GEOMETRY OF THE HITCHIN MAP

by

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Declaration

I hereby declare that the investigation presented in this thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part of a degree/diploma at this or any other Institution/University.

SARBESWAR

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SARBESWAR

Contents

Chapter 1

Introduction

In this chapter we will explain the main results of the thesis using as little mathematical background as possible. We will always work over the complex numbers, i.e., all manifolds will be complex manifolds. Also we assume that manifolds are projective, i.e., there is an embedding in $\mathbb{P}^n_{\mathbb{C}}$.

Let X be a smooth projective curve of genus $g \geq 2$.

In [7] D.Mumford proved that the set of isomorphism classes of stable vector bundles of rank n and degree d on X has a natural structure of a nonsingular quasi-projective variety of dimension $n^2(g-1)+1$. C.S.Seshadri in [12], by introducing the notion of S-equivalence of semi-stable bundles, constructed a normal projective variety $U(n,d)$ which is a compactification of the space of stable bundles of rank n and degree d. If n and d are coprime, there is no distinction between stable and semi-stable bundles and $U(n,d)$ is itself nonsingular.

In [8] S.Ramanan and M.S.Narasimhan gave an explicit description of the moduli space of stable vector bundles of rank 2 and odd degree over a genus 2 curve. In particular, they showed that the moduli space of stable vector bundles of rank 2 with fixed determinant of odd degree over a smooth curve of genus 2 is isomorphic to a smooth intersection of two quadrics in $\mathbb{P}^5$.

In [5] N.J.Hitchin defined a stable pair (E,φ) on X, as a vector bundle E on X, together with a homomorphism $\varphi : E \longrightarrow E \otimes K$ of vector bundles, where K is the canonical line bundle over X, such that for any φ-invariant proper subbundle F of E, the inequality $\mu(F) < \mu(E)$ holds (where $\mu =$ slope $= \frac{\text{degree}}{\text{rank}}$). It has been proved that the set of all isomorphism classes of stable pairs of rank 2 over a compact Riemann surface can be given the structure of a quasi-projective variety which has the coarse moduli property.

Hitchin also defined a map of the moduli space of stable pairs of rank 2 to the affine space $H^0(X,K^2)$ by mapping (E,φ) to $\det \varphi \in H^0(X,K^2)$. This is known as the *Hitchin map*.

In [9] N.Nitsure constructed a coarse moduli scheme within the algebraic category in a more general setup, and showed that the *Hitchin map* is proper.

It is obvious that the pair (E,φ) is stable if E is stable. But in general (E,φ) may be stable as a pair without E being stable. We fix a smooth projective curve X of genus 2 and a line bundle δ of degree 1 over X. In this thesis, we will study how non-stable bundles occur in a stable pair.

Define

$$\tilde{X} := \{\xi \in J \text{ such that } H^0(X, K \otimes \xi^{-2} \otimes \delta^{-1}) \neq 0\},$$

where J denotes the Jacobian of line bundles over X of degree zero. We prove the following theorem.

Theorem 1.0.1. *Let X be a smooth projective curve over $\mathbb{C}$ of genus 2. Then the moduli space of stable Higgs pairs over X of rank 2 and of odd degree contains the cotangent bundle of the moduli space of stable vector bundles over X of rank 2 and of odd degree as an open dense subset whose complement is isomorphic to a bundle $\mathcal{G}$ over $\tilde{X}$, with fibres isomorphic to a two-sheeted cover of $H^0(X, K^2)$ ramified along a subspace of Co-dimension 1, where $\tilde{X}$ is as above.*

We have remarked that the moduli space of rank 2 stable vector bundles over X with fixed determinant δ is isomorphic to a smooth intersection Y of two quadrics in $\mathbb{P}^5$ ([8]). It is also known that the Jacobian of the curve X is isomorphic to the variety of lines in Y ([2]).

Let $\mathbb{P}(W)$ be the smooth pencil of quadrics in $\mathbb{P}^5$ containing Y. Consider pairs (Q, V_1), where Q is a quadric in the pencil and V_1 is an irreducible component of the variety of planes contained in Q. It is known ([11]) that a pair (Q, V_1) can be identified with a point on the curve X. For each pair (Q, V_1) we define an involution $\iota_{(Q,V_1)}$ on the variety of lines in Y as follows:

For each line l there exists a unique plane Λ in V_1 containing l and its intersection with Y gives another line l'. We define $\iota_{(Q,V_1)}(l) = l'$. Therefore we get a Kummer surface for each pair (Q, V_1) as the quotient by the involution $\iota_{(Q,V_1)}$. Thus we have a family of Kummer surfaces parametrised by X.

We will define *special* lines in Y as double points of any such Kummer surface. Thus the variety $\tilde{X}$ of special lines form a 16-sheeted cover of X. Then we prove:

Lemma 1.0.2. *1) Let l be a special line in Y then $T^*Y \mid_l \simeq \mathcal{O}(-1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(-2)$.*

2)If l is a non special line then $T^*Y \mid_l \simeq \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-2)$.

On the other hand, let the Jacobian of degree zero line bundles over X be denoted by J. We say a point $\xi \in J$ is *special* if $H^0(X, K \otimes \xi^{-2} \otimes \delta^{-1}) \neq 0$. To each $\xi \in J$ we will associate the line $\mathbb{P}(H^1(X, \xi^{-2} \otimes \delta^{-1}))$ and prove that it in fact gives a line in Y and then prove:

Theorem 1.0.3. *The decomposition of the tangent bundle to the moduli space of rank 2 stable vector bundles with fixed determinant δ over X restricted to the line $l = \mathbb{P}(H^1(X, \xi^{-2} \otimes \delta^{-1}))$*

for a special point $\xi \in J$ *is*

$$\mathcal{O}(1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-2)$$

and for a generic point $\xi \in J$ *it is*

$$\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-2).$$

Then by the Theorem 1.0.3 and Lemma 1.0.2 we conclude that the line in the intersection of two smooth quadrics in $\mathbb{P}^5$, given by a special line bundle of degree zero is a special line defined as double points of Kummer surfaces.

We define a point in Y as special point if it lies on a special line. We can also define special points as follows:

Definition 1.0.4. *A point* p *is called a special point if the projective tangent space to* Y *at* p *contains fewer than 4 lines in* J*.*

Define $\Delta = \{p \in Y : \mathbb{P}(P^{\perp})$ contains fewer than 4 lines of $J\}$, where $P^{\perp} := \{v \in V; Q(v,w) = 0$ for all $Q \in W$ and $w \in P\}$, where P denote the vector space associated to p.

i.e., Δ is the set of special points in Y.

Definition 1.0.5. *A rank 2 stable vector bundle* V *with fixed determinant* δ *is called special if it contains a degree zero line bundle* ξ^{-1} *with* $H^0(X, K \otimes \xi^{-2} \otimes \delta^{-1}) \neq 0$.

In other words in the identification of the moduli space of rank 2 stable bundles with fixed determinant δ, with Y, V corresponds to a special point in Y (as definition 4.3.7).

Definition 1.0.6. *A stable vector bundle is called very stable if it admits no non-zero nilpotent Higgs field.*

We prove the following Lemma:

Lemma 1.0.7. *A rank 2 stable vector bundle* V *with fixed determinant* δ *over* X *is special if and only if it admits a non-zero nilpotent Higgs field.*

Therefore Δ is isomorphic to the variety of stable bundles which are not very stable.

Then we prove the following theorem:

Theorem 1.0.8. *There is a subspace* S *in the moduli space of stable Higgs bundles with two fibrations over* $\tilde{X}$ *one via* $\mathcal{G}$ *and other via* $\mathcal{F}_1$*.*

Lastly we give a geometric description of the Hitchin map ([5]) restricted to the cotangent bundle to the moduli space of stable bundles.

We have remarked that the cotangent bundle of the moduli space of stable bundles is contained in the moduli of Higgs pairs as an open dense subset. We will prove that the Hitchin map gives a range of quadrics in the cotangent bundle. Let $\mathcal{M}$ denote the moduli space of stable vector bundles with fixed determinant δ. If E is a point in $\mathcal{M}$, then the cotangent space to $\mathcal{M}$ at E can be identified with $H^0(X,\ \mathrm{Ad}E \otimes K)$, where $\mathrm{Ad}E$ denote the bundle of trace-free homomorphisms. Therefore the *Hitchin map*, i.e., the evaluation map

$$H^0(X,\ \mathrm{Ad}E \otimes K) \otimes \mathcal{O} \longrightarrow \mathrm{Ad}E \otimes K,$$

composed with the determinant map $\mathrm{Ad}E \otimes K \longrightarrow K^2$ gives a map $H^0(X,\ \mathrm{Ad}E \otimes K) \otimes \mathcal{O} \longrightarrow K^2$. In other words if p_i' denotes the projection from $\mathcal{M} \times \mathbb{P}^1$, where $\mathbb{P}^1 = X/\iota$, (ι being the hyperelliptic involution on X), to the i-th factor, then it gives a homomorphism

$$f' : (p_1')^* T^* \longrightarrow (p_1')^* T \otimes (p_2')^* \mathcal{O}(2), \tag{1.0.1}$$

where T denote the tangent bundle to the moduli space of stable bundles. Taking adjoint of this we will show that it gives a pencil of quadrics on $T \otimes \mathcal{O}(-1)$, i.e., a homomorphism

$$f : (p_1')^*(T \otimes \mathcal{O}(-1)) \longrightarrow (p_1')^*(T^* \otimes \mathcal{O}(1)) \otimes (p_2')^* \mathcal{O}_{\mathbb{P}^1}(1). \tag{1.0.2}$$

On the other hand we will take a smooth intersection Y of two quadrics in $\mathbb{P}^5$ and let $\mathbb{P}(W)$ denote the pencil of quadrics in $\mathbb{P}^5$ passing through Y.

Restriction of this pencil to the projective tangent bundle of Y gives a pencil of quadrics on $\tilde{T}Y$ with $\mathcal{O}(-1) \subset \tilde{T}Y$ as null space. From the exact sequence of bundles on Y

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \tilde{T}Y \longrightarrow TY \otimes \mathcal{O}(-1) \longrightarrow 0,$$

where $\tilde{T}Y$ and TY denote the projective and abstract tangent bundle to Y respectively, we therefore get a pencil of quadrics on $TY \otimes \mathcal{O}_Y(-1)$.

In other words if p_i' denote the projection from $Y \times \mathbb{P}^1$, where $\mathbb{P}^1 = \mathbb{P}(W)$, to the i-th factor then we get a morphism

$$\tilde{f} : (p_1')^* TY \otimes \mathcal{O}_Y(-1) \longrightarrow (p_1')^* T^*Y \otimes \mathcal{O}_Y(1) \otimes (p_2')^* \mathcal{O}_{\mathbb{P}^1}(1). \tag{1.0.3}$$

Finally identifying the moduli space of stable bundles $\mathcal{M}$ with a smooth intersection of two quadrics in $\mathbb{P}^5$ ([8]) we prove :

Theorem 1.0.9. *The morphisms f in* (1.0.2) *and $\tilde{f}$ in* (1.0.3) *are same.*

Therefore from the above theorem we can identify

$$W = H^0(\mathbb{P}^1, \mathcal{O}(1)) \text{ with } H^0(X, K)$$

and the geometry of the Hitchin map can be described using above Theorem as follows: For each element $w \in W$ we get a quadratic form on $T\mathcal{M} \otimes \mathcal{O}(-1)$. Dualizing we will get a range of quadratic forms which can be identified with W^*, on the cotangent bundle of the moduli space, i.e., to every point s of the cotangent bundle

$$w^* \mapsto q_{w^*}(s, s),$$

where q_{w^*} denotes the quadratic form corresponding to $w^* \in W^*$, defines a quadratic form on W^*. i.e., an element of

$$S^2(W) = H^0(X, K^2)$$

which is the Hitchin map on the cotangent bundle to the moduli space we wanted.

Remark 1.0.10. *If we fix a point y in the moduli space of stable bundles and consider the projective space corresponding to the cotangent space at y then the geometry of the Hitchin map will be clear. In this situation a Hitchin point (a point in the Hitchin space $\mathbb{P}(H^0(X, K^2))$) can be thought as two quadrics Q_1 and Q_2 in the pencil. Then by the above discussion these two quadrics will give two conics in the cotangent space at y. Then the fibre over this Hitchin point to the cotangent space at y are the points contained in both conics.*

Dually a point in the cotangent space gives a line in the tangent space and the fibre of the Hichin map over a Hitchin point given by Q_1 and Q_2 are the lines in the tangent space which touch the conics given by Q_1 and Q_2.

We conclude the introduction by indicating the organisation of the thesis.
In next chapter, we collect the preliminaries that are required to understand the thesis.
We review the basics of Higgs bundles and quadratic geometry.

In chapter 2 we consider the stable trace-free Higgs fields associated to a non-stable rank 2 vector bundle of odd degree and prove that the set of such Higgs fields forms a two sheeted cover of $H^0(X; K^2)$.

In chapter 3 we define special lines and special points in a smooth intersection of two quadrics in $\mathbb{P}^5$ in one hand and special points in the Jacobian of degree zero line bundles over X and special points in the moduli space of rank 2 stable vector bundles with fixed determinant of odd degree on the other. Then we prove that spacial lines in the smooth intersection of two quadrics in $\mathbb{P}^5$ correspond to special points in the Jacobian and special points correspond to special points in the moduli space.

In chapter 4 we describe the geometry of the Hitchin map restricted to the cotangent bundle of the moduli space of stable vector bundles.

Chapter 2

Preliminaries

In this chapter we will collect the preliminaries which are essential for the rest of the thesis. In particular we will review the basic definitions and theorems on Higgs bundles on the one hand and quadritic geometry on the other.

2.1 Notation and Convention

Here are some notations and conventions we will use in this thesis.

$\mathbb{P}(V)$ denote the projective space associated to V. The points of $\mathbb{P}(V)$ are one dimensional subspaces of V.

We use the same notation for a quadratic form and the associated symmetric bilinear form interchangeably.

We use the notation Q for quadric and q for quadratic form associated to it.

2.2 Moduli of Higgs Pairs

Definition 2.2.1. *A Higgs pair over a smooth projective curve X of genus g consists of a vector bundle E over X and a linear map $\varphi : E \longrightarrow E \otimes L$, where L is a fixed line bundle over X; the map φ is called the Higgs field.*

Definition 2.2.2. *A Higgs pair (E, φ) is called semistable (resp. stable) if for any subbundle E' of E which is φ invariant in the sense that $\varphi(E') \subset E' \otimes L$, we have $\mu(E') \leq \mu(E)$ (resp. $< \mu(E)$), where $\mu(E)$ is the slope of E, namely, $\frac{\text{degree of } E}{\text{rank of } E}$.*

It is clear that if E is a semistable (stable) bundle then (E, φ) is a semistable (stable) Higgs pair for all φ.

2.2.1 S-equivalence of semistable Higgs pairs

Proposition 2.2.3. *Let $(E, \varphi : E \longrightarrow E \otimes L)$ be a semistable Higgs pair. Then there exists a sequence of φ invariant subbundles $0 \subset E_1 \subset E_2 \subset ..... \subset E_l = E$ such that $\mu(E_i/E_{i-1}) = \mu(E)$ for each $i = 1, 2, ..., l$ and each pair $(E_i/E_{i-1}, \varphi_i)$ is stable where $\varphi_i : E_i/E_{i-1} \longrightarrow E_i/E_{i-1} \otimes L$ is induced from φ. Moreover the associate pair $gr(E, \varphi) = \bigoplus(E_i/E_{i-1}, \varphi_i)$ is determined up to isomorphism by (E, φ).*

Proof. : [9, Proposition, 4.1] □

Definition 2.2.4. *Two Higgs bundles $(E, \varphi), (E', \varphi')$ are said to be isomorphic if there is an isomorphism $f : E \longrightarrow E'$ and the following diagram commutes;*

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi'} \\ E \otimes L & \xrightarrow{(f \otimes Id)} & E' \otimes L. \end{array}$$

Definition 2.2.5. *Two semistable Higgs pair (E, φ) and (E', φ') are called S-equivalent if the associated pairs $gr(E, \varphi)$ and $gr(E', \varphi')$ are isomorphic.*

Remark 2.2.6. *If (E, φ) is semistable then $gr(E, \varphi)$ is also semistable.*

Theorem 2.2.7. *There exists a scheme $\mathcal{M}(r, d, L)$ which is a coarse moduli space for S-equivalent classes of semistable Higgs pairs (E, φ) over X where $r = rank\ E$, $d = degree\ E$, and L is a fixed line bundle over X. Moreover the isomorphism classes of stable pairs form an open subscheme $\mathcal{M}'(r, d, L)$ of $\mathcal{M}(r, d, L)$.*

Proof. : [9, Theorem, 5.10] □

If E is a semistable (stable) bundle over X and φ is the zero morphism then the pair (E, φ) is semistable (stable). Therefore the moduli space of S-equivalence classes of semistable (stable) vector bundles over X is contained in $\mathcal{M}(r, d, L)$ $(\mathcal{M}'(r, d, L))$.

Now a morphism $\varphi : E \longrightarrow E \otimes L$ can be identified with a section of the bundle $(\text{End}E \otimes L)$. If we take the line bundle L as the canonical line bundle K then φ belongs to $H^0(X, \text{End}E \otimes K)$ which is, by Serre Duality, isomorphic to $H^1(X, \text{End}E)^*$. If $\mathcal{M}$ is the moduli space of equivalence classes of stable vector bundles over X then the tangent space to $\mathcal{M}$ at a point E can be canonically identified with $H^1(X, \text{End}E)$. Therefore φ can be identified with an element of the cotangent space of $\mathcal{M}$ at E. Thus the cotangent bundle of the moduli space of equivalence classes of stable bundles is contained in the moduli space of the s-equivalence classes of stable Higgs pairs. It is obvious that a semistable Higgs pair is stable if the degree and rank of the bundle are coprime.

It can also happen that a non-stable bundle occurs in a stable Higgs pair.

Example 2.2.8. *Let K be the canonical line bundle over X, and $K^{\frac{1}{2}}$ be a line bundle over X such that $K^{\frac{1}{2}} \otimes K^{\frac{1}{2}} \simeq K$.*

Now consider the vector bundle $V = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$ and

$$\varphi = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad \in H^0(X, EndV \otimes K).$$

Clearly V is not a stable vector bundle but (V, φ) is a stable Higgs pair.This is because $K^{-\frac{1}{2}}$ is the only φ invariant subbundle and it is of negative degree.

2.3 Spectral Curve and Hitchin Map

Let L be a line bundle over X (with a nonzero section). Let $s = (s_k)$, where s_k is a section of $L^{\otimes k}$ for $k = 1, 2, 3, ..., n$. Then we will construct a scheme X_s and a finite morphism $\pi : X_s \longrightarrow X$.

Let $p : \mathbb{P}(\mathcal{O} \oplus L) \longrightarrow X$ be the natural projection and $\mathcal{O}(1)$ the relatively ample line bundle.

Then $p_*(\mathcal{O}(1)) \simeq \mathcal{O} \oplus L^*$ which has a canonical section namely the constant section 1 of $\mathcal{O}$. This gives a section of $\mathcal{O}(1)$ over $\mathbb{P}(\mathcal{O} \oplus L)$ (as $H^0(\mathcal{O}(1)) = H^0(p_*(\mathcal{O}(1))))$ which we will denote by y. On the other hand $p_*(p^*L \otimes \mathcal{O}(1))$ is by projection formula isomorphic to $L \otimes p_*(\mathcal{O}(1)) \simeq L \otimes (\mathcal{O} \oplus L^*) \simeq L \oplus \mathcal{O}$. Hence it also has a canonical section and we denote the corresponding section of $p^*L \otimes \mathcal{O}(1)$ by x.

Now consider the section $x^n + p^*s_1 x^{n-1} y + p^*s_2 x^{n-2} y^2 + ... + p^*s_n y^n$ of $p^*L^n \otimes \mathcal{O}(n)$.

Let X_s denote its zero scheme. It is then clear that the restriction π of p to X_s is finite and that at any point v of X the fibre over v is the subscheme of $\mathbb{P}^1$ given by

$x^n + a_1 x^{n-1} y + ... + a_n y^n = 0$ where (x, y) is a homogeneous co-ordinate system and a_i is the value of s_i identifying the fibre of L at $\mathbb{P}(\mathcal{O} \oplus L)$ with the residue field at v.

Let E be a vector bundle over X of rank n, L a line bundle over X and $\varphi : E \longrightarrow E \otimes L$ a homomorphism of vector bundles. Then one can define its trace as an element of $\Gamma(L)$. More generally, its characteristic co-efficient $a_i \in \Gamma(X, L^i)$ for $0 \leq i \leq n$ may be defined by setting $a_i = (-1)^i Trace \wedge^i \varphi$.

Proposition 2.3.1. *Let X be a smooth projective curve over an algebraically closed field of characteristic 0 and L be any line bundle having nonzero section over X. Let $s = (s_i)$ be an n-tuple with sections s_i of L^i for $1 \leq i \leq n$. Assume that the corresponding scheme X_s as constructed above is integral. Then there is a bijective correspondence between isomorphism classes of pairs (E, φ) where E is a vector bundle over X of rank n and $\varphi : E \longrightarrow E \otimes L$ a homomorphism of vector bundles with characteristic co-efficients (s_i) and the isomorphism classes of torsion free sheaves of rank 1 over X_s.*

The correspondence is given by associating to any line bundle M over X_s the sheaf $\pi_*(M)$ on X and the natural homomorphism $\pi_*(M) \longrightarrow L \otimes \pi_*(M) \simeq \pi_*(\pi^* L \otimes M)$ given by the section x of $\pi^*(L)$.

Proof. See [1, Proposition, 3.6]. □

Let $\mathcal{M}(n, d, L)$ be the moduli space of S-equivalence classes of semistable Higgs pairs. Let $\mathcal{H} = \bigoplus_{i=1}^{n} H^0(X, L^i)$. The space $\mathcal{H}$ is known as *Hitchin space.*

Proposition 2.3.2. *If (E, φ) and (E', φ') are S-equivalent, then their characteristic polynomials are same.*

Proof. See [9, Proposition, 4.4]. □

Therefore we have a morphism from the moduli space $\mathcal{M}(n, d, L)$ to the space $\mathcal{H}$, namely the map which takes a Higgs pair (E, φ) to its characteristic co-efficients in $\mathcal{H}$. This map is known as the *Hitchin map.*

Let X be a smooth curve over an algebraically closed field of characteristic 0 of genus $g \geq 2$. We fix the line bundle L to be K, where K is the canonical line bundle over X and denote the total space of the canonical line bundle by Y. Then the genus of the curve X_s can be obtained by the adjunction formula

$$K_Y.X_s + X_s^2 = 2g(X_s) - 2,$$

where K_Y is the canonical line bundle over Y. Since Y is a symplectic manifold, $K_Y = 0$ and X_s is in the linear system nK_X. The zero section is in the linear system K_X and has self-intersection number

$$K_X^2 = 2g - 2.$$

Therefore

$$2.g(X_s) - 2 = 2n^2(g - 1),$$

i,e,

$$g(X_s) = n^2(g - 1) + 1.$$

Let us concentrate on Higgs pairs of rank 2 over a smooth curve X of genus g over an algebraically closed field of characteristic zero and the fixed line bundle to be the canonical line bundle K over X. In this case the Hitchin space $\mathcal{H}$ is $H^0(X, K) \oplus H^0(X, K^2)$. If we consider trace-free Higgs fields then the corresponding Hitchin space is $H^0(X, K^2)$.

Theorem 2.3.3. *Let X be a smooth curve over the complex field $\mathbb{C}$ of genus $g \geq 2$. The moduli space of all stable pairs (V, φ), where V is a rank 2 vector bundle of fixed determinant and odd degree and φ is a trace-free section of $EndV \otimes K$, is a smooth manifold of dimension $6(g-1)$.*

Proof. See [5, Theorem, 5.8]. □

Theorem 2.3.4. *Let $\mathcal{M}_H$ be the moduli space of stable Higgs pairs (V, φ) over X, where V is a vector bundle of rank 2 and odd degree with fixed determinant and φ is a trace free section of $EndV \otimes K$. Then the map,*

$$det : \mathcal{M}_H \longrightarrow H^0(X, K^2)$$

satisfies the following property:
(1) det is proper;
(2) det is surjective;
(3) If $q \in H^0(X, K^2)$ is a quadratic differential with simple zeros, then $det^{-1}(q)$ is biholomorphically isomorphic to the Prym variety of the double covering of X determined by q;
(4) The cotangent bundle of the moduli space of the stable vector bundles of rank 2 and odd degree with fixed determinant over X lies naturally in $\mathcal{M}_H$ as the complement of an analytic set of co-dimension at least g.

Proof. See [5, Theorem, 8.1], [6, Theorem. 6.1]. □

Theorem 2.3.4 says that the cotangent bundle of the moduli space of stable bundles is an open dense subset of the moduli of stable Higgs pairs.
Therefore one may ask what bundles are not stable but occur in a stable pair. The following proposition answers this question in the rank 2 case.

Proposition 2.3.5. *Let X be a compact Riemann surface of genus $g \geq 1$. A rank 2 vector bundle V with fixed determinant δ of odd degree, occurs in a stable pair (V, φ) if and only if one of the following holds:*

(1) V is stable;

(2) V is not stable and $dimH^0(X, L_V^{-2} \otimes K \otimes \delta)$ is greater than 1, where L_V is the unique subbundle of V with $degreeL_V > \frac{1}{2}$ $degreeV$.

(3) V is decomposable as $V = L_V \oplus (L_V^* \otimes \delta)$ and $dimH^0(X, L_V^{-2} \otimes K \otimes \delta) = 1$.

Proof. As we have already remarked, if the rank and the degree are coprime, stability and semistability of bundles are the same by our hypothesis. If V is stable, then clearly (V, φ) is stable for all φ.

Assume V is not stable. Then there exists a unique line subbundle L of V with degree$(L) > \frac{1}{2}$ degree(V) and V is an extension of $L^* \otimes \delta$ by L.

Consider the exact sequence

$$0 \longrightarrow L \longrightarrow V \longrightarrow L^* \otimes \delta \longrightarrow 0. \tag{2.3.1}$$

Tensoring by the canonical line bundle K we have

$$0 \longrightarrow L \otimes K \longrightarrow V \otimes K \longrightarrow L^* \otimes \delta \otimes K \longrightarrow 0.$$

From the above two exact sequences we get a subbundle

$$K \otimes L^2 \otimes \delta^* \subset \mathrm{End}_0 V \otimes K$$

of trace-free endomorphisms which vanish on L. Since

$$\text{degree } (K \otimes L^2 \otimes \delta^*) > 2g - 2,$$

this bundle has a non-zero section. Therefore the only subbundle invariant by all $\varphi \in H^0(X; K \otimes L^2 \otimes \delta^*)$ is L. Hence in particular if any line subbundle of V is invariant for all $\varphi \in H^0(X;\ \mathrm{End}_0 V \otimes K)$ then it must be L.

Let us consider the exact sequence of vector bundles

$$0 \longrightarrow F \longrightarrow \mathrm{End}_0 V \otimes K \longrightarrow L^{-2} \otimes K \otimes \delta \longrightarrow 0, \tag{2.3.2}$$

where F is the kernel of the surjective homomorphism $\mathrm{End}_0 V \otimes K \longrightarrow L^{-2} \otimes K \otimes \delta$, and can be thought as the bundle of endomorphisms $V \longrightarrow V \otimes K$ which preserves the exact sequence 2.3.1.

Sections of F are the sections $\varphi \in H^0(X, \mathrm{End}_0 V \otimes K)$ which leave L invariant. If V does not occur in a stable pair then for all $\varphi \in H^0(X;\ \mathrm{End}_0 V \otimes K)$, L is invariant. Therefore

$$H^0(X, F) = H^0(X, \mathrm{End}_0 V \otimes K).$$

Thus from the long exact sequence of 2.3.2 we have the coboundary map:

$$\partial : H^0(X, L^{-2} \otimes K \otimes \delta) \longrightarrow H^1(X, F) \tag{2.3.3}$$

is injective. Again consider the exact sequence of vector bundles

$$0 \longrightarrow K \otimes L^2 \otimes \delta^* \longrightarrow F \longrightarrow K \longrightarrow 0. \tag{2.3.4}$$

Since degree $(K \otimes L^2 \otimes \delta^*) > 2g-2$, we have $H^1(X; K \otimes L^2 \otimes \delta^*) = 0$. Therefore from the cohomology sequence of 2.3.4 we have

$$\vartheta : H^1(X; F) \simeq H^1(X; K) \simeq \mathbb{C}.$$

Thus if $\dim H^0(X; L^{-2} \otimes K \otimes \delta) \geq 2$, then ∂ can never be injective, and we have a contradiction to the assumption that V occurs in a stable pair which proves the case (2).

The map

$$\vartheta\partial : H^0(X, L^{-2} \otimes K \otimes \delta) \longrightarrow H^1(X, K)$$

is given by the product with the extension class $e \in H^1(X, L^2 \otimes \delta^*)$ defining V. By Serre duality, this is surjective if $e \neq 0$. Therefore if $\dim H^0(X, L^{-2} \otimes K \otimes \delta) = 1$ then the map ∂ fails to be injective if V is the trivial extension. This provides case (3).

Conversely, let V be a bundle which is not covered by Case (1-3). Then V is a non-stable bundle with unique line subbundle L such that degree$L > \frac{1}{2}$ degree$V$ and $\dim H^0(X; L^{-2} \otimes K \otimes \delta) \leq 1$, the equality occurring when V is not the trivial extension. Therefore from the long exact sequence of cohomology of the sequence 2.3.2 it follows that $H^0(X; F) = H^0(X; \text{End}_0 V \otimes K)$. Thus all trace-free endomorphisms $V \longrightarrow V \otimes K$ leave L invariant. Hence V does not occur in a stable pair.

□

2.4 Quadrics in $\mathbb{P}^N$ and Hyperelliptic Curves

Theorem 2.4.1. *A smooth quadric Q of dimension m contains no linear spaces of dimension strictly greater than $m/2$. On the other hand:*

(1) If $m = 2g+1$, then Q contains an irreducible $(g+1)(g+2)/2$ dimensional family of n-planes while;

(2) If $m = 2g$, then Q contains two irreducible components of $g(g+1)/2$ dimensional family of n-planes. Moreover for any two g-planes $\Lambda, \Lambda' \subset Q$, dim $(\Lambda \cap \Lambda') \equiv g(2)$
if and only if Λ and Λ' belong to the same family.

(3) In case (2) for every $(g-1)$-plane contained in Q, there exist two g-planes in Q containing it and these belong to opposite families.

Proof. See [3] [4]. □

2.5 Intersection of two smooth quadrics in $\mathbb{P}^n(V)$ with dim $(V) = (n+1)$

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2.5.1 Pencil of Quadrics

If $\mathbb{P}^n = \mathbb{P}^n(V)$ where V is a vector space over $\mathbb{C}$ of dimension (n+1), then any quadric in $\mathbb{P}^n$ can be thought of a quadratic form on V, i,e, an element of $S^2(V^*)$.

Definition 2.5.1. *A pencil of quadratic forms Φ on V is a projective line $\mathbb{P}^1_\Phi \subset \mathbb{P}(S^2(V^*))$.*

Proposition 2.5.2. *Let $Y = Q_1 \cap Q_2$ be an intersection of two quadrics Q_1 and Q_2 in $\mathbb{P}^n$. Then the following are equivalent:*

(1) Y is nonsingular and of codimension 2 in $\mathbb{P}^n$ and Q_i are non-degenerate.

(2) There exists a basis of V, orthogonal for all $Q_\lambda = \lambda_1 Q_1 + \lambda_2 Q_2$ where $(\lambda_1, \lambda_2) \in \mathbb{P}^1$ such that

$$Q_1(\sum x_i e_i) = \sum x_i^2$$
$$Q_2(\sum x_i e_i) = \sum \lambda_i x_i^2$$

with $\lambda_i \neq \lambda_j$ for $i \neq j; \lambda_i{}'s \neq 0$.

Further more, the basis $\{e_i\}$ is unique up to changes of sign and order.

Proof. The projective tangent space $T_y(Q_1)$ at any $y \in Y$ to Q_1 is $v^{\perp q_1}$, where $v \in V \setminus 0$ represents y. Since $Q_1 \cap Q_2$ (as a scheme) is smooth by hypothesis, $T_y(Q_1 \cap Q_2)$ is the intersection of $T_y(Q_1)$ and $T_y(Q_2)$. If we denote by $v^\perp$ the subspace of V given by $\{w \in V : q_1(v,w) = 0$ and $q_2(v,w) = 0\} = \{w \in V : q(v,w) = 0$ for all q in the pencil $\}$, then $v^\perp$ is of codimension 2.

Consider the endomorphism of V given by $q_2^{-1} \circ q_1$, where $q_i{}'$s are the homomorphisms $V \longrightarrow V^*$ given by q_i. We claim that its eigen-values are of multiplicity 1. For if $W \subset V$ is an eigen space of dim ≥ 2, then since $q_{1|_W} = \lambda q_{2|_W}$ for some λ, there exists an $w \in W \setminus 0$ such that $q_1(w,w) = q_2(w,w) = 0$. Thus there exists $y \in Y$ represented by w such that $y^{\perp q_1} = y^{\perp q_2}$, thus $y^\perp$ is of co-dimension 1 in V, which is a contradiction. This implies that $q_2^{-1} \circ q_1$ is diagonalisable with distinct eigen-values. Hence there is a basis $e_1, e_2, ..., e_n$ of V such that $q_1 e_i = \lambda_i q_2 e_i$ and $\lambda_i{}'$s are distinct. But $q_1(e_i, e_j) = q_1(e_j, e_i)$ which implies that $\lambda_i q_2(e_i, e_j) = \lambda_j q_2(e_j, e_i)$. But $\lambda_i \neq \lambda_j$. Hence $q_2(e_i, e_j) = 0$ if $i \neq j$. Since q_1 is non-degenerate we can choose e_i's as orthonormal with respect to q_1. Therefore Q_1 and Q_2 are of the form stated in the theorem.

(2) $\Longrightarrow$ (1) is obvious. □

Theorem 2.5.3. *Let Y be a nonsingular intersection of two quadrics and let $\dim Y = 2g-1$. Let Gr be the Grassmanian of r-planes of $\mathbb{P} = \mathbb{P}^{2g+1}$ and let $S \subset Gr$ be the subvariety of r-planes lying on Y. Then S is nonsingular, reduced and of dimension $(2g-2r-1)(r+1)$.*

The proof of this theorem can be found in [11]. Here we will give a different proof.

Proof. We will prove the theorem by induction on r. Clearly the theorem is true for $r = 0$. Let us assume the theorem is true for all $r' < r$.

Define $I := \{(p, \Lambda_r) : p \in \Lambda_r \subset Y\}$, where Λ_r denotes r-plane, as a subspace of $Y \times Gr(r+1, 2g+2)$. The restriction to I of the projection map to the second factor gives a surjective map from I onto S with fibres isomorphic to $\mathbb{P}^r$ and the fibres over $p \in Y$ of the projection map to the first factor are the r-planes in Y passing through p.

If $p \in \Lambda_r \subset Y$ then $\Lambda_r \subset T_pY$, where T_pY denotes the projective tangent space to Y at p. Therefore $\Lambda_r \subset Y \cap T_pY$. But since $Y \cap T_pY$ is a cone with vertex p and base, a smooth intersection of quadrics Y' in $\mathbb{P}^{2g-2}$, r-planes passing through p in Y is isomorphic to $S' := \{(r-1)-$ planes in $Y'\}$. By induction hypothesis S' is smooth, reduced and of dimension $2r(g-r-1)$ and since Y is smooth and every fibre is smooth of constant dimension, I is smooth. Therefore the dimension of I is

$$2r(g-r-1) + 2g - 1 = (r+1)(2g-2r-1) + r.$$

Therefore the dimension of S is

$$(r+1)(2g-2r-1) + r - r = (r+1)(2g-2r-1)$$

and the smoothness of S follows from the fact that I is smooth and the fibres of the surjective morphism $I \longrightarrow S$ are smooth and of constant dimension.

□

Now we will construct a hyperelliptic curve of genus g in terms of intersection of quadrics and variety of linear spaces, contained in it.

Let V be a vector space of dimension $2g+2$. Consider the projective space $\mathbb{P}(V) = \mathbb{P}^{2g+1}$. Let q_1 and q_2 be two non-degenerate quadratic forms in $\mathbb{P}^{2g+1}$ and let $\mathbb{P}^1_\Phi$ be the pencil consisting the quadratic forms $\{q_\lambda\}_{\lambda \in \mathbb{P}^1}$ of the form $\lambda_1 q_1 + \lambda_2 q_2$ for $\lambda = (\lambda_1, \lambda_2) \in \mathbb{P}^1$.

The family of g planes fit together in the following way: let

$$\text{Gen }(\Phi) \subset \mathbb{P}^1_\Phi \times \text{ Gr },$$

where $\text{Gr} = \text{Gr}(g+1, V)$ is the usual Grassmanian, be defined by

$$\text{Gen}(\Phi) = \{(\lambda, E) : q_\lambda \mid_E = 0\}.$$

It is obvious that the first projection $\text{Gen}(\Phi) \longrightarrow \mathbb{P}^1$ has as fibre over λ, the variety of g planes of q_λ.

Theorem 2.5.4. *$Gen(\Phi)$ is nonsingular, and the morphism $p_1 : Gen(\Phi) \longrightarrow \mathbb{P}^1$ has the Stein factorization*

$$\begin{array}{ccc} Gen(\Phi) & \xrightarrow{\;p\;} & C \\ {\scriptstyle p_1}\big\downarrow & \swarrow{\scriptstyle q} & \\ \mathbb{P}^1 & & \end{array}$$

where C is nonsingular, q is a double covering ramified precisely in $Sing(\Phi)$, where $Sing(\Phi)$ denotes the singular quadrics in the pencil, and p is smooth. In particular the fibres of p_1 are pairs of families of g planes contained in a quadric.

Proof. See [11, Theorem, 1.10] □

If we set $C := \{(Q_\lambda, V_1), \lambda \in \mathbb{P}^1, V_1$ is an irrducible component of g-planes in $Q_\lambda\}$

Then by above the theorem, C is a curve of genus g.

Remark 2.5.5. *(1) Here the ramification points are those $\lambda \in \mathbb{P}^1$ for which the corresponding quadric Q_λ in the pencil is singular.*

By Proposition 2.5.2 we can take

$$Q_1 = \sum_{i=1}^{2g+2} x_i^2 \text{ and } Q_2 = \sum_{i=1}^{2g+2} \lambda_i x_i^2.$$

Therefore the singular quadrics in the pencil are precisely,

$$\lambda_i Q_1 - Q_2, i = 1, 2, ..., 2g + 2.$$

(2) Any hyperelliptic curve can be obtained this way.

The geometry of the Jacobian of a hyperelliptic curve of any genus has described in [2] which says that:

Theorem 2.5.6. *Let X be a smooth projective hyperelliptic curve of arbitrary genus g over an algebraically closed field of characteristic zero. Then the Jacobian J^g of isomorphism classes of the line bundle of degree g over X is canonically isomorphic to the variety of all $(g-1)$ planes contained in the intersection of two smooth quadrics in $\mathbb{P}^{2g+1}$ which defines the curve X as above.*

Here we will give only the map which assign to $\alpha \in J^g$ a $(g-1)$ plane. The details of the proof can be found in [2].

Fix a line bundle ξ of degree $2g+1$ over X and fix an isomorphism $\xi \otimes \iota^*\xi \approx h^{2g+1}$, where ι is the hyperelliptic involution and h is the hyperelliptic line bundle.

Let $\alpha \in J^g$ and an isomorphism $\alpha \otimes \iota^*\alpha \approx h^g$ be given. Consider the exact

$$0 \longrightarrow \alpha^2 \otimes h^{-g} \otimes \iota^*\xi \otimes \mathcal{O}(-W) \longrightarrow \alpha^2 \otimes h^{-g} \otimes \iota^*\xi \longrightarrow \alpha^2 \otimes h^{-g} \otimes \iota^*\xi_{|W} \longrightarrow 0,$$

where W is the set of Weierstrass points. Since $\mathcal{O}(W) \simeq h^{g+1}, H^0(X;\alpha^2 \otimes h^{-g} \otimes \iota^*\xi \otimes \mathcal{O}(-W)) = 0$ and hence we can consider $H^0(X, \alpha^2 \otimes h^{-g} \otimes \iota^*\xi)$ as a subspace of $\Sigma_{w\in W}\xi_w$.

Set $\mathbb{P}^1 = X/\iota$. On $(\Sigma\xi_\omega)_{\mathbb{P}^1}$ define a quadratic form by canonical map $\xi_\omega^2 \simeq h_\omega^{2g+1} \simeq \tau_\omega^{2g+1}$, where τ is the hyperplane bundle over $\mathbb{P}^1$. Now the space of sections of τ^{2g+1} vanishing at all points of $W \setminus \omega$ is isomorphic to τ_ω^{2g+1} via evaluation map at ω. Compose the inverse of this isomorphism with the evaluation map

$$(H^0(\mathbb{P}^1, \tau^{2g+1}))_{\mathbb{P}^1} \longrightarrow \tau^{2g+1},$$

we get a quadratic form on $\Sigma\xi_\omega$ with values in τ^{2g+1}.

For each $y \in \mathbb{P}^1 - W$, the above quadratic form restricts to the subspace $H^0(X, \alpha^2 \otimes h^{-g} \otimes \iota^*\xi)$ as follows.

Let s be a section of $\alpha^2 \otimes h^{-g} \otimes \iota^*\xi$. Then $s \otimes \iota^*s$ is an ι -invariant section of h^{2g+1} and by evaluation at y we get a quadratic form with values in τ_y^{2g+1}.

Now consider the following commutative diagram

$$\begin{array}{ccc} H^0(X, \alpha^2 \otimes h^{-g} \otimes \iota^*\xi) & \longrightarrow & H^0(\mathbb{P}^1, \tau^{2g+1}) \\ \downarrow & & \downarrow \\ \Sigma\xi_w & \longrightarrow & \Sigma\tau_w^{2g+1}. \end{array}$$

Here the top horizontal map is described above and the lower horizontal map is the quadratic map obtained by squaring ξ_w and identifying $\xi_w^2 = (\xi \otimes \iota^*\xi)_w$ with τ_w^{2g+1}.

Using exact sequence defined by the divisor $(y + \iota y)$

$$0 \longrightarrow \alpha^2 \otimes h^{-g} \otimes \iota^*\xi \otimes h^{-1} \longrightarrow \alpha^2 \otimes h^{-g} \otimes \iota^*\xi \longrightarrow (\alpha^2 \otimes h^{-g} \otimes \iota^*\xi)_{|(y+\iota y)} \longrightarrow 0.$$

we can consider $H^0(X, \alpha^2 \otimes h^{-g} \otimes \iota^*\xi \otimes h^{-1})$ as a subspace of $H^0(X, \alpha^2 \otimes h^{-g} \otimes \iota^*\xi)$ and from the above exact sequence and the commutative diagram it follows that $H^0(X, \alpha^2 \otimes h^{-g} \otimes \iota^*\xi \otimes h^{-1})$ is contained in each quadraic defined by $y \in \mathbb{P}^1$. This defines the map we needed.

Let Q_1 and Q_2 be two smooth quadrics is $\mathbb{P}(V)$, where V is a 6 dimensional vector space over $\mathbb{C}$. Set $Y = Q_1 \cap Q_2$.

Definition 2.5.7. $\Sigma := \{y \in Y :$ *such that* T_yQ_2 *is tangent to* $Q_1\}$

We will now give an explicit description (as in [3]) of $\Sigma \subset Y$. In fact we will show that Σ is a smooth intersection of three quadrics Q_1, Q_2 and say, Q_3.

As in Proposition 2.5.2 there is a basis for V such that the quadrics Q_1 and Q_2 can be taken as

$$Q_1 = \sum_{i=1}^{6} x_i^2, \text{ and } Q_2 = \sum_{i=1}^{6} \lambda_i x_i^2,$$

where λ_i's are distinct non-zero scalar. Let $x = (a_1, a_2, ..., a_6)$ be an element of Σ. Then T_xQ_2 is tangent to Q_1 at some point $y = (b_1, b_2, ..., b_6) \in Q_1$.

But T_xQ_2 is given by the linear equation $\sum_{i=1}^{6} a_i\lambda_i x_i$. Similarly T_yQ_1 is given by $\sum_{i=1}^{6} b_i x_i$. Therefore T_xQ_2 is tangent to Q_1 at y, i.e.,

$$T_xQ_2 = T_yQ_1, \text{ if and only if there is a } c \in \mathbb{C}^*, \text{ with}$$

$b_i = ca_i\lambda_i$. Since $y \in Q_1$,

$$\sum_{i=1}^{6} b_i^2 = 0, \text{ which implies that } c^2 \sum_{i=1}^{6} a_i^2 \lambda_i^2 = 0.$$

Therefore $x \in Q_3$, where Q_3 is the quadric in $\mathbb{P}^5$ given by $\sum_{i=1}^{6} \lambda_i^2 x_i^2$.

Thus $\Sigma = Q_1 \cap Q_2 \cap Q_3$.

The smoothness of the intersection is obvious.

Alternative proof:

Let $x = [x_0, ..., x_5]$ be homogeneous coordinates on $\mathbb{P}^5$, and suppose that Q_1 and Q_2 are given as the loci

$$q_1(x, x) = 0 \text{ and } q_2(x, x) = 0,$$

where q_1 and q_2 also denote the bilinear forms corresponding to the quadrics Q_1 and Q_2 respectively. Then Q_i defines an isomorphism $q_i : \mathbb{P}(V) \longrightarrow \mathbb{P}(V^*)$ taking $x \longrightarrow q_i(x, .)$. Restriction $\mathcal{G}_{q_i}$ of the map to q_i is the map takes $x \in Q_i \longrightarrow T_xQ_i$, known as Gauss map for q_i. The image of the Gauss map is a hypersurface in $\mathbb{P}(V^*)$ given by

$$Q_i^* = (q_i^{-1}(x^*, x^*) = 0),$$

where x^* denote the projective tangent space T_xQ_i We see from this that for $x \in Q_2, T_x(Q_2)$ will be tangent to Q_1 if and only if

$$\mathcal{G}_{Q_2}(x) \in Q_1^*,$$

i.e, when

$$q_1^{-1}(q_2(x,.), q_2(x,.)) = 0,$$

which implies that

$$q_2 q_1^{-1} q_2(x,x) = 0.$$

The surface Σ is thus cut out by the quadric hypersurface

$$Q_3 = (q_2 q_1^{-1} q_2(x,x) = 0).$$

We claim now that in fact the intersection

$$\Sigma = Q_1 \cap Q_2 \cap Q_3$$

is everywhere transverse. To see this, suppose that for some $x \in Q_1 \cap Q_2 \cap Q_3$ the hyperplanes $T_x(Q_1), T_x(Q_2)$ and $T_x(Q_3)$ were linearly dependent, i.e., that the points

$$\mathcal{G}_{Q_1}(x) = q_1(x,.), \mathcal{G}_{Q_2}(x) = q_2(x,.) \text{ and } \mathcal{G}_{Q_3}(x) = q_2 q_1^{-1} q_2(x,.)$$

in $\mathbb{P}^{5*}$ lay on a line. The three points

$$x, x' = q_1^{-1} q_2(x,.) \text{ and } x'' = (q_1^{-1} q_2)^2(x,.)$$

would then likewise be collinear in $\mathbb{P}^5$. Since $\mathcal{G}_{Q_2}(x) \in Q_1^*$ and $\mathcal{G}_{Q_1}(x) = q_1(x,.), \mathcal{G}_{Q_2}(x) = q_2(x,.)$ and $\mathcal{G}_{Q_3}(x) = q_2 q_1^{-1} q_2(x,.)$ are colinear and hence $\mu_1 \mathcal{G}_{Q_1}(x) + \mu_2 \mathcal{G}_{Q_2}(x) + \mu_3 \mathcal{G}_{Q_3}(x) = 0$, all three points x, x', x'' lie on Q_1. Thus the line L they span would lie on Q_1. But now the linear transformation

$$M : x \mapsto q_1^{-1} q_2(x,.)$$

taking Q_1 into Q_1 takes x and x' (distinct, since by hypothesis $q_1(x,.) \neq q_2(x,.)$ for any $x \in Q_1 \cap Q_2$) into L, and so takes L into itself; thus $L \subset Q_1 \cap Q_2$. M must have a fixed point y somewhere on L. i.e., for some $y \in L$,

$$q_1(y,.) = q_2(y,.).$$

But since $L \subset Q_1 \cap Q_2$, this implies that Q_1 and Q_2 are tangent at y, a contradiction.

Chapter 3

Stable Higgs bundles associated to a non-stable bundle

From now on, we will assume X is a smooth projective curve of genus 2, over complex field. Let K be the canonical line bundle over X. Consider Higgs fields with values in K. In this chapter we will describe the stable Higgs bundles associated to a non stable vector bundle of rank 2 over X. We fix a line bundle δ over X of degree 1.

Let V be a nonstable vector bundle over X which occurs in a stable Higgs pair with $\wedge^2 V = \delta$. Let L_V be a subbundle of V of degree ≥ 1. We have already remarked that it is unique. Then we have the exact sequence

$$0 \longrightarrow L_V \longrightarrow V \longrightarrow L_V^* \otimes \delta \longrightarrow 0.$$

Tensoring with the canonical line bundle K we get the following exact sequence

$$0 \longrightarrow L_V \otimes K \longrightarrow V \otimes K \longrightarrow L_V^* \otimes \delta \otimes K \longrightarrow 0.$$

If φ is a Higgs field such that (V, φ) is a stable pair, then clearly φ is nonzero and from the above two exact sequences it follows that

$$Hom(L_V, L_V^* \otimes \delta \otimes K) \neq 0$$

$$i.e., H^0(X, L_V^{-2} \otimes \delta \otimes K) \neq 0.$$

Now since degree$L_V \geq 1$, and $H^0(X, L_V^{-2} \otimes \delta \otimes K) \neq 0$ it follows that

$$\text{degree } L_V = 1.$$

By Proposition 2.3.5, we have,

$$H^0(L_V^{-2} \otimes \delta \otimes K) \geq 1.$$

But degree $(L_V^{-2} \otimes \delta \otimes K) = 1$. Therefore $L_V^{-2} \otimes \delta \otimes K = \mathcal{O}(y)$ for some $y \in X$.
Thus dim $H^0(X, L_V^{-2} \otimes \delta \otimes K) = 1$ which is the case (3) of Proposition 2.3.5.
Therefore nonstable bundles which occur in stable Higgs pairs are of the form

$$L \oplus (L^* \otimes \delta)$$

where L is a line bundle of degree 1 satisfying the equation

$$K \otimes L^{-2} \otimes \delta = \mathcal{O}(y) \tag{3.0.1}$$

for some $y \in X$.

Modulo tensoring with a line bundle of order 2, the relevant vector bundles are thus parametrized by the points of X itself.

Now for a given point $y \in X$, there are exactly $2^4 = 16$ solutions of the equation (3.0.1). Therefore the variety of nonstable vector bundles which occur in a stable Higgs pair over X is a 16-fold covering $\tilde{X}$ of X.

Nonstable bundles which occur in a stable pair having been determined, naturally the question arises that given a nonstable vector bundle V of the form described above, what are the Higgs fields φ such that pair (V, φ) is stable.

We will answer this question in this section. Here we will describe all the Higgs field φ associated to a nonstable vector bundle $V = L \oplus (L^* \otimes \delta)$ over X such that the pair (V, φ) is a stable pair.

We take $L = \xi \otimes \delta$, where ξ is a line bundle of degree 0 over X such that

$$\xi^{-2} \otimes K \otimes \delta^{-1} = \mathcal{O}(y)$$

for some $y \in X$. Then $V = \xi^{-1} \oplus (\xi \otimes \delta)$ is a non stable vector bundle. Let $\varphi : \xi^{-1} \oplus (\xi \otimes \delta) \longrightarrow (\xi^{-1} \otimes K) \oplus (\xi \otimes K \otimes \delta)$ be a nonzero trace-free endomorphism.
Then φ can be written as a matrix

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

where $a \in H^0(X, K)$, $b \in H^0(X, K \otimes \xi^{-2} \otimes \delta^{-1})$ and $c \in H^0(X, K \otimes \xi^2 \otimes \delta)$.

The only subbundle of V with degree greater than 1/2 is $\xi \otimes \delta$ itself. Therefore (V, φ) is stable if $b \neq 0$.

Since $b \in H^0(X, K \otimes \xi^{-2} \otimes \delta^{-1})$ and $\dim H^0(X; K \otimes \xi^{-2} \otimes \delta^{-1}) = \dim H^0(X; \mathcal{O}(y)) = 1$, we can take $b = b_0$ up to nonzero scalar where b_0 is a fixed nonzero element in $H^0(X, K \otimes \xi^{-2} \otimes \delta^{-1})$ vanishing at y.

$$\text{Now } c \in H^0(X, K \otimes \xi^2 \otimes \delta) = H^0(X, K \otimes K \otimes \mathcal{O}(-y)) = H^0(X, K \otimes \mathcal{O}(\iota y))$$

where ι is the hyperelliptic involution on X. Therefore c can be taken as $c_0 s$, where c_0 is a section of $\mathcal{O}(\iota y)$ that vanishes at ιy and s is a section of $H^0(X, K)$. Therefore φ can be written as

$$\begin{bmatrix} a & c_1 b_0 \\ c_0 s & -a \end{bmatrix}$$

where c_1 is a non-zero scalar. Now the automorphism group of $V = (\xi^{-1} \oplus \xi \otimes \delta)$ acts on the space of Higgs fields by conjugation.

An automorphism of V of determinant 1 can be written as a matrix

$$\begin{bmatrix} \lambda_1 & 0 \\ \nu & \lambda_1^{-1} \end{bmatrix}$$

where $\lambda_1 \in H^0(X, \mathcal{O}) \setminus 0 \simeq \mathbb{C}^*$, $\nu \in H^0(X, \xi^2 \otimes \delta) = H^0(X, K \otimes \mathcal{O}(-y)) = H^0(X, \mathcal{O}(\iota y))$. We may take $\nu = kc_0$. Therefore an automorphism of V can be written as a matrix

$$\begin{bmatrix} \lambda_1 & 0 \\ kc_0 & \lambda_1^{-1} \end{bmatrix}.$$

Now the action of $\begin{bmatrix} \lambda & 0 \\ kc_0 & \lambda^{-1} \end{bmatrix}$ on $\begin{bmatrix} a & c_1 b_0 \\ c_0 s & -a \end{bmatrix}$ is given by

$$\begin{bmatrix} \lambda & 0 \\ kc_0 & \lambda^{-1} \end{bmatrix} \cdot \begin{bmatrix} a & b_0 \\ c_0 s & -a \end{bmatrix} \cdot \begin{bmatrix} \lambda & 0 \\ kc_0 & \lambda^{-1} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \lambda & 0 \\ kc_0 & \lambda^{-1} \end{bmatrix} \cdot \begin{bmatrix} a & c_1 b_0 \\ c_0 s & -a \end{bmatrix} \cdot \begin{bmatrix} \lambda^{-1} & 0 \\ -kc_0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & 0 \\ kc_0 & \lambda^{-1} \end{bmatrix} \cdot \begin{bmatrix} a\lambda^{-1} - kc_1 c_0 b_0 & c_1 b_0 \lambda \\ c_0 s \lambda^{-1} + akc_0 & -a\lambda \end{bmatrix}$$

$$= \begin{bmatrix} a - kc_1 c_0 b_0 \lambda & c_1 b_0 \lambda^2 \\ a\lambda^{-1} kc_0 - k^2 c_1 c_0^2 b_0 + c_0 s \lambda^{-2} + ak\lambda^{-1} c_0 & -a + kc_1 c_0 b_0 \lambda \end{bmatrix}.$$

Let $b_0c_0 = s_0 \in \Gamma(X, K)$.

Then the above matrix is

$$= \begin{bmatrix} a - \lambda k c_1 s_0 & c_1 b_0 \lambda^2 \\ 2\lambda^{-1} k a c_0 - k^2 c_1 c_0 s_0 + \lambda^{-2} c_0 s & -a + \lambda k c_1 s_0 \end{bmatrix}$$

Therefore two Higgs fields ,

$$\begin{bmatrix} a_1 & c_1 b_0 \\ c_0 s_1 & -a_1 \end{bmatrix}, \begin{bmatrix} a_2 & c_2 b_0 \\ c_0 s_2 & -a_2 \end{bmatrix}$$

are equivalent if and only if for some $\lambda \in \mathbb{C}^*$, we have

$$a_2 = a_1 - k\lambda c_1 s_0$$

$$\text{and } s_2 = \lambda^{-2} s_1 - k^2 c_1 s_0 + 2\lambda^{-1} k a_1$$

$$\text{and } c_1 \lambda^2 = c_2.$$

It is clear that given an element $(a, s, c) \in H^0(X, K) \times H^0(X, K) \times \mathbb{C}^*$ we can associated a Higgs field namely,

$$\begin{bmatrix} a & cb_0 \\ c_0 s & -a \end{bmatrix}$$

Therefore we have a surjective homomorphism from $H^0(X, K) \times H^0(X, K) \times \mathbb{C}^*$ to the space of Higgs fields.

Lemma 3.0.8. *Under the above equivalence the space of Higgs fields on V is isomorphic to the two sheeted covering of $S^2(H^0(X, K))$, ramified along $s_0.H^0(X; K)$.*

Proof. Let D denote the space of Higgs fields. We define a map from D to $S^2(H^0(X, K))$ by

$$\begin{bmatrix} a & c_1 b_0 \\ c_0 s & -a \end{bmatrix} \mapsto a^2 + c_1 s s_0.$$

Clearly this is surjective.

In fact if $s_1^2, s_1 s_2, s_2^2$ are a basis of $S^2(H^0(X, K))$ then,

$$\begin{bmatrix} s_1 & b_0 \\ 0 & s_1 \end{bmatrix} \mapsto s_1^2, \begin{bmatrix} s_2 & b_0 \\ 0 & s_2 \end{bmatrix} \mapsto s_2^2, \text{ and since } s_0 = \mu_1 s_1 + \mu_2 s_2 \text{ for some } \mu_1, \mu_2 \in \mathbb{C}$$

$$\begin{bmatrix} 0 & b_0 \\ c_0 s_1 & 0 \end{bmatrix} \mapsto \mu_1 s_1^2 + \mu_2 s_1 s_2$$

But

$$\begin{bmatrix} a_1 & c_1b_0 \\ c_0s_1 & -a_1 \end{bmatrix} \text{ and } \begin{bmatrix} -a_1 & c_1b_0 \\ c_0s_1 & a_1 \end{bmatrix}$$

gives the same element in $S^2(H^0(X,K))$ by this map.

Let we define an action of $\mathbb{Z}_2$ on D by taking

$$\begin{bmatrix} a & c_1b_0 \\ c_0s & -a \end{bmatrix} \mapsto \begin{bmatrix} -a & c_1b_0 \\ c_0s & a \end{bmatrix}$$

Let $D' = D/\mathbb{Z}_2$, then clearly we have an well-defined surjective map from D' to $S^2(H^0(X,K))$.

Now we will prove that this map is injective .

Let

$$a_1^2 + c_1s_0s_1 = a_2^2 + c_2s_0s_2$$

$$\Rightarrow (a_1^2 - a_2^2) = s_0(c_2s_2 - c_1s_1)$$

$$\Rightarrow (a_1 + a_2)(a_1 - a_2) = s_0(c_2s_2 - c_1s_1).$$

Since $S^2(H^0(X,K))$ is unique factorization domain ,

either $a_1 - a_2 = s_0$ and $a_1 + a_2 = c_2s_2 - c_1s_1$

Or, $a_1 + a_2 = s_0$ and $a_1 - a_2 = c_2s_2 - c_1s_1$

Now $a_1 - a_2 = s_0 \Rightarrow a_2 = a_1 - s_0$
and $a_1 + a_2 = c_2s_2 - c_1s_1 \Rightarrow c_2s_2 = c_1s_1 + (a_1 - s_0) + a_1$

$\Rightarrow s_2 = \frac{2a_1}{c_2} - \frac{s_0}{c_2} + \frac{c_1s_1}{c_2}$

Which means

$$\begin{bmatrix} a_1 & c_1b_0 \\ c_0s_1 & -a_1 \end{bmatrix} \text{ and } \begin{bmatrix} a_2 & c_2b_0 \\ c_0s_2 & -a_2 \end{bmatrix}$$

are equivalent under the equivalence defined earlier.
The argument for the other case is similar.

Therefore $D' \simeq S^2(H^0(X,K))$. Hence the proof is complete. □

Therefore the space of stable Higgs pairs with nonstable vector bundles is isomorphic to a bundle over $\tilde{X}$ with fibres isomorphic to D, where $\tilde{X}$ has been defined earlier and D is a two sheeted covering of $H^0(X, K^2)$.

From the above Lemma and the previous discussion we have the following theorem.

Theorem 3.0.9. *Let X be a smooth projective curve over $\mathbb{C}$ of genus 2. Then the moduli space of stable Higgs pairs over X of rank 2 with fixed determinant of odd degree contains the cotangent bundle of the moduli space of stable vector bundles over X of rank 2 and of odd degree as an open dense subset whose complement is isomorphic to a bundle $\mathcal{G}$ over $\tilde{X}$, with fibres isomorphic to D where $\tilde{X}$ and D as above.*

Chapter 4

Kummer Surface and Special line bundles

To each point of the curve X, one can associate a Kummer surface defining an involution on the Jacobian of X corresponding to a point of X. Using theorem 2.5.6 we will identify the Jacobian of X with lines in the intersection of two smooth quadrics in $\mathbb{P}^5$. We will define here the special lines in the Jacobian as the fixed points of involutions (in other words, double points of Kummer surfaces parametized by X) and define special line bundles of degree zero on the other hand. We will prove that under the above they correspond to one another.

4.1 Kummer Surface

Definition 4.1.1. *Let X be a smooth curve over an algebraically closed field of characteristic 0 of genus 2 and J be its Jacobian. Then the Kummer surface is the quotient of the Jacobian J by the involution $a \mapsto -a$, that is, by identifying each point with its inverse under the group law.*

Remark 4.1.2. *Note that Kummer surface is not smooth. It has singularity as double points at the fixed point of the involution ι.*

We have already seen that the points of the curve X can be identified with pairs $(Q, V_1)$where Q is a quadric in the pencil $\mathbb{P}(W)$ generated by Q_1 and Q_2 and V_1 is an irreducible component of the variety of 2-planes contained in Q.

Also by Theorem 2.5.6 the Jacobian of the curve X can be identified with the space of lines contained in all quadrics of the pencil $\mathbb{P}(W)$.

Let $J = \{\text{lines contained in all quadrics in the pencil}\}$ and (Q, V_1) be a point on X. We define an involution on J using (Q, V_1). Let $L \in J$ be an element. Then by Theorem 2.4.1 there exists a unique plane in V_1 containing L, say Λ. The plane Λ intersects all other quadrics of the pencil $\mathbb{P}(W)$ in a pair of lines, namely L and say, $L_{(Q,V_1)}$.
Therefore the map $L \longrightarrow L_{(Q,V_1)}$ defines an involution on $\iota_{(Q,V_1)}$ on J and the quotient of J by the involution $\iota_{(Q,V_1)}$ gives a Kummer surface defined by the point (Q, V_1), which can be thought of as a surface in $\mathbb{P}^3$ where $\mathbb{P}^3$ is the space of 2-planes contained in Q. This Kummer surface can be classified by the planes in Q such that the restriction of the planes at the other quadratic forms in the pencil has $rank \leq 2$. Moreover the planes whose restriction has rank exactly 1, form the singular locus of the surface.

Clearly we have the obvious surjective map from the Jacobian J onto the Kummer surface $\mathcal{K}_{\iota_{(Q,V_1)}}$ and the fibres over singular locus are those lines $L \in J$ such that the unique plane $\Lambda \in V_1$ containing L has the property $\Lambda \cap Q' = 2L$ for all $Q' \in \mathbb{P}(W)$.

Definition 4.1.3. *The lines in J which are singular points of a Kummer surface for some point (Q, V_1) of the curve defined above are called special lines.*

4.2 Restriction of cotangent bundle to a line

In this section we will prove that the restriction of the co-tangent bundle of $Q_1 \cap Q_2$ to a *special line* is isomorphic to $\mathcal{O}(-1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(-2)$ and to a non-special line, $\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-2)$.

As before let $\mathbb{P}(W)$ denote the pencil of quadrics in $\mathbb{P}^5$ passing through the intersection $Q_1 \cap Q_2$ where Q_1 and Q_2 are two non-degenerate quadrics in $\mathbb{P}^5$. We denote the space $Q_1 \cap Q_2$ by Y.

Lemma 4.2.1. *If l is a special line and L be the corresponding vector space, then $L^\perp = \{v \in V$ with $Q(v, w) = 0$ for all $Q \in \mathbb{P}(W)$ and $w \in L\}$ is of dimension 3.*

Proof. Since l is a special line, there exist a unique quadric $Q \in \mathbb{P}(W)$ and a plane Λ contained in Q containing l with $\Lambda \cap Q' = 2l$ for all $Q' \in \mathbb{P}(W) \setminus Q$.
Now since Λ is contained in Q, the projective tangent space to $T_p(Q)$ to Q at any point p of Λ contains Λ. In particular, at any point p of l, $T_p(Q)$ contains Λ. Therefore $\Lambda \subset \cap_{p \in l} T_p Q$, on the other hand, since $\Lambda \cap Q' = 2l$ for all $Q' \in \mathbb{P}(W) \setminus Q$, Λ is tangent to Q' at every point of l. Therefore $\Lambda \subset \cap_{p \in l} T_p Q'$ for all $Q' \in \mathbb{P}(W) \setminus Q$ and hence $\Lambda \subset \cap_{p \in l} T_p(Q \cap Q')$. Since $\cap_{p \in L} T_p(Q \cap Q')$ has dimension at most 2, $\cap_{p \in L} T_p(Q \cap Q') = \Lambda$. i.e., $\mathbb{P}(L^\perp) = \Lambda$. Thus $L^\perp$ is of dimension 3.

□

Lemma 4.2.2. *1) Let l be a special line on Y then $T^*Y\mid_l \simeq \mathcal{O}(-1)\oplus\mathcal{O}(1)\oplus\mathcal{O}(-2)$.*

2)If l is a non-special line then $T^*Y\mid_l\simeq \mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O}(-2)$.

Proof. 1)Let p be a point on Y then we denote by $P^{\perp}$ the vector subspace $\{v\in V$ with $q(v,P)=0$ for all $q\in\mathbb{P}(W)\}$ where P denotes the vector space associated to the point p. Then the cotangent space $T_p^*(Y)$ to Y at p can be identified with $(P^{\perp}/P)^*\otimes P$. Let E_1 be the bundle on l with fibre $P^{\perp}$ for $p\in l$ and $\mathcal{O}(-1)$ be the hyperplane bundle. Now from the exact sequence

$$0\longrightarrow P^{\perp}\longrightarrow V\longrightarrow W^*\otimes P^*\longrightarrow 0,$$

where the last map is given by $v\longrightarrow q(v,.)_{|P}$, we get an exact sequence of bundles

$$0\longrightarrow E_1\longrightarrow V\otimes\mathcal{O}\longrightarrow W^*\otimes\mathcal{O}(1)\longrightarrow 0.$$

Therefore the determinant of the bundle E_1 is isomorphic to $\mathcal{O}(-2)$.

Let E_2 be the bundle over l with fibre $P^{\perp}/P$. Then the determinant of E_2 is isomorphic to $\mathcal{O}(-1)$.

If l is a special line then $L^{\perp}$ is of dimension 3 and in the non-special case $L^{\perp}=L$, where L is the associated vector space. Therefore if l is a special line then the line bundle E_3 with fibre $P^{\perp}/L^{\perp}$ is isomorphic to $\mathcal{O}(-2)$. Also the line bundle E_4 with fibre L/P is isomorphic to $\mathcal{O}(1)$. Now we have an obvious surjective map

$$\psi: E_2^*\otimes\mathcal{O}(-1)\longrightarrow E_4^*\otimes\mathcal{O}(-1).$$

Since degree of $E_2^*\otimes\mathcal{O}(-1)$ is -2 and degree of $E_4^*\otimes\mathcal{O}(-1)$ is also -2, degree of $ker\psi$ is zero. Again $ker\psi$ contains the line bundle $E_3^*\otimes\mathcal{O}(-1)$ which is of degree 1 and therefore the quotient has degree -1 and since $H^1(l,\mathcal{O}(2))=0$, $ker\psi$ splits as $\mathcal{O}(1)\oplus\mathcal{O}(-1)$ again since $H^1(l,\mathcal{O}(3))$ and $H^1(l,\mathcal{O}(1)=0$, the bundle $E_2^*\otimes\mathcal{O}(-1)$ splits as $\mathcal{O}(1)\oplus\mathcal{O}(-1)\oplus\mathcal{O}(-2)$.

2). First we observe the following fact:

If $0\subset V_1\subset V_2\subset V$ be a flag of vector spaces and q a nonsingular quadratic form on V then

$$V_1^{\perp q}\text{ can be identified with }(V/V_1)^*,$$

where $V_1^{\perp q}$ denotes the set of vectors in V perpendicular to V_1 with respect to the quadratic form q.

But we have the following exact sequence

$$0\longrightarrow V_2/V_1\longrightarrow V/V_1\longrightarrow V/V_2\longrightarrow 0,$$

taking dual we get the following exact sequence

$$0 \longrightarrow (V/V_2)^* \longrightarrow (V/V_1)^* \longrightarrow (V_2/V_1)^* \longrightarrow 0,$$

from which we get

$$(V_1^{\perp q}/V_2^{\perp q}) \simeq (V_2/V_1)^*.$$

Let l be a general line in Y, Q a nonsingular quadric $\in \mathbb{P}(W)$ and p be a point on l.
Then by the above observation we have

$$P^{\perp q}/L^{\perp q} \simeq (L/P)^*,$$

where P, L are the corresponding vector spaces for p, l respectively.
From the following two exact sequences

$$0 \longrightarrow P^{\perp} \longrightarrow P^{\perp q} \longrightarrow P^{\perp q}/P^{\perp} \longrightarrow 0$$

and

$$0 \longrightarrow L^{\perp} \longrightarrow L^{\perp q} \longrightarrow L^{\perp q}/L^{\perp} \longrightarrow 0,$$

we have a map from $P^{\perp}/L^{\perp} \longrightarrow P^{\perp q}/L^{\perp q}$ with kernel $(P^{\perp} \cap L^{\perp q})/L^{\perp}$.

But for a general line l, we have $L^{\perp} = L$ and $P^{\perp}$ and $L^{\perp q}$ are contained in $P^{\perp q}$.

$$\text{Hence dim } (P^{\perp} \cap L^{\perp Q}) \geq 3.$$

If $P^{\perp} = L^{\perp q}$ then for any $p' \in l$ dimension of $(P^{\perp} \cap P'^{\perp}) \geq 3$, a contradiction, because for a general line l, we have $P^{\perp} \cap P'^{\perp} = L$.
Therefore the kernel of the above map is one dimensional and hence the map $P^{\perp}/L^{\perp} \longrightarrow P^{\perp q}/L^{\perp q}$ is surjective. Let E_1, E_2, E_3, E_4 and E_5 be the bundles with fibres $(P^{\perp}/L)^*, (P^{\perp}/P)^*, (L/P)^*$, $(P^{\perp q}/L^{\perp q})^*$ and $(P^{\perp} \cap L^{\perp q}/L)^*$ respectively. Then we have the following exact sequences

$$0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0$$

and

$$0 \longrightarrow E_4 \longrightarrow E_1 \longrightarrow E_5 \longrightarrow 0.$$

Since E_4 and E_5 are line bundles of degree 1 and $H^1(\mathbb{P}^1, \mathcal{O}) = 0$, E_1 splits as $\mathcal{O}(1) \oplus \mathcal{O}(1)$ and again since E_3 is a line bundle of degree -1, and $H^1(\mathbb{P}^1, \mathcal{O}(2)) = 0$, E_2 splits as

$$\mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(-1).$$

Therefore the restriction of the cotangent bundle to Y to $\mathbb{P}(L)$ for a general line splits as

$$(\mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(-1)) \otimes \mathcal{O}(-1)$$

$$= \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-2).$$

□

4.3 Special line bundle of degree zero and relation with geometric special line

In this section we will define *special* line bundles of degree zero over a curve X of genus 2 and show that every special line bundle gives uniquely a *special* line.

Let J denote the Jacobian of degree zero line bundles over X and δ a fixed line bundle of degree 1 over X.

Definition 4.3.1. *A line bundle $\xi \in J$ is said to be special if $H^0(X, K \otimes \xi^{-2} \otimes \delta^{-1}) \neq 0$, where K is the canonical line bundle over X.*

Now $H^0(X, K \otimes \xi^{-2} \otimes \delta^{-1}) \neq 0$ implies that $K \otimes \xi^{-2} \otimes \delta^{-1} = \mathcal{O}(y)$ for some $y \in X$. In other words, the *special* points in J are the solution of the equations of the form:

$$K \otimes \xi^{-2} \otimes \delta^{-1} = \mathcal{O}(y), \tag{4.3.1}$$

for some $y \in X$. For any $y \in X$ we can define an involution ι_y on the Jacobian of X namely,

$$\xi \longrightarrow K \otimes \delta^{-1} \otimes \mathcal{O}(-y) \otimes \xi^{-1}$$

clearly this is an involution on the Jacobian of X and the fixed points of this involution constitute the solution space of the above equation.

Lemma 4.3.2. *Let ξ be a line bundle over X of degree zero. Then any non-trivial extension*

$$0 \longrightarrow \xi^{-1} \longrightarrow E \longrightarrow \xi \otimes \delta \longrightarrow 0$$

of $\xi \otimes \delta$ by ξ^{-1} is stable, where δ is a line bundle of degree one over X.

Proof. If E is not stable then there exists a line subbundle ζ of positive degree. If $\varphi : \zeta \longrightarrow E$ is the inclusion then composing φ with the surjection $E \longrightarrow \xi \otimes \delta$ we get a non-zero homomorphism $\zeta \longrightarrow \xi \otimes \delta$ which is only possible if ζ is of degree one. In that case this homomorphism is an isomorphism, which implies that the sequence splits, a contradiction to the assumption that E is a non-trivial extension. □

Lemma 4.3.3. *Let E, E' be two non-trivial extensions of F by F'. Then E and E' are isomorphic as bundles if $\delta(E) = \lambda\delta(E')$ for some $\lambda \in \mathbb{C}^*$, where $\delta(E) \in H^1(X, Hom(F, F'))$ corresponding to the extension E. Moreover, if every non-zero homomorphism of E and E' is an isomorphism and the only endomorphisms of E and E' are scalars, then E and E' are isomorphic if and only if $\delta(E) = \lambda\delta(E')$ for some $\lambda \in \mathbb{C}^*$.*

Proof. See [8, Lemma, 3.3] □

Let ξ be a line bundle of degree zero. Then the set of extensions of $\xi \otimes \delta$ by ξ^{-1} is classified by $H^1(X, \xi^{-2} \otimes \delta^{-1})$. Therefore by Lemmas (4.3.2 and 4.3.3) the isomorphism classes of stable bundles of rank two with determinant δ corresponding to non-trivial extensions of above type can be classified by the projective line $\mathbb{P}(H^1(X, \xi^{-2} \otimes \delta^{-1}))$. But it is known that the moduli space of stable vector bundles of rank 2 and fixed determinant of odd degree over a smooth projective curve of genus 2 is classified by the intersection of two smooth quadrics say, $Y = Q_1 \cap Q_2$ in $\mathbb{P}^5$.

Therefore by the universal property of moduli space and the above lemma we have an injective morphism:

$$\mathbb{P}(H^1(X, \xi^{-2} \otimes \delta^{-1})) \longrightarrow Y \longrightarrow \mathbb{P}^5$$

But Y is a complete intersection in $\mathbb{P}^5$, therefore $\omega_Y = \mathcal{O}_Y(2 + 2 - 6) = \mathcal{O}(-2)$, i.e, degree of the tangent bundle of Y is 2 and hence the pullback of the line bundle $\mathcal{O}(2)$ in $\mathbb{P}^5$ to Y is ω_Y^{-1}. Again since the restriction of the tangent bundle to $\mathbb{P}(H^1(X, \xi^{-2} \otimes \delta^{-1}))$ can be identified with $R^1(p_*)(\mathrm{Ad}E)$, where E is the universal family of vector bundles over X parametrised by $H^1(X, \xi^{-2} \otimes \delta^{-1})$ and p is the projection from $X \times \mathbb{P}(H^1(X, \xi^{-2} \otimes \delta^{-1}))$ to $\mathbb{P}(H^1(X, \xi^{-2} \otimes \delta^{-1}))$. Therefore the pullback of the line bundle $\mathcal{O}(2)$ in $\mathbb{P}^5$ to $\mathbb{P}(H^1(X, \xi^{-2} \otimes \delta^{-1}))$ is the determinant of the bundle $R^1(p_*)(\mathrm{Ad}E)$.

Let ξ be a line bundle of degree zero. Let E be the family of bundles over X parametrised by the extensions of $\xi \otimes \delta$ by ξ^{-1}. Therefore we have the following exact sequence of bundles over $X \times \mathbb{P}(H^1(X, \xi^{-2} \otimes \delta^{-1}))$:

$$0 \longrightarrow \xi^{-1} \longrightarrow E \longrightarrow \xi \otimes \delta \otimes \mathcal{O}(-1) \longrightarrow 0. \tag{4.3.2}$$

We denote by $\mathrm{Ad}E$, the bundle of endomorphisms of trace zero and $\mathrm{Ad}'E$ the bundle of endomorphisms of trace zero which preserves the exact sequence. From the above exact sequence we have the following exact sequences:

$$0 \longrightarrow \mathrm{Ad}'E \longrightarrow \mathrm{Ad\ E} \longrightarrow \xi^2 \otimes \delta \otimes \mathcal{O}(-1) \longrightarrow 0 \tag{4.3.3}$$

and

$$0 \longrightarrow \xi^{-2} \otimes \delta^{-1} \otimes \mathcal{O}(1) \longrightarrow \mathrm{Ad}'E \longrightarrow \mathcal{O} \longrightarrow 0. \tag{4.3.4}$$

Applying p_* to the exact sequence (4.3.3) and (4.3.4) and using the fact that E_e is stable for all $e \in \mathbb{P}^1$ and therefore $p_*\ \mathrm{Ad}'E = p_*\ \mathrm{Ad}E = 0$,

we have the following exact sequences

$$0 \longrightarrow H^0(X, \xi^2 \otimes \delta) \otimes \mathcal{O}(-1) \longrightarrow R^1 p_*\ \mathrm{Ad}'E \longrightarrow R^1 p_*\ \mathrm{Ad}E \longrightarrow H^1(X, \xi^2 \otimes \delta) \otimes \mathcal{O}(-1) \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O} \longrightarrow H^1(X, \xi^{-2} \otimes \delta^{-1}) \otimes \mathcal{O}(1) \longrightarrow R^1 p_* \text{ Ad}'E \longrightarrow H^1(X; \mathcal{O}) \otimes \mathcal{O} \longrightarrow 0.$$

From the above two exact sequences it is clear that degree of $R^1 p_* \text{ Ad}E$ is two. Therefore the pullback of the line bundle $\mathcal{O}(2)$ in $\mathbb{P}^5$ to $\mathbb{P}(H^1(X, \xi^{-2} \otimes \delta^{-1}))$ is $\mathcal{O}(2)$ and hence $\mathbb{P}(H^1(X, \xi^{-2} \otimes \delta^{-1}))$ gives a line in $\mathbb{P}^5$ contained in Y.

Tensoring (4.3.3) by $\mathcal{O}(-1)$ and using the fact that E_e is stable for all $e \in \mathbb{P}^1$ and therefore $p_* \text{ Ad}'E = p_* \text{ Ad}E = 0$ and $H^1(\text{ Ad}E) = \Gamma(R^1 p_* \text{ Ad}E)$ (by Leray Spectral Sequence) we have the following exact sequence:

$$0 \longrightarrow \Gamma(R^1 p_*(\text{ Ad}'E \otimes p^*\mathcal{O}(-1))) \longrightarrow \Gamma(R^1 p_*(\text{ Ad E} \otimes p^*\mathcal{O}(-1))) \longrightarrow \Gamma(\xi^2 \otimes \delta) \otimes H^1(\mathbb{P}^1, \mathcal{O}(-2)) \tag{4.3.5}$$

$$\longrightarrow H^2(\text{ Ad}'E \otimes p^*\mathcal{O}(-1)) \longrightarrow H^2(\text{ Ad}E \otimes p^*\mathcal{O}(-1)) \longrightarrow H^1(X, \xi^2 \otimes \delta) \otimes H^1(\mathbb{P}^1, \mathcal{O}(-2)) \longrightarrow 0$$

and tensoring (4.3.3) by $\mathcal{O}(-2)$ we have the following exact sequence:

$$0 \longrightarrow \Gamma(R^1 p_*(\text{ Ad}'E \otimes p^*\mathcal{O}(-2))) \longrightarrow \Gamma(R^1 p_*(\text{ Ad E} \otimes p^*\mathcal{O}(-2))) \longrightarrow \Gamma(\xi^2 \otimes \delta) \otimes H^1(\mathbb{P}^1, \mathcal{O}(-3)) \tag{4.3.6}$$

$$\longrightarrow H^2(\text{ Ad}'E \otimes p^*\mathcal{O}(-2)) \longrightarrow H^2(\text{ Ad}E \otimes p^*\mathcal{O}(-2)) \longrightarrow H^1 X, \xi^2 \otimes \delta) \otimes H^1(\mathbb{P}^1, \mathcal{O}(-3)) \longrightarrow 0.$$

Also tensoring (4.3.4) by $p^*\mathcal{O}(-1)$ and considering the long exact sequence of cohomology we have

$$\Gamma(R^1 p_*(\text{ Ad}'E \otimes p^*\mathcal{O}(-1))) \simeq H^1(X, \xi^{-2} \otimes \delta^{-1}) \text{ and } H^2(\text{ Ad}'E \otimes p^*\mathcal{O}(-1)) = 0. \tag{4.3.7}$$

Again tensoring (4.3.4) by $p^*\mathcal{O}(-2)$ we have,

$$\Gamma(R^1 p_*(\text{ Ad}'E \otimes p^*\mathcal{O}(-2))) \simeq H^1(\mathbb{P}^1, \mathcal{O}(-2)) \text{ and } H^2(\text{ Ad}'E \otimes p^*\mathcal{O}(-2)) \simeq H^1(X, \mathcal{O}). \tag{4.3.8}$$

Therefore from the exact sequence (4.3.5) and (4.3.7) it follows that

$$\dim(\Gamma(R^1 p_*(\text{ Ad}E \otimes p^*\mathcal{O}(-1))) = \dim(H^1(X, \xi^{-2} \otimes \delta^{-1}) + \dim(\Gamma(\xi^2 \otimes \delta) \otimes H^1(\mathbb{P}^1, \mathcal{O}(-2).$$

Therefore if ξ is a special point in J then

$$\dim(\Gamma(R^1 p_*(\text{ Ad}E \otimes p^*\mathcal{O}(-1)))) = 3$$

and if ξ is not special, then

$$\dim(\Gamma(R^1 p_*(\text{ Ad}E \otimes p^*\mathcal{O}(-1)))) = 2$$

If ξ is not special then from (4.3.6) and (4.3.8) we have,

$$\Gamma(R^1p_*(\ \mathrm{Ad}E \otimes p^*\mathcal{O}(-2))) \simeq \Gamma(R^1p_*(\ \mathrm{Ad}'E \otimes p^*\mathcal{O}(-2))) \simeq H^1(\mathbb{P}^1, \mathcal{O}(-2)).$$

Therefore for a generic line bundle ξ,

$$\dim(\Gamma(R^1p_*(\ \mathrm{Ad}E \otimes p^*\mathcal{O}(-2)))) = 1$$

The moduli space of vector bundles of rank 2 and with fixed determinant of degree 1 over X is classified by the intersection of two quadrics say, $Y = Q_1 \cap Q_2$ in $\mathbb{P}^5$ and each point $\xi \in J$ gives a line $l = \mathbb{P}(H^1(X, \xi^{-2} \otimes \delta^{-1}))$ in Y and since the tangent space at a point e to the moduli space can be identified by $H^1(X,\ \mathrm{Ad}E)$, the restriction of the tangent bundle to the moduli space at a line l can be identified by $R^1p_*(\ \mathrm{Ad}E)$, where E is as above.

Therefore $\dim(\Gamma(l, T\mid_l \otimes \mathcal{O}(-1))) = 2$ for a general line defined by a general point in J and is 3 for a special line and $\dim(\Gamma(l, T\mid_l \otimes \mathcal{O}(-2))) = 1$ for general line .

Since any bundle over $\mathbb{P}^1$ splits as a direct sum of line bundles and the degree of the tangent bundle restricted to a line l is 2, it has a decomposition as follows:

$$\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(2-a-b)$$

Therefore $T\mid_l \otimes \mathcal{O}(-1)$ and $T\mid_l \otimes \mathcal{O}(-2)$, where T denotes the tangent bundle to the moduli space of vector bundles, have decomposition as

$$\mathcal{O}(a-1) \oplus \mathcal{O}(b-1) \oplus \mathcal{O}(1-a-b) \text{ and } \mathcal{O}(a-2) \oplus \mathcal{O}(b-2) \oplus \mathcal{O}(-a-b)$$

respectively. Now using the fact that $\Gamma(T\mid_l \otimes \mathcal{O}(-1))$ and $\Gamma(T\mid_l \otimes \mathcal{O}(-2))$ have dimensions 2 and 1 respectively for general lines and $\Gamma(T\mid_l \otimes \mathcal{O}(-1))$ is of dimension 3 for special lines, we have the following decomposition of the tangent bundle restricted to a general line :

$$\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(2)$$

and to a special line it is either

$$\mathcal{O}(-1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(2).$$

or

$$\mathcal{O}(-1) \oplus \mathcal{O} \oplus \mathcal{O}(3).$$

It is known that J^2, the Jacobian of degree 2 line bundles over X is isomorphic to the lines in Y and J^2 is canonically isomorphic to J. Therefore J is isomorphic to the variety of lines in Y. By the Lemma 4.2.2 and the above computation we conclude that the restriction of the above isomorphism to genaral lines gives an isomorphism between the general lines and general line bundles of degree 0 and so its complements. Therefore we have the following theorem:

Theorem 4.3.4. *The decomposition of the tangent bundle to the moduli space of vector bundles of rank 2 and fixed determinant δ over X restricted to the line $l = \mathbb{P}(H^1(X, \xi^{-2} \otimes \delta{-}1))$ for a special point $\xi \in J$ is*

$$\mathcal{O}(1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(2)$$

and for a generic point $\xi \in J$ it is

$$\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(2)$$

For the projective space $\mathbb{P}^5(V)$ we have the following exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow V \otimes \mathcal{O} \longrightarrow T \otimes \mathcal{O}(-1) \longrightarrow 0,$$

where T denote the tangent bundle.Therefore restricting it to the subvariety Y we get the exact sequence

$$0 \longrightarrow \mathcal{O}_Y(-1) \longrightarrow (V \otimes \mathcal{O}) \mid_Y \longrightarrow T \mid_Y \otimes \mathcal{O}_Y(-1) \longrightarrow 0.$$

But it is clear the projective tangent bundle $\tilde{T}Y$ has a natural embedding in the projective tangent bundle $V \otimes \mathcal{O}$ over $\mathbb{P}^5$ and the image of the surjective map in the above exact sequence is exactly $TY \otimes \mathcal{O}_Y(-1)$. Therefore we have

$$0 \longrightarrow \mathcal{O}_Y(-1) \longrightarrow \tilde{T}Y \longrightarrow T \otimes \mathcal{O}_Y(-1) \longrightarrow 0$$

and restricting to the line $\mathbb{P}^1$ we have the following exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \tilde{T}Y \mid_{\mathbb{P}^1} \longrightarrow TY \mid_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow 0.$$

If $\mathbb{P}^1 = l$ is a *special* line then $TY \mid_l$ has a splitting (by the Theorem 4.3.4) as $\mathcal{O}(-1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)$. Therefore $\tilde{T}Y \mid_{\mathbb{P}^1}$ is a nontrivial extension of $(\mathcal{O}(-1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)) \otimes \mathcal{O}(-1)$ by $\mathcal{O}(-1)$. But $H^1(\mathcal{O}(1))$ and $H^1(\mathcal{O}(-1))$ is zero and $H^1(\mathcal{O}(-2)$ has a nonzero canonical element, the non-trivial extension corresponds to the canonical element in $H^1(\mathcal{O}(-2))$ and since over $\mathbb{P}^1$ we have an exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow L \otimes \mathcal{O} \longrightarrow \mathcal{O}(1) \longrightarrow 0,$$

where L is the vector space corresponding to the line l the nonzero canonical element gives the nontrivial extension which is $L \otimes \mathcal{O}$.

Therefore We have $\tilde{T}Y \mid_l = \mathcal{O}^{\oplus 3} \oplus \mathcal{O}(-2)$. similarly by the same argument for a generic line we have $\tilde{T}Y \mid_l = \mathcal{O}^{\oplus 2} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

Therefore by the above calculation we have the following proposition:

Proposition 4.3.5. *The projective tangent bundle $\tilde{T}Y$ to Y restricted to a generic line (resp. special line) in Y is isomorphic to $\tilde{T}Y \mid_l = \mathcal{O}^{\oplus 2} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ (resp. $\tilde{T}Y \mid_l = \mathcal{O}^{\oplus 3} \oplus \mathcal{O}(-2)$).*

Definition 4.3.6. *A point p in the intersection Y of two quadrics Q_1 and Q_2 in $\mathbb{P}^5$ is said to be a special point if it lies on a special line in Y.*

It is obvious that if l is a line passing through p and contained in Y, then the projective tangent space at p to Y contains the line l.
The restriction of a quadric $Q \in \mathbb{P}(W)$ to the projective tangent space at a point p to Y is a cone with vertex p and with base a conic. Now any two conics in a plane generally intersect in four points. All the conics pass through these four points. Therefore all the cones intersect in four lines joining the vertex p and the intersection points of the conics.
Therefore generically the projective tangent space at p to Y contains exactly four lines in Y.

Now if p is a *special* point then there is a special line l containing p. In that case since we have $L^{\perp} \cap Y = 2L$, $L^{\perp}$ touches every base conic in the pencil. Hence the number of intersection points of the base conics in this case is ≤ 3, one of them with multiplicity 2.

Therefore in this case $\mathbb{P}(P^{\perp})$ contains less than 4 lines. Therefore we can define the special points also as

Definition 4.3.7. *A point p is called a special point if the projective tangent space to Y at p contains fewer than 4 lines in Y.*

Define $\Delta = \{p \in Y : \mathbb{P}(P^{\perp})$ contains fewer than 4 lines of $Y\}$

Remark 4.3.8. *It is known ([3], page 793) that Δ is a surface of degree 32.*

Definition 4.3.9. *A rank 2 stable vector bundle V with fixed determinant δ is called special if it contains a degree zero line bundle ξ^{-1} with ξ special.*

In other words in the identification of the moduli space of rank 2 stable bundles with fixed determinant δ, with Y, V corresponds to a special point in Y (as definition 4.3.7).

Definition 4.3.10. *A stable vector bundle is called very stable if it admits no non-zero nilpotent Higgs field.*

Lemma 4.3.11. *A stable vector bundle V of rank 2 and determinant δ over X is special if and only if it admits a nonzero nilpotent Higgs field.*

Proof. Let $\varphi : V \longrightarrow V \otimes K$ be non-zero nilpotent. Then $\ker\varphi$ and image φ are non-zero. Since V and $V \otimes K$ are stable and $\mu(V \otimes K) = \frac{5}{2}$, degree($\ker\varphi$) is zero or -1.

Case(1). degree($\ker\varphi$) $= 0$. Let $\ker\varphi := \xi^{-1}$, where ξ is a degree zero line bundle over X. Then we have the following diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \xi^{-1} & \longrightarrow & V & \longrightarrow & \xi \otimes \delta & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \varphi} & & & & \\ 0 & \longrightarrow & \xi^{-1} \otimes K & \longrightarrow & V \otimes K & \longrightarrow & \xi \otimes \delta \otimes K & \longrightarrow & 0 \end{array}$$

Clearly the induced map $\xi^{-1} \longrightarrow \xi \otimes \delta \otimes K$ is zero. Therefore φ takes ξ^{-1} to $\xi^{-1} \otimes K$ which gives a section of K. Since φ is nilpotent and trace φ is zero, $\varphi^2 = 0$. Thus induced map $\xi \otimes \delta \longrightarrow \xi \otimes \delta \otimes K$ is zero.

This implies that φ factors through a map $\xi \otimes \delta \longrightarrow \xi^{-1} \otimes K$.

This means that $\xi^{-2} \otimes K \otimes \delta^{-1}$ has a non-zero section.

i.e., ξ is a *special* by definition 4.3.1 and hence V is *special* (4.3.9).

Case(2). Degree(kerφ) $= -1$:

Since degree(kerφ) $= -1$, image φ is of degree 2 line subbundle of $V \otimes K$. Therefore (image) $\varphi \otimes K^{-1}$ is a degree zero line subbundle of V. Again since trace $\varphi = 0$ and φ is nilpotent, φ^2 is zero and therefore (image) $\varphi \otimes K^{-1}$ is a degree zero line subbundle of V contained in the kernel of φ, a contradiction.

Conversely if V is *special* then it can be obtained as an extension

$$0 \longrightarrow \xi^{-1} \longrightarrow V \longrightarrow \xi \otimes \delta \longrightarrow 0,$$

where ξ is a line bundle of degree zero satisfying $H^0(X, \xi^{-2} \otimes \delta^{-1} \otimes K) \neq 0$. Therefore we have a nonzero homomorphism $\xi \otimes \delta \longrightarrow \xi^{-1} \otimes K$ which gives a nonzero nilpotent map $V \longrightarrow V \otimes K$. Hence the proof is complete. □

Therefore by the above Lemma it follows that the variety of stable bundles which are not very stable is isomorphic to Δ.

Now we shall construct a bundle over $\tilde{X}$ with fibre at a point ξ isomorphic to $H^1(X, \xi^{-2} \otimes \delta^{-1})$.

Consider the following commutative diagram

$$\begin{array}{ccc} X \times J^1 & \xrightarrow{\tau} & J \times J^1 \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle \mu} \\ X & \xrightarrow{t} & J^1, \end{array}$$

where $\tau(x, \eta) = (L_x \otimes \eta^{-1}, \eta)$, $x \in X, \eta \in J^1$, $\mu(j, \eta) = j \otimes \eta, j \in J, \eta \in J^1$ and t is the natural inclusion of X in J^1. Since $\mu^{-1}(tX) = \tau(X \times J^1)$ it is clear that the line bundle on $J \times J^1$ defined by the divisor $T = \tau(X \times J^1)$ is just $\mu^*(L_\theta)$ where θ is the divisor in J^1 defined by the embedding of X in J^1.

Let $\sigma : J \times J^1 \longrightarrow J \times J^1$ be the morphism given by $(j, \eta) = (j^{-1}, \eta)$ for $j \in J, \eta \in J^1$. In [8], Lemma(6.4) it has been proved that the family of line bundles on X parametrised by J^1 defined as $M_1 = \tau^* \sigma^* L_T$ assigns to each $\eta \in J^1$ a bundle isomorphic to η^2.

Consider the map

$$m : X \times J \longrightarrow X \times J^1$$

defined by $(x, \xi) \longrightarrow (x, \xi \otimes \delta)$. Then the bundle $m^*(M_1)$ is a family of line bundles on X parametrised by J assigns to each $\xi \in J$ a line bundle isomorphic to $\xi^2 \otimes \delta^2$.

Consider the bundle $M = p_1^*\delta \otimes (m^*(M_1))^*$, which is a family of line bundles on X parametrised by J assigns to each $\xi \in J$ a line bundle $\xi^{-2} \otimes \delta^{-1}$.

Let $\tilde{M}$ denote the bundle M restricted on $X \times \tilde{X}$. Since $H^1(X; \xi^{-2} \otimes \delta^{-1})$ is of constant dimension as ξ varies over $\tilde{X}$, the 1st direct image $R^1(p_2)_*\tilde{M}$, where p_2 is the projection of $X \times \tilde{X}$ to the second factor, is locally free (Semicontinuity Theorem).

Let $\mathcal{F}$ denote the vector bundle $R^1(p_2)_*\tilde{M}$. By [10], there is a family of vector bundles parametrized by $\mathbb{P}(\mathcal{F})$. By Lemmas 4.3.2 and 4.3.3 this family gives stable bundles. Therefore by the universal property of the moduli space we have a morphism

$$\mathbb{P}(\mathcal{F}) \longrightarrow Y.$$

Clearly the image of this map is isomorphic to Δ. It is known (4.3.8) that Δ is a smooth surface of degree 32.

Let $\mathcal{F}_1$ denote the restriction of the cotangent bundle of Y to Δ.

Since by Theorem 3.0.9 the bundle $\mathcal{G}$ and $\mathcal{F}_1$ is embedded in the moduli space of stable Higgs bundle, there is a subspace S with two fibration over $\tilde{X}$. Therefore we have,

Theorem 4.3.12. *There is a subspace S in the moduli space of stable Higgs bundles with two fibrations over $\tilde{X}$ one via $\mathcal{G}$ and other via $\mathcal{F}_1$.*

Chapter 5

Geometry Of The Hitchin Map

In chapter 2, section 2.3 we have defined the Hitchin map on the moduli space of Higgs bundles. In this section we will describe the geometry of the map.

Let X be a smooth curve of genus 2. Then as in Chapter 2, section 2.3 the Hitchin map is a proper map from the moduli space of stable Higgs pairs (E, φ) over X, where E is vector bundle of rank 2 and of degree 1 with fixed determinant δ and φ is a trace-free section of $\mathrm{End}E \otimes K$ onto $H^0(X, K^2)$, where K is the canonical line bundle over X.

It is known that the cotangent bundle of the moduli space of stable vector bundles is an open dense subset of the moduli space of stable Higgs bundles.

Let $\mathcal{M}$ denote the moduli space of stable vector bundles of rank 2 with determinant δ. Then the cotangent space to $\mathcal{M}$ at a point E can be identified with $H^0(X,\ \mathrm{Ad}E \otimes K)$, where $\mathrm{Ad}E \otimes K$ denotes the vector bundle of trace-free endomorphisms from $E \longrightarrow E \otimes K$.

As defined earlier the restriction of the Hitchin map to the cotangent space at a point E is a quadratic map from

$$H^0(X,\ \mathrm{Ad}E \otimes K) \longrightarrow H^0(X, K^2).$$

Lemma 5.0.13. *If E is a vector bundle of rank 2 and*

$$dim\ H^0(X, \xi^{-1} \otimes E)) \geq 2,$$

then either $E \approx \xi \oplus \xi$, or $dim\ H^0(\xi^{-1} \otimes L_x^{-1} \otimes E)) \neq 0$ for some $x \in X$.

Proof. Assume that $E \neq \xi \oplus \xi$. Consider the map

$$\xi \otimes H^0(X, \xi^{-1} \otimes E)) \longrightarrow \xi \otimes (\xi^{-1} \otimes E) \approx E.$$

Since, by assumption, this is not an isomorphism, there exists a nonzero homomorphism $\varphi : \xi \longrightarrow E$ such that $\varphi(x) = 0$, for some $x \in X$. This homomorphism factors through a homomorphism $\xi \otimes \mathcal{O}(x) \longrightarrow E$, proving our assertion. $\square$

Lemma 5.0.14. *Let E be any stable vector bundle of rank 2 and degree 1. Then there exists a degree zero line bundle ξ such that $H^0(X, Hom(\xi^{-1}, E)) \neq 0$.*

Proof. By the Riemann-Roch theorem, we see that $\dim H^0(X, L \otimes E)) \geq 3$, for any line bundle L of degree 2. Let W be a subspace of $H^0(X, L \otimes E))$ of dimension 3. Associate to each $x \in X$, the subspace H_x of sections of $L \otimes E$ in W which vanish at x. Then the subspace H_x is contained in the image under the injective map

$$H^0(X, L \otimes E \otimes L_x^{-1}) \longrightarrow H^0(X, L \otimes E).$$

If H_x is of dimension ≥ 2, this implies that $\dim H^0(X, L \otimes E \otimes L_x^{-1}) \geq 2$ and our assertion follows from Lemma 5.0.13. We shall therefore assume that H_x is of dimension 1 for all x. This means that we have a non-constant morphism $X \longrightarrow \mathbb{P}(W)$. Since X has genus 2 it cannot be an embedding. Therefore there exist at least one point σ in the image with atleast two points in the fiber. Since E is stable, its fiber contains exactly two points say, $y, z \in X$. Then σ is the image of a section of $L \otimes L_y^{-1} \otimes L_z^{-1} \otimes E$ by the natural map $L \otimes L_y^{-1} \otimes L_z^{-1} \otimes E \longrightarrow L \otimes E$. This completes the proof of the Lemma. □

Lemma 5.0.15. *Let E be a fixed vector bundle of rank 2 over X with determinant δ. Let $AdE \otimes K$ be the bundle of trace-free endomorphisms from E to $E \otimes K$, where K is the canonical line bundle over X. Then the rank of the evaluation map*

$$H^0(X,\ AdE \otimes K) \otimes \mathcal{O} \longrightarrow\ AdE \otimes K$$

is ≥ 2 at every point of X.

Proof. If the rank of the above map is < 2 at some point $x \in X$ and s_1, s_2 are two sections of $\mathrm{Ad}\, E \otimes K$ vanishing at x, then $\det\,(s_1)$ and $\det\,(s_2)$ vanish at x with multiplicity 2, and since $\det\,(s_i), i = 1, 2$ are sections of $K^{\otimes 2}$, they vanishes also at ιx with multiplicity 2, where ι denotes the hyperelliptic involution on X. Therefore for any $y \in X \setminus \{x, \iota x\}$, $\det\,(\lambda_1 s_1 + \lambda_2 s_2)(y) \neq 0$, where λ_i are arbitrary scalars(not both zero). But as the space of sections s such that $\det(s_y) = 0$ is a hypersurface, some linear combination of $s_1(y)$ and $s_2(y)$ has determinant 0 which is a contradiction. This proves our lemma. □

Lemma 5.0.16. *Let E be a stable vector bundle over X of rank 2 and determinant δ. Then there are exactly 4 line bundles of degree zero contained in E, for non-special E and if E is special then the number of distinct line bundles of degree zero contained in it is less than 4.*

Proof. Let ξ^{-1} be a line bundle over X of degree zero, contained in E (which exists by above Lemma 5.0.14). Then any non trivial extension of $\xi \otimes \delta$ by ξ^{-1} is stable by Lemma 4.3.2 and two nontrivial extensions are isomorphic if and only if one is the scalar multiple of the other

(Lemma 4.3.3). Therefore the bundle E gives an element e in $\mathbb{P}(H^1(X, \xi^{-2} \otimes \delta^{-1}))$ where $H^1(X, \xi^{-2} \otimes \delta^{-1}))$ is the classifying space of extensions.

Let ξ be non-special. Consider the base point free line bundle $K \otimes \xi^2 \otimes \delta$, where K is the canonical line bundle over X. This gives rise to a morphism

$$\pi : X \longrightarrow \mathbb{P}((H^0(X, K \otimes \xi^2 \otimes \delta))^*) \simeq \mathbb{P}(H^1(X, \xi^{-2} \otimes \delta^{-1}))$$

given by mapping each $x \in X$ on the point in $\mathbb{P}(H^1(X, \xi^{-2} \otimes \delta^{-1}))$ corresponding to the kernel of the map

$$H^1(X, \xi^{-2} \otimes \delta^{-1}) \longrightarrow (H^1(X, \xi^{-2} \otimes \delta^{-1} \otimes \mathcal{O}(x)).$$

The degree of the map is the degree of the line bundle $K \otimes \xi^2 \otimes \delta$, which is 3.

Therefore $\pi^{-1}(e)$, where e is the point in $(\mathbb{P}H^1(X, \xi^{-2} \otimes \delta^{-1}))$ corresponding to the bundle E, will give three points say, $x_i, i = 1, 2, 3$. We claim that they are distinct. If not, then let $x_1 = x_2 = x$. Since $\mathcal{O}(x_1 + x_2 + x_3) = K \otimes \xi^2 \otimes \delta$, $K \otimes \xi^2 \otimes \delta \otimes \mathcal{O}(-2x)$ has a section. Thus the line bundle $\xi^{-1} \otimes \delta^{-1} \otimes \mathcal{O}(x)$ is special.

Consider the exact sequence

$$0 \longrightarrow \xi^{-1} \longrightarrow E \longrightarrow \xi \otimes \delta \longrightarrow 0.$$

Then the natural map $\xi \otimes \delta \otimes \mathcal{O}(-x) \longrightarrow \xi \otimes \delta$ factors through $\xi \otimes \delta \otimes \mathcal{O}(-x) \longrightarrow E$ (as the corresponding point e is in the kernel of the map $H^1(X, \xi^{-2} \otimes \delta^{-1}) \longrightarrow (H^1(X, \xi^{-2} \otimes \delta^{-1} \otimes \mathcal{O}(x)))$. Thus E is special, a contradiction.

Therefore there are 4 distinct line bundles of degree zero namely, $\xi^{-1}, \xi \otimes \delta \otimes \mathcal{O}(-x_i), i = 1, 2, 3$ contained in E.

Conversely if η be a line bundle of degree zero contained in E, composing with the surjective map of the exact sequence

$$0 \longrightarrow \xi^{-1} \longrightarrow E \longrightarrow \xi \otimes \delta \longrightarrow 0,$$

we have a morphism $\eta \longrightarrow \xi \otimes \delta$. If this map is zero then the map $\eta \longrightarrow E$ factors through $\eta \longrightarrow \xi^{-1}$, since both are of degree zero, they are isomorphism. If the map is not zero then we have $\eta \otimes \mathcal{O}(x) \simeq \xi \otimes \delta$, for some $x \in X$. That is $\eta \simeq \xi \otimes \delta \otimes \mathcal{O}(-x)$.

If ξ is special then $\xi^2 \otimes \delta = \mathcal{O}(x)$, for some $x \in X$. In that case the line bundle $K \otimes \xi^2 \otimes \delta$ is not base point free, its base locus is $\{x\}$ and therefore tensoring it by the line bundle $\mathcal{O}(-x)$ we get the base point free canonical line bundle K. Using the isomorphism of $H^0(X, K)$ and $H^0(X, K \otimes \xi^2 \otimes \delta)$ we again have a morphism,

$$\pi : X \longrightarrow \mathbb{P}(H^0(X, K)^*) \simeq \mathbb{P}(H^0(X, K \otimes \xi^2 \otimes \delta)^*)) \simeq \mathbb{P}(H^1(X, \xi^{-2} \otimes \delta^{-1})),$$

given by mapping to each $y \in X$ on the point in $\mathbb{P}(H^1(X, \xi^{-2} \otimes \delta^{-1}))$ corresponding to the kernel of the map

$$H^1(X, \mathcal{O}) \longrightarrow H^1(X, \mathcal{O}(y)).$$

Note that the degree of the map in this case is 2. Therefore $\pi^{-1}(e)$, where e is the point in $\mathbb{P}(H^1(X, \xi^{-2} \otimes \delta^{-1}))$ associated to E will give 2 points, say y_1 and y_2. Then by the same argument as earlier it follows that E contains at most 3 line bundles of degree zero.

□

Remark 5.0.17. *If $y_1 = y_2 = y$ then $\mathcal{O}(2y) = K$, i.e., y is a Weierstrass point and the line bundle $\xi^{-1} \otimes \delta^{-1} \otimes \mathcal{O}(y)$ is also special and E contains the line bundle $\xi \otimes \delta \otimes \mathcal{O}(-y)$. Therefore in this case the number of distinct line subbundles of degree zero is 2.*
If $\xi^2 \otimes \delta = \mathcal{O}(w)$, where w is a Weierstrass point and E is the bundle corresponding to the extension class $\pi(w)$ then E contains only one line bundle of degree zero.

Let ξ and η be two distinct line subbundles of E of degree zero over X. Then we have the following exact sequence

$$0 \longrightarrow \xi \oplus \eta \longrightarrow E \longrightarrow \mathcal{O}_x \longrightarrow 0,$$

for some $x \in X$. It is also clear from the above exact sequence that $\xi \otimes \eta \simeq \delta \otimes \mathcal{O}(-x)$. Let i_ξ denotes the natural map

$$\xi \longrightarrow \xi \otimes \mathcal{O}(x).$$

Consider the map

$$\xi \oplus \eta \longrightarrow (\xi \oplus \eta) \otimes \mathcal{O}(x),$$

given by $\{i_\xi, -i_\eta\}$. Clearly this map vanishes at x.

Now the following lemma says that the above map factors through $E \longrightarrow (\xi \oplus \eta) \otimes \mathcal{O}(x)$.

Lemma 5.0.18. *Let V and W be two vector bundles over X of same rank and*

$$0 \longrightarrow V \longrightarrow W \longrightarrow \mathcal{O}_D \longrightarrow 0$$

for some divisor D over X. Then the natural map $V \longrightarrow V \otimes \mathcal{O}(D)$ factors through $W \longrightarrow V \otimes \mathcal{O}(D)$.

Proof. From the exact sequence

$$0 \longrightarrow V \longrightarrow W \longrightarrow \mathcal{O}_D \longrightarrow 0,$$

we have an exact sequence

$$0 \longrightarrow W^* \longrightarrow V^* \longrightarrow \operatorname{Ext}^1(\mathcal{O}_D, \mathcal{O}) \longrightarrow 0.$$

On the other hand, from the exact sequence

$$0 \longrightarrow \mathcal{O}(-D) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_D \longrightarrow 0,$$

we have $\mathrm{Ext}^1(\mathcal{O}_D, \mathcal{O}) \simeq \mathcal{O}_D(D)$. Therefore we have the following exact sequence

$$0 \longrightarrow W^* \longrightarrow V^* \longrightarrow \mathcal{O}_D(D) \longrightarrow 0.$$

Tensoring with $V \otimes \mathcal{O}(D)$ we get

$$0 \longrightarrow W^* \otimes V \otimes \mathcal{O}(D) \longrightarrow V^* \otimes V \otimes \mathcal{O}(D) \longrightarrow \mathcal{O}_D(D)) \otimes V \otimes \mathcal{O}(D) \longrightarrow 0.$$

Now the lemma is a consequence of the above exact sequence. □

Now composing the map $E \longrightarrow (\xi \oplus \eta) \otimes \mathcal{O}(x)$ with the morphism $(\xi \oplus \eta) \otimes \mathcal{O}(x) \longrightarrow E \otimes \mathcal{O}(x)$, we get a trace-free morphism $E \longrightarrow E \otimes \mathcal{O}(x)$. Again composing it with the natural map $E \otimes \mathcal{O}(x) \longrightarrow E \otimes K$, we get a trace-free morphism $E \longrightarrow E \otimes K$ vanishing at ιx.

Therefore each pair (ξ, η) of degree zero line bundles contained in E gives a point $x \in X$ such that the map

$$H^0(X,\ \mathrm{Ad}E \otimes K) \otimes \mathcal{O} \longrightarrow\ \mathrm{Ad}E \otimes K$$

fails to be of maximal rank at ιx, with $\xi \otimes \eta = \delta \otimes \mathcal{O}(-x)$.

Let E be a stable vector bundle of rank 2 with determinant δ. Then by the Lemma 5.0.16, E contains 4 line bundles counted with multiplicity of degree zero over X, say, $\xi_1 = \xi^{-1}, \xi_{1+i} = \xi \otimes \delta \otimes \mathcal{O}(-x_i), i = 1, 2, 3$, where $x_i \in X$. From the proof of the Lemma 5.0.16, it follows that

$$\mathcal{O}(x_1 + x_2 + x_3) = K \otimes \xi^2 \otimes \delta \tag{5.0.1}$$

and

$$\Pi \xi_i = K^{-1} \otimes \delta^2. \tag{5.0.2}$$

Let x_{ij} denote the point given by the pair (ξ_i, ξ_j), such that the map

$$H^0(X,\ \mathrm{Ad}E \otimes K) \otimes \mathcal{O} \longrightarrow\ \mathrm{Ad}E \otimes K$$

fails to be of maximal rank at ιx_{ij}. Then we have $\xi_i \otimes \xi_j = \delta \otimes \mathcal{O}(-x_{ij})$. Therefore,

$$\Pi(\xi_i \otimes \xi_j) = \delta^6 \otimes \mathcal{O}(-\Sigma x_{ij}).$$

Hence from the equation (5.0.2), it follows that $\mathcal{O}(\Sigma x_{ij}) = K^3$.

Now we have $\xi_1 \otimes \xi_{i+1} = \delta \otimes \mathcal{O}(-x_i), i = 1, 2, 3$ and $\xi_i \otimes \xi_j (i, j \neq 1) = \delta \otimes \mathcal{O}(-x_{ij})$, which gives

$$\xi^2 \otimes \delta \otimes \mathcal{O}(-x_{i-1} - x_{j-1}) = \mathcal{O}(-x_{ij}).$$

But we have from (5.0.1)

$$\mathcal{O}(x_{i-1} + x_{j-1}) = K \otimes \xi^2 \otimes \delta \otimes \mathcal{O}(-x_{9-i-j-1}).$$

Therefore we have,

$$\xi^2 \otimes \delta \otimes K^{-1} \otimes \xi^{-2} \otimes \delta^{-1} \otimes \mathcal{O}(x_{9-i-j-1}) = \mathcal{O}(-x_{ij}).$$

Hence $x_{ij} = \iota x_{9-i-j-1}$.

Therefore the divisor of degree 6 where the map

$$H^0(X, \ \mathrm{Ad}E \otimes K) \otimes \mathcal{O} \longrightarrow \ \mathrm{Ad}E \otimes K \tag{5.0.3}$$

fails to be of maximal rank is given by $\sum_{i=1}^{3} x_i + \sum_{i=1}^{3} \iota x_i$.

Let us fix a point $x \in X$ and let $\mathrm{Ad}\mathcal{E}$ denote the universal bundle of trace-free endomorphisms from $\mathcal{E} \longrightarrow \mathcal{E}$ over $\mathcal{M} \times X$, where $\mathcal{M}$ denotes the moduli space of stable rank 2 bundle with fixed determinant δ over X. Consider the map

$$p_{1*}(\ \mathrm{Ad}\mathcal{E} \otimes p_2^* K) \longrightarrow (\ \mathrm{Ad}\mathcal{E}_x \otimes p_2^* K_x) \otimes \mathcal{O}, \tag{5.0.4}$$

where p_i denotes the i-th projection. The map fails to be of maximal rank along a section of $\mathcal{O}_{\mathcal{M}}(2)$. Now if ξ is a degree zero line bundle such that $\xi^2 \otimes \delta \neq \mathcal{O}(\iota x)$, then there is a unique stable vector bundle of rank 2 with determinant δ corresponding to the unique extension class of $\xi \otimes \delta$ by ξ^{-1} given by the kernel of the map $H^1(X; \xi^{-2} \otimes \delta^{-1}) \longrightarrow H^1(X; \xi^{-2} \otimes \delta^{-1} \otimes \mathcal{O}(\iota x))$, containing ξ^{-1} and $\xi \otimes \delta \otimes \mathcal{O}(-\iota x)$, in which case the map in (5.0.4) fails to be of maximal rank at E.

If $\xi^2 \otimes \delta = \mathcal{O}(\iota x)$ then $H^0(X, \xi^{-2} \otimes \delta^{-1} \otimes \mathcal{O}(\iota x)) \neq 0$, from the following two exact sequences

$$0 \longrightarrow \xi^{-1} \longrightarrow E \longrightarrow \xi \otimes \delta \longrightarrow 0$$

and

$$0 \longrightarrow \xi^{-1} \otimes \mathcal{O}(\iota x) \longrightarrow E \otimes \mathcal{O}(\iota x) \longrightarrow \xi \otimes \delta \otimes \mathcal{O}(\iota x) \longrightarrow 0,$$

we get a nonzero trace free morphism from $E \longrightarrow E \otimes \mathcal{O}(\iota x)$, for all stable bundles corresponding to nontrivial extensions of $\xi \otimes \delta$ by ξ^{-1}. Composing with the natural map $E \otimes \mathcal{O}(\iota x) \longrightarrow E \otimes K$ we get a trace-free section of $\mathrm{Ad}E \otimes K$ vanishing at x. In other words the map in (5.0.4) is of lower rank at each point corresponding to the nontrivial extensions of $\xi \otimes \delta$ by ξ^{-1}.

Set $D_x = \{E \in \mathcal{M} : \xi^{-1}, \xi \otimes \delta \otimes \mathcal{O}(-\iota x) \subset E : \xi \in J\}$, where J is the Jacobian of X.

We define an involution ι_x on the Jacobian J by $\xi \longrightarrow \xi^{-1} \otimes \delta \otimes \mathcal{O}(-x)$. Then J/ι_x is a Kummer surface K_x associated to x. Let Z_x denote its double points.

As in [8], we identify the moduli space $\mathcal{M}$ with intersection of two smooth quadric Q_1 and Q_2 in $\mathbb{P}^5$. Also we identify the line bundles of degree zero over X with the lines in $Y = Q_1 \cap Q_2$ as in Chapter 4. Now for each E in D_x there exists a unique pair $\xi^{-1}, \xi \otimes \delta \otimes \mathcal{O}(-\iota x) \subset E$ in J and hence a point in $K_{\iota x}$. In other words there exist exactly two lines l_1 and l_2 in Y containing the point $y \in Y$ associated to E and a plane Λ generated by l_1 and l_2, contained in the quadric, say $Q_{\iota x} = Q_x$ associated to $\iota x \in X$ such that $\Lambda \cap Q_1$ is degenerate, where Q_1 is a non-degenerate quadric in the pencil other than Q_x.

Therefore Λ is tangent to Q_1 at ιx.

Now D_x can also be defined as follows

$$D_x := \Sigma := \{y \in Y : \text{ there exists a 2-plane } \Lambda \subset Q_{\iota x} \text{ such that } \Lambda \text{ is tangent to } Q_1 \text{ at } y\}.$$

Therefore we have a morphism

$$b_{\iota x} : \Sigma \longrightarrow K_{\iota x}$$

such that $\Sigma \setminus b_{\iota x}^{-1}(Z_{\iota x})$ is isomorphic to $K_{\iota x} \setminus Z_{\iota x}$ and over the double points the fibres are isomorphic to $\mathbb{P}^1$.

The following Lemma characterises Σ.

Lemma 5.0.19. *For $y \in Y$,*

$$y \in \Sigma \textit{ if and only if } T_y Q_1 \textit{ is tangent to } Q_{\iota x}.$$

Proof. By Proposition 2.5.2, we can take Q_1 to be of the form $\sum_{i=1}^{6} x_i^2$ and $Q_x = Q_{\iota x}$ as $\sum_{i=1}^{6} \lambda_i x_i^2$.

Let $T_y Q_1$ is a tangent to $Q_{\iota x}$ at y', i.e., $T_y Q_1 = T_{y'} Q_x$. If y and y' are presented by $(a_1, ..., a_6)$ and $(b_1, ..., b_6)$ respectively, then we have $a_i = c b_i \lambda_i$ for some non-zero scalar. Since $y \in Y$, $\sum_{i=1}^{6} a_i^2 = \sum_{i=1}^{6} c b_i \lambda_i = 0$, i.e., $y' \in T_{y'} Q_x$. Therefore the line joining y and y' is contained in $T_{y'} Q_x \cap Q_x$. Thus there exists a plane, say, Λ, contained in $Q_{\iota x} \cap T_{y'} Q_{\iota x}$. Therefore $\Lambda \subset T_{y'} Q_{\iota x} = T_y Q_1$.

i.e, Λ is a tangent to Q_1 at y; thus $y \in \Sigma$.

Conversely, let Λ be a plane contained in Q_x such that $\Lambda \subset T_y Q_1$, then the quadric threefold $T_y Q_1 \cap Q_{\iota x}$ contains the plane Λ and therefore it must be singular; thus $T_y Q_1$ must be tangent to $Q_{\iota x}$ somewhere. □

But in Chapter 2 it has been shown that Σ is a smooth intersection of three quadrics in $\mathbb{P}^5$ and therefore it gives the line bundle $\mathcal{O}_{\mathcal{M}}(2)$ over the moduli space $\mathcal{M}$. Hence

$$\mathcal{O}(D_x) = \mathcal{O}_{\mathcal{M}}(2).$$

Therefore the map in (5.0.4) has lower rank exactly along the divisor D_x.

Hence the projection map

$$p_1^*(p_{1*}(\text{ Ad}\mathcal{E} \otimes p_2^* K)) \longrightarrow \text{ Ad}\mathcal{E} \otimes p_2^* K$$

fails to be of maximal rank along the divisor

$$\mathcal{D} = \{(E, x) \in \mathcal{M} \times X : \xi, \eta \subset E \text{ with } \xi \otimes \eta = \delta \otimes \mathcal{O}(\iota x) \text{ for some } \xi, \eta \in J\}.$$

Let $h : X \longrightarrow \mathbb{P}^1$ be the hyperelliptic map. Fix a stable vector bundle E of rank 2 with determinant δ. Taking the direct image of the map

$$H^0(X,\ \mathrm{Ad}E \otimes K) \otimes \mathcal{O} \longrightarrow K^2,$$

we have

$$H^0(X,\ \mathrm{Ad}E \otimes K) \otimes \mathcal{O} \longrightarrow \mathcal{O}(2),$$

which gives a range of conics on $H^0(X,\ \mathrm{Ad}E \otimes K)$. Therefore we have the following exact sequence

$$0 \longrightarrow H^0(X,\ \mathrm{Ad}E \otimes K) \otimes \mathcal{O} \longrightarrow H^0(X,\ \mathrm{Ad}E \otimes K)^* \otimes \mathcal{O}(2) \longrightarrow \mathcal{O}(D') \longrightarrow 0$$

where D' is of the form $2y_1 + 2y_2 + 2y_3, y_i \in \mathbb{P}^1$ such that $h^{-1}(y_i) = \{x_i, \iota x_i\}$, where x_i are the points of X where the map

$$H^0(X,\ \mathrm{Ad}E \otimes K) \otimes \mathcal{O} \longrightarrow\ \mathrm{Ad}E \otimes K$$

has lower rank.

Therefore the natural map

$$H^0(X,\ \mathrm{Ad}E \otimes K) \otimes \mathcal{O} \longrightarrow H^0(X,\ \mathrm{Ad}E \otimes K) \otimes \mathcal{O}(3)$$

factors through $H^0(X,\ \mathrm{Ad}E \otimes K)^* \otimes \mathcal{O}(2) \longrightarrow H^0(X,\ \mathrm{Ad}E \otimes K) \otimes \mathcal{O}(3)$ which gives a pencil of conics on $H^0(X,\ \mathrm{Ad}E \otimes K)^*$.

Let $\wp$ denotes the map

$$Id_{\mathcal{M}} \times h : \mathcal{M} \times X \longrightarrow \mathcal{M} \times \mathbb{P}^1$$

and p'_i denotes the i-th projection from $\mathcal{M} \times \mathbb{P}^1$ to the i-th factor.

Now consider the quadratic map

$$p_1^*(p_{1*}(\ \mathrm{Ad}\mathcal{E} \otimes p_2^* K)) \longrightarrow p_2^*(h^*\mathcal{O}(2)).$$

Taking 1st direct image under the map $\wp$ we have the following quadratic map

$$\wp_*(p_1^*(p_{1*}(\ \mathrm{Ad}\mathcal{E} \otimes p_2^* K))) \longrightarrow {p'_2}^*(\mathcal{O}(2)).$$

Therefore we have the following exact sequence

$$0 \longrightarrow \wp_*(p_1^*(p_{1*}(\ \mathrm{Ad}\mathcal{E} \otimes p_2^* K))) \longrightarrow (\wp_*(p_1^*(p_{1*}(\ \mathrm{Ad}\mathcal{E} \otimes p_2^* K))))^* \otimes p'_2(\mathcal{O}(2)) \longrightarrow \mathcal{O}_{\mathcal{D}'} \longrightarrow 0,$$

where $\mathcal{O}(\mathcal{D}') = \mathcal{O}(2\mathcal{D}'')$ and $\mathcal{O}(\mathcal{D}'') = {p'_1}^*\mathcal{O}_{\mathcal{M}}(2) \otimes {p'_2}^*\mathcal{O}(3)$.

Therefore the natural map

$$\wp_*(p_1^*(p_{1*}(\ \mathrm{Ad}\mathcal{E} \otimes p_2^* K))) \longrightarrow \wp_*(p_1^*(p_{1*}(\ \mathrm{Ad}\mathcal{E} \otimes p_2^* K))) \otimes \mathcal{O}(\mathcal{D}''),$$

factors through

$$\wp_*(p_1^*(p_{1*}(\ \mathrm{Ad}\mathcal{E}\otimes p_2^*K))))^*\otimes p_2'(\mathcal{O}(2))\longrightarrow \wp_*(p_1^*(p_{1*}(\ \mathrm{Ad}\mathcal{E}\otimes p_2^*K)))\otimes\mathcal{O}(\mathcal{D}''), \qquad (5.0.5)$$

which give the morphism

$$f:\wp_*(p_1^*(p_{1*}(\ \mathrm{Ad}\mathcal{E}\otimes p_2^*K))))^*\otimes {p_1'}^*\mathcal{O}_{\mathcal{M}}(-1)\longrightarrow \wp_*(p_1^*(p_{1*}(\ \mathrm{Ad}\mathcal{E}\otimes p_2^*K))))\otimes {p_1'}^*\mathcal{O}_{\mathcal{M}}(1)\otimes {p_2'}^*\mathcal{O}(1). \qquad (5.0.6)$$

Taking the direct image of the map p_1' we get a pencil of quadrics on $(p_{1*}(\ \mathrm{Ad}\mathcal{E}\otimes p_2^*K))^*\otimes\mathcal{O}_{\mathcal{M}}(-1)$.

On the other hand let Y be a smooth intersection of two quadrics Q_1 and Q_2 in $\mathbb{P}^5$ and $\mathbb{P}(W)$ denote the pencil of quadrics in $\mathbb{P}^5$ passing through Y and TY denote its tangent bundle. Then we have the following Lemma ;

Lemma 5.0.20. *dim* $H^0(Y,s^2(T^*Y\otimes\mathcal{O}_Y(1)))=2$

Proof. Let V is a 6-dimensional vector space and Q_1,Q_2 be two non-degenerate quadrics in $\mathbb{P}(V)$ and $Y=Q_1\cap Q_2$. i.e., Y is a complete intersection of two quadrics Q_1 and Q_2 in $\mathbb{P}(V)$.

Therefore det $(T^*Y)=\mathcal{O}_Y(-2)$. The projective tangent bundle to $\mathbb{P}(V)$ is trivial.

Let $\tilde{T}$ denotes the projective tangent bundle to Y, then we have the following exact sequence

$$0\longrightarrow\tilde{T}\longrightarrow V\otimes\mathcal{O}_Y\longrightarrow\mathcal{N}_Y\longrightarrow 0, \qquad (5.0.7)$$

where $\mathcal{N}_Y$ is the projective normal bundle.

Claim: $\mathcal{N}_Y\simeq\mathcal{O}_Y(1)\oplus\mathcal{O}_Y(1)$.

Consider the quadric Q_1 in $\mathbb{P}(V)$, then degree$T^*Q_1=-4$ and hence $\det(\tilde{T}Q_1)\simeq\mathcal{O}_{Q_1}(-1)$.

From the exact sequence on Q_1

$$0\longrightarrow\tilde{T}Q_1\longrightarrow V\otimes\mathcal{O}_{Q_1}\longrightarrow\mathcal{N}_{Q_1}\longrightarrow 0$$

it is clear that $\mathcal{N}_{Q_1}\simeq\mathcal{O}_{Q_1}(1)$. on the other hand degree $\tilde{T}Y=-2$ and therefore from the exact sequence

$$0\longrightarrow\tilde{T}Y\longrightarrow\tilde{T}Q_1\mid_Y\longrightarrow\mathcal{N}_{Q_1}\mid_{Q_2}\longrightarrow 0,$$

we have $\mathcal{N}_{Q_1}\mid_{Q_2}=\mathcal{O}_Y(1)$.

Now we have the following diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
 & & \tilde{T}Y & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \longrightarrow & \tilde{T}Q_1 \mid_Y & \longrightarrow & V \otimes \mathcal{O}_Y & \longrightarrow & \mathcal{O}_Y(1) & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & \mathcal{O}_Y(1) & & & & & & \\
 & & \downarrow & & & & & & \\
 & & o & & & & & &
\end{array}
$$

From the exact sequence (5.0.7) and the above diagram we have

$$0 \longrightarrow \mathcal{O}_Y(1) \longrightarrow \mathcal{N}_Y \longrightarrow \mathcal{O}_Y(1) \longrightarrow 0.$$

Using the rationality of Y and hence $H^1(Y, \mathcal{O}) = 0$ we can conclude our claim.

Thus we have the following exact sequence

$$0 \longrightarrow \tilde{T} \longrightarrow V \otimes \mathcal{O}_Y \longrightarrow \mathcal{O}_Y(1) \oplus \mathcal{O}_Y(1) \longrightarrow 0.$$

Taking dual we get

$$0 \longrightarrow \mathcal{O}_Y(-1) \oplus \mathcal{O}_Y(-1) \longrightarrow V \otimes \mathcal{O}_Y \longrightarrow \tilde{T}^* \longrightarrow 0,$$

which gives the surjective map from

$$S^2(V \otimes \mathcal{O}_Y) \longrightarrow S^2(\tilde{T}^*).$$

Let G be its kernel. Then we have the following two exact sequences

$$0 \longrightarrow G \longrightarrow S^2(V \otimes \mathcal{O}_Y) \longrightarrow S^2(\tilde{T}^*) \longrightarrow 0 \tag{5.0.8}$$

and

$$0 \longrightarrow S^2(\mathcal{O}_Y(-1) \oplus \mathcal{O}_Y(-1)) \longrightarrow G \longrightarrow \tilde{T}^* \otimes (\mathcal{O}_Y(-1) \oplus \mathcal{O}_Y(-1)) \longrightarrow 0. \tag{5.0.9}$$

Since Y is a complete intersection in $\mathbb{P}^5$, by Lefschetz theorem on hyperplane section, we have $H^0(Y, T^*) = 0$ and $H^1(Y, T^*)$ is of dimension 1. The long exact sequence of cohomologies of the short exact sequence

$$0 \longrightarrow T^* \longrightarrow \tilde{T}^* \otimes \mathcal{O}_Y(-1) \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

gives the following exact sequence

$$0 \longrightarrow H^0(Y, \tilde{T}^* \otimes \mathcal{O}(-1)) \longrightarrow H^0(Y, \mathcal{O}) \longrightarrow H^1(Y, T^*) \longrightarrow H^1(Y, \tilde{T}^* \otimes \mathcal{O}(-1)) \longrightarrow 0.$$

But the bundle $\tilde{T}^* \otimes \mathcal{O}(-1)$ is the nontrivial canonical extension of $\mathcal{O}$ by T^*, therefore

$$H^1(Y, T^*) \simeq H^0(Y, \mathcal{O}).$$

Therefore $H^i(Y, \tilde{T} \otimes \mathcal{O}(-1)) = 0, i = 0, 1.$

From the long exact sequence of the cohomologies of the exact sequence in (5.0.9) it follows that $H^i(Y, G) = 0, i = 0, 1.$ and therefore from the exact sequence (5.0.8) we have

$$H^0(Y, S^2(V \otimes \mathcal{O})) \simeq H^0(Y, S^2(\tilde{T}^*)).$$

On the other hand we have an exact sequence

$$0 \longrightarrow T^* \otimes \mathcal{O}(1) \longrightarrow \tilde{T}^* \longrightarrow \mathcal{O}(1) \longrightarrow 0,$$

which gives as before the following two exact sequences

$$0 \longrightarrow F \longrightarrow S^2(\tilde{T}^*) \longrightarrow \mathcal{O}(2) \longrightarrow 0 \tag{5.0.10}$$

and

$$0 \longrightarrow S^2(T^* \otimes \mathcal{O}(1)) \longrightarrow F \longrightarrow T^* \otimes \mathcal{O}(2) \longrightarrow 0. \tag{5.0.11}$$

Since Y is embedded in $\mathbb{P}(V)$, we have an surjection from

$$H^0(Y, S^2(\tilde{T}^*)) \simeq H^0(\mathbb{P}(V), \mathcal{O}(2)) \longrightarrow H^0(Y, \mathcal{O}_Y(2))$$

But the dimension of $H^0(\mathbb{P}(V), \mathcal{O}(2))$ is 21 and as Y is the intersection of two non-degenerate quadrics, $\dim H^0(Y, \mathcal{O}_Y(2)$ is 19.Therefore from the long exact sequence of cohomologies of the exact sequence (5.0.10) we have $\dim H^0(Y, F)$ is 2.

Hence from the exact sequence (5.0.11) it is clear that $\dim H^0(Y, S^2(T^* \otimes \mathcal{O}(1)))$ is at most 2.

Let $\mathbb{P}(W)$ be the pencil of quadratic form on $\mathbb{P}(V)$ defined by the quadrics Q_1, and Q_2. Then it gives a pencil of quadratic forms on $\tilde{T}$ along the fibre.

From the exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \tilde{T} \longrightarrow T \otimes \mathcal{O}(-1) \longrightarrow 0$$

and by the definition of Y in $\mathbb{P}(V)$, $\mathcal{O}_Y(-1)$ is orthogonal to $\tilde{T}$ with respect to the pencil $\mathbb{P}(W)$.Therefore it gives a pencil of quadratic form along the fibre of $TY \otimes \mathcal{O}(-1)$. Hence $W \subset H^0(Y, S^2(T^* \otimes \mathcal{O}(1)))$.

Therefore $\dim H^0(Y, S^2(T^* \otimes \mathcal{O}(1)))$ is at least 2.

Therefore $\dim H^0(Y, S^2(T^* \otimes \mathcal{O}(1))) = 2.$

$\square$

Now restriction to the projective tangent bundle to Y of the pencil W gives a pencil and hence on $TY \otimes \mathcal{O}_Y(-1)$ (as in above Lemma).

i.e., if p'_i denote the i-th projection from $Y \times \mathbb{P}^1$, where $\mathbb{P}^1 = \mathbb{P}(W)$, to the i-th factor then we get a morphism

$$\tilde{f} : (p'_1)^* TY \otimes \mathcal{O}_Y(-1) \longrightarrow (p'_1)^* T^*Y \otimes \mathcal{O}_Y(1) \otimes (p'_2)^* \mathcal{O}_{\mathbb{P}^1}(1). \tag{5.0.12}$$

Therefore if we identify the moduli space $\mathcal{M}$ of stable bundles as a smooth intersection Y of two quadrics in $\mathbb{P}^5$ then by the above Lemma we conclude that the pencil given by f and $\tilde{f}$ are same. In other words we have the following Theorem;

Theorem 5.0.21. *The morphisms f in* (5.0.6) *and $\tilde{f}$ in* (5.0.12) *are same.*

Therefore from the above theorem we can identify

$$W = H^0(\mathbb{P}^1, \mathcal{O}(1)) \text{ with } H^0(X, K)$$

and the geometry of the Hitchin map can be described using above Theorem as follows: For each element $w \in W$ we get a quadratic form on $T\mathcal{M} \otimes \mathcal{O}(-1)$. Dualizing we will get a range of quadratic forms which can be identified with W^*, on the cotangent bundle of the moduli space, i.e., to every point s of the cotangent bundle

$$w^* \mapsto q_{w^*}(s, s),$$

where q_{w^*} denotes the quadratic form corresponding to $w^* \in W^*$, defines a quadratic form on W^*. i.e., an element of

$$S^2(W) = H^0(X, K^2)$$

which is the Hitchin map on the cotangent bundle to the moduli space we wanted.

Remark 5.0.22. *If we fix a point y in the moduli space of stable bundles and consider the projective space corresponding to the cotangent space at y then the geometry of the Hitchin map will be clear. In this situation a Hitchin point (a point in the Hitchin space $\mathbb{P}(H^0(X, K^2))$) can be thought as two quadrics Q_1 and Q_2 in the pencil. Then by the above discussion these two quadrics will give two conics in the cotangent space at y. Then the fibre over this Hitchin point to the cotangent space at y are the points contained in both conics.*

Dually a point in the cotangent space gives a line in the tangent space and the fibre of the Hichin map over a Hitchin point given by Q_1 and Q_2 are the lines in the tangent space which touch the conics given by Q_1 and Q_2.

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