SOME PROBLEMS IN TRANSCENDENTAL NUMBER THEORY

By

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Veekesh Kumar

List of Publications arising from the thesis

Journal

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3. "On the transcendence of certain real numbers", Veekesh Kumar and Bill Mance, *Bull. Aust. Math. Soc.*,**2019**, *99 no 3*, 392-402.

4. "Linear independence of certain numbers", Veekesh Kumar, *Arch. Math. (Basel)*, **2019**, *112*, 377-385.

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Conferences:

1. Leuca 2016 Celebrating Michel Waldschmidt's 70th birthday, during 13-17, June, **2016** at Patu (Lecce), Italy.

2. ICCGNFRT during 12-15, September, **2017** at Harish-Chandra Research Institute, Allahabad and delivered a talk on *Transcendence criterion for Cantor series*.

3. Attended the School on Modular forms during 12-24, February, **2018** at Kerala School of Mathematics, Kozhikode and delivered a talk on *Linear independence of certain numbers*.

Veekesh Kumar

Dedicated to

MY GRAND FATHER

Shree.Lajjaram

and

MY MOTHER

Smt.Sukhdevi

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Contents

Abstract

This thesis deals with some applications of well known Schmidt's Subspace theorem in Transcendental Number Theory and linear independence of values of Jacobi theta-constants.

In 2004, Adamczewski, Bugeaud and Luca provided a transcendence criterion for a real number written in b-ary expansion. In the first problem, we extend this criterion under the assumption of Subspace Lang's conjecture for a much wider class of irrational numbers.

The Q-ary expansion, which is a generalization of b-ary expansion, there are many known results about irrationality of Cantor series (or Q-ary expansion). In the second problem, we study transcendence criterion for these series analogous to b-ary expansion transcendence results, using the Subspace theorem.

The problem of finding the transcendence nature of an infinite series is very challenging. In the third problem, we study transcendence of certain infinite sums and certain infinite products as an application of the Subspace theorem. Our results extend the works of Erdős and Straus, 1974, Hančl and Rucki, 2005. In particular, we have shown that at least one of the real numbers

$$\sum_{n=1}^{\infty} \frac{\sigma(n)}{b_1 b_2 \cdots b_n}, \quad \sum_{n=1}^{\infty} \frac{\phi(n)}{b_1 b_2 \cdots b_n}, \quad \sum_{n=1}^{\infty} \frac{d_n}{b_1 b_2 \cdots b_n}$$

is transcendental, where $\phi(n)$ is the Euler totient function, $\sigma(n)$ is the sum of the divisors of n and $(d_n)_n$ is any sequence of integers satisfying $|d_n| < n^{\frac{1}{2}-\delta}$ for all large n with $d_n \neq 0$ for infinitely many values n.

In the last part of the thesis, we study the $\mathbb{Q}$-linear independence of certain b-ary expansions. The proof of this result involves many estimates and heavily depend on the theory of Uniform distribution mod 1.

As an immediate consequence of this result, we have the following corollary: For $\tau = \frac{i \log b}{\pi}$, the real numbers

$$1, \quad \theta_3(a_1\tau), \quad \theta_3(a_2\tau), \ldots, \theta_3(a_m\tau)$$

are $\mathbb{Q}$-linearly independent for distinct positive integers $a_1, \ldots, a_m$, where $\theta_3(\tau) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}$.

CHAPTER 1

Introduction

In this chapter we introduce basic definitions and properties of Cantor series expansion (or Q-ary expansion), Rational approximation of algebraic numbers, Continued fraction and Uniform distribution mod 1.

1.1 The Q-ary expansion of real numbers

For this section we closely follow [36] (see also [9]), [13], [18] and [39]).

Let $Q = (b_n)_{n\geq 1}$ be a sequence of positive integers, with $b_n \geq 2$ for all integers $n \geq 1$. We define an *Q-ary expansion* as follows. A *Q-ary expansion*, which is denoted by $c_0.c_1c_2\cdots c_n\cdots$, and it is defined by a series of the form

$$c_0 + \frac{c_1}{b_1} + \frac{c_2}{b_1 b_2} + \cdots + \frac{c_n}{b_1 b_2 b_3 \cdots b_n} + \cdots \tag{1.1}$$

for some integer $c_0 \in \mathbb{Z}$ and $c_n \in \{0, 1, \ldots, b_n - 1\}$ for all integers $n \geq 1$, together with the condition that $c_n < b_n - 1$ for infinitely many integers n. The Q-ary expansion is, now-a-days, known as *Cantor series* with respect to Q.

Theorem 1.1.1 *Let $Q = (b_n)_{n\geq 1}$ sequence of positive integers, with $b_n \geq 2$ for all integers $n \geq 1$. Every Q-ary expansion converges to a real number. Conversely, every real number has an unique Q-ary expansion.*

Before giving the proof of theorem, we need the following observation.

Proposition 1.1.2

$$\sum_{i=1}^{\infty} \frac{b_{n+i}-1}{b_{n+1}\cdots b_{n+i}} = 1.$$

Proof. First we see that

$$\sum_{i=1}^{N} \frac{b_{n+i}-1}{b_{n+1}\cdots b_{n+i}} = 1 - \frac{1}{b_{n+1}\cdots b_{n+N}}.$$

Therefore, by letting $N \to \infty$, we get

$$\sum_{i=1}^{\infty} \frac{b_{n+i}-1}{b_{n+1}\cdots b_{n+i}} = 1.$$

Proof of Theorem 1.1.1. Consider a Q-ary expansion of the form (1.1). For each integer $n \geq 1$, we define n-th partial sum

$$s_n = c_0 + \frac{c_1}{b_1} + \frac{c_2}{b_1 b_2} + \cdots + \frac{c_n}{b_1 \cdots b_n}.$$

From the non-negativity of c_i's, it is clear that

$$c_0 \leq s_1 \leq s_2 \leq \cdots \leq s_n.$$

Also, we see that

$$\begin{aligned} s_n &\leq c_0 + \frac{b_1 - 1}{b_1} + \frac{b_2 - 1}{b_1 b_2} + \cdots + \frac{b_n - 1}{b_1 \cdots b_n} \\ &= c_0 + 1 - \frac{1}{b_1 \cdots b_n} \end{aligned}$$

$$< c_0 + 1.$$

Hence, for each $n \geq 1$, we get

$$c_0 \leq s_n < c_0 + 1$$

Thus, $(s_n)_n$ is bounded and increasing sequence of real numbers. Therefore, it converges its supremum which is a real number.

Let α be any real number. We shall prove that α has an Q-ary expansion. We define the integers $c_0, c_1, \ldots, c_n, \ldots$ and a sequence of real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$ as follows.

$$c_0 = [\alpha] \qquad \text{and} \quad \alpha_1 = \alpha - [\alpha].$$

For each $n \geq 1$, define

$$c_n = [b_n \alpha_n] \quad \text{and} \quad \alpha_{n+1} = b_n \alpha_n - [b_n \alpha_n].$$

We have defined $c_0, c_1, \ldots$ and a sequence of real numbers $\alpha_1, \alpha_2, \ldots$. Using above equations we can write

$$\begin{aligned} \alpha = c_0 + \alpha_1 &= c_0 + \frac{\alpha_2}{b_1} + \frac{[b_1 \alpha_1]}{b_1} \\ &= c_0 + \frac{c_1}{b_1} + \frac{\alpha_2}{b_1} \\ &= c_0 + \frac{c_1}{b_1} + \frac{c_2}{b_1 b_2} + \frac{\alpha_3}{b_1 b_2}. \end{aligned}$$

By continuing this process we obtain

$$\alpha = c_0 + \frac{c_1}{b_1} + \cdots + \frac{c_n}{b_1 \cdots b_n} + \frac{\alpha_{n+1}}{b_1 \cdots b_n}. \tag{1.2}$$

We claim that $c_0.c_1c_2\dots c_n\dots$ represent α. In order to prove this we need to prove the following two statements.

(1) The integers $c_n \in \{0, 1, 2, \dots, b_n - 1\}$, for all $n \geq 1$.

(2) The series

$$c_0 + \sum_{n=1}^{\infty} \frac{c_n}{b_1 \cdots b_n} = \alpha.$$

Since α_n is the fractional part of $b_{n-1}\alpha_{n-1}$, we have

$$0 \leq \alpha_n < 1 \iff 0 \leq b_n\alpha_n < b_n \implies 0 \leq [b_n\alpha_n] < b_n \quad \text{for all} \quad n = 1, 2, 3, \dots.$$

Thus, by the construction of c_n, we get $c_n \in \{0, 1, \dots, b_n - 1\}$. This proves (1).

Now we proof (2). We define

$$s_n = c_0 + \frac{c_1}{b_1} + \frac{c_2}{b_1b_2} + \cdots + \frac{c_n}{b_1 \cdots b_n}. \tag{1.3}$$

By (1.2) and (1.3), we get

$$0 \leq \alpha - s_n = \frac{\alpha_{n+1}}{b_1 \cdots b_n}.$$

Since $0 \leq \alpha_n < 1$ and $b_n \geq 2$ for all $n = 1, 2, \dots$, we have

$$0 \leq \alpha - \alpha_n = \frac{\alpha_{n+1}}{b_1 \cdots b_n} < \frac{1}{2^n}$$

for all n. Hence, $c_0 + \sum_{i=1}^{\infty} \frac{c_n}{b_1 \cdots b_n} = \alpha$.

Next we prove that $c_n < b_n - 1$ for infinitely many values of n. By the construction of $c_1, c_2, \dots, c_n, \dots$ and $\alpha_1, \alpha_2, \dots, \alpha_n, \dots$, we have

$$\alpha = c_0 + \frac{c_1}{b_1} + \cdots + \frac{c_N}{b_1 \cdots b_N} + \frac{\alpha_{N+1}}{b_1 \cdots b_N}. \tag{1.4}$$

Suppose $c_N = b_N - 1$ for all $n > N$. Then we have

$$\alpha = c_o + \sum_{i=1}^{N} \frac{c_i}{b_1 \cdots b_i} + \sum_{i=N+1}^{\infty} \frac{b_i - 1}{b_1 \cdots b_i}$$
$$= c_o + \sum_{i=1}^{N} \frac{c_i}{b_1 \cdots b_i} + \frac{1}{b_1 \cdots b_N} \sum_{i=1}^{\infty} \frac{b_{N+i} - 1}{b_{N+1} \cdots b_{N+i}}.$$

Thus by Proposition 1.1.2 and (1.4), we get $\alpha_{N+1} = 1$, which is a contradiction because $0 \le \alpha_n < 1$ for all n.

Uniqueness. Suppose we have another Q-ary expansion of α say,

$$\alpha = c_0 + \sum_{n=1}^{\infty} \frac{c_n}{b_1 \cdots b_n} = c_0' + \sum_{i=1}^{\infty} \frac{c_n'}{b_1 \cdots b_n}.$$

Since $c_n' < b_n - 1$ for infinitely many values of n, we can easily see that

$$\sum_{n=1}^{\infty} \frac{c_n'}{b_1 \cdots b_n} < 1.$$

This implies that $c_0' = [\alpha]$ and also we have $c_0 = [\alpha]$. Thus we get $c_0 = c_0'$. Therefore, we assume that there exists an integer N such that

$$c_n = c_n' \qquad \text{for all} \;\; n = 1, 2, \ldots, N \quad \text{and} \quad c_{N+1} \ne c_{N+1}'.$$

Hence, we have

$$\sum_{n=N+1}^{\infty} \frac{c_n}{b_1 \cdots b_n} = \sum_{n=N+1}^{\infty} \frac{c_n'}{b_1 \cdots b_n}.$$

Since both the series converges absolutely, and by rearranging the terms we get

$$\left| \sum_{n=N+2}^{\infty} \frac{c_n - c_n'}{b_1 \cdots b_n} \right| = \frac{|c_{N+1} - c_{N+1}'|}{b_1 \cdots b_{N+1}} \ge \frac{1}{b_1 \cdots b_{N+1}}.$$

By using the fact that $c_n, c'_n < b_n - 1$ for infinitely many values of n, we obtain

$$\begin{aligned}\left|\sum_{n=N+2}^{\infty} \frac{c_n - c'_n}{b_1 \cdots b_n}\right| &< \sum_{n=N+2}^{\infty} \frac{b_n - 1}{b_1 \cdots b_n} \\ &= \frac{1}{b_1 \cdots b_{N+1}} \sum_{n=1}^{\infty} \frac{b_{N+1+n} - 1}{b_{N+2} \cdots b_{N+1+n}} \\ &= \frac{1}{b_1 \cdots b_{N+1}},\end{aligned}$$

which is a contradiction, and hence the theorem. □

Remark 1.1.1 *In the case $b_n = b$ for all n, the resulting Q-ary expansion is the classical b-ary expansion of α.*

We introduce notion of *shift operator* as following.

Definition 1.1.1 *A map σ defined by the following way*

$$\sigma^0(\alpha) = \alpha - c_0 = \sum_{n=1}^{\infty} \frac{c_n}{b_1 \cdots b_n}$$

and

$$\sigma(\alpha) = \sigma(\sigma^0(\alpha)) = \sigma\left(\sum_{n=1}^{\infty} \frac{c_n}{b_1 \cdots b_n}\right) = \sum_{n=2}^{\infty} \frac{c_n}{b_2 \cdots b_n}$$

is called the shift operator.

It is easy to see that for every non-negative integer n,

$$\sigma^n(\alpha) = \sigma^n(\sigma^0(\alpha)) = \sigma^n\left(\sum_{n=1}^{\infty} \frac{c_n}{b_1 \cdots b_n}\right) = \sum_{k=n+1}^{\infty} \frac{c_k}{b_{n+1} \cdots b_k}.$$

Therefore, we conclude that

$$\sigma^0(\alpha) = \alpha - c_0 = \sum_{i=1}^{n} \frac{c_i}{b_1 \cdots b_i} + \frac{1}{b_1 \cdots b_n}\sigma^n(\alpha).$$

When $\alpha \in [0,1)$, this equality can be written as

$$\alpha = \sum_{i=1}^{n} \frac{c_i}{b_1 \cdots b_i} + \frac{1}{b_1 \cdots b_n} \sigma^n(\alpha). \tag{1.5}$$

The following theorem is a necessary and sufficient conditions for rationality.

Theorem 1.1.3 *A real number $\alpha \in [0,1)$ written as (1.1) is rational if and only if the cardinality of the set $\{\sigma^n(\alpha) : n \in \mathbb{N} \cup \{0\}\}$ is finite.*

Proof. Let α be rational, say $\alpha = \frac{p}{q}$, where $p < q$ and $(p,q) = 1$. Consider the sequence $(\sigma^n(\alpha))_n$ generated by *shift operator*. That is

$$\begin{aligned}
\sigma^0(\alpha) &= \alpha, \\
\sigma(\alpha) &= b_1\alpha - c_1, \\
&\vdots \\
\sigma^n(\alpha) &= b_n\sigma^{n-1}(\alpha) - c_n.
\end{aligned}$$

From the expression (1.5), it follows that

$$\sigma^n(\alpha) = \frac{pb_1 \cdots b_n - q(c_1b_2 \cdots b_n + \cdots + c_{n-1}b_n + c_n)}{q} = \frac{p_n}{q},$$

for some non-negative integer p_n.

Since $\alpha \in (0,1)$, we have $\sigma^n(\alpha) \in [0,1)$ for all $n \geq 0$. Therefore, we conclude that the sequence $(p_n)_n$ takes values in the set $\{0, 1, \ldots, q-1\}$ and hence, the set $\{\sigma^n(\alpha) : n \geq 0\}$ is finite. Thus there exists a number $m \in \mathbb{N}$ such that $p_{n+m} = p_n$.

Conversely, suppose there exist integers $n \geq 0$ and $m \geq 1$ such that $\sigma^n(\alpha) =$

$\sigma^{n+m}(\alpha)$. That is, from (1.5) it follows that

$$\begin{aligned} b_1 \cdots b_n \left(\alpha - \sum_{i=1}^{n} \frac{c_i}{b_1 \cdots b_i} \right) &= b_1 \cdots b_{n+m} \left(\alpha - \sum_{i=1}^{n+m} \frac{c_i}{b_1 \cdots b_i} \right) \\ &= b_1 \cdots b_{n+m} \left(\alpha - \sum_{i=1}^{n} \frac{c_i}{b_1 \cdots b_i} - \sum_{i=n+1}^{n+m} \frac{c_i}{b_1 \cdots b_i} \right). \end{aligned}$$

Hence, by re-arranging the term in this above equality, we get

$$\alpha = \sum_{i=1}^{n} \frac{c_i}{b_1 \cdots b_i} - \frac{b_{n+1} \cdots b_{n+m}}{b_{n+1} \cdots b_{n+m} - 1} \sum_{i=n+1}^{n+m} \frac{c_i}{b_1 \cdots b_i},$$

which in turns implies that α is rational. This proves the assertion. □

As a consequence of Theorem 1.1.3, we have another necessary and sufficient condition for rationality.

Theorem 1.1.4 *A real number α is rational if and only if there exists $q \in \mathbb{N}$ such that*

$$\mathit{card} \left\{ \sigma^n(\alpha) : \sigma^n(\alpha) \in \left(\frac{r}{q}, \frac{r+1}{q} \right) \right\}$$

is finite for all integers $r \geq 1$.

Proof. Suppose α is rational. Then by Theorem 1.1.3, we get that the cardinality of the set $\{\sigma^n(\alpha) : n \geq 0\}$ is finite. Therefore any interval $\left(\frac{r}{q}, \frac{r+1}{q}\right)$ contains at most finitely many $\sigma^n(\alpha)$ in it.

Conversely, suppose there exists an integer $q \geq 1$ such that

$$\text{card} \left\{ \sigma^n(\alpha) : \sigma^n(\alpha) \in \left(\frac{r}{q}, \frac{r+1}{q} \right) \right\} < \infty$$

holds for all $r \geq 1$.

Suppose not, assume $\alpha \notin \mathbb{Q}$. By the relation (1.5), we have

$$b_1 \cdots b_n \alpha - p_n = \sigma^n(\alpha)$$

for some non-negative integer p_n. Therefore, if α is irrational then $\sigma^n(\alpha)$ is irrational for every $n \geq 1$. Moreover, for distinct positive integers $n_1 < n_2$, the numbers $\sigma^{n_1}(\alpha)$ and $\sigma^{n_2}(\alpha)$ are distinct. To see this, suppose for positive integers $n_1 \neq n_2$,

$$\sigma^{n_1}(\alpha) = \sigma^{n_2}(\alpha).$$

This is equivalent to

$$b_1 \cdots b_{n_1} \alpha - p_{n_1} = b_1 \cdots b_{n_2} \alpha - p_{n_2} \iff (b_1 \cdots b_{n_1} - b_1 \cdots b_{n_2})\alpha = p_{n_1} - p_{n_2},$$

which in turns implies that α is rational. This contradicts to the fact that α is irrational. Therefore, we conclude that for distinct n_1 and n_2 corresponding values $\sigma^{n_1}(\alpha)$ and $\sigma^{n_2}(\alpha)$ are distinct. Thus $\sigma^n(\alpha)$ is irrational for every integer $n \geq 0$ and $0 \leq \sigma^n(\alpha) < 1$ for all $n \geq 0$. Let $q \geq 1$ be any integer. We partitioning the interval $[0, 1)$ into q interval as follows;

$$\left[\frac{r}{q}, \frac{r+1}{q}\right) \qquad r = 0, 1, 2, \ldots, q-1.$$

Since $\sigma^n(\alpha)$ is irrational and $\sigma^n(\alpha) \in [0, 1)$ for all integers $n \geq 0$, there exists $1 \leq r \leq q-1$ and an infinite sequence $(n_i)_{i \geq 1}$ of positive integers such that

$$\sigma^{n_i}(\alpha) \in \left(\frac{r}{q}, \frac{r+1}{q}\right) \quad \text{for all} \quad i \geq 1.$$

This implis that

$$\operatorname{card}\left\{\sigma^n(\alpha) : \sigma^n(\alpha) \in \left(\frac{r}{q}, \frac{r+1}{q}\right)\right\} = \infty,$$

which is a contradiction to the assumption that the cardinality of this set is finite. Therefore, we conclude that α is rational and hence the theorem. □

Remark 1.1.2 *The contrapositive statement of Theorem 1.1.4 is as follows: α is irrational if and only if for every integer $q \geq 1$, there exist an integer $r \leq q-1$ and an infinite sequence (n_i) of positive integers such that*

$$\sigma^{n_i}(\alpha) \in \left(\frac{r}{q}, \frac{r+1}{q}\right) \quad \text{for all} \quad i \geq 1.$$

This was intially proved by Oppenheim [39]. As an consequence of Remark 1.1.2, we have the following Corollary.

Corollary 1.1.5 *If $c_n > 0$ for infinitely many natural numbers n and if there exists a subsequence $(i_n)_n$ of positive integers satisfying*

$$\lim_{n\to\infty} \frac{c_{i_n}}{b_{i_n}} = 0,$$

then the infinite series given by (1.1) is irrational.

Proof. Since $c_n > 0$ for infinitely many values of n and $\lim_{n\to\infty} \frac{c_{i_n}}{b_{i_n}} = 0$ implies that $c_n < b_n - 1$ for infinitely many values of n. Therefore, we can easily see that

$$0 \leq \frac{c_{i_n}}{b_{i_n}} < \sigma^{i_{n-1}}(\alpha) < \frac{c_{i_n}}{b_{i_n}} + \frac{1}{b_{i_n}}.$$

Since $\frac{c_{i_n}}{b_{i_n}} \to 0$ as $n \to \infty$, for any positive integer q, there exists a positive integer

n_0 such that for all $n \geq n_0$,

$$0 < \sigma^{i_n}(\alpha) < \frac{1}{q}.$$

Now in Remark 1.1.2, by taking $r = 0$ and $n_i = i_n$, we get the result. □

We have the following necessary and sufficient condition for rationality, when the sequence $Q = (b_n)_n$ is periodic.

Proposition 1.1.6 *Let $Q = (b_n)_{n\geq 1}$ be an eventually periodic sequence of positive integers, with $b_n \geq 2$ for all integers $n \geq 1$. Then α given by (1.1) is rational if and only if the sequence $(c_n)_n$ is periodic.*

Proof. First we proof this proposition when $b_n = b$ for all $n \geq 1$. Then by using this we prove for general periodic sequence $Q = (b_n)_n$. Let

$$\alpha = a_0.c_1c_2\dots c_N a_1a_2\dots a_\ell a_1a_2\dots a_\ell\dots$$

be a b-ary expansion of α which is eventually periodic with period ℓ. We shall rewrite α in a series form as follows.

$$\begin{aligned}
\alpha &= a_0 + \sum_{m=1}^{N}\frac{c_m}{b^m} + \frac{1}{b^N}\left(\sum_{m=1}^{\ell}\frac{a_m}{b^m} + \frac{1}{b^\ell}\sum_{m=1}^{\ell}\frac{a_m}{b^m} + \cdots\right) \\
&= a_0 + \sum_{m=1}^{N}\frac{c_m}{b^m} + \frac{1}{b^N}\left(\sum_{m=1}^{\ell}\frac{a_m}{b^m}\right)\left(1 + \frac{1}{b^\ell} + \frac{1}{b^{2\ell}} + \cdots\right) \\
&= a_0 + \sum_{m=1}^{N}\frac{c_m}{b^m} + \frac{1}{b^N}\left(\sum_{m=1}^{\ell}\frac{a_m}{b^m}\right)\left(\frac{1}{1-\frac{1}{b^\ell}}\right) \\
&= a_0 + \sum_{m=1}^{N}\frac{c_m}{b^m} + \frac{1}{b^{N-\ell}(b^\ell - 1)}\left(\sum_{m=1}^{\ell}\frac{a_m}{b^m}\right) \in \mathbb{Q}.
\end{aligned}$$

Hence, if $\alpha \in \mathbb{R}_{\geq 0}$ has eventually periodic b-ary expansion, then $\alpha \in \mathbb{Q}$.

Suppose α is a positive rational number. We shall prove that it has eventually periodic b-ary expansion. Since α is a rational number, we let $\alpha = p/q$ where p and q are positive integers such that $(p, q) = 1$. If $p > q$, then $\alpha = m + p'/q$ where m is an integer and $p' < q$ with $(p', q) = 1$. Note that eventually periodicity of p'/q implies the eventually periodicity of p/q. Hence, without loss of generality, we can assume that $p < q$ and $(p, q) = 1$.

Also, note that for any integer $k \geq 1$, we see that the rational number $(b^k p)/q$ is eventually periodic if and only if the rational number p/q is eventually periodic, because, if we multiply by b^k with p/q, then, we get a rational number whose digits are same with p/q but moved by k places. Hence, it is enough to assume that the rational number $\alpha = p/q$ satisfies $(p, q) = 1$, $p < q$ and $(q, b) = 1$.

Since $(b, q) = 1$, b is an element of the multiplicative group, $(\mathbb{Z}/q\mathbb{Z})^*$. Let ℓ be the order of b in $(\mathbb{Z}/q\mathbb{Z})^*$. Therefore, we have,

$$b^\ell \equiv 1 \pmod{q} \text{ and } b^m \not\equiv 1 \pmod{q} \text{ for all } 1 \leq m < \ell.$$

Write

$$b^\ell = 1 + kq \text{ for some integer } k.$$

Hence, $kq > b^{\ell-1}$. Now, we consider

$$\begin{aligned} \alpha &= \frac{p}{q} = \frac{kp}{kq} = \frac{kp}{b^\ell - 1} \\ &= \frac{kp}{b^\ell} \frac{1}{1 - \frac{1}{b^\ell}} \\ &= \frac{kp}{b^\ell} \left(1 + \frac{1}{b^\ell} + \frac{1}{b^{2\ell}} + \cdots \right). \end{aligned}$$

Since $kp < kq < b^\ell$, we conclude that the number of digits in the b-ary expansion of the integer pk is at most ℓ. Suppose $kp = b_1 b^{r-1} + b_2 b^{r-2} + \cdots + b_r$ for some

$r \le \ell$ and $b_i \in \{0, 1, \dots, b-1\}$. Then, we have

$$\frac{kp}{b^\ell} = 0.00\dots 0b_1b_2\dots b_r = 0.c_1c_2\dots c_\ell, \text{ where } c_i = 0 \text{ for all } 1 \le i \le \ell - r.$$

Thus,

$$\alpha = \frac{kp}{b^\ell}\left(1 + \frac{1}{b^\ell} + \frac{1}{b^{2\ell}} + \cdots\right) = 0.c_1c_2\dots c_\ell c_1c_2\dots c_\ell \cdots .$$

Now we consider the general eventually periodic sequence $Q = (b_n)_n$ of positive integers with $b_n \ge 2$ for all integers $n \ge 1$. Then there exist positive integers $\ell \ge 2$ and $N \ge 1$ such that $b_{\ell+n} = b_n$ for all $n \ge N$. It is enough to prove to the assertion for $N = 1$ because $b_1 \cdots b_{N-1}\alpha - M$ is rational for some integer M if and only if α is rational. Thus we assume that

$$b_{\ell+n} = b_n \quad \text{for all} \quad n \ge 1. \tag{1.6}$$

Then, by (1.1) and (1.6), we have

$$\begin{aligned}
\alpha &= \left(\frac{c_1}{b_1} + \frac{c_2}{b_1b_2} + \cdots + \frac{c_\ell}{b_1\cdots b_\ell}\right) + \left(\frac{c_{\ell+1}}{b_1^2\cdots b_{\ell+1}} + \cdots\right) + \cdots \\
&= \left(\frac{c_1b_2\cdots b_\ell + \cdots + c_\ell}{b_1\cdots b_\ell}\right) + \left(\frac{c_{\ell+1}b_2\cdots b_n + \cdots}{(b_1\cdots b_\ell)^2}\right) + \cdots \\
&= \sum_{m=0}^{\infty} \frac{(c_{\ell m+1}b_2\cdots b_\ell + c_{\ell m+2}b_3\cdots b_\ell + \cdots + c_{\ell m+\ell})}{(b_1\cdots b_\ell)^{m+1}}.
\end{aligned}$$

Put $B = b_1\cdots b_\ell$. Since $c_i \in \{0, 1, \dots, b_i - 1\}$, we see that $c_{\ell m+i} \in \{0, 1, \dots, b_i - 1\}$. Therefore, we see that

$$c_{\ell m+1}b_2\cdots b_\ell + c_{\ell m+2}b_3\cdots b_\ell + \cdots + c_{\ell m+\ell} \le b_1\cdots b_\ell - 1.$$

Thus we obtain a usual B-ary expansion of α given by

$$\alpha = \sum_{m=0}^{\infty} \frac{C_m}{B^{m+1}},$$

where $C_m = c_{\ell m+1} b_2 \cdots b_\ell + c_{\ell m+2} b_3 \cdots b_\ell + \cdots + c_{\ell m+\ell}$ for all $m \geq 0$.

If $(c_n)_n$ is periodic, then the sequence $(C_m)_m$ is also periodic. Therefore, we can easily see that α is rational. Conversely, let α be rational, say $\alpha = \frac{p}{q}$. Then by the above argument, the sequence $(C_m)_m = (c_{\ell m+1} b_2 \cdots b_\ell + c_{\ell m+2} b_3 \cdots b_\ell + \cdots + c_{\ell m+\ell})_m$ is eventually periodic. That is there exist positive integers N and N_0 such that

$$C_m = C_{m+N} \quad \text{for all} \quad m \geq N_0.$$

It is equivalent to

$$c_{\ell m+1} b_2 \cdots b_\ell + c_{\ell m+2} b_3 \cdots b_\ell + \cdots + c_{\ell m+\ell} = c_{\ell m+\ell N+1} b_2 \cdots b_\ell + \cdots + c_{\ell m+\ell N+\ell}$$

for all $m \geq N_0$. By dividing $b_1 \cdots b_\ell$ on both sides , we get

$$\frac{c_{\ell m+1}}{b_1} + \frac{c_{\ell m+2}}{b_1 b_2} + \cdots + \frac{c_{\ell m+\ell}}{b_1 \cdots b_\ell} = \frac{c_{\ell m+\ell N+1}}{b_1} + \cdots + \frac{c_{\ell m+\ell N+\ell}}{b_1 \cdots b_\ell}.$$

By rewriting this equality, we have

$$c_{\ell m+1} - c_{\ell m+\ell N+1} = \frac{c_{\ell m+\ell N+2} - c_{\ell m+2}}{b_2} + \cdots + \frac{c_{\ell m+\ell N+\ell} - c_{\ell m+\ell}}{b_2 \cdots b_\ell}. \tag{1.7}$$

We note that the left-hand side of (1.7) is an integer. Since

$$c_{\ell m+\ell N+i-c_{\ell m+i}} < b_i, \quad \text{for all} \quad i \geq 2,$$

we see that the right hand side of (1.7)

$$\left|\frac{c_{\ell m+\ell N+2}-c_{\ell m+2}}{b_2}+\cdots+\frac{c_{\ell m+\ell N+\ell}-c_{\ell m+\ell}}{b_2\cdots b_\ell}\right|<\left(1-\frac{1}{b_2\cdots b_\ell}\right)<1.$$

Therefore we conclude that

$$c_{\ell m+1}=c_{\ell m+\ell N+1}.$$

Consequently, by (1.7) we get

$$c_{\ell m+2}-c_{\ell m+\ell N+2}=\frac{c_{\ell m+\ell N+3}-c_{\ell m+3}}{b_3}+\cdots+\frac{c_{\ell m+\ell N+\ell}-c_{\ell m+\ell}}{b_3\cdots b_\ell}.$$

By the similar process as above we obtain

$$c_{\ell m+i}=c_{\ell m+\ell N+i}\quad \text{for all}\quad i\geq 2.$$

Hence, by continuing this process we get

$$c_{\ell m+i}=c_{\ell m+\ell N+i}\quad \text{for all}\quad i\geq 2.$$

This implies that the sequence $(c_n)_n$ is eventually periodic and hence the assertion □

Now we introduce the notion of Condensation.

Condensation. Let $(i_n)_{n\geq 1}$ be a given sequence of positive integers such that $1\leq i_1<i_2<\cdots<i_n<\cdots$. Let

$$\frac{c_1}{b_1}+\frac{c_2}{b_1b_2}+\cdots+\frac{c_{i_1}}{b_1b_2\cdots b_{i_1}}=\frac{C_1}{B_1}.$$

For any integer $m \geq 2$, we let

$$\frac{c_{i_{m-1}+1}}{b_{i_{m-1}+1}} + \frac{c_{i_{m-1}+2}}{b_{i_{m-1}+1} b_{i_{m-1}+2}} + \cdots + \frac{c_{i_m}}{b_{i_{m-1}+1} b_{i_{m-1}+2} \cdots b_{i_m}} = \frac{C_m}{B_m},$$

where $B_1 = b_1 b_2 \ldots b_{i_1}$ and for all integers $m \geq 2$, $B_m = b_{i_{m-1}+1} b_{i_{m-1}+2} \ldots b_{i_m}$. Thus, the Cantor series (1.1) reduces to

$$c_0 + \frac{C_1}{B_1} + \frac{C_2}{B_1 B_2} + \cdots + \frac{C_m}{B_1 B_2 B_3 \cdots B_m} + \cdots ,$$

with $B_m \geq 2$ and $0 \leq C_m \leq B_m - 1$ for all $m \geq 1$.

By the above method, we get another Cantor series expansion for the same real number with respect to the new sequence $Q = (B_n)_n$. We call this procedure a *Condensation*. Now, we have another necessary and sufficient conditions for rationality is as follows.

Lemma 1.1.2 *The following is a necessary and sufficient condition for a real number given by* (1.1) *to be a rational number. There exists a condensation and there exist co-prime integers h and k satisfying $0 \leq h < k$ such that,*

$$C_n = \frac{h}{k}(B_n - 1) \text{ for all integers } n \geq N,$$

for some natural number N.

Proof. Suppose there exist a positive integer N and co-prime integers $0 \leq h \leq k$ such that

$$C_n = \frac{h}{k}(B_n - 1) \text{ for all integers } n \geq N.$$

Then, we have

$$\frac{C_N}{B_N} + \frac{C_{N+1}}{B_N B_{N+1}} + \cdots + \frac{C_{N+m}}{B_N B_{N+1} \cdots B_{N+m}} = \frac{h}{k}.$$

Therefore, the real number α given by

$$c_0 + \frac{C_1}{B_1} + \frac{C_2}{B_1 B_2} + \cdots + \frac{C_{N-1}}{B_1 B_2 \cdots B_{N-1}} + \frac{h}{k},$$

which is a rational number. Conversely, let α be rational, say $\alpha = \frac{p}{q}$ with $(p,q) = 1$. By the definition of $\sigma^n(\alpha)$, for each integer $n \geq 1$, we have

$$b_n \sigma^n(\alpha) - c_n = \sigma^{n+1}(\alpha).$$

Therefore, if α is rational then $\sigma^n(\alpha)$ is also rational. By Theorem 1.1.4, the set $\{\sigma^n(\alpha) | n \in \mathbb{N}\}$ is finite. Thus there exists an increasing sequence $(n_\ell)_\ell$ of positive integers such that

$$\sigma^{n_1}(\alpha) = \sigma^{n_2}(\alpha) = \cdots = \sigma^{n_\ell}(\alpha) = \ldots.$$

Since α is rational, there exists integers $0 \leq h < k$ with $(h,k) = 1$ such that

$$\sigma^{n_\ell}(\alpha) = \frac{h}{k}, \quad (h,k) = 1.$$

Let us write a condensation of α with respect to the sequence $(n_\ell)_\ell$.

$$\begin{aligned}
\frac{C_1}{B_1} &= \frac{c_1}{b_1} + \cdots + \frac{c_{n_1}}{b_1 \cdots b_{n_1}} \\
&\vdots \\
\frac{C_m}{B_m} &= \frac{c_{n_{m-1}+1}}{b_{n_{m-1}+1}} + \cdots + \frac{c_{n_m}}{b_{n_{m-1}+1} \cdots b_{n_m}}
\end{aligned}$$

and

$$\alpha = \frac{C_1}{B_1} + \frac{C_2}{B_1 B_2} + \cdots + \frac{C_m}{B_1 \cdots B_m} + \cdots.$$

By the definition of *shift operator*, we note that

$$\sigma^{n_1}(\alpha) = \frac{C_2}{B_2} + \frac{C_3}{B_2 B_3} + \cdots$$
$$\vdots$$
$$\sigma^{n_\ell}(\alpha) = \frac{C_{\ell+1}}{B_{\ell+1}} + \frac{C_{\ell+2}}{B_{\ell+1}B_{\ell+2}} + \cdots .$$

Since $\sigma^{n_\ell}(\alpha) = \frac{h}{k}$ for all $\ell \geq 1$, by the relation $\sigma^{n_{\ell+1}}(\alpha) = B_{\ell+1}\sigma^{n_\ell}(\alpha) - C_{\ell+1}$, we conclude that

$$C_{\ell+1} = \frac{h}{k}(B_{\ell+1} - 1), \qquad \text{for all} \quad \ell = 2, 3, \ldots,$$

which clearly implies that $k|(B_{\ell+1} - 1)$. This proves the lemma. □

As an consequence of Lemma 1.1.2, we have the following Corollary which is first proved by Cantor himself.

Corollary 1.1.7 *Let $Q = (b_n)_{n\geq 1}$ be a sequence of positive integers, with $b_n \geq 2$ for all integers $n \geq 1$. Suppose for every integer q divides $b_1 b_2 \cdots b_n$ for all sufficiently large values of n. Then α given by (1.1) is rational if and only if there exists a positive integer N such that $c_n = 0$ for all $n > N$.*

Proof. If $c_n = 0$ for all $n > N$, then clearly we see that α is rational. Conversely, let α be rational, say $\alpha = \frac{p}{q}$. Then by Lemma 1.1.2 there exist co-prime integer $0 \leq h < k$ and positive integer N such that

$$C_n = \frac{h}{k}(B_n - 1), \text{ for all integers } n \geq N,$$

where $B_n = b_1 \ldots b_n$. Since $k|(B_n - 1)$ for all $n \geq N$ and by the hypothesis $k|B_n$ for all sufficiently large values of n, we conclude that $k = 1$ and hence $h = 0$. This implies that that $C_n = 0$ for all $n \geq N$. Hence, by the definition of C_n, we conclude that $c_n = 0$ for all sufficiently large values of n. This proves the assertion. □

For further results about rationality or irrationality can be found [13], [18] and [19].

1.2 The rational approximation of algebraic irrational real numbers

A real number α is said to be an *algebraic number* if there exists a non-zero polynomial $P(X) \in \mathbb{Z}[X]$ such that $P(\alpha) = 0$. Otherwise, the real number is called *a transcendental number.*

In 1844, Liouville proved a classical theorem concerning the rational approximation of algebraic real numbers.

Theorem 1.2.1 *(Liouville) Let α be an algebraic number of degree $d \geq 2$. Then there exist a positive constant $c(\alpha)$ such that for all rational numbers $\frac{p}{q}$ with $(p,q) = 1$, $q > 0$, we have*

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{c(\alpha)}{q^d}.$$

Proof. For the proof see [33], pp 1.

Liouville was the first who constructed explicit examples of transcendental by using this Theorem 1.2.1. In particular he proved that the real number

$$\sum_{n=1}^{\infty} \frac{1}{10^{n!}}$$

is transcendental.

In 1955 K. F. Roth [42] established the following famous improvement of Liouville's theorem.

Theorem 1.2.2 *Let α be an algebraic number and ϵ be a positive real number. Then, the inequality*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\epsilon}}$$

has only finitely many integer solution in (p, q), with $q > 0$.

Using this Theorem, we can conclude that the real number

$$\sum_{n=1}^{\infty} \frac{1}{10^{3^n}}$$

is transcendental.

In order to study the further improvements in this directions, we need to introduce p-adic absolute values on a finite extension K over $\mathbb{Q}$. First, we shall define the p-adic absolute value on $\mathbb{Q}$ and then we shall extend this absolute value for a finite extension K over $\mathbb{Q}$.

Let p be a prime number in $\mathbb{Z}$. Let x/y be any rational number where $x \in \mathbb{Z}\backslash\{0\}$, $y \geq 1$ integer and $(x, y) = 1$. We define

$$\mathrm{ord}_p(x/y) = \begin{cases} n & \text{if } p^n \| x \\ -n & \text{if } p^n \| y. \end{cases}$$

Then, the p-adic absolute value on $\mathbb{Q}$, denoted by $|\cdot|_p$ and defined as

$$\left|\frac{x}{y}\right|_p = \left(\frac{1}{p}\right)^{\mathrm{ord}_p(x/y)} \quad \text{and } |0|_p = 0.$$

In this set up, the usual absolute value $|\cdot|$ on $\mathbb{Q}$ is denoted by $|\cdot|_\infty$.

Now, let $K/\mathbb{Q}$ be a number field and $\mathcal{O}_K$ be its ring of integers. Then, for any prime number $p \in \mathbb{Z}$, the ideal $p\mathcal{O}_K$ in $\mathcal{O}_K$ can be factored into product of prime ideals as

$$p\mathcal{O}_K = \mathfrak{p}_1^{e_1} \dots \mathfrak{p}_g^{e_g}$$

with $e_i \geq 1$ integers and $\mathfrak{p}_i$ are prime ideals in $\mathcal{O}_K$. Hence $p\mathcal{O}_K \subset \mathfrak{p}_i$ for all $i = 1, 2, \dots, g$. In this situation, we say $\mathfrak{p}_i | p$ ($\mathfrak{p}_i$ divides p) for all $i = 1, 2, \dots, g$.

Since K is the quotient field of $\mathcal{O}_K$, any $\alpha \in K$ can be written as $\alpha = x/y$ where $x, y \in \mathcal{O}_K$ with $\gcd(x\mathcal{O}_K, y\mathcal{O}_K) = \mathcal{O}_K$. Therefore, for any $\alpha \in K$ and for a given prime ideal $\mathfrak{p}$ in $\mathcal{O}_K$, we can define

$$\operatorname{ord}_{\mathfrak{p}}(\alpha) = \begin{cases} n & \text{if } \mathfrak{p}^n \| x\mathcal{O}_K \\ -n & \text{if } \mathfrak{p}^n \| y\mathcal{O}_K \end{cases}$$

Also, for any non-zero prime ideal $\mathfrak{p}$ in $\mathcal{O}_K$, the norm of $\mathfrak{p}$ denoted by $N\mathfrak{p}$ and defined by $N\mathfrak{p} = |\mathcal{O}_K/\mathfrak{p}|$, cardinality of the quotient ring (which is known to be finite). Now, we can extend the p-adic absolute value for any $\alpha \in K\backslash\{0\}$ as

$$|\alpha|_p = \left|\frac{x}{y}\right|_p = \prod_{\mathfrak{p}|p} \left(\frac{1}{N\mathfrak{p}}\right)^{\operatorname{ord}_{\mathfrak{p}}(\alpha)}.$$

If $p = \infty$, then we define

$$|\alpha|_\infty = |N_{K/\mathbb{Q}}(\alpha)|,$$

where $N_{K/\mathbb{Q}}(\alpha)$ is the norm of α (which is nothing but the product of all the Galois conjugates of α) in $K/\mathbb{Q}$. With these definitions, one can check the product formula

$$|\alpha|_\infty \prod_p |\alpha|_p = 1$$

holds for all $\alpha \in K\backslash\{0\}$.

In 1957, Ridout [41], extend Theorem 1.2.2 and he proved the following theorem.

Theorem 1.2.3 *(Ridout's Theorem) Let α be an algebraic number and $\epsilon > 0$. Let S be a finite set of distinct primes. Then the inequality*

$$\left(\prod_{\ell \in S} |p|_\ell . |q|_\ell\right) \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\epsilon}}$$

has only finitely many integer solutions in (p, q) with $q \geq 1$.

1.3 The Schmidt subspace theorem

A multidimensional generalization of Roth's and Ridout's theorem is known as *the Schmidt subspace theorem.* It is obtained by Wolfgang M. Schmidt in 1972 (see for instance [44], page 176 and [45]). This theorem plays a crucial role in this thesis.

Theorem 1.3.1 *(Subspace Theorem) Let $n > 1$ be an integer and let $L_1, \ldots, L_n$ be given linearly independent linear forms in n-variables, whose coefficients are algebraic numbers. Let $\epsilon > 0$ be given. Then the set*

$$T_1 = \left\{(y_1, \ldots, y_n) \in \mathbb{Z}^n : \prod_{i=1}^{n} |L_i(y_1, \ldots, y_n)| < (\max\{|y_1|, \ldots, |y_n|\})^{-\epsilon}\right\}$$

is contained in a finite union of proper subspaces of $\mathbb{Q}^n$.

As a consequence of the Subspace theorem, we have the following important Corollary.

Corollary 1.3.2 *For any given integer $m \geq 2$, let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be real numbers. Let $\delta > 0$ be a real number such that $\delta > \frac{1}{m}$. Suppose there exist infinitely many $(m+1)$-tuples $(p_{1n}, p_{2n}, \ldots p_{mn}, q_n)$ of integers satisfying $q_n \neq 0$ and*

$$\left|\alpha_i - \frac{p_{in}}{q_n}\right| < \frac{1}{q_n^{1+\delta}}, \qquad \textit{for} \ \ 1 \leq i \leq m. \tag{1.8}$$

Then either the real numbers $1, \alpha_1, \alpha_2, \ldots, \alpha_m$ are $\mathbb{Q}$-linearly dependent or at least one of α_i's is transcendental.

Proof. Without loss of generality we can assume that the absolute value of the real numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$ is less than 1. By (1.8), we have

$$\begin{aligned} |q_n\alpha_1 - p_{1n}| &< \frac{1}{q_n^\delta}; \\ |q_n\alpha_2 - p_{2n}| &< \frac{1}{q_n^\delta} \\ &\vdots \\ |q_n\alpha_m - p_{mn}| &< \frac{1}{q_n^\delta}. \end{aligned} \tag{1.9}$$

Suppose the real numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$ are algebraic. In order to finish the proof we prove that $1, \alpha_1, \alpha_2, \ldots, \alpha_m$ are $\mathbb{Q}$-linearly dependent.

In order to prove $1, \alpha_1, \alpha_2, \ldots, \alpha_m$ are $\mathbb{Q}$-linearly dependent, we shall apply Theorem 1.3.1. Consider linear forms with algebraic coefficients

$$\begin{aligned} L_0(X_1, X_2, \ldots, X_m) &= X_1, \\ L_1(X_1, X_2, \ldots, X_m) &= \alpha_1 X_1 - X_2, \\ &\vdots \\ L_m(X_1, X_2, \ldots, X_m) &= \alpha_m X_1 - X_m. \end{aligned} \tag{1.10}$$

Clearly, the above linear forms are linearly independent.

To apply Theorem 1.3.1, we need to compute the quantity

$$\prod_{i=0}^{m} |L_i(p_{1n}, p_{2n}, \ldots, p_{mn}, q_n)|.$$

Thus, from (1.9) and (1.10), we conclude that

$$\prod_{i=0}^{m} |L_i(p_{1n}, p_{2n}, \ldots, p_{mn}, q_n)| < (\max(|p_{1n}|, |p_{2n}|, \ldots, |p_{mn}|, |q_n|))^{-\delta'},$$

holds for infinitely many integers n and for some $\delta' > 0$. Hence, by Theorem 1.3.1, all these non-zero integer lattice points $X^{(n)} = (p_{1n}, p_{2n}, \ldots, p_{mn}, q_n)$ lie only in finitely many proper subspaces of $\mathbb{Q}^{m+1}$. Therefore, there exists a proper subspace of $\mathbb{Q}^{m+1}$ containing these integer lattice points. That is, there exists a non-zero tuple $(z_1, z_2, \ldots, z_{m+1}) \in \mathbb{Z}^{m+1}$ such that

$$z_1 q_n + z_2 p_{1n} + z_3 p_{2n} + \cdots + z_{m+1} p_{mn} = 0$$

holds for infinitely many values of n. This implies that

$$\lim_{n\to\infty} \left(z_1 + z_2 \frac{p_{1n}}{q_n} + z_3 \frac{p_{2n}}{q_n} + \cdots + z_{m+1} \frac{p_{mn}}{q_n} \right) = 0$$
$$\implies z_1 + z_2\alpha_1 + z_3\alpha_2 + \cdots + z_{m+1}\alpha_m = 0.$$

Hence, $1, \alpha_1, \ldots, \alpha_m$ are $\mathbb{Q}$-linearly dependent. This proves the assertion.

In 1977, Schlickewei [43] generalized Theorem 1.3.1 for number fields. More precisely, we have the following theorem.

Theorem 1.3.3 *(H. P. Schlickewei) Let S_f be a finite subset of prime numbers and let $S = S_f \cup \{\infty\}$. Let $n > 1$ be an integer. For every prime $p \in S$, $L_{1,p}, \ldots, L_{n,p}$ be the given linearly independent linear forms in n-variables whose coefficients are algebraic numbers. Let $\epsilon > 0$ be given. Then the set*

$$T_2 = \left\{ (y_1, \ldots, y_n) \in \mathbb{Z}^n : \prod_{p\in S} \prod_{i=1}^{n} |L_{i,p}(y_1, \ldots, y_n)|_p < (\max\{|y_1|, \ldots, |y_n|\})^{-\epsilon} \right\}$$

is contained in a finite union of proper subspaces of $\mathbb{Q}^n$.

Extension of Theorem 1.3.3 is an analogue conjecture of Lang stated by Dixit, Rath and Shankar [10] over any number fields. For our purposes, we state the conjecture over $\mathbb{Q}$.

Conjecture 1. *Let* S_f *be a finite subset of prime numbers and let* $S = S_f \cup \{\infty\}$. *Let* $n > 1$ *be an integer. For every prime* $p \in S$, *let the given linear forms* $L_{1,p}, \ldots, L_{n,p}$ *in* n*-variables, whose coefficients are algebraic numbers, be linearly independent. Let* $\epsilon > 0$ *be given. Then the set*

$$T_3 = \left\{ (y_1, \ldots, y_n) \in \mathbb{Z}^n : \prod_{p \in S} \prod_{i=1}^{n} |L_{i,p}(y_1, \ldots, y_n)|_p \leq \frac{1}{\log^{n-1+\epsilon}(\max\{|y_1|, \ldots, |y_n|\})} \right\}$$

is contained in a finite union of proper subspaces of $\mathbb{Q}^n$.

1.4 Continued fraction

For this section we follows [32].

A *simple continued fraction* is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}} \tag{1.10}$$

where each $a_i \in \mathbb{Z}$ and $a_i \geq 1$ for $i \geq 1$. It is also denoted by $[a_0; a_1, a_2, \ldots, a_n, \ldots\ldots]$. If this expression contains a finite number of terms, it is called a *finite simple continued fraction*. The finite continued fraction $\frac{p_k}{q_k} = [a_0; a_1, a_2, \ldots, a_k]$, $0 \leq k \leq n$ is called the *kth convergent* of $[a_0; a_1, a_2, \ldots, a_n]$. Let $(a_n)_{n\geq 0}$ be an infinite sequence of integers with $a_i \geq 0$ for $i \geq 1$. Now we define the infinite simple continued fraction $[a_0; a_1, a_2, \ldots]$ is the limit to be the sequence $\left(\frac{p_n}{q_n}\right)_{n\geq 0}$.

A finite simple continued fraction represents a rational number. Conversely, by using Euclidean algorithm, we can prove that every rational number can be expressed as a finite continued fraction. Similarly, every infinite simple continued fraction is irrational, and every irrational number can be expressed as an infinite simple continued fraction.

Definition. A simple continued fraction is called *periodic with period k* for an integer $k \geq 1$, if there exists a positive integer N such that $a_n = a_{n+k}$ for all $n \geq N$. We denote such a continued fraction by $[a_0; a_1, a_2, \ldots, \overline{a_{N-1}, a_N, \ldots, a_{N+k-1}}]$.

Lagrange [32] proved a necessary and sufficient condition for a real number α written as (1.10) to be a quadratic irrational if and only if its continued fraction is periodic.

As an consequence of continued fraction theory, we have the following result.

Theorem 1.4.1 *Let m be a positive integer which is not a perfect square. Let n be the period of the continued fraction of $\sqrt{m}$ and p_r/q_r be the r-th convergent of the continued fraction of $\sqrt{m}$. Then, we have,*

$$p_{n-1}^2 - mq_{n-1}^2 = 1$$

and all other positive solutions of the Pell equation $X^2 - mY^2 = 1$ are given by

$$x_\ell + y_\ell\sqrt{m} = (p_{n-1} + q_{n-1}\sqrt{m})^\ell, \text{ for all integers } \ell \geq 1.$$

Proof. For the proof see [32], pp 110. We have the following application of Theorem 1.4.1.

Lemma 1.4.2 *Let* $1 \le a_1 < a_2$ *be two integers such that* $a_1 a_2$ *is not a perfect square. Then, there exist infinitely many pairs* (X, Y) *of positive integers satisfying*

$$X^2 - a_1 a_2 Y^2 = 1 \quad and \quad X \equiv 1 \ mod \ a_2. \tag{1.12}$$

Proof. We consider the following Pell's equation,

$$X^2 - a_1 a_2 Y^2 = 1. \tag{1.13}$$

By Theorem 1.4.1, the equation (1.13) has a non-trivial positive integer solution $(p_{n-1}, q_{n-1}) \neq (\pm 1, 0)$ where p_{n-1}/q_{n-1} is the $(n-1)$-th convergent of the continued fraction $\sqrt{a_1 a_2}$ and its period is n. Also, note that $p_{n-1}^2 \equiv 1 \pmod{a_2}$. Also, by Theorem 1.4.1, we know that all the other positive solutions to (1.13) are given by $X_\ell + \sqrt{a_1 a_2} Y_\ell = (p_{n-1} + \sqrt{a_1 a_2} q_{n-1})^\ell$ for all integers $\ell \ge 1$. By taking $\ell = 2k$ for all $k \ge 1$, then $X_{2k} + Y_{2k}\sqrt{a_1 a_2} = (X_k + \sqrt{a_1 a_2} Y_k)^2 = X_k^2 + a_1 a_2 Y_k^2 + 2\sqrt{a_1 a_2} X_k Y_k$ is a solution of (1.13). Since any solution (X_k, Y_k) of (1.13) satisfies $X_k^2 \equiv 1 \pmod{a_2}$ and by the choice of $X_{2k} = X_k^2 + a_1 a_2 Y_k^2$, clearly we see that $X_{2k} \equiv 1 \pmod{a_2}$. Thus, we found infinitely many solutions (X_{2k}, Y_{2k}) of (1.12). This proves the lemma.

1.5 Uniform distribution modulo 1

For this section we follows [20], [22] and [32]. We start with the following definition.

Definition 1.5.1 We say that the sequence (x_n) of real numbers is *uniformly* *distributed mod* 1, if for any subset $E = [a, b]$ of $[0, 1)$, we have that

$$\lim_{N\to\infty} \frac{\text{card}\{n|1 \leq n \leq N, \{x_n\} \in E\}}{N} = b - a \tag{1.14}$$

where $\{x_n\}$ denote the fractional part of x_n.

The condition (1.14) is saying that if the sequence (x_n) is uniformly distributed mod 1, then for any interval $I = (c, d)$ of $[0, 1)$, there exist infinitely many positive integer n such that

$$c < \{x_n\} < d.$$

Note that if a sequence (x_n) is uniformly distributed mod 1, then the sequence $(\{x_n\})_n$ is dense in $[0, 1)$. If not, then there exists a subset $E = (a, b)$ of $[0, 1)$ such that $E \cap (\{x_n\}) = \phi$. This implies that

$$\lim_{N\to\infty} \frac{\text{card}\{n|1 \leq n \leq N, \{x_n\} \in E\}}{N} = 0,$$

which is a contradiction.

The following result gives a necessary and sufficient condition for a sequence $(x_n)_n$ to be uniformly distributed mod 1.

Theorem 1.5.2 *(Weyl's Criterion) A sequence $(x_n)_n$ is uniformly distributed mod 1 if and only if for every non-zero integer m, we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i m x_n} = 0.$$

Behaviour of the sequence $(n\alpha)_n$. This depends on choice of α. If α is rational, say $\alpha = \frac{h}{k}$, then the sequence $\{n\alpha\}$ takes k distinct values

$$0, \left\{\frac{h}{k}\right\}, \left\{\frac{2h}{k}\right\}, \ldots, \left\{\frac{(k-1)h}{k}\right\}.$$

Therefore, by (1.14) we conclude that the sequence $(n\alpha)_n$ is not uniformly distributed mod 1.

If α is an irrational, then situation is completely different. By applying Weyl's criterion, we conclude that the sequence $(n\alpha)_n$ is uniformly distributed mod 1. In particular this implies that the sequence $(\{n\alpha\})$ is dense in $[0, 1)$.

Now we generalize the definition of uniform distribution mod 1 in higher dimensions.

Definition 1.5.3 Let $s \geq 1$ be an integer. We say that the sequence $(x_n)_{n\geq 1}$ in $\mathbb{R}^s$ is *uniformly distributed mod* 1, if for any subset $E = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_s, b_s]$ of $[0, 1)^s$, we have that

$$\lim_{N\to\infty} \frac{\text{card}\{n|1 \leq n \leq N, \{x_n\} \in E\}}{N} = \prod_{i=1}^{s}(b_i - a_i),$$

where $\{x_n\}$ denote the fractional parts of each co-ordinates of x_n.

For the vectors $x = (a_1, a_2, \dots, a_s)$ and $y = (b_1, b_2, \dots, b_s)$, we define $< x, y >= a_1b_1 + \cdots + a_sb_s$ the standard inner product on $\mathbb{R}^s$. Now we have the following analogues of Theorem 1.5.2.

Theorem 1.5.4 *(Weyl's Criterion) A sequence $(x_n)_{n\geq 1}$ in $\mathbb{R}^s$ is uniformly distributed mod* 1 *if and only if for every non-zero lattice point $h \in \mathbb{Z}^s$, we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i <h, x_n>} = 0.$$

Proof. For the proof see [22], pp 48.

We have the following application of Theorem 1.5.4.

Corollary 1.5.5 *Let α_1, $\alpha_2, \dots, \alpha_r$ be real numbers such that $1, \alpha_1, \dots, \alpha_r$ are*

$\mathbb{Q}$-linearly independent. Then the sequence of points

$$(\{n\alpha_1\}, \{n\alpha_2\}, \ldots \{n\alpha_r\})_{n\geq 1}$$

is uniformly distributed mod 1 in $\mathbb{R}^s$.

Proof. Since $1, \alpha_1, \ldots, \alpha_r$ are $\mathbb{Q}$-linearly independent, for any non-zero lattice point $h \in \mathbb{Z}^r$, we have $< h, \alpha >$ is an irrational number, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$. Thus, by Theorem 1.5.2, we conclude that the sequence $(n < h, \alpha >)_n$ is uniformly distributed mod 1. This implies that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i < h, x_n >} = 0,$$

for all non-zero lattice point $h \in \mathbb{Z}^r$. This proves the assertion.

CHAPTER 2

Subspace Lang conjecture and some remarks on a transcendental criterion

Let $b \geq 2$ be an integer and α be a non-zero real number written in b-ary expansion. In 2004, Adamczewski, Bugeaud and Luca [1] provided a criterion for an irrational number to be a transcendental number using b-ary expansion. In this chapter, under the assumption of the Subspace Lang's conjecture, we extend this criterion for much wider class of irrational numbers. The results of this chapter have been published in [27].

2.1 Introduction

We recall a definition of b-ary expansion which we had discussed earlier in Chapter 1.

Let $b \geq 2$ be an integer. We say a non-zero real number α is *written in base* b, if there exist $a_0 \in \mathbb{Z}$ and non-negative integers $a_1, a_2, \ldots, a_k, \ldots$ with $0 \leq a_k \leq b-1$ such that

$$\alpha = a_0 + \frac{a_1}{b} + \frac{a_2}{b^2} + \cdots + \frac{a_k}{b^k} + \cdots . \tag{2.1}$$

Definition 2.1.1 *A b-ary expansion $\alpha = a_0.a_1a_2\ldots a_n\ldots$ is said to be eventually periodic, if there exists a positive integer N such that*

$$a_{N+k} = a_k \quad \textit{for all } k \geq N_0$$

for some integer $N_0 \geq 1$. The least positive integer N satisfying the above condition is called period.

This b-ary expansion of real number gives a necessary and sufficient criterion for it to be a rational number. In fact, it says the following. A positive real number α has a eventually periodic b-ary expansion if and only if $\alpha \in \mathbb{Q}$. We will prove this result in the next section.

In 2004, Adamczewski, Bugeaud and Luca [2] (see also in [3]) proved the following.

Theorem A. ([2] and [3]) *Let $b \geq 2$ be an integer. Let $\alpha \in [0, 1)$ be a non-zero real number satisfying (2.1). Suppose there exists $\epsilon > 0$ and there exist infinitely many 3-tuples (j_n, k_n, ℓ_n) of natural numbers satisfying*

$$a_{j_n+i} = a_{j_n+k_n+i} \textit{ for all } i = 1, 2, \ldots, \ell_n \textit{ and for all } n = 1, 2, \ldots \tag{2.2}$$

and

$$\epsilon(j_n + k_n) \leq \ell_n \leq k_n \text{ for all } n = 1, 2, \ldots. \tag{2.3}$$

Then α is either a rational number or a transcendental number.

We first remark about the parameters, mainly, k_n and ℓ_n in Theorem A as follows.

Remark 2.1.1 In Theorem A, first one notes that the sequence $\{k_n\}$ is unbounded. If not, the sequence $\{k_n\}$ is bounded. That is, there exists a constant $K > 0$ such that $k_n \leq K$ for all $n = 1, 2, \ldots$. Therefore, by (2.3), it is clear that $\ell_n \leq K$ for all $n = 1, 2, \ldots$ and hence

$$\epsilon(j_n + k_n) \leq k_n \leq K \implies j_n \leq j_n + k_n \leq K\epsilon^{-1}$$

for all integers $n \geq 1$. This means that k_n, ℓ_n and j_n are bounded. Therefore, the number of tuples (j_n, k_n, ℓ_n) is finite, which is a contradiction. In [2], they consider two cases, namely, the sequence $\{k_n\}$ is bounded or otherwise. In the first case, they use Ridout's theorem [41] to prove Theorem A.

Remark 2.1.2 By Remark 4.2.4 and (2.3), we see that the sequence $\{\ell_n\}$ is also unbounded.

Let $\alpha > 1$ be a positive real number and U be a finite word on alphabet $\{0, 1\}$. Then, we define the word U^α by concatenating words $U^{[\alpha]}U'$, where U' is the prefix of the word U of length $\lceil(\alpha - [\alpha])|U|\rceil$. Here $|U|$ denotes the length of the word, $[x]$ denotes the integral part of x and $\lceil x \rceil$ denotes the least integer $\geq x$ of the real number x. Such a word U^α we call as α-power. When we say that α-powers occurs in binary expansion of some real number it means that the word U^α defined above occurs in the binary expansion for every word U. As an application of Theorem A, in 2008, Adamczewski and Rampersad [4] proved that every algebraic irrational contains infinitely many occurrences of 7/3-powers in its binary expansion. In the same paper, they proved that every algebraic number contains either infinitely many occurrences of squares or infinitely many occurrences of one of the blocks 010 or 02120 in its ternary expansion.

By (2.1), we see that ℓ_n varies between $\epsilon(j_n + k_n)$ and k_n. In this chapter, we look at the lower bound for ℓ_n. More precisely, we prove the following theorem.

Theorem 2.1.2 *Let $b \geq 2$ be an integer. Let $\alpha \in [0, 1)$ be a non-zero real number satisfying (2.1) and $\epsilon > 0$ be given. Suppose there exist infinitely many 3-tuples (j_n, k_n, ℓ_n) of natural numbers satisfying (2.2) and*

$$\frac{(2+\epsilon)(\log(j_n + k_n) + \log\log b)}{\log b} \leq \ell_n \leq k_n. \tag{2.4}$$

If the Subspace-Lang's Conjecture is true (see Conjecture 1 of Chapter 1), then α is either rational or transcendental.

Note that the lower bound $\dfrac{(2+\epsilon)(\log(j_n + k_n) + \log\log b)}{\log b}$ is much smaller than that of $\epsilon(j_n+k_n)$ and hence allowing a wider class of real numbers to satisfy the hypothesis of Theorem 1 compared to the result of Adamczewski, Bugeaud and Luca [1] at the expense of an unproven hypothesis.

To illustrate Theorem 2.1.2, we take $u_n = n[\log n]^2$ for all integers $n \geq 1$ and let

$$\alpha = \sum_{n\geq 1} \frac{1}{10^{u_n}}$$

written in base 10. By Theorem 2.1.2, it follows that α is a transcendental number. To see this, first we observe that the block of zeroes in between $\frac{1}{10^{u_n}}$ and $\frac{1}{10^{u_{n+1}}}$ is of length $u_{n+1} - u_n - 1 = (n+1)[\log(n+1)]^2 - n[\log n]^2 - 1 \geq 100[\log n]$ for all sufficiently large integer n and hence α cannot be a rational number. Now, for all large enough integers n, we let $\ell_n = 60[\log n]$, $j_n = n[\log n]^2$ and

$k_n = (n+1)[\log(n+1)]^2 - n[\log n]^2$. Hence, we get, $j_n + k_n = (n+1)[\log(n+1)]^2$. Therefore, there does not exist any $\epsilon > 0$ such that $\epsilon(j_n + k_n) \leq \ell_n$ for infinitely many values of n. However, we note that

$$\log(j_n + k_n) = \log(n+1) + 2\log([\log(n+1)]) \leq 16[\log n]$$

for all sufficiently large value of n. By taking $\epsilon = 1$ in the statement of Theorem 2.1.2, we see that ℓ_n satisfies the required lower bound and hence by Theorem 2.1.2, α must be a transcendental number, provided the Subspace-Lang's conjecture is true.

2.2 Preliminaries

In this section, we shall prove basic Propositions.

Proposition 2.2.1 *Let r be any non-zero integer. Let $p_1, p_2, \ldots, p_s$ be the distinct prime divisors of r. Then*

$$\prod_{p \in \{\infty, p_1, p_2, \ldots, p_s\}} |r|_p = 1.$$

Proof. Since $p_1, p_2, \ldots, p_s$ are only prime divisors of r, we can write

$$r = p_1^{e_1} \cdots p_s^{e_s}, \quad \text{where} \quad e_j \geq 0, \quad \text{for} \quad 1 \leq j \leq s.$$

Let p be any prime other than $p_1, \ldots, p_s$. By the definition of p-adic absolute values as discussed in Chapter 1, we have $|r|_p = 1$ and hence

$$\prod_{p \in \{\infty, p_1, p_2, \ldots, p_s\}} |r|_p = |r|_\infty \cdot \frac{1}{p_1^{e_1}} \cdots \frac{1}{p_r^{e_r}} = 1.$$

□

2.3 Proof of Theorem 2.1.2

Proof of Theorem 2.1.2. Given that there exist $\epsilon > 0$ and a sequence $(j_n, k_n, \ell_n)_n$ of 3-tuples of natural numbers satisfying (2.2) and (2.4).

Using these conditions, we shall construct a sequence $\{\alpha_n\}_n$ of rational numbers as follows. For each $n \geq 1$, we let

$$\alpha_n = 0.a_1a_2 \dots a_{j_n}\overline{a_{j_n+1} \dots a_{j_n+k_n}},$$

where $\overline{a_{j_n+1} \dots a_{j_n+k_n}}$ means this block of digits are repeating. Hence, by Proposition 1.1.6, α_n is a rational number for all integers $n \geq 1$. Thus, we get,

$$\begin{aligned} \alpha_n &= \frac{a_1}{b} + \dots + \frac{a_{j_n}}{b^{j_n}} + \left(\frac{a_{j_n+1}}{b^{j_n+1}} + \dots + \frac{a_{j_n+k_n}}{b^{j_n+k_n}}\right)\left(1 + \frac{1}{b^{k_n}} + \frac{1}{b^{2k_n}} + \dots\right) \\ &= \frac{a_1}{b} + \dots + \frac{a_{j_n}}{b^{j_n}} + \left(\frac{a_{j_n+1}}{b^{j_n+1}} + \dots + \frac{a_{j_n+k_n}}{b^{j_n+k_n}}\right)\frac{b^{k_n}}{(b^{k_n}-1)} = \frac{p_n}{b^{j_n}(b^{k_n}-1)} \end{aligned}$$

for some integer p_n for all integers $n \geq 1$. Therefore,

$$b^{j_n}(b^{k_n} - 1)\alpha_n = p_n \text{ for all integers } n \geq 1. \tag{2.5}$$

Since $a_{j_n+1} = a_{j_n+k_n+1}, \dots, a_{j_n+\ell_n} = a_{j_n+k_n+\ell_n}$, we get,

$$\begin{aligned} |\alpha - \alpha_n| &= \left|\alpha - \frac{p_n}{(b^{k_n}-1)b^{j_n}}\right| \leq \frac{|a_{j_n+k_n+\ell_n+1} - a_{j_n+1}|}{b^{j_n+k_n+\ell_n+1}} + \dots \\ &< \frac{b-1}{b^{j_n+k_n+\ell_n+1}}\left(1 + \frac{1}{b} + \dots\right) \\ &\leq \left(\frac{b-1}{b^{j_n+k_n+\ell_n+1}}\right)\left(\frac{b}{b-1}\right) = \frac{1}{b^{j_n+k_n+\ell_n}}. \end{aligned}$$

Thus,

$$|\alpha - \alpha_n| < \frac{1}{b^{j_n+k_n+\ell_n}} \text{ for all integers } n \geq 1. \tag{2.6}$$

Consider

$$\begin{aligned} \left|b^{j_n+k_n}\alpha - b^{j_n}\alpha - p_n\right| &= \left|b^{j_n+k_n}\alpha - b^{j_n}\alpha - \left(b^{j_n}(b^{k_n}-1)\right)\alpha_n\right| \\ &= b^{j_n}(b^{k_n}-1)|\alpha - \alpha_n| \\ &< \frac{b^{j_n}(b^{k_n}-1)}{b^{j_n+k_n+\ell_n}} < \frac{1}{b^{\ell_n}}. \end{aligned}$$

Thus, we get,

$$\left|b^{j_n+k_n}\alpha - b^{j_n}\alpha - p_n\right| < \frac{1}{b^{\ell_n}}. \tag{2.7}$$

By Remark 4.2.4, we have seen that the sequence $\{k_n\}$ is unbounded. Without loss of generality, we shall assume that

$$k_1 < k_2 < \ldots < k_n < \ldots \text{ such that } k_n \to \infty \text{ as } n \to \infty. \tag{2.8}$$

Since we want to prove α is either rational or transcendental, we shall assume that α is not a transcendental number (and hence it is an algebraic number). Now to finish the proof, we need to prove α is a rational number.

In order to prove α is a rational number, we shall apply the Subspace - Lang Conjecture (Conjecture 1). Let $S = \{\infty\} \cup \{p : p \text{ is a prime and } p|b\}$ be a finite subset of prime numbers which includes the infinite prime. For each prime $q \in S$, we need to define linearly independent linear forms with algebraic coefficients. Consider

$$L_{1,\infty}(X_1, X_2, X_3) = X_1$$

$$L_{2,\infty}(X_1, X_2, X_3) = X_2$$

$$L_{3,\infty}(X_1, X_2, X_3) = \alpha X_1 - \alpha X_2 - X_3. \tag{2.9}$$

Clearly, as α is algebraic, the linear form $L_{i,\infty}$ is with algebraic coefficients for all $i = 1, 2, 3$. Since the determinant of the coefficient matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & -\alpha & -1 \end{pmatrix}$$

is non-zero, we conclude that $L_{1,\infty}, L_{2,\infty}$ and $L_{3,\infty}$ are linearly independent linear forms. Now, for any finite prime $p \in S$, we define

$$\begin{aligned} L_{1,p}(X_1, X_2, X_3) &= X_1 \\ L_{2,p}(X_1, X_2, X_3) &= X_2 \\ L_{3,p}(X_1, X_2, X_3) &= X_3. \end{aligned} \tag{2.10}$$

Clearly, the linear forms $L_{1,p}, L_{2,p}$ and $L_{3,p}$ are linearly independent.

For any integer $n \geq 1$, we let

$$\mathbf{x}^{(n)} = \left(b^{j_n+k_n}, b^{j_n}, p_n\right) \in \mathbb{Z}^3. \tag{2.11}$$

Since $\alpha_n \in (0, 1)$ for all $n = 1, 2, \ldots$, we see that $p_n < b^{j_n+k_n}$ and hence, we get

$$\max\{|b^{j_n+k_n}|, |b^{j_n}|, |p_n|\} \leq b^{j_n+k_n} \text{ for all } n = 1, 2, \ldots, \tag{2.12}$$

since $|x_i|_p \leq 1$ for all $x_i \in \mathbb{Z}\backslash\{0\}$ and for all finite primes p.

In order to apply the Subspace-Lang Conjecture, we need to compute the quantity

$$\prod_{i=1}^{3}\prod_{q\in S}\left|L_{i,q}\left(\mathbf{x}^{(n)}\right)\right|_q. \tag{2.13}$$

First note that

$$\prod_{i=1}^{3}|L_{i,\infty}((b^{j_n+k_n},b_n^j,p_n))|_\infty = b^{j_n+k_n}b^{j_n}\left|\alpha b^{j_n+k_n}-\alpha b^{j_n}-p_n\right|_\infty. \tag{2.14}$$

For any prime $q\in S\backslash\{\infty\}$, we have

$$\prod_{i=1}^{3}|L_{i,q}((b^{j_n+k_n},b_n^j,p_n))|_q = |b^{j_n+k_n}|_q|b^{j_n}|_q|p_n|_q. \tag{2.15}$$

Therefore, by (2.14) and Proposition 2.2.1, we get

$$\begin{aligned}\prod_{i=1}^{3}\prod_{p\in S}|L_{i,p}(\mathbf{x}^{(n)})|_p &= \prod_{p\in S}|b^{j_n+k_n}|_p\prod_{p\in S}|b^{j_n}|_p|\alpha b^{j_n+k_n}-\alpha b^{j_n}-p_n|\prod_{p\in S\backslash\{\infty\}}|p_n|_p\\ &\leq |\alpha b^{j_n+k_n}-\alpha b^{j_n}-p_n|.\end{aligned}$$

Thus, by (2.7) we get

$$\prod_{i=1}^{3}\prod_{p\in S}|L_{i,p}(\mathbf{x}^{(n)})|_p<\frac{1}{b^{\ell_n}}. \tag{2.16}$$

By (2.4), we see that

$$\frac{(2+\epsilon)(\log(j_n+k_n)+\log\log b)}{\log b}\leq \ell_n \iff \log^{(2+\epsilon)}(b^{j_n+k_n}) \leq b^{\ell_n}.$$

Thus, from (2.12) we have

$$\log^{2+\epsilon}\left(\max\{|b^{j_n+k_n}|,|b^{j_n}|,|p_n|\}\right)\leq b^{\ell_n}.$$

By putting this information in (2.16), we get

$$\prod_{i=1}^{3}\prod_{p\in S}|L_{i,p}(\mathbf{x}^{(n)})|_p \leq \frac{1}{\log^{2+\epsilon}(\max\{|b^{j_n+k_n}|,|b^{j_n}|,|p_n|\})}.$$

Thus for all n, the non zero lattice points $\mathbf{x}^{(n)} \in \mathbb{Z}^3$ satisfies the hypothesis of Subspace-Lang Conjecture. Thus, by Conjecture 1, the integer lattice points $\mathbf{x}^{(n)} = (b^{j_n+k_n}, b^{j_n}, p_n)$ lie in finitely many proper subspaces of $\mathbb{Q}^3$ for all integer $n \geq 1$. Therefore, there exists a proper subspace of $\mathbb{Q}^3$ containing the integer lattice point $\mathbf{x}^{(n)} = (b^{j_n+k_n}, b^{j_n}, p_n)$ for infinitely many values of n. That is, there exist integers a_1, a_2 and a_3 with $(a_1, a_2, a_3) \neq (0,0,0)$ such that

$$a_1 b^{j_n+k_n} + a_2 b^{j_n} + a_3 p_n = 0 \tag{2.17}$$

holds true for infinitely many values of n's.

First note that $a_3 \neq 0$. If not, we assume that $a_3 = 0$. Then, clearly, we have $a_1 \neq 0$ and $a_2 \neq 0$. Thus, for infinitely many values of n's, we get

$$a_1 b^{j_n+k_n} + a_2 b^{j_n} = 0 \implies b^{k_n} = -a_2/a_1,$$

which implies that k_n is bounded for these values of n (which in turn prove that $(k_n)_n$ is bounded), which is a contradiction to (2.8). Therefore, we conclude that $a_3 \neq 0$. Now, by (2.17), we consider

$$\begin{aligned}
\lim_{n\to\infty}\left(a_1\frac{b^{j_n+k_n}}{b^{j_n}(b^{k_n}-1)} + a_2\frac{b^{j_n}}{b^{j_n}(b^{k_n}-1)} + a_3\frac{p_n}{b^{j_n}(b^{k_n}-1)}\right) &= 0\\
\lim_{n\to\infty}\left(a_1\frac{b^{j_n+k_n}}{b^{j_n}(b^{k_n}-1)} + a_2\frac{b^{j_n}}{b^{j_n}(b^{k_n}-1)} + a_3\alpha_n\right) &= 0\\
\implies \lim_{n\to\infty}\left(\frac{a_1 b^{k_n}}{b^{k_n}-1} + \frac{a_2}{b^{k_n}-1} + a_3\alpha_n\right) &= 0\\
\implies a_1 + a_3\alpha &= 0,
\end{aligned}$$

which implies α is a rational number. This proves the theorem. $\square$

CHAPTER 3

On transcendence criterion for Cantor series

Let $Q = (b_n)_n$ be a sequence of positive integers with $b_n \geq 2$ for all $n \geq 1$. In this chapter we study the transcendence of certain Cantor series (or Q-ary expansions) with respect to the base Q. The results of this chapter has been published in [23].

3.1 Introduction

We recall a definition of Q-ary expansion which we had discussed already in Chapter 1.

Let $Q = (b_n)_n$ be a sequence of positive integers with $b_n \geq 2$ for all integers $n \geq 1$. We say a non-zero real number α has *Q-ary expansion* (or Cantor series expansion with respect to Q), if there exist $c_0 \in \mathbb{Z}$ and non-negative integers $c_1, c_2, \ldots, c_n, \ldots$ with $0 \leq c_n \leq b_n - 1$ for all integers $n \geq 1$ such that

$c_0.c_1c_2\cdots c_n\cdots$, and it is defined by a series of the form

$$c_0 + \frac{c_1}{b_1} + \frac{c_2}{b_1b_2} + \cdots + \frac{c_n}{b_1b_2b_3\cdots b_n} + \cdots . \tag{3.1}$$

Irrationality of Cantor series has a vast history. In 1869, Cantor was the first who studied the irrationality of this series and he gave the following necessary and sufficient condition; *let α be a real number given by* (3.1). *Suppose for every integer $q \geq 1$, there exists an integer n such that q divides $B_n = b_1 \ldots b_n$ and $c_n > 0$ for infinitely many values of n. Then α is an irrational number.*

In 1954, Oppenheim [9] proved a necessary and sufficient condition for a real number α written as (3.1) to be an irrational number as follows. *For each integer $q \geq 1$, there exists an integer r and a subsequence $(i_n)_{n\geq 1}$ of natural numbers such that*

$$\frac{r}{q} < \alpha_{i_n} < \frac{r+1}{q} \text{ for each } n = 1, 2, \ldots$$

where

$$\alpha_i := \frac{c_i}{b_i} + \frac{c_{i+1}}{b_ib_{i+1}} + \frac{c_{i+2}}{b_ib_{i+1}b_{i+2}} + \cdots$$

which is related to α for all integers $i \geq 1$. In 1971, Erdős and Straus [13] studied the irrationality of the series

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{b_1b_2\cdots b_n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\sigma(n)}{b_1b_2\cdots b_n},$$

where $\phi(n)$ denotes the Euler totient function and $\sigma(n) = \sum_{d|n} d$.

Later Hančl and Tijdemann [18] (see also Hančl and Tijdemann [19]) generalized the result of Oppenheim and also improved some of the result of Erdős and Straus.

As we have seen in Chapter 1, the Q-ary expansion which is defined by (3.1) is the generalization to the usual b-ary expansion. There are many transcendence results for b-ary expansion using the Subspace theorem. We mentioned one of the result here due to Adamczewski and Bugeaud [1]. In order to state this result and further results of this Chapter, we need to fix some notations.

Definition 3.1.1 Two real numbers α and α' are said to be Q-equivalent (respectively, b-equivalent) if their Q-ary expansions (respectively, b-ary expansions) have the same tail.

Theorem B.([1]) *Let $b \geq 2$ be an integer. Let $\mathbf{a} = (c_k)_{k\geq 1}$ and $\mathbf{a}' = (c'_k)_{k\geq 1}$ be the given sequences of integers, where $c_k \in \{0, 1, 2, \dots, b-1\}$ and $c'_k \in \{0, 1, 2, \dots, b-1\}$, for all integers $k \geq 1$. Suppose there exist infinitely many 3-tuples (j_m, j'_m, k_m) of natural numbers satisfying*

(1) for each $m \geq 1$, the block $.c_1 c_2 \dots c_{j_m} c_{j_m+1} \dots c_{j_m+k_m}$ is a prefix of $\mathbf{a}$;

(2) for each $m \geq 1$, the block $.c'_1 c'_2 \dots c'_{j'_m} c_{j_m+1} \dots c_{j_m+k_m}$ is a prefix of $\mathbf{a}'$;

(3) the sequences $\left(\dfrac{j_m}{k_m}\right)_m$ and $\left(\dfrac{j'_m}{k_m}\right)_m$ are bounded from above.

Then, either at least one of the real numbers

$$\alpha = \sum_{k=1}^{\infty} \frac{c_k}{b^k}, \quad \alpha' = \sum_{k=1}^{\infty} \frac{c'_k}{b^k}$$

is transcendental or α and α' are b-equivalent.

In this chapter we generalize this result for Q-ary expansions.

3.2 Main Results

We start with the following result.

Theorem 3.2.1 *Let $Q = (b_n)_{n\geq 1}$ be a bounded sequence of positive integers with $b_n \geq 2$ for all integers $n \geq 1$. Let $\mathbf{a} = (c_k)_{k\geq 1}$ and $\mathbf{a}' = (c'_k)_{k\geq 1}$ be the given sequences of integers with $c_k \in \{0, 1, 2, \ldots, b_k - 1\}$ and $c'_k \in \{0, 1, 2, \ldots, b_k - 1\}$, for all $k \geq 1$. Suppose there exist infinitely many distinct 2-tuples (j_m, k_m) of natural numbers satisfying*

(1) for each $m \geq 1$, the block $.c_1c_2 \ldots c_{j_m}c_{j_{m+1}} \ldots c_{j_m+k_m}$ is a prefix of $\mathbf{a}$;

(2) for each $m \geq 1$, the block $.c'_1c'_2 \ldots c'_{j_m}c_{j_{m+1}} \ldots c_{j_m+k_m}$ is a prefix of $\mathbf{a}'$;

(3) the sequence $\left(\frac{j_m}{k_m}\right)_m$ is bounded from above.

Then, either the real number

$$\alpha - \alpha' = \sum_{k=1}^{\infty} \frac{c_k}{b_1b_2 \cdots b_k} - \sum_{k=1}^{\infty} \frac{c'_k}{b_1b_2 \cdots b_k}$$

is rational or a transcendental number.

To illustrate Theorem 3.2.1, we take the sequence $Q = (b_n)_n$ consists of positive integers such that $b_n \in \{b_1, b_2\}$ and consider the set

$$\mathcal{S}_1 = \{2^m + 3^m : m \in \mathbb{N}\}.$$

Then we define

$$c_n = \begin{cases} 1; & \text{if } n \in \mathcal{S}_1 \\ 0; & \text{otherwise} \end{cases},$$

and $c'_n = 2c_n$ for all $n \geq 1$. Now we consider the Q-ary expansions

$$\alpha = \sum_{n=1}^{\infty} \frac{c_n}{b_1 \cdots b_n} = \sum_{n\in\mathcal{S}_1} \frac{1}{b_1 \cdots b_n} = \sum_{m=1}^{\infty} \frac{1}{b_1^{u_m} b_2^{v_m}}$$

and

$$\alpha' = 2\alpha = \sum_{m=1}^{\infty} \frac{2}{b_1^{u_m} b_2^{v_m}}$$

with $u_m + v_m = 2^m + 3^m$. By Theorem 3.2.1, it follows that α is a transcendental number. To see this, first we observe that the block of zeroes in between $\frac{1}{b_1^{u_m} b_2^{v_m}}$ and $\frac{1}{b_1^{u_{m+1}} b_2^{v_{m+1}}}$, which is of length $u_{m+1} + v_{m+1} - u_m - v_m - 1 = 2^m + 2 \times 3^m - 1$ for all natural numbers $m \geq 1$ and hence $\alpha' - \alpha = \alpha$ is an irrational number. Now, for all large enough integers m, we let $j_m = u_m + v_m$ and $k_m = 3^m$. We easily see that the sequences $(c_m)_m, (c'_m)_m, (j_m)_m$ and $(k_m)_m$ satisfies the hypothesis of Theorem 3.2.1. Hence, by Theorem 3.2.1, α must be a transcendental number.

Theorem 3.2.1 can be improved in the case when the sequence $Q = (b_n)_n$ is periodic. More precisely, we have the following theorem.

Theorem 3.2.2 *Let $Q = (b_n)_{n \geq 1}$ be an eventually periodic sequence of positive integers with $b_n \geq 2$ for all integers $n \geq 1$. Let $\mathbf{a} = (c_k)_{k \geq 1}$ and $\mathbf{a}' = (c'_k)_{k \geq 1}$ be the given sequences of integers with $c_k \in \{0, 1, 2, \dots, b_k - 1\}$ and $c'_k \in \{0, 1, 2, \dots, b_k - 1\}$, for all $k \geq 1$. Suppose there exist infinitely many distinct 3-tuples (j_m, j'_m, k_m) of natural numbers satisfying*

(1) for each $m \geq 1$, the block $.c_1 c_2 \dots c_{j_m} c_{j_{m+1}} \dots c_{j_m + k_m}$ is a prefix of $\mathbf{a}$;

(2) for each $m \geq 1$, the block $.c'_1 c'_2 \dots c'_{j'_m} c_{j_{m+1}} \dots c_{j_m + k_m}$ is a prefix of $\mathbf{a}'$;

(3) the sequences $\left(\frac{j_m}{k_m}\right)_m$ and $\left(\frac{j'_m}{k_m}\right)_m$ are bounded from above.

Then, either at least one of the real numbers

$$\alpha = \sum_{k=1}^{\infty} \frac{c_k}{b_1 b_2 \cdots b_k}, \quad \alpha' = \sum_{k=1}^{\infty} \frac{c'_k}{b_1 b_2 \cdots b_k}$$

is transcendental or α and α' are Q-equivalent.

To illustrate Theorem 3.2.2, we take the sequence $Q = (b_n)_n$ where b_n is defined as

$$b_n = \begin{cases} b_1; & \text{if } n \text{ is odd} \\ b_2; & \text{if } n \text{ is even,} \end{cases}$$

and consider the set

$$\mathcal{S}_2 = \{2^m + 1 : m \in \mathbb{N}\}.$$

Then we define

$$c_n = \begin{cases} 1; & \text{if } n \in \mathcal{S}_2 \\ 0; & \text{otherwise,} \end{cases}$$

and $c'_n = 2c_n$ for all $n \geq 1$. Now we consider the Q-ary expansions

$$\alpha = \sum_{n=1}^{\infty} \frac{c_n}{b_1 \cdots b_n} = \sum_{n \in \mathcal{S}_2} \frac{1}{b_1 \cdots b_n} = \sum_{n \in \mathcal{S}_2} \frac{1}{b_1^{[n/2]+1} b_2^{[n/2]}} = \sum_{m=1}^{\infty} \frac{1}{b_1^{(2^{m-1}+1)} b_2^{2^{m-1}}}$$

and

$$\alpha' = \sum_{m=1}^{\infty} \frac{2}{b_1^{(2^{m-1}+1)} b_2^{2^{m-1}}}.$$

By Theorem 3.2.2, it follows that α is a transcendental number. To see this, first we observe that the block of zeroes in between $\frac{1}{b_1^{(2^{m-1}+1)} b_2^{2^{m-1}}}$ and $\frac{1}{b_1^{2^m+1} b_2^{2^m}}$, which is of length $2 \cdot 2^m - 2 \cdot 2^{m-1} - 1 = 2^m - 1$ for all integers $m \geq 1$ and hence $\alpha' - \alpha$ can not be a rational number. This implies that α and α' are non-equivalent. Now, for all large enough integers m, we let $j_m = j'_m = 2^m + 1$ and $k_n = 2^{m-2}$. We easily see that the sequences $(c_m)_m$, $(c'_m)_m$, $(j_m)_m$, $(j'_m)_m$ and $(k_m)_m$, satisfies the hypothesis of Theorem 3.2.2. Hence, by Theorem 3.2.2, α must be a transcendental number.

In Theorem 3.2.1, we had assumed that the sequence $Q = (b_n)_{n \geq 1}$ is bounded. If the sequence $Q = (b_n)_{n \geq 1}$ is not bounded, can one obtain a similar conclusion for Cantor series with respect to Q? In the following theorem, we address this question and obtain a result conditionally.

Theorem 3.2.3 *Let $Q = (b_n)_{n\geq 1}$ be a sequence of positive integers with $b_n \geq 2$ for all integers $n \geq 1$ and let δ be any positive real number. Suppose there exists an infinite subset T consists of natural numbers N such that*

$$(b_1 b_2 \cdots b_N)^{\frac{1}{2}+\delta} \leq b_{N+1}. \tag{3.2}$$

Let $\mathbf{a} = (c_k)_{k\geq 1}$ and $\mathbf{a}' = (c'_k)_{k\geq 1}$ be the given sequences of integers with $c_k \in \{0, 1, 2, \ldots, b_k - 1\}$ and $c'_k \in \{0, 1, 2, \ldots, b_k - 1\}$, for all $k \geq 1$. If one of the following conditions, namely,

(1) the sequences $(c_n)_n$ and $(c'_n)_n$ are bounded;

(2) Suppose there exist co-prime integers h and k with $0 < h \leq k$ such that the sequence $(kc_n - hb_n)_n$ is bounded and the sequence $(c'_n)_n$ is bounded;

(3) Suppose there exist co-prime integers h and k with $0 < h \leq k$ such that the sequence $(kc_n - hb_n)_n$ is bounded and there exist co-prime integers h' and k' with $0 < h' \leq k'$ such that the sequence $(k'c'_n - h'b_n)_n$ is bounded;

is true, then either at least one of the real numbers

$$\alpha = \sum_{k=1}^{\infty} \frac{c_k}{b_1 b_2 \cdots b_k}, \quad \alpha' = \sum_{k=1}^{\infty} \frac{c'_k}{b_1 b_2 \cdots b_k}$$

is transcendental or $1, \alpha$ and α' are $\mathbb{Q}$-linearly dependent.

The proof of all these results based on the Subspace theorem, which is stated in Chapter 1.

As an application of Theorem 3.2.3, we have the following corollary.

Corollary 3.2.1 *Let $Q = (b_n)_n$ be a sequence of positive integers with $b_n \geq 2$ for all integer $n \geq 1$ such that $(b_1 b_2 \cdots b_m)^{\frac{1}{2}+\delta} \leq b_{m+1}$, for infinitely many natural numbers m. Let*

$$\alpha = \sum_{k=1}^{\infty} \frac{c_k}{b_1 b_2 \cdots b_k}, \quad \alpha' = \sum_{k=1}^{\infty} \frac{c'_k}{b_1 b_2 \cdots b_k}$$

be two algebraic numbers in the interval $(0,1)$. Then, either at least one of the sequences $\mathbf{a} = (c_k)_{k\geq 1}$ and $\mathbf{a}' = (c'_k)_{k\geq 1}$ is unbounded or $1, \alpha$ and α' are $\mathbb{Q}$-linearly dependent.

To illustrate Theorem 3.2.3 in Case 1, we take $b_n = 2^{u_n}$, where $u_n = 2^n$ for all $n \geq 1$ and

$$c_n = \begin{cases} 1; & \text{if } n = u_1 + u_2 + \cdots + u_m, \text{ for some } m \\ 0; & \text{otherwise,} \end{cases}$$

and

$$c'_n = \begin{cases} 0; & \text{if } n = u_1 + u_2 + \cdots + u_{m+1}, \text{ for some } m \\ 1; & \text{otherwise.} \end{cases}$$

Then consider the Q-ary expansions

$$\alpha = \sum_{n=1}^{\infty} \frac{c_n}{b_1 \cdots b_n} = \sum_{m=1}^{\infty} \frac{1}{2^{u_1+u_2+\cdots+u_m}}$$

and

$$\alpha' = \sum_{n=1}^{\infty} \frac{c'_n}{b_1 \cdots b_n} = \alpha' = \sum_{m=1}^{\infty} \frac{1}{2^{u_1+u_2+\cdots+u_{m+1}}}.$$

To illustrate Theorem 3.2.3 in Case 2, we take b_n as in Case 1 and define $c_n = 2^{u_n} - 2$ and $c'_n = 1$ for all $n \geq 1$.

To illustrate Theorem 3.2.3 in Case 3, we take b_n as in Case 1 and define $c_n = 2^{u_n} - 2$ and $c'_n = 2^{u_n} - 3$ for all $n \geq 1$. Then consider the Q-ary expansions

$$\alpha = \sum_{n=1}^{\infty} \frac{c_n}{b_1 \cdots b_n} \quad \text{and} \quad \alpha' = \sum_{n=1}^{\infty} \frac{c'_n}{b_1 \cdots b_n}.$$

We easily see that the sequences $(b_n)_n$, $(c_n)_n$ and $(c'_n)_n$ in all the cases satisfying the hypothesis of Theorem 3.2.3. Hence, by Theorem 3.2.3, either at least one of the real numbers α and α' is transcendental or $1, \alpha$ and α' are $\mathbb{Q}$-linearly dependent.

3.3 Proofs of Theorem 3.2.1, 3.2.2 and 3.2.3

Proof of Theorem 3.2.1. Suppose the given real number $\alpha - \alpha'$ is algebraic. In order to finish the proof we need to prove that the real number $\alpha - \alpha'$ is rational.

Since the sequence $(b_n)_n$ is bounded, it takes only finitely many distinct values, say, $b_1, b_2, \ldots, b_r$ for some integer $r \geq 1$. WLOG we assume that $b_1 < b_2 \cdots < b_r$ Hence, for each integer $m \geq 1$, we can write

$$b_1 b_2 \cdots b_m = b_1^{i_m^{(1)}} b_2^{i_m^{(2)}} b_3^{i_m^{(3)}} \cdots b_r^{i_m^{(r)}}, \tag{3.3}$$

for some non-negative integers $i_m^{(1)}, i_m^{(2)}, \ldots, i_m^{(r)}$ with $i_m^{(1)} + i_m^{(2)} + \cdots + i_m^{(r)} = m$.

Suppose the sequence $(j_m)_m$ is bounded. Then, there exists a positive integer s and an infinite set $\mathcal{N}_1$ of distinct positive integers such that $j_m = s$, for all $m \in \mathcal{N}_1$. Therefore, for infinitely many integers $m \in \mathcal{N}_1$, the block

$$c_{j_m+1} c_{j_m+2} \cdots c_{j_m+k_m}$$

is the prefix of $b_1 b_2 \cdots b_s \alpha - c_1 c_2 \ldots c_s$ and $b_1 b_2 \cdots b_s \alpha' - c'_1 c'_2 \ldots c'_s$. This implies that

$$\begin{aligned} & b_1 b_2 \cdots b_s \alpha - c_1 c_2 \dots c_s = b_1 b_2 \cdots b_s \alpha' - c'_1 c'_2 \dots c'_s \\ \iff & (\alpha - \alpha') b_1 b_2 \cdots b_s = c_1 c_2 \dots c_s - c'_1 c'_2 \dots c'_s \end{aligned}$$

which in turns implies that the real number $\alpha - \alpha'$ is rational. Now we assume that the sequence $(j_m)_m$ is unbounded. Then, there exists an infinite set $\mathcal{N}_2$ of positive integers such that

$$j_{m_1} < j_{m_2} < \ldots < j_{m_k} < \ldots$$

for any $m_k \in \mathcal{N}_2$. This implies that there exist sequences $(\ell_m)_m$ and $(\ell'_m)_m$ of integers such that

$$\left| b_1 b_2 \cdots b_{j_m} (\alpha - \alpha') - \ell_m + \ell'_m \right| = \left| \left(\frac{c_{j_m+k_m+1} - c'_{j_m+k_m+1}}{b_{j_m+1} \cdots b_{j_m+k_m+1}} + \frac{c_{j_m+k_m+2} - c'_{j_m+k_m+2}}{b_{j_m+1} \cdots b_{j_m+k_m+1} b_{j_m+k_m+2}} + \cdots \right) \right|.$$

Since $c_k, c'_k \in \{0, 1, \ldots b_k - 1\}$, for all $k \geq 1$, we get

$$\begin{aligned} \left| b_1 b_2 \cdots b_{j_m} (\alpha - \alpha') - \ell_m + \ell'_m \right| &= \left| \left(\frac{c_{j_m+k_m+1} - c'_{j_m+k_m+1}}{b_{j_m+1} \cdots b_{j_m+k_m+1}} + \frac{c_{j_m+k_m+2} - c'_{j_m+k_m+2}}{b_{j_m+1} \cdots b_{j_m+k_m+1} b_{j_m+k_m+2}} + \cdots \right) \right| \\ &\leq \frac{1}{b_{j_m+1} \cdots b_{j_m+k_m}} \left(\frac{b_{j_m+k_m+1} - 1}{b_{j_m+k_m+1}} + \frac{b_{j_m+k_m+2} - 1}{b_{j_m+k_m+1} b_{j_m+k_m+2}} + \cdots \right) \\ &= \frac{1}{b_{j_m+1} \cdots b_{j_m+k_m}} \sum_{i=1}^{\infty} \frac{b_{j_m+k_m+i} - 1}{b_{j_m+k_m+1} \cdots b_{j_m+k_m+i}}. \end{aligned}$$

By Proposition 1.1.2, the sum

$$\sum_{i=1}^{\infty} \frac{b_{j_m+k_m+i} - 1}{b_{j_m+k_m+1} \cdots b_{j_m+k_m+i}} = 1.$$

Thus, we get

$$\left|b_1 b_2 \cdots b_{j_m}(\alpha - \alpha') - \ell_m + \ell'_m\right| \le \frac{1}{b_{j_m+1} b_{j_m+2} \cdots b_{j_m+k_m}}, \tag{3.4}$$

holds for all positive integers $m \in \mathcal{N}_2$. Now we can rewrite (3.4) as follows

$$\left|\alpha - \alpha' - \frac{(\ell_m - \ell'_m)}{b_1 b_2 \cdots b_{j_m}}\right| \le \frac{1}{(b_1 b_2 \cdots b_{j_m})(b_{j_m+1} b_{j_m+2} \cdots b_{j_m+k_m})} \tag{3.5}$$

Thus, from (3.3) and (3.5), we get

$$\left|\alpha - \alpha' - \frac{(\ell_m - \ell'_m)}{b_1^{i_{j_m}^{(1)}} b_2^{i_{j_m}^{(2)}} \cdots b_r^{i_{j_m}^{(r)}}}\right| \le \frac{1}{\left(b_1^{i_{j_m}^{(1)}} b_2^{i_{j_m}^{(2)}} \cdots b_r^{i_{j_m}^{(r)}}\right)\left(b_1^{i_{k_m}^{(1)}} b_2^{i_{k_m}^{(2)}} \cdots b_r^{i_{k_m}^{(r)}}\right)},$$

where $i_{j_m}^{(1)} + i_{j_m}^{(2)} + \cdots + i_{j_m}^{(r)} = j_m$ and $i_{k_m}^{(1)} + i_{k_m}^{(2)} + \cdots + i_{k_m}^{(r)} = k_m$. Since the sequence $\left(\frac{j_m}{k_m}\right)$ is bounded above, there exists a positive real number M such that $j_m \le M k_m$, which is equivalent to

$$\left(i_{j_m}^{(1)} + i_{j_m}^{(2)} + \cdots + i_{j_m}^{(r)}\right) \le M\left(i_{k_m}^{(1)} + i_{k_m}^{(2)} + \cdots + i_{k_m}^{(r)}\right).$$

This implies that

$$\left(b_1^{i_{j_m}^{(1)}} b_2^{i_{j_m}^{(2)}} \cdots b_r^{i_{j_m}^{(r)}}\right) \le \left(b_1^{i_{k_m}^{(1)}} b_2^{i_{k_m}^{(2)}} \cdots b_r^{i_{k_m}^{(r)}}\right)^{\ell r M},$$

where ℓ is a non-negative integer such that $b_1^{\ell-1} < b_r \le b_1^{\ell}$. Hence, there exists a positive real number ϵ such that

$$\left|\alpha - \alpha' - \frac{(\ell'_m - \ell_m)}{b_1^{i_{j_m}^{(1)}} b_2^{i_{j_m}^{(2)}} \cdots b_r^{i_{j_m}^{(r)}}}\right| \le \frac{1}{\left(b_1^{i_{j_m}^{(1)}} b_2^{i_{j_m}^{(2)}} \cdots b_r^{i_{j_m}^{(r)}}\right)^{1+\epsilon}},$$

holds for every $m \in \mathcal{N}_2$. Let $S = \cup_{i=1}^{r}\{p : p|b_i\}$ be a finite set of primes. Then

$$\left(\prod_{p\in S} |\ell'_m - \ell_m|_p \left|b_1^{i_{j_m}^{(1)}} b_2^{i_{j_m}^{(2)}} \cdots b_r^{i_{j_m}^{(r)}}\right|_p\right) \left|\alpha - \alpha' - \frac{(\ell'_m - \ell_m)}{b_1^{i_{j_m}^{(1)}} b_2^{i_{j_m}^{(2)}} \cdots b_r^{i_{j_m}^{(r)}}}\right| < \frac{1}{\left(b_1^{i_{j_m}^{(1)}} b_2^{i_{j_m}^{(2)}} \cdots b_r^{i_{j_m}^{(r)}}\right)^{2+\epsilon}},$$

since $|x|_p \leq 1$ for every $x \in \mathbb{Z}$ and $\prod_{p\in S}\left|b_1^{i_{j_m}^{(1)}} b_2^{i_{j_m}^{(2)}} \cdots b_r^{i_{j_m}^{(r)}}\right|_p = \frac{1}{\left(b_1^{i_{j_m}^{(1)}} b_2^{i_{j_m}^{(2)}} \cdots b_r^{i_{j_m}^{(r)}}\right)}$. Hence, by Theorem 1.2.3 in Chapter 1, we get $(\alpha - \alpha')$ is a transcendental number, which is a contradiction to the assumption that both α and α' are algebraic numbers. Therefore, we have

$$\left|\alpha - \alpha' - \frac{(\ell'_m - \ell_m)}{b_1^{i_{j_m}^{(1)}} b_2^{i_{j_m}^{(2)}} \cdots b_r^{i_{j_m}^{(r)}}}\right| = 0,$$

for all large $m \in \mathcal{N}_2$. This implies that $\alpha - \alpha'$ is a rational number. □

Proof of Theorem 3.2.2. By the hypothesis, the sequence $Q = (b_n)$ is an eventually periodic sequence of positive integers with $b_n \geq 2$ for all integers $n \geq 1$. Without loss of generality, we shall assume that the sequence Q is eventually periodic with period 2 and the proof of the general case is verbatim. That is, there exists an integer $N \geq 1$ satisfying

$$b_{N+i} = \begin{cases} d_1; & \text{if } i \text{ is odd} \\ d_2; & \text{if } i \text{ is even} \end{cases} \tag{3.6}$$

for some positive integers d_1 and d_2 in Q. Then, by (3.6), we have,

$$\alpha = \frac{c_1}{b_1} + \frac{c_2}{b_1 b_2} + \cdots + \frac{c_N}{b_1 b_2 \cdots b_N} + \frac{c_{N+1}}{b_1 b_2 \cdots b_N d_1} + \frac{c_{N+2}}{b_1 b_2 \cdots b_N d_1 d_2} + \cdots .$$

Similarly, we have

$$\alpha' = \frac{c'_1}{b_1} + \frac{c'_2}{b_1 b_2} + \cdots + \frac{c'_N}{b_1 b_2 \cdots b_N} + \frac{c'_{N+1}}{b_1 b_2 \cdots b_N d_1} + \frac{c'_{N+2}}{b_1 b_2 \cdots b_N d_1 d_2} + \cdots .$$

Now we multiply α and α' by $b_1 \cdots b_N$, we get

$$b_1 \dots b_N \alpha = m + \frac{c_{N+1}}{d_1} + \frac{c_{N+2}}{d_1 d_2} + \frac{c_{N+3}}{d_1^2 d_2} + \frac{c_{N+4}}{d_1^2 d_2^2} + \cdots ,$$

and

$$b_1 \cdots b_N \alpha' = m' + \frac{c'_{N+1}}{d_1} + \frac{c'_{N+2}}{d_1 d_2} + \frac{c'_{N+3}}{d_1^2 d_2} + \frac{c'_{N+4}}{d_1^2 d_2^2} + \cdots .$$

In order to prove that α and α' are either Q-equivalent or at least one of them is transcendental, it is enough to prove either $\theta = b_1 \cdots b_N \alpha - m$ and $\theta' = b_1 \cdots b_N \alpha' - m'$ are Q-equivalent or at least one of them is transcendental. For simplicity, we write $c_{N+i} = a_i$ and $c'_{N+i} = a'_i$ for all integers $i \geq 1$ and $d_1 = b_1$, $d_2 = b_2$. Then, we get,

$$\theta = \frac{a_1}{b_1} + \frac{a_2}{b_1 b_2} + \cdots + \frac{a_{j_m}}{b_1^{j_{1m}} b_2^{j_{2m}}} + \frac{a_{j_m+1}}{b_1^{j_{1m}+1} b_2^{j_{2m}}} + \cdots + \frac{a_{j_m+k_m}}{b_1^{j_{1m}+k_{1m}} b_2^{j_{2m}+k_{2m}}} + \cdots ,$$

where $j_{1m} + j_{2m} = j_m$, $k_{1m} + k_{2m} = k_m$ for all integers $m = 1, 2, \ldots$ and $0 \leq a_{2i+1} \leq b_1 - 1$ and $0 \leq a_{2i} \leq b_2 - 1$ for all integers $i = 0, 1, 2, \ldots$.

Similarly, corresponding to α' we get

$$\theta' = \frac{a'_1}{b_1} + \frac{a'_2}{b_1 b_2} + \cdots + \frac{a'_{j'_m}}{b_1^{j'_{1m}} b_2^{j'_{2m}}} + \frac{a'_{j'_m+1}}{b_1^{j'_{1m}+1} b_2^{j'_{2m}}} + \cdots + \frac{a'_{j'_m+k_m}}{b_1^{j'_{1m}+k_{1m}} b_2^{j'_{2m}+k_{2m}}} + \cdots ,$$

where $j'_{1m} + j'_{2m} = j'_m$, $k_{1m} + k_{2m} = k_m$ for all integers $m = 1, 2, \ldots$ and $0 \leq a'_{2i+1} \leq b_1 - 1$ and $0 \leq a'_{2i} \leq b_2 - 1$ for all integers $i = 0, 1, 2, \ldots$.

Now, we define the sequences $(\theta_m)_m$ and $(\theta'_m)_m$ of rational numbers as follows.

For each integer $m \geq 1$, we define

$$\theta_m = \frac{a_1}{b_1} + \frac{a_2}{b_1 b_2} + \cdots + \frac{a_{j_m+1}}{b_1^{j_{1m}+1} b_2^{j_{2m}}} + \cdots + \frac{a_{j_m+k_m}}{b_1^{j_{1m}+k_{1m}} b_2^{j_{2m}+k_{2m}}} = \frac{p_m}{b_1^{j_{1m}+k_{1m}} b_2^{j_{2m}+k_{2m}}},$$

for some positive integer p_m, and

$$\theta'_m = \frac{a'_1}{b_1} + \frac{a'_2}{b_1 b_2} + \cdots + \frac{a'_{j'_m+1}}{b_1^{j'_{1m}+1} b_2^{j'_{2m}}} + \cdots + \frac{a'_{j'_m+k_m}}{b_1^{j'_{1m}+k_{1m}} b_2^{j'_{2m}+k_{2m}}} = \frac{p'_m}{b_1^{j'_{1m}+k_{1m}} b_2^{j'_{2m}+k_{2m}}},$$

for some positive integer p'_m.

Now, we consider

$$|\theta - \theta_m| = \left| \theta - \frac{p_m}{b_1^{j_{1m}+k_{1m}} b_2^{j_{2m}+k_{2m}}} \right| \leq \frac{C}{b_1^{j_{1m}+k_{1m}+1} b_2^{j_{2m}+k_{2m}+1}},$$

where C is some fixed positive constant. This is equivalent to,

$$\left| \theta b_1^{j_{1m}+k_{1m}} b_2^{j_{2m}+k_{2m}} - p_m \right| \leq C', \tag{3.7}$$

for some positive constant C'.

Similarly, we get

$$\left| \theta b_1^{j'_{1m}+k_{1m}} b_2^{j'_{2m}+k_{2m}} - p'_m \right| \leq C'', \tag{3.8}$$

for some positive constant C''.

For every prime p, we see that ,

$$\begin{aligned} |p_m - p'_m|_p &= \left| b_1^{j_{1m}+k_{1m}} b_2^{j_{2m}+k_{2m}} \theta_m - b_1^{j'_{1m}+k_{1m}} b_2^{j'_{2m}+k_{2m}} \theta'_m \right|_p \\ &= \left| b_1^{k_{1m}} b_2^{k_{2m}} \left(b_1^{j_{1m}} b_2^{j_{2m}} \theta_m - b_1^{j'_{1m}} b_2^{j'_{2m}} \theta'_m \right) \right|_p \end{aligned}$$

$$\leq \left|b_1^{k_{1m}} b_2^{k_{2m}}\right|_p \left|\left(b_1^{j_{1m}} b_2^{j_{2m}} \theta_m - b_1^{j'_{1m}} b_2^{j'_{2m}} \theta'_m\right)\right|_p .$$

By the assumption $a_{j_m+i} = a'_{j'_m+i}$ for all $1 \leq i \leq k_m$, we see that the quantity $\left(b_1^{j_{1m}} b_2^{j_{2m}} \theta_m - b_1^{j'_{1m}} b_2^{j'_{2m}} \theta'_m\right)$ is an integer. Therefore, by using the fact that $|x|_p \leq 1$, for every non-zero integer x, we obtain

$$\begin{aligned} |p_m - p'_m|_p &\leq \left|b_1^{k_{1m}} b_2^{k_{2m}}\right|_p \left|\left(b_1^{j_{1m}} b_2^{j_{2m}} \theta_m - b_1^{j'_{1m}} b_2^{j'_{2m}} \theta'_m\right)\right|_p \\ &\leq |{b_1}^{k_{1m}} {b_2}^{k_{2m}}|_p \leq |{b_1}^{k_{1m}}|_p |{b_2}^{k_{2m}}|_p. \end{aligned} \tag{3.9}$$

Suppose we assume that θ and θ' are both algebraic numbers. Now to finish the proof of the theorem, we need to prove θ and θ' are Q-equivalent.

In order to prove that θ and θ' are Q-equivalent, we shall apply Theorem 1.3.3 in Chapter 1. Let

$$S = \{p : p|b_1\} \cup \{p : p|b_2\} \cup \{\infty\}$$

be the finite subset of primes, which includes infinite prime. For $p = \infty$, we define the following linearly independent linear forms with algebraic coefficients,

$$\begin{aligned} L_{1,\infty}(X_1, X_2, X_3, X_4) &= \theta X_1 - X_3 \\ L_{2,\infty}(X_1, X_2, X_3, X_4) &= \theta' X_2 - X_4 \\ L_{3,\infty}(X_1, X_2, X_3, X_4) &= X_1 \\ L_{4,\infty}(X_1, X_2, X_3, X_4) &= X_2. \end{aligned}$$

For any prime $p \in S$ other than ∞, we define the following linearly independent linear forms with integer coefficients

$$L_{1,p}(X_1, X_2, X_3, X_4) = X_1$$
$$L_{2,p}(X_1, X_2, X_3, X_4) = X_2$$
$$L_{3,p}(X_1, X_2, X_3, X_4) = X_3$$
$$L_{4,p}(X_1, X_2, X_3, X_4) = X_3 - X_4.$$

For any positive integer $m \geq 1$, we let

$$\mathbf{x}^{(m)} = (b_1^{j_{1m}+k_{1m}} b_2^{j_{2m}+k_{2m}}, b_1^{j'_{1m}+k_{1m}} b_2^{j'_{2m}+k_{2m}}, p_m, p'_m) \in \mathbb{Z}^4.$$

For applying Theorem 1.3.3, we need to compute the quantity

$$\prod_{p\in S}\prod_{i=1}^{4}|L_{i,p}(\mathbf{x}^{(m)})|_p = \prod_{i=1}^{4}|L_{i,\infty}(\mathbf{x}^{(m)})|_\infty \prod_{p\in S\setminus\{\infty\}}\prod_{i=1}^{4}|L_{i,p}(\mathbf{x}^{(m)})|_p.$$

By (3.7) and (3.8), we get

$$\prod_{p\in S}\prod_{i=1}^{4}|L_{i,p}(\mathbf{x}^{(m)})|_p < C'C''\prod_{p\in S}\left|b_1^{j_{1m}+k_{1m}} b_2^{j_{2m}+k_{2m}}\right|_p \left|b_1^{j'_{1m}+k_{1m}} b_2^{j'_{2m}+k_{2m}}\right|_p$$
$$\cdot \prod_{p\in S\setminus\{\infty\}} |p_m|_p|p'_m|_p|p_m - p'_m|_p.$$

By Proposition 2.2.1 (the product formula) and the fact that $|p_m|_p, |p'_m|_p \leq 1$, we get

$$\prod_{p\in S}\prod_{i=1}^{4}|L_{i,p}(\mathbf{x}^{(m)})|_p < C'C'' \prod_{p\in S\setminus\{\infty\}} |p_m - p'_m|_p.$$

Thus, from (3.9) we have

$$\prod_{p\in S}\prod_{i=1}^{4}|L_{i,p}(\mathbf{x}^{(m)})|_p < \frac{C'C''}{b_1^{k_{1m}}b_2^{k_{2m}}}.$$

Since the sequence $\left(\frac{j_m}{k_m}\right)_m$ and $\left(\frac{j'_m}{k_m}\right)_m$ are bounded, there exists a positive real number M such that $j_m \leq Mk_m$, $j'_m \leq Mk_m$. By the periodicity of the sequence $(b_n)_n$, we note that

$$|j_{1m} - j_{2m}| \leq 1 \quad \text{and} \quad |k_{1m} - k_{2m}| \leq 1.$$

Therefore, we have

$$\frac{j_m}{2} - 1 \leq j_{im} \leq \frac{j_m}{2} + 1, \quad \text{for} \;\; i = 1, 2,$$

and

$$\frac{k_m}{2} - 1 \leq k_{im} \leq \frac{k_m}{2} + 1, \quad \text{for} \;\; i = 1, 2.$$

Hence,

$$\begin{aligned} b_1^{j_{1m}+k_{1m}}b_2^{j_{2m}+k_{2m}} &\leq b_1^{\frac{j_m}{2}+1+k_{1m}}b_2^{\frac{j_m}{2}+1+k_{2m}} \\ &\leq b_1^{\frac{Mk_m}{2}+1+k_{1m}}b_2^{\frac{Mk_m}{2}+1+k_{2m}} \\ &\leq b_1^{Mk_{1m}+1+k_{1m}}b_2^{Mk_{2m}+1+k_{2m}} \\ &\leq \left(b_1^{k_{1m}}b_2^{k_{2m}}\right)^{2(M+1)}. \end{aligned}$$

Similarly, we get

$$b_1^{j'_{1m}+k_{1m}}b_2^{j'_{2m}+k_{2m}} \leq \left(b_1^{k_{1m}}b_2^{k_{2m}}\right)^{2(M+1)}.$$

Let $0 < \epsilon < \frac{1}{2(M+1)}$ be a real number. Then by these inequalities, we have

$$\frac{1}{b_1^{k_{1m}} b_2^{k_{2m}}} \leq \max\{b_1^{j_{1m}+k_{1m}} b_2^{j_{2m}+k_{2m}}, b_1^{j'_{1m}+k_{1m}} b_2^{j'_{2m}+k_{2m}}\}^{-\epsilon}.$$

This implies that

$$\prod_{p \in S} \prod_{i=1}^{4} |L_{i,p}(\mathbf{x}^{(m)})|_p \leq \max\{b_1^{j_{1m}+k_{1m}} b_2^{j_{2m}+k_{2m}}, b_1^{j'_{1m}+k_{1m}} b_2^{j'_{2m}+k_{2m}}\}^{-\epsilon'}.$$

holds for all large positive integer m and some $0 < \epsilon' < \epsilon$. Since $\theta, \theta' \in [0, 1)$, we have $p_m, p'_m \leq \max\{b_1^{j_{1m}+k_{1m}} b_2^{j_{2m}+k_{2m}}, b_1^{j'_{1m}+k_{1m}} b_2^{j'_{2m}+k_{2m}}\}$. Thus, we have

$$\prod_{p \in S} \prod_{i=1}^{4} |L_{i,p}(\mathbf{x}^{(m)})|_p \leq \max\{b_1^{j_{1m}+k_{1m}} b_2^{j_{2m}+k_{2m}}, b_1^{j'_{1m}+k_{1m}} b_2^{j'_{2m}+k_{2m}}, p_m, p'_m\}^{-\epsilon}$$

holds for all large positive integer m. Hence, for all large positive integers m, the non-zero integer lattice point

$$\mathbf{x}^{(m)} = (b_1^{j_{1m}+k_{1m}} b_2^{j_{2m}+k_{2m}}, b_1^{j'_{1m}+k_{1m}} b_2^{j'_{2m}+k_{2m}}, p_m, p'_m) \in \mathbb{Z}^4$$

satisfies the hypothesis of Theorem 1.3.3. Therefore, for all large positive integers m, the non-zero integer lattice points $\mathbf{x}^{(m)} \in \mathbb{Z}^4$ lie in finitely many proper subspaces of $\mathbb{Q}^4$. Hence, there exists a non-zero integer quadruple (z_1, z_2, z_3, z_4) such that

$$z_1 b_1^{j_{1m}+k_{1m}} b_2^{j_{2m}+k_{2m}} + z_2 b_1^{j'_{1m}+k_{1m}} b_2^{j'_{2m}+k_{2m}} + z_3 p_m + z_4 p'_m = 0,$$

holds for infinitely many values of m. Rest of the proof follows from the proof of Theorem 2 in [1]. We shall omit the proof here. □

Proof of Theorem 3.2.3. We shall consider the following three cases.

Case 1. The sequences $(c_n)_n$ and $(c'_n)_n$ are bounded.

We define the sequences $(\alpha_N)_N$ and $(\alpha'_N)_N$ of rational numbers as follows. For each integer $N \geq 1$, we define

$$\alpha_N = \sum_{n=1}^{N} \frac{c_n}{b_1 b_2 \cdots b_n} = \frac{p_N}{b_1 b_2 \cdots b_N},$$

for some positive integer p_N, and

$$\alpha'_N = \sum_{n=1}^{N} \frac{c'_n}{b_1 b_2 \cdots b_n} = \frac{p'_N}{b_1 b_2 \cdots b_N},$$

for some positive integer p'_N. Now, we consider,

$$|\alpha - \alpha_N| = \left|\alpha - \frac{p_N}{b_1 b_2 \cdots b_N}\right| \leq \frac{1}{b_1 b_2 \cdots b_{N+1}}\left(c_{N+1} + \frac{c_{N+2}}{b_{N+2}} + \cdots\right). \tag{3.10}$$

Since the sequence $(c_n)_n$ is bounded, there exists a positive constant C such that $c_n \leq C$ for all positive integer $n \geq 1$. Then, by (3.10), we have,

$$\left|\alpha - \frac{p_N}{b_1 b_2 \cdots b_N}\right| \leq \frac{C}{b_1 b_2 \cdots b_N b_{N+1}}\left(1 + \frac{1}{b_1} + \frac{1}{b_1^2} + \cdots\right) = \frac{C b_1}{(b_1 - 1) b_1 b_2 \cdots b_N b_{N+1}}$$

for all positive integers $N \geq 1$.

By (3.2), we get

$$\left|\alpha - \frac{p_N}{b_1 b_2 \cdots b_N}\right| \leq \frac{C b_1}{(b_1 - 1)(b_1 b_2 \cdots b_N)^{\frac{3}{2}+\delta}} = \frac{C'}{(b_1 b_2 \cdots b_N)^{\frac{3}{2}+\delta}}$$

holds for all $N \in T$, where $C' = \frac{C b_1}{b_1 - 1}$. Thus, we get

$$|b_1 b_2 \cdots b_N \alpha - p_N| \leq \frac{C'}{(b_1 b_2 \cdots b_N)^{\frac{1}{2}+\delta}}, \tag{3.11}$$

for all $N \in T$. Similarly, we get

$$|b_1 b_2 \cdots b_N \alpha' - p'_N| \leq \frac{C'}{(b_1 b_2 \cdots b_N)^{\frac{1}{2}+\delta}}, \tag{3.12}$$

for all integers $N \in T$.

We shall assume that both α and α' are algebraic. Now to finish the proof, we need to prove $1, \alpha$ and α' are $\mathbb{Q}$-linearly dependent.

In order to prove that $1, \alpha$ and α' are $\mathbb{Q}$-linearly dependent, we shall apply Theorem 1.3.1 in Chapter 1. Consider the linear forms with algebraic coefficients

$$\begin{aligned} L_1(X_1, X_2, X_3) &= X_1 \\ L_2(X_1, X_2, X_3) &= \alpha X_1 - X_2, \\ L_3(X_1, X_2, X_3) &= \alpha' X_1 - X_3. \end{aligned} \tag{3.13}$$

Clearly, the above linear forms are linearly independent.

For any $N \in T$, we let

$$\mathbf{x}^{(N)} = (b_1 b_2 \cdots b_N, p_N, p'_N) \in \mathbb{Z}^3.$$

Since $\alpha_N < 1$, $\alpha'_N < 1$ for all integers $N \in T$, we see that $p_N < b_1 b_2 \cdots b_N$, $p'_N < b_1 b_2 \cdots b_N$ and hence, we get

$$\max\{b_1 b_2 \cdots b_N, p_N, p'_N\} \leq b_1 b_2 \cdots b_N.$$

For applying Theorem 1.3.1, we need to compute the quantity,

$$\prod_{i=1}^{3} |L_{i,p}(\mathbf{x}^{(N)})|.$$

First, by (3.13), we note that

$$\prod_{i=1}^{3}|L_i(\mathbf{x}^{(N)})| = |b_1b_2\cdots b_N||b_1b_2\cdots b_N\alpha - p_N||b_1b_2\cdots b_N\alpha' - p'_N| \qquad (3.14)$$

Thus, from (3.11), (3.12) and (3.14) we conclude that,

$$\prod_{i=1}^{3}|L_i(\mathbf{x}^{(N)})| < \Big(\max\{b_1b_2\cdots b_N, p_N, p'_N\}\Big)^{-\delta'},$$

holds for all large $N \in T$ and for some $\delta' > 0$. Hence, for all large $N \in T$, the non-zero integer lattice point $\mathbf{x}^{(N)} = (b_1b_2\cdots b_N, p_N, p'_N) \in \mathbb{Z}^3$ satisfies the hypothesis of Theorem 1.3.1. Therefore, the non-zero integer lattice point $\mathbf{x}^{(N)} = (b_1b_2\cdots b_N, p_N, p'_N) \in \mathbb{Z}^3$ lie only in finitely many proper subspaces of $\mathbb{Q}^3$ for all large $N \in T$. Since T is an infinite subset, by the Dirichlet box principle, there exists a proper subspace of $\mathbb{Q}^3$ containing the integer lattice point $\mathbf{x}^{(N)} = (b_1b_2\cdots b_N, p_N, p'_N) \in \mathbb{Z}^3$ for infinitely many values of $N \in T$. That is, there exists a non-zero tuple $(z_1, z_2, z_3) \in \mathbb{Z}^3$ such that

$$z_1b_1b_2\cdots b_N + z_2p_N + z_3p'_N = 0$$

holds for infinitely many values of $N \in T$. This implies that

$$\lim_{N\to\infty, N\in T}\left(z_1 + z_2\frac{p_N}{b_1b_2\cdots b_N} + z_3\frac{p'_N}{b_1b_2\cdots b_N}\right) = 0$$
$$\implies z_1 + z_2\alpha + z_3\alpha' = 0.$$

Hence, $1, \alpha$ and α' are $\mathbb{Q}$-linearly dependent. This proves the assertion.

Case 2. Suppose there exist co-prime integers h and k with $0 < h \leq k$ such that the sequences $(kc_n - hb_n)_n$ and $(c'_n)_n$ are bounded.

We define the sequences $(\alpha_N)_N$ and $(\alpha'_N)_N$ of rational numbers as follows.

For each integer $N \geq 1$, we define

$$\alpha_N = \frac{c_1}{b_1} + \frac{c_2}{b_1 b_2} + \cdots + \frac{c_N}{b_1 b_2 \cdots b_N} + \frac{\frac{h}{k}(b_{N+1}-1)}{b_1 b_2 \cdots b_{N+1}} + \frac{\frac{h}{k}(b_{N+2}-1)}{b_1 b_2 \cdots b_{N+2}} + \cdots ,$$

and

$$\alpha'_N = \frac{c'_1}{b_1} + \frac{c'_2}{b_1 b_2} + \cdots + \frac{c'_N}{b_1 b_2 \cdots b_N}.$$

By Lemma 1.1.2, the real numbers α_N and α'_N are rational for all integer $N \geq 1$. Hence, we get

$$\alpha_N = \frac{c_1}{b_1} + \cdots + \frac{c_N}{b_1 b_2 \cdots b_N} + \frac{h}{k b_1 b_2 \cdots b_N}\left(\sum_{i=1}^{\infty} \frac{b_{N+i}-1}{b_{N+1}\cdots b_{N+i}}\right) = \frac{p_N}{k b_1 b_2 \ldots b_N},$$

for some integer p_N, and

$$\alpha'_N = \frac{p'_N}{b_1 b_2 \cdots b_N}$$

for some integer p'_N. Now we consider,

$$|\alpha - \alpha_N| \leq \frac{1}{b_1 b_2 \cdots b_{N+1}}\left(\frac{|c_{N+1} - \frac{h}{k}(b_{N+1}-1)|}{1} + \frac{|c_{N+2} - \frac{h}{k}(b_{N+2}-1)|}{b_{N+2}} + \cdots\right).$$

Since both the sequences $(kc_n - hb_n)_n$ and $(c'_n)_n$ are bounded, there exists a positive constant C such that $|kc_n - hb_n| \leq C$ and $|c'_n| \leq C$ for all $n \geq 1$. Hence, we get

$$|\alpha - \alpha_N| \leq \frac{C+h}{k b_1 b_2 \cdots b_{N+1}}\left(1 + \frac{1}{b_1} + \frac{1}{b_1^2} + \cdots + \cdots\right) = \frac{b_1(C+h)}{(b_1-1)k b_1 b_2 \cdots b_N b_{N+1}},$$

for all positive integers $N \geq 1$.

By (3.2), we have

$$\left|\alpha - \frac{p_N}{k b_1 b_2 \cdots b_N}\right| \leq \frac{b_1(C+h)}{k(b_1-1)(b_1 b_2 \cdots b_N)^{\frac{3}{2}+\delta}} = \frac{C'}{k(b_1 b_2 \cdots b_N)^{\frac{3}{2}+\delta}}$$

for all $N \in T$, where $C' = \frac{b_1(C+h)}{b_1-1}$. Thus we get,

$$|kb_1b_2\cdots b_N\alpha - p_N| \le \frac{C'}{(kb_1b_2\cdots b_N)^{\frac{1}{2}+\delta}},$$

for all $N \in T$.

Now we consider

$$\left|\alpha' - \frac{p'_N}{b_1b_2\cdots b_N}\right| \le \frac{C}{b_1b_2\cdots b_N b_{N+1}}\left(1+\frac{1}{b_1}+\frac{1}{b_1^2}+\cdots\right) = \frac{Cb_1}{(b_1-1)b_1b_2\cdots b_N b_{N+1}}.$$

By using the same argument as in the Case 1, we get the following

$$|b_1b_2\ldots b_N\alpha' - p'_N| \le \frac{C''}{(b_1b_2\cdots b_N)^{\frac{1}{2}+\delta}},$$

where $C'' = \frac{Cb_1}{b_1-1}$.

We shall assume that both of α and α' are algebraic. Now to finish the proof, we need to prove $1, \alpha$ and α' are $\mathbb{Q}$-linearly dependent.

In order to prove that $1, \alpha$ and α' are $\mathbb{Q}$-linearly dependent, we shall apply Theorem 1.3.1 in Chapter 1. Consider the linear forms with algebraic coefficients,

$$L_1(X_1, X_2, X_3) = X_1$$
$$L_2(X_1, X_2, X_3) = k\alpha X_1 - X_2,$$
$$L_3(X_1, X_2, X_3) = \alpha' X_1 - X_3.$$

Clearly, the above linear forms are linearly independent.

For any $N \in T$, we let

$$\mathbf{x}^{(N)} = (b_1b_2\cdots b_N, \quad p_N, \quad p'_N) \in \mathbb{Z}^3.$$

The remaining proof is similar to the proof of Case 1 and we omit the details here.

Case 3. Suppose there exist co-prime integers h and k with $0 < h \le k$ such that the sequence $(kc_n - hb_n)_n$ is bounded and there exist co-prime integers h' and k' with $0 < h' \le k'$ such that the sequence $(k'c'_n - h'b_n)_n$ is bounded.

We define the sequences $(\alpha_N)_N$ and $(\alpha'_N)_N$ of rational numbers as follows. For each positive integer $N \ge 1$, we define

$$\alpha_N = \frac{c_1}{b_1} + \frac{c_2}{b_1 b_2} + \cdots + \frac{c_N}{b_1 b_2 \cdots b_N} + \frac{\frac{h}{k}(b_{N+1}-1)}{b_1 b_2 \cdots b_{N+1}} + \frac{\frac{h}{k}(b_{N+2}-1)}{b_1 b_2 \cdots b_{N+2}} + \cdots ,$$

and

$$\alpha'_N = \frac{c'_1}{b_1} + \frac{c'_2}{b_1 b_2} + \cdots + \frac{c'_N}{b_1 b_2 \cdots b_N} + \frac{\frac{h'}{k'}(b_{N+1}-1)}{b_1 b_2 \cdots b_{N+1}} + \frac{\frac{h'}{k'}(b_{N+2}-1)}{b_1 b_2 \cdots b_{N+2}} + \cdots .$$

By Lemma 1.1.2, the real numbers α_N, α'_N are rational for each integer $N \ge 1$. Hence, we get,

$$\alpha_N = \frac{p_N}{k b_1 b_2 \cdots b_N}, \quad \alpha'_N = \frac{p'_N}{k' b_1 b_2 \cdots b_N}$$

for some integers p_N and p'_N. Now we consider,

$$|\alpha - \alpha_N| \le \frac{1}{b_1 b_2 \cdots b_{N+1}} \left(\frac{|c_{N+1} - \frac{h}{k}(b_{N+1}-1)|}{1} + \frac{|c_{N+2} - \frac{h}{k}(b_{N+2}-1)|}{b_{N+2}} + \cdots \right).$$

Since both the sequences $(kc_n - hb_n)_n$ and $(k'c'_n - h'b_n)_n$ are bounded, there exists a positive constant C such that $|kc_N - hb_N| \le C$ and $|k'c'_N - h'b_N| \le C$. Then, we have

$$|\alpha - \alpha_N| \le \frac{C+h}{k b_1 b_2 \cdots b_{N+1}} \left(1 + \frac{1}{b_1} + \frac{1}{b_1^2} + \cdots \right) = \frac{b_1(C+h)}{(b_1 - 1) k b_1 b_2 \cdots b_N b_{N+1}},$$

for all integer $N \geq 1$. By (3.2), we get

$$\left|\alpha - \frac{p_N}{kb_1b_2\cdots b_N}\right| \leq \frac{b_1(C+h)}{k(b_1-1)(b_1b_2\cdots b_N)^{\frac{3}{2}+\delta}} = \frac{C'}{k(b_1b_2\cdots b_N)^{\frac{3}{2}+\delta}}, \tag{3.15}$$

for all $N \in T$, where $C' = \frac{b_1(C+h)}{b_1-1}$. Thus, from (3.15), we get

$$|kb_1b_2\cdots b_N\alpha - p_N| \leq \frac{C'}{(kb_1b_2\cdots b_N)^{\frac{1}{2}+\delta}}, \tag{3.16}$$

for all $N \in T$. Similarly, we get

$$|k'b_1b_2\cdots b_N\alpha' - p'_N| \leq \frac{C''}{(k'b_1b_2\cdots b_N)^{\frac{1}{2}+\delta}}, \tag{3.17}$$

for all $N \in T$, where $C'' = \frac{b_1(C+h')}{b_1-1}$

We shall assume that both of α and α' are algebraic. Now to finish the proof, we need to prove $1, \alpha$ and α' are $\mathbb{Q}$-linearly dependent.

In order to prove that $1, \alpha, \alpha'$ are $\mathbb{Q}$-linearly dependent, we shall apply Theorem 1.3.1 Chapter 1. Consider the linear forms with algebraic coefficients,

$$\begin{aligned} L_1(X_1, X_2, X_3) &= X_1 \\ L_2(X_1, X_2, X_3) &= k\alpha X_1 - X_2, \\ L_3(X_1, X_2, X_3) &= k'\alpha' X_1 - X_3. \end{aligned} \tag{3.18}$$

Clearly, the above linear forms are linearly independent.

For any $N \in T$, we let

$$\mathbf{x}^{(N)} = (b_1b_2\cdots b_N, \quad p_N, \quad p'_N) \in \mathbb{Z}^3.$$

Since $\alpha_N < 1$ and $\alpha'_N < 1$ for all $N \in T$, we see that $p_N < kb_1b_2\cdots b_N$,

$p'_N < k'b_1b_2\cdots b_N$ and hence, we get

$$\max\{b_1b_2\cdots b_N, p_N, p'_N\} \leq \max\{kb_1b_2\cdots b_N, k'b_1b_2\cdots b_N\}. \tag{3.19}$$

For applying Theorem 1.3.1, we need to compute the quantity,

$$\prod_{i=1}^{3} |L_i(\mathbf{x}^{(N)})|. \tag{3.20}$$

By (3.18), we observe that

$$\prod_{i=1}^{3} |L_i(\mathbf{x}^{(N)})| = |b_1b_2\cdots b_N||kb_1b_2\cdots b_N\alpha - p_N||k'b_1b_2\cdots b_N\alpha' - p_N|. \tag{3.21}$$

Thus, from (3.16) and (3.17), we have

$$\begin{aligned}\prod_{i=1}^{3} |L_i(\mathbf{x}^{(N)})| &\leq \frac{1}{(kk')^{\frac{1}{2}+\delta}} \frac{C'^2}{(b_1b_2\cdots b_N)^{2\delta}} \\ &< \Big(\max\{kb_1b_2\cdots b_N, k'b_1b_2\cdots b_N\}\Big)^{-\delta'}\end{aligned}$$

for some $\delta' > 0$. Hence, by (3.19) we conclude that

$$\begin{aligned}\prod_{i=1}^{3} |Li(\mathbf{x}^{(N)})| &< \Big(\max\{kb_1b_2\cdots b_N, k'b_1b_2\cdots b_N\}\Big)^{-\delta'} \\ &\leq \Big(\max\{b_1b_2\cdots b_N, p_N, p'_N\}\Big)^{-\delta'}\end{aligned}$$

holds for all large $N \in T$. Hence, for infinitely many values of $N \in T$, the non-zero lattice point $\mathbf{x}^{(N)} = (b_1b_2\cdots b_N, p_N, p'_N) \in \mathbb{Z}^3$ satisfies the hypothesis of Theorem 1.3.1. Thus, the integer lattice points $\mathbf{x}^{(N)} = (b_1b_2\cdots b_N, p_N, p'_N) \in \mathbb{Z}^3$ lie in finitely many proper subspaces of $\mathbb{Q}^3$ for infinitely values of $N \in T$. Therefore, there exists a proper subspace of $\mathbb{Q}^3$, containing the integer lattice point $\mathbf{x}^{(N)} = (b_1 b_2 \cdots b_N, p_N, p'_N) \in \mathbb{Z}^3$ for infinitely many values of $N \in T$. That is, there exists a non-zero triple $(z_1, z_2, z_3) \in \mathbb{Z}^3$ such that

$$z_1 b_1 b_2 \cdots b_N + z_2 p_N + z_3 p'_N = 0$$

holds for infinitely many values of $N \in T$. This implies that

$$\begin{aligned} \lim_{N\to\infty, N\in T} \left(z_1 + z_2 \frac{p_N}{b_1 b_2 \cdots b_N} + z_3 \frac{p'_N}{b_1 b_2 \cdots b_N} \right) &= 0 \\ \implies z_1 + z_2\alpha + z_3\alpha' = 0. \end{aligned}$$

Hence, $1, \alpha$ and α' are $\mathbb{Q}$-linearly dependent. This proves the assertion. $\square$

CHAPTER 4

On transcendence of certain real numbers

In this chapter we deal with the arithmetic nature of certain infinite sums and products of the form

$$\sum_{n=1}^{\infty} \frac{c_n}{b_n} \quad \text{and} \quad \prod_{n=1}^{\infty} \left(1 + \frac{c_n}{b_n}\right).$$

The content of this chapter is published in [24].

4.1 Introduction

In the literature, there are several methods to prove the transcendence of an infinite series. By Mahler's method [35], one can prove the transcendence of certain infinite sums. In 2001, Adhikari *et al.* [5], by an application of Baker's theory of linear forms in logarithms of algebraic numbers, they showed that the following series are transcendental;

$$\sum_{n=0}^{\infty} \frac{1}{(3n+1)(3n+2)(3n+3)}, \quad \sum_{n=1}^{\infty} \frac{\chi(n)}{n} \text{ and } \sum_{n=1}^{\infty} \frac{F_n}{n2^n}.$$

In the same year, Hančl [14] and Nyblom [37] (see also [38]) studied the transcendence of infinite series by invoking Roth's Theorem. In 2004, using the subspace theorem, Adamczewski, Bugeaud, and Luca [1] proved a transcendence criterion for a real number based on its b-ary expansion. In Chapter 3, we also proved some transcendence results for a real number based on its Q-ary expansion.

In 1974, Erdős and Straus [13] studied the linear independence of certain Cantor series expansions. In particular, they proved the following result.

Theorem 4.1.1 *Let $Q = (b_n)_{n\geq 1}$ be a sequence of positive integers with $b_n \geq 2$ for all integers $n \geq 1$ and let $\delta > \frac{1}{3}$ be any positive real number. Suppose that for all sufficiently large values of N, we have*

$$(b_1 b_2 \cdots b_N)^{\delta} \leq b_{N+1}.$$

Then the real numbers

$$1, \quad \sum_{n=1}^{\infty} \frac{\sigma(n)}{b_1 b_2 \cdots b_n}, \quad \sum_{n=1}^{\infty} \frac{\phi(n)}{b_1 b_2 \cdots b_n}, \quad \sum_{n=1}^{\infty} \frac{d_n}{b_1 b_2 \cdots b_n}$$

are $\mathbb{Q}$-linearly independent, where $\phi(n)$ denotes the Euler totient function, $\sigma(n) = \sum_{d|n} d$ and $(d_n)_n$ is any sequence of integers satisfying $|d_n| < n^{\frac{1}{2}-\delta}$ for all large n and $d_n \neq 0$ for infinitely many values n.

In 2005, J. Hančl and Rucki [17] gave sufficient conditions under which an infinite sum is transcendental. We mention one of their results here.

Theorem 4.1.2 *Let $\delta > 0$ be a real number. Let $(b_n)_n$ and $(c_n)_n$ be sequences* *of positive integers such that*

$$\limsup_{n\to\infty} \frac{b_{n+1}}{(b_1 b_2 \cdots b_n)^{2+\delta}} \frac{1}{c_{n+1}} = \infty \quad \text{and} \quad \liminf_{n\to\infty} \frac{b_{n+1}}{b_n} \frac{c_n}{c_{n+1}} > 1.$$

Then the real number $\alpha = \sum_{n=1}^{\infty} \frac{c_n}{b_n}$ *is transcendental.*

In this Chapter we extend Theorem 4.1.1 and the results in [17]. Moreover, we study the transcendence of certain infinite products.

4.2 Main results

We prove the following results.

Theorem 4.2.1 *Let* $Q = (b_n)_{n\geq 1}$ *be a sequence of positive integers with* $b_n \geq 2$ *for all integers* $n \geq 1$ *and let* $\delta > \frac{1}{3}$ *be any positive real number. Suppose that for all sufficiently large values of* N *we have*

$$\sigma(N+1)(b_1 b_2 \cdots b_N)^{\delta} \leq b_{N+1}. \tag{4.1}$$

Then at least one of the real numbers

$$\beta_1 = \sum_{n=1}^{\infty} \frac{\sigma(n)}{b_1 b_2 \cdots b_n}, \quad \beta_2 = \sum_{n=1}^{\infty} \frac{\phi(n)}{b_1 b_2 \cdots b_n}, \quad \beta_3 = \sum_{n=1}^{\infty} \frac{d_n}{b_1 b_2 \cdots b_n}$$

is transcendental, where d_n *as in Theorem 4.1.1.*

In order to state further results, we first fix some notation.

Let $\delta > 0$ and $\epsilon > 0$ be given real numbers. For any given integer $m \geq 2$, let $(c_{i,n})_n$, $i = 1, 2, \ldots, m$ be sequences of non-zero integers. Consider the following two conditions on a sequence $(b_n)_n$ of positive integers.

$$\limsup_{n\to\infty} \frac{b_{n+1}}{(b_1 b_2 \cdots b_n)^{1+\delta}} \frac{1}{c_{i,n+1}} = \infty \tag{4.2}$$

$$\liminf_{n\to\infty} \frac{b_{n+1}}{b_n} \frac{c_{i,n}}{c_{i,n+1}} > 1 \qquad \text{for all} \qquad i \in \{1, 2, \ldots, m\}. \tag{4.3}$$

We may now state our results.

Theorem 4.2.2 *For any given integer $m \geq 2$, let $\delta > \frac{1}{m}$ be a real number. Let $(c_{i,n})_n$, $i = 1, 2, \ldots, m$ and $(b_n)_n$ be sequences of positive integers satisfying (4.2) and (4.3). Then either at least one of the real numbers*

$$\beta_1 = \sum_{n=1}^{\infty} \frac{c_{1,n}}{b_n}, \quad \beta_2 = \sum_{n=1}^{\infty} \frac{c_{2,n}}{b_n}, \ldots, \beta_m = \sum_{n=1}^{\infty} \frac{c_{m,n}}{b_n}$$

is transcendental or $1, \beta_1, \beta_2, \ldots, \beta_m$ are $\mathbb{Q}$-linearly dependent.

Theorem 4.2.2 extends to large class of sequences compared to Theorem 4.1.2. However, the assertion of Theorem 4.2.2 is weaker.

Corollary 4.2.1 *Let $(b_n)_n$ be a sequence of positive integers such that $b_1 = 2$ and*

$$b_{n+1} = (b_1 b_2 \cdots b_n + 1)^2, \quad \textit{for all integers } n \geq 1.$$

Then at least one of the real numbers

$$\sum_{n=1}^{\infty} \frac{1}{b_n} \quad \textit{and} \quad \sum_{n=1}^{\infty} \frac{d(n)}{b_n}$$

is transcendental, where $d(n) = \sum_{r|n} 1$.

If $b_{n+1} = (b_1 b_2 \cdots b_n + 1)^2$, then it is clear that for $0 < \delta < 1$

$$\limsup_{n\to\infty} \frac{b_{n+1}}{(b_1 b_2 \cdots b_n)^{1+\delta}} \frac{1}{c_{i,n+1}} = \infty$$

and

$$\liminf_{n\to\infty} \frac{b_{n+1}}{b_n} \frac{c_{i,n}}{c_{i,n+1}} > 1,$$

where $c_{1,n} = 1$ and $c_{2,n} = d(n)$. On other hand we see that $(b_1 \cdots b_n)^{2+\delta} > b_{n+1}$ for any choice of $\delta > 0$. Therefore, by Theorem 4.1.2, we can not conclude the transcendence of any of the numbers

$$\sum_{n=1}^{\infty} \frac{1}{b_n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{d(n)}{b_n}$$

However, by Corollary 4.2.1, one can prove that one of these numbers is transcendental.

Theorem 4.2.2 can be strengthened by saying that at least one of the β_i's is transcendental under an additional assumptions on the growth rate of the sequences $(c_{i,n})_n$ and $(b_n)_n$. More precisely, we have the following theorem.

Theorem 4.2.3 *For any given integer $m \geq 2$, let $\delta > \frac{1}{m}$ be a real number. Let $(c_{i,n})_n$, $i = 1, 2, \ldots, m$ and $(b_n)_n$ be sequences of positive integers satisfying (4.2) and (4.3). Further, suppose that*

$$1 \leq \liminf_{n\to\infty} b_n^{\frac{1}{(m+1)^n}} < \limsup_{n\to\infty} b_n^{\frac{1}{(m+1)^n}} < \infty \qquad \textit{and}$$

$$\lim_{n\to\infty} \frac{c_{i,n}}{c_{j,n}} = 0, \qquad \textit{for all} \qquad i, j \in \{1, 2, \ldots, m\}, i > j.$$

Then at least one of the real numbers

$$\beta_1 = \sum_{n=1}^{\infty} \frac{c_{1,n}}{b_n}, \quad \beta_2 = \sum_{n=1}^{\infty} \frac{c_{2,n}}{b_n}, \ldots, \beta_m = \sum_{n=1}^{\infty} \frac{c_{m,n}}{b_n}$$

is transcendental.

Using the same notation used in (4.2) and (4.3), we consider two more conditions on a sequence of positive integers $(b_n)_n$.

$$\limsup_{n\to\infty} \frac{b_{n+1}}{(b_1 b_2 \cdots b_n)^{1+\delta+\frac{1}{\epsilon}}} \frac{1}{c_{i,n+1}} = \infty \tag{4.4}$$

and for all sufficiently large n

$$\sqrt[1+\epsilon]{\frac{b_{n+1}}{c_{i,n+1}}} \geq \sqrt[1+\epsilon]{\frac{b_n}{c_{i,n}}} + 1 \quad \text{for all} \quad i \in \{1, 2, \ldots, m\} \tag{4.5}$$

We first remark about the conditions (4.2), (4.4) and (4.3), (4.5) as follows.

Remark 4.2.4 First we note that (4.4) $\implies$ (4.2). Since

$$\frac{b_{n+1}}{(b_1 b_2 \cdots b_n)^{1+\delta+\frac{1}{\epsilon}}} \frac{1}{c_{i,n+1}} \leq \frac{b_{n+1}}{(b_1 b_2 \cdots b_n)^{1+\delta+}} \frac{1}{c_{i,n+1}} \quad \text{for all} \quad n,$$

we have

$$\limsup_{n\to\infty} \frac{b_{n+1}}{(b_1 b_2 \cdots b_n)^{1+\delta+\frac{1}{\epsilon}}} \frac{1}{c_{i,n+1}} \leq \limsup_{n\to\infty} \frac{b_{n+1}}{(b_1 b_2 \cdots b_n)^{1+\delta+}} \frac{1}{c_{i,n+1}}.$$

Therefore, we conclude that (4.4) $\implies$ (4.2). Now we see that the condition (4.5) not always implies (4.3). By (4.5), we have

$$\sqrt[1+\epsilon]{\frac{b_{n+1}}{c_{i,n+1}}} \geq \sqrt[1+\epsilon]{\frac{b_n}{c_{i,n}}} + 1$$

for all sufficiently large n. This implies that

$$\sqrt[1+\epsilon]{\frac{b_{n+1}}{c_{i,n+1}}} > \sqrt[1+\epsilon]{\frac{b_n}{c_{i,n}}}.$$

By raising $(1+\epsilon)$-power on this inequality, we get

$$\frac{b_{n+1}}{c_{i,n+1}} > \frac{b_n}{c_{i,n}} \iff \frac{b_{n+1}}{b_n}\frac{c_{i,n}}{c_{i,n+1}} > 1$$

for all sufficiently large n. This implies that

$$\liminf_{n\to\infty} \frac{b_{n+1}}{b_n}\frac{c_{i,n}}{c_{i,n+1}} \geq 1.$$

Therefore, in the case when this lim inf equals to 1, (4.5) does not implies (4.3). But on other hand, (4.3) implies (4.5). To see this, by (4.3), we have

$$\sqrt[1+\varepsilon]{\frac{b_{n+1}}{c_{i,n+1}}} > \sqrt[1+\varepsilon]{(1+\delta)\frac{b_n}{c_{i,n}}}.$$

Then, since $\frac{b_n}{c_{i,n}}$ tends to ∞ with n and $\sqrt[1+\varepsilon]{1+\delta} > 1$, we deduce that for n large enough we have

$$\sqrt[1+\varepsilon]{1+\delta}.\sqrt[1+\varepsilon]{\frac{b_n}{c_{i,n}}} > \sqrt[1+\varepsilon]{\frac{b_n}{c_{i,n}}} + 1.$$

Hence, by these inequalities we conclude that

$$\sqrt[1+\varepsilon]{\frac{b_{n+1}}{c_{i,n+1}}} > \sqrt[1+\varepsilon]{(1+\delta)\frac{b_n}{c_{i,n}}} > \sqrt[1+\varepsilon]{\frac{b_n}{c_{i,n}}} + 1.$$

This proves the assertion.

Theorem 4.2.5 *For any given integer $m \geq 2$, let δ and ϵ be positive real numbers such that $\frac{\delta\epsilon}{1+\epsilon} > \frac{1}{m}$. Let $(c_{i,n})_n$, $i = 1, 2, \ldots, m$ and $(b_n)_n$ be sequences of* *positive integers satisfying* (4.4) *and* (4.5)*. Then at least one of the real numbers*

$$\beta_1 = \sum_{n=1}^{\infty} \frac{c_{1,n}}{b_n}, \quad \beta_2 = \sum_{n=1}^{\infty} \frac{c_{2,n}}{b_n}, \ldots, \beta_m = \sum_{n=1}^{\infty} \frac{c_{m,n}}{b_n}$$

is transcendental or $1, \beta_1, \beta_2, \ldots, \beta_m$ *are* $\mathbb{Q}$*-linearly dependent.*

Theorem 4.2.7 can be generalised by saying at least one of the β_i's is transcendental under some additional assumptions on the growth of the sequences $(c_{i,n})_n$ and $(b_n)_n$. More precisely, we have the following theorem.

Theorem 4.2.6 *For any given integer* $m \geq 2$*, let* δ *and* ϵ *be positive real numbers such that* $\frac{\delta\epsilon}{1+\epsilon} > \frac{1}{m}$*. Let* $(c_{i,n})_n$*,* $i = 1, 2, \ldots, m$ *and* $(b_n)_n$ *be sequences of positive integers satisfying* (4.4) *and* (4.5)*. Further, suppose that*

$$1 \leq \liminf_{n\to\infty} b_n^{\frac{1}{(m+1)^n}} < \limsup_{n\to\infty} b_n^{\frac{1}{(m+1)^n}} < \infty \qquad \text{and}$$

$$\lim_{n\to\infty} \frac{c_{i,n}}{c_{j,n}} = 0, \qquad \text{for all} \qquad i, j \in \{1, 2, \ldots, m\}, i > j.$$

Then at least one of the real numbers

$$\beta_1 = \sum_{n=1}^{\infty} \frac{c_{1,n}}{b_n}, \quad \beta_2 = \sum_{n=1}^{\infty} \frac{c_{2,n}}{b_n}, \ldots, \beta_m = \sum_{n=1}^{\infty} \frac{c_{m,n}}{b_n}$$

is transcendental.

Theorem 4.2.7 *For any given integer* $m \geq 2$*, let* $\delta > \frac{1}{m}$ *be a real number. Let* $(c_{i,n})_n$*,* $i = 1, 2, \ldots, m$ *and* $(b_n)_n$ *be sequences of positive integers same as in Theorem 4.2.6. Then at least one of the real numbers*

$$\beta_1 = \prod_{n=1}^{\infty} \left(1 + \frac{c_{1,n}}{b_n}\right), \quad \beta_2 = \prod_{n=1}^{\infty} \left(1 + \frac{c_{2,n}}{b_n}\right), \ldots, \beta_m = \prod_{n=1}^{\infty} \left(1 + \frac{c_{m,n}}{b_n}\right)$$

is transcendental.

4.3 Preliminaries

We state with the following result of Hančl [15] which will be useful for our purposes.

Theorem 4.3.1 *For a given integer $m \geq 2$, let $(b_n)_n$ be a sequence of positive integers such that*

$$1 \leq \liminf_{n\to\infty} b_n^{\frac{1}{(m+1)^n}} < \limsup_{n\to\infty} b_n^{\frac{1}{(m+1)^n}} < \infty \quad \text{and} \quad b_n \geq n^{1+\epsilon}$$

holds for all large values of n and for some $\epsilon > 0$. Let $(c_{i,n})_n$, $i = 1, 2, \ldots, m$ be sequences of positive integers satisfying

$$\lim_{n\to\infty} \frac{c_{i,n}}{c_{j,n}} = 0 \quad \text{for all} \quad 1 \leq i < j \leq m; \text{ and}$$
$$c_{i,n} < 2^{(\log b_n)^\alpha} \quad \text{for some fixed } \alpha > 0 \text{ and for all large enough } n.$$

Then the real numbers

$$1, \quad \sum_{n=1}^{\infty} \frac{c_{1,n}}{b_n}, \ldots, \sum_{n=1}^{\infty} \frac{c_{m,n}}{b_n}$$

are $\mathbb{Q}$-linearly independent.

In [16], Hančl, Kolouch and Novotný [16] proved the following theorem for infinite products.

Theorem 4.3.2 *For any given integer $m \geq 2$, let $(c_{i,n})_n$, $i = 1, 2, \ldots, m$ and $(b_n)_n$ be sequences of positive integers satisfying*

$$\lim_{n\to\infty} \frac{c_{i,n}}{c_{j,n}} = 0, \quad \text{for all} \quad 1 \leq i < j \leq m; \text{ and}$$
$$c_{i,n} < b_n^{\frac{1}{\log^{1+\epsilon} \log b_n}}, \text{ for all large enough } n.$$

Then the real numbers

$$1, \quad \prod_{n=1}^{\infty}\left(1+\frac{c_{1,n}}{b_n}\right), \dots, \prod_{n=1}^{\infty}\left(1+\frac{c_{m,n}}{b_n}\right)$$

are $\mathbb{Q}$-linearly independent.

4.4 Proof of Theorems

Proof of Theorem 4.2.1. We define the sequences $(\beta_{1,N})_N$, $(\beta_{2,N})_N$, and $(\beta_{3,N})_N$ of rational numbers as follows. For each integer $N \geq 1$ and for all $i = 1, 2$ and 3, we define

$$\beta_{i,N} = \sum_{n=1}^{N} \frac{f_i(n)}{b_1 b_2 \cdots b_n} = \frac{p_{i,N}}{b_1 b_2 \cdots b_N},$$

for some positive integer $p_{i,N}$, where $f_1(n) = \sigma(n)$, $f_2(n) = \phi(n)$, and $f_3(n) = d_n$. By (4.1), and using the fact that $\sigma(N+1) > d_{N+1}$ and $\sigma(N+1) > \phi(N+1)$, we get

$$\left|\beta_i - \frac{p_{i,N}}{b_1 b_2 \cdots b_N}\right| < \frac{1}{(b_1 b_2 \cdots b_N)^{1+\delta'}}$$

for infinitely many values of $N \in T$ and for some $\delta' > \frac{1}{3}$.

By taking $\alpha_i = \beta_i$, $p_{iN} = p_{i,N}$ for $1 \leq i \leq 3$ and $q_N = b_1 b_2 \cdots b_N$, in Corollary 1.3.2 of Chapter 1, with $N \in T$. Then by Corollary 1.3.2, we get either $1, \beta_1$, β_2 and β_3 are $\mathbb{Q}$-linearly dependent or at least one of them is transcendental. But by Theorem 4.1.1, they are $\mathbb{Q}$-linearly independent. Hence, we conclude that one of them is transcendental. This proves the theorem.

Proof of Theorem 4.2.2. For each integer $1 \leq i \leq m$, we define the sequence $(\beta_{i,N})_N$ of rational numbers as follows. For each integer $N \geq 1$, we define

$$\beta_{i,N} = \sum_{n=1}^{N} \frac{c_{i,n}}{b_n} = \frac{p_{i,N}}{b_1 b_2 \cdots b_N}$$

for some positive integer $p_{i,N}$. By (4.3), there exists a real number $A > 1$ and a positive constant N_0 such that for all positive integer $N > N_0$, we have

$$\frac{1}{A}.\frac{c_{i,N}}{b_N} > \frac{c_{i,N+1}}{b_{N+1}}.$$

Therefore, inductively, we get for every N with $N > N_0$

$$\frac{1}{A^p}.\frac{c_{i,N}}{b_N} > \frac{c_{i,N+p}}{b_{N+p}}$$

for any natural number p. Hence, for all sufficiently large positive integers N, we have

$$\begin{aligned} \left|\beta_i - \frac{p_{i,N}}{b_1 b_2 \cdots b_N}\right| &= \left|\sum_{n=1}^{\infty} \frac{c_{i,n}}{b_n} - \sum_{n=1}^{N} \frac{c_{i,n}}{b_n}\right| = \left|\sum_{n=N+1}^{\infty} \frac{c_{i,n}}{b_n}\right| \\ &= \left(\frac{c_{i,N+1}}{b_{N+1}} + \frac{c_{i,N+2}}{b_{N+2}} + \cdots\right) \\ &< \frac{c_{i,N+1}}{b_{N+1}}\left(1 + \frac{1}{A} + \frac{1}{A^2} + \cdots\right) = \frac{c_{i,N+1}}{b_{N+1}}\frac{A}{A-1}. \end{aligned}$$

Choose $M > \frac{A}{A-1}$. Then, by (4.2), there exist infinitely many integers N such that

$$\frac{1}{M(b_1 b_2 \cdots b_N)^{1+\delta}} > \frac{c_{i,N+1}}{b_{N+1}}.$$

Hence, we get

$$\left|\beta_i - \frac{p_{i,N}}{b_1 b_2 \cdots b_N}\right| < \frac{c_{i,N+1}}{b_{N+1}}\frac{A}{A-1} \leq \frac{1}{(b_1 b_2 \cdots b_N)^{1+\delta}}$$

holds for infinitely many positive integers N.

By taking $\alpha_i = \beta_i$ and $p_{in} = p_{i,n}$ for $1 \leq i \leq m$ in Corollary 1.3.2 of Chapter 1, we get that either $1, \beta_1, \beta_2, \ldots, \beta_m$ are $\mathbb{Q}$-linearly dependent or at least one of the β_i's is transcendental.

Proof of Theorem 4.2.3. By Theorem 4.2.2, we get that either $1, \beta_1, \beta_2, \ldots, \beta_m$ are $\mathbb{Q}$-linearly dependent or at least one β_i is transcendental. Since the sequences $(c_{i,n})_n$ and $(b_n)_n$ satisfy the hypotheses of Theorem 4.3.1, we obtain that $1, \beta_1, \beta_2, \ldots, \beta_m$ are $\mathbb{Q}$-linearly independent. Therefore, we conclude that at least one of β_i's is transcendental. This proves the theorem.

Proof of Theorem 4.2.5. For each integer $1 \leq i \leq m$, we define the sequence $(\beta_{i,N})_N$ of rational numbers as follows. For each integer $N \geq 1$, we define

$$\beta_{i,N} = \sum_{n=1}^{N} \frac{c_{i,n}}{b_n} = \frac{p_{i,N}}{b_1 b_2 \ldots b_N},$$

for some positive integer $p_{i,N}$. By (4.5) and by mathematical induction, we get for all sufficiently large integers N and every integer r

$$\sqrt[1+\epsilon]{\frac{b_{N+r}}{c_{i,N+r}}} \geq \sqrt[1+\epsilon]{\frac{b_N}{c_{i,N}}} + r.$$

Hence

$$\frac{b_{N+r}}{c_{i,N+r}} \geq \left(\sqrt[1+\epsilon]{\frac{b_N}{c_{i,N}}} + r \right)^{1+\epsilon}. \tag{4.6}$$

Also, for all real $x > 1$, we know that

$$\sum_{s=0}^{\infty} \frac{1}{(x+s)^{1+\epsilon}} < \int_{x-1}^{\infty} \frac{dy}{y^{1+\epsilon}} = \frac{1}{\epsilon(x-1)^{\epsilon}}. \tag{4.7}$$

By (4.6) and (4.7), for infinitely many N, we get

$$\begin{aligned}
\left| \beta_i - \frac{p_{i,N}}{b_1 b_2 \cdots b_N} \right| &= \left| \sum_{n=1}^{\infty} \frac{c_{i,n}}{b_n} - \sum_{n=1}^{N} \frac{c_{i,n}}{b_n} \right| = \left| \sum_{n=N+1}^{\infty} \frac{c_{i,n}}{b_n} \right| = \left(\frac{c_{i,N+1}}{b_{N+1}} + \frac{c_{i,N+2}}{b_{N+2}} + \cdots \right) \\
&\leq \left(\sqrt[1+\epsilon]{\frac{b_{N+1}}{c_{i,N+1}}} \right)^{-(1+\epsilon)} + \left(\sqrt[1+\epsilon]{\frac{b_{N+1}}{c_{i,N+1}}} + 1 \right)^{-(1+\epsilon)} + \cdots
\end{aligned}$$

$$< \frac{1}{\epsilon}\left(\sqrt[1+\epsilon]{\frac{b_{N+1}}{c_{i,N+1}}}-1\right)^{-\epsilon}.$$

Since, by (4.4), we have $\lim_{n\to\infty}\left(\frac{b_n}{c_{i,n}}\right)=\infty$, there exists a positive constant C which does not depend on N such that

$$\begin{aligned}\left|\beta_i-\frac{p_{i,N}}{b_1b_2\cdots b_N}\right| &< \frac{1}{\epsilon}\left(\sqrt[1+\epsilon]{\frac{b_{N+1}}{c_{i,N+1}}}-1\right)^{-\epsilon} < \frac{C}{\epsilon}\left(\sqrt[1+\epsilon]{\frac{b_{N+1}}{c_{i,N+1}}}\right)^{-\epsilon}\\ &= \frac{C}{\epsilon}\left(\frac{c_{i,N+1}}{b_{N+1}}\right)^{\frac{\epsilon}{1+\epsilon}}.\end{aligned}$$

Choose $M>\frac{C}{\epsilon}$. Then by (4.4), there are infinitely many integers N such that

$$\frac{1}{M(b_1b_2\cdots b_N)^{1+\delta+\frac{1}{\epsilon}}}>\frac{c_{i,N+1}}{b_{N+1}}.$$

This implies that

$$\left|\beta_i-\frac{p_{i,N}}{b_1b_2\cdots b_N}\right|<\frac{1}{(b_1b_2\cdots b_n)^{1+\frac{\delta\epsilon}{1+\epsilon}}}$$

holds for infinitely many positive integers N. The rest of the proof follows verbatim as the proof of Theorem 4.2.2.

Proof of Theorem 4.2.7. For each integer $1\le i\le m$, we define the sequence $(\beta_{i,N})_N$ of rational numbers as follows. For each integer $N\ge 1$, we define

$$\beta_{i,N}=\prod_{n=1}^{N}\left(1+\frac{c_{i,n}}{b_n}\right)=\frac{p_{i,N}}{b_1b_2\ldots b_N}$$

for some positive integer $p_{i,N}$. Consider

$$\left|\beta_i-\frac{p_{i,N}}{b_1b_2\cdots b_N}\right|=\prod_{n=1}^{\infty}\left(1+\frac{c_{i,n}}{b_n}\right)-\prod_{n=1}^{N}\left(1+\frac{c_{i,n}}{b_n}\right)$$

$$= \prod_{n=1}^{N}\left(1+\frac{c_{i,n}}{b_n}\right)\left(\prod_{n=N+1}^{\infty}\left(1+\frac{c_{i,n}}{b_n}\right)-1\right). \tag{4.8}$$

By the hypothesis, for all sufficiently large values of N, we have

$$\prod_{n=N+1}^{\infty}\left(1+\frac{c_{i,n}}{b_n}\right) < 1+2\sum_{n=N+1}^{\infty}\frac{c_{i,n}}{b_n}.$$

This implies that

$$\left(\prod_{n=N+1}^{\infty}\left(1+\frac{c_{i,n}}{b_n}\right)-1\right) < 2\sum_{n=N+1}^{\infty}\frac{c_{i,n}}{b_n}.$$

Thus, by (4.8) we have

$$\left|\beta_i-\frac{p_{i,N}}{b_1b_2\dots b_N}\right| \le 2\prod_{n=1}^{N}\left(1+\frac{c_{i,n}}{b_n}\right)\left(\sum_{n=N+1}^{\infty}\frac{c_{i,n}}{b_n}\right).$$

By the similar argument as in the proof of Theorem 4.2.2, from (4.3), we conclude that for all sufficiently large positive integers N,

$$\left|\beta_i-\frac{p_{i,N}}{b_1b_2\cdots b_N}\right| \le 2\prod_{n=1}^{N}\left(1+\frac{c_{i,n}}{b_n}\right)\left(\sum_{n=N+1}^{\infty}\frac{c_{i,n}}{b_n}\right) < 2\prod_{n=1}^{N}\left(1+\frac{c_{i,n}}{b_n}\right)\frac{c_{i,N+1}}{b_{N+1}}\frac{A}{A-1},$$

for some constant $A > 1$. Hence, by (4.2) we obtain

$$\left|\beta_i-\frac{p_{i,N}}{b_1b_2\cdots b_N}\right| < \prod_{n=1}^{N}\left(1+\frac{c_{i,n}}{b_n}\right)\frac{1}{(b_1b_2\cdots b_N)^{1+\delta}}, \tag{4.9}$$

for infinitely many values of N. Since, by the assumption, $\frac{c_{i,n}}{b_n} \le 1$ for $n \ge 1$, we have

$$\prod_{n=1}^{N}\left(1+\frac{c_{i,n}}{b_n}\right) < 2^N$$

for all integer $N \geq 1$. Therefore, by (4.9), we have

$$\left|\beta_i - \frac{p_{i,N}}{b_1 b_2 \cdots b_N}\right| < \frac{2^N}{(b_1 b_2 \dots b_N)^{1+\delta}}.$$

Since the sequence $(b_n)_n$ grows like a doubly exponential sequence, we can find δ' with $\frac{1}{m} < \delta' < \delta$ such that

$$\frac{2^N}{(b_1 b_2 \cdots b_N)^{1+\delta}} < \frac{1}{(b_1 b_2 \cdots b_N)^{1+\delta'}}.$$

Therefore, for each $1 \leq i \leq m$, we get,

$$\left|\beta_i - \frac{p_{i,N}}{b_1 b_2 \cdots b_N}\right| < \frac{1}{(b_1 b_2 \cdots b_N)^{1+\delta'}}$$

holds true for infinitely many values of N. The rest of the proof follows verbatim as the proof of Theorem 4.2.2.

Proof of Corollary 4.2.1. By taking $c_{1,n} = 1$ and $c_{2,n} = d(n)$, we see that these sequences satisfy the hypothesis of Theorem 4.2.2. Hence, by Theorem 4.2.2, we get that either $1, \alpha$ and α' are $\mathbb{Q}$-linearly dependent or at least one of α and α' is transcendental. In order to finish the proof of this corollary, we shall prove that $1, \alpha$ and α' are $\mathbb{Q}$-linearly independent. Suppose that these numbers are $\mathbb{Q}$-linearly dependent. Then, there exist integers z_0, z_1 and z_2 not all zero such that

$$z_0 + z_1 \sum_{n=1}^{\infty} \frac{1}{b_n} + z_2 \sum_{n=1}^{\infty} \frac{d(n)}{b_n} = 0.$$

This equality we can rewrite as follows

$$z_0 + z_1 \sum_{n=1}^{N} \frac{1}{b_n} + z_2 \sum_{n=1}^{N} \frac{d(n)}{b_n} = -\left(z_1 \sum_{n=N+1}^{\infty} \frac{1}{b_n} + z_2 \sum_{n=N+1}^{\infty} \frac{d(n)}{b_n} \right).$$

By multiplying $b_1 b_2 \ldots b_N$ on both sides, we get

$$b_1 b_2 \cdots b_N \left(z_0 + z_1 \sum_{n=1}^{N} \frac{1}{b_n} + z_2 \sum_{n=1}^{N} \frac{d(n)}{b_n} \right) = -b_1 \cdots b_N \left(\sum_{n=N+1}^{\infty} \frac{z_1}{b_n} + \sum_{n=N+1}^{\infty} \frac{z_2 d(n)}{b_n} \right).$$

We note that the left-hand side of the above equality is an integer. Now, we claim the following.

Claim. The quantity

$$\left| -b_1 b_2 \cdots b_N \left(z_1 \sum_{n=N+1}^{\infty} \frac{1}{b_n} + z_2 \sum_{n=N+1}^{\infty} \frac{d(n)}{b_n} \right) \right| \to 0 \quad \text{as} \quad N \to \infty.$$

In order to prove the claim, we estimate the above quantity as follows. Consider

$$\left| -b_1 b_2 \cdots b_N \left(z_2 \sum_{n=N+1}^{\infty} \frac{d(n)}{b_n} \right) \right| \le |z_2 b_1 \cdots b_N| \left(\frac{d(N+1)}{b_{N+1}} + \frac{d(N+2)}{b_{N+2}} + \cdots \right).$$

Using $d(n) = O(n)$ and $b_{n+1} = (b_1 \cdots b_n + 1)^2$, we have

$$\begin{aligned}
\left| -b_1 b_2 \cdots b_N \left(z_2 \sum_{n=N+1}^{\infty} \frac{d(n)}{b_n} \right) \right| &\le |z_2 b_1 \cdots b_N| \left(\frac{d(N+1)}{b_{N+1}} + \frac{d(N+2)}{b_{N+2}} + \cdots \right) \\
&< C b_1 \cdots b_N \left(\frac{N+1}{(b_1 \cdots b_N + 1)^2} + \frac{N+2}{(b_1 \cdots b_{N+1} + 1)^2} + \cdots \right) \\
&< C b_1 \cdots b_N \left(\frac{N+1}{(b_1 \cdots b_N)^2} + \frac{N+2}{(b_1 \cdots b_{N+1})^2} + \frac{N+3}{(b_1 \cdots b_{N+2})^2} + \cdots \right) \\
&< C \left(\frac{N+1}{b_1 \cdots b_N} + \frac{N+2}{(b_1 \cdots b_N)^4} + \frac{N+3}{(b_1 \cdots b_N)^6} + \cdots \right) \\
&= C \left(\frac{N+1}{b_1 \cdots b_N} \right) + \frac{C}{b_1 \cdots b_N} \left(\frac{N+2}{(b_1 \cdots b_N)^3} + \frac{N+3}{(b_1 \cdots b_N)^5} + \cdots + \right) \\
&< C \left(\frac{N+1}{b_1 \cdots b_N} \right) + \frac{C}{b_1 \cdots b_N}.
\end{aligned}$$

Since $b_{n+1} = (b_1 \cdots b_n + 1)^2$, we see that $b_n > 2^{2^{n-1}}$ for all $n \ge 2$ and hence

$$\left| -b_1 b_2 \cdots b_N \left(z_2 \sum_{n=N+1}^{\infty} \frac{d(n)}{b_n} \right) \right| \to 0 \quad \text{as} \quad N \to \infty. \tag{4.10}$$

Similarly we get

$$\left| -b_1 b_2 \cdots b_N \left(z_1 \sum_{n=N+1}^{\infty} \frac{1}{b_n} \right) \right| \to 0 \quad \text{as} \quad N \to \infty. \tag{4.11}$$

Thus, by (4.10) and (4.11), we get the claim. Hence, we have

$$z_0 + z_1 \sum_{n=1}^{N} \frac{1}{b_n} + z_2 \sum_{n=1}^{N} \frac{d(n)}{b_n} = 0 \tag{4.12}$$

for all sufficiently large values of N. By multiplying $b_1 b_2 \ldots b_{N-1}$, we get

$$\begin{aligned} \left| b_1 b_2 \cdots b_{N-1} \left(z_0 + z_1 \sum_{n=1}^{N-1} \frac{1}{b_n} + z_2 \sum_{n=1}^{N-1} \frac{d(n)}{b_n} \right) \right| &= \left| \frac{-b_1 b_2 \cdots b_{N-1}(z_1 + z_2 d(N)}{b_N} \right| \\ &< \frac{|z_1| + |z_2| d(N)}{(b_1 b_2 \cdots b_{N-1} + 1)}. \end{aligned}$$

Clearly, the left hand side is an integer. Since $\frac{(|z_1|+|z_2|d(N))}{b_1 b_2 \cdots b_{N-1}+1} \to 0$ as $N \to \infty$, we get

$$0 \leq \left| \frac{(|z_1| + |z_2| d(N))}{b_1 b_2 \cdots b_{N-1} + 1} \right| < 1$$

for all sufficiently large values of N. Thus, we have

$$z_0 + z_1 \sum_{n=1}^{N-1} \frac{1}{b_n} + z_2 \sum_{n=1}^{N-1} \frac{d(n)}{b_n} = 0. \tag{4.13}$$

Hence, by (4.12) and (4.13), we get

$$\frac{z_1 + z_2 d(N)}{b_N} = 0 \iff z_1 + z_2 d(N) = 0$$

for all sufficiently large values of N. First we note that $z_1 \neq 0$. If not, we assume that $z_1 = 0$. Since $d(n) \neq 0$ for all integer $n \geq 1$, we obtain $z_2 = 0$. This implies that $z_0 = z_1 = z_2 = 0$, which is a contradiction to the assumption that not all z_0, z_1 and z_2 are zero. Hence, $\frac{1}{d_N} = \frac{-z_2}{z_1}$ for all sufficiently large values of N. This implies that the sequence $(d(n))_n$ is eventually constant. This gives a contradiction to the fact that it has at least two limit points.

CHAPTER 5

On linear independence of certain numbers

In this chapter, we study the linear independence of certain infinite sums. In particular, linear independence of the values of Jacobi theta-constant at different points. The results of this chapter is in the article [25] and [26].

5.1 Introduction

For a complex number $\tau \in \mathbb{H} := \{\tau \in \mathbb{C} : \text{Im}(\tau) > 0\}$, the Jacobi theta-constant (see for instance [28]) is defined as follows;

$$\theta_3(\tau) = 1 + 2\sum_{n=1}^{\infty} q^{n^2},$$

where $q = e^{i\pi\tau}$.

In 1997, D. Bertrand [7] proved that for any algebraic number q with $0 <$

$|q| < 1$, the following numbers, namely, $\theta_3(\tau), \theta_3'(\tau)$ and $\theta_3''(\tau)$ are algebraically independent (see also [11] and [34]). In particular, it follows that for any algebraic number q with $0 < |q| < 1$, the following numbers, namely,

$$\sum_{n=1}^{\infty} q^{n^2}, \quad \sum_{n=1}^{\infty} n^2 q^{n^2} \text{ and } \sum_{n=1}^{\infty} n^4 q^{n^2}$$

are algebraically independent over $\mathbb{Q}$. If we put $\tau = \frac{i \log b}{\pi}$, for some integer $b \geq 2$, then $q = 1/b$ which is algebraic and $0 < |q| < 1$. Therefore, we get

$$\sum_{n=1}^{\infty} \frac{1}{b^{n^2}}, \quad \sum_{n=1}^{\infty} \frac{n^2}{b^{n^2}} \text{ and } \sum_{n=1}^{\infty} \frac{n^4}{b^{n^2}}$$

are algebraically independent $\mathbb{Q}$.

In 2011, H. Kaneko [21] proved that for any integer $b \geq 2$ and for any real number $\alpha > 4$, the following numbers, namely,

$$1, \quad \sum_{n=1}^{\infty} \frac{1}{b^{[n^\alpha]}}, \quad \left(\sum_{n=1}^{\infty} \frac{1}{b^{[n^\alpha]}} \right)^2 \text{ and } \left(\sum_{n=1}^{\infty} \frac{1}{b^{[n^\alpha]}} \right)^3$$

are $\mathbb{Q}$-linearly independent, where $[n^\alpha]$ is the integral part of n^α.

In 2014, F. Luca and Y. Tachiya [29] proved the following: If b is an integer with $|b| \geq 2$, then for any integer $\ell \geq 1$, the real numbers

$$1, \quad \sum_{n=1}^{\infty} \frac{1}{b^n - 1}, \quad \sum_{n=1}^{\infty} \frac{1}{b^{n^2} - 1}, \ldots, \sum_{n=1}^{\infty} \frac{1}{b^{n^\ell} - 1}$$

are $\mathbb{Q}$-linearly independent.

Recetly in 2018, Elsner et.al., proved the followng: let τ be any complex number with $\mathrm{Im}(\tau) > 0$ such that $e^{i\pi\tau}$ is algebraic. Let $m, n \geq 1$ be distinct positive integers. Then the numbers $\theta_3(m\tau)$ and $\theta_3(n\tau)$ are algebraically independent over $\mathbb{Q}$.

On the other hand, in the same year, C. Elsner and Y. Tachiya proved the following result: let ℓ, m and n be positive integers. Then, the numbers $\theta_3(\ell\tau), \theta_3(m\tau)$ and $\theta_3(n\tau)$ are algebraically dependent over $\mathbb{Q}$.

Though these three values of Jacobi theta-constants are algebraicaly dependent over $\mathbb{Q}$, these may be linearly independent over $\mathbb{Q}$. In this chapter we partially answer this question and proved the following result.

Theorem 5.1.1 *Let $b \geq 2$ and $D \geq 1$ be integers and let $1 \leq a_1 < a_2$ be two integers such that $a_1 a_2$ is not a perfect square. Let $k \geq 0$ be an integer and $f, g : \mathbb{N} \to \mathbb{N}$ be functions such that $f(n) = O(n^k)$ (respectively, $g(n) = O(n^k)$). Then the real numbers*

$$1, \quad \sum_{n=1}^{\infty} \frac{f(n)}{b^{a_1 n^2}}, \quad \sum_{n=1}^{\infty} \frac{g(n)}{b^{a_2 n^2}},$$

are $\mathbb{Q}(\sqrt{-D})$-linearly independent.

Later in the same year we proved the more general result. More precisely we proved the following theorem.

Theorem 5.1.2 *Let $k \geq 2$, $b \geq 2$ and $1 \leq a_1 < a_2 < \cdots < a_m$ be integers such that $\sqrt[k]{a_i/a_j} \notin \mathbb{Q}$ for any $i \neq j$. Then the real numbers*

$$1, \quad \sum_{n=1}^{\infty} \frac{1}{b^{a_1 n^k}}, \quad \sum_{n=1}^{\infty} \frac{1}{b^{a_2 n^k}}, \ldots, \sum_{n=1}^{\infty} \frac{1}{b^{a_m n^k}}$$

are $\mathbb{Q}(\sqrt{-D})$-linearly independent.

As an immediate consequence by putting $k = 2$, we have the following corollary.

Corollary 5.1.1 *Let $b \geq 2$ be an integer and $1 \leq a_1 < a_2 < \cdots < a_m$ be integers such that $\sqrt{a_i/a_j} \notin \mathbb{Q}$ for any $i \neq j$. Set $\tau = \frac{i \log b}{\pi}$. Then the real numbers*

$$1, \quad \theta_3(a_1\tau), \quad \theta_3(a_2\tau), \ldots, \theta_3(a_m\tau)$$

are $\mathbb{Q}(\sqrt{-D})$-linearly independent.

5.2 Preliminaries

Proposition 5.2.1 *Let a_1 and a_2 be positive integers such a_1a_2 is not a perfect square. Then, there exist infinitely many pairs (N_0, N_1) of positive integers satisfying*

$$a_1N_1^2 - a_2N_0^2 = 2N_0 \tag{5.1}$$

and

$$a_2(N_0+1)^2 - a_1N_1^2 = 2a_2N_0 + a_2 - 2N_0. \tag{5.2}$$

In particular, as $N_0 \to \infty$, for any integer $k \geq 1$, we get

$$a_1N_1^2 - a_2N_0^2 - k\log N_0 \to \infty \text{ and } a_2(N_0+1)^2 - a_1N_1^2 - k\log N_0 \to \infty.$$

Proof. Consider

$$a_1N_1^2 - a_2N_0^2 = 2N_0$$

We rewrite this equality as follows

$$a_1N_1^2 - a_2\left(N_0^2 + \frac{2N_0}{a_2}\right) = 0.$$

Now by adding and subtracting $\frac{1}{a_2^2}$ in the bracket on the left hand side, we get

$$a_1N_1^2 - a_2\left(N_0 - \frac{1}{a_2}\right)^2 + \frac{1}{a_2} = 0.$$

By multiplying a_2 on both sides, we obtain

$$(a_2N_0 - 1)^2 - a_1a_2N_1^2 = 1.$$

Set $X = a_2N_0 - 1$ and $Y = N_1$. Now this equation turns out to be

$$X^2 - a_1a_2Y^2 = 1.$$

Thus from Lemma 1.4.2 of Chapter 1, we get infinitely many solutions of this Pell's equation given by (X_{2k}, Y_{2k}) such that $X_{2k} \equiv 1 \pmod{a_2}$. Therefore for every $k \geq 1$, we take

$$N_0 = \frac{X_{2k} - 1}{a_2} \quad \text{and} \quad N_1 = Y_{2k}$$

and these are solutions of (5.1). This proves the assertion. □

Lemma 5.2.1 *Let α and β be given real numbers such that $1, \alpha$ and β are $\mathbb{Q}$-linearly independent. Then, for any integer $D \geq 1$, they are $\mathbb{Q}(\sqrt{-D})$-linearly independent.*

Proof. If not, they are $\mathbb{Q}(\sqrt{-D})$-linearly dependent. Then there exist integers a_1, b_1, a_2, b_2, a_3 and b_3 such that

$$a_1 + b_1 i\sqrt{D} + (a_2 + b_2 i\sqrt{D})\alpha + (a_3 + b_3 i\sqrt{D})\beta = 0.$$

Since α and β are real numbers, we conclude that

$$a_1 + a_2\alpha + a_3\beta = 0 \text{ and } b_1 + b_2\alpha + b_3\beta = 0.$$

Since $1, \alpha$ and β are $\mathbb{Q}$-linearly independent, we get $a_1 = a_2 = a_3 = 0$ and $b_1 = b_2 = b_3 = 0$, which proves the lemma. □

Proposition 5.2.2 *Let $a_1, a_2, \ldots, a_m$ be positive integers such that $1, \left(\frac{a_1}{a_2}\right)^{\frac{1}{k}}, \ldots, \left(\frac{a_1}{a_m}\right)^{\frac{1}{k}}$ are $\mathbb{Q}$-linearly independent. Then there exist infinitely many positive integers N such that*

$$\frac{1}{\sqrt[k]{10^{a_m}+1}} < \left\{\left(\frac{a_1}{a_i}\right)^{\frac{1}{k}} N\right\} < \frac{1}{\sqrt[k]{10^{a_m}}},$$

holds, for all $i \in \{2, 3, \ldots, m\}$.

Proof. Since $1, \left(\frac{a_1}{a_2}\right)^{\frac{1}{k}}, \ldots, \left(\frac{a_1}{a_m}\right)^{\frac{1}{k}}$ are $\mathbb{Q}$-linearly independent, by Corollary 1.5.5 Chapter 1, page 22, the sequence

$$\left(\left\{n\left(\frac{a_1}{a_2}\right)^{\frac{1}{k}}\right\}, \ldots, \left\{n\left(\frac{a_1}{a_m}\right)^{\frac{1}{k}}\right\}\right), n = 1, 2, 3, \ldots$$

is uniformly distributed mod 1. Take a subset

$$E = \left[\frac{1}{\sqrt[k]{\ell+1}}, \frac{1}{\sqrt[k]{\ell}}\right]^s \qquad \text{of } [0,1)^s.$$

Since the quantity

$$\left(\frac{1}{\sqrt[k]{\ell}} - \frac{1}{\sqrt[k]{\ell+1}}\right) > 0,$$

by the definition of a sequence uniformly distributed mod 1, there exist infinitely many natural numbers N such that

$$\frac{1}{\sqrt[k]{\ell+1}} < \left\{\left(\frac{a_1}{a_i}\right)^{\frac{1}{k}} N\right\} < \frac{1}{\sqrt[k]{\ell}}.$$

□

Proposition 5.2.3 *For any integer $m \geq 2$, let $1 \leq a_1 < a_2 < \cdots < a_m$ be integers. Then*

$$a_{i+1} > \left(\frac{10^{a_m}+1}{10^{a_m}}\right) a_i, \qquad \text{for all} \quad i = 1, 2, \ldots, m-1.$$

Proof. By noticing that $a_i < a_m < 10^{a_m}$ and using the fact that $a_i < a_{i+1}$, we get that

$$a_i + \frac{a_i}{10^{a_m}} < a_i + 1 \leq a_{i+1}.$$

This proves the assertion. □

We have the following theorem due to Besicovitch [8].

Theorem 5.2.2 *Let $a_1, a_2, \ldots, a_m$ be distinct postive integers and $n_1, \ldots, n_m$ be positive integers such that $\sqrt[n_i]{a_i}$ is irrational for $1 \leq i \leq m$. Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be the positive real roots of the equations*

$$X^{n_1} - a_1 = 0, \quad X^{n_2} - a_2 = 0, \ldots, X^{n_m} - a_m = 0$$

respectively. Let $P(X_1, \ldots, X_m)$ be a non-zero polynomial with rational coefficients such that degree$_{X_i}$ $P \leq n_i - 1$ for all $i = 1, 2, \ldots, m$. Then

$$P(\alpha_1, \alpha_2, \ldots, \alpha_m) \neq 0.$$

We also need the following Lemma, which can be easily deduce from Theorem 5.2.2.

Lemma 5.2.3 *Let $a_1, a_2, \ldots, a_m$ be distinct natural numbers, and $k \geq 2$ be an integer. Then the following three statements are equivalent:*

(1) $\sqrt[k]{a_1}, \ldots, \sqrt[k]{a_m}$ *are* $\mathbb{Q}$*-linearly independent,*

(2) $\sqrt[k]{a_i/a_j}$ *is irrational for any* $i \neq j$*;*

(3) $\frac{1}{\sqrt[k]{a_1}}, \ldots, \frac{1}{\sqrt[k]{a_m}}$ *are* $\mathbb{Q}$*-linearly independent.*

Proof. (1) $\implies$ (2). Suppose the real numbers $\sqrt[k]{a_1}, \ldots, \sqrt[k]{a_m}$ are $\mathbb{Q}$-linearly independent. To prove, for any $i \neq j$, $\sqrt[k]{a_i/a_j}$ is irrational.

Let $1 \leq i, j \leq m$ such that $i \neq j$ be given integers. WLOG we assume that $i < j$. Since

$$\sqrt[k]{a_1}, \ldots, \sqrt[k]{a_i}, \ldots, \sqrt[k]{a_j}, \ldots, \sqrt[k]{a_m}$$

are $\mathbb{Q}$-linearly independent. This implies that

$$\sqrt[k]{a_1/a_j}, \ldots, \sqrt[k]{a_i/a_j}, \ldots, 1, \ldots, \sqrt[k]{a_m/a_j}$$

are $\mathbb{Q}$-linearly independent. In particular, $\sqrt[k]{a_i/a_j}$ is irrational for any $i \neq j$.

(2) $\implies$ (1). Given that $\sqrt[k]{a_i/a_j}$ is irrational for any $i \neq j$. This implies at most only one of the $\sqrt[k]{a_1}, \ldots, \sqrt[k]{a_m}$ is rational. To see this, suppose $\sqrt[k]{a_i}$ and $\sqrt[k]{a_j}$ are rational for some $1 \leq i < j \leq m$, then $\sqrt[k]{a_i/a_j}$ can not be irrational. Now we divide in two cases.

Case 1. Suppose all $\sqrt[k]{a_1}, \ldots, \sqrt[k]{a_m}$ are irrational.

Therefore, take $\alpha = \sqrt[k]{a_1}, \ldots, \sqrt[k]{a_m}$ are the positive real roots of

$$X^k - a_1 = 0, \ldots, X^k - a_m = 0,$$

respectively. Let $P(x_1, x_2, \ldots, x_m) = c_1 x_1 + \cdots + c_m x_m$ be non-zero linear form with rational coefficients. Then by Theorem 5.2.2,

$$P(\alpha, \ldots, \alpha_m) \neq 0.$$

This implies that the numbers $\sqrt[k]{a_1}, \ldots, \sqrt[k]{a_m}$ are $\mathbb{Q}$-linearly independent.

Case 2. Suppose one of $\sqrt[k]{a_i}$'s is rational. Let say $\sqrt[k]{a_m}$ is rational and $\sqrt[k]{a_i}$ are irrational for all $1 \leq i \leq m-1$. Let

$$P(x_1, x_2, \ldots, x_{m-1}) = c_1 x_1 + \cdots + c_{m-1} x_{m-1} + \sqrt[k]{a_m} c_m,$$

where $(c_1, \ldots, c_m) \in \mathbb{Z}^m \backslash \{(0, 0, \ldots, 0)\}$. Then by Theorem 5.2.2, we conclude that

$$P(\sqrt[k]{a_1}, \ldots, \sqrt[k]{a_{m-1}}) = c_1 \sqrt[k]{a_1} + \cdots + c_{m-1} \sqrt[k]{a_{m-1}} + c_m \sqrt[k]{a_m} \neq 0.$$

This implies that the numbers $\sqrt[k]{a_1}, \ldots, \sqrt[k]{a_m}$ are $\mathbb{Q}$-linearly independent and hence the lemma.

Now we prove (3) $\implies$ (2). Since $\frac{1}{\sqrt[k]{a_1}}, \ldots, \frac{1}{\sqrt[k]{a_m}}$ are $\mathbb{Q}$-linearly independent, for any $1 \leq i < j \leq r$, we see that $\sqrt[k]{a_i/a_j}$ is irrational.

(2) $\implies$ (3). Suppose $\sqrt[k]{a_i/a_j}$ is irrational for any $i \neq j$. This implies at most one of the numbers $\sqrt[k]{a_1}, \ldots \sqrt[k]{a_m}$ is rational. Here we have taken $\sqrt[k]{a_i}$ is the positive real root of $X^k - a_i = 0$ for all i. Let $P(X_1, \ldots, X_m) = c_1 X_1 + \cdots + c_m X_m$ be a non-zero linear form with rational coefficients.

Claim. $P\left(\frac{1}{\sqrt[k]{a_1}}, \ldots, \frac{1}{\sqrt[k]{a_m}}\right) \neq 0.$

In order to prove claim, consider

$$\begin{aligned} P\left(\frac{1}{\sqrt[k]{a_1}}, \ldots, \frac{1}{\sqrt[k]{a_m}}\right) &= c_1 \frac{1}{\sqrt[k]{a_1}} + \cdots + c_m \frac{1}{\sqrt[k]{a_m}} \\ &= \frac{c_1 \sqrt[k]{a_2 \cdots a_m} + \cdot + c_m \sqrt[k]{a_1 \cdots a_{m-1}}}{\sqrt[k]{a_1 \cdots a_m}}. \end{aligned}$$

Write $b_i = a_1 a_2 \cdots a_{i-1} a_{i+1} \cdots a_m$ for all $i = 1, 2, \ldots, m$. Under these notations, we have

$$P\left(\frac{1}{\sqrt[k]{a_1}}, \ldots, \frac{1}{\sqrt[k]{a_m}}\right) = \frac{c_1\sqrt[k]{b_1} + \cdots + c_m\sqrt[k]{b_m}}{\sqrt[k]{a_1\cdots a_m}}.$$

Hence, in order to prove that $P\left(\frac{1}{\sqrt[k]{a_1}}, \ldots, \frac{1}{\sqrt[k]{a_m}}\right) \neq 0$. It is enough to prove

$$P(\sqrt[k]{b_1}, \ldots, \sqrt[k]{b_m}) \neq 0.$$

Note that for $1 \leq i < j \leq m$, we have

$$\sqrt[k]{\frac{b_i}{b_j}} = \sqrt[k]{\frac{a_1\cdots a_{i-1}a_{i+1}\cdots a_m}{a_1\cdots a_{j-1}a_{j+1}\cdots a_m}} = \sqrt[k]{\frac{a_j}{a_i}}$$

is irrational. Thus we have the following situation; $b_1, \ldots, b_m$ are distinct positive integers such that $\sqrt[k]{b_i/b_j}$ is irrational for all $1 \leq i < j \leq m$ with $i \neq j$. Therefore, by the implication (2) $\implies$ (1), we conclude that $\sqrt[k]{b_1}, \ldots, \sqrt[k]{b_m}$ are $\mathbb{Q}$-linearly independent. Hence,

$$P(\sqrt[k]{b_1}, \ldots, \sqrt[k]{b_m}) \neq 0 \implies P\left(\sqrt[k]{b_1}, \ldots, \sqrt[k]{b_m}\right) \neq 0.$$

This implies that the numbers $\frac{1}{\sqrt[k]{a_1}}, \ldots, \frac{1}{\sqrt[k]{a_m}}$ are $\mathbb{Q}$-linearly independent, and hence the lemma. □

5.3 Proofs of Theorem 5.1.1 and 5.1.2

Proof of Theorem 5.1.1. Let $f, g : \mathbb{N} \to \mathbb{N}$ be functions such that $f(n) = O(n^k) = g(n)$ for some integer $k \geq 0$. By Lemma 5.2.1, it is enough to prove the linear independence of the numbers

$$1, \quad \sum_{n=1}^{\infty} \frac{f(n)}{b^{a_1 n^2}}, \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{g(n)}{b^{a_2 n^2}},$$

over $\mathbb{Q}$.

Suppose that these numbers are $\mathbb{Q}$-linearly dependent. Then there exist integers c_0, c_1 and c_2 not all zero such that

$$c_0 + c_1 \sum_{n=1}^{\infty} \frac{f(n)}{b^{a_1 n^2}} + c_2 \sum_{n=1}^{\infty} \frac{g(n)}{b^{a_2 n^2}} = 0. \tag{5.3}$$

First we note that both the numbers

$$\sum_{n=1}^{\infty} \frac{f(n)}{b^{a_1 n^2}} \quad \textit{and} \quad \sum_{n=1}^{\infty} \frac{g(n)}{b^{a_2 n^2}}$$

are irrational. If not, we assume that the real number $\sum_{n=1}^{\infty} \frac{f(n)}{b^{a_1 n^2}} = \frac{p}{q}$. Then, we have

$$\frac{p}{q} = \sum_{n=1}^{N} \frac{f(n)}{b^{a_1 n^2}} + \sum_{n=N+1}^{\infty} \frac{f(n)}{b^{a_1 n^2}}.$$

By multiplying $b^{a_1 N^2}$ on both sides, we get

$$p - q\left(b^{a_1 N^2} \sum_{n=1}^{N} \frac{f(n)}{b^{a_1 n^2}}\right) = q\left(b^{a_1 N^2} \sum_{n=N+1}^{\infty} \frac{f(n)}{b^{a_1 n^2}}\right) = q\left(\sum_{n=N+1}^{\infty} \frac{f(n)}{b^{a_1 n^2 - a_1 N^2}}\right).$$

Note that left hand side is an integer. Since $f(n) > 0$ for all n and $f(n) = O(n^k)$, we see that

$$0 < p - q\left(b^{a_1 N^2} \sum_{n=1}^{N} \frac{f(n)}{b^{a_1 n^2}}\right) \to 0 \quad \text{as} \quad N \to \infty$$

which is a contradiction. Therefore the number $\sum_{n=1}^{\infty} \frac{f(n)}{b^{a_1 n^2}}$ is irrational. Similarly, we can prove that the number $\sum_{n=1}^{\infty} \frac{g(n)}{b^{a_2 n^2}}$ is irrational. This fact implies that both c_1 and c_2 in (5.3), are non-zero. Now, we rewrite (5.3) as

$$c_0 + \left(\sum_{n=1}^{N_1} c_1 \frac{f(n)}{b^{a_1 n^2}} + c_2 \sum_{n=1}^{N_0} \frac{g(n)}{b^{a_2 n^2}}\right) = \left(-c_1 \sum_{N_1+1}^{\infty} \frac{f(n)}{b^{a_1 n^2}} - c_2 \sum_{n=N_0+1}^{\infty} \frac{g(n)}{b^{a_2 n^2}}\right) \tag{5.4}$$

By multiplying $b^{a_1N_1^2}$ on both sides of (5.4), we get

$$b^{a_1N_1^2}\left(c_0+c_1\sum_{n=1}^{N_1}\frac{f(n)}{b^{a_1n^2}}+c_2\sum_{n=1}^{N_0}\frac{g(n)}{b^{a_2n^2}}\right)=b^{a_1N_1^2}\left(-c_1\sum_{N_1+1}^{\infty}\frac{f(n)}{b^{a_1n^2}}-c_2\sum_{n=N_0+1}^{\infty}\frac{g(n)}{b^{a_2n^2}}\right). \tag{5.5}$$

By Proposition 5.2.1, there exist infinitely many positive integers N_0 and N_1 such that $a_1N_1^2-a_2N_0^1=2N_0>0$, we see that the left-hand side of (5.5) is an integer. Now, we claim the following.

Claim. The quantity

$$b^{a_1N_1^2}\left(-c_1\sum_{N_1+1}^{\infty}\frac{f(n)}{b^{a_1n^2}}-c_2\sum_{n=N_0+1}^{\infty}\frac{g(n)}{b^{a_2n^2}}\right)\to 0 \quad \text{as} \quad N_0\to\infty.$$

In order to prove Claim, we estimate the above quantity as follows. Consider

$$\left|b^{a_1N_1^2}\left(c_2\sum_{n=N_0+1}^{\infty}\frac{g(n)}{b^{a_2n^2}}\right)\right| \quad \leq \quad |c_2|b^{a_1N_1^2}\left(\frac{|g(N_0+1)|}{b^{a_2(N_0+1)^2}}+\frac{|g(N_0+2)|}{b^{a_1(N_0+2)^2}}+\cdots\right).$$

Using $g(n)=O(n^k)$ and (5.2), we have

$$\begin{aligned}\left|b^{a_1N_1^2}\left(c_2\sum_{n=N_0+1}^{\infty}\frac{g(n)}{b^{a_2n^2}}\right)\right| &\leq C\left(\frac{(N_0+1)^k}{b^{[a_2(N_0+1)^2-a_1N_1^2]}}+\frac{(N_0+2)^k}{b^{[a_2(N_0+2)^2-a_1N_1^2]}}+\cdots\right)\\ &\leq \frac{C(N_0+1)^k}{b^{2(a_2-1)N_0}}\left(1+\frac{1}{b}+\frac{1}{b^2}+\cdots\right).\end{aligned}$$

Hence,

$$b^{a_1N_1^2}\left(c_2\sum_{n=N_0+1}^{\infty}\frac{f(n)}{b^{a_2n^2}}\right)\to 0 \quad \text{as} \quad N_0\to\infty. \tag{5.6}$$

Similarly, we get

$$b^{a_1N_1^2}\left(c_1\sum_{n=N_1+1}^{\infty}\frac{f(n)}{b^{a_1n^2}}\right)\to 0 \quad \text{as} \quad N_0\to\infty. \tag{5.7}$$

Thus, by (5.6) and (5.7), we get Claim.

Hence, by Claim we have,

$$c_0 + c_1 \sum_{n=1}^{N_1} \frac{f(n)}{b^{a_1 n^2}} + c_2 \sum_{n=1}^{N_0} \frac{g(n)}{b^{a_2 n^2}} = 0. \tag{5.8}$$

Then, multiplying both sides by $b^{a_2 N_0^2}$, we get,

$$\begin{aligned} \sum_{n=1}^{N_0} \frac{c_2 g(n) b^{a_2 N_0^2}}{b^{a_2 n^2}} &= b^{a_2 N_0^2} \left(-c_0 - c_1 \sum_{n=1}^{N_1} \frac{f(n)}{b^{a_1 n^2}} \right) \\ &= b^{a_2 N_0^2} \left(-c_0 - c_1 \sum_{n=1}^{N_1 - 1} \frac{f(n)}{b^{a_1 n^2}} \right) - c_1 \frac{f(N_1)}{b^{a_1 N_1^2 - a_2 N_0^2}}. \end{aligned} \tag{5.9}$$

By the relation $a_1 N_1^2 - a_2 N_0^2 = 2N_0$, we see that $a_1(N_1 - 1)^2 - a_2 N_0^2 \leq 0$. Therefore first term on right hand side is an integer. Since c_1 and c_2 are non-zero, we estimate second term on right hand side of (5.9), we get

$$|c_1| \frac{f(N_1)}{b^{a_1 N_1^2 - a_2 N_0^2}} = |c_1| \frac{f(N_1)}{b^{2N_0}} \leq |c_1| \frac{f(N_1)}{b^{C N_1}},$$

where C is some fixed positive real number. Since $f : \mathbb{N} \to \mathbb{N}$ and $f(n) = O(n^k)$, for some integer $k \geq 0$ and c_1 is not zero, we have

$$0 < |c_1| \frac{f(N_1)}{b^{a_1 N_1^2 - a_2 N_0^2}} \leq |c_1| \frac{f(N_1)}{b^{C N_1}} = O\left(\frac{N_1^k}{b^{C N_1}} \right).$$

By choosing N_1 sufficiently large, we get

$$0 < |c_1| \frac{f(N_1)}{b^{a_1 N_1^2 - a_2 N_0^2}} < 1.$$

Thus, the equality (5.9) is impossible, which turns implies that the equality (5.8) is not possible. This proves the theorem. □

Proof of Theorem 5.1.2. By Lemma 5.2.1, it is enough to prove the linear independence of the numbers

$$1, \quad \sum_{n=1}^{\infty} \frac{1}{b^{a_1 n^k}}, \quad \sum_{n=1}^{\infty} \frac{1}{b^{a_2 n^k}}, \ldots, \sum_{n=1}^{\infty} \frac{1}{b^{a_m n^k}}$$

over $\mathbb{Q}$. Suppose that these numbers are $\mathbb{Q}$-linearly dependent. Then there exist integers $c_0, c_1, \ldots, c_m$ not all zero such that

$$c_0 + c_1 \sum_{n=1}^{\infty} \frac{1}{b^{a_1 n^k}} + c_2 \sum_{n=1}^{\infty} \frac{1}{b^{a_2 n^k}} + \cdots + c_m \sum_{n=1}^{\infty} \frac{1}{b^{a_m n^k}} = 0. \tag{5.10}$$

Since $\sqrt[k]{a_i/a_j} \notin \mathbb{Q}$ for any $i \neq j$, by Lemma 5.2.3, we get $1, \left(\frac{a_1}{a_2}\right)^{\frac{1}{k}}, \ldots, \left(\frac{a_1}{a_m}\right)^{\frac{1}{k}}$ are also $\mathbb{Q}$-linearly independent. Thus, by Proposition 5.2.2, there exist infinitely many positive integers N_1 such that

$$\frac{1}{\sqrt[k]{10^{a_m}} + 1} < \left\{ \left(\frac{a_1}{a_i}\right)^{\frac{1}{k}} N_1 \right\} < \frac{1}{\sqrt[k]{10^{a_m}}}, \tag{5.11}$$

for $i = 2, 3, \ldots, m$.

If $N_i = \left[\left(\frac{a_1}{a_i}\right)^{\frac{1}{k}} N_1 \right]$ for $i = 2, 3, \ldots m$, then

$$a_1 N_1^k - a_i N_i^k > a_1 N_1^k - a_i \left(\frac{a_1}{a_i}\right) N_1^k = 0.$$

These inequalities imply that quantity in the left hand side of the equality

$$b^{a_1 N_1^k} \left(c_0 + \sum_{n=1}^{N_1} \frac{c_1}{b^{a_1 n^k}} + \cdots + \sum_{n=1}^{N_m} \frac{c_m}{b^{a_m n^k}} \right) = -b^{a_1 N_1^k} \left(\sum_{n=N_1+1}^{\infty} \frac{c_1}{b^{a_1 n^k}} + \cdots + \sum_{n=N_m+1}^{\infty} \frac{c_m}{b^{a_m n^k}} \right) \tag{5.12}$$

is an integer.

By (5.11), we have $\frac{1}{\sqrt[k]{10^{a_m+1}}} < \left\{ \left(\frac{a_1}{a_i} \right)^{\frac{1}{k}} N_1 \right\} < \frac{1}{\sqrt[k]{10^{a_m}}}$ and

$$\left(\frac{a_1}{a_i} \right)^{\frac{1}{k}} N_1 - \frac{1}{\sqrt[k]{10^{a_m}}} < N_i < \left(\frac{a_1}{a_i} \right)^{\frac{1}{k}} N_1 - \frac{1}{\sqrt[k]{10^{a_m}+1}}, \qquad i = 2, \dots, m.$$

Note that for all $i = 2, 3, \dots, m$

$$\begin{aligned} a_i (N_i + 1)^k - a_1 N_1^k &> a_i \left(\left(\frac{a_1}{a_i} \right)^{\frac{1}{k}} N_1 - \frac{1}{\sqrt[k]{10^{a_m}}} + 1 \right)^k - a_1 N_1^k \\ &= \left(\sqrt[k]{a_1} N_1 + \sqrt[k]{a_i} - \left(\frac{a_i}{10^{a_m}} \right)^{\frac{1}{k}} \right)^k - a_1 N_1^k \\ &= \left(\sqrt[k]{a_1} N_1 + \frac{\sqrt[k]{a_i}}{\sqrt[k]{10^{a_m}}} (\sqrt[k]{10^{a_m}} - 1) \right)^k - a_1 N_1^k. \end{aligned}$$

Since $a_m > 1$ for all $i = 2, 3, \dots, m$, we have

$$\begin{aligned} a_i (N_i + 1)^k - a_1 N_1^k &> \left(\sqrt[k]{a_1} N_1 + \frac{\sqrt[k]{a_i}}{\sqrt[k]{10^{a_m}}} (\sqrt[k]{10^{a_m}} - 1) \right)^k - a_1 N_1^k \\ &> \left(\sqrt[k]{a_1} N_1 + \left(\frac{a_i}{10^{a_m}} \right)^{\frac{1}{k}} \right)^k - a_1 N_1^k > c N_1, \end{aligned} \tag{5.13}$$

for some positive real number $c > 0$ not depending on N_1.

We note that the left-hand side of (5.12) is an integer. Now, we claim the following.

Claim. The quantity

$$-b^{a_1 N_1^k} \left(\sum_{n=N_1+1}^{\infty} \frac{c_1}{b^{a_1 n^k}} + \sum_{n=N_2+1}^{\infty} \frac{c_2}{b^{a_2 n^k}} + \cdots + \sum_{n=N_m+1}^{\infty} \frac{c_m}{b^{a_m n^k}} \right) \to 0 \quad \text{as} \quad N_1 \to \infty.$$

In order to prove the claim, we estimate the above quantity as follows. Consider

$$\left| -b^{a_1 N_1^k} \left(\sum_{n=N_i+1}^{\infty} \frac{c_i}{b^{a_i n^k}} \right) \right| \le |c_i| \left(\frac{1}{b^{a_i (N_i+1)^k - a_1 N_1^k}} + \frac{1}{b^{a_i (N_i+2)^k - a_1 N_1^k}} + \cdots \right).$$

By (5.13), we have

$$\left|-b^{a_1 N_1^k}\left(\sum_{n=N_i+1}^{\infty}\frac{c_i}{b^{a_i n^k}}\right)\right| \leq \frac{|c_i|}{b^{cN_1}}\left(\frac{1}{b}+\frac{1}{b^2}+\cdots\right)$$

for $i = 2, 3, \ldots, m$. Hence,

$$-b^{a_1 N_1^k}\left(\sum_{n=N_i+1}^{\infty}\frac{c_i}{b^{a_i n^k}}\right) \to 0 \quad \text{as} \quad N_1 \to \infty \tag{5.14}$$

for all $i \geq 2$. Since $a_1(N_1+1)^k - a_1 N_1^k > 2N_1$, we get

$$-b^{a_1 N_1^k}\left(\sum_{n=N_1+1}^{\infty}\frac{c_1}{b^{a_1 n^k}}\right) \to 0 \quad \text{as} \quad N_1 \to \infty. \tag{5.15}$$

Thus, by (5.14) and (5.15), we get the Claim. Thus for all large enough N_1, we conclude that

$$c_0 + \sum_{n=1}^{N_1}\frac{c_1}{b^{a_1 n^k}} + \sum_{n=1}^{N_2}\frac{c_2}{b^{a_2 n^k}} + \cdots + \sum_{n=1}^{N_m}\frac{c_m}{b^{a_m n^k}} = 0. \tag{5.16}$$

Choose N_1 large enough integer such that (5.16) holds true. Set $r_1 = [\frac{N_1}{10^{a_m}}]$ and $r_2 = [p_2 N_1]$, where p_2 is positive real number satisfying

$$p_2 < \min_{2\leq r\leq m-1}\left\{\left(\frac{a_{r+1}}{10^{a_m}+1}\right)^{\frac{1}{k}} - \left(\frac{a_r}{10^{a_m}}\right)^{\frac{1}{k}}\right\}.$$

By Proposition 5.2.3, we see that $a_{r+1} > \left(\frac{10^{a_m}+1}{10^{a_m}}\right) a_r$ holds for all $r = 2, 3, \ldots, m-1$. This is equivalent to

$$\frac{a_{r+1}}{10^{a_m}+1} > \frac{a_r}{10^{a_m}},$$

which in turns implies that such p_2 exists. Multiply by $b^{a_1 N_1^k - r_1}$ both sides of

(5.16) to get

$$c_0 b^{a_1 N_1^k - r_1} + \sum_{n=1}^{N_1} \frac{c_1 b^{a_1 N_1^k - r_1}}{b^{a_1 n^k}} + \sum_{n=1}^{N_2} \frac{c_2 b^{a_1 N_1^k - r_1}}{b^{a_2 n^k}} + \cdots + \sum_{n=1}^{N_m} \frac{c_m b^{a_1 N_1^k - r_1}}{b^{a_m n^k}} = 0.$$

This implies that

$$\frac{c_1}{b^{r_1}} = -c_0 b^{a_1 N_1^k - r_1} - \sum_{n=1}^{N_1 - 1} \frac{c_1 b^{a_1 N_1^k - r_1}}{b^{a_1 n^k}} - \sum_{n=1}^{N_2} \frac{c_2 b^{a_1 N_1^k - r_1}}{b^{a_2 n^k}} - \cdots - \sum_{n=1}^{N_m} \frac{c_m b^{a_1 N_1^k - r_1}}{b^{a_m n^k}}. \tag{5.17}$$

First we note that

$$a_1 N_1^k - r_1 - a_1 (N_1 - 1)^k > k a_1 N_1^{k-1} - a_1 \left(\frac{k(k-1)}{2} \right) N_1^{k-2} + \cdots + (-1)^{k+1} a_1 - \frac{N_1}{10^{a_m}}.$$

Since $k a_1 - \frac{1}{10^{a_m}} > 0$, for all $k \geq 2$ we get

$$a_1 N_1^k - r_1 - a_1 (N_1 - 1)^k > 0,$$

holds for sufficiently large value of N_1. For $i = 2, 3, \ldots, m$, by the definition of N_i's and (5.11) we have

$$\begin{aligned} a_1 N_1^k - r_1 - a_i N_i^k &> a_1 N_1^k - a_i \left(\left(\frac{a_1}{a_i} \right)^{\frac{1}{k}} N_1 - \frac{1}{\sqrt[k]{10^{a_m} + 1}} \right)^k - \frac{N_1}{10^{a_m}} \\ &= a_1 N_1^k - \left(\sqrt[k]{a_1} N_1 - \left(\frac{a_i}{10^{a_m} + 1} \right)^{\frac{1}{k}} \right)^k - \frac{N_1}{10^{a_m}}. \end{aligned}$$

By the formula $(x-y)^n = x^n - n x^{n-1} y + \cdots + (-1)^n y^n$, we see that the quantity

$$a_1 N_1^k - \left(\sqrt[k]{a_1} N_1 - \left(\frac{a_i}{10^{a_m} + 1} \right)^{\frac{1}{k}} \right)^k$$

is a polynomial in variable N_1 of degree $k-1$ with leading coefficient $k a_1^{\frac{k-1}{k}} \left(\frac{a_i}{10^{a_m}+1} \right)^{\frac{1}{k}} >$

0. For $k \geq 3$, we can easily see that

$$a_1 N_1^k - r_1 - a_i N_i^k = a_1 N_1^k - \left(\sqrt[k]{a_1} N_1 - \left(\frac{a_i}{10^{a_m} + 1} \right)^{\frac{1}{k}} \right)^k - \frac{N_1}{10^{a_m}} > 0$$

for sufficiently large values of N_1. To prove this inequality for $k = 2$, we need to verify that

$$2\sqrt{a_1} \left(\frac{a_i}{10^{a_m} + 1} \right)^{\frac{1}{2}} - \frac{1}{10^{a_m}} > 0.$$

It can be easily seen as follows

$$\begin{aligned} \left(2\sqrt{a_1} \left(\frac{a_i}{10^{a_m} + 1} \right)^{\frac{1}{2}} - \frac{1}{10^{a_m}} \right)^2 &= \frac{4a_1 a_i}{10^{a_m} + 1} + \frac{1}{10^{2a_m}} - \frac{4\sqrt{a_1 a_i}}{10^{a_m}\sqrt{10^{a_m} + 1}} \\ &= \frac{4a_1 a_i 10^{2a_m} + 10^{a_m} + 1 - (4\sqrt{a_1 a_i}\sqrt{10^{a_m} + 1})10^{a_m}}{10^{2a_m}(10^{a_m} + 1)} > 0. \end{aligned}$$

Hence, we conclude that for all $k \geq 2$ and $i = 2, 3, \ldots, m$, we have

$$a_1 N_1^k - r_1 - a_i N_i^k > 0.$$

Thus, by these inequalties we obtain that the left hand side of (5.17) tends to 0 as $N_1 \to \infty$ but the right hand side is always an integer for all N_1. Therefore, we conclude that $c_1 = 0$ and (5.16) becomes

$$c_0 + \sum_{n=1}^{N_2} \frac{c_2}{b^{a_2 n^k}} + \cdots + \sum_{n=1}^{N_m} \frac{c_m}{b^{a_m n^k}} = 0.$$

Again, this equality can be written in the form

$$\frac{c_2}{b^{r_2}} = -c_0 b^{a_2 N_2^k - r_2} - \sum_{n=1}^{N_2 - 1} \frac{c_2 b^{a_2 N_2^k - r_2}}{b^{a_2 n^k}} - \cdots - \sum_{n=1}^{N_m} \frac{c_m b^{a_2 N_2^k - r_2}}{b^{a_m n^k}}. \tag{5.18}$$

Notice that

$$a_2N_2^k - a_2(N_2-1)^k - r_2 = a_2kN_2^{k-1} - a_2\left(\frac{k(k-1)}{2}\right)N_2^{k-2} + \cdots + (-1)^{k+1} - [p_2N_1]$$
$$> a_2k\left(\left(\frac{a_1}{a_2}\right)^{\frac{1}{k}}N_1 - 1\right)^{k-1} - \left(\frac{a_2k(k-1)}{2}\right)\left(\left(\frac{a_1}{a_2}\right)^{\frac{1}{k}}N_1\right)^{k-2} - p_2N_1.$$

We see that the right hand side of this inequality is a polynomial in the variable N_1 of degree $k-1$ with the leading coefficient $ka_2^{\frac{1}{k}}a_1^{\frac{k-1}{k}}$. Since a_1 and a_2 are positive integers, we have $ka_2^{\frac{1}{k}}a_1^{\frac{k-1}{k}} > 1$. Therefore for all sufficiently large values of N_1, we have

$$a_2N_2^k - a_2(N_2-1)^k - r_2 > a_2k\left(\frac{a_1}{a_2}\right)^{\frac{1}{k}}N_1^{k-1} - a_2\left(\frac{k(k-1)}{2}\right) + \cdots + (-1)^{k+1} - p_2N_1 > 0,$$

and for $i \geq 3$,

$$\begin{aligned} a_2N_2^k - r_2 - a_iN_i^k &> \left(\sqrt[k]{a_1}N_1 - \left(\frac{a_2}{10^{a_m}}\right)^{\frac{1}{k}}\right)^k - \left(\sqrt[k]{a_1}N_1 - \left(\frac{a_i}{10^{a_m}+1}\right)^{\frac{1}{k}}\right)^k - r_2 \\ &> ka_1^{\frac{k-1}{k}}\left(\left(\frac{a_i}{10^{a_m}+1}\right)^{\frac{1}{k}} - \left(\frac{a_2}{10^{a_m}}\right)^{\frac{1}{k}}\right)N_1^{k-1} + \cdots \\ &+ (-1)^k\left(\frac{a_2}{10^{a_m}} - \frac{a_i}{10^{a_m}+1}\right) - p_2N_1. \end{aligned}$$

Hence, by applying the similar argument as we have done before, for all $k \geq 3$ we get

$$a_2N_2^k - r_2 - a_iN_i^k > 0,$$

holds for sufficiently large values of N_1. For the case $k = 2$, we see that

$$\begin{aligned} a_2N_2^2 - r_2 - a_iN_i^2 &> \left(\sqrt{a_1}N_1 - \left(\frac{a_2}{10^{a_m}}\right)^{\frac{1}{2}}\right)^2 - \left(\sqrt{a_1}N_1 - \left(\frac{a_i}{10^{a_m}+1}\right)^{\frac{1}{2}}\right)^2 - r_2 \\ &= 2\sqrt{a_1}\left(\left(\frac{a_i}{10^{a_m}+1}\right)^{\frac{1}{2}} - \left(\frac{a_2}{10^{a_m}}\right)^{\frac{1}{2}}\right)N_1 - p_2N_1 + \frac{a_2}{10^{a_m}} - \frac{a_i}{10^{a_m}+1} \end{aligned}$$

$$\geq \; 2\sqrt{a_1}\left(\left(\frac{a_3}{10^{a_m}+1}\right)^{\frac{1}{2}} - \left(\frac{a_2}{10^{a_m}}\right)^{\frac{1}{2}}\right)N_1 - p_2N_1 + \frac{a_2}{10^{a_m}} - \frac{a_i}{10^{a_m}+1}.$$

By the choice of p_2, we get

$$a_2N_2^2 - r_2 - a_iN_i^2 > 2p_2\sqrt{a_1}N_1 - p_2N_1 + \frac{a_2}{10^{a_m}} - \frac{a_i}{10^{a_m}+1} > 0$$

holds for all sufficiently large values of N_1 and hence, we obtain that for all $i \geq 3$

$$a_2N_2^k - r_2 - a_iN_i^k > 0$$

holds for all sufficiently large values of N_1. By these inequalities we easily see that the right hand side of (5.18) is an integer. But the left hand side tends to zero as $N_1 \to \infty$. Consequently, $c_2 = 0$ and we get

$$c_0 + \sum_{n=1}^{N_3} \frac{c_3}{b^{a_3n^k}} + \cdots + \sum_{n=1}^{N_m} \frac{c_m}{b^{a_mn^k}} = 0.$$

By continuing this process we get $c_r = 0$ for all $r = 1, 2, \ldots, i$ and we get

$$c_0 + \sum_{n=1}^{N_{i+1}} \frac{c_{i+1}}{b^{a_{i+1}n^k}} + \cdots + \sum_{n=1}^{N_m} \frac{c_m}{b^{a_mn^k}} = 0.$$

This equality can be written in the form

$$\frac{c_{i+1}}{b^{r_2}} = -c_0b^{a_{i+1}N_{i+1}^k - r_2} - \sum_{n=1}^{N_{i+1}-1} \frac{c_{i+1}b^{a_{i+1}N_{i+1}^k - r_2}}{b^{a_{i+1}n^k}} - \cdots - \sum_{n=1}^{N_m} \frac{c_mb^{a_{i+1}N_{i+1}^k - r_2}}{b^{a_mn^k}}. \quad (5.19)$$

By applying the similar argument, we note that

$$a_{i+1}N_{i+1}^k - a_{i+1}(N_{i+1}-1)^k - r_2 > 0,$$

and for all $j = 2, \ldots, m - i$,

$$
\begin{aligned}
a_{i+1}N_{i+1}^k - r_2 - a_{i+j}N_{i+j}^k &> \left(\sqrt[k]{a_1}N_1 - \left(\frac{a_{i+1}}{10^{a_m}}\right)^{\frac{1}{k}}\right)^k - \left(\sqrt[k]{a_1}N_1 - \left(\frac{a_{i+j}}{10^{a_m}+1}\right)^{\frac{1}{k}}\right)^k - r_2 \\
&> k a_1^{\frac{k-1}{k}} \left(\left(\frac{a_{i+j}}{10^{a_m}+1}\right)^{\frac{1}{k}} - \left(\frac{a_i}{10^{a_m}}\right)^{\frac{1}{k}}\right) N_1^{k-1} + \cdots \\
&+ (-1)^k \left(\frac{a_i}{10^{a_m}} - \frac{a_{i+j}}{10^{a_m}+1}\right) - p_2 N_1.
\end{aligned}
$$

Hence, by using similar argument as previous page for the case $k = 2$ and $k \geq 3$, we get

$$a_{i+1}N_{i+1}^k - r_2 - a_{i+j}N_{i+j}^k > 0$$

holds for all sufficiently large values of N_1 and $j = 2, \ldots, m-i$. Hence, by (5.19) we arrive at $c_i = 0$, for all $i = 1, 2, \ldots, m$ and from (5.10) we derive $c_0 = 0$, which is a contradiction. This proves the theorem. □

5.4 Concluding remarks

1. Let $f_1, f_2, \ldots, f_m : \mathbb{N} \to \mathbb{Z}$ be functions with polynomial growth. For any integer $m \geq 2$, let $1 \leq a_1 < a_2 < \cdots < a_m$ be integers such that $\sqrt[k]{a_i/a_j} \notin \mathbb{Q}$ for any $i \neq j$. Then, by applying the same method that applied in the proof of Theorem 5.1.2, one can prove the following general statement; the real numbers

$$1, \quad \sum_{n=1}^{\infty} \frac{f_1(n)}{b^{a_1 n^k}}, \quad \sum_{n=1}^{\infty} \frac{f_2(n)}{b^{a_2 n^k}}, \ldots, \sum_{n=1}^{\infty} \frac{f_m(n)}{b^{a_m n^k}}$$

are $\mathbb{Q}$-linearly independent.

2. It is reasonable to expect that the same conclusion of Theorem 5.1.2 can be achieved under the assumption that $a_1 < a_2 < \cdots < a_m$ are any positive

integers instead of $\sqrt[k]{a_i/a_j} \notin \mathbb{Q}$ for any $i \neq j$.

3. In Theorem 5.1.2, if we take $\tau = \frac{i \log b}{\pi}$, then it satisfies that $e^{i\pi\tau} \in \overline{\mathbb{Q}}$. This condition naturally suggests to ask the following question: let $\tau \in \mathbb{H}$ such that $e^{i\pi\tau} \in \overline{\mathbb{Q}}$. Then is it true that the numbers

$$\theta_3(a_1\tau), \ldots, \theta_3(a_m\tau),$$

are $\mathbb{Q}$-linearly independent ?.

For instance, the above question can be answered in the following Case. Let $m = 3$, $a_1 = 2^n$, $a_2 = 2^{n+1}$ and $a_3 = 2^{n+2}$. Since $\tau \in \mathbb{H}$ such that $e^{i\pi\tau} \in \overline{\mathbb{Q}}$ and the known identities, namely,

$$2\theta_3^2(2\tau) = \theta_3^2(\tau) + \theta_4^2(\tau) \quad \text{and} \quad 2\theta_3(4\tau) = \theta_3(\tau) + \theta_4(\tau)$$

we can see that

$$\theta_3(2^n\tau), \quad \theta_3(2^{n+1}\tau), \quad \theta_3(2^{n+2}\tau)$$

are $\overline{\mathbb{Q}}$-linearly independent.

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