ZEROS OF GENERAL L-FUNCTIONS ON THE CRITICAL LINE

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DECLARATION

I, hereby declare that the investigtion presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Krishnan Rajkumar

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Abstract

We study the gaps between consecutive zeros on the critical line for the Riemann zeta function $\zeta(s)$ and certain generalisations of $\zeta(s)$, namely, the Epstein zeta function and the Selberg class of functions. We first give a simplified exposition of a result of Ivić and Jutila on the large gaps between consecutive zeros of $\zeta(s)$ on the critical line. We then present a generalization of this result to the case of the Epstein zeta function $\zeta_Q(s)$ associated to a certain binary, positive definite, integral quadratic form $Q(x, y)$. We next establish the analogue of Hardy's theorem, namely that there are infinitely many zeros on the critical line, for degree 2 elements of the Selberg class of L-functions whose Dirichlet coefficients satisfy certain mild growth conditions. We conclude with a conditional version of Hardy's theorem for the degree $d > 2$ elements of the Selberg class.

Contents

List of Publications

- (with Anirban Mukhopadhyay and Kotyada Srinivas) On the zeros of functions in the Selberg class, *Funct. Approx. Comment. Math.* 38 (2008), part 2, 121–130.

- (with Anirban Mukhopadhyay and Kotyada Srinivas) On the zeros of the Epstein zeta function. *Proceedings in honour of Prof. T. C. Vasudevan,* RMS Lecture Note Series, No. 15 (2011), pp. 73–87.

- (with Debashish Bose, C.P. Anil Kumar and Shobha Madan) On Fuglede's conjecture for three intervals, *Online J. Anal. Comb.* No. 5 (2010), 24 pp.

- (with Kotyada Srinivas) Zeros of functions in the Selberg class with degree $d > 2$. *Preprint.*

Chapter 1

Introduction

The study of the analytical properties of the Riemann zeta function $\zeta(s)$, is central to number theory. This function was invented by Euler to study the infinitude of prime numbers. Riemann showed that $\zeta(s)$, originally defined as a Dirichlet series for $\operatorname{Re} s > 1$, admits an analytic continuation for all complex values excluding $s = 1$ and made the far-reaching observation that the location of the complex zeros of $\zeta(s)$ is intimately connected to the distribution of the primes. One of the most important open problems in mathematics, the Riemann hypothesis, asserts that all the complex zeros of $\zeta(s)$ lie on the line $\operatorname{Re} s = 1/2$ (the critical line). A proof or disproof of this statement will have immense implications in many areas of mathematics.

As a first step towards the Riemann hypothesis, Hardy [15] showed that $\zeta(s)$ has infinitely many zeros on the critical line. The main idea of Hardy was extended by several authors to show the infinitude of zeros of other classical zeta functions on the critical line (see below for a discussion).

Research on the critical zeros of $\zeta(s)$ can be broadly classified into two themes:

Theme A (Local Results): Here one studies the gap between consecutive zeros of $\zeta(s)$ on the critical line.

Theme B (Global Results): Here one is interested in the proportion (suitably defined) of complex zeros that lie on the critical line.

Theme A has been studied extensively from the time of Hardy and Littlewood. In a classical paper [16], Hardy and Littlewood showed that, given T sufficently large, there is a zero of $\zeta(\frac{1}{2}+it)$ with t in the interval $[T, T+H]$ if $H \geq T^{1/4+\epsilon}$ for any $\epsilon > 0$. This was subsequently improved to $H \geq T^{1/6+\epsilon}$ by Balasubramanian [1] and $H \geq T^{1/6}(\log T)^{5+\epsilon}$ by Mozer [33], [34]. The next major improvement was by Karatsuba [27] who showed that the result holds with $H \geq T^{5/32}(\log T)^2$. Ivić ([19], p. 261) generalized Karatsuba's method by using the theory of exponent pairs and improved the result to $H \geq T^{0.1559...+\epsilon}$ (note that $5/32 = 0.15625$). Huxley and Watt [17] and Watt [51] obtained the latest improvements in the exponent of T to the values $23/148 = 0.1554\ldots$ and $229/1476 = 0.1551\ldots$, respectively, by constructing new exponent pairs and using Ivić's method mentioned above.

As for the results belonging to Theme B, we mention the well-known theorem of Selberg [46] which states that the function $\zeta(\frac{1}{2}+it)$ has $\geq \frac{A}{2\pi}T\log T$ zeros in the interval $[0, T]$, for some effectively computable constant $A > 0$. This means that a positive proportion of the complex zeros of $\zeta(s)$ lie on the critical line, because, by the Riemann - van Mangoldt formula ([50], p.214), the number of zeros in the critical strip with ordinates in $[0, T]$ is asymptotic to $\frac{1}{2\pi}T\log T$. Subsequent research on this problem focussed on determining the best possible value of A and we list the series of results obtained so far: Levinson [29], [30] $A = 0.3405, 0.3474$; Lou and Yao [31] $A = 0.3484$; Conrey [6], [7] $A = 0.3658, 0.4088$; Bui, Conrey and Young [4] $A = 0.4105$; Feng [11] $A = 0.4128$.

In this thesis, we shall address questions belonging to Theme A for the Riemann zeta function and a certain generalisation of $\zeta(s)$, namely, the Epstein zeta function. We shall then discuss the analogue of Hardy's theorem for the Selberg class of functions.

A word about the notation used throughout the thesis: the symbols $f(x) \ll g(x)$, $f(x) = O(g(x))$ and $g(x) \gg f(x)$ will be interchangeably used to mean that there exists a constant $C > 0$ such that $|f(x)| \le C|g(x)|$ for all x; the symbol $f(x) \asymp g(x)$ means that both $f(x) \ll g(x)$ and $f(x) \gg g(x)$ is true; and $f(x) = o(g(x))$ means that for every $\epsilon > 0$, $|f(x)| < \epsilon|g(x)|$ for x sufficiently large.

This thesis consists of five chapters. In the current chapter, we shall give a brief introduction and statement of the results contained in this thesis.

In Chapter 2, we shall address the following question

Question C: *Given T sufficiently large and $V > 0$, how many consecutive zeros of $\zeta(s)$ are there on the critical line with ordinates in $[0, T]$ which are atleast V distance apart?*

Let $R_1 := R_1(T, V)$ be the number of gaps which are larger than V between consecutive zeros of $\zeta(\frac{1}{2} + it)$ with ordinates between 0 and T.

Note that $R_1 \ll TV^{-1}$ trivially. Karatsuba [27] showed that $R_1 \ll TV^{-3/2}$ for $V = T^\epsilon$ for any $\epsilon > 0$. Ivić and Jutila [20] made a substantial improvement and proved the following theorem, which remains the best result known till date.

Theorem 1. *Let T, V be positive real numbers. Then the following estimates hold uniformly in T and V*

$$R_1 \ll TV^{-2} \log T, \tag{1.1}$$

and

$$R_1 \ll TV^{-3} \log^5 T. \tag{1.2}$$

In Chapter 2, we shall present a simplified exposition of this result, with our proof following along the lines of [20] to a substantial degree. We streamline the proof by dividing it into three steps - a local estimate, a global estimate and arithmetical methods. Our presentation of the first two steps will be based on [20], while the last step will be substantially new and relies on standard estimates of certain sums involving the divisor function.

In Chapter 3, we shall address the analogue of Question C for the Epstein zeta function $\zeta_Q(s)$ associated to a certain binary, positive definite, integral quadratic form Q. The properties of $\zeta_Q(s)$ relevant to our discussion shall be discussed in this chapter.

Questions related to Theme A with $\zeta(s)$ replaced with $\zeta_Q(s)$ have also been studied extensively. It was proved by Potter and Titchmarsh [40] that, given T sufficently large, there is a zero of $\zeta_Q(\frac{1}{2}+it)$ with t in the interval $[T, T+H]$ if $H \geq T^{1/2+\epsilon}$ for any $\epsilon > 0$. Sankaranarayanan [45] sharpened this by showing the same for $H \gg T^{1/2} \log T$. The latest result was due to Jutila and Srinivas [21], who reduced the exponent of T below 1/2, by proving that the result holds with $H \geq T^{5/11+\epsilon}$.

In Chapter 3, we shall prove the following theorem as an answer to Question C for the Epstein zeta function.

Let $R_2 := R_2(T, V)$ denote the number of gaps of length at least V between consecutive zeros of $\zeta_Q(\frac{1}{2}+it)$ with ordinates in the interval $[0, T]$. Then we have

Theorem 2. *Let Q be a binary, positive definite, integral, quadratic form with discriminant $-\triangle$ such that $\sqrt{\triangle}$ is irrational. Let $\epsilon > 0$ be sufficiently small and T, V be positive real numbers. Then*

$$R_2 \ll T^{1+\epsilon} V^{-2}, \tag{1.3}$$

where the constant in $\ll$ may depend only on ϵ.

This result was given by Mukhopadhyay, Srinivas and the author in [35]. Our proof of this result in Chapter 3 will be an exposition of the method in [35]. We have simplified the proof considerably and modelled it on that of Theorem 1. It will also be divided into three steps - a local estimate, a global estimate and arithmetical methods. The main tool to derive the local estimate will be a transformation formula for Dirichlet polynomials due to Jutila and Srinivas [21]. The global estimate will be identical to that in Chapter 2. The last step of using arithmetical methods will be entirely based on [35].

We remark that Theorem 2 should be contrasted with another result of Ivić and Jutila [20] regarding the corresponding problem for the L-function $L_f(s)$ associated to a cusp form f for the full modular group with real Fourier coefficients. For such $L_f(s)$, we let $R_3 := R_2(T,V)$ denote the number of gaps of length at least V between consecutive zeros of $L_f(\frac{1}{2}+it)$ with ordinates in the interval $[0,T]$. Then it is proved in [20] that uniformly in the range $\log^5 T \ll V$, we have

$$R_3 \ll TV^{-2}\log T, \qquad \text{and}$$
$$R_3 \ll T^2V^{-6}\log^5 T.$$

In Chapter 4, we shall discuss Hardy's theorem for general L-functions of a certain type. Hardy's ideas in [15] to show the infinitude of critical zeros of $\zeta(s)$ were extended by several authors to other classical zeta functions. For instance, Potter and Titchmarsh [40] proved the analogue of Hardy's theorem for the Epstein zeta function $\zeta_Q(s)$ of a positive definite integral quadratic form; Chandrasekharan and Narasimhan [5] proved it for the ideal class zeta functions of quadratic number fields.

It is natural, therefore, to ask the following question

Question D: *What is the most general class of functions to which one can extend Hardy's theorem?*

In this chapter, we shall answer Question D in the context of the Selberg class of functions, denoted by $\mathcal{S}$.

We recall that the Selberg class $\mathcal{S}$ is the class of functions satisfying the following five axioms:

(i) $F(s)$ can be written as an absolutely convergent Dirichlet series $\sum_{n\geq 1} a(n)n^{-s}$ for $\sigma > 1$;

(ii) $(s-1)^m F(s)$ is an entire function of finite order for some integer $m \geq 0$;

(iii) $F(s)$ satisfies a functional equation of the form

$$\Phi(s) = \omega \bar{\Phi}(1-s), \tag{1.4}$$

where

$$\Phi(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s)$$

with $Q > 0, \lambda_j > 0, \Re\mu_j \geq 0$ and $|\omega| = 1$. Here we denote $\bar{f}(s) = \overline{f(\bar{s})}$;

(iv) The Dirichlet coefficients $a(n)$ satisfy the Ramanujan conjecture $a(n) \ll n^{\epsilon}$ for every $\epsilon > 0$;

(v) $\log F(s) = \sum_{n\geq 1} b(n)n^{-s}$ where $b(n) = 0$ unless $n = p^m$ for some prime p and integer $m \geq 1$ and $b(n) \ll n^{\theta}$ for some $\theta < 1/2$.

For excellent surveys of $\mathcal{S}$ and its properties, see Kaczorowski and Perelli [23] and Perelli [39].

The *degree* of a function $F \in \mathcal{S}$ is defined by $d = 2\sum_{j=1}^{r} \lambda_j$, where λ_j are as in the functional equation (1.4) of F. For example, $\zeta(s)$ has degree 1, the L-function $L_f(s)$

associated with a cusp form f has degree 2 and the Dedekind zeta function $\zeta_K(s)$ associated with a number field K has degree $d = [K : \mathbb{Q}]$.

The degree is an important invariant associated to F and we have $d \geq 0$ by definition. There are two important open problems connected with the degree. The first is the question, *is d an integer?* The other problem is to *classify all functions in $\mathcal{S}$ of given degree d.*

The answer to the first question is known to be affirmative in the range $0 < d < 1$ from the work of Bochner [2], Conrey and Ghosh [8], Kaczorowski and Perelli [24] and in the range $1 < d < 2$ by the deep results of Kaczorowski and Perelli [25], [26].

Regarding the second problem, it is known that the only function in $\mathcal{S}$ with $d = 0$ is the constant function 1 (see Bochner [2], Conrey and Ghosh [8], Kaczorowski and Perelli [24]). The classification of degree 1 functions in $\mathcal{S}$ is also complete (see Kaczorowski and Perelli [24], Soundararajan [49])- these are $\zeta(s)$, the Dirichlet L-functions $L(s, \chi)$ associated with primitive characters χ and their imaginary shifts $L(s + i\,\omega, \chi)$ with ω real.

The answer to Question D in case of degree 1 elements of $\mathcal{S}$ is therefore affirmative. In fact, stronger results belonging to Theme B are known. In other words, it is known that a positive proportion of zeros lie on the critical line for all degree 1 functions (see [46], [52]) as mentioned earlier in the context of $\zeta(s)$.

Concerning the degree 2 functions in $\mathcal{S}$, the only known examples are the products of two degree 1 functions, like $\zeta^2(s)$ and the Dedekind zeta function of a quadratic number field K, $\zeta_K(s)$; the L-function $L_f(s)$ associated with a suitable normalized modular form f which is either holomorphic (cusp form) or non-holomorphic (Maass form); and the imaginary shifts of any of these functions which are entire. Hardy's theorem is known to be true for all the known elements of degree 2. Like in the case of degree 1, stronger results belonging to Theme B are also known (see [13],[14]).

The classification of *all* degree 2 functions in $\mathcal{S}$ has not yet been achieved. Hence, the answer to Question D for any general degree 2 function in $\mathcal{S}$ becomes significant.

The known results on Question D for general degree 2 functions are all based on some unproved hypotheses. For example, Gritsenko [12] considers distinct primitive functions $F_1, \ldots F_n$ in $\mathcal{S}$ of degree 2 which satisfy certain conditions on the growth of both their Dirichlet coefficients and their Euler factors. Then, assuming the Selberg orthogonality conjecture (see [47]), it is shown that the linear combination $\sum_{j=1}^{N} b_j Z_{F_j}(t)$ where b_j are real, has at least $T\exp(\sqrt{\log\log\log T})$ zeros in $[T, 2T]$. Here $Z_{F_j}(t)$ is the analogue for F_j of Hardy's function $Z(t)$.

Bombieri and Hejhal [3] consider functions $L_1(s), \ldots, L_n(s)$ which satisfy the same functional equation, are orthogonal in the sense of Selberg, satisfy the Riemann Hypothesis and a certain conjecture on the distribution of the zeros. Then they show that almost all zeros of the linear combination $\sum_{j=1}^{N} b_j L_j(s)$ where b_j are real, lie on the critical line and are simple.

Now, we shall state the main result of Chapter 4 which answers Question D for the class of general degree 2 elements of $\mathcal{S}$ satisfying certain mild conditions.

For $F \in \mathcal{S}$ satisfying (1.4), we define the conductor $q = (2\pi)^d Q^2 \prod_{j=1}^{r} \lambda_j^{2\lambda_j}$ and the internal shift $\theta = 2\,\mathrm{Im}(\sum_{j=1}^{r} \mu_j)$, which are also invariants associated with F (see [23]).

Theorem 3. *Let $F \in S$ be any function with degree $d = 2$, $\theta = 0$, $\sqrt{q}$ irrational and the Dirichlet coefficients $a(n)$ of F satisfying*

$$\sum_{n \leq x} |a(n)|^2 = O(x), \tag{1.5}$$

then $F(s)$ has infinitely many zeros on the critical line.

This result was given by Mukhopadhyay, Srinivas and the author in [36]. Our proof of this result will follow the method of [36] to a large extent. The main difference will be in the use of the saddle point theorem. Here we use a version of this theorem due to Krätzel [28] and thereby get sharper error terms in our estimations compared to [36].

In Chapter 5 we study the analogue of Hardy's theorem for functions of degree $d > 2$. The answer to this question is not known for any classical L-function, except in special cases where $\zeta(s)$ appears as a factor. This happens, for instance, for the Dedekind zeta function $\zeta_K(s)$ of a number field K with solvable Galois group $\mathrm{Gal}[K : \mathbb{Q}]$.

The main result of Chapter 5 is the following attempt to answer Question D under some conditions.

Theorem 4. *Let $F \in \mathcal{S}$ with degree $2 \leq d < 4$, conductor q and internal shift $\theta = 0$. Let T, H be sufficiently large such that $T^{\frac{d}{4}+\epsilon} \ll H \ll T$ for some $\epsilon > 0$. Let $q_* = q^{1/2}(2\pi)^{-d/2}$, $G = H/\log T$ and $\psi(u) = \exp(-u^2G^{-2})$ be a smoothing function. If the condition*

$$\sum_{|(n/q_*)^{\frac{2}{d}} - T| \leq H} \frac{a(n)}{n^{\frac{1}{2}-\frac{1}{d}}} \psi\big((n/q_*)^{\frac{2}{d}} - T\big) \exp\left(-i\frac{d}{2}(n/q_*)^{\frac{2}{d}}\right) = o(G) \tag{1.6}$$

holds, then $F(\frac{1}{2} + it)$ has a zero for some $t \in [T - H, T + H]$.

This result is an unpublished work of Srinivas and the author [41]. In Chapter 5, we shall give a proof of this result by generalizing the methods of Chapter 4 to the case when $d \geq 2$.

We wish to draw the attention of the reader to the philosophy behind the organization of the various chapters and the methods used therein. The guiding theme is the progession from the theory of a special L-function to the theory of a general L-function. This is accompanied by a decline in strength of the corresponding results on the critical zeros.

We start with the special case of the Riemann zeta function, which is a prototype of all L-functions; move on to the Epstein zeta function, which is a particular example of a degree 2 function in the extended Selberg class (see [23], [39]); study a general degree 2 function in the Selberg class; and finally, consider a general element of the Selberg class of degree larger than 2.

The results that we obtain will range from comparably strong estimates on the number of large gaps between consecutive critical zeros for $\zeta(s)$ and $\zeta_Q(s)$; existence results about consecutive zeros for degree 2 elements of $\mathcal{S}$; and conditional results for elements of $\mathcal{S}$ for degree larger than 2.

The analytical properties of these functions that we will use, summarized at the beginning of each chapter, will be strikingly similar.

All the proofs have structural features in common with each other. The first steps of all proofs involve local estimates, which when combined with suitable arithmetical estimates yield the best known results regarding Theme A for the corresponding functions. The global estimates which appear in Step II of Chapters 2 and 3 are virtually identical, though they are not needed in Chapters 4 and 5. The final and crucial steps in each chapter are completed using arithmetical methods.

However, this is the point where the similarities end and the staggering variety of the arithmetical methods give an indication of the crucial difficulties involved in assimilating various special cases into the picture of a general L-function. We hope that the reader will be convinced that the last word has not been said on either of the problems considered in this thesis. We wish that the reader will consider this thesis as an invitation to partake in the study of the zeros of general L-functions on the critical line.

Chapter 2

Large gaps for the Riemann zeta function

We start by recalling the main properties of the Riemann zeta function. It is defined by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

in the half-plane $\operatorname{Re} s = \sigma > 1$. It has an analytic continuation to the entire complex plane excluding $s = 1$, where it has a simple pole with residue 1. It satisfies the functional equation

$$\pi^{-\frac{1}{2}s}\Gamma\left(\tfrac{s}{2}\right)\zeta(s) = \pi^{-\frac{1}{2}(1-s)}\Gamma\left(\tfrac{1-s}{2}\right)\zeta(1-s), \tag{2.1}$$

where $\Gamma(s)$ is the well known gamma function. The above equation can also be rewritten in the form $\zeta(s) = \chi(s)\zeta(1-s)$ where

$$\chi(s) = \pi^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}. \tag{2.2}$$

The functional equation (2.1) implies that $|\chi(\frac{1}{2}+it)| = 1$.

The Hardy's function $Z(t)$ is defined as

$$Z(t) = \chi(\tfrac{1}{2} + it)^{-1/2}\zeta(\tfrac{1}{2} + it). \tag{2.3}$$

The function $Z(t)$ was introduced by Hardy to show that the Riemann zeta function has infinitely many zeros on the critical line. Again, the functional equation (2.1) implies that for real values of t, we have $Z(t)$ is real and $|Z(t)| = |\zeta(\frac{1}{2} + it)|$.

Lastly, $\zeta(s)$ has an Euler product, given by

$$\zeta(s) = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1},$$

where the product is over all primes p and is valid for $\operatorname{Re} s > 1$.

As mentioned in Chapter 1, the Riemann Hypothesis is the statement that the zeros of $\zeta(s)$ which lie in the critical strip $0 \le \sigma \le 1$ have real part $1/2$. This is one of the most important open problems in the field of number theory.

In the next section, we shall state several supplementary results which we shall need in the course of the proof of Theorem 1. This proof will be covered in the last section.

2.1 Lemmas

The first result in this section is a special case of the well-known Hölder's inequality.

Lemma 2.1. *Let ψ,f be in $C[a, b]$ with ψ positive and let k be a positive integer, then the following holds*

$$\left|\int_a^b f(u)\psi(u)du\right|^{2k} \le \left(\int_a^b \psi(u)du\right)^{2k-1} \left(\int_a^b |f(u)|^{2k}\psi(u)du\right).$$

The next result is the first derivative estimate for an oscillatory integral. The form that we state is Lemma 4.2 of Titchmarsh [50].

Lemma 2.2. *Let r be in $C^1[a,b]$ such that $r'(x)$ is monotonic with minimum modulus $m = min_{[a,b]}|r'(x)|$, then the following holds*

$$\int_a^b e^{ir(x)}dx \leq \frac{4}{m}.$$

Ramachandra showed (see [44], Chap. 2) that the first power mean of a generalized Dirichlet series satisfying certain conditions cannot be too small. The following lemma from Balasubramanian [1] is a particular case of this general theorem, which is quite useful in obtaining lower bounds of this type, even in short intervals.

Lemma 2.3. *Let $B(s) = \sum_{n=1}^{\infty} b_n n^{-s}$ be any Dirichlet series satisfying the following conditions:*

(i) not all b_n's are zero;

(ii) the function can be continued analytically in $\sigma \geq a$, $|t| \geq t_0$, and in this region $B(s) = O((|t|+10)^A)$.

Then for every $\epsilon > 0$, we have

$$\int_T^{T+H} |B(\sigma+it)|dt \gg H$$

for all $H \geq (\log T)^\epsilon$, $T \geq T_0(\epsilon)$, and $\sigma > a$.

Now we state an approximate functional equation for $Z(t)$ (see [50], pp.89).

Lemma 2.4. *Let t be sufficiently large, $P(t) = \sqrt{t/(2\pi)}$ and write $\chi(\frac{1}{2}+it)^{-1/2} = e^{i\theta(t)}$. Then*

$$(2.4) \qquad Z(t) = 2 \sum_{n \le P(t)} n^{-\frac{1}{2}} \cos(\theta(t) - t \log n) + O(t^{-\frac{1}{4}}).$$

The following lemma is a quantitative statement of the decay properties of the smoothing function $\psi(u) = \exp(u^2 G^{-2})$.

Lemma 2.5. *Assume all the conditions of Lemma 2.4, and let* $0 < V \ll t^{1/4}$*,* $L = 4 \log t$*,* $G = VL^{-1/2}$*,* $X \geq t^{1/2} LV^{-1}$*. Then for any integer* n *with* $0 < n < P(t) - X$*, we have*

$$(2.5) \qquad \int_{-V/4}^{V/4} e^{i\theta(t+u)} n^{-i(t+u)} \psi(u) du \ll G t^{-1/2}.$$

Proof. We start with the well known Stirling's formula for the Γ-function which states that in any fixed vertical strip $-\infty < \alpha \leq \sigma \leq \beta < \infty$,

$$(2.6) \qquad \Gamma(\sigma + it) = (2\pi)^{1/2} t^{\sigma + it - 1/2} e^{-\frac{\pi}{2} t + \frac{\pi}{2} i(\sigma - 1/2) - it} (1 + O(1/t)) \quad \text{as } t \to \infty.$$

Now, we use this to estimate $\chi(s)$ as

$$\chi(\sigma + it) = P(t)^{1 - 2\sigma - 2it} e^{-\frac{\pi}{2} i(\sigma - \frac{1}{2})} e^{it} (1 + O(1/t)).$$

Noting that $\theta(t) = \arg \chi(\frac{1}{2} + it)^{-1/2}$, we get the standard approximation ([50], pp. 78)

$$\begin{aligned} \theta(t + u) &= (t + u) P(t + u) - \tfrac{1}{2}(t + u) - \tfrac{\pi}{4} + O(\tfrac{1}{t}) \\ &= \theta(t) + u \log P(t) + O(\tfrac{u^2}{t}). \end{aligned}$$

Using this approximation, the left side of (2.5) becomes

$$(2.7)\qquad \int_{-V/4}^{V/4} e^{i\theta(t+u)} n^{-i(t+u)} \psi(u)du \ll \left| \int_{-V/4}^{V/4} e^{iu\log\left(\frac{P(t)}{n}\right)} \psi(u)du \right| + O(GV^2t^{-1}).$$

The error term above is $O(Gt^{-1/2})$ by the condition on V. Now we use the standard estimate for the complementary error function $\mathrm{erfc}(x)$ (see [37], Sec 7.12)

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-s^2} ds \ll \frac{1}{x} e^{-x^2},$$

to replace the integral on the right of (2.7) by the full integral

$$(2.8)\qquad \int_{-\infty}^{\infty} e^{iu\log\left(\frac{P(t)}{n}\right)} \psi(u)du = \sqrt{\pi} G \exp\left(-\tfrac{1}{4} G^2 \log^2\left(P(t)/n\right)\right),$$

which is the Fourier transform of $\psi(u) = e^{-u^2G^{-2}}$ (see [37], Sec 1.14(vii)).

By the condition on n, we have $\log(P(t)/n) \geq X/P(t) \geq L/V$. Hence, combining (2.7) and (2.8) we get

$$\int_{-V/4}^{V/4} e^{i\theta(t+u)} n^{-i(t+u)} \psi(u)du \ll G\exp(-L/4) + O(Gt^{-1/2}) \ll Gt^{-1/2},$$

as claimed. □

The last lemma is a summary of the arithmetic facts that we will need during the course of the proof of Theorem 1. Part (a) is a weaker form of an asymptotic estimate due to Ingham [18] which is

$$\sum_{n \leq N} d(n)d(n+s) \sim \tfrac{6}{\pi^2} \sigma_{-1}(s) N \log^2 N,$$

as $N \to \infty$ for a fixed $s \neq 0$. Here $d(n)$ is the divisor function and $\sigma_{-1}(n) = \sum_{\delta | n} \delta^{-1}$. Several authors have worked on the deep problem of improving the range of s (relative to N) where this asymptotic formula holds and these results have important applications in the theory of the Riemann zeta function. The best result so far is due to Duke, Friedlander and Iwaniec [10] where they establish that the estimate holds in the range $s \ll N^{3/2-\epsilon}$. We shall only need an upper bound of this form in the range $s \ll N$, and we present an elementary proof of this.

Part (b) is a standard estimate of the mean value of a multiplicative function.

Lemma 2.6. *Let N be sufficiently large.*

(a) For $s \ll N$ a nonnegative integer, we have

$$\sum_{n \le N} d(n)d(n+s) \ll \begin{cases} \sigma_{-1}(s) N \log^2 N, & s \neq 0 \\ N \log^3 N, & s = 0 \end{cases}$$

(b) We have

$$\sum_{s \ll N} \frac{\sigma_{-1}(s)}{s} \ll \log N.$$

Proof of Part (a). Using the inequality $d(n) \le 2 \sum_{\substack{m|n \\ m \le \sqrt{n}}} 1$, we get

$$\sum_{n \le N} d(n)d(n+s) \ll \sum_{n \le N} \sum_{\substack{d_1|n,\, d_2|(n+s) \\ d_1 \le \sqrt{n},\, d_2 \le \sqrt{n+s}}} 1$$

$$\ll \sum_{d_1, d_2 \ll \sqrt{N}} \sum_{\substack{n \le N \\ d_1|n,\, d_2|(n+s)}} 1. \tag{2.9}$$

Note that $d_1|n$, $d_2|(n+s) \Rightarrow \delta|s$, $\delta|n$ where $\delta = (d_1, d_2)$. Summing over δ in the last sum of (2.9) and writing $d_1 = e_1\delta$, $d_2 = e_2\delta$, $n = m\delta$ gives

$$\sum_{\substack{\delta|s \\ \delta\ll N}} \sum_{\substack{e_1,e_2\ll\sqrt{N}/\delta \\ (e_1,e_2)=1}} \sum_{\substack{m\le N/\delta \\ m\equiv 0(e_1) \\ m\equiv -s/\delta(e_2)}} 1. \tag{2.10}$$

By the Chinese remainder theorem, any two consecutive solutions of the congruence conditions in the last sum in (2.10) will differ by e_1e_2 as $(e_1, e_2) = 1$ in the previous sum. Thus the innermost sum in (2.10) is bounded by $\frac{N}{\delta e_1 e_2} + O(1)$. Using this in (2.10) and dropping the condition $(e_1, e_2) = 1$ gives the upper bound

$$\begin{aligned}
&\ll N \sum_{\substack{\delta|s \\ \delta\ll N}} \frac{1}{\delta} \sum_{e_1,e_2\ll\sqrt{N}/\delta} \frac{1}{e_1e_2} + O\big(\sum_{\delta\ll N} \sum_{e_1,e_2\ll\sqrt{N}/\delta} 1\big) \\
&\ll N \sum_{\substack{\delta|s \\ \delta\ll N}} \frac{1}{\delta} \log^2 N + O\big(\sum_{\delta\ll N} \frac{N}{\delta^2}\big) \\
&\ll N \log^2 N \sum_{\substack{\delta|s \\ \delta\ll N}} \frac{1}{\delta} + O(N). \qquad (2.11)
\end{aligned}$$

Noting that the sum in the main term of (2.11) is $\sigma_{-1}(s)$ when $s \neq 0$ and $O(\log N)$ when $s = 0$, we get the required reuslt of Part (a).

Proof of Part (b). Note that

$$\begin{aligned}
\sum_{s\ll N} \frac{\sigma_{-1}(s)}{s} &= \sum_{s\ll N} \frac{1}{s} \sum_{\delta|s} \frac{1}{\delta} \\
&= \sum_{\delta\ll N} \frac{1}{\delta} \sum_{\substack{s\ll N \\ \delta|s}} \frac{1}{s}.
\end{aligned}$$

Writing $s = t\delta$ in the last sum above, we get the bound

$$\begin{aligned} &\ll \sum_{\delta \ll N} \frac{1}{\delta^2} \sum_{t \ll N/\delta} \frac{1}{t} \\ &\ll \sum_{\delta \ll N} \frac{1}{\delta^2} \log N \\ &\ll \log N, \end{aligned}$$

as required. □

2.2 Proof of Theorem 1

We start by recalling the statement of Theorem 1: Let T, V be positive real numbers and $R_1 := R_1(T, V)$ be the number of gaps of length at least V between consecutive zeros of $\zeta(\frac{1}{2} + it)$ with ordinate between 0 and T. Then the following estimates hold uniformly in T and V

$$R_1 \ll TV^{-2} \log T, \tag{2.12}$$

and

$$R_1 \ll TV^{-3} \log^5 T. \tag{2.13}$$

We first remark that when $V \ll \log T$, the trivial estimate $R_1 \leq TV^{-1}$ is better than (2.12) and (2.13). Next, by the result of Balasubramanian [1] mentioned in Chapter 1, $R_1 = 0$ whenever $V \gg T^{1/6+\epsilon}$. Hence the theorem trivially holds in the region $V \gg T^{1/4}$. For the rest of this chapter, we shall assume that V lies in the range $\log T \ll V \ll T^{1/4}$.

We shall also restrict ourselves to proving the theorem for consecutive zeros of

$\zeta(\frac{1}{2}+it)$ with ordinates lying in $[T, 2T]$. This is clearly equivalent to the statement of theorem by a dyadic division of the original interval.

For ease of exposition, we shall divide the proof of Theorem 1 into three steps. In the first step we shall consider the smoothed integral of Hardy's function over a short interval of size $V/2$. We shall estimate it from below and above, whenever the interval is part of a gap between consecutive zeros of $\zeta(\frac{1}{2}+it)$. This will give a local estimate valid in these gaps. In the second step, we shall average this local estimate over the entire interval $[T, 2T]$ after using Hölder's inequality, to get a global estimate connecting R_1 with the moments of a certain zeta sum. The last step will be the crucial application of arithmetic methods to estimate the second and fourth moments and complete the proof of the theorem.

Let $L = 4\log T$, $G = VL^{-1/2}$ be a parameter which is slightly smaller than V.

Step I: Local Estimate

Let $\psi(u) = \exp(-u^2G^{-2})$ be a smoothing function. Now, consider the integral

$$I_1(t) = \int_{-V/4}^{V/4} Z(t+u)\psi(u)du. \tag{2.14}$$

Let τ_1 and τ_2 be the ordinates of consecutive zeros of $\zeta(s)$ on the critical line such that $\tau_1, \tau_2 \in [T, 2T]$ and $\tau_2 - \tau_1 \geq V$. Note τ_1 and τ_2 are consecutive real zeros of $Z(u)$ and hence for

$$t \in \left(\tau_1 + \tfrac{V}{4}, \tau_2 - \tfrac{V}{4}\right), \tag{2.15}$$

we deduce that the integrand in (2.14) does not change sign.

Thus, for the range (2.15), we bound $I_1(t)$ from below as

$$|I_1(t)| = \left| \int_{-V/4}^{V/4} Z(t+u)\psi(u)du \right| = \int_{-V/4}^{V/4} |Z(t+u)|\psi(u)du$$
$$\gg \int_{-G}^{G} |Z(t+u)|du \gg G. \tag{2.16}$$

The last step is by Ramachandra's theorem for general Dirichlet series (Lemma 2.3). Alternately, since $|Z(t)| = |\zeta(\frac{1}{2}+it)|$, we can use Ramachandra's lower bound for the first moment of $|\zeta(\frac{1}{2}+it)|$ in short intervals (see [43]) to get a sharper bound ($\gg GL^{1/4}$). We have also used the fact $\psi(u) \geq e^{-1}$ when $|u| \leq G$.

For the upper bound estimation of $I_1(t)$, we first use the approximate functional equation (Lemma 2.4) to replace Hardy's function in the integrand by a zeta sum to get

$$I_1(t) \ll \left| \int_{-V/4}^{V/4} e^{i\theta(t+u)} \sum_{n \leq P(t+u)} n^{-\frac{1}{2}-i(t+u)}\psi(u)du \right| + o(G), \tag{2.17}$$

where $P(t) = \sqrt{t/(2\pi)}$.

Let $X = (2T)^{1/2}LV^{-1}$. Now we use the properties of the smoothing function (Lemma 2.5) to remove the terms $n < P(t+u) - X$ from the zeta sum in (2.17) and estimate their contribution as

$$O\left(GT^{-1/2} \sum_{n<P(2T)-X} n^{-\frac{1}{2}} \right) = O(GT^{-1/2+1/4}) = o(G).$$

Thus the terms $n < P(t+u) - X$ in (2.17) can be absorbed into the error term, and we get

$$I_1(t) \ll \int_{-V/4}^{V/4} |\Sigma(t+u)|\, \psi(u)du + o(G), \tag{2.18}$$

where

$$(2.19) \qquad \Sigma(v) = \sum_{P(v)-X \le n \le P(v)} n^{-\frac{1}{2}-iv}.$$

Hence, for t in the range (2.15), we can combine the bounds (2.16) and (2.18) to conclude

$$(2.20) \qquad G \ll \int_{-V/4}^{V/4} |\Sigma(t+u)| \, \psi(u) du.$$

This is the local estimate valid in the large gaps between consecutive zeros of $\zeta(\frac{1}{2}+it)$ on the critical line.

Step II: Global Estimate

Let $\mathcal{T}$ be the set of $t \in [T + \frac{V}{4}, 2T - \frac{V}{4}]$ for which (2.20) holds. Recall that R_1 is the number of pairs of consecutive zeros of $\zeta(s)$ on the critical line with ordinates $\tau_1, \tau_2 \in [T, 2T]$, such that $\tau_2 - \tau_1 \ge V$.

From Step I we see that the range (2.15) is contained in $\mathcal{T}$, for each such pair τ_1 and τ_2. Thus $R_1 V \ll \mu(T)$ where μ is the usual Lebesgue measure.

Now, applying Hölder's inequality (Lemma 2.1) to (2.20), we get for all $t \in \mathcal{T}$ and k any positive integer

$$G^{2k} \ll \left(\int_{-V/4}^{V/4} |\Sigma(t+u)|^{2k} \psi(u) du \right) \left(\int_{-V/4}^{V/4} \psi(u) du \right)^{2k-1}.$$

Bounding the second integral by $\int_{\mathbb{R}} \psi(u) du \ll G$, we get

$$(2.21) \qquad G \ll \int_{-V/4}^{V/4} |\Sigma(t+u)|^{2k} \psi(u) du,$$

for all $t \in \mathcal{T}$.

Integrating (2.21) over $\mathcal{T}$, we get the estimate

$$\begin{aligned} R_1 V G \ll \mu(\mathcal{T}) G &\ll \int_{\mathcal{T}} \int_{-V/4}^{V/4} |\Sigma(t+u)|^{2k} \psi(u)\, du\, dt \\ &\ll \int_{T+V/4}^{2T-V/4} \int_{-V/4}^{V/4} |\Sigma(t+u)|^{2k} \psi(u)\, du\, dt \\ &\ll \int_{T}^{2T} |\Sigma(s)|^{2k} \left(\int_{-V/4}^{V/4} \psi(u) du \right) ds. \end{aligned}$$

Bounding the inner integral by G as before, we finally obtain

$$R_1 V \ll \int_{T}^{2T} |\Sigma(t)|^{2k}\, dt, \tag{2.22}$$

for k any positive integer. We call this the global estimate, since we are counting the large gaps between consecutive zeros of $\zeta(\frac{1}{2}+it)$ in the entire interval $[T, 2T]$.

Step III: Arithmetical Methods

We shall first analyze (2.22) with $k = 1$, which leads to an estimation of the second moment of the integrand. Expanding the square of the modulus of $\Sigma(t)$ (see (2.19)) in (2.22), we get

$$R_1 V \ll \sum_{n_1, n_2} \frac{1}{(n_1 n_2)^{1/2}} \int_{I(n_1,n_2)} \exp\left(it \log\left(\frac{n_1}{n_2} \right) \right) dt, \tag{2.23}$$

where the sum ranges over $P_1 < n_1, n_2 < P_2$ with $P_1 = P(T) - X$ and $P_2 = P(2T)$. Here $I(n_1, n_2) = \bigcap_{i=1,2} [2\pi n_i^2, 2\pi(n_i + X)^2] \bigcap [T, 2T]$ is an interval, which is non-empty only when $|n_1 - n_2| \le X$.

Now, we subdivide the sum in (2.23) into S_1 and S_2 corresponding to the ranges $n_1 = n_2$ and $n_1 \ne n_2$, respectively. The reason for this subdivision is that, in the first case (the "diagonal" case) the integral in (2.23) will be trivial, while in the other "nondiagonal" case, we will use the first derivative estimate Lemma 2.2 for the integral.

For estimating S_1, we note that $|I(n_1, n_1)| \ll \min(T, n_1 X) \ll P_2 X \ll TLV^{-1}$ which will be the trivial estimate for the integral. Noting that $n_1 \asymp T^{1/2}$ and the number of terms is at most $P_2 \ll T^{1/2}$, we get

$$S_1 \ll T^{1/2} \frac{1}{T^{1/2}} \frac{TL}{V} \ll \frac{TL}{V}. \tag{2.24}$$

As for S_2, we use Lemma 2.2 and obtain that the integral in (2.23) to be $\ll 1/|\log(n_1/n_2)| \ll (n_1 + n_2)/|n_1 - n_2|$. Now, as $n_1, n_2 \asymp T^{1/2}$ and $|n_2 - n_1| \le X$, we get

$$\begin{aligned} S_2 &\ll \frac{1}{T^{1/2}} \sum_{1 \le r \le X} \sum_{\substack{n_1 \\ n_2 = n_1 + r}} \frac{n_1 + n_2}{r} \ll \sum_{1 \le r \le X} \frac{1}{r} \sum_{n_1} 1 \\ &\ll \sum_{r \le X} \frac{1}{r} P_2 \ll T^{\frac{1}{2}} L. \end{aligned} \tag{2.25}$$

Using (2.24) and (2.25) in (2.23), we get

$$R_1 \ll \frac{TL}{V^2},$$

which is the first part (2.12) of our theorem.

We next turn to (2.22) with the value $k = 2$. Expanding the 4^{th} power of $|\Sigma(t)|$ in the integrand we get

$$R_1 V \ll \sum_{n_1, n_2, n_3, n_4} \frac{1}{(n_1 n_2 n_3 n_4)^{1/2}} \int_{I(\mathbf{n})} \exp\left(it \log\left(\frac{n_1 n_2}{n_3 n_4}\right)\right) dt, \tag{2.26}$$

where the sum now ranges over $P_1 < n_i < P_2$ for $i = 1, \dots 4$ and $I(\mathbf{n}) = I(n_1, n_2, n_3, n_4) = \bigcap_{i=1}^{4}[2\pi n_i^2, 2\pi(n_i + X)^2] \bigcap [T, 2T]$ is an interval. Here $I(\mathbf{n})$ is non-empty only when $|n_1 - n_i| \leq X$ for $i = 2, 3, 4$.

Now, denoting $r(\mathbf{n}) = n_1 + n_2 - n_3 - n_4$, we subdivide the sum in (2.26) into S_3 and S_4 corresponding to the ranges $|r(\mathbf{n})| \geq 2X^2/P_1$ and $|r(\mathbf{n})| < 2X^2/P_1$, respectively. In both cases, we will use Lemma 2.2 to bound the integral in (2.26) depending only on $s(\mathbf{n})$ where

$$s(\mathbf{n}) = n_1 n_2 - n_3 n_4. \tag{2.27}$$

The motivation behind this subdivision is that, in order to count the number of solutions n_i to $s(\mathbf{n}) = s$ for a fixed s, the first range will reduce this problem to solving a linear equation, while in the second range we will tackle the more difficult problem of solving a quadratic equation by invoking Lemma 2.6.

We first make the observation that

$$s(\mathbf{n}) = n_1 r(\mathbf{n}) - (n_3 - n_1)(n_4 - n_1). \tag{2.28}$$

For the estimation of S_3, note that $|n_1 r(\mathbf{n})| \geq 2X^2$ while $|(n_3 - n_1)(n_4 - n_1)| \leq X^2$. Hence the identity (2.28) gives us $|s(\mathbf{n})| \gg P_1 |r(\mathbf{n})|$.

Using Lemma 2.2, the integral in (2.26) can hence be bounded in this range by

$$\frac{1}{|\log\big((n_1 n_2)/(n_3 n_4)\big)|} \ll \frac{n_1 n_2 + n_3 n_4}{|s(\mathbf{n})|} \ll \frac{T^{1/2}}{|r(\mathbf{n})|}.$$

Next, we claim that the number of solutions of $r(\mathbf{n}) = r$ for any fixed r is $O(T^{1/2} X^2)$. This is because, we can first choose n_1 arbitrarily with $O(T^{1/2})$ choices, and then depending on n_1 we choose n_2 and n_3 arbitrarily with $O(X^2)$ choices. Finally n_4

has at most one choice to get $r(\mathbf{n}) = r$.

Thus we get

$$S_3 \ll \frac{1}{T^{(1/2)2}} \sum_{\frac{X^2}{P_1} \ll r \ll X} \sum_{\substack{n_i \\ r(\mathbf{n})=r}} \frac{T^{1/2}}{r} \ll \frac{1}{T^{1/2}} \sum_{r} \frac{1}{r} T^{1/2} X^2 \ll \frac{TL^3}{V^2}. \tag{2.29}$$

For the estimation of S_4, we first estimate the number of solutions, $H(s)$, to $s(\mathbf{n}) = s$ in this range. Observe that $H(s)$ is also the number of solutions to (2.28) for the values $n_1, r(\mathbf{n}), n_3 - n_1, n_4 - n_1$ with appropriate restrictions. This is because there is a bijection between the tuples (n_1, n_2, n_3, n_4) and $(n_1, r(\mathbf{n}), n_3 - n_1, n_4 - n_1)$.

The number of solutions $(n_3 - n_1,\ n_4 - n_1)$ to the equation $m = (n_3 - n_1)(n_4 - n_1)$ is bounded trivially by $d(m)$ (the divisor function). Similarly the number of solutions to $m + s = n_1 r(\mathbf{n})$ is bounded by $d(m + s)$.

Noting that, in this range, $n_1 r(\mathbf{n}), (n_3 - n_1)(n_4 - n_1), s(\mathbf{n}) \ll X^2$, we get

$$H(s) \ll \sum_{m \ll X^2} d(m) d(m+s) \ll \begin{cases} \sigma_{-1}(s) X^2 L^2, & s \neq 0 \\ X^2 L^3, & s = 0 \end{cases} \tag{2.30}$$

by using Lemma 2.6(a).

Now we subdivide S_4 further into the ranges $s(\mathbf{n}) \neq 0$ and $s(\mathbf{n}) = 0$ and call the sums $S_{4,1}$ and $S_{4,2}$ respectively. These correspond to the "nondiagonal" and "diagonal" cases, respectively. The integral in (2.26) will be bounded by using Lemma 2.2 in the first case and trivially in the other.

For the estimation of $S_{4,1}$, we use Lemma 2.2 for the integral in (2.26) to get

$$\begin{aligned} S_{4,1} &\ll \frac{1}{T^{(1/2)2}} \sum_{1\le s\ll X^2} H(s)\frac{n_1n_2+n_3n_4}{s} \ll \sum_s \frac{H(s)}{s} \\ &\ll X^2L^2 \sum_{s\ll X^2} \frac{\sigma_{-1}(s)}{s} \ll \frac{TL^5}{V^2}, \end{aligned} \tag{2.31}$$

by using Lemma 2.6(b).

As for $S_{4,2}$, we take the trivial estimate $O(T)$ for the integral in (2.26) and use (2.30) to get

$$S_{4,2} \ll \frac{1}{T^{(1/2)2}} H(0)T \ll X^2L^3 \ll \frac{TL^5}{V^2}. \tag{2.32}$$

Using (2.29),(2.31) and (2.32) in (2.26), we get

$$R_1 \ll TV^{-3}\log^5 T.$$

This completes the proof of the second part (2.13) of the theorem. □

In the next chapter, we shall consider the analogue of this question for the Epstein zeta function.

Chapter 3

Large gaps for the Epstein zeta function

The Epstein zeta-function $\zeta_Q(s)$ associated to a binary, positive definite, integral quadratic form $Q(x, y) = ax^2 + bxy + cy^2$ is defined by the Dirichlet series

$$\zeta_Q(s) = \sum_{(x,y)\neq(0,0)} \frac{1}{Q(x,y)^s} = \sum_{n=1}^{\infty} \frac{r_Q(n)}{n^s}, \tag{3.1}$$

in the region $\sigma > 1$, where $r_Q(n)$ is the number of integer solutions to the equation $Q(x, y) = n$. It is well known that $\zeta_Q(s)$ can be analytically continued to the entire complex plane except for $s = 1$ where it has a simple pole with residue $2\pi/\sqrt{\Delta}$ where $\Delta = 4ac - b^2 > 0$. Note that the discriminant of $Q(x, y)$ is $-\Delta = b^2 - 4ac < 0$.

It satisfies the functional equation

$$\left(\frac{\sqrt{\Delta}}{2\pi}\right)^s \Gamma(s)\zeta_Q(s) = \left(\frac{\sqrt{\Delta}}{2\pi}\right)^{1-s} \Gamma(1-s)\zeta_Q(1-s). \tag{3.2}$$

We will use (3.2) in the form $\zeta_Q(s) = \gamma(s)\zeta_Q(1-s)$ where $\gamma(s)$ is defined by

$$\gamma(s) = \left(\frac{\sqrt{\triangle}}{2\pi}\right)^{1-2s} \frac{\Gamma(1-s)}{\Gamma(s)}. \tag{3.3}$$

We next define the function

$$W(t) = \gamma(\tfrac{1}{2} + it)^{-1/2} \zeta_Q(\tfrac{1}{2} + it). \tag{3.4}$$

The function $W(t)$ will play the same role that Hardy's function $Z(t)$ (see (2.3)) does in Chapter 2. The functional equation for $\zeta_Q(s)$ implies that for real values of t, we have $W(t)$ is real and $|W(t)| = |\zeta_Q(\frac{1}{2} + it)|$, analogous to the properties of $Z(t)$. Thus the zeros of $\zeta_Q(s)$ on the critical line correspond to the real zeros of $W(t)$.

We suppose throughout the rest of chapter that Δ is not a square, so that $\sqrt{\triangle}$ is irrational; other cases like, for e.g., $\triangle = 4$ related to the form $Q(x,y) = x^2 + y^2$, are either easier or well-known.

From the classical theory of binary quadratic forms, it is well known that, for any $\epsilon > 0$, $r_Q(n) \ll n^{\epsilon}$ for sufficiently large n depending on ϵ. This is the Ramanujan hypothesis for $\zeta_Q(s)$ which will be used extensively in this chapter.

Unlike the case of $\zeta(s)$ or $\zeta_K(s)$, the Epstein zeta function does not have an Euler product in general and does not satisfy the analogue of the Riemann hypothesis, in general.

The next section will contain several supplementary results and the proof of Theorem 2 will be presented in the last section.

3.1 Lemmas

The first result in this section is a generalization of Lemma 2.2 (see Titchmarsh [50], Lemma 4.3).

Lemma 3.1. *Let $f(x)$ and $g(x)$ be real functions, $f'(x)/g(x)$ monotonic with minimum modulus $m = min_{[a,b]}|f'(x)/g(x)| > 0$. Then*

$$\left|\int_a^b g(x)e^{if(x)}dx\right| \leq \frac{4}{m} \tag{3.5}$$

Next, we state an approximate functional equation for $\zeta_Q(\frac{1}{2}+it)$ which is a direct adaptation of Lemma 3 in [22] (with $X = t^3$).

Lemma 3.2. *For $t \geq 2$, we have*

$$\begin{aligned}\zeta_Q(\tfrac{1}{2}+it) = \sum_{n\leq t^3} r_Q(n)n^{-\frac{1}{2}-it} + \frac{1}{\log 2}\sum_{t^3<n\leq 2t^3} r_Q(n)\log\left(\tfrac{2t^3}{n}\right)n^{-\frac{1}{2}-it}\\ + O(t^{-1/2}).\end{aligned} \tag{3.6}$$

The next lemma is an analogue of Lemma 2.5 and is a statement about the decay properties of the Gaussian smoothing function $\psi(u) = \exp(-u^2G^{-2})$ that we encountered in Chapter 2.

Lemma 3.3. *Let t be sufficiently large, $t^{2\epsilon} \ll V \ll t^{1/2-\epsilon}$, $L = A\log t$, $G = VL^{-1/2}$, $Y = t^{1+\epsilon}V^{-1}$. Define the function $P_1(u) = u\sqrt{\triangle}/(2\pi)$. Then for any integer n with $|n - P_1(t)| > Y$, we have*

$$\int_{-V/4}^{V/4} \gamma(1/2+i(t+u))^{-\frac{1}{2}}n^{-i(t+u)}\psi(u)du \ll Gt^{-A/16}. \tag{3.7}$$

Proof. Using Stirling's formula (2.6), we get

$$\gamma(\sigma+it) = P_1(t)^{1-2\sigma-2it}e^{-\pi i(\sigma-1/2)}e^{2it}(1+O(1/t)).$$

For the proof of (3.7), we first consider the case $n > P_1(t) + Y$. We view the integral to be estimated as a complex integral over the rectangular contour with vertices $\pm V/4$, $\pm V/4 - iG$.

On the vertical sides, where $u = \pm V/4 - iw$ with $w \in [0, G]$, we estimate the first two factors in the integrand as follows

$$\begin{aligned}\gamma(\tfrac{1}{2} + i(t+u))^{-\frac{1}{2}} n^{-i(t+u)} &\ll \left(\frac{P_1(t \pm \frac{V}{4})}{n}\right)^w \\ &\ll \left(\frac{P_1(t) \pm \frac{\sqrt{\Delta}}{8\pi} V}{n}\right)^w < 1,\end{aligned}$$

The last inequality follows because of the condition on n and the fact that $V = o(Y)$ by the condition on V. As for the other factor, we have

$$e^{-\frac{u^2}{G^2}} \ll e^{-\frac{(V/4)^2 - w^2}{G^2}} \ll e^{-A \log t/16}.$$

Hence the value of the integral on the vertical sides becomes $O(GT^{-A/16})$.

On the horizontal side in the lower half plane, let $u = v - iG$ with $v \in [-\frac{V}{4}, \frac{V}{4}]$. Here $e^{-(u/G)^2}$ is bounded and the other factors are estimated as

$$\begin{aligned}\gamma(\tfrac{1}{2} + i(t+u))^{-\frac{1}{2}} n^{-i(t+u)} &\ll \left(\frac{n}{P_1(t+v)}\right)^{-G} \ll \exp\left(-G \log\left(\frac{n}{P_1(t+v)}\right)\right) \\ &\ll \exp\left(-\frac{GY}{P_1(t)}\right) \ll \exp(-t^{\epsilon'}) \ll \exp(-A \log t/16),\end{aligned}$$

for any ϵ' such that $0 < \epsilon' < \epsilon$. Hence the value of the integral on the horizontal side is also estimated as $O(GT^{-A/16})$ and this completes the proof of this case.

For $n < P_1(t) - Y$, we take the rectangular contour with vertices $\pm V/4$, $\pm V/4 + iG$ in the upper half plane and argue in a similar way to estimate the value of the integral to be $O(GT^{-A/16})$. □

The following lemma is a transformation formula which is a special case of a more general formula due to Jutila and Srinivas ([21] Lemma 3.2).

Let $\chi(s)$ be as defined in (2.2).

Lemma 3.4. *Let t be sufficiently large and $t^{1/2+\epsilon} \ll Y \ll t^{1-\epsilon}$. Let $r = h/k$ a positive rational number with $(h,k) = 1$, such that $|\sqrt{\Delta} - k/h| \asymp h^{-2}$ and $h, k \asymp \sqrt{t/Y}$. Let*

$$M_j = \frac{t\sqrt{\Delta}}{2\pi} + (-1)^j 2Y, \qquad m_j = \left| M_j - \frac{t}{2\pi r} \right|$$
$$M_j' = \frac{t\sqrt{\Delta}}{2\pi} + (-1)^j Y, \qquad m_j' = \left| M_j' - \frac{t}{2\pi r} \right|$$

for $j = 1, 2$. For a constant $\triangle_0$, define

$$n_j = n_j(t) = \triangle_0 h^2 {m_j}^2 {M_j}^{-1}, \quad n_j' = n_j'(t) = \triangle_0 h^2 (m_j')^2 (M_j')^{-1}. \tag{3.8}$$

Then there exists a certain sufficiently smooth weight function $\eta(x)$ with support $[M_1, M_2]$ satisfying $\eta(x) = 1$ for $x \in [M_1', M_2']$, and functions $w_j(x)$ piecewise continuous and bounded in the interval $[0, n_j]$, such that

$$\sum_{n=1}^{\infty} \eta(n) r_Q(n) n^{-1/2-it} = C_0 (hkt)^{-\frac{1}{4}} r^{it} \chi\left(\tfrac{1}{2} + it\right) \sum_{j=1}^{2} (-1)^j \sum_{n<n_j} w_j(n) \rho(n) n^{-\frac{1}{4}}$$
$$\times \exp(2\pi i C_1 n) \left(1 + C_2 \frac{n}{t}\right)^{-\frac{1}{4}} \exp\left(i(-1)^{j-1} \left(2t\phi(C_2 \frac{n}{t}) + \frac{\pi}{4}\right)\right)$$
$$+ O(t^{-\epsilon/3}), \tag{3.9}$$

where C_0 is a constant, $C_1 = \bar{h}\bar{\triangle}_0/k - 1/(2hk\triangle_0)$, $C_2 = \pi/(2hk\triangle_0)$ and

$$\phi(x) = \operatorname{arcsinh}(x^{1/2}) + (x + x^2)^{1/2}. \tag{3.10}$$

Here $\rho(n) = \rho(n, Q, h/k)$ is a certain arithmetical function defined in terms of a *binary, positive definite, integral quadratic form $Q^*(x,y)$ depending on Q and k, such that $\rho(n) \ll r_{Q^*}(n)$.*

Proof. We first note that $M_j, M_j' \asymp t$ and

$$m_j = \left|(-1)^j \frac{t}{2\pi}\left(\sqrt{\Delta} - \frac{1}{r}\right) + 2Y\right| \asymp \max(Th^{-2}, Y) \asymp Y. \tag{3.11}$$

Similarly, we get $m_j' \asymp Y$. Thus the conditions

$$M_1 < \tfrac{t}{2\pi r} < M_2, \quad m_1 \asymp m_2, \quad m_j \asymp m_j',$$
$$k \ll {M_1}^{1/2-\epsilon}, \quad t^\epsilon \max(t^{1/2}r^{-1}, hk) \ll m_1 \ll {M_1}^{1-\epsilon},$$

are satisfied because of the condition on Y. Next, for the integer J specified by the Lemma 3.2 in [21], U is given by $JU = \min(m_1', m_2')$. Hence $U \asymp Y$ and satisifies the condition $U \gg r^{-1}t^{1/2+\epsilon}$.

Thus all conditions of Lemma 3.2 in [21] are satisfied and we get the transformation formula as required, except that the first term of [21]

$$2\pi\triangle^{-1/2}k^{-2}G_Q(k,-h)r^{-1/2} \ll k^{-1} \ll (Y/t)^{1/2} \ll t^{-\epsilon/2},$$

goes into the error term in (3.9). Here we have used $|G_Q(k,-h)| \leq (\triangle, k)k \ll k$ (see [48], Lemma 1).

The error term of [21] is $h^2k^{-1}{m_1}^{1/2}t^{-3/2}U \log t \ll (Y/t)^{1/2} \ll t^{-\epsilon/2}\log t$ and hence this is also absorbed into the error term in (3.9). $\square$

3.2 Proof of Theorem 2

We start by recalling the statement of Theorem 2: Let Q be a binary, positive definite, integral, quadratic form with discriminant $-\triangle$ such that $\sqrt{\triangle}$ is irrational. Let $\epsilon > 0$ be sufficiently small, T, V be positive real numbers and $R_2 := R_2(T, V)$ denote the number of gaps of length at least V between consecutive zeros of $\zeta_Q(\frac{1}{2}+it)$ with ordinates in the interval $[0, T]$. Then

$$R_2 \ll T^{1+2\epsilon}V^{-2}, \tag{3.12}$$

where the constant in $\ll$ may depend only on ϵ.

We remark that the dependence in ϵ in (3.12) has been chosen in order to simplify the calculations that appear during the course of the proof.

We note that when $V \ll T^{2\epsilon}$, the trivial estimate $R_2 \leq TV^{-1}$ is better than (3.12). Next, by the result of Jutila and Srinivas [21] mentioned in Chapter 1, we have $R_2 = 0$ whenever $V \gg T^{5/11+\epsilon}$. Hence the theorem trivially holds in the region $V \gg T^{1/2-\epsilon}$. For the rest of this chapter, we shall therefore assume that V lies in the range $T^{2\epsilon} \ll V \ll T^{1/2-\epsilon}$.

We shall also restrict ourselves to proving the result for consecutive zeros of $\zeta_Q(\frac{1}{2}+it)$ with ordinates lying in $[T, 2T]$ because it clearly implies (3.12).

As in Chapter 2, we shall divide the proof of Theorem 2 into three steps. In the first step we shall consider the smoothed integral of $W(t)$ over a short interval. We shall estimate it from below and above, whenever the interval is part of a gap between consecutive zeros, to get a local estimate valid in these gaps. In the second step, we shall average this local estimate over the entire interval $[T, 2T]$, to get a global estimate connecting R_2 with the moments of an exponential sum. The last step will be the application of arithmetic methods to estimate the second moment and complete the proof of the theorem.

For the rest of this chapter, we shall denote $L = A \log T$, $G = VL^{-1/2}$ and $\psi(u) = \exp(-u^2G^{-2})$. Here $A > 0$ is a large constant to be chosen later.

Step I: Local Estimate

We start by considering the integral

$$I_2(t) = \int_{-V/4}^{V/4} W(t+u)\psi(u)du. \tag{3.13}$$

Let τ_1 and τ_2 be the ordinates of consecutive zeros of $\zeta_Q(s)$ on the critical line such that $\tau_1, \tau_2 \in [T, 2T]$ and $\tau_2 - \tau_1 \geq V$. Note that τ_1 and τ_2 are consecutive real zeros of $W(u)$ and hence for

$$t \in \left(\tau_1 + \tfrac{V}{4}, \tau_2 - \tfrac{V}{4}\right), \tag{3.14}$$

it follows that the integrand in (3.13) does not change sign. Thus, for the range (3.14) we bound $I_2(t)$ from below as follows

$$\begin{aligned} |I_2(t)| &= \left| \int_{-V/4}^{V/4} W(t+u)\psi(u)du \right| = \int_{-V/4}^{V/4} |W(t+u)|\psi(u)du \\ &\gg \int_{-G}^{G} |W(t+u)|du \gg G. \end{aligned} \tag{3.15}$$

The last step follows from Ramachandra's estimate for general Dirichlet series (see Lemma 2.3). We have also used the fact that $\psi(u) \geq e^{-1}$ when $|u| \leq G$.

Now we shall focus on the upper bound estimation of $I_2(t)$. First, we define $P_1(u) = u\sqrt{\Delta}/(2\pi)$ and the parameter Y by

$$VY = T^{1+\epsilon}. \tag{3.16}$$

Let $\eta(x)$ be a bounded function (to be chosen later) with the property

$$\eta(x) = \begin{cases} 1 & \text{if } x \in [P_1(t) - Y, P_1(t) + Y] \\ 0 & \text{if } |x - P_1(t)| > 2Y \end{cases} \tag{3.17}$$

Now we rewrite Lemma 3.2 by using the function $\eta(x)$ as follows

$$\begin{aligned} \zeta_Q\left(\tfrac{1}{2} + it\right) &= \sum_{n \le 2t^3} r_Q(n)\eta(n) n^{-\frac{1}{2}-it} + \sum_{n \le 2t^3} r_Q(n)(1 - \eta(n))c(t, n) n^{-\frac{1}{2}-it} \\ &\quad + O(t^{-1/2}). \end{aligned} \tag{3.18}$$

where $c(t, n) = 1$ if $n \le t^3$ and $c(t, n) = \log(2t^3/n)/\log 2$ otherwise. We have used the fact that $\eta(n) = 0$ for $t^3 < n$ which follows from (3.17).

Recall that $W(t) = \gamma(\frac{1}{2} + it)^{-\frac{1}{2}}\zeta_Q(\frac{1}{2} + it)$. Using this expression in the integrand of $I_2(t)$ and replacing $\zeta_Q(\frac{1}{2} + it)$ by (3.18), we get

$$\begin{aligned} I_2(t) &= \int_{-V/4}^{V/4} \gamma(\tfrac{1}{2} + i(t+u))^{-\frac{1}{2}} \Bigg(\sum_{n \le 2t^3} \eta(n) r_Q(n) n^{-\frac{1}{2}-i(t+u)} \Bigg) \psi(u) du \\ &\quad + \sum_{n \le 2t^3} (1-\eta(n)) r_Q(n) n^{-\frac{1}{2}} c(t, n) \int_{-V/4}^{V/4} \gamma(\tfrac{1}{2} + i(t+u))^{-\frac{1}{2}} n^{-i(t+u)} \psi(u) du \\ &\quad + o(G). \end{aligned} \tag{3.19}$$

Now, we note (3.17) implies that $1 - \eta(n)$ is supported in the range $|n - P_1(t)| > Y$. Hence the conditions of Lemma 3.3 hold for the nonzero terms of second sum in (3.19) and the integral in (3.19) is $O(GT^{-A/16})$. Thus, using $r_Q(n) \ll n^\epsilon$ and $c(t, n), 1 - \eta(n) \ll 1$, the second sum in (3.19) can be estimated as

$$O\Bigg(\sum_{n \le 2t^3} n^{-1/2+\epsilon} GT^{-A/16} \Bigg) = O(GT^{3/2+3\epsilon-A/16}) = o(G),$$

if we choose A sufficiently large (for e.g., $A = 32$ should do).

Thus, by noting that $|\gamma(\frac{1}{2}+it)^{-\frac{1}{2}}|=1$ (coming from the functional equation (3.2)) and using the previous estimate, we arrive at the upper bound

$$I_2(t) \ll \int_{-V/4}^{V/4} \left| \sum_{n=1}^{\infty} \eta(n) r_Q(n) n^{-1/2-i(t+u)} \right| \psi(u) du + o(G). \tag{3.20}$$

By our assumption that $\sqrt{\Delta}$ is a quadratic irrational, we can choose $r = h/k$ with $(h,k)=1$, such that $|\sqrt{\Delta} - k/h| \asymp h^{-2}$ and $h, k \asymp \sqrt{T/Y}$. Besides, the assumption $T^{2\epsilon} \ll V \ll T^{1/2-\epsilon}$ gives $T^{1/2+2\epsilon} \ll Y \ll T^{1-\epsilon}$ because of the condition (3.16). Thus, the conditions of Lemma 3.4 are satisfied.

Now we choose η to be the function given by Lemma 3.4 and use the transformation formula (3.9) for the sum in (3.20) to obtain

$$I_2(t) \ll \int_{-V/4}^{V/4} |\Sigma_1(t+u)| \psi(u) du + o(G), \tag{3.21}$$

where $\Sigma_1(u)$ is as follows

$$\begin{aligned} \Sigma_1(u) = {} & (hku)^{-1/4} r^{iu} \chi\left(1/2+iu\right) \sum_{j=1}^{2} (-1)^j \sum_{n<n_j(u)} w_j(n) \rho(n) n^{-1/4} \\ & \times \exp(2\pi i C_1 n) \left(1 + C_2 n/u\right)^{-1/4} \exp\left(i(-1)^{j-1}\left(2u\phi(C_2 n/u) + \pi/4\right)\right), \end{aligned} \tag{3.22}$$

with $C_1 = \bar{h}\bar{\triangle}_0/k - 1/(2hk\triangle_0)$, and $C_2 = \pi/(2hk\triangle_0)$. Note that the error term $O(t^{-\epsilon/3})$ from Lemma 3.4 leads to a contribution of $O(GT^{-\epsilon/3})$ to (3.21) which can be absorbed into the error term.

Hence, for t in the range (3.14), we can combine the bounds (3.15) and (3.21) to conclude

$$G \ll \int_{-V/4}^{V/4} |\Sigma_1(t+u)|\, \psi(u) du. \tag{3.23}$$

Step II: Global Estimate

This step is an exact replica of Step II of the previous chapter. Hence, without repeating the details, we conclude that we can use the local estimate (3.23) for t in the range (3.14) to get the following global estimate over the entire interval $[T, 2T]$

$$R_2 V \ll \int_T^{2T} |\Sigma_1(t)|^{2k}\, dt, \tag{3.24}$$

where R_2 is the number of pairs of consecutive zeros of $\zeta_Q(s)$ on the critical line with ordinates $\tau_1, \tau_2 \in [T, 2T]$, such that $\tau_2 - \tau_1 \geq V$.

Step III: Arithmetical Methods

We shall focus on estimating the second moment of $\Sigma_1(t)$ over the interval $[T, 2T]$ for the rest of this chapter. In other words, we shall analyze (3.24) with $k = 1$. Expanding the square of the modulus of $\Sigma_1(t)$ (defined in (3.22)) in the integrand in (3.24) we get

$$\begin{aligned} R_2 V \ll\ & (hk)^{-1/2} T^{1/2} \sum_{j=1,2} \sum_{n<n_j(T_0)} |w_j(n)\rho(n)|^2 n^{-1/2} \\ & +\ (hkT)^{-1/2} \sum_{j=1,2} \sum_{\substack{m,n<n_j(T_0) \\ m\neq n}} |w_j(m)\rho(m)w_j(n)\rho(n)|(mn)^{-1/4} \\ & \times \int ((1 + C_2 m/u)^{-1/4} (1 + C_2 n/u)^{-1/4} \\ & \times \exp\left(i(-1)^{j-1} 2u\left(\phi\left(C_2 m/u\right) - \phi\left(C_2 n/u\right)\right)\right) du \\ =\ & S^{(1)} + S^{(2)}, \text{say.} \end{aligned} \tag{3.25}$$

In $S^{(2)}$, the integral will be over an appropriate subinterval of $[T, 2T]$ depending on m and n. We also assume that $n_j(T_0) = \max\{n_j(u) : u \in [T, 2T]\}$.

We note that by the definition (3.8) of $n_j(T_0)$ and the estimates $h \asymp \sqrt{T/Y}$, $M_j \asymp T$,

$m_j \asymp Y$ (from Lemma 3.4), we get

$$n_j = n_j(T_0) \asymp Y. \tag{3.26}$$

For estimating $S^{(1)}$, we will need an estimate on the mean-square of the coefficients $r_{Q^*}(n)$. Since $r_{Q^*}(n) \ll n^{\epsilon}$, we have the trivial estimate

$$\sum_{n \le x} r_{Q^*}^2(n) = O\left(x^{1+\epsilon}\right). \tag{3.27}$$

In fact, for certain quadratic forms much better results are known in this direction (see [42], [38]).

Now, since $\rho(n) \ll r_{Q^*}(n)$, and $w_j(n) \ll 1$ (Lemma 3.4), (3.27) implies by partial summation that the inner sum in $S^{(1)}$ is

$$\sum_{n<n_j} |w_j(n)\rho(n)|^2 n^{-1/2} \ll n_j^{1/2+\epsilon}, \tag{3.28}$$

where $n_j = n_j(T_0)$. Hence, using (3.26), we get the estimate

$$S^{(1)} \ll (hk)^{-1/2} T^{1/2} Y^{1/2+\epsilon} \ll Y^{1+\epsilon} \ll T^{1+\epsilon} Y^{\epsilon} V^{-1} \ll T^{1+2\epsilon} V^{-1}. \tag{3.29}$$

For estimating $S^{(2)}$, we use the first derivative estimate (Lemma 3.1) with $g(x) = ((1 + C_2 m/x)(1 + C_2 n/x))^{-1/4}$ and $f(x) = 2x(\phi(C_2 m/x) - \phi(C_2 n/x))$. From the definition (3.10) of ϕ we get

$$\begin{aligned} f'(x) &= 2 \operatorname{arcsinh}\left((C_2 m/x)^{1/2}\right) - 2 \operatorname{arcsinh}\left((C_2 n/x)^{1/2}\right) \\ &\quad + 4\left((C_2 m/x)\left(1 + C_2 m/x\right)\right)^{1/2} - 4\left((C_2 n/x)\left(1 + C_2 n/x\right)\right)^{1/2}. \end{aligned}$$

Therefore $f'(x)/g(x)$ is monotonic in the interval $[T, 2T]$, after replacing $f(x)$ by

$-f(x)$ if necessary. We also have

$$|f'(x)/g(x)| \gg (C_2/T)^{1/2}|\sqrt{m}-\sqrt{n}|,$$

in the interval $[T, 2T]$.

Hence, by Lemma 3.1, we get that the integral in $S^{(2)}$ (see (3.25)) is bounded by

$$(T/C_2)^{1/2}|\sqrt{m}-\sqrt{n}|^{-1} \ll (hkT)^{1/2}|\sqrt{m}-\sqrt{n}|^{-1},$$

because $C_2 \asymp 1/(hk)$ from the definition (3.22).

Using this estimate in (3.25) we get

$$\begin{aligned} S^{(2)} &\ll \sum_j \sum_{\substack{m,n<n_j \\ m\neq n}} |w_j(m)\rho(m)w_j(n)\rho(n)|(mn)^{-1/4}|\sqrt{m}-\sqrt{n}|^{-1} \\ &\ll \sum_j \sum_{n<m<n_j} |w_j(n)\rho(n)w_j(m)\rho(m)|n^{-1/4}m^{1/4}(m-n)^{-1} \\ &\ll \sum_j \sum_{h<n_j} h^{-1} \sum_{n<n_j-h} |w_j(n)\rho(n)w_j(n+h)\rho(n+h)|n^{-1/4}(n+h)^{1/4}, \end{aligned} \tag{3.30}$$

where $n_j = n_j(T_0)$ as before.

Using the Cauchy-Schwarz inequality, the inner sum over n in (3.30) is bounded by

$$\begin{aligned} &\left(\sum_{n<n_j} |w_j(n)\rho(n)|^2 n^{-1/2}\right)^{1/2} \left(\sum_{n+h<n_j} |w_j(n+h)\rho(n+h)|^2 (n+h)^{1/2}\right)^{1/2} \\ &\qquad \ll \left(n_j^{1/2+\epsilon} n_j^{3/2+\epsilon}\right)^{\frac{1}{2}} = n_j^{1+\epsilon}, \end{aligned}$$

by using the method of partial summation.

Combining this with $\sum_{h<n_j} h^{-1} \ll \log n_j$ in (3.30), we get

$$S^{(2)} \ll \sum_j n_j^{1+\epsilon} \log n_j \ll Y^{1+\epsilon} \log Y \ll T^{1+2\epsilon} V^{-1}. \tag{3.31}$$

Here, we have used the fact that $Y^\epsilon \log Y \ll T^\epsilon$.

Using (3.29) and (3.31) in (3.25), we get

$$R_2 \ll T^{1+2\epsilon} V^{-2}.$$

This completes the proof of the theorem. □

In the next chapter we shall discuss the problem of showing that there are infinitely many zeros on the critical line for functions in the Selberg class.

Chapter 4

Hardy's theorem for the Selberg class: $d = 2$

Let $F(s) \in \mathcal{S}$ be a function in the Selberg class. We shall summarize the properties of $F(s)$ that will be relevant to this chapter. These are derived from the axioms defining the Selberg class which we have stated in the first chapter.

$F(s)$ has an absolutely convergent Dirichlet series expansion

$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s},$$

in the half-plane $\operatorname{Re} s = \sigma > 1$. $F(s)$ has an analytic continuation to the entire complex plane, except at $s = 1$ where it has a possible pole of order $m \geq 0$. It satisfies a functional equation of the form

$$\Phi(s) = \omega \bar{\Phi}(1 - s), \tag{4.1}$$

where

$$\Phi(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s)$$

with $Q > 0, \lambda_j > 0, \operatorname{Re}\mu_j \geq 0$ and $|\omega| = 1$ and $\bar{\Phi}(s) = \overline{\Phi(\bar{s})}$. This functional equation can be rewritten in the form $F(s) = \Delta(s)\bar{F}(1-s)$ where

$$\Delta(s) = \omega Q^{1-2s} \prod_{j=1}^{r} \frac{\Gamma(\lambda_j(1-s) + \overline{\mu_j})}{\Gamma(\lambda_j s + \mu_j)}. \tag{4.2}$$

Next we define the function $Z_F(t)$, which is the analogue of Hardy's function $Z(t)$ (see 2.3), which played a central role in Chapter 2.

$$Z_F(t) = \Delta(\tfrac{1}{2} + it)^{-1/2} F(\tfrac{1}{2} + it). \tag{4.3}$$

In a similar way as for $Z(t)$, the functional equation (4.1) implies that for real values of t, we have $Z_F(t)$ is real and $|Z_F(t)| = |F(\frac{1}{2} + it)|$. Thus the zeros of $F(s)$ on the critical line correspond to the real zeros of $Z_F(t)$.

An Euler product expansion for $F(s)$ exists in the form

$$F(s) = \prod_p \left(1 + \frac{a(p)}{p^s} + \frac{a(p^2)}{p^{2s}} + \frac{a(p^3)}{p^{3s}} + \cdots\right)$$

valid for $\sigma > 1$. This means that the Dirichlet coefficients $a(n)$ are multiplicative. Multiplicativity of the coefficients will be crucial in the proof of Theorem 3.

The Ramanujan hypothesis for $F(s)$ holds in the form

$$a(n) = O(n^{\epsilon}), \tag{4.4}$$

for any $\epsilon > 0$ and all sufficently large n (depending on ϵ). This will also be used extensively in the proof of Theorem 3.

Lastly, we recall the definitions of invariants associated with $F(s)$ (see [23]): the degree $d = 2\sum_{j=1}^{r} \lambda_j$, the ξ-invariant $\xi = 2\sum_{j=1}^{r} (\mu_j - 1/2)$, the internal shift $\theta =$

$\operatorname{Im}\xi$ and the conductor $q = (2\pi)^d Q^2 \prod_{j=1}^r \lambda_j^{2\lambda_j}$. We note that though these invariants are defined based on the data in (4.1), they are independent of the particular form of the functional equation and depend only on $F(s)$.

We shall state several supplementary results in the next section and detail the proof of Theorem 3 in the last section.

4.1 Lemmas

We first state an asymptotic formula for $\Delta(s)$ that follows directly from Stirling's formula (see (2.6) of Chapter 2) applied to each Γ-factor in the definition (4.2).

Lemma 4.1. *The following asymptotic formula holds uniformly in any fixed vertical strip $-\infty < \alpha \le \sigma \le \beta < \infty$ as $t \to \infty$*

$$\Delta(s) = \omega_1 (q_* t^{d/2})^{1-2\sigma-2it} t^{-i\theta} e^{idt}(1 + O(1/t)), \tag{4.5}$$

where $\omega_1 = \omega e^{-\frac{\pi}{2}i(d/2+\operatorname{Re}\xi)+i\theta} \prod_{j=1}^r \lambda_j^{-2i \operatorname{Im}\mu_j}$, with $|\omega_1| = 1$, and the parameter $q_* = \sqrt{q}\,(2\pi)^{-d/2}$.

Thus, we have

$$\Delta^{-1/2}(s) = \omega_1 (q_* t^{d/2})^{\sigma-1/2+it} t^{i\theta/2} e^{-idt/2}(1 + O(1/t)), \tag{4.6}$$

where ω_1 is a constant with modulus 1.

Next, we state the following uniform convexity estimates.

Lemma 4.2. *For any $\epsilon > 0$ and t sufficiently large, the following growth estimates* *hold*

$$\Delta^{-1/2}(s)F(s) = \begin{cases} O(t^{d/4+\epsilon}) & 0 \le \sigma \le 1 \\ O(t^{(\sigma-1/2)d/2+\epsilon}) & 1 < \sigma \le 1+\delta \end{cases} \tag{4.7}$$

Proof. Note that

$$F(s) = O(t^{\epsilon}),$$

holds in the region $\sigma \ge 1$ because of the absolutely convergent Dirichlet series in $\sigma > 1$ and by continuity on the boundary when t is sufficiently large. This implies, by the functional equation $F(s) = \Delta(s)\bar{F}(1-s)$ and the estimate (4.5), that

$$F(s) = O(t^{(1-2\sigma)d/2+\epsilon}),$$

in the region $\sigma \le 0$. Using the well known Phragmen - Lindelöf convexity theorem for vertical strips and the above two estimates, we get

$$F(s) = O(t^{(1-\sigma)d/2+\epsilon})$$

in the region $0 \le \sigma \le 1$.

Combining these estimates for $F(s)$ with the estimate (4.6) for $\Delta^{-1/2}(s)$ proves the lemma. □

The following lemma is a version of the saddle point theorem for exponential integrals. This is an adaptation of Lemma 2.5 of Krätzel ([28], p.44).

Lemma 4.3. *Let $f \in C^3[a,b]$ such that $f'(t)$ is monotonic, $f'(c) = 0$ for $a < c < b$, $|f''(t)| \asymp \lambda_2$ and $0 < |f'''(t)| \ll \lambda_3$. Let $g \in C^2[a,b]$ with $|g(t)| \le G$, $|g'(t)| \le G_1$. Suppose that $f'^6(t) - 8f''(c)f''^2(t)(f(t)-f(c))^3$ and $g''(t)$ have only bounded number* *of zeros in* $[a, b]$*. Then,*

$$\int_a^b g(t)e^{if(t)}dt = c_1 \frac{g(c)}{\sqrt{f''(c)}} e^{if(c)} + O\left(\frac{G_1}{\lambda_2}\right) + O\left(G(b-a)\frac{\lambda_3^2}{\lambda_2^3}\right) + O\left(G\frac{\lambda_3}{\lambda_2^2}\right) + O\left(G \min\left(\frac{1}{|f'(a)|} + \frac{1}{|f'(b)|}, \frac{1}{\sqrt{\lambda_2}}\right)\right),$$

for a constant c_1 *depending only on the sign of* f''*.*

Proof. The statement is the same as Lemma 2.5 of [28], except that we have replaced the condition $g'(t)$ is monotonic with the condition that $g''(t)$ has a bounded number of zeros. In other words, $[a, b]$ can be subdivided into finitely many subintervals, in each of which $g'(t)$ is monotonic. The proof given in [28] goes through without any changes. □

The last lemma that we state is the an important result on exponential sums with multiplicative coefficients. This is Theorem 1 of Daboussi and Delange [9].

Lemma 4.4. *Let* f *be a multiplicative arithmetical function satisfying the condition* $\sum_{n\leq x} |f(n)|^2 = O(x)$*. Then, for every irrational* α*, we have*

$$\sum_{n\leq x} f(n)e^{2\pi i n\alpha} = o(x). \tag{4.8}$$

We remark that the estimate $o(x)$ in (4.8) can be improved to $O(x(\log\log x)^{-1/2})$ by a careful analysis of the method in [9]. In fact, under the additional condition that $|f(p)| \leq A$ for all primes p, Montgomery and Vaughan [32] improved the estimate to $O(x \log^{-1} x)$.

In passing, we remark that for α rational, the estimate (4.8) was established in [9] under the restriction that $|f(n)| \leq 1$ and some additional conditions. However, in the general case, the Dirichlet coefficients of functions in $\mathcal{S}$ do not satisfy these restrictions.

4.2 Proof of Theorem 3

We start by recalling the statement of Theorem 3: Let $F \in S$ be any function with degree $d = 2$, $\theta = 0$, $\sqrt{q}$ irrational and the Dirichlet coefficients $a(n)$ satisfying

$$\sum_{n \le x} |a(n)|^2 = O(x). \tag{4.9}$$

Then $F(s)$ has infinitely many zeros on the critical line.

We first recall that the zeros of $F(s)$ on the critical line correspond to the real zeros of $Z_F(t)$. Let us assume that $F(s)$ has only finitely many zeros on the critical line. This implies that $Z_F(t)$ has no zeros in the interval $[T, 2T]$ where T is sufficiently large.

Consider the integral

$$I_3 = \int_T^{2T} Z_F(t)dt. \tag{4.10}$$

The proof will proceed by using the assumption on $Z_F(t)$ to estimate $|I_3|$ from below and above, and obtain a contradiction. The methods will be analogous to Step I of Chapters 2 and 3. However, the main difference will be that it will not suffice to express the upper bound as an integral of a finite sum (see (2.18) and (3.21)). The saddle point method will be used to transform this integral into an explicit exponential sum and the crucial result of Daboussi and Delange (Lemma 4.4) will be applied to this sum to complete the upper bound estimation.

By our assumption on $Z_F(t)$, we can say that the integrand in (4.10) is of constant sign. Hence

$$|I_3| = \int_T^{2T} |Z_F(t)|dt = \int_T^{2T} |F(\tfrac{1}{2} + it)|dt.$$

Now, we invoke Ramachandra's theorem by taking $H = T$ in Lemma 2.3. Thus, we obtain the lower bound

$$|I_3| \gg T. \tag{4.11}$$

For the upper bound estimation, we remark that, unlike the case of $\zeta(s)$ and $\zeta_Q(s)$ in the previous chapters, there is no known approximate functional equation for a general function $F(s) \in \mathcal{S}$. Hence we will need to move the line of integration to the region where $F(s)$ has a Dirichlet series expansion.

Accordingly, we first write the integral I_3 as

$$I_3 = -i \int_{1/2+iT}^{1/2+i2T} \Delta^{-1/2}(s)F(s)ds. \tag{4.12}$$

Next, we move the line of integration to $\sigma = 1 + \delta$, (where $\delta > 0$ is a small positive constant less than 1/2) and apply Cauchy's theorem to the integral

$$\int \Delta^{-1/2}(s)F(s)ds,$$

along the rectangle with sides $\sigma = 1/2$, $\sigma = 1 + \delta$, $t = T$ and $t = 2T$.

By Cauchy's theorem, the integral (4.12) reduces to

$$\int_{T}^{2T} \Delta^{-1/2}(1 + \delta + it)F(1 + \delta + it)dt, \tag{4.13}$$

with an error $O(T^{1/2+\delta+\epsilon}) = o(T)$ coming from the horizontal lines, by using the growth estimate in Lemma 4.2 (with $d = 2$).

Expanding $F(1 + \delta + it)$ as a Dirchlet series and using the asymptotic formula (4.6) (with $d = 2$ and $\theta = 0$) in (4.13), we see that the expression (4.13) is a constant multiple of

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^{1+\delta}} \int_T^{2T} t^{1/2+\delta} \left(\frac{q_* t}{en}\right)^{it} \big(1+O(1/t)\big) dt. \tag{4.14}$$

The contribution of the $O-$term in (4.14) is trivially $O(T^{1/2+\delta}) = o(T)$.

Collecting all the above estimates, we obtain

$$I_3 \ll \sum_{n=1}^{\infty} \frac{a(n)}{n^{1+\delta}} \int_T^{2T} g(t) e^{if(t)} dt + o(T), \tag{4.15}$$

where $f(t) = t\log(\frac{q_* t}{en})$ and $g(t) = t^{1/2+\delta}$.

Now we subdivide the sum in (4.15) into sub-intervals

$$1 \le n \le q_* T - 1, \qquad q_* T + 1 \le n \le 2q_* T - 1, \qquad 2q_* T + 1 \le n,$$

and denote the corresponding sums over these ranges as $\Sigma_1, \Sigma_2, \Sigma_3$ respectively. The reason for this subdivision is that in the first and last ranges we will use the first derivative estimate (Lemma 3.1), while in the middle range we will use the saddle point theorem (Lemma 4.3).

Notice that there are atmost 4 integers in the range of (4.15) which have not been included in the above ranges. Noting that $n \asymp q_* T$ for these integers and using the Ramanujan hypothesis (4.4), their contribution to (4.15) is

$$O\bigg(\frac{n^{\epsilon}}{n^{1+\delta}} T^{1/2+\delta} T\bigg) = O\big(T^{1/2+\epsilon}\big) = o(T).$$

The sum Σ_1 is estimated by using Lemma 3.1. Note that here $f'(x) = \log(q_* t/n) \gg \log(q_* T/n)$ and $g(x) \asymp T^{1/2+\delta}$ and the condition on monotonicity of f'/g is satisfied by subdiving $[T, 2T]$ into two parts if necessary.

$$\Sigma_1 = O\left(T^{1/2+\delta} \sum \frac{|a(n)|}{n^{1+\delta} \log(q_* T/n)}\right). \tag{4.16}$$

We then further sub-divide the range of (4.16) into two sub-sums as follows

$$1 \leq n \leq q_* T/2, \qquad q_* T/2 < n \leq q_* T - 1,$$

and denote the corresponding sums by Σ_{11} and Σ_{12} respectively. This is done in order to simplify the estimation of the sum in (4.16), so that the log factor is bounded in the first range and $n^{1+\delta}$ is bounded in the other range.

For estimating Σ_{11}, we note that $\log(q_* T/n) \geq \log 2$ in this range and hence we get

$$\Sigma_{11} = O\left(T^{1/2+\delta} \sum \frac{|a(n)|}{n^{1+\delta}}\right) = O\left(T^{1/2+\delta}\right) = o(T).$$

For estimating Σ_{12}, we use the inequality $\log(q_* T/n) \geq (q_* T - n)/q_* T$, $n \asymp q_* T$ and (4.4) to obtain

$$\Sigma_{12} = O\left(T^{1/2+\delta} \sum \frac{|a(n)|}{n^{1+\delta}} \frac{q_* T}{q_* T - n}\right) = O\left(T^{1/2+\epsilon} \sum \frac{1}{q_* T - n}\right). \tag{4.17}$$

Observe that the last sum in (4.17) is

$$\sum (q_* T - n)^{-1} = \sum_{1 \leq k < q_* T/2} (k + f)^{-1} = O(\log T),$$

where $f = q_* T - \lfloor q_* T \rfloor \geq 0$ is the fractional part of $q_* T$. Using this in (4.17) gives

$$\Sigma_{12} = O(T^{1/2+\epsilon} \log T) = o(T).$$

This concludes the estimate

$$\Sigma_1 = o(T). \tag{4.18}$$

Estimating Σ_3 is similar to that of Σ_1 except that we use $|f'(t)| \gg \log(n/(2q_* T))$ in Lemma 3.1. Hence, we find in a similar manner that

$$\Sigma_3 = o(T). \tag{4.19}$$

We now estimate the main contribution to the upper bound (4.15) coming from the sum Σ_2. This is estimated by using the saddle point theorem Lemma 4.3. Here we have $c = n/q_*$, $\lambda_2 = T^{-1}$, $\lambda_3 = T^{-2}$, $G = T^{1/2+\delta}$ and $G_1 = T^{-1/2+\delta}$. Hence we get

$$\begin{aligned} \Sigma_2 \quad &= \quad c_2 \sum a(n) e^{-in/q_*} + O\left(\sum \frac{|a(n)|}{n^{1+\delta}} T^{1/2+\delta} \right) + \\ &\quad + O\left(\sum \frac{|a(n)|}{n^{1+\delta}} T^{1/2+\delta} E_n \right), \end{aligned} \tag{4.20}$$

where c_2 is a constant and E_n is

$$E_n = \log\left(\tfrac{n}{q_* T}\right)^{-1} + \log\left(\tfrac{2q_* T}{n}\right)^{-1}.$$

The first O-term in (4.20) is estimated by noting that the sum is convergent to get $O(T^{1/2+\delta}) = o(T)$.

The estimation of the second O-term in (4.20) is done in the same way as was done for Σ_{12} to conclude that it is $O(T^{1/2+\epsilon} \log T) = o(T)$.

Finally, the upper bound estimation reduces to the crucial exponential sum

$$\sum_{q_*T+1\le n\le 2q_*T-1} a(n)e^{-2\pi in/\sqrt{q}} \,.$$

Here, we have replaced $q_* = \sqrt{q}(2\pi)^{-1}$ in the main exponential sum in (4.20).

By the multiplicativity of the Dirichlet coefficients $a(n)$, the assumption (4.9) on $a(n)$, the irrationality of $\sqrt{q}$ and Lemma 4.4, we get this sum to be $o(T)$. This concludes the estimate

$$\Sigma_2 = o(T). \tag{4.21}$$

Thus, using (4.18), (4.19) and (4.21) in (4.15), we get

$$I_3 = o(T). \tag{4.22}$$

Thus from (4.11) and (4.22), we derive a contradiction for T sufficiently large. Thus, $F(s)$ has infinitely many zeros on the critical line. This completes the proof of the theorem. □

In the next chapter, we shall study the same problem for functions in $\mathcal{S}$ with degree $d > 2$.

Chapter 5

Hardy's theorem for the Selberg class: $d > 2$

The problem of showing that there are infinitely many zeros on the critical line for any L-function of degree $d > 2$ is an extremely challenging one. In this chapter, we shall discuss a conditional approach to this problem by generalizing the methods of Chapter 4 pertaining to the degree 2 case.

We first recall the statement of Theorem 4: Let $F \in \mathcal{S}$ with degree $2 \leq d < 4$, conductor q and internal shift $\theta = 0$. Let T, H be sufficiently large such that $T^{\frac{d}{4}+\epsilon} \ll H \ll T$ for some $\epsilon > 0$, $q_* = q^{1/2}(2\pi)^{-d/2}$ and $G = H/\log T$. Let $\psi(u) = \exp(-u^2 G^{-2})$ be a smoothing function. If the condition

$$\sum_{|(n/q_*)^{2/d} - T| \leq H} \frac{a(n)}{n^{\frac{1}{2}-\frac{1}{d}}} \psi\big((n/q_*)^{2/d} - T\big) \exp\left(-i\frac{d}{2}(n/q_*)^{2/d}\right) = o(G) \tag{5.1}$$

holds, then $F(\frac{1}{2} + it)$ has a zero for some $t \in [T - H, T + H]$.

We recall that the zeros of $F(s)$ on the critical line correspond to the real zeros of $Z_F(t)$, which is the analogue of Hardy's function for $F(s)$ (see (4.3)). Let us assume that $F(\frac{1}{2}+it)$, and hence $Z_F(t)$, has no zeros with t in the interval $[T-H, T+H]$.

Consider the smoothed integral

$$I_4 = \int_{-H}^{H} Z_F(T+u)\psi(u)du, \tag{5.2}$$

The method of proof will be similar to that of Theorem 3. The proof proceeds by using our assumption on $Z_F(t)$ to estimate I_4 from above and below and arrive at a contradiction. Both the estimates are established by generalizing the corresponding methods of Chapter 4 to deal with the smoothing factor in the integral.

By our assumption on $Z_F(t)$, we get that the integrand in (5.2) does not change sign in the range of integration. Hence, we get the lower bound

$$|I_4| \gg \int_{-G}^{G} |Z_F(T+u)|du = \int_{-G}^{G} |F(\tfrac{1}{2}+i(T+u))|du \gg G. \tag{5.3}$$

by noting that $\psi(u) \gg 1$ when $|u| \leq G$ for the first inequality and using Ramachandra's theorem (Lemma 2.3) to get the last inequality.

For the upper bound estimation of I_4, we first convert (5.2) into a complex integral and move the line of integration to line $\operatorname{Re} s = 1+\delta$, where $F(s)$ has a Dirichlet series expansion (here $\delta > 0$ is a small constant to be chosen later).

$$\begin{aligned} I_4 &= -i\int_{\frac{1}{2}+i(T-H)}^{\frac{1}{2}+i(T+H)} \Delta^{-1/2}(s)F(s)e^{(s-\frac{1}{2}-iT)^2G^{-2}}ds \\ &= -i\int_{\frac{1}{2}+i(T-H)}^{1+\delta+i(T-H)} -i\int_{1+\delta+i(T+H)}^{\frac{1}{2}+i(T+H)} -i\int_{1+\delta+i(T-H)}^{1+\delta+i(T+H)}. \end{aligned} \tag{5.4}$$

We bound the first two integrals in (5.4) - the "horizontal" integrals - by noting that

$$\Delta^{-1/2}(s)F(s) \ll T^{\frac{d}{2}(\frac{1}{2}+\delta)+\epsilon},$$

in the range of integration. This follows from Lemma 4.2. Also, writing $s = \sigma + i(T \pm H)$, we observe that

$$\exp\left((s - \tfrac{1}{2} - iT)^2 G^{-2}\right) = \exp\left((-H^2 + (\sigma - \tfrac{1}{2})^2 \pm i(2\sigma - 1)H)G^{-2}\right)$$
$$\ll \exp(-\tfrac{1}{2}H^2 G^{-2}) \ll T^{-\frac{1}{2}\log T}.$$

Hence, the first two integrals in (5.4) give an error of $o(1)$.

Now, in the last integral of (5.4)- the "vertical" integral - we expand $F(s)$ as a Dirichlet series to get

$$I_4 \ll \sum_{n=1}^{\infty} \frac{a(n)}{n^{1+\delta}} \int_{-H}^{H} \Delta^{-1/2}(1 + \delta + i(T+u)) n^{-i(T+u)} e^{(\frac{1}{2}+\delta+iu)^2 G^{-2}} du. \tag{5.5}$$

We now recall the asymptotic formula (4.6) (with $\theta = 0$)

$$\Delta^{-1/2}(1 + \delta + i(T+u)) = \omega_1 (q_*(T+u)^{\frac{d}{2}})^{\frac{1}{2}+\delta+i(T+u)} e^{-i\frac{d}{2}(T+u)} (1 + O(T^{-1})), \tag{5.6}$$

for a constant $\omega_1 \neq 0$. Using (5.6) in (5.5) we get

$$\begin{aligned} I_4 \ll & \sum_{n} \frac{a(n)}{n^{1+\delta}} \int_{-H}^{H} (T+u)^{d(\frac{1}{4}+\frac{\delta}{2})} \left((\tfrac{T+u}{e})^{\frac{d}{2}} \tfrac{q_*}{n} \right)^{i(T+u)} e^{(\frac{1}{2}+\delta+iu)^2 G^{-2}} du \\ & + O(HT^{d(\frac{1}{4}+\frac{\delta}{2})-1}), \end{aligned} \tag{5.7}$$

where the error term in (5.7) comes from the O-term in (5.6) and a trivial estimation of the corresponding integral.

Expanding the exponential factor in the integral in (5.7) and multiplying with a suitable constant $\ll \exp((\frac{1}{2} + \delta)^2 G^{-2}) \ll 1$, we get

$$I_4 \ll \sum_{n} \frac{a(n)}{n^{1+\delta}} \int_{-H}^{H} g(u) e^{if(u)} du + O(HT^{d(\frac{1}{4}+\frac{\delta}{2})-1}), \tag{5.8}$$

where $g(u)$ and $f(u)$ are the functions

$$g(u) = (T+u)^{d(\frac{1}{4}+\frac{\delta}{2})} e^{-u^2 G^{-2}}, \quad f(u) = \tfrac{d}{2}(T+u)\log\left(\tfrac{T+u}{e}\left(\tfrac{q_*}{n}\right)^{\frac{2}{d}} c_0\right), \tag{5.9}$$

and c_0 is as follows

$$c_0 = \exp(\tfrac{4}{d}(\tfrac{1}{2}+\delta)G^{-2}). \tag{5.10}$$

We now subdivide the sum over n in (5.8) into three ranges

$$n \le n_0 - 1, \qquad n_0 + 1 \le n \le n_1 - 1, \qquad n \ge n_1 + 1,$$

and call the corresponding sums $\Sigma^{(1)}, \Sigma^{(2)}, \Sigma^{(3)}$, respectively, where

$$n_0 = q_* c_0^{\frac{d}{2}} (T-H)^{\frac{d}{2}}, \qquad \text{and} \qquad n_1 = q_* c_0^{\frac{d}{2}} (T+H)^{\frac{d}{2}}. \tag{5.11}$$

The reason for this subdivision is that in the first and last ranges we will use the first derivative estimate for the integral in (5.8), while for the middle range we will use the saddle point theorem.

Thus (5.8) becomes

$$I_4 \ll \Sigma^{(1)} + \Sigma^{(2)} + \Sigma^{(3)} + O(HT^{d(\frac{1}{4}+\frac{\delta}{2})-1}). \tag{5.12}$$

Note that there may be atmost 4 integers from (5.8) which have not been covered in the above subdivision. Since $n \asymp T^{d/2}$ for these integers and the Ramanujan hypothesis holds in the form $a(n) \ll n^{\delta}$, their contribution to I_4 is

$$O\left(\frac{T^{\frac{d}{2}\delta}}{T^{\frac{d}{2}(1+\delta)}} HT^{d(\frac{1}{4}+\frac{\delta}{2})}\right) = O(HT^{d(\frac{1}{4}+\frac{\delta}{2})-1}),$$

as $d \geq 2$ and this is absorbed into the error term in (5.12).

The condition $d < 4$ implies that we can choose δ small enough so that $d(\frac{1}{4} + \frac{\delta}{2}) < 1$ and hence the error term in (5.12) becomes $o(G)$.

The estimation of $\Sigma^{(1)}$ is an application of Lemma 3.1 to the integral in (5.8). Note that $f'(u) = \frac{d}{2}\log((T+u)(\frac{q_*}{n})^{\frac{2}{d}}c_0) \gg \log(n_0/n)$ (see (5.11)) and $g(u) \ll T^{d(\frac{1}{4}+\frac{\delta}{2})}$. The condition on monotonicity of f'/g is satisfied by subdiving the range of integration into 2 parts if necessary. Hence, we get

$$\begin{aligned} \Sigma^{(1)} &\ll T^{d(\frac{1}{4}+\frac{\delta}{2})} \sum_{n \leq n_0 - 1} \frac{|a(n)|}{n^{1+\delta}\log(n_0/n)} \\ &\ll T^{d(\frac{1}{4}+\frac{\delta}{2})} \left(\sum_{n \leq n_0/2} \frac{|a(n)|}{n^{1+\delta}\log 2} + \sum_{n_0/2 < n \leq n_0 - 1} \frac{|a(n)|}{n^{1+\delta}\log(n_0/n)} \right). \end{aligned} \tag{5.13}$$

Note that the first sum in (5.13) is convergent, and in the second sum we use $n \asymp n_0$, $\log(n_0/n) \gg (n_0 - n)/n_0$ and $a(n) \ll n^{\delta}$, to get

$$\Sigma^{(1)} \ll T^{d(\frac{1}{4}+\frac{\delta}{2})} \left(1 + \sum_{n_0/2 < n \leq n_0 - 1} \frac{1}{n_0 - n}\right) \ll T^{d(\frac{1}{4}+\frac{\delta}{2})} \log T = o(G), \tag{5.14}$$

if we choose $\frac{d}{2}\delta < \epsilon$ because $G \log T = H \gg T^{\frac{d}{4}+\epsilon}$. This is the second condition on the choice of δ.

For $\Sigma^{(3)}$, we proceed just as in the case of $\Sigma^{(1)}$, except that we use $|f'(u)| \geq \log(n/n_1)$ in Lemma 3.1, to conclude that

$$\Sigma^{(3)} = o(G). \tag{5.15}$$

Finally, for $\Sigma^{(2)}$, we use Lemma 4.3 for the integral in (5.8). Note that the saddle point u_0 (the value such that $f'(u_0) = 0$) satisfies $1 = (T + u_0)(\frac{q_*}{n})^{\frac{2}{d}}c_0$.

We also have the bounds $f'' \asymp T^{-1}$, $f''' \asymp T^{-2}$, $g(u) \ll M$ where $M = T^{d(\frac{1}{4}+\frac{\delta}{2})}$ and $g'(u) = g(u)((T+u)^{-1} - 2uG^{-2}) \ll MHG^{-2} = MH^{-1}\log^2 T$.

Thus applying Lemma 4.3 to the integral in (5.8) in this range, we get

$$\int_{-H}^{H} g(u)e^{if(u)}du = c_2 \frac{g(u_0)}{f''(u_0)^{1/2}} e^{if(u_0)} + O\left(MH^{-1}T\log^2 T\right) + O\left(MHT^{-1}\right) + O\left(M\right) + O(M(|f'(-H)|^{-1} + |f'(H)|^{-1})), \tag{5.16}$$

for some constant c_2.

Now, recall that $\Sigma^{(2)}$ is the sum in (5.8) restricted to the range $n_0+1 \le n \le n_1 - 1$. To estimate $\Sigma^{(2)}$, we will replace the integral in (5.8) by the expression (5.16) and estimate the contribution of each term.

First, we will estimate the contribution of the O-terms in (5.16) to the sum $\Sigma^{(2)}$. We note that the first O-term dominates the next two. Hence it is enough to consider only the first and the last O-terms in (5.16).

Note that the number of terms in the range of $\Sigma^{(2)}$ is $\ll (T+H)^{\frac{d}{2}} - (T-H)^{\frac{d}{2}} \ll T^{\frac{d}{2}-1}H$.

The contribution of the first O-term in (5.16) to $\Sigma^{(2)}$ is estimated trivially by using $a(n) \ll n^{\delta}$ and noting that $n \asymp T^{d/2}$ in this range, to get

$$O\left(T^{\frac{d}{2}-1}H \frac{T^{\frac{d}{2}\delta}}{T^{\frac{d}{2}(1+\delta)}} \frac{T^{d(\frac{1}{4}+\frac{\delta}{2})}T\log^2 T}{H}\right) = O(T^{d(\frac{1}{4}+\frac{\delta}{2})}\log^2 T) = o(G), \tag{5.17}$$

by our choice of δ.

For the contribution of the last error term in (5.16) to $\Sigma^{(2)}$, we shall estimate the part involving $f'(-H)$, the other one involving $f'(H)$ being handled similarly. We write $n = n_0 + m$ with $1 \le m \ll T^{\frac{d}{2}-1}H$ in this range. Recall that $|f'(-H)| =$

$\log(\frac{n}{n_0}) \gg \frac{m}{n_0}$. Using this bound and proceeding as in the previous estimate (5.17), the contribution to $\Sigma^{(2)}$ is

$$O\left(\frac{T^{d(\frac{1}{4}+\frac{\delta}{2})+\frac{d}{2}\delta}}{T^{\frac{d}{2}(1+\delta)}}\sum_m \frac{n_0}{m}\right) = O(T^{d(\frac{1}{4}+\frac{\delta}{2})}\log T) = o(G), \tag{5.18}$$

where the sums are over $1 \leq m \ll T^{\frac{d}{2}-1}H$.

The contribution of the main term in (5.16) to $\Sigma^{(2)}$ is a constant multiple of

$$\sum_{n_0+1\leq n\leq n_1-1} \frac{a(n)}{n^{\frac{1}{2}-\frac{1}{d}}} \exp\left(-i\frac{d}{2}c_0^{-1}(n/q_*)^{\frac{2}{d}} - \frac{(T-c_0^{-1}(n/q_*)^{\frac{2}{d}})^2}{G^2}\right). \tag{5.19}$$

We shall remove the dependence on c_0 in (5.19) both in the exponential factor as well as in the range. First, from (5.10) we have $c_0^{-1} = 1 + O(G^{-2})$ and hence the argument of the exponential in (5.19) is

$$-i\frac{d}{2}(n/q_*)^{\frac{2}{d}} + O(TG^{-2}) - \frac{(T-(n/q_*)^{\frac{2}{d}})^2}{G^2} + O(T^2G^{-4}) \tag{5.20}$$

Thus, using (5.20) in the exponential factor of (5.19) results in

$$\sum_{n_0+1\leq n\leq n_1-1} \frac{a(n)}{n^{\frac{1}{2}-\frac{1}{d}}} \exp\left(-i\frac{d}{2}(n/q_*)^{\frac{2}{d}} - \frac{(T-(n/q_*)^{\frac{2}{d}})^2}{G^2}\right), \tag{5.21}$$

with an error

$$O\left(T^{\frac{d}{2}-1}H\frac{T^{\frac{d}{2}\delta}}{T^{\frac{d}{2}(\frac{1}{2}-\frac{1}{d})}}\frac{T}{G^2}\right) = O\left(\frac{T^{\frac{d}{4}+\frac{1}{2}+\frac{d}{2}\delta}\log^2 T}{H}\right) = o(G), \tag{5.22}$$

since $T^{\frac{d}{4}+\frac{d}{2}\delta} = o(H)$ by the choice of δ and $T^{1/2}\log^2 T \ll G$.

Now, to remove the dependence on c_0 in the range of (5.21), we note that for the terms where $|(n/q_*)^{2/d} - T| > H$, the value of the smoothing function $\exp(-(T-(n/q_*)^{2/d})^2G^{-2}) \ll \exp(-H^2G^{-2}) \ll T^{-\log T}$. Thus the contribution of these terms to $\Sigma^{(2)}$ is $o(1)$. Hence the sum (5.21) reduces to

$$\sum_{|(n/q_*)^{2/d}-T|\leq H} \frac{a(n)}{n^{\frac{1}{2}-\frac{1}{d}}}\psi\big((n/q_*)^{2/d}-T\big)\exp\left(-i\frac{d}{2}(n/q_*)^{2/d}\right)=o(G), \tag{5.23}$$

the last bound following from the hypothesis (5.1) of the theorem.

Combining (5.17), (5.18), (5.22) and (5.23), we get

$$\Sigma^{(2)}=o(G). \tag{5.24}$$

Hence, using (5.14), (5.15) and (5.24) in (5.12), we complete the upper bound estimate

$$I_4=o(G). \tag{5.25}$$

Thus, from the lower bound (5.3) and the upper bound (5.25), we derive a contradiction. This contradicts our assumption and therefore establishes the truth of the theorem.

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