.3.10 in (2.76) and (2.77) respectively and then writing them as a single inequality, we get

$$\pm \int_{E_j^\pm} \Omega(t) \left| \psi_j \left(\frac{1}{2} + it \right) \right|^2 dt \geq \frac{1}{2} |a_j| \operatorname{meas}(S_j) + o(T) \qquad (T \to \infty).$$

By the Cauchy-Schwarz inequality, we deduce that

$$\frac{1}{2} |a_j| \operatorname{meas}(S_j) + o(T) \leq \operatorname{meas}(E_j^\pm)^{\frac{1}{2}} \left(\int_{S_j} (\Omega(t))^2 \left| \psi_j \left(\frac{1}{2} + it \right) \right|^4 dt \right)^{\frac{1}{2}}.$$

Using Proposition 2.3.8 on the right-hand side of the above inequality and then using part (b) of Lemma 2.3.6 we have

$$\operatorname{meas}(E_j^\pm) \gg \frac{\operatorname{meas}(S_j)^2}{rT} \gg \frac{T}{r^3}. \tag{2.78}$$

From the definition of the sets S_j, $j = 1, \ldots, r$, it is clear that $S_k \cap S_l = \emptyset$ for all $k, l \in \{1, 2, \ldots, r\}$ and $k \neq l$. Hence summing over j in (2.78) gives

$$\operatorname{meas}(I^\pm(T, T)) \gg \frac{T}{r^2}.$$

Proof of Theorem 2.2.3. Let χ be a character modulo q and ψ be a Dirichlet polynomial which is defined in the same way as we defined ψ_j in (2.12). We recall Lemma 2.3.7 which says that for $X = T^\theta$ and $1 \leq q \leq T^{\frac{1}{5} - 8\theta - \varepsilon}$ with $0 < \theta < \frac{1}{40} - \frac{\varepsilon}{8}$, we have

$$\int_T^{T+V} Z(t, \chi)^2 \left| \psi \left(\frac{1}{2} + it \right) \right|^4 dt \ll V, \tag{2.79}$$

where $V = T^b$, $3/5 \leq b \leq 1$. Although for fixed q, we can take b to be smaller than $3/5$, in the present prove it suffices to consider b in the range $[3/5, 1]$. Next, for $2 \leq H \leq T$, we want to obtain an upper bound for $I(T,H,\chi)$, which is defined by

$$I(T,H,\chi)=\int_T^{T+H} Z(t,\chi)\left|\psi\left(\frac{1}{2}+it\right)\right|^2 dt.$$

From the definition of $Z(t,\chi)$ and the fact that $\overline{\psi(s)}=\psi(1-s)$ we can write

$$I(T,H,\chi)=\frac{1}{i}\int_{\frac{1}{2}+iT}^{\frac{1}{2}+i(T+H)} L(s,\chi)\rho(s,\chi)^{-\frac{1}{2}}\psi(s)\psi(1-s)ds.$$

We replace the above path of integration by a rectangular path going from $\frac{1}{2}+iT$ to $\frac{1}{2}+i(T+H)$ via the points $c+iT$ and $c+i(T+H)$, where $c=1+\frac{1}{\log T}$. We also write

$$I(T,H,\chi)=I_h(T,H,\chi)+I_v(T,H,\chi),$$

where $I_h(T,H,\chi)$ and $I_v(T,H,\chi)$ are respectively the total contributions to the integral $I(T,H,\chi)$ along the horizontal and vertical paths. The Stirling's formula for the gamma function (see [77, corollary II.0.13]) gives

$$\rho(\sigma+it,\chi)=\mathfrak{w}(\chi)\left(\frac{2\pi}{qt}\right)^{\sigma+it-\frac{1}{2}} e^{it+\pi i(1-2\mathfrak{a})/4}\left\{1+\mathrm{O}\left(\frac{1}{t}\right)\right\}. \tag{2.80}$$

Also, we get for $\sigma\geq -1$,

$$\psi(s,\chi)\ll max\{X^{1-\sigma},\log X\}. \tag{2.81}$$

Using these two facts and Lemma 2.3.1 we get

$$\begin{aligned}I_h(T,H,\chi)&\ll\int_{\frac{1}{2}}^{c}(qT)^{(1-\sigma)/2}(qT)^{\frac{1}{2}(\sigma-\frac{1}{2})}max\{X^{1-\sigma},\log X\}X^{\sigma}\log T\,d\sigma\\&\ll q^{\frac{1}{4}}XT^{\frac{1}{4}}\log X\,\log T.\end{aligned}$$

As we know that the Dirichlet L-series $L(s,\chi)$ converges absolutely on the vertical line segment $\{c+it : T \le t \le T+H\}$, we can write the integral along the vertical segment as:

$$\begin{aligned} I_v(T,H,\chi) &= \mathfrak{w}(\chi) \int_T^{T+H} \sum_{n\ge1} \frac{\chi(n)}{n^{c+it}} \sum_{l,k\le X} \frac{\beta(l)\beta(k)}{l^{1-c-it}k^{c+it}} \left(\frac{qt}{2\pi}\right)^{(c+it-\frac{1}{2})/2} \\ &\qquad e^{-i(t+\pi(1-2\mathfrak{a})/4)/2}\left(1+\mathrm{O}\left(\frac{1}{t}\right)\right) dt \\ &= \mathfrak{w}(\chi) \int_T^{T+H} \sum_{n\ge1} \frac{\chi(n)}{n^{c+it}} \sum_{l,k\le X} \frac{\beta(l)\beta(k)}{l^{1-c-it}k^{c+it}} \left(\frac{qt}{2\pi}\right)^{(c+it-\frac{1}{2})/2} e^{-i(t+\pi(1-2\mathfrak{a})/4)/2} dt \\ &\qquad + O\left(q^{\frac{1}{4}} \int_T^{T+H} \left| \sum_{n\ge1} \frac{\chi(n)}{n^{c+it}} \sum_{l,k\le X} \frac{\beta(l)\beta(k)}{l^{1-c-it}k^{c+it}} \right| t^{(c-\frac{1}{2})/2-1}\, dt \right). \end{aligned} \tag{2.82}$$

Using (2.81), (2.80) and Lemma 2.3.1, the expression of the error term in (2.82) is

$$\begin{aligned} &\ll q^{\frac{1}{4}} \int_T^{T+H} \log T \log X X^{1+(1/\log T)} t^{(c-\frac{1}{2})/2-1} dt \\ &\ll q^{\frac{1}{4}} X T^{\frac{1}{4}} \log X \log T. \end{aligned}$$

Without the constant coefficient, the main term in (2.82) can be written as:

$$\sum_{n\ge1}\sum_{k\le X}\sum_{l\le X} \frac{\chi(n)\beta(k)\beta(l)l^{c-1}}{(nk)^c} \int_T^{T+H} \left(\frac{qt}{2\pi}\right)^{(c-\frac{1}{2})/2} \exp\left(\frac{it}{2}\log\left(\frac{qtl^2}{2\pi e n^2 k^2}\right)\right) dt. \tag{2.83}$$

By using Lemma 1.11.1, we estimate the integral of the exponential function and get

$$\int_T^{T+H} \left(\frac{qt}{2\pi}\right)^{(c-\frac{1}{2})/2} \exp\left(\frac{it}{2}\log\left(\frac{qtl^2}{2\pi e n^2 k^2}\right)\right) dt \ll q^{\frac{1}{4}} T^{\frac{3}{4}}.$$

Since $|\chi(n)| \le 1$ and $|\beta(m)| \le 1$, we conclude that the expression (2.83) is

$$\ll q^{\frac{1}{4}} T^{\frac{3}{4}} X (\log T)^2.$$

Hence,

$$I(T,H,\chi) \ll q^{\frac{1}{4}} T^{\frac{3}{4}} X (\log T)^2. \tag{2.84}$$

Lastly, we want to obtain a lower bound for $J(T,H,\chi)$, which is defined by

$$J(T,H,\chi) := \int_T^{T+H} |Z(t,\chi)| \left|\psi\left(\frac{1}{2}+it\right)\right|^2 dt.$$

We can see that the above expression satisfies the inequality

$$\int_T^{T+H} |Z(t,\chi)| \left|\psi\left(\frac{1}{2}+it\right)\right|^2 dt \geq \left|\int_T^{T+H} L(\frac{1}{2}+it,\chi)\psi\left(\frac{1}{2}+it\right)^2 dt\right|.$$

Using the approximate functional equation (1.49), we estimate the integral appearing in the right-hand side of the above inequality, that is

$$\begin{aligned}\int_T^{T+H} L\left(\frac{1}{2}+it,\chi\right)\psi\left(\frac{1}{2}+it\right)^2 dt = \int_T^{T+H} \sum_{m\leq qT/\pi} \frac{\chi(m)}{m^{\frac{1}{2}+it}} \sum_{n\leq X^2} \frac{b(n)}{n^{\frac{1}{2}+it}} dt \\ + \mathrm{O}\left(\sqrt{\frac{q}{T}} \int_T^{T+H} \left|\sum_{n\leq X} \beta(n) n^{-\frac{1}{2}-it}\right|^2 dt\right). \quad (2.85)\end{aligned}$$

The first term on the right-hand side of the equation (2.85) is

$$\begin{aligned}\int_T^{T+H} \sum_{m\leq qT/\pi} \frac{\chi(m)}{m^{\frac{1}{2}+it}} \sum_{n\leq X^2} \frac{b(n)}{n^{\frac{1}{2}+it}} dt &= H + \sum_{\substack{m\leq qT/\pi, n\leq X^2 \\ mn\geq 1}} \frac{\chi(m)b(n)}{\sqrt{mn}} \int_T^{T+H} (mn)^{-it} dt \\ &= H + \mathrm{O}\left(\sum_{\substack{m\leq qT/\pi, n\leq X^2 \\ mn>1}} \frac{d(n)}{\sqrt{mn}\log(mn)}\right) \\ &= H + \mathrm{O}(\sqrt{qT} X \log X). \quad (2.86)\end{aligned}$$

Now, by applying Theorem 1.11.5 we can write

$$\begin{aligned}\int_T^{T+H} \left|\sum_{n\leq X} \beta(n) n^{-\frac{1}{2}-it}\right|^2 dt &= H \sum_{n\leq X} \frac{|\beta(n)|^2}{n} + \mathrm{O}\left(\sum_{n\leq X} |\beta(n)|^2\right) \quad (2.87) \\ &= \mathrm{O}(H\log X) + \mathrm{O}(X).\end{aligned}$$

Using (2.86), (2.87) in (2.85), we get

$$J(T,H,\chi) \geq H + \mathrm{O}(\sqrt{qT}X\log X). \tag{2.88}$$

Now, by using (2.79),(2.84),(2.88) in (2.4) we get the result.

2.5 Generalized Davenport-Heilbronn function

In Theorem 2.2.1, the a_j's are real numbers. Analogouasly, we can consider the case when the a_j's are complex numbers. In Theorem 2.6.2, we provide an answer in a few cases. Let $\chi_1, \chi_2, \ldots, \chi_r$ be primitive Dirichlet characters having same parity modulo a conductor q and $c_1, c_2, \ldots, c_r$ be complex numbers such that

$$\sum_{j=1}^{r} c_j\chi_j(n) \text{ is real and } \quad \sum_{j=1}^{r} c_j\chi_j(n) = \pm\sum_{j=1}^{r} c_j\overline{\chi_j(n)}\mathfrak{w}(\chi_j) \text{ for all } n \in \mathbb{Z}/q\mathbb{Z}, \tag{2.89}$$

where $\mathfrak{w}$ is defined in (1.30). Let us write

$$f(s) = \sum_{j=1}^{r} c_j L(s,\chi_j) \tag{2.90}$$

and

$$F(t) = i^{-a}\rho_1\left(\frac{1}{2}+it\right)^{-\frac{1}{2}} f\left(\frac{1}{2}+it\right), \tag{2.91}$$

where

$$\rho_1(s) := \left(\frac{\pi}{q}\right)^{s-\frac{1}{2}} \frac{\Gamma((1-s+\mathfrak{a})/2)}{\Gamma((s+\mathfrak{a})/2)} \tag{2.92}$$

and

$$a := \begin{cases} 0 & \text{if } \sum_{j=1}^{r} c_j\chi_j(n) = \sum_{j=1}^{r} c_j\overline{\chi_j(n)}\mathfrak{w}(\chi_j), \\ 1 & \text{if } \sum_{j=1}^{r} c_j\chi_j(n) = -\sum_{j=1}^{r} c_j\overline{\chi_j(n)}\mathfrak{w}(\chi_j). \end{cases}$$

To make sense of a later result (see Theorem 2.6.2), using the functional equation of Dirichlet L-series and (2.89), we deduce that $F(t)$ is a real valued function for real t. Thus, we prove the following lemma:

Lemma 2.5.1 *Under the condition (2.89), we have* $\overline{F(t)} = F(t)$.

Proof. Using the functional equation of $L(s,\chi_j)$'s we get

$$f(s) = \sum_{j=1}^{r} c_j \mathfrak{w}(\chi_j)\rho_1(s)L(1-s,\overline{\chi_j}).$$

By applying equation (2.89) we get the functional equation for $f(s)$ in the form

$$f(s) = \pm\rho_1(s)f(1-s), \tag{2.93}$$

where the positive or negative sign is determined according as $a = 0$ or 1.

For real t, we have

$$\overline{F(t)} = i^a\overline{\rho_1}\left(\frac{1}{2}+it\right)^{-\frac{1}{2}}\overline{f}\left(\frac{1}{2}+it\right).$$

One can rewrite

$$\overline{\rho_1}\left(\frac{1}{2}+it\right)^{-\frac{1}{2}} = \left(\frac{\pi}{q}\right)^{\frac{it}{2}}\frac{\overline{\Gamma\left(\left(\frac{1}{2}+it+\mathfrak{a}\right)/2\right)}}{\left|\Gamma\left(\left(\frac{1}{2}+it+\mathfrak{a}\right)/2\right)\right|}.$$

The functional equation gives

$$\overline{f}\left(\frac{1}{2}+it\right) = \pm\left(\frac{\pi}{q}\right)^{-it}\frac{\overline{\Gamma\left(\left(\frac{1}{2}-it+\mathfrak{a}\right)/2\right)}}{\overline{\Gamma\left(\left(\frac{1}{2}+it+\mathfrak{a}\right)/2\right)}}\overline{f\left(\frac{1}{2}-it\right)}.$$

So, we get

$$\overline{F(t)} = \pm i^a\left(\frac{\pi}{q}\right)^{-\frac{it}{2}}\frac{\overline{\Gamma\left(\left(\frac{1}{2}-it+\mathfrak{a}\right)/2\right)}}{\left|\Gamma\left(\left(\frac{1}{2}+it+\mathfrak{a}\right)/2\right)\right|}\overline{f\left(\frac{1}{2}-it\right)} = F(t).$$

Hence $F(t)$ is real.

Now we give some examples of meromorphic functions $f(s)$ which satisfy the conditions (2.89).

Example 2.5.2 (Davenport and Heilbronn, 1936) The Davenport-Heilbronn function is a nice example of such a meromorphic function. The Davenport-Heilbronn function is given by

$$\Upsilon(s)=\frac{1-i\kappa}{2}L(s,\chi_1)+\frac{1+i\kappa}{2}L(s,\chi_2),$$

where $\kappa:=\frac{\sqrt{10-2\sqrt{5}}-2}{\sqrt{5}-1}$, χ_1 is the character modulo 5 such that $\chi_1(2)=i$ and $\chi_2=\overline{\chi_1}$.

In the literature, the Davenport-Heilbronn function has been studied extensively. In 1936, Davenport and Heilbronn [24] proved that the above function has infinitely many zeros on the critical line and also infinitely many zeros in the half-plane $\sigma>1$. In 1981, Voronin [83] showed that the number of odd order zeros in the interval $[0,T]$ on the line $\mathrm{Re}(s)=\frac{1}{2}$ is at least

$$c\,T\exp\left(\frac{1}{20}(\log\log\log\log T)^{\frac{1}{2}}\right),\qquad \text{where } c \text{ is a positive constant.}$$

In 1990, Karatsuba [44] proved that for sufficiently large T, the number of odd order zeros of the Davenport-Heilbronn function in the interval $[0,T]$ on the line $\mathrm{Re}\,(s)=\frac{1}{2}$ is at least $T(\log T)^{\frac{1}{2}-\varepsilon}$, where $0<\varepsilon<\frac{1}{100}$. In 1992, he [46] slightly improved the lower bound by replacing the factor $(\log T)^{-\varepsilon}$ by $e^{-c\sqrt{\log\log T}}$, $c>0$. In the same year, he [45] proved that the function $\Omega(t)$, defined in (2.3), has at least $T(\log T)^{\frac{2}{\varphi(q)}-\varepsilon}$ many zeros on the line $\mathrm{Re}\,(s)=\frac{1}{2}$, where $q=\mathrm{lcm}(q_1,q_2,\ldots,q_r)$. In 2015, Tam [75] proved the following; let $\varepsilon,\varepsilon_1>0$ be any fixed positive constants, $X\geq X_0(\varepsilon,\varepsilon_1)$, $H=X^{\varepsilon_1}$. Moreover, let Y be the set of $T, X\leq T\leq 2X$, such that the number of zeros of the function $\Omega(t)$ in $[T,T+H]$ is strictly less than $cH(\log T)^{\frac{1}{\varphi(q)}-\varepsilon}$, where Ω is defined in (2.3). Then the measure of Y is less than or equal to $X^{1-\frac{1}{2}\varepsilon_1}$. Under the same conditions, he also proved that for $M=[XH^{-1}]$, the number of segments $[mH,(m+1)H], M<m\leq 2M$, that contain less than $cH(\log T)^{\frac{1}{\varphi(q)}-\varepsilon}$ zeros of $\Omega(t)$, does not exceed $M^{1-\frac{1}{2}\varepsilon_1}$. In 2017, Gritsenko [31] proved that the Davenport-Heilbronn function has at least $T(\log T)^{\frac{1}{2}+\frac{1}{16}-\varepsilon}$ many zeros in

$[0, T]$ on the line $\mathrm{Re}\,(s) = \frac{1}{2}$. Further, he [32] improved the above lower bound as

$$\gg T(\log T)^{\frac{1}{2}+\frac{1}{12}-\varepsilon}.$$

Our next example is due to R. C. Vaughan.

Example 2.5.3 In 2015, R. C. Vaughan [82] considered the following functions:

$$\vartheta_1(s) = \frac{L(s,\chi) + \mathfrak{w}(\chi)L(s,\overline{\chi})}{1+\mathfrak{w}(\chi)}, \qquad \vartheta_2(s) = \frac{L(s,\chi) - \mathfrak{w}(\chi)L(s,\overline{\chi})}{1-\mathfrak{w}(\chi)},$$

where χ is any Dirichlet character modulo q such that $\mathfrak{w}(\chi) \neq -1$ for ϑ_1 and $\mathfrak{w}(\chi) \neq 1$ for ϑ_2. He proved that the number of zeros of ϑ_1, ϑ_2 in the region $\{s : \mathrm{Re}(s) > 1, |\mathrm{Im}(s)| \leq T\} \asymp T$. Note that ϑ_1 and ϑ_2 satisfy (2.89). Hence, the results of Theorems 2.6.1 and 2.6.2 are true for ϑ_1 and ϑ_2.

In the next example, we construct an infinite family of meromorphic functions which are complex linear combinations of Dirichlet L-series and satisfy the condition (2.89). Here we generalize the Davenport-Heilbronn function.

Example 2.5.4 (Generalized Davenport-Heilbronn function) For any positive integer q, let $A_q := \{\chi_1, \chi_2, \ldots, \chi_{2r}\}$ be a collection of complex primitive Dirichlet characters modulo q having the same parity such that if $\chi \in A_q$ then $\overline{\chi} \in A_q$. Without loss of generality let us consider $\overline{\chi_j} = \chi_{r+j}$ for $j = 1, 2, \ldots, r$. Let $c_1, c_2, \ldots, c_{2r}$ be the complex numbers such that $c_j = c_{r+j}\mathfrak{w}(\chi_{r+j})$. Then, one can check that c_j and χ_j satisfy (2.89). For $1 \leq j \leq r$ we have $|\mathfrak{w}(\chi_j)| = 1$, and hence we can write $\mathfrak{w}(\chi_j) = e^{i\theta_j}$ where θ_j's are fixed real numbers depending upon q and lie between 0 to π. So, we choose

$$c_j = a_j\left(1 - i\tan\frac{\theta_j}{2}\right) \text{ and } c_{r+j} = a_j\left(1 + i\tan\frac{\theta_j}{2}\right) \text{ for } j = 1, 2, \ldots, r,$$

where a_j's are real numbers. Hence we get a family of meromorphic functions of the following form;

$$\lambda(s) = \sum_{j=1}^{r} a_j \left(1 - i\tan\frac{\theta_j}{2}\right) L(s, \chi_j) + \sum_{j=1}^{r} a_j \left(1 + i\tan\frac{\theta_j}{2}\right) L(s, \chi_{r+j}). \tag{2.94}$$

2.6 Results on complex linear combination

The first theorem of this section is concerning the zeros of $f(s)$ on the line $\mathrm{Re}(s) = \frac{1}{2}$ or the distribution of zeros of $F(t)$ in short intervals, where f and F are defined in (2.90) and (2.91) respectively.

Theorem 2.6.1 *Let q be the conductor of the characters χ_j, $j = 1, \ldots, r$ such that the Euler function at q i.e. $\varphi(q) > 2$. Let $0 < \varepsilon < \min\left\{\frac{2}{\varphi(q)}, \frac{1}{100}\right\}$ and $0 < \varepsilon' < \frac{1}{100}$ be arbitrarily small fixed positive real numbers. Consider T to be sufficiently large such that $T > T_0(\varepsilon, \varepsilon')$ where $T_0(\varepsilon, \varepsilon')$ is some positive number that depends on ε, ε' and suppose $H = T^{\frac{27}{82}+\varepsilon'}$. Then $f(s)$ has at least $H(\log T)^{\frac{2}{\varphi(q)}-\varepsilon}$ odd order zeros along the critical line on the interval $[T, T+H]$.*

Now, in Lemma 2.5.1 we proved $F(t)$ is a real function and so one can ask the extension of Theorem 2.2.1 for $F(t)$. The next theorem is the affirmative answer of it concerning the distribution of signs of $F(t)$:

Theorem 2.6.2 *For sufficiently large T and any small $\varepsilon > 0$, let $1 \leq q \leq T^{\frac{1}{5}-\varepsilon}$ and r may grow with T but not faster than $\mathrm{O}(e^{(1-\varepsilon)(\log\log T)^{\frac{1}{4}}})$. Then we have*

$$\mathrm{meas}\left(\{t \in [T, 2T] : F(t) > 0\}\right) \gg \frac{T}{r^2}, \quad \mathrm{meas}\left(\{t \in [T, 2T] : F(t) < 0\}\right) \gg \frac{T}{r^2},$$

where the implied constants are independent of r, a_j's and q.

From Theorem 2.6.1, we deduce the following corollary :

Corollary 2.6.3 *Let λ be as in (2.94) and q be the conductor of χ_j such that $\varphi(q) > 2$. Let $0 < \varepsilon < \min\left(\frac{2}{\varphi(q)}, \frac{1}{100}\right)$ and $0 < \varepsilon' < \frac{1}{100}$ be arbitrarily small fixed positive real* *numbers. Consider T to be sufficiently large such that $T > T_0(\varepsilon, \varepsilon')$, where $T_0(\varepsilon, \varepsilon')$ is some positive number that depends on ε and ε' and $H = T^{\frac{27}{82}+\varepsilon'}$. Then, $\lambda(s)$ has at least $H(\log T)^{\frac{2}{\varphi(q)}-\varepsilon}$ odd order zeros along the critical line on the interval $[T, T+H]$.*

Remark 2.6.4 For $r = 2$ and $q = 5$, the above theorem recovers the main theorem of [44].

Since the functions from Example 2.5.3 satisfy (2.89), we have the following corollary:

Corollary 2.6.5 *Let ϑ_1, ϑ_2 be as in Example 2.5.3. Then ϑ_1 and ϑ_2 have at least $H(\log T)^{\frac{2}{\varphi(q)}-\varepsilon}$ odd order zeros along the critical line on the interval $[T, T+H]$.*

2.7 Proof of results on complex linear combination

Proof of Theorem 2.6.1. To prove Theorem 2.6.1, we follow the methods of A. A. Karatsuba [44], [45, Theorem 3] . In [45, Theorem 3], Karatsuba gave a lower bound for the number of sign changes of the real valued function $\Omega(t)$ on the interval $[T, T+H]$, where Ω is as given in (2.3). It is observed that $\Omega(t)$ is a real linear combination of Z-functions which are themselves real functions. In our case, we consider complex linear combinations of Z-functions such that the linear combinations are real valued function. To study the sign changes of $\Omega(t)$, Karatsuba considered the following:

$$\prod_{p\equiv\pm1 (\mathrm{mod}\ K)} \left(1-\frac{1}{p^s}\right)^{\frac{1}{2}} = \sum_{n=1}^{\infty} \frac{\alpha_1(n)}{n^s}, \qquad \mathrm{Re}(s) > 1, \tag{2.95}$$

where $K = \mathrm{lcm}(q_1, q_2, \ldots, q_r)$ and $\alpha(n)$'s are complex numbers. Denote

$$\beta_1(n) := \begin{cases} \alpha(n)\left(1-\frac{\log n}{\log X}\right), & 1 \le n < X = T^{\frac{\varepsilon'}{100}}, \\ 0, & n \ge X. \end{cases}$$

Let us consider the Dirichlet polynomials $g_j(s)$, defined by

$$g_j(s) = \sum_{n<X} \frac{\beta_1(n)}{n^s}, \qquad j = 1, 2, \ldots, r.$$

Then take $g_j^2(s)$ as a mollifier of $L(s,\chi_j)$. Since the functions $g_j(s)$, $1 \le j \le r$ are all equal, we can write $g_j(s) = g(s)$. Now write

$$\Theta(t) := \Omega(t)\left|g\left(\frac{1}{2}+it\right)\right|^2.$$

Let E be a subset of the interval $(T, T+H]$ such that if $t \in E \subset (T, T+H]$, then

$$\begin{aligned}
&\int_0^{h_1}\cdots\int_0^{h_1}|\Theta(t+u_1+\cdots+u_r)|du_1\cdots du_r \\
&> \left|\int_0^{h_1}\cdots\int_0^{h_1}\Theta(t+u_1+\cdots+u_r)du_1\cdots du_r\right|,
\end{aligned} \tag{2.96}$$

where $h_1 = \frac{h}{r}$, $h = \frac{A}{\log T}$, $A = c_1 r\log T(\log X)^{-2\gamma}$, $r = [c\log\log T]$, c, c_1 are two constants bigger than 1, and $\gamma = 1/\varphi(q)$. Observe that the inequality (2.96) implies that $\Theta(t)$ has at least one sign change in the interval $(t, t+rh_1) = (t, t+h)$. So, finding a lower bound for the number of odd order zeros of $\Omega(t)$ in $[T, T+H]$ is equivalent to finding the lower bound for the ratio $\frac{\mathrm{meas}(E)}{h}$. For an arbitrary fixed real number $a \in (0,1)$, define

$$\begin{aligned}
I_1 &:= \int_E\left(\int_0^{h_1}\cdots\int_0^{h_1}|\Theta(t+u_1+\cdots+u_r)|du_1\cdots du_r\right)^a dt,\\
I_2 &:= \int_T^{T+H}\left|\int_0^{h_1}\cdots\int_0^{h_1}\Theta(t+u_1+\cdots+u_r)du_1\cdots du_r\right|^a dt,\\
I_3 &:= \int_T^{T+H}\left(\int_0^{h_1}\cdots\int_0^{h_1}|\Theta(t+u_1+\cdots+u_r)|du_1\cdots du_r\right)^a dt.
\end{aligned}$$

From the definition of E it follows (see [45, p. 494-495]) that

$$I_1 + I_2 \ge I_3. \tag{2.97}$$

As the upper bound for I_1 involves $\mathrm{meas}(E)$, by replacing the lower bound for I_3 and upper bound for I_1, I_2 we obtain the lower bound for $\mu(E)$. In our case, consider $q_1 =$

$q_2 = \cdots = q_r = q$, $K = q$. We write

$$\Theta_1(t) := F(t)\left|g\left(\frac{1}{2}+it\right)\right|^2,$$

where F is the real valued function which is given in (2.91). Note that $\Theta_1(t)$ is the complex analogue of $\Theta(t)$. Let us define

$$I_1' := \int_E \left(\int_0^{h_1} \cdots \int_0^{h_1} |\Theta_1(t+u_1+\cdots+u_r)|du_1\cdots du_r\right)^a dt,$$
$$I_2' := \int_T^{T+H} \left|\int_0^{h_1} \cdots \int_0^{h_1} \Theta_1(t+u_1+\cdots+u_r)du_1\cdots du_r\right|^a dt,$$
$$I_3' := \int_T^{T+H} \left(\int_0^{h_1} \cdots \int_0^{h_1} |\Theta_1(t+u_1+\cdots+u_r)|du_1\cdots du_r\right)^a dt.$$

Proceeding in a similar way as in (2.97), we obtain $I_1'+I_2' \geq I_3'$. Now, we need to estimate I_1', I_2', I_3'. We obtain such estimates by following the method of A. A. Karatsuba [45, eq:(69)–(82)]. The only new things that would appear in these estimates are some new absolute constants which depend on r and c_j. Note that the remaining part of Karatsuba's proof (see eq: (83) at the end of the proof in [45]), remains the same for any absolute constant. Thus, the proof of our theorem follows.

Proof of Theorem 2.6.2. We have $\rho(s,\chi) = \mathfrak{w}(\chi)\rho_1(s)$ (see (1.45) and (2.92)), and this allows us to write

$$F(t) = \sum_{j=1}^{r} c_j(\chi_j)Z(t,\chi_j),$$

where $c_j(\chi_j) = i^{-a}c_j\sqrt{\mathfrak{w}(\chi_j)}$ are complex numbers having the same modulus as c_j, $1 \leq j \leq r$ and F is the real valued function which is given in (2.91). One can observe that if we replace a_j by $c_j(\chi_j)$, $1 \leq j \leq r$ in Proposition 2.3.8, 2.3.9 and 2.3.10, then the order of T will remain the same in the integral values of these propositions. Then by following the proof of Theorem 2.2.1 we obtain the required result.

Remark 2.7.1 We expect a better bound in Proposition 2.3.8. More precisely, $J(T) \ll ra_j^2\mu(S_j)$, where $J(T)$ is the mean square given in Proposition 2.3.8. This improved bound will lead to an improvement in the estimates of Theorem 2.2.1 and 2.6.2 as $\mu(I_F^{\pm}(T,H)) \gg \frac{T}{r}$.

Chapter 3

Higher Dimensional Dedekind Sums and Twisted Mean Values of Dirichlet L-functions

3.1 Introduction

Let χ be a Dirichlet character modulo $q \geq 2$. In Chapters 1 and 2, we have studied many analytic properties of the Dirichlet L-function $L(s, \chi)$. The arithmetic of special values of such L-functions at $\frac{1}{2}$ or integers is a central topic of interest and their evaluation in terms of Bernoulli numbers leads to a variety of new insights and generalizations. Moreover, the connection between the class number of a quadratic field and $L(1, \chi)$ has turned out to have important consequences. The mean square average of special values of L-functions when χ ranges over all non-trivial or all odd or all even characters modulo q has a very rich literature. We mention some of them here. In 1981, Heath-Brown [37] proved that

$$\frac{1}{\varphi(q)} \sum_{\chi \bmod q} |L(1/2, \chi)|^2 = \frac{1}{q} \sum_{k|q} \mu(k/q) T(k),$$

where $T(k)$ has the following asymptotic expansion

$$T(k) = k\left(\log\frac{k}{8\pi}+\gamma\right)+2\zeta^2\left(\frac{1}{2}\right)\sqrt{k}+\sum_{m=0}^{2N-1} c_m k^{-\frac{m}{2}}+\mathrm{O}(k^{-N}),$$

for $N \geq 1$. Here, γ is the Euler constant and c_m, $m = 0, 1, \ldots, 2N-1$ are some real constants. For $\sigma = 1$, the asymptotic formula

$$\frac{1}{\varphi(q)}\sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}} |L(1,\chi)|^2 = \frac{\pi^2}{6}\prod_{p|q}\left(1-\frac{1}{p^2}\right)-\frac{\varphi(q)}{q^2}\left(\log q+\sum_{p|q}\frac{\log p}{p-1}\right)^2+\mathrm{o}\left(\log\log q\right)$$

was proved by Wen P. Zhang (see [50]). In fact, one can study more general mean values which are twisted by a character as follows:

$$M(q, c; m, n) = \frac{2}{\varphi(q)}\sum_{\substack{\chi \bmod q\\ \chi(-1)=(-1)^m}} \chi(c)L(m,\chi)L(n,\overline{\chi}), \tag{3.1}$$

where c, q, m and n are positive integers and c is coprime to q. Note that for a fixed positive integer m, the sum in (3.1) varies over either even characters or odd characters depending upon whether m is even or odd respectively. For non-integers m and n, the mean values in (3.1) are very interesting and these have been studied extensively in [20, 9]. For any positive integers $c > 1$, $q > 2$ and $(c, q) = 1$, S. R. Louboutin [55] obtained that

$$M(q, c; 1, 1) = \frac{\pi^2}{6c}\frac{\varphi(q)}{q}\left(\prod_{p|q}\left(1+\frac{1}{p}\right)-\frac{3c}{q}\right)-\frac{2\pi^2}{q^2}\sum_{d|q} d\mu(q/d)s(d, c),$$

where $s(d,\ c)$ is a finite trigonometric sum, called Dedekind sum and this is defined as follows: Let a be an integer and b be a natural number with $(a,\ b) = 1$. The classical Dedekind sum $s(a,\ b)$ is defined as:

$$s(a,\ b) = \sum_{k=1}^{b}((k/b))((ak/b)) = \frac{1}{4b}\sum_{k=1}^{b-1}\cot\frac{\pi k}{b}\cot\frac{\pi ka}{b}.$$

Here $((x))$ is the sawtooth function which is defined as:

$$((x)) = \begin{cases} x - [x] - 1/2, & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}; \\ 0, & \text{if } x \in \mathbb{Z}, \end{cases}$$

where $[\cdot]$ denotes the greatest integer function, see [64, p. 1]. Thus, the mean value $M(q,\, c;\, 1,\, 1)$ depends on the closed formula for finite trigonometric sums. In the same paper, Louboutin [55] posed an open question for the general formula for (3.1). In [53], Liu studied some special cases, namely, the mean values $M(p,\, c;\, 1,\, n)$ and $M(p,\, c;\, 2,\, n)$ for $c = 1,\, 2,\, 3,\, 4$, where p is a prime number, and proved a few identities using character sums and Bernoulli polynomials. Our main aim is to generalize Liu's result and obtain closed formulas for (3.1) in some more cases. For this purpose, firstly we study closed formulas for certain general trigonometric sums. Along the way, we will also study the behavior of certain trigonometric Dirichlet series.

Many finite trigonometric sums, evidently, do not have a closed form. However, they may possess beautiful reciprocity theorems. The most famous reciprocity theorem for trigonometric sums is undoubtedly the one which is equivalent to the reciprocity theorem for Dedekind sums. Dedekind proved the following reciprocity law for these sums.

Theorem 3.1.1 *If a and b are relatively prime, then*

$$s(a,\, b) + s(b,\, a) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a}\right).$$

Originally, Dedekind sums appeared in the theory of modular forms (see [2]). Because of their important applications, mainly in number theory, Dedekind sums have been studied extensively by several authors in a variety of contexts, e.g. [11, 29, 81, 86]. These sums also have interesting applications in other fields, namely, in connection with class numbers, lattice point problems, topology, and algebraic geometry (see [3, 64, 61, 80]).

Dedekind sums have various generalizations. The generalization of the Dedekind sums which we mention here is due to Zagier [85]. From topological considerations, he arrived naturally at expressions of the following kind:

Definition 3.1.2 Let n be a positive integer, and $a_1,\ a_2,\dots,\ a_m$ be integers which are coprime to n and m be an even positive integer. The higher dimensional Dedekind sum $d(n; a_1, a_2, \dots, a_m)$ is defined by

$$d(n; a_1,\ a_2,\ \dots,\ a_m) = (-1)^{\frac{m}{2}} \sum_{k=1}^{n-1} \cot\frac{\pi k a_1}{n} \cdots \cot\frac{\pi k a_m}{n}. \tag{3.2}$$

When $m = 2$, higher dimensional Dedekind sums become the classical Dedekind sums, up to some factor, namely,

$$d(n;\ a_1,\ a_2) = -4ns(a_1 a_2^{-1},\ n),$$

where a_2^{-1} is the inverse of $a_2 \pmod{n}$. Hence, the above sum in (3.2) generalizes the classical Dedekind sum. The higher dimensional Dedekind sums $d(n;\ a_1,\ a_2,\dots,\ a_m)$ possess a reciprocity law only if the integers $a_1,\ a_2,\dots,\ a_m$ are pairwise coprime. In this case, the reciprocity law was proved in [85, Section 3] and is as follows:

$$\sum_{j=1}^{m} \frac{1}{a_j} d(a_j;\ a_1,\dots,\ \hat{a}_j,\dots,\ a_m) = 1 - \frac{l_m(a_0,\dots,\ a_m)}{a_0 \cdots a_m}, \tag{3.3}$$

where $(a_1,\dots,\ \hat{a}_j,\dots,\ a_m)$ means that a_j is omitted from the sequence $(a_1,\ a_2,\dots,\ a_m)$, and $l_m(a_0,\dots,\ a_m)$ is the coefficient of x^m in the power series:

$$\prod_{j=0}^{m} \frac{a_j x}{\tanh a_j x} = \prod_{j=0}^{m} \left(1 + \frac{1}{3} a_j^2 x^2 - \frac{1}{45} a_j^2 x^4 + \frac{2}{945} a_j^6 x^6 - \cdots\right).$$

Later, we will see that the mean values $M(q,\ c;\ m,\ n)$ can be evaluated in terms of higher dimensional Dedekind sums. In other words, the mean value $M(q,\ c;\ m,\ n)$ is computable if the values of $d(q;\ 1,\dots,\ 1,\ c,\dots,\ c)$ are known, where 1 is repeated j times and c is repeated k times with $j+k$ an even integer for $1 \le j \le m$ and $1 \le k \le n$. We denote such higher dimensional Dedekind sums of dimension $j+k$ by $S(q,\, c;\, j,\, k)$, i.e.,

$$S(q,\, c;\, j,\, k) = (-1)^{\frac{j+k}{2}} \sum_{l=1}^{q-1} \left(\cot \frac{\pi l}{q}\right)^j \left(\cot \frac{\pi c l}{q}\right)^k.$$

In [85, p.165-166], D. Zagier computed special values as well as general formulas for four dimensional Dedekind sums. We derive formulas for $S(d,\, c;\, j,\, k)$, for $c = 2,\, 4$ and of dimension $j+k$ (see Proposition 3.2.7). By using these higher dimensional Dedekind sums, we shall compute the mean values in (3.1) for any odd positive integer q and some positive integers m and n. For example, the following are the new formulas for mean values of the type (3.1):

$$\begin{aligned}
&M(q,\, 2;\, 2,\, 4) = -11\pi^6(\varphi_4(q)+70\varphi_2(q))/1080q^6, \\
&M(q,\, 2;\, 2,\, 6) = -\pi^8(5\varphi_6(q)+371\varphi_4(q)-27685\varphi_2(q))/10800q^8, \\
&M(q,\, 4;\, 3,\, 3) = -\pi^6(\varphi_4(q)+490\varphi_2(q)+360\varphi_1(\chi_4,\, q))/360q^6, \\
&M(q,\, 4;\, 5,\, 3) = -\pi^8(2\varphi_6(q)-35\varphi_4(q)-56252\varphi_2(q)-27720\varphi_1(\chi_4,\, q))/15120q^8,
\end{aligned} \tag{3.4}$$

where $\chi_4(d)$ is the nontrivial character modulo 4,

$$\varphi_k(\chi_4,\, q) := q^k \chi_4(q) \prod_{p|q} \left(1 - \frac{\chi_4(p)}{p^k}\right) \text{ and } \varphi_k(m) := m^k \prod_{p|m} \left(1 - \frac{1}{p^k}\right).$$

The function $\varphi_k(m)$ is known as Jordan's totient function of order k. To get closed formulas of $S(q,\, c;\, j,\, k)$ for $c = 2$ and 4, we will establish a trigonometric formula for $\cot^{2n} x \sec^m 2x$, where x is a real number and $m,\, n$ are positive integers. Also, we use this trigonometric formula to study some trigonometric Dirichlet series.

Let us consider the following trigonometric Dirichlet series:

$$\xi_s(\tau) := \sum_{n=1}^{\infty} \frac{\cot(\pi n \tau)}{n^s}, \quad \psi_s(\tau) := \sum_{n=1}^{\infty} \frac{\sec(\pi n \tau)}{n^s}.$$

Over the years, several mathematicians [6, 48, 7, 73] studied the properties such as convergence and irrationality of these trigonometric Dirichlet series as well as calculated special values of these Dirichlet series. Recently, A. Straub [73] studied the more general trigonometric Dirichlet L-series

$$\psi_s^{a,b}(\tau) := \sum_{n=1}^{\infty} \frac{trig^{a,b}(\pi n\tau)}{n^s},$$

where $trig^{a,b} = \sec^a \csc^b$ and a, b are integers. Straub proved that:

Theorem 3.1.3 ([73], Theorem 1.1) *For integers a, b and s, the Dirichlet series $\psi_s^{a,\,b}(\tau)$ converges if $s \geq \max(a,\ b,\ 1)+1$, and for such s, we have $\psi_s^{a,\,b}(\tau) \in \pi^s \mathbb{Q}(\tau)$, provided that τ is a real quadratic irrational and s, b have the same parity. In addition, if $\tau^2 \in \mathbb{Q}$ and $a+b > 0$, then $\psi_s^{a,\,b}(\tau) \in (\pi\tau)^s \mathbb{Q}$.*

Note that the series $\psi_s^{a,\,b}(\tau)$ also converges when τ is a real algebraic irrational and $s \geq \max(a,\ b,\ 1)+1$. In the same paper [73], Straub showed that all Dirichlet series $\sum_{n=1}^{\infty} f(\pi n\tau)/n^s$ of the appropriate parity, with $f(\tau)$ being an arbitrary product of elementary trigonometric functions, can be evaluated as an (simple) algebraic multiple of π^s whenever τ is a real quadratic irrational. Further, he raised the following question:

Question 1. Can this be extended to the series such as

$$\sum_{n=1}^{\infty} \frac{\cot(\pi n\tau_1)\cdots\cot(\pi n\tau_k)}{n^s}, \tag{3.5}$$

where $\tau_1, \ldots,\ \tau_k$ are quadratic (or algebraic) irrationals ?

Here we provide a partial answer to this question. We first obtain an identity for trigonometric products of the form $\sec^m 2x \cot^{2n} x$ (see Theorem 3.2.1) and give a couple of applications of the same. One application addresses Question 1 and other gives a formula for $S(d,\ c;\ j,\ k)$.

3.2 Results

Following are the main results in this chapter.

Theorem 3.2.1 *Let x be a real number and $m,\ n$ be any positive integers. Then we have*

$$\cot^{2n} x \sec^m 2x = \begin{cases} \sum\limits_{i=0}^{n} a_i(m,\ n)(\cot x)^{2(n-i)} + \sum\limits_{j=1}^{m} b_j(m,\ n)\sec^j 2x, & \text{if } m \geq n, \\ \sum\limits_{i=0}^{m} a_i(n,\ m)(\sec 2x)^{(m-i)} + \sum\limits_{j=1}^{n} b_j(n,\ m)\cot^{2j} x, & \text{if } m \leq n, \end{cases}$$

where the coefficients $a_i(m,\ n)$ and $b_j(m,\ n)$ are defined by

$$a_i(m,\ n) = \begin{cases} 1, & \text{if } i = 0, \\ 2m, & \text{if } i = 1, \\ \sum\limits_{j=-1}^{i-3}\{c(j,\ i-1)+c(j,\ i)\}2^{j+2}\binom{m}{j+2} + 2^i\binom{m}{i}, & \text{if } 2 \leq i \leq n-1, \\ 1+2c(-1,\ n-1)(m-1) + \sum\limits_{j=0}^{n-3} c(j,\ n-1)2^{j+2}\binom{m}{j+2}, & \text{if } i = n, \end{cases}$$

and

$$b_j(m,\ n) = \begin{cases} 1, & \text{if } 1 \leq j \leq m-1, \\ \sum\limits_{k=-1}^{n-2} c(k,\ n)2^{k+2}\binom{m-j}{k+1}, & j = m. \end{cases}$$

The coefficients $c(j,\ i)$ are defined by

$$c(j,\ i) = \begin{cases} \frac{1-(-1)^i}{2}, & \text{if } j = -1, \\ [\frac{i}{2}], & \text{if } j = 0, \\ \frac{i(i-2)}{4}, & \text{if } j = 1 \text{ and } i \text{ is even}, \\ \left(\frac{i-1}{2}\right)^2, & \text{if } j = 1 \text{ and } i \text{ is odd}, \end{cases}$$

and for $2 \le j \le i-2$,

$$c(j,\ i) = c(j-1,\ i-1) + c(j,\ i-1) = \sum_{l=2}^{i-j} \binom{i-1-l}{j-1} \left[\frac{l}{2}\right].$$

As we have

$$\tan^{2n} x \sec^m 2x = (-1)^m \cot^{2n}\left(\frac{\pi}{2} - x\right) \sec^m 2\left(\frac{\pi}{2} - x\right),$$

Theorem 3.2.1 implies the following:

Corollary 3.2.2 *For a real number x and positive integers m and n, we have*

$$\tan^{2n} x \sec^m 2x = \begin{cases} (-1)^m \sum_{i=0}^{n} a_i(m,\ n)(\tan x)^{2(n-i)} + \sum_{j=1}^{m} b_j(m,\ n)(-1)^{m+j} \sec^j 2x, & \text{if } m \ge n, \\ \sum_{i=0}^{m} a_i(n,\ m)(-1)^i (\sec 2x)^{(m-i)} + (-1)^m \sum_{j=1}^{n} b_j(n,\ m) \tan^{2j} x, & \text{if } m \le n, \end{cases}$$

where $a_i(m,\ n)$ *and* $b_j(m,\ n)$ *are defined as in Theorem 3.2.1.*

In the following example, we give a concrete expression for $\cot^{2n} x \sec^m 2x$ for some particular values of m and n.

Example 3.2.3 (a) $\cot^4 x \sec^3 2x = \cot^4 x + 6\cot^2 x + 5 + \sec^3 2x + 4\sec^2 2x + 8\sec 2x.$

(b) $\cot^6 x \sec^3 2x = \cot^6 x + 6\cot^4 x + 18\cot^2 x + 13 + \sec^3 2x + 6\sec^2 2x + 18\sec 2x.$

Theorem 3.2.1 is useful in proving the convergence and irrationality of the following series.

$$\Psi_s^1(m,\ 2n,\ \tau) := \sum_{k=1}^{\infty} \frac{\sec^m(2\pi k\tau)\cot^{2n}(\pi k\tau)}{k^s},\ m \in \mathbb{N},\ n \in \mathbb{Z}, \tag{3.6}$$

$$\Psi_s^2(m,\ l,\ n,\ \tau) := \sum_{k=1}^{\infty} \frac{\cot^m(\pi k\tau)\cot^l(2\pi k\tau)\cot^n(4\pi k\tau)}{k^s},\ l,\ m,\ n \in \mathbb{N},\ l+m+n \text{ even.}$$

It is evident that Ψ_s^2 is a particular case of (3.5).

Theorem 3.2.4 *Let* τ *be an algebraic irrational number. Then* *(i)* $\Psi_s^1(m,\ 2n,\ \tau)$ *converges absolutely for* $s \geq \max(m,\ 2n)+1$.

(ii) $\Psi_s^2(m,\ l,\ n,\ \tau)$ *converges absolutely for* $s \geq m+l+n+1$.

Moreover, for $i = 1,\ 2$, *we have* $\Psi_s^i \in \pi^s\mathbb{Q}(\tau)$, *and if* $\tau^2 \in \mathbb{Q}$, *then* $\Psi_s^i \in (\pi\tau)^s\mathbb{Q}$.

From the proof of Theorem 3.2.4, for $i = 1,\ 2$, we find that Ψ_s^i's can be expressed as a rational linear combination of $\psi_s^{a,\ 0}(2\tau)$ and $\psi_s^{0,\ 2b}(\tau)$, where a and b are non-negative integers. From the proof of Theorem 3.1.3 in [73], it has been seen that for any a and b, the series $\psi_s^{a,\ 0}(2\tau)$ and $\psi_s^{0,\ 2b}(\tau)$ can be evaluated at the points of convergence. We note that $\psi_s^{0,\ 2b}(\tau) = \Psi_s^1(0,\ 2n,\ \tau)+\zeta(s)$, and therefore, from [73, Example 1.2] and [48, eq. 4.4], we get

$$\Psi_4^1(1,\ 2,\ \sqrt{5}) = \frac{2536}{18045}\pi^4, \quad \Psi_4^2(1,\ 1,\ 0, \sqrt{11}) = \frac{-\pi^4}{1386}, \quad \Psi_4^2(1,\ 0,\ 1,\ \sqrt{5}) = \frac{-9061}{2394}\frac{\pi^4}{60}.$$

Next, we want to formulate a closed formula for certain trigonometric sums. The study of finite trigonometric sums has a long history. Many mathematicians have extensively studied the closed forms of these sums [8, 15, 25, 84]. Given positive integers $m,\ n$ and d, we provide a closed formula for the following trigonometric sum:

$$A(d;\ m,\ \pm 2n) := \sum_{a=1}^{d-1} \sec^m\left(\frac{2\pi a}{d}\right)\cot^{\pm 2n}\left(\frac{\pi a}{d}\right). \tag{3.7}$$

In order to derive these sums, our main ingredients are Theorem 3.2.1 and the following trigonometric sums:

$$F(d,\ \pm 2n) := \sum_{a=1}^{d-1}\left(\cot\frac{\pi a}{d}\right)^{\pm 2n} \text{ and } G(d,\ m) := \sum_{a=1}^{d-1}\left(\sec\frac{2\pi a}{d}\right)^{m}.$$

Note that, for odd m, $F(d,\ \pm m) = 0$. This follows from the following fact:

$$\sum_{a=1}^{d-1}\left(\cot\frac{\pi a}{d}\right)^{\pm m} = \sum_{a=1}^{d-1}\left(-\cot\frac{\pi(d-a)}{d}\right)^{\pm m} = -F(d,\ \pm m).$$

Thus, we need to obtain closed formulas for $F(d,\ \pm 2n)$ and $G(d,\ m)$ for positive integers m and n. We obtain such formulas in the following proposition:

Proposition 3.2.5 *Let $N = \min(d-1,\ m)$, where d is odd, m is even and*

$$M = \begin{cases} \min(m,\ d), & \text{if } m \text{ is odd}, \\ \min(m-1,\ d), & \text{if } m \text{ is even}, \end{cases}$$

i.e, N is even and M is odd. Then we have the following formulas:

$$F(d,\ m) = m \sum_{\substack{(k_2,\ k_4,\dots,\ k_N) \\ 2k_2+4k_4+\cdots+Nk_N=m}} (-1)^{k_4+\cdots+k_{4\left[\frac{N}{4}\right]}} \frac{(k_2+\cdots+k_N-1)!}{k_2!\cdots k_N!} \prod_{i=1}^{\frac{N}{2}} \left(\frac{1}{d}\binom{d}{2i+1}\right)^{k_{2i}},$$

$$F(d,\ -m) = m \sum_{\substack{(k_2,\ k_4,\dots,\ k_N) \\ 2k_2+4k_4+\cdots+Nk_N=m}} (-1)^{k_4+\cdots+k_{4\left[\frac{N}{4}\right]}} \frac{(k_2+\cdots+k_N-1)!}{k_2!\cdots k_N!} \prod_{i=1}^{\frac{N}{2}} \binom{d}{2i}^{k_{2i}},$$

$$G(d,\ m) = -1 + m \sum_{\substack{(k_1,\ k_3,\dots,\ k_M) \\ k_1+3k_3+\cdots+Mk_M=m}} (-1)^{k_1+\cdots+k_M} \frac{(k_1+\cdots+k_M-1)!}{k_1!\cdots k_M!} \times \prod_{\substack{r=1 \\ r \text{ odd}}}^{M} \left(\frac{-d2^r(-1)^{\frac{d-r}{2}}}{d+r}\binom{\frac{d+r}{2}}{\frac{d-r}{2}}\right)^{k_r}.$$

From Proposition 3.2.5, we list out some formulas in [22, Table 1, 2], namely,

$$\begin{aligned} F(d,\ -8) &= d + \left(17d^8 - 112d^6 + 308d^4 - 528d^2\right)/315, \\ F(d,\ 10) &= -d + \left(2d^{10} - 66d^8 + 946d^6 - 7898d^4 + 55737d^2 + 44838\right)/93555, \\ G(d,\ 7) &= -1 + \chi_4(d)\left(1327d^7 - 305d^5 + 1813d^3 + 525d\right)/3360. \end{aligned}$$

Hence, by Theorem 3.2.1 and Corollary 3.2.2, we get the formulas of (3.7) as follows:

Corollary 3.2.6 *For positive integers m, n and d, the following identities hold:*

$$A(d;\ m,\ n) = \begin{cases} \sum_{i=0}^{n} a_i(m,\ n)F(d,\ 2n-2i) + \sum_{j=1}^{m} b_j(m,\ n)G(d,\ j) \text{ if } m \geq n, \\ \sum_{i=0}^{m} a_i(n,\ m)G(d,\ m-i) + \sum_{j=1}^{n} b_j(n,\ m)F(d,\ 2j) \text{ if } m \leq n, \end{cases}$$

and

$$A(d;\ m,\ -n) = \begin{cases} (-1)^m \sum_{i=0}^{n} a_i(m,\ n)F(d,\ -2n+2i) + \sum_{j=1}^{m} b_j(m,\ n)(-1)^{m+j}G(d,\ j) \text{ if } m \geq n, \\ \sum_{i=0}^{m} a_i(n,\ m)(-1)^i G(d,\ m-i) + (-1)^m \sum_{j=1}^{n} b_j(n,\ m)F(d,\ -2j) \text{ if } m \leq n, \end{cases}$$

where $a_i(m,\ n), b_j(m,\ n)$ are defined as in Theorem 3.2.1.

We compute a list of formulas of $A(d;\ m,\ \pm 2n)$ for some positive integers m and integers n in [22, Table-5]. For example,

$$A(d;\ 2,\ 6) = 6\chi_4(d)d + 2(d^6 + 21d^4 - 1134d^2 - 3991)/945,$$
$$A(d;\ 3,\ -4) = \chi_4(d)(d^3 + 17d)/2 - (26d^2 + d^4)/3.$$

In the next result, we express higher dimensional Dedekind sums $S(d,\ c;\ m,\ n)$ for $c = 2$ and 4 as a linear sum $F(d,\ \pm 2j)$ and $A(d;\ j,\ \pm 2k)$, where the variables j and k depend on m and n.

Proposition 3.2.7 *For positive integers m, n, and d with odd d, we have*

$$S(d,\ 2;\ m,\ n) = \frac{(-1)^{\frac{m+n}{2}}}{2^n} \sum_{i=0}^{n} \binom{n}{i} (-1)^i F(d,\ m+n-2i), \tag{3.8}$$

and

$$S(d,\ 4;\ m,\ n) = \frac{1}{4^n} \sum_{n_1+n_2+n_3=n} \binom{n}{n_1, n_2, n_3} (-1)^{n_1+n_2} 3^{n_2} 2^{n_3} A(d;\ n_3,\ m-n+2n_1). \tag{3.9}$$

We have computed some special values of $S(d,\ c;\ m,\ n)$, where $4 \le m+n \le 10$ in [22, Table-3, 7], which are new formulas in the literature. Some of these are

$$S(d,\ 2;\ 5,\ 3) = d + \frac{3d^8 - 170d^6 + 3759d^4 - 53280d^2 - 63712}{113400},$$
$$S(d,\ 4;\ 6,\ 2) = d + \frac{3d^8 - 380d^6 + 12579d^4 + 20430d^2 + 340200\chi_4(d)d - 599632}{226800},$$
$$S(d,\ 4;\ 3,\ 3) = d + \frac{-2d^6 + 357d^4 + 3780\chi_4(d)d^3 - 3948d^2 + 3780\chi_4(d)d - 64447}{60480}.$$

Our next result shows the connection between the mean values $M(q,\ c,\ m,\ n)$ and the higher dimensional Dedekind sums.

Theorem 3.2.8 *Let $m+n$ be even for $m \ge 1$ and $n \ge 1$. Then*

$$M(q,\ c;\ m+1,\ n+1) = B_q(m,\ n) \sum_{\substack{1\le r_1\le m\\ 1\le r_2\le n}} S(r_1,\ r_2) \sum_{\substack{0\le j_1,\ j_2\le r_i-1\\ j_1+j_2=even}} C(j_1,\ j_2) R_q(c,\ j_1,\ j_2),$$

where $B_q(m,\ n) = \left(\frac{2\pi}{q}\right)^{m+n+2} \frac{(-1)^{\frac{3m+3n+2}{2}}}{8m!n!}$, $S(r_1,\ r_2) = \frac{r_1!r_2!}{2^{r_1+r_2}} \left\{ {m \atop r_1} \right\} \left\{ {n \atop r_2} \right\} (-1)^{r_1+r_2}$, and the notation $\left\{ {a \atop b} \right\} := \frac{1}{b!} \sum_{j=0}^{b} (-1)^{b-j} \binom{b}{j}\ j^a$ is called Stirling number of the second kind, $C(j_1,\ j_2) = \binom{r_1-1}{j_1}\binom{r_2-1}{j_2}$, and whenever j_1+j_2 is even, we have

$$R_q(c,\ j_1,\ j_2) = \sum_{\substack{l|q\\ l\neq 1}} \mu\left(\frac{q}{l}\right) \big(S(l,\ c;\ j_1,\ j_2) - S(l,\ c;\ j_1+2,\ j_2)$$
$$-S(l,\ c;\ j_1,\ j_2+2) + S(l,\ c;\ j_1+2,\ j_2+2)\big),$$

$M(q,\ c;\ 1,\ 1) = \frac{\pi^2}{2q^2} s(q,\ c)$, where $s(q,\ c)$ is the classical Dedekind sum.

As a consequence of Theorem 3.2.8, we list the mean values $M(q,\ c;\ m,\ n)$ whenever q is odd, $c = 2,\ 4$ and $2 \le m,\ n \le 8$ in [22, Table-4,7] respectively. We have already mentioned these formulas in (3.4). For $c = 1$, one can determine the mean values $M(q,\ 1;\ m,\ n)$ from Theorem 3.2.8 using only cotangent sums evaluated from Proposition 3.2.5.

In Section 3.3, we state and prove some results these are necessary ingredients in the proof of the main results of this chapter.

3.3 Preliminaries

The following two lemmas are useful in proving Theorem 3.2.8.

Lemma 3.3.1 ([57], Proposition 3) *Let $\chi(-1) = (-1)^{m+1}$, where χ is a Dirichlet character modulo q. Then*

$$L(m+1,\,\chi) = \frac{(-1)^m \pi^{m+1}}{2q^{m+1}m!}\sum_{k=1}^{q-1}\chi(k)\cot^{(m)}\frac{\pi k}{q}.$$

Lemma 3.3.2 *For $(ab,\,q) = 1$, we have*

$$\frac{2}{\varphi(q)}\sum_{\substack{\chi \\ \chi(-1)=-1}}\chi(a)\overline{\chi(b)} = \begin{cases} 1, & \text{if } b \equiv a \pmod q, \\ -1, & \text{if } b \equiv -a \pmod q, \\ 0, & \text{otherwise}, \end{cases} \tag{3.10}$$

and

$$\frac{2}{\varphi(q)}\sum_{\substack{\chi \\ \chi(-1)=1}}\chi(a)\overline{\chi(b)} = \begin{cases} 1, & \text{if } b \equiv \pm a \pmod q, \\ 0, & \text{otherwise}, \end{cases} \tag{3.11}$$

where χ is a Dirichlet character modulo q.

We get the identity (3.10) as a special case of the identity given in [56, p. 1541]. We deduce (3.11) using (3.10) and the orthogonality relation (1.13) among the Dirichlet characters modulo q.

Lemma 3.3.3 ([1], Lemma 2.1) *For any integer $n > 1$,*

$$\left(\frac{d}{dx}\right)^n \cot x = (2i)^n(\cot x - i)\sum_{k=1}^{n}\frac{k!}{2^k}\left\{ n \atop k \right\}(i\cot x - 1)^k,$$

where $\left\{ \cdot \atop \cdot \right\}$ is the Stirling number defined in Proposition 3.2.8.

We will next see three lemmas which will be uesd to establish formulas in Proposition 3.2.5.

Lemma 3.3.4 ([30], Girard-Waring formula, Eq: 8) *Let $f(x) = \sum\limits_{r=0}^{n} a_r x^{n-r}$, $a_0 = 1$, be a polynomial of degree n, and $\alpha_1, \ldots, \alpha_n$ be its zeros. Then, for $m \geq 0$, we have*

$$\sum_{r=1}^{n} \alpha_i^m = m \sum (-1)^{\sum\limits_{i=1}^{n} k_i} \frac{\left(\sum\limits_{i=1}^{n} k_i - 1\right)!}{\prod\limits_{i=1}^{n} k_i} \prod_{i=1}^{n} a_i^{k_i},$$

where the sum is over all non-negative integers k_i such that $k_1 + 2k_2 + \cdots + nk_n = m$.

Lemma 3.3.5 *Let d be an odd integer. Then*

(i) $\tan\left(\frac{\pi a}{d}\right)$ *is algebraic over $\mathbb{Q}$ for $0 \leq a \leq d-1$ and satisfies the polynomial*

$$P_d(x) = \operatorname{Im}\left((1+ix)^d\right) = \sum_{k=0}^{\frac{d-1}{2}} (-1)^k \binom{d}{2k+1} x^{2k+1} = \sum_{r=0}^{d} a_r x^{d-r}$$

of degree d.

(ii) $\cot\left(\frac{\pi a}{d}\right)$ *is algebraic over $\mathbb{Q}$ for $1 \leq a \leq d-1$ and satisfies the polynomial*

$$P_d(x) = \frac{(x+i)^d - (x-i)^d}{2id} = \sum_{r=1}^{d} a_r x^{d-r}$$

of degree $(d-1)$.

Proof. In [13, p. 10], Calcut proved that for $0 \leq a \leq d-1$, $\tan\left(\frac{\pi a}{d}\right)$ are the zeros of the polynomial

$$p(x) = \operatorname{Im}\left((1+ix)^d\right) = \sum_{k=0}^{\frac{d-1}{2}} (-1)^k \binom{d}{2k+1} x^{2k+1},$$

if d is odd. Rearranging the coefficients gives us the identity (i).

Since $\cot\theta = \frac{1}{\tan\theta}$ and $\cot\left(\frac{\pi a}{d}\right)$ are the zeros of the polynomial $x^d p\left(\frac{1}{x}\right)$, the identity in (ii) follows.

The result in (ii) is also mentioned in [56, Eq: 12].

Now, we find a polynomial for which the zeros look like $\sec\left(\frac{2\pi a}{d}\right)$.

Lemma 3.3.6 *Let d be an odd integer and a be an integer with $1 \le a \le d$. Then $\sec\left(\frac{2\pi a}{d}\right)$ are the zeros of the polynomial*

$$g_d(x) = -x^d + \frac{d}{2}\sum_{r=0}^{\frac{d-1}{2}} \frac{(-1)^r}{d-r}\binom{d-r}{r} 2^{d-2r}x^{2r} = \sum_{r=0}^{d} a_r x^{d-r}.$$

Proof. We deduce this lemma from the definition of the Chebyshev polynomial. The n^{th} degree Chebyshev polynomial of the first kind $T_n(x)$ is defined by the identity

$$T_n(\cos\theta) = \cos(n\theta), \quad n \in \mathbb{Z},\ \theta \in \mathbb{R}. \tag{3.12}$$

Alternatively, $T_n(x)$ can be expressed as

$$T_n(x) = \frac{n}{2}\sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^r}{n-r}\binom{n-r}{r}(2x)^{n-2r}. \tag{3.13}$$

From (3.12), we have $T_n\left(\cos\left(\frac{2\pi k}{n}\right)\right) = 1,\ k = 1,\ 2,\ldots,\ n$. This implies that $\cos\left(\frac{2\pi k}{n}\right)$ $(k = 1,\ 2,\ldots,\ n)$ are the zeros of $T_n - 1$. Hence, $\sec\left(\frac{2\pi k}{n}\right)$, $k = 1,\ 2,\ldots,\ n$, are the zeros of the polynomial

$$g_n(x) := x^n\left(T_n\left(\frac{1}{x}\right) - 1\right).$$

Therefore, for odd d, from (3.13) we write

$$g_d(x) = -x^d + \frac{d}{2}\sum_{r=0}^{\frac{d-1}{2}} \frac{(-1)^r}{d-r}\binom{d-r}{r} 2^{d-2r}x^{2r}.$$

3.4 Proof of results

Proof of Theorem 3.2.1. First, we note the identity

$$\sec 2x \cot^2 x = 1 + \sec 2x + \cot^2 x. \tag{3.14}$$

Now, we split the proof into two cases:

Case (a). $m \geq n$.

We claim that

$$\cot^{2n} x \sec^m 2x = \sum_{i=0}^{n} a_i(m,\, n)(\cot x)^{2(n-i)} + \sum_{j=1}^{m} b_j(m,\, n) \sec^j 2x. \tag{3.15}$$

For $m = n = 1$, this is same as the identity (3.14). For $n = 1$, and for any positive integer m, the following formula can be derived easily

$$\cot^2 x \sec^m 2x = \cot^2 x + 1 + \sec^m 2x + 2 \sum_{j=1}^{m-1} \sec^j 2x. \tag{3.16}$$

Since $a_0(m,\, 1) = a_1(m,\, 1) = b_m(m,\, 1) = 1$, and $b_k(m,\, 1) = 2$ for $1 \leq k \leq m-1$ and for all positive integers m, from (3.16), it is clear that for $n = 1$ and any positive integer m, (3.15) is true. Now, to prove our claim (3.15), we use induction on n. Let us assume that the result holds for all $k \leq n$, i.e.,

$$\cot^{2k} x \sec^m 2x = \sum_{i=0}^{k} a_i(m,\, k)(\cot x)^{2(k-i)} + \sum_{j=1}^{m} b_j(m,\, k) \sec^j 2x \tag{3.17}$$

for all $k \leq n < m$.

Now, for $k = n$, multiplying both sides of (3.17) by $\cot^2 x$, we get

$$A(n+1,\, m) = \sum_{i=0}^{n} a_i(m,\, n)(\cot x)^{2(n+1-i)} + \sum_{j=1}^{m} b_j(m,\, n) \cot^2 x \sec^j 2x, \tag{3.18}$$

where $A(m,\ k) := \cot^{2k} x \sec^m 2x$ for all $m,\ k$. Using (3.16) in (3.18), we write

$$A(m,\ n+1) = \sum_{i=0}^{n} a_i(m,\ n)(\cot x)^{2(n+1-i)} + \sum_{j=1}^{m} b_j(m,\ n)(\cot^2 x + 1) + \sum_{j=1}^{m} b_j(m,\ n)\left(\sec^j 2x + 2\sum_{l=1}^{j-1} \sec^l 2x\right).$$

Rearranging the coefficients of powers of $\cot^2 x$ and $\sec 2x$, we get

$$A(m,\ n+1) = \sum_{i=0}^{n-1} a_i(m,\ n)\cot^{2(n+1-i)} x + \left(a_n(m,\ n) + \sum_{j=1}^{m} b_j(m,\ n)\right)\cot^2 x + \sum_{j=1}^{m} b_j(m,\ n) + \sum_{j=1}^{m-1}\left(b_j(m,\ n) + 2\sum_{l=j+1}^{m} b_l(m,\ n)\right)\sec^j 2x + b_m(m,\ n)\sec^m 2x.$$

From the definition of $a_i(m,\ n)$, we have $a_i(m,\ n) = a_i(m,\ n-1)$ for $0 \le i \le n-2$. Thus, $a_i(m,\ n+1) = a_i(m,\ n)$, for $0 \le i \le n-1$. Now,

$$\begin{aligned} a_n(m,\ n) + \sum_{j=1}^{m} b_j(m,\ n) &= 2 + 2c(-1,\ n-1)(m-1) + \sum_{j=0}^{n-3} c(j,\ n-1)2^{2+j}\binom{m}{j+2} \\ &\quad + \sum_{j=1}^{m-1} b_j(m,\ n) \\ &= 2 + 2(m-1)(c(-1,\ n-1) + c(-1,\ n)) + \sum_{j=0}^{n-3}(c(j,\ n-1) \\ &\quad + c(j,\ n))2^{2+j}\binom{m}{j+2}. \end{aligned}$$

Since $c(-1,\ n-1) + c(-1,\ n) = 1$, we have

$$2 + 2(m-1)(c(-1,\ n-1) + c(-1,\ n)) = 2m = 2(c(-1,\ n-1) + c(-1,\ n))\binom{m}{1}.$$

Hence,

$$a_n(m,\ n) + \sum_{j=1}^{m} b_j(m,\ n) = a_n(m,\ n+1).$$

Now using the definition of $b_j(m, n)$, we get

$$\sum_{j=1}^{m} b_j(m, n) = 1 + \sum_{j=1}^{m-1} b_j(m, n) = a_{n+1}(m, n+1).$$

Now, the part $2\sum_{l=j+1}^{m} b_l(m, n)$ in the coefficients of $\sec^j 2x$ can be simplified as

$$\begin{aligned} 2\sum_{l=j+1}^{m} b_l(m, n) &= 2b_m(m, n) + 2\sum_{l=j+1}^{m-1}\sum_{i=-1}^{n-2} c(i, n)2^{i+2}\binom{m-l}{i+1} \\ &= 2 + 4c(-1, n)(m-j-1) + \sum_{i=0}^{n-2} c(i, n)2^{i+3}\sum_{l=j+1}^{m-1}\binom{m-l}{i+1}. \end{aligned}$$

Now, replacing $c(i, n)$ by $c(i+1, n+1) - c(i+1, n)$ for $0 \le i \le n-2$ and using the 'hockey-stick' identity $\sum\limits_{i=k}^{n}\binom{i}{k} = \binom{n+1}{k+1}$ for $k \le n$, we get

$$\sum_{i=0}^{n-2} c(i, n)2^{i+3}\sum_{l=j+1}^{m-1}\binom{m-l}{i+1} = \sum_{j=1}^{n-1} c(j, n+1)2^{j+2}\binom{m-k}{j+1} - \sum_{j=1}^{n-1} c(j, n)2^{j+2}\binom{m-k}{j+1}.$$

This gives

$$\begin{aligned} b_j(m, n) + 2\sum_{l=j+1}^{m} b_l(m, n) &= 2 + 4c(-1, n)(m-j-1) + 2c(-1, n) + 4c(0, n)(m-k) \\ &\quad -2c(-1, n+1) - 4c(0, n+1)(m-k) + b_k(m, n+1). \end{aligned}$$

Using the definition of the sequences $c(-1, k)$ and $c(0, k)$, for any positive integer k, we get

$$\begin{aligned} &2 + 4c(-1, n)(m-j-1) + 2c(-1, n) + 4c(0, n)(m-k) - 2c(-1, n+1) \\ &\qquad - 4c(0, n+1)(m-k) = 0. \end{aligned}$$

Hence,

$$b_j(m, n) + 2\sum_{l=j+1}^{m} b_l(m, n) = b_k(m, n+1).$$

Thus, by mathematical induction, we conclude (3.15).

Case (b). $n \geq m$.

In this case, we claim that the following holds:

$$\cot^{2n} x \sec^m 2x = \sum_{i=0}^{m} a_i(n,\ m)(\sec 2x)^{m-i} + \sum_{j=1}^{n} b_j(n,\ m)\cot^{2j} x.$$

Again, for $m = n = 1$, this holds as in the previous case. Now, for $m = 1$ and for any n, we have

$$\cot^{2n} x \sec 2x = \sec^2 2x + 1 + \cot^{2n} x + 2\sum_{j=1}^{n-1} \cot^{2j} x.$$

Using the induction principle again, as done in the previous case, we conclude our claim. This completes the proof of this theorem.

Proof of Theorem 3.2.4. (i). By using Theorem 3.2.1, we rewrite the series $\Psi_s^1(m,\ 2n, \tau)$ as a rational linear combination of $\Psi_s^1(j,\ 0,\ \tau)$ and $\Psi_s^1(0,\ 2i,\ \tau)$, i.e.,

$$\Psi_s^1(m,\ 2n,\ \tau) = \begin{cases} \sum\limits_{i=0}^{n} a_i(m,\ n)\Psi_s^1(0,\ 2(n-i),\ \tau) + \sum\limits_{j=1}^{m} b_j(m,\ n)\Psi_s^1(j,\ 0,\ \tau), \text{ if } m \geq n, \\ \sum\limits_{i=0}^{m} a_i(n,\ m)\Psi_s^1(m-i,\ 0,\ \tau) + \sum\limits_{j=1}^{n} b_j(n,\ m)\Psi_s^1(0,\ 2j,\ \tau), \text{ if } m \leq n, \end{cases} \tag{3.19}$$

and

$$\Psi_s^1(m,\ -2n,\ \tau) = \begin{cases} (-1)^m \sum\limits_{i=0}^{n} a_i(m,\ n)\Psi_s^1(0,\ -2(n-i),\ \tau) + \sum\limits_{j=1}^{m} b_j(m,\ n)(-1)^{m+j}\Psi_s^1(j,\ 0,\ \tau), \text{ if } m \geq n, \\ \sum\limits_{i=0}^{m} a_i(n,\ m)(-1)^i\Psi_s^1(m-i,\ 0,\ \tau) + \sum\limits_{j=1}^{n} b_j(n,\ m)(-1)^m\Psi_s^1(0,\ -2j,\ \tau), \text{ if } m \leq n. \end{cases} \tag{3.20}$$

Note that, in (3.19), the highest power of $\sec 2x$ is m and the highest power of $\cot x$ is $2n$. On putting $\cot^{2n} x = \sum_{j=0}^{n} \binom{n}{j} \csc^{2j} x$ in the series $\Psi_s^1(0,\ 2n,\ \tau)$, we find that the highest power of $\csc x$ is $2n$. Similarly, in equation (3.20), the highest power of $\sec x$ is $m + 2n$.

Finally, by applying Theorem 3.1.3, we conclude the result.

(ii). We start with the following trigonometric identity

$$\begin{aligned}\cot^m x \cot^n(2x) &= \frac{\cot^m x}{2^n}(\cot x - \tan x)^n \\ &= \frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}(-1)^i(\cot x)^{m+n-2i}. \end{aligned}\tag{3.21}$$

On writing $4\cot 4x = \cot x - 3\tan x - \tan x \sec 2x$, and using multinomial theorem, we get

$$\cot^m x \cot^n 4x = \frac{1}{4^n}\sum_{n_1+n_2+n_3=n}\binom{n}{n_1,\ n_2,\ n_3}(-1)^{n_1+n_2}3^{n_2}2^{n_3}\sec^{n_3} 2x(\cot x)^{m-n+2n_1}.\tag{3.22}$$

We expand the series $\Psi_s^2(m,\ l,\ n,\ \tau)$, using (3.21) and (3.22), to get

$$\Psi_s^2(m,\ l,\ n,\ \tau) = \frac{1}{2^l 4^n}\sum_{i=0}^{l}\sum_{n_1+n_2+n_3=n}\binom{l}{i}\binom{n}{n_1,\ n_2,\ n_3}(-1)^{n_1+n_2+i}3^{n_2}2^{n_3}\Psi_s^1(n_3,\ m+l-n+2n_1-2i,\ \tau).$$

For $0 \le i \le l$ and $0 \le n_1,\ n_3 \le n$, $\max(n_3,\ m+l-n+2n_1-2i) = m+n+l$. Hence, using the convergence condition of Ψ_s^1, we conclude the result.

Lastly, let s be a positive integer such that Ψ_s^i's are convergent for $i = 1,\ 2$. From the proof of (i) and (ii), one can easily see that Ψ_s^2 can be expressed as a rational linear combination of the Ψ_s^1's. From (3.20), $\Psi_s^1(m,\ n,\ \tau)$ is a rational linear combination of the $\psi_s^{a,\ b}$'s, where $a,\ b$ can be chosen suitably. Hence, the irrationality of Ψ_s^i, $i = 1,\ 2$, follows from the irrationality of $\psi_s^{a,\ b}$. Therefore, our result follows from the last part of Theorem 3.1.3.

Proof of Proposition 3.2.5. The results can be proved by applying Lemma 3.3.4 to Lemmas 3.3.5 and 3.3.6.

Proof of Proposition 3.2.7. We can establish the identity (3.8) from (3.21) and the identity (3.9) from (3.22).

Proof of Theorem 3.2.8. By using Lemma 3.3.1, we write

$$M(q,\ c;\ m+1,\ n+1) = A_q(m,\ n)\frac{2}{\varphi(q)}\sum_{\substack{\chi(\bmod q)\\ \chi(-1)=(-1)^{m+1}}}\chi(c)\sum_{k=1}^{q-1}\sum_{l=1}^{q-1}\chi(k)\overline{\chi(l)}\cot^{(m)}\left(\frac{\pi k}{q}\right)\cot^{(n)}\left(\frac{\pi l}{q}\right),$$

where $A_q(m,\, n) = \frac{(-1)^{m+n}\pi^{m+n+2}}{4q^{m+n+2}m!n!}$. Now, by interchanging the order of summations, we get

$$\frac{M(q,\, c;\, m+1,\, n+1)}{A_q(m,\, n)} = \sum_{l=1}^{q-1}\sum_{k=1}^{q-1} \cot^{(m)}\left(\frac{\pi k}{q}\right)\cot^{(n)}\left(\frac{\pi l}{q}\right)\left[\frac{2}{\varphi(q)}\sum_{\substack{\chi\,(\mathrm{mod\, q})\\ \chi(-1)=(-1)^{m+1}}} \chi(ck)\overline{\chi(l)}\right].$$

Using Lemma 3.3.2 and removing the congruence condition from the sums, we have

$$\frac{M(q,\, c;\, m+1,\, n+1)}{A_q(m,\, n)} = \begin{cases} L_1(m,\, n) - L_2(m,\, n), & \text{if } m \text{ is even,}\\ L_1(m,\, n) + L_2(m,\, n), & \text{if } m \text{ is odd,}\end{cases}$$

where

$$L_1(m,\, n) = \sum_{\substack{k=1\\ (k,\, q)=1}}^{q-1} \cot^{(m)}\left(\frac{\pi k}{q}\right)\cot^{(n)}\left(\frac{c\pi k}{q}\right),$$

and

$$L_2(m,\, n) = \sum_{\substack{k=1\\ (k,\, q)=1}}^{q-1} \cot^{(m)}\left(\frac{\pi k}{q}\right)\cot^{(n)}\left(\frac{-c\pi k}{q}\right).$$

Hence, we have

$$\frac{M(q,\, c;\, m+1,\, n+1)}{A_q(m,\, n)} = \begin{cases} \sum\limits_{\substack{k=1\\ (k,\, q)=1}}^{q-1} \cot^{(m)}(\frac{\pi ck}{q})\left[\cot^{(n)}(\frac{\pi ck}{q}) - \cot^{(n)}(\frac{-\pi ck}{q})\right], & \text{if } m \text{ even,}\\ \sum\limits_{\substack{k=1\\ (k,\, q)=1}}^{q-1} \cot^{(m)}(\frac{\pi ck}{q})\left[\cot^{(n)}(\frac{\pi ck}{q}) + \cot^{(n)}(\frac{-\pi ck}{q})\right], & \text{if } m \text{ odd.}\end{cases}$$

Now, using the identity $\cot^{(m)}(-\theta) = (-1)^{m-1}\cot^{(m)}(\theta)$ for any real θ, we get

$$M(q,\, c;\, m+1,\, n+1) = \begin{cases} 2A_q(m,\, n) \sum\limits_{\substack{k=1\\ (k,\, q)=1}}^{q-1} \cot^{(m)}(\frac{\pi k}{q})\cot^{(n)}(\frac{\pi ck}{q}), & \text{if } m+n \text{ is even,}\\ 0, & \text{otherwise.}\end{cases}$$

This gives $M(q,\, c;\; 1,\; 1) = \frac{\pi^2}{2q^2} \sum\limits_{\substack{k=1\\ (k,\, q)=1}}^{q-1} \cot\left(\frac{\pi k}{q}\right)\cot\left(\frac{\pi ck}{q}\right)$, for $m = n = 0$.

For $m,\; n \geq 1$, we expand the derivatives of cot function in terms of the cot function by using Lemma 3.3.3. We modify the formula in Lemma 3.3.3 as

$$\cot^{(m)} x = -i(2i)^m \csc^2 x \sum_{k=1}^{n} \frac{k!}{2^k} \left\{ {n \atop k} \right\} (-1)^k (1 - i\cot x)^{k-1}.$$

Using the above expression, for $m,\ n \geq 1$ with $m+n$ even, we get

$$M(q,\ c;\ m+1,\ n+1) = 2A_q(m,\ n) \sum_{\substack{k=1 \\ (k,\ q)=1}}^{q-1} -(2i)^{m+n} \csc^2\left(\frac{\pi k}{q}\right) \csc^2\left(\frac{\pi ck}{q}\right)$$
$$\times \sum_{r_1=1}^{m} \sum_{r_2=1}^{n} S(r_1,\ r_2) \left(\sum_{\substack{0 \leq j_1 \leq r_i - 1 \\ i=1,2 \\ j_1+j_2 \ even}} C(j_1,\ j_2)(-1)^{\frac{j_1+j_2}{2}} \cot^{j_1}\left(\frac{\pi k}{q}\right) \cot^{j_2}\left(\frac{\pi ck}{q}\right) \right).$$

By interchanging the order of summations, we have

$$M(q,\ c;\ m+1,\ n+1) = B_q(m,\ n) \sum_{r_1=1}^{m} \sum_{r_2=1}^{n} S(r_1,\ r_2) \sum_{\substack{0 \leq j_1 \leq r_i - 1 \\ i=1,\ 2 \\ j_1+j_2=even}} C(j_1,\ j_2) R'_q(c,\ j_1,\ j_2), \tag{3.23}$$

where

$$R'_q(c,\ j_1,\ j_2) := (-1)^{\frac{j_1+j_2}{2}} \sum_{\substack{k=1 \\ (k,\ q)=1}}^{q-1} \csc^2\left(\frac{\pi k}{q}\right) \csc^2\left(\frac{\pi ck}{q}\right) \cot^{j_1}\left(\frac{\pi k}{q}\right) \cot^{j_2}\left(\frac{\pi ck}{q}\right).$$

Removing the coprimality condition of $R'_q(c,\ j_1,\ j_2)$, we have

$$R'_q(c,\ j_1,\ j_2) = (-1)^{\frac{j_1+j_2}{2}} \sum_{\substack{l \mid q \\ l \neq 1}} \mu\left(\frac{q}{l}\right) \sum_{k=1}^{l-1} \left(\csc \frac{\pi k}{l}\right)^2 \left(\csc \frac{\pi ck}{l}\right)^2 \left(\cot \frac{\pi k}{l}\right)^{j_1} \left(\cot \frac{\pi ck}{l}\right)^{j_2},$$

where μ is the Möbius function. Now replacing $\csc^2$ by $1+\cot^2$ and simplifying we get $R'_q(c,\ j_1,\ j_2) = R_q(c,\ j_1,\ j_2)$.

This thesis does not contain any Conclusion Chapter

Summary

In this thesis, we study few problems from the theory of Dirichlet L-functions. It consists of two articles. The first one concerns the distribution of the values of a linear combination of Dirichlet L-functions along the critical line. The other article concerns formulas for quadratic twisted mean values of Dirichlet L-functions at positive integers and values of trigonometric Dirichlet L-series.

In Chapter 1 we go through the general background in the theory of the Riemann zeta function and Dirichlet L-functions. In particular, we focus on some of their properties. For example, we study their zeros, mean values, mollification, and more importantly Selberg's central limit theorem.

Chapter 2 contains the major part of this thesis and we present here the first paper[23]. In 2016, S. M. Gonek and A. Ivić [28] showed that the Lebesgue measure of the subset of the real line on which the Hardy's Z-function takes positive values, and respectively negative values, in the interval $(T, 2T]$ is $\gg T$, for all large enough T. The aim of Chapter 2 is basically to show that analogous results can be shown for any real linear combination and some special type of complex linear combination of Z-functions associated with Dirichlet L-functions for different characters. Another aim of this Chapter is a generalization of a result of A. A. Karatsuba [45] on the lower bound for the number of odd order zeros of any real linear combination of Z-functions associated with Dirichlet L-functions for different characters. We generalize this result to the case of complex-linear combination.

In Chapter 3, we present the second paper [21], where we obtain a connection between quadratic twisted mean values of Dirichlet L-functions at positive integers and D. Zagier's higher dimensional Dedekind sums. We establish some new formulas for higher dimensional Dedekind sums and thereby derived some explicit formulas of quadratic twisted mean values of Dirichlet L-function. Along the way, we investigate some special cases of a question of A. Straub on values of trigonometric Dirichlet L- series.

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Thesis Highlight

Name of the Student: Mithun Kumar Das
Name of the CI/OCC: Harish-Chandra Research Institute
Enrolment No.: MATH08201304005
Thesis Title: Some Topics on Dirichlet L-functions
Discipline: Mathematical Sciences
Sub-Area of Discipline: Analytic Number Theory
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The Hardy Z-function is the real-valued function is defined to study the Riemann zeta function. In 2016, Gonek and Ivic showed that the Lebesgue measure of the subset of the real line on which the Hardy Z-function takes positive values, and respectively negative values, in the interval [T, 2Tis at least cT, for some positive constant c. We have shown that the analogous results hold for any real linear combination and some special type of complex linear combination of Z-functions associated with Dirichlet L-functions for different characters. Moreover, a result by A. A. Karatsuba on the lower bound for the number of odd order zeros of any real linear combination of Z-functions associated with Dirichlet L-functions for different characters has been extended for any complex-linear combination of Z-functions associated with Dirichlet L-functions.

The general formula for the quadratic twisted mean values of Dirichlet L-functions is an open question and it is posed by Louboutin. A connection between quadratic twisted mean values of Dirichlet L-functions at positive integers and Zagier's higher dimensional Dedekind sums has been established. Moreover, a few new formulas for higher dimensional Dedekind sums have been obtained and thereby derived some explicit formulas of quadratic twisted mean values of Dirichlet L-function. Furthermore, some special cases of a question of Straub on values of trigonometric Dirichlet L-series have been investigated.