PROBLEMS IN THE THEORY OF CONNECTIONS IN ALGEBRAIC GEOMETRY

By

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Anoop Singh

List of Publications arising from the thesis

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Anoop Singh

To

my parents

Maya Singh

and Arun Kumar Singh

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Anoop Singh

Contents

TABLES AND FIGURES

This thesis does not contain any tables and figures.

CONCLUSION

The research in the thesis aimed to answer the questions regarding the existence of relative holomorphic connections in a vector bundle over a family of compact complex manifolds, and to determine algebro-geometric invariants of the moduli space of logarithmic connections over a compact Riemann surface. The thesis contains six chapters covering the topics in the theory of connections in algebraic geometry. The first chapter is about the introduction of the thesis. The second, third and fourth chapters are about relative connections in sheaves of modules over ringed spaces, and their holomorphic aspects. The last two chapters contains a description of some invariants of moduli spaces of logarithmic connections on compact Riemann surfaces. So, the thesis can be considered in two parts. In the first part, we show that the relative Chern classes of a holomorphic vector bundle over a family of compact and Kähler manifolds vanish if the bundle admits a relative holomorphic connection. Next, we give a sufficient condition for the existence of the relative holomorphic connections in a holomorphic vector bundle over a family of connected compact complex manifolds.

In the second part of the thesis, we consider the moduli space of logarithmic connections with fixed residues over a compact Riemann surface. This moduli space is known to be a quasi-projective variety. We show that it is embedded in a projective space so that the complement is a hyperplane. We, then compute the Picard group of this moduli space and show that the moduli space of logarithmic connections with fixed determinant is isomorphic to the set of integers. Next, we show that the moduli space does not admit any non-constant algebraic functions, although it admits non-constant holomorphic functions. We also characterise the algebraic functions on the moduli space of logarithmic connections with arbitrary residues over a compact Riemann surface.

Summary

This thesis is about two topics in the theory of connections in algebraic geometry. It is firstly about relative connections in sheaves of modules over ringed spaces, and their holomorphic aspects. Secondly, it contains a description of some invariants of moduli spaces of logarithmic connections on compact Riemann surfaces.

The thesis is divided into two parts. In the first part, we show that the relative Chern classes of a holomorphic vector bundle over a family of compact and Kähler manifolds vanish if the bundle admits a relative holomorphic connection. Next, we give a sufficient condition for the existence of the relative holomorphic connections in a holomorphic vector bundle over a family of connected compact complex manifolds.

In the second part, we consider the moduli space of logarithmic connections with fixed residues over a compact Riemann surface. This moduli space is known to be a quasi-projective variety. We show that it is embedded in a projective space so that the complement is a hyperplane. We, then compute the Picard group of this moduli space and show that the moduli space of logarithmic connections with fixed determinant is isomorphic to the set of integers. Next, we show that the moduli space does not admit any non-constant algebraic functions, although it admits non-constant holomorphic functions. We also characterise the algebraic functions on the moduli space of logarithmic connections with arbitrary residues over a compact Riemann surface.

Chapter 1

Introduction

In the theory of holomorphic vector bundles over a complex manifold, the notion of holomorphic connection plays an important role. But unlike in differentiable set up, holomorphic connections in a holomorphic vector bundle need not exist. Atiyah [Ati57] introduced the notion of holomorphic connections in principal bundles over a complex manifold. A theorem due to Atiyah and Weil, [Ati57], [Wei38] which is known as the *Atiyah-Weil criterion*, says that a holomorphic vector bundle over a compact Riemann surface admits a holomorphic connection if and only if the degree of each indecomposable component of the holomorphic vector bundle is zero (see [BR05] for an exposition of the Atiyah-Weil criterion); this criterion generalizes to holomorphic principal bundles over a compact Riemann surface [AB02]. Also, if a holomorphic vector bundle over a compact Kähler manifold admits a holomorphic connection, then all the Chern classes vanish. These aforementioned theorems gives rise to a natural questions about the existence of relative holomorphic connections in a holomorphic vector bundle over a family of compact connected complex manifolds.

The Picard group of a moduli space is a very important invariant while studying the classification problems for algebro-geometric objects. The Picard group of moduli space of vector bundles have been studied extensively by several algebraic geometers, [DN89], [Ram73], [Bho99] to name a few. Also, the Picard group and algebraic functions for the moduli space of logarithmic connections singular exactly at one point of the compact Riemann surface has been studied in [BR05]. In [Seb11], the algebraic functions on the moduli space of rank one logarithmic connections singular at finitely many points have been computed.

This thesis is divided into two parts. The first part consists of chapters 2, 3, 4 and deals with problem related to the existence of relative holomorphic connections in a holomorphic vector bundle over a complex analytic family and the computation of the relative Chern classes of a holomorphic vector bundle under certain conditions. We give a sufficient condition for the existence of the relative holomorphic connection in a holomorphic vector bundle over a complex analytic family. We define the relative Chern classes of a complex vector bundle and show that the relative Chern classes of a holomorphic vector bundle over a family of compact and Kähler manifolds vanish if the bundle admits a relative holomorphic connection.

The second part of the thesis consists of chapters 5 and 6 which deals with compactification, computation of the Picard group and computation of algebraic functions for the moduli space of logarithmic connections singular over a finite subset of a compact Riemann surface. We describe a compactification for the moduli space such that the complement is a hyperplane at infinity. We show that the moduli space of logarithmic connections over a compact Riemann surface with fixed residues do not admit any non-constant algebraic functions. On the other hand, it admits non-constant holomorphic functions.

1.1 Structure of this thesis

The thesis is organised as follows.

In Chapter 2, we give an introduction to the theory of relative connections in a sheaf of modules. We define the relative derivations, relative connections, relative connections on associated sheaf of modules, covariant derivative and connection-curvature matrices in the set-up of ringed spaces. We define curvature form associated to the relative connection and prove *Bianchi's* first and second identity. We also establish relation between different relative connections, their covariant derivative and curvature forms.

In Chapter 3, we introduce the notion of finite order relative differential operators between sheaves of modules. We define the connection algebra of a morphism between ringed spaces. We show that the category of sheaves of modules over connection algebra is equivalent to category of sheaves of modules with relative connections. We also define a symbol of a first order relative differential operator.

In Chapter 4, we define the notion of complex analytic family. We describe relative Atiyah algebra and Atiyah class of a holomorphic vector bundle over a complex analytic family. We define the notion of relative Chern classes of a complex vector bundle over a complex analytic family. We show one of the main result that the relative Chern classes of a holomorphic vector bundle over a family of compact and Kähler manifolds vanish if the bundle admits a relative holomorphic connection. And, finally we give a sufficient condition for the existence of relative holomorphic connections.

In Chapter 5, we give an outline of the construction of the moduli space of meromorphic and logarithmic connections singular over a smooth normal crossing divisor of a smooth complex projective variety with a fixed ample line bundle. We start with recalling the construction of the moduli space of coherent sheaf and then describe the Simpson's construction of the moduli space of coherent Λ-modules, where we restrict ourselves to coherent Λ^{Mero} and Λ^{Log}-modules. Finally, we restrict ourselves to the moduli space of logarithmic connections over a compact Riemann surface with fixed residues, and study its compactification.

In Chapter 6, we define the Picard group of a scheme and compute the Picard group of moduli space of logarithmic connection with fixed residues. We show that the Picard group of the moduli space of the logarithmic connections with fixed residues and fixed determinant is isomorphic to $\mathbf{Z}$. We prove that the only algebraic functions on the moduli space are constant functions. We also characterise the algebraic functions on the moduli space of logarithmic connections with arbitrary residues.

Chapter 2

Basics on relative connections

In [Kos86], Koszul studied 'Differential Calculus' in the frame work of commutative algebra which can be reformulated in sheaf theoretic manner following [GD67] and [Ram05]. This chapter provides the basics for the latter chapters of the thesis. In this chapter, we develop the formal machinery of differential calculus in the theory of ringed spaces, in the relative context.

2.1 Relative connections

In this section, we define the notion of relative derivations (or S-derivations) and relative connections (or S-connections) following [GD67] and relative connection (or S-connection). A *ringed space* is a pair $(X, \mathscr{O}_X)$ where X is a topological space, and $\mathscr{O}_X$ is a sheaf of rings on X.

Let $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be two ringed spaces. A *morphism of ringed spaces* from $(X, \mathscr{O}_X)$ to $(Y, \mathscr{O}_Y)$ is a pair $(f, \tilde{f})$, where

1. $f : X \to Y$ is a continuous map.

2. $\tilde{f}$ is an assignment which attaches to each open subset V of Y, a homomorphism of rings $\tilde{f}_V : \mathscr{O}_Y(V) \to \mathscr{O}_X(f^{-1}(V))$, such that for every pair (V, V') of open subsets of

Y with $V \supset V'$, the diagram

$$\begin{array}{ccc} \mathscr{O}_Y(V) & \xrightarrow{\tilde{f}_V} & \mathscr{O}_X(f^{-1}(V)) \\ {\scriptstyle \rho^V_{V'}}\downarrow & & \downarrow{\scriptstyle \rho^{f^{-1}(V)}_{f^{-1}(V')}} \\ \mathscr{O}_Y(V') & \xrightarrow[\tilde{f}_{V'}]{} & \mathscr{O}_X(f^{-1}(V')) \end{array}$$

commutes, where $\rho^V_{V'} : \mathscr{O}_Y(V) \to \mathscr{O}_Y(V')$ is the restriction morphismvg.

Throughout this chapter, we shall assume that $(X, \mathscr{O}_X)$, $(S, \mathscr{O}_S)$ are two ringed spaces, and $(\pi, \pi^\sharp) : (X, \mathscr{O}_X) \longrightarrow (S, \mathscr{O}_S)$ is a morphism between them.

2.1.1 Relative derivation

Definition 2.1.1

1. Let $\mathscr{F}, \mathscr{G}$ be two $\mathscr{O}_X$-modules. A morphism

$$\alpha : \mathscr{F} \to \mathscr{G}$$

of sheaves of abelian groups is said to be S-linear if for every open subset $V \subset S$, for every open subset $U \subset \pi^{-1}(V)$, for every $t \in \mathscr{F}(U)$ and for every $s \in \mathscr{O}_S(V)$, we have

$$\alpha(\rho_{U,\pi^{-1}(V)}(\pi^\sharp_V(s))t) = \rho_{U,\pi^{-1}(V)}(\pi^\sharp_V(s))\alpha(t)\,,$$

where

$$\rho_{U,\pi^{-1}(V)} : \mathscr{O}_X(\pi^{-1}(V)) \longrightarrow \mathscr{O}_X(U)$$

is the restriction map. We denote by $\mathscr{H}om_S(\mathscr{F}, \mathscr{G})$ the sheaf of S-linear morphism from $\mathscr{F}$ to $\mathscr{G}$.

We denote $\rho_{U,\pi^{-1}(V)}(\pi^\sharp_V(s))$ by $s|_U$.

2. For the following definition see [GD67] (Chapitre IV, **16.5**).

Let $\mathscr{F}$ be an $\mathscr{O}_X$-module. A **relative derivation** or S-derivation from $\mathscr{O}_X$ to $\mathscr{F}$ is a morphism

$$\delta : \mathscr{O}_X \to \mathscr{F}$$

of sheaves of abelian groups which satisfies the following conditions:

(a) δ is an S-linear morphism.

(b) (Leibniz rule) For every open subset $U \subset X$, and for every $a, b \in \mathscr{O}_X(U)$, we have

$$\delta_U(ab) = a\delta_U(b) + \delta_U(a)b\,.$$

The set of all S-derivation from $\mathscr{O}_X$ to $\mathscr{F}$ form a left $\mathscr{O}_X(X)$-module denoted by

$$\mathrm{Der}_S(\mathscr{O}_X, \mathscr{F})\,.$$

For every open subset $U \subset X$, we note that $\mathrm{Der}_S(\mathscr{O}_X|_U, \mathscr{F}|_U)$ is a left $\mathscr{O}_X(U)$-module. For every open subset $U \subset X$, the assignment

$$U \longmapsto \mathrm{Der}_S(\mathscr{O}_X|_U, \mathscr{F}|_U)$$

is a sheaf of $\mathscr{O}_X$-modules and it is denoted by $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{F})$.

Let $\mathscr{E}nd_S(\mathscr{F})$ denote the sheaf of S-linear endomorphism on $\mathscr{F}$. Then $\mathscr{E}nd_S(\mathscr{F})$ is an $\mathscr{O}_X$-module. In particular, if we take $\mathscr{F} = \mathscr{O}_X$, then $\mathscr{E}nd_S(\mathscr{O}_X)$ is a sheaf of Lie algebras with respect to the bracket operation defined as follows.

$$[\xi, \eta] = \xi \circ \eta - \eta \circ \xi$$

for every open subset $U \subset X$ and for all $\xi,\, \eta \in \mathscr{E}nd_S(\mathscr{O}_X)(U)$. We note that $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$ is a Lie subalgebra of $\mathscr{E}nd_S(\mathscr{O}_X)$.

2.1.2 Relative connections on modules

Definition 2.1.2

1. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module. An S-connection or **relative connection** on $\mathscr{F}$ is an $\mathscr{O}_X$–module homomorphism

$$D : \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X) \longrightarrow \mathscr{E}nd_S(\mathscr{F})$$

such that for every open subset U of X and for every $\xi \in \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)(U)$, the $\mathscr{O}_X(U)$–module homomorphism

$$D_U : \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)(U) \longrightarrow \mathscr{E}nd_S(\mathscr{F})(U),$$

sending $\xi \longmapsto (D_U)_\xi$ satisfies the Leibniz rule which says that

$$((D_U)_\xi)_V(ag) = \xi|_V(a)g + a((D_U)_\xi)_V(g)$$

for every open subset V of U, for all $a \in \mathscr{O}_X(V)$ and $g \in \mathscr{F}(V)$.

2. If $\pi : X \longrightarrow S$ is a holomorphic map of complex manifolds, and $\mathscr{F}$ a holomorphic vector bundle over X, we call D a *holomorphic S-connection* or *relative holomorphic connection* . Similarly, If $\pi : X \longrightarrow S$ is a smooth map of smooth manifolds and $\mathscr{F}$ a smooth vector bundle, we call D a *smooth S-connection* or *relative smooth connection* .

3. An $\mathscr{O}_X$-module with S-connection is a pair $(\mathscr{F}, D)$, where $\mathscr{F}$ is an $\mathscr{O}_X$-module and D is an S-connection on $\mathscr{F}$. A morphism

$$\Phi : (\mathscr{F}, D^{\mathscr{F}}) \to (\mathscr{E}, D^{\mathscr{E}}),$$

of $\mathscr{O}_X$-modules with S-connection is an $\mathscr{O}_X$-linear map

$$\Phi : \mathscr{F} \to \mathscr{E}$$

such that

$$D_\xi^{\mathscr{E}} \circ \Phi = \Phi \circ D_\xi^{\mathscr{F}},$$

for every local section ξ of $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$. We thus get a category $\mathbf{MC}(X/S)$ of $\mathscr{O}_X$-modules with S-connections.

Remark 2.1.3

1. We note that

$$(D_U)_\xi : \mathscr{F}|_U \longrightarrow \mathscr{F}|_U$$

is an S-linear endomorphism, where S-linearity is with respect to $\pi|_U : U \longrightarrow S$. To avoid the cumbersome notation $(D_U)_\xi$, we shall simply denote it by D_ξ.

2. The inclusion map

$$\varepsilon : \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X) \hookrightarrow \mathscr{E}nd_S(\mathscr{O}_X)$$

is an S-connection on the $\mathscr{O}_X$-module $\mathscr{O}_X$, and it is called the *canonical S-connection* on $\mathscr{O}_X$.

Proposition 2.1.4 *Let $\mathscr{F}$ be a free $\mathscr{O}_X$-module, and let $(e_i)_{i\in I}$ be an $\mathscr{O}_X$-basis of $\mathscr{F}$. Then for every family $(\omega_i)_{i\in I}$ of elements of $\mathrm{H}^0(X, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F}))$, there exists a unique S-connection D on $\mathscr{F}$ such that*

$$D_\xi(e_i) = \omega_i(\xi) \tag{2.1}$$

for every open set $U \subset X$, for all $\xi \in \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)(U)$ and $i \in I$.

Proof. Suppose D is such a connection. Let U be an open subset of X and $u \in \mathscr{F}(U)$.

Then write

$$u = \sum_{i \in I} a_i e_i,$$

where $(a_i)_{i \in I}$ is a finitely supported family of elements of $\mathscr{O}_X(U)$. Then from the definition of S-connection, for every $\xi \in \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)(U)$, we have

$$\begin{aligned} D_\xi(u) &= \sum_{i \in I} D_\xi(a_i e_i) \\ &= \sum_{i \in I} (\xi(a_i)e_i + a_i \omega_i(\xi)) \end{aligned}$$

Thus, D is unique. Moreover, if we define D as above, it is a S-connection on $\mathscr{F}$.

There exist $\mathscr{O}_X$-modules which do not admit any S-connection. We will see that in following example

Example 2.1.5 Let k be a non-zero commutative ring. let $X = \mathrm{Spec}(k[T])$ and $S = \mathrm{Spec}(k)$ be affine schemes, and let $\pi : X \to S$ be the natural morphism between them. Then $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$ is a free $\mathscr{O}_X$-module of rank 1, with $\{\xi\}$ as an $\mathscr{O}_X$-basis, where $\xi = \frac{d}{dT}$.

Consider k as $A = k[T]$-algebra, through the ring homomorphism $\varepsilon : A \to k$ defined by $\varepsilon(f) = f(0)$, where $f \in A$. Then ε will induce a morphism of schemes $\tilde{\varepsilon} : \mathrm{Spec}(k) \to \mathrm{Spec}(k[T])$ and thus giving an $\mathscr{O}_X$-module structure on $\tilde{\varepsilon}_* \mathscr{O}_S = \mathscr{F}$. Now, $\mathscr{O}_X$-module $\mathscr{F}$ does not admit any S-connection.

Suppose that $\mathscr{F}$ admits an S-connection D. Then, since $\varepsilon(T) = 0$, taking global sections, we have

$$0 = D_\xi(\varepsilon(T)) = D_\xi(T.1) = \xi(T).1 + T.D_\xi(1) = 1 + \varepsilon(T).D_\xi(1) = 1.$$

which is a contradiction because $k \neq 0$.

Definition 2.1.6 (G-torsor) Let G be a group. A G-torsor , or a torsor of G is a non-empty set E, together with a right action

$$E \times G \to E,$$

sending $(x,g) \mapsto xg$ of G on E, which is free and transitive, where $x \in E$ and $g \in G$. If G is additively written we will write $x+g$ instead of xg.

Remark 2.1.7 Let $\mathscr{C}(\mathscr{F})$ be the set of all S-connection on $\mathscr{F}$. For any S-connection D on $\mathscr{F}$, and any $\mathscr{O}_X$-linear map

$$h : \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X) \to \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{F})$$

the morphism

$$D+h : \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X) \to \mathscr{E}nd_S(\mathscr{F})$$

defined by sending a local section ξ of $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$ to $D_\xi + h(\xi)$, is also an S-connection on $\mathscr{F}$. Now, the abelian group

$$G = \mathrm{H}^0(X, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{F})))$$

acts on $\mathscr{C}(\mathscr{F})$ from right via the map $(D,h) \mapsto D+h$. This action is free and transitive.

Thus, if $\mathscr{F}$ admits an S-connection, then $\mathscr{C}(\mathscr{F})$ is a G-torsor, where

$$G = \mathrm{H}^0(X, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{F})))$$

.

2.1.3 Relative connections on the associated modules

Let $(\mathscr{F}_i)_{i \in I}$ be a family of $\mathscr{O}_X$-modules, and for each $i \in I$, let D^i be an S-connection on $\mathscr{F}_i$. Then, in this section, we will see that the various $\mathscr{O}_X$-modules obtained from $(\mathscr{F}_i)_{i \in I}$

by functorial construction, has natural S-connections.

1. **Direct Sum**

   If

$$\mathscr{F} = \bigoplus_{i \in I} \mathscr{F}_i,$$

   and if we define

$$D_\xi(u) = (D^i_\xi(u_i))_{i \in I}$$

   for all sections ξ of $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$ and all $u = (u_i)_{i \in I}$ of $\mathscr{F}$, then we get an S-connection D on $\mathscr{F}$. In particular, if we take each $\mathscr{F}_i$ to be $\mathscr{O}_X$ and each D^i to be ε, the canonical S-connection on $\mathscr{O}_X$, then every free $\mathscr{O}_X$-module has a canonical S-connection.

2. **Tensor products**

   Suppose $I = \{1, 2, \cdots, p\}$, where p is an integer ≥ 1. Then for every open subset U of X, and for each $\xi \in \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)(U)$, there exists a unique S-linear endomorphism D_ξ of $\mathscr{F}_1 \bigotimes_{\mathscr{O}_X} \cdots \bigotimes_{\mathscr{O}_X} \mathscr{F}_p$ such that on the presheaf level it is given by the formula

$$D_\xi(s_1 \otimes_{\mathscr{O}_X} \cdots \otimes_{\mathscr{O}_X} s_p) = \sum_{i=1}^{p} s_1 \otimes_{\mathscr{O}_X} \cdots \otimes_{\mathscr{O}_X} s_{i-1} \otimes_{\mathscr{O}_X} D^i_\xi(s_i) \otimes_{\mathscr{O}_X} s_{i+1} \otimes_{\mathscr{O}_X} \cdots \otimes_{\mathscr{O}_X} s_p$$

   for every $s_1 \otimes_{\mathscr{O}_X} \cdots \otimes_{\mathscr{O}_X} s_p \in \mathscr{F}_1(U) \bigotimes_{\mathscr{O}_X(U)} \cdots \bigotimes_{\mathscr{O}_X(U)} \mathscr{F}_p(U)$. This gives an S-connection on $\mathscr{F}_1 \bigotimes_{\mathscr{O}_X} \cdots \bigotimes_{\mathscr{O}_X} \mathscr{F}_p$.

   Suppose that

$$\mathscr{F}_1 = \mathscr{F}_2 = \cdots = \mathscr{F}_p = \mathscr{F},$$

   and denote $\mathscr{F}_1 \bigotimes_{\mathscr{O}_X} \cdots \bigotimes_{\mathscr{O}_X} \mathscr{F}_p$ by $T^p_{\mathscr{O}_X}(\mathscr{F})$.

   Equip $T^0_{\mathscr{O}_X}(\mathscr{F}) = \mathscr{O}_X$ with the canonical S-connection ε on $\mathscr{O}_X$, and for each $p \geq 1$, equip $T^p_{\mathscr{O}_X}(\mathscr{F})$ with the S-connection induced by the S-connection D on $\mathscr{F}$; this S-connection on $T^p_{\mathscr{O}_X}(\mathscr{F})$ will be denoted by D^p. Recall that the tensor algebra of $\mathscr{O}_X$-module $\mathscr{F}$ is a graded $\mathscr{O}_X$-algebra

$$T_{\mathscr{O}_X}(\mathscr{F}) = \bigoplus_{p\in\mathbf{N}} T^p_{\mathscr{O}_X}(\mathscr{F}).$$

Let D be the S-connection on the $\mathscr{O}_X$-module $T_{\mathscr{O}_X}(\mathscr{F})$, which is the direct sum (see 1) of the connections. It is called the induced connection on $T_{\mathscr{O}_X}(\mathscr{F})$.

Remark 2.1.8 On tensor algebra $T_{\mathscr{O}_X}(\mathscr{F})$, we have

$$D_\xi(s\otimes t) = D_\xi(s)\otimes t + s\otimes D_\xi(t),$$

for all local sections ξ of $\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X)$ and local sections s,t of $T_{\mathscr{O}_X}(\mathscr{F})$.

3. **Submodule and quotient module**

Let $\mathscr{F}$ be an $\mathscr{O}_X$-module with an S-connection D, and let $\mathscr{G}$ be an $\mathscr{O}_X$-submodule of $\mathscr{F}$. Let $\mathscr{H}$ denote the quotient $\mathscr{O}_X$-module $\mathscr{F}/\mathscr{G}$. Suppose that for every section ξ of $\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X)$, we have $D_\xi(\mathscr{G})\subset\mathscr{G}$. Then D will induce an S-connection on $\mathscr{G}$ and on $\mathscr{H}$.

4. **Symmetric algebra and exterior Algebra**

Let D be an S-connection on $\mathscr{F}$. The S-connection on the tensor algebra $T_{\mathscr{O}_X}(\mathscr{F})$ induced by D will also be denoted by D. Let $\mathscr{I}$ denote the two sided ideal sheaf of $T_{\mathscr{O}_X}(\mathscr{F})$ described as follows:

for every open subset U of X, let $\mathscr{I}(U)$ be the two sided ideal in $T_{\mathscr{O}_X}(\mathscr{F})(U)$ generated by elements of the form $s\otimes t - t\otimes s$, where $s,t\in\mathscr{F}(U)$. Then $D_\xi(\mathscr{I})\subset\mathscr{I}$ for all sections ξ of $\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X)$. Thus, by above point 3, we get an S-

connection D on the symmetric algebra

$$Sym_{\mathscr{O}_X}(\mathscr{F}) = T_{\mathscr{O}_X}(\mathscr{F})/\mathscr{I}$$

of $\mathscr{F}$.

Similarly, let $\mathscr{J}$ denote the two sided ideal sheaf of $T_{\mathscr{O}_X}(\mathscr{F})$ generated by the local sections $s \otimes s$ of $T^2_{\mathscr{O}_X}(\mathscr{F})$, where s is a local section of $\mathscr{F}$. Then we have $D_\xi(\mathscr{J}) \subset \mathscr{J}$ for all local sections ξ of $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$, and hence a connection on the exterior algebra

$$\Lambda_{\mathscr{O}_X}(\mathscr{F}) = T_{\mathscr{O}_X}(\mathscr{F})/\mathscr{J}$$

of $\mathscr{F}$ is obtained.

Remark 2.1.9

(a) For all $p \in \mathbf{N}$, we have

$$D_\xi(Sym^p_{\mathscr{O}_X}(\mathscr{F})) \subset Sym^p_{\mathscr{O}_X}(\mathscr{F}),$$

where $Sym^p_{\mathscr{O}_X}(\mathscr{F})$ is the p-th graded component of the symmetric algebra $Sym_{\mathscr{O}_X}(\mathscr{F})$. Consequently, we get an S-connection on $Sym^p_{\mathscr{O}_X}(\mathscr{F})$. Similarly, we get an S-connection on $\Lambda^p_{\mathscr{O}_X}(\mathscr{F})$.

(b) We have

$$D_\xi(ss') = D_\xi(s)s' + sD_\xi(s')$$

for all local sections ξ of $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$ and s, s' of $Sym_{\mathscr{O}_X}(\mathscr{F})$, and

$$D_\xi(t \wedge t') = D_\xi(t) \wedge t' + t \wedge D_\xi(t')$$

for all local sections ξ of $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$ and t, t' of $\Lambda_{\mathscr{O}_X}(\mathscr{F})$.

5. **S-Connection on $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})$**

Let $\mathscr{F}$, $\mathscr{G}$ be $\mathscr{O}_X$-modules with S-connections $D^{\mathscr{F}}$ and $D^{\mathscr{G}}$ respectively. For every local section ξ of $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$, let D_ξ be the S-linear endomorphism of the $\mathscr{O}_X$-module $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})$,which is defined by

$$D_\xi(h) = D^{\mathscr{G}}_\xi \circ h - h \circ D^{\mathscr{F}}_\xi,$$

for all local sections h of $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})$. Then the morphism

$$D = \xi \mapsto D_\xi : \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X) \to \mathscr{E}nd_S(\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}))$$

is an S-connection on $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})$.

Remark 2.1.10

(a) If $\mathscr{F} = \mathscr{G}$, and $D^{\mathscr{F}} = D^{\mathscr{G}}$, then the above S-connection D on $\mathscr{E}nd_S(\mathscr{F})$ is given by

$$D_\xi(h) = [D^{\mathscr{F}}_\xi, h] = D^{\mathscr{G}}_\xi \circ h - h \circ D^{\mathscr{G}}_\xi$$

for all local sections h of $\mathscr{E}nd_S(\mathscr{F})$.

(b) If $\mathscr{G} = \mathscr{O}_X$, and if $D^{\mathscr{G}}$ is the canonical connection on $\mathscr{O}_X$, then the above connection on $\mathscr{F}^* = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{O}_X)$ is given by

$$D_\xi(f) = \xi \circ f - f \circ D^{\mathscr{F}}_\xi$$

for all local sections f of $\mathscr{F}^*$.

6. **S-Connection on $\mathscr{O}_X$-module of $\mathscr{O}_X$-multilinear maps**

Let $p \geq 1$ be an integer, and let $\mathscr{F}_1, \mathscr{F}_2, \cdots, \mathscr{F}_p, \mathscr{G}$ be $\mathscr{O}_X$-modules with S-connections $D^1, D^2, \cdots, D^p, D^{\mathscr{G}}$ respectively. For every open subset U of X, define

$$\mathscr{L}_{\mathscr{O}_X}(\mathscr{F}_1, \cdots, \mathscr{F}_p; \mathscr{G})(U) := L_{\mathscr{O}_X|_U}(\mathscr{F}_1|_U, \cdots, \mathscr{F}_p|_U; \mathscr{G}|_U),$$

where $L_{\mathscr{O}_X|_U}(\mathscr{F}_1|_U, \cdots, \mathscr{F}_p|_U; \mathscr{G}|_U)$ is the $\mathscr{O}_X(U)$-module of $\mathscr{O}_X|_U$-multilinear maps from $\mathscr{F}_1|_U \times \mathscr{F}_2|_U \times \cdots \times \mathscr{F}_p|_U$ to $\mathscr{G}|_U$. The sheaf of $\mathscr{O}_X$-multilinear maps is denoted by $\mathscr{L}_{\mathscr{O}_X}(\mathscr{F}_1, \cdots, \mathscr{F}_p; \mathscr{G})$. For every local section ξ of $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$, let D_ξ be the S-linear endomorphism of the $\mathscr{O}_X$-module $\mathscr{L}_{\mathscr{O}_X}(\mathscr{F}_1, \cdots, \mathscr{F}_p; \mathscr{G})$ defined by

$$D_\xi(\omega)(u_1, u_2, \ldots, u_p) = D^{\mathscr{G}}_\xi(\omega(u_1, u_2, \ldots, u_p)) - \sum_{i=1}^{p} \omega(u_1, \ldots, u_{i-1}, D^i_\xi(u_i), u_{i+1}, \ldots, u_p),$$

for all local sections ω of $\mathscr{L}_{\mathscr{O}_X}(\mathscr{F}_1, \cdots, \mathscr{F}_p; \mathscr{G})$ and local sections $(u_1, u_2, \cdots, u_p)$ of $\mathscr{F}_1 \times \mathscr{F}_2 \times \cdots \times \mathscr{F}_p$. Then the morphism

$$D = \xi \longmapsto D_\xi : \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X) \longrightarrow \mathscr{E}nd_S(\mathscr{L}_{\mathscr{O}_X}(\mathscr{F}_1, \cdots, \mathscr{F}_p; \mathscr{G}))$$

is an S-connection on $\mathscr{L}_{\mathscr{O}_X}(\mathscr{F}_1, \cdots, \mathscr{F}_p; \mathscr{G})$.

Remark 2.1.11 Let $\mathscr{L}^p_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ denote the $\mathscr{O}_X$-module $\mathscr{L}_{\mathscr{O}_X}(\mathscr{F}_1, \cdots, \mathscr{F}_p; \mathscr{G})$, where $\mathscr{F}_1 = \cdots = \mathscr{F}_p = \mathscr{F}$. Let D be the S-connection on $\mathscr{L}^p_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ induced by $D^{\mathscr{F}}$ and $D^{\mathscr{G}}$. Let $\mathscr{S}ym^p_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ (respectively, $\mathscr{A}lt^p_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$) denote the $\mathscr{O}_X$-submodule of $\mathscr{L}^p_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ consisting of symmetric (respectively, alternating) $\mathscr{O}_X$-multilinear maps from $\mathscr{F}^p$ to $\mathscr{G}$. Then

$$D_\xi(\mathscr{S}ym^p_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})) \subset \mathscr{S}ym^p_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$$

(respectively, $D_\xi(\mathscr{A}lt^p_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})) \subset \mathscr{A}lt^p_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$). Therefore, D induces an S-

connection on the $\mathscr{O}_X$-submodules

$$\mathscr{S}ym^p_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}) \quad \text{and} \quad \mathscr{A}lt^p_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})$$

of $\mathscr{L}^p_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})$.

7. **Compatibility of multilinear maps and S-connections**

Let $p \geq 1$, and let

$$\mathscr{F}_1, \cdots, \mathscr{F}_p, \mathscr{G}$$

be $\mathscr{O}_X$-modules with S-connections $D^1, \cdots, D^p, D^{\mathscr{G}}$ respectively. Let

$$\mu : \mathscr{F}_1 \times \mathscr{F}_2 \times \cdots \times \mathscr{F}_p \longrightarrow \mathscr{G}$$

be an $\mathscr{O}_X$-multilinear map. We say that $D^1, D^2, \cdots, D^p, D^{\mathscr{G}}, \mu$ are compatible if for every local section ξ of $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$, and for all local sections $(u_1, u_2, \cdots, u_p)$ of $\mathscr{F}_1 \times \mathscr{F}_2 \times \cdots \times \mathscr{F}_p$, we have

$$D^{\mathscr{G}}_{\xi}(\mu(u_1,\cdots,u_p)) = \sum_{i=1}^{p} \mu(u_1,\cdots,u_{i-1},D^i_{\xi}(u_i),u_{i+1},\ldots,u_p)\,.$$

The following proposition is straight-forward to prove.

Proposition 2.1.12 *Let $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{H}$ be $\mathscr{O}_X$-modules, and let*

$$\mu : \mathscr{F} \times \mathscr{G} \longrightarrow \mathscr{H}$$

be a $\mathscr{O}_X$-bilinear map. Let $\mathscr{K}$ be any $\mathscr{O}_X$-module and $p \geq 1$, $q \geq 1$. Then, we have a $\mathscr{O}_X$-bilinear map

$$\wedge : \mathscr{A}lt^p_{\mathscr{O}_X}(\mathscr{K},\mathscr{F}) \times \mathscr{A}lt^q_{\mathscr{O}_X}(\mathscr{K},\mathscr{G}) \longrightarrow \mathscr{A}lt^{p+q}_{\mathscr{O}_X}(\mathscr{K},\mathscr{H})$$

defined by

$$\alpha\wedge\beta(u_1,\ldots,u_{p+q}) = \sum_{\sigma\in S(p,q)} sgn(\sigma)\mu(\alpha(u_{\sigma(1)},\ldots,u_{\sigma(p)}),\beta(u_{\sigma(p+1)}\ldots,u_{\sigma(p+q)}))$$

for all local sections α *of* $\mathscr{A}lt^p_{\mathscr{O}_X}(\mathscr{K},\mathscr{F})$, *and* β *of* $\mathscr{A}lt^q_{\mathscr{O}_X}(\mathscr{K},\mathscr{G})$, *where* $S(p,q)$ *is the set of all* (p,q)*-shuffles, that is, the set of all permutation* $\sigma\in S_{p+q}$ *such that* $\sigma(1)<\cdots<\sigma(p)$ *and* $\sigma(p+1)<\cdots<\sigma(p+q)$.

The construction in Proposition 2.1.12 produces the following corollary.

Corollary 2.1.13 *Let* $D^{\mathscr{F}}, D^{\mathscr{G}}$ *and* $D^{\mathscr{H}}$ *be S-connections on* $\mathscr{F},\mathscr{G}$ *and* $\mathscr{H}$ *respectively, which are compatible with* μ. *Let* $D^{\mathscr{K}}$ *be an S-connection on* $\mathscr{K}$. *Denote the induced connections on* $\mathscr{A}lt^p_{\mathscr{O}_X}(\mathscr{K},\mathscr{F})$, $\mathscr{A}lt^p_{\mathscr{O}_X}(\mathscr{K},\mathscr{G})$ *and* $\mathscr{A}lt^{p+1}_{\mathscr{O}_X}(\mathscr{K},\mathscr{H})$ *by* $D^{\mathscr{F}}$, $D^{\mathscr{G}}$ *and* $D^{\mathscr{H}}$ *respectively. Then* $D^{\mathscr{F}}$, $D^{\mathscr{G}}$, $D^{\mathscr{H}}$ *and* $\wedge$ *are compatible, that is,*

$$D^{\mathscr{H}}_\xi(\alpha\wedge\beta) = D^{\mathscr{F}}_\xi(\alpha)\wedge\beta+\alpha\wedge D^{\mathscr{G}}_\xi(\beta)$$

for all local sections ξ *of* $\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X)$, α *of* $\mathscr{A}lt^p_{\mathscr{O}_X}(\mathscr{K},\mathscr{F})$ *and* β *of* $\mathscr{A}lt^q_{\mathscr{O}_X}(\mathscr{K},\mathscr{G})$.

2.2 The relative Lie derivative

2.2.1 The relative Lie derivative associated with a relative connection

Let $\mathscr{F}$ be an $\mathscr{O}_X$-module and D an S-connection on $\mathscr{F}$. Let $p\geq 1$ be an integer, U an open subset of X, $\xi\in\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X)(U)$, and $\alpha\in\mathscr{L}^p_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{F})(U)$. Then the map

$$\theta_\xi(\alpha):(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X)(U))^p\longrightarrow\mathscr{F}(U),$$

defined by

$$\theta_\xi(\alpha)(\eta_1,\cdots,\eta_p) = D_\xi(\alpha(\eta_1,\cdots,\eta_p))-\sum_{i=1}^p\alpha(\eta_1,\cdots,\eta_{i-1},[\xi,\eta_i],\eta_{i+1},\cdots,\eta_p)$$

for all $\eta_1, \cdots, \eta_p \in \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)(U)$, is an $\mathscr{O}_X(U)$-multilinear map. Moreover, the map

$$\theta_\xi : \mathscr{L}^p_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F})(U) \longrightarrow \mathscr{L}^p_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F})(U)$$

is S-linear, because D_ξ is S-linear.

Definition 2.2.1 The S-linear morphism

$$\theta : \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X) \longrightarrow \mathscr{E}nd_S(\mathscr{L}^p_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F}))$$

defined above, is called the **relative Lie derivation** in degree p associated with D.

Remark 2.2.2

1. The relative Lie derivation satisfy the followings:

   - $\theta_\xi(\alpha + \beta) = \theta_\xi(\alpha) + \theta_\xi(\beta)$,
   - $\theta_\xi(a\alpha) = \xi(a)\alpha + a\theta_\xi(\alpha)$,
   - $\theta_{\xi+\zeta}(\alpha) = \theta_\xi(\alpha) + \theta_\zeta(\alpha)$, and
   - $\theta_{s\xi}(\alpha) = s\theta_\xi(\alpha)$

   for all local sections α, β of $\mathscr{L}^p_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F})$, ξ, ζ of $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$, a of $\mathscr{O}_X$, and s of $\mathscr{O}_S$.

2. If α is alternating (respectively, symmetric), then so is $\theta_\xi(\alpha)$, that is,

   $$\theta_\xi(\mathscr{A}lt^p_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F})) \subset \mathscr{A}lt^p_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F})$$

   (respectively, $\theta_\xi(\mathscr{S}ym^p_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F})) \subset \mathscr{S}ym^p_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F})$).

2.2.2 The relative Lie derivative and the exterior product

Let $\mathscr{F}$, $\mathscr{G}$, and $\mathscr{H}$ be $\mathscr{O}_X$-modules equipped with S-connections $D^{\mathscr{F}}$, $D^{\mathscr{G}}$ and $D^{\mathscr{H}}$ respectively. Let

$$\mu : \mathscr{F} \times \mathscr{G} \longrightarrow \mathscr{H}$$

be an $\mathscr{O}_X$-bilinear map. Take integers $p \geq 1$ and $q \geq 1$. Then, we have an $\mathscr{O}_X$-bilinear map

$$\wedge : \mathscr{A}lt^p_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F}) \times \mathscr{A}lt^q_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{G})$$

$$\longrightarrow \mathscr{A}lt^{p+q}_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{H}).$$

Suppose that $D^{\mathscr{F}}$, $D^{\mathscr{G}}$, $D^{\mathscr{H}}$ and μ are compatible, that is,

$$D^{\mathscr{H}}_\xi(\mu(u,v)) = \mu(D^{\mathscr{F}}_\xi(u),v) + \mu(u, D^{\mathscr{G}}_\xi(v))$$

for all local sections ξ of $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$, u of $\mathscr{F}$, and v of $\mathscr{G}$.

Then we have

$$\theta_\xi(\alpha \wedge \beta) = \theta_\xi(\alpha) \wedge \beta + \alpha \wedge \theta_\xi(\beta) \tag{2.2}$$

where the relative Lie derivations are associated with their respective relative connections, while α and β are local sections of their respective $\mathscr{O}_X$-modules.

2.3 Covariant derivative

2.3.1 Covariant derivative with respect to a relative connection

Let $\mathscr{F}$ be an $\mathscr{O}_X$-module and $\xi \in \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)(U)$, where $U \subset X$ is an open subset, and $p \geq 2$ an integer. For each

$$\alpha \in \mathscr{L}^p_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F})(U),$$

define $(\iota_\xi(\alpha))_U \in \mathscr{L}^{p-1}_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F})(U)$ by

$$(\iota_\xi(\alpha))_U(\eta_1, \cdots, \eta_{p-1}) = \alpha(\xi, \eta_1, \cdots, \eta_{p-1})$$

for all $\eta_1, \cdots, \eta_{p-1} \in \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)(U)$. When α is of degree 0, we define $\iota_\xi(\alpha)_U = 0$. We call $(\iota_\xi(\alpha))_U$ the **relative interior product** of ξ and α over U. This yields an $\mathscr{O}_X$-module homomorphism

$$\iota : \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X) \longrightarrow \mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}^p_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F}), \mathscr{L}^{p-1}_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F}))$$

defined by $\iota_U(\xi)(\alpha) = \iota_\xi(\alpha)_U$, for every open subset U of X.

The interior product satisfies the following properties mentioned in the following

Lemma 2.3.1

1. *$\iota_{\xi+\eta} = \iota_\xi + \iota_\eta$, for all local sections ξ and η of $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$.*

2. *$\iota_{a\xi} = a\iota_\xi$, for all local sections a of $\mathscr{O}_X$ and ξ of $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$.*

3. *If D is an S-connection on $\mathscr{F}$, and θ the associated relative Lie derivative, then for all local sections ξ, η of $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$,*

$$\theta_\xi \circ \iota_\eta - \iota_\eta \circ \theta_\xi = \iota_{[\xi,\eta]}\,.$$

4. *If α is a local section of $\mathscr{A}lt^p_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$, then $\iota_\xi(\iota_\xi(\alpha)) = 0$.*

5. *Let $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{H}$ be $\mathscr{O}_X$-modules equipped with S-connections $D^{\mathscr{F}}$, $D^{\mathscr{G}}$ and $D^{\mathscr{H}}$ respectively. Let $\mu : \mathscr{F} \times \mathscr{G} \longrightarrow \mathscr{H}$ be an $\mathscr{O}_X$-bilinear map. Let $p \geq 1$ and $q \geq 1$ be integers. Then, we have an $\mathscr{O}_X$-bilinear map*

$$\wedge : \mathscr{A}lt^p_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F}) \times \mathscr{A}lt^q_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{G})$$

$$\longrightarrow \mathscr{A}lt^{p+q}_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{H})\,.$$

Suppose that $D^{\mathscr{F}}$, $D^{\mathscr{G}}$, $D^{\mathscr{H}}$ *and* μ *are compatible, that is,*

$$D^{\mathscr{H}}_{\xi}(\mu(u,v)) = \mu(D^{\mathscr{F}}_{\xi}(u),v)+\mu(u,D^{\mathscr{G}}_{\xi}(v))$$

for all local sections ξ *of* $\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X)$*,* u *of* $\mathscr{F}$*, and* v *of* $\mathscr{G}$*. Then*

$$\iota_{\xi}(\alpha\wedge\beta) = \iota_{\xi}(\alpha)\wedge\beta+(-1)^p\alpha\wedge\iota_{\xi}(\beta)\,, \tag{2.3}$$

where the Lie derivations are associated with their respective connections, while α *and* β *are local sections of their respective* $\mathscr{O}_X$*-modules.*

Proposition 2.3.2 *Let* D *be an* S*-connection on an* $\mathscr{O}_X$*-module* $\mathscr{F}$*. Then, there exists a unique family of* S*-linear morphism*

$$d = d_p : \mathscr{A}lt^p_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{F}) \longrightarrow \mathscr{A}lt^{p+1}_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{F})\,,$$

where $p\in\mathbb{N}$*, such that*

$$\theta_{\xi} = d_{p-1}\circ\iota_{\xi}+\iota_{\xi}\circ d_p \tag{2.4}$$

for all local sections ξ *of* $\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X)$*, where* $d_{-1}=0$ *by convention.*

Proof. **Uniqueness:** Suppose that we have two families $(d_p)_{p\in\mathbf{N}}$ and $(d'_p)_{p\in\mathbf{N}}$ of S-linear morphisms satisfying (2.4). We shall prove by induction on p that $d_p = d'_p$ for all $p\in\mathbf{N}$. First note that for every $p\in\mathbf{N}$, we have

$$(d_{p-1}-d'_{p-1})\iota_{\xi}+\iota_{\xi}(d_p-d'_p)=0 \tag{2.5}$$

for every local section ξ of $\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X)$. If $p=0$, then $\iota_{\xi}(\alpha)=0$, for every local sections ξ of $\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X)$ and α of $\mathscr{L}^0_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{F})=\mathscr{F}$. Therefore, from

(2.5), for every local sections ξ of $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$ and α of $\mathscr{F}$, we have

$$\begin{aligned} 0 &= (d'_{-1} - d_{-1})(\iota_\xi(\alpha)) \\ &= \iota_\xi(d_0 - d'_0)(\alpha) \\ &= d_0(\alpha)(\xi) - d'_0(\alpha)(\xi) \end{aligned}$$

Thus, $d_0 = d'_0$.

Suppose that $p \geq 1$, and $d_r = d'_r$ for all $0 \leq r \leq p-1$. Then, for every local sections $\xi, \eta_1, \ldots, \eta_{p-1}$ of $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$, and α of $\mathscr{L}^p_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F})$, using (2.5) and induction hypothesis, we have

$$\begin{aligned} d_p(\alpha)(\xi, \eta_1, \ldots, \eta_{p-1}) &= \iota_\xi(d_p(\alpha))(\eta_1, \ldots, \eta_{p-1}) \\ &= \iota_\xi(d'_p(\alpha))(\eta_1, \ldots, \eta_{p-1}) \\ &= d'_p(\alpha)(\xi, \eta_1, \ldots, \eta_{p-1}) \end{aligned}$$

Thus, $d_p = d'_p$, for every $p \in \mathbf{N}$.

Existence: We will show existence by induction on p. Let α be a local section of $\mathscr{F} = \mathscr{A}lt^0_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F})$. Then we will define d_0 so that $\iota_\xi(d_0\alpha) = \theta_\xi(\alpha)$, for every local section ξ of $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$. Now, by definition (2.2.1) of θ, we have $\theta_\xi(\alpha) = D_\xi(\alpha)$. Thus, for every local section α of $\mathscr{L}^0_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F}) = \mathscr{F}$, and ξ of $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$ we define

$$d_0(\alpha)(\xi) = D_\xi(\alpha). \tag{2.6}$$

Suppose that d_r has been defined for $r = 0, 1, \ldots, p-1$ such that (2.1.2) is satisfied. Then for every local sections

$\eta_1, \ldots, \eta_{p+1}$ of $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$ and α of $\mathscr{L}^p_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F})$, define

$$d_p\alpha(\eta_1, \ldots, \eta_{p+1}) = \theta_{\eta_1}(\alpha)(\eta_2, \ldots, \eta_{p+1}) - d_{p-1}(\iota_{\eta_1}(\alpha))(\eta_2, \ldots, \eta_{p+1}). \tag{2.7}$$

Now, $d_p(\alpha)$ is linear in $\eta_2, \ldots, \eta_{p+1}$ follows from the induction assumption and the multilinearity of $\theta_{\eta_1}(\alpha)$. Moreover, we show that $d_p(\alpha)$ is an alternating form, this means that for every local sections $\eta_1, \ldots, \eta_{p+1}$ of $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$

$$d_p(\eta_1, \ldots, \eta_{p+1}) = 0,$$

whenever $\eta_i = \eta_j$ for $i \neq j$. Since $\theta_{\eta_1}(\alpha)$ and $d_{p-1}(\iota_{\eta_1}(\alpha))$ are alternating forms, it suffices to show that

$$\theta_{\eta_1}(\alpha)(\eta_2, \ldots, \eta_{p+1}) = d(\iota_{\eta_1}(\alpha))(\eta_2, \ldots, \eta_{p+1})$$

when $\eta_1 = \eta_2$. Now, under the assumption that $\eta_1 = \eta_2$, we have

$$\begin{aligned} \theta_{\eta_2}(\alpha)(\eta_2, \ldots, \eta_{p+1}) &= \iota_{\eta_2}\theta_{\eta_2}(\alpha)(\eta_3, \ldots, \eta_{p+1}) \\ &= \theta_{\eta_2}(\iota_{\eta_2}(\alpha))(\eta_3, \ldots, \eta_{p+1}), \end{aligned}$$

where the last equality follows from (2.4.6). Now, using induction hypothesis, we have

$$\begin{aligned} \theta_{\eta_2}(\alpha)(\eta_2, \ldots, \eta_{p+1}) &= \theta_{\eta_2}(\iota_{\eta_2}(\alpha))(\eta_3, \ldots, \eta_{p+1}) \\ &= (d_{p-2}\iota_{\eta_2}(\iota_{\eta_2}(\alpha)) + \iota_{\eta_2}(\alpha)d_{p-1}(\iota_{\eta_2}(\alpha)))(\eta_3, \ldots, \eta_{p+1}) \\ &= d_{p-1}(\iota_{\eta_2}(\alpha))(\eta_2, \eta_3, \ldots, \eta_{p+1}) \end{aligned}$$

The linearity of $d_p(\alpha)$ in η_1 follows from the fact that $d_p(\alpha)$ is alternating and linearity in other variables. This completes the proof.

Definition 2.3.3 The family $(d_p)_{p\in\mathbf{N}}$ of S-linear morphism in Proposition 2.3.2 is called the **covariant derivative** associated with the S-connection D.

We usually write

$$\mathscr{A}lt^{\bullet}_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{F}) = \oplus_{p\in\mathbf{N}}\mathscr{A}lt^{p}_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{F})$$

and d instead of d_p.

2.3.2 Explicit formula for the covariant derivative

Proposition 2.3.4 *Let D be an S-connection on an $\mathscr{O}_X$-module $\mathscr{F}$. Then, the covariant derivative d with respect to D is given by*

$$d(\alpha)(\xi_1,\cdots,\xi_{p+1}) = \sum_{i=1}^{p+1}(-1)^{i+1}D_{\xi_i}(\alpha(\xi_1,\cdots,\hat{\xi}_i,\dots,\xi_{p+1}))$$

$$+\sum_{1\le i<j\le p+1}(-1)^{(i+1)}\alpha([\xi_i,\xi_j],\xi_1,\dots,\hat{\xi}_i,\dots,\hat{\xi}_j,\dots,\xi_{p+1})$$

for all local sections α of $\mathscr{A}lt^{p}_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{F})$ and $\xi_1,\cdots,\xi_{p+1}$ of $\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X)$.

Proof. Define d as above. Then for every local sections η_1 of $\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X)$ and α of $\mathscr{A}lt^{p}_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{F})$, we have

$$\iota_{\eta_1}(d(\alpha)) + d(\iota_{\eta_1}(\alpha)) = \theta_{\eta_1}(\alpha)$$

Now, the proposition follows from the fact that the covariant derivative is unique.

Proposition 2.4.4 gives the following:

Corollary 2.3.5 *The following three hold.*

1. $d(\alpha)(\xi) = D_\xi(\alpha)$ *for all local sections α of $\mathscr{F}$ and ξ of $\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X)$.*

2. $d(\alpha)(\xi,\eta) = D_\xi(\alpha(\eta)) - D_\eta(\alpha(\xi)) - \alpha([\xi,\eta])$, *for all local sections α of $\mathscr{A}lt^{2}_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{F})$ and ξ,η of $\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X)$.*

3. $d(\alpha)(\xi,\eta,\nu) = \sum_{cyclic}(D_\xi(\alpha(\eta,\nu)) - \alpha([\xi,\eta],\nu))$.

2.3.3 Covariant derivative and exterior product

Proposition 2.3.6 *Let $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{H}$ be $\mathscr{O}_X$-modules equipped with S connections $D^{\mathscr{F}}$, $D^{\mathscr{G}}$ and $D^{\mathscr{H}}$ respectively. Let*

$$\mu : \mathscr{F} \times \mathscr{G} \longrightarrow \mathscr{H}$$

be an $\mathscr{O}_X$-bilinear map. Suppose that $D^{\mathscr{F}}$, $D^{\mathscr{G}}$, $D^{\mathscr{H}}$ and μ are compatible. Then for all local sections α of $\mathscr{A}lt^p_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F})$ and β of $\mathscr{A}lt^\bullet_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F})$, we have

$$d(\alpha \wedge \beta) = d(\alpha) \wedge \beta + (-1)^p \alpha \wedge d(\beta)\,. \tag{2.8}$$

Proof. We prove it by induction on $p+q$, where q is the degree of β. When $p+q=0$, then (2.8) is satisfied because it expresses the compatibility of the product with covariant derivative. Suppose that the theorem is proved for $p+q=r-1$. We have

$$d(\alpha\wedge\beta)(\eta_1,\ldots,\eta_{p+q+1}) = \iota_{\eta_1} d(\alpha\wedge\beta)(\eta_2,\ldots,\eta_{p+q+1})$$

From (2.4), we have

$$\iota_{\eta_1} d(\alpha\wedge\beta) = \theta_{\eta_1}(\alpha\wedge\beta) - d(\iota_{\eta_1}(\alpha\wedge\beta))$$

Now, from (2.2), (2.3) and induction hypothesis, we have

$$\begin{aligned}
\iota_{\eta_1} d(\alpha\wedge\beta) &= (\theta_{\eta_1}\alpha)\wedge\beta+\alpha\wedge(\theta_{\eta_1}\beta)-d(\iota_{\eta_1}(\alpha)\wedge\beta+(-1)^p\alpha\wedge\iota_{\eta_1}(\beta))\\
&= (\theta_{\eta_1}\alpha)\wedge\beta+\alpha\wedge(\theta_{\eta_1}\beta)-d(\iota_{\eta_1}(\alpha))\wedge\beta-(-1)^{p-1}\iota_{\eta_1}\alpha\wedge d\beta\\
&-(-1)^p d\alpha\wedge\iota_{\eta_1}(\beta)-\alpha\wedge d\iota_{\eta_1}(\beta)\\
&= \iota_{\eta_1} d\alpha\wedge\beta-(-1)^p d\alpha\wedge\iota_{\eta_1}(\beta)+\alpha\wedge\iota_{\eta_1} d\beta-(-1)^{p-1}\iota_{\eta_1}\alpha\wedge d\beta\\
&= \iota_{\eta_1}(d\alpha\wedge\beta)+(-1)^p((\iota_{\eta_1}\alpha)\wedge d\beta+(-1)^p\alpha\wedge\iota_{\eta_1} d\beta)\\
&= \iota_{\eta_1}(d\alpha\wedge\beta)+(-1)^p\iota_{\eta_1}(\alpha\wedge d\beta)\\
&= \iota_{\eta_1}(d\alpha\wedge\beta+(-1)^p\alpha\wedge d\beta)
\end{aligned}$$

This completes the proof.

Proposition 2.3.6 gives the following:

Corollary 2.3.7 *Let $\mathscr{F}$ be an $\mathscr{O}_X$-module with connection D. Then*

$$d(a\alpha) = d(a)\wedge\alpha+ad(\alpha)$$

for all local sections a of $\mathscr{O}_X$ and α of $\mathscr{A}lt^{\bullet}_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{F})$, where $d(a)$ is the covariant derivative of a with respect to the canonical connection on $\mathscr{O}_X$.

2.4 The curvature form

Let D be an S-connection on an $\mathscr{O}_X$-module $\mathscr{F}$, and let

$$d:\mathscr{A}lt^{\bullet}_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{F})\longrightarrow\mathscr{A}lt^{\bullet+1}_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{F})$$

be the covariant derivative associated with D. Then the map

$$d \circ d = d^2 : \mathscr{A}lt^{\bullet}_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F}) \longrightarrow \mathscr{A}lt^{\bullet+2}_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F})$$

is called the curvature operator of D, and it will be denoted by R.

Let α be a local section of $\mathscr{A}lt^0_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F}) = \mathscr{F}$. Then

$$R(\alpha) = d(d(\alpha))$$

is a local section of $\mathscr{A}lt^2_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F})$.

Let ξ and η be local sections of $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$. Then

$$\begin{aligned} R(\alpha)(\xi,\eta) &= d(d(\alpha))(\xi,\eta) \\ &= D_\xi(d(\alpha)(\eta)) - D_\eta(d(\alpha)(\xi)) - d(\alpha)([\xi,\eta]) \\ &= D_\xi(D_\eta(\alpha)) - D_\eta(D_\xi(\alpha)) - D_{[\xi,\eta]}(\alpha). \end{aligned}$$

Thus, for every open subset U of X and for all sections $\xi, \eta \in \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)(U)$, we get an $\mathscr{O}_X|_U$-module homomorphism

$$K_U(\xi,\eta) : \mathscr{F}|_U \longrightarrow \mathscr{F}|_U$$

defined by

$$K_U(\xi, \eta) = D_\xi \circ D_\eta - D_\eta \circ D_\xi - D_{[\xi,\eta]}. \tag{2.9}$$

Moreover, we have

1. $K_U(\xi,\eta) = -K_U(\eta,\xi)$
2. $K_U(\xi+\xi',\eta) = K_U(\xi,\eta) + K_U(\xi',\eta)$
3. $K_U(f\xi,\eta)\alpha = fK_U(\xi,\eta)\alpha$ for every local sections f of $\mathscr{O}_X$, and α of $\mathscr{F}$.

Hence, these K_U together define an $\mathscr{O}_X$-bilinear map

$$K : \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X) \times \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X) \longrightarrow \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{F}) . \tag{2.10}$$

From above properties of K it is an alternating map.

Definition 2.4.1 The alternating $\mathscr{O}_X$-bilinear map K defined above is called the **curvature form** of D. We say the S-connection is **flat** if the curvature form is identically zero, that is, if the map

$$D : \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X) \to \mathscr{E}nd_S(\mathscr{F})$$

is a homomorphism of Lie algebras.

Example 2.4.2 Let $\mathscr{F} = \mathscr{O}_X$, and D be the canonical connection ε on $\mathscr{O}_X$. Then

$$D : \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X) \hookrightarrow \mathscr{E}nd_S(\mathscr{O}_X)$$

is the inclusion map. Since, $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$ is a Lie subalgebra of $\mathscr{E}nd_S(\mathscr{O}_X)$, D is a homomorphism of Lie subalgebra. Thus, D is a flat connection.

There are S-connections which are not flat.

Example 2.4.3 Let $S = \text{Spec}(k)$ be an affine scheme, where k be a non-zero commutative ring, and $X = \text{Spec}(k[T_1, T_2])$. Then $\mathscr{O}_X$-module $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$ is free, and the set $\{\xi_1, \xi_2\}$ is an $\mathscr{O}_X$-basis of $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$, where $\xi_1 = \frac{\partial}{\partial T_1}$ and $\xi_2 = \frac{\partial}{\partial T_2}$. Let $\mathscr{F} = \mathscr{O}_X$, and let

$$\omega : \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X) \to \mathscr{O}_X \tag{2.11}$$

be the unique $\mathscr{O}_X$-linear map such that $\omega(\xi_1) = T_2$ and $\omega(\xi_2) = 1$. Then, since 1 is an $\mathscr{O}_X$-basis of $\mathscr{O}_X$, by Proposition 2.1.4, there exists a unique S-connection D on $\mathscr{O}_X$ such that $D_\xi(1) = \omega(\xi)$, for every local section ξ of $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$ Thus, $D_{\xi_1}(1) = \omega(\xi_1) = T_2$

and $D_{\xi_2}(1) = \omega(\xi_2) = 1$. So, we have

$$\begin{aligned}
K(\xi_1, \xi_2)(1) &= D_{\xi_1}(D_{\xi_2}(1)) - D_{\xi_2}(D_{\xi_1}(1)) - D_{[\xi_1, \xi_2]}(1) \\
&= D_{\xi_1}(1) - D_{\xi_2}(T_2.1) \\
&= T_2 - \xi_2(T_2).1 - T_2.D_{\xi_2}(1) \\
&= T_2 - 1.1 - T_2.1 \\
&= -1 \neq 0
\end{aligned}$$

Therefore, the S-connection D on $\mathscr{O}_X$ is not flat.

The following lemma gives the relation between relative Lie derivation and curvature form. This follows from straight forward verification.

Lemma 2.4.4 *Let D be an S-connection on $\mathscr{F}$. Then,*

$$\theta_\xi(\theta_\eta(\alpha)) - \theta_\eta(\theta_\xi(\alpha)) = \theta_{[\xi,\eta]}(\alpha) + K(\xi, \eta)(\alpha), \tag{2.12}$$

for all local sections ξ, η of $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$ and α of $\mathscr{A}lt^2_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F})$.

Note that $K(\xi, \eta)(\alpha)$ is defined when α is a local section of $\mathscr{F}$, it will again makes sense if we replace $\mathscr{F}$ by any $\mathscr{O}_X$-module and in particular $\mathscr{A}lt^2_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F})$. Further, the following lemma establishes the relation between covariant derivative and curvature form

Lemma 2.4.5 *Let D be an S-connection on $\mathscr{F}$ and let $p \in \mathbf{N}$. Then,*

$$\theta_\xi(d(\alpha)) - d(\theta_\xi(\alpha)) = \iota_\xi(K) \wedge \alpha \tag{2.13}$$

for every local sections ξ, η of $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$ and α of $\mathscr{A}lt^2_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F})$, *where, the wedge product on the R.H.S. of* (2.13) *is the* $\mathscr{O}_X$*-bilinear map*

$$\wedge : \mathscr{A}lt^p_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{F})) \times \mathscr{A}lt^q_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F})$$

$$\longrightarrow \mathscr{A}lt^{p+q}_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F}).$$

induced by the $\mathscr{O}_X$*-bilinear map*

$$\mu : \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{F}) \times \mathscr{F} \to \mathscr{F} \tag{2.14}$$

defined by $(f,u) \mapsto f(u)$*, for every local section* f *of* $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{F})$ *and* u *of* $\mathscr{F}$*.*

Proof. We prove this by induction on the degree of α, that is, p. If α is of degree zero, then (2.13) reduces to

$$D_\xi(d\alpha(\eta)) - d\alpha([\xi,\eta]) - (dD_\xi)\alpha(\eta) = (\iota_\xi K \wedge \alpha)\eta$$

that is nothing but

$$D_\xi D_\eta(\alpha) - D_{[\xi,\eta]}(\alpha) - D_\eta D_\xi \alpha = (\iota_\xi K \wedge \alpha)\eta,$$

which is nothing but the definition of K. Suppose that the lemma is true for degree $< p$.

Now, consider

$$\begin{aligned}
\iota_\eta \theta_\xi d\alpha - \iota_\eta d\theta_\xi \alpha - \iota_\eta(\iota_\xi K \wedge \alpha) &= \theta_\xi \iota_\eta d\alpha - \iota_{[\xi,\eta]} d\alpha + d\iota_\eta \theta_\xi \alpha \\
&- \theta_\eta \theta_\xi \alpha - (\iota_\eta \iota_\xi K) \wedge \alpha + (\iota_\xi K) \wedge (\iota_\eta \alpha) \\
&= -\theta_\xi d\iota_\eta \alpha + \theta_\xi \theta_\eta - \iota_{[\xi,\eta]} d\alpha + d\theta_\xi \iota_\eta \alpha \\
&- d\iota_{[\xi,\eta]} \alpha - \theta_\eta \theta_\xi \alpha - (\iota_\eta \iota_\xi K) \wedge \alpha + (\iota_\xi K) \wedge (\iota_\eta \alpha) \\
&= -\theta_\xi d\iota_\eta \alpha + d\theta_\xi \theta_\eta \alpha + \theta_\xi \theta_\eta \alpha \\
&- \theta_\eta \theta_\xi \alpha - \theta_{[\xi,\eta]} \alpha - \iota_\eta(\iota_\xi K) \wedge \alpha + (\iota_\xi K) \wedge (\iota_\eta \alpha) \\
&= -\theta_\xi d\iota_\eta \alpha + d\theta_\xi \iota_\eta \alpha + (\iota_\xi K) \wedge (\iota_\eta \alpha) \\
&= 0,
\end{aligned}$$

where the last equality follows from induction hypothesis.

Remark 2.4.6 Let D be an S-connection on $\mathscr{F}$. Then the morphisms ι_ξ, θ_ξ, and d are of degree -1, 0, and 1 respectively, between the graded $\mathscr{O}_X$-module

$$\mathscr{A}lt^\bullet_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F}).$$

Moreover, if ϕ and ψ are S-linear endomorphisms of $\mathscr{A}lt^\bullet_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F})$ of degree p and q respectively, define

$$[\phi, \psi] = \phi \circ \psi - (-1)^{pq} \psi \circ \phi. \tag{2.15}$$

Then, we have

1. $[\theta_\xi, \iota_\eta] = \iota_{[\xi,\eta]}$ by Lemma 2.3.1.
2. $[d, \iota_\xi] = \theta_\xi$ by characterising property of d.
3. $[\theta_\xi, \theta_\eta] = \theta_{[\xi,\eta]} + K(\xi, \eta)$.

4. $[\theta_\xi, d] = 0$ if the curvature form $K = 0$.

5. $[\iota_\xi, \iota_\eta] = 0$.

Proposition 2.4.7 (Bianchi's first identity) *Let D be an S-connection on $\mathscr{F}$. Then,*

$$R(\alpha) = K \wedge \alpha \tag{2.16}$$

for all local section α of $\mathscr{A}lt^\bullet_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{F}))$, where the wedge product is with respect to $\mathscr{O}_X$-bilinear map

$$\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{F}) \times \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{F}) \to \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{F})$$

defined by composition $(g, f) \mapsto g \circ f$.

Proof. We shall prove it by induction on the degree of α. If α is of degree zero, that is, it is a local section of $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{F})$, then by definition of curvature form K, we have

$$dd\alpha(\xi, \eta) = K(\xi, \eta)(\alpha). \tag{2.17}$$

Suppose that the lemma is true for the degree $< p$. Now, consider

$$\begin{aligned}
\iota_\xi dd\alpha &= \theta_\xi d\alpha - d\iota_\xi d\alpha \\
&= d\theta_\xi \alpha + (\iota_\xi K) \wedge \alpha - d\theta_\xi \alpha + dd\iota_\xi \alpha \quad \text{by Lemma 2.4.5} \\
&= (\iota_\xi K) \wedge \alpha + K \wedge \iota_\xi(\alpha) \;\; \text{by induction hypothesis} \\
&= \iota_\xi (K \wedge \alpha).
\end{aligned}$$

Since ξ is arbitrary local section of $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$, thus, we have the lemma.

Proposition 2.4.8 (Bianchi's second identity) *Let D be an S-connection on $\mathscr{F}$. Then,*

$$dK = 0, \tag{2.18}$$

on $\mathscr{A}lt^3_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{F}))$*, where d is covariant derivative associated to the S-connection D.*

Proof. The proof is straightforward verification using Corollary 2.3.5, (3) and Jacobi's identity.

2.5 Connection and curvature matrices

Let $\mathscr{F}$ be a locally free $\mathscr{O}_X$-module of finite rank r. Let U be an open subset of X such that $\mathscr{F}|_U$ is a free $\mathscr{O}_X|_U$-module. Let $s=(s_1,\ldots,s_r)$ be an $\mathscr{O}_X|_U$-basis of $\mathscr{F}|_U$. For each $\xi\in\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X)(U)$, define an $r\times r$ matrix

$$\omega(\xi)=(\omega_{ij}(\xi))_{1\le i,j\le r}$$

of elements of $\mathscr{O}_X(U)$ by the equation

$$D_\xi(s_j)=\sum_{i=1}^{r}\omega_{ij}(\xi)s_i(1\le j\le r).$$

We, thus get, for all $i,j\in\{1,\ldots,r\}$ an element ω_{ij} of

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{O}_X)(U)=\mathscr{A}lt^1_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{O}_X)(U).$$

This gives an $r\times r$ matrix $\omega=(\omega_{ij})_{1\le i,j\le r}$, where entries are sections of

$$\mathscr{A}lt^1_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{O}_X)$$

over U.

It is called the **connection matrix** of D with respect to s. Considering $s=(s_1,\ldots,s_r)$ as a row vector, ω is the unique $r\times r$ matrix over $\mathscr{A}lt^1_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{O}_X)(U)$ such that

$$D_\xi(s) = s\omega(\xi)$$

for all $\xi \in \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)(U)$. If

$$u = \sum_{j=1}^{r} a_j s_j \in \mathscr{F}(U),$$

where $a_j \in \mathscr{O}_X(U)$, then

$$D_\xi(u) = s(\xi(a) + \omega(\xi)a)$$

for all $\xi \in \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)(U)$. If

$$d = d_{X/S} : \mathscr{A}lt^0_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{O}_X) = \mathscr{O}_X \to \mathscr{A}lt^1_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{O}_X)$$

is the covariant derivative associated with canonical connection on $\mathscr{O}_X$, then

$$D_\xi(u) = s(d(a) + \omega(\xi)a)$$

for all $\xi \in \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)(U)$.

Let $t = (t_1, \ldots, t_r)$ be another $\mathscr{O}_X|_U$-basis of $\mathscr{F}|_U$, and

$$t_j = \sum_{i=1}^{r} a_{ij} s_i,$$

for $1 \leq j \leq r$. Then the matrix $a = (a_{ij})_{1 \leq i,j \leq r}$ is an element of $GL_r(\mathscr{O}_X(U))$.

Let ω' be the connection matrix of D with respect to t. Then

$$\omega' = a^{-1}da + a^{-1}\omega a.$$

Let K be the curvature form of D. For all $\xi, \eta \in \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)(U)$, let

$$\Omega(\xi, \eta) = (\Omega_{ij}(\xi, \eta))_{1 \leq i,j \leq r}$$

be the $r \times r$ matrix over $\mathscr{O}_X(U)$, defined by

$$K(\xi,\eta)(s_j) = \sum_{i=1}^{r} \Omega_{ij}(\xi,\eta)s_i$$

for $1 \leq j \leq r$.

We, thus get for all $i, j \in \{1,\dots,r\}$ an element Ω_{ij} of $\mathscr{A}lt^2_{\mathscr{O}_X}\big(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{O}_X\big)(U)$. This gives an $r \times r$ matrix $\Omega = (\Omega_{ij})_{1\leq i,j\leq r}$, where entries are sections of

$$\mathscr{A}lt^2_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{O}_X)$$

over U. It is called the **curvature matrix** of D with respect to s. Considering $s = (s_1,\dots,s_r)$ as a row vector, Ω is the unique $r \times r$ matrix over $\mathscr{A}lt^2_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{O}_X)(U)$ such that

$$K(\xi,\eta)s = s\Omega(\xi,\eta)$$

for all $\xi,\eta \in \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)(U)$.

We have

$$\Omega = d\omega + \omega \wedge \omega.$$

If $t = (t_1,\dots,t_r)$ be another $\mathscr{O}_X|_U$-basis of $\mathscr{F}|_U$ as above, and if Ω' is the curvature matrix of D with respect to t, then

$$\Omega' = a^{-1}\Omega a,$$

where, $a = (a_{ij})_{1\leq i,j\leq r}$ as before.

2.6 Torsion form

Consider the sheaf $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$ of S-derivations on $\mathscr{O}_X$. Let D be an S-connection on the $\mathscr{O}_X$-module $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$. Let

$$\tau : \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X) \to \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$$

be the identity morphism. Then τ is a section of

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X)) = \mathscr{A}lt^1_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X))$$

.

Consider the covariant derivative

$$d : \mathscr{A}lt^\bullet_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X)) \longrightarrow \mathscr{A}lt^\bullet_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X))$$

associated with the connection D on $\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X)$. We thus get a section

$$T = d(\tau)$$

of $\mathscr{A}lt^2_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X))$.

Definition 2.6.1 The alternating $\mathscr{O}_X$-bilinear map

$$T : \mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X) \times \mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X) \to \mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X)$$

defined above is called the **torsion form** of the S-connection D on $\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X)$. We say that an S-connection D on $\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X)$ is torsion free if $T = 0$.

Now, we have Bianchi's first identity in this particular case stated as follows.

Proposition 2.6.2 (Bianchi's first identity) *Let D be an S-connection on $\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X)$. Then*

$$d(T)(\xi,\eta,\nu) = \sum_{cyclic} K(\xi,\eta)(\nu)$$

for every local sections of ξ,η,ν of $\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X)$

Proof. The proof follows from Proposition 2.4.7.

Let D be an S-connection on $\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X)$, $\mathscr{F}$ be an $\mathscr{O}_X$-module, $D^{\mathscr{F}}$ an S-connection on $\mathscr{F}$. Let $p \geq 1$. Then, there is an induced S-connection D on the $\mathscr{O}_X$-module

$$\mathscr{L}^p_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F})$$

(see section 2.1.3 (6)) defined by

$$D_\xi(\omega)(u_1, u_2, \ldots, u_p) = D^{\mathscr{F}}_\xi(\omega(u_1, u_2, \ldots, u_p)) - \sum_{i=1}^{p} \omega(u_1, \ldots, u_{i-1}, D_\xi(u_i), u_{i+1}, \ldots, u_p),$$

where ω is a section of

$$\mathscr{L}^p_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F})$$

and $\xi, u_1, \ldots, u_p$ are sections of $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$.

Define an S-linear morphism

$$P : \mathscr{L}^p_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F}) \to \mathscr{L}^{p+1}_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F})$$

by

$$\begin{aligned} P(\omega)(u_1, u_2, \ldots, u_{p+1}) &= D_{u_1}(\omega)(u_2, \ldots, u_{p+1})) \\ &- \sum_{i=2}^{p+1} \omega(u_2, \ldots, u_{i-1}, D_{u_1}(u_i), u_{i+1}, \ldots, u_{p+1}), \end{aligned}$$

for every local section ω of $\mathscr{L}^p_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F})$, and $u_1, \ldots, u_{p+1}$ of $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$. The S-linear morphism P defined above may not map $\mathscr{O}_X$-submodule

$$\mathscr{A}lt^p_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F})$$

of $\mathscr{L}^p_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F})$ to the $\mathscr{O}_X$-submodule $\mathscr{A}lt^{p+1}_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F})$ of $\mathscr{L}^{p+1}_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F})$. We therefore define

$$d' : \mathscr{L}^p_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F}) \to \mathscr{L}^{p+1}_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{F})$$

by

$$d'(\omega)(u_1,u_2,\ldots,u_{p+1}) = \sum_{i=1}^{p+1}(-1)^{i+1}P(\omega)(u_i,u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_{p+1}),$$

where ω and $u_i's$ are as above. Then

$$d'(\mathscr{A}lt^p_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{F}) \subset \mathscr{A}lt^{p+1}_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{F}).$$

Proposition 2.6.3 *Let D be an S-connection on $\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X)$. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module, and $D^{\mathscr{F}}$ an S-connection on $\mathscr{F}$. Let*

$$d' : \mathscr{A}lt^p_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{F}) \to \mathscr{A}lt^{p+1}_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{F})$$

be the graded S-linear morphism of degree 1 *defined above. Suppose that the connection D on $\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X)$ is torsion free. Then d' equals the covariant derivative associated with the connection $D^{\mathscr{F}}$ on $\mathscr{F}$.*

Proof. First observe that $\iota_\xi \circ d' + d' \circ \iota_\xi = \theta_\xi$, where ξ is a local section of $\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X)$ and θ is the relative Lie derivative associated to $D^{\mathscr{F}}$. Now, proposition follows prom Proposition 2.3.2.

Corollary 2.6.4 (Bianchi's second identity) *Let D be a torsion free connection on $\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X)$. Then*

$$\sum_{cyclic} D_\xi(K)(\eta,\nu) = 0,$$

for every local sections ξ,η,ν of $\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X)$. Here, K is the curvature form associated to the connection D.

Proof. By Proposition 2.4.8, $d(K) = 0$. Since D is torsion free, from previous Proposition

2.6.3, $d'(K)=0$. Now, corollary follows from the fact that

$$d'(K)(\xi,\eta,\nu)=\sum_{cyclic} D_\xi(K)(\eta,\nu),$$

for every local sections ξ,η,ν of $\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X)$.

2.7 Relations between different relative connections

We will investigate the relations between covariant derivatives, and curvature forms associated with two different S-connection on the same $\mathscr{O}_X$-module $\mathscr{F}$.

Let $\mathscr{F}$ be an $\mathscr{O}_X$-module. Let D and D' be two S-connections on $\mathscr{F}$. Let

$$d:\mathscr{A}lt^\bullet_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{F})\longrightarrow \mathscr{A}lt^\bullet_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{F}),$$

(respectively,

$$d':\mathscr{A}lt^\bullet_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{F})\longrightarrow \mathscr{A}lt^\bullet_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{F}))$$

be the covariant derivative associated with D (respectively, D').

Let $K,K'\in\mathscr{A}lt^2_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{F}))(X)$ be the curvature forms of D and D' respectively.

Let

$$h:\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X)\to\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{F})$$

be the map $\xi\mapsto h_\xi$ defined by

$$h_\xi=D'_\xi-D_\xi.$$

Then h_ξ is an $\mathscr{O}_X$-linear, hence h gives sections of $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{F}))$.

Proposition 2.7.1

$$d'(\alpha)=d(\alpha)+h\wedge\alpha,$$

where α is a local section of $\mathscr{A}lt^{\bullet}_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{F})$, and the wedge product on the RHS is the one induced by the $\mathscr{O}_X$-linear map

$$\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{F})\times\mathscr{F}\to\mathscr{F}$$

sending $(f,u)\mapsto f(u)$.

Note that h is a section of the sheaf

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{F}))=\mathscr{A}lt^{1}_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{F})).$$

Therefore, $h\wedge\alpha$ is a section of $\mathscr{A}lt^{p+1}_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{F})$, whenever α is a section of $\mathscr{A}lt^{p}_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{F})$.

Proof. Let ξ be a section of $\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X)$. Then, by definition of relative Lie derivative,

$$\begin{aligned}\theta'_\xi(\alpha)&=\theta_\xi(\alpha)+h_\xi\wedge\alpha\\&=d(\iota_\xi(\alpha))+\iota_\xi(d(\alpha))+\iota_\xi(h)\wedge\alpha.\end{aligned}$$

Since, h is of degree 1, we have

$$\iota_\xi(h\wedge\alpha)=\iota_\xi(h)\wedge\alpha-h\wedge\iota_\xi(\alpha)$$

.

Therefore,

$$\begin{aligned}\theta'_\xi(\alpha)&=d(\iota_\xi(\alpha))+\iota_\xi(d(\alpha))+\iota_\xi(h\wedge\alpha)+h\wedge\iota_\xi(\alpha)\\&=\tilde{d}(\iota_\xi(\alpha))+\iota_\xi(\tilde{d}(\alpha)),\end{aligned}$$

where $\tilde{d}(\beta)=d(\beta)+h\wedge\beta$, and β is a section of $\mathscr{A}lt^{\bullet}_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{F})$. But, by definition of covariant derivative, d' is the unique S-linear endomorphism on

$\mathscr{A}lt^{\bullet}_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X),\mathscr{F})$ of degree 1, such that $\theta'_\xi = d'\circ\iota_\xi + \iota_\xi\circ d'$. It follows that $d' = \tilde{d}$. This completes the prof.

Similarly, we have the formula which gives the relationship between the different curvature forms K and K', mentioned as follows.

Proposition 2.7.2 $K' = K + h\wedge h + d(h)$.

Proof. From the definition of the curvature form (2.9), for the local sections ξ and η of $\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X)$, we have

$$\begin{aligned}
K'(\xi,\eta) &= D'_\xi D'_\eta - D'_\eta D'_\xi - D'_{[\xi,\eta]} \\
&= (D_\xi + h_\xi)(D_\eta + h_\eta) - (D_\eta + h_\eta)(D_\xi + h_\xi) - D_{[\xi,\eta]} - h_{[\xi,\eta]} \\
&= K(\xi,\eta) + h_\xi h_\eta - h_\eta h_\xi + D_\xi h_\eta - D_\eta h_\xi - h_{[\xi,\eta]} \\
&= K(\xi,\eta) + h_\xi h_\eta - h_\eta h_\xi + dh(\xi,\eta) \\
&= K(\xi,\eta) + (h\wedge h)(\xi,\eta) + dh(\xi,\eta).
\end{aligned}$$

This completes the proof of the proposition.

Chapter 3

Relative differential operators and connection algebra

This chapter introduces the theory of relative differential operators and symbol of a relative differential operator. The chapter also describes the connection algebra of a morphism between ringed spaces. In section 3.1, we shall see the functorial property of the differential operators. In section 3.2, we will introduce the connection algebra $\mathscr{C}_S(X)$ of a morphism $\pi : X \to S$ of ringed spaces and show that the category of $\mathscr{C}_S(X)$-modules is equivalent to the category relative connections. In section 3.3, we introduce the symbol of a first order differential operator.

3.1 Relative differential operators

In this section, we recall the definition of S- differential operators and some of its properties.

Definition 3.1.1 Let $(\pi, \pi^\sharp) : (X, \mathscr{O}_X) \to (S, \mathscr{O}_S)$ be a morphism of ringed spaces, and $\mathscr{F}$, $\mathscr{G}$ be two $\mathscr{O}_X$ -modules. Let $k \geq 0$ be an integer. For the case $k = 0$, we define a relative differential operator or an S-differential operator of order 0 to be an $\mathscr{O}_X$-linear map from $\mathscr{F}$ to $\mathscr{G}$. A **relative differential operator** or an S-**differential operator** of order k is a morphism

$$P : \mathscr{F} \to \mathscr{G}$$

of sheaves of abelian groups such that

1. P is an S-linear morphism.

2. for every open subset U of X and for every $f \in \mathscr{O}_X(U)$, the bracket

$$[P|_U, f] : \mathscr{F}|_U \to \mathscr{G}|_U$$

   defined as

$$[P|_U, f]_V(s) = P_V(f|_V s) - f|_V P_V(s) \tag{3.1}$$

   an S-differential operator of order $k-1$, for every open subset V of U, and for all $s \in \mathscr{F}(V)$.

The sheaf $\mathscr{H}om_S(\mathscr{F}, \mathscr{G})$ of S-linear morphism form $\mathscr{F}$ to $\mathscr{G}$ (see Definition 2.1.1) has $\mathscr{O}_X$-bimodule structure defined as follows; For every local sections f of $\mathscr{O}_X$, and P of $\mathscr{H}om_S(\mathscr{F}, \mathscr{G})$, the S-linear morphisms fP and Pf are given by

$$fP(\alpha) = f(P(\alpha)),$$

and

$$(Pf)(\alpha) = P(f\alpha),$$

where α is a local section of $\mathscr{F}$. Unless otherwise mentioned, we will consider $\mathscr{H}om_S(\mathscr{F}, \mathscr{G})$ as a left $\mathscr{O}_X$- module.

Let $\mathrm{Diff}_S^k(\mathscr{F}, \mathscr{G})$ denote the set of all S-differential operator of order k. For every open subset U of X, the assignment

$$U \mapsto \mathrm{Diff}_S^k(\mathscr{F}|_U, \mathscr{G}|_U)$$

is the sheaf of S-differential operators of order k. This sheaf is denoted by

$$\mathscr{D}iff^k_S(\mathscr{F},\mathscr{G}) \subset \mathscr{H}om_S(\mathscr{F},\mathscr{G}),$$

which is an $\mathscr{O}_X$-subbimodule of $\mathscr{H}om_S(\mathscr{F},\mathscr{G})$. Unless otherwise mentioned, we will consider $\mathscr{D}iff^k_S(\mathscr{F},\mathscr{G})$ as a left $\mathscr{O}_X$- module. When we use left and right structures simultaneously, we will write $\mathscr{D}iff^k_S(\mathscr{F},\mathscr{G})^L$ and $\mathscr{D}iff^k_S(\mathscr{F},\mathscr{G})^R$ for left and right $\mathscr{O}_X$-module structures, respectively.

Given an S-linear morphism

$$P : \mathscr{F} \to \mathscr{G}.$$

For every open subset $U \subset X$ and for every $f \in \mathscr{O}_X(U)$, we define

$$\Delta_f(P|_U) = [P|_U, f]. \tag{3.2}$$

Thus, we get an S-linear morphism

$$\Delta_f : \mathscr{H}om_S(\mathscr{F},\mathscr{G})(U) \to \mathscr{H}om_S(\mathscr{F},\mathscr{G})(U),$$

defined in (3.2).

We avoid cumbersome notation and just write $\Delta_f(P)$ for a local section f of $\mathscr{O}_X$.

Now, for local sections $f_0,\dots,f_k$ of $\mathscr{O}_X$, define

$$\Delta_{f_0,\dots,f_k} = \Delta_{f_k} \circ \cdots \circ \Delta_{f_0} \tag{3.3}$$

Lemma 3.1.2 *Let f and g be two local sections of $\mathscr{O}_X$. Then for any S-differential operator P from $\mathscr{F}$ to $\mathscr{G}$, we have*

1. $\Delta_{f,g} = \Delta_{g,f}$

2. $\Delta_{fg}(P) = \Delta_f(P)g + f\Delta_g(P)$

Proof.

1. For an S-differential operator P, we have by definition

$$\begin{aligned}\Delta_{f,g}(P) &= [[P,f],g]\\ &= [Pf - fP, g]\\ &= (Pf - fP)(g) - g(Pf - fP)\\ &= P(fg) - fP(g) - gP(f) + gfP\\ &= (Pg - gP)(f) - f(Pg - gP)\\ &= [[P,g],f]\\ &= \Delta_{g,f}(P)\end{aligned}$$

2. By definition, we have $\Delta_{fg}(P) = [P, fg]$. Now,

$$\begin{aligned}\Delta_{fg}(P) &= [P, fg]\\ &= P(fg) - fgP\\ &= P(fg) - fPg + fPg - fgP\\ &= (Pf - fP)(g) + f(Pg - gP)\\ &= \Delta_f(P)g + f\Delta_g(P)\end{aligned}$$

Proposition 3.1.3 *$P : \mathscr{F} \to \mathscr{G}$ is an S-differential operator of order $\leq k$ if and only if $\Delta_{f_0,\ldots,f_k}(P) = 0$ for every local sections $f_0,\ldots,f_k$ of $\mathscr{O}_X$.*

Proof. It is an easy verification that follows from the above definition.

We have following chain of inclusions,

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}) = \mathscr{D}iff_S^0(\mathscr{F},\mathscr{G}) \subset \ldots \subset \mathscr{D}iff_S^k(\mathscr{F},\mathscr{G})$$

$$\subset \mathscr{D}iff_S^{k+1}(\mathscr{F},\mathscr{G}) \subset \ldots \subset \mathscr{H}om_S(\mathscr{F},\mathscr{G})$$

Define

$$\mathscr{D}iff_S^{\infty}(\mathscr{F},\mathscr{G}) = \cup_{k\geq 0}\mathscr{D}iff_S^k(\mathscr{F},\mathscr{G}).$$

If $\mathscr{F} = \mathscr{G} = \mathscr{O}_X$, then $\mathscr{D}iff_S^{\infty}(\mathscr{F},\mathscr{G})$ is denoted by $\mathscr{D}_{X/S}$ and is called the sheaf of rings of differential operators, that we shall see as an standard example of sheaf of rings of differential operators in Chapter 5, subsection 5.2.1. Furthermore, if the base $S = \mathrm{Spec}(k)$, then we denote it by $\mathscr{D}_{X/k}$ or simply by $\mathscr{D}_X$.

Now we will explore functorial property of S-differential operators. Let us fix an $\mathscr{O}_X$-module $\mathscr{F}$. Let

$$P : \mathscr{F} \to \mathscr{G}$$

be an S-differential operator of order $\leq k$ and let

$$\alpha : \mathscr{G} \to \mathscr{G}'$$

be an $\mathscr{O}_X$-module homomorphism. Then the composition

$$\alpha \circ P : \mathscr{F} \to \mathscr{G}'$$

is also an S-differential operator of order $\leq k$, and we get a $\mathscr{O}_X$-bimodule homomorphism

$$\mathscr{D}iff_S^k(\mathscr{F},\alpha) : \mathscr{D}iff_S^k(\mathscr{F},\mathscr{G}) \to \mathscr{D}iff_S^k(\mathscr{F},\mathscr{G}')$$

We denote the category of $\mathscr{O}_X$-modules by $\mathbf{MOD}_X$ and category of $\mathscr{O}_X$-bimodules by $\mathbf{BiMOD}_X$. Thus, for a fixed $\mathscr{O}_X$-module $\mathscr{F}$, we have a covariant functor

$$\mathscr{D}iff_S^k(\mathscr{F},*) : \mathbf{MOD}_X \to \mathbf{BiMOD}_X$$

Similarly, fix an $\mathscr{O}_X$-module $\mathscr{G}$. Let

$$P : \mathscr{F} \to \mathscr{G}$$

be a differential operator of order $\leq k$ and let

$$\beta : \mathscr{F}' \to \mathscr{F}$$

be an $\mathscr{O}_X$-module homomorphism. Then the composition

$$P \circ \beta : \mathscr{F}' \to \mathscr{G}$$

is an S-differential operator of order $\leq k$, and we get an $\mathscr{O}_X$-bimodule homomorphism

$$\mathscr{D}iff_S^k(\beta, \mathscr{G}) : \mathscr{D}iff_S^k(\mathscr{F}, \mathscr{G}) \to \mathscr{D}iff_S^k(\mathscr{F}', \mathscr{G}).$$

Thus, for a fixed $\mathscr{O}_X$-module $\mathscr{G}$, we have a contravariant functor

$$\mathscr{D}iff_S^k(*, \mathscr{G}) : \mathbf{MOD}_X \to \mathbf{BiMOD}_X$$

Now consider the $\mathscr{O}_X$-modules $\mathscr{D}iff_S^k(\mathscr{F}, \mathscr{G})^L$ and $\mathscr{D}iff_S^k(\mathscr{F}, \mathscr{G})^R$ with their respective left and right module structures. We have an isomorphism

$$\mathscr{D}iff_S^k(\mathscr{F}, \mathscr{G})^L \cong \mathscr{D}iff_S^k(\mathscr{F}, \mathscr{G})^R$$

of sheaves of abelian groups. Define the identity map

$$\mathbf{1}_{LR} : \mathscr{D}iff_S^k(\mathscr{F}, \mathscr{G})^L \to \mathscr{D}iff_S^k(\mathscr{F}, \mathscr{G})^R$$

of sheaves of abelian groups. Then $\mathbf{1}_{LR}$ is an S-differential operator of order $\leq k$. Let us verify it for the case $k = 1$. $\mathbf{1}_{LR}$ is an S-linear map. From Proposition 3.1.3, it is enough to check that $\Delta_{f_0,f_1}(\mathbf{1}_{LR}) = 0$, for every local sections f_0, f_1 of $\mathscr{O}_X$, and that is an easy verification. Note that $\mathbf{1}_{LR}$ is not an $\mathscr{O}_X$-linear map.

Similarly, we have another identity map

$$\mathbf{1}_{RL} : \mathscr{D}iff^k_S(\mathscr{F},\mathscr{G})^R \to \mathscr{D}iff^k_S(\mathscr{F},\mathscr{G})^L$$

of sheaves of abelian groups, which is an S-differential operator of order $\leq k$.

3.2 The connection algebra

Let $\pi : X \to S$ be a morphism of ringed spaces. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module. Let $U \subset X$ be an open set and $f \in \mathscr{O}_X(U)$, we define the homothety

$$\mu^{\mathscr{F}}_f : \mathscr{F}|_U \to \mathscr{F}|_U,$$

as follows: for every open subset $V \subset U$, the map

$$\mu^{\mathscr{F}}_f|_V : \mathscr{F}(V) \to \mathscr{F}(V), \tag{3.4}$$

is given by

$$\mu^{\mathscr{F}}_f|_V(s) = f|_V.s, \tag{3.5}$$

for every $s \in \mathscr{F}(V)$.

Let $\mathscr{F}$ and $\mathscr{G}$ be $\mathscr{O}_X$-modules. We have already seen that the set $\mathscr{H}om_S(\mathscr{F},\mathscr{G})$ has natural structure of $(\mathscr{O}_X,\mathscr{O}_X)$-bimodule. We can reinterpret this $(\mathscr{O}_X,\mathscr{O}_X)$-bimodule structure in terms of composition with homothety map (3.4), namely for every local section f of $\mathscr{O}_X$ we define

$$fP = \mu^{\mathscr{G}}_f \circ P,$$

$$Pf = P \circ \mu_f^{\mathscr{F}},$$

for all $P \in \mathscr{H}om_S(\mathscr{F}, \mathscr{G})(U)$.

Moreover, the bracket defined in (3.1) and (3.2) can be described as

$$[P, f] = P \circ \mu_f^{\mathscr{F}} - \mu_f^{\mathscr{G}} \circ P, \tag{3.6}$$

where P and f are as above.

Whenever $\mathscr{F}$ is the structure sheaf $\mathscr{O}_X$, we denote $\mu_f^{\mathscr{O}_X}$ by μ_f for every local section f of $\mathscr{O}_X$.

Proposition 3.2.1 *Let $P : \mathscr{O}_X \to \mathscr{O}_X$ be a first order S-differential operator if and only if for every open subset $U \subset X$, $P|_U - \mu_{P_U(1_U)}$ is an S-derivation, where $1_U \in \mathscr{O}_X(U)$. In particular,*

$$\mathscr{D}iff_S^1(\mathscr{O}_X, \mathscr{O}_X) \cong \mathscr{O}_X \oplus \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X) \tag{3.7}$$

Proof. Let U be an open subset of X, and $\xi = P|_U - \mu_{P_U(1_U)}$. Let $f \in \mathscr{O}_X(U)$. The map

$$[P|_U, \mu_f] : \mathscr{O}_X|_U \to \mathscr{O}_X|_U$$

is $\mathscr{O}_U$-linear if and only if

$$[P|_U, \mu_f]_V(g) = g[P|_U, \mu_f]_V(1_V),$$

for every open subset $V \subset U$, $g \in \mathscr{O}_X(V)$ and $1_V \in \mathscr{O}_X(V)$. For f and g as above, we also have

$$[P|_U, \mu_f](g) = \xi(fg) - f\xi(g),$$

and

$$g[P|_U, \mu_f](1_V) = g\xi(f).$$

Thus, ξ is an S-derivation.

Let $\mathscr{T}$ denote the tensor algebra of the $(\mathscr{O}_X,\mathscr{O}_X)$- bimodule $\mathscr{D}iff^1_S(\mathscr{O}_X,\mathscr{O}_X)$. As a sheaf of abelian groups,

$$\mathscr{T} = \bigoplus_{p\geq 0} \mathscr{T}^p,$$

where $\mathscr{T}^0 = \mathscr{O}_X$ and $\mathscr{T}^p = \mathscr{D}iff^1_S(\mathscr{O}_X,\mathscr{O}_X) \otimes_{\mathscr{O}_X} \mathscr{T}^{p-1}$

For each integer $p \geq 0$, let

$$\iota_p : \mathscr{T}^p \to \mathscr{T}$$

denote the canonical injection.

The pair $(X,\mathscr{T})$ is a ringed space such that the map

$$\iota_0 : \mathscr{T}^0 = \mathscr{O}_X \to \mathscr{T} \tag{3.8}$$

is a morphism between sheaves of rings.

Universal property of the tensor algebra $\mathscr{T}$:

For every ringed space $(X,\mathscr{B})$, morphism of sheaves of rings

$$\phi : \mathscr{O}_X \to \mathscr{B},$$

and $(\mathscr{O}_X,\mathscr{O}_X)$-bimodule homomorphism

$$f : \mathscr{D}iff^1_S(\mathscr{O}_X,\mathscr{O}_X) \to \mathscr{B},$$

where $\mathscr{B}$ is considered an $(\mathscr{O}_X,\mathscr{O}_X)$-bimodule through ϕ, there is a unique morphism of sheaves of rings

$$g : \mathscr{T} \to \mathscr{B}$$

such that

$$g|_{\mathscr{O}_X} = \phi,$$

and

$$g|_{\mathscr{D}iff^1_S(\mathscr{O}_X,\mathscr{O}_X)} = f.$$

Definition 3.2.2 Let $\mathscr{J}$ denote the two sided ideal of $\mathscr{T}$ described as follows. For every open subset $U \subset X$, let $\mathscr{J}(U)$ denote the two sided ideal of in $\mathscr{T}(U)$ generated by the elements $1 - \mathbf{1}_{\mathscr{O}_X(U)}$ where $1 \in \mathscr{O}_X(U) = \mathscr{T}^0(U)$, and

$$\mathbf{1}_{\mathscr{O}_X(U)} : \mathscr{O}_X(U) \to \mathscr{O}_X(U)$$

is the identity map. The sheaf

$$\mathscr{C}_S(X) = \mathscr{T}/\mathscr{J}$$

of rings is called the **connection algebra** of the ringed space $(X, \mathscr{O}_X)$ over $(S, \mathscr{O}_S)$.

Let

$$\vartheta : \mathscr{T} \to \mathscr{C}_S(X)$$

be the canonical projection. For each $p \in \mathbf{N}$, let

$$\vartheta_p = \vartheta \circ \iota_p : \mathscr{T}^p \to \mathscr{C}_S(X). \tag{3.9}$$

In particular, $\vartheta_0 : \mathscr{T}^0 = \mathscr{O}_X \to \mathscr{C}_S(X)$ is a ring homomorphism. Let $\mathscr{C}_S(X) - \mathbf{Mod}$ denote the category of $\mathscr{C}_S(X)$-modules. We will construct an isomorphism of categories from $\mathscr{C}_S(X) - \mathbf{Mod}$ to $\mathbf{MC}(X/S)$[see Chapter 2; Definition 2.1.2 (3)].

Let $\mathscr{H}$ be a $\mathscr{C}_S(X)$-module. Consider $\mathscr{H}$ as an $\mathscr{O}_X$-module through the morphism $\vartheta_0 : \mathscr{T}^0 = \mathscr{O}_X \to \mathscr{C}_S(X)$.

For every local section ξ of $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$, define a map

$$D^{\mathscr{H}}_{\xi} : \mathscr{H} \to \mathscr{H} \tag{3.10}$$

by

$$D^{\mathscr{H}}_{\xi}(u) = \vartheta(\xi)u, \tag{3.11}$$

where u is a local section of $\mathscr{H}$.

Lemma 3.2.3 *The map $D^{\mathscr{H}}_{\xi}$ is an S-linear for every local section ξ of $\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X)$. Moreover, the map*

$$D^{\mathscr{H}} := \xi \mapsto D^{\mathscr{H}}_{\xi} : \mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X) \to \mathscr{E}nd_S(\mathscr{H})$$

is an S-connection on the $\mathscr{O}_X$-module $\mathscr{H}$, for every local section ξ of $\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X)$.

Proof. Let $D = D^{\mathscr{H}}$. Then

$$D_{\xi+\eta}(u) = D_{\xi}(u) + D_{\eta}(u),$$

and

$$D_{\xi}(u+v) = D_{\xi}(u) + D_{\xi}(v),$$

for every local setion ξ,η of $\mathscr{D}er_S(\mathscr{O}_X,\mathscr{O}_X)$, and u,v of $\mathscr{H}$. To complete the proof, we want to show that

$$D_{f\xi}(u) = fD_{\xi}(u),$$

and

$$D_{\xi}(fu) = \xi(f)u + fD_{\xi}(u).$$

From the definition of $D^{\mathscr{H}}$ in (3.11), we have

$$D_{f\xi}(u) = \vartheta(f\xi)u = \vartheta(f)\vartheta(\xi)(u) = f\vartheta(\xi)(u) = fD_{\xi}(u),$$

and

$$\begin{aligned}
D_\xi(fu) &= \vartheta(\xi)\vartheta(f)u \\
&= \vartheta(\xi f)u \\
&= \vartheta(\xi\circ\mu_f)(u) \\
&= \vartheta(\xi(f)\mathbf{1}_{\mathscr{O}_U}+f\xi)u \\
&= \vartheta(\xi(f))\vartheta(\mathbf{1}_{\mathscr{O}_U})u+\vartheta(f)\vartheta(\xi)(u) \\
&= \xi(f)u+fD_\xi(u).
\end{aligned}$$

Lemma 3.2.4 *Let $\mathscr{E}$ and $\mathscr{F}$ be two $\mathscr{C}_S(X)$-modules. Then a morphism*

$$\Psi:\mathscr{E}\to\mathscr{F}$$

is a $\mathscr{C}_S(X)$-module homomorphism if and only if it is a morphism of $\mathscr{O}_X$-modules with S-connections from $(\mathscr{E},D^{\mathscr{E}})$ to $(\mathscr{F},D^{\mathscr{F}})$.

Proof. Let U be an open subset of X. Then the ring $\mathscr{T}(U)$ is generated by $\mathscr{O}_X(U)\cup\mathscr{D}iff^1_S(\mathscr{O}_X,\mathscr{O}_X)(U)$. From Proposition 3.2.1, every element of $\mathscr{D}iff^1_S(\mathscr{O}_X,\mathscr{O}_X)(U)$ is of the form $\mu_f+\xi$, where f and ξ are the local sections of $\mathscr{O}_U$ and $\mathscr{D}er_S(\mathscr{O}_U,\mathscr{O}_U)$ respectively. Hence, the ring $\mathscr{T}(U)$ is generated by $\mathscr{O}_X(U)\cup\{\mathbf{1}_{\mathscr{O}_X(U)}\}\cup\mathscr{D}er_S(\mathscr{O}_U,\mathscr{O}_U)$. Therefore, the ring $\mathscr{C}_S(X)(U)$ is generated by $\vartheta(\mathscr{O}_X(U))\cup\vartheta(\mathscr{D}er_S(\mathscr{O}_U,\mathscr{O}_U))$. Thus, a morphism of sheaves of abelian groups $\Psi:\mathscr{E}\to\mathscr{F}$ is $\mathscr{C}_S(X)$-linear if and only if $\Psi(\vartheta(f)u)=\vartheta(f)\Psi(u)$ and $\Psi(\vartheta(\xi)u)=\vartheta(\xi)\Psi(u)$, where f and ξ are as above. These conditions are equivalent to $\Psi(fu)=f\Psi(u)$ and $\Psi(D^{\mathscr{E}}_\xi(U))=D^{\mathscr{F}}_\xi(\Psi(u))$. Thus, Ψ is $\mathscr{C}_S(X)$-linear if and only if it is a morphism of $\mathscr{O}_X$-modules with S-connections.

Thus, we have a functor

$$F:\mathscr{C}_S(X)-\mathbf{Mod}\longrightarrow\mathbf{MC}(X/S) \tag{3.12}$$

which sends $\mathscr{E} \mapsto (\mathscr{E}, D^{\mathscr{E}})$, follows from Lemma 3.2.3, and gives a bijection on morphisms, that is,

$$F : \mathrm{Hom}_{\mathscr{C}_S(X)-\mathbf{Mod}}(\mathscr{E}, \mathscr{F}) \to \mathrm{Hom}_{\mathbf{MC}(X/S)}((\mathscr{E}, D^{\mathscr{E}}), (\mathscr{F}, D^{\mathscr{F}})) \tag{3.13}$$

which sends $\Psi \mapsto \Psi$, is a bijection, follows from Lemma 3.2.4.

Theorem 3.2.5 *The functor* $F : \mathscr{C}_S(X) - \mathbf{Mod} \longrightarrow \mathbf{MC}(X/S)$ *is an isomorphism of categories.*

Proof. From Lemma 3.2.4, F is fully faithful. To show that the object function of F is a bijection. Suppose that $\mathscr{E}$ and $\mathscr{F}$ are two $\mathscr{C}_S(X)$-modules, and $(\mathscr{E}, D^{\mathscr{E}}) = (\mathscr{F}, D^{\mathscr{F}})$. Then, the sheaves $\mathscr{E}$ and $\mathscr{F}$ are equal, $D^{\mathscr{E}} = D^{\mathscr{F}}$, and the identity map $\mathbf{1}_{\mathscr{E}} : \mathscr{E} \to \mathscr{F}$ is a morphism of $\mathscr{O}_X$-modules with S-connections from $(\mathscr{E}, D^{\mathscr{E}})$ to $(\mathscr{F}, D^{\mathscr{F}})$. By Lemma 3.2.4, $\mathbf{1}_{\mathscr{E}} : \mathscr{E} \to \mathscr{F}$ is a $\mathscr{C}_S(X)$-module morphism. Therefore, the $\mathscr{C}_S(X)$-modules $\mathscr{E}$ and $\mathscr{F}$ are equal. Thus, the object function F is injective.

Let $(\mathscr{E}, D)$ be an $\mathscr{O}_X$-module with connections. Then the map

$$\psi : \mathscr{O}_X \to \mathscr{E}nd_{\mathbf{Z}}(\mathscr{E}) \tag{3.14}$$

defined by $f \mapsto \mu_f$ is a morphism of sheaves of rings, where f is a local section of $\mathscr{O}_X$. Define

$$\phi : \mathscr{D}iff^1_S(\mathscr{O}_X, \mathscr{O}_X) \to \mathscr{E}nd_{\mathbf{Z}}(\mathscr{E}), \tag{3.15}$$

by $\phi(\mu_f + \xi) = \mu_f + D_\xi$,where f and ξ are local sections of $\mathscr{O}_X$ and $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$, respectively. Then

$$\phi(P+Q) = \phi(P) + \phi(Q),$$

where P and Q are first order S-differential operators on $\mathscr{O}_X$. If g is a local section of $\mathscr{O}_X$,

$$\begin{aligned}\phi(g(\mu_f+\xi))u &= \phi(\mu_{gf}+g\xi)u\\ &= gfu+D_{g\xi}(u)\\ &= gfu+gD_{\xi}(u)\\ &= g\phi(\mu_f+\xi)(u).\end{aligned}$$

Therefore, ϕ is a left $\mathscr{O}_X$-module homomorphism. Moreover,

$$\begin{aligned}\phi((\mu_f+\xi)g)u &= \phi(\mu_{fg+\xi(g)}+g\xi)(u)\\ &= fgu+\xi(g)u+gD_{\xi}(u)\\ &= fgu+D_{\xi}(gu)\\ &= \phi(\mu_f+\xi)(gu).\end{aligned}$$

Therefore, ϕ is a right $\mathscr{O}_X$-module homomorphism. Thus, ϕ is an $(\mathscr{O}_X,\mathscr{O}_X)$-bimodule homomorphism. By universal property of $\mathscr{T}$, there exits a unique morphism

$$\nu:\mathscr{T}\to\mathscr{E}nd_{\mathbf{Z}}(\mathscr{E})$$

of sheaves of rings such that

$$\nu|_{\mathscr{O}_X}=\psi$$

and

$$\nu|_{\mathscr{D}iff^1_S(\mathscr{O}_X,\mathscr{O}_X)}=\phi.$$

Take a local section u of $\mathscr{E}$, then

$$\nu(1-\mathbf{1}_{\mathscr{O}_X})(u)=\nu(1)u-\nu(\mathbf{1}_{\mathscr{O}_X})(u)=u-\phi(\mathbf{1}_{\mathscr{O}_X})u=u-\phi(\mu_1)(u)=u-u=0.$$

Therefore, there exists a unique morphism

$$\chi : \mathscr{C}_S(X) \to \mathscr{E}nd_{\mathbf{Z}}(\mathscr{E})$$

such that

$$\chi \circ \vartheta = \nu.$$

The scalar multiplication law

$$\mathscr{C}_S(X) \times \mathscr{E} \to \mathscr{E} \tag{3.16}$$

$(\gamma, u) \mapsto \chi(\gamma)(u)$ defines a $\mathscr{C}_S(X)$-module structure on $\mathscr{E}$. Now, it is easy to check that the $\mathscr{C}_S(X)$-module $\mathscr{E}$ maps to $(\mathscr{E}, D)$ under F. Thus, F is surjective. This completes the proof.

3.3 Symbol of a first order relative differential operator

Given a first order S-differential operator $P : \mathscr{F} \longrightarrow \mathscr{G}$, define a morphism of abelian sheaves

$$\theta : \mathscr{O}_X \longrightarrow \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$$

by

$$\theta_U(f) = [P|_U, f]$$

for every open subset $U \subset X$ and $f \in \mathscr{O}_X(U)$. Then θ is an S-derivation. For every $W \subset S$, $V \subset \pi^{-1}(W)$, $s \in \mathscr{O}_S(W)$, $t \in \mathscr{O}_X(V)$, and $u \in \mathscr{F}(V)$, we have, by S-linearity of

P,

$$\begin{aligned}\theta_V(s|_V t)(u) &= [P|_V, s|_V t](u)\\ &= P(s|_V t)(u) - s|_V tP(u)\\ &= (s|_V)P(t)u - s|_V tP(u)\\ &= (s|_V)[P|_V, t](u)\,.\end{aligned}$$

Thus θ is an S-linear morphism.

Next, to verify θ satisfies Leibniz rule, for every $f, g \in \mathscr{O}_X(U)$, where $U \subset X$ is any open subset, and for every $u \in \mathscr{F}(U)$, we have

$$[P|_U, f](gu) = P(fgu) - fP(gu)$$

which gives

$$P(fgu) = g[P, f](u) + fP(gu).$$

Now,

$$\begin{aligned}\theta(fg)(u) &= [P, fg](u)\\ &= P(fgu) - fgP(u)\\ &= g[P, f](u) + fP(gu) - fgP(u)\\ &= g\theta(f)(u) + f\theta(g)(u)\\ &= (g\theta(f) + f\theta(g))(u).\end{aligned}$$

Thus, $\theta(fg) = \theta(f)g + f\theta(g)$.

Hence, we have the following:

Proposition 3.3.1 *Let $\mathscr{F}$ and $\mathscr{G}$ be $\mathscr{O}_X$-modules and*

$$P : \mathscr{F} \longrightarrow \mathscr{G}$$

a first order S-differential operator. Then there exists a unique S-derivation

$$\theta : \mathscr{O}_X \longrightarrow \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})$$

such that

$$\theta_U(f) = [P|_U, f]$$

for every open subset U of X and for every $f \in \mathscr{O}_X(U)$.

The S-derivation θ defined above is called the **symbol of** P and it will be denoted by $\sigma_1(P)$.

Every $\mathscr{O}_X$-module homomorphism is a first order S-differential operator. Therefore, $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})$ is an $\mathscr{O}_X$-submodule of $\mathscr{D}iff^1_S(\mathscr{F},\mathscr{G})$. Let

$$\iota : \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}) \longrightarrow \mathscr{D}iff^1_S(\mathscr{F},\mathscr{G})$$

be the inclusion morphism. Thus, we have an exact sequence of $\mathscr{O}_X$-modules

$$0 \longrightarrow \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}) \xrightarrow{\iota} \mathscr{D}iff^1_S(\mathscr{F},\mathscr{G}) \xrightarrow{\sigma_1} \mathscr{D}er_S(\mathscr{O}_X, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}))\,. \quad (3.17)$$

Chapter 4

Relative Chern classes and existence of relative holomorphic connections

This chapter deals with the theory of relative holomorphic connections in a holomorphic vector bundle over a family of compact connected complex manifolds. We define the relative Chern classes of a complex vector bundle and describe their properties. In subsection 4.1.1, we define the notion of complex analytic family of compact complex manifolds and describe relative holomorphic tangent and cotangent bundles. For more details, we refer excellent texts [KS58], [KS86] and [Voi04] on these topics. In subsection 4.1.3, we establish symbol exact sequence and define the Atiyah algebra of a holomorphic vector bundle over a complex analytic family.

In subsection 4.2.1, we define the smooth relative forms on the differentiable family of smooth manifolds and describe the sheaf of relative de Rham cohomology. In subsection 4.2.2, we define the notion of relative Chern classes of a complex vector bundle over a complex analytic family, and show its functorial property, that the total relative Chern class is compatible with the pullback functor. We show one of the main results of this chapter in Theorem 4.2.8, which states that if the fibres of complex analytic family are compact and Kähler and a holomorphic vector bundle over this complex analytic family admits a relative holomorphic connection, then all the relative Chern classes vanish. In the last section 4.3, we give a sufficient condition for the existence of a relative holomorphic connections on a holomorphic vector bundle over a complex analytic family.

4.1 Analytic theory of families of complex manifolds

4.1.1 Complex analytic families

Definition 4.1.1 ([KS58],[KS86]) Let $(S, \mathscr{O}_S)$ be a complex manifold of dimension n. For each $t \in S$, let there be given a compact connected complex manifold X_t of fixed dimension l. The set $\{X_t \mid t \in S\}$ of compact connected complex manifolds is called a *complex analytic family of compact connected complex manifolds* , if there is a complex manifold $(X, \mathscr{O}_X)$ and a surjective holomorphic map $\pi : X \longrightarrow S$ of complex manifolds with connected fibers such that the followings hold:

1. $\pi^{-1}(t) = X_t$ for all $t \in S$,
2. $\pi^{-1}(t)$ is a compact connected complex submanifold of X for all $t \in S$, and
3. the rank of the Jacobian matrix of π is equal to n at each point of X.

In other words, $\pi : X \longrightarrow S$ is a surjective holomorphic proper submersion, such that $\pi^{-1}(t) = X_t$ is connected for every $t \in S$ (see also [Voi04]).

4.1.2 Relative holomorphic tangent and cotangent bundles

Let $\pi : X \longrightarrow S$ be a surjective holomorphic submersion of complex manifolds with connected fibers such that $\dim(X) = m$ and $\dim(S) = n$. For any $t \in t$, the fiber $\pi^{-1}(t)$ will be denoted by X_t. Let

$$d\pi_S : TX \longrightarrow \pi^* TS$$

be the differential of π. The subbundle

$$T(X/S) := \mathrm{Ker}(d\pi_S) \subset TX$$

is called the relative tangent bundle for π. Thus we have a short exact sequence

$$0 \longrightarrow \mathscr{T}_{X/S} \stackrel{\iota}{\longrightarrow} \mathscr{T}_X \stackrel{d\pi_S}{\longrightarrow} \pi^*\mathscr{T}_S \longrightarrow 0 \tag{4.1}$$

of $\mathscr{O}_X$-modules and $\mathscr{O}_X$-linear maps.

The dual $T(X/S)^*$ of the relative tangent bundle is called the relative cotangent bundle and it is denoted by $\Omega^1(X/S)$. The sheaf of holomorphic sections of relative cotangent bundle $\Omega^1(X/S)$ will also be denoted by $\Omega^1_{X/S}$. Dualising the short exact sequence in (4.1), we get a short exact sequence

$$0 \longrightarrow \pi^*\Omega^1_S \stackrel{d\pi_S^*}{\longrightarrow} \Omega^1_X \stackrel{\iota^*}{\longrightarrow} \Omega^1_{X/S} \longrightarrow 0\,. \tag{4.2}$$

The relative tangent sheaf $\mathscr{T}_{X/S}$ and the relative cotangent sheaf $\Omega^1_{X/S}$ are locally free $\mathscr{O}_X$-modules of rank $l = m - n$.

From Proposition 2.3.2, there exists a unique S-derivation

$$d_{X/S} : \mathscr{O}_X \longrightarrow \Omega^1_{X/S}.$$

For any integer $r \geq 1$, define $\Omega^r_{X/S} = \Lambda^r\Omega^1_{X/S}$, which is called the sheaf of holomorphic relative r-forms on X over S. We have the short exact sequence

$$0 \longrightarrow \pi^*\Omega^1_S \otimes \Omega^{r-1}_X \longrightarrow \Omega^r_X \longrightarrow \Omega^r_{X/S} \longrightarrow 0 \tag{4.3}$$

which is derived from the short exact sequence in (4.2).

Theorem 4.1.2 *There exists canonical $\pi^{-1}\mathscr{O}_S$-linear maps $\partial^r_{X/S} : \Omega^r_{X/S} \longrightarrow \Omega^{r+1}_{X/S}$, called the relative exterior derivative, satisfying the following:*

1. $\partial^0_{X/S} = d_{X/S} : \mathscr{O}_X \longrightarrow \Omega^1_{X/S}$,

2. $\partial^{r+1}_{X/S} \circ \partial^r_{X/S} = 0$, *and*

3. $\partial^{r+s}_{X/S}(\alpha\wedge\beta) = \partial^r_{X/S}\alpha\wedge\beta + (-1)^r\alpha\wedge\partial^s_{X/S}\beta$, *for all local sections* α *of* $\Omega^r_{X/S}$ *and* β *of* $\Omega^s_{X/S}$.

Proof. This basically follows from Proposition 2.3.2. First note that the sheaf

$$\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)$$

is canonically isomorphic to the holomorphic relative tangent sheaf $\mathscr{T}_{X/S}$ as an $\mathscr{O}_X$-module, and hence

$$\mathscr{A}lt^r_{\mathscr{O}_X}(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X), \mathscr{O}_X)$$

is canonically isomorphic to the sheaf $\Omega^r_{X/S}$ as an $\mathscr{O}_X$-module. In view of the canonical S-connection in $\mathscr{O}_X$, by Proposition 2.3.2, canonical S-linear map exists satisfying (1), and by Proposition 2.3.6, it satisfies (3). Finally, (2) follows from Corollary 2.3.5(2).

4.1.3 Relative Atiyah algebra

Proposition 4.1.3 (Symbol exact sequence) *Let* $\pi : X \longrightarrow S$ *be a holomorphic proper submersion of complex manifolds with connected fibers, and let* $\mathscr{F}$ *and* $\mathscr{G}$ *be two locally free* $\mathscr{O}_X$*-modules of rank* r *and* p *respectively. Then*

$$0 \longrightarrow \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}) \stackrel{\iota}{\longrightarrow} \mathscr{D}iff^1_S(\mathscr{F},\mathscr{G}) \stackrel{\sigma_1}{\longrightarrow} \mathscr{D}er_S(\mathscr{O}_X, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})) \longrightarrow 0$$

is an exact sequence of $\mathscr{O}_X$*-modules.*

Proof. It is enough to show that σ_1 is surjective. Let $\theta \in (\mathscr{D}er_S(\mathscr{O}_X, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})))_x$ with $x \in X$. We have to show that there exists a first order S-differential operator P defined near x, such that $(\sigma_1)_x(P_x) = \theta$, where

$$P_x \in (\mathscr{D}iff^1_S(\mathscr{F},\mathscr{G}))_x$$

is the germ of P at x. Let $(U, \phi = (z_1, \cdots, z_l, z_{l+1}, \cdots, z_{l+n}))$ be a holomorphic chart on X around x, and let $s = (s_1, \cdots, s_r)$ and $t = (t_1, \cdots, t_p)$ be holomorphic frames of $\mathscr{F}$ and $\mathscr{G}$ respectively, on U. We may assume that θ is the germ at x of a section u of $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}))$ over U. Since

$$\mathscr{H}om_{\mathscr{O}_X}(\Omega^1_{X/S}, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})) \cong \mathscr{D}er_S(\mathscr{O}_X, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})),$$

u can be considered as a section of $\mathscr{H}om_{\mathscr{O}_X}(\Omega^1_{X/S}, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}))$ over U, that is,

$$u : \Omega^1_{X/S}|_U \longrightarrow \mathscr{H}om_{\mathscr{O}_U}(\mathscr{F}|_U, \mathscr{G}|_U)$$

is an $\mathscr{O}_U$-module homomorphism. As $\{dz_\alpha \mid 1 \leq \alpha \leq l\}$ is an $\mathscr{O}_U$-basis of $\Omega^1_{X/S}|_U$, there exists a uniquely determined function $b^\alpha_{ij} \in \mathscr{O}_X(U)$, where $1 \leq i \leq p$, $1 \leq j \leq r$ and $1 \leq \alpha \leq l$, such that

$$u(dz_\alpha)(s_j) = \sum_{i=1}^{p} b^\alpha_{ij} t_i.$$

Define $P : \mathscr{F}|_U \longrightarrow \mathscr{G}|_U$ by

$$P(\sum_{j=1}^{r}(f_j s_j) = \sum_{i,j,\alpha} b^\alpha_{ij} \frac{\partial f_j}{\partial z_\alpha} t_i.$$

Then P is S-linear, because for any $V \subset \pi^{-1}(W) \bigcap U$, where $W \subset S$ is any open subset, and any $g \in \mathscr{O}_S(W)$, we have

$$\frac{\partial g \circ \pi}{\partial z_\alpha} = d\pi_x(\frac{\partial}{\partial z_\alpha})(g) = 0$$

for all $\alpha = 1, \cdots, l$. The bracket operation $[P, f]$ is $\mathscr{O}_U$-linear. Thus P is a first order S-differential operator, that is, $P \in \mathscr{D}iff^1_S(\mathscr{F}, \mathscr{G})(U)$. Let $V \subset U$ and $\xi \in \Omega^1_{X/S}(V)$. Then

$$\xi = \sum_{\alpha=1}^{l} \xi_\alpha dz_\alpha,$$

where $\xi_\alpha \in \mathscr{O}_U(V)$. Then by the construction of the symbol map, and the universal property of $(\Omega^1_{X/S}, d_{X/S})$, we have

$$\sigma_1(P)_V(\xi) = \sum_\alpha \xi_\alpha \sigma_1(P)_V(dz_\alpha) = \sum_\alpha \xi_\alpha [P, z_\alpha]\,. \tag{4.4}$$

From the definition of P it follows that $P(s_j) = 0$ for all j, and hence

$$[P, z_\alpha](s_j) = P(z_\alpha s_j) = \sum_i b^\alpha_{ij} t_i = u_V(dz_\alpha)(s_j)\,.$$

Therefore, $[P, z_\alpha] = u(dz_\alpha)$, and hence (4.4) becomes

$$\sigma_1(P)_V(\xi) = \sum_\alpha \xi_\alpha u_V(dz_\alpha) = u_V(\xi)\,.$$

This proves that $\sigma_1(P)_U = u$, that is, $(\sigma_1)_x(P_x) = \theta$.

Let E be a locally free $\mathscr{O}_X$-module. By Proposition 4.1.3, we have a short exact sequence of $\mathscr{O}_X$-modules

$$0 \longrightarrow \mathscr{E}nd_{\mathscr{O}_X}(E) \xrightarrow{\ \iota\ } \mathscr{D}i\!f\!f^1_S(E,E) \xrightarrow{\ \sigma_1\ } \mathscr{D}er_S(\mathscr{O}_X, \mathscr{E}nd_{\mathscr{O}_X}(E)) \longrightarrow 0\,.$$

For any S-derivation $\xi : \mathscr{O}_X \longrightarrow \mathscr{O}_X$, let $\widetilde{\xi} : \mathscr{O}_X \longrightarrow \mathscr{E}nd_{\mathscr{O}_X}(E)$ be the map defined by $a \longmapsto \xi(a)\mathbf{1}_E$, where a is a local sections of $\mathscr{O}_X$. Then $\widetilde{\xi}$ is an S-derivation. Thus, we have an $\mathscr{O}_X$-module homomorphism

$$\Psi : \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X) \longrightarrow \mathscr{D}er_S(\mathscr{O}_X, \mathscr{E}nd_{\mathscr{O}_X}(E))$$

defined by $\xi \longmapsto \widetilde{\xi}$. Note that Ψ is an injective homomorphism.

Define

$$\mathscr{A}t_S(E) = \sigma_1^{-1}(\Psi(\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)))\,,$$

which is an $\mathscr{O}_X$-module and for every open subset U of X. Note that $\mathscr{A}t_S(E)(U)$ consists of first order S-differential operator $P \in \mathscr{D}i\!f\!f^1_S(E,E)(U)$ such that $(\sigma_1)_U(P) = \Psi(\xi)$

for some $\xi \in \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)(U)$, which is equivalent to the assertion that $\sigma_1(P)(a) = \xi(a)\mathbf{1}_E$ or $[P, a] = \xi(a)\mathbf{1}_E$, for all $a \in \mathscr{O}_X(U)$ and for some $\xi \in \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)(U)$.

Let $P, Q : E \to E$ be two S-linear morphisms. We define the bracket $[P, Q]$ by

$$[P, Q] = P \circ Q - Q \circ P. \tag{4.5}$$

Then $[P, Q] : E \to E$ is also an S-linear morphism, and $\mathscr{E}nd_S(E)$ is a sheaf of Lie algebras over S with respect to the Lie bracket (4.5). Now, $\mathscr{A}t_S(E)$ is a sheaf of Lie subalgebras of $\mathscr{E}nd_S(E)$ over S, and it is called the **relative Atiyah algebra** of E.

Moreover, we have a short exact sequence

$$0 \longrightarrow \mathscr{E}nd_{\mathscr{O}_X}(E) \xrightarrow{\iota} \mathscr{A}t_S(E) \xrightarrow{\sigma_1} \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X) \longrightarrow 0 \tag{4.6}$$

of $\mathscr{O}_X$-modules, which is called the **Atiyah sequence** .

Proposition 4.1.4 *Let $\pi : X \longrightarrow S$ be a holomorphic proper submersion of complex manifolds with connected fibers, and let E be a holomorphic vector bundle over X. Then E admits an holomorphic S-connection if and only if the Atiyah sequence in* (4.6) *splits holomorphically.*

Proof. Suppose that the Atiyah sequence in (4.6) splits holomorphically, that is, there exists an $\mathscr{O}_X$-module homomorphism

$$\nabla : \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X) \longrightarrow \mathscr{A}t_S(E)$$

such that $\sigma_1 \circ \nabla = \mathbf{1}_{\mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)}$. Then, for every open subset $U \subset X$ and for every $\xi \in \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)(U)$, this $\nabla_U(\xi)$ is a first order S-differential operator such that $\sigma_1(\nabla_U(\xi))(a) = \xi'(a)\mathbf{1}_E$ for some $\xi' \in \mathscr{D}er_S(\mathscr{O}_X, \mathscr{O}_X)(U)$ and for every $a \in \mathscr{O}_X(U)$. This implies that $\xi = \xi'$, because the Atiyah sequence splits. We have $[\nabla_U(\xi), a] = \xi(a)\mathbf{1}_E$, which can be expressed as

$$\nabla_U(\xi)(as) = a\nabla_U(\xi)(s) + \xi(a)s$$

for every $s \in E(U)$. Thus $\nabla_U(\xi)$ satisfies Leibniz rule, and since $\mathscr{A}t_S(E)$ is an $\mathscr{O}_X$-submodule of $\mathscr{E}nd_S(\mathscr{E})$, it follows that ∇ is actually an S-connection on E.

The converse follows from the fact that any S-connection satisfies Leibniz rule, because it gives an splitting of Atiyah exact sequence.

The extension class of the Atiyah exact sequence (4.6) of a holomorphic vector bundle E over X is an element of $\mathrm{H}^1(X, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{T}_{X/S}, \mathscr{E}nd_{\mathscr{O}_X}(E)))$. This extension class is called the **relative Atiyah class** of E, and it is denoted by $\mathrm{at}_S(E)$. Note that

$$\mathrm{H}^1(X, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{T}_{X/S}, \mathscr{E}nd_{\mathscr{O}_X}(E))) \cong \mathrm{H}^1(X, \Omega^1_{X/S}(\mathscr{E}nd_{\mathscr{O}_X}(E)))\,.$$

Proposition 4.1.4 has the following corollary:

Corollary 4.1.5 *A holomorphic vector bundle E on E admits a holomorphic S-connection if and only if its relative Atiyah class* $\mathrm{at}_S(E) \in \mathrm{H}^1(X, \Omega^1_{X/S}(\mathscr{E}nd_{\mathscr{O}_X}(E)))$ *is zero.*

4.1.4 Induced family of holomorphic connections

As before, $\pi : X \longrightarrow S$ is a surjective holomorphic proper submersion with connected fibers. Let $\varpi : E \longrightarrow X$ be a holomorphic vector bundle. For every $t \in S$, the restriction of E to $X_t = \pi^{-1}(t)$ is denoted by E_t. Let U be an open subset of X and $s : U \longrightarrow E$ a holomorphic section. We denote by $r_t(s)$ the restriction of s to $X_t \cap U$, whenever $U \cap X_t \neq \emptyset$. Clearly, $r_t(s)$ is a holomorphic section of E_t over $U \cap X_t$. The map $r_t : s \longmapsto r_t(s)$ induces, therefore, a homomorphism of $\mathbf{C}$-vector spaces from E to E_t, which is denoted by the same symbol r_t (the restriction map r_t is discussed in[KS58, p. 343] and [KS60, p. 58]). Also, X_t is a complex submanifold of X, so $\mathscr{O}_X|_{X_t} = \mathscr{O}_{X_t}$. We also have the restriction map $r_t : \mathscr{E}nd_{\mathscr{O}_X}(E) \longrightarrow \mathscr{E}nd_{\mathscr{O}_{X_t}}(E_t)$.

Similarly, if $P : E \longrightarrow F$ is a first order S-differential operator, where F is a holomorphic vector bundle over X, then the restriction map $r_t : E_t \longrightarrow F_t$ gives rise to a first order differential operator $P_t : E_t \longrightarrow F_t$ for every $t \in S$. Thus, we have the restriction map

$$r_t : \mathscr{D}iff^1_S(E,F) \longrightarrow \mathscr{D}iff^1_{\mathbf{C}}(E_t,F_t).$$

In particular, for $E = F$, we have the restriction map

$$r_t : \mathscr{D}iff^1_S(E,E) \longrightarrow \mathscr{D}iff^1_{\mathbf{C}}(E_t,E_t)$$

for every $t \in S$. Since, the restriction of the relative tangent bundle $T(X/S)$ to each fiber X_t of π is the tangent bundle $T(X_t)$ of the fiber X_t, we have the restriction map $r_t : \mathscr{T}_{X/S} \longrightarrow \mathscr{T}_{X_t}$.

Now, for every $t \in S$, the restriction maps gives a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{E}nd_{\mathscr{O}_X}(E) & \longrightarrow & \mathscr{A}t_S(E) & \xrightarrow{\sigma_1} & \mathscr{T}_{X/S} & \longrightarrow & 0 \\ & & \downarrow r_t & & \downarrow r_t & & \downarrow r_t & & \\ 0 & \longrightarrow & \mathscr{E}nd_{\mathscr{O}_{X_t}}(E_t) & \longrightarrow & \mathscr{A}t(E_t) & \xrightarrow{\sigma_{1t}} & \mathscr{T}_{X_t} & \longrightarrow & 0 \end{array} \tag{4.7}$$

where the bottom sequence is the Atiyah sequence of the holomorphic vector bundle E_t over X_t (see (4.6)) and σ_{1t} is the restriction of the symbol map σ_1.

Suppose that E admits a holomorphic S-connection, which is equivalent to saying that the relative Atiyah sequence in (4.6) splits holomorphically. If

$$\nabla : \mathscr{T}_{X/S} \longrightarrow \mathscr{A}t_S(E)$$

is a holomorphic splitting of the relative Atiyah sequence in (4.6), then for every $t \in S$, the restriction of ∇ to $\mathscr{T}_{X_t}$ gives an $\mathscr{O}_{X_t}$-module homomorphism

$$\nabla_t : \mathscr{T}_{X_t} \longrightarrow \mathscr{A}t(E_t).$$

Now, ∇_t is a holomorphic splitting of the Atiyah sequence of the holomorphic vector bundle E_t, which follows from the fact that the restriction maps r_t defined above are surjective. Thus, we have the following:

Proposition 4.1.6 *Let $\pi : X \longrightarrow S$ be a surjective holomorphic proper submersion with connected fibers and $\varpi : E \longrightarrow X$ a holomorphic vector bundle. Let D be a holomorphic S-connection on E. Then for every $t \in S$, we have a holomorphic connection D_t on the holomorphic vector bundle $E_t \longrightarrow X_t$. In other words, we have a family*

$$\{D_t \mid t \in S\}$$

of holomorphic connections on the holomorphic family of vector bundles

$$\{E_t \longrightarrow X_t \mid t \in S\}.$$

4.2 Families of smooth manifolds

4.2.1 Smooth relative tangent bundle and smooth relative r-forms

As before, $\pi : X \longrightarrow S$ is a complex analytic family of complex manifolds. Consider π as a C^∞ map between real manifolds. We denote the **smooth relative tangent bundle** by $T^{\mathbf{R}}(X/S)$, while its sheaf of smooth sections is denoted by $\mathscr{T}^{\mathbf{R}}_{X/S}$. Similarly, there is a **smooth relative cotangent bundle** denoted by $A^1_{\mathbf{R}}(X/S)$ and its sheaf of smooth sections, which is denoted by $\mathscr{A}^1_{\mathbf{R}}(X/S)$. Define

$$T^*(X/S)_{\mathbf{C}} = A^1_{\mathbf{R}}(X/S) \otimes_{\mathbf{R}} \mathbf{C} = A^1_{\mathbf{R}}(X/S)_{\mathbf{C}},$$

which is nothing but the complexification of the smooth relative cotangent bundle $A^1_{\mathbf{R}}(X/S)$.

A smooth section of $A^1_{\mathbf{R}}(X/S)_{\mathbf{C}}$ is called a *complex valued smooth* 1*-form on X relative to S*, or a *complex valued smooth relative* 1*-form on X over S*. We denote the sheaf of smooth sections of $A^1_{\mathbf{R}}(X/S)_{\mathbf{C}}$ by $\mathscr{A}^1_{X/S}$; also, denote the sheaf of complex valued smooth function on X by $\mathscr{C}^\infty_X$. Then $\mathscr{A}^1_{X/S}$ is an $\mathscr{C}^\infty_X$-module, and there exists a unique S-derivation

$$d_{X/S} : \mathscr{C}^\infty_X \longrightarrow \mathscr{A}^1_{X/S}.$$

The kernel $\mathscr{K}er(d_{X/S})$ of $d_{X/S}$ is the sheaf of complex valued smooth functions on X, which are constant along the fibers X_t, for all $t \in S$, that is,

$$\mathscr{K}er(d_{X/S}) = \pi^{-1}\mathscr{C}^\infty_S,$$

where $\mathscr{C}^\infty_S$ is the sheaf of complex valued smooth functions on S.

Similarly, we can define the complex valued smooth relative r-forms on X over S. A smooth section of $\Lambda^r T^*(X/S)_{\mathbf{C}}$ is called a *complex valued smooth relative r-form* on X over S. Denote the sheaf of smooth sections of $\Lambda^r T^*(X/S)_{\mathbf{C}}$ by $\mathscr{A}^r_{X/S}$. The following analog Theorem 4.1.2 is derived using Proposition 2.3.2 again.

Theorem 4.2.1 *There exist canonical S-linear maps $\delta^r_{X/S} : \mathscr{A}^r_{X/S} \longrightarrow \mathscr{A}^{r+1}_{X/S}$ called relative exterior derivative satisfying the following:*

1. $\delta^0_{X/S} = d_{X/S} : \mathscr{C}^\infty_X \to \mathscr{A}^1_{X/S}$,
2. $\delta^{r+1}_{X/S} \circ \delta^r_{X/S} = 0$, *and*
3. $\delta_{X/S}(\alpha \wedge \beta) = \delta_{X/S}\alpha \wedge \beta + (-1)^r \alpha \wedge \delta_{X/S}\beta$ *for all local sections* α *of* $\mathscr{A}^r_{X/S}$ *and* β *of* $\mathscr{A}^s_{X/S}$.

Proof. First note that the sheaf $\mathscr{D}er_S(\mathscr{C}^\infty_X, \mathscr{C}^\infty_X)$ is canonically isomorphic to the relative tangent sheaf $\mathscr{T}^{\mathbf{R}}_{X/S}$ as an $\mathscr{C}^\infty_X$-module, and hence $\mathscr{A}lt^r_{\mathscr{C}^\infty_X}(\mathscr{D}er_S(\mathscr{C}^\infty_X, \mathscr{C}^\infty_X), \mathscr{C}^\infty_X)$ is canonically isomorphic to the sheaf $\mathscr{A}^r_{X/S}$ as an $\mathscr{C}^\infty_X$-module. Considering the canonical S-connection in $\mathscr{C}^\infty_X$, by Proposition 2.3.2, there is a canonical S-linear map that satisfies (1), and by Proposition 2.3.6, it satisfies (3). Finally, (2) follows using By Corollary 2.3.5(2).

Henceforth, we shall denote $\delta^r_{X/S}$ by $d_{X/S}$, for all $r \geq 0$. By the relative Poincaré lemma, there is an exact sequence

$$0 \longrightarrow \pi^{-1}\mathscr{C}^\infty_S \longrightarrow \mathscr{C}^\infty_X \xrightarrow{d_{X/S}} \mathscr{A}^1_{X/S} \xrightarrow{d_{X/S}} \cdots \xrightarrow{d_{X/S}} \mathscr{A}^{2l}_{X/S} \longrightarrow 0$$

of $\mathscr{C}^\infty_X$-module and S-linear maps. Thus we have a smooth relative de Rham complex $(\mathscr{A}^\bullet_{X/S}, d_{X/S})$, which is a resolution of the sheaf $\pi^{-1}\mathscr{C}^\infty_S$.

For every integer $p \geq 0$ and for every open subset $V \subset S$, the assignment

$$V \longmapsto \mathbb{H}^p(\pi^{-1}(V), \mathscr{A}^\bullet_{X/S}|_{\pi^{-1}(V)})$$

is a presheaf of $\pi_*\mathscr{C}^\infty_X(V) = \mathscr{C}^\infty_X(\pi^{-1}(V))$-module, where

$$\mathbb{H}^q(\pi^{-1}(V), \mathscr{A}^\bullet_{X/S}|_{\pi^{-1}(V)})$$

denotes the hypercohomolgy group of $\pi^{-1}(V) \subset X$ with values in $\mathscr{A}^\bullet_{X/S}|_{\pi^{-1}(V)}$. The sheafification of this presheaf is a $\mathscr{C}^\infty_S$-module, and it is denoted by $\mathbb{R}^p\pi_*(\mathscr{A}^\bullet_{X/S})$.

The sheaf $\mathbb{R}^p\pi_*(\mathscr{A}^\bullet_{X/S})$ of $\mathscr{C}^\infty_S$-module is called the sheaf of relative de Rham cohomology, and it is denoted by $\mathscr{H}^p_{dR}(X/S)$. Since, $\mathscr{A}^\bullet_{X/S}$ is an acyclic resolution of $\pi^{-1}\mathscr{C}^\infty_S$, we have the following:

Proposition 4.2.2 *Let $\pi : X \longrightarrow S$ be a holomorphic proper submersion of complex manifolds with connected fibers. Then*

$$\mathscr{H}^p_{dR}(X/S) \cong R^p\pi_*(\pi^{-1}\mathscr{C}^\infty_S),$$

where $R^p\pi_*(\pi^{-1}\mathscr{C}^\infty_S)$ is the higher direct image sheaf of $\pi^{-1}\mathscr{C}^\infty_S$ on S.

Proof. Since, for each $p \geq 0$, the sheaf $\mathscr{A}^p_{X/S}$ is fine, from the definition of hypercohomology, we have

$$\mathbb{H}^p(\pi^{-1}(V), \mathscr{A}^\bullet_{X/S}|_{\pi^{-1}(V)}) \cong \mathrm{H}^p(\Gamma(\pi^{-1}(V), \mathscr{A}^\bullet_{X/S}|_{\pi^{-1}(V)}))$$

for every open subset $V \subset S$. Also, $\mathscr{A}^\bullet_{X/S}$ is an acyclic resolution of $\pi^{-1}\mathscr{C}^\infty_S$. Thus, we have

$$\mathrm{H}^p(\pi^{-1}(V), \pi^{-1}\mathscr{C}^\infty_S) \cong \mathrm{H}^p(\Gamma(\pi^{-1}(V), \mathscr{A}^\bullet_{X/S}|_{\pi^{-1}(V)}))$$

for every open subset $V \subset S$. Now, the proposition follows from the definition of higher direct image sheaves.

Note that $\mathscr{H}^p_{dR}(X/S)$ is locally free $\mathscr{C}^\infty_S$-module.

Proposition 4.2.3 (Pullback of smooth relative forms) *Suppose we have the following commutative diagram*

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ T & \xrightarrow{g} & S \end{array} \tag{4.8}$$

of complex manifolds and holomorphic maps, where π, and π' are surjective holomorphic proper submersions. Then, for every open subset $U \subset X$, and every smooth relative differential form $\omega \in \mathscr{A}^r_{X/S}(U)$, the pullback $\widetilde{f^*}(\omega)$ is an element of $\mathscr{A}^r_{Y/T}(f^{-1}(U))$.

Proof. Given the commutative diagram in (4.8), we have the following commutative diagrams:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi^*\mathscr{A}^1_S & \longrightarrow & \mathscr{A}^1_X & \longrightarrow & \mathscr{A}^1_{X/S} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi^* g^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle \widetilde{f^*}} & & \\ 0 & \longrightarrow & \pi^*\mathscr{A}^1_T & \longrightarrow & \mathscr{A}^1_Y & \longrightarrow & \mathscr{A}^1_{Y/T} & \longrightarrow & 0 \end{array} \tag{4.9}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi^*\mathscr{A}^1_S \otimes_{\mathscr{C}^\infty_{X,\mathbf{C}}} \mathscr{A}^{r-1}_X & \longrightarrow & \mathscr{A}^r_X & \longrightarrow & \mathscr{A}^r_{X/S} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi^* g^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle \widetilde{f^*}} & & \\ 0 & \longrightarrow & \pi^*\mathscr{A}^1_T \otimes_{\mathscr{C}^\infty_{Y,\mathbf{C}}} \mathscr{A}^{r-1}_Y & \longrightarrow & \mathscr{A}^r_Y & \longrightarrow & \mathscr{A}^r_{Y/T} & \longrightarrow & 0 \end{array} \tag{4.10}$$

Thus, given any $\omega \in \mathscr{A}^1_{X/S}(U)$, from (4.9), we get that $\widetilde{f^*}(\omega)$ is an element of $\mathscr{A}^1_{Y/T}(f^{-1}(U))$, and similarly, from (4.10) it follows that for any smooth relative r-form $\omega \in \mathscr{A}^r_{X/S}(U)$, we have $\widetilde{f}^*(\omega) \in \mathscr{A}^r_{Y/T}(f^{-1}(U))$.

4.2.2 Smooth relative connection and relative Chern classes

In this section, we define the relative Chern class of a differentiable family of complex vector bundles $\varpi : E \longrightarrow X$ of rank r. For each $t \in S$, the restriction of E to $X_t = \pi^{-1}(t)$ will be denoted by E_t.

We follow section 2.5 and substitute E in place of $\mathscr{F}$ there. Let D be a smooth S-connection on E. Let (U_α, h_α) be a trivialization of E over $U_\alpha \subset X$. Let R be the S-curvature form for D, and let $\Omega_\alpha = (\Omega_{ij\alpha})$ be the curvature matrix of D over U_α, as defined in section 2.5, so $\Omega_{ij\alpha} \in \mathscr{A}^2_{X/S}(U_\alpha)$. We have $\Omega_\beta = g^{-1}_{\alpha\beta}\Omega_\alpha g_{\alpha\beta}$, where

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}_r(\mathbf{C})$$

is the change of frame matrix (transition function), which is a smooth map.

Consider the adjoint action of $\mathrm{GL}_r(\mathbf{C})$ on it Lie algebra $\mathfrak{gl}_r(\mathbf{C}) = \mathrm{M}_r(\mathbf{C})$. Let f be a $\mathrm{GL}_r(\mathbf{C})$-invariant homogeneous polynomial on $\mathfrak{gl}_r(\mathbf{C})$ of degree p. Then, we can associate a unique p-multilinear symmetric map $\widetilde{f}$ on $\mathfrak{gl}_r(\mathbf{C})$ such that $f(X) = \widetilde{f}(X, \cdots, X)$, for all $X \in \mathfrak{gl}_r(\mathbf{C})$. Define

$$\gamma_\alpha = \widetilde{f}(\Omega_\alpha, \cdots, \Omega_\alpha) = f(\Omega_\alpha) \in \mathscr{A}^{2p}_{X/S}(U_\alpha).$$

Since f is $\mathrm{GL}_r(\mathbf{C})$-invariant, it follows that γ_α is independent of the choice of frame, and hence it defines a global smooth relative differential form of degree $2p$, which we denote by the symbol $\gamma \in \mathscr{A}^{2p}_{X/S}(X)$.

The following theorem has been shown in [Kob87, Chapter II, section 2, p. 36] in absolute context.

Theorem 4.2.4 *Let $\pi : X \longrightarrow S$ be a surjective holomorphic proper submersion of com-*

plex manifolds with connected fibers and $\varpi : E \longrightarrow X$ *a differentiable family of complex vector bundle. Let* D *be a smooth* S*-connection on* E*. Suppose that* f *is a* $\mathrm{GL}_r(\mathbf{C})$*-invariant polynomial function on* $\mathfrak{gl}_r(\mathbf{C})$ *of degree* p*. Then the following hold:*

1. $\gamma = f(\Omega_\alpha)$ *is* $d_{X/S}$*-closed, that is,* $d_{X/S}(\gamma) = 0$*.*
2. *The image* $[\gamma]$ *of* γ *in the relative de Rham cohomology group*

$$\mathrm{H}^{2p}(\Gamma(X, \mathscr{A}^{\bullet}_{X/S})) = \mathrm{H}^{2p}(X, \pi^{-1}\mathscr{C}^{\infty}_S)$$

is independent of the smooth S*-connection* D *on* E*.*

Proof. To show 1, we need following well known fact:

A symmetric p-multilinear form $\tilde{f}$ *on* $\mathfrak{gl}_r(\mathbf{C})$ *is* $GL_r(\mathbf{C})$*-invariant if and only if*

$$\sum_{j=1}^{p} \tilde{f}(X_1, \ldots, [X_j, Y], \ldots, X_p) = 0 \tag{4.11}$$

for all $X_1, \ldots, X_k, Y \in \mathfrak{gl}_r(\mathbf{C})$.

Now, using the symmetric p-multilinearity of $\tilde{f}$, and above fact, we have the following,

$$\begin{aligned} d_{X/S}(\gamma) &= d_{X/S} f(\Omega_\alpha) \\ &= d_{X/S} \tilde{f}(\Omega_\alpha, \ldots, \Omega_\alpha) \\ &= \sum \tilde{f}(\Omega_\alpha, \ldots, d_{X/S}\Omega_\alpha, \ldots, \Omega_\alpha) \\ &= \sum \tilde{f}(\Omega_\alpha, \ldots, [\Omega_\alpha, \omega_\alpha], \ldots, \Omega_\alpha) \\ &= 0. \end{aligned}$$

To prove 2, let D_0, D_1 be two smooth S-connections in E. Let ω_i, Ω_i, for $i \in \{0,1\}$, be their connection and curvature matrices, respectively. Let

$$\gamma_i = f(\Omega_i) \in \mathscr{A}^{2p}_{X/S}(X)$$

for $i \in \{0,1\}$. We want to show that there exists a smooth relative $(2p-1)$-form φ on X, such that $\gamma_1 - \gamma_0 = d_{X/S}(\varphi)$. Let

$$D_t = D_0 + t(D_1 - D_0), \tag{4.12}$$

for $0 \leq t \leq 1$. Then D_t is a smooth S-connection in E. Let ω_t and Ω_t be the connection and curvature matrices, respectively. Then

$$\omega_t = \omega_0 + t(\omega_1 - \omega_0) = \omega_0 + t\alpha, \tag{4.13}$$

where $\alpha = \omega_1 - \omega_0$ is the smooth relative 1-form. Note that $\alpha = \omega_1 - \omega_0$ being difference of two connection forms is $GL(r, \mathbf{C})$-invariant, and hence does not depend on the frame, yields a global smooth relative 1-form, which we again denote by α. Now from 4.13, we have

$$\begin{aligned} \Omega_t &= d_{X/S}(\omega_t) + \omega_t \wedge \omega_t \\ &= d_{X/S}(\omega_0) + t d_{X/S}(\alpha) + \omega_t \wedge \omega_t. \end{aligned}$$

Differentiating Ω_t, partially with respect to parameter t, we get

$$\frac{\partial}{\partial t}(\Omega_t) = d_{X/S}\alpha + \alpha \wedge \omega_t + \omega_t \wedge \alpha = D_t(\alpha) \tag{4.14}$$

Further, $\tilde{f}(\alpha, \Omega_t, \ldots, \Omega_t)$ defines a 1-parameter family of smooth relative $(2p-1)$-form on X, that is, $\tilde{f}(\alpha, \Omega_t, \ldots, \Omega_t) \in \mathscr{A}^{2p-1}_{X/S}(X)$. Now, differentiating $\tilde{f}(\Omega_t, \ldots, \Omega_t)$, partially with respect to t, we have

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{f}(\Omega_t,\dots,\Omega_t) &= p\tilde{f}(\frac{\partial}{\partial t}\Omega_t,\Omega_t,\dots,\Omega_t)\\ &= p\tilde{f}(D_t\alpha,\Omega_t,\dots,\Omega_t)\\ &= pD_t\tilde{f}(\alpha,\Omega_t,\dots,\Omega_t)\\ &= pd_{X/S}\tilde{f}(\alpha,\Omega_t,\dots,\Omega_t)\end{aligned}$$

We set

$$\varphi = p\int_0^1 \tilde{f}(\alpha,\Omega_t,\dots,\Omega_t)\mathrm{d}t \in \mathscr{A}^{2p-1}_{X/S}(X).$$

Hence,

$$\begin{aligned}d_{X/S}\varphi &= p\int_0^1 d_{X/S}\tilde{f}(\alpha,\Omega_t,\dots,\Omega_t)\mathrm{d}t\\ &= \int_0^1 \frac{\partial}{\partial t}\tilde{f}(\Omega_t,\dots,\Omega_t)\mathrm{d}t\\ &= \gamma_1-\gamma_0.\end{aligned}$$

This completes the proof of 2.

Define homogeneous polynomials f_p on $\mathfrak{gl}_r(\mathbf{C})$, of degree $p = 1, 2, \cdots, r$, to be the coefficient of λ^p in the following expression:

$$\det(\lambda\mathrm{I}+\frac{\sqrt{-1}}{2\pi}A) = \Sigma_{j=0}^{r}\lambda^{r-j}f_j(\frac{\sqrt{-1}}{2\pi}A),$$

where $f_0(\frac{\sqrt{-1}}{2\pi}A) = 1$ while $f_r(\frac{\sqrt{-1}}{2\pi}A)$ is the coefficient of λ^0. These polynomials $f_1, \cdots, f_r$ are $GL_r(\mathbf{C})$-invariant, and they generate the algebra of $\mathrm{GL}_r(\mathbf{C})$-invariant polynomials on $\mathfrak{gl}_r(\mathbf{C})$. We now define the *p-th cohomology class* as follows:

$$c_p^S(E) = [f_p(\frac{\sqrt{-1}}{2\pi}\Omega)] \in \mathrm{H}^{2p}(\Gamma(X, \mathscr{A}^{\bullet}_{X/S}))$$

for $p = 0, 1, \cdots, r$.

The relative de Rham cohomology sheaf

$$\mathscr{H}^p_{dR}(X/S) \cong R^p\pi_*(\pi^{-1}\mathscr{C}^\infty_S)$$

on S is by definition the sheafification of the presheaf

$$V \longmapsto \mathrm{H}^p(\pi^{-1}(V), \pi^{-1}\mathscr{C}^\infty_S|_{\pi^{-1}(V)}),$$

for open subset $V \subset S$. Therefore, we have a natural homomorphism

$$\rho : \mathrm{H}^{2p}(X, \pi^{-1}\mathscr{C}^\infty_S) \longrightarrow \mathscr{H}^{2p}_{dR}(X/S)(S)$$

which maps $c^S_p(E)$ to $\rho(c^S_p(E)) \in \mathscr{H}^{2p}_{dR}(X/S)(S)$.

Define

$$C^S_p(E) = \rho(c^S_p(E)).$$

We call $C^S_p(E)$ the **p-th relative Chern class** of E over S .

Let

$$C^S(E) = \sum_{p\geq 0} C^S_p(E) \in \mathscr{H}^*_{dR}(X/S)(S) = \oplus_{k\geq 0}\mathscr{H}^k_{dR}(X/S)(S)$$

be the *total relative Chern class* of E.

The following proposition has been proved in [Wel80, Chapter III, Theorem 3.6] for the total Chern class of a vector bundle.

Proposition 4.2.5 *Let $E \xrightarrow{\varpi} X \xrightarrow{\pi} S$ be as in Theorem 4.2.4. Let $\pi' : Y \longrightarrow T$ be a surjective holomorphic proper submersion, such that the following diagram*

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ T & \xrightarrow{g} & S \end{array} \tag{4.15}$$

is commutative, where $f : Y \longrightarrow X$ and $g : T \longrightarrow S$ are holomorphic maps. Then

$$f^*(C^S(E)) = C^T(f^*E),$$

where $C^S(E)$ is the total relative Chern class of E over S, and $C^T(f^*E)$ is the total relative Chern class of f^*E over T.

Proof. Let D be smooth S-connection in E. It is enough to define a smooth T-connection D^* in f^*E, such that $f^*\Omega = \Omega^*$, where Ω^* is the curvature matrix of D^*. Let $e = (e_1, \cdots, e_r)$ be a frame of E over an open subset U of X. Then, we have $e^* = (e_1^*, \cdots, e_r^*)$, where $e_i^* = f \circ e_i^* : f^{-1}(U) \longrightarrow E$ is a frame of f^*E over $f^{-1}(U)$. If $a : U \longrightarrow \mathrm{GL}_r(\mathbf{C})$ is a change of frame over U, then

$$f^*a = a^* = a \circ f : f^{-1}(U) \longrightarrow \mathrm{GL}_r(\mathbf{C})$$

is a change of frame in f^*E over $f^{-1}(U)$. Now, we define S-connection matrix

$$\omega^* = f^*\omega = [\widetilde{f^*}\omega_{ij}],$$

where, $\omega_{ij} \in \mathscr{A}^1_{X/S}(U)$, and $\widetilde{f^*} : \mathscr{A}^1_{X/S} \longrightarrow \mathscr{A}^1_{Y/T}$ is the pullback map of the relative forms as in Proposition 4.2.3. Moreover, if ω' is the connection matrix with respect to the frame $e' = e.a$ and ω'^* is the pullback of ω' under f^*, then

$$\omega'^* = a^{*-1}\omega^* a^* + a^{*-1} da^*.$$

Thus, if we consider $D^* = d_{Y/T} + \omega^*$, then from above compatibility condition, this defines a smooth T-connection in f^*E. Let Ω^* be the curvature form of D^*. Then

$$\Omega^* = d_{Y/T}\omega^* + \omega^* \wedge \omega^* = f^*\Omega.$$

Now, consider the homogeneous polynomial f_p of degree p as defined above. The p-th cohomology class is

$$c_p^T(f^*E) = [f_p(\frac{\sqrt{-1}}{2\pi}\Omega^*)] = f^*[c_p^S(E)]$$

for all $p \geq 0$, which is the pullback of the cohomology class $[c_p^S(E)]$, where

$$f^* : \mathrm{H}^{2p}(\Gamma(X, \mathscr{A}^\bullet_{X/S})) \longrightarrow \mathrm{H}^{2p}(\Gamma(Y, \mathscr{A}^\bullet_{Y/T}))$$

is the morphism of **C**-vector spaces induced by the commutative diagram (4.15). Further, we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}^{2p}(X, \pi^{-1}\mathscr{C}^\infty_S) & \xrightarrow{\rho} & \mathscr{H}^{2p}_{dR}(X/S)(S) \\ \downarrow{f^*} & & \downarrow{f^*} \\ \mathrm{H}^{2p}(Y, \pi^{-1}\mathscr{C}^\infty_T) & \xrightarrow{\rho} & \mathscr{H}^{2p}_{dR}(Y/T)(T) \end{array} \tag{4.16}$$

which implies that $f^*(C_p^S(E)) = C_p^T(f^*E)$. This completes the proof.

In particular, taking $T = \{t\} \subset S$, g to be the inclusion map $t \hookrightarrow S$, $Y = X_t$, $\pi' = \pi|_{X_t} : X_t \longrightarrow T$ and f to be the inclusion map $j : X_t \hookrightarrow X$, by Proposition 4.2.5, we have the following:

Corollary 4.2.6 *For every $t \in S$, there is a natural map*

$$j^* : \mathscr{H}^{2p}_{dR}(X/S)(S) \longrightarrow \mathrm{H}^{2p}(X_t, \mathbf{C})$$

which maps the p-th relative Chern class of E to the p-th Chern class of the smooth vector bundle $E_t \longrightarrow X_t$, that is, $j^*(C_p^S(E)) = c_p(E_t)$.

The following topological proper base change theorem is given in [God64, p. 202, Remark 4.17.1] and [Del70, p. 19, Corollary 2.25]:

Theorem 4.2.7 (Topological proper base change) *Let $f : X \longrightarrow S$ be a proper continuous map of Hausdorff topological spaces. Suppose that S is locally compact, and $\mathscr{F}$ is* *a sheaf of abelian groups on X. Then for all $t \in S$, we have a canonical isomorphism*

$$(R^p f_* \mathscr{F})_t \simeq \mathrm{H}^p(f^{-1}(t), \mathscr{F}|_{f^{-1}(t)})$$

of abelian groups.

Note that $\mathscr{H}^p_{dR}(X/S)$ is a locally free $\mathscr{C}^\infty_S$-module, and from Theorem 4.2.7 we have a $\mathbf{C}$-vector space isomorphism

$$\eta : \mathscr{H}^p_{dR}(X/S)_t \otimes_{\mathscr{C}^\infty_{S,t}} k(t) \longrightarrow \mathrm{H}^p(X_t, \mathbf{C}) \tag{4.17}$$

for every $t \in S$.

Theorem 4.2.8 *Let $\pi : X \longrightarrow S$ be a surjective holomorphic proper submersion with connected fibers, such that $\pi^{-1}(t) = X_t$ is compact Kähler manifold for every $t \in S$. Let $\varpi : E \longrightarrow X$ be a holomorphic vector bundle. Suppose that E admits a holomorphic S-connection. Then all the relative Chern classes $C^S_p(E) \in \mathscr{H}^{2p}_{dR}(X/S)(S)$ of E over S are zero.*

Proof. Let D be a holomorphic S-connection on E. From Proposition 4.1.6 it follows that for every $t \in S$, there is a holomorphic connection D_t in E_t. Since X_t is a compact complex manifold of Kähler type, from Theorem **4** in [Ati57, p. 192] it follows that all the Chern classes $c_p(E_t)$ of E_t are zero. From Corollary 4.2.6 and the isomorphism in (4.17) we have the following commutative diagram;

$$\begin{array}{ccc} \mathscr{H}^{2p}_{dR}(X/S)(S) & \longrightarrow & \mathscr{H}^{p}_{dR}(X/S)_t \otimes_{\mathscr{C}^\infty_{S,t}} k(t) \\ & \searrow^{j^*} & \downarrow \eta \\ & & \mathrm{H}^{2p}(X_t, \mathbf{C}) \end{array}$$

Now,

$$\eta(C^S_p(E)_t \otimes 1) = j^*(C^S_p(E)) = c_p(E_t) = 0,$$

which implies that $C_p^S(E)_t \otimes 1 = 0$, for every $t \in S$, because η is an isomorphism. Thus, we have $C_p^S(E) = 0$. This completes the proof.

4.3 A sufficient condition for holomorphic connection

Given a surjective holomorphic proper submersion $\pi : X \longrightarrow S$ with connected fibers and a holomorphic vector bundle $\varpi : E \longrightarrow X$, Proposition 4.1.6 gives a necessary condition for the existence of a holomorphic S-connection on E, namely the vector bundle $E_t = E|_{X_t} \longrightarrow X_t$ should admit a holomorphic connection for every $t \in S$.

If for every $t \in S$ the vector bundle E_t admits a holomorphic connection, it is natural to ask whether E admits a holomorphic S-connection. We will present a sufficient condition for the existence of holomorphic S-connection on E.

Theorem 4.3.1 *Let $E \xrightarrow{\varpi} X$ be a holomorphic vector bundle. Suppose that for every $t \in S$, there is a holomorphic connection on the holomorphic vector bundle $\varpi|_{E_t} : E_t \longrightarrow X_t$, and*

$$\mathrm{H}^1(S, \pi_*(\Omega^1_{X/S}(\mathscr{E}nd_{\mathscr{O}_X}(E)))) = 0\,.$$

Then, E admits a holomorphic S-connection.

Proof. Consider the relative Atiyah exact sequence in (4.6). Tensoring it by $\Omega^1_{X/S}$ produces the exact sequence

$$0 \longrightarrow \Omega^1_{X/S}(\mathscr{E}nd_{\mathscr{O}_X}(E)) \longrightarrow \Omega^1_{X/S}(\mathscr{A}t_S(E)) \xrightarrow{q} \Omega^1_{X/S} \otimes \mathscr{T}_{X/S} \longrightarrow 0\,. \tag{4.18}$$

Note that $\mathscr{O}_X \cdot \mathrm{Id} \subset \mathrm{End}(\mathscr{T}_{X/S}) = \Omega^1_{X/S} \otimes \mathscr{T}_{X/S}$. Define

$$\Omega^1_{X/S}(\mathscr{A}t'_S(E)) := q^{-1}(\mathscr{O}_X \cdot \mathrm{Id}) \subset \Omega^1_{X/S}(\mathscr{A}t_S(E))\,,$$

where q is the projection in (4.18). So we have the short exact sequence of sheaves

$$0 \longrightarrow \Omega^1_{X/S}(\mathscr{E}nd_{\mathscr{O}_X}(E)) \longrightarrow \Omega^1_{X/S}(\mathscr{A}t'_S(E)) \xrightarrow{q} \mathscr{O}_X \longrightarrow 0 \tag{4.19}$$

on X, where $\Omega^1_{X/S}(\mathscr{A}t'_S(E))$ is constructed above. Let

$$\Phi : \mathbb{C} \subset \mathrm{H}^0(X, \mathscr{O}_X \cdot \mathrm{Id}) \longrightarrow \mathrm{H}^1(X, \Omega^1_{X/S}(\mathscr{E}nd_{\mathscr{O}_X}(E))) \tag{4.20}$$

be the homomorphism in the long exact sequence of cohomologies associated to the exact sequence in (4.19). The relative Atiyah class $\mathrm{at}_S(E)$ (see Corollary 4.1.5) coincides with $\Phi(1) \in \mathrm{H}^1(X, \Omega^1_{X/S}(\mathscr{E}nd_{\mathscr{O}_X}(E)))$. Therefore, from Corollary 4.1.5 it follows that E admits a holomorphic S-connection if and only if

$$\Phi(1) = 0. \tag{4.21}$$

To prove the vanishing statement in (4.21), first note that $\mathrm{H}^1(X, \Omega^1_{X/S}(\mathscr{E}nd_{\mathscr{O}_X}(E)))$ fits in the exact sequence

$$\mathrm{H}^1(S, \pi_*(\Omega^1_{X/S}(\mathscr{E}nd_{\mathscr{O}_X}(E)))) \xrightarrow{\beta_1} \mathrm{H}^1(X, \Omega^1_{X/S}(\mathscr{E}nd_{\mathscr{O}_X}(E))) \xrightarrow{q_1} \mathrm{H}^0(S, R^1\pi_*(\Omega^1_{X/S}(\mathscr{E}nd_{\mathscr{O}_X}(E)))), \tag{4.22}$$

where π is the projection of X to S.

The given condition that for every $t \in S$, there is a holomorphic connection on the holomorphic vector bundle $\varpi|_{E_t} : E_t \longrightarrow X_t$, implies that

$$q_1(\Phi(1)) = 0,$$

where q_1 is the homomorphism in (4.22). Therefore, from the exact sequence in (4.22) we conclude that

$$\Phi(1) \in \beta_1(\mathrm{H}^1(S, \pi_*(\Omega^1_{X/S}(\mathscr{E}nd_{\mathscr{O}_X}(E))))).$$

Finally, the given condition that $\mathrm{H}^1(S, \pi_*(\Omega^1_{X/S}(\mathscr{E}nd_{\mathscr{O}_X}(E)))) = 0$. implies that $\Phi(1) = 0$. Since (4.21) holds, the vector bundle E admits a holomorphic S-connection.

Take $\pi : X \longrightarrow S$ to be a surjective holomorphic proper submersion of relative dimension one, so $\pi^{-1}(t)$ is a compact connected Riemann surface, for every $t \in S$. Then, by the *Atiyah-Weil* criterion given in [Ati57], [Wei38] and [BR05] (Theorem **6.12**), we have the following:

Corollary 4.3.2 *Let $\pi : X \longrightarrow S$ be a surjective holomorphic proper submersion such that $\pi^{-1}(t) = X_t$ is a compact connected Riemann surface for every $t \in S$. Let $\varpi : E \longrightarrow X$ be a holomorphic vector bundle. Suppose that for every $t \in S$, the degree of the indecomposable components of E_t are zero and*

$$\mathrm{H}^1(S, \pi_*(\Omega^1_{X/S}(\mathscr{E}nd_{\mathscr{O}_X}(E)))) = 0\,.$$

Then, E admits a holomorphic S-connection.

where q is the projection in (4.18). So we have the short exact sequence of sheaves

$$0 \longrightarrow \Omega^1_{X/S}(\mathscr{E}nd_{\mathscr{O}_X}(E)) \longrightarrow \Omega^1_{X/S}(\mathscr{A}t'_S(E)) \xrightarrow{q} \mathscr{O}_X \longrightarrow 0 \tag{4.19}$$

on X, where $\Omega^1_{X/S}(\mathscr{A}t'_S(E))$ is constructed above. Let

$$\Phi : \mathbb{C} \subset \mathrm{H}^0(X, \mathscr{O}_X \cdot \mathrm{Id}) \longrightarrow \mathrm{H}^1(X, \Omega^1_{X/S}(\mathscr{E}nd_{\mathscr{O}_X}(E))) \tag{4.20}$$

be the homomorphism in the long exact sequence of cohomologies associated to the exact sequence in (4.19). The relative Atiyah class $\mathrm{at}_S(E)$ (see Corollary 4.1.5) coincides with $\Phi(1) \in \mathrm{H}^1(X, \Omega^1_{X/S}(\mathscr{E}nd_{\mathscr{O}_X}(E)))$. Therefore, from Corollary 4.1.5 it follows that E admits a holomorphic S-connection if and only if

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$$\Phi(1) \in \beta_1(\mathrm{H}^1(S, \pi_*(\Omega^1_{X/S}(\mathscr{E}nd_{\mathscr{O}_X}(E))))).$$

Finally, the given condition that $\mathrm{H}^1(S, \pi_*(\Omega^1_{X/S}(\mathscr{E}nd_{\mathscr{O}_X}(E)))) = 0$. implies that $\Phi(1) = 0$. Since (4.21) holds, the vector bundle E admits a holomorphic S-connection.

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Corollary 4.3.2 *Let $\pi : X \longrightarrow S$ be a surjective holomorphic proper submersion such that $\pi^{-1}(t) = X_t$ is a compact connected Riemann surface for every $t \in S$. Let $\varpi : E \longrightarrow X$ be a holomorphic vector bundle. Suppose that for every $t \in S$, the degree of the indecomposable components of E_t are zero and*

$$\mathrm{H}^1(S, \pi_*(\Omega^1_{X/S}(\mathscr{E}nd_{\mathscr{O}_X}(E)))) = 0\,.$$

Then, E admits a holomorphic S-connection.

Chapter 5

Moduli space of meromorphic and logarithmic connections

The main objective of this chapter is to give an outline of the construction of the moduli spaces of meromorphic and logarithmic connections over a smooth complex projective variety with fixed ample line bundle.

The moduli spaces of sheaves have been studied extensively by several algebraic geometers. We recall the notion of stability of sheaves, that was first introduced by Mumford for vector bundles on curves and later generalised to sheaves on higher dimensional varieties by Takemoto, Maruyama, Gieseker and Simpson, et al. In section 5.1, we follow [HL10], [Mar77], [Mar78], to recall the GIT construction of moduli spaces of sheaves.

Simpson [SimI94] introduced the notion of sheaf of rings of differential operators, denoted by Λ, over a projective scheme and constructed quasi-projective moduli schemes of (semi)-stable coherent Λ-modules. We give a brief treatment for the sheaf of rings of differential operators Λ^{Mero} and Λ^{Log}, such that Λ^{Mero}-module (respectively, Λ^{Log}-module) corresponds to the sheaf of modules with meromorphic (respectively, logarithmic) connections. In [Mac11] and [Mach11], Machu has studied the moduli space of meromorphic connections. In section 5.2, we give a brief introduction to the sheaf of rings of differential operator Λ, stability of Λ-modules, and construction of moduli space of semistable Λ-modules. In subsection 5.2.4, we will talk about the irreduciblity of the moduli spaces of meromorphic connections and logarithmic connections over a compact Riemann surface. In section 5.3, we define the residue of a logarithmic connection on a holomorphic vector bundle over a compact Riemann surface and restrict ourselves to the moduli space of logarithmic connection with fixed residues. In subsection 5.3.2, we describe a natural compactification of the moduli space of logarithmic connections with fixed residues whose underlying vector bundle is stable.

5.1 The moduli space of sheaves

5.1.1 Stability of sheaves

In this section, we recall the notion of stability of coherent sheaves over a projective scheme. Let X be a projective scheme over a field k with very ample invertible sheaf $\mathscr{O}_X(1)$.

Definition 5.1.1

1. The **support** of a coherent sheaf E over X is defined as

$$\operatorname{Supp}(E) = \{x \in X | E_x \neq 0\}.$$

   The dimension of the set $\operatorname{Supp}(E)$ is called the dimension of the sheaf E and is denoted by $\dim(E)$.

2. A coherent sheaf E is said to be **pure** of dimension d if $\dim(F) = d$ for all non-trivial coherent subsheaves $F \subset E$.

Note that since E is a coherent sheaf over X, the support $\operatorname{Supp}(E)$ is a closed subset of X.

Recall that the Euler characteristic of a coherent sheaf E is given by

$$\chi(E) = \sum_{i\geq 0}(-1)^i h^i(X,E),$$

where

$$h^i(X,E) = \dim_k(\mathrm{H}^i(X,E)).$$

Since we have already fixed an ample line bundle $\mathscr{O}_X(1)$ on X, the **Hilbert polynomial** P_E is given by

$$m \mapsto \chi(E \otimes \mathscr{O}_X(m)).$$

In particular, P_E can be uniquely written in the form

$$P_E(m) = \sum_{i=0}^{\dim(E)} \alpha_i(E)\frac{m^i}{i!}$$

with $\alpha_i(E) \in \mathbf{Q}$ for $i = 0,\ldots,\dim(E)$. Moreover, if $E \neq 0$ the leading coefficient $\alpha_{\dim(E)}(E)$, called the *multiplicity*, is always positive. Also, the degree of X with respect to $\mathscr{O}_X(1)$ is given by $\alpha_{\dim(X)}(\mathscr{O}_X)$.

Definition 5.1.2 The rank $\mathrm{rk}(E)$ of a coherent sheaf E of dimension $d = \dim(X)$ is defined by

$$\mathrm{rk}(E) = \frac{\alpha_d(E)}{\alpha_d(\mathscr{O}_X)}.$$

Note that, $\mathrm{rk}(E)$ need not be an integer.

Definition 5.1.3 For a coherent sheaf E of dimension $d = \dim(X)$, the reduced Hilbert polynomial p_E of E is defined by

$$p_E(m) = \frac{P_E(m)}{\alpha_d(E)}.$$

We define an order on the polynomial ring $\mathbf{Q}[T]$ as follows: Let $f,g \in \mathbf{Q}[T]$ be two polynomials with rational coefficients. Then $f \leq g$ if and only if there exists a natural number N such that $f(m) \leq g(m)$ for all $m \geq N$. Analogously, $f < g$ if and only if there exists a natural number N such that $f(m) < g(m)$ for all $m \geq N$.

Definition 5.1.4 Let E be a coherent sheaf of dimension $d = \dim(X)$. Then we have following definitions:

1. E is said to be semistable (respectively, stable) if the following conditions are satisfied:

   (a) E is pure.

   (b) for any proper subsheaf $F \subset E$ one has $p_F \leq p_E$ (respectively, $p_F < p_E$).

2. The degree of E is defined by

$$\deg(E) = \alpha_{d-1}(E) - \mathrm{rk}(E)\alpha_{d-1}(\mathscr{O}_X),$$

   and its slope by

$$\mu(E) = \frac{\deg(E)}{\mathrm{rk}(E)}.$$

3. E is said to be μ-semistable(respectively, μ-stable) if

   (a) E is pure.

   (b) for any proper subsheaf $F \subset E$ one has $\mu(F) \leq \mu(E)$ (respectively, $\mu(F) < \mu(E)$).

The notion of stability in Definition 5.1.4 (1), defined using the reduced Hilbert polynomial, is called Gieseker-stability.

Lemma 5.1.5 *Let E be a coherent sheaf, pure of dimension $d = \dim(X)$. Then, we have*

$$E \textit{ is } \mu-\textit{stable} \Rightarrow E \textit{ is stable} \Rightarrow E \textit{ is semistable} \Rightarrow E \textit{ is } \mu-\textit{semistable}$$

5.1.2 Representable and corepresentable functors

Let $\mathscr{C}$ be a category, $\mathscr{C}^{op}$ the opposite category, that is, the category having same objects and reversed arrows, and let $\mathscr{C}'$ be the functor category whose objects are functors

$$\mathscr{C}^{op} \longrightarrow \mathbf{Set}$$

from $\mathscr{C}^{op}$ to the category **Set** of sets, and whose morphisms are the morphisms between functors also called natural transformations. For any object $X \in Ob(\mathscr{C})$, we have a functor

$$h_X : \mathscr{C}^{op} \to \mathbf{Set}$$

defined as $Y \mapsto \mathrm{Hom}(Y,X)$. Thus, we get an object in $\mathscr{C}'$.

The Yoneda Lemma states that the functor $h : \mathscr{C} \to \mathscr{C}'$ sending $X \mapsto h_X$ embeds $\mathscr{C}$ as a full subcategory into $\mathscr{C}'$.

Definition 5.1.6 A functor $F \in ob(\mathscr{C}')$ is called **representable** if there exists an object $X \in Ob(\mathscr{C})$ and an isomorphism $F \cong h_X$.

Now, assume that the category $\mathscr{C}$ admits fibre products, then so does $\mathscr{C}'$.

Definition 5.1.7

1. A functor $F \in ob(\mathscr{C}')$ is said to be **corepresentable** if there exists an object $X \in ob(\mathscr{C})$ and a $\mathscr{C}'$-morphism

$$\alpha : F \to h_X$$

such that for any $\mathscr{C}'$-morphism

$$\alpha' : F \to h_{X'},$$

there exists unique $\mathscr{C}'$-morphism

$$\beta : h_X \to h_{X'}$$

such that

$$\beta \circ \alpha = \alpha'.$$

In that case, we say that F is corepresented by X.

2. $F \in ob(\mathscr{C}')$ is said to be **universally corepresentable** if there exits a morphism

$$\alpha : F \to h_X,$$

and for any morphism

$$\phi : h_Y \to h_X,$$

the fibre product $T = h_Y \times_{h_X} F$ is corepresented by Y. In that case, we say that F is universally corepresented by $\alpha : F \to h_X$.

Lemma 5.1.8 *Let $\mathscr{C}$ be a category and let*

$$F_1, F_2 : \mathscr{C}^{op} \longrightarrow \textbf{\textit{Set}}$$

be two functors. Let $M_1, M_2 \in \mathrm{Ob}(\mathscr{C})$ corepresents F_1 and F_2 respectively. Let

$$\Phi : F_1 \to F_2$$

be a natural transformation. Then there exists a unique morphism

$$\psi : M_1 \to M_2$$

such that the following diagram

$$\begin{array}{ccc} F_1 & \xrightarrow{\Phi} & F_2 \\ \downarrow & & \downarrow \\ h_{M_1} & \xrightarrow{\psi\circ\_} & h_{M_2} \end{array}$$

commutes.

Proof. Consider the composition of natural transformations

$$F_1 \xrightarrow{\Phi} F_2 \to h_{M_2}$$

Since M_1 corepresents F_1, this composition factors as

$$F_1 \to h_{M_1} \to h_{M_2}.$$

By Yoneda's lemma, this transformation $h_{M_1} \to h_{M_2}$ is induced by a unique morphism $\psi : M_1 \to M_2$.

There are notions of coarse and fine moduli spaces which we define in general setting. For the definitions, we follow [New78]. Let $\mathscr{T}$ a collection of objects of $\mathscr{C}$, and suppose that we are given a notion of equivalence (for example, isomorphism of objects) of objects in $\mathscr{T}$, a notion of families of objects (depends on a particular moduli problem) in $\mathscr{T}$ parametrized by schemes, a notion of equivalence of families parametrised by a scheme S, and a notion of pullback of families through a morphism of schemes.

Definition 5.1.9 A *moduli problem* or *moduli functor* is a contravariant functor from the category **Sch** of schemes to the category **Set** of sets , that is, we have a functor

$$\mathscr{F} : (\mathbf{Sch})^{op} \to \mathbf{Set}.$$

A moduli space for a given moduli problem $\mathscr{F}$ is a scheme M whose (k-valued) points are in bijection with the set of equivalence classes of objects in $\mathscr{T}$, and reflects the structure of the families of objects in $\mathscr{T}$. This can be made precise in two ways. First is the notion of *fine moduli space*.

Definition 5.1.10 A *fine moduli space* for the moduli problem $\mathscr{F}$ is a representing object of $\mathscr{F}$, that is, a pair consisting of a scheme M and a natural isomorphism $\Phi : \mathscr{F} \to h_M$.

The above definition is equivalent to the following definition.

Definition 5.1.11 A fine moduli space consists of a scheme M and a family U parametrised by M such that, for every family E parametrised by a scheme S, there is a unique morphism $\phi : S \to M$ such that the families E and ϕ^*U are equivalent. Such a family U is called a *universal* family for the moduli problem $\mathscr{F}$.

Example 5.1.12 We give example of moduli functors. Let X be a projective scheme over an algebraically closed field k of characteristic zero with a fixed very ample invertible sheaf $\mathscr{O}_X(1)$. Let $\mathbf{Sch}/k$ be the category of schemes over k. For a fixed polynomial $P \in \mathbf{Q}[z]$, define a functor

$$\mathscr{N} = \mathscr{N}_P^X : (\mathbf{Sch}/k)^{op} \to \mathbf{Set} \tag{5.1}$$

as follows;

If $S \in Ob(\mathbf{Sch}/k)$, define $\mathscr{N}(S)$ to be the set of isomorphism classes of S-flat families $F \to X \times S$ of semistable sheaves such that for $F|_s$ has Hilbert polynomial P for every $s \in S$. For any morphism $f : S' \to S$ in $\mathbf{Sch}/k$, define $\mathscr{N}(f)$ to be the map obtained by pulling-back sheaves via $f_X = \mathbf{1}_X \times f$, that is,

$$\mathscr{N}(f) : \mathscr{N}(S) \to \mathscr{N}(S'), \qquad [F] \mapsto [f_X^* F].$$

Further, let $F \in \mathscr{N}(S)$ be an S-flat family of semistable sheaves, and L a line bundle on S. Let $p_S : X \times S \to S$ denote the canonical projection map. Then $F \otimes p_S^* L$ is also an S-flat family, and the fibres F_s and $(F \otimes p_S^* L)_s = F_s \otimes_{k(s)} L_s$ are isomorphic for each $s \in S$.

Therefore, we can consider another functor

$$\mathscr{M} = \mathscr{M}_P^X = \mathscr{N} / \sim, \tag{5.2}$$

where $\sim$ is an equivalence relation defined as follows:

$F \sim F'$ for $F, F' \in \mathscr{N}(S)$ if and only if $F \cong F' \otimes p_S^* L$ for some $L \in \mathrm{Pic}(S)$.

The functor $\mathscr{M}$ is called the moduli functor of semistable sheaves on X with fixed polynomial P. If we take families of stable sheaves, we get an open subfunctor $\mathscr{M}^s \subset \mathscr{M}$ follows from [Mar76].

Remark 5.1.13 In general, the functor $\mathscr{M}$ need not be representable. In fact, there are very few classification problems for which a fine moduli scheme exists. By only asking for a natural transformation $\Phi : \mathscr{F} \to h_M$ which is universal and bijection over $\mathrm{Spec}(k)$, we get a reasonable notion of a coarse moduli space.

Definition 5.1.14 A *coarse moduli space* for $\mathscr{F}$ is a scheme M together with a natural transformation

$$\Phi : \mathscr{F} \to h_M$$

such that

1. The map $\Phi_{\mathrm{Spec}(k)} : \mathscr{F}(\mathrm{Spec}(k)) \to h_M(\mathrm{Spec}(k))$ is bijective.
2. For any scheme N and any natural transformation $\Psi : \mathscr{F} \to h_N$, there exists a unique natural transformation

$$\eta : h_M \to h_N$$

such that $\Psi = \eta \circ \Phi$.

In other words, M corepresents $\mathscr{F}$.

We briefly describe the Grothendieck's Quot-scheme following [HL10, Chapter 2].

Let k be a field and S a scheme of finite type over k. We consider the category $\mathbf{Sch}/S$ of schemes over S. Let

$$\pi : X \longrightarrow S$$

be a projective morphism. Let $\mathscr{O}_X(1)$ be a π-ample line bundle on X, that is, $\mathscr{O}_X(1)$ restricted to every fibre X_s is an ample line bundle, where $s \in S$. Fix a polynomial $P \in \mathbf{Q}[z]$. Let $\mathscr{E}$ be a coherent module over X. Define a functor

$$\mathscr{Q} = \mathbf{Quot}_{X/S} : (\mathbf{Sch}/S)^{op} \longrightarrow \mathbf{Set} \tag{5.3}$$

as follows:

for an object $g : T \to S \in Ob(\mathbf{Sch}/S)$, define $\mathscr{Q}(T)$ to be the set of all T-flat coherent quotient sheaves

$$\mathscr{E}_T = g_X^* \mathscr{E} \longrightarrow \mathscr{W}$$

with Hilbert polynomial P, where $g_X : X \times_S T \to X$ is the natural projection. For an S-morphism

$$h : T' \to T,$$

let

$$\mathscr{Q}(h) : \mathscr{Q}(T) \to \mathscr{Q}(T')$$

be the map that sends $\mathscr{E}_T \to \mathscr{W}$ to $\mathscr{E}'_T \to h_X^* \mathscr{W}$

Theorem 5.1.15 *[HL10, Theorem 2.2.4] The functor*

$$\mathscr{Q} = \mathbf{Quot}_{X/S}(\mathscr{E}, P) : (\mathbf{Sch}/S)^{op} \longrightarrow \mathbf{Set},$$

as defined in (5.3), is represented by a projective S-scheme

$$\pi : Quot_{X/S}(\mathscr{E}, P) \to S.$$

5.1.3 The construction

We will assume the notions of algebraic groups, group action, quotient for group actions, and linearisation of sheaves. We use and cite results directly from [MF82] and [New78] related to GIT.

Definition 5.1.16 Let E be a semistable sheaf of dimension $d = \dim(X)$. A Jordan-Hölder filtration of E is a filtration

$$0 = E_0 \subset E_1 \subset \ldots \subset E_l = E,$$

such that the factors $\mathrm{gr}_i(E) = E_i/E_{i-1}$ are stable with reduced Hilbert polynomial p_E.

Jordan-Hölder filtration always exists but need not be unique. Nevertheless, up to an isomorphism the sheaf $\mathrm{gr}(E) = \oplus_i \mathrm{gr}_i(E)$ does not depend on the choice of the Jordan-Hölder filtration.

Definition 5.1.17 Two semistable sheaves E_1 and E_2 with the same reduced Hilbert polynomial are called S-equivalent if $\mathrm{gr}(E_1) \cong \mathrm{gr}(E_2)$.

Lemma 5.1.18 *Let $\mathscr{M}$ be the moduli functor as defined in (5.2). Suppose that the moduli functor $\mathscr{M}$ is corepresented by M. Then S-equivalent semistable sheaves correspond to same closed points in M.*

Proof. For the proof see [HL10, Lemma 4.1.2].

For the existence of a coarse moduli space for the functor $\mathscr{M}$ as defined in (5.2), the family of all semistable sheaves on X should be bounded. If we fix the a polynomial $P \in \mathbf{Q}[z]$, then from [HL10, Theorem 3.3.7], the family of semistable sheaves with Hilbert polynomial P is bounded. We will recall the definition of the boundedness of the family of coherent sheaves.

Definition 5.1.19 Let X be a projective scheme over a field k, and $\mathscr{G}$ a family of isomorphism classes of coherent sheaves on X. We say that $\mathscr{G}$ is **bounded**, if there exists a

k-scheme S of finite type and a coherent sheaf F of $\mathscr{O}_{X\times S}$-module on $X\times S$ such that the given family $\mathscr{G}$ is contained in the set $\{F_s|\ s$ a closed point in $S.\}$

We require the notion of m-regularity of a coherent sheaf, where m is an integer, to get a k-scheme S of finite type and $\mathscr{O}_{X\times S}$-module F to ensure the boundedness for $\mathscr{M}$.

Definition 5.1.20 Let F be a coherent sheaf over X.

1. F is called m-regular, if

$$\mathrm{H}^i(X, F(m-i)) = 0 \ \text{ for all } i > 0,$$

   where m is an integer.

2. Define an integer

$$\mathrm{reg}(F) = \inf\{m \in \mathbf{Z} \mid F \text{ is } m-\text{regular}\}.$$

   The integer $\mathrm{reg}(F)$ associated with F is called the Castelnuovo-Mumford regularity of F.

Lemma 5.1.21 *[HL10, Lemma 1.7.6] Let $(F_i)_{i\in I}$ be a family of coherent sheaves on X. Then the following are equivalent:*

1. *The family $(F_i)_{i\in I}$ is bounded.*

2. *The set $\{P_{F_i}\}_{i\in I}$, consists of Hilbert polynomials of F_i for every $i\in I$, is finite, and*

$$\mathrm{reg}(F_i) \leq c$$

   for all $i\in I$, for some $c\in\mathbf{Z}$.

3. *The set $\{P_{F_i}\}_{i\in I}$ consists of Hilbert polynomials of F_i for every $i\in I$, is finite, and*

there is a coherent sheaf F such that we have surjective homomorphisms

$$F \longrightarrow F_i,$$

for every $i \in I$.

Theorem 5.1.22 *[HL10, Theorem 3.3.7] Let X be a projective scheme over a field k with fixed ample line bundle $\mathscr{O}_X(1)$. Let $P \in \mathbf{Q}[z]$ be a polynomial. The family of semistable sheaves over X with Hilbert polynomial P is bounded.*

Now, we give an outline of the construction of the moduli space of semistable sheaves. Let k be an algebraically closed field of characteristic zero, and X a projective scheme over k with fixed ample line bundle $\mathscr{O}_X(1)$. We fix a polynomial $P \in \mathbf{Q}[z]$. For every coherent sheaf F, there is an integer m such that F is m-regular, follows from the Serre's vanishing theorem. Therefore, $F(m)$ is globally generated. In that case, we consider those coherent sheaves F over X, for which $P(m) = \dim_k(\mathrm{H}^0(X, F(m)))$.

Let $V = \mathrm{H}^0(X, F(m))$, and $\mathscr{H} = V \otimes_k \mathscr{O}_X(-m)$. Then there is a surjection

$$\rho : \mathscr{H} \to F$$

obtained by the canonical evaluation map. The above surjection map $\rho : \mathscr{H} \to F$ defines a closed point $[\rho : \mathscr{H} \to F] \in Quot(\mathscr{H}, P)$. There is an open subset $R \subset Quot(\mathscr{H}, P)$ consists of those points which arise from the quotients $[\mathscr{H} \to E]$, where E is semistable and the induced map

$$V \to \mathrm{H}^0(X, \mathscr{H}(m)) \to \mathrm{H}^0(X, E(m))$$

is an isomorphism. Openness of R follows from the fact that E is semistable and the semicontinuity theorem for cohomologies for the induced map. Further, let $R^s \subset R$ denote the open subscheme which parametrises the stable sheaves.

Thus, the set of all semistable sheaves with fixed Hilbert polynomial P is parametrised by R. There are choices for the bases of the vector space V, which will give a natural action of the group $GL(V)$ on the scheme $Quot(\mathscr{H},P)$ from the right, defined as follows:

$$[\rho]\cdot g = [\rho \circ g],$$

where $\rho \in Quot(\mathscr{H},P)$, $g \in GL(V)$. Now, R is invariant under this action, and the isomorphism classes of semisatble sheaves are given by the set $R(k)/GL(V)(k)$. Note that we have already defined the moduli functor $\mathscr{M}$ in (5.2).

Theorem 5.1.23 *M corepresents the moduli functor $\mathscr{M}$ if and only if the morphism*

$$R \longrightarrow M$$

is a categorical quotient for the $GL(V)$-action. Similarly,

$$R^s \longrightarrow M^s$$

is a categorical quotient if and only if M^s corepresents $\mathscr{M}^s$. Therefore, we have $M = R//GL(V)$ and $M^s = R^s//GL(V)$.

Proof. See [HL10, Lemma 4.3.1].

Finally, we have

Theorem 5.1.24 *The functor $\mathscr{M}$ is corepresented by a projective scheme M. The Closed points of M are in bijection with S-equivalence classes of semistable sheaves with Hilbert polynomial P. Moreover, there is open subset M^s of M that universally corepresents the functor $\mathscr{M}^s$.*

Proof. See [HL10, Theorem 4.3.4]. Also, see [SimI94, Theorem 1.21].

In other words, functor $\mathscr{M}$ has a coarse moduli scheme M over k.

5.2 Moduli space of meromorphic and logarithmic connections

In this section, we will sketch the construction of the moduli space of meromorphic, and logarithmic connections. The foundation for the construction of these moduli spaces have been developed by Simpson [SimI94] and Nitsure [Nit93]. Nitsure [Nit93] showed that there exists a coarse moduli scheme which parametrises the integrable logarithmic connections with poles along a normal crossing divisor over a smooth complex projective variety. Simpson [SimI94] produced more general approach to construct the moduli space not only for regular integrable connections, but also Higgs bundles, Hicthin pair, integrable connections along a foliation, Deligne's τ-connections and so on.

We first recall the definition of a complex space. Let B be an open subset in $\mathbf{C}^n$ and let $\mathscr{J}$ be an ideal sheaf in the sheaf $\mathscr{O}_B$ of holomorphic functions, which is of "finite type" on B, that is, for every point $z \in B$ there exists an open neighborhood $U \subset B$ of z and functions $f_i \in \mathscr{O}_B(U)$ $(1 \leq i \leq k)$ such that the sheaf $\mathscr{J}$ is generated over U by f_i's, that is,

$$\mathscr{J}(V) = \mathscr{O}(V) f_1|_V + \cdots + \mathscr{O}(V) f_k|_V,$$

for every open subset V of U. The quotient sheaf $\mathscr{O}_B/\mathscr{J}$ is a sheaf of rings on B. We consider its support $Y = Supp(\mathscr{O}_B/\mathscr{J})$, that is the set of all points $z \in B$, where $\mathscr{J}_z \neq \mathscr{O}_z$. So, locally Y is the zero set of finitely many holomorphic functions. The restriction

$$\mathscr{O}_Y := (\mathscr{O}_B/\mathscr{J})|_Y$$

of $\mathscr{O}_B/\mathscr{J}$ is a sheaf of rings on Y. The ringed space $(Y, \mathscr{O}_Y)$ is called a *complex model space* in B.

Definition 5.2.1 Let $(X, \mathscr{O}_X)$ be a $\mathbf{C}$-ringed space such that X is a Hausdorff space. We call $(X, \mathscr{O}_X)$ a *complex space* if every point of X has an open neighborhood U such that the open $\mathbf{C}$-ringed subspace $(U, \mathscr{O}_U)$ of $(X, \mathscr{O}_X)$ is isomorphic to a complex model space.

Let X be a complex space and let S be a divisor of X, that is, S is a hypersurface of X.

Definition 5.2.2 We say that S is a normal crossing divisor if locally there exist coordinates $z_1,\ldots,z_n$ on X such that S is defined by the monomial equation $z_1\cdots z_r = 0$ for a positive integer r which naturally depends on the considered open set. We say that S is a smooth normal crossing divisor if its each irreducible component is smooth.

Let

$$\Omega_X^k(*S) = \varinjlim_{n\in\mathbf{Z}} \Omega_X^k(nS)$$

denote the sheaf of meromorphic k-forms with poles along S.

Definition 5.2.3 Let E be a holomorphic vector bundle over X. A **meromorphic connection** in E is a $\mathbf{C}$- linear map

$$D : E \to E \otimes \Omega_X^1(*S)$$

satisfying the following

$$D(fs) = df \otimes s + fD(s)$$

where f and s is a local section of $\mathscr{O}_X$ and E, respectively.

Definition 5.2.4 ([Sai80], [Voi05]) Let $\Omega_X^k(\log S)$ be the subsheaf of the sheaf $\Omega_X^k(*S)$ of meromorphic forms on X, called the sheaf of logarithmic k-forms singular over S, defined by the condition:

If β is a meromorphic differential form on U, holomorphic on $U\setminus S\cap U$, $\beta \in \Omega_X^k(\log S)|_U$ if β admits a pole of order at most 1 along (each component of) S, and same holds for $d\beta$.

Definition 5.2.5 Let E be a holomorphic vector bundle over X. A **logarithmic connection** in E is a $\mathbf{C}$-linear map

$$D : E \to E \otimes \Omega_X^1(\log S)$$

satisfying the Leibniz rule

$$D(fs) = df \otimes s + fD(s)$$

where f and s is a local section of $\mathscr{O}_X$ and E, respectively (see [Del70] for the details).

5.2.1 Sheaves of rings of differential operators

In this section, we follow [SimI94] to recall the definition of sheaves of rings of differential operators. We define the notion in algebraic set up. The analytic sheaf of rings of differential operators on a complex space can be defined in exactly the same way.

Definition 5.2.6 Let X be a scheme of finite type over $\mathbf{C}$. A sheaf of rings of differential operators on X over $\mathbf{C}$ is a sheaf Λ of associative $\mathscr{O}_X$-algebras, with a filtration

$$\Lambda_0 \subset \Lambda_1 \subset \ldots \Lambda_i \subset \Lambda_{i+1} \subset \ldots$$

by subsheaves of abelian groups satisfying following properties:

1. $\Lambda = \cup_{i=0}^{\infty} \Lambda_i$ and $\Lambda_i \Lambda_j \subset \Lambda_{i+j}$.
2. The image of the morphism $\mathscr{O}_X \to \Lambda$ is equal to Λ_0.
3. The image of the constant sheaf $\mathbf{C}_X$ in $\mathscr{O}_X = \Lambda_0$ is contained in the centre of Λ.
4. The left and right $\mathscr{O}_X$-module structures on $\mathrm{gr}_i(\Lambda) := \Lambda_i / \Lambda_{i-1}$ are same.
5. The sheaves of $\mathscr{O}_X$-modules $\mathrm{gr}_i(\Lambda)$ are coherent.
6. The sheaf of graded $\mathscr{O}_X$-algebra $\mathrm{gr}(\Lambda) = \bigoplus_{i=0}^{\infty} \mathrm{gr}_i(\Lambda)$ is generated by $\mathrm{gr}_1(\Lambda)$ in the sense that the morphism sheaves

   $$\mathrm{gr}_1(\Lambda) \otimes_{\mathscr{O}_X} \cdots \otimes_{\mathscr{O}_X} \mathrm{gr}_1(\Lambda) \to \mathrm{gr}_i(\Lambda)$$

   is surjective.

The sheaf of rings of differential operators $\mathscr{D}_{X/\mathbf{C}}$ as described in Chapter 3, section 3.1, is an obvious example.

Definition 5.2.7 We say that a sheaf of rings of differential operators Λ on X is *almost polynomial* if the following properties is satisfied:

1. $\Lambda_0 = \mathscr{O}_X$
2. $\mathrm{gr}_1(\Lambda)$ is locally free $\mathscr{O}_X$-module .
3. The graded ring $\mathrm{gr}(\Lambda)$ is the symmetric algebra $\mathrm{Symm}^{\bullet}(\mathrm{gr}_1(\Lambda))$ over $\mathrm{gr}_1(\Lambda)$.

Moreover, Λ is said to be *polynomial* if $\Lambda \cong \mathrm{gr}(\Lambda)$.

Definition 5.2.8 An almost polynomial sheaf of rings of differential operators Λ is said to be *split* if there exists a morphism

$$\zeta : \mathrm{gr}_1(\Lambda) \longrightarrow \Lambda_1$$

of left $\mathscr{O}_X$-modules which gives the splitting of the following short exact sequence

$$0 \to \Lambda_0 \to \Lambda_1 \to \mathrm{gr}_1(\Lambda) \to 0.$$

We will provide another description of the split almost polynomial sheaf Λ of rings of differential operators as follows.

Let Ξ be the set of the triples (H, δ, γ) which satisfy following properties.

1. H is a locally free sheaf of $\mathscr{O}_X$-modules on X.
2. $\delta : \mathscr{O}_X \to H$ is a $\mathbf{C}$-derivation.
3. Define $\Lambda_1^{H,\delta} := H^* \oplus \mathscr{O}_X$, and $\Lambda_0^{H,\delta} = \mathscr{O}_X$. Equip $\Lambda_1^{H,\delta}$ the left $\mathscr{O}_X$- module structure by

$$g \cdot (s, f) = (gs, gf)$$

and the right $\mathscr{O}_X$- module structure given by

$$(s,f)\cdot g = (gs, f + s(\delta(g))).$$

4. Equip $\bigwedge^2 H$ with same left and right $\mathscr{O}_X$-module structure. Suppose

$$\gamma : H \longrightarrow \bigwedge^2 H \otimes_{\mathscr{O}_X} \Lambda_1^{H,\delta} = (\bigwedge^2 H \otimes_{\mathscr{O}_X} H^*) \oplus (\bigwedge^2 H \otimes_{\mathscr{O}_X} \mathscr{O}_X)$$

is a morphism of right $\mathscr{O}_X$-modules such that the composition with the projection into $\bigwedge^2 H \otimes_{\mathscr{O}_X} H^*$ is equal to the canonical map

$$\gamma_1 : H \longrightarrow \bigwedge^2 H \otimes_{\mathscr{O}_X} H^*.$$

Note that

$$\bigwedge^2 H \otimes_{\mathscr{O}_X} H^* \cong \mathscr{H}om_{\mathscr{O}_X}(H, \bigwedge^2 H),$$

therefore the canonical map

$$\gamma_1 : H \longrightarrow \bigwedge^2 H \otimes_{\mathscr{O}_X} H^* = \mathscr{H}om_{\mathscr{O}_X}(H, \bigwedge^2 H)$$

is given by

$$\gamma_1(u)(v) = u \wedge v,$$

where u and v are sections of H.

Now, consider the map

$$p_1 : H^* \otimes_{\mathscr{O}_X} H^* \longrightarrow \mathscr{H}om_{\mathscr{O}_X}(H, \Lambda_1^{H,\delta})$$

defined as

$$p_1(s \otimes t)(u) = ((s \wedge t) \otimes \mathbf{1}_{\Lambda_1^{H,\delta}})(\gamma(u)),$$

where s and t are sections of H^*, and u is a section of H. Note that $(\bigwedge^2 H)^* = \bigwedge^2 H^*$, therefore $s \wedge t : \bigwedge^2 H \to \mathscr{O}_X$ is given by $(s\wedge t)(u \wedge v) = s(u)t(v) - s(v)t(u)$. Now, $(s\wedge t)\otimes \mathbf{1}_{\Lambda_1^{H,\delta}}$ is a map from $\bigwedge^2 H \otimes \Lambda_1^{H,\delta} \to \Lambda_1^{H,\delta}$, and $\gamma(u)$ is a section of $\bigwedge^2 H \otimes \Lambda_1^{H,\delta}$. Thus, applying $(s\wedge t)\otimes \mathbf{1}_{\Lambda_1^{H,\delta}}$ on $\gamma(u)$ we get a section of $\Lambda_1^{H,\delta}$.

Further, we define two other maps

$$p_2 : H^* \otimes_{\mathscr{O}_X} H^* \longrightarrow \mathscr{H}om_{\mathscr{O}_X}(H, \mathscr{O}_X) \subset \mathscr{H}om_{\mathscr{O}_X}(H, \Lambda_1^{H,\delta}),$$

by

$$p_2(s\otimes t)(u) = -t(\delta(s(u))) + s(\delta(t(u))),$$

and

$$p_3 : H^* \otimes_{\mathscr{O}_X} H^* \longrightarrow \mathscr{H}om_{\mathscr{O}_X}(H, H^*) \subset \mathscr{H}om_{\mathscr{O}_X}(H, \Lambda_1^{H,\delta}),$$

by

$$p_3(s\otimes t)(u) = -s(u)t + t(u)s,$$

where s,t are sections of H^* and u is a section of H.

5. Define a bracket

$$\{\ ,\ \}_\gamma : H^* \otimes_{\mathbf{C}} H^* \to \mathscr{H}om_{\mathscr{O}_X}(H, \Lambda_1^{H,\delta})$$

by the formula

$$\{s,t\}_\gamma(u) = p_1(s\otimes t)(u) + p_2(s\otimes t)(u) + p_3(s\otimes t)(u).$$

where s and t are local sections of H^* and u is a local section of H. Note that p_2 takes the value in $\mathscr{H}om_{\mathscr{O}_X}(H, \mathscr{O}_X) = H^*$. We show that $p_1 + p_3$ takes the value in $\mathscr{H}om_{\mathscr{O}_X}(H, \mathscr{O}_X)$, that is, the projection of $p_1 + p_3$ into $\mathscr{H}om_{\mathscr{O}_X}(H, H^*)$ is zero. Since we are taking projections into $\mathscr{H}om_{\mathscr{O}_X}(H, H^*)$, we will use γ_1. Note that

$(s\wedge t)\otimes \mathbf{1}_{\Lambda_1^{H,\delta}})(\gamma_1(u)$ is a section of H^*. Now, we have

$$\begin{aligned}(p_1(s\otimes t)(u))(v) &= ((s\wedge t)\otimes \mathbf{1}_{\Lambda_1^{H,\delta}})(\gamma_1(u))(v)\\ &= (s\wedge t)(u\wedge v)\\ &= s(u)t(v)-s(v)t(u)\\ &= (s(u)t-t(u)s)(v)\\ &= -p_3(s\otimes t)(u)(v).\end{aligned}$$

Therefore, we have

$$p_1(s\otimes t)(u)+p_3(s\otimes t)(u)=0,$$

and sections s,t of H^* and u of H are arbitrary.

Thus, the bracket $\{,\}_\gamma$ takes the values in $\mathscr{H}om_{\mathscr{O}_X}(H,\mathscr{O}_X)=H^*$. This bracket is antisymmetric and satisfies the Leibniz formula, that is, for s,t sections of H^* and u a section of H, we have

$$\begin{aligned}\{s,ft\}_\gamma(u) &= (s\wedge ft)(\gamma(u))-ft(\delta(s(u)))+s(\delta(ft(u)))-s(u)(ft)+ft(u)s\\ &= f(s\wedge t)(\gamma(u))-ft(\delta(s(u)))+s((\delta f)t(u)))+fs(\delta(t(u)))\\ &\quad -s(u)(ft)+ft(u)s\\ &= s(\delta(f)t(u))+f\{s,t\}_\gamma(u)\end{aligned}$$

Thus, $\{s,ft\}_\gamma = s(\delta(f)t(.))+f\{s,t\}_\gamma$.

6. The bracket $\{\ ,\ \}_\gamma$ satisfied the Jacobi identity.

The symbol morphism of a differential operator has been described in Chapter 3, section 3.3.

Theorem 5.2.9 *[SimI94, Theorem 2.11] Suppose that (Λ, ζ) is a split almost polynomial sheaf of rings of differential operators. Then there exists a unique triple (H, δ, γ) in Ξ and an isomorphism*

$$\eta : \mathrm{gr}_1(\Lambda) \cong H^*,$$

such that δ corresponds to the symbol morphism and the bracket $\{\ ,\ \}_\gamma$ gives the commutator of the elements under the isomorphism

$$\Lambda_1 \cong H^* \oplus \mathscr{O}_X = \Lambda_1^{H,\delta}$$

given by the splitting. Conversely, suppose that $(H, \delta, \gamma) \in \Xi$. Then there exists a split almost polynomial sheaf of rings of differential operators $(\Lambda^{H,\delta,\gamma}, \zeta)$ together with an isomorphism

$$\eta : \mathrm{gr}_1(\Lambda^{H,\delta,\gamma}) \cong H^*$$

such that δ corresponds to the symbol and γ corresponds to the commutator of the elements of under the isomorphism

$$\Lambda_1^{H,\delta,\gamma} \cong H^* \oplus \mathscr{O}_X = \Lambda_1^{H,\delta}$$

given by the splitting.

Moreover, if (Λ, ζ) is any other split almost polynomial sheaf of rings of differential operators corresponding to (H, δ, γ) under the previous paragraph, then there is a unique isomorphism $\Lambda \cong \Lambda^{H,\delta,\gamma}$ compatible with the splitting and the isomorphism η.

Let $(H, \delta, \gamma) \in \Xi$. From above Theorem 5.2.9, there is a a split polynomial sheaf Λ of rings of differential operators corresponding to the triple (H, δ, γ). Suppose that E is a Λ-module. Then, we have morphism of sheaves

$$D : E \to H \otimes_{\mathscr{O}_X} E$$

defined by

$$D(e)(\lambda) = \zeta(\lambda)(e),$$

where e is a local section of E and λ is a local section of H^*, which satisfies the Leibniz rule,

$$D(fe) = fD(e) + \delta(f) \otimes e$$

for local sections e of E and f of $\mathscr{O}_X$.

Any such map D gives a left $\mathscr{O}_X$-module homomorphism

$$\phi : \Lambda_1 \otimes_{\mathscr{O}_X} E \to E$$

defined as

$$\phi((h, f) \otimes e) = fD(e)(h),$$

where $(h, f) \in \Lambda_1 \cong H^* \oplus \mathscr{O}_X$. Thus, we have following maps

$$E \xrightarrow{D} H \otimes_{\mathscr{O}_X} E \xrightarrow{\gamma \otimes \mathbf{1}_E} K \otimes_{\mathscr{O}_X} \Lambda_1 \otimes_{\mathscr{O}_X} E \xrightarrow{\mathbf{1}_K \otimes \phi} K \otimes_{\mathscr{O}_X} E$$

whose composition $(\mathbf{1}_K \otimes \phi) \circ (\gamma \otimes \mathbf{1}_E) \circ D$ is zero. From this observation, we get that the connections under consideration are integrable.

In different set up we will get different Λ and H, which is described as follows:

Let X be a smooth projective variety over $\mathbf{C}$, and S a smooth normal crossing divisor on X. Let $\mathscr{D}_X$ denote the sheaf of rings of differential operators on X. Then $\mathscr{D}_X$ is a split almost polynomial sheaf of rings of differential operators. In fact, $\mathscr{D}_X$ is polynomial. Let $\mathscr{T}_X$ denote the tangent sheaf over X. Then $\mathscr{D}_X$ is isomorphic to the symmetric algebra $\mathrm{Symm}^\bullet(\mathscr{T}_X)$. We have $H = \Omega^1_X$ and δ is the canonical derivation. Since the $\mathscr{O}_X$-coherent Λ-modules are automatically locally free over $\mathscr{O}_X$. Thus, a Λ-module consists of a locally free sheaf E with an integrable connection $D : E \to E \otimes \Omega^1_X$.

We are mainly interested in the construction of the moduli space of integrable meromorphic connections and integrable logarithmic connections singular over a smooth normal crossing divisor. For that we shall describe two types of split almost polynomial sheaf of rings of differential operators.

I. Define $\mathscr{T}_X(-S) = \mathscr{T}_X \otimes_{\mathscr{O}_X} \mathscr{O}_X(-S)$, that is, $\mathscr{T}_X(-S)$ is a subsheaf of $\mathscr{T}_X$ consisting of vector fields on X which vanishes on S, where $\mathscr{O}_X(-S)$ is the ideal sheaf of $\mathscr{O}_X$. Let $\Lambda^{\mathrm{Mero}} \subset \mathscr{D}_X$ denote the split almost polynomial sheaf of rings of differential operator generated by $\mathscr{O}_X$ and $\mathscr{T}_X(-S)$. In this case,

$$H = \Omega^1_X(*S),$$

and δ is the universal derivation from the sheaf of meromorphic functions on X to $\Omega^1_X(*S)$. And a Λ^{Mero}-module E is a an $\mathscr{O}_X$-module E with a meromorphic connection on E singular over S.

II. Let $\mathscr{T}_X(\log S) \subset \mathscr{T}_X$ denote the subsheaf of the tangent sheaf dual to $\Omega^1_X(\log S)$ the sheaf of logarithmic differentials singular over S. Let $\Lambda^{\mathrm{Log}} \subset \mathscr{D}_X$ denote the split almost polynomial sheaf of rings of differential operators generated by $\mathscr{O}_X$ and $\mathscr{T}_X(\log S)$. In this case,

$$H = \Omega^1_X(\log S),$$

and δ is the usual derivation from $\mathscr{O}_X$ to $\Omega^1_X(\log S)$. In a similar way, a Λ^{Log}-module E is nothing but an $\mathscr{O}_X$-module E with a logarithmic connections singular over S.

5.2.2 Stability of Λ-modules

To construct the moduli space, we need to define the notion of (semi)stable objects in the category of the sheaves of Λ- modules. The definitions are the same as in the previous section 5.1.1, except that we consider only those subsheaves F preserved under the action of Λ. In what follows, Λ stands for both Λ^{Mero} and Λ^{Log}.

Let X be a smooth projective space over $\mathbf{C}$ with a fixed ample line bundle $\mathscr{O}_X(1)$ and

Λ be a sheaf of rings of differential operators over X. By a pure coherent Λ-module E, we mean that pure coherent $\mathscr{O}_X$-module equipped with a Λ-module structure, and hence the Hilbert polynomial and the rank of E are that of underlying $\mathscr{O}_X$-module.

Definition 5.2.10 A coherent Λ-module E of rank $r > 0$ is said to be a semistable (respectively, stable) if the following conditions are satisfied:

1. E is pure.

2. For any Λ-submodule $F \subset E$ with $0 < \mathrm{rk}(F) < \mathrm{rk}(E)$, there exists an N such that

$$\frac{P_F(n)}{\mathrm{rk}(F)} \leq \frac{P_E(n)}{\mathrm{rk}(E)}$$

(respectively, $<$) for $n \geq N$, where $P_E(n)$ denote the Hilbert polynomial of E.

A coherent Λ-module E is said to be μ-semistable (respectively, μ-stable) if E is pure and for any proper Λ-submodule F of E, we have

$$\mu(F) \leq \mu(E)$$

(respectively, $\mu(F) < \mu(E)$).

5.2.3 The construction

Suppose E is a semistable Λ-module. Then there exists a unique filtration by Λ-submodules such that the successive quotients are direct sum of stable Λ-modules with the same reduced Hilbert polynomial. As usual define $\mathrm{gr}(E)$ to be the direct sum of the quotients in this filtration.

Definition 5.2.11 We say that two semistable Λ-modules, E_1 and E_2 with same reduced Hilbert polynomial, are called S-equivalent if $\mathrm{gr}(E_1) \cong \mathrm{gr}(E_2)$.

We shall state the following lemma, which is about the boundedness of the family of semistable Λ- modules with fixed Hilbert polynomial P.

Lemma 5.2.12 *Let l be an integer such that $\mathrm{gr}_1(\Lambda)\otimes_{\mathscr{O}_X}\mathscr{O}_X(l)$ is generated by global sections. Then for any semistable Λ- module E of rank r, and any $\mathscr{O}_X$-submodule $F\neq 0$, we have*

$$\mu(F)\leq\mu(E)+lr.$$

Proof. See the proof of Lemma 3.3 of [SimI94], which also works for Λ^{Mero}.

Corollary 5.2.13 *The set of semistable Λ-modules on X with given Hilbert polynomial P is bounded.*

Proof. See [SimI94], Corollary 3.4.

From the boundedness of the set of all semistable Λ-modules with fixed Hilbert polynomial, we will get a parametrizing scheme for the family of semistable Λ-modules stated in the following theorem.

Theorem 5.2.14 *[SimI94, Theorem 3.8] For a fixed polynomial P, there exists a positive integer $N_0(\Lambda,P)>0$ depending on P and Λ such that for any $N\geq N_0$, any scheme S of finite type over $\mathbf{C}$, and any S-flat semistable Λ-module E with fixed Hilbert polynomial P, we have*

$$\mathrm{H}^i(X,E_s(N))=0 \quad \text{for } i>0,$$

$$\dim_k(\mathrm{H}^0(X,E_s(N)))=P(N)$$

and $E_s(N)$ is generated by global sections, for every $s\in S$.

Let N_0 be chosen as above. Choose any $N\geq N_0$. Then the functor which associates to every $\mathbf{C}$-scheme S the set of isomorphism classes of pairs (E,α), where E is a semistable Λ-module with Hilbert polynomial P on $X_S=X\times_{\mathbf{C}}S$ and

$$\alpha:\mathscr{O}_S^{P(N)}\to\mathrm{H}^0(X_S/S,E(N))$$

is an isomorphism, is represented by a quasi-projective scheme Q over $\mathbf{C}$.

The construction of the scheme Q in Theorem 5.2.14 has been made in several steps.

We summarise the steps involved in the construction as follows:

1. Take the Grothendieck's Quot scheme

$$Q' = Quot_{X/\mathbf{C}}(\mathscr{O}_X^{P(N)}(-N), P)$$

parametrising the quotients

$$\mathscr{O}_X^{P(N)}(-N) \to E \to 0$$

with Hilbert polynomial P.

2. Now, over Q', construct another scheme Q'' which parametrises the family of morphisms

$$\Lambda_1 \otimes_{\mathscr{O}_X} E \longrightarrow E,$$

defining the structure of Λ-modules on the quotients E parametrised by Q'.

3. Finally, Q is the open subscheme of Q'' parametrising the semistable Λ-modules.

Under the same hypothesis and notation as in Theorem 5.2.14, Q is invariant under $\mathrm{SL}(P(N))$.

Define a functor

$$\mathscr{M}(\Lambda, P) : (\mathbf{Sch}/\mathbf{C})^{op} \longrightarrow \mathbf{Set}$$

as follows:

If $S \in Ob(\mathbf{Sch}/\mathbf{C})$, define $\mathscr{M}(\Lambda, P)(S)$ to be the set of isomorphism classes of semistable Λ-modules on $X \times_{\mathbf{C}} S$, flat over S with Hilbert polynomial P.

For any morphism $f : S' \to S$ in $\mathbf{Sch}/\mathbf{C}$, define $\mathscr{M}(\Lambda, P)(f)$ to be the map obtained by pulling-back sheaves via $f_X = \mathbf{1}_X \times f$, that is,

$$\mathscr{M}(\Lambda, P)(f) : \mathscr{M}(\Lambda, P)(S) \to \mathscr{M}(\Lambda, P)(S'), \qquad [F] \mapsto [f_X^* F].$$

Theorem 5.2.15 *[SimI94, Theorem 4.7] Let* $\mathrm{M}(\Lambda, P) = Q/\mathrm{SL}(P(N))$ *be the GIT quotient. Then, there is a morphism of functors* $\phi : \mathscr{M}(\Lambda, P) \to h_{\mathrm{M}(\Lambda,P)}$ *such that* $(\mathrm{M}(\Lambda, P), \phi)$ *universally corepresents the functor* $\mathscr{M}(\Lambda, P)$*. The following properties are satisfied.*

1. $\mathrm{M}(\Lambda, P)$ *is a quasiprojective variety over* $\mathbf{C}$.
2. *The geometric points of* $\mathrm{M}(\Lambda, P)$ *represents the S-equivalence classes of semistable* Λ*-modules with the Hilbert polynomial* P.
3. *The closed orbits in* Q *are in* 1*-to-*1 *correspondence with the semisimple objects.*
4. *There is an open subset* $\mathrm{M}^s(\Lambda, P) \subset \mathrm{M}(\Lambda, P)$ *whose points represents the isomorphism class of stable* Λ*-modules.*

5.2.4 Irreducibility of the moduli spaces

We consider the moduli space of meromorphic and logarithmic connections singular over a finite subset of a compact Riemann surface.

Let X be a compact Riemann surface. Then, the moduli space of meromorphic (respectively, logarithmic) connections, singular over a finite subset of X, will be denoted by $\mathscr{M}_{Mero}(n,d)$ (respectively, $\mathscr{M}_{Log}(n,d)$).

From Theorem 5.2.15, the moduli spaces $\mathscr{M}_{Mero}(n,d)$ and $\mathscr{M}_{Log}(n,d)$ are quasi-projective schemes over $\mathbf{C}$.

Theorem 5.2.16 *If* X *is a compact Riemann surface of genus* $g \geq 2$*. Then the moduli spaces* $\mathscr{M}_{Mero}(n,d)$ *and* $\mathscr{M}_{Log}(n,d)$ *are normal irreducible varieties.*

Proof. See [SimII 94, Theorem 11.1].

5.3 Moduli space of logarithmic connections over a compact Riemann surface

In this section, we restrict ourselves to the moduli space of logarithmic connections over a compact Riemann surface.

Let X be a compact Riemann surface of genus $g \geq 2$, and

$$S = \{x_1, \ldots, x_m\}$$

be a finite subset consisting of distinct points of X. We denote by

$$S = x_1 + \cdots + x_m,$$

the reduced effective divisor on X associated with the set S. In this case the sheaf of logarithmic 1-forms singular over S will become $\Omega_X^1(\log S) = \Omega_X^1 \otimes \mathscr{O}_X(S)$.

5.3.1 Residue of a logarithmic connection

Let E be a holomorphic vector bundle on X of rank $n \geq 1$. We will denote the fibre of E over any point $x \in X$ by $E(x)$.

We have already defined the notion of logarithmic connection (Definition 5.2.5) in a holomorphic vector bundle E singular over S.

Let D be a logarithmic connection in E singular over S. For any $x_\beta \in S$, the fiber $\Omega_X^1 \otimes \mathscr{O}_X(S)(x_\beta)$ is canonically identified with $\mathbf{C}$ by sending a meromorphic form to its residue at x_β.

Let $v \in E(x_\beta)$ be any vector in the fiber of E over x_β. Let U be an open set around x_β and let

$$s : U \longrightarrow E$$

be a holomorphic section of E over U such that

$$s(x_\beta) = v.$$

Consider the following composition

$$\Gamma(U,E) \to \Gamma(U, E \otimes \Omega^1_X \otimes \mathscr{O}_X(S)) \to E \otimes \Omega^1_X \otimes \mathscr{O}_X(S)(x_\beta) = E(x_\beta), \tag{5.4}$$

where the equality is given because of the identification $\Omega^1_X \otimes \mathscr{O}_X(S)(x_\beta) = \mathbf{C}$.

Let t be a uniformiser at x_β on U. In other words, the coordinate system (U,t) is centered at x_β [War83], that is

$$t(x_\beta) = 0.$$

Suppose that $\sigma \in \Gamma(U,E)$ such that

$$\sigma(x_\beta) = 0.$$

Then

$$\sigma = t\sigma'$$

for some $\sigma' \in \Gamma(U,E)$. Now,

$$\begin{aligned} D(\sigma) = D(t\sigma') &= tD(\sigma') + dt \otimes \sigma' \\ &= tD(\sigma') + t(\frac{dt}{t} \otimes \sigma'), \end{aligned}$$

and

$$D(\sigma)(x_\beta) = 0.$$

Thus, we have a well defined endomorphism, denoted by

$$Res(D, x_\beta) \in \mathrm{End}(E)(x_\beta) = \mathrm{End}(E(x_\beta)) \tag{5.5}$$

that sends v to $D(s)(x_\beta)$.

Definition 5.3.1 This endomorphism $Res(D,x_\beta)$ defined above is called the **residue** of the logarithmic connection D at the point $x_\beta \in S$.

If D is a logarithmic connection in E singular over S and $\theta \in \mathrm{H}^0(X,\Omega^1_X \otimes \mathrm{End}(E))$, then $D+\theta$ is also a logarithmic connection in E, singular over S. Also, we have

$$Res(D,x_\beta) = Res(D+\theta,x_\beta),$$

for every $x_\beta \in S$.

Conversely, if D and D' are two logarithmic connections on E singular over S with

$$Res(D,x_\beta) = Res(D',x_\beta), \tag{5.6}$$

then $D' = D+\theta$, where $\theta \in \mathrm{H}^0(X,\Omega^1_X \otimes \mathrm{End}(E))$.

Thus, the space of all logarithmic connections, in a given holomorphic vector bundle E, singular over S, and satisfying (5.6), is an affine space modelled over $\mathrm{H}^0(X,\Omega^1_X \otimes \mathrm{End}(E))$.

For every $i = 1,\ldots,m$, fix $\lambda_i \in \mathbf{Q}$ such that $n\lambda_i \in \mathbf{Z}$, where n is the rank of the vector bundle E. By a pair (E,D) over X, we mean that

1. E is a holomorphic vector bundle of degree d and rank n over X.

2. D is a logarithmic connection in E singular over S with residues

$$Res(D,x_i) = \lambda_i \mathbf{1}_{E(x_i)}$$

   for all $i = 1,\ldots,m$.

Then from [Oht82, Theorem 3], we have

$$d + n\sum_{j=1}^{m} \lambda_i = 0 \tag{5.7}$$

Lemma 5.3.2 *Let (E,D) be a pair over X as described above. Suppose that F is a holomorphic subbundle of E such that the restriction $D' = D|_F$ of D to F is a logarithmic connection in F singular over S. Then*

$$Res(D', x_j) = \lambda_j \mathbf{1}_{F(x_j)}$$

for all $j = 1, \ldots, m$.

Proof. Follows from the definition of residues.

Definition 5.3.3 A logarithmic connection D in a holomorphic vector bundle E is called **irreducible** if F is a holomorphic subbundle of E with

$$D(F) \subset \Omega_X^1(\log S) \otimes F,$$

then either $F = E$ or $F = 0$.

Proposition 5.3.4 *Let (E,D) be a pair over X as described above. Suppose that n and d are mutually coprime. Then D is irreducible.*

Proof. Let $0 \neq F$ be a holomorphic subbundle of E invariant under D, that is, $D(F) \subset F \otimes \Omega_X^1(\log S)$. Set $D' = D|_F$. Then from Lemma (5.3.2) $Res(D', x_i) = \lambda_i \mathbf{1}_{F(x_i)}$, and from [Oht82] Theorem 3, we have

$$\text{degree}(F) + r \sum_{i=1}^{m} \lambda_i = 0, \tag{5.8}$$

where r denotes the rank of F. From (5.7) and (5.8), we get that $\mu(F) = \mu(E)$. Since F is a subbundle of E, if rank of F is less than rank of E, we get that $n|d$, which is a contradiction. Thus $F = E$.

5.3.2 Compactification of the moduli space

Let $\mathscr{M}_{lc}(n,d)$ denote the moduli space which parametrizes the isomorphic class of pairs (E,D) as described in previous section. We say that two pairs (E,D) and (E',D') of rank n and degree d are isomorphic if there exists an isomorphism

$$\Phi : E \to E'$$

such that the following diagram

$$\begin{array}{ccc} E & \xrightarrow{D} & \tilde{E} \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi\otimes \mathbf{1}_{\Omega^1_X(\log S)}} \\ E' & \xrightarrow{D'} & \tilde{E}' \end{array} \tag{5.9}$$

commutes, where $\tilde{E} = E \otimes \Omega^1_X(\log S)$ and $\tilde{E}' = E' \otimes \Omega^1_X(\log S)$.

Henceforth, we will assume following conditions

1. d and n are mutually coprime.

2. for each $i = 1,\dots,m$, $\lambda_i \in \mathbf{Q}$ such that $n\lambda_i \in \mathbf{Z}$.

3. $d,n,\lambda_1,\dots,\lambda_m$ satisfies following relation

$$d + n\sum_{i=1}^{m} \lambda_i = 0. \tag{5.10}$$

Under the above conditions, from the Proposition 5.3.4, every logarithmic connection (E,D) in $\mathscr{M}_{lc}(n,d)$ is irreducible. Since the singular points of $\mathscr{M}_{lc}(n,d)$ corresponds to reducible logarithmic connections [BR05], the moduli space $\mathscr{M}_{lc}(n,d)$ is smooth. Since we have assumed that genus $g \geq 2$, from Theorem 5.2.16, the moduli space $\mathscr{M}_{lc}(n,d)$ is irreducible. Thus, $\mathscr{M}_{lc}(n,d)$ is an irreducible smooth quasi-projective variety over $\mathbf{C}$.

Let $\mathscr{M}'_{lc}(n,d)$ denote the subset of $\mathscr{M}_{lc}(n,d)$ parametrising the logarithmic connections (E,D) with the underlying vector bundle E stable. Then, from [Mar76, Theorem

2.8(A)] $\mathscr{M}'_{lc}(n,d)$ is a Zariski open subset of $\mathscr{M}_{lc}(n,d)$. Moreover, since $\mathscr{M}_{lc}(n,d)$ irreducible, $\mathscr{M}'_{lc}(n,d)$ is Zariski dense open subset of $\mathscr{M}_{lc}(n,d)$.

Fix a holomorphic line bundle L on X of degree d. Fix a logarithmic connection D_L on L singular over S with residues

$$Res(D_L,x_i) = n\lambda_i,$$

for every $i = 1,\ldots,m$.

Let $\mathscr{M}_{lc}(n,L)$ denote the moduli space parametrising all pairs (E,D) satisfying the following properties:

1. E is a holomorphic vector bundle of rank n over X with $\bigwedge^n E \cong L$.
2. D is a logarithmic connection on E singular over S with $Res(D,x_i) = \lambda_i \mathbf{1}_{E(x_i)}$, for every $i = 1,\ldots,m$.
3. $(\bigwedge^n E,\tilde{D}) \cong (L,D_L)$, where $\tilde{D}$ is a logarithmic connection in $\bigwedge^n E$ induced by D.

Then $\mathscr{M}_{lc}(n,L)$ is a closed subvariety of $\mathscr{M}_{lc}(n,d)$. Define

$$\mathscr{M}'_{lc}(n,L) = \mathscr{M}_{lc}(n,L) \cap \mathscr{M}'_{lc}(n,d).$$

Then $\mathscr{M}'_{lc}(n,L)$ Zariski open subset of $\mathscr{M}_{lc}(n,L)$.

In particular, if we take

$$L_0 = \otimes_{i=1}^{m} \mathscr{O}_X(-n\lambda_i x_i)$$

and D_{L_0} the logarithmic connection defined by the de Rham differential, then D_{L_0} is singular over S with residues

$$Res(D_{L_0},x_i) = n\lambda_i$$

for all $i = 1,\ldots,m$. For the pair (L_0,D_{L_0}), we denote the moduli spaces $\mathscr{M}_{lc}(n,L)$ and $\mathscr{M}'_{lc}(n,L)$ by $\mathscr{M}_{lc}(n,L_0)$ and $\mathscr{M}'_{lc}(n,L_0)$ respectively.

Let $\mathscr{U}(n,d)$ denote the moduli space of all stable vector bundles of rank n and degree d over X. Then $\mathscr{U}(n,d)$ is an irreducible smooth complex projective variety of dimension $n^2(g-1)+1$ (see [Ram73]).

Let

$$p : \mathscr{M}'_{lc}(n,d) \to \mathscr{U}(n,d) \tag{5.11}$$

be the forgetful map which forgets its logarithmic structure. Then, from Lemma 5.1.8 p is a morphism of algebraic varieties.

Let $E \in \mathscr{U}(n,d)$. Then E is indecomposable. Since d,n satisfy equation (5.10), from [BDP18], Proposition 1.2, E admits a logarithmic connection D singular over S, with residues $Res(D,x_j) = \lambda_j \mathbf{1}_{E(x_j)}$ for all $j = 1,\dots,m$.

Thus, the pair (E,D) is in the moduli space $\mathscr{M}'_{lc}(n,d)$, and hence p is surjective.

To prove the compactification of $\mathscr{M}_{lc}(n,d)$, we need the notion of torsors. We recall the definition of torsors and will show that the map

$$p : \mathscr{M}'_{lc}(n,d) \to \mathscr{U}(n,d)$$

is an $\Omega^1_{\mathscr{U}(n,d)}$-*torsor* on $\mathscr{U}(n,d)$, where $\Omega^1_{\mathscr{U}(n,d)}$ denotes the holomorphic cotangent bundle over $\mathscr{U}(n,d)$.

Definition 5.3.5 Let M be a connected complex manifold. Let

$$\pi : \mathscr{V} \to M$$

be a holomorphic vector bundle.

A $\mathscr{V}$-*torsor* on M is a holomorphic fiber bundle $p : Z \to M$, and holomorphic map from the fiber product

$$\varphi : Z \times_M \mathscr{V} \to Z$$

such that

1. $p \circ \varphi = p \circ p_Z$, where p_Z is the natural projection of $Z \times_M \mathscr{V}$ to Z,

2. the map $Z \times_M \mathscr{V} \to Z \times_M Z$ defined by $p_Z \times \varphi$ is an isomorphism,

3. $\varphi(\varphi(z,v),w) = \varphi(z,v+w)$.

Note that the isomorphic classes of $\mathscr{V}$-torsors over M are parametrized by $\mathrm{H}^1(M,\mathscr{V})$.

Proposition 5.3.6 *Let $p : \mathscr{M}'_{lc}(n,d) \to \mathscr{U}(n,d)$ be the map as defined in (5.11). Then $\mathscr{M}'_{lc}(n,d)$ is an $\Omega^1_{\mathscr{U}(n,d)}$-*torsor *on $\mathscr{U}(n,d)$.*

Proof. Let $E \in \mathscr{U}(n,d)$. Then $p^{-1}(E) \subset \mathscr{M}'_{lc}(n,d)$ is an affine space for $\mathrm{H}^0(X,\Omega^1_X \otimes \mathrm{End}(E))$ and the fiber of the cotangent bundle

$$\pi : \Omega^1_{\mathscr{U}(n,d)} \to \mathscr{U}(n,d)$$

at E is isomorphic to $\mathrm{H}^0(X,\Omega^1_X \otimes \mathrm{End}(E))$, that is,

$$\Omega^1_{\mathscr{U}(n,d),E} \cong \mathrm{H}^0(X,\Omega^1_X \otimes \mathrm{End}(E)).$$

There is a natural action of $\Omega^1_{\mathscr{U}(n,d),E}$ on $p^{-1}(E)$, that is,

$$\Omega^1_{\mathscr{U}(n,d),E} \times p^{-1}(E) \to p^{-1}(E)$$

sending (ω, D) to $\omega + D$. This action on the fibre is faithful and transitive. This action will induce a holomorphic map on the fibre product

$$\varphi : \Omega^1_{\mathscr{U}(n,d)} \times_{\mathscr{U}(n,d)} \mathscr{M}'_{lc}(n,d) \to \mathscr{M}'_{lc}(n,d), \tag{5.12}$$

which satisfies the above conditions in the definition of the torsor.

Remark 5.3.7 Note that $p : \mathscr{M}'_{lc}(n,d) \to \mathscr{U}(n,d)$ as defined in (5.11) is a fibre bundle (not a vector bundle) with fibre $p^{-1}(E)$ which is an affine space modelled over

$\mathrm{H}^0(X, \Omega^1_X \otimes \mathrm{End}(E))$. Moreover, from Proposition 5.3.6, $\mathscr{M}'_{lc}(n,d)$ is a $\Omega^1_{\mathscr{U}(n,d)}$-*torsor* on $\mathscr{U}(n,d)$. We know that the dual of an affine space is a vector space over $\mathbf{C}$. Similarly, the dual of a torsor is a vector bundle. We use this fact to construct an algebraic vector bundle over $\mathscr{U}(n,d)$.

Theorem 5.3.8 *There exists an algebraic vector bundle*

$$\pi : \Xi \to \mathscr{U}(n,d)$$

such that $\mathscr{M}'_{lc}(n,d)$ *is embedded in* $\mathbf{P}(\Xi)$ *with* $\mathbf{P}(\Xi) \setminus \mathscr{M}'_{lc}(n,d)$ *as the hyperplane at infinity.*

Proof. For any $E \in \mathscr{U}(n,d)$, the fiber $p^{-1}(E)$ is an affine space modelled on $\mathrm{H}^0(X, \Omega^1_X \otimes \mathrm{End}(E))$. The dual

$$p^{-1}(E)^\vee = \{\varphi : p^{-1}(E) \to \mathbf{C} \mid \varphi \text{ is an affine linear map}\}$$

is a vector space over $\mathbf{C}$.

Let

$$\pi : \Xi \to \mathscr{U}(n,d)$$

be the algebraic vector bundle such that for every Zariski open subset U of $\mathscr{U}(n,d)$, a section of Ξ over U is an algebraic function

$$f : p^{-1}(U) \to \mathbf{C}$$

whose restriction to each fiber $p^{-1}(E)$, is an element of $p^{-1}(E)^\vee$. Then, a fiber $\Xi(E) = \pi^{-1}(E)$ of Ξ at $E \in \mathscr{U}(n,d)$ is $p^{-1}(E)^\vee$. Let $(E,D) \in \mathscr{M}'_{lc}(n,d)$, and define a map

$$\Phi_{(E,D)} : p^{-1}(E)^\vee \to \mathbf{C},$$

by

$$\Phi_{(E,D)}(\varphi) = \varphi[(E,D)],$$

which is nothing but the evaluation map. Now, the kernel $\mathrm{Ker}(\Phi_{(E,D)})$ defines a hyperplane in $p^{-1}(E)^\vee$ denoted by $H_{(E,D)}$. Let $\mathbf{P}(\Xi)$ be a projective bundle defined by hyperplanes in the fiber $p^{-1}(E)^\vee$, that is, we have

$$\tilde{\pi} : \mathbf{P}(\Xi) \to \mathscr{U}(n,d) \tag{5.13}$$

induced from π.

Define a map

$$\iota : \mathscr{M}'_{lc}(n,d) \to \mathbf{P}(\Xi) \tag{5.14}$$

by sending (E,D) to the equivalence class of $H_{(E,D)}$, which is clearly an open embedding. Set $Z = \mathbf{P}(\Xi) \setminus \mathscr{M}'_{lc}(n,d)$. Then $\tilde{\pi}^{-1}(E) \cap Z$ is a projective hyperplane in $\tilde{\pi}^{-1}(E)$ for every $E \in \mathscr{U}(n,d)$, and hence Z is a hyperplane at infinity. This completes the proof.

Consider the moduli space $\mathscr{M}'_{lc}(n,L)$ as described above and $\mathscr{U}_L(n,d) \subset \mathscr{U}(n,d)$ be the moduli space of stable vector bundles with $\bigwedge^n E \cong L$. Similarly, we have a natural morphism

$$p_0 : \mathscr{M}'_{lc}(n,L) \to \mathscr{U}_L(n,d), \tag{5.15}$$

which sends $(E,D) \mapsto E$.

Let $\Omega^1_{\mathscr{U}_L(n,d)}$ denote the holomorphic cotangent bundle on $\mathscr{U}_L(n,d)$. Then, we have following proposition.

Proposition 5.3.9 *Let $p_0 : \mathscr{M}'_{lc}(n,L) \to \mathscr{U}_L(n,d)$ be the map as defined in (5.15). Then $\mathscr{M}'_{lc}(n,L)$ is a $\Omega^1_{\mathscr{U}_L(n,d)}$-*torsor *on $\mathscr{U}_L(n,d)$.*

Proof. First note that for any $E \in \mathscr{U}_L(n,d)$, the holomorphic cotangent space $\Omega^1_{\mathscr{U}_L(n,d),E}$ at E is isomorphic to $\mathrm{H}^0(X, \Omega^1_X \otimes \mathrm{ad}(E))$, where $\mathrm{ad}(E) \subset \mathrm{End}(E)$ is the subbundle consists of endomorphism of E whose trace is zero. Also, $p_0^{-1}(E)$ is an affine space modelled over

$\mathrm{H}^0(X, \Omega^1_X \otimes \mathrm{ad}(E))$. Thus, there is a natural action of $\Omega^1_{\mathscr{U}_L(n,d),E}$ on $p_0^{-1}(E)$, that is,

$$\Omega^1_{\mathscr{U}_L(n,d),E} \times p_0^{-1}(E) \to p_0^{-1}(E)$$

sending (ω, D) to $\omega + D$, which is faithful and transitive.

Proposition 5.3.10 *There exists an algebraic vector bundle $\pi : \Xi' \to \mathscr{U}_L(n,d)$ such that $\mathscr{M}'_{lc}(n,L)$ is embedded in $\mathbf{P}(\Xi')$ with $\mathbf{P}(\Xi') \setminus \mathscr{M}'_{lc}(n,L)$ as the hyperplane at infinity.*

Proof. The proof is exactly similar to the proof of the Theorem 5.3.8.

Chapter 6

Picard group and functions for the moduli space

In present chapter, our aim is to study algebraic functions and Picard group for the moduli space of logarithmic connections over a compact Riemann surface.

In section 6.1, we study the Picard group of the moduli space of logarithmic connections singular over a finite subset of a compact Riemann surface whose underlying vector bundle is stable, see Theorem 6.1.1. We also compute the Picard group of the moduli space of logarithmic connections with fixed determinant, see Proposition 6.1.2. In subsection 6.2.1, we study the algebraic functions on the moduli space of logarithmic connections with fixed residues and show that the moduli space does not admit any non-constant algebraic function. In subsection 6.2.2, we describe the holomorphic structure on the moduli space. Using Deligne's extension theorem [Del70], we show that the moduli space is biholomorphic to the Betti moduli space. We deduce that the moduli space admits a non-constant holomorphic function. In subsection 6.2.3, we characterise the algebraic functions on the moduli space of logarithmic connections with arbitrary residues.

6.1 Picard group of the moduli space of logarithmic connections

The theory of Picard groups on schemes has been developed in several excellent references, for instance [Liu02], [Bos13]. Here, we recall the definition of Picard group of a scheme. Let X be a scheme over a field k. Recall that an $\mathscr{O}_X$-module L is said to be an invertible sheaf on X if it is locally free sheaf of rank 1, that is, there exists an open covering $(U_i)_{i\in I}$ of X such that $L|_{U_i} \cong \mathscr{O}_X|_{U_i}$, for every $i \in I$. Invertible sheaves are also termed as Line bundles.

Let $\mathrm{Pic}(X)$ denote the set of isomorphism classes of invertible sheaves on X. First note that if L_1 and L_2 are two invertible sheaves on X, then $L_1 \otimes_{\mathscr{O}_X} L_2$ is also an invertible sheaf. Thus, the tensor product of invertible sheaves defines a law of composition on $\mathrm{Pic}(X)$. It is obvious that this law is commutative and associative, and $\mathscr{O}_X$ is the identity. Let $L^\vee := \mathscr{H}om_{\mathscr{O}_X}(L, \mathscr{O}_X)$ denote the dual of the invertible sheaf L. There is a canonical morphism of $\mathscr{O}_X$ modules

$$f : L^\vee \otimes_{\mathscr{O}_X} L \to \mathscr{O}_X,$$

which is given, on the presheaf level, by

$$\phi \otimes s \mapsto \phi(s),$$

where ϕ and s are local sections of $L^\vee$ and L, respectively. This f is an isomorphism if $L = \mathscr{O}_X$, and this is again valid if L is locally free. Therefore, $L^\vee$ is the inverse of L. Thus $\mathrm{Pic}(X)$ is a commutative group, called the **Picard group** of the scheme X.

Let $f : X \to Y$ be a morphism of schemes and L be an invertible sheaf on Y. Then, the pull-back f^*L, of the invertible sheaf L, is an invertible sheaf on X. This will induce a group homomorphism

$$\mathrm{Pic}(f) = f^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X).$$

Thus, $\mathrm{Pic}(\_)$ is a functor from the category of schemes to the category of abelian groups.

In Chapter 5 section 5.3, we have discussed irreducibility and compactification of the moduli space $\mathscr{M}'_{lc}(n,d)$. In this section, we will compute the Picard group of the same moduli space.

Let us recall that we have a morphism $p : \mathscr{M}'_{lc}(n,d) \to \mathscr{U}(n,d)$ of varieties as defined in Chapter 5, (5.11). The morphism p induces a homomorphism

$$p^* : \mathrm{Pic}(\mathscr{U}(n,d)) \to \mathrm{Pic}(\mathscr{M}'_{lc}(n,d)) \tag{6.1}$$

of Picard groups, that sends an algebraic line bundle ξ over $\mathscr{U}(n,d)$ to an algebraic line bundle $p^*\xi$ over $\mathscr{M}'_{lc}(n,d)$.

Theorem 6.1.1 *The homomorphism* $p^* : \mathrm{Pic}(\mathscr{U}(n,d)) \to \mathrm{Pic}(\mathscr{M}'_{lc}(n,d))$ *is an isomorphism of groups.*

Proof. First we show that p^* in (6.1) is injective. Let $\xi \to \mathscr{U}(n,d)$ be an algebraic line bundle such that $p^*\xi$ is a trivial line bundle over $\mathscr{M}'_{lc}(n,d)$. Giving a trivialization of $p^*\xi$ is equivalent to giving a nowhere vanishing section of $p^*\xi$ over $\mathscr{M}'_{lc}(n,d)$. Fix $s \in \mathrm{H}^0(\mathscr{M}'_{lc}(n,d), p^*\xi)$ a nowhere vanishing section. Take any point $E \in \mathscr{U}(n,d)$. Then,

$$s|_{p^{-1}(E)} : p^{-1}(E) \to \xi(E)$$

is a nowhere vanishing map. Notice that $p^{-1}(E) \cong \mathbf{C}^N$ and $\xi(E) \cong \mathbf{C}$, where $N = n^2(g-1)+1$. Now, any nowhere vanishing algebraic function on an affine space $\mathbf{C}^N$ is a constant function, that is, $s|_{p^{-1}(E)}$ is a constant function and hence corresponds to a non-zero vector $\alpha_E \in \xi(E)$. Since s is constant on each fiber of p, the trivialization s of $p^*\xi$ descends to a trivialization of the line bundle ξ over $\mathscr{U}(n,d)$, and hence giving a nowhere vanishing section of ξ over $\mathscr{U}(n,d)$. Thus, ξ is a trivial line bundle over $\mathscr{U}(n,d)$.

Now, we show that p^* is surjective.

Let $\theta \to \mathscr{M}'_{lc}(n,d)$ be an algebraic line bundle. Since $\mathscr{M}'_{lc}(n,d) \hookrightarrow \mathbf{P}(\Xi)$ [follows from the Theorem 5.3.8], we can extend θ to a line bundle θ' over $\mathbf{P}(\Xi)$. Further, from the morphism

$$\tilde{\pi} : \mathbf{P}(\Xi) \to \mathscr{U}(n,d)$$

in (5.13), and from [Har77, Chapter III, Exercise 12.5, p.n. 291.],we have

$$\operatorname{Pic}(\mathbf{P}(\Xi)) \cong \tilde{\pi}^* \operatorname{Pic}(\mathscr{U}(n,d)) \oplus \mathbf{Z}\mathscr{O}_{\mathbf{P}(\Xi)}(1) \tag{6.2}$$

Therefore,

$$\theta' = \tilde{\pi}^* L \otimes \mathscr{O}_{\mathbf{P}(\Xi)}(l), \tag{6.3}$$

where L is a line bundle over $\mathscr{U}(n,d)$ and $l \in \mathbf{Z}$. Since $Z = \mathbf{P}(\Xi) \setminus \mathscr{M}'_{lc}(n,d)$ is the hyperplane at infinity, again from (6.2) the line bundle $\mathscr{O}_{\mathbf{P}(\Xi)}(Z)$ associated to the divisor Z can be expressed as

$$\mathscr{O}_{\mathbf{P}(\Xi)}(Z) = \tilde{\pi}^* L_1 \otimes \mathscr{O}_{\mathbf{P}(\Xi)}(1) \tag{6.4}$$

for some line bundle L_1 over $\mathscr{U}(n,d)$. Now, from (6.3) and (6.4), we get

$$\theta' = \tilde{\pi}^* (L \otimes (L_1^\vee)^{\otimes l}) \otimes \mathscr{O}_{\mathbf{P}(\Xi)}(lY).$$

Since, the restriction of the line bundle $\mathscr{O}_{\mathbf{P}(\Xi)}(Z)$ to the complement $\mathbf{P}(\Xi) \setminus Z = \mathscr{M}'_{lc}(n,d)$ is the trivial line bundle and restriction of $\tilde{\pi}$ to $\mathscr{M}'_{lc}(n,d)$ is the map p defined in (5.11), therefore, we have

$$\theta = p^* (L \otimes (L_1^\vee)^{\otimes l}).$$

This completes the proof.

Now, consider the morphism

$$p_0 : \mathscr{M}'_{lc}(n,L) \to \mathscr{U}_L(n,d)$$

as defined in (5.15). The morphism p_0 induces a homomorphism of Picard groups, and we have

Proposition 6.1.2 *The homomorphism* $p_0^* : \mathrm{Pic}(\mathscr{U}_L(n,d)) \to \mathrm{Pic}(\mathscr{M}'_{lc}(n,L))$ *defined by* $\xi \mapsto p_0^*\xi$ *is an isomorphism of groups.*

Proof. In view of the Proposition 5.3.10, the proof is exactly similar to the proof of the Theorem 6.1.1.

6.2 Functions on the moduli space

6.2.1 Algebraic functions on the moduli space with fixed residues

Consider the moduli space $\mathscr{U}_L(n,d)$. Then, from [Ram73, Proposition 3.4, (ii)], we have

$$\mathrm{Pic}(\mathscr{U}_L(n,d)) \cong \mathbf{Z}.$$

Thus, in view of Proposition 6.1.2, we have

$$\mathrm{Pic}(\mathscr{M}'_{lc}(n,L)) \cong \mathbf{Z}. \tag{6.5}$$

Let Θ be the ample generator of the group $\mathrm{Pic}(\mathscr{U}_L(n,d))$. We have the *symbol exact sequence* ([Ati57], [BR08]) for the holomorphic line bundle Θ given as follows,

$$0 \to \mathscr{E}nd_{\mathscr{O}_{\mathscr{U}_L(n,d)}}(\Theta) \xrightarrow{\iota} \mathscr{D}iff^1(\Theta,\Theta) \xrightarrow{\sigma} T\mathscr{U}_L(n,d) \otimes \mathscr{E}nd_{\mathscr{O}_{\mathscr{U}_L(n,d)}}(\Theta) \to 0, \tag{6.6}$$

where $\mathscr{D}iff^1(\Theta,\Theta)$ denotes the sheaf of first order holomorphic differential operator from Θ to itself, and $T\mathscr{U}_L(n,d)$ is the holomorphic tangent bundle over $\mathscr{U}_L(n,d)$. Since Θ is a holomorphic line bundle, the *symbol exact sequence* (6.6) becomes

$$0 \to \mathscr{O}_{\mathscr{U}_L(n,d)} \xrightarrow{\iota} \mathrm{At}(\Theta) \xrightarrow{\sigma} T\mathscr{U}_L(n,d) \to 0, \tag{6.7}$$

because in that case $\mathrm{At}(\Theta) = \mathscr{D}\!\mathit{iff}^1(\Theta, \Theta)$, where $\mathrm{At}(\Theta)$ is the Atiyah bundle associated to Θ.

Dualising the exact sequence (6.7), we get a short exact sequence,

$$0 \to \Omega^1_{\mathscr{U}_L(n,d)} \xrightarrow{\sigma^*} \mathrm{At}(\Theta)^* \xrightarrow{\iota^*} \mathscr{O}_{\mathscr{U}_L(n,d)} \to 0. \tag{6.8}$$

Consider $\mathscr{O}_{\mathscr{U}_L(n,d)}$ as trivial line bundle $\mathscr{U}_L(n,d) \times \mathbf{C}$. Let

$$s : \mathscr{U}_L(n,d) \to \mathscr{U}_L(n,d) \times \mathbf{C}$$

be a holomorphic map defined by $E \mapsto (E, 1)$. Then s is a holomorphic section of the trivial line bundle $\mathscr{U}_L(n,d) \times \mathbf{C}$.

Let $S = \mathrm{Im}(s) \subset \mathscr{U}_L(n,d) \times \mathbf{C}$ be the image of s. Then $S \to \mathscr{U}_L(n,d)$ is a fibre bundle. Consider the inverse image $\iota^{*-1}S \subset \mathrm{At}(\Theta)^*$, and denote it by $\mathscr{C}(\Theta)$. Then for every open subset $U \subset \mathscr{U}_L(n,d)$, a holomorphic section of $\mathscr{C}(\Theta)|_U$ over U gives a holomorphic splitting of (6.7). For instance, suppose $\gamma : U \to \mathscr{C}(\Theta)|_U$ is a holomorphic section. Then γ will be a holomorphic section of $\mathrm{At}(\Theta)^*|_U$ over U, because $\mathscr{C}(\Theta) = \iota^{*-1}S \subset \mathrm{At}(\Theta)^*$. Since $\gamma \circ \iota = \iota^*(\gamma) = \mathbf{1}_U$, so we get a holomorphic splitting γ of (6.7). Thus, $\Theta|_U$ admits a holomorphic connection. Conversely, given any holomorphic splitting of (6.7) over an open subset $U \subset \mathscr{U}_L(n,d)$, we get a holomorphic section of $\mathscr{C}(\Theta)|_U$.

Let

$$\psi : \mathscr{C}(\Theta) \to \mathscr{U}_L(n,d) \tag{6.9}$$

be the canonical projection. Then using the short exact sequence (6.8), $\mathscr{C}(\Theta)$ is a $\Omega^1_{\mathscr{U}_L(n,d)}$-*torsor* on $\mathscr{U}_L(n,d)$

Proposition 6.2.1 *There is an isomorphism of algebraic varieties*

$$f : \mathscr{C}(\Theta) \to \mathscr{M}'_{lc}(n,L) \tag{6.10}$$

such that $p_0 \circ f = \psi$, *where* p_0 *and* ψ *are defined in* (5.15) *and* (6.9) *respectively.*

Proof. We know that an isomorphism class of $\Omega^1_{\mathscr{U}_L(n,d)}$-torsor over $\mathscr{U}_L(n,d)$ is given by a cohomology class in $\mathrm{H}^1(\mathscr{U}_L(n,d),\Omega^1_{\mathscr{U}_L(n,d)})$. Since $\mathscr{C}(\Theta)$ and $\mathscr{M}'_{lc}(n,L)$ are $\Omega^1_{\mathscr{U}_L(n,d)}$-torsors, let $\alpha,\beta \in \mathrm{H}^1(\mathscr{U}_L(n,d),\Omega^1_{\mathscr{U}_L(n,d)})$ be the cohomology class corresponding to $\mathscr{C}(\Theta)$ and $\mathscr{M}'_{lc}(n,L)$ respectively. Since the

$$\dim_{\mathbf{C}}(\mathrm{H}^1(\mathscr{U}_L(n,d),\Omega^1_{\mathscr{U}_L(n,d)})) = 1,$$

there exists $c \in \mathbf{C}$ such that $\beta = c\,\alpha$. Thus, $\mathscr{C}(\Theta)$ and $\mathscr{M}'_{lc}(n,L)$ are isomorphic as a fibre bundle over $\mathscr{U}_L(n,d)$. Now, to complete the proof, it is sufficient to show that $\alpha \neq 0$ and $\beta \neq 0$. Θ being an ample line bundle, its first Chern class $c_1(\Theta) \neq 0$ and $\alpha = c_1(\Theta)$. From [BR98, Theorem 2.11], we conclude that $\beta \neq 0$.

Let $\alpha_j \in \mathbf{Q}$, for $j = 1,\ldots,m$, such that $n\alpha_j \in \mathbf{Z}$ and

$$d + n\sum_{j=1}^{m} \alpha_j = 0.$$

Fix a holomorphic line bundle L of degree d, and fix a logarithmic connection D'_L on L singular over S with residues $Res(D'_L, x_j) = n\alpha_j$ for $j = 1,\ldots,m$.

Let $\mathscr{V}_{lc}(n,L)$ denote the moduli space parametrising all pairs (E,D) satisfying following properties

1. E is a holomorphic vector bundle of rank n over X with $\bigwedge^n E \cong L$.
2. D is a logarithmic connection on E singular over S with $Res(D,x_i) = \alpha_i \mathbf{1}_{E(x_i)}$, for every $i = 1,\ldots,m$.
3. $(\bigwedge^n E, \tilde{D}) \cong (L, D_L)$, where $\tilde{D}$ is a logarithmic connection in $\bigwedge^n E$ induced by D.

Let $\mathscr{V}'_{lc}(n,L)$ denote the subset of $\mathscr{V}_{lc}(n,L)$ parametrising (E,D) with underlying vector bundle E stable.

Corollary 6.2.2 *There is an isomorphism between $\mathscr{M}'_{lc}(n,L)$ and $\mathscr{V}'_{lc}(n,L)$.*

Proof. From above Proposition 6.2.1 both the varieties are isomorphic to $\mathscr{C}(\Theta)$.

Corollary 6.2.3 *$\mathscr{M}_{lc}(n,L)$ and $\mathscr{V}_{lc}(n,L)$ are birationally equivalent.*

Proof. Since $\mathscr{M}'_{lc}(n,L)$ and $\mathscr{V}'_{lc}(n,L)$ are dense open subset of $\mathscr{M}_{lc}(n,L)$ and $\mathscr{V}_{lc}(n,L)$, respectively, and $\mathscr{M}_{lc}(n,L)$ and $\mathscr{V}_{lc}(n,L)$ are irreducible quasi-projective varieties over $\mathbf{C}$, from Corollary 6.2.2 we are done.

We will show that $\mathscr{M}'_{lc}(n,L)$ does not admit any non-constant algebraic function. In view of Proposition 6.2.1, it is enough to show that $\mathscr{C}(\Theta)$ does not have any non constant algebraic function.

Theorem 6.2.4 *Assume that genus$(X) \geq 3$. Then*

$$\mathrm{H}^0(\mathscr{C}(\Theta), \mathscr{O}_{\mathscr{C}(\Theta)}) = \mathbf{C}. \tag{6.11}$$

Proof. Let $\mathrm{At}(\Theta)$ be the Atiyah bundle over $\mathscr{U}_L(n,d)$ associated to ample line bundle Θ as described in (6.7), and $\mathbf{P}(\mathrm{At}(\Theta))$ be the projectivization of $\mathrm{At}(\Theta)$, that is, $\mathbf{P}(\mathrm{At}(\Theta))$ parametrises hyperplanes in $\mathrm{At}(\Theta)$. Let $\mathbf{P}(T\mathscr{U}_L)$ be the projectivization of the tangent bundle $T\mathscr{U}_L(n,d)$. Notice that $\mathbf{P}(T\mathscr{U}_L)$ is a subvariety of $\mathbf{P}(\mathrm{At}(\Theta))$, and $\mathbf{P}(T\mathscr{U}_L)$ is the zero locus of the of a section of the tautological line bundle $\mathscr{O}_{\mathbf{P}(\mathrm{At}(\Theta))}(1)$. Now, observe that

$$\mathscr{C}(\Theta) = \mathbf{P}(\mathrm{At}(\Theta)) \setminus \mathbf{P}(T\mathscr{U}_L).$$

Then we have

$$\mathrm{H}^0(\mathscr{C}(\Theta), \mathscr{O}_{\mathscr{C}(\Theta)}) = \varinjlim_k \mathrm{H}^0(\mathbf{P}\mathrm{At}(\Theta), \mathscr{O}_{\mathbf{P}\mathrm{At}(\Theta)}(k)) = \varinjlim_k \mathrm{H}^0(\mathscr{U}_L(n,d), \mathscr{S}^k\mathrm{At}(\Theta)) \tag{6.12}$$

where $\mathscr{S}^k\mathrm{At}(\Theta)$ denotes the k-th symmetric powers of $\mathrm{At}(\Theta)$. Consider the symbol operator

$$\sigma : \mathrm{At}(\Theta) \to T\mathscr{U}_L(n,d) \tag{6.13}$$

given in (6.7). This induces a morphism

$$\mathscr{S}^k(\sigma) : \mathscr{S}^k\mathrm{At}(\Theta) \to \mathscr{S}^k T\mathscr{U}_L(n,d) \tag{6.14}$$

of k-th symmetric powers. Now, because of the following composition

$$\mathscr{S}^{k-1}\mathrm{At}(\Theta) = \mathscr{O}_{\mathscr{U}_L(n,d)} \otimes \mathscr{S}^{k-1}\mathrm{At}(\Theta) \hookrightarrow \mathrm{At}(\Theta) \otimes \mathscr{S}^{k-1}\mathrm{At}(\Theta) \to \mathscr{S}^k\mathrm{At}(\Theta),$$

we have

$$\mathscr{S}^{k-1}\mathrm{At}(\Theta) \subset \mathscr{S}^k\mathrm{At}(\Theta) \quad \text{for all } k \geq 1. \tag{6.15}$$

Thus, we get a short exact sequence of vector bundles over $\mathscr{U}_L(n,d)$,

$$0 \to \mathscr{S}^{k-1}\mathrm{At}(\Theta) \to \mathscr{S}^k\mathrm{At}(\Theta) \xrightarrow{\mathscr{S}^k(\sigma)} \mathscr{S}^k T\mathscr{U}_L(n,d) \to 0. \tag{6.16}$$

In other words, we get a filtration

$$0 \subset \mathscr{S}^0\mathrm{At}(\Theta) \subset \mathscr{S}^1\mathrm{At}(\Theta) \subset \ldots \subset \mathscr{S}^{k-1}\mathrm{At}(\Theta) \subset \mathscr{S}^k\mathrm{At}(\Theta) \subset \ldots \tag{6.17}$$

such that

$$\mathscr{S}^k\mathrm{At}(\Theta)/\mathscr{S}^{k-1}\mathrm{At}(\Theta) \cong \mathscr{S}^k T\mathscr{U}_L(n,d) \quad \text{for all } k \geq 1. \tag{6.18}$$

Above filtration in (6.17) gives following increasing chain of $\mathbf{C}$-vector spaces

$$\mathrm{H}^0(\mathscr{U}_L(n,d), \mathscr{O}_{\mathscr{U}_L(n,d)}) \subset \mathrm{H}^0(\mathscr{U}_L(n,d), \mathscr{S}^1\mathrm{At}(\Theta)) \subset \ldots \tag{6.19}$$

To prove (6.11), it is enough to show that

$$\mathrm{H}^0(\mathscr{U}_L(n,d), \mathscr{S}^{k-1}\mathrm{At}(\Theta)) \cong \mathrm{H}^0(\mathscr{U}_L(n,d), \mathscr{S}^k\mathrm{At}(\Theta)) \quad \text{for all } k \geq 1. \tag{6.20}$$

Since,

$$\frac{\mathscr{S}^k \mathrm{At}(\Theta)}{\mathscr{S}^{k-2} \mathrm{At}(\Theta)} \cong \frac{\mathscr{S}^k T\mathscr{U}_L(n,d)}{\mathscr{S}^{k-1} T\mathscr{U}_L(n,d)},$$

we have following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{S}^{k-1}\mathrm{At}(\Theta) & \longrightarrow & \mathscr{S}^k\mathrm{At}(\Theta) & \xrightarrow{\mathscr{S}^k(\sigma)} & \mathscr{S}^k T\mathscr{U}_L(n,d) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{S}^{k-1}T\mathscr{U}_L(n,d) & \longrightarrow & \frac{\mathscr{S}^k\mathrm{At}(\Theta)}{\mathscr{S}^{k-2}\mathrm{At}(\Theta)} & \longrightarrow & \mathscr{S}^k T\mathscr{U}_L(n,d) & \longrightarrow & 0 \end{array} \tag{6.21}$$

which gives rise to a following commutative diagram of long exact sequences

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathrm{H}^0(\mathscr{U}_L(n,d), \mathscr{S}^k T\mathscr{U}_L(n,d)) & \xrightarrow{\delta_k'} & \mathrm{H}^1(\mathscr{U}_L(n,d), \mathscr{S}^{k-1}\mathrm{At}(\Theta)) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \mathrm{H}^0(\mathscr{U}_L(n,d), \mathscr{S}^k T\mathscr{U}_L(n,d)) & \xrightarrow{\delta_k} & \mathrm{H}^1(\mathscr{U}_L(n,d), \mathscr{S}^{k-1} T\mathscr{U}_L(n,d)) & \longrightarrow & \cdots \end{array} \tag{6.22}$$

To show (6.20), it is enough to prove that the boundary operator δ_k' is injective for all $k \geq 1$, which is equivalent to showing that the boundary operator

$$\delta_k : \mathrm{H}^0(\mathscr{U}_L(n,d), \mathscr{S}^k T\mathscr{U}_L(n,d)) \to \mathrm{H}^1(\mathscr{U}_L(n,d), \mathscr{S}^{k-1} T\mathscr{U}_L(n,d)) \tag{6.23}$$

is injective for every $k \geq 1$.

Now, we will describe δ_k using the first Chern class $c_1(\Theta) \in \mathrm{H}^1(\mathscr{U}_L(n,d), T^*\mathscr{U}_L(n,d))$ of the ample line bundle Θ over $\mathscr{U}_L(n,d)$ .The cup product with $kc_1(\Theta)$ gives rise to a homomorphism

$$\mu : \mathrm{H}^0(\mathscr{U}_L(n,d), \mathscr{S}^k T\mathscr{U}_L(n,d)) \to \mathrm{H}^1(\mathscr{U}_L(n,d), \mathscr{S}^k T\mathscr{U}_L(n,d) \otimes T^*\mathscr{U}_L(n,d)) \tag{6.24}$$

Also, we have a canonical homomorphism of vector bundles

$$\beta : \mathscr{S}^k T\mathscr{U}_L(n,d) \otimes T^*\mathscr{U}_L(n,d) \to \mathscr{S}^{k-1} T\mathscr{U}_L(n,d)$$

which induces a morphism of **C**-vector spaces

$$\beta^* : \mathrm{H}^1(\mathscr{U}_L(n,d), \mathscr{S}^k T\mathscr{U}_L(n,d) \otimes T^*\mathscr{U}_L(n,d)) \to \mathrm{H}^1(\mathscr{U}_L(n,d), \mathscr{S}^{k-1} T\mathscr{U}_L(n,d)). \tag{6.25}$$

So, we get a morphism

$$\tilde{\mu} = \beta^* \circ \mu : \mathrm{H}^0(\mathscr{U}_L(n,d), \mathscr{S}^k T\mathscr{U}_L(n,d)) \to \mathrm{H}^1(\mathscr{U}_L(n,d), \mathscr{S}^{k-1} T\mathscr{U}_L(n,d)). \tag{6.26}$$

Then $\tilde{\mu} = \delta_k$. It is sufficient to show that $\tilde{\mu}$ is injective.

Moreover, we have natural projection

$$\eta : T^*\mathscr{U}_L(n,d) \to \mathscr{U}_L(n,d) \tag{6.27}$$

and

$$\eta_*\eta^*\mathscr{O}_{\mathscr{U}_L(n,d)} = \bigoplus_{k\geq 0} \mathscr{S}^k T\mathscr{U}_L(n,d). \tag{6.28}$$

Thus, we have

$$\mathrm{H}^j(T^*\mathscr{U}_L(n,d), \mathscr{O}_{T^*\mathscr{U}_L(n,d)}) = \bigoplus_{k\geq 0} \mathrm{H}^j(\mathscr{U}_L(n,d), \mathscr{S}^k T\mathscr{U}_L(n,d)) \quad \text{for all } j \geq 0. \tag{6.29}$$

Now, we use Hitchin fibration to compute $\mathrm{H}^j(T^*\mathscr{U}_L(n,d), \mathscr{O}_{T^*\mathscr{U}_L(n,d)})$. Let

$$h : T^*\mathscr{U}_L(n,d) \to B_n = \bigoplus_{i=2}^{n} \mathrm{H}^0(X, K_X^i) \tag{6.30}$$

be the Hitchin map defined by sending a pair (E,ϕ) to $\sum_{i=2}^n trace(\phi^i)$. Notice that the base of the Hithcin map h in (6.2.1) is a vector space over **C** of dimension $n^2(g-1)+1$.

A generic fibre of h is of the form $h^{-1}(b) = A \setminus F$, where A is some abelian variety and F is a subvariety of A with $\operatorname{codim}(F,A) \geq 3$, where $b \in B_n$.

Let $\mathscr{M}_{Higgs}(n,L)$ denote the moduli space of stable Higgs bundles of rank n and determinant L. Then $T^*\mathscr{U}_L(n,d)$ is an open dense subset of $\mathscr{M}_{Higgs}(n,L)$ with complement of $T^*\mathscr{U}_L(n,d)$ in $\mathscr{M}_{Higgs}(n,L)$ has codimension at least 2.

In fact we have the Hitchin fibration

$$h : \mathscr{M}_{Higgs}(n,L) \to B_n = \bigoplus_{i=2}^{n} \mathrm{H}^0(X, K_X^i)$$

which is a proper map and a generic fibre is an abelian variety (for more details on Hitchin fibration see [BNR89], [Hit87] and for abelian varieties see [LB92]).

Let $g : T^*\mathscr{U}_L(n,d) \to \mathbf{C}$ be an algebaric function and let $Z = \mathscr{M}_{Higgs}(n,L) \setminus T^*\mathscr{U}_L(n,d)$. Since $\mathrm{codim}(Z, \mathscr{M}_{Higgs}(n,L)) \geq 2$, using Hartog's theorem g can be extended to an algebraic function $\tilde{g}$ on $\mathscr{M}_{Higgs}(n,L)$. Since h is proper and every generic fibre is connected, we have

$$h_* \mathscr{O}_{\mathscr{M}_{Higgs}(n,L)} = \mathscr{O}_{B_n}.$$

Thus, there exists a unique algebraic function $\hat{g} : B_n \to \mathbf{C}$ such that

$$\tilde{g} = \hat{g} \circ h.$$

Set $\mathscr{B} = \mathrm{d}(\mathrm{H}^0(B_n, \mathscr{O}_{B_n})) \subset \mathrm{H}^0(B_n, \Omega^1_{B_n})$ the space of all exact algebraic 1-form. Define a map

$$\theta : \mathrm{H}^0(T^*\mathscr{U}_L(n,d), \mathscr{O}_{T^*\mathscr{U}_L(n,d)}) \to \mathscr{B} \tag{6.31}$$

by $g \mapsto d\hat{g}$, where $\hat{g}$ is the function which is defined by descent of g as above. Then θ is an isomorphism.

From (6.29) and (6.31), we have

$$\theta : \bigoplus_{k \geq 0} \mathrm{H}^0(\mathscr{U}_L(n,d), \mathscr{S}^k T\mathscr{U}_L(n,d)) \to \mathscr{B} \tag{6.32}$$

which is an isomorphism.

Let $T_h = T_{T^*\mathscr{U}_L(n,d)/B_n} = \mathscr{K}er(dh)$ be the relative tangent sheaf on $T^*\mathscr{U}_L(n,d)$, where $dh : T(T^*\mathscr{U}_L(n,d)) \to h^*TB_n$ is a morphism of bundles.

Note that $\mathrm{H}^0(B_n, \Omega^1_{B_n}) \subset \mathrm{H}^0(T^*\mathscr{U}_L(n,d), T_h)$, and hence from (6.32), we have an injective homomorphism

$$\nu : \mathscr{B} = \oplus_{k\geq 0}\theta(\mathrm{H}^0(\mathscr{U}_L(n,d), \mathscr{S}^k T\mathscr{U}_L(n,d))) \to \mathrm{H}^0(T^*\mathscr{U}_L(n,d), T_h). \tag{6.33}$$

Consider the morphism

$\mathrm{H}^0(T^*\mathscr{U}_L(n,d), T_h) \to \mathrm{H}^1(T^*\mathscr{U}_L(n,d), T_h \otimes T^*T^*\mathscr{U}_L(n,d))$ defined by taking cup product with the first Chern class $c_1(\eta^*\Theta) \in \mathrm{H}^1(T^*\mathscr{U}_L(n,d), T^*T^*\mathscr{U}_L(n,d))$.

Using the pairing $T_h \otimes T^*T^*\mathscr{U}_L(n,d) \to \mathscr{O}_{T^*\mathscr{U}_L(n,d)}$, we get a homomorphism

$$\psi : \mathrm{H}^0(T^*\mathscr{U}_L(n,d), T_h) \to \mathrm{H}^1(T^*\mathscr{U}_L(n,d), \mathscr{O}_{T^*\mathscr{U}_L(n,d)}) \tag{6.34}$$

Since $c_1(\eta^*\Theta) = \eta^*(c_1\Theta)$, we have

$$k\psi \circ \nu \circ \theta(\omega_k) = \tilde{\mu}(\omega_k), \tag{6.35}$$

for all $\omega_k \in \mathrm{H}^0(\mathscr{U}_L(n,d), \mathscr{S}^k T\mathscr{U}_L(n,d)))$. Since ν and θ are injective homomorphisms, it is enough to show that $\psi|_{\nu(\mathscr{B})}$ is injective homomorphism. Let $\omega \in \mathscr{B} \setminus \{0\}$ be a non-zero exact 1-form. Choose $b \in B_n$ such that $\omega(b) \neq 0$. As previously discussed $h^{-1}(b) = A \setminus F$, where A is an abelian variety and F is a subvariety of A such that $\mathrm{codim}(F,A) \geq 3$. Now, $\psi(\nu(\omega)) \in \mathrm{H}^1(T^*\mathscr{U}_L(n,d), \mathscr{O}_{T^*\mathscr{U}_L(n,d)})$ and we have restriction map $\mathrm{H}^1(T^*\mathscr{U}_L(n,d), \mathscr{O}_{T^*\mathscr{U}_L(n,d)}) \to \mathrm{H}^1(h^{-1}(b), \mathscr{O}_{h^{-1}(b)})$. Since $\omega(b) \neq 0$, $\psi(\nu(\omega)) \in \mathrm{H}^1(h^{-1}(b), \mathscr{O}_{h^{-1}(b)})$. Because of the following isomorphisms

$$\mathrm{H}^1(h^{-1}(b), \mathscr{O}_{h^{-1}(b)}) \cong \mathrm{H}^1(A, \mathscr{O}_A) \cong \mathrm{H}^0(A, TA),$$

it follows that $\psi(\nu(\omega)) \neq 0$. This completes the proof.

Since $\mathscr{M}'_{lc}(n,L)$ is a open dense subset of $\mathscr{M}_{lc}(n,L)$, we have following

Corollary 6.2.5 $\mathrm{H}^0(\mathscr{M}_{lc}(n,L), \mathscr{O}_{\mathscr{M}_{lc}(n,L)}) = \mathbf{C}$.

6.2.2 Holomorphic functions on the moduli space

Let $X_0 = X \setminus S$ and $x_0 \in X_0$. Let U_j be a simply connected open set in $X_0 \cup \{x_j\}$ containing x_0 and x_j. Then

$$\pi_1(U_j \setminus \{x_j\}, x_0) \cong \mathbf{Z},$$

where 1 corresponds to the anticlockwise loop around x_j. We have a natural group homomorphism

$$h_j : \pi_1(U_j \setminus \{x_j\}, x_0) \to \pi_1(X_0, x_0).$$

for all $j = 1, \ldots, m$. Suppose that $h_j(1) = \gamma_j$ for all $j = 1, \ldots, m$. Then $\pi_1(X_0, x_0)$ admits a presentation with $2g+m$ generators $a_1, b_1, \ldots, a_g, b_g, \gamma_1, \ldots, \gamma_m$ with relation

$$\Pi_{i=1}^{g}[a_i, b_i]\Pi_{j=1}^{m}\gamma_j = 1.$$

Let $(E, D) \in \mathscr{M}_{lc}(n, L_0)$. Then D determines a holomorphic (flat) connection on the holomorphic vector bundle $E|_{X_0}$ restricted to X_0. Since $Res(D, x_j) = \lambda_j \mathbf{1}_{E(x_j)}$, for $j = 1, \ldots, m$, the image of γ_j under the monodromy representation is the $n \times n$ diagonal matrix with $\exp(-2\pi\sqrt{-1}\lambda_j)$ (see, [Del70, p.79, Proposition 3.11]). Let

$$\mathscr{R}_g \subset \mathrm{Hom}(\pi_1(X_0, x_0), \mathrm{SL}(n, \mathbf{C}))$$

denote the space of those representations

$$\rho : \pi_1(X_0, x_0) \to \mathrm{SL}(n, \mathbf{C})$$

such that

$$\rho(\gamma_j) = \exp(-2\pi\sqrt{-1}\lambda_j)\mathrm{I}_{n\times n}$$

for all $j = 1, \ldots, m$, where $\mathrm{I}_{n\times n}$ denotes the $n \times n$ identity matrix. Since the logarithmic connection D is irreducible, any representation in $\mathscr{R}_g$ is irreducible. Consider the action of $\mathrm{SL}(n,\mathbf{C})$ on $\mathscr{R}_g$ by conjugation, that is, for any $T \in \mathrm{SL}(n,\mathbf{C})$ and $\rho \in \mathscr{R}_g$ the action is defined by

$$\rho.T = T^{-1}\rho T.$$

Let

$$\mathscr{B}_g = \mathscr{R}_g/\mathrm{SL}(n,\mathbf{C})$$

be the quotient space for the conjugation action. The algebraic structure of $\mathscr{R}_g$ induces an algebraic structure on $\mathscr{B}_g$. In literature, $\mathscr{B}_g$ is known as **Betti moduli space** (see [SimI94], [SimII 94]) and it is irreducible smooth quasi-projective variety over $\mathbf{C}$. Thus, we have a holomorphic map

$$\Phi : \mathscr{M}_{lc}(n,L_0) \to \mathscr{B}_g \tag{6.36}$$

sending (E,D) to the equivalence class of its monodromy representaion under the conjugation action of $\mathrm{SL}(n,\mathbf{C})$.

For the inverse map of Φ, let $\rho \in \mathscr{B}_g$ and let (E_ρ,∇_ρ) be the flat holomorphic vector bundle over X_0 associated to ρ. Then E_ρ over X_0 extends to a holomorphic vector bundle $\overline{E_\rho}$ over X, and the connection ∇_ρ on E_ρ extends to a connection $\overline{\nabla_\rho}$ such that $(\overline{E_\rho},\overline{\nabla_\rho}) \in \mathscr{M}_{lc}(n,L_0)$(See [BM87, p.159, Theorem 4.4]). Thus, Φ is a biholomorphism.

We show that the moduli space $\mathscr{M}_{lc}(n,L_0)$ admits a non-constant holomorphic function. Consider the Betti moduli space $\mathscr{B}_g$ described above, which is an affine variety.

Let $\gamma_j \in \pi_1(X_0,x_0)$. Define a function

$$f_{jk} : \mathscr{B}_g \to \mathbf{C}$$

by $\rho \mapsto \mathrm{trace}(\rho(\gamma_j)^k)$ for $k \in \mathbf{N}$. Then f_{jk} are non-constant algebraic functions on $\mathscr{B}_g$ for $j = 1,\ldots,m$ and $k \in \mathbf{N}$. Thus, we have

Proposition 6.2.6 *$\mathscr{M}_{lc}(n,L_0)$ is not isomorphic to B_g as algebraic varieties.*

Since $\mathscr{M}_{lc}(n, L_0)$ is biholomorphic to $\mathscr{B}_g$,

$$f_{jk} \circ \Phi : \mathscr{M}_{lc}(n, L_0) \to \mathbf{C}$$

are non-constant holomorphic functions for all $j = 1, \ldots, m$ and $k \in \mathbf{N}$. Thus, $\mathscr{M}_{lc}(n, L_0)$ admits non-constant holomorphic functions.

6.2.3 Algebraic functions on the Moduli space with arbitrary residues

Let X be a compact Riemann surface of genus$(g) \geq 3$ and

$$S = \{x_1, \ldots, x_m\}$$

be a subset of distinct points of X as in previous section. By a pair (E, D) over X, we mean that

1. E is a holomorphic vector bundle over X of degree d and rank n.
2. n and d are mutually coprime.
3. D is a logarithmic connection in E singular over S.

Now, given such a pair (E, D), from [Oht82, Theorem 3], we have

$$d + \sum_{j=1}^{m} \mathrm{Tr}(Res(D, x_j)) = 0, \tag{6.37}$$

where $Res(D, x_j) \in \mathrm{End}(E(x_j))$, for all $j = 1, \ldots, m$.

Let $\mathscr{N}_{lc}(n, d)$ be the moduli space which parametrises isomorphism class of pairs (E, D). Then $\mathscr{N}_{lc}(n, d)$ is a separated quasi-projective scheme over $\mathbf{C}$ [Nit93]. Let $\mathscr{N}'_{lc}(n, d)$ be a subset of $\mathscr{N}_{lc}(n, d)$ parametrising (E, D) with underlying vector bundle E stable. Let (E, D) and (E, D') be two points in $\mathscr{N}'_{lc}(n, d)$. Then

$$D - D' \in \mathrm{H}^0(X, \mathrm{End}(E) \otimes \Omega^1_X(\log S)). \tag{6.38}$$

Next, for $\theta \in \mathrm{H}^0(X, \mathrm{End}(E) \otimes \Omega^1_X(\log S))$, we have $(E, D+\theta) \in \mathscr{N}'_{lc}(n,d)$. Notice the difference between the affine spaces when residue is fixed and otherwise. Thus, the space of all logarithmic connections D on a given stable vector bundle E singular over S, is an affine space modelled over $\mathrm{H}^0(X, \mathrm{End}(E) \otimes \Omega^1_X(\log S))$. Let

$$q : \mathscr{N}'_{lc}(n,d) \to \mathscr{U}(n,d) \tag{6.39}$$

be the natural projection defined by sending (E,D) to E. Given $E \in \mathscr{U}(n,d)$. Choose a set of complex numbers $\alpha_1, \ldots, \alpha_m$ which satisfies the following equation

$$d + n \sum_{j=1}^{m} \alpha_j = 0. \tag{6.40}$$

Since E is stable, from [BDP18, Proposition 1.2], E admits a logarithmic connection D singular over S. Thus, q is a surjective map, and dimension of each fibre $q^{-1}(E)$ is $n^2(g-1+m)$.

Now, fix a pair (L, D_L), where L is a holomorphic vector bundle of degree d and D_L is a fixed logarithmic connections on L singular over S. Let $\mathscr{N}_{lc}(n,L)$ denote the moduli space parametrising all pairs (E,D) satisfying the following properties:

1. E is a holomorphic vector bundle of rank n and degree d with $\bigwedge^n E \cong L$, and n and d are mutually coprime.

2. D is a logarithmic connection in E singular over S with $Res(D, x_j) \in Z(\mathfrak{gl}(n, \mathbf{C}))$, and $\mathrm{Tr}(Res(D, x_j)) \in \mathbf{Z}$, where $Z(\mathfrak{gl}(n, \mathbf{C}))$ denotes the centre of $\mathfrak{gl}(n, \mathbf{C})$.

3. The logarithmic connection on $\bigwedge^n E$ induced by D coincides with the given logarithmic connection D_L on L.

Then, Lemma 5.3.2 holds for such a pair (E,D), and by Proposition 5.3.4, (E,D) is irreducible.

Let $\mathcal{N}_{lc}'(n,L)$ be the subset of $\mathcal{N}_{lc}(n,L)$ whose underlying vector bundle is stable. Let

$$q_0 : \mathcal{N}_{lc}'(n,L) \to \mathcal{U}_L(n,d) \tag{6.41}$$

be the natural projection sending (E,D) to E. Note that $q_0 : \mathcal{N}_{lc}'(n,L) \to \mathcal{U}_L(n,d)$ is not a $\Omega^1_{\mathcal{U}_L(n,d)}$-torsor, and therefore we cannot apply the same technique as in previous section 6.2.1 to compute the algebraic functions on $\mathcal{N}_{lc}'(n,L)$.

Next, let

$$V = \{(\alpha_1,\ldots,\alpha_m) \in \mathbf{C}^m |\ n\alpha_j \in \mathbf{Z} \text{ and } d + n\sum_{j=1}^{m} \alpha_j = 0\}$$

Define a map

$$\Phi : \mathcal{N}_{lc}'(n,L) \to V \tag{6.42}$$

by $(E,D) \mapsto (\mathrm{Tr}(Res(D,x_1))/n,\ldots,\mathrm{Tr}(Res(D,x_m))/n)$.

Theorem 6.2.7 *Every algebraic function on $\mathcal{N}_{lc}'(n,L)$ factors through the surjective map*

$$\Phi : \mathcal{N}_{lc}'(n,L) \to V$$

as defined in (6.42).

Proof. Let $(\alpha_1,\ldots,\alpha_m) \in V$. Then $\Phi^{-1}((\alpha_1,\ldots,\alpha_m))$ is the moduli space of logarithmic connections with fixed residues $\alpha_j \mathbf{1}_{E(x_j)}$, which is isomorphic to $\mathcal{M}_{lc}'(n,L)$ follows from Corollary 6.2.2. Let

$$g : \mathcal{N}_{lc}'(n,d) \to \mathbf{C}$$

be an algebraic function. Then g restricted to each fibre of Φ is an algebraic function on the moduli space isomorphic to $\mathcal{M}_{lc}'(n,L)$. Now, from Theorem 6.2.4, g is constant on each fibre and thus defining a function from $V \to \mathbf{C}$. This completes the proof.

Similarly, we define a map

$$\Psi : \mathcal{N}_{lc}(n,L) \to V \tag{6.43}$$

by $(E,D) \mapsto (\mathrm{Tr}(Res(D,x_1))/n, \ldots, \mathrm{Tr}(Res(D,x_m))/n)$. We have following

Theorem 6.2.8 *Every algebraic function on $\mathcal{N}_{lc}(n,L)$ factors through the surjective map*

$$\Psi : \mathcal{N}_{lc}(n,L) \to V$$

as defined in (6.43).

Proof. Let $g : \mathcal{N}_{lc}(n,L) \to \mathbf{C}$ be an algebraic function. Then restriction of g to each fibre of Ψ is a constant function, follows from Corollary 6.2.5, and hence defining a function from $V \to \mathbf{C}$.

Main Conventions and Notations

For a morphism $\pi : X \to S$ of ringed spaces and two $\mathscr{O}_X$-modules $\mathscr{F}, \mathscr{G}$, $\mathscr{H}om_S(\mathscr{F}, \mathscr{G})$ will denote the sheaf of S-linear morphism from $\mathscr{F}$ to $\mathscr{G}$. If $\mathscr{F} = \mathscr{G}$, then $\mathscr{E}nd_S(\mathscr{F})$ will denote the sheaf $\mathscr{H}om_S(\mathscr{F}, \mathscr{G})$. $\mathscr{D}er_S(\mathscr{O}_X, \mathscr{F})$ will denote the sheaf of S-derivations from $\mathscr{O}_X$ to $\mathscr{F}$. We will denote by $\mathscr{C}_S(X)$ the connection algebra of X over S. $\mathrm{MC}(X/S)$ will denote the category of $\mathscr{O}_X$-modules with S-connections. For a non-negative integer k, $\mathscr{D}iff_S^k(\mathscr{F}, \mathscr{G})$ will denote the sheaf of S-differential operator of order $\leq k$ from $\mathscr{F}$ to $\mathscr{G}$ and $\mathscr{D}_{X/S}$ will denote the sheaf of ring of S- differential operators on $\mathscr{O}_X$.

For $\pi : X \to S$, a holomorphic map of complex manifolds, and a holomorphic vector bundle E over X, we will use the following notations: $\mathscr{T}_{X/S}$ and $\Omega^1_{X/S}$ will respectively denote the relative holomorphic tangent and cotangent sheaves. $\mathscr{A}t_S(E)$ and $\mathrm{at}_S(E)$ will respectively denote the relative Atiyah algebra and Atiyah class of E.

For $\pi : X \to S$, smooth morphism of smooth manifolds, and for a smooth complex vector bundle E, we will use the following notations: $\mathscr{C}_X^\infty$ will denote the sheaf of smooth functions on X. $\mathscr{A}^r_{X/S}$ will denote the sheaf of complex valued smooth relative r-forms on X over S. For non-negative integer p, $\mathscr{H}^p_{dR}(X/S)$ will denote the relative de Rham cohomology sheaf and $C_p^S(E)$ will denote the p-th relative Chern class of E over S.

For a smooth normal crossing divisor S of a complex space (or a complex smooth projective variety) X, $\Omega^k(*S)$ (respectively, $\Omega^k(\log S)$) will denote the sheaf of meromorphic (respectively, logarithmic) k-forms with poles along S.

For a scheme X, $\mathrm{Pic}(X)$ will denote the Picard group of X.

For a finite subset S of a compact Riemann surface X, we will denote by $\mathscr{M}_{lc}(n,d)$

the moduli space of logarithmic connections of degree d and rank n, singular over S with fixed residues. $\mathscr{M}'_{lc}(n,d)$ will denote the subset of $\mathscr{M}_{lc}(n,d)$ whose underlying vector bundle is stable. For a holomorphic line bundle L over X, we will denote by $\mathscr{M}_{lc}(n,L)$ the moduli space of logarithmic connections with fixed determinant L. $\mathscr{M}'_{lc}(n,L)$ will denote the subset of $\mathscr{M}_{lc}(n,L)$, whose underlying vector bundle is stable. $\mathscr{U}(n,d)$ will denote the moduli space of stable vector bundles over X of degree d and rank n. $\mathscr{U}_L(n,d)$ will denote the subset of $\mathscr{U}_(n,d)$ with fixed determinant L.

The symbols $\mathbf{N},\mathbf{Z},\mathbf{Q},\mathbf{R}$ and $\mathbf{C}$ will denote the sets of natural numbers, integers, rational numbers, real numbers and complex numbers respectively.

The trace of a square matrix A will be denoted by $\mathrm{Tr}(A)$.

The symbols $\mathscr{F}_x$ and $\mathscr{F}(x)$ will respectively denote the stalk and fibre of a sheaf $\mathscr{F}$ at the point x.

Bibliography

[Ati57] M. F. Atiyah. Complex analytic connections in fibre bundles, *Trans. Amer. Math.Soc.* 85, 181–207, 1957.

[AB02] H. Azad and I. Biswas. On holomorphic principal bundles over a compact Riemann surface admitting a flat connection, *Math. Ann.* 322, 333–346,2002.

[BNR89] A. Beauville, M. S. Narasimhan, S. Ramanan. Spectral curves and the generalised theta divisor, *Jour. reine angew. Math.* 398, 169-179, 1989.

[Bho99] U. N. Bhosle. Picard group of the moduli space of vector bundles,*Math. Ann.* 314 245-263,1999.

[BDP18] I. Biswas, A. Dan, A. Paul. Criterion for logarithmic connections with prescribed residues. *manuscripta math.*155:77–88, 2018.

[BR98] I. Biswas, N. Raghavendra. Curvature of the determinant bundle and the Kähler form over the moduli of parabolic bundles for a family of pointed curves, *Asian Jour. Math.* 2, 303-324, 1998.

[BR05] I. Biswas, N. Raghavendra. Line bundles over a moduli space of logarithmic connections on a Riemann surface. *Geom. Funct. Anal*, 15 780-808, 2005.

[BR08] I. Biswas, N. Raghavendra. The Atiyah- Weil criterion for holomorphic connections, *Indian J. pure appl. Math.*,39(1):3-47, 2008.

[BS20] I. Biswas, Anoop Singh. On the relative connections. *Comm. Algebra* 48, no. 4, 1452–1475, 2020.

[BM87] A. Borel, P.-P. Grivel,B. Kaup, A. Haefliger, B. Malgrange, F. Ehlers. *Algebraic D-modules*. Perspectives in Mathematics, 2. Academic Press, Inc., Boston, MA, xii+355 pp, 1987.

[Bos13] S. Bosch. *Algebraic Geometry and Commutative Algebra.* Springer-Verlag London, 2013.

[Del70] P. Deligne. *Équations différentielles á points singuliers réguliers*, Springer Lecture Notes in Mathematics, 163, 1970.

[DN89] J.-M. Drezet and M. S. Narasimhan. Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques. *Invent. Math.*, 97(1):53–94, 1989.

[GD67] A. Grothendieck, J. Dieudonné. *EGA IV. Étude locale des schémas et des morphismes de schémas. IV*, Inst. Hautes Études Sci. Publ. Math. 32, 5–361, 1967.

[God64] R. Godement. *Topologie alg ébrique et théorie des faisceaux. Troisième édition revue et corrigèe. Publications de l'Institut de Mathématique de l'Université de Strasbourg*, XIII. *Actualités Scientifiques et Industrielles*, No. 1252. Hermann, Paris, viii+283 pp. 1973.

[Har77] R. Hartshorne. *Algebraic geometry. Graduate Texts in Mathematics*, Vol. 52. Springer-Verlag, New York-Heidelberg, 1977.

[Hit87] N. J. Hitchin. Stable bundles and integrable systems, *Duke Math. Jour.* 54, 91-114,1987.

[HL10] D. Huybrechts and M. Lehn. The geometry of moduli spaces of sheaves Second edition. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2010.

[Kos86] J. L. Koszul. *Lectures on fiber bundles and differential geometry*, Springer-Verlag Berlin Heidelberg GmbH, 1986.

[Kob87] S. Kobayashi. *Differential geometry of complex vector bundles*, Iwanami Shoten, Princeton University Press, 1987.

[KS58] K. Kodaira and D. C. Spencer, On deformations of complex analytic structures, I, II, *Annals of Math.* **67**, 328–466, 1958.

[KS60] K. Kodaira and D. C. Spencer. On deformations of complex analytic structures, III, *Annals of Math.* **71**, 43–76, 1960.

[KS86] K. Kodaira. *Complex manifolds and deformations of complex structures* Grundlehren der Math. Wiss. 283, Springer. 1986.

[LB92] H. Lange, Ch. Birkenhake. *Complex abelian varieties*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 302. Springer-Verlag, Berlin, 1992.

[Liu02] Qing Liu. *Algebraic geometry and arithmetic curves, volume 6 of Oxford Graduate Texts in Mathematics.* Oxford University Press, Oxford, 2002. Translated from the French by Reinie Erné, Oxford Science Publications.

[Mac11] F.-X. Machu. Local structure of moduli spaces. *Internat. J. Math.* 22, no. 12, 1683–1709, 2011.

[Mach11] F.-X. Machu. Kuranishi spaces of meromorphic connections, *Illinois J. Math.* 55, no. 2, 509–541, 2011.

[Mar76] M. Maruyama. Openness of a family of torsion free sheaves, *Jour. Math. Kyoto Univ.* 16, 627-637, 1976.

[Mar77] M. Maruyama. Moduli of stable sheaves, I. *J. Math. Kyoto Univ.* 17, no. 1, 91–126, 1977.

[Mar78] M. Maruyama. Moduli of stable sheaves, II.*J. Math. Kyoto Univ.* 18, no. 3, 557–614, 1978.

[MF82] D. Mumford, J. Fogarty, F. Kirwan, *Geometric invariant theory.* Erg. Math. 34, 2nd ed., Springer Verlag, Berlin, Heidelberg, 1994.

[Nit93] N. Nitsure. Moduli of semistable logarithmic connections, *Jour. Amer. Math. Soc.* 6, 597-609, 1993.

[New78] P. E. Newstead. *Introduction to moduli problems and orbit spaces*, volume 51 of *Tata Institute of Fundamental Research Lectures on Mathematics and Physics*. Tata Institute of Fundamental Research, Bombay; by the Narosa Publishing House, New Delhi, 1978.

[Oht82] M. Ohtsuki. A residue formula for Chern classes associated with logarithmic connections. *Tokyo J. Math.* Vol.5, No. 1, 1982.

[Ram73] S. Ramanan. The moduli space of vector bundles over an algebraic curve, *Math. Ann.* 200, 69-84, 1973.

[Ram05] S. Ramanan. *Global calculus. Graduate Studies in Mathematics, 65.* American Mathematical Society, Providence, RI, xii+316 pp, 2005.

[SimI94] C. T. Simpson. Moduli of representations of fundamental group of a smooth projective variety, I, *Inst. Hautes Études Sci. Publ. Math.* 79, 47-129, 1994.

[SimII 94] C. T. Simpson. Moduli of representations of fundamental group of a smooth projective variety, II, *Inst. Hautes Études Sci. Publ. Math.* 80, 5-79, 1994.

[Sai80] Kyoji Saito. Theory of logarithmic differential forms and logarithmic vector fields. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* 27, no. 2, 265-291, 1980.

[Seb11] R. Sebastian. Torelli theorems for moduli space of logarithmic connections and parabolic bundles, *manuscripta math.* 136, 249-271, 2011.

[Voi04] Claire Voisin. *Hodge theory and complex algebraic geometry. I, Cambridge Studies in Advanced Mathematics, vol. 76*, Cambridge University Press, Cambridge, 2002. Reprinted 2004. Translated from the French original by Leila Schneps.

[Voi05] Claire Voisin. *Hodge theory and complex algebraic geometry. II, Cambridge Studies in Advanced Mathematics, vol. 77*, Cambridge University Press, Cambridge, 2003. Reprinted 2005. Translated from the French original by Leila Schneps.

[War83] Frank W. Warner. *Foundations of differentiable manifolds and Lie groups*, volume 94 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1983. Corrected reprint of the 1971 edition.

[Wei38] A. Weil, Généralisation des fonctions abéliennes, *J. Math. Pures Appl.* 17, 47–87, 1938.

[Wel80] R. O. Wells, Jr. *Differential analysis on complex manifolds*, volume 65 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, second edition, 1980.

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Thesis Highlight

Name of the Student : Anoop Singh

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Enrolment No : MATH08201404001

Thesis Title : Problems in the theory of connections in algebraic geometry

Discipline : Mathematical Sciences

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This thesis is about two topics in the theory of connections in algebraic geometry. It is firstly about the relative connections in sheaves of modules over ringed spaces, and their holomorphic aspects. Secondly, it contains a description of some invariants of moduli space of logarithmic connections singular over a finite subset of a compact Riemann surface. Here are the highlights of my thesis results.

1. We give a sufficient condition for the existence of relative holomorphic connections in a holomorphic vector bundle over a family of connected compact complex manifolds.

2. We recall the definition of relative Chern classes of a holomorphic vector bundle and show that the relative Chern classes of a holomorphic vector bundle over a family of compact Kähler manifolds vanish if the bundle admits a relative holomorphic connection.

3. To show that there exists a natural compactification of the moduli space of logarithmic connections singular over a finite subset of a compact Riemann surface.

4. To show that the Picard group of the moduli space of logarithmic connections with fixed determinant is isomorphic to the set of integers.

5. To show that the moduli space of logarithmic connections does not admit any non-constant algebraic function, although it admits non-constant holomorphic functions.

6. We also characterize the algebraic functions on the moduli space of logarithmic connections with arbitrary residues over a compact Riemann surface.