ACYCLIC, k-INTERSECTION EDGE COLOURINGS AND ORIENTED COLOURING.

by

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I, hereby declare that the investigation presented in this thesis has been carried out by me. The work is original and the work has not been submitted earlier as a whole or in part for a degree/diploma at this or any other Institution or University.

Narayanan N

...प्रकृत्या इदं...
(to the nature)

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Abstract

In this thesis, we study three graph colouring problems. The main theme of this thesis is the acyclic edge colouring problem. We also study two other problems, namely, the k-intersection edge colouring and oriented vertex colouring.

In the acyclic edge colouring problem, we are required to find the minimum number of colours χ'_a that suffices to colour the edges of a graph properly such that the union of any two colour classes forms a forest. The acyclic edge colouring conjecture (due to Fiamcik [Fia78] and also independently to Alon, Sudakov and Zaks [ASZ01]) states that it is possible to colour the edges of any graph G acyclically with at most $\Delta(G) + 2$ colours. It is considered to be a difficult problem as very little is known about exact or tight estimates of this invariant even for highly structured classes of graphs.

In the first part of this thesis, we obtain improved upper bounds on χ'_a for some classes of graphs. We also show that certain classes of graphs satisfy the acyclic edge colouring conjecture. Some of these results are obtained making use of probabilistic arguments, while the others are proved making use of structural properties of the underlying graphs.

The second part deals with a related problem that we call the k-intersection edge colouring. Here, one seeks to find the minimum number of colours that are sufficient to colour the edges such that for any pair of adjacent vertices, the number of common colours received on the edges incident on them is at most k. We obtain an upper bound of $O(\Delta^2/k)$ and show that this bound is indeed tight for complete graphs.

In the third part, we look at the oriented vertex colouring of graphs. An oriented k-colouring of an oriented graph $\overrightarrow{G}$ is a mapping $\mathcal{C} : V(\overrightarrow{G}) \mapsto [k]$ so that (i) $\mathcal{C}(x) \neq \mathcal{C}(y) \quad \forall (x, y) \in A(\overrightarrow{G})$ and (ii) $\mathcal{C}(x) = \mathcal{C}(w) \implies \mathcal{C}(y) \neq \mathcal{C}(z) \quad \forall (x, y), (z, w) \in A(\overrightarrow{G})$. The oriented chromatic number χ_o of an oriented graph $\overrightarrow{G}$ is the smallest k such that there is an oriented k-colouring. The oriented chromatic number for an undirected graph G is the maximum $\chi_o(\overrightarrow{G})$ over all orientations $\overrightarrow{G}$ of G. We obtain improved upper and lower bounds on oriented chromatic number for certain classes of graphs and products of graphs.

Contents

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Publications

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[6] Rahul Muthu, Narayanan N, and C R Subramanian. Improved bounds on acyclic edge colouring. *Discrete Mathematics*, 307(23):3063–3069, 2007.

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Preface

Graph theory is the study of graphs. The origins of graph theory dates back to Leonhard Euler, who published the first paper in graph theory in 1736. A graph is an abstraction of a set of entities (called vertices) with a binary relation between them (called edges). For example, the set of entities may be a set of cities or network hubs, persons, computer processes, or mobile towers and so on. A binary relation between a pair of cities may be that they are connected by a direct road. In the case of mobile towers, one may say that two towers are related if the range of areas they cover intersect. A typical problem would be to find out whether it is possible to start from any city and reach any other. In the case of mobile network, given a set of towers and the areas they cover, one may need to find out the minimum number of frequency bands that need to be allocated so that the same frequency is not used on adjacent towers. There is practically an infinitude of such real life problems. Graph theory tries to abstract out such problems and to find a solution to the problem in a mathematical setting. This makes it immensely useful in dealing with many problems from different branches of science or industry.

Some of the areas where graph theory is greatly used include network analysis, VLSI design, molecular chemistry, DNA sequencing, molecular biology, study of evolution graphs, development of algorithms and study of neural networks. With the advent of computer science, graph theory came to prominence and has become an essential tool.

Many problems in graph theory are of an extremal nature. For example, one may need to find the minimum (maximum) cardinality of a set of vertices satisfying certain conditions. For example, a central question is to find out the maximum number of vertices in a graph such that there are no edges between any pair. Another type of extremal problem is to partition the vertices or edges so that the partition satisfies certain properties.

Graph colouring is a branch of graph theory which deals with such partitioning problems. For example, suppose that we have a world map and we would like to colour the countries so that if two countries share a boundary line, then they need to get different colours. We can translate the map to a graph by letting countries be represented by vertices and two vertices are made adjacent if and only if the corresponding countries share a boundary line. Then the problem of map colouring is equivalent to the problem of vertex colouring its graph. Hence the original map colouring now reduces to vertex colouring of the associated graph.

Another important problem is the scheduling problem. In this, each important citizen in a city belongs to several committees. Each committee is to meet once, but two committees having a common member cannot meet at the same time. How many time slots are needed? In graph theoretic terms, vertices stand for committees and an edge between two vertices means that the corresponding committees have some members in common. Once again a minimum vertex colouring provides us the minimum number of time slots that are necessary for scheduling the meetings.

There are many variations of graph colouring notions. Many of them have applications in various fields. We, in this thesis, study three different types of graph colouring problems.

Outline of the thesis

Here, we give an outline of the results presented in this thesis. In this work, we study three graph colouring problems, namely, acyclic edge colouring, k-intersection edge colouring and oriented vertex colouring.

The first chapter introduces the relevant notations, notions and definitions that we encounter in the later chapters. The rest of the thesis is divided into three parts, each dedicated to one of the three problems. The first part, which forms the main part, deals with the acyclic edge colouring problem. In Chapter 2, we formally introduce the problem of acyclic edge colouring and list some of the earlier results. Chapter 3 presents an introduction to The Probabilistic Method and then we make use of one of its powerful tools known as the Lovász Local Lemma to obtain improved upper bounds on χ'_a. These results were presented in the conference GRACO 2005 [4]. A modified version appears in [6]. This work was done in collaboration with Rahul Muthu and it appears in his thesis also.

In Chapter 4, we make use of some structural arguments to verify the acyclic edge colouring conjecture for the class of outerplanar graphs. We also sketch an algorithm to obtain such a colouring that runs in $O(|V| \log \Delta)$ time. These results were presented at AAIM 2007 [5].

Chapter 5 deals with graphs of maximum average degree at most 6 and gives improved upper bounds on χ'_a for such graphs. This in particular includes improved bounds for planar graphs, two and three fold graphs etc. We make use of The Discharging Method to obtain some nice structural properties of graphs of maximum average degree ≤ 6. This work was done in collaboration with Anna Fiedorwicz and Mariusz Hałushczak during my visit to the University of Zielona Göra, Poland. These results appear in [1].

In Chapter 6, we look at a few other classes of graphs and show that they satisfy the acyclic edge colouring conjecture. These include the Harary graphs, minimally 2-connected graphs and fully subdivided graphs [3].

The second part (Chapter 7) deals with another edge colouring problem, namely, the k-intersection edge colouring. Once again, we make use of probabilistic arguments and tools like the Chernoff bounds and Lovász Local Lemma to obtain a tight bound for this colouring notion. This result was presented in the conference CID 2007. This result appears in [7].

The final part deals with the oriented colouring problem. This chapter contains recent results we have obtained for some classes of graphs. These results are presented in Chapter 8. This is a joint work with N.R. Aravind [2].

We conclude the thesis with some remarks and open problems in Chapter 9.

How to read this thesis

The 'pdf' version of this thesis contains hyperlinked text and index for ease of navigation and reference. Any reference, figure, index or definition that is hyperlinked appears in a small box. The reader can left-click on the box to navigate directly to the page in which it is first defined or to the bibliography pages. The 'preliminaries' chapter also contains the important terms that we define in the right margin of the text. The term appears on the right side of the line in which it is defined. There is a glossary page for notations and short definitions. The reader can get a quick definition from here as well as click on the corresponding page number (first in the list) to go to the page where it is first defined.

Another convention used is the use of two QED symbols. A square filled black stands for the end of arguments for the main proof. An unfilled square denotes the end of arguments for the proof of any claim within a main proof.

यथा शिखा मयूराणां नागानां मणयो यथा ।
तथा वेदाङ्गशास्त्राणां गणितं मूर्धनि स्थितं ॥

As the crest on peacock's head and gem on a snake,
mathematics stands at the top of all sciences.
200AD

1 Preliminaries

Graph theory is one of the most important branches of modern mathematics. Graph colouring is one of the best known and widely studied areas of graph theory. The huge popularity it enjoys is not just owing to the fact that it is a colourful subject but to the fact that many practically important problems are easily expressed as a graph colouring problem. It has come to be one of the highly fertile topics in modern graph theory with numerous core theorems and applications to other science streams. In general terms, a colouring is a labelled partition of the elements of a set where elements with the same label ('colour') belong to the same set.

Many well-known theorems or conjectures in mathematics are very easy to state, but are notoriously difficult to come up with a proof or a counterexample. Graph theory also has many such problems. The foremost in the list of such problems is the planar Four Colouring Problem (FCP) which says that any planar map can be coloured using at most four colours. The FCP has generated so much interest that the contributions due to its study are overwhelming. Several techniques that are used in modern graph theory were developed to solve the FCP. It took more than 125 years of work by many people before it was finally solved in 1976 by Kenneth Appel and Wolfgang Haken [AH76].

In this thesis, we focus on three graph colouring notions. One of these contains a long standing open problem which is open for more than 30 years. This thesis contributes to these research areas by improving the known results for some classes of graphs. The improvements we obtain are achieved by making use of tools from the famous Probabilistic Method pioneered by Paul Erdös, the Discharging Method developed exquisitely for the proof of FCP and some structural properties of the classes of graphs we consider.

1.1 Definitions and Notation

In this section, we introduce some basic terms and notations. For any term which is not defined here, we direct the reader to the standard text book "Graph Theory" by Reinhard Diestel [Die00].

An *undirected graph* or simply a *graph* $G = (V, E)$ is a pair of sets $V = V(G)$ and $E = E(G)$. The elements of V are known as the *vertices* of G and that of E are called the *edges*. Whenever it is clear from the context, we omit the specification of the graph and simply write V or E. The elements of E are two element subsets of V. Thus every edge $e = (u, v)$ is an unordered pair, where $u, v \in V$. Pictorially, one represents a graph by drawing a dot or circle for each vertex and a line segment joining two vertices whenever there is an edge between them. All graphs that we consider in this thesis are assumed to be finite and simple unless stated otherwise. An example of a graph is given in Figure 1.1. *graph* *edge*

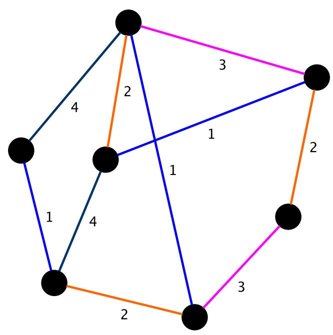

Figure 1.1: *A graph on 7 vertices with coloured edges.*

The number of vertices in a graph G is its *order*. A vertex v is said to be *adjacent* to a vertex u if $(u, v) \in E$. Similarly a vertex u and an edge e are *incident* with one another if u belongs to e. The set of edges incident with a vertex v is denoted by $E(v)$. Two edges *order* are *adjacent* if they share a common vertex.

degree

The *degree* of a vertex v, is the number of edges incident with it. We denote the degree of a vertex by $d(v)$ (sometimes $d_G(v)$ to specify the graph). The *maximum* degree and *minimum* degree of a graph G are denoted $\Delta(G)$ and $\delta(G)$ (or simply Δ and δ) respectively. A graph is said to be *regular* if $\delta(G) = \Delta(G)$. Vertices of degree k, at least k, and at most k are respectively called a k-vertex, k^+-vertex, and k^--vertex. A graph is called *k-degenerate*, if every induced subgraph contains a k^--vertex.

path

A *path* in a graph is a sequence of distinct vertices $u = v_0, v_1, \ldots, v_k = v$ where for every $i < k$, $(v_i, v_{i+1}) \in E$. The length of a path is the number of edges in it. A path from u to v is called a $u - v$ path. A path on k vertices is denoted by P_k. A *cycle* of length k is a sequence $v_0, v_1, \ldots, v_k = v_0$ of vertices such that all but the end vertices are distinct and $(v_i, v_{i+1}) \in E$ for every i. We denote a cycle on k vertices by C_k. An edge that joins two non-consecutive vertices of a cycle is called a *chord* of the cycle.

subgraph

We say that H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In addition, if H contains all the edges $(u, v) \in E(G)$ with $u, v \in V(H)$, then H is an *induced subgraph* of G. We say that $V(H)$ induces H in G. An *edge induced* subgraph of a graph G is a subset of edges of G together with vertices that are the endpoints of the given edges.

connected

We say that G is a *connected graph,* if for all pairs u, v of vertices, G contains a $u - v$ path. A maximal connected induced subgraph is known as a *connected component* of G. A *cut-vertex* is a vertex whose removal increases the number of connected components of the graph. For a graph G, a maximal connected induced subgraph without cut-vertices is called a *block* of G. A *2-connected* graph is a connected graph on three or more vertices without any cut-vertex. It is well-known that in a 2-connected graph, for every pair u, v of vertices, there is a cycle passing through both of them. A graph without cycles is called a *forest*. Further, if it is connected, then it is called a *tree*.

forest

A pair of edges are *independent* if they are not adjacent. A set of pairwise independent edges is called a *matching*. If a set of vertices are pairwise non-adjacent, they form an *independent set*. The length of a shortest cycle, if any, in a graph is called its *girth*, denoted by $g(G)$.

girth

If all the vertices in a graph are pairwise adjacent it is a *complete graph*. A complete graph of order n is denoted K_n. If an induced subgraph on k vertices is complete, it is called a *k-clique*.

clique

The *line graph* $L(G)$ of a graph G is the graph with $E(G)$ as its vertex set and a pair of vertices in $L(G)$ being adjacent whenever the corresponding edges are adjacent in G.

A graph is *embedded* in the plane if its vertices are mapped to distinct points in $\mathbb{R}^2$ and edges are mapped to simple Jordan curves connecting the corresponding points. Further, we require that no two curves share any points except possibly an end-point. Such a mapping is called a planar embedding of the graph.

A graph is a *planar graph* if it can be embedded in the *plane*. A planar graph together with an embedding is called a *plane* graph. Given any plane graph G, notice that $\mathbb{R}^2 \setminus G$ is open; its connected regions are the *faces* of G. Let F be the set of faces in any planar embedding of a graph G. The well-known Euler's formula states the following fact. *planar graph*

Fact 1.1.1 (Euler's Formula). *Every connected plane graph $G = (V, E)$ satisfies the formula $|V| - |E| + |F| = +2$.*

Using Euler's formula, one can show that any planar graph has at most $3|V| - 6$ edges.

We say that a vertex *lies* on a face, if it is part of an edge forming the boundary of the face. A graph is called an *outerplanar graph* if it has a planar embedding such that every vertex lies on the outer/unbounded face. *outerplanar*

A *directed graph* G is a pair (V, A) where the elements of A are ordered 2-element subsets of V. Elements of A are called *arcs*. The arcs (x, y) and and (y, x) are opposite arcs. An orientation of a simple undirected graph G is a digraph obtained by giving every edge of G one of its two possible orientations. An *oriented graph* is an orientation of some simple undirected graph. In short, a simple directed graph without any pair of opposite arcs is an oriented graph. A pair of vertices x and y in a directed graph are said to be *adjacent* if at least one of the arcs (x, y), (y, x) belongs to A. *directed graph* *oriented graph*

Let $G = (V, E)$ and $H = (V', E')$ be two graphs. We say that G and H are *isomorphic* to each other, if there exist a bijection $\varphi : V \mapsto V'$ such that $(x, y) \in E$ *if and only if* $(\varphi(x), \varphi(y)) \in E'$ for all $x, y \in V$.

A *homomorphism* from a graph G to a graph H, is a map $\psi : V(G) \mapsto V(H)$ that preserves the edge relations. That is, $(x, y) \in E(G) \implies (\psi(x), \psi(y)) \in E(H)$. Analogously, isomorphism and homomorphism can be defined between a pair of oriented graphs. *homomorphism*

Operations on Graphs

One can combine two or more graphs to get bigger graphs. We define some of these operations below.

Definition 1.1.1 (Cartesian Product). Let $G = (V_1, E_1)$ and $H = (V_2, E_2)$ be two graphs. Their *cartesian product*, denoted $G\Box H$, is the graph (V, E) where $V = V_1 \times V_2$ and $\big([u_1, u_2], [v_1, v_2]\big) \in E$ if and only if **either** $u_1 = v_1$ and $(u_2, v_2) \in E_2$ **or** $u_2 = v_2$ and $(u_1, v_1) \in E_1$. *cartesian product*

Definition 1.1.2 (Strong Product). Let $G = (V_1, E_1)$ and $H = (V_2, E_2)$ be two graphs. Their *strong product* denoted $G \boxtimes H$ is the graph (V, E) where $V = V_1 \times V_2$ and $\big([u_1, u_2], [v_1, v_2]\big) \in E$ if and only if **either** $u_1 = v_1$ and $(u_2, v_2) \in E_2$ **or** $u_2 = v_2$ and $(u_1, v_1) \in E_1$ **or** $(u_1, v_1) \in E_1$ and $(u_2, v_2) \in E_2$. *strong product*

The cartesian and strong products are both associative and commutative up to isomorphism. Further, there is a unique prime factorisation (UPF) of any connected graph into cartesian (strong) product of prime graphs [IK00]. Here, a graph is a prime graph if it cannot be expressed as the cartesian (strong) product of 2 or more other graphs.

The *directed cartesian product* and *directed strong product* are defined analogously with the edge set being replaced by the arc set.

1.2 Graph Parameters and Colourings

Here we define some graph parameters that we study or make use of in the thesis.

The average degree d_{avg} of a graph G is the ratio of the total degree to the number of vertices. That is, $d_{avg}(G) = \sum_{v\in V} d_G(v)/|V|$.

Definition 1.2.1 (Maximum Average Degree). The *maximum average degree* of a graph G denoted $mad(G)$ is the quantity defined as *$mad(G)$*

$$mad(G) = \max_{H\subseteq G} d_{avg}(H)$$

Observe that any class of graphs having bounded *mad* is closed under subgraph inclusion. For any positive integer r, let $[r]$ denote the set $\{1, 2 \ldots, r\}$.

A proper *vertex r-colouring* of a graph $G = (V, E)$ is a map $\mathcal{C} : V \mapsto [r]$ such that $\mathcal{C}(u) \neq \mathcal{C}(v)$ whenever u and v are adjacent. The elements of $[r]$ are known as *colours*. The *chromatic number* is the smallest integer r for which there is a proper vertex r-colouring of G. The chromatic number of G is denoted by $\chi(G)$. Notice that a proper vertex r-colouring partitions the vertex set V into r labelled independent sets. *chromatic number, χ*

Similarly, a proper *edge r-colouring* is a map $\mathcal{C} : E \mapsto [r]$ such that $\mathcal{C}(e) \neq \mathcal{C}(f)$ whenever e and f are adjacent. That is, a proper edge r-colouring partitions the edge set E into r matchings. The minimum value of r for which G admits a proper edge r-colouring is called the *chromatic index* of G and is denoted by $\chi'(G)$. In general, chromatic number and chromatic index respectively denotes the minimum number of colours required to properly colour the vertices or edges of a graph. *chromatic index*

Definition 1.2.2. A proper edge colouring of a graph G in which the union of any two colour classes forms a forest, is called an *acyclic edge colouring* of G. Here, a *colour class* stands for the set of all edges coloured with a given colour. *acyclic edge colouring*

The minimum number of colours sufficient to colour the edges of a graph acyclically is known as the *acyclic chromatic index* of G. It is denoted $\chi'_a(G)$ or $a'(G)$. We follow the first notation in this thesis. χ'_a

Acyclic vertex colouring is an analogous notion that requires one to properly colour the vertices of a graph such that the subgraph induced by vertices of any two colours is a forest. It is to be noted that an acyclic edge colouring of a graph corresponds to an acyclic vertex colouring of its line graph and vice versa.

With respect to a proper edge colouring, we say that a vertex (say v) *sees* a colour x, if any of the edges incident with v is coloured x.

Definition 1.2.3. Given $1 \leq k \leq \Delta$, a *k-intersection edge colouring* of a graph G is a proper edge colouring of G in which the number of common colours seen by any pair of adjacent vertices is at most k. *k-intersection edge colouring*

In other words, if we use $\mathcal{C}(v)$ to denote the list of colours seen by a vertex v in an edge colouring $\mathcal{C}$, then for every edge (u, v), we require that $|\mathcal{C}(u) \cap \mathcal{C}(v)| \leq k$. The minimum number of colours sufficient for such a colouring is called the *k-intersection chromatic index* and is denoted by $\chi'_k(G)$. χ'_k

Definition 1.2.4. An *oriented r-colouring* of an oriented graph $\overrightarrow{G}$ is a mapping $\mathcal{C} : V \mapsto [r]$ such that **(i)** $\mathcal{C}(x) \neq \mathcal{C}(y)$ for any arc (x, y) and **(ii)** $\mathcal{C}(x) = \mathcal{C}(w)$ only if $\mathcal{C}(y) \neq \mathcal{C}(z)$, for all pairs of arcs (x, y) and (z, w). *oriented colouring*

Less formally, it means that if there is an arc from a *red* vertex to a *blue* vertex in the coloured graph, then there cannot be an arc from a *blue* vertex to a *red* vertex.

For an oriented graph $\overrightarrow{G}$, the minimum integer r such that there is an oriented r-colouring of $\overrightarrow{G}$ is known as its *oriented chromatic number* and is denoted by $\chi_o(\overrightarrow{G})$. The *oriented chromatic number* $\chi_o(G)$ of an undirected graph G is defined as the maximum oriented chromatic number among all possible orientations $\overrightarrow{G}$ of G. That is $\chi_o(G) = \max_{\overrightarrow{G}} \left(\chi_o(\overrightarrow{G})\right)$ $\chi_o(\overrightarrow{G})$

Part I

Acyclic Edge Colouring

अपूर्वः कोऽपि कोशोऽयं विद्यते तव भारति ।
व्ययते वृद्धिमायाति क्षयमायाति सञ्चयात् ॥

Oh goddess, you have the most peculiar wealth
(of knowledge). It increases as you spend/share
and the less you spend the less you will have.

2 Acyclic Edge Colouring: Related Results

In this chapter, we formally introduce the acyclic edge colouring problem and present a brief sketch of some previous work.

2.1 The Problem

In the problem of acyclic edge colouring, we are required to map the edges of a graph $G = (V, E)$ to the first k natural numbers so that the following holds.

Let $\mathcal{C} : E \to \{1, 2, \ldots, k\}$. Then we require that (**i**) for any pair of adjacent edges e, f, we have $\mathcal{C}(e) \neq \mathcal{C}(f)$ and (**ii**) for any pair of colours i and j, the graph induced by the edges coloured i or j is acyclic. In other words, the union of any pair of colour classes forms a forest. It also means that a proper edge colouring is acyclic if and only if every cycle receives at least 3 colours. A graph with a proper edge colouring that is not acyclic is given in Figure 2.1. The same graph with an acyclic edge colouring is given in 2.2. We are interested in estimating the acyclic chromatic index $\chi'_a(G)$, which is the smallest integer k such that G has an edge k-colouring which is acyclic.

Since we are concerned with edge colourings, it is to be noted that any proper edge colouring of a graph G requires at least $\Delta = \Delta(G)$ colours since each edge incident to a vertex of degree Δ should get a different colour. Vizing proved in [Viz64] that one can always properly edge colour a graph with at most $\Delta + 1$ colours. His arguments are constructive leading to a linear time algorithm to obtain such a colouring.

Some facts are immediate. To ensure properness, any graph requires at least Δ colours. Observe that if a graph is regular, $\chi'_a \geq \Delta + 1$. Otherwise, all the vertices see all the colours and hence every pair of colour classes decompose into cycles. It is a known fact that for complete graphs of prime order $\chi'_a = \Delta + 1$. On the other hand, for complete graphs of even order $\chi'_a(K_{2n}) \geq \Delta + 2 = 2n + 1$. This is because at most one colour class can contain n edges (a perfect matching) and all others can contain at most $n - 1$ edges each. If not, the union of any two perfect matchings decomposes into cycles. A simple edge counting argument shows that the claim holds. See [ASZ01] for further details.

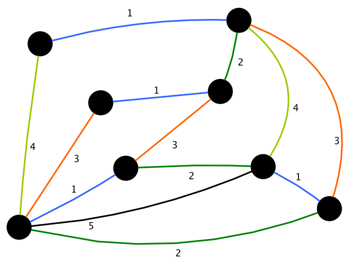

Figure 2.1: *An edge colouring which is not acyclic*

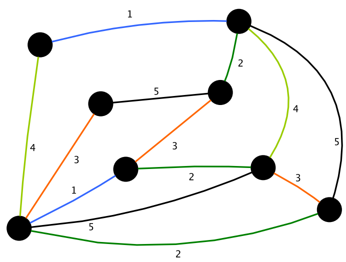

Figure 2.2: *An acyclic edge colouring of the same graph*

2.2 Related Work

The notions of acyclic colourings were introduced by Grünbaum in [Grü73]. The acyclic edge colouring was studied by Fiamcik (in the late 70's) who also proposed a conjecture on the same (which we state later). Since any 2-coloured cycle has a pair of edges of the same colour joined by an edge, it is easy to see that any distance-2 edge colouring (where any two edges joined by another edge are not coloured the same) is acyclic. Since there is always a distance-2 colouring using at most $O(\Delta^2)$ colours, it is easy to see that $\chi'_a = O(\Delta^2)$.

Burnstein [Bur79] proved that any graph of maximum degree at most 4 can be acyclically vertex coloured using at most 5 colours. This implies that any subcubic graph (graphs with degree bounded by 3) can be acyclically edge coloured using at most 5 colours. This is because the line graph of a subcubic graph has maximum degree at most 4. Let us recall that the acyclic edge colouring of a graph is an acyclic vertex colouring of its line graph.

In 1991, Alon, McDiarmid and Reed [AMR91] proved a linear upper bound of 64Δ for χ'_a, using probabilistic arguments. This was later improved to 16Δ using essentially the same arguments, but with a tighter analysis by Molloy and Reed [MR98].

Theorem 2.2.1. $\chi'_a(G) \leq 16\Delta$ *for any graph* G.

Fiamcik proposed the following conjecture [Fia78] in 1978. It was also independently proposed by Alon, Sudakov and Zaks [ASZ01] in 2001.

Conjecture 2.2.1. *For any graph* G, $\chi'_a(G) \leq \Delta + 2$.

If true, this conjecture has some interesting consequences. If it holds for the classes of complete graphs, then the *perfect 1-factorisation conjecture* and *perfect near-1 factorisation conjecture* are also true. Precisely, the perfect 1-factorisation conjecture states that the edges of K_{2n} $\forall n \geq 2$ can be edge decomposed into $2n - 1$ perfect matchings so that the union of any pair of them forms a hamiltonian cycle. Similarly, the perfect near-1 factorisation conjecture says that the edges of K_{2n+1} has a decomposition into $2n + 1$ matchings of size n so that the union of any pair forms a hamiltonian path. These conjectures hold if and only if $\chi'_a(K_{2n+1}) = 2n + 1$ for all n. For further details, refer [ASZ01].

Alon, et al. were led to propose the above conjecture after observing the following results both appearing in the same paper.

Theorem 2.2.2. *For any graph with girth* $g \geq 2000\Delta \log \Delta$, $\chi'_a(G) \leq \Delta + 2$.

Theorem 2.2.3. *Let* $G \in G_{n,\Delta}$ *be a random* Δ*-regular graph on* n *labelled vertices. Then asymptotically almost surely,* $\chi'_a(G) \leq \Delta + 2$.

Both of the above results are obtained using Lovász Local Lemma (LLL). In the same paper, it is also shown that for any graph with girth at least $c \log \Delta$, $\chi'_a \leq 2\Delta + 2$, where c is a small constant. Nešetřil and Wormald improved the above upper bound for random regular graphs to $\Delta + 1$ in [NW05]. One feature of the above results is that they are not constructive and do not easily yield efficient algorithms. Very few constructive results are known. Some of them are the following.

San Skulrattanakulchai [Sku04] obtained a linear time algorithm to colour any subcubic graph using at most 5 colours. Subramanian [Sub06] proposed and analysed a greedy heuristic which obtains an acyclic edge colouring in $O(mn\Delta^2(\log \Delta)^2)$ time and uses $O(\Delta \log \Delta)$ colours on any graph.

Alon and Zaks [AZ02] obtained a hardness result that even for subcubic graphs, it is NP-complete to determine exactly the value of χ'_a.

3
Girth and $\chi'_a(G)$

In this chapter, we obtain an improved upper bound for the acyclic chromatic index assuming the girth g to be at least some constant. This improves the previous 16Δ bound mentioned before. We make use of probabilistic arguments, especially the Lovász Local Lemma (LLL). In the first part, we give an introduction to the probabilistic method and introduce LLL.

3.1 The Probabilistic Method

In combinatorics, modern graph theory and number theory, the probabilistic method has attained the status of one of the strongest and most important of tools. This is more relevant when one is interested in showing the existence of a combinatorial object with certain properties. Since many of the graph colouring problems are precisely about existence, the probabilistic method has become a very powerful tool in dealing with colouring problems.

Loosely speaking, the idea behind the probabilistic method is that, "to show the existence of a combinatorial object with given properties, it is enough if we show that a randomly chosen object (from a suitable probability distribution) has the desired properties with positive probability".

More formally, the probabilistic method says the following.

A *finite probability space* is a finite set Ω together with a function $\Pr : \Omega \to [0, 1]$, such that $\sum_{x\in\Omega} \Pr(x) = 1$. Subsets of Ω are called events and for any $A \subseteq \Omega$, $\Pr(A) = \sum_{x\in A} \Pr(x)$. Suppose that the elements of Ω are a set of combinatorial objects and $\mathcal{P}$ is a property. If $\Pr(x \text{ does not satisfy } \mathcal{P}) < 1$, then it follows that there exists an $x \in \Omega$ such that x satisfies $\mathcal{P}$.

It is to be noted that the probabilistic method, in general, provides a short and simple (although not necessarily constructive) proof of an existential statement. An important task in applying the probabilistic method is to design an appropriate random experiment and carry out a probabilistic analysis. We present some of the tools we use and illustrate the application of this principle in the following.

The first tool we present here is the union bound on the probability. It states that the probability of a disjunction of events is upper bounded by the sum of the probabilities of the events. i.e., $\Pr(\bigcup_i E_i) \leq \sum_i \Pr(E_i)$.

A classic example of how this fact can be effectively used is the following problem. Suppose we need to find the Ramsey number $R(k)$, defined as the smallest integer n such that any 2-colouring of the edges of the complete graph K_n using colours say *red* and *blue*, always contains a monochromatic clique of size k. We use the union bound to show a lower bound of $R(k) \geq 2^{k/2}$. It is to be noted that, the best known constructive lower bound is $\exp\left((1+o(1))\log^2 k/\log\log k\right)$ [FW81].

Theorem. (Erdös) [AS00] $R(k) \geq 2^{k/2}$ for all $k \geq 3$.

Proof: Colour the edges of K_n red or blue independently and uniformly at random. Let E_S be the event that the induced clique on a given set S of k vertices is monochromatic (i.e., all of its edges are coloured with one colour). Since each of the $\binom{k}{2}$ edges has a probability $1/2$ of getting coloured a given colour, and since the colourings are independent, we have $\Pr(E_S) = 2 \times 2^{-\binom{k}{2}}$. We now upper bound the probability that there is at least one monochromatic k-clique. There are $\binom{n}{k}$ distinct ways to select S and we can use the sub-additivity property to get,

$$\Pr(\bigcup E_S) \leq \sum \Pr(E_S) = \binom{n}{k} 2^{1-\binom{k}{2}}$$

If $n < 2^{k/2}$, we can show that $\binom{n}{k}2^{1-\binom{k}{2}} < 1$ and hence with positive probability no event E_S occurs. Therefore there has to be at least one colouring in which no monochromatic k-clique is present. This completes the proof. ∎

Now, we introduce another tool from probability theory, which we use in this thesis. It is a lemma known as The Lovász Local Lemma (LLL). The LLL is very useful in situations where the events of interest are not necessarily independent, but are almost

independent in the sense that each event is dependent only on events which have only a local relationship with the given event. Thus, the tool introduced by Láslo Lovász ensures that if the dependencies are only local, then all the events fail to hold simultaneously with positive probability provided the conditions of the lemma are satisfied. This means that, if we define the events in such a way that the failure of all them together implies the existence of the desired combinatorial object, we can apply LLL.
Formally, we have the following lemma.

Lemma 3.1.1 (The Lovàsz Local Lemma). *Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be events in a probability space Ω and suppose that each event A_i is mutually independent of all events in $\mathcal{A} - (\{A_i\} \bigcup \mathcal{D}_i)$, for some $\mathcal{D}_i \subseteq \mathcal{A}$. If there exist $x_1, \ldots, x_n \in (0, 1)$ such that*

$$\Pr(A_i) \leq x_i \prod_{A_j \in \mathcal{D}_i} (1 - x_j), \quad 1 \leq i \leq n$$

Then $\Pr(\overline{A_1} \wedge \ldots \wedge \overline{A_n}) > 0$.

In a typical application of LLL, we design a random experiment and define events A_i in such a way that the A_i are bad events whose absence ensures the existence of the desired combinatorial object. Then we show that these events satisfy the requirements of LLL.

3.2 Introduction

In this chapter, we prove improved upper bounds on the acyclic chromatic index $\chi'_a(G)$ for graphs where the girth g is lower bounded by a small constant. We recall the fact that the best upper bound known for χ'_a for all graphs to date is 16Δ. Specifically, we prove the following.

Theorem 3.2.1. *If G is a graph such that $g(G) \geq 9$, then $\chi'_a(G) \leq 6\Delta$.*

That is, we could reduce the bound from 16Δ to 6Δ. If we relax the girth to a slightly higher value, we can further improve the bound as the following theorem states.

Theorem 3.2.2. *If G is such that $g(G) \geq 220$, then $\chi'_a(G) \leq 4.52\Delta$.*

We prove these results using the probabilistic method. The assumed lower bound on girth is not surprising and perhaps natural for a simple random experiment, since for any uniformly chosen random edge colouring, even cycles of short length are more likely to be two-coloured than cycles of large length. We discuss this in detail in the next section when we try to formalise this phenomenon.

3.3 Getting below the barrier — limited improperness

As we mentioned in the previous chapter, 16Δ is the best upper bound known for χ'_a and the particular random experiment used has the inherent limitation that one cannot hope to improve the bound much more. We noticed that it was mainly because of the properness requirement. The inequalities in the application of the Local Lemma corresponding to the properness constraints are the main bottlenecks which require $4e\Delta$ colours to go through. Here, we attempt to cross this barrier with a slight modification to the event specification by relaxing the properness requirement, but in a limited sense. Afterwards, we take care of the improperness with extra colours.

We define the type of colouring we are looking for as follows.

Definition 3.3.1. For a positive integer η, a colouring of the edges of a graph such that the following holds is called a *η-improper acyclic edge colouring*. Here, the integer η is the measure of improperness.

i Given any vertex v and colour x, at most η edges incident to v are coloured x.

ii There are no proper bichromatic cycles

iii No cycle in the graph is monochromatically coloured.

In the following part of this section we prove that the minimum number of colours needed for such a colouring is upper bounded by $c\Delta$, $1 < c < 3$ assuming a small lower bound for the girth.

Lemma 3.3.1. *For a graph G with girth at least 9, there exists a 2-improper acyclic colouring using at most 3Δ colours.*

Lemma 3.3.2. *For a graph G with girth at least 220, there exists a 4-improper acyclic colouring using at most 1.13Δ colours.*

3.3.1 Proof of Lemmas 3.3.1 and 3.3.2

We provide a common proof for both of the lemmas. In the following proof, we let the improperness η to be a constant which we fix later.

Proof: We assume that $\Delta \geq 4$ in the rest of the arguments. Recall that for the case $\Delta < 4$, we have $\chi'_a \leq \Delta + 2$. This bound is stronger than what we want to prove.

We prove the theorem using probabilistic arguments, making use of the Lovász Local Lemma.

We consider the following random experiment. Each edge chooses a colour uniformly and independently at random from a set of $c\Delta$ colours, where c is specified later. We define the following three types of *unfavourable* events. As explained below, in the absence of these events, the colouring obtained is an η-improper acyclic edge colouring.

Type I For a set of $\eta + 1$ edges $\{e_1, \ldots, e_{\eta+1}\}$ incident on a vertex u, let $E_{e_1,\ldots,e_{\eta+1}}$ be the event that all of them receive the same colour. We call this an event of Type I.

Type II Let $E_{C,2k}$ denote the event that an even cycle C of length $2k$ is properly coloured with 2 colours. We call this an event of Type II.

Type III Let $E_{C,\ell}$ denote the event that a cycle C of length ℓ is coloured monochromatically. We call this an event of Type III.

Claim 3.3.1. *Suppose $\mathcal{C}$ be a colouring such that none of the above events hold. We claim that the resultant colouring is an η-improper acyclic edge colouring.*

Proof: It is easy to see that the absence of events of Type I implies that condition (i) is satisfied. Similarly absence of Type II event ensures that no even cycle is properly two-coloured. Since no odd cycle can be properly two-coloured, we have condition (ii). One can see that the absence of Type III event clearly implies condition (iii). ■

In order to apply the local lemma, we need to find good estimates for the probabilities of each event, and also for the number of other events of each Type which can possibly influence any given event. For the above random experiment, an event $\mathcal{E}$ is mutually independent of a set $\mathcal{B}$ of other events if the set of edges on which $\mathcal{E}$ depends is disjoint from the set of edges on which the events in $\mathcal{B}$ depend. Hence, we calculate the number of events of each type that depend on a given edge, and multiply by the number of edges in $\mathcal{E}$ to get an upper bound on the number of events influencing $\mathcal{E}$. The following two lemmas present the estimates.

The proof of Lemma 3.3.3 is straightforward. Lemma 3.3.4 is also not difficult and uses standard arguments (see [AMR91] for details). For the sake of completeness, we give the arguments for the second case of Lemma 3.3.4 below.

Lemma 3.3.3. *The probabilities of events can be bounded as follows:*

1. *For each Type I event $E_{e_1,\ldots,e_{\eta+1}}$, we have* $\Pr(E_{e_1,\ldots,e_{\eta+1}}) = \frac{1}{(c\Delta)^\eta}$.
2. *For each Type II event $E_{C,2k}$ where length of C is $2k$,* $\Pr(E_{C,2k}) \leq \frac{1}{(c\Delta)^{2k-2}}$
3. *For each Type III event $E_{C,\ell}$ where C is of length ℓ,* $\Pr(E_{C,\ell}) = \frac{1}{(c\Delta)^{\ell-1}}$.

Lemma 3.3.4. *The following holds for any given edge e:*

1. *Less than $\frac{2\Delta^\eta}{\eta!}$ events of Type I depend on e.*
2. *Less than Δ^{2k-2} events of Type II depend on e.*
3. *Less than $\Delta^{\ell-2}$ events of Type III depend on e.*

We give arguments for the case 2 of Lemma 3.3.4. To see why Δ^{2k-2} is in fact an upper bound on the events that depend on a given edge, we fix an edge e. An event of Type II depend on e only if e is part of the associated cycle of length $2k$. Hence we can upper bound the number of events that depend on e with the maximum number of cycles of length $2k$ that contain e. To form such a cycle, we need a path of length $2k-2$ starting from one end point of e and the other end point should be joined to the endpoint of the path. There are at most Δ^{2k-2} possibilities for such a path since at each vertex there are less than Δ choices. Other cases follow from similar arguments.

In order to apply the Lovász Local Lemma, let $x_0 = 1/(\alpha\Delta)^\eta$, $x_k = 1/(\beta\Delta)^{2k-2}$ and $y_\ell = 1/(\gamma\Delta)^{\ell-1}$, be the values associated with events of Types I, II and III respectively, where $\alpha,\beta,\gamma > 1$ are constants to be determined later. Recall that we use g to denote girth. We conclude that, with positive probability none of the above events occur, provided $\forall k \geq \lceil \frac{g}{2} \rceil, \ell \geq g$, the following inequalities hold true.

$$\frac{1}{(c\Delta)^\eta} \leq x_0(1-x_0)^{(\eta+1)\frac{2\Delta^\eta}{\eta!}} \prod_{\theta \geq \lceil \frac{g}{2} \rceil} (1-x_\theta)^{(\eta+1)\Delta^{2\theta-2}} \prod_{\lambda \geq g} (1-y_\lambda)^{(\eta+1)\Delta^{\lambda-2}}$$

$$\frac{1}{(c\Delta)^{2k-2}} \leq x_k(1-x_0)^{2k\frac{2\Delta^\eta}{\eta!}} \prod_{\theta \geq \lceil \frac{g}{2} \rceil} (1-x_\theta)^{2k\Delta^{2\theta-2}} \prod_{\lambda \geq g} (1-y_\lambda)^{2k\Delta^{\lambda-2}}$$

$$\frac{1}{(c\Delta)^{\ell-1}} \leq y_\ell(1-x_0)^{\ell\frac{2\Delta^\eta}{\eta!}} \prod_{\theta \geq \lceil \frac{g}{2} \rceil} (1-x_\theta)^{\ell\Delta^{2\theta-2}} \prod_{\lambda \geq g} (1-y_\lambda)^{\ell\Delta^{\lambda-2}}$$

Let $f(z) = (1-\frac{1}{z})^z$. It is well-known that $(1-\frac{1}{z})^z \uparrow \frac{1}{e}$. Defining

$$\Lambda = \min\left\{ f({x_0}^{-1}),\ \min_{\theta \geq \lceil \frac{g}{2} \rceil} f({x_\theta}^{-1}),\ \min_{\lambda \geq g} f({y_\lambda}^{-1}) \right\},\ \text{ it follows that}$$

$$(1-x_0)^{\frac{2\Delta^\eta}{\eta!}} = \left(1-\frac{1}{(\alpha\Delta)^\eta}\right)^{\frac{2\Delta^\eta}{\eta!}} = \left(\left(1-\frac{1}{(\alpha\Delta)^\eta}\right)^{(\alpha\Delta)^\eta}\right)^{\frac{2}{\eta!\alpha^\eta}} \geq \Lambda^{\frac{2}{\eta!\alpha^\eta}}.$$

Similarly,

$$\prod_{\theta \geq \lceil \frac{g}{2} \rceil} (1-x_\theta)^{\Delta^{2\theta-2}} = \prod_{\theta \geq \lceil \frac{g}{2} \rceil} \left(1-\frac{1}{(\beta\Delta)^{2\theta-2}}\right)^{\Delta^{2\theta-2}} \geq \prod_{\theta \geq \lceil \frac{g}{2} \rceil} \Lambda^{\beta^{-(2\theta-2)}} \geq \Lambda^{S_1}$$

where

$$S_1 = \sum_{\theta \geq \lceil \frac{g}{2} \rceil} \frac{1}{\beta^{2\theta-2}} \leq \frac{1}{(\beta^2-1)\beta^{2\lceil \frac{g}{2} \rceil - 4}}, \quad \text{and}$$

$$\prod_{\lambda \geq g} (1-y_\lambda)^{\Delta^{\lambda-2}} = \prod_{\lambda \geq g} \left(1-\frac{1}{(\gamma\Delta)^{\lambda-1}}\right)^{\Delta^{\lambda-2}} \geq \prod_{\lambda \geq g} \Lambda^{\gamma^{-(\lambda-1)}/\Delta} \geq \Lambda^{S_2}$$

where

$$S_2 = \sum_{\lambda \geq g} \frac{1}{\Delta\gamma^{\lambda-1}} \leq \frac{1}{\Delta\gamma^{g-2}(\gamma-1)}.$$

Thus, taking roots on both sides and simplifying, the three inequalities required by local lemma are satisfied $\forall k \geq \lceil \frac{g}{2} \rceil$, $\ell \geq g$, provided

$$\frac{1}{c} \leq \frac{1}{\alpha}\Lambda^{\frac{\eta+1}{\eta}\Upsilon}, \qquad \frac{1}{c} \leq \frac{1}{\beta}\Lambda^{\frac{2k}{2k-2}\Upsilon} \quad \text{and} \quad \frac{1}{c} \leq \frac{1}{\gamma}\Lambda^{\frac{\ell}{\ell-1}\Upsilon} \tag{3.1}$$

where

$$\Upsilon = \frac{2}{\eta!\alpha^\eta} + \frac{1}{(\beta^2-1)\beta^{2\lceil \frac{g}{2} \rceil - 4}} + \frac{1}{\Delta\gamma^{g-2}(\gamma-1)}.$$

Now to prove Lemma 3.3.1, we set specific values to α, β, γ and η. Set $\eta = 2$ and

$\alpha = \beta = \gamma = 2$ and making use of the facts $g \geq 9$ and $\Delta \geq 4$, we have $\Lambda \geq (1 - \frac{1}{64})^{64} \geq 0.3649$. It can easily be verified that the above inequalities (3.1) are satisfied by setting $c = 2.951$. This proves Lemma 3.3.1.

Similarly, by setting $\eta = 4$, $\alpha = 1.02$, $\beta = 1.04$ and $\gamma = 1.04$, and since $g \geq 220$ and $\Delta \geq 4$, we have $\Lambda \geq (1 - \frac{1}{256})^{256} \geq 0.3671$. By choosing $c = 1.13$, we notice that the inequalities (3.1) are satisfied. Hence Lemma 3.3.2 follows. ∎

3.3.2 Proof of Theorems 3.2.1 and 3.2.2

Once we have an η-improper acyclic edge colouring, we have a deterministic procedure to take care of the improperness. Suppose that we have an η-improper acyclic edge colouring $\mathcal{C}$. Notice that from conditions (i) and (iii), the colouring $\mathcal{C}$ now has the property that, each *colour class* (recall that a colour class is a set of edges receiving the same colour) is a forest of maximum degree at most η.

For each colour α used in $\mathcal{C}$, we properly edge colour the forest formed by the colour class corresponding to α with η colours $\alpha_1, \ldots, \alpha_\eta$. Let the resultant colouring be $\mathcal{C}'$.

Claim 3.3.2. *The colouring $\mathcal{C}'$ obtained above is proper as well as acyclic.*

Proof. We know that every forest of maximum degree at most d can be properly edge coloured using d colours. Hence, after the splitting, the colouring $\mathcal{C}'$ is proper.

The colouring $\mathcal{C}'$ is acyclic, since it was obtained from $\mathcal{C}$ by splitting each colour class and the union of any two colour classes of $\mathcal{C}$ does not form properly 2-coloured cycles. □

Now, from Lemma 3.3.1 we have an η-improper acyclic edge colouring with improperness 2 using less than 3Δ colours provided the girth is at least 9. Using the above procedure, we use at most $2 \times 3\Delta = 6\Delta$ colours to obtain an acyclic edge colouring of G.

Similarly, from Lemma 3.3.2, we have a 1.13Δ colouring with improperness 4 if the girth is at least 220. Again, by using the procedure above, we get a $4 \times 1.13\Delta \leq 4.52\Delta$ acyclic edge colouring.

Further improvements on $\chi'_a(G)$, which can be obtained (with this experiment) by strengthening the girth requirement are only marginal as long as we focus on constant lower bounds on girth. This has been verified by us using computers.

3.4 A general relation between girth and χ'_a

In this section, we present a relation between girth and χ'_a. This quantifies the inverse dependence of χ'_a on girth, mentioned before.

We notice that in the simple random experiment to bound the acyclic chromatic index, the main obstacles are even cycles of short length. This is because the probability that a cycle is 2-coloured depends inversely on the length of the cycle. The smaller cycles have a higher probability of becoming 2-coloured than larger cycles. On the other hand, if the length of the shortest cycle is “reasonably high”, it leads to good upper bounds on acyclic chromatic index. The following theorem captures this observation formally by exhibiting an inverse dependence of the bound on girth.

Theorem 3.4.1. *There are absolute constants c_1, $c_2 > 0$ such that, for any G with $g \geq c_1 \log \Delta$ we have,*

$$\chi'_a(G) \leq \Delta + 1 + \left\lceil c_2 \left(\frac{\Delta \log \Delta}{g} \right) \right\rceil$$

An even cycle is called *half-monochromatic* with respect to a colouring if one of its *halves* (a set of alternate edges) is monochromatic. Notice that, this definition includes bichromatic cycles also.

Proof: (Theorem 3.4.1)

For the sake of simplicity in the analysis, we write g in the form $c_1 \Delta^{\varepsilon} \log \Delta$, where $\varepsilon \geq 0$ and where c_1 is mentioned in Theorem 3.4.1. We can assume without loss of generality that $\varepsilon \leq 1$, because when ε exceeds 1, by choosing a large value of c_1, $\chi'_a(G) \leq \Delta + 2$ as shown in [ASZ01]. As before, we assume that $\Delta \geq 4$, without loss of generality.

The proof consists of an initial deterministic phase followed by a random phase. We begin by obtaining a proper edge colouring of G using $\Delta + 1$ colours by applying Vizing's arguments. We, then randomly recolour some of the edges with a new set of colours, and show that with positive probability, the colouring obtained is proper and acyclic. This random experiment is a slight modification of the one used in [ASZ01].

The random colouring is obtained as follows:

1. Obtain a proper colouring $\mathcal{C} : E \to S_1 = \{1, \ldots, \Delta + 1\}$.
2. In the second phase we do the following:
   - Activate each edge independently with probability $p = \frac{1}{\Delta^{\varepsilon}}$.
   - Each activated edge chooses a new colour uniformly at random and independently, from the set $S_2 = \{1', \ldots, (a\Delta^{1-\varepsilon})'\}$, where $a > 1$ is a constant to be determined later.

Denote the resulting random colouring by $\mathcal{C}'$. With respect to $\mathcal{C}'$, we define the following *unfavourable* events.

1. For a pair of incident edges e and f, let $E_{e,f}$ denote the event that they are both activated and recoloured with the same new colour. We call this an event of Type I.
2. Let $E_{C,2k}$ denote the event that a bichromatic cycle C of length $2k$ in $\mathcal{C}$ is undisturbed in the recolouring process. Call this a Type II event.
3. Let $E_{C,2\ell}$ denote the event that a half-monochromatic cycle C of length 2ℓ in $\mathcal{C}$ becomes bichromatic by retaining the same colour on a half and receiving a common new colour on the other half, a Type III event.
4. Let $E_{C,2m}$ denote the Type IV event where an even length cycle C of length $2m$ becomes properly bichromatic with 2 of the new colours.

We claim that the absence of Type $I-IV$ events imply that the colouring $\mathcal{C}'$ is proper and is also acyclic. Since $\mathcal{C}$ is proper, the absence of events of Type I ensures that $\mathcal{C}'$ is also proper. The absence of events of Type II, III and IV ensure respectively, (i) the absence of bichromatic cycles using both colours from S_1, (ii) one colour from each of S_1 and S_2 and (iii) both colours from S_2. It is therefore sufficient to show the absence of the above four types of events which we do by using Lovász Local Lemma.

To apply the local lemma we need estimates for the probabilities of each event, and for the number of events of each type possibly influencing a given event. As before, we calculate the number of events of each type that depend on a single edge and multiply by the number of edges in any event to get an upper bound on the total dependency. The following two lemmas present the estimated bounds.

Lemma 3.4.2. *The probabilities of events are as follows: For each*

1. *event $E_{f,g}$ of Type I,* $\Pr(E_{f,g}) = \frac{p^2}{a\Delta^{1-\varepsilon}} = \frac{1}{a\Delta^{1+\varepsilon}}$.
2. *event $E_{C,2k}$ of Type II,* $\Pr(E_{C,2k}) = (1-p)^{2k} \le e^{\frac{-2k}{\Delta^{\varepsilon}}}$.
3. *event $E_{C,2\ell}$ of Type III,* $\Pr(E_{C,2\ell}) \le \frac{2p^{\ell}(1-p)^{\ell}}{(a\Delta^{1-\varepsilon})^{\ell-1}} < \frac{2a\Delta^{1-\varepsilon}}{(a\Delta)^{\ell}}$.
4. *event $E_{C,2m}$ of Type IV,* $\Pr(E_{C,2m}) = p^{2m}\binom{a\Delta^{1-\varepsilon}}{2}\frac{2}{(a\Delta^{1-\varepsilon})^{2m}} < \frac{(a\Delta^{1-\varepsilon})^2}{(a\Delta)^{2m}}$.

Lemma 3.4.3. *The following is true for any given edge e:*

1. *Less than 2Δ events of Type I depend on e.*
2. *Less than Δ events of Type II depend on e.*
3. *Less than $2\Delta^{\ell-1}$ events of Type III depend on e, for each $\ell \ge 2$.*
4. *Less than Δ^{2m-2} events of Type IV depend on e, for each $m \ge 2$.*

To apply Lovász Local Lemma, let $x_0 = 1/(\alpha\Delta^{1+\varepsilon})$, $x_1 = 1/(\beta\Delta^{1+2\varepsilon})$, $y_\ell = (2a\Delta^{1-\varepsilon})/(\gamma\Delta)^{\ell}$ and $z_m = (a\Delta^{1-\varepsilon})^2/((\delta\Delta)^{2m})$ be the values associated with events of Type I, II, III and IV, where lengths of cycles III and IV are 2ℓ and $2m$, respectively. Here $\alpha, \beta, \gamma, \delta > 1$ are real values to be determined later. We conclude that with positive probability none of the above events occur, provided $\forall k, \ell, m \ge \lceil \frac{g}{2} \rceil$, we have

$$\frac{1}{a\Delta^{1+\varepsilon}} \le x_0(1-x_0)^{4\Delta}(1-x_1)^{2\Delta} \prod_{\theta \ge \lceil \frac{g}{2} \rceil} (1-y_\theta)^{4\Delta^{\theta-1}} \prod_{\lambda \ge \lceil \frac{g}{2} \rceil} (1-z_\lambda)^{2\Delta^{2\lambda-2}}$$

$$e^{\frac{-2k}{\Delta^{\varepsilon}}} \le x_1(1-x_0)^{4k\Delta}(1-x_1)^{2k\Delta} \prod_{\theta \ge \lceil \frac{g}{2} \rceil} (1-y_\theta)^{4k\Delta^{\theta-1}} \prod_{\lambda \ge \lceil \frac{g}{2} \rceil} (1-z_\lambda)^{2k\Delta^{2\lambda-2}}$$

$$\frac{2a\Delta^{1-\varepsilon}}{(a\Delta)^{\ell}} \le y_\ell(1-x_0)^{4\ell\Delta}(1-x_1)^{2\ell\Delta} \prod_{\theta \ge \lceil \frac{g}{2} \rceil} (1-y_\theta)^{4\ell\Delta^{\theta-1}} \prod_{\lambda \ge \lceil \frac{g}{2} \rceil} (1-z_\lambda)^{2\ell\Delta^{2\lambda-2}}$$

$$\frac{(a\Delta^{1-\varepsilon})^2}{(a\Delta)^{2m}} \le z_m(1-x_0)^{4m\Delta}(1-x_1)^{2m\Delta} \prod_{\theta \ge \lceil \frac{g}{2} \rceil} (1-y_\theta)^{4m\Delta^{\theta-1}} \prod_{\lambda \ge \lceil \frac{g}{2} \rceil} (1-z_\lambda)^{2m\Delta^{2\lambda-2}}$$

Setting $\alpha = \beta = \gamma = \delta = 1000$ and $a = 4000$ and using the fact that $(1-\frac{1}{z})^z \ge \frac{1}{4}$ $\forall z \ge 2$ we have,

$$(1-x_0)^{2\Delta} \geq \left(\frac{1}{4}\right)^{2\Delta x_0} = \left(\frac{1}{4}\right)^{\frac{2}{\alpha\Delta^{\varepsilon}}}; \quad (1-x_1)^{\Delta} \geq \left(\frac{1}{4}\right)^{\Delta x_1} = \left(\frac{1}{4}\right)^{\frac{1}{\beta\Delta^{2\varepsilon}}}$$

$$\prod_{\theta\geq\lceil\frac{g}{2}\rceil}(1-y_\theta)^{2\Delta^{\theta-1}} \geq \left(\frac{1}{4}\right)^{\mathcal{S}_1} \quad \text{and} \quad \prod_{\lambda\geq\lceil\frac{g}{2}\rceil}(1-z_\lambda)^{\Delta^{2\lambda-2}} \geq \left(\frac{1}{4}\right)^{\mathcal{S}_2}$$

where

$$\mathcal{S}_1 = \sum_{\theta\geq\lceil\frac{g}{2}\rceil} 2y_\theta\Delta^{\theta-1} = \frac{4a}{\Delta^{\varepsilon}}\sum_{\theta\geq\lceil\frac{g}{2}\rceil}\frac{1}{\gamma^{\theta}} \leq \frac{4a}{\Delta^{\varepsilon}\gamma^{\lceil\frac{g}{2}\rceil-1}(\gamma-1)}$$

and

$$\mathcal{S}_2 = \sum_{\lambda\geq\lceil\frac{g}{2}\rceil} z_\lambda\Delta^{2\lambda-2} = \frac{a^2}{\Delta^{2\varepsilon}}\sum_{\lambda\geq\lceil\frac{g}{2}\rceil}\frac{1}{(\delta)^{2\lambda}} \leq \frac{a^2}{\Delta^{2\varepsilon}\delta^{2\lceil\frac{g}{2}\rceil-2}(\delta^2-1)}.$$

Let $\mathcal{P}_i$, $\mathcal{N}_i$ and x_i denote, respectively, the probabilities, number of edges and local lemma constants associated with events of type i. We can see that, as in the previous proof, the inequalities required by local lemma are satisfied provided

$$\mathcal{P}_i \leq x_i\left(\frac{1}{4}\right)^{\mathcal{N}_i\Upsilon}, \quad \forall i \tag{3.2}$$

where

$$\Upsilon = \frac{2}{\alpha\Delta^{\varepsilon}} + \frac{1}{\beta\Delta^{2\varepsilon}} + \frac{4a}{\Delta^{\varepsilon}\gamma^{\lceil\frac{g}{2}\rceil-1}(\gamma-1)} + \frac{a^2}{\Delta^{2\varepsilon}\delta^{\lceil\frac{g}{2}\rceil-2}(\delta^2-1)}$$

By choosing c_1 suitably large, we can verify that $\Delta^{\varepsilon}\Upsilon \leq \frac{1}{125}$ and each of the inequalities (3.2) are satisfied. As a result, the inequalities corresponding to local lemma are also satisfied. Finally fixing $c_2 = a \cdot c_1$, the theorem is proved. ■

3.5 Remarks

We brought down the upper bound on $\chi'_a(G)$ from 16Δ to 4.6Δ, assuming the girth to be at least a small constant. We believe that, with a more careful analysis it should be possible to remove the girth assumption.

As we mentioned earlier, it is the short cycles which are difficult to deal with when we use probabilistic arguments, since they have a higher probability of becoming bichromatic as compared to long cycles. Similarly, when we try to kill bichromatic cycles in a proper colouring by randomly recolouring some of the edges with a set of new colours, short cycles have a high probability of survival. It would be interesting to find a way of handling short cycles using probabilistic arguments.

3.6 A note on the claimed 9Δ bound in [MR02]

The proof of $\chi'_a(G) \leq 9\Delta$ given in [MR02] is based on applying a specialised version of Lovász Local Lemma, to the following random experiment: choose a colour for each edge independently and uniformly at random, from a set C of $a\Delta$ colours for some $a > 1$. It is easy to see that the requirements of the local lemma are not met in the proof given.

We give below an argument explaining why the proof is not easily rectifiable even if we ignore the acyclicity requirements and only want to ensure properness. More precisely, we show that any proof, which is based on applying Local Lemma on the random experiment stated above and which assumes a uniform value for all the constants (associated with various events), requires that $a \geq 4e$. It is natural to assume that the constants are uniformly the same unless one wants to look at proofs which make use of the structure of the graph under consideration.

With respect to the random experiment, consider an unfavourable event that a pair of incident edges e, f receive the same colour. Denote it by $E_{e,f}$. Clearly, $\Pr(E_{e,f}) = \frac{1}{a\Delta}$ and the number of other events which may influence a given event is at most 4Δ. Let x_0 be the uniform constant chosen for all events.

Applying the local lemma, we see that none of these bad events hold, if $\frac{1}{a\Delta} \leq x_0(1-x_0)^{4\Delta}$. Write x_0 as $\frac{1}{\alpha\Delta}$. It follows that the inequality of the local lemma holds only if $\frac{1}{a} \leq \frac{1}{\alpha}(1-\frac{1}{\alpha\Delta})^{4\Delta} \leq \frac{1}{\alpha}e^{-4/\alpha}$. Let $f(\alpha) = \frac{1}{\alpha}e^{-4/\alpha}$. To find the extrema, we have $f'(\alpha) = -\frac{1}{\alpha^2}e^{-4/\alpha} + \frac{4}{\alpha^3}e^{-4/\alpha} = 0$, which yields $\alpha^* = 4$. Since $f'(\alpha) = \left(\frac{4}{\alpha^2} - \frac{1}{\alpha}\right)f(\alpha)$, we get $f''(\alpha) = \left(\frac{4}{\alpha^2} - \frac{1}{\alpha}\right)f'(\alpha) + \left(\frac{1}{\alpha^2} - \frac{8}{\alpha^3}\right)f(\alpha)$. Since $f''(\alpha^*) = -\frac{1}{64e} < 0$, the maximum value of $f(\alpha)$ namely $\frac{1}{4e}$ is achieved at $\alpha = 4$. Hence we need to have $\frac{1}{a} \leq \frac{1}{4e}$ or equivalently, $a \geq 4e$.

4

Outerplanar graphs

In this chapter, we show that the acyclic edge colouring conjecture holds true for the class of outerplanar graphs. In fact an even stronger statement is true. That is, if G is an outerplanar graph then $\chi'_a(G) \leq \Delta(G) + 1$. This bound is tight and we also obtain an $O(n \log \Delta)$ time algorithm to produce such a colouring. We make use of several structural properties of outerplanar graphs in the proof.

4.1 Introduction

Recall that a graph is called *planar* if it can be embedded on the plane. A planar graph is *outerplanar*, if there is a planar embedding such that all the vertices are incident on the outer face. With respect to an acyclic edge colouring $\mathcal{C}$, let $\mathcal{C}(v)$ denote the set of colours used by the edges incident on any specific vertex $v \in V(G)$. The standard notation $[k]$ denotes the set $\{1, \ldots, k\}$. Recall that a graph is k-degenerate, if it contains a k^--vertex and the graph obtained by removing all the k^--vertices is also k-degenerate. We recall that a graph is *outerplanar* if there exists a planar embedding of G in which all the vertices lie on the unbounded face.

Formally, we prove the following result.

Theorem 4.1.1. *Let $G = (V, E)$ be an outerplanar graph of maximum degree Δ. Then, $\chi'_a(G) \leq \Delta + 1$. Further, an acyclic $(\Delta + 1)$- edge colouring can be obtained in $O\big(n \log \Delta\big)$ time where n denotes the number of vertices in G.*

Our proof is constructive and we describe an $O\big(n \log \Delta\big)$ time algorithm to obtain a colouring in section 4.3. For bounded values of Δ, the complexity is linear in n.

4.2 Colouring

We first state certain facts relating to outerplanar graphs and relating to acyclic edge colourings. The first four results are standard and can be found in [Wes01].

Fact 4.2.1. *Any subgraph of an outerplanar graph is also outerplanar.*

Fact 4.2.2. *Every outerplanar graph contains a* 2^-*-vertex.*

Fact 4.2.3. *If* G *is an outerplanar graph on 4 or more vertices, then* G *has two non-adjacent* 2^-*-vertices.*

Fact 4.2.4. *Any two-connected outerplanar graph has a unique hamiltonian cycle.*

Fact 4.2.5. *If* $\chi'_a(G) = \eta$ *and if* G *has* $H_1, \ldots, H_k$ *as its blocks, then an acyclic* η*-edge colouring of* G *can be obtained from any collection of acyclic* η*-edge colourings of* $H_1, \ldots, H_k$ *after suitably permuting the colours of edges incident at cut-vertices.*

The above fact can be easily seen as follows. We notice that in the block decomposition of a graph, two blocks share at most one vertex between them. Thus the block graph looks like a tree. Once the blocks are coloured acyclically, one can combine different blocks as follows. Start by processing any arbitrary block. For each of the adjacent block, recolour the single shared vertex with the colour assigned in the processed part by swapping colour classes within the new block if necessary. Due to the tree structure, one can repeat this without conflicts for all the blocks.

The following lemma gives some insight into the structural properties of outerplanar graphs.

Lemma 4.2.1. *Every two-connected outerplanar graph* G *has a 2-vertex adjacent to a* 4^-*-vertex.*

Proof: Since G is two-connected and outerplanar, there is always a vertex of degree 2 in G (using Fact 4.2.2). Noting that no 2-vertex in G can be part of any chord edge of the unique hamiltonian cycle of G (Fact 4.2.4), it follows that every vertex in G can have at most 2 neighbours of degree exactly 2.

Supposing that each vertex of degree 2 has no neighbour having degree at most 4, consider the outerplanar subgraph H obtained by deleting all vertices of degree 2 from G (Fact 4.2.1). The degree of any vertex in H is at least 3. This contradicts Fact 4.2.2.

Thus our claim holds. ■

A stronger version of the above is shown in [HK01] (Lemma 4.2.2 in this reference) which implies Lemma 4.2.1 but the proof is more complicated.

Lemma 4.2.2 (See [HK01]). *Let G be an outerplanar graph with minimum degree at least 2. Then G satisfies one of the following two properties:*

(a) There exists a vertex of degree 2 having a neighbour of degree ≤ 3.

(b) There exist two vertices of degree 2 having a common neighbour of degree 4.

We frequently use the following powerful extension lemma in the proof.

Lemma 4.2.3 (Extension Lemma). *Let G be any graph with maximum degree* Δ *and* u *be a degree* 2 *vertex in* G *with neighbours* v *and* w. *Let* $\mathcal{C}$ *be any* $[k]$*-acyclic edge colouring of* $G \setminus u$, *for some* $k > \Delta$. *If* $|\mathcal{C}(v) \cup \mathcal{C}(w)| < k$, *then* $\mathcal{C}$ *can be extended to get an acyclic edge colouring of G without using additional colours.*

Proof: Colour the edge (u, v) using any colour $c \in [k] \setminus \big(\mathcal{C}(v) \cup \mathcal{C}(w)\big)$. For the edge (u, w), use some arbitrary colour $c' \in [k] \setminus \big(\mathcal{C}(w) \cup \{c\}\big)$. Note that $|\mathcal{C}(w)| \leq \Delta - 1$ and hence c' can always be found. Since $c \notin \mathcal{C}(w)$, the colouring of the edges (u, v) and (u, w) cannot introduce any (c, c')-coloured cycle. ■

We now prove our main result.

Proof:[Theorem 4.1.1] We use induction on the number of vertices. If G is a graph on at most three vertices, then clearly, $\chi'_a(G) \leq \Delta + 1$. Assume the statement is true for each outerplanar graph on fewer than $|V(G)|$ vertices. Using Fact 4.2.5, we can assume without loss of generality that G is 2-connected and hence has a unique hamiltonian cycle.

By Lemma 4.2.2, either (i) G has a vertex u of degree 2 having a neighbour with degree at most 3 or (ii) G has two vertices u and x each of degree *two* having a common neighbour of degree 4. In each case, u is the vertex which is removed for applying inductive hypothesis. For the rest of the proof, we assume that v and w denote the two neighbours of u with degree of v being at most 4.

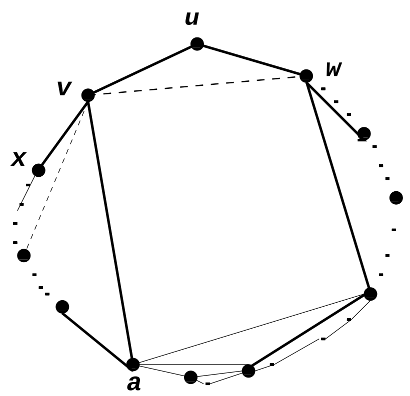

Figure 4.1: *Outerplanar graph with vertex u of degree 2*

By inductive hypothesis, $G \setminus u$ can be acyclically edge coloured using colours from $[\Delta(G \setminus u) + 1] \subseteq [\Delta + 1]$. Let $\mathcal{C}(v)$ and $\mathcal{C}(w)$ be the respective sets of colours used on edges incident at v and w in the acyclic colouring of $G \setminus u$. We have the following cases.

Case 1: $[\Delta(G \setminus u) = \Delta - 1]$ Using inductive hypothesis, $G \setminus u$ can be acyclically edge coloured using colours from $[\Delta]$. Since $\Delta + 1 \notin \mathcal{C}(v) \cup \mathcal{C}(w)$, we can apply the Extension Lemma and obtain an acyclic $[\Delta + 1]$-colouring of G.
Hence, for the rest of the proof, we may assume that $\Delta(G \setminus u) = \Delta$.

Case 2: $[(v, w) \notin E]$ In this case, we consider the graph $H = (G \setminus u) \cup (v, w)$. We have $\Delta(H) \leq \Delta$ and H is 2-connected and outerplanar. By inductive hypothesis, we have an acyclic $(\Delta + 1)$-colouring of H from which we get an acyclic $(\Delta + 1)$-colouring of $G \setminus u$. In this colouring, the colour used on (v, w) is missing from $\mathcal{C}(v) \cup \mathcal{C}(w)$ and hence we can apply the Extension Lemma and obtain an acyclic colouring of G with $[\Delta + 1]$.

Henceforth, we assume that $\Delta(G \setminus u) = \Delta$ and also that $(v, w) \in E$.

Case 3: $[d_v = 3]$ Since $(v, w) \in E$, $\mathcal{C}(v) \cap \mathcal{C}(w) \neq \emptyset$. Then $|\mathcal{C}(v) \cup \mathcal{C}(w)| \leq |\mathcal{C}(v)| + |\mathcal{C}(w)| - 1 \leq 2 + (\Delta - 1) - 1 < \Delta + 1$. By applying the Extension Lemma, we get an acyclic colouring of G with $[\Delta + 1]$.

Case 4: $[d_v = 4]$ By Lemma 4.2.2, there exists a vertex x such that u and x are both degree 2 vertices in G having v as a common neighbour. Since there can be no chord edge incident at a degree 2 vertex of a 2-connected outerplanar graph, it follows that u, v and x appear in that order in the unique hamiltonian cycle of G.

Using the Extension Lemma, we can assume without loss of generality that $\mathcal{C}(v) \cup \mathcal{C}(w) = [\Delta + 1]$. Also, $|\mathcal{C}(v) \cap \mathcal{C}(w)| \geq 1$ since $(v, w) \in E$. Since, $|\mathcal{C}(v)| = 3$ and $|\mathcal{C}(w)| \leq \Delta - 1$, it follows that $|\mathcal{C}(v) \cap \mathcal{C}(w)| = 1$. Without loss of generality, we assume that $\mathcal{C}(v) = \{1, 2, 3\}$ and $\mathcal{C}(w) = \{3, 4, \ldots, \Delta + 1\}$ and 3 is used on (v, w). See Fig. 4.1.

Without loss of generality, assume that the hamiltonian cycle edge (v, x) is coloured with 1. Since x has degree 2 in G, for some $c \in \mathcal{C}(w) \setminus \{3\}$, there is no c-coloured edge incident at x and hence there is no $(1, c)$-coloured path joining v and w in $(G \setminus u) \setminus (v, w)$. Thus, we can safely colour (u, v) with c and (u, w) with 1 to get an acyclic edge colouring of G. This completes the proof of the bound stated in Theorem 4.1.1. The description and analysis of the algorithm are presented in the next section. ∎

4.3 Algorithmic aspects

The proof of $\chi'_a(G) \leq \Delta + 1$ given above for outerplanar graphs is constructive. In this section, we show how to implement the various steps involved efficiently, leading to an $O\big(n \log \Delta\big)$ time algorithm. Here, n denotes the number of vertices. We use m to denote the number of edges. The procedure is described in the pseudo-code BlockColOP(B) given below. In order to keep the discussion simple and brief, we assume that the input B is a 2-connected outerplanar graph (also known as a block). This does not result in any loss of generality for the following reasons.

First, it is easy to see that blocks (maximal 2-connected subgraphs) of an outerplanar graph $G = (V, E)$ are also outerplanar. Moreover, since $|E| \leq 2|V|$ for outerplanar graphs, the blocks and cut-vertices of G can be computed in $O(n)$ time using standard search techniques like DFS. Once this is done, we compute the block-cutpoint graph $BC(G)$ of G in $O(n)$ time. This is a bipartite graph $H = (A, B, F)$ where A is the set of articulation vertices of G and B is the set of blocks of G. For each $a \in A$, $b \in B$, we join them by an edge if and only if $a \in V(b)$. It is easy to verify that $BC(G)$ is a forest in general and is a tree if G is connected.

We now invoke BlockColOP(B) for each block B and obtain an acyclic colouring of B in $O(m_B \log \Delta_B)$ time where m_B denotes the number of edges in B and Δ_B denotes the maximum degree of B. Since $m = \sum_B m_B$, this takes a total of $O(m \log \Delta) = O(n \log \Delta)$ time.

Now, since articulation vertices are shared by more than one block, we need to permute the colourings of edges incident at articulation vertices so as to remove potential conflicts among edges incident at articulation vertices. For this purpose, we first root the tree $BC(G)$ at an articulation vertex a of G and order the articulation vertices and blocks of G based on the preorder-traversal order of $BC(G)$.

For each articulation vertex a of G considered in this order, let B_0 be the parent of a and $B_1, \ldots, B_k$ be its children. When we come to process a, we distribute the remaining colours of $[\Delta + 1]$ *not* used on the edges from B_0 which are incident at a, to the remaining edges (from other blocks $B_1, \ldots B_k$) incident at a. For each block B considered in this order, we permute colour classes of edges in B so as to match the colours used on edges incident at a (the parent of B) with those distributed by a. This takes care of conflicts at each articulation vertex a of G. It is easy to see that this can be achieved in $O(n)$ time with suitable data structures.

Hence it suffices to show that 2-connected outerplanar graphs can be acyclically edge coloured in $O(n \log \Delta)$ time. Now onwards, we assume that G is a 2-connected outerplanar graph.

We assume the adjacency list representation for storing G. The two occurrences of each edge (i, j) (one in $Adj[i]$ and the other in $Adj[j]$) are linked to each other. The set of colours used so far on edges incident at a vertex u are stored in a height balanced binary search tree (BST) $Col(u)$ ordered by the colour values. In addition, we assume two queues $Q3$ and $Q4$ where $Q3$ is a queue on those vertices of degree 2 having a neighbour of degree ≤ 3. $Q4$ is a queue on those vertices of degree 2 having a neighbour v of degree 4 such that v has a neighbour of degree 2. All degrees are with respect to the graph being considered in the current recursive invocation. All data structures mentioned before are assumed to be globally available in each recursive call.

4.3.1 Correctness and Complexity

Since BlockColOP is essentially the proof of Theorem 4.1.1 stated as an algorithm, the correctness follows immediately. So we focus on the complexity of the algorithm.

By adding the edge (v, w) to $B \setminus u$ whenever required, we ensure that the input graph to each recursive call is always 2-connected. Also, since each recursive call works on a graph with one vertex less than its parent call, there are at most n recursive calls.

One can build $Q3$ and $Q4$ initially once in the first invocation of BlockColOP in $O(n)$ time by scanning the adjacency lists. After this, for each recursive call, we only need to update $Q3$ and $Q4$ and do not need to compute them from scratch. It is easy to check that this update can be done in $O(1)$ time. Hence, total time required in all recursive calls for Step 4 is $O(n)$.

After u has been found in Step 4, checking each of the **if** conditions in Steps 5, 10 and 14 can be done in $O(1)$ time per recursive call. Step 7 involves removing the colour of the edge (v, w) from each of $Col(v)$ and $Col(w)$ *if* (v, w) is not part of $E(B)$ and has been explicitly added to B' to make it 2-connected. This can be done in $O(\log \Delta)$ time per recursive call.

We now need to estimate the time required for an application of the Extension Lemma. Recall from its proof that we need to find a colour $c \notin \big(Col(v) \cup Col(w)\big)$ and also a colour $c' \notin \big(Col(w) \cup \{c\}\big)$. For $j = 1, 2, \ldots$, we keep finding the *j-th smallest* colour which is *not* in $Col(w)$ until we find one which is not also in $Col(v)$. Since there exists such a colour and since $|Col(v)| \leq 3$, we don't need to go beyond $j = 4$. For each j, the j-th smallest colour which is absent from $Col(w)$ can be found in $O(\log \Delta)$ time by maintaining the size of each subtree at its root in the BST associated with $Col(w)$. Similarly, one can find c' also. Thus, the total time required for all applications of Extension Lemma is $O(n \log \Delta)$ since there are at most n recursive calls. Step 16 is similar to applying to Extension Lemma and this also requires the same time on the whole.

Thus, the overall time required by BlockColOP(B) is $O(n \log \Delta)$ in the worst case. Hence, an arbitrary outerplanar graph can be acyclically edge coloured using at most $\Delta + 1$ colours in $O(n \log \Delta)$ time.

Algorithm BlockColOP(B)

1: if B is a single edge (u, v) then
2:   colour (u, v) with 1 and RETURN.
3: end if
4: Find a vertex u having exactly two neighbours v and w in B such that *either* (i) degree of v in B is at most 3 *or* (ii) degree of v in B is exactly 4 and v has a neighbour x having degree 2 in B.
5: if $\Delta(B \setminus u) < \Delta(B)$ or if $(v, w) \notin E(B)$ then

6: Obtain an acyclic $(\Delta(B')+1)$ colouring of $B' = (B \setminus u) \cup \{(v,w)\}$ by invoking BlockColOP(B').
7: From this, obtain an acyclic $(\Delta(B')+1)$ colouring of $B \setminus u$.
8: Applying the Extension Lemma, obtain an acyclic $(\Delta(B)+1)$ colouring of B and RETURN.
9: end if
10: if the degree of v in B is exactly 3 then
11: Obtain an acyclic $(\Delta(B \setminus u)+1)$-colouring of $B \setminus u$ by invoking BlockColOP($B \setminus u$).
12: Applying the Extension Lemma, obtain an acyclic $(\Delta(B)+1)$ colouring of B and RETURN.
13: end if
14: if the degree of v in B is exactly 4 then
15: Obtain an acyclic $(\Delta(B \setminus u)+1)$-colouring of $B \setminus u$ by invoking BlockColOP($B \setminus u$).
16: Colour (u,w) with the colour used for (v,x) and colour (u,v) with a colour $c \in C_w \setminus C_x$ to obtain an acyclic $(\Delta(B)+1)$ colouring of B and RETURN.
17: end if

4.4 Conclusions

As mentioned earlier, it is NP-hard to determine $\chi'_a(G)$ even for 2-degenerate graphs. The class of graphs we have studied here (outerplanar graphs) are a non-trivial subclass of 2-degenerate graphs. We have also obtained tight estimates on $\chi'_a(G)$ for a few other subclasses of 2-degenerate graphs. Recently Basavaraju and Chandran have shown that for all 2-degenerate graphs, $\chi'_a \leq \Delta + 1$ (Manuscript). An interesting algorithmic question is to design (if it is possible) a linear, that is $O(n)$, time algorithm for $(\Delta+1)$-acyclic edge colouring of outerplanar graphs. It is also possible to look at the case of bounded tree-width graphs with this approach.

Another interesting problem is to extend these results to planar graphs, which we do in the following chapter with relaxed bounds.

5

Planar and 3-fold graphs

In this chapter, we consider graphs having number of edges linear in terms of the number of vertices. Specifically, we prove that, for the class of planar graphs and 3-fold graphs, χ'_a is at most $2\Delta + 29$. We also show that $\chi'_a \leq \Delta + 6$ for two-fold graphs and triangle-free planar graphs. Our proof is based on the discharging method, well-known for its role in the proof of the four colour theorem. In the first section, we introduce the idea behind the discharging method and demonstrate how it is generally applied with the help of an example. In the next section, we formally state our results without proofs. In the following section, we present the proof arguments. Finally, we conclude the chapter with some open questions and remarks.

5.1 The Discharging Method

The discharging method plays an important role in modern graph theory. The central ideas behind the method are the notion of reducible configurations and the notion of discharging to prove the unavoidability of these configurations. The notion of reducible configurations has been used before also, the first known use being by P. Wernicke [Wer04]. But its full potential was demonstrated by H. Heesch, who developed the idea of discharging. A configuration is any set of substructures. Given the hypothesis that a class of graphs satisfy a certain graph property, a configuration is called *reducible* if it cannot appear in any minimum counterexample to the hypothesis. We say that a set of configurations are *unavoidable* in a family of graphs, if every member of the family contains at least one of the configurations as a substructure. Thus, if we can show that a class of graphs has a set of reducible configurations that are unavoidable, it implies that there cannot be any minimal counterexample and thus the hypothesis has to be true. The discharging method has come to prominence since its initial use in the proof of the Planar Four Colour Theorem by Appel and Hacken [AH76].

5.1.1 Why Discharging?

In the following and for most practical purposes, we consider *charges* as real numbers. The discharging method essentially consists of two phases.

In the initial phase, we assign charges to graph components like vertices, edges, faces, corners etc. This phase is known as the *charging phase*. We then find the total charge by using some known relationship involving these components. For example, in the case of planar graphs, one may make use of Euler's formula to find the sum of charges. Other than Euler's formula, one may also use some other relation between the number of edges and vertices (as we do in our case) or any other relationship that works best.

In the second phase, we redistribute the charges according to a set of carefully formulated rules depending on what we intend to show. This process is called *discharging* due to its similarity to the discharging of electrical charges in a physical system. Thus the method is known as the discharging method.

After the discharging step, we look at the distribution of charges. Based on this and using some structural properties of the graphs, we arrive at some conclusions.

5.1.2 Example: Two simple applications of the discharging method

We give two simple examples to demonstrate the technique of the discharging method. The first example is a well-known fact and its purpose is to illustrate the powerful idea behind the discharging procedure.

Suppose we need to prove the well known fact that every planar graph contains a vertex of degree at most five. It suffices to prove it for maximal planar graphs. Given any maximal planar graph G, we give charges to the vertices and faces of the graph G as follows. To each vertex v, assign a charge of $d(v) - 6$. To each face f of G, we assign a charge $2|f| - 6$. Here, $|f|$ denotes the number of edges in f.

Now the total charge becomes

$$\sum_{v \in V}(d(v) - 6) + \sum_{f \in F}(2|f| - 6) = 2|E| - 6|V| + 4|E| - 6|F| = 6(|E| - |V| - |F|) = -12$$

We make use of the facts that for any graph, the sum of degrees of its vertices is twice the number of edges and for any maximal planar graph, the sum of sizes of faces is again twice the number of edges. We get the last equality from the famous Euler's formula that any planar graph satisfies the equation $|V| + |F| = |E| + 2$.

Therefore we know that the total charge is negative. Since the graph is maximal planar, each face is of length at least 3 and hence $2|f| - 6 \geq 2 * 3 - 6 \geq 0$. That is, each face has non-negative charge. Since the total charge is negative, there has to be some vertex whose charge is negative. This is possible only for vertices of degree at most 5.

We give another simple example to demonstrate how the discharging phase helps in more complicated situations. An edge is called *light*, if the sum of the degrees of its endpoints is at most 11.

Theorem. Every planar graph with minimum degree 5 contains a light edge.

Proof. Initially, we pre-process the graph as follows. Consider any plane embedding of the graph. Triangulate each face by adding edges. This operation cannot introduce light edges, since the endpoints of new edges have degrees at least six each. Since the degree of vertices do not decrease, no old edge becomes light.

After this step, we assign a charge of $6 - d(v)$ to each vertex v. To each face f, we assign a charge of $6 - 2|f|$. The total charge is calculated as follows.

$$\sum_{v} (6 - d_v) + \sum_{f} (6 - 2|f|) = 6|V| - 2|E| + 6|F| - 4|E| = 6(|V| + |F| - |E|) = 12$$

Once again, notice that the selection of charges is such that their summation yields the Euler's formula. This trick is often used to keep the calculations simple when dealing with planar graphs.

In the discharging phase we do the following. Each degree five vertex sends a charge of $1/5$ to each of its neighbours. Before the discharging phase, only vertices of degree five had positive charges. After the discharging, the charge of a vertex v becomes at most $6 - d(v) + d(v)/5 = 6 - \frac{4}{5}d(v)$, which is positive only if the degree of v is at most seven. Note that any vertex that is positively charged after the discharging phase should have received some charge from its neighbours. Since the total charge which remains unchanged is 12, there are some vertices which are positively charged after the discharging procedure.

Let x be any such vertex. The degree of x belongs to $\{5, 6, 7\}$. If degree of x is either five or six, it means that the vertex became positively charged by receiving charge from a degree five neighbour. This implies that there is a light edge involving x. On the other hand if x has degree seven, then at least six of its neighbours have to be of degree five to make it positively charged. Since the graph is a planar triangulation, two of the degree five neighbours must be adjacent implying that there is a light edge. □

5.2 Introduction

Let us recall some basic terms and facts that we use in this chapter. A graph $G = (V, E)$ on n-vertices ($|V| = n$) and m-edges ($|E| = m$) is *planar* if it has a planar embedding. A planar graph together with such an embedding is called a *plane* graph. A face f in a plane graph is a closed (including the outer) region bounded by a set of edges. We denote by $|f|$ the number of edges in a face f and F denotes the set of all faces in a given embedding. A planar graph without cycles of length three is a *triangle-free* planar graph. A planar graph has at most $3n - 6$ edges and a triangle-free planar graph has at most $2n - 4$ edges.

A graph is *k-fold*, if there exists a partition of its edge set into k forests. It follows from the definition that a 2-fold graph has at most $2n-2$ edges and a 3-fold graph has at most $3n - 3$ edges. It is a well known fact that for any graph G, $\sum_{v \in V} d(v) = 2|E| = 2m$. *k-fold*

A vertex of degree k is called a k-vertex. Vertices of degree at most and at least k are respectively called k^- and k^+ vertices. Let us denote the set of k-vertices adjacent to a vertex v with $\mathcal{N}_k(v)$. The notations $\mathcal{N}_{k-}(v)$ and $\mathcal{N}_{k+}(v)$ are analogously defined. We call a graph property *hereditary*, if the property holds for every subgraph as well. Recall that for any graph G, the maximum average degree $mad(G)$ is the quantity defined as

$$mad(G) = \max_{H \subseteq G} \sum_{v \in V(H)} d(v)/|V(H)|.$$

In the following, we show improved upper bounds for χ'_a for the classes of planar graphs, 3-fold graphs, triangle-free planar graphs and 2-fold graphs. We also give upper bounds for graphs with maximum average degree at most 6.

Theorem 5.2.1. *Let G be such that for any $H \subseteq G$, we have $|E(H)| \leq 2|V(H)| - 1$. Then $\chi'_a(G) \leq \Delta(G) + 6$.*

One of our main results is for the classes of triangle-free planar graphs and 2-fold graphs which follows directly from the above theorem. Specifically, it states the following.

Theorem 5.2.2. *Let G be a 2-fold graph or a triangle-free planar graph. Then $\chi'_a(G) \leq \Delta + 6$.*

The proof is immediate from the above theorem and is given below.

Proof:

Form their definitions, it is easy to see that the properties of being k-fold, being planar and being triangle-free are hereditary. That is, the property holds for any subgraph also. We know that for 2-fold graphs, the maximum number of edges is bounded as $|E(G)| \leq 2|V(G)| - 2$. Similarly for any triangle-free planar graph of order at least 3, $|E(G)| \leq 2|V(G)| - 4$ which can be proved using Euler's formula.

Thus the conditions of Theorem 5.2.1 are fulfilled. Hence for the above classes of graphs $\chi'_a(G) \leq \Delta(G) + 6$. ■

Our next result is the following.

Theorem 5.2.3. *Let G be a graph such that for any subgraph $H \subseteq G$, we have $|E(H)| \leq 3|V(H)| - 1$. Then, $\chi'_a(G) \leq 2\Delta(G) + 29$.*

Our main result on planar graphs follows as a corollary of the above theorem. Specifically, we prove that for the classes of planar graphs and 3-fold graphs the following result holds. The result is not tight, but it is the best upper bound known.

Theorem 5.2.4. *Let G be a graph which is either a planar graph or a 3-fold graph. Then $\chi'_a(G) \leq 2\Delta + 29$.*

Once again the proof is immediate from the above theorem and is given below.

Proof: Recall the well-known fact that for any planar graph G, $|E(G)| \leq 3|V(G)| - 6$. Again planarity is a hereditary property and thus the bound holds for any subgraph. Similarly, 3-fold graphs are also hereditary. We also know that any such graph, being the union of three forests satisfy $|E(G)| \leq 3|V(G)| - 3$.

Therefore the conditions of Theorem 5.2.3 are satisfied for both classes of graphs and we get the desired result. ■

We may generalise the above result by using the concept of maximum average degree.

Theorem 5.2.5. *Let G be a graph with* $mad(G) < 6$*. Then* $\chi'_a(G) \leq 2\Delta + 29$.

The proof, which follows easily from Theorem 5.2.3 is given below.
Proof: By the very definition of maximum average degree, we notice that it is a hereditary property. That is, if the *mad* of a graph is strictly less than 6, then all subgraphs also have it as an upper bound.

We notice that $mad(G) < 6$ implies that the total degree of G less than $6n$, and hence the number of edges is strictly less than $3n$ and hence is at most $3n - 1$. Since this property also holds for all subgraphs of G, the conditions of Theorem 5.2.3 are satisfied and as before, the result follows immediately. ■

The theorems 5.2.1 and 5.2.3 are proved by making use of the discharging method. We obtain a set of reducible configurations for acyclic edge colouring the corresponding classes of graphs. We then show that this set of configurations are unavoidable in the class by using the discharging method.

5.3 Digging out some structure — the discharging method

We make use of the structure enforced on the graphs by the hereditary property that the number of edges is linearly bounded. We use the discharging method to fine tune the structure to suit our needs. That is, to enable us to extend a partial acyclic colouring. The following lemmas are instrumental.

Lemma 5.3.1. *Let G be a graph such that* $|E(G)| \leq 2|V(G)| - 1$ *and* $\delta(G) \geq 2$*. Then G contains at least one of the following configurations:*

Conf1: A 2-vertex adjacent to a 5^--vertex.

Conf2: A 3-vertex adjacent to at least *two* 5^--vertices.

Conf3: A 6-vertex adjacent to at least *five* 3^--vertices.

Conf4: A 7-vertex adjacent to *seven* 3^--vertices.

Conf5: A vertex v such that at least $d(v) - 3$ of its neighbours are 3^--vertices with one of them having degree *two*.

Lemma 5.3.2. *Let G be any graph such that $|E(G)| \leq 3|V(G)| - 1$ and $\delta(G) \geq 3$. Then at least one of the following configurations occurs in G:*

Conf1: A vertex of degree 3 having a 11^- neighbour.

Conf2: A vertex of degree 4 having at least *two* 11^- neighbours.

Conf3: A vertex of degree 5 having at least *three* 11^- neighbours.

Conf4: A vertex v, $12 \leq d(v) \leq 14$ such that $|\mathcal{N}_{5^-}(v)| \geq d(v) - 2$.

Conf5: A vertex of degree 15 having at least *fourteen* 5^- neighbours.

Conf6: A vertex v such that $16 \leq d(v) \leq 17$, and all of its neighbours are 5^- vertices.

Conf7: A vertex v such that $|\mathcal{N}_{5^-}(v)| \geq d(v) - 5$, one of the neighbours being of degree *three*.

In the remaining part of this section, we prove the above two lemmas.

5.3.1 Proof of lemma 5.3.1

Proof: Let G, be any graph satisfying the hypothesis. We have $m \leq 2n - 1$ and $\delta(G) \geq 2$. We use the discharging method in the following.

Charging Phase

In this phase we assign to each vertex v, a charge $\Gamma(v) = d(v) - 4$. We then obtain the total charge

$$\sum_{v \in V} \Gamma(v) = \sum_{v} d(v) - 4 = \sum_{v} d(v) - 4n = 2m - 4n \leq -2$$

using the fact that $m \leq 2n - 1$.

Discharging Phase

After the charging, we proceed to let the vertices discharge according to the following rule.

Discharging Rule:

Each vertex, if its degree is at least six, *sends a charge of +1 to each degree two neighbour and a charge of $+\frac{1}{2}$ to each of its degree three neighbours.*

Let the new charge of each vertex v be $\Gamma'(v)$. Since the total charge in the system is conserved,

$$\sum_v \Gamma'(v) = \sum_v \Gamma(v) = -2 \tag{5.1}$$

We now prove Lemma 5.3.1 by way of contradiction; i.e., we show that if any graph satisfying the hypothesis does not satisfy Lemma 5.3.1, then $\sum_v \Gamma'(v) > 0$ contradicting Equation (5.1) of conservation of charges.

We now analyse the possible values of $\Gamma'(v)$ if Lemma 5.3.1 does not hold.

- In the case *Conf1* does not occur in the graph, for each vertex of degree two, both of its neighbours are of degree at least six and thus gives a charge of +1 each to v. Therefore $\Gamma'(v) = -2 + 1 \cdot |\mathcal{N}_{6^+}(v)| = -2 + 2 = 0$.

- When $d(v) = 3$ and *Conf2* does not hold in G, we have $\Gamma'(v) = -1 + \frac{1}{2} \cdot |\mathcal{N}_{6^+}(v)| \geq -1 + \frac{1}{2} \cdot 2 = 0$.

- Vertices with degree 4 and degree 5 do not participate in the discharging and thus retain their charges which are non-negative.

- For a degree six vertex v, $\Gamma'(v) = 2 - 1 \cdot |\mathcal{N}_2(v)| - \frac{1}{2} \cdot |\mathcal{N}_3(v)|$. We have two cases. In the first case, we assume that $|\mathcal{N}_2(v)| = \phi$. In this case, since *Conf3* is prohibited, $|\mathcal{N}_3(v)| \leq 4$ and hence $\Gamma'(v) = 2 - \frac{1}{2} \cdot |\mathcal{N}_3(v)| \geq 2 - \frac{1}{2} \cdot 4 = 0$. Otherwise the vertex v has a degree two neighbour and the absence of *Conf5* assures that there are at most two 3^- neighbours. This implies that $\Gamma'(v) \geq 2 - 2 = 0$.

- Similarly, when $d(v) = 7$, we have $\Gamma'(v) = 3 - 1 \cdot |\mathcal{N}_2(v)| - \frac{1}{2} \cdot |\mathcal{N}_3(v)|$. First we assume that v has a neighbour of degree two. Then, because *Conf5* does not occur, we see that v can have at most $d(v) - 4 = 7 - 4 = 3$ neighbours of degree at most three. Therefore $\Gamma'(v) \geq 3 - 3 = 0$. In the other case where v does not have a degree two neighbour, we have $\Gamma'(v) = 3 - \frac{1}{2} \cdot |\mathcal{N}_3(v)| \geq 3 - \frac{1}{2} \cdot 6 = 0$, since *Conf4* does not occur.

- Finally, consider any vertex v of degree at least eight. Then if v has a degree two neighbour and *Conf5* is not allowed, the vertex v can have at most $d(v) - 4$ neighbours of degree at most three and therefore $\Gamma'(v) \geq 0$. For the case when v does not have a neighbour of degree two, we have $\Gamma'(v) = d(v) - 4 - \frac{1}{2} \cdot |\mathcal{N}_3(v)| \geq d(v) - 4 - \frac{1}{2} \cdot d(v) = \frac{1}{2} \cdot d(v) - 4 \geq 0$.

Thus we observe that if at least one of the five configurations does not occur, every vertex v has a non-negative charge. Therefore the total charge also has to be non-negative. But this contradicts equation (5.1). This completes the proof. ∎

5.3.2 Proof of lemma 5.3.2

Proof: This proof is along the same lines as the previous one. We let G to be any graph satisfying the hypothesis. Therefore we have, $\delta(G) \geq 3$ and $m \leq 3n - 1$.

Charging Phase

As earlier, we assign a charge $\Gamma(v) = d(v) - 6$ to each vertex $v \in V$.

Again we have, the total charge

$$\sum_{v \in V} \Gamma(v) = \sum_{v} d(v) - 6 = \sum_{v} d(v) - 6n = 2m - 6n \leq -2$$

since $m \leq 3n - 1$.

Discharging Phase

As in the previous case, we set discharging rules as given below.

Discharging Rule:

Every vertex of degree at least 12 sends a charge of $+1$ *to each of the adjacent* 3*-vertices, a charge of* $+\frac{2}{3}$ *to every* 4*-vertex in the neighbourhood, and* $+\frac{1}{3}$ *to each* 5*-vertex in its neighbourhood.*

As earlier, we denote the new charge of each vertex v by $\Gamma'(v)$. Since the total charge is conserved we have,

$$\sum_{v} \Gamma'(v) = \sum_{v} \Gamma(v) = -2 \tag{5.2}$$

We now show that any graph satisfying the hypothesis of the lemma must contain at least one of the seven configurations by way of contradiction; i.e., we show that the charges of each vertex becomes non-negative, contradicting (5.2).

We now look at the possible values of $\Gamma'(v)$ depending on the degree of v assuming that the lemma does not hold. We have the following cases.

- Suppose $d(v) = 3$ and *Conf1* is not present in the graph. Then, $\Gamma'(v) = -3 + 1 \cdot |\mathcal{N}_{12^+}(v)| = -3 + 3 = 0$.

- If $d(v) = 4$, then $\Gamma'(v) = -2 + \frac{2}{3} \cdot |\mathcal{N}_{12^+}(v)| \geq -2 + \frac{2}{3} \cdot 3 = 0$, because *Conf2* is not present.

- Assume that $d(v) = 5$ and *Conf3* does not occur. Then we have, $\Gamma'(v) = -1 + \frac{1}{3} \cdot |\mathcal{N}_{12^+}(v)| \geq -1 + \frac{1}{3} \cdot 3 = 0$.

- For vertices v such that $6 \leq d(v) \leq 11$, the charge remain intact and thus $\Gamma'(v) = \Gamma(v) \geq 0$.

- Now suppose that $12 \leq d(v) \leq 14$. We have $\Gamma'(v) = d(v) - 6 - 1 \cdot |\mathcal{N}_3(v)| - \frac{2}{3} \cdot |\mathcal{N}_4(v)| - \frac{1}{3} \cdot |\mathcal{N}_5(v)|$ by the discharging rule. If we assume that the vertex v has a neighbour of degree 3, then, since *Conf7* does not occur in the graph, the number of 5^- neighbours that the vertex v can have is at most $d(v) - 6$ and hence $\Gamma'(v) \geq 0$. Otherwise, we observe that $\Gamma'(v) \geq d(v) - 6 - \frac{2}{3} \cdot |\mathcal{N}_{5^-}(v)| \geq d(v) - 6 - \frac{2}{3} \cdot (d(v) - 3) \geq 0$, since *Conf4* is absent.

- When $d(v) = 15$, $\Gamma'(v) = 9 - |\mathcal{N}_3(v)| - \frac{2}{3} \cdot |\mathcal{N}_4(v)| - \frac{1}{3} \cdot |\mathcal{N}_5(v)|$. As in the previous case, if we assume that the vertex v is adjacent to a 3-vertex, then, since *Conf7* is prohibited, we can see that v can have at most $d(v) - 6$ neighbours of degree at most 5 implying that $\Gamma'(v) \geq 0$. On the other hand, when v is not adjacent to a 3-vertex we have $\Gamma'(v) \geq 9 - \frac{2}{3} \cdot |\mathcal{N}_{5^-}(v)| \geq 9 - \frac{2}{3} \cdot 13 \geq 0$, because *Conf5* does not hold.

- For the cases $16 \leq d(v) \leq 17$, we notice that $\Gamma'(v) = d(v) - 6 - |\mathcal{N}_3(v)| - \frac{2}{3} \cdot |\mathcal{N}_4(v)| - \frac{1}{3} \cdot |\mathcal{N}_5(v)|$. If v is adjacent to a 3-vertex, then the absence of *Conf7* implies that v can be adjacent to at most $d(v) - 6$ vertices of degree at most 5 and hence $\Gamma'(v) \geq 0$. If v does not have a 3-vertex neighbour, $\Gamma'(v) \geq d(v) - 6 - \frac{2}{3} \cdot |\mathcal{N}_{5^-}(v)| \geq d(v) - 6 - \frac{2}{3} \cdot (d(v) - 1) \geq 0$, since *Conf6* does not occur.

- Finally, when $d(v) \geq 18$, $\Gamma'(v) = d(v) - 6 - |\mathcal{N}_3(v)| - \frac{2}{3} \cdot |\mathcal{N}_4(v)| - \frac{1}{3} \cdot |\mathcal{N}_5(v)|$. Supposing that v is adjacent to a 3-vertex and *Conf7* does not occur, we see that the vertex v can be adjacent to at most $d(v) - 6$ vertices of degree at most 5 once again implying $\Gamma'(v) \geq 0$. Again, if v is not adjacent to a 3-vertex, then $\Gamma'(v) \geq d(v) - 6 - \frac{2}{3} \cdot |\mathcal{N}_{5^-}(v)| \geq d(v) - 6 - d(v) \cdot \frac{2}{3} \geq 0$.

Since the charge $\Gamma'(.)$ is non-negative for every vertex, we obtain a contradiction with equation (5.2). ■

5.4 Extending the partial colouring: reducibility

In the sequel we frequently use the following notations.

Let $\mathcal{C}$ denote an acyclic edge colouring of a graph G using k colours, for some integer k. We denote by $\mathcal{C}(v)$ the set of colours assigned by $\mathcal{C}$ to the edges incident to the vertex v. For any subset $W \subseteq V(G)$ we define $\mathcal{C}(W) = \bigcup_{w \in W} \mathcal{C}(w)$. The colour of an edge (u, v) in the colouring $\mathcal{C}$ is denoted by $\mathcal{C}(uv)$.

Let u, v be two distinct vertices of G. The set of neighbours w of the vertex v in G for which $\mathcal{C}(vw) \in \mathcal{C}(u)$ is denoted $W_G(vu)$. Notice that the order of v and u is important here and that the set $W_G(vu)$ could be empty. Recall that a k-colouring is a colouring where the colours are from $[k]$.

The next lemma is a modification of the Extension Lemma (Lemma 4.2.3) from the previous chapter. The graph obtained from G by deleting the edge (u, v) is denoted $G - uv$.

Lemma 5.4.1. *Let G be a graph, $(u, v) \in E(G)$ and let $\mathcal{C}$ be an acyclic k-colouring of $G - uv$. If $|\mathcal{C}(v) \cup \mathcal{C}(u) \cup \mathcal{C}(W_{G-uv}(vu))| < k$, then the colouring $\mathcal{C}$ can be extended to G without using additional colours.*

Proof: It is enough to colour the edge (u, v) with any colour α from the set $[k] - (\mathcal{C}(u) \cup \mathcal{C}(v) \cup \mathcal{C}(W_{G-uv}(vu)))$, to obtain an acyclic k-colouring of G. ■

5.4.1 Proof of Theorem 5.2.1

Proof: We prove the theorem by showing that all the five unavoidable configurations mentioned in Lemma 5.3.1 are reducible. This implies that there is no minimum counterexample to the theorem since at least one of the configurations in present. Suppose that $\mathcal{H}$ is a minimum counterexample to the theorem (one with the number of edges being a minimum). Let $k = \Delta(\mathcal{H}) + 6$.

Recalling the arguments given in the introduction, we may assume without loss of generality that $\mathcal{H}$ is 2-connected. Otherwise, we can obtain an acyclic k-colouring of each of its blocks and combine them (by renaming some colours if needed) to get an acyclic k-colouring of the entire graph. If $\mathcal{H}$ is 2-connected, $\delta(\mathcal{H}) \geq 2$, which we assume for the rest of the proof.

According to Lemma 5.4.1 it suffices if we show that there exists an edge (u, v) for which $|\mathcal{C}(v) \cup \mathcal{C}(u) \cup \mathcal{C}(W_{\mathcal{H}-uv}(vu))| < k$. We show the existence of such an edge in the following. We consider a number of cases, depending on which of the five unavoidable configurations occur in $\mathcal{H}$. In each case we point out such an edge and we apply Lemma 5.4.1 proving the reducibility.

By Lemma 5.3.1 the graph $\mathcal{H}$ contains at least one of the five given configurations.

Configuration 1 : When *Conf1* is present, $\mathcal{H}$ contains a 2-vertex x adjacent to some 5^--vertex y. Let z be the remaining neighbour of x. Moreover, let $\mathcal{H}' = \mathcal{H} - xz$. Since $\mathcal{H}$ was a minimum counterexample, $\chi'_a(\mathcal{H}') \leq k$. Let $\mathcal{C}$ be any acyclic k-colouring of $\mathcal{H}'$ and let $\mathcal{C}(x) = a$. We have the following two cases.

Case 1: If $|\mathcal{C}(z) \cap \{a\}| = 0$ then $|\mathcal{C}(z) \cup \{a\}| \leq \Delta(\mathcal{H})$. Therefore, applying Lemma 5.4.1, $\mathcal{H}$ has an acyclic k-colouring, contradicting the assumption.

Case 2: The other case is when $|\mathcal{C}(z) \cap \{a\}| = 1$. Using the fact that $d_{\mathcal{H}}(y) \leq 5$ we see that $|\mathcal{C}(y)| \leq 5$. Therefore $|\mathcal{C}(z) \cup \mathcal{C}(y)| \leq \Delta(\mathcal{H}) + 3$. Again, using the generalised extension lemma we extend the colouring $\mathcal{C}$ to an acyclic k-colouring of $\mathcal{H}$, a contradiction.

Configuration 2 : This means that there is some 3-vertex x in $\mathcal{H}$ adjacent to two 5^--vertices, say z_1 and z_2. Consider y to be the third neighbour of x. Let $\mathcal{H}' = \mathcal{H} - xz_1$. Since $\mathcal{H}'$ has less edges than $\mathcal{H}$, $\chi'_a(\mathcal{H}') \leq k$ due to the minimality of $\mathcal{H}$. Let $\mathcal{C}$ be an acyclic k-colouring of $\mathcal{H}'$. We have the following cases.

Case 1: $|\mathcal{C}(z_1) \cap \mathcal{C}(x)| \leq 1$.

Here, we have $|\mathcal{C}(x) \cup \mathcal{C}(z_1) \cup \mathcal{C}(W_{\mathcal{H}'}(xz_1))| \leq \Delta(\mathcal{H}) + 4$. Using extension lemma, we extend the colouring $\mathcal{C}$ to an acyclic k-colouring of $\mathcal{H}$, a contradiction.

Case 2: $|\mathcal{C}(z_1) \cap \mathcal{C}(x)| = 2$.

Subcase 2.1 When $\mathcal{C}(xy) \notin \mathcal{C}(z_2)$, we have $|\mathcal{C}(z_1) \cup \mathcal{C}(z_2) \cup \mathcal{C}(x)| \leq \Delta(\mathcal{H}) + 3$. Therefore we recolour (in $\mathcal{H}'$) the edge (x, z_2) with a colour $\alpha \notin \mathcal{C}(z_1) \cup \mathcal{C}(z_2) \cup \mathcal{C}(x)$, obtaining an acyclic k-colouring $\mathcal{C}'$ of $\mathcal{H}'$ reducing it to the previous case.

Subcase 2.2 $\mathcal{C}(xy) \in \mathcal{C}(z_2)$.

*Case*2.2.a If $\mathcal{C}(xz_2) \notin \mathcal{C}(y)$, then $|\mathcal{C}(x) \cup \mathcal{C}(y) \cup \mathcal{C}(z_1)| \leq \Delta(\mathcal{H}) + 3$. Thus we recolour, the edge (x, y) in $\mathcal{H}$ with a colour $\alpha \notin \mathcal{C}(x) \cup \mathcal{C}(y) \cup \mathcal{C}(z_1)$ to obtain an acyclic k-colouring $\mathcal{C}'$ of $\mathcal{H}'$ and we are in the first case.
*Case*2.2.b If $\mathcal{C}(xz_2) \in \mathcal{C}(y)$, then $|\mathcal{C}(x) \cup \mathcal{C}(z_1) \cup \mathcal{C}(z_2) \cup \mathcal{C}(y)| \leq \Delta(\mathcal{H}) + 5$ and using Lemma 5.4.1, we extend the colouring $\mathcal{C}$ to an acyclic k-colouring of $\mathcal{H}$, obtaining a contradiction.

Configuration 3 : The occurrence of *Conf3* implies that there is a 6-vertex x adjacent to five 3^--vertices. Let any two of them be z, z_1. Let y be the remaining neighbour of x. Also let $\mathcal{H}' = \mathcal{H} - xz$. Due to the minimality of $\mathcal{H}$, we have $\chi'_a(\mathcal{H}') \leq k$. Let $\mathcal{C}$ be any acyclic k-colouring of $\mathcal{H}'$.
Case 1: $|\mathcal{C}(x) \cap \mathcal{C}(z)| \leq 1$ or $\mathcal{C}(xy) \notin \mathcal{C}(z)$. It follows $|\mathcal{C}(x) \cup \mathcal{C}(z) \cup \mathcal{C}(W_{\mathcal{H}'}(xz))| \leq \Delta(\mathcal{H}) + 5$ and hence by Lemma 5.4.1 we extend the colouring $\mathcal{C}$ to an acyclic k-colouring of $\mathcal{H}$, a contradiction.

Case 2: $|\mathcal{C}(x) \cap \mathcal{C}(z)| = 2$ and $\mathcal{C}(xy) \in \mathcal{C}(z)$. Assume without loss of generality that $\mathcal{C}(xz_1) \in \mathcal{C}(z)$.
Subcase 2.1 If $|\mathcal{C}(y) \cap \mathcal{C}(x)| = 1$, we recolour the edge (x, y) (in $\mathcal{H}'$) with a colour $\alpha \notin \mathcal{C}(x) \cup \mathcal{C}(y)$ to obtain an acyclic k-colouring $\mathcal{C}'$ and it reduces to the previous case.
Subcase 2.2 If $|\mathcal{C}(y) \cap \mathcal{C}(x)| \geq 2$, then $|\mathcal{C}(x) \cup \mathcal{C}(z) \cup \mathcal{C}(W_{\mathcal{H}'}(xz))| \leq \Delta(\mathcal{H}) + 5$ and by Lemma 5.4.1 we extend the colouring $\mathcal{C}$ to an acyclic k-colouring of $\mathcal{H}$, again a contradiction.

Configuration 4 : If $\mathcal{H}$ has a 7-vertex x adjacent to seven 3^--vertices, then let z be one of its neighbours and let $\mathcal{H}' = \mathcal{H} - xz$. Since $\mathcal{H}'$ has less edges than $\mathcal{H}$, $\chi'_a(\mathcal{H}') \leq k$. Let $\mathcal{C}$ be an acyclic k-colouring of $\mathcal{H}'$. We observe that $|\mathcal{C}(x) \cup \mathcal{C}(z) \cup \mathcal{C}(W_{\mathcal{H}'}(xz))| \leq 10$. Recall that W is the set of neighbours w of x in $\mathcal{H}'$ for which $\mathcal{C}(xw) \in \mathcal{C}(z)$. Therefore, according to Lemma 5.4.1 and since $\Delta(\mathcal{H}) \geq 7$, $\mathcal{H}$ has an acyclic k-colouring, a contradiction.

Configuration 5 : If none of the cases 1-4 occurs, then there must be a vertex x in $\mathcal{H}$ such that at least $d_{\mathcal{H}}(x) - 3$ of its neighbours are 3^--vertices and one of them, say z, is of degree 2. Let us consider the graph $\mathcal{H}' = \mathcal{H} - xz$. Since $\mathcal{H}'$ has less edges than $\mathcal{H}$, we have $\chi'_a(\mathcal{H}') \leq k$. Let $\mathcal{C}$ be an acyclic k-colouring of $\mathcal{H}'$. Let $\mathcal{C}(z) = \{a\}$, $\mathcal{C}_1 = \{\mathcal{C}(xy) : y \in \mathcal{N}_{\mathcal{H}}(x)\ \&\ d_{\mathcal{H}}(y) > 3\}$, and $\mathcal{C}_2 = \{\mathcal{C}(xy) : y \in \mathcal{N}_{\mathcal{H}}(x) - \{z\}\ \&\ d_{\mathcal{H}}(y) \leq 3\}$.

Case 1: If $a \notin \mathcal{C}_1$, then $|\mathcal{C}(x) \cup \mathcal{C}(z) \cup \mathcal{C}(w)| \leq \Delta(\mathcal{H}) + 1$, where w is the neighbour of x in $\mathcal{H}'$ satisfying $\mathcal{C}(xw) = a$. Therefore, from Lemma 5.4.1, $\mathcal{H}$ has an acyclic k-colouring, a contradiction.
Case 2: On the other hand if $a \in \mathcal{C}_1$, let y be the neighbour of z in $\mathcal{H}'$. There is a colour $\alpha \notin \mathcal{C}_1 \cup \mathcal{C}(y)$ such that we can recolour (in the graph $\mathcal{H}'$) the edge (z, y) with α, obtaining in this way an acyclic k-colouring $\mathcal{C}'$ of $\mathcal{H}'$ and it reduces to the previous case. ■

5.4.2 Proof of Theorem 5.2.3

Proof: Following the idea of the previous proof, let us assume that $\mathcal{H}$ is a minimal counterexample to Theorem 5.2.3. For brevity, in the following of the section k stands for $2\Delta(\mathcal{H}) + 29$.

We know from Lemma 5.4.1 that it is enough if we show that there exists some edge (u, v) for which $|\mathcal{C}(v) \cup \mathcal{C}(u) \cup \mathcal{C}(W_{\mathcal{H}-uv}(vu))| < k$.

Claim 5.4.1. *Any graph with $\delta \leq 2$ cannot be a minimum counterexample to the lemma.*

Proof: Let $\mathcal{H}$ be a minimum counterexample with $\delta(H) \leq 2$. Let e be an edge incident to a minimum degree vertex. By the minimality of $\mathcal{H}$, we have $\chi'_a(\mathcal{H} - e) \leq k$. It is easy to see that the requirement of Lemma 5.4.1 is met for the edge e and we can extend any k-acyclic colouring of $\mathcal{H} - e$ to $\mathcal{H}$. ■

Hence we may assume without loss of generality that $\delta(\mathcal{H}) \geq 3$. Therefore, applying Lemma 5.3.2, $\mathcal{H}$ contains at least one of the 7 configurations.

We consider different cases depending on which of the 7 unavoidable configurations occur in $\mathcal{H}$. As in the previous proof we show the existence of a suitable edge in order to apply Lemma 5.4.1 thus proving the reducibility by way of contradiction.

Configuration 1 : There exists a 3-vertex x, adjacent to a 11^--vertex y in $\mathcal{H}$. Let us assume that z is another neighbour of x (in $\mathcal{H}$) and let $\mathcal{H}' = \mathcal{H} - xz$. From the fact that $\mathcal{H}$ is the minimal counter example we see that $\mathcal{H}'$ has an acyclic k-colouring, say $\mathcal{C}$. Moreover, since $d_{\mathcal{H}}(y) \leq 11$, we have $|\mathcal{C}(z) \cup \mathcal{C}(x) \cup \mathcal{C}(W_{\mathcal{H}}(xz))| \leq 2\Delta(\mathcal{H}) + 8$. According to Lemma 5.4.1, it follows that $\mathcal{H}$ has an acyclic k-colouring, a contradiction.

Configuration 2 : If $\mathcal{H}$ contains a 4-vertex x adjacent to at least two 11^--vertices say y_1, y_2, let z be any other neighbour of x (in $\mathcal{H}$) and let $\mathcal{H}' = \mathcal{H} - xz$. Since $\mathcal{H}$ is a minimal counter example, $\mathcal{H}'$ has an acyclic k-colouring. Moreover, because $d_{\mathcal{H}}(y_1), d_{\mathcal{H}}(y_2) \leq 11$, we have $|\mathcal{C}(z) \cup \mathcal{C}(x) \cup \mathcal{C}(W_{\mathcal{H}'}(xz))| \leq 2\Delta(\mathcal{H}) + 18$. From Lemma 5.4.1 it follows that $\mathcal{H}$ has an acyclic k-colouring, a contradiction.

Configuration 3 : If $\mathcal{H}$ has a 5-vertex x adjacent to at least three 11^--vertices y_1, y_2, y_3, then let z be any other neighbour of x. Let $\mathcal{H}' = \mathcal{H} - xz$. As before, $\mathcal{H}'$ has an acyclic k-colouring $\mathcal{C}$. We also have $d_{\mathcal{H}}(y_1)$, $d_{\mathcal{H}}(y_2), d_{\mathcal{H}}(y_3) \leq 11$ and $\Delta(\mathcal{H}) \geq 5$ implying $|\mathcal{C}(z) \cup \mathcal{C}(x) \cup \mathcal{C}(W_{\mathcal{H}'}(xz))| \leq 2\Delta(\mathcal{H}) + 28$. As in the previous case, it follows that $\mathcal{H}$ has an acyclic k-colouring, a contradiction.

Configuration 4 : If $\mathcal{H}$ has a vertex x of degree 12 or 13 or 14, adjacent to at least $d_{\mathcal{H}}(x) - 2$ vertices of degree at most 5 then let z be any of them and let $\mathcal{H}' = \mathcal{H} - xz$. From the fact that $\mathcal{H}$ is the minimal counterexample we have that $\mathcal{H}'$ has an acyclic k-colouring $\mathcal{C}$. From the fact that all, except at most two, neighbours of x in $\mathcal{H}$ are of degree at most 5 we have $|\mathcal{C}(z) \cup \mathcal{C}(x) \cup \mathcal{C}(W_{\mathcal{H}'}(xz))| \leq 2\Delta(\mathcal{H}) + 19$. According to Lemma 5.4.1 it follows that $\mathcal{H}$ has an acyclic k-colouring, a contradiction.

Configuration 5 : If there is a 15-vertex x adjacent to at least fourteen 5^--vertices, then let z be any of them and let $\mathcal{H}' = \mathcal{H} - xz$. As before $\mathcal{H}'$ has an acyclic k-colouring $\mathcal{C}$. We notice that $|\mathcal{C}(z) \cup \mathcal{C}(x) \cup \mathcal{C}(W_{\mathcal{H}'}(xz))| \leq \Delta(\mathcal{H}) + 25 \leq 2\Delta(\mathcal{H}) + 10$, because $\Delta(\mathcal{H}) \geq 15$. It follows from Lemma 5.4.1 that $\mathcal{H}$ has an acyclic k-colouring, a contradiction.

Configuration 6 : If $\mathcal{H}$ contains a vertex x of degree either 16 or 17, such that all its neighbours are 5^--vertices, then let z be any of this neighbours and let $\mathcal{H}' = \mathcal{H} - xz$. From the fact that $\mathcal{H}$ is the minimal counterexample we have that $\mathcal{H}'$ has an acyclic k-colouring $\mathcal{C}$. Moreover, since all neighbours of x in $\mathcal{H}$ are of degree at most 5 we have $|\mathcal{C}(z) \cup \mathcal{C}(x) \cup \mathcal{C}(W_{\mathcal{H}'}(xz))| \leq 32 \leq 2\Delta(\mathcal{H})$. According to Lemma 5.4.1 it follows that $\mathcal{H}$ has an acyclic k-colouring, a contradiction.

Configuration 7 : If $\mathcal{H}$ has a vertex x, having at least $d_{\mathcal{H}}(x) - 5$ neighbours of degree at most 5, one of which say z, is of degree 3, then let $\mathcal{H}' = \mathcal{H} - xz$. From the fact that $\mathcal{H}$ is the minimal counterexample, $\mathcal{H}'$ has an acyclic k-colouring $\mathcal{C}$. Let $\mathcal{C}_1 = \{\mathcal{C}(xy) : y \in \mathcal{N}_{\mathcal{H}}(x) \text{ and } d_{\mathcal{H}}(y) > 5\}$, $\mathcal{C}_2 = \{\mathcal{C}(xy) : y \in \mathcal{N}_{\mathcal{H}}(x),\ y \neq z \text{ and } d_{\mathcal{H}}(y) \leq 5\}$.

If $\mathcal{C}(z) \cap \mathcal{C}_1 = \emptyset$, then $|\mathcal{C}(z) \cup \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}(W_{\mathcal{H}'}(xz))| \leq \Delta(\mathcal{H}) + 7$. From Lemma 5.4.1

it follows that $\mathcal{H}$ has an acyclic k-colouring, a contradiction.

If $\mathcal{C}(z) \cap \mathcal{C}_1 \neq \emptyset$, then let z_1, z_2 be neighbours of z in $\mathcal{H}'$. If $\mathcal{C}(zz_1) \in \mathcal{C}_1$, then there is a colour $\alpha \notin \mathcal{C}_1 \cup \mathcal{C}(z_1) \cup \mathcal{C}(z_2) \cup \mathcal{C}(z)$, with which we recolour the edge (z, z_1) with α, to obtain an acyclic k-colouring $\mathcal{C}'$ of $\mathcal{H}'$. It follows from the fact that $|\mathcal{C}_1 \cup \mathcal{C}(z_1) \cup \mathcal{C}(z_2) \cup \mathcal{C}(z)| \leq 2\Delta(\mathcal{H}) + 4$. Further, if $\mathcal{C}(zz_2) \in \mathcal{C}_1$, then there is a colour $\beta \notin \mathcal{C}_1 \cup \mathcal{C}(z_1) \cup \mathcal{C}(z_2) \cup \mathcal{C}(z) \cup \{\alpha\}$, which can be used to recolour the edge (z, z_2) to obtain an acyclic k-colouring $\mathcal{C}''$ of $\mathcal{H}'$. In each situation the problem reduces to the previous case. ∎

The main results follow immediately from the above, as proved in the introduction.

5.5 Algorithmic Aspects

One of the advantages of using The Discharging Method is that it is inherently algorithmic in nature. Our proof is no different. The Lemmas 5.3.1 and 5.3.2 guarantees the existence of certain types of vertices. We make note of the fact that such a vertex can be efficiently identified and thus we can recursively find an ordering of vertices by deleting the identified vertices. Once we have such an ordering, we can proceed to apply the extension lemma and obtain an acyclic colouring of the original graph inductively.

5.6 Remarks

In this chapter, we have made use of the discharging method to get some information on the local structure for graphs with a linear number of edges like planar graphs and 2-fold graphs. This was exploited to improve the bounds on the acyclic chromatic index of such graphs. It might be interesting to obtain similar bounds for other classes of graphs where the number of edges is linear. It is also interesting to see if the bound for planar graphs can be improved further. We also notice that our proof is constructive and yield efficient algorithm to produce a colouring. If we make use of similar techniques to obtain any linear upper bound, it can lead to efficient algorithms.

6 Some more graphs satisfying AEC

6.1 Introduction

In this chapter, we consider minimally 2-connected graphs, subdivided graphs and Harary graphs. These classes of graphs are interesting as they possess some nice structural properties.

Definition 6.1.1. For $n \geq k \geq 1$ such that $n \geq 2k+1$, the Harary graph $\mathcal{H}_{n,k}$ is defined as $\mathcal{H}_{n,k} = (\{0, 1, \ldots, n-1\}, \{(i, i+j \bmod n) \mid 1 \leq j \leq k\})$.

Note that $\Delta(\mathcal{H}_{n,k}) = 2k$ if $n \geq 2k+1$. If $n = 2k+1$, $\mathcal{H}_{n,k}$ becomes the complete graph K_n. In the following, we first show that the graphs $\mathcal{H}_{n,2}$ are acyclically edge colourable with at most 6 colours. An example of a Harary graph is given in Figure 6.1. Since $\Delta(\mathcal{H}_{n,2}) = 4$, this proves the acyclic edge colouring conjecture for this class. Formally,

Theorem 6.1.1. *The class of graphs $\mathcal{H}_{n,2}$ are acyclically edge colourable with* 6 *colours.*

Definition 6.1.2. A 2-connected graph is *minimally 2-connected* if for every edge e, $G-e$ contains a cut-vertex.

We show that minimally 2-connected graphs can be acyclically edge coloured using at most $\Delta+1$ colours. It is also called a *block line critical* graph [Plu68] or a *chordless graph* since every cycle is without chords as shown in [Plu68].

Theorem 6.1.2. *Let G be a minimally 2-connected graph. Then $\chi'_a(G) \leq \Delta(G)+1$.*

Definition 6.1.3. We call a graph *fully subdivided*, if it is obtained by replacing every edge of a simple graph by a path of length two (by introducing a new vertex.)

We then show that the fully subdivided graphs also can be coloured using at most $\Delta + 1$ colours. Specifically, we prove the following.

Theorem 6.1.3. *If G is fully subdivided, then $\chi'_a(G) \leq \Delta + 1$.*

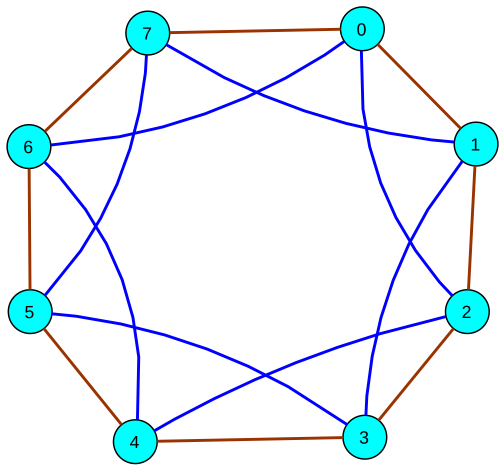

Figure 6.1: *An example of an $\mathcal{H}_{n,2}$ graph*

6.2 Algorithmic proofs of the above theorems

Here we provide algorithmic proofs of the above results. We do not describe the algorithm explicitly, but it is easy to see that the methods implicitly provide efficient algorithms.

6.2.1 Harary Graphs

Proof: (Theorem 6.1.1)
Consider the Harary graph $G = \mathcal{H}_{n,2}$ on n vertices $0, 1, 2, \ldots, n-1$. We colour the graph in two phases. In the initial phase, we colour the graph so that it is proper and 'almost acyclic' and then rectify the colouring in the next phase by recolouring a few edges. Initially, we colour the graph G as follows.
First Phase:

First of all, we assume that $n > 6$ to avoid some technicalities. For smaller values of n, we can easily verify that the result is true. We also note that all the additions and subtractions in the following are with respect to modulo n arithmetic unless stated otherwise.

1. All the edges of the form $(i, i+1)$ are coloured with the colour $i \bmod 3 + 1$, $0 \leq i \leq n-1$.
2. The edges $(i, i+2)$ are coloured with 4 if **either** $i \equiv 0 \bmod 4$ **or** $i \equiv 1 \bmod 4$
3. The edges $(i, i+2)$ are coloured with 5 if **either** $i \equiv 2 \bmod 4$ **or** $i \equiv 3 \bmod 4$
4. If $n \equiv 1 \bmod 3$, then recolour the edge $(n-1, 0)$ with colour 2.
5. If $n \equiv 0 \bmod 4$, recolour the edges $(n-2, 0)$ and $(1, 3)$ with colour 6.
6. If $n \equiv 1 \bmod 4$, recolour the edge $(1, 3)$ with colour 6.
7. If $n \equiv 2 \bmod 4$, recolour the edges $(0, 2)$ and $(1, 3)$ with colour 6.
8. If $n \equiv 3 \bmod 4$, recolour the edge $(n-2, 0)$ with colour 6.

This forms the first phase of the colouring.

As exemplified in Figure 6.1, the Harary graph $\mathcal{H}_{n,2}$ consists of a hamiltonian cycle $C_n = (0, 1, 2, \ldots, n-1, 0)$ on n vertices and every other edge is a chord of this cycle between vertices that are at distance 2 from each other in this cycle. From now on, the edges of the hamiltonian cycle C_n are called *hamiltonian edges* and other edges as *chord edges*. We call the cycles formed by chord edges alone as *chord cycles*. We observe the following fact.

Fact 6.2.1.

- *When* $n \equiv 0 \bmod 4$*, the chord edges form two disjoint cycles of even length. One cycle visits all the even vertices and another cycle visits all the odd vertices.*
- *When* $n \equiv 1 \bmod 4$*, the chord edges form an odd length hamiltonian cycle that visits alternate vertices consecutively.*
- *Also when* $n \equiv 2 \bmod 4$*, the chord edges form two disjoint cycles of odd length, again one that visits all the even vertices and another one that visits all the odd vertices.*
- *When* $n \equiv 3 \bmod 4$*, the chord edges form an odd length hamiltonian cycle.*

Claim 6.2.1. *The colouring obtained after the first phase is proper. Further,* C_n *and the chord cycles are 3-coloured.*

Proof. In Step 1. of the above phase, we colour the edges of C_n using the colours 1,2 and 3. This yields a proper colouring except when $n \equiv 1 \bmod 3$, in which case the edges $(n-1, 0)$ and $(0, 1)$ are both coloured 1. In this case, we recolour the edge $(n-1, 0)$ with colour 2 (Step 4). This makes the colouring of the edges of C_n proper. Figure 6.2 shows all three cases of the colouring of C_n.

Now, we look at the chord edges. It follows from the description of the colouring that the chord edges are coloured with a set of colours disjoint from those used for the hamiltonian edges. Hence we need to only focus on the properness of the colouring restricted to chord edges.

When $n \equiv 0 \bmod 4$, observe that each of the even length chord cycles are coloured properly in Steps 2 and 3 with the colours 4 and 5. We then recolour the edges (Step 5) $(n-2, 0)$ and $(1, 3)$ so that these cycles become acyclic (3-coloured) as well.

When $n \equiv 1 \bmod 4$, we observe that chord cycle is almost proper except that the edges $(1, 3)$ and $(n-1, 1)$ get the same colour 4 (Steps 2,3), making it improper. We recolour the edge $(1, 3)$ (Step 6) using colour 6 to make the colouring of the chord cycle proper. This also makes it 3-coloured. See Figure 6.3 for illustration of the above two cases.

When $n \equiv 2 \bmod 4$, we observe that the edges $(n-2, 0)$, $(0, 2)$, $(n-1, 1)$ and $(1, 3)$ are coloured with the same colour 4 in Steps 2 and 3. Thus the colouring is improper. With the recolouring (Step 7) of the edges $(0, 2)$ and $(1, 3)$, we see that the colouring becomes proper and the chord cycles get 3-coloured.

When $n \equiv 3 \bmod 4$, we notice that the chord cycle is properly coloured except for the edges $(n-2,0)$ and $(0,2)$ that gets the same colour 4 making it improper. We recolour the edge $(n-2,0)$ in Step 8 using colour 6. Thus the colouring becomes proper and chord cycle gets 3-coloured. See Figure 6.4 for illustration. □

Thus, we see that the colouring obtained in the first phase is proper. Also the hamiltonian cycle C_n and the cycles formed by chord edges are 3-coloured. It is still possible that there are bicoloured cycles using both chord edges and hamiltonian edges. In the second phase, we recolour a few edges so that the resulting colouring becomes acyclic. During this process, we ensure that the colouring continues to be proper so that we get a proper acyclic edge colouring of $\mathcal{H}_{n,2}$. We notice and make use of the following fact frequently.

Fact 6.2.2. *All the hamiltonian edges are coloured so that any pair of hamiltonian edges coloured the same are separated by two hamiltonian edges except in the following 2 cases.*

1. When $n \equiv 1 \bmod 3$, the edges $(n-3, n-2)$, $(n-1, 0)$ and $(1, 2)$ are coloured 2.

2. When $n \equiv 2 \bmod 3$, the edges $(n-2, n-1)$ and $(0, 1)$ are coloured 1 and the edges $(n-1, 0)$ and $(1, 2)$ are coloured 2.

We call such pairs of hamiltonian edges listed in Cases 1 and 2 above as *abnormal* pairs. We now describe the recolouring phase. Notice that abnormal pairs are coloured either 1 or 2.

Second Phase:

To show that there are no bicoloured edges using both hamiltonian and chord edges, we make use of the fact (except for the exceptions mentioned in Fact 6.2.2) that two hamiltonian edges of the same colour are separated by at least 2 edges along the hamiltonian cycle C_n. As stated before, this property is not satisfied by *abnormal pairs*. These abnormal pairs can potentially lead to bichromatic cycles. We take care of these potential bichromatic cycles by recolouring at most one edge as given below.

Recolouring

9. $n \equiv 1 \bmod 4$: In this case, irrespective of the value of $n \bmod 3$, recolour the edge $(n-1, 0)$ using colour 6.

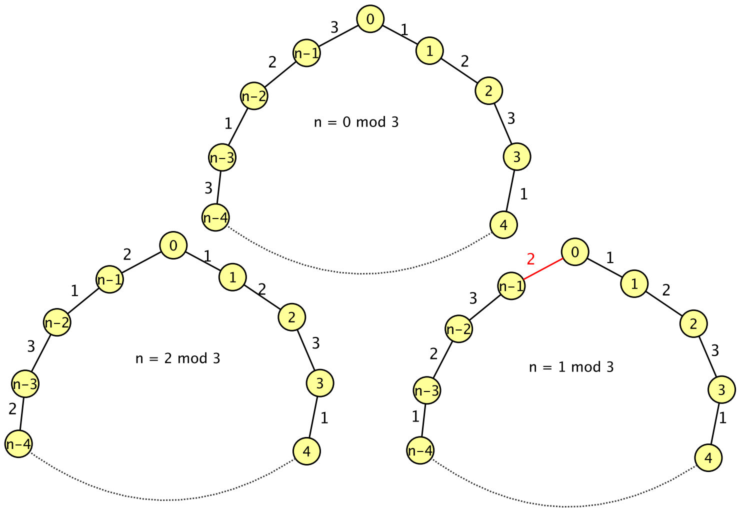

Figure 6.2: *The colouring of the hamiltonian cycle (cases modulo 3)*

10. $n = 2 \bmod 4$: In this case, if we also have that $n \equiv 2 \bmod 3$, recolour the edge $(n-2, n-1)$ with 6.

This forms the recolouring phase. We call the resulting colouring $\mathcal{C}$.

We observe that in Steps 4-8 of the first phase and 9-10 of the recolouring phase, a constant number of edges gets recoloured. We call the edges that gets recoloured in these steps as *exceptional edges*.

We claim that the recolouring phase ensures there are no 2-coloured cycles that use *abnormal pairs*. We also claim that there are no bichromatic cycles using *exceptional edges*. Both of these claims are proved by showing that any bichromatic path of length at least 4 (smallest possible length for any bichromatic cycle) that uses *exceptional edges* or *abnormal pairs* has a terminating vertex (since one of the two colours of the path being not available) and hence cannot lead to a bichromatic cycle.

Claim 6.2.2. *The colouring $\mathcal{C}$ is such that there are no bichromatic cycles using* abnormal pairs *or* exceptional edges.

Proof. The following observation helps to simplify the proof arguments.

Fact 6.2.3. *An abnormal pair of edges cannot lead to a bichromatic cycle unless all the four endpoints of the pair see at least one colour in common.*

If any such colour is present, we call it an *eligible* colour for that abnormal pair.

We refer to a bichromatic path (of length at least 4) that uses colours x and y as an x-y *bipath*. We say that an x-y bipath *terminates* at a vertex u if u belongs to the path and one of the colours x or y is not seen by u.

Notice that there are no bichromatic cycles that uses only the colours 1,2,3 or only the colours 4,5 as they only appear either on the hamiltonian cycle C_n or chord cycles which are shown to be 3-coloured. Hence we do not focus on such pairs of colours. We also observe that there are no abnormal pairs when $n \equiv 0 \bmod 3$.

Thus, in the following four cases, we consider each abnormal pair (coloured either 1 or 2) and see if any of the colours 4, 5, or 6 is *eligible*. For each eligible colour, we show that the bipath using the eligible colour has a terminating vertex, establishing the claim. The Figures 6.3, 6.4, and 6.2 may help to verify the cases.

Case 1: $n \equiv 0 \bmod 4$

In this case, the only *exceptional edges* are $(1, 3)$ and $(n - 2, 0)$ both coloured 6 and $(n - 1, 0)$ coloured 2 (if $n \equiv 1 \bmod 3$ as well). Colour 6 is used only on 2 edges and there is no 4-cycle involving the edges coloured 6 (since $n > 6$). Thus colour 6 cannot lead to a bichromatic cycle.

Notice that when $n \equiv 1 \bmod 3$ the abnormal pairs are $\{(n - 3, n - 2), (n - 1, 0)\}$ and $\{(n - 1, 0), (1, 2)\}$ all coloured 2. As vertex 0 does not see colour 5, colour 5 is *not eligible* for both of the pairs. Since 4 is missing from vertex 1, it is not eligible for the pair $\{(n - 1, 0), (1, 2)\}$. Thus 4 is *eligible* only for the pair $\{(n - 3, n - 2), (n - 1, 0)\}$ which leads to the 2-4 bipath $\{n - 4, n - 2, n - 3, n - 1, 0, 2, 1\}$ terminating at 1.

When $n \equiv 2 \bmod 3$ the abnormal pairs are $\{(n - 2, n - 1), (0, 1)\}$ coloured 1 and $\{(n - 1, 0), (1, 2)\}$ coloured 2. As before, colours 4 and 5 are *not eligible* for the pair $\{(n - 1, 0), (1, 2)\}$. Similarly, since vertex 1 is missing colour 4 and vertex 0 is missing colour 5, neither is eligible for the abnormal pair $\{(n - 2, n - 1), (0, 1)\}$.

This shows for this case that *abnormal pairs* or *exceptional edges* are not part of bichromatic cycles.

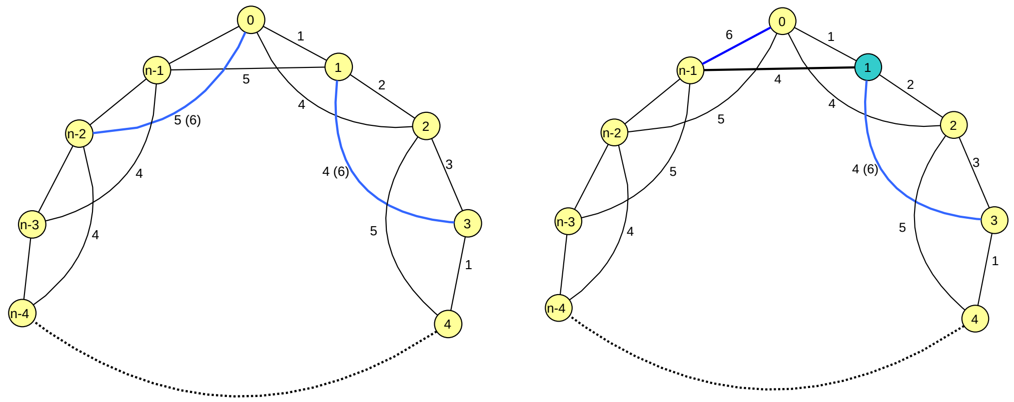

Figure 6.3: *The colouring when* $n \equiv 0 \bmod 4$ *and* $n \equiv 1 \bmod 4$ *resp.*

Case 2: $n \equiv 1 \bmod 4$

Notice that the recolouring (Step 9) does not affect the properness. Also notice that the hamiltonian cycle C_n continues to be acyclic. See Figure 6.3 for illustration.

The only exceptional edges are $(n-1, 0)$ and $(1, 3)$ both coloured 6. Again, it is easy to see that they do not lead to a 4-cycle and hence do not lead to bicoloured cycles.

Since the edge $(n-1, 0)$ gets recoloured with 6, the edges coloured 2 do not become abnormal pairs. Therefore the only abnormal pair that occurs is $(n-2, n-1)$ and $(0, 1)$, both coloured 1, when $n \equiv 2 \bmod 3$ (case 2 of Fact 6.2.2). As colour 5 is missing at vertex 1, it is *not eligible*. The only eligible colour is 4, leading to the 1-4 bipath $\{2, 0, 1, n-1, n-2, n-4\}$ terminating at vertex 2.

As before, this shows that *abnormal pairs* or *exceptional edges* are not part of bichromatic cycles.

Case 3: $n \equiv 2 \bmod 4$

The only exceptional edges are $(0, 2)$, $(1, 3)$ and when $n \equiv 2 \bmod 3$, $(n-2, n-1)$ all coloured 6 **and** $(n-1, 0)$ coloured 2 (if $n \equiv 1 \bmod 3$).

When $n \equiv 1 \bmod 3$, the abnormal pairs are $\{(n-3, n-2), (n-1, 0)\}$ and $\{(n-1, 0), (1, 2)\}$ all coloured 2. As colour 5 is missing from vertex 0, it is *not eligible* for either

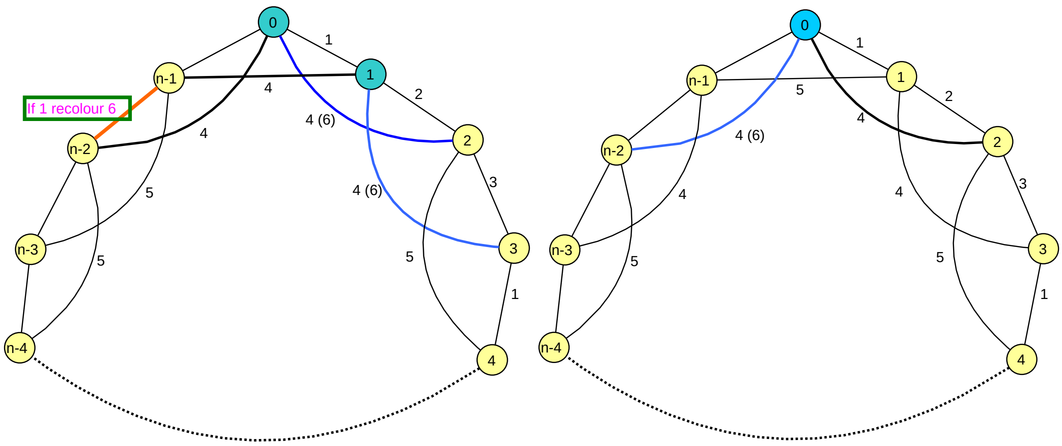

Figure 6.4: *The colouring when* $n \equiv 2 \bmod 4$ *and* $n \equiv 3 \bmod 4$ *resp.*

abnormal pair. Similarly colour 6 is *not eligible* as it is missing from vertex $n-1$. Colour 4 is *eligible* for both pairs leading to the 2-4 bipath $\{n-5, n-3, n-2, 0, n-1, 1, 2\}$ terminating at 2.

When $n \equiv 2 \bmod 3$, the only abnormal pair is $\{(n-1, 0), (1, 2)\}$ coloured 2. Notice that since we recolour $(n-2, n-1)$ by 6 (Step 10) whenever it is coloured 1, there are no abnormal pairs coloured 1.

As before, 5 is *not eligible*. Colour 4 is *not eligible* as it is absent at vertex 2. Colour 6 is eligible (since we recolour $(n-2, n-1)$), leading to the 2-6 bipath $\{3, 1, 2, 0, n-1, n-2\}$ terminating at 3.

Since colour 6 appears on both C_n and chord cycles in this particular case, to see that there are no bicoloured cycles involving the colour 6, we also notice that the 6-4 bipath $\{3, 1, n-1, n-2, 0, 2\}$ terminates at 2 , the 6-3 bipath $\{1, 3, 2, 0, \ldots\}$ terminates at 1 and finally, the 1-6 bipath $\{2, 0, 1, 3, 4\}$ terminates at 2. There is no 6-5 bipath.

As before, this establishes that *abnormal pairs* or *exceptional edges* are not part of bichromatic cycles.

Case 4: $n \equiv 3 \bmod 4$

In this final case, the only exceptional edges are $(n-2, 0)$ coloured 6 and $(n-1, 0)$

coloured 2 (when $n \equiv 1 \bmod 3$). Since we recolour only one edge with 6, it cannot be part of any bicoloured cycle.

When $n \equiv 1 \bmod 3$, the abnormal pairs are $\{(n-3, n-2), (n-1, 0)\}$ and $\{(n-1, 0), (1, 2)\}$ all coloured 2. Colour 5 is missing from vertex 0, and is *not eligible* for either pair. Colour 4 is *not eligible* for the pair $\{(n-3, n-2), (n-1, 0)\}$ as it is missing from $n-2$. It is *eligible* for the pair $\{(n-1, 0), (1, 2)\}$ leading to the 2-4 bipath $\{3, 1, 2, 0, n-1, n-3, n-2\}$ that terminates at 3.

When $n \equiv 2 \bmod 3$, the abnormal pairs are $\{(n-2, n-1), (0, 1)\}$ coloured 1 and $\{(n-1, 0), (1, 2)\}$ coloured 2. Since colour 4 is missing from $n-2$ and colour 5 is missing from 0, *neither is eligible* for the pair $\{(n-2, n-1), (0, 1)\}$. The case for $\{(n-1, 0), (1, 2)\}$ follows from the arguments above (for $n \equiv 1 \bmod 3$) as the colouring of chord edges remains the same.

This proves the claim for this final case and establishes that *abnormal pairs* and *exceptional edges* are not part of bichromatic cycles. □

Thus we have a proper colouring $\mathcal{C}$ in which the hamiltonian cycle C_n and the chord cycles are properly coloured with at least 3 colours. We have also proved that there are no bichromatic cycles that uses the *abnormal* pairs or *exceptional* edges. Thus, we only need to show that there are no bicoloured cycles that use both hamiltonian edges and normal chord edges. This we do in the following.

Claim 6.2.3. *The proper colouring $\mathcal{C}$ is also acyclic.*

Proof. In the following, we show that there are no bicoloured cycles that uses both hamiltonian edges and chord edges. Since there are no bichromatic cycles that uses colour 6 (since exceptional edges cannot be part of them), we only need to consider the edges coloured 1-5 for the rest of the argument. Since colours $1, 2, 3$ appears only on hamiltonian edges and $4, 5$ only on chord edges, it suffices if we show that there are no bichromatic cycles that uses hamiltonian and chord edges alternately. We also make use of the fact that there are no bicoloured cycles that uses abnormal pairs or exceptional edges.

Suppose that there is some 2-coloured cycle formed by chord edges and hamiltonian edges. We know that none of its edges can be an *exceptional* edge. We also notice that any hamiltonian edge $(j, j+1)$ which is not *exceptional* is coloured with $j \bmod 3 + 1$. Start from some vertex i (of this cycle) so that the edge $(i, i+1)$ is a hamiltonian edge of this cycle. This edge is coloured with the colour $\alpha = i \bmod 3 + 1$. The chord edge from $i + 1$ (coloured $\beta \in \{4, 5\}$), can lead to either the vertex $i - 1$ or the vertex $i + 3$.

Suppose that the chord leads to the vertex $i - 1$. In this case, we cannot continue the cycle through the edge $(i - 1, i)$ since it will create an improper triangle, a contradiction. The only other hamiltonian edge is $(i-2, i-1)$ which is coloured $(i-2) \bmod 3+1 \neq \alpha$. This means that we cannot continue the bicoloured cycle, a contradiction.

In the other case, the vertex that we reach is $i + 3$. Notice that the edge $(i + 3, i + 4)$ is coloured $i \bmod 3 + 1 = \alpha$. This leads to the vertex $i + 4$. If the β coloured chord is $(i+4, i+2)$, then the argument is similar to the previous case leading to a contradiction. Otherwise, the forward chord $(i + 4, i + 6)$ is coloured β. We notice that continuing this way, to reach the vertex i (so as to complete the bicoloured cycle), we need either $(i - 2, i)$ or $(i, i + 2)$ be coloured β. The former is not possible since $(i + 4, i + 6)$ and $(i - 2, i)$ are coloured differently in the colouring $\mathcal{C}$. The remaining possibility is that $(i, i + 2)$ is coloured β. Noting that the vertex $(i + 2)$ does not see the colour α (since exceptional edges, if any, are not part of bicoloured cycles), we arrive at a contradiction. □

Since $\mathcal{C}$ is proper and acyclic, it follows that any graph in $\mathcal{H}_{n,2}$ can be coloured with at most 6 colours and thus satisfies the acyclic edge colouring conjecture. ■

6.3 Minimally 2-connected graphs

In this section, we establish Theorem 6.1.2 which states that any minimally 2-connected graph can be acyclically edge coloured with $\Delta + 1$ colours..

We know that every minimally 2-connected graph is 2-degenerate with minimum degree 2 [Bol78]. Let G be a minimally 2-connected graph and let S be the set of vertices of degree 2 in G. We call a path $P = (v_1, v_2, \ldots, v_t)$, $t \geq 3$, in G an S-path, if each of its intermediate vertices $v_2, \ldots, v_{t-1}$ are of degree 2. We have the following results due to Plummer [Plu68] and independently due to Dirac [Dir67].

Theorem 6.3.1 ([Dir67, Bol78]). *Let G be a minimally 2-connected graph. Then for any edge e, $G - e$ decomposes into blocks such that each block is either an edge or a minimally 2-connected subgraph of G.*

Theorem 6.3.2 ([Plu68]). *Let G be a minimally 2-connected graph which is not a cycle. Let S be the set of all degree 2 vertices of G. Then $G - S$ is a forest with components $T_1, T_2, \ldots, T_s$, $s \geq 2$, such that there is no S-path joining two vertices of the same tree T_i.*

The following lemma follows from the above theorem.

Lemma 6.3.3. *Every minimally 2-connected graph G contains some vertex x such that at least $d(x) - 1$ of its neighbours are 2-vertices.*

Proof:

If $\Delta(G) = 2$, then G is a cycle and the lemma is trivial. Otherwise, $\Delta(G) \geq 3$ and from Theorem 6.3.2, we have that $G - S$ is a forest with at least 2 components. Since each component is a tree, it contains some vertex of degree at most 1. Let x be any such vertex. Observe that each neighbour of x other than its unique neighbour (if any) in its component belongs to S and thus has degree 2 in G. See Figure 6.5 for illustration. The lemma follows. ∎

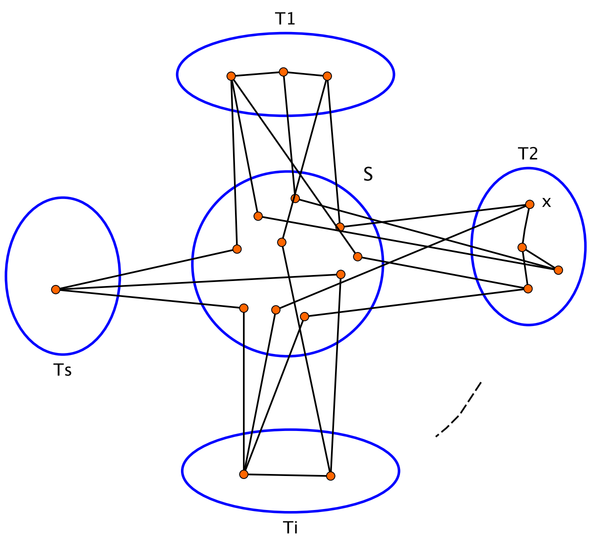

Figure 6.5: *Structure of minimally 2-connected graphs*

We are now ready to prove Theorem 6.1.2.

Proof: We prove the theorem using contradiction. Assume that there is a counter example to Theorem 6.1.2. Let H be a counter example with the least number of edges (minimum counter example) and let S be the set of degree 2 vertices of H.

From Lemma 6.3.3, H contains a vertex u such that all except *possibly* one of its neighbours (say x) are degree 2 vertices. Let v be any of its degree 2 neighbours. From Theorem 6.3.1, $H - uv$ has a cut vertex, and each block of $H - uv$ is either an edge or a minimally 2-connected, induced subgraph. By the minimality of H, each of these blocks can be acyclically edge coloured with $\Delta + 1$ colours and we know (Fact 4.2.5) that if each block of a graph G has an acyclic edge r-colouring, then G also has an acyclic edge r-colouring. Let $\mathcal{C}$ be an acyclic $(\Delta + 1)$-edge colouring of $H - uv$.

Let w be the other neighbour of the 2-vertex v and let $\mathcal{C}(vw) = \alpha$. Note that $|\mathcal{C}(u) \cup \mathcal{C}(vw)| < \Delta + 1$. If α is not seen by the vertex u, then we can use any of the available colours to colour (u, v) thereby extending the colouring to H. See Figure 6.6 for an illustration.

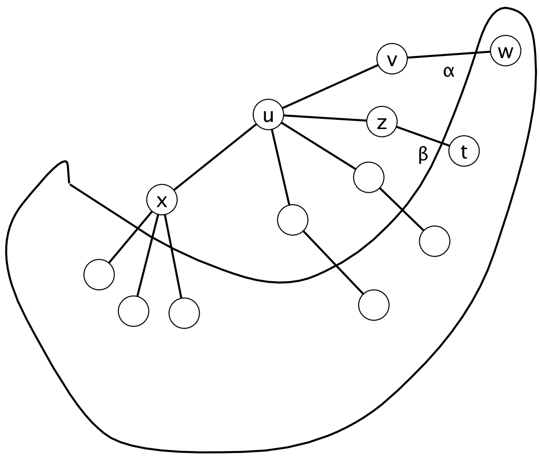

Figure 6.6: *Structure of H in the neighbourhood of u.*

If α is seen by u, then $|\mathcal{C}(u) \cup \mathcal{C}(vw)| < \Delta$ (α is used at both endpoints). We have the following two cases.

Case 1: x does not exist or $\mathcal{C}(ux) \neq \alpha$.

In this case, there is a neighbour $z \neq x$ such that $\mathcal{C}(uz) = \alpha$. Let t be the other neighbour of z and $\mathcal{C}(zt) = \beta$. Notice that assigning an available colour different from β to (u, v) cannot create bichromatic cycles. Since $|\mathcal{C}(u) \cup \mathcal{C}(vw)| < \Delta$, we have at least one colour distinct from β available for the edge (u, v). Therefore we can extend the colouring to (u, v) also.

Case 2: x exists and $\mathcal{C}(ux) = \alpha$.

There are at most Δ colours seen by the vertex w (this is true for any vertex). Thus there is at least one colour free (not used) at w. Since (u, v) is uncoloured, the edge (v, w) can be recoloured with some available colour γ different from α. This reduces the problem to one of the previous cases (γ is not seen by u **or** $\mathcal{C}(ux) \neq \gamma$).

Thus H cannot be a counterexample to the theorem leading to a contradiction. The theorem follows. ∎

6.3.1 Fully Subdivided Graphs

Proof: (Theorem 6.1.3) We provide an algorithmic proof of the theorem.

Our algorithm proceeds iteratively by extending a partial (possibly empty) colouring. At each step, it extends the partial colouring to one or more edges till the graph is entirely coloured. We use a maximum of $\Delta + 1$ colours for the colouring.

We fix an ordering of the edges (arbitrary) and colour them in that order. We notice that in every fully subdivided graph, every edge joins a vertex of degree at most Δ and a vertex of degree two introduced in the subdivision. We call the vertices introduced in the subdivision as *new* vertices. The remaining vertices are referred to as *old* vertices.

At step i, we need to colour edge $e_i = (u_i, v_i)$ where u_i is an old vertex and v_i is a new vertex. Notice that the uncoloured edge e_i has at most $\Delta - 1$ edges adjacent to it at u_i and exactly one at v_i. There are two cases.

Suppose that the two sets of colours seen by the endpoints of e_i do not have any common colour. In this case, there are at most Δ colours used at u_i and v_i together and we have at least one colour available for colouring e_i. Since there are no common colours at the endpoints of e_i, there cannot be any bichromatic cycle passing through it. This completes the first case and we are able to extend the partial colouring.

Suppose that there is a common colour c used at both end points of e_i. We notice that there are at most $\Delta - 1$ colours used at either of the endpoints of e_i together and we have at least 2 colours available for colouring e_i. Let f be the edge incident to u_i that is coloured c. Then the only possible bicoloured cycle through e_i uses c and the colour of the other unique edge adjacent to f. Hence we have at least one colour available to extend the partial colouring to include e_i.

Thus we are able to inductively extend the partial acyclic edge colouring to the whole graph using at most $\Delta + 1$ colours. ■

Part II

k-intersection Edge Colouring

7

k intersection edge colouring.

In this chapter, we introduce the notion of k-intersection edge colouring and provide matching upper and lower bounds on the chromatic index. This problem arose while we were trying to find if there is any general relation between different edge colouring problems and the corresponding chromatic indices. We use two tools from the probabilistic method namely the Chernoff bound and the Lovász Local Lemma to obtain the upper bound. We obtain a matching lower bound by showing that certain graphs require that many colours.

7.1 Introduction

We start with recalling the definition of k-intersection edge colouring. Given $1 \leq k \leq \Delta$, a *k-intersection edge colouring* (k-IEC),of a graph $G = (V, E)$ is a proper edge colouring in which the number of common colours seen by any pair of adjacent vertices is at most k. In other words, it is a proper edge colouring of a graph, in which no more than k colours seen by any vertex u is also seen by a fixed neighbour of u.

Formally, a colouring $\mathcal{C}$ is a k-IEC , if given any vertex u, we have $|\mathcal{C}(u) \cap \mathcal{C}(v)| \leq k$ for every edge $e = (u, v) \in E$. Recall that $\mathcal{C}(x)$ is the set of colours seen by the vertex x in the colouring $\mathcal{C}$. We are interested in bounding the *k-intersection chromatic index* $\chi'_k(G)$, which is the minimum number of colours that suffices to have a k intersection colouring of G.

We know that in a proper edge colouring, the number of common colours between a pair of adjacent vertices can be as high as $\Delta(G)$. We have seen that for any graph G, the chromatic index is at most $\Delta + 1$ from Vizing's argument [Viz64]. So by letting k to be Δ, the k-IEC becomes the usual proper edge colouring.

A distance-2 edge colouring is a proper edge colouring where edges coloured the same are separated by a path on at least 2 edges. The minimum number of colours that suffices for such a colouring of G is the distance-2 chromatic index of G. If a proper edge colouring is also a distance-2 edge colouring, it is a 1-IEC. That is, the number of common colours seen by adjacent vertices is exactly 1. It is easy to see that there are graphs that require $\Omega(\Delta^2)$ colours in any distance-2 edge colouring. So by restricting k to be 1, we know that the k-IEC becomes equivalent to a distance 2 edge colouring and thus requires $\Omega(\Delta^2)$ colours.

Now consider the case when $k = \Delta - 1$ and graphs are regular. This means that for each $(u, v) \in E$, the k-IEC $\mathcal{C}$ is such that $|\mathcal{C}(u) \cap \mathcal{C}(v)| \leq k - 1$ or equivalently, $\mathcal{C}(u) \neq \mathcal{C}(v)$. In other words, each pair of adjacent vertices see a different set of colours. This type of colouring is called an *adjacent vertex distinguishing* edge colouring or AVD colouring for short.

We saw that we need $\Omega(\Delta^2)$ colours in the worst-case for a distance-2 edge colouring, while $\Delta + 1$ is an upper bound for a proper edge colouring. The concept of k-intersection edge colouring simultaneously generalises all the above notions by allowing the maximum number of common colours to be bounded by some k, which lies between 1 and Δ, inclusive of both. Our study is motivated by the interest to know what happens to the chromatic index when the maximum number of common colours varies from 1 to Δ.

It follows from the definition that for any proper edge colouring of a graph G, we need at least Δ colours. V. G. Vizing in [Viz64] showed that there is a proper $\Delta + 1$ edge colouring for any graph. His proof is constructive and immediately provides a deterministic polynomial time algorithm to obtain such a colouring.

We make use of Vizing's result as well as two tools from probabilistic method, namely the Lovász Local Lemma (Symmetric and General forms) and the Chernoff bound. See [AS00, MR02, EL75] for further details on the probabilistic tools.

Specifically, we prove the following.

Theorem 7.1.1. *Let*

$$f_k(\Delta) = \max_{G:\Delta(G)=\Delta} \{\chi'_k(G)\}$$

then $f_k(\Delta) = \Theta(\frac{\Delta^2}{k})$.

We give two lemmas which together imply the above theorem. Recall that $K_{\Delta+1}$ denotes the complete graph on $\Delta + 1$ vertices.

Lemma 7.1.2. *For any graph G with maximum degree Δ,*

1. $\chi'_k(G) \leq \lceil \frac{2\Delta}{k} \rceil (\Delta + 1)$, if $20 \log \Delta \leq k \leq \Delta$.
2. $\chi'_k(G) \leq \frac{22\Delta^2}{k}$, if $1 \leq k \leq \Delta$.

Lemma 7.1.3. $\chi'_k(K_{\Delta+1}) \geq \frac{\Delta^2}{2k}$

Note that statement 2 of Lemma 7.1.2 itself implies the upper bound of Theorem 7.1.1. But Statement 1 improves the constant from 22 to 2 and its proof is based on a different colouring argument. For both statements, it may be possible to optimise the constants involved. Here, the logarithms are the natural logarithms with respect to the base e.

7.2 Upper Bound

Proof:[**Proof (Lemma 7.1.2: Part 1)**] We first handle the case $k \geq 20 \log \Delta$ by making use of the symmetric form of Lovász Local Lemma (stated below).

Lemma 7.2.1 (Symmetric form of Lovász Local Lemma). *Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ be events in a probability space. Suppose that each event $\mathcal{A}_i$ is mutually independent of all but at most d other events $\mathcal{A}_j$. Further assume that* $\Pr(\mathcal{A}_i) \leq p$ *for* $1 \leq i \leq n$. *If*

$$ep(d + 1) \leq 1$$

then $\Pr(\bigwedge_{i=1}^{n} \bar{\mathcal{A}}_i) > 0$.

First, we use Vizing's theorem to obtain a proper $(\Delta+1)$-edge colouring $\mathcal{C}'$. We then design the following random experiment. For each edge coloured a, we assign a new colour chosen independently and uniformly at random from the set $\{a_1, a_2, \ldots, a_\eta\}$, where η is to be fixed later. Let the resulting colouring be $\mathcal{C}$. Let $e_{u,i}$ denote the edge coloured i incident (if any) to the vertex u in the original colouring $\mathcal{C}'$. Given two edges $e_{u,i}, e_{v,i}$ coloured i in $\mathcal{C}'$, the probability that they receive the same colour in $\mathcal{C}$ is $1/\eta$. Let $\mathcal{C}'(u)$ denote the set of colours used on edges incident to vertex u in the initial colouring $\mathcal{C}'$. For any edge $e = (u, v) \in E(G)$, let $s_e = |\mathcal{C}'(u) \cap \mathcal{C}'(v)| - 1$. Let ζ_e stand for the number of common colours i (other than the colour of e) such that edges $e_{u,i}$ and $e_{v,i}$ get the same colour in the new colouring $\mathcal{C}$. A "bad" event is that for some edge $e = (u, v)$, the vertices u and v have at least $k + 1$ colours in common in the new colouring $\mathcal{C}$, or equivalently $\zeta_e \geq k$. Absence of every such event implies that we have the desired colouring. We have $Exp(\zeta_e) = \dfrac{s_e}{\eta}$.

Let $B(n, p)$ denote the sum of n independently and identically distributed indicator variables each having expectation p. By the well-known Chernoff bound (see [AS00, MR95]), we have

$$\Pr(B(n, p) \geq (1 + \epsilon)np) \leq \left(\frac{e^{\epsilon}}{(1 + \epsilon)^{1+\epsilon}}\right)^{np} \text{ for any } \epsilon > 0.$$

Note that $\zeta_e = B(s_e, 1/\eta)$. Further, ζ_e is stochastically dominated by $B(\Delta, 1/\eta)$. We now set $\eta = \lceil \frac{2\Delta}{k} \rceil$. For the sake of simplicity, we ignore the ceilings (without affecting the correctness of the arguments) and treat $\eta = \frac{2\Delta}{k}$. Hence, we have

$$\Pr(\zeta_e \geq k) \leq \Pr(B(\Delta, 1/\eta) \geq k) \leq e^{\frac{-(0.38)k}{2}}.$$

Using the assumption that $k \geq 20 \log \Delta$, we get

$$\Pr(\zeta_e \geq k) \leq e^{-(3.8) \log \Delta} \leq \Delta^{-3.8}.$$

Thus, we obtain an upper bound on the probability of having more than k colours in common between the endpoints of any fixed edge. Since two such events are dependent only if they share an edge, each event is mutually independent of all but at most $2\Delta(\Delta - 1)$ other events. Now, to apply the symmetric form of Lovász Local Lemma, we need to verify that

$$e \times \Delta^{-3.8} \times (2\Delta(\Delta - 1) + 1) \leq 1$$

which holds if

$$\frac{2e\Delta^2}{\Delta^{3.8}} \leq 1 \iff \frac{2e}{\Delta^{1.8}} \leq 1 \iff (2e)^{5/9} \leq \Delta$$

which is true for every $\Delta \geq 3$.

Hence, the result follows for the case $k \geq 20 \log \Delta$ by fixing $\eta = \lceil \frac{2\Delta}{k} \rceil$. Since any number strictly less than but sufficiently close to 2 suffices, a more tight analysis shows that we can do away with the additive sub-linear term. We skip the details. ■

Now we look at the case when $k < 20 \log \Delta$ where the above arguments fail because the uniform upper bound on the probabilities $\Pr(\zeta_e \geq k)$ is not sufficiently small to apply the symmetric version of Local Lemma.

Proof: (Lemma 7.1.2: Part 2) We assume that $\Delta \geq 6$, since otherwise k is at most 5 and even a distance-2 colouring suffices, since the number of colours used would be at most $2\Delta(\Delta - 1) + 1 < 22\Delta^2/k$ for the range $k \leq \Delta \leq 5$. For proving Part 2, we use the most general form of Lovász Local Lemma as given below.

Lemma 7.2.2 (The Lovàsz Local Lemma (general form)). *Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be events in a probability space Ω such that each event A_i is mutually independent of all events in $\mathcal{A} - (\{A_i\} \cup \mathcal{D}_i)$, for some $\mathcal{D}_i \subseteq \mathcal{A}$. Also suppose that there exist $x_1, \ldots, x_n \in (0, 1)$ such that*

$$\Pr(A_i) \leq x_i \prod_{A_j \in \mathcal{D}_i} (1 - x_j), \quad 1 \leq i \leq n$$

Then $\Pr(\overline{A_1} \wedge \ldots \wedge \overline{A_n}) > 0$.

Consider the following random experiment. Colour each edge uniformly and independently at random with one of the $c = 22\Delta^2/k$ colours. We define the following two types of *bad* events.

Type *I* A pair of incident edges e, f receive the same colour. Denote this event by $\mathcal{E}_{e,f}$.

Type *II* For an edge $e = (u, v)$, and sets $S_1 \subseteq \mathcal{E}(u) \setminus \{e\}$, $S_2 \subseteq \mathcal{E}(v) \setminus \{e\}$, with $|S_1| = |S_2| = k$, these edges are properly coloured with a set of k colours. We denote this event by $\mathcal{E}_{e,S_1,S_2}$.

Suppose that the random colouring $\mathcal{C}$ is such that none of the above events hold. Then $\mathcal{C}$ is proper, and further, no two adjacent vertices share more than k colours in common. We show that this happens with positive probability. In order to apply LLL, observe that the probabilities of each type of event can be upper bounded as in the following lemma.

Lemma 7.2.3. *For each event*

1. $\mathcal{E}_{e,f}$ *of Type I,* $\Pr(\mathcal{E}_{e,f}) = \dfrac{1}{c}$.

2. $\mathcal{E}_{e,s_1,s_2}$ *of Type II,* $\Pr(\mathcal{E}_{e,s_1,s_2}) = \dfrac{\binom{c}{k}(k!)^2}{c^{2k}} \leq \dfrac{k!}{c^k}$.

We obtain an upper bound on the number of events of any type whose outcome depends on a given edge. We then multiply this by the number of edges whose colouring affects a given event $\mathcal{E}$ to get an upper bound on the number of other events of each type on which $\mathcal{E}$ could possibly depend.

Lemma 7.2.4. *For an edge e, the following holds true.*

- *at most* $2\Delta - 2$ *events of Type I depend on e.*
- *at most* $2\Delta\binom{\Delta-1}{k}\binom{\Delta-2}{k-1} < \frac{2\Delta^{2k}}{k!(k-1)!}$ *events of Type II depend on e.*

Proof: Since the edges are coloured independently, the outcome of any event depends exclusively on the colouring of the set of edges on which it is defined. It follows that the colour received by a given edge influences exactly those events whose definitions are based on that edge.

The event of Type I is based on a pair of incident edges. Thus, to compute the number of events of Type I, whose outcome is dependent on an edge e, we have to compute the number of pairs of incident edges to which e belongs. The number of such pairs can be easily seen to be at most 2Δ. Similarly, an event of Type II is defined on a set of $2k$ edges, incident on a fixed edge, k at each endpoint. The number of such sets of $2k$ edges containing e is at most $2\Delta\binom{\Delta-1}{k}\binom{\Delta-2}{k-1}$. This is explained as follows. For an edge $f = (w, x)$, we need to select an edge e incident to either w or to x. There are at most 2Δ possibilities for this. We then need to choose $k - 1$ edges for the set (containing f) from among the remaining at most $\Delta - 2$ edges and choose another k edges from the other endpoint of e. ■

Observe that the outcomes of events of Types I and II depend respectively on exactly 2 and $2k$ participating edges. In order to apply the Local Lemma we now need to choose real numbers associated with the bad events which ensures that the inequalities are satisfied. We choose $x = \dfrac{2}{c}$ and $y = \dfrac{k!2^k}{c^k}$ as the real numbers associated with events of Type I, II respectively.

To verify that the colouring $\mathcal{C}$ satisfies the required properties, it is sufficient to show that the following inequalities hold:

$$\frac{1}{c} \le \frac{2}{c}\left(1-\frac{2}{c}\right)^{2\times 2\Delta}\left(1-\frac{k!2^k}{c^k}\right)^{2\times \frac{2\Delta^{2k}}{k!(k-1)!}} \tag{7.1}$$

and

$$\frac{k!}{c^k} \le \frac{k!2^k}{c^k}\left(1-\frac{2}{c}\right)^{2k\times 2\Delta}\left(1-\frac{k!2^k}{c^k}\right)^{2k\times \frac{2\Delta^{2k}}{k!(k-1)!}}. \tag{7.2}$$

Simplifying and taking roots, we see that both, (1) and (2), follow from

$$1 \le 2\left(1-\frac{2}{c}\right)^{4\Delta}\left(1-\frac{k!2^k}{c^k}\right)^{\frac{4\Delta^{2k}}{k!(k-1)!}}, \tag{7.3}$$

Substituting $c = 22\Delta^2/k$, (3) is equivalent to

$$1 \le 2\left(1-\frac{k}{11\Delta^2}\right)^{4\Delta}\left(1-k!\left(\frac{k}{11\Delta^2}\right)^k\right)^{\frac{4\Delta^{2k}}{k!(k-1)!}}, \tag{7.4}$$

and further to

$$1 \le 2\beta_1^{\frac{4k}{11\Delta}}\beta_2^{\frac{4}{(k-1)!}\left(\frac{k}{11}\right)^k}, \tag{7.5}$$

where,

$$\beta_1 = \left(1-\frac{k}{11\Delta^2}\right)^{\frac{11\Delta^2}{k}} \text{ and } \beta_2 = \left(1-\frac{k!k^k}{(11)^k\Delta^{2k}}\right)^{\frac{(11)^k\Delta^{2k}}{(k!k^k)}}.$$

By using the assumptions $\Delta \ge 6$ and $k \le \Delta$, one can verify that $\beta_1, \beta_2 \ge 1/4$ and also that the sum of the exponents of β_1 and β_2 in (5) is at most $1/2$. This establishes inequality (5). It follows that there exist a k-intersection edge colouring of G using $22\Delta^2/k$ colours. ■

We can also show that if $1 \le k \le \log\Delta$, then $\chi'_k(G) \le \frac{13\Delta^2}{k}$. The proof follows from essentially the same argument making use of the fact that k is at most $\log\Delta$. Hence the upper bound can be improved to $\frac{13\Delta^2}{k}$ for all graphs irrespective of the value of k.

7.3 A Lower Bound

This section proves the Lemma 7.1.3 to obtain a matching lower bound.

Proof:(Lemma 7.1.3)

To show that the bound is tight, we show that complete graphs require at least $\frac{\Delta^2}{2k}$ colours. As before, we use the standard notation $K_{\Delta+1}$ to denote the complete graph on $\Delta + 1$ vertices. Without loss of generality, assume that $V(K_{\Delta+1}) = \{1, 2, \ldots, \Delta + 1\}$. The set of edges incident to a vertex v is denoted $E(v)$. Recall that $\mathcal{C}(v)$ is the set of colours seen by v in $\mathcal{C}$.

Consider any k-intersection edge colouring $\mathcal{C}$ of $K_{\Delta+1}$. Starting from vertex 1, we scan the vertices in increasing order. There are Δ colours used on $E(1)$. Since $|\mathcal{C}(1) \cap \mathcal{C}(2)| \leq k$, at least $\Delta - k$ colours should have been used on $E(2)$ which are not used for $E(1)$. Similarly, $E(3)$ should use at least $\Delta - 2k$ colours which are not used on $E(1) \cup E(2)$. Continuing in this way, we see that a minimum of $\Delta + \Delta - k + \ldots + \Delta - \lfloor \frac{\Delta}{k} \rfloor \times k$ colours are required to colour $K_{\Delta+1}$. Using s to denote $\lfloor \Delta/k \rfloor$, the summation above can be re-written as

$$\Delta(s + 1) - k(1 + 2 + \ldots + s) = (s + 1)(\Delta - ks/2) \geq (s + 1)(\Delta/2) \geq \frac{\Delta^2}{2k}.$$

This establishes the lemma and lower bound. ■

7.4 Remarks

It would be interesting to know, whether the lower bound is tight for other classes of graphs like bicliques (complete bipartite graphs). An interesting problem is to design an efficient algorithm which produces a k-intersection edge colouring that matches the upper bounds derived here.

We have mentioned that the constants for the upper bounds can be improved by assuming that k is at most $\log \Delta$. Similarly it may be possible to improve the lower bound drastically. We believe that with some tight analysis, one could show that the upper and lower bounds can be brought closer. This also is an interesting open problem.

Part III

Oriented Colouring

8
Oriented Colouring

8.1 Introduction

The concept of oriented colouring was introduced by Bruno Courcelle in [Cou94]. Since then, many researchers have worked on the problem due to its importance in mobile communication and VLSI design. Even for simple classes of graphs like 2-dimensional grid graphs or planar graphs, we do not know tight bounds on the value of this parameter. In the next section, we start with a few important definitions and then proceed to present the results we have obtained.

8.1.1 Definitions and Results

An oriented graph $\overrightarrow{G} = (V, A)$ is an orientation of the edges of a simple undirected graph $G = (V, E)$. That is, $\overrightarrow{G}$ does not contain loops or opposite arcs. An oriented k-colouring of an oriented graph is a partition of its vertex set into k labelled subsets such that no two adjacent vertices belong to the same subset, and all the arcs between a pair of subsets have the same orientation. Precisely, an *oriented k-colouring* of an oriented graph $\overrightarrow{G}$ is a mapping $\mathcal{C} : V \mapsto [k]$ such that **(i)** $\mathcal{C}(x) \neq \mathcal{C}(y)$ for any arc $(x, y) \in A(\overrightarrow{G})$ and **(ii)** $\mathcal{C}(x) = \mathcal{C}(w)$ only if $\mathcal{C}(y) \neq \mathcal{C}(z)$ for all arcs (x, y) and (z, w) in $A(\overrightarrow{G})$. Notice that $\mathcal{C}(x)$ in this chapter stands for the colour of the vertex x w.r.t. the colouring $\mathcal{C}$.

The *oriented chromatic number* of an oriented graph $\overrightarrow{G}$ is the smallest $k \in \mathbb{N}$ that admits an oriented vertex k-colouring of $\overrightarrow{G}$ and is denoted by $\chi_o(\overrightarrow{G})$. One can also view an oriented k-colouring as a homomorphism from $\overrightarrow{G}$ to a suitable oriented graph on k vertices. A *homomorphism* from a directed graph $\overrightarrow{G}$ to a directed graph $\overrightarrow{H}$ is a mapping that preserves the arcs. That is, $\phi : V(\overrightarrow{G}) \mapsto V(\overrightarrow{H})$ is a homomorphism if $(\phi(u), \phi(v)) \in A(\overrightarrow{H})$ for every arc (u, v) in $A(\overrightarrow{G})$. Hence we note that $\chi_o(\overrightarrow{G})$ is the smallest order of an oriented graph $\overrightarrow{H}$ such that there is a homomorphism from $\overrightarrow{G}$ to $\overrightarrow{H}$. The oriented chromatic number of an undirected graph G denoted $\chi_o(G)$, is the maximum $\chi_o(\overrightarrow{G})$ over over all orientations $\overrightarrow{G}$ of G. The work [Sop01] of Sopena contains a number of interesting results on this problem.

An automorphism of an oriented graph $\overrightarrow{G}$ is a bijection from $V(\overrightarrow{G})$ to itself that preserves edges, non-edges and directions of the edges. If an automorphism does not map any vertex to itself, we call it a *non-fixing automorphism*.

Recall that K_2 denotes an edge and P_k denotes an undirected path on k vertices. Also recall the definitions of Cartesian product and Strong product of graphs defined in Chapter 1.

For any graph G, let $G^n_\square$ and $G^n_\boxtimes$ denote respectively the cartesian and strong products of G with itself n times. The graph $\mathcal{H}_d = K_2{}^d_\square$ is called the *hypercube* of dimension d. In other words $\mathcal{H}_d$ is the cartesian product of d edges. The *d-dimensional hyper grid (mesh)* denoted $\mathcal{M}_d$ is the cartesian product of d paths. The graph $\mathcal{M}_{m,n} = P_m \square P_n$ is called an *$m \times n$ grid*. We call the graph $\mathcal{S}_{m,n} = P_m \boxtimes P_n$ an *$m \times n$ strong-grid*.

A tournament is an orientation of an undirected complete graph. Let n be a prime number of the form $4k + 3$. Let $c_1, c_2, \ldots, c_d$ be the non-zero quadratic residues of n. It is known that $d = \frac{n-1}{2}$. Define a directed graph $\overrightarrow{T}_n = T(n; c_1, \ldots, c_d)$ over $V = \{0, 1, \ldots, n-1\}$ as follows. For every $x, y \in V,\ x \neq y$, (x, y) is an arc if $y = x + c_i$ for some $i \in [d]$. It is well-known that $\overrightarrow{T}_n$ is a tournament and is called the *Paley tournament*.

A graph G is *arc transitive* if for any two arcs e, f in G, there exists an automorphism mapping e to f. In other words an arc-transitive graph is a graph such that any two arcs are equivalent under some element of its automorphism group. It is a well-known fact [Fri70] that a Paley tournament is arc transitive.

We obtain the following results on $\chi_o(G)$ when G is a product of undirected graphs or oriented graphs. We also propose a conjecture.

Theorem 8.1.1. *Let $\overrightarrow{G}$ be an oriented graph and $\overrightarrow{T}$ be a Paley tournament such that $\chi_o(\overrightarrow{G}) = |\overrightarrow{T}|$. Let $\overrightarrow{P}_k$ be any orientation of P_k. Assume that there is a homomorphism $\phi : V(\overrightarrow{G}) \mapsto V(\overrightarrow{T})$. Then $\chi_o(\overrightarrow{G} \square \overrightarrow{P}_k) = \chi_o(\overrightarrow{G})$, $\forall k \geq 2$.*

Corollary 8.1.1.1. *For the oriented product $\overrightarrow{\mathcal{H}}_d$ of d oriented edges, we have $\chi_o(\overrightarrow{\mathcal{H}}_d) = 3$.*

Theorem 8.1.2. *For any undirected graph G,*

1. $\chi_o(G \square P_k) \leq (2k-1)\chi_o(G),\ \forall k \geq 2.$

2. $\chi_o(G \square C_k) \leq 2k\chi_o(G),\ \forall k \geq 3.$

Theorem 8.1.3. *For the strong product of undirected paths, we have the following.*

1. $8 \leq \chi_o(\mathcal{S}_{2,n}) \leq 11$

2. $10 \leq \chi_o(\mathcal{S}_{3,n}) \leq 67$

We believe and conjecture that the Theorem 8.1.1 above can be strengthened to the following.

Conjecture 8.1.1. *Let $\overrightarrow{H}$ be an arc transitive oriented graph having a non-fixing automorphism. Then, if $\phi : V(G) \mapsto V(H)$ is a homomorphism such that $\chi_o(\overrightarrow{G}) = |\overrightarrow{H}|$, then $\chi_o(\overrightarrow{G} \square \overrightarrow{P}_k) = \chi_o(\overrightarrow{G})$.*

8.2 Proofs

In this section we prove the various results claimed above. We start with the proof of Theorem 8.1.1.

8.2.1 Proof of theorem 8.1.1

Proof: Let $\overrightarrow{G}$ be an oriented graph $\overrightarrow{T}$ be a Paley tournament satisfying the conditions of the theorem. Now consider the automorphism $\pi(i) = i + 1 \bmod p$, where p is the order of the Paley tournament. It is not difficult to see that π and hence π^{-1} are both non-fixing automorphisms. From the definition of π and the fact that 1 is a quadratic residue of p, it follows that $(u, \pi(u)) \in A(\overrightarrow{T})$ for every $u \in V(\overrightarrow{T})$.

Now, we colour the graph $\overrightarrow{G} \square \overrightarrow{P}_k$ as follows. Let $\overrightarrow{G}_i,\ i = 0, 1, \ldots, k-1$, be the i^{th} copy of $\overrightarrow{G}$ in $\overrightarrow{G} \square \overrightarrow{P}_k$. We colour inductively in the order $\overrightarrow{G}_0, \overrightarrow{G}_1, \ldots$. We colour the copy $\overrightarrow{G}_0$ with the homomorphism ϕ. To colour $\overrightarrow{G}_i$, $i \geq 1$, consider the orientation of the arcs between $\overrightarrow{G}_{i-1}$ and $\overrightarrow{G}_i$. If they are from $\overrightarrow{G}_{i-1}$ to $\overrightarrow{G}_i$, each vertex $x \in \overrightarrow{G}_i$ is coloured with $\pi(c_x)$ where c_x is the colour of x in $\overrightarrow{G}_{i-1}$. On the other hand, if the arcs are from $\overrightarrow{G}_i$ to $\overrightarrow{G}_{i-1}$, each vertex $x \in \overrightarrow{G}_i$ is coloured with $\pi^{-1}(c_x)$.

We claim that the above colouring is an oriented colouring of the graph $\vec{G} \,\square\, \vec{P}_k$. Each $\vec{G}_i$ mapped to $\vec{T}$ by the homomorphism $\sigma_{i-1}\sigma_{i-2}\ldots\sigma_1\phi$ where each σ_i, $i \geq 1$ is either π or π^{-1} depending on the orientation of $\vec{P}_k$. For any vertex u in $\vec{G}$, let u_i and c_{u_i} denote the vertex corresponding to u and its colour within $\vec{G}_i$. If $(i, i+1) \in A(\vec{P}_k)$, then in our colouring, $(c_{u_i}, c_{u_{i+1}}) \in A(\vec{T})$. Similarly, if $(i+1, i) \in A(\vec{P}_k)$, then we have $(c_{u_{i+1}}, c_{u_i}) \in A(\vec{T})$. Thus, we have extended the homomorphism $\phi : \vec{G} \mapsto \vec{T}$ to a homomorphism from $\vec{G} \,\square\, \vec{P}_k$ into $\vec{T}$. Hence $\chi_o(\vec{G} \,\square\, \vec{P}_k) = \chi_o(\vec{G})$. ∎

8.2.2 Proof of Theorem 8.1.2

Proof: We now prove that $\chi_o(G \,\square\, P_k) \leq (2k-1)\chi_o(G)$. Fix any arbitrary orientation of the product. By definition, any orientation $\vec{G}$ of G can be coloured with $\chi_o(G)$ colours. For each of the k oriented (perhaps differently) copies $\vec{G}_0, \vec{G}_1, \ldots, \vec{G}_{k-1}$ of G in $G \,\square\, P_k$, we initially colour the vertices of $\vec{G}_i$ using a distinct set of $\chi_o(G)$ colours. Now starting from $\vec{G}_0$, we inductively recolour each $\vec{G}_i$ as follows. To colour $\vec{G}_i$, consider the copies $\vec{G}_i$ and $\vec{G}_{i+1}$. For each colour c used in $\vec{G}_i$, consider the set $C_0(c)$ of vertices coloured c in $\vec{G}_i$ which has arcs going to $\vec{G}_{i+1}$ and the set $C_1(c)$ of vertices coloured c having arcs coming from $\vec{G}_{i+1}$. Now we split the colour class corresponding to c into c_0 and c_1. We repeat this for every colour in $\vec{G}_i$. Notice that this ensures that there are no pair of colours c (used in $\vec{G}_i$) and d (used in $\vec{G}_{i+1}$) having arcs in both directions between colour classes of c and d. Thus we have used at most $2\chi_o(G)$ colours in the copy $\vec{G}_i$. We perform this operation on each $\vec{G}_i, \forall 0 \leq i < k-1$. Note that splitting the colour classes of $\vec{G}_i$ does not introduce violations w.r.t. edges between $\vec{G}_{i-1}$ and $\vec{G}_i$. Thus we have used at most $(2k-1)\chi_o(G)$ colours. It is easily seen that the entire colouring is oriented and proper.

Notice that the above argument can be easily and directly extended to the case of products with cycles as well except that we need to perform the doubling in all the k copies. Thus, for any graph G, $\chi_o(G \,\square\, C_k) \leq 2k\chi_o(G)$. ∎

8.2.3 Proof of Theorem 8.1.3

Proof: First we show that $8 \leq \chi_o(S_{2,n}) \leq 11$. We obtain the upper bound by establishing the existence of a homomorphism from any orientation $\vec{S}_{2,n}$ of $S_{2,n}$ into the Paley tournament $\vec{T}_{11}$.

Consider the Figure 8.1. We map the vertices of $\vec{S}_{2,n}$ to $\vec{T}_{11}$ inductively. We assume that the vertices are coloured from the left up to and including the vertices x and y. Now we show that, for all possible orientations of the dotted arcs across, we can extend the partial homomorphism (colouring) to the vertices a and b.

We make use of the fact mentioned earlier that the Paley tournament $\vec{T}_{11}$ is arc transitive. Hence we may assume, without loss of generality, that the vertices x and y are coloured with 0 and 1 respectively.

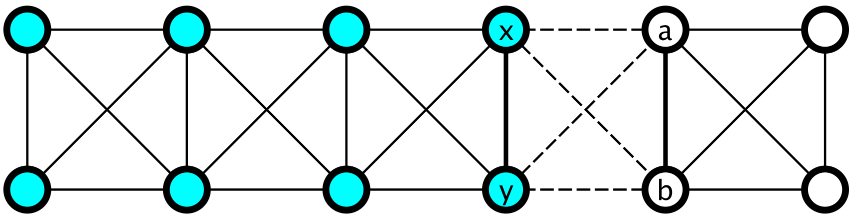

Figure 8.1: *The partially coloured* $2 \times n$ *grid*

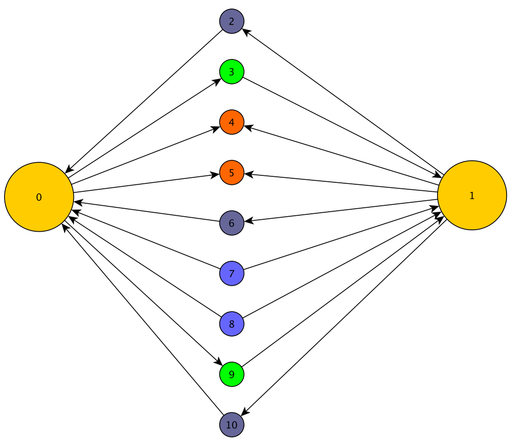

Figure 8.2: *The orientations of 2-ears*

Consider the dotted 2-paths x-a-y and x-b-y. We have the following two cases.

Case 1: The 2-paths x-a-y and x-b-y are identically oriented. We can see from Figure 8.2 that for every possible orientation of a 2-path, there are at least 2 vertices p and q such that the ears 0-p-1 and 0-q-1 are identically oriented. Now we colour a and b suitably by looking at the orientations of the arcs between a and b and between p and q.

Case 2: The orientations of x-a-y and x-b-y are not the same. Now we notice that we have at least 2 possible colours (say $\{r, s\}$) satisfying the orientation of x-a-y as well as a disjoint set of 2-colours (say $\{t, u\}$) for x-b-y. Now it is easy to check that between $\{r, s\}$ and $\{t, u\}$, there is at least one arc which satisfies the orientation of (a, b). Hence we can inductively extend the colouring to $(\overrightarrow{S}_{2,n})$.

The lower bound is explicit from the oriented graph depicted in Figure 8.3 which requires 8 colours in any oriented colouring.

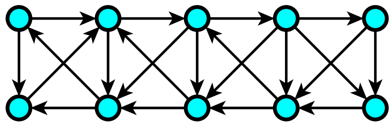

Figure 8.3: *2x5 graph requiring 8 colours.*

8.2.3.1 The second result

We now prove that $10 \leq S_{3,n} \leq 67$. Here we have a huge gap between the upper and lower bounds. Once again, we map the vertices to $\overrightarrow{T}_{67}$ to show the upper bound. The lower bound follows from the fact that, many orientations of $S_{3,5}$ (e.g. Figure 8.4) requires at least 10 colours.

Let $\overrightarrow{G}$ be any orientation of $S_{3,n}$. As before, we construct a homomorphism from $\overrightarrow{G}$ to a Paley tournament, namely $\overrightarrow{T}_{67}$. We first recall some definitions. An orientation vector of size m is a sequence $\alpha = (\alpha_1, ..., \alpha_m)$ in $\{0, 1\}^m$. Given a sequence $X = (x_1, ..., x_m)$ of vertices in an oriented graph $\overrightarrow{G}$, an α-successor of X is a vertex y such that for each i, $(x_i, y) \in A(\overrightarrow{G})$ if $\alpha_i = 1$ and $(y, x_i) \in A(\overrightarrow{G})$ if $\alpha_i = 0$. We say that an oriented graph has the property $P(m, k)$ if for any sequence X of m distinct vertices in

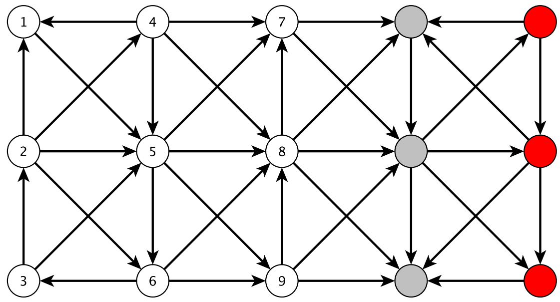

Figure 8.4: *An orientation of $S_{3,5}$ that requires 10 colours.*

the graph and for any orientation vector α in $\{0, 1\}^m$, there are at least k α-successors of X. Note that the property $P(m, k)$ implies the property $P(n, k)$ for all $n < m$. Paley tournaments are good candidates for satisfying such properties and in particular, we shall use the following fact:

Fact 8.2.1. *$\vec{T}_{67}$ satisfies $P(4, 1)$ [Esp07] as well as $P(2, 2)$ [BKN+99].*

We now map $S_{3,n}$ to $\vec{T}_{67}$ using induction on n.

Base case: $n = 1$. In this case, $\vec{G}$ is just an oriented path on 3 vertices and it is easy to map the 3 vertices of $\vec{G}$ to 3 distinct vertices in $\vec{T}_{67}$ using the $P(2, 2)$ property.

Induction step: Assume that the subgraph induced by the vertices $(i, j) : i \in \{0, 1, 2\}, j \in \{0, 1, ..., n-2\}$ are mapped to $\vec{T}_{67}$. Now $(0, n-1)$ has exactly 2 neighbours which have already been mapped (coloured) to distinct vertices in $\vec{T}_{67}$ and since $\vec{T}_{67}$ has the property $P(2, 2)$, we can extend the mapping so that $(0, n-1)$ is mapped to a vertex in $\vec{T}_{67}$ that is different from the image of $(2, n-2)$. We now see that $(1, n-1)$ has four neighbours that have already been mapped to four distinct vertices in $\vec{T}_{67}$ and using the property $P(4, 1)$ of $\vec{T}_{67}$, we extend the mapping to the vertex $(1, n-1)$. We can now map $(2, n-1)$ to $\vec{T}_{67}$ as well since it has three distinct coloured neighbours and we make use of the property $P(3, 1)$ of $\vec{T}_{67}$. This completes the proof of part (ii) of Theorem 8.1.3. ■

अनेकसंशयोच्छेदि परोक्षार्थस्य दर्शकं ।
सर्वस्य लोचनं शास्त्रं यस्य नास्त्यन्ध एव सः ॥

It is by asking questions that one finds the hidden truth. Science is the eye of all. A person without it is really blind.

9 Remarks and Open problems

In this chapter, we present some observations on the colouring problems and its current status. We also note the technical difficulties we faced in dealing with these problems and of these what we expect to be easy and what we expect to be difficult. We conclude the section with a list of a few open problems.

9.1 Remarks on Acyclic Edge Colouring

We are interested in finding χ'_a, i.e., the minimum number of colours needed to acyclically edge colour a given graph G. It has been known for some time that a linear number of colours is sufficient. The acyclic edge colouring conjecture due to Fiamcik [Fia78] (also mentioned in [ASZ01]) predicts that $\Delta + 2$ colours are sufficient for any graph of maximum degree Δ. The conjecture has been shown to be true for graphs having a high girth as well as for almost all regular graphs.

We show that the conjecture is true for some other classes of graphs. We provide a tight bound for the class of outerplanar graphs, harary graphs $\mathcal{H}_{n,2}$ and the class of fully subdivided graphs in this work. We improve the upper bound for graphs with girth at least 9 using probabilistic arguments. We then provide improved upper bounds for the classes of planar graphs, three-fold graphs, triangle-free planar graphs and two-fold graphs making use of the discharging method.

We had shown (earlier) that the conjecture holds true for partial 2-trees, and cartesian products of paths and cycles. This work, done jointly with my colleague Rahul Muthu forms a part of his Ph.D. thesis and the interested reader can refer to [MNS06] and [MNS] for further details.

The following table gives a collection of classes of graphs together with known upper bounds on χ'_a for its members. The first column mentions the graph class, the second lists the best known upper bound, third column states if it is known to satisfy the acyclic edge colouring conjecture or not, and the last column indicates if the members of the class contain at most a linear number of edges (in terms of its order).

Class	**Upper bound χ'_a**	**Conjecture?**	**Sparse?**
Outerplanar	$\Delta + 1$	True	Yes.
Partial 2-tree	$\Delta + 1$	True	Yes, $\|E\| \leq 2\|V\|$
Partial torus	$\Delta + 1$	True	Yes
K_p, p a prime.	$\Delta + 1$	True	No
$g \geq 2000\Delta \log \Delta$	$\Delta + 2$	True	?
Random Regular	a.a.s. $\Delta + 1$	a.a.s. True.	Yes.
2-degenerate	$\Delta + 1$	True	Yes
Planar Graphs	$2\Delta + 29$	Open	Yes
3-fold graphs	$2\Delta + 29$	Open	Yes
triangle-free planar	$\Delta + 6$	Open	Yes
2-fold graph	$\Delta + 6$	Open	Yes
K_n	$\Delta + O(\Delta^{2/3})$	Open	No
$G : \|E\| \leq cn$	16Δ	Open	Yes
G arbitrary	16Δ	Open	No

We can see from Table 9.1 that all the classes that are known to satisfy the conjecture, except for the class of complete graphs of prime order, are reasonably sparse. We make a special mention of the a.a.s result on random graphs which was proved using Lovász Local Lemma. The proof makes use of a typical property of random regular graphs that, for a given degree d, any two small cycles (of length less than some constant depending on d), in a random d-regular graph are separated by a long path (again a constant depending on d).

We feel that it is reasonable to believe in the conjecture for the cases when the number of edges is linear in the number of vertices. We also expect that it might be possible to prove a $(1 + o(1))\Delta$ bound for the general graphs.

9.1.1 Some typical difficulties

In this section, we present some technical difficulties that arise while trying to obtain good bounds on acyclic chromatic index. These observations are based on the obstacles we faced while working on this problem.

We feel that one of the major difficulties in the acyclic edge colouring comes from the fact that many of the standard techniques are not easily applicable. For example, the Kempe chain swaps, one of the widely used tools used in modifying a partial colouring, is not easily applicable in the case of acyclic edge colouring. This is because the change in the colour of one edge affects all the cycles passing through it, and thus have a global (involving all colour classes) effect rather than a local effect as in the case of some of the other colouring problems. In fact, a small Kempe chain flip can cause a lot of new 2-coloured cycles getting created.

Another difficulty comes when we try to extend a partial colouring to include new edges. The difficulty comes because even if we assume that the set of colours used are entirely disjoint (i.e., the set of colours in the coloured part and the newly selected set of edges are disjoint) and the colouring of each part is individually acyclic, they can still introduce bichromatic cycles which uses edges from both of the parts alternately. We still do not know any way to take care of this. Due to this, iterative procedures, which have been successfully used in many edge colouring problem, are not easily applicable here.

9.1.2 Open problems

There are many interesting and challenging problems that one may look into. We are interested in the following questions.

In Chapter 5, we noticed that one can use the discharging method to get some good structural properties and make use of them to design an algorithm to obtain a good bound on the acyclic chromatic index and also produce an acyclic edge colouring in polynomial time. We believe that there is scope for improving the bounds further and also for obtaining such colourings. One interesting problem is to improve the bound for planar graphs. Similarly, it would be interesting to show that the conjectured $\Delta + 2$ bound is true for the class of triangle-free planar or 2-fold graphs.

We believe that one can use the discharging method on classes of graphs which has a linear number of edges to obtain improved bounds in a similar manner.

Another interesting problem is to remove the assumed lower bound on girth from the upper bounds obtained in Chapter 3.

9.2 Remarks on k-intersection edge colouring

As it is easy to see, this problem generalises and connects many edge-colouring problems together. This enables one to generalise the framework where these practically important problems can be applied.

An open problem would be to find whether the lower bound is tight for other classes of graphs like bicliques (complete bipartite graphs). There is a simple algorithm to obtain a k-intersection edge colouring for $K_{m,n}$ using mn/k colours. This shows that for $m \ll n$, $\chi'_k(K_{m,n}) \leq \frac{mn}{k} \ll \frac{n^2}{k} = \frac{\Delta^2}{k}$. Another interesting problem is to find an efficient algorithm which obtains a colouring that matches the upper bound of $O(\frac{\Delta^2}{k})$. Yet another problem would be to generalise this concept to uniform hypergraphs.

9.3 Remarks on Oriented Colouring

There is a plethora of open problems related to oriented colouring. Even for the class of two-dimensional grids, we only know that the oriented chromatic number is between 8 and 11. For the planar graphs we only know that it is between 16 and 80. Any improvement along these directions would be very interesting.

In this work, we obtain some upper bounds for the cartesian product of a graph with paths (directed as well as undirected). For the undirected case, the bound seems far from optimal. It would be interesting to find a better upper bound for the same. Also for the strong product of paths, when both paths are longer, we do not have any good upper bound. One can also try to bound the oriented chromatic number of the cartesian product of more than two paths or cycles (higher dimensional grids or tori).

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Glossary

Notation	Description	Page List
(u, v)	Edge (arc) between u and v in an undirected (directed) graph	5
$E(v)$	The set of edges incident with a vertex v	5
$G \Box H$	Cartesian product of G and H	8
$G \boxtimes H$	Strong product of G and H	8
K_n	Complete graph on n vertices	6
$\mathcal{C}(uv)$	The colour of the edge (u, v) in the colouring $\mathcal{C}$	47
$\chi'_a(G)$	Smallest integer k such that there is a proper edge k-colouring that is acyclic	9
$\mathcal{C}(v)$	Set of colours seen by v in the edge colouring $\mathcal{C}$	9
$\chi'_k(G)$	k-intersection chromatic index	9
$\chi_o(G)$	Oriented chromatic number of G	10
$d(v)$ **or** $d_G(v)$	The degree of v in G	6
k**-intersection edge colouring**	A proper edge colouring of a graph in which the number of common colours seen by any pair of adjacent vertices is at most k	9
k**-vertex**	A vertex of degree k	6
k^+**-vertex**	A vertex of degree at least k	6
k^-**-vertex**	A vertex of degree at most k	6
x-y **bipath**	A bichromatic path of length ≥ 4 that uses colours x and y	59
acyclic edge colouring	A proper edge colouring where the union of any two colour classes form an acyclic subgraph	9
adjacent vertices	u and v are adjacent if $(u, v) \in E(G)$	5
adjacent edges	Edges e and f are adjacent if they share a common vertex	5

Notation	Description	Page List
block	A maximal connected subgraph without cut-vertices	6
cartesian product	$G \square H$ so that $V(G \square H) = V(G) \times V(H)$ and $([u_1, u_2], [v_1, v_2]) \in E(G \square H)$ if **either** $u_1 = v_1$ and $(u_2, v_2) \in E(H)$ **or** $u_2 = v_2$ and $(u_1, v_1) \in E(G)$	8
chord	An edge that joins two non-consecutive vertices of a cycle	6, 53
connected graph	A graph in which there is a path between any pair of vertices	6
cut-vertex	A vertex whose removal increases the number of connected components of the graph	6
cycle	A sequence $v_0, v_1, \ldots, v_k = v_0$ of vertices such that all except the end vertices are distinct and $(v_i, v_{i+1}) \in E$ for each i	6
degree	Number of edges incident with a vertex	6
directed graph	A pair (V, A) where the elements of A are 2-element ordered subsets of V	7
edge	An element of E in $G = (V, E)$	5
edge r-colouring	A map $\mathcal{C} : E \mapsto [r]$ such that $\mathcal{C}(e) \neq \mathcal{C}(f)$ whenever e and f are adjacent	9
forest	A graph having no cycles is a forest	6
graph (directed graph)	An ordered pair of sets $G = (V, E)$ where E is a collection of 2-element (ordered) subsets of V	5
homomorphism	A mapping between the vertices of two graphs that preserves the edge (arc) relations	7

Notation	Description	Page List
minimally 2-connected	A 2-connected graph G so that for any edge e, $G-e$ contains a cut-vertex	53
order	The number of vertices in a graph G	5
oriented r-colouring	A mapping $\mathcal{C} : V(G) \mapsto [k]$ so that (i) $\mathcal{C}(x) \neq \mathcal{C}(y)$ $\forall (x,y) \in A(G)$ and (ii) $\mathcal{C}(x) = \mathcal{C}(w) \implies \mathcal{C}(y) \neq \mathcal{C}(z)$ $\forall (x,y),(z,w) \in A(G)$	9
oriented chromatic number	Smallest $r \in \mathbb{N}$ that admits an oriented vertex r-colouring of $\overrightarrow{G}$. For undirected graph G, it is the maximum χ_o over all orientations.	10
oriented graph	An orientation of the edges of a simple graph	7
outerplanar graph	A planar graph that has a planar embedding with all the vertices on the outer face.	7
path	A sequence of distinct vertices $u = v_0, v_1, \ldots, v_k = v$ where $(v_i, v_{i+1}) \in E \, \forall i$	6
planar graph	Graph that can be embedded on a plane	7
strong product	$G \boxtimes H$ such that $V(G \boxtimes H) = V(G) \times V(H)$ and $([u_1,u_2],[v_1,v_2]) \in E(G \boxtimes H)$ if **either** $u_1 = v_1$ and $(u_2,v_2) \in E(H)$ **or** $u_2 = v_2$ and $(u_1,v_1) \in E(G)$ **or** $(u_1,v_1) \in E(G)$ and $(u_2,v_2) \in E(H)$	8
subgraph	$H \subseteq G$ if $E(H) \subseteq E(G)$ and $V(H) \subseteq V(G)$	6
vertex	An element of V in $G = (V, E)$	5
vertex r-colouring	A map $\mathcal{C} : V \mapsto [r]$ such that $\mathcal{C}(u) \neq \mathcal{C}(v)$, $\forall (u,v) \in E$	8

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