FORBIDDEN SUBGRAPH COLORINGS, ORIENTED COLORINGS AND INTERSECTION DIMENSIONS OF GRAPHS

by

N.R. Aravind

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-- **Date :**
Chairman : R. Balasubramanian

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-- **Date :**
Member : Sundar Vishwanathan

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-- **Date :**
Guides : R. Balasubramanian, C.R. Subramanian

DECLARATION

I hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and the work has not been submitted earlier as a whole or in part for a degree/diploma at this or any other Institution or University.

N.R. Aravind

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Abstract

This thesis deals mainly with two related coloring problems - forbidden subgraph colorings and oriented colorings. The former deals with proper colorings of vertices or edges of a graph with constraints on the union of color classes. A well-known example is the acyclic vertex coloring in which we require a proper coloring such that the union of any two color classes is acyclic. Other well-studied examples include the acyclic edge coloring and star coloring. Our focus in this thesis is a generalization of these special types of colorings.

Oriented coloring deals with colorings of oriented graphs (directed graphs obtained by orienting each edge of a simple undirected graph). Specifically, an oriented coloring is a homomorphism to an oriented graph, the vertices of the target graph being considered as the colors assigned to the vertices of the source graph.

For both of these problems, we want to find good upper bounds for the number of colors required for such colorings.

In this thesis, we find upper bounds for forbidden subgraph chromatic numbers in terms of the maximum degree. For the union of two color classes, we show the asymptotic tightness of our bounds by a probabilistic contstruction. We then show that the oriented chromatic number of a graph can be bounded in terms of the forbidden subgraph chromatic numbers. In conjunction with our afore-mentioned results, this allowed us to prove improved bounds on oriented chromatic numbers of graphs on surfaces.

Specifically, we obtained the following results:

- Given a family $\mathcal{F}$ of connected graphs each having at least m edges, the vertices of any graph of maximum degree Δ can be properly colored using $O(\Delta^{1+\frac{1}{m-1}})$ colors so that in the union of any 2 color classes, there is no copy of H for any $H \in \mathcal{F}$.
- Any graph of genus g has oriented chromatic number at most $2^{g^{1/2+o(1)}}$.

We also consider edge colorings of graphs with restrictions on the union of color classes. While edge colorings can simply be considered as vertex colorings of the line graph, it is usually the case that they are often quite different in

nature. Indeed, we found a general upper bound which shows that the bounds for edge colorings with similar restrictions as those on vertex colorings often require substantially fewer colors in terms of the maximum degree.

In particular, we showed that using just $O(\Delta)$ colors, (where Δ is the maximum degree), we can properly color the edges of a graph with any (or even all) of the following constraints:

(i) the union of any 2 color classes is a forest (this is a known result due to Alon, McDiarmid and Reed);

(ii) the union of any 3 color classes is outerplanar;

(iii) the union of any 4 color classes has treewidth at most 2;

(iv) the union of any 5 color classes is planar;

(v) the union of any 6 color classes is 5-degenerate.

We obtain the above bounds as an application of a special case of the Lovász Local Lemma which we derive and show that these bounds can be constructivized by the algorithm obtained by Moser and Tardos in [MT10].

Finally, we also study the intersection dimension of graphs. In contrast to coloring problems where we partition the graph into smaller pieces, the problem here is the following: Given a graph class $\mathcal{A}$ and a graph G, express G as the intersection of some supergraphs on the vertex set of G, subject to the condition that each of these supergraphs belongs to the class $\mathcal{A}$. The least number of supergraphs needed is the intersection dimension of G with respect to the class $\mathcal{A}$. A well-known example of such a parameter is the boxicity of a graph, which is the least number of interval graphs whose intersection is the given graph.

We show that the intersection dimension of graphs with respect to several hereditary classes can be bounded as a function of the maximum degree. As an interesting special case, we show that the circular dimension of a graph with maximum degree Δ is at most $O(\Delta\frac{\log\Delta}{\log\log\Delta})$. We also obtained bounds in terms of treewidth.

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Glossary

We use standard terminology from Bollobas [B.B05], Diestel [Die05] and West [Wes01]. For easy reference, we give below the definitions of some terms used in this thesis.

Acyclic chromatic number of a graph G: The minimum number of colors used in any acyclic vertex coloring of G. It is denoted by $a(G)$.

Acyclic chromatic index of a graph G: The minimum number of colors used in any acyclic edge coloring of G. It is denoted by $a'(G)$.

Acyclic edge coloring: A proper coloring of the edges of a graph such that the union of any two color classes forms a forest.

Acyclic vertex coloring: A proper coloring of the vertices of a graph such that the union of any two color classes induces a forest.

Adjacent vertices: Two vertices joined by an edge.

Bipartite graph: A graph whose vertices can be partitioned into two independent sets; equivalently, a 2-colorable graph.

Chordal graph: A graph having no induced cycle of length at least 4.

Chromatic index of a graph G: The smallest integer k such that the graph admits a proper edge coloring using k colors. It is denoted by $\chi'(G)$.

Chromatic number of a graph G: The smallest integer k such that the graph admits a proper vertex coloring using k colors. It is denoted by $\chi(G)$.

Clique of a graph: A set of vertices which are pairwise adjacent.

Clique number of a graph G: The maximum size of a clique in G; it is denoted by $\omega(G)$.

Complement of a graph: The complement of $G = (V, E)$, denoted by G^c is the graph (V, E') where $E' = \binom{V}{2} \setminus E$.

Complete bipartite graph $K_{s,t}$: A graph whose vertex set is a union of two disjoint independent sets of size s and t, and each vertex in one set is adjacent to every vertex in the other.

Complete graph: A simple graph in which any two vertices are adjacent.

Complete multipartite graph or complete l-partite graph $K_{n_1,\ldots,n_l}$: A graph whose vertex set consists of l independent sets $S_1, \ldots, S_l$ of sizes $n_1, \ldots, n_l$ respectively, and whose edge set is $\cup_{1 \leq i < j \leq l}\{(u, v) : u \in S_i, v \in S_j\}$.

Component of a graph: A maximal connected induced subgraph.

Connected graph: A graph in which any two vertices are connected by a path.

Cycle: An alternating sequence of vertices and edges with no repetitions of vertices except the first and the last vertex, where each edge is incident with its preceding and succeeding vertices.

Degeneracy of a graph G: max $\{\delta(H) : H$ is a subgraph of $G\}$

Degree of a vertex v in a graph G: The number of edges incident with v in G. It is denoted by $d(v)$ or $d_G(v)$.

Disconnected graph: A graph with more than one component.

Distance between a pair of vertices: The length of a shortest path between the vertices.

Distance-two coloring of a graph G: A proper coloring of G such that any two vertices which are at distance at most two in G get different colors, equivalently a proper coloring of G^2.

Forest: A graph having no cycles.

$\mathcal{F}$-free graph G: If $\mathcal{F}$ is a family of graphs, then G is $\mathcal{F}$-free if there is no $H \in \mathcal{F}$ which is isomorphic to a subgraph of G.

Girth of a graph: Length of a shortest cycle, if there is any cycle.

Graph class or graph family: A collection of graphs closed under isomorphism.

Hereditary family of graphs $\mathcal{G}$: If $G \in \mathcal{G}$ and H is an induced subgraph of G, then $H \in \mathcal{G}$.

Hypergraph : $G = (V, E)$ where E is a collection of subsets of V. G is k-uniform if every element of E has size k.

Independent set of a graph: A set of vertices no two of which are adjacent.

Induced subgraph on a vertex subset W of G: The subgraph with vertex set W and edge set consisting of edges of G with both the ends in W.

Isomorphic graphs: Two graphs,say $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ that have an isomorphism between them. That is, there exists a bijective function $f : V_1 \to V_2$ such that $(u, v) \in E_1$ if and only if $(f(u), f(v)) \in E_2$.

$(j, \mathcal{F})$-subgraph coloring: A proper coloring of the vertices of a graph such that the subgraph induced by the union of any j color classes is $\mathcal{F}$-free.

$(j, \mathcal{F})$-subgraph chromatic number of a graph G: The minimum number of colors used in any $(j, \mathcal{F})$-subgraph coloring of G. It is denoted by $\chi_{j,\mathcal{F}}(G)$.

$(j, \mathcal{F})$-edge coloring: A proper coloring of the edges of a graph in which the subgraph formed by the union of any j color classes is $\mathcal{F}$-free.

$(j, \mathcal{F})$-chromatic index of a graph G: The minimum number of colors used in any $(j, \mathcal{F})$-edge coloring of G. It is denoted by $\chi'_{j,\mathcal{F}}(G)$.

(j, k)-treewidth coloring: A proper coloring of the vertices of a graph such that the subgraph induced by the union of any j color classes has treewidth at most k.

(j, k)-treewidth chromatic number of a graph G: The minimum number of colors used in any (j, k)-treewidth coloring of G.

Length of a cycle: The number of edges in the cycle.

Length of a path: The number of edges in the path.

Maximum degree of a graph G : Max $\{d_G(v) : v \in V(G)\}$. It is denoted by $\Delta(G)$.

Minimum degree of a graph G : Min $\{d_G(v) : v \in V(G)\}$. It is denoted by $\delta(G)$.

Minor of a graph G: A graph obtained from G by a sequence of edge deletions, edge contractions and vertex removals.

Minor-closed family: A family $\mathcal{F}$ of graphs such that if a graph G is in $\mathcal{F}$, then any minor of G is also in $\mathcal{F}$.

Neighbor of a vertex v: Any vertex adjacent to v.

Neighborhood of a vertex v: The set of neighbors of v.

Order of a graph: The number of vertices in a graph.

Oriented graph: A graph $\vec{G}$ obtained by orienting each edge of an undirected graph G, equivalently a directed graph with exactly one direction per edge.

Oriented coloring: A homomorphism from an oriented graph to another oriented graph, with the vertices of the latter considered as the colors of the vertices of the former.

Oriented chromatic number of $\vec{G}$: Denoted by $\chi_o(\vec{G})$, it is the smallest oriented graph to which $\vec{G}$ has a homomorphism. For an undirected graph G, it is the maximum of $\chi(\vec{G})$ over all orientations $\vec{G}$ of G; it is denoted by $\chi_o(G)$.

Path: An alternating sequence of vertices and edges with no repetitions where each edge is incident with its preceding and succeeding vertices. A path with u and v as terminal vertices is called an **(u, v)-path**.

Perfect graph: A *perfect graph* is a graph G such that for every induced subgraph H of G, $\chi(H) = \omega(H)$.

Power of a graph: The kth power of a graph G is $G^k = (V, E^k)$, where $(u, v) \in E^k$ if and only if $d_G(u, v) \leq k$.

Proper coloring or k-coloring or proper k-vertex coloring of a graph: An assignment of k colors to the vertices of a graph such that no two adjacent vertices receive the same color.

Regular graph: A graph in which all the vertices have same degree. If the common degree is k, then the graph is called **k-regular**.

Simple graph: A graph with no multiple edges or loops.

Star: A graph of the form $K_{1,t}$ is called a star.

Star coloring: A proper coloring of the vertices of a graph such that the union of any two color classes induces a star forest.

Star chromatic number of a graph G: The minimum number of colors used in any star coloring of G. It is denoted by $\chi_s(G)$.

Star forest: A disjoint union of stars is called a star forest.

Subgraph of a graph G: A graph H whose vertices and edges are all in G.

Sum or join of two vertex disjoint graphs G_1 and G_2: The graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{(x, y) : x \in V(G_1), y \in V(G_2)\}$.

Tournament: An oriented graph with exactly one oriented edge between every pair of vertices.

Tree: A connected graph having no cycles.

Union of two vertex disjoint graphs G_1 and G_2: The graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$.

Universal vertex of a graph: A vertex which is adjacent to every other vertex.

Notation:

$a(\Delta) = \max\{a(G) : \Delta(G) = \Delta\}$.

$a'(\Delta) = \max\{a'(G) : \Delta(G) = \Delta\}$.

$\chi_s(\Delta) = \max\{\chi_s(G) : \Delta(G) = \Delta\}$.

$\chi_{j,\mathcal{F}}(\Delta) = \max\{\chi_{j,\mathcal{F}}(G) : \Delta(G) = \Delta\}$.

$\chi^{tw}_{j,k}(\Delta) = \max\{\chi^{tw}_{j,k}(G) : \Delta(G) = \Delta\}$.

$\chi'_{j,\mathcal{F}}(\Delta) = \max\{\chi_{j,\mathcal{F}}(G) : \Delta(G) = \Delta\}$.

$\theta(j, \mathcal{F}) = \sup_{H \in \mathcal{F}} \frac{(|V(H)|-2)}{(|E(H)|-j)}$.

$D(\mathcal{F}) = \min_{H \in \mathcal{F}}(|E(H) - |V(H)|)$.

For a positive real number x, $\log^* x = \max\{i \geq 1 : \underbrace{\log\log\ldots\log}_{i \text{ times}} x \geq 1\}$.

1

Introduction

The origin of graph theory goes back to the 18th century. In 1736, the famous Swiss mathematician Leonhard Euler was presented with the problem of the seven bridges of Konigsberg (see Fig 1.1). The problem was to cross the seven bridges exactly once each.

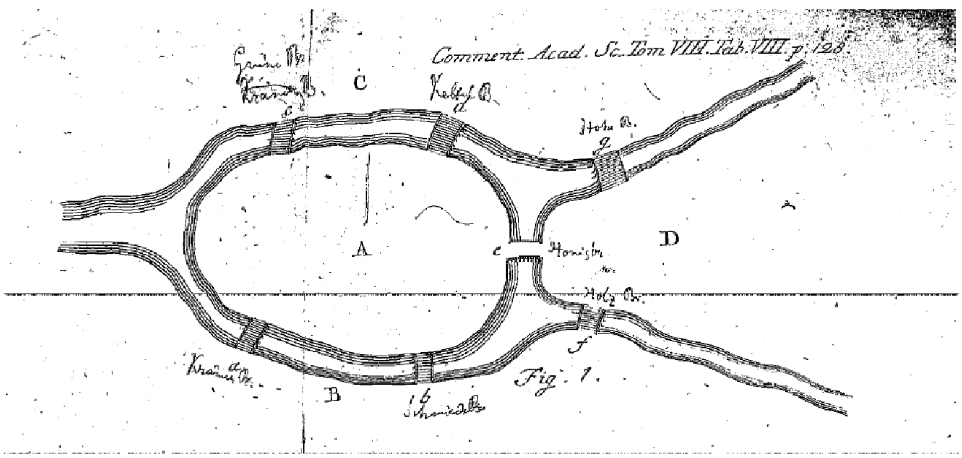

Figure 1.1: Illustration of the Konigsberg bridge problem in Euler's paper

Euler showed that doing this is impossible by observing that in any such tour, any region which is connected to an odd number of other regions (i.e. having odd "degree") must be a starting or ending point. Since there were more than two regions with odd degree, no such tour was possible.

Euler's reasoning involved objects (regions) and the pairwise connections (bridges) between them. This gave rise to the idea of an abstract graph which consists of a set of objects called *vertices* and a set of pairwise connections between them, each connection being called an *edge*.

Arguably, the next major impetus to graph theory came from the four color problem - now the Four Color Theorem. This problem is attributed to Francis Guthrie who asked Augustus de Morgan if it was always possible to color any map (drawn on a plane) using only 4 colors so that adjacent regions get different colors. Translated into graph-theoretic language, the problem is to prove that the vertices of any planar graph can be 4-colored so that adjacent vertices get different colors (this condition is known as *properness* and colorings which obey it are called *proper colorings*).

Very soon after the conjecture was made, Heawood showed that 5 colors is always sufficient for such a coloring. But the intriguing problem of using only 4 colors remained a challenge for more than a century before it was finally settled in 1976 by Appel and Haken, with the proof requiring the help of a computer to verify an enormous number of cases.

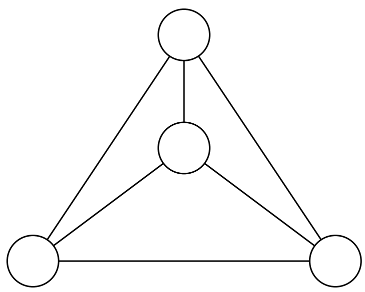

Figure 1.2: K_4 requires 4 colors

The attempts to solve the four-color theorem led to a huge amount of work in graph coloring as well in other areas of graph theory. Connections to combinatorics, number theory and other branches of mathematics were found and the abundance of problems that graph theory offers has kept it alive since.

The interest in graph theory increased multifold with the advent of computers and computer science. Not only did graph theory offer a plethora of natural computational problems, several graph algorithms were found to have widespread practical applications. Special types of graphs, notably trees and tree-like graphs are the building blocks of many data structures.

Graph coloring itself remains a major area of study and one reason for this is the practical applications it has found, notably in scheduling problems. For example, consider the following problem: There is a set of processors which must use a set of identical resources to complete some jobs. However, certain pairs of processors are not allowed to share a resource. The problem is to minimize the number of resources used. This can be modeled as a graph coloring problem by building the following graph: Assign a vertex for each processor and an edge for "mutually exclusive" pairs of processors. The problem of minimal allocation of resources is then the same as a proper coloring of the graph constructed, using a minimum number of colors.

The *chromatic number* $\chi(G)$ of a graph G is the least number of colors used in any proper coloring of G. Thus in modern graph-theoretic language, the four color theorem says that $\chi(G) \leq 4$ if G is planar. The maximum degree of a graph is usually denoted by $\Delta(G)$ and we sometimes use just Δ to indicate that we are considering an arbitrary graph with $\Delta(G) = \Delta$. It is easy to obtain the bound $\chi(G) \leq \Delta + 1$. We fix a set of $\Delta + 1$ colors and try to color the vertices one by one in any order. At any stage, we will not be able to color a vertex only if all its neighbors have used up all the colors. But this is impossible since the number of neighbors is strictly less than the number of colors. Thus all the vertices can be colored if $\Delta + 1$ colors are available.

Apart from the proper coloring notion mentioned so far, several variants of coloring have also been studied. Some of these variants relax the condition that each color class should induce an independent set. An *arboreal coloring* of a graph, for example, requires that each color class induces a forest. On the other hand, there are variants such as acyclic coloring which impose restictions on the union of every few color classes in addition to the requirement of properness.

An *acyclic vertex coloring* (introduced in [Grü73], see also [AB76]) of $G = (V, E)$ is a proper coloring of V in which the subgraph induced by the union of any two color classes is acyclic. Equivalently, it is a proper coloring which admits no two-colored cycle. The *acyclic chromatic number* $a(G)$ is the least k such that G admits an acyclic vertex coloring using k colors. Yet another variant is a *star coloring* of a graph - this is a proper coloring of the vertices of a graph such that the union of any two color classes induces a collection of vertex disjoint stars. Both acyclic coloring and star coloring have applications in computing the Hessians and Jacobians of matrices (see [GTMP07] for details).

Another example is the *distance-2 (vertex) coloring* of G. It is a coloring of the vertices such that any two vertices whose distance is at most 2 do not get the same color. This can be translated into a proper coloring of the graph G^2 of G obtained by connecting all pairs of vertices at distance at most two in G. The minimum number of colors sufficient for such a coloring, i.e., $\chi(G^2)$ is a parameter that is closely related to the span of a radio-coloring of a graph [FNPS05] and is hence related to applications in mobile communication.

All these variants of coloring have one thing in common - they are proper colorings with restrictions on the union of any few (typically two) color classes. Further these restrictions are expressed by means of a set of forbidden subgraphs. In this thesis, we study the problem of obtaining bounds on the chromatic number associated with such colorings when an arbitrary family of graphs is forbidden in the union of every few color classes.

So far, we have mentioned only vertex colorings and variants of these. An equally interesting and well-studied area is that of edge colorings.

A *proper edge coloring* is a coloring of the edges of a graph so that adjacent edges get distinct colors. Edge colorings also have applications in scheduling problems, but of a different kind. The minimum number of colors required for a proper edge coloring of a graph G is known as its *chromatic index* and is denoted by $\chi'(G)$. Like the chromatic number, the chromatic index can also be bounded as a function of the maximum degree. Vizing [Viz64] proved that the chromatic index of a graph G is at most $\Delta(G)+1$ and the complete graphs show that this bound is tight. Indeed, for every graph G, all the edges adjacent incident on a vertex of maximum degree must get pairwise distinct colors in any proper edge coloring, so $\chi'(G) \geq \Delta(G)$ for all graphs. This is in contrast to vertex colorings - the chromatic number is not lower-bounded by any function of maximum degree. Indeed, bipartite graphs (graphs with chromatic number two) can have arbitrarily large maximum degree.

As in the case of vertex colorings, several variants of edge coloring have been studied. The restriction that the union of any two color classes must be a forest is again a well-known example. This is known as an *acyclic edge coloring* and the minimum number of colors used in any acyclic edge coloring of a graph G, known as its *acyclic chromatic index*, is denoted by $a'(G)$. A generalization of this was studied in [GGW06] and bounds for the associated chromatic number obtained in terms of the maximum degree of a graph. In this thesis, we study a natural generalization of edge colorings where we place restrictions on the union of every few color classes. For example, we could require a coloring where the union of any *three* color classes has treewidth at most two.

The variants of proper vertex colorings mentioned above are also related to *oriented colorings*. An *oriented graph* is a directed graph obtained by orienting each edge of a simple, undirected graph. We will use the term *arc* to denote a directed edge. For an undirected graph, a proper coloring using t colors can also be considered as a homomorphism to K_t, the clique on t vertices. Oriented coloring is a natural generalization of this definition for oriented graphs. An oriented coloring of an oriented graph $\vec{G}$ is a homomorphism from $\vec{G}$ to another oriented graph $\vec{H}$, whose vertex set we consider to be the set of colors. In other words, it is a mapping $f : \vec{G} \to \vec{H}$ such that for every pair of vertices u and v in $\vec{G}$, there is an arc from u to v in $\vec{G}$ only if there is an arc from $f(u)$ to $f(v)$ in $\vec{H}$.

The minimum number of vertices in any target graph $\vec{H}$ admitting a homomorphism from $\vec{G}$, is called the *oriented chromatic number* of $\vec{G}$ and is denoted by $\chi_o(\vec{G})$. The oriented chromatic number is also defined for undirected graphs - for an undirected graph G, it is the maximum of the oriented chromatic numbers $\chi(\vec{G})$ over all possible orientations $\vec{G}$ of G and is denoted by $\chi_o(G)$.

The oriented chromatic number of a family of graphs is the maximum of the oriented chromatic numbers of its members. The oriented chromatic number of planar graphs is known to be between 17 and 80, the upper bound being obtained in [RS94] as a consequence of a relation between the oriented chromatic number and the acyclic chromatic number (of a graph). The lower bound was obtained by Marshall in [Mar07]. Upper bounds for oriented chromatic numbers were also obtained for triangle-free planar graphs [Och04], for 2-outerplanar graphs [EO07], for arbitrary graphs in terms of maximum degree [KSZ97], maximum average degree [BKN+99] and in terms of treewidth [Sop97]. Similar to proper vertex colorings and edge colorings, oriented colorings are also known to have applications in task assignment problems; an example of such an application is presented in [CD06].

Results

We now provide an outline of the main results obtained in this thesis. The full details can be found in the respective chapters.

Forbidden subgraph vertex colorings The main contribution of this thesis is to obtain bounds for coloring the vertices of a graph such that the union of every few color classes does not contain as a subgraph, any graph from a fixed set of forbidden graphs.

Specifically, we obtain the following result. For any positive integer j and a family $\mathcal{F}$ of graphs, there is a constant $C = C(j, \mathcal{F})$ such that the following holds: Every graph of maximum degree Δ can be properly colored using $C\Delta^{\frac{k-1}{k-j}}$ colors so that the union of any j color classes has no graph from $\mathcal{F}$ as a subgraph. Here, k is the minimum number of vertices in any member of $\mathcal{F}$.

When $j = 2$, we obtain the following improvement. Given a family $\mathcal{F}$ of connected graphs each having at least m edges, any graph of maximum degree Δ can be colored using $O(\Delta^{1+\frac{1}{m-1}})$ colors so that in the union of any 2 color classes, there is no copy of H for any $H \in \mathcal{F}$. This generalizes known upper bounds for acyclic chromatic numbers ([AMR91]) and star chromatic numbers ([FRR04]). This bound is also shown to be nearly tight by a probabilistic construction.

Forbidden subgraph edge colorings Given a positive integer j and a family $\mathcal{F}$ of graphs, we consider the problem of properly coloring the edges of a graph (using a minimum number of colors) so that in the union of any j color classes, there is no copy of H. We show that any such graph of maximum degree Δ can always be colored in such a way using $O(\Delta^{\{max(1,\theta)\}})$ colors, where $\theta = \theta(j, \mathcal{F})$ is a parameter defined by $\theta = \sup_{H\in\mathcal{F}} \frac{(|V(H)|-2)}{(|E(H)|-j)}$.

As interesting special cases, we find that using $O(\Delta)$ colors, where Δ is the maximum degree, we can properly color the edges of a graph so that the following hold (even simultaneously):

(i) the union of any 2 color classes is a forest (this is the result of Alon et al in [AMR91]);

(ii) the union of any 3 color classes is outerplanar;

(iii) the union of any 4 color classes has treewidth at most 2;

(iv) the union of any 5 color classes is planar;

(v) the union of any 16 color classes is 5-degenerate.

We obtain the above bounds as an application of a special case of the Lovasz Local Lemma which we derive and show that the colorings obtained can be constructivized by the algorithm obtained by Moser and Tardos in [MT10]. We also obtain a general result for coloring the vertices of a hypergraph with constraints on the union of every few color classes.

Oriented coloring We obtain upper bounds for the oriented chromatic number of an arbitrary graph in terms of its generalized chromatic numbers, in particular the $(2, k)$-treewidth chromatic number which is the least number of colors required to color the vertices of a graph so that the union of any two color classes has treewidth at most k. Generalizing a result of Alon et al. in [AMS96], we prove that graphs of genus g have $(2, k)$-treewidth chromatic number $O(g^{\frac{1}{2}+\frac{1}{8k/3+2}})$ and use this result to show that graphs of genus g have oriented chromatic number at most $2^{g^{1/2+o(1)}}$.

Intersection Dimension Fix a graph property P. Given a graph G, what is the minimum k such that G can be expressed as the intersection of k graphs with property P? This minimum value is called the *intersection dimension* of G (w.r.t. property P) (see [KT94]) and generalizes the notions of *boxicity* ($P =$ set of interval graphs) and *circular dimension* (P=set of circular-arc graphs). We obtain upper bounds on the intersection dimenstion of arbitrary graphs with respect to several hereditary properties in terms of the maximum degree. In particular, we prove that the circular dimension of graphs of maximum degree Δ is $O(\Delta\frac{\log \Delta}{\log \log \Delta})$.

Outline of the thesis

In Chapter 2, we obtain bounds for generalized vertex colorings with constraints on the union of every few color classes. We obtain this as a consequence of a more general result on partitioning of the vertices of a hypergraph which in turn is obtained by deriving a special form of the Lovász Local Lemma.

In Chapter 3, we focus our attention on colorings with restrictions on the union of any *two* color classes. We first obtain lower bounds and then find upper bounds which are nearly tight.

In Chapter 4, we relate forbidden subgraph colorings and oriented colorings, obtaining a bound for the oriented chromatic number in terms of the former (chromatic number). We also obtain bounds for the $(2, k)$-treewidth chromatic numbers of graphs on surfaces. Again, we use a probabilistic argument to show that this bound is nearly tight. We then use these $(2, k)$-treewidth chromatic numbers to obtain bounds on the oriented chromatic number of graphs of bounded genus.

In Chapter 5, we study generalized edge colorings once again using the Lovász Local lemma as our tool. We obtain bounds in terms of the maximum degree and for several interesting special cases, we show that the bounds are in fact linear in terms of the maximum degree.

In Chapter 6, we prove results on the intersection dimension of a graph in terms of maximum degree. As an interesting special case, we obtain improved bounds for the ciruclar dimension of arbitrary graphs in terms of maximum degree.

Finally, in Chapter 7, we summarize our results and conclude with some open problems.

2

Technical Background

In this chapter, we present some technical concepts in graph theory and related results which we will use later.

2.1 Graph minors and treewidth

Given a graph $G = (V, E)$ and an edge $e = (u, v)$ of G, the removal of the edge e produces the graph $G - e = (V, E - \{e\})$. The *contraction* of the edge e produces the graph $G/e = (V - \{u, v\} + \{w\}, E')$ and E' consists of the edges in $G - \{u, v\}$ as well as edges between w and all vertices in $N_G(u) \cup N_G(v)$. A graph H is a *minor* of G (written $H \triangleleft G$) if H is obtained from G by a sequence of edge removals, vertex deletions and edge contractions.

A family of graphs is said to be *minor-closed* (or closed under minors) if for every graph G in the family, any minor of G also belongs to the family. Such a family is said to be *properly minor-closed* if it is a proper subset of the set of all graphs. Several natural graph families are closed under minors and hence the notion of graph minors has become fundamental to studying graph properties.

An important example of a minor-closed family is the family of *planar graphs*. Planar graphs are graphs whose vertices can be identified with points on a plane in such a way that the edges can be identified with pairwise non-intersecting arcs joining the points associated with the vertices.

A fundamental result in the theory of graph minors is the result of Robertson and Seymour (see [Die05] for details) that any proper minor-closed family of graphs is characterized by a finite set of forbidden minors. That is, a family $\mathcal{F}$ of graphs is closed under the operation of taking minors if and only if there exists a finite set $\mathcal{S}$ of graphs $\mathcal{S} = \{H_1, \ldots, H_s\}$ such that $\mathcal{F}$ consists exactly of those graphs which do not contain a copy of any graph from $\mathcal{S}$ as a minor. For example, a well-known result of Kuratowski characterizes planar graphs as precisely those graphs which do not contain K_5 or $K_{3,3}$ as a minor.

We now define the treewidth of a graph which is a parameter that measures how "tree-like" the graph is.

Given a graph G, a tree decomposition of G is a pair (T, X), where T is a tree with vertex set I and X is a collection of subsets $\{X_i : i \in I\}$ of the vertex set of G, satisfying the following three properties:

- $\bigcup_i X_i = V$;
- for every edge (u, v) of G, there is some X_i containing both u and v;
- for every vertex u of G, the subgraph of T induced on $\{i \in I : u \in X_i\}$ is a subtree.

The *width* of the tree decomposition is defined to be $max_i(|X_i| - 1)$. The *treewidth* of a graph G is defined to be the minimum width of any tree decomposition of G. A connected graph of treewidth at most k is also known as a partial k-tree. There are also other equivalent characterizations of treewidth, some of which are stated and used in Chapters 3 and 4.

2.2 Graph Classes

In this section, we define some well-known classes of graphs and mention some known results relating to them.

A *perfect graph* is a graph G such that for every induced subgraph H of G, $\chi(H) = w(H)$. An equivalent characterization is that a perfect graph is one which does not contain an odd hole or an odd anti-hole. An odd hole is an induced odd cycle on at least 5 vertices and an odd anti-hole is the complement of an odd hole. The equivalence of these two characterizations was a long-standing open problem suggested by Berge in 1960. It was known as the Strong Perfect Graph Conjecture (now the strong Perfect Graph Theorem) and it was settled affirmatively by Chudnovsky, Robertson, Seymour and Thomas in 2002 (see [CRST06]).

A *chordal graph* is a graph in which there are no induced cycles of length four or more. Chordal graphs form a proper subclass of perfect graphs. Chordal graphs can be recognized in linear time (see [RLT76]).

An *interval graph* is the intersection graph of a multiset of closed intervals on the real line. Formally, a graph $G = (V, E)$ is an interval graph if there is a mutliset $\{I(u) : u \in V\}$ of intervals such that for any two vertices u and v, $(u, v) \in E$ if and only if $I(u) \cap I(v) \neq \emptyset$. Interval graphs are a proper subclass of chordal graphs and hence are perfect. Interval graphs have a forbidden subgraph characterization [LB62] and can also be recognized in linear time (see [BL76]).

A *circular-arc graph* is the intersection graph of a multiset of closed arcs of a circle. Let S be the set of all closed arcs of the unit circle in the plane. Formally, a graph $G = (V, E)$ is a circular-arc graph if there is a function $I : V \to S$ such that for any two distinct vertices u and v, $(u, v) \in E$ if and only if $I(u) \cap I(v) \neq \emptyset$. Circular-arc graphs form a strict superclass of interval graphs. Despite their similarity to interval graphs, they are not necessarily perfect and there is no known explicit characterization of circular-arc graphs in terms of forbidden subgraphs. However, they can also be recognized in linear time, as shown by McConnell in [McC03].

A *permutation graph* is the intersection graph of a finite family of line segements that connect two parallel lines in the Euclidean plane. Equivalently, given a permutation π of $1, 2, \ldots, n$, the permutation graph corresponding to π consists of the vertex set $\{1, 2, \ldots, n\}$ and edges connecting two vertices i and j if $i < j$ and $\pi^{-1}(i) > \pi^{-1}(j)$. Permutation graphs also form a subclass of perfect graphs.

A *split graph* is a graph in which the vertex set can be partitioned into a clique and an independent set. Split graphs form a proper subclass of chordal graphs.

2.3 Graphs on surfaces

It is known that a drawing of a planar graph on a plane is "equivalent" to a drawing on the sphere S^2 (sphere in three dimensions), since the points of the plane can be homemomorphically mapped to points of S^2. An *embedding* of a graph G on a surface S is defined to be a representation of a graph on S such that the vertices of G are mapped to points on S and the edges of G are mapped to arcs in S in such a way that two arcs representing touching edges do not intersect each other. Thus a planar graph is one which admits an embedding on the sphere S^2.

Consider the surface obtained by adding a "handle" to the sphere as in Fig 1. This surface is known as the torus. If a graph G is not planar, we can ask whether it can be embedded on a torus. If not, can we always add more handles to obtain a surface on which G can be embedded? It turns out that the answer is yes. The surface S_g obtained by adding g handles to a sphere in 3-space is said to have *genus* g. The *genus of a graph* G is the least g such that G can be embedded on S_g.

Figure 2.1: A sphere with a handle, i.e. a torus

The surfaces S_g (and those which are homeomorphic to them) are known to be the only closed and connected orientable surfaces. There are also non-orientable surfaces. The interested reader can refer to [GT01] for details.

For a graph G embedded on S_g, where g is the genus of G, the Euler characterisitc of G with respect to a fixed embedding, is defined to be the quantity $v-e+f$, where v and e denote the number of vertices and edges (respectively) of G and f denotes the number of faces in the embedding. Euler's polyhedral formula states that the Euler characteristic is always a constant for any surface and is in fact, determined by the genus of the surface by the following relation: $v-e+f=2-2g$.

2.4 Random graphs

In their seminal paper [ER59], Erdös and Renyi introduced the $G(n,p)$ model of random graphs. In this model, a graph is randomly chosen by fixing a set of n labeled vertices and picking each of the $\binom{n}{2}$ unordered pairs as an edge independently with probability p.

Random graphs have found several applications in graph theory. One of its suprising applications emerged when Erdös showed the existence of graphs with high girth and high chromatic number. Since then, the use of random graphs to show the existence of graphs with a desired property has become a standard technique. In many cases, the only proofs of existence are based on the random graph approach and explicit constructions often turn out to be quite difficult.

In this thesis, we will use random graphs to prove the existence of graphs with high forbidden subgraph chromatic numbers.

We shall need the following well-known result on the degrees of random graphs. For a proof, see for example [B.B85].

Define μ by $\mu=(n-1)p \approx np$. Then, if Δ denotes the maximum degree of a random graph drawn from $G(n,p)$, we have

$$\mathbf{Pr}\left(\mu/2 \leq \Delta \leq 2\mu\right) \rightarrow 1 \text{ as } n \rightarrow \infty \tag{2.1}$$

provided $\mu \rightarrow \infty$ as $n \rightarrow \infty$.

In other words, the maximum degree of a random graph is almost surely close to its expected value. A similar bound on the maximum degree can also be obtained for a random bipartite graph. There are also other random graph models, such as random regular graphs, random geometric graphs etc, but in this thesis, we use only the $G(n,p)$ model.

3

Generalized vertex colorings

3.1 Introduction

The notion of acyclic (vertex) coloring was first introduced by Grünbaum [Grü73] in 1973. Acyclic coloring is a proper vertex coloring of G such that there are no two-colored cycles. Equivalently, the union of any two color classes must induce a forest. The minimum number of colors used by any such coloring is called the acyclic chromatic number of G and is denoted by $\chi_a(G)$. In [Grü73], Grünbaum showed that any planar graph can be acyclically colored using 9 colors and proposed the conjecture that every planar graph has an acyclic coloring using 5 colors. A series of improvements ([Mit74], [AB77], [Kos76]) on this bound followed in subsequent years and Borodin [Bor06] finally settled the conjecture in 2006.

A different problem was posed by Erdös in 1976 (see [AB76]). He conjectured that graphs of maximum degree Δ can be acyclically colored using $o(\Delta^2)$ colors. This problem was solved by Alon, McDiarmid and Reed [AMR91] in 1991, when they showed that for any graph G of maximum degree Δ, $\chi_a(G) \leq c\Delta^{4/3}$, where c is some absolute constant. They also showed that this bound is almost tight by giving a probabilistic construction of graphs which require $\Omega(\Delta^{4/3}/\log \Delta^{1/3})$ colors for any acyclic coloring.

The above result is the starting point of our work. In [AMR91], it was noted that the same method could be extended to avoiding paths of fixed length in the union of two color classes. Recall that a star coloring of a graph is a proper coloring in which a path on four vertices is forbidden in the union of any two color classes and the minimum number of colors that would guarantee such a coloring is called its star chromatic number. In [FRR04], Fertin, Raspaud and Reed obtained an upper bound of $O(\Delta^{3/2})$ for the star chromatic number and this bound was also shown to be nearly tight. A natural question to ask is whether these results can be extended for proper colorings in which we forbid an *arbitrary but fixed family of graphs* in the union of 2 color classes and more generally in the union of any j color classes where $j \geq 2$ is any natural number.

In this chapter, we will obtain some general bounds for such colorings in terms of the maximum degree and in the next chapter, obtain nearly tight bounds when the restriction is on the union of two color classes. We first give the formal definition of the general coloring notion which we consider.

Definition 3.1 *Given two graphs G and H, we say that G is H-free if G has no isomorphic copy of H as a subgraph (not necessarily induced). Given a family $\mathcal{F}$ of graphs, we say that G is $\mathcal{F}$-free if G is H-free for each $H \in \mathcal{F}$.*

Definition 3.2 *Let j be a positive integer and $\mathcal{F}$ be a family of connected graphs of (usual) chromatic number at most j such that for each $H \in \mathcal{F}$, $|V(H)| > j$. We define a $(j, \mathcal{F})$-*subgraph coloring *(or just $(j, \mathcal{F})$ coloring) to be a proper coloring of the vertices of a graph G so that the subgraph of G induced by the union of any j color classes is $\mathcal{F}$-free. We denote by $\chi_{j,\mathcal{F}}(G)$ the minimum number of colors sufficient to guarantee a $(j, \mathcal{F})$-subgraph coloring of G.*

Remark: We require $j < |V(H)|$ for each $H \in \mathcal{F}$ because otherwise if G contains a copy of H such that $j \geq |V(H)|$, no proper coloring of $V(G)$ would be a $(j, \mathcal{F})$-subgraph coloring. Also if $j < |V(H)|$ for each $H \in \mathcal{F}$, we are guaranteed of at least one $(j, \mathcal{F})$ coloring, namely the trivial coloring in which each vertex gets a distinct color. We include the condition that the chromatic number of H be at most j because otherwise any proper coloring would automatically forbid H in the union of j color classes and we can remove such a graph H from $\mathcal{F}$.

We also define $\chi_{j,\mathcal{F}}(\Delta) = max\{\chi_{j,\mathcal{F}}(G) : \Delta(G) = \Delta\}$. It can be seen that a proper coloring of the power graph G^j is a $(j, \mathcal{F})$ coloring of G and so Δ^j+1 is a trivial upper bound on $\chi_{j,\mathcal{F}}(G)$ if $\Delta(G) = \Delta$. Thus, $\chi_{j,\mathcal{F}}(\Delta)$ exists and is well-defined. It can also be verified that $\chi_{j,\mathcal{F}}(\Delta)$ is an increasing function of Δ.

An acyclic coloring is thus the same as a $(2, \mathcal{F})$-subgraph coloring for $\mathcal{F} = \{C_2, C_4, C_6 \ldots\}$ where C_i denotes a cycle on i vertices. Likewise, a star coloring is the same as a $(2, \{P_4\})$ coloring and a distance-two coloring is the same as a $(2, \{P_3\})$-coloring, where P_i denotes a path on i vertices.

The coloring notion we have described was first considered in its entire generality by Nesetril and Ossona de Mendez in [NdM06]. The coloring problem defined in [NdM06] is in fact even more general - it allows us to consider several pairs $(j, \mathcal{F})$ simultaneously. We will follow suit and consider such a general coloring later. While their focus was to show that some of these chromatic numbers are bounded for proper minor-closed families, our results are in the form of bounds in terms of the maximum degree for arbitrary graphs. Further we also consider a more general problem in this chapter - that of partitioning the vertices of hypergraphs with constraints on the unions of parts.

In [NdM06], it was proved that some of the chromatic numbers associated with such general colorings are bounded for proper minor-closed families of graphs. For suitably chosen constraints, this general notion specializes to known restricted colorings like acyclic colorings, star colorings, etc.

Another motivation to study this problem is its connection to oriented colorings. In [RS94], Raspaud and Sopena proved that the oriented chromatic number can be bounded as a function of the acyclic chromatic number. They then used this to show that the oriented chromatic number of planar graphs is at most 80. By extending their proof arguments, we show later that the oriented chromatic number can in fact be bounded as a function of $(2, \mathcal{F})$-chromatic numbers. Thus, a study of the $(2, \mathcal{F})$-chromatic numbers presents itself as a possible way to obtain improved bounds on oriented chromatic numbers for special graph classes. In Chapter 4, such improved bounds are indeed obtained and the connection between the two types of colorings is explored in more detail.

We now consider a special type of coloring where we require $\mathcal{F}$ to be a special class of graphs and obtain upper bounds on the corresponding chromatic numbers. These bounds will yield bounds on $\chi_{j,\mathcal{F}}(G)$ for arbitrary families $\mathcal{F}$ as a consequence.

Definition 3.3 *Let j and k be positive integers such that $j \leq k$. We define a* (j,k)-coloring *of a graph G to be a proper coloring of the vertices of G such that in the union of any j color classes, each connected component has size at most k. We denote by $\chi^{con}_{j,k}(G)$ the minimum number of colors sufficient to obtain a (j,k)-coloring of G.*

Note that a (j,k)-coloring is the same as a $(j,\mathcal{F})$-subgraph coloring if we choose $\mathcal{F}$ to be the set of all connected graphs on $k+1$ vertices. We also define $\chi^{con}_{j,k}(\Delta) = max\{\chi^{con}_{j,k}(G) : \Delta(G) = \Delta\}$; this is well-defined since it is a special case of the well-defined parameter $\chi_{j,\mathcal{F}}(\Delta)$.

First, using probabilistic arguments, we obtain the following upper bound on $\chi^{con}_{j,k}(G)$ of any graph in terms of its maximum degree Δ, which is one of the main results of our paper.

Theorem 3.4 *Let j, k be given positive integers such that $j \leq k$. Then there exists a constant $C = C(j,k)$ such that for any graph G of maximum degree Δ, $\chi^{con}_{j,k}(G) \leq C\Delta^{\frac{k}{k+1-j}}$.*

The above theorem immediately leads to an upper bound for $(j,\mathcal{F})$-subgraph colorings.

Theorem 3.5 *Let j be a positive integer and $\mathcal{F}$ be a family of connected graphs of chromatic number at most j. Let k (with $k > j$) denote $min_{H\in\mathcal{F}}|V(H)|$, i.e. k is the size of the smallest graph in $\mathcal{F}$. Then there exists a constant $C = C(j,k)$ such that for any graph of maximum degree Δ, $\chi_{j,\mathcal{F}}(G) \leq C\Delta^{\frac{k-1}{k-j}}$.*

By choosing $\mathcal{F} = \{P_4\}$ where P_4 is a path of length 3 on 4 vertices and by noting that a $(2,\mathcal{F})$-subgraph coloring is the same as a star coloring, we notice that the bound of $O\left(\Delta^{3/2}\right)$ on star chromatic number obtained in [FRR04] follows as a consequence of Theorem 3.5. On the other hand, we see that a bound of $O\left(\Delta^{3/2}\right)$

also applies to acyclic chromatic number, where $\mathcal{F} = \{C_4, C_6, \ldots\}$, and which is known to have a $O(\Delta^{4/3})$ upper bound.

We thus see that the bounds of Theorem 3.5 are not necessarily tight always and we can possibly obtain improvements by making use of the structure of the members of $\mathcal{F}$.

In the next section (Section 3.2), we prove our first main result, namely Theorem 3.4. In Section 3.3, we define and study colorings with constraints on the treewidth of the union of some color classes. In Section 3.4, we discuss the generalizations to forbidding several families simultaneously. In Section 3.5, we present some generalizations to constrained hypergraph colorings.

3.2 Proof of Theorem 3.4

The Lovász Local Lemma is a powerful probabilistic tool, introduced by Erdos and Lovász in their paper [EL75]. Qualitatively, it says the following: given a set of events, if each event depends on only a few other events (this is quantified by the exact statement), then the probability that none of them occur is greater than zero.

The following general form of the Local Lemma was obtained by J.Spencer and is necessary when dealing with asymmetric events, which will often be the case.

Lemma 3.6 *(see [AS92]) Let $\{A_1, A_2, ..., A_n\}$ be a family of events in an arbitrary probability space. Let the graph $H = (V, E)$ on the nodes $1, 2, \ldots, n$ be a dependency digraph for the events A_i; that is, assume that for each i, $Pr(A_i) = Pr(A_i|B_S)$ for any $S \subset M$, where $M = \{A_j : (i, j) \notin E\}$ and B_S denotes the event that all the events in S hold and none of the events in $M \setminus S$ hold. If there are reals $0 \leq y_i < 1$ such that for all i,*

$$Pr(A_i) \leq y_i \prod_{(i,j)\in E} (1 - y_j)$$

then

$$Pr(\cap(\overline{A_i})) \geq \prod_{i=1}^{n}(1 - y_i) > 0$$

so that with positive probability no event A_i occurs.

We now prove the following explicit version of Theorem 3.4.

Proposition 3.7 *Let j, k be given positive integers such that $j \leq k$. Then for any graph G of maximum degree Δ, $\chi_{j,k}^{con}(G) < \lceil C\Delta^{\frac{k}{k+1-j}} \rceil$ where $C = C(j,k) = (4(k+1)(12j)^{k+1})^{\frac{1}{k+1-j}}$.*

Proof of Proposition 3.7:

When $j = 1$, a (j, k)-coloring is also a proper coloring and the converse is also true. In this case, $\chi_{1,k}^{con}(G) = \chi(G) \leq \Delta + 1 \leq C\Delta$ since $C(1, k) \geq 12$. Hence, without loss of generality, we assume that $j \geq 2$. Now, let $x = \lceil C\Delta^{\frac{k}{k+1-j}} \rceil$ where $C = C(j,k) = (4(k+1)(12j)^{k+1})^{\frac{1}{k+1-j}}$.

Let $f : V \to \{1, 2, ..., x\}$ be a random vertex coloring of G, where for each vertex $v \in V$ independently, the color $f(v) \in \{1, 2, ..., x\}$ is chosen uniformly at random. It suffices to prove that with positive probability, f is a (j, k)-coloring of G. To this end, we define a family of bad events whose total failure implies that f is a (j, k)-coloring and use the Lovász Local Lemma to show that with positive probability none of them occur.

The events we consider are of the following two types.

a) **Type I**: For each pair of adjacent vertices u and v, let $A_{u,v}$ be the event that $f(u) = f(v)$.

b) **Type II**: For every connected induced subgraph L of $V(G)$ such that $|L| = k + 1$, let B_L be the event that the vertices in L are colored using at most j colors in the coloring by f.

Now we can see that if none of the events of the above two types occur, then f is a (j, k)-coloring.

Since no event of Type I occurs, the coloring is proper. Since no event of Type II occurs, the union of any j color classes cannot have a connected subgraph on $k + 1$ vertices.

It remains to show that with positive probability none of these events happen. To prove this, we apply Lemma 3.6. Any event of either of the two types is mutually independent of all events that do not share a vertex in common with the given event.

To enable the application of Local Lemma, we need to estimate the number of events of each type possibly influencing any given event. This estimate is given in the following two simple lemmas. First, we recall the following known fact from [LJK03].

Fact 3.8 *The number of mutually non-isomorphic (or unlabeled) trees on n vertices is at most 4^n.*

Proof This fact is proved in Chapter 8 of [LJK03]. We give an outline of this proof for the sake of completion.

Embed an unlabeled tree in the plane without crossing edges and draw an extra copy of each edge by its side. Fix any vertex with degree one as the root. Start from the root and complete an Eulerian traversal of the edges by always following the rule of traversing the clockwise next edge incident at a vertex. Encode this traversal by representing each edge by a 1 if it takes the traversal to an unvisited vertex and by a 0 otherwise. One can verify that this encoding is an injective one-to-many mapping of unlabeled trees into binary strings of length $2(n-1)$. Since the number of binary strings of length $2(n-1)$ is $4^{n-1} \leq 4^n$, the result is proved. ∎

Lemma 3.9 *Let v be an arbitrary vertex of the graph $G = (V, E)$. Then the following two statements hold.*

(i) v belongs to at most Δ edges of G.

(ii) v belongs to at most $(k+1)4^{k+1}\Delta^k$ connected induced subgraphs of size $k+1$ in $V(G)$.

Proof of Lemma 3.9

Part (i) follows from the fact that $\Delta(G) = \Delta$.

Part (ii) can be seen as follows: Let $\mathcal{G}(v, k+1)$ be the set of $(k+1)$-element connected induced subgraphs in G containing v and let $\mathcal{T}(v, k+1)$ be the set of $(k+1)$-element trees in G containing v. Each tree in $\mathcal{T}(v, k+1)$ can be identified with a unique connected induced subgraph of G and each connected induced subgraph in $\mathcal{G}(v, k+1)$ has at least one tree in $\mathcal{T}(v, k+1)$ which is identified with it. Thus $|\mathcal{G}(v, k+1)| \leq |\mathcal{T}(v, k+1)|$. We now find an upper bound for $|\mathcal{T}(v, k+1)|$. Since there are at most 4^{k+1} non-isomorphic trees on $k+1$ vertices (by Fact 3.8), there are at most 4^{k+1} choices for choosing the non-isomorphic structure of a tree in $\mathcal{T}(v, k+1)$. Once this is fixed, we now have to embed this tree in G. The number of choices for the position of v in the tree is $k+1$. Now the remaining vertices in the unlabeld tree can be embedded in at most Δ^k ways. To see this, we observe that there are at most Δ choices for each neighbor of v in the chosen tree. Once these are fixed, the number of choices for each vertex at distance 2 from v is again at most Δ. Repeating this process, we can see that the number of choices for embedding all the vertices (other than v) is at most Δ^k. ∎

Lemma 3.10 *For $i, j \in \{I, II\}$, the (i, j)-th entry of the table given below is an upper bound on the number of events of type j in which can possibly influence an event of type i.*

	I	$II(B_{L'})$
I	2Δ	$2(k+1)4^{k+1}\Delta^k$
$II(B_L)$	$(k+1)\Delta$	$(k+1)^2 4^{k+1}\Delta^k$

The lemma follows from Lemma 3.9 and the fact that any event is mutually independent of all other events which do not share any vertex with the given event.

We now estimate the probability of occurrence of each type of event.

Fact 3.11 *(i) For each type I event A, $Pr(A) = \frac{1}{x}$.*
(ii) For each type II event B, $Pr(B) \leq \frac{j^{k+1}}{x^{k+1-j}}$.

The number of ways in which a $(k+1)$-element set can be colored using at most j colors is at most $\binom{x}{j} j^{k+1} \leq x^j j^{k+1}$. This proves (ii).

We now define the weights y_i to enable us to apply Lemma 3.6.

For an event A of type I, $y_A = \frac{9}{x}$. For an event B of type II, $y_B = \frac{(3j)^{k+1}}{x^{k+1-j}}$. It follows from the definition of x that $y_B < 1$.

By Lemma 3.6, Lemma 3.10 and Fact 3.11, it suffices to verify the following two inequalities.

$$\frac{1}{x} \leq \frac{9}{x}\left(1 - \frac{9}{x}\right)^{2\Delta}\left(1 - \frac{(3j)^{k+1}}{x^{k+1-j}}\right)^{2(k+1)4^{k+1}\Delta^k} \tag{3.1}$$

$$\frac{j^{k+1}}{x^{k+1-j}} \leq \frac{(3j)^{k+1}}{x^{k+1-j}}\left(1 - \frac{9}{x}\right)^{(k+1)\Delta}\left(1 - \frac{(3j)^{k+1}}{x^{k+1-j}}\right)^{(k+1)^2 4^{k+1}\Delta^k} \tag{3.2}$$

We observe that (3.2) is equivalent to (3.1). This can be seen by taking both sides of inequality (3.2) to the $2/(k+1)$-th power after canceling the term j^{k+1}/x^{k+1-j} on each side. Thus it is sufficient to prove (3.1).

In (3.1), we substitute $x = C\Delta^{\frac{k}{k+1-j}}$ where $C = C(j,k) = (4(k+1)(12j)^{k+1})^{\frac{1}{k+1-j}}$ and using the fact that $(1 - \frac{1}{z})^z \geq 1/4$ for all $z \geq 2$, we see that it is sufficient to prove:

$$\frac{1}{9} \leq 4^{-\frac{18\Delta}{x}} 4^{-1/2}$$

Since $x \geq 18\Delta$ for $j \geq 2$, the above inequality is true.

Thus by the Lovász Local Lemma, with probability greater than zero none of the bad events occurs and hence there exists a (j,k)-coloring using $O(\Delta^{\frac{k}{k+1-j}})$ colors. This completes the proof of Proposition 3.7 and hence of Theorem 3.4. ∎

3.3 Low treewidth coloring

In this section, we consider a specialization of forbidden subgraph colorings obtained by restricting the union of color classes to be a graph of bounded treewidth. From this, we obtain the notion of (low) treewidth coloring. This naturally generalizes the acyclic vertex coloring which requires the union of two color classes to have treewidth at most 1.

Low treewidth colorings have been studied in [DDO+04], where the authors prove the following result: For any fixed graph H and a positive integer k, there exists a constant $C = C(H, k)$ such that any graph that does not contain H as a minor can be vertex-partitioned into C parts, so that for all $j \leq k$, the union of any j parts has treewidth at most $j - 1$. In contrast to obtaining bounds for minor-closed families, our focus will be to obtain bounds for treewidth chromatic numbers of arbitrary graphs in terms of the maximum degree.

To begin, we recall one of many equivalent definitions of the treewidth of a graph. The treewidth of a graph G is the minimum k such that G is a subgraph of a k-tree. A k-tree is a graph obtained by starting with a complete graph on $k+1$ vertices and then iteratively adding a new vertex and joining it (by an edge) to each member of some k-clique in the partial graph obtained so far.

Definition 3.12 *Let j, k be positive integers such that $j \leq k + 1$. We define a* (j, k)-treewidth (vertex) coloring *of a graph $G = (V, E)$ to be a proper coloring of $V(G)$ such that the subgraph induced by the union of any j color classes has treewidth at most k. We denote by $\chi^{tw}_{j,k}(G)$ the minimum number of colors required for a (j, k)-treewidth coloring of G.*

Remark: We require $j \leq k+1$ because otherwise if G contains a clique on $k+2$ vertices, then no proper coloring of $V(G)$ would be a (j, k)-treewidth coloring. Also if $j \leq k+1$ we are guaranteed of at least one (j, k)-treewidth coloring, namely the trivial coloring in which each vertex gets a distinct color.

We also define $\chi^{tw}_{j,k}(\Delta) = max\{\chi^{tw}_{j,k}(G) : \Delta(G) = \Delta\}$. This is a well-defined parameter, as it is a special case of $\chi_{j,\mathcal{F}}(\Delta)$.

We note that a (j, k)-treewidth coloring is the same as a $(j, \mathcal{F})$-subgraph coloring where $\mathcal{F}$ is the set of all j-colorable graphs of treewidth $k + 1$. Also, an acyclic coloring is the same as a $(2, 1)$-treewidth coloring.

In this section, we prove that Theorem 3.5 also leads to the following upper bounds for (j, k)-treewidth colorings.

Theorem 3.13 *Let j, k be given positive integers such that $j \leq k + 1$. Then,* (*i*) *there exists a constant* $C = C(j,k)$ *such that for any graph* G *of maximum degree* Δ, $\chi^{tw}_{j,k}(G) < C\Delta^{\frac{kj+1}{kj+1-(j-1)^2}}$. *In particular, for each* $k \geq 3$, *we have* $\chi^{tw}_{2,k}(G) \leq C\Delta^{(1+\frac{1}{2k})}$.

(*ii*) *When* $j = k = 2$, *we have the following better bound* $\chi^{tw}_{2,2}(\Delta) = O(\Delta^{8/7})$. *This is the minimum number of colors sufficient to ensure that any two color classes induces a graph of treewidth at most 2.*

We first show that Part (i) of Theorem 3.13 follows from Theorem 3.5. For this, it only remains to obtain a lower bound on the number of vertices in any j-colorable graph H whose treewidth is at least $k+1$. All such graphs are forbidden for a (j,k)-treewidth coloring. We make use of the following easy to prove observation.

Proposition 3.14 *Let* H *be a complete* j*-partite graph* $K_{m_1,\ldots,m_j}$ *where we assume that* $m_1 \leq \ldots \leq m_j$. *Then,* $tw(H) = m_1 + m_2 + \ldots + m_{j-1}$.

Proof of Proposition 3.14 A graph is *chordal* if it has no induced cycle of length 4 or more. A *chordal completion* of a graph $G = (V,E)$ is any super graph $G' = (V,F)$, $E \subseteq F$, which is also chordal. It is well known (see [RS86]) that the treewidth of a graph G is exactly one less than the minimum value of the maximum clique size $\omega(G')$ of any chordal completion G' of G.

Let $C_1, \ldots, C_j$ be the j color classes of H with $|C_j| = m_j$. Let m denote the sum $m_1 + \ldots + m_j$. Any chordal completion H' of H should have enough edges to make each (except possibly one, say, C_i) of the color classes a complete subgraph. Also, to minimize the value of $\omega(H')$, we need to maintain C_i as an independent set in H'. Hence $\omega(H') = m - m_i + 1$. This value is minimized when $i = j$. Our claim follows from this observation. ∎

Proof of Part (*i*) **of Theorem 3.13:** Fix a j-colorable graph H whose treewidth is at least $k+1$ and having a minimum number of vertices. Suppose H has a j-coloring with color classes $C_1, \ldots, C_j$, where we assume without loss of generality, that $|C_1| \leq \ldots \leq |C_j|$. Since adding edges does not decrease treewidth, we can assume without loss of generality that H is a complete j-partite graph. For each i, let m_i denote $|C_i|$. Then, by the previous proposition, we have $\sum_{i<j} m_i \geq k+1$ and hence $|V(H)| = \sum_{i \leq j} m_i \geq (k+1)j/(j-1)$. Applying this fact to Theorem 3.5,

we obtain (after simplifications) that $\chi^{tw}_{j,k}(G) \leq c\left(\Delta^{\frac{kj+1}{kj+1-(j-1)^2}}\right)$ for some absolute positive constant c. This proves Part (i). ∎

For proving Part (ii), we shall need the following well-known result:

Fact 3.15 *([WC83]) A graph has treewidth at most 2 if and only if it has no subgraph which is isomorphic to a subdivision of K_4.*

We remark that in [BLS] also, an equivalent statement may be found, where the paper of Wald and Colbourn referred to above is cited. We now prove Part (ii) of Theorem 3.13 in the following specific form.

Proposition 3.16 *Let $G = (V, E)$ be a graph with maximum degree Δ. Then $\chi^{tw}_{2,2}(G) \leq 25\Delta^{8/7}$.*

Proof of Proposition 3.16:

Put $\alpha = 6/7$; $x = \lceil c_1 c_2 \Delta^{2-\alpha} \rceil$ where $c_1, c_2 > 1$ are constants to be chosen later so that $c_1 c_2 = 25$.

Let $f : V \to \{1, 2, ..., x\}$ be a random vertex coloring of G, where for each vertex $v \in V$ independently, the color $f(v) \in \{1, 2, ..., x\}$ is chosen uniformly at random. It suffices to prove that with positive probability, the union of any two color classes has no subdivision of K_4 and hence has treewidth at most 2. To ensure this, we define a family of bad events which correspond to proper two-colorings of bipartite subdivisions of K_4 in G, then apply the Lovász Local Lemma to show that with positive probability none of them occur, and conclude that since none of them occur f is a (2,2)-treewidth coloring. The events we consider are of the following six types.

a) **Type I**: For each pair of adjacent vertices u and v, let $A_{u,v}$ be the event that $f(u) = f(v)$.

Absence of Type I events ensure properness, so, by Fact 3.15, we need only to ensure each 2-colorable subdivision of K_4 which is present in G is not 2-colored.

To reduce the number of bipartite K_4 subdivisions we need to consider, we use a notion similar to the one employed in [AMR91] and [AMR92]. Recall that when counting the number of copies of a forbidden graph H containing a given vertex, we allow Δ choices for embedding a vertex some of whose neighbors have already been embedded. However, if at least two neighbors of a vertex v are already embedded, then we would like to bound the number of choices for v in G to be a smaller function of Δ, say Δ^α. This can be achieved if non-adjacent pairs which have more than Δ^α common neighbors are distinctly colored, since this would ensure that copies of H containing such pairs would use at least 3 colors. We now apply this idea.

A pair of non-adjacent vertices is called a *special pair* if they have more than Δ^α common neighbours.

b) **Type II**: For each pair of special vertices u and v, let $B_{u,v}$ be the event that $f(u) = f(v)$.

If we forbid all events of Types I and II, then it suffices to only ensure that those bipartite K_4 subdivisions are not 2-colored, which do NOT have a triple (u, v, w) such that $\{u, v\}$ forms a special pair and w is one of their common neighbors. This is because any K_4 subdivision having such a triple will be colored with at least 3 colors.

Henceforth, we only focus on bipartite (that is, 2-colorable) K_4 subdivisions which do not have such a triple described before.

Note that every bipartite subdivision of K_4 should have at least 6 vertices. Also note that the graphs H_1, H_2 and $\{H_3, H_4\}$ which we consider below, are the only non-isomorphic bipartite subdivisions of K_4 on 6,7 and 8 vertices respectively.

c) **Type III**:

For each subgraph $H_1(v_0, v_1, v_2, v_3, v_4, v_5)$ of the form shown below (Figure 1), in which whenever $i = j \pmod 2$, v_i and v_j are non-adjacent and not a special pair, let $C_1\{v_0, v_1, v_2, v_3, v_4, v_5\}$ be the event that H is properly two-colored by f, i.e, $f(v_0) = f(v_2) = f(v_4)$ and $f(v_1) = f(v_3) = f(v_5)$.

d) **Type IV**:

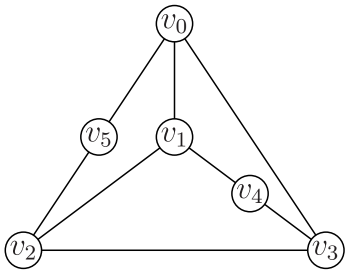

Figure 3.1: H_1

For each subgraph $H_2(v_0, v_1, v_2, v_3, v_4, v_5, v_6)$ of the form shown below (Figure 2), in which if $i = j \pmod 2$ v_i and v_j are non-adjacent and not a special pair, let $C_2\{v_0, v_1, v_2, v_3, v_4, v_5, v_6\}$ be the event that H is properly two-colored by f, i.e, $f(v_0) = f(v_2) = f(v_4) = f(v_6)$ and $f(v_1) = f(v_3) = f(v_5)$.

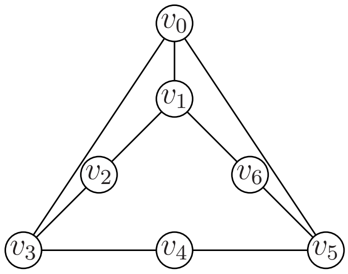

Figure 3.2: H_2

e) **Type V**:

For each of the two subgraphs $H_3(v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7)$ and $H_4(v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7)$ of the forms shown below (Figure 3), in which if $i = j \pmod 2$ v_i and v_j are non-adjacent and not a special pair, let $C_3\{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ be the event that H is properly two-colored by f, i.e, $f(v_0) = f(v_2) = f(v_4) = f(v_6)$ and $f(v_1) = f(v_3) = f(v_5) = f(v_7)$.

f) **Type VI**:

For $l \geq 9$ and each bipartite subdivision H_l of K_4 of size l, let $D_{l,V(H_l)}$ be the event that the vertices of H_l are properly two-colored in the f-coloring.

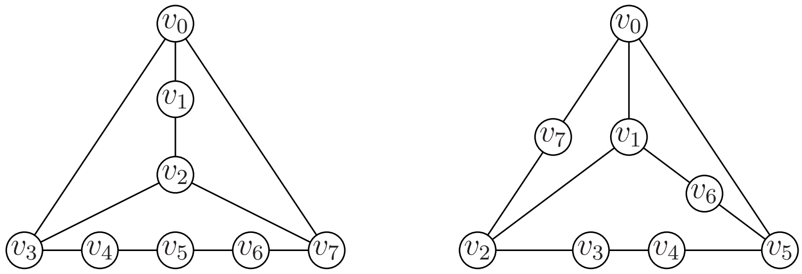

Figure 3.3: H_3 and H_4

From the arguments given above, it follows that if none of the events of the six Types I, II, III, IV, IV and VI described above occurs, then f is a (2,2)-treewidth coloring.

It remains to show that with positive probability none of these events happen. To prove this we apply the Lovász Local Lemma. We construct a dependency graph H whose nodes are all the events of all the six types, in which two nodes X_S and Y_T (where X and Y are one the A, B, C, D events and X and Y respectively depend on the colors of vertices in S and T) are adjacent if and only if $S \cap T \neq \emptyset$.

We need to estimate the number of nodes of each type in H adjacent to any given node. This estimate is given in the following two simple lemmas.

Lemma 3.17 *Let v be an arbitrary vertex of the graph $G = (V, E)$. Then the following four statements hold.*

(i) v belongs to at most Δ edges of G.

(ii) The number of special pairs containing v is at most $\Delta^{2-\alpha}$.

(iii) For each $t \in \{1, 2\}$, the number of subgraphs of G isomorphic to H_t and containing v is at most $8\Delta^{t+1+3\alpha}$. The number of subgraphs of G isomorphic to H_3(or H_4) and containing v is at most $8\Delta^{4+3\alpha}$.

(iv) For $l \geq 9$, the number of subgraphs of G on l vertices isomorphic to some bipartite subdivison of K_4 and containing v is at most $\frac{l^6 \cdot \Delta^{l-1}}{120}$.

Proof of Lemma 3.17

Part (i) follows from the fact that $\Delta(G) = \Delta$.

Part (ii) follows from the fact that there are at most Δ^2 induced paths of length 2 starting from v and for each special pair $\{u, v\}$ there are more than Δ^α induced paths of length 2 leading to u. Thus the number of special pairs containing v is at most $\frac{\Delta^2}{\Delta^\alpha} = \Delta^{2-\alpha}$.

Proof of Part (iii): Consider the case $t = 3$. There are at most 8 ways of identifying v with a vertex in H_3. Suppose v is identified with v_0. The there are at most Δ choices each for v_3 and v_7. Once these are fixed there are at most Δ choices for each of v_4 and v_6. Now there are at most Δ^α choices for each of v_2 and v_5 since neither v_3 and v_7 nor v_4 and v_6 form a special pair. Now since v_0 and v_2 do not form a special pair, there are at most Δ^α choices for v_1. Thus there are at most $\Delta^{4+3\alpha}$ ways of embedding H_3 in G so that it contains v in the position of v_0. A similar analysis shows that in each of the other five cases, there are at most $\Delta^{4+3\alpha}$ ways of embedding H_3 in G so that it contains v in a fixed position. This proves (iii) for $t = 3$. The proofs for the cases $t \in \{1, 2, 4\}$ are similar.

Proof of Part (iv) : Note that the number of mutually non-isomorphic bipartite subdivisions of K_4 on l vertices is at most the number of ordered partitions of $l-4$ into six non-negative integers. The latter number is well-known to be $\binom{l+1}{5} \leq l^5/120$. For any such bipartite subdivision H_l, v can be one of the l vertices in H_l. Thus there are at most l ways to fix the position of v in H_l. Since H_l is connected, there is a spanning tree T which is a subgraph of H_l with v as the root and we fix one such spanning tree. Once v is fixed, for each of its neighbors in H_l, i.e. the nodes in the first level in T, there are at most Δ choices. Similarly, once these node are fixed, the nodes in the next level have at most Δ choices each. Thus the number of copies of H_l is at most $l\Delta^{l-1}$. Multiplying this by the number of possible H_ls, we prove Part (iv).

Lemma 3.18 *For $i, j \in \{I, II, III, IV, V, VI\}$ the (i, j) entry of the table M given below is an upper bound on the number of nodes of type j in the dependency graph H which are adjacent to a node of type i in H. The upper bound for the number of events of types Y that can influence an event of type X is obtained by multiplying the number of vertices in the event of type X by the bound obtained in Lemma 3.17 for the number of events of type Y that contain a given vertex.*

	I	II	III	IV	V	$VI(D_{l,V(H_l)})$
I	2Δ	$2\Delta^{2-\alpha}$	$16\Delta^{2+3\alpha}$	$16\Delta^{3+3\alpha}$	$32\Delta^{4+3\alpha}$	$2l^6\Delta^{l-1}/120$
II	2Δ	$2\Delta^{2-\alpha}$	$16\Delta^{2+3\alpha}$	$16\Delta^{3+3\alpha}$	$32\Delta^{4+3\alpha}$	$2l^6\Delta^{l-1}/120$
III	6Δ	$6\Delta^{2-\alpha}$	$48\Delta^{2+3\alpha}$	$48\Delta^{3+3\alpha}$	$96\Delta^{4+3\alpha}$	$6l^6\Delta^{l-1}/120$
IV	7Δ	$7\Delta^{2-\alpha}$	$56\Delta^{2+3\alpha}$	$56\Delta^{3+3\alpha}$	$112\Delta^{4+3\alpha}$	$7l^6\Delta^{l-1}/120$
V	8Δ	$8\Delta^{2-\alpha}$	$64\Delta^{2+3\alpha}$	$64\Delta^{3+3\alpha}$	$128\Delta^{4+3\alpha}$	$8l^6\Delta^{l-1}/120$
$VI(D_{k,V(H_k)})$	$k\Delta$	$k\Delta^{2-\alpha}$	$8k\Delta^{2+3\alpha}$	$8k\Delta^{3+3\alpha}$	$16k\Delta^{4+3\alpha}$	$kl^6\Delta^{l-1}/120$

Fact 3.19 *(i) For each type I event A, $Pr(A) = \frac{1}{x}$.*
(ii) For each type II event B, $Pr(B) = \frac{1}{x}$.
(iii) For each type III event C, $Pr(C_1) \leq \frac{1}{x^4}$.
(iv) For each type IV event D, $Pr(C_2) \leq \frac{1}{x^5}$.
(v) For each type V event E, $Pr(C_3) \leq \frac{1}{x^6}$.
(vi) For each type VI event D_l, $(l \geq 9)$, $Pr(D_l) \leq \frac{1}{x^{l-2}}$.

We now define the weights y_i to apply the Lemma 3.6.

Recall that c_1 and c_2 are positive constants such that $c_1c_2 = 25$. We choose $c_1 = 6.25$ and $c_2 = 4$.

For an event A of type I, $y_A = \frac{c_2}{x}$. For an event B of type II, $y_B = \frac{c_2}{x}$. For an event of the form C_t, $t \in \{1,2,3\}$, $y_{C_t} = \frac{{c_2}^{\frac{t+3}{2}}}{x^{t+3}}$. For an event of the form D_l of type VI, $y_{D_l} = \frac{{c_2}^{\frac{l-2}{2}}}{x^{l-2}}$.

Let $T_2 = \left(1 - \frac{c_2}{x}\right)$, $T_3 = \left(1 - \frac{{c_2}^2}{x^4}\right)$, $T_4 = \left(1 - \frac{{c_2}^{2.5}}{x^5}\right)$, $T_5 = \left(1 - \frac{{c_2}^3}{x^6}\right)$, $T_6 = \left(1 - \frac{{c_2}^{\frac{l-2}{2}}}{x^{l-2}}\right)$.

By Lemma 3.6, Lemma 3.18 and Fact 3.19, it suffices to verify the following two inequalities, where the first inequality corresponds to events of types I and II and the 2nd inequality to events of types III,IV,V and VI.

$$\frac{1}{x} \leq \frac{c_2}{x} T_2^{2\Delta+2\Delta^{2-\alpha}} T_3^{16\Delta^{2+3\alpha}} T_4^{16\Delta^{3+3\alpha}} T_5^{32\Delta^{4+3\alpha}} \prod_{l \geq 9} T_6^{2l^6\Delta^{l-1}/120} \tag{3.3}$$

For $k \geq 6$,

$$\frac{1}{x^{k-2}} \leq \frac{c_2^{\frac{k-2}{2}}}{x^{k-2}} {T_2}^{k\Delta+k\Delta^{2-\alpha}} {T_3}^{8k\Delta^{2+3\alpha}} {T_4}^{8k\Delta^{3+3\alpha}} {T_5}^{16k\Delta^{4+3\alpha}} \prod_{l\geq 9} {T_6}^{kl^6\Delta^{l-1}/120} \tag{3.4}$$

Simplifying (3.3), we get:

$${T_2}^{\Delta+\Delta^{2-\alpha}} {T_3}^{8\Delta^{2+3\alpha}} {T_4}^{8\Delta^{3+3\alpha}} {T_5}^{16\Delta^{4+3\alpha}} \prod_{l\geq 9} {T_6}^{l^6\Delta^{l-1}/120} \geq \frac{1}{\sqrt{c_2}} \tag{3.5}$$

Simplifying (3.4), we get:
For $k \geq 6$,

$${T_2}^{\Delta+\Delta^{2-\alpha}} {T_3}^{8\Delta^{2+3\alpha}} {T_4}^{8\Delta^{3+3\alpha}} {T_5}^{16\Delta^{4+3\alpha}} \prod_{l\geq 9} {T_6}^{l^6\Delta^{l-1}/120} \geq {c_2}^{\frac{1}{k}-\frac{1}{2}} \tag{3.6}$$

Clearly, proving (3.6) for $k = 6$ is sufficient to prove both inequalities (3.5) and (3.6). We now substitute $c_1 = 6.25$ $c_2 = 4$. This yields R.H.S. of (3.6) (for $k = 6$) $= \frac{1}{4}^{\frac{1}{3}}$.

Consider the L.H.S. of (3.6) (for $k = 6$). Substituting $x = c_1c_2\Delta^{2-\alpha}$ and using the fact that $(1 - \frac{1}{z})^z \geq 1/4$ for all $z \geq 2$, we deduce that L.H.S. of (4) is at least $(\frac{1}{4})^{S_1}$, where

$$S_1 = \left(\frac{2}{c_1}\right) + \left(\frac{8}{(c_1\sqrt{c_2})^4\Delta^{6-7\alpha}}\right) + \left(\frac{8}{(c_1\sqrt{c_2})^5\Delta^{7-8\alpha}}\right) + \left(\frac{16}{(c_1\sqrt{c_2})^6\Delta^{8-9\alpha}}\right) + S_2$$

and $S_2 = \sum_{l\geq 9}\left(\frac{l^6}{120(c_1\sqrt{c_2})^{l-2}\Delta^{(l-3)-(l-2)\alpha}}\right)$

$$S_1 \leq \frac{2}{c_1} + \frac{24}{(c_1\sqrt{c_2})^4} + \sum_{l\geq 9} \frac{l^6}{120(c_1\sqrt{c_2})^{l-2}} \text{ (using } \alpha = 6/7 \text{ and } c_1\sqrt{c_2} > 12)$$

$$\leq \frac{2}{6.25} + \frac{2}{12^3} + \sum_{l\geq 9} \frac{1}{60 * 2^{l-2}} \text{ (using } c_2 = 4,\ c_1 = 6.25 \text{ and } 2(6.25)^{l-2} \geq l^6 \text{ for } l \geq 9)$$

Thus,

$$S_1 \leq \frac{2}{6.25} + \frac{2}{12^3} + \frac{1}{60 * 2^6} \text{ which is smaller than } \tfrac{1}{3}$$

Hence inequality (3.6) is proved.

Thus by the Lovász Local Lemma, with probability greater than zero none of the bad events occurs and hence there exists a $(2,2)$-coloring using $O(\Delta^{\frac{8}{7}})$ colors. This completes the proof of Proposition 3.16 and hence of Theorem 3.13.

3.4 Extensions to colorings with several families forbidden simultaneously

It is also possible to extend our results to more restricted colorings where we require simultaneously for several pairs $(j_i, \mathcal{F}_i)$ $(i = 1, \ldots, l)$ that the union of *any* j_i color classes has no copy of any member of $\mathcal{F}_i$. Such colorings are precisely the kind of colorings considered by Nešetřil and Ossona de Mendez in [NdM06] for families of H-minor-free graphs. This notion generalizes the kind of colorings studied by DeVos, et. al. in [DDO+04] for families of H-minor-free graphs and discussed in the beginnning of Section 3.3. For some types of such generalized colorings, Nešetřil and Ossona de Mendez prove in [NdM06] that the associated chromatic numbers are bounded for any proper minor-closed family of graphs. See also [Zhu09] for some related work on some similar colorings by Zhu. However, we obtain bounds which work for any arbitrary graph G. We first formally define these colorings.

Definition 3.20 *Let $\mathcal{P} = \{(j_1, \mathcal{F}_1), \ldots, (j_l, \mathcal{F}_l)\}$ be a set of $l \geq 1$ pairs such that for each $i \leq l$, j_i is a positive integer and $\mathcal{F}_i$ is a family of connected graphs of (usual) chromatic number at most j_i such that for each $H \in \mathcal{F}_i$, $|V(H)| > j_i$. We define a $\mathcal{P}$-subgraph coloring to be a proper coloring of the vertices of a graph G so that, for each $i \leq l$, the subgraph of G induced by the union of any j_i color classes does not contain an isomorphic copy of H as a subgraph, for each $H \in \mathcal{F}_i$. We denote by $\chi_{\mathcal{P}}(G)$ the minimum number of colors sufficient for a $\mathcal{P}$-subgraph coloring of G.*

As before (i.e. when $\mathcal{P}$ consists of only one pair), we shall first consider colorings in which we restrict the size of every connected component in the union of color classes and then derive, as a consequence, bounds for the $\mathcal{P}$-colorings defined above.

Definition 3.21 *Let $\mathcal{T} = \{(j_1, k_1), \ldots, (j_l, k_l)\}$ where the j_i's and k_i's are positive integers such that $j_i \leq k_i$ for each $i \in \{1, \ldots, l\}$. We define a $\mathcal{T}$-coloring to be a proper coloring of the vertices of a graph so that in the union of any j_i color classes, each connected component has size at most k_i for each $i \in \{1, \ldots, l\}$. We denote by $\chi_{\mathcal{T}}^{con}(G)$ the minimum number of colors sufficient for a $\mathcal{T}$-coloring of $V(G)$.*

We now present the main results of this section.

Theorem 3.22 *Let $\mathcal{T} = \{(j_1, k_1), \ldots, (j_l, k_l)\}$ where the j_is and k_is are positive integers such that $j_i \leq k_i$ for each $i \in \{1, \ldots l\}$. Then there exists a constant $C = C(\mathcal{T})$ such that for any graph G of maximum degree Δ, $\chi_{\mathcal{T}}^{con}(G) \leq C\Delta^{max_i \frac{k_i}{k_i+1-j_i}}$ where we choose*

$$C = C(\mathcal{T}) = \max_i \left(4l(k_i+1)(12j_i)^{k_i+1}\right)^{\frac{1}{k_i+1-j_i}}.$$

We skip the proof of the above theorem as it is based on an application of the Lovász Local Lemma and is very similar to the proof of Theorem 3.4. The above theorem immediately leads to an upper bound for $\mathcal{P}$-subgraph colorings.

Corollary 3.23 *Let $\mathcal{P} = \{(j_1, \mathcal{F}_1), \ldots, (j_l, \mathcal{F}_l)\}$ be a set of $l \geq 1$ pairs such that for each $i \leq l$, j_i is a positive integer and $\mathcal{F}_i$ is a family of connected graphs of (usual) chromatic number at most j_i such that for each $H \in \mathcal{F}_i$, $|V(H)| > j_i$. Let k_i (with $k_i > j_i$) denote the size of the smallest graph in $\mathcal{F}_i$. Then there exists a constant $C = C((j_1, k_1), \ldots, (j_l, k_l))$ such that for any graph G of maximum degree Δ, $\chi_{\mathcal{P}}(G) \leq C\Delta^{max_i \frac{k_i-1}{k_i-j_i}}$.*

By setting $\mathcal{P}_l = \{(1, \mathcal{F}_1), \ldots, (l, \mathcal{F}_l)\}$ where $\mathcal{F}_i$ is the set of all i-colorable (usual coloring) graphs of treewidth i, for each $i \leq l$, we can get upper bounds on the the type of colorings studied by DeVos, et. al. in [DDO$^+$04]. The proof of the following result follows essentially from the proof arguments of Part (i) of Theorem 3.13 (on low treewidth colorings).

Corollary 3.24 *For $l \geq 1$, let $\chi_{\mathcal{P}_l}(G)$ denote the minimum number of colors sufficient to obtain a proper coloring of $V(G)$ so that the union of any $j \leq l$ color classes forms a subgraph of treewidth at most $j-1$. Then, there exists a constant $C = C(l)$ such that for any graph of maximum degree Δ, $\chi_{\mathcal{P}_l}(G) \leq C\dot{\Delta}^{l-1+(1/l)}$.*

Note that the problem of testing whether $\chi_{j,\mathcal{F}}(G) \leq k$ for an input graph G and input parameter k is NP-complete even for some fixed $(j, \mathcal{F})$ (examples : $(1, \mathcal{F}_1)$, $(2, \mathcal{F}_2)$ where $\mathcal{F}_1 = \{K_2\}$ and $\mathcal{F}_2$ is the set of cycles). It would be interesting to determine the computational complexity of this problem for other pairs $(j, \mathcal{F})$.

3.5 Special form of Lovász Local Lemma and hypergraph colorings

We now derive a special form of the Lovász Local Lemma, using which we generalize the results of Section 3.2 to hypergraph colorings with constraints. We also show that this special version of LLL is efficiently constructive (provided there is a polynomial time algorithm for detecting a forbidden event). Here, we measure the efficiency with respect to n, the number of independent random variables. We will derive this from the constructive version of the Lovász Local Lemma proved by Moser and Tardos in [MT10], which we state below.

Theorem 3.25 *([MT10]) Let $\mathcal{P}$ be a finite set of mutually independent random variables in a probability space. Let $\mathcal{A}$ be a finite set of events determined by these variables. For each event $A \in \mathcal{A}$, let $\Gamma_{\mathcal{A}}[A]$ denote the set of events in $\mathcal{A}$ such that A is mutuallly independent of all events in $\mathcal{A} \setminus (\Gamma_{\mathcal{A}}[A] \cup \{A\})$. If there exists an assignmment of real values $x : \mathcal{A} \rightarrow (0, 1)$ such that*

$$\forall A \in \mathcal{A} : Pr[A] \leq x(A) \prod_{B \in \Gamma_{\mathcal{A}}[A]} (1 - x(B)),$$

then there exists an assignment of values to the variable $\mathcal{P}$ so that none of the events in $\mathcal{A}$ holds. Moreover, there is a randomized algorithm that resamples an event

$A \in \mathcal{A}$ at most an expected $x(A)/(1-x(A))$ times before it finds such an evaluation. Thus, the total expected number of resampling steps is at most $\sum_{A\in\mathcal{A}} \frac{x(A)}{1-x(A)}$.

We now state the special form of the Lovász Local Lemma.

Lemma 3.26 *(Special case of Lovász Local Lemma) Consider a finite collection $\mathcal{A}$ of events determined by n inpedendent random variables. Suppose that the events can be partitioned into types $1, 2, \ldots, k$ such that the following hold:*

(i) For any $i \in \{1, 2, \ldots, k\}$, each event of type i is determined by exactly a_i random variables and occurs with probability at most p_i.

(ii) Every random variable influences at most $b_i \geq 1$ events of type i, for every $i \in \{1, 2, \ldots, k\}$.

Suppose that **(A) :** $\sum_i 2^{(a_i+1)} b_i p_i \leq 1$ *holds. Then,*

$$Pr_{A\in\mathcal{A}}(\cap(\overline{A})) > 0$$

i.e. with positive probability none of the events holds. In particular, if the number of different types of events is k and $k2^{a_i+1} b_i p_i \leq 1$ for each $i \in [k]$, then with positive probability, none of the events in $\mathcal{A}$ hold.

Further, suppose that there is a polynomial (in n) time algorithm which, given an assignment for the random variables, determines if any event in $\mathcal{A}$ occurs and finds one such event. Then, there is a randomized algorithm, whose expected running time is polynomial in n, for finding an assignment of values to the random variables such that no forbidden event occurs.

We now derive the proof of the above lemma from Theorem 3.25.

Proof of Lemma 3.26: Let k be the number of types of events. From assumption **(A)**, it follows that $2^{a_i} p_i \leq 1/2$ for each $i \in [k]$. Now, for each $i \in [1, k]$ and each event of type i, we choose the same common value of $x_i = c_i p_i$ where $c_i = 2^{a_i}$. It now suffices to show that

$$p_i \;\leq\; c_i p_i \prod_{j\in[k]} (1 - c_j p_j)^{a_i b_j}, \;\; \text{for each} \;\; i \in [1, k] \;\ldots\; (I)$$

Using the well-known fact $(1-\frac{1}{z})^z \geq 1/4$ for each real $z \geq 2$, we see that (I) follows if

$$1 \;\leq\; c_i 4^{-a_i \sum_j c_j b_j p_j}$$

which is true if

$$a_i \sum_j c_j b_j p_j \;\leq\; \log_4 c_i = \frac{a_i}{2} \Leftrightarrow \sum_j 2 c_j b_j p_j \;\leq\; 1.$$

The last inequality is true by our assumption **(A)**.

For each event A of type i, since $x_A \leq 1/2$, we have $x_A/(1-x_A) \leq 2^{a_i+1}p_i$. Also, each random variable influences at most b_i events of type i, so that the number of events of type i is at most nb_i. Thus, $\sum_A \frac{x_A}{1-x_A} \leq \sum_i n2^{a_i+1}b_i p_i$, and the latter sum is at most n, by assumption **(A)** of the lemma. Hence, from Theorem 3.25, it follows that there is a randomized algorithm with polynomial expected time for finding such an assignment to the random variables. This completes the proof of Lemma 3.26. ∎

We now state our result on hypergraph colorings, the proof of which fits naturally into the framework of Lemma 3.26.

Theorem 3.27 *Let U be a finite universe of elements. Let $\mathcal{F}_1, \mathcal{F}_2, ...\mathcal{F}_t$ be families of subsets of U such that for each $i \in \{1, \ldots, t\}$, the family $\mathcal{F}_i$ is a_i-uniform, that is, consists of sets of size a_i. Let $a_i \geq 2$ for each i and let k_i, $i \in \{1, \ldots, t\}$ be positive integers such that $k_i \leq a_i$. Suppose that each element in U appears in at most b_i sets in $\mathcal{F}_i$.*

Let $S = max_i\{k_i(8^{a_i}tb_i)^{\frac{1}{a_i-k_i}}\}$.

Then U can be colored using S colors so that no set in $\mathcal{F}_i$ is contained in the union of any k_i color classes.

Proof of Theorem 3.27 Each element in U is assigned one of the S colors independently and uniformly at random. Let p_i be the probability that a given set in the ith family is contained in the union of some k_i color classes.
Clearly $p_i \leq \binom{S}{k_i}(k_i/S)^{a_i}$.

Applying Lemma 3.26, we see that if the inequality

$$\sum_i 2\binom{S}{k_i}(2k_i/S)^{a_i}b_i \leq 1$$

holds, then with positive probability none of the sets in $\mathcal{F}_i$ is contained in the union of any k_i color classes for each i.

In particular, if each term of the summand is at most $1/t$, the inequality holds. Using this and the fact that $\binom{S}{k_i}$ is at most $(\frac{eS}{k_i})^{k_i}$, we see that if $S = max_i\{k_i[2tb_i(2e)^{a_i}]^{\frac{1}{a_i-k_i}}\}$, then the inequality is satisfied. Since we have $a_i \geq 2$, we have $2(2e)^{a_i} < 8^{a_i}$. This proves the theorem. ∎

We note that the bounds of Theorem 3.4 can also be obtained as a consequence of Theorem 3.27 by choosing $U = V(G)$, $\mathcal{F}_2 = E(G)$, $\mathcal{F}_2 = \{S \subset V(G) : |S| = k+1, G[S] \text{ is connected}\}$, $k_1 = 1$, $k_2 = j$, $a_1 = 2$, $b_1 = \Delta$, $a_2 = k+1$, and $b_2 = (k+1)4^{k+1}\Delta^k$. Further, there is a randomized (expected) polynomial time algorithm to obtain such a coloring. For example, one can obtain efficiently a star coloring of a graph of maximum degree Δ using at most $O(\Delta^{3/2})$ colors.

3.6 Conclusions and Open Problems

We proved an upper bound of $O(\Delta^{\frac{k}{k+1-j}})$ for (j,k)-coloring of graphs of maximum degree Δ and used this to obtain upper bounds for forbidden subgraph colorings and as a special case, for low treewidth colorings. But in these colorings, forbidding all connected graphs on $k+1$ vertices is often a stronger requirement than what is expected and does not make use of the structure of the individual members of the forbidden family and so there is scope for further improving the upper bounds on the corresponding chromatic numnbers for several *specific* families of forbidden graphs.

In the next chapter, we will provide lower bounds on the maximum value (for a given Δ) of the respective chromatic numbers for the case $j = 2$ and obtain improved upper bounds that are nearly tight.

The algorithmic aspects of forbidden subgraph colorings are wide open. While we saw that our bounds can be constructivized by the algorithm of Moser and Tardos, there are many unanswered questions. For instance, the decision versions of several special cases of these colorings, such as acyclic and star coloring, are known to be NP-complete, but it is not known if the NP-completeness holds uniformly for the decision version of every $(j, \mathcal{F})$ pair, though we can expect the answer to be yes.

Assuming that these problems are computationally hard, an interesting question is that of approximating the chromatic numbers associated with them. In the case of proper coloring, it is known that the chromatic number is unlikely to be approximated within a multiplicative factor of $n^{1-\epsilon}$ for any $\epsilon > 0$ (see [FK98]). However, given the promise that a graph is 3-chromatic, there are algorithms (see [KMS94]) which can find a n^{α}-coloring for some fixed α in polynomial time. It would be interesting to obtain similar or stronger results when we are given a graph which is promised to have a small forbidden subgraph chromatic number.

4

Tight bounds on (2,$\mathcal{F}$)-subgraph colorings

4.1 Introduction

In this chapter, we focus on proper colorings with constraints on the unions of two color classes. In this case, we are able to obtain nearly tight upper bounds on $\chi_{2,\mathcal{F}}(\Delta)$.

In the previous chapter, we obtained the bound of $O\left(\Delta^{\frac{k-1}{k-j}}\right)$ on $\chi_{j,\mathcal{F}}(\Delta)$, where $k = \min_{H \in \mathcal{F}} |V(H)|$. For $j = 2$, this yields $\chi_{2,\mathcal{F}}(\Delta) = O(\Delta^{1+\frac{1}{k-2}})$.

However, this bound is not asymptotically optimal: for example, in the case of acyclic coloring, we have $j = 2$, $\mathcal{F} = \{C_4, C_6, \ldots\}$ and $k = 4$ and hence we get a bound of $O(\Delta^{3/2})$ but as mentioned earlier, a bound of $O(\Delta^{4/3})$ was proved in [AMR91]. We will generalize the ideas in [AMR91] to obtain nearly tight bounds for $\chi_{2,\mathcal{F}}(\Delta)$ for any arbitrary family $\mathcal{F}$.

Before presenting the improved upper bound, we first obtain a lower bound on $\chi_{2,\mathcal{F}}(\Delta)$.

4.2 Lower bound

The following lower bound is a generalization of a lower bound on the maximum value of acyclic chromatic numbers that was proved in [AMR91].

Theorem 4.1 *Given a connected bipartite graph H with m edges, for every sufficiently large Δ, there exist graphs G of maximum degree at most Δ such that*

$\chi_{2,\{H\}}(G) \geq C\frac{\Delta^{1+\frac{1}{m-1}}}{(\log\Delta)^{1/(m-1)}}$ *for some positive constant* C. *Hence, for any family* $\mathcal{F}$ *of connected bipartite graphs, we have* $\chi_{2,\mathcal{F}}(\Delta) = \Omega\left(\frac{\Delta^{1+\frac{1}{m-1}}}{(\log\Delta)^{1/(m-1)}}\right)$, *where* m *is the minimum number of edges in any member of* $\mathcal{F}$.

Proof of Theorem 4.1

The proof is based on analyzing a random graph $G(n,p)$ for a suitably chosen value of p and is a generalization of the proof arguments used by Alon, McDiarmid and Reed [AMR91] for acyclic colorings.

Let $V = \{1, 2, ..., n\}$ be a set of n labelled vertices.

Choose $p = c(\frac{\log n}{n})^{\frac{1}{m}}$, where $c > 0$ is a constant, independent of n, to be chosen later, and let $G = G_{n,p} = (V, E)$ be a random graph on V obtained by choosing each pair of distinct members of V independently to be an edge with probability p. Let Δ be the maximum degree of G. Recall from Chapter 2 that

$$\mathbf{Pr}\left(\mu/2 \leq \Delta \leq 2\mu\right) \to 1 \text{ as } n \to \infty \tag{4.1}$$

where $\mu = (n-1)p = cn^{1-\frac{1}{m}}(\log n)^{\frac{1}{m}}$.

Let H be the bipartite graph in Theorem 4.1 and $V(H) = X \sqcup Y$ be a bipartition into independent sets X and Y such that $r = max\{|X|, |Y|\}$.

We first claim that for any fixed partition of $V = V(G)$ into $s \leq n/r$ disjoint parts, the probability that this partition is a $(2, \{H\})$-coloring of G is at most $(1-p^m)^{\binom{n/r^2}{2}}$.

Let $V_1, ..., V_s$ be the parts of the partition. For each V_i, remove at most $r-1$ smallest (with respect to some fixed linear ordering of V) vertices to obtain a V_i' such that $|V_i'| \equiv 0 \pmod r$. The number of removed vertices is at most $s(r-1) \leq n(r-1)/r$ so that the graph induced by the union of the V_i's has at least n/r vertices. Now partition each V_i' into subsets of size r so that we get at least $\lceil n/r \rceil$ vertices partitioned into subsets $U_1, U_2, ..., U_k$ of cardinality r each, where $k \geq n/r^2$. For each i, j such that $1 \leq i < j \leq k$, the probability that $U_i \bigcup U_j$ does not contain a copy of H is at most $1-p^m$. Since all these $\binom{k}{2}$ events are mutually independent, the probability that the union of any 2 color classes does not contain a copy of H is at most $(1-p^m)^{\binom{n/r^2}{2}}$ and this probability is an upper bound on the required probability thereby proving the claim in the preceding paragraph.

The total number of partitions of V is at most n^n. Hence the probablity that there exists a partition $V = V_1 \cup \ldots \cup V_s$ ($s \leq n/r$) which forms a $(2, \{H\})$-subgraph coloring is at most

$$n^n(1-p^m)^{\binom{n/r^2}{2}} < exp\left(n\log n - \binom{n/r^2}{2}p^m\right)$$

Since $p = c(\log n/n)^{\frac{1}{m}}$, we choose c such that $c^m > 2r^4$, so that this probability is $o(1)$.

Therefore, $Pr[\chi_{2,\{H\}}(G) > n/r] \rightarrow 1$ as $n \rightarrow \infty$.

Combining this with (4.1), we see that there exist graphs G such that $\Delta = \Delta(G) \leq 2cn^{1-\frac{1}{m}}(\log n)^{\frac{1}{m}}$ and $\chi_{2,\{H\}}(G) > n/r$. Hence, $\chi_{2,\{H\}}(\Delta) = \Omega\left(\frac{\Delta^{1+\frac{1}{m-1}}}{(\log n)^{\frac{1}{m-1}}}\right) = \Omega\left(\frac{\Delta^{1+\frac{1}{m-1}}}{(\log \Delta)^{\frac{1}{m-1}}}\right)$ using $\log \Delta = \Omega(\log n)$.

This completes the proof of Theorem 4.1.

We mention that the above lower bound can be extended to bipartite graphs with a slight modification of the above argument by considering a random bipartite graph $G \in G(n, n, p)$ obtained by including each of the n^2 edges independently with probability p between two independent sets of size n each.

Applying Theorem 4.1 to $(2, k)$-colorings (see Definition 3.3) by choosing $\mathcal{F}$ to be the set of all connected graphs on $k+1$ vertices, we get the following result.

Corollary 4.2 $\chi^{con}_{2,k}(\Delta) = \Omega\left(\frac{\Delta^{\frac{k}{k-1}}}{(\log \Delta)^{1/(k-1)}}\right)$.

We see that when $j = 2$, Theorem 3.4 is tight up to polylogarithmic factors. Theorem 3.5 on the other hand is not tight uniformly for every family $\mathcal{F}$, even for the case $j = 2$. This is not surprising because the proof of Theorem 3.5 does not make use of the structure of the members of $\mathcal{F}$.

We will now use Theorem 4.1 to obtain lower bounds on $\chi^{tw}_{2,k}(\Delta)$. This requires us to present a characterization of treewidth due to Seymour and Thomas [ST93].

Definition 4.3 *Let $G = (V, E)$ be a graph. Two subsets $W_1, W_2 \subset V$ are said to* touch *if they have at least one vertex in common or if there is some edge $(w_1, w_2) \in E$ such that $w_1 \in W_1$, $w_2 \in W_2$. A set B of mutually touching connected vertex sets is called a* bramble. *A* hitting set *for B is a set which intersects every element of B. The* order *of a bramble B is the size of a minimum hitting set for B. The bramble number of G is the maximum order of all brambles of G.*

Theorem 4.4 *(Seymour and Thomas [ST93]) Let k be a non-negative integer. A graph has treewidth k if and only if it has bramble number $k + 1$.*

Corollary 4.5 *If G has a bramble of order k, $tw(G) \geq k - 1$.*

The lower bound of Theorem 4.1 yields the following lower bound on $\chi^{tw}_{2,k}(\Delta)$.

Theorem 4.6 *For any given $k \geq 2$, there is a positive constant $C = C(k)$ such that for all suffficiently large values of Δ, there exist graphs G of maximum degree at most Δ such that $\chi^{tw}_{2,k}(G) \geq C \frac{\Delta^{1+\frac{2}{k^2+5k}}}{(\log \Delta)^{\frac{2}{k^2+5k}}}$.*

Remark: Note that for $k = 2$, the above theorem implies that the upper bound in Part (ii) of Theorem 3.13 is tight up to polylogarithmic factors.

Proof of Theorem 4.6 Observe that any $(2, k)$-treewidth coloring is also a $(2, \{H\})$-subgraph coloring for any bipartite graph H of treewidth more than k. Hence, by Theorem 4.1, it suffices to prove that there exists a bipartite graph H having treewidth greater than k and having $(k^2 + 5k + 2)/2$ edges.

Consider the bipartite graph $H = (V, E)$ where

$$V = \{a_1, a_2, ..., a_{k+1}\} \cup \{b_1, b_2, ..., b_{k+1}\} \text{ and}$$

$$E = \{ (a_i, b_j) : 1 \le i \le j \le k+1 \} \bigcup \{ (a_i, b_1) : 2 \le i \le k+1 \}.$$

The number of edges in this graph is $\binom{k+1}{2} + 2k + 1 = (k^2 + 5k + 2)/2$.

Consider the following bramble B in H.

$$B = \{\{a_1\}, \{b_1\}\} \bigcup \{ \{a_i, b_i\} : 2 \le i \le k+1 \}.$$

It is clear that any hitting set of B has to have size at least $k+2$. Hence by Corollary 4.5, $tw(H) \ge k+1$. This completes the proof of Theorem 4.6.

In the following section, we present improved upper bounds on $(2, \mathcal{F})$-chromatic numbers.

4.3 Upper bound

We saw that the lower bound of Theorem 4.1 in the previous section and the upper bound of Theorem 3.5 in Chapter 3 need not match. We are thus motivated to find tighter upper bounds for the $(2, \mathcal{F})$-chromatic numbers. In particular, Theorem 4.1 makes us wonder if we can replace the exponent $\frac{k-1}{k-2}$ in Theorem 3.5 (for $j = 2$) by the value $\frac{m}{m-1}$, where $k = min_{H \in \mathcal{F}} |V(H)|$ by $m = min_{H \in \mathcal{F}} |E(H)|$. It turns out that this is indeed possible as the following result shows.

Theorem 4.7 *Let $\mathcal{F}$ be a family of connected bipartite graphs on 3 or more vertices such that the minimum number of edges in any member of $\mathcal{F}$ is m. Then, for any graph G of maximum degree Δ, $\chi_{2,\mathcal{F}}(G) \le C\Delta^{1+\frac{1}{m-1}}$ where $C = C(\mathcal{F}) = 64(m+1)^3 s$ and s is the number of bipartite graphs in $\mathcal{F}$ on at most m vertices.*

In view of Theorem 4.1, for every *fixed* family $\mathcal{F}$, the upper bound of Theorem 4.7 is tight within a multiplicative factor of $O((\log \Delta)^{1/(m-1)})$.

The key idea in the following proof is to *reduce the number of dependencies* of some of the bad events. This is done by adding some other bad events in the form of monochromatic special subsets. Special subsets are independent subsets of vertices that have a 'large' number of common neighbors. These are defined in such a way that the number of dependencies involving them is not too large, but avoiding them enables us to reduce the number of dependencies involving the original bad events. This in turn helps us to reduce the bound on the number of colors used. For an illustration, recall the notion of special pairs introduced in tne proof of Theorem 3.13. The proof will make this idea clear and we present it now.

Proof of Theorem 4.7:

Choose $x = \lceil C\Delta^{1+\beta} \rceil$ where $\beta = \frac{1}{m-1}$ and $C = C(\mathcal{F}) = 64(m+1)^3 s$.

Let $f : V \rightarrow \{1, 2, ..., x\}$ be a random vertex coloring of G, where for each vertex $v \in V$ independently, the color $f(v) \in \{1, 2, ..., x\}$ is chosen independently and uniformly at random. It suffices to prove that with positive probability, f is a $(2, \mathcal{F})$-coloring of G. To this end, we define a family of bad events whose total failure implies a $(2, \mathcal{F})$-coloring and use the Lovasz Local Lemma (as stated in Lemma 3.26) to show that with positive probability none of them occur. The events we consider are of the following types.

a) **Type 1**: For each pair of adjacent vertices u and v, let $A_{u,v}$ be the event that $f(u) = f(v)$.

To reduce the number of copies of forbidden subgraphs we need to consider, we define a notion which helps us generalize the "special pair" technique employed in [AMR91]. An independent subset of k vertices is called a *special k-set* if there are more than $\Delta^{1-(k-1)\beta}$ vertices adjacent to each of the k vertices.

We say that an independent subset S of the vertices is *good* if for every vertex $v \in S$ and for any $k \in [2, m]$, the set of neighbors of v does not contain any special k-set as a subset.

For each $k \in [2, m]$, we define the following events:

b) **Types 2,k**: For each special set S of k vertices, let $B_k(S)$ be the event that the vertices of S are colored with one common color by f.

c) **Type 3**: For each connected bipartite induced subgraph L of $V(G)$ such that $|V(L)| = m + 1$, let C_L be the event that the vertices in L are properly colored using at most 2 colors in the coloring by f.

Let the bipartite members of $\mathcal{F}$ of size at most m be H_1, H_2,...,H_s where $s = s(\mathcal{F})$ is the number of such members. For each $i \in [1, s]$, we define the following Type $4, i$ events:

d) **Type 4,i**: For each good subset S of vertices of G such that $G[S]$ is bipartite and contains H_i as a spanning subgraph, let $D_i(S)$ be the event that the random coloring f uses at most 2 colors on the vertices of S.

If we forbid all events of Types 1 and $(2, k)$, then for any $S \subseteq V$ such that (i) $G[S]$ contains some H_i as a spanning subgraph and (ii) S is not a good set, there should be some $v \in S$ and some $k \in [2, m]$ such that $N_S(v)$ contains a special k-set which is not monochromatically colored (since events of Type 2,k are forbidden) and hence f uses at least 3 colors on S.

Thus, it follows that if none of the events of the above types occur, then f is a $(2, \mathcal{F})$-coloring. We first estimate upper bounds on the probabilities of each type of events.

(i) For each Type 1 event A, $p_1 = Pr(A) = \frac{1}{x}$.
(ii) For each Type $(2, k)$ event B_k, $p_{2,k} = Pr(B_k) = \frac{1}{x^{k-1}}$.
(iii) For each Type 3 event C, $p_3 = Pr(C) \leq \frac{1}{x^{m-1}}$.
(iv) For each Type $(4, i)$ event D_i, $p_{4,i} = Pr(D_i) \leq \frac{2^{n_i}}{x^{n_i-2}}$.

Note that any of the events defined above is mutually independent of all events that do not share a vertex in common with the given event. Thus, it suffices to estimate the number of events of each type containing a given vertex. This estimate is given in the following simple lemma.

Claim 1 *Let v be an arbitrary vertex of the graph $G = (V, E)$. Then the following statements hold.*

(i) v belongs to at most Δ edges of G.

(ii) For each $k \in [2, m]$, the number of special k-sets containing v is at most $\Delta^{(k-1)(1+\beta)}$.

(iii) v belongs to at most $(m + 1)4^{m+1}\Delta^m$ connected induced subgraphs of size $m + 1$ in $V(G)$.

(iv) For each $i \in [1, s]$, v belongs to at most $n_i\Delta^{(n_i-2)(1+\beta)}$ subgraphs isomorphic to H_i where $n_i = |V(H_i)|$ and such that the vertex set of the subgraph is good.

Proof of Claim 1

Part (i) follows from the definition of Δ as the maximum degree in G.

Part (ii) follows from the fact that there are at most Δ^k induced stars of size $k+1$ in G, with v as a leaf, and for each special k-set there are more than $\Delta^{1-(k-1)\beta}$ centers of the $k+1$-star. Thus the number of special k-sets containing v is at most $\frac{\Delta^k}{\Delta^{1-(k-1)\beta}} = \Delta^{(k-1)(1+\beta)}$.

Part (iii) has already been established as part of the proof of Proposition 3.7 in Chapter 3.

Part (iv) can be seen as follows: The position of v in H_i has at most n_i choices. Once v is identified with a vertex of H_i, the number of ways of embedding the remaining vertices can be bounded as follows: consider a sequence $v_2, ..., v_{n_i}$ of the remaining vertices of H_i such that each vertex has atleast one neighbour to its left in the sequence. Clearly this is possible since H_i is connected. Let t_l denote the number of vertices to the left of v_l and adjacent to it. Once the vertices to the left of v_l are embedded in G, the number of ways of identifying v_l in G is at most $\Delta^{1-(t_l-1)\beta}$ because there is no special t_l set among these vertices. Thus the number of ways of embedding the remaining vertices of H_i in G is at most $\Delta^{\sum_{l=2}^{n_i}[1-(t_l-1)\beta]}$. Using the fact that $\sum_{l=2}^{n_i} t_l = |E(H_i)| \geq m$ and $\beta = \frac{1}{m-1}$, we see that $\sum_{l=2}^{n_i}[1-(t_l-1)\beta] \leq (n_i-1)(1+\beta) - m\beta = (n_i-2)(1+\beta)$. This proves Part (iv) and completes the proof of Claim 1.

Since an event is independent of all other events with which it does not share a vertex, we see that the assumptions of Lemma 3.26 hold with the following values of a_is and b_is.

Type 1 : $a_1 = 2$, $b_1 = \Delta$.

Type 2,k : $a_{2,k} = k$, $b_{2,k} = \Delta^{(k-1)(1+\beta)}$ for each $k \in [2, m]$.

Type 3 : $a_3 = m+1$, $b_3 = (m+1)4^{m+1}\Delta^m$.

Type 4,i : $a_{4,i} = n_i$, $b_{4,i} = n_i\Delta^{(n_i-2)(1+\beta)}$ for each $i \in [1, s]$.

By Lemma 3.26, to prove that with positive probability none of the "bad" events hold, it suffices to verify the following inequality:

$$8\frac{\Delta}{x} + \sum_{k=2}^{m} 2^{(k+1)} \frac{\Delta^{(k-1)(1+\beta)}}{x^{k-1}} + 2(m+1)8^{m+1}\frac{\Delta^m}{x^{m-1}} + \sum_{i=1}^{s} 2n_i 4^{n_i} \frac{\Delta^{(n_i-2)(1+\beta)}}{x^{n_i-2}} \leq 1$$

We now substitute $x = C\Delta^{1+\frac{1}{m-1}}$ where $C = 64(m+1)^3 s$. Using the facts that $\beta = \frac{1}{m-1}$ and $n_i \leq m$ for $i \in [1, s]$, we see that it suffices to verify:

$$\frac{1}{8m^3 s} + \frac{1}{32ms} + \frac{2(m+1)8^{m+1}}{(4m+4)^{3m-3}s} + \frac{1}{4m^2} \leq 1$$

The above inequality can easily be seen to be true for any $m \geq 2$, $s \geq 1$.

Thus by Lemma 3.26, with positive probability, none of the bad events occurs and hence there exists a $(2, \mathcal{F})$-coloring using $O(\Delta^{1+\frac{1}{m-1}})$ colors. This completes the proof of Theorem 4.7.

4.4 Concluding Remarks

We obtained nearly tight upper bounds on $\chi_{2,\mathcal{F}}(\Delta)$. However, narrowing the poly-log factor gap is an interesting and challenging problem that is still open, even for acyclic vertex coloring. Another unresolved question is whether the upper bound of $O(\Delta^{\frac{k-1}{k-j}})$ for $\chi_{j,\mathcal{F}}(\Delta)$ ($k = min_{H\in\mathcal{F}}|H|$) is tight for $j \geq 3$. The lower bound technique used for $j = 2$ does not seem to work for $j \geq 3$ and it would be interesting to prove such bounds.

5

Oriented coloring

5.1 Introduction

The concept of oriented coloring was introduced by Bruno Courcelle in [Cou94]. Since then, many researchers have worked on the problem, partly because of its applications in task assignment problems [CD06].

Sopena, in ([Sop97]), studied the notion of oriented chromatic number for oriented graphs. Recall that the oriented chromatic number of an oriented graph $\vec{G}$ is the minimum number of vertices of an oriented graph $\vec{H}$ such that there is a homomorphism from $\vec{G}$ to $\vec{H}$. The oriented chromatic number of $\vec{G}$ is denoted by $\chi_o(\vec{G})$ and the oriented chromatic number of an undirected graph G, denoted by $\chi_o(G)$ is the maximum oriented chromatic number of $\vec{G}$ taken over all orientations $\vec{G}$ of G. Upper bounds for the oriented chromatic number have been obtained in terms of the maximum degree and also for special families of graphs such as trees, planar graphs, partial k-trees [Sop97], for triangle-free planar graphs [Och04], for 2-outerplanar graphs [EO07], for arbitrary graphs in terms of maximum degree [KSZ97], maximum average degree [BKN$^+$99] and in terms of treewidth [Sop97]. The following two results, in particular, are relevant to the main results of this chapter. They are :

(B1) The result of Sopena in [Sop97] that, for every $r \geq 1$, every partial r-tree has oriented chromatic number at most $(r+1)2^r$.

(B2) The result of Raspaud and Sopena in [RS94] that if a graph has acyclic chromatic number at most k, then $\chi_o(G) \leq k2^{k-1}$.

5.1.1 Our Results

We generalize the result (B2) by obtaining a relationship connecting the oriented chromatic number $\chi_o(G)$ of graphs and the $(j, \mathcal{F})$-subgraph chromatic numbers $\chi_{j,\mathcal{F}}(G)$ introduced and studied in Chapters 3 and 4. In particular, we relate the oriented chromatic number and the $(2, r)$-treewidth chromatic number and show that $\chi_o(G) \leq k\left((r+1)2^r\right)^{k-1}$ for any graph G having $(2, r)$-treewidth chromatic number at most k. We recall that the latter parameter is the least number of colors in any proper vertex coloring which is such that the subgraph induced by the union of any two color classes has treewidth at most r.

We also generalize a result of Alon, Mohar and Sanders [AMS96] on the acyclic chromatic number of graphs on surfaces to $(2, \mathcal{F})$-subgraph chromatic numbers. For certain families $\mathcal{F}$, we prove that $\chi_{2,\mathcal{F}}(G) = O(\gamma^{m/(2m-1)})$ for any graph G of Euler characteristic $-\gamma$, where $\gamma \geq 0$. Here, $m = min\{|E(H)| : H \in \mathcal{F}\}$. We also show that this bound is nearly tight. We then use this result to show that graphs of genus g have oriented chromatic number at most $2^{O(g^{1/2+\epsilon})}$ for every fixed $\epsilon > 0$. This improves the currently best known bound of $2^{O(g^{4/7})}$ which follows from the result of [AMR91] (see subsection 5.1.4). We also refine the proof of a bound on $\chi_o(G)$ (in terms of maximum degree) obtained by Kostochka, Sopena and Zhu in [KSZ97] to obtain an improved bound on $\chi_o(G)$. In the following subsections of this section, we present the formal statements (without proofs) of the main results of this chapter.

5.1.2 Relating $\chi_{j,\mathcal{F}}(G)$ and $\chi_o(G)$

In this subsection, we state the following connection between $(j, \mathcal{F})$-subgraph colorings and oriented colorings. This result generalizes and was inspired by the connection between $a(G)$ and $\chi_o(G)$ established in [RS94]. Recall that for a family $\mathcal{F}$ of connected graphs, $Forb(\mathcal{F}) = \{G : G \text{ is } \mathcal{F} - \text{free}\}$.

Theorem 5.1 *Let $\mathcal{F}$ be a family of connected graphs. Suppose there exists a natural number t such that $\chi_o(F) \leq t$, for each $F \in Forb(\mathcal{F})$. Suppose $j \geq 2$. Then, for any graph $G \notin Forb(\mathcal{F})$ with $\chi_{j,\mathcal{F}}(G) \leq k$, its oriented chromatic number $\chi_o(G)$ is at most $kt^{\lceil \frac{2k-j}{j} \rceil}$ if j is even and is at most $kt^{\lceil \frac{2k-j+1}{j-1} \rceil}$ if j is odd.*

In Section 5.2, we prove this theorem. By specializing to $j = 2$, we get the following theorem. This specialization is stated separately again since it plays an important role in other results of this chapter.

Theorem 5.2 *Let $\mathcal{F}$ be a family of connected bipartite graphs. Suppose there exists a t such that $\chi_o(F) \leq t$, for each $F \in Forb(\mathcal{F})$. Then, for any graph $G \notin Forb(\mathcal{F})$ with $\chi_{2,\mathcal{F}}(G) \leq k$, its oriented chromatic number $\chi_o(G)$ is at most kt^{k-1}.*

We now specialize Theorem 5.2 by choosing $\mathcal{F}$ to be the set of all connected bipartite graphs of treewidth $r + 1$ and apply the bound **(B1)** (mentioned before) on the oriented chromatic number of partial r-trees to obtain the following result as a consequence.

Corollary 5.3 *For $r \geq 1$, let G be any graph with a $(2, r)$-treewidth chromatic number at most k. Then G has oriented chromatic number at most $k((r + 1)2^r)^{k-1}$.*

5.1.3 $(2, \mathcal{F})$-subgraph colorings of graphs on surfaces

It is known from the Map Color Theorem of Ringel and Youngs [RY68] that the chromatic number of an arbitrary surface of Euler characteristic $-\gamma$ is $\Theta(\gamma^{1/2})$. Using the upper bound of $O(\Delta^{4/3})$ bound on $a(\Delta)$, Alon, Mohar and Sanders proved in [AMS96] that the acyclic chromatic number of a (simple) graph embeddable on a surface of characteristic $-\gamma(\leq 0)$ is at most $100\gamma^{\frac{4}{7}} + 10^4$. It was also shown that this bound is nearly tight.

Generalizing these arguments and by using the bound of Theorem 5.1, we prove that this result can be extended to $(2, \mathcal{F})$-colorings as well provided that $\mathcal{F}$ does not contain connected graphs with pendant vertices. Our next main result in this chapter is this extension. Specifically, we prove (using essentially the arguments of [AMS96]) the following statement.

Theorem 5.4 *Let $\mathcal{F}$ be a family of connected bipartite graphs on at least 4 vertices each having minimum degree at least 2. Let m be the smallest number of edges of any member of $\mathcal{F}$. If G is a (simple) graph embeddable on a surface of Euler characteristic $-\gamma \leq 0$, then $\chi_{2,\mathcal{F}}(G) \leq A\gamma^{\frac{m}{2m-1}} + B$ where A and B are constants depending only on $\mathcal{F}$.*

When $\mathcal{F} = \{C_4, C_6, \ldots\}$ corresponding to the acyclic chromatic number, we have $m = 4$ and $m/(2m-1) = 4/7$ and the result is consistent with the bound of [AMS96]. By choosing $\mathcal{F} = \mathcal{F}_r$ where $\mathcal{F}_r$ is the set of all minimal connected bipartite graphs of treewidth $r+1$, we get the following consequence of Theorem 5.4.

Corollary 5.5 *If G is a simple graph embeddable on a surface of Euler characteristic $-\gamma \leq 0$, then, $\chi_{2,r}^{tw}(G) \leq A\gamma^{\frac{m_r}{2m_r-1}} + B$ for every $r \geq 1$. Here, A and B are suitable absolute positive constants and m_r denotes the minimum number of edges in any member of $\mathcal{F}_r$.*

We also establish that the upper bound of Theorem 5.4 is tight upto a *polylog*(γ) multiplicative factor. This generalizes a similar tightness result presented in [AMS96] for acyclic chromatic numbers.

Theorem 5.6 *Let $\mathcal{F}$ and m be as described in Theorem 5.4. For every sufficiently large $\gamma \geq 0$, there is a graph G embeddable on a surface (orientable or non-orientable) with Euler characteristic $-\gamma$ such that $\chi_{2,\mathcal{F}}(G) \geq c\gamma^{\frac{m}{2m-1}}/(\log\gamma)^{1/(2m-1)}$ for some positive constant c which depends only on $\mathcal{F}$.*

5.1.4 Oriented chromatic numbers of graphs on surfaces

For graphs of Euler characteristic $-\gamma \leq 0$, by combining the upper bound of $O(\gamma^{4/7})$ on oriented chromatic number (obtained in [AMS96]) with the bound **(B2)** of [RS94] (mentioned before), we get an upper bound of $O(\gamma^{4/7}2^{O(\gamma^{4/7})}) = 2^{O(\gamma^{4/7})}$ for the oriented chromatic number $\chi_o(G)$. The next main result of this chapter is an improvement of this bound and is obtained by combining Corollary 5.3 and Corollary 5.5. Recall that Corollary 5.3 is a generalization of bound **(B2)** and Corollary 5.5 is a generalization of the bound obtained in [AMS96].

Theorem 5.7 *Let $r \geq 0$ be any fixed integer. There exists a positive constant c_r and a positive integer m_r, both depending only on r, such that the following holds: For any simple graph G embeddable on a surface of Euler characteristic $-\gamma \leq 0$,*

$$\chi_o(G) \leq c_r(\gamma^{\frac{m_r}{2m_r-1}})((r+1)2^r)^{O(\gamma^{\frac{m_r}{2m_r-1}})} \leq 2^{O(\gamma^{m_r/(2m_r-1)})}$$

. Here, $m_r = min\{E(H) : tw(H) > r\}$. It can be seen that $m_r \geq r+1$, so that $m_r \to \infty$. Thus for every $\epsilon > 0$, there exists c_ϵ such that $\chi_o(G) \leq 2^{c_\epsilon \gamma^{(1/2)+\epsilon}}$.

Proof Follows as a consequence of combining Corollary 5.3 and Corollary 5.5 with the bound **(B1)** (mentioned earlier). ∎

Note that this significantly improves the bound $2^{O(\gamma^{4/7})}$ mentioned before.

5.1.5 An improved bound on the oriented chromatic number

In [KSZ97], Kostochka, Sopena and Zhu showed that the oriented chromatic number of any graph G of maximum degree Δ is at most $2\Delta^2 2^\Delta$. They prove this result by showing (with the help of probabilistic arguments) the existence of a tournament on $t = 2\Delta^2 2^\Delta$ vertices possessing a nice property which enables one to obtain an oriented coloring of any orientation of G with t colors.

We show that this proof can in fact be refined so that we obtain the following improvement of this result.

Theorem 5.8 *If G is any graph of maximum degree Δ and degeneracy d, then its oriented chromatic number $\chi_o(G)$ is at most $16\Delta d 2^d$.*

This replaces a factor $\Delta 2^\Delta$ by $d2^d$ and will result in a better bound for those G having $d \ll \Delta$.

5.1.6 Outline of this chapter

We prove Theorem 5.1 in Section 5.2. Theorems 5.4 and 5.6 are proved in Section 5.3. In Section 5.4, we prove Theorem 5.8. Finally, in Section 5.5, we conclude with some remarks and open problems.

5.2 Relating $\chi_{j,\mathcal{F}}(G)$ and $\chi_o(G)$

We now prove Theorem 5.1 which relates oriented chromatic number and the forbidden subgraph colorings.

Proof of Theorem 5.1 Let $G = (V, E)$ be an undirected graph such that $G \notin Forb(\mathcal{F})$ and let $\vec{G} = (V, A)$ be an arbitrary orientation of $E(G)$. Since $G \notin Forb(\mathcal{F})$, we have $k \geq \chi_{j,\mathcal{F}}(G) \geq j + 1$. Let $V_1, ..., V_k$ be the color classes of V with respect to a $(j, \mathcal{F})$-subgraph coloring c of $V(G)$ using k colors. Let $\mathcal{T}$ be the collection of subsets obtained by partitioning $[1, k]$ into at most $\lceil \frac{k}{\lfloor j/2 \rfloor} \rceil$ subsets of size at most $\lfloor j/2 \rfloor$ each. Note that $|\mathcal{T}|$ is at most $\lceil \frac{2k}{j} \rceil$ if j is even and is at most $\lceil \frac{2k}{j-1} \rceil$ if j is odd. Let $\mathcal{S}$ be the collection defined by

$$\mathcal{S} = \{T \cup T' : T, T' \in \mathcal{T},\ T \neq T'\}.$$

It follows that

(i) Each $S \in \mathcal{S}$ is a set of size at most j.

(ii) for every $l, m \in [1, k]$, there exists a $S \in \mathcal{S}$ with $l, m \in S$,

(iii) for each $i \in [k]$, i is a member of at most $\lceil \frac{k}{\lfloor j/2 \rfloor} \rceil - 1$ sets in $\mathcal{S}$. Let $\mathcal{S}_i$ be defined by $\mathcal{S}_i = \{S \in \mathcal{S} : i \in S\}$.

For each $S \in \mathcal{S}$, let $\vec{G}_S$ denote the induced subgraph $\vec{G}[\cup_{i \in S} V_i]$. Clearly $G_S \in Forb(\mathcal{F})$, since $(V_1, \ldots, V_k)$ is a $(j, \mathcal{F})$-subgraph coloring.

Let c_S be an oriented coloring of $\vec{G}_S$ using at most t colors.

Assume an ordering $\{S_1, S_2, \ldots\}$ on the members of $\mathcal{S}$. We now define a new coloring ϕ of $V(G)$: Fix any i and let $\mathcal{S}_i = \{S_{i_1}, \ldots, S_{i_l}\}$ be the members of $\mathcal{S}_i$ where we have $l \leq \lceil \frac{k}{\lfloor j/2 \rfloor} \rceil - 1$. For each $v \in V_i$,

$$\phi(v) = \{c(v), (c_{S_{i_1}}(v), S_i), ..., (c_{S_{i_l}}(v), S_l)\}.$$

Clearly, ϕ is a proper coloring of $V(\vec{G})$ because of the component c. We now prove that it is an oriented coloring. If it is not an oriented coloring, then there are four vertices x, y, z, t of $\vec{G}$ such that $(x, y) \in A$ and $(z, t) \in A$ with $\phi(x) = \phi(t)$ and $\phi(y) = \phi(z)$. By the definition of ϕ, x and t (respectively y and z) belong to the same V_i (respectively V_j) where $i = c(x) = c(t)$ and $j = c(y) = c(z)$. Let S be any set in $\mathcal{S}$ containing i and j where $S \in \mathcal{S}_i \cap \mathcal{S}_j$ and $x, y, z, t \in V(\vec{G}_s)$. For each $u \in \{x, y, z, t\}$, the pair $(c_S(u), S) \in \phi(u)$. By the definition of ϕ, we have $c_S(x) = c_S(t)$ and $c_S(y) = c_S(z)$. But this contradicts the fact that c_S is an oriented coloring of $\vec{G}_S$.

The number of possible values of $\phi(v)$ is at most $kt^{\lceil \frac{k}{\lfloor j/2 \rfloor} \rceil - 1}$. This number is $kt^{\lceil \frac{2k-j}{j} \rceil}$ if j is even and is $kt^{\lceil \frac{2k-j+1}{j-1} \rceil}$ if j is odd. This proves Theorem 5.1.

5.3 $(2, \mathcal{F})$-subgraph colorings of graphs on surfaces

By applying the bound of Theorem 4.7 which holds for general graphs, we obtain a bound on $\chi_{2,\mathcal{F}}(G)$ for graphs embeddable on surfaces, provided the members of $\mathcal{F}$ have minimum degree at least 2. This bound was stated in Theorem 5.4 and is proved in this section.

The proof is essentially the proof of [AMS96] extended to a more general setting. Hence, we do not provide the complete proof but only provide the sketch to give an idea of the proof.

5.3.1 Proof of Theorem 5.4

We follow the proof of [AMS96]. Assume the theorem is false for a surface S with Euler characteristic $-\gamma \leq 0$, and let G be a graph embeddable on it, with a minimum number of vertices, which is a minimal counterexample to the theorem. Let H be G with (possibly multiple) edges added to triangulate S. Clearly $deg_G(v) \leq deg_H(v)$ for all vertices v of G. Suppose $V(G) = V(H) = \{v_1, ..., v_n\}$, where $deg_H(v_1) \leq deg_H(v_2) \leq \ldots \leq deg_H(v_n)$. If $\gamma = 0$, define $h_1 = 0$ and $h_2 = 0$. Otherwise, define $h_1 := \lceil c\gamma^{\frac{m}{2m-1}} \rceil$ and $h_2 := \lfloor 6\gamma/h_1 \rfloor$ $(\leq 6\gamma^{\frac{m-1}{2m-1}}/c)$, where c is an absolute constant, to be chosen later. Let $d := deg(v_{n-h_1})$. The proof will split on the size of d.

Case I: $d \leq (4/3)h_2 + 9$. In this case, the induced subgraph of G on $\{v_1, ..., v_n\}$ has maximum degree at most d, and thus has a $(2, \mathcal{F})$-subgraph coloring using at most $\lceil Cd^{m/(m-1)} \rceil$ colors, by Theorem 5.2. Coloring the remaining vertices of G with h_1 new colors that have not been used before gives a $(2, \mathcal{F})$-subgraph coloring of G with at most

$$\lceil C((4/3)h_2 + 9)^{m/(m-1)} \rceil + h_1 \leq C(8\gamma^{(m-1)/(2m-1)}/c + 9)^{m/(m-1)} + 1 + c\gamma^{m/(2m-1)} + 1$$

colors. An appropriate choice of constant values (independent of γ) for A, B and c shows that this is smaller than $A\gamma^{m/(2m-1)} + B$, implying that in this case G cannot be a counterexample.

Case II: $d \geq (4/3)h_2 + (28/3)$. We charge each vertex as follows. Define $charge'(v_i) = 6 - deg_H(v_i)$ for $1 \leq i \leq n - h_1$, and $charge'(v_i) = -deg_H(v_i)/4$ for $n - h_1 + 1 \leq i \leq n$.

As shown in [AMS96],

$$\sum_{1 \leq i \leq n} charge'(v_i) = \left(\sum_{i \leq n - h_1} 6 - deg_H(v_i) + \sum_{i > n - h_1} -deg_H(v_i)/4 \right) > 0.$$

Following [AMS96], we define new charges $charge''(v)$ for each vertex by the following discharging rules. (i) Send a charge of 1/2 from each vertex of degree 4 to each of its neighbors of degree at least 8. (ii) Send a charge of 1/4 from each vertex of degree 5 to each of its neighbors of degree at least 7. The degrees are with respect to H. By conservation of total charges, we have $\sum_{i \leq n} charge''(v_i) > 0$. Hence for some j, we have $charge''(v_j) > 0$.

Using the definition of $charge''(v_j)$, we see that $deg_H(v_j) \neq 6$. Now consider the following cases :

Case 1: $deg_H(v_j) \leq 3$. Then, $deg_G(v_j) \leq 3$ and we delete v_j from G and join every pair of its neighbors by an edge (if it is not there) in the embedding of $G - v_j$. Since G is a counter example on minimum number of vertices, $G - v_j$ is $(2, \mathcal{F})$-colorable using the allowed number of colors where neighbors of v_j get different colors. Now we can extend this coloring by coloring v_j with any permissible color and it will continue to be a $(2, \mathcal{F})$-coloring of G contradicting our assumption.

Case 2: $deg_H(v_j) = 4$. In this case, v_j should have a neighbor v_k with $deg_H(v_j) \leq 7$. Let K be the graph obtained by removing v_j and making every pair of neighbors other than v_k adjacent. From a $(2, \mathcal{F})$-coloring of K, we can obtain a $(2, \mathcal{F})$-coloring of G by assigning v_j with any color not used on its neighbors or the neighbors of v_k. This contradicts our assumption.

Case 3: $deg_H(v_j) = 5$. Now $charge(v_j) = 1$, thusv_j must have two neighbors, say v_k and v_m of degree at most 6. Let K be G with v_j deleted, and edges added so that the neighbors ofv_j in G (except possibly v_k, v_m are pairwise adjacent. Give K a $(2, \mathcal{F})$-coloring by induction, this can be extended to G by coloring v_j with a color different form each of its neighbors as well as the neighbors of v_k and v_m.

As shown in [AMS96], the other cases reduce to the three previous ones. This completes the proof.

Remark: For any graph G, $\chi^{tw}_{2,r}(G) = \chi(G)$ when $r = tw(G)$. When r becomes large, the bound of Corollary 5.5 approaches the Heawood bound of $O(g^{1/2})$ for the chromatic number of genus g (fixed g) graphs. Hence, the upper bound of Corollary 5.5 approximates the Heawood bound more closely in the case of graphs of large treewidth.

5.3.2 Proof of Theorem 5.6

The proof is based on an approach similar to the one used in [AMS96]. It uses the following lemma whose proof follows from the proof of Theorem 4.1 presented in Chapter 4 of this thesis. The proof is based on analyzing a random graph $G(n, p)$ for a suitably chosen p.

Lemma 5.9 *Let $\mathcal{F}$ and m be as described in Theorem 5.6. Let $G = G(n, p)$ be the random graph on $\{1, \dots, n\}$ where each potential edge is chosen independently with probability $p = c\left(\frac{\log n}{n}\right)^{1/m}$ for a suitable positive constant c which depends only on $\mathcal{F}$. Then, almost surely, G is connected and has at most $cn^{(2m-1)/m}(\log n)^{1/m}$ edges and satisfies $\chi_{2,\mathcal{F}}(G) = \Omega(n)$.*

Let G be a connected graph on at most $O(n^{(2m-1)/m}(\log n)^{1/m})$ edges and satisfying $\chi_{2,\mathcal{F}}(G) = \Omega(n)$ as guaranteed by Lemma 5.9. Let G be embedded on a surface of characteristic $-\gamma$ for the smallest $\gamma \geq 0$ possible. Let $e = |E(G)|$. By an application of Euler's formula, one can show (as shown in [AMS96]) that $\gamma > n^{(2m-1)/m}$, and hence $\log \gamma > (2m-1)(\log n)/m$ and also that $\gamma = O\left(n^{(2m-1)/m}(\log \gamma)^{1/m}\right)$. Hence, $\chi_{2,\mathcal{F}}(G) = \Omega(n) = \Omega\left(\gamma^{m/(2m-1)}/(\log \gamma)^{1/(2m-1)}\right)$.

5.4 Proof of Theorem 5.8

As in [KSZ97], we prove (using probabilistic arguments) the following lemma. Before that, we recall the following notation from [KSZ97]. For an oriented graph $G = (V, A)$ and a subset $I = \{x_1, \ldots, x_i\}$ of V and a vertex $v \in V \setminus I$ such that v is adjacent to each x_j, we use $F(I, v, G)$ to denote the vector $a = (a_1, \ldots, a_i)$ where, for each $j \leq i$, $a_j = 1$ if $(x_j, v) \in A$ and $a_j = -1$ if $(v, x_j) \in A$.

Lemma 5.10 *Let d, k be positive integers with $d \leq k$ and $k \geq 5$. There exists a tournament $T = (V, A)$ on $t = 16kd2^d$ vertices with the following property :*

For each $i, 0 \leq i \leq d$, for each $I \subseteq V$, $|I| = i$, and for each $a \in \{1, -1\}^i$, there exist at least $kd + 1$ vertices $v \in V \setminus I$ with $F(I, v, T) = a$.

Proof of Lemma 5.10 : Consider a random tournament $T = (V, A)$ on t vertices obtained by randomly and independently orienting each edge of K_t (complete undirected graph on t vertices) in one of the two directions with equal probability.

Fix an $i \leq d$ and fix any $I \subseteq V$ of size i. Also, fix a vector $a \in \{1, -1\}^i$. Define the random variable

$$X_{I,a} \;=\; |\{u \in V \setminus I : F(I, u, T) = a\ \}|.$$

It is easy to verify that $X_{I,a}$ is the sum of $t - i$ independent and identically distributed indicator random variables each having the common expectation 2^{-i}. Hence it follows that

$$\mu_{I,a} \;=\; E(X_{I,a}) \;=\; (t-i)2^{-i} \;\geq\; (t-d)2^{-d}.$$

Also, by the well-known Chernoff-Hoeffding bounds (see Chapter 4 of [MR95]), it also follows, using $k \geq 5$ and $d \geq 2$, that

$$\begin{aligned} \mathbf{Pr}(X_{I,a} \leq kd) &= \mathbf{Pr}(X_{I,a} - \mu_{I,a} \leq kd - \mu_{I,a}) \\ &\leq e^{-\mu_{I,a}(1-kd/\mu_{I,a})^2/3} \;\leq\; e^{-\mu_{I,a}/4} \;\leq\; e^{-(3.75)kd}. \end{aligned}$$

Hence, for the event $\mathcal{E}$ defined by $\mathcal{E} = \exists I, a : |I| \leq d,\ X_{I,a} \leq kd$, we have

$$\begin{aligned}\mathbf{Pr}(\mathcal{E}) &\leq d \cdot \binom{t}{d} \cdot 2^d \cdot e^{-(3.75)kd} \\ &\leq e^{-d\left((3.75)k - \ln(2e) - \frac{\ln d}{d} - \frac{\ln t}{d}\right)} \\ &\leq e^{-d\left((3.75)k - \ln(2e) - \frac{\ln d}{d} - \ln 16 - d(\ln 2) - \ln k\right)} < 1\end{aligned}$$

where the last strict inequality uses the definition of t and the assumption $k \geq 5$, $d \geq 2$. This shows that, with positive probability, there exists a tournament with desired properties, completing the proof of the lemma.

We now give the proof of Theorem 5.8 where we shall make use of Lemma 5.10.

Proof of Theorem 5.8 Let $G = (V, E)$ be any graph of maximum degree Δ and degeneracy d. If $d \leq 1$, then G is a forest and hence its $\chi_o(G) \leq 3$ as shown in [Sop97]. For $d \geq 2$ and $\Delta \leq 4$, the result follows from a bound of $(2\Delta - 1)2^{2\Delta-2}$ derived in [Sop97]. Hence, we assume that $\Delta \geq 5$ and $d \geq 2$. Consider a linear ordering $(v_n, \ldots, v_1)$ of V such that for each $i \leq n$, v_i has at most d neighbors in the subgraph G_i induced by $V_i = \{v_1, \ldots, v_i\}$. Let T be the tournament on $t = 16kd2^d$ vertices specified in Lemma 5.10, with $k = \Delta$. Let G' be any orientation of G. We inductively color vertices of G' in the order $(1, \ldots, n)$ in such a way that after the coloration of the first m vertices :

(1) the partial coloring $f(v_1), \ldots, f(v_m)$ is a valid oriented coloring of G'_m using vertices of T ;

(2) for each v_j with $j > m$, all neighbors of v_j in V_m are colored with distinct colors.

Now, we need to color v_{m+1} so that (1) and (2) hold for $f(v_{m+1})$ as well. For this, let $\{y_1, \ldots, y_i\} \subseteq V_m$ be the neighbors of v_{m+1} in V_m each colored with distinct colors (because of (2)) from $I = \{f(y_1), \ldots, f(y_i)\}$. Note that $i \leq d$. Let $a = F(\{y_1, \ldots, y_i\}, v_{m+1}, G'_{m+1})$. Let $K = \{w \in V(T) \setminus I : F(I, w, T) = a\}$. By Lemma 5.10, we know that $|K| \geq kd + 1$. Now, there can be at most kd paths of the form (v_{m+1}, u, v_j) such that $u \in V \setminus V_{m+1}$ is a neighbor of v_{m+1} in G and $v_j, j \leq m$ is a neighbor of u in V_m. Let $B \subseteq V_m$ be the set of all such v_j's and let

$f(B)$ be the set of their colors with $|f(B)| \leq kd$. Now, color v_{m+1} with any color from $K \setminus f(B)$ and one can easily check that $f(v_{m+1})$ satisfies both (1) and (2), thus extending the coloring inductively. This proves Theorem 5.8.

5.5 Conclusions and Open Problems

We obtained a relation between forbidden subgraph colorings and oriented colorings. In particular, we obtained an upper bound for the oriented chromatic number in terms of low treewidth chromatic numbers and found an upper bound of $O(2^{g^{1/2+o(1)}})$ for the oriented chromatic number of graphs of genus g. However, we believe that this bound is not tight. In fact, we believe in the following conjecture:

Conjecture : There exist absolute positive constants c_1, c_2 such that : if G is a graph of genus at most g, then $\chi_o(G) \leq c_1 2^{c_2\sqrt{g}}$.

Further, it would be interesting to obtain bounds for the (j,k)-treewidth chromatic number (for graphs of bounded genus), when $j > 2$. We also pose the following interesting and challenging open problem.

Open Problem : Determine if there is a k such that $\chi^{tw}_{2,k}(G) \leq 4$ for all planar graphs G and find the smallest such k if it exists.

Note that if we replace 4 by 5 in the above inequality, then the answer is yes for $k = 1$ since it has been shown by Borodin [Bor79] that $a(G) \leq 5$ for any planar graph G. Also, this bound is tight as Grünbaum [Grü73] obtained an infinite family of planar graphs having no acyclic 4-coloring.

6

Generalized edge colorings

6.1 Introduction

A proper edge coloring is a labeling of the edges of a graph such that touching edges (i.e. edges sharing a common endpoint) do not get the same color. The minimum number of colors sufficient for a proper edge coloring of a graph G is called the chromatic index and is denoted by $\chi'(G)$. This is a well-studied parameter and it is known by a theorem of Vizing [Viz64] (see also [Wes01]) that $\chi'(G)$ is always at most $\Delta(G) + 1$ where $\Delta(G)$ denotes the maximum degree of any vertex in G.

Several variants of edge colorings have been studied, many of them naturally arising as variants of vertex colorings of line graphs. An interesting example is acyclic edge coloring introduced in chapter 1. Recall that this is a proper coloring of the edges of a graph such that there are no bichromatic cycles and that the minimum number of colors required for such a coloring of a graph G is known as its acyclic edge chromatic index and is denoted by $a'(G)$. It was conjectured in [ASZ01] that $a'(G)$ is at most $\Delta + 2$ for any graph G of maximum degree Δ. Currently the best known upper bound is 16Δ which was obtained by Molloy and Reed in [MR98]. A *distance-2 edge coloring* or a strong edge coloring is a proper edge coloring in which edges adjacent to a common edge must also get distinct colors. It can be seen that a distance-2 edge coloring can be obtained using $O(\Delta^2)$ colors for any graph of maximum degree Δ.

In this chapter, we study $(j, \mathcal{F})$-edge colorings introduced in Chapter 1 and which generalize the above-mentioned types of colorings. As in the case of vertex colorings, we obtain bounds in terms of the maximum degree, using the Lovász Local Lemma as a tool in the proof arguments.

Before we state the main results of this chapter, we formally define the generalized notion of restricted edge colorings.

Definition 6.1 *Let $\mathcal{F}$ be a family of connected graphs on 3 or more vertices and j be a positive integer such that $j < min_{H\in\mathcal{F}}(|E(H)|)$. We define a $(j,\mathcal{F})$ edge coloring of a graph G to be a proper coloring of $E(G)$ such that the subgraph of G induced by the union of any j color classes does not contain an isomorphic copy of H as a subgraph, for each $H \in \mathcal{F}$. We denote by $\chi'_{j,\mathcal{F}}(G)$ the minimum number of colors required for a $(j,\mathcal{F})$-edge coloring of G and also call it the $(j,\mathcal{F})$-chromatic index of G.*

Remark: We require $j < |E(H)|$ for each $H \in F$ because otherwise if G contains a copy of H such that $j \geq |E(H)|$, no proper coloring of $E(G)$ would be a $(j,\mathcal{F})$-edge coloring. Also if $j < |E(H)|$ for each $H \in F$, we are guaranteed at least one (j,k)-coloring, namely the trivial coloring in which each edge gets a distinct color.)

We also define $\chi'_{j,\mathcal{F}}(\Delta) = max\{\chi_{j,\mathcal{F}}(G) : \Delta(G) = \Delta\}$. As will be proved later (Theorem 6.2), $\chi'_{j,\mathcal{F}}(G)$ can be upper-bounded by a function of $\Delta = \Delta(G)$ and hence $\chi'_{j,\mathcal{F}}(\Delta)$ exists and is a well-defined parameter.

Notation: For a positive integer j and a family $\mathcal{F}$ of graphs such that $j < E(H)$ for each $H \in \mathcal{F}$, we define and use $\theta(j,\mathcal{F})$ to denote the expression below:

$$\sup_{H\in\mathcal{F}}\frac{(|V(H)|-2)}{(|E(H)|-j)}.$$

The following is our main theorem of this chapter.

Theorem 6.2 *Let $\mathcal{F}$ be a family of connected graphs on 3 or more vertices and let j be a positive integer such that $j < min_{H\in\mathcal{F}}(|E(H)|)$. Let $\theta = \theta(j,\mathcal{F})$. Then there exists a constant $C = C(j,\mathcal{F})$ such that for any graph G of maximum degree Δ, $\chi'_{j,\mathcal{F}}(G) \leq C\Delta^{max(\theta,1)}$. Equivalently, $\chi_{j,\mathcal{F}}(\Delta) = O(\Delta^{max(\theta,1)})$.*

As mentioned before, the acyclic chromatic index of graphs of maximum degree Δ is at most $O(\Delta)$. This naturally leads to the general question of determining those $(j, \mathcal{F})$ pairs for which $\chi'_{j,\mathcal{F}}(\Delta) = O(\Delta)$. The following corollary of the previous theorem provides a partial answer to this question.

Corollary 6.3 *Let $\mathcal{F}$ be a family of connected graphs on 3 or more vertices and let $D = D(\mathcal{F}) = min_{H \in \mathcal{F}}(|E(H) - |V(H)|)$. Then there exists a constant $C = C(\mathcal{F})$ such that for any graph G of maximum degree Δ and for any $j \leq D + 2$, $\chi'_{j,\mathcal{F}}(G) \leq \lceil C\Delta \rceil$.*

For acyclic edge coloring, $D = 0$ since $\mathcal{F}$ is the set of all even cycles and thus, a linear upper bound on $a'(\Delta)$ follows.

In Section 5.2, we present the proof of Theorem 5.2 and in Section 5.3, we present some interesting consequences of both Theorem 5.2 and Corollary 5.3. In Section 5.4, we also present extensions to avoiding several families simultaneously and in Section 5.5, we preesnt some ways to obtain improved bounds on $(j, \mathcal{F})$-chromatic indices.

6.2 Proof of results

To prove Theorem 6.2, we will use the non-symmetric form of Lovász Local Lemma stated as Lemma 3.6 in Chapter 3. We note that Theorem 6.2 can also be obtained as a consequence of Theorem 3.27 given in Chapter 3, but present the following proof as an explicit application of the non-symmetric form of Lovász Local Lemma.

We prove the following explicit version of Theorem 6.2.

Proposition 6.4 *Let $\mathcal{F}$ be a family of graphs on 3 or more vertices and j be a positive integer as in Theorem 1.2. Let $\theta = \theta(j, \mathcal{F}) = max_{H \in \mathcal{F}} \frac{(|V(H)|-2)}{(|E(H)|-j)}$. Then for any graph G of maximum degree Δ, $\chi'_{j,\mathcal{F}}(G) < \lceil (C\Delta)^{max(\theta,1)} \rceil$ where $C = C(j, \mathcal{F}) = 200 \cdot 2^{6j+6D}(3j)^{2j}$ where $D = D(\mathcal{F}) = min_{H \in \mathcal{F}}(|E(H) - |V(H)|)$.*

Proof of Proposition 6.4:

Let $G = (V, E)$ be the given graph. Without loss of generality, we assume that $j \geq 2$. When $j = 1$, any $(j, \mathcal{F})$ coloring is the same as a proper edge coloring of G which always exists with $\Delta + 1$ colors by Vizing's theorem. Henceforth, we assume that $j \geq 2$.

Put $x = \lceil (C\Delta)^{max(\theta,1)} \rceil$ where $C = 200 \cdot (2)^{6j+6D} \cdot (3j)^{2j}$.

Let $f : E \rightarrow \{1, 2, ..., x\}$ be a random edge coloring of G, where for each edge $e \in E$ independently, the color $f(e) \in \{1, 2, ..., x\}$ is chosen uniformly at random. It suffices to prove that with positive probability, f is a $(j, \mathcal{F})$ edge coloring of G. To this end, we define a family of bad events whose absence implies that the random coloring is a $(j, \mathcal{F})$ edge coloring and use the Lovász local lemma to show that with positive probability none of these events occur.

The events we consider are of the following two types.

a) **Type I**: For each pair of touching edges $e_1 = (u, v)$ and $e_2 = (u, w)$, let A_{e_1,e_2} be the event that $f(e_1) = f(e_2)$.

We define $\alpha = \frac{1}{\theta}$. The definition of the Type II event depends on whether $\alpha < 1$ or $\alpha \geq 1$.

Case $\alpha < 1$:
b) **Type II**: For each connected subgraph L of $V(G)$ such that $|E(L)| = max\{|V(L)| - 1, \lceil \alpha(|V(L)| - 2) + j \rceil\}$, let B_L be the event that the edges in L are colored using at most j colors in the coloring by f.
Note that for each $H \in \mathcal{F}$, we have $|E(H)| \leq |V(H)| - 1$ and $|E(H)| \leq \lceil \alpha(|V(H)| - 2) + j \rceil$ and hence the absence of type II events in this case ensures that the union of j color classes cannot have a copy of any member of $\mathcal{F}$.

Case $\alpha \geq 1$:
b) **Type II**: For each connected subgraph L of $V(G)$ such that $|E(L)| = |V(L)| + D$, let B_L be the event that the edges in L are colored using at most j colors in the coloring by f. Note that in this case $D \leq 0$. Also, for each $H \in \mathcal{F}$, we have $|E(H)| \leq |V(H)| + D$ and thus the absence of type II events in this case ensures that the union of j color classes cannot have a copy of any member of $\mathcal{F}$.

Thus we see that if none of the events of the two types above occurs, then f is a $(j, \mathcal{F})$-edge coloring. It remains to show that with positive probability none of these events happen. To prove this we apply the local lemma. Any event of either of the two types is mutually independent of all events that do not share an edge in common with the given event.

We need to estimate the number of events of each type possibly influencing any given event. This estimate is given in the following two simple lemmas.

Lemma 6.5 *Let $e = (u, v)$ be an arbitrary edge of the graph $G = (V, E)$. Then the following two statements hold.*

(i) e touches at most 2Δ edges in G.

(ii) e belongs to at most $2k^{2j+2D+1}4^k\Delta^{k-2}$ subgraphs of $V(G)$ on k vertices which are as in a Type II event.

Proof Part (i) follows from the fact that $\Delta(G) = \Delta$.

Part (ii) can be seen as follows: If $\alpha < 1$, let $\mathcal{G}(e, k)$ be the set of connected subgraphs (containing e) in G on k vertices and having $\max\{k-1, \lceil\alpha(k-2)+j\rceil\}$ edges. If $\alpha \geq 1$, let $\mathcal{G}(e, k)$ be the set of connected subgraphs (containing e) in G on k vertices and having $k + D$ edges. Let $\mathcal{T}(e, k)$ be the set of k-vertex trees in G containing e with some arbitrary linear order imposed on them.

If $\alpha < 1$, each tree in $\mathcal{T}(e, k)$ is a subgraph of at most

$$\binom{\binom{k}{2}}{\max\{0, \lceil\alpha(k-2)+j\rceil - (k-1)\}} \leq k^{2j-2}$$

connected subgraphs in $\mathcal{G}(e, k)$ on the same set of vertices. If $\alpha \geq 1$, each tree in $\mathcal{T}(e, k)$ is a subgraph of at most $\binom{\binom{k}{2}}{D+1} \leq k^{2D+2}$ connected subgraphs in $\mathcal{G}(e, k)$ on the same set of vertices. Each connected subgraph H in $\mathcal{G}(e, k)$ has at least one tree in $\mathcal{T}(e, k)$ the smallest (with respect to the assumed linear ordering) of which is identified with H. Thus $|\mathcal{G}(e, k)| \leq k^{2j+2D}|\mathcal{T}(e, k)|$, irrespective of whether $\alpha < 1$ or $\alpha \geq 1$.

We now find an upper bound for $|\mathcal{T}(e, k)|$. Since there are at most 4^k unlabeled trees on k vertices (see Chapter 8 of [LJK03]), there are at most 4^k choices for choosing the unlabeled structure of a tree in $\mathcal{T}(e, k)$. Once this unlabeled structure is fixed, we now have to embed this unlabeled tree in G. The number of ways of identifying edge e with an edge in the unlabeled tree is at most $2(k-1) < 2k$. Now the remaining vertices in the unlabeled tree can be embedded in at most

Δ^{k-2} ways. To see this, we observe that there are Δ choices for each neighbor of v in the chosen unlabeled tree. Once these are fixed, the number of choices for a neighbor of each first neighbor is again Δ. Repeating this process, we can see that the number of choices for embedding all the vertices (other than u,v) is at most Δ^{k-2}. This proves (ii).

Lemma 6.6 *For $\{i, j\} \in \{I, II\}$ the (i, j)-th entry of the table given below is an upper bound on the number of events of type j which can possibly influence an event of type i.*

	I	$II(B_{L'})$
I	4Δ	$4l^{2j+2D+1}4^l\Delta^{l-2}$
$II(B_L)$	$2m\Delta$	$2ml^{2j+2D+1}4^l\Delta^{l-2}$

Here, m is the number of edges in L and l is the number of vertices in L'. The lemma follows from Lemma 6.5 and the fact that any event is mutually independent of all other events which do not share any edge with the given event. We now estimate the probability of occurrence of each type of event.

Fact 6.7 *(i) For each type I event A, $Pr(A) = \frac{1}{x}$.*
(ii) For each type II event B_L, $Pr(B_L) \leq \frac{j^m}{x^{m-j}}$, where $m = |E(L)|$.

The number of ways in which m edges can be colored using at most j colors from $\{1, 2, ..., x\}$ is at most $\binom{x}{j} j^m \leq x^j j^m$. This proves (ii).

We now define the constants y_i to enable us to apply the Local Lemma. For an event A of type I, we define $y_A = \frac{9}{x}$. For an event B_L of type II, we define $y_{B_L} = \frac{(3j)^m}{x^{m-j}}$, where $m = |E(L)|$.

If $\alpha < 1$, $|E(L)| - j \geq \alpha(|V(L)| - 2)$ for each forbidden j-colored graph L and using $x > 3j$, we note that $y_{B_L} \leq \frac{(3j)^{j+\alpha(k-2)}}{x^{\alpha(k-2)}}$ where $k = |V(L)|$.

If $\alpha \geq 1$, then $|E(L)| - j \geq |V(L)| - 2$ for each forbidden j-colored graph L and hence $y_{B_L} = \frac{(3j)^{k+D}}{x^{k+D-j}} \leq \frac{(3j)^{k+j-2+D-j+2}}{x^{k-2+D-j+2}} \leq \frac{(3j)^{k+j-2}}{x^{k-2}}$, where $k = |V(L)|$. Here we used $x > 3j$ and also the fact that $D \geq j - 2$ whenever $\alpha \geq 1$.

In either case, by substituting $x = (C\Delta)^{max(\theta,1)}$, we find that $y_{B_L} \leq \frac{(3j)^{k+j-2}}{(C\Delta)^{k-2}}$ and hence $(1 - y_{B_L}) \geq 1 - \frac{(3j)^{j+k-2}}{(C\Delta)^{k-2}}$.

By Lemma 3.6, Lemma 6.6 and Fact 6.7, it thus suffices to verify the following two inequalities.

$$\frac{1}{x} \leq \frac{9}{x}\left(1 - \frac{9}{x}\right)^{4\Delta} \prod_{l \geq 3} (1 - y_{B'_L})^{4l^{2j+2D+1}4^l\Delta^{l-2}} \tag{6.1}$$

$$\frac{j^m}{x^{m-j}} \leq \frac{(3j)^m}{x^{m-j}}\left(1 - \frac{9}{x}\right)^{2md} \prod_{l \geq 3} (1 - y_{B'_L})^{2ml^{2j+2D+1}4^l\Delta^{l-2}}, \quad \forall m \geq 3 \tag{6.2}$$

We see that (6.2) is equivalent to (6.1). Thus it is sufficient to prove (6.1).

In (6.1), we substitute $x = (C\Delta)^{max(\theta,1)}$ where $C = 200 \cdot (2)^{6j+6D} \cdot (3j)^{2j}$ and using the known fact that $(1 - \frac{1}{z})^z \geq 1/4$ for all $z \geq 2$, as well as the fact that $(1 - y_{B_{L'}}) \geq 1 - \frac{(3j)^{j+l-2}}{(C\Delta)^{(l-2)}}$ we see that it is sufficient to prove:

$$\frac{1}{9} \leq 4^{-\frac{36\Delta}{x}} 4^{-S}$$

where

$$S = \sum_{l \geq 3} \frac{(3j)^{j+l-2} \cdot 4^{l+1} \cdot l^{2j+2D+1}}{200^{l-2} \cdot 2^{(6j+6D)(l-2)} \cdot (3j)^{(2j)(l-2)}}$$

Using the fact that

$$j + l - 2 \leq 2j(l-2), \quad \forall j \geq 2,\ l \geq 3$$

and also the fact that

$$l^{2j+2D+1} < 2^{(2j+2D)l} \leq 2^{(6j+6D)(l-2)}, \quad \forall j \geq 2,\ l \geq 3,\ D \geq -1,$$

we get

$$S \ \leq \ \sum_{l \geq 3} \frac{4^{l+1}}{200^{l-2}} \ = \ \frac{64}{49} < \frac{4}{3}.$$

We thus find that it is sufficient to prove:

$$\frac{1}{9} \leq 4^{-\frac{36\Delta}{x}} 4^{-\frac{4}{3}}$$

Since $x \geq 216\Delta$, the above inequality is true.

Thus by Lovász Local Lemma, with positive probability, none of the bad events occur and hence a $(j, \mathcal{F})$ edge coloring exists using $O(\Delta^{max(\theta,1)})$ colors. This completes the proof of Proposition 6.4 and hence of Theorem 6.2. ∎

6.2.1 Free $(j, \mathcal{F})$ edge colorings

Suppose, in Definition 6.1, we do not explicitly insist that the edge coloring be proper. We call such a coloring a free (from having to be proper) $(j, \mathcal{F})$ edge coloring. We use the notation $f\chi'_{j,\mathcal{F}}(G)$ to denote the corresponding free chromatic index. It follows that there is an analogue of Proposition 6.4 corresponding to free $(j, \mathcal{F})$ edge colorings also. It is given below without proof since the proof is very similar to that of Proposition 6.4.

Proposition 6.8 *Let $\mathcal{F}, j$, $\theta = \theta(j, \mathcal{F})$, $D = D(\mathcal{F})$, $C = C(j, \mathcal{F})$ be all the same as defined in Proposition 6.4 except that $C(1, \mathcal{F})$ is redefined to be $7200 \cdot 2^{6+6D}$. Then, for any graph G of maximum degree d, the free $(j, \mathcal{F})$ chromatic index is bounded as $\chi'_{j,\mathcal{F}}(G) \leq (C\Delta)^{\theta}$.*

By setting $j = 1$ and $\mathcal{F} = \{K_{t,t}\}$, we see that $\theta(j, \mathcal{F}) = \frac{2t-2}{t^2-1} = \frac{2}{t+1}$ and hence $E(K_n)$ can be partitioned into $O(n^{2/(t+1)})$ parts so that each part has no copy of $K_{t,t}$. This strengthens a well-known fact in extremal graph theory (see [ES74]), namely, that there is a $K_{t,t}$-free graph on n vertices having $\Omega(n^{2-2/(t+1)})$ edges. In particular, it follows that there is an edge-coloring of K_n using $O(n^{1/2})$ colors so that each color class is triangle-free.

6.3 Consequences

We now apply Theorem 6.2 and Corollary 6.3 to some interesting families of graphs to obtain the results in the following table.

Restriction on the union of color classes	j	$\mathcal{F}$	$\theta(j,\mathcal{F})$	Bound on $\chi'_{j,\mathcal{F}}(\Delta)$
Planar	5	Subdivisions of $K_{3,3}$ and K_5	1	$O(d)$
""	6	""	4/3	$O(\Delta^{4/3})$
""	7	""	2	$O(\Delta^2)$
""	8	""	4	$O(\Delta^4)$
Outerplanar	3	Subdivisions of K_4 and $K_{2,3}$	1	$O(\Delta)$
""	4	""	3/2	$O(\Delta^{3/2})$
""	5	""	3	$O(\Delta^3)$
Treewidth at most 2	4	Subdivisions of K_4	1	$O(\Delta)$
""	5	" "	2	$O(\Delta^2)$
Treewidth at most k for $k \geq 2$	$k+2$	Edge minimal graphs of treewidth more than k	1	$O(\Delta)$
k-degenerate graphs	$\frac{k^2+k+2}{2}$	Edge minimal graphs that are non-k-degenerate	1	$O(\Delta)$
k-colorable graphs	$\frac{k^2-k+2}{2}$	Edge-critical $(k+1)$-chromatic graphs	1	$O(\Delta)$
Genus at most g	$2g+3$	Edge minimal graphs of genus more than g	1	$O(\Delta)$

Justification for some entries :

1. Planarity restriction :

   Note that any subdivision of K_5 is a graph on $5+k$ vertices and $10+k$ edges for some $k \geq 0$. Similarly, any subdivision of $K_{3,3}$ is a graph on $6+l$ vertices and $9+l$ edges for some $l \geq 0$. Hence $\theta(j,\mathcal{F}) = \sup_{k,l\geq 0}\left\{\frac{3+k}{10-j+k}, \frac{4+l}{9-j+l}\right\}$.This value is atmost 1 if $j \leq 5$ and is 4/3 for $j=6$ and is 2 for $j=7$ and is 4 for $j=8$. This proves the entries in the table.

2. Outerplanarity restriction :

Note that any subdivision of K_4 is a graph on $4+k$ vertices and $6+k$ edges for some $k \geq 0$. Similarly, any subdivision of $K_{2,3}$ is a graph on $5+l$ vertices and $6+l$ edges for some $l \geq 0$. Hence $\theta(j, \mathcal{F}) = \sup_{k,l \geq 0} \left\{ \frac{2+k}{6-j+k}, \frac{3+l}{6-j+l} \right\}$. This value is atmost 1 if $j \leq 3$ and is $3/2$ for $j = 4$ and is 3 for $j = 5$. This proves the entries in the table.

3. k-degeneracy restriction :

   Any connected minimal (with respect to edge deletion) graph of degeneracy $k+1$ is a graph on v vertices for some $v \geq k+2$ and has minimum degree $k+1$ and hence has at least $v(k+1)/2$ edges. Thus, $D \geq (k+2)(k-1)/2$ amd hence for $j \leq \frac{(k+2)(k-1)}{2} + 2 = \frac{k^2+k+2}{2}$, we can apply Corollary 6.3 to deduce that $O(\Delta)$ colors suffice.

4. k-colorablility restriction :

   Any connected minimal (with respect to edge deletion) graph of chromatic number $k+1$ is a graph on v vertices for some $v \geq k+1$ and has minimum degree at least k and hence has at least $vk/2$ edges. Thus, $D \geq (k+1)(k-2)/2$ and hence for $j \leq \frac{(k+1)(k-2)}{2} + 2 = \frac{k^2-k+2}{2}$, we can apply Corollary 6.3 to deduce that $O(d)$ colors suffice.

5. Treewidth at most k :

   It can be shown by a simple inductive argument that any connected graph on v vertices and having treewidth more than k contains at least $v+k$ edges provided $k \geq 2$. This shows that for $j \leq k+2$, $\theta(\mathcal{F}) \leq 1$.

6. Genus at most g :

   By Euler's polyhedral formula, the number of edges in a graph of genus at least $g+1$ and having v vertices is at least $v + 2g + 1$. Thus $D(\mathcal{F}) = \min_{H \in \mathcal{F}}(|E(H) - |V(H)|) \geq 2g+1$. Hence, by Corollary 6.3, for $j \leq 2g+3$, $O(\Delta)$ colors suffice.

6.4 Extensions to colorings with several families forbidden simultaneously

We can also extend our results to more restricted edge colorings where we require simultaneously for several pairs $(j_i, \mathcal{F}_i)$ $(i = 1, \ldots, s)$ that the union of *any* j_i color classes has no copy of any member of $\mathcal{F}_i$. The vertex versions of such colorings were considered by Nešetřil and Ossona de Mendez in [NdM06] for families of H-minor-free graphs. A slightly relaxed notion (where we don't insist on properness) was studied by DeVos, et. al. in [DDO$^+$04] for families of H-minor-free graphs. However, we obtain bounds which work for any arbitrary graph G. We first formally define these colorings.

Definition 6.9 *Let $\mathcal{P} = \{(j_1, \mathcal{F}_1), \ldots, (j_s, \mathcal{F}_s)\}$ be a set of $s \geq 1$ pairs such that for each $i \leq s$, j_i is a positive integer and $\mathcal{F}_i$ is a family of connected graphs such that $j_i < |E(H)|$ for each $H \in \mathcal{F}_i$. We define a $\mathcal{P}$-edge coloring to be a proper edge coloring of G so that, for each $i \leq s$, the union of any j_i color classes does not contain an isomorphic copy of H as a subgraph, for each $H \in \mathcal{F}_i$. We denote by $\chi'_{\mathcal{P}}(G)$ the minimum number of colors sufficient for a $\mathcal{P}$-edge coloring of G.*

Note : Similarly, one can define the free version (without explicitly insisting on properness) of a $\mathcal{P}$-edge coloring and denote the corresponding chromatic index by $f\chi'_{\mathcal{P}}(G)$.

We now present the main result of this section. We skip the proof of the following theorem as it is based on an application of the Local Lemma and is similar to the proofs of Theorem 6.2 and Proposition 6.8.

Theorem 6.10 *Let $\mathcal{P} = \{(j_1, \mathcal{F}_s), \ldots, (j_s, \mathcal{F}_s)\}$ be a set of $s \geq 1$ pairs such that for each $i \leq s$, j_i is a positive integer and $\mathcal{F}_i$ is a family of connected graphs such that for each $j_i < |E(H)|$ for each $H \in \mathcal{F}_i$. Define*

$$\theta_i = \theta(j_i, \mathcal{F}_i) = \sup_{H \in \mathcal{F}_i} \frac{(|V(H)| - 2)}{(|E(H)| - j_i)},\ \forall i \leq s,$$

$$D_i = D(\mathcal{F}_i) = \min_{H \in \mathcal{F}_i}(|E(H) - |V(H)|),\ \forall i \leq s,$$

$$C_i = C(j_i, \mathcal{F}_i) = 200s \cdot 2^{6j_i+6D_i} \cdot (3j_i)^{2j_i},\ \forall i \leq s,$$

$$\theta = \max_{i \leq s} \theta_i,\ \ C = \max_{i \leq s} C_i.$$

Then, for any graph G of maximum degree d, $\chi'_{\mathcal{P}}(G) \leq (C\Delta)^{\max(\theta,1)}$. Also, in the case of $\mathcal{P}$-free colorings, we have $f\chi'_{\mathcal{P}}(G) \leq (C\Delta)^{\theta}$ with C_i being redefined as $C_i = 7200s \cdot 2^{6(D_i+1)}$ if $j_i = 1$.

By setting $\mathcal{P}_s = \{(1, \mathcal{F}_1), \ldots, (s, \mathcal{F}_s)\}$ where $\mathcal{F}_i$ is the set of all i colorable (usual edge coloring) graphs of treewidth $i+1$, for each $i \leq s$, we get upper bounds on the the type of edge colorings studied by DeVos, et. al. in [DDO+04].

Corollary 6.11 *For $s \geq 1$, let $\chi'_{\mathcal{P}_s}(G)$ denote the minimum number of colors sufficient to obtain a proper edge coloring of G so that the union of any $j \leq s$ color classes forms a subgraph of treewidth at most j. Then, there exists a constant $C = C(s)$ such that for any graph of maximum degree Δ, $\chi'_{\mathcal{P}_s}(G) \leq C\dot{\Delta}$.*

Remark : It is essential that s (the number of distinct j's) of Theorem 6.10 is finite. If we allow s to be infinite, then it is possible that the corresponding chromatic number may not be bounded by a function of maximum degree Δ alone. For example, if $\mathcal{P} = \{(k-1, \{P_k\}) : k \geq 2\}$ (P_k is a path on k edges), then $\chi'_{\mathcal{P}}(P_n) = n$ for every $n \geq 2$ while maximum degree is 2.

Generalized acyclic edge colorings :

This notion was introduced in [GGW06] and is a generalization of the acyclic edge colorings. For any $r \geq 3$, the r-acyclic chromatic index $a'_r(G)$ is the minimum number colors sufficient to properly color the edges of G so that every k-cycle uses at least $\min\{r, k\}$ colors, for every $k \geq 3$. Note that this specializes to the standard acyclic chromatic index when $r = 3$. Let $a'_r(\Delta) = max\{a'_r(G) : \Delta(G) = \Delta\}$. In [GP05], it is shown that for every fixed $r \geq 4$, $a'_r(\Delta) = O(\Delta^{\lfloor r/2 \rfloor})$.

This result follows as a corollary of Theorem 6.10. Let $l = \lfloor r/2 \rfloor + 1$. Let $\mathcal{P}$ be defined by

$$\mathcal{P} = \{\ (2, P_3), (3, P_4), \ldots, (l-1, P_l), (r-1, \{C_k : k > r\})\ \}.$$

Here, P_k denotes a path on k edges and C_k denotes a cycle on k edges. The first $l-2$ pairs forbid any path having $k \leq l$ edges being colored with fewer than k colors. This, in turn, implies that any cycle C_k on $k \leq r$ edges is colored with k colors. The last pair takes care of the remaining cycles. Thus, every $\mathcal{P}$-edge coloring is also a generalized r-acylic edge coloring. It is easy to see that

$$\forall k,\ 3 \leq k \leq l,\ \ \theta(k-1, P_k) = k-1 \leq \lfloor r/2 \rfloor,$$

$$\theta(r-1, \{C_k : k > r\}) = \sup_{k \geq 1} \frac{r+k-2}{k+1} = \frac{r-1}{2} \leq \lfloor r/2 \rfloor.$$

Applying Theorem 6.10, for each fixed $r \geq 3$, we have $a'_r(\Delta) \leq \chi'_{\mathcal{P}}(\Delta) = O(\Delta^{\lfloor r/2 \rfloor})$. The upper bound is tight upto a constant factor as shown in [GP05].

Note that if, instead of defining $\mathcal{P}$ as above, we had used the natural definition of

$$\mathcal{P} = \{\ (2, C_3), (3, C_4), \ldots, (r-1, \{C_k : k \geq r\})\ \},$$

we would have only obtained a bound of $O(\Delta^{r-2})$. In fact, our choice of $\mathcal{P}$ was motivated by the choice of bad events used in [GP05]. This shows that it sometimes helps to upper bound a more restrictive coloring. We formally state and apply this observation in the following subsection.

6.5 Improving some of the table entries

For a connected graph H, let $dl(H)$ denote the diameter of the line graph of H. This means that any two edges in H are part of a path in H on at most $dl(H)+1$ edges. Note that if an edge coloring (proper or free) of G is such that any path in G on k (for each $k \leq dl(H)+1$) edges uses exactly k colors, then any copy of H in G must use at least $|E(H)|$ colors. Otherwise, there must be two edges in a copy of H colored the same and since these are part of some path on $k \leq dl(H)+1$ edges, this path must use at most $k-1$ colors, a contradiction. This, in turn, implies that for any $j < |E(H)|$, any j color classes of this coloring does not have a copy of H. This is a more restricted coloring than forbidding a copy of H in any j color classes. But, this may result in a better bound. By applying Theorem 6.10 to this observation, we get the following refinement of Theorem 6.2.

Theorem 6.12 *Let $\mathcal{F}$ be a fixed family of connected graphs and let j be a positive integer such that $j < \min_{H\in\mathcal{F}}(|E(H)|)$. Let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ be a fixed partition of $\mathcal{F}$ where $\mathcal{F}_1$ is finite. Let $\theta_2 = \theta(j, \mathcal{F}_2)$ and $\theta_1 = \max_{H\in\mathcal{F}_1} \min(dl(H), \theta(j, \{H\}))$ where $dl(H)$ is the diameter of the line graph of H. Then, there exists a constant $C = C(j, \mathcal{F}_1, \mathcal{F}_2)$ such that for any graph G of maximum degree Δ, we have*

(i) $\chi'_{j,\mathcal{F}}(G) \leq C\Delta^{\max(1,\theta_1,\theta_2)}$;

(ii) $f\chi'_{j,\mathcal{F}}(G) \leq C\Delta^{\max(\theta_1,\theta_2)}$;

The motivation for this theorem is that for a suitable choice of the partition $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, it may be that $\max\{\theta_1, \theta_2\} < \theta(j, \mathcal{F})$ resulting in an asymptotic improvement of the bound. This is illustrated in the following two improvements on entries in Table 1 in the previous section.

1. For the planarity restriction with $j = 8$, we can improve the upper bound to $O(\Delta^2)$ from the $O(\Delta^4)$ presented before. Write $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, where $\mathcal{F}_1$ is the set of all subdivisions of $K_{3,3}$ with at most one subdivision and $\mathcal{F}_2 = \mathcal{F}\backslash\mathcal{F}_2$. $\mathcal{F}_1$ has exactly two members and for each of them, the diameter of the corresponding line graph $L(H)$ is 2 and hence $\theta_1 = 2$.

   We have:

   $$\theta(8, \mathcal{F}_2) = \sup_{k\geq 0, l\geq 2}\left\{\frac{3+k}{10-8+k}, \frac{4+l}{9-8+l}\right\} = 2.$$

   Thus, by Theorem 6.12, we can properly color the edges of a graph of maximum degree Δ using $O(\Delta^2)$ colors so that the union of any 8 color classes is planar.

2. For the outerplanarity restriction with $j = 5$, write $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, where $\mathcal{F}_1$ is the set of all subdivisions of $K_{2,3}$ with at most one subdivision and $\mathcal{F}_2 = \mathcal{F} \setminus \mathcal{F}_1$. For each of the two members in $\mathcal{F}_1$, the diameter of the corresponding line graph $L(H)$ is 2 and hence $\theta_1 = 2$.

   We have:

   $$\theta(5, \mathcal{F}_2) = \sup_{k\geq 0, l\geq 2}\left\{\frac{2+k}{6-5+k}, \frac{3+l}{6-5+l}\right\} = 2.$$

Thus, by Theorem 6.12, we can properly color the edges of a graph of maximum Δ using $O(\Delta^2)$ colors so that the union of any 5 color classes is outerplanar.

3. If we take $\mathcal{F} = \{K_l\}$ ($l \geq 5$) and set $j = \binom{l}{2} - 1$, then $\theta(j, \mathcal{F}) = l - 2 \geq 3$, $dl(K_l) = 2$, $\mathcal{F}_2 = \emptyset$ and $\theta_1 = 2$. Theorem 6.2, on the other hand, only provides a bound of $O(\Delta^{l-2})$ since $\theta(j, K_l) = l - 2$.

The example 3 given above motivates the following special case of Theorem 6.12 which provides an improvement of Theorem 6.2 for finite families $\mathcal{F}$. It is explicitly stated below for the sake of completion.

Theorem 6.13 *Let $\mathcal{F}$ be a* finite *family of connected graphs and let j be a positive integer such that $j < min_{H \in \mathcal{F}} |E(H)|$. Let $\theta_1 = \theta_1(j, \mathcal{F})$ be defined as*

$$\theta_1(j, \mathcal{F}) = \max\{\min(dl(H), \theta(j, \{H\}) : H \in \mathcal{F}\}.$$

Then, there exists a constant $C = C(j, \mathcal{F})$ such that for any graph G of maximum degree Δ, we have

(i) $\chi'_{j,\mathcal{F}}(G) \leq C\Delta^{\max(1,\theta_1)}$.

(ii) $f\chi'_{j,\mathcal{F}}(G) \leq C\Delta^{\theta_1}$.

6.6 Another strengthening and list analogues

We can further strengthen the asymptotic behavior of the upper bounds (as a power of Δ) on optimal free colorings in some cases. Given a pair $(j, \mathcal{F})$ with usual meanings, define $K(H)$, for each $H \in \mathcal{F}$, as any connected induced subgraph K of H with $|E(K)| > j$ and having the least possible value of $\frac{|V(K)|-2}{|E(K)|-j}$. Define $\mathcal{F}' = \{K(H) : H \in \mathcal{F}\}$. Define $\theta_S(j, \mathcal{F}) = \theta(j, \mathcal{F}')$.

Then, any $(j, \mathcal{F}')$ edge coloring (proper or free) is also a $(j, \mathcal{F})$ edge coloring (proper or free). Also, $\theta(j, \mathcal{F}') \leq \theta(j, \mathcal{F})$ and the inequality can be strict possibly. As a result, one can in fact substitute $\theta_S(j, \mathcal{F})$ in place of $\theta(j, \mathcal{F})$ in Proposition 6.4 and Proposition 6.8.

However, it is easily verified that

$$\frac{|V(K(H))|-2}{|E(K(H))|-j} < \frac{|V(H)|-2}{|E(H)|-j} \text{ only if } \frac{|V(H)|-2}{|E(H)|-j} < 1.$$

Hence, the possibility of an asymptotic improvement by using $\theta_S(j,\mathcal{F})$ is ruled out for proper $(j,\mathcal{F})$ chromatic indices. However, the asymptotic improvement is possible for upper bounds on free $(j,\mathcal{F})$ chromatic indices. For example, consider the graph F on $[5]=\{1,\ldots,5\}$ where the subset $[4]$ induces a K_4 and 5 is adjacent to only 4. Then $\theta(2,\{F\})=3/5$ but $\theta_S(2,\{F\})=1/2$. Thus, using $\theta_S(j,\mathcal{F})$ (in place of $\theta(j,\mathcal{F})$) allows us to get an improved bound of $O(\Delta^{1/2})$. Also, this strengthening can be extended to colorings forbidding several pairs of $(j,\mathcal{F})$ simultaneously.

The strengthening of Theorem 6.12 is not always achieved by the strengthening outlined above. It was noted in Section 5.5 that Theorem 6.12 achieves asymptotically the bound of $O(\Delta^2)$ on $\chi_{j,K_l}(\Delta)$ for $j=\binom{l}{2}-1$. But this bound is not achieved by the strengthening of this section, since $\mathcal{F}'=\{K_l\}$.

List analogues : It can be verified that our proofs (based on probabilistic arguments) can in fact easily be adapted to work for the list analogues of the $(j,\mathcal{F})$ edge colorings and chromatic indices. In the list version, each edge is given a list of colors and we are interested in determining the minimum size of any list which guarantees (irrespective of the actual contents of the lists) the existence of a $(j,\mathcal{F})$-edge coloring of G. We refer to the minimum size as the *list* $(j,\mathcal{F})$-chromatic index of G (or the *list* $\mathcal{P}$-chromatic index of G). Hence it follows that each of the Propositions 6.4 and 6.8 and Theorems 6.10, 6.12 and 6.13 holds true even if we replace the chromatic index by its list analogue in the statement.

6.7 Conclusions and Open Problems

We considered a generalization of some known edge colorings like acyclic edge colorings and obtained upper bounds on the chromatic index in terms of the maximum degree Δ. We have not tried to optimize the constants mentioned in the statements and it is very likely that the constants can be brought down further to small values.

For several $(j, \mathcal{F})$ edge colorings, the bounds are actually $O(\Delta)$, thereby showing that imposing additional restrictions involving any few color classes does not necessarily increase the required number of colors asymptotically. Obviously, these bounds are tight within a constant factor for such colorings. It would be interesting to establish the tightness (at least within a constant or a polylog multiplicative factor) of other super linear upper bounds.

It would also be interesting to obtain constructive (that is, deterministically and algorithmically efficiently realizable) bounds which match the bounds presented in this paper for some specific pairs $(j, \mathcal{F})$. For some colorings, there is an asymptotic gap between existential and deterministically constructible bounds. For example, acyclic chromatic index of any graph is at most 16Δ but the currently known deterministically constructible bound (see [Sub06]) is only shown to be $O(\Delta \log \Delta)$.

However, the recent breakthrough result of Moser and Tardos [MT10] on a constructive version of Lovász Local Lemma can be applied to the proof arguments of Theorem 6.2 resulting in a *randomized* algorithm with a polynomial expected running time for obtaining a $(j, \mathcal{F})$-edge coloring matching the upper bound. The details will appear elsewhere.

Another interesting direction is to explore improvements in the bounds for random graphs or for random regular graphs. Such results have been obtained for acyclic edge coloring in [NW05] where it was shown that the acyclic chromatic index of a random d-regular graph is at most $d + 1$ with high probability.

7

Intersection dimension

7.1 Introduction

In [MF89], Cozzens and Roberts introduced the idea of dimensional properties of graphs. They termed a graph class or graph property P as dimensional if any graph can be written as the intersection of graphs from P, i.e., for any graph $G = (V, E)$, there are k graphs $\{G_i = (V, E_i) \in P : 1 \leq i \leq k\}$ (for some k) such that $E = \cap_i E_i$.

Given a dimensional property $\mathcal{A}$, the minimum number k such that a graph G can be written as the intersection of k graphs in the class $\mathcal{A}$ is defined as the *intersection dimension* of G with respect to $\mathcal{A}$ and is denoted by $dim_{\mathcal{A}}(G)$.

In [KT94], Kratochvil and Tuza showed that a property P is dimensional if and only if all complete graphs and all complete graphs minus an edge are in P. They also proved that for any dimensional hereditary property $\mathcal{A}$, either $dim_{\mathcal{A}}(G) = 1$ for every G or it can take arbitrarily large values. However, it may still be possible to express $dim_{\mathcal{A}}(G)$ in terms of other invariants of G.

Some interesting specializations of intersection dimension include the boxicity of a graph (with respect to the class of interval graphs), cubicity (with respect to unit interval graphs), circular dimension (with respect to circular arc graphs), overlap dimension (with respect to overlap graphs) and permutation dimension (with respect to permutation graphs). Of these, boxicity is the most well-studied and various results on boxicity for special graph classes are known. For example, in

[Tho86], it was shown that every planar graph has boxicity at most 3. Bounds have also been obtained for graphs of bounded treewidth [CS07] and graphs of bounded maximum degree [CFS08]. Circular dimension was first studied by Feinberg in [Fei79], where the value of circular dimension was determined exactly for the class of complete partite graphs. However, while the boxicity of a graph provides an upper bound on circular dimension, tighter bounds for circular dimension were not known.

In this chapter, we obtain bounds for the intersection dimension of a graph with respect to certain hereditary properties in terms of its maximum degree. We also show that for such properties, the intersection dimension is bounded for graphs in a proper minor closed family and in particular, for graphs of bounded treewidth. We also obtain improved bounds for special cases, notably the circular dimension and permutation dimension. The proofs of these bounds are based on relating the intersection dimension with forbidden subgraph colorings, in particular, frugal colorings.

This chapter is organized as follows: In Section 7.2, we present the basic results of this chapter relating intersection dimension (with respect to certain hereditary classes) and forbidden subgraph colorings. Section 7.3 contains improved bounds on intersection dimension in terms of maximum degree obtained by using frugal colorings. In Section 7.4, we obtain an improved bound for the circular dimension.

7.2 Some Definitions and Lemmas

We first need a few preliminaries.

Definition 7.1 *Following [KT94], we say that a class $\mathcal{A}$ of graphs has the Full Degree Completion (FDC) property if for any graph $G = (V, E)$ in $\mathcal{A}$, the graph obtained by adding a universal vertex (i.e. a vertex adjacent to all of V) also belongs to $\mathcal{A}$.*

Definition 7.2 *The Zykov sum of two graphs with disjoint vertex sets is formed by taking the union of the two graphs and adding all edges between the graphs. We say that a class $\mathcal{A}$ of graphs has the Zykov Sum property (or ZS property) if the Zykov sum of any two graphs in $\mathcal{A}$ is also in $\mathcal{A}$.*

It can be verified that if a hereditary graph class satisfies the Zykov sum property, then it also satisfies the FDC property. In their paper [KT94], Kratochvil and Tuza proved the following lemmas which we shall need.

Lemma 7.3 *([KT94]) Let $\mathcal{A}$ be a class of graphs satisfying the FDC requirement. Suppose $G = (V, E)$ is a graph and $G_i = (V_i, E_i)$, $i = 1, 2, ...k$ are induced subgraphs of G such that each nonedge of G is present as a nonedge in some G_i. Then, $dim_{\mathcal{A}}(G) \leq \sum_{i=1}^{k} dim_{\mathcal{A}}(G_i)$.*

Lemma 7.4 *([KT94]) Let $\mathcal{A}$ be a class of graphs satisfying the Zykov sum property. If $G = (V, E)$ is a graph and $G_{ij} = (V_{ij}, E_{ij})$, $i = 1, 2, \ldots, k$, $j = 1, \ldots, l_i$, are induced subgraphs of G such that (i) each nonedge of G is present as a nonedge in some G_{ij} and (ii) for every i, the vertex sets $V_{ij}, j = 1, 2..., l_i$ form a partition of V. Then $dim_{\mathcal{A}}(G) \leq \sum_{i=1}^{k} \max_{1 \leq j \leq l_i} dim_{\mathcal{A}} G_{ij}$.*

Definition 7.5 *We denote by $\mathcal{G}(\mathcal{F})$ the set of all graphs which do not contain any graph in $\mathcal{F}$ as an induced subgraph.*

Remark: Recall that we used $Forb(\mathcal{F})$ to denote the set of all graphs which do not contain any graph in $\mathcal{F}$ as a subgraph. In contrast to this, a graph in $\mathcal{G}(\mathcal{F})$ cannot contain a graph from $\mathcal{F}$ only as an induced subgraph. Thus $Forb(\mathcal{F}) \subset \mathcal{G}(\mathcal{F})$.

Using Lemma 7.3 and Lemma 7.4, we now obtain a result which connects intersection dimension and $(2, \mathcal{F})$-subgraph colorings.

Theorem 7.6 *Let $\mathcal{A}$ be a hereditary class of graphs which is closed under disjoint union and having the FDC property. Let $\mathcal{F}$ be a family of connected graphs and suppose there exists a constant $t = t(\mathcal{F})$ such that for all graphs $H \in Forb(\mathcal{F})$, the intersection dimension of H with respect to the class $\mathcal{A}$ is at most t. Then for any graph G, $dim_{\mathcal{A}}(G) \leq t\binom{\chi_{2,\mathcal{F}}(G)}{2}$. Further, if $\mathcal{A}$ has the Zykov sum property, then $dim_{\mathcal{A}}(G) \leq t\chi_{2,\mathcal{F}}(G)$.*

Proof of Theorem 7.6: Let G be any graph and let $C_1, ..., C_k$ be the color classes in a $(2, \mathcal{F})$-subgraph coloring of G using $k = \chi_{2,\mathcal{F}}(G)$ colors.

For all $i \neq j$, let $G_{i,j}$ be the subgraph of G induced by the union of the color classes C_i and C_j. We have $G_{i,j} \in Forb(\mathcal{F})$ and hence $dim_{\mathcal{A}}(G_{i,j}) \leq t$. Also, each nonedge of G is present as a non-edge in some $G_{i,j}$. Hence, by Lemma 7.3, $dim_{\mathcal{A}}(G) \leq \sum_{1 \leq i < j \leq k} dim_{\mathcal{A}}(G_{i,j}) \leq t\binom{\chi_{2,\mathcal{F}}(G)}{2}$.

Suppose that $\mathcal{A}$ also satifies the Zykov sum property. Consider an optimal $(2, \mathcal{F})$-subgraph coloring of G as before, with $C_1, \ldots, C_k$ being the color classes. Now consider a proper edge coloring of K_k using k colors. Let $M_1, \ldots, M_k$ be the matchings forming the k color classes in this edge coloring. For each i, let $\mathcal{H}_i = \{G_{i,j}\}_j$ be a collection of induced subgraphs of G obtained as follows: For each matching edge (l, m) in M_i, include the induced subgraph formed by the union of color classes C_l and C_m in $\mathcal{H}_i$. If, for $l \in \{1, \ldots, k\}$, the vertex l is unmatched in M_i, include the subgraph induced by the single color class C_l in $\mathcal{H}_i$. Clearly, the vertex sets of $G_{i,j}$ form a partition of V for each i. Also, each non-edge of G is present as a non-edge in some $G_{i,j}$. Further, for all i, j, $G_{i,j} \in Forb(\mathcal{F})$. Applying Lemma 7.4, we get $dim_{\mathcal{A}}(G) \leq kt = t\chi_{2,\mathcal{F}}(G)$. This proves Theorem 7.6.

Any hereditary class of graphs which is closed under disjoint union and which has the FDC property, must contain all star forests. We now use some results of Albertson et al. [ACK+04] and Nesetril and Ossona de Mendez [NdM03] on the star chromatic number in conjunction with Theorem 7.6 to obtain the following corollary.

Corollary 7.7 *Let $\mathcal{A}$ be a non-trivial hereditary class of graphs which is closed under disjoint union. Then, for any graph G,*

(a) if $\mathcal{A}$ satisfies the FDC property, then $dim_{\mathcal{A}}(G) \leq \binom{\chi_s(G)}{2}$;

(b) if $\mathcal{A}$ satisfies the Zykov sum property, then $dim_{\mathcal{A}}(G) \leq \chi_s(G)$.

In particular, if $\mathcal{A}$ satisfies the FDC property, then there exist constants c_1, c_2, c_3 such that the following hold:

(i) for any graph G of maximum degree Δ, $dim_{\mathcal{A}}(G) \leq c_1\Delta^3$;

(ii) for any graph G of treewidth t, $dim_{\mathcal{A}}(G) \leq c_2 t^4$;

(iii) for any fixed graph H, there exists a constant c_H depending only on H such that for all H-minor free graphs G, $dim_{\mathcal{A}}(G) \leq c_H$.

(iv) for any graph G of genus $g > 0$, $dim_{\mathcal{A}}(G) \leq c_3 g^{6/5}$.

Further, if $\mathcal{A}$ satisfies the Zykov sum property, then there exist constants c_4, c_5 such that the following hold:

(i) if G is a graph of maximum degree Δ, $dim_{\mathcal{A}}(G) \leq c_4 \Delta^{3/2}$;

(ii) if G has treewidth t, $dim_{\mathcal{A}}(G) \leq \frac{(t+2)(t+1)}{2}$;

(iii) if G has genus $g > 0$, $dim_{\mathcal{A}}(G) \leq c_5 g^{3/5}$.

Proof

Statements (a) and (b) follow from Theorem 7.6 and the observation that $\mathcal{A}$ contains all star forests, that is, disjoint unions of stars.

The remaining results follow from the following upper bounds on star chromatic numbers.

- $\chi_s(\Delta) = O(\Delta^{3/2})$ ([ACK+04]).
- If graph G has treewidth at most t, then $\chi_s(G) \leq (t+2)(t+1)/2$ ([FRR04]).
- For any fixed graph H, there is a constant d_H such that for any H-minor free graph G, $\chi_s(G) \leq d_H$ ([NdM03]).
- For a graph G of genus g, $\chi_s(G) \leq c_6 g^{3/5}$, where c_6 is some absolute constant ([MS08]).

This completes the proof of Corollary 7.7. ∎

7.3 Improved bounds

In this section, we considerably improve the bounds of Corollary 7.7 by combining Theorem 7.6 with the following result of Molloy and Reed [MR09] on frugal colorings.

Theorem 7.8 *([MR09]) There exists a postiive constant Δ_0 such that every graph G of maximum degree $\Delta \geq \Delta_0$ can be properly colored using $\Delta+1$ colors so that any vertex has at most β neighbors in any color class, where $\beta = \lfloor a(\log \Delta)/(\log\log \Delta) \rfloor$ and a is some absolute positive constant.*

Notation: Let $\mathcal{A}$ be a hereditary and dimensional class of graphs satisfying the FDC property and closed under disjoint union. For such classes, and for any positive real number t, we define $dim_{\mathcal{A}}(t) = max\{dim_{\mathcal{A}}(G) : \Delta(G) \leq t\}$. By Corollary 7.7, $dim_{\mathcal{A}}(t)$ is well-defined.

By combining Theorem 7.6 with Theorem 7.8, we obtain the the following result.

Theorem 7.9 *Let $\mathcal{A}$ be a hereditary class of graphs closed under disjoint union and satisfying the FDC property. Then for all sufficiently large Δ and some positive constant B, the following holds.*

- $dim_{\mathcal{A}}(\Delta) \leq \Delta^2(\log\Delta)^2 \cdot B^{\log^*\Delta}$;
- *If $\mathcal{A}$ satisfies the Zykov sum property as well, then:*
  $dim_{\mathcal{A}}(\Delta) \leq \Delta(\log\Delta) \cdot B^{\log^*\Delta}$;
- *In particular, if $\mathcal{A}$ is the class of all permutation graphs, then for any graph G, $dim_{\mathcal{A}}(G) \leq \Delta(\log\Delta) \cdot B^{\log^*\Delta}$.*

Proof

Let G be a graph of maximum degree $\Delta \geq \Delta_0$, as in Theorem 7.8. We apply Theorem 7.6 with $\mathcal{F} = \{K_{1,\beta+1}\}$ where $\beta = \lfloor a(\log\Delta)/(\log\log\Delta)\rfloor$, a being the constant in Theorem 7.8. By Theorem 7.8, $\chi_{2,\mathcal{F}}(\Delta) \leq \Delta + 1$. Applying Theorem 7.6, we get $dim_{\mathcal{A}}(G) \leq \binom{\Delta+1}{2} dim_{\mathcal{A}}(\beta)$. Thus, we get

$$dim_{\mathcal{A}}(\Delta) \leq \binom{\Delta+1}{2} dim_{\mathcal{A}}\left(\left\lfloor\frac{a\log\Delta}{\log\log\Delta}\right\rfloor\right) \leq \Delta^2 dim_{\mathcal{A}}\left(\left\lfloor\frac{a\log\Delta}{\log\log\Delta}\right\rfloor\right)$$

For $x > e$, we define

$$f(x) = \left\lfloor\frac{a\log x}{\log\log x}\right\rfloor$$

and for $i \geq 1$,

$$f^{i+1}(x) = \left\lfloor\frac{a\log f^i(x)}{\log\log f^i(x)}\right\rfloor$$

Let $k = max\{i : f^i(\Delta) \geq e^{e^a}\}$. Note that $f^{i+1}(\Delta) \leq \lfloor\log f^i(\Delta)\rfloor$ for $i \leq k$. Hence $k \leq \log^*\Delta$.

We have

$$\begin{aligned} dim_{\mathcal{A}}(\Delta) &\le \Delta^2 dim_{\mathcal{A}}(f(\Delta)) \\ &\le \Delta^2 (f(\Delta))^2 dim_{\mathcal{A}}(f^2(\Delta)) \\ &\le \dots \\ &\le \Delta^2 \left(\prod_{1\le l\le k} (f^i(\Delta))^2 \right) dim_{\mathcal{A}}(\lfloor e^{e^a} \rfloor) \end{aligned}$$

We now bound the product

$$S = \prod_{1\le l\le k} (f^i(\Delta))$$

Using the fact that $f^{i+1}(\Delta) \le \log f^i(\Delta)$ for $i \le k$, we get

$$S \le \left(\frac{a\log\Delta}{\log\log\Delta}\right)\left(\frac{a\log\log\Delta}{\log\log f(\Delta)}\right)\left(\frac{a\log\log f(\Delta)}{\log\log f^2(\Delta)}\right)\cdots\left(\frac{a\log\log f^{k-2}(\Delta)}{\log\log f^{k-1}(\Delta)}\right)$$

Thus,

$$S \le a^k \log\Delta$$

Hence, we get

$$dim_{\mathcal{A}}(\Delta) \le c\Delta^2 (\log\Delta)^2 \cdot a^{2\log^*\Delta}$$

where $c = c_1 e^{3e^a}$ and c_1 is the constant mentioned in Corollary 7.7.

If $\mathcal{A}$ satisfies the Zykov sum property, applying Theorem 7.6 yields:

$$dim_{\mathcal{A}}(\Delta) \le (\Delta+1) dim_{\mathcal{A}}\left(\left\lfloor \frac{a\log\Delta}{\log\log\Delta} \right\rfloor\right) \le 2\Delta dim_{\mathcal{A}}\left(\left\lfloor \frac{a\log\Delta}{\log\log\Delta} \right\rfloor\right)$$

It is easily seen that in this case, a similar analysis as above gives $\dim\mathcal{A}(\Delta) \le \Delta\log\Delta B^{\log^*\Delta}$ for some positive constant B. This completes the proof of Theorem 7.9. ∎

The assumption of closure under disjoint union used in Theorems 7.6 and 7.9

is essential, as otherwise the dimension number need not always be expressed as a function of the maximum degree as the following examples illustrate.

Unbounded dimension with only the FDC assumption: Consider the class of graphs consisting of cliques and cliques minus edges. This is the intersection of all dimensional classes satisfying the FDC property. The intersection dimension of a graph G with respect to this class is $|E(G^c)|$, which is not bounded by any function of the maximum degree.

Unbounded dimension with the Zykov Sum assumption: The Zykov sum property carries over intersection and thus we can consider the smallest dimensional class of graphs with ZS property. This class is in fact the set of all cliques plus cliques minus a matching (of any size). It is easy to see that the intersection dimension of a graph G with respect to this class is in fact $\chi'(G^c)$. This shows that for classes satisfying the ZS property too, the intersection dimension need not always be bounded by a function of the maximum degree.

7.4 Circular dimension - A Special Case

Circular arc graphs (shortly, CA graphs) are defined as the intersection graphs of closed arcs of a circle. Despite their similarity to interval graphs (which are a subclass of CA graphs), these need not be perfect graphs while interval graphs are also perfect graphs. Also, no complete forbidden induced subgraph characterization is known for the class CA. The class CA is clearly dimensional and hereditary. The corresponding interesection dimension is known as the circular dimension or CA-dimension and is denoted by $dim_{CA}(G)$.

Since the class of circular arcs is a superclass of interval graphs, it follows that for any graph G, $dim_{CA}(G) \leq boxicity(G)$. However, while $O(\Delta^2)$ is the best known ([Esp09]) asymptotic upper bound on the boxicity of an arbitrary graph of maximum degree Δ, an asymptotically tight upper bound is still unknown. However, for CA dimension, we shall obtain an upper bound on $dim_{CA}(G)$ that is nearly linear in Δ.

Lemma 7.10 *Let G be a split graph such that every clique vertex has at most t neighbors in the independent set. Then, G has circular dimension at most $t+1$.*

Proof of Lemma 7.10 Form $t+1$ CA graphs $G_1, ..., G_t$ with $G = G_0 \cap G_1 \cap ... \cap G_t$ as follows. Assume, without loss of generality, that $I = \{1, \ldots, n\}$ is the independent set in G. Consider $n+1$ distinct points on the unit circle and label them consecutively with $0, 1, \ldots, n$, traversing in the clockwise direction. In each G_k $(0 \leq k \leq t)$, each $i \in I$ is identified with the closed circular arc consisting of just the point i on the circle. Define $i_0 = 0$. For any clique vertex u with $r \geq 1$ neighbors in I, say $i_1 < i_2 < \ldots i_r$, and for any s, $0 \leq s \leq r$, we identify u with the closed circular arc (clockwise) joining i_{s+1} with i_s (modulo $r+1$) in the graph G_s. For $s > r$, identify u in G_s with the circular arc used in G_r. If u has no neighbor in I, then identify u with the closed arc consisting of just the point i_0, in each G_s $(0 \leq s \leq t)$. It can be verified that $E(G) = E(G_0) \cap E(G_1) \cap ... \cap E(G_t)$ and that each G_i is a split graph. This proves the lemma.

Theorem 7.11 *The circular dimension satisfies: $dim_{CA}(\Delta) = O(\Delta \frac{\log \Delta}{\log \log \Delta})$.*

Proof of Theorem 7.11 Using Theorem 7.8, we obtain a $\beta = O(\frac{\log \Delta}{\log \log \Delta})$-frugal coloring of $V(G)$ using $k = \Delta + 1$ colors. Let $V_1, \ldots, V_k$ be the color classes. We now form k split supergraphs $G_1, \ldots, G_k$ where G_i is obtained from G by making $G[V - V_i]$ a complete graph. It can be seen that $E(G) = E(G_1) \cap \ldots \cap E(G_k)$. Now we apply Lemma 7.10 to each G_i and deduce that $dim_{CA}(G_i) \leq \beta + 1$ and hence $dim_{CA}(G) \leq k(\beta + 1) = O(\Delta \frac{\log \Delta}{\log \log \Delta})$. This proves the theorem. ∎

In this context, we recall the following lower bound on circular dimension, obtained by Shearer in [She80].

Theorem 7.12 *There exist graphs on n vertices for which the circular dimension is at least $\Omega(\frac{n}{\log_2 n})$.*

7.5 Concluding Remarks:

We were able to obtain bounds in terms of maximum degree for several hereditary properties. But the tightness of bounds in several cases is yet to be established. The computational complexity of intersection dimension is also not well-studied.

In particular, we have the following open problems:

- What is the asymptotically best bound for circular dimension in terms of maximum degree?
- It is known that testing whether a graph has boxicity 2 is NP-complete. Is computing the intersection dimension NP-complete with respect to any fixed nontrivial graph property?

8

Conclusions

8.1 Summary

In this thesis, we studied the notion of forbidden subgraph vertex colorings and its applications to oriented colorings and intersection dimension. We proved that any graph can be properly vertex-colored using $C\Delta^{\frac{k-1}{k-j}}$ colors so that the union of any j color classes is a member of $Forb(\mathcal{F})$, where $\mathcal{F}$ is a family of connected j-colorable graphs on k or more vertices and $C = C(j, \mathcal{F})$ is a constant which depends only on j and $\mathcal{F}$. When $j = 2$, we obtained an improved upper bound of $O(\Delta^{1+\frac{1}{m-1}})$ on $\chi_{2,\mathcal{F}}(\Delta)$ (where m is the minimum number of edges in any member of $\mathcal{F}$). We also showed by a probabilistic construction that this bound is nearly tight. Our upper bounds were based on combining probabilistic arguments using the Lovász Local Lemma and some counting arguments.

We also obtained a relationship between oriented chromatic numbers and $(j, \mathcal{F})$-subgraph chromatic numbers. By obtaining bounds on the treewidth chromatic numbers of graphs in terms of their genus, we showed that the oriented chromatic numnber of any graph of genus $g > 0$ is bounded by $O(2^{g^{1/2+o(1)}})$.

For forbidden subgraph edge colorings, we again obtained bounds in terms of the maximum degree. For several interesting graph families $\mathcal{F}$, we showed that properly coloring the edges of any graph so that the union of every few color classes is a member of $Forb(\mathcal{F})$ can be done using just $O(\Delta)$ colors.

We also studied the intersection dimension of graphs with respect to several hereditary properties. By relating intersection dimension with forbidden subgraph vertex colorings, particularly star coloring and frugal colorings, we obtained bounds on intersection dimensions with respect to certain hereditary properties in terms of maximum degree. In particular, we showed that the circular dimension of any graph of maximum degree Δ is at most $O(\Delta \frac{\log \Delta}{\log \log \Delta})$.

8.2 Future Directions

While the upper bounds on $\chi_{j,\mathcal{F}}(\Delta)$ were shown to be nearly tight, removing the polylog factors is a challenging open problem. Obtaining good lower bounds on $\chi_{j,\mathcal{F}}(\Delta)$ for $j > 2$ is also an interesting open problem.

In the case of edge colorings, obtaining any lower bound on forbidden subgraph chromatic indices even for $j = 2$ would be interesting.

For graph families $\mathcal{F}$ with every member of $\mathcal{F}$ having minimum degree at least two, we obtained bounds on $(2, \mathcal{F})$- subgraph chromatic numbers in terms of the genus of a graph. It is an open problem to obtain such bounds when $\mathcal{F}$ is an arbitrary family. Obtaining lower bounds is also an intersesting line of study.

Obtaining lower bounds on intersection dimensions in terms of maximum degree as well as upper bounds for arbitrary hereditary properties are challenging problems as well.

Finally, studying the asymptotics of generalized chromatic numbers, oriented chromatic numbers and intersection dimensions, of random graphs ($G(n, p)$ model or random regular graphs) is another direction of future research.

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List of Publications

The work presented in this thesis is based on the following publications:

- **Bounds on vertex colorings with restrictions on the union of color classes** N.R. Aravind and C.R. Subramanian (2011). Journal of Graph Theory, vol 66, No. 3, pages 213-234.
- **Bounds on edge colorings with restrictions on the union of color classes** N.R. Aravind and C.R. Subramanian (2010). SIAM Journal of Discrete Mathematics, vol 24, No. 3, pages 841-852.
- **Forbidden subgraph coloring and the oriented chromatic number** N.R. Aravind and C.R. Subramanian. Proceedings of 4th International Workshop on Combinatorial Algorithms (IWOCA) 2009. Springer-Verlag LNCS Volume 4393, pages 477–488.
- **Intersection dimension and maximum degree** N.R. Aravind and C.R. Subramanian. Proceedings of Latin-American Algorithms, Graphs and Optimization Symposium (LAGOS) 2009. ENDM volume 35, pages 353-358.