SOME PROBLEMS ON ARITHMETIC FUNCTIONS

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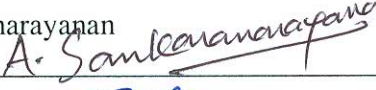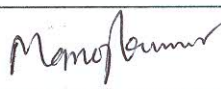
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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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To My Parents and My Teachers

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Contents

Summary

This thesis consists of three chapters, each of which studies a problem on the asymptotic behaviour of the partial sums of arithmetic functions. The main results of the first chapter extends to arithmetical functions in several variables the asymptotic formulae with error terms obtained by B. Saha for the partial sums of an arithmetic function with an absolutely convergent Ramanujan expansion, assuming suitable decay conditions for the coefficients of this expansion.

The principal result of the second chapter of this thesis is an explicit asymptotic formula for the partial sums of a real valued multiplicative function subject to certain conditions which may, intuitively, be viewed as requiring that the function takes more non-negative values than negative values on the set of prime numbers.

Finally, in the third and final chapter of this thesis we obtain an asymptotic formula for the partial sums of the arithmetical function $n \mapsto \frac{\delta_k^r(n)}{n^{\sigma+r}}$, where $\delta_k(n)$ denotes the greatest divisor of n which is coprime to a given integer k, σ is any real number and $r \geq 1$ is an integer. This result extends and improves an old result of P.N. Ramachandran on this subject.

List of Key Notations

C   The set of complex numbers

R   The set of real numbers

Q   The set of rational numbers

Z   The set of integers

N   The set of natural numbers, that is, integers ≥ 1.

$\Omega(n)$ The number of prime divisors n with multiplicity.

$\omega(n)$ The number of distinct prime divisors of n.

$\phi(n)$ Euler's totient function

$\Lambda(n)$ The Von Mangoldt function

$\mu(n)$ The Möbius function

$\tau(n)$ The number of divisors of n.

$\zeta(s)$ The Riemann zeta function

CHAPTER 1

Ramanujan expansions and partial sums

In this chapter we derive asymptotic formulae for the partial sums of arithmetic functions of two or more variables with absolutely convergent Ramanujan expansions, assuming suitable decay conditions on the coefficients of these expansions.

1.1 Introduction

For any integers $q \geq 1$ and $n \geq 1$, the *Ramanujan sum $c_q(n)$*, named for S. Ramanujan, is defined by the relation

$$c_q(n) = \sum_{a \in (\mathbb{Z}/q\mathbb{Z})^*} e^{\frac{2\pi i a n}{q}} . \tag{1.1}$$

In the pioneering work [24], Ramanujan obtained expansions in terms of these sums for a variety of arithmetic functions. For this reason an arithmetic function of one variable, that is, a map f from $\mathbf{N}$ to $\mathbf{C}$, is said to have a *Ramanujan expansion* if there exists a sequence of complex numbers $\{a_q\}_{q \geq 1}$ such that for each integer $n \geq 1$ the series

$$\sum_{q \geq 1} a_q c_q(n) \tag{1.2}$$

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converges to $f(n)$. The sequence $\{a_q\}_{q \geq 1}$ is then called a sequence of *Ramanujan coefficients* for f. If the series (1.2) is absolutely convergent for all integers $n \geq 1$, the arithmetic function f is said to have an absolutely convergent Ramanujan expansion. The following Ramanujan expansions are given in [24] for real $s > 0$:

$$\frac{\sigma_s(n)}{n^s} = \zeta(s+1) \sum_{q=1}^{\infty} \frac{c_q(n)}{q^{s+1}}, \qquad\qquad \frac{\phi_s(n)}{n^s} = \frac{1}{\zeta(s+1)} \sum_{q=1}^{\infty} \frac{\mu(q)}{\phi_{s+1}(q)} c_q(n),$$

$$r(n) = \pi \sum_{q=1}^{\infty} \frac{(-1)^{q-1}}{2q-1} c_{2q-1}(n), \qquad\qquad 0 = \sum_{q=1}^{\infty} \frac{c_q(n)}{q}.$$

where $\sigma_s(n) = \sum_{d|n} d^s$, and $\phi_s(n) = n^s \prod_{p|n} (1 - 1/p^s)$ for any real s, ζ is the Riemann zeta function and $r(n)$ is the number of representations of n as the sum of two squares. Note that the first of these expansions is absolutely convergent and the last one shows that a given arithmetic function may admit more than one sequence of Ramanujan coefficients.

The study of the existence of Ramanujan expansions for arithmetic functions and their convergence properties has been carried out by a number of authors; see [4, 10, 12, 25, 26, 31]. The works [18, 20] of L.G. Lucht and M.R. Murty respectively are recent surveys on Ramanujan expansions.

R. Bellman [2] appears to have been among the earliest authors to observe that the existence of an absolutely convergent Ramanujan expansion for an arithmetic function provides an alternative to the standard methods for studying the growth of the partial sums of this function. Indeed, if an arithmetic function f admits (1.2) as its Ramanujan expansion and if this series is absolutely convergent then we have

$$\sum_{1 \leq n \leq N} f(n) = \sum_{1 \leq n \leq N} \sum_{q \geq 1} a_q c_q(n) = \sum_{q \geq 1} a_q \sum_{1 \leq n \leq N} c_q(n) . \tag{1.3}$$

When $q = 1$ we have $\sum_{1 \leq n \leq N} c_q(n) = N$. However, when $q \geq 2$ there is significant cancella-

tion in the sum over n so that one may hope to obtain the asymptotic formula

$$\sum_{1 \leq n \leq N} f(n) \sim a_1 N \text{ as } N \to +\infty. \tag{1.4}$$

In the recent literature this idea has been revived by several authors, with stronger assumptions on the Ramanujan coeffcients, to obtain asymptotic formulae with remainder terms for the partial sums of various arithmetic functions and their convolutions. In particular, B. Saha's Main Theorem in [25] and Theorem 4 in [26] give asymptotic formulae with remainder terms for $\sum_{1 \leq n \leq N} f(n)$ for an arithmetic function f under the assumption that f has a Ramanujan expansion whose coefficients a_q satisfy respectively the conditions

$$\left| a_q \right| \ll \frac{1}{q^{1+\delta}} \text{ and } \left| a_q \right| \ll \frac{1}{q \log^\alpha q} \tag{1.5}$$

for some $\delta > 0$ and $\alpha > 2$ and all $q \geq 1$, which are easily seen to guarantee the absolute convergence of this expansion.

All of the aforementioned articles, however, are concerned with Ramanujan expansions for arithmetic functions in one variable. It appears that very few results are known on Ramanujan expansions of arithmetic functions of two or more variables, with the exception of the recent papers of N. Ushiroya [36] and L. Tóth [35].

Our main purpose in this chapter is to extend the results of B. Saha cited above to arithmetic functions of two variables with absolutely convergent Ramanujan expansions as considered by Ushiroya [36]. We begin by giving the precise statements of our results in the following section. These statements correct various errors in the statements of the results in [30]. The proofs of our results are given in Section 1.4 using preliminaries recalled in Section 1.3. This is followed by Section 1.5, where we describe how our results may be extended to arithmetic functions in more than two variables. Finally, in Section 1.6 we discuss a simple and elementary method that yields better results in specific cases than our general Theorem 1.2.1 given in the following section.

1.2 Results

Here on, for the sake of brevity, we will supress the lower end point of a summation range when this lower end point is 1. Thus we will often write $\sum_{n \le x}$ in place of $\sum_{1 \le n \le x}$ etc..

An arithmetic function of two variables is a map $f : \mathbf{N} \times \mathbf{N} \mapsto \mathbf{C}$. Such a function f is said to have an *absolutely convergent Ramanujan expansion* if there exists a family of complex numbers $\{a_{q_1,q_2}\}$ with (q_1, q_2) varying over $\mathbf{N} \times \mathbf{N}$ such that for each $(n_1, n_2) \in \mathbf{N} \times \mathbf{N}$ the double series

$$\sum_{(q_1,q_2) \in \mathbf{N} \times \mathbf{N}} a_{q_1,q_2} \, c_{q_1}(n_1) c_{q_2}(n_2) \tag{1.6}$$

is absolutely convergent and its sum is $f(n_1, n_2)$. The family $\{a_{q_1,q_2}\}$, with $(q_1, q_2) \in \mathbf{N} \times \mathbf{N}$, is then called a family of *Ramanujan coefficients* for f. Note that unlike in the single variable case we only consider absolutely convergent Ramanujan expansions in the two variable, and later, in the several variable case.

Our first result is an extension of the Main Theorem of [25] to arithmetic functions of two variables. Theorem 1.5.1 of Section 1.5 gives the several variable version of this result.

Theorem 1.2.1. *Suppose that $\{a_{q_1,q_2}\}$, with $(q_1, q_2) \in \mathbf{N} \times \mathbf{N}$, is a family of complex numbers satisfying the condition*

$$\left| a_{q_1,q_2} \right| \ll \frac{1}{[q_1, q_2]^{1+\delta}} \tag{1.7}$$

for some $\delta > 0$ and all $(q_1, q_2) \in \mathbf{N} \times \mathbf{N}$, where $[q_1, q_2]$ denotes the least common multiple of q_1 and q_2. Then the series (1.6) is absolutely convergent for every $(n_1, n_2) \in \mathbf{N} \times \mathbf{N}$. If moreover $\{a_{q_1,q_2}\}$ is a family of Ramanujan coefficients for an arithmetic function of two variables f then for any integer $N \ge 1$ we have

$$\sum_{n_1, n_2 \le N} f(n_1, n_2) = \begin{cases} a_{1,1} N^2 + O(N^{2-\delta} (\log eN)^{\frac{14-7\delta}{2}}) & \text{if } \delta \le 1, \\ a_{1,1} N^2 + O(N) & \text{if } \delta > 1. \end{cases} \tag{1.8}$$

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The implied constants in the $\ll$ and O symbols above depend only on δ. Here and elsewhere in this thesis e denotes the familiar numerical constant $2.712\ldots$.

In the following theorem we weaken the decay condition on the Ramanujan coefficients. This result extends Theorem 4 of [26] to arithmetic functions of two variables. Theorem 1.5.2 of Section 1.5 gives the several variable version of this result.

Theorem 1.2.2. *Suppose that* $\{a_{q_1,q_2}\}$, *with* $(q_1,q_2) \in \mathbf{N} \times \mathbf{N}$, *is a family of complex numbers satisfying the condition*

$$\left| a_{q_1,q_2} \right| \ll \frac{1}{[q_1,q_2](\log e[q_1,q_2])^\gamma} \tag{1.9}$$

for some real number $\gamma > 8$ *and all* $(q_1,q_2) \in \mathbf{N} \times \mathbf{N}$. *Then the series* (1.6) *is absolutely convergent for every* $(n_1,n_2) \in \mathbf{N} \times \mathbf{N}$. *If moreover* $\{a_{q_1,q_2}\}$ *is a family of Ramanujan coefficients for an arithmetic function of two variables* f *then for any integer* $N \geq 1$ *we have*

$$\sum_{n_1,n_2 \leq N} f(n_1,n_2) = a_{1,1}N^2 + O\left(\frac{N^2}{(\log eN)^{\gamma-8}} \right).$$

The implied constants in the $\ll$ and O symbols in the statement of Theorem 1.2.2 depend only on γ.

B. Saha applies the Main Theorem of [25] to the Ramanujan expansions of the (single variable) functions $n \mapsto \frac{\sigma_s(n)}{n^s}$ and $n \mapsto \frac{\phi_s(n)}{n^s}$ to obtain the Corollaries 1 and 2 of [25]. We may similarly apply Theorem 1.2.1 to obtain analogous results in the two variable case. More precisely, Examples 3.8 and 3.11 of Ushiroya [36] give the following Ramanujan expansions for arithmetic functions in two variables, where we have put $\delta - 1$ in place of s used in [36]. For any real number $\delta > 0$ we have

$$\frac{\sigma_{\delta-1}((n_1,n_2))}{(n_1,n_2)^{\delta-1}} = \zeta(1+\delta) \sum_{q_1,q_2=1}^{\infty} \frac{1}{[q_1,q_2]^{1+\delta}} c_{q_1}(n_1)c_{q_2}(n_2) \tag{1.10}$$

and

$$\frac{\phi_{\delta-1}((n_1,n_2))}{(n_1,n_2)^{\delta-1}} = \frac{1}{\zeta(1+\delta)} \sum_{q_1,q_2=1}^{\infty} \frac{\mu([q_1,q_2])}{\phi_{1+\delta}([q_1,q_2])} c_{q_1}(n_1)c_{q_2}(n_2), \tag{1.11}$$

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where (n_1, n_2) denotes the greatest common divisor of n_1 and n_2.

Let

$$f_1(n_1, n_2) = \frac{\sigma_{\delta-1}((n_1, n_2))}{(n_1, n_2)^{\delta-1}} \text{ and } f_2(n_1, n_2) = \frac{\phi_{\delta-1}(n_1, n_2)}{((n_1, n_2))^{\delta-1}}.$$

We assert that Ramanujan coefficients of functions f_1 and f_2 in the expansions given by (1.10) and (1.11) satisfy the condition (1.7) of Theorem 1.2.1. This is plainly the case for f_1 by (1.10), while for f_2 our assertion follows from (1.11) and the inequality

$$\phi_{1+\delta}(n) = n^{1+\delta} \prod_{p|n} \left(1 - \frac{1}{p^{1+\delta}}\right) \geq n^{1+\delta} \prod_{p \geq 2} \left(1 - \frac{1}{p^{1+\delta}}\right) = \frac{n^{1+\delta}}{\zeta(1+\delta)}. \qquad (1.12)$$

Consequently, we immediately obtain the following corollaries to Theorem 1.2.1. However, one may obtain sharper forms of these corollaries by a simple and elementary method, described in Section 1.6. It is also shown in that section that this method also yields sharper forms of Corollaries 1 and 2 of [25].

Corollary 1.2.3. *Let $\delta > 0$ be a given real number. Then for any integer $N \geq 1$ we have*

$$\sum_{n_1, n_2 \leq N} \frac{\sigma_{\delta-1}((n_1, n_2))}{(n_1, n_2)^{\delta-1}} = \begin{cases} \zeta(1+\delta)N^2 + O(N^{2-\delta}(\log eN)^{\frac{14-7\delta}{2}}) & \text{if } \delta \leq 1, \\ \\ \zeta(1+\delta)N^2 + O(N) & \text{if } \delta > 1. \end{cases}$$

Corollary 1.2.4. *Let $\delta > 0$ be a given real number. Then for any integer $N \geq 1$ we have*

$$\sum_{n_1, n_2 \leq N} \frac{\phi_{\delta-1}((n_1, n_2))}{(n_1, n_2)^{\delta-1}} = \begin{cases} \frac{N^2}{\zeta(1+\delta)} + O(N^{2-\delta}(\log eN)^{\frac{14-7\delta}{2}}) & \text{if } \delta \leq 1, \\ \\ \frac{N^2}{\zeta(1+\delta)} + O(N) & \text{if } \delta > 1. \end{cases}$$

1.3 Preliminaries

We record here some results which will be used to prove the theorems stated in the preceding section. We begin with the well-known partial summation formula :

Proposition 1.3.1. *Let a be an arithmetic function, $x \geq 1$ a real number and let $f : [1,x] \to \mathbf{C}$ be a complex valued function with continuous derivative f' on $[1,x]$. Then we have that*

$$\sum_{n \leq x} a(n)f(n) = A(x)f(x) - \int_1^x A(t)f'(t)dt,$$

where

$$A(t) = \sum_{n \leq t} a(n).$$

Proof. See the proof of Theorem 1.14 of [3]. □

The following pair of corollaries put Proposition 1.3.1 into easily applicable forms.

Corollary 1.3.2. *Let α, β and δ be real numbers with $\beta \geq 0$ and let $\{a(q)\}_{q \geq 1}$ be a sequence of real numbers ≥ 0 satisfying*

$$\sum_{q \leq t} a(q) \ll t^{1+\alpha}(\log t)^\beta \tag{1.13}$$

for all $t \geq 2$. Then for any real numbers V, U with $1 \leq V \leq U$ we have

$$\sum_{V < q \leq U} \frac{a(q)}{q^{1+\delta}} \ll \begin{cases} U^{\alpha-\delta}(\log U)^\beta & \text{for } \delta < \alpha, \\ (\log U)^{\beta+1} & \text{for } \delta = \alpha, \\ V^{\alpha-\delta}(\log eV)^\beta & \text{for } \delta > \alpha. \end{cases} \tag{1.14}$$

In particular, when $\delta > \alpha$ the series $\sum_{1 \leq q} \frac{a(q)}{q^{1+\delta}}$ converges.

In (1.14) the implied constants depend only on α, β, δ and the implied constant in (1.13).

Proof. Since $1 \leq V$ and $V < q$, the left hand side of (1.14) is independent of $a(1)$, which we may take to be 0. Then (1.13) holds for all $t \geq 1$. Let us now set $A(t) = \sum_{q \leq t} a(q)$ for $t \geq 1$. Then we have

$$\sum_{V < q \leq U} \frac{a(q)}{q^{1+\delta}} = \sum_{q \leq U} \frac{a(q)}{q^{1+\delta}} - \sum_{q \leq V} \frac{a(q)}{q^{1+\delta}} = \frac{A(U)}{U^{1+\delta}} - \frac{A(V)}{V^{1+\delta}} + (1+\delta)\int_V^U \frac{A(t)}{t^{2+\delta}}dt, \tag{1.15}$$

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where we have applied Proposition 1.3.1 twice to obtain the second equality. On using $A(t) \ll t^{1+\alpha}(\log t)^\beta$ we then deduce that

$$\sum_{V < q \leq U} \frac{a(q)}{q^{1+\delta}} \ll \frac{(\log U)^\beta}{U^{\delta-\alpha}} + \frac{(\log V)^\beta}{V^{\delta-\alpha}} + \int_V^U \frac{(\log t)^\beta}{t^{1+\delta-\alpha}} dt \ . \tag{1.16}$$

Let us temporarily write I for the integral on the right hand side of (1.16). When $\delta < \alpha$ we have

$$I \leq (\log U)^\beta \int_1^U t^{\alpha-\delta-1} dt \leq \frac{1}{\alpha-\delta}(\log U)^\beta U^{\alpha-\delta}. \tag{1.17}$$

When $\delta = \alpha$ we have $I = \frac{1}{1+\beta}((\log U)^{\beta+1} - (\log V)^{\beta+1}) \leq (\log U)^{\beta+1}$, since $\beta \geq 0$. Since $V \leq U$ and $\beta \geq 0$, the second term on the right hand side of (1.16) does not exceed the first when $\delta \leq \alpha$. These remarks together with (1.17) and (1.16) give (1.14) in the first two cases. When $\delta > \alpha$ we have

$$I \leq \int_1^{+\infty} \frac{(\log Vt)^\beta}{(Vt)^{1+\delta-\alpha}} d(Vt) \leq \frac{(\log eV)^\beta}{V^{\delta-\alpha}} \int_1^{+\infty} \frac{(\log et)^\beta}{t^{1+\delta-\alpha}} dt, \tag{1.18}$$

where the first inequality results from extending the range of integration from $[V,U]$ to $[V,\infty)$ and then using the change of variables $t \mapsto Vt$, while the second inequality results on noting that $\log Vt \leq (\log eV)(\log et)$, valid for any $V,t \geq 1$. Note that the last integral in (1.18) is convergent since $\delta > \alpha$, and is independent of V. Since $a(q) \geq 0$ for all $q \geq 1$, we get on combining (1.18) and (1.16) that

$$\sum_{V < q \leq U} \frac{a(q)}{q^{1+\delta}} \leq \lim_{U \to +\infty} \sum_{V < q \leq U} \frac{a(q)}{q^{1+\delta}} \ll \frac{(\log eV)^\beta}{V^{\delta-\alpha}} \tag{1.19}$$

when $\delta > \alpha$. The second inequality in (1.19) tells us that $\sum_{1 \leq q} \frac{a(q)}{q^{1+\delta}}$ converges in this case.

$\square$

Corollary 1.3.3. *Let α, β, γ be real numbers and let $\{a(q)\}_{q \geq 1}$ be as in Corollary 1.3.2. Then for any real numbers V, U with $2 \leq V \leq U$ we have*

$$\sum_{V \leq q \leq U} \frac{a(q)}{q(\log q)^\gamma} \ll \begin{cases} \dfrac{U^\alpha}{(\log U)^{\gamma-\beta}} + 1 & \text{for } \alpha > 0, \gamma > \beta, \\[2ex] \dfrac{1}{(\log V)^{\gamma-\beta-1}} & \text{for } \alpha = 0, \gamma > \beta + 1. \end{cases} \tag{1.20}$$

In particular, when $\alpha = 0$ and $\gamma > \beta + 1$, the series $\sum_{2 \leq q} \frac{a(q)}{q(\log q)^\gamma}$ converges.

In (1.20) the implied constants depend only on α, β, γ and the implied constant in (1.13).

Proof. As in the proof of the preceding corollary, we may suppose that $a(1) = 0$ and hence that $A(t) = \sum_{q \leq t} a(q) \ll t^{1+\alpha} (\log t)^\beta$ for all $t \geq 1$. Let u be such that $\sqrt{e} \leq u < V$. By means of Proposition 1.3.1 we then obtain

$$\sum_{u < q \leq U} \frac{a(q)}{q(\log q)^\gamma} \ll \frac{U^\alpha}{(\log U)^{\gamma-\beta}} + \frac{u^\alpha}{(\log u)^{\gamma-\beta}} + \int_u^U \frac{t^{\alpha-1}}{(\log t)^{\gamma-\beta}}\, dt \,, \tag{1.21}$$

where we have used

$$\left(\frac{1}{t(\log t)^\gamma}\right)' = -\frac{1}{t^2(\log t)^\gamma} - \frac{\gamma}{t^2(\log t)^{\gamma+1}} \ll \frac{1}{t^2(\log t)^\gamma}\,, \tag{1.22}$$

for t in $[u, U]$, since $\log t \geq \frac{1}{2}$ for such t. On letting $u \to V$ in (1.21) we conclude that

$$\sum_{V \leq q \leq U} \frac{a(q)}{q(\log q)^\gamma} \ll \frac{U^\alpha}{(\log U)^{\gamma-\beta}} + \frac{V^\alpha}{(\log V)^{\gamma-\beta}} + \int_V^U \frac{t^{\alpha-1}}{(\log t)^{\gamma-\beta}}\, dt \tag{1.23}$$

for any V, U with $2 \leq V \leq U$. Suppose now that $\alpha > 0$, $\gamma > \beta$ and for $t > 1$ let us set $\varphi(t) = \frac{t^\alpha}{(\log t)^{\gamma-\beta}}$. Then we see that

$$\frac{\alpha t^{\alpha-1}}{2(\log t)^{\gamma-\beta}} \leq \varphi'(t) = \frac{\alpha t^{\alpha-1}}{(\log t)^{\gamma-\beta}} - \frac{(\gamma-\beta)t^{\alpha-1}}{(\log t)^{\gamma-\beta+1}} \tag{1.24}$$

when $t \geq V_0 = \exp(\frac{2(\gamma-\beta)}{\alpha}) \geq 1$. Thus when $V_0 \leq V$ we have from (1.23) and (1.24) that

$$\sum_{V \leq q \leq U} \frac{a(q)}{q(\log q)^\gamma} \ll \varphi(U) + \varphi(V) + \int_V^U \varphi'(t)\, dt = 2\varphi(U)\,. \tag{1.25}$$

When $V < V_0$ the left hand side of (1.25) can be written as

$$\sum_{V \le q < V_0} \frac{a(q)}{q(\log q)^\gamma} + \sum_{V_0 \le q \le U} \frac{a(q)}{q(\log q)^\gamma} \ll \frac{V_0^{1+\alpha}(\log V_0)^\beta}{(\log 2)^\gamma} + \varphi(U), \qquad (1.26)$$

using (1.13) and (1.25) with $V = V_0$ and since $V \ge 2$ and $a(q) \ge 0$. The bound in (1.20) for $\alpha > 0$ now follows from (1.25) and (1.26). When $\alpha = 0$ and $\gamma > \beta + 1$ we obtain from (1.23) that

$$\sum_{V \le q \le U} \frac{a(q)}{q(\log q)^\gamma} \le \lim_{U \to +\infty} \sum_{V \le q \le U} \frac{a(q)}{q(\log q)^\gamma} \ll \frac{1}{(\log V)^{\gamma-\beta}} + \int_V^\infty \frac{1}{t(\log t)^{\gamma-\beta}} \, dt \qquad (1.27)$$

for any V, U with $2 \le V \le U$, since $a(q) \ge 0$. To conclude the second case of (1.20) from (1.27) it remains only to note that the integral in (1.27) evaluates to $\frac{1}{(\gamma-\beta-1)(\log V)^{\gamma-\beta-1}}$ since $\gamma > \beta + 1$ and $\frac{\log 2}{(\log V)^{\gamma-\beta}} \le \frac{1}{(\log V)^{\gamma-\beta-1}}$. Also, it follows from the second inequality in (1.27) that $\sum_{2 \le q} \frac{a(q)}{q(\log q)^\gamma}$ converges in this case.

$\square$

We next give some simple bounds for $c_q(n)$ and its partial sums over n for a given q. All of our bounds are easy consequences of the following classical relation (see [21], page 110) :

$$c_q(n) = \sum_{d|(q,n)} \mu(\frac{q}{d})d, \qquad (1.28)$$

for all integers $q, n \ge 1$, where (q,n) is the gcd of q and n and μ is the Möbius function. We shall write $\sigma(n)$ in place of $\sigma_1(n) = \sum_{d|n} d$. Then an application of the triangle inequality to the sum on the right hand side of (1.28) immediately gives

Lemma 1.3.4. *For all integers $n \ge 1$ and $q \ge 1$ we have*

$$|c_q(n)| \le \sigma(n). \qquad (1.29)$$

Now we note that for all integers $N \ge 1$ and $q \ge 1$ we have using (1.28) and an interchange of summations that

$$\sum_{n \leq N} c_q(n) = \sum_{d|q} \mu(\frac{q}{d})d \sum_{\substack{n \leq N, \\ d|n.}} 1 = \sum_{d|q} \mu(\frac{q}{d})d \left[\frac{N}{d}\right]. \qquad (1.30)$$

Lemmas 1.3.5 and 1.3.6 below follow from (1.30). These lemmas are versions of Lemma 1 and Lemma 2 of [7] respectively.

Lemma 1.3.5. *For all integers $N \geq 1$ and $q \geq 1$ we have*

$$\left| \sum_{n \leq N} c_q(n) \right| \leq N\tau(q). \qquad (1.31)$$

Proof. This results from (1.30) on using the triangle inequality and $d \left[\frac{N}{d}\right] \leq d \cdot \frac{N}{d} = N$.

$\square$

Lemma 1.3.6. *For all integers $N \geq 1$ and $q \geq 2$ we have*

$$\left| \sum_{n \leq N} c_q(n) \right| \leq \sigma(q). \qquad (1.32)$$

Proof. We note that $\left[\frac{N}{d}\right] = \frac{N}{d} + \varepsilon_d(N)$ with $-1 < \varepsilon_d(N) \leq 0$ for all integers $d, N \geq 1$. Thus from (1.30) we have

$$\sum_{n \leq N} c_q(n) = \sum_{d|q} \mu(\frac{q}{d})d \left(\frac{N}{d} + \varepsilon_d(N) \right) = \sum_{d|q} \mu(\frac{q}{d})d\varepsilon_d(N), \qquad (1.33)$$

since the Möbius inversion formula tells us that $\sum_{d|q} \mu(\frac{q}{d}) = 0$ when $q \geq 2$. Applying the triangle inequality we then get

$$\left| \sum_{n \leq N} c_q(n) \right| = \left| \sum_{d|q} \mu(\frac{q}{d})d\varepsilon_d(N) \right| \leq \sum_{d|q} d = \sigma(q). \qquad (1.34)$$

$\square$

We now discuss various products on arithmetic functions with an emphasis on multiplicativity. We recall that an arithmetic function (of one variable) f is said to be multiplicative if $f(1) = 1$ and we have $f(mn) = f(m)f(n)$ whenever $m, n \in \mathbf{N}$ are coprime.

When f and g are arithmetic functions we write $f \cdot g$ for the arithmetic function $n \mapsto f(n)g(n)$. Further, we write $f * g$ for the Dirichlet convolution of f and g defined by $n \mapsto \sum_{d|n} f(d)g(\frac{n}{d})$. When f and g are multiplicative functions so are $f \cdot g$ and $f * g$.

Let $r \geq 1$ be an integer and $f_1, f_2, \ldots, f_r$ be arithmetic functions. Then $f_1 \times f_2 \times \ldots \times f_r$ shall denote the arithmetic function defined by the relation

$$f_1 \times f_2 \times \ldots \times f_r(n) = \sum_{[m_1, m_2, \ldots, m_r] = n} f_1(m_1) f_2(m_2) \ldots f_r(m_r), \qquad (1.35)$$

for all $n \in \mathbf{N}$, where the summation is over all r-tuples $(m_1, m_2, \ldots, m_r)$ such that the lcm of the m_i, denoted by $[m_1, m_2, \ldots, m_r]$, is n. We also write $\times_{i \leq r} f_i$ in place of $f_1 \times f_2 \times \ldots \times f_r$, which is sometimes called the lcm product of the f_i. Likewise, we write $\prod_{1 \leq i \leq r} f_i$ in place of $f_1 \cdot f_2 \cdots f_r$.

We now have following classical but apparently not so well-known lemma, which goes back to Von Sterneck (see footnote on page 723 of D.H. Lehmer [15]) and D.H. Lehmer [16].

Lemma 1.3.7. *Let $r \geq 1$ be an integer and $f_1, f_2, \ldots, f_r$ be arithmetic functions. If each of $f_1, f_2, \ldots, f_r$ is multiplicative then so is $f_1 \times f_2 \times \ldots \times f_r$.*

Proof. To show that an arithmetic function f is multiplicative, it suffices to show that $1 * f$ is multiplicative, where 1 is the arithmetic function defined by $n \mapsto 1$ for all $n \in \mathbf{N}$. Indeed, if $1 * f$ is multiplicative then $f = \mu * 1 * f$ by the Möbius inversion formula and since μ is multiplicative, it would follow that so is f.

For any $n \in \mathbf{N}$ we have identity

$$\sum_{m|n} \sum_{[m_1, m_2, \ldots, m_r] = m} f_1(m_1) f_2(m_2) \ldots f_r(m_r) = \prod_{1 \leq i \leq r} \sum_{m_i|n} f_i(m_i) \qquad (1.36)$$

which is easily verified by opening the product on the right hand side and regrouping terms according to lcm. This identity can be rewritten as

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$$1 * (f_1 \times f_2 \times \ldots \times f_r) = \prod_{1 \leq i \leq r} 1 * f_i \,. \qquad (1.37)$$

Since each f_i is multiplicative, so is the right hand side of (1.37) and consequently, so is its left hand side, as required.

□

Finally, we recall a basic bound for the partial sums of non-negative multiplicative functions.

Lemma 1.3.8. *Let f be a non-negative multiplicative function such that for all real $x \geq 1$ we have*

$$\frac{1}{x} \sum_{p \leq x} f(p) \log p \leq \kappa, \qquad (1.38)$$

$$\sum_{p \leq x} \sum_{v \geq 2} \frac{f(p^v) \log(p^v)}{p^v} \leq \kappa' \qquad (1.39)$$

for some real numbers κ and κ'. Then for all $x \geq 1$ we have

$$\sum_{n \leq x} f(n) \ \leq \ e^{\kappa'} (\kappa' + \kappa + 1) \frac{x}{\log ex} \exp\left(\sum_{p \leq x} \frac{f(p)}{p} \right).$$

Proof. See the proof of Theorem 4.22 in [3]. □

As an application of the preceding pair of lemmas we have :

Proposition 1.3.9. *Suppose that $r \geq 1$ is an integer, A, B, ℓ are real numbers and that $f_1, f_2, \ldots, f_r$ are non-negative multiplicative functions such that $f_i(p) \leq \ell$ and $f_i(p^v) \leq Av^B$ for $1 \leq i \leq r$, all integers $v \geq 2$ and prime numbers p. Then we have that*

$$\sum_{n \leq x} f_1 \times f_2 \times \ldots \times f_r(n) \ \ll x (\log ex)^{(\ell+1)^r - 2} \,. \qquad (1.40)$$

The implied constant in (1.40) depends only on A, B, r and ℓ.

Proof. By Lemma 1.3.7 we have that $\times_{i \leq r} f_i$ is a non-negative multiplicative function. We will deduce (1.40) by applying Lemma 1.3.8 to this function. To this end we set $f = \times_{i \leq r} f_i$

and note that for any prime number p and integer $v \geq 1$ we have

$$f(p^v) = \sum_{\substack{[m_1,m_2,...,m_r]=p^v}} f_1(m_1)f_2(m_2)\ldots f_r(m_r) = \sum_{\substack{(v_1,v_2,...,v_r), \\ 0 \leq v_i \leq v, \\ \max(v_1,v_2,...,v_r)=v.}} \prod_{1 \leq i \leq r} f_i(p^{v_i}) . \quad (1.41)$$

In particular with $v = 1$ we have

$$f(p) = \sum_{\substack{I \subseteq \{1,2,...,r\}, \\ I \neq \phi.}} \prod_{i \in I} f_i(p) . \quad (1.42)$$

Consequently we obtain that

$$f(p) \leq \sum_{1 \leq k \leq r} \binom{r}{k} \ell^k = (\ell+1)^r - 1 \quad (1.43)$$

for all primes p and therefore, on using the Chebyshev bound $\sum_{p \leq x} \log p \leq (\log 4)x$ for $x \geq 2$, that

$$\frac{1}{x} \sum_{p \leq x} f(p) \log p \leq \log 4((\ell+1)^r - 1)) = \kappa. \quad (1.44)$$

Now we note that

$$\sum_{\substack{(v_1,v_2,...,v_r), \\ 0 \leq v_i \leq v, \\ \max(v_1,v_2,...,v_r)=v.}} 1 = \sum_{\substack{(v_1,v_2,...,v_r), \\ 0 \leq v_i \leq v.}} 1 - \sum_{\substack{(v_1,v_2,...,v_r), \\ 0 \leq v_i \leq v-1.}} 1 = (v+1)^r - v^r. \quad (1.45)$$

By the mean value theorem $(v+1)^r - v^r \leq r(v+1)^{r-1} \leq 2^{r-1} r v^{r-1}$ for any integer $v \geq 1$. Thus we have from (1.41) that

$$f(p^v) = \sum_{\substack{(v_1,v_2,...,v_r), \\ 0 \leq v_i \leq v, \\ \max(v_1,v_2,...,v_r)=v.}} \prod_{1 \leq i \leq r} f_i(p^{v_i}) \leq r 2^{r-1} v^{r-1} A^r v^{rB} = r 2^{r-1} A^r v^{r(B+1)-1}, \quad (1.46)$$

using which and the easily verfied bound $\log n \le 2\sqrt{n}$, valid for any integer $n \ge 1$, with $n = p^v$ we deduce that

$$\sum_{p \le x}\sum_{v \ge 2} \frac{f(p^v)\log(p^v)}{p^v} \le r(2A)^r \sum_{v \ge 2}\sum_{p \ge 2} \frac{v^{r(B+1)-1}}{p^{v-\frac{1}{2}}} = \kappa' < +\infty. \qquad (1.47)$$

To conclude (1.40) from Lemma 1.3.8, (1.44) and (1.47) it remains only to note that

$$\sum_{p \le x} \frac{f(p)}{p} \le ((\ell+1)^r - 1)\sum_{p \le x} \frac{1}{p} \le ((\ell+1)^r - 1)(\log\log ex + C) \qquad (1.48)$$

for an absolute constant C, where first inequality follows from (1.43) and the second from the classical Mertens bound.

$\square$

Corollary 1.3.10. *For all integers $r \ge 1$ and real numbers $x \ge 1$ we have*

$$\sum_{n \le x}\sum_{[m_1,m_2,\ldots,m_r]=n} 1 \ll x(\log ex)^{2^r - 2}. \qquad (1.49)$$

Corollary 1.3.11. *For all integers $r \ge 1$ and real numbers $x \ge 1$ we have*

$$\sum_{n \le x}\sum_{[m_1,m_2,\ldots,m_r]=n} \tau(m_1)\tau(m_2)\ldots\tau(m_r) \ll x(\log ex)^{3^r - 2}. \qquad (1.50)$$

The implied constants in (1.49) and (1.50) depend on r alone.

Proofs of the Corollaries. For Corollary 1.3.10 we apply Proposition 1.3.9 with each $f_i = 1$ so that we have $f_i(p^v) = 1$ for $1 \le i \le r$, all integers $v \ge 1$ and primes numbers p. Thus we may take $\ell = 1$, $A, B = 1$ when applying the proposition.

To obtain Corollary 1.3.11 from Proposition 1.3.9 we take each $f_i = \tau$ so that we have $f_i(p^v) = v + 1$ for $1 \le i \le r$, all integers $v \ge 1$ and prime numbers p. We may therefore set $\ell = 2$, $A = 2$ and $B = 1$ when applying the proposition.

$\square$

Our final proposition is yet another application of Lemmas 1.3.7 and 1.3.8, only slightly different from the corollaries above.

Proposition 1.3.12. *For all integers $r \geq 1$ and real numbers $x \geq 1$ we have*

$$\sum_{n \leq x} \sum_{[m_1, m_2, \ldots, m_r] = n} \sigma(m_1)\sigma(m_2)\ldots\sigma(m_r) \ll x^{r+1} . \tag{1.51}$$

The implied constant in (1.51) depends on r alone.

Proof. We have

$$\sum_{n \leq x} \sum_{[m_1, m_2, \ldots, m_r] = n} \sigma(m_1)\sigma(m_2)\ldots\sigma(m_r) \leq x^r \sum_{n \leq x} \frac{1}{n^r} \sum_{[m_1, m_2, \ldots, m_r] = n} \sigma(m_1)\sigma(m_2)\ldots\sigma(m_r) . \tag{1.52}$$

Let us temporarily define the arithmetic function λ by

$$\lambda(n) = \frac{1}{n^r} \sum_{[m_1, m_2, \ldots, m_r] = n} \sigma(m_1)\sigma(m_2)\ldots\sigma(m_r) \tag{1.53}$$

for any $n \in \mathbf{N}$. Since σ is a multiplicative function, so is λ by Lemma 1.3.7. Further, for any prime number p we have

$$\lambda(p) = \frac{1}{p^r} \sum_{\substack{I \subseteq \{1,2,\ldots,r\}, \\ I \neq \emptyset.}} \prod_{i \in I} \sigma(p) = \frac{1}{p^r} \sum_{1 \leq k \leq r} \binom{r}{k}(p+1)^k = \frac{(p+2)^r - 1}{p^r} \tag{1.54}$$

and consequently that

$$\lambda(p) \leq \frac{(p+2)^r}{p^r} \leq 1 + \frac{r2^r}{p} , \tag{1.55}$$

since the mean value theorem gives $(p+2)^r - p^r \leq 2r(p+2)^{r-1} \leq r2^r p^{r-1}$. Using the first inequality in (1.55) we therefore have

$$\frac{1}{x} \sum_{p \leq x} \lambda(p) \log p \leq (\log 4)2^r = \kappa \tag{1.56}$$

for any $x \geq 1$, by the Chebyshev bound $\sum_{p \leq x} \log p \leq (\log 4)x$ and $(1 + \frac{2}{p})^r \leq 2^r$, valid for all

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primes p. Also, using the second inequality in (1.55) and the Mertens bound we get

$$\sum_{p\leq x}\frac{\lambda(p)}{p} \leq \sum_{p\leq x}\frac{1}{p}+\sum_{p\leq x}\frac{r2^r}{p^2} \leq \log\log ex+C(r) \qquad (1.57)$$

for some real number $C(r)$ depending only on r. Finally, we note that for all integers $v \geq 1$ and prime numbers p we have

$$\lambda(p^v) = \frac{1}{p^{rv}} \sum_{\substack{(v_1,v_2,...,v_r),\\ 0\leq v_i\leq v,\\ \max(v_1,v_2,...,v_r)=v.}} \sigma(p^{v_1})\sigma(p^{v_2})\dots\sigma(p^{v_r}) \leq (v+1)^r \sum_{\substack{(v_1,v_2,...,v_r),\\ 0\leq v_i\leq v,\\ \max(v_1,v_2,...,v_r)=v.}} 1 \quad (1.58)$$

since $\sigma(p^{v_i}) \leq (v_i+1)p^{v_i} \leq (v+1)p^v$ when $v_i \leq v$. Using (1.45) and the mean value theorem as before we then conclude that

$$\lambda(p^v) \leq r(v+1)^{2r-1} \leq 2^{2r-1}rv^{2r-1} \qquad (1.59)$$

for any integer $v \geq 1$. Therefore we have

$$\sum_{p\leq x}\sum_{v\geq 2}\frac{\lambda(p^v)\log(p^v)}{p^v} \leq 2^r r \sum_{v\geq 2}\sum_{p\geq 2}\frac{v^{2r-1}}{p^{v-\frac{1}{2}}} = \kappa' < +\infty. \qquad (1.60)$$

using again the inequality $\log n \leq 2\sqrt{n}$ with $n = p^v$. We now apply Lemma 1.3.8 taking account of (1.56), (1.60) and (1.57) to obtain

$$\sum_{n\leq x}\lambda(n) \ll x, \qquad (1.61)$$

which on recalling (1.53) and combining with (1.52) yields (1.51).

$\square$

Remark 1.3.13. The upper bounds given by (1.49), (1.50) and (1.51) are optimal up to the implied constants. Indeed, they can be refined to asymptotic formulae. This can be seen using the results of the following chapter and, in the case of (1.49) and (1.50), using Theorem 1 of [17], for instance.

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1.4 Proof of Theorems 1.2.1 and 1.2.2

1.4.1 Absolute Convergence

We begin by verifying here that if $\{a_{q_1,q_2}\}$, with (q_1,q_2) varying over $\mathbf{N} \times \mathbf{N}$, is a family of complex numbers satisfying either of the conditions (1.7) or (1.9) then the double series (1.6) is absolutely convergent. Since (1.9) is the weaker of the two conditions, it suffices to verify our assertion under this condition. For any $(n_1,n_2) \in \mathbf{N} \times \mathbf{N}$ we then have using (1.9) and the bound (1.29) that

$$\sum_{(q_1,q_2) \in \mathbf{N} \times \mathbf{N}} |a_{q_1,q_2}||c_{q_1}(n_1)||c_{q_2}(n_2)| \ll |a_{1,1}| + \sum_{2 \leq [q_1,q_2]} \frac{\sigma(n_1)\sigma(n_2)}{[q_1,q_2](\log[q_1,q_2])^\gamma}. \tag{1.62}$$

Now we note that

$$\sum_{2 \leq [q_1,q_2]} \frac{1}{[q_1,q_2](\log[q_1,q_2])^\gamma} = \sum_{2 \leq q} \frac{\sum_{[q_1,q_2]=q} 1}{q(\log q)^\gamma}. \tag{1.63}$$

By Corollary 1.3.10 with $r = 2$ we have $\sum_{k \leq t} \sum_{[q_1,q_2]=k} 1 \ll t(\log t)^2$. It then follows from Corollary 1.3.3 that the above series converges when $\gamma > 3$, which is certainly the case. This proves the required assertion.

From here on we present our proofs of Theorems 1.2.1 and 1.2.2 in a manner that allows immediate generalisation of these results to the case of arithmetic functions in more than two variables, taken up in Section 1.5, the final section of this chapter.

We begin with a decomposition of the series (1.6).

1.4.2 An Intial Decomposition

In this subsection we suppose only that the series (1.6) is an absolutely convergent Ramanujan expansion for an arithmetic function of two variables f. The absolute convergence of (1.6) allows us, in particular, to rearrange the terms of this series without changing its sum. We will use this remark at several places in what follows without further comment.

For any subset I of $\{1,2\}$, let us write $\mathscr{F}(I)$ for the set of (q_1,q_2) in $\mathbf{N} \times \mathbf{N}$ such that $q_i \neq 1$ if $i \in I$ and $q_i = 1$ if $i \notin I$. Then the family of sets $\mathscr{F}(I)$, with I varying over the subsets of $\{1,2\}$, form a partition of $\mathbf{N} \times \mathbf{N}$. Consequently, for any $(n_1,n_2) \in \mathbf{N} \times \mathbf{N}$ we have

$$f(n_1,n_2) = \sum_{(q_1,q_2)\in\mathbf{N}\times\mathbf{N}} a_{q_1,q_2}\, c_{q_1}(n_1)c_{q_2}(n_2) = \sum_{I\subseteq\{1,2\}} \sum_{(q_1,q_2)\in\mathscr{F}(I)} a_{q_1,q_2}\, c_{q_1}(n_1)c_{q_2}(n_2)\,. \tag{1.64}$$

For any integer $N \geq 1$ we now sum both sides of the above relation over $n_1, n_2 \leq N$. Then after an obvious interchange of summations and on remarking that for any subset I of $\{1,2\}$ we have $\sum_{n_i \leq N} c_{q_i}(n_i) = N$ when $i \in \{1,2\} \setminus I$, since for such i we have $q_i = 1$ and hence $c_{q_i}(n_i) = 1$ for all integers $n_i \geq 1$, we deduce that

$$\sum_{n_1,n_2\leq N} f(n_1,n_2) = \sum_{I\subseteq\{1,2\}} N^{2-|I|} \sum_{(q_1,q_2)\in\mathscr{F}(I)} a_{q_1,q_2} \prod_{i\in I} \sum_{n_i\leq N} c_{q_i}(n_i)\,. \tag{1.65}$$

For any subset I of $\{1,2\}$ let us set

$$B_I = \sum_{(q_1,q_2)\in\mathscr{F}(I)} a_{q_1,q_2} \prod_{i\in I} \sum_{n_i\leq N} c_{q_i}(n_i)\,. \tag{1.66}$$

Note that $B_\phi = a_{1,1}$ since $\mathscr{F}(\phi) = \{(1,1)\}$. We therefore have from (1.65) that

$$\sum_{n_1,n_2\leq N} f(n_1,n_2) = a_{1,1}N^2 + \sum_{\substack{I\subseteq\{1,2\},\\ I\neq\phi.}} N^{2-|I|} B_I\,. \tag{1.67}$$

1.4.3   Proof of Theorem 1.2.1

In the Section 1.4.1 we have seen that if a family of complex numbers $\{a_{q_1,q_2}\}$, with (q_1,q_2) varying over $\mathbf{N} \times \mathbf{N}$, satisfies (1.7) then the series (1.6) is absolutely convergent. Thus under the hypotheses of Theorem 1.2.1 we have (1.67). We shall presently prove this theorem by estimating the sums B_I on the right hand side for (1.67). We will do this by extending an argument of B. Saha, who estimated essentially the same sums for I with cardinality $|I| = 1$ to prove the Main Theorem of [25].

We begin with the following simple remarks. Firstly, for any non-empty I we have by (1.7) and the triangle inequality applied to the right hand side of (1.66) that

$$B_I \ll \sum_{q \geq 2} \frac{1}{q^{1+\delta}} \sum_{\substack{(q_1,q_2) \in \mathscr{F}(I), \\ [q_1,q_2]=q.}} \prod_{i \in I} |\sum_{n_i \leq N} c_{q_i}(n_i)| \,, \tag{1.68}$$

on noting that $[q_1,q_2] \geq 2$ when $(q_1,q_2) \in \mathscr{F}(I)$ for non-empty I, since each $q_i \geq 2$ for $i \in I$. This also means that we may use (1.32) to estimate the sums of $c_{q_i}(n_i)$ over n_i in (1.68).

Secondly, for any non-empty subset I of $\{1,2\}$ with cardinality $|I| = r$, any non-negative arithmetic function f and any integer $q \geq 1$ we have

$$\sum_{\substack{(q_1,q_2) \in \mathscr{F}(I), \\ [q_1,q_2]=q.}} \prod_{i \in I} f(q_i) \leq \sum_{\substack{(m_1,\ldots,m_r) \in \mathbf{N}^r, \\ [m_1,\ldots,m_r]=q.}} f(m_1) \cdots f(m_r) \,. \tag{1.69}$$

When f is $n \mapsto \sigma(n)$, respectively $n \mapsto \tau(n)$, we shall write $a_{|I|}(q)$, respectively $b_{|I|}(q)$, to denote the right hand side of the above relation.

Now let $\delta > 0$ be given and let us decompose the sum on the right hand side of (1.67) as

$$\sum_{\substack{I \subseteq \{1,2\}, \\ I \neq \phi.}} N^{2-|I|} B_I = \sum_{\substack{I \subseteq \{1,2\}, \\ I \neq \phi, \\ |I| < \delta.}} N^{2-|I|} B_I + \sum_{\substack{I \subseteq \{1,2\}, \\ I \neq \phi, \\ |I| = \delta.}} N^{2-|I|} B_I + \sum_{\substack{I \subseteq \{1,2\}, \\ I \neq \phi, \\ |I| > \delta.}} N^{2-|I|} B_I \,. \tag{1.70}$$

We first estimate the sums B_I occuring in the first sum on the right hand side of (1.70). Thus

let I be a non-empty subset of $\{1,2\}$ with $|I| < \delta$. Then on applying (1.32) to estimate the sums over n_i in (1.68) we get

$$B_I \ll \sum_{q \geq 2} \frac{1}{q^{1+\delta}} \sum_{\substack{(q_1,q_2) \in \mathscr{F}(I), \\ [q_1,q_2]=q.}} \prod_{i \in I} \sigma(q_i) \ll \sum_{q \geq 2} \frac{a_{|I|}(q)}{q^{1+\delta}}, \tag{1.71}$$

on using (1.69) with f taken to be the function $n \mapsto \sigma(n)$ for the second inequality. Since $\sum_{q \leq t} a_{|I|}(q) \ll t^{|I|+1}$ from Proposition 1.3.12 with $r = |I|$ and since $\delta > |I|$, it follows from Corollary 1.3.2 that the last series in (1.71) converges. Consequently, we have

$$B_I \ll 1 \text{ when } I \neq \phi \text{ and } |I| < \delta. \tag{1.72}$$

Now we estimate B_I when $|I| \geq \delta$. To this end, we let $\lambda \geq 2$ be any real number. We use (1.32) to estimate the sums over n_i in (1.68) when $2 \leq q \leq \lambda$ and (1.31) when $\lambda < q$. Then on applying (1.69) with f taken to be $n \mapsto \sigma(n)$ and $n \mapsto \tau(n)$ respectively we get

$$B_I \ll \sum_{2 \leq q \leq \lambda} \frac{a_{|I|}(q)}{q^{1+\delta}} + N^{|I|} \sum_{\lambda < q} \frac{b_{|I|}(q)}{q^{1+\delta}}. \tag{1.73}$$

We first bound the second sum on the right hand side of (1.73). We do this by noting that $\sum_{q \leq t} b_{|I|}(q) \ll t (\log t)^{3^{|I|}-2}$ from Corollary 1.3.11 with $r = |I|$ and using the third case of (1.14) with $V = \lambda$ and letting $U \to +\infty$. Turning to the first sum on the right hand side of (1.73), we bound this sum when $|I| = \delta$ by noting as above that $\sum_{q \leq t} a_{|I|}(q) \ll t^{|I|+1}$ from Proposition 1.3.12 and using the second case of (1.14) in Corollary 1.3.2 with $V = 1$ and $U = \lambda$. When $|I| > \delta$ we bound this sum similarly except that we use the first case of (1.14) in Corollary 1.3.2.

When $|I| = \delta$ we then obtain from (1.73) that

$$B_I \ll \log \lambda + \frac{N^{|I|} (\log \lambda)^{3^{|I|}-2}}{\lambda^{|I|}} \tag{1.74}$$

for all $\lambda \geq 2$, which we apply with $\lambda = (eN)^{\frac{|I|+3^{|I|}-2}{|I|}} \geq 2$, since $N, |I| \geq 1$. This gives

$$B_I \ll \log eN + 1 \ll \log eN \text{ when } \delta = |I|. \tag{1.75}$$

When $|I| > \delta$ we get

$$B_I \ll \lambda^{|I|-\delta} + \frac{N^{|I|}(\log \lambda)^{3^{|I|}-2}}{\lambda^\delta} \tag{1.76}$$

for all $\lambda \geq 2$, in which we take $\lambda = eN(\log eN)^{\frac{3^{|I|}-2}{|I|}} \geq 2$, since $N, |I| \geq 1$. This gives

$$B_I \ll N^{|I|-\delta}(\log eN)^{u_{|I|}(\delta)} \text{ when } |I| > \delta, \tag{1.77}$$

where $u_k(\delta)$ is defined for any integer $k \geq 1$ and $\delta > 0$ by

$$u_k(\delta) = \left(1 - \frac{\delta}{k}\right)(3^k - 2). \tag{1.78}$$

With a view to the proof of the generalisation of Theorem 1.2.1 that we give in Section 1.5, it will be convinient to proceed from here on by setting $\ell = 2$ and estimating the first sum on the right hand side of (1.70) as

$$\sum_{\substack{I \subseteq \{1,2\}, \\ I \neq \phi, \\ |I| < \delta.}} N^{\ell-|I|}B_I \ll \sum_{\substack{I \subseteq \{1,2\}, \\ I \neq \phi, \\ |I| < \delta.}} N^{\ell-|I|} \leq N^{\ell-1} \sum_{\substack{I \subseteq \{1,2\}, \\ I \neq \phi, \\ |I| < \delta}} 1, \tag{1.79}$$

where the first inequality follows from (1.72) and the second since $|I| \geq 1$ when $I \neq \phi$. For the second sum on the right hand side of (1.70) we have using (1.75) that

$$\sum_{\substack{I \subseteq \{1,2\}, \\ I \neq \phi, \\ |I| = \delta.}} N^{\ell-|I|}B_I \ll N^{\ell-\delta} \log eN \sum_{\substack{I \subseteq \{1,2\}, \\ I \neq \phi, \\ |I| = \delta.}} 1 \tag{1.80}$$

Likewise, on using (1.77) we have for the third sum on the right hand side of (1.70) that

$$\sum_{\substack{I \subseteq \{1,2\}, \\ I \neq \phi, \\ |I| > \delta.}} N^{\ell-|I|} B_I \ll \sum_{\substack{I \subseteq \{1,2\}, \\ I \neq \phi, \\ |I| > \delta.}} N^{\ell-\delta} (\log eN)^{u_{|I|}(\delta)} \leq N^{\ell-\delta} (\log eN)^{u_\ell(\delta)} \sum_{\substack{I \subseteq \{1,2\}, \\ I \neq \phi, \\ |I| > \delta.}} 1, \qquad (1.81)$$

since $k \mapsto u_k(\delta)$ increases with k when $k > \delta$ for a given δ and since $|I| \leq \ell$.

When $\delta > 1$ there is a $C(\delta, \ell) \geq 0$, depending only on δ and ℓ, such that $N^{\ell-\delta}(\log eN)^{u_\ell(\delta)} \leq C(\delta, \ell) N^{\ell-1}$ and $N^{\ell-\delta} \log eN \leq C(\delta, \ell) N^{\ell-1}$ for all $N \geq 1$, since these bounds are valid with $C(\delta, \ell) = 1$ when N is large enough, depending only on δ and ℓ. Thus on combining (1.79), (1.80) and (1.81) with (1.70) we get

$$\sum_{\substack{I \subseteq \{1,2\}, \\ I \neq \phi.}} N^{2-|I|} B_I \ll N^{\ell-1} \text{ when } \delta > 1. \qquad (1.82)$$

When $\delta = 1$ the first sum on the right hand side of (1.70) is 0 since it is an empty sum. Also, we have

$$1 \leq u_k(1) \text{ for all integers } k \geq 2 \qquad (1.83)$$

and in particular when $k = \ell$. We therefore have from (1.80) and (1.81) together with (1.70) that

$$\sum_{\substack{I \subseteq \{1,2\}, \\ I \neq \phi.}} N^{2-|I|} B_I \ll N^{\ell-1} (\log eN)^{u_\ell(1)} \text{ when } \delta = 1. \qquad (1.84)$$

When $0 < \delta < 1$ the first and second sums on the right hand side of (1.70) are 0 since they are empty sums. We then conclude from (1.81) and (1.70) that

$$\sum_{\substack{I \subseteq \{1,2\}, \\ I \neq \phi.}} N^{2-|I|} B_I \ll N^{\ell-1} (\log eN)^{u_\ell(\delta)} \text{ when } 0 < \delta < 1. \qquad (1.85)$$

Finally, we note that $u_\ell(\delta) = \frac{14-7\delta}{2}$ from (1.78), since $\ell = 2$. Theorem 1.2.1 now follows

from (1.82), (1.84) and (1.85) taken together with (1.67).

1.4.4 Proof of Theorem 1.2.2

Intuitively speaking, the condition (1.9) corresponds to the condition (1.7) with δ "infinitesimally" close to 0. Thus to obtain Theorem 1.2.2 from (1.67) we estimate the sums B_I as in the case $\delta < |I|$ of the proof of Theorem 1.2.1 above. In effect, for any non-empty subset I of $\{1,2\}$, using (1.9) and the triangle inequality we have from (1.66) that

$$B_I \ll \sum_{q \geq 2} \frac{1}{q(\log q)^\gamma} \sum_{\substack{(q_1,q_2) \in \mathscr{F}(I), \\ [q_1,q_2]=q.}} \prod_{i \in I} |\sum_{n_i \leq N} c_{q_i}(n_i)| . \tag{1.86}$$

As in the derivation of (1.73) from (1.71), we let $\lambda \geq 2$ be any real number. We then use (1.32) to estimate the sums over n_i in (1.86) when $2 \leq q \leq \lambda$ and (1.31) when $\lambda < q$. Finally, from (1.69) with $f = a_{|I|}, b_{|I|}$ we get

$$B_I \ll \sum_{2 \leq q \leq \lambda} \frac{a_{|I|}(q)}{q(\log q)^\gamma} + N^{|I|} \sum_{\lambda < q} \frac{b_{|I|}(q)}{q(\log q)^\gamma} . \tag{1.87}$$

We bound the first sum on the right hand side of (1.87) by remarking that $\sum_{q \leq t} a_{|I|}(q) \ll t^{|I|+1}$ from Proposition 1.3.12 with $r = |I|$ and using the first case of (1.20) in Corollary 1.3.3 with $V = 2$ and $U = \lambda$. We then bound the second sum on the right hand side of (1.87) by recalling that $\sum_{q \leq t} b_{|I|}(q) \ll t(\log t)^{3^{|I|}-2}$ from Corollary 1.3.11 with $r = |I|$ and using the second case of (1.20) with $V = \lambda$ and letting $U \to +\infty$. These cases of (1.20) are applicable since $\gamma > 3^\ell - 1 \geq 3^{|I|} - 2 + 1$, where $\ell = 2$ as before. We then deduce that

$$B_I \ll \frac{\lambda^{|I|}}{(\log \lambda)^\gamma} + 1 + \frac{N^{|I|}}{(\log \lambda)^{\gamma-3^{|I|}+1}} \tag{1.88}$$

for all $\lambda \geq 2$, in which we take $\lambda = eN \geq 2$, since $N \geq 1$. This gives for any non-empty subset I of $\{1,2\}$ the bound

$$B_I \ll \frac{N^{|I|}}{(\log eN)^\gamma} + 1 + \frac{N^{|I|}}{(\log eN)^{\gamma-3^{|I|}+1}} \ll \frac{N^{|I|}}{(\log eN)^{\gamma-3^\ell+1}} \tag{1.89}$$

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where for the second inequality we used $|I| \leq \ell = 2$ and $1 \leq (([s]+1)!)^z \frac{(eN)^z}{(\log eN)^{sz}}$, valid for any $s, z \geq 0$ and $N \geq 1$, with $s = \frac{\gamma - 3^{|I|} + 1}{|I|}$ and $z = |I|$. Theorem 1.2.2 follows on substituting the last term of (1.89) in place of B_I in (1.67) and using $\ell = 2$.

1.5 The Case of Several Variables

Let $m \geq 2$ be an integer. Then an arithmetic function of m variables is a map $f : \mathbf{N}^m \mapsto \mathbf{C}$. Such a function f is said to have an *absolutely convergent Ramanujan expansion* if there exists a family of complex numbers $\{a_{q_1,q_2,\ldots,q_m}\}$ with $(q_1, q_2, \ldots, q_m)$ varying over $\mathbf{N}^m$ such that for each $(n_1, n_2, \ldots, n_m) \in \mathbf{N}^m$ the series

$$\sum_{(q_1,q_2,\ldots,q_m)\in\mathbf{N}^m} a_{q_1,q_2,\ldots,q_m} \prod_{1\leq i\leq m} c_{q_i}(n_i) \tag{1.90}$$

is absolutely convergent and its sum is $f(n_1, n_2, \ldots, n_m)$. The family $\{a_{q_1,\ldots,q_m}\}$, with $(q_1, q_2, \ldots, q_m) \in \mathbf{N}^m$, is then called a family of *Ramanujan coefficients* for f.

Theorem 1.5.1. *Let $m \geq 2$ be an integer and suppose that $\{a_{q_1,q_2,\ldots,q_m}\}$, with the $(q_1, q_2, \ldots, q_m)$ varying over $\mathbf{N}^m$, is a family of complex numbers satisfying the condition*

$$\left|a_{q_1,q_2,\ldots,q_m}\right| \ll \frac{1}{[q_1, q_2, \ldots, q_m]^{1+\delta}} \tag{1.91}$$

for some $\delta > 0$ and all $(q_1, q_2, \ldots, q_m) \in \mathbf{N}^m$, where $[q_1, q_2, \ldots, q_m]$ denotes the least common multiple of $q_1, q_2, \ldots, q_m$. Then the series (1.90) is absolutely convergent for every $(n_1, n_2, \ldots, n_m) \in \mathbf{N}^m$. If moreover $\{a_{q_1,q_2,\ldots,q_m}\}$ is family of Ramanujan coefficients for an arithmetic function of m variables f then for any integer $N \geq 1$ we have

$$\sum_{n_1,n_2,\ldots,n_m\leq N} f(n_1, n_2, \ldots, n_m) = \begin{cases} a_{1,1,\ldots,1}N^m + O(N^{m-\delta}(\log eN)^{u_m(\delta)}) & \text{if } \delta \leq 1, \\ \\ a_{1,1,\ldots,1}N^m + O(N^{m-1}) & \text{if } \delta > 1. \end{cases}$$

where $u_m(\delta) = (1 - \frac{\delta}{m})(3^m - 2)$.

The implied constants in the $\ll$ and O symbols in the statement of Theorem 1.5.1 depend only on δ and m. Since $u_2(\delta)$ can be written as $\frac{14-7\delta}{2}$, Theorem 1.5.1 indeed reduces to Theorem 1.2.1 when $m = 2$.

Theorem 1.5.2. *Let $m \geq 2$ be an integer and suppose that $\{a_{q_1,q_2,\ldots,q_m}\}$, with $(q_1, q_2, \ldots, q_m) \in \mathbf{N}^m$, is a family of complex numbers satisfying the condition*

$$\left| a_{q_1,q_2,\ldots,q_m} \right| \ll \frac{1}{[q_1,q_2,\ldots,q_m](\log e[q_1,q_2,\ldots,q_m])^\gamma} \tag{1.92}$$

for some real number $\gamma > 3^m - 1$ and all $(q_1, q_2, \ldots, q_m) \in \mathbf{N}^m$. Then the series (1.90) is absolutely convergent for every $(n_1, n_2, \ldots, n_m) \in \mathbf{N}^m$. If moreover $\{a_{q_1,q_2,\ldots,q_m}\}$ is family of Ramanujan coefficients for an arithmetic function of m variables f then for any integer $N \geq 1$ we have

$$\sum_{n_1,n_2,\ldots,n_m \leq N} f(n_1, n_2, \ldots, n_m) = a_{1,1,\ldots,1} N^m + O\left(\frac{N^m}{(\log eN)^{\gamma - 3^m + 1}} \right).$$

The implied constants in the $\ll$ and O symbols in the statement of Theorem 1.5.2 depend only on γ and m. Plainly, putting $m = 2$ in Theorem 1.5.2 gives Theorem 1.2.2.

Proofs of Theorem 1.5.1 and Theorem 1.5.2. We begin, as before, by verifying that if $\{a_{q_1,q_2,\ldots,q_m}\}$, with $(q_1, q_2, \ldots, q_m)$ varying over $\mathbf{N}^m$, is a family of complex numbers satisfying either of the conditions (1.91) or (1.92) then the series (1.90) is absolutely convergent. By what we have said in Subsection 1.4.1 it is evident, using (1.29), that this reduces to checking that the series

$$\sum_{2 \leq [q_1,q_2,\ldots,q_m]} \frac{1}{[q_1,q_2,\ldots,q_m](\log[q_1,q_2,\ldots,q_m])^\gamma} = \sum_{2 \leq q} \frac{\sum_{[q_1,q_2,\ldots,q_m]=q} 1}{q(\log q)^\gamma} \tag{1.93}$$

is convergent when $\gamma > 3^m + 1$. We have $\sum_{k \leq t} \sum_{[q_1,q_2,\ldots,q_m]=k} 1 \ll t(\log t)^{2^m - 2}$ from Corollary 1.3.10 with $r = m$. Since $3^m + 1 > 2^m - 2 + 1$ our assertion follows from Corollary 1.3.3.

For any subset I of $\{1, 2, \ldots, m\}$, let us now write $\mathscr{F}(I)$ for the set of $(q_1, q_2, \ldots, q_m)$ in $\mathbf{N}^m$

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such that $q_i \neq 1$ if $i \in I$ and $q_i = 1$ if $i \notin I$. Then the family of sets $\mathscr{F}(I)$, with I varying over the subsets of $\{1, 2, \ldots, m\}$, form a partition of $\mathbf{N}^m$. By the absolute convergence of (1.90) we then have for any $(n_1, n_2, \ldots, n_m) \in \mathbf{N}^m$ that

$$f(n_1, n_2, \ldots, n_m) = \sum_{I \subseteq \{1,2,\ldots,m\}} \sum_{(q_1,q_2,\ldots,q_m) \in \mathscr{F}(I)} a_{q_1,q_2,\ldots,q_m} \prod_{1 \le i \le m} c_{q_i}(n_i) \,. \tag{1.94}$$

Then for any subset I of $\{1, 2, \ldots, m\}$ we set

$$B_I = \sum_{(q_1,q_2,\ldots,q_m) \in \mathscr{F}(I)} a_{q_1,q_2,\ldots,q_m} \prod_{i \in I} \sum_{n_i \le N} c_{q_i}(n_i) \,. \tag{1.95}$$

and obtain from (1.94) that

$$\sum_{n_1,n_2 \le N} f(n_1, \ldots, n_m) = a_{1,1,\ldots,1} N^m + \sum_{\substack{I \subseteq \{1,2,\ldots,m\}, \\ I \neq \phi.}} N^{m-|I|} B_I \,. \tag{1.96}$$

on remarking that $B_\phi = a_{1,1,\ldots,1}$ since $\mathscr{F}(\phi) = \{(1, 1, \ldots, 1)\}$ and $\sum_{n \le N} c_q(n) = N$ when $q = 1$.

To obtain Theorem 1.5.1 it is only required to modify the proof of Theorem 1.2.1 given in Subsection 1.4.3 by making the obvious modifications, that is, by replacing $\{1, 2\}$ with $\{1, 2, \ldots, m\}$, $[q_1, q_2]$ with $[q_1, q_2, \ldots, q_m]$, ℓ with m and $\ell = 2$ with $\ell = m$, etc. at every occurence and using (1.96) in place of (1.67). The same modifications incorporated into the proof of Theorem 1.2.2 in Subsection 1.4.4 yield Theorem 1.5.2.

$\square$

1.6 Remarks on Corollaries 1.2.3 and 1.2.4

Theorem 1.2.1 is a general result applicable to any arithmetic function of two variables with an absolutely convergent Ramanujan expansion whose coefficients satisfy (1.7). This generality is also its weakness in that the exponent of the logarithmic factors on the right hand

side of (1.8) can be substantially improved for specific arithmetic functions f. We will illustrate this remark in this section by giving sharper forms of Corollaries 1.2.3 and 1.2.4 by a straightforward and elementary method. We will also show that the version of Corollary 1.2.3 given below as Proposition 1.6.1 is optimal. It follows from this that except for the aforementioned logarithmic factors, Theorem 1.2.1 is optimal. We end this section (and the chapter) by noting that the same elementary method gives sharper forms of Corollaries 1 and 2 of B. Saha [25]. We begin with the following auxiliary lemma.

Lemma 1.6.1. *Let $X \geq 1$ be a real number and let $d \geq 1$ be an integer. Then the number $a_d(X)$ of pairs of integers (n_1, n_2) satisfying $1 \leq n_1, n_2 \leq X$ and $(n_1, n_2) = d$ is given by*

$$a_d(X) = \sum_{k \geq 1} \mu(k) \left[\frac{X}{kd} \right]^2 . \tag{1.97}$$

Proof. For any real $X \geq 1$ we have

$$[X]^2 = \sum_{d \geq 1} a_d(X) . \tag{1.98}$$

Also we have $a_d(X) = a_1(\frac{X}{d})$ for all integers $d \geq 1$ since the map $(n_1, n_2) \mapsto (\frac{n_1}{d}, \frac{n_2}{d})$ is a bijection from the set of pairs of integers (n_1, n_2) satisfying $1 \leq n_1, n_2 \leq X$ and $(n_1, n_2) = d$ to the set of pairs of integers (n_1, n_2) satisfying $1 \leq n_1, n_2 \leq \frac{X}{d}$ and $(n_1, n_2) = 1$. From (1.98) we then deduce that

$$[X]^2 = \sum_{d \geq 1} a_1 \left(\frac{X}{d} \right) \tag{1.99}$$

for all real $X \geq 1$. By a classical version of the Möbius inversion formula we then have that

$$a_1(X) = \sum_{k \geq 1} \mu(k) \left[\frac{X}{k} \right]^2 \tag{1.100}$$

for all real $X \geq 1$. Putting $\frac{X}{d}$ in place of X in the above relation and using $a_d(X) = a_1(\frac{X}{d})$ we obtain (1.97).

$\square$

Here is the optimal form of Corollary 1.2.3.

Proposition 1.6.1. *Let $\delta > 0$ be a given real number. Then for any integer $N \geq 1$ we have*

$$\sum_{n_1,n_2 \leq N} \frac{\sigma_{\delta-1}((n_1,n_2))}{(n_1,n_2)^{\delta-1}} = \begin{cases} \zeta(1+\delta)N^2 + O(N^{2-\delta}) & \text{if } \delta < 1, \\[2mm] \zeta(1+\delta)N^2 + O(N\log eN) & \text{if } \delta = 1, \\[2mm] \zeta(1+\delta)N^2 + O(N) & \text{if } \delta > 1. \end{cases} \quad (1.101)$$

Proof. We have

$$\sum_{n_1,n_2 \leq N} \frac{\sigma_{\delta-1}((n_1,n_2))}{(n_1,n_2)^{\delta-1}} = \sum_{d \leq N} \frac{\sigma_{\delta-1}(d)}{d^{\delta-1}} a_d(N) = \sum_{d \leq N} \frac{\sigma_{\delta-1}(d)}{d^{\delta-1}} \sum_{k \geq 1} \mu(k) \left[\frac{N}{kd}\right]^2 \quad (1.102)$$

where the second equality follows from (1.97) of the preceding lemma. Plainly, when $kd > N$ we have $\left[\frac{N}{kd}\right] = 0$, so that on setting $m = kd$ we may deduce from (1.102) after an interchange of summation that

$$\sum_{n_1,n_2 \leq N} \frac{\sigma_{\delta-1}((n_1,n_2))}{(n_1,n_2)^{\delta-1}} = \sum_{1 \leq m \leq N} \left[\frac{N}{m}\right]^2 \sum_{kd=m} \mu(k) \frac{\sigma_{\delta-1}(d)}{d^{\delta-1}}. \quad (1.103)$$

From the definition of $\sigma_{\delta-1}$ we see that

$$\sum_{kd=m} \mu(k) \frac{\sigma_{\delta-1}(d)}{d^{\delta-1}} = \sum_{kd=m} \mu(k) \sum_{\ell|d} \frac{\ell^{\delta-1}}{d^{\delta-1}} = \sum_{kd=m} \mu(k) \sum_{\ell|d} \frac{1}{\ell^{\delta-1}} = \frac{1}{m^{\delta-1}}, \quad (1.104)$$

where the third equality follows on using $\frac{d}{\ell} \mapsto \ell$ for divisors ℓ of d and the last equality follows from the Möbius inversion formula. On combining (1.103) and (1.104) we obtain

$$\sum_{n_1,n_2 \leq N} \frac{\sigma_{\delta-1}((n_1,n_2))}{(n_1,n_2)^{\delta-1}} = \sum_{1 \leq m \leq N} \frac{1}{m^{\delta-1}} \left[\frac{N}{m}\right]^2 \quad (1.105)$$

from which we easily conclude that

$$\sum_{n_1,n_2 \leq N} \frac{\sigma_{\delta-1}((n_1,n_2))}{(n_1,n_2)^{\delta-1}} = N^2 \sum_{m \geq 1} \frac{1}{m^{1+\delta}} - E_1(\delta) - E_2(\delta) \qquad (1.106)$$

where

$$E_1(\delta) = N^2 \sum_{N < m} \frac{1}{m^{1+\delta}} \quad \text{and} \quad E_2(\delta) = \sum_{1 \leq m \leq N} \frac{1}{m^{\delta-1}} \left(\left(\frac{N}{m}\right)^2 - \left[\frac{N}{m}\right]^2 \right) \qquad (1.107)$$

Let us verify that

$$N^{2-\delta} \ll E_1(\delta) \ll N^{2-\delta}. \qquad (1.108)$$

Indeed, since we have

$$\int_{N+1}^{\infty} \frac{dt}{t^{1+\delta}} \leq \sum_{N < m} \frac{1}{m^{1+\delta}} \leq \sum_{N \leq m} \frac{1}{m^{1+\delta}} \leq \frac{1}{N^{1+\delta}} + \int_{N}^{\infty} \frac{dt}{t^{1+\delta}} . \qquad (1.109)$$

we see that

$$\frac{2^{\delta} N^{2-\delta}}{\delta} \leq \frac{N^2}{\delta(N+1)^{\delta}} \leq E_1(\delta) \leq \frac{N^2}{N^{1+\delta}} + \frac{N^2}{\delta N^{\delta}} \leq \left(1 + \frac{1}{\delta}\right) N^{2-\delta} \qquad (1.110)$$

using $\delta > 0$ and $N + 1 \geq \frac{N}{2}$, which implies (1.108). Turning to $E_2(\delta)$, we note that

$$0 \leq E_2(\delta) \leq 2N \sum_{1 \leq m \leq N} \frac{1}{m^{\delta}} \leq 2N \left(1 + \int_{1}^{N} \frac{dt}{t^{\delta}}\right), \qquad (1.111)$$

since with $a = \frac{N}{m}, b = \left[\frac{N}{m}\right]$ we have $0 \leq a - b \leq 1$ and hence that $a^2 - b^2 = (a-b)(a+b) \leq 2a$. The integral in (1.111) is bounded above by $\frac{N^{1-\delta}}{1-\delta}$ when $0 < \delta < 1$, by $\frac{1}{\delta-1}$ when $\delta > 1$ and is equal to $\log N$ when $\delta = 1$. The relation (1.101) now follows on combining (1.106) with (1.108) and (1.111). Let us now verify that (1.101) is optimal. To do this we first note from (1.106) that

$$\left| \sum_{n_1,n_2 \leq N} \frac{\sigma_{\delta-1}((n_1,n_2))}{(n_1,n_2)^{\delta-1}} - \zeta(1+\delta)N^2 \right| = E_1(\delta) + E_2(\delta), \qquad (1.112)$$

since $E_1(\delta), E_2(\delta) \geq 0$. In light of (1.108), it thus suffices to show that

$$E_2(\delta) \gg N \log N \text{ when } \delta = 1 \qquad (1.113)$$

and that

$$E_2(\delta) \gg N \text{ for infinitely many } N \text{ when } \delta > 1. \qquad (1.114)$$

Let us take up (1.114) first, which we will in fact show holds for all $\delta > 0$. Indeed, for any such δ we have

$$E_2(\delta) \geq N \sum_{1 \leq m \leq N} \left(\frac{N}{m} - \left[\frac{N}{m} \right] \right) \frac{1}{m^\delta} \qquad (1.115)$$

since with $a = \frac{N}{m}$, $b = \left[\frac{N}{m} \right]$ as before we have $a^2 - b^2 = (a-b)(a+b) \geq a(a-b)$. For any integer $N \geq 1$ we have $\frac{N}{m} - \left[\frac{N}{m} \right] \geq \frac{1}{m}$ when m does not divide N. Thus when N is a prime number we see that

$$\sum_{1 \leq m \leq N} \left(\frac{N}{m} - \left[\frac{N}{m} \right] \right) \frac{1}{m^\delta} \geq \sum_{2 \leq m \leq N-1} \frac{1}{m^{1+\delta}} = \zeta(1+\delta) - 1 - \sum_{m \geq N} \frac{1}{m^{1+\delta}}. \qquad (1.116)$$

Since the last sum tends to 0 as $N \to \infty$, we then conclude from (1.115) that (1.114) holds for all large enough prime numbers N, even when $\delta > 0$. Moving to (1.113), we see that when $\delta = 1$, the right hand side of (1.115) can be written as

$$\frac{N}{2} \sum_{1 \leq m \leq N} \frac{1}{m} + \sum_{1 \leq m \leq N} \frac{N}{m} \psi \left(\frac{N}{m} \right), \qquad (1.117)$$

where $\psi(t) = t - [t] - \frac{1}{2}$ for any real t. From page 335 of [3] we have that

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$$\sum_{\sqrt{N} < m \leq N} \frac{N}{m} \psi\left(\frac{N}{m}\right) \ll N .\tag{1.118}$$

Since a trivial estimate yields

$$\left| \sum_{1 \leq m \leq \sqrt{N}} \frac{N}{m} \psi\left(\frac{N}{m}\right) \right| \leq \frac{N}{2} \sum_{1 \leq m \leq \sqrt{N}} \frac{1}{m} ,\tag{1.119}$$

it now follows that when $\delta = 1$ we have

$$E_2(\delta) \geq N \sum_{\sqrt{N} \leq m \leq N} \frac{1}{m} + O(N) \gg N \log N,\tag{1.120}$$

as asserted in (1.113).

$\square$

Proposition 1.6.2. *Let $\delta > 0$ be a given real number. Then for any integer $N \geq 1$ we have*

$$\sum_{n_1,n_2 \leq N} \frac{\phi_{\delta-1}((n_1,n_2))}{(n_1,n_2)^{\delta-1}} = \begin{cases} \frac{N^2}{\zeta(1+\delta)} + O(N^{2-\delta}) & \text{if } \delta < 1, \\[2ex] \frac{N^2}{\zeta(1+\delta)} + O(N \log eN) & \text{if } \delta = 1, \\[2ex] \frac{N^2}{\zeta(1+\delta)} + O(N) & \text{if } \delta > 1. \end{cases}\tag{1.121}$$

This is an improved form of Corollary 1.2.4.

Proof. We apply the same method as in the proof of the preceding proposition. Indeed, using the relation

$$\phi_s(n) = n^s \sum_{d|n} \frac{\mu(d)}{d^s}\tag{1.122}$$

and arguing as before we get

$$\sum_{n_1,n_2 \leq N} \frac{\phi_{\delta-1}((n_1,n_2))}{(n_1,n_2)^{\delta-1}} = \sum_{1 \leq m \leq N} \frac{\mu(m)}{m^{\delta-1}} \left[\frac{N}{m}\right]^2\tag{1.123}$$

from which we conclude by an application of the triangle inequality that

$$\sum_{n_1,n_2 \leq N} \frac{\phi_{\delta-1}((n_1,n_2))}{(n_1,n_2)^{\delta-1}} = N^2 \sum_{m \geq 1} \frac{\mu(m)}{m^{1+\delta}} + O(E_1(\delta) + E_2(\delta)), \qquad (1.124)$$

where $E_1(\delta)$ and $E_2(\delta)$ are given by (1.107). The proposition now follows on recalling that $\sum_{m \geq 1} \frac{\mu(m)}{m^{1+\delta}} = \frac{1}{\zeta(1+\delta)}$ and using (1.108) and (1.111).

<div style="text-align: right">□</div>

To conclude, we give improved forms of Corollaries 1 and 2 of B. Saha [25]. They are, respectively, the following pair of propositions.

Proposition 1.6.3. *Let $\delta > 0$ be a given real number. Then for any integer $N \geq 1$ we have*

$$\sum_{n \leq N} \frac{\sigma_\delta(n)}{n^\delta} = \begin{cases} \zeta(1+\delta)N + O(N^{1-\delta}) & \text{if } \delta < 1, \\\\ \zeta(1+\delta)N + O(\log eN) & \text{if } \delta = 1, \\\\ \zeta(1+\delta)N + O(1) & \text{if } \delta > 1. \end{cases} \qquad (1.125)$$

Proof. The principle of the method is the same as in the proofs of Propositions 1.6.1 and 1.6.2, the details even simpler. We have

$$\sum_{n \leq N} \frac{\sigma_\delta(n)}{n^\delta} = \sum_{n \leq N} \sum_{m|n} \frac{1}{m^\delta} = \sum_{m \leq N} \frac{1}{m^\delta} \sum_{\substack{n \leq N, \\ m|n.}} 1 = \sum_{m \leq N} \frac{1}{m^\delta} \left[\frac{N}{m}\right] \qquad (1.126)$$

Consequently we obtain

$$\sum_{n \leq N} \frac{\sigma_\delta(n)}{n^\delta} = N \sum_{1 \leq m} \frac{1}{m^{1+\delta}} - N \sum_{N < m} \frac{1}{m^{1+\delta}} - \sum_{1 \leq m \leq N} \frac{1}{m^\delta} \left(\frac{N}{m} - \left[\frac{N}{m}\right]\right), \qquad (1.127)$$

from which the proposition follows on noting that for any $\delta > 0$ we have

$$0 \leq \sum_{N < m} \frac{1}{m^{1+\delta}} \leq \frac{1}{N^{1+\delta}} + \int_N^\infty \frac{dt}{t^{1+\delta}} \leq \left(1 + \frac{1}{\delta}\right) N^{-\delta} \qquad (1.128)$$

and

$$0 \leq \sum_{1 \leq m \leq N} \frac{1}{m^\delta} \left(\frac{N}{m} - \left[\frac{N}{m} \right] \right) \leq \sum_{1 \leq m \leq N} \frac{1}{m^\delta} \leq 1 + \int_1^N \frac{dt}{t^\delta}. \tag{1.129}$$

and remarking, as before, that the last integral is bounded above by $\frac{N^{1-\delta}}{1-\delta}$ when $0 < \delta < 1$, by $\frac{1}{\delta-1}$ when $\delta > 1$ and is equal to $\log N$ when $\delta = 1$.

□

Proposition 1.6.4. *Let $\delta > 0$ be a given real number. Then for any integer $N \geq 1$ we have*

$$\sum_{n \leq N} \frac{\phi_\delta(n)}{n^\delta} = \begin{cases} \frac{N}{\zeta(1+\delta)} + O(N^{1-\delta}) & \text{if } \delta < 1, \\[2mm] \frac{N}{\zeta(1+\delta)} + O(\log eN) & \text{if } \delta = 1, \\[2mm] \frac{N}{\zeta(1+\delta)} + O(1) & \text{if } \delta > 1. \end{cases} \tag{1.130}$$

Proof. The same method as in the proof of the preceding proposition except that one uses (1.122) in place of $\frac{\sigma_\delta(n)}{n^\delta} = \frac{1}{n^\delta} \sum_{d|n} d^\delta$.

□

CHAPTER 2

Partial sums of mildly oscillating multiplicative functions

In this chapter we obtain an asymptotic formula with an explicit error term for the partial sums of a real valued multiplicative function that, intuitively speaking, takes more non-negative values than negative values on the set of primes.

2.1 Introduction

We recall that a multiplicative function is a function $f : \mathbf{N} \to \mathbf{C}$ such that $f(mn) = f(m)f(n)$ whenever $(m,n) = 1$. In general, the behaviour of a multiplicative function can be very irregular. As taking averages smoothens out fluctuations, it is reasonable to expect that the partial sums of f behave more tractably than f. The study of the growth of the partial sums of multiplicative functions has been a major topic of research for many years (see [6],[8], [14], [37], [19]). In particular, B. V. Levin and A. S. Fainleib [6] and E. Wirsing [37] obtained fundamental results on asymptotic expansions of partial sums of multiplicative functions. Later, O. Ramaré extended the result of Levin and Fainleib to multiplicative functions that are not necessarily supported on squarefree integers and obtained an explicit result. This result of Ramaré is stated and proved as Theorem 21.1 in [23]. Subsequently, D.S. Ramana and O. Ramaré extended Ramaré's result in the direction taken by Wirsing [37], that is, to

non-negative multiplicative functions f for which $f(p)$ is, roughly, $\kappa \geq 0$ on the average over the primes. This result, given as Theorem 21.2 in [23] is, however, not explicit.

Our first objective in this chapter, which is based on [29], is to make Theorem 21.2 of [23] mentioned above explicit. This is done in Theorem 2.3.3. We then extend this result to multiplicative functions that are not necessarily non-negative but satisfy certain conditions which amount to requiring that f takes more non-negative values than negative values, on the average over the primes. More precisely, we prove the following theorem, which is the main result of this chapter. This theorem is stated using the $\mathscr{O}^*$ notation. We write $a = \mathscr{O}^*(b)$ for $|a| \leq b$.

Theorem 2.1.1. *Let f be a real valued multiplicative function and suppose that there exist non-negative real numbers $\kappa, \kappa', B_1, B_2, B_1', B_2'$ and A_2' with $1 + \kappa - \kappa' > 0$ such that*

$$\sum_{\substack{p \geq 2, v \geq 1 \\ p^v \leq Q}} f(p^v) \log(p^v) = \kappa Q + \mathscr{O}^* \left(\frac{B_1 Q}{(\log 2Q)^2} + B_2 \right), \qquad (2.1)$$

$$\sum_{\substack{p \geq 2, v \geq 1 \\ p^v \leq Q}} |f(p^v)| \log(p^v) = \kappa' Q + \mathscr{O}^* \left(\frac{B_1' Q}{(\log 2Q)^2} + B_2' \right), \qquad (2.2)$$

$$\sum_{\substack{p \geq 2 \\ p^{v+k} \leq Q}} \sum_{\substack{v,k \geq 1}} |f(p^v)||f(p^k)| \log(p^v) + \sum_{p \geq 2} \sum_{\substack{v \geq 2 \\ p^v \leq Q}} |f(p^v)| \log(p^v) \leq A_2' \sqrt{Q} \qquad (2.3)$$

for all real $Q \geq 2$. Then we have

$$\sum_{d \leq D} f(d) = \kappa D C_{\kappa,f} (\log D)^{\kappa - 1} + \mathscr{O}^* \left(\frac{10(2 + 2\kappa - \kappa')}{(1 + \kappa - \kappa')} C_{\kappa',|f|} R_f \gamma_{|f|} D (\log D)^{\kappa' - 2} \right) \quad (2.4)$$

for any real $D > \exp(2(B_1' + B_2' + 2A_2' + \kappa'))$, where

$$\gamma_{|f|} = \kappa' + B_1' + B_2' + 1, \qquad (2.5)$$

$$R_f = 2(B_1 + B_2 + 2A_2' + \kappa + 1)\left(1 + 2(\kappa' + 1)\exp(\kappa' + 1)\right), \qquad (2.6)$$

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$$C_{\kappa(f)} = \frac{1}{\Gamma(\kappa+1)} \prod_{p \geq 2} \left(\left(1 - \frac{1}{p}\right)^{\kappa} \sum_{v \geq 0} \frac{f(p^v)}{p^v} \right), \tag{2.7}$$

$$C_{\kappa'(|f|)} = \frac{1}{\Gamma(\kappa'+1)} \prod_{p \geq 2} \left(\left(1 - \frac{1}{p}\right)^{\kappa'} \sum_{v \geq 0} \frac{|f(p^v)|}{p^v} \right). \tag{2.8}$$

The main contribution to the sums on the left hand sides of (2.1) and (2.2) come from the primes, that is, from $v = 1$. Thus κ and κ' can be thought of as the average values of f and $|f|$ respectively on the primes. Then $\frac{1}{2}(\kappa' - \kappa)$ is, intuitively, the average value of $\frac{1}{2}(|f| - f)$, the negative part of f. Therefore, the condition $1 + \kappa - \kappa' > 0$, which is the same as $\frac{1}{2} > \frac{1}{2}(\kappa' - \kappa)$, can be viewed as requiring that f takes more non-negative values than negative values on the set of prime numbers or that it "oscillates mildly" on this set.

We prove Theorem 2.1.1 in Section 2.4. This result stands on an unpublished theorem of A. Saldana which is, however, available online as [33]. For the convenience of the reader, we have stated and proved Saldana's theorem in Section 2.3 as Theorem 2.3.2. In Section 2.5, which is the final section of this chapter, we give a pair of applications of our Theorem 2.1.1. In the following section, as in the preceding chapter, we collect together various preliminaries that we will use subsequently.

2.2 Auxiliary results

We begin with an application of integration by parts.

Lemma 2.2.1. *For an arithmetical function f suppose that there exist non-negative constants κ, A, M_1 and M_2 such that*

$$\sum_{\substack{p \geq 2, v \geq 1 \\ p^v \leq x}} f(p^v) \log(p^v) = \kappa x + \mathcal{O}^* \left(M_1 \frac{x}{(\log 2x)^2} + M_2 \right), \tag{2.9}$$

$$\sum_{p \geq 2} \sum_{\substack{v, k \geq 1 \\ p^{v+k} \leq x}} |f(p^v)||f(p^k)| \log(p^v) + \sum_{p \geq 2} \sum_{\substack{v \geq 2 \\ p^v \leq x}} |f(p^v)| \log(p^v) \leq A\sqrt{x} \tag{2.10}$$

for all real $x \geq 2$. Then we have from (2.9) *that*

$$\sum_{\substack{p \geq 2, v \geq 1 \\ p^v \leq x}} \frac{f(p^v) \log(p^v)}{p^v} = \kappa \log x + \mathscr{O}^*(M_1 + M_2 + \kappa), \tag{2.11}$$

for all real $x \geq 2$ and this relation in turn gives

$$\sum_{\substack{p \geq 2, v \geq 1 \\ p^v \leq x}} \frac{f(p^v)}{p^v} = \kappa \log\left(\frac{e \log x}{\log 2}\right) + a_f + \mathscr{O}^*\left(\frac{2(M_1 + M_2 + \kappa)}{\log x}\right) \tag{2.12}$$

for all real $x \geq 2$, where a_f is a real number depending only on f given by the formula (2.19). *Finally, we have from* (2.10) *that*

$$\sum_{p \geq 2} \sum_{v,k \geq 1} \frac{|f(p^v)||f(p^k)| \log(p^v)}{p^{v+k}} + \sum_{p \geq 2} \sum_{v \geq 2} \frac{|f(p^v)| \log(p^v)}{p^v} \leq 2A. \tag{2.13}$$

Proof. Indeed, for any subset S of the natural numbers, an arithmetical function g and a real number $x \geq 1$ we have from Proposition 1.3.1 that

$$\sum_{\substack{n \in S, \\ n \leq x.}} \frac{g(n)}{n} = \frac{1}{x} \sum_{\substack{n \in S, \\ n \leq x.}} g(n) + \int_1^x \frac{1}{t^2} \sum_{\substack{n \in S, \\ n \leq t.}} g(n) \, dt. \tag{2.14}$$

Taking S to be the set of prime powers, that is, the set of p^v with p prime and $v \geq 1$ and $g(n) = f(n) \log n$ for all natural numbers n we get from (2.14) and (2.9) that

$$\sum_{\substack{p \geq 2, v \geq 1 \\ p^v \leq x}} \frac{f(p^v) \log(p^v)}{p^v} = \kappa + \mathscr{O}^*\left(\frac{M_1}{(\log 2x)^2} + \frac{M_2}{x}\right)$$

$$+ \kappa \int_2^x \frac{dt}{t} + \mathscr{O}^*\left(M_1 \int_2^x \frac{dt}{t(\log 2t)^2} + M_2 \int_2^x \frac{dt}{t^2}\right),$$

for $x \geq 2$, on noting that in this case the integrand in the integral on the right hand side of (2.14) is 0 for $1 \leq t < 2$. Consequently, for $x \geq 2$ we have

$$\left| \sum_{\substack{p \geq 2, \nu \geq 1 \\ p^\nu \leq x}} \frac{f(p^\nu) \log(p^\nu)}{p^\nu} - \kappa \log x \right| \leq \kappa(1 - \log 2) + M_1 \left(\frac{1}{(\log 2x)^2} + \int_2^x \frac{dt}{t(\log 2t)^2} \right)$$

$$+ M_2 \left(\frac{1}{x} + \int_2^x \frac{dt}{t^2} \right),$$

from which (2.11) follows on remarking that $\frac{1}{(\log 2x)^2} + \int_2^x \frac{dt}{t(\log 2t)^2} \leq 1$ and $\frac{1}{x} + \int_2^x \frac{dt}{t^2} \leq 1$ for all $x \geq 2$. To obtain (2.13) we apply (2.14) with S taken to be the set of p^ℓ with p prime and $\ell \geq 2$ and set

$$g(n) = \underset{\substack{(\nu, k), \\ n = p^{\nu+k}.}}{{\sum}'} |f(p^\nu) f(p^k)| \log(p^\nu)$$

for all natural numbers n, where the prime over the summation indicates that the sum is restricted to such pairs (ν, k) satisfying either $\nu \geq 1, k \geq 1$ or $\nu \geq 2, k = 0$. Then

$$\sum_{\substack{n \in S, \\ n \leq x.}} g(n) = \sum_{p \geq 2} \sum_{\substack{\nu, k \geq 1 \\ p^{\nu+k} \leq x}} |f(p^\nu) f(p^k)| \log(p^\nu) + \sum_{\substack{\nu \geq 2 \\ p^\nu \leq x}} |f(p^\nu)| \log(p^\nu)$$

and it follows from (2.14) and (2.10) that for all $x \geq 1$ we have

$$\sum_{p \geq 2} \underset{\substack{\nu, k \geq 1 \\ p^{\nu+k} \leq x}}{{\sum}'} \frac{|f(p^\nu) f(p^k)| \log(p^\nu)}{p^{\nu+k}} \leq \frac{A}{\sqrt{x}} + \int_1^x \frac{A}{t^{\frac{3}{2}}} dt \leq 2A. \tag{2.15}$$

Finally, to verify (2.12) we note that using Proposition 1.3.1 for any subset S of the natural numbers and an arithmetical function g we have

$$\sum_{\substack{n \in S, \\ 2 \leq n \leq x.}} \frac{g(n)}{\log n} = \frac{1}{\log x} \sum_{\substack{n \in S, \\ 2 \leq n \leq x.}} g(n) + \int_2^x \frac{1}{t(\log t)^2} \sum_{\substack{n \in S, \\ 2 \leq n \leq t.}} g(n) \, dt \tag{2.16}$$

for all real $x \geq 2$. We now take S to be the set of prime powers and $g(n) = \frac{f(n) \log n}{n}$ for all natural numbers n and observe (2.11) gives

$$\sum_{\substack{n \in S, \\ 2 \le n \le x.}} g(n) = \sum_{\substack{p \ge 2, v \ge 1 \\ p^v \le x}} \frac{f(p^v)\log(p^v)}{p^v} = \kappa \log x + \mathscr{O}^*(M_1 + M_2 + \kappa). \qquad (2.17)$$

Now we set

$$\sum_{\substack{p \ge 2, v \ge 1 \\ p^v \le x}} \frac{f(p^v)\log(p^v)}{p^v} = \kappa \log x + h(x) \qquad (2.18)$$

for all $x \ge 2$. Then we have $|h(x)| \le M_1 + M_2 + \kappa$ for all $x \ge 2$ from (2.17) and consequently that $\int_2^\infty \frac{h(t)}{t(\log t)^2}$ converges. We define a_f to be this integral. More explicitly, we have

$$a_f = \int_2^\infty \left(\sum_{\substack{p \ge 2, v \ge 1 \\ p^v \le t}} \frac{f(p^v)\log(p^v)}{p^v} - \kappa \log t \right) \frac{dt}{t(\log t)^2}. \qquad (2.19)$$

Using (2.18) in (2.16) with g and S as in (2.17), we get

$$\sum_{\substack{p \ge 2, v \ge 1 \\ p^v \le x}} \frac{f(p^v)}{p^v} = \kappa + \frac{h(x)}{\log x} + \kappa \int_2^x \frac{dt}{t \log t} + \int_2^x \frac{h(t)}{t(\log t)^2} dt$$

$$= \kappa \log \left(\frac{e \log x}{\log 2} \right) + a_f + \mathscr{O}^* \left(\frac{|h(x)|}{\log x} + \int_x^\infty \frac{|h(t)|}{t(\log t)^2} dt \right), \qquad (2.20)$$

since $\int_2^x \frac{h(t)}{t(\log t)^2} dt = a_f - \int_x^\infty \frac{h(t)}{t(\log t)^2} dt$. The relation (2.12) results from (2.20) on estimating the error terms using the bound for $h(x)$ given above.

$\square$

Our next lemma is a basic observation in the study of mean values of multiplicative functions.

Lemma 2.2.2. *Let f be a multiplicative function (not necessarily real valued). Then for any real $D \ge 1$ we have*

$$\sum_{d \leq D} f(d) \log d = \sum_{\ell \leq D} f(\ell) \sum_{\substack{p \geq 2, v \geq 1, \\ p^v \leq \frac{D}{\ell}.}} f(p^v) \log(p^v)$$

$$- \sum_{\ell \leq D} f(\ell) \sum_{\substack{p \geq 2, v \geq 1, k \geq 1, \\ (p,\ell)=1, \\ p^{v+k} \leq \frac{D}{\ell}.}} f(p^v) f(p^k) \log(p^v). \tag{2.21}$$

Proof. Using the fundamental theorem of arithmetic we have that

$$\sum_{d \leq D} f(d) \log d = \sum_{d \leq D} f(d) \sum_{p^v \| d} \log p^v = \sum_{\substack{p \geq 2, v \geq 1, \\ p^v \leq D.}} \log(p^v) \sum_{\substack{d \leq D, \\ p^v \| d.}} f(d) , \tag{2.22}$$

where $p^v \| d$ means $p^v | d$ and $(p, \frac{d}{p^v}) = 1$. On writing $d = \ell p^v$ for each $d \leq D$ and p^v such that $p^v \| d$, we have $(\ell, p) = 1$ and $f(d) = f(\ell) f(p^v)$, since f is multiplicative. Using this in the last sum in (2.22) and interchanging summations we obtain

$$\sum_{d \leq D} f(d) \log d = \sum_{\ell \leq D} f(\ell) \sum_{\substack{p \geq 2, v \geq 1, \\ (p,\ell)=1, \\ p^v \leq \frac{D}{\ell}.}} f(p^v) \log(p^v) . \tag{2.23}$$

Now we note that for any $D, \ell \geq 1$ we have

$$\sum_{\substack{p \geq 2, v \geq 1 \\ (p,\ell)=1 \\ p^v \leq \frac{D}{\ell}}} f(p^v) \log(p^v) = \sum_{\substack{p \geq 2, v \geq 1 \\ p^v \leq \frac{D}{\ell}}} f(p^v) \log(p^v) - \sum_{\substack{p \geq 2, v \geq 1 \\ p|\ell \\ p^v \leq \frac{D}{\ell}}} f(p^v) \log(p^v) .$$

Substituting this into the right hand side of (2.23) we obtain

$$\sum_{\ell \leq D} f(\ell) \sum_{\substack{p \geq 2, v \geq 1 \\ (p,\ell)=1, \\ p^v \leq \frac{D}{\ell}}} f(p^v) \log(p^v) = \sum_{\ell \leq D} f(\ell) \sum_{\substack{p \geq 2, v \geq 1 \\ p^v \leq \frac{D}{\ell}}} f(p^v) \log(p^v)$$

$$- \sum_{\ell \leq D} f(\ell) \sum_{\substack{p \geq 2, v \geq 1 \\ p|\ell \\ p^v \leq \frac{D}{\ell}}} f(p^v) \log(p^v) . \tag{2.24}$$

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In the lower sums on the right hand side of (2.24) we set, like before, $\ell = \ell' p^k$ with $(\ell', p^k) = 1$ for each $\ell \leq D$ and prime p such that $p | \ell$. Then we have $f(\ell) = f(\ell') f(p^k)$ since f is multiplicative and get

$$
\begin{aligned}
\sum_{\ell \leq D} f(\ell) \sum_{\substack{p \geq 2, v \geq 1 \\ (p, \ell) = 1, \\ p^v \leq \frac{D}{\ell}}} f(p^v) \log(p^v) = {} & \sum_{\ell \leq D} f(\ell) \sum_{\substack{p \geq 2, v \geq 1 \\ p^v \leq \frac{D}{\ell}}} f(p^v) \log(p^v) \\
& - \sum_{\ell' \leq D} f(\ell') \sum_{\substack{p \geq 2, v \geq 1, k \geq 1, \\ (p, \ell') = 1, \\ p^{v+k} \leq \frac{D}{\ell'}}} f(p^v) f(p^k) \log(p^v) .
\end{aligned}
\tag{2.25}
$$

Changing ℓ' to ℓ in the lower sums on the right hand side of the above relation and combining it with (2.23) we obtain (2.21), as required. $\qquad \square$

As a first application of Lemma 2.2.2, we relate the summatory function of a non-negative multiplicative function f to that of the multiplicative function $d \mapsto \frac{f(d)}{d}$. Our result is a minor variant of Theorem 4.22 of [3], given as Lemma 1.3.8 of the preceding chapter, and of Theorem 9.2 of [23].

Lemma 2.2.3. *Let f be a non-negative multiplicative function and K and K' be non-negative real numbers such that*

$$
\sum_{\substack{p \geq 2, v \geq 1 \\ p^v \leq Q.}} f(p^v) \log(p^v) \leq KQ + K'
\tag{2.26}
$$

for all real $Q \geq 2$. Then for all real $D \geq 1$ we have

$$
\sum_{d \leq D} f(d) \leq \frac{(K + K' + 1)D}{\log eD} \sum_{d \leq D} \frac{f(d)}{d} .
\tag{2.27}
$$

Proof. We follow [3], page 190 *et seq.*. The inequality $1 + a \leq \exp(a)$, valid for all real a, gives with $a = \log\left(\frac{D}{d}\right)$ for any real $D \geq 1$ and integer $1 \leq d \leq D$, the relation

$$\log eD = 1 + \log D \leq \log d + \frac{D}{d} \ . \tag{2.28}$$

Since f is non-negative we then deduce that

$$\log eD \sum_{d \leq D} f(d) \leq \sum_{d \leq D} f(d) \log d + D \sum_{d \leq D} \frac{f(d)}{d} \ . \tag{2.29}$$

From Lemma 2.2.2 we have

$$\sum_{d \leq D} f(d) \log d \leq \sum_{\ell \leq D} f(\ell) \sum_{\substack{p \geq 2, v \geq 1, \\ p^v \leq \frac{D}{\ell}.}} f(p^v) \log(p^v) \tag{2.30}$$

where we have dropped the second term on the right hand side of (2.21), allowed since f is non-negative. We now substitute (2.26) with $Q = \frac{D}{\ell}$ into the right hand side of (2.30), change ℓ to d and combine the resulting relation with (2.29) to get

$$\log eD \sum_{d \leq D} f(d) \leq KD \sum_{d \leq D} \frac{f(d)}{d} + K' \sum_{d \leq D} f(d) + D \sum_{d \leq D} \frac{f(d)}{d} \ . \tag{2.31}$$

Finally, we observe that since f is non-negative, $\sum_{d \leq D} f(d) \leq D \sum_{d \leq D} \frac{f(d)}{d}$. Using this in the right hand side of above relation and dividing throughout by $\log eD$ we obtain (2.27). □

The preceding lemma was deduced from (2.29) which gives an upper bound for the partial sums of a non-negative arithmetical function f in terms of the partial sums of the arithmetical function $n \mapsto f(n) \log n$. Partial summation in fact gives a more general equality as follows :

Lemma 2.2.4. *For any arithmetical function f we have*

$$(\sum_{d \leq D} f(d)) \log D = \sum_{d \leq D} f(d) \log d + \int_1^D \frac{\sum_{d \leq t} f(d)}{t} dt \tag{2.32}$$

for all real $D \geq 1$.

Proof. For any real $D \geq 1$, the partial summation formula of Proposition 1.3.1 yields

$$\sum_{d \leq D} f(d) \log d = (\sum_{d \leq D} f(d)) \log D - \int_1^D \frac{\sum_{d \leq t} f(d)}{t} dt \ . \tag{2.33}$$

from which (2.32) follows after a rearrangement of terms.

$\square$

In Section 2.3, we will apply Lemma 2.2.4 together with Lemma 2.2.2 and Lemma 2.2.3. The error terms resulting in these applications will be estimated by the explicit bounds given by the following lemma.

Lemma 2.2.5. *Let f be a non-negative arithmetical function and γ_f be a real number such that*

$$\sum_{d \leq D} f(d) \leq \frac{\gamma_f D}{\log eD} \sum_{d \leq D} \frac{f(d)}{d} \tag{2.34}$$

for all real $D \geq 1$. Then we have the inequalities

$$\sum_{\ell \leq D} \frac{f(\ell)}{\ell (\log \left(\frac{2D}{\ell} \right))^2} \leq \frac{9 \gamma_f}{\log eD} \sum_{d \leq D} \frac{f(d)}{d}, \tag{2.35}$$

$$\sum_{\ell \leq D} \frac{f(\ell)}{\sqrt{\ell}} \leq \frac{4 \gamma_f \sqrt{D}}{\log eD} \sum_{d \leq D} \frac{f(d)}{d}, \tag{2.36}$$

$$\int_1^D \frac{\sum_{d \leq t} f(d)}{t} dt \leq \frac{3 \gamma_f D}{2(\log eD)} \sum_{d \leq D} \frac{f(d)}{d} \tag{2.37}$$

for all $D \geq 1$.

Proof. We first prove (2.35). We begin by noting that

$$\sum_{\ell \leq D} \frac{f(\ell)}{\ell (\log \left(\frac{2D}{\ell} \right))^2} \leq \frac{1}{(\log 2)^2} \sum_{\ell \leq D} \frac{f(\ell)}{\ell (\log \left(\frac{eD}{\ell} \right))^2} \tag{2.38}$$

since $\log \left(\frac{2D}{\ell} \right) \geq \log 2 (\log \left(\frac{eD}{\ell} \right))$ for $1 \leq \ell \leq D$. To bound the sum on the right hand side of (2.38) via partial summation we temporarily set $\varphi(t) = \frac{1}{t(\log(\frac{eD}{t}))^2}$ for $1 \leq t \leq D$. Then on remarking that

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$$\varphi'(t) = -\frac{1}{t^2(\log\left(\frac{eD}{t}\right))^2} + \frac{2}{t^2(\log\left(\frac{eD}{t}\right))^3} \tag{2.39}$$

and applying Proposition 1.3.1 we get

$$\begin{aligned}
\sum_{\ell\leq D}\frac{f(\ell)}{\ell(\log\left(\frac{eD}{\ell}\right))^2} &= \varphi(D)\sum_{d\leq D}f(d) - \int_1^D \varphi'(t)\sum_{d\leq t}f(d)\,dt \\
&= \frac{\sum_{d\leq D}f(d)}{D} + \int_1^D \frac{\sum_{d\leq t}f(d)}{t^2(\log\left(\frac{eD}{t}\right))^2}dt - 2\int_1^D \frac{\sum_{d\leq t}f(d)}{t^2(\log\left(\frac{eD}{t}\right))^3}\,dt.
\end{aligned} \tag{2.40}$$

Since f is non-negative, the integrand in the last integral in the lower line of (2.40) is non-negative. Consequently, we have that

$$\sum_{\ell\leq D}\frac{f(\ell)}{\ell(\log\left(\frac{eD}{\ell}\right))^2} \leq \frac{\sum_{d\leq D}f(d)}{D} + \int_1^D \frac{\sum_{d\leq t}f(d)}{t^2(\log\left(\frac{eD}{t}\right))^2}\,dt\ . \tag{2.41}$$

We now use the bound (2.34) in the form

$$\sum_{d\leq t}f(d) \leq \frac{\gamma_f t}{\log et}\sum_{d\leq D}\frac{f(d)}{d} \quad \text{for } 1\leq t\leq D, \tag{2.42}$$

which is valid, again, since f is non-negative (*cf.* the remark following (2.2), page 80 of [9]). Substituting (2.42) in the right hand side of (2.41) we then obtain

$$\sum_{\ell\leq D}\frac{f(\ell)}{\ell(\log\left(\frac{eD}{\ell}\right))^2} \leq \gamma_f\left(\frac{1}{\log eD} + \int_1^D \frac{1}{t(\log\left(\frac{eD}{t}\right))^2\log et}\,dt\right)\sum_{d\leq D}\frac{f(d)}{d} \tag{2.43}$$

It remains to estimate the integral on the right hand side of (2.43). To do this we set $a = \log eD$ and note using the change of variables $\log et \mapsto u$ that

$$\int_1^D \frac{1}{t(\log\left(\frac{eD}{t}\right))^2(\log et)}\,dt = \int_1^a \frac{du}{(1+a-u)^2 u}\ . \tag{2.44}$$

We then evaluate the integral on the right hand side of the above relation using the partial fraction decomposition

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$$\frac{1}{(1+a-u)^2 u} = \frac{1}{(1+a)(1+a-u)^2} + \frac{1}{(1+a)^2(1+a-u)} + \frac{1}{(1+a)^2 u}. \tag{2.45}$$

Thus on integrating both sides of this relation we get

$$\int_1^a \frac{du}{(1+a-u)^2 u} = \frac{1}{(1+a)}\left(1 - \frac{1}{a}\right) + \frac{2\log a}{(1+a)^2}, \tag{2.46}$$

from which and (2.44) we conclude that

$$\int_1^D \frac{1}{t(\log(\frac{eD}{t}))^2(\log et)} dt = \frac{\log D}{(\log e^2 D)(\log eD)} + \frac{2\log eD}{(\log e^2 D)^2} \leq \frac{3}{\log eD}, \tag{2.47}$$

since $\log D \leq \log e^2 D$ and $(\log eD)^2 \leq (\log e^2 D)^2$ for all $D \geq 1$. Using (2.47) in (2.43), combining the result with (2.38) and finally remarking that $\frac{4}{(\log 2)^2} \leq 9$, we obtain (2.35).

We now turn to (2.36). Again, by means of Proposition 1.3.1 and the bound (2.42) we have

$$\begin{aligned}
\sum_{\ell \leq D} \frac{f(\ell)}{\sqrt{\ell}} &= \frac{\sum_{\ell \leq D} f(\ell)}{\sqrt{D}} + \frac{1}{2}\int_1^D \frac{\sum_{\ell \leq t} f(\ell)}{t^{\frac{3}{2}}} dt \\
&\leq \gamma_f\left(\frac{\sqrt{D}}{\log eD} + \int_1^D \frac{1}{(2\log et)\sqrt{t}} dt\right) \sum_{d \leq D} \frac{f(d)}{d}.
\end{aligned} \tag{2.48}$$

Since $2\log et \geq \log e^2 t$ when $t \geq 1$ we see that

$$\int_1^D \frac{1}{(2\log et)\sqrt{t}} dt \leq \int_1^D \frac{1}{(\log e^2 t)\sqrt{t}} dt = \frac{1}{e}\int_1^{\log(e\sqrt{D})} \frac{e^u}{u} du, \tag{2.49}$$

where for the equality we have used the change of variable $\log e^2 t \mapsto 2u$. Now we remark that for any $\lambda \geq 1$ we have

$$\int_1^\lambda \frac{e^u}{u} du \leq \frac{3e^\lambda}{2\lambda}. \tag{2.50}$$

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Indeed, the derivative of the function $g : \lambda \mapsto \frac{3e^{\lambda}}{2\lambda} - \int_1^{\lambda} \frac{e^u}{u} \, du$ on $[1, \infty)$ is given by $g'(\lambda) = \lambda \mapsto \frac{3e^{\lambda}}{2\lambda} \left(1 - \frac{1}{\lambda}\right) - \frac{e^{\lambda}}{\lambda}$. Thus g has a unique local minimum on $[1, \infty)$ at $\lambda = 3$. It is then easily seen that $g(3) \leq g(\lambda)$ for all $\lambda \geq 1$. Consequently, to verify (2.50) it suffices to check that $0 \leq g(3)$, which follows on noting that

$$\int_1^3 \frac{e^u}{u} \, du = \int_1^2 \frac{e^u}{u} \, du + \int_2^3 \frac{e^u}{u} \, du \leq \frac{e^2}{2} + \frac{e^3 - e^2}{2} \leq \frac{e^3}{2}, \tag{2.51}$$

since $u \mapsto \frac{e^u}{u}$ is an increasing function on $[1, 2]$ and $\int_2^3 \frac{e^u}{u} du \leq \frac{1}{2} \int_2^3 e^u du$. On putting $\lambda = \log(e\sqrt{D})$ in (2.50), taking account of (2.49) and (2.48) and remarking that $\log e\sqrt{D} \geq \frac{1}{2} \log eD$ we obtain (2.36).

Finally, we take up (2.37). We have from (2.34) and (2.42) that

$$\int_1^D \frac{\sum_{d \leq t} f(d)}{t} dt \leq \gamma_f \left(\int_1^D \frac{dt}{\log et} \right) \sum_{d \leq D} \frac{f(d)}{d} \tag{2.52}$$

for all $D \geq 1$. Using the change of variables $\log et \mapsto u$ and putting $\lambda = \log eD$ we have

$$\int_1^D \frac{dt}{\log et} = \frac{1}{e} \int_1^{\lambda} \frac{e^u}{u} \, du \leq \frac{3e^{\lambda}}{2\lambda} = \frac{3D}{2 \log eD}, \tag{2.53}$$

on recalling (2.50). The inequalities (2.53) and (2.52) yield (2.37).

$\square$

We conclude this section with a lemma that allows us to affirm that the infinite products that appear in the statement of Theorem 2.1.1 and other results of this chapter do converge.

Lemma 2.2.6. *Let f be an arithmetical function and κ, M, A be non-negative real numbers such that*

$$\sum_{\substack{p \geq 2, v \geq 1 \\ p^v \leq Q}} \frac{f(p^v) \log(p^v)}{p^v} = \kappa \log Q + \mathcal{O}^*(M), \tag{2.54}$$

$$\sum_{p \geq 2} \sum_{v, k \geq 1.} \frac{|f(p^v)||f(p^k)| \log(p^v)}{p^{v+k}} + \sum_{p \geq 2} \sum_{v \geq 2.} \frac{|f(p^v)| \log(p^v)}{p^v} \leq A, \tag{2.55}$$

where (2.54) *holds for all* $Q \geq 2$. *Then the product*

$$\prod_{p \geq 2} \left(\left(1 - \frac{1}{p}\right)^{\kappa} \sum_{v \geq 0} \frac{f(p^v)}{p^v} \right) \tag{2.56}$$

converges.

Proof. First we observe from (2.54) together with (2.12) of Lemma 2.2.1 that

$$\sum_{\substack{p \geq 2, v \geq 1 \\ p^v \leq Q}} \frac{f(p^v)}{p^v} = \kappa \log \log Q + a_f + \kappa \log \left(\frac{e}{\log 2}\right) + \mathcal{O}^* \left(\frac{2M}{\log Q}\right) \tag{2.57}$$

for all $Q \geq 2$, where a_f is given by (2.19). Since $\log p^v \geq \log 2$ for all $p \geq 2, v \geq 1$, it follows from (2.55) that

$$\sum_{p \geq 2} |\sum_{v \geq 1} \frac{f(p^v)}{p^v}|^2 \leq \sum_{p \geq 2} \sum_{v,k \geq 1} \frac{|f(p^v)||f(p^k)|}{p^{v+k}} \leq \sum_{p \geq 2} \sum_{v,k \geq 1.} \frac{|f(p^v)||f(p^k)|\log(p^v)}{p^{v+k}\log 2} \leq \frac{A}{\log 2}. \tag{2.58}$$

Thus for each prime number p, the sum $u_p = \sum_{v \geq 1} \frac{f(p^v)}{p^v}$ converges and we have that

$$\sum_{p \geq 2} |u_p|^2 \leq \frac{A}{\log 2}. \tag{2.59}$$

Also, from (2.55) again, we have

$$|\sum_{p \leq Q} \sum_{p^v > Q} \frac{f(p^v)}{p^v}| \leq \sum_{p \geq 2} \sum_{\substack{v \geq 2, \\ p^v > Q.}} \frac{|f(p^v)|}{p^v} \leq \frac{1}{\log Q} \sum_{p \geq 2} \sum_{v \geq 2.} \frac{|f(p^v)|\log(p^v)}{p^v} \leq \frac{A}{\log Q}. \tag{2.60}$$

Combining this with (2.57) using the triangle inequality we obtain

$$\sum_{p \leq Q} u_p = \kappa \log \log Q + a_f + \kappa \log \left(\frac{e}{\log 2}\right) + \mathcal{O}^* \left(\frac{2M + A}{\log Q}\right) \tag{2.61}$$

for all $Q \geq 2$. From (2.61) and Mertens formula (Theorem 1.10, part I of [34]) we then

conclude that $\sum_{p\geq2}(u_p - \frac{\kappa}{p})$ converges. Further, if we set $v_p = -\frac{\kappa u_p}{p}$ for each prime p then we have

$$(1 - \frac{\kappa}{p})(1 + u_p) = 1 + u_p - \frac{\kappa}{p} + v_p\,.$$

Moreover, $\sum_{p\geq2}|v_p| \leq \kappa\sqrt{\frac{A}{\log 2}}\sqrt{\sum_{p\geq2}\frac{1}{p^2}}$ by the Cauchy-Schwarz inequality and (2.59). We then deduce from Theorem 4.3, Part III of [34] that the product $\prod_{p\geq2\kappa}(1 - \frac{\kappa}{p})(1 + u_p)$ converges. This implies and that the product (2.56), which is the same as $\prod_{p\geq2}(1 - \frac{1}{p})^\kappa(1 + u_p)$, converges. To see this it suffices to note that

$$\frac{\prod_{p\geq2\kappa}(1 - \frac{1}{p})^\kappa(1 + u_p)}{\prod_{p\geq2\kappa}(1 - \frac{\kappa}{p})(1 + u_p)} = \prod_{p\geq2\kappa} \exp\left(\kappa\log(1 - \frac{1}{p}) - \log(1 - \frac{\kappa}{p})\right) \qquad (2.62)$$

converges, which in turn follows from the inequality

$$|\kappa\log(1 - \frac{1}{p}) - \log(1 - \frac{\kappa}{p})| \leq \frac{\kappa + \kappa^2}{p^2}, \qquad (2.63)$$

obtainable from (4.6), page 477 of [34] since $\frac{\kappa}{p} \leq \frac{1}{2}$, and the fact that $\sum_p \frac{1}{p^2}$ converges.

$\square$

2.3   Some Asymptotic Formulae

We begin with Theorem 21.1 of [23], which gives an explicit asymptotic formula for the partial sums of a non-negative multiplicative function f such that $f(p)$ behaves, intuitively speaking, on the average over the primes as $\frac{\kappa'}{p}$ for some fixed $\kappa' \geq 0$. This requirement is expressed by the condition (2.64) below.

Theorem 2.3.1. *Let f be a non-negative multiplicative function. Suppose there exist non-negative real numbers κ', L_1' and A_1' such that*

$$\sum_{\substack{p\geq2,v\geq1 \\ p^v\leq Q}} f(p^v)\log(p^v) = \kappa'\log Q + \mathscr{O}^*(L_1') \qquad (2.64)$$

$$\sum_{p\geq 2}\sum_{v,k\geq 1} f(p^v)f(p^k)\log(p^v) + \sum_{p\geq 2}\sum_{v\geq 2} f(p^v)\log(p^v) \leq A_1' \qquad (2.65)$$

where (2.64) *holds for all real $Q \geq 2$. Then for all real $D \geq \exp(2(L_1' + A_1'))$ we have*

$$\sum_{d\leq D} f(d) = C_{\kappa'(f)}(\log D)^{\kappa'}\left(1 + \mathcal{O}^*\left(\frac{R_f}{\log D}\right)\right),$$

where

$$R_f = 2(L_1' + A_1')\left(1 + 2(\kappa' + 1)\exp(\kappa' + 1)\right), \qquad (2.66)$$

$$C_{\kappa'(f)} = \frac{1}{\Gamma(\kappa' + 1)}\prod_{p\geq 2}\left(\left(1 - \frac{1}{p}\right)^{\kappa'}\sum_{v\geq 0} f(p^v)\right). \qquad (2.67)$$

Proof. See page 178 *et seq.* of [23]. Our hypotheses are stronger : the second sum on left hand side of (2.65) does not appear in the corresponding hypothesis of Theorem 21.2 of [23]. The boundedness of this sum, as expressed by (2.65) is, however, only required to ensure the convergence of the product defining $C_{\kappa'(f)}$ above, which now follows on applying Lemma 2.2.6 to the function $n \mapsto nf(n)$. Note that this function satisfies the conditions of Lemma 2.2.6 on account of (2.64) and (2.65). For the rest of the assertions of Theorem 2.3.1, the proof is the same as in that of Theorem 21.1 of [23].

$\square$

The theorem that follows is, up to minor modifications, Théorème 1 of A. Saldana [33], mentioned in the final paragraph of Section 2.1. This result generalises Theorem 2.3.1 to multiplicative functions f that are "mildly oscillating" on the primes (see the paragraph following the statement of Theorem 2.1.1). Indeed, on taking f to be non-negative real valued and $\kappa = \kappa'$, $L = L'$, $A_1 = A_1'$ in Theorem 2.3.2 we recover Theorem 2.3.1 except for the minor factor $(2 + \kappa)$ in the error term. On the other hand, Theorem 2.3.1 is an important step in the proof of the result below.

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Theorem 2.3.2. *Let f be a real valued multiplicative function. Suppose there exist non-negative real numbers κ, κ', L, L' and A_1 with $1 + \kappa - \kappa' > 0$ satisfying*

$$\sum_{\substack{p \geq 2, v \geq 1 \\ p^v \leq Q}} f(p^v) \log(p^v) = \kappa \log Q + \mathscr{O}^*(L), \tag{2.68}$$

$$\sum_{\substack{p \geq 2, v \geq 1 \\ p^v \leq Q}} |f(p^v)| \log(p^v) = \kappa' \log Q + \mathscr{O}^*(L'), \tag{2.69}$$

$$\sum_{p \geq 2} \sum_{v, k \geq 1} |f(p^k) f(p^v)| \log(p^v) + \sum_{p \geq 2} \sum_{v \geq 2} |f(p^v)| \log(p^v) \leq A_1, \tag{2.70}$$

where (2.68) and (2.69) hold for all real $Q \geq 2$. Then for all real $D \geq \exp 2(L' + A_1)$ we have

$$\sum_{d \leq D} f(d) = C_{\kappa(f)} (\log D)^\kappa \left(1 + \mathscr{O}^* \left(\left(\frac{2 + 2\kappa - \kappa'}{1 + \kappa - \kappa'} \right) \cdot \frac{C_{\kappa'(|f|)}}{C_{\kappa(f)}} \cdot \frac{R_f}{(\log D)^{1 + \kappa - \kappa'}} \right) \right) \tag{2.71}$$

where

$$R_f = 2(L + A_1) \left(1 + 2(\kappa' + 1) \exp(\kappa' + 1) \right), \tag{2.72}$$

$$C_{\kappa(f)} = \frac{1}{\Gamma(\kappa + 1)} \prod_{p \geq 2} \left(\left(1 - \frac{1}{p} \right)^\kappa \sum_{v \geq 0} f(p^v) \right), \tag{2.73}$$

$$C_{\kappa'(|f|)} = \frac{1}{\Gamma(\kappa' + 1)} \prod_{p \geq 2} \left(\left(1 - \frac{1}{p} \right)^{\kappa'} \sum_{v \geq 0} |f(p^v)| \right). \tag{2.74}$$

Proof. We follow A. Saldana [33] with some changes in the exposition. The proof will be completed on page 62. It will be convinient to set

$$F(D) = \sum_{d \leq D} f(d) \text{ and } \bar{F}(D) = \sum_{d \leq D} |f(d)| \tag{2.75}$$

for any real $D \geq 1$. Since f is a multiplicative function, $|f|$ is a non-negative multiplicative function. Moreover, the conditions (2.69) and (2.70) tell us that $|f|$ satisfies the hypotheses

of Theorem 2.3.1 from which we then have that

$$\bar{F}(D) = C_{\kappa'(|f|)}(\log D)^{\kappa'}\left(1 + \mathscr{O}^*\left(\frac{R_{|f|}}{\log D}\right)\right) \tag{2.76}$$

when $D \geq \exp(2(L' + A_1))$, where we have set

$$R_{|f|} = 2(L_1' + A_1)\left(1 + 2(\kappa' + 1)\exp(\kappa' + 1)\right). \tag{2.77}$$

The relation (2.33) from the proof of Lemma 2.2.4 can be rewritten as

$$\sum_{d \leq D} f(d)\log d = F(D)\log D - \int_1^D \frac{F(t)}{t}dt, \tag{2.78}$$

for any real $D \geq 1$. We will apply Lemma 2.2.2 to treat the left hand side of (2.78). Thus from the relation (2.21) we have

$$\sum_{d \leq D} f(d)\log d = \sum_{\ell \leq D} f(\ell) \sum_{\substack{p \geq 2, v \geq 1, \\ p^v \leq \frac{D}{\ell}.}} f(p^v)\log(p^v)$$

$$- \sum_{\ell \leq D} f(\ell) \sum_{\substack{p \geq 2, v \geq 1, k \geq 1, \\ (p,\ell)=1, \\ p^{v+k} \leq \frac{D}{\ell}.}} f(p^v)f(p^k)\log(p^v). \tag{2.79}$$

On using (2.68) and the triangle inequality we see that

$$\sum_{\ell \leq D} f(\ell) \sum_{\substack{p \geq 2, v \geq 1, \\ p^v \leq \frac{D}{\ell}.}} f(p^v)\log(p^v) = \kappa \sum_{\ell \leq D} f(\ell)\log\left(\frac{D}{\ell}\right) + \mathscr{O}^*\left(L\bar{F}(D)\right). \tag{2.80}$$

Since $\log\left(\frac{D}{\ell}\right) = \log D - \log \ell$, it follows from (2.78) that

$$\sum_{\ell \leq D} f(\ell)\log\left(\frac{D}{\ell}\right) = F(D)\log D - \sum_{d \leq D} f(d)\log d = \int_1^D \frac{F(t)}{t}dt, \tag{2.81}$$

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so that (2.80) can be rewritten as

$$\sum_{\ell \leq D} f(\ell) \sum_{\substack{p \geq 2, v \geq 1, \\ p^v \leq \frac{D}{\ell}.}} f(p^v) \log(p^v) = \kappa \int_1^D \frac{F(t)}{t} dt + \mathcal{O}^* (L\bar{F}(D)). \tag{2.82}$$

Now an application of the triangle inequality together with (2.70) shows that

$$| \sum_{\ell \leq D} f(\ell) \sum_{\substack{p \geq 2, v \geq 1, k \geq 1, \\ (p,\ell)=1, \\ p^{v+k} \leq \frac{D}{\ell}.}} f(p^v) f(p^k) \log(p^v)| \leq A_1 \bar{F}(D), \tag{2.83}$$

on ignoring the conditions $(p, \ell) = 1$ and $p^{v+k} \leq \frac{D}{\ell}$. From (2.83), (2.82), (2.79) and (2.78) we derive the fundamental relation

$$F(D) \log D = (1 + \kappa) \int_1^D \frac{F(t)}{t} dt + \mathcal{O}^* ((L+A_1)\bar{F}(D)) \tag{2.84}$$

for all real $D \geq 1$. To exploit this relation let us temporarily set

$$\varphi(t) = \frac{1}{(\log t)^{\kappa+1}} \int_1^t \frac{F(u)}{u} du \text{ for all real } t > 1. \tag{2.85}$$

Then the function $\varphi : t \mapsto \varphi(t)$ is a continuous function on $(1, +\infty)$ that is differentiable at all points of this interval except the integers. Indeed, when t is not an integer we have

$$\varphi'(t) = \frac{F(t)}{t(\log t)^{\kappa+1}} - \frac{(1+\kappa)}{t(\log t)^{\kappa+2}} \int_1^t \frac{F(u)}{u} du, \tag{2.86}$$

from which we also see that φ' is continuous on the complement of the set of integers in $(1, +\infty)$ and moreover, using (2.84) we have that

$$t(\log t)^{\kappa+2} \varphi'(t) = F(t) \log t - (1+\kappa) \int_1^t \frac{F(u)}{u} du = \mathcal{O}^* ((L+A_1)\bar{F}(t)) \tag{2.87}$$

when $t \geq \exp(2(L'+A_1)) = t_0, t \notin \mathbf{Z}^+ \cup \{0\}$, from which using (2.76) we conclude that

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$$t(\log t)^{\kappa+2}\varphi'(t) \ll (\log t)^{\kappa'} \quad \text{when } t \geq t_0, t \notin \mathbf{Z}^+ \cup \{0\}. \tag{2.88}$$

Consequently, we obtain that

$$\int_1^{+\infty} \varphi'(t)\,dt = \int_1^{t_0} \varphi'(t)\,dt + \int_{t_0}^{+\infty} \varphi'(t)\,dt \ll 1 + \int_{t_0}^{+\infty} \frac{dt}{t(\log t)^{2+\kappa-\kappa'}} \ll 1, \tag{2.89}$$

taking account of the condition $1 + \kappa - \kappa' > 0$. In other words, φ' is integrable on $(1, +\infty)$. Further, the function $\psi : t \mapsto \varphi(t) - \int_1^t \varphi'(u)du$ is a continuous function on $(1, +\infty)$ which is differentiable at each t in $(1, +\infty)$ that is not an integer and, for such t, we have $\psi'(t) = \varphi'(t) - \varphi'(t) = 0$. By a variant of the classical mean value theorem (see (8.5.2), page 160 and the remark following the proof of (8.6.1), page 162 of [5]) it follows that ψ is a constant function on $(1, +\infty)$. Thus there exists a real number C_0 such that

$$\varphi(t) = \int_1^t \varphi'(u)du + C_0 \text{ for all } t \in (1, +\infty). \tag{2.90}$$

On account of (2.89) we may write $\int_1^t \varphi'(u)du = \int_1^{+\infty} \varphi'(u)du - \int_t^{+\infty} \varphi'(u)du$ for all t in $(1, +\infty)$. Using (2.87) we then get

$$\varphi(t) = \int_1^{+\infty} \varphi'(u)\,du + C_0 + \mathscr{O}^* \left((L+A_1) \int_t^{+\infty} \frac{\bar{F}(u)}{u(\log u)^{\kappa+2}}\,du \right) \tag{2.91}$$

when $t \geq t_0$. On recalling the definition of $\varphi(t)$ from (2.85) and writing C_1 for $\int_1^{+\infty} \varphi'(u)\,du + C_0$ we then conclude that

$$\int_1^t \frac{F(u)}{u}du = C_1(\log t)^{\kappa+1} + \mathscr{O}^* \left((L+A_1)(\log t)^{\kappa+1} \int_t^{+\infty} \frac{\bar{F}(u)}{u(\log u)^{\kappa+2}}\,du \right) \tag{2.92}$$

for all $t \geq t_0$, which we combine with (2.84) and divide throughout the resulting relation by $\log t$ to deduce that

$$F(t) = (1+\kappa)C_1(\log t)^{\kappa} + \mathcal{O}^*\left((L+A_1)\left(\frac{\bar{F}(t)}{\log t} + (1+\kappa)(\log t)^{\kappa}\int_t^{+\infty}\frac{\bar{F}(u)}{u(\log u)^{\kappa+2}}du\right)\right)$$

$$(2.93)$$

for all $t \geq t_0 = \exp(2(L'+A_1))$. Let us estimate the error term in the above relation using (2.76). We set

$$a(\kappa) = 1 + 2(\kappa+1)\exp(\kappa+1) \text{ for } \kappa \geq 0 \tag{2.94}$$

and note from (2.77) that

$$R_{|f|} = (\log t_0)a(\kappa'). \tag{2.95}$$

Thus for $t \geq t_0$ we have $\frac{R_{|f|}}{\log t_0} > 1$ and therefore for such t the bound (2.76) gives

$$\bar{F}(t) \leq C_{\kappa'(|f|)}(\log t)^{\kappa'}\left(1+a(\kappa')\right). \tag{2.96}$$

Let us temporarily write $g(t)$ for the right hand side of the above relation. Then it follows from (2.93) that

$$F(t) = (1+\kappa)C_1(\log t)^{\kappa} + \mathcal{O}^*\left((L+A_1)\left(\frac{g(t)}{\log t} + (1+\kappa)(\log t)^{\kappa}\int_t^{+\infty}\frac{g(u)}{u(\log u)^{\kappa+2}}du\right)\right)$$

$$(2.97)$$

for $t \geq t_0$. Now we have

$$\frac{g(t)}{\log t} + (1+\kappa)(\log t)^{\kappa}\int_t^{+\infty}\frac{g(u)}{u(\log u)^{\kappa+2}}du = C_{\kappa'(|f|)}(1+a(\kappa'))\left(\frac{2+2\kappa-\kappa'}{1+\kappa-\kappa'}\right)\frac{1}{(\log t)^{1-\kappa'}}$$

$$(2.98)$$

on evaluating the integral on the left hand side of (2.98) taking account of the condition $1+\kappa-\kappa' > 0$. Also, we have $1+a(\kappa') \leq 2a(\kappa')$, since $1 \leq a(\kappa')$, and $R_f = 2(L+A_1)a(\kappa')$

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from (2.72). We then conclude from (2.97) that

$$F(t) = (\log t)^\kappa \left((1+\kappa)C_1 + \mathcal{O}^* \left(\left(\frac{2+2\kappa-\kappa'}{1+\kappa-\kappa'} \right) \cdot \frac{C_{\kappa'(|f|)}R_f}{(\log t)^{1+\kappa-\kappa'}} \right) \right) \qquad (2.99)$$

for all $t \geq t_0 = \exp(2(L'+A_1))$. We shall now identify $(1+\kappa)C_1$ as $C_{\kappa(f)}$ of (2.73). The argument is standard. For any real $s > 0$ we have by the partial summation formula of Proposition 1.3.1 that

$$\sum_{1 \leq n \leq N} \frac{f(n)}{n^s} = \frac{F(N)}{N^s} + s \int_1^N \frac{F(t)}{t^{s+1}} dt. \qquad (2.100)$$

Since $1+\kappa-\kappa' > 0$, it follows from (2.99) that

$$F(t) \sim (1+\kappa)C_1 (\log t)^\kappa \text{ as } t \to +\infty, \qquad (2.101)$$

from which we see that $\frac{F(N)}{N^s} \to 0$ and that the integral on the right hand side of (2.100) converges as $N \to +\infty$. We therefore obtain

$$\sum_{n \geq 1} \frac{f(n)}{n^s} = s \int_1^{+\infty} \frac{F(t)}{t^{s+1}} dt \qquad (2.102)$$

for all real $s > 0$. We write $L(f,s)$ for the left hand side of this relation, which is called the Dirichlet series associated to f. For $t \geq t_0$ we set

$$F(t) = (\log t)^\kappa \left((1+\kappa)C_1 + h(t) \right). \qquad (2.103)$$

Then it follows from (2.99) that $|h(t)| \ll (\log t)^{-(1+\kappa-\kappa')}$ for $t \geq t_0$ so that $h : t \mapsto h(t)$ is a bounded measurable function on $[t_0, +\infty)$ that tends to 0 as $t \to +\infty$. We write

$$\int_1^{+\infty} \frac{F(t)}{t^{s+1}} dt = \int_1^{t_0} \frac{F(t)}{t^{s+1}} dt + \int_{t_0}^{+\infty} \frac{(\log t)^\kappa \left((1+\kappa)C_1 + h(t) \right)}{t^{s+1}} dt . \qquad (2.104)$$

We multiply throughout the above relation by $s^{\kappa+1}$ and make the change of variable $t \mapsto e^{\frac{t}{s}}$

in the second integral on its right hand side to get using (2.102) that

$$s^{\kappa}L(f,s) = s^{1+\kappa}\int_1^{t_0}\frac{F(t)}{t^{s+1}}dt + \int_{s\log t_0}^{+\infty}t^{\kappa}e^{-t}\left((1+\kappa)C_1 + h(e^{\frac{t}{s}})\right)dt. \qquad (2.105)$$

Since $F(t) \leq \bar{F}(t_0)$ for $t \leq t_0$, we have $s\int_1^{t_0}\frac{F(t)}{t^{s+1}}dt \leq \bar{F}(t_0)\int_1^{+\infty}\frac{s}{t^{s+1}}dt \leq \bar{F}(t_0)$. We then obtain from (2.105) that

$$s^{\kappa}L(f,s) = \int_{s\log t_0}^{+\infty}t^{\kappa}e^{-t}\left((1+\kappa)C_1 + h(e^{\frac{t}{s}})\right)dt + \mathscr{O}^*(s^{\kappa}\bar{F}(t_0)) \qquad (2.106)$$

for all real $s > 0$. Now we let $s \to 0^+$ in the above relation. Since h is bounded on $[t_0, +\infty)$ and $h(e^{\frac{t}{s}}) \to 0$ as $s \to 0^+$ for each t in $[t_0, +\infty)$, it follows from (2.106) by an application of the dominated covergence theorem that

$$\lim_{s \to 0^+}s^{\kappa}L(f,s) = (1+\kappa)C_1\int_0^{+\infty}t^{\kappa}e^{-t}dt = (1+\kappa)C_1\Gamma(1+\kappa), \qquad (2.107)$$

by the definition of the Gamma function Γ. On recalling that for the Riemann zeta function ζ we have $\lim_{s\to 0^+}s\zeta(s+1) = 1$ and $\zeta(s+1) \neq 0$ for $s > 0$ we obtain

$$\lim_{s \to 0^+}s^{\kappa}L(f,s) = \lim_{s \to 0^+}s^{\kappa}\zeta(s+1)^{\kappa}L(f,s)\zeta(s+1)^{-\kappa} = \lim_{s \to 0^+}L(f,s)\zeta(s+1)^{-\kappa}. \qquad (2.108)$$

Finally, using the Euler product formula for ζ we write

$$L(f,s)\zeta(s+1)^{-\kappa} = \prod_{p\geq 2}\left(\left(1-\frac{1}{p^{s+1}}\right)^{\kappa}\sum_{v\geq 0}\frac{f(p^v)}{p^{vs}}\right) \qquad (2.109)$$

when $s > 0$. As remarked in the proof of Theorem 2.3.1, it follows from Lemma 2.2.6 and the conditions (2.68) and (2.70) that the right hand side of (2.109) for $s = 0$, which is by definition $C_{\kappa(f)}$, converges as well. In particular, $C_{\kappa(f)} \neq 0$. Moreover, we have that the right hand side of (2.109) converges uniformly as $s \to 0^+$. Thus on passing to this limit in (2.109) and combining the result with (2.108), (2.107) and (2.73) we deduce that

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$$(1+\kappa)C_1 = \frac{1}{(\Gamma(\kappa+1)} \prod_{p\geq 2}\left(\left(1-\frac{1}{p}\right)^{\kappa}\sum_{v\geq 0}f(p^v)\right) = C_{\kappa(f)}. \qquad (2.110)$$

Substituting this into (2.99) and changing t to D we obtain (2.71) after a rearrangement of terms.

$\square$

The next theorem is the analogue of Theorem 2.3.1 for non-negative multiplicative functions f such that $f(p)$ behaves, intuitively speaking, on the average over the primes as κ', for some fixed $\kappa' \geq 0$. This requirement is expressed by the condition (2.111) below. Our principal result, Theorem 2.1.1, is a generalisation of this theorem to multiplicative functions f that are "mildly oscillating" on the primes.

Theorem 2.3.3. *Let f be a non-negative multiplicative function and suppose that there exist non-negative real numbers κ', A_2', B_1' and B_2' such that for all $Q \geq 1$ we have*

$$\sum_{\substack{p\geq 2, v\geq 1 \\ p^v\leq Q}} f(p^v)\log(p^v) = \kappa'Q + \mathscr{O}^*\left(B_1'\frac{Q}{(\log 2Q)^2} + B_2'\right), \qquad (2.111)$$

$$\sum_{\substack{p\geq 2 \\ p^{v+k}\leq Q.}}\sum_{\substack{v,k\geq 1,}} f(p^v)f(p^k)\log(p^v) + \sum_{p\geq 2}\sum_{\substack{v\geq 2, \\ p^v\leq Q.}} f(p^v)\log(p^v) \leq A_2'\sqrt{Q}. \qquad (2.112)$$

Then we have that

$$\sum_{d\leq D} f(d) = \kappa'C_{\kappa'(f)}D(\log D)^{\kappa'-1} + \mathscr{O}^*\left(10C_{\kappa'(f)}R_f\gamma_f D(\log D)^{\kappa'-2}\right), \qquad (2.113)$$

for all real numbers $D > \exp(2(B_1' + B_2' + 2A_2' + \kappa'))$ where

$$\gamma_f = \kappa' + B_1' + B_2' + 1, \qquad (2.114)$$

$$R_f = 2(B_1' + B_2' + 2A_2' + \kappa' + 1)\left(1 + 2(\kappa'+1)\exp(\kappa'+1)\right), \qquad (2.115)$$

$$C_{\kappa'(f)} = \frac{1}{\Gamma(\kappa'+1)}\prod_{p\geq 2}\left(\left(1-\frac{1}{p}\right)^{\kappa'}\sum_{v\geq 0}\frac{f(p^v)}{p^v}\right). \qquad (2.116)$$

Proof. This theorem is an explicit version of Theorem 21.2 of [23]. Our proof for the most part follows the method of [23], making various bounds explicit with the aid of the preliminary results that we have already obtained. The proof will be completed on page 66.

Let us first dispose of the convergence of the product $C_{\kappa'(f)}$. Indeed, we have from (2.111) and (2.112) together with (2.11) and (2.13) of Lemma 2.2.1 that

$$\sum_{\substack{p \geq 2, v \geq 1 \\ p^v \leq Q}} \frac{f(p^v) \log p^v}{p^v} = \kappa' \log Q + \mathscr{O}^* \left(B_1' + B_2' + \kappa' \right), \tag{2.117}$$

$$\sum_{p \geq 2} \sum_{v,k \geq 1.} \frac{f(p^v) f(p^k) \log(p^v)}{p^{v+k}} + \sum_{p \geq 2} \sum_{v \geq 2} \frac{f(p^v) \log(p^v)}{p^v} \leq 2A_2'. \tag{2.118}$$

Then it follows from Lemma 2.2.6 that the product $C_{\kappa'(f)}$ converges. Now we note from (2.111) that

$$\sum_{\substack{p \geq 2, v \geq 1 \\ p^v \leq Q}} f(p^v) \log(p^v) \leq (\kappa' + B_1')Q + B_2', \tag{2.119}$$

for $Q \geq 2$, since then $\log 2Q \geq 1$. As f is non-negative, (2.27) of Lemma 2.2.3 applied to f gives

$$\sum_{d \leq D} f(d) \leq \frac{\gamma_f D}{\log eD} \sum_{d \leq D} \frac{f(d)}{d} \tag{2.120}$$

for all $D \geq 1$, with γ_f as in (2.114). We propose to improve the upper bound (2.120) to the asymptotic formula (2.113). As in the proof of Theorem 2.3.2, we begin by recalling that (2.32) of Lemma 2.2.4 gives

$$(\sum_{d \leq D} f(d)) \log D = \sum_{d \leq D} f(d) \log d + \int_1^D \frac{\sum_{d \leq t} f(d)}{t} dt \tag{2.121}$$

for all real $D \geq 1$ and use (2.21) of Lemma 2.2.2 to treat the sum $\sum_{d \leq D} f(d) \log d$ in (2.121). To this end, we note that for any $1 \leq \ell \leq D$, our hypotheses (2.111) and (2.112) give

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$$\sum_{\substack{p \geq 2, \nu \geq 1, \\ p^\nu \leq \frac{D}{\ell}.}} f(p^\nu) \log(p^\nu) = \frac{\kappa' D}{\ell} + \mathscr{O}^* \left(\frac{B_1' D}{\ell (\log\left(\frac{2D}{\ell}\right))^2} + B_2' \right), \tag{2.122}$$

$$\sum_{\substack{p \geq 2, \nu, k \geq 1 \\ (p, \ell) = 1, \\ p^{\nu+k} \leq \frac{D}{\ell}}} f(p^\nu) f(p^k) \log(p^\nu) \leq A_2' \sqrt{\frac{D}{\ell}}, \tag{2.123}$$

where to obtain (2.123) from (2.112) we have ignored the condition (p, ℓ) on the left hand side of (2.123), as we may since f non-negative. Substituting (2.122) into the first sum on the right hand side of (2.21) and applying the triangle inequality taking account of (2.123) we deduce that

$$\begin{aligned}
\sum_{d \leq D} f(d) \log d &= \kappa' D \sum_{\ell \leq D} \frac{f(\ell)}{\ell} \\
&+ \mathscr{O}^* \left(B_1' D \sum_{\ell \leq D} \frac{f(\ell)}{\ell (\log\left(\frac{2D}{\ell}\right))^2} + B_2' \sum_{\ell \leq D} f(\ell) + A_2' \sqrt{D} \sum_{\ell \leq D} \frac{f(\ell)}{\sqrt{\ell}} \right).
\end{aligned} \tag{2.124}$$

On combining (2.124) with (2.121) we then derive that

$$\begin{aligned}
(\sum_{d \leq D} f(d)) \log D &= \kappa' D \sum_{\ell \leq D} \frac{f(\ell)}{\ell} \\
&+ \mathscr{O}^* \left(B_1' D \sum_{\ell \leq D} \frac{f(\ell)}{\ell (\log\left(\frac{2D}{\ell}\right))^2} + B_2' \sum_{\ell \leq D} f(\ell) + A_2' \sqrt{D} \sum_{\ell \leq D} \frac{f(\ell)}{\sqrt{\ell}} \right) \\
&+ \int_1^D \frac{\sum_{d \leq t} f(d)}{t} dt
\end{aligned} \tag{2.125}$$

for all $D \geq 1$. We now use the bound (2.120) and the bounds that follow from it, given by Lemma 2.2.5, to estimate the integral and the sums in error term of (2.125). On dividing throughout the resulting relation by $\log D$ we obtain

$$\sum_{d \leq D} f(d) = \left(\kappa' + \mathcal{O}^* \left(\frac{(9B_1' + B_2' + 4A_2' + 2)\gamma_f}{\log D} \right) \right) \frac{D}{\log D} \sum_{d \leq D} \frac{f(d)}{d} \qquad (2.126)$$

for all $D > 1$, where we have changed ℓ to d and used $\log eD > \log D$ in the error term. The relations (2.117) and (2.118) tell us that the multiplicative function $n \mapsto \frac{f(n)}{n}$ satisfies the hypotheses of Theorem 2.3.1. This theorem then gives

$$\sum_{d \leq D} \frac{f(d)}{d} = C_{\kappa'(f)} (\log D)^{\kappa'} \left(1 + \mathcal{O}^* \left(\frac{R}{\log D} \right) \right), \qquad (2.127)$$

when

$$D > \exp(2(B_1' + B_2' + 2A_2' + \kappa')) \geq 1, \qquad (2.128)$$

where $C_{\kappa'(f)}$ is given by (2.116) in this case and

$$R = 2(B_1' + B_2' + 2A_2' + \kappa') \left(1 + 2(\kappa' + 1)\exp(\kappa' + 1) \right). \qquad (2.129)$$

Substituting (2.127) into the right hand side of (2.126) we conclude that for any D satisfying (2.128) we have

$$\sum_{d \leq D} f(d) = C_{\kappa'(f)} D (\log D)^{\kappa'-1} \left(1 + \mathcal{O}^* \left(\frac{R}{\log D} \right) \right) \left(\kappa' + \mathcal{O}^* \left(\frac{S}{\log D} \right) \right), \qquad (2.130)$$

where for brevity we have temporarily set $S = (9B_1' + B_2' + 4A_2' + 2)\gamma_f$. Also, on writing $a(\kappa')$ for $1 + 2(\kappa' + 1)\exp(\kappa' + 1)$ as in the proof of Theorem 2.3.2, we see that $\frac{R}{\log D} < a(\kappa')$ when D satisfies (2.128). Thus on multiplying out the last two terms on the right hand side of (2.130) we obtain

$$\sum_{d \leq D} f(d) = C_{\kappa'(f)} D (\log D)^{\kappa'-1} \left(\kappa' + \mathcal{O}^* \left(\frac{\kappa' R + S + a(\kappa')S}{\log D} \right) \right). \qquad (2.131)$$

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With R_f as in (2.115) we have $\kappa' R < R_f \gamma_f$, since $\kappa' \leq \gamma_f$ and $R < R_f$. Also, we have $(1 + a(\kappa'))S \leq 9R_f \gamma_f$, since $1 + a(\kappa') < 2a(\kappa')$ as $1 < a(\kappa')$. Using these bounds in the error term of (2.131) we deduce that

$$\sum_{d \leq D} f(d) = C_{\kappa'(f)} D(\log D)^{\kappa'-1} \left(\kappa' + \mathscr{O}^* \left(\frac{10R_f \gamma_f}{\log D} \right) \right), \tag{2.132}$$

when D satisfies (2.128), from which (2.113) follows after an obvious rearrangement of terms.

$\square$

2.4   Proof of Theorem 2.1.1

The proof is largely similar to the proof of Theorem 2.3.3, which may be looked up for any details missing here.

We have from (2.1), (2.2) and (2.3) together with (2.11) and (2.13) of Lemma 2.2.1 that

$$\sum_{\substack{p \geq 2, v \geq 1 \\ p^v \leq Q}} \frac{f(p^v) \log p^v}{p^v} = \kappa \log Q + \mathscr{O}^* (B_1 + B_2 + \kappa), \tag{2.133}$$

$$\sum_{\substack{p \geq 2, v \geq 1 \\ p^v \leq Q}} \frac{|f(p^v)| \log p^v}{p^v} = \kappa' \log Q + \mathscr{O}^* (B'_1 + B'_2 + \kappa'), \tag{2.134}$$

$$\sum_{p \geq 2} \sum_{v,k \geq 1.} \frac{|f(p^v)||f(p^k)| \log(p^v)}{p^{v+k}} + \sum_{p \geq 2} \sum_{v \geq 2} \frac{|f(p^v)| \log(p^v)}{p^v} \leq 2A'_2. \tag{2.135}$$

It follows from these relations and Lemma 2.2.6 that the products $C_{\kappa(f)}$ and $C_{\kappa'(|f|)}$ of (2.7), (2.8) converge. From (2.27) of Lemma 2.2.3 applied to the non-negative multiplicative function $|f|$ we get

$$\sum_{d \leq D} |f(d)| \leq \frac{\gamma_{|f|} D}{\log eD} \sum_{d \leq D} \frac{|f(d)|}{d} \tag{2.136}$$

for all $D \geq 1$, with $\gamma_{|f|}$ as in (2.5). Next, we have from (2.32) that

$$\left(\sum_{d\leq D} f(d)\right)\log D = \sum_{d\leq D} f(d)\log d + \int_1^D \frac{\sum_{d\leq t} f(d)}{t}\,dt \qquad (2.137)$$

for all real $D \geq 1$. Now we note that for any $1 \leq \ell \leq D$, our hypotheses (2.1) and (2.3) give

$$\sum_{\substack{p\geq 2,\nu\geq 1,\\ p^\nu\leq \frac{D}{\ell}.}} f(p^\nu)\log(p^\nu) = \frac{\kappa D}{\ell} + \mathscr{O}^*\left(\frac{B_1 D}{\ell(\log\left(\frac{2D}{\ell}\right))^2} + B_2\right), \qquad (2.138)$$

$$\sum_{\substack{p\geq 2,\nu,k\geq 1\\ (p,\ell)=1,\\ p^{\nu+k}\leq \frac{D}{\ell}}} |f(p^\nu)||f(p^k)|\log(p^\nu) \leq A_2'\sqrt{\frac{D}{\ell}}. \qquad (2.139)$$

Using these relations in the right hand side of (2.21) of Lemma 2.2.2 together with the triangle inequality we get

$$\sum_{d\leq D} f(d)\log d = \kappa D \sum_{\ell\leq D} \frac{f(\ell)}{\ell}$$
$$+ \mathscr{O}^*\left(B_1 D \sum_{\ell\leq D} \frac{|f(\ell)|}{\ell(\log\left(\frac{2D}{\ell}\right))^2} + B_2 \sum_{\ell\leq D} |f(\ell)| + A_2'\sqrt{D} \sum_{\ell\leq D} \frac{|f(\ell)|}{\sqrt{\ell}}\right). \qquad (2.140)$$

On combining (2.140) with (2.137) and using the triangle inequality we obtain

$$\left(\sum_{d\leq D} f(d)\right)\log D = \kappa D \sum_{\ell\leq D} \frac{f(\ell)}{\ell}$$
$$+ \mathscr{O}^*\left(B_1 D \sum_{\ell\leq D} \frac{|f(\ell)|}{\ell(\log\left(\frac{2D}{\ell}\right))^2} + B_2 \sum_{\ell\leq D} |f(\ell)| + A_2'\sqrt{D} \sum_{\ell\leq D} \frac{|f(\ell)|}{\sqrt{\ell}}\right)$$
$$+ \mathscr{O}^*\left(\int_1^D \frac{\sum_{d\leq t} |f(d)|}{t}\,dt\right)$$

$$(2.141)$$

for all $D \geq 1$. We now use the bound (2.136) and those deduced from it in Lemma 2.2.5 to estimate the integral and the sums in error term of (2.141). On dividing throughout the

resulting relation by $\log D$ we obtain

$$\sum_{d \leq D} f(d) = \frac{D}{\log D} \left(\kappa \sum_{d \leq D} \frac{f(d)}{d} + \mathscr{O}^* \left(\frac{(9B_1 + B_2 + 4A'_2 + 2)\gamma_{|f|}}{\log D} \sum_{d \leq D} \frac{|f(d)|}{d} \right) \right) \quad (2.142)$$

for all $D > 1$, where we have changed ℓ to d and used $\log eD > \log D$ in the error term. The relations (2.133), (2.134) and (2.135) tell us that the multiplicative functions $n \mapsto \frac{f(n)}{n}$ and $n \mapsto \frac{|f(n)|}{n}$ satisfy the hypotheses of Theorem 2.3.2 and Theorem 2.3.1 respectively. From the former we obtain

$$\sum_{d \leq D} \frac{f(d)}{d} = C_{\kappa(f)}(\log D)^\kappa + \mathscr{O}^* \left(\left(\frac{2 + 2\kappa - \kappa'}{1 + \kappa - \kappa'} \right) C_{\kappa'(|f|)} R_f (\log D)^{\kappa' - 1} \right) \quad (2.143)$$

when

$$D > \exp(2(B'_1 + B'_2 + A'_2)) \geq 1, \quad (2.144)$$

with $R_f, C_{\kappa(f)}, C_{\kappa'(|f|)}$ as in (2.6), (2.7) and (2.8). Now with

$$R'_f = 2(B'_1 + B'_2 + 2A'_2 + \kappa')a(\kappa') \quad (2.145)$$

where, as before, $a(\kappa') = 1 + 2(\kappa' + 1)\exp(\kappa' + 1)$, we have from Theorem 2.3.1 that

$$\sum_{d \leq D} \frac{|f(d)|}{d} = C_{\kappa'(|f|)}(\log D)^{\kappa'} \left(1 + \mathscr{O}^* \left(\frac{R'_f}{\log D} \right) \right) \leq C_{\kappa'(|f|)}(\log D)^{\kappa'}(1 + a(\kappa')), \quad (2.146)$$

when D satisfies (2.144), since for such D we have $\frac{R'_f}{\log D} < a(\kappa')$ from (2.145). Using (2.143) and (2.146) in (2.142) we conclude that

$$\sum_{d \leq D} f(d) = \kappa C_{\kappa(f)} D (\log D)^{\kappa-1} + \mathscr{O}^* \left(\left(\frac{2+2\kappa-\kappa'}{1+\kappa-\kappa'} \right) D C_{\kappa'(|f|)} \kappa R_f (\log D)^{\kappa'-2} \right) \tag{2.147}$$
$$+ \mathscr{O}^* \left(C_{\kappa'(|f|)} D (\log D)^{\kappa'-2} (1 + a(\kappa'))(9B_1 + B_2 + 4A_2' + 2) \gamma_{|f|} \right).$$

We have $1 + a(\kappa') < 2a(\kappa')$ since $1 < a(\kappa')$ and consequently that

$$(1 + a(\kappa'))(9B_1 + B_2 + 4A_2' + 2) \leq 2(9B_1 + B_2 + 4A_2' + 2)a(\kappa') \leq 9R_f$$

from (2.6). Also, on dividing (2.133) and (2.134) throughout by $\log Q$, using the triangle inequality and letting $Q \to +\infty$, we see that $\kappa \leq \kappa'$ and therefore that $\kappa R_f \leq \kappa' R_f < R_f \gamma_{|f|}$. Finally, we note that $1 < \frac{2+2\kappa-\kappa'}{1+\kappa-\kappa'}$. With these remarks (2.4) follows from (2.147).

2.5 Applications

We give here two results to illustrate Theorem 2.1.1. Let us first observe that the condition (2.3) holds true for a wide class of multiplicative functions, namely those that are bounded on the prime powers.

Proposition 2.5.1. *If f is an arithmetic function such that there are real numbers A, B satisfisfying $|f(p^\nu)| \leq A\nu^B$ for all primes p and integers $\nu \geq 1$ then for all real numbers $Q \geq 1$ we have that*

$$\sum_{p \geq 2} \sum_{\substack{\nu,k \geq 1 \\ p^{\nu+k} \leq Q}} |f(p^\nu) f(p^k)| \log(p^\nu) + \sum_{p \geq 2} \sum_{\substack{\nu \geq 2 \\ p^\nu \leq Q}} |f(p^\nu)| \log(p^\nu) \ll \sqrt{Q}, \tag{2.148}$$

where the implied constant can be given explicitly in terms of A and B.

Proof. Since $|f(p^\nu)| \leq A\nu^B$, the left hand side of (2.148) does not exceed

$$A^2 \sum_{\substack{p \geq 2 \\ p^{v+k} \leq Q}} \sum_{v,k \geq 1} v^B k^B \log(p^v) + A \sum_{p \geq 2} \sum_{\substack{v \geq 2 \\ p^v \leq Q}} v^B \log(p^v)$$

$$= A^2 \sum_{p \geq 2} \log p \sum_{\substack{n \geq 2, \\ p^n \leq Q}} \sum_{\substack{k \geq 1, v \geq 1 \\ k+v=n.}} v^{B+1} k^B + A \sum_{p \geq 2} \log p \sum_{\substack{v \geq 2 \\ p^v \leq Q}} v^{B+1} \tag{2.149}$$

$$= \sum_{n \geq 2} (An^{B+1} + A^2 \sum_{\substack{k \geq 1, v \geq 1 \\ k+v=n.}} v^{B+1} k^B) \theta(Q^{\frac{1}{n}}),$$

where $\theta(x) = \sum_{p \leq x} \log p$. By means of the trivial bound

$$An^{B+1} + A^2 \sum_{\substack{k \geq 1, v \geq 1 \\ k+v=n.}} v^{B+1} k^B \leq An^{B+1} + A^2 n^{2(B+1)} \leq (A+1)^2 n^{2(B+1)} \tag{2.150}$$

we then conclude that the left hand side of (2.148) does not exceed

$$(A+1)^2 \sum_{n \geq 2} n^{2(B+1)} \theta(Q^{\frac{1}{n}}) . \tag{2.151}$$

Now we note that

$$\sum_{n \geq 2} n^{2(B+1)} \theta(Q^{\frac{1}{n}}) = 4^{B+1} \theta(Q^{\frac{1}{2}}) + \mathcal{O}^* \left(\theta(Q^{\frac{1}{3}}) \sum_{2 \leq n \leq \frac{\log Q}{\log 2}} n^{2(B+1)} \right)$$

$$= 4^{B+1} \theta(Q^{\frac{1}{2}}) + \mathcal{O}^* \left(\theta(Q^{\frac{1}{3}}) \left(\frac{\log Q}{\log 2} \right)^{2B+3} \right) \tag{2.152}$$

By (3.32), page 70 of [27], we have $\theta(x) \leq 1.02x$ for $x \geq 1$. Also, we have $Q^{\frac{1}{6}} \geq \frac{(\log Q)^{2B+3}}{[6(2B+3)]!}$ when $Q \geq 1$. Consequently, we have

$$(A+1)^2 \sum_{n \geq 2} n^{2(B+1)} \theta(Q^{\frac{1}{n}}) \leq (A+1)^2 (5^{B+1} + 1.02 \, [6(2B+3)]!) \sqrt{Q}, \tag{2.153}$$

from which the desired conclusion follows on recalling that the left hand side of (2.148) does not exceed (2.151). $\qquad\square$

$\qquad\square$

The following is our first application Theorem 2.1.1.

Theorem 2.5.1. *Let a and q be positive integers such that $(a,q) = 1$, $\phi(q) > 2$ and let P^* be the set of prime numbers congruent to a modulo q. If for any prime number p and integer $k \geq 1$ we set $\Omega(p^k;a,q) = k$ when p is in P^* and $\Omega(p^k;a,q) = 0$ otherwise, then we have that*

$$\sum_{d \leq D} \prod_{p^v || d} (-1)^{\Omega(p^v;a,q)} = \frac{(1 - \frac{2}{\phi(q)})CD}{(\log D)^{\frac{2}{\phi(q)}}} + O\left(\frac{D}{\log D}\right) \qquad (2.154)$$

for all $D > 1$, where the implied constant in the O symbol is an unspecified real number depending on q and

$$C = \frac{1}{\Gamma\left(2 - \frac{2}{\phi(q)}\right)} \prod_{p \geq 2} \left(\left(1 - \frac{1}{p}\right)^{1 - \frac{2}{\phi(q)}} \sum_{v \geq 0} \frac{f(p^v)}{p^v}\right).$$

Proof. We set $f(n) = \prod_{p^v || n} (-1)^{\Omega(p^v;a,q)}$ for any integer $n \geq 1$. Since for any prime p we have $f(p) = 1$ when p is not in P^* and $f(p) = -1$ when p is in P^*, it follows that

$$\sum_{\substack{p \geq 2, v \geq 1 \\ p^v \leq Q}} f(p^v) \log(p^v) = \sum_{p \leq Q} \log p - 2 \sum_{\substack{p \leq Q, \\ p \in P^*}} \log p + \sum_{\substack{p \geq 2, v \geq 2, \\ p^v \leq Q}} f(p^v) \log p^v. \qquad (2.155)$$

From (11.36) of Corollary (11.21) of [21] we have

$$\sum_{\substack{p \leq Q, \\ p \in P^*}} \log p = \sum_{\substack{p \leq Q, \\ p \equiv a[q]}} \log p = \frac{Q}{\phi(q)} + O\left(\frac{Q}{(\log 2Q)^2}\right), \qquad (2.156)$$

where the implied constant in the O symbol is not explicit. On the other hand, since $|f| = 1$, Proposition 2.5.1 gives $|\sum_{\substack{p \geq 2, v \geq 2, \\ p^v \leq Q}} f(p^v) \log p^v| \ll \sqrt{Q}$. Since we have that $\frac{7Q}{(\log 2Q)^2} \geq \sqrt{Q}$ for $Q \geq 1$ we therefore obtain from (2.155) that

$$\sum_{\substack{p \geq 2, v \geq 1 \\ p^v \leq Q}} f(p^v) \log(p^v) = \left(1 - \frac{2}{\phi(q)}\right) Q + O\left(\frac{Q}{(\log 2Q)^2}\right) \tag{2.157}$$

for all $Q \geq 1$. Since $|f| = 1$ we have using the prime number theorem that

$$\sum_{\substack{p \geq 2, v \geq 1 \\ p^v \leq Q}} |f(p^v)| \log(p^v) = \sum_{\substack{p \geq 2, v \geq 1 \\ p^v \leq Q}} \log(p^v) = Q + O\left(\frac{Q}{(\log 2Q)^2}\right) \tag{2.158}$$

for all $Q \geq 1$. It follows from (2.157), (2.158) and (2.148) that f satisfies (2.1), (2.2)and (2.3) with $\kappa = 1 - \frac{2}{\phi(q)}$ and $\kappa' = 1$. Finally, we have $1 + \kappa - \kappa' > 0$ since $\phi(q) > 2$. Therefore Theorem 2.1.1 is applicable and yields (2.154).

$\square$

Our second and final application of Theorem 2.1.1 is the following result.

Theorem 2.5.1. *Let a and q be positive integers such that $(a,q) = 1$ and $\phi(q) > 2$. Further, let P^* be the set of prime numbers congruent to m modulo q and let A^* be the set of positive integers coprime to each prime number in P^*. Then we have*

$$\sum_{\substack{n \leq x, \\ n \in A^*.}} \tau(n) = Cx(\log x)^{\left(1 - \frac{2}{\phi(q)}\right)} + O(x \log \log x) \tag{2.159}$$

for all real $x \geq 2$. Here the implied constant in the O symbol is an unspecified real number depending on q and

$$C = \frac{1}{\Gamma\left(2 - \frac{2}{\phi(q)}\right)} \prod_{p \geq 2} \left(\left(1 - \frac{1}{p}\right)^{\left(1 - \frac{2}{\phi(q)}\right)} \sum_{v \geq 0} \frac{\alpha_{P^*}(p^v)}{p^v}\right) \tag{2.160}$$

where $\alpha_{P^*}(p^v) = 1$ if p is not in P^* and $v \geq 1$, $\alpha_{P^*}(p) = -1$ if p is in P^* and finally $\alpha_{P^*}(p^v) = 0$ if p is in P^* and $v \geq 2$.

Proof. Since the multiplicative functions α_{P^*} and f of the preceding theorem agree on the set of prime numbers and $|\alpha_{P^*}| \leq 1$, it follows from (2.157), (2.158) and Proposition 2.5.1

that α_{P^*} also satisfies the (2.1), (2.2) and (2.3) with $\kappa = 1 - \frac{2}{\phi(q)}$ and $\kappa' = 1$. Also, as before, we have $1 + \kappa - \kappa' > 0$ since $\phi(q) > 2$. Therefore Theorem 2.1.1 is applicable and yields

$$\sum_{d \leq D} \alpha_{P^*}(d) = \frac{(1 - \frac{2}{\phi(q)})CD}{(\log D)^{\frac{2}{\phi(q)}}} + O\left(\frac{D}{\log D}\right) \tag{2.161}$$

for all $D \geq 2$, where C is as (2.160) and the implied constant in the O symbol is an unspecified real number depending on q. From the definition of α_{P^*} we have for any $n \in A^*$ that $\sum_{d|n} \alpha_{P^*}(d) = \tau(n)$ and for $n \notin A^*$ we have $\sum_{d|n} \alpha_{P^*}(d) = 0$. Hence

$$\sum_{\substack{n \leq x \\ n \in A^*}} \tau(n) = \sum_{n \leq x} \sum_{dm=n} \alpha_{P^*}(d). \tag{2.162}$$

By means of the Dirichlet hyperbola method we then see that

$$\sum_{\substack{n \leq x \\ n \in A^*}} \tau(n) = S_1 + S_2 - S_3, \tag{2.163}$$

where

$$S_1 = \sum_{m \leq \sqrt{x}} \alpha_{P^*}(m) \sum_{d \leq \frac{x}{m}} 1, \qquad S_2 = \sum_{d \leq \sqrt{x}} \sum_{m \leq \frac{x}{d}} \alpha_{P^*}(m),$$

and

$$S_3 = \sum_{m \leq \sqrt{x}} 1 \sum_{d \leq \sqrt{x}} \alpha_{P^*}(d).$$

Using the asymptotic formula (2.161) and with κ as above we then see that

$$
\begin{aligned}
S_1 &= \sum_{m \le \sqrt{x}} \alpha_{P^*}(m) \left[\frac{x}{m} \right] = x \sum_{m \le \sqrt{x}} \frac{\alpha_{P^*}(m)}{m} + O(x(\log x)^{\kappa-1}) \\
&= x \left(\frac{\sum_{m \le \sqrt{x}} \alpha_{P^*}(m)}{\sqrt{x}} + \int_1^{\sqrt{x}} \frac{\sum_{m \le t} \alpha_{P^*}(m)}{t^2} dt \right) + O(x(\log x)^{\kappa-1}) \\
&= x \int_1^{\sqrt{x}} \frac{\sum_{m \le t} \alpha_{P^*}(m)}{t^2} dt + O(x(\log x)^{\kappa-1}) \\
&= \kappa C x \int_1^{\sqrt{x}} (\log t)^{\kappa-1} \frac{dt}{t} + O(x \log \log x) \\
&= \frac{C}{2^{\kappa}} x (\log x)^{\kappa} + O(x \log \log x)) \, .
\end{aligned}
\tag{2.164}
$$

Similarly, we have

$$
\begin{aligned}
S_2 &= \kappa C \sum_{m \le \sqrt{x}} \frac{x}{m} \log \left(\frac{x}{m} \right)^{\kappa-1} + O \left(\sum_{m \le \sqrt{x}} \frac{x}{m \log \left(\frac{x}{m} \right)} \right) \\
&= \kappa C x \int_1^{\sqrt{x}} \log \left(\frac{x}{t} \right)^{\kappa-1} \frac{dt}{t} + O(x) \\
&= C \left(1 - \frac{1}{2^{\kappa}} \right) x (\log x)^{\kappa} + O(x) \, .
\end{aligned}
\tag{2.165}
$$

Finally, we see that

$$
S_3 \ll \sqrt{x} \cdot \sqrt{x} (\log x)^{\kappa-1} \ll x (\log x)^{\kappa-1} \, .
\tag{2.166}
$$

The formula (2.159) results from (2.163), (2.164), (2.165) and (2.166).

$\square$

CHAPTER 3

Weighted partial sums of the greatest divisor of n coprime to k

For any integer $n \geq 1$, let $\delta_k(n)$ denote the greatest divisor of n coprime to a given integer $k \geq 2$. In this chapter we give for any integer $r \geq 1$ and real number σ an asymptotic formula for $\sum_{n \leq x} \delta_k^r(n) n^{-\sigma - r}$ as $x \to +\infty$.

3.1 Introduction

For any integers $k \geq 2$ and $n \geq 1$ let us set

$$\delta_k(n) = \max\{d : d|n \text{ and } (d,k) = 1\}. \tag{3.1}$$

Thus $\delta_k(n)$ is the greatest divisor of n coprime to k. If $n = \prod_p p^{v_p(n)}$ is the prime factorisation of an integer $n \geq 1$, then we have $\delta_k(n) = \prod_{(p,k)=1} p^{v_p(n)}$. Using this it immediately follows that function $\delta_k : n \mapsto \delta_k(n)$ is a completely multiplicative function, that is, $\delta_k(mn) = \delta_k(m)\delta_k(n)$ for all integers $m, n \geq 1$. For any integer $k \geq 1$ and complex numbers s let us also define $J_s(k)$ and $\alpha(s,k)$ by

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$$J_s(k) = \sum_{d|k} \mu(d) \left(\frac{k}{d}\right)^s = k^s \prod_{p|k} \left(1 - \frac{1}{p^s}\right), \tag{3.2}$$

$$\alpha(s,k) = \sum_{p|k} \frac{\log p}{p^s - 1}. \tag{3.3}$$

We recall that function $J_s : k \mapsto J_s(k)$ on the natural numbers is called the Jordan totient function and has been denoted by ϕ_s in Chapter 1. Finally, in this chapter we shall write $\nu(k)$ for the number of square free divisors of an integer k.

The function $n \mapsto \delta_k(n)$ for a given integer $k \geq 2$ is the subject of a number of works in the literature, for instance see [32, 22, 1]. The earliest work on δ_k appears to be that of D. Suryanarayana who used the convolution method to obtain, on page 154 of [32], an asymptotic formula for the partial sums $\sum_{n \leq x} \frac{\delta_k(n)}{n^\sigma}$ when σ is an integer ≥ 0. Subsequently, P.N. Ramachandran [22], again using the convolution method, but somewhat differently from Suryanarayana, obtained the following theorem, which covers all real σ. Here and in what follows, $\zeta(s)$ denotes the value of the Riemann zeta function ζ at the complex number s.

Theorem 3.1.1. *Let σ be a real number, $k \geq 2$ an integer. Then for any real $x \geq 1$ we have*

$$\sum_{n \leq x} \frac{\delta_k(n)}{n^{\sigma+1}} = \begin{cases} \frac{k\varphi(k)}{(1-\sigma)J_2(k)} x^{1-\sigma} + O\left(\frac{\nu(k)}{x^\sigma(1-\sigma)} \log k\right) & (\sigma < 1), \\ \frac{k\varphi(k)}{J_2(k)} (\log x + \gamma + \alpha(1,k) - \alpha(2,k)) + O\left(\frac{\nu(k)}{x} \log k\right) & (\sigma = 1), \\ \frac{kJ_\sigma(k)\zeta(\sigma)}{J_{\sigma+1}(k)} + \frac{kx^{1-\sigma}}{\sigma-1} \frac{\varphi(k)}{J_2(k)} + O\left(\frac{\nu(k)}{x^\sigma(1-\sigma)} \log k\right) & (\sigma > 1). \end{cases} \tag{3.4}$$

Remark 3.1.1. Let us note that Ramachandran [22] uses t in place of our σ and orders the three cases considered above differently. Further, $J_2(k), \alpha(1,k), \alpha(2,k)$ above are $J(k), \alpha(k)$ and $\beta(k)$ respectively in [22]. Also, the statement of Ramachandran's theorem on page 336 of [22] appears to contain errors. Indeed, for the reasons we give in Remark 3.2.1, the conditions $\sigma < 1$ and $\sigma > 1$ above should in fact read $\sigma \leq 0$ and $\sigma > 0, \sigma \neq 1$ respectively.

In this chapter, which is based on joint work with Tadaki Igawa [11], we extend Ramachan-

dran's theorem stated above to powers of the function δ_k, that is, to the functions δ_k^r, for any integer $r \geq 1$. Our result is the following :

Theorem 3.1.2. *Let σ be a real number and $r \geq 1$, $k \geq 2$ be integers. Then for any real $x \geq 1$ we have*

$$
\sum_{n \leq x} \frac{\delta_k^r(n)}{n^{\sigma+r}} = \begin{cases} \frac{k^r \varphi(k)}{J_{1+r}(k)} \frac{x^{1-\sigma}}{(1-\sigma)} + O\left(\frac{v(k)}{x^\sigma}\right) & (\sigma \leq 0), \\[2ex] \frac{k^r \varphi(k)}{J_{1+r}(k)} \left(\log x + \gamma + \alpha(1,k) - \alpha(1+r,k)\right) + O\left(\frac{v(k)}{x}\right) & (\sigma = 1), \\[2ex] \frac{k^r J_\sigma(k) \zeta(\sigma)}{J_{\sigma+r}(k)} + \frac{k^r \varphi(k)}{J_{1+r}(k)} \frac{x^{1-\sigma}}{(1-\sigma)} + O\left(\frac{\beta(\sigma)v(k)}{x^\sigma}\right) & (\sigma > 0, \ \sigma \neq 1), \end{cases}
$$

$$\tag{3.5}$$

where $\beta(\sigma) = 2$ when $0 < \sigma \leq \frac{1}{2}$ and $\beta(\sigma) = \max(1, \frac{1}{\sigma})$ when $\frac{1}{2} < \sigma$. The implied constant in the O symbols in (3.5) can be taken to be 1 in each case.

Taking account of Remark 3.1.1, even for $r = 1$ the error terms of our formulae in (3.5) improve on the corresponding ones in Ramachandran's Theorem 3.1.1 in their dependence on k by a factor of $\log k$ in each case. Moreover, our error terms are completely explicit and the error term for the case $\sigma > 0$, $\sigma \neq 1$ has a finite limit when $\sigma \to 1$.

3.2 Ramachandran's method

We shall prove Theorem 3.1.2 here by modifying the method of Ramachandran [22]. With k and r as in the theorem, let us set

$$
a_n^{(r)} = \sum_{d \mid n} \mu\left(\frac{n}{d}\right) \frac{\delta_k^r(d)}{d^r} \tag{3.6}
$$

for any integer $n \geq 1$. Then the Möbius inversion formula gives $\frac{\delta_k^r(n)}{n^r} = \sum_{d \mid n} a_d^{(r)}$ for all integers $n \geq 1$, from which we see that

$$
\sum_{n \leq x} \frac{\delta_k^r(n)}{n^{\sigma+r}} = \sum_{n \leq x} \frac{1}{n^\sigma} \left(\sum_{d \mid n} a_d^{(r)}\right) = \sum_{d \leq x} \frac{a_d^{(r)}}{d^\sigma} \sum_{m \leq \frac{x}{d}} \frac{1}{m^\sigma}. \tag{3.7}
$$

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We shall obtain Theorem 3.1.2 from (3.7) with the aid of the lemmas given below.

Preliminary Lemmas

We begin by recalling that the Riemann zeta function ζ is defined by $\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$ for complex numbers s with $\text{Re}(s) > 1$ and extends as a meromorphic function to the entire complex plane. This extension, also denoted by ζ, has a unique pole on the complex plane, a simple pole at $s = 1$, where it has the limited Laurent expansion

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1). \tag{3.8}$$

Lemma 3.2.1. *For any real σ and $x \geq 1$ we have that*

$$\sum_{n\leq x} \frac{1}{n^\sigma} = \begin{cases} \frac{x^{1-\sigma}}{1-\sigma} + O(x^{-\sigma}) & (\sigma \leq 0), \\ \zeta(\sigma) + \frac{x^{1-\sigma}}{1-\sigma} + O(x^{-\sigma}) & (\sigma > 0, \sigma \neq 1). \end{cases} \tag{3.9}$$

The implied constant in the O symbols above can be taken to be 1 in each case.

Proof. These relations are well-known consequences of the classical Euler-Maclaurin formula (See Section 1.7.2, page 19 of [3]). Only the assertion on the implied constant needs some care to verify. Let us check this when $\sigma \leq 0$. For a continuously differentiable function f on $[1,x]$ the Euler-Maclaurin formula gives

$$\sum_{n\leq x} f(n) = \int_1^x f(t)dt + f(1) - \{x\}f(x) + \int_1^x \{t\}f'(t)dt \tag{3.10}$$

where $\{t\} = t - [t]$. If $f \geq 0$ and $f' \geq 0$ on $[1,x]$ then $0 \leq \int_1^x \{t\}f'(t)dt \leq f(x) - f(1)$ and $-f(x) \leq -\{x\}f(x) \leq 0$. For such f we then have from (3.10) that

$$-f(x) + f(1) \leq \sum_{n\leq x} f(n) - \int_1^x f(t)dt \leq f(x) \tag{3.11}$$

When $\sigma \leq 0$, f with $f(t) = t^{-\sigma}$ satisfies the stated conditions, so that (3.11) with this f gives

$$-\frac{1}{x^{\sigma}}+1 \leq \sum_{n \leq x} \frac{1}{n^{\sigma}} - \frac{x^{1-\sigma}}{1-\sigma} + \frac{1}{1-\sigma} \leq \frac{1}{x^{\sigma}} \tag{3.12}$$

on evaluating $\int_1^x \frac{1}{t^{\sigma}} dt$. Since $\sigma \leq 0$ we have $-1 \leq -\frac{1}{1-\sigma} \leq 0$. The first case of (3.2.1) follows on adding this relation and (3.12). The second case of (3.2.1) is obtained by putting $s = \sigma$ in Theorem 3.57, page 98 of [3].

$\square$

Remark 3.2.1. Lemma 3.2.1 corresponds to Lemma 1, page 336 of [22]. However, the statement of the latter lemma contains errors that can confuse a reader of [22]. Indeed, in case (a) of Lemma 1 of [22], the condition $t > 1$ should be replaced with $t \leq 0$. Also, in (c) of this lemma $\frac{x^{1-t}}{t-1}$ should be replaced with $\frac{x^{1-t}}{1-t}$ and the condition $t > 1$ with $t > 0, t \neq 1$. We have not given the case corresponding to (b) of the Lemma, which in our notation would be the case $\sigma = 1$, since we will not use it.

Proposition 3.2.1. *For any real numbers $0 < u \leq 1$ and $\sigma > 0$ we have*

$$1 + \left|\frac{u^{1-\sigma}-1}{1-\sigma}\right| \leq \beta(\sigma)u^{-\sigma}, \tag{3.13}$$

where $\beta(\sigma)$ is as defined in the statement of Theorem 3.1.2.

Proof. For any $\sigma > 0, \sigma \neq 1$, we have

$$1 + \left|\frac{u^{1-\sigma}-1}{1-\sigma}\right| = 1 + \int_u^1 \frac{dt}{t^{\sigma}} = 1 + \int_1^{\frac{1}{u}} t^{\sigma-2} dt \leq 1 + u^{-\sigma}\int_1^{\infty} \frac{dt}{t^2} \leq 2u^{-\sigma}, \tag{3.14}$$

where the second equality results from the change of variables $t \mapsto \frac{1}{t}$. On recalling the definition of $\beta(\sigma)$ we see that (3.14) verifies (3.13) when $0 < \sigma < \frac{1}{2}$. Now for any $v \geq 1$ let us temporarily set $\varphi(v) = \max(1, \frac{1}{\sigma})v^{\sigma} - \int_1^v t^{\sigma-2} dt - 1$. Then we have that $\varphi'(v) = \max(\sigma, 1)v^{\sigma-1} - v^{\sigma-2} \geq 0$ for all $v \geq 1$ and any $\sigma > 0$. Thus $\varphi(v) \geq \varphi(1) \geq 0$ for all $v \geq 1$ and any $\sigma > 0$. In particular, we have $0 \leq \varphi(\frac{1}{u})$ for all $\sigma > 0$. This together with the equalities in (3.14) verifies (3.13) for $\sigma > \frac{1}{2}$.

$\square$

Lemma 3.2.2. *For any real numbers $x \geq 1$ and $\sigma > 0, \sigma \neq 1$, we have*

$$\left| \zeta(\sigma) + \frac{1}{1-\sigma} \left(\frac{x}{d} \right)^{1-\sigma} \right| \leq \beta(\sigma) \left(\frac{d}{x} \right)^{\sigma}, \tag{3.15}$$

for all $d > x$.

Proof. We set $u = \frac{x}{d}$, so that $0 < u < 1$, and note using the triangle inequality that we have

$$\left| \zeta(\sigma) + \frac{1}{1-\sigma} \left(\frac{x}{d} \right)^{1-\sigma} \right| \leq \left| \zeta(\sigma) - \frac{1}{\sigma-1} \right| + \left| \frac{u^{1-\sigma}-1}{1-\sigma} \right|. \tag{3.16}$$

To obtain (3.15) from (3.16) and (3.13), it suffices to observe that

$$0 \leq \zeta(\sigma) - \frac{1}{\sigma-1} \leq 1 \tag{3.17}$$

when $0 < \sigma, \sigma \neq 1$, which is well-known. We nevertheless recall the proof for completeness. Using the Euler-Maclaurin formula (3.10) with $f(t) = t^{-s}$, where s is any complex number with $\mathrm{Re}\,(s) > 1$, and any real $x \geq 1$ and then letting $x \to +\infty$ we get

$$\zeta(s) = \frac{1}{s-1} + 1 - s \int_1^\infty \frac{\{t\}}{t^{s+1}} \, dt \tag{3.18}$$

for all complex s with $\mathrm{Re}\,(s) > 1$. This relation extends ζ as a meromorphic function to $\mathrm{Re}\,(s) > 0$, since the integral on its right hand side extends as holomorphic function to this half-plane as well. The bounds in (3.17) follow on putting $s = \sigma > 0, \sigma \neq 1$ in (3.18) and noting that $0 \leq \sigma \int_1^\infty \frac{\{t\}}{t^{\sigma+1}} \, dt \leq 1$ for such σ.

$\square$

It is by means of the inequality (3.15) that we gain over Ramachandran [22]. In the following pair of lemmas we extend some observations from [22] for $r = 1$ to the case $r \geq 1$.

Lemma 3.2.3 (cf. [22, p. 336]). *Let $a_n^{(r)}$ be as in (3.6) and for an integer $k \geq 2$, let $\mathscr{K}$ be the set of the natural numbers all of whose prime divisors divide k. Then for any integer $n \geq 1$*

we have that

$$
a_n^{(r)} = \begin{cases} \frac{1}{n^r} \prod_{p|n} (1 - p^r) & (n \in \mathcal{K}), \\[2ex] 0 & (n \notin \mathcal{K}). \end{cases} \tag{3.19}
$$

Proof. Since $n \mapsto \frac{\delta_k^r(n)}{n^r}$ is a multiplicative function, it follows from (3.6) that so is the function $n \mapsto a_n^{(r)}$. Also, the right hand side of (3.19) defines a multiplicative function. It therefore suffices to check this relation when $n = p^m$ for any prime p and integer $m \geq 1$. For such n the relation $n \in \mathcal{K}$ is equivalent to $p|k$. Now we note that for any divisor d of p^m we have

$$
\frac{\delta_k^r(d)}{d^r} = \begin{cases} \frac{1}{d^r} & \text{when } p \text{ divides } k, \\[2ex] 1 & \text{when } p \text{ does not divide } k. \end{cases} \tag{3.20}
$$

Substituting this into (3.6) and using the Möbius inversion formula we immediately obtain (3.19) for $n = p^m$, as required. $\qquad \square$

For the next lemma we recall the defintions of $J_s(k)$ and $\alpha(\sigma, k)$ from (3.2) and (3.3) respectively.

Lemma 3.2.4 (cf. [22, p. 337, Lemma 2]). *Let $k \geq 2$ and $r \geq 1$ be integers and let σ be real number. Then we have*

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{a_n^{(r)}}{n^\sigma} = \frac{k^r J_\sigma(k)}{J_{\sigma+r}(k)}$ *when $\sigma > -r$ and in particular* $\displaystyle\sum_{n=1}^{\infty} \frac{a_n^{(r)}}{n} = \frac{k^r \phi(k)}{J_{1+r}(k)}$,

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{a_n^{(r)} \log n}{n} = -\frac{k^r \phi(k)}{J_{1+r}(k)} (\alpha(1,k) - \alpha(1+r,k))$,

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{|a_n^{(r)}|}{n^\sigma} = \prod_{p|k} \left(1 + \frac{p^r - 1}{p^{\sigma+r} - 1}\right)$ *when $\sigma > -r$,*

(d) $\displaystyle\sum_{n=1}^{\infty} |a_n^{(r)}| = v(k)$.

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Proof. Let us first verify (c). From (3.19) of Lemma 3.2.3 we see that $|a_n^{(r)}| \leq 1$ for all $n \geq 1$. Therefore the Dirichlet series $\sum_{n=1}^{\infty} \frac{|a_n^{(r)}|}{n^s}$ converges absolutely for all complex s with $\operatorname{Re} s > 1$. Since the function $n \mapsto |a_n^{(r)}|$ is multiplicative we have using (3.19) again that

$$\sum_{n=1}^{\infty} \frac{|a_n^{(r)}|}{n^s} = \prod_p \sum_{m \geq 0} \frac{|a_{p^m}^{(r)}|}{p^{ms}} = \prod_{p|k} \left(1 + (p^r - 1) \sum_{m \geq 1} \frac{1}{p^{(s+r)m}} \right) = \prod_{p|k} \left(1 + \frac{p^r - 1}{p^{s+r} - 1} \right) \quad (3.21)$$

for complex numbers s with $\operatorname{Re} s > 1$. The last product over $p|k$ in (3.21) is a holomorphic function of s for $\operatorname{Re} s > -r$. Consequently, we have from (3.21) that the Dirichlet series with non-negative coefficents $\sum_{n=1}^{\infty} \frac{|a_n^{(r)}|}{n^s}$ extends as a holomorphic function of s to $\operatorname{Re} s > -r$. By a classical theorem of Landau (see Corollary 4.45, page 207 of [3]) we then conclude that this series converges for $\operatorname{Re} s > -r$ and that (3.21) holds for all such complex numbers s. On putting $s = \sigma$ in (3.21) for real $\sigma > -r$ we get (c) and on putting $s = 0$, allowed since $r \geq 1$, and noting that $2^{\omega(k)} = \nu(k)$ we get (d).

To verify (a) we note from (c) that the Dirichlet series $\sum_{n=1}^{\infty} \frac{a_n^{(r)}}{n^s}$ is absolutely convergent for $\operatorname{Re} s > -r$ and since $n \mapsto a_n^{(r)}$ is a multiplicative function we have using (3.19) that

$$\sum_{n=1}^{\infty} \frac{a_n^{(r)}}{n^s} = \prod_p \sum_{m \geq 0} \frac{a_{p^m}^{(r)}}{p^{ms}} = \prod_{p|k} \left(1 + (1 - p^r) \sum_{m \geq 1} \frac{1}{p^{(s+r)m}} \right) = \prod_{p|k} \left(1 + \frac{1 - p^r}{p^{s+r} - 1} \right) \quad (3.22)$$

for $\operatorname{Re} s > -r$. We now obtain the first part of (a) from (3.22) on noting that

$$\prod_{p|k} \left(1 + \frac{1 - p^r}{p^{s+r} - 1} \right) = \prod_{p|k} \left(\frac{p^{s+r} - p^r}{p^{s+r} - 1} \right) = \prod_{p|k} \left(\frac{1 - \frac{1}{p^s}}{1 - \frac{1}{p^{s+r}}} \right) = \frac{k^r J_s(k)}{J_{s+r}(k)}, \quad (3.23)$$

where the last equality is immediate from the definition of $J_s(k)$ given by (3.2). Putting $\sigma = 1$ in the first part of (a) we get the second part, since $J_1(k) = \phi(k)$. Equating the derivatives of $s \mapsto \sum_{n=1}^{\infty} \frac{a_n^{(r)}}{n^s}$ and $s \mapsto \frac{k^r J_s(k)}{J_{s+r}(k)}$ we get

$$\sum_{n=1}^{\infty} \frac{a_n^{(r)} \log n}{n^s} = -\frac{k^r J_s(k)}{J_{s+r}(k)} \left(\frac{J_s(k)'}{J_s(k)} - \frac{J_{s+r}(k)'}{J_{s+r}(k)} \right) \tag{3.24}$$

for $\mathrm{Re}\, s > -r$. From the definition of $J_s(k)$ we see that

$$\frac{J_s(k)'}{J_s(k)} = \log k + \sum_{p|k} \frac{\log p}{p^s - 1} = \log k + \alpha(s,k) \tag{3.25}$$

and hence that

$$\frac{J_s(k)'}{J_s(k)} - \frac{J_{s+r}(k)'}{J_{s+r}(k)} = \alpha(s,k) - \alpha(s+r,k) \tag{3.26}$$

for any complex number s. We obtain (b) from (3.24) and (3.26) on putting $s = 1$ in these relations. $\qquad\square$

Proof of Theorem 3.1.2 completed

We continue from the relation (3.7). It will be clear that the implied constants in the O symbols in the remainder of this proof can all taken to be 1. Using the case $\sigma \leq 0$ of Lemma 3.2.1 for the sum over m in (3.7) we get

$$\sum_{n \leq x} \frac{\delta_k^r(n)}{n^{\sigma+r}} = \frac{x^{1-\sigma}}{1-\sigma} \sum_{d \leq x} \frac{a_d^{(r)}}{d} + O\left(x^{-\sigma} \sum_{d \leq x} |a_d^{(r)}| \right). \tag{3.27}$$

when $\sigma \leq 0$. Now we note that

$$\left| \frac{x^{1-\sigma}}{1-\sigma} \sum_{d=1}^{\infty} \frac{a_d^{(r)}}{d} - \frac{x^{1-\sigma}}{1-\sigma} \sum_{d \leq x} \frac{a_d^{(r)}}{d} \right| \leq \frac{x^{1-\sigma}}{1-\sigma} \sum_{d>x} \frac{|a_d^{(r)}|}{d} \leq x^{-\sigma} \sum_{d>x} |a_d^{(r)}|, \tag{3.28}$$

since $\frac{1}{1-\sigma} \leq 1$ when $\sigma \leq 0$. Combining (3.27) and (3.28) we then deduce that

$$\sum_{n \leq x} \frac{\delta_k^r(n)}{n^{\sigma+r}} = \frac{x^{1-\sigma}}{1-\sigma} \sum_{d=1}^{\infty} \frac{a_d^{(r)}}{d} + O\left(x^{-\sigma} \sum_{d=1}^{\infty} |a_d^{(r)}| \right), \tag{3.29}$$

from which the case $\sigma \leq 0$ of Theorem 3.1.2 follows on recalling (a) and (d) of Lemma 3.2.4. We now turn to the third case of Theorem 3.1.2, namely, when $\sigma > 0, \sigma \neq 1$. We use

the corresponding case of Lemma 3.2.1 for the sum over m in (3.7) to get

$$\sum_{n \leq x} \frac{\delta_k^r(n)}{n^{\sigma+r}} = \sum_{d \leq x} \frac{a_d^{(r)}}{d^\sigma} \left(\zeta(\sigma) + \frac{1}{1-\sigma} \left(\frac{x}{d}\right)^{1-\sigma} \right) + O\left(x^{-\sigma} \sum_{d \leq x} |a_d^{(r)}| \right). \tag{3.30}$$

By means of the triangle inequality we have

$$\left| \left(\sum_{d=1}^\infty - \sum_{d \leq x} \right) \frac{a_d^{(r)}}{d^\sigma} \left(\zeta(\sigma) + \frac{1}{1-\sigma} \left(\frac{x}{d}\right)^{1-\sigma} \right) \right| \leq \sum_{d > x} \frac{|a_d^{(r)}|}{d^\sigma} \left| \zeta(\sigma) + \frac{1}{1-\sigma} \left(\frac{x}{d}\right)^{1-\sigma} \right|. \tag{3.31}$$

Using (3.15) of Lemma 3.2.2 for the right hand side of (3.31) we obtain

$$\left| \left(\sum_{d=1}^\infty - \sum_{d \leq x} \right) \frac{a_d^{(r)}}{d^\sigma} \left(\zeta(\sigma) + \frac{1}{1-\sigma} \left(\frac{x}{d}\right)^{1-\sigma} \right) \right| \leq \beta(\sigma) x^{-\sigma} \sum_{d > x} |a_d^{(r)}|. \tag{3.32}$$

On combining this relation with (3.30) and noting that $1 \leq \beta(\sigma)$ for all $\sigma > 0$, we then deduce that

$$\sum_{n \leq x} \frac{\delta_k^r(n)}{n^{\sigma+r}} = \sum_{d=1}^\infty \frac{a_d^{(r)}}{d^\sigma} \left(\zeta(\sigma) + \frac{1}{1-\sigma} \left(\frac{x}{d}\right)^{1-\sigma} \right) + O\left(\beta(\sigma) x^{-\sigma} \sum_{d \geq 1} |a_d^{(r)}| \right). \tag{3.33}$$

The case $\sigma > 0, \sigma \neq 1$ of (3.5) results from (3.33) and the relations in (a) and (d) of Lemma 3.2.4. To obtain the case $\sigma = 1$ in (3.5) we let $\sigma \to 1$ in (3.33). Note that $0 < \beta(\sigma) x^{-\sigma} \leq 2$ for all $\sigma > 0$. We therefore have from (3.32) that the first sum on the right hand side of (3.33) converges uniformly in σ for $\sigma > 0$. This justifies interchanging $\lim_{\sigma \to 1}$ with the summation in the first sum on the right hand side of (3.33). Now we note that

$$\lim_{\sigma \to 1} \left(\zeta(\sigma) + \frac{1}{1-\sigma} \left(\frac{x}{d} \right)^{1-\sigma} \right) = \lim_{\sigma \to 1} \left(\zeta(\sigma) - \frac{1}{\sigma-1} + \frac{\left(\frac{x}{d} \right)^{1-\sigma} - 1}{1-\sigma} \right) \tag{3.34}$$

$$= \gamma + \log \left(\frac{x}{d} \right),$$

on using (3.8). Also, we have $\lim_{\sigma \to 1} \beta(\sigma) = 1$. Consequently, we obtain from (3.33) that

$$\sum_{n \le x} \frac{\delta_k^r(n)}{n^{1+r}} = \sum_{d=1}^{\infty} \frac{a_d^{(r)}}{d} \left(\gamma + \log \left(\frac{x}{d} \right) \right) + O \left(\frac{1}{x} \sum_{d \ge 1} |a_d^{(r)}| \right) \tag{3.35}$$

from which we conclude the case $\sigma = 1$ of (3.5) by means of (a), (b) and (d) of Lemma 3.2.4.

$\square$

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