EQUALITY OF ELEMENTARY ORBITS AND ELEMENTARY SYMPLECTIC ORBITS

By

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As members of the Viva Voce Board, we recommend that the dissertation prepared by **Pratyusha Chattopadhyay** entitled "Equality of Elementary Orbits and Elementary Symplectic Orbits" may be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and the work has not been submitted earlier as a whole or in part for a degree/diploma at this or any other Institution or University.

Pratyusha Chattopadhyay

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Abstract

Aim of this thesis is to show a bijection between the orbit spaces of unimodular rows under the action of the elementary linear group and the orbit spaces of unimodular rows under the action of the elementary symplectic group. We also establish a relative version of it with respect to an ideal. We then generalise this result and show that the orbit space of unimodular rows of a projective module under the action of the group of elementary transvections is in bijection with the orbit space of unimodular rows of a projective module under the action of the group of elementary symplectic transvections with respect to an alternating form.

Let $(Q, \langle , \rangle)$ be a symplectic module with hyperbolic rank ≥ 1 (which means that there is a summand $\mathbb{H}(R)$). We use the above equalities to improve the injective stability bound for $K_1\mathrm{Sp}(R)$ and $\mathrm{Sp}(Q, \langle , \rangle)/\mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle , \rangle)$.

Contents

List of Publications

[1] P. Chattopadhyay and R.A. Rao. Elementary symplectic orbit and improved k_1-stability. (to appear in Journal of K-Theory).

[2] R. Basu, P. Chattopadhyay, and R.A. Rao. Some remarks on symplectic injective stability. (to appear in Proceedings of the American Mathematical Society).

[3] H. Apte, P. Chattopadhyay, and R.A. Rao. ——————. (in preparation).

[4] P. Chattopadhyay and R.A. Rao. Equality of linear and symplectic orbits. (in preparation).

Chapter 1

Introduction

We always work over a commutative ring R with 1. I denotes an ideal of R. We use the notation $(P, \langle , \rangle)$ to denote a symplectic R-module where P is a finitely generated projective R-module of even rank and $\langle , \rangle : P \times P \longrightarrow R$ is a non-degenerate (i.e, $P \cong P^*$ by $x \longrightarrow \langle x, \rangle$) alternating bilinear form. Also Q represents $(R^2 \perp P)$ with induced form on $(\mathbb{H}(R) \perp P)$ and $Q[X]$ represents $(R[X]^2 \perp P[X])$ with induced form on $(\mathbb{H}(R[X]) \perp P[X])$. Here $\mathbb{H}(R)$ denotes $(R \oplus R^*)$, with a unique non-degenerate bilinear form, namely $\langle (a_1, b_1), (a_2, b_2) \rangle = a_1 b_2 - a_2 b_1$.

Now we will make a chapter wise summary.

Chapter 2

In this chapter we recall the definitions of finitely presented module, extended module, unimodular row, relative unimodular row with respect to an ideal, elementary groups (linear, symplectic, orthogonal), relative elementary groups (linear, symplectic, orthogonal) with respect to an ideal and alternating matrix, commutator identities satisfied by the (standard) elementary generators of the elementary groups. Here we also fix some notations and state our assumptions. Then we state some well known results and give a proof of a few of them.

Chapter 3

In chapter 3 we first state D. Quillen's famous Local Global principle which says the following:

Quillen's Local-Global Principle: ([24], Theorem 1)

Let M be a finitely presented module over $R[X]$. If $M_{\mathfrak{m}}$ is extended $R_{\mathfrak{m}}[X]$-module for each maximal ideal $\mathfrak{m}$ of R, then M is extended from R. $\square$

We next state L.N. Vaserstein's "action version" of Quillen's Local Global principle ([18], Chapter 3, Theorem 2.5) which says

Let $n \geq 3$ and $v(X) \in \mathrm{Um}_n(R[X])$. If $v(X) \in v(0)\mathrm{GL}_n(R_{\mathfrak{m}}[X])$, for all maximal ideals $\mathfrak{m}$ of R, then $v(X) \in v(0)\mathrm{GL}_n(R[X])$. □

and also state R.A. Rao's similar result for the elementary linear group ([25], Theorem 2.3) which says

Let $v(X) \in \mathrm{Um}_n(R[X])$, $n \geq 3$. Suppose for all maximal ideals $\mathfrak{m}$ in R, $v(X) \in v(0)\mathrm{E}_n(R_{\mathfrak{m}}[X])$. Then $v(X) \in v(0)\mathrm{E}_n(R[X])$. □

The aim of this chapter is to prove a relative (w.r.t. an extended ideal) elementary (linear and symplectic) action version of L.N. Vaserstein and R.A. Rao's result, which says

Theorem 1: (**Local Global Principle w.r.t. an Extended Ideal**:)
(see Theorem 3.2.3)

Let $n \geq 3$. Let I be an ideal of R and $v(X) \in \mathrm{Um}_n(R[X], I[X])$. If for all maximal (or even prime) ideals $\mathfrak{m}$ of R, $v(X)_{\mathfrak{m}} \in v(0)_{\mathfrak{m}}\mathrm{E}(n, I_{\mathfrak{m}}[X])$, then

$$v(X) \;\in\; v(0)\;\mathrm{E}(n, R[X], I[X]).$$

□

This theorem plays a crucial role in the thesis. Also, we state and prove a stronger version of the above theorem (see Theorem 3.3.5) in this chapter.

Chapter 4

L.N. Vaserstein showed in ([29], Lemma 5.5) that

For any natural number n and any alternating matrix φ from $\mathrm{GL}_{2n}(R)$,

$$e_1\;(\mathrm{E}_{2n}(R)) \;=\; e_1\;(\mathrm{E}_{2n}(R)\;\cap\;\mathrm{Sp}_{\varphi}(R)),$$

where $\mathrm{Sp}_{\varphi}(R)$ denotes the isotropy group of φ, i.e.

$$\mathrm{Sp}_{\varphi}(R) \;=\; \{\alpha \;\in\; \mathrm{SL}_{2n}(R) \mid \alpha^t\;\varphi\;\alpha \;=\; \varphi\}.$$

□

In this chapter we concentrate on the special case when $\varphi = \psi_n$, the standard alternating matrix. In this special case the proof is much easier to establish. In this case following L.N. Vaserstein's proof one observes that

For any natural number $n \geq 2$, $e_1 \mathrm{E}_{2n}(R) = e_1 \mathrm{ESp}_{2n}(R)$. $\square$

The above lemma means that if v is the first row of an elementary matrix of even size then it is also the first row of an elementary symplectic matrix. This led us to query whether the orbit space of unimodular rows under the action of the elementary subgroup is in bijective correspondence with the orbit space of unimodular rows under the action of the elementary symplectic group. In this chapter, we prove that this is so, and also establish the relative version. In particular,

Theorem 2(a): (see Theorem 4.1.1)

Let R be a commutative ring and let $v \in \mathrm{Um}_{2n}(R)$, then

$$v\ \mathrm{E}_{2n}(R) \ = \ v\ \mathrm{ESp}_{2n}(R),$$

for $n \geq 2$. $\square$

Theorem 2(b): (see Theorem 4.2.2)

Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Let $v \in \mathrm{Um}_{2n}(R, I)$, then

$$v\ \mathrm{E}_{2n}(R, I) \ = \ v\ \mathrm{ESp}_{2n}(R, I),$$

for $n \geq 3$. $\square$

Chapter 5

In this chapter we define the **Elementary symplectic group with respect to an alternating matrix** following the lead of L.N. Vaserstein.

In ([29], Lemma 5.4) L.N. Vaserstein obtained the following:

Let n be a natural number and φ be an alternating matrix from $\mathrm{GL}_{2n}(R)$. Let us assume

$$\varphi \ = \ \begin{pmatrix} 0 & -c \\ c^t & \nu \end{pmatrix} \ \textit{and}\ \varphi^{-1} \ = \ \begin{pmatrix} 0 & d \\ -d^t & \mu \end{pmatrix},$$

where $c, d \in R^{2n-1}$. For any v from R^{2n-1} we have

$$\begin{aligned} \alpha \ &= \ \alpha(\varphi, v) \ = \ I_{2n-1} + d^t v \nu, \\ \beta \ &= \ \beta(\varphi, v) \ = \ I_{2n-1} - \mu v^t c. \end{aligned}$$

It can be easily checked that

$$\begin{pmatrix} 1 & 0 \\ \alpha v^t & \alpha \end{pmatrix}, \begin{pmatrix} 1 & v \\ 0 & \beta \end{pmatrix}$$

belong to $\mathrm{E}_{2n}(R) \cap \mathrm{Sp}_\varphi(R)$. $\square$

The above lemma emboldened us to set

$$L_\varphi(v) = \begin{pmatrix} 1 & 0 \\ \alpha v^t & \alpha \end{pmatrix}, \; R_\varphi(v) = \begin{pmatrix} 1 & v \\ 0 & \beta \end{pmatrix},$$

for $v \in R^{2n-1}$. We say that the subgroup of $\mathrm{Sp}_\varphi(R)$ generated by the elements $L_\varphi(v), R_\varphi(v)$, for $v \in R^{2n-1}$ is the **elementary symplectic group** $\mathrm{ESp}_\varphi(R)$ **with respect to the alternating matrix** φ.

Let I be an ideal of R. The relative elementary group $\mathrm{ESp}_\varphi(I)$ is a subgroup of $\mathrm{ESp}_\varphi(R)$ generated as a group by the elements $L_\varphi(v), R_\varphi(v)$, where $v \in I^{2n-1}$.

The relative elementary group $\mathrm{ESp}_\varphi(R, I)$ is the normal closure of $\mathrm{ESp}_\varphi(I)$ in $\mathrm{ESp}_\varphi(R)$.

We established dilation principle, Local Global principle, action version of Local Global principle for both $\mathrm{ESp}_{\varphi\otimes R[X]}(R[X])$ and relative group $\mathrm{ESp}_{\varphi\otimes R[X]}(R[X], I[X])$. Using action version of Local Global principle we show the following:

Theorem 3(a): (see Theorem 5.11.1)

Let φ be an alternating matrix of Pfaffian 1. Then the natural map

$$\mathrm{Um}_{2n}(R)/\mathrm{ESp}_\varphi(R) \longrightarrow \mathrm{Um}_{2n}(R)/\mathrm{E}_{2n}(R),$$

is bijective for $n \geq 2$. $\square$

Theorem 3(b): (see Theorem 5.11.2)

Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Let φ be an alternating matrix of Pfaffian 1 such that $\varphi \equiv \psi_n \pmod I$. Then the natural map

$$\mathrm{Um}_{2n}(R, I)/\mathrm{ESp}_\varphi(R, I) \longrightarrow \mathrm{Um}_{2n}(R, I)/\mathrm{E}_{2n}(R, I)$$

is bijective for $n \geq 3$. $\square$

We recall the definition of transvections, elementary transvections of a finitely generated R-module, symplectic transvections, elementary symplectic transvections of symplectic module due to H. Bass and study these subgroups of automorphisms generated by them.

Let M be a finitely generated R-module. Let $q \in M$ and $\pi \in M^* = \mathrm{Hom}(M, R)$, with $\pi(q) = 0$. Let $\pi_q(p) := \pi(p)q$. An automorphism of the form $1 + \pi_q$ is called a **transvection** of M, if either $q \in \mathrm{Um}(M)$ or $\pi \in \mathrm{Um}(M^*)$. Collection of transvections of M is denoted by $\mathrm{Trans}(M)$. This forms a subgroup of $\mathrm{Aut}(M)$.

Let M be a finitely generated R module. The automorphisms of $N = (R \perp M)$ of the form

$$\begin{aligned}(a,p) &\mapsto (a, p + ax),\\ (a,p) &\mapsto (a + \tau(p), p),\end{aligned}$$

where $x \in M$ and $\tau \in M^*$ are called **elementary transvections** of N.

Let I be an ideal of R. The group of **relative transvections** w.r.t. an ideal I is generated by the transvections of the form $1 + \pi_q$, where either $q \in \mathrm{Um}(IM), \pi \in \mathrm{Um}(M^*)$, or $q \in \mathrm{Um}(M), \pi \in \mathrm{Um}(IM^*)$. The group of relative transvections is denoted by $\mathrm{Trans}(M, IM)$.

Let I be an ideal of R. The elementary transvections of $N = (R \perp M)$ of the form

$$\begin{aligned}(a,p) &\mapsto (a, p + ax),\\ (a,p) &\mapsto (a + \tau(p), p),\end{aligned}$$

where $x \in IM$ and $\tau \in (IM)^*$ are called **relative elementary transvections** w.r.t. an ideal I, and the group generated by them is denoted by $\mathrm{ETrans}(IN)$. The normal closure of $\mathrm{ETrans}(IN)$ in $\mathrm{ETrans}(N)$ is denoted by $\mathrm{ETrans}(N, IN)$.

The group of isometries of $(P, \langle , \rangle)$ is denoted by $\mathrm{Sp}(P, \langle , \rangle)$.

In [7] Bass has defined a **symplectic transvection** of a symplectic module $(P, \langle , \rangle)$ to be an automorphism of the form

$$\sigma(p) = p + \langle u, p\rangle v + \langle v, p\rangle u + \alpha\langle u, p\rangle u,$$

where $\alpha \in R$ and $u, v \in P$ are fixed elements with $\langle u, v\rangle = 0$.

The symplectic transvections of Q of the form

$$\begin{aligned}(a,b,p) &\mapsto (a, b-\langle p,q\rangle+\alpha a, p+aq),\\ (a,b,p) &\mapsto (a+\langle p,q\rangle-\alpha b, b, p+bq),\end{aligned}$$

where $a, b \in R$ and $p, q \in P$ are called **elementary symplectic transvections**.

The subgroup of $\mathrm{Sp}(P, \langle,\rangle)$ generated by the symplectic transvections is denoted by $\mathrm{Trans}_{\mathrm{Sp}}(P, \langle,\rangle)$, whereas the subgroup of $\mathrm{Trans}_{\mathrm{Sp}}(Q, \langle,\rangle)$ generated by elementary symplectic transvections is denoted by $\mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle,\rangle)$.

The group of relative symplectic transvections w.r.t. an ideal I is generated by the symplectic transvecions of the form

$$\sigma(p) = p + \langle u,p\rangle v + \langle v,p\rangle u + \alpha\langle u,p\rangle u,$$

where $\alpha \in I$ and $u \in P$, $v \in IP$ are fixed elements with $\langle u,v\rangle = 0$.

The group generated by relative symplectic transvections, as above, is denoted by $\mathrm{Trans}_{\mathrm{Sp}}(P, IP, \langle,\rangle)$.

The elementary symplectic transvections of Q of the form

$$\begin{aligned}(a,b,p) &\mapsto (a, b-\langle p,q\rangle+\alpha a, p+aq),\\ (a,b,p) &\mapsto (a+\langle p,q\rangle-\beta b, b, p+bq),\end{aligned}$$

where $\alpha, \beta \in I$ and $q \in IP$ are called relative elementary symplectic transvections w.r.t. an ideal I.

The subgroup of $\mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle,\rangle)$ generated by relative elementary symplectic transvections is denoted by $\mathrm{ETrans}_{\mathrm{Sp}}(IQ, \langle,\rangle)$. The relative group $\mathrm{ETrans}_{\mathrm{Sp}}(Q, IQ, \langle,\rangle)$ is the normal closure of $\mathrm{ETrans}_{\mathrm{Sp}}(IQ, \langle,\rangle)$ in the group $\mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle,\rangle)$.

For both the groups we establish dilation principle, Local Global principle and action version of Local Global principle in absolute case and relative case.

Using these principles we get the main results of this chapter. They are the following:

Theorem 4(a): (see Theorem 5.10.3)

Let $(P, \langle,\rangle_\varphi)$ be a symplectic R-module with P free R-module of rank $2n$, $n \geq 1$. Let $\langle u,v\rangle = u\varphi v^t$, where φ is an alternating matrix of Pfaffian 1. Then

$$\mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle,\rangle_\varphi) \;=\; \mathrm{ESp}_{\psi_1 \perp \varphi}(R).$$

$\square$

Theorem 4(b): (see Theorem 5.10.4)

Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Let $(P, \langle , \rangle_\varphi)$ be a symplectic R-module with P free R-module of rank $2n$, $n \geq 2$. Let $\langle u, v \rangle = u\varphi v^t$, where φ is an alternating matrix of Pfaffian 1 such that $\varphi \equiv \psi_n \pmod{I}$. Then

$$\mathrm{ETrans}_{\mathrm{Sp}}(Q, IQ, \langle , \rangle_\varphi) \quad = \quad \mathrm{ESp}_{\psi_1 \perp \varphi}(R, I).$$

$\square$

Using dilation principle, Local Global principle and action version of Local Global principle we deduce the **global version** of Theorem 2(a) and Theorem 2(b).

Theorem 5(a): (see Theorem 5.11.3)

Let $(P, \langle , \rangle)$ be a symplectic R-module with P finitely generated projective module of rank $2n$, with $n \geq 1$ and $v = (a, b, p) \in \mathrm{Um}(Q)$. Then

$$(a, b, p)\ \mathrm{ETrans}(Q) \quad = \quad (a, b, p)\ \mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle , \rangle).$$

$\square$

Theorem 5(b): (see Theorem 5.11.4)

Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Let $(P, \langle , \rangle)$ be a symplectic R-module with P finitely generated projective module of rank $2n$, with $n \geq 2$. Let $v = (a, b, p) \in \mathrm{Um}(Q, IQ)$. Then

$$(a, b, p)\ \mathrm{ETrans}(Q, IQ) \quad = \quad (a, b, p)\ \mathrm{ETrans}_{\mathrm{Sp}}(Q, IQ, \langle , \rangle).$$

Here we also assume that of each maximal ideal $\mathfrak{m}$ of R, the alternating form $\langle , \rangle$ corresponds to the alternating matrix $\varphi_\mathfrak{m}$, where $\varphi_\mathfrak{m} \equiv \psi_n \pmod{I}$, over the local ring $R_\mathfrak{m}$. $\square$

Remark: *All the theorems appeared in this chapter can also be proved for invertible alternating matrix need not be of Pfaffian 1. The details are left to the reader.*

Chapter 6

In this chapter we recall W. van der Kallen's definition of Excision ring in ([15], (3.19)).

The Excision ring $(\mathbb{Z} \oplus I)$: If I is an ideal in R, one can construct the ring $\mathbb{Z} \oplus I$

with multiplication defined by $(n \oplus i)(m \oplus j) = (nm \oplus nj + mi + ij)$, for $m, n \in \mathbb{Z}$, $i, j \in I$.

We also recall W. van der Kallen's Excision theorem in the linear case.

Excision Theorem:([15], Theorem 3.21)

Let $n \geq 3$ and let I be an ideal in R. Then the natural maps

$$\frac{\mathrm{Um}_n(\mathbb{Z} \oplus I, 0 \oplus I)}{\mathrm{E}_n(\mathbb{Z} \oplus I, 0 \oplus I)} \longrightarrow \frac{\mathrm{Um}_n(R, I)}{\mathrm{E}_n(R, I)},$$
$$\frac{\mathrm{Um}_n(\mathbb{Z} \oplus I, 0 \oplus I)}{\mathrm{E}_n(\mathbb{Z} \oplus I, 0 \oplus I)} \longrightarrow \frac{\mathrm{Um}_n(\mathbb{Z} \oplus I)}{\mathrm{E}_n(\mathbb{Z} \oplus I)},$$

are bijective. □

The goal of this chapter is to establish a symplectic analogue of W. van der Kallen's Excision theorem, which appears next.

Theorem 6: (see Theorem 6.3.2)

Let R be a commutative ring with $R = 2R$, ane let I be an ideal of R. Then the natural maps

$$\Phi : \frac{\mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)}{\mathrm{ESp}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)} \longrightarrow \frac{\mathrm{Um}_{2n}(R, I)}{\mathrm{ESp}_{2n}(R, I)},$$
$$\Psi : \frac{\mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)}{\mathrm{ESp}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)} \longrightarrow \frac{\mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I)}{\mathrm{ESp}_{2n}(\mathbb{Z}[1/2] \oplus I)},$$

are bijective for $n \geq 3$. □

Here, using the above theorem we recapture Theorem 2(b), which says that $v\mathrm{E}_{2n}(R, I) = v\mathrm{ESp}_{2n}(R, I)$, for an ideal I of R, when $R = 2R$.

We also establish the following Excision theorem for the group of elementary symplectic transvections for a free (projective) module.

Definition: Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Let us consider the excision ring $\mathbb{Z}[1/2] \oplus I$. The standard alternating matrix of size $2n$ over the ring $\mathbb{Z}[1/2] \oplus I$ is defined inductively as

$$\widehat{\psi_n} = \widehat{\psi_{n-1}} \perp \widehat{\psi_1},$$

where

$$\widehat{\psi_1} = \begin{pmatrix} (0,0) & (1,0) \\ (-1,0) & (0,0) \end{pmatrix}.$$

Theorem 7: (see Theorem 6.6.3)

Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Let us consider the excision ring $\mathbb{Z}[1/2] \oplus I$. Let $\big((\mathbb{Z}[1/2] \oplus I)^{2n-2}, \langle , \rangle_{\varphi^*}\big)$ be a symplectic $(\mathbb{Z}[1/2] \oplus I)$-module, where φ^* be an alternating matrix over the ring $\mathbb{Z}[1/2] \oplus I$ and $\varphi^* \equiv \widehat{\psi_{n-1}} \pmod{0 \oplus I}$. Then the natural maps

$$\frac{\mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)}{\mathrm{ETrans}_{\mathrm{Sp}}\big((\mathbb{Z}[1/2] \oplus I)^{2n}, (0 \oplus I)^{2n}, \langle , \rangle_{\varphi^*}\big)} \xrightarrow{\eta} \frac{\mathrm{Um}_{2n}(R, I)}{\mathrm{ETrans}_{\mathrm{Sp}}(R^{2n}, I^{2n}, \langle , \rangle_{\varphi^*})},$$
$$\frac{\mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)}{\mathrm{ETrans}_{\mathrm{Sp}}\big((\mathbb{Z}[1/2] \oplus I)^{2n}, (0 \oplus I)^{2n}, \langle , \rangle_{\varphi^*}\big)} \xrightarrow{\delta} \frac{\mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I)}{\mathrm{ETrans}_{\mathrm{Sp}}\big((\mathbb{Z}[1/2] \oplus I)^{2n}, \langle , \rangle_{\varphi^*}\big)},$$

are bijective for $n \geq 3$. □

Chapter 7

Here we recall the definition of stable general linear group, stable special linear group, stable range and stable dimension.

H. Bass, J. Milnor, J-P. Serre began the study of the stabilization for the linear group $\mathrm{GL}_n(R)/\mathrm{E}_n(R)$, for $n \geq 3$. In ([6], Corollary 11.3), they showed that:

Suppose that the maximal spectrum of a commutative ring R is a Noetherian space of dimension $\leq d$. Then the map

$$\frac{\mathrm{GL}_n(R)}{\mathrm{E}_n(R)} \longrightarrow \frac{\mathrm{GL}_{n+1}(R)}{\mathrm{E}_{n+1}(R)}$$

is an isomorphism of groups for all $n \geq d + 3$. □

In ([4], §11) Bass conjectured the following:

Conjecture due to Bass:
Let R be a commutative ring with 1 and Jacobson-Krull dimension of R is d. Then the map

$$\frac{\mathrm{GL}_n(R)}{\mathrm{E}_n(R)} \longrightarrow \frac{\mathrm{GL}_{n+1}(R)}{\mathrm{E}_{n+1}(R)}$$

is an isomorphism for $n \geq d + 2$.

In [34], L.N. Vaserstein proved the above conjecture.

After this in ([26]), R.A. Rao and W. van der Kallen began the study of whether the stabilization bound above improves for special rings; and they could show (see [26], Theorem 1)

Let A be a non-singular affine algebra of Krull dimension $d \geq 2$ over a perfect C_1-field. Let $\sigma \in \mathrm{SL}_{d+1}(A)$ and $(1 \perp \sigma) \in \mathrm{E}_{d+2}(A)$. Then σ is homotopic to identity, i.e, there exists a $\rho(X) \in \mathrm{SL}_{d+1}(A[X])$ such that $\rho(1) = \sigma$ and $\rho(0) = Id$. $\square$

Then, as a consequence they showed that

If A is a non-singular affine algebra of Krull dimension $d \geq 2$ over a perfect C_1-field, then the natural map

$$\frac{\mathrm{SL}_n(A)}{\mathrm{E}_n(A)} \longrightarrow \frac{\mathrm{SL}_{n+1}(A)}{\mathrm{E}_{n+1}(A)}$$

is injective for $n \geq d+1$. $\square$

L.N. Vaserstein in [35] considered the symplectic, orthogonal and the unitary K_1-functors, and obtained stabilization theorems for them. In ([35], Theorem 3.3) he showed that:

The natural map

$$\varphi_{n,n+1} : \frac{\mathrm{Sp}_{2n}(R)}{\mathrm{ESp}_{2n}(R)} \longrightarrow \frac{\mathrm{Sp}_{2n+2}(R)}{\mathrm{ESp}_{2n+2}(R)}$$

is an isomorphism for $2n \geq 2d+4$. Here d is the stable dimension of R. $\square$

R. Basu and R.A. Rao showed, in ([9], Theorem 1), the following:

Let R be a non-singular affine algebra over a perfect C_1-field of odd Krull dimension $d \geq 2$. Let $\sigma \in \mathrm{Sp}_{2n}(R)$ and $(I_2 \perp \sigma) \in \mathrm{ESp}_{2n+2}(R)$. Then σ is homotopic to identity, i.e, there exists $\rho(X) \in \mathrm{Sp}_{2n}(R[X])$ such that $\rho(1) = \sigma$ and $\rho(0) = Id$. $\square$

And as a consequence they showed (see [9], Theorem 2)

If R is a non-singular affine algebra over a perfect C_1-field of odd Krull dimension $d \geq 2$, then the natural map

$$\varphi_{n,n+1} : \frac{\mathrm{Sp}_{2n}(R)}{\mathrm{ESp}_{2n}(R)} \longrightarrow \frac{\mathrm{Sp}_{2n+2}(R)}{\mathrm{ESp}_{2n+2}(R)}$$

is an isomorphism for $2n \geq d+1$. $\square$

Using Theorem 2(b) we can reprove this result. Moreover we show the following:

Theorem 8: (see Theorem 7.1.15)

Let R be a finitely generated algebra of even Krull dimension $d \geq 4$ over K, where $K = \mathbb{Z}$ or F or $\overline{F}$ and char(K)$\neq 2$. (Here F is a finite field and $\overline{F}$ is its algebraic closure.) Let $\sigma \in \mathrm{Sp}_d(R)$ and $(I_2 \perp \sigma) \in \mathrm{ESp}_{d+2}(R)$. Then σ is (stably elementary symplectic) homotopic to the identity. In fact, $\sigma = \rho(1)$, and $\rho(0) = Id$, for some

$$\rho(X) \;\; \in \;\; \mathrm{Sp}_d(R[X]) \; \cap \; \mathrm{ESp}_{d+2}(R[X]).$$

$\square$

As a consequence of the above result we show that:

Theorem 9: (see Corollary 7.1.16)

Let R be a finitely generated non-singular algebra of even Krull dimension $d \geq 4$ over K, where K is either a finite field or the algebraic closure of a finite field and char(K)$\neq 2$. Let $\sigma \in \mathrm{Sp}_d(R)$ and $(I_2 \perp \sigma) \in \mathrm{ESp}_{d+2}(R)$. Then σ belongs to $\mathrm{ESp}_d(R)$. In particular,

$$\frac{\mathrm{Sp}_{2n}(R)}{\mathrm{ESp}_{2n}(R)} \;\; \longrightarrow \;\; \frac{\mathrm{Sp}_{2n+2}(R)}{\mathrm{ESp}_{2n+2}(R)}$$

is injective for $2n \geq d$. $\square$

We also show the following:

Theorem 10: (see Theorem 7.2.7)

Let R be a commutative ring of dimension d. Let us assume $R = 2R$. Let $(P\langle,\rangle)$ be a symplectic R-module with P finitely generated projective module of even rank $\geq max\{2, d-3\}$. Let $Q = (R^2 \perp P)$, and $\widehat{Q} = (R^2 \perp Q)$. Let $\sigma \in \mathrm{Sp}(Q, \langle,\rangle)$, and $(I_2 \;\perp\; \sigma) \in \mathrm{ETrans}_{\mathrm{Sp}}(\widehat{Q}), \langle,\rangle)$. Then σ is (stably elementary symplectic) homotopic to the identity. In fact, $\sigma = \rho(1)$, and $\rho(0) = Id$, for some

$$\rho(X) \;\; \in \;\; \mathrm{Sp}(Q[X], \langle,\rangle) \; \cap \; \mathrm{ETrans}_{\mathrm{Sp}}(\widehat{Q}[X], \langle,\rangle).$$

Here we assume that over the local ring $R_{\mathfrak{m}}$, where $\mathfrak{m}$ is a maximal ideal of R, the alternating form $\langle,\rangle$ corresponds to the alternating matrix $\varphi_{\mathfrak{m}}$, where $\varphi_{\mathfrak{m}} \equiv \psi_n \pmod{I}$. $\square$

The next theorem is a consequence of the above theorem. This gives an improvement for Basu-Rao (see Theorem 7.1.8) estimate in the module case over finitely generated rings.

Theorem 11: (see Corollary 7.2.9)

Let R be a finitely generated non-singular algebra of dimension d over K, where K is either a finite field or the algebraic closure of a finite field. Let us assume $R = 2R$. Let $(P.\langle,\rangle)$ be a symplectic R-module with P a finitely generated projective module of even rank $\geq max\{2, d-3\}$. Let $Q = (R^2 \perp P)$, and $\widehat{Q} = (R^2 \perp Q)$. Let $\sigma \in \mathrm{Sp}(Q, \langle,\rangle)$ and $(I_2 \perp \sigma) \in \mathrm{ETrans}_{\mathrm{Sp}}(\widehat{Q}, \langle,\rangle)$. Then σ belongs to $\mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle,\rangle)$.

Here we assume that over the local ring $R_{\mathfrak{m}}$, where $\mathfrak{m}$ is a maximal ideal of R, the alternating form $\langle,\rangle$ corresponds to the alternating matrix $\varphi_{\mathfrak{m}}$, where $\varphi_{\mathfrak{m}} \equiv \psi_n \pmod{I}$. □

Chapter 2

Preliminaries

In this chapter we will recall a few definitions, fix some notations, state few known results as well as state some preliminary results with their proofs, which will be used throughout this thesis.

2.1 Definitions and Notations

Definition 2.1.1 An R-module M is said to be **finitely presented** if there is an exact sequence

$$R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0,$$

for suitable natural numbers n, m.

Definition 2.1.2 An $R[X]$-module M will be called **extended** from R if it is isomorphic to $R[X] \otimes_R N$ for some R-module N.

Definition 2.1.3 A row $v = (v_1, \ldots, v_n) \in R^n$ is said to be **unimodular** (of length n) if there is a row vector $w = (w_1, \ldots, w_n)$ from R^n such that

$$\langle v, w \rangle = v \cdot w^t = v_1 w_1 + \cdots + v_n w_n = 1.$$

$\mathrm{Um}_n(R)$ will denote the set of all unimodular rows $v \in R^n$.

Definition 2.1.4 Let I be an ideal of R. A row is said to be **relative unimodular** w.r.t. I if it is unimodular and congruent to $e_1 = (1, 0, \ldots, 0)$ modulo I.

$\mathrm{Um}_n(R, I)$ will denote the set of all relative unimodular rows w.r.t. I of length n. If $I = R$, then $\mathrm{Um}_n(R, I)$ is $\mathrm{Um}_n(R)$.

Definition 2.1.5 Let P be a finitely generated projective R-module of rank n. An element v in P is said to be **unimodular** if for any maximal ideal $\mathfrak{m}$ of R, we have $v_\mathfrak{m} \in \mathrm{Um}_n(R_\mathfrak{m})$. The collection of unimodular elements of P is denoted by $\mathrm{Um}(P)$. Note that $\mathrm{Um}(P, IP)$ denotes the collection of elements from $\mathrm{Um}(P)$ such that $v_\mathfrak{m} \in \mathrm{Um}_n(R_\mathfrak{m}, I_\mathfrak{m})$. The set $\mathrm{Um}(P, IP)$ is the collection of all **relative unimodular** elements w.r.t. an ideal I of R.

Definition 2.1.6 The set of all $n \times n$ invertible matrices, whose entries come from a ring R, is a group under matrix multiplication. This group is called **General Linear** group of size n. (This group is denoted by $\mathrm{GL}_n(R)$.)

Notation 2.1.7 The group $\mathrm{GL}_n(R)$ of invertible matrices acts on R^n in a natural way: $v \longrightarrow v\sigma$, if $v \in R^n$, $\sigma \in \mathrm{GL}_n(R)$. This map preserves $\mathrm{Um}_n(R)$ (see Lemma 2.2.2). So $\mathrm{GL}_n(R)$ acts on $\mathrm{Um}_n(R)$. Note that any subgroup G of $\mathrm{GL}_n(R)$ also acts on $\mathrm{Um}_n(R)$. Let $v, w \in \mathrm{Um}_n(R)$, we denote $v \sim_\mathrm{G} w$ or $v \in wG$, if there is a $g \in \mathrm{G}$ such that $v = wg$.

Definition 2.1.8 Let R be a commutative ring with 1. The set of all $n \times n$ invertible matrices, with determinant 1 is a group which is called **Special Linear** group. (This is a subgroup of $\mathrm{GL}_n(R)$ and denoted by $\mathrm{SL}_n(R)$.)

Definition 2.1.9 Let $\mathrm{E}_n(R)$ denote the subgroup of $\mathrm{SL}_n(R)$ consisting of all *elementary* matrices, i.e. those matrices which are a finite product of the *(standard) elementary generators*

$$E_{ij}(\lambda) \;=\; I_n + e_{ij}(\lambda),\ 1 \le i \ne j \le n,\ \lambda \in R,$$

where $e_{ij}(\lambda) \in \mathrm{M}_n(R)$ has at most one non-zero entry λ in its (i, j)-th position. The group $\mathrm{E}_n(R)$ is called **Elementary Group**.

In the thesis, if α denotes an $m \times n$ matrix, then we let α^t denote its *transpose* matrix. This is of course an $n \times m$ matrix. However, we will mostly be working with square matrices, or rows and columns. Also in the sequel $\mathrm{GL}_n(R, I)$ denotes the kernel of the map $\mathrm{GL}_n(R) \longrightarrow \mathrm{GL}_n(R/I)$ and $\mathrm{SL}_n(R, I)$ denotes the kernel of the map $\mathrm{SL}_n(R) \longrightarrow \mathrm{SL}_n(R/I)$.

Definition 2.1.10 Let I be an ideal of R. The **relative elementary group** $\mathrm{E}_n(I)$ is the subgroup of $\mathrm{E}_n(R)$ generated as a group by the elements $\mathrm{E}_{ij}(x)$, $x \in I$, $1 \leq i \neq j \leq n$. The **relative elementary group** $\mathrm{E}_n(R, I)$ is the normal closure of $\mathrm{E}_n(I)$ in $\mathrm{E}_n(R)$.

(Equivalently, $\mathrm{E}_n(R, I)$ denotes the smallest normal subgroup of $\mathrm{E}_n(R)$ containing the element $E_{21}(x)$, where $x \in I$. Also $\mathrm{E}_n(R, I)$ is generated as a group by the elements $\mathrm{E}_{ij}(a)\mathrm{E}_{ji}(x)\mathrm{E}_{ij}(-a)$, with $a \in R$, $x \in I$, and $1 \leq i \neq j \leq n$, provided $n \geq 3$ (see Lemma 2.2.29).)

Definition 2.1.11 $\mathbf{E}_n^1(R, I)$ is the subgroup of $\mathrm{E}_n(R)$ generated by the elements $E_{1i}(a)$, where $a \in R$, and $E_{i1}(x)$, where $x \in I$, $2 \leq i \leq n$.

Definition 2.1.12 **Symplectic Group** $\mathbf{Sp}_{2n}(R)$**:** The group of all invertible $2n \times 2n$ matrices

$$\{\alpha \in \mathrm{GL}_{2n}(R) \mid \alpha^t \psi_n \alpha = \psi_n\},$$

where ψ_n is the alternating matrix $\sum_{i=1}^{n} e_{2i-1,2i} - \sum_{i=1}^{n} e_{2i,2i-1}$ (corresponding to the standard symplectic form). By $\mathrm{Sp}_{2n}(R, I)$ we denote the kernel of the map $\mathrm{Sp}_{2n}(R) \longrightarrow \mathrm{Sp}_{2n}(R/I)$.

Notation 2.1.13 Let σ denote the permutation of the natural numbers given by $\sigma(2i) = 2i - 1$ and $\sigma(2i - 1) = 2i$.

Definition 2.1.14 We define, for $1 \leq i \neq j \leq 2n$, $z \in R$,

$$se_{ij}(z) = \begin{cases} 1_{2n} + e_{ij}(z) & \text{if } i = \sigma(j), \\ 1_{2n} + e_{ij}(z) - (-1)^{i+j} e_{\sigma(j)\sigma(i)}(z) & \text{if } i \neq \sigma(j) \text{ and } i < j. \end{cases}$$

It is easy to check that all these generators belong to $\mathrm{Sp}_{2n}(R)$. We call them the *(standard) elementary symplectic* matrices over R and the subgroup of $\mathrm{Sp}_{2n}(R)$ generated by them is called the **Elementary Symplectic group** $\mathrm{ESp}_{2n}(R)$.

Definition 2.1.15 Let I be an ideal of R. The **relative elementary group** $\mathrm{ESp}_{2n}(I)$ is a subgroup of $\mathrm{ESp}_{2n}(R)$ generated as a group by the elements $se_{ij}(x)$, $x \in I$ and $1 \leq i \neq j \leq 2n$.

The **relative elementary group** $\mathrm{ESp}_{2n}(R, I)$ is the normal closure of $\mathrm{ESp}_{2n}(I)$ in $\mathrm{ESp}_{2n}(R)$.

(Equivalently, $\mathrm{ESp}_{2n}(R,I)$ is the smallest normal subgroup of $\mathrm{ESp}_{2n}(R)$ containing the element $se_{21}(x)$, where $x \in I$. Also $\mathrm{ESp}_{2n}(R,I)$ is generated as a group by the elements $se_{ij}(a)se_{ji}(x)se_{ij}(-a)$, with $a \in R$, $x \in I$, $1 \leq i \neq j \leq 2n$, provided $n \geq 3$ (see Lemma 2.2.29).)

Definition 2.1.16 $\mathrm{ESp}^1_{2n}(R,I)$ is the subgroup of $\mathrm{ESp}_{2n}(R)$ generated by the elements $se_{1i}(a)$, where $a \in R$, and $se_{i1}(x)$, where $x \in I$, $2 \leq i \leq 2n$.

Definition 2.1.17 Orthogonal Group $\mathrm{O}_{2n}(R)$: The group of all invertible $2n \times 2n$ matrices

$$\{\alpha \ \in \ \mathrm{GL}_{2n}(R) \mid \alpha^t\widetilde{\psi_n}\alpha \ = \ \widetilde{\psi_n}\},$$

where $\widetilde{\psi_n}$ is the symmetric matrix $\sum\limits_{i=1}^{n} e_{2i-1,2i} + \sum\limits_{i=1}^{n} e_{2i,2i-1}$ (corresponding to the standard hyperbolic form). By $\mathrm{O}_{2n}(R,I)$ we denote the kernel of the map $O_{2n}(R) \longrightarrow O_{2n}(R/I)$.

Definition 2.1.18 We define for $1 \leq i \neq j \leq 2n$, $z \in R$,

$$oe_{ij}(z) \ = \ 1_{2n} + e_{ij}(z) - e_{\sigma(j)\sigma(i)}(z), \text{ if } i \neq \sigma(j), \text{ and } i < j.$$

It is easy to check that all these matrices belong to $\mathrm{O}_{2n}(R)$. We call them the *(standard) elementary orthogonal* matrices over R and the subgroup of $\mathrm{O}_{2n}(R)$ generated by them is called the **Elementary Orthogonal group** $\mathrm{EO}_{2n}(R)$.

Definition 2.1.19 Let I be an ideal of R. The **relative elementary group** $\mathrm{EO}_{2n}(I)$ is a subgroup of $\mathrm{EO}_{2n}(R)$ generated as a group by the elements $oe_{ij}(x)$, $x \in I$ and $1 \leq i \neq j \leq 2n$. The **relative elementary group** $\mathrm{EO}_{2n}(R,I)$ is the normal closure of $\mathrm{EO}_{2n}(I)$ in $\mathrm{EO}_{2n}(R)$. (Equivalently, $\mathrm{EO}_{2n}(R,I)$ is generated as a group by $oe_{ij}(a)oe_{ji}(x)oe_{ij}(-a)$, with $a \in R$, $x \in I$, $i \neq j$, provided $n \geq 3$.)

Definition 2.1.20 $\mathrm{EO}^1_{2n}(R,I)$ is the subgroup of $\mathrm{EO}_{2n}(R)$ generated by the elements $oe_{1i}(a)$, where $a \in R$, and $oe_{i1}(x)$, where $x \in I$, $3 \leq i \leq 2n$.

Notation 2.1.21 We fix some notations.

- $\mathrm{M}(n,R)$ will denote the set of all $n \times n$ matrices.
- $\mathrm{G}(n,R)$ will denote either the linear group $\mathrm{GL}_n(R)$, the symplectic group $\mathrm{Sp}_{2m}(R)$, or the orthogonal group $\mathrm{O}_{2m}(R)$, **for** $n = 2m$.

• Now onwards, $\mathrm{E}(n,R)$ will denote either of the elementary subgroups $\mathrm{E}_n(R)$, $\mathrm{ESp}_{2m}(R)$ or $\mathrm{EO}_{2m}(R)$. The standard elementary generators of $\mathrm{E}(n,R)$ are denoted by $ge_{ij}(a)$, $a \in R$.

• Let I be an ideal in R. Let $\mathrm{G}(n,R,I)$ denote the relative linear groups $\mathrm{GL}_n(R,I)$, $\mathrm{SL}_n(R,I)$, the relative symplectic group $\mathrm{Sp}_{2m}(R,I)$, or the relative orthogonal group $\mathrm{O}_{2m}(R,I)$.

• $\mathrm{E}(n,I)$ is a subgroup of $\mathrm{E}(n,R)$ generated as a group by the elements $ge_{ij}(x)$, where $x \in I$, and $1 \leq i \neq j \leq n$.

• $\mathrm{E}(n,R,I)$ denotes the corresponding relative elementary subgroups $\mathrm{E}_n(R,I)$, $\mathrm{ESp}_{2m}(R,I)$, $\mathrm{EO}_{2m}(R,I)$, respectively. These are the normal closures of the subgroups $\mathrm{E}(n,I)$ in $\mathrm{E}(n,R)$, which are also known to be generated by the elements $ge_{ij}(a)ge_{ji}(x)ge_{ij}(-a)$, $a \in R, x \in I$, and $1 \leq i \neq j \leq n$ (see Lemma 2.2.29).

• $\mathrm{E}^1(n,R,I)$ is a subgroup of $\mathrm{E}(n,R)$, generated by the elements $ge_{1i}(a)$, where $a \in R$ and $ge_{i1}(x)$, where $x \in I$, $2 \leq i \leq n$ in the linear and symplectic case, and $3 \leq i \leq n$ in the orthogonal case.

• In the symplectic case we set $\widetilde{v} = v\psi_m$, where ψ_m is the standard symplectic form, and in the orthogonal case we set $\widetilde{v} = v\widetilde{\psi}_m$, where $\widetilde{\psi}_m$ is the standard hyperbolic form.

• Let $\alpha \in \mathrm{G}(n,R)$ and $\beta \in \mathrm{G}(m,R)$, then by $\alpha \perp \beta$ we denote the matrix

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \quad \in \quad \mathrm{G}(n+m,R).$$

• Let a, b be two elements of $\mathrm{M}(n,R)$. The symbol $[a,b]$ represents the element $aba^{-1}b^{-1}$ and known as commutator of the elements a and b. Also we fix a notation ${}^{a}b = aba^{-1}$.

In this thesis we shall have the

Blanket Assumption:

1. $n \geq 3$ in the linear case and $n = 2m$ with $m \geq 3$ in the symplectic case and in the orthogonal case.

Here we state few commutator identities for the standard elementary generators of $\mathrm{E}(n,R)$.

• For $\mathrm{E}_n(R)$, when $n \geq 3$:

$$\begin{aligned}
[E_{ij}(a), E_{jk}(b)] &= E_{ik}(ab) \quad \text{if } \ i \neq k, \\
[E_{ij}(a), E_{kl}(b)] &= id \qquad\quad \text{if } \ i \neq l, \ \text{and } j \neq k.
\end{aligned}$$

• For $\mathrm{ESp}_{2m}(R)$, for $m \geq 3$:

$$\begin{aligned}
[se_{ik}(a), se_{kj}(b)] &= se_{ij}(ab) && \text{if } \ i \neq j, \sigma(j), \\
[se_{ik}(a), se_{k\sigma(i)}(b)] &= se_{i\sigma(i)}(2ab), && \\
[se_{1j}(a), se_{1l}(b)] &= id && \text{if } \ l \neq \sigma(j), \\
[se_{1j}(a), se_{1l}(b)] &= se_{12}((-1)^{j+1}2ab) && \text{if } \ l = \sigma(j), j \neq 2, \\
[se_{i1}(a), se_{k1}(b)] &= id && \text{if } \ k \neq \sigma(i), \\
[se_{i1}(a), se_{k1}(b)] &= se_{21}((-1)^{i}2ab) && \text{if } \ k = \sigma(i), i \neq 2.
\end{aligned}$$

Definition 2.1.22 A matrix from $\mathrm{M}_n(R)$ is said to be **alternating** if it has the form $\nu - \nu^t$, where $\nu \in \mathrm{M}_n(R)$. It follows that its diagonal elements are zeros.

2.2 Preliminary Results

The most useful property of the standard elementary generators of the classical linear, symplectic and orthogonal groups is the following **linear property**:

Lemma 2.2.1 *For all $a, b \in R$, $ge_{ij}(a+b) = ge_{ij}(a)ge_{ij}(b)$.*

Proof: Follows by an easy verification. □

Lemma 2.2.2 *The natural action of $\mathrm{GL}_n(R)$ on R^n preserves $\mathrm{Um}_n(R)$.*

Proof: Let $v \in \mathrm{Um}_n(R)$ and let $g \in \mathrm{GL}_n(R)$. We need to show vg is in $\mathrm{Um}_n(R)$. Let $w \in R^n$ be such that $\langle v, w \rangle = 1$. Therefore $\langle vg, w(g^{-1})^t \rangle = 1$ and hence $vg \in \mathrm{Um}_n(R)$. □

Lemma 2.2.3 *Let α be in $\mathrm{E}(n, R)$. Then there exists $\alpha(X) \in \mathrm{E}(n, R[X])$ such that $\alpha(1) = \alpha$, and $\alpha(0) = Id$.*

Proof: Let $\alpha = \prod_{k=1}^{r} ge_{i_k j_k}(a_k)$, where $a_k \in R$. Let us define

$$\alpha(X) \;=\; \prod_{k=1}^{r} ge_{i_k j_k}(a_k X).$$

Clearly $\alpha(X) \in \mathrm{E}(n, R[X])$. Note that $\alpha(1) = \alpha$, and $\alpha(0) = Id$. $\square$

Lemma 2.2.4 *Let M be an R-module and let $\alpha(X), \beta(X) \in \mathrm{Aut}(M[X])$, with $\alpha(0) = Id$, $\beta(0) = Id$. Let a be a non-nilpotent element in R. Let $\alpha(X)_a = \beta(X)_a$ in $\mathrm{Aut}(M_a[X])$. Then $\alpha(a^N X) = \beta(a^N X)$ in $\mathrm{Aut}(M[X])$, for $N \gg 0$.*

Proof: Using $\alpha(0) - \beta(0) = 0$, we get $\alpha(X) - \beta(X) = X\gamma(X)$, for some $\gamma(X) \in \mathrm{Aut}(M[X])$. Also $\alpha(X)_a - \beta(X)_a = 0$ in $\mathrm{Aut}(M_a[X])$, i.e, $\big(\alpha(X) - \beta(X)\big)_a = 0$, i.e, $\big(X\gamma(X)\big)_a = 0$. Hence $a^N\big(X\gamma(X)\big) = 0$, in $\mathrm{Aut}(M[X])$, for some $N \gg 0$. Therefore

$$\alpha(a^N X) - \beta(a^N X) \;=\; a^N X \gamma(a^N X) \;=\; 0,$$

in $\mathrm{Aut}(M[X])$, for $N \gg 0$. $\square$

Lemma 2.2.5 ([31], Lemma 1.3): *Let $v = (v_1, \dots, v_n) \in \mathrm{Um}_n(R)$ and let $u = (u_1, \dots, u_n) \in R^n$ be such that $\sum_{i=1}^{n} v_i u_i = 1$. Let $\varphi \colon R^n \to R$ be the map sending $e_i \mapsto v_i$ where $e_1, \dots, e_n$ is the natural basis for R^n. Then, for $w = (w_1, \dots, w_n) \in \ker(\varphi)$, $w = \sum_{i<j} a_{ij}(v_j e_i - v_i e_j)$, $a_{ij} \in R$.*

See ([14], Page 18, Lemma 4.6) or ([20], Proposition 5.1.1) for an alternative proof. $\square$

Lemma 2.2.6 ([31], Corollary 1.2): *Let $n \geq 3$ and I be an ideal of R. Let $v \in R^n$ and $w \in I^n$ be such that $\langle w, v\rangle = 0$. If $w_i = 0$, for some $1 \leq i \leq n$, then $\mathrm{I}_n + v^T w \in \mathrm{E}_n(R, I)$.* $\square$

Lemma 2.2.7 (Suslin): (see [20], Corollary 5.1.3) *Let $n \geq 3$. Let $v, w \in R^n$ be such that $v \in \mathrm{Um}_n(R)$ and $\langle w, v\rangle = 0$. Then $\mathrm{I}_n + v^T w \in \mathrm{E}_n(R)$.* $\square$

Lemma 2.2.8 *Let $n \geq 3$ and I be an ideal of R. Let $v \in \mathrm{Um}_n(R)$ and $w \in I^n$ such that $\langle w, v\rangle = 0$. Then $I_n + v^t w \in \mathrm{E}_n(R, I)$.*

Proof: Let $v = (v_1, \ldots, v_n) \in R^n$, and $w = (w_1, \ldots, w_n) \in I^n$. Let $u \in R^n$ be such that $\sum v_i u_i = 1$. Using Lemma 2.2.5 we get

$$\begin{aligned} w &= \sum w_i e_i = \sum_{i \neq j} v_j (w_i u_j - w_j u_i) e_i \\ &= \sum_{i<j} (w_i u_j - w_j u_i)(v_j e_i - v_i e_j) \\ &= \sum_{i<j} a_{ij} (v_j e_i - v_i e_j), \end{aligned}$$

where $a_{ij} \in I$. Now

$$\begin{aligned} I_n + v^t w &= I_n + \sum_{i<j} a_{ij} v^t (v_j e_i - v_i e_j) \\ &= \prod_{i<j} \big(I_n + a_{ij} v^t (v_j e_i - v_i e_j) \big). \end{aligned}$$

Each term appeared in the above product is in $\mathrm{E}_n(R, I)$ (see Lemma 2.2.6). Hence we established the claim. $\square$

Lemma 2.2.9 ([36], Lemma 8) *Let R be an associative ring with identity and let I be a two sided ideal in R. Then $\mathrm{E}_n(R, I) = [\mathrm{E}_n(R), \mathrm{E}_n(I)]$, for $n \geq 3$.* $\square$

Corollary 2.2.10 ([31], Corollary 1.4) *For $n \geq 3$, $\mathrm{E}_n(R, I)$ is a normal subgroup of $\mathrm{GL}_n(R)$.* $\square$

Lemma 2.2.11 ([29], Lemma 2.7(a)) *Let R be an associative ring with 1. Then $\mathrm{E}_n(R)$ is generated by the matrices of the form*

$$\begin{pmatrix} 1 & v \\ 0 & I \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ v^t & I \end{pmatrix},$$

where $v \in R^{2n-1}$. $\square$

Remark 2.2.12 *Note that if $v = (v_1, \ldots, v_{2n-1}) \in R^{2n-1}$, we have*

$$\begin{pmatrix} 1 & v \\ 0 & I \end{pmatrix} = \prod_{i=2}^{2n} E_{1i}(v_{i-1}),$$

$$\begin{pmatrix} 1 & 0 \\ v^t & I \end{pmatrix} = \prod_{i=2}^{2n} E_{i1}(v_{i-1}),$$

and hence $\mathrm{E}_n(R)$ *is generated by the elements of the form* $E_{1i}(a), E_{i1}(b)$*, where* $a, b \in R$*, and* $2 \leq i \leq n$*.*

Lemma 2.2.13 ([29], Lemma 2.7(b)) *Let* R *be an associative ring with 1,* I *be an ideal of* R*, and* $n \geq 3$ *be a natural number. Then* $\mathrm{E}_n(R, I^2) \subseteq \mathrm{E}_n(I)$*, where* I^2 *is a two sided ideal of* R *consisting of sums of elements of the form* ab *where* $a, b \in I$*.*

Proof: Let $\beta = E_{ij}(z) \in \mathrm{E}_n(I^2)$ and $\alpha = E_{kl}(z') \in \mathrm{E}_n(R)$. We need to show $\alpha\beta\alpha^{-1} \in \mathrm{E}_n(I)$. If $(i, j) \neq (l, k)$, then the matrix $\alpha\beta\alpha^{-1}$ splits in to product of elementary matrices from $\mathrm{E}_n(I)$. If $(i, j) = (l, k)$, we choose $r \leq n$ different from i, j and write $z = a_1b_1 + \cdots + a_sb_s$. Now we can write

$$\beta = E_{ij}(z) = \prod_{t=1}^{s} [E_{ir}(a_t), E_{rj}(b_t)],$$

and

$$\alpha\beta\alpha^{-1} = \prod_{t=1}^{r} [\alpha E_{ir}(a_t)\alpha^{-1}, \alpha E_{tj}(b_t)\alpha^{-1}] \in \mathrm{E}_n(I).$$

Hence the lemma is proved. □

Lemma 2.2.14 ([21], Lemma 2.5) **(Whitehead's Lemma):**

$$\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \in \mathrm{E}_2(R),$$

whenever u *is a unit in* R*. Moreover*

$$\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \in \mathrm{E}_2(R, I),$$

whenever u *is unit in* R *and* $u \equiv 1 \pmod{I}$*.*

Proof: To prove this lemma we need to consider the following equation:

$$\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} = E_{21}(u^{-1}-1)E_{12}(1)E_{21}(u-1)E_{12}(-1)E_{12}(1-u^{-1}),$$

and hence the proof follows. ☐

Lemma 2.2.15 *Let $(R, \mathfrak{m})$ be a local ring. Then for any $v \in \mathrm{Um}_n(R)$, $v \in e_1\mathrm{E}_n(R)$.*

Proof. Let $v = (v_1, \ldots, v_n) \in \mathrm{Um}_n(R)$. Since R is a local ring this forces one of the v_i to be unit in R. Therefore there exists $E \in \mathrm{E}_n(R)$ such that $vE = (0, \ldots, 0, v_i, 0, \ldots, 0)$. Note that

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \in \mathrm{E}_2(R),$$

and hence there exists $E' \in \mathrm{E}_n(R)$ such that $vEE' = (v_i, 0, \ldots, 0)$. Now

$$vEE'\begin{pmatrix} v_i^{-1} & 0 & 0 \\ 0 & v_i & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix} = (1, 0, \ldots, 0),$$

and hence using Lemma 2.2.14 we get, $v \in e_1\mathrm{E}_n(R)$. ☐

Lemma 2.2.16 *Let $(R, \mathfrak{m})$ be a local ring. Then for any $v \in \mathrm{Um}_{2n}(R)$, $v \in e_1\mathrm{ESp}_{2n}(R)$.*

Proof: Let $v = (v_1, \ldots, v_n) \in \mathrm{Um}_n(R)$. Since R is a local ring this forces one of the v_i to be unit in R. Therefore there exists $E \in \mathrm{ESp}_{2n}(R)$ such that $vE = (0, \ldots, 0, v_i, 0, \ldots, 0)$. Namely we choose

$$E = se_{i1}(-v_1v_i^{-1})\ldots se_{i2n}(-v_1v_{2n}^{-1})se_{i\sigma(i)}(*),$$

for a suitable element $* \in R$. Note that

$$(0, \ldots, 0, v_i, 0, \ldots, 0)se_{i1}(1)se_{1i}(-1) = (v_i, 0, \ldots, 0).$$

Now

$$vE\; se_{i1}(1)se_{1i}(-1)\begin{pmatrix} v_i^{-1} & 0 & 0 \\ 0 & v_i & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix} \;=\; (1,0,\ldots,0),$$

and hence using Lemma 2.2.14 we get, $v \in e_1\mathrm{ESp}_{2n}(R)$. □

Lemma 2.2.17 *Let I be an ideal in R. Let $v = (1+x_1, x_2, \ldots, x_{2n}) \in \mathrm{Um}_{2n}(R, I)$. Let $1 + x_1$ be a unit in R. Then there exists*

$$g \;\in\; \mathrm{ESp}_{2n}(R,I)\ \big(\subseteq \mathrm{E}_{2n}(R,I)\big)$$

such that $v = e_1 g$.

Proof: Since $1 + x_1 = u$ is a unit in the ring R, it is easy to show as in the proof of Lemma 2.2.16 that there exists $g^* \in \mathrm{ESp}_{2n}(R, I)$ such that $vg^* = (u, 0, \ldots, 0)$. We have

$$(u,0,\ldots,0) \;=\; e_1\begin{pmatrix} u & 0 & 0 \\ 0 & u^{-1} & 0 \\ 0 & 0 & I_{2n-2} \end{pmatrix} \;\in\; e_1\mathrm{ESp}_{2n}(R,I),$$

by Lemma 2.2.14. Let us take

$$g \;=\; E_{21}(u^{-1}-1)E_{12}(1)E_{21}(u-1)E_{12}(-1)E_{12}(1-u^{-1})(g^*)^{-1}.$$

Clearly $g \in \mathrm{ESp}_{2n}(R, I)$ and $v = e_1 g$. □

Corollary 2.2.18 *Let $(R, \mathfrak{m})$ be a local ring and I be a ideal of R. Let $v \in \mathrm{Um}_n(R, I)$, then $v \in e_1\mathrm{E}_n(R, I)$.*

Corollary 2.2.19 *Let $(R, \mathfrak{m})$ be a local ring and I be a ideal of R. Let $v \in \mathrm{Um}_{2n}(R, I)$, then $v \in e_1\mathrm{ESp}_{2n}(R, I)$.*

Lemma 2.2.20 *Let I be an ideal of R and let $\varepsilon \in \mathrm{E}(n, R/I^k, I/I^k)$, for some positive integer $k > 1$. Then there exists $\varepsilon_0 \in \mathrm{E}(n, R, I)$ (depending on k) such that $\overline{\varepsilon_0} = \varepsilon$. Here 'bar' means reduction modulo I^k.*

Proof: Let $\varepsilon = \prod_{k=1}^{r} ge_{i_kj_k}(\overline{a_k})ge_{j_ki_k}(\overline{x_k})ge_{i_kj_k}(-\overline{a_k})$, where $a_k \in R$ and $x_k \in I$, for $1 \leq k \leq r$ (see Lemma 2.2.29). Let us set

$$\varepsilon_0 \;=\; \prod_{k=1}^{r} ge_{i_kj_k}(a_k)ge_{j_ki_k}(x_k)ge_{i_kj_k}(-a_k) \;\in\; \mathrm{E}(n, R, I).$$

Clearly $\overline{\varepsilon_0} = \varepsilon$. □

Lemma 2.2.21 *Let I be an ideal of R and $v \in \mathrm{Um}_{2n}(R, I)$. Then there exists $\varepsilon_k \in \mathrm{ESp}_{2n}(R, I)$ such that $v\varepsilon_k \in \mathrm{Um}_{2n}(R, I^k)$, for any positive integer k.*

Proof: Let $v = (1 + i_1, i_2, \ldots, i_{2n})$ and let

$$\overline{v} \;=\; (\overline{1} + \overline{i}_1, \overline{i}_2, \ldots, \overline{i}_{2n}) \;\in\; \mathrm{Um}_{2n}(R/I^k, I/I^k).$$

Here *'bar'* means reduction modulo I^k. As $\overline{i}_1$ is nilpotent in R/I^k, $\overline{1} + \overline{i}_1$ is a unit in R/I^k. By Lemma 2.2.17, there exists $\varepsilon \in \mathrm{ESp}_{2n}(R/I^k, I/I^k)$ such that $\overline{v}\varepsilon = (\overline{1}, \overline{0}, \ldots, \overline{0})$ in $(R/I^k)^{2n}$. Using Lemma 2.2.20 we get a $\varepsilon_k \in \mathrm{ESp}_{2n}(R, I)$ such that $\overline{\varepsilon_k} = \varepsilon$ and $\overline{v\varepsilon_k} = (\overline{1}, \overline{0}, \ldots, \overline{0})$ in $(R/I^k)^{2n}$. So $v\varepsilon_k = (1 + x_1, x_2, \ldots, x_{2n})$, where $x_1, \ldots, x_{2n} \in I^k$. □

Lemma 2.2.22 ([16], Lemma 1.5) *Let $n \geq 2$, and I be an ideal of R. Let $a \in I, v \in R^{2n}$, or $a \in R, v \in I^{2n}$. Then $I_{2n} + av^t\widetilde{v} \in \mathrm{ESp}_{2n}(R, I)$.* □

Lemma 2.2.23 ([16], Lemma 1.10) *Let $n \geq 2$, and I be an ideal of R. Let $v \in I^{2n}$, and $w \in \mathrm{Um}_{2n}(R)$ be such that $\widetilde{v}w^t = 0$. Then $I_{2n} + v^t\widetilde{w} + w^t\widetilde{v} \in \mathrm{ESp}_{2n}(R, I)$.* □

Lemma 2.2.24 ([16], Lemma 1.11) *When $n \geq 2$, $\mathrm{ESp}_{2n}(R, I)$ is a normal subgroup of $\mathrm{Sp}_{2n}(R)$.* □

The following Lemma is proved in a similar way as Lemma 2.2.13. We include the proof for completeness.

Lemma 2.2.25 *Let I be an ideal of R. Assume that $R = 2R$. Then $\mathrm{ESp}_{2n}(R, I^2)$ is a subset of $\mathrm{ESp}_{2n}(I)$, for $n \geq 3$.*

Proof: Let $z^* = \sum a_t b_t$ with $a_t, b_t \in I$. Let $\beta = se_{ij}(z^*) \in \mathrm{ESp}_{2n}(I^2)$ and $\alpha = se_{kl}(z) \in \mathrm{ESp}_{2n}(R)$, for some $z \in R$. It suffices to show that $\alpha\beta\alpha^{-1} \in \mathrm{ESp}_{2n}(I)$. If $(i,j) \neq (l,k)$ and $(i,j) \neq (\sigma(k),\sigma(l))$, then the matrix $\alpha\beta\alpha^{-1}$ splits into a product of elementary matrices from $\mathrm{ESp}_{2n}(I)$.

When $(i,j) = (l,k)$ or $(i,j) = (\sigma(k),\sigma(l))$ we need to consider the following two cases:

Case (1): In this case $i \neq \sigma(j)$. We can choose $r \leq 2n$ different from $i, j, \sigma(i), \sigma(j)$. Now,

$$\beta \;=\; se_{ij}(z^*) \;=\; \prod_t [se_{ir}(a_t), se_{rj}(b_t)]$$

and hence

$$\alpha\beta\alpha^{-1} \;=\; \prod_t [\alpha se_{ir}(a_t)\alpha^{-1}, \alpha se_{rj}(b_t)\alpha^{-1}] \;\in\; \mathrm{ESp}_{2n}(I).$$

Case (2): In this case $i = \sigma(j)$. We can choose $r \leq 2n$ different from i and $\sigma(i)$. We have

$$\beta \;=\; se_{i\sigma(i)}(z^*) \;=\; \prod_t [se_{ir}(a_t/2), se_{r\sigma(i)}(b_t)],$$

and hence

$$\alpha\beta\alpha^{-1} \;=\; \prod_t [\alpha se_{ir}(a_t/2)\alpha^{-1}, \alpha se_{r\sigma(i)}(b_t)\alpha^{-1}] \;\in\; \mathrm{ESp}_{2n}(I).$$

Therefore the claim is established. □

Remark 2.2.26 *The calculation in Case 1 in the above proof says that we need to choose an integer r, which is different from $i, j, \sigma(i), \sigma(j)$, and hence these matrices should have size at least 5. But these matrices are of even size. Therefore we need to assume $n \geq 3$, where $2n$ is the size of the group $\mathrm{ESp}_{2n}(R)$.*

Lemma 2.2.27 *Let $\alpha(X) \in \mathrm{E}(n, R[X])$ and $\alpha(0) = Id$. Then,*

$$\alpha(X) \;=\; \prod_{k=1}^{r} \gamma_k \; ge_{i_k j_k}(X h_k(X)) \; \gamma_k^{-1},$$

where $\gamma_k \in \mathrm{E}(n, R)$.

Proof: Let $\alpha(X) = \prod_{k=1}^{r} ge_{i_k j_k}(f_k(X))$, where $f_k(X) = f_k(0) + Xh_k(X)$, for some $h_k(X) \in R[X]$. Therefore we have

$$\begin{aligned} \alpha(X) &= \prod_{k=1}^{r} ge_{i_k j_k}(f_k(0)) ge_{i_k j_k}(Xh_k(X)) \\ &= \prod_{k=1}^{r} \gamma_k ge_{i_k j_k}(Xh_k(X))\gamma_k^{-1}, \end{aligned}$$

where $\gamma_l = \prod_{k=1}^{l} ge_{i_k j_k}(f_l(0))$. □

Lemma 2.2.28 ([8], Corollary 3.9): If $\varepsilon = \varepsilon_1\varepsilon_2\ldots\varepsilon_r$, where each ε_j is a (standard) elementary generator, then

$$\varepsilon ge_{pq}(X^{2^r m}Y)\varepsilon^{-1} = \prod_{t=1}^{k} ge_{p_t q_t}(X^m h_t(X,Y)),$$

for $h_t(X,Y) \in R[X,Y]$. □

Following lemma is due to L.N. Vaserstein. However our proof imitates W. van der Kallen's proof in the linear case (see [15], Lemma 2.2).

Lemma 2.2.29 ([29], §2): Let $n \geq 3$ in the linear case and $n \geq 6$ in the symplectic case. Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Let $\mathrm{E}(n,R,I)$ be the smallest normal subgroup of $\mathrm{E}(n,R)$ containing the elements $ge_{21}(x)$, $x \in I$. Then $\mathrm{E}(n,R,I)$ is generated by the elements $ge_{ij}(a)ge_{ji}(x)ge_{ij}(-a)$, where $a \in R, x \in I$, and $1 \leq i \neq j \leq n$.

Proof: Let G_1 denote the smallest normal subgroup of $\mathrm{E}(n,R)$ containing the elements $ge_{21}(x)$, with $x \in I$ and G_2 denote the subgroup which is generated by the elements $ge_{ij}(a)ge_{ji}(x)ge_{ij}(-a)$, with $a \in R, x \in I$, and $1 \leq i \neq j \leq n$. Let us set the following notation:

$$S_{ij} = \{ge_{ij}(a)ge_{ji}(x)ge_{ij}(-a) : a \in R, x \in I, 1 \leq i \neq j \leq n\}.$$

Note that $G_2 = \langle S_{ij} : i \neq j\rangle$, subgroup of $\mathrm{E}(n,R)$ generated by all possible $S_{ij}, i \neq j$. Clearly $S_{12} \subseteq G_1$. Let us consider an element from $S_{1j}, j \neq 1,2$, of the form $ge_{1j}(a)ge_{j1}(x)ge_{1j}(-a)$. Now

$$\begin{aligned} ge_{1j}(a)ge_{j1}(x)ge_{1j}(-a) &= {}^{ge_{1j}(a)}[ge_{j2}(1), ge_{21}(x)] \\ &= [{}^{ge_{1j}(a)}ge_{j2}(1), {}^{ge_{1j}(a)}ge_{21}(x)] \\ &= [\alpha_1, \alpha_2] \\ &= \alpha_1\alpha_2\alpha_1^{-1}\alpha_2^{-1}, \end{aligned}$$

where $\alpha_1 = {}^{ge_{1j}(a)} ge_{j2}(1)$ and $\alpha_2 = {}^{ge_{1j}(a)} ge_{21}(x)$. Clearly $\alpha_2 \in G_1$, and hence $\alpha_1\alpha_2\alpha_1^{-1} \in G_1$, as G_1 is a normal subgroup of $\mathrm{E}(n, R)$. Therefore $S_{1j} \subseteq G_1$, for $2 < j \leq n$.

Let us now consider an element $ge_{ij}(a)ge_{ji}(x)ge_j(-a) \in S_{ij}$, and $i, j \neq 1$. Note that

$$\begin{aligned} ge_{ij}(a)ge_{ji}(x)ge_j(-a) &= {}^{ge_{ij}(a)}[ge_{j1}(*), ge_{1i}(1)] \\ &= [{}^{ge_{ij}(a)}ge_{j1}(*), {}^{ge_{ij}(a)}ge_{1i}(1)] \\ &= [\beta_1, \beta_2] \ = \ \beta_1\beta_2\beta_1^{-1}\beta_2^{-1}, \end{aligned}$$

where $*$ is an element of the ideal I, $\beta_1 = {}^{ge_{ij}(a)} ge_{j1}(*)$ and $\beta_2 = {}^{ge_{ij}(a)} ge_{1i}(1)$. Clearly $\beta_1 \in G_1$ and hence $\beta_2\beta_1^{-1}\beta_2^{-1} \in G_1$, as G_1 is a normal subgroup of $\mathrm{E}(n, R)$. Therefore $S_{ij} \subseteq G_1$, for $i, j \neq 1$.

Here we consider an element of the form $ge_{i1}(a)ge_{1i}(x)ge_{i1}(-a)$ from S_{i1} for $i \geq 2$. Now

$$\begin{aligned} ge_{i1}(a)ge_{1i}(x)ge_{i1}(-a) &= {}^{ge_{i1}(a)}[ge_{1j}(1), ge_{ji}(*)] \\ &= [{}^{ge_{i1}(a)}ge_{1j}(1), {}^{ge_{i1}(a)}ge_{ji}(*)] \\ &= [\gamma_1, \gamma_2] \\ &= \gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}, \end{aligned}$$

where $*$ is an element of the ideal I, $\gamma_1 = {}^{ge_{i1}(a)} ge_{1j}(1)$ and $\gamma_2 = {}^{ge_{i1}(a)} ge_{ji}(*)$. Note that $\gamma_2 \in G_1$ and hence $\gamma_1\gamma_2\gamma_1^{-1} \in G_1$, as G_1 is a normal subgroup of $\mathrm{E}(n, R)$. Therefore $S_{i1} \subseteq G_1$, for $i \geq 2$. All the above set inclusions give us $\langle S_{ij} : i \neq j \rangle \subseteq G_1$, i.e, $G_2 \subseteq G_1$.

Note that $ge_{21}(x) \in G_2$. For showing the other inclusion we need to show G_2 is a normal subgroup of $\mathrm{E}(n, R)$, i.e, we need to show

$$hge_{ij}(a)ge_{ji}(x)ge_{ij}(-a)h^{-1} \quad \in \quad G_2,$$

for $h \in \mathrm{E}(n, R)$. It suffices to show $hge_{ij}(a)ge_{ji}(x)ge_{ij}(-a)h^{-1} \in G_2$, for standard elementary generator h of the group $\mathrm{E}(n, R)$. Exploiting commutator identities we get this inclusion and hence we have $G_1 = G_2$. □

Lemma 2.2.30 (see [15], Lemma 2.2) *Let $n \geq 3$ in the linear case and $n \geq 6$ in the symplectic case. Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Then the following sequence is exact*

$$1 \longrightarrow \mathrm{E}(n, R, I) \longrightarrow \mathrm{E}^1(n, R, I) \longrightarrow \mathrm{E}^1(n, R/I, 0) \longrightarrow 1.$$

Thus $\mathrm{E}(n, R, I)$ *equals* $\mathrm{E}^1(n, R, I) \cap \mathrm{G}(n, R, I)$.

Proof: Let $f : \mathrm{E}^1(n, R, I) \longrightarrow \mathrm{E}^1(n, R/I, 0)$. Note that

$$\ker(f) \quad \subseteq \quad \mathrm{E}^1(n, R, I) \cap \mathrm{G}(n, R, I).$$

Let $M = \prod ge_{j1}(x_j)ge_{1i}(a_i) \in \mathrm{E}^1(n, R, I) \cap \mathrm{G}(n, R, I)$. Note that $M \in \mathrm{G}(n, R, I)$ implies

$$\begin{aligned} \bar{M} &= \prod ge_{j1}(0)ge_{1i}(\bar{a}_i) \\ &= \prod ge_{1i}(\bar{a}_i) = I_n, \end{aligned}$$

i.e, $M \in \ker(f)$. Therefore $\ker(f) = \mathrm{E}^1(n, R, I) \cap \mathrm{G}(n, R, I)$.

Now we shall prove that $\ker(f) = \mathrm{E}(n, R, I)$. Let

$$E \quad = \quad \prod_{k=1}^{r} ge_{j_k1}(x_k)ge_{1i_k}(a_k) \quad \in \quad \ker(f).$$

Note that E can be written as $ge_{j_11}(x_1) \prod_{k=2}^{r} \gamma_k ge_{j_k1}(x_k)\gamma_k^{-1}$, where γ_l is equal to $\prod_{k=1}^{l-1} ge_{1i_k}(a_k) \in \mathrm{E}(n, R)$, and hence $\ker(f) \subseteq \mathrm{E}(n, R, I)$. To establish the reverse inclusion we need to show $\mathrm{E}(n, R, I) \subseteq \mathrm{E}^1(n, R, I)$. It suffices to show $\mathrm{E}^1(n, R, I)$

contains the set

$$S_{ij} \;=\; \{ge_{ij}(a)ge_{ji}(x)ge_{ij}(-a) : a \in R, x \in I\},$$

for all i, j, with $i \neq j$. First we state the following identities

$$[gh, k] \;=\; \big({}^{g}[h,k]\big)[g,k], \tag{2.1}$$

$$[g, hk] \;=\; [g,h]\big({}^{h}[g,k]\big), \tag{2.2}$$

$${}^{g}[h,k] \;=\; [{}^{g}h, {}^{g}k], \tag{2.3}$$

where ${}^{g}h$ denotes ghg^{-1} and $[g,h] = ghg^{-1}h^{-1}$. In the following computation we show that $\mathrm{E}^1(n,R,I)$ contains S_{ij}, if it contains S_{ik} and S_{jk}. We write $*$ for some element of I and of $\mathrm{E}^1(n,R,I)$. Now

$$\begin{aligned}
{}^{ge_{ij}(a)}ge_{ji}(x) &= {}^{ge_{ij}(a)}[ge_{jk}(1), ge_{ki}(*)] \\
&= [{}^{ge_{ij}(a)}ge_{jk}(1), {}^{ge_{ij}(a)}ge_{ki}(*)] \\
&= [ge_{ik}(a)ge_{jk}(1), ge_{ki}(*)ge_{kj}(*)] \\
&= {}^{ge_{ik}(a)}[ge_{jk}(1), ge_{ki}(*)ge_{kj}(*)][ge_{ik}(a), ge_{ki}(*)ge_{kj}(*)] \\
&= {}^{ge_{ik}(a)}ge_{ji}(*)\big({}^{ge_{ik}(a)ge_{ki}(*)}[ge_{jk}(1), ge_{kj}(*)]\big) \\
&\quad [ge_{ik}(a), ge_{ki}(*)]ge_{kj}(*)ge_{ij}(*) \\
&= (*)^{ge_{ik}(a)}\big(ge_{ki}(*)ge_{ji}(*)[ge_{jk}(1), ge_{kj}(*)]\big) \\
&\quad [ge_{ik}(a), ge_{ki}(*)](*) \\
&= (*)[ge_{ik}(a), ge_{ki}(*)](*)[ge_{jk}(1), ge_{kj}(*)](*) \\
&\quad [ge_{ik}(a), ge_{ki}(*)](*),
\end{aligned}$$

which lies in the group generated by $\mathrm{E}^1(n,R,I)$, S_{ik} and S_{jk}. Similarly, if $\mathrm{E}^1(n,R,I)$ contains S_{ki} and S_{kj} then it contains S_{ji}. Note that $\mathrm{E}^1(n,R,I)$ contains S_{12}, S_{13} and hence it contains S_{23}, S_{32}, S_{21}, S_{31}, and so on. $\square$

Remark 2.2.31 *In the above two lemmas (Lemma 2.2.29 and Lemma 2.2.30) we require the assumption $R = 2R$ when we prove the result for elementary symplectic group, i.e, $\mathrm{E}(n,R,I) = \mathrm{ESp}_n(R,I)$. We do not require this assumption for the elementary linear group.*

The following lemma is due to L.N. Vaserstein. We include the proof for completeness.

Lemma 2.2.32 ([29], Lemma 5.4) *Let $n \geq 2$ be a natural number and φ be an alternating matrix from $\mathrm{GL}_{2n}(R)$. Then for any v from R^{2n-1} there exists α, β in $\mathrm{E}_{2n-1}(R)$ such that*

$$\begin{pmatrix} 1 & 0 \\ \alpha v^t & \alpha \end{pmatrix}, \; \begin{pmatrix} 1 & v \\ 0 & \beta \end{pmatrix}$$

belong to $\mathrm{E}_{2n}(R) \cap \mathrm{Sp}_\varphi(R)$, where $\mathrm{Sp}_\varphi(R)$ denotes the isotropy group of φ, i.e,

$$\mathrm{Sp}_\varphi(R) \;=\; \{\alpha \;\in\; \mathrm{SL}_{2n}(R) \mid \alpha^t \varphi \alpha \;=\; \varphi\}.$$

Proof: We write

$$\varphi = \begin{pmatrix} 0 & -c \\ c^t & \nu \end{pmatrix}, \; \varphi^{-1} = \begin{pmatrix} 0 & d \\ -d^t & \mu \end{pmatrix},$$

where $c, d \in R^{2n-1}$. From the equality $\varphi\varphi^{-1} = 1$ we see that $cd^t = 1$, $\nu d^t = 0$, $c\mu = 0$ and $d^t c + \mu\nu = I_{2n-1}$. Let

$$\begin{aligned} \alpha &= \alpha(\varphi, v) = I_{2n-1} + d^t v \nu, \\ \beta &= \beta(\varphi, v) = I_{2n-1} - \mu v^t c. \end{aligned}$$

Notice that $\alpha \in \mathrm{E}_{2n-1}(R)$ since $v\nu \cdot d^t = 0$ and $d \in \mathrm{Um}_{2n-1}(R)$. Similarly $\beta \in \mathrm{E}_{2n-1}(R)$ since $c \cdot \mu v^t = 0$ and $c \in \mathrm{Um}_{2n-1}(R)$ (see Lemma 2.2.7). We have

$$\begin{aligned} L_\varphi(v) &:= \begin{pmatrix} 1 & 0 \\ \alpha v^t & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} 1 & 0 \\ v^t & I_{n-1} \end{pmatrix}, \\ R_\varphi(v) &:= \begin{pmatrix} 1 & v \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & I_{n-1} \end{pmatrix}. \end{aligned}$$

Now $(1 \perp \alpha), (1 \perp \beta) \in \mathrm{E}_n(R)$ and

$$\begin{pmatrix} 1 & 0 \\ v^t & I_{n-1} \end{pmatrix}, \; \begin{pmatrix} 1 & v \\ 0 & I_{n-1} \end{pmatrix} \in \mathrm{E}_n(R)$$

in view of Lemma 2.2.11. And hence $L_{\varphi}(v), R_{\varphi}(v) \in \mathrm{E}_n(R)$. The inclusions

$$\begin{pmatrix} 1 & 0 \\ \alpha v^t & \alpha \end{pmatrix}, \begin{pmatrix} 1 & v \\ 0 & \beta \end{pmatrix} \in \mathrm{Sp}_{\varphi}(R)$$

are verified immediately. □

Lemma 2.2.33 *Let $n \geq 2$ and $\varepsilon \in \mathrm{E}_{2n}(R)$. Then there exists $\rho \in \mathrm{E}_{2n-1}(R)$ such that $(1 \perp \rho)\varepsilon \in \mathrm{ESp}_{2n}(R)$.*

Proof: Let $\varepsilon = \varepsilon_r \ldots \varepsilon_1$, where each ε_i is of the form

$$\begin{pmatrix} 1 & v \\ 0 & I \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ v^t & I \end{pmatrix},$$

where $v = (a_1, \ldots, a_{2n-1}) \in R^{2n-1}$ (see Lemma 2.2.11). We prove the result using induction on r. If $r = 0$ there is nothing to prove. Let $r \geq 1$. Let us assume the result is true for $r-1$, i.e, when $\varepsilon = \varepsilon_{r-1} \ldots \varepsilon_1$, then there exists a $\delta \in \mathrm{E}_{2n-1}(R)$ such that $(1 \perp \delta)\varepsilon \in \mathrm{ESp}_{2n}(R)$. Now we prove the result when number of generator of ε is r. Now,

$$\begin{aligned} L_{\psi_n}(v) &:= \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} 1 & 0 \\ v^t & I_{2n-1} \end{pmatrix} = \prod_{i=2}^{2n} se_{i1}(a_{i-1}), \\ R_{\psi_n}(v) &:= \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & I_{2n-1} \end{pmatrix} = \prod_{i=2}^{2n} se_{1i}(a_{i-1}). \end{aligned}$$

Note that $\alpha = \alpha(\psi_n, v), \beta = \beta(\psi_n, v) \in \mathrm{E}_{2n-1}(R)$. Let us set γ equal to either α or β depending on the form of ε_1. Now,

$$\begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix} \varepsilon_1 \in \mathrm{ESp}_{2n}(R),$$

and each

$$\beta_i = \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix} \varepsilon_i \begin{pmatrix} 1 & 0 \\ 0 & \gamma^{-1} \end{pmatrix}$$

is of the form

$$\begin{pmatrix} 1 & v \\ 0 & I \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ v^t & I \end{pmatrix}.$$

Now we have

$$\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & \gamma^{-1} \end{pmatrix} \beta_r \ldots \beta_2 \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix} \varepsilon_1.$$

By induction hypothesis $(1 \perp \delta)\beta_r \ldots \beta_2 \in \mathrm{ESp}_{2n}(R)$, for some $\delta \in \mathrm{E}_{2n-1}(R)$. Hence $(1 \perp \rho)\varepsilon \in \mathrm{ESp}_{2n}(R)$, where $\rho = \delta^{-1}\gamma \in \mathrm{E}_{2n-1}(R)$. □

Lemma 2.2.34 ([29], Lemma 5.5) *For any natural number $n \geq 2$ and any alternating matrix φ from* $\mathrm{GL}_{2n}(R), e_1(\mathrm{E}_{2n}(R)) = e_1(\mathrm{E}_{2n}(R) \cap \mathrm{Sp}_\varphi(R))$. □

We will only use Lemma 2.2.34 in the special case when $\varphi = \psi_n$. In this special case the proof is much easier to establish. In this case following L.N. Vaserstein's proof one can show that

Lemma 2.2.35 *For any natural number $n \geq 2$, $e_1\mathrm{E}_{2n}(R) = e_1\mathrm{ESp}_{2n}(R)$.*

Proof: One way inclusion is obvious. To show $e_1\mathrm{E}_{2n}(R) \subseteq e_1\mathrm{ESp}_{2n}(R)$ let us choose $v \in e_1\mathrm{E}_{2n}(R)$ such that $v = e_1\varepsilon_r \ldots \varepsilon_1$, where $\varepsilon_r \ldots \varepsilon_1 \in \mathrm{E}_{2n}(R)$ and each ε_i is of the form

$$\begin{pmatrix} 1 & v_i \\ 0 & I_{2n-1} \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ v_i^t & I_{2n-1} \end{pmatrix},$$

where $v_i \in R^{2n-1}$(see Lemma 2.2.11). By induction on r we will show that $v \in e_1\mathrm{ESp}_{2n}(R)$. If $r = 0$ we have nothing to prove. Let $r \geq 1$. Let us assume the result is true for $r-1$, i.e, $e_1\varepsilon_{r-1} \ldots \varepsilon_1 \in e_1\mathrm{ESp}_{2n}(R)$. Now we prove the result when $v = e_1\varepsilon_r \ldots \varepsilon_1$. By Lemma 2.2.33, we get γ in $\mathrm{E}_{2n-1}(R)$ such that

$$\begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix} \varepsilon_1 \in \mathrm{ESp}_{2n}(R),$$

and

$$v = e_1\beta_r \ldots \beta_2 \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix} \varepsilon_1,$$

where each

$$\beta_i \;=\; \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix} \varepsilon_i \begin{pmatrix} 1 & 0 \\ 0 & \gamma^{-1} \end{pmatrix}.$$

Note that each β_i is of the form

$$\begin{pmatrix} 1 & v \\ 0 & I_{2n-1} \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ v & I_{2n-1} \end{pmatrix}.$$

By induction hypothesis we have $e_1\beta_r \dots \beta_2 \in e_1\mathrm{ESp}_{2n}(R)$ and hence v is in $e_1\mathrm{ESp}_{2n}(R)$. □

Lemma 2.2.36 *Let $n \geq 2$ and let I be an ideal of R. Let $\varepsilon \in \mathrm{E}^1_{2n}(R, I)$. Then there exists a ρ such that $\rho^t \in \mathrm{E}^1_{2n-1}(R, I)$ and $(1 \perp \rho)\varepsilon \in \mathrm{ESp}^1_{2n}(R, I)$.*

Proof: Let $\varepsilon = \varepsilon_r \dots \varepsilon_1$, where each ε_i is of the form

$$\begin{pmatrix} 1 & v \\ 0 & I_{2n-1} \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ w^t & I_{2n-1} \end{pmatrix},$$

where $v = (a_1, \dots, a_{2n-1}) \in R^{2n-1}$, and $w = (b_1, \dots, b_{2n-1}) \in I^{2n-1}$ (see Lemma 2.2.11). We prove the result using induction on r. If $r = 0$ there is nothing to prove. Let $r \geq 1$. Let us assume the result is true for $r - 1$, i.e, when $\varepsilon = \varepsilon_{r-1} \dots \varepsilon_1$, then there exists a δ such that $\delta^t \in \mathrm{E}^1_{2n-1}(R, I)$ and $(1 \perp \delta)\varepsilon \in \mathrm{ESp}^1_{2n}(R, I)$. Now we prove the result when number of generators of ε is r. Let

$$\begin{aligned} \alpha = \alpha(\psi_n, w) &= I_{2n-1} + (-e_1)^t w \begin{pmatrix} 0 & 0 \\ 0 & \psi_{n-1} \end{pmatrix} \\ &= E_{12}(b_3)E_{13}(-b_2) \dots E_{12n-1}(-b_{2n-2}), \end{aligned}$$

and

$$\begin{aligned} \beta \;=\; \beta(\psi_n, v) &= I_{2n-1} - \begin{pmatrix} 0 & 0 \\ 0 & \psi^t_{n-1} \end{pmatrix} v^t(-e_1) \\ &= E_{21}(a_3)E_{31}(-a_2) \dots E_{2n-11}(-a_{2n-2}). \end{aligned}$$

Note that $\alpha^t, \beta^t \in \mathrm{E}^1_{2n-1}(R,I)$. Also note that

$$\begin{aligned}
\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} 1 & 0 \\ w^t & I \end{pmatrix} &= \prod_{i=2}^{2n} se_{i1}(b_{i-1}) \in \mathrm{ESp}^1_{2n}(R,I), \\
\begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & I \end{pmatrix} &= \prod_{i=2}^{2n} se_{1i}(a_{i-1}) \in \mathrm{ESp}^1_{2n}(R,I).
\end{aligned}$$

We can set γ to be α or β depending on the form of ε_1 such that $\gamma^t \in \mathrm{E}^1_{2n-1}(R,I)$ and

$$\begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix} \varepsilon_1 \in \mathrm{ESp}^1_{2n}(R,I)$$

Therefore we have,

$$\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & \gamma^{-1} \end{pmatrix} \beta_r \ldots \beta_2 \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix} \varepsilon_1,$$

where each

$$\beta_i = \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix} \varepsilon_i \begin{pmatrix} 1 & 0 \\ 0 & \gamma^{-1} \end{pmatrix} \in \mathrm{E}^1_{2n}(R,I).$$

By induction hypothesis there exists δ such that $\delta^t \in \mathrm{E}^1_{2n-1}(R,I)$ and $(1 \perp \delta)\beta_r \ldots \beta_2 \in \mathrm{ESp}^1_{2n}(R,I)$. Let us set $\rho = \delta^{-1}\gamma$. Clearly $\rho^t \in \mathrm{E}^1_{2n-1}(R,I)$ and $(1 \perp \rho)\varepsilon \in \mathrm{ESp}^1_{2n}(R,I)$. □

Lemma 2.2.37 *Let $n \geq 2$ and let I be an ideal of R. Let $\varepsilon \in \mathrm{E}_{2n}(R,I)$, $n \geq 2$. Then there exists $\rho \in \mathrm{E}_{2n-1}(R,I)$ such that $(1 \perp \rho)\varepsilon \in \mathrm{ESp}_{2n}(R,I)$.*

Proof: We have $\varepsilon \in \mathrm{E}_n(R,I) = \mathrm{E}^1_{2n}(R,I) \cap \mathrm{GL}_{2n}(R,I)$ (see Lemma 2.2.30). Using Lemma 2.2.36 we get a ρ such that $\rho^t \in \mathrm{E}^1_{2n-1}(R,I)$ and $(1 \perp \rho)\varepsilon = \alpha$, where $\alpha \in \mathrm{ESp}^1_{2n}(R,I)$. We have,

$$\begin{aligned}
\bar{\varepsilon} &= (1 \perp \bar{\rho})^{-1}\bar{\alpha} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & *_1 & I_{2n-2} \end{pmatrix} \begin{pmatrix} 1 & *_2 & *_3 \\ 0 & 1 & 0 \\ 0 & \tilde{*_3} & I_{2n-2} \end{pmatrix} \\
&= I_{2n} \pmod{I}.
\end{aligned}$$

Here *'bar'* means reduction modulo I. Comparing the entries we get, $*_1, *_2, *_3$ are zero modulo I. Hence $\rho \in \mathrm{GL}_{2n-1}(R,I)$ and $\alpha \in \mathrm{Sp}_{2n}(R,I)$. We get, via Lemma 2.2.30,

$$\begin{aligned} \alpha &\in \mathrm{ESp}^1_{2n}(R,I) \cap \mathrm{Sp}_{2n}(R,I) &= \mathrm{ESp}_{2n}(R,I), \\ \rho^t &\in \mathrm{E}^1_{2n-1}(R,I) \cap \mathrm{GL}_{2n-1}(R,I) &= \mathrm{E}_{2n-1}(R,I). \end{aligned}$$

and hence $\rho \in \mathrm{E}_{2n-1}(R,I)$. Therefore

$$(1 \perp \rho)\varepsilon \quad = \quad \alpha \ \in \ \mathrm{ESp}_{2n}(R,I).$$

□

We use this to prove the next Lemma, which is a relative version of a special case of an (as yet) unpublished result of R.A. Rao and R.G. Swan.

Corollary 2.2.38 (Rao-Swan): *Let $n \geq 2$ and $\varepsilon \in \mathrm{E}_{2n}(R,I)$. Then*

$$\varepsilon^t \psi_n \varepsilon \quad = \quad (1 \perp \varepsilon_0)^t \psi_n (1 \perp \varepsilon_0),$$

for some $\varepsilon_0 \in \mathrm{E}_{2n-1}(R,I)$.

Proof: Note that $\varepsilon^{-1} \in \mathrm{E}_{2n}(R,I)$. Using Lemma 2.2.37 we get $\varepsilon_0 \in \mathrm{E}_{2n-1}(R,I)$ such that $(1 \perp \varepsilon_0)\varepsilon^{-1} \in \mathrm{ESp}_{2n}(R,I)$, and hence

$${\varepsilon^{-1}}^t (1 \perp \varepsilon_0)^t \ \psi_n \ (1 \perp \varepsilon_0)\varepsilon^{-1} \quad = \quad \psi_n.$$

Therefore we have

$$\begin{aligned} \varepsilon^t \psi_n \varepsilon &= \varepsilon^t \ \{{\varepsilon^{-1}}^t (1 \perp \varepsilon_0)^t\} \ \psi_n \ \{(1 \perp \varepsilon_0)\varepsilon^{-1}\} \ \varepsilon \\ &= (1 \perp \varepsilon_0)^t \psi_n (1 \perp \varepsilon_0). \end{aligned}$$

□

Chapter 3

Quillen's Local-Global Principle

We begin this chapter with D. Quillen's famous Local-Global principle (see Theorem 3.1.2). Quillen proved it using his Splitting lemma (see Lemma 3.1.1) for invertible elements in the fibre of $X = 0$. Before stating the Splitting Lemma, let us fix a notation.

Let $(1+XR[X])^*$ denotes the group of invertible elements in the polynomial ring $R[X]$ which are congruent to 1 modulo X.

3.1 Splitting Lemma and Related Results

Lemma 3.1.1 Quillen's Splitting Lemma: ([24], Lemma 1)
Let A be an algebra over R, let $f \in R$, and let $\theta \in (1 + XA_f[X])^*$. Then there exists an integer $k \geq 0$ such that for any $g_1, g_2 \in R$, with $g_1 - g_2 \in f^k R$, there exists $\psi(X) \in (1 + XA[X])^*$ such that $\psi(X)_f = \theta(g_1 X)\theta(g_2 X)^{-1}$. □

Theorem 3.1.2 Quillen's Local-Global Principle: ([24], Theorem 1)
Let M be a finitely presented module over $R[X]$. If $M_{\mathfrak{m}}$ is extended $R_{\mathfrak{m}}[X]$-module for each maximal ideal $\mathfrak{m}$ of R, then M is extended from R. □

In [31] A. Suslin stated and proved an analogue for the elementary matrices of the above theorem of Quillen. Suslin's proof was inspired by the ideas in the proof of Quillen. To prove his Local-Global result Suslin first proved a lemma, which is similar to the above Splitting lemma, for elementary matrices. Here we state the lemma as well as Suslin's elementary matrices analogue of Quillen's Local-Global principle.

Lemma 3.1.3 Splitting Lemma for Elementary Matrices: ([31], Lemma 3.4) *Let* $\alpha(X) \in \mathrm{GL}_n(R[X])$, *and let* $\alpha(0) = Id$, $n \geq 3$. *Let* $a \in R$ *be non-nilpotent and also assume* $\alpha(X)_a \in \mathrm{E}_n(R_a[X])$. *Then there exists a natural number* m *such that* $\alpha(cX)\alpha(dX)^{-1} \in \mathrm{E}_n(R[X])$ *for* $c \equiv d \pmod{a^m}$. $\square$

Theorem 3.1.4 Local-Global Principle for Elementary Matrices: ([31], Theorem 3.1)
Let $\alpha(X)$*is in* $\mathrm{GL}_n(R[X])$, *with* $\alpha(0) = Id$, $n \geq 3$. *Then* $\alpha(X)$ *lies in* $\mathrm{E}_n(R[X])$ *if and only if for each maximal ideal* $\mathfrak{m}$ *of* R *the canonical image of* $\alpha(X)$ *in* $\mathrm{GL}_n(R_{\mathfrak{m}}[X])$ *lies in* $\mathrm{E}_n(R_{\mathfrak{m}}[X])$. $\square$

After Suslin, in [16] V.I. Kopeĭko stated and proved an elementary symplectic matrices analogue of Quillen's Local-Global principle. Here we state Kopeĭko's result.

Theorem 3.1.5 Local-Global Principle for Elementary Symplectic Matrices: ([16], Theorem 3.6)
Let $\alpha(X) \in \mathrm{Sp}_{2n}(R[X])$, *with* $\alpha(0) = Id$, $n \geq 2$. *Then* $\alpha(X) \in \mathrm{ESp}_{2n}(R[X])$ *if and only if for any maximal ideal* $\mathfrak{m}$ *of* R, *the canonical image of* $\alpha(X)$ *in* $\mathrm{Sp}_{2n}(R_{\mathfrak{m}}[X])$ *lies in* $\mathrm{ESp}_{2n}(R_{\mathfrak{m}}[X])$. $\square$

Now we talk about the **action version** of Quillen's Local Global Principle due to L.N. Vaserstein.

In a letter to H. Bass, L.N. Vaserstein gave a short proof of an "action version" of Quillen's well known Local Global Principle (see Theorem 3.1.6),

Theorem 3.1.6 (**L.N. Vaserstein**) ([18], Chapter 3, Theorem 2.5)
Let $n \geq 3$ *and* $v(X) \in \mathrm{Um}_n(R[X])$. *If* $v(X) \in v(0)\mathrm{GL}_n(R_{\mathfrak{m}}[X])$, *for all maximal ideals* $\mathfrak{m}$ *of* R, *then* $v(X) \in v(0)\mathrm{GL}_n R([X])$. $\square$

R.A. Rao proved similar result as above for the elementary linear group.

Theorem 3.1.7 (**R.A. Rao**) ([25], Theorem 2.3)
Let $v(X) \in \mathrm{Um}_n(R[X])$, $n \geq 3$. *Suppose for all maximal ideals* $\mathfrak{m}$ *in* R, $v(X) \in v(0)\mathrm{E}_n(R_{\mathfrak{m}}[X])$. *Then* $v(X) \in v(0)\mathrm{E}_n(R[X])$. $\square$

3.2 Local Global Principle

We prove a relative (w.r.t. an extended ideal) elementary (linear, symplectic) action version of Theorem 3.1.7 below. We first state and prove the essential steps of L.N. Vaserstein's Local Global principle for action on $\mathrm{Um}_n(R[X])$, and a few preliminary lemmas.

Note that in this and the next section we establish results for elementary linear and elementary symplectic groups, but not for elementary orthogonal group (though the results are also true in this case, as shown in [1]).

Lemma 3.2.1 *Let $n \geq 3$. Let I be an ideal of R and S be a multiplicatively closed set in R. Let $\alpha(X) \in \mathrm{E}(n, I_S[X])$, with $\alpha(0) = Id$. Then there exists $\alpha^*(X) \in \mathrm{E}(n, R[X], I[X])$ such that $\alpha^*(X)$ localises to $\alpha(sX)$, for some $s \in S$, with $\alpha^*(0) = Id$.*

Proof: Since there are only finitely many denominators involved, there exists $t \in S$ such that $\alpha(X) \in \mathrm{E}(n, I_t[X])$. Let

$$\alpha(X) \;=\; \prod_{k=1}^{r} ge_{i_k j_k}(h_k(X)),$$

where $h_k(X) = h_k(0) + X\widetilde{f}_k(X)$. Given that $\alpha(0) = Id$. Using Lemma 2.2.27 we get,

$$\alpha(X) \;=\; \prod_{k=1}^{r} \gamma_k \; ge_{i_k j_k}(Xf_k(X)/t^s) \; \gamma_k^{-1},$$

where $\gamma_l = \prod_{k=1}^{l} ge_{i_k j_k}(h_k(0)) \in \mathrm{E}(n, I_t)$ and $f_k(X) \in I[X]$.

Case (A): Linear case, i.e, when $\mathrm{E}(n, I[X]) = \mathrm{E}_n(I[X])$.

Let $v_{i_k}^t$ be the i_k-th column of γ_k and w_{j_k} be the j_k-th row of γ_k^{-1}. Therefore,

$$\alpha(X) \;=\; \prod_{k=1}^{r} \left(\mathrm{I}_n + Xf_k(X)/t^s v_{i_k}^t w_{j_k}\right).$$

Here $v_{i_k}, w_{j_k} \in R_t^n$. Let us set $v_{i_k} = (1/t^s)v_{i_k}^*, w_{j_k} = (1/t^s)w_{j_k}^*$, for some $s \geq 0$, with $v_{i_k}^*, w_{j_k}^* \in R^n$. We can write,

$$\alpha(X) = \prod_{k=1}^{r} \left(\mathrm{I}_n + Xf_k(X)/t^{3s}(v_{i_k}^*)^t w_{j_k}^*\right).$$

Let us take $N = 3s$ and define,

$$\alpha^*(X) = \prod_{k=1}^{r} \left(\mathrm{I}_n + Xf_k(t^N X)(v_{i_k}^*)^t w_{j_k}^*)\right).$$

Clearly $\alpha^*(X) \in \mathrm{E}_n(R[X], I[X])$, as $f_k(t^N X) \in I[X]$ (see Lemma 2.2.8), and localises to $\alpha(t^N X)$.

Case (B): Symplectic case, i.e, when $\mathrm{E}(n, I[X]) = \mathrm{ESp}_{2m}(I[X])$.

Let σ be the permutation defined before Definition 2.1.14. For any row vector v we define $\widetilde{v} = v\psi_n$. Let $v_{i_k}^t$ and $v_{\sigma(j_k)}^t$ be the i_k-th and $\sigma(j_k)$-th columns of γ_k respectively. Then $\widetilde{v}_{i_k}$ and $\widetilde{v}_{\sigma(j_k)}$ are the $\sigma(i_k)$-th and j_k-th rows of γ_k^{-1} respectively. If $i_k = \sigma(j_k)$ then,

$$\gamma_k se_{i_k j_k}(Xf_k(X)/t^s)\gamma_k^{-1} = I_{2m} + (Xf_k(X)/t^s)v_{i_k}^t \widetilde{v}_{i_k} \in \mathrm{ESp}_{2m}(I_t[X]).$$

If $i_k \neq \sigma(j_k)$ and $i_k < j_k$ then, $\gamma_k se_{i_k j_k}(Xf_k(X)/t^s)\gamma_k^{-1}$

$$= I_{2m} + (Xf_k(X)/t^s)v_{\sigma(j_k)}^t \widetilde{v}_{i_k} + (Xf_k(X)/t^s)v_{i_k}^t \widetilde{v}_{\sigma(j_k)} \in \mathrm{ESp}_{2m}(I_t[X]).$$

If $v \in R_t^{2m}$, then $v = (1/t^s)v^*$, for some $s \geq 0$, with $v^* \in R^{2m}$. Let us define,

$$a_k = \begin{cases} I_{2m} + (Xf_k(X)/t^{3s})v_{i_k}^{*t}\widetilde{v}_{i_k}^*, & \text{if } i_k = \sigma(j_k), \\ I_{2m}, & \text{otherwise.} \end{cases}$$

$$b_k = \begin{cases} I_{2m} + (Xf_k(X)/t^{3s})v_{\sigma(j_k)}^{*t}\widetilde{v}_{i_k}^* + & \\ (Xf_k(X)/t^{3s})v_{i_k}^{*t}\widetilde{v}_{\sigma(j_k)}^*, & \text{if } i_k \neq \sigma(j_k), i_k < j_k, \\ I_{2m}, & \text{otherwise.} \end{cases}$$

Note that $\alpha(X) = \prod_{k=1}^{r} a_k b_k$. Let us take $N = 3s$ and define,

$$\widetilde{a}_k = \begin{cases} I_{2m} + (Xf_k(t^N X))v_{i_k}^{*t}\widetilde{v}_{i_k}^*, & \text{if } i_k = \sigma(j_k), \\ I_{2m}, & \text{otherwise.} \end{cases}$$

$$\widetilde{b}_k = \begin{cases} I_{2m} + (Xf_k(t^N X))v_{\sigma(j_k)}^{*t}\widetilde{v}_{i_k}^* + & \\ (Xf_k(t^N X))v_{i_k}^{*t}\widetilde{v}_{\sigma(j_k)}^*, & \text{if } i_k \neq \sigma(j_k), i_k < j_k, \\ I_{2m}, & \text{otherwise.} \end{cases}$$

Define

$$\alpha^*(X) \quad = \quad \prod_{k=1}^{r} \widetilde{a}_k \widetilde{b}_k.$$

It is easy to see $\alpha^*(X) \in \mathrm{ESp}_{2m}(R[X], I[X])$, as $f_k(t^N X) \in I[X]$ (see Lemma 2.2.22 and Lemma 2.2.23). $\alpha^*(X)$ localises to $\alpha(t^N X)$. $\square$

The following argument of L.N. Vaserstein is standard (see [18], Chapter III, Proposition 2.3):

Lemma 3.2.2 *Let $n \geq 3$. Let I be an ideal of R and S be a multiplicatively closed set in R. Let $v(X) \in \mathrm{Um}_n(R[X])$ and let $v(X) \in v(0)\mathrm{E}(n, I_S[X])$. Then there is an s in S such that for any a in R,*

$$v(X + asT) \quad \in \quad v(X)\mathrm{E}(n, R[X,T], I[X,T]).$$

Proof: Let $\alpha(X) \in \mathrm{E}(n, I_S[X])$ such that $v(X)\alpha(X) = v(0)$. Let

$$\beta(X,T) \quad = \quad \alpha(X+T)\alpha(X)^{-1} \;\in\; \mathrm{E}(n, I_S[X,T]).$$

Then

$$\begin{aligned} v(X+T)\beta(X,T) \quad &= \quad v(X+T)\alpha(X+T)\alpha(X)^{-1} \\ &= \quad v(0)\alpha(X)^{-1} \\ &= \quad v(X) \text{ in } R_S[X,T]^{2n}. \end{aligned}$$

Since $\beta(X,0) = Id$, we can find $\beta^*(X,T) \in \mathrm{E}(n, R[X,T], I[X,T])$ which localises to $\beta(X, sT)$ for some $s \in S$ with $\beta^*(X,0) = Id$ (see Lemma 3.2.1). Then in $R[X,T]^n$ we have,

$$v(X+sT)\beta^*(X,T) - v(X) \quad = \quad Tw(X,T),$$

for some $w(X,T)$ which localises to 0. Thus for some $s^* \in S$, and for all $a \in R$, we get,

$$v(X + ass^*T)\beta^*(X, as^*T) - v(X) \;=\; Tas^*w(X, as^*T) \;=\; 0.$$

$\square$

Now we prove our main theorem of this section, which plays a crucial role in this thesis.

Theorem 3.2.3 Local Global Principle w.r.t. an Extended Ideal: *Let $n \geq 3$. Let I be an ideal of R and $v(X) \in \mathrm{Um}_n(R[X], I[X])$. If for all maximal (or even prime) ideals $\mathfrak{m}$ of R, $v(X)_{\mathfrak{m}} \in v(0)_{\mathfrak{m}}\mathrm{E}(n, I_{\mathfrak{m}}[X])$, then*

$$v(X) \;\in\; v(0)\ \mathrm{E}(n, R[X], I[X]).$$

Proof: By assumption $v(X)_{\mathfrak{m}} \in v(0)_{\mathfrak{m}}\mathrm{E}(n, I_{\mathfrak{m}}[X])$, for all maximal (or all prime) ideals $\mathfrak{m}$ of R. Using Lemma 3.2.2 it follows that, for each maximal ideal $\mathfrak{m}$ of R, there exists $s_k \in R \setminus \mathfrak{m}$ such that for all $a \in R$,

$$v(X + as_kT) \;\in\; v(X)\ \mathrm{E}(n, R[X,T], I[X,T]). \tag{3.1}$$

Note that the ideal generated by $s_k's$ is the whole ring R. Therefore there exists $s_{k_1}, \ldots, s_{k_r}$ such that $a_1 s_{k_1} + \cdots + a_r s_{k_r} = 1$ where $a_i \in R$, for $1 \leq i \leq r$.

In equation (3.1) replacing X by $a_2 s_{k_2} X + \cdots + a_r s_{k_r} X$ and as_kT by $a_1 s_{k_1} X$ we get,

$$\begin{aligned} v(X) &= v(a_1 s_{k_1} X + a_2 s_{k_2} X + \cdots + a_r s_{k_r} X) \\ &\in v(a_2 s_{k_2} X + \cdots + a_r s_{k_r} X)\ \mathrm{E}(n, R[X], I[X]). \end{aligned}$$

Again in equation (3.1) replacing X by $a_3 s_{k_3} X + \cdots + a_r s_{k_r} X$ and as_kT by $a_2 s_{k_2} X$ we get,

$$v(a_2 s_{k_2} X + \cdots + a_r s_{k_r} X) \;\in\; v(a_3 s_{k_3} X + \cdots + a_r s_{k_r} X)\ \mathrm{E}(n, R[X], I[X]).$$

Continuing in this way we get, $v(a_r s_{k_r} X+0) \in v(0)\mathrm{E}(n, R[X], I[X])$. Combining all these we get,

$$v(X) \;\in\; v(0)\, \mathrm{E}(n, R[X], I[X]),$$

and hence the result is proved. □

Remark 3.2.4 The above Theorem is sufficient to prove Theorem 4.2.2, our main result in the free case. However to prove Theorem 5.11.4, a projective module analogue of Theorem 4.2.2, we need a stronger version of Theorem 3.2.3. This version was independently observed earlier in [1] by using Suslin's theory of special forms being elementary. Here we use commutator laws to prove those result. We state and prove the theorems in the next section.

3.3 Local Global Principle: A Stronger Version

Lemma 3.3.1 *Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Let $n \geq 3$ in the linear case and $n \geq 6$ in the symplectic case. Let $\varepsilon = \varepsilon_1 \ldots \varepsilon_r$ be an element in $\mathrm{E}^1(n, R, I)$, where each ε_k is a (standard) elementary generator. Also $ge_{ij}(Xf(X))$ is a standard elementary generator of $\mathrm{E}^1(n, R[X], I[X])$. Then*

$$\varepsilon\; ge_{ij}(Y^{4^r} X f(Y^{4^r} X))\; \varepsilon^{-1} \;=\; \prod_{t=1}^{s} ge_{i_t j_t}(Y h_t(X,Y)),$$

where either $i_t = 1$ or $j_t = 1$ and $h_t(X,Y) \in R[X,Y]$, when $i_t = 1$; $h_t(X,Y) \in I[X,Y]$ when $j_t = 1$.

Proof: Given that $ge_{ij}(Xf(X)) \in \mathrm{E}^1(n, R[X], I[X])$. First assume $i = 1$ and $f(X) \in R[X]$. We prove the result using induction on r, the number of generators of ε. Let $r = 1$ and $\varepsilon = ge_{pq}(a)$. Note that when $p = 1, a \in R$, and when $q = 1, a \in I$.

Case (1): Let $(p,q) = (1,j))$. In this case

$$ge_{1j}(a)\; ge_{1j}(Y^4 X f(Y^4 X))\; ge_{1j}(-a) \;=\; ge_{1j}(Y^4 X f(Y^4 X)).$$

Case (2): Let $(p,q) = (1,k)), k \neq j$. In this case

$$ge_{1k}(a)\; ge_{1j}(Y^4 X f(Y^4 X))\; ge_{1k}(-a) \;=\; ge_{1j}(Y^4 X f(Y^4 X)).$$

Case (3): Let $(p,q)=(k,1)), k\neq j$. In this case

$$\begin{aligned}
& ge_{k1}(a)\ ge_{1j}(Y^4Xf(Y^4X))\ ge_{k1}(-a)\\
= \ & ge_{kj}(*Y^4Xf(Y^4X))\ ge_{1j}(Y^4Xf(Y^4X))\\
= \ & [ge_{k1}(*Y^2),\ ge_{1j}(Y^2Xf(Y^4X))]\ ge_{1j}(Y^4Xf(Y^4X)),
\end{aligned}$$

where * is an element of I.

Case (4): Let $(p,q)=(j,1))$. Let us choose $k\neq 1,2,j,\sigma(j)$. In this case

$$\begin{aligned}
& ge_{j1}(a)\ ge_{1j}(Y^4Xf(Y^4X))\ ge_{j1}(-a)\\
= \ & ge_{j1}(a)\ [ge_{1k}(Y^2Xf(Y^4X)),\ ge_{kj}(Y^2)]\ ge_{j1}(-a)\\
= \ & [ge_{jk}(Y^2X*f(Y^4X))\ ge_{1k}(Y^2Xf(Y^4X)),\ ge_{k1}(-Y^2*)\\
& ge_{kj}(Y^2)]\\
= \ & ge_{jk}(Y^2X*f(Y^4X))\ ge_{1k}(Y^2Xf(Y^4X))\ ge_{k1}(-Y^2*)\ ge_{kj}(Y^2)\\
& ge_{1k}(-Y^2Xf(Y^4X))\ ge_{jk}(-Y^2X*f(Y^4X))\ ge_{kj}(-Y^2)\\
& ge_{k1}(Y^2*)\\
= \ & ge_{jk}(Y^2X*f(Y^4X))\ ge_{1k}(Y^2Xf(Y^4X))\ ge_{k1}(-Y^2*)\ ge_{kj}(Y^2)\\
& ge_{1k}(-Y^2Xf(Y^4X))\ [ge_{j1}(-Y*), ge_{1k}(YXf(Y^4X))]\ ge_{kj}(-Y^2)\\
& ge_{k1}(Y^2*)\\
= \ & ge_{jk}(Y^2X*f(Y^4X))\ ge_{1k}(Y^2Yf(Y^4X))\ ge_{k1}(-Y^2*)\ ge_{kj}(Y^2)\\
& ge_{1k}(-Y^2Xf(Y^4X))\ ge_{kj}(-Y^2)\ ge_{kj}(Y^2)\\
& [ge_{j1}(-Y*),\ ge_{1k}(YXf(Y^4X))]\ ge_{kj}(-Y^2)\ ge_{k1}(Y^2*)\\
= \ & [ge_{j1}(Y*),\ ge_{1k}(YXf(Y^4X))]\ ge_{1k}(Y^2Xf(Y^4X))\ ge_{k1}(-Y^2*)\\
& ge_{1k}(-Y^2Xf(Y^4X))\ ge_{1j}(Y^4Xf(Y^4X))\ [ge_{k1}(-Y^3*)\\
& ge_{j1}(-Y*),\ ge_{1j}(-Y^3Xf(Y^4X))\ ge_{1k}(YXf(Y^4X))]\ ge_{k1}(Y^2*),
\end{aligned}$$

where * is an element of I.

Hence the result is true when $r=1$. We show the case $j=1$ by carrying out similar calculations. Let us assume the result is true when the number of generators is $r-1$, i.e,

$$\begin{aligned} &\varepsilon_2 \ldots \varepsilon_r \; ge_{ij}(Y^{4^{r-1}} X f(Y^{4^{r-1}}(X)) \; \varepsilon_r^{-1} \ldots \varepsilon_2^{-1} \\ &= \; \textstyle\prod_{t=1}^{s} ge_{i_t j_t}(Y h_t(X,Y)), \end{aligned}$$

where either $i_t = 1$ or $j_t = 1$. Note that $h_t(X,Y) \in R[X,Y]$, when $i_t = 1$ and $h_t(X,Y) \in I[X,Y]$, when $j_t = 1$.

Now we prove the result when the number of generators is r. We have

$$\begin{aligned} &\varepsilon_1 \; \varepsilon_2 \ldots \varepsilon_r \; ge_{ij}(Y^{4^r} X f(Y^{4^r}(X)) \; \varepsilon_r^{-1} \ldots \varepsilon_2^{-1} \; \varepsilon_1^{-1} \\ &= \; \varepsilon_1 \left(\textstyle\prod_{t=1}^{s} ge_{i_t j_t}(Y^4 h_t(X,Y)) \right) \varepsilon_1^{-1} \\ &= \; \textstyle\prod_{t=1}^{s} \varepsilon_1 \; ge_{i_t j_t}(Y^4 h_t(X,Y)) \; \varepsilon_1^{-1}. \end{aligned}$$

We now repeat the calculation under *Cases 1, 2, 3, 4* to conclude that

$$\varepsilon ge_{ij}(Y^{4^r} X f(Y^{4^r} X))\varepsilon^{-1} \; = \; \prod_{t=1}^{s} ge_{i_t j_t}(Y h_t(X,Y)),$$

where either $i_t = 1$ or $j_t = 1$. Here $h_t(X,Y) \in R[X,Y]$, when $i_t = 1$ and $h_t(X,Y) \in I[X,Y]$, when $j_t = 1$. □

Theorem 3.3.2 *Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Let $n \geq 3$ in the linear case and $n \geq 6$ in the symplectic case. Let a be a non-nilpotent element in R and $\alpha(X)$ be in $\mathrm{E}^1(n, R_a[X], I_a[X])$, with $\alpha(0) = Id$. Then there exists $\alpha^*(X) \in \mathrm{E}^1(n, R[X], I[X])$ such that $\alpha^*(X)$ localises to $\alpha(bX)$, for some $b \in (a^N)$, $N \gg 0$, with $\alpha^*(0) = Id$.*

Proof: Let $\alpha(X) = \prod_{k=1}^{r} ge_{i_k j_k}(f_k(X))$ is in $\mathrm{E}^1(n, R_a[X], I_a[X])$, and $f_k(X) = f_k(0) + X g_k(X)$. Using Lemma 2.2.27 we get

$$\alpha(X) \; = \; \prod_{k=1}^{r} \gamma_k \; ge_{i_k j_k}(X g_k(X)) \; \gamma_k^{-1},$$

where $\gamma_l = \prod_{k=1}^{l} ge_{i_k j_k}(f_k(0)) \in \mathrm{E}^1(n, R_a, I_a)$. Using Lemma 3.3.1 we can say that

$$\alpha(Y^{4^r} X) \; = \; \prod_{k=1}^{r} \Big(\prod_{t=1}^{s} ge_{i_t j_t}(Y h_t(X,Y))/a^m) \Big),$$

where either $i_t = 1$ or $j_t = 1$. Note that $h_t(X,Y) \in R[X,Y]$, when $i_t = 1$ and $h_t(X,Y) \in I[X,Y]$, when $j_t = 1$, and m is a natural number. Let us choose $N = m + N'$ and define

$$\alpha^*(X,Y) \quad = \quad \prod_{k=1}^{r}\Big(\prod_{t=1}^{s} ge_{i_t j_t}(a^{N'}Y h_t(X, a^N Y))\Big).$$

Clearly $\alpha^*(X,Y) \in \mathrm{E}^1(n, R[X,Y], I[X,Y])$ and

$$\alpha((a^N Y)^{4^r} X) \quad = \quad \alpha^*(X,Y).$$

Substituting $Y = 1$, we get $\alpha(bX) = \alpha^*(X)$, for $b \in (a^N)$, $N \gg 0$. Note that $\alpha^*(X) \in \mathrm{E}^1(n, R[X], I[X])$, with $\alpha^*(0) = Id$. $\square$

Theorem 3.3.3 *Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Let $n \geq 3$ in the linear case and $n \geq 6$ in the symplectic case. Let a be a non-nilpotent element in R and $\alpha(X)$ be in $\mathrm{E}(n, R_a[X], I_a[X])$, with $\alpha(0) = Id$. Then there exists $\alpha^*(X) \in \mathrm{E}(n, R[X], I[X])$ such that $\alpha^*(X)$ localises to $\alpha(bX)$, for some $b \in (a^N)$, $N \gg 0$, with $\alpha^*(0) = Id$.*

Proof: Follows from the previous theorem and Lemma 2.2.30, which says that $\mathrm{E}(n, R, I) = \mathrm{E}^1(n, R, I) \cap \mathrm{G}(n, R, I)$. $\square$

Theorem 3.3.4 *Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Let $n \geq 3$ in the linear case and $n \geq 6$ in the symplectic case. Let $\alpha(X) \in \mathrm{G}(n, R[X], I[X])$, with $\alpha(0) = Id$. If $\alpha(X)_{\mathfrak{m}}$ belongs to $\mathrm{E}(n, R_{\mathfrak{m}}[X], I_{\mathfrak{m}}[X])$, for all maximal ideals $\mathfrak{m}$ of R then, $\alpha(X) \in \mathrm{E}(n, R[X], I[X])$.*

Proof: One can suitably choose an element $a_{\mathfrak{m}}$ from $R \setminus \mathfrak{m}$ such that $\alpha(X)_{a_{\mathfrak{m}}} \in \mathrm{E}(n, R_{a_{\mathfrak{m}}}[X], I_{a_{\mathfrak{m}}}[X])$. Let us define

$$\beta(X,Y) \quad = \quad \alpha(X+Y)_{a_{\mathfrak{m}}} \alpha(Y)^{-1}_{a_{\mathfrak{m}}}.$$

Clearly

$$\beta(X,Y) \quad \in \quad \mathrm{E}(n, R_{a_{\mathfrak{m}}}[X,Y], I_{a_{\mathfrak{m}}}[X,Y]),$$

and $\beta(0,Y) = Id$. Therefore $\beta(b_{\mathfrak{m}}X, Y) \in \mathrm{E}(n, R[X,Y], I[X,Y])$, where $b_{\mathfrak{m}} \in (a_{\mathfrak{m}}^N)$, for some $N \gg 0$ (see Theorem 3.3.3). The ideal generated by $b_{\mathfrak{m}}$'s is the whole ring

R. Therefore we have $c_1 b_{\mathfrak{m}_1} + \cdots + c_k b_{\mathfrak{m}_k} = 1$, where $c_i \in R$, for $1 \le i \le k$. Note that $\beta(c_i b_{\mathfrak{m}_i} X, Y) \in \mathrm{E}(n, R[X,Y], I[X,Y])$, for $1 \le i \le k$. Hence

$$\alpha(X) \;=\; \prod_{i=1}^{k} \beta(c_i b_{\mathfrak{m}_i} X, T_i)\beta(c_k b_{\mathfrak{m}_k}, 0) \;\in\; \mathrm{E}(n, R[X], I[X]),$$

where $T_i = c_{i+1} b_{\mathfrak{m}_{i+1}} X + \cdots + c_k b_{\mathfrak{m}_k} X$. $\square$

Theorem 3.3.5 *Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Let $n \ge 3$ in the linear case and $n \ge 6$ in the symplectic case. Let $v(X) \in \mathrm{Um}_n(R[X], I[X])$. If for all maximal ideal $\mathfrak{m}$ of R, $v(X)_{\mathfrak{m}} \in v(0)_{\mathfrak{m}} \mathrm{E}(n, R_{\mathfrak{m}}[X], I_{\mathfrak{m}}[X])$, then*

$$v(X) \in v(0)\mathrm{E}(n, R[X], I[X]).$$

Proof: For each maximal ideal $\mathfrak{m}$ of R, we get $\alpha_{(\mathfrak{m})}(X) \in \mathrm{E}(n, R_{\mathfrak{m}}[X], I_{\mathfrak{m}}[X])$ such that

$$v(X)\alpha_{(\mathfrak{m})}(X) \;=\; v(0).$$

Let us define

$$\beta(X,T) \;=\; \alpha_{(\mathfrak{m})}(X+T)\ \alpha_{(\mathfrak{m})}(X)^{-1}.$$

Clearly $\beta(X,T) \in \mathrm{E}(n, R_{\mathfrak{m}}[X,T], I_{\mathfrak{m}}[X,T])$. Note that there are only finitely many denominators involved, and hence there exists $a_{\mathfrak{m}} \in R \setminus \mathfrak{m}$ such that $\beta(X,T)$ is in $\mathrm{E}(n, R_{a_{\mathfrak{m}}}[X,T], I_{a_{\mathfrak{m}}}[X,T])$. Also $\beta(X,0) = Id$. This implies $\beta(X, b_{\mathfrak{m}} T) \in \mathrm{E}(n, R[X,T], I[X,T])$ for suitable $b_{\mathfrak{m}} \in (a_{\mathfrak{m}}^N)$, $N \gg 0$ (see Theorem 3.3.3). Now,

$$\begin{aligned} v(X + b_{\mathfrak{m}}T)\ \beta(X, b_{\mathfrak{m}}T) &= v(X + b_{\mathfrak{m}}T)\ \alpha_{(\mathfrak{m})}(X + b_{\mathfrak{m}}T)\ \alpha_{(\mathfrak{m})}(X)^{-1} \\ &= v(0)\ \alpha_{(\mathfrak{m})}(X)^{-1} \\ &= v(X). \end{aligned}$$

Note that the ideal generated by $b_{\mathfrak{m}}$'s is the whole ring R. Therefore $c_1 b_{\mathfrak{m}_1} + \cdots + c_k b_{\mathfrak{m}_k} = 1$, where $c_i \in R$, for $1 \le i \le k$. In the above equation replacing $b_{\mathfrak{m}}T$

by $c_1 b_{\mathfrak{m}_1} X$ and X by $c_2 b_{\mathfrak{m}_2} X + \cdots + c_k b_{\mathfrak{m}_k} X$ we get,

$$\begin{aligned} v(X) &= v(b_{\mathfrak{m}_1} X + b_{\mathfrak{m}_2} X + \cdots + b_{\mathfrak{m}_k} X) \\ &\in v(b_{\mathfrak{m}_2} X + \cdots + b_{\mathfrak{m}_k} X) \ \mathrm{E}(n, R[X], I[X]). \end{aligned}$$

Again in the above equation replacing X by $b_{\mathfrak{m}_3} X + \cdots + b_{\mathfrak{m}_k} X$ and $b_{\mathfrak{m}} T$ by $b_{\mathfrak{m}_2} X$ we get,

$$v(b_{\mathfrak{m}_2} X + \cdots + b_{\mathfrak{m}_k} X) \in v(b_{\mathfrak{m}_3} X + \cdots + b_{\mathfrak{m}_k} X) \ \mathrm{E}(n, R[X], I[X]).$$

Continuing in this way we get

$$v(b_{m_k} X + 0) \in v(0) \ \mathrm{E}(n, R[X], I[X]).$$

Combining all these we get

$$v(X) \in v(0) \ \mathrm{E}(n, R[X], I[X]).$$

$\square$

Chapter 4

Equality of Orbits

L.N. Vaserstein showed in [29] that if v is the first row of an elementary matrix of even size then it is also the first row of an elementary symplectic matrix (see Lemma 2.2.34). This led us to query whether the orbit space of unimodular rows under the action of the elementary subgroup is in bijective correspondence with the orbit space of unimodular rows under the action of the elementary symplectic group. In this chapter, we prove that this is so, and also establish the relative version, $v\mathrm{E}_{2n}(R,I) = v\mathrm{ESp}_{2n}(R,I)$, for an ideal I of R, when $R = 2R$.

4.1 The absolute case

In this section we prove that the set of orbits of the action of the elementary symplectic group on all unimodular rows is the same as the set of orbits of the action of the elementary linear group on all unimodular rows.

Theorem 4.1.1 *Let R be a commutative ring and let $v \in \mathrm{Um}_{2n}(R)$, then $v\mathrm{E}_{2n}(R) = v\mathrm{ESp}_{2n}(R)$, for $n \geq 2$.*

Proof: Let $v^*_{ij}(X) = vE_{ij}(X)$. Let $\mathfrak{m}$ be a maximal ideal of R. Using Lemma 2.2.15 we get, $v_{\mathfrak{m}} = e_1 E$, for some $E \in \mathrm{E}_{2n}(R_{\mathfrak{m}})$. Using Lemma 2.2.35 we get,

$$v_{\mathfrak{m}} \;=\; e_1 E \;=\; e_1 \tilde{E},$$

where $\tilde{E} \in \mathrm{ESp}_{2n}(R_{\mathfrak{m}})$. Also

$$v_{ij}^*(X)_{\mathfrak{m}} \;=\; v_{\mathfrak{m}}E_{ij}(X)_{\mathfrak{m}} \;=\; e_1 E E_{ij}(X)_{\mathfrak{m}} \;=\; e_1\tilde{F}(X),$$

where $\tilde{F}(X) \in \mathrm{ESp}_{2n}(R_{\mathfrak{m}}[X])$ (see Lemma 2.2.35). Therefore,

$$\begin{aligned} v_{ij}^*(X)_{\mathfrak{m}} &= v_{\mathfrak{m}}E_{ij}(X)_{\mathfrak{m}} \\ &= e_1 E E_{ij}(X)_{\mathfrak{m}} \\ &= e_1\tilde{F}(X) \\ &= e_1\tilde{E}\tilde{E}^{-1}\tilde{F}(X) \\ &= v_{\mathfrak{m}}\tilde{E}^{-1}\tilde{F}(X) \\ &\in v_{ij}^*(0)_{\mathfrak{m}}\mathrm{ESp}_{2n}(R_{\mathfrak{m}}[X]). \end{aligned}$$

Hence, $v_{ij}^*(X)_{\mathfrak{m}} \in v_{ij}^*(0)_{\mathfrak{m}}\mathrm{ESp}_{2n}(R_{\mathfrak{m}}[X])$, for all maximal ideal $\mathfrak{m}$ of R. By Theorem 3.2.3 (when $I = R$)(or see the main theorem in [8]), $v_{ij}^*(X) \in v_{ij}^*(0)\mathrm{ESp}_{2n}(R[X])$; whence also to $v_{ij}^*(\lambda)$, for any $\lambda \in R$. Hence the result follows. $\square$

Theorem 4.1.2 *The natural map*

$$\frac{\mathrm{Um}_{2n}(R)}{\mathrm{ESp}_{2n}(R)} \longrightarrow \frac{\mathrm{Um}_{2n}(R)}{\mathrm{E}_{2n}(R)}$$

is bijective for $n \geq 2$.

Proof: The proof follows from Theorem 4.1.1 $\square$

4.2 The Relative Case

In this section we prove a relative version (see Theorem 4.2.2), with respect to an ideal I in R, of the above Theorem 4.1.1. Vaserstein's Lemma (Lemma 2.2.37) and Local Global principle w.r.t. an extended ideal (see Theorem 3.2.3) will play a crucial role in the proof of the relative version. Local Global principle w.r.t. an extended ideal will be used to prove the Lemma 4.2.1. Vaserstein's Lemma and Lemma 4.2.1 will be employed to prove Theorem 4.2.2.

Lemma 4.2.1 *Let $n \geq 3$. Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Let $v \in \mathrm{Um}_{2n}(R, I^2)$. If $\rho \in \mathrm{E}_{2n-1}(R, I)$, then $v\,(1 \perp \rho) \in v\,\mathrm{ESp}_{2n}(R, I)$.*

Proof: Let $\rho(X) \in \mathrm{E}_{2n-1}(R[X], I[X])$, with $\rho(1) = \rho$ and $\rho(0) = Id$ (see Lemma 2.2.3). Let $v = (1 + a_1, a_2, \ldots, a_{2n})$, with $a_i \in I^2$, for $1 \leq i \leq 2n$. Let us assume $V(X) = v(1 \perp \rho(X))$. Note that $e_1 V(X) = 1 + a_1$. Let $\mathfrak{m}$ be a maximal ideal of R.

If $I \subset \mathfrak{m}$, then $(1 + a_1)_\mathfrak{m}$ is unit in $R_\mathfrak{m}$. Using Lemma 2.2.17 we get $g(\mathfrak{m})(X) \in \mathrm{ESp}_{2n}(R_\mathfrak{m}[X], I_\mathfrak{m}[X]^2)$ such that $V(X)_\mathfrak{m} = e_1 g(\mathfrak{m})(X)$.

If $I \not\subseteq \mathfrak{m}$, then either $(1 + a_1)_\mathfrak{m}$ is a unit or for some i_0, $1 < i_0 \leq 2n$, $a_{i_0} \notin \mathfrak{m}$. In either case, since $I_\mathfrak{m} = R_\mathfrak{m}$, by Lemma 2.2.15 and Lemma 2.2.35 we have,

$$\begin{aligned} V(X)_\mathfrak{m} &\in e_1 \mathrm{E}_{2n}(R_\mathfrak{m}[X]) \\ &= e_1 \mathrm{ESp}_{2n}(R_\mathfrak{m}[X]) \\ &= e_1 \mathrm{ESp}_{2n}(R_\mathfrak{m}[X], I_\mathfrak{m}[X]^2). \end{aligned}$$

Therefore $V(X)_\mathfrak{m} \in e_1 \mathrm{ESp}_{2n}(R_\mathfrak{m}[X], I_\mathfrak{m}[X]^2)$, for each maximal ideal $\mathfrak{m}$ of R. Also, we will have $g(\mathfrak{m})_0$ from $\mathrm{ESp}_{2n}(R_\mathfrak{m}, I_\mathfrak{m}^2)$ such that $V(0)_\mathfrak{m} g(\mathfrak{m})_0 = e_1$. Therefore $V(X)_\mathfrak{m} \in V(0)_\mathfrak{m} g(\mathfrak{m})_0 \mathrm{ESp}_{2n}(R_\mathfrak{m}[X], I_\mathfrak{m}[X]^2)$, for each maximal ideal $\mathfrak{m}$ of R. Using Lemma 2.2.25 we get, $V(X)_\mathfrak{m} \in V(0)_\mathfrak{m} \mathrm{ESp}_{2n}(I_\mathfrak{m}[X])$, for each maximal ideal $\mathfrak{m}$ of R.

Using Theorem 3.2.3, we get $V(X) \in V(0)\mathrm{ESp}_{2n}(R[X], I[X])$. Substituting $X = 1$ we get $v(1 \perp \rho) \in v\mathrm{ESp}_{2n}(R, I)$. $\square$

Theorem 4.2.2 *Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Let $v \in \mathrm{Um}_{2n}(R, I)$, then $v\ \mathrm{E}_{2n}(R, I) = v\ \mathrm{ESp}_{2n}(R, I)$, for $n \geq 3$.*

Proof: It suffices to show the left hand side is contained in the right hand side. The reverse inclusion is obvious. Let $\varepsilon \in \mathrm{E}_{2n}(R, I)$. Using Lemma 2.2.21 we get ε_1 from $\mathrm{ESp}_{2n}(R, I)$ such that $v\varepsilon\varepsilon_1 \in \mathrm{Um}_{2n}(R, I^2)$. Using Lemma 2.2.37 we get ρ in $\mathrm{E}_{2n-1}(R, I)$ with $\varepsilon\varepsilon_1(1 \perp \rho) \in \mathrm{ESp}_{2n}(R, I)$. Now

$$v\varepsilon = v\varepsilon\varepsilon_1(1 \perp \rho)\ (1 \perp \rho)^{-1}\ \varepsilon_1^{-1}.$$

We have $v\varepsilon\varepsilon_1(1 \perp \rho)$ is in $\mathrm{Um}_{2n}(R, I^2)$. Hence by Lemma 4.2.1,

$$[v\varepsilon\varepsilon_1(1 \perp \rho)]\ (1 \perp \rho)^{-1} \in v\ \mathrm{ESp}_{2n}(R, I)$$

Let

$$v\varepsilon\varepsilon_1(1 \perp \rho)\ (1 \perp \rho)^{-1} = v\beta,$$

where β is in $\mathrm{ESp}_{2n}(R, I)$. Therefore $v\varepsilon = v\beta\varepsilon_1^{-1} \in v\mathrm{ESp}_{2n}(R, I)$. □

Now we are in a position to give a proof of relative version of Theorem 4.1.2 using the above lemmas.

Theorem 4.2.3 *Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Then the natural map*

$$\frac{\mathrm{Um}_{2n}(R, I)}{\mathrm{ESp}_{2n}(R, I)} \longrightarrow \frac{\mathrm{Um}_{2n}(R, I)}{\mathrm{E}_{2n}(R, I)}$$

is bijective for $n \geq 3$.

Proof: It is easy to show that the above map is surjective. To show the map is injective let us consider v, w from $\mathrm{Um}_{2n}(R, I)$ and g from $\mathrm{E}_{2n}(R, I)$, such that $vg = w$. We need to show w is in the $\mathrm{ESp}_{2n}(R, I)$-orbit of v. Using Theorem 4.2.2 we get h from $\mathrm{ESp}_{2n}(R, I)$, such that $vg = vh$ and hence $w = vh$. □

Chapter 5

Equality of Orbits: A Global Version

Symplectic transvections were defined by H. Bass in 1964 in [4], and L.N. Vaserstein defined certain symplectic transvections of a free module in 1974 in [29]. In this chapter we will relate these two objects.

Here we define Elementary Symplectic group with respect to an alternating matrix of Pfaffian 1, following the lead of L.N. Vaserstein. We then prove a Local-Global principle for this group. We also recall the definition of the group of elementary transvections and the group of elementary symplectic transvections with respect to an alternating form, due to H. Bass and prove Local-Global principle for these groups. Our main theorem is that the Elementary Symplectic group of Vaserstein and the group of elementary symplectic transvections of Bass are the same when we are dealing with the free case. Thus, the group of elementary symplectic transvections of H. Bass may be regarded as the globalization of the L.N. Vaserstein's elementary symplectic group.

As a consequence of the Local Global principles established, we generalise the theorems of previous chapters and show that the orbit space of unimodular rows of a projective module under the action of the group of elementary transvections is in bijection with the orbit space of unimodular rows of a projective module under the action of the group of elementary symplectic transvections with respect to an alternating form.

5.1 Elementary Symplectic Group $\mathrm{ESp}_\varphi(R)$

Definition 5.1.1 The group of all invertible $2n \times 2n$ matrices

$$\{\alpha \in \mathrm{GL}_{2n}(R) \mid \alpha^t \varphi \alpha = \varphi\},$$

where φ is an alternating matrix of Pfaffian 1 is called **Symplectic Group $\mathbf{Sp}_\varphi(R)$ With Respect To An Invertible Alternating Matrix φ.**

Definition 5.1.2 Let $v \in R^{2n-1}$. Following Lemma 2.2.32 we can define

$$\begin{aligned} L_\varphi(v) &= \begin{pmatrix} 1 & 0 \\ \alpha v^t & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ v^t & \alpha \end{pmatrix} \quad \text{and} \\ R_\varphi(v) &= \begin{pmatrix} 1 & v \\ 0 & \beta \end{pmatrix}. \end{aligned}$$

Here φ is an invertible alternating matrix of the form

$$\varphi = \begin{pmatrix} 0 & -c \\ c^t & \nu \end{pmatrix}, \text{ and } \varphi^{-1} = \begin{pmatrix} 0 & d \\ -d^t & \mu \end{pmatrix},$$

where $c, d \in R^{2n-1}$, and

$$\begin{aligned} \alpha &= \alpha(\varphi, v) = I_{2n-1} + d^t v \nu, \\ \beta &= \beta(\varphi, v)` = I_{2n-1} - \mu v^t c. \end{aligned}$$

By Lemma 2.2.32 it follows that all these matrices belong to $\mathrm{Sp}_\varphi(R)$. The subgroup of $\mathrm{Sp}_\varphi(R)$ generated by $L_\varphi(v)$ and $R_\varphi(v)$, for $v \in R^{2n-1}$ is called the **elementary symplectic group** $\mathrm{ESp}_\varphi(R)$ **with respect to the alternating matrix** φ **of Pfaffian 1** . This definition is due to L.N. Vaserstein.

Definition 5.1.3 Let I be an ideal of R. The **relative elementary group** $\mathrm{ESp}_\varphi(I)$ is a subgroup of $\mathrm{ESp}_\varphi(R)$ generated as a group by the elements $L(v)$ and $R(v)$, where $v \in I^{2n-1}$.

The **relative elementary group** $\mathrm{ESp}_\varphi(R, I)$ is the normal closure of $\mathrm{ESp}_\varphi(I)$ in $\mathrm{ESp}_\varphi(R)$.

Definition 5.1.4 Let I be an ideal in R. The **relative group** $\mathrm{ESp}^1_\varphi(R, I)$ is a subgroup of $\mathrm{ESp}_\varphi(R)$ generated by the elements of the form $R(v)$ and $L(w)$, where $v \in R^{2n-1}$ and $w \in I^{2n-1}$.

Lemma 5.1.5 *For the standard alternating matrix ψ_n,*

$$\begin{aligned} \mathrm{ESp}_{\psi_n}(R) &= \mathrm{ESp}_{2n}(R), \\ \mathrm{ESp}_{\psi_n}(R, I) &= \mathrm{ESp}_{2n}(R, I), \\ \mathrm{ESp}^1_{\psi_n}(R, I) &= \mathrm{ESp}^1_{2n}(R, I), \end{aligned}$$

for $n \geq 3$.

Proof: In the proof of Lemma 2.2.33 we have seen

$$R_{\psi_n}(v) = \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & I \end{pmatrix} = \prod_{i=2}^{2n} se_{1i}(a_{i-1}), \tag{5.1}$$

$$L_{\psi_n}(v) = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} 1 & 0 \\ v^t & I \end{pmatrix} = \prod_{i=2}^{2n} se_{i1}(a_{i-1}), \tag{5.2}$$

where $v = (a_1, \ldots, a_{2n-1}) \in R^{2n-1}$. Therefore we have

$$\mathrm{ESp}_{\psi_n}(R) \subseteq \mathrm{ESp}_{2n}(R).$$

Note that $se_{1i}(a), se_{j1}(b) \in \mathrm{ESp}_{\psi_n}(R)$. For $i, j \neq 1$,

$$\begin{aligned} se_{ij}(a) &= [\ se_{i1}(*),\ se_{1j}(1)\ ] \\ &= [\ L_{\psi_n}(*e_{i-1}),\ R_{\psi_1}(e_{j-1})\ ] \\ &\in \mathrm{ESp}_{\psi_n}(R), \end{aligned}$$

where * is an element of R, and hence $\mathrm{ESp}_{2n}(R) \subseteq \mathrm{ESp}_{\psi_n}(R)$. Therefore the first equality is established.

To show the second equality let us first show $\mathrm{ESp}_{\psi_n}(R, I) \subseteq \mathrm{ESp}_{2n}(R, I)$. It is enough to show that an element of the form $T_{\psi_n}(v)\, S_{\psi_n}(w)\, T_{\psi_n}(v)^{-1}$ is in $\mathrm{ESp}_{2n}(R, I)$, for $v \in R^{2n-1}$ and $w \in I^{2n-1}$. Here T_{ψ_n} and S_{ψ_n} are L_{ψ_n} or R_{ψ_n}. Using the definition of $\mathrm{ESp}_{2n}(R, I)$ and equations (5.1), (5.2) we get

$$T_{\psi_n}(v)\ S_{\psi_n}(w)\ T_{\psi_n}(v)^{-1} \in \mathrm{ESp}_{2n}(R, I),$$

and hence $\mathrm{ESp}_{\psi_n}(R, I) \subseteq \mathrm{ESp}_{2n}(R, I)$.

To show the other inclusion we recall the equivalent definition of the relative group which says that $\mathrm{ESp}_{2n}(R, I)$ is the smallest normal subgroup of $\mathrm{ESp}_{2n}(R)$ containing $se_{21}(x)$, where $x \in I$ (see Lemma 2.2.29). We need to show

$$g\ se_{21}(x)\ g^{-1} \quad \in \quad \mathrm{ESp}_{\psi_n}(R, I),$$

where $g \in \mathrm{ESp}_{2n}(R) = \mathrm{ESp}_{\psi_n}(R)$. Hence $g\ se_{21}(x)\ g^{-1} \in \mathrm{ESp}_{\psi_n}(R, I)$ and

$$\mathrm{ESp}_{2n}(R, I) \quad \subseteq \quad \mathrm{ESp}_{\psi_n}(R, I).$$

Therefore the second equality is established.

Generators of $\mathrm{ESp}^1_{\psi_n}(R, I)$ is of the form $R_{\psi_n}(v)$, $L_{\psi_n}(w)$, where $v \in R^{2n-1}$ and $w \in I^{2n-1}$. By equations (5.1) and (5.2) we have $R_{\psi_n}(v), L_{\psi_n}(w)$ are in $\mathrm{ESp}^1_{2n}(R, I)$, hence $\mathrm{ESp}^1_{\psi_n}(R, I) \subseteq \mathrm{ESp}^1_{2n}(R, I)$. On the other hand generators of the group $\mathrm{ESp}^1_{2n}(R, I)$ are of the form $se_{1i}(a), se_{j1}(x)$, where $a \in R$ and $x \in I$. Using equations (5.1) and (5.2) we get $se_{1i}(a) = R_{\psi_n}(ae_{i-1})$, and $se_{j1}(x) = L_{\psi_n}(xe_{j-1})$, hence $\mathrm{ESp}^1_{2n}(R, I) \subseteq \mathrm{ESp}^1_{\psi_n}(R, I)$. Therefore the third equality is established. □

Lemma 5.1.6 *Let φ and φ^* be two alternating matrices of Pfaffian 1 such that $\varphi = (1 \perp \varepsilon)^t\ \varphi^*\ (1 \perp \varepsilon)$, for some $\varepsilon \in \mathrm{E}_{2n-1}(R)$. Then we have*

$$\begin{aligned}
\mathrm{Sp}_{\varphi}(R) &= (1 \perp \varepsilon)^{-1}\ \mathrm{Sp}_{\varphi^*}(R)\ (1 \perp \varepsilon), \\
\mathrm{ESp}_{\varphi}(R) &= (1 \perp \varepsilon)^{-1}\ \mathrm{ESp}_{\varphi^*}(R)\ (1 \perp \varepsilon).
\end{aligned}$$

Proof: First we will show $(1 \perp \varepsilon)^{-1}\ \mathrm{Sp}_{\varphi^*}(R)\ (1 \perp \varepsilon) \subseteq \mathrm{Sp}_{\varphi}(R)$. Let $\rho \in \mathrm{Sp}_{\varphi^*}(R)$ i.e, $\rho^t \varphi^* \rho = \varphi^*$ (by definition of symplectic group w.r.t. an alternating matrix). Now

$$\begin{aligned}
& (1 \perp \varepsilon)^t\ \rho^t\ (1 \perp \varepsilon)^{-1^t}\ \varphi\ (1 \perp \varepsilon)^{-1}\ \rho(1 \perp \varepsilon) \\
= \ & (1 \perp \varepsilon)^t\ \rho^t\ (1 \perp \varepsilon)^{-1^t}\ \{(1 \perp \varepsilon)^t\ \varphi^*\ (1 \perp \varepsilon)\}\ (1 \perp \varepsilon)^{-1}\ \rho\ (1 \perp \varepsilon) \\
= \ & (1 \perp \varepsilon)^t\ \varphi^*\ (1 \perp \varepsilon) \\
= \ & \varphi,
\end{aligned}$$

and hence $(1 \perp \varepsilon)^{-1}\ \mathrm{Sp}_{\varphi^*}(R)\ (1 \perp \varepsilon) \subseteq \mathrm{Sp}_{\varphi}(R)$. Similarly we will be able to show $(1 \perp \varepsilon)\ \mathrm{Sp}_{\varphi}(R)\ (1 \perp \varepsilon)^{-1} \subseteq \mathrm{Sp}_{\varphi^*}(R)$. Therefore

$$\mathrm{Sp}_{\varphi}(R) \quad = \quad (1 \perp \varepsilon)^{-1}\mathrm{Sp}_{\varphi^*}(R)(1 \perp \varepsilon).$$

We also have

$$(1 \perp \varepsilon)\ L_{\varphi}(v)\ (1 \perp \varepsilon)^{-1} = L_{\varphi^*}(v\varepsilon^t), \tag{5.3}$$
$$(1 \perp \varepsilon)\ R_{\varphi}(v)\ (1 \perp \varepsilon)^{-1} = R_{\varphi^*}(v\varepsilon^{-1}), \tag{5.4}$$

and hence $\mathrm{ESp}_{\varphi}(R) = (1 \perp \varepsilon)^{-1}\ \mathrm{ESp}_{\varphi^*}(R)\ (1 \perp \varepsilon)$. $\square$

Lemma 5.1.7 *Let φ and φ^* be two alternating matrices of Pfaffian 1 such that $\varphi = (1 \perp \varepsilon)^t\ \varphi^*\ (1 \perp \varepsilon)$, for some $\varepsilon \in \mathrm{E}_{2n-1}(R, I)$. Then we have*

$$\begin{aligned} \mathrm{ESp}_{\varphi}(R, I) &= (1 \perp \varepsilon)^{-1}\ \mathrm{ESp}_{\varphi^*}(R, I)\ (1 \perp \varepsilon), \\ \mathrm{ESp}^1_{\varphi}(R, I) &= (1 \perp \varepsilon)^{-1}\ \mathrm{ESp}^1_{\varphi^*}(R, I)\ (1 \perp \varepsilon). \end{aligned}$$

Proof: To prove the above equalities we use definitions of $\mathrm{ESp}_{\varphi}(R, I)$, $\mathrm{ESp}^1_{\varphi}(R, I)$ and the equations (5.3), (5.4). $\square$

Lemma 5.1.8 *Let $(R, \mathfrak{m})$ be a local ring and I be an ideal of R. Let φ be an alternating matrix of Pfaffian 1 over R, and $\varphi \equiv \psi_n \pmod I$. Then φ is of the form*

$$(1 \perp \varepsilon)^t\ \psi_n\ (1 \perp \varepsilon),$$

for some $\varepsilon \in \mathrm{E}_{2n-1}(R, I)$.

Proof: We will prove the result using induction on n. When φ is of size 2×2, the result is true. Let us assume the result is true for alternating matrix of size $2(n-1) \times 2(n-1)$, i.e, for an alternating matrix φ^* of size $2(n-1) \times 2(n-1)$, we have η from $\mathrm{E}_{2n-3}(R, I)$ such that

$$\varphi^* \ = \ (1 \perp \eta)^t\ \psi_{n-1}\ (1 \perp \eta).$$

We will prove the result for alternating matrix φ of size $2n \times 2n$. Let

$$\varphi \ = \ \begin{pmatrix} 0 & a \\ -a^t & \alpha \end{pmatrix} \ \equiv \ \psi_n \pmod I,$$

where $a \in \mathrm{Um}_{2n-1}(R, I)$ and α is alternating matrix of size $(2n-1) \times (2n-1)$. Note that

$$\alpha \;\equiv\; \begin{pmatrix} 0 & 0 \\ 0 & \psi_{n-1} \end{pmatrix} \pmod{I}.$$

As R is local ring we have $a = e_1\beta$, where $\beta \in \mathrm{E}_{2n-1}(R, I)$ (see Corollary 2.2.18). Hence

$$(1 \perp \beta^t)^{-1}\,\varphi\,(1 \perp \beta)^{-1} \;=\; \begin{pmatrix} 0 & e_1 \\ -e_1^t & \gamma \end{pmatrix},$$

where $\gamma = (\beta^t)^{-1}\,\alpha\,\beta^{-1}$. Note that γ is an alternating matrix. Therefore γ can be written as $\left(\begin{smallmatrix} 0 & b \\ -b^t & \varphi^* \end{smallmatrix}\right)$. Note that $\overline{\gamma} = (\overline{\beta}^t)^{-1}\overline{\alpha}\overline{\beta}^{-1} \equiv \left(\begin{smallmatrix} 0 & 0 \\ 0 & \psi_{n-1} \end{smallmatrix}\right) \pmod{I}$, and hence $b \in I^{2n-2}$ and $\varphi^* \equiv \psi_{n-1} \pmod{I}$. Now

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -b\varphi^{*-1} \\ 0 & 0 & I_{2n-2} \end{pmatrix} (1 \perp \beta^t)^{-1}\,\varphi\,(1 \perp \beta)^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -b\varphi^{*-1} \\ 0 & 0 & I_{2n-2} \end{pmatrix}^t = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \varphi^* \end{pmatrix}.$$

Let us call the matrix

$$((I_3 \perp \eta)^{-1})^t \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -b\varphi^{*-1} \\ 0 & 0 & I_{2n-2} \end{pmatrix} (1 \perp \beta^t)^{-1} = ((1 \perp \varepsilon)^{-1})^t.$$

Note that $\varepsilon \in \mathrm{E}_{2n-1}(R, I)$. Using induction hypothesis we get

$$\begin{aligned} & ((1 \perp \varepsilon)^{-1})^t\,\varphi\,(1 \perp \varepsilon)^{-1} \\ = \;& ((I_3 \perp \eta)^{-1})^t \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \varphi^* \end{pmatrix} (I_3 \perp \eta)^{-1} \\ = \;& \psi_n, \end{aligned}$$

and hence $\varphi = (1 \perp \varepsilon)^t \psi_n (1 \perp \varepsilon)$. Therefore the result is established. $\square$

Remark 5.1.9 *As a particular case of the previous lemma we get that when* $(R, \mathfrak{m})$ *is a local ring and* φ *is an alternating matrix of Pfaffian 1 over R, then there exists* $\varepsilon \in \mathrm{E}_{2n-1}(R)$ *such that*

$$\varphi \;=\; (1 \perp \varepsilon)^t \; \psi_n \; (1 \perp \varepsilon).$$

Remark 5.1.10 *The condition that the alternating matrices, in this thesis, are of Pfaffian one can be extended to all invertible alternating matrices by observing that an invertible alternating matrix over a local ring which is congruent to* $u\,\psi_1 \;\perp\; \psi_{n-1}$ $(\mathrm{mod}\ I)$*, where* $u =$ *Pfaffian* φ*, is of the form* $E^t(u\psi_1 \perp \psi_{n-1})E$*, for some relative elementary matrix* E*. Only slight modifications in the proofs given below are needed, which is an easy exercise.*

Remark 5.1.11 *Let* φ *be an alternating matrix of Pfaffian* 1*, over R. Let us consider the local ring* $R_{\mathfrak{m}}$*, where* $\mathfrak{m}$ *be a maximal ideal of R. We will get* $\varepsilon(\mathfrak{m}) \in \mathrm{E}_{2n-1}(R_{\mathfrak{m}})$ *such that over* $R_{\mathfrak{m}}$ *we have*

$$\varphi \;=\; (1 \perp \varepsilon(\mathfrak{m}))^t \; \psi_n \; (1 \perp \varepsilon(\mathfrak{m}))$$

(see remark 5.1.9). Let a be the product of denominators of all the entries of $\varepsilon(\mathfrak{m})$*. Clearly a is not in* $\mathfrak{m}$*. Hence we get* ε *from* $\mathrm{E}_{2n-1}(R_a)$ *such that*

$$\varphi \;=\; (1 \perp \varepsilon)^t \; \psi_n \; (1 \perp \varepsilon).$$

Also, when dealing with relative case w.r.t. an ideal I **of** R**, we will always assume that the alternating matrix** φ **of Pfaffian 1 is congruent to** $\psi_n(\mathrm{mod}\ I)$**. Therefore over the local ring** $R_{\mathfrak{m}}$**, we have**

$$\varphi \;=\; (1 \perp \varepsilon(\mathfrak{m}))^t \; \psi_n \; (1 \perp \varepsilon(\mathfrak{m})),$$

for some $\varepsilon(\mathfrak{m}) \in \mathrm{E}_{2n-1}(R_{\mathfrak{m}}, I_{\mathfrak{m}})$ *(see Lemma 5.1.8).* **Since there are only finitely many denominators, we can find** a **not in** $\mathfrak{m}$ **such that**

$$\varphi \;=\; (1 \perp \varepsilon)^t \; \psi_n \; (1 \perp \varepsilon),$$

where $\varepsilon \in \mathrm{E}_{2n-1}(R_a, I_a)$**. We will constantly use this fact without even referring to it !**

Lemma 5.1.12 *Let φ be an alternating matrix of Pfaffian 1 of the form $(1 \perp \varepsilon)^t\ \psi_n\ (1 \perp \varepsilon)$, for some $\varepsilon \in \mathrm{E}_{2n-1}(R, I)$. Then*

$$\mathrm{ESp}_{\varphi}(R, I) \;=\; \mathrm{ESp}^1_{\varphi}(R, I)\ \cap\ \mathrm{Sp}_{\varphi}(R, I),$$

for $n \geq 3$.

Proof:

$$\begin{aligned}
\mathrm{ESp}_{\varphi}(R, I) &= (1 \perp \varepsilon)^{-1}\ \mathrm{ESp}_{\psi_n}(R, I)\ (1 \perp \varepsilon) \\
&= (1 \perp \varepsilon)^{-1}\ \mathrm{ESp}_{2n}(R, I)\ (1 \perp \varepsilon) \\
&= (1 \perp \varepsilon)^{-1}\ \Big(\mathrm{ESp}^1_{2n}(R, I)\ \cap\ \mathrm{Sp}_{2n}(R, I)\Big)\ (1 \perp \varepsilon) \\
&= (1 \perp \varepsilon)^{-1}\ \Big(\mathrm{ESp}^1_{\psi_n}(R, I)\ \cap\ \mathrm{Sp}_{\psi_n}(R, I)\Big)\ (1 \perp \varepsilon) \\
&= \big((1 \perp \varepsilon)^{-1}\ \mathrm{ESp}^1_{\psi_n}(R, I)\ (1 \perp \varepsilon)\big)\ \cap \\
&\quad\ \big((1 \perp \varepsilon)^{-1}\ \mathrm{Sp}_{\psi_n}(R, I)\big)\ (1 \perp \varepsilon)\big) \\
&= \mathrm{ESp}^1_{\varphi}(R, I)\ \cap\ \mathrm{Sp}_{\varphi}(R, I).
\end{aligned}$$

The third equality follows from Lemma 2.2.30. □

5.2 Dilation Principle for $\mathrm{ESp}_{\varphi}(R)$

Lemma 5.2.1 *Let $n \geq 2$. Let φ be an alternating matrix of Pfaffian 1. Let $a \in R$ be non-nilpotent and $\varphi = (1 \perp \varepsilon)^t\ \psi_n\ (1 \perp \varepsilon)$, for some $\varepsilon \in \mathrm{E}_{2n-1}(R_a)$ over the ring R_a. Let $\alpha(X) \in \mathrm{ESp}_{\varphi \otimes R_a[X]}(R_a[X])$, with $\alpha(0) = Id$. Then there exists $\alpha^*(X) \in \mathrm{ESp}_{\varphi \otimes R[X]}(R[X])$ such that $\alpha^*(X)$ localises to $\alpha(bX)$, for some $b \in (a^N)$, $N \gg 0$ and $\alpha^*(0) = Id$.*

Proof: $\alpha(X)$ can be written as $\prod_{t=1}^{s} T_{\varphi}(g_t(X))$, where T_{φ} is L_{φ} or R_{φ}, and $g_t(X) \in (R_a[X])^{2n-1}$. Having $\varphi = (1 \perp \varepsilon)^t\ \psi_n\ (1 \perp \varepsilon)$, with some $\varepsilon \in \mathrm{E}_{2n-1}(R_a)$, will allow us to write

$$\begin{aligned}
\alpha(X) &= \prod_{t=1}^{s} (1 \perp \varepsilon)^{-1}\ T_{\psi_n}(f_t(X))\ (1 \perp \varepsilon) \\
&= (1 \perp \varepsilon)^{-1}\ (\prod_{t=1}^{s} T_{\psi_n}(f_t(X)))\ (1 \perp \varepsilon) \\
&= (1 \perp \varepsilon)^{-1}\ \eta(X)\ (1 \perp \varepsilon),
\end{aligned}$$

where $f_t(X) = g_t(X)\ \varepsilon^t$, if $T_\varphi = L_\varphi$, and $f_t(X) = g_t(X)\ \varepsilon^{-1}$, if $T_\varphi = R_\varphi$, and $\eta(X) \in \mathrm{ESp}_{2n}(R_a[X])$ (see Lemma 5.1.5 and Lemma 5.1.6). Note that $\eta(0) = Id$, as $\alpha(0) = Id$. Therefore,

$$\eta(X) \;=\; \prod_{k=1}^{r} \gamma_k\ se_{i_k j_k}(Xh_k(X)/a^s)\ \gamma_k^{-1},$$

where $\gamma_k \in \mathrm{ESp}_{2n}(R_a)$ and $h_k(X) \in R[X]$ (see Lemma 2.2.27). Now,

$$\begin{aligned}
\eta(Y^{2^{r+1}}X) &= \prod_{k=1}^{r} \gamma_k\ se_{i_k j_k}(Y^{2^{r+1}}Xh_k(Y^{2^{r+1}}X)/a^s)\ \gamma_k^{-1}\\
&= \prod_{t=1}^{l} se_{p_t q_t}(Y^2 u_t(X,Y)/a^s)\\
&= \prod_{t=1}^{l} [se_{p_t 1}(Y),\ se_{1 q_t}(Yu_t(X,Y)/a^s)],
\end{aligned}$$

where $u_t(X,Y) \in R[X,Y]$. The second equality above follows from Lemma 2.2.28. Let us take $N = M^{2^{r+1}}$, where $M = M' + s$ be a natural number. We define

$$\alpha^*(X,Y) \;=\; \prod_{t=1}^{s} [L_\varphi(a^M Y e_{p_t-1}(\varepsilon^t)^{-1}),\ R_\varphi(a^{M'} Y u_t(X, a^M Y) e_{q_t-1}\varepsilon)],$$

where $\alpha^*(X,Y) \in \mathrm{ESp}_{\varphi\otimes R[X,Y]}(R[X,Y])$, for $N \gg 0$. Note that

$$\begin{aligned}
\alpha(a^N X Y^{2^{r+1}}) &= \alpha((a^M Y)^{2^{r+1}} X)\\
&= (1 \perp \varepsilon)^{-1}\ \eta((a^M Y)^{2^{r+1}} X)\ (1 \perp \varepsilon)\\
&= (1 \perp \varepsilon)^{-1} \prod_{t=1}^{s} [se_{p_t 1}(a^M Y),\ se_{1 q_t}(a^M Y u_t(X, a^M Y)/a^s)]\\
&\quad\ (1 \perp \varepsilon)\\
&= \prod_{t=1}^{s} [L_\varphi(a^M Y e_{p_t-1}(\varepsilon^t)^{-1}),\ R_\varphi(a^{M'} Y u_t(X, a^M Y) e_{q_t-1}\varepsilon)].
\end{aligned}$$

Substituting $Y = 1$ we get $\alpha^*(X) = \alpha(bX)$, for $b \in (a^N)$, $N \gg 0$ (see Lemma 2.2.4). Observe that $\alpha^*(X) \in \mathrm{ESp}_{\varphi\otimes R[X]}(R[X])$, and $\alpha^*(0) = Id$. $\square$

Now we prove dilation principle for $\mathrm{ESp}_{\varphi\otimes R[X]}(R[X], I[X])$.

Lemma 5.2.2 *Let $n \geq 3$. Let R be a commutative ring with $R = 2R$, and let I*

be an ideal of R. Let φ be an alternating matrix of Pfaffian 1. Let $a \in R$ be a non-nilpotent element, and $\varphi = (1 \perp \varepsilon)^t\ \psi_n\ (1 \perp \varepsilon)$, for some $\varepsilon \in \mathrm{E}_{2n-1}(R_a, I_a)$ over the ring R_a. Let $\alpha(X) \in \mathrm{ESp}_{\varphi\otimes R_a[X]}(R_a[X], I_a[X])$, with $\alpha(0) = Id$. Then there exists $\alpha^*(X) \in \mathrm{ESp}_{\varphi\otimes R[X]}(R[X], I[X])$ such that $\alpha^*(X)$ localises to $\alpha(bX)$, for some $b \in (a^N)$, $N \gg 0$, and $\alpha^*(0) = Id$.

Proof: We have $\alpha(X) = (1 \perp \varepsilon)^{-1}\ \eta(X)\ (1 \perp \varepsilon)$, where $\eta(X)$ belongs to $\mathrm{ESp}_{2n}(R_a[X], I_a[X])$ (see Lemma 5.1.5 and Lemma 5.1.7). Note that $\eta(0) = Id$, as $\alpha(0) = Id$. Using dilation principle for $\mathrm{ESp}_{2n}(R[X], I[X])$ (see Theorem 3.3.3), we get an $\eta^*(X) \in \mathrm{ESp}_{2n}(R[X], I[X])$ such that $\eta^*(X)$ localises to $\eta(b'X)$, for $b' \in (a^N)$, $N \gg 0$, with $\eta^*(0) = Id$, and $\eta(b'X) \in \mathrm{ESp}_{2n}(R[X], I[X])$. Let $\alpha^*(X)$ be an element of $\mathrm{ESp}_{\varphi\otimes R[X]}(R[X], I[X])$ such that

$$\begin{aligned} \alpha^*(X)_a &= (1 \perp \varepsilon)^{-1}\ \eta^*(X)_a\ (1 \perp \varepsilon) \\ &= (1 \perp \varepsilon)^{-1}\ \eta(b'X)\ (1 \perp \varepsilon) \\ &= \alpha(b'X), \end{aligned}$$

over R_a. Using Lemma 2.2.4 we can say $\alpha^*(X)$ localises to $\alpha(bX)$, for $b \in (a^N)$, $N \gg 0$, and $\alpha^*(0) = Id$. □

5.3 Local Global Principle for $\mathrm{ESp}_\varphi(R)$

Lemma 5.3.1 *Let φ be an alternating matrix of Pfaffian 1, of size at least 4, over R. Let $\alpha(X) \in \mathrm{Sp}_{\varphi\otimes R[X]}(R[X])$ and $\alpha(0) = Id$. If for each maximal ideal $\mathfrak{m}$ of R, $\alpha(X)_{\mathfrak{m}} \in \mathrm{ESp}_{\varphi\otimes R_{\mathfrak{m}}[X]}(R_{\mathfrak{m}}[X])$, then $\alpha(X) \in \mathrm{ESp}_{\varphi\otimes R[X]}(R[X])$.* □

We now state and prove a relative version of Lemma 5.3.1. The above lemma is a particular case of Lemma 5.3.2 when $I[X] = R[X]$.

Lemma 5.3.2 *Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Let φ be an alternating matrix of Pfaffian 1, of size at least 6, over R and let $\varphi \equiv \psi_n \pmod{I}$. Let $\alpha(X) \in \mathrm{Sp}_{\varphi\otimes R[X]}(R[X], I[X])$, with $\alpha(0) = Id$. If for each maximal ideal $\mathfrak{m}$ of R, $\alpha(X)_{\mathfrak{m}} \in \mathrm{ESp}_{\varphi\otimes R_{\mathfrak{m}}[X]}(R_{\mathfrak{m}}[X], I_{\mathfrak{m}}[X])$, then $\alpha(X) \in \mathrm{ESp}_{\varphi\otimes R[X]}(R[X], I[X])$.*

Proof: For each maximal ideal $\mathfrak{m}$ of R one can suitably choose an element $a_{\mathfrak{m}}$ from $R \setminus \mathfrak{m}$ such that $\alpha(X)_{a_{\mathfrak{m}}} \in \mathrm{ESp}_{\varphi\otimes R_{a_{\mathfrak{m}}}[X]}(R_{a_{\mathfrak{m}}}[X], I_{a_m}[X])$ and also $\varphi = (1 \perp$

$\varepsilon)^t\ \psi_n\ (1 \perp \varepsilon)$, for some $\varepsilon \in \mathrm{E}_{2n-1}(R_{a_{\mathfrak{m}}}, I_{a_{\mathfrak{m}}})$. Let us define

$$\beta(X,Y) \;=\; \alpha(X+Y)_{a_{\mathfrak{m}}}\ \alpha(Y)^{-1}_{a_{\mathfrak{m}}}.$$

It is clear that

$$\beta(X,Y) \;\in\; \mathrm{ESp}_{\varphi\otimes R_{a_{\mathfrak{m}}}[X,Y]}(R_{a_{\mathfrak{m}}}[X,Y], I_{a_{\mathfrak{m}}}[X,Y])$$

and $\beta(0,Y) = Id$. Therefore $\beta(b_{\mathfrak{m}}X,Y) \in \mathrm{ESp}_{\varphi\otimes R[X,Y]}(R[X,Y], I[X,Y])$, where $b_{\mathfrak{m}} \in (a_m^N)$ for $N \gg 0$ (see Lemma 5.2.2). The ideal generated by $b_{\mathfrak{m}}$'s is the whole ring R. Hence we have $c_1 b_{\mathfrak{m}_1} + \cdots + c_k b_{\mathfrak{m}_k} = 1$, where $c_i \in R$, for $1 \le i \le k$. Note that $\beta(c_i b_{\mathfrak{m}_i} X, Y) \in \mathrm{ESp}_{\varphi\otimes R[X,Y]}(R[X,Y], I[X,Y])$, for $1 \le i \le k$. Now,

$$\alpha(X) \;=\; \prod_{i=1}^{k} \beta(b_{\mathfrak{m}_i}X, T_i)\ \beta(b_{\mathfrak{m}_k}, 0) \;\in\; \mathrm{ESp}_{\varphi\otimes R[X]}(R[X], I[X]),$$

where $T_i = c_{i+1} b_{\mathfrak{m}_{i+1}} X + \cdots + c_k b_{\mathfrak{m}_k} X$. $\square$

Now we prove a action version of above Local Global principle.

Theorem 5.3.3 *Let $n \ge 2$ and $v(X) \in \mathrm{Um}_{2n}(R[X])$. Let φ be an alternating matrix of Pfaffian 1 over R. If for each maximal ideal $\mathfrak{m}$ of R, $v(X) \in v(0)\mathrm{ESp}_{\varphi\otimes R_{\mathfrak{m}}[X]}(R_{\mathfrak{m}}[X])$, then*

$$v(X) \;\in\; v(0)\mathrm{ESp}_{\varphi\otimes R[X]}(R[X]).$$

$\square$

We establish a relative version of Theorem 5.3.3 below. The above theorem can be treated as a particular case of Theorem 5.3.4 when $I[X] = R[X]$.

Theorem 5.3.4 *Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Let $n \ge 3$ and $v(X) \in \mathrm{Um}_{2n}(R[X], I[X])$. Let φ be an alternating matrix of Pfaffian 1 over R, and let $\varphi \equiv \psi_n \pmod{I}$. If for each maximal ideal $\mathfrak{m}$ of R, $v(X) \in v(0)\mathrm{ESp}_{\varphi\otimes R_{\mathfrak{m}}[X]}(R_{\mathfrak{m}}[X], I_{\mathfrak{m}}[X])$, then*

$$v(X) \;\in\; v(0)\mathrm{ESp}_{\varphi\otimes R[X]}(R[X], I[X]).$$

Proof: For each maximal ideal $\mathfrak{m}$ of R, we get $\alpha_{(\mathfrak{m})}(X) \in \mathrm{ESp}_{\varphi \otimes R_{\mathfrak{m}}[X]}(R_{\mathfrak{m}}[X], I_{\mathfrak{m}}[X])$ such that

$$v(X)\alpha_{(\mathfrak{m})}(X) = v(0).$$

Let us define

$$\beta(X,T) = \alpha_{(\mathfrak{m})}(X+T)\ \alpha_{(\mathfrak{m})}(X)^{-1}.$$

Clearly $\beta(X,T)$ is in $\mathrm{ESp}_{\varphi \otimes R_{\mathfrak{m}}[X,T]}(R_{\mathfrak{m}}[X,T], I_{\mathfrak{m}}[X,T])$. Since there are only finitely many denominators involved, there exists $a_{\mathfrak{m}} \in R \setminus \mathfrak{m}$ such that $\beta(X,T)$ is in $\mathrm{ESp}_{\varphi \otimes R_{a_{\mathfrak{m}}}[X,T]}(R_{a_{\mathfrak{m}}}[X,T], I_{a_{\mathfrak{m}}}[X,T])$. Also $\beta(X,0) = Id$. This implies $\beta(X, b_{\mathfrak{m}}T) \in \mathrm{ESp}_{\varphi \otimes R[X,T]}(R[X,T], I[X,T])$, for suitable $b_{\mathfrak{m}} \in (a_{\mathfrak{m}}^N)$, $N \gg 0$ (see Lemma 5.2.2). Now,

$$\begin{aligned} v(X + b_{\mathfrak{m}}T)\ \beta(X, b_{\mathfrak{m}}T) &= v(X + b_{\mathfrak{m}}T)\ \alpha_{(\mathfrak{m})}(X + b_{\mathfrak{m}}T)\ \alpha_{(\mathfrak{m})}(X)^{-1} \\ &= v(0)\ \alpha_{(\mathfrak{m})}(X)^{-1} \\ &= v(X). \end{aligned}$$

Note that the ideal generated by $b_{\mathfrak{m}}$'s is the whole ring R. Therefore $c_1 b_{\mathfrak{m}_1} + \cdots + c_k b_{\mathfrak{m}_k} = 1$, where $c_i \in R$, for $1 \leq i \leq k$. In the above equation replacing X by $c_2 b_{\mathfrak{m}_2} X + \cdots + c_k b_{\mathfrak{m}_k} X$ and $b_{\mathfrak{m}}T$ by $c_1 b_{\mathfrak{m}_1} X$ we get,

$$\begin{aligned} v(X) &= v(b_{\mathfrak{m}_1} X + b_{\mathfrak{m}_2} X + \cdots + b_{\mathfrak{m}_k} X) \\ &\in v(b_{\mathfrak{m}_2} X + \cdots + b_{\mathfrak{m}_k} X)\ \mathrm{ESp}_{\varphi \otimes R[X]}(R[X], I[X]). \end{aligned}$$

Again in the above equation replacing X by $b_{\mathfrak{m}_3} X + \cdots + b_{\mathfrak{m}_k} X$ and $b_{\mathfrak{m}}T$ by $b_{\mathfrak{m}_2} X$ we get,

$$v(b_{\mathfrak{m}_2} X + \cdots + b_{\mathfrak{m}_k} X) \in v(b_{\mathfrak{m}_3} X + \cdots + b_{\mathfrak{m}_k} X)\ \mathrm{ESp}_{\varphi \otimes R[X]}(R[X], I[X]).$$

Continuing in this way we get

$$v(b_{m_k} X + 0) \in v(0)\ \mathrm{ESp}_{\varphi \otimes R[X]}(R[X], I[X]).$$

Combining all these we get

$$v(X) \;\in\; v(0)\ \mathrm{ESp}_{\varphi\otimes R[X]}(R[X], I[X]).$$

$\square$

5.4 Transvection Group

Following H.Bass one can define transvections of a finitely generated R-module as follows:

Definition 5.4.1 Let M be a finitely generated R-module. Let $q \in M$ and $\pi \in M^* = \mathrm{Hom}(M, R)$, with $\pi(q) = 0$. Let $\pi_q(p) := \pi(p)q$. An automorphism of the form $1 + \pi_q$ is called a **transvection** of M, if either $q \in \mathrm{Um}(M)$ or $\pi \in \mathrm{Um}(M^*)$. Collection of transvections of M is denoted by $\mathrm{Trans}(M)$. This forms a subgroup of $\mathrm{Aut}(M)$.

Definition 5.4.2 Let M be a finitely generated R module. The automorphisms of $N = (R \perp M)$ of the form

$$\begin{aligned}(a,p) &\mapsto (a, p + ax),\\ (a,p) &\mapsto (a + \tau(p), p),\end{aligned}$$

where $x \in M$ and $\tau \in M^*$ are called **elementary transvections** of N. Let us denote the first automorphism by E_x and the second one by E^*_τ. It can be verified that these are transvecions of N. Let us consider $\pi(t, y) = t$ and $q = (0, x)$ to get E_x. Next we can consider $\pi(a, p) = \tau(p)$, where $\tau \in M^*$ and $q = (1, 0)$ to get E^*_τ. The subgroup of $\mathrm{Trans}(N)$ generated by elementary transvections is denoted by $\mathrm{ETrans}(N)$.

Definition 5.4.3 Let I be an ideal of R. The group of **relative transvections** w.r.t. an ideal I is generated by the transvections of the form $1 + \pi_q$, where either $q \in \mathrm{Um}(IM), \pi \in \mathrm{Um}(M^*)$, or $q \in \mathrm{Um}(M), \pi \in \mathrm{Um}(IM^*)$. The group of relative transvections is denoted by $\mathrm{Trans}(M, IM)$.

Definition 5.4.4 Let I be an ideal of R. The elementary transvections of $N = (R \perp M)$ of the form E_x, E^*_τ, where $x \in IM$ and $\tau \in (IM)^*$ are called **relative**

elementary transvections w.r.t. an ideal I, and the group generated by them is denoted by $\mathrm{ETrans}(IN)$. The normal closure of $\mathrm{ETrans}(IN)$ in $\mathrm{ETrans}(N)$ is denoted by $\mathrm{ETrans}(N, IN)$.

Lemma 5.4.5 *Let M be a free R module of rank $n \geq 3$, and $N = (R \perp M)$. Then*

$$\mathrm{Trans}(M) = \mathrm{E}_n(R),$$
$$\mathrm{ETrans}(N) = \mathrm{Trans}(N) = \mathrm{E}_{n+1}(R).$$

Proof: Let $M = R^n$. Note that $\pi_q : R^n \longrightarrow R \longrightarrow R^n$, and hence $1 + \pi_q = I_n + v^t w$, for some $v, w \in R^n$, with either v or w unimodular and $\langle v, w \rangle = 0$. Therefore $\mathrm{Trans}(M) \subseteq \mathrm{E}_n(R)$ (see Lemma 2.2.7).

A standard elementary generator of the group $\mathrm{E}_n(R)$ can be expressed as $I_n + a e_i^t e_j$, where $1 \leq i \neq j \leq n$, and $a \in R$. Hence $\mathrm{E}_n(R) \subseteq \mathrm{Trans}(R)$, which implies $\mathrm{Trans}(R) = \mathrm{E}_n(R)$.

One can observe that when $M = R^n$, the matrices correspond to the elementary transvections E_x and E_τ^* of $N = (R \perp M)$ are of the form

$$\begin{pmatrix} 1 & x \\ 0 & I_n \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ y^t & I_n \end{pmatrix},$$

respectively, where $x, y \in R^n$, and hence $\mathrm{ETrans}(N) \subseteq \mathrm{E}_{n+1}(R)$. Note that $\mathrm{E}_{n+1}(R)$ is generated by the matrices of the form $\left(\begin{smallmatrix} 1 & x \\ 0 & I_n \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ y^t & I_n \end{smallmatrix}\right)$ (see Lemma 2.2.11), and hence $\mathrm{E}_{n+1}(R) \subseteq \mathrm{ETrans}(N)$. Therefore $\mathrm{E}_{n+1}(R) = \mathrm{ETrans}(N)$. Also $\mathrm{ETrans}(N) \subseteq \mathrm{Trans}(N)$ and hence

$$\mathrm{E}_{n+1}(R) = \mathrm{ETrans}(N) \subseteq \mathrm{Trans}(N) = \mathrm{E}_{n+1}(R).$$

Therefore we have the second equality. $\square$

Lemma 5.4.6 *Let I be an ideal of R and M be a free R module of rank $n \geq 2$, and $N = (R \perp M)$. Then*

$$\mathrm{ETrans}(N, IN) = \mathrm{Trans}(N, IN) = \mathrm{E}_{n+1}(R, I).$$

Proof: Note that when M is a free R module, an element of $\mathrm{Trans}(N, IN)$ looks like $I_{n+1} + v^t w$, for some $v, w \in R^{n+1}$, with either v or w unimodular and belongs to

I^{n+1} ($\subseteq R^{n+1}$). Also $\langle v, w\rangle = 0$. Therefore $\text{Trans}(N, IN) \subseteq \text{E}_{n+1}(R, I)$ (see Lemma 2.2.8).

For a free R-module M, the elements of $\text{ETrans}(N, IN)$ are of the form

$$\begin{pmatrix} 1 & a \\ 0 & I_n \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x^t & I_n \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & I_n \end{pmatrix}^{-1},$$
$$\begin{pmatrix} 1 & 0 \\ b^t & I_n \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & I_n \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b^t & I_n \end{pmatrix}^{-1},$$

where $a, b \in M = R^n$, and $x, y \in I^n(\subseteq R^n)$. Hence $\text{ETrans}(N, IN) \subseteq \text{E}_{n+1}(R, I)$. By Lemma 2.2.11 and Lemma 2.2.29 we have $\text{E}_{n+1}(R, I) \subseteq \text{ETrans}(N, IN)$, hence $\text{ETrans}(N, IN) = \text{E}_{n+1}(R, I)$. We have

$$\text{E}_{n+1}(R, I) = \text{ETrans}(N, IN) \subseteq \text{Trans}(N, IN) \subseteq \text{E}_{n+1}(R, I),$$

and hence the result follows. □

5.5 Dilation for Elementary Transvections

Lemma 5.5.1 ([2], Proposition 3.1) *Let M be a finitely generated module of R and a be non-nilpotent element of R such that M_a be free R_a-module of rank at least 2. Let $N = (R \perp M)$. Let $\alpha(X) \in \text{ETrans}(N_a[X])$, with $\alpha(0) = Id$. Then there exists $\alpha^*(X) \in \text{ETrans}(N[X])$ such that $\alpha^*(X)$ localises to $\alpha(bX)$ for some $b \in (a^N)$, $N \gg 0$ and $\alpha^*(0) = Id$.* □

Next we will establish a relative version of the above dilation principle (Lemma 5.5.1).

Lemma 5.5.2 *Let I be an ideal of R and let M be a finitely generated module of R. Let a be non-nilpotent element of R such that M_a be free R_a-module of rank at least 2. Let $N = (R \perp M)$. Let $\alpha(X) \in \text{ETrans}(N_a[X], IN_a[X])$, with $\alpha(0) = Id$. Then there exists $\alpha^*(X) \in \text{ETrans}(N[X], IN[X])$ such that $\alpha^*(X)$ localises to $\alpha(bX)$ for some $b \in (a^N)$, $N \gg 0$ and $\alpha^*(0) = Id$.*

Proof: Given that M_a is a free R_a-module. Using Lemma 5.4.6 we get that $\text{ETrans}(N_a[X], IN_a[X]) = \text{E}_{n+1}(R_a[X], I_a[X])$. Now we use dilation principle for the group $\mathrm{E}_{n+1}(R[X], I[X])$ (see Theorem 3.3.3) to get $\beta^*(X) \in \mathrm{E}_{n+1}(R[X], I[X])$ such that $\beta^*(X)_a = \alpha(b'X)$, for some $b' \in (a^N)$, $N \gg 0$.

Let us choose $\alpha^*(X)$ from $\mathrm{ETrans}(N[X], IN[X])$ such that $\alpha^*(X)_a = \beta^*(X)_a$, over the ring $R_a[X]$. Using Lemma 2.2.4 we can say that $\alpha^*(X)$ localises to $\alpha(bX)$ for some $b \in (a^N)$, $N \gg 0$ and $\alpha^*(0) = Id$. $\square$

5.6 Local Global Principle for Elementary Transvections

Lemma 5.6.1 ([2], Proposition 3.6) *Let M be a finitely generated projective R-module of rank $n \geq 2$, and $N = (R \perp M)$. Let $\alpha(X) \in \mathrm{Aut}(N[X])$, with $\alpha(0) = Id$. If for each maximal ideal $\mathfrak{m}$ of R, $\alpha(X)_{\mathfrak{m}}$ is in $\mathrm{ETrans}(N_{\mathfrak{m}}[X])$, then $\alpha(X) \in \mathrm{ETrans}(N[X])$.* $\square$

We state and prove a relative version of above Local Global principle. Local Global principle in the absolute case (Lemma 5.6.1) is a particular case of the Local Global principle in the relative case (Lemma 5.6.2) when $I[X] = R[X]$.

Lemma 5.6.2 *Let I be an ideal of R and let M be a finitely generated projective R-module of rank $n \geq 2$. Let $N = (R \perp M)$. Let $\alpha(X) \in \mathrm{Aut}(N[X])$, with $\alpha(0) = Id$. If for each maximal ideal $\mathfrak{m}$ of R, $\alpha(X)_{\mathfrak{m}} \in \mathrm{ETrans}(N_{\mathfrak{m}}[X], IN_{\mathfrak{m}}[X])$, then $\alpha(X) \in \mathrm{ETrans}(N[X], IN[X])$.*

Proof: One can suitably choose an element $a_{\mathfrak{m}}$ from $R \setminus \mathfrak{m}$ such that $\alpha(X)_{a_{\mathfrak{m}}} \in \mathrm{ETrans}(N_{a_{\mathfrak{m}}}[X], IN_{a_{\mathfrak{m}}}[X])$. Let us define

$$\beta(X, Y) \;=\; \alpha(X+Y)_{a_{\mathfrak{m}}}\ \alpha(Y)_{a_{\mathfrak{m}}}^{-1}.$$

Clearly

$$\beta(X, Y) \;\in\; \mathrm{ETrans}(N_{a_{\mathfrak{m}}}[X, Y], IN_{a_{\mathfrak{m}}}[X, Y]),$$

and $\beta(0, Y) = Id$. Therefore $\beta(b_{\mathfrak{m}}X, Y) \in \mathrm{ETrans}(N[X, Y], IN[X, Y])$, where $b_{\mathfrak{m}} \in (a_{\mathfrak{m}}^N)$, for some $N \gg 0$ (see Lemma 5.5.2). The ideal generated by $b_{\mathfrak{m}}$'s is the whole ring R. Therefore we have $c_1 b_{\mathfrak{m}_1} + \cdots + c_k b_{\mathfrak{m}_k} = 1$, where $c_i \in R$, for $1 \leq i \leq k$. Note that $\beta(c_i b_{\mathfrak{m}_i} X, Y) \in \mathrm{ETrans}(N[X, Y], IN[X, Y])$, for $1 \leq i \leq k$. Hence

$$\alpha(X) \;=\; \prod_{i=1}^{k} \beta(c_i b_{\mathfrak{m}_i} X, T_i)\ \beta(c_k b_{\mathfrak{m}_k}, 0) \;\in\; \mathrm{ETrans}(N[X], IN[X]),$$

where $T_i = c_{i+1} b_{\mathfrak{m}_{i+1}} X + \cdots + c_k b_{\mathfrak{m}_k} X$. $\square$

Now we prove action version of above Local Global principle.

Theorem 5.6.3 *Let M be a finitely generated projective R-module of rank $n \geq 2$, and $N = (R \perp M)$. Let $q(X) = (a(X), p(X)) \in \mathrm{Um}(N[X])$. If for each maximal ideal $\mathfrak{m}$ of R, $q(X) \in q(0)\mathrm{ETrans}(N_{\mathfrak{m}}[X])$, then*

$$q(X) \ \in \ q(0)\ \mathrm{ETrans}(N[X]).$$

$\square$

Here we establish a relative version of the above theorem. Theorem 5.6.3 is a particular case of Theorem 5.6.4 when $I[X] = R[X]$.

Theorem 5.6.4 *Let I be an ideal of R and let M be a finitely generated projective R-module of rank $n \geq 2$. Let $N = (R \perp M)$. If for each maximal ideal $\mathfrak{m}$ of R, $q(X) \in q(0)\mathrm{ETrans}(N_{\mathfrak{m}}[X], IN_{\mathfrak{m}}[X])$, where $q(X) = (a(X), p(X))$ is in $\mathrm{Um}(N[X], IN[X])$, then*

$$q(X) \ \in \ q(0)\ \mathrm{ETrans}(N[X], IN[X]).$$

Proof: For each maximal ideal $\mathfrak{m}$ of R, we get $\alpha_{(\mathfrak{m})}(X) \in \mathrm{ETrans}(N_{\mathfrak{m}}[X], IN_{\mathfrak{m}}[X])$ such that

$$q(X)\ \alpha_{(\mathfrak{m})}(X) \ = \ q(0).$$

Let us define

$$\beta(X,T) \ = \ \alpha_{(\mathfrak{m})}(X+T)\ \alpha_{(\mathfrak{m})}(X)^{-1}.$$

Clearly $\beta(X,T)$ is in $\mathrm{ETrans}(N_{\mathfrak{m}}[X,T], IN_{\mathfrak{m}}[X,T])$. Since there are only finitely many denominators involved, there exists $a_{\mathfrak{m}} \in R \setminus \mathfrak{m}$ such that $\beta(X,T)$ is in $\mathrm{ETrans}(N_{a_{\mathfrak{m}}}[X,T], IN_{a_{\mathfrak{m}}}[X,T])$. Also $\beta(X,0) = Id$. This implies $\beta(X, b_{\mathfrak{m}}T) \in \mathrm{ETrans}(N[X,T], IN[X,T])$ for suitable $b_{\mathfrak{m}} \in (a_{\mathfrak{m}}^N)$, $N \gg 0$ (see Lemma 5.5.2). Now,

$$\begin{aligned} q(X+b_{\mathfrak{m}}T)\ \beta(X,b_{\mathfrak{m}}T) &= q(X+b_{\mathfrak{m}}T)\ \alpha_{(\mathfrak{m})}(X+b_{\mathfrak{m}}T)\ \alpha_{(\mathfrak{m})}(X)^{-1} \\ &= q(0)\ \alpha_{(\mathfrak{m})}(X)^{-1} \\ &= q(X). \end{aligned}$$

Note that the ideal generated by $b_{\mathfrak{m}}$'s is the whole ring R. Therefore $c_1 b_{\mathfrak{m}_1} + \cdots + c_k b_{\mathfrak{m}_k} = 1$, where $c_i \in R$, for $1 \leq i \leq k$. In the above equation replacing X by $c_2 b_{\mathfrak{m}_2} X + \cdots + c_k b_{\mathfrak{m}_k} X$ and $b_{\mathfrak{m}} T$ by $c_1 b_{\mathfrak{m}_1} X$ we get,

$$\begin{aligned} q(X) &= q(b_{\mathfrak{m}_1} X + b_{\mathfrak{m}_2} X + \cdots + b_{\mathfrak{m}_k} X) \\ &\in q(b_{\mathfrak{m}_2} X + \cdots + b_{\mathfrak{m}_k} X) \text{ ETrans}(N[X], IN[X]). \end{aligned}$$

Again in the above equation replacing X by $b_{\mathfrak{m}_3} X + \cdots + b_{\mathfrak{m}_k} X$ and $b_{\mathfrak{m}} T$ by $b_{\mathfrak{m}_2} X$ we get,

$$q(b_{\mathfrak{m}_2} X + \cdots + b_{\mathfrak{m}_k} X) \in q(b_{\mathfrak{m}_3} X + \cdots + b_{\mathfrak{m}_k} X) \text{ ETrans}(N[X], IN[X]).$$

Continuing in this way we get

$$q(b_{m_k} X + 0) \in q(0) \text{ ETrans}(N[X], IN[X]).$$

Combining all these we get

$$q(X) \in q(0) \text{ ETrans}(N[X], IN[X]).$$

□

Proposition 5.6.5 *Let M be a finitely generated projective R-module of rank at least 2, and $N = (R \perp M)$. Then*

$$\text{Trans}(N) = \text{ETrans}(N).$$

Proof: Note that $\text{ETrans}(N) \subseteq \text{Trans}(N)$. Let us consider an element $\alpha \in \text{Trans}(N)$. There exists $\alpha(X) \in \text{Trans}(N[X])$ such that $\alpha(1) = \alpha$ and $\alpha(0) = Id$. Let $\mathfrak{m}$ be a maximal ideal of R. We have $\alpha(X)_{\mathfrak{m}} \in \text{Trans}(N_{\mathfrak{m}}[X]) = \text{ETrans}(N_{\mathfrak{m}}[X])$ (see Lemma 5.4.5). This is true for all maximal ideal $\mathfrak{m}$ of R and hence by Lemma 5.6.1 we have $\alpha(X)$ is in $\text{ETrans}(N[X])$. Substituting $X = 1$ we get $\alpha \in \text{ETrans}(N)$, and hence $\text{Trans}(N) \subseteq \text{ETrans}(N)$. □

Similarly we can prove the following:

Proposition 5.6.6 *Let I be an ideal of R. Let M be a finitely generated projective*

R-module of rank at least 2, and $N = (R \perp M)$. *Then*

$$\mathrm{Trans}(N, IN) \;=\; \mathrm{ETrans}(N, IN).$$

Proof: Note that $\mathrm{ETrans}(N, IN) \subseteq \mathrm{Trans}(N, IN)$. Let us consider an element $\alpha \in \mathrm{Trans}(N, IN)$. There exists $\alpha(X) \in \mathrm{Trans}(N[X], IN[X])$ such that $\alpha(1) = \alpha$ and $\alpha(0) = Id$. Let $\mathfrak{m}$ be a maximal ideal of R. We have $\alpha(X)_{\mathfrak{m}} \in \mathrm{Trans}(N_{\mathfrak{m}}[X], IN_{\mathfrak{m}}[X]) = \mathrm{ETrans}(N_{\mathfrak{m}}[X], IN_{\mathfrak{m}}[X])$ (see Lemma 5.4.6). This is true for all maximal ideal $\mathfrak{m}$ of R and hence by Lemma 5.6.2 we have $\alpha(X)$ is in $\mathrm{ETrans}(N[X], IN[X])$. Substituting $X = 1$ we get $\alpha \in \mathrm{ETrans}(N, IN)$, and hence $\mathrm{Trans}(N, IN) \subseteq \mathrm{ETrans}(N, IN)$. □

5.7 Symplectic Modules and Symplectic Transvections

Definition 5.7.1 A **symplectic** R**-module** is a pair $(P, \langle , \rangle)$, where P is a finitely generated projective R-module of even rank and $\langle , \rangle : P \times P \longrightarrow R$ is a non-degenerate (i.e, $P \cong P^*$ by $x \longrightarrow \langle x, \rangle$) alternating bilinear form.

Definition 5.7.2 Let $(P_1, \langle , \rangle_1)$ and $(P_2, \langle , \rangle_2)$ be two symplectic R-modules. Their **orthogonal sum** is the pair $(P, \langle , \rangle)$, where $P = P_1 \oplus P_2$ and the inner product is defined by $\langle (p_1, p_2), (q_1, q_2) \rangle = \langle p_1, q_1 \rangle_1 + \langle p_2, q_2 \rangle_2$.

There is a unique non-degenerate bilinear form $\langle , \rangle$ on the R-module $\mathbb{H}(R) = R \oplus R^*$, namely $\langle (a_1, b_1), (a_2, b_2) \rangle = a_1 b_2 - a_2 b_1$.

Now onwards Q **will denote** $(R^2 \perp P)$ **with induced form on** $(\mathbb{H}(R) \perp P)$**, and** $Q[X]$ **will denote** $(R[X]^2 \perp P[X])$ **with induced form on** $(\mathbb{H}(R[X]) \perp P[X])$**.**

Definition 5.7.3 An **isometry** of a symplectic module $(P, \langle , \rangle)$ is an automorphism of P which fixes the bilinear form. The group of isometries of $(P, \langle , \rangle)$ is denoted by $\mathrm{Sp}(P, \langle , \rangle)$.

Definition 5.7.4 In [7] Bass has defined a **symplectic transvection** of a symplectic module P to be an automorphism of the form

$$\sigma(p) \;=\; p + \langle u, p \rangle v + \langle v, p \rangle u + \alpha \langle u, p \rangle u,$$

where $\alpha \in R$ and $u, v \in P$ are fixed elements with $\langle u, v \rangle = 0$. It is easy to check that $\langle \sigma(p), \sigma(q) \rangle = \langle p, q \rangle$ and σ has an inverse

$$\tau(p) \quad = \quad p - \langle u, p \rangle v - \langle v, p \rangle u - \alpha \langle u, p \rangle u.$$

The subgroup of $\mathrm{Sp}(P, \langle , \rangle)$ generated by the symplectic transvections is denoted by $\mathrm{Trans}_{\mathrm{Sp}}(P, \langle , \rangle)$ (see [33], Page 35).

Definition 5.7.5 The symplectic transvections of $Q = (R^2 \perp P)$ of the form

$$\begin{aligned} (a, b, p) &\mapsto (a, b - \langle p, q \rangle + \alpha a, p + aq), \\ (a, b, p) &\mapsto (a + \langle p, q \rangle - \beta b, b, p + bq), \end{aligned}$$

where $\alpha, \beta \in R$ and $q \in P$, are called **elementary symplectic transvections**.

The elementary symplectic transvections are symplectic transvections on Q. Take $(u, v) = ((0, 1, 0), (0, 0, q))$ and $(u, v) = ((-1, 0, 0), (0, 0, q))$ respectively to get the above two transvections of Q.

The subgroup of $\mathrm{Trans}_{\mathrm{Sp}}(Q, \langle , \rangle)$ generated by elementary symplectic transvections is denoted by $\mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle , \rangle)$.

Definition 5.7.6 Let I be an ideal of R. The group of **relative symplectic transvections** w.r.t. an ideal I is generated by the symplectic transvecions of the form

$$\sigma(p) \quad = \quad p + \langle u, p \rangle v + \langle v, p \rangle u + \alpha \langle u, p \rangle u,$$

where $\alpha \in I$ and $u \in P$, $v \in IP$ are fixed elements with $\langle u, v \rangle = 0$.

The group generated by relative symplectic transvections is denoted by $\mathrm{Trans}_{\mathrm{Sp}}(P, IP, \langle , \rangle)$.

Definition 5.7.7 The elementary symplectic transvections of Q of the form

$$\begin{aligned} (a, b, p) &\mapsto (a, b - \langle p, q \rangle + \alpha a, p + aq), \\ (a, b, p) &\mapsto (a + \langle p, q \rangle - \beta b, b, p + bq), \end{aligned}$$

where $\alpha, \beta \in I$ and $q \in IP$ are called **relative elementary symplectic transvections** w.r.t. an ideal I.

The subgroup of $\mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle , \rangle)$ generated by relative elementary symplectic transvections is denoted by $\mathrm{ETrans}_{\mathrm{Sp}}(IQ, \langle , \rangle)$. The normal closure of $\mathrm{ETrans}_{\mathrm{Sp}}(IQ, \langle , \rangle)$

in $\mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle , \rangle)$ is denoted by $\mathrm{ETrans}_{\mathrm{Sp}}(Q, IQ, \langle , \rangle)$.

Definition 5.7.8 The subgroup of $\mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle , \rangle)$ generated by

$$\begin{aligned}(a, b, p) &\mapsto (a, b - \langle p, \widetilde{q}\rangle + \alpha a, p + a\widetilde{q}),\\ (a, b, p) &\mapsto (a + \langle p, q\rangle - \alpha b, b, p + bq),\end{aligned}$$

with $\widetilde{q} \in IP$, is denoted by $\mathrm{ETrans}^1_{\mathrm{Sp}}(Q, IQ, \langle , \rangle)$.

Remark 5.7.9 *Let P be a free R-module and $\langle p, q\rangle = p\varphi q^t$, where φ be an alternating matrix with Pfaffian 1.*

In this case the symplectic transvection

$$\sigma(p) = p + \langle u, p\rangle v + \langle v, p\rangle u + \alpha\langle u, p\rangle u$$

corresponds to the following matrix:

$$(I_{2n} - v^t u\varphi - u^t v\varphi)(I_{2n} - \alpha u^t u\varphi);$$

and the group generated by them is denoted by $\mathrm{Trans}_{\mathrm{Sp}}(P, \langle , \rangle_\varphi)$.

Also in this case $\mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle , \rangle_\varphi)$ *will be generated by the matrices of the form*

$$\begin{aligned}\rho_\varphi(q, \alpha) &= \begin{pmatrix} 1 & 0 & 0\\ \alpha & 1 & q\varphi\\ q^t & 0 & I_{2n}\end{pmatrix},\\ \mu_\varphi(q, \beta) &= \begin{pmatrix} 1 & -\beta & -q\varphi\\ 0 & 1 & 0\\ 0 & q^t & I_{2n}\end{pmatrix}.\end{aligned}$$

Note that for $q = (q_1, \ldots, q_{2n}) \in R^{2n}$, and for the standard alternating matrix ψ_n, we have

$$\rho_\psi(q, \alpha) = se_{21}(\alpha) \prod_{i=3}^{2n+2} se_{i1}(q_{i-2}), \tag{5.5}$$

$$\mu_\psi(q, \beta) = se_{12}(\beta) \prod_{i=3}^{2n+2} se_{1i}((-1)^i q_{\sigma(i-2)}). \tag{5.6}$$

Lemma 5.7.10 *Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Let P be a free R-module of rank $2n$, $n \geq 2$. Let $\varphi = \psi_n$, the standard alternating matrix, then*

$$\begin{aligned} \mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle , \rangle_{\psi_n}) &= \mathrm{ESp}_{2n+2}(R), \\ \mathrm{ETrans}_{\mathrm{Sp}}(Q, IQ, \langle , \rangle_{\psi_n}) &= \mathrm{ESp}_{2n+2}(R, I), \\ \mathrm{ETrans}^1_{\mathrm{Sp}}(Q, IQ, \langle , \rangle_{\psi_n}) &= \mathrm{ESp}^1_{2n+2}(R, I). \end{aligned}$$

Proof: From equations (5.5) and (5.6) it follows that

$$\mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle , \rangle_{\psi_n}) \subseteq \mathrm{ESp}_{2n+2}(R).$$

Using the commutator identities for the (standard) elementary generators of the group $\mathrm{ESp}_{2n+2}(R)$ it follows that $\mathrm{ESp}_{2n+2}(R)$ is generated by the elements $se_{1i}(a), se_{j1}(b)$, $1 < i \neq j \leq 2n$, $a, b \in R$, and hence $\mathrm{ESp}_{2n+2}(R) \subseteq \mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle , \rangle_{\psi_n})$. Therefore the first equality holds.

To show the second equality let us first show $\mathrm{ETrans}_{\mathrm{Sp}}(Q, IQ, \langle , \rangle_{\psi_n})$ is a subset of $\mathrm{ESp}_{2n+2}(R, I)$. It is enough to show that an element of the form

$$t_{\psi_n}(q_1, \alpha)\ s_{\psi_n}(q_2, \beta)\ t_{\psi_n}(q_1, \alpha)^{-1}$$

is in $\mathrm{ESp}_{2n}(R, I)$, for $q_1 \in R^{2n}$, $q_2 \in I^{2n}$, $\alpha \in R$ and $\beta \in I$. Here t_{ψ_n} and s_{ψ_n} are ρ_{ψ_n} or μ_{ψ_n}. Using the definition of $\mathrm{ESp}_{2n}(R, I)$ and equations (5.5), (5.6) we get

$$t_{\psi_n}(q_1, \alpha)\ s_{\psi_n}(q_2, \beta)\ t_{\psi_n}(q_1, \alpha)^{-1} \in \mathrm{ESp}_{2n}(R, I)$$

and hence

$$\mathrm{ETrans}_{\mathrm{Sp}}(Q, IQ, \langle , \rangle_{\psi_n}) \subseteq \mathrm{ESp}_{2n}(R, I).$$

To show the other inclusion we recall the equivalent definition of the relative group which says that $\mathrm{ESp}_{2n}(R, I)$ is the smallest normal subgroup of $\mathrm{ESp}_{2n}(R)$ containing $se_{21}(x)$, where $x \in I$ (see Lemma 2.2.29). We need to show

$$g\ se_{21}(x)\ g^{-1} \in \mathrm{ETrans}_{\mathrm{Sp}}(Q, IQ, \langle , \rangle_{\psi_n}),$$

where $g \in \mathrm{ESp}_{2n}(R) = \mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle , \rangle_{\psi_n})$. Therefore $g \; se_{21}(x) \; g^{-1}$ belongs to $\mathrm{ETrans}_{\mathrm{Sp}}(Q, IQ, \langle , \rangle_{\psi_n})$ and

$$\mathrm{ESp}_{2n}(R, I) \;\; \subseteq \;\; \mathrm{ETrans}_{\mathrm{Sp}}(Q, IQ, \langle , \rangle_{\psi_n}),$$

and hence the second equality is established.

Generators of $\mathrm{ETrans}^1_{\mathrm{Sp}}(Q, IQ, \langle , \rangle_{\psi_n})$ is of the form $\rho_{\psi_n}(q_1, \alpha)$, $\mu_{\psi_n}(q_2, \beta)$, where $q_1 \in I^{2n-1}$, $\alpha \in I$, $q_2 \in R^{2n-1}$ and $\beta \in R$. By equations (5.5) and (5.6) we have $\rho_{\psi_n}(q_1, \alpha), \mu_{\psi_n}(q_2, \beta) \in \mathrm{ESp}^1_{2n}(R, I)$, and hence

$$\mathrm{ETrans}^1_{\mathrm{Sp}}(Q, IQ, \langle , \rangle_{\psi_n}) \;\; \subseteq \;\; \mathrm{ESp}^1_{2n}(R, I).$$

On the other hand generators of $\mathrm{ESp}^1_{2n}(R, I)$ are of the form $se_{1i}(a), se_{j1}(x)$, where $a \in R$ and $x \in I$. Using equations (5.5) and (5.6) we get $se_{1i}(a), se_{j1}(x) \in \mathrm{ETrans}^1_{\mathrm{Sp}}(Q, IQ, \langle , \rangle_{\psi_n})$, and hence $\mathrm{ESp}^1_{2n}(R, I) \subseteq \mathrm{ETrans}^1_{\mathrm{Sp}}(Q, IQ, \langle , \rangle_{\psi_n})$. Therefore the third equality is established. □

Lemma 5.7.11 *Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Let P be a free R-module of rank $2n$, $n \geq 2$. Let $\varphi = \psi_n$, the standard alternating matrix, then*

$$\begin{aligned} \mathrm{Trans}_{\mathrm{Sp}}(Q, \langle , \rangle_{\psi_1 \perp \psi_n}) &= \mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle , \rangle_{\psi_n}) = \mathrm{ESp}_{2n+2}(R), \\ \mathrm{Trans}_{\mathrm{Sp}}(Q, IQ, \langle , \rangle_{\psi_1 \perp \psi_n}) &= \mathrm{ETrans}_{\mathrm{Sp}}(Q, IQ, \langle , \rangle_{\psi_n}) = \mathrm{ESp}_{2n+2}(R, I), \end{aligned}$$

Proof: Using Lemma 2.2.22 and Lemma 2.2.23 it follows that

$$\begin{aligned} \mathrm{Trans}_{\mathrm{Sp}}(Q, \langle , \rangle_{\psi_1 \perp \psi_n}) &\subseteq \mathrm{ESp}_{2n+2}(R), \\ \mathrm{Trans}_{\mathrm{Sp}}(Q, IQ, \langle , \rangle_{\psi_1 \perp \psi_n}) &\subseteq \mathrm{ESp}_{2n+2}(R, I). \end{aligned}$$

Also

$$\begin{aligned} \mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle , \rangle_{\psi_n}) &\subseteq \mathrm{Trans}_{\mathrm{Sp}}(Q, \langle , \rangle_{\psi_1 \perp \psi_n}), \\ \mathrm{ETrans}_{\mathrm{Sp}}(Q, IQ, \langle , \rangle_{\psi_n}) &\subseteq \mathrm{Trans}_{\mathrm{Sp}}(Q, IQ, \langle , \rangle_{\psi_1 \perp \psi_n}). \end{aligned}$$

Therefore using previous lemma we have

$$\mathrm{ESp}_{2n+2}(R) \; = \; \mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle , \rangle_{\psi_n}) \subseteq \mathrm{Trans}_{\mathrm{Sp}}(Q, \langle , \rangle_{\psi_1 \perp \psi_n}) \; \subseteq \; \mathrm{ESp}_{2n+2}(R),$$

and hence the first sequence of equalities follow. The second sequence of equalities follow similarly. □

Lemma 5.7.12 *Let P be a free R-module of rank $2n$. Let $(P, \langle , \rangle_{\varphi})$ and $(P, \langle , \rangle_{\varphi^*})$ be two symplectic R-modules with $\varphi = (1 \perp \varepsilon)^t\ \varphi^*\ (1 \perp \varepsilon)$, for some $\varepsilon \in \mathrm{E}_{2n-1}(R)$. Then*

$$\begin{aligned}\mathrm{Trans}_{\mathrm{Sp}}(P, \langle , \rangle_{\varphi}) &= (1 \perp \varepsilon)^{-1}\ \mathrm{Trans}_{\mathrm{Sp}}(P, \langle , \rangle_{\varphi^*})\ (1 \perp \varepsilon),\\ \mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle , \rangle_{\varphi}) &= (I_3 \perp \varepsilon)^{-1}\ \mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle , \rangle_{\varphi^*})\ (I_3 \perp \varepsilon)).\end{aligned}$$

Proof: In the free case for symplectic transvections we have

$$\begin{aligned}&(I_{2n} - v^t u\varphi - u^t v\varphi)(I_{2n} - \alpha u^t u\varphi)\\ = \ &(1 \perp \varepsilon)^{-1}\ (I_{2n} - \tilde{v}^t\tilde{u}\varphi^* - \tilde{u}^t\tilde{v}\varphi^*)\ (I_{2n} - \alpha\tilde{u}^t\tilde{u}\varphi^*)\ (1 \perp \varepsilon),\end{aligned}$$

where $\tilde{u} = u(1 \perp \varepsilon)^t$ and $\tilde{v} = v(1 \perp \varepsilon)^t$. Hence the first equality follows.

For elementary symplectic transvections we have

$$\begin{aligned}&(I_2 \perp (1 \perp \varepsilon))^{-1}\rho_{\varphi^*}(q, \alpha)(I_2 \perp (1 \perp \varepsilon))\\ = \ &(I_3 \perp \varepsilon)^{-1}\rho_{\varphi^*}(q, \alpha)(I_3 \perp \varepsilon)\\ = \ &\rho_{\varphi}(q(1 \perp \varepsilon^t)^{-1}, \alpha),\end{aligned}$$

and

$$\begin{aligned}&(I_2 \perp (1 \perp \varepsilon))^{-1}\mu_{\varphi^*}(q, \beta)(I_2 \perp (1 \perp \varepsilon))\\ = \ &(I_3 \perp \varepsilon)^{-1}\mu_{\varphi^*}(q, \beta)(I_3 \perp \varepsilon)\\ = \ &\mu_{\varphi}(q(1 \perp \varepsilon^t)^{-1}, \beta),\end{aligned}$$

and hence the second equality follows. □

Lemma 5.7.13 *Let I be an ideal of R and P be a free R-module of rank $2n$. Let $(P, \langle , \rangle_{\varphi})$ and $(P, \langle , \rangle_{\varphi^*})$ be two symplectic R-modules with $\varphi = (1 \perp \varepsilon)^t\ \varphi^*\ (1 \perp \varepsilon)$, for some $\varepsilon \in \mathrm{E}_{2n-1}(R, I)$. Then*

$$\begin{aligned}\mathrm{Trans}_{\mathrm{Sp}}(P, IP, \langle , \rangle_{\varphi}) &= (1 \perp \varepsilon)^{-1}\ \mathrm{Trans}_{\mathrm{Sp}}(P, IP, \langle , \rangle_{\varphi^*})\ (1 \perp \varepsilon),\\ \mathrm{ETrans}_{\mathrm{Sp}}(Q, IQ, \langle , \rangle_{\varphi}) &= (I_3 \perp \varepsilon)^{-1}\ \mathrm{ETrans}_{\mathrm{Sp}}(Q, IQ, \langle , \rangle_{\varphi^*})\ (I_3 \perp \varepsilon),\\ \mathrm{ETrans}^1_{\mathrm{Sp}}(Q, IQ, \langle , \rangle_{\varphi}) &= (I_3 \perp \varepsilon)^{-1}\ \mathrm{ETrans}^1_{\mathrm{Sp}}(Q, IQ, \langle , \rangle_{\varphi^*})\ (I_3 \perp \varepsilon).\end{aligned}$$

Proof: Using the three equations appear in the proof of Lemma 5.7.12, we get these equalities. □

Proposition 5.7.14 *Let $(P, \langle , \rangle_\varphi)$ be a symplectic R-module with P free of rank 2n. Let $\varphi = (1 \perp \varepsilon)^t\ \psi_n\ (1 \perp \varepsilon)$, for some $\varepsilon \in \mathrm{E}_{2n-1}(R)$. Then*

$$\mathrm{Trans}_{\mathrm{Sp}}(Q, \langle , \rangle_{\psi_1 \perp \varphi}) \;=\; \mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle , \rangle_\varphi) \;=\; \mathrm{ESp}_{\psi_1 \perp \varphi}(R).$$

Proof: Using Lemma 5.1.5, Lemma 5.1.6, Lemma 5.7.10 and Lemma 5.7.12 we get,

$$\begin{aligned}
\mathrm{Trans}_{\mathrm{Sp}}(Q, \langle , \rangle_{\psi_1 \perp \varphi}) &= (I_3 \perp \varepsilon)^{-1}\ \mathrm{Trans}_{\mathrm{Sp}}(Q, \langle , \rangle_{\psi_1 \perp \psi_n})\ (I_3 \perp \varepsilon) \\
&= (I_3 \perp \varepsilon)^{-1}\ \mathrm{ESp}_{2+2n}(R)\ (I_3 \perp \varepsilon) \\
&= (I_3 \perp \varepsilon)^{-1}\ \mathrm{ESp}_{\psi_1 \perp \psi_n}(R)\ (I_3 \perp \varepsilon) \\
&= \mathrm{ESp}_{\psi_1 \perp \varphi}(R),
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle , \rangle_{\varphi}) &= (I_3 \perp \varepsilon)^{-1}\ \mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle , \rangle_{\psi_n})\ (I_3 \perp \varepsilon) \\
&= (I_3 \perp \varepsilon)^{-1}\ \mathrm{ESp}_{2+2n}(R)\ (I_3 \perp \varepsilon) \\
&= (I_3 \perp \varepsilon)^{-1}\ \mathrm{ESp}_{\psi_1 \perp \psi_n}(R)\ (I_3 \perp \varepsilon)) \\
&= \mathrm{ESp}_{\psi_1 \perp \varphi}(R),
\end{aligned}$$

and hence the sequence of equalities are established. □

Now we state and prove a relative version of the above proposition.

Proposition 5.7.15 *Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Let $(P, \langle , \rangle_\varphi)$ be a symplectic R-module with P free of rank 2n, $n \geq 2$. Let $\varphi = (1 \perp \varepsilon)^t\ \psi_n\ (1 \perp \varepsilon)$, for some $\varepsilon \in \mathrm{E}_{2n-1}(R, I)$. Then*

$$\mathrm{Trans}_{\mathrm{Sp}}(Q, IQ, \langle , \rangle_{\psi_1 \perp \varphi}) \;=\; \mathrm{ETrans}_{\mathrm{Sp}}(Q, IQ, \langle , \rangle_\varphi) \;=\; \mathrm{ESp}_{\psi_1 \perp \varphi}(R, I).$$

Proof: Using Lemma 5.1.5, Lemma 5.1.7, Lemma 5.7.10 and Lemma 5.7.13 we get,

$$\begin{aligned}
\text{Trans}_{\text{Sp}}(Q, IQ, \langle , \rangle_{\psi_1 \perp \varphi}) &= (I_3 \perp \varepsilon)^{-1} \, \text{Trans}_{\text{Sp}}(Q, IQ, \langle , \rangle_{\psi_1 \perp \psi_n}) \, (I_3 \perp \varepsilon) \\
&= (I_3 \perp \varepsilon)^{-1} \, \text{ESp}_{2+2n}(R, I) \, (I_3 \perp \varepsilon) \\
&= (I_3 \perp \varepsilon)^{-1} \, \text{ESp}_{\psi_1 \perp \psi_n}(R, I) \, (I_3 \perp \varepsilon) \\
&= \text{ESp}_{\psi_1 \perp \varphi}(R, I),
\end{aligned}$$

and

$$\begin{aligned}
\text{ETrans}_{\text{Sp}}(Q, IQ, \langle , \rangle_{\varphi}) &= (I_3 \perp \varepsilon)^{-1} \, \text{ETrans}_{\text{Sp}}(Q, IQ, \langle , \rangle_{\psi_n}) \, (I_3 \perp \varepsilon) \\
&= (I_3 \perp \varepsilon)^{-1} \, \text{ESp}_{2+2n}(R, I) \, (I_3 \perp \varepsilon) \\
&= (I_3 \perp \varepsilon)^{-1} \, \text{ESp}_{\psi_1 \perp \psi_n}(R, I) \, (I_3 \perp \varepsilon)) \\
&= \text{ESp}_{\psi_1 \perp \varphi}(R, I),
\end{aligned}$$

and hence the sequence of equalities are established. □

Remark 5.7.16 *In view of above two lemmas, for any symplectic module $(P, \langle , \rangle_{\varphi})$ over a local ring $(R, \mathfrak{m})$, we have*

$$\begin{aligned}
\text{Trans}_{\text{Sp}}(Q, \langle , \rangle_{\psi_1 \perp \varphi}) &= \text{ETrans}_{\text{Sp}}(Q, \langle , \rangle_{\varphi}) = \text{ESp}_{\psi_1 \perp \varphi}(R), \\
\text{Trans}_{\text{Sp}}(Q, IQ, \langle , \rangle_{\psi_1 \perp \varphi}) &= \text{ETrans}_{\text{Sp}}(Q, IQ, \langle , \rangle_{\varphi}) = \text{ESp}_{\psi_1 \perp \varphi}(R, I).
\end{aligned}$$

Here I is an ideal of the ring R.

5.8 Dilation for Elementary Symplectic Transvections

Proposition 5.8.1 *Let $(P, \langle , \rangle)$ be a symplectic R-module with P finitely generated projective R-module of rank $2n$, $n \geq 1$. Let $a \in R$ be non-nilpotent and $(P_a, \langle , \rangle_{\varphi})$ be a symplectic module with P_a be a free R_a-module. Also let $\varphi = (1 \perp \varepsilon)^t \, \psi_n \, (1 \perp \varepsilon)$, for some $\varepsilon \in \text{E}_{2n-1}(R_a)$ over the ring R_a. Let $\alpha(X) \in \text{ETrans}_{\text{Sp}}(Q_a[X], \langle , \rangle_{\varphi})$, with $\alpha(0) = Id$. Then there exists $\alpha^*(X) \in \text{ETrans}_{\text{Sp}}(Q[X], \langle , \rangle)$ such that $\alpha^*(X)$ localises to $\alpha(bX)$ for some $b \in (a^N)$, $N \gg 0$, and $\alpha^*(0) = Id$.*

Proof: Let $\alpha(X) = \prod_{l=1}^{s} t_{\varphi}(g_l(X), \alpha_l(X))$, where t_{φ} is either ρ_{φ} or μ_{φ} and $g_l(X) \in (R_a[X])^{2n}$, $\alpha_l(X) \in R_a[X])$. Having $\varphi = (1 \perp \varepsilon)^t \, \psi_n \, (1 \perp \varepsilon)$, with some $\varepsilon \in$

$\mathrm{E}_{2n-1}(R_a)$, will allow us to write

$$\begin{aligned}\alpha(X) &= \prod_{l=1}^{s}(I_3 \perp \varepsilon)^{-1} t_{\psi_n}(f_l(X), \alpha_l(X))(I_3 \perp \varepsilon) \\ &= (I_3 \perp \varepsilon)^{-1}\Big(\prod_{l=1}^{s} t_{\psi_n}(f_l(X), \alpha_l(X))\Big)(I_3 \perp \varepsilon) \\ &= (I_3 \perp \varepsilon)^{-1}\eta(X)(I_3 \perp \varepsilon),\end{aligned}$$

where $f_l(X) = g_l(X)(1 \perp \varepsilon^t)$, and $\eta(X) \in \mathrm{ESp}_{2n+2}(R_a[X])$ (see Lemma 5.7.10 and Lemma 5.7.12). Note that $\eta(0) = Id$, as $\alpha(0) = Id$. Therefore,

$$\eta(X) = \prod_{k=1}^{r} \gamma_k \; se_{i_k j_k}(X h_k(X)) \; \gamma_k^{-1},$$

where $f_k(X) = f(0) + X h_k(X)$ and $\gamma_k \in \mathrm{ESp}_{2n+2}(R_a)$ (see Lemma 2.2.27). Now,

$$\begin{aligned}\eta(Y^{2^{r+1}}X) &= \prod_{k=1}^{r} \gamma_k se_{i_k j_k}(Y^{2^{r+1}} X h_k(Y^{2^{r+1}}X))\gamma_k^{-1} \\ &= \prod_{t=1}^{l} se_{p_t q_t}(Y^2 u_t(X,Y) \\ &= \prod_{t=1}^{l}[se_{p_t 1}(Y), se_{1 q_t}(Y u_t(X,Y)],\end{aligned}$$

where $u_t(X,Y) \in R_a[X,Y]$. The second equality above follows from Lemma 2.2.28. Taking $N = M^{2^{r+1}}$ we get,

$$\begin{aligned}\alpha(a^N X Y^{2^{r+1}}) &= \alpha((a^M Y)^{2^{r+1}} X) \\ &= (I_3 \perp \varepsilon)^{-1}\eta((a^M Y)^{2^{r+1}} X)(I_3 \perp \varepsilon) \\ &= (I_3 \perp \varepsilon)^{-1}\prod_{t=1}^{s}[se_{p_t 1}(a^M Y), se_{1 q_t}(a^M Y u_t(X,Y))](I_3 \perp \varepsilon).\end{aligned}$$

Note that

$$se_{p_t 1}(a^M Y) = \begin{cases}\rho_{\psi_n}(0, a^M Y) & \text{if } \; p_t = 2, \\ \rho_{\psi_n}(a^M Y e_{p_t - 2}, 0) & \text{if } \; p_t \geq 3,\end{cases}$$

and

$$se_{1q_t}(a^M Y u_t(X,Y)) = \begin{cases} \mu_{\psi_n}(0, a^M Y u_t(X,Y)) & \text{if } \ q_t = 2, \\ \mu_{\psi_n}(a^M Y u_t(X,Y) e_{\sigma(q_t-2)}, 0) & \text{if } \ q_t \geq 3. \end{cases}$$

Also

$$\begin{gathered} (I_3 \perp \varepsilon)^{-1}) \rho_{\psi_n}(0, a^M Y)(I_3 \perp \varepsilon) \\ = \rho_{\varphi}(0, a^M Y), \\ (I_3 \perp \varepsilon)^{-1}) \rho_{\psi_n}(a^M Y e_{p_t-2}, 0)(I_3 \perp \varepsilon) \\ = \rho_{\varphi}(a^M Y e_{p_t-2}(1 \perp \varepsilon^t)^{-1}), 0), \end{gathered}$$

and

$$\begin{gathered} (I_3 \perp \varepsilon)^{-1}) \mu_{\psi_n}(0, a^M Y u_t(X,Y))(I_3 \perp \varepsilon) \\ = \mu_{\varphi}(0, a^M Y u_t(X,Y)), \\ (I_3 \perp \varepsilon)^{-1}) \mu_{\psi_n}(a^M Y u_t(X,Y) e_{\sigma(q_t-2)}, 0)(I_3 \perp \varepsilon) \\ = \mu_{\varphi}(a^M Y u_t(X,Y) e_{\sigma(q_t-2)}(1 \perp \varepsilon^t)^{-1}). \end{gathered}$$

Let us fix some notations here.

$$\overline{\rho}(p_t) = \begin{cases} \rho_{\varphi}(0, a^M Y) & \text{if } p_t = 2, \\ \rho_{\varphi}(a^M Y e_{p_t-2}(1 \perp \varepsilon^t)^{-1}), 0) & \text{if } p_t \geq 3, \end{cases}$$

and

$$\overline{\mu}(q_t) = \begin{cases} \mu_{\varphi}(0, a^M Y u_t(X,Y)) & \text{if } q_t = 2, \\ \mu_{\varphi}(a^M Y u_t(X,Y) e_{\sigma(q_t-2)}(1 \perp \varepsilon^t)^{-1}), 0) & \text{if } q_t \geq 3. \end{cases}$$

Note that

$$\alpha(a^N X Y^{2^{r+1}}) \quad = \quad \prod_{t=1}^{s} [\overline{\rho}(p_t), \overline{\mu}(q_t)].$$

Let $\alpha^*(X) \in \mathrm{ETrans}_{\mathrm{Sp}}(Q[X], \langle , \rangle)$ be such that

$$\alpha^*(X)_a \;=\; \prod_{t=1}^{s} [\overline{\rho}(p_t), \overline{\mu}(q_t)].$$

Using Lemma 2.2.4 we can claim that $\alpha^*(X)$ localises to $\alpha(bX)$, for some $b \in (a^N)$, $N \gg 0$, and $\alpha^*(0) = Id$. $\square$

Next we state and prove a relative version of Proposition 5.8.1. We can prove Proposition 5.8.1 in the way we prove Proposition 5.8.2, without involving commutator identities.

Proposition 5.8.2 *Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Let $(P, \langle , \rangle)$ be a symplectic R-module with P finitely generated projective R-module of rank $2n$, $n \geq 2$. Let $a \in R$ be non-nilpotent and $(P_a, \langle , \rangle_\varphi)$ be a symplectic module with P_a be a free R_a-module. Also let $\varphi = (1 \perp \varepsilon)^t \; \psi_n \; (1 \perp \varepsilon)$, for some $\varepsilon \in \mathrm{E}_{2n-1}(R_a, I_a)$ over the ring R_a. Let $\alpha(X) \in \mathrm{ETrans}_{\mathrm{Sp}}(Q_a[X], IQ_a[X], \langle , \rangle_\varphi)$, with $\alpha(0) = Id$. Then there exists $\alpha^*(X) \in \mathrm{ETrans}_{\mathrm{Sp}}(Q[X], IQ[X], \langle , \rangle)$ such that $\alpha^*(X)$ localises to $\alpha(bX)$, for some $b \in (a^N)$, $N \gg 0$, and $\alpha^*(0) = Id$.*

Proof: By Lemma 5.7.15 we have

$$\mathrm{ETrans}_{\mathrm{Sp}}(Q_a[X], IQ_a[X], \langle , \rangle_\varphi) \;=\; \mathrm{ESp}_{\psi_1 \perp \varphi}(R_a[X], I_a[X]),$$

and using dilation principle for $\mathrm{ESp}_{\psi_1 \perp \varphi}(R[X], I[X])$ (see Lemma 5.2.2) we get a $\beta(X) \in \mathrm{ESp}_{\psi_1 \perp \varphi}(R[X], I[X])$ such that $\beta(X)_a = \alpha(bX)$, for some $b \in (a^N)$. Now we choose a $\alpha^*(X)$ from $\mathrm{ETrans}_{\mathrm{Sp}}(Q[X], IQ[X], \langle , \rangle)$ such that $\alpha^*(X)_a = \beta(X)_a$. Using Lemma 2.2.4 we claim that $\alpha^*(X)$ localises to $\alpha(bX)$, for some $b \in (a^N)$, $N \gg 0$, and $\alpha^*(0) = Id$. $\square$

5.9 Local Global principle for $\mathrm{ETrans}_{\mathrm{Sp}}(Q)$

Lemma 5.9.1 *Let $(P, \langle , \rangle)$ be a symplectic R-module with P finitely generated projective module of rank $2n$, $n \geq 1$. Let $\alpha(X) \in \mathrm{Sp}(Q[X], \langle . \rangle)$, with $\alpha(0) = Id$. If for each maximal ideal $\mathfrak{m}$ of R, $\alpha(X)_\mathfrak{m} \in \mathrm{ETrans}_{\mathrm{Sp}}(Q_\mathfrak{m}[X], \langle , \rangle_{\varphi_\mathfrak{m}})$, then $\alpha(X) \in \mathrm{ETrans}_{\mathrm{Sp}}(Q[X], \langle , \rangle)$.* $\square$

Next we state and prove a relative version of the above lemma. Lemma 5.9.1 can be treated as a particular case of Lemma 5.9.2, when $I[X] = R[X]$.

Lemma 5.9.2 *Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Let $(P, \langle , \rangle)$ be a symplectic R-module with P finitely generated projective module of rank $2n$, $n \geq 2$. Let $\alpha(X) \in \mathrm{Sp}(Q[X], \langle . \rangle)$, with $\alpha(0) = Id$. If for each maximal ideal $\mathfrak{m}$ of R, $\alpha(X)_{\mathfrak{m}} \in \mathrm{ETrans}_{\mathrm{Sp}}(Q_{\mathfrak{m}}[X], IQ_{\mathfrak{m}}[X], \langle , \rangle_{\varphi_{\mathfrak{m}}})$, where $\varphi_{\mathfrak{m}} \equiv \psi_n \pmod{I}$, then*

$$\alpha(X) \quad \in \quad \mathrm{ETrans}_{\mathrm{Sp}}(Q[X], IQ[X], \langle , \rangle).$$

Proof: Let $\mathfrak{m}$ be a maximal ideal of R. One can suitably choose an element $a_{\mathfrak{m}}$ from $R \setminus \mathfrak{m}$ such that $\alpha(X)_{a_{\mathfrak{m}}} \in \mathrm{ETrans}_{\mathrm{Sp}}(Q_{a_{\mathfrak{m}}}[X], IQ_{a_{\mathfrak{m}}}[X], \langle , \rangle_{\varphi_{a_{\mathfrak{m}}}})$, where $\varphi_{a_{\mathfrak{m}}}$ is the alternating matrix with Pfaffian 1 corresponding to the alternating form $\langle , \rangle_{\varphi_{a_{\mathfrak{m}}}}$. Also $\varphi_{\mathfrak{m}} = (1 \perp \varepsilon)^t \ \psi_n \ (1 \perp \varepsilon)$, for some $\varepsilon \in \mathrm{E}_{2n-1}(R_{a_{\mathfrak{m}}}, I_{a_{\mathfrak{m}}})$. Let us define

$$\beta(X, Y) \quad = \quad \alpha(X + Y)_{a_{\mathfrak{m}}} \ \alpha(Y)_{a_{\mathfrak{m}}}^{-1}.$$

Clearly

$$\beta(X, Y) \quad \in \quad \mathrm{ETrans}_{\mathrm{Sp}}(Q_{a_{\mathfrak{m}}}[X, Y], IQ_{a_{\mathfrak{m}}}[X, Y], \langle , \rangle_{\varphi_{a_{\mathfrak{m}}}}),$$

and $\beta(0, Y) = Id$. Therefore $\beta(b_{\mathfrak{m}}X, Y) \in \mathrm{ETrans}_{\mathrm{Sp}}(Q[X, Y], IQ[X, Y], \langle , \rangle)$, where $b_{\mathfrak{m}} \in (a_{\mathfrak{m}}^N)$ for $N \gg 0$ (see Proposition 5.8.2). The ideal generated by $b_{\mathfrak{m}}$'s is the whole ring R. Therefore we have $c_1 b_{\mathfrak{m}_1} + \cdots + c_k b_{\mathfrak{m}_k} = 1$, where $c_i \in R$, for $1 \leq i \leq k$. Note that $\beta(c_i b_{\mathfrak{m}_i} X, Y) \in \mathrm{ETrans}_{\mathrm{Sp}}(Q[X, Y], IQ[X, Y], \langle , \rangle)$, for $1 \leq i \leq k$. Now,

$$\alpha(X) \quad = \quad \prod_{i=1}^{k} \beta(b_{\mathfrak{m}_i} X, T_i) \ \beta(b_{\mathfrak{m}_k}, 0) \ \in \ \mathrm{ETrans}_{\mathrm{Sp}}(Q[X], IQ[X], \langle , \rangle)$$

where $T_i = c_{i+1} b_{\mathfrak{m}_{i+1}} X + \cdots + c_k b_{\mathfrak{m}_k} X$. □

Here we state and prove action version of above Local Global principle.

Theorem 5.9.3 *Let $(P, \langle , \rangle)$ be a symplectic R-module with P finitely generated projective module of rank $2n$, $n \geq 1$. Let $q(X) = (a(X), b(X), p(X)) \in \mathrm{Um}(Q[X])$. If for each maximal ideal $\mathfrak{m}$ of R, $q(X) \in q(0)\mathrm{ETrans}_{\mathrm{Sp}}(Q_{\mathfrak{m}}[X], \langle , \rangle_{\varphi_{\mathfrak{m}}})$, then*

$$q(X) \quad \in \quad q(0) \ \mathrm{ETrans}_{\mathrm{Sp}}(Q[X], \langle , \rangle).$$

□

Next we state and prove a relative version of the above theorem. Theorem 5.9.3 can be treated as a particular case of Theorem 5.9.4, when $I[X] = R[X]$.

Theorem 5.9.4 *Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Let $(P, \langle , \rangle)$ be a symplectic R-module with P finitely generated projective module of rank $2n$, $n \geq 2$. Let $q(X) = (a(X), b(X), p(X))$ is in $\mathrm{Um}(Q[X], IQ[X])$. If for each maximal ideal $\mathfrak{m}$ of R, we have $q(X) \in q(0)\mathrm{ETrans}_{\mathrm{Sp}}(Q_{\mathfrak{m}}[X], IQ_{\mathfrak{m}}[X], \langle , \rangle_{\varphi_{\mathfrak{m}}})$, where $\varphi_{\mathfrak{m}} \equiv \psi_n \pmod{I}$, then*

$$q(X) \in q(0)\ \mathrm{ETrans}_{\mathrm{Sp}}(Q[X], IQ[X], \langle , \rangle).$$

Proof: For each maximal ideal $\mathfrak{m}$ of R, we will find an element $\alpha_{(\mathfrak{m})}(X)$ from $\mathrm{ETrans}_{\mathrm{Sp}}(Q_{\mathfrak{m}}[X], IQ_{\mathfrak{m}}[X], \langle , \rangle_{\varphi \otimes R_{\mathfrak{m}}[X]})$ such that

$$q(X)\ \alpha_{(\mathfrak{m})}(X) = q(0).$$

Let us define

$$\beta(X,T) = \alpha_{(\mathfrak{m})}(X+T)\ \alpha_{(\mathfrak{m})}(X)^{-1}.$$

Clearly $\beta(X,T)$ is in $\mathrm{ETrans}_{\mathrm{Sp}}(Q_{\mathfrak{m}}[X,T], IQ_{\mathfrak{m}}[X,T], \langle , \rangle_{\varphi_{\mathfrak{m}}})$. Since there are only finitely many denominators, there exists $a_{\mathfrak{m}} \in R \setminus \mathfrak{m}$ such that $\beta(X,T)$ is in $\mathrm{ETrans}_{\mathrm{Sp}}(Q_{a_{\mathfrak{m}}}[X,T], IQ_{\mathfrak{m}}[X,T], \langle , \rangle_{\varphi_{a_{\mathfrak{m}}}})$. Also $\beta(X,0) = Id$. This implies $\beta(X, b_{\mathfrak{m}}T)$ is in $\mathrm{ETrans}_{\mathrm{Sp}}(Q[X,T], IQ[X,T], \langle , \rangle)$, for suitable $b_{\mathfrak{m}} \in (a_{\mathfrak{m}}^N)$ (see Proposition 5.8.2). Now,

$$\begin{aligned} q(X + b_{\mathfrak{m}}T)\ \beta(X, b_{\mathfrak{m}}T) &= q(X + b_{\mathfrak{m}}T)\ \alpha_{(\mathfrak{m})}(X + b_{\mathfrak{m}}T)\ \alpha_{(\mathfrak{m})}(X)^{-1} \\ &= q(0)\ \alpha_{(\mathfrak{m})}(X)^{-1} \\ &= q(X) \end{aligned}$$

Note that the ideal generated by $b_{\mathfrak{m}}$'s is the whole ring R. Therefore $c_1 b_{\mathfrak{m}_1} + \cdots + c_k b_{\mathfrak{m}_k} = 1$, where $c_i \in R$, for $1 \leq i \leq k$. In the above equation replacing $b_{\mathfrak{m}}T$ by $c_1 b_{\mathfrak{m}_1} X$ and X by $c_2 b_{\mathfrak{m}_2} X + \cdots + c_k b_{\mathfrak{m}_k} X$ we get,

$$\begin{aligned} q(X) &= q(c_1 b_{\mathfrak{m}_1} X + \cdots + c_k b_{\mathfrak{m}_k} X) \\ &\in q(c_2 b_{\mathfrak{m}_2} X + \cdots + c_k b_{\mathfrak{m}_k} X)\ \mathrm{ETrans}_{\mathrm{Sp}}(Q[X,T], IQ[X,T], \langle , \rangle). \end{aligned}$$

In the above equation replacing $b_{\mathfrak{m}}T$ by $c_2 b_{\mathfrak{m}_2} X$ and X by $c_3 b_{\mathfrak{m}_3} X + \cdots + c_k b_{\mathfrak{m}_k} X$ we get

$$\begin{aligned} & q(c_2 b_{\mathfrak{m}_2} X + \cdots + c_k b_{\mathfrak{m}_k} X) \\ \in \;& q(c_3 b_{\mathfrak{m}_3} X + \cdots + c_k b_{\mathfrak{m}_k} X) \ \mathrm{ETrans}_{\mathrm{Sp}}(Q[X,T], IQ[X,T], \langle , \rangle). \end{aligned}$$

Continuing in this way we get

$$q(X) \in q(0) \ \mathrm{ETrans}_{\mathrm{Sp}}(Q[X], IQ[X], \langle , \rangle).$$

□

5.10 Equality of $\mathrm{Trans}_{\mathrm{Sp}}(Q, \langle , \rangle)$, $\mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle , \rangle)$ and $\mathrm{ESp}_{\varphi}(R)$

In this section we establish equality of the above mentioned groups.

Theorem 5.10.1 *Let $(P, \langle , \rangle)$ be a symplectic R-module with P finitely generated projective module of rank $2n$, $n \geq 1$. Then*

$$\mathrm{Trans}_{\mathrm{Sp}}(Q, \langle , \rangle) = \mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle , \rangle).$$

Proof: By definition $\mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle , \rangle) \subseteq \mathrm{Trans}_{\mathrm{Sp}}(Q, \langle , \rangle)$. We need to show $\mathrm{Trans}_{\mathrm{Sp}}(Q, \langle , \rangle) \subseteq \mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle , \rangle)$. Let $\alpha \in \mathrm{Trans}_{\mathrm{Sp}}(Q, \langle , \rangle)$. There exists $\alpha(X)$ in $\mathrm{Trans}_{\mathrm{Sp}}(Q[X], \langle , \rangle_{\varphi \otimes R[X]})$ such that $\alpha(1) = \alpha$ and $\alpha(0) = Id$. For each maximal ideal $\mathfrak{m}$ of R, one has

$$\mathrm{Trans}_{\mathrm{Sp}}(Q_{\mathfrak{m}}[X], \langle , \rangle_{\psi_1 \perp \varphi_{\mathfrak{m}}}) = \mathrm{ETrans}_{\mathrm{Sp}}(Q_{\mathfrak{m}}[X], \langle , \rangle_{\varphi_{\mathfrak{m}}})$$

(see Remark 5.7.16). Hence $\alpha(X)_{\mathfrak{m}} \in \mathrm{ETrans}_{\mathrm{Sp}}(Q_{\mathfrak{m}}[X], \langle , \rangle_{\varphi_{\mathfrak{m}}})$, for each maximal ideal $\mathfrak{m}$ of R. Using Lemma 5.9.1 we get $\alpha(X)$ is in $\mathrm{ETrans}_{\mathrm{Sp}}(Q[X], \langle , \rangle)$ and hence substituting $X = 1$ we get $\alpha \in \mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle , \rangle)$. □

Theorem 5.10.2 *Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Let $(P, \langle , \rangle)$ be a symplectic R-module with P finitely generated projective module of rank $2n$, $n \geq 2$. Also assume for any maximal ideal $\mathfrak{m}$ of R, the alternating form $\langle , \rangle$ corresponds to the alternating matrix $\varphi_{\mathfrak{m}}$, where $\varphi_{\mathfrak{m}} \equiv \psi_n \pmod{I}$, over the ring*

$R_{\mathfrak{m}}$. *Then*

$$\mathrm{Trans}_{\mathrm{Sp}}(Q, IQ, \langle , \rangle) \;\;=\;\; \mathrm{ETrans}_{\mathrm{Sp}}(Q, IQ, \langle , \rangle).$$

Proof: We have $\mathrm{ETrans}_{\mathrm{Sp}}(Q, IQ, \langle , \rangle) \subseteq \mathrm{Trans}_{\mathrm{Sp}}(Q, IQ, \langle , \rangle)$. We need to show $\mathrm{Trans}_{\mathrm{Sp}}(Q, IQ, \langle , \rangle) \subseteq \mathrm{ETrans}_{\mathrm{Sp}}(Q, IQ, \langle , \rangle)$. Let $\alpha \in \mathrm{Trans}_{\mathrm{Sp}}(Q, IQ, \langle , \rangle)$. There exists $\alpha(X)$ in $\mathrm{Trans}_{\mathrm{Sp}}(Q[X], IQ[X]\langle , \rangle_{\varphi \otimes R[X]})$ such that $\alpha(1) = \alpha$ and $\alpha(0) = Id$. For each maximal ideal $\mathfrak{m}$ of R, one has

$$\mathrm{Trans}_{\mathrm{Sp}}(Q_{\mathfrak{m}}[X], IQ_{\mathfrak{m}}[X], \langle , \rangle_{\varphi_{\mathfrak{m}}}) \;\;=\;\; \mathrm{ETrans}_{\mathrm{Sp}}(Q_{\mathfrak{m}}[X], IQ_{\mathfrak{m}}[X], \langle , \rangle_{\varphi_{\mathfrak{m}}})$$

(see Remark 5.7.16). Hence $\alpha(X)_{\mathfrak{m}} \in \mathrm{ETrans}_{\mathrm{Sp}}(Q_{\mathfrak{m}}[X], IQ_{\mathfrak{m}}[X], \langle , \rangle_{\varphi \otimes R_{\mathfrak{m}}[X]})$, for each maximal ideal $\mathfrak{m}$ of R. Therefore from Lemma 5.9.2 it follows that $\alpha(X)$ is in $\mathrm{ETrans}_{\mathrm{Sp}}(Q[X], IQ[X], \langle , \rangle)$ and hence substituting $X = 1$ we get the result. $\square$

We now come to main theorems of this chapter.

Theorem 5.10.3 *Let $(P, \langle , \rangle_{\varphi})$ be a symplectic R-module with P free of rank $2n$, $n \geq 1$. Let $\langle u, v \rangle = u \varphi v^t$, where φ is an alternating matrix of Pfaffian 1. Then*

$$\mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle , \rangle_{\varphi}) \;\;=\;\; \mathrm{ESp}_{\psi_1 \perp \varphi}(R).$$

Proof: Let $\delta \in \mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle , \rangle_{\varphi})$. There exists a

$$\delta(X) \;\;\in\;\; \mathrm{ETrans}_{\mathrm{Sp}}(Q[X], \langle , \rangle_{\varphi})$$

such that $\delta(1) = \delta$ and $\delta(0) = Id$. For any maximal ideal $\mathfrak{m}$ in R,

$$\delta(X)_{\mathfrak{m}} \;\;\in\;\; \mathrm{ETrans}_{\mathrm{Sp}}(Q_{\mathfrak{m}}[X], \langle , \rangle_{\varphi_{\mathfrak{m}}}) \;\;=\;\; \mathrm{ESp}_{\psi_1 \perp \varphi}(R_{\mathfrak{m}}[X])$$

(see Remark 5.7.16). By Lemma 5.3.1 it follows that

$$\delta(X) \;\;\in\;\; \mathrm{ESp}_{\psi_1 \perp \varphi}(R[X]),$$

and hence $\delta \in \mathrm{ESp}_{\psi_1 \perp \varphi}(R)$.

Let $\omega \in \mathrm{ESp}_{\psi_1 \perp \varphi}(R)$. There exists $\omega(X) \in \mathrm{ESp}_{\psi_1 \perp \varphi}(R[X])$ such that $\omega(1) = \omega$ and $\omega(0) = Id$. For any maximal ideal $\mathfrak{m}$ in R,

$$\omega(X)_{\mathfrak{m}} \;\;\in\;\; \mathrm{ESp}_{\psi_1 \perp \varphi}(R_{\mathfrak{m}}[X]) \;\;=\;\; \mathrm{ETrans}_{\mathrm{Sp}}(Q_{\mathfrak{m}}[X], \langle , \rangle_{\varphi_{\mathfrak{m}}})$$

(see Remark 5.7.16). By Lemma 5.9.1 it follows that

$$\omega(X) \quad \in \quad \mathrm{ETrans}_{\mathrm{Sp}}(Q[X], \langle , \rangle_{\varphi}),$$

and hence $\omega \in \mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle , \rangle_{\varphi})$. $\square$

Arguing exactly in the similar way we establish a relative version of the above theorem. We state the relative version of Theorem 5.10.3 below.

Theorem 5.10.4 *Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Let $(P, \langle , \rangle_{\varphi})$ be a symplectic R-module with P free R-module of rank $2n$, $n \geq 2$. Let $\langle u, v\rangle = u\varphi v^t$, where φ is an alternating matrix of Pfaffian 1 such that $\varphi \equiv \psi_n \pmod I$. Then*

$$\mathrm{ETrans}_{\mathrm{Sp}}(Q, IQ, \langle , \rangle_{\varphi}) \quad = \quad \mathrm{ESp}_{\psi_1 \perp \varphi}(R, I).$$

$\square$

Remark 5.10.5 *Let $(P, \langle , \rangle_{\varphi})$ be a symplectic R-module with P free R-module of rank $2n$, $n \geq 1$. Let $\langle u, v\rangle = u\varphi v^t$, where φ is an alternating matrix of Pfaffian 1. Then*

$$\mathrm{Trans}_{\mathrm{Sp}}(Q, \langle , \rangle_{\psi_1 \perp \varphi}) \quad = \quad \mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle , \rangle_{\varphi}) \quad = \quad \mathrm{ESp}_{\psi_1 \perp \varphi}(R).$$

Moreover, let us assume $R = 2R$, and I be an ideal of R. Let P free R-module of rank $2n$, $n \geq 2$, and let $\varphi \equiv \psi_n \pmod I$. Then

$$\mathrm{Trans}_{\mathrm{Sp}}(Q, IQ, \langle , \rangle_{\varphi}) \quad = \quad \mathrm{ETrans}_{\mathrm{Sp}}(Q, IQ, \langle , \rangle_{\varphi}) \quad = \quad \mathrm{ESp}_{\psi_1 \perp \varphi}(R, I).$$

5.11 Equality of orbits

In this section we establish main result of the thesis regarding equality of orbits.

Theorem 5.11.1 *Let φ be an alternating matrix of Pfaffian 1. Then the natural map*

$$\mathrm{Um}_{2n}(R)/\mathrm{ESp}_{\varphi}(R) \quad \longrightarrow \quad \mathrm{Um}_{2n}(R)/\mathrm{E}_{2n}(R),$$

is bijective for $n \geq 2$. $\square$

Now we establish a relative version of the above theorem. Theorem 5.11.1 can be treated as a particular case of Theorem 5.11.2, when $I = R$.

Theorem 5.11.2 *Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Let φ be an alternating matrix of Pfaffian 1 such that $\varphi \equiv \psi_n \pmod{I}$. Then the natural map*

$$\mathrm{Um}_{2n}(R, I)/\mathrm{ESp}_{\varphi}(R, I) \longrightarrow \mathrm{Um}_{2n}(R, I)/\mathrm{E}_{2n}(R, I)$$

is bijective for $n \geq 3$.

Proof: It is easy to show that the map is surjective. To show injectivity the map we need to consider $v, w \in \mathrm{Um}_{2n}(R, I)$ and $g \in \mathrm{E}_{2n}(R, I)$ such that $vg = w$. We have to show w is in the same $\mathrm{ESp}_{\varphi}(R, I)$-orbit of v. Let $g(X)$ be in $\mathrm{E}_{2n}(R[X], I[X])$ such that $g(1) = g$, and $g(0) = Id$ (see Lemma 2.2.3). Let us define

$$V(X) = v\, g(X).$$

For each maximal ideal $\mathfrak{m}$ of R we have $\varphi = (1 \perp \varepsilon(\mathfrak{m}))^t\, \psi_n(1 \perp \varepsilon(\mathfrak{m}))$, for some $\varepsilon(\mathfrak{m}) \in \mathrm{E}_{2n-1}(R_{\mathfrak{m}}, I_{\mathfrak{m}})$, over the ring $R_{\mathfrak{m}}$. We define

$$V^{(\mathfrak{m})}(X) = v\, g_{\mathfrak{m}}(X)\, (1 \perp \varepsilon(\mathfrak{m}))^{-1}.$$

Note that $V^{(\mathfrak{m})}(0) = v\ (1 \perp \varepsilon(\mathfrak{m}))^{-1}$. We have

$$V^{(\mathfrak{m})}(X) \in V^{(\mathfrak{m})}(0)\ \mathrm{E}_{2n}(R_{\mathfrak{m}}[X], I_{\mathfrak{m}}[X]).$$

Using Theorem 4.2.2 we can say

$$V^{(\mathfrak{m})}(X) \in V^{(\mathfrak{m})}(0)\ \mathrm{ESp}_{2n}(R_{\mathfrak{m}}[X], I_{\mathfrak{m}}[X]).$$

Therefore there exists $h(X) \in \mathrm{ESp}_{2n}(R_{\mathfrak{m}}[X], I_{\mathfrak{m}}[X])$ such that

$$V^{(\mathfrak{m})}(X)\ h(X) = V^{(\mathfrak{m})}(0).$$

This implies $vg_{\mathfrak{m}}(X)(1 \perp \varepsilon(\mathfrak{m}))^{-1}h(X) = v(1 \perp \varepsilon(\mathfrak{m}))^{-1}$, which means

$$V_{\mathfrak{m}}(X) = V_{\mathfrak{m}}(0)\ (1 \perp \varepsilon(\mathfrak{m}))^{-1}\ h(X)^{-1}\ (1 \perp \varepsilon(\mathfrak{m})),$$

i.e, $V_{\mathfrak{m}}(X) \in V_{\mathfrak{m}}(0)\ \mathrm{ESp}_{\varphi}(R_{\mathfrak{m}}[X], I_{\mathfrak{m}}[X])$ for each maximal ideal $\mathfrak{m}$ in R (see Lemma 5.1.5 and Lemma 5.1.7). Using Theorem 5.3.4 we get an $\alpha(X) \in \mathrm{ESp}_{\varphi}(R[X], I[X])$ such that $V(X) = V(0)\alpha(X)$. Substituting $X = 1$ we get $vg = v\alpha(1)$ where $\alpha(1) \in \mathrm{ESp}_{\varphi}(R, I)$. Hence w is in the same $\mathrm{ESp}_{\varphi}(R, I)$ orbit of v. □

Theorem 5.11.3 *Let $(P, \langle , \rangle)$ be a symplectic R-module with P finitely generated projective module of rank $2n$, $n \geq 1$ and $v = (a, b, p) \in \mathrm{Um}(Q)$. Then*

$$(a, b, p)\ \mathrm{ETrans}(Q) \quad = \quad (a, b, p)\ \mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle , \rangle).$$

□

Here we state and prove a relative version of the above theorem. Theorem 5.11.3 can be treated as a particular case of Theorem 5.11.4, when $I = R$.

Theorem 5.11.4 *Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Let $(P, \langle , \rangle)$ be a symplectic R-module with P finitely generated projective module of rank $2n$, $n \geq 2$. Let $v = (a, b, p) \in \mathrm{Um}(Q, IQ)$. Then*

$$(a, b, p)\ \mathrm{ETrans}(Q, IQ) \quad = \quad (a, b, p)\ \mathrm{ETrans}_{\mathrm{Sp}}(Q, IQ, \langle , \rangle).$$

Here we also assume that of each maximal ideal $\mathfrak{m}$ of R, the alternating form $\langle , \rangle$ corresponds to the alternating matrix $\varphi_{\mathfrak{m}}$, where $\varphi_{\mathfrak{m}} \equiv \psi_n \pmod{I}$, over the local ring $R_{\mathfrak{m}}$.

Proof: Let $\alpha \in \mathrm{ETrans}(Q, IQ)$. Let us choose $\alpha(X)$ from $\mathrm{ETrans}(Q[X], IQ[X])$ such that $\alpha(1) = \alpha$ and $\alpha(0) = Id$. Let us define $V(X) = (a, b, p)\alpha(X)$. Note that $V(0) = (a, b, p)$, and

$$V(X) \quad \in \quad V(0)\ \mathrm{ETrans}(Q[X], IQ[X]).$$

Let $\mathfrak{m}$ be a maximal ideal of R. Over $R_{\mathfrak{m}}$, we have $\varphi_{\mathfrak{m}} = (1 \perp \varepsilon(\mathfrak{m}))^t\ \psi_n\ (1 \perp \varepsilon(\mathfrak{m}))$, where $\varepsilon(\mathfrak{m}) \in \mathrm{E}_{2n}(R_{\mathfrak{m}}, I_{\mathfrak{m}})$. Let us define $W(X) = V(X)\ (1 \perp \varepsilon(\mathfrak{m}))^{-1}$. We have

$$\begin{aligned} W(X) &\in W(0)\ \mathrm{E}_{2n+2}(R_{\mathfrak{m}}[X], I_{\mathfrak{m}}[X]) \\ &= W(0)\ \mathrm{ESp}_{2n+2}(R_{\mathfrak{m}}[X], I_{\mathfrak{m}}[X]) \\ &= W(0)\ \mathrm{ETrans}_{\mathrm{Sp}}(Q_{\mathfrak{m}}[X], IQ_{\mathfrak{m}}[X]), \langle , \rangle_{\psi_n}), \end{aligned}$$

and hence

$$\begin{aligned} V(X) &\in V(0)\ (1 \perp \varepsilon(\mathfrak{m}))^{-1}\ \text{ETrans}_{\text{Sp}}(Q_{\mathfrak{m}}[X], IQ_{\mathfrak{m}}[X], \langle , \rangle_{\psi_n})\ (1 \perp \varepsilon(\mathfrak{m})) \\ &= V(0)\ \text{ETrans}_{\text{Sp}}(Q_{\mathfrak{m}}[X], IQ_{\mathfrak{m}}[X], \langle , \rangle_{\varphi_{\mathfrak{m}}}). \end{aligned}$$

This is true for all maximal ideal $\mathfrak{m}$ of R, and hence by Theorem 5.9.4 we get $V(X) \in V(0)\ \text{ETrans}_{\text{Sp}}(Q[X], I[X], \langle , \rangle)$. Substituting $X = 1$ we get

$$(a, b, p)\ \alpha \ \in \ (a, b, p)\ \text{ETrans}_{\text{Sp}}(Q, IQ, \langle , \rangle).$$

Now we consider β from $\text{ETrans}_{\text{Sp}}(Q, IQ, \langle , \rangle)$. Let $\beta(X)$ be an element of $\text{ETrans}_{\text{Sp}}(Q[X], IQ[X], \langle , \rangle)$ such that $\beta(1) = \beta$ and $\beta(0) = Id$. We define $V(X) = (a, b, p)\beta(X)$. Note that

$$V(X) \ \in \ V(0)\ \text{ETrans}_{\text{Sp}}(Q[X], IQ[X], \langle , \rangle).$$

Let $\mathfrak{m}$ be a maximal ideal of R. Over the local ring $R_{\mathfrak{m}}$, we define $W(X) = V(X)\ (1 \perp \varepsilon(\mathfrak{m}))^{-1}$. We have

$$\begin{aligned} W(X) &\in W(0)\ (1 \perp \varepsilon(\mathfrak{m}))\ \text{ETrans}_{\text{Sp}}(Q_{\mathfrak{m}}[X], IQ_{\mathfrak{m}}[X], \langle , \rangle_{\varphi_{\mathfrak{m}}})\ (1 \perp \varepsilon(\mathfrak{m}))^{-1} \\ &= W(0)\ \text{ETrans}_{\text{Sp}}(Q_{\mathfrak{m}}[X], IQ_{\mathfrak{m}}[X], \langle , \rangle_{\psi_n}) \\ &= W(0)\ \text{ESp}_{2n+2}(R_{\mathfrak{m}}[X], I_{\mathfrak{m}}[X]) \\ &= W(0)\ \text{E}_{2n+2}(R_{\mathfrak{m}}[X], I_{\mathfrak{m}}[X]), \end{aligned}$$

and hence

$$\begin{aligned} V(X) &\in V(0)\ \text{E}_{2n+2}(R_{\mathfrak{m}}[X], I_{\mathfrak{m}}[X]) \\ &= V(0)\ \text{ETrans}(Q_{\mathfrak{m}}[X], IQ_{\mathfrak{m}}[X]). \end{aligned}$$

This is true for all maximal ideal $\mathfrak{m}$ of R, and hence by Theorem 5.6.4 we have $V(X) \in V(0)\text{ETrans}(Q[X], IQ[X])$. Substituting $X = 1$, we get

$$(a, b, p)\ \beta \ \in \ (a, b, p)\ \text{ETrans}(Q, IQ).$$

□

Chapter 6

Excision Theorem

In this chapter we first recall the Excision Theorem of W. van der Kallen (see Theorem 6.1.1). We also recall the definition of the Excision ring in the linear case and its properties. We shall then prove similar results (see Theorem 6.3.2), following the lead of van der Kallen in Theorem 6.1.1, for the relative elementary symplectic groups. We also prove Excision theorem for the elementary symplectic transvection group in the free case.

The Excision theorem (Theorem 6.1.1) of van der Kallen enables one to transform a problem on unimodular rows from the relative case ($I \neq R$) to the absolute case.

6.1 Excision Ring and Excision Theorem

The Excision ring ($\mathbb{Z} \oplus I$): If I is an ideal in R, one can then construct the ring $\mathbb{Z} \oplus I$ with multiplication defined by $(n \oplus i)(m \oplus j) = (nm \oplus nj + mi + ij)$, for $m, n \in \mathbb{Z}$, $i, j \in I$. The maximal spectrum of the ring $\mathbb{Z} \oplus I$ is Noetherian, being the union of finitely many subspaces of dimension $\leq \dim(R)$. There is also a natural homomorphism $\varphi : \mathbb{Z} \oplus I \longrightarrow R$ given by $(m \oplus i) \mapsto m + i \in R$.

Theorem 6.1.1 (**W. van der Kallen**) ([15], Theorem 3.21)
Let I be an ideal in R. Then the natural maps

$$\frac{\mathrm{Um}_n(\mathbb{Z} \oplus I, 0 \oplus I)}{\mathrm{E}_n(\mathbb{Z} \oplus I, 0 \oplus I)} \longrightarrow \frac{\mathrm{Um}_n(R, I)}{\mathrm{E}_n(R, I)},$$
$$\frac{\mathrm{Um}_n(\mathbb{Z} \oplus I, 0 \oplus I)}{\mathrm{E}_n(\mathbb{Z} \oplus I, 0 \oplus I)} \longrightarrow \frac{\mathrm{Um}_n(\mathbb{Z} \oplus I)}{\mathrm{E}_n(\mathbb{Z} \oplus I)},$$

are bijective for $n \geq 3$. □

6.2 Discussion on The Excision Theorem

The critical input in W. van der Kallen's proof of Excision theorem is that if $v = (1+a_1, a_2, \ldots, a_n) \in \mathrm{Um}_n(R, I), n \geq 3$, then he observed that

$$\begin{array}{rcl} v\ E_{ij}(t) & = & v\ E_{1j}(a_i t)\ E_{ij}(-a_1 t), \quad \text{for} \quad 2 \leq i \neq j \leq n, \\ v\ E_{i1}(t) & = & v\ E_{ij}(t)\ E_{j1}(1)\ E_{ij}(-t)\ E_{j1}(-1). \end{array}$$

Using these formulae he was able to transform a problem on elementary orbit to a problem on relative elementary orbit.

We first tried this direct approach in the case of elementary symplectic orbits. One can make similar observations with $se_{ij}(t)$, in very special cases (of the short roots). First we will discuss it.

Let $v = (1+a_1, a_2, \ldots, a_{2n}) \in \mathrm{Um}_{2n}(R, I), n \geq 2$. Let i be an odd integer. Then,

$$v\ se_{ii+1}(t) = v\ se_{1i+1}(a_i t + a_1 a_i t)\ se_{ii+1}(-2a_1 t - a_1^2 t)\ se_{12}(-a_i^2 t).$$

Proof:

$$\begin{array}{c} v \\ \Big\downarrow {\scriptstyle se_{1i+1}(a_i t + a_1 a_i t)} \\ (1+a_1, a_2 + a_i^2 t + a_1 a_i^2 t, \ldots, a_{i+1} + a_i t + 2a_1 a_i t + a_1^2 a_i t, \ldots, a_{2n}) \\ \Big\downarrow {\scriptstyle se_{ii+1}(-2a_1 t - a_1^2 t)} \\ (1+a_1, a_2 + a_i^2 t + a_1 a_i^2 t, \ldots, a_{i+1} + a_i t, \ldots, a_{2n}) \\ \Big\downarrow {\scriptstyle se_{12}(-a_i^2 t)} \\ (1+a_1, a_2, \ldots, a_{i+1} + a_i t, \ldots, a_{2n}). \end{array}$$

□

Again let $v = (1+a_1, a_2, \ldots, a_{2n}) \in \mathrm{Um}_{2n}(R, I), n \geq 2$. Let i be an even integer. Then,

$$v\ se_{ii-1}(t) = v\ se_{1i-1}(a_i t + a_1 a_i t)\ se_{ii-1}(-2a_1 t - a_1^2 t)\ se_{12}(a_i^2 t).$$

Proof:

$$\begin{array}{c} v \\ \Big\downarrow{\scriptstyle se_{1i-1}(a_it+a_1a_it)} \\ (1+a_1, a_2 - a_i^2t - a_1a_i^2t, \ldots, a_{i-1} + a_it + 2a_1a_it + a_1^2a_it, \ldots, a_{2n}) \\ \Big\downarrow{\scriptstyle se_{ii-1}(-2a_1t-a_1^2t)} \\ (1+a_1, a_2 - a_i^2t - a_1a_i^2t, \ldots, a_{i-1} + a_it, \ldots, a_{2n}) \\ \Big\downarrow{\scriptstyle se_{12}(a_i^2t)} \\ (1+a_1, a_2, \ldots, a_{i-1} + a_it, \ldots, a_{2n}). \end{array}$$

□

However we were unable to get direct proofs as above in other cases (of the long roots).

6.3 Symplectic Analogue of The Excision Theorem

Here we will see again that the Local Global principle w.r.t. an extended ideal (Theorem 3.2.3) plays an important role in the proof of symplectic analogue of the Excision Theorem. Theorem 3.2.3 along with Lemma 2.2.17, Lemma 2.2.21 and Lemma 2.2.25 will be employed to prove the following lemma.

Lemma 6.3.1 *Let $n \geq 3$. Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Let $v \in \mathrm{Um}_{2n}(R, I)$. Then for $t \in R$,*

$$v\ se_{ij}(t) \ \in \ v\ \mathrm{ESp}_{2n}(R, I),$$

when $i \neq 1, j \neq 2$.

Proof: Using Lemma 2.2.21 we get ε_0 from $\mathrm{ESp}_{2n}(R, I)$ such that $v\varepsilon_0 = (1 + x_1, x_2, \ldots, x_{2n})$, where $x_1, \ldots, x_{2n} \in I^4$. Let $v^* = v\varepsilon_0$. For any maximal ideal $\mathfrak{m}$ of R there exists $E(\mathfrak{m}) \in \mathrm{ESp}_{2n}(R_{\mathfrak{m}}, I^4_{\mathfrak{m}})$ such that $v^*_{\mathfrak{m}} = e_1\mathrm{E}(\mathfrak{m})$ by Lemma 2.2.17. (Note here that when $I^4 \nsubseteq \mathfrak{m}$, the relative group $\mathrm{ESp}_{2n}(R_{\mathfrak{m}}, I^4_{\mathfrak{m}}) = \mathrm{ESp}_{2n}(R_{\mathfrak{m}})$. So one can either appeal to L.N. Vaserstein's lemma, or infer it independently as in the proof of Lemma 2.2.17). Using Lemma 2.2.25 we get $\mathrm{ESp}_{2n}(R_{\mathfrak{m}}, I^4_{\mathfrak{m}}) \subseteq \mathrm{E}(n, I^2_{\mathfrak{m}})$. Let us define $V(Y) = v^* se_{ij}(tY)$. Note that $V(0) = v^*$. We have,

$$\begin{aligned} v^*_{\mathfrak{m}}\; se_{ij}(tY) &= e_1\; \mathrm{E}(\mathfrak{m})\; se_{ij}(tY) \\ &= e_1\; se_{ij}(tY)\; se_{ij}(-tY)\; \mathrm{E}(\mathfrak{m})\; se_{ij}(tY) \\ &= e_1\; se_{ij}(-tY)\; \mathrm{E}(\mathfrak{m})\; se_{ij}(tY) \\ &= v^*_{\mathfrak{m}}\; \mathrm{E}(\mathfrak{m})^{-1}\; se_{ij}(-tY)\; \mathrm{E}(\mathfrak{m})\; se_{ij}(tY). \end{aligned}$$

Let us fix a notation $\mathrm{E}(\mathfrak{m})^{-1}\; se_{ij}(-tY)\; \mathrm{E}(\mathfrak{m})\; se_{ij}(tY) = E$. Note that $E \in \mathrm{ESp}_{2n}(R_{\mathfrak{m}}[Y], I_{\mathfrak{m}}[Y]^2)$. By Lemma 2.2.25 we have $\mathrm{ESp}_{2n}(R_{\mathfrak{m}}[Y], I_{\mathfrak{m}}[Y]^2)$ is a subset of $\mathrm{ESp}_{2n}(I_{\mathfrak{m}}[Y])$. Therefore $V(Y)_{\mathfrak{m}} \in V(0)_{\mathfrak{m}}\; \mathrm{ESp}_{2n}(I_{\mathfrak{m}}[Y])$. By Theorem 3.2.3, there exists $\mathrm{E}_0(Y) \in \mathrm{ESp}_{2n}(R[Y], I[Y])$ such that $V(Y) = V(0)\mathrm{E}_0(Y)$. Put $Y = 1$, to get $v^* se_{ij}(t) = v^*\mathrm{E}_0(1)$, where $\mathrm{E}_0(1) \in \mathrm{ESp}_{2n}(R, I)$, i.e. $v\varepsilon_0 se_{ij}(t) = v\varepsilon_0\mathrm{E}_0(1)$. Hence $v se_{ij}(t) \in v\mathrm{ESp}_{2n}(R, I)$. □

Theorem 6.3.2 which appears next, is a symplectic analogue of W. van der Kallen's Excision theorem. We now prove the following theorem.

Theorem 6.3.2 *Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Then the natural maps*

$$\begin{aligned} \Phi : \frac{\mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)}{\mathrm{ESp}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)} &\longrightarrow \frac{\mathrm{Um}_{2n}(R, I)}{\mathrm{ESp}_{2n}(R, I)}, \\ \Psi : \frac{\mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)}{\mathrm{ESp}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)} &\longrightarrow \frac{\mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I)}{\mathrm{ESp}_{2n}(\mathbb{Z}[1/2] \oplus I)}, \end{aligned}$$

are bijective for $n \geq 3$.

Proof: The map Φ is surjective because of the same reason for which the first map in Theorem 6.1.1 is surjective; we sketch the proof for completeness.

We have to show that given $[v] \in \mathrm{Um}_{2n}(R, I)/\mathrm{ESp}_{2n}(R, I)$, there exists a $[w] \in \mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)/\mathrm{ESp}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)$ such that $[w]$ maps to $[v]$ by the natural map. We can think of v as a vector from $(\mathbb{Z}[1/2] \oplus I)^{2n}$. Note that $v \equiv e_1 \pmod{0 \oplus I}$. We will try to show $v \in \mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)$. For this we need to show there exists $y \in (\mathbb{Z}[1/2] \oplus I)^{2n}$ such that $\langle v, y \rangle = 1$.

(Note that $(b_1, b_2, \ldots, b_{2n}) \in \mathrm{Um}_{2n}(R)$ and hence $(b_1, b_2^2, \ldots, b_{2n}^2)$ is in $\mathrm{Um}_{2n}(R)$. Because if $(b_1, b_2^2, \ldots, b_{2n}^2) \notin \mathrm{Um}_{2n}(R)$, then ideal generated by

$$\{b_1, b_2^2, \ldots, b_{2n}^2\} \subseteq \mathfrak{m}_R,$$

for some maximal ideal of R. But $b_i^2 \in \mathfrak{m}_R$ implies $b_i \in \mathfrak{m}_R$ since $\mathfrak{m}_R$ is prime ideal. Therefore $\{b_1, b_2, \ldots, b_{2n}\} \subseteq \mathfrak{m}_R$, a contradiction.)

Let $v = (1 + x_1, x_2, \ldots, x_{2n}) \in \mathrm{Um}_{2n}(R, I)$. Take

$$u = (1 + x_1, x_2^2, \ldots, x_{2n}^2) \quad \in \quad \mathrm{Um}_{2n}(R, I).$$

Then $\bar{u} = (1, 0, \ldots, 0) \in (R/I)^{2n}$. Here *'bar'* means reduction modulo I. Now $u \in \mathrm{Um}_{2n}(R, I)$ implies there exists $w = (w_1, w_2, \ldots, w_{2n}) \in R^{2n}$ such that $\langle u, w \rangle = 1$. Now $\langle \bar{u}, \bar{w} \rangle = \bar{1}$ implies $\langle e_1, \bar{w} \rangle = \bar{1}$ and hence $\bar{w}_1 = \bar{1}$. So $w_1 = 1 + y_1$, where $y_1 \in I$, and $\langle u, w \rangle = 1$ implies

$$(1 + x_1)(1 + y_1) + x_2^2 w_2 + \cdots + x_{2n}^2 w_{2n} = 1.$$

Let us take $y = (1+y_1, x_2 w_2, \ldots, x_{2n} w_{2n})$. This $y \in (\mathbb{Z}[1/2] \oplus I)^{2n}$ and $\langle v, y \rangle = 1$. Therefore $v \in \mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)$.

To show Ψ is surjective we need to show for any $v \in \mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I)$, there exists $g \in \mathrm{ESp}_{2n}(\mathbb{Z}[1/2] \oplus I)$ such that $vg \in \mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)$. From the surjectivity of the second map in Theorem 6.1.1 it follows that, for any $v \in \mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I)$, there exists $\acute{g} \in \mathrm{E}_{2n}(\mathbb{Z}[1/2] \oplus I)$ such that

$$v\acute{g} \in \mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I).$$

Now from Theorem 4.1.1 it follows that, there exists $g \in \mathrm{ESp}_{2n}(\mathbb{Z}[1/2] \oplus I)$, such that $vg = v\acute{g}$. Hence Ψ is surjective.

To show Ψ is injective we need to consider $v, w \in \mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)$ and $g \in \mathrm{ESp}_{2n}(\mathbb{Z}[1/2] \oplus I)$ such that $vg = w$. We have to show w is in the $\mathrm{ESp}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)$-orbit of v. Let

$$\begin{aligned} g &= \prod_{k=1}^{r} se_{i_k j_k}(a_k \oplus x_k) \\ &= \prod_{k=1}^{r} se_{i_k j_k}(0 \oplus x_k) se_{i_k j_k}(a_k \oplus 0) \\ &= se_{i_1 j_1}(0 \oplus x_1) \prod_{k=2}^{r} \gamma_k se_{i_k j_k}(0 \oplus x_k) \gamma_k^{-1} \Big(\prod_{k=1}^{r} se_{i_k j_k}(a_k \oplus 0) \Big) \\ &= g_1 g_2, \end{aligned}$$

where $\gamma_l = \prod_{k=1}^{l-1} se_{i_k j_k}(a_k \oplus 0)$ and hence $\gamma_l \in \mathrm{ESp}_{2n}(\mathbb{Z}[1/2] \oplus 0)$. Note that here

$$\begin{aligned} g_1 &= se_{i_1 j_1}(0 \oplus x_1) \prod_{k=2}^{r} \gamma_k \; se_{i_k j_k}(0 \oplus x_k) \; \gamma_k^{-1}, \\ g_2 &= \prod_{k=1}^{r} se_{i_k j_k}(a_k \oplus 0). \end{aligned}$$

Clearly $g_1 \in \mathrm{ESp}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)$ and $g_2 \in \mathrm{ESp}_{2n}(\mathbb{Z}[1/2])$. We also have $\overline{vg} = \overline{w}$. Here *'bar'* means modulo the ideal $0 \oplus I$. Therefore we have $\bar{v}\bar{g}_1\bar{g}_2 = \bar{w}$, i.e, $e_1\overline{g_2} = e_1 = e_1 g_2$. Hence we have

$$g_2 = \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & * \\ * & 0 & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & * \\ * & 0 & I_{2n-2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha \end{pmatrix},$$

where α is in $\mathrm{Sp}_{2n-2}(\mathbb{Z}[1/2]) = \mathrm{ESp}_{2n-2}(\mathbb{Z}[1/2])$. To see that v and w are in the same $\mathrm{ESp}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)$ orbit we use Lemma 6.3.1, replacing R by $\mathbb{Z}[1/2] \oplus I$.

To see the map Φ is injective we now consider the following commutative diagram of the orbit spaces and the natural maps between them:

$$\begin{array}{ccc} \dfrac{\mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)}{\mathrm{ESp}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)} & \xrightarrow{\Psi} & \dfrac{\mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I)}{\mathrm{ESp}_{2n}(\mathbb{Z}[1/2] \oplus I)} \\ \Big\downarrow \Psi_2 & & \Big\downarrow \Psi_1 \\ \dfrac{\mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)}{\mathrm{E}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)} & \xrightarrow{\Psi_3} & \dfrac{\mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I)}{\mathrm{E}_{2n}(\mathbb{Z}[1/2] \oplus I)}. \end{array}$$

Clearly $\Psi_1 \circ \Psi = \Psi_3 \circ \Psi_2$. Note that we have proved that Ψ is injective. The injectivity of Ψ_1 follows from Theorem 4.1.1. Therefore we have $\Psi_1 \circ \Psi$ is injective. This implies $\Psi_3 \circ \Psi_2$ is injective and hence Ψ_2 is injective.

We now consider another commutative diagram:

$$\begin{array}{ccc} \dfrac{\mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)}{\mathrm{ESp}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)} & \xrightarrow{\Phi} & \dfrac{\mathrm{Um}_{2n}(R, I)}{\mathrm{ESp}_{2n}(R, I)} \\ \Big\downarrow \Phi_2 & & \Big\downarrow \Phi_1 \\ \dfrac{\mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)}{\mathrm{E}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)} & \xrightarrow{\Phi_3} & \dfrac{\mathrm{Um}_{2n}(R, I)}{\mathrm{E}_{2n}(R, I)}. \end{array}$$

We have $\Phi_1 \circ \Phi = \Phi_3 \circ \Phi_2$. Note that $\Phi_2 = \Psi_2$ and hence Φ_2 is injective. The injectivity of Φ_3 follows from Theorem 6.1.1. Therefore $\Phi_3 \circ \Phi_2$ is injective. This implies $\Phi_1 \circ \Phi$ is injective and hence Φ is injective. $\square$

6.4 Equality of Orbits and Excision

We can recapture Theorem 4.2.3 which is a relative version of Theorem 4.1.2, as an application of symplectic analogue of the Excision Theorem. Here we establish our claim.

Theorem 6.4.1 *Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Then the natural map*

$$\frac{\mathrm{Um}_{2n}(R,I)}{\mathrm{ESp}_{2n}(R,I)} \longrightarrow \frac{\mathrm{Um}_{2n}(R,I)}{\mathrm{E}_{2n}(R,I)}$$

is bijective for $n \geq 3$.

Proof: Consider the following commutative diagram:

$$\begin{array}{ccc} \dfrac{\mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I)}{\mathrm{ESp}_{2n}(\mathbb{Z}[1/2] \oplus I)} & \xrightarrow{\Omega_1} & \dfrac{\mathrm{Um}_{2n}(R,I)}{\mathrm{ESp}_{2n}(R,I)} \\ \downarrow{\scriptstyle \Omega_3} & & \downarrow{\scriptstyle \Omega_2} \\ \dfrac{\mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I)}{\mathrm{E}_{2n}(\mathbb{Z}[1/2] \oplus I)} & \xrightarrow{\Omega_4} & \dfrac{\mathrm{Um}_{2n}(R,I)}{\mathrm{E}_{2n}(R,I)}. \end{array}$$

The bijectivity of Ω_1 and Ω_3 follows from Theorem 6.3.2 and Theorem 4.1.2 respectively. Further, the bijectivity of Ω_4 is immediate from Theorem 6.1.1. These three bijections together implies that the map Ω_2 is bijective. $\square$

6.5 Suresh linear relation property for a group G

Let G be a functor from commutative rings R (in which 2 is invertible) to groups. A group $G(R)$ is said to satisfy **Suresh linear relation property** if it has a set of generators $x_\alpha(b)$, for α in some indexing set, $b \in R$ and

$$x_\alpha(a+b) \;=\; x_\alpha(a/2)\, x_\alpha(b)\, x_\alpha(a/2),$$

for all $a, b \in R$.

In [27], Amit Roy generalized Eichler's construction in [11] (over fields) by defining orthogonal transformations of a quadratic module with a hyperbolic summand.

In ([28], Lemma 1.2, Lemma 1.3) V. Suresh showed that the Eichler orthogonal transformations defined by A. Roy satisfied this linear property.

Here we show that generators of the elementary symplectic group $\mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle , \rangle_\varphi)$, where $Q = R^2 \perp P$, also satisfies similar linear relations as above when P is a free R module of rank $2n$. Let $v_1, v_2 \in R^{2n}$ and $\alpha_1, \alpha_2 \in R$. Then

$$\begin{aligned}
& \rho_\varphi(v_1/2, \alpha_1/2)\ \rho_\varphi(v_2, \alpha_2)\ \rho_\varphi(v_1/2, \alpha_1/2) \\
= & \begin{pmatrix} 1 & 0 & 0 \\ \alpha_1/2 & 1 & (v_1/2)\varphi \\ v_1^t/2 & 0 & I_{2n} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \alpha_2 & 1 & v_2\varphi \\ v_2^t & 0 & I_{2n} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \alpha_1/2 & 1 & (v_1/2)\varphi \\ v_1^t/2 & 0 & I_{2n} \end{pmatrix} \\
= & \begin{pmatrix} 1 & 0 & 0 \\ \alpha_1/2 + \alpha_2 + (v_1/2)\varphi v_2^t & 1 & (v_1/2)\varphi + v_2\varphi \\ v_1^t/2 + v_2^t & 0 & I_{2n} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \alpha_1/2 & 1 & (v_1/2)\varphi \\ v_1^t/2 & 0 & I_{2n} \end{pmatrix} \\
= & \begin{pmatrix} 1 & 0 & 0 \\ \alpha_1/2 + \alpha_2 + \alpha_1/2 & 1 & (v_1/2)\varphi + v_2\varphi + (v_1/2)\varphi \\ v_1^t/2 + v_2^t + v_1^t/2 & 0 & I_{2n} \end{pmatrix} \\
= & \begin{pmatrix} 1 & 0 & 0 \\ \alpha_1 + \alpha_2 & 1 & v_1\varphi + v_2\varphi \\ v_1^t + v_2^t & 0 & I_{2n} \end{pmatrix} \\
= & \rho_\varphi(v_1 + v_2, \alpha_1 + \alpha_2),
\end{aligned}$$

and

$$\begin{aligned}
& \mu_\varphi(v_1/2, \alpha_1/2)\ \mu_\varphi(v_2, \alpha_2)\ \mu_\varphi(v_1/2, \alpha_1/2) \\
= & \begin{pmatrix} 1 & -\alpha_1/2 & (v_1/2)\varphi \\ 0 & 1 & 0 \\ 0 & v_1^t/2 & I_{2n} \end{pmatrix} \begin{pmatrix} 1 & -\alpha_2 & v_2\varphi \\ 0 & 1 & 0 \\ 0 & v_2^t & I_{2n} \end{pmatrix} \begin{pmatrix} 1 & -\alpha_1/2 & (v_1/2)\varphi \\ 0 & 1 & 0 \\ 0 & v_1^t/2 & I_{2n} \end{pmatrix} \\
= & \begin{pmatrix} 1 & -\alpha_1/2 - \alpha_2 + (v_1/2)\varphi v_2^t & (v_1/2)\varphi + v_2\varphi \\ 0 & 1 & 0 \\ 0 & v_1^t/2 + v_2 & I_{2n} \end{pmatrix} \begin{pmatrix} 1 & -\alpha_1/2 & (v_1/2)\varphi \\ 0 & 1 & 0 \\ 0 & v_1^t/2 & I_{2n} \end{pmatrix} \\
= & \begin{pmatrix} 1 & -\alpha_1/2 - \alpha_2 - \alpha_1/2 & (v_1/2)\varphi + v_2\varphi + (v_1/2)\varphi \\ 0 & 1 & 0 \\ 0 & v_1^t/2 + v_2 + v_1^t/2 & I_{2n} \end{pmatrix} \\
= & \mu_\varphi(v_1 + v_2, \alpha_1 + \alpha_2).
\end{aligned}$$

6.6 Excision Theorem for Symplectic Transvection Group

Now we are ready to prove Excision theorem for elementary symplectic transvection group. Before that we state a preliminary lemma which will be required in the proof.

Lemma 6.6.1 *Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Let $v \in \mathrm{Um}_{2n}(R, I)$. Let $\alpha \in \mathrm{ETrans}_{\mathrm{Sp}}(R^{2n}, \langle , \rangle_{\varphi})$ such that $e_1\alpha = e_1$. Here φ is the alternating matrix corresponding to the alternating form $\langle , \rangle$ and $\varphi \equiv \psi_{n-1} \pmod{I}$. Then*

$$v\alpha \;\in\; v\ \mathrm{ETrans}_{\mathrm{Sp}}(R^{2n}, I^{2n}, \langle , \rangle_{\varphi}),$$

for $n \geq 3$.

Proof: Let us choose $\alpha(X)$ from $\mathrm{ETrans}_{\mathrm{Sp}}(R[X]^{2n}, \langle , \rangle_{\varphi})$, with $\alpha(1) = \alpha$, and $\alpha(0) = Id$ (see Lemma 2.2.3). Let us set $V(X) = v\alpha(X)$. Note that $V(0) = v$, and hence $V(X) \in V(0)\ \mathrm{ETrans}_{\mathrm{Sp}}(R[X]^{2n}, \langle , \rangle_{\varphi})$. Let $\mathfrak{m}$ be a maximal ideal of R. Over the local ring $R_{\mathfrak{m}}$, we have

$$V(X) \;\in\; V(0)\ \mathrm{ETrans}_{\mathrm{Sp}}(R_{\mathfrak{m}}[X], \langle , \rangle_{\varphi_{\mathfrak{m}}}),$$

and $\varphi_{\mathfrak{m}} = (1 \perp \varepsilon(\mathfrak{m}))^t\ \psi_{n-1}\ (1 \perp \varepsilon(\mathfrak{m}))$, for some $\varepsilon(\mathfrak{m}) \in \mathrm{E}_{2n-3}(R_{\mathfrak{m}}, I_{\mathfrak{m}})$ (see Lemma 5.1.8). Note that by Lemma 5.7.12 we have

$$\begin{aligned}&\mathrm{ETrans}_{\mathrm{Sp}}(R_{\mathfrak{m}}[X]^{2n}, \langle , \rangle_{\varphi_{\mathfrak{m}}})\\ &= (I_3 \perp \varepsilon(\mathfrak{m}))^{-1}\ \mathrm{ETrans}_{\mathrm{Sp}}(R_{\mathfrak{m}}[X]^{2n}, \langle , \rangle_{\psi_{n-1}})\ (I_3 \perp \varepsilon(\mathfrak{m})),\end{aligned}$$

and hence

$$\begin{aligned} V(X) &\in V(0)\ (I_3 \perp \varepsilon(\mathfrak{m}))^{-1}\ \mathrm{ETrans}_{\mathrm{Sp}}(R_{\mathfrak{m}}[X]^{2n}, \langle , \rangle_{\psi_{n-1}})\ (I_3 \perp \varepsilon(\mathfrak{m}))\\ &= V(0)\ (1 \perp \varepsilon(\mathfrak{m})))^{-1}\ \mathrm{ESp}_{2n}(R_{\mathfrak{m}}[X])\ (1 \perp \varepsilon(\mathfrak{m}))) \end{aligned}$$

(see Lemma 5.7.10). Let us set $\beta(X) = (I_3 \perp \varepsilon(\mathfrak{m}))\ \alpha(X)_{\mathfrak{m}}\ (I_3 \perp \varepsilon(\mathfrak{m}))^{-1}$ and define $W(X) = V(X)\ (I_3 \perp \varepsilon(\mathfrak{m}))^{-1}$. Note that $W(0)_{\mathfrak{m}}\ \beta(X) = W(X)_{\mathfrak{m}}$, where $\beta(X)$ is in $\mathrm{ESp}_{2n}(R_{\mathfrak{m}}[X])$ and $e_1\beta(X) = e_1$. By Lemma 6.3.1 we get that $W(0)_{\mathfrak{m}}\ \beta(X) \in$

$W(0)_{\mathfrak{m}}\ \mathrm{ESp}_{2n}(R_{\mathfrak{m}}[X], I_{\mathfrak{m}}[X])$, hence $W(X)_{\mathfrak{m}} \in W(0)_{\mathfrak{m}}\ \mathrm{ESp}_{2n}(R_{\mathfrak{m}}[X], I_{\mathfrak{m}}[X])$, i.e,

$$v\ \alpha_{\mathfrak{m}}(X)\ (I_3 \perp \varepsilon(\mathfrak{m}))^{-1} \quad \in \quad v\ (I_3 \perp \varepsilon(\mathfrak{m}))^{-1}\ \mathrm{ESp}_{2n}(R_{\mathfrak{m}}[X], I_{\mathfrak{m}}[X]),$$

and hence

$$v\ \alpha_{\mathfrak{m}}(X) \quad \in \quad v\ (I_3 \perp \varepsilon(\mathfrak{m}))^{-1}\ \mathrm{ESp}_{2n}(R_{\mathfrak{m}}[X], I_{\mathfrak{m}}[X])\ (I_3 \perp \varepsilon(\mathfrak{m})),$$

i.e, $V(X)_{\mathfrak{m}} \in V(0)_{\mathfrak{m}}\mathrm{ETrans}_{\mathrm{Sp}}(R_{\mathfrak{m}}[X]^{2n}, I_{\mathfrak{m}}[X]^{2n}, \langle , \rangle_{\varphi_{\mathfrak{m}}})$. This is true for all maximal ideal $\mathfrak{m}$ of R. Using Theorem 5.9.4 we get

$$V(X) \quad \in \quad V(0)\ \mathrm{ETrans}_{\mathrm{Sp}}(R[X]^{2n}, I[X]^{2n}, \langle , \rangle_{\varphi}).$$

Substituting $X = 1$ we get $v\alpha \in v\ \mathrm{ETrans}_{\mathrm{Sp}}(R^{2n}, I^{2n}, \langle , \rangle_{\varphi})$. $\square$

Definition 6.6.2 *Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Let us consider the excision ring $\mathbb{Z}[1/2] \oplus I$. The standard alternating matrix of size $2n$ over the ring $\mathbb{Z}[1/2] \oplus I$ is defined inductively as*

$$\widehat{\psi_n} \quad = \quad \widehat{\psi_{n-1}} \perp \widehat{\psi_1},$$

where

$$\widehat{\psi_1} \quad = \quad \begin{pmatrix} (0,0) & (1,0) \\ (-1,0) & (0,0) \end{pmatrix}.$$

Theorem 6.6.3 Excision Theorem for the Group of Elementary Symplectic Transvections:
Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Let us consider the excision ring $\mathbb{Z}[1/2] \oplus I$. Let $\big((\mathbb{Z}[1/2] \oplus I)^{2n-2}, \langle , \rangle_{\varphi^*}\big)$ be a symplectic $(\mathbb{Z}[1/2] \oplus I)$-module, where φ^* be an alternating matrix over the ring $\mathbb{Z}[1/2] \oplus I$ and $\varphi^* \equiv \widehat{\psi_{n-1}} \pmod{0 \oplus I}$. Then the natural maps

$$\frac{\mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)}{\mathrm{ETrans}_{\mathrm{Sp}}\big((\mathbb{Z}[1/2] \oplus I)^{2n}, (0 \oplus I)^{2n}, \langle , \rangle_{\varphi^*}\big)} \xrightarrow{\eta} \frac{\mathrm{Um}_{2n}(R, I)}{\mathrm{ETrans}_{\mathrm{Sp}}(R^{2n}, I^{2n}, \langle , \rangle_{\varphi^*})},$$

$$\frac{\mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)}{\mathrm{ETrans}_{\mathrm{Sp}}\big((\mathbb{Z}[1/2] \oplus I)^{2n}, (0 \oplus I)^{2n}, \langle , \rangle_{\varphi^*}\big)} \xrightarrow{\delta} \frac{\mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I)}{\mathrm{ETrans}_{\mathrm{Sp}}\big((\mathbb{Z}[1/2] \oplus I)^{2n}, \langle , \rangle_{\varphi^*}\big)},$$

are bijective for $n \geq 3$.

Proof: Clearly η is surjective (follows from the surjectivity of the first map in Theorem 6.3.2). To show δ is sucjective we need to show for $v \in \mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I)$ there exists a $g \in \mathrm{ETrans}_{\mathrm{Sp}}\big((\mathbb{Z}[1/2] \oplus I)^{2n}, \langle , \rangle_{\varphi^*}\big)$ such that $vg \in \mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)$. From the Excision theorem in the linear case (see Theorem 6.1.1) it follows that there exists $g^* \in \mathrm{E}_{2n}(\mathbb{Z}[1/2] \oplus I)$ such that $vg^* \in \mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)$. By Theorem 5.10.3 and Theorem 5.11.1 we have $v\ \mathrm{E}_{2n}(R) = v\ \mathrm{ESp}_{\widehat{\psi_1} \perp \varphi^*}(R) = v\ \mathrm{ETrans}_{\mathrm{Sp}}(R^{2n}, \langle , \rangle_{\varphi^*})$ and hence there exists a $g \in \mathrm{ETrans}_{\mathrm{Sp}}\big((\mathbb{Z}[1/2] \oplus I)^{2n}, \langle , \rangle_{\varphi *}\big)$ such that $vg = vg^* \in \mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)$.

To show δ is injective we need to consider $v, w \in \mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)$ and $g \in \mathrm{ETrans}_{\mathrm{Sp}}\big((\mathbb{Z}[1/2] \oplus I)^{2n}, \langle , \rangle_{\varphi^*}\big)$ such that $vg = w$. We have to show w is in the $\mathrm{ETrans}_{\mathrm{Sp}}\big((\mathbb{Z}[1/2] \oplus I)^{2n}, 0 \oplus I)^{2n}, \langle , \rangle_{\varphi^*}\big)$-orbit of v. Let

$$\begin{aligned}
g &= \prod_{k=1}^{r} t_{\varphi^*}(a_k \oplus x_k, \alpha_k \oplus \beta_k) \\
&= \prod_{k=1}^{r} t_{\varphi^*}(0 \oplus x_k/2, 0 \oplus \beta_k/2)\ t_{\varphi^*}(a_k \oplus 0, \alpha_k \oplus 0)\ t_{\varphi^*}(0 \oplus x_k/2, 0 \oplus \beta_k/2) \\
&= t_{\varphi^*}(0 \oplus x_k/2, 0 \oplus x_k/2) \prod_{k=2}^{r} \gamma_k\ t_{\varphi^*}(0 \oplus x_k/2, 0 \oplus x_k/2)\ \gamma_k^{-1} \\
&\quad \Big(\prod_{k=1}^{r} t_{\varphi^*}(a_k \oplus 0, \alpha_k \oplus 0)\Big) \\
&= g_1\ g_2,
\end{aligned}$$

where t_{φ^*} is ρ_{φ^*} or μ_{φ^*}, and $\gamma_l = \prod_{k=1}^{l-1} t_{\varphi^*}(a_k \oplus 0, \alpha_k \oplus 0)$. Therefore γ_l is in $\mathrm{ETrans}_{\mathrm{Sp}}\big((\mathbb{Z}[1/2] \oplus 0)^{2n}, \langle , \rangle_{\varphi^*}\big)$. Note that here

$$\begin{aligned}
g_1 &= t_{\varphi^*}(0 \oplus x_k/2, 0 \oplus \beta_k/2) \prod_{k=2}^{r} \gamma_k\ t_{\varphi^*}(0 \oplus x_k/2, 0 \oplus \beta_k/2)\ \gamma_k^{-1}, \\
g_2 &= \prod_{k=1}^{r} t_{\varphi^*}(a_k \oplus 0, \alpha_k \oplus 0).
\end{aligned}$$

Clearly g_1 is in the relative group $\mathrm{ETrans}_{\mathrm{Sp}}\big((\mathbb{Z}[1/2] \oplus I)^{2n}, (0 \oplus I)^{2n}, \langle , \rangle_{\varphi^*}\big)$ and g_2 is in $\mathrm{ETrans}_{\mathrm{Sp}}\big((\mathbb{Z}[1/2])^{2n}, \langle , \rangle_{\varphi^*}\big)$. We also have $\overline{vg} = \overline{w}$. Here *'bar'* means modulo the ideal $(0 \oplus I)$. Therefore we have $\bar{v}\ \bar{g_1}\ \bar{g_2} = \bar{w}$, i.e, $e_1 \overline{g_2} = e_1 = e_1\ g_2$, and hence $v\ g_1\ g_2 \in v\ \mathrm{ETrans}_{\mathrm{Sp}}\big((\mathbb{Z}[1/2] \oplus I)^{2n}, (0 \oplus I)^{2n}, \langle , \rangle_{\varphi^*}\big)$ (see Lemma 6.6.1), i.e, $w \in v\ \mathrm{ETrans}_{\mathrm{Sp}}\big((\mathbb{Z}[1/2] \oplus I)^{2n}, (0 \oplus I)^{2n}, \langle , \rangle_{\varphi^*}\big)$.

To see the map η is injective we now consider the following commutative diagrams of the orbit spaces and the natural maps between them:

$$\begin{array}{ccc}
\dfrac{\mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)}{\mathrm{ETrans}_{\mathrm{Sp}}\big((\mathbb{Z}[1/2] \oplus I)^{2n}, (0 \oplus I)^{2n}, \langle , \rangle_{\varphi^*}\big)} & \xrightarrow{\delta} & \dfrac{\mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I)}{\mathrm{ETrans}_{\mathrm{Sp}}\big((\mathbb{Z}[1/2] \oplus I)^{2n}, \langle , \rangle_{\varphi^*}\big)} \\
\Big\downarrow \delta_2 & & \Big\downarrow \delta_1 \\
\dfrac{\mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)}{\mathrm{E}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)} & \xrightarrow{\delta_3} & \dfrac{\mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I)}{\mathrm{E}_{2n}(\mathbb{Z}[1/2] \oplus I)}.
\end{array}$$

Clearly $\delta_1 \circ \delta = \delta_3 \circ \delta_2$. Note that we have proved that δ is injective. The injectivity of δ_1 follows from Theorem 5.10.3 and Theorem 5.11.1. Therefore we have $\delta_1 \circ \delta$ is injective. This implies $\delta_3 \circ \delta_2$ is injective and hence δ_2 is injective.

We now consider another commutative diagram:

$$\begin{array}{ccc}
\dfrac{\mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)}{\mathrm{ETrans}_{\mathrm{Sp}}\big((\mathbb{Z}[1/2] \oplus I)^{2n}, (0 \oplus I)^{2n}, \langle , \rangle_{\varphi^*}\big)} & \xrightarrow{\eta} & \dfrac{\mathrm{Um}_{2n}(R, I)}{\mathrm{ETrans}_{\mathrm{Sp}}(R^{2n}, I^{2n}, \langle , \rangle_{\varphi^*}))} \\
\Big\downarrow \eta_2 & & \Big\downarrow \eta_1 \\
\dfrac{\mathrm{Um}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)}{\mathrm{E}_{2n}(\mathbb{Z}[1/2] \oplus I, 0 \oplus I)} & \xrightarrow{\eta_3} & \dfrac{\mathrm{Um}_{2n}(R, I)}{\mathrm{E}_{2n}(R, I)}.
\end{array}$$

We have $\eta_1 \circ \eta = \eta_3 \circ \eta_2$. Note that $\eta_2 = \delta_2$ and hence η_2 is injective. The injectivity of η_3 follows from W.van der Kallen's Excision theorem (see Theorem 6.1.1). Therefore $\eta_3 \circ \eta_2$ is injective. This implies $\eta_1 \circ \eta$ is injective and hence η is injective. $\square$

Chapter 7

Injective Stability

7.1 Decrease in injective stability for $K_1\mathrm{Sp}(R)$

In this chapter first we would like to recall the definition of $K_1(R)$. Given $\alpha \in \mathrm{M}_n(R)$ and $\beta \in \mathrm{M}_m(R)$, then

$$\alpha \perp \beta \;=\; \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \;\in\; \mathrm{M}_{n+m}(R).$$

Using the above definition one has a natural inclusion

$$\mathrm{GL}_n(R) \;\subset\; \mathrm{GL}_{n+1}(R) \;\subset\; \ldots$$

defined by $\alpha \in \mathrm{GL}_n(R)$ goes to $(1 \perp \alpha) \in \mathrm{GL}_{n+1}(R)$. Stable linear group $\mathrm{GL}(R)$ is defined as $\bigcup_n \mathrm{GL}_n(R)$. In an obvious and unique way a group structure is defined on $\mathrm{GL}(R)$ which coincides with the group structures on $\mathrm{GL}_n(R)$ when restricted to $\mathrm{GL}_n(R)$, for all n. We also recall definitions of the subgroups

$$\begin{aligned} \mathrm{E}(R) &= \bigcup_n \mathrm{E}_n(R), \\ \mathrm{SL}(R) &= \bigcup_n \mathrm{SL}_n(R), \end{aligned}$$

of $\mathrm{GL}(R)$.

By Lemma 2.2.9 we have $\mathrm{E}(R) = [\mathrm{E}(R), \mathrm{E}(R)] = [\mathrm{GL}(R), \mathrm{GL}(R)]$. In particular, $\mathrm{E}(R)$ is a normal subgroup of $\mathrm{GL}(R)$.

Definition 7.1.1

$$
\begin{aligned}
K_1(R) &= \frac{\mathrm{GL}(R)}{\mathrm{E}(R)}, \\
SK_1(R) &= \frac{\mathrm{SL}(R)}{\mathrm{E}(R)}.
\end{aligned}
$$

H. Bass, J. Milnor, J-P. Serre began the study of the stabilization for the linear group $\mathrm{GL}_n(R)/\mathrm{E}_n(R)$, for $n \geq 3$, where R is a commutative ring with identity. In [6], they showed the following:

Corollary 7.1.2 ([6], Corollary 11.3) *Suppose that the maximal spectrum of a commutative ring R is Noetherian space of dimension $\leq d$. Then the map*

$$
\frac{\mathrm{GL}_n(R)}{\mathrm{E}_n(R)} \longrightarrow K_1(R)
$$

is an isomorphism of groups for all $n \geq d+3$. □

Bass-Milnor-Serre also showed that $\mathrm{K}_1(R) = \mathrm{GL}_3(R)/\mathrm{E}_3(R)$, when $d = 1$. So the natural question of improving the stability bound arises. In ([4], §11) Bass conjectured the following:

The dimension of the maximal spectrum under the Zariski topology is called the *Jacobson-Krull dimension.*

Conjecture of Bass:
Let R be a commutative ring with 1 and Jacobson-Krull dimension of R is d. Then the map

$$
\frac{\mathrm{GL}_n(R)}{\mathrm{E}_n(R)} \longrightarrow \frac{\mathrm{GL}_{n+1}(R)}{\mathrm{E}_{n+1}(R)}
$$

is an isomorphism for $n \geq d+2$.

In [34], L.N. Vaserstein proved the above conjecture for an algebra A, which is finite as a module over a commutative ring R, and whose spectrum of maximal ideals is a Noetherian space of dim d.

In [26], R.A. Rao and W. van der Kallen began the study of whether the stabilization bound above improves in the case of finitely generated algebras A over a field k. Such rings A are called *affine algebras.* Note that by Hilbert's Nullstellensatz, for such rings the *Krull dimension* and the *Jacobson-Krull dimension* (i.e. the dimension of the maximal ideal spectrum under the Zariski topology) coincide.

An affine algebra A will be called *non-singular* if $A_\wp$ is a regular local ring, for every prime ideal $\wp$ of A. The well-known Jacobian criterion gives an effective method to determine whether a given ideal $I = (f_1, \ldots, f_r)$ of $k[X_1, \ldots, X_n]$ gives rise to a non-singular algebra $k[X_1, \ldots, X_n]/I$.

Definition 7.1.3 C_1**-field**:
A field F is called a C_1 field if for any homogeneous polynomial $f(x_1, \ldots, x_n)$ in $F[x_1, \ldots, x_n]$ of degree d (d is any positive integer), where $n > d$ has a nontrivial zero in F^n.

Example of C_1-field due to Tsen-Lang: If F is an algebraically closed field and E is a function field in one variable over F, then E is a C_1-field.

For more examples of C_1-field one can see [12].

R.A. Rao and W. van der Kallen showed the following:

Theorem 7.1.4 ([26], Theorem 1)
Let A be a non-singular affine algebra of Krull dimension $d \geq 2$ over a perfect C_1-field. Let $\sigma \in \mathrm{SL}_{d+1}(A)$ and $(1 \perp \sigma) \in \mathrm{E}_{d+2}(A)$. Then σ is homotopic to identity, i.e, there exists a $\rho(X) \in \mathrm{SL}_{d+1}(A[X])$ such that $\rho(1) = \sigma$ and $\rho(0) = Id$. $\square$

As a consequence of the above result they showed the following:

Theorem 7.1.5 ([26], Theorem 1)
If A is a non-singular affine algebra of Krull dimension $d \geq 2$ over a perfect C_1-field, then the natural map

$$\frac{\mathrm{SL}_n(A)}{\mathrm{E}_n(A)} \longrightarrow \frac{\mathrm{SL}_{n+1}(A)}{\mathrm{E}_{n+1}(A)}$$

is injective for $n \geq d+1$. $\square$

Thus, the set of all non-singular affine algebras over a perfect C_1-field, of Krull dimension d, became an important subclass of the set of all commutative rings of Jacobson-Krull dimension d, over which one could seek improvement in K_0 and K_1 results. (The famous theorems of Suslin in [30], [32] first showed that stabilization results of H. Bass for K_0 improved over such rings. This is the key why one expect to hope for improvement for K_1 for this class of rings.)

L.N. Vaserstein in [35] considered the symplectic, orthogonal and the unitary K_1-functors, and obtained stabilization theorems for them. These results have been sharpened and extended to other groups in [3].

We restrict ourselves to the symplectic case here.

Theorem 7.1.6 ([35], Theorem 3.3)
The natural map

$$\varphi_{n,n+1} : \frac{\mathrm{Sp}_{2n}(R)}{\mathrm{ESp}_{2n}(R)} \longrightarrow \frac{\mathrm{Sp}_{2n+2}(R)}{\mathrm{ESp}_{2n+2}(R)}$$

is an isomorphism for $2n \geq 2d + 4$. Here d is the stable dimension of R. □

R. Basu and R.A. Rao showed, in particular, the following:

Theorem 7.1.7 ([9], Theorem 1)
Let R be a non-singular affine algebra over a perfect C_1-field of odd Krull dimension $d \geq 2$. Let $\sigma \in \mathrm{Sp}_{2n}(R)$ and $(I_2 \perp \sigma) \in \mathrm{ESp}_{2n+2}(R)$. Then σ is homotopic to identity, i.e, there exists $\rho(X) \in \mathrm{Sp}_{2n}(R[X])$ such that $\rho(1) = \sigma$ and $\rho(0) = Id$. □

As a consequence they showed that

Theorem 7.1.8 ([9], Theorem 2)
If R is a non-singular affine algebra over a perfect C_1-field of odd Krull dimension $d \geq 2$, then the natural map

$$\varphi_{n,n+1} : \frac{\mathrm{Sp}_{2n}(R)}{\mathrm{ESp}_{2n}(R)} \longrightarrow \frac{\mathrm{Sp}_{2n+2}(R)}{\mathrm{ESp}_{2n+2}(R)}$$

is an isomorphism for $2n \geq d + 1$. □

In this section we are going to reprove this result. Moreover via our main result we show that there is a further decrease in the injective stabilization bound (for the symplectic K_1) of a non-singular affine algebra over a finite field of characteristic not equal to 2 (or its algebraic closure). We show that if the field is a finite field of characteristic not equal to 2 (or its algebraic closure) then the bound improves to $2n \geq d$, provided d is even ≥ 4.

We would like to recall the surjective stability estimates first. We begin with a definition:

Definition 7.1.9 Stable Range: *Let R be a commutative ring. The following concept was introduced by H. Bass:*

(R_m) *for every* $(a_1, \ldots, a_m) \in \mathrm{Um}_m(R)$*, there are* $x_i \in R$*,* for $1 \leq i \leq m-1$*, such that* $(a_1 + x_1 a_m, \ldots, a_{m-1} + x_{m-1} a_m) \in \mathrm{Um}_{m-1}(R)$*. The condition* (R_m) *implies* (R_{m+1})*, for every* $m > 0$*. Moreover, for any* $n \geq m$*, the condition* (R_m) implies (R_n)*, with* $x_i = 0$*, for* $i \geq m$*. By stable range* $\mathrm{sr}(R)$ *of a ring* R *we mean the least* n *such that* (R_n) *holds.*

Definition 7.1.10 Stable Dimension: *The stable dimension of a ring R is the integer one less than the stable range. It is denoted by* $\mathrm{sdim}(R)$*. If R is a Noetherian ring of Krull dimension d, then* $\mathrm{sdim}(R) \leq d+1$*.*

The following is well-known:

Lemma 7.1.11 *Let I be an ideal of R and* $\mathrm{sr}(R) \leq t$*. Let us assume $t \geq 2$. Then* $\mathrm{Um}_n(R, I) = e_1 \mathrm{E}_n(R, I)$*, for $n \geq t$.*

Proof: Let $v = (a_1, \ldots, a_{n-1}, a_n) \in \mathrm{Um}_n(R, I)$, then $w = (a_1, \ldots, a_{n-1}, a_n^2)$ is in $\mathrm{Um}_n(R, I)$. Since $\mathrm{sr}(R) \leq t$, there exists $b_i \in R$ such that

$$w^* \;=\; (a_1 + b_1 a_n^2, \ldots, a_{n-1} + b_{n-1} a_n^2) \;\in\; \mathrm{Um}_{n-1}(R, I).$$

There exists $E_1 \in \mathrm{E}_n(R, I)$ such that $vE_1 = (w^*, a_n)$. Let us consider $(w^*, a_n) \in \mathrm{Um}_n(\mathbb{Z} \oplus I)$, with $w^* \in \mathrm{Um}_{n-1}(\mathbb{Z} \oplus I)$. Clearly there exists an elementary matrix $E_2 \in \mathrm{E}_n(\mathbb{Z} \oplus I)$ such that $(w^*, a_n)E_2 = e_1$. By W. van der Kallen's Excision theorem (see Theorem 6.1.1) we get an $E_3 \in \mathrm{E}_n(R, I)$ such that $(w^*, a_n)E_3 = e_1$. Hence $vE_1E_3 = e_1$, where $E_1E_3 \in \mathrm{E}_n(R, I)$. $\square$

Proposition 7.1.12 *Let R be a Noetherian commutative ring of odd Krull dimension $d \geq 3$. Assume $R = 2R$. Let* $\sigma \in \mathrm{Sp}_{d+1}(R)$ *and* $(I_2 \perp \sigma) \in \mathrm{ESp}_{d+3}(R)$*. Then σ is (stably elementary symplectic) homotopic to the identity, i.e. there exists a $\rho(X)$ in* $\mathrm{Sp}_{d+1}(R[X])$*, such that $\sigma = \rho(1)$, and $\rho(0) = Id$.*

Proof: Let $\alpha(X) \in \mathrm{ESp}_{d+3}(R[X])$ be such that $(I_2 \perp \sigma) = \alpha(1)$ and $\alpha(0) = Id$ (see Lemma 2.2.3). Let $e_1\alpha(X) = v(X)$. Therefore

$$v(X) \;\in\; \mathrm{Um}_{d+3}(R[X], (X^2 - X)).$$

The Krull dimension of $R[X]$ is $d+1$. Note that $R[X]$ is Noetherian and hence $\text{sdim}(R[X]) \le d+2$. This implies $\text{sr}(R[X]) \le d+3$. Therefore,

$$\begin{aligned}\text{Um}_{d+3}(R[X],(X^2-X)) &= e_1\text{E}_{d+3}(R[X],(X^2-X)) \\ &= e_1\text{ESp}_{d+3}(R[X],(X^2-X)).\end{aligned}$$

(First equality follows form Lemma 7.1.11 and second equality follows from Theorem 4.2.3.) Let

$$\varepsilon(X) \in \text{ESp}_{d+3}(R[X],(X^2-X))$$

be such that $v(X) = e_1\varepsilon(X)$.

Let us define $\beta(X) = \alpha(X)\varepsilon(X)^{-1}$. Clearly $e_1\beta(X) = e_1$ and $\beta(X) \in \text{ESp}_{d+3}(R[X])$. This implies $\beta(X)$ is of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ * & 1 & * \\ * & 0 & \beta^*(X) \end{pmatrix},$$

where $\beta^*(X) \in \text{Sp}_{d+1}(R[X])$. Now $(I_2 \perp \sigma) = \alpha(1) = \beta(1)$ since $\varepsilon(1) = I_{d+3}$. Therefore $\beta^*(1) = \sigma, \beta^*(0) = Id$; and hence σ is (stably elementary symplectic) homotopic to the identity. □

The next result is proven in the linear case in ([37], Theorem 3.3), using H. Lindel's insight in [19], when R is a localization of an affine algebra over a field K at a non-singular point. This can be extended to any regular ring $(R,\mathfrak{m})$ containing a field, or if characteristic $R/\mathfrak{m} \notin \mathfrak{m}$,[2] using the deep approximation theorem of D. Popescu in [22]. (Also see [23], Corollary 4.4.) The argument imitated in the symplectic case is outlined in [9] and asserts:

Theorem 7.1.13 ([9], Theorem 3.8) *Let $(R,\mathfrak{m})$ be a regular local ring. Assume that R contains a field, or characteristic $R/\mathfrak{m} \notin \mathfrak{m}$.*[2] *Then*

$$\text{Sp}_{2n}(R[X]) = \text{ESp}_{2n}(R[X]),$$

for $n \ge 2$. □

Corollary 7.1.14 *Let R be a non-singular affine algebra of odd Krull dimension*

$d \geq 5$ *over a field* K*. Also assume* $R = 2R$*. Then*

$$\mathrm{Sp}_{d+1}(R) \ \cap \ \mathrm{ESp}_{d+3}(R) \ = \ \mathrm{ESp}_{d+1}(R).$$

Proof: It suffices to show that the left hand side is contained in the right hand side. The reverse inclusion is obvious. Let $\sigma \in \mathrm{Sp}_{d+1}(R) \cap \mathrm{ESp}_{d+3}(R)$. By Proposition 7.1.12, σ is stably elementary symplectic homotopic to the identity, i.e, there exists $\alpha(X) \in \mathrm{Sp}_{d+1}(R[X])$ such that $\alpha(1) = \sigma$ and $\alpha(0) = Id$. Now $\alpha_{\mathfrak{m}}(X) \in \mathrm{Sp}_{d+1}(R_{\mathfrak{m}}[X])$, for all maximal ideals $\mathfrak{m}$ in R. By Theorem 7.1.13 we have $\mathrm{Sp}_{d+1}(R_{\mathfrak{m}}[X]) = \mathrm{ESp}_{d+1}(R_{\mathfrak{m}}[X])$, for all maximal ideals $\mathfrak{m}$ of R . Therefore $\alpha(X)_{\mathfrak{m}} \in \mathrm{ESp}_{d+1}(R_{\mathfrak{m}}[X])$, for all maximal ideals $\mathfrak{m}$ in R. Theorem 3.1.5 implies $\alpha(X) \in \mathrm{ESp}_{d+1}(R[X])$. We have $\sigma = \alpha(1) \in \mathrm{ESp}_{d+1}(R)$. $\square$

Theorem 7.1.15 *Let* R *be a finitely generated algebra of even Krull dimension* $d \geq 4$ *over* K*, where* $K = \mathbb{Z}$ *or* F *or* $\overline{F}$ *and char(K)*$\neq 2$*. (Here* F *is a finite field and* $\overline{F}$ *is its algebraic closure.) Let* $\sigma \in \mathrm{Sp}_d(R)$ *and* $(I_2 \perp \sigma) \in \mathrm{ESp}_{d+2}(R)$*. Then* σ *is (stably elementary symplectic) homotopic to the identity. In fact,* $\sigma = \rho(1)$*, and* $\rho(0) = Id$*, for some*

$$\rho(X) \ \in \ \mathrm{Sp}_d(R[X]) \ \cap \ \mathrm{ESp}_{d+2}(R[X]).$$

Proof: Let $\alpha(X) \in \mathrm{ESp}_{d+2}(R[X])$ be such that $(I_2 \perp \sigma) = \alpha(1)$, and $\alpha(0) = Id$. Let $e_1\alpha(X) = v(X)$. Therefore

$$v(X) \ \in \ \mathrm{Um}_{d+2}(R[X], (X^2 - X)).$$

By ([29], Corollary 20.4),

$$\begin{aligned} \mathrm{Um}_{d+2}(R[X], (X^2 - X)) \ &= \ e_1\mathrm{E}_{d+2}(R[X], (X^2 - X)) \\ &= \ e_1\mathrm{ESp}_{d+2}(R[X], (X^2 - X)). \end{aligned}$$

(The last equality follows from Theorem 4.2.2.) Let

$$\varepsilon(X) \ \in \ \mathrm{ESp}_{d+2}(R(X), (X^2 - X))$$

be such that $v(X) = e_1\varepsilon(X)$.

Let us define $\beta(X) = \alpha(X)\varepsilon(X)^{-1}$. Clearly $e_1\beta(X) = e_1$ and $\beta(X) \in \mathrm{ESp}_{d+2}(R[X])$.

This implies $\beta(X)$ is of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ * & 1 & * \\ * & 0 & \beta^*(X) \end{pmatrix},$$

where $\beta^*(X) \in \mathrm{Sp}_d(R[X])$. Now $(I_2 \perp \sigma) = \alpha(1) = \beta(1)$ since $\varepsilon(1) = I_{d+2}$. Therefore $\beta^*(1) = \sigma, \beta^*(0) = Id$; and hence σ is (stably elementary symplectic) homotopic to identity. $\square$

Corollary 7.1.16 *Let R be a finitely generated non-singular algebra of even Krull dimension $d \geq 4$ over K, where K is either a finite field or the algebraic closure of a finite field and char(K)$\neq 2$. Let $\sigma \in \mathrm{Sp}_d(R)$ and $(I_2 \perp \sigma) \in \mathrm{ESp}_{d+2}(R)$. Then σ belongs to $\mathrm{ESp}_d(R)$.*

Proof: From the proof of Theorem 7.1.15 it follows that $\sigma = \beta^*(1)$ for some $\beta^*(X) \in \mathrm{Sp}_d(R[X])$, with $\beta^*(0) = Id$. We know $\mathrm{Sp}_d(R_{\mathfrak{m}}[X]) = \mathrm{ESp}_d(R_{\mathfrak{m}}[X])$, for all maximal ideals $\mathfrak{m}$ of R (see Theorem 7.1.13). This implies $\beta^*(X) \in \mathrm{ESp}_d(R_{\mathfrak{m}}[X])$, for all maximal ideals $\mathfrak{m}$ in R. By Theorem 3.1.5, $\beta^*(X) \in \mathrm{ESp}_d(R[X])$. Hence $\sigma = \beta^*(1)$ belongs to $\mathrm{ESp}_d(R)$. $\square$

7.2 Decrease in injective stability for $\mathrm{Sp}(Q, \langle , \rangle)/\mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle , \rangle)$

Final goal of this section is to give an improvement for Basu-Rao (see Theorem 7.1.8) estimate in the module case over finitely generated rings. For this purpose we state and prove a few preliminary results. While dealing with the results in the relative case w.r.t. an ideal I of a ring R, we will always assume that over the local ring $R_{\mathfrak{m}}$, where $\mathfrak{m}$ is a maximal ideal, the alternating form $\langle , \rangle$ corresponds to the alternating matrix $\varphi_{\mathfrak{m}}$, where $\varphi_{\mathfrak{m}} \equiv \psi_n \pmod{I}$.

Theorem 7.2.1 ([5], Theorem 3.4, Page 183) *Let R be a commutative ring of dim d. Let I be an ideal of R and P be a projective module of rank $\geq d+1$. Let $\tilde{Q} = R \perp P$. Let $v_1, v_2 \in \mathrm{Um}(\tilde{Q})$ and $v_1 \equiv v_2 \pmod{I\tilde{Q}}$. Then there exists $\beta \in \mathrm{ETrans}(\tilde{Q}, I\tilde{Q})$ such that $v_1\beta = v_2$.* $\square$

Lemma 7.2.2 *Let R be commutative ring of dimension d, and let I be an ideal of R. Let us assume $R = 2R$. Let $(P, \langle , \rangle)$ be a symplectic R-module with P finitely generated projective module of even rank $\geq max\{4, d-1\}$, and let $Q = R^2 \perp P$.*

Let $v_1, v_2 \in \mathrm{Um}(Q)$ *and* $v_1 \equiv v_2 \pmod{IQ}$. *Then there exists* $\beta \in \mathrm{ETrans}_{\mathrm{Sp}}(Q, IQ)$ *such that* $v_1\beta = v_2$.

Proof: Follows from Theorem 7.2.1 and Theorem 5.11.4. □

Let us recall the property $P_r(R, I)$ introduced in ([17]). Let R be a commutative ring and I be an ideal of R. Let P be a projective module of rank $\geq r$ and let $\tilde{Q} = R \perp P$. We say $P_r(R, I)$ holds if $\mathrm{ETrans}(\tilde{Q}, I\tilde{Q})$ acts transitively on the set of unimodular elements $(a, x) \in \mathrm{Um}(\tilde{Q})$ with the property $(a, x) \equiv (1, 0) \pmod{I\tilde{Q}}$. We can similarly introduce $P_r(R, I)$ for the group of elementary symplectic transvections $\mathrm{ETrans}_{\mathrm{Sp}}$. In the next two lemmas 'bar' will denote modulo the ideal $nil(R)$.

Lemma 7.2.3 ([17], Remark 2.3): $P_r(\overline{R}, \overline{I})$ *implies* $P_r(R, I)$.

Proof: Let $(a, p) \in \mathrm{Um}(\tilde{Q})$ with the property $(a, p) \equiv (1, 0) \pmod{I\tilde{Q}}$. We have $(\overline{a}, \overline{p}) \in \mathrm{Um}(\overline{\tilde{Q}})$ and $(\overline{a}, \overline{p}) \equiv (\overline{1}, \overline{0}) \pmod{\overline{I\tilde{Q}}}$. Given that $P_r(\overline{R}, \overline{I})$ holds and hence there exists $\alpha \in \mathrm{ETrans}(\overline{\tilde{Q}}, \overline{I\tilde{Q}})$ such that $(\overline{a}, \overline{x})\alpha = (\overline{1}, \overline{0})$. We know the map

$$\mathrm{ETrans}(\tilde{Q}, I\tilde{Q}) \quad \longrightarrow \quad \mathrm{ETrans}(\overline{\tilde{Q}}, \overline{I\tilde{Q}})$$

is surjective. Therefore there exists $\alpha_0 \in \mathrm{ETrans}(\tilde{Q}, I\tilde{Q})$ such that $\overline{\alpha_0} = \alpha$ and we have $(\overline{a}, \overline{p})\overline{\alpha_0} = (\overline{1}, \overline{0})$. Hence $(a, p)\alpha_0 = (1, 0) \pmod{nil(R)}$. We may assume $(a, p) \equiv (1, 0) \pmod{nil(R)}$. Therefore we have $(a, p) \equiv (1, 0) \pmod{nil(R) \cap I\tilde{Q}}$. Let $(a, p) = (1 + a', p)$, where $a' \in nil(R) \cap I$ and $p \in \big(nil(R)\tilde{Q} \cap I\tilde{Q}\big)$. We define $v(X) = (1 + a'X, pX)$. For any maximal ideal $\mathfrak{m}$ of R, $P_{\mathfrak{m}}$ will be a free $R_{\mathfrak{m}}$-module and $v(X)_{\mathfrak{m}} \in \mathrm{Um}_n(R_{\mathfrak{m}}[X], I_{\mathfrak{m}}[X])$, where $n \geq r + 1$. Note that $a' \in nil(R)$ and hence $1 + a'X$ is a unit in $R[X]$. Therefore by Lemma 2.2.17 we get $\beta(X) \in \mathrm{E}_n(R_{\mathfrak{m}}[X], I_{\mathfrak{m}}[X])$ such that $v(X)\beta(X) = (1, 0) = v(0)$. We have

$$\mathrm{E}_n(R_{\mathfrak{m}}[X], I_{\mathfrak{m}}[X]) \quad = \quad \mathrm{ETrans}(\tilde{Q}_{\mathfrak{m}}[X], I\tilde{Q}_{\mathfrak{m}}[X])$$

(see Lemma 5.4.6). Therefore

$$v(X) \quad \in \quad v(0)\ \mathrm{ETrans}(\tilde{Q}_{\mathfrak{m}}[X], I\tilde{Q}_{\mathfrak{m}}[X]),$$

and this is true for all maximal ideal $\mathfrak{m}$ of R. Hence using Theorem 5.6.4 we claim

$$v(X) \quad \in \quad v(0)\ \mathrm{ETrans}(\tilde{Q}[X], I\tilde{Q}[X]).$$

Substituting $X = 1$ we get the result. $\square$

Lemma 7.2.4 *$P_r(\overline{R}, \overline{I})$ for $\mathrm{ETrans}_{\mathrm{Sp}}$ implies $P_r(R, I)$ for $\mathrm{ETrans}_{\mathrm{Sp}}$.*

Proof: Follows from Lemma 7.2.3 and Theorem 5.11.4. $\square$

Theorem 7.2.5 ([17], Theorem 2.4): *Let R be a finitely generated ring of dimension $d \geq 2$, and I be an ideal of R. Let P be a projective module of rank $\geq d$, and $\tilde{Q} = R \perp P$. Let $(a, x) \in \mathrm{Um}(\tilde{Q})$ with the property $(a, x) \equiv (1, 0) \pmod{I\tilde{Q}}$. Then there exists $\alpha \in \mathrm{ETrans}(\tilde{Q}, I\tilde{Q})$ such that $(a, x)\alpha = (1, 0)$.*

Proof: Let $\mathfrak{m}$ be a maximal ideal of R. Consider the free module $P_{\mathfrak{m}}$ of $R_{\mathfrak{m}}$. Also $(a_{\mathfrak{m}}, x_{\mathfrak{m}}) \in \mathrm{Um}_n(R_{\mathfrak{m}}, I_{\mathfrak{m}})$, where $n \geq d+1$. By Corollary 2.2.18 we get $\beta \in \mathrm{E}_n(R_{\mathfrak{m}}, I_{\mathfrak{m}})$ such that $(a_{\mathfrak{m}}, x_{\mathfrak{m}})\beta = (1, 0)$. Let us choose $\beta(X)$ from $\mathrm{E}_n(R_{\mathfrak{m}}[X], I_{\mathfrak{m}}[X])$ such that $\beta(1) = \beta$ and $\beta(0) = Id$ (see Lemma 2.2.3). Let us define $v(X) = (a_{\mathfrak{m}}, x_{\mathfrak{m}})\beta(X)$. Note that $v(1) = (1, 0)$ and $v(0) = (a, x)$ and

$$v(X) \quad \in \quad v(0)\ \mathrm{E}_n(R_{\mathfrak{m}}[X], I_{\mathfrak{m}}[X]).$$

We have $\mathrm{E}_n(R_{\mathfrak{m}}[X], I_{\mathfrak{m}}[X]) = \mathrm{ETrans}(\tilde{Q}_{\mathfrak{m}}[X], I\tilde{Q}_{\mathfrak{m}}[X])$ (see Lemma 5.4.6). Therefore

$$v(X) \quad \in \quad v(0)\ \mathrm{ETrans}(\tilde{Q}_{\mathfrak{m}}[X], I\tilde{Q}_{\mathfrak{m}}[X]),$$

and this is true for all maximal ideal $\mathfrak{m}$ of R. Hence using Theorem 5.6.4 we claim

$$v(X) \quad \in \quad v(0)\ \mathrm{ETrans}(\tilde{Q}[X], I\tilde{Q}[X]).$$

Substituting $X = 1$ we get the result. $\square$

Theorem 7.2.6 *Let R be a finitely generated ring of dimension $d \geq 2$, and let I be an ideal of R. Let us assume $R = 2R$. Let $(P, \langle , \rangle)$ be a symplectic R-module with P finitely generated projective module of even rank $\geq max\{4, d-1\}$, and $Q = R^2 \perp P$. Let $(a, x) \in \mathrm{Um}(Q)$ with the property $(a, x) \equiv (1, 0) \pmod{IQ}$. Then there exists $\alpha \in \mathrm{ETrans}_{\mathrm{Sp}}(Q, IQ)$ such that $(a, x)\alpha = (1, 0)$.*

Proof: Follows from Theorem 7.2.5 and Theorem 5.11.4. $\square$

Theorem 7.2.7 *Let R be a commutative ring of dimension d. Let us assume $R = 2R$. Let $(P\langle , \rangle)$ be a symplectic R-module with P finitely generated projective module* *of even rank* $\geq max\{2, d-3\}$. *Let* $Q = (R^2 \perp P)$, *and let* $\widehat{Q} = (R^2 \perp Q)$. *Let* $\sigma \in \mathrm{Sp}(Q, \langle , \rangle)$ *and* $(I_2 \perp \sigma) \in \mathrm{ETrans}_{\mathrm{Sp}}(\widehat{Q}), \langle , \rangle)$. *Then* σ *is (stably elementary symplectic) homotopic to the identity. In fact,* $\sigma = \rho(1)$, *and* $\rho(0) = Id$, *for some*

$$\rho(X) \;\in\; \mathrm{Sp}(Q[X], \langle , \rangle) \;\cap\; \mathrm{ETrans}_{\mathrm{Sp}}(\widehat{Q}[X], \langle , \rangle).$$

Proof: Let us choose $\alpha(X)$ from $\mathrm{ETrans}_{\mathrm{Sp}}(\widehat{Q}[X], \langle , \rangle)$, such that $\alpha(1) = I_2 \perp \sigma$, and $\alpha(0) = Id$. Let $e_1\alpha(X) = v(X) \in \mathrm{Um}(\widehat{Q}[X], (X^2 - X)\widehat{Q}[X])$. Also $e_1 = (1, 0, 0) \in \mathrm{Um}(\widehat{Q}[X], (X^2 - X)\widehat{Q}[X])$. Therefore by Lemma 7.2.2, we have $\beta(X) \in \mathrm{ETrans}_{\mathrm{Sp}}(\widehat{Q}[X], (X^2 - X)\widehat{Q}[X], \langle , \rangle)$, such that $v(X)\beta(X) = (1, 0, 0)$, i.e, $e_1\alpha(X)\beta(X) = e_1$. Let us call the product $\alpha(X)\beta(X) = \delta(X)$. Since $\delta(X) \in \mathrm{ETrans}_{\mathrm{Sp}}(\widehat{Q}[X], \langle , \rangle)$, we have $\langle e_1\delta(X), e_2\delta(X)\rangle = \langle e_1, e_2\rangle = 1$, and hence $e_2\,\delta(X) = (a(X), 1, q(X)) = u(X)$ (say). Note that $\delta(0) = Id$, hence $u(0) = (0, 1, 0)$. Also $\delta(1) = Id$, hence $u(1) = (0, 1, 0)$, i.e, $u(X) \equiv (0, 1, 0)$ (mod $(X^2 - X)$). Let $\mathfrak{m}$ be a maximal ideal of R and $\varphi_{\mathfrak{m}}$ be the alternating matrix over $R_{\mathfrak{m}}$, which corresponds to the alternating bilinear form $\langle , \rangle$. Let us choose an element

$$\gamma(X) \;=\; \begin{pmatrix} 1 & 0 & 0 \\ -a(X) & 1 & -q(X) \\ {\varphi_{\mathfrak{m}}^{-1}}^t q(X)^t & 0 & I \end{pmatrix}$$

$$= \; \rho(q(X)\varphi_{\mathfrak{m}}^{-1}, -a(X))$$

from $\mathrm{ETrans}_{\mathrm{Sp}}(\widehat{Q}_{\mathfrak{m}}[X], (X^2 - X)\widehat{Q}_{\mathfrak{m}}[X], \langle , \rangle)$, such that $e_1\,\gamma(X) = e_1$ and $u(X)\,\gamma(X) = e_2 = u(0)$. This is true for all maximal ideals $\mathfrak{m}$ of R. By Theorem 5.9.4 there is a $\widetilde{\gamma}(X) \in \mathrm{ETrans}_{\mathrm{Sp}}(\widehat{Q}[X], (X^2 - X)\widehat{Q}[X], \langle , \rangle)$ such that $e_1\,\widetilde{\gamma}(X) = e_1$ and $u(X)\,\widetilde{\gamma}(X) = e_2$. Let us call $\delta(X)\widetilde{\gamma}(X) = \eta(X)$. Clearly $\eta(X) \in \mathrm{ETrans}_{\mathrm{Sp}}(\widehat{Q}[X], \langle , \rangle)$, and $e_1\,\eta(X) = e_1; e_2\,\eta(X) = e_2$. Let $\eta(X) = I_2 \perp \rho(X)$, where $\rho(X) = \eta(X)|_{Q[X]}$. Note that $\rho(X) \in \mathrm{Sp}(Q[X], \langle , \rangle)$, and

$$\begin{aligned} I_2 \perp \rho(1) \;&=\; \eta(1) \;=\; \delta(1)\widetilde{\gamma}(1) \\ &=\; \alpha(1)\beta(1)\widetilde{\gamma}(1) \\ &=\; \alpha(1) \;=\; (I_2 \perp \sigma), \end{aligned}$$

and $\rho(0) = Id$. $\square$

Theorem 7.2.8 ([9], Theorem 3.13) *Let $(R, \mathfrak{m})$ be a regular local ring. Assume that R contains a field, or characteristic of $R \setminus \mathfrak{m} \notin \mathfrak{m}^2$. Then*

$$\mathrm{Sp}(R[X]^{2n+2}, \langle , \rangle_{\varphi_{\mathfrak{m}}}) \quad = \quad \mathrm{ETrans}_{\mathrm{Sp}}(R[X]^{2n+2}, \langle , \rangle_{\varphi_{\mathfrak{m}}}),$$

for $n \geq 1$, where $\varphi_{\mathfrak{m}}$ is the associated matrix of the alternating bilinear form $\langle , \rangle$. $\square$

The next corollary improves Basu-Rao (see Theorem 7.1.8) estimate in the module case over finitely generated rings.

Corollary 7.2.9 *Let R be a finitely generated non-singular algebra of dimension d over K, where K is either a finite field or the algebraic closure of a finite field. Let us assume $R = 2R$. Let $(P.\langle , \rangle)$ be a symplectic R-module with P finitely generated projective module of even rank $\geq max\{2, d-3\}$. Let $Q = (R^2 \perp P)$, and $\widehat{Q} = (R^2 \perp Q)$. Let $\sigma \in \mathrm{Sp}(Q, \langle , \rangle)$ and $(I_2 \perp \sigma) \in \mathrm{ETrans}_{\mathrm{Sp}}(\widehat{Q}, \langle , \rangle)$. Then σ belongs to $\mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle , \rangle)$.*

Proof: From the proof of Theorem 7.2.7 it follows that $\sigma = \rho(1)$ for some $\rho(X) \in \mathrm{Sp}(Q[X], \langle , \rangle)$, with $\rho(0) = Id$. Using Theorem 7.2.8 we get that $\mathrm{Sp}(R_{\mathfrak{m}}[X]^{2n+2}, \langle , \rangle_{\varphi_{\mathfrak{m}}})$ $=$ $\mathrm{ETrans}_{\mathrm{Sp}}(R_{\mathfrak{m}}[X]^{2n+2}, \langle , \rangle_{\varphi_{\mathfrak{m}}})$, for all maximal ideals $\mathfrak{m}$ of R. This implies $\rho(X) \in \mathrm{ETrans}_{\mathrm{Sp}}(R_{\mathfrak{m}}[X]^{2n+2}, \langle , \rangle_{\varphi_{\mathfrak{m}}})$, for all maximal ideals $\mathfrak{m}$ in R. By Lemma 5.9.1, $\rho(X) \in \mathrm{ETrans}_{\mathrm{Sp}}(R[X], \langle , \rangle)$. Hence $\sigma = \rho(1)$ belongs to $\mathrm{ETrans}_{\mathrm{Sp}}(R, \langle , \rangle)$. $\square$

Remark 7.2.10 *We believe that Corollary 7.2.9 should also hold for finitely generated rings of dimension ≥ 2 in view of results in [13].*

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