REPRESENTATIONS AND CONJUGACY CLASSES OF GENERAL LINEAR GROUPS OVER PRINCIPAL IDEAL LOCAL RINGS OF LENGTH TWO

By

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The Institute of Mathematical Sciences, Chennai

A thesis submitted to the
Board of Studies in Mathematical Sciences

In partial fulfillment of requirements
For the Degree of

DOCTOR OF PHILOSOPHY

of

HOMI BHABHA NATIONAL INSTITUTE

January 2010

Homi Bhabha National Institute

Recommendations of the Viva Voce Board

As members of the Viva Voce Board, we recommend that the dissertation prepared by Pooja Singla entitled " Representations and Conjugacy Classes of General Linear Groups over Principal Ideal Local Rings of Length Two" may be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

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Declaration

I hereby declare that the investigation presented in this thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part of a degree/diploma at this or any other Institution/University.

Pooja Singla

Dedicated to my parents
and my siblings.

To my teachers ....

Gurur Brahma Gurur Vishnu Gurur Devo Maheshwarah
Gurur Sakshat Param Brahma Tasmai Sri Guruve Namaha.

Abstract

We study the irreducible complex representations and conjugacy classes of general linear groups over principal ideal local rings of length two with a fixed finite residue field. We construct a canonical correspondence between the irreducible representations of all such groups which preserves dimensions and a canonical correspondence between the conjugacy classes of all such groups which preserves cardinalities. For general linear groups of order three and four over these rings, we construct all the irreducible representations. We show that the the problem of constructing all the irreducible representations of all general linear groups over these rings is not easier than the problem of constructing all the irreducible representations of the general linear groups over principal ideal local rings of arbitrary length in the function field case.

Acknowledgements

I am indebted to my guide Amritanshu Prasad. This thesis would never have been possible but for his erudite and stimulating guidance and constructive criticism. I have learnt a lot from him. I thank him for the fruitful mathematical discussions throughout this project.

It is a great pleasure to thank Uri Onn for many fruitful discussions, and in particular, for the proof of Lemma 4.4.2, which he provided through a personal communication.

I am also grateful to Alexander Stasinski for reading the first draft of parts of this thesis carefully and for providing very useful feedback.

I thank all the faculty members of IMSc for their help and cooperation at various stages of my work. My special thanks to Prof. N. S. Noorie who inspired me for higher studies and provided his invaluble guidance whenever I felt the need.

I am indebted to my many student colleagues for providing a stimulating and fun environment in which to learn and grow. I am especially grateful to Anisa, Sachin, Kavita, Ved, Sundar, Umesh, Ajay, Basudha, Nutan, Swagata, Alok, Prem, Vivek, Akshay for being great friends and motivators during all these years.

Words can not express my gratitude to my beloved parents to whom I owe much more than what words can say. My siblings Puneet and Tanu deserve special thanks for keeping me updated with their day to day lives and providing much required humour and liveliness at times.

How can I forget to thank the Badminton and TT players at IMSc, playing with them was such a big fun, moreover it always provided lots of energy and relaxation throughout my stay at IMSc.

The work leading to this thesis was carried out completely at the Institute of Mathematical Sciences, Chennai. I would like to thank the institute for its financial support and the wonderful infrastructure and working environment that it provides to its members.

Above all, I thank Almighty for the kind blessings.

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List of Publications

- On Representations of General Linear Groups over Principal Ideal Local Rings of Length Two - to appear in *Journal of Algebra*.

Chapter 1

Introduction

1.1 Overview

Let F be a non-Archimedean local field with ring of integers $\mathcal{O}$. Let $\wp$ be the unique maximal ideal of $\mathcal{O}$ and π be a fixed uniformizer of $\wp$. Assume that the residue field $\mathcal{O}/\wp$ is finite. The typical examples of such rings of integers are $\mathbf{Z}_p$ (the ring of p-adic integers) and $\mathbf{F}_q[[t]]$ (the ring of formal power series with coefficients over a finite field). We denote by $\mathcal{O}_\ell$ the reduction of $\mathcal{O}$ modulo $\wp^\ell$, i.e. $\mathcal{O}_\ell = \mathcal{O}/\wp^\ell$. Let Λ_k denote the set of partitions with k parts, namely, non-increasing finite sequences $(\ell_1, \ell_2, \ldots, \ell_k)$ of positive integers, and let $\Lambda = \cup\Lambda_k$. Since $\mathcal{O}$ is a principal ideal domain with a unique maximal ideal $\wp$, every finite $\mathcal{O}$-module is of the form $\oplus_{i=1}^{k}\mathcal{O}_{\ell_i}$, where ℓ_i's can be arranged so that $\lambda = (\ell_1, \ell_2, \ldots, \ell_k) \in \Lambda_k$. Let $\mathrm{M}_\lambda = \oplus_{i=1}^{k}\mathcal{O}_{\ell_i}$ and

$$\mathrm{G}_{\lambda,F} = \mathrm{Aut}_{\mathcal{O}}(\mathrm{M}_\lambda).$$

We write G_λ instead of $G_{\lambda,F}$ whenever field F is clear from the context. If $\mathrm{M}_\lambda = \mathcal{O}_\ell^n$ for some natural number n, then the group G_λ consists of invertible matrices of order n with entries in the ring $\mathcal{O}_\ell$, so we use the notation $\mathrm{GL}_n(\mathcal{O}_\ell)$ for G_λ in this case.

The representation theory of the finite groups G_λ has attracted the attention of many mathematicians. We give a brief history of this problem. Green [Gre55] calculated the characters of the irreducible representations of $\mathrm{GL}_n(\mathcal{O}_1)$. Several authors, e.g., Frobenius [Fro96], Rohrbach [Roh32], Kloosterman [Klo46a, Klo46b], Tanaka [Tan67], Kutzko [Kut73], Nobs [Nob76], Nobs-Wolfart [NW74], Nagornyi [Nag76] and Stasinski [Sta08] studied the representations of the $\mathrm{SL}_2(\mathcal{O}_\ell)$ and $\mathrm{GL}_2(\mathcal{O}_\ell)$. Nagornyi [Nag78] obtained partial results regarding the representations of $\mathrm{GL}_3(\mathcal{O}_\ell)$ and Onn [Onn08] constructed all the irreducible representations of the groups

$G_{(\ell_1,\ell_2)}$. Recently, Avni-Klopsch-Onn-Voll [AKOV09] have announced results about the representation theory of the groups $SL_3(\mathbf{Z}_p)$.

In another direction, it was observed that, being maximal compact subgroups, $GL_n(\mathcal{O})$ play an important role in the representation theory of the groups $GL_n(F)$. Further, every continuous representation of $GL_n(\mathcal{O})$ factors through one of the natural homomorphisms $GL_n(\mathcal{O}) \to GL_n(\mathcal{O}_\ell)$. This brings the study of irreducible representations of groups $GL_n(\mathcal{O}_\ell)$ to the forefront. Various questions regarding the complexity of the problem of determining irreducible representations of these groups were asked. For example Nagornyi [Nag78] proved that that this problem contains the matrix pair problem. Aubert-Onn-Prasad-Stasinski [AOPS] proved that, for $F = \mathbf{F}_q((t))$, constructing all irreducible representations of $GL_n(\mathcal{O}_2)$ for all n is equivalent to constructing all irreducible representations of $G_{\lambda,\mathbf{F}_{q^m}((t))}$ for all λ and m (see also Section 3.2).

Motivated by Lusztig's work for finite groups of Lie type, Hill [Hil93] partitioned all the irreducible representations of groups $GL_n(\mathcal{O}_\ell)$ into geometric conjugacy classes and reduced the study of irreducible representations of $GL_n(\mathcal{O}_\ell)$ to the study of its nilpotent characters. In later publications [Hil94, Hil95a, Hil95b], he succeeded in constructing many irreducible representations (namely strongly-semisimple, semisimple, regular etc.) for these groups. Following the techniques used in the representation theory of groups $GL_n(\mathcal{O}_1)$ and $GL_n(F)$, various notions like cuspidality and supercuspidality were introduced for representations of $GL_n(\mathcal{O})$ (for more on this see [AOPS]), but a complete understanding of the irreducible representations of the groups $GL_n(\mathcal{O}_\ell)$ for $\ell \geq 2$ remains elusive.

From the available results, it was observed that methods of constructing irreducible representations of groups G_λ do not depend on the particular ring of integers $\mathcal{O}$, but depend only on the residue field. This led Onn to conjecture [Onn08, Conjecture 1.2] that

Conjecture 1.1.1 *The isomorphism type of the group algebra $\mathbb{C}[G_\lambda]$ depends only on λ and $q = |\mathcal{O}/\wp|$.*

1.2 The Main Results

We discuss the method of constructing complex irreducible representations of the groups $GL_n(\mathcal{O}_2)$ with the help of Clifford theory and reduce this problem to constructing irreducible representations of certain subgroups of $GL_n(\mathcal{O}_1)$. This enables us to give an affirmative answer to the above conjecture for $GL_n(\mathcal{O}_2)$. The groups $GL_n(\mathcal{O}_2)$, for distinct rings of integers $\mathcal{O}$ are not necessarily isomorphic, even when the residue fields are isomorphic. For example; for a natural number n and a prime p, the group $\mathrm{GL}_n(\mathbf{F}_p[[t]]/t^2)$ is a semi-direct product of the groups $\mathrm{M}_n(\mathbf{F}_p)$ and $\mathrm{GL}_n(\mathbf{F}_p)$, but on the other hand $\mathrm{GL}_n(\mathbf{Z}_p/p^2\mathbf{Z}_p)$ is not unless $n = 1$ or $(n,p) = (2,2),(2,3)$ or $(3,2)$ (Sah [Sah77, p. 22], Ginosar [Gin01]). Our main emphasis is on proving that all of their irreducible representations can be constructed in a uniform way. We also succeed in showing that representation theory of groups $G_{\lambda,\mathbf{F}_{q^m}((t))}$ plays a vital role in representation theory of groups $\mathrm{GL}_n(\mathcal{O}_2)$ for any $\mathcal{O}$, in the sense that if we know irreducible representations of the groups $G_{\lambda,\mathbf{F}_{p^m}((t))}$ for all positive integers m, we can determine all the representations of $\mathrm{GL}_n(\mathcal{O}_2)$.

More precisely, let F and F' be local fields with rings of integers $\mathcal{O}$ and $\mathcal{O}'$ respectively such that their residue fields are finite and isomorphic (with a fixed isomorphism). Let $\wp$ and $\wp'$ be the maximal ideals of $\mathcal{O}$ and $\mathcal{O}'$ respectively. As described earlier, $\mathcal{O}_2$ and $\mathcal{O}'_2$ denote the rings $\mathcal{O}/\wp^2$ and $\mathcal{O}'/\wp'^2$ respectively. We prove

Theorem 1.2.1 *There exists a canonical bijection between the irreducible representations of $GL_n(\mathcal{O}_2)$ and those of $GL_n(\mathcal{O}'_2)$, which preserves dimensions.*

Definition 1 *(Representation Zeta function) Let G be a finite group. The representation zeta function of G is the function*

$$R_G(\mathcal{D}) = \sum_{\rho\in \mathrm{Irr}G} \mathcal{D}^{\dim\rho} \in \mathbb{Z}[\mathcal{D}]$$

In view of the above definition, Theorem 1.2.1 implies that

Corollary 1.2.2 *The representation zeta functions of $GL_n(\mathcal{O}_2)$ and $GL_n(\mathcal{O}'_2)$ are equal.*

In other words, the representation zeta function depends on the ring only through the order of its residue field.

Concerning the complexity of the problem of constructing irreducible representations of groups $\mathrm{GL}_n(\mathcal{O}_2)$, we obtain the following generalisation of [AOPS, Theorem 6.1].

Theorem 1.2.3 *Let $\mathcal{O}$ be the ring of integers of a non-Archimedean local field F, such that residue field has cardinality q. Then the problem of constructing irreducible representations of the following groups are equivalent:*

1. *$GL_n(\mathcal{O}_2)$ for all $n \in \mathbf{N}$.*
2. *$G_{\lambda,E}$ for all partitions λ and all unramified extensions E of $\mathbf{F}_q((t))$.*

We construct all the irreducible representations of $\mathrm{GL}_2(\mathcal{O}_2)$, $\mathrm{GL}_3(\mathcal{O}_2)$ and $\mathrm{GL}_4(\mathcal{O}_2)$. As mentioned earlier, the representation theory of $\mathrm{GL}_2(\mathcal{O}_2)$ is already known. Partial results regarding the representations of $\mathrm{GL}_3(\mathcal{O}_2)$ have been obtained by Nagornyi [Nag78] but the representation theory of $\mathrm{GL}_4(\mathcal{O}_2)$ seems completely novel. We find that

Theorem 1.2.4 *The number and dimensions of irreducible representations of groups $GL_3(\mathcal{O}_2)$ and $GL_4(\mathcal{O}_2)$ are polynomials in $\mathbb{Q}[q]$.*

This theorem proves the strong version of Onn's conjecture [Onn08, Conjecture 1.3] for the groups $\mathrm{GL}_3(\mathcal{O}_2)$ and $\mathrm{GL}_4(\mathcal{O}_2)$.

The equality of the number of irreducible representations with the number of conjugacy classes for finite groups suggests that the question of determining irreducible representations is in some sense parallel to the question of finding conjugacy classes of these groups. Just like the problem of representations of groups G_λ, the problem of finding conjugacy classes of these groups is also very hard. The complexity of this problem for groups $\mathrm{GL}_n(\mathcal{O}_\ell)$ is best described by quoting Hill [Hil95a] from one of his publications,

One cannot expect to find a good general description of the conjugacy classes of $GL_n(\mathcal{O}_\ell)$. For, if one did have a classification of all the conjugacy classes for groups $GL_n(\tilde{\mathcal{O}}_\ell)$ for all $\ell \geq 1$ then one would also have a classification of the indecomposable $\tilde{\mathcal{O}}_r$-lattices for all cyclic p-groups. One knows from Gudivok-Pogorilyak [GP89] that this is a wild problem.

Nevertheless, the classification problem of similarity classes of matrices over rings has been studied by many authors. We give a brief history of this problem. The similarity classes of $\mathrm{GL}_n(\mathcal{O}_1)$ are well understood in terms of their rational canonical forms for a long time and are discussed, for example in Dickson [Dic59]. Davis [Dav68] has shown using Hensel's method, that two matrices in $M_n(\mathbf{Z}/p^\ell\mathbf{Z})$ which are zeroes of a common polynomial whose reduction modulo p has no repeated roots are similar if and only if their reductions modulo p are similar. In a similar direction, using an extension of the Sylow theorems, Pomfret [Pom73] has shown that matrices in $\mathrm{GL}_n(\mathcal{O}_\ell)$ whose orders are coprime to the characteristic of the residue field are similar if and only if their images in $\mathrm{GL}_n(\mathcal{O}_1)$ similar. Given any two matrices α and α' in $\mathrm{SL}_n(\mathbf{Z}_p)$, Appelgate and Onishi [AO82] have given an explicit method to determine a positive integer ℓ such that α and α' are conjugate in $\mathrm{SL}_n(\mathbf{Z}_p)$ if and only if they are conjugate in $\mathrm{SL}_n(\mathbf{Z}/p^\ell\mathbf{Z})$, thereby reducing the conjugacy problem in the uncountable group $\mathrm{SL}_n(\mathbf{Z}_p)$ to a finite one. Nechaev [Nec83] has classified the similarity classes in the case $n = 3$ and $\ell = 2$ and generalising his result Avni-Onn-Prasad-Vaserstein [AOPV09] have classified the similarity classes of groups $\mathrm{GL}_3(\mathcal{O}_\ell)$ and $M_3(\mathcal{O}_\ell)$. Motivated by Theorem 1.2.1, our next question concerns the class equations of groups $\mathrm{GL}_n(\mathcal{O}_\ell)$ for distinct ring of integers $\mathcal{O}$. With all the notations as above, we prove

Theorem 1.2.5 *There exists a bijection between the conjugacy classes of group $GL_n(\mathcal{O}_2)$ and those of $GL_n(\mathcal{O}'_2)$ which preserves the sizes of conjugacy classes.*

In other words, class equation depends on the ring only through the order of its residue field.

1.3 Organization of the thesis

Chapter 2 is devoted to the discussion of a few basic results, for example we recall Clifford theory from the representation theory of finite groups and review the similarity classes of $M_n(\mathbf{F}_q)$. At the end of this chapter we discuss the centralizer algebras of matrices, namely the set of matrices that commute with a given matrix. We describe explicitly the centralizer of certain specific matrices in $\mathrm{M}_n(\mathbf{F}_q)$, and in $\mathrm{GL}_n(\mathbf{F}_q)$. We also discuss a few general results regarding the centralizers of matrices over arbitrary rings. All the results discussed in this chapter are either well known or are elementary in nature.

In Chapter 3, we set up basic notation that we use throughout the thesis and discuss the action of groups $\mathrm{GL}_n(\mathcal{O}_2)$ on the characters of a normal subgroup $K = \mathrm{Ker}(\mathrm{GL_n}(\mathcal{O}_2) \mapsto \mathrm{GL_n}(\mathcal{O}_1))$. With the help of results of Chapter 2, we prove Theorems 1.2.1 and state a few corollaries. At the end of this chapter we prove the Theorem 1.2.3, which concerns the complexity of the problem of determining irreducible representations of groups $\mathrm{GL}_n(\mathcal{O}_2)$.

Chapter 4, is fully devoted to applications of Theorem 1.2.1. We begin this chapter with a discussion regarding the relation between the representation zeta functions of the groups $\mathrm{GL}_n(\mathcal{O}_2)$ and those of centralizer subgroups in $\mathrm{GL}_n(\mathcal{O}_1)$. We then construct the irreducible representations of the groups $\mathrm{GL}_2(\mathcal{O}_2), \mathrm{GL}_3(\mathcal{O}_2)$, $G_{(2,1,1)}$, and $\mathrm{GL}_4(\mathcal{O}_2)$ and describe their representation zeta functions . In particular we prove Theorem 1.2.4.

In Chapter 5, we present the proof of Theorem 1.2.5. For the proof, we use the analysis of centralizers from Chapter 2. At the end of this chapter, which also happens to be our last chapter, we give a few interesting questions naturally arising out of the work in this thesis.

Chapter 2

Preliminaries

In this chapter, we set up notation and discuss a few basic results that we shall use in later chapters. In the first section, we recall two basic results from representation theory. For basic definitions and other results of representation theory we refer Curtis-Reiner [CR62] and Serre [Ser78]. In the second section, we recall the structure theorem for $\mathbf{F}_q[t]$-modules which are finite dimensional as $\mathbf{F}_q$-vector spaces and the primary decomposition of matrices. The third section is devoted to the centralizer of a given matrix, namely the set of matrices in $M_n(\mathbf{F}_q)$ and $\mathrm{GL}_n(\mathbf{F}_q)$ that commute with a given matrix. We also discuss some results of a general nature in this section.

2.1 Preliminaries from Representation Theory

If G is a group we use $\mathrm{Irr}(G)$ to denote set of isomorphism classes of irreducible representations of group G. Let N be normal subgroup of G. Then G acts on $\mathrm{Irr}(N)$ by $\rho \mapsto \rho^g$, where

$$\rho^g(x) = \rho(gxg^{-1}), \text{ for all } x \in N \text{ and } g \in G.$$

The following theorem is known as *Clifford Theory*.

Theorem 2.1.1 *Let G be a finite group and N be a normal subgroup. For any irreducible representation ρ of N, let $T(\rho) = \{g \in G \mid \rho^g = \rho\}$ denote the stabilizer of ρ. Then the following hold*

1. *If π is an irreducible representation of G such that $\langle \pi|_N, \rho\rangle \neq 0$, then $\pi|_N = e(\oplus_{\rho' \in \Omega} \rho')$ where Ω is the orbit of ρ under the action of G on $\mathrm{Irr}(N)$ and e is a positive integer.*

2. *Suppose that* ρ *is an irreducible representation of* N. *Let*

$$A = \{\theta \in \mathrm{Irr}(T(\rho)) | \langle Res_N^{T(\rho)}\theta, \rho\rangle \neq 0\}$$
$$B = \{\pi \in \mathrm{Irr}(G) | \langle Res_N^{G}\pi, \rho\rangle \neq 0\}$$

Then

$$\theta \to \mathrm{Ind}_{T(\rho)}^{G}(\theta)$$

is a bijection of A *onto* B.

3. *Let* H *be a subgroup of* G *containing* N, *and suppose that* ρ *is an irreducible representation of* N *which has an extension* $\tilde{\rho}$ *to* H *(i.e.* $\tilde{\rho}|_N = \rho$*). Then the representations* $\chi \otimes \tilde{\rho}$ *for* $\chi \in \mathrm{Irr}(H/N)$ *are irreducible, distinct for distinct* χ *and*

$$\mathrm{Ind}_N^H(\rho) = \oplus_{\chi \in \mathrm{Irr}(H/N)} \chi \otimes \tilde{\rho}.$$

Proof: See for example, 6.2, 6.11, and 6.17 respectively in Isaacs [Isa76]. □

Lemma 2.1.2 *Let* G *be a finite group with two subgroups* N *and* M, *such that* N *is normal in* G *and* $G = N.M$. *If* ψ_1 *and* ψ_2 *are one dimensional representations of* N *and* M *respectively such that* $\psi_1(mnm^{-1}) = \psi_1(n)$ *for all* $m \in M$, $n \in N$ *and* $\psi_1|_{N\cap M} = \psi_2|_{N\cap M}$, *then* $\psi_1.\psi_2$ *defined by* $\psi_1.\psi_2(n.m) := \psi_1(n)\psi_2(m)$ *is the unique one dimensional representation of* G *extending both* ψ_1 *and* ψ_2

Proof: We prove that $\psi_1.\psi_2$ is a one dimensional representation of G.

1. $\psi_1.\psi_2$ is well defined: Suppose $nm = n'm'$, where $n, n' \in N$ and $m.m' \in M$. Then $n'^{-1}n = m'm^{-1} \in M \cap N$.

$$\begin{aligned}
\psi_1.\psi_2(nm) &= \psi_1(n)\psi_2(m) \\
&= \psi_1(n'n'^{-1}n).\psi_2(mm'^{-1}m') \\
&= \psi_1(n').\psi_1(n'^{-1}nmm'^{-1}).\psi_2(m') \\
&= \psi_1(n').\psi_2(m') \\
&= \psi_1\psi_2(n'm')
\end{aligned}$$

2. $\psi_1.\psi_2$ is a homomorphism:

$$\begin{aligned}
\psi_1.\psi_2(nmn'm') &= \psi_1.\psi_2(nmn'm^{-1}mm') \\
&= \psi_1(n)\psi_1(mn'm^{-1})\psi_2(mm') \\
&= \psi_1(n)\psi_1(n')\psi_2(m)\psi_2(m') \\
&= \psi_1.\psi_2(nm)\psi_1.\psi_2(n'm')
\end{aligned}$$

Thus $\psi_1.\psi_2$ is a well defined one dimensional representation of G. Let ν be any other one dimensional representation of G extending ψ_1 and ψ_2. Then $\nu(nm) = \nu(n.1)\nu(1.m) = \psi_1(n)\psi_2(m)$ for all $n \in N$ and $m \in M$. This proves the uniqueness of extension. □

2.2 Primary Decomposition and Jordan Canonical Form

In this section we discuss the structure of $\mathbf{F}_q[t]$-modules which are finite dimensional as $\mathbf{F}_q$-vector space and the primary decomposition of matrices. We use this to describe Jordan canonical forms for the matrices whose characteristic polynomials split over $\mathbf{F}_q$. For this section, wherever required we have reproduced material from the lecture notes of Prasad [Pra07], which are available online.

Given a matrix $A \in M_n(\mathbf{F}_q)$, for every vector $x \in \mathbf{F}_q^n$ and every polynomial $f(t) \in \mathbf{F}[t]$ define $f(t)x = f(A)x$. This endows $\mathbf{F}_q^n$ a structure of an $\mathbf{F}_q[t]$ module, which we denote by M^A.

The following Lemma, whose proof is quite straightforward, relates the similarity problem for matrices with isomorphism problem for $\mathbf{F}_q[t]$-modules.

Lemma 2.2.1 *Two matrices A and B are similar if and only if the modules M^A and M^B are isomorphic.*

The next theorem discusses the structure of $\mathbf{F}_q[t]$-modules.

Theorem 2.2.2 *Let M be a $\mathbf{F}_q[t]$ module which is also a finite dimensional $\mathbf{F}_q$-vector space. For every monic irreducible polynomial $f(t) \in \mathbf{F}_q[t]$, let M_f be the vector subspace consisting of elements m of M such that $f(A)^k.m = 0$ for some integer k. Then M_f is $\mathbf{F}_q[t]$ submodule, non-zero for only finitely many irreducible monic polynomials $f(t) \in \mathbf{F}_q[t]$, and*

$$M = \oplus_f M_f,$$

the sum being taken over all the irreducible monic polynomials f for which $M_f \neq 0$.

Let $f(t) \in \mathbf{F}_q[t]$ be an irreducible monic polynomial.

Definition 2 (f-primary module) *An $\mathbf{F}_q[t]$-module M is called* f-primary *if $M = M_f$.*

Theorem 2.2.3 (Structure of a Primary module) *If M^A is an f-primary $\mathbf{F}_q[t]$-module, then there exists a non-decreasing sequence of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ such that*

$$M \cong \mathbf{F}_q[t]/f(t)^{\lambda_1} \oplus \mathbf{F}_q[t]/f(t)^{\lambda_2} \oplus \cdots \oplus \mathbf{F}_q[t]/f(t)^{\lambda_k},$$

where $f(t)^{\lambda_1+\lambda_2+\cdots+\lambda_k}$ is the characteristic polynomial, and $f(t)^{\lambda_1}$ is the minimal polynomial of A.

The preceding theorems are the *Structure theorems for $\mathbf{F}_q[t]$-modules.* For proofs, see Bourbaki [Bou03, A.VII.31]. We use these theorems to obtain primary decomposition and Jordan canonical forms for matrices over $\mathbf{F}_q$.
Notation: If A_i's for $1 \leq i \leq l$ are matrices, then we denote by $\oplus_i A_i$ the block diagonal matrix

$$\begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_l \end{bmatrix}$$

We call this *direct sum* of matrices. In the same spirit, If $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_l$ are the sets of matrices then $\oplus_{i=1}^{l} \mathcal{A}_i$ denotes the set of matrices $\{\oplus_{i=1}^{l} A_i | \ A_i \in \mathcal{A}_i\}$.

Definition 3 (f-primary Matrix) *A matrix with entries in $\mathbf{F}_q$ is called* f-primary *if its characteristic polynomial is a power of f.*

Theorem 2.2.4 (Primary Decomposition) *Every matrix $A \in M_n(\mathbf{F}_q)$ is similar to a matrix of the form*

$$\oplus_f A_f.$$

where A_f is an f-primary matrix, and the sum is over the irreducible factors of the characteristic polynomial of A. Moreover, for every f, the similarity class of A_f is uniquely determined by the similarity class of A.

Definition 4 (Split Matrix) *A matrix with entries in $\mathbf{F}_q$ is called* split *if its characteristic polynomial splits over $\mathbf{F}_q$. By abusing notation, we also say that the matrix* splits *over $\mathbf{F}_q$.*

Definition 5 (Elementary Jordan Blocks) *For a natural number n and an element a,* elementary Jordan block $J_n(a)$ *is the matrix*

$$\begin{bmatrix} a & 1 & 0 & 0 & \cdots & 0 \\ 0 & a & 1 & 0 & \cdots & 0 \\ 0 & 0 & a & 1 & & \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ & & & & a & 1 \\ 0 & 0 & \cdots & & 0 & a \end{bmatrix}_{n\times n}$$

Theorem 2.2.5 (Jordan Canonical Form for Split Matrices) *Every split matrix $A \in M_n(\mathbf{F}_q)$, up to the rearrangement of the a_i's, is similar to a unique matrix of the form*

$$\oplus_i J_{\lambda(a_i)}(a_i)$$

where $\lambda(a_i) = (\lambda_1(a_i), \lambda_2(a_i), \ldots, \lambda_{k_i}(a_i))$ is a partition and

$$J_{\lambda(a_i)}(a_i) = \begin{bmatrix} J_{\lambda_1(a_i)}(a_i) & 0 & \cdots & 0 \\ 0 & J_{\lambda_2(a_i)}(a_i) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{\lambda_{k_i}(a_i)}(a_i) \end{bmatrix}$$

and each $J_{\lambda_j(a_i)}(a_i)$ is an elementary Jordan block with eigenvalue a_i.

Proof: Let A be a split matrix with the characteristic polynomial $(t - a_1)^{r_1}(t-a_2)^{r_2}\ldots(t-a_l)^{r_l}$ and M^A be the corresponding $\mathbf{F}_q[t]$-module. Then by Structure Theorem of $\mathbf{F}_q[t]$-modules.

$$M^A \cong M_{a_1} \oplus M_{a_2} \oplus \cdots \oplus M_{a_l}$$

with each,

$$M_{a_i} \cong M_{a_i}^{\lambda_1(a_i)} \oplus M_{a_i}^{\lambda_2(a_i)} \oplus \cdots \oplus M_{a_i}^{\lambda_{k_i}(a_i)}$$

where, for $i \in \{1, \ldots, l\}$ and $j \in \{1, \ldots, k_i\}$, $M_{a_i}^{\lambda_j(a_i)} \cong \mathbf{F}_q[t]/(t - a_i)^{\lambda_j(a_i)}$ for some partition $\{\lambda_1(a_i), \lambda_2(a_i), \ldots, \lambda_{k_i}(a_i)\}$. For each $M_{a_i}^{\lambda_j(a_i)}$, choose the basis $\{1, t - a_i, (t - a_i)^2, \cdots, (t - a_i)^{\lambda_j(a_i)-1}\}$. Then Lemma 2.2.1 gives the required results. □

2.3 Centralizers

In this section, we prove a few elementary results regarding centralizers, namely the matrices that commute with a given matrix. In particular, for a commutative ring R with identity, we describe explicitly centralizers of specific matrices in $M_n(R)$ and $\mathrm{GL}_n(R)$ (see Lemmas 2.3.4 and 2.3.5).

Definition 6 (Centralizer of an element) *Let L be a semigroup under multiplication and l be an element of L. Assume that T is a subset of L. Then* centralizer *of l in T, $Z_T(l)$, is the set of elements of T that commute with l, i.e.,*

$$Z_T(l) = \{t \in T \mid tl = lt\}$$

Remark 1 *If T is a group, then $Z_T(l)$ is a subgroup of T.*

The proofs of Lemmas 2.3.1-2.3.4 involve simple matrix multiplications, so we leave these for reader.

Lemma 2.3.1 *Let R be a commutative ring with unity and a and b be two elements of R such that $a - b$ is invertible in R. Assume that A and B are two upper triangular matrices such that all the diagonal entries of A are equal to a and those of B are equal to b. Then there does not exist any non-zero matrix X over R such that $XA = BX$.*

Lemma 2.3.2 *Let R be commutative ring with unity. Let $a_1, a_2, \ldots, a_l$ be elements of R such that for all $i \neq j$, $a_i - a_j$ is invertible in R. Let $A = \oplus_{i=1}^{l} A_i$ be a square matrix of order n, where A_i's are upper triangular matrices of order n_i. Assume that all diagonal entries of A_i are equal to a_i. Then,*

$$Z_{GL_n(R)}(A) = \oplus_{i=1}^{l} Z_{GL_{n_i}(R)}(A_i)$$

Definition 7 (Principal Nilpotent Matrix) *A square matrix of order n is called Principal Nilpotent, if it is*

$$N_n = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & & 0 & 1 \\ 0 & \cdots & & 0 & 0 \end{bmatrix}_{n \times n}$$

In the sequel we use the notation N_n for the principal nilpotent matrix of order n.

Let $n_1, n_2, \ldots, n_l$ be a sequence of natural numbers, such that $n = n_1 + n_2 + \cdots + n_l$. Let R be a commutative ring with unity and let $A = \oplus_{i=1}^{l} N_{n_i}$. We now describe the centralizer algebras $Z_{M_n(R)}(A)$.

Definition 8 (Upper Toeplitz Matrix) *A square matrix of order n is called Upper Toeplitz, if it is of the form*

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ 0 & a_1 & a_2 & & a_{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & \cdots & & a_1 & a_2 \\ 0 & \cdots & 0 & 0 & a_1 \end{bmatrix}_{n\times n}$$

Lemma 2.3.3 *Let R be a ring with unity. Assume that N_n and N_m are principal nilpotent matrices of order n and m respectively. Then the matrices X over R such that $XN_m = N_nX$ are of the form*

$$X = \begin{cases} \begin{bmatrix} 0_{n\times m-n} & T_{n\times n} \end{bmatrix} & \text{if } n \leq m \\ \\ \begin{bmatrix} T_{m\times m} \\ 0_{n-m\times m} \end{bmatrix} & \text{if } n \geq m \end{cases}$$

where, for a natural number s, $T_{s\times s}$ is an upper Toeplitz Matrix of order s over the ring R.

This Lemma motivates the following definition of rectangular upper Toeplitz matrix.

Definition 9 (Rectangular Upper Toeplitz Matrix) *A matrix of order $n \times m$, over a ring R is called a Rectangular upper Toeplitz matrix if it is of the form*

$$\begin{bmatrix} 0_{n\times m-n} & T_{n\times n} \end{bmatrix} \text{ if } n \leq m \quad \text{or} \quad \begin{bmatrix} T_{m\times m} \\ 0_{n-m\times m} \end{bmatrix} \text{ if } n \geq m$$

where $T_{s\times s}$, for a natural number s, is the upper Toeplitz matrix of order s.

Lemma 2.3.4 *Let $n_1, n_2, \ldots, n_l$ be a sequence of natural numbers, such that $n = n_1 + n_2 + \cdots + n_l$. Let R be a commutative ring with identity and let $A = \oplus_{i=1}^{l} N_{n_i}$. Then the centralizer, $Z_{M_n(R)}(A)$ of A in $M_n(R)$ consists of matrices of the form*

$$\begin{bmatrix} T_{n_1\times n_1} & T_{n_1\times n_2} & \cdots & T_{n_1\times n_l} \\ T_{n_2\times n_1} & T_{n_2\times n_2} & \cdots & T_{n_2\times n_l} \\ \vdots & \vdots & & \vdots \\ T_{n_l\times n_1} & T_{n_l\times n_2} & \cdots & T_{n_l\times n_l} \end{bmatrix}$$

where $T_{n_i\times n_j}$ for all i, j are rectangular upper Toeplitz matrices.

Definition 10 (Block Upper Toeplitz Matrix) *Let $T_{n_i\times n_j}$ be a rectangular upper Toeplitz matrix of order $n_i \times n_j$, over the ring R. A matrix of the form*

$$\begin{bmatrix} T_{n_1\times n_1} & T_{n_1\times n_2} & \cdots & T_{n_1\times n_l} \\ T_{n_2\times n_1} & T_{n_2\times n_2} & \cdots & T_{n_2\times n_l} \\ \vdots & \vdots & & \vdots \\ T_{n_l\times n_1} & T_{n_l\times n_2} & \cdots & T_{n_l\times n_l} \end{bmatrix}$$

is called block upper Toeplitz matrix of order $(n_1, n_2, \ldots, n_l)$ over the ring R.

In the following lemma we relate the group of automorphisms $\mathrm{G}_{\lambda,\mathbf{F}_q((t))}$ with the centralizers in $\mathrm{GL}_n(\mathbf{F}_q)$.

Lemma 2.3.5 *Let $A = \oplus_{i=1}^{k} N_{\lambda_i}$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition. Then the following groups are isomorphic:*

1. *The group of automorphisms, $\mathrm{G}_{\lambda,\mathbf{F}_q((t))} = \mathrm{Aut}_{\mathcal{O}}(\mathcal{O}_{\lambda_1} \oplus \mathcal{O}_{\lambda_2} \oplus \cdots \oplus \mathcal{O}_{\lambda_k})$.*
2. *The centralizer $Z_{GL_n(\mathbf{F}_q)}(A)$.*
3. *The set of invertible block upper Toeplitz matrices of order $(\lambda_1, \lambda_2, \cdots, \lambda_k)$ over $\mathbf{F}_q$.*

Proof: Lemma 2.3.4 implies that groups (2) and (3) are actually equal. We prove isomorphism between (1) and (3). Every $f \in \mathrm{G}_\lambda$ can be thought as an invertible matrix of the form

$$\begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1k} \\ f_{21} & f_{22} & \cdots & f_{2k} \\ \vdots & \vdots & & \vdots \\ f_{k1} & f_{k2} & \cdots & f_{kk} \end{bmatrix}$$

with each $f_{ij} \in \mathrm{End}_{\mathcal{O}}(\mathcal{O}_{\lambda_i}, \mathcal{O}_{\lambda_j})$. Hence it is enough to prove that there is an isomorphism between $\mathrm{End}_{\mathcal{O}}(\mathcal{O}_{\lambda_i}, \mathcal{O}_{\lambda_j})$ and the group of rectangular Toeplitz matrices of order $\lambda_i \times \lambda_j$ over $\mathbf{F}_q$ taking composition to matrix multiplication. We prove isomorphism only for $\lambda_i = \lambda_1$ and $\lambda_j = \lambda_2$ with $\lambda_1 \geq \lambda_2$, leaving the rest to the reader.

Let $\mathcal{T}_{\lambda_1,\lambda_2}$ be the set of rectangular upper Toeplitz matrices of order $\lambda_1 \times \lambda_2$ over $\mathbf{F}_q$. Define a map $\mathrm{End}_{\mathcal{O}}(\mathcal{O}_{\lambda_1}, \mathcal{O}_{\lambda_2}) \to \mathcal{T}_{\lambda_1,\lambda_2}$ by

$$f \mapsto \left[\begin{array}{c|c} 0_{(\lambda_1-\lambda_2)\times\lambda_2} & A \end{array}\right]_{\lambda_1\times\lambda_2}$$

where

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_{\lambda_2-1} & a_{\lambda_2} \\ 0 & a_1 & a_2 & & a_{\lambda_2-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & \cdots & & a_1 & a_2 \\ 0 & \cdots & 0 & 0 & a_1 \end{bmatrix}_{\lambda_2\times\lambda_2},$$

and the elements $a_1, a_2, \ldots, a_{\lambda_2}$ are determined by the expression $f(1) = a_1 + a_2\pi + \cdots + a_{\lambda_2}\pi^{\lambda_2-1}$. It is straightforward to see that this map gives the required isomorphism. □

Another description of centralizers of a matrix A in $M_n(\mathbf{F}_q)$ is the following: each $X \in Z_{M_n(\mathbf{F}_q)}(A)$, defines an $\mathbf{F}_q[t]$-module endomorphism $\phi_X : M^A \to M^A$ given by $\phi_X(m) = X(m)$. The map $X \mapsto \phi_X$ gives an isomorphism

$$Z_{M_n(\mathbf{F}_q)}(A) \cong \mathrm{End}_{\mathbf{F}_q[t]}(M^A, M^A) \tag{2.3.1}$$

A consequence of Lemma 2.3.5 is the following,

Theorem 2.3.6 *Let $f(t)$ be an irreducible polynomial of degree d in $\mathbf{F}_q[t]$, and suppose that for some matrix A and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \Lambda_k$,*

$$M^A \cong \mathbf{F}_q[t]/f(t)^{\lambda_1} \oplus \mathbf{F}_q[t]/f(t)^{\lambda_2} \oplus \cdots \oplus \mathbf{F}_q[t]/f(t)^{\lambda_k}.$$

Let $\mathcal{T}_\lambda(F)$ denote the set of block upper Toeplitz matrices of order $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ over F. Then

1. $Z_{GL_n(\mathbf{F}_q)}(A) \cong G_{\lambda,\mathbf{F}_{q^d}((t))}$ as groups.
2. $Z_{M_n(\mathbf{F}_q)}(A) \cong \mathcal{T}_\lambda(\mathbf{F}_{q^d})$ as rings.

This theorem follows easily from Lemma 2.3.5 and the following Theorem.

Theorem 2.3.7 *Let $f(t) \in \mathbf{F}_q[t]$ be an irreducible polynomial of degree n and let E denote the field $\mathbf{F}_q[t]/f(t)$. Then the rings $\mathbf{F}_q[t]/f(t)^r$ and $E[x]/x^r$ are isomorphic.*

Proof: For the proof we need the following well known Lemma,

Lemma 2.3.8 (Hensel) *Let $f(t)$ be an irreducible polynomial over $\mathbf{F}_q[t]$. Then, for each positive integer r, there exists $q_r(t) \in \mathbf{F}_q[t]$ such that $q_r(t) \equiv t(\text{mod } f(t))$, and $f(q_r(t)) \equiv 0(\text{mod } f(t)^r)$.*

Proof of Lemma: The proof is by induction on r. When $r = 1$, take $q_1(t) = t$. Suppose that $q_{r-1}(t) \in \mathbf{F}_q[t]$ such that

$$q_{r-1}(t) \equiv t \pmod{f(t)} \text{ and } f(q_{r-1}(t)) \equiv 0 \pmod{f(t)^{r-1}}.$$

Then by using the Taylor expansion, for any $g(t) \in \mathbf{F}_q[t]$,

$$f(q_{r-1}(t) + f(t)^{r-1}g(t)) \equiv f(q_{r-1}(t)) + f(t)^{r-1}g(t)f'(q_{r-1}(t)) \pmod{f(t)^r}.$$

$q_{r-1}(t) \equiv t \pmod{f(t)}$, implies that $f'(q_{r-1}(t)) \equiv f'(t) \pmod{f(t)}$. Since $f'(t)$ is not divisible by $f(t)$ and $f(t)$ is irreducible, there exists $r(t), s(t) \in \mathbf{F}_q[t]$ such that $f'r + fs = 1$, which means that $f'(t)r(t) \equiv 1 \pmod{f(t)}$. Since $f(q_{r-1}(t)) \equiv 0 \pmod{f(t)^{r-1}}$, there exists $f_1(t) \in \mathbf{F}_q[t]$ such that

$$f(q_{r-1}(t)) = f(t)^{r-1}f_1(t)$$

When $g(t) = -f_1(t)r(t)$ and $q_r(t) = q_{r-1}(t) + f(t)^{r-1}g(t)$, one has

$$q_r(t) \equiv t \pmod{f(t)} \text{ and } f(q_r(t)) \equiv 0 \pmod{f(t)^r}$$

This completes the proof of the lemma.

Let $q_r(t)$ be as in Hensel's lemma. Then $\delta(y) = q_r(t)$ and $\delta(x) = f(t)$ gives rise to a well-defined ring homomorphism

$$\delta : \mathbf{F}_q[x,y]/(f(y), x^r) \to \mathbf{F}_q[t]/f(t)^r$$

Since $q_r(t) \equiv t \pmod{f(t)}$ and both $q_r(t)$ and $f(t)$ lie in the image of δ and hence t also lies in the image of δ. Therefore δ is a surjective map. Further, both $\mathbf{F}_q[x,y]/(f(y), x^r)$ and $\mathbf{F}_q[t]/f(t)^r$ are nr dimensional $\mathbf{F}_q$-vector spaces. Therefore δ is an isomorphism of rings. □

Lemma 2.3.9 *For A as in the Theorem 2.3.6,*

$$\dim_{\mathbf{F}_q}(Z_{M_n(\mathbf{F}_q)}(A)) = d(\lambda_1 + 3\lambda_2 + \cdots + (2k-1)\lambda_k)$$

Proof: Follows easily from (2.3.1) and Theorem 2.3.6. □

Definition 11 (Companion Matrix) *Let $f(t) = t^n - a_{n-1}t^{n-1} - \cdots - a_1 t - a_0$. Then the companion matrix of f is the $n \times n$ matrix*

$$C_f = \begin{pmatrix} 0 & 0 & \cdots & 0 & a_0 \\ 1 & 0 & \cdots & 0 & a_1 \\ 0 & 1 & \cdots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_{n-1} \end{pmatrix}$$

We use Theorem 2.3.7 to give another canonical form of matrices.

Theorem 2.3.10 (Block Jordan Canonical Form) *Let $A \in M_n(\mathbf{F}_q)$. Then A can be written as a block diagonal matrix with blocks of the form*

$$J_r(f) = \begin{pmatrix} C_f & I & 0 & \cdots & 0 & 0 \\ 0 & C_f & I & \cdots & 0 & 0 \\ 0 & 0 & C_f & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & C_f & I \\ 0 & 0 & 0 & \cdots & 0 & C_f \end{pmatrix}_{rd \times rd}$$

where d is the degree of f, an irreducible factor of the characteristic polynomial of A, C_f is the companion matrix of f, and r is a positive integer. Up to rearrangement of blocks, this canonical form is unique.

Proof: By the structure theorems of $\mathbf{F}_q[t]$-modules, we can assume that A is f-primary, for some irreducible monic polynomial f. Let $E = \mathbf{F}_q[x]/f(x)$. By the structure theorems and Theorem 2.3.7, there exists a partition λ such that

$$M^A \cong E[y]/y^{\lambda_1} \oplus E[y]/y^{\lambda_2} \oplus \cdots \oplus E[y]/y^{\lambda_t}$$

In the notation of the proof of Theorem 2.3.7, let $\theta(t) = t - q(t)$, where we write q for q_i for some i. Then $\theta(t) \in (f(t))$. But $\theta \notin (f(t))^2$, for if it did, we would have

$$\begin{aligned} f(t) &= f(\theta(t) + q(t)) \\ &\cong f(q(t)) + \theta(t) f'(q(t)) \mod (f(t)^2) \\ &= 0 \mod (f(t)^2), \end{aligned}$$

a contradiction. Therefore, $\theta(t) = \alpha f(t)$, where α is unit in $\mathbf{F}_q[t]/f(t)^{\lambda_i}$. In the isomorphism

$$\mathbf{F}_q[t]/(f(t)^r) \to E[y]/y^r = \mathbf{F}_q[x, y]/(y^r, f(x))$$

$t \mapsto \alpha y + x$. Since A acts by t, with respect to the basis of $E[y]/y^{\lambda_i}$ over $\mathbf{F}_q$ given by

$$\begin{array}{cccc} (\alpha y)^{\lambda_i - 1}, & (\alpha y)^{\lambda_i - 1} x, & \ldots, & (\alpha y)^{\lambda_i - 1} x^{d-1}, \\ (\alpha y)^{\lambda_i - 2}, & (\alpha y)^{\lambda_i - 2} x, & \ldots, & (\alpha y)^{\lambda_i - 2} x^{d-1}, \\ \vdots & \vdots & \ldots & \vdots \\ 1, & x, & \ldots, & x^{d-1} \end{array}$$

The matrix of multiplication by $t = \alpha y + x$ is $J_{\lambda_i}(f)$.

□

Definition 12 (Regular Matrix) *A matrix over $\mathbf{F}_q$ is called regular, if its characteristic polynomial is equal to its minimal polynomial.*

Definition 13 (Simple matrix) *A matrix is called simple if its characteristic polynomial is irreducible.*

In the following theorem, we determine the centralizer algebra of regular matrices.

Theorem 2.3.11 *Let $A \in \mathrm{M}_n(\mathbf{F}_q)$, then centralizer $Z_{M_n(\mathbf{F}_q)}(A)$ of A in $M_n(\mathbf{F}_q)$ is the algebra $\mathbf{F}_q[A]$ if and only if A is regular, and in this case, $\dim_{\mathbf{F}_q} Z_{M_n(\mathbf{F}_q)}(A) = n$. Furthermore, the centralizer $Z_{M_n(\mathbf{F}_q)}(A)$ is a field if and only if A is simple.*

Proof: Let $p_1(t)^{a_1} p_2(t)^{a_2} \ldots p_k(t)^{a_k}$ be the characteristic polynomial of A, where $p_1(t), p_2(t), \ldots, p_k(t)$ are distinct irreducible polynomials in $\mathbf{F}_q[t]$. If A is regular, by the structure theorems of $\mathbf{F}_q[t]$-modules,

$$M^A \cong \mathbf{F}_q[t]/(p_1(t))^{a_1} \oplus \mathbf{F}_q[t]/(p_2(t))^{a_2} \oplus \cdots \oplus \mathbf{F}_q[t]/(p_k(t))^{a_k}.$$

Then

$$\mathbf{F}_q[t]/(p_1(t))^{a_1} \oplus \mathbf{F}_q[t]/(p_2(t))^{a_2} \oplus \cdots \oplus \mathbf{F}_q[t]/(p_k(t))^{a_k} \cong \mathrm{End}_{\mathbf{F}_q[t]}(M^A, M^A).$$

Therefore by (2.3.1), $\dim_{\mathbf{F}_q}(Z_{M_n(\mathbf{F}_q)}(A)) = n$. The algebra $\mathbf{F}_q[A] \subseteq Z_{M_n(\mathbf{F}_q)}(A)$. If A is regular then $\dim_{\mathbf{F}_q}(\mathbf{F}_q[A]) = n$. Therefore $\mathbf{F}_q[A] = Z_{M_n(\mathbf{F}_q)}(A)$. Conversely if A is not regular then by using Lemma 2.3.9, it is easy to see that $\dim_{\mathbf{F}_q}(Z_{M_n(\mathbf{F}_q)}(A))$ is strictly greater than n, furthermore in this case $\dim_{\mathbf{F}_q}(\mathbf{F}_q[A])$ is strictly less than n. This proves the first part of the theorem. Further, if A is simple then by (2.3.1) its centralizer algebra is a field. Conversely assume that A is not simple. Without loss of generality we can assume that A is f-primary. For, if not then for distinct polynomials f and f', $\mathrm{End}(M_f, M_{f'}) = 0$ implies that centralizer algebra is not a field. The result for f-primary case follows from part(1) of Theorem 2.3.6.

□

Another proof of this lemma, which does not use language of modules, is presented in Suprunenko-Tyshkevich [ST66].

In the next theorem, we give the centralizers of primary matrices which are in their block Jordan canonical form.

Theorem 2.3.12 *Let* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ *be a partition and* $A = \oplus_{i=1}^{t} J_{\lambda_i}(f)$, *where* f *is an irreducible polynomial and* $J_{\lambda_i}(f)$ *is as described in Theorem 2.3.10. Then the centralizer of* $Z_{M_n(\mathbf{F}_q)}(A)$ *consists of block upper Toeplitz matrices of type* $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ *over the ring* $\mathbf{F}_q[C_f]$.

Proof: In the proof of Theorem 2.3.10, a linear endomorphism of $\oplus_{i=1}^{l} E[y]/y^{\lambda_i}$ commutes with matrix A if and only if it is an $\mathbf{F}_q[t]$-module homomorphism, and hence, if and only if it is $E[y]/y^r$-module homomorphism, which just means that it commutes with the matrices of multiplication by x and y, namely with the matrices

$$\oplus_{i=1}^{l} \alpha^{-1} \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & 0 & \cdots & 0 & I \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \text{ and } \oplus_{i=1}^{l} \begin{pmatrix} C_f & 0 & 0 & \cdots & 0 \\ 0 & C_f & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & 0 & \cdots & C_f & 0 \\ 0 & 0 & 0 & \cdots & C_f \end{pmatrix}$$

Any matrix commuting with these two matrices is easily seen to be block upper Toeplitz matrix of type $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ with entries as matrices commuting with C_f. Since f is irreducible, by Theorem 2.3.11 any matrix commutes with C_f if and only it belongs to $\mathbf{F}_q[C_f]$ □

Chapter 3

Representations of $GL_n(\mathcal{O}_2)$

In this chapter, we apply Clifford theory to the groups $\mathrm{GL}_n(\mathcal{O}_2)$ to prove Theorem 1.2.1 and a few corollaries. We then prove Theorem 1.2.3, which concerns the complexity of the problem of constructing all the irreducible representations of the groups $\mathrm{GL}_n(\mathcal{O}_2)$.

Throughout this chapter, by a character we mean a one dimensional representation, unless stated otherwise. For any abelian group A, we denote by $\hat{A}$ the group of its characters. Recall from Chapter 1, that F is a non-Archimedean local field with ring of integers $\mathcal{O}$, $\wp$ is the unique maximal ideal of $\mathcal{O}$ and π is a fixed uniformizer of $\wp$. Further $\mathcal{O}_\ell$ denotes the reduction of $\mathcal{O}$ modulo $\wp^\ell$, i.e. $\mathcal{O}_\ell = \mathcal{O}/\wp^\ell$ and the residue field $\mathcal{O}_1 = \mathcal{O}/\wp$ is assumed to be finite of cardinality q, so we also occasionally use the notation $\mathbf{F}_q$ for the residue field.

3.1 Proof of Theorem 1.2.1:

Let $\kappa : \mathrm{GL}_n(\mathcal{O}_2) \to \mathrm{GL}_n(\mathcal{O}_1)$ be the natural quotient map and $K = \mathrm{Ker}(\kappa)$. Then $A \mapsto I + \pi A$ induces an isomorphism $M_n(\mathcal{O}_1) \tilde{\to} K$. Fix a non-trivial additive character $\psi : \mathcal{O}_1 \to \mathbf{C}^*$ and for any $A \in \mathrm{M}_n(\mathcal{O}_1)$ define $\psi_A : K \to \mathbf{C}^*$ by

$$\psi_A(I + \pi X) = \psi(\mathrm{Tr}(AX)).$$

Then $A \mapsto \psi_A$ gives an isomorphism $M_n(\mathcal{O}_1) \tilde{\to} \hat{K}$. The group $\mathrm{GL}_n(\mathcal{O}_2)$ acts on $M_n(\mathcal{O}_1)$ by conjugation via its quotient $\mathrm{GL}_n(\mathcal{O}_1)$, and therefore on $\hat{K}$.

For $\alpha \in \mathrm{GL}_n(\mathcal{O}_2)$ and $\psi_A \in \hat{K}$, we have

$$\begin{aligned}\psi_A^{\alpha}(I+\pi X) &= \psi_A(I+\pi\alpha X\alpha^{-1}) \\ &= \psi(\mathrm{Tr}(A\kappa(\alpha)X\kappa(\alpha)^{-1})) \\ &= \psi(\mathrm{Tr}(\kappa(\alpha)^{-1}A\kappa(\alpha)X)) \\ &= \psi_{\kappa(\alpha)^{-1}A\kappa(\alpha)}(I+\pi X) \qquad (3.1.1)\end{aligned}$$

Thus the action of $\mathrm{GL}_n(\mathcal{O}_2)$ on the characters of K transforms to its conjugation (inverse) action on elements of $\mathrm{M}_n(\mathcal{O}_1)$. To prove the theorem, we apply Clifford Theory (Theorem 2.1.1) to the group $G = \mathrm{GL}_n(\mathcal{O}_2)$ and normal subgroup $N = K$. Recall that for any character $\rho \in \hat{K}$, $T(\rho) = \{g \in \mathrm{GL}_n(\mathcal{O}_2) \mid \rho^g = \rho\}$ is the stabilizer of ρ in $\mathrm{GL}_n(\mathcal{O}_2)$. By (3.1.1), for each $\psi_A \in \hat{K}$,

$$T(\psi_A) = \kappa^{-1}(Z_{\mathrm{GL}_n\mathcal{O}_1}(A)) \qquad (3.1.2)$$

Fix a section $s : \mathcal{O}_1 \to \mathcal{O}_2$ of the natural quotient map $\mathcal{O}_2 \to \mathcal{O}_1$ such that $s(0) = 0$ and $s(1) = 1$. By applying s entry-wise, we obtain a map $\mathfrak{s} : M_n(\mathcal{O}_1) \to M_n(\mathcal{O}_2)$. Observe that the restriction of $\mathfrak{s}$ to $\mathrm{GL}_n(\mathcal{O}_1)$ defines a section of κ. For any matrix $A \in M_n(\mathcal{O}_1)$, let $Z_{\mathrm{GL}_n(\mathcal{O}_2)}(\mathfrak{s}(A))$ be the centralizer of $\mathfrak{s}(A)$ in $\mathrm{GL}_n(\mathcal{O}_2)$.

Lemma 3.1.1 *Assume that A is a split matrix and is in its Jordan canonical form, then*

$$Z_{GL_n(\mathcal{O}_1)}(A) = \kappa(Z_{GL_n(\mathcal{O}_2)}(\mathfrak{s}(A)))$$

Proof: Let $\alpha = \kappa(t)$ for some $t \in Z_{\mathrm{GL}_n(\mathcal{O}_2)}(\mathfrak{s}(A))$. Then by definition, t satisfies $t\mathfrak{s}(A) = \mathfrak{s}(A)t$, which along with the fact that κ is a homomorphism implies $\kappa(t)A = A\kappa(t)$. Hence $\alpha = \kappa(t) \in Z_{\mathrm{GL}_n(\mathcal{O}_1)}(A)$. This proves $\kappa(Z_{\mathrm{GL}_n(\mathcal{O}_2)}(\mathfrak{s}(A))) \subseteq Z_{\mathrm{GL}_n(\mathcal{O}_1)}(A)$. For the reverse inclusion, since A is a split matrix, by Theorems 2.2.4 and 2.2.5,

$$A = \oplus_{i=1}^{l} A_i$$

where each A_i is a split primary matrix and is of the form

$$A_i = \oplus_{j=1}^{l_i} J_{\lambda_{ij}}(a_i)$$

with a_i's being distinct elements of the field $\mathcal{O}_1$. By using $s(0) = 0$ and $s(1) = 1$, we obtain $\mathfrak{s}(A_i)$, for all i is an upper triangular matrix with all diagonal entries equal to $s(a_i)$ and $\mathfrak{s}(A) = \oplus_{i=1}^{l}(\mathfrak{s}(A_i))$. Further $a_i \neq a_j$

imply that $s(a_i) - s(a_j)$ are invertible elements of the ring $\mathcal{O}_2$. Therefore by Lemma 2.3.2,

$$Z_{\mathrm{GL}_n(\mathcal{O}_2)}(\mathfrak{s}(A)) = \oplus_{i=1}^{l} Z_{\mathrm{GL}_{n_i}(\mathcal{O}_2)}(\mathfrak{s}(A_i)) \tag{3.1.3}$$

Since we also have

$$Z_{\mathrm{GL}_n(\mathcal{O}_1)}(A) = \oplus_{i=1}^{l} Z_{\mathrm{GL}_{n_i}(\mathcal{O}_1)}(A_i),$$

it is sufficient to prove $Z_{\mathrm{GL}_n(\mathcal{O}_1)}(A) \subseteq \kappa(Z_{\mathrm{GL}_n(\mathcal{O}_2)}(\mathfrak{s}(A)))$ when A is a split primary.

Split Primary Case: Now we may assume that A is a split primary matrix and is in its Jordan canonical form. Theorems 2.2.4 and 2.2.5 give that $A = aI_n + (\oplus_{i=1}^{t} N_{n_i})$ for some $a \in \mathcal{O}_1$. Let $\alpha \in Z_{\mathrm{GL}_n(\mathcal{O}_1)}(A)$, by Lemma 2.3.5, α is an invertible block Toeplitz matrix of order $(n_1, n_2, ..., n_t)$ over the ring $\mathcal{O}_1$. Our choice of section s ensures that, $\mathfrak{s}(A) = s(a)I_n + (\oplus_{i=1}^{t} N_{n_i})$, and $\mathfrak{s}(\alpha)$ is an invertible block Toeplitz matrix of order $(n_1, n_2, \ldots, n_t)$ over the ring $\mathcal{O}_2$. But then by Lemma 2.3.4 $\mathfrak{s}(\alpha) \in Z_{\mathrm{GL}_n(\mathcal{O}_2)}(\mathfrak{s}(A))$. Hence $\alpha = \kappa(\mathfrak{s}(\alpha)) \in \kappa(Z_{\mathrm{GL}_n(\mathcal{O}_2)}(\mathfrak{s}(A)))$. □

From the proof of above lemma we obtain,

Corollary 3.1.2 *If A is a split matrix and is in its Jordan canonical form, then $\alpha \in Z_{GL_n(\mathcal{O}_1)}(A)$ if and only if $\mathfrak{s}(\alpha) \in Z_{GL_n(\mathcal{O}_2)}(\mathfrak{s}(A))$.*

Corollary 3.1.3 *If A is a split matrix and is in its Jordan canonical form, then $T(\psi_A) = KZ_{GL_n(\mathcal{O}_2)}(\mathfrak{s}(A))$.*

Proof: The inclusion $T(\psi_A) \subseteq KZ_{\mathrm{GL}_n(\mathcal{O}_2)}(\mathfrak{s}(A))$ follows from (3.1.2) and Lemma 3.1.1. □

Proposition 3.1.4 *For a given $A \in M_n(\mathcal{O}_1)$, there exists a character χ of $T(\psi_A)$ such that $\chi|_K = \psi_A$ (such a character χ is called an extension of ψ_A).*

Proof: It follows from (3.1.1) that orbits of the action of $\mathrm{GL}_n(\mathcal{O}_2)$ on K are the same as orbits of $M_n(\mathcal{O}_1)$ under the action of $\mathrm{GL}_n(\mathcal{O}_1)$, namely the similarity classes. It is easy to see that, if we can extend the character ψ_A from K to $T(\psi_A)$, then we can extend any $\psi_{A'}$ in the orbit of ψ_A under the action of $\mathrm{GL}_n(\mathcal{O}_2)$ on $T(\psi_A)$. So to prove the proposition, it is enough to choose A as a representative of similarity class of $\mathrm{M}_n(\mathcal{O}_1)$ and to extend the corresponding character ψ_A from K to $T(\psi_A)$. We prove existence of this extension in three steps:

Step 1: A is Split Primary

Let A be a split primary matrix with unique eigenvalue $a \in \mathcal{O}_1$. Replace A by a matrix in its similarity class of the form $\oplus_{i=1}^{l} J_{\lambda_i}(a)$, where each $J_{\lambda_i}(a)$ is an elementary Jordan block.

We define a character $\psi_a : \mathcal{O}_1 \to \mathbb{C}^*$ by $\psi_a(x) = \psi(ax)$. The map $x \mapsto 1 + \pi x$ gives an isomorphism from $\mathcal{O}_1$ onto the subgroup $1 + \pi\mathcal{O}_1$ of the multiplicative group $\mathcal{O}_2^*$. Choose $\chi \in \hat{\mathcal{O}}_2^*$ such that $\chi(1+\pi x) = \psi_a(x)$ for all $x \in \mathcal{O}_1$. Define a character $\tilde{\chi} : Z_{\mathrm{GL}_n(\mathcal{O}_2)}(\mathfrak{s}(A)) \to \mathbb{C}^*$ by $\tilde{\chi}(x) = \chi(\det(x))$.

Lemma 3.1.5 *The character $\tilde{\chi}$ of $Z_{GL_n(\mathcal{O}_2)}(\mathfrak{s}(A))$ satisfies*

$$\tilde{\chi}|_{K \cap Z_{GL_n(\mathcal{O}_2)}(\mathfrak{s}(A))} = \psi_A|_{K \cap Z_{GL_n(\mathcal{O}_2)}(\mathfrak{s}(A))}$$

Proof: By Lemma 2.3.5, $K \cap Z_{\mathrm{GL}_n(\mathcal{O}_2)}(\mathfrak{s}(A)) = I + \pi Z_{\mathrm{M}_n(\mathcal{O}_1)}(A)$. If $X = (x_{ij}) \in Z_{\mathrm{M}_n(\mathcal{O}_1)}(A)$, then by Lemma 2.3.4, X is a block upper Toeplitz matrix. Therefore $\mathrm{Tr}(AX) = a(x_{11} + x_{22} + \cdots + x_{nn})$. We have

$$\begin{aligned} \psi_A(I + \pi X) &= \psi(\mathrm{Tr}(AX)) \\ &= \psi(a(x_{11} + x_{22} + \cdots + x_{nn})) \\ &= \chi(\det(I + \pi X)) = \tilde{\chi}(I + \pi X) \end{aligned}$$

□

Applying Lemma 2.1.2 to the group $T(\psi_A)$ with its subgroups K and $Z_{\mathrm{GL}_n(\mathcal{O}_2)}(\mathfrak{s}(A))$, and characters $\psi_1 = \psi_A$ and $\psi_2 = \tilde{\chi}$ we obtain that the character $\psi_A.\tilde{\chi}$ is an extension of ψ_A from K to $T(\psi_A)$.

Step 2: A is split

Let A be a split matrix with distinct eigenvalues $a_1, a_2, \ldots, a_l$. Then by Theorem 2.2.4, A can be written as $\oplus_{i=1}^{l} A_i$, where each A_i is a split primary matrix, say of order n_i, and has a unique eigenvalue a_i. We may assume that each A_i is in its Jordan canonical form. Then by (3.1.3),

$$K \cap Z_{\mathrm{GL}_n(\mathcal{O}_2)}(\mathfrak{s}(A)) = \oplus_{i=1}^{l} (K \cap Z_{\mathrm{GL}_{n_i}(\mathcal{O}_2)}(\mathfrak{s}(A_i)))$$

As in the Step 1, define the characters $\tilde{\chi}_i$ of $Z_{\mathrm{GL}_{n_i}(\mathcal{O}_2)}(\mathfrak{s}(A_i))$ such that

$$\tilde{\chi}_i|_{K \cap Z_{\mathrm{GL}_{n_i}(\mathcal{O}_2)}(\mathfrak{s}(A_i))} = \psi_{A_i}|_{K \cap Z_{\mathrm{GL}_{n_i}(\mathcal{O}_2)}(\mathfrak{s}(A_i))}$$

Then $\tilde{\chi} = \tilde{\chi}_1 \times \tilde{\chi}_2 \times \ldots \tilde{\chi}_l$ is a character of $Z_{\mathrm{GL}_n(\mathcal{O}_2)}(\mathfrak{s}(A))$, such that

$$\tilde{\chi}|_{K \cap Z_{\mathrm{GL}_n(\mathcal{O}_2)}(\mathfrak{s}(A))} = \psi_A|_{K \cap Z_{\mathrm{GL}_n(\mathcal{O}_2)}(\mathfrak{s}(A))}$$

Again by Lemma 2.1.2, $\psi_A.\tilde{\chi}$ is an extension of ψ_A from K to $T(\psi_A)$.

Step 3 : General case

Let $\tilde{\mathcal{O}}_1$ be a splitting field for the characteristic polynomial of A and let $\tilde{\mathcal{O}}_2$ be the corresponding unramified extension of $\mathcal{O}_2$. Let $\tilde{K} = \mathrm{Ker}(\mathrm{GL_n}(\tilde{\mathcal{O}}_2) \to \mathrm{GL_n}(\tilde{\mathcal{O}}_1))$ under the natural quotient map, and $\tilde{\psi} : \tilde{\mathcal{O}}_1 \to \mathbb{C}^*$ be a character such that $\tilde{\psi}|_{\mathcal{O}_1} = \psi$. Then $\tilde{\psi}_A : \tilde{K} \to \mathbb{C}^*$, defined by

$$\tilde{\psi}_A(I + \pi X) = \tilde{\psi}(\mathrm{Tr}(AX))$$

is a character of $\tilde{K}$. Let $\tilde{T}(\psi_A)$ be the stabilizer of $\tilde{\psi}_A$ in $\mathrm{GL}_n(\tilde{\mathcal{O}}_2)$. Since A splits over $\tilde{\mathcal{O}}_1$, by Step 2, there exists a character $\tilde{\chi} : \tilde{T}(\psi_A) \to \mathbb{C}^*$, such that

$$\tilde{\chi}|_{\tilde{K}} = \tilde{\psi}_A.$$

Define a character $\chi : T(\psi_A) \to \mathbb{C}^*$ by $\chi = \tilde{\chi}|_{T(\psi_A)}$. Then χ is an extension of ψ_A to $T(\psi_A)$. This completes the proof of Proposition 3.1.4. □

Fix an extension χ_A of ψ_A from K to $T(\psi_A)$ and let $\mathcal{S}$ denote the set of similarity classes of $\mathrm{M}_n(\mathcal{O}_1)$. By (3.1.2), the groups $T(\psi_A)/K$ and $Z_{\mathrm{GL}_n(\mathcal{O}_1)}(A)$ are isomorphic. Therefore by Clifford Theory, there exists a bijection between the sets

$$\coprod_{A \in \mathcal{S}} \{\mathrm{Irr}(Z_{\mathrm{GL}_n(\mathcal{O}_1)}(A))\} \quad \longleftrightarrow \quad \mathrm{Irr}(\mathrm{GL_n}(\mathcal{O}_2)), \tag{3.1.4}$$

given by,

$$\phi \mapsto \mathrm{Ind}_{T(\psi_A)}^{\mathrm{GL}_n(\mathcal{O}_2)}(\chi_A \otimes \phi). \tag{3.1.5}$$

As $[\mathrm{GL}_n(\mathcal{O}_2) : T(\psi_A)] = [\mathrm{GL}_n(\mathcal{O}_1) : Z_{\mathrm{GL}_n(\mathcal{O}_1)}(A)]$, this already proves that there exists a bijection between the sets $\mathrm{Irr}(\mathrm{GL_n}(\mathcal{O}_2))$ and $\mathrm{Irr}(\mathrm{GL_n}(\mathcal{O}'_2))$ which preserves dimensions. To prove that this bijection is canonical we need to do little more work.

Let $\mathcal{O}'$ be the ring of integers of another non-Archimedean local field F', such that residue fields of both $\mathcal{O}$ and $\mathcal{O}'$ are isomorphic. We fix an isomorphism ϕ between their residue fields. From now onwards we shall assume that section $s : \mathcal{O}_1 \to \mathcal{O}_2$ satisfies $s(0) = 0$ and $s|_{\mathcal{O}_1^*}$ is multiplicative. The existence and uniqueness of this section is proved in, for example, Serre [Ser79, Prop. 8]. In the sequel, this unique section will be called *the multiplicative section* of $\mathcal{O}_1$ (or of $\mathcal{O}_1^*$) (depending on the domain). Given above isomorphism ϕ,

Lemma 3.1.6 *There exists a canonical isomorphism between groups* $\hat{\mathcal{O}}_2^*$ *and* $\hat{\mathcal{O}}_2'^*$.

Proof: Let $s : \mathcal{O}_1^* \to \mathcal{O}_2^*$ and $s' : \mathcal{O}_1'^* \to \mathcal{O}_2'^*$ be the multiplicative sections of $\mathcal{O}_1^*$ and $\mathcal{O}_1'^*$ respectively. Then the following exact sequences split,

$$0 \longrightarrow \mathcal{O}_1 \xrightarrow[i]{} \mathcal{O}_2^* \underset{s}{\rightleftarrows} \mathcal{O}_1^* \longrightarrow 1$$

$$0 \longrightarrow \mathcal{O}_1' \xrightarrow[i']{} \mathcal{O}_2'^* \overset{s'}{\rightleftarrows} \mathcal{O}_1'^* \longrightarrow 1$$

The uniqueness of the sections s and s' implies the existence of a unique isomorphism $f : \mathcal{O}_2^* \to \mathcal{O}_2'^*$ such that $f \circ s = s' \circ \phi$ and $f \circ i = i' \circ \phi$. This gives a canonical isomorphism between $\hat{\mathcal{O}}_2^*$ and $\hat{\mathcal{O}}_2'^*$. □

Let $K' = \mathrm{Ker}(\mathrm{GL_n}(\mathcal{O}_2') \to \mathrm{GL_n}(\mathcal{O}_1'))$. Then

$$K \cong K' \cong \mathrm{M}_n(\mathcal{O}_1)$$

hence the set $\{\psi_A \mid A \in \mathrm{M}_n(\mathcal{O}_1)\}$ can also be thought as the set of characters of K'. Let $T'(\psi_A)$ denote the stabilizer of character ψ_A in $\mathrm{GL}_n(\mathcal{O}_2')$. By (3.1.2), groups $T'(\psi_A)/K'$ and $T(\psi_A)/K$ are canonically isomorphic. Further to prove that there exists a canonical bijection between $\mathrm{Irr}(\mathrm{GL_n}(\mathcal{O}_2))$ and $\mathrm{Irr}(\mathrm{GL_n}(\mathcal{O}_2'))$, it is sufficient to prove that for a given $A \in \mathrm{M}_n(\mathcal{O}_1)$, and an extension $\chi_A : T(\psi_A) \to \mathbf{C}^*$ of ψ_A, there exists a canonical extension $\chi_A' : T'(\psi_A) \to \mathbf{C}^*$ of ψ_A from K' to $T'(\psi_A)$. For that, in steps 1 and 2 of the proof of Proposition 3.1.4, choose the character of $\mathcal{O}_2'^*$ by using the given character of $\mathcal{O}_2^*$ and canonical isomorphism between $\hat{\mathcal{O}}_2^*$ and $\hat{\mathcal{O}}_2'^*$; the rest of the argument follows easily for these steps. For step 3, any isomorphism between $\tilde{\mathcal{O}}_1$ and $\tilde{\mathcal{O}}_1'$ that extends $\phi : \mathcal{O}_1 \to \mathcal{O}_1'$ is defined uniquely up to an element of the Galois group. To complete the proof for this step, we observe that, if $\tilde{\psi}$ is an extension of ψ from $\mathcal{O}_1$ to $\tilde{\mathcal{O}}_1$ and γ is an element of Galois group $\mathrm{Gal}(\tilde{\mathcal{O}}_1/\mathcal{O}_1)$ then by definition $(\tilde{\psi} \circ \gamma)_A = \tilde{\psi}_A \circ \gamma$ where on the right side γ is thought as scalar matrix with all its diagonal entries equal to γ. Choose $\tilde{\chi}_A$ an extension of $\tilde{\psi}_A$ from $\tilde{K}$ to $\tilde{T}(\psi_A)$ and let $\tilde{\gamma}$ be a lift of γ from $\mathrm{Gal}(\tilde{\mathcal{O}}_1/\mathcal{O}_1)$ to $\mathrm{Gal}(\tilde{\mathrm{F}}/\mathrm{F})$, which takes the maximal ideal of $\tilde{\mathcal{O}}$ to itself (for existence of this see [MR067, p. 26]), then $\tilde{\chi} \circ \tilde{\gamma}$ (again $\tilde{\gamma}$ is thought as scalar matrix with diagonal entries equal to $\tilde{\gamma}$) extends $\tilde{\psi}_A \circ \gamma$. The restrictions of $\tilde{\chi}_A$ and $\tilde{\chi}_A \circ \tilde{\gamma}$ to $T(\psi_A)$ coincide. This completes the proof of Theorem 1.2.1.

Corollary 3.1.7 *The isomorphism type of group algebra $\mathbf{C}[GL_n(\mathcal{O}_2)]$ depends only on the cardinality of the residue field of $\mathcal{O}$.*

This can be restated as,

Corollary 3.1.8 *The number and dimensions of irreducible representations of groups $GL_n(\mathcal{O}_2)$ depends only on the residue field.*

The equivalence between number of conjugacy classes and irreducible representations further gives,

Corollary 3.1.9 *The number of conjugacy classes of groups $GL_n(\mathcal{O}_2)$ depend only on the cardinality of residue fields.*

Remark 2 *In Chapter 5, we will sharpen this result by showing that the class equation of $GL_n(\mathcal{O}_2)$ depends only on the cardinality of the residue field.*

3.2 Complexity of the problem

In this section we comment on the complexity of the problem of constructing all the irreducible representations of $\mathrm{GL}_n(\mathcal{O}_2)$.

In this context, Aubert-Onn-Prasad-Stasinski have proved the following ([AOPS, Theorem 6.1])

Theorem 3.2.1 *Let $F = \mathbf{F}_q((t))$ be a local function field. Then the problems of constructing all the irreducible representations of the following are equivalent:*

1. *$G_{2^n,F}$ for all $n \in \mathbf{N}$.*
2. *$G_{k^n,F}$ for all $k, n \in \mathbf{N}$.*
3. *$G_{\lambda,E}$ for all partitions λ and all unramified extensions E of F.*

Proof: Clearly (3) $\implies$ (2) $\implies$ (1), we prove that (1) $\implies$ (3). The group $G_{2^n,F}$ is semi-direct product of $\mathrm{GL}_n(\mathcal{O}_1)$ by $\mathrm{M}_n(\mathcal{O}_1)$. By Clifford theory the problem of constructing the irreducible representations of $G_{2^n,F}$ is equivalent to the problem of constructing irreducible representations of the centralizers in $\mathrm{GL}_n(\mathcal{O}_1)$ of characters of $M_n(\mathcal{O}_1)$. By (3.1.2) these are the same as the centralizers $Z_{\mathrm{GL}_n(\mathcal{O}_1)}(A)$, for $A \in M_n(\mathcal{O}_1)$. The result now follows from (2.3.1) and Theorem 2.3.6. □

The above, combined with Theorem 1.2.1, proves the following,

Theorem 3.2.2 *Let $\mathcal{O}$ be the ring of integers of a non-Archimedean local field F, such that the residue field has cardinality q. Then the problems of constructing irreducible representations of the following groups are equivalent:*

1. *$GL_n(\mathcal{O}_2)$ for all $n \in \mathbf{N}$.*
2. *$G_{\lambda,E}$ for all partitions λ and all unramified extensions E of $\mathbf{F}_q((t))$.*

Chapter 4

Applications

In this chapter, we discuss a few applications of the theory developed so far. In particular, we discuss the relation between the representation zeta function of $\mathrm{GL}_n(\mathcal{O}_2)$ and those of centralizers in $\mathrm{GL}_n(\mathcal{O}_1)$. We also construct all the irreducible representations of the groups $\mathrm{GL}_2(\mathcal{O}_2)$, $\mathrm{GL}_3(\mathcal{O}_2)$, $\mathrm{GL}_4(\mathcal{O}_2)$ and obtain their representation zeta functions.

Recall the following definition from Chapter 1,

Definition 14 *(Representation Zeta function) Let G be a finite group. The representation zeta function of G is the function*

$$R_G(\mathcal{D}) = \sum_{\rho \in \mathrm{Irr} G} \mathcal{D}^{\dim \rho} \in \mathbb{Z}[\mathcal{D}]$$

4.1 Representation zeta function of $\mathrm{GL}_n(\mathcal{O}_2)$

Let $\mathcal{S}$ be the set of similarity classes of $\mathrm{M}_n(\mathcal{O}_1)$. From (3.1.5) it is clear that representations of centralizers play an important role in determining irreducible representations of $\mathrm{GL}_n(\mathcal{O}_2)$. Moreover we obtain the following relation between their representation zeta functions.

$$R_{\mathrm{GL}_n(\mathcal{O}_2)}(\mathcal{D}) = \sum_{A \in \mathcal{S}} R_{Z_{\mathrm{GL}_n(\mathcal{O}_1)}(A)}(\mathcal{D}^{[\mathrm{GL}_n(\mathcal{O}_1):Z_{\mathrm{GL}_n(\mathcal{O}_1)}(A)]}) \tag{4.1.1}$$

where, $[\mathrm{GL}_n(\mathcal{O}_1) : Z_{\mathrm{GL}_n(\mathcal{O}_1)}(A)] = [\mathrm{GL}_n(\mathcal{O}_2) : T(\psi_A)]$ is the index of $Z_{\mathrm{GL}_n(\mathcal{O}_1)}(A)$ in $\mathrm{GL}_n(\mathcal{O}_1)$. Following Green [Gre55], a similarity class c of $\mathrm{M}_n(\mathcal{O}_1)$ can be denoted by the symbol

$$c = (\ldots, f^{\nu_c(f)}, \ldots)$$

where f is an irreducible polynomial appearing in the characteristic polynomial of c and $\nu_c(f)$ is the partition associated with f in the canonical form of c.

Let $c = (\ldots, f^{\nu_c(f)}, \ldots)$. Let d be a positive integer, and let ν be a partition other than zero. Let $r_c(d, \nu)$ be the number of f appearing in the characteristic polynomial of c with degree d and $\nu_c(f) = \nu$. Let $\rho_c(\nu)$ be the partition

$$\{\underbrace{n, n, \ldots, n}_{r_c(n,\nu)}, \underbrace{n-1, n-1, \ldots, n-1}_{r_c(n-1,\nu)}, \ldots\}$$

Then two classes b and c are of the same *type* if and only if $\rho_b(\nu) = \rho_c(\nu)$ for each non zero partition ν. By abusing notation we shall also say matrices of class c and d have same type.

Let ρ_ν be a partition-valued function on the nonzero partitions ν (ρ_ν may take value zero). The condition for ρ_ν to describe a type of $\mathrm{M}_n(\mathcal{O}_1)$ is

$$\sum_\nu |\rho_\nu||\nu| = n$$

The total number $t(n)$ of functions ρ_ν satisfying above expression is independent of q, and so is the number of types of $\mathrm{M}_n(\mathcal{O}_1)$ (for large enough q). The following lemma (which is easy) underlines the importance of types in the calculation of the representation zeta functions of the groups $\mathrm{GL}_n(\mathcal{O}_2)$:

Lemma 4.1.1 *If matrices A and B in $M_n(\mathcal{O}_1)$ are of same* type *then their centralizers are isomorphic.*

Let $\mathcal{T}$ denote the set of representatives of types of $\mathrm{M}_n(\mathcal{O}_1)$ and for each $A \in \mathcal{T}$, let n_A be the total number of similarity classes of type A. The expression (4.1.1) simplifies to

$$R_{\mathrm{GL}_n(\mathcal{O}_2)}(\mathcal{D}) = \sum_{A \in \mathcal{T}} n_A R_{Z_{\mathrm{GL}_n(\mathcal{O}_1)}(A)}(\mathcal{D}^{[\mathrm{GL}_n(\mathcal{O}_1):Z_{\mathrm{GL}_n(\mathcal{O}_1)}(A)]}) \qquad (4.1.2)$$

Summarising the discussion so far, to determine the irreducible representations of groups $\mathrm{GL}_n(\mathcal{O}_2)$, it is sufficient to determine the representations of the centralizers $Z_{\mathrm{GL}_n(\mathcal{O}_1)}(A)$ where A varies over the set of types of $\mathrm{M}_n(\mathcal{O}_1)$. But determining representations of groups $Z_{\mathrm{GL}_n(\mathcal{O}_1)}(A)$ for general n is still an open problem. We discuss representations of these groups for $n = 2$, $n = 3$, and $n = 4$.

Table 4.1: Group $\mathrm{GL}_2(\mathcal{O}_2)$

Type A	Number of similarity classes of given type (n_A)	Isomorphism type of centralizer $Z_{\mathrm{GL}_2(\mathcal{O}_1)}(A)$	*Index* $[\mathrm{GL}_2(\mathcal{O}_1) : Z_{\mathrm{GL}_2(\mathcal{O}_1)}(A)]$
$\begin{pmatrix} \rho^a & 0 \\ 0 & \rho^a \end{pmatrix}$	q	$\mathrm{GL}_2(\mathcal{O}_1)$	1
$\begin{pmatrix} \rho^a & 0 \\ 0 & \rho^b \end{pmatrix}$	$\frac{1}{2}q(q-1)$	$\mathcal{O}_1^* \times \mathcal{O}_1^*$	$q(q+1)$
$\begin{pmatrix} \rho^a & 1 \\ 0 & \rho^a \end{pmatrix}$	q	$\mathcal{O}_2^*$	q^2-1
$\begin{pmatrix} \sigma^a & 0 \\ 0 & \sigma^{aq} \end{pmatrix}$	$\frac{1}{2}q(q-1)$	$\mathbf{F}_{q^2}^*$	q^2-q

The element ρ and σ are primitive elements of $\mathbf{F}_q$ and $\mathbf{F}_{q^2}$ respectively, such that $\rho = \sigma^{q+1}$.

4.2 Representations of $\mathrm{GL}_2(\mathcal{O}_2)$

The irreducible representations of groups $\mathrm{GL}_2(\mathcal{O}_2)$ are already described by Nagornyi [Nag76] and Onn [Onn08]. Since it falls out of our discussion very easily and is used in representation theory of groups $\mathrm{GL}_4(\mathcal{O}_2)$, we add its brief description also. For representation theory of groups $\mathrm{GL}_n(\mathcal{O}_1)$, we refer Green [Gre55] and Steinberg [Ste51]. In Table 4.1 we describe types of $M_2(\mathcal{O}_1)$ (set of 2×2 matrices over $\mathcal{O}_1$) with their centralizers. To determine centralizers, wherever required, we have used Theorems 2.3.11 and 2.3.6.

Table 4.1 provides complete data required for the groups $\mathrm{GL}_2(\mathcal{O}_2)$. By 4.1.2, we obtain the following representation zeta function for $\mathrm{GL}_2(\mathcal{O}_2)$:

$$\begin{aligned} R_{\mathrm{GL}_2(\mathcal{O}_2)}(\mathcal{D}) &= qR_{\mathrm{GL}_2(\mathcal{O}_1)}(\mathcal{D}) + \frac{1}{2}q(q-1)^3\mathcal{D}^{q(q+1)} + q^2(q-1)\mathcal{D}^{q^2-1} \\ &\quad + \frac{1}{2}q(q+1)(q-1)^2\mathcal{D}^{q^2-q} \end{aligned} \tag{4.2.1}$$

where

$$\begin{aligned} R_{\mathrm{GL}_2(\mathcal{O}_1)}(\mathcal{D}) &= (q-1)\mathcal{D} + (q-1)\mathcal{D}^q + \frac{1}{2}(q-1)(q-2)\mathcal{D}^{q+1} \\ &\quad + \frac{1}{2}q(q-1)\mathcal{D}^{q-1} \end{aligned} \tag{4.2.2}$$

is the representation zeta function of the group $\mathrm{GL}_2(\mathcal{O}_1)$ (see Steinberg [Ste51]).

4.3 Representations of $\mathrm{GL}_3(\mathcal{O}_2)$

In this section we describe representations of groups $\mathrm{GL}_3(\mathcal{O}_2)$. Partial results in this direction are already given by Nagornyi [Nag78]. In Table 4.2 we describe the types in $\mathrm{M}_3(\mathcal{O}_1)$ and their centralizers.

The irreducible representations of all the centralizers appearing in Table 4.2 except $G_{2,1}$ are either very easy or well known. Onn [Onn08, Theorem 4.1] has described all the irreducible representations of groups $G_{(\ell,1)}$ for $\ell > 1$. As a consequence, we have

Lemma 4.3.1 *The representation zeta function of the group $G_{(2,1)}$ is*

$$R_{G_{(2,1)}}(\mathcal{D}) = (q-1)^2\mathcal{D} + (q^2-1)\mathcal{D}^{q-1} + (q-1)^3\mathcal{D}.^q$$

Collecting all the pieces together, we obtain the expression for representation zeta function of $\mathrm{GL}_3(\mathcal{O}_2)$:

$$\begin{aligned}
\mathcal{R}_{\mathrm{GL}_3(\mathcal{O}_2)}(\mathcal{D}) \quad = \quad & q\mathcal{R}_{\mathrm{GL}_3(\mathcal{O}_1)}(\mathcal{D}) + q(q-1)^2\mathcal{R}_{\mathrm{GL}_2(\mathcal{O}_1)}(\mathcal{D}^{q^2(q^2+q+1)}) \\
& + \frac{1}{6}q(q-2)(q-1)^4\mathcal{D}^{q^3(q+1)(q^2+q+1)} + \\
& q^2(q-1)^3\mathcal{D}^{q^2(q^3-1)(q+1)} + q\mathcal{R}_{\mathrm{G}_{(2,1)}}(\mathcal{D}^{(q^3-1)(q+1)}) \\
& + q^3(q-1)\mathcal{D}^{q(q^3-1)(q^2-1)} \\
& + \frac{1}{2}q^2(q-1)^2(q^2-1)\mathcal{D}^{q^3(q^3-1)} \\
& + \frac{1}{3}q(q^2-1)(q^3-1)\mathcal{D}^{q^3(q-1)^2(q+1)} \qquad (4.3.1)
\end{aligned}$$

where,

$$\begin{aligned}
\mathcal{R}_{\mathrm{GL}_3(\mathcal{O}_1)}(\mathcal{D}) \quad = \quad & (q-1)\mathcal{D} + (q-1)\mathcal{D}^{q^2+q} + (q-1)\mathcal{D}^{q^3} \\
& (q-1)(q-2)\mathcal{D}^{q^2+q+1} + (q-1)(q-2)\mathcal{D}^{q(q^2+q+1)} \\
& + \frac{1}{6}(q-1)(q-2)(q-3)\mathcal{D}^{(q+1)(q^2+q+1)} \\
& + \frac{1}{2}q(q-1)^2\mathcal{D}^{(q-1)(q^2+q+1)} \\
& + \frac{1}{3}q(q-1)(q+1)\mathcal{D}^{(q+1)(q-1)^2} \qquad (4.3.2)
\end{aligned}$$

as obtained by Steinberg [Ste51].

Table 4.2: Group $\mathrm{GL}_3(\mathcal{O}_2)$

Type A	Number of similarity classes of given type (n_A)	Isomorphism type of centralizer $Z_{\mathrm{GL}_3(\mathcal{O}_1)}(A)$	*Index* $[\mathrm{GL}_3(\mathcal{O}_1) : Z_{\mathrm{GL}_3(\mathcal{O}_1)}(A)]$
$\begin{pmatrix} \rho^a & 0 & 0 \\ 0 & \rho^a & 0 \\ 0 & 0 & \rho^a \end{pmatrix}$	q	$\mathrm{GL}_3(\mathcal{O}_1)$	1
$\begin{pmatrix} \rho^a & 0 & 0 \\ 0 & \rho^a & 0 \\ 0 & 0 & \rho^b \end{pmatrix}$	$q(q-1)$	$\mathcal{O}_1^* \times \mathrm{GL}_2(\mathcal{O}_1)$	$q^2(q^2+q+1)$
$\begin{pmatrix} \rho^a & 0 & 0 \\ 0 & \rho^b & 0 \\ 0 & 0 & \rho^c \end{pmatrix}$	$\frac{1}{6}q(q-1)(q-2)$	$\mathcal{O}_1^* \times \mathcal{O}_1^* \times \mathcal{O}_1^*$	$q^3(q+1)(q^2+q+1)$
$\begin{pmatrix} \rho^a & 1 & 0 \\ 0 & \rho^a & 0 \\ 0 & 0 & \rho^b \end{pmatrix}$	$q(q-1)$	$\mathcal{O}_1^* \times \mathcal{O}_2^*$	$q^2(q^3-1)(q+1)$
$\begin{pmatrix} \rho^a & 1 & 0 \\ 0 & \rho^a & 0 \\ 0 & 0 & \rho^a \end{pmatrix}$	q	$G_{(2,1)}$	$(q^3-1)(q+1)$
$\begin{pmatrix} \rho^a & 1 & 0 \\ 0 & \rho^a & 1 \\ 0 & 0 & \rho^a \end{pmatrix}$	q	$\mathcal{O}_3^*$	$q(q^3-1)(q^2-1)$
$\begin{pmatrix} \rho^a & 0 & o \\ 0 & \sigma^a & 0 \\ 0 & 0 & \sigma^{aq} \end{pmatrix}$	$\frac{1}{2}q^2(q-1)$	$\mathcal{O}_1^* \times \mathbf{F}_{q^2}^*$	$q^3(q^3-1)$
$\begin{pmatrix} \tau & 0 & 0 \\ 0 & \tau^{bq} & 0 \\ 0 & 0 & \tau^{bq^2} \end{pmatrix}$	$\frac{1}{3}q(q^2-1)$	$\mathbf{F}_{q^3}^*$	$q^3(q-1)^2(q+1)$

The elements ρ, σ and τ are primitive elements of $\mathbf{F}_q$, $\mathbf{F}_{q^2}$ and $\mathbf{F}_{q^3}$ respectively, such that $\rho = \sigma^{q+1} = \tau^{q^2+q+1}$.

4.4 Representations of $\mathrm{GL}_4(\mathcal{O}_2)$

In this section we discuss representation theory of groups $\mathrm{GL}_4(\mathcal{O}_2)$. Table 4.3 describes the data required for it.

In Table 4.3, we give all the data required for the representations of $\mathrm{GL}_4(\mathcal{O}_2)$. Among the centralizers appearing in this Table only the results regarding the representations of the group $G_{(2,1,1)}$ are not clear from our discussion so far. We follow a method of Uri Onn to compute the representation zeta function of this group.

The following proposition (belonging to the theory of finite Heisenberg groups) is a part of Proposition 8.3.3 of Bushnell-Fröhlich [BF83].

Proposition 4.4.1 *Let N be a normal subgroup of G, with $V = G/N$ an elementary finite abelian p-group so also viewed a finite dimensional vector space over $\mathbf{F}_p$. Let $\chi : N \mapsto \mathbb{C}^*$ be a nontrivial character such that G stabilizes χ. Assume furthermore that $h_\chi(g_1N, g_2N) = \langle g_1N, g_2N\rangle_\chi = \chi([g_1, g_2])$ is an alternating nondegenerate bilinear form on V. Then there exists a unique irreducible representation ρ_χ of G such that $\rho_\chi|_N$ is χ-isotypic. Moreover,* $\dim(\rho_\chi)^2 = [G : N]$.

Proof: Let V_1 be a maximal isotropic subspace of V under h_χ and let G_1 be its inverse image in G. Then G_1 is a maximal abelian group in G. Choose a character χ_1 of G_1 such that $\chi_1|_N = \chi$. Define

$$\rho_\chi = \mathrm{Ind}_{G_1}^{G}(\chi_1).$$

Then $\dim(\rho_\chi) = [G : G_1] = [G : N]^{\frac{1}{2}}$. Let $g \in G_1, g \notin N$, Then, using the formula for the induced representation,

$$\mathrm{Tr}(\rho_\chi(g)) = \sum_{x \in G/G_1} \chi_1(x^{-1}gx) = \chi_1(g) \sum_{x \in G/G_1} h_\chi(x, g)$$

Now, $x \mapsto h_\chi(x, g)$ is a non-trivial character of G/G_1, and the sum is therefore zero; $\mathrm{Tr}(\rho_\chi(g)) = 0$ for $g \in G_1, g \notin N$. Since V_1 is maximal isotropic space, so same result holds for $g \in G, g \notin G_1$. On the other hand, if $n \in N$, then $\mathrm{Tr}(\rho_\chi(n)) = \chi(n).\deg(\rho_\chi)$. Thus $\rho_\chi|_N$ is a multiple of χ. Moreover,

$$\sum_{g \in G} \mathrm{Tr}(\rho_\chi(g))\mathrm{Tr}(\rho_\chi(g^{-1})) = |G|$$

Thus ρ_χ is irreducible. Above calculation of character ρ_χ implies that it does not depend on the choice of G_1 and χ_1 so $\mathrm{Ind}_N^G(\chi) = [G : N]^{1/2}\rho_\chi$. Further

Frobenius reciprocity implies that ρ_χ is the only irreducible representation of G whose restriction to N contains χ.

□

Lemma 4.4.2 *The representation zeta function of the group $G_{(2,1,1)}$ is*

$$\mathcal{R}_{G_{(2,1,1)}}(\mathcal{D}) = (q-1)^2\mathcal{R}_{GL_2(\mathcal{O}_1)}(\mathcal{D}^{q^2}) + (q-1)\mathcal{R}_{(\mathcal{O}_1^2\times\mathcal{O}_1^2)\rtimes \mathrm{G}_{(1,1)}}(\mathcal{D}),$$

where

$$\begin{aligned}\mathcal{R}_{(\mathcal{O}_1^2\times\mathcal{O}_1^2)\rtimes \mathrm{G}_{(1,1)}}(\mathcal{D}) &= \mathcal{R}_{(1,1)}(\mathcal{D}) + 2(q-1)\mathcal{D}^{q^2-1} + (q-1)^2\mathcal{D}^(q^2-1)q \\ &\quad +(q+2)\mathcal{D}^{(q^2-1)(q-1)}. \qquad (4.4.1)\end{aligned}$$

Proof: Using the notation in the proof of Lemma 2.3.5, let H be the kernel of map $\mathrm{G}_{(2,1,1)} \to \mathrm{G}_1 \times \mathrm{G}_{(1,1)}$,

$$g \mapsto \left(g_{11}(\mathrm{mod}\wp), \begin{pmatrix} g_{22} & g_{23} \\ g_{32} & g_{33}\end{pmatrix}\right).$$

Then

$$H = I + \begin{bmatrix} \wp & \wp & \wp \\ \mathcal{O}_1 & & \\ \mathcal{O}_1 & & \end{bmatrix}$$

The centre of H, i.e. $Z(H) \cong \mathcal{O}_1$. Firstly we claim that H has $q-1$ irreducible representations of dimension q^2 which lie above the non-trivial characters of $Z(H)$. We identify $Z(H)$ with its dual by $z \mapsto \psi_z(.) = \psi(\mathrm{Tr}(z.))$. H stabilizes these characters of $Z(H)$ and furthermore, each of the non-trivial characters gives rise to an alternating non-degenerate bilinear form $\langle h_1Z(H), h_2Z(H)\rangle_{\psi_z} = \psi(\mathrm{Tr}(z[h_1,h_2]))$ on $H/Z(H)$. Proposition 4.4.1 gives $q-1$ pairwise inequivalent irreducible representations of dimension $|H/Z(H)|^{1/2} = q^2$. This proves the claim. Furthermore, the group $\mathrm{G}_{2,1,1}$ stabilizes each of these representations of H. Let $\rho_\chi \in \widehat{H}$ be such a representation lying over a non-trivial character $\chi \in \widehat{Z(H)}$. We claim that the representation ρ_χ can be extended to $\mathrm{G}_{(2,1,1)}$. Let $H^i = Z(H)\times\mathcal{O}_1\times\mathcal{O}_1$ be the pre-image in H of the maximal isotropic subgroup $\mathcal{O}_1\times\mathcal{O}_1$ for the above bilinear form. Let χ^i be any extension of χ to H^i. Indeed the subgroup $G_2\times G_{(1,1)}$ stabilizes both H^i and χ^i. Let $\tilde{\chi}$ be an extension of χ to $G_2\times G_{(1,1)}$. Then by Lemma 2.1.2, $\chi^i.\tilde{\chi}(a.b) = \chi^i(a)\tilde{\chi}(b)$ for all $a\in H^i$ and $b \in G_2\times G_{(1,1)}$ is a well defined linear character of $H^i.(G_2\times G_{(1,1)})$. By the proof of Proposition 4.4.1, ρ_χ does not depend on the choice of isotropy group and the extension χ^i. Therefore for the induced representation

$$\rho_{\chi^i} = \mathrm{ind}_{H^i(G_2 \times G_{(1,1)})}^{H.G_2 \times G_{(1,1)}}(\chi^i),$$

$\rho_\chi \leq \rho_{\chi^i}$, and as $\dim\rho_\chi = \dim\rho_{\chi^i} = q^2$ we conclude that ρ_{χ^i} is an extension of ρ_χ to $\mathrm{G}_{(2,1,1)}$. By the Clifford theory, it follows that all the representations of $\mathrm{G}_{(2,1,1)}$ which lie above ρ_χ are of the form $\{\rho_{\chi^i}.\phi \,|\, \phi \in \mathrm{G}_{(2,1,1)}/H\}$. Hence the contribution to representation zeta function of $\mathrm{G}_{(2,1,1)}$ from these representations is $(q-1)\mathcal{R}_{\mathcal{O}_1^* \times \mathrm{GL}_2(\mathcal{O}_1)}(\mathcal{D}^{q^2})$. The remaining representations correspond to representations of H whose central character is trivial, that is, representations pulled back from $((\mathcal{O}_1^2 \times \mathcal{O}_1^2) \rtimes G_{(1,1)}) \times \mathcal{O}_1^*$. The action of $G_{(1,1)}$ on $\mathcal{O}_1^2 \times \mathcal{O}_1^2$ is given by

$$\begin{pmatrix} 1 & \\ & D \end{pmatrix} \begin{pmatrix} 1 & \pi v \\ w & I \end{pmatrix} \begin{pmatrix} 1 & \\ & D^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \pi v D^{-1} \\ Dw & I \end{pmatrix},$$

After a choice of identification of $\mathcal{O}_1^2 \times \mathcal{O}_1^2$ with its dual: $\langle (\hat{v}, \hat{w}), (v, w) \rangle = \psi(v\hat{v} + w\hat{w})$, we get

$$g^{-1}(\hat{v}, \hat{w}) = (D^{-1}\hat{v}, \hat{w}D), \text{ where } g = \begin{pmatrix} 1 & \\ & D \end{pmatrix},$$

and the orbits and stabilizers of this action are given by are given by

	Orbits	Stabilizers
(1)	$\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \end{pmatrix}\right]$	$\mathrm{G}_{(1,1)}$
(2)	$\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \mathcal{O}_1^2 \setminus 0 & 0 \end{pmatrix}\right]$	$\mathcal{O}_1 \rtimes \mathcal{O}_1^*$
(3)	$\left[\mathcal{O}_1^2 \setminus \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \end{pmatrix}\right]$	$\mathcal{O}_1 \rtimes \mathcal{O}_1^*$
(4)	$\left[\mathcal{O}_1^2 \setminus \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathcal{O}_1^* \end{pmatrix}\right]$	$\mathcal{O}_1$
(5)	$\left[\mathcal{O}_1^2 \setminus \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} u^* & \mathcal{O}_1 \end{pmatrix}\right], u^* \in \mathcal{O}_1^*$	$\mathcal{O}_1^*$

Collecting all the pieces we get the desired result. □

We obtain the following representation zeta function of $\mathrm{GL}_4(\mathcal{O}_2)$:

$$\begin{aligned}
\mathcal{R}_{\mathrm{GL}_4(\mathcal{O}_2)}(\mathcal{D}) \;=\; & q\mathcal{R}_{\mathrm{GL}_4(\mathcal{O}_1)}(\mathcal{D}) \\
& +q\mathcal{R}_{\mathrm{G}_{(2,1,1)}}(\mathcal{D}^{(q^2+1)(q^3-1)(q+1)}) \\
& +q\mathcal{R}_{\mathrm{G}_{(2,2)}}(\mathcal{D}^{q(q^4-1)(q^3-1)}) + q\mathcal{R}_{\mathrm{G}_{(3,1)}}(\mathcal{D}^{q^2(q^4-1)(q^3-1)(q+1)}) \\
& +q^4(q-1)\mathcal{D}^{q^3(q^4-1)(q^3-1)(q^2-1)} + q^3(q-1)^3\mathcal{D}^{q^4(q^4-1)(q^3-1)(q+1)} \\
& +q(q-1)^2\mathcal{R}_{G_{(2,1)}}(\mathcal{D}^{q^3(q^2+1)(q+1)^2(q^3-1)}) \\
& +q(q-1)^2\mathcal{R}_{\mathrm{GL}_3(\mathcal{O}_1)}(\mathcal{D}^{q^3(q+1)(q^2+1)}) \\
& +\frac{1}{2}q(q-1)\mathcal{R}_{\mathrm{GL}_2(\mathcal{O}_1)\times\mathrm{GL}_2(\mathcal{O}_1)}(\mathcal{D}^{q^4(q^2+1)(q^2+q+1)}) \\
& +q^2(q-1)^2\mathcal{R}_{\mathrm{GL}_2(\mathcal{O}_1)}(\mathcal{D}^{q^4(q^2+q+1)(q^4-1)}) \\
& +\frac{1}{2}q^3(q-1)^3\mathcal{D}^{q^4(q+1)(q^4-1)(q^3-1)} \\
& +\frac{1}{2}q(q-1)^3(q-2)\mathcal{R}_{\mathrm{GL}_2(\mathcal{O}_1)}(\mathcal{D}^{q^5(q+1)(q^2+1)(q^2+q+1)}) \\
& +\frac{1}{2}q^2(q-1)^4(q-2)\mathcal{D}^{q^5(q^2+1)(q+1)^2(q^3-1)} \\
& +\frac{1}{24}q(q-1)^5(q-2)(q-3)\mathcal{D}^{q^6(q^3+q^2+q+1)(q+1)(q^2+q+1)} \\
& +\frac{1}{2}q^2(q-1)(q^2-1)\mathcal{R}_{\mathrm{GL}_2(\mathcal{O}_1)}(\mathcal{D}^{q^5(q^2+1)(q^3-1)}) \\
& +\frac{1}{2}q^3(q-1)^2(q^2-1)\mathcal{D}^{q^5(q^3-1)(q^4-1)} \\
& +\frac{1}{4}q^2(q-1)^4(q^2-1)\mathcal{D}^{q^6(q+1)(q^2+1)(q^3-1)} \\
& +\frac{1}{2}q(q-1)\mathcal{R}_{\mathrm{GL}_2(\mathbf{F}_{q^2})}(\mathcal{D}^{q^4(q-1)(q^3-1)}) \\
& +\frac{1}{2}q^3(q-1)^2(q+1)\mathcal{D}^{q^4(q^4-1)(q^3-1)(q-1)} \\
& +\frac{1}{8}(q^2-q)(q^2-q-2)(q^2-1)^2\mathcal{D}^{q^6(q^2+1)(q^3-1)(q-1)} \\
& +\frac{1}{3}q^2(q-1)(q^2-1)(q^3-1)\mathcal{D}^{q^6(q^4-1)(q^2-1)} \\
& +\frac{1}{4}q^2(q^2-1)(q^4-1)\mathcal{D}^{q^6(q-1)(q^2-1)(q^3-1)} \qquad (4.4.2)
\end{aligned}$$

Where,

$$
\begin{aligned}
\mathcal{R}_{\mathrm{GL}_4(\mathcal{O}_1)}(\mathcal{D}) \quad = \quad & (q-1)\mathcal{D} + (q-1)\mathcal{D}^{q(q^2+q+1)} + (q-1)\mathcal{D}^{q^2(q^2+1)} \\
& +(q-1)\mathcal{D}^{q^3(q^2+q+1)} + (q-1)\mathcal{D}^{q^6} \\
& +(q-1)(q-2)\mathcal{D}^{(q+1)(q^2+1)} + (q-1)(q-2)\mathcal{D}^{q(q+1)^2(q^2+1)} \\
& +(q-1)(q-2)\mathcal{D}^{q^3(q+1)(q^2+1)} \\
& +\frac{1}{2}(q-1)(q-2)(q-3)\mathcal{D}^{(q+1)(q^2+1)(q^2+q+1)} \\
& +\frac{1}{2}(q-1)(q-2)(q-3)\mathcal{D}^{q(q+1)(q^2+1)(q^2+q+1)} \\
& +\frac{1}{24}(q-1)(q-2)(q-3)(q-4)\mathcal{D}^{(q+1)^2(q^2+1)(q^2+q+1)} \\
& +\frac{1}{4}q(q-1)^2(q-2)\mathcal{D}^{(q-1)(q+1)(q^2+1)(q^2+q+1)} \\
& +\frac{1}{3}q(q-1)^2(q+1)\mathcal{D}^{(q-1)^2(q+1)^2(q^2+1)} \\
& +\frac{1}{2}(q-1)(q-2)\mathcal{D}^{(q^2+1)(q^2+q+1)} \\
& +\frac{1}{2}(q-1)(q-2)\mathcal{D}^{q^2(q^2+1)(q^2+q+1)} \\
& +(q-1)(q-2)\mathcal{D}^{q(q^2+1)(q^2+q+1)} \\
& +\frac{1}{2}q(q-1)^2\mathcal{D}^{(q-1)(q^2+1)(q^2+q+1)} \\
& +\frac{1}{2}q(q-1)^2\mathcal{D}^{q(q-1)(q^2+1)(q^2+q+1)} \\
& +\frac{1}{8}q(q-1)(q+1)(q-2)\mathcal{D}^{(q-1)^2(q^2+1)(q^2+q+1)} \\
& \frac{1}{4}q^2(q-1)(q+1)\mathcal{D}^{(q+1)(q-1)^3(q^2+q+1)} \\
& +\frac{1}{2}q(q-1)\mathcal{D}^{q^2(q-1)^2(q^2+q+1)} \\
& +\frac{1}{2}q(q-1)\mathcal{D}^{(q-1)^2(q^2+q+1)} \qquad (4.4.3)
\end{aligned}
$$

(See Steinberg [Ste51]) and $\mathcal{R}_{G_{(2,1)}}(\mathcal{D})$, $\mathcal{R}_{G_{(3,1)}}(\mathcal{D})$, $\mathcal{R}_{G_{(2,1,1)}}$ are described in Lemmas 4.3.1 and 4.4.2.

This completes our discussion regarding representations of groups $\mathrm{GL}_2(\mathcal{O}_2)$, $\mathrm{GL}_3(\mathcal{O}_2)$ and $\mathrm{GL}_4(\mathcal{O}_2)$.

Remark 3 *The representation zeta function $R_G(\mathcal{D})$ for $\mathcal{D} = 1$ gives the number of conjugacy classes of G. From above, we can easily obtain the* *number of conjugacy classes of groups $GL_2(\mathcal{O}_2)$, $GL_3(\mathcal{O}_2)$ and $GL_4(\mathcal{O}_2)$. Number of conjugacy classes of $GL_2(\mathcal{O}_2)$ and $GL_3(\mathcal{O}_2)$ is already known, see Avni-Onn-Prasad-Vaserstein [AOPV09].*

Table 4.3: Group $\mathrm{GL}_4(\mathcal{O}_2)$

Type A	Number of similarity classes of given type (n_A)	Isomorphism type of centralizer $Z_{\mathrm{GL}_4(\mathcal{O}_1)}(A)$	*Index* $[\mathrm{GL}_4(\mathcal{O}_1) : Z_{\mathrm{GL}_4(\mathcal{O}_1)}(A)]$
$\begin{pmatrix} \rho^a & 0 & 0 & 0 \\ 0 & \rho^a & 0 & 0 \\ 0 & 0 & \rho^a & 0 \\ 0 & 0 & 0 & \rho^a \end{pmatrix}$	q	$\mathrm{GL}_4(\mathcal{O}_1)$	1
$\begin{pmatrix} \rho^a & 0 & 0 & 0 \\ 1 & \rho^a & 0 & 0 \\ 0 & 0 & \rho^a & 0 \\ 0 & 0 & 0 & \rho^a \end{pmatrix}$	q	$G_{(2,1,1)}$	$(q^2+1)(q^3-1)(q+1)$
$\begin{pmatrix} \rho^a & 0 & 0 & 0 \\ 1 & \rho^a & 0 & 0 \\ 0 & 0 & \rho^a & 0 \\ 0 & 0 & 1 & \rho^a \end{pmatrix}$	q	$\mathrm{G}_{(2,2)}$	$q(q^4-1)(q^3-1)$
$\begin{pmatrix} \rho^a & 0 & 0 & 0 \\ 1 & \rho^a & 0 & 0 \\ 0 & 1 & \rho^a & 0 \\ 0 & 0 & 0 & \rho^a \end{pmatrix}$	q	$G_{(3,1)}$	$q^2(q^4-1)(q^3-1)(q+1)$
$\begin{pmatrix} \rho^a & 0 & 0 & 0 \\ 1 & \rho^a & 0 & 0 \\ 0 & 1 & \rho^a & 0 \\ 0 & 0 & 1 & \rho^a \end{pmatrix}$	q	$\mathcal{O}_4^*$	$q^3(q^4-1)(q^3-1)(q^2-1)$
$\begin{pmatrix} \rho^a & 0 & 0 & 0 \\ 0 & \rho^a & 0 & 0 \\ 0 & 0 & \rho^a & 0 \\ 0 & 0 & 0 & \rho^b \end{pmatrix}$	$q(q-1)$	$\mathrm{GL}_3(\mathcal{O}_1) \times \mathcal{O}_1^*$	$q^3(q+1)(q^2+1)$
$\begin{pmatrix} \rho^a & 0 & 0 & 0 \\ 1 & \rho^a & 0 & 0 \\ 0 & 0 & \rho^a & 0 \\ 0 & 0 & 0 & \rho^b \end{pmatrix}$	$q(q-1)$	$G_{(2,1)} \times \mathcal{O}_1^*$	$q^3(q^2+1)(q+1)^2(q^3-1)$
$\begin{pmatrix} \rho^a & 0 & 0 & 0 \\ 1 & \rho^a & 0 & 0 \\ 0 & 1 & \rho^a & 0 \\ 0 & 0 & 0 & \rho^b \end{pmatrix}$	$q(q-1)$	$\mathcal{O}_3^* \times \mathcal{O}_1^*$	$q^4(q^4-1)(q^3-1)(q+1)$
$\begin{pmatrix} \rho^a & 0 & 0 & 0 \\ 0 & \rho^a & 0 & 0 \\ 0 & 0 & \rho^b & 0 \\ 0 & 0 & 0 & \rho^b \end{pmatrix}$	$\frac{1}{2}q(q-1)$	$\mathrm{GL}_2(\mathcal{O}_1) \times \mathrm{GL}_2(\mathcal{O}_1)$	$q^4(q^2+1)(q^2+q+1)$

Type A	Number of similarity classes of given type (n_A)	Isomorphism type of centralizer $Z_{\mathrm{GL}_4(\mathcal{O}_1)}(A)$	*Index* $[\mathrm{GL}_4(\mathcal{O}_1) : Z_{\mathrm{GL}_4(\mathcal{O}_1)}(A)]$
$\begin{pmatrix} \rho^a & 0 & 0 & 0 \\ 1 & \rho^a & 0 & 0 \\ 0 & 0 & \rho^b & 0 \\ 0 & 0 & 0 & \rho^b \end{pmatrix}$	$q(q-1)$	$\mathcal{O}_2^* \times \mathrm{GL}_2(\mathcal{O}_1)$	$q^4(q^2+q+1)(q^4-1)$
$\begin{pmatrix} \rho^a & 0 & 0 & 0 \\ 1 & \rho^a & 0 & 0 \\ 0 & 0 & \rho^b & 0 \\ 0 & 0 & 1 & \rho^b \end{pmatrix}$	$\frac{1}{2}q(q-1)$	$\mathcal{O}_2^* \times \mathcal{O}_2^*$	$q^4(q+1)(q^4-1)(q^3-1)$
$\begin{pmatrix} \rho^a & 0 & 0 & 0 \\ 0 & \rho^a & 0 & 0 \\ 0 & 0 & \rho^b & 0 \\ 0 & 0 & 0 & \rho^c \end{pmatrix}$	$\frac{1}{2}q(q-1)(q-2)$	$\mathrm{GL}_2(\mathcal{O}_1) \times \mathcal{O}_1^* \times \mathcal{O}_1^*$	$q^5(q+1)(q^2+1)(q^2+q+1)$
$\begin{pmatrix} \rho^a & 0 & 0 & 0 \\ 1 & \rho^a & 0 & 0 \\ 0 & 0 & \rho^b & 0 \\ 0 & 0 & 0 & \rho^c \end{pmatrix}$	$\frac{1}{2}q(q-1)(q-2)$	$\mathcal{O}_2^* \times \mathcal{O}_1^* \times \mathcal{O}_1^*$	$q^5(q^2+1)(q+1)^2(q^3-1)$
$\begin{pmatrix} \rho^a & 0 & 0 & 0 \\ 0 & \rho^b & 0 & 0 \\ 0 & 0 & \rho^c & 0 \\ 0 & 0 & 0 & \rho^d \end{pmatrix}$	$\frac{1}{24}q(q-1)(q-2)(q-3)$	$\mathcal{O}_1^* \times \mathcal{O}_1^* \times \mathcal{O}_1^* \times \mathcal{O}_1^*$	$q^6(q^3+q^2+q+1)(q+1)(q^2+q+1)$
$\begin{pmatrix} \rho^a & 0 & 0 & 0 \\ 0 & \rho^a & 0 & 0 \\ 0 & 0 & \sigma^b & 0 \\ 0 & 0 & 0 & \sigma^{bq} \end{pmatrix}$	$\frac{1}{2}q^2(q-1)$	$\mathrm{GL}_2(\mathcal{O}_1) \times \mathbf{F}_{q^2}^*$	$q^5(q^2+1)(q^3-1)$
$\begin{pmatrix} \rho^a & 0 & 0 & 0 \\ 1 & \rho^a & 0 & 0 \\ 0 & 0 & \sigma^b & 0 \\ 0 & 0 & 0 & \sigma^{bq} \end{pmatrix}$	$\frac{1}{2}q^2(q-1)$	$\mathcal{O}_2^* \times \mathbf{F}_{q^2}^*$	$q^5(q^3-1)(q^4-1)$
$\begin{pmatrix} \rho^a & 0 & 0 & 0 \\ 0 & \rho^b & 0 & 0 \\ 0 & 0 & \sigma^c & 0 \\ 0 & 0 & 0 & \sigma^{cq} \end{pmatrix}$	$\frac{1}{4}q^2(q-1)^2$	$\mathcal{O}_1^* \times \mathcal{O}_1^* \times \mathbf{F}_{q^2}^*$	$q^6(q+1)(q^2+1)(q^3-1)$
$\begin{pmatrix} \sigma^a & 0 & 0 & 0 \\ 0 & \sigma^{aq} & 0 & 0 \\ 0 & 0 & \sigma^a & 0 \\ 0 & 0 & 0 & \sigma^{aq} \end{pmatrix}$	$\frac{1}{2}q(q-1)$	$\mathrm{GL}_2(\mathbf{F}_{q^2})$	$q^4(q-1)(q^3-1)$
$\begin{pmatrix} \sigma^a & 0 & 0 & 0 \\ 0 & \sigma^{aq} & 0 & 0 \\ 1 & 0 & \sigma^a & 0 \\ 0 & 1 & 0 & \sigma^{aq} \end{pmatrix}$	$\frac{1}{2}q(q-1)$	$\mathbf{F}_{q^2} \times \mathbf{F}_{q^2}^*$	$q^4(q^4-1)(q^3-1)(q-1)$
$\begin{pmatrix} \sigma^a & 0 & 0 & 0 \\ 0 & \sigma^{aq} & 0 & 0 \\ 0 & 0 & \sigma^b & 0 \\ 0 & 0 & 0 & \sigma^{bq} \end{pmatrix}$	$\frac{1}{8}q(q-1)(q^2-q-2)$	$\mathbf{F}_{q^2}^* \times \mathbf{F}_{q^2}^*$	$q^6(q^2+1)(q^3-1)(q-1)$

Type A	Number of similarity classes of given type (n_A)	Isomorphism type of centralizer $Z_{\mathrm{GL}_4(\mathcal{O}_1)}(A)$	*Index* $[\mathrm{GL}_4(\mathcal{O}_1) : Z_{\mathrm{GL}_4(\mathcal{O}_1)}(A)]$
$\begin{pmatrix} \rho^a & 0 & 0 & 0 \\ 0 & \tau^b & 0 & 0 \\ 0 & 0 & \tau^{bq} & 0 \\ 0 & 0 & 0 & \tau^{bq^2} \end{pmatrix}$	$\frac{1}{3}q^2(q^2-1)$	$\mathcal{O}_1^* \times \mathbf{F}_{q^3}^*$	$q^6(q^4-1)(q^2-1)$
$\begin{pmatrix} \omega^a & 0 & 0 & 0 \\ 0 & \omega^a q & 0 & 0 \\ 0 & 0 & \omega^{aq^2} & 0 \\ 0 & 0 & 0 & \omega^{aq^3} \end{pmatrix}$	$\frac{1}{4}q^2(q^2-1)$	$\mathbf{F}_{q^4}^*$	$q^6(q-1)(q^2-1)(q^3-1)$

The elements ρ, σ, τ, and ω are primitive elements of $\mathbf{F}_q$, $\mathbf{F}_{q^2}$, $\mathbf{F}_{q^3}$ and $\mathbf{F}_{q^4}$ respectively, such that $\rho = \sigma^{q+1} = \tau^{q^2+q+1} = \omega^{q^3+q^2+q+1}$ and $\sigma = \omega^{q^2+1}$.

Chapter 5

Conjugacy classes of $\mathrm{GL}_n(\mathcal{O}_2)$

In this chapter, we prove Theorem 1.2.5. At the end of this chapter, we give a few natural questions regarding conjugacy classes of the groups G_λ which arise from the discussion in this thesis.

5.1 Proof of Theorem 1.2.5

Throughout this chapter, by canonical form of a matrix, we mean its block Jordan canonical form (see Theorem 2.3.10).

Lemma 5.1.1 *Let $X \in GL_n(\mathcal{O}_1)$ be in its canonical form. Then there exists a section $\mathfrak{s} : GL_n(\mathcal{O}_1) \to GL_n(\mathcal{O}_2)$ of the canonical surjective map $\kappa : GL_n(\mathcal{O}_2) \to GL_n(\mathcal{O}_1)$ such that for all $Y \in Z_{GL_n(\mathcal{O}_1)}(X)$, we have $\mathfrak{s}(Y)\mathfrak{s}(X) = \mathfrak{s}(X)\mathfrak{s}(Y)$. Further, this section is canonically defined on the centralizer $Z_{GL_n(\mathcal{O}_1)}(X)$.*

Proof: We prove it in two steps:

Step 1: X splits: If the matrix X splits its block Jordan canonical form is the same as its Jordan canonical form.

We use the canonical multiplicative section s of $\mathcal{O}_1$ entry-wise to obtain a canonical section $\mathfrak{s} : \mathrm{GL}_n(\mathcal{O}_1) \to \mathrm{GL}_n(\mathcal{O}_2)$ of κ. By Lemmas 2.3.2 and 2.3.3, if $Y \in Z_{\mathrm{GL}_n(\mathcal{O}_1)}(X)$, then it is in block Toeplitz form. Since s satisfies $s(0) = 0$ and $s(1) = 1$, $\mathfrak{s}(X)$ is again in Jordan canonical form and $s(Y)$ is in block Toeplitz form. Applying Lemmas 2.3.2 and 2.3.3 once again, we see that $\mathfrak{s}(X)$ and $\mathfrak{s}(Y)$ commute.

Step 2: X does not split: In this case, X is a block diagonal matrix with blocks of the form

$$J_r(f) = \begin{pmatrix} C_f & I & 0 & \cdots & 0 & 0 \\ 0 & C_f & I & \cdots & 0 & 0 \\ 0 & 0 & C_f & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & C_f & I \\ 0 & 0 & 0 & \cdots & 0 & C_f \end{pmatrix}_{rd \times rd}$$

where d is the degree of f, an irreducible factor of the characteristic polynomial of X, C_f is the companion matrix of f, and r is a positive integer. Up to rearrangement of blocks, this canonical form is unique. For distinct irreducible polynomials f and f'

$$\mathrm{Hom}_{\mathbf{F}_q[x]}(\mathbf{F}_q[x]/f(x), \mathbf{F}_q[x]/f'(x)) = 0.$$

Therefore there does not exist any nonzero matrix A such that $AC_f = C_{f'}A$. Since we are interested only in the centralizer of X, so we may assume that $X = \oplus_{i=1}^{l} J_{\lambda_i}(f)$.

Let $\tilde{\mathcal{O}_1}$ be the splitting field of $f(x)$ and $\tilde{\mathcal{O}_2}$ be the corresponding unramified extension of $\mathcal{O}_2$. Then by Theorem 2.3.11, $\tilde{\mathcal{O}_1} \cong \mathcal{O}_1[C_f]$. Taking $\mathfrak{s}$ as before, we also have $\tilde{\mathcal{O}_2} \cong \mathcal{O}_2[\mathfrak{s}(C_f)]$. These isomorphisms allows us to view $GL_t(\tilde{\mathcal{O}_1})$ and $GL_t(\tilde{\mathcal{O}_2})$ as subgroups of $\mathrm{GL}_n(\mathcal{O}_1)$ and $\mathrm{GL}_n(\mathcal{O}_2)$ consisting of block matrices with blocks in $\mathcal{O}_1[C_f]$ and $\mathcal{O}_2[\mathfrak{s}(C_f)]$ respectively. By Theorem 2.3.12, the centralizer $Z_{\mathrm{GL}_n(\mathcal{O}_1)}(X)$ consists of block upper Toeplitz matrices of order $(\lambda_1, \lambda_2, \dots, \lambda_l)$ over the ring $\mathcal{O}_1[C_f]$, hence is a subgroup of $\mathrm{GL}_t(\tilde{\mathcal{O}_1})$. As an element of $\mathrm{GL}_t(\tilde{\mathcal{O}_1})$, the matrix X splits over $\tilde{\mathcal{O}_1}$. Therefore by Step 1, there exists a canonical section $\tilde{\mathfrak{s}} : \mathrm{GL}_t(\tilde{\mathcal{O}_1}) \to \mathrm{GL}_t(\tilde{\mathcal{O}_2})$

$$\begin{array}{ccc} \mathrm{GL}_t(\tilde{\mathcal{O}_2}) & \underset{\longrightarrow}{\overset{\tilde{\mathfrak{s}}}{\dashleftarrow}} & \mathrm{GL}_t(\tilde{\mathcal{O}_1}) \\ \big\downarrow & & \big\downarrow \\ \mathrm{GL}_n(\mathcal{O}_2) & \xrightarrow{\kappa} & \mathrm{GL}_n(\mathcal{O}_1) \end{array}$$

such that for all $Y \in Z_{\mathrm{GL}_n(\mathcal{O}_1)}(X)$, $\tilde{\mathfrak{s}}(X)\tilde{\mathfrak{s}}(Y) = \tilde{\mathfrak{s}}(Y)\tilde{\mathfrak{s}}(X)$. Then any section $\mathrm{GL}_n(\mathcal{O}_1) \to \mathrm{GL}_n(\mathcal{O}_2)$ extending $\tilde{\mathfrak{s}}$ satisfies the required condition. □

Proof of Theorem 1.2.5 Any conjugacy class of $\mathrm{GL}_n(\mathcal{O}_2)$ under the natural quotient map maps onto a conjugacy class of $\mathrm{GL}_n(\mathcal{O}_1)$. Let $\mathcal{C}$ denote the set of conjugacy classes of $\mathrm{GL}_n(\mathcal{O}_1)$. As representatives of conjugacy classes, we take matrices in their block Jordan canonical forms and therefore may assume that $\mathcal{C}$ is the set of such matrices. For each $X \in \mathcal{C}$, let A_X denote the set of conjugacy classes of $\mathrm{GL}_n(\mathcal{O}_2)$ that maps onto the conjugacy class of X. Then

$$\text{Conjugacy classes of } \mathrm{GL}_n(\mathcal{O}_2) = \cup_{X\in\mathcal{C}} A_X.$$

Similarly, for the group $\mathrm{GL}_n(\mathcal{O}_2')$, let A'_X denote the set of conjugacy classes that maps onto the conjugacy class of X in $\mathrm{GL}_n(\mathcal{O}_1)$. We claim: *for all $X \in \mathcal{C}$, there exists a canonical bijection between A_X and A'_X which preserves the sizes of conjugacy classes.*

For given $X \in \mathcal{C}$, we choose sections $\mathfrak{s} : \mathrm{GL}_n(\mathcal{O}_1) \to \mathrm{GL}_n(\mathcal{O}_2)$ and $\mathfrak{s}' : \mathrm{GL}_n(\mathcal{O}_1) \to \mathrm{GL}_n(\mathcal{O}_2')$ as in Lemma 5.1.1.

Let $g = \mathfrak{s}(A)i(B)$ then

$$\begin{aligned} g\mathfrak{s}(X)i(Y)g^{-1} &= \mathfrak{s}(A)i(B)\mathfrak{s}(X)i(Y)i(B)^{-1}\mathfrak{s}(A)^{-1} \\ &= \mathfrak{s}(A)\mathfrak{s}(X)\mathfrak{s}(X)^{-1}i(B)\mathfrak{s}(X)i(Y)i(B)^{-1}\mathfrak{s}(A)^{-1} \\ &= \mathfrak{s}(A)\mathfrak{s}(X)\mathfrak{s}(A)^{-1}\mathfrak{s}(A)\mathfrak{s}(X)^{-1}i(B)\mathfrak{s}(X)i(Y)i(B)^{-1}\mathfrak{s}(A)^{-1} \\ &= \mathfrak{s}(A)\mathfrak{s}(X)\mathfrak{s}(A)^{-1}i({}^{A}({}^{X^{-1}}B + Y - B)) \end{aligned}$$

In particular, if $g\mathfrak{s}(X)i(Y_1)g^{-1} = \mathfrak{s}(X)i(Y_2)$, then A commutes with X, and therefore, by Lemma 5.1.1 $\mathfrak{s}(A)$ commutes with $\mathfrak{s}(X)$. Therefore, $\mathfrak{s}(X)i(Y_1)$ and $\mathfrak{s}(X)i(Y_2)$ are conjugate if and only if

$$Y_2 = {}^{A}({}^{X^{-1}}B + Y_1 - B) \tag{5.1.1}$$

for some $A \in Z_{GL_n(\mathcal{O}_1)}X$ and $B \in M_n(\mathcal{O}_1)$. Also, a necessary and sufficient condition that g commutes with $s(X)i(Y)$ is

$$A \in Z_{GL_n(\mathcal{O}_1)}X \text{ and } Y = {}^{A}({}^{X^{-1}}B + Y - B) \tag{5.1.2}$$

The conditions (5.1.1) and (5.1.2) are the same whether we work in $\mathrm{GL}_n(\mathcal{O}_2)$ or in $\mathrm{GL}_n(\mathcal{O}_2')$. Let $i' : M_n(\mathcal{O}_1) \to \mathrm{GL}_n(\mathcal{O}_2')$ be the inclusion map. By (5.1.1), $\mathfrak{s}(X)i(Y_1)$ and $\mathfrak{s}(X)i(Y_2)$ are conjugate if and only if $\mathfrak{s}'(X)i'(Y_1)$ and $\mathfrak{s}'(X)i'(Y_2)$ are conjugate. Therefore, the map from A_X to A'_X taking the conjugacy class of $\mathfrak{s}(X)i(Y)$ to the conjugacy class of $\mathfrak{s}'(X)i'(Y)$ is a well defined size preserving bijection of conjugacy classes. □

5.2 Further questions on conjugacy classes

From our discussion on irreducible representations and conjugacy classes of groups $\mathrm{GL}_n(\mathcal{O}_2)$, we observed that both representation zeta function and class equations of the groups $\mathrm{GL}_n(\mathcal{O}_2)$ depend on the ring $\mathcal{O}$ only through the order of residue field. This suggests that in other places where results have been proved for representations, they should also be tractable for conjugacy classes. So we are tempted to ask the following questions for the conjugacy classes of the more general groups G_λ.

Question 5.2.1 *Does the class equations of groups* $\mathrm{G}_{\lambda,F}$ *depend on ring of integers of* F *only through the order of its residue field?*

This is a question parallel to the Onn's Conjecture 1.1.1.

Since we are able to give precisely all the representations of groups $\mathrm{GL}_4(\mathcal{O}_2)$, we may hope that following question is also tractable (though we have not been able to answer it so far)

Question 5.2.2 *What are the conjugacy classes of groups* $GL_4(\mathcal{O}_2)$.

In this thesis we did not look at the questions regarding finding the explicit class equations of groups $\mathrm{GL}_n(\mathcal{O}_2)$ but we expect that doing calculations to find the class equations explicitly may be an interesting question in itself. Another interesting question is regarding the complexity of the problem of finding conjugacy classes of these groups. Motivated by Theorem 1.2.3, we ask

Question 5.2.3 *Are the problems of finding the conjugacy classes of following groups related, if yes how?*

1. $G_{2^n,F_q((t))}$ *for all* $n \in \mathbf{N}$.
2. $G_{k^n,F_q((t))}$ *for all* $k, n \in \mathbf{N}$.
3. $G_{\lambda,E}$ *for all partitions* λ *and all unramified extensions* E *of* $F_q((t))$.
4. $G_{2^n,F}$ *for any non-Archimedean field* F.

Bibliography

[AKOV09] Nir Avni, Benjamin Klopsch, Uri Onn, and Christopher Voll. On representation zeta functions of groups and a conjecture of Larsen-Lubotzky. arXiv:0912.4739, 2009.

[AO82] Harry Appelgate and Hironori Onishi. The similarity problem for 3×3 integer matrices. *Linear Algebra Appl.*, 42:159–174, 1982.

[AOPS] Ann-Marie Aubert, Uri Onn, Amritanshu Prasad, and Alexander Stasinski. On cuspidal representations of general linear groups over discrete valuation rings. To appear in the Israel Journal of Mathematics. arXiv:0706.0058.

[AOPV09] Nir Avni, Uri Onn, Amritanshu Prasad, and Leonid Vaserstein. Similarity classes of 3×3 matrices over a local principal ideal ring. *Comm. Algebra*, 37(8):2601–2615, 2009.

[BF83] Colin J. Bushnell and Albrecht Fröhlich. *Gauss sums and p-adic division algebras*, volume 987 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1983.

[Bou03] Nicolas Bourbaki. *Algebra II. Chapters 4–7*. Translated from the French by P. M. Cohn and J. Howie. Springer-Verlag, Berlin, 2003.

[CR62] Charles W. Curtis and Irving Reiner. *Representation theory of finite groups and associative algebras*. Pure and Applied Mathematics, Vol. XI. Interscience Publishers, 1962.

[Dav68] Ronald W. Davis. Certain matrix equations over rings of integers. *Duke Math. J.*, 35:49–59, 1968.

[Dic59] Leonard E. Dickson. *Algebraic theories*. Dover Publications Inc., New York, 1959.

[Fro96] G. F. Frobenius. Über Gruppencharactere. *Berliner Sitzungsberichte*, 1896.

[Gin01] Yuval Ginosar. On splitting of the canonical map: $\mathrm{mod}(p)\colon \mathrm{GL}(n,(p)) \to \mathrm{GL}(n,p)$. *Comm. Algebra*, 29(12):5879–5881, 2001.

[GP89] P. M. Gudivok and V. I. Pogorilyak. Representation of finite p-groups over local rings of positive characteristic. *Dokl. Akad. Nauk Ukrain. SSR Ser. A*, (2):5–8, 85, 1989.

[Gre55] J. A. Green. The characters of the finite general linear groups. *Trans. Amer. Math. Soc.*, 80:402–447, 1955.

[Hil93] Gregory Hill. A Jordan decomposition of representations for $\mathrm{GL}_n(\mathcal{O})$. *Comm. Algebra*, 21(10):3529–3543, 1993.

[Hil94] Gregory Hill. On the nilpotent representations of $\mathrm{GL}_n(\mathcal{O})$. *Manuscripta Math.*, 82(3-4):293–311, 1994.

[Hil95a] Gregory Hill. Regular elements and regular characters of $\mathrm{GL}_n(\mathcal{O})$. *J. Algebra*, 174(2):610–635, 1995.

[Hil95b] Gregory Hill. Semisimple and cuspidal characters of $\mathrm{GL}_n(\mathcal{O})$. *Comm. Algebra*, 23(1):7–25, 1995.

[Isa76] I. Martin Isaacs. *Character theory of finite groups*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1976.

[Klo46a] H. D. Kloosterman. The behaviour of general theta functions under the modular group and the characters of binary modular congruence groups. I. *Ann. of Math. (2)*, 47:317–375, 1946.

[Klo46b] H. D. Kloosterman. The behaviour of general theta functions under the modular group and the characters of binary modular congruence groups. II. *Ann. of Math. (2)*, 47:376–447, 1946.

[Kut73] Philip C. Kutzko. The characters of the binary modular congruence group. *Bull. Amer. Math. Soc.*, 79:702–704, 1973.

[MR067] *Algebraic number theory*. Proceedings of an instructional conference organized by the London Mathematical Society (a NATO Advanced Study Institute) with the support of the Inter national Mathematical Union. Edited by J. W. S. Cassels and A. Fröhlich. Academic Press, London, 1967.

[Nag76] S. V. Nagornyi Complex representations of the group $\mathrm{GL}(2, Z/p^n Z)$. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 64:95–103, 161, 1976.

[Nag78] S. V. Nagornyi Complex representations of the general linear group of degree three modulo a power of a prime. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 75:143–150, 197–198, 1978.

[Nec83] A. A. Nechaev. Similarity of matrices over a commutative local Artinian ring. *Trudy Sem. Petrovsk.*, (9):81–101, 1983.

[Nob76] Alexandre Nobs. La série déployée et la série non ramifiée de $\mathrm{GL}_2(\mathcal{O})$. *C. R. Acad. Sci. Paris Sér. A-B*, 283(6):Aii, A297–A300, 1976.

[NW74] Alexandre Nobs and Jürgen Wolfart. Darstellungen von SL(2, $\mathbf{Z}/p^\lambda\mathbf{Z}$) und Thetafunktionen. I. *Math. Z.*, 138:239–254, 1974.

[Onn08] Uri Onn. Representations of automorphism groups of finite $\mathfrak{o}$-modules of rank two. *Adv. Math.*, 219(6):2058–2085, 2008.

[Pom73] J. Pomfret. Similarity of matrices over finite rings. *Proc. Amer. Math. Soc.*, 37:421–422, 1973.

[Pra07] Amritanshu Prasad. Representations of $GL_2(\mathbf{F}_q)$ and $SL_2(\mathbf{F}_q)$, and some remarks on $GL_n(\mathbf{F}_q)$. Notes from lectures given at the *Advanced Instructional School on Representation Theory and its Applications*, held at the Bhaskaracharya Pratishthana and University of Pune. Available from `http://www.imsc.res.in/~amri`, July 2007.

[Roh32] H. Rohrbach. Die Charactere der binären Kongruenzgruppen mod p^2. *Scriften d. math. Seminars d. Univ. Berlin 1*, 1932.

[Sah77] Chih Han Sah. Cohomology of split group extensions. II. *J. Algebra*, 45(1):17–68, 1977.

[Ser78] Jean-Pierre Serre. *Représentations linéaires des groupes finis.* Hermann, Paris, revised edition, 1978.

[Ser79] Jean-Pierre Serre. *Local Fields.* Springer-Verlag, Newyork, 1979.

[ST66] D. A. Suprunenko and R. I. Tyškevič. *Perestanovochnye matritsy.* Nauka i Tehnika, Minsk, 1966.

[Sta08] Alexander Stasinski. The smooth representations of $\mathrm{GL}_2(\mathcal{O})$. arXiv:0807.4684, 2008.

[Ste51] Robert Steinberg. The representations of $\mathrm{GL}(3, q)$, $\mathrm{GL}(4, q)$, $\mathrm{PGL}(3, q)$, and $\mathrm{PGL}(4, q)$. *Canadian J. Math.*, 3:225–235, 1951.

[Tan67] Shun'ichi Tanaka. Irreducible representations of the binary modular congruence groups $\bmod p^\lambda$. *J. Math. Kyoto Univ.*, 7:123–132, 1967.