ON SOME PROBLEMS IN ADDITIVE COMBINATORICS AND RELATED AREAS

By

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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DEDICATIONS

Dedicated to

MY BABA AND MAA

And

MY TEACHERS

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Summary

This thesis deals with some problems on the regularity of Diophantine equation, weighted zero-sum constants, determination of 2-color zero-sum generalized Schur numbers and fractionally dense sets of $\mathbb{R}$ and $\mathbb{C}$. The thesis consists of five chapters.

In 2006, Fox and Kleitman proved that, the equation $L_n(b): \quad (x_1 - y_1) + (x_2 - y_2) + \cdots + (x_n - y_n) = b$ is not $2n$-regular. They further conjectured that, for each $n \in \mathbb{N}$, there is $b_n \in \mathbb{N}$ such that the equation $(x_1 - y_1) + (x_2 - y_2) + \cdots + (x_n - y_n) = b_n$ is $(2n-1)$-regular. In other words, $\max\limits_{b \in \mathbb{N}} dor(L_n(b)) = 2n-1$. In the first chapter, we consider the non-linear equation

$$\mathsf{Q}_n(B) : (x_1 - y_1)(x_2 - y_2) \ldots (x_n - y_n) = B.$$

For every $r \geq 1$, we produce a positive integer $B = B(r)$ for which $\mathsf{Q}_n(B)$ is r-regular. In particular, $\max\limits_{B \in \mathbb{N}} dor(\mathsf{Q}_n(B)) = \infty$.

Let n be a positive integer and consider the weight set $C_n = \{a^3 \pmod{n} \mid a \in (\mathbb{Z}/n\mathbb{Z})^*\}$. In the second chapter, we prove an upper bound of $D_{C_n}(\mathbb{Z}/n\mathbb{Z})$ and $\mathsf{E}_{C_n}(\mathbb{Z}/n\mathbb{Z})$.

We study the $\{\pm 1\}$-weighted zero-sum constants in the third chapter. For an odd prime p, we prove a conditional result about the constant $\mathsf{s}_{\{\pm 1\}}((\mathbb{Z}/p\mathbb{Z})^3)$. Also, if k is an even divisor of $p-1$ and A is the subgroup of $(\mathbb{Z}/p\mathbb{Z})^*$ of order k, then we obtain an upper bound of $\mathsf{s}_{3,A}((\mathbb{Z}/p\mathbb{Z})^{k+1})$.

Robertson considered a weaker version of regularity and introduced the *zero-sum generalized Schur number* and the 2-*color zero-sum generalized Schur number*. In the fourth chapter, we determine the exact value of 2-color zero-sum generalized Schur number. We also generalize these two constants in a more general way and compute some of their exact values.

We study some fractionally dense sets of $\mathbb{R}$ and $\mathbb{C}$ in the fifth chapter. Let K be an algebraic number field such that K is not a subfield of $\mathbb{R}$ and $\mathcal{O}_K$ its ring of integers. For any subset A of $\mathbb{N}$ (respectively, $\mathcal{O}_K$), we say that A is *fractionally dense* in $\mathbb{R}_{>0}$ (respectively, in $\mathbb{C}$), if the *quotient set* $R(A) = \{a/a' : a, a' \in A\}$ is dense in $\mathbb{R}_{>0}$ (respectively, in $\mathbb{C}$). We prove some subsets of $\mathbb{N}$ and any non-zero ideal of an order of an imaginary quadratic field are fractionally dense in $\mathbb{R}_{>0}$ and $\mathbb{C}$, respectively.

CHAPTER 1

Regularity of certain Diophantine equation

An equation L is said to be r-regular if for every r-coloring of the positive integers, there exists a monochromatic solution to the equation L. Rado [73] conjectured that for every positive integer r, there exists a linear equation which is $(r-1)$-regular but not r-regular. Alexeev and Tsimerman ([17]) and Golowich ([51]) independently proved this conjecture for two different families of linear equations. In this chapter we show that, for every pair of positive integers r and n, there exists a positive integer $B = B(r, a_{m,i})$ such that the Diophantine equation

$$\prod_{m=1}^{n}\left(\sum_{i=1}^{k_m} a_{m,i}x_{m,i} - \sum_{j=1}^{l_m} b_{m,j}y_{m,j}\right) = B$$

is r-regular, where $a_{m,i}$ and $b_{m,j}$ are positive integers satisfying

$$\sum_{i=1}^{k_m} a_{m,i} = \sum_{j=1}^{l_m} b_{m,j} \qquad \textit{for all } m = 1, \ldots, n.$$

The content of this chapter is published in the article [82].

1.1 Definitions and some early results

In this section, we discuss some Ramsey type results on the set of integers. We closely follow [1], [62] and [73] for this section. For two natural numbers M and N with $M < N$, the interval $[M, N]$ means the set $\{M, M+1, M+2, \ldots, N\}$.

Definition 1.1.1 Let r be a positive integer and S a subset of $\mathbb{N}$. An *r-coloring* of S is a mapping $\chi : S \to \{1, 2, \ldots, r\}$.

Definition 1.1.2 Let S be a subset of $\mathbb{N}$ and $\chi : S \to \{1, 2, \ldots, r\}$ an r-coloring of S. A subset A of S is said to be *monochromatic* under χ if χ is constant on A.

Remark 1.1.3 Let S be a subset of $\mathbb{N}$ and $\chi : S \to \{1, 2, \ldots, r\}$ an r-coloring of S. Writing $S = \chi^{-1}(1) \cup \chi^{-1}(2) \ldots \cup \chi^{-1}(r)$, this r-coloring of S induces an r-partition of the set S and also conversely. Thus a subset A of S is monochromatic under χ, if $A \subset \chi^{-1}(i)$ for some $i \in \{1, 2, \ldots, r\}$.

The classical Ramsey theorem is a generalization of the following pigeonhole principle.

Theorem 1.1.4 *(Generalized Pigeonhole Principle) Let A and $B = \{b_1, b_2, \ldots, b_r\}$ be two finite nonempty sets. Let $f : A \to B$ be a function and $a_1, a_2, \ldots, a_r$ non-negative integers such that $|A| = a_1 + a_2 + \cdots + a_r - r + 1$ holds. Then there exists some integer $i \in [1, r]$ such that $|f^{-1}(b_i)| \geq a_i$.*

Proof. If possible, suppose that $|f^{-1}(b_i)| \leq (a_i - 1)$ for all $i = 1, 2, \ldots, r$. Then

$$|A| = \sum_{i=1}^{r} |f^{-1}(b_i)| \leq \sum_{i=1}^{r} (a_i - 1) = \sum_{i=1}^{r} a_i - r,$$

a contradiction to our assumption. □

Definition 1.1.5 Let S be a finite subset of $\mathbb{N}$ and let k be a positive integer such that $k \le |S|$. A subset T of S with $|T| = k$ is said to be a *k-subset of S*.

First we state the original Ramsey Theorem as follows.

Theorem 1.1.6 *(Ramsey) Let k, r and $l \ge k$ be positive integers. Then there exists a smallest positive integer $R = R(k, r, l)$ such that for any r-coloring of the k-subsets of $[1, R]$, there is an l-subset L of $[1, R]$ all of whose k-subsets are of the same color.*

In the rest of this section, we state three classical theorems in Ramsey theory on the integers. The first theorem is due to van der Waerden [97], which he proved in 1927.

Theorem 1.1.7 *(van der Waerden Theorem) Let k and r be two positive integers. Then there exists a smallest positive integer $W(k; r)$ such that for every r-coloring of $[1, W(k; r)]$, there is a monochromatic arithmetic progression of length k.*

The following is an equivalent statement of van der Waerden Theorem, known as the *infinite version of van der Waerden Theorem.*

Theorem 1.1.8 *Let k and r be any two positive integers and $\mathbb{N} = X_1 \cup X_2 \cup \ldots \cup X_r$ an r-partition. Then at least one of the X_i's contains an arithmetic progression of length k.*

In other words, if $\mathbb{N}$ is partitioned into r parts then at least one part contains arbitrary long arithmetic progression. We note that there exists a 2-partition of $\mathbb{N}$ such that none of the parts contains an arithmetic progression of infinite length.

Erdős and Turan [35] conjectured that if the 'size' of a set is 'big', then it contains arbitrary long arithmetic progression. To state the precise statement, we need the following definition.

Definition 1.1.9 For a subset A of $\mathbb{N}$, we define the *upper natural density* $\overline{d}(A)$ of A by

$$\overline{d}(A) = \limsup_{N\to\infty} \frac{|A \cap [1, N]|}{N}.$$

Conjecture 1.1.10 (Erdős-Turan) Let A be a subset of $\mathbb{N}$ such that $\overline{d}(A) > 0$. Then A contains an arithmetic progression of arbitrary length.

In 1953, Roth [80] proved that any subset A of $\mathbb{N}$ with $\overline{d}(A) > 0$ contains a three-term arithmetic progression. Later, Szemerédi [94] first improved Roth's result by proving that A contains a four-term arithmetic progression and finally in [95], he proved the general Erdős-Turan conjecture.

Theorem 1.1.11 *[95] Let k be a positive integer and δ a real number such that $0 < \delta \leq 1$. Then there exists a smallest positive integer $M = M(k, \delta)$ such that every subset A of $[1, M]$ with density $\frac{|A|}{M} \geq \delta$ contains an arithmetic progression of length k.*

We call $M(k, \delta)$ as the *Szemerédi number* corresponding to k and δ.

Remark 1.1.12 For any two positive integers k and r, let $\chi : E = [1, M(k, \frac{1}{r})] \to \{c_1, c_2, \dots, c_r\}$ be an r-coloring of E. Then by the pigeonhole principle, there exists a monochromatic subset S of E such that $\frac{|S|}{|E|} \geq \frac{1}{r}$. Therefore, by the definition of Szemerédi number, S contains an arithmetic progression of length k. Since the set S is monochromatic, we get a monochromatic arithmetic progression of length k. Thus $W(k, r) \leq M(k, \frac{1}{r})$.

For a positive integer n and a function $f : \mathbb{N} \to \mathbb{N}$, let

$$f^{(n)}(x) = \underbrace{f(f(\ldots(f}_{n-times}(x))))$$

be the composition of f with itself n times.

Let $(f_i)_{i\in\mathbb{N}} : \mathbb{N} \to \mathbb{N}$ be a family of functions such that $f_1(k) = 2k$ and $f_{i+1}(k) = f_i^{(k)}(1)$ for all $i \in \mathbb{N}$. We say the functions $f_2(k)$, $f_3(k)$ and $f_4(k)$ as $exp_2(k)$, $tower(k)$ and $wow(k)$ respectively. To give some idea about the magnitude of these functions, we present some of their values.

We have $exp_2(k) = f_2(k) = 2^k$, $tower(k) = f_3(k) = \underbrace{2^{2^{2^{2^{\cdot^{\cdot^{\cdot}}}}}}}_{k \text{ many } 2's}$, a tower of k many 2's and $wow(1) = 2$, $wow(2) = 2^2 = 4$ and $wow(3) = tower(4) = 65536$.

Also we define the *Ackermann function* as $ack(k) = f_k(k)$. The original proof of van der Waerden's theorem gives the upper bound $W(k, 2) \leq ack(k)$. Graham asked the question whether $W(k, 2) \leq tower(k)$ or not. In 1987, Shelah [90] proved that $W(k, 2) \leq wow(k)$ which is quite an improvement from the previous bound. Later Gowers [52] proved that

Theorem 1.1.13 *For any positive integer k and r, we have*

$$W(k, r) \leq 2^{2^{f(k,r)}}, \text{ where } f(k, r) = r^{2^{2^{k+9}}}.$$

When $r = 2$, the Gowers' bound is much small in magnitude than $tower(k)$, thereby answering Graham's question positively.

van der Waerden Theorem (Theorem 1.1.7) has the following consequence:

Lemma 1.1.14 *Let ℓ and r be two positive integers. Then there exists a smallest positive integer $n = n(\ell, r)$ such that for every r-coloring of $[1, n]$, there are positive integers $a_1 < a_2 < \ldots < a_\ell$ satisfying $\sum_{i=1}^{\ell} a_i \leq n$ with the property* *that for any subset I of $\{1, 2, \ldots, \ell\}$ the color of the element $\sum_{i \in I} a_i$ depends only on maximum value of I (denoted by $max(I)$).*

Using Lemma 1.1.14 we have the following theorem of Folkman and Sanders [53, 85] .

Theorem 1.1.15 *Let ℓ and r be two positive integers. Then there exists a smallest positive integer $m = m(\ell, r)$ such that for every r-coloring of $[1, m]$, there are positive integers $a_1 < a_2 < \ldots < a_\ell$ satisfying $\sum_{i=1}^{\ell} a_i \leq m$ such that for all subset I of $\{1, 2, \ldots, \ell\}$ the color of the elements $\sum_{i \in I} a_i$ are same.*

Proof. Let ℓ and r be two positive integers and $\chi : [1, n((\ell-1)r+1, r)] \to \{c_1, c_2, \ldots, c_r\}$ any r-coloring. By Lemma 1.1.14, there are positive integers $a_1 < a_2 < \ldots < a_{(\ell-1)r+1}$ satisfying $\sum_{i=1}^{(\ell-1)r+1} a_i \leq n$ such that for any subset I of $\{1, 2, \ldots, (\ell-1)r+1\}$ the color of the element $\sum\limits_{i \in I} a_i$ depends only on $\max(I)$.

Let $A = \{a_1, a_2, \ldots, a_{(\ell-1)r+1}\}$ which is of cardinality $|A| = \underbrace{\ell + \ell + \cdots + \ell}_{r-times} - r + 1$ and let $\chi|_A : A \to \{c_1, c_2, \ldots, c_r\}$ be the restriction function. Then by Theorem 1.1.4, there exists some integer $i \in [1, r]$ such that $|\chi|_A^{-1}(c_i)| \geq \ell$. In other words, there exist ℓ different integers $i_1, i_2, \ldots, i_\ell$ in $[1, (\ell-1)r+1]$ such that the set $A_i = \{a_{i_1}, a_{i_2}, \ldots, a_{i_\ell}\}$ is monochromatic of color c_i.

Clearly for any subset J of $\{i_1, i_2, \ldots, i_\ell\}$, the color of $\sum\limits_{j \in J} a_j$ is c_i, as the color depends only on max $(J) \in \{i_1, i_2, \ldots, i_\ell\}$. This proves the existence of such an m. □

Remark 1.1.16 In Theorem 1.1.15, it is clear that $m(\ell, r) \leq n((\ell-1)r+1, r)$.

Let us consider the case $\ell = 2$. Then Theorem 1.1.15 implies that for a positive integer r there is a smallest positive integer $m = m(2, r)$ such that for every r-coloring of $[1, m]$, there are positive integers $a_1 < a_2$ satisfying $a_1 + a_2 \leq m$ and the set $\{a_1, a_2, a_1 + a_2\}$ is monochromatic. In other words, for every

r-coloring of $[1, m]$ the equation $x + y = z$ has a solution in $[1, m]$ such that the set $\{x, y, z\}$ is monochromatic.

More generally, we have the following definition.

Definition 1.1.17 Let $\chi : \mathbb{N} \to \{1, 2, \dots, r\}$ be an r-coloring of $\mathbb{N}$ and L an equation or a system of equations in n variables. A solution $(x_1, x_2, \dots, x_n)$ of L is said to be a *monochromatic solution* to L if the set $\{x_1, x_2, \dots, x_n\}$ is monochromatic under χ.

The second classical theorem in Ramsey theory is due to Schur [89], which he proved in 1916. It is a special case of Theorem 1.1.15, as explained above.

Theorem 1.1.18 *Let r be a positive integer. Then there exists a smallest positive integer $S(r)$ such that for every r-coloring of $[1, S(r)]$ there is a monochromatic solution to the equation $x + y = z$.*

The numbers $S(r)$ are known as *Schur numbers*. Let us look at the case when $r = 2$. Then $S(2) > 4$, because if we take the 2-coloring $\chi : [1, 4] \to \{1, 2\}$ such that $\chi(1) = \chi(4) = 1$ and $\chi(2) = \chi(3) = 2$, then it is not possible to find a monochromatic solution to $x + y = z$ in $[1, 4]$.

Now consider any 2-coloring of $[1, 5]$ say, with colors red and blue. Without loss of generality, we can assume that 1 is colored by red. Suppose there is no monochromatic solution in $[1, 5]$. Since $(1, 1, 2)$ is a solution, we must color 2 by blue. Again since $(2, 2, 4)$ is a solution, we must color 4 by red. Since $(1, 4, 5)$ is a solution, we must color 5 by blue. But now either $(1, 3, 4)$ or $(2, 3, 5)$ is a monochromatic solution. This proves that $S(2) = 5$.

The only known values of Schur numbers are $S(1) = 2$, $S(2) = 5$, $S(3) = 14$ and $S(4) = 45$.

Schur gave a general bound of $S(r)$ for every positive integer r, which is $\frac{3^r+1}{2} \leq S(r) \leq 3r\,!$.

The third classical theorem in Ramsey Theory is due to Rado [73]. Here we state an abridged version, which is a generalization of Schur's theorem. Before that we need the following definition.

Definition 1.1.19 Let r be a positive integer and L an equation or a system of equations. Then L is said to be r-*regular* if for every r-coloring of $\mathbb{N}$ there exists a monochromatic solution to L. L is said to be *regular* if it is r-regular for all $r \geq 1$.

Thus Schur's theorem can be stated as "the equation $x + y = z$ is regular".

We state Rado's theorem which characterizes the regular linear homogeneous equations on $\mathbb{Z}$.

Theorem 1.1.20 *[73] Let $n \geq 2$ be a positive integer and $c_1, \ldots, c_n$ be non-zero integers. Then the linear Diophantine equation $c_1x_1 + \cdots + c_nx_n = 0$ is regular if and only if $\sum_{i \in I} c_i = 0$ for some non-empty subset I of $\{1, \ldots, n\}$.*

Let L be an equation or a system of equations which is not regular. Then by definition, there exists an integer $\ell \geq 2$ such that L is not ℓ-regular. Also by the definition of r-regularity, it is clear that if L is r-regular, then it is m-regular for all $1 \leq m \leq r$. Therefore, if L is not regular, then there exists a largest natural number r for which L is r-regular. This natural thing leads to the following definition.

Definition 1.1.21 Let L be an equation or a system of equations. The *degree of regularity* of L is defined to be infinite if L is regular, or else, it is the largest positive integer r such that L is r-regular. We denote the degree of regularity of L by $dor(L)$.

1.2 Introduction and Motivation of the Problem

In this section, we state two conjectures of Rado and their progress till now. These conjectures motivated us to pursue in this direction.

For a positive integer $n \geq 2$ and non-zero integers $c_1, c_2, \ldots, c_n$, let us consider the linear Diophantine equation

$$L: \qquad c_1x_1 + \cdots + c_nx_n = 0 \tag{1.1}$$

such that $\sum\limits_{i\in I} c_i \neq 0$ for all $I \subset [1, n]$.

Theorem 1.1.20 implies that L is not regular, that is there exists a natural number r such that L is not r-regular. In fact we prove that L is not $(p-1)$-regular where p is a prime which satisfies the property that $p \nmid \sum_{i\in I} c_i$ for all $I \subset [1, n]$.

In order to prove this, we first define the *super modulo color* S_p for a prime number p as follows.

Definition of super modulo color S_p : Let p be a prime number. Any $x \in \mathbb{Q}^*$ can be uniquely written as

$$x = \frac{p^j a}{b}, \qquad \text{for some } j, a \in \mathbb{Z}, b \in \mathbb{N}, p \nmid ab \text{ and } \gcd(a, b) = 1.$$

We define the *super modulo color* $S_p : \mathbb{Q}^* \to \{1, 2, \ldots, p-1\}$ on $\mathbb{Q}^*$ by

$$S_p(x) = \frac{a}{b} \pmod{p}.$$

Clearly S_p is a $(p-1)$-coloring on $\mathbb{Q}^*$ with the property that

$$S_p(x) = S_p(y) \Leftrightarrow S_p(\alpha x) = S_p(\alpha y) \text{ for all } \alpha \in \mathbb{Q}^*. \tag{1.2}$$

Let p be the least prime number satisfying $p \nmid \sum_{i \in I} c_i$ for every $I \subset [1, n]$. Then we prove that L has no monochromatic solution under S_p.

Suppose that $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ is a monochromatic solution to L under the S_p coloring. Then $(\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$ is also a monochromatic solution to L for any $\alpha \in \mathbb{Q}^*$. Therefore we can assume that gcd $(x_1, x_2, \ldots, x_n) = 1$.

Now without loss of generality, we can assume that there exists a $k \in \mathbb{N}$ with $1 \leq k \leq n$ such that

$$p \nmid x_i \text{ for } 1 \leq i \leq k \text{ and}$$

$$p \mid x_i \text{ for } k+1 \leq i \leq n.$$

Therefore from (1.1), we get

$$c_1x_1 + \cdots + c_kx_n \equiv c_1x_1 + \cdots + c_kx_k \pmod{p} \equiv \left(\sum_{i=1}^{k} c_i\right) \overline{x_1} \equiv 0 \pmod{p}.$$

As $p \nmid x_1$, we get that $\displaystyle\sum_{i=1}^{k} c_i \equiv 0 \pmod{p}$. This is a contradiction to the choice of p. Hence L is not $(p-1)$-regular. □

Remark 1.2.1 Let $n \geq 2$ be a positive integer and $c_1, \ldots, c_n$ non-zero integers such that $\sum_{i \in I} c_i \neq 0$ for every subset I of $[1, n]$. Then the linear equation $L : c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0$ is not $(p-1)$-regular for a prime $p \nmid \sum_{i \in I} c_i$ for every subset I of $[1, n]$. In other words, for such a prime

$$dor(L) \leq p - 2.$$

However, if we change c_i's then the prime p also changes. Thus the degree of regularity of L, in the above method, depends not only on the number of variables n but also on the coefficients $c_1, c_2, \ldots, c_n$ of L.

In his thesis [73], Rado conjectured that for a fixed natural number n, the degree of regularity of a non-regular n-variable linear homogeneous equation depends only on n not on the equation. This conjecture is known as *Rado's Boundedness Conjecture*. More generally, Rado's Boundedness Conjecture is as follows.

Conjecture 1.2.2 [73] For all positive integers m and n, there exists a positive integer $k(m, n)$ depending only on m and n, such that if a system of m linear equations (not necessarily homogeneous) in n-variables, say, $L_{m,n}$ is non-regular, then $L_{m,n}$ is not $k(m, n)$-regular.

Remark 1.2.3 In the above conjecture, the value $k(m, n)$ does not change even if we change $L_{m,n}$ to $L'_{m,n}$.

Rado [73] proved that the Conjecture 1.2.2 is true if it is true for linear equations. That is, if Conjecture 1.2.2 is true for all n-variable linear equations, then Conjecture 1.2.2 is true for all system of m-linear equations in n-variables. Further Fox and Kleitman [36] proved that it is enough to prove Conjecture 1.2.2 for a linear homogeneous equation. More precisely, they proved the following.

Theorem 1.2.4 *[36] Suppose for each positive integer n there exists a positive integer $k(n)$, depending only on n, such that if any n-variable linear homogeneous equation L is non-regular, then L is not $k(n)$-regular. Then Conjecture 1.2.2 is true.*

In other words, if L is a non-regular n-variable linear homogeneous equation, then

$$dor(L) \leq k(n) - 1.$$

Fox and Kleitman [36] have proved the conjecture for $n = 3$. In particular they have proved the following theorem.

Theorem 1.2.5 *Let a_1, a_2 and a_3 be non-zero integers and $L : a_1x_1 + a_2x_2 + a_3x_3 = 0$ a non-regular equation. Then*

$$dor(L) \leq 23.$$

In other words, for every non-zero integers a_1, a_2 and a_3, if $L : a_1x_1 + a_2x_2 + a_3x_3 = 0$ is 24-regular then L is regular.

In his thesis [73], Rado also made the following conjecture about the exact degree of regularity of linear equations.

Conjecture 1.2.6 For each positive integer r, there exists a linear equation which is $(r-1)$-regular but not r-regular. In other words, for each positive integer r there exists a linear equation L such that $dor(L) = r - 1$.

This conjecture was open for a long time until it was proved by Alexeev and Tsimerman [17]. Precisely they proved the following:

Theorem 1.2.7 *For each positive integer r, the equation L_r*

$$\sum_{i=1}^{r-1} \frac{2^i}{2^i - 1} x_i = \left(-1 + \sum_{i=1}^{r-1} \frac{2^i}{2^i - 1} \right) x_0$$

is $(r-1)$-regular but not r-regular.

Prior to Alexeev and Tsimerman's proof, Fox and Radoičić [37] proved that the equation

$$x_1 + 2x_2 + \cdots + 2^{r-2}x_{r-1} = 2^{r-1}x_r$$

is not r-regular. Fox and Radoičić [37] also conjectured that,

Conjecture 1.2.8 For each positive integer r, the equation

$$x_1 + 2x_2 + \cdots + 2^{r-2}x_{r-1} = 2^{r-1}x_r$$

is $(r-1)$-regular.

Conjecture 1.2.8 was proved by Alexeev, Fox and Graham in [16] for each positive integer $r \leq 7$. Later Golowich [51] proved this conjecture for all r. Therefore by combining the results of Fox and Radoičić and Golowich, we get an alternative proof of Conjecture 1.2.6.

In [36], Fox and Kleitman also considered similar type of problems of determining the exact degree of regularity of some linear equations. In particular, they proved the following theorem.

Theorem 1.2.9 *If b and n are positive integers, then the equation $L_n(b)$: $(x_1 - y_1) + (x_2 - y_2) + \cdots + (x_n - y_n) = b$ is not $2n$-regular.*

They further conjectured that their theorem is tight in the sense that,

Conjecture 1.2.10 [36] For each positive integer n, there exists a positive integer b_n such that the equation $(x_1 - y_1) + (x_2 - y_2) + \cdots + (x_n - y_n) = b_n$ is $(2n-1)$-regular. In other words, $\max\limits_{b \in \mathbb{N}} dor(L_n(b)) = 2n - 1$.

For more information about this conjecture we refer to the articles [4, 9, 88]. Also Conjecture 1.2.10 is proved in [88] recently.

1.3 Non-linear equations

Let n and B be two positive integers. We consider the non-linear Diophantine equation

$$\mathsf{Q}_n(B) : (x_1 - y_1)(x_2 - y_2) \ldots (x_n - y_n) = B.$$

Put $s = \lfloor \sqrt[n]{B} \rfloor + 1$, and consider the coloring of $\mathbb{N}$ by modulo s-coloring. Then we claim that $\mathsf{Q}_n(B)$ has no monochromatic solution under this coloring. If possible, suppose that $x_1, y_1, \ldots, x_n, y_n$ are of same color under the modulo coloring and satisfy the equation $\mathsf{Q}_n(B)$. Then $x_1, y_1, \ldots, x_n, y_n$ are all congruent mod s, and hence $(x_1 - y_1)(x_2 - y_2)\ldots(x_n - y_n)$ is divisible by s^n. Therefore $(x_1 - y_1)(x_2 - y_2)\ldots(x_n - y_n)$ cannot be equal to B since $s^n > B$. For every $r \geq 1$, our aim is to produce some $B = B(r)$ for which $\mathsf{Q}_n(B)$ is r-regular.

When $n = 2$ the equation

$$\mathsf{Q}_2(B): \qquad (x_1 - y_1)(x_2 - y_2) = B$$

was considered in [5]. More precisely, it is proved that

Theorem 1.3.1 *Let r be a positive integer and $B = B(r) = N!r!$, where $N = M(r! + 1, \frac{1}{r})$ is the Szemerédi number corresponding to $k = r! + 1$ and $\delta = \frac{1}{r}$ as defined in Theorem 1.1.11. Then the equation $\mathsf{Q}_2(B)$ is r-regular. In particular,*

$$\max_{B \in \mathbb{N}} dor(\mathsf{Q}_2(B)) = \infty.$$

Note that $\max\limits_{B \in \mathbb{N}} dor(L_2(B)) = 3$, where $L_2(B)$ is the additive analogue of $\mathsf{Q}_2(B)$.

Theorem 1.3.1 can be easily generalized as follows.

Corollary 1.3.2 *Let r and k be two positive integers. Then the equation*

$$\mathsf{Q}_{2k}(B): (x_1 - y_1)(x_2 - y_2)\ldots(x_{2k} - y_{2k}) = B$$

with $B = (N!r!)^k$ is r-regular where $N = M(r! + 1, \frac{1}{r})$.

Proof. By Theorem 1.3.1, the equation $\mathsf{Q}_2(B')$ where $B' = N!r!$ is r-regular.

Since $\mathsf{Q}_{2k}(B)$ has k number of pairs, by applying Theorem 1.3.1 to each pair, we get the desired result. $\square$

Now we state our main results of this chapter as follows.

Theorem 1.3.3 *Let r and k be two positive integers and $N = M(r\,! + 1, \frac{1}{r})$ the Szemerédi number. Then the Diophantine equation $(x_1 - y_1)(x_2 - y_2)\dots(x_{2k+1} - y_{2k+1}) = (N\,!)^{2k} r\,!$ is r-regular.*

A generalization of Theorem 1.3.3 is considered in the next theorem, where we show that,

Theorem 1.3.4 *Let r, k_1, l_1, k_2 and l_2 be positive integers and also let $a_1, \dots a_{k_1}$, $b_1, \dots b_{l_1}$, $c_1, \dots c_{k_2}$ and $d_1, \dots d_{l_2}$ be positive integers such that*

$$A = \sum_{i=1}^{k_1} a_i = \sum_{j=1}^{l_1} b_j \text{ and } B = \sum_{i=1}^{k_2} c_i = \sum_{j=1}^{l_2} d_j.$$

Then, by taking $N = M(r\,! + 1, \frac{1}{r})$, the equation

$$\left(\sum_{i=1}^{k_1} a_i x_i - \sum_{j=1}^{l_1} b_j y_j\right)\left(\sum_{i=1}^{k_2} c_i z_i - \sum_{j=1}^{l_2} d_j w_j\right) = ABN\,!r\,!$$

is r-regular.

We also have the following corollary.

Corollary 1.3.5 *Let r, n, k_m and l_m be positive integers and $N = M(r\,! + 1, \frac{1}{r})$ the Szemerédi number. Then the equation*

$$\prod_{m=1}^{n}\left(\sum_{i=1}^{k_m} a_{m,i} x_{m,i} - \sum_{j=1}^{l_m} b_{m,j} y_{m,j}\right) = \prod_{m=1}^{n} A_m (N\,!)^{n-1} r\,! \qquad (1.3)$$

with

$$A_m = \sum_{i=1}^{k_m} a_{m,i} = \sum_{j=1}^{l_m} b_{m,j} \qquad \text{for all } m = 1, \ldots, n$$

is r-regular.

We also prove the r-regularity of the equation (1.3) but for a smaller B than in Corollary 1.3.5. This shows that the choice of B is not unique.

Theorem 1.3.6 *Let r, n, k_m and l_m be positive integers and $N = M(r\,! + 1, \frac{1}{r})$ the Szemerédi number. Then the equation*

$$\prod_{m=1}^{n} \left(\sum_{i=1}^{k_m} a_{m,i} x_{m,i} - \sum_{j=1}^{l_m} b_{m,j} y_{m,j} \right) = \prod_{m=1}^{n} c_{m,1} (N\,!)^{n-1} r\,!$$

with

$$A_m = \sum_{i=1}^{k_m} a_{m,i} = \sum_{j=1}^{l_m} b_{m,j} = B_m \qquad \text{for all } m = 1, \ldots, n$$

is r-regular, where $c_{m,1} = min\,\{a_{m,1}, a_{m,2}, \ldots, a_{m,k_m}, b_{m,1}, b_{m,2}, \ldots, b_{m,l_m}\}$ for all $m = 1, 2, \ldots, n$.

1.4 Preliminaries

We need the following elementary lemma about the preservation of density under partitions of finite sets.

Lemma 1.4.1 *Let A and E be two non-empty finite sets with $A \subseteq E$ and $\delta = \frac{|A|}{|E|}$ the density of A in E. Let $E = E_1 \sqcup \ldots \sqcup E_r$ be an r-partition of E. Then there exists an index i, $1 \leq i \leq r$ such that $\frac{|A \cap E_i|}{|E_i|} \geq \delta$.*

Proof. Assume for a contradiction that $\frac{|A \cap E_i|}{|E_i|} < \delta$ for each $i = 1, 2, \ldots, r$. The partition of E gives

$$A = A \cap E = (A \cap E_1) \sqcup \ldots \sqcup (A \cap E_r).$$

Therefore $|A| = \sum_{i=1}^{r} |A \cap E_i| < \sum_{i=1}^{r} \delta |E_i| = \delta |E| = |A|$, a contradiction. $\square$

1.5 Proof of Main Theorems

1.5.1 Proof of Theorem 1.3.3

We prove the result when $k = 1$. The other cases follow in the similar line of arguments.

We prove that for every positive integer r the equation

$$(x_1 - y_1)(x_2 - y_2)(x_3 - y_3) = (N\,!)^2 r\,!$$

is r-regular.

Let $\Delta : E = [1, (r+1)(N\,!)^2] \to \{c_1 \ldots c_r\}$ be an arbitrary r-coloring of the interval E. Then there exists a monochromatic subset S of E such that $\frac{|S|}{|E|} \geq \frac{1}{r}$.

To finish the proof, we show that

$$(N\,!)^2 r\,! \in (S - S)(S - S)(S - S).$$

Now we divide E into subintervals of length N. Since $\frac{|S|}{|E|} \geq \frac{1}{r}$, by Lemma 1.4.1, there exists one subinterval A of E of length N such that $\frac{|S \cap A|}{|A|} = \frac{|S \cap A|}{N} \geq \frac{1}{r}$. Since $N = M(r\,! + 1, \frac{1}{r})$, by the definition, there exist two natural numbers s and d such that the arithmetic progression $\{s, s+d, \ldots, s + r\,!d\}$ is contained in $S \cap A$. Since A is an interval of length N, we conclude that $d \leq N$. Also we get $[1, r\,!]d \subset S - S$, and in particular $d \in (S - S)$.

Let $m = \frac{(N!)^2}{d^2}$ and consider the partition of E mod m. That is

$$E = E_0 \sqcup E_1 \sqcup \ldots \sqcup E_{m-1}$$

where $E_i = \{\ell \in E | \ell \equiv i \pmod{m}\}$ for all $i = 0, 1, \ldots, m-1$.

Clearly, $|E_i| \geq \frac{|E|}{m} = \frac{(r+1)(N!)^2}{(N!)^2/d^2} = (r+1)d^2$ for all i.

Again, we divide each E_i into d^2 sub parts and hence

$$E = \bigsqcup_{i=1}^{d^2} E_{0,i} \sqcup \bigsqcup_{i=1}^{d^2} E_{1,i} \sqcup \ldots \sqcup \bigsqcup_{i=1}^{d^2} E_{m-1,i}$$

where

$$E_{0,i} = \{((i-1)r+i)m, ((i-1)r+i+1)m, ((i-1)r+i+2)m, \ldots, ((i-1)r+i+r)m\}$$

for all $1 \leq i \leq d^2$, and

$$E_{a,i} = \{a+((i-1)r+i-1)m, a+((i-1)r+i)m, a+((i-1)r+i+1)m, \ldots, a+((i-1)r+i+r-1)m\}$$

for all $1 \leq a \leq m-1$ and for all $1 \leq i \leq d^2$.

Since $S \subset E$ and $\frac{|S|}{|E|} \geq \frac{1}{r}$, by Lemma 1.4.1, there exist integers $0 \leq a \leq m-1$ and $1 \leq i \leq d^2$ such that

$$\frac{|S \cap E_{a,i}|}{|E_{a,i}|} \geq \frac{1}{r} \Leftrightarrow |S \cap E_{a,i}| \geq \frac{|E_{a,i}|}{r} = \frac{r+1}{r} > 1.$$

Since $|S \cap E_{a,i}| > 1$, there exist ℓ_1' and ℓ_2' with $\ell_1' < \ell_2'$ such that $\{a + \ell_1' m, a + \ell_2' m\} \subset S \cap E_{a,i}$. Clearly

$$a + \ell_1' m = a + (ir + \ell_1)m \text{ and } a + \ell_2' m = a + (ir + \ell_2)m$$

where $0 \leq \ell_1 < \ell_2 \leq r$. Now we put $b = a + irm$ to get $\{b + \ell_1 m, b + \ell_2 m\} \subset S \cap E_{a,i}$ where $0 \leq \ell_1 < \ell_2 \leq r$.

Let $u = (\ell_2 - \ell_1) \in [1, r]$. Then $\frac{r\,!}{u} \in [1, r\,!]$, and since $[1, r\,!]d \subset S - S$, we get $\frac{r\,!}{u} d \in (S - S)$. Since $b + \ell_1 m$ and $b + \ell_2 m \in S$, we see that $(\ell_2 - \ell_1)m = um = u\frac{(N\,!)^2}{d^2} \in S - S$.

Therefore $(N\,!)^2 r\,! = u\frac{(N\,!)^2}{d^2} \centerdot \frac{r\,!}{u} d \centerdot d \in (S - S)(S - S)(S - S)$. Since every element of S is of same color, we have a monochromatic solution to the equation $(x_1 - y_1)(x_2 - y_2)(x_3 - y_3) = (N\,!)^2 r\,!$. $\square$

1.5.2 Proof of Theorem 1.3.4.

We prove that for every positive integer r, the equation

$$\left(\sum_{i=1}^{k_1} a_i x_i - \sum_{j=1}^{l_1} b_j y_j\right)\left(\sum_{i=1}^{k_2} c_i z_i - \sum_{j=1}^{l_2} d_j w_j\right) = ABN\,!r\,!$$

with

$$A = \sum_{i=1}^{k_1} a_i = \sum_{j=1}^{l_1} b_j \text{ and } B = \sum_{i=1}^{k_2} c_i = \sum_{j=1}^{l_2} d_j,$$

is r-regular.

Consider an arbitrary r-coloring $\Delta : E = [1, (r+1)(N\,!)] \to \{c_1 \ldots c_r\}$. Then there is a monochromatic subset S of E such that $\frac{|S|}{|E|} \geq \frac{1}{r}$. We show that the equation has a solution in S.

Now we divide E into subintervals of length N. Since $\frac{|S|}{|E|} \geq \frac{1}{r}$, by Lemma 1.4.1, there exists one subinterval A of E of length N such that $\frac{|S \cap A|}{|A|} = \frac{|S \cap A|}{N} \geq \frac{1}{r}$. Since $N = M(r\,! + 1, \frac{1}{r})$, by the definition, there exist two natural numbers s and d such that the arithmetic progression $\{s, s + d, \ldots, s + r\,!d\}$ is contained in $S \cap A$. Since A is an interval of length N, we conclude that $d \leq N$. Also we get $s + [1, r\,!]d \subset S$.

Let $m = \frac{(N!)}{d}$ and consider the partition of E mod m. That is

$$E = E_0 \sqcup E_1 \sqcup \ldots \sqcup E_{m-1}$$

where $E_i = \{\ell \in E | \ell \equiv i \pmod{m}\}$ for all $i = 0, 1, \ldots, m-1$.

Clearly, $|E_i| \geq \frac{|E|}{m} = \frac{(r+1)(N!)}{(N!)/d} = (r+1)d$ for all i.

Again, we divide each E_i into d sub parts and hence

$$E = \bigsqcup_{i=1}^{d} E_{0,i} \sqcup \bigsqcup_{i=1}^{d} E_{1,i} \sqcup \ldots \sqcup \bigsqcup_{i=1}^{d} E_{m-1,i}$$

where

$$E_{0,i} = \{((i-1)r+i)m, ((i-1)r+i+1)m, ((i-1)r+i+2)m, \ldots, ((i-1)r+i+r)m\}$$

for all $1 \leq i \leq d$, and

$$E_{a,i} = \{a+((i-1)r+i-1)m, a+((i-1)r+i)m, a+((i-1)r+i+1)m, \ldots, a+((i-1)r+i+r-1)m\}$$

for all $1 \leq a \leq m-1$ and for all $1 \leq i \leq d$.

By the same argument given in the proof of Theorem 1.3.3, there exist positive integers b, ℓ_1 and ℓ_2 such that $0 \leq \ell_1 < \ell_2 \leq r$ and $\{b + \ell_1 m, b + \ell_2 m\} \subset S \cap E_{a,i}$, for some integers a and i satisfying $0 \leq a \leq m-1$ and $1 \leq i \leq d$.

Let $t = (\ell_2 - \ell_1) \in [1, r]$. Then $\frac{r!}{t} \in [1, r!]$, and since $s + [1, r!]d \subset S$, we get $s + \frac{r!}{t}d \in S$.

Now we find x_i, y_j, z_i and w_j taking values from S and satisfying our desired equation. For showing this, put

$$x_i \quad = \quad s + \frac{r!}{t}d \qquad \text{for } i = 1, \ldots, k_1,$$

$$\begin{aligned} y_j &= s && \text{for } j = 1, \ldots, l_1, \\ z_i &= b + \ell_2 m && \text{for } i = 1, \ldots, k_2, \\ w_j &= b + \ell_1 m && \text{for } j = 1, \ldots, l_2. \end{aligned}$$

Using the given hypothesis

$$A = \sum_{i=1}^{k_1} a_i = \sum_{j=1}^{l_1} b_j \text{ and } B = \sum_{i=1}^{k_2} c_i = \sum_{j=1}^{l_2} d_j,$$

we get that

$$\begin{aligned} \left(\sum_{i=1}^{k_1} a_i x_i \right. &- \left. \sum_{j=1}^{l_1} b_j y_j\right) \left(\sum_{i=1}^{k_2} c_i z_i - \sum_{j=1}^{l_2} d_j w_j\right) \\ &= \left(\sum_{i=1}^{k_1} a_i (s + \frac{r\,!}{t} d) - \sum_{j=1}^{l_1} b_j s\right) \left(\sum_{i=1}^{k_2} c_i (b + \ell_2 m) - \sum_{j=1}^{l_2} d_j (b + \ell_1 m)\right) \\ &= A \left(\frac{r\,!}{t}\right) d \,.\, B(\ell_2 - \ell_1) m \\ &= A \left(\frac{r\,!}{t}\right) d \,.\, Bt \left(\frac{N\,!}{d}\right) \\ &= ABN\,! r\,! \end{aligned}$$

This proves the theorem. □

1.5.3 Proof of Theorem 1.3.6.

We prove the result when $n = 2$. The general case follows in the similar line of arguments. We prove that for every positive integer r, the equation

$$\left(\sum_{i=1}^{k_1} a_{1,i} x_{1,i} - \sum_{j=1}^{l_1} b_{1,j} y_{1,j}\right) \left(\sum_{i=1}^{k_2} a_{2,i} x_{2,i} - \sum_{j=1}^{l_2} b_{2,j} y_{2,j}\right) = c_{1,1} c_{2,1} N\,! r\,!$$

is r-regular, where $c_{1,1} = \min\{a_{1,1}, a_{1,2}, \ldots, a_{1,k_1}, b_{1,1}, b_{1,2}, \ldots, b_{1,l_1}\}$ and $c_{2,1} = \min\{a_{2,1}, a_{2,2}, \ldots, a_{2,k_2}, b_{2,1}, b_{2,2}, \ldots, b_{2,l_2}\}$.

Consider an arbitrary r-coloring $\Delta : E = [1, (r+1)(N!)] \to \{c_1 \ldots c_r\}$. Then there is a monochromatic subset S of E such that $\frac{|S|}{|E|} \geq \frac{1}{r}$. We show that the equation has a solution in S.

Now we divide E into subintervals of length N. Since $\frac{|S|}{|E|} \geq \frac{1}{r}$, by Lemma 1.4.1, there exists one subinterval A of E of length N such that $\frac{|S \cap A|}{N} \geq \frac{1}{r}$. Since $N = M(r! + 1, \frac{1}{r})$, by the definition, there exist two natural numbers s and d such that the arithmetic progression $\{s, s+d, \ldots, s+r!d\}$ is contained in $S \cap A$. Since A is an interval of length N, we conclude that $d \leq N$. Also we get $s + [1, r!]d \subset S$.

Let $m = \frac{(N!)}{d}$ and consider the partition of E mod m. Again by a similar argument as in the proof of Theorems 1.3.3 and 1.3.4, there exist positive integers b, ℓ_1 and ℓ_2 such that $0 \leq \ell_1 < \ell_2 \leq r$ and $\{b + \ell_1 m, b + \ell_2 m\} \subset S$.

Let $t = (\ell_2 - \ell_1) \in [1, r]$. Then $\frac{r!}{t} \in [1, r!]$. Since $s + [1, r!]d \subset S$, we get $s + \frac{r!}{t}d \in S$.

Now we find $x_{m,i}$'s and $y_{m,j}$'s which take values from S and satisfy our desired equation.

For showing this, put

$$
\begin{array}{rcll}
x_{1,1} & = & s + \dfrac{r!}{t}d, & \\
x_{1,i} & = & s & \text{for } i = 2, \ldots, k_1, \\
y_{1,j} & = & s & \text{for } j = 1, \ldots, l_1, \\
x_{2,i} & = & b + \ell_2 m & \text{for } i = 1, \ldots, k_2, \\
y_{2,1} & = & b + \ell_1 m & \text{and} \\
y_{2,j} & = & b + \ell_2 m & \text{for } j = 2, \ldots, l_2
\end{array}
$$

Now without loss of generality we can assume that $c_{1,1} = a_{1,1}$ and $c_{2,1} = b_{2,1}$, because this choice covers all the cases. Also since $A_m = B_m$ for all m, we can write

$$\begin{aligned}
\left(\sum_{i=1}^{k_1} a_{1,i}x_{1,i}\right. &- \left.\sum_{j=1}^{l_1} b_{1,j}y_{1,j}\right)\left(\sum_{i=1}^{k_2} a_{2,i}x_{2,i} - \sum_{j=1}^{l_2} b_{2,j}y_{2,j}\right) \\
&= \left(a_{1,1}x_{1,1} - a_{1,1}y_{1,1} + \sum_{i=2}^{k_1} a_{1,i}x_{1,i} - (b_{1,1} - a_{1,1})y_{1,1} - \sum_{j=2}^{l_1} b_{1,j}y_{1,j}\right) \\
&\quad \left(\sum_{i=2}^{k_2} a_{2,i}x_{2,i} + b_{2,1}x_{2,1} - b_{2,1}y_{2,1} - \sum_{j=2}^{l_2} b_{2,j}y_{2,j} + (a_{2,1} - b_{2,1})x_{2,1}\right) \\
&= \left(a_{1,1}\left(s + \frac{r\,!}{t}d\right) - a_{1,1}s + \sum_{i=2}^{k_1} a_{1,i}s - (b_{1,1} - a_{1,1})s - \sum_{j=2}^{l_1} b_{1,j}s\right) \\
&\quad \left(\sum_{i=2}^{k_2} a_{2,i}(b + \ell_2 m) + b_{2,1}(b + \ell_2 m) - b_{2,1}(b + \ell_1 m)\right. \\
&\quad \left. - \sum_{j=2}^{l_2} b_{2,j}(b + \ell_2 m) + (a_{2,1} - b_{2,1})(b + \ell_2 m)\right) \\
&= a_{1,1}\left(\frac{r\,!}{t}d\right) \cdot b_{2,1}(\ell_2 - \ell_1)m \\
&= a_{1,1}\left(\frac{r\,!}{t}d\right) \cdot b_{2,1}t\left(\frac{N\,!}{d}\right) \\
&= c_{1,1}c_{2,1}N\,!r\,!.
\end{aligned}$$

This proves the theorem. $\square$

CHAPTER 2

Generalization of some weighted zero-sum theorems

Let G be a finite abelian group of exponent n and let A be a non-empty subset of $[1, n-1]$. The Davenport constant with weight A, denoted by $D_A(G)$, and is defined to be the least positive integer ℓ such that any sequence over G of length ℓ has a non-empty A-weighted zero-sum subsequence. The constant $\mathsf{E}_A(G)$ is defined to be the least positive integer ℓ such that any sequence over G of length ℓ has an A-weighted zero-sum subsequence of length $|G|$. In this chapter, we determine an upper bound of $D_A(\mathbb{Z}/n\mathbb{Z})$ and $\mathsf{E}_A(\mathbb{Z}/n\mathbb{Z})$ where A is the set of all cubes in $(\mathbb{Z}/n\mathbb{Z})^*$. The content of this chapter is in the article [87].

2.1 Definitions and early results

Let G be a finite abelian group (written additively) and $\exp(G)$ the *exponent* of the group G. We denote the free abelian monoid with basis G by $\mathcal{F}(G)$. That is, every element $S \in \mathcal{F}(G)$ has a unique representation of the form

$$S = \prod_{g\in G} g^{\mathsf{v}_g(S)} \qquad \text{with } \mathsf{v}_g(S) \in \mathbb{N}_0.$$

An element S of $\mathcal{F}(G)$ is called a *sequence over* G (here our notation is consistent with [42, 48, 49]).

Definition 2.1.1 Let $S = g_1 g_2 \ldots g_t = \prod_{g\in G} g^{\mathsf{v}_g(S)}$ be a sequence over G. Then $\mathsf{v}_g(S)$ is called the *multiplicity of* g *in* S and $|S| = t = \sum_{g\in G} \mathsf{v}_g(S)$ is called the *length of* S.

Definition 2.1.2 Let S and T be two sequences over G. Then T is said to be a *subsequence of* S if $\mathsf{v}_g(T) \le \mathsf{v}_g(S)$ for all $g \in G$. We denote a subsequence T of S by $T \mid S$.

For a subsequence T of S, we denote the sequence obtained after deleting T from S by ST^{-1}.

Definition 2.1.3 Let $S = g_1 g_2 \ldots g_t = \prod_{g\in G} g^{\mathsf{v}_g(S)}$ be a sequence over G. Then

$$\sigma(S) = \sum_{i=1}^{t} g_i = \sum_{g\in G} \mathsf{v}_g(S) g \in G$$

is said to be *the sum* of S. A sequence S over G is said to be *a zero-sum sequence* if $\sigma(S) = 0$, where 0 is the identity element of G. We denote the monoid of *zero-sum sequences over* G by $\mathcal{B}(G) = \{S \in \mathcal{F}(G) \mid \sigma(S) = 0\}$.

2.1.1 Classical zero-sum constants

In this section, we define some classical combinatorial invariants that are attached to a finite abelian group G and state some of their results. We first define the *Davenport constant* $D(G)$ of a finite abelian group G. Though this constant is named after Davenport, it was first studied by Rogers [78] in 1963.

Definition 2.1.4 Let G be a finite abelian group. The *Davenport constant* $D(G)$ is defined to be the least positive integer ℓ such that any sequence over G of length ℓ has a non-empty zero-sum subsequence.

The motivation to study this constant is factorization in algebraic number fields (see [78]). Let K be an algebraic number field with the ring of integers $\mathcal{O}_K$ and the class group $\mathcal{C}_K$. Let $x \in \mathcal{O}_K$ be an irreducible element. Since $\mathcal{O}_K$ is a Dedekind domain, we can factor the principal ideal $x\mathcal{O}_K$ into a product of finitely many prime ideals $\mathcal{P}_i$'s in $\mathcal{O}_K$, say

$$x\mathcal{O}_K = \mathcal{P}_1\mathcal{P}_2 \ldots \mathcal{P}_k.$$

Then, it is known that the length k in the decomposition of any principal ideal $x\mathcal{O}_K$ generated by an irreducible element x into prime ideals is bounded above by the Davenport constant of the ideal class group $\mathcal{C}_K$.

Later this constant has found important roles in graph theory (see [21], [28] and [42]) and in the proof of the infinitude of Carmichael numbers by Alford, Granville and Pomerance [18].

By the structure theorem of finite abelian groups, we get $G \cong (\mathbb{Z}/n_1\mathbb{Z}) \times (\mathbb{Z}/n_2\mathbb{Z}) \times \cdots \times (\mathbb{Z}/n_d\mathbb{Z})$ for some $n_1, n_2, \ldots, n_d$ with $n_1 \mid n_2 \mid \cdots \mid n_d$. If we write $D^*(G) = 1 + \sum_{i=1}^{d}(n_i - 1)$, then it is easy to see that $D^*(G) \leq D(G) \leq |G|$. The equality $D(G) = |G|$ holds if and only if G is a cyclic group. Olson ([71], [72]) proved that $D(G) = D^*(G)$ for all finite abelian groups of rank 2 and for all p-groups. It is also known that $D(G) > D^*(G)$ for infinitely many finite abelian groups G of rank $d > 3$ (see [50]).

For a finite abelian group G of exponent n, Emde Boas and Kruyswijk [27] proved that

$$D(G) \leq n\left(1 + \log\frac{|G|}{n}\right). \tag{2.1}$$

This was again proved by Alford, Granville and Pomerance [18], Meshulam [66] and Rath, Srilakshmi and Thangadurai [74].

We have the following conjectures in the literature.

1. For a finite abelian group G of rank 3 and for the group $G = (\mathbb{Z}/n\mathbb{Z})^d$, it is conjectured that $D(G) = D^*(G)$ ([39] and [41]).

2. For the group $G = (\mathbb{Z}/n_1\mathbb{Z}) \times (\mathbb{Z}/n_2\mathbb{Z}) \times \cdots \times (\mathbb{Z}/n_d\mathbb{Z})$ with $n_1|n_2|\cdots|n_d$, it is conjectured that $D(G) \leq \sum_{i=1}^{d} n_i$ ([69]).

In the direction of Conjecture 2, the following upper bound for the Davenport constant was observed in [31].

Theorem 2.1.5 *Let $G = (\mathbb{Z}/n_1\mathbb{Z}) \times (\mathbb{Z}/n_2\mathbb{Z}) \times \cdots \times (\mathbb{Z}/n_d\mathbb{Z})$ be a finite abelian group with $n_1 \mid n_2 \mid \cdots \mid n_d$. Then*

$$D(G) \leq n_d + n_{d-1} + (c(3)-1)n_{d-2} + (c(4)-1)n_{d-3} + \cdots + (c(d)-1)n_1 + 1,$$

where $c(i)$ is the Alon-Dubiner constant (see [19]), which depends only on the rank of the group $(\mathbb{Z}/n_d\mathbb{Z})^i$.

We now define the *Erdős-Ginzburg-Ziv constant* $\mathsf{s}(G)$ for a finite abelian group G.

Definition 2.1.6 Let G be a finite abelian group with $\exp(G) = n$. The *Erdős-Ginzburg-Ziv constant* $\mathsf{s}(G)$ is defined to be the least positive integer ℓ such that any sequence over G of length ℓ has a zero-sum subsequence of length n.

The first prototype theorem in the zero-sum area is the following theorem of Erdős-Ginzburg-Ziv [34] which can be stated as follows.

Theorem 2.1.7 $\mathsf{s}(\mathbb{Z}/n\mathbb{Z}) = 2n - 1$.

They only proved $\mathsf{s}(\mathbb{Z}/n\mathbb{Z}) \leq 2n - 1$, that is, any sequence over $\mathbb{Z}/n\mathbb{Z}$ of length $(2n - 1)$ has a subsequence of length n whose sum is zero in $\mathbb{Z}/n\mathbb{Z}$. Also, the sequence $\underbrace{0 \cdot 0 \cdot \ldots \cdot 0}_{n-1} \cdot \underbrace{1 \cdot 1 \cdot \ldots \cdot 1}_{n-1}$ over $\mathbb{Z}/n\mathbb{Z}$ of length $(2n - 2)$ has no subsequence of length n whose sum is zero in $\mathbb{Z}/n\mathbb{Z}$. Therefore, we get $\mathsf{s}(\mathbb{Z}/n\mathbb{Z}) \geq 2n - 1$.

For the group $G = (\mathbb{Z}/n\mathbb{Z})^2$, Kemnitz [61] had conjectured that $\mathsf{s}(G) = 4n - 3$. In 2000, using a polynomial method, Rónyai [79] came very close to it by proving that $\mathsf{s}((\mathbb{Z}/p\mathbb{Z})^2) \leq 4p-2$ and then Gao proved $\mathsf{s}((\mathbb{Z}/p^\ell\mathbb{Z})^2) \leq 4p^\ell - 2$, for a prime p. Finally in 2007, Reiher [75] proved the conjecture.

Till now the exact value of the constant $\mathsf{s}((\mathbb{Z}/n\mathbb{Z})^d)$ where $d \geq 3$ is unknown. However, for all positive integers n and d, we have the following trivial bound.

$$(n-1)2^d + 1 \leq \mathsf{s}((\mathbb{Z}/n\mathbb{Z})^d) \leq (n-1)n^d + 1.$$

For all odd integers $n \geq 3$, Elsholtz [33] proved a lower bound as

$$\mathsf{s}((\mathbb{Z}/n\mathbb{Z})^d) \geq (1 \cdot 125)^{\lfloor \frac{d}{3} \rfloor}(n-1)2^d + 1.$$

Thus for $d \geq 3$, this is an improvement of the trivial lower bound.

In the other direction, Alon and Dubiner [19] proved that there is an absolute constant $c > 0$ such that for all n and d,

$$\mathsf{s}((\mathbb{Z}/n\mathbb{Z})^d) \leq (cd \log_2 d)^d n.$$

However, it has been conjectured [19] that there is an absolute constant c such that for all n and d,

$$\mathsf{s}((\mathbb{Z}/n\mathbb{Z})^d) \leq c^d n.$$

For further readings in this direction we refer to the articles [19, 28, 32, 42]. For more information about the Davenport constant and its application, we refer to the excellent monographs [48] and [49].

We end this section with the definition of another combinatorial constant.

Definition 2.1.8 Let G be a finite abelian group. The constant $\mathsf{E}(G)$ is defined to be the least positive integer ℓ such that any sequence over G of length ℓ has a zero-sum subsequence of length $|G|$.

By Theorem 2.1.7, it follows that $\mathsf{E}(\mathbb{Z}/n\mathbb{Z}) = 2n - 1$. Moreover, Gao [38] proved that, for a finite abelian group G the constants $D(G)$ and $\mathsf{E}(G)$ are related by the relation

$$\mathsf{E}(G) = D(G) + |G| - 1. \tag{2.2}$$

Thus for a finite abelian group G, if we want to compute the exact value of $\mathsf{E}(G)$ then it is enough to compute the exact value of $D(G)$.

2.1.2 Weighted zero-sum constants

In this section, we define weighted generalization of the above zero-sum constants. These generalizations were first considered (see [6], [7], [13], [96]) about twelve years back, which became popular (see [54], [55], [63], [98]) and the results have found some applications (see [56], [64]) as well. First we define the following.

Definition 2.1.9 Let G be a finite abelian group of exponent n and let A be a non-empty subset of $[1, n-1]$. A sequence $S = g_1 g_2 \dots g_k \in \mathcal{F}(G)$ is said to be an *A-weighted zero-sum sequence* if there exist $a_1, \dots, a_k$ in A such that

$$\sum_{i=1}^{k} a_i g_i = 0.$$

Definition 2.1.10 Let G be a finite abelian group of exponent n and let A be a non-empty subset of $[1, n-1]$. The *Davenport constant of G with weight A*, denoted by $D_A(G)$, is defined to be the least positive integer ℓ such that any sequence over G of length ℓ has a non-empty A-weighted zero-sum subsequence.

Like the classical Davenport constant, Halter-Koch [56] found that the Davenport constant with weight $\{\pm 1\}$ is related to the norms of principal ideals in quadratic number field.

Theorem 2.1.11 *Let K be a quadratic number field with $\mathcal{O}_K$ and $\mathcal{C}_K$ be its ring of integers and the class group respectively. Then $D_{\{\pm 1\}}(\mathcal{C}_K)$ is the smallest positive integer ℓ with the property that, if $q_1, q_2, \dots, q_\ell$ are pairwise coprime positive integers such that their product $q = q_1 q_2 \cdots q_\ell$ is the norm of an ideal of $\mathcal{O}_K$, then there exists a divisor $d > 1$ of q which is the norm of a principal ideal of $\mathcal{O}_K$.*

Notice that when $A = \{1\}$, the constant $D_A(G)$ is the classical Davenport constant $D(G)$. Similarly, for a finite abelian group G of exponent n and a non-empty subset A of $[1, n-1]$, we define the constants $\mathsf{s}_A(G)$ and $\mathsf{E}_A(G)$ (as introduced in [3], [7] and [96]) as follows.

Definition 2.1.12 Let G be a finite abelian group of exponent n and let A be a non-empty subset of $[1, n-1]$. The *weighted Erdős-Ginzburg-Ziv constant* $\mathsf{s}_A(G)$ is defined to be the least integer ℓ such that any sequence over G of length ℓ has an A-weighted zero-sum subsequence of length n.

Definition 2.1.13 Let G be a finite abelian group of exponent n and let A be a non-empty subset of $[1, n-1]$. The constant $\mathsf{E}_A(G)$ is defined to be the least integer ℓ such that any sequence over G of length ℓ has an A-weighted zero-sum subsequence of length $|G|$.

Once again, when $A = \{1\}$, the constants $\mathsf{s}_A(G)$ and $\mathsf{E}_A(G)$ are the constants $\mathsf{s}(G)$ and $\mathsf{E}(G)$ respectively.

In [7], by introducing $\mathsf{E}_A(G)$ and $D_A(G)$ for the group $\mathbb{Z}/n\mathbb{Z}$, it was proved that

Theorem 2.1.14 *If $A = \{\pm 1\}$, then $D_A(\mathbb{Z}/n\mathbb{Z}) = 1 + \lfloor \log_2 n \rfloor$ and $\mathsf{E}_A(\mathbb{Z}/n\mathbb{Z}) = \mathsf{s}_A(\mathbb{Z}/n\mathbb{Z}) = n + \lfloor \log_2 n \rfloor$, where for a real number x, $\lfloor x \rfloor$ denotes the largest integer $\leq x$.*

Therefore the weighted generalization of Gao's relation (2.2) namely,

$$\mathsf{E}_A(G) = D_A(G) + |G| - 1 \tag{2.3}$$

holds true for the group $G = \mathbb{Z}/n\mathbb{Z}$ and the weight set $A = \{\pm 1\}$. For every finite abelian group G and weight set A, this relation has been expected by Adhikari and Rath [13] and conjectured by Thangadurai [96]. It has been proved by Yuan and Zeng [98] for cyclic groups. Later Grynkiewicz, Marchan and Ordaz [55] proved the relation (2.3) for a general finite abelian group G and weight set A.

2.2 Introduction and Motivation of the Problem

For a positive integer n, we consider the weight set $A = \{a \pmod{n} \mid a \in (\mathbb{Z}/n\mathbb{Z})^*\}$. We denote the number of prime factors of n counted with multiplicity (respectively, without multiplicity) by $\Omega(n)$ (respectively, $\omega(n)$).

In [7], it was proved that $\mathsf{E}_A(\mathbb{Z}/n\mathbb{Z}) \geq n + \Omega(n)$. They also conjectured that it is the exact value. This conjecture has been proved in [54] and [63] independently. More precisely,

Theorem 2.2.1 *Let $n > 1$ be an integer. Then $D_A(\mathbb{Z}/n\mathbb{Z}) = 1 + \Omega(n)$ and $\mathsf{E}_A(\mathbb{Z}/n\mathbb{Z}) = n + \Omega(n)$.*

Now let us consider the weight set to be the set of squares mod n, namely $R_n = \{a^2 \pmod{n} \mid a \in (\mathbb{Z}/n\mathbb{Z})^*\}$.

In [13], Adhikari and Rath considered the problem of determining the exact value of the constants $D_{R_p}(\mathbb{Z}/p\mathbb{Z})$ and $\mathsf{E}_{R_p}(\mathbb{Z}/p\mathbb{Z})$ for a prime p. More precisely, they proved

Theorem 2.2.2 *Let p be a prime. Then $D_{R_p}(\mathbb{Z}/p\mathbb{Z}) = 3$ and $\mathsf{s}_{R_p}(\mathbb{Z}/p\mathbb{Z}) = \mathsf{E}_{R_p}(\mathbb{Z}/p\mathbb{Z}) = p + 2$.*

Adhikari, David and Urroz [8] extended the above theorem for a square-free positive integer n which is coprime to 6.

Theorem 2.2.3 *Let n be a square-free positive integer such that $(n, 6) = 1$. Then $D_{R_n}(\mathbb{Z}/n\mathbb{Z}) = 1 + 2\omega(n)$ and $\mathsf{E}_{R_n}(\mathbb{Z}/n\mathbb{Z}) = n + 2\omega(n)$.*

They also proved a lower bound for a general n as follows.

Theorem 2.2.4 *Let $n > 1$ be a positive integer. Then $D_{R_n}(\mathbb{Z}/n\mathbb{Z}) \geq 1 + 2\Omega(n)$ and $\mathsf{E}_{R_n}(\mathbb{Z}/n\mathbb{Z}) \geq n + 2\Omega(n)$.*

Later Chintamani and Moriya [30] extended some results of [8]. More precisely they proved the following theorem.

Theorem 2.2.5 *Let $n = 5^\ell \prod_{i=2}^{k} p_i^{\alpha_i}$, where $\ell, \alpha_i \geq 0$ with $p_i \geq 7$ primes for each $i \in \{2, 3, \ldots, k\}$. Let $m \geq 3\omega(n) + 1$ and let $S = x_1 \cdot x_2 \cdot \ldots \cdot x_{m+2\Omega(n)+\ell}$ be a sequence over $\mathbb{Z}/n\mathbb{Z}$. Then there exists a subsequence $x_{i_1} \cdot x_{i_2} \cdot \ldots \cdot x_{i_m}$ and $a_1, a_2, \ldots, a_m \in R_n$ such that $\sum_{j=1}^{m} a_j x_{i_j} \equiv 0 \pmod{n}$. In particular,*

$$\mathsf{E}_{R_n}(\mathbb{Z}/n\mathbb{Z}) \leq n + 2\Omega(n) + \ell.$$

Remark 2.2.6 As a consequence, if n is an integer such that $gcd(30, n) = 1$, then combining Theorem 2.2.5 and Theorem 2.2.4, we get $E_{R_n}(\mathbb{Z}/n\mathbb{Z}) = n + 2\Omega(n)$.

In this chapter, we prove similar results as in [8, 13, 30] but for the weight set to be the set of all cubes in $(\mathbb{Z}/n\mathbb{Z})^*$.

For a positive integer n, we denote the set of all cubes in $(\mathbb{Z}/n\mathbb{Z})^*$ by $C_n = \{a^3 \pmod{n} \mid a \in (\mathbb{Z}/n\mathbb{Z})^*\}$. We prove the following theorems.

Theorem 2.2.7 *Let p be an odd prime and C_p the set of all cubic residues modulo p. Then we have*

(i) $D_{C_p}(\mathbb{Z}/p\mathbb{Z}) \leq 4$, and

(ii) $E_{C_p}(\mathbb{Z}/p\mathbb{Z}) \leq p + 3$.

Theorem 2.2.8 *Let $n = n_1 n_2$ be an odd integer such that $n_1 = \prod_{i=1}^{r} p_i^{e_i}$ and $n_2 = \prod_{j=1}^{s} q_j^{f_j}$ with primes $p_i \equiv 1 \pmod{3}$ and $q_j \equiv 2 \pmod{3}$ and $7 \nmid n$. Then we have*

(i) $D_{C_n}(\mathbb{Z}/n\mathbb{Z}) \leq 3\Omega(n_1) + \Omega(n_2) + 1$, and

(ii) $E_{C_n}(\mathbb{Z}/n\mathbb{Z}) \leq n + 3\Omega(n_1) + \Omega(n_2)$.

More generally, when the prime 7 is involved, we have the following theorem.

Theorem 2.2.9 *Let $n = 7^l n_1 n_2$ be an odd integer such that $n_1 = \prod_{i=1}^{r} p_i^{e_i}$ and $n_2 = \prod_{j=1}^{s} q_j^{f_j}$ with primes $p_i \equiv 1 \pmod 3$ and $q_j \equiv 2 \pmod 3$ and $7 \nmid n_1$. Then we have*

(i) $D_{C_n}(\mathbb{Z}/n\mathbb{Z}) \leq 3\Omega(n_1) + \Omega(n_2) + 5l + 1$, *and*

(ii) $E_{C_n}(\mathbb{Z}/n\mathbb{Z}) \leq n + 3\Omega(n_1) + \Omega(n_2) + 5l$.

2.3 Preliminaries

In this section we state some basic definitions and some preliminary lemmas.

Definition 2.3.1 Let a and m be two integers. Then a is said to be a *primitive root modulo m* if the residue class of $a \pmod m$ generates the group $(\mathbb{Z}/m\mathbb{Z})^*$.

Definition 2.3.2 Let m and n be two positive integers and let a be a integer such that $\gcd(a, m) = 1$. Then a is said to be an *n-th power residue modulo m* if $x^n \equiv a \pmod m$ is solvable.

Lemma 2.3.3 *[60] Let m be a positive integer which possesses primitive roots and let a be a unit in $\mathbb{Z}/m\mathbb{Z}$. Then a is an n-th power residue modulo m if and only if $a^{\frac{\phi(m)}{d}} \equiv 1 \pmod m$, where $d = \gcd(\phi(m), n)$, and ϕ is the Euler totient function.*

Let p^r be an odd prime power. Then we know that the group $(\mathbb{Z}/p^r\mathbb{Z})^*$ is a cyclic group and hence p^r possesses a primitive root. Now by taking $n = 3$, we have

$$d = \gcd(\phi(p^r), 3) = \begin{cases} 1\ ; & \text{if } p \equiv 2 \pmod 3 \text{ or } p = 3 \text{ and } r = 1 \\ 3\ ; & \text{if } p \equiv 1 \pmod 3 \text{ or } p = 3 \text{ and } r > 1 \end{cases}$$

Let p be an odd prime number such that $p \equiv 2 \pmod 3$. Then by Lemma 2.3.3, every element of $(\mathbb{Z}/p^r\mathbb{Z})^*$ is a cubic residue modulo p^r. Thus in this case, $C_{p^r} = (\mathbb{Z}/p^r\mathbb{Z})^*$.

Let p be a prime number such that $p \equiv 1 \pmod 3$ and let g be a generator of the cyclic group $(\mathbb{Z}/p^r\mathbb{Z})^*$. Then $C_{p^r} = \{g^3, g^{2.3}, \ldots, g^{\frac{p^r-p^{r-1}}{3}.3} = 1\}$ forms a subgroup of $(\mathbb{Z}/p^r\mathbb{Z})^*$ of order $\frac{p^r-p^{r-1}}{3}$.

Also $C_3 = (\mathbb{Z}/3\mathbb{Z})^*$, and C_{3^r} is the subgroup of $(\mathbb{Z}/3^r\mathbb{Z})^*$ of order $\frac{3^r-3^{r-1}}{3}$, where $r > 1$.

The following lemma is a consequence of the Chinese Remainder Theorem.

Lemma 2.3.4 *[60] Suppose that $m = 2^e p_1^{e_1} \ldots p_\ell^{e_\ell}$. Then $x^n \equiv a \pmod m$ is solvable if and only if the system of congruences $x^n \equiv a \pmod{2^e}$, $x^n \equiv a \pmod{p_1^{e_1}}$, …, $x^n \equiv a \pmod{p_\ell^{e_\ell}}$ is solvable.*

For an abelian group G and a non-empty subset A of G, we define the *Stabilizer of A* as follows.

Definition 2.3.5 Let G be an abelian group and let A be a non-empty subset of G. Then the stabilizer of A, denoted by $Stab(A)$, is defined as $Stab(A) = \{x \in G : x + A = A\}$.

Lemma 2.3.6 *Let G be an abelian group and let A be a non-empty subset of G. Then*

(i) $Stab(A)$ is a subgroup of G.

(ii) $Stab(A) = G$ *if and only if* $A = G$.

(iii) *Let* $H = Stab(A)$ *and* $\psi : G \to G/H$ *the natural map. Then* $\psi(A)$ *has the trivial stabilizer in* G/H.

We require the following theorem which is due to Kneser (see [70]).

Theorem 2.3.7 *Let* G *be a finite abelian group and* $A_1, A_2, \ldots, A_n$ *non-empty subsets of* G. *If* $H = Stab(A_1 + A_2 + \cdots + A_n)$, *then*

$$|A_1 + A_2 + \cdots + A_n| \geq |A_1 + H| + |A_2 + H| + \cdots + |A_n + H| - (n-1)|H|.$$

We also use the following remark which was proved in [30].

Remark 2.3.8 Let m and n be two positive integers such that m divides n and $\psi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ a surjective ring homomorphism. Then $(\mathbb{Z}/m\mathbb{Z})^* = \psi\left((\mathbb{Z}/n\mathbb{Z})^*\right)$ and hence $C_m = \psi(C_n)$.

We now prove some lemmas which are useful to prove our main theorems.

Lemma 2.3.9 *Let* $\alpha \geq 1$ *be an integer and* $n = p^\alpha$ *where* p *is a prime such that* $p \equiv 1 \pmod 3$ *with* $p \geq 13$. *Let* $S = x_1 \cdot x_2 \cdot \ldots \cdot x_m$ *be a sequence over* $\mathbb{Z}/n\mathbb{Z}$ *such that at least four elements of* S *are units in* $\mathbb{Z}/n\mathbb{Z}$. *Then there exist* $a_1, a_2, \ldots, a_m \in C_n$ *such that* $\sum_{i=1}^m a_i x_i \equiv 0 \pmod n$.

Proof. As $p \equiv 1 \pmod 3$, we know that $|C_n| = \frac{p^\alpha - p^{\alpha-1}}{3}$. Now without loss of generality, let x_1, x_2, x_3 and x_4 be units in $\mathbb{Z}/n\mathbb{Z}$ and $H = Stab(x_1C_n + x_2C_n + x_3C_n + x_4C_n)$ the stabilizer of $(x_1C_n + x_2C_n + x_3C_n + x_4C_n)$. Then the group $(\mathbb{Z}/n\mathbb{Z})/H$ is cyclic, say $\mathbb{Z}/k\mathbb{Z}$, where $k = p^\beta$, $\beta \leq \alpha$.

Consider the natural homomorphism $\psi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/k\mathbb{Z}$ with kernel H. By

Remark 2.3.8, we have $\psi(C_n) = C_k$, and hence by Lemma 2.3.6 (iii), we have

$$\{\psi(0)\} = Stab\left(\psi\left(\sum_{i=1}^{4} x_i C_n\right)\right) = Stab\left(\sum_{i=1}^{4} \psi(x_i) C_k\right). \tag{2.4}$$

Since $\psi(x_i)$ is a unit in $\mathbb{Z}/k\mathbb{Z}$ for all $i = 1, 2, 3$ and 4, by Theorem 2.3.7 we get,

$$\left|\sum_{i=1}^{4} \psi(x_i) C_k\right| \geq 4|C_k| - 3 = \left(\frac{4(p^\beta - p^{\beta-1})}{3} - 3\right) \geq p^\beta.$$

Since $p \geq 13$, we have $\frac{4(p-1)}{3} - 3 \geq p$, and $\left(\frac{4(p^\beta - p^{\beta-1})}{3} - 3\right) = \left(\frac{4p^{\beta-1}(p-1)}{3} - 3\right) \geq p^{\beta-1}(p+3) - 3 = p^\beta + 3p^{\beta-1} - 3 \geq p^\beta$.

Thus $\sum_{i=1}^{4} \psi(x_i) C_k = \mathbb{Z}/k\mathbb{Z}$ and hence $Stab\left(\sum_{i=1}^{4} \psi(x_i) C_k\right) = \mathbb{Z}/k\mathbb{Z}$, which gives $\mathbb{Z}/k\mathbb{Z} = \{\psi(0)\}$ by (2.4). Therefore $Stab(x_1 C_n + x_2 C_n + x_3 C_n + x_4 C_n) = H = \mathbb{Z}/n\mathbb{Z}$, and by Lemma 2.3.6 (ii) we get $x_1 C_n + x_2 C_n + x_3 C_n + x_4 C_n = \mathbb{Z}/n\mathbb{Z}$. Thus we can write $a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 = -(x_5 + \cdots + x_m)$, where $a_1, a_2, a_3, a_4 \in C_n$. □

For the prime 7, we have the following lemma, whose proof is similar to the proof of Lemma 2.3.9.

Lemma 2.3.10 *Let $\alpha \geq 1$ be an integer and $n = 7^\alpha$. Let $S = x_1 \cdot x_2 \cdot \ldots \cdot x_m$ be a sequence over $\mathbb{Z}/n\mathbb{Z}$ such that at least six elements of S are units in $\mathbb{Z}/n\mathbb{Z}$. Then there exist $a_1, a_2, \ldots, a_m \in C_n$ such that $\sum_{i=1}^{m} a_i x_i \equiv 0 \pmod{n}$.*

We also need the following lemma which was proved by Griffiths [54].

Lemma 2.3.11 *Let p^a be an odd prime power and $S = x_1 \cdot x_2 \cdot \ldots \cdot x_m$ a sequence over $\mathbb{Z}/p^a\mathbb{Z}$ such that at least two elements of S are units in $\mathbb{Z}/p^a\mathbb{Z}$. Then there exist $a_1, a_2, \ldots, a_m \in (\mathbb{Z}/p^a\mathbb{Z})^*$ such that $\sum_{i=1}^{m} a_i x_i \equiv 0 \pmod{p^a}$.*

We also need the following theorem by Chevally and Warning.

Theorem 2.3.12 *Let p be a prime number and let F be a finite field of characteristic p. For $i = 1, 2, \ldots, m$, let $f_i \in F[x_1, x_2, \ldots, x_n]$ be a non-zero polynomial of degree d_i over the field F. Let N denotes the number of n-tuples $(y_1, y_2, \ldots, y_n)$ of elements of F such that*

$$f_i(y_1, y_2, \ldots, y_n) = 0,$$

for all $i = 1, 2, \ldots, m$. If $d_1 + d_2 + \cdots + d_m < n$, then

$$N \equiv 0 \pmod{p}.$$

In particular if $N \geq 1$, then there is a non-zero simultaneous solution over F.

2.4 Proof of Main Theorems

2.4.1 Proof of Theorem 2.2.7

Proof of Theorem 2.2.7. Let p be an odd prime such that either $p = 3$ or $p \equiv 2 \pmod{3}$. Then by the discussion below Lemma 2.3.3, we get $C_p = (\mathbb{Z}/p\mathbb{Z})^*$. Hence in this case, $E_{C_p}(\mathbb{Z}/p\mathbb{Z}) = p + 1$ and $D_{C_p}(\mathbb{Z}/p\mathbb{Z}) = 2$ by Theorem 2.2.1.

Now let p be a prime such that $p \equiv 1 \pmod{3}$. Then C_p is a proper subgroup of $(\mathbb{Z}/p\mathbb{Z})^*$ of order $\frac{p-1}{3}$. Let $s_1 \cdot s_2 \cdot \ldots \cdot s_{p+3}$ be a sequence over $(\mathbb{Z}/p\mathbb{Z})$ of length $p + 3$. Consider the following system of homogeneous equations over the finite field $\mathbb{F}_p$

$$\sum_{i=1}^{p+3} s_i x_i^3 = 0 \textit{ and } \sum_{i=1}^{p+3} x_i^{p-1} = 0.$$

Clearly this system has the trivial solution $(0, 0, \ldots, 0)$ and the sum of the degrees of the equations is $3+(p-1) = p+2 < p+3$, the number of variables. Thus by Theorem 2.3.12, we get that there is a nontrivial solution $(y_1, y_2, \ldots, y_{p+3})$ of the above system. If we write $I = \{i : y_i \neq 0\}$, by the first equation it follows that $\sum_{i\in I} a_i s_i = 0$ where a_i's are cubic residues of $(\mathbb{Z}/p\mathbb{Z})^*$. As $p \geq 7$, by Fermat's little theorem, from the last equation we get $|I| = p$. Hence $E_{C_p}(\mathbb{Z}/p\mathbb{Z}) \leq p+3$, and from the relation (2.3) we get $D_{C_p}(\mathbb{Z}/p\mathbb{Z}) \leq 4$. □

2.4.2 Proof of Theorem 2.2.8

Theorem 2.2.8 is an easy corollary of the next proposition. We use Lemma 2.3.9, Lemma 2.3.11 and the Chinese Remainder Theorem to prove the proposition.

Proposition 2.4.1 *Let $n = n_1 n_2$ be an odd integer such that $n_1 = \prod_{i=1}^{r} p_i^{e_i}$ and $n_2 = \prod_{j=1}^{s} q_j^{f_j}$ with primes $p_i \equiv 1 \pmod 3$ and $q_j \equiv 2 \pmod 3$ and $7 \nmid n$. Let $m \geq 4\omega(n_1)+2\omega(n_2)$ and $S = x_1 \cdot x_2 \cdot \ldots \cdot x_{m+3\Omega(n_1)+\Omega(n_2)}$ be a sequence over $\mathbb{Z}/n\mathbb{Z}$. Then there exists a subsequence $x_{i_1} \cdot x_{i_2} \cdot \ldots \cdot x_{i_m}$ of S and $a_1, a_2, \ldots, a_m \in C_n$ such that $\sum_{j=1}^{m} a_j x_{i_j} \equiv 0 \pmod n$.*

Proof. We proceed by induction on $\Omega(n)$. When $\Omega(n) = 1$, then n is a prime, say $n = p$. If $p \equiv 1 \pmod 3$ then by our assumption, $p \geq 13$. By Lemma 2.3.9 (with $\alpha = 1$), if a sequence $S = x_1 \cdot x_2 \cdot \ldots \cdot x_{m+3}$ over $\mathbb{Z}/p\mathbb{Z}$ has at least four non-zero elements, then there are $a_i \in C_p$ for $i = 1, 2, \ldots, m$ such that $\sum_{i=1}^{m} a_i x_i \equiv 0 \pmod p$. Otherwise there is a subsequence $S_1 = x_{i_1} \cdot x_{i_2} \cdot \ldots \cdot x_{i_m}$ of S of length m such that all elements of S_1 are divisible by p. Hence $\sum_{j=1}^{m} a_j x_{i_j} \equiv 0 \pmod p$ for any choice of $a_i \in C_p$.

Now if $p \equiv 2 \pmod 3$, then we know that $C_p = (\mathbb{Z}/p\mathbb{Z})^*$. Hence by Lemma 2.3.11, if a sequence $S = x_1 \cdot x_2 \cdot \ldots \cdot x_{m+1}$ over $\mathbb{Z}/p\mathbb{Z}$ has at least two non-zero elements, then there are $a_i \in C_p$ for $i = 1, 2, \ldots, m$ such that $\sum_{i=1}^{m} a_i x_i \equiv 0 \pmod p$. Otherwise there is a subsequence $S_1 = x_{i_1} \cdot x_{i_2} \cdot \ldots \cdot x_{i_m}$ of S of length m such that all elements of S_1 are divisible by p. Hence $\sum_{j=1}^{m} a_j x_{i_j} \equiv 0 \pmod p$ for any choice of $a_i \in C_p$. This proves the proposition when $\Omega(n) = 1$.

Suppose now that $\Omega(n) \geq 2$ and the result is true for any odd integer $N = N_1N_2$ such that $7 \nmid N$ where each prime divisor of N_1 is congruent to 1 modulo 3 and each prime divisor of N_2 is congruent to 2 modulo 3 with $\Omega(N_1) < \Omega(n_1)$ or $\Omega(N_2) < \Omega(n_2)$. Let $S = x_1 \cdot x_2 \cdot \ldots \cdot x_{m+3\Omega(n_1)+\Omega(n_2)}$ be a sequence over $\mathbb{Z}/n\mathbb{Z}$.

Case 1. There exists a prime $p_t \mid n_1$ such that the sequence S contains at most three elements which are co prime to p_t.

In this case, we remove those elements and consider the subsequence S_1 of S of length at least $m + 3\Omega(\frac{n_1}{p_t}) + \Omega(n_2)$ all of whose elements are zero modulo p_t. Since $m \geq 4\omega(\frac{n_1}{p_t}) + 2\omega(n_2)$, by the induction hypothesis, there is a subsequence $x_{i_1} \cdot x_{i_2} \cdot \ldots \cdot x_{i_m}$ of S_1 and $a_1, a_2, \ldots, a_m \in C_{\frac{n}{p_t}}$ such that

$$\sum_{j=1}^{m} a_j \frac{x_{i_j}}{p_t} \equiv 0 \pmod{n/p_t}.$$

Since $m' = n/p_t$ divides n, define the natural map $\psi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m'\mathbb{Z}$. By Remark 2.3.8, we see that $\psi(C_n) = C_{\frac{n}{p_t}}$. Thus for each a_j there exists $a_j' \in C_n$ such that $\psi(a_j') = a_j$ i.e. $a_j' \equiv a_j \pmod{n/p_t}$. Hence

$$\sum_{j=1}^{m} a_j' \frac{x_{i_j}}{p_t} \equiv 0 \pmod{n/p_t}.$$

Therefore,

$$\sum_{j=1}^{m} a_j' x_{i_j} \equiv 0 \pmod{n}.$$

Case 2. There is a prime $q_t \mid n_2$ such that the sequence S contains at most one element which is co prime to q_t.

In this case, we remove this element and consider the subsequence S_1 of S of length at least $m + 3\Omega(n_1) + \Omega(\frac{n_2}{q_t})$ all of whose elements are zero modulo q_t. Since $m \geq 4\omega(n_1) + 2\omega(\frac{n_2}{q_t})$, by the induction hypothesis, there is a subsequence

$x_{i_1} \cdot x_{i_2} \cdot \ldots \cdot x_{i_m}$ of S_1 and $a_1, a_2, \ldots, a_m \in C_{\frac{n}{q_t}}$ such that

$$\sum_{j=1}^{m} a_j \frac{x_{i_j}}{q_t} \equiv 0 \pmod{n/q_t}.$$

As in the Case 1, using Remark 2.3.8, there exist $a_1^{'}, a_2^{'}, \ldots, a_m^{'} \in C_n$ such that $\sum_{j=1}^{m} a_j^{'} x_{i_j} \equiv 0 \pmod{n}$.

Case 3. For all primes $p_i \mid n_1$, the sequence S contains at least four unit elements modulo p_i and for all primes $q_j \mid n_2$, the sequence S contains at least two unit elements modulo q_j.

In this case, without loss of generality, let $S_1 = x_1 \cdot x_2 \cdot \ldots \cdot x_t$ be a subsequence of length $t \leq 4\omega(n_1) + 2\omega(n_2) \leq m$ such that S_1 has at least four units modulo each prime p_i and at least two units modulo each prime q_j. Now let us extend this subsequence S_1 to a subsequence $S_2 = x_1 \cdot x_2 \cdot \ldots \cdot x_m$ of S of length m. Then by Lemma 2.3.9 and Lemma 2.3.11, for each p_i and q_j, we have

$$\sum_{k=1}^{m} a_k^i x_k \equiv 0 \pmod{p_i^{e_i}} \text{ and } \sum_{k=1}^{m} b_k^j x_k \equiv 0 \pmod{q_j^{f_j}},$$

where $a_k^i \in C_{p_i^{e_i}}$ and $b_k^j \in C_{q_j^{f_j}}$. Now the result follows from the Chinese Remainder Theorem and Lemma 2.3.4. □

Proof of Theorem 2.2.8. Since $n = n_1 n_2$ is an odd integer such that $n_1 = \prod_{i=1}^{r} p_i^{e_i}$ and $n_2 = \prod_{j=1}^{s} q_j^{f_j}$ with primes $p_i \equiv 1 \pmod{3}$ and $q_j \equiv 2 \pmod{3}$ and $7 \nmid n$, we have $n \geq 4\omega(n_1) + 2\omega(n_2)$. Hence by Proposition 2.4.1, any sequence over $\mathbb{Z}/n\mathbb{Z}$ of length $n + 3\Omega(n_1) + \Omega(n_2)$ has a C_n-weighted zero-sum subsequence of length n. This proves that $E_{C_n}(\mathbb{Z}/n\mathbb{Z}) \leq n + 3\Omega(n_1) + \Omega(n_2)$, and the relation (2.3) gives $D_{C_n}(\mathbb{Z}/n\mathbb{Z}) \leq 3\Omega(n_1) + \Omega(n_2) + 1$. □

2.4.3 Proof of Theorem 2.2.9

Theorem 2.2.9 is an easy corollary of the next proposition. We use Lemma 2.3.9, Lemma 2.3.10, Lemma 2.3.11 and the Chinese Remainder Theorem to prove the proposition.

Proposition 2.4.2 *Let $n = 7^l n_1 n_2$ be an odd integer such that $n_1 = \prod_{i=1}^{r} p_i^{e_i}$ and $n_2 = \prod_{j=1}^{s} q_j^{f_j}$ with primes $p_i \equiv 1 \pmod 3$ and $q_j \equiv 2 \pmod 3$ and $7 \nmid n_1$. Let $m \geq 4\omega(7^l n_1) + 2\omega(n_2) + 2$ and $S = x_1 \cdot x_2 \cdot \ldots \cdot x_{m+3\Omega(n_1)+\Omega(n_2)+5l}$ be a sequence over $\mathbb{Z}/n\mathbb{Z}$. Then there exists a subsequence $x_{i_1} \cdot x_{i_2} \cdot \ldots \cdot x_{i_m}$ of S and $a_1, a_2, \ldots, a_m \in C_n$ such that $\sum_{j=1}^{m} a_j x_{i_j} \equiv 0 \pmod n$.*

Proof. We prove the proposition by induction on $\Omega(n)$. Suppose $\Omega(n) = 1$. Then n is prime, say $n = p$.

If $p = 7$ then $m \geq 6$. If a sequence $S = x_1 \cdot x_2 \cdot \ldots \cdot x_{m+5}$ has at least six non-zero elements modulo 7, then by Lemma 2.3.10 we get a C_7-weighted zero-sum subsequence of length m. Otherwise there is a subsequence S_1 of S of length m such that all elements of S_1 are divisible 7, and hence we are done.

Suppose $p \neq 7$ and $p \equiv 1 \pmod 3$. Then $p \geq 13$. By Lemma 2.3.9 (with $\alpha = 1$), if a sequence $S = x_1 \cdot x_2 \cdot \ldots \cdot x_{m+3}$ over $\mathbb{Z}/p\mathbb{Z}$ has at least four non-zero elements, then there are $a_1, a_2, \ldots, a_m \in C_p$ such that $\sum_{i=1}^{m} a_i x_i \equiv 0 \pmod p$. Otherwise there is a subsequence $S_1 = x_{i_1} \cdot x_{i_2} \cdot \ldots \cdot x_{i_m}$ of S of length m such that all elements of S_1 are divisible by p. Hence $\sum_{j=1}^{m} a_j x_{i_j} \equiv 0 \pmod p$ for any choice of $a_j \in C_p$.

Now if $p \equiv 2 \pmod 3$, then we know that $C_p = (\mathbb{Z}/p\mathbb{Z})^*$. Hence by Lemma 2.3.11, if a sequence $S = x_1 \cdot x_2 \cdot \ldots \cdot x_{m+1}$ over $\mathbb{Z}/p\mathbb{Z}$ has at least two non-zero elements, then there are $a_i \in C_p$ for $i = 1, 2, \ldots, m$ such that $\sum_{i=1}^{m} a_i x_i \equiv 0 \pmod p$. Otherwise there is a subsequence $S_1 = x_{i_1} \cdot x_{i_2} \cdot \ldots \cdot x_{i_m}$ of S of length m such that all elements of S_1 are divisible by p. Hence $\sum_{j=1}^{m} a_j x_{i_j} \equiv 0 \pmod p$

for any choice of $a_i \in C_p$. This proves the proposition when $\Omega(n) = 1$.

Now we assume that $\Omega(n) > 1$ and the result is true for all odd integer $N = 7^L N_1 N_2$ such that $7 \nmid N_1$, where each prime divisor of N_1 is congruent to 1 modulo 3 and each prime divisor of N_2 is congruent to 2 modulo 3 with $L < l$ or $\Omega(N_1) < \Omega(n_1)$ or $\Omega(N_2) < \Omega(n_2)$. Let $S = x_1 \cdot x_2 \cdot \ldots \cdot x_{m+3\Omega(n_1)+\Omega(n_2)+5l}$ be a sequence over $\mathbb{Z}/n\mathbb{Z}$.

Case 1. There is a prime divisor $p_t \equiv 1 \pmod 3$ of n_1 such that at most three elements of S are co prime to p_t.

Let S_1 be the subsequence of S after removing those elements. Then the length of S_1 is at least $m + 3\Omega(\frac{n_1}{p_t}) + \Omega(n_2) + 5l$ and every element of S_1 is divisible by p_t. Since $m \geq 4\omega(\frac{7^l n_1}{p_t}) + 2\omega(n_2) + 2$, by the induction hypothesis we get a subsequence $x_{i_1} \cdot x_{i_2} \cdot \ldots \cdot x_{i_m}$ of S_1 and $a_1, a_2, \ldots, a_m \in C_{\frac{n}{p_t}}$ such that

$$\sum_{j=1}^{m} a_j \frac{x_{i_j}}{p_t} \equiv 0 \pmod{n/p_t}.$$

Since $m^{'} = n/p_t$ divides n, define the natural map $\psi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m^{'}\mathbb{Z}$. By Remark 2.3.8, we see that $\psi(C_n) = C_{\frac{n}{p_t}}$. Thus for each $a_j \in C_{\frac{n}{p_t}}$ there exists $a_j^{'} \in C_n$ such that $\psi(a_j^{'}) = a_j$ i.e. $a_j^{'} \equiv a_j \pmod{n/p_t}$. Hence

$$\sum_{j=1}^{m} a_j^{'} \frac{x_{i_j}}{p_t} \equiv 0 \pmod{n/p_t}.$$

Therefore,

$$\sum_{j=1}^{m} a_j^{'} x_{i_j} \equiv 0 \pmod{n}.$$

Case 2. There is a prime divisor q_t of n_2 such that at most one element of S is co prime to q_t.

Let S_1 be the subsequence of S after removing this element. Then the length of S_1 is at least $m+3\Omega(n_1)+\Omega(\frac{n_2}{q_t})+5l$ and every element of S_1 is divisible by q_t. Since $m \geq 4\omega(7^l n_1)+2\omega(\frac{n_2}{q_t})+2$, by induction hypothesis, we get a subsequence $x_{i_1} \cdot x_{i_2} \cdot \ldots \cdot x_{i_m}$ of S_1 and $a_1, a_2, \ldots, a_m \in C_{\frac{n}{q_t}}$ such that

$$\sum_{j=1}^{m} a_j \frac{x_{i_j}}{q_t} \equiv 0 \pmod{n/q_t}.$$

As in the Case 1, using Remark 2.3.8, we get $a_j' \in C_n$ such that $\sum_{j=1}^{m} a_j' x_{i_j} \equiv 0 \pmod{n}$.

Case 3. The sequence S contains at most five non-zero elements modulo 7.

Let S_1 be the subsequence of S obtained by removing these terms. Then S_1 has at least $m + 3\Omega(n_1) + \Omega(n_2) + 5(l-1)$ many elements and every element of S_1 is divisible by 7. Since $m \geq 4\omega(\frac{7^l n_1}{7}) + \omega(n_2) + 2$, by applying induction hypothesis, we get a subsequence $x_{i_1} \cdot x_{i_2} \cdot \ldots \cdot x_{i_m}$ of S_1 and $a_1, a_2, \ldots, a_m \in C_{\frac{n}{7}}$ such that

$$\sum_{j=1}^{m} a_j \frac{x_{i_j}}{7} \equiv 0 \pmod{n/7}.$$

As in the Case 1, using Remark 2.3.8, we get $a_j' \in C_n$ such that $\sum_{j=1}^{m} a_j' x_{i_j} \equiv 0 \pmod{n}$.

Case 4. The sequence S contains at least six units modulo 7, at least four units modulo each p_i and at least two units modulo each q_j.

In this case, without loss of generality, we can assume that $S_1 = x_1 \cdot x_2 \cdot \ldots \cdot x_t$ is a subsequence of length $t \leq 4\omega(7^l n_1) + 2\omega(n_2) + 2 \leq m$ such that S_1 has at least six units modulo 7, at least four units modulo each p_i and at least two units modulo each q_j. Now let us extend this subsequence S_1 to a subsequence $S_2 = x_1 \cdot x_2 \cdot \ldots \cdot x_m$ of S of length m. Then by Lemma 2.3.9, Lemma 2.3.10

and Lemma 2.3.11, for each p_i and q_j, we have

$$\sum_{k=1}^{m} a_k^0 x_k \equiv 0 \pmod{7^l}$$

$$\sum_{k=1}^{m} a_k^i x_k \equiv 0 \pmod{p_i^{e_i}} \text{ and } \sum_{k=1}^{m} b_k^j x_k \equiv 0 \pmod{q_j^{f_j}},$$

where $a_k^i \in C_{p_i^{e_i}}$, $b_k^j \in C_{q_j^{f_j}}$ and $a_k^0 \in C_{7^l}$. Now the result follows from the Chinese Remainder Theorem and Lemma 2.3.4. □

Proof of Theorem 2.2.9. Since $n = 7^l n_1 n_2$ is an odd integer such that $n_1 = \prod_{i=1}^{r} p_i^{e_i}$ and $n_2 = \prod_{j=1}^{s} q_j^{f_j}$ with primes $p_i \equiv 1 \pmod 3$ and $q_j \equiv 2 \pmod 3$ and $7 \nmid n_1$, we have $n \geq 4\omega(7^l n_1) + 2\omega(n_2) + 2$. Hence by Proposition 2.4.2, any sequence over $\mathbb{Z}/n\mathbb{Z}$ of length $n+3\Omega(n_1)+\Omega(n_2)+5l$ has a C_n-weighted zero-sum subsequence of length n. This proves that $E_{C_n}(\mathbb{Z}/n\mathbb{Z}) \leq n+3\Omega(n_1)+\Omega(n_2)+5l$, and the relation (2.3) gives $D_{C_n}(\mathbb{Z}/n\mathbb{Z}) \leq 3\Omega(n_1) + \Omega(n_2) + 5l + 1$. □

CHAPTER 3

$\{\pm 1\}$-weighted zero-sum constants for some finite abelian groups of higher ranks and modification of a polynomial method of Rónyai

Let p be an odd prime. In this chapter, we prove a conditional result about $\mathsf{s}_{\{\pm 1\}}((\mathbb{Z}/p\mathbb{Z})^3)$. We also modify a polynomial method of Rónyai to prove that for an odd prime p and for a positive even integer $k \geq 2$ which divides $p-1$, if A is the subgroup of $(\mathbb{Z}/p\mathbb{Z})^*$ of order k, then any sequence over $(\mathbb{Z}/p\mathbb{Z})^{k+1}$ of length $4p + \frac{p-1}{k} - 1$ contains an A-weighted zero-sum subsequence of length $3p$. The content of this chapter is published in the article [15].

3.1 Introduction

Let G be a finite abelian group of exponent n and let A be a non-empty subset of $[1, n-1]$. In the previous chapter we have defined the weighted Davenport constant $D_A(G)$ and the weighted Erdős-Ginzburg-Ziv constant $\mathsf{s}_A(G)$. It was also mentioned that, when $A = \{1\}$, the constants $D_A(G)$ and $\mathsf{s}_A(G)$ are $D(G)$ and $\mathsf{s}(G)$ respectively. Here we turn our attention to the results when the weight set A is $\{\pm 1\}$. For a general finite abelian group, the following bound has been proved in [10].

Theorem 3.1.1 *Let $G = (\mathbb{Z}/n_1\mathbb{Z}) \times (\mathbb{Z}/n_2\mathbb{Z}) \times \cdots \times (\mathbb{Z}/n_r\mathbb{Z})$ be a finite abelian group with $1 < n_1 \mid n_2 \mid \cdots \mid n_r$. Then*

$$\sum_{i=1}^{r} \lfloor \log_2 n_i \rfloor + 1 \leq D_{\{\pm 1\}}(G) \leq \lfloor \log_2 |G| \rfloor + 1.$$

In particular, when G is a cyclic group or a 2-group, the upper and lower bounds coincide and we get the exact value. Later Marchan, Ordaz and Schmid [65] established the following generalization of Theorem 3.1.1.

Theorem 3.1.2 *Let $G = (\mathbb{Z}/n_1\mathbb{Z}) \times (\mathbb{Z}/n_2\mathbb{Z}) \times \cdots \times (\mathbb{Z}/n_r\mathbb{Z})$ be a finite abelian group. Then*

$$\sum_{i=1}^{r} \lfloor \log_2 n_i \rfloor + 1 \leq D_{\{\pm 1\}}(G) \leq \lfloor \log_2 |G| \rfloor + 1.$$

The difference to the above mentioned theorem with Theorem 3.1.1 is that we do not require the n_i's to satisfy the condition $n_1 \mid n_2 \mid \cdots \mid n_r$. Theorem 3.1.2 may give better lower bound than Theorem 3.1.1. We illustrate this by an example.

We have $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/(3 \cdot 11 \cdot 23)\mathbb{Z} \cong \mathbb{Z}/(3 \cdot 11)\mathbb{Z} \times \mathbb{Z}/(3 \cdot 23)\mathbb{Z}$. The former decomposition gives $(1+9)+1=11$ as lower bound, whereas the latter gives $(5+6)+1=12$ as lower bound.

Let n and d be two positive integers and G the group $(\mathbb{Z}/n\mathbb{Z})^d$. Then Theorem 3.1.1 gives,

$$d\lfloor \log_2 n\rfloor + 1 \leq D_{\{\pm 1\}}(G) \leq \lfloor d\log_2 n\rfloor + 1.$$

We state some known results for the constant $\mathsf{s}_{\{\pm 1\}}(G)$.

1. [3] Let n be an odd positive integer. Then we have $\mathsf{s}_{\{\pm 1\}}((\mathbb{Z}/n\mathbb{Z})^2) = 2n-1$. Also $\mathsf{s}_{\{\pm 1\}}((\mathbb{Z}/2\mathbb{Z})^2) = \mathsf{s}((\mathbb{Z}/2\mathbb{Z})^2) = 5$.

2. [10] For finite abelian groups of even exponent and fixed rank, we have $\mathsf{s}_{\{\pm 1\}}(G) = \exp(G) + \log_2|G| + O(\log_2\log_2|G|) \quad \text{as} \quad \exp(G)\to\infty.$

For the sake of completeness, we state the known results for other weights. Let p be a prime and G a finite abelian p-group. Then the following theorem has been proved in [10].

Theorem 3.1.3 *Let p be a prime and G an abelian p-group with $|G|>1$. Let A be a subset of $[1,\exp(G)]\setminus p\mathbb{Z}$ such that any two distinct elements of A are incongruent modulo p. Then for every positive integer k, any sequence over G of length $p^k - 1 + \lceil \frac{D(G)}{|A|}\rceil$ contains a non-empty A-weighted zero-sum sequence of length divisible by p^k. Thus, if $|A|(\exp(G)-1) \geq D(G)-1$ (which happens if $|A|$ is at least the rank of G), then we have*

$$\mathsf{s}_A(G) \leq \exp(G) - 1 + \left\lceil \frac{D(G)}{|A|}\right\rceil.$$

Under the conditions of Theorem 3.1.3, using the group-ring method, Thangadurai [96] showed that $D_A(G) \leq \left\lceil \frac{D(G)}{|A|}\right\rceil$. We can deduce this result as a corollary to Theorem 3.1.3 also. For, let S be a sequence over G of length $\left\lceil \frac{D(G)}{|A|} \right\rceil$ and consider the sequence $0^{\exp(G)-1}S$. Then Theorem 3.1.3 implies an A-weighted zero-sum subsequence of length $\exp(G)$ and deleting the 0's from this subsequence we get a non-empty A-weighted zero-sum subsequence of S.

Let n and d be two positive integers. The exact values of $D_A((\mathbb{Z}/n\mathbb{Z})^d)$ and $\mathsf{s}_A((\mathbb{Z}/n\mathbb{Z})^d)$ are known for several set of weights $A \subset [1, n-1]$.

Theorem 3.1.4 *Let n and d be two positive integers and p an odd prime.*

1. *[2, 54, 63] If $A = (\mathbb{Z}/p\mathbb{Z})^*$ then $D_A((\mathbb{Z}/p\mathbb{Z})^d) = d+1$, and $\mathsf{s}_A((\mathbb{Z}/p\mathbb{Z})^d) = p+d$ for $d < p$. For a composite number n, if $A = (\mathbb{Z}/n\mathbb{Z})^*$ then $D_A(\mathbb{Z}/n\mathbb{Z}) = 1 + \Omega(n)$ and $\mathsf{s}_A(\mathbb{Z}/n\mathbb{Z}) = n + \Omega(n)$.*

2. *[2, 8, 13] Let A be the set of quadratic residues mod p. Then $D_A(\mathbb{Z}/p\mathbb{Z}) = 3$ and $\mathsf{s}_A(\mathbb{Z}/p\mathbb{Z}) = p+2$. More generally, if $d \leq \frac{p-1}{2}$ then $\mathsf{s}_A((\mathbb{Z}/p\mathbb{Z})^d) = p+2d$. Also, if n is a square-free positive integer such that $(n,6) = 1$ and $R = \{x^2 \mid x \in \mathbb{Z}/n\mathbb{Z}\}$, then $D_R(\mathbb{Z}/n\mathbb{Z}) = 1 + 2\omega(n)$ and $\mathsf{s}_R(\mathbb{Z}/n\mathbb{Z}) = n + 2\omega(n)$.*

3. *[8, 13] Let r be a positive integer such that $1 < r < p$ and $A = \{1, 2, \ldots, r\}$. Then $D_A(\mathbb{Z}/p\mathbb{Z}) = \lceil \frac{p}{r} \rceil$ and $D_A((\mathbb{Z}/p\mathbb{Z})^d) \leq \lceil \frac{d(p-1)+1}{r} \rceil$, where for a real number x, $\lceil x \rceil$ denotes the smallest integer $\geq x$. For a composite number n, if $A = \{1, 2, \ldots, r\}$ where $1 < r < n$, then $D_A(\mathbb{Z}/n\mathbb{Z}) = \lceil \frac{n}{r} \rceil$ and $\mathsf{s}_A(\mathbb{Z}/n\mathbb{Z}) = n - 1 + \lceil \frac{n}{r} \rceil$.*

One main result of this chapter is the problem of determining the value of the constant $\mathsf{s}_{\{\pm 1\}}((\mathbb{Z}/p\mathbb{Z})^3)$. We prove the following result.

Theorem 3.1.5 *Let p be an odd prime. Then*

1. *$\mathsf{s}_{\{\pm 1\}}((\mathbb{Z}/p\mathbb{Z})^3) \geq 4p - 3$ and*

2. *Any sequence* $S = g_1 \cdot g_2 \cdot \ldots \cdot g_{4p-3}$ *over* $(\mathbb{Z}/p\mathbb{Z})^3$ *of length* $4p-3$ *with at least five zero-elements has a subsequence* $g_{i_1} \cdot \ldots \cdot g_{i_p}$ *of* S *of length* p *such that*

$$\omega_1 g_{i_1} + \cdots + \omega_p g_{i_p} = 0$$

where $\omega_i \in \{\pm 1\}$.

Remark 3.1.6 Let p be an odd prime. Then we have, $D_{\{\pm 1\}}((\mathbb{Z}/p\mathbb{Z})^3) \leq \lfloor 3\log_2 p \rfloor + 1$. Let $p > \lfloor 3\log_2 p \rfloor$ and S a sequence over $(\mathbb{Z}/p\mathbb{Z})^3$ of length p. If T is a maximal $\{\pm 1\}$-weighted zero-sum subsequence of S, then $|S| - |T| \leq \lfloor 3\log_2 p \rfloor$. Because otherwise, $|ST^{-1}| = |S| - |T| \geq \lfloor 3\log_2 p \rfloor + 1$ and so ST^{-1} has a non-empty $\{\pm 1\}$-weighted zero-sum subsequence T_1 of ST^{-1} and thus TT_1 is a $\{\pm 1\}$-weighted zero-sum subsequence of S, contradicting the maximality of T. Hence, given a sequence over $(\mathbb{Z}/p\mathbb{Z})^3$ of length $p + \lfloor 3\log_2 p \rfloor$ with $\lfloor 3\log_2 p \rfloor$ zeros, there must be a $\{\pm 1\}$-weighted zero-sum subsequence of length p.

In the next section, we further generalize the Erdős-Ginzburg-Ziv constant and compute their value in some cases. We use a suitable modification of a polynomial method of Rónyai [79] to prove our results.

3.2 A further generalisation of weighted Erdős-Ginzburg-Ziv Constant

Let G be a finite abelian group of exponent n. Here we define a generalization of the constant $\mathsf{s}(G)$ which was first studied in [14] and [40].

Definition 3.2.1 Let G be a finite abelian group of exponent n. For any integer $m \geq 1$, the constant $\mathsf{s}_m(G)$ is the least positive integer ℓ such that any sequence over G of length ℓ has a zero-sum subsequence of length mn.

In 2006, Gao and Thangadurai [43] studied this constant for the groups $(\mathbb{Z}/n\mathbb{Z})^3$ and $(\mathbb{Z}/n\mathbb{Z})^4$. In particular, using combinatorial technique they proved the following theorems.

Theorem 3.2.2 *Let $p \geq 5$ be an odd prime. Then we have*

1. $5p + \frac{p-1}{2} - 3 \leq \mathsf{s}_2((\mathbb{Z}/p\mathbb{Z})^3) \leq 6p - 3$,
2. $6p - 3 \leq \mathsf{s}_3(\mathbb{Z}/p\mathbb{Z})^3) \leq 8p - 7$,
3. $\mathsf{s}_m(\mathbb{Z}/p\mathbb{Z})^3) = mp + 3p - 3$ *for every* $m \geq 4$,
4. $\mathsf{s}_2((\mathbb{Z}/3\mathbb{Z})^3) = 13$, $15 \leq \mathsf{s}_3((\mathbb{Z}/3\mathbb{Z})^3) \leq 17$ *and* $\mathsf{s}_m((\mathbb{Z}/3\mathbb{Z})^3) = 3m + 6$ *for all* $m \geq 4$, *and*
5. $\mathsf{s}_m((\mathbb{Z}/2\mathbb{Z})^3) = 2m + 3$ *for every* $m \geq 2$.

Theorem 3.2.3 *Let m be a positive integer and $p \geq 7$ a prime. Then we have*

$$\mathsf{s}_{6m}(\mathbb{Z}/p\mathbb{Z})^4) \leq 6(m + 1)p - 4.$$

We further generalize this combinatorial constant with weights as follows.

Definition 3.2.4 Let G be a finite abelian group of exponent n and let A be a non-empty subset of $[1, n-1]$. For any integer $m \geq 1$, the constant $\mathsf{s}_{m,A}(G)$ is the least positive integer ℓ such that any sequence over G of length ℓ has an A-weighted zero-sum subsequence of length mn.

Clearly when $m = 1$, we get $\mathsf{s}_{1,A}(G) = \mathsf{s}_A(G)$.

Recently, Adhikari and Mazumdar [11] considered the rank 3 case and they proved the following result.

Theorem 3.2.5 *For an odd prime p, we have* $\mathsf{s}_{3,\{\pm 1\}}((\mathbb{Z}/p\mathbb{Z})^3) \leq \frac{9p-3}{2}$.

In another paper [12] they also proved the following result for elementary abelian p-groups of even rank.

Theorem 3.2.6 *Let p be an odd prime and $k \geq 3$ a divisor of $p-1$. Let θ be an element of order k of the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^*$ and A the subgroup of $(\mathbb{Z}/p\mathbb{Z})^*$ generated by θ. Then we have*

$$\mathsf{s}_{3,A}((\mathbb{Z}/p\mathbb{Z})^{2k}) \leq 5p-2.$$

Here in this chapter, we prove a result similar to Theorem 3.2.6 for elementary abelian p-groups of odd rank. More precisely, we prove the following.

Theorem 3.2.7 *Let p be an odd prime and $k \geq 2$ an even integer which divides $p-1$. Let θ be an element of order k of the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^*$ and A the subgroup of $(\mathbb{Z}/p\mathbb{Z})^*$ generated by θ. Then we have*

$$\mathsf{s}_{3,A}((\mathbb{Z}/p\mathbb{Z})^{k+1}) \leq 4p + \frac{p-1}{k} - 1.$$

Remark 3.2.8 If $(e_1, e_2, \ldots, e_{k+1})$ is a basis of $(\mathbb{Z}/p\mathbb{Z})^{k+1}$, then the sequence

$$\mathbf{0}^{3p-1}\prod_{i=1}^{k+1} e_i$$

has no A-weighted zero-sum subsequence of length $3p$. Therefore, we have

$$\mathsf{s}_{3,A}((\mathbb{Z}/p\mathbb{Z})^{k+1}) \geq 3p + k + 1.$$

Hence, for general k, there is a gap between this lower bound and the upper bound given by the theorem above. However if $k = p-1$, then the lower bound obtained above is $3p+k+1 = 4p$ and the upper bound obtained in the theorem is also $4p + \frac{p-1}{k} - 1 = 4p$.

Remark 3.2.9 Since p is an odd prime, $2 \mid (p-1)$ and therefore, by putting $k = 2$ in Theorem 3.2.7, we get Theorem 3.2.5.

3.3 Proof of Theorem 3.1.5

Proof of Theorem 3.1.5(1). Let p be an odd prime. We show that there exists a sequence over $(\mathbb{Z}/p\mathbb{Z})^3$ of length $4p-4$ which has no $\{\pm 1\}$-weighted zero-sum subsequence of length p.

Let (e_1, e_2, e_3) be a basis of $(\mathbb{Z}/p\mathbb{Z})^3$ and $e_0 = e_1 + e_2 + e_3$. Consider the sequence

$$S = \prod_{i=0}^{3} e_i^{p-1}$$

over $(\mathbb{Z}/p\mathbb{Z})^3$ of length $4p-4$. This sequence S has no $\{\pm 1\}$-weighted zero-sum subsequence of length p. Since to obtain $(0,0,0)$, we have to add an e_i with its additive inverse or we have to add the sum of e_1, e_2 and e_3 with the additive inverse of e_0, each involves an even number of elements in the subsequence. Therefore $\mathsf{s}_{\{\pm 1\}}(G) \geq 4p-3$. $\square$

Proof of Theorem 3.1.5(2). Let p be an odd prime and $S = g_1 \cdots g_{4p-3}$ any sequence over $(\mathbb{Z}/p\mathbb{Z})^3$ of length $4p-3$ with at least five zero-elements. We have to prove that S has a subsequence $g_{i_1} \cdots g_{i_p}$ of length p such that

$$\omega_1 g_{i_1} + \cdots + \omega_p g_{i_p} = 0 \quad \text{where} \quad \omega_i \in \{\pm 1\}.$$

Let $z_1, \ldots, z_5$ be the zero-elements in the sequence. Without loss of the generality, we assume that the remaining $4p-8$ elements are $g_1, \ldots, g_{4p-8}$.

If $p \leq 5$, then we trivially get a zero-sum subsequence of length p with the

z_i's. Hence we assume that $p > 5$.

Key step. Consider the $3p$ elements $g_1, \ldots, g_{3p-3}, z_1, z_2, z_3$ (which is possible, since for $p > 5$, $4p - 8 > 3p - 3$) and rewrite them as

$$a_1, b_1, c_1, \ldots, a_p, b_p, c_p,$$

where a_p, b_p, c_p are the elements z_1, z_2, z_3.

If the sums $a_i + b_j + c_k$ corresponding to the distinct triples (i, j, k) are all distinct, then they are all the p^3 elements of the group $(\mathbb{Z}/p\mathbb{Z})^3$. We add the sum of remaining $p-3$ elements of the sequence S to each of these distinct three-element sums. Clearly, these p-element sums are distinct and they are the p^3 elements of the group $(\mathbb{Z}/p\mathbb{Z})^3$. Hence one of these sums must be zero. Thus we get a zero-sum subsequences of S of length p.

If the sums $a_i+b_j+c_k$ are not all distinct, then two 3-sums are same and we get a non-empty $\{\pm 1\}$-weighted zero-sum subsequence T_1 of S not involving a_p, b_p and c_p. (For instance, if $a_1+b_1+c_p = a_2+b_3+c_4$, we have $a_1+b_1-a_2-b_3-c_4 = 0$ as $c_p = z_3 = 0$.) We observe that, $1 \leq |T_1| \leq 6$.

We remove the elements of T_1 from the sequence $g_1 \ldots g_{3p-3}$ and replace them by $|T_1|$ elements from $g_{3p-2}, \ldots, g_{4p-8}$ (which are $p-5$ in number).

We repeat the above mentioned "key-step" and stop when we reach the stage $p - 5 \leq |T_1 \cup T_2 \cup \cdots \cup T_r| \leq p$.

We adjoin some elements from $z_1, \ldots, z_5$ with $T_1 \cup T_2 \cup \cdots \cup T_r$ to get a $\{\pm 1\}$-weighted zero-sum subsequence of S of length p. □

Remark 3.3.1 It is easy to observe that the proof of the above theorem goes through if, instead of five zero-elements, the sequence $S = g_1 \cdots g_{4p-3}$ has three zero-elements and a pair of elements g_i, g_j such that either $g_i = g_j$ or $g_i = -g_j$.

Remark 3.3.2 For any odd integer $d \geq 3$, we can modify the counter example of Theorem 3.1.5(1) to get $\mathsf{s}_{\{\pm 1\}}((\mathbb{Z}/p\mathbb{Z})^d) \geq (d+1)p - d$.

Also we can generalize Theorem 3.1.5(2) as follows:

Let $d \geq 3$ be a positive integer and $p \geq \lfloor d \log_2 p \rfloor$ any prime. Then any sequence $S = g_1 \cdot g_2 \cdot \ldots \cdot g_{(d+1)p-d}$ over $(\mathbb{Z}/p\mathbb{Z})^d$ of length $(d+1)p - d$ with at least $2d - 1$ zero-elements has a subsequence $g_{i_1} \cdot g_{i_2} \cdot \ldots \cdot g_{i_p}$ of S of length p such that

$$\epsilon_1 g_{i_1} + \cdots + \epsilon_p g_{i_p} = 0 \quad \textit{where} \quad \epsilon_i \in \{\pm 1\}.$$

3.4 Proof of Theorem 3.2.7

We start with some lemmas. The following lemma has been proved in [12]; we record it here.

Lemma 3.4.1 *Let p be an odd prime and let k be a divisor of $p-1$. Let θ be an element of order k of $(\mathbb{Z}/p\mathbb{Z})^*$ and $D = \{0, \theta, \theta^2, \ldots, \theta^k = 1\}$. For a positive integer m, let*

$$\mathcal{C} = \{\text{functions} \quad f : D^m \to (\mathbb{Z}/p\mathbb{Z})\}$$

be the vector space over the field $\mathbb{Z}/p\mathbb{Z}$. Then the monomials $\prod_{1 \leq i \leq m} x_i^{r_i}$, $r_i \in [0, k]$ constitute a basis of $\mathcal{C}$ over $\mathbb{Z}/p\mathbb{Z}$.

Proof. It is easy to observe that the dimension of the space spanned by the monomials $\prod_{i=1}^m x_i^{r_i}$, $r_i \in [0, k]$ over $\mathbb{Z}/p\mathbb{Z}$ is $(k+1)^m$ which is the same as the dimension of the $\mathbb{Z}/p\mathbb{Z}$ vector space $\mathcal{C}$. Therefore it is sufficient to verify that every element of $\mathcal{C}$ can be expressed as a $\mathbb{Z}/p\mathbb{Z}$-linear combination of the monomials $\prod_{1 \leq i \leq m} x_i^{r_i}$, $r_i \in [0, k]$. Since the space $\mathcal{C}$ is generated by the characteristic functions, it is enough to prove the required representation for the characteristic functions.

For a point $(x_1, x_2, \ldots, x_m)$ in D^m, and a subset W of $[1, m]$, we consider the following functions

$$f_{0,W}(x_1, x_2, \ldots, x_m) := \prod_{j \in W} (1 - x_j^k),$$

$$f_{k,W}(x_1, x_2, \ldots, x_m) := \prod_{j \in W} x_j k^{-1}(1 + x_j + \cdots + x_j^{k-1}) \text{ and}$$

$$f_{t,W}(x_1, x_2, \ldots, x_m) := \prod_{j \in W} \frac{\prod_{i \neq t}(x_j - \theta^i)}{\prod_{i \neq t}(\theta^t - \theta^i)} \quad \text{for every } t \in [1, k-1].$$

Let $W_0, W_1, \ldots, W_k$ be a partition of $[1, m]$. Then the function

$$f_{W_0, W_1, \ldots, W_k}(x_1, x_2, \ldots, x_m) := \prod_{t \in [0,k]} f_{t, W_t}(x_1, x_2, \ldots, x_m)$$

takes the value 1 precisely at the point $(x_1, x_2, \ldots, x_m)$ of D^m where $x_j = 0$ for $j \in W_0$, $x_j = \theta^t$ for $j \in W_t$ for $t \in [1, k]$. By expanding the right hand side we get an expression in the required form and we are through. □

Lemma 3.4.2 *Let p be an odd prime and $k \geq 2$ an even integer which divides $p - 1$. Let θ be an element of $(\mathbb{Z}/p\mathbb{Z})^*$ of order k and $A = \{\theta, \theta^2, \ldots, \theta^k = 1\}$ the subgroup of $(\mathbb{Z}/p\mathbb{Z})^*$ generated by θ. Let $S = \prod_{i=1}^{t} w_i$ be a sequence over $(\mathbb{Z}/p\mathbb{Z})^{k+1}$ of length $t = 2p + \dfrac{p-1}{k} - 1$. Then S has an A-weighted zero-sum subsequence of length either p or $2p$.*

Proof. For all integers $i = 1, 2, \ldots, t$, we let $w_i = (a_{i1}, a_{i2}, \ldots, a_{i(k+1)}) \in (\mathbb{Z}/p\mathbb{Z})^{k+1}$.

We consider the following system of equations over $\mathbb{Z}/p\mathbb{Z}$.

$$\sum_{i=1}^{t} a_{i1}x_i^{\frac{p-1}{k}} = 0, \qquad \sum_{i=1}^{t} a_{i2}x_i^{\frac{p-1}{k}} = 0, \ldots, \sum_{i=1}^{t} a_{i(k+1)}x_i^{\frac{p-1}{k}} = 0$$

and

$$\sum_{i=1}^{t} x_i^{p-1} = 0.$$

Note that the sum of the degrees of the polynomials is $(k+1)\frac{p-1}{k} + (p-1) = 2p + \frac{p-1}{k} - 2 < 2p + \frac{p-1}{k} - 1 = t$, the number of variables.

Since the above system has the trivial zero solution, by Theorem 2.3.12, there exists a non-zero solution $(y_1, y_2, \ldots, y_t) \in (\mathbb{Z}/p\mathbb{Z})^t$ of the above system.

If we write $I = \{i : y_i \not\equiv 0 \pmod{p}\}$, then from the first $(k+1)$ equations, we get

$$\sum_{i \in I} y_i^{(p-1)/k}(a_{i1}, a_{i2}, \ldots, a_{(k+1)i}) = (0, 0, \ldots, 0)$$

and from the last equation, we get $|I| \equiv 0 \pmod{p}$. Since $y_i \not\equiv 0 \pmod{p}$ for all $i \in I$, we see that $y_i^{(p-1)/k} \in A$. Since $t < 3p$ we get either $|I| = p$ or $|I| = 2p$. Hence, we conclude that the sequence S has an A-weighted zero-sum subsequence of length either p or $2p$.

Corollary 3.4.3 *Let p be an odd prime and $k \geq 2$ an even integer which divides $p-1$. Let θ be an element of $(\mathbb{Z}/p\mathbb{Z})^*$ of order k and $A = \{\theta, \theta^2, \ldots, \theta^k = 1\}$ the subgroup of $(\mathbb{Z}/p\mathbb{Z})^*$ generated by θ. Let $S = \prod_{i=1}^{t} w_i$ be a sequence over $(\mathbb{Z}/p\mathbb{Z})^{k+1}$ of length $t = 3p + \dfrac{p-1}{k} - 1$. Then S has an A-weighted zero-sum subsequence of length $2p$.*

Proof. Since the given sequence S is of length $t = 3p + \frac{p-1}{k} - 1$ over $(\mathbb{Z}/p\mathbb{Z})^{k+1}$, it has an A-weighted zero-sum subsequence T of length either p or $2p$ by Lemma 3.4.2. If T is of length $2p$, then we are done. Otherwise, consider the deleted sequence ST^{-1} which is of length

$$3p + \frac{p-1}{k} - 1 - p = 2p + \frac{p-1}{k} - 1$$

and hence, by Lemma 3.4.2, we get that ST^{-1} has an A-weighted zero-sum subsequence T_1 of length either p or $2p$. If $|T_1| = 2p$, then we are done. If $|T_1| = p$, then TT_1 is of length $2p$ and it is the required subsequence.

Corollary 3.4.4 *Let p be an odd prime and $k \geq 2$ an even integer which divides $p-1$. Let θ be an element of $(\mathbb{Z}/p\mathbb{Z})^*$ of order k and $A = \{\theta, \theta^2, \ldots, \theta^k = 1\}$ the subgroup of $(\mathbb{Z}/p\mathbb{Z})^*$ generated by θ. Let $S = \prod_{i=1}^{t} w_i$ be a sequence over $(\mathbb{Z}/p\mathbb{Z})^{k+1}$ of length $t = 4p + \dfrac{p-1}{k} - 1$. If S has an A-weighted zero-sum subsequence of length p, then it has an A-weighted zero-sum subsequence of length $3p$.*

Proof. Since S has an A-weighted zero-sum subsequence T of length p, consider the deleted sequence ST^{-1} which is of length $3p+\frac{p-1}{k}-1$. Therefore, by Corollary 3.4.3, we get an A-weighted zero-sum subsequence T_1 of ST^{-1} of length $2p$. Hence, TT_1 is the required zero-sum subsequence. □

Proof of Theorem 3.2.7. Let p be an odd prime and $k \geq 2$ an even integer such that k divides $p-1$. Let θ be an element of order k of the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^*$ and A the subgroup of $(\mathbb{Z}/p\mathbb{Z})^*$ generated by θ. We have to show that

$$\mathsf{s}_{3,A}((\mathbb{Z}/p\mathbb{Z})^{k+1}) \leq 4p + \frac{p-1}{k} - 1.$$

Note that D as defined in Lemma 3.4.1 is $A \cup \{0\}$.

Let $S = \prod_{i=1}^{m} w_i$ be a sequence over $(\mathbb{Z}/p\mathbb{Z})^{k+1}$ of length $m = 4p + \frac{p-1}{k} - 1$. For all $i = 1, 2, \ldots, m$, we let $w_i = (a_{i1}, a_{i2}, \ldots, a_{i(k+1)}) \in (\mathbb{Z}/p\mathbb{Z})^{k+1}$. We prove that S has an A-weighted zero-sum subsequence of length $3p$.

If possible, suppose that S has no A-weighted zero-sum subsequence of length $3p$. Therefore, by Corollary 3.4.4, S cannot have any A-weighted zero-sum subsequence of length p. Thus, if $T = \prod_{j=1}^{\ell} w_{i_j}$ is a subsequence of S of length $\ell = 3p$ or p, then for any $(z_1, \ldots, z_\ell) \in A^\ell$, we have

$$z_1 w_{i_1} + \cdots + z_\ell w_{i_\ell} \not\equiv (0, 0, \ldots, 0) \pmod p. \tag{3.1}$$

In order to get a contradiction, we need to invoke Lemma 3.4.1. For this purpose, we shall introduce some polynomials as follows. Let

$$\sigma(x_1, x_2, \ldots, x_m) = \sum_{\substack{I \subset [1,m], \\ |I| = p}} \prod_{i \in I} x_i^k,$$

be the p-th elementary symmetric polynomial of the variables $x_1^k, x_2^k, \ldots, x_m^k$. We also consider the following polynomials, $P_1(x_1, x_2, \ldots, x_m) =$

$$\left(\left(\sum_{i=1}^m a_{i1} x_i\right)^{p-1} - 1\right)\left(\left(\sum_{i=1}^m a_{i2} x_i\right)^{p-1} - 1\right) \cdots \left(\left(\sum_{i=1}^m a_{i(k+1)} x_i\right)^{p-1} - 1\right),$$

$$P_2(x_1, x_2, \ldots, x_m) = \left(\left(\sum_{i=1}^m x_i^k\right)^{p-1} - 1\right),$$

$$P_3(x_1, x_2, \ldots, x_m) = (\sigma(x_1, x_2, \ldots, x_m) - 2)(\sigma(x_1, x_2, \ldots, x_m) - 4)$$

and

$$P(x_1, x_2, \ldots, x_m) = P_1(x_1, x_2, \ldots, x_m) P_2(x_1, x_2, \ldots, x_m) P_3(x_1, x_2, \ldots, x_m).$$

First, we note that

$$\deg(P) = (k+1)(p-1) + k(p-1) + 2kp = 4kp + p - 1 - 2k. \tag{3.2}$$

Claim. $P(\alpha_1, \dots, \alpha_m) = 0$ for all $(\alpha_1, \dots, \alpha_m) \in D^m \backslash \{(0, 0, \dots, 0)\}$ and $P(0, 0, \dots, 0) = 8$.

Let $\alpha = (\alpha_1, \dots, \alpha_m) \in D^m \backslash \{(0, 0, \dots, 0)\}$ be an arbitrary element.

If the number of non-zero entries of α is not a multiple of p and if we take $I = \{1 \leq i \leq m : \alpha_i \neq 0\}$, then

$$\left(\left(\sum_{i=1}^{m} \alpha_i^k\right)^{p-1} - 1\right) = \left(\left(\sum_{i \in I} \alpha_i^k\right)^{p-1} - 1\right) = 0$$

by Fermat's Little Theorem and hence we get $P_2(\alpha_1, \dots, \alpha_m) = 0$.

If the number of non-zero entries of α is either p or $3p$, then by (3.1), we get $P_1(\alpha_1, \dots, \alpha_m) = 0$.

If the number of non-zero entries of α is $2p$, then $\sigma(\alpha) = \binom{2p}{p} = 2 \in \mathbb{Z}/p\mathbb{Z}$ and if the number of non-zero entries of α is $4p$, then $\sigma(\alpha) = \binom{4p}{p} = 4 \in \mathbb{Z}/p\mathbb{Z}$. Therefore, if the number of non-zero entries of α is either $2p$ or $4p$, then $P_3(\alpha_1, \dots, \alpha_m) = 0$. Therefore the polynomial $P(x_1, x_2, \dots, x_m)$ vanishes at all the points of D^m, except at $(0, 0, \dots, 0)$ and $P(0, 0, \dots, 0) = 8$. This proves the claim.

We now consider the function $P : D^m \to \mathbb{Z}/p\mathbb{Z}$ in $\mathcal{C}$ given by the polynomial $P(\alpha_1, \dots, \alpha_m)$.

Now let $R = 8(1 - x_1^k)(1 - x_2^k) \dots (1 - x_m^k) \in (\mathbb{Z}/p\mathbb{Z})[x_1, \dots, x_m]$. Then $R(\alpha_1, \dots, \alpha_m) = 0$ for all $\alpha = (\alpha_1, \dots, \alpha_m) \in D^m \backslash \{(0, 0, \dots, 0)\}$ and $R(0, \dots, 0) = 8$.

Therefore, the functions $P(x_1, \dots, x_m)$ and $R(x_1, \dots, x_m)$ are equal as elements in $\mathcal{C}$.

By Lemma 3.4.1, we know that $\mathcal{C}$ has a special basis consisting of monomials of the form $\prod_{1 \leq i \leq m} x_i^{r_i}$, $r_i \in [0, k]$. Now, we write P as a linear combination of these basis elements by replacing each x_i^{tk+r} for some integers $t \geq 1$ and $r \in [1, k]$

by x_i^r and let Q be the polynomial obtained in this way. Also in this process, the degree of the polynomial Q is not increased. Hence by (3.2) we get, deg $Q \leq 4kp + p - 1 - 2k$. Clearly, as elements in $\mathcal{C}$, the functions P and Q are the same. Hence, Q and R are the same as elements in $\mathcal{C}$.

However, deg $R = mk = 4kp + p - 1 - k > 4kp + p - 1 - 2k \geq$ deg Q.

This leads to a nontrivial relation among the basis elements consisting of the monomials $\prod_{1 \leq i \leq m} x_i^{r_i}$, which is impossible. □

Remark 3.4.5 Let p be an odd prime and $k \geq 2$ an even integer which divides $p-1$. Let θ be an element of $(\mathbb{Z}/p\mathbb{Z})^*$ of order k and $A = \{\theta, \theta^2, \ldots, \theta^k = 1\}$ the subgroup of $(\mathbb{Z}/p\mathbb{Z})^*$ generated by θ. Let $S = \prod_{i=1}^{t} w_i$ be a sequence over $(\mathbb{Z}/p\mathbb{Z})^{k+1}$ of length $t = 5p + \dfrac{p-1}{k} - 1$. Then by applying Corollary 3.4.3 twice, we get an A-weighted zero-sum subsequence of S of length $4p$. In other words,

$$\mathsf{s}_{4,A}((\mathbb{Z}/p\mathbb{Z})^{k+1}) \leq 5p + \frac{p-1}{k} - 1.$$

Remark 3.4.6 Let p be an odd prime and $k \geq 2$ an even integer which divides $p-1$. Let θ be an element of $(\mathbb{Z}/p\mathbb{Z})^*$ of order k and $A = \{\theta, \theta^2, \ldots, \theta^k = 1\}$ the subgroup of $(\mathbb{Z}/p\mathbb{Z})^*$ generated by θ. Let $S = \prod_{i=1}^{t} w_i$ be a sequence over $(\mathbb{Z}/p\mathbb{Z})^{k+1}$ of length $t = 6p + \dfrac{p-1}{k} - 1$. Then S has an A-weighted zero-sum subsequence of length $5p$. In other words,

$$\mathsf{s}_{5,A}((\mathbb{Z}/p\mathbb{Z})^{k+1}) \leq 6p + \frac{p-1}{k} - 1.$$

Proof. Since the given sequence S over $(\mathbb{Z}/p\mathbb{Z})^{k+1}$ is of length $t = 6p + \frac{p-1}{k} - 1$, S has an A-weighted zero-sum subsequence T of length $3p$ by Theorem 3.2.7. Now consider the deleted sequence ST^{-1} which is of length $3p + \frac{p-1}{k} - 1$ and hence, by Corollary 3.4.3, we get that ST^{-1} has an A-weighted zero-sum subsequence

T_1 of length $2p$. Then TT_1 is of length $5p$ and it is the required subsequence. $\square$

Remark 3.4.7 By repeatedly taking out A-weighted zero-sum subsequences of length $2p$ by using Corollary 3.4.3, and using Remark 3.4.6, we get bounds for $\mathsf{s}_{m,A}((\mathbb{Z}/p\mathbb{Z})^{k+1})$ for odd integers $m \geq 7$. Similarly using Corollary 3.4.3 and Remark 3.4.5, we get bounds for $\mathsf{s}_{m,A}((\mathbb{Z}/p\mathbb{Z})^{k+1})$ for even integers $m \geq 6$. More precisely, for all integers $m \geq 6$, we have

$$\mathsf{s}_{m,A}((\mathbb{Z}/p\mathbb{Z})^{k+1}) \leq (m+1)p + \frac{p-1}{k} - 1.$$

CHAPTER 4

The Determination of Zero-sum ℓ-Generalized Schur Numbers

Let k and r be two positive integers such that r divides k and $\mathcal{E}$ the equation $x_1 + \cdots + x_{k-1} = x_k$. The 2-color r-zero-sum generalized Schur number $S_{\mathfrak{z},2}(k;r)$ is defined to be the least positive integer t such that for any 2-coloring $\chi : [1,t] \to \{0,1\}$ there exists a solution $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_k)$ to the equation $\mathcal{E}$ satisfying $\sum_{i=1}^{k} \chi(\hat{x}_i) \equiv 0 \pmod{r}$. In a recent paper [76], the question of determining the exact value of $S_{\mathfrak{z},2}(k;4)$ was posed. In this chapter, we show that, $S_{\mathfrak{z},2}(k,r) = kr-2r+1$ for all positive integers k and r with $r \mid k$ and $k \geq 2r$. We also generalize this constant and compute its exact value for some cases. The content of this chapter is published in the articles [77] and [81].

4.1 Introduction

Let r be a positive integer. We recall few definitions from Chapter 1.

Definition 4.1.1 Let S be a subset of $\mathbb{N}$ and $\chi : S \to \{1, 2, \ldots, r\}$ an r-coloring of S. A subset A of S is said to be *monochromatic* under χ if χ is constant on A.

Definition 4.1.2 Let $\chi : \mathbb{N} \to \{1, 2, \ldots, r\}$ be an r-coloring of $\mathbb{N}$ and L an equation or a system of equations in n variables. A solution $(x_1, x_2, \ldots, x_n)$ of L is said to be a *monochromatic solution* to L, if the set $\{x_1, x_2, \ldots, x_n\}$ is monochromatic under χ.

Definition 4.1.3 Let L be an equation or a system of equations. L is said to be r*-regular* if for every r-coloring of $\mathbb{N}$ there exists a monochromatic solution to L. It is said to be *regular* if it is r-regular for all $r \geq 1$.

We also recall the following theorems from Chapter 1 as well.

Theorem 4.1.4 *Let r be a positive integer. Then there exists a smallest positive integer $S(r)$ such that for every r-coloring of $[1, S(r)]$ there is a monochromatic solution to the equation $x + y = z$.*

Theorem 4.1.5 *[73] Let $k \geq 2$ be a positive integer and $c_1, \ldots, c_k$ non-zero integers. Then the linear Diophantine equation $c_1x_1 + \cdots + c_kx_k = 0$ is regular if and only if $\sum_{i \in I} c_i = 0$ for some non-empty subset I of $\{1, \ldots, k\}$.*

Let $k \geq 2$ and $r \geq 1$ be two positive integers and $c_1, \ldots, c_k$ non-zero integers satisfying $\sum_{i \in I} c_i = 0$ for some non-empty subset $I \subseteq \{1, \ldots, k\}$. Then by Theorem 4.1.5, there exists a smallest positive integer $S(k; r, c_1, \ldots, c_k)$ such that for every r-coloring of $[1, S(k; r, c_1, \ldots, c_k)]$ there is a monochromatic solution to the equation $c_1x_1 + \cdots + c_kx_k = 0$.

Thus, we can define the *generalized Schur number* as follows.

Definition 4.1.6 Let $k \geq 2$ and $r \geq 1$ be two positive integers. Then there exists a least positive integer $S(k;r)$, called *generalized Schur number*, such that for every r-coloring of $[1, S(k;r)]$ there is a monochromatic solution to the equation $x_1 + x_2 + \cdots + x_{k-1} = x_k$.

Indeed, Theorem 4.1.5 proves that the number $S(k,r)$ exists and is finite. In [20], Beutelspacher and Brestovansky proved the exact value $S(k;2) = k^2 - k - 1$. But in general, it is very difficult to find the exact value of this number. In this chapter, we discuss a weaker version of this constant and compute its value. We need the following definition.

Definition 4.1.7 Let r be a positive integer. We say that a set of integers $\{a_1, a_2, \ldots, a_n\}$ is *r-zero-sum* if $\sum_{i=1}^{n} a_i \equiv 0 \pmod{r}$.

In [76], Robertson replaced the "*monochromatic property*" in the definition of generalized Schur number by the "*zero-sum property*" and introduced the *zero-sum generalized Schur number.*

Definition 4.1.8 Let k and r be two positive integers such that r divides k. We define the *zero-sum generalized Schur number* $S_{\mathfrak{z}}(k;r)$ to be the least positive integer t such that for any r-coloring $\chi : [1,t] \to \{0, 1, \ldots, r-1\}$ there exists a solution $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_k)$ to the equation $x_1 + \cdots + x_{k-1} = x_k$ satisfying $\displaystyle\sum_{i=1}^{k} \chi(\hat{x}_i) \equiv 0 \pmod{r}$.

Notation. We denote the equation $x_1 + \cdots + x_{k-1} = x_k$ by $\mathcal{E}$.

When r divides k, the zero-sum property is weaker than the monochromatic property in the sense that, any monochromatic solution to $\mathcal{E}$ is an r-zero-sum solution. Let $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_k)$ be a monochromatic solution to the equation $\mathcal{E}$ with respect to an r-coloring χ. Then $\sum_{i=1}^{k} \chi(\hat{x}_i) = k\chi(\hat{x}_1) \equiv 0 \pmod{r}$, and hence the solution is an r-zero-sum solution. Hence, we get, $S_{\mathfrak{z}}(k;r) \leq S(k;r)$, and therefore $S_{\mathfrak{z}}(k;r)$ is finite. But when $r \nmid k$, coloring every positive integers with the color 1 does not admit an r-zero-sum solution to $\mathcal{E}$. Hence we always assume that $r \mid k$.

In [76], Robertson calculated lower bounds of this number for some r. In particular, he proved the following result.

Theorem 4.1.9 *[76] Let k and r be two positive integers such that r divides k. Then*

$$S_{\mathfrak{z}}(k;r) \geq \begin{cases} 3k-3 & \textit{when } r=3; \\ 4k-5 & \textit{when } r=4; \\ 2(k^2-k-1) & \textit{when } r=k \textit{ is an odd positive integer.} \end{cases}$$

He also asked the following questions.

1. Is it true that $S_{\mathfrak{z}}(k;3) = 3k-3$ for $k \geq 6$ and $S_{\mathfrak{z}}(k;4) = 4k-5$ for $k \geq 8$?

2. Is it true that $S_{\mathfrak{z}}(k;k)$ is of order k^2?

Recently in [67], E. Metz showed the exact values of this constant for $r = 3, 4$. Moreover, he proved the following results.

Theorem 4.1.10 *[67] Let r and k be two positive integers such that r divides k and $k \geq 2r$. Then*

$$S_{\mathfrak{z}}(k;r) \leq \begin{cases} kr-r & ; r \textit{ is an odd prime} \\ 4k-5 & ; r=4 \\ kr-\sum_{i=1}^{t}(p_i-1)-1 & ; r \geq 6 \textit{ and } r=p_1\dots p_t \\ & \quad \textit{be the prime decomposition of } r \\ & \quad \textit{and } p_i\textit{'s are not necessarily distinct .} \end{cases}$$

Theorem 4.1.11 *[67] Let r and k be two positive integers such that r divides k. Then*

$$S_{\mathfrak{z}}(k;r) \geq \begin{cases} kr - r & ; r \text{ is an odd integer} \\ kr - r - 1 & ; r \text{ is an even integer.} \end{cases}$$

In the same article [76], Robertson introduced another constant meant only for 2-colorings, but keeping the r-zero-sum notion.

Definition 4.1.12 Let k and r be two positive integers such that r divides k. The 2-*color zero-sum generalized Schur number* $S_{\mathfrak{z},2}(k;r)$ is defined to be the least positive integer t such that every 2-coloring $\chi : [1,t] \to \{0,1\}$ admits a solution $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_k)$ to the equation $\mathcal{E}$ satisfying $\sum_{i=1}^{k} \chi(\hat{x}_i) \equiv 0 \pmod{r}$.

Since any 2-coloring of $[1, S_{\mathfrak{z}}(k;r)]$ is also an r-coloring (for $r \geq 2$), we see that $S_{\mathfrak{z},2}(k;r) \leq S_{\mathfrak{z}}(k;r)$ and hence $S_{\mathfrak{z},2}(k;r)$ is finite. Furthermore, in the case when $k = r$ we recover the generalized Schur number $S(k;2)$

In [76], Robertson proved the following theorem related to this 2-color zero-sum generalized Schur number.

Theorem 4.1.13 *[76] Let k and r be two positive integers such that r divides k. Then*

$$S_{\mathfrak{z},2}(k;r) = \begin{cases} 2k - 3; & \text{if } r = 2 \\ 3k - 5; & \text{if } r = 3 \text{ and } k \neq 3 \\ k^2 - k - 1; & \text{if } r = k \end{cases}$$

He also asked the question that

1. What is the exact value of $S_{\mathfrak{z},2}(k;4)$?

We note that the exact values of $S_{\mathfrak{z},2}(k;r)$ for $r = 2, 3$ and $S_{\mathfrak{z},2}(r,r)$ do not show any obvious generalization to $S_{\mathfrak{z},2}(k;r)$ for any k which is a multiple of r.

However, the computations given in [76] when $r = 4$ and $k = 4, 8, 12$ and when $r = 5$ and $k = 5, 10, 15$ were enough for us to conjecture a general formula, which turns out to be true. To this end, by Theorem 4.1.14 below, we answer the above question and, more generally, determine the exact values of $S_{\mathfrak{z},2}(k; r)$.

Theorem 4.1.14 *[77] Let k and r be two positive integers such that r divides k and $k \geq 2r$. Then $S_{\mathfrak{z},2}(k; r) = rk - 2r + 1$.*

We further generalize the zero-sum generalized Schur number and 2-color zero-sum generalized Schur number. But before that we fix the following notation.

Notation: Let r and k be two positive integers such that r divides k. For given integers $\ell \in [1, k]$, $v \in \left[0, \lfloor \frac{k-1}{2r} \rfloor\right]$ and $\epsilon \in \{0, 1\}$, we set the linear homogeneous equation as follows.

$$\mathcal{E}_v^{(\ell,\epsilon)} : x_1 + \cdots + x_{k-(rv+\epsilon)} = x_{k-(rv+\epsilon-1)} + \cdots + x_{k-1} + \ell x_k. \tag{4.1}$$

Definition 4.1.15 Let r, m and $k \geq 2$ be positive integers such that r divides k. Let $v \in \left[0, \lfloor \frac{k-1}{2r} \rfloor\right]$ and $\epsilon \in \{0, 1\}$ be two integers and ℓ a positive integer such that $\ell \in [1, k]$. Then the *zero-sum ℓ-generalized Schur number* $S_{\mathfrak{z},m}^{(\ell,\epsilon)}(k; r; v)$ is defined to be the least positive integer t such that for every m-coloring $\chi : [1, t] \to \{0, 1, \ldots, m-1\}$ there is a solution $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_k)$ to the equation $\mathcal{E}_v^{(\ell,\epsilon)}$ satisfying $\sum_{i=1}^{k} \chi(\hat{x}_i) \equiv 0 \pmod{r}$.

For integers $\ell \in [1, k]$, $v \in \left[1, \lfloor \frac{k-1}{2r} \rfloor\right]$ and $\epsilon \in \{0, 1\}$ or $\ell \in [1, k-1], v = 0$ and $\epsilon = 1$, the linear equation $\mathcal{E}_v^{(\ell,\epsilon)}$ satisfies the condition of Theorem 4.1.5 and hence $S_{\mathfrak{z},m}^{(\ell,\epsilon)}(k; r; v)$ are finite for any k, r and m.

We note that when $\ell = \epsilon = 1$, $v = 0$ and $m = r$, the constant $S_{\mathfrak{z},r}^{(1,1)}(k; r; 0)$ is the zero-sum generalized Schur number $S_{\mathfrak{z}}(k; r)$. Also, when $\ell = \epsilon = 1$, $v = 0$

and $m = 2$, the constant $S^{(1,1)}_{\mathfrak{z},2}(k; r; 0)$ is the 2-color zero-sum generalized Schur number $S_{\mathfrak{z},2}(k; r)$.

In general, we don't know the exact values of $S_{\mathfrak{z}}(k; r)$. Thus, the study of this constant in more general way (like; Definition 4.1.15) may shed some light towards the exact values of $S_{\mathfrak{z}}(k; r)$.

First note that, when $\ell = k-1, v = 0$ and $\epsilon = 1$, the equation $\mathcal{E}^{(k-1,1)}_0$: $x_1 + x_2 + \cdots + x_{k-1} = (k-1)x_k$ satisfies the condition of Theorem 4.1.5. Also $(1, 1, \ldots, 1)$ satisfies the equation together with the r-zero-sum condition. Thus we see that $S^{(k-1,1)}_{\mathfrak{z},m}(k; r; 0) = 1$ for each $m \geq 2$.

The equation $\mathcal{E}^{(k,1)}_0 : x_1 + \cdots + x_{k-1} = kx_k$ does not satisfy the condition of Theorem 4.1.5. Hence, we can not conclude the finiteness of the constant $S^{(k,1)}_{\mathfrak{z},2}(k; r; 0)$. However, we prove the finiteness of this constant by calculating the exact value as follows.

Theorem 4.1.16 *[81] Let r and k be positive integers such that r divides k and $k \geq 2$. Then*

$$S^{(k,1)}_{\mathfrak{z},2}(k; r; 0) = \begin{cases} 3 & ; r = 2 \text{ and } k \geq 4 \\ 4 & ; r \geq 3 \text{ and } k = r \text{ or } k = 2r \\ 3 & ; r \geq 3 \text{ and } k \geq 3r. \end{cases}$$

Now we move on to the case when v is not zero in the equation $\mathcal{E}^{(\ell,\epsilon)}_v$.

Theorem 4.1.17 *[81] Let k be an even positive integer and $v \in [1, \lfloor \frac{k-1}{4} \rfloor]$ an integer. Then*

$$S^{(1,1)}_{\mathfrak{z},2}(k; 2; v) \leq \left(\frac{k}{2} - 2v\right).$$

In the following theorem, we compute the exact value of the constant $S^{(\ell,\epsilon)}_{\mathfrak{z},m}(k; r; v)$ when $\epsilon = 0$, $l = 1$ and $m = r$.

Theorem 4.1.18 *[81] Let r and k be two positive integers such that r divides k and $v \in [1, \lfloor \frac{k-1}{2r} \rfloor]$ an integer. Then*

$$S_{\mathfrak{z},r}^{(1,0)}(k;r;v) = \frac{k}{r} - \left\lfloor \frac{(v-1)k}{vr} \right\rfloor - 1.$$

4.2 Proof of the Theorems

4.2.1 Proof of Theorem 4.1.14

We start by presenting a pair of lemmas useful for proving our upper bounds.

Lemma 4.2.1 *Let k and r be positive integers such that r divides k and $k \geq 2r$. Let $\chi : [1, rk-2r+1] \to \{0,1\}$ be a 2-coloring such that $\chi(1) = \chi(r-1) = 0$. Then there exists an r-zero-sum solution to equation $\mathcal{E} : x_1 + x_2 + \cdots + x_{k-1} = x_k$ under χ.*

Proof. Consider the solution $(1, 1, \ldots, 1, k-1)$ to the equation $\mathcal{E}$. If $\chi(k-1) = 0$, then, this solution is an r-zero-sum solution as $\chi(1) = 0$, and we are done. Hence, we assume that $\chi(k-1) = 1$.

Next, we look at the solution

$$(\underbrace{1, \ldots, 1}_{k-r}, \underbrace{k-1, k-1, k-1}_{r-1}, rk-2r+1).$$

Since $\chi(1) = 0$ and $\chi(k-1) = 1$, we can assume that $\chi(rk-2r+1) = 0$; otherwise, we have exactly r integers of color 1 and hence we get the solution is r-zero-sum.

Since $(1, r, r, \ldots, r, rk-2r+1)$ is a solution to $\mathcal{E}$, we can assume that $\chi(r) = 1$.

Finally, consider the solution

$$(\underbrace{r-1,\ldots,r-1}_{r-1},\underbrace{r,\ldots,r}_{k-r},rk-2r+1).$$

Since $\chi(r-1)=0, \chi(r)=1$, $\chi(rk-2r+1)=0$, and $r \mid k$, this solution is an r-zero-sum solution, thereby proving the lemma. $\square$

Lemma 4.2.2 *Let k and r be positive integers such that r divides k and $k \geq 2r$. Let $\chi : [1, rk-2r+1] \to \{0,1\}$ be a 2-coloring such that $\chi(1)=0$ and $\chi(r-1)=1$. If one of the following*

(a) $\chi(k-1)=0$;

(b) $\chi(k)=0$;

(c) $\chi(rk-2r+1)=1$;

(d) $\chi(r)=0$

(e) $\chi(k-2)=1$;

(f) $\chi(rk-2r-1)=0$.

holds, then there exists an r-zero-sum solution to the equation $\mathcal{E}$.

Proof. We prove each possibility separately; however, the order in which we enumerated is unimportant.

(a) Consider the solution $(1,1,\ldots,1,k-1)$ to the equation $\mathcal{E}$. If $\chi(k-1)=0$ then this solution is an r-zero sum.

(b) By considering the solution $(r-1,\ldots,r-1,(r-1)(k-1))$, we can assume that $\chi((r-1)(k-1))=0$. Using this in $(\underbrace{1,\ldots,1}_{k-r+1},\underbrace{k,\ldots,k}_{r-2},(r-1)(k-1))$ along with the assumption that $\chi(k)=0$, we have an r-zero-sum solution.

(c) From part (a), we may assume that $\chi(k-1)=1$. Looking at $(r-1,\dots,r-1,k-1,rk-2r+1)$, since $\chi(k-1)=\chi(r-1)=1$, and we assume that $\chi(rk-2r+1)=1$, we have an r-zero sum solution.

(d) From part (c), we may assume that $\chi(rk-2r+1)=0$. With this assumption, we see that the solution $(1,r,\dots,r,rk-2r+1)$ is an r-zero-sum when $\chi(r)=0$.

(e) From parts (d) and (c), we may assume $\chi(r)=1$ and $\chi(rk-2r+1)=0$. Under these assumptions, we find that the solution $(\underbrace{1,\dots,1}_{k-r-1},r,\underbrace{k-2,\dots,k-2}_{r-1},rk-2r+1)$ is an r-zero-sum solution when $\chi(k-2)=1$.

(f) By considering the solution $(\underbrace{1,\dots,1}_{r-1},\underbrace{r,\dots,r}_{k-2r},\underbrace{2r-3,\dots,2r-3}_{r},rk-2r-1)$ and using $r\mid k$, we have an r-zero-sum solution when $\chi(rk-2r-1)=0$. $\square$

Proof of Theorem 4.1.14. Let k and r be two positive integers such that r divides k and $k\geq 2r$. Then, we have to show that

$$S_{\mathfrak{z},2}(k;r)=rk-2r+1.$$

We start with the lower bound.

Lower bound. To prove that $S_{\mathfrak{z},2}(k;r)>rk-2r$, we consider the 2-coloring $\chi:[1,rk-2r]\to\{0,1\}$ defined by $\chi(i)=0$ for $1\leq i\leq k-2$ and $\chi(i)=1$ for $k-1\leq i\leq rk-2r$. Assume, for a contradiction, that χ admits an r-zero-sum solution $(\hat{x}_1,\hat{x}_2,\dots,\hat{x}_k)$ to the equation $\mathcal{E}$. Then $\chi(\hat{x}_i)=1$ for some $i\in\{1,2,\dots,k\}$; otherwise the solution is monochromatic of color 0, but $\sum_{i=1}^{k-1}\hat{x}_i\geq k-1$, meaning that $\hat{x}_k$ cannot be of color 0.

Since the solution $(\hat{x}_1,\hat{x}_2,\dots,\hat{x}_k)$ is an r-zero-sum and not monochromatic of color 0, we must have $\chi(\hat{x}_j)=1$ for at least r of the $\hat{x}_j$'s. Since the minimum integer under χ that is of color 1 is $k-1$, this gives us

$$\sum_{i=1}^{k-1} \hat{x}_i \geq (r-1)(k-1) + (k-r)1 = (rk - 2r + 1) > rk - 2r,$$

which is outside the domain, a contradiction. Hence, χ does not admit an r-zero-sum solution to $\mathcal{E}$ and we conclude that $S_{\mathfrak{z},2}(k; r) \geq rk - 2r + 1$.

We now move on to the upper bound.

Upper bound. We let $\chi : [1, rk - 2r + 1] \to \{0, 1\}$ be an arbitrary 2-coloring. We may assume that $\chi(1) = 0$ since χ admits an r-zero-sum solution if and only if the induced coloring $\overline{\chi}$ defined by $\overline{\chi}(i) = 1 - \chi(i)$ also does so.

The cases $r = 2, 3$ have been done by Theorem 4.1.13. Hence, we may assume that $r \geq 4$. We handle the case $r = 4$ separately; we start with this case.

We show that $4k - 7$ serves as an upper bound for $S_{\mathfrak{z}}(k; 4)$. Consider the following solution to $\mathcal{E}$:

$$(1, 1, 1, \underbrace{2, \ldots, 2}_{k-8}, 3, 3, k, k, 4k - 7).$$

Noting that $r - 1 = 3$ and $rk - 2r + 1 = 4k - 7$, by Lemmas 4.2.1 and 4.2.2, we may assume $\chi(3) = 1$, $\chi(k) = 1$, and $\chi(4k - 7) = 0$. Since k is a multiple of 4 and $k \geq 8$, we see that $k - 8$ is also a multiple of 4. Hence, the color of 2 does not affect whether or not this solution is 4-zero-sum. Of the integers which are not equal to 2, we have exactly four of them of color 1. Hence, this solution is 4-zero-sum. This, along with the lower bound above, proves that $S_{\mathfrak{z},2}(k; 4) = 4k - 7$.

We now move on to the cases where $r \geq 5$. We proceed by assuming that no r-zero-sum solution occurs under an arbitrary 2-coloring $\chi : [1, rk - 2r + 1] \to \{0, 1\}$. From Lemmas 4.2.1 and 4.2.2, we may assume the following table of colors holds.

color 0	color 1
1	$r-1$
$k-2$	r
$rk-2r+1$	$k-1$
	k
	$rk-2r-1.$

Since

$$(\underbrace{1,\ldots,1}_{k-r},\underbrace{k-2,\ldots,k-2}_{r-2},k+r-3,rk-2r+1)$$

is a solution to $\mathcal{E}$, we deduce that $\chi(k+r-3)=1$. Since

$$(\underbrace{1,\ldots,1}_{k-r},\underbrace{k,\ldots,k}_{r-2},k+r-3,rk-3)$$

is another solution to $\mathcal{E}$, as $\chi(k+r-3)=1$, we may assume that $\chi(rk-3)=0$. In turn, we use this in the solution

$$(\underbrace{2,\ldots,2}_{k-r-1},r,r-1,\underbrace{k,\ldots,k}_{r-2},rk-3)$$

to deduce that $\chi(2)=1$. By Modifying this last solution slightly, we have

$$(\underbrace{3,\ldots,3}_{k-r-1},r,r,r,\underbrace{k,\ldots,k}_{r-3},rk-3)$$

is a solution to $\mathcal{E}$ and hence we assume that $\chi(3)=1$. Finally, since $r\geq 5$, we have

$$(\underbrace{2,\ldots,2}_{k-2r+6},\underbrace{3,\ldots,3}_{r-5},\underbrace{k-1,\ldots,k-1}_{r-2},rk-2r-1)$$

is a solution to $\mathcal{E}$. We see that this solution is a monochromatic solution of color 1, and hence it is an r-zero-sum solution, which is a contradiction. This proves that $S_{\mathfrak{z},2}(k;r) \leq rk - 2r + 1$ for $r \geq 5$, which together with the lower bound, gives us $S_{\mathfrak{z},2}(k;r) = rk - 2r + 1$, thereby completing the proof. □

4.2.2 Proof of Theorem 4.1.16.

Let r and k be positive integers such that r divides k and $k \geq 2$. Here we consider the equation $\mathcal{E}_0^{(k,1)} : x_1 + \cdots + x_{k-1} = kx_k$. We have to show that

$$S_{\mathfrak{z},2}^{(k,1)}(k;r;0) = \begin{cases} 3 & ; r = 2 \text{ and } k \geq 4 \\ 4 & ; r \geq 3 \text{ and } k = r \text{ or } k = 2r \\ 3 & ; r \geq 3 \text{ and } k \geq 3r. \end{cases}$$

Case I: ($r = 2$ and $k \geq 4$)

Let $\chi : [1,3] \to \{0,1\}$ be any 2-coloring. We may assume that $\chi(1) = 0$, since χ admits a 2-zero sum solution if and only if $\hat{\chi}$ defined by $\hat{\chi}(i) = 1 - \chi(i)$ does. Considering the solution $(\underbrace{1,\ldots,1}_{\text{(k-2)-times}},2,1)$ and using the color of 1, we can conclude $\chi(2) = 1$. Since $2\chi(3) + (k-3)\chi(2) + \chi(2) \equiv 0 \pmod 2$, we see that $(3,3,\underbrace{2,\ldots,2}_{\text{(k-3)-times}},2)$ is a 2-zero sum solution to the equation $\mathcal{E}_0^{(k,1)}$. This proves that, in this case, $S_{\mathfrak{z},2}^{(k,1)}(k;2;0) \leq 3$.

For proving the lower bound, we consider the coloring $\chi : [1,2] \to \{0,1\}$ defined by $\chi(1) = 0$ and $\chi(2) = 1$. If $(x_1,\ldots,x_k)$ is a solution to $\mathcal{E}_0^{(k,1)}$, then $x_k \neq 2$ because $x_1 + \cdots + x_{k-1} \leq 2(k-1)$ where as $kx_k = 2k$. If $x_k = 1$, then $kx_k = k$ and hence the only solution to the equation $\mathcal{E}_0^{(k,1)}$ is $(\underbrace{1,\ldots,1}_{\text{(k-2)-times}},2,1)$, which does not satisfy 2-zero-sum condition. Hence, we conclude $S_{\mathfrak{z},2}^{(k,1)}(k;2;0) = 3$.

Case II: ($r \geq 3$ and $k = r$ or $r \geq 3$ and $k = 2r$)

Let $\chi : [1, 4] \to \{0, 1\}$ be any 2-coloring. We may assume that $\chi(1) = 0$. Looking at the solution $(\underbrace{1, \ldots, 1}_{\text{(k-2)-times}}, 2, 1)$ and using the color of 1, we conclude that $\chi(2) = 1$. Now, considering the solution $(\underbrace{2, \ldots, 2}_{\text{(k-3)-times}}, 3, 3, 2)$ and using the color of 2, we conclude that $\chi(3) = 0$. Again, consider the solution $(\underbrace{2, \ldots, 2}_{\text{(k-2)-times}}, 4, 2)$. If $\chi(4) = 1$, then we are done. If $\chi(4) = 0$, then the solution $(\underbrace{3, \ldots, 3}_{\text{(k-4)-times}}, 4, 4, 4, 3)$ is an r-zero-sum solution to the equation $\mathcal{E}_0^{(k,1)}$, and we get $S_{\mathfrak{z},2}^{(k,1)}(k; r; 0) \leq 4$.

Now it remains to prove the lower bound. We consider the coloring $\chi : [1, 3] \to \{0, 1\}$ defined by $\chi(1) = 0 = \chi(3)$ and $\chi(2) = 1$.

Subcase I. ($r \geq 3$ and $k = r$)

Since $k = r$, first we observe that an r-zero sum solution to the equation $\mathcal{E}_0^{(k,1)}$ is a monochromatic solution and vice versa.

Second, by taking $x_i = 2$ for all $i = 1, 2, \ldots, k$, we cannot get any solution to the equation $\mathcal{E}_0^{(k,1)}$. Also, since $kx_k \geq r$, by taking only $x_i = 3$ or $x_i = 1$, we cannot get any solution to the equation $\mathcal{E}_0^{(k,1)}$. Hence, we can conclude $S_{\mathfrak{z},2}^{(k,1)}(k; r; 0) \geq 4$.

Subcase II. ($r \geq 3$ and $k = 2r$)

By taking $x_k = 1$, the only possible solution is $(\underbrace{1, \ldots, 1}_{\text{(k-2)-times}}, 2, 1)$, which is not an r-zero sum solution. Again, by taking $x_k = 2$, the only possible solution to the equation $\mathcal{E}_0^{(k,1)}$ is $(3, 3, \underbrace{2, \ldots, 2}_{\text{(k-3)-times}}, 2)$, which is not an r-zero sum solution under the coloring χ. Finally, if we take $x_k = 3$, then $kx_k = 3k$ and $x_1 + \cdots + x_{k-1} \leq 3k - 3$. Hence, there is no solution with $x_k = 3$ also. Thus we get, $S_{\mathfrak{z},2}^{(k,1)}(k; r; 0) \geq 4$.

Case III: ($r \geq 3$ and $k \geq 3r$)

Suppose for a contradiction that there exists a 2-coloring $\chi : [1,3] \to \{0,1\}$ for which $\mathcal{E}_0^{(k,1)}$ does not have any r-zero-sum solution for some $r \geq 3$. We may assume that $\chi(1) = 0$. Looking at the solution $(\underbrace{1,\ldots,1}_{\text{(k-2)-times}}, 2, 1)$ and using the color of 1, we can conclude $\chi(2) = 1$. Now, considering the solution $(\underbrace{2,\ldots,2}_{\text{(k-3)-times}}, 3, 3, 2)$ and using the color of 2, we can conclude $\chi(3) = 0$.

If k is an odd multiple of r, then we observe that $\frac{k-r}{2}$ is a positive integer. Hence we see that

$$(\underbrace{2,\ldots,2}_{(r-1)\text{-times}}, \underbrace{1,\ldots,1}_{\left(\frac{k-r}{2}-1\right)\text{-times}}, \underbrace{3,\ldots,3}_{\left(\frac{k-r}{2}+1\right)\text{-times}}, 2)$$

is an r-zero sum solution to the equation $\mathcal{E}_0^{(k,1)}$.

If k is an even multiple of r, then $\frac{k-2r}{2}$ is a positive integer as $k \geq 3r$. Thus we see that $(\underbrace{2,\ldots,2}_{(2r-1)\text{-times}}, \underbrace{1,\ldots,1}_{(\frac{k-2r}{2}-1)\text{-times}}, \underbrace{3,\ldots,3}_{(\frac{k-2r}{2}+1)\text{-times}}, 2)$ is an r-zero sum solution to the equation $\mathcal{E}_0^{(k,1)}$, which is a contradiction. Hence we have $S_{\mathfrak{z},2}^{(k,1)}(k;r;0) \leq 3$.

For the lower bound, we consider the coloring $\chi : [1,2] \to \{0,1\}$ defined by $\chi(1) = 0$ and $\chi(2) = 1$. Note that $x_k \neq 2$ as $kx_k = 2k$ and $x_1 + \cdots + x_{k-1} \leq 2k-2$. Therefore, $x_k = 1$ and hence $kx_k = k$. Thus there is only one solution, namely, $(\underbrace{1,\ldots,1}_{\text{(k-2)-times}}, 2, 1)$, to the equation $\mathcal{E}_0^{(k,1)}$, which does not satisfy r-zero-sum condition. This proves the lower bound and the theorem. $\square$

4.2.3 Proof of Theorem 4.1.17.

Let k be an even positive integer and $v \in [1, \lfloor \frac{k-1}{4} \rfloor]$. Here we consider the equation $\mathcal{E}_v^{(1,1)} : x_1 + \cdots + x_{k-(2v+1)} = x_{k-2v} + \cdots + x_k$. We have to show that

$$S_{\mathfrak{z},2}^{(1,1)}(k;2;v) \leq \left(\frac{k}{2} - 2v\right).$$

Let us denote $t := \left(\frac{k}{2} - 2v\right)$ and $\chi : [1, t] \to \{0, 1\}$ be any 2-coloring of the interval $[1, t]$. Since $v \leq \frac{k-1}{4}$ and k is even, we get $k \geq 4v + 2$.

Clearly,

$$\left(\underbrace{1, \ldots, 1}_{(k-4v)\text{-times}}, \underbrace{\left(\frac{k}{2} - 2v\right), \ldots, \left(\frac{k}{2} - 2v\right)}_{4v\text{-times}}\right)$$

is a solution to the equation

$$x_1 + \cdots + x_{k-(2v+1)} = x_{k-2v} + \cdots + x_k. \tag{4.2}$$

Also, since k is even, we get

$$\sum_{i=1}^{k} \chi(x_i) = (k - 4v)\chi(1) + 4v\chi\left(\frac{k}{2} - 2v\right) \equiv 0 \pmod 2.$$

Therefore, $S^{(1,1)}_{\mathfrak{z},2}(k; 2; v) \leq (\frac{k}{2} - 2v)$. □

4.2.4 Proof of Theorem 4.1.18.

Let r and k be two positive integers such that r divides k and $v \in [1, \left\lfloor\frac{k-1}{2r}\right\rfloor]$ an integer. Here we consider the equation

$$\mathcal{E}^{(1,0)}_v : x_1 + \cdots + x_{k-vr} = x_{k-vr+1} + \cdots + x_k. \tag{4.3}$$

We have to show that

$$S^{(1,0)}_{\mathfrak{z},r}(k; r; v) = \frac{k}{r} - \left\lfloor\frac{(v-1)k}{vr}\right\rfloor - 1.$$

Let us denote $s := \frac{k}{r} - \left\lfloor\frac{(v-1)k}{vr}\right\rfloor - 1$ for simplicity.

Case I: $(k = 2vr)$

In this case, the number of variables in both sides of the (4.3) are equal and hence $(1,\ldots,1)$ is an r-zero sum solution. Thus we get, $S^{(1,0)}_{\mathfrak{z},r}(k;r;v)=1$, as desired.

Case II: $(k>2vr)$

Since $r|k$, by division algorithm we write $k-2vr=vrt+ir$ for some non-negative integers t and i with $i\in[1,v]$. Therefore, we get

$$\begin{aligned} k=vrt+(2v+i)r &\Leftrightarrow (v-1)k=v(v-1)rt+(2v+i)(v-1)r \\ &\Leftrightarrow \frac{(v-1)k}{vr}=(v-1)t+\frac{(v-1)(2v+i)}{v} \\ &\Leftrightarrow \left\lfloor\frac{(v-1)k}{vr}\right\rfloor=(v-1)t+(2v+i-3). \end{aligned}$$

Hence, we have $s=\frac{k}{r}-\left\lfloor\frac{(v-1)k}{vr}\right\rfloor-1=\frac{k}{r}-(v-1)t-(2v+i-3)-1$.

Lower bound. For proving the lower bound we show that the equation (4.3) does not have any solution in the interval $[1,s-1]$. First, we observe that

$$\begin{aligned} x_{k-vr+1}+\cdots+x_k &\le vr(s-1) \\ &= vr\left(\frac{k}{r}-\left\lfloor\frac{(v-1)k}{vr}\right\rfloor-1-1\right) \\ &= vr\left(\frac{k}{r}-(v-1)t-(2v+i-3)-1-1\right) \\ &= vk-v(v-1)rt-v(2v+i-3)r-vr-vr \\ &= vk-(v-1)k+(2v+i)(v-1)r-v(2v+i-3)r-2vr \\ &= k-(2v+i)r+3vr-2vr=k-vr-ir \end{aligned}$$

but $x_1+\cdots+x_{k-vr}\ge k-vr$. Thus we get $S^{(1,0)}_{\mathfrak{z},r}(k;r;v)\ge\frac{k}{r}-\left\lfloor\frac{(v-1)k}{vr}\right\rfloor-1$.

Upper bound. Let $\chi:[1,s]\to\{0,1,\ldots,r-1\}$ be an arbitrary r-coloring of

$[1, s]$. We consider the solution

$$(x_1, x_2, \ldots, x_{k-vr}, x_{k-vr+1}, \ldots, x_k) = (\underbrace{1, \ldots, 1}_{(k-vr)\text{-times}}, \underbrace{s, \ldots, s}_{ir\text{-times}}, \underbrace{s-1, \ldots, s-1}_{(vr-ir)\text{-times}}) \tag{4.4}$$

to the equation (4.3).

Since $r \mid k$, we see that

$$\sum_{i=1}^{k} \chi(x_i) = (k - vr)\chi(1) + ir(\chi(s)) + (vr - ir)\chi(s-1) \equiv 0 (\mathrm{mod}\ r).$$

Therefore, the solution (4.4) satisfies the r-zero sum condition, and we get $S_{\mathfrak{z},r}^{(1,0)}(k; r; v) \leq \frac{k}{r} - \left\lfloor \frac{(v-1)k}{vr} \right\rfloor - 1.$ $\square$

CHAPTER 5

On fractionally dense sets

Let K be an algebraic number field such that K is not a subfield of $\mathbb{R}$ and $\mathcal{O}_K$ its ring of integers. For any subset A of $\mathbb{Z}$ (respectively, $\mathcal{O}_K$), we define $R(A)$ to be the set of all numbers $\frac{a}{a'}$ such that both a and a' lie in A and we call the set $R(A)$ to be the quotient set of A. If $A \subset \mathbb{N}$ (respectively, $\mathcal{O}_K$) and $R(A)$ is dense in $\mathbb{R}_{>0}$ (respectively, in $\mathbb{C}$), then we say that A is fractionally dense in $\mathbb{R}_{>0}$ (respectively, in $\mathbb{C}$). In this formulation, for example, we can say that $\mathbb{N}$ is fractionally dense in $\mathbb{R}_{>0}$. We prove some subsets of natural numbers $\mathbb{N}$ and any non-zero ideal of an order of an imaginary quadratic field are fractionally dense in $\mathbb{R}_{>0}$ and $\mathbb{C}$, respectively. The content of this chapter is published in the article [29].

5.1 Introduction

Definition 5.1.1 Let A and B be two subsets of the set of all integers $\mathbb{Z}$. We define the corresponding *set of quotients* $R(A,B)$ by

$$R(A,B) = \left\{ \frac{a}{b} \mid a \in A, b(\neq 0) \in B \right\}.$$

Let A be a subset of $\mathbb{N}$ and m a positive integer. We define the sets $mA := \{ma : a \in A\}$, $A + m := \{a + m : a \in U\}$ and $A^{(n)} := \{a^n : a \in A\}$. The proof of the following lemma is straightforward.

Lemma 5.1.2 *Let A and B be two subsets of $\mathbb{N}$.*

(i) $R(A,B)$ is dense in $\mathbb{R}_{>0}$ if and only if $R(B,A)$ is dense in $\mathbb{R}_{>0}$.

(ii) Let m and n be two positive integers. If $R(A,B)$ is dense in $\mathbb{R}_{>0}$, then $R(mA,nB)$ is dense in $\mathbb{R}_{>0}$.

(iii) Let m and n be two positive integers. If $R(A,B)$ is dense in $\mathbb{R}_{>0}$, then $R(A+m,B+n)$ is dense in $\mathbb{R}_{>0}$.

(iv) Let n be a positive integer. If $R(A,B)$ is dense in $\mathbb{R}_{>0}$, then $R(A^{(n)},B^{(n)})$ is dense in $\mathbb{R}_{>0}$.

When $B = A$, we denote the set $R(A,A)$ by $R(A)$ and call it the *quotient set of* A. For example, $R(\mathbb{N}) = \mathbb{Q}_{>0}$ and $R(\mathbb{Z}) = \mathbb{Q}$.

Definition 5.1.3 Let A be a subset of $\mathbb{Z}$ (respectively, $\mathbb{N}$). We say A is *fractionally dense* in $\mathbb{R}$ (respectively, $\mathbb{R}_{>0}$) if $R(A)$ is dense in $\mathbb{R}$ (respectively, $\mathbb{R}_{>0}$).

In this formulation, for example, we can say that $\mathbb{Z}$ is fractionally dense in $\mathbb{R}$ and $\mathbb{N}$ is fractionally dense in $\mathbb{R}_{>0}$.

The major open problem is to characterize all the subsets of $\mathbb{Z}$ (respectively, $\mathbb{N}$) which are fractionally dense in $\mathbb{R}$ (respectively, $\mathbb{R}_{>0}$). In this direction, many results have already been obtained in [22], [23], [24], [25], [26] [47], [57], [58], [83], [92] and [93]. This problem has also been considered in the p-adic set-up in [45], [46], [68] and [86].

Indeed, the most interesting set, namely, the set of all prime numbers $\mathbb{P}$ is proved to be fractionally dense in $\mathbb{R}_{>0}$ (see [58] and [83]). In [92], it is proved that the set of all prime numbers in a given arithmetic progression is also fractionally dense in $\mathbb{R}_{>0}$. In this chapter, along with the other results, we generalize this fact.

In [22], it is proved that for a given natural number $b \geq 2$, the set of all natural numbers whose base b representation begins with the digit 1 is fractionally dense if and only if $b = 2, 3$ and 4. In the following theorem, we generalize this result.

Theorem 5.1.4 *Let $b \geq 2$ be a given integer and let a and c be integers satisfying $1 \leq a < c \leq b$. Consider the subset*

$$A = \bigcup_{k=0}^{\infty} [ab^k, cb^k) \cap \mathbb{N}$$

of $\mathbb{N}$. Then, the following statements are true:

1. *If $ab < c^2$, then the set $B = A \cup \{b^k : k = 0, 1, 2, \ldots\}$ is fractionally dense in $\mathbb{R}_{>0}$.*
2. *If A is fractionally dense in $\mathbb{R}_{>0}$, then $a^2 b \leq c^2$.*

When we put $a = 1$ (respectively, $a = 1$ and $c = 2$) in Theorem 5.1.4, we recover the earlier results proved in [22]. Also, an easy corollary is as follows.

Corollary 5.1.5 *Let $b \geq 3$ be a given integer and let a be an integer satisfying $2 \leq a \leq b - 1$. Then the set of all integers whose base b representation begins* *with the digit a together with b^k for all $k = 0, 1, \dots$ is fractionally dense in $\mathbb{R}_{>0}$, if $a = b - 1$ or $b - 2$.*

Definition 5.1.6 For $A \subset \mathbb{N}$ and for every $x > 1$, we define $A(x) = \{a \in A : a \leq x\}$. We say that A has a *natural density* $d(A)$, if

$$d(A) = \lim_{n\to\infty} \frac{|A(n)|}{n},$$

provided the limit exists. A subset A of $\mathbb{N}$ is said to have *lower natural density* $\underline{d}(A)$, if

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{|A(n)|}{n}.$$

In [22] and [93], it was proved that if a subset $A \subset \mathbb{N}$ satisfies $\underline{d}(A) \geq \frac{1}{2}$, then A is fractionally dense in $\mathbb{R}_{>0}$. In the following theorem, we consider those subsets A which satisfy $d(A) > 0$.

Theorem 5.1.7 *Let U and V be two subsets of $\mathbb{N}$ such that $d(U)$ exists and equals $\gamma > 0$. Then $R(U, V)$ is dense in $\mathbb{R}_{>0}$ if and only if V is infinite.*

Note that, if A is a subset of $\mathbb{N}$ such that $d(A) > 0$, then A must be an infinite set. Here we give an alternative proof of the following corollary, which was first proved by Šalát in [83] and again in [57], by taking $V = U$ in Theorem 5.1.7.

Corollary 5.1.8 *Let U be a subset of $\mathbb{N}$ such that $d(U)$ exists and is positive. Then U is fractionally dense in $\mathbb{R}_{>0}$.*

Now we define the *relative density* of a subset A of the set of all prime numbers $\mathbb{P}$ as follows.

Definition 5.1.9 A subset A of $\mathbb{P}$ has *relative density* $\delta(A)$, if

$$\delta(A) = \lim_{x\to\infty} \frac{|A(x)\cap\mathbb{P}|}{\pi(x)},$$

provided the limit exists. Here $\pi(x)$ denotes the number of primes p with $p \le x$.

It readily follows from the definition that if $\delta(A) > 0$, then A must be an infinite subset of $\mathbb{P}$.

Let a and $m \ge 2$ be two positive integers such that $\gcd(a,m) = 1$. Then the set $D(a,m)$ of all prime numbers p with $p \equiv a \pmod{m}$ has relative density $\delta(D(a,m)) = 1/\phi(m)$, by *Dirichlet's Prime Number Theorem.* Motivated by many examples of subsets of $\mathbb{P}$, we have the following general theorem.

Theorem 5.1.10 *Let U be a subset of $\mathbb{P}$ such that $\delta(U)$ exists and equals $\gamma > 0$ and V a subset of $\mathbb{N}$. Then $R(U,V)$ is dense in $\mathbb{R}_{>0}$ if and only if V is infinite.*

By taking $V = U$ in Theorem 5.1.10, we have the following corollary.

Corollary 5.1.11 *Let U be a subset of $\mathbb{P}$ such that $\delta(U)$ exists and equals $\gamma > 0$. Then U is fractionally dense in $\mathbb{R}_{>0}$.*

The following theorem is first proved in [92]. This can be seen as a corollary to Theorem 5.1.10. However, we give an alternative proof, using the distribution of prime numbers in some special intervals.

Theorem 5.1.12 *Let a, b, m and n be given positive integers with $m, n \ge 2$ such that $gcd(a,m) = gcd(b,n) = 1$. Then the set*

$$R(D(a,m), D(b,n)) = \left\{ \frac{p}{q} \quad : \quad p \in D(a,m), \quad q \in D(b,n) \right\}$$

is dense in $\mathbb{R}_{>0}$.

In the literature, there is a natural generalization of this concept to the set of all complex numbers $\mathbb{C}$. Let K be an algebraic number field such that K is not a subfield of $\mathbb{R}$ and $\mathcal{O}_K$ its ring of integers. A subset A of $\mathcal{O}_K$ is said to be *fractionally dense* in $\mathbb{C}$, if its quotient set $R(A)$ is dense in $\mathbb{C}$.

When $K = \mathbb{Q}(i)$ with $i = \sqrt{-1}$, Garcia ([44]) proved that the set of all prime elements in $\mathcal{O}_K = \mathbb{Z}[i]$ is fractionally dense in $\mathbb{C}$. This has been generalized to arbitrary number fields by Sittinger in [91].

In [22], [24] and [93], it has been proved that if $\mathbb{N}$ is partitioned into two subsets, then at least one of them is fractionally dense in $\mathbb{R}_{>0}$. But, we observe that $\mathbb{Z}$ has a two-partition like $\mathbb{Z} = \mathbb{N} \cup (\mathbb{Z} \setminus \mathbb{N})$ with neither $\mathbb{N}$ nor $(\mathbb{Z} \setminus \mathbb{N})$ is fractionally dense in $\mathbb{R}$. Since $\mathbb{Z}[\sqrt{-d}]$ is a discrete subset of $\mathbb{C}$, it is quite natural to ask the same kind of questions for some particular type of subsets of $\mathbb{Z}[\sqrt{-d}]$. In this chapter, we study the non-zero ideals of the order $\mathbb{Z}[\sqrt{-d}]$ of imaginary quadratic fields.

More precisely, we prove the following theorem, which is a generalization of a result in [22].

Theorem 5.1.13 *Let $d > 0$ be a square-free integer and let $\mathfrak{a}$ be a non-zero ideal in $\mathbb{Z}[\sqrt{-d}]$. Let $\mathfrak{a} = C \cup D$ be a two-partition of $\mathfrak{a}$. Then either C or D is fractionally dense in $\mathbb{C}$.*

Indeed, the result in Theorem 5.1.13 is optimal in the following sense.

Theorem 5.1.14 *Let K be an algebraic number field not entirely contained in $\mathbb{R}$ with $\mathcal{O}_K$ its ring of integers. Let $\mathfrak{a}$ be a non-empty subset of $\mathcal{O}_K$. Then there exist pairwise disjoint non-empty subsets A, B and C of $\mathfrak{a}$ such that none of them is fractionally dense in $\mathbb{C}$ and $\mathfrak{a} = A \cup B \cup C$.*

Remark 5.1.15 The method we adapt to prove Theorem 5.1.14 goes through not only for an algebraic number field, but also for $\mathbb{C}$ in general. More precisely, one can prove the following statement. *There exist three disjoint subsets A, B and C such that $\mathbb{C} = A \cup B \cup C$ and none of them is fractionally dense in $\mathbb{C}$.*

The next theorem exhibits an infinite subset of prime elements in $\mathbb{Z}[\sqrt{-d}]$ which is not fractionally dense in $\mathbb{C}$. For that we assume additionally that $\mathbb{Z}[\sqrt{-d}]$ is a principal ideal domain.

Theorem 5.1.16 *Let $d = 1$ or 2. Then there exists an infinite set A of prime elements in $\mathbb{Z}[\sqrt{-d}]$ which is not fractionally dense in $\mathbb{C}$.*

5.2 Preliminaries

In the preceding section, we have defined the set $D(a, m)$ for any two positive integers a and $m \geq 2$. By *Dirichlet's Prime Number Theorem*, $D(a, m)$ is an infinite set if and only if $gcd(a, m) = 1$. For any real number $x > 1$, $\pi(a, m, x)$ denotes the number of all primes $p \equiv a \pmod{m}$ with $p \leq x$.

Theorem 5.2.1 *(Dirichlet Prime Number Theorem) Let a and $m \geq 2$ be two positive integers such that $gcd(a, m) = 1$. For any real number $x > 1$, we define*

$$G(x) = \frac{\pi(a, m, x)}{x/\phi(m) \log x},$$

where $\phi(m)$ denotes the Euler's phi function. Then we have

$$\lim_{x \to \infty} G(x) = 1.$$

The following lemma proves the existence of primes in certain arithmetic progressions in some special intervals.

Lemma 5.2.2 *Let a and $m \geq 2$ be two positive integers such that $gcd(a, m) = 1$ and $\alpha > 1$ a given real number. Then there exists a positive integer $m_0 = m_0(\alpha)$, depending only on α, such that for all integers $n \geq m_0$, we have*

$$[\alpha^n, \alpha^{n+1}] \cap D(a, m) \neq \emptyset.$$

Proof. For all real number $x > 1$, let

$$L(x) = \frac{\log G(x)}{\log \alpha} = \log_\alpha(G(x))$$

where $G(x)$ is as defined in Theorem 5.2.1. By Theorem 5.2.1, we know that $\lim_{x\to\infty} G(x) = 1$ and hence we have $\lim_{x\to\infty} L(x) = 0$. Therefore, there exists an integer $n_0 > 0$ such that

$$L(\alpha^{n+1}) - L(\alpha^n) > -\frac{1}{2} \text{ for every integer } n \geq n_0. \tag{5.1}$$

Suppose there exists a strictly increasing sequence $\{r_n\}_n$ of natural numbers such that

$$[\alpha^{r_n}, \alpha^{r_n+1}] \cap D(a, m) = \emptyset.$$

Therefore we get, $\pi(a, m, \alpha^{r_n}) = \pi(a, m, \alpha^{r_n+1})$ and hence,

$$L(\alpha^{r_n+1}) - L(\alpha^{r_n}) = \log_\alpha\left(\frac{G(\alpha^{r_n+1})}{G(\alpha^{r_n})}\right) = \log_\alpha\left(\frac{r_n+1}{\alpha r_n}\right) = \epsilon(r_n) - 1, \tag{5.2}$$

where $\epsilon(r_n) = \log_\alpha\left(\frac{r_n+1}{r_n}\right)$. Since $\lim_{n\to\infty} r_n = \infty$, we get $\lim_{n\to\infty} \epsilon(r_n) = 0$. Thus there exists an integer $n_1 > 0$ such that for all integers $n \geq n_1$, we have $\epsilon(r_n) < 1/2$.

Put $m_0 = \max\{n_0, n_1\}$. Then, by (5.2), for all r_n with $n \geq m_0$, we get

$$L(\alpha^{r_n+1}) - L(\alpha^{r_n}) = \epsilon(r_n) - 1 < -\frac{1}{2},$$

which is a contradiction to (5.1). This proves the lemma. □

We also need the following number field version of the Bertrand's postulate which is due to Hulse and Ram Murty (see [59]).

Lemma 5.2.3 (Bertrand's postulate for number fields) *Let K be an algebraic number field with $\mathcal{O}_K$ its ring of integers. Then there exists a smallest number $B_K > 1$ such that for every $x > 1$, we can find a prime ideal $\mathfrak{p}$ in $\mathcal{O}_K$ whose norm $N(\mathfrak{p})$ lies in $[x, B_K x]$.*

5.3 Proof of Main Theorems

5.3.1 Proof of Theorem 5.1.4

Given that a, b and c are integers satisfying $1 \leq a < c \leq b$ and the set

$$A = \bigcup_{k=0}^{\infty} [ab^k, cb^k) \cap \mathbb{N}.$$

(1) Assume that $ab < c^2$. Then, we prove that the set $B = A \cup \{b^k : k = 0, 1, 2, \ldots\}$ is fractionally dense in $\mathbb{R}_{>0}$.

Claim 1. $\displaystyle\bigcup_{k\in\mathbb{Z}} \left(\left[\frac{ab^k}{c}, ab^k\right) \cup \left[ab^k, cb^k\right) \right) = \bigcup_{k\in\mathbb{Z}} \left[\frac{ab^k}{c}, cb^k\right) = (0, \infty).$

The condition $ab < c^2$ implies that any two consecutive intervals of the form $\left[\frac{ab^k}{c}, cb^k\right)$ and $\left[\frac{ab^{k+1}}{c}, cb^{k+1}\right)$ have non-empty intersection. Note that,

$cb^k \to \infty$ as $k \to \infty$, and $\frac{ab^k}{c} \to 0$ as $k \to -\infty$. Therefore, we get

$$(0, \infty) \subset \bigcup_{k \in \mathbb{Z}} \left[\frac{ab^k}{c}, ab^k\right) \cup \left[ab^k, cb^k\right)$$

and hence Claim 1 follows.

Let $\xi \in \mathbb{R}_{>0}$ be any element and let $\epsilon > 0$ be given. We prove that there exists $\alpha \in R(B)$ such that $|\xi - \alpha| < \epsilon$. By Claim 1, either $\xi \in [ab^k, cb^k)$ for some k or $\xi \in [\frac{ab^k}{c}, ab^k)$ for some k.

Case 1. Let $\xi \in [ab^k, cb^k)$ for some integer k.

Let $\epsilon > 0$ be given. Then there exists a sufficiently large positive integer j such that $a < b^j\epsilon$. Since $\xi \in [ab^k, cb^k)$, we get,

$$ab^{j+k} \leq b^j\xi < cb^{j+k}.$$

If we choose $\ell = \lfloor b^j\xi - ab^{j+k} \rfloor \geq 0$, then we have

$$ab^{j+k} + \ell \leq b^j\xi \leq a(b^{j+k} + 1) + \ell \tag{5.3}$$

with

$$0 \leq \ell \leq (c - a)b^{j+k} - 1. \tag{5.4}$$

By (5.3) we get,

$$0 \leq b^j\xi - (ab^{j+k} + \ell) \leq a \quad \Longrightarrow \quad 0 \leq \xi - \frac{ab^{j+k} + \ell}{b^j} \leq \frac{a}{b^j} < \epsilon$$

By (5.4), we note that $ab^{j+k} + \ell \geq ab^{j+k}$ and $ab^{j+k} + \ell < cb^{j+k}$ and hence the element $\dfrac{ab^{j+k} + \ell}{b^j} = \alpha \in R(B)$, as desired.

Case 2. Let $\xi \in \left[\dfrac{ab^k}{c}, ab^k\right)$ for some integer k.

Since the proof is similar to Case 1, we omit the proof here. Hence, we conclude that B is fractionally dense in $\mathbb{R}_{>0}$. This proves the first assertion.

(2) If possible, suppose $c^2 < a^2 b$. We show that A is not fractionally dense in $\mathbb{R}_{>0}$.

Let $x, y \in A$ be arbitrary elements. Then, by the definition of A, there exist non-negative integers k_1 and k_2 such that

$$x \in [ab^{k_1}, cb^{k_1}) \text{ and } y \in [ab^{k_2}, cb^{k_2}).$$

Therefore, we get

$$\frac{a}{c} b^{k_1 - k_2} < \frac{x}{y} \leq \frac{c}{a} b^{k_1 - k_2}.$$

Hence every element of $R(A)$ lies in the interval of the form

$$I_\ell = \left(\frac{a}{c} b^\ell, \frac{c}{a} b^\ell \right]$$

for some $\ell \in \mathbb{Z}$.

Since by the assumption, $c^2 < a^2 b$, we get $\frac{c}{a} = \frac{c^2}{ca} < \frac{a^2 b}{ca} = \frac{ab}{c}$. Therefore, for any integers j and k with $j < k$, we have

$$\frac{c}{a} b^j < \frac{a}{c} b^{j+1} \leq \frac{a}{c} b^k.$$

Thus we get, $I_j \cap I_k = \emptyset$ for all integers j and k such that $j < k$. Hence the interval $\left(\frac{c}{a} b^j, \frac{a}{c} b^{j+1} \right]$ is non-empty and

$$R(A) \cap \left(\frac{c}{a} b^j, \frac{a}{c} b^{j+1} \right] = \emptyset.$$

This implies that A is not fractionally dense in $\mathbb{R}_{>0}$. $\square$

5.3.2 Proof of Theorem 5.1.7

It is given that U and V are subsets of $\mathbb{N}$ such that $d(U)$ exists and $d(U) = \gamma > 0$.

Suppose V is an infinite subset of $\mathbb{N}$. For a positive real number X, let

$$U(X) := \#\{u \in U : u \leq X\}$$

counts the number of elements of U less than or equal to X. Since U has natural density $\gamma > 0$, we have,

$$\lim_{X\to\infty} \frac{U(X)}{X} = \gamma > 0 \iff U(X) = \gamma X + o(X) \text{ for all large enough } X$$

where $o(X)$ stands for a nonnegative function $g(X)$ such that $g(X)/X \to 0$ as $X \to \infty$.

Let a and b be any two real numbers satisfying $0 < a < b$. We need to prove that there exist $u \in U$ and $v \in V$ such that $a < \frac{u}{v} \leq b$. We have

$$\lim_{X\to\infty} \frac{U(aX)}{U(bX)} = \lim_{X\to\infty} \frac{\gamma aX + o(aX)}{\gamma bX + o(bX)} = \frac{a}{b} < 1.$$

Put $2\epsilon = 1 - \frac{a}{b}$. Since $a < b$, we see that $\epsilon > 0$. For this ϵ, there exists X_0 such that

$$\left| U(aX) - \frac{a}{b} U(bX) \right| < \epsilon U(bX)$$

holds true for all $X \geq X_0$. This implies

$$U(aX) < \left(\frac{a}{b} + \epsilon\right) U(bX) < U(bX)$$

for all $X \geq X_0$. In other words, for all $X \geq X_0$, there exists $u \in U$ such that $aX < u \leq bX$.

Since V is infinite, we can choose $v \in V$ such that $v \geq X_0$. Therefore, by the above observation, there exists $u \in U$ such that $av < u \leq bv$ holds true. In other words, we have $a < \dfrac{u}{v} \leq b$. Hence, we conclude that $R(U, V)$ is dense in $\mathbb{R}_{>0}$.

Conversely, if possible, suppose that V is finite, say, $V = \{v_1, \ldots, v_k\}$. Then

$$R(U, V) = A_1 \cup \ldots \cup A_k$$

where $A_j = \left\{ \dfrac{u}{v_j} : u \in U \right\}$ for all $j = 1, \ldots, k$.

Since $U \subset \mathbb{N}$, we see that U is a discrete subset of $\mathbb{R}_{>0}$. Hence, each of the sets A_j is discrete and therefore, being a finite union of discrete sets, $R(U, V)$ is also discrete. Hence, $R(U, V)$ is not dense in $\mathbb{R}_{>0}$, a contradiction. This proves that V is infinite. $\square$

5.3.3 Proof of Theorem 5.1.10

It is given that U is a subset of $\mathbb{P}$ such that $\delta(U)$ exists and equals $\gamma > 0$.

Suppose V is an infinite subset of $\mathbb{N}$. For any positive real number X, we let

$$U(X) = \# \{u \in U \ : \ u \leq X\}$$

which counts the number of element of U less than or equal to X. Since $\delta(U) = \gamma > 0$, for all large enough X, we have

$$U(X) = \gamma\pi(X) + o(\pi(X)).$$

Therefore, for any real numbers $0 < a < b$, we see that

$$\lim_{X\to\infty} \frac{U(aX)}{U(bX)} = \frac{a}{b} < 1.$$

The rest of the proof is verbatim to the proof of Theorem 5.1.7 and hence we omit the proof here. $\square$

5.3.4 Proof of Theorem 5.1.13

Let $d > 0$ be a square-free integer and let $\mathfrak{a}$ be a non-zero ideal of $\mathbb{Z}[\sqrt{-d}]$. Let a and b be two elements of $\mathbb{Z}[\sqrt{-d}]$ such that $\{a, b\}$ is an integral basis of $\mathfrak{a}$ and $\mathfrak{a} = C \cup D$ the given two-partition of $\mathfrak{a}$. We show that either C or D is fractionally dense in $\mathbb{C}$. Note that, if C is finite then D is infinite and vice versa.

Case 1. C is finite.

Let $C = \{\alpha_1, \ldots, \alpha_r\}$. The quotient set of $\mathfrak{a}$ is

$$R(\mathfrak{a}) = \left\{ \frac{ax + by}{ax' + by'} \mid x, y, x', y' \in \mathbb{Z}[\sqrt{-d}] \right\}.$$

Now, we see that

$$R(\mathfrak{a}) = R(C \cup D) = R(D) \cup A_1 \cup \ldots \cup A_r, \tag{5.5}$$

where

$$A_j = \left\{ \frac{\alpha_j}{\beta} \ : \ \beta \in \mathfrak{a} \right\} \cup \left\{ \frac{\beta}{\alpha_j} \ : \ \beta \in \mathfrak{a} \right\}$$

for all $j = 1, 2, \ldots, r$. Since $\mathbb{Z}[\sqrt{-d}]$ is discrete in $\mathbb{C}$, we see that A_j's are nowhere dense subsets in $\mathbb{C}$. Since $R(\mathfrak{a})$ is dense in $\mathbb{C}$, we see that $R(D) \cup A_1 \cup \ldots \cup A_r$ is dense in $\mathbb{C}$, where $A_1 \cup \ldots \cup A_r$ is a nowhere dense subset in $\mathbb{C}$.

If $R(D)$ is not dense in $\mathbb{C}$, then there exists an open ball B such that $B \cap$

$R(D) = \emptyset$. Therefore $B \subset A_1 \cup \ldots \cup A_r$ which is a contradiction, as $A_1 \cup \ldots \cup A_r$ has empty interior. Hence, D is fractionally dense in $\mathbb{C}$.

Case 2. Both the sets C and D are infinite subsets of $\mathfrak{a}$.

Suppose that neither C nor D is fractionally dense in $\mathbb{C}$. Then there exists $\epsilon > 0$ and non-zero complex numbers α and β such that

$$B(\alpha, \epsilon) \cap R(C) = \emptyset \text{ and } B(\beta, \epsilon) \cap R(D) = \emptyset, \tag{5.6}$$

where $B(z, r)$ denotes the open ball of radius r, centered at z in the complex plane. Now, choose a sufficiently large positive integer n_0 satisfying

$$\frac{\begin{pmatrix} |(1+\sqrt{-d})(a+b)| + |(1+\sqrt{-d})\beta(a+b)| \\ + |(1+\sqrt{-d})\alpha\beta(a+b)| \end{pmatrix}^2}{n_0} < \epsilon \tag{5.7}$$

and

$$\frac{\left(|(1+\sqrt{-d})\alpha(a+b)| + |(1+\sqrt{-d})\alpha\beta(a+b)|\right)^2}{n_0} < \epsilon. \tag{5.8}$$

Once n_0 is chosen, as both C and D are infinite sets, we can find $\gamma \in C$ satisfying

$$|\gamma|^2 > n_0|\alpha|^2, \quad |\gamma|^2 > n_0|\beta|^2 \text{ and } |\gamma|^2 > n_0|\alpha\beta|^2 \tag{5.9}$$

together with the following constraint

$$D_1 \cap D \neq \emptyset, \tag{5.10}$$

where

$$D_1 = \{\gamma \pm a, \gamma \pm b, \gamma \pm a \pm b\}.$$

To see this fact, suppose, if possible, that for every $\gamma \in C$ satisfying (5.9), we have $D_1 \cap D = \emptyset$. This implies that D is bounded. Since $\mathbb{Z}[\sqrt{-d}]$ is discrete, it follows that D is finite, which is a contradiction. Also, note that all the elements of D_1 can be written as $\gamma \pm \epsilon a \pm \epsilon^{'} b$ for some $\epsilon, \epsilon^{'} \in \{0,1\}$ such that $(\epsilon, \epsilon') \neq (0,0)$.

Now, write the complex number

$$\frac{\gamma}{\alpha\beta} = \gamma_1 a + \gamma_2 b, \text{ for some } \gamma_1 = x_1 + \sqrt{-d} y_1, \gamma_2 = x_2 + \sqrt{-d} y_2$$

such that $x_1, y_1, x_2, y_2 \in \mathbb{R}$ and define

$$s = (\langle x_1 \rangle + \sqrt{-d}\langle y_1 \rangle)a + (\langle x_2 \rangle + \sqrt{-d}\langle y_2 \rangle)b.$$

where

$$\langle x \rangle = \begin{cases} \lceil x \rceil; & \text{if } x > 0 \\ \lfloor x \rfloor; & \text{if } x < 0 \end{cases}$$

and $\lceil x \rceil$ is the ceiling of x and $\lfloor x \rfloor$ is the floor of x. Note that

$$s = \frac{\gamma}{\alpha\beta} \pm (\epsilon_1 \pm \sqrt{-d}\epsilon_1^{'})a \pm (\epsilon_2 \pm \sqrt{-d}\epsilon_2^{'})b \in \mathfrak{a}, \tag{5.11}$$

for some $\epsilon_1, \epsilon_1^{'}, \epsilon_2, \epsilon_2^{'} \in [0,1)$.

Claim 1. $s \notin C \cup D$

If we prove the above claim, then we get a contradiction to the fact that $s \in \mathfrak{a} = C \cup D$. Hence, to finish the proof of this theorem, it is enough to prove the claim. Since $s \in \mathfrak{a} = C \cup D$ and $C \cap D = \emptyset$, the element s lies inside C or D but not both. If possible, we assume that $s \in C$.

Now we write $\alpha s = \delta_1 a + \delta_2 b$ for some $\delta_1 = x_3 + \sqrt{-d} y_3$, $\delta_2 = x_4 + \sqrt{-d} y_4$

such that x_3, y_3, x_4, $y_4 \in \mathbb{R}$ and define

$$t = (\langle x_3 \rangle + \sqrt{-d}\langle y_3 \rangle)a + (\langle x_4 \rangle + \sqrt{-d}\langle y_4 \rangle)b,$$

where $\langle x \rangle$ is defined as above. Then

$$t = \alpha s \pm (\epsilon_3 \pm \sqrt{-d}\epsilon_3')a \pm (\epsilon_4 \pm \sqrt{-d}\epsilon_4')b \in \mathfrak{a} \tag{5.12}$$

for some $\epsilon_3, \epsilon_3', \epsilon_4, \epsilon_4' \in [0, 1)$. Let $d(z_1, z_2)$ denote the usual distance function in $\mathbb{C}$ and we estimate the distance between t/s and α as follows.

Since, by (5.9), the inequality $|s|^2 \geq \left|\frac{\gamma}{\alpha\beta}\right|^2 > n_0$ holds, we see that

$$\begin{aligned} d\left(\frac{t}{s}, \alpha\right)^2 &= \left|\frac{t - \alpha s}{s}\right|^2 \\ &= \left|\frac{(\epsilon_3 \pm \sqrt{-d}\epsilon_3')a \pm (\epsilon_4 \pm \sqrt{-d}\epsilon_4')b}{s}\right|^2 \\ &\leq \left|\frac{(1 + \sqrt{-d})(a + b)}{s}\right|^2 < \epsilon, \end{aligned}$$

by (5.7). If $t \in C$, then $t/s \in R(C)$. Therefore by (5.6), we conclude that $t \notin C$, which implies $t \in D$.

Now we calculate the distance between the elements of the form δ/t for any $\delta \in D_1$ and β as follows. Let $\delta \in D_1$ be an arbitrary element and consider

$$\begin{aligned} d\left(\frac{\delta}{t}, \beta\right)^2 = \frac{|\delta - \beta t|^2}{|t|^2} &= \frac{\left|\delta - \beta(\alpha s \pm (\epsilon_3 \pm \sqrt{-d}\epsilon_3')a \pm (\epsilon_4 \pm \sqrt{-d}\epsilon_4')b)\right|^2}{|t|^2} \\ &= \frac{\left|\begin{array}{l} \delta \pm \beta((\epsilon_3 \pm \sqrt{-d}\epsilon_3')a \pm (\epsilon_4 \pm \sqrt{-d}\epsilon_4')b) \\ \quad - \alpha\beta\left(\frac{\gamma}{\alpha\beta} \pm (\epsilon_1 \pm \sqrt{-d}\epsilon_1')a \pm (\epsilon_2 \pm \sqrt{-d}\epsilon_2')b\right) \end{array}\right|^2}{|t|^2} \end{aligned}$$

$$= \frac{\left| \begin{array}{c} \pm\epsilon a \pm \epsilon' b + \beta((\epsilon_3 \pm \sqrt{-d}\epsilon_3')a \pm (\epsilon_4 \pm \sqrt{-d}\epsilon_4')b) \\ + \alpha\beta(\pm(\epsilon_1 \pm \sqrt{-d}\epsilon_1')a \pm (\epsilon_2 \pm \sqrt{-d}\epsilon_2')b) \end{array} \right|^2}{|t|^2}$$

$$< \epsilon$$

by (5.11), (5.12) and using the estimate

$$|t|^2 \geq |\alpha s|^2 \geq \left|\frac{\gamma}{\beta}\right|^2 > n_0$$

together with the inequality (5.7). Note that the above inequality is true for all $\delta \in D_1$. By (5.10), we know that $|D_1 \cap D| \geq 1$ and hence there exists a $\delta \in D_1$ such that $\delta \in D$ also. For this δ, we get $\frac{\delta}{t} \in B(\beta, \epsilon) \cap R(D)$, which is a contradiction. Therefore, we conclude that $s \notin C$ and hence $s \in D$. Again,

we write $\beta s = \delta_1' a + \delta_2' b$ for some $\delta_1' = x_3' + \sqrt{-d}y_3'$, $\delta_2' = x_4' + \sqrt{-d}y_4'$ such that $x_3', y_3', x_4', y_4' \in \mathbb{R}$ and consider

$$t' = (\langle x_3' \rangle + \sqrt{-d}\langle y_3' \rangle)a + (\langle x_4' \rangle + \sqrt{-d}\langle y_4' \rangle)b,$$

where $\langle x \rangle$ is defined similarly as above. Hence,

$$t' = \beta s \pm (\epsilon_5 \pm \sqrt{-d}\epsilon_5')a \pm (\epsilon_6 \pm \sqrt{-d}\epsilon_6')b \in \mathfrak{a}$$

for some $\epsilon_5, \epsilon_5', \epsilon_6, \epsilon_6' \in [0, 1)$ and we get

$$|t'|^2 \geq |\beta s|^2 \geq \left|\frac{\gamma}{\alpha}\right|^2 > n_0$$

by (5.9). Again, by the similar arguments, we can show that

$$d\left(\frac{t'}{s},\beta\right)^2<\epsilon$$

and conclude $t'\in C$ as $B(\beta,\epsilon)\cap R(D)=\emptyset$.

Now, we consider

$$\begin{aligned}
d\left(\frac{\gamma}{t'},\alpha\right)^2 &= \frac{\left|\gamma-\alpha(\beta s\pm(\epsilon_5\pm\sqrt{-d}\epsilon_5')a\pm(\epsilon_6\pm\sqrt{-d}\epsilon_6')b)\right|^2}{|t'|^2}\\
&= \frac{\left|\gamma-\alpha\beta s+\alpha(\pm(\epsilon_5\pm\sqrt{-d}\epsilon_5')a\pm(\epsilon_6\pm\sqrt{-d}\epsilon_6')b)\right|^2}{|t'|^2}\\
&= \frac{\left|\begin{array}{c}\gamma-\alpha\beta\left(\dfrac{\gamma}{\alpha\beta}\pm(\epsilon_1\pm\sqrt{-d}\epsilon_1')a\pm(\epsilon_2\pm\sqrt{-d}\epsilon_2')b\right)\\ +\alpha(\pm(\epsilon_5\pm\sqrt{-d}\epsilon_5')a\pm(\epsilon_6\pm\sqrt{-d}\epsilon_6')b)\end{array}\right|^2}{|t'|^2}\\
&= \frac{\left|\begin{array}{l}\alpha\beta(\pm(\epsilon_1\pm\sqrt{-d}\epsilon_1')a\pm(\epsilon_2\pm\sqrt{-d}\epsilon_2')b)\\ \qquad+\alpha\left(\pm(\epsilon_5\pm\sqrt{-d}\epsilon_5')a\pm(\epsilon_6\pm\sqrt{-d}\epsilon_6')b\right)\end{array}\right|^2}{|t'|^2}\\
&< \epsilon
\end{aligned}$$

by (5.8) and the above estimate. Thus we get,

$$\frac{\gamma}{t'}\in B(\alpha,\epsilon)\cap R(C),$$

which is a contradiction again. This proves the Claim 1 and the theorem. □

5.3.5 Proof of Theorem 5.1.14

We want to find a three-partition of the set $\mathfrak{a}$ such that none of the part is fractionally dense in $\mathbb{C}$. If $\mathfrak{a}$ is finite, then there is nothing to prove. Now if $\mathfrak{a}$ is infinite, let us consider the sets

$$A=\bigcup_{k=0}^{\infty}\left\{a+ib : a+ib\in\mathfrak{a} \text{ and } a^2+b^2\in[5^k,2\cdot 5^k)\right\},$$

$$B=\bigcup_{k=0}^{\infty}\left\{a+ib : a+ib\in\mathfrak{a} \text{ and } a^2+b^2\in[2\cdot 5^k,3\cdot 5^k)\right\} \text{ and}$$

$$C=\bigcup_{k=0}^{\infty}\left\{a+ib : a+ib\in\mathfrak{a} \text{ and } a^2+b^2\in[3\cdot 5^k,5\cdot 5^k)\right\}.$$

It is easy to observe that $\mathfrak{a}=A\cup B\cup C\subseteq\mathcal{O}_K$ with $A\cap B=\emptyset$, $B\cap C=\emptyset$ and $C\cap A=\emptyset$.

Claim 1. C is not fractionally dense in $\mathbb{C}$.

First note that if $\frac{p}{q}\in R(C)$, then $\frac{p}{q}$ lies in an annulus of the form

$$B_\ell=\left\{x+iy\in\mathbb{C} : \frac{3}{5}5^\ell<x^2+y^2<\frac{5}{3}5^\ell\right\},$$

for some integer ℓ.

Since, for any integer j and k with $j<k$, the inequality $\dfrac{5}{3}\cdot 5^j<\dfrac{3}{5}\cdot 5^k$ is true, we get $B_j\cap B_k=\emptyset$. Thus for any integer ℓ, the set

$$M=\{x+iy\in\mathbb{C} \ : \ \frac{5}{3}5^\ell<x^2+y^2<\frac{3}{5}5^{\ell+1}\}$$

is non-empty and satisfies $M\cap R(C)=\emptyset$. Therefore C is not fractionally dense in $\mathbb{C}$.

Similarly, we can prove that neither A nor B is fractionally dense in $\mathbb{C}$. This completes the proof of the theorem. $\square$

5.3.6 Proof of Theorem 5.1.16

When $d = 1$ or 2, it is well-known that $\mathbb{Z}[\sqrt{-d}]$ is the ring of integers of $\mathbb{Q}(\sqrt{-d})$ and it is a principal ideal domain. We construct an infinite subset A of prime elements in $\mathbb{Z}[\sqrt{-d}]$ which is not fractionally dense in $\mathbb{C}$.

By Lemma 5.2.3, there exists a smallest number $B > 1$ such that for every real number $x > 1$, we can find a prime ideal $\mathfrak{p}$ of $\mathbb{Z}[\sqrt{-d}]$ whose norm $N(\mathfrak{p}) \in [x, Bx]$. Since $\mathbb{Z}[\sqrt{-d}]$ is a principal ideal domain, every prime ideal $\mathfrak{p}$ is generated by a prime element, say, $\alpha_{\mathfrak{p}}$ and $N(\mathfrak{p}) = N(\alpha_{\mathfrak{p}})$. Thus, we conclude that for every real number $x > 1$, there exists a prime element $\alpha \in \mathbb{Z}[\sqrt{-d}]$ whose norm $N(\alpha) \in [x, Bx]$.

In other words, for each natural number $n > 1$, there exists a prime element $\alpha_n \in \mathbb{Z}[\sqrt{-d}]$ whose norm $N(\alpha_n) \in [B^{2n-1}, B^{2n}]$. Let A be the subset of $\mathbb{Z}[\sqrt{-d}]$ which consists precisely those α_n's. Clearly the set A is infinite.

Claim: A is not fractionally dense in $\mathbb{C}$.

Let $1 < m < n$ be any given integers. Then by the above argument, we know that $N(\alpha_m) \in [B^{2m-1}, B^{2m}]$ and $N(\alpha_n) \in [B^{2n-1}, B^{2n}]$. Therefore, we get

$$N(\alpha_m) \leq B^{2m} \leq B^{2(n-1)} < B^{2n-1} \leq N(\alpha_n).$$

Hence, we get

$$\frac{N(\alpha_m)}{N(\alpha_n)} < \frac{1}{B} \text{ and } \frac{N(\alpha_n)}{N(\alpha_m)} > B.$$

Thus the annulus

$$AN = \left\{ z \in \mathbb{C} \ : \ \sqrt{\frac{1}{B}} < |z| < \sqrt{B} \right\}$$

does not contain any element of $R(A)$. This proves the claim and the theorem.

$\square$

5.3.7 Proof of Theorem 5.1.12

For given positive integers a, b, m and n with $m \geq 2$, $n \geq 2$ and $\gcd(a, m) = 1 = \gcd(b, n)$, let

$$R(D(a,m), D(b,n)) = \left\{ \frac{p}{q} \quad : p \in D(a,m), \quad q \in D(b,n) \right\}$$

be a subset of $\mathbb{R}_{>0}$. To prove that $R(D(a,m), D(b,n))$ is dense in $\mathbb{R}_{>0}$, it is enough to show that $R(D(a,m), D(b,n)) \cap [c,d] \neq \emptyset$ for every non-empty interval $[c,d]$ of $\mathbb{R}_{>0}$. In other words, we prove that $D(a,m) \cap [qc, qd] \neq \emptyset$ for some prime $q \in D(b,n)$.

Let c and d be any two positive real numbers such that $c < d$. We choose a real number $\alpha > 1$ with $\alpha^2 < \dfrac{d}{c}$. Then by Lemma 5.2.2, there exists an integer $m_0 = m_0(\alpha)$ such that for all integers $k \geq m_0$, we have $D(a,m) \cap [\alpha^k, \alpha^{k+1}] \neq \emptyset$.

Since $D(b,n)$ is infinite, we choose a prime $q \in D(b,n)$ such that $q > \dfrac{\alpha^{m_0}}{c}$. Observe that

$$\log_\alpha(dq) - \log_\alpha(cq) = \log_\alpha \left(\frac{d}{c} \right) > \log_\alpha \alpha^2 = 2.$$

Thus, there exists an integer ℓ such that the interval $[\ell, \ell+1]$ is contained in the interval $[\log_\alpha(cq), \log_\alpha(dq)]$ whence

$$[\alpha^\ell, \alpha^{\ell+1}] \subset [cq, dq].$$

Since $\alpha^\ell \geq cq > \alpha^{m_0}$, we get $\ell > m_0$. Hence, there exists a prime $p \in D(a,m) \cap [\alpha^\ell, \alpha^{\ell+1}]$. This proves the theorem. $\square$

Bibliography

[1] S. D. Adhikari, *Aspects of combinatorics and combinatorial number theory*, Narosa Publishing House, New Delhi, 2002.

[2] S. D. Adhikari, A. A. Ambily and B. Sury, *Zero-sum problems with subgroup weights*, Proc. Indian Acad. Sci. (Math. Sci.), **120 (3)** (2010), 259–266.

[3] S. D. Adhikari, R. Balasubramanian, F. Pappalardi and P. Rath, *Some zero-sum constants with weights*, Proc. Indian Acad. Sci. (Math. Sci.), **118 (2)** (2008), 183–188.

[4] S. D. Adhikari, L. Boza, S. Eliahou, M. P. Revuelta and M. I. Sanz, *Equation-regular sets and the Fox-Kleitman conjecture*, Discrete Math., **341 (2)** (2018), 287–298.

[5] S. D. Adhikari, L. Boza, S. Eliahou, M. P. Revuelta and M. I. Sanz, *On the degree of regularity of a certain quadratic Diophantine equation,* European J. Combin., **70** (2018), 50–60.

[6] S. D. Adhikari and Y. G. Chen, *Davenport constant with weights and some related questions II*, J. Combin. Theory Ser. A, **115 (1)** (2008), 178–184.

[7] S. D. Adhikari, Y. G. Chen, J. B. Friedlander, S. V. Konyagin and F. Pappalardi, *Contributions to zero-sum problems*, Discrete Math., **306 (1)** (2006), 1–10.

[8] S. D. Adhikari, C. David and J. J. Urroz, *Generalizations of some zero-sum theorems*, Integers, **8** (2008), A52, 11 pp.

[9] S. D. Adhikari and S. Eliahou, *On a conjecture of Fox and Kleitman on the degree of regularity of a certain linear equation*, Combinatorial and additive number theory., Springer Proc. Math. Stat., **220**, Springer, Cham, (2017), 1–8.

[10] S. D. Adhikari, D. J. Grynkiewicz and Z. W. Sun, *On weighted zero-sum sequences*, Adv. in Appl. Math., **48 (3)** (2012), 506–527.

[11] S. D. Adhikari and E. Mazumdar, *Modifications of some methods in the study of zero-sum constants*, Integers, **14** (2014), A25, 15 pp.

[12] S. D. Adhikari and E. Mazumdar, *The polynomial method in the study of zero-sum theorems*, Int. J. Number Theory, **11 (5)** (2015), 1451–1461.

[13] S. D. Adhikari and P. Rath, *Davenport constant with weights and some related questions*, Integers, **6** (2006), A30, 6 pp.

[14] S. D. Adhikari and P. Rath, *Remarks on some zero-sum problems*, Expo. Math., **21 (3)** (2003), 185–191.

[15] S. D. Adhikari, B. Roy and S. Sarkar, *Weighted zero-sums for some finite abelian groups of higher ranks*, In: Nathanson M. (eds) Combinatorial and additive number theory. III. CANT 2018., Springer Proc. Math. Stat., **297** (2020), Springer, Cham, 1–12.

[16] B. Alexeev, J. Fox and R. Graham, *On minimal colorings without monochromatic solutions to a linear equation*, Combinatorial number theory, de Gruyter, Berlin (2007), 1–22.

[17] B. Alexeev and J. Tsimerman, *Equations resolving a conjecture of Rado on partition regularity*, J. Combin. Theory Ser. A, **117 (7)** (2010), 1008–1010.

[18] W. R. Alford, A. Granville and C. Pomerance, *There are infinitely many Carmichael numbers*, Ann. of Math.(2), **139 (3)** (1994), no. 3, 703–722.

[19] N. Alon and M. Dubiner, *A lattice point problem and additive number theory*, Combinatorica, **15 (3)** (1995), 301–309.

[20] A. Beutelspacher and W. Brestovansky, *Generalized Schur Numbers*, Lecture Notes in Mathematics **969** (1982), 30–38.

[21] A. Bialostocki and P. Dierker, *On the Erdős-Ginzburg-Ziv theorem and the Ramsey numbers for stars and matchings*, Discrete Math. **110 (1-3)** (1992), 1–8.

[22] B. Brown, M. Dairyko, S. R. Garcia, B. Lutz and M. Someck, *Four quotient set gems*, Amer. Math. Monthly, **121 (7)** (2014), 590–599.

[23] J. Bukor, P. Erdős, T. Šalát and J. T. Tóth, *Remarks on the (R)-density of sets of numbers. II*, Math. Slovaca, **47 (5)** (1997), 517–256.

[24] J. Bukor, T. Šalát and J. T. Tóth, *Remarks on R-density of sets of numbers*, Tatra Mt. Math. Publ., **11** (1997), 159–165.

[25] J. Bukor and J. T. Tóth, *On accumulation points of ratio sets of positive integers*, Amer. Math. Monthly, **103 (6)** (1996), 502–504.

[26] J. Bukor and J. T. Tóth, *On some criteria for the density of the ratio sets of positive integers*, JP J. Algebra Number Theory Appl., **3 (2)** (2003), 277–287.

[27] P. van Emde Boas and D. Kruswjik, *A combinatorial problem on finite abelian groups*, Math. Centrum Amsterdam Afd. Zuivere Wisk (1967), 27 pp.

[28] Y. Caro, *Zero-sum problems - a survey*, Discrete Math., **152 (1-3)** (1996), 93–113.

[29] J. Chattopadhyay, B. Roy and S. Sarkar, *On fractionally dense sets*, Rockey Mountain J. Math., **49 (3)** (2019), 743–760.

[30] M. N. Chintamani, and B. K. Moriya, *Generalizations of some zero sum theorems*, Proc. Indian Acad. Sci. (Math. Sci.), **122 (1)** (2012), 15–21.

[31] M. N. Chintamani, B. K. Moriya, W. D. Gao, P. Paul and R. Thangadurai, *New upper bounds for the Davenport and for the Erdős-Ginzburg-Ziv constants*, Arch. Math.(Basel), **98 (2)** (2012), 133–142.

[32] Y. Edel, C. Elsholtz, A. Geroldinger, S. Kubertin, L. Rackham, *Zero-sum problems in finite abelian groups and affine caps*, Q. J. Math., **58 (2)** (2007), 159–186.

[33] C. Elsholtz, *Lower bounds for multidimensional zero sums*, Combinatorica, **24 (3)**, (2004), 351–358.

[34] P. Erdős, A. Ginzburg, and A. Ziv, *Theorem in additive number theory*, Bull. Res. Council Israel Sect.F, **10F (1)** (1961), 41–43.

[35] P. Erdős and P. Turan, *On some sequences of integers*, J. London Math. Soc., **11 (4)** (1936), 261–264.

[36] J. Fox and D. J. Kleitman, *On Rado's boundedness conjecture*, J. Combin. Theory Ser. A, **113 (1)** (2006), 84–100.

[37] J. Fox and R. Radoičić, *On the degree of regularity of generalized van der Waerden triples*, Integers, **5 (1)** (2005) A32, 6 pp.

[38] W. D. Gao, *A combinatorial problem on finite abelian groups*, J. Number Theory, **58 (1)** (1996), 100–103.

[39] W. D. Gao, *On Davenport's constant of finite abelian groups with rank three*, Discrete Math., **222 (1-3)** (2000), 111–124.

[40] W. D. Gao, *On zero-sum subsequence of restricted size. II*, Discrete Math., **271 (1-3)** (2003), 51–59.

[41] W. D. Gao and A. Geroldinger, *Zero-sum problems and coverings by proper cosets*, European J. Combin., **24 (5)** (2003), 531–549.

[42] W. Gao and A. Geroldinger, *Zero-sum problems in finite abelian groups: a survey*, Expo. Math., **24 (4)** (2006), 337–369.

[43] W. D. Gao and R. Thangadurai, *On zero-sum sequences of prescribed length*, Aequationes Math., **72 (5)** (2006), 201–212.

[44] S. R. Garcia, *Quotients of Gaussian primes*, Amer. Math. Monthly, **120 (9)** (2013), 851–853.

[45] S. R. Garcia, Y. X. Hong, F. Luca, E. Pinsker, C. Sanna, E. Schechter and A. Starr, *p-adic quotient sets*, Acta Arith., **179 (2)** (2017), 163–184.

[46] S. R. Garcia and F. Luca, *Quotients of Fibonacci numbers*, Amer. Math. Monthly, **123 (10)** (2016), 1039–1044.

[47] S. R. Garcia, V. Selhorst-Jones, D. E. Poore and N. Simon, *Quotient sets and Diophantine equations*, Amer. Math. Monthly, **118 (8)** (2011), 704–711.

[48] A. Geroldinger, *Additive group theory and non-unique factorizations*, in Combinatorial Number Theory and Additive Group Theory, in: Adv. Courses Math. CRM Barcelona, Eds. A. Geroldinger, I. Ruzsa, Birkhauser Verlag, Basel, 2009.

[49] A. Geroldinger and F. Halter-Koch, *Non-unique factorizations: Algebraic, combinatorial and analytic theory*, Pure and Applied Mathematics(Boca Raton), **278**, Chapman & Hall/CRC, Boca Raton, FL, 2006.

[50] A. Geroldinger and R. Schneider, *On Davenport's constant*, J. Combin. Theory, Ser. A, **61 (1)** (1992), 147–152.

[51] N. Golowich, *Resolving a conjecture on degree of regularity of linear homogeneous equations*, Electron. J. Combin., **21 (3)** (2014) Paper 3.28, 8 pp.

[52] W. T. Gowers, *A new proof of Szemerédi's theorem*, Geom. Funct. Anal., **11 (3)** (2001), 465–588.

[53] R. L. Graham, B. L. Rothschild and J. H. Spencer, *Ramsey Theory, 2nd Edition*, Wiley, (1990).

[54] S. Griffiths, *The Erdős-Ginzburg-Ziv theorem with units*, Discrete Math., **308 (23)** (2008), 5473–5484.

[55] D. J. Grynkiewicz, L. E. Marchan and O. Ordaz, *A weighted generalization of two theorems of Gao*, Ramanujan J., **28 (3)** (2012), 323–340.

[56] F. Halter-Koch, *Arithmetical interpretation of weighted Davenport constants*, Arch. Math. (Basel), **103 (2)** (2014), 125–131.

[57] S. Hedman and D. Rose, *Light subsets of* $\mathbb{N}$ *with dense quotient sets*, Amer. Math. Monthly, **116 (7)** (2009), 635–641.

[58] D. Hobby and D. M. Silberger, *Quotients of primes*, Amer. Math. Monthly, **100 (1)** (1993), 50–52.

[59] T. A. Hulse and M. Ram Murty, *Bertrand's postulates for number fields*, Colloq. Math., **147 (2)**, (2017) 165–180.

[60] K. Ireland and M. Rosen, *A classical introduction to modern number theory*, Springer, 2nd Edition.

[61] A. Kemnitz, *On a lattice point problem*, Ars Combin., **16b** (1993), 151–160.

[62] B. M. Landman and A. Robertson, *Ramsey Theory on the integers, 2nd Edition*, American Mathematical Society (2004).

[63] F. Luca, *A generalization of a classical zero-sum problem*, Discrete Math., **307 (13)** (2007), 1672–1678.

[64] L. E. Marchan, O. Ordaz, I. Santos and W. A. Schmid, *Multi-wise and constrained fully weighted Davenport constants and interactions with coding theory*, J. Combin. Theory Ser. A, **135** (2015), 237–267.

[65] L. E. Marchan, O. Ordaz and W. A. Schmid, *Remarks on the plus-minus weighted Davenport constant*, Int. J. Number Theory, **10 (5)** (2014), 1219–1239.

[66] R. Meshulam, *An uncertainty inequality and zero subsums*, Discrete Math., **84 (2)** (1990), 197–200.

[67] E. Metz, *Upper and lower bounds on zero-sum generalized Schur numbers*, arxiv:1803.03851v1.

[68] P. Miska and C. Sanna, *On the p-adic denseness of the quotient set of a polynomial image*, J. Number Theory, **197** (2019) 218–227.

[69] W. Narkiewicz and J. Śliwa, *Finite abelian groups and factorization problems. II*, Colloq. Math., **46 (1)** (1982), 115–122.

[70] M. B. Nathanson, *Additive Number Theory: Inverse Problems and the Geometry of Sumsets*, Springer, 1996.

[71] J. E. Olson, *A combinatorial problem on finite Abelian groups. I*, J. Number Theory, **1** (1969), 8–10.

[72] J. E. Olson, *A combinatorial problem on finite Abelian groups. II*, J. Number Theory, **1** (1969), 195–199.

[73] R. Rado, *Studien zur Kombinatorik*, Math.l Z., **36** (1933), 424–480.

[74] P. Rath, K. Srilakshmi and R. Thangadurai, *On Davenport's constant*, Int. J. Number Theory, **4 (1)** (2008), 107–115.

[75] C. Reiher, *On Kemnitz' conjecture concerning lattice-points in the plane*, Ramanujan J., **13 (1-3)** (2007), 333–337.

[76] A. Robertson, *Zero-sum generalized Schur numbers*, J. Comb. Number Theory, **10 (1)** (2018), 51–62.

[77] A. Robertson, B. Roy and S. Sarkar, *The determination of 2-color zero-sum generalized Schur numbers*, Integers, **18** (2018), A96, 7 pp.

[78] K. Rogers, *A Combinatorial problem in Abelian groups,* Proc. Cambridge Philos. Soc., **59** (1963), 559–562.

[79] L. Rónyai, *On a conjecture of Kemnitz*, Combinatorica, **20 (4)** (2000), 569–573.

[80] K. F. Roth, *On certain sets of integers*, J. London Math. Soc., **28** (1953), 104–109.

[81] B. Roy and S. Sarkar, *On determination of zero-sum ℓ-generalized Schur numbers for some linear equations*, J. Comb. Number Theory, **11 (1)** (2020).

[82] B. Roy and S. Sarkar, *Regularity of certain diophantine equations*, Proc. Indian Acad. Sci. Math. Sci., **129 (2)** (2019), Art. 19, 8 pp.

[83] T. Šalát, *On ratio sets of sets of natural numbers*, Acta Arith., **15** (1968/69), 273–278.

[84] T. Šalát, *Quotientbasen und (R)-dichte Mengen. (German)*, Acta Arith., **19** (1971), 63–78.

[85] J. H. Sanders, *A generalization of Schur's Theorem*, Ph.D. thesis, Yale University Press (1968), 41 pp.

[86] C. Sanna, *The quotient set of k-generalised Fibonacci numbers is dense in $\mathbb{Q}_p$*, Bull. Austral. Math. Soc., **96 (1)** (2017), 24–29.

[87] S. Sarkar, *Generalization of some weighted zero-sum theorems*, (communicated).

[88] T. Schoen and K. Taczala, *The degree of regularity of the equation $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i + b$*, Mosc. J. Combin. Number Theory, **7 (2)** (2017), 74–93.

[89] I. Schur, *Über die Kongruenz* $x^m + y^m \equiv z^m \pmod{p}$, Jahresber. Deutsch. Math.-Verein., **25** (1916), 114–117.

[90] S. Shelah, *Primitive recursive bounds for van der Waerden numbers*, J. Amer. Math. Soc., **1 (3)** (1988), 683–697.

[91] B. D. Sittinger, *Quotients of primes in a quadratic number ring*, arXiv:1607.08319 (2016).

[92] P. Starni, *Answers to two questions concerning quotients of primes*, Amer. Math. Monthly, **102 (4)** (1995), 347–349.

[93] O. Strauch and J. T. Tóth, *Asymptotic density of* $A \subset \mathbb{N}$ *and density of the ratio set* $R(A)$, Acta Arith., **87 (1)** (1998), 67–78.

[94] E. Szemerédi, *On sets of integers containing no four elements in arithmetic progression*, Acta Math. Acad. Sci. Hungar., **20** (1969), 89–104.

[95] E. Szemerédi, *On sets of integers containing no* k *elements in arithmetic progression*, Acta Arith., **27** (1975), 199–245.

[96] R. Thangadurai, *A variant of Davenport's constant*, Proc. Indian Acad. Sci. (Math. Sci.), **117 (2)** (2007), 147 –158.

[97] B. L. Waerden, *Beweis einer Baudetschen Vermutung*, Nieuw Arch. Wisk., **15** (1927), 212–216.

[98] P. Yuan, X. Zeng, *Davenport constant with weights*, European J. Combin., **31 (3)** (2010), 677–680.

Thesis Highlight

Name of the Student: Subha Sarkar
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Thesis Title: On Some Problems in Additive Combinatorics and Related Areas
Discipline: Mathematical Sciences
Date of viva voce: 09/07/2020

This thesis is about some problems on the regularity of Diophantine equation, weighted zero-sum constants attached to a finite abelian group, determination of zero-sum l-generalized Schur numbers and fractionally dense sets. The highlights of the thesis are furnished below.

An equation L is said to be r-regular, if for every r-coloring of positive integers there is a monochromatic solution to the equation L. First, for every r, we have provided a non-linear equation which is r-regular. We have used Szemeredi number to prove the result. This type of results is important towards the settlement of the famous Rado's boundedness Conjecture. Then we had moved on to the problem of determining the values of some weighted zero-sum constants attached to a finite abelian group. Here we proved upper bounds of these constants for some particular weight sets, and in some cases we proved the exact value. Next, we had moved on to the problem of determining the values of zero-sum generalized Schur number and 2-color zero-sum generalized Schur number. Robertson considered a weaker version of regularity of an equation and introduced these numbers. Here we have considered a problem of him and addressed it completely. We also generalize these constants in a more general way, namely zero-sum l-generalized Schur number and compute some of their exact values. Finally, we studied some fractionally dense sets of the set of all positive real numbers and the set of all complex numbers. Hobby and Silberger first proved that the set of all prime numbers is fractionally dense in the set of all positive real numbers. Here we have given some more examples of sets which are fractionally dense. We also proved that, for any 2-partition of a non-zero ideal of an order of an imaginary quadratic field, there exists at least one part which is fractionally dense in the set of all complex numbers.