ASYMPTOTIC BEHAVIOUR OF SUM OF COEFFICIENTS OF A CLASS OF DIRICHLET SERIES

By

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Prof. Kalyan Chakraborty

Guide

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“Infinite patience, infinite purity and infinite perseverance are the secret of success in a good cause.”

- Swami Vivekananda

“Only in the world of mathematics do two negatives multiply into a positive.”

- Abby Morel

“The progress of our knowledge of numbers is advanced not only by what we already know about them but also by realizing what we yet do not know about them.”

- Waclaw Sierpinski

Dedicated to

My Family

List of Publications arising from the thesis

Journal

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Soumyarup Banerjee

DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Soumyarup Banerjee

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Soumyarup Banerjee

Contents

The proof of theorem 0.4.2 was obtained in [5]. This is a joint work with Shigeru Kanemitsu. The fourth chapter of the thesis contains the details of the proof of this theorem.

Suppose two summatory functions are given as

$$A(x) = \sum_{n \le x} a(n) = M_A(x) + E_A(x), \;\; B(x) = \sum_{n \le x} b(n) = M_B(x) + E_B(x)$$

and the error terms satisfy some estimates that allow the intermediate procedures.

Let $G_A(s)$ (resp. $G_B(s)$) indicate the generating function of $\{a(n)\}$ (resp. $\{b(n)\}$) where we take $s = \sigma + it$ for the complex variable. These generating functions satisfy some conditions as described by Lau in [38].

Here the question which arises is, is it possible to express Lau's convolution theorem [38] in terms of Stieltjes integral with the basic assumptions of Lau? The answers are affirmative. The following theorem answers the above questions. Lau's theorem is an easy consequence of the following theorem.

Theorem 0.4.2 *For sufficiently small* $\epsilon' > 0$, *let*

$$y = y(x) = \exp(-c(\epsilon')\mathcal{N}(x)).$$

Then we have

$$C(x) = \frac{1}{2\pi i}\int_{\mathcal{H}} G_B(s)G_A(s)x^s \frac{\mathrm{d}s}{s} + \int_{u \le y} A(x/u)\,\mathrm{d}B(u) - \int_0^{x/y} M_A(u)\,\mathrm{d}M_B(x/u)$$
$$- G_{A,\alpha}(0)B(y) + E_C(x),$$

where

$$E_C(x) = O(x^a\delta(x)), \quad \delta(x) = \delta_c(x) = \exp(-c\mathcal{N}(x)).$$

Here $G_{A,\alpha}(0) = G_A(0)$ *if* $\alpha < 0$ *and* 0 *otherwise.*

such a formula has been treated according as the generating function has no logarithmic singularity or is generated by the complex power of a certain zeta-function (the so-called Selberg type divisor problem). However, the general Abel-Tauber theorem [3] makes it possible to treat them as one Abel-Tauber process at a stretch. Moreover it gives an asymptotic formula for Riesz sums of higher order.

Since many arithmetic functions are given as a Dirichlet convolution of two arithmetic functions, whose asymptotic behaviors are known, it is essential to deduce the asymptotic formula for the summatory function of Dirichlet convolutions. In literature there are general theorems on asymptotic formulas for convolutions (cf. [30], [43]). There is a far-reaching theorem of Y. -K. Lau [38] which gives a rather precise asymptotic formula for the summatory function of the Dirichlet convolution of two arithmetical functions. J. P. Tull in [52] developed a general method for obtaining asymptotic formulas for the summatory function of the convolution of two arithmetic functions $a(n)$ and $b(n)$ whose summatory functions $A(x)$ and $B(x)$ satisfy asymptotic formulas. Indeed, his method is more general than just for summatory function but can also treat the Stieltjes resultant.

Definition 0.4.1 (Stieltjes resultant) The Stieltjes resultant C of A and B can be defined as

$$C(x) = (A \times B)(x) = \int_{u \le x} \int_{v \le x/u} \mathrm{d}A(v)\,\mathrm{d}B(u) = \int_a^x A(x/u)\,\mathrm{d}B(u),$$

provided the integral exists and for all $x \in \mathbb{R}^+$, $C(x)$ lies between the limits $\lim\limits_{h \to \mp 0} C(x \pm h)$.

$$\sum_{n=1}^{\infty} \lambda_f(n)\overline{\lambda_g(n)} \exp(-ny) = \begin{cases} R_1 + \mathcal{P}(y) + O(y^{\epsilon}) & \textit{if } \ f = g, \\ \mathcal{P}(y) + O(y^{\epsilon}) & \textit{otherwise.} \end{cases}$$

where the residual term

$$R_1 = \frac{24}{\pi^2 y} Sin\{\pi/2(1+2ir)\} \parallel f \parallel^2$$

and

$$\mathcal{P}(y) = \sum_{\rho} \frac{\Gamma(\rho/2)L(\rho/2, f \otimes g)}{\zeta'(\rho) y^{\rho/2}}$$

where, $\rho = x + iy$ is running through all the non-trivial zeros of the ζ-function. This sum is decomposed into pieces so that the terms for which

$$|y_1 - y_2| < \exp\left(-A\frac{y_1}{\log y_1}\right) + \exp\left(-A\frac{y_2}{\log y_2}\right),$$

where A is a suitable positive constant, are included in the same piece.

The proof of theorem 0.3.1 were obtained in [4]. Mainly Cauchy residue theorem and functional equation of Rankin-Selberg L- function are used to proof the above theorem. This is the joint work with Kalyan Chakraborty. The third chapter of the thesis contains the details of the proof of this theorem.

0.4 Asymptotic behaviour of arithmetical convolutions

In number theory one of the most important tools is an asymptotic formula for the summatory function of a given arithmetic function. In existing literature, and g by $\lambda_f(n)$ and $\lambda_g(n)$ with the Fourier series expansion;

$$f(z) = y^{1/2} \sum_{n \neq 0} \lambda_f(n) K_{ir}(2\pi |n| y) e(nx) \tag{1}$$

and

$$g(z) = y^{1/2} \sum_{n \neq 0} \lambda_g(n) K_{iq}(2\pi |n| y) e(nx) \tag{2}$$

respectively. Here K_{ir} and K_{iq} are the modified Bessel function of the second kind. Also we use, $z = x + iy$ and $e(x) = e^{2\pi i x}$.

Let $\| f \|$ denote the norm of f with respect to the Petersson inner product. We consider the following Dirichlet series associated to f and g with the eigen values $(1/4 + r^2)$ and $(1/4 + q^2)$ respectively:

$$R(s, f \otimes g) := \sum_{n=1}^{\infty} \frac{\lambda_f(n) \overline{\lambda_g(n)}}{n^s}.$$

The **Rankin Selberg L- function** associated to f and g is defined as:

$$L(s, f \otimes g) := \zeta(2s) \sum_{n=1}^{\infty} \frac{\lambda_f(n) \overline{\lambda_g(n)}}{n^s}.$$

Now we recall **grand simplicity hypothesis** which is the main assumption in our result. It tells that all the non trivial zeros of a Dirichlet L-function are simple. We are going to state our main result here.

Theorem 0.3.1 *Let f and g be Maass cusp forms which are normalized Hecke eigenforms over the full modular group with Fourier series expansion as in* (1) *and* (2) *respectively. Assume that the grand simplicity hypothesis holds. Then for any positive y,*

0.3 Asymptotic behaviour of a series à la Zagier

In 1981, Zagier [59] conjectured that the inverse Mellin transform of symmetric square L- function attached to Ramanujan's tau function has an asymptotic expansion in terms of the zeros of the Riemann zeta function. He considered the series:

$$a_0(y) := \sum_{n=0}^{\infty} \tau^2(n) exp(-ny)$$

and plotted the graph of $a_0(y)$, where he found the oscillatory behaviour of the series. More importantly, he mentioned that the asymptotic expansion of the above series can actually be used to evaluate the non-trivial zeros of $\zeta(s)$. Later, the above conjecture was proved by Hafner and Stopple [24]. They found the asymptotic expansion which in particular shows the oscillatory behaviour of the above series.

Now the natural question arises in this direction about the asymptotic behaviour of the series $\sum_{n=1}^{\infty} c^2(n) \exp(-nz)$ when $z \to 0^+$, where $c(n)$ is the nth Fourier coefficient of any cusp form f over $\Gamma = SL_2(\mathbb{Z})$.

Chakraborty et. al. in [10] answered the above question. They have shown that the above series also can be expressed in terms of the non-trivial zeros of the Riemann zeta function. Recently, this result has been further extended [11] for any cusp form over congruence subgroups of $SL_2(\mathbb{Z})$.

Now one can ask similar kind of question in Maass form set up. An affirmative answer has been obtained in this direction.

Let f and g be two Maass cusp form which are normalized Hecke eigenform over Γ with the eigenvalues $(1/4 + r^2)$ and $(1/4 + q^2)$ respectively corresponding to the hyperbolic Laplace operator Δ . We denote the Fourier coefficients of f

Now the natural question comes here about the expression of the asymptotic formula of $D(x;\boldsymbol{\alpha})$. Is it possible to get the error term in terms of special functions ?

An affirmative answer to the above question has been obtained in the following theorem.

Theorem 0.2.1 *For* $\boldsymbol{\alpha}=(\alpha_1,\cdots,\alpha_\varkappa)\in(0,1)^\varkappa$, *we have*

$$\begin{aligned}
D(x;\boldsymbol{\alpha}) =&\mathrm{P}(x)-\frac{(e^{-\frac{i\pi}{2}})^\varkappa}{(2\pi)^\varkappa}\sum_{n=1}^{\infty}\frac{d_{+\varkappa,-0}(n)}{n}V\left((2\pi)^\varkappa\left(e^{\frac{\pi i}{2}}\right)^\varkappa nx\middle|1,\cdots,1,0\right)\\
&-\binom{\varkappa}{1}\frac{(e^{-\frac{i\pi}{2}})^{\varkappa-2}}{(2\pi)^\varkappa}\sum_{n=1}^{\infty}\frac{d_{+(\varkappa-1),-1}(n)}{n}V\left((2\pi)^\varkappa\left(e^{\frac{\pi i}{2}}\right)^{\varkappa-2} nx\middle|1,\cdots,1,0\right)\\
&-\binom{\varkappa}{2}\frac{(e^{-\frac{i\pi}{2}})^{\varkappa-4}}{(2\pi)^\varkappa}\sum_{n=1}^{\infty}\frac{d_{+(\varkappa-2),-2}(n)}{n}V\left((2\pi)^\varkappa\left(e^{\frac{\pi i}{2}}\right)^{\varkappa-4} nx\middle|1,\cdots,1,0\right)\\
&-\cdots\\
&-\frac{(e^{\frac{i\pi}{2}})^\varkappa}{(2\pi)^\varkappa}\sum_{n=1}^{\infty}\frac{d_{+0,-\varkappa}(n)}{n}V\left((2\pi)^\varkappa\left(e^{-\frac{\pi i}{2}}\right)^\varkappa nx\middle|1,\cdots,1,0\right).
\end{aligned}$$

Here $\mathrm{P}(x)=\mathrm{P}_{\boldsymbol{\alpha}}(x)$ *is the residual function which is the sum of the residues of the weighted generating function*

$$\varphi(s)\frac{x^s}{s}=\zeta(s,\alpha_1)\cdots\zeta(s,\alpha_\varkappa)\frac{x^s}{s}$$

at $s=0$ *and* 1.

In particular, for $\varkappa=2$ we obtain modified bessel function of second kind in the error term. The proof of theorem 0.2.1 were obtained in [55]. The main tools used are the Cauchy residue theorem and Perron's formula to prove the above theorem. This is a joint work with N. L. Wang. The second chapter of the thesis contains the details of the proof of this theorem.

where m, n, p, q are integers with $0 \leq m \leq q$, $0 \leq n \leq p$ and $a_i - b_j \notin \mathbb{N}$ for $1 \leq i \leq p$, $1 \leq j \leq q$.

- The **modified Bessel function** of second kind $K_\nu(z)$ is a special case of the **Meijer G-function**. It can be defined by

$$K_\nu(z) = 2^{\nu-1} z^{-\nu} G^{2,0}_{0,2}\left(z \,\middle|\, \begin{matrix} - \\ \nu, 0 \end{matrix} \right).$$

- Voronoï-Steen function $V = V(x; a_1, \cdots, a_n)$ (cf. [48]) is defined by

$$\frac{1}{2\pi i} \int_0^\infty x^s V(x; a_1, \cdots, a_n) \frac{\mathrm{d}x}{x} = \Gamma(s + a_1) \cdots \Gamma(s + a_n).$$

Let $\varkappa$ be a positive integer ≥ 2 and let $\{\lambda_n\}$ denote the strictly increasing sequence of numbers of the form

$$\lambda_n = (n_1 + \alpha_1) \cdots (n_\varkappa + \alpha_\varkappa),$$

with $n_j \in \mathbb{N} \cup \{0\}$. Let $\tilde{d}(\lambda_n)$ be the multiplicity of λ_n i.e,

$$\tilde{d}(\lambda_n) = \tilde{d}_\varkappa(\lambda_n) = \sum_{\substack{(n_1+\alpha_1)\cdots(n_\varkappa+\alpha_\varkappa)=\lambda_n \\ n_j \in \mathbb{N}\cup\{0\}}} 1.$$

As a generalization of the Piltz divisor problem, one may consider the summatory function

$$D(x; \boldsymbol{\alpha}) = {\sum_{\lambda_n \leq x}}' \tilde{d}(\lambda_n) = {\sum_{\substack{(n_1+\alpha_1)\cdots(n_\varkappa+\alpha_\varkappa)\leq x \\ n_j \in \mathbb{N}\cup\{0\}}}}' 1.$$

the asymptotic result with the main term the area πr^2 and the error term of the order of r. Dirichlet considered the corresponding problem for a hyperbola $xy = r$ and succeeded in obtaining an asymptotic formula with the error term of the order of r. Estimating the error term has been known as the Gauss circle problem and Dirichlet's divisor problem, respectively. As is known, $r(n)$ and $d(n)$ have generating Dirichlet series

$$\sum_{n=1}^{\infty} \frac{r(n)}{n^s} = \zeta_Q(s), \quad \sum_{n=1}^{\infty} \frac{d(n)}{n^s} = \zeta^2(s)$$

for $\mathrm{Re}\, s = \sigma > 1$, where $Q = Q(x, y) = x^2 + y^2$ and $\zeta_Q(s) = \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{1}{(m^2+n^2)^s}$ and $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ are the Epstein zeta-function and the Riemann zeta-function, respectively.

Voronoï introduced a new phase not only into the lattice point problem but also into the fields where there is a zeta-function, as expressing the error term in terms of a series of special functions and in particular, Bessel functions. Some of the generalizations are in higher dimensions, such as the $\varkappa$-dimensional sphere problem associated to the Epstein zeta-functions and the Piltz divisor problem associated to $\zeta^{\varkappa}(s)$. There has been a vast development in this area due to E. Landau, A. Z. Walfisz, A. A. Walfisz. K. Chandrasekharan and R. Narasimhan [13], B. Berndt et al from the point of view of functional equations satisfied by the associated zeta-functions.

0.2.1 Some Special Functions

- The Meijer G-function is defined by the following line integral

$$G_{p,q}^{m,n}\left(z \middle| \begin{matrix} a_1, .., a_p \\ b_1, .., b_q \end{matrix}\right) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j - s) \prod_{j=1}^{n} \Gamma(1 - a_j + s) z^s}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + s) \prod_{j=n+1}^{p} \Gamma(a_j - s)} \mathrm{d}s,$$

Synopsis

0.1 Introduction

This thesis is concerned about a study of asymptotic behaviour of coefficients of a class of Dirichlet series. These works are done during my stay at Harish-Chandra Research Institute as a research scholar. The thesis is divided in four chapters. Basic notations, definitions and some important results has been introduced in the first chapter which will be used throughout the thesis. The second chapter deals with a generalization of the Dirichlet divisor problem in the case of the product of two or more Hurwitz zeta-functions. The summary of this part is given in Section 0.2. In the third chapter we discuss about an asymptotic behaviour for the summatory function (with exponential weights) of the coefficients of the Rankin-Selberg L-function. The summary of this part is given in Section 0.3. The fourth chapter contains the discussion about general convolution theorems. We are able to express Lau's convolution theorem with Stieltjes integral. The summary of this chapter is given in Section 0.4.

0.2 Shifted Divisor Problem

Counting lattice points in a domain has been a fascinating subject and already Gauss considered the lattice points in a circle with radius r, say, and enunciated

Theorem 1.5.14 (Hecke) *Let $f \in S_k(\Gamma)$. The function $L(s, f)$ can be analytically continued to an entire function and satisfies the functional equation*

$$(2\pi)^{-s}\Gamma(s)L(s, f) = i^k(2\pi)^{-(k-s)}\Gamma(k-s)L(k-s, f).$$

The proof of this result can be found in [40, Theorem 5.3.7, p. 66].

It is important to note that "Riemann Hypothesis" can be extended for Hecke L-function which is sometimes known as "Grand Riemann Hypothesis."

1.5.3 Hecke L-function

Hecke introduced generating Dirichlet series associated to the Fourier coefficients of modular forms which is sometimes known as Hecke L-function.

Definition 1.5.12 Let $f \in S_k$ with Fourier expansion $f(z) = \sum_{n=1}^{\infty} c(n)e^{2\pi inz}$ and $s = \sigma + it$ be a complex variable. Then the function

$$L(s, f) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}$$

is said to be the Hecke L-function associated to the cusp form f.

The Series $L(s, f)$ converges absolutely for $\sigma > k/2 + 1$.

Hecke studied a certain kind of averaging operator which is known as **Hecke operator**. For all $n \in \mathbb{N}$ the **Hecke operator** T_n takes a modular form in M_k to another modular form in M_k.

A modular form is said to be a **Hecke eigenform** if it is an eigen vector for Hecke operators T_n for all $n \in \mathbb{N}$.

Proposition 1.5.13 *Let $f \in S_k$ with Fourier coefficient $c(n)$. Then the following are equivalent:*

(i) *f is a Hecke eigenform with $c(1) = 1$.*

(ii) *$L(s, f)$ has an Euler product expansion*

$$L(s, f) = \prod_p (1 - c(p)p^{-s} + p^{k-1-2s})^{-1}.$$

The proof of this proposition can be found in [19, Theorem 5.9.2, pp. 201].

where χ is a Dirichlet character and $s \in \mathbb{C}$ with $\Re(s) > 1$.

In particular, for the trivial character χ the associated Dirichlet L-function is the Riemann zeta function. Since Dirichlet characters are completely multiplicative, $L(s, \chi)$ will have an Euler product representation as follows:

$$L(s, \chi) = \prod_{p \in \mathbb{P}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1},$$

for $\Re(s) > 1$.

Theorem 1.5.9 (i) *For the principal character* $\chi_1 \pmod{N}$ *the* L*-function* $L(s, \chi_1)$ *is analytic everywhere except for a simple pole at* $s = 1$ *with residue* $\frac{\phi(k)}{k}$*, where* ϕ *is an Euler totient function.*
(ii) *If* $\chi \neq \chi_1$*,* $L(s, \chi)$ *is an entire function of* s*.*

$L(s, \chi)$ also satisfies a suitable functional equation. Riemann Hypothesis can be extended for Dirichlet L-function which is known as generalized Riemann hypothesis. This conjecture was first formulated by Adolf Piltz in 1884.

Conjecture 1.5.10 (Generalized Riemann Hypothesis (GRH)) For every Dirichlet character χ, all the non-trivial zeros of $L(s, \chi)$ lie on the line $\Re(s) = 1/2$.

The following conjecture will also be used in the results.

Conjecture 1.5.11 (Grand Simplicity Hypothesis (GSH)) The (positive) imaginary parts of non-trivial zeros of $L(s, \chi)$ with χ running over all primitive Dirichlet characters are linearly independent over $\mathbb{Q}$ (see Rubinstien and Sarnak [45]).

In particular, GSH implies that all the non-trivial zeros of $\zeta(s)$ are simple.

Conjecture 1.5.6 (Riemann Hypothesis (RH)) All the non-trivial zeros of $\zeta(s)$ lie on the line $\Re(s) = 1/2$.

1.5.2 Dirichlet L-function

Dirichlet introduced an important example of Dirichlet series in 1837 to prove the celebrated theorem on "primes in arithmetic progressions."

Definition 1.5.7 (Dirichlet characters) Let $N \in \mathbb{N}$. A Dirichlet character $\chi \pmod N$ is a homomorphism

$$\chi : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*.$$

We extend the definition of χ to all natural numbers by setting

$$\chi(n) = \begin{cases} \chi \;\; (n \pmod N)), & \text{if} \quad \gcd(n, N) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

If $\chi(n) = 1$ for all $n \in \mathbb{N}$, then we call χ a trivial character. The principal character χ_1 is that which has the properties

$$\chi_1(n) = \begin{cases} 1 & \text{if} \quad (n, N) = 1 \\ 0 & \text{if} \quad (n, N) > 1 \end{cases}$$

Dirichlet characters are completely multiplicative functions.

Definition 1.5.8 (Dirichlet L-function) A Dirichlet L-series is a function of the form

$$L(s, \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

This series is absolutely convergent for $\Re(s) > 1$ and has the Euler product representation

$$\zeta(s) = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Definition 1.5.3 (Gamma function) The classical Gamma function is defined by

$$\Gamma(s) := \int_0^\infty x^{s-1} e^{-x} \mathrm{d}x$$

for $s \in \mathbb{C}$ with $\Re(s) > 0$.

It satisfies the functional equation $\Gamma(s+1) = s\Gamma(s)$ and can be analytically continued to a meromorphic function on the complex plane with poles at non-positive integers.

Theorem 1.5.4 *The Riemann Zeta function $\zeta(s)$ can be analytically continued to the whole complex plane except for a simple pole at $s = 1$, and it satisfies the functional equation*

$$\xi(s) = \xi(1-s),$$

where

$$\xi(s) := \frac{s(s-1)}{2} \pi^{-s/2} \Gamma(s/2) \zeta(s).$$

One can find a proof of this celebrated result in [51]. $\zeta(s)$ has trivial zeros at $s = -2n$ for $n \in \mathbb{N}$, which arise due to the poles of $\Gamma(s/2)$. The following theorem can be proved from the Euler product of $\zeta(s)$.

Theorem 1.5.5 *The function $\zeta(s)$ has no zeros with $\Re(s) \geq 1$.*

One of the most important conjectures in mathematics is the Riemann hypothesis, which is about the non-trivial zeros of $\zeta(s)$.

form

$$\sum_{n=1}^{\infty} \frac{r_2(n)}{n^s} = \zeta_Q(s) \quad \text{for} \quad \Re(s) > 1,$$

where $Q = Q(x, y) = x^2 + y^2$ is a positive definite binary quadratic form and

$$\zeta_Q(s) = \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{1}{(m^2+n^2)^s}$$

is the Epstein zeta-function.

Euler [22] discovered an important theorem in 1737, which can be taken as a definition of Euler product.

Theorem 1.5.1 *Assume* $\sum_{n=1}^{\infty} f(n)n^{-s}$ *converges absolutely for* $\Re(s) > \sigma_a$. *If* f *is multiplicative we have*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left[1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \cdots\right] \quad \textit{if} \quad \Re(s) > \sigma_a,$$

and if f *is completely multiplicative we have*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \frac{1}{1 - f(p)p^{-s}} \quad \textit{if} \quad \Re(s) > \sigma_a.$$

1.5.1 Riemann zeta function

One of the important examples of Dirichlet series is the Riemann zeta function which was studied by Bernhard Riemann in the year 1859.

Definition 1.5.2 (Riemann Zeta function) Let $s \in \mathbb{C}$. The Riemann Zeta function is defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{for} \quad \Re(s) > 1.$$

k in $\mathrm{SL}(2,\mathbb{Z})$. Then

$$\sum_{n\leq x} c(n) \ll x^{\frac{k-1}{2}+\frac{1}{4}+\epsilon}.$$

In their seminal work, Chandrasekharan and Narasimhan ([13], [14]) showed that the Classical Conjecture is true. In other words, partial sums of coefficients of cusp forms appear to satisfy a Gauss circle problem type growth bound.

Further, in Gauss circle problem one estimates the partial sum of $r_2(n)$, the number of representation of n as a sum of two squares. The coefficients $r_2(n)$ appear as the Fourier coefficients of a modular form $\theta^2(z)$(cf. [19, pp. 11]). Hence the analogy between this partial sum with the partial sum of Fourier coefficients of cusp form is very strong. However θ^2 is not cuspidal, so there are some differences.

1.5 Dirichlet Series

In number theory, one of the most useful tools is Dirichlet series. In this section we will discuss some basic notions and some important examples of the series.

Consider the series of the form

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s},$$

where $f(n)$ is an arithmetical function. These are called **Dirichlet series** with coefficients $f(n)$.

For any arithmetic function $f(n)$ we can associate a Dirichlet series as above. In this case we call this series **generating Dirichlet series**. For example in the case of Gauss circle problem $r_2(n)$ has a generating Dirichlet series of the

1.4 Connection between the Gauss circle problem and Modular forms

We have from Ramanujan conjecture that individual $\tau(n)$ satisfy the bound

$$\tau(n) \ll n^{\frac{11}{2}+\epsilon}$$

and numerical experimentation might lead one to conjecture that

$$\sum_{n\le x} \tau(n) \ll x^{\frac{11}{2}+\frac{1}{4}+\epsilon}.$$

just like in Hardy's estimate in the Gauss circle problem 1/4 comes in the exponent of the partial sum of Ramanujan tau function. Ramanujan's conjecture can be extended to include all modular and automorphic forms. For a cusp form f of weight k in $\mathrm{SL}(2,\mathbb{Z})$, the conjecture states that

$$c(n) \ll n^{\frac{k-1}{2}+\epsilon}.$$

This is now known as the Ramanujan-Petersson conjecture, and it is true for integral weight, holomorphic cusp forms over full modular group $\mathrm{SL}(2,\mathbb{Z})$ as a consequence of Deligne's proof of the Weil conjecture.

For a general cusp form f of weight k in $\mathrm{SL}(2,\mathbb{Z})$, we expect an analogous conjecture for $\sum_{n\le x} c(n)$ to hold, which we can refer to as the "classical conjecture."

Conjecture 1.4.1 Let $f(z) = \sum_{n=1}^{\infty} c(n)q^n$ be a holomorphic cusp form of weight

The set of all modular forms (resp. cusp forms) of weight k over $\mathrm{SL}(2,\mathbb{Z})$ forms a vector space. We denote this space by M_k (resp. S_k).

Example 1.3.1 (Ramanujan delta function) One of the popular examples of modular forms is the Ramanujan delta function which is defined as

$$\Delta(z) = q\prod_{n\geq 1}(1-q^n)^{24},$$

where $q = e^{2\pi i z}$. It is a holomorphic cusp form of weight 12 i.e. $\Delta(z) \in S_{12}$. It has a Fourier expansion

$$\Delta(z) = \sum_{n=1}^{\infty}\tau(n)e^{2\pi i n z},$$

where $\tau(n)$ is known as the **Ramanujan tau function**.

In 1916, Ramanujan [44] studied this function and stated the following conjectures:

1. $\tau(n)$ is a multiplicative function.

2. $\tau(p^{r+1}) = \tau(p)\tau(p^r) - p^{11}\tau(p^{r-1})$.

3. $|\tau(p)| \leq p^{11/2}$.

The first two properties were proved by Mordell [39] in 1917 and the third one was proved by Deligne [18] in 1974.

as follows:

$$\mathrm{SL}(2,\mathbb{Z}) : \mathfrak{H} \longrightarrow \mathfrak{H}$$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} : z \longmapsto \frac{az+b}{cz+d}.$$

Definition 1.3.1 (Modular form) A complex-valued function f on $\mathfrak{H}$ is said to be a modular form of weight k over full modular group $\mathrm{SL}(2,\mathbb{Z})$ if it satisfies the following properties:

(i) The function f is a holomorphic on $\mathfrak{H}$.

(ii) For $z \in \mathfrak{H}$, $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \quad \forall\, \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z})$.

Note that $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ implies $f(z+1) = f(z)$. Hence these functions will have a Fourier series expansion of the form

$$f(z) = \sum_{n=0}^{\infty} c(n) q^n,$$

where $q = e^{2\pi i z}$ and $z \in \mathfrak{H}$.

(iii) The function f must be holomorphic as $z \to i\infty$.

Definition 1.3.2 (Cusp form) A modular form f of weight k for $\mathrm{SL}(2,\mathbb{Z})$, is said to be a cusp form if it vanishes at the cusp $i\infty$ i.e., if $c(0) = 0$. Then the Fourier series expansion of f can be written as

$$f(z) = \sum_{n=1}^{\infty} c(n) q^n.$$

$$\Delta(x) \ll \begin{cases} x^{139/429+\epsilon} & \text{(Kolesnik, 1985)} \\ x^{23/73+\epsilon} & \text{(Huxley, 1993)} \\ x^{131/416}(\log x)^{\frac{18637}{8320}} & \text{(Iwaniec, Mozzoici, 2003)} \end{cases}$$

In the year 1919, Hardy ([25], [26, pp. 243-263]) obtained a celebrated identity which states that

$$\Delta(x) = \sqrt{x}\sum_{n\geq 1}\frac{r_2(n)}{n^{1/2}}J_1(2\pi\sqrt{nx})$$

where

$$J_\nu(z) := \sum_{n\geq 0}\frac{(-1)^n}{\Gamma(n+1)\Gamma(\nu+n+1)}(z/2)^{\nu+2n}$$

is the ordinary Bessel function of first kind. This identity is known as the "Hardy identity".

1.3 Modular forms

In this section, we recall some basic notions related to classical modular forms.

Let $\mathfrak{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ be the **Poincaré upper half plane**. Let k be an even positive integer. Let

$$\mathrm{SL}(2,\mathbb{Z}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Big| a,b,c,d \in \mathbb{Z},\ ad-bc=1 \right\}.$$

Now $\mathrm{SL}(2,\mathbb{Z})$ acts on the upper half plane $\mathfrak{H}$ by linear fractional transformation,

1.2 Gauss circle problem

In the year 1798, Gauss [23] proposed a simple and innocent question that how many integer lattice points lie inside or on the circle of radius $\sqrt{x}$? Intuitively,

$$\#\{(p,q) \in \mathbb{Z}^2 |\, p^2 + q^2 \leq x\} \approx \pi x$$

This can be made rigorous by thinking of each lattice point as being the center of a 1×1 square in the plane and counting those squares fully contained within the circle and those squares lying on the boundary of the circle. Using this line of thought, Gauss [23] enunciated the asymptotic result with the main term, the area of the circle πx and the error term of order $\sqrt{x}$. Let

$$r_2(n) := \#\{(x,y) \in \mathbb{Z}^2 |\, x^2 + y^2 = n\}.$$

Setting $r_2(0) = 1$, we can say that $\sum_{n \leq x} r_2(n)$ is equal to the number of lattice points lying within or on the circle of radius $\sqrt{x}$. Thus we have

$$\sum_{n \leq x} r_2(n) = \pi x + \Delta(x)$$

where $\Delta(x)$ is the error term. Estimating this error term $\Delta(x)$ is known as **Gauss circle problem**. Over 100 years there was no further development in estimating the error term. It was Sierpinski [46] who got the first improvement in the year 1906. Then many improvements have been obtained during the past century. The history of estimating $\Delta(x)$ is as follows :

$$\Delta(x) \ll \begin{cases} x^{1/3} & \text{(Sierpinski, 1906)} \\ \Omega(x^{1/4}) & \text{(Hardy, 1917)} \end{cases}$$

Similarly, an arithmetic function g is multiplicative if it satisfies

$$f(mn) = f(m)f(n),$$

for m, n relatively prime. If this property holds for all m and n, then f is said to be completely multiplicative. For example, $f(n) = n^{-s}$ with $s \in \mathbb{C}$, is completely multiplicative.

We will write $f(n) = O(g(n))$, for two arithmetic functions f and g, if there is a constant K such that $|f(n)| \leq Kg(n)$ for all $n \in \mathbb{N}$. Sometimes we also use the notation $\ll$ and write $f(n) \ll g(n)$ to indicate the same thing.

Many arithmetical functions fluctuate considerably as n increases and it is often difficult to determine their behaviour for large n. For example, consider $d(n)$, the number of divisors of n. This function takes on the value 2 infinitely often when n is prime and it also takes on arbitrary large values when n has a large number of divisors. Hence it is fruitful to study the asymptotic behaviour of the arithmetic functions.

In this context, it is important to study about the partial sums of an arbitrary function f i.e, to study the sum $\sum_{k=1}^{n} f(k)$. Sometimes it is convenient to replace the upper index n by an arbitrary positive real number x and to consider instead sums of the form

$$\sum_{k \leq x} f(k).$$

Here the index k varies from 1 to $[x]$, the greatest integer $\leq x$. If $0 < x < 1$ the sum is empty and we assign it the value 0.

CHAPTER 1

Background

We introduce basic notations, definitions and some important results in this chapter, which will be used throughout the thesis. We only present what is relevant to this thesis, and is by no means a complete overview of the subject.

1.1 Introduction

Let $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ denote the set of natural numbers, integers, rational numbers, real and complex numbers respectively. The set of prime numbers is denoted by $\mathbb{P}$. For $z \in \mathbb{C}$, $\Re(z)$ denotes the real part of z and $\Im(z)$ denotes the imaginary part of z. We use the symbol $\#$ for the cardinality of a set.

Definition 1.1.1 (Arithmetic function) A real or complex valued function f defined on $\mathbb{N}$ is called an arithmetic function. An arithmetic function f is said to be additive if it satisfies

$$f(mn) = f(m) + f(n),$$

for m, n relatively prime. If this property holds for all m and n, then f is said to be completely additive. For example, $f(n) = \log n$ is completely additive.

or

$$D(x;\alpha,1) = \mathrm{P}_{\alpha,1}(x) + \frac{\sqrt{x}}{2}\sum_{n=1}^{\infty}\frac{d_\alpha(n)+d_{-\alpha}(n)}{\sqrt{n}}Y_1(-4\pi\sqrt{nx})$$
$$-\frac{\sqrt{x}}{\pi}\sum_{n=1}^{\infty}\frac{d_\alpha(n)+d_{-\alpha}(n)}{\sqrt{n}}K_1(4\pi\sqrt{nx})$$
$$-\frac{i\sqrt{x}}{2}\sum_{n=1}^{\infty}\frac{d_\alpha(n)+d_{-\alpha}(n)}{\sqrt{n}}J_1(-4\pi\sqrt{nx}).$$

where $d_\alpha(n) = d_{\alpha,1}(n) = \sum_{l|n} e^{2\pi i l\alpha}$ *defined by* (2.10) *and*

$$P_{\alpha,1}(x) = x\log x + x\left[\left(-\frac{\Gamma'}{\Gamma}(\alpha)\right)+\gamma-1\right]+\frac{1}{2}\alpha-\frac{1}{4}.$$

Here γ *is the Euler constant.*

Proof. In (2.19) the generating function is $\zeta(s,\alpha)\zeta(s)$.

Using the functional equation we have

$$\zeta(s,\alpha)\zeta(s) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\pi^{-\frac{1}{2}+s}\{e^{-\frac{\pi i}{2}(1-s)}l_{1-s}(\alpha)+e^{\frac{\pi i}{2}(1-s)}l_{1-s}(-\alpha)\}\zeta(1-s).$$

Proceeding similarly to that of Theorem 2.6.1, we complete the proof of the corollary.

Theorem 2.7.1.

Theorem 2.7.3 (Corrected version of Nakajima's theorem) *For* $0 < \alpha, \beta \le 1$,

$$D(x;\alpha,\beta) = \mathrm{P}(x) + \Delta(x;\alpha,\beta), \tag{2.27}$$

where

$$\begin{aligned}\mathrm{P}(x) = \mathrm{P}_{\alpha,\beta}(x) = x\log x &+ \left\{\left(-\frac{\Gamma'}{\Gamma}(\alpha)\right) + \left(-\frac{\Gamma'}{\Gamma}(\beta)\right) - 1\right\}x \\ &+ \zeta(0,\alpha)\zeta(0,\beta)\end{aligned} \tag{2.28}$$

and

$$\begin{aligned}\Delta(x;\alpha,\beta) = &\frac{\sqrt{x}}{2}\sum_{n=1}^{\infty}\frac{d_{\alpha,\beta}(n)+d_{-\alpha,-\beta}(n)}{\sqrt{n}}Y_1(-4\pi\sqrt{nx}) \\ &-\frac{\sqrt{x}}{\pi}\sum_{n=1}^{\infty}\frac{d_{-\alpha,\beta}(n)+d_{\alpha,-\beta}(n)}{\sqrt{n}}K_1(4\pi\sqrt{nx}) \\ &-\frac{i\sqrt{x}}{2}\sum_{n=1}^{\infty}\frac{d_{\alpha,\beta}(n)+d_{-\alpha,-\beta}(n)}{\sqrt{n}}J_1(-4\pi\sqrt{nx}),\end{aligned} \tag{2.29}$$

where the coefficients are defined in section 2.5.

It is of some interest to consider the case where one of the perturbation parameters is 1. The following corollary deals with it.

Corollary 2.7.4 *For* $0 < \alpha \le 1$, *we have*

$$\begin{aligned}D(x;\alpha,1) = \mathrm{P}_{\alpha,1}(x) &+ \frac{i\sqrt{x}}{\pi}\sum_{n=1}^{\infty}\frac{d_{\alpha}(n)+d_{-\alpha}(n)}{\sqrt{n}}K_1(4\pi i\sqrt{nx}) \\ &- \frac{\sqrt{x}}{\pi}\sum_{n=1}^{\infty}\frac{d_{\alpha}(n)+d_{-\alpha}(n)}{\sqrt{n}}K_1(4\pi\sqrt{nx}),\end{aligned}$$

Theorem 2.7.1 *For* $0 < \alpha, \beta < 1$,

$$D(x;\alpha,\beta) = \mathrm{P}(x) + \Delta(x;\alpha,\beta) \tag{2.24}$$

where

$$\mathrm{P}(x) = \mathrm{P}_{\alpha,\beta}(x) = x\log x + \left\{\left(-\frac{\Gamma'}{\Gamma}(\alpha)\right) + \left(-\frac{\Gamma'}{\Gamma}(\beta)\right) - 1\right\}x + \zeta(0,\alpha)\zeta(0,\beta)$$

and

$$\begin{aligned}\Delta(x;\alpha,\beta) =& \frac{i\sqrt{x}}{\pi}\sum_{n=1}^{\infty}\frac{d_{\alpha,\beta}(n)+d_{-\alpha,-\beta}(n)}{\sqrt{n}}K_1(4\pi i\sqrt{nx}) \\ &-\frac{\sqrt{x}}{\pi}\sum_{n=1}^{\infty}\frac{d_{-\alpha,\beta}(n)+d_{\alpha,-\beta}(n)}{\sqrt{n}}K_1(4\pi\sqrt{nx}).\end{aligned} \tag{2.25}$$

Remark 2.7.2 The K-Bessel function is related to other Bessel functions via

$$Y_\nu(iz) = e^{\frac{\pi i(\nu+1)}{2}}I_\nu(z) - \frac{2}{\pi}e^{-\frac{\pi i\nu}{2}}K_\nu(z) \tag{2.26}$$

where $\arg z \in (-\pi, \frac{\pi}{2}]$. Let $J_\nu(z)$ denote the Bessel function of the first kind. Then

$$J_\nu(iz) = e^{\frac{\pi i\nu}{2}}I_\nu(z).$$

Now for $\nu = 1$, this gives

$$\begin{aligned}K_1(-iz) &= \frac{\pi}{2}\left(-J_1(z) - iY_1(z)\right), \\ K_1(iz) &= \frac{\pi}{2}\left(-J_1(-z) - iY_1(-z)\right).\end{aligned}$$

Hence the corrected version of Nakajima's result follows immediately from *where*

$$\mathrm{P}(x) = x\log x + \left\{ \left(-\frac{\Gamma'}{\Gamma}(\alpha_1)\right) + \left(-\frac{\Gamma'}{\Gamma}(\alpha_2)\right) - 1 \right\} x + \left(\frac{1}{2}-\alpha_1\right)\left(\frac{1}{2}-\alpha_2\right).$$

(ii) *For* $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in (0,1)^3$, *we have*

$$\begin{aligned} D(x;\boldsymbol{\alpha}) = \mathrm{P}(x) &- \frac{(e^{-\frac{i\pi}{2}})^3}{(2\pi)^3}\sum_{n=1}^{\infty}\frac{d_{+3,-0}(n)}{n} V\left((2\pi)^3\left(e^{\frac{\pi i}{2}}\right)^3 nx \middle| 1,1,0\right) \\ &- 3\frac{e^{-\frac{i\pi}{2}}}{(2\pi)^3}\sum_{n=1}^{\infty}\frac{d_{+2,-1}(n)}{n} V\left((2\pi)^3 e^{\frac{\pi i}{2}} nx \middle| 1,1,0\right) \\ &- 3\frac{e^{\frac{i\pi}{2}}}{(2\pi)^3}\sum_{n=1}^{\infty}\frac{d_{+1,-2}(n)}{n} V\left((2\pi)^3 e^{-\frac{\pi i}{2}} nx \middle| 1,1,0\right) \\ &- \frac{(e^{\frac{i\pi}{2}})^3}{(2\pi)^3}\sum_{n=1}^{\infty}\frac{d_{+0,-3}(n)}{n} V\left((2\pi)^3\left(e^{-\frac{\pi i}{2}}\right)^3 nx \middle| 1,1,0\right) \end{aligned}$$

with $\mathrm{P}(x) = \mathrm{P}_{\boldsymbol{\alpha}}(x) = \mathrm{P}_{\alpha_1,\alpha_2,\alpha_3}(x)$ *is the residual function* (2.14).

2.7 Two-dimensional case

In this section we simplify the error term for the case of $\varkappa = 2$. Let us recall the inverse Heaviside integral

$$G^{2,0}_{0,2}\left(z \,\middle|\, \begin{matrix} - \\ a, b \end{matrix}\right) = 2\, z^{\frac{1}{2}(a+b)} K_{a-b}\big(2\sqrt{z}\,\big)\,, \tag{2.23}$$

where $K_\kappa(z)$ is the modified Bessel function of the second kind which is often referred to as the K-Bessel function. If one uses this in Corollary 2.6.2, (i); it entails:

we transform the first integral as

$$
\begin{aligned}
I_1 =& \frac{1}{2\pi i}\int_{(-a)} \frac{\Gamma(1-s)^{\varkappa}}{(2\pi)^{\varkappa(1-s)}}(e^{-\frac{\pi i}{2}(1-s)})^{\varkappa} l_{1-s}(\alpha_1)\cdots l_{1-s}(\alpha_{\varkappa})\frac{x^s}{s}\mathrm{d}s \\
=& -\frac{1}{(2\pi)^{\varkappa}}e^{-\frac{\pi i}{2}\varkappa}\sum_{n=1}^{\infty}\frac{d_{+\varkappa,-0}(n)}{n}G^{\varkappa,0}_{0,\varkappa}\left((2\pi)^{\varkappa}(e^{\frac{\pi i}{2}})^{\varkappa}nx \,\middle|\, \begin{matrix} - \\ 1,1,\cdots,1,0\end{matrix}\right) \\
=& -\frac{(e^{-\frac{\pi i}{2}})^{\varkappa}}{(2\pi)^{\varkappa}}\sum_{n=1}^{\infty}\frac{d_{+\varkappa,-0}(n)}{n}V\left((2\pi)^{\varkappa}(e^{\frac{\pi i}{2}})^{\varkappa}nx \,\middle|\, 1,1,\cdots,1,0\right),
\end{aligned}
$$

The second integral as

$$
\begin{aligned}
I_2 =& \frac{1}{2\pi i}\int_{(-a)} \frac{\Gamma(1-s)^{\varkappa}}{(2\pi)^{\varkappa(1-s)}}(e^{-\frac{\pi i}{2}(1-s)})^{\varkappa-2}\{l_{1-s}(-\alpha_1)\cdots l_{1-s}(\alpha_{\varkappa})+\cdots \\
&+ l_{1-s}(\alpha_1)\cdots l_{1-s}(-\alpha_{\varkappa})\}\frac{x^s}{s}\mathrm{d}s \\
=& -\binom{\varkappa}{1}\frac{1}{(2\pi)^{\varkappa}}e^{-\frac{\pi i}{2}(\varkappa-2)}\sum_{n=1}^{\infty}\frac{d_{+(\varkappa-1),-1}(n)}{n}G^{\varkappa,0}_{0,\varkappa}\left((2\pi)^{\varkappa}(e^{\frac{\pi i}{2}})^{\varkappa-2}nx \,\middle|\, \begin{matrix} - \\ 1,1,\cdots,1,0\end{matrix}\right) \\
=& -\binom{\varkappa}{1}\frac{(e^{-\frac{\pi i}{2}})^{\varkappa-2}}{(2\pi)^{\varkappa}}\sum_{n=1}^{\infty}\frac{d_{+(\varkappa-1),-1}(n)}{n}V\left((2\pi)^{\varkappa}(e^{\frac{\pi i}{2}})^{\varkappa-2}nx \,\middle|\, 1,1,\cdots,1,0\right).
\end{aligned}
$$

Similarly, by transforming the other integrals, finally we can conclude our main result.

In particular, Theorem 2.6.1 with $\varkappa = 2,3$ amounts to:

Corollary 2.6.2 (i) *For* $\boldsymbol{\alpha} = (\alpha_1,\alpha_2) \in (0,1)^2$, *we have*

$$
\begin{aligned}
D(x;\boldsymbol{\alpha}) = \mathrm{P}(x) &+ \frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{d_{\alpha_1,\alpha_2}(n)}{n}V\left((2\pi)^2 e^{\pi i}nx \,\middle|\, 1,0\right) \\
&- \frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{d_{\alpha_1,-\alpha_2}(n)+d_{-\alpha_1,\alpha_2}(n)}{n}V\left((2\pi)^2 nx \,\middle|\, 1,0\right) \\
&+ \frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{d_{-\alpha_1,-\alpha_2}(n)}{n}V\left((2\pi)^2 e^{-\pi i}nx \,\middle|\, 1,0\right),
\end{aligned}
$$

is the error term.

At this stage one can apply the standard procedure of applying the functional equation (2.11) followed by the interchange of summation and integration. Thus we arrive onto ρ times integration of the Voronoï-Steen function and we have sums with the coefficient

$$G^{\varkappa+1,1}_{1,\varkappa}\left((2\pi)^{\varkappa}(e^{\frac{\pi i}{2}})^{\varkappa}nx \,\middle|\, \begin{matrix} 1 \\ 1,1,\cdots,1,-\rho \end{matrix}\right). \tag{2.20}$$

It would be of some interest to express the resulting G-functions in explicit form as in [32], [58] etc., but here we stick to the 0th order Riesz sum. Hence we apply the differentiation under the integral sign ρ times since we are assured of the uniform convergence of the integral of the form $I : \frac{1}{2\pi i}\int_{(-a)}\Gamma(1-s)^{\varkappa}\frac{w^s}{s}\mathrm{d}s$. Finally we differentiate the residual function $P^{\rho}(x)$ under the integral sign to arrive at the main term. We now Use (2.11) and the fact that $l_s(1-\alpha) = l_s(-\alpha)$ to obtain

$$\begin{aligned}\varphi(s) &= \zeta(s,\alpha_1)\zeta(s,\alpha_2)\cdots\zeta(s,\alpha_{\varkappa})\\ &= \frac{\Gamma(1-s)^{\varkappa}}{(2\pi)^{\varkappa(1-s)}}\Big[(e^{-\frac{\pi i}{2}(1-s)})^{\varkappa}l_{1-s}(\alpha_1)l_{1-s}(\alpha_2)\cdots l_{1-s}(\alpha_{\varkappa})\\ &+(e^{-\frac{\pi i}{2}(1-s)})^{\varkappa-1}e^{\frac{\pi i}{2}(1-s)}\{l_{1-s}(-\alpha_1)l_{1-s}(\alpha_2)\cdots l_{1-s}(\alpha_{\varkappa})\\ &+l_{1-s}(\alpha_1)l_{1-s}(-\alpha_2)\cdots l_{1-s}(\alpha_{\varkappa})+\cdots+l_{1-s}(\alpha_1)l_{1-s}(\alpha_2)\cdots l_{1-s}(-\alpha_{\varkappa})\}\\ &+\cdots\cdots\cdots+(e^{\frac{\pi i}{2}(1-s)})^{\varkappa}l_{1-s}(-\alpha_1)l_{1-s}(-\alpha_2)\cdots l_{1-s}(-\alpha_{\varkappa})\Big].\end{aligned} \tag{2.21}$$

We compute the error term where we denote the resulting integrals by I_1, $I_2,\cdots,I_{\varkappa}$. Now using the basic property of Γ function

$$\frac{\Gamma(1-s)^{\varkappa}}{s} = -\Gamma(1-s)^{\varkappa-1}\Gamma(-s), \tag{2.22}$$

for $\sigma > 1$. We consider the Riesz sum of order ρ

$$D^{\rho}(x;\boldsymbol{\alpha}) = \frac{1}{\Gamma(\rho+1)} \sum_{\lambda_n \le x} (x-\lambda_n)^{\rho} \tilde{d}(\lambda_n), \tag{2.16}$$

where $\rho \in \mathbb{N}$ and $\rho > \frac{1}{2}\varkappa(1+a)+1$ for some $a > 0$.

Now using generalized Perron's formula (2.12) the ρ order Riesz sum (2.16) can be written as

$$D^{\rho}(x;\boldsymbol{\alpha}) = \frac{1}{2\pi i} \int_{(c)} \varphi(s) \frac{1}{s(s+1)\cdots(s+\rho)} x^{s+\rho}\, \mathrm{d}s, \tag{2.17}$$

where $\varphi(s)$ is defined by (2.15).

We apply the Cauchy residue theorem as follows. Take a rectangle with vertices at $s = c - iT$, $s = c + iT$, $s = -a + iT$ and $s = -a - iT$, for $0 < T < \infty$. Since the order ρ of the Riesz sum satisfies the condition in (2.16), hence by lemma 2.5.2 the horizontal integrals will vanish as $T \to \infty$. At the same time, the condition on the order ρ assures the absolute convergence of the vertical integral along $s = -a + it$. Thus we let $T \to \infty$ and express the initial integral by the residual function and the resulting vertical integral, which is the meaning of 'moving the line of integration' from (c) to $(-a)$:

$$D^{\rho}(x;\boldsymbol{\alpha}) = P^{\rho}(x) + \Delta^{\rho}(x;\boldsymbol{\alpha}) \tag{2.18}$$

where $P^{\rho}(x)$ is the residual function and

$$\Delta^{\rho}(x;\boldsymbol{\alpha}) = \frac{1}{2\pi i} \int_{(-a)} \zeta(s,\alpha_1)\zeta(s,\alpha_2)\cdots\zeta(s,\alpha_{\varkappa}) \frac{1}{s(s+1)\cdots(s+\rho)} x^{s+\rho}\, \mathrm{d}s. \tag{2.19}$$

It can also be expressed in terms of a G-function

$$V(x;a_1,\cdots,a_n)=G^{n,0}_{0,n}\left(x\,\middle|\,\begin{matrix}-\\ a_1,\dots,a_n\end{matrix}\right).$$

2.6 Main Result

We have the following theorem for the summatory function (2.2) of the $\varkappa$-dimensional shifted divisor function.

Theorem 2.6.1 *For $\boldsymbol{\alpha}=(\alpha_1,\cdots,\alpha_\varkappa)\in(0,1)^\varkappa$, we have*

$$\begin{aligned}
D(x;\boldsymbol{\alpha}) =&\mathrm{P}(x)-\frac{(e^{-\frac{i\pi}{2}})^\varkappa}{(2\pi)^\varkappa}\sum_{n=1}^{\infty}\frac{d_{+\varkappa,-0}(n)}{n}V\left((2\pi)^\varkappa\left(e^{\frac{\pi i}{2}}\right)^\varkappa nx\middle|1,\cdots,1,0\right)\\
&-\binom{\varkappa}{1}\frac{(e^{-\frac{i\pi}{2}})^{\varkappa-2}}{(2\pi)^\varkappa}\sum_{n=1}^{\infty}\frac{d_{+(\varkappa-1),-1}(n)}{n}V\left((2\pi)^\varkappa\left(e^{\frac{\pi i}{2}}\right)^{\varkappa-2} nx\middle|1,\cdots,1,0\right)\\
&-\binom{\varkappa}{2}\frac{(e^{-\frac{i\pi}{2}})^{\varkappa-4}}{(2\pi)^\varkappa}\sum_{n=1}^{\infty}\frac{d_{+(\varkappa-2),-2}(n)}{n}V\left((2\pi)^\varkappa\left(e^{\frac{\pi i}{2}}\right)^{\varkappa-4} nx\middle|1,\cdots,1,0\right)\\
&-\cdots\\
&-\frac{(e^{\frac{i\pi}{2}})^\varkappa}{(2\pi)^\varkappa}\sum_{n=1}^{\infty}\frac{d_{+0,-\varkappa}(n)}{n}V\left((2\pi)^\varkappa\left(e^{-\frac{\pi i}{2}}\right)^\varkappa nx\middle|1,\cdots,1,0\right),
\end{aligned}\tag{2.13}$$

where $\mathrm{P}(x)=\mathrm{P}_{\boldsymbol{\alpha}}(x)$ is the residual function which is the sum of the residues of the weighted generating function

$$\varphi(s)\frac{x^s}{s}=\zeta(s,\alpha_1)\cdots\zeta(s,\alpha_\varkappa)\frac{x^s}{s}\tag{2.14}$$

at $s=0$ and 1.

Proof. Let

$$\varphi(s):=\sum_{\lambda_n}\frac{\tilde{d}(\lambda_n)}{\lambda_n^s}=\zeta(s,\alpha_1)\cdots\zeta(s,\alpha_\varkappa)\tag{2.15}$$

which is also called the Riesz sum of order κ with $\varphi(w) = \sum_{\lambda_n} \frac{\alpha_n}{\lambda_n^w}$. Here $\int_{(c)}$ indicates the Bromwich integral along the vertical line $\Re(s) = c$.

2.5.2 Some special functions

We recall the following special functions which will be used in the sequel.

Definition 2.5.4 (Meijer G-function) Meijer G-function is defined by the following line integral

$$G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} a_1, .., a_p \\ b_1, .., b_q \end{matrix} \right) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s) z^s}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)} \mathrm{d}s,$$

where m, n, p, q are integers with $0 \leq m \leq q$, $0 \leq n \leq p$ and $a_i - b_j \notin \mathbb{N}$ for $1 \leq i \leq p$, $1 \leq j \leq q$.

The **modified Bessel function** of second kind $K_\nu(z)$ can be expressed in terms of a **Meijer G-function** such as

$$K_\nu(z) = 2^{\nu-1} z^{-\nu} G_{0,2}^{2,0}\left(z \,\middle|\, \begin{matrix} - \\ \nu, 0 \end{matrix} \right).$$

Definition 2.5.5 (Voronoï-Steen function) Voronoï-Steen function $V(x; a_1, \cdots, a_n)$ is defined by

$$\frac{1}{2\pi i} \int_0^\infty x^s V(x; a_1, \cdots, a_n) \frac{\mathrm{d}x}{x} = \Gamma(s + a_1) \cdots \Gamma(s + a_n).$$

are negative. For example

$$d_{+2,-1} = d_{+2,-1}(n) = \sum_{\substack{m_j \in \mathbb{N} \\ m_1 m_2 m_3 = n}} e^{2\pi i(\alpha_1 m_1 + \alpha_2 m_2 - \alpha_3 m_3)}.$$

The Hurwitz zeta-function $\zeta(s, \alpha)$ is related to the Lerch zeta-function $l_s(\alpha)$ by the functional equation (cf. [1, pp. 257]):

$$\zeta(1-s, \alpha) = \frac{\Gamma(s)}{(2\pi)^s}\{e^{-\frac{\pi i}{2}s} l_s(\alpha) + e^{\frac{\pi i}{2}s} l_s(1-\alpha)\}. \tag{2.11}$$

2.5.1 Riesz Sum

Riesz means were introduced by M. Riesz [28] and have been studied in connection with summability of Fourier series and of Dirichlet series (cf. [9], [12], [33]). For a given increasing sequence $\{\lambda_n\}$ of real numbers and a given sequence $\{\alpha_n\}$ of complex numbers, the Riesz sum of order κ is defined by

$$\begin{aligned} \mathcal{A}^\kappa(x) = \mathcal{A}^\kappa_\lambda(x) &= {\sum_{\lambda_n \le x}}' (x - \lambda_n)^\kappa \alpha_n \\ &= \kappa \int_0^x (x-t)^{\kappa-1} \mathcal{A}_\lambda(t) \mathrm{d}t \\ &= \kappa \int_0^x (x-t)^{\kappa-1} \mathrm{d}\mathcal{A}_\lambda(t) \end{aligned}$$

with $\mathcal{A}_\lambda(x) = \mathcal{A}^0_\lambda(x) = {\sum_{\lambda_n \le x}}' \alpha_n$, where the prime on the summation sign means that when $\lambda_n = x$, the corresponding term is to be halved. Sometimes normalized $\frac{1}{\Gamma(\kappa+1)}\mathcal{A}^\kappa(x)$ can be expressed in terms of generalized Perron's formula

$$\frac{1}{\Gamma(\kappa+1)} {\sum_{\lambda_n \le x}}' \alpha_n (x - \lambda_n)^\kappa = \frac{1}{2\pi i} \int_{(C)} \frac{\Gamma(w)\varphi(w) x^{\kappa+w}}{\Gamma(w+\kappa+1)} \mathrm{d}w, \tag{2.12}$$

for $\sigma := \operatorname{Re} s > 1$.

In particular for $\alpha = 1$, it reduces to **Riemann zeta function** i.e. $\zeta(s) = \zeta(s, 1)$.

Lemma 2.5.2 *For $\Re(s) \leq 1$, we have*

$$\zeta(s, \alpha) = O\left(|t|^{\tau(\sigma)} \log |t|\right), \tag{2.8}$$

where

$$\tau(\sigma) = \begin{cases} \frac{1}{2}(1-\sigma) & \text{for } 0 \leq \sigma \leq 1 \\ \frac{1}{2} - \sigma & \text{for } \sigma \leq 0. \end{cases}$$

Definition 2.5.3 (Lerch zeta function) The Lerch zeta-function is defined by

$$l_s(\alpha) = \sum_{n=1}^{\infty} \frac{e^{2\pi i n \alpha}}{n^s}$$

for $\sigma > 1, \alpha \in \mathbb{R}$ (or $s = 1, 0 < \alpha < 1$). It is also known as **polylogarithm function**.

For $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_\varkappa) \in (0,1]^\varkappa \cap \mathbb{R}^\varkappa$, we can write the product of $\varkappa$ Lerch zeta-functions as

$$l_s(\alpha_1) \cdots l_s(\alpha_\varkappa) = \sum_{n=1}^{\infty} \frac{d_{\boldsymbol{\alpha}}(n)}{n^s}, \quad \sigma > 1, \tag{2.9}$$

where

$$d_{\boldsymbol{\alpha}}(n) = d_{\alpha_1, \cdots, \alpha_\varkappa}(n) = \sum_{\substack{m_j \in \mathbb{N} \\ m_1 \cdots m_\varkappa = n}} e^{2\pi i (\alpha_1 m_1 + \cdots + \alpha_\varkappa m_\varkappa)}. \tag{2.10}$$

We write $d_{+(\varkappa - r), -r}(n)$ to indicate $d_{\boldsymbol{\alpha}}(n)$ with $(\varkappa - r)$ α_i's are positive and r α_i's

$$-\frac{i\sqrt{x}}{2}\sum_{n=1}^{\infty}\frac{d_{\alpha,-\beta}(n)-d_{-\alpha,\beta}(n)}{\sqrt{n}}J_1(4\pi\sqrt{nx}), \tag{2.6}$$

where the coefficients are to be defined below.

Remark 2.4.2 Nakajima applies the 0th order Perron's formula to express $D(x;\alpha,\beta)$ as

$$D(x;\alpha,\beta)=\frac{1}{2\pi i}\int_{(c)}\zeta(s,\alpha)\zeta(s,\beta)\frac{x^s}{s}\,\mathrm{d}s \quad (c>1). \tag{2.7}$$

It was warned, however, e.g. in Davenport [16, pp. 104-105] that applying the 0th order Perron formula is problematic because there is no guarantee that the interchange of summation and integration is legitimate and that to stick to the 0th order Perron formula, one has to apply the truncated formula as can be found in many textbooks. Nakajima made the same mistake in his another paper [42] which was subsequently corrected and improved by Banerjee and Mehta [2].

The common procedure is to apply higher order Riesz sums as in many previous investigations including Landau [37], A.A.Walfisz [54], and Chandrasekharan and Narasimhan [13, 106-111]. Then the final result for the 0th Riesz sum can be obtained by differencing or in most cases by differentiating as long as the differentiated series is uniformly convergent.

2.5 Preliminaries

Definition 2.5.1 (Hurwitz zeta-function) The Hurwitz zeta-function $\zeta(s,\alpha)$ is defined for $0<\alpha\le 1$ by

$$\zeta(s,\alpha)=\sum_{n=0}^{\infty}\frac{1}{(n+\alpha)^s}$$

where the prime on the summation sign means that the term corresponding to $(n_1+\alpha_1)\cdots(n_\varkappa+\alpha_\varkappa)=x$ is halved. Here the main objective is to estimate the error term of $D(x;\boldsymbol{\alpha})$ in terms of a special function.

2.4.1 Work of Nakajima

Nakajima [41] considered the above problem when $\varkappa=2$. For $0<\alpha\leq 1$ and $0<\beta\leq 1$, he mainly considered the summatory function

$$D(x;\alpha,\beta)=\sideset{}{'}\sum_{\substack{(m+\alpha)(n+\beta)\leq x\\ m,n\in\mathbb{N}\cup\{0\}}} 1 \tag{2.3}$$

where the prime on the summation sign means that the term corresponding to $(n_1+\alpha_1)\cdots(n_\varkappa+\alpha_\varkappa)=x$ is halved.

Theorem 2.4.1 (Nakajima) *For* $0<\alpha,\beta\leq 1$,

$$D(x;\alpha,\beta)=\mathrm{P}(x)+\Delta(x;\alpha,\beta), \tag{2.4}$$

where

$$\begin{aligned}\mathrm{P}(x)=\mathrm{P}_{\alpha,\beta}(x)=x\log x \;&+\; \left\{\left(-\frac{\Gamma'}{\Gamma}(\alpha)\right)+\left(-\frac{\Gamma'}{\Gamma}(\beta)\right)-1\right\}x\\ &+\; \zeta(0,\alpha)\zeta(0,\beta)\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\Delta(x;\alpha,\beta)=&-\frac{\sqrt{x}}{2}\sum_{n=1}^{\infty}\frac{d_{\alpha,\beta}(n)+d_{-\alpha,-\beta}(n)}{\sqrt{n}}Y_1(4\pi\sqrt{nx})\\ &-\frac{\sqrt{x}}{\pi}\sum_{n=1}^{\infty}\frac{d_{\alpha,-\beta}(n)+d_{-\alpha,\beta}(n)}{\sqrt{n}}K_1(4\pi\sqrt{nx})\end{aligned}$$

bola if we shift the origin to a fixed coordinate (α, β)?

Nakajima [41] introduced this problem in 1993 but his work contains some technical errors. In this chapter we correct the result of Nakajima and generalize the problem in higher dimensions.

2.4 Shifted divisor problem

Let $\varkappa$ be a positive integer ≥ 2, and $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_\varkappa) \in [0,1)^\varkappa \cap \mathbb{R}^\varkappa$. We are mainly concerned in counting the number of lattice points of the shifted hyperbola i.e, to find

$$\#\{(x_1, x_2, \cdots x_\varkappa) \in \mathbb{Z}^\varkappa : (x_1+\alpha_1)(x_2+\alpha_2)\cdots(x_\varkappa+\alpha_\varkappa) \leq x\}.$$

Note that the problem reduces to generalized divisor problem for $\boldsymbol{\alpha} = 0$. Let $\{\lambda_n\}$ be a strictly increasing sequence of real numbers of the form

$$\lambda_n = (n_1+\alpha_1)(n_2+\alpha_2)\cdots(n_\varkappa+\alpha_\varkappa) \qquad \text{for } n_j \in \mathbb{N}\cup\{0\}.$$

Let us define the shifted generalized divisor function $\tilde{d}(\lambda_n)$ as;

$$\tilde{d}(\lambda_n) = \tilde{d}_\varkappa(\lambda_n) = \sum_{\substack{(n_1+\alpha_1)\cdots(n_\varkappa+\alpha_\varkappa)=\lambda_n \\ n_j\in\mathbb{N}\cup\{0\}}} 1. \tag{2.1}$$

As a generalization of the generalized divisor problem, one may consider the summatory function

$$D(x;\boldsymbol{\alpha}) = {\sum_{\lambda_n\leq x}}' \tilde{d}(\lambda_n) = {\sum_{\substack{(n_1+\alpha_1)\cdots(n_\varkappa+\alpha_\varkappa)\leq x \\ n_j\in\mathbb{N}\cup\{0\}}}}' 1, \tag{2.2}$$

The estimation of $\Delta_k(x)$ for $k = 3$ is known as **Piltz divisor problem**. Set $\Delta_3(x) = O(x^{a_3+\epsilon})$. The history of estimating a_3 is as follows :

$$a_3 = \begin{cases} 1/2 & \text{(Hardy, Littlewood, 1922)} \\ 43/87 & \text{(Walfisz, 1925)} \\ 37/75 & \text{(Atkinson, 1941)} \\ 14/29 & \text{(Ming-i, 1958)} \\ 8/17 & \text{(Ming-i, Fang, 1962)} \\ 5/11 & \text{(Jing-run, 1965)} \\ 43/96 & \text{(Kolesnik, 1981)} \end{cases}$$

Several mathematicians have studied **general divisor problem** for $k \geq 4$.

Theorem 2.3.1 [30, Thm. 13.2] *Set $\Delta_k(x) = O(x^{a_k+\epsilon})$. Then for any $\epsilon > 0$ we have*

$$\begin{aligned} a_k &= \frac{3k-4}{4k} && (for \quad 4 \leq k \leq 8) \\ a_9 &= 35/54,\ a_{10} = 41/60,\ a_{11} = 7/10, \\ a_k &= \frac{k-2}{k+2} && (for \quad 12 \leq k \leq 25) \\ a_k &= \frac{k-1}{k+4} && (for \quad 26 \leq k \leq 50) \\ a_k &= \frac{31k-98}{32k} && (for \quad 51 \leq k \leq 57) \\ a_k &= \frac{7k-34}{7k} && (for \quad k \geq 58) \end{aligned}$$

In this context naturally one can ask the following :

Question 2.3.2 *How many lattice points are there inside or on the the hyper-*

$$a = \begin{cases} 346/1067 & \text{(Kolensik, 1973)} \\ 35/108 & \text{(Kolesnik, 1982)} \\ 7/22 & \text{(Iwaniec, Mozocci, 1988)} \\ 23/73 & \text{(Huxley, 2003)} \end{cases}$$

2.3 Generalized divisor problem

It is natural to generalize **Dirichlet divisor problem** by considering the summatory function

$$\sum_{n \le x}{}' d_k(n)$$

where $d_k(n)$ is the number of ways n may be written as a product of k given factors, so that $d_1(n) \equiv 1$ and $d_2(n) \equiv d(n)$. Estimating the error term $\Delta_k(x)$ of the summatory function is known as **generalized divisor problem**. The error term $\Delta_k(x)$ can be written in the form

$$\Delta_k(x) := \sum_{n \le x}{}' d_k(n) - xP_{k-1}(\log x) - \frac{(-1)^k}{2^k}$$

where $P_{k-1}(t)$ is a suitable polynomial in t of degree k, and one has in fact

$$P_{k-1}(\log x) = \operatorname*{Res}_{s=1} \frac{x^{s-1}\zeta^k(s)}{s}.$$

The connection between $d_k(n)$ and $\zeta^k(s)$ is a natural one,

$$\zeta^k(s) = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s} \qquad (\Re(s) > 1).$$

Theorem 2.2.1 (Voronoï) *Let $x > 0$, we have*

$$\sum_{n\le x}{}' d(n) = x\log x + (2\gamma - 1)x + \frac{1}{4} - \sqrt{x}\sum_{k=1}^{\infty}\frac{d(k)}{\sqrt{k}}\left(Y_1\left(4\pi\sqrt{xk}\right) + \frac{2}{\pi}K_1\left(4\pi\sqrt{xk}\right)\right).$$

where $\sum'$ means that the corresponding term to be halved when $n = x$, γ is Euler's constant. Y_1 and K_1 denotes the Bessel function of second kind and modified Bessel function of second kind respectively.

We have the asymptotic formula in the following form for $x > 0$,

$$\sum_{n\le x}{}' d(n) = x\log x + (2\gamma - 1)x + \frac{1}{4} + \Delta(x)$$

where $\Delta(x)$ is an error term. The Dirichlet divisor problem basically asks for the correct order of magnitude of $\Delta(x)$ as $x \to \infty$. Let us set, $\Delta(x) = O(x^{a+\epsilon})$. Many improvements have been obtained in determining the value of a during the past century or more. For every $\epsilon > 0$, the history of estimating a is as follows :

$$a = \begin{cases} 33/100 & \text{(Corput, 1923)} \\ 37/112 & \text{(Littlewood, Hardy, 1925)} \\ 163/494 & \text{(Walfisz, 1927)} \\ 27/82 & \text{(Nieland, 1928)} \\ 15/46 & \text{(Titchmarch, 1934)} \\ 13/40 & \text{(Hua, 1942)} \\ 12/37 & \text{(Chen, 1963)} \end{cases}$$

many researches since the beginning of the 20th century. cf. e.g. [51]. Some of the generalizations are in the higher dimensions. As a generalization of the Gauss circle problem, Kendall and Rankin [34] considered the lattice points in a random sphere by shifting the centre from the origin and obtained the Bessel series expression. Later, Chandrasekharan and Narasimhan [15] obtained the Bessel series expression for the higher dimensional random sphere problem. Our aim here is to consider corresponding problem for a hyperbola as a generalization of Dirichlet divisor problem.

2.2 Divisor problem

Dirichlet considered the problem of counting the lattice points in a hyperbola in the year 1849. Let $d(n)$ denotes the divisor function i.e, $d(n) = \sum_{d|n} 1$. For all $x \geq 1$, we have

$$\sum_{n\leq x} d(n) = \sum_{n\leq x} \sum_{d|n} 1 = \#\{(p,q) \in \mathbb{Z}^2 | pq \leq x\}.$$

Dirichlet obtained an asymptotic formula of $\sum_{n\leq x} d(n)$ with a main term

$$x\log x + (2\gamma - 1)x + \frac{1}{4},$$

where γ is the Euler's constant and an error term of order $\sqrt{x}$. Estimating the error term of the summatory function $\sum_{n\leq x} d(n)$ is known as Dirichlet divisor problem or Dirichlet hyperbola problem.

In the year 1904, Voronoï [53] introduced a new phase into the lattice point problem. He was able to express the error term in terms of a Bessel functions.

CHAPTER 2

Shifted Divisor Problem

This chapter deals with a certain generalization of the Dirichlet divisor problem. This problem is also known as Dirichlet hyperbola problem as the main goal of this problem is to count the number of lattice points inside or on the hyperbola. Here we are mainly concerned in counting the number of lattice points of the hyperbola of higher dimension after shifting the origin to an arbitrary coordinate. The results of this chapter have been published in [55].

2.1 Introduction

Counting lattice points in a domain has been a fascinating subject initiated by Gauss when he considered the problem of counting the lattice points in a circle. Dirichlet considered the corresponding problem for a hyperbola and succeeded in obtaining an asymptotic formula with the error term. Estimating the error term has been known as the Gauss circle problem and the Dirichlet's divisor problem respectively. Voronoï [53] who introduced a new phase into the lattice point problem by expressing the error term in terms of special functions, and in particular Bessel functions. Both the Gauss circle problem and the Dirichlet divisor problem along with their generalizations have been a driving force for

Now we put $w = 1 - s$, and so we have $\mathfrak{R}(w) > 1$. Again, using Proposition 3.4.2, we get

$$|I_2| \ll \frac{1}{2\pi i} \int_{\nu - i\infty}^{\nu + i\infty} \left| B(1-w) y^{w-1} \right| \mathrm{d}w,$$

where $\nu = 1 - \mu$ and $\nu > 1$. Finally, Stirling formula, i.e., lemma 3.6.1, gives

$$\begin{aligned} |I_2| &\ll \lim_{T\to\infty} \int_{-T}^{T} |B(1-\nu-it)| y^{\nu-1} \mathrm{d}t \\ &\ll \lim_{T\to\infty} \int_{-T}^{T} |t|^{\nu} e^{-\frac{1}{2}\pi|t|} y^{\nu-1} \mathrm{d}t \\ &\ll \Gamma(\nu+1) y^{\nu-1}. \end{aligned}$$

Therefore, $|I_2| = O(y^{\epsilon})$ as we can choose $\nu = 1 + \epsilon$, where ϵ is arbitrary small positive number. This completes the proof.

Remark 3.6.4 As in the classical case, this result too can be extended to forms for congruence subgroups with of course added complications. We leave it for the interested reader.

$$|y_1 - y_2| < \exp\left(-A\frac{y_1}{\log y_1}\right) + \exp\left(-A\frac{y_2}{\log y_2}\right),$$

where A is a suitable positive constant, are included in the same bracket.

We now shift our attention to the following integral:

$$I_2 := \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} \Gamma(s)R(s, f\otimes g)y^{-s}\mathrm{d}s.$$

On the line of integral $\Re(s) = \mu < 0$, we use the functional equation of the Rankin-Selberg L-function, to get $\Re(1-s) > 1$, where Rankin-Selberg L-function is absolutely convergent. From the functional equation of the Rankin-Selberg L-function we have

$$R(s, f\otimes g) = \frac{L_\infty(1-s, f\otimes g)\zeta(2-2s)}{L_\infty(s, f\otimes g)\zeta(2s)}R(1-s, f\otimes g). \tag{3.23}$$

Then using the functional equation of zeta function, we get

$$\begin{aligned}\Gamma(s)R(s, f\otimes g) &= \frac{(\Gamma(s))^2 L_\infty(1-s, f\otimes g)\zeta(2-2s)}{\pi^{\frac{4s-1}{2}}\Gamma(\frac{1-2s}{2})L_\infty(s, f\otimes g)\zeta(1-2s)}R(1-s, f\otimes g)\\ &= \pi^{\frac{4s-3}{2}}B(s)\frac{\zeta(2-2s)}{\zeta(1-2s)}R(1-s, f\otimes g), \end{aligned}\tag{3.24}$$

where

$$B(s) = \frac{\Gamma(s)^2\Gamma(\frac{1-s+ir+iq}{2})\Gamma(\frac{1-s+ir-iq}{2})\Gamma(\frac{1-s-ir+iq}{2})\Gamma(\frac{1-s-ir-iq}{2})}{\Gamma(\frac{1-2s}{2})\Gamma(\frac{s+ir+iq}{2})\Gamma(\frac{s+ir-iq}{2})\Gamma(\frac{s-ir+iq}{2})\Gamma(\frac{s-ir-iq}{2})}.$$

As $\zeta(s)$ is convergent for $\Re(s) > 1$, we have

$$|I_2| \ll \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} \left|B(s)R(1-s, f\otimes g)y^{-s}\right|\mathrm{d}s.$$

and I_1 is just a special case of I_2 by taking f in place of g.

When $f = g$, $L(s, f \otimes g)$ has a simple pole at $s = 1$. Hence the residue becomes

$$\begin{aligned} R_1 &= \operatorname*{Res}_{s=1} \Gamma(s) \frac{L(s, f \otimes f)}{\zeta(2s)} y^{-s} \\ &= \operatorname*{Res}_{s=1} \Gamma(s) \frac{\Lambda(s, f \otimes f)}{\zeta(2s) L_\infty(s, f \otimes f)} y^{-s} \\ &= \lim_{s \to 1} \left\{ \frac{(s-1)\Gamma(s)\Lambda(s, f \otimes f)}{\zeta(2s) L_\infty(s, f \otimes f)} y^{-s} \right\}. \end{aligned}$$

Now we use proposition 3.4.1 and obtain

$$\begin{aligned} R_1 &= \frac{4\pi}{\zeta(2)y} Sin\{\pi/2(1+2ir)\} \parallel f \parallel^2 \\ &= \frac{24}{\pi y} Sin\{\pi/2(1+2ir)\} \parallel f \parallel^2 . \end{aligned} \tag{3.20}$$

Next, we will concentrate on the infinite residual term $\mathcal{P}(y)$. Let ρ be any arbitrary non-trivial zero of $\zeta(s)$. If we assume the *grand simplicity hypothesis*, which implies that, the non-trivial zeros of $\zeta(s)$ are simple, then

$$\begin{aligned} \mathcal{P}(y) &= \sum_\rho \operatorname*{Res}_{s=\rho/2} \Gamma(s) \frac{L(s, f \otimes g)}{\zeta(2s)} y^{-s} \\ &= \sum_\rho \frac{\Gamma(\rho/2) L(\rho/2, f \otimes g)}{\zeta'(\rho) y^{\rho/2}}. \end{aligned} \tag{3.21}$$

In general, if n_ρ is the multiplicity of ρ, then

$$\mathcal{P}(y) = \sum_\rho \frac{1}{(n_\rho - 1)!} \frac{\mathrm{d}^{n_\rho - 1}}{\mathrm{d}s^{n_\rho - 1}} \left\{ \frac{(s-\rho/2)^{n_\rho} \Gamma(s) L(s, f \otimes g)}{\zeta(2s) y^s} \right\} \Big|_{s=\rho/2}, \tag{3.22}$$

where $\rho = x + iy$ is running through all the non-trivial zeros of $\zeta(s)$ and the sum is decomposed into bracket so that the terms for which ing of infinitely many terms contributed by the non-trivial zeros of $\zeta(2s)$.

Firstly we prove that the horizontal integrals

$$H_1 = \frac{1}{2\pi i}\int_{\lambda+iT}^{\mu+iT}\Gamma(s)R(s,f\otimes g)y^{-s}\mathrm{d}s$$

and

$$H_2 = \frac{1}{2\pi i}\int_{\mu-iT}^{\lambda-iT}\Gamma(s)R(s,f\otimes g)y^{-s}\mathrm{d}s,$$

vanish as $T\to\infty$. We have

$$\begin{aligned} H_1 &= \frac{1}{2\pi i}\int_{\lambda+iT}^{\mu+iT}\Gamma(s)\frac{L(s,f\otimes g)}{\zeta(2s)}y^{-s}\mathrm{d}s \\ &= \frac{1}{2\pi i}\int_{\lambda}^{\mu}\Gamma(\sigma+iT)\frac{L(\sigma+iT,f\otimes g)}{\zeta(2(\sigma+iT))y^{\sigma+iT}}\mathrm{d}\sigma. \end{aligned}$$

Using lemmas 3.6.1, 3.6.2 and 3.6.3, we can write

$$|H_1| \ll |T|^A \exp\left(A_2T - \frac{1}{2}\pi|T|\right),$$

where A is a constant. Thus $H_1\to 0$ as $T\to\infty$. Similarly, one can show that $H_2\to 0$ as $T\to\infty$.

Therefore, (3.18) gives

$$\sum_{n=1}^{\infty}\lambda_f(n)\overline{\lambda_g(n)}\exp(-ny) = \begin{cases} R_1+\mathcal{P}(y)+I_1 & \text{if } f=g, \\ \mathcal{P}(y)+I_2 & \text{otherwise,} \end{cases} \tag{3.19}$$

where

$$I_2 = \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\Gamma(s)R(s,f\otimes g)y^{-s}\mathrm{d}s$$

Proof. We refer [31] for the proof.

3.6.1 Proof of Theorem 3.5.1

The inverse Mellin transform for the Γ-function is

$$\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}\Gamma(s)x^{-s}\mathrm{d}s = \begin{cases} e^{-x} & \text{if} \quad \lambda > 0, \\ e^{-x}-1 & \text{if} \quad -1 < \lambda < 0. \end{cases} \tag{3.16}$$

Now using (3.16), we can write

$$\begin{aligned}\sum_{n=1}^{\infty}\lambda_f(n)\overline{\lambda_g(n)}\exp(-ny) &= \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}\Gamma(s)R(s,f\otimes g)y^{-s}\mathrm{d}s \\ &= \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}\Gamma(s)\frac{L(s,f\otimes g)}{\zeta(2s)}y^{-s}\mathrm{d}s,\end{aligned} \tag{3.17}$$

where we choose $\lambda > 1$. Now by proposition 3.4.1, $R(s, f\otimes g)$ has a simple pole at $s = 1$ if $f = g$. If we assume generalized Riemann hypothesis, then we can see that $\zeta(2s)$ has infinitely many non-trivial zeros on the line $\mathfrak{R}(s) = 1/4$, and thus above integral has infinitely many poles on the line $\mathfrak{R}(s) = 1/4$.

Now we will choose a contour in such a way so that the poles lie inside the contour. For a large positive real number T, we consider the contour $\mathcal{C}$ determined by the line segments $[\lambda - iT, \lambda + iT]$, $[\lambda + iT, \mu + iT]$, $[\mu + iT, \mu - iT]$ and $[\mu - iT, \lambda - iT]$ where $\mu < 0$.

We then appeal to Cauchy's residue theorem and obtain

$$\frac{1}{2\pi i}\int_{\mathcal{C}}\Gamma(s)\frac{L(s,f\otimes g)}{\zeta(2s)}y^{-s}\mathrm{d}s = \begin{cases} R_1 + \mathcal{P}(y) & \text{if} \quad f = g, \\ \mathcal{P}(y) & \text{otherwise} \end{cases} \tag{3.18}$$

where $R_1 = \operatorname*{Res}_{s=1}\Gamma(s)\frac{L(s,f\otimes f)}{\zeta(2s)}y^{-s}$ and $\mathcal{P}(y)$ denotes the residual function consist-

$$|\Gamma(\sigma + iT)| = \sqrt{2\pi}|T|^{\sigma-1/2}e^{-\frac{1}{2}\pi|T|}\left(1 + O\left(\frac{1}{|T|}\right)\right) \tag{3.13}$$

as $|T| \to \infty$.

Proof. We refer [31, p. 151] for a proof of this well known result.

Lemma 3.6.2 *In the vertical strip* $-1 < \sigma < 2$*, we have*

$$\frac{1}{|\zeta(\sigma + iT)|} < e^{A_2 T} \tag{3.14}$$

for some suitable positive constant A_2*.*

Proof. We have from [51, p. 218, Equation(9.7.3)],

$$\log|\zeta(\sigma + iT)| \geq \sum_{|T-\gamma|\leq 1} \log|T - \gamma| + O(\log T).$$

Now let us choose a sequence of positive numbers T tending to infinity such that $|T - \gamma| > \exp(-A_1\gamma/\log\gamma)$ for every ordinate γ of a zero of $\zeta(s)$, where A_1 is some suitable positive constant. Then

$$\log|\zeta(\sigma + iT)| \geq -\sum_{|T-\gamma|\leq 1} A_1\frac{\gamma}{\log\gamma} + O(\log T) > -A_2 T$$

where $A_2 < \pi/4$ if A_1 is small enough. Thus the conclusion.

Lemma 3.6.3 *In a vertical strip* $a \leq \sigma \leq b$*, for large values of* $|T|$*, there exist a suitable constant* $A(\sigma)$ *(depending on* σ*) such that*

$$|L(\sigma + iT, f \otimes g)| \ll |T|^{A(\sigma)+\epsilon} \tag{3.15}$$

for any $\epsilon > 0$.

Theorem 3.5.1 *Let f and g be two Maass cusp forms which are normalized Hecke eigenforms over the full modular group with the Fourier series expansions as in (3.6) and (3.7) respectively. Assume that the non-trivial zeros of $\zeta(s)$ are simple. Then for any positive real y,*

$$\sum_{n=1}^{\infty} \lambda_f(n)\overline{\lambda_g(n)} \exp(-ny) = \begin{cases} R_1 + \mathcal{P}(y) + O(y^{\epsilon}) & \textit{if } f = g, \\ \mathcal{P}(y) + O(y^{\epsilon}) & \textit{otherwise.} \end{cases} \tag{3.12}$$

where the residual term

$$R_1 = \frac{24}{\pi y} Sin\{\pi/2(1+2ir)\} \parallel f \parallel^2$$

and

$$\mathcal{P}(y) = \sum_{\rho} \frac{\Gamma(\rho/2)L(\rho/2, f \otimes g)}{\zeta'(\rho) y^{\rho/2}}$$

where, $\rho = x + iy$ is running through all the non-trivial zeros of ζ-function. This sum is decomposed into bracket so that the terms for which

$$|y_1 - y_2| < \exp\left(-A\frac{y_1}{\log y_1}\right) + \exp\left(-A\frac{y_2}{\log y_2}\right),$$

where A is a suitable positive constant, are included in the same bracket.

3.6 Proof of Theorem 3.5.1

We require few lemmas to complete the proof of our main theorem.

Lemma 3.6.1 (Stirling's formula for Γ function) *In a vertical strip $\alpha \leq \sigma \leq \beta$ $(s = \sigma + iT)$,* Let

$$\Lambda(s, f \otimes g) = L_\infty(s, f \otimes g) L(s, f \otimes g), \tag{3.11}$$

where

$$L_\infty(s, f \otimes g) = \pi^{-2s}\Gamma(\frac{s+ir+iq}{2})\Gamma(\frac{s-ir+iq}{2})$$
$$\Gamma(\frac{s+ir-iq}{2})\Gamma(\frac{s-ir-iq}{2}).$$

The following proposition provides the analytic behaviour and the functional equation satisfied by the Rankin-Selberg L-function.

Proposition 3.4.1 *$\Lambda(s, f \otimes g)$ is absolutely convergent for $\Re(s) > 1$. It can be analytically continued to the whole complex plane except for finitely many poles. It has a simple pole at $s = 1$ if $f = g$ and has no poles otherwise. It also satisfies the functional equation:*

$$\Lambda(s, f \otimes g) = \Lambda(1 - s, f \otimes g).$$

The residue of Λ at $s = 1$ is $4 \parallel f \parallel^2$ if $f = g$.

Proof. We refer [8] and [31] for the proof.

The next proposition will be useful to estimate the error terms.

Proposition 3.4.2 *If $\sigma > 1$, then $|R(\sigma + iT, f \otimes g)| \leq R(\sigma, f \otimes g) < \infty$ for all values of T.*

3.5 Statement of the result

We are going to state our main result here.

coefficients of f and g by $\lambda_f(n)$ and $\lambda_g(n)$ with the Fourier series expansions:

$$f(z) = y^{1/2} \sum_{n\neq 0} \lambda_f(n) K_{ir}(2\pi|n|y) e(nx) \tag{3.6}$$

and

$$g(z) = y^{1/2} \sum_{n\neq 0} \lambda_g(n) K_{iq}(2\pi|n|y) e(nx) \tag{3.7}$$

respectively. It is normalized by setting $\lambda_f(1) = \lambda_g(1) = 1$. Here K_{ir} and K_{iq} are the modified Bessel function of the second kind, $z = x + iy$ and $e(x) = e^{2\pi ix}$.

The Petersson inner Product is given by :

$$\langle f, g\rangle = \int_{\Gamma(1)\backslash\mathcal{H}} f(z)\overline{g(z)} \frac{\mathrm{d}x\mathrm{d}y}{y^2}. \tag{3.8}$$

This integral is well defined because the integrand $f(z)\overline{g(z)}$ and the measure $\frac{dxdy}{y^2}$ are both invariant under the action of $\Gamma(1)$. This integral converges absolutely as the cusp forms are rapidly decreasing functions at each cusps. Let $\| f \|$ denote the norm of f with respect to the inner product.

3.4.2 Rankin-Selberg L-function

Our interest lies in the following Dirichlet series associated to f and g with the eigen values $(1/4 + r^2)$ and $(1/4 + q^2)$ respectively:

$$R(s, f \otimes g) := \sum_{n=1}^{\infty} \frac{\lambda_f(n)\overline{\lambda_g(n)}}{n^s}. \tag{3.9}$$

The **Rankin-Selberg L- function** associated to f and g is defined as:

$$L(s, f \otimes g) := \zeta(2s) \sum_{n=1}^{\infty} \frac{\lambda_f(n)\overline{\lambda_g(n)}}{n^s}. \tag{3.10}$$

holds (A is a suitable positive constant) are included in the same bracket. Also, $\mu_n = 4\pi^2 n$ *and* $\beta_n = \frac{1}{\sqrt{\pi}}|c^2(n)|$.

3.3.3 Remarks

This result has been further extended for any cusp form over any congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$ in [11]. The rest of the chapter is mainly concerned in studying the asymptotic behaviour of the same Lambert series in the Maass form set up. Unlike in the previous works, a much simpler looking expression has been obtained which avoids the celebrated relation of Shimura, that is, (3.3).

3.4 Maass form

A Maass form f for $\Gamma(1) = \mathrm{SL}(2,\mathbb{Z})$ is a smooth function on $\mathcal{H}$ such that,

1)$f(\gamma z) = f(z)\ \forall\ \gamma \in \Gamma(1)$ and $z \in \mathcal{H}$.

2)f is an eigen function of the non-Euclidean Laplacian operator

$$\Delta = -y^2\left(\frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y}\right).$$

3)There exists a positive integer N such that $f(x+iy) = O(y^N)$ as $y \to \infty$.

In particular f is a **cusp form** if

$$\int_0^1 f(z+x)\mathrm{d}x = 0.$$

3.4.1 Petersson inner product

Let f and g be two Maass cusp forms which are Hecke eigenforms over $\Gamma(1)$ with $\Delta(f) = (1/4+r^2)f$ and $\Delta(g) = (1/4+q^2)g$ respectively. We denote the Fourier

3.3.2 Result of Chakraborty et al.

Chakraborty et al. [10] generalized this result for arbitrary cusp forms over the full modular group. They used the functional equation of the Rankin-Selberg L-function to prove that the Lambert series $\sum_{n=1}^{\infty} |c^2(n)| \exp(-ny)$ also has an asymptotic expansion and it can be expressed in terms of the non-trivial zeros of $\zeta(s)$ when $y \to 0^+$, where $c(n)$ is the nth Fourier coefficient of any cusp form f over $\mathrm{SL}(2,\mathbb{Z})$. They expressed the error terms in terms of confluent hypergeometric function of second kind $U(a,b,z)$, which has an integral representation of the form

$$U(a,b,z) = \frac{1}{\Gamma(a)} \int_0^{\infty} e^{-zt} t^{a-1} (1+t)^{b-a-1} \mathrm{d}t.$$

Their main result is :

Theorem 3.3.2 [10, Thm. 1.1] *Let $f \in S_k(SL(2,\mathbb{Z}))$ with $f(\tau) = \sum_{n=1}^{\infty} c(n) e^{2\pi i n\tau}$. Assume that the non-trivial zeros of $\zeta(s)$ are simple. Then for positive real z,*

$$\sum_{n=1}^{\infty} |c^2(n)| e^{-4\pi^2 nz} = \frac{\Gamma(k) D(k)}{(4\pi^2 z)^k \zeta(2)} + \mathrm{P}(z) + z^{1-2k} \sum_{n=1}^{\infty} \beta_n e^{-\frac{\mu_n}{z}} U\left(-\frac{1}{2}, k, \frac{\mu_n}{z}\right), \tag{3.5}$$

where

$$\mathrm{P}(z) = \sum_{\rho} \frac{\Gamma(\frac{\rho}{2}+k-1)\zeta(\frac{\rho}{2}) D(\frac{\rho}{2}+k-1)}{\zeta'(\rho)(4\pi^2 z)^{\frac{\rho}{2}+k-1}}$$

with $\rho = x+iy$ runs over the non-trivial zeros of $\zeta(s)$ and the sum over ρ involves bracketing the terms so that the terms for which

$$|y-y'| < \exp\left(-\frac{Ay}{\log y}\right) + \exp\left(-\frac{Ay'}{\log y'}\right)$$

3.3 Previous works on Zagier's conjecture and its generalization

In this section we briefly recall the works been done on Zagier's conjecture and its generalization.

3.3.1 Work of Hafner and Stopple

Hafner and Stopple [24] considered the L-function associated to the Ramanujan delta function

$$L(s,\Delta)=\sum_{n=1}^{\infty}\tau(n)n^{-s}=\prod_{p}\left(1-\alpha_p p^{-s}\right)^{-1}\left(1-\beta_p p^{-s}\right)^{-1}.$$

The associated symmetric square L-function is,

$$\begin{aligned}D(s)&=L(s,\mathrm{Sym}^2\Delta)\\&=\prod_{p}\left(1-\alpha_p^2 p^{-s}\right)^{-1}\left(1-\alpha_p\beta_p p^{-s}\right)^{-1}\left(1-\beta_p^2 p^{-s}\right)^{-1}.\end{aligned}$$

They proved the following result.

Theorem 3.3.1 [24, Corr. (2.3)] *Assume all the non-trivial zeros of $\zeta(s)$ to be simple. For $z\to 0^+$,*

$$\begin{aligned}&\sum_{n=1}^{\infty}\tau^2(n)e^{-zn}=12\Gamma(11)z^{-12}\\&+z^{-11-1/4}\sum_{\rho}z^{1/4-\rho/2}\Gamma\left(\frac{\rho}{2}+11\right)\frac{\zeta(\rho/2)}{\zeta'(\rho)}D\left(\frac{\rho}{2}+11\right)\\&+O(z^{-11+1/2})\end{aligned}\tag{3.4}$$

where ρ runs through the non-trivial zeros of $\zeta(s)$.

the form

$$L(s,f)=\prod_{p}(1-\alpha(p)p^{-s})^{-1}(1-\beta(p)p^{-s})^{-1}$$

where $\alpha(p)$ and $\beta(p)$ are two complex numbers such that $\alpha(p)+\beta(p)=a(p)$ and $\alpha(p)\beta(p)=p^{k-1}$.

Definition 3.2.1 (Symmetric square L-function) Let $f\in S_k(\Gamma)$ be a normalized Hecke eigen form. Then the symmetric square L-function associated to f can be defined as

$$D(s)=L(s,Sym^2f)=\prod_{p}(1-\alpha^2(p)p^{-s})^{-1}(1-\alpha(p)\beta(p)p^{-s})^{-1}(1-\beta^2(p)p^{-s})^{-1}.$$

Shimura [47] gave the analytic continuation and functional equation of the symmetric square L-function.

Definition 3.2.2 (Rankin-Selberg L-function) Let f and g be modular forms of weight k over $\mathrm{SL}(2,\mathbb{Z})$ with Fourier expansion $f(z)=\sum_{n=0}^{\infty}a(n)e^{2\pi inz}$ and $g(z)=\sum_{n=0}^{\infty}b(n)e^{2\pi inz}$ respectively where at least one of f or g is cuspidal. Then the Rankin-Selberg L-function associated to f and g is defined as

$$L(s,f\otimes g)=\zeta(2s-2k+2)\sum_{n=1}^{\infty}\frac{a(n)b(n)}{n^s},$$

where $\Re(s)$ is sufficiently large.

A nice relation between the symmeric square L-function and the Rankin-Selberg L-function associated to cusp form f was obtained by Shimura [47] in the year 1975. He established that

$$\zeta(2s-2k+2)L(s,f\otimes f)=\zeta(s-k+1)D(s). \tag{3.3}$$

was proved by Hafner and Stopple [24]. They found the asymptotic expansion which in particular shows the oscillatory behaviour of the above Lambert series (assuming the Riemann hypothesis). Chakraborty et. al. [10] studied :

$$b_0(y) := \sum_{n=0}^{\infty} |c^2(n)| exp(-ny)$$

for $y \to 0^+$, where $c(n)$ is the n-th Fourier coefficient of any cusp form over the full modular group. They have shown that the above series also can be expressed in terms of the non-trivial zeros of the Riemann zeta function. Recently, this result has been further extended [11] for any cusp form over congruence subgroups of $\mathrm{SL}(2,\mathbb{Z})$. In this chapter, we deal with the same series associated to any cuspidal Maass Hecke eigenform for the full modular group and obtain the main term of the series in terms of non-trivial zeros of ζ-function. Unlike in the previous works, we obtain a much simpler looking expression not involving the symmetric square L-function.

3.2 L-function associated to a cusp form

Let $f \in S_k$ be a normalized Hecke eigenform with Fourier expansion $f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi inz}$ and $s = \sigma + it$ be a complex variable. Then the Hecke L-function associated to cusp form f is

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}.$$

The series $L(s,f)$ converges absolutely for $\sigma > k/2+1$. It follows from the proposition 1.5.13 that the Euler product expansion of $L(s,f)$ can be written in

Definition 3.1.2 (Lambert Series) A Lambert series is a series of the form

$$S(q) = \sum_{n=1}^{\infty} a(n)\frac{q^n}{1-q^n},$$

where $a(n)$ is any arithmetic function and $q \in \mathbb{C}$.

This series is absolutely convergent for $|q| < 1$ and represents an analytic function. Moreover, the power series of this function can be obtained by formal rearrangement of the series. Expanding naturally, we have

$$S(q) = \sum_{n=1}^{\infty} a(n) \sum_{k=1}^{\infty} q^{nk} = \sum_{n=1}^{\infty} b(n)q^n,$$

where $b(n) = \sum_{d|n} a(d)$. If we choose $q = \exp(-z)$, where z is a positive real number, then the Lambert series becomes

$$S(z) = \sum_{n=1}^{\infty} b(n)\exp(-nz).$$

Lambert series of various nature have been studied extensively and in this chapter we will see the asymptotic expansion of one interesting Lambert series.

In 1981, Zagier [59] conjectured that the inverse Mellin transform of the symmetric square L- function attached to Ramanujan's tau function has an asymptotic expansion in terms of the zeros of the Riemann zeta function. He considered the Lambert series:

$$\sum_{n=0}^{\infty} \tau^2(n)exp(-ny)$$

and mentioned that the asymptotic expansion of the series as $y \to 0^+$ can actually be used to evaluate the non-trivial zeros of $\zeta(s)$. Later, the above conjecture where r is any positive real number. This formula turned out to be incorrect as the contribution of non-trivial zeros of the Riemann zeta function $\zeta(s)$ was missing. Later, the corrected version of this formula was obtained by Hardy and Littlewood, which can be stated as follows :

Theorem 3.1.1 (Ramanujan, Hardy, Littlewood) [27, p. 156, Section 2.5] *Let α and β be two positive real numbers such that $\alpha\beta = \pi$. Assume that the series $\sum_\rho \frac{\Gamma\left(\frac{1-\rho}{2}\right)}{\zeta'(\rho)}\beta^\rho$ converges, where ρ runs through all non-trivial zeros of $\zeta(s)$, and non-trivial zeros of $\zeta(s)$ are simple. Then*

$$\sqrt{\alpha}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-\alpha^2/n^2} - \sqrt{\beta}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-\beta^2/n^2} = -\frac{1}{2\sqrt{\beta}}\sum_{\rho}\frac{\Gamma\left(\frac{1-\rho}{2}\right)}{\zeta'(\rho)}. \tag{3.2}$$

One can look into Berndt [6, p. 470] and Titchmarsh [51, p. 219] for the related works. Dixit [20] obtained a character analogue of the Ramanujan-Hardy-Littlewoood identity. Recently, Dixit et al. [21] gave one variable generalization of the above identity and analogues of these identities to Hecke forms. More importantly, it can be shown that the series on the right hand side of (3.2) is convergent if the terms ρ are in the same bracket for which

$$|\Im(\rho) - \Im(\rho')| < \exp\left(-A\frac{\rho}{\log \Im(\rho)}\right) + \exp\left(-A\frac{\rho'}{\log(\Im\rho')}\right),$$

where A is a positive constant (cf. [27, p. 158] and [51, p. 220]). We still do not know whether this series is convergent without the condition of bracketing the terms but the general belief is that the series will converge in the ordinary sense too.

Now we shall give the definition of Lambert series.

CHAPTER 3

Asymptotic behaviour of a series à la Zagier

Hafner and Stopple proved a conjecture of Zagier on the asymptotic expansion of a Lambert series involving Ramanujan's tau function with the main term involving the nontrivial zeros of the Riemann zeta function. Recently, Chakraborty et. al. have extended this result to any cusp forms over the full modular group and also over any congruence subgroups. The aim in this chapter is to study the asymptotic behaviour of a similar Lambert series involving the coefficients of Maass cusp forms over the full modular group. The contents of this chapter have been published in [4].

3.1 Introduction

Ramanujan's contribution has influenced many areas of number theory. During his stay at Cambridge, he obtained the following identity which is of interest to us.

$$\sum_{n=1}^{\infty} \frac{\mu(n)e^{-r/n^2}}{n} = \sqrt{\frac{\pi}{r}} \sum_{n=1}^{\infty} \frac{\mu(n)e^{-\pi^2/n^2 r}}{n}, \tag{3.1}$$

which can be continued analytically over the whole plane and it satisfies the functional equation

$$\Lambda_E(s,\chi) := \left(\frac{2\pi}{f\sqrt{N_E}}\right)^{-s}\Gamma(s)L_E(s,\chi) = c\Lambda_E(2-s,\chi), \tag{4.51}$$

where c is a certain constant.

As another example, we take up material in [17] on power moments of the zeta-functions along the line in the critical strip. Let $N(x)$ denote the number of cubic characters whose conductor $\leq x$ and consider the Z-function

$$Z(s) = \frac{1}{N(x)}\sum_{f\leq x}|L_E(s,\chi)|^{2k}. \tag{4.52}$$

Let $L(E\otimes E,s) = \sum_{n=1}^{\infty}\frac{a(n)}{n^{s+1}}$ be the Rankin-Selberg L-function associated with $L_E(s)$. Then we have a decomposition

$$Z(s) = Z^*(s)\Phi(s), \quad Z^*(s) = L(E\otimes E, 2(s-2))^{k^2} \tag{4.53}$$

and $\Phi(s)$ is analytic for $\sigma > \frac{1}{3}$.

Example 4.6.2 In the following examples, the asymptotic formulas for summatory functions can be obtained not only by our theorem but also by simpler theorems stated in [30].

1. Let β be a multiplicative function defined by $\beta(1) = 1$, $\beta(n) = \alpha_1 \cdots \alpha_r$ for $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$. Given any Dirichlet character mod k, Knopmacher [35] studies the generating function

$$Z(s) = \sum_{n=1}^{\infty} \frac{\chi(n)\beta(n)}{n^s} = Z^*(s)\Phi(s), \tag{4.47}$$

where

$$Z^*(s) = L(s,\chi)L(2s,\chi^2)L(3s,\chi^3) \qquad \text{and} \qquad \Phi(s) = \frac{1}{L(6s,\chi^6)}.$$

2. Let E be an elliptic curve defined over $\mathbb{Q}$ with conductor N_E and let

$$Z(s) = L_E(s) = \prod_{p \nmid N_E} \left(1 - \frac{a_p}{p^s} + \frac{1}{p^{2s-1}}\right)^{-1} \prod \left(1 - \frac{a_p}{p^s}\right)^{-1} = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \tag{4.48}$$

be the L-function associated with E for $\operatorname{Re} s = \sigma > 2$. It can be continued meromorphically over the whole plane in the case considered here including the case where the curve has complex multiplication. It has a decomposition

$$Z(s) = Z^*(s)\Phi(s), \quad Z^*(s) = \frac{1}{L_h(s)}, \quad \Phi(s) = \zeta(s)\zeta(s-1) \tag{4.49}$$

where $L_h(s)$ is the Hecke L-function with Grössencharacter. Cf. [7].

Let χ be a primitive character of conductor f. The twist of L-function is defined by

$$L_E(s,\chi) = \sum_{n=1}^{\infty} \frac{a(n)\chi(n)}{n^s} \tag{4.50}$$

$$Z(s) = \frac{Z_1(c_1 s) \cdots Z_k(c_k s)}{Z_{k+1}(c_{k+1} s)}, \tag{4.44}$$

where for $j = 1, 2, \cdots, k$ we have $c_j > 0$ and $c_{k+1} \geq c_j$. An asymptotic formula for the summatory functions of the coefficients can be obtained either from Theorem 4.4.1 or from Theorem 4.2.1 with some suitable conditions on the zeta-functions. We state some examples of this class of generating functions.

Example 4.6.1 ([29, Satz 11]) [3, Theorem 1.3] gives a general decomposition with a logarithmic singularity of generating Dirichlet series which has a representation of the form

$$Z(s) = Z^*(s)\Phi(s), \quad Z^*(s) = \exp\left(\sum_{j=1}^{h} \tau_j(s) \log L(s, \chi_j)\right) \tag{4.45}$$

where $\Phi(s)$ is regular and non-vanishing for $\sigma > 1/2$. In this case, Theorem 4.4.1 gives an asymptotic formula while Theorem 4.2.1 can give an asymptotic formula with unknown coefficients save for the leading one.

On [50, p. 256]), the special case is stated of the Gaussian field as the problem of the number of ideals whose norms are integers [49] and the leading coefficient is determined.

In [29, p. 251] with extension degree = 2, a generalization of the Dirichlet divisor problem is considered. The generating Dirichlet series $Z(s)$ has the representation

$$Z(s) = Z^*(s)\Phi(s) \tag{4.46}$$

where

$$Z^*(s) = \zeta_k(s)^2 \zeta_K(s) \quad \text{and} \quad \Phi(s) = \frac{1}{\zeta_K(2s)}.$$

In these cases, both theorems can give a concrete asymptotic formula.

4.5.2 Proof of Lau's Theorem

Lau's theorem is an easy consequence of Theorem 4.4.1. We have

$$\int_{u\le y} A(x/u)\mathrm{d}B(u) = \sum_{n\le y} b(n)A\left(\frac{x}{n}\right)$$
$$= \sum_{n=1}^{\infty} b(n)M_A\left(\frac{x}{n}\right) - \sum_{n>y} b(n)M_A\left(\frac{x}{n}\right) + \sum_{n\le y} b(n)E_A\left(\frac{x}{n}\right), \tag{4.42}$$

where the second sum is

$$\int_y^{\infty} M_A\left(\frac{x}{u}\right)\,\mathrm{d}B(u) = \int_y^{\infty} M_A\left(\frac{x}{u}\right)\,\mathrm{d}M_B(u) + E_1(x).$$

Hence with the change of variable, (4.42) becomes

$$\int_{u\le y} A(x/u)\mathrm{d}B(u)$$
$$= \sum_{n=1}^{\infty} b(n)M_A\left(\frac{x}{n}\right) + \sum_{n\le y} b(n)E_A\left(\frac{x}{n}\right) - \int_y^{\infty} M_A\left(\frac{x}{u}\right)\,\mathrm{d}M_B(u) + E_1(x)$$
$$= \sum_{n=1}^{\infty} b(n)M_A\left(\frac{x}{n}\right) + \sum_{n\le y} b(n)E_A\left(\frac{x}{n}\right) + \int_0^{x/y} M_A(u)\,\mathrm{d}M_B\left(\frac{x}{u}\right) + E_1(x). \tag{4.43}$$

Now substituting (4.43) in (4.16) and for $B(y) = \sum_{n\le y} b(n)$, we deduce Lau's theorem up to the error estimates.

4.6 Examples

In this section we discuss the case of any quotient with possibly one zeta function in the denominator and with a few zeta functions in the numerator such as

Now w lies on the contour $\mathcal{H}_Y(a,\delta)$, so that $a-Y\le \operatorname{Re} s\le a+\delta$. If $\operatorname{Re} w=\lambda=\eta$, we have $G_{A,\eta}(0)=0$ and $\operatorname{Re} w-\operatorname{Re} s=\eta-a-\delta>0$ by (4.5). Hence in (4.36) we may regard $t_0=0$, so that

$$\int_0^{t_0} E_A(t)t^{-s-1}\,\mathrm{d}t=\frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} G_A(w)\frac{\mathrm{d}w}{w(w-s)} \tag{4.37}$$

on regarding x/y as t_0. Adding (4.36) and (4.37) leads to

$$\begin{aligned}&\int_0^{x/y} E_A(t)t^{-s-1}\,\mathrm{d}t \qquad\qquad (4.38)\\ &=\frac{1}{2\pi i}\int_{(\alpha)}\frac{G_A(w)}{w(w-s)}\left(\frac{x}{y}\right)^{w-s}\mathrm{d}w+s^{-1}G_{A,\alpha}(0)\left(t_0^{-s}-\left(\frac{y}{x}\right)^s\right)\\ &+\frac{1}{2\pi i}\left\{\int_{(\eta)}-\int_{(\alpha)}\right\}\frac{G_A(w)}{w(w-s)}\,\mathrm{d}w.\end{aligned}$$

Since the last summand equals $s^{-1}G_A(s)-s^{-1}G_{A,\alpha}(0)$, with $t_0=1$, it becomes

$$\begin{aligned}&\int_0^{x/y} E_A(t)t^{-s-1}\,\mathrm{d}t \qquad\qquad (4.39)\\ &=\frac{1}{2\pi i}\int_{(\alpha)}\frac{G_A(w)}{w(w-s)}\left(\frac{x}{y}\right)^{w-s}\mathrm{d}w+s^{-1}\left(G_A(s)-G_{A,\alpha}(0)\left(\frac{y}{x}\right)^s\right).\end{aligned}$$

Substituting (4.39) in (4.31), we find that

$$J=\frac{1}{2\pi i}\int_{\mathcal{H}} G_B(s)\left(G_A(s)x^s-G_{A,\alpha}(0)y^s\right)\frac{\mathrm{d}s}{s}+E_4(x), \tag{4.40}$$

where $E_4(x)$ is absorbed in the error term (4.9).

Substituting (4.40) in (4.29) and noting

$$\frac{1}{2\pi i}\int_{\mathcal{H}} G_B(s)y^s\frac{\mathrm{d}s}{s}=\sum_{n\le y} b(n)=B(y) \tag{4.41}$$

completes the proof of Theorem 4.4.1 up to the error estimates.

where $G_{A,\lambda}(0) = G_A(0)$ if $\lambda < 0$ and 0 otherwise, and

$$I(\lambda, T, t) = \frac{1}{2\pi i}\left\{ \int_{2-iT}^{\lambda-iT} + \int_{\lambda+iT}^{2+iT} + \int_{2-i\infty}^{2-iT} + \int_{2+iT}^{2+i\infty} \right\} G_A(w)\frac{t^w}{w}\,\mathrm{d}w. \tag{4.33}$$

The application of Perron's formula in finite integral form is to be understood as in Davenport [16, pp. 104-105], i.e. as the truncated one applied with a proper error term which, however, can be absorbed in the error term. Indeed, in Lau's work the infinite integral is replaced by finite integral.

Under these conditions, for $\alpha \le \lambda \le \eta$, it follows by (4.2) that

$$E_A(t) = \frac{1}{2\pi i}\int_{\lambda-iT}^{\lambda+iT} G_A(w)t^w \frac{\mathrm{d}w}{w} + G_{A,\lambda}(0) + I(\lambda, T, t). \tag{4.34}$$

Substituting (4.34) in (4.31) and interchanging the integration, we are to evaluate the integral

$$\int_0^{x/y} t^{w-s-1}\,\mathrm{d}t, \tag{4.35}$$

which can be improper at $t = 0$ and is proper only for $\lambda = \eta$ in view of (4.5). However we are to choose $\lambda = \alpha$ for extracting part of the main term, and so we need to divide the integral inside the curly bracket in (4.31) into two parts. There is no need to split at $t = 1$ and at any midpoint $t = t_0 > 0$, we have

$$\begin{aligned}&\int_{t_0}^{x/y} E_A(t)t^{-s-1}\,\mathrm{d}t = \frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT} G_A(w)\frac{\mathrm{d}w}{w}\int_{t_0}^{x/y} t^{w-s-1}\,\mathrm{d}t\\ &+ G_{A,\alpha}(0)\int_{t_0}^{x/y} t^{-s-1}\,\mathrm{d}t + \int_{t_0}^{x/y} I(\alpha, T, t)t^{-s-1}\,\mathrm{d}t\\ &= \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} G_A(w)\left(\left(\frac{x}{y}\right)^{w-s} - t_0^{w-s}\right)\frac{\mathrm{d}w}{w(w-s)}\\ &+ s^{-1}G_{A,\alpha}(0)\left(t_0^{-s} - \left(\frac{y}{x}\right)^{s}\right).\end{aligned} \tag{4.36}$$

$$+ E_2(x). \tag{4.27}$$

Hence adding (4.27) to (4.26), we see that the integrals simplify to

$$C(x) = \int_{u\leq y} A(x/u)\mathrm{d}B(u) + \int_0^{1-} M_A(u)\,\mathrm{d}M_B\left(\frac{x}{u}\right) - \int_{1-}^{x/y} E_A(u)\,\mathrm{d}M_B\left(\frac{x}{u}\right)$$
$$- \int_0^{x/y} M_A(u)\,\mathrm{d}M_B\left(\frac{x}{u}\right) + E_2(x). \tag{4.28}$$

Now note that for $0 < t < 1$, we have $A(t) = 0$ and so $M_A(t) = -E_A(t)$ for $0 < t < 1$. Hence combining second and the third integral we have the integral over $(0, x/y)$, so that

$$C(x) = \int_{u\leq y} A(x/u)\mathrm{d}B(u) - \int_0^{x/y} M_A(u)\,\mathrm{d}M_B\left(\frac{x}{u}\right) + J + E_2(x) \tag{4.29}$$

where

$$J = -\int_0^{x/y} E_A(t)\,\mathrm{d}M_B\left(\frac{x}{t}\right). \tag{4.30}$$

Substituting (4.7) in (4.30), we derive that

$$J = \frac{1}{2\pi i}\int_{\mathcal{H}} G_B(s)x^s\Big\{\int_0^{x/y} E_A(t)t^{-s-1}\,\mathrm{d}t\Big\}\,\mathrm{d}s. \tag{4.31}$$

We apply the Riesz sum of order 0 referred to as Perron's formula and then from the residue theorem we have

$$\begin{aligned} A(t) &= \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} G_A(w)\frac{t^w}{w}\,\mathrm{d}w \\ &= M_A(t) + G_{A,\lambda}(0) + \frac{1}{2\pi i}\int_{\lambda-iT}^{\lambda+iT} G_A(w)t^w\frac{\mathrm{d}w}{w} + I(\lambda,T,t). \end{aligned} \tag{4.32}$$

Thus we have

$$C(x) = S_1 + S_2, \tag{4.24}$$

where

$$S_1 = S_1(x,y) = \int_{u\le y} A(x/u)\mathrm{d}B(u)$$

and

$$S_2 = S_2(x,y) = \int_{u\le \frac{x}{y}} \mathrm{d}A(u) \int_{y<v\le \frac{x}{u}} \mathrm{d}B(v).$$

The dissection (4.23) is the hyperbola method and can be derived from the summatory function for the convolution $a * b$:

$$C(x) = (A\times B)(x) = \sum_{n\le x}(a*b)(n) = \sum_{mn\le x} a(m)b(n). \tag{4.25}$$

Now applying (4.11), we have

$$S_1 = \int_{v\le y} \mathrm{d}B(v) \int_{u\le \frac{x}{v}} \mathrm{d}A(u) = \int_{u\le y} A(x/u)\mathrm{d}B(u). \tag{4.26}$$

On the other hand, by (4.10) we have

$$\begin{aligned}
S_2 &= \int_{u\le \frac{x}{y}} \mathrm{d}A(u) \int_{y<v\le \frac{x}{u}} \mathrm{d}B(v)\\
&= \int_{u\le \frac{x}{y}} \mathrm{d}A(u) \int_{v\le \frac{x}{u}} \mathrm{d}B(v) - \int_{u\le \frac{x}{y}} \mathrm{d}A(u) \int_{v\le y} \mathrm{d}B(v)\\
&= \int_{1-}^{x/y} \left(B\left(\frac{x}{u}\right) - B(y)\right) \mathrm{d}A(u)\\
&= \left[\left(B\left(\frac{x}{u}\right) - B(y)\right) A(u)\right]_{1-}^{x/y} - \int_{1-}^{x/y} A(u)\,\mathrm{d}\left(B\left(\frac{x}{u}\right) - B(y)\right)\\
&= -\int_{1-}^{x/y} M_A(u)\,\mathrm{d}M_B\left(\frac{x}{u}\right) - \int_{1-}^{x/y} E_A(u)\,\mathrm{d}M_B\left(\frac{x}{u}\right) + E_2(x)\\
&= \int_0^{1-} M_A(u)\,\mathrm{d}M_B\left(\frac{x}{u}\right) - \int_{1-}^{x/y} E_A(u)\,\mathrm{d}M_B\left(\frac{x}{u}\right) - \int_0^{x/y} M_A(u)\,\mathrm{d}M_B\left(\frac{x}{u}\right)
\end{aligned}$$

$$E_4(x) = \frac{1}{2\pi i} \int_{(\alpha)} \frac{G_A(w)}{w(w-s)} \left(\frac{x}{y}\right)^{w-s} \mathrm{d}w. \tag{4.20}$$

Proof. $E_1(x) = \int_y^\infty M_A\left(\frac{x}{t}\right) \mathrm{d}(O(t^a \delta_{c_0}(t)) << \frac{x}{y} y^a \exp(-c_0 \mathcal{N}(y)) << \delta_{c_1}(x)$. We have for $y = \exp(-c(\epsilon')\mathcal{N}(x))$,

$$E_2(x) = x^a \delta_{c_0}(x) \sum_{m \le x/y} \frac{|a(m)|}{m^a}. \tag{4.21}$$

Note that by assumption, for $r > 0$ and $t > 0$

$$S(t) = \sum_{m \le t} \frac{|a(m)|}{m^{1+r}} = O(1).$$

Now applying partial summation formula, we deduce that the sum in (4.21) is $<< O(1)\left(\frac{x}{y}\right)^{r+1-a}$. Hence $E_2(x) << x^a \delta_{c_2}(x)$.

Finally, by (4.4) with $\delta = \frac{1}{\log x}$

$$E_4(x) = x^\alpha y^{a-\alpha} (\log y)^{c'} \int_{(\alpha)} \frac{|G_A(w)|}{|w|^2} |\mathrm{d}w|. \tag{4.22}$$

The integral is finite by (4.3) and hence taking $y = \exp(-c(\epsilon')\mathcal{N}(x))$, we have $E_4(x) = x^a \delta_{c_6(\epsilon)}(x)$.

4.5.1 Proof of Theorem 4.4.1

Set $y = y(x) = \exp(-c(\epsilon')\mathcal{N}(x))$ for any $\epsilon' > 0$. We divide the integral in (4.13) into two parts $n \le y$ and $n > y$ to deduce that

$$\begin{aligned} C(x) &= \int_{v \le y} \int_{u \le \frac{x}{v}} + \int_{y < v \le x} \int_{u \le \frac{x}{v}} \\ &= \int_{v \le y} \mathrm{d}B(v) \int_{u \le \frac{x}{v}} \mathrm{d}A(u) + \int_{u \le \frac{x}{y}} \mathrm{d}A(u) \int_{y < v \le \frac{x}{u}} \mathrm{d}B(v). \end{aligned} \tag{4.23}$$

4.4 Main result

We have the following main result under the same assumptions as those of Lau.

Theorem 4.4.1 *For sufficiently small* $\epsilon' > 0$*, let* $y = y(x) = \exp(-c(\epsilon')\mathcal{N}(x))$*, we have*

$$C(x) = \sum_{n \le x}(a * b)(n) = \frac{1}{2\pi i}\int_{\mathcal{H}} G_B(s)G_A(s)x^s \frac{\mathrm{d}s}{s} + \int_{u \le y} A(x/u)\,\mathrm{d}B(u) - \int_0^{x/y} M_A(u)\,\mathrm{d}M_B(x/u) - G_{A,\alpha}(0)B(y) + E_C(x), \tag{4.16}$$

where

$$E_C(x) = O(x^a\delta(x)), \quad \delta(x) = \delta_c(x) = \exp(-c\mathcal{N}(x)), \tag{4.17}$$

$G_{A,\alpha}(0) = G_A(0)$ *if* $\alpha < 0$ *and* 0 *otherwise. For notation, cf. §4.2.*

4.5 Proof of the main result

The following lemma is necessary to prove the above theorem.

Lemma 4.5.1 *Suppose*

$$M_A(x) = O(x(\log x)^{c'}), x \to \infty; \quad M_A(x) = O(x^\eta), x \to +0 \tag{4.18}$$

and that

$$E_B(x) = O(x^a\delta_{c_0}(x)), \quad \delta_{c_0}(x) = \exp(-c_0\mathcal{N}(x)). \tag{4.19}$$

Then all the error terms $E_j(x)$ *are absorbed in the error term* (4.9).

$$E_1(x) = \int_y^\infty M_A\left(\frac{x}{t}\right)\mathrm{d}E_B(t), \quad E_2(x) = \sum_{m \le x/y} a(m)O\left(E_B\left(\frac{x}{m}\right)\right),$$

4.3.1 Contribution of Tull

J. P. Tull [52] developed a general method for obtaining asymptotic formulas for the summatory function of the convolution of two arithmetic functions $a(n)$ and $b(n)$ whose summatory functions $A(x)$ and $B(x)$ satisfy asymptotic formulas. Indeed, his method is more general and can treat the Stieltjes resultant. Given two functions A and B defined for $x \geq 1$ of bounded variation on each bounded interval, one can define the **Stieltjes resultant** C of A and B on the basis of local-global principle by

$$C(x) = (A \times B)(x) = \sum_{n \leq x} (a * b)(n) = \sum_{mn \leq x} a(m)b(n) = \int \int_{uv \leq x} \mathrm{d}A(u)\mathrm{d}B(v). \tag{4.13}$$

Remark 4.3.2 We may also express the Stieltjes resultant C of A and B as

$$C(x) = (A \times B)(x) = \int_{u \leq x} \int_{v \leq x/u} \mathrm{d}A(v)\,\mathrm{d}B(u) = \int_1^x A(x/u)\,\mathrm{d}B(u), \tag{4.14}$$

whenever the integral exits and for all $x \in \mathbb{R}^+$, $C(x)$ lies between the limits $\lim_{h \to \mp 0} C(x \pm h)$.

Remark 4.3.3 If $A(1) = B(1) = 0$, then (4.14) may be also written as

$$C(x) = (B \times A)(x) = \int_1^x B(x/u)\,\mathrm{d}A(u). \tag{4.15}$$

Hence it is better to define the summatory function as $A(x) = \sum_{n<x} a(n)$. Cf. Widder [56] [57, pp.83-91].

4.3 Stieltjes Integral

One introduces the Stieltjes integral in almost verbatim to that of the Riemann integral for bounded functions f, g defined on the bounded interval $[a, b]$. The only difference is that one uses g for the difference $x_{j+1} - x_j : g(x_{j+1}) - g(x_j)$ i.e, one can write Stieltjes integral in the form

$$\int_a^b f(x) \mathrm{d}g(x).$$

The following proposition outlines some fundamental properties of Stieltjes integral.

Proposition 4.3.1 *(i) The Stieltjes integral exists if f is continuous and g is of bounded variation.*

(ii) The formula for integration by parts holds true.

$$\int_a^b f(x)\, \mathrm{d}g(x) = [f(x)g(x)]_a^b - \int_a^b g(x)\, \mathrm{d}f(x), \tag{4.10}$$

provided that f is continuous and g is of bounded variation or g is continuous and f is of bounded variation.

(iii) If g is a step function with jumps a_n at x_n, the Stieltjes integral reduces to the sum:

$$\int_a^x f(x)\, \mathrm{d}g(x) = \sum_{a < x_n \le x} f(x_n) a_n. \tag{4.11}$$

(iv) If f is continuous and g is differentiable, then the Stieltjes integral reduces to the Riemann integral:

$$\int_a^b f(x)\, \mathrm{d}g(x) = \int_a^b f(x) g'(x)\, \mathrm{d}x. \tag{4.12}$$

where $\delta < |Y|$ satisfies $0 < \delta < \eta - a$, so that

$$a + \delta < \eta, \quad \alpha < a - Y. \tag{4.5}$$

(4) As $x \to \infty$, we have

$$B(x) = \sum_{n \le x} b(n) = M_B(x) + O(x^a \exp(-c_0 \mathcal{N}(x))) \tag{4.6}$$

where

$$M_B(x) = \frac{1}{2\pi i} \int_{\mathcal{H}} G_B(s) x^s \frac{\mathrm{d}s}{s}. \tag{4.7}$$

The function $\mathcal{N}(x)$ is positive, non-decreasing which satisfies:

(i) $\mathcal{N}(x)/\log_2 x \to \infty$ and (ii) $\mathcal{N}(x^\gamma) \gg_\gamma \mathcal{N}(x)$ for some $\gamma \in (0,1)$. $\mathcal{H} = \mathcal{H}_Y(a, \delta)$ indicates the truncated Hankel contour starting and ending at $a - Y$ and surrounding $s = a$ with the circle of radius δ.

Under the above conditions, we have the following theorem.

Theorem 4.2.1 (Lau) *For any sufficiently small $\epsilon' > 0$, let $y = y(x) = \exp(-c(\epsilon')\mathcal{N}(x))$, we have*

$$C(x) = \sum_{n \le x} (a * b)(n) = \sum_{n=1}^{\infty} b(n) M_A\left(\frac{x}{n}\right) + \frac{1}{2\pi i} \int_{\mathcal{H}} G_B(s) G_A(s) x^s \frac{\mathrm{d}s}{s}$$
$$+ \sum_{n \le y} b(n) \left(E_A\left(\frac{x}{n}\right) - G_{A,\alpha}(0)\right) + E_C(x), \tag{4.8}$$

where

$$E_C(x) = O(x^a \delta(x)), \quad \delta(x) = \delta_c(x) = \exp(-c\mathcal{N}(x)), \tag{4.9}$$

$G_{A,\alpha}(0) = G_A(0)$ *if* $\alpha < 0$ *and* 0 *otherwise.*

4.2 Result of Lau

In this section we will briefly discuss the result of Lau. The following basic assumptions were taken by Lau to prove the far-reaching theorem. Let $a(n)$ and $b(n)$ be two complex-valued arithmetic functions. Let us consider three fixed numbers $0 \leq a < \eta < 1$ and $0 \neq \alpha < a$ where the upper limit 1 can be replaced by any number greater than 1. Suppose two summatory functions are given as

$$A(x) = \sum_{n \leq x} a(n) = M_A(x) + E_A(x), \tag{4.1}$$

and

$$B(x) = \sum_{n \leq x} b(n) = M_B(x) + E_B(x) \tag{4.2}$$

where $M_A(x)$, $M_B(x)$ are the main terms and $E_A(x)$, $E_B(x)$ are the error terms. Let $G_A(s)$ (resp. $G_B(s)$) indicate the generating Dirichlet series of $\{a(n)\}$ (resp. $\{b(n)\}$) where we take $s = \sigma + it$ for the complex variable. These generating series satisfy some conditions as described in [38].

(1) $G_A(s)$ (resp. $G_B(s)$) is absolutely convergent for $\sigma > 1$ (resp. $\sigma > \alpha$).

(2) $G_A(s)$ has analytic continuation on $\sigma > \alpha - \epsilon$ (resp. $\sigma > \alpha$) and satisfies

$$G_A(s) << |t|^{1-\varepsilon}, \quad \alpha \leq \sigma \leq 1^-. \tag{4.3}$$

(3) Let $U(a, \delta, Y)$ denote an open connected set containing the line segment $[a - Y, a]$ and the closed disc of radius δ centered at a and $U^- = U^-(a, \delta, Y)$ is the set $U - [-\infty, a]$. $G_B(s)$ has analytic continuation in a wider region and it is analytic on U^- and satisfies

$$G_B(s) << \delta^{-c}, \quad |s - a| = \delta, \quad s \in U^- \tag{4.4}$$

$$B(x) = \sum_{n<x} b_n = O(x^{\beta+\epsilon}), \qquad \sum_{n<x} |b_n| = O(x^{\tau+\epsilon})$$

for all $x \geq 1$*, then for each* $\epsilon > 0$*,*

$$C(x) = \sum_{n<x} c_n = O(x^{\omega+\epsilon})$$

with

$$\omega = \frac{\rho\tau - \alpha\beta}{\rho + \tau - \alpha - \beta}.$$

In the year 1958, J. P. Tull ([52]) developed a new method introducing Stieltjes resultant which generalized the result of Landau [36]. He mainly considered the summatory functions $A(x)$ and $B(x)$ which are representable in the form

$$\sum_{\mu=1}^{h} x^{a_\mu} P_\mu(\log x) + O(x^\alpha \log^t(x+1))$$

where a_μ are complex numbers and the P_μ are polynomial functions. This method offers a new tool to attack on a certain class of lattice point problems.

Recently, Y. -K. Lau [38] obtained a far-reaching theorem, which gives a rather precise asymptotic formula for the summatory function of the Dirichlet convolution of two arithmetical functions. In this chapter, our aim is to elucidate Lau's theorem [38] in the light of the Stieltjes resultant. Although it looks like there are some miraculous cancellations occurring in the process in [38], we shall show that the cancellations are necessitated by the argument based on Stieltjes integration.

The general three-dimensional divisor problem consists of estimating the function $\Delta(a,b,c;x)$, which may be considered as the error term in the asymptotic formula

$$\begin{aligned} D(a,b,c;x) &= \sum_{n\le x} d(a,b,c;n) \\ &= P(a,b,c;x) + \Delta(a,b,c;x) \end{aligned}$$

where $P(a,b,c;x)$ is the main term.

The associated Dirichlet series of the function $d(a,b,c;n)$ is

$$H(s) = \sum_{n=1}^{\infty} \frac{d(a,b,c;n)}{n^s} = \zeta(as)\zeta(bs)\zeta(cs).$$

The main term $P(a,b,c;x)$ is the residual function which is the sum of residues at the poles of the function $\zeta(as)\zeta(bs)\zeta(cs)\frac{x^s}{s}$ and the error term $\Delta(a,b,c;x)$ can be obtained after applying the **convolution theorem** using the asymptotic formulas of the function whose associated Dirichlet series are $\zeta(as)$, $\zeta(bs)$ and $\zeta(cs)$.

Landau [36] obtained a result on the asymptotic formulas for the summatory function of the convolution of two arithmetic functions $a(n)$ and $b(n)$ whose summatory functions $A(x)$ and $B(x)$ satisfy asymptotic formulas which can be stated as follows :

Theorem 4.1.2 (Landau) *Given non-negative real numbers α, β, ρ, τ with $\alpha \le \rho$, $\beta \le \tau$, $\rho \ge \beta$, $\tau \ge \alpha$, $\rho+\tau-\alpha-\beta > 0$, if for each $\epsilon > 0$*

$$A(x) = \sum_{n<x} a_n = O(x^{\alpha+\epsilon}), \qquad \sum_{n<x} |a_n| = O(x^{\rho+\epsilon})$$

Definition 4.1.1 (Dirichlet convolution) Let $a(n)$ and $b(n)$ be two arithmetic functions. Then the Dirichlet convolution $c(n)$ is defined by

$$c(n) = (a * b)(n) = \sum_{d|n} a(n/d)b(d) = \sum_{d|n} b(n/d)a(d).$$

Many arithmetic functions are given as a Dirichlet convolution of two arithmetic functions, whose asymptotic behaviors are known. Thus it is essential to deduce the asymptotic formula for the summatory function of Dirichlet convolutions. In the literature, there are general theorems on asymptotic formulas for convolutions and these are known as general convolution theorems.

4.1.1 Convolution theorem

A common technique while dealing with the asymptotic formula for the sum $C(x) = \sum_{n\leq x} c(n)$ is to express suitably $c(n)$ in the form of Dirichlet convolution and then to estimate $C(x)$ using asymptotic formulas for the summatory functions $A(x) = \sum_{n\leq x} a(n)$ and $B(x) = \sum_{n\leq x} b(n)$.

If the generating series (cf. §1.5.2) $F(s) = \sum\limits_{n=1}^{\infty} a(n)n^{-s}$ and $G(s) = \sum\limits_{n=1}^{\infty} b(n)n^{-s}$ both converge absolutely in the half space $\sigma > \sigma_1$ respectively, then in this half plane,

$$H(s) = \sum_{n=1}^{\infty} c(n)n^{-s} = F(s)G(s).$$

Example 4.1.1 The following example illustrates the importance of convolution theorems. Let $d(a, b, c; n)$ be the number of representations of n as $n = n_1^a n_2^b n_3^c$ where n_1, n_2 and n_3 are natural numbers, that is

$$d(a, b, c; n) = \sum_{n=n_1^a n_2^b n_3^c} 1$$

CHAPTER 4

Asymptotic behaviour of arithmetical convolutions

In the pursuit of the number-theoretic nature of a given set, one defines an arithmetic function and considers its average behaviour in view of the fact that independent values are rather singular. In this chapter, we are interested in the asymptotic formula for the summatory function of the arithmetic function which are given as the coefficients of a product of two generating Dirichlet series, i.e. they are Dirichlet convolution of the respective coefficients. Our main purpose is to elucidate a result of Lau in the light of Stieltjes resultant and give some applications which involve a possible logarithmic singularities. The contents of this chapter have appeared in [5].

4.1 Introduction

We start with the definition of Dirichlet convolution which is basically the binary operation defined for arithmetic functions. It is one of the most important tools in number theory developed by Dirichlet.

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