L_0-TYPES COMMON TO A BOREL-DE SIEBENTHAL DISCRETE SERIES AND ITS ASSOCIATED HOLOMORPHIC DISCRETE SERIES

By

Pampa Paul

Enrollment No. : MATH10200704001

The Institute of Mathematical Scieces , Chennai

A thesis submitted to the

Board of Studies in Mathematical Sciences

In partial fulfillment of requirements

For the Degree of

DOCTOR OF PHILOSOPHY

of

HOMI BHABHA NATIONAL INSTITUTE

February, 2013

Homi Bhabha National Institute

Recommendations of the Viva Voce Board

As members of the Viva Voce Board, we recommend that the dissertation prepared by **Pampa Paul** entitled **"L_0-Types Common to a Borel-de Siebenthal Discrete Series and Its Associated Holomorphic Discrete Series"** may be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

____ **Date :**
(External Examinar)

____ **Date :**
Pralay Chatterjee (Member)

____ **Date :**
D. S. Nagaraj (Member)

____ **Date :**
K. N. Raghavan (Member)

____ **Date :**
Parameswaran Sankaran (Chairman/Guide)

Final approval and acceptance of this dissertation is contingent upon the candidate's submission of the final copies of the dissertation to HBNI.

DECLARATION

I hereby certify that I have read this dissertation prepared under my direction and recommend that it may be accepted as fulfilling the dissertation requirement.

____ **Date :**

Guide: Prof. Parameswaran Sankaran

STATEMENT BY AUTHOR

This dissertation has been submitted in partial fulfillment of requirements for an advanced degree at Homi Bhabha National Institute (HBNI) and is deposited in the Library to be made available to borrowers under rules of the HBNI.

Brief quotations from this dissertation are allowable without special permisiion, provided that accurate acknowledgement of source is made. Requests for permisiion for extended quotation from or reproduction of this manuscriptin whole or in part may be granted by the Competent Authority of HBNI when in his or her judgement the proposed use of the material is in the interests of scholarship. In all other instances, however, permission must be obtained from the author.

Pampa Paul

DECLARATION

I, hereby declare that the investigtion presented in the thesis has been carried out by me. The work is original and has notbeen submitted eralier as a whole or in part for a degree / diploma at this or any other Institution / University.

Pampa Paul

ABSTRACT

Let G_0 be a simply connected non-compact real simple Lie group with maximal compact subgroup K_0. Let $T_0 \subset K_0$ be a maximal torus. Assume that $\mathrm{rank}(G_0) = \mathrm{rank}(K_0)$ so that G_0 has discrete series representations. We denote by $\mathfrak{g}, \mathfrak{k}$, and $\mathfrak{t}$, the complexifications of the Lie algebras $\mathfrak{g}_0, \mathfrak{k}_0$ and $\mathfrak{t}_0$ of G_0, K_0 and T_0 respectively. Denote by Δ the root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$. There exists a positive root system known as the Borel-de Siebenthal positive system such that there is exactly one non-compact simple root, denoted ν. We assume that G_0/K_0 is not Hermitian. In this case one has a partition $\Delta = \cup_{-2\leq i\leq 2}\Delta_i$ where $\alpha \in \Delta$ belongs to Δ_i precisely when the coefficient of ν in α when expressed as a sum of simple roots is equal to i. Let G be the simply connected complexification of G_0. Denote by L_0 and $\bar{L}_0$, the centralizer in K_0 of a certain circle subgroup S_0 of T_0 and its image in G (under the homomorphism $p : G_0 \longrightarrow G$ defined by the inclusion $\mathfrak{g}_0 \hookrightarrow \mathfrak{g}$) respectively so that the root system of (L_0, T_0) is Δ_0. Any $\bar{L}_0$-representation is regarded as an L_0-representation via p.

Let γ be the highest weight of an irreducible representation of $\bar{L}_0$ such that $\gamma + \rho_{\mathfrak{g}}$ is negative on $\Delta_1 \cup \Delta_2$. Here $\rho_{\mathfrak{g}}$ denotes half the sum of positive roots of $\mathfrak{g}$. Then $\gamma+\rho_{\mathfrak{g}}$ is the Harish-Chandra parameter of a discrete series representation $\pi_{\gamma+\rho_{\mathfrak{g}}}$ of G_0 called a Borel-de Siebenthal discrete series representation. The K_0-finite part of $\pi_{\gamma+\rho_{\mathfrak{g}}}$ is admissible for a simple factor $K_1 \subset K_0$. It turns out that $S_0 \subset K_1$ and $K_1/L_1 = K_0/L_0$ is a Hermitian symmetric space where $L_1 = L_0 \cap K_1$. One has a Hermitian symmetric pair of non-compact type $(K_0^*, \bar{L}_0)$ dual to the pair (K_0, L_0). The element γ also determines a holomorphic discrete series representation $\pi_{\gamma+\rho_{\mathfrak{k}}}$ of K_0^*.

In this thesis we address the following question: Does there exist common L_0-types between the Borel-de Siebenthal discrete series representation $\pi_{\gamma+\rho_{\mathfrak{g}}}$ and the holomorphic discrete series representation $\pi_{\gamma+\rho_{\mathfrak{k}}}$? We settle this question completely in the quaternionic case, namely, when $\mathfrak{k}_1 \cong \mathfrak{su}(2)$. In the general case, affirmative answer is obtained under the following two hypotheses—(i) there exists a (non-constant) relative invariant for the prehomogeneous space $(L_0^{\mathbb{C}}, \mathfrak{u}_1)$, where $\mathfrak{u}_1$ is the representation of L_0 on the normal space at the identity coset for the (holomorphic) imbedding $K_0/L_0 \hookrightarrow G_0/L_0$, and, (ii) the longest element $w_{\mathfrak{k}}^0$ of the Weyl group of K_0 normalizes L_0. The proof uses, among others, a decomposition theorem of Schmid and Littelmann's path model.

ACKNOWLEDGEMENT

First I thank my advisor Dr. Parameswaran Sankaran and my co-advisor Dr. K.N. Raghavan for their guidance in completing this thesis. It would not have come to existence without their help.

I take this opportunity to thank Dr. R. Parthasarathy, Dr. D.S. Nagaraj and Dr. Pralay Chatterjee for their support and encouragement.

I would like to thank my M.Sc. teachers Dr. Khitish Chattopadhyay and Dr. Mantu Saha for their support and encouragement.

I thank all the academic, administrative and technical members of the Institute of Mathematical Sciences for giving a conducive environment to pursue research.

I thank my friends specially Madhushree and Mubeena for their support.

Last but not the least I thank my family for giving me the freedom to pursue research. This thesis is dedicated to them.

Contents

LIST OF PUBLICATION(S)

Pampa Paul, K.N. Raghavan, Parameswaran Sankaran, L_0-types common to a Borel-de Siebenthal discrete series and its associated holomorphic discrete series, C. R. Acad. Sci. Paris **350** (2012), no. 23–24, 1007–1009 .

Chapter 1

INTRODUCTION

Let G_0 be a simply connected non-compact real simple Lie group with maximal compact subgroup K_0. Assume that $\text{rank}(G_0) = \text{rank}(K_0)$ so that G_0 has discrete series representations. If G_0/K_0 is Hermitian symmetric, one has a relatively simple discrete series of G_0, namely the holomorphic discrete series of G_0. Now assume that G_0/K_0 is not Hermitian symmetric space. In this case, one has the class of Borel-de Siebenthal discrete series of G_0 defined in a manner analogous to the holomorphic discrete series. Let L_0 be the centralizer in K_0 of a certain circle subgroup of K_0. It turns out that K_0/L_0 is an irreducible compact Hermitian symmetric space. See §2.4.3 of Chapter 2. Let K_0^* be the dual of K_0 with respect to L_0. Then K_0^*/L_0 is an irreducible non-compact Hermitian symmetric space dual to K_0/L_0.

In this thesis, to each Borel-de Siebenthal discrete series representation of G_0, we will associate a holomorphic discrete series representation of K_0^*. See §3.2 of Chapter 3. The main aim of this thesis is to compare the restrictions to the compact subgroup L_0 of G_0 which is also a maximal compact subgroup of K_0^*, of a Borel-de Siebenthal discrete series representation and its associated holomorphic discrete series representation under certain conditions. In fact we address the following question: Does there exist common L_0-types between a Borel-de Siebenthal discrete series representation and its associated holomorphic discrete series representation? We settle this question completely in the so called quaternionic case. See Theorem 1.0.1. In the general case, affirmative answer is obtained under the following two hypotheses—(i) there exists a (non-constant) relative invariant for the prehomogeneous space $(L_0^{\mathbb{C}}, \mathfrak{u}_1)$, where $\mathfrak{u}_1$ is the representation of L_0 on the normal space at the identity coset for the (holomorphic) imbedding $K_0/L_0 \hookrightarrow G_0/L_0$ (see §4.3 of Chapter 4), and, (ii) the longest element $w_{\mathfrak{k}}^0$ of the Weyl group of K_0 normalizes L_0. See Theorem 1.0.2. The proof uses, among others, a decomposition theorem of Schmid and Littelmann's path model which are discussed in §2.5 and §2.6 of Chapter 2 respectively. We also discuss L_0-admissibility of a Borel-de Siebenthal discrete series representation of G_0. See Proposition 1.0.3.

Borel-de Siebenthal Discrete Series

Let G_0 be a simply connected non-compact real simple Lie group and let K_0 be a maximal compact subgroup of G_0. Let $T_0 \subset K_0$ be a maximal torus. Assume that rank(K_0) =rank(G_0) so that G_0 has discrete series representations. Note that T_0 is a Cartan subgroup of G_0 as well. Also the condition rank(K_0)=rank(G_0) implies that K_0 is the fixed point set of a Cartan involution of G_0. We shall denote by $\mathfrak{g}_0, \mathfrak{k}_0$, and $\mathfrak{t}_0$ the Lie algebras of G_0, K_0, and T_0 respectively and by $\mathfrak{g}, \mathfrak{k}$, and $\mathfrak{t}$ the complexifications of $\mathfrak{g}_0, \mathfrak{k}_0$, and $\mathfrak{t}_0$ respectively. Let Δ be the root system of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{t}$. Let Δ^+ be a Borel-de Siebenthal positive system so that the set of simple roots Ψ has exactly one non-compact root ν. The Killing form B of $\mathfrak{g}$ determines a non-degenerate symmetric bilinear pairing $\langle\ ,\ \rangle : \mathfrak{t}^* \times \mathfrak{t}^* \longrightarrow \mathbb{C}$ which is normalized so that $\langle \nu, \nu\rangle = 2$.

When G_0/K_0 is a Hermitian symmetric space, one has a partition $\Delta = \cup_{-1\le i\le 1}\Delta_i$ where $\alpha \in \Delta$ belongs to Δ_i precisely when the coefficient $n_\nu(\alpha)$ of ν in α when expressed as a sum of simple roots is equal to i, and the set of compact and non-compact roots of $\mathfrak{g}_0$ are Δ_0 and $\Delta_1 \cup \Delta_{-1}$ respectively. Let $\Delta_0^\pm = \Delta^\pm \cap \Delta_0$. Then $\Delta^+ = \Delta_0^+ \cup \Delta_1$. The root system and the induced positive system of $(\mathfrak{k}, \mathfrak{t})$ are Δ_0 and Δ_0^+ respectively. If γ is the highest weight of an irreducible representation of K_0 such that $\langle \gamma + \rho_\mathfrak{g}, \alpha\rangle < 0$ for all $\alpha \in \Delta_1$, then $\gamma + \rho_\mathfrak{g}$ is the Harish-Chandra parameter of a holomorphic discrete series representation $\pi_{\gamma+\rho_\mathfrak{g}}$ of G_0. The K_0-finite part of $\pi_{\gamma+\rho_\mathfrak{g}}$ is described as $\oplus_{n\ge 0} E_\gamma \otimes S^n(\mathfrak{u}_{-1})$ where E_γ is the irreducible K_0-representation with highest weight γ, $\mathfrak{u}_{-1} = \oplus_{\alpha\in\Delta_{-1}}\mathfrak{g}_\alpha$, $\mathfrak{g}_\alpha$ being the root space for $\alpha \in \Delta$ and $S^n(\mathfrak{u}_{-1})$ is the n-th symmetric power of $\mathfrak{u}_{-1}$. See §2.4.2 of Chapter 2 and also [8], [21].

Assume that G_0/K_0 is *not* a Hermitian symmetric space. This is equivalent to the requirement that the centre of K_0 is discrete. Then there exists a partition $\Delta = \cup_{-2\le i\le 2}\Delta_i$ where $\alpha \in \Delta$ belongs to Δ_i precisely when the coefficient $n_\nu(\alpha)$ of ν in α when expressed as a sum of simple roots is equal to i. Denote by μ the highest root; then $\mu \in \Delta_2$. The set of compact and non-compact roots of $\mathfrak{g}_0$ are $\Delta_0 \cup \Delta_2 \cup \Delta_{-2}$ and $\Delta_1 \cup \Delta_{-1}$ respectively. Let $\Delta_0^\pm = \Delta^\pm \cap \Delta_0$. Then $\Delta^+ = \Delta_0^+ \cup \Delta_1 \cup \Delta_2$. The root system of $\mathfrak{k}$ is $\Delta_\mathfrak{k} = \Delta_0 \cup \Delta_2 \cup \Delta_{-2}$, and the induced positive system of $\Delta_\mathfrak{k}$ is obtained as $\Delta_\mathfrak{k}^+ = \Delta_0^+ \cup \Delta_2$. Let G be the simply connected complexification of G_0. The inclusion $\mathfrak{g}_0 \hookrightarrow \mathfrak{g}$ defines a homomorphism $p : G_0 \longrightarrow G$. Let $Q \subset G$ be the parabolic subgroup with Lie algebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}_{-1} \oplus \mathfrak{u}_{-2}$, where $\mathfrak{u}_i = \sum_{\alpha\in\Delta_i}\mathfrak{g}_\alpha$ $(-2 \le i \le 2)$, $\mathfrak{g}_\alpha$ being the root space for $\alpha \in \Delta$, and $\mathfrak{l} = \mathfrak{t} \oplus \mathfrak{u}_0$. Let L be the Levi subgroup of Q; thus $Lie(L) = \mathfrak{l}$. Then $\bar{L}_0 := p(G_0) \cap Q$ is a real form of L and $L_0 := p^{-1}(\bar{L}_0)$ is the centralizer in K_0 of a circle subgroup of T_0. Note that G_0/L_0 is an open orbit of the complex flag manifold G/Q, K_0/L_0 is an irreducible Hermitian symmetric space of compact type and $G_0/L_0 \longrightarrow G_0/K_0$ is a fibre bundle projection with fibre K_0/L_0.

The Borel-de Siebenthal discrete series of G_0, whose systematic study carried out by Ørsted and Wolf [19], is defined analogously to the holomorphic discrete series as follows: Let γ be the highest weight of an irreducible representation E_γ of $\bar{L}_0$ such that $\langle \gamma + \rho_\mathfrak{g}, \alpha\rangle < 0$ for all $\alpha \in \Delta_1 \cup \Delta_2$. Here $\rho_\mathfrak{g}$ denotes half the sum of positive roots of $\mathfrak{g}$. The Borel-de Siebenthal discrete series representation $\pi_{\gamma+\rho_\mathfrak{g}}$ is the discrete series representation of G_0 for which the Harish-Chandra parameter is $\gamma + \rho_\mathfrak{g}$. Ørsted and Wolf proved that the K_0-finite part of $\pi_{\gamma+\rho_\mathfrak{g}}$ is in fact K_1-admissible, where K_1 is the simple factor of K_0 corresponding to the simple ideal $\mathfrak{k}_1$ of $\mathfrak{k}_0$ such that $\mathfrak{k}_1^\mathbb{C}$ contains the root space

$\mathfrak{g}_\mu$. They described the K_0-finite part of $\pi_{\gamma+\rho_\mathfrak{g}}$ in terms of the Dolbeault cohomology as $\oplus_{m\geq 0} H^s(K_0/L_0; \mathbb{E}_\gamma \otimes \mathbb{S}^m(\mathfrak{u}_{-1}))$ where $s = \dim_\mathbb{C} K_0/L_0$, $\mathbb{E}_\gamma$ and $\mathbb{S}^m(\mathfrak{u}_{-1})$ denote the holomorphic vector bundles associated to the irreducible L_0-module E_γ and the m-th symmetric power $S^m(\mathfrak{u}_{-1})$ of the irreducible L_0-module $\mathfrak{u}_{-1}$ respectively. See Theorem 2.4.1 in Chapter 2.

R. Parthasarathy [20] obtained essentially the same description as above in a more general context that includes holomorphic and Borel-de Siebenthal discrete series as well as certain limits of discrete series representations. We give a brief description of his results in the §2.4.4 of Chapter 2.

The K_1-admissibility of the Borel-de Siebenthal discrete series also follows from the work of Kobayashi [15] who obtained a criterion for the admissibility of restriction of certain representations to reductive subgroups in a more general context.

Associated Holomorphic Discrete Series

Recall that K_0/L_0 is an irreducible compact Hermitian symmetric space. Let K be the connected Lie subgroup of G with Lie algebra $\mathfrak{k}$. Let K_0^* be the dual of $p(K_0)$ in K with respect to $\bar{L}_0$ so that $K_0^*/\bar{L}_0$ is the non-compact irreducible Hermitian symmetric space dual to K_0/L_0. Note that $\mathfrak{k} = Lie(K_0^*) \otimes_\mathbb{R} \mathbb{C}$ and that $\mathfrak{t} \subset \mathfrak{l}$ is a Cartan subalgebra of $\mathfrak{k}$. The sets of compact and non-compact roots of $(\mathfrak{k}, \mathfrak{t})$ are Δ_0 and $\Delta_2 \cup \Delta_{-2}$ respectively. $\Delta_\mathfrak{k}^+$ is a positive root system of $(\mathfrak{k}, \mathfrak{t})$. Let ϵ denote lowest element of Δ_2 (that is, $\beta \geq \epsilon$ for all $\beta \in \Delta_2$). Then the unique non-compact simple root of $\Delta_\mathfrak{k}^+$ is ϵ. The positive system $\Delta_\mathfrak{k}^+$ is a Borel-de Siebenthal positive system for K_0^*.

Since the space $K_0^*/\bar{L}_0$ is Hermitian symmetric, the group K_0^* admits holomorphic discrete series. See §2.4.2 of Chapter 2.

Let $\gamma + \rho_\mathfrak{g}$ be the Harish-Chandra parameter of a Borel-de Siebenthal discrete series representation of G_0. Thus γ is the highest weight of an irreducible $\bar{L}_0$-representation and $\langle \gamma + \rho_\mathfrak{g}, \beta \rangle < 0$ for all $\beta \in \Delta_1 \cup \Delta_2$. This implies $\langle \gamma + \rho_\mathfrak{k}, \beta \rangle < 0$ for all $\beta \in \Delta_2$. Here $\rho_\mathfrak{k}$ denotes half the sum of roots in $\Delta_\mathfrak{k}^+$. Thus, $\gamma + \rho_\mathfrak{k}$ is the Harish-Chandra parameter for a holomorphic discrete series representation $\pi_{\gamma+\rho_\mathfrak{k}}$ of K_0^*. This is the holomorphic discrete series representation associated to the Borel-de Siebenthal discrete series representation $\pi_{\gamma+\rho_\mathfrak{g}}$ of G_0. See §3.2 of Chapter 3 for details.

Main Results

It is a natural question to ask which L_0-types are common to the Borel-de Siebenthal discrete series representation $\pi_{\gamma+\rho_\mathfrak{g}}$ and the corresponding holomorphic discrete series representation $\pi_{\gamma+\rho_\mathfrak{k}}$. We shall answer this question completely when $\mathfrak{k}_1 \cong \mathfrak{su}(2)$, the so-called quaternionic case. See Theorem 1.0.1. In the non-quaternionic case, we obtain complete results assuming that (i) there exists a (non-constant) relative invariant for the prehomogeneous space $(L_0^\mathbb{C}, \mathfrak{u}_1)$ or equivalently, there exists a non-trivial one dimensional

L_0-subrepresentation in the symmetric algebra $S^*(\mathfrak{u}_{-1})$ and (ii) the longest element of the Weyl group of K_0 preserves Δ_0. See Theorem 1.0.2 below. Note that the second condition is trivially satisfied in the quaternionic case. The existence of non-trivial one dimensional L_0-submodule in the symmetric algebra $S^*(\mathfrak{u}_{-1})$ greatly simplifies the task of detecting occurrence of common L_0-types. The classification of Borel-de Siebenthal positive systems for which such one dimensional L_0-subrepresentations exist has been carried out by Ørsted and Wolf [19, §4].

We now state the main results of this thesis.

Theorem 1.0.1 *We keep the above notations. Suppose that Lie*$(K_1) \cong \mathfrak{su}(2)$. *If* $\mathfrak{g}_0 = \mathfrak{so}(4,1)$ *or* $\mathfrak{sp}(1,l-1), l > 1$, *then there are at most finitely many* L_0*-types common to* $\pi_{\gamma+\rho_{\mathfrak{g}}}$ *and* $\pi_{\gamma+\rho_{\mathfrak{k}}}$. *Moreover, if* $\dim E_\gamma = 1$ *then there are no common* L_0*-types.*

Suppose that $\mathfrak{g}_0 \neq \mathfrak{so}(4,1)$ *or* $\mathfrak{sp}(1,l-1), l > 1$. *Then each* L_0*-type in the holomorphic discrete series representation* $\pi_{\gamma+\rho_{\mathfrak{k}}}$ *occurs in the Borel-de Siebenthal discrete series representation* $\pi_{\gamma+\rho_{\mathfrak{g}}}$ *with* infinite *multiplicity.*

The Theorem 1.0.1 is proved in Chapter 6. The cases $G_0 = SO(4,1), Sp(1,l-1)$ are exceptional among the quaternionic cases in that these are precisely the cases for which the prehomogeneous space $(L_0^{\mathbb{C}}, \mathfrak{u}_1)$ has no (non-constant) relative invariants—equivalently $S^m(\mathfrak{u}_{-1}), m \geq 1$, has no one dimensional L_0-subrepresentation. In the non-quaternionic case, we have the following result.

Theorem 1.0.2 *With the above notations, suppose that (i)* $w^0_{\mathfrak{k}}(\Delta_0) = \Delta_0$ *where* $w^0_{\mathfrak{k}}$ *is the longest element of the Weyl group of* $\mathfrak{k}$ *(with respect to the positive system* $\Delta^+_{\mathfrak{k}}$*), and, (ii) there exists a* 1*-dimensional* L_0*-submodule in* $S^m(\mathfrak{u}_{-1})$ *for some* $m \geq 1$. *Then there are infinitely many* L_0*-types common to* $\pi_{\gamma+\rho_{\mathfrak{g}}}$ *and* $\pi_{\gamma+\rho_{\mathfrak{k}}}$ *each of which occurs in* $\pi_{\gamma+\rho_{\mathfrak{g}}}$ *with infinite multiplicity. Moreover, if* $\dim E_\gamma = 1$, *then* $\pi_{\gamma+\rho_{\mathfrak{k}}}$ *itself occurs in* $\pi_{\gamma+\rho_{\mathfrak{g}}}$ *with infinite multiplicity.*

The Theorem 1.0.2 is proved in Chapter 7. We recall, in Proposition 4.3.1, the Borel-de Siebenthal root orders for which condition (ii) of the above theorem holds. We obtain in Proposition 2.5.2 a criterion for condition (i) to hold. For the convenience of the reader we indicate the result in §4.2 in the non-quaternionic cases.

The second part of Theorem 1.0.1 is a particular case of Theorem 1.0.2 (when *Lie*$(K_1) \cong \mathfrak{su}(2)$, the common L_0-types are all in $\pi_{\gamma+\rho_{\mathfrak{k}}}$). The proof of Theorem 1.0.1 involves only elementary considerations. But the proof of Theorem 1.0.2 involves much deeper results and arguments.

The existence (or non-existence) of one dimensional L_0-submodules in $\oplus_{m\geq 1} S^m(\mathfrak{u}_{-1})$ is closely related to the L_0-admissibility of $\pi_{\gamma+\rho_{\mathfrak{g}}}$. Note that Theorem 1.0.2 implies that, under the condition $w^0_{\mathfrak{k}}(\Delta_0) = \Delta_0$, the restriction of the Borel-de Siebenthal discrete series representation is not L_0-admissible when $\sum_{m>0} S^m(\mathfrak{u}_{-1})$ has one dimensional subrepresentations. When $\mathfrak{k}_1 \cong \mathfrak{su}(2)$ and $\sum_{m>0} S^m(\mathfrak{u}_{-1})$ has no one dimensional submodule, the Borel-de Siebenthal discrete series representation is L_0-admissible. In fact we shall establish the following result which is proved in Chapter 5.

Proposition 1.0.3 *Suppose that $S^m(\mathfrak{u}_{-1})$ has a one dimensional L_0-subrepresentation for some $m \geq 1$, then the Borel-de Siebenthal discrete series representation $\pi_{\gamma+\rho_{\mathfrak{g}}}$ is not L_0'-admissible where $L_0' = [L_0, L_0]$. The converse holds if $\mathfrak{k}_1 \cong \mathfrak{su}(2)$.*

We also obtain, in §3.2 a result on the L_0' admissibility of the holomorphic discrete series representation $\pi_{\gamma+\rho_{\mathfrak{k}}}$ of K_0^*. Note that any holomorphic discrete series representation of K_0^* is L_0-admissible. (It is even T_0-admissible; see §2.4.2 or [21]).

Combining Theorems 1.0.1 and 1.0.2, we see that there are infinitely many L_0-types common to $\pi_{\gamma+\rho_{\mathfrak{g}}}$ and $\pi_{\gamma+\rho_{\mathfrak{k}}}$ whenever $S^m(\mathfrak{u}_{-1})$ has a one dimensional L_0-submodule for some $m \geq 1$ and $w_{\mathfrak{k}}^0(\Delta_0) = \Delta_0$.

We make use of the description of the K_0-finite part of the Borel-de Siebenthal discrete series, obtained by Ørsted and Wolf in terms of the Dolbeault cohomology of the flag variety K_0/L_0 with coefficients in the holomorphic bundle associated to the L_0-represenation $E_\gamma \otimes S^m(\mathfrak{u}_{-1})$. This will be recalled in §2.4.3 of Chapter 2. Proof of Theorem 1.0.2 crucially makes use of a result of Schmid [23] on the decomposition of the L_0-representation $S^m(\mathfrak{u}_{-2})$ and Littelmann's path model [17], [18].

Chapter 2

PRELIMINARIES

As described in the introduction, corresponding to a Borel-de Siebenthal discrete series representation of a simply connected non-compact real simple Lie group G_0, there exists a holomorphic discrete series representation of a connected non-compact semisimple real Lie group K_0^* dual to a maximal compact subgroup of G_0. The aim of this thesis is to compare the restrictions to a certain compact reductive subgroup L_0 of G_0 which is maximal compact in K_0^*, of a Borel-de Siebenthal discrete series representation and its associated holomorphic discrete series representation under certain conditions. For this purpose, we recall in this chapter certain well known definitions and results. In §2.1, we discuss some basic notions of representation theory including admissible representations and discrete series representations. In §2.2, we discuss Riemannian globally symmetric spaces and its duality and irreducibility. §2.3 deals with Hermitian symmetric spaces, particularly bounded symmetric domains. In §2.4, the notions of Borel-de Siebenthal positive system, the holomorphic discrete series, and the Borel-de Siebenthal discrete series are discussed. §2.5 deals with Schmid's theorem and its application. In §2.6, we discuss about Littelmann's path model. In this thesis, it is assumed that the reader is familiar with differentiable manifolds and Lie groups [24]; the structure of finite dimensional Lie algebras and the theory of finite dimensional representations of compact Lie groups ([10, Chapter III], [12]) as well as the abstract theory of compact groups [13, Sections 5, 6 of Chapter I].

2.1 Basic notions of representation theory

We follow [13] for this section.

Let G be a topological group. A **representation of** G on a complex Hilbert space $V(\neq 0)$ is a homomorphism $\Phi : G \longrightarrow B(V)^*$, $B(V)^*$ be the group of all bounded linear operators on V with bounded inverses, such that the action map $G \times V \longrightarrow V$ is continuous.

Let G be a locally compact topological group and $V = L^2(G, d_l x)$, where the measure is a left invariant Haar measure. For $g \in G$, define $\Phi(g)f(x) = f(g^{-1}x)$ for all $f \in V = L^2(G, d_l x)$. Then Φ is a representation of G on V, called the **left regular representation** of G. The **right regular representation** of G is given by $\Phi'(g)f(x) = f(xg)$ on $L^2(G, d_r x)$ (the measure is a right invariant Haar measure).

A vector subspace U of V is called **invariant** under Φ if $\Phi(g)U \subseteq U$ for all $g \in G$. The representation Φ is called **irreducible** if it has no closed invariant subspaces other than 0 and V.

The representation Φ is **unitary** if $\Phi(g)$ is a unitary operator on V for all $g \in G$. For a unitary representation the orthogonal complement $U^\perp$ of a closed invariant subspace U is a closed invariant subspace.

Two representations of G, Φ on V and Φ' on V', are **equivalent** if there is a bounded linear map $E : V \longrightarrow V'$ with bounded inverse such that $\Phi'(g)E = E\Phi(g)$ for all $g \in G$. If Φ and Φ' are unitary, they are **unitarily equivalent** if they are equivalent via an operator E that is unitary.

A **matrix coefficient** of Φ is a function $G \longrightarrow \mathbb{C}$ defined as $g \mapsto (\Phi(g)v, w)$, where $v, w \in V$ and $(\ , \ )$ is the inner product on V.

2.1.1 C^∞ vectors

Now assume that G is a Lie group and Φ is a representation of G on a Hilbert space V. Let $\mathfrak{g}$ be the Lie algebra of G.

A function $f : U \longrightarrow E$, where U is an open set in $\mathbb{R}^n$ and E is a topological vector space, is differentiable at $x_0 \in U$ if there is a (necessarily unique) linear map $f'(x_0) : \mathbb{R}^n \longrightarrow E$ such that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{\|x - x_0\|} = 0$$

Now $\mathrm{End}(\mathbb{R}^n, E)$ is a topological vector space in a natural way, since $\mathbb{R}^n$ is finite dimensional. If f is differentiable at each point of U, then $x \to f'(x)$ is a map from U into $\mathrm{End}(\mathbb{R}^n, E)$. We say that f is of class C^1 if $x \to f'(x)$ is continuous, of class C^2 if $x \to f'(x)$ is of class C^1, and so on. We say f is of **class** C^∞ if f is of class C^k for all $k \geq 1$.

The above definitions can be carried over to a smooth manifold in the obvious way.

A vector $v \in V$ is said to be a C^∞ **vector** for the representation Φ if $g \to \Phi(g)v$ is of class C^∞. The set of C^∞ vectors is denoted by $C^\infty(\Phi)$ (or V^∞). Evidently $C^\infty(\Phi)$ is a vector subspace of V.

Now we will associate to Φ, a representation ϕ of $\mathfrak{g}$ on $C^\infty(\Phi)$ as follows:
Let $v \in C^\infty(\Phi)$ and let

$$f(x) = \Phi(\exp X)v \quad \text{for} \quad X \in \mathfrak{g}$$

Then f is of class C^∞. Put

$$\phi(X)(v) = f'(0)(X)$$

Then

$$\begin{aligned}\phi(X)(v) &= f'(0)(c'_X(0)), \text{ where } c_X(t) = tX \ \text{ for } t \in \mathbb{R} \\ &= (f \circ c_X)'(0) \\ &= \lim_{t\to 0} \frac{f \circ c_X(t) - f \circ c_X(0)}{t} \\ &= \lim_{t\to 0} \frac{\Phi(\exp tX)v - v}{t}\end{aligned}$$

So $\phi(X)$ is a linear map from $C^\infty(\Phi)$ into V depends linearly on X.

The map ϕ has the following properties:
• $\phi(X)(C^\infty(\Phi)) \subset C^\infty(\Phi)$ for all $X \in \mathfrak{g}$ and $\phi : \mathfrak{g} \longrightarrow \mathrm{End}_{\mathbb{C}}(C^\infty(\Phi))$ is a representation of $\mathfrak{g}$.
• If Φ is unitary, then $\phi(X)$ is skew-Hermitian for all $X \in \mathfrak{g}$.
• $\Phi(g)(C^\infty(\Phi)) \subset C^\infty(\Phi)$ for $g \in G$ and $\Phi(g) \circ \phi(X) \circ \Phi(g)^{-1} = \phi(Ad(g)X)$ for $X \in \mathfrak{g}$ and $g \in G$.
• $C^\infty(\Phi)$ is dense in V.
See [13, Chapter III] for proofs of the above properties.

2.1.2 Admissible representations

Let $\mathfrak{g}$ be a reductive Lie algebra over $\mathbb{R}$. Let θ be an involutive automorphism of $\mathfrak{g}$ and $\mathfrak{k}$ and $\mathfrak{p}$ be the subspaces of $\mathfrak{g}$ corresponding to the eigenvalues 1 and -1 respectively. Assume that $\theta|_{[\mathfrak{g},\mathfrak{g}]}$ is a Cartan involution of $[\mathfrak{g},\mathfrak{g}]$. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and K the connected Lie subgroup of G corresponding to $\mathfrak{k}$. Assume :
(i) K is compact,
(ii) The map $(k, X) \mapsto k.\exp X (k \in K, X \in \mathfrak{p})$ is a diffeomorphism of $K \times \mathfrak{p}$ onto G.
Note that with the above assumptions, K is a maximal compact subgroup of G.
In this thesis, by a connected reductive Lie group G with maximal compact subgroup K, we always mean that G and K satisfy the conditions given above. See [25, Section 1.1.5] for detailed exposition. Note that these conditions are satisfied when G is a finite cover of a connected reductive *linear* Lie group.

Let G be a connected reductive Lie group with maximal compact subgroup K. Let π be a representaton of G on a Hilbert space V. A vector $v \in V$ is called K**-finite** if $\pi(K)v$ spans a finite dimensional subspace of V. Let V_K denote the subspace of K-finite vectors in V. The associated representation of $\mathfrak{g}$ on V^∞ is denoted by the same notation π. We set $V_0 = V^\infty \cap V_K$. Then $\pi(X)(V_0) \subset V_0$ for all $X \in \mathfrak{g}$. Consequently a representation π leads to a representation of $\mathfrak{g}$ on V_0. Also V_0 is a K-representation in such a way that $\pi(k)\pi(X)(v) = \pi(Ad(k)X)\pi(k)(v)$ for $k \in K, X \in \mathfrak{g}, v \in V_0$. The representation V_0 is said to be the associated $(\mathfrak{g}, K)$**-module** of π.

When K acts by unitary operators, by the Peter-Weyl Theorem, we have

$$\pi|_K \cong \sum_{\tau \in \hat{K}} n_\tau \tau \tag{2.1}$$

where the sum is a Hilbert sum, $\hat{K}$ is the unitary dual of K, that is, the set of equivalence classes of irreducible unitary representations of K and n_τ is the multiplicity of τ in $\pi|_K$.

Note that $n_\tau = \dim(\mathrm{Hom}_K(\tau, \pi|_K))$ and is a non-negative integer or is $+\infty$. The equivalence classes τ with $n_\tau \neq 0$ are called the **K-types** that occur in π. It is obvious from (2.1) that the subspace of all K-finite vectors is dense.

A representation π of a connected reductive Lie group G on a Hilbert space V is called **admissible** if K acts by unitary operators and if each $\tau \in \hat{K}$ occurs with finite multiplicity in $\pi|_K$.

Theorem 2.1.1 *[13, Th 8.1, Ch. VIII] Let π be an irreducible unitary representation of a connected reductive Lie group G on a Hilbert space V. Then the multiplicity n_τ of the K-type τ in $\pi|_K$ satisfies $n_\tau \leq \dim \tau$ for every $\tau \in \hat{K}$.*

So, by the above theorem, irreducible unitary representations are admisible.

For an admissible representation π, every K-finite vector is a C^∞ vector that is $V_0 = V_K$, and $\pi(X)(V_K) \subseteq V_K$ for all $X \in \mathfrak{g}$. For an admissible representation π, V_K is the associated $(\mathfrak{g}, K)$-module of π.

Two admissible representations π and π' of G are called **infinitesimally equivalent** if the associated $(\mathfrak{g}, K)$-modules of K-finite vectors are algebraically equivalent (that is, if there is a linear isomorphism commuting with the action of $\mathfrak{g}$).

If π is an admissible representation of G on V and U is a closed G-invariant subspace of V, then evidently the K-finite vectors in U form a $\mathfrak{g}$-invariant subspace dense in U. The following theorem suggests a converse result. (Note that, the closure of a $\mathfrak{g}$-invariant subspace of C^∞ vectors need not be G-invariant. For example, consider the left regular representation of $\mathbb{R}$ on $L^2(\mathbb{R})$. Then $U :=$ the subspace of members of $C^\infty_{\mathrm{com}}(\mathbb{R})$ with support in $[0, 1]$, is a subspace of C^∞ vectors for the left regular representation of $\mathbb{R}$. U is invariant under the Lie algebra action but the closure of U in $L^2(\mathbb{R})$ is not invariant under the group action).

Theorem 2.1.2 *[13, Th. 8.9, Ch. VIII] If G is a connected reductive Lie group and π is an admissible representation of G on a Hilbert space V, then the closure in V of any $\mathfrak{g}$-invariant subspace of V_K is G-invariant.*

As a corollary we obtain the following:

Corollary 2.1.3 *[13, Cor. 8.10, Ch. VIII] If π is an admissible representation of G on V, then the closed G-invariant subspaces U of V are in one-one correspondence with the $\mathfrak{g}$-invariant subspaces U_K of V_K, the correspondence being $U_K = U \cap V_K$ and $U = \bar{U}_K$.*

Hence for an admissible representation π of G on V, $\pi(G)$ has no non-trivial closed invariant subspace in V *if and only if* $\pi(\mathfrak{g})$ has no non-trivial invariant subspace in V_K. The representation π is called **irreducible admissible** if any one of the equivalent conditions is satisfied for π.

If π is an admissible representation of G on V, then for $u \in V_K$ and $X \in \mathfrak{g}$ (regarded X as a left invariant vector field on G),

$$\begin{aligned}
X(\pi(g)u, v) &= Xc_{u,v}(g), \text{ where } c_{u,v}(g) = (\pi(g)u, v) \text{ for } g \in G \\
&= X_g(c_{u,v}) \\
&= d\gamma_g\Big(\frac{d}{dt}\Big|_{t=0}\Big)(c_{u,v}), \text{ where } \gamma_g(t) = g \exp tX \text{ for } t \in \mathbb{R} \\
&= \frac{d}{dt}\Big|_{t=0}(c_{u,v} \circ \gamma_g) \\
&= \frac{d}{dt}(\pi(g \exp tX)u, v)\Big|_{t=0} \\
&= \frac{d}{dt}(\pi(g)\pi(\exp tX)u, v)\Big|_{t=0} \\
&= \frac{d}{dt}(\pi(\exp tX)u, \pi(g)^* v)\Big|_{t=0} \\
&= (\pi(g)\pi(X)u, v)
\end{aligned}$$

Hence

$$D(\pi(g)u, v) = (\pi(g)\pi(D)u, v) \tag{2.2}$$

for all $D \in U(\mathfrak{g}^{\mathbb{C}})$, where $U(\mathfrak{g}^{\mathbb{C}})$ denotes the universal enveloping algebra of the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$.

A matrix coefficient $g \mapsto (\pi(g)u, v)$ $(u, v \in V)$ is said to be ***K*-finite** if $u, v \in V_K$. Equation (2.2) gives the action of $U(\mathfrak{g}^{\mathbb{C}})$ on K-finite matrix coefficients.

If π and π' are infinitesimally equivalent admissible representations of G, then they have the same set of K-finite matrix coefficients. Conversely, if π and π' are irreducible admissible representations of G with a non-zero K-finite matrix coefficient in common, then they are infinitesimally equivalent. See [13, Cor. 8.8 and Cor. 8.12 in Ch. VIII].

If π is an irreducible admissible representation of G on V and $L : V_K \longrightarrow V_K$ is a linear operator commuting with $\pi(\mathfrak{g})$, then by Schur lemma, L is scalar. Hence for an irreducible admissible representation π of G, each member of the centre $Z(\mathfrak{g}^{\mathbb{C}})$ of $U(\mathfrak{g}^{\mathbb{C}})$ acts as a scalar operator on the space of K-finite vectors of π. In fact there exists an algebra homomorphism $\chi_\pi : Z(\mathfrak{g}^{\mathbb{C}}) \longrightarrow \mathbb{C}$ such that $\pi(z) = \chi_\pi(z).\mathrm{Id}$ for all $z \in Z(\mathfrak{g}^{\mathbb{C}})$. The homomorphism χ_π is called the **infinitesimal character** of π. The action of $U(\mathfrak{g}^{\mathbb{C}})$ on K-finite matrix coefficients given by the equation (2.2) suggests that the K-finite matrix coefficients of an irreducible admissible representation are eigenfunctons of $Z(\mathfrak{g}^{\mathbb{C}})$.

An admissible representation π of G on a Hilbert space V is said to be **infinitesimally unitary** if V_K admits an inner product with respect to which $\pi(\mathfrak{g})$ acts by skew-Hermitian operators. Evidently a unitary representation is infinitesimally unitary. There is one-one correspondence between irreducible unitary representations upto unitary equivalence and infinitesimally unitary irreducible admissible representations upto infinitesimal equivalence. See [13, Cor. 9.2, Th. 9.3 in Ch. IX].

2.1.3 Verma module and Harish-Chandra isomorphism

Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Let Δ denote the set of non-zero roots of $(\mathfrak{g}, \mathfrak{h})$. Choose a positive system Δ^+ of Δ. Define $\mathfrak{n}_+ := \sum_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$ and $\mathfrak{n}_- := \sum_{\alpha\in\Delta^+} \mathfrak{g}_{-\alpha}$, $\mathfrak{g}_\alpha$ being the root space for $\alpha \in \Delta$. Then we have, $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Let $\mathfrak{b}$ denote the Borel subalgebra $\mathfrak{h} \oplus \mathfrak{n}_+$ and let $\rho := \frac{1}{2}\sum_{\alpha\in\Delta^+} \alpha$. For any $\lambda \in \mathfrak{h}^*$, let $\mathbb{C}_{\lambda-\rho}$ denote the one dimensional $\mathfrak{h}$-module on which $\mathfrak{h}$ acts by the function $\lambda - \rho$. Then $\mathbb{C}_{\lambda-\rho}$ is a $\mathfrak{b}$-module by extending the action of $\mathfrak{n}_+$ trivally on $\mathbb{C}_{\lambda-\rho}$. Hence $\mathbb{C}_{\lambda-\rho}$ is a left $U(\mathfrak{b})$-module, where $U(\mathfrak{b})$ denotes the universal enveloping algebra of $\mathfrak{b}$. Note that the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ is a right $U(\mathfrak{b})$-module with the usual multiplication. Define $V(\lambda) := U(\mathfrak{g})\otimes_{U(\mathfrak{b})}\mathbb{C}_{\lambda-\rho}$. Then $V(\lambda)$ is a left $U(\mathfrak{g})$-module and hence is a $\mathfrak{g}$-module and is called a **Verma module** with highest weight $\lambda - \rho$. By PBW theorem, we have $U(\mathfrak{g}) \cong U(\mathfrak{n}_-) \otimes_{\mathbb{C}} U(\mathfrak{b})$ as vector spaces. Hence $V(\lambda)$ is isomorphic to $U(\mathfrak{n}_-) \otimes_{\mathbb{C}} \mathbb{C}_{\lambda-\rho}$ as a vector space. Note that $\lambda - \rho$ is the highest weight of $V(\lambda)$ and its multiplicity is 1 in $V(\lambda)$. Also $V(\lambda)$ has a unique irreducible quotient and we denote it by $L(\lambda)$. *If $\lambda - \rho$ is a dominant integral weight, then $L(\lambda)$ is the finite dimensional irreducible $\mathfrak{g}$-module with highest weight $\lambda - \rho$.*

Let $Z(\mathfrak{g})$ denote the centre of $U(\mathfrak{g})$ and let v be a non-zero element of $\mathbb{C}_{\lambda-\rho}$. Then for any $z \in Z(\mathfrak{g})$, $z.(1 \otimes v)$ is a weight vector in $V(\lambda)$ of weight $\lambda - \rho$. Since the multiplicity of $\lambda - \rho$ is 1 in $V(\lambda)$, we have $z.(1 \otimes v)$ is a scalar multiple of $1 \otimes v$. Hence there exists a function $\chi_\lambda : Z(\mathfrak{g}) \longrightarrow \mathbb{C}$ such that $z.(1\otimes v) = \chi_\lambda(z)(1\otimes v)$ for all $z \in Z(\mathfrak{g})$. Note that χ_λ is a algebra homomorphism and is called the **character** determined by λ. Again since $Z(\mathfrak{g})$ is the centre of $U(\mathfrak{g})$ and $V(\lambda) = U(\mathfrak{g}).(1 \otimes v)$, $Z(\mathfrak{g})$ acts on $V(\lambda)$ by the character χ_λ. Hence $Z(\mathfrak{g})$ acts on any submodule and quotient module of $V(\lambda)$ by the same character χ_λ. So if $\lambda - \rho$ is a dominant integral weight, then $Z(\mathfrak{g})$ acts on the irreducible finite dimensional $\mathfrak{g}$-module $L(\lambda)$ by the character χ_λ.

For any $\alpha \in \Delta$, choose $E_\alpha(\neq 0) \in \mathfrak{g}_\alpha$. Define

$$\mathcal{P} := \sum_{\alpha\in\Delta^+} U(\mathfrak{g})E_\alpha, \text{ and}$$
$$\mathcal{N} := \sum_{\alpha\in\Delta^+} E_{-\alpha}U(\mathfrak{g}).$$

Then we have,

Theorem 2.1.4 *[14, Prop. 5.34, Ch. V] (i) $U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathcal{P} + \mathcal{N})$, where $U(\mathfrak{h})$ denote the universal enveloping algebra of $\mathfrak{h}$.*

(ii) Also any member of $Z(\mathfrak{g})$ has its $\mathcal{P} + \mathcal{N}$ component in $\mathcal{P}$.

Let $\bar{\gamma} : Z(\mathfrak{g}) \longrightarrow U(\mathfrak{h})$ be the projection map on $U(\mathfrak{h})$ component. Define a linear map $\tau : \mathfrak{h} \longrightarrow U(\mathfrak{h})$ by

$$\tau(H) = H - \rho(H)1 \quad \text{for all} \quad H \in \mathfrak{h}.$$

Then τ can be extended to an algebra homomorphism on $U(\mathfrak{h})$, by the universal property of $U(\mathfrak{h})$ and we denote the extended map on $U(\mathfrak{h})$ by the same notation τ. The **Harish-Chandra map** γ is defined by

$$\gamma = \tau \circ \bar{\gamma}.$$

Theorem 2.1.5 *(Harish-Chandra) [14, Th. 5.44, Ch. V] The Harish-Chandra map γ is an algebra isomorphism of $Z(\mathfrak{g})$ onto the algebra $U(\mathfrak{h})^W$, where W is the Weyl group of* $(\mathfrak{g}, \mathfrak{h})$ *and* $U(\mathfrak{h})^W := \{H \in U(\mathfrak{h}) \ : \ wH = H \text{ for } w \in W\}$.

Note that for any $\lambda \in \mathfrak{h}^*$, $\lambda(\gamma(z)) = (\lambda - \rho)(\bar{\gamma}(z))$ for all $z \in Z(\mathfrak{g})$ (here we have taken the algebra homomorphism $U(\mathfrak{h}) \longrightarrow \mathbb{C}$ defined by λ, by the universal property of $U(\mathfrak{h})$). In view of Theorem 2.1.4, we have $\chi_\lambda(z) = \lambda(\gamma(z))$ for all $z \in Z(\mathfrak{g})$. Hence for $\lambda, \mu \in \mathfrak{h}^*$, we have $\chi_\lambda = \chi_\mu$ *if and only if* $\mu = w\lambda$ for some $w \in W$, using Theorem 2.1.5 and some little work. Also any algebra homomorphism $Z(\mathfrak{g}) \longrightarrow \mathbb{C}$ is of the form χ_λ for some $\lambda \in \mathfrak{h}^*$. See [14, Th 5.62, Ch. V].

2.1.4 Discrete series representations

Let G be a connected reductive Lie group with maximal compact subgroup K. For an irreducible unitary representation π of G on V, the following conditions are equivalent : [13, Prop. 9.6, Ch. IX]

(1) Some non-zero K-finite matrix coefficient of π is in $L^2(G)$.

(2) All the matrix coefficients of π are in $L^2(G)$.

(3) The representation π is equivalent to a direct summand of the left regular representation of G on $L^2(G)$.

When these conditions are satisfied π is said to be **square integrable** and we say that π is a **discrete series representation**. By definition, a **discrete series** of G is the equivalence class of an irreducible unitary square integrable representation of G.

When π is a discrete series representation of G, there exists a positive number d_π such that

$$\int_G (\pi(x)u_1, v_1)\overline{(\pi(x)u_2, v_2)}\, dx = d_\pi^{-1}(u_1, u_2)\overline{(v_1, v_2)}$$

for all $u_1, u_2, v_1, v_2 \in V$. d_π is called the **formal degree** of π.

For G compact, every irreducible unitary representation is a discrete series representation and is finite dimensional. If Haar measure has total mass 1, then the formal degree is the degree of the representation, by Schur orthogonality. See [13, Section 5 of Chapter I].

If π is a discrete series representation of G, then the Plancherel measure for the decomposition of $L^2(G)$ assigns mass d_π to the one point set $\{\pi\}$ in the unitary dual $\hat{G}$ and vice-versa. See [7].

Recall that the rank of a Lie group G is, by definition, the dimension of any Cartan subalgebra of Lie(G).

Theorem 2.1.6 *[13, Th 12.20, Ch. XII] Let G be a connected semisimple Lie group with finite centre and let K be a maximal compact subgroup of G. Then G has discrete series representations if and only if rank (G) = rank (K).*

Note that if G admits a discrete series representation, then G cannot be a complex Lie group, since rank $(G) = 2$ rank (K).

Let G be a connected semisimple Lie group with finite centre. Let K be a maximal compact subgroup of G. Assume that rank (G) = rank (K). Denote by $\mathfrak{g}$, the Lie algebra of G and by $\mathfrak{k} \subset \mathfrak{g}$, the Lie algebra of K. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition. Let $\mathfrak{t} \subset \mathfrak{k}$ be a maximal abelian subalgebra of $\mathfrak{k}$. Then $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$ as well. Let

$$\Delta = \text{roots of } (\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}}), \text{ and}$$

$$\Delta_{\mathfrak{k}} = \text{roots of } (\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}}).$$

Since root spaces are one dimensional and

$$[\mathfrak{t}^{\mathbb{C}}, \mathfrak{k}^{\mathbb{C}}] \subseteq \mathfrak{k}^{\mathbb{C}} \text{ and } [\mathfrak{t}^{\mathbb{C}}, \mathfrak{p}^{\mathbb{C}}] \subseteq \mathfrak{p}^{\mathbb{C}},$$

each root space is contained either in $\mathfrak{k}^{\mathbb{C}}$ or in $\mathfrak{p}^{\mathbb{C}}$. The roots in Δ are called **compact** or **non-compact** accordingly. Clearly $\Delta_{\mathfrak{k}}$ is the set of compact roots. Let Δ_n be the set of non-compact roots. That is, $\Delta_n = \Delta \setminus \Delta_{\mathfrak{k}}$. Let $W_{\mathfrak{g}}$ and $W_{\mathfrak{k}}$ be the Weyl groups of Δ and $\Delta_{\mathfrak{k}}$ respectively. Then $W_{\mathfrak{k}} \subset W_{\mathfrak{g}}$. Let $\langle\ ,\ \rangle$ be the positive definite symmetric bilinear form on $(i\mathfrak{t})^*$ induced from the Killing form of $\mathfrak{g}^{\mathbb{C}}$.

Theorem 2.1.7 *[13, Th. 9.20 in Ch. IX, Th. 12.21 in Ch. XII] Let G be a connected semisimple Lie group with finite centre and K be a maximal compact subgroup of G. Assume that rank (G) = rank (K). Let $\lambda \in (i\mathfrak{t})^*$ be* **non-singular** *relative to Δ, that is, $\langle\lambda, \alpha\rangle \neq 0$ for all $\alpha \in \Delta$. Define Δ^+ by*

$$\Delta^+ := \{\alpha \in \Delta \ :\ \langle\lambda, \alpha\rangle > 0\} \tag{2.3}$$

Define $\Delta_{\mathfrak{k}}^+ = \Delta^+ \cap \Delta_{\mathfrak{k}}$. Let

$$\rho_{\mathfrak{g}} = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha \ \text{ and } \ \rho_{\mathfrak{k}} = \frac{1}{2}\sum_{\alpha\in\Delta_{\mathfrak{k}}^+}\alpha.$$

If $\lambda + \rho_{\mathfrak{g}}$ is **analytically integral** *(that is, $\lambda + \rho_{\mathfrak{g}}$ is the differential of a smooth function on the Cartan subgroup of G corresponding to $\mathfrak{t}$), then there exists a discrete series representation π_λ of G with the following properties :*

(a) π_λ has infinitesimal character χ_λ (recall from §2.1.3 that χ_λ is the character of the Verma module of $\mathfrak{g}^{\mathbb{C}}$ with highest weight $\lambda - \rho_{\mathfrak{g}}$).

(b) $\pi_\lambda|_K$ contains with multiplicity one the K-type with highest weight

$$\Lambda = \lambda + \rho_{\mathfrak{g}} - 2\rho_{\mathfrak{k}}.$$

(c) If Λ' is the highest weight of a K-type in $\pi_\lambda|_K$, then Λ' is of the form

$$\Lambda' = \Lambda + \sum_{\alpha \in \Delta^+} n_\alpha \alpha \ \text{ for integers } n_\alpha \geq 0.$$

Two such representations π_λ and $\pi_{\lambda'}$ are unitarily equivalent if and only if $\lambda = w\lambda'$ for some $w \in W_{\mathfrak{k}}$.
Upto equivalence these are the all discrete series representations of G.

The λ as above, is called the **Harish-Chandra parameter** and Λ is called the **Blattner parameter** of the discrete series representation π_λ of G. The positive system Δ^+ defined by the equation (2.4.2) is called the **Harish-Chandra root order** corresponding to λ.

All the parameters $w\lambda$ for $w \in W_{\mathfrak{g}}$ give the same infinitesimal character. According to the theorem, exactly $|W_{\mathfrak{g}}|/|W_{\mathfrak{k}}|$ of the discrete series representations $\pi_{w\lambda}$ are mutually inequivalent.

2.2 Riemannian symmetric spaces

We follow [10] for this section.

Let M be a connected Riemannian manifold. M is called **Riemannian globally symmetric** if each $p \in M$ is an isolated fixed point of an involutive isometry s_p of M.

Examples

(i) $\mathbb{R}^n$ ($n \geq 1$) with the usual metric, is a Riemannian globally symmetric space. For $p \in \mathbb{R}^n$, s_p is given by $s_p(x) = 2p - x$ for all $x \in \mathbb{R}^n$.

(ii) S^n ($n \geq 1$) with the Riemannian metric induced from $\mathbb{R}^{n+1}$, is a Riemannian globally symmetric space. Let $v_0 \in S^n$ and $\{v_0, v_1, \ldots, v_n\}$ be an orthonormal basis of $\mathbb{R}^{n+1}$ extending v_0. Define $s_{v_0} : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$ by $s_{v_0}(v_0) = v_0$, $s_{v_0}(v_i) = -v_i$ ($1 \leq i \leq n$) and then extend linearly. Then $s_{v_0}(S^n) = S^n$ and v_0 is an isolated fixed point of the involutive isometry $s_{v_0}|_{S^n}$ of S^n.

(iii) The upper half plane $= \{z \in \mathbb{C} : \ \mathrm{Im} z > 0\}$ with the Poincaré metric is Riemannian globally symmetric. For the point i, $s_i : z \mapsto -\frac{1}{z}$ is an involutive isometry. Since the isometry group $\mathrm{PSL}_2(\mathbb{R})$ of the upper half plane acts transitively, so each point is an isolated fixed point of an involutive isometry.

Let M be a connected Riemannian manifold. Let N_0 be a neighbourhood of 0 in $\mathcal{T}_pM$ (where $\mathcal{T}_pM$ is the tangent space of M at p) such that if $v \in N_0$, then $tv \in N_0$ for all $t \in [-1, 1]$ and Exp_p is a diffeomorphism of N_0 onto a neighbourhood of p in M (See [10, Chapter I] or [5, Chapter 3]). Let $N_p := \mathrm{Exp}_p N_0$. For each $q \in N_p$ there exists a geodesic $\gamma(t)$ in N_p with $\gamma(0) = p$ and $\gamma(1) = q$. Put $q' = \gamma(-1)$. Then the mapping $s_p : N_p \longrightarrow N_p$ defined by $q \mapsto q'$ is a diffeomorphism and is called the **geodesic symmetry** with respect to p. Note that ${s_p}^2 = \mathrm{Id}$ and $(ds_p)_p = -\mathrm{Id}$. If for every $p \in M$, s_p is an isometry, then M is called a **Riemannian locally symmetric space**.

If M is a Riemannian globally symmetric space, then for $p \in M$, an involutive isometry s_p is the geodesic symmetry on a normal neighbourhood N_p of p (that is N_p is a neighbourhood of p defined as above). So there is only one such s_p and M is a Riemannian locally symmetric space. Also M is a complete Riemannian manifold. Let $I(M)$ denote the group of isometries of M. With the compact-open topology, $I(M)$ is a Lie group and the action of $I(M)$ on M is smooth. Since M is complete any two points $p, q \in M$ can be joined by a minimal geodesic. If m is the midpoint of this geodesic, then $s_m(p) = q$. In particular $I(M)$ acts transitively on M. Let $I_0(M)$ denote the connected component of M. Since M is connected, $I_0(M)$ itself acts transitively on M. If $p \in M$ and K denotes the isotropy subgroup of $I_0(M)$ at p, then $I_0(M)/K$ is diffeomorphic to M. Also K is compact and there exists an involutive automorphism $\sigma : G \longrightarrow G$ defined by

$$\sigma(g) = s_p g s_p \ \text{ for all } \ g \in G$$

such that $(K_\sigma)_0 \subset K \subset K_\sigma$, where K_σ is the subgroup of G of all fixed points of σ and $(K_\sigma)_0$ is the connected component of K_σ. The group K contains no normal subgroup of G other than $\{e\}$. See [10, Th. 3.3, Ch. IV].

2.2.1 Riemannian symmetric pair

Let G be a connected Lie group and K a closed subgroup of G. The pair (G, K) is called a **Riemannian symmetric pair** if

(i) there exists an involutive automorphism σ of G such that $(K_\sigma)_0 \subset K \subset K_\sigma$, where K_σ is the set of fixed points of σ and $(K_\sigma)_0$ is the connected component of K_σ, and

(ii) $\mathrm{Ad}_G(K)$ is compact.

If (G, K) is a Riemannian symmetric pair, then in each G-invariant Riemannian structure on G/K (such Riemannian structure exist), the manifold G/K is a Riemannian globally symmetric space. The involutive isometry s_0 at $0 = eK \in G/K$ is given by

$$s_0(gK) = \sigma(g)K \ \text{ for all } \ gK \in G/K$$

where σ is an involutive automorphism of G such that $(K_\sigma)_0 \subset K \subset K_\sigma$. See [10, Prop. 3.4, Ch. IV].

A compact connected Lie group G can always be regarded as a Riemannian globally symmetric space as follows :
The mapping $(g_1, g_2) \mapsto (g_2, g_1)$ is an involutive automorphism of $G \times G$, whose fixed point set is $G^* = \{(g, g) : g \in G\}$. Hence the pair $(G \times G, G^*)$ is a Riemannian symmetric pair. The manifold $G \times G/G^*$ is diffeomorphic to G via the diffeomorphism given by

$$(g_1, g_2)G^* \mapsto g_1 {g_2}^{-1}$$

A Riemannian structure on $G \times G/G^*$ is $G \times G$-invariant *if and only if* the corresponding Riemannian structure on G is invariant under left and right translations. So G is a Riemannian globally symmetric space in each bi-invariant Riemannian structure. See [10, §6, Ch. IV].

2.2.2 Orthogonal symmetric Lie algebra and Riemannian globally symmetric space

Note that each Riemannian globally symmetric space gives rise to a pair $(\mathfrak{g}, s)$, where
(i) $\mathfrak{g}$ is a Lie algebra over $\mathbb{R}$,
(ii) s is an involutive automorphism of $\mathfrak{g}$,
(iii) $\mathfrak{k}$, the set of fixed points of s, is a compactly imbedded subalgebra of $\mathfrak{g}$, [1] and
(iv) $\mathfrak{k} \cap \mathfrak{z} = \{0\}$, $\mathfrak{z}$ denotes the centre of $\mathfrak{g}$.

A pair $(\mathfrak{g}, s)$ with the properties (i), (ii), (iii) is called an **orthogonal symmetric Lie algebra**. It is said to be **effective** if, in addition, (iv) holds. A **pair** (G, K), where G is a connected Lie group with Lie algebra $\mathfrak{g}$ and K is a Lie subgroup of G with Lie algebra $\mathfrak{k}$, is said to be **associated** with the orthogonal symmetric Lie algebra $(\mathfrak{g}, s)$.

Let $(\mathfrak{g}, s)$ be an effective orthogonal symmtric Lie algebra. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the decomposition of $\mathfrak{g}$ into the eigenspaces of s for the eigenvalues $+1$ and -1 respectively.

(a) If $\mathfrak{g}$ is compact and semisimple, $(\mathfrak{g}, s)$ is said to be of the **compact type**.

(b) If $\mathfrak{g}$ is non-compact, semisimple and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$, then $(\mathfrak{g}, s)$ is said to be of the **non-compact type**.

(c) If $\mathfrak{p}$ is an abelian ideal in $\mathfrak{g}$, then $(\mathfrak{g}, s)$ is said to be of the **Euclidean type**.

Let $(\mathfrak{g}, s)$ be an orthogonal symmtric Lie algebra and suppose the pair (G, K) is associated with $(\mathfrak{g}, s)$. The pair is said to be of the **compact type**, **non-compact type** or **Euclidean type** according to the type of $(\mathfrak{g}, s)$. Let M be a Riemannian globally symmetric space. Then M is said to be of the **compact type**, **non-compact type** or **Euclidean type** according to the type of the Riemannian symmetric pair $(I_0(M), K)$, K being the isotropy subgroup of $I_0(M)$ at some point in M.

The decomposition of an effective orthogonal symmetric Lie algebra into three others, which are of the compact type, non-compact type and Euclidean type respectively, leads to the decomposition of a simply connected Riemannian globally symmetric space M as

$$M = M_0 \times M_- \times M_+$$

where M_0 is a Euclidean space and M_- and M_+ are Riemannian globally symmetric spaces of compact type and non-compact type respectively. See [10, Prop. 4.2, Ch. V].

Let (G, K) be a pair of non-compact type. Then K is connected, closed and contains the centre Z of G. K is compact *if and only if* Z is finite. In this case, K is a maximal compact subgroup of G. Also the pair (G, K) is a Riemannian symmetric pair. If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the corresponding Cartan decomposition, then the map $\phi : \mathfrak{p} \times K \longrightarrow G$ given by

$$\phi(X, k) = (\exp X).k \ \text{ for } \ X \in \mathfrak{p}, \ k \in K$$

is a diffeomorphism. Hence the Riemannian globally symmetric space G/K is diffeomorphic with $\mathfrak{p}$ and so G/K is simply connected. See [10, Th. 1.1, Ch. VI].

---

[1] For a Lie algebra $\mathfrak{g}$ over $\mathbb{R}$, let $\text{Int}(\mathfrak{g})$ denote the connected Lie subgroup of $GL(\mathfrak{g})$ with Lie algebra $\text{ad}_{\mathfrak{g}}(\mathfrak{g}) \subset \text{End}(\mathfrak{g})$. A Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ is called **compactly imbedded** in $\mathfrak{g}$ if the connected Lie subgroup of $\text{Int}(\mathfrak{g})$ corresponding to the Lie algebra $\text{ad}_{\mathfrak{g}}(\mathfrak{k})$ is compact. A Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ is called **compact** if $\mathfrak{g}$ is compactly imbedded in itself.

Note : Since $\mathfrak{g}$ is semisimple, the Lie algebra $\mathfrak{k}$ is compact. Hence $\mathfrak{k}$ can be written as $\mathfrak{k} = \mathfrak{k}_s \oplus \mathfrak{k}_a$, where the ideals $\mathfrak{k}_s$ and $\mathfrak{k}_a$ are semisimple and abelian respectively (see [10, Prop. 6.6(ii), Ch. II]). Let K_s and K_a denote the corresponding connected Lie subgroups of K. The group K_a can be written as $K_a = T \times V$, where T and V are connected Lie subgroups of K_a which are isomorphic to the torus and the Euclidean space respectively. Define $K' := K_sT$. Then K' is the unique maximal compact subgroup of K. This group is also maximal compact in G. See [10, Th. 2.2(i), Ch. VI].

So if G is a connected semisimple Lie group with maximal compact subgroup H and rank (G) = rank (H), then rank (G) = rank (K') (where K' is defined as above), for any two maximal compact subgroups of a connected semisimple Lie group are conjugate under an inner automorphism of G (see [10, Th. 2.2(ii), Ch. VI]). But rank (K') = rank $(K)-$ dim(V). Since rank $(K) \leq$ rank (G), so rank (G) = rank (K') implies rank (K) = rank (K'). Hence $V = \{0\}$ and $K = K'$. So K is a maximal compact subgroup of G. Therefore the centre of G is finite.

Let (G, K) be a pair of compact type. Then K is closed. See [10, Prop. 3.6, Ch. IV]. If K is connected then G/K is a Riemannian globally symmetric space in each G-invariant Riemannian metric on G/K. See [10, page 349, Ch. VII].

For a Riemannian globally symmetric space G/K, the following theorem describes $I_0(G/K)$:

Theorem 2.2.1 *[10, Th. 4.1(i), Ch. V] Let (G, K) be a Riemannian symmetric pair and $M := G/K$.*

Suppose that G is semisimple and acts effectively on the Riemannian globally symmetric space M. Then $G = I_0(M)$ (as Lie groups).

More generally, if G is semisimple and if N denotes the kernel of the action of G on M, then $G/N = I_0(M)$.

The duality

Let $(\mathfrak{g}, s)$ be an orthogonal symmetric Lie algebra and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the decomposition of $\mathfrak{g}$ into the eigenspaces of s corresponding to the eigenvalues $+1$ and -1 respectively. Let $\mathfrak{g}^{\mathbb{C}}$ denote the complexification of $\mathfrak{g}$. Define $\mathfrak{g}^* := \mathfrak{k} \oplus i\mathfrak{p}$ to be the subspace of $\mathfrak{g}^{\mathbb{C}}$. Then $\mathfrak{g}^*$ is a Lie subalgebra of $\mathfrak{g}^{\mathbb{C}}$ over $\mathbb{R}$. The mapping $s^* : T + iX \mapsto T - iX$ ($T \in \mathfrak{k},\ X \in \mathfrak{p}$) is an involutive automorphism of $\mathfrak{g}^*$. The pair $(\mathfrak{g}^*, s^*)$ is an orthogonal symmetric Lie algebra, called the **dual** of $(\mathfrak{g}, s)$. Then $(\mathfrak{g}, s)$ is the dual of $(\mathfrak{g}^*, s^*)$. If $(\mathfrak{g}, s)$ is of the compact type, $(\mathfrak{g}^*, s^*)$ is of the non-compact type and conversely. If $(\mathfrak{g}_1, s_1)$ is isomorphic to $(\mathfrak{g}_2, s_2)$, then $(\mathfrak{g}_1^*, s_1^*)$ is isomorphic to $(\mathfrak{g}_2^*, s_2^*)$. [2] See [10, Prop. 2.1, Ch. V].

The following proposition shows that the non-compact real forms of a complex semisimple Lie algebra $\mathfrak{g}^{\mathbb{C}}$ (up to conjugacy) are in one-one correspondence with the involutive automorphisms of a compact real form of $\mathfrak{g}^{\mathbb{C}}$ (up to conjugacy).

[2] $(\mathfrak{g}_1, s_1)$ is said to be **isomorphic** to $(\mathfrak{g}_2, s_2)$ if there exists an isomorphism $\phi : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ such that $\phi \circ s_1 = s_2 \circ \phi$.

Proposition 2.2.2 *[10, Prop. 2.2, Ch. V] Let $\mathfrak{g}$ be a compact semisimple Lie algebra. Let s_1 and s_2 be two involutive automorphisms of $\mathfrak{g}$ with corresponding duals $\mathfrak{g}_1^*$ and $\mathfrak{g}_2^*$ respectively. Then s_1 and s_2 are conjugate within the group $Aut(\mathfrak{g})$ if and only if $\mathfrak{g}_1^*$ and $\mathfrak{g}_2^*$ are conjugate under an automorphism of $\mathfrak{g}^{\mathbb{C}}$.*

A pair (G_1, K_1) is called **dual** to a pair (G_2, K_2) if the corresponding orthogonal symmetric Lie algebras are dual to each other. Let M_1 and M_2 be two Riemannian globally symmetric spaces. M_1 is called **dual** to M_2 if the pairs $(I_0(M_1), K_1)$ and $(I_0(M_2), K_2)$ are dual to each other, where K_1 (respectively K_2) is the isotropy subgroup of $I_0(M_1)$ (respectively $I_0(M_2)$) at some point in M_1 (respectively in M_2).

Irreducibility

Let $(\mathfrak{g}, s)$ be an orthogonal symmetric Lie algebra, $\mathfrak{k}$ and $\mathfrak{p}$ be the eigenspaces of s for the eigenvalues $+1$ and -1 respectively. One says that $(\mathfrak{g}, s)$ is **irreducible** if the following two conditions are satisfied :

(i) $\mathfrak{g}$ is semisimple and $\mathfrak{k}$ contains no non-zero ideal of $\mathfrak{g}$, and

(ii) the algebra $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$ acts irreducibly on $\mathfrak{p}$.

Note that $(\mathfrak{g}, s)$ is irreducible *if and only if* the dual $(\mathfrak{g}^*, s^*)$ is irreducible.

Let (G, K) be a pair associated with $\mathfrak{g}, s)$. One says that (G, K) is **irreducible** if $(\mathfrak{g}, s)$ is so. A Riemannian globally symmetric space M is called **irreducible** if the pair $(I_0(M), K)$ is irreducible, K being the isotropy subgroup of $I_0(M)$ at some point in M. Any simply connected Riemannian globally symmetric space of the compact type or the non-compact type is the direct product of irreducible Riemannian globally symmetric spaces of the same type (the type of M). See [10, Prop. 5.5, Ch. VIII]. Let (G, K) be an irreducible Riemannian symmetric pair. Then all G-invariant Riemannian structures on G/K coincide except for a constant factor. We can therefore always assume that this Riemannian structure is induced by $+B$ or $-B$, where B is the Killing form of $\mathfrak{g}$.

The irreducible orthogonal symmetric Lie algebras of the compact type are :

I. $(\mathfrak{g}, s)$ *where $\mathfrak{g}$ is a compact simple Lie algebra and s any involutive automorphism of* $\mathfrak{g}$.

II. $(\mathfrak{g}, s)$ *where the compact Lie algebra $\mathfrak{g}$ is the direct sum $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ of simple ideals which are interchanged by an involutive automorphism s of $\mathfrak{g}$.*

See [10, Th. 5.3, Ch. VIII].

The irreducible orthogonal symmetric Lie algebras of the non-compact type are :

III. $(\mathfrak{g}, s)$ *where $\mathfrak{g}$ is a simple non-compact Lie algebra over $\mathbb{R}$, the complexification $\mathfrak{g}^{\mathbb{C}}$ is a simple Lie algebra over $\mathbb{C}$ and s is an involutive automorphism of $\mathfrak{g}$ such that the fixed points form a compactly imbedded subalgebra.*

IV. $(\mathfrak{g}, s)$ *where $\mathfrak{g} = \mathfrak{l}^{\mathbb{R}}$, $\mathfrak{l}$ being a simple Lie algebra over $\mathbb{C}$. Here s is the conjugation of $\mathfrak{g}$ with respect to a maximal compactly imbedded subalgebra.*

Furthermore, if $(\mathfrak{g}^*, s^*)$ denotes the dual of $(\mathfrak{g}, s)$, then

$(\mathfrak{g}, s)$ *is of type III* if and only if $(\mathfrak{g}^*, s^*)$ *is of type I,*

$(\mathfrak{g}, s)$ *is of type IV* if and only if $(\mathfrak{g}^*, s^*)$ *is of type II.*

See [10, Th. 5.4, Ch. VIII].

Let M be an irreducible Riemannian globally symmetric space and $(\mathfrak{g}, s)$ be the orthogonal symmetric Lie algebra associated with M. The space M is said to be of **type** *i (i = I, II, III, IV)* if $(\mathfrak{g}, s)$ is of type i in the notation given above.

Note that *the Riemannian globally symmetric spaces of type IV are the spaces G/U, where G is a connected Lie group whose Lie algebra is $\mathfrak{g}^{\mathbb{R}}$ where $\mathfrak{g}$ is a simple Lie algebra over $\mathbb{C}$, and U is a maximal compact subgroup of G. The Riemannian metric on G/U is G-invariant and is uniquely determined (up to a factor) by this condition.* Clearly the Riemannian globally symmetric spaces of type *IV* are simply connected.

The Riemannian globally symmetric spaces of type II are the simple , compact, connected Lie groups. The Riemannian metric on a such group is invariant under left and right translations and is uniquely determined (up to a factor) by this condition. See [10, Prop. 1.2, Ch. X]. The simply connected Riemannian globally symmetric spaces of type II are the simply connected, compact, simple Lie groups with the (up to a factor) left and right translations invariant Riemannian metric.

The classification of involutive automorphisms of compact simple Lie algebras (up to conjugacy) leads to the classification of irreducible orthogonal symmetric Lie algebras of type *I* and hence of type *III* (up to isomorphism). These lead to the É. Cartan's classification of simply connected irreducible Riemannian globally symmetric spaces of type *I* and *III*. See [10, Ch. X] for details.

2.3 Hermitian symmetric spaces

We follow [10] for this section.

Let M be a connected complex manifold. A Riemannian structure on M is called a **Hermitian structure** if the complex structure on each tangent space is an isometry. Let M be a connected complex manifold with a Hermitian structure. M is said to be a **Hermitian symmetric space** if M is a Riemannian globally symmetric space and for each point $p \in M$, the involutive isometry s_p is holomorphic.

The complex vector space $\mathbb{C}^n$ $(n \geq 1)$, the Riemann sphere S^2, the upper half plane with the Poincaré metric are examples of Hermitian symmetric spaces.

Let $A(M)$ denote the set of all holomorphic isometries of M. Then $A(M)$ is a closed subgroup of $I(M)$. The group $A(M)$ acts transitively on M, since it contains all the symmetries. Hence $A_0(M)$, the identity component of A(M), also acts transitively on M. Therefore M is diffeomorphic to $A_0(M)/K$, K being the isotropy subgroup of $A_0(M)$ at some point $p \in M$. Note that the pair $(A_0(M), K)$ is a Riemannian symmetric pair.

Conversely, let (G, K) be a Riemannian symmetric pair and Q be a G-invariant Riemannian structure on $M = G/K$. Suppose J is an endomorphism of the tangent space $\mathcal{T}_0(M)$ at $0 = eK$ such that
(i) $J^2 = -\mathrm{Id}$,

(ii) $Q_0(JX, JY) = Q_0(X, Y)$ for $X, Y \in \mathcal{T}_0(M)$,

(iii) J commutes with each element of $\mathrm{Ad}_G(K)$.

Then J defines a unique complex structure on M such that the action of G on M is holomorphic and the induced complex structure on $\mathcal{T}_0(M)$ is J, the Riemannian structure Q is Hermitian and M is a Hermitian symmetric space. See [10, Prop. 4.2, Ch VIII].

For a Hermitian symmetric space M, $A_0(M)$ is not necessarily equals to $I_0(M)$. For example, if $M = \mathbb{C}^2$, then $A_0(M)$ and $I_0(M)$ are different. But one of the groups $A_0(M)$ and $I_0(M)$ is semisimple implies the groups are the same, that is $A_0(M) = I_0(M)$ [10, Lemma 4.3, Ch VIII].

A Hermitian symmetric space M is said to be of the **compact type** (respectively of the **non-compact type**) if the Riemannian symmetric pair $(A_0(M), K)$ is of the compact type (respectively of the non-compact type), K being the isotropy subgroup at some point $p \in M$. A Hermitian symmetric space of the compact type or non-compact type is simply connected [10, Th. 4.6, Ch. VIII]. A simply connected Hermitian space M can be decomposed as

$$M = M_0 \times M_- \times M_+,$$

where $M_0 = \mathbb{C}^n$ for some integer $n \geq 0$, M_- and M_+ are Hermitian symmeric spaces of the compact type and non-compact type respectively. See [10, Prop. 4.4, Ch. VIII].

Let $(\mathfrak{g}, s)$ be an orthogonal symmetric Lie algebra and M_1, M_2 be Riemannian globally symmetric spaces associated with $(\mathfrak{g}, s)$. It may happen that one of them is a Hermitian symmetric space but the other is not. For example, the Riemann sphere S^2 and the two dimensional real projective space $\mathbb{RP}^2$ are associated with the same orthogonal symmetric Lie algebra. Note that S^2 is a Hermitian symmetric space but $\mathbb{RP}^2$ is not. But there is exactly one simply connected Riemannian globally symmetric space associated with an orthogonal symmetric Lie algebra.

Let M be an irreducible simply connected Riemannian globally symmetric space and $(\mathfrak{g}, s)$ be the orthogonal symmetric Lie algebra associated with M. Then M is a Hermitian symmetric space **if and only if** the fixed point set $\mathfrak{k}$ of s has non-zero centre. So in particular, a Riemannian globally symmetric space of type *II* or *IV* cannot be Hermitian symmetric. If $(\mathfrak{g}, s)$ is an orthogonal symmetric Lie algebra associated with an irreducible Hermitian symmetric space, then the centre of the fixed point set $\mathfrak{k}$ of s is one dimensional. Any Hermitian symmetric space of the compact type (respectively, non-compact type) can be decomposed as a product of irreducible Hermitian symmetric spaces of the compact type (respectively, non-compact type) [10, Prop. 5.5, Ch. VIII].

2.3.1 Bounded symmetric domains

A **domain** in $\mathbb{C}^N$ (for some positive integer N) is an open connected subset of $\mathbb{C}^N$. A bounded domain D of $\mathbb{C}^N$ is said to a **bounded symmetric domain** if each point $p \in D$ is an isolated fixed point of an involutive holomorphic diffemorphism of D.

If D is a bounded domain, there exists a Riemannian structure coming from the *Bergman metric* on D [10, page 369, Chapter VIII] which is a Hermitian structure and, with respect to this metric, any holomorphic diffeomorphism of D is an isometry. In fact, any bounded symmetric domain equipped with the Bergman metric is a Hermitian symmetric space of the non-compact type [10, Th 7.1(i), Ch. VIII].

Conversely, let M be a Hermitian symmeric space of non-compact type and $(\mathfrak{g}_0, s)$ be the orthogonal symmetric Lie algebra associated with M. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the Cartan decomposition corresponding to the Cartan involution s. Let $\mathfrak{c}_0$ denote the centre of $\mathfrak{k}_0$. Then $\mathfrak{c}_0 \neq \{0\}$ and we have $\mathfrak{z}_{\mathfrak{g}_0}(\mathfrak{c}_0) = \mathfrak{k}_0$. Let $\mathfrak{t}_0$ be a maximal abelian subspace of $\mathfrak{k}_0$. Then $\mathfrak{c}_0 \subset \mathfrak{t}_0$ and $\mathfrak{t}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$. Let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$, $\mathfrak{t}$, $\mathfrak{c}$ denote the complexifications of $\mathfrak{g}_0$, $\mathfrak{k}_0$, $\mathfrak{p}_0$, $\mathfrak{t}_0$, $\mathfrak{c}_0$ respectively. Then $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$ as well as of $\mathfrak{g}$. Let

$$\Delta := \text{the roots of } (\mathfrak{g}, \mathfrak{t}) \ \text{ and } \ \Delta_{\mathfrak{k}} := \text{the roots of } (\mathfrak{k}, \mathfrak{t}).$$

Note that $\Delta_{\mathfrak{k}} \subset \Delta$. Let $\Delta_n := \Delta \setminus \Delta_{\mathfrak{k}}$. A root is compact (respectively non-compact) if it is in $\Delta_{\mathfrak{k}}$ (respectively in Δ_n). Note that a root α is compact *if and only if* α vanishes identically on $\mathfrak{c}$. Choose a basis of $i\mathfrak{c}_0$ and extend this to a basis B of $i\mathfrak{t}_0$. Now consider the lexicographic ordering of the dual of $i\mathfrak{t}_0$ with respect to the basis B. This ordering will introduce an ordering of Δ. Let Δ^+ denote the set of positive roots in Δ with respect to this ordering. The positive system of Δ^+ is defined to be a **special positive system**. Let $\Delta_n^+ := \Delta^+ \cap \Delta_n$. Define

$$\mathfrak{p}_+ := \sum_{\alpha \in \Delta_n^+} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{p}_- := \sum_{-\alpha \in \Delta_n^+} \mathfrak{g}_\alpha,$$

$\mathfrak{g}_\alpha$ being the root space for $\alpha \in \Delta$. Then $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are abelian, $[\mathfrak{k}, \mathfrak{p}_+] \subset \mathfrak{p}_+$, $[\mathfrak{k}, \mathfrak{p}_-] \subset \mathfrak{p}_-$ and $\mathfrak{p} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$.

Let G be the simply connected Lie group with Lie algebra $\mathfrak{g}$ and G_0, K_0, K, P_+, P_- be the connected Lie subgroups of G corresponding to the subalgebras $\mathfrak{g}_0$, $\mathfrak{k}_0$, $\mathfrak{k}$, $\mathfrak{p}_+$, $\mathfrak{p}_-$ respectively. Note that $M = G_0/K_0$. The map exponential induces a diffeomorphism of $\mathfrak{p}_+$ onto P_+ and of $\mathfrak{p}_-$ onto P_-.

The Harish-Chandra decomposition

Note that P_+KP_- is an open submanifold of G with $G_0 \subset P_+KP_-$, G_0KP_- is open in P_+KP_-, $G_0 \cap KP_- = K_0$ and there exists a bounded open connected subset D of $\mathfrak{p}_+$ such that

$$G_0KP_- = (\exp D)KP_-.$$

So there exists a holomorphic diffeomorphism of $M = G_0/K_0$ onto D.

Let $\mathfrak{u} = \mathfrak{k}_0 \oplus i\mathfrak{p}_0$ be the dual of $(\mathfrak{g}_0, s)$ in $\mathfrak{g}$ and U be the connected Lie subgroup of G corresponding to the subalgebra $\mathfrak{u}$. The mapping $uK_0 \mapsto uKP_-$ is a holomorphic diffeomorphism of U/K_0 onto G/KP_-. Therefore the Hermitian symmetric space $M = G_0/K_0$ is an open submanifold of its dual U/K_0.

2.4 Holomorphic discrete series and Borel-de Siebenthal discrete series

2.4.1 Borel-de Siebenthal positive root system

Let $(\mathfrak{g}_0, s)$ be an irreducible orthogonal symmetric Lie algebra of the non-compact type and $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the Cartan decomposition corresponding to the Cartan involution s. Let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$ denote the complexifications of $\mathfrak{g}_0$, $\mathfrak{k}_0$, $\mathfrak{p}_0$ respectively. Assume that rank $(\mathfrak{g}_0) =$ rank $(\mathfrak{k}_0)$. Then note that $\mathfrak{g}$ is simple. Fix a maximal abelian subalgebra $\mathfrak{t}_0$ of $\mathfrak{k}_0$. Then $\mathfrak{t}_0$ is also a Cartan subalgebra of $\mathfrak{g}_0$. Let $\mathfrak{t}$ be the complexification of $\mathfrak{t}_0$. The construction of Borel-de Siebenthal [2] provides a positive root system of $(\mathfrak{g}, \mathfrak{t})$ known as a **Borel-de Siebenthal positive system** such that the corresponding simple system Ψ contains exactly one non-compact root ν and the coefficient of ν in the highest root μ when expressed as a sum of simple roots is 1 or 2. Let $n_\nu(\alpha)$ denote the coefficient of the non-compact simple root ν in a root α when expressed as a sum of simple roots.

If the orthogonal symmetric Lie algebra $(\mathfrak{g}_0, s)$ is associated with a Hermitian symmetric space, then $n_\nu(\mu) = 1$, $\mathfrak{k} = \mathfrak{t} \oplus \mathfrak{u}_0$ and $\mathfrak{p} = \mathfrak{u}_{-1} \oplus \mathfrak{u}_1$, where $\mathfrak{u}_i = \sum_{n_\nu(\alpha)=i} \mathfrak{g}_\alpha$ for $-1 \le i \le 1$, $\mathfrak{g}_\alpha$ being the root space for the root α of $(\mathfrak{g}, \mathfrak{t})$. Note that $\Psi \setminus \{\nu\}$ is a simple root system of $(\mathfrak{k}, \mathfrak{t})$.

Otherwise, $n_\nu(\mu) = 2$, $\mathfrak{k} = \mathfrak{u}_{-2} \oplus \mathfrak{t} \oplus \mathfrak{u}_2$ and $\mathfrak{p} = \mathfrak{u}_{-1} \oplus \mathfrak{u}_1$, where $\mathfrak{u}_i = \sum_{n_\nu(\alpha)=i} \mathfrak{g}_\alpha$ for $-2 \le i \le 2$. In this case, $(\Psi \setminus \{\nu\}) \cup \{-\mu\}$ is a simple root system of $(\mathfrak{k}, \mathfrak{t})$.

Note : In the first case, a positive root system of $(\mathfrak{g}, \mathfrak{t})$ is a special positive system *if and only if* it is a Borel-de Siebenthal positive system.

Conversely, let $\mathfrak{g}$ be a complex simple Lie algebra. Choose a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ and a simple root system of $(\mathfrak{g}, \mathfrak{t})$. Let μ denote the highest root.

If there exists a simple root ν such that $n_\nu(\mu) = 1$, then $\mathfrak{g} = \mathfrak{u}_{-1} \oplus \mathfrak{l} \oplus \mathfrak{u}_1$, where $\mathfrak{l} = \mathfrak{t} \oplus \mathfrak{u}_0$, the $\mathfrak{u}_i$ are defined as above. Define $\mathfrak{k} := \mathfrak{l}$ and $\mathfrak{p} := \mathfrak{u}_{-1} \oplus \mathfrak{u}_1$. Note that rank $(\mathfrak{k}) =$ rank $(\mathfrak{g})$. There exists a unique (up to an inner automorphism of $\mathfrak{g}$) irreducible orthogonal symmetric Lie algebra $(\mathfrak{g}_0, s)$ of the non-compact type such that $\mathfrak{g}$ and $\mathfrak{k}$ are the complexifications of $\mathfrak{g}_0$ and $\mathfrak{k}_0$ respectively, $\mathfrak{k}_0$ being the fixed point set of s. Also the chosen simple root system is the simple system of a Borel-de Siebenthal positive system of $\mathfrak{g}_0$ with the non-compact simple root ν. The orthogonal symmetric Lie algebra $(\mathfrak{g}_0, s)$ is associated with a Hermitian symmetric space.

If there exists a simple root ν such that $n_\nu(\mu) = 2$, then $\mathfrak{g} = \mathfrak{u}_{-2} \oplus \mathfrak{u}_{-1} \oplus \mathfrak{l} \oplus \mathfrak{u}_1 \oplus \mathfrak{u}_2$, with $\mathfrak{l} = \mathfrak{t} \oplus \mathfrak{u}_0$. Define $\mathfrak{k} := \mathfrak{u}_{-2} \oplus \mathfrak{l} \oplus \mathfrak{u}_2$ and $\mathfrak{p} := \mathfrak{u}_{-1} \oplus \mathfrak{u}_1$. Like as above, rank $(\mathfrak{k}) =$ rank $(\mathfrak{g})$ and there exists a unique (up to an inner automorphism of $\mathfrak{g}$) irreducible orthogonal symmetric Lie algebra $(\mathfrak{g}_0, s)$ of the non-compact type such that $\mathfrak{g}$ and $\mathfrak{k}$ are the complexifications of $\mathfrak{g}_0$ and $\mathfrak{k}_0$ respectively, $\mathfrak{k}_0$ being the fixed point set of s. The chosen simple root system is the simple system of a Borel-de Siebenthal positive system of $\mathfrak{g}_0$ with the non-compact simple root ν. In this case, $\mathfrak{k}_0$ is semisimple.

2.4.2 Holomorphic discrete series

Let G_0 be a connected non-compact semisimple Lie group with finite centre and K_0 be a maximal compact subgroup of G_0. Then (G_0, K_0) is a pair of non-compact type. Let $\mathfrak{g}_0$, $\mathfrak{k}_0$ denote the Lie algebras of G_0 and K_0 respectively. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the corresponding Cartan decomposition. Assume that G_0/K_0 is a Hermitian symmetric space. Let $\mathfrak{c}_0$ denote the centre of $\mathfrak{k}_0$. Then $\mathfrak{c}_0 \neq \{0\}$ and $\mathfrak{z}_{\mathfrak{g}_0}(\mathfrak{c}_0) = \mathfrak{k}_0$. Let $\mathfrak{t}_0$ be a maximal abelian subspace of $\mathfrak{k}_0$. Then as in §2.3.1, $\mathfrak{t}_0$ is maximal abelian in $\mathfrak{g}_0$. Let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$, $\mathfrak{t}$, $\mathfrak{c}$ denote the complexifications of $\mathfrak{g}_0$, $\mathfrak{k}_0$, $\mathfrak{p}_0$, $\mathfrak{t}_0$, $\mathfrak{c}_0$ respectively. Then $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$ as well as of $\mathfrak{g}$. That is rank (G_0) = rank (K_0) so that G_0 has discrete series representations. As in §2.3.1, let Δ^+ be a special positive system of $(\mathfrak{g}, \mathfrak{t})$. Let $\Delta^+_{\mathfrak{k}}$ and Δ^+_n denote the set of all positive compact roots and positive non-compact roots respectively. Define $\rho_{\mathfrak{g}} := \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$. Suppose $\Lambda \in \mathfrak{t}^*$ is analytically integral such that

$$\langle \Lambda, \alpha\rangle \geq 0 \text{ for all } \alpha \in \Delta^+_{\mathfrak{k}} \text{ and } \langle \Lambda + \rho_{\mathfrak{g}}, \beta\rangle < 0 \text{ for all } \beta \in \Delta^+_n. \tag{2.4}$$

where $\langle\ ,\ \rangle$ denote the positive definite symmetric bilinear form on $(i\mathfrak{t}_0)^*$ induced from the Killing form of $\mathfrak{g}$.

Note that $\Lambda + \rho_{\mathfrak{g}}$ is non-singular and is a Harish-Chandra parameter of a discrete series representation $\pi_{\Lambda+\rho_{\mathfrak{g}}}$ of G_0 which is called a **holomorphic discrete series representation** of G_0. The Harish-Chandra root order corresponding to $\Lambda+\rho_{\mathfrak{g}}$ is $\Delta^+_{\mathfrak{k}}\cup\Delta^-_n$, where $\Delta^-_n = -\Delta^+_n$. Therefore the Blattener parameter of $\pi_{\Lambda+\rho_{\mathfrak{g}}}$ is $\Lambda + \rho_{\mathfrak{g}} + \dfrac{1}{2}\displaystyle\sum_{\alpha\in\Delta^+_{\mathfrak{k}}\cup\Delta^-_n}\alpha - \sum_{\alpha\in\Delta^+_{\mathfrak{k}}}\alpha = \Lambda$ (see Theorem 2.1.7(b) in §2.1.4).

The space of K_0-finite vectors of a holomorphic discrete series representation $\pi_{\Lambda+\rho_{\mathfrak{g}}}$ is described as $\oplus_{n\geq 0}E_\Lambda \otimes S^n(\mathfrak{p}_-)$, where E_Λ is the irreducible K_0-representation with highest weight Λ, $\mathfrak{p}_- = \sum_{\alpha\in\Delta^-_n}\mathfrak{g}_\alpha$ as in §2.3.1 and $S^n(\mathfrak{p}_-)$ denotes the n-th symmetric power of $\mathfrak{p}_-$. See [8] and also [21]. Hence the $(\mathfrak{g}, K_0)$-module associated with $\pi_{\Lambda+\rho_{\mathfrak{g}}}$ is the irreducible quotient of the Verma module of $\mathfrak{g}$ with highest weight Λ with respect to the positive system Δ^+.

2.4.3 Borel-de Siebenthal discrete series

In this section we describe Borel-de Siebenthal discrete series. *The notations introduced here will be used from Chapter 3 onwards unless otherwise stated explicitly.*

Let G_0 be a simply connected non-compact real simple Lie group with maximal compact subgroup K_0. Assume that

(i) rank (G_0) = rank (K_0) (hence G_0 has discrete series representations), and

(ii) G_0/K_0 is not Hermitian symmetric that is, K_0 is semisimple.

Let $\mathfrak{g}_0$, $\mathfrak{k}_0$ denote the Lie algebras of G_0, K_0 respectively and $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the corresponding Cartan decomposition. Let $\mathfrak{t}_0$ be a Cartan subalgebra of $\mathfrak{k}_0$, which is also a Cartan subalgebra of $\mathfrak{g}_0$. Let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$, $\mathfrak{t}$ denote the complexifications of $\mathfrak{g}_0$, $\mathfrak{k}_0$, $\mathfrak{p}_0$, $\mathfrak{t}_0$ respectively. Let Δ be the root system of $(\mathfrak{g}, \mathfrak{t})$, $\Delta^+ \subset \Delta$ be a Borel-de Siebenthal positive system and Ψ the set of simple roots. Let $\alpha \in \Delta$ be any root and let $n_\nu(\alpha)$ be the coefficient of ν (the non-compact simple root) when α is expressed as a sum of simple roots. Note that $n_\nu(\mu) = 2$, where μ denotes the highest root. One has a partition of the set of roots Δ into subsets $\Delta_i, i = 0, \pm 1, \pm 2$ where $\Delta_i \subset \Delta$ defined to be $\{\alpha \in \Delta \mid n_\nu(\alpha) = i\}$. Note that $\Delta_{\mathfrak{k}} = \Delta_0 \cup \Delta_2 \cup \Delta_{-2}$ and $\Delta_n = \Delta_1 \cup \Delta_{-1}$, where $\Delta_{\mathfrak{k}}$ and Δ_n are the set of compact roots and non-compact roots respectively. Define $\mathfrak{q} := \mathfrak{l} \oplus \mathfrak{u}_{-1} \oplus \mathfrak{u}_{-2}$, where $\mathfrak{l} = \mathfrak{t} \oplus \mathfrak{u}_0$, the $\mathfrak{u}_i$ are defined as in §2.4.1. The Killing form B of $\mathfrak{g}$ determines a positive definite symmetric bilinear form $\langle\ ,\ \rangle$ on $(i\mathfrak{t}_0)^*$ which is normalized so that $\langle \nu, \nu \rangle = 2$. Let $\nu^* \in (i\mathfrak{t}_0)^*$ be the fundamental weight of $(\mathfrak{g}, \mathfrak{t})$ corresponding to the simple root ν of Ψ and $h_{\nu^*} \in i\mathfrak{t}_0$ be such that $\nu^*(h) = \langle h, h_{\nu^*} \rangle$ for all $h \in i\mathfrak{t}_0$. Then the centre of $\mathfrak{l}$ is $\mathbb{C}h_{\nu^*}$. Let G denote the simply connected complexification of G_0. The inclusion $i : \mathfrak{g}_0 \hookrightarrow \mathfrak{g}$ defines a homomorphism $p : G_0 \longrightarrow G$. Let Q, L, K be the connected Lie subgroups of G corresponding to the subalgebras $\mathfrak{q}$, $\mathfrak{l}$. $\mathfrak{k}$ respectively.

Note that $\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{u}_2 \oplus \mathfrak{u}_{-2}$. Let $\theta = \mathrm{Ad}_K(\exp \frac{i\pi}{2} h_{\nu^*}) = \exp(\mathrm{ad}_{\mathfrak{k}}(\frac{i\pi}{2} h_{\nu^*})) \in \mathrm{Aut}(\mathfrak{k})$. Since

$$\mathrm{ad}_{\mathfrak{k}}(\frac{i\pi}{2} h_{\nu^*})(X) = 0 \quad \text{for all} \quad X \in \mathfrak{l}, \text{ and}$$

$$\begin{aligned} \mathrm{ad}_{\mathfrak{k}}(\frac{i\pi}{2} h_{\nu^*})(Y) &= i\pi Y \ \text{ for } \ Y \in \mathfrak{u}_2, \\ &= -i\pi Y \ \text{ for } \ Y \in \mathfrak{u}_{-2}. \end{aligned}$$

We have $\theta(X) = X$ for all $X \in \mathfrak{l}$ and $\theta(Y) = -Y$ for all $Y \in \mathfrak{u}_2 \oplus \mathfrak{u}_{-2}$. Hence $\theta^2 = \mathrm{Id}$. Notice that $\frac{i\pi}{2} h_{\nu^*} \in \mathfrak{k}_0$. Therefore $\theta(\mathfrak{k}_0) \subset \mathfrak{k}_0$ and $(\mathfrak{k}_0, \theta|_{\mathfrak{k}_0})$ is an orthogonal symmetric Lie algebra of the compact type. Let $\mathfrak{l}_0$ be the set of fixed points of $\theta|_{\mathfrak{k}_0}$. Then $\mathfrak{l}_0$ is a real form of $\mathfrak{l}$. Let L_0 be the centralizer in K_0 of the circle subgroup $S_{\nu^*} := \{exp(ith_{\nu^*}) \ : \ t \in \mathbb{R}\}$ of K_0. Then L_0 is connected and $\mathrm{Lie}(L_0) = \mathfrak{l}_0$. Define $\bar{L}_0 := p(L_0)$.

The Borel-de Siebenthal discrete series of G_0, whose systematic study was carried out by Ørsted and Wolf [19], is defined analogously to the holomorphic discrete series as follows: Let γ be the highest weight of an irreducible representation E_γ of $\bar{L}_0$ such that $\gamma + \rho_{\mathfrak{g}}$ is negative on $\Delta_1 \cup \Delta_2$. Here $\rho_{\mathfrak{g}}$ denotes half the sum of roots in Δ^+. The **Borel-de Siebenthal discrete series representation** $\pi_{\gamma+\rho_{\mathfrak{g}}}$ is the discrete series representation of G_0 for which the Harish-Chandra parameter is $\gamma + \rho_{\mathfrak{g}}$.

Let $\mathfrak{k}_1^{\mathbb{C}}$ denote the simple ideal of $\mathfrak{k}$ that contains the root space $\mathfrak{g}_\mu$. It is the complexification of the Lie algebra $\mathfrak{k}_1$ of a compact Lie group K_1 which is a simple factor of K_0. It turns out that $\mathfrak{u}_2, \mathfrak{u}_{-2} \subset \mathfrak{k}_1^{\mathbb{C}}$. Let $\mathfrak{k}_2$ be the ideal of $\mathfrak{k}_0$ such that $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$. We let $\mathfrak{l}_j^{\mathbb{C}} = \mathfrak{k}_j^{\mathbb{C}} \cap \mathfrak{l}$, $j = 1, 2$. Note that $\mathfrak{k}_2^{\mathbb{C}} = \mathfrak{l}_2^{\mathbb{C}}$ and so $\mathfrak{l}_2^{\mathbb{C}}$ is semisimple. Thus the centre of $\mathfrak{l}$ is contained in $\mathfrak{l}_1^{\mathbb{C}}$. Let $L_1 \subset K_1$ be the centralizer of $S_{\nu^*} \subset K_1$. Then $L_1 \subset L_0$ and $Lie(L_1) =: \mathfrak{l}_1$ is a compact real form of $\mathfrak{l}_1^{\mathbb{C}}$. Let K_2 be the connected Lie subgroup of K_0 with Lie algebra $\mathfrak{k}_2$. Then $K_0 = K_1 \times K_2$ as K_0 is simply connected. Also $L_0 = L_1 \times K_2$. It will be convenient to set $L_2 := K_2$.

The map $p : G_0 \longrightarrow G$ defines a smooth map $G_0/L_0 \subset G/Q$, since $\mathfrak{l}_0 \subset \mathfrak{q}$. Since $\dim_{\mathbb{R}}(G_0/L_0) = \dim_{\mathbb{R}}(\mathfrak{u}_1 + \mathfrak{u}_2) = 2\dim_{\mathbb{C}}(G/Q)$, we conclude that G_0/L_0 is an open domain of the complex flag variety G/Q. Note that one has a fibre bundle projection $G_0/L_0 \longrightarrow G_0/K_0$ with fibre K_0/L_0. Note that K_0/L_0 is a Riemannian globally symmetric space which is isomorphic with the complex flag variety $K/(K \cap Q)$. With this complex structure, K_0/L_0 is a Hermitian symmetric space. Since $K_0/L_0 = K_1/L_1$, it is irreducible. We shall denote the identity coset of any homogeneous space by o. The holomorphic tangent bundles of K_0/L_0 and G/Q are the bundles associated to the $\bar{L}_0$-modules $\mathfrak{u}_2$ and $\mathfrak{u}_1 \oplus \mathfrak{u}_2$ respectively, since we have the isomorphisms of tangent spaces $\mathcal{T}_o K_0/L_0 = \mathfrak{u}_2$ and $\mathcal{T}_o G/Q = \mathfrak{u}_1 \oplus \mathfrak{u}_2$ of $\bar{L}_0$-modules. Hence the normal bundle to the imbedding $K_0/L_0 \hookrightarrow G/Q$ is the bundle associated to the representation of $\bar{L}_0$ on $\mathfrak{u}_1$.

We regard any $\bar{L}_0$ representation as an L_0-representation via the covering projection $p|_{L_0}$. Any L_0-representation we consider in this thesis arises from an $\bar{L}_0$-representation and so *we shall abuse notation and simply write* L_0 *for* $\bar{L}_0$ *as well*. Define $Y := K_0/L_0$.

We recall the following result due to Parthasarathy [20] (see §2.4.4 below) and Ørsted and Wolf [19]. Let γ be the highest weight of an irreducible finite dimensional complex representation of L_0 on E_γ and suppose that $\langle \gamma + \rho_{\mathfrak{g}}, \alpha \rangle < 0$ for all $\alpha \in \Delta_1 \cup \Delta_2$.

Theorem 2.4.1 *(Parthasarathy [20], Ørsted and Wolf [19]) The K_0-finite part of the Borel-de Siebenthal discrete series representation $\pi_{\gamma+\rho_{\mathfrak{g}}}$ is isomorphic to $\oplus_{m\geq 0} H^s(Y; \mathbb{E}_\gamma \otimes \mathbb{S}^m(\mathfrak{u}_{-1}))$ where $s = \dim Y$ and moreover, it is K_1-admissible.*

The K_1-admissibility of the Borel-de Siebenthal discrete series also follows from the work of Kobayashi [16] who obtained a criterion for the admissibility of restriction of certain representations to reductive subgroups in a more general context.

The set $\Delta_{\mathfrak{k}}$ is the root system of $\mathfrak{k}$ with respect to the Cartan subalgebra $\mathfrak{t}$ for which $(\Psi\setminus\{\nu\})\cup\{-\mu\}$ is a set of simple roots defining a positive system of roots, namely, $\Delta_0^+\cup\Delta_{-2}$. On the other hand $(\mathfrak{k}, \mathfrak{t})$ inherits a positive root system from $(\mathfrak{g}, \mathfrak{t})$, namely, $\Delta_{\mathfrak{k}}^+ := \Delta_0^+ \cup \Delta_2$. Let ϵ denote the lowest root of Δ_2 (so that $\beta \geq \epsilon$ for all $\beta \in \Delta_2$). Then $\Psi_{\mathfrak{k}} := (\Psi \setminus \{\nu\}) \cup \{\epsilon\}$ is the set of simple roots in $\Delta_{\mathfrak{k}}^+$.[3] Lemma 2.4.2 brings out the relation between these two positive system. Also $\Delta_{\mathfrak{l}} := \Delta_0$ is the root system of $(\mathfrak{l}, \mathfrak{t})$ for which $\Psi_{\mathfrak{l}} := \Psi \setminus \{\nu\}$ is the set of simple roots defining the positive system $\Delta_{\mathfrak{l}}^+ := \Delta_0^+$. Let $w_{\mathfrak{k}}^0$ (respectively, $w_{\mathfrak{l}}^0$) denote the longest element of the Weyl group of $(\mathfrak{k}, \mathfrak{t})$ (respectively, of $(\mathfrak{l}, \mathfrak{t})$) with respect to the positive system $\Delta_{\mathfrak{k}}^+$ (respectively, $\Delta_{\mathfrak{l}}^+$).

Write $\gamma = \gamma_0 + t\nu^*$, where $\langle \gamma_0, \nu^* \rangle = 0$. The assumption that γ is an $\mathfrak{l}$-dominant integral weight and that $\gamma + \rho_{\mathfrak{g}}$ is negative on positive roots of $\mathfrak{g}$ complementary to those of $\mathfrak{l}$ implies that t is 'sufficiently negative'. That is, t is real and it satisfies the conditions (see [19, Theorem 2.12]): [4]

$$t < -1/2\langle \gamma_0 + \rho_{\mathfrak{g}}, \mu \rangle \quad \text{and} \quad t < -\langle \gamma_0 + \rho_{\mathfrak{g}}, w_{\mathfrak{l}}^0(\nu) \rangle. \tag{2.5}$$

The adjoint action of L_0 on $\mathfrak{g}$ yields L_0-representations on $\mathfrak{u}_i, i = \pm 1, \pm 2$, which are irreducible. The highest (resp. lowest) weights of $\mathfrak{u}_{-2}, \mathfrak{u}_{-1}$, $j = 1, 2$, are $-\epsilon, -\nu$ (resp. $-\mu, w_{\mathfrak{l}}^0(-\nu)$) respectively. Let $\epsilon^* \in (i\mathfrak{t}_0)^*$ be the fundamental weight of $(\mathfrak{k}, \mathfrak{t})$ corresponding to the simple root ϵ of the simple system $\Psi_{\mathfrak{k}}$ and let $w_Y := w_{\mathfrak{k}}^0 w_{\mathfrak{l}}^0$.

Lemma 2.4.2 *(i)* $\epsilon^* = ||\epsilon||^2 \nu^*/4$.
(ii) $w_Y(\Delta_0^+ \cup \Delta_{-2}) = \Delta_0^+ \cup \Delta_2$, $\Psi_{\mathfrak{k}} = w_Y((\Psi \setminus \{\nu\}) \cup \{-\mu\})$.
(iii) If $\lambda \in \mathfrak{t}^*$, then $\lambda = \lambda' + a\nu^*$ where $a = \langle \lambda, \nu^* \rangle / ||\nu^*||^2$ and $\lambda' \in (\mathfrak{t} \cap [\mathfrak{l}, \mathfrak{l}])^* = \{\nu^*\}^\perp$.
(iv) The sum $\sum_{\beta\in\Delta_2} \beta = c\epsilon^*$ where $c = s||\epsilon||^2/2||\epsilon^*||^2$ (with $s = |\Delta_2|$) is an integer.

[3] Ørsted and Wolf [19] denote by $\Psi_{\mathfrak{k}}$ the set $(\Psi \setminus \{\nu\}) \cup \{-\mu\}$.
[4] The decomposition of $\gamma = \gamma_0 + t\nu^*$ used in [19, Theorem 2.12] is different.

Proof: We will only prove (iv), the proofs of the remaining parts being straightforward.

Observe that if E is a finite dimensional representation of $\mathfrak{l}$, then the sum λ of all weights of E, counted with multiplicity, is a multiple of ϵ^*. This follows from the fact that the top-exterior $\Lambda^{\dim(E)}(E)$ is a one dimensional representation of $\mathfrak{l}$ isomorphic to $\mathbb{C}_\lambda$. Applying this to $\mathfrak{u}_2$, we obtain that $\sum_{\beta\in\Delta_2}\beta = c\epsilon^*$. Clearly c is an integer since the β are roots of $\mathfrak{k}$ and so $\sum_{\beta\in\Delta_2}\beta$ is in the weight lattice of $\mathfrak{k}$. $\square$

2.4.4 Realization of Borel-de Siebenthal Discrete Series from Parthasarathy's Construction in [20]

Here we give a brief description of Parthasarathy's [20] results on his construction of certain unitarizable $(\mathfrak{g}, K_0)$-modules, which includes those associated to the Borel-de Siebenthal discrete series. We also explain how to obtain the description of Borel-de Siebenthal discrete series due to Ørsted and Wolf as Borel-de Siebenthal discrete series from Parthasarathy's results.

Let G_0 be a non-compact real semisimple Lie group with finite centre and let K_0 be a maximal compact subgroup of G_0. Assume that G_0 contains a compact Cartan subgroup $T_0 \subset K_0$. Let P be a positive root system of $(\mathfrak{g}, \mathfrak{t})$ and let $\mathfrak{p}_+$ (resp. $\mathfrak{p}_-$) equal $\sum \mathfrak{g}_\alpha$ where the sum is over positive (respectively negative) non-compact roots. Suppose that $[\mathfrak{p}_+, [\mathfrak{p}_+, \mathfrak{p}_+]] = 0$. Let B denote the Borel subgroup of $K = K_0^{\mathbb{C}}$ such that $Lie(B) = \mathfrak{t} \oplus \sum \mathfrak{g}_\alpha$ where the sum is over positive compact roots. Let $P_\mathfrak{k}$ and P_n denote the set of compact and non-compact roots in P respectively.

Write $\rho = (1/2)\sum_{\alpha\in P}\alpha$ and $w_\mathfrak{k}, w_\mathfrak{g}$ the longest element of the Weyl groups of $\mathfrak{k}$ and $\mathfrak{g}$ with respect to the positive systems $P_\mathfrak{k}$ and P respectively. Let λ be the highest weight of an irreducible representation of K_0 such that the following "regularity" conditions hold: (i) $\lambda + \rho$ is dominant for $\mathfrak{g}$, and, (ii) $H^j(K/B; \Lambda^q(\mathfrak{p}_-) \otimes \mathbb{L}_{\lambda+2\rho}) = 0$ for all $0 \le j < d, 0 \le q \le \dim \mathfrak{p}_-$ where $d := \dim_\mathbb{C} K/B$ and $\mathbb{L}_\varpi$ denotes the holomorphic line bundle over K/B associated to a character ϖ of T extended to a character of B in the usual way. From [11, Lemma 9.1] we see that condition (ii) holds for λ since $[\mathfrak{p}_+, [\mathfrak{p}_+, \mathfrak{p}_+]] = 0$. Parthasarathy shows that the $\mathfrak{k}$-module structure on $\oplus_{m\ge0}H^d(K/B; \mathbb{L}_{\lambda+2\rho} \otimes S^m(\mathfrak{p}_+))$ extends to a $\mathfrak{g}$-module structure which is unitarizable.

Suppose that $\lambda + \rho$ is regular dominant for $\mathfrak{g}$ so that condition (i) holds. Then, the $\mathfrak{g}$-module $\oplus_{m\ge0}H^d(K/B; \mathbb{L}_{\lambda+2\rho} \otimes S^m(\mathfrak{p}_+))$ is the K_0-finite part of a discrete series representation π with Harish-Chandra parameter $\lambda + \rho$ and Harish-Chandra root order P. The Blattner parameter is $\lambda + 2\rho_n$. See [20, p.3-4].

Now start with a Borel-de Siebenthal positive system Δ^+ where G_0 is further assumed to be simply-connected and simple. Assume also that G_0/K_0 is not Hermitian symmetric. The Harish-Chandra root order for the Borel-de Siebenthal discrete series $\pi_{\gamma+\rho_\mathfrak{g}}$ is $\Delta_0^+ \cup \Delta_{-1} \cup \Delta_{-2}$. The Blattner parameter for $\pi_{\gamma+\rho_\mathfrak{g}}$ is $\gamma + \sum_{\beta\in\Delta_2}\beta$. Thus, setting $P := \Delta_0^+ \cup \Delta_{-1} \cup \Delta_{-2}$, we have $P_n = \Delta_{-1}$, $\mathfrak{p}_+ = \mathfrak{u}_{-1}$ and $[\mathfrak{p}_+, [\mathfrak{p}_+, \mathfrak{p}_+]] = 0$ holds.

Finally, we have the isomorphism [20, equation (9.20)]

$$H^d(K/B; \mathbb{L}_{\lambda+2\rho} \otimes \mathbb{S}^m(\mathfrak{p}_+)) \cong H^s(Y; \mathbb{E}_{\lambda+2\rho_n} \otimes \mathbb{E}_\kappa \otimes \mathbb{S}^m(\mathfrak{p}_+))$$

of K-representations where $\kappa = \sum_{\beta\in\Delta_{-2}} \beta$. Note that $\mathbb{E}_\kappa$ is the canonical line bundle of Y. From Parthasarathy's description of the K_0-finite part of the discrete series representation $\pi_{\lambda+\rho}$ and using the above isomorphism we have

$$\begin{aligned}(\pi_{\lambda+\rho})_{K_0} &= \oplus_{m\geq 0} H^d(K/B; \mathbb{L}_{\lambda+2\rho} \otimes \mathbb{S}^m(\mathfrak{p}_+)) \\ &\cong \oplus_{m\geq 0} H^s(Y; \mathbb{E}_{\lambda+2\rho_n} \otimes \mathbb{E}_\kappa \otimes \mathbb{S}^m(\mathfrak{p}_+)) \\ &= \oplus_{m\geq 0} H^s(Y; \mathbb{E}_{\lambda+2\rho_n+\kappa} \otimes \mathbb{S}^m(\mathfrak{p}_+)) \\ &= \oplus_{m\geq 0} H^s(Y; \mathbb{E}_\gamma \otimes \mathbb{S}^m(\mathfrak{u}_{-1}))\end{aligned}$$

where $\gamma := \lambda + 2\rho_n + \kappa$. Note that $\gamma + \rho_{\mathfrak{g}} = \lambda + 2\rho_n + \kappa + \rho_{\mathfrak{g}} = \lambda + \rho$. Therefore, by [19], the module in the last line is the K_0-finite part of $\pi_{\gamma+\rho_{\mathfrak{g}}}$. Hence we see that Parthasarathy's description of $(\pi_{\gamma+\rho_{\mathfrak{g}}})_{K_0}$ agrees with that of Ørsted and Wolf.

2.5 A theorem of Schmid

Let $\mathfrak{g}_0$ be a non-compact simple Lie algebra over $\mathbb{R}$ with $\mathfrak{g}$:= the complexification of $\mathfrak{g}_0$, is simple. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be a Cartan decomposition with the corresponding Cartan involution s_0. Then $(\mathfrak{g}_0, s_0)$ is an orthogonal symmetric Lie algebra of type *III*. Assume that $\mathfrak{c}_0 := \mathfrak{z}_{\mathfrak{k}_0}$ is non-zero. Let G_0 be a connected Lie group with Lie algebra $\mathfrak{g}_0$ and K_0 be a Lie subgroup of G_0 corresponding to the subalgebra $\mathfrak{k}_0$. Then the orthogonal symmetric Lie algebra $(\mathfrak{g}_0, s_0)$ is associated with the irreducible Hermitian symmetric space G_0/K_0 of the non-compact type. Let $\mathfrak{t}_0$ be a maximal abelian subalgebra of $\mathfrak{k}_0$. Then $\mathfrak{c}_0 \subset \mathfrak{t}_0$ and $\mathfrak{t}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$. Let $\mathfrak{k}$, $\mathfrak{p}$, $\mathfrak{t}$, $\mathfrak{c}$ denote the complexifications of $\mathfrak{k}_0$, $\mathfrak{p}_0$, $\mathfrak{t}_0$, $\mathfrak{c}_0$ respectively. Let Δ := the set of non-zero roots of $(\mathfrak{g}, \mathfrak{t})$ and $\Delta_{\mathfrak{k}}$, Δ_n denote the set of compact and non-compact roots in Δ respectively. Let Δ^+ be a special positive system of Δ as in §2.3.1 with $\Psi := \{\psi_1, \psi_2, \cdots \psi_n\}$, the set of simple roots in Δ^+. Then Ψ contains exactly one non-compact root, say ϵ. Let $\Delta_{\mathfrak{k}}^+ := \Delta^+ \cap \Delta_{\mathfrak{k}}$ and $\Delta_n^+ := \Delta^+ \cap \Delta_n$. Let μ denote the highest root of $\mathfrak{g}$. Then $\mu \in \Delta_n^+$. Define $\mathfrak{p}_+ := \sum_{\beta\in\Delta_n^+} \mathfrak{g}_\beta$ and $\mathfrak{p}_- := \sum_{-\beta\in\Delta_n^+} \mathfrak{g}_\beta$. We have $\mathfrak{p}_+$, $\mathfrak{p}_-$ are abelian; $[\mathfrak{k}, \mathfrak{p}_+] \subset \mathfrak{p}_+$, $[\mathfrak{k}, \mathfrak{p}_-] \subset \mathfrak{p}_-$ and $\mathfrak{p} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$. In fact $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are irreducible K_0-modules with highest weights μ and $-\epsilon$ respectively. Let $\langle\ ,\ \rangle$ be the positive definite symmetric bilinear form on $(i\mathfrak{t}_0)^*$ induced from the Killing form of $\mathfrak{g}$. Let β_1, $\beta_2 \in -(\Delta_n^+)$. Then β_1, β_2 are called **strongly orthogonal roots** if $\beta_1 + \beta_2$ and $\beta_1 - \beta_2$ are not roots of $(\mathfrak{g}, \mathfrak{t})$. Since $\mathfrak{p}_-$ is abelian, $\beta_1 + \beta_2$ is not a root. Hence β_1 is strongly orthogonal to β_2 *if and only if* $\langle\beta_1, \beta_2\rangle = 0$. Let Γ be a maximal set of strongly orthogonal roots in $-(\Delta_n^+)$. The cardinality of Γ equals the rank of G_0/K_0, that is, the maximum dimension of the Euclidean space that can be imbedded in G_0/K_0 as a totally geodesic submanifold. See [10, §6 of Chapter V, Cor. 7.6 of Chapter VIII].

We now consider a specific maximal set $\Gamma \subset -(\Delta_n^+)$ of strongly orthogonal roots whose elements $\gamma_1, \ldots, \gamma_r$ are inductively defined as follows. Fix an ordering of the simple roots in Δ^+ and consider the induced lexicographic ordering on Δ. Now let $\gamma_1 := -\epsilon$, the highest root in $-(\Delta_n^+)$. Having defined $\gamma_1, \ldots, \gamma_i$, let γ_{i+1} be the highest root in $-(\Delta_n^+)$

which is orthogonal to $\gamma_j, 1 \leq j \leq i$.

Denote by E_γ the irreducible K_0-representation with highest weight γ. We have the following decomposition theorem due to Schmid [23], which is a far reaching generalization of the fact that the symmetric power of the defining representation of the special unitary group is irreducible. See [14, Theorem 10.25].

Theorem 2.5.1 *(Schmid [23]) With the above notations, one has the decomposition $S^m(\mathfrak{p}_-)$ as an K_0-representation*

$$S^m(\mathfrak{p}_-) = \oplus E_{a_1\gamma_1+\cdots+a_r\gamma_r}$$

where the sum is over all partitions $a_1 \geq \cdots \geq a_r \geq 0$ of m. □

Let ϵ^* be the fundamental weight corresponding to the non-compact simple root ϵ and $\mathfrak{c}^*$ be the dual space of $\mathfrak{c}$. Note that $\mathfrak{c}^* = \mathbb{C}\epsilon^*$. Hence E_γ is one dimensional precisely when $\gamma = k\epsilon^*$ for some integer k. Now we see from the above theorem that $S^m(\mathfrak{p}_-)$ admits a one dimensional K_0-subrepresentation precisely when there exists non-negative integers $a_1 \geq \cdots \geq a_r \geq 0$ such that $\sum a_i\gamma_i = c_0\epsilon^*$ for some constant c_0. The first part of the following proposition gives a criterion for this to happen.

Proposition 2.5.2 *(i) Let $\Gamma = \{\gamma_1, \ldots, \gamma_r\}$ be the maximal set of strongly orthogonal roots obtained as above. Let $w^0_{\mathfrak{g}}$ denote the longest element of the Weyl group of $(\mathfrak{g}, \mathfrak{t})$ with respect to the positive root system Δ^+. Suppose that $w^0_{\mathfrak{g}}(-\epsilon) = \epsilon$. Then $\sum_{1\leq i\leq r}\gamma_i = -2\epsilon^*$. Conversely, if $\sum_{1\leq i\leq r} a_i\gamma_i$ is a non-zero multiple of ϵ^* where $a_i \in \mathbb{Z}$, then $a_i = a_j\ \forall 1 \leq i, j \leq r$, and, $w^0_{\mathfrak{g}}(\epsilon) = -\epsilon$.*

(ii) Moreover, for any $1 \leq j \leq r$, if the coefficient of a compact simple root α of $\mathfrak{g}$ in the expression of $\sum_{1\leq i\leq j}\gamma_i$ is non-zero, then $\sum_{1\leq i\leq j}\gamma_i$ is orthogonal to α (without any assumption on $w^0_{\mathfrak{g}}$).

Proof: Our proof involves a straightforward verification using the classification of irreducible Hermitian symmetric pairs of non-compact type. See [10, §6, Ch. X]. We follow the labelling conventions of Bourbaki [4, Planches I-VII] and make use of the description of the root system, especially in cases E-III and E-VII. Note that $-w^0_{\mathfrak{g}}$ induces an automorphism of the Dynkin diagram of $\mathfrak{g}$. In particular, $-w^0_{\mathfrak{g}}(\epsilon) = \epsilon$ when the Dynkin diagram of $\mathfrak{g}$ admits no symmetries.

Case A III: $(\mathfrak{g}_0, \mathfrak{k}_0) = (\mathfrak{su}(p, q), \mathfrak{s}(\mathfrak{u}(p) \times \mathfrak{u}(q))), p \leq q$. The simple roots are $\psi_i = \varepsilon_i - \varepsilon_{i+1}$, $1 \leq i \leq p+q-1$. If $p+q > 2$, then $-w^0_{\mathfrak{g}}$ induces the order 2 automorphism of the Dynkin diagram of $\mathfrak{g}$, which is of type A_{p+q-1}. Thus $-w^0_{\mathfrak{g}}(\psi_j) = \psi_{p+q-j}$ in any case. The simple non-compact root is $\epsilon = \psi_p = \varepsilon_p - \varepsilon_{p+1}$, all other simple roots are compact roots. Therefore $-w^0_{\mathfrak{g}}(\psi_p) = \psi_p$ if and only if $p = q$. On the other hand, the set of negative non-compact roots $-(\Delta^+_n) = \{\varepsilon_j - \varepsilon_i \mid 1 \leq i \leq p < j \leq p+q\}$ and $\Gamma = \{\gamma_j := \varepsilon_{p+j} - \varepsilon_{p-j+1} \mid 1 \leq j \leq p\}$. If $p = q$, then $\sum_{1\leq j\leq p}\gamma_j = \sum_{1\leq j\leq q}\varepsilon_{p+j} - \sum_{1\leq j\leq p}\varepsilon_{p-j+1}$. Using the fact that $\sum_{1\leq i\leq p+q}\varepsilon_i = 0$, we see that $\sum_{1\leq j\leq p}\gamma_j = -2(\sum_{1\leq j\leq p}\varepsilon_j) = -2\epsilon^*$ if $p = q$.

For the converse part, assume that $\sum_j a_j\gamma_j = m\epsilon^*, m \neq 0$. It is evident when $p < q$ that $\sum a_j\gamma_j$ is not a multiple of ϵ^* (since ε_{p+q} does not occur in the sum). Since the $\gamma_j, 1 \leq j \leq p$, are linearly independent, the uniqueness of the expression of ϵ^* as a linear combination of the γ_j implies that $a_j = a_1$ for all j.

To prove (*ii*), note that $\gamma_1 = -\epsilon$ and $\gamma_j = -(\epsilon + \psi_{p-j+1} + \cdots + \psi_{p-1} + \psi_{p+1} + \cdots + \psi_{p+j-1})$, $2 \leq j \leq p$. So the only compact simple roots whose coefficients are non-zero in the expression of $\sum_{1\leq i\leq j}\gamma_i (j > 1)$ are ψ_i $(p - j + 1 \leq i \leq p + j - 1,\ i \neq p)$. Note that $\sum_{1\leq i\leq j}\gamma_i = -(\varepsilon_{p-j+1} + \cdots + \varepsilon_p - \varepsilon_{p+1} - \cdots - \varepsilon_{p+j})$. Hence $\langle\sum_{1\leq i\leq j}\gamma_i, \psi_i\rangle = 0$ for all $p - j + 1 \leq i \leq p + j - 1, i \neq p$.

Case D III: $(\mathfrak{so}^*(2p), \mathfrak{u}(p)), p \geq 4$. The simple roots are $\psi_i = \varepsilon_i - \varepsilon_{i+1}$, $1 \leq i \leq p-1$ and $\psi_p = \varepsilon_{p-1} + \varepsilon_p$. In this case the only non-compact simple root $\epsilon = \psi_p = \varepsilon_{p-1} + \varepsilon_p$; $\epsilon^* = (1/2)(\sum_{1\leq j\leq p}\varepsilon_j)$. The set of non-compact positive roots is $\Delta_n^+ = \{\varepsilon_i + \varepsilon_j \mid 1 \leq i < j \leq p\}$ and $\Gamma = \{\gamma_j = -(\varepsilon_{p-2j+1} + \varepsilon_{p-2j+2}) \mid 1 \leq j \leq \lfloor p/2\rfloor\}$. So $\sum_{1\leq j\leq\lfloor p/2\rfloor}\gamma_j = -2\epsilon^*$ if p is even. On the other hand $w_{\mathfrak{g}}^0$ maps ϵ to $-\epsilon$ precisely when p is even.

When p is odd, it is readily seen that $\sum_j a_j\gamma_j$ is not a non-zero multiple of ϵ^* since ε_1 does not occur in the sum.

To prove (*ii*), note that $\gamma_1 = -\epsilon$ and $\gamma_j = -(\epsilon + \psi_{p-2j+1} + 2\psi_{p-2j+2} + \cdots + 2\psi_{p-2} + \psi_{p-1})$, $2 \leq j \leq \lfloor p/2\rfloor$. So the only compact simple roots whose coefficients are non-zero in the expression of $\sum_{1\leq i\leq j}\gamma_i (j > 1)$ are ψ_i $(p - 2j + 1 \leq i \leq p - 1)$. Note that $\sum_{1\leq i\leq j}\gamma_i = -(\varepsilon_{p-2j+1} + \cdots + \varepsilon_p)$. Hence $\langle\sum_{1\leq i\leq j}\gamma_i, \psi_i\rangle = 0$ for all $p - 2j + 1 \leq i \leq p - 1$.

Case BD I (rank= 2): $(\mathfrak{so}(2, p), \mathfrak{so}(2) \times \mathfrak{so}(p)), p > 2$. We have $\epsilon = \psi_1 = \varepsilon_1 - \varepsilon_2, \epsilon^* = \varepsilon_1$ and $w_{\mathfrak{g}}^0(\epsilon) = -\epsilon$. Now $\Delta_n^+ = \{\varepsilon_1 \pm \varepsilon_j \mid 2 \leq j \leq p\} \cup \{\varepsilon_1\}$ if p is odd and is equal to $\{\varepsilon_1 \pm \varepsilon_j \mid 2 \leq j \leq p\}$ if p is even. For any $p, \Gamma = \{\gamma_1 = -(\varepsilon_1 - \varepsilon_2), \gamma_2 = -(\varepsilon_1 + \varepsilon_2)\}$. Clearly $a_1\gamma_1 + a_2\gamma_2 = m\epsilon^*$ if and only if $a_1 = a_2$. Since in this case rank is 2 and $\gamma_1 + \gamma_2 = -2\epsilon^*$, (*ii*) is obvious.

Case C I: $(\mathfrak{sp}(p, \mathbb{R}), \mathfrak{u}(p)), p \geq 3$. The simple roots are $\psi_i = \varepsilon_i - \varepsilon_{i+1}$, $1 \leq i \leq p - 1$ and $\psi_p = 2\varepsilon_p$. We have $\epsilon = 2\varepsilon_p, \epsilon^* = \sum_{1\leq j\leq p}\varepsilon_j$, and $w_{\mathfrak{g}}^0(\epsilon) = -\epsilon$. Also $\Delta_n^+ = \{\varepsilon_i + \varepsilon_j \mid 1 \leq i \leq j \leq p\}$. Therefore $\Gamma = \{\gamma_j := -2\varepsilon_{p-j+1} \mid 1 \leq j \leq p\}$. Evidently $\sum_{1\leq j\leq p}\gamma_j = -2\epsilon^*$.

The converse part is obvious in this case.

To prove (*ii*), note that $\gamma_1 = -\epsilon$ and $\gamma_j = -(\epsilon + 2\psi_{p-j+1} + \cdots + 2\psi_{p-1})$, $2 \leq j \leq p$. So the only compact simple roots whose coefficients are non-zero in the expression of $\sum_{1\leq i\leq j}\gamma_i (j > 1)$ are ψ_i $(p - j + 1 \leq i \leq p - 1)$. Note that $\sum_{1\leq i\leq j}\gamma_i = -2(\varepsilon_{p-j+1} + \cdots + \varepsilon_p)$. Hence $\langle\sum_{1\leq i\leq j}\gamma_i, \psi_i\rangle = 0$ for all $p - j + 1 \leq i \leq p - 1$.

Case E III: $(\mathfrak{e}_{6,-14}, \mathfrak{so}(10)\oplus\mathfrak{so}(2))$. The simple roots are $\psi_1 = (1/2)(\varepsilon_8 - \varepsilon_6 - \varepsilon_7 + \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5), \psi_2 = \varepsilon_1 + \varepsilon_2, \psi_3 = \varepsilon_2 - \varepsilon_1, \psi_4 = \varepsilon_3 - \varepsilon_2, \psi_5 = \varepsilon_4 - \varepsilon_3, \psi_6 = \varepsilon_5 - \varepsilon_4$. In this case the rank is 2, $\epsilon = \psi_1 = (1/2)(\varepsilon_8 - \varepsilon_6 - \varepsilon_7 + \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5)$, and $\epsilon^* = (2/3)(\varepsilon_8 - \varepsilon_7 - \varepsilon_6)$. We have $-w_{\mathfrak{g}}^0(\epsilon) = \psi_6 \neq \epsilon$. Now $\Delta_2 = \{(1/2)(\varepsilon_8 - \varepsilon_7 - \varepsilon_6 + \sum_{1\leq i\leq 5}(-1)^{s(i)}\varepsilon_i) \mid s(i) = 0, 1, \sum_i s(i) \equiv 0 \bmod 2\}$. There are five roots in Δ_{-2} which are orthogonal to $\gamma_1 = -\epsilon$. Among these the highest is $\gamma_2 = -(1/2)(\varepsilon_8 - \varepsilon_6 - \varepsilon_7 - \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 - \varepsilon_5)$. Thus $\Gamma = \{\gamma_1, \gamma_2\}$. Now $a_1\gamma_1 + a_2\gamma_2$ is not a multiple of ϵ^* for any $a_1, a_2 \geq 0$ unless $a_1 = a_2 = 0$.

Note that $\gamma_2 = -(\epsilon + \psi_2 + 2\psi_3 + 2\psi_4 + \psi_5)$, $\gamma_1 + \gamma_2 = -(\varepsilon_8 - \varepsilon_7 - \varepsilon_6 - \varepsilon_5)$. Hence $\langle \gamma_1 + \gamma_2, \psi_i \rangle = 0$ for all $2 \leq i \leq 5$.

Case E VII: $(\mathfrak{e}_{7,-25}, \mathfrak{e}_6 \oplus \mathfrak{so}(2))$. The simple roots are $\psi_1 = (1/2)(\varepsilon_8 - \varepsilon_6 - \varepsilon_7 + \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5), \psi_2 = \varepsilon_1 + \varepsilon_2, \psi_3 = \varepsilon_2 - \varepsilon_1, \psi_4 = \varepsilon_3 - \varepsilon_2, \psi_5 = \varepsilon_4 - \varepsilon_3, \psi_6 = \varepsilon_5 - \varepsilon_4, \psi_7 = \varepsilon_6 - \varepsilon_5$. In this case rank= 3, $\epsilon = \psi_7 = \varepsilon_6 - \varepsilon_5, \epsilon^* = \varepsilon_6 + (1/2)(\varepsilon_8 - \varepsilon_7), w^0_{\mathfrak{k}}(-\epsilon) = \epsilon$. $\Delta^+_n = \{\varepsilon_6 - \varepsilon_j, \varepsilon_6 + \varepsilon_j, 1 \leq j \leq 5\} \cup \{\varepsilon_8 - \varepsilon_7\} \cup \{(1/2)(\varepsilon_8 - \varepsilon_7 + \varepsilon_6 + \sum_{1\leq j\leq 5}(-1)^{s(j)}\varepsilon_j) \mid s(j) = 0, 1, \sum_j s(j) \equiv 1 \bmod 2\}$. Now $\Gamma = \{\gamma_1 = \varepsilon_5 - \varepsilon_6, \gamma_2 = -\varepsilon_5 - \varepsilon_6, \gamma_3 = \varepsilon_7 - \varepsilon_8\}$ and we have $\gamma_1 + \gamma_2 + \gamma_3 = -2\epsilon^*$. The converse part is easily established.

We have $\gamma_2 = -(\epsilon + \psi_2 + \psi_3 + 2\psi_4 + 2\psi_5 + 2\psi_6), \gamma_1 + \gamma_2 = -2\varepsilon_6$. Hence $\langle \gamma_1 + \gamma_2, \psi_i \rangle = 0$ for all $2 \leq i \leq 6$. Also $\gamma_1 + \gamma_2 + \gamma_3 = -2\epsilon^*$. So (*ii*) is proved. □

As a corollory we obtain the following.

Proposition 2.5.3 *Suppose that G_0 and K_0 are as above and K'_0 be the connected Lie subgroup of K_0 corresponding to the semisimple ideal $[\mathfrak{k}_0, \mathfrak{k}_0]$ of $\mathfrak{k}_0$. Let $\pi_{\gamma+\rho_{\mathfrak{g}}}$ be a holomorphic discrete series representation of G_0, where $\rho_{\mathfrak{g}} = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$. If $w^0_{\mathfrak{g}}(\epsilon) = -\epsilon$, then the K_0-finite part $(\pi_{\gamma+\rho_{\mathfrak{g}}})_{K_0}$ of $\pi_{\gamma+\rho_{\mathfrak{g}}}$ is not K'_0-admissible. Conversely, if a holomorphic discrete series representation $\pi_{\gamma+\rho_{\mathfrak{g}}}$ of G_0 is not K'_0-admissible, then $w^0_{\mathfrak{g}}(\epsilon) = -\epsilon$.*

Proof: One has the following description of $(\pi_{\gamma+\rho_{\mathfrak{g}}})_{K_0}$ due to Harish-Chandra: $(\pi_{\gamma+\rho_{\mathfrak{g}}})_{K_0} = \oplus_{m\geq 0} E_\gamma \otimes S^m(\mathfrak{p}_-)$. Suppose that $w^0_{\mathfrak{g}}(\epsilon) = -\epsilon$. Then by Proposition 2.5.2 and Schmid's theorem 2.5.1 we see that $E_\gamma \otimes E_{-a\epsilon^*}$ occurs in $(\pi_{\gamma+\rho_{\mathfrak{g}}})_{K_0}$ for infinitely many values of a. Since $E_{-\epsilon^*}$ is one dimensional, it is trivial as an K'_0-representation. Hence $(\pi_{\gamma+\rho_{\mathfrak{g}}})_{K_0}$ is not K'_0-admissible.

Conversely, since $\pi_{\gamma+\rho_{\mathfrak{g}}}$ is not K'_0-admissible, in view of Proposition 5.1.1 we have, $(\pi_{\gamma+\rho_{\mathfrak{g}}})_{K_0}$ is not K'_0-admissible. Suppose that $w^0_{\mathfrak{g}}(-\epsilon) \neq \epsilon$. Any K'_0-type in $(\pi_{\gamma+\rho_{\mathfrak{g}}})_{K_0}$ is of the form $E_{\sum a_j\gamma_j+\kappa}$ (considered as K'_0-module) for some weight κ of E_γ. Since the set of weights of E_γ is finite, $(\pi_{\gamma+\rho_{\mathfrak{g}}})_{K_0}$ is not K'_0 admissible implies $S^*(\mathfrak{p}_-)$ is not K'_0 admissible. If $E_{\sum a_j\gamma_j} \cong E_{\sum b_j\gamma_j}$ as K'_0-modules, then $\sum(a_j - b_j)\gamma_j$ is a multiple of ϵ^*. Proposition 2.5.2 implies that $a_j = b_j, 1 \leq j \leq r$. □

We conclude this section with the following remark.

Remark 2.5.4 Let Γ be the set of strongly orthogonal roots as in Proposition 2.5.2 and suppose that $w^0_{\mathfrak{g}}(\epsilon) = -\epsilon$. Then:
(i) It follows from the explicit description of Γ in each case that $w^0_{\mathfrak{k}}(\gamma_j) = \gamma_{r+1-j} = -w_0(\gamma_j), 1 \leq j \leq r$, where $w^0_{\mathfrak{k}}$ is the longest element of $(\mathfrak{k}, \mathfrak{t})$ with respect to the positive system $\Delta^+_{\mathfrak{k}}$ and $w_0 = w^0_{\mathfrak{g}} w^0_{\mathfrak{k}}$. In particular $-\mu \in \Gamma$.
(ii) For any w in the Weyl group of $(\mathfrak{k}, \mathfrak{t})$, $\sum_{\gamma\in\Gamma} w(\gamma) = w(\sum_{\gamma\in\Gamma}\gamma) = -2w(\epsilon^*) = -2\epsilon^*$.
(iii) Note that $\|\gamma_i\| = \|\epsilon\|, 1 \leq i \leq r$. This property holds even without the assumption that $w^0_{\mathfrak{g}}(\epsilon) = -\epsilon$.

2.6 Littelmann's path model

Although Littelmann has constructed his path model in the generality of complex symmetrizable Kac-Moody algebras, we shall contend ourselves with the finite dimensional case. Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Let X denote the weight lattice of $(\mathfrak{g}, \mathfrak{h})$ and $\langle\ ,\ \rangle$ be the positive definite symmetric bilinear form on $X \otimes_{\mathbb{Z}} \mathbb{R}$ induced from the Killing form of $\mathfrak{g}$. By a, say, closed interval $[a, b] \subset \mathbb{Q}$, we mean the set $\{t \in \mathbb{Q} \ : \ a \leq t \leq b\}$ and define $\Pi := \{\pi : [0,1] \longrightarrow X \otimes_{\mathbb{Z}} \mathbb{Q} \ : \ \pi$ is piecewise linear with $\pi(0) = 0\}$. Two paths in Π are considered equivalent if one can be obtained from the other by a piecewise linear order preserving reparametrization of the interval $[0, 1]$. (We regard members of Π as equivalence classes of paths.) Let Δ^+ be a positibe root system of $(\mathfrak{g}, \mathfrak{h})$ and Ψ the set of simple roots in Δ^+. For an element w of the Weyl group W of $(\mathfrak{g}, \mathfrak{h})$ and a path $\pi \in \Pi$, let $w(\pi)$ be the path given by $w(\pi)(t) := w(\pi(t))$ for all $t \in [0, 1]$. Let $\pi_1, \pi_2 \in \Pi$. The concatenation of two paths π_1 and π_2, denoted by $\pi_1 * \pi_2$, is defined by

$$\pi_1 * \pi_2(t) := \begin{cases} \pi_1(2t) & \text{if } 0 \leq t \leq 1/2 \text{ and } t \in \mathbb{Q}; \\ \pi_1(1) + \pi_2(2t-1), & \text{if } 1/2 \leq t \leq 1 \text{ and } t \in \mathbb{Q}. \end{cases}$$

For $\alpha \in \Psi$, the Littelmann's **root operators** e_α **and** f_α on Π will be defined now. For that, let $\pi \in \Pi$ and $h_\alpha : [0, 1] \longrightarrow \mathbb{Q}$ be the function defined by

$$h_\alpha(t) := \frac{2\langle \pi(t), \alpha \rangle}{\langle \alpha, \alpha \rangle} \ \text{ for } \ t \in [0, 1].$$

Let $m_\alpha = \min\{h_\alpha(t) \ : \ t \in [0, 1]\}$. Then m_α is attained by h_α.

Definition of e_α **:** Let $t_1 \in [0, 1]$ be minimal such that $h_\alpha(t_1) = m_\alpha$.
If $m_\alpha \leq -1$ that is, $h_\alpha(0) - m_\alpha \geq 1$; fix $t_0 \in [0, t_1)$ maximal such that $h_\alpha(t) \geq m_\alpha + 1$ for $t \in [0, t_0]$. Note that $h_\alpha(t_0) = m_\alpha + 1$, $h_\alpha(t_1) = m_\alpha$, and for any $\epsilon > 0$, there exists $t \in (t_0, t_0 + \epsilon)$ such that $m_\alpha < h_\alpha(t) < m_\alpha + 1$. Choose a partition $t_0 = s_0 < s_1 < \cdots < s_r = t_1$ of $[t_0, t_1]$ such that either
(i) $h_\alpha(s_{i-1}) = h_\alpha(s_i)$ and $h_\alpha(t) \geq h_\alpha(s_{i-1})$ for $t \in [s_{i-1}, s_i]$, or
(ii) h_α is strictly decreasing on $[s_{i-1}, s_i]$ with $h_\alpha(t) \geq h_\alpha(s_{i-1})$ for $t \leq s_{i-1}$.
Setting $s_{-1} := 0$, $s_{r+1} := 1$ and $\pi_i(t) := \pi(s_{i-1} + t(s_i - s_{i-1})) - \pi(s_{i-1})$ for all $t \in [s_{i-1}, s_i]$, for $0 \leq i \leq r + 1$; we have $\pi = \pi_0 * \pi_1 * \cdots * \pi_{r+1}$.
Define

$$e_\alpha(\pi) := \begin{cases} 0 & \text{if } m_\alpha > -1, \\ \pi_0 * \eta_1 * \eta_2 * \cdots * \eta_r * \pi_{r+1} & \text{otherwise}; \end{cases} \tag{2.6}$$

where $\eta_i = \pi_i$ if $h_\alpha\Big|_{[s_{i-1}, s_i]}$ is as in (i) and $\eta_i = s_\alpha(\pi_i)$ if $h_\alpha\Big|_{[s_{i-1}, s_i]}$ is as in (ii).

Definition of f_α **:** Let $t_0 \in [0, 1]$ be maximal such that $h_\alpha(t_0) = m_\alpha$.
If $h_\alpha(1) - m_\alpha \geq 1$, fix $t_1 \in (t_0, 1]$ minimal such that $h_\alpha(t) \geq m_\alpha + 1$ for $t \in [t_1, 1]$. Note that $h_\alpha(t_0) = m_\alpha$, $h_\alpha(t_1) = m_\alpha + 1$, and for any $t_1 - t_0 > \epsilon > 0$, there exists $t \in (t_0, t_1 - \epsilon)$ such that $m_\alpha < h_\alpha(t) < m_\alpha + 1$. Choose a partition $t_0 = s_0 < s_1 < \cdots < s_r = t_1$ of $[t_0, t_1]$ such that either
(i) $h_\alpha(s_{i-1}) = h_\alpha(s_i)$ and $h_\alpha(t) \geq h_\alpha(s_{i-1})$ for $t \in [s_{i-1}, s_i]$, or

(ii) h_α is strictly increasing on $[s_{i-1}, s_i]$ with $h_\alpha(t) \geq h_\alpha(s_i)$ for $t \geq s_i$.
Setting $s_{-1} := 0,\ s_{r+1} := 1$ and $\pi_i(t) := \pi(s_{i-1} + t(s_i - s_{i-1})) - \pi(s_{i-1})$ for all $t \in [s_{i-1}, s_i]$, for $0 \leq i \leq r+1$; we have $\pi = \pi_0 * \pi_1 * \cdots * \pi_{r+1}$.
Define

$$f_\alpha(\pi) := \begin{cases} 0 & \text{if } h_\alpha(1) - m_\alpha < 1, \\ \pi_0 * \eta_1 * \eta_2 * \cdots * \eta_r * \pi_{r+1} & \text{otherwise;} \end{cases} \tag{2.7}$$

where $\eta_i = \pi_i$ if $h_\alpha\Big|_{[s_{i-1},s_i]}$ is as in (i) and $\eta_i = s_\alpha(\pi_i)$ if $h_\alpha\Big|_{[s_{i-1},s_i]}$ is as in (ii).

2.6.1 Some properties of the root operators

1. If $e_\alpha\pi \neq 0$, then $e_\alpha\pi(1) = \pi(1) + \alpha$. Similarly if $f_\alpha\pi \neq 0$, then $f_\alpha\pi(1) = \pi(1) - \alpha$.

2. If $e_\alpha\pi \neq 0$, then $f_\alpha e_\alpha\pi = \pi$ and if $f_\alpha\pi \neq 0$, then $e_\alpha f_\alpha\pi = \pi$.
In fact if $e_\alpha(\pi) \neq 0$, then for the path $e_\alpha\pi$, the minimum value of the function $\bar{h}_\alpha$ defined by $t \mapsto \dfrac{2\langle e_\alpha\pi(t), \alpha\rangle}{\langle\alpha,\alpha\rangle}$, is $m_\alpha + 1$. Therefore $\bar{h}_\alpha(1) - (m_\alpha + 1) = h_\alpha(1) + 2 - m_\alpha - 1 = h_\alpha(1) - m_\alpha + 1 \geq 1$, since m_α is the minimum value of the function h_α. So $f_\alpha e_\alpha\pi \neq 0$. Note that if t_0 and t_1 are as in the definition of e_α, then t_0 is maximal such that $\bar{h}_\alpha(t_0) = m_\alpha + 1$ and $t_1 \in (t_0, 1]$ is minimal such that $\bar{h}_\alpha(t) \geq m_\alpha + 2$ for $t \in [t_1, 1]$. In the interval $[t_0, t_1]$, the behaviour of the function $\bar{h}_\alpha$ is as in (i) or (ii) (in the definition of f_α) according as h_α behaves as in (i) or (ii) (in the definition of e_α). Hence $f_\alpha e_\alpha\pi = \pi$.
The proof of the other part is similar.

3. $e_\alpha^n\pi = 0$ *if and only if* $n > |m_\alpha|$ and $f_\alpha^n\pi = 0$ *if and only if* $n > \dfrac{2\langle\pi(1),\alpha\rangle}{\langle\alpha,\alpha\rangle} - m_\alpha$.

4. Let $\pi \in \Pi$ be such that $\pi(1) \in X$. Let n_1 and n_2 be maximal such that $e_\alpha^{n_1}\pi \neq 0$ and $f_\alpha^{n_2}\pi \neq 0$. Then $\dfrac{2\langle\pi(1),\alpha\rangle}{\langle\alpha,\alpha\rangle} = n_2 - n_1$.

5. $e_\alpha\pi = 0$ for all $\alpha \in \Psi$ *if and only if* the image of the path π shifted by ρ that is, $\mathrm{Im}(\rho + \pi)$ is contained in the interior of the dominant Weyl chamber, where $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$.

See [18] for further properties of the root operators.

Let λ be a dominant integral weight and $\pi \in \Pi$ be a path such that $\pi(1) = \lambda$ and $\mathrm{Im}(\pi)$ is completely contained in the dominant Weyl chamber. For such a π, let B_π denote the set of all non-zero paths in Π by applying the monomials in the root operators f_α, e_β (α, $\beta \in \Psi$) on π. Then π is the only path in B_π which lies completely in the dominant Weyl chamber and any element of B_π is of the form $f_{\alpha_1} f_{\alpha_2} \ldots f_{\alpha_s}\pi$, for some $\alpha_1, \alpha_2, \cdots, \alpha_s \in \Psi$. See [18].

For $\lambda \in X$, denote by π_λ the path $t \mapsto t\lambda$. If λ is a dominant integral weight, we will denote by B_λ the set B_{π_λ}. If $\eta \in B_\lambda$, we say that η is an LS-path of shape λ. See [17]. Note that if w is an element of the Weyl group W of $(\mathfrak{g}, \mathfrak{h})$, then $w(\pi_\lambda) = \pi_{w\lambda}$.

Proposition 2.6.1 *If λ is a dominant integral weight and π_λ is the path $t \mapsto t\lambda$, then for any $w \in W$, $w(\pi_\lambda)$ is an LS-path of shape λ.*

Proof: We will prove this by induction on the length $l(w)$ of the Weyl group element w.

If $l(w) = 1$ that is, $w = s_\alpha$ for some $\alpha \in \Psi$, then $s_\alpha(\pi_\lambda) = \pi_{s_\alpha \lambda}$. Indeed, if $a_{\lambda,\alpha} := \dfrac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}$ is 0, then $s_\alpha(\pi_\lambda) = \pi_\lambda$. Otherwise, $s_\alpha(\pi_\lambda) = f^{a_{\lambda,\alpha}} \pi_\lambda$, by property (3) of the root operators. In any case, $s_\alpha(\pi_\lambda)$ is an LS-path of shape λ.

Assume that $n > 1$ and the proposition is true for all Weyl group elements of length $n - 1$.

Let w be a Weyl group element with $l(w) = n$. Write $w = s_\alpha w_1$, where $\alpha \in \Psi$ and w_1 is an element of the Weyl group with $l(w_1) = n - 1$. By induction hypothesis, $w_1(\pi_\lambda)$ is an LS-path of shape λ. This implies $w_1 \lambda = \lambda - \sum_{\beta \in \Psi} n_\beta \beta$, where the n_β are non-negative integers. Hence $a_{w_1\lambda,\, \alpha} := \dfrac{2\langle w_1 \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} = \dfrac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} - \sum_{\beta \in \Psi} n_\beta \dfrac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \geq 0$, since $\langle \lambda, \alpha \rangle \geq 0$ and $\langle \beta, \alpha \rangle \leq 0$ for all $\alpha \in \Psi$. If $a_{w_1\lambda,\, \alpha} = 0$, then $w(\pi_\lambda) = s_\alpha(\pi_{w_1\lambda}) = \pi_{w_1\lambda}$ is an LS-path of shape λ. Otherwise $w(\pi_\lambda) = s_\alpha(\pi_{w_1\lambda}) = f_\alpha^{a_{w_1\lambda,\, \alpha}} \pi_{w_1\lambda}$, which is also an LS-path of shape λ. This completes the proof. □

Proposition 2.6.2 *Let λ_1, λ_2 be two dominant integral weights and π_1, $\pi_2 \in \Pi$ be $\mathfrak{g}$-dominant paths with $\pi_1(1) = \lambda_1$ and $\pi_2(1) = \lambda_2$. Assume that $\eta_1 \in B_{\pi_1}$ and $\eta_2 \in B_{\pi_2}$. Then for $\alpha \in \Psi$,*

$$f_\alpha(\eta_1 * \eta_2) = \begin{cases} (f_\alpha \eta_1) * \eta_2, & \textit{if } f_\alpha^n \eta_1 \neq 0 \textit{ and } e_\alpha^n \eta_2 = 0 \textit{ for some } n \geq 1; \\ \eta_1 * (f_\alpha \eta_2), & \textit{otherwise.} \end{cases} \tag{2.8}$$

Similarly,

$$e_\alpha(\eta_1 * \eta_2) = \begin{cases} \eta_1 * (e_\alpha \eta_2), & \textit{if } e_\alpha^n \eta_2 \neq 0 \textit{ and } f_\alpha^n \eta_1 = 0 \textit{ for some } n \geq 1; \\ (e_\alpha \eta_1) * \eta_2, & \textit{otherwise.} \end{cases} \tag{2.9}$$

Proof: Denote by m_1, the minimum of the function $t \mapsto \dfrac{2\langle \eta_1(t), \alpha \rangle}{\langle \alpha, \alpha \rangle}$, $t \in [0, 1]$ and by m_2, the minimum of $t \mapsto \dfrac{2\langle \eta_1(t), \alpha \rangle}{\langle \alpha, \alpha \rangle}$, $t \in [0, 1]$. Note that m_1, m_2 and $\eta_1(1)$ are all integers (see [18]) and

$$f_\alpha(\eta_1 * \eta_2) = \begin{cases} (f_\alpha \eta_1) * \eta_2, & \text{if } m_1 < \dfrac{2\langle \eta_1(1), \alpha \rangle}{\langle \alpha, \alpha \rangle} + m_2; \\ \eta_1 * (f_\alpha \eta_2), & \text{otherwise.} \end{cases}$$

Now $f_\alpha^n \eta_1 \neq 0$ and $e_\alpha^n \eta_2 = 0$ *if and only if* $-m_2 = |m_2| < n \leq \dfrac{2\langle \eta_1(1), \alpha \rangle}{\langle \alpha, \alpha \rangle} - m_1$, by property (3) of the root operators; which in turn equivalent to the condition $m_1 < \dfrac{2\langle \eta_1(1), \alpha \rangle}{\langle \alpha, \alpha \rangle} + m_2$.

The proof of the other part is similar. This completes the proof. □

2.6.2 Applications to representation theory

For a dominant integral weight λ, let V_λ denote the finite dimensional irreducible $\mathfrak{g}$-module with highest weight λ.

Theorem 2.6.3 *(Littelmann [18]) Let λ be a dominant integral weight. If $\pi \in \Pi$ be a $\mathfrak{g}$-dominant path with $\pi(1) = \lambda$, then Char $V_\lambda = \sum_{\eta \in B_\pi} e^{\eta(1)}$.*

Theorem 2.6.4 *(Littelmann [18]) Let λ_1 and λ_2 be two dominant integral weights. Let $\pi_1, \pi_2 \in \Pi$ be two paths such that $\pi_1(1) = \lambda_1$, $\pi_2(1) = \lambda_2$ and π_1, π_2 are $\mathfrak{g}$-dominant. Then the tensor product $V_{\lambda_1} \otimes V_{\lambda_2}$ of the finite dimensional irreducible $\mathfrak{g}$-modules V_{λ_1} and V_{λ_2} decomposes as*

$$V_{\lambda_1} \otimes V_{\lambda_2} \cong \oplus V_{\lambda_1 + \eta(1)},$$

where the sum is over all paths $\eta \in B_{\pi_2}$ such that $\pi_1 * \eta$ is $\mathfrak{g}$-dominant.

Let $\mathfrak{l}$ be a Levi subalgebra of $\mathfrak{g}$. Thus $\mathfrak{l}$ is a reductive subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$. The positive root system Δ^+ induces a positive root system of $(\mathfrak{l}, \mathfrak{h})$. If κ is a dominant integral weight of $\mathfrak{l}$, let E_κ denote the finite dimensional irreducible $\mathfrak{l}$-module with highest weight κ.

Theorem 2.6.5 *(Littelmann [18]) Let $\mathfrak{l}$ be a Levi subalgebra of the Lie algebra $\mathfrak{g}$ as above and λ be a dominant integral weight of $\mathfrak{g}$. If $\pi \in \Pi$ be a $\mathfrak{g}$-dominant path with $\pi(1) = \lambda$, then V_λ as an $\mathfrak{l}$-module can be decomposed as*

$$V_\lambda \cong \oplus E_{\eta(1)}, \tag{2.10}$$

where the sum is over all $\mathfrak{l}$-dominant paths in B_π.

Chapter 3

HOLOMORPHIC DISCRETE SERIES ASSOCIATED TO A BOREL-DE SIEBENTHAL DISCRETE SERIES

Unless explicitly stated, from here onwards we keep the notations of §2.4.3. In §3.1, we discuss the irreducible bounded symmetric domain dual to $Y = K_0/L_0 \cong K/K \cap Q$. In §3.2, we will see that for every Borel-de Siebenthal discrete series representation of G_0, there is a naturally associated holomorphic discrete series representation of K_0^* which is the dual of K_0 in K.

3.1 Hermitian symmetric space dual to Y

Recall that $Y = K_0/L_0 = K_1/L_1$ is an irreducible Hermitian symmetric space of the compact type. Also recall that $\theta = \mathrm{Ad}_K(\exp \frac{i\pi}{2} h_{\nu^*})$ and $(\mathfrak{k}_0, \theta|_{\mathfrak{k}_0})$ is an orthogonal symmetric Lie algebra of the compact type with $\mathfrak{l}_0$ the set of fixed points of $\theta|_{\mathfrak{k}_0}$. Notice that $\theta(\mathfrak{k}_1) \subset \mathfrak{k}_1$ and $\mathfrak{l}_1$ is the set of fixed points of $\theta|_{\mathfrak{k}_1}$. Hence $(\mathfrak{k}_1, \theta|_{\mathfrak{k}_1})$ is an irreducible orthogonal symmetric Lie algebra of the compact type and is associated with Y. Let $\mathfrak{k}_0^* \subset \mathfrak{k}$ (respectively, $\mathfrak{k}_1^* \subset \mathfrak{k}_1^{\mathbb{C}}$) denote the non-compact real form of $\mathfrak{k}$ (respectively, $\mathfrak{k}_1^{\mathbb{C}}$) dual to $(\mathfrak{k}_0, \theta|_{\mathfrak{k}_0})$ (respectively, $(\mathfrak{k}_1, \theta|_{\mathfrak{k}_1})$). We have $\mathfrak{k}_0^* = \mathfrak{k}_1^* \oplus \mathfrak{k}_2$. Let K_0^* denote the connected Lie subgroup of K with Lie algebra $\mathfrak{k}_0^*$ and K_1^* the connected Lie subgroup of K_0^* corresponding to the Lie subalgebra $\mathfrak{k}_1^*$. We have $K_0^* = K_1^* K_2$ and $X := K_0^*/L_0 = K_1^*/L_1$ (denoting $\bar{L}_0$, L_0 by the same notation L_0 and similarly for L_1) is an irreducible Hermitian symmetric space of the non-compact type dual to Y .

A well-known result of Harish-Chandra (see [10, Ch. VIII] or §2.3.1) is that X is naturally imbedded as a bounded symmetric domain in $\mathfrak{u}_2 = \mathcal{T}_o(Y)$, the holomorphic tangent space at $o = eK_0$ of Y. Denote by $\mathcal{U}_{\pm 2} \subset K$ the (unipotent) Lie subgroup of K with Lie algebra $\mathfrak{u}_{\pm 2} \subset \mathfrak{k}$. Then the exponential map is a diffeomorphism from $\mathfrak{u}_{\pm 2}$ onto $\mathcal{U}_{\pm 2}$. The image $\mathcal{U}_2$ in $K/(L.\mathcal{U}_{-2})$ is an open neighbourhood of o in $K/(L.\mathcal{U}_{-2}) \cong Y$. Thus X is imbedded in Y as an open complex analytic submanifold. See §2.3.1.

3.2 Holomorphic discrete series associated to a Borel-de Siebenthal discrete series

Recall that $\mathfrak{k} = \mathfrak{k}_0^* \otimes_{\mathbb{R}} \mathbb{C}$ and that $\mathfrak{t} \subset \mathfrak{l}$ is a Cartan subalgebra of $\mathfrak{k}$. The sets of compact and non-compact roots of $(\mathfrak{k}_0^*), \mathfrak{t}_0)$ are Δ_0 and $\Delta_2 \cup \Delta_{-2}$ respectively. The unique non-compact simple root of $\Psi_{\mathfrak{k}}$ is $\epsilon \in \Delta_2$.

Note that the group K_0^* admits holomorphic discrete series. See §2.4.2 or [13, Theorem 6.6, Chapter VI]. The positive system $\Delta_{\mathfrak{k}}^+$ is a special positive system of $(\mathfrak{k}, \mathfrak{t})$ as in §2.4.2.

Let $\gamma + \rho_{\mathfrak{g}}$ be the Harish-Chandra parameter for a Borel-de Siebenthal discrete series representation of G_0. Thus γ is the highest weight of an irreducible L_0-representation and $\langle \gamma + \rho_{\mathfrak{g}}, \beta \rangle < 0$ for all $\beta \in \Delta_1 \cup \Delta_2$. Clearly $\langle \gamma + \rho_{\mathfrak{k}}, \alpha \rangle > 0$ for all positive compact roots $\alpha \in \Delta_0^+$. We claim that $\langle \gamma + \rho_{\mathfrak{k}}, \beta \rangle < 0$ for all positive non-compact roots $\beta \in \Delta_2$. To see this, let $\beta_i \in \Delta_i, i = 1, 2$. Observe that $\beta_1 + \beta_2$ is not a root and so $\langle \beta_1, \beta_2 \rangle \geq 0$. It follows that $\langle \rho_{\mathfrak{k}}, \beta_2 \rangle = \langle \rho_{\mathfrak{g}} - 1/2 \sum_{\beta_1 \in \Delta_1} \beta_1, \beta_2 \rangle = \langle \rho_{\mathfrak{g}}, \beta_2 \rangle - 1/2 \sum_{\beta_1 \in \Delta_1} \langle \beta_1, \beta_2 \rangle \leq \langle \rho_{\mathfrak{g}}, \beta_2 \rangle$. So $\langle \gamma + \rho_{\mathfrak{k}}, \beta \rangle \leq \langle \gamma + \rho_{\mathfrak{g}}, \beta \rangle < 0$ for all $\beta \in \Delta_2$. Thus $\gamma + \rho_{\mathfrak{k}}$ *is the Harish-Chandra parameter for a holomorphic discrete series representation* $\pi_{\gamma+\rho_{\mathfrak{k}}}$ *of* K_0^**, which is naturally associated to the Borel-de Siebenthal discrete series representation* $\pi_{\gamma+\rho_{\mathfrak{g}}}$ *of* G_0.

The L_0-finite part of $\pi_{\gamma+\rho_{\mathfrak{k}}}$ equals $E_\gamma \otimes S^*(\mathfrak{u}_{-2})$, where E_γ is the irreducible L_0-representation with highest weight γ (see §2.4.2). Write $\gamma = \lambda + \kappa$ where λ and κ are dominant weights of $\mathfrak{l}_1^{\mathbb{C}}$ and $\mathfrak{l}_2^{\mathbb{C}}$ respectively. We have $E_\gamma = E_\lambda \otimes E_\kappa$. Hence $(\pi_{\gamma+\rho_{\mathfrak{k}}})_{L_0} = E_\kappa \otimes (E_\lambda \otimes S^*(\mathfrak{u}_{-2})) = E_\kappa \otimes (\pi_{\lambda+\rho_{\mathfrak{k}_1^{\mathbb{C}}}})_{L_1}$, where $\pi_{\lambda+\rho_{\mathfrak{k}_1^{\mathbb{C}}}}$ is the holomorphic discrete series representation of K_1^* with Harish-Chandra parameter $\lambda + \rho_{\mathfrak{k}_1^{\mathbb{C}}}$.

We have $(\pi_{\gamma+\rho_{\mathfrak{k}}})_{L_0} = E_\kappa \otimes (\pi_{\lambda+\rho_{\mathfrak{k}_1^{\mathbb{C}}}})_{L_1}$. Therefore $\pi_{\gamma+\rho_{\mathfrak{k}}}$ is L_0'-admissible if and only if $\pi_{\lambda+\rho_{\mathfrak{k}_1^{\mathbb{C}}}}$ is L_1'-admissible, where L_0' (respectively, L_1') denote the connected Lie subgroup of L_0 (respectively, L_1) corresponding to the semisimple ideal $[\mathfrak{l}_0, \mathfrak{l}_0]$ (respectively, $[\mathfrak{l}_1, \mathfrak{l}_1]$) of $\mathfrak{l}_0$ (respectively $\mathfrak{l}_1$). Since K_1 is *simple*, and since $w_{\mathfrak{k}}^0(\epsilon) = w_{\mathfrak{k}_1^{\mathbb{C}}}^0(\epsilon)$, it follows from the Proposition 2.5.3 of Chapter 2 that $\pi_{\gamma+\rho_{\mathfrak{k}}}$ *is* L_0' *admissible if and only if* $w_{\mathfrak{k}}^0(\epsilon) \neq -\epsilon$.

Chapter 4

TWO INVARIANTS ASSOCIATED TO A BOREL-DE SIBENTHAL POSITIVE SYSTEM

In this chapter we shall associate to a Borel-de Siebenthal positive system two invariants. One of them is the first Chern class of the Hermitian symmetric space $Y = K_0/L_0 = K/K \cap Q$ (with notations as in §2.4.3). The other is the degree of the algebra generator of the algebra of relative invariants of $(\mathfrak{u}_1, L)$. See §4.3. The relation between them will play a crucial role in our proof of Theorem 1.0.2.

Recall that G_0 is a simply connected non-compact real simple Lie group with maximal compact subgroup K_0 such that
(i) rank (G_0) = rank (K_0), and
(ii) G_0/K_0 is not a Hermitian symmetric space.
Recall that $Y = K_0/L_0 = K_1/L_1$ is an irreducible Hermitian symmetric space of the non-compact type.

4.1 Spin structure on Y

We have seen in Lemma 2.4.2 that the sum $\sum_{\beta\in\Delta_2}\beta = c\epsilon^*$, where c is an integer. The parity of c will be relevant for our purposes. We give an interpretation of it in terms of the existence of spin structures on Y. The cohomology group $H^2(Y;\mathbb{Z})$ is naturally isomorphic to $\mathbb{Z}[\epsilon^*] \cong \mathbb{Z}$, the quotient of the weight lattice of K_0 by the weight lattice of L_0. If λ is an integral weight of K_0 its class in $H^2(Y;\mathbb{Z})$ is denoted by $[\lambda]$. Thus $[\lambda] = 2(\langle\lambda,\epsilon\rangle/||\epsilon||^2)[\epsilon^*]$. The holomorphic tangent bundle $\mathcal{T}Y$ is the bundle associated to the L_0-representation $\mathfrak{u}_2 = \sum_{\beta\in\Delta_2}\mathfrak{g}_\beta$. This implies that $c_1(Y)$, first Chern class of Y, equals $\sum_{\beta\in\Delta_2}[\beta] = c[\epsilon^*] \in H^2(Y;\mathbb{Z})$. Consequently Y admits a spin structure if and only if c is even. The value of c can be explicitly computed. (See, for example, [1, §16].) This leads to the following conclusion. The complex Grassmann variety $\mathbb{C}G_p(\mathbb{C}^{p+q}) = SU(p+q)/S(U(p)\times U(q))$ admits a spin structure if and only if $p+q$ is even and that the complex quadric $SO(2+p)/SO(2)\times SO(p)$ admits a spin structure precisely when p is even. The orthogonal Grassmann variety $SO(2p)/U(p)$ admits a spin structure for all p. The symplectic Grassmann variety $Sp(p)/U(p)$ admits a spin structure if and only if p is odd. The Hermitian symmetric spaces $E_6/(Spin(10)\times SO(2))$ and $E_7/(E_6\times SO(2))$ admit spin structures.

4.2 Classification of Borel-de Siebenthal root orders

The complete classification of Borel-de Siebenthal root orders is given in [19, §3]. But it will be convenient to recall here, in brief, their classification. We list the quaternionic and non-quaternionic cases separately.

Let $\mathfrak{g}_0$ be a non-compact real simple Lie algebra with maximal compactly imbedded subalgebra $\mathfrak{k}_0$ such that rank $(\mathfrak{g}_0)$ = rank $(\mathfrak{k}_0)$ and $\mathfrak{k}_0$ is semisimple.
Having fixed a fundamental Cartan subalgebra $\mathfrak{t}_0 \subset \mathfrak{g}_0$; a positive root system of $(\mathfrak{g},\mathfrak{t})$ containing exactly one non-compact simple root ν, is Borel-de Siebenthal if the coefficient of ν in the highest root is 2. Conversely, let $\mathfrak{g}$ be a complex simple Lie algebra. Choose a Cartan subalgebra $\mathfrak{t}\subset\mathfrak{g}$ and a positive root system of $(\mathfrak{g},\mathfrak{t})$. If there exists a simple root ν whose coefficient in the highest root is 2, then ν determines uniquely (up to an inner automorphism) a non-compact real form $\mathfrak{g}_0$ of $\mathfrak{g}$ satisfying the conditions given above such that the positive system is a Borel-de Siebenthal positive system of $\mathfrak{g}_0$.

If Ψ is the set of simple roots of a Borel-de Siebenthal positive system of $\mathfrak{g}_0$ and $\nu\in\Psi$ is the unique non-compact root, we denote the Borel-de Siebenthal root order by (Ψ,ν). Corresponding to $\mathfrak{g}_0$, we can have several Borel-de Siebenthal root orders. Given one such, we have its negative $(-\Psi,-\nu)$. We list below the Borel-de Siebenthal root orders up to sign changes.

The quaternionic case is characterized by the property that highest root μ is orthogonal to *all* the compact simple roots and hence $-\mu$ is adjacent to the simple non-compact root ν in the extended Dynkin diagram of $\mathfrak{g}$.

As in [19], we shall follow Bourbaki's notation [4] in labeling the simple roots of $\mathfrak{g}$. We point out the simple root which is non-compact for $\mathfrak{g}_0$ and the compact Lie subalgebras $\mathfrak{k}_1,\mathfrak{l}_1,\mathfrak{l}_2=\mathfrak{k}_2\subset\mathfrak{k}_0$. We also point out, based on Proposition 4.3.1 below, whether the algebra $\mathcal{A}:=\mathcal{A}(\mathfrak{u}_1,L)$ of relative invariants is $\mathbb{C}$ or $\mathbb{C}[f]$. In the latter case we indicate the value of $\deg(f)$. See [19] for a more detailed analysis.

We also indicate the non-compact dual Hermitian symmetric space $X:=Y^*$. In the non-quaternionic cases we point out whether or not $w^0_{\mathfrak{k}}(\Delta_0)=\Delta_0$ (equivalently $w^0_{\mathfrak{k}}(\epsilon)=-\epsilon$). For a proof see Proposition 2.5.2 in Chapter 2.

Borel-de Siebenthal root orders.
(a) *Quaternionic type:* We have $\mathfrak{k}_1=\mathfrak{su}(2),\mathfrak{l}_1=\mathfrak{so}(2)=i\mathbb{R}\nu^*$. Also $Y=\mathbb{P}^1$. $X=Y^*=SU(1,1)/U(1)$, the unit disk in $\mathbb{C}$. The condition $w^0_{\mathfrak{k}}(\epsilon)=-\epsilon$ is trivially valid.

1. $\mathfrak{g}_0=\mathfrak{so}(4,2l-3), l>2$. Then $\mathfrak{g}$ is of type B_l and $\nu=\psi_2$. $\mathfrak{l}_2=\mathfrak{sp}(1)\oplus\mathfrak{so}(2l-3)$. $\mathcal{A}=$

$\mathbb{C}[f], \deg(f) = 4$.

2. $\mathfrak{g}_0 = \mathfrak{so}(4,1)$. Then $\mathfrak{g}$ is of type B_2, $\nu = \psi_2, \mathfrak{l}_2 = \mathfrak{sp}(1)$. $\mathcal{A} = \mathbb{C}$.

3. $\mathfrak{g}_0 = \mathfrak{sp}(1, l-1), l > 1$. Then $\mathfrak{g}$ is of type $C_l, \nu = \psi_1, \mathfrak{l}_2 = \mathfrak{sp}(l-1)$. $\mathcal{A} = \mathbb{C}$.

4. $\mathfrak{g}_0 = \mathfrak{so}(4, 2l-4), l > 4$. $\mathfrak{g}$ is of type $D_l, \nu = \psi_2, \mathfrak{l}_2 = \mathfrak{sp}(1) \oplus \mathfrak{so}(2l-4)$. $\mathcal{A} = \mathbb{C}[f], \deg(f) = 4$.

5. $\mathfrak{g}_0 = \mathfrak{so}(4,4)$. $\mathfrak{g}$ is of type $D_4, \nu = \psi_2, \mathfrak{l}_2 = \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$. $\mathcal{A} = \mathbb{C}[f], \deg(f) = 4$.

6. $\mathfrak{g}_0 = \mathfrak{g}_{2;A_1,A_1}$, the split real form of the exceptional Lie algebra of type G_2. $\mathfrak{g} = \mathfrak{g}_2, \nu = \psi_2, \mathfrak{l}_2 = \mathfrak{sp}(1)$. $\mathcal{A} = \mathbb{C}[f], \deg(f) = 4$.

7. $\mathfrak{g}_0 = \mathfrak{f}_{4;A_1,C_3}$, the split real form of the exceptional Lie algebra of type F_4. $\mathfrak{g} = \mathfrak{f}_4, \nu = \psi_1, \mathfrak{l}_2 = \mathfrak{sp}(3)$. $\mathcal{A} = \mathbb{C}[f], \deg(f) = 4$.

8. $\mathfrak{g}_0 = \mathfrak{e}_{6;A_1,A_5,2}$. $\mathfrak{g} = \mathfrak{e}_6$, the exceptional Lie algebra. $\nu = \psi_2, \mathfrak{l}_2 = \mathfrak{su}(6)$. $\mathcal{A} = \mathbb{C}[f], \deg(f) = 4$.

9. $\mathfrak{g}_0 = \mathfrak{e}_{7;A_1,D_6,1}$. $\mathfrak{g} = \mathfrak{e}_7, \nu = \psi_1, \mathfrak{l}_2 = \mathfrak{so}(12)$. $\mathcal{A} = \mathbb{C}[f], \deg(f) = 4$.

10. $\mathfrak{g}_0 = \mathfrak{e}_{8;A_1,E_7}$. $\mathfrak{g} = \mathfrak{e}_8, \nu = \psi_8, \mathfrak{l}_2^{\mathbb{C}} = \mathfrak{e}_7$. $\mathcal{A} = \mathbb{C}[f], \deg(f) = 4$.

(b) *Non-quaternionic type:*

1. $\mathfrak{g}_0 = \mathfrak{so}(2p, 2l-2p+1), 2 < p < l, l > 3$. $\mathfrak{g}$ is of type B_l, $\nu = \psi_p, \mathfrak{k}_1 = \mathfrak{so}(2p), \mathfrak{l}_1 = \mathfrak{u}(p), \mathfrak{l}_2 = \mathfrak{so}(2l-2p+1)$. The variety $Y = SO(2p)/U(p), X = SO^*(2p)/U(p)$. $w_{\mathfrak{k}}^0(\epsilon) = -\epsilon$ if and only if p is even. $\mathcal{A} = \mathbb{C}[f]$ (with $\deg(f) = 2p$) if and only if $3p \leq 2l+1$.

2. $\mathfrak{g}_0 = \mathfrak{so}(2l, 1), l > 2$. $\mathfrak{g}$ is of type B_l, $\nu = \psi_l, \mathfrak{k}_0 = \mathfrak{k}_1 = \mathfrak{so}(2l), \mathfrak{l}_1 = \mathfrak{u}(l)$. The variety $Y = SO(2l)/U(l), X = SO^*(2l)/U(l)$. $w_{\mathfrak{k}}^0(\epsilon) = -\epsilon$ if and only if l is even. $\mathcal{A} = \mathbb{C}$.

3. $\mathfrak{g}_0 = \mathfrak{sp}(p, l-p)$, $l > 2, 1 < p < l$. $\mathfrak{g}$ is of type C_l, $\nu = \psi_p, \mathfrak{k}_1 = \mathfrak{sp}(p), \mathfrak{l}_1 = \mathfrak{u}(p), \mathfrak{l}_2 = \mathfrak{sp}(l-p)$, and $Y = Sp(p)/U(p), X = Sp(p, \mathbb{R})/U(p)$. $w_{\mathfrak{k}}^0(\epsilon) = -\epsilon$. $\mathcal{A} = \mathbb{C}[f]$, (with $\deg(f) = p$) if and only if $3p \leq 2l$ and p even.

4. $\mathfrak{g}_0 = \mathfrak{so}(2l-4,4), l > 4$. $\mathfrak{g}$ is of type D_l, $\nu = \psi_{l-2}, \mathfrak{k}_1 = \mathfrak{so}(2l-4), \mathfrak{l}_1 = \mathfrak{u}(l-2), \mathfrak{l}_2 = \mathfrak{su}(2)\oplus\mathfrak{su}(2)$. The variety $Y = SO(2l-4)/U(l-2), X = SO^*(2l-4)/U(l-2)$. $w^0_{\mathfrak{k}}(\epsilon) = -\epsilon$ if and only if l is even. $\mathcal{A} = \mathbb{C}$ if $l > 6$. When $l = 5, 6$, $\mathcal{A} = \mathbb{C}[f]$ with $\deg(f) = 6, 8$ respectively.

5. $\mathfrak{g}_0 = \mathfrak{so}(2p, 2l-2p), 2 < p < l-2, l > 5$. $\mathfrak{g}$ is of type D_l, $\nu = \psi_p, \mathfrak{k}_1 = \mathfrak{so}(2p), \mathfrak{l}_1 = \mathfrak{u}(p)$, $\mathfrak{l}_2 = \mathfrak{so}(2l-2p)$. $Y = SO(2p)/U(p), X = SO^*(2p)/U(p)$. $w^0_{\mathfrak{k}}(\epsilon) = -\epsilon$ if and only if p is even. $\mathcal{A} = \mathbb{C}[f]$ (with $\deg(f) = 2p$) if and only if $3p \leq 2l$.

6. $\mathfrak{g}_0 = \mathfrak{f}_{4;B_4}$, the real form of $\mathfrak{f}_4$ having $\mathfrak{k}_0 \cong \mathfrak{so}(9)$ as a maximal compactly embedded subalgebra. $\nu = \psi_4$ and $\mathfrak{k}_0 = \mathfrak{k}_1, \mathfrak{l}_1 = i\mathbb{R}\nu^* \oplus \mathfrak{so}(7)$. $Y = SO(9)/(SO(7)\times SO(2)), X = SO_0(2,7)/(SO(2)\times SO(7))$. $w^0_{\mathfrak{k}}(\epsilon) = -\epsilon$. $\mathcal{A} = \mathbb{C}[f], \deg(f) = 2$.

7. $\mathfrak{g}_0 = \mathfrak{e}_{6;A_1,A_5,1}$, a real form of $\mathfrak{e}_6$ with $\nu = \psi_3$. $\mathfrak{k}_1 = \mathfrak{su}(6), \mathfrak{l}_1 = \mathfrak{su}(5) \oplus i\mathbb{R}\nu^*, \mathfrak{l}_2 = \mathfrak{su}(2)$. $Y = \mathbb{P}^5, X = SU(1,5)/S(U(1)\times U(5))$. $w^0_{\mathfrak{k}}(\epsilon) \neq -\epsilon$. $\mathcal{A} = \mathbb{C}$.

8. $\mathfrak{g}_0 = \mathfrak{e}_{7;A_1,D_6,2}$, a real form of $\mathfrak{e}_7$ with $\nu = \psi_6$. $\mathfrak{k}_1 = \mathfrak{so}(12), \mathfrak{l}_1 = \mathfrak{so}(10)\oplus i\mathbb{R}\nu^*, \mathfrak{l}_2 = \mathfrak{sp}(1)$. $Y = SO(12)/SO(2)\times SO(10), X = SO_0(2,10)/(SO(2)\times SO(10))$. $w^0_{\mathfrak{k}}(\epsilon) = -\epsilon$. $\mathcal{A} = \mathbb{C}$.

9. $\mathfrak{g}_0 = \mathfrak{e}_{7;A_7}$, a real form of $\mathfrak{e}_7$ with $\nu = \psi_2$. $\mathfrak{k}_0 = \mathfrak{k}_1 = \mathfrak{su}(8)$, $\mathfrak{l}_1 = \mathfrak{su}(7) \oplus i\mathbb{R}\nu^*$. The variety $Y = \mathbb{P}^7, X = SU(1,7)/S(U(1)\times U(7))$. $w^0_{\mathfrak{k}}(\epsilon) \neq -\epsilon$. $\mathcal{A} = \mathbb{C}[f], \deg(f) = 7$.

10. $\mathfrak{g}_0 = \mathfrak{e}_{8;D_8}$, a real form of $\mathfrak{e}_8$ with $\nu = \psi_1$. $\mathfrak{k}_0 = \mathfrak{k}_1 = \mathfrak{so}(16)$, $\mathfrak{l}_1 = i\mathbb{R}\nu^* \oplus \mathfrak{so}(14)$. $Y = SO(16)/SO(2)\times SO(14), X = SO_0(2,14)/(SO(2)\times SO(14))$. $w^0_{\mathfrak{k}}(\epsilon) = -\epsilon$. $\mathcal{A} = \mathbb{C}[f], \deg(f) = 8$.

4.3 Relative invariants of $(\mathfrak{u}_1, L)$

The action of $L = L_0^{\mathbb{C}}$ on $\mathfrak{u}_1$ is known to have a Zariski dense orbit. See [14, Th 10.19, Ch X]. It follows that the coordinate ring $\mathbb{C}[\mathfrak{u}_1] = S^*(\mathfrak{u}_{-1})$ has no non-constant invariant functions, that is, $\mathbb{C}[\mathfrak{u}_1]^L = \mathbb{C}$. However, it is possible that $\mathfrak{u}_1$ has non-zero relative invariants, that is, an $h \in \mathbb{C}[\mathfrak{u}_1]$ such that $x.h = \chi(x)h, x \in L$, for some rational character $\chi : L \longrightarrow \mathbb{C}^*$. It can be seen that the subalgebra $\mathcal{A}(\mathfrak{u}_1, L) \subset \mathbb{C}[\mathfrak{u}_1]$ of all relative invariants is either $\mathbb{C}$ or is a polynomial algebra $\mathbb{C}[f]$ for a suitable (non-zero) homogeneous polynomial function $f \in \mathbb{C}[\mathfrak{u}_1]$. It is clear that a homogeneous function h belongs to $\mathcal{A}(\mathfrak{u}_1, L)$ if and only if $\mathbb{C}h$ is an L-submodule of $S^m(\mathfrak{u}_{-1})$ where $m = \deg(h)$. Ørsted and Wolf [19] determined when $\mathcal{A}(\mathfrak{u}_1, L)$ is a polynomial algebra $\mathbb{C}[f]$ and described in such cases the generator f in detail. See also [22].

Proposition 4.3.1 *Let Δ^+ be a Borel-de Siebenthal positive system of $(\mathfrak{g}, \mathfrak{t})$ listed above. If $\mathfrak{g}_0 = \mathfrak{so}(4, 1), \mathfrak{sp}(1, l-1)$(with $l > 1$), $\mathfrak{e}_{6;A_1,A_5,1}, \mathfrak{e}_{7;A_1,D_6,2}, \mathfrak{g}_0 = \mathfrak{so}(2p, r)$ with $p > r \geq 1$, $\mathfrak{g}_0 = \mathfrak{sp}(p, q)$ where $p > 2q > 0$ or p is odd, then $\mathcal{A}(\mathfrak{u}_1, L) = \mathbb{C}$. In all the remaining cases $\mathcal{A}(\mathfrak{u}_1, L) = \mathbb{C}[f]$, a polynomial algebra where* $\deg(f) > 0$. □

In the case when $\mathfrak{g}_0 = \mathfrak{so}(2l, 1)$, or $\mathfrak{sp}(1, l-1)$, the L_0-representation $S^m(\mathfrak{u}_{-1})$ is irreducible for all $m \geq 0$.

Proof: Only the irreducibility of the L_0-module $S^m(\mathfrak{u}_{-1})$ when $\mathfrak{g}_0 = \mathfrak{so}(2l, 1), \mathfrak{sp}(1, l-1)$ needs to be established as the remaining assertions have already been established in [19, §4].

When $\mathfrak{g}_0 = \mathfrak{so}(2l, 1), L_0' \cong SU(l)$ and $\mathfrak{u}_{-1}$, as an L_0'-representation, is isomorphic to the standard representation. Hence $S^m(\mathfrak{u}_{-1})$ is irreducible as an L_0'-module—consequently as an L_0-module—for all m.

When $\mathfrak{g}_0 = \mathfrak{sp}(1, l-1), L_0' = Sp(l-1)$. Again $\mathfrak{u}_{-1}$, as an L_0'-representation, is isomorphic to the standard representation of $Sp(l-1)$ (of dimension $2l-2$). Using the Weyl dimension formula, it follows that for any $m \geq 1$, $S^m(\mathfrak{u}_{-1})$ is irreducible as L_0'-module and hence as an L_0-module. □

Remark 4.3.2 The centre $\mathbb{C}H_{\nu^*} \subset \mathfrak{l}$ acts via the character $-\nu^*/||\nu^*||^2 = -||\epsilon||^2\epsilon^*/(4||\epsilon^*||^2)$ on the irreducible $\mathfrak{l}$-representation $\mathfrak{u}_{-1}$ and hence by $-k||\epsilon||^2\epsilon^*/(4||\epsilon^*||^2)$ on $S^k(\mathfrak{u}_{-1})$ for all k. Suppose that $\mathcal{A}(\mathfrak{u}_1, L) = \mathbb{C}[f]$ where $f \in S^k(\mathfrak{u}_{-1})$ with $\deg(f) = k > 0$. Let $E_{q\epsilon^*} = \mathbb{C}f$ be the one-dimensional subrepresentation of $S^k(\mathfrak{u}_{-1})$. Then $q = -k||\epsilon||^2/(4||\epsilon^*||^2)$.

When $\mathfrak{g}_0 = \mathfrak{sp}(p, l-p), 2 \leq p \leq 2(l-p)$ with p even, it turns out that $k = \deg(f) = p$ from [19, §4]. In this case $||\epsilon||^2 = 4$, $\epsilon^* = \nu^*$ and $||\epsilon^*||^2 = p$. Hence $q = -1$.

When $\mathfrak{g}_0 = \mathfrak{f}_{4,B_4}$, $k = \deg(f) = 2$ from [19, §4]. In view of our normalization $||\nu||^2 = 2$, using [4, Planche VIII], a straightforward calculation leads to $||\epsilon^*||^2 = ||\nu^*||^2 = 2, ||\epsilon||^2 = 4$ and so $q = -1$.

It follows from §4.1 that when Y does not admit a spin structure and $\mathcal{A}(\mathfrak{u}_1, L) = \mathbb{C}[f]$, the value of q is odd.

In fact it turns out that in all the remaining cases for which $\mathcal{A}(\mathfrak{u}_1, L) = \mathbb{C}[f]$, the number q is even. In view of §4.1 we interpret this as follows: Denote by $\mathcal{K}_Y$ the canonical bundle of Y and let $\mathbb{E}$ denote the line bundle over Y determined by the L_0-representation $E := \mathbb{C}f$. Then the line bundle $\mathcal{K}_Y \otimes \mathbb{E}$ always admits a square root, that is, $\mathcal{K}_Y \otimes \mathbb{E} = \mathcal{L} \otimes \mathcal{L}$ for a (necessarily unique) line bundle $\mathcal{L}$ over Y.

4.4 K_0-types of a Borel-de Siebenthal discrete series representation of G_0

Let $\gamma + \rho_{\mathfrak{g}}$ be the Harish-Chandra parameter of a Borel-de Siebenthal discrete series representation $\pi_{\gamma+\rho_{\mathfrak{g}}}$ of G_0. Ørsted and Wolf described the K_0-finite part of the Borel-de Siebenthal discrete series representation $\pi_{\gamma+\rho_{\mathfrak{g}}}$ in terms of the Dolbeault cohomology as $\oplus_{m\geq 0} H^s(K_0/L_0; \mathbb{E}_\gamma \otimes \mathbb{S}^m(\mathfrak{u}_{-1}))$ where $s = \dim_{\mathbb{C}} K_0/L_0$, $\mathbb{E}_\gamma$ and $\mathbb{S}^m(\mathfrak{u}_{-1})$ denote the holomorphic vector bundles associated to the irreducible L_0-module E_γ and the m-th symmetric power $S^m(\mathfrak{u}_{-1})$ of the irreducible L_0-module $\mathfrak{u}_{-1}$ respectively. See Theorem 2.4.1 in §2.4.3.

The highest weight of any irreducible L_0-submodule of $E_\gamma \otimes S^m(\mathfrak{u}_{-1})$ is of the form $\gamma + \phi$ where ϕ is a weight of $S^m(\mathfrak{u}_{-1})$. Thus $\phi = \alpha_1 + \cdots + \alpha_m$ for suitable α_i in Δ_{-1} (not necessarily distinct). Now if $\alpha \in \Delta_{-1}$ and $\beta \in \Delta_2$, then $\beta - \alpha$ is not a root. Hence $\langle \alpha, \beta \rangle \leq 0$ for all $\alpha \in \Delta_{-1}, \beta \in \Delta_2$. It follows that $\langle \gamma + \rho_{\mathfrak{k}}, \beta \rangle \leq \langle \gamma + \rho_{\mathfrak{g}}, \beta \rangle$ and $\langle \phi, \beta \rangle \leq 0$ for all $\beta \in \Delta_2$. Since $\langle \gamma + \rho_{\mathfrak{g}}, \beta \rangle < 0$ for all $\beta \in \Delta_2$, therefore $\langle \gamma + \rho_{\mathfrak{k}}, \beta \rangle < 0$ and $\langle \gamma + \phi + \rho_{\mathfrak{k}}, \beta \rangle < 0$ for all $\beta \in \Delta_2$. Hence, by the Borel-Weil-Bott theorem ([3], also see [6, Th. 1.6.8, Ch. 1]), the highest weight of $H^s(Y; \mathbb{E}_{\gamma+\phi})$ equals $w_Y(\gamma + \phi + \rho_{\mathfrak{k}}) - \rho_{\mathfrak{k}}$, since $w_Y(\Delta_0 \cup \Delta_{-2}) = \Delta_0 \cup \Delta_2$. See Lemma 2.4.2 in §2.4.3. We shall make use of this in this thesis without explicit reference to it.

Chapter 5

L_0-ADMISSIBILITY OF THE BOREL-DE SIEBENTHAL DISCRETE SERIES

We begin this chapter by establishing, in §5.1, Proposition 5.1.1 which implies that there is no loss of generality in confining our attention throughout to the K_0-finite part of the Borel-de Siebenthal series rather than the discrete series itself when the K_0-finite part is L_0-admissible. *Up to the end of proof of Proposition 5.1.1 we shall use the symbols G_0, K_0, L_0 etc., in a more general context described in §5.1.* In §5.2, we discuss the L_0-admissibility of a Borel-de Siebenthal discrete series representation of G_0 and prove Proposition 1.0.3.

5.1 A general result

Let K_0 be a maximal compact subgroup of a connected semisimple Lie group G_0 with finite centre and let π be a unitary K_0-admissible representation of G_0 on a separable complex Hilbert space $\mathcal{H}$. Denote by $\mathcal{H}_{K_0}$ the K_0-finite vectors of $\mathcal{H}$ and by π_{K_0} the restriction of π to $\mathcal{H}_{K_0}$. Thus $\mathcal{H}_{K_0}$ is dense in $\mathcal{H}$.

Proposition 5.1.1 *Suppose that π_{K_0} is L_0-admissible where L_0 is a closed subgroup of K_0. Then any finite dimensional L_0-subrepresentation of π is contained in $\mathcal{H}_{K_0}$. In particular, π is L_0-admissible.*

Proof: To see this, suppose that $v \in \mathcal{H}$ is contained in an irreducible (finite dimensional) L_0-submodule of $\mathcal{H}$. Then $\sum_{1\le i\le m} c_i\pi(x_i)v_0 = v$ for some L_0-highest weight vector v_0 of weight, say, λ, for suitable $x_i \in L_0, c_i \in \mathbb{C}$. Let $\{v_j\}$ be an orthonormal basis of $\mathcal{H}$ consisting of L_0-weight vectors, obtained by taking union of certain orthonormal bases of L_0-isotypic components of $\mathcal{H}_{K_0}$. Write $v_0 = \sum_j a_j v_j$. It is readily seen that a_j is zero unless v_j is an L_0-*highest* weight vector of weight λ. This means that v_0 belongs to the L_0-isotypic component of π_{K_0} having highest weight λ. Since π_{K_0} is L_0-admissible, it follows that $v_0 \in \mathcal{H}_{K_0}$. Hence $v \in \mathcal{H}_{K_0}$. □

5.2 Restriction of a Borel-de Siebenthal discrete series representation to L_0

For the rest of this chapter we keep the notations of §2.4.3. We denote by L_0' and L_1', the connected Lie subgroups of L_0 and L_1 corresponding to the semisimple ideals $[\mathfrak{l}_0, \mathfrak{l}_0]$, $[\mathfrak{l}_1, \mathfrak{l}_1]$ of $\mathfrak{l}_0$ and $\mathfrak{l}_1$ respectively. Any irreducible finite dimensional complex representation E of $L_0 = L_1 \times L_2$ is isomorphic to a tensor product $E_1 \otimes E_2$ where E_j is an irreducible representation of $L_j, j = 1, 2$. In particular, if E_1 is one dimensional, then it is trivial as an L_1' representation and L_1 acts on E_1 via a character $\chi : L_1/L_1' \longrightarrow \mathbb{S}^1$. If E_2 one dimensional, then it is trivial as an L_2-representation.

Applying this observation to $S^k(\mathfrak{u}_{-1})$ we see that one-dimensional L_0-subrepresentations of $S^k(\mathfrak{u}_{-1})$ are all of the form $\mathbb{C}h$ where $h \in S^k(\mathfrak{u}_{-1})$ a weight vector which is invariant under the action of $L_1' \times L_2$. That is, h is a relative invariant of $(\mathfrak{u}_1, L)$. See §4.3. If $h \in S^k(\mathfrak{u}_{-1})$ is a relative invariant, then so is h^j for any $j \geq 1$. If $\chi = \sum_{\alpha\in\Delta_{-1}} r_\alpha \alpha, r_\alpha \geq 0$ is the weight of a relative invariant h, then, as L_0' acts trivially on $\mathbb{C}h$, we see that χ is a multiple of ν^*.

When $\mathfrak{k}_1 \cong \mathfrak{su}(2)$ we have $L_1 \cong \mathbb{S}^1$. Let π be a representation of G_0 on a separable Hilbert space $\mathcal{H}$. For example, π is a Borel-de Siebenthal representation. We have the following:

Lemma 5.2.1 *Suppose that π is K_1-admissible where $\mathfrak{k}_1 = \mathfrak{su}(2)$. Then π is L_0-admissible if and only if π is L_2-admissible.*

Proof: We need only prove that L_0 admissibility of π implies the L_2 admissibility. Note that $L_0' = L_2$. Assume that π is not L_2 admissible. Say E is a L_2 type which occurs in π with infinite multiplicity. In view of Proposition 5.1.1 and since $L_0' = L_2$, the L_2-type E actually occurs in π_{K_0} with infinite multiplicity. Then, denoting the irreducible K_1-representation of dimension $d + 1$ by U_d, we deduce from K_1-admissibility of π that the irreducible K_0-representations $U_{d_j} \otimes E$ occurs in π where (d_j) is a strictly increasing sequence of natural numbers. Without loss of generality we assume that all the d_j are of same parity. Notice that U_c as an L_1-module, is a submodule of U_d, if $c \leq d$ and $c \equiv d$ mod 2. It follows that the L_0-type $U_{d_1} \otimes E$ occurs in *every* summand of $\oplus_{j\geq 1} U_{d_j} \otimes E$. Thus π is not L_0-admissible. □

Proof of Proposition 1.0.3: Let $h \in S^k(\mathfrak{u}_{-1})$ be a relative invariant for $(\mathfrak{u}_1, L)$ with weight $\chi = r\nu^*$. Denote by $\mathcal{L}$ the holomorphic line bundle $K_0 \times_{L_0} \mathbb{C}h \longrightarrow K_0/L_0 = Y$. Then $\mathcal{L} = \mathbb{E}_\chi$ and so $\mathbb{E}_\gamma \otimes \mathcal{L}^{\otimes j} = \mathbb{E}_{\gamma+j\chi}$ is a subbundle of the bundle $\mathbb{E}_\gamma \otimes \mathbb{S}^{jk}(\mathfrak{u}_{-1})$ for all $j \geq 1$. Hence the K_0-module $H^s(Y; \mathbb{E}_{\gamma+j\chi})$ occurs in the Borel-de Siebenthal discrete series $\pi_{\gamma+\rho_\mathfrak{g}}$. The lowest weight of the K_0-module $H^s(Y; \mathbb{E}_{\gamma+j\chi})$ is $w_\mathfrak{l}^0(\gamma + j\chi + \rho_\mathfrak{k}) - w_\mathfrak{k}^0\rho_\mathfrak{k} =$

$w^0_{\mathfrak{l}}(\gamma_0) + (tv^* + jrv^*) + \sum_{\alpha\in\Delta_2}\alpha$ where $\chi = rv^*$. As observed above, $\sum_{\alpha\in\Delta_2}\alpha = 2sv^*/||v^*||^2$. Since v^* is in the centre of $\mathfrak{l}$, the irreducible L'_0 representation with lowest weight $w^0_{\mathfrak{l}}(\gamma_0)$, namely E_{γ_0}, occurs in $H^s(Y;\mathbb{E}_{\gamma+j\chi})$ for *all* $j \geq 1$. It follows that $\pi_{\gamma+\rho_{\mathfrak{g}}}$ is not L'_0-admissible.

It remains to prove the converse assuming $\mathfrak{k}_1 \cong \mathfrak{su}(2)$. We shall suppose that $\pi_{\gamma+\rho_{\mathfrak{g}}}$ is not L'_0-admissible and that $S^m(\mathfrak{u}_{-1})$ has no one-dimensional L'_0-submodules and arrive at a contradiction. By Lemma 5.2.1, $\pi_{\gamma+\rho_{\mathfrak{g}}}$ is not L_0-admissible. By Proposition 5.1.1, the K_0-finite part of $\pi_{\gamma+\rho_{\mathfrak{g}}}$ is not L_0-admissible. In view of Proposition 4.3.1 we have $\mathfrak{g}_0 = \mathfrak{so}(4,1)$ or $\mathfrak{sp}(1,l-1)$ and the L_0-module $S^m(\mathfrak{u}_{-1})$ is irreducible for all m. The highest weight of $S^m(\mathfrak{u}_{-1})$ as an L_2-module is $m(-v - av^*)$ where av^* is the character by which $L_1 = L_0/L_2 \cong \mathbb{S}^1$ acts on $\mathfrak{u}_{-1}$.

Now $H^1(\mathbb{P}^1;\mathbb{E}_\gamma \otimes \mathbb{S}^m(\mathfrak{u}_{-1})) = H^1(\mathbb{P}^1;\mathbb{E}_{(t+ma)v^*} \otimes \mathbb{E}_{-mv-mav^*} \otimes \mathbb{E}_{\gamma_0}) = H^1(\mathbb{P}^1;\mathbb{E}_{(t+ma)v^*}) \otimes E_{-mv-mav^*} \otimes E_{\gamma_0}$ as a $K_1 \times L_2$-module. Since the K_0-finite part of $\pi_{\gamma+\rho_{\mathfrak{g}}}$ is not L_0-admissible, there exist a b and an L_2-dominant integral weight λ such that the L_0-type $E = E_{bv^*} \otimes E_\lambda$ occurs in $H^1(\mathbb{P}^1;\mathbb{E}_{(t+ma)v^*}) \otimes E_{-mv-mav^*} \otimes E_{\gamma_0}$ for infinitely many distinct values of m. This implies that E_λ occurs in $E_{-mv-mav^*} \otimes E_{\gamma_0}$ for infinitely many values of m. The highest weights of L_2-types occurring in $E_{-mv-mav^*} \otimes E_{\gamma_0}$ are all of the form $-mv - mav^* + \kappa_m$ where κ_m is a weight of E_{γ_0}. Thus $\lambda = -mv - mav^* + \kappa_m$ for infinitely many m. Since E_{γ_0} is finite dimensional, it follows that for some weight κ of E_{γ_0}, we have $\lambda - \kappa = -mv - mav^*$ for infinitely many values of m, which is absurd. □

Chapter 6

COMMON L_0-TYPES IN THE QUATERNIONIC CASE

As usual we keep the notations of §2.4.3. In this chapter we focus on the quaternionic case, namely, when $\mathfrak{k}_1 = \mathfrak{su}(2)$. This case is characterized by the property that $-\mu$ is connected to ν in the extended Dynkin diagram of $\mathfrak{g}$. In this case $\Delta_2 = \{\mu\}, L_1 \cong \mathbb{S}^1, Y = \mathbb{P}^1, L_2 = [L_0, L_0] = L_0'$, and, $\mathfrak{l}' = [\mathfrak{l}, \mathfrak{l}] = \mathfrak{l}_2^{\mathbb{C}}$. Also, since both μ and ν^* are orthogonal to $\mathfrak{l}_2^{\mathbb{C}}$, μ is a non-zero multiple of ν^*. Write $\mu = d\nu^*$. Since $\mu = 2\nu + \beta$ where β is a linear combinations of roots of $\mathfrak{l}_2^{\mathbb{C}}$, we obtain $||\mu||^2 = d\langle \nu^*, \mu\rangle = d\langle \nu^*, 2\nu\rangle = d||\nu||^2 = 2d$ as $||\nu||^2 = 2$. Since $s_\nu(\mu) = \mu - d\nu$ is a root and since $\mu - 3\nu$ is not a root, we must have $d = 1$ or 2. For example, when $\mathfrak{g}_0 = \mathfrak{so}(4, 2l-3)$ or the split real form of the exceptional Lie algebra $\mathfrak{g}_2$, we have $d = 1$, whereas when $\mathfrak{g}_0 = \mathfrak{sp}(1, l-1)$, we have $d = 2$.

Clearly $\mathfrak{k}_1^{\mathbb{C}} = \mathfrak{g}_\mu \oplus \mathbb{C}h_\mu \oplus \mathfrak{g}_{-\mu} \cong \mathfrak{sl}(2, \mathbb{C})$, where $h_\mu \in (i\mathfrak{t}_0)^*$ is such that $\langle h, h_\mu\rangle = \mu(h)$ for $h \in (i\mathfrak{t}_0)^*$. The fundamental weight of $\mathfrak{k}_1^{\mathbb{C}}$ equals $\mu^* := \mu/2 = d\nu^*/2$. We shall denote by U_k the $(k+1)$-dimensional $\mathfrak{k}_1^{\mathbb{C}}$-module with highest weight $k\mu^* = dk\nu^*/2$. Also, $\mathbb{C}_\chi$ denotes the one dimenional $\mathfrak{l}_1^{\mathbb{C}}$-module corresponding to a character $\chi \in \mathbb{C}\nu^*$.

In §6.1 the 'sufficiently negativity' condition (2.5) for the quaternionic case is discussed. The Theorem 1.0.1 is proved in §6.2.

6.1 'Sufficiently negativity' condition in the quaternionic case

Let $\gamma = \gamma_0 + t\nu^*$ where γ_0 is a dominant integral weight of $\mathfrak{l}' = \mathfrak{l}_2^{\mathbb{C}}$ and t satisfies the 'sufficiently negative' condition (2.5), that is,

$$t < -1/2\langle \gamma_0 + \rho_{\mathfrak{g}}, \mu\rangle \ \text{ and } \ t < -\langle \gamma_0 + \rho_{\mathfrak{g}}, w_{\mathfrak{l}}^0(\nu)\rangle.$$

We have the following lemma.

Lemma 6.1.1 *Suppose that $\mathfrak{k}_1 = \mathfrak{su}(2)$, $\gamma = \gamma_0 + t\nu^*$ where γ_0 is an $\mathfrak{l}'$-dominant weight.*

Then t satisfies the 'sufficient negativity' condition (2.5) if and only if the following inequalities hold:

$$t < -\frac{d}{4}(|\Delta_1| + 2), \text{ and } t < -\langle \gamma_0, w_{\mathfrak{t}}^0(\nu)\rangle - (1/2)(\sum a_i \|\psi_i\|^2)$$

where $w_{\mathfrak{t}}^0(\nu) = \sum a_i \psi_i$ is the highest root in Δ_1.

Proof: Since γ_0 is a dominant integral weight of $\mathfrak{t}' = \mathfrak{t}_2^{\mathbb{C}}$ and since $\mu = d\nu^*$ is orthogonal to $\mathfrak{t}_2^{\mathbb{C}}$, we have $\langle \gamma_0, \mu\rangle = 0$. Since $\rho_{\mathfrak{g}} = (1/2)\sum_{\alpha\in\Delta^+}\alpha$, we get $\langle \rho_{\mathfrak{g}}, \mu\rangle = (d/2)(\sum_{\alpha\in\Delta_0^+}\langle\alpha,\nu^*\rangle + \sum_{\alpha\in\Delta_1}\langle\alpha,\nu^*\rangle + \sum_{\alpha\in\Delta_2}\langle\alpha,\nu^*\rangle) = (d/2)(|\Delta_1| + 2|\Delta_2|)$, since $\langle\alpha,\nu^*\rangle = i\langle\nu,\nu^*\rangle = i$ whenever $\alpha \in \Delta_i, i = 0, 1, 2$. Since $|\Delta_2| = 1$, we have $t < -(1/2)\langle\gamma_0 + \rho_{\mathfrak{g}}, \mu\rangle$ if and only if $t < -(d/4)(|\Delta_1| + 2)$.

Now $w_{\mathfrak{t}}^0(\nu) = \sum a_j\psi_j$ is the highest weight of $\mathfrak{u}_1$, which is indeed the highest root in Δ_1. Therefore $\langle \rho_{\mathfrak{g}}, w_{\mathfrak{t}}^0(\nu)\rangle = \langle \sum \psi_i^*, \sum a_j\psi_j\rangle = (1/2)(\sum a_i\|\psi_i\|^2)$. This completes the proof. □

6.2 Proof of Theorem 1.0.1

We now prove Theorem 1.0.1.

Proof of Theorem 1.0.1 : Write $\mathfrak{u}_{-1} = E_1 \otimes E_2$ where E_i is an irreducible L_i-module. By our hypothesis $L_1 \cong \mathbb{S}^1 = \{\exp(i\lambda H_\mu) | \lambda \in \mathbb{R}\}$ and so E_1 is 1-dimensional, given by the character $-\nu^*/\|\nu^*\|^2 = -\mu^*$. On the other hand, the highest weight of E_2 is $-(\nu - \mu^*)$. Hence $E_2 \cong E_{\mu^*-\nu}$. Since E_1 is one dimensional, we have $S^m(\mathfrak{u}_{-1}) = \mathbb{C}_{-m\mu^*} \otimes S^m(E_{\mu^*-\nu})$. On the other hand $\mathfrak{u}_{-2}$ is 1-dimensional and is isomorphic as an L_0-module to $\mathbb{C}_{-\mu} = \mathbb{C}_{-2\mu^*}$. Therefore $S^m(\mathfrak{u}_{-2}) = \mathbb{C}_{-2m\mu^*}$.

The vector bundle $\mathbb{E}$ over $Y = K_1/L_1$ associated to any L_2 representation space E is clearly isomorphic to the product bundle $Y \times E \longrightarrow Y$. Therefore the bundle $\mathbb{E}_\gamma \otimes \mathbb{S}^m(\mathfrak{u}_{-1})$ over $Y = \mathbb{P}^1$ is isomorphic to $\mathbb{E}_{(2t/d-m)\mu^*} \otimes E_{\gamma_0} \otimes S^m(E_{\mu^*-\nu})$. It follows that $H^1(Y; \mathbb{E}_\gamma \otimes \mathbb{S}^m(\mathfrak{u}_{-1})) \cong H^1(Y; \mathbb{E}_{(2t/d-m)\mu^*}) \otimes E_{\gamma_0} \otimes S^m(E_{\mu^*-\nu}) \cong U_{-2t/d+m-2} \otimes E_{\gamma_0} \otimes S^m(E_{\mu^*-\nu})$. By Theorem 2.4.1 we conclude that

$$(\pi_{\gamma+\rho_{\mathfrak{g}}})_{K_0} = \oplus_{m\geq 0} U_{(m-2t/d-2)} \otimes E_{\gamma_0} \otimes S^m(E_{\mu^*-\nu}). \tag{6.1}$$

We now turn to the description of the holomorphic discrete series $\pi_{\gamma+\rho_{\mathfrak{k}}}$ of $K_0^* = K_1^* K_2$. Now recall (see §2.4.2) the following description of the holomorphic discrete series of K_1^* determined by $t\nu^* = (2t/d)\mu^*$, namely, $(\pi_{(2t/d)\mu^*+\rho_{\mathfrak{k}_1^{\mathbb{C}}}})_{L_1} = \oplus_{r\geq 0}\mathbb{C}_{(2t/d)\mu*} \otimes S^r(\mathfrak{u}_{-2}) = \oplus_{r\geq 0}\mathbb{C}_{(2t/d-2r)\mu^*}$. It follows that the holomorphic discrete series of K_0^* determined by γ is

$$(\pi_{\gamma+\rho_{\mathfrak{k}}})_{L_0} = \oplus_{r\geq 0}\mathbb{C}_{(2t/d-2r)\mu^*} \otimes E_{\gamma_0}. \tag{6.2}$$

Comparing (6.1) and (6.2) we observe that there exists an L_0-type common to $(\pi_{\gamma+\rho_{\mathfrak{g}}})_{K_0}$ and $\pi_{\gamma+\rho_{\mathfrak{k}}}$ *if and only if* the following two conditions hold:

(a) E_{γ_0} occurs in $E_{\gamma_0} \otimes S^m(E_{\mu^*-\nu})$.
(b) Assuming that (a) holds for some $m \geq 0$, $(2t/d - 2r)\mu^*$ occurs as a weight in $U_{m-2t/d-2}$ for some r, that is, $2t/d - 2r = (m - 2t/d - 2) - 2i$ for some $0 \leq i \leq (m - 2t/d - 2)$.

First suppose that $\mathfrak{g}_\circ = \mathfrak{so}(4,1)$ or $\mathfrak{sp}(1, l-1), l > 1$. In view of Proposition 1.0.3 and Proposition 5.1.1, the Borel-de Siebenthal discrete series $\pi_{\gamma+\rho_\mathfrak{g}}$ is L_0-admissible and any L_0-type in $\pi_{\gamma+\rho_\mathfrak{g}}$ is contained in $(\pi_{\gamma+\rho_\mathfrak{g}})_{K_0}$. Also $S^m(E_{\mu^*-\nu})$ is irreducible with highest weight $m(\mu^* - \nu)$ (see Proposition 4.3.1). Recall that the highest weights of irreducible sub representations which occur in a tensor product $E_\lambda \otimes E_\kappa$ of two irreducible representations of $\mathfrak{l}_2^\mathbb{C}$ are all of the form $\theta + \kappa$ where θ is a weight of E_λ. So if (a) holds, then $\gamma_0 = m(\mu^* - \nu) + \theta$, for some weight θ of E_{γ_0}. This implies $\gamma_0 - \theta = m(\mu^* - \nu)$, which holds for atmost finitely many m since the number of weights of E_{γ_0} is finite. So by (a), there are atmost finitely many L_0-types common to $\pi_{\gamma+\rho_\mathfrak{g}}$ and $\pi_{\gamma+\rho_\mathfrak{k}}$.

Moreover, if $\gamma_0 = 0$, then the trivial L_0-representation E_{γ_0} occurs in $E_{\gamma_0} \otimes S^m(E_{\mu^*-\nu}) = E_{m(\mu^*-\nu)}$ only when $m = 0$. Since $2t/d - 2r \leq 2t/d < 2t/d + 2$ for all $r \geq 0$, $(2t/d - 2r)\mu^*$ cannot be a weight of $U_{-2t/d-2}$ for all $r \geq 0$. So in view of (a) and (b), there are no common L_0-types between $\pi_{\gamma+\rho_\mathfrak{g}}$ and $\pi_{\gamma+\rho_\mathfrak{k}}$.

Now suppose that $\mathfrak{g}_\circ \neq \mathfrak{so}(4,1), \mathfrak{sp}(1, l-1), l > 1$. In view of Proposition 4.3.1, we see that $\mathcal{A}(\mathfrak{u}_1, L) = \mathbb{C}[f]$, where f is a relative invariant (hence is a homogeneous polynomial) of positive degree, say of degree k. Then the trivial module is a sub module of the L_0-module $S^{jk}(E_{\mu^*-\nu})$ for all $j \geq 0$. So E_{γ_0} occurs in $E_{\gamma_0} \otimes S^{jk}(E_{\mu^*-\nu})$ for all $j \geq 0$. That is (a) holds.
Let r be a non negative integer. Then $(2t/d - 2r)\mu^*$ is a weight of $U_{jk-2t/d-2}$ for some $j \geq 0$ if and only if $2t/d - 2r = (jk - 2t/d - 2) - 2i$ for some $0 \leq i \leq (jk - 2t/d - 2)$ if and only if jk is even and $jk \geq 2(r+1)$.

So in view of (a) and (b), each L_0-type in $\pi_{\gamma+\rho_\mathfrak{k}}$ occurs in $\pi_{\gamma+\rho_\mathfrak{g}}$ with infinite multiplicity. This completes the proof. □

Chapter 7

PROOF OF THEOREM 1.0.2

Recall from §2.4.3 that G_0 is a simply connected non-compact real simple Lie group and K_0 is a maximal compact subgroup of G_0 such that rank (G_0) = rank (K_0) and G_0/K_0 is not Hermitian symmetric. Also recall that $Y = K_0/L_0$ is an irreducible Hermitian symmetric space of the compact type with the non-compact dual $X = K_0^*/L_0$. The Δ_0^+ is a positive system of $(\mathfrak{l}, \mathfrak{t})$ with $\Psi_{\mathfrak{l}} = \Psi \setminus \{\nu\}$ the set of simple roots and $\Delta_0^+ \cup \Delta_2$ is a positive system of $(\mathfrak{k}, \mathfrak{t})$ with $\Psi_{\mathfrak{k}} = (\Psi \setminus \{\nu\}) \cup \{\epsilon\}$ the set of simple roots. The simple root ϵ is the unique non-compact root in $\Psi_{\mathfrak{k}}$. If $w_{\mathfrak{k}}^0(\epsilon) = -\epsilon$, then $w_{\mathfrak{k}}^0(\Delta_0^+) = \Delta_0^-, w_{\mathfrak{k}}^0(\Delta_2) = \Delta_{-2}$ and $w_Y(\Delta_0^+) = \Delta_0^+, w_Y(\Delta_2) = \Delta_{-2}$, where $w_Y = w_{\mathfrak{k}}^0 w_{\mathfrak{l}}^0$. Hence $w_Y^2(\Delta_0^+ \cup \Delta_2) = \Delta_0^+ \cup \Delta_2$. This implies $w_Y^2 = \mathrm{Id}$. Also $w_{\mathfrak{k}}^0(\epsilon) = -\epsilon$ implies $w_Y(\epsilon^*) = -\epsilon^*$. Let $\Gamma = \{\gamma_1, \ldots, \gamma_r\} \subset \Delta_{-2}$ be the maximal set of strongly orthogonal roots obtained as in §2.5. If $\gamma + \rho_{\mathfrak{g}}$ is the Harish-Chandra parameter of a Borel-de Siebenthal discrete series representation $\pi_{\gamma+\rho_{\mathfrak{g}}}$ of G_0, then the K_0 finite part $(\pi_{\gamma+\rho_{\mathfrak{g}}})_{K_0}$ of $\pi_{\gamma+\rho_{\mathfrak{g}}}$ is isomorphic to $\oplus_{m\geq 0} H^s(Y; \mathbb{E}_\gamma \otimes \mathbb{S}^m(\mathfrak{u}_{-1}))$. See Theorem 2.4.1. The L_0 finite part $(\pi_{\gamma+\rho_{\mathfrak{k}}})_{L_0}$ of the associated holomorphic discrete series representation $\pi_{\gamma+\rho_{\mathfrak{k}}}$ of K_0^* is isomorphic to $E_\gamma \otimes S^*(\mathfrak{u}_{-2})$. See §3.2 in Chapter 3.

In §7.1, we establish three lemmas which will be needed in the proof of Theorem 1.0.2. We shall use Littelmann's path model described in §2.6 to prove these lemmas. *Up to the end of proof of Lemma 7.1.3 we shall use the symbols π, π_λ, etc., paths in the sense of Littelmann and are not to be confused with discrete series.* The main result of this thesis Theorem 1.0.2 is proved in §7.2.

7.1 Branching rule using Littelmann's path model

Recall from §2.6.1 that π_λ denotes the path $t \mapsto t\lambda$, $0 \leq t \leq 1$, for an integral weight λ of $\mathfrak{k}$. If in addition λ is dominant, then $w(\pi_\lambda) = \pi_{w\lambda}$ is an LS-path of shape λ for any element w in the Weyl group of $(\mathfrak{k}, \mathfrak{t})$. We also have the action of Littelmann's root operator f_α $(\alpha \in \Psi_{\mathfrak{k}})$ on the concatenation of two paths. See (2.8) in the Proposition 2.6.2.

We denote by V_λ (respectively E_κ), the finite dimensional irreducible representation of $\mathfrak{k}$ (respectively $\mathfrak{l}$) with highest weight λ (respectively κ). If V is a $\mathfrak{k}$-representation, we shall denote by $\mathrm{Res}_{\mathfrak{l}}(V)$ its restriction to $\mathfrak{l}$. Since $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{k}$, we have the branching rule (2.10) of $\mathrm{Res}_{\mathfrak{l}}(V)$.

Lemma 7.1.1 *(i) The restriction $Res_{\mathfrak{l}}(V_{m\epsilon^*})$ to $\mathfrak{l}$ of the irreducible $\mathfrak{k}$-representation $V_{m\epsilon^*}$ contains $Res_{\mathfrak{l}}(V_{(m-p)\epsilon^*}) \otimes \mathbb{C}_{p\epsilon^*}$ for $0 \le p \le m$.*
(ii) Suppose that $w^0_{\mathfrak{k}}(\Delta_0) = \Delta_0$. Then $Res_{\mathfrak{l}}(V_{m\epsilon^*})$ contains $Res_{\mathfrak{l}}(V_{(m-p)\epsilon^*}) \otimes \mathbb{C}_{-p\epsilon^*}$.

Proof: (i) Note that $\pi_{m\epsilon^*}$ equals the concatenation $\pi_{(m-p)\epsilon^*} * \pi_{p\epsilon^*}$.

Let τ be an LS-path of shape $(m-p)\epsilon^*$ which is $\mathfrak{l}$-dominant. Then $\tau = f_{\alpha_q} \cdots f_{\alpha_1}\pi_{(m-p)\epsilon^*}$ for some sequence $\alpha_1, \ldots, \alpha_q$ of simple roots in $\Psi_{\mathfrak{k}}$. Then $f_{\alpha_i} \ldots f_{\alpha_1}(\pi_{(m-p)\epsilon^*}) \neq 0$ for $1 \le i \le q$. It follows that $f_{\alpha_q} \ldots f_{\alpha_1}(\pi_{m\epsilon^*}) = f_{\alpha_q} \ldots f_{\alpha_1}(\pi_{(m-p)\epsilon^*} * \pi_{p\epsilon^*}) = f_{\alpha_q} \ldots f_{\alpha_1}(\pi_{(m-p)\epsilon^*}) * \pi_{p\epsilon^*} = \tau * \pi_{p\epsilon^*}$ since $e_\alpha(\pi_{p\epsilon^*}) = 0$. Thus we see that if τ is any $\mathfrak{l}$-dominant LS-path of shape $(m-p)\epsilon^*$, then $\tau * \pi_{p\epsilon^*}$ is an LS-path of shape $m\epsilon^*$. It is clear that $\tau * \pi_{p\epsilon^*}$ is $\mathfrak{l}$-dominant. Since $E_{\tau * \pi_{p\epsilon^*}(1)} = E_{\tau(1)+p\epsilon^*} \cong E_{\tau(1)} \otimes \mathbb{C}_{p\epsilon^*}$ and since for any path σ, $\sigma * \pi_{p\epsilon^*} = \tau * \pi_{p\epsilon^*}$ implies $\sigma = \tau$, it follows that $\mathrm{Res}_{\mathfrak{l}}(V_{m\epsilon^*})$ contains $\mathrm{Res}_{\mathfrak{l}}(V_{(m-p)\epsilon^*}) \otimes \mathbb{C}_{p\epsilon^*}$ in view of (2.10).

(ii) Suppose that $w^0_{\mathfrak{k}}(\Delta_0) = \Delta_0$. This is equivalent to the condition that $w^0_{\mathfrak{k}}(\epsilon^*) = -\epsilon^*$, which in turn is equivalent to the requirement that $V_{q\epsilon^*}$ is self-dual as a $\mathfrak{k}$-representation for all $q \ge 1$. Since $\mathrm{Res}_{\mathfrak{l}}(V_{(m-p)\epsilon^*}) \otimes \mathbb{C}_{p\epsilon^*}$ is contained in $V_{m\epsilon^*}$, so is its dual. That is, $\mathrm{Res}_{\mathfrak{l}}(V_{(m-p)\epsilon^*}) \otimes \mathbb{C}_{-p\epsilon^*}$ is contained in $\mathrm{Res}_{\mathfrak{l}}(V_{m\epsilon^*})$. □

Lemma 7.1.2 *Let $0 \le p_r \le \cdots \le p_1 \le p_0 \le m$ be a sequence of integers. Then $Res_{\mathfrak{l}}V_{m\epsilon^*}$ contains E_κ where $\kappa = m\epsilon^* + p_1\gamma_1 + \cdots p_r\gamma_r$. Moreover, if $w^0_{\mathfrak{k}}(\Delta_0) = \Delta_0$, then E_λ occurs in $Res_{\mathfrak{l}}V_{m\epsilon^*}$ where $\lambda = (m-2p_0)\epsilon^* - (\sum_{1\le j\le r} p_j\gamma_{r+1-j})$.*

Proof: Recall that $\gamma_1 = -\epsilon$. Since the γ_i are pairwise orthogonal we see that $s_{\gamma_i}s_{\gamma_j} = s_{\gamma_j}s_{\gamma_i}$. Also since $\gamma_j \in \Delta_{-2}$, $\langle\epsilon^*, \gamma_i\rangle = \langle\epsilon^*, -\epsilon\rangle = -||\epsilon||^2/2$. As noted in Remark 2.5.4(iii), all the γ_i have the same length: $||\gamma_i|| = ||\epsilon||$. Using these facts a straightforward computation yields that $s_{\gamma_i}(\epsilon^*) = \epsilon^* + \gamma_i$, $s_{\gamma_i}(\gamma_j) = \gamma_j$ for $1 \le i, j \le r, i \neq j$. Defining $p_{r+1} = 0$, it follows that $s_{\gamma_1} \cdots . s_{\gamma_j}(\pi_{(p_j - p_{j+1})\epsilon^*}) =: \pi_j$ is the straight-line path of weight $(p_j - p_{j+1})(\epsilon^* + \gamma_1 + \cdots + \gamma_j)$ and hence we have $f_{I_j}(\pi_{(p_j-p_{j+1})\epsilon^*}) = \pi_j$ for a suitable monomial in root operators f_{I_j} of simple roots of $\mathfrak{k}$ for all $2 \le j \le r$. So, writing $\pi_{m\epsilon^*} = \pi_{p_r\epsilon^*} * \pi_{(p_{r-1}-p_r)\epsilon^*} * \cdots * \pi_{(p_2-p_3)\epsilon^*} * \pi_{(m-p_2)\epsilon^*}$ we have $f_{I_r}(\pi_{m\epsilon^*}) = \pi_r * \pi_{(p_{r-1}-p_r)\epsilon^*} * \cdots * \pi_{(p_2-p_3)\epsilon^*} * \pi_{(m-p_2)\epsilon^*}$, in view of (2.8). Clearly $f_\epsilon(\pi_j) = 0$ for all $2 \le j \le r$. Also in view of the Proposition 2.5.2(ii), if the coefficient of a compact simple root α of $\mathfrak{k}$ in the expression of $\sum_{1\le i\le j}\gamma_i$ is non zero, then $f_\alpha(\pi_j) = 0$. Now for a simple root α of $\mathfrak{k}$, if f_α is involved in the expression of f_{I_j}, then the coefficient of α in the expression of $\sum_{1\le i\le(j+1)}\gamma_i$ is non zero. Hence $f_\alpha(\pi_{j+1}) = 0$ for $2 \le j \le r-1$. Therefore $f_{I_2} \ldots f_{I_r}(\pi_{m\epsilon^*}) = \pi_r * \pi_{r-1} * \cdots * \pi_2 * \pi_{(m-p_2)\epsilon^*}$, in view of (2.8). Since $f_\epsilon(\pi_j) = 0$ for all $2 \le j \le r$ and $f_\epsilon^{p_1-p_2}(\pi_{(m-p_2)\epsilon^*}) = \pi_{(p_1-p_2)(\epsilon^*-\epsilon)} * \pi_{(m-p_1)\epsilon^*}$, we obtain $\tau := f_\epsilon^{p_1-p_2} f_{I_2} \ldots f_{I_r}(\pi_{m\epsilon^*}) = \pi_r * \cdots * \pi_2 * \pi_{(p_1-p_2)(\epsilon^*-\epsilon)} * \pi_{(m-p_1)\epsilon^*}$, again by (2.8). The break-points and the terminal point of τ are $p_r(\epsilon^* + \gamma_1 + \cdots + \gamma_r)$, $p_{r-1}(\epsilon^* + \gamma_1 + \cdots + \gamma_{r-1}) + p_r\gamma_r$, $p_{r-2}(\epsilon^* + \gamma_1 + \cdots + \gamma_{r-2}) + p_{r-1}\gamma_{r-1} + p_r\gamma_r, \ldots, p_2(\epsilon^* + \gamma_1 + \gamma_2) + p_3\gamma_3 + \cdots + p_r\gamma_r$, $p_1(\epsilon^* + \gamma_1) + p_2\gamma_2 + \cdots + p_r\gamma_r$ and $m\epsilon^* + p_1\gamma_1 + p_2\gamma_2 + \cdots + p_r\gamma_r$. All these are $\mathfrak{l}$-dominant weights (since $p_1 \ge p_2 \ge \cdots \ge$

$p_r \geq 0$) and so we conclude that τ is an $\mathfrak{l}$-dominant LS-path. Hence by the branching rule, $E_{m\epsilon^*+p_1\gamma_1+p_2\gamma_2+\cdots+p_r\gamma_r}$ occurs in $V_{m\epsilon^*}$.

Now suppose $w^0_{\mathfrak{k}}(\Delta_0) = \Delta_0$. By Lemma 7.1.1, we have $\mathrm{Res}_{\mathfrak{l}}V_{m\epsilon^*}$ contains $\mathrm{Res}_{\mathfrak{l}}V_{p_0\epsilon^*} \otimes E_{(m-p_0)\epsilon^*}$. By what has been proved already $\mathrm{Res}_{\mathfrak{l}}V_{p_0\epsilon^*}$ contains $E_{p_0\epsilon^*+p_1\gamma_1+p_2\gamma_2+\cdots+p_r\gamma_r} =: E$. Since $V_{p_0\epsilon^*}$ is self-dual, $\mathrm{Hom}(E, \mathbb{C})$ is contained in $\mathrm{Res}_{\mathfrak{l}}V_{p_0\epsilon^*}$. The highest weight of $\mathrm{Hom}(E, \mathbb{C})$ is $-p_0\epsilon^* - \sum_{1\leq j\leq r} p_j w^0_{\mathfrak{l}}(\gamma_j) = -p_0\epsilon^* - p_1\gamma_r - p_2\gamma_{r-1} + \cdots - p_r\gamma_1$ using Remark 2.5.4(i). Tensoring with $E_{(m-p_0)\epsilon^*}$ we conclude that E_λ occurs in $\mathrm{Res}_{\mathfrak{l}}V_{m\epsilon^*}$ with $\lambda = (m - 2p_0)\epsilon^* - p_r\gamma_1 - p_{r-1}\gamma_2 - \cdots - p_2\gamma_{r-1} - p_1\gamma_r$. □

Write $\gamma = \gamma_0 + t\epsilon^*$ with $\langle\gamma_0, \mu\rangle = 0$. Then γ_0 is $\mathfrak{k}$-integral weight and t is an integer (γ being a $\mathfrak{k}$-integral weight). Also γ is $\mathfrak{l}$-dominant implies that γ_0 is $\mathfrak{l}$-dominant. Since $\langle\gamma + \rho_{\mathfrak{k}}, \mu\rangle < 0$, we have $t < -2\langle\rho_{\mathfrak{k}}, \mu\rangle/||\epsilon||^2$. Assuming $w^0_{\mathfrak{k}}(\epsilon) = -\epsilon$, we get $\langle w_Y(\gamma_0), \alpha\rangle \geq 0$ when α is in Δ^+_0 and $\langle w_Y(\gamma_0), \epsilon\rangle = 0$. So $w_Y(\gamma_0)$ is $\mathfrak{k}$-dominant integral weight.

Lemma 7.1.3 *With the above notation, suppose that $w^0_{\mathfrak{k}}(\epsilon) = -\epsilon$ and that E_τ is a subrepresentation of $Res_{\mathfrak{l}}(V_{m\epsilon^*})$. Then $E_{\gamma_0+w_Y(\tau)}$ is a subrepresentation of $Res_{\mathfrak{l}}(V_{w_Y(\gamma_0)+m\epsilon^*})$.*

Proof: Let π denote the path $\pi_{m\epsilon^*} * \pi_{w_Y(\gamma_0)}$. Then $\mathrm{Im}(\pi)$ is contained in the dominant Weyl chamber (of $\mathfrak{k}$) and $\pi(1) = w_Y(\gamma_0) + m\epsilon^*$. Since E_τ is contained in $\mathrm{Res}_{\mathfrak{l}}(V_{m\epsilon^*})$, there exist a sequence $\alpha_1, \ldots, \alpha_k$ of simple roots of $\mathfrak{k}$ such that $f_{\alpha_1} \ldots f_{\alpha_k}(\pi_{m\epsilon^*}) =: \eta$ is $\mathfrak{l}$-dominant path with $\eta(1) = \tau$. Since $\pi_{w_Y(\gamma_0)}$ is $\mathfrak{k}$-dominant path, $\theta := f_{\alpha_1} \ldots f_{\alpha_k}(\pi) = \eta * \pi_{w_Y(\gamma_0)}$, in view of (2.8). Clearly θ is $\mathfrak{l}$-dominant and $\theta(1) = \tau + w_Y(\gamma_0)$. Hence by the branching rule (2.10), $E_{w_Y(\gamma_0)+\tau}$ occurs in $\mathrm{Res}_{\mathfrak{l}}(V_{w_Y(\gamma_0)+m\epsilon^*})$.

Let $\Phi : K_0 \longrightarrow GL(V_{\lambda_0})$ be the representation, where $\lambda_0 := w_Y(\gamma_0) + m\epsilon^*$. Then $\phi := d\Phi : \mathfrak{k}_\circ \longrightarrow End(V_{\lambda_0})$. For $k \in K_0$ and $X \in \mathfrak{k}_\circ$, we have

$$\Phi(k^{-1}) \circ \phi(X) \circ \Phi(k) = \phi(Ad(k^{-1})X) \tag{7.1}$$

Let $v \in V_{\lambda_0}$ is a weight vector of weight $\lambda := w_Y(\gamma_0) + \tau$ such that it is a highest weight vector of E_λ. Now $w_Y = (Ad(k)|_{it_\circ})^*$ for some $k \in N_{K_0}(T_0)$. Then $\Phi(k)v$ is a weight vector of weight $w_Y(\lambda)$ and it is killed by all root vectors X_α ($\alpha \in \Delta^+_0$), in view of (7.1); since $w_Y(\Delta^+_0) = \Delta^+_0$. Hence $\Phi(k)v$ is a highest weight vector of an irreducible L_0- submodule of $\mathrm{Res}_{\mathfrak{l}}(V_{\lambda_0})$. Therefore $E_{w_Y(\lambda)} = E_{\gamma_0+w_Y(\tau)}$ occurs in $\mathrm{Res}_{\mathfrak{l}}(V_{\lambda_0})$. □

7.2 Proof of Theorem 1.0.2

We are now ready to prove Theorem 1.0.2.

Proof of Theorem 1.0.2: Write $\gamma = \gamma_0 + t\epsilon^*$ where $\langle\gamma_0, \mu\rangle = 0$.

We have

$$(\pi_{\gamma+\rho_{\mathfrak{k}}})_{L_0} = E_\gamma \otimes S^*(\mathfrak{u}_{-2}) = \oplus(E_\gamma \otimes E_{a_1\gamma_1+\cdots+a_r\gamma_r})$$

where the sum is over all integers $a_1 \geq \cdots \geq a_r \geq 0$. (In view of Theorem 2.5.1).
So $(\pi_{\gamma+\rho_{\mathfrak{k}}})_{L_0}$ contains $E_{\gamma+a_1\gamma_1+\cdots+a_r\gamma_r}$, for all integers $a_1 \geq \cdots \geq a_r \geq 0$.

Let $k \geq 1$ be the least integer such that $S^k(\mathfrak{u}_{-1})$ has one-dimensional L_0-subrepresentation, which is necessarily of the form $E_{q\epsilon^*}$ for some $q < 0$. Now $(\pi_{\gamma+\rho_\mathfrak{g}})_{K_0}$ contains $\oplus_{j\geq 0} H^s(Y; \mathbb{E}_{\gamma+jq\epsilon^*})$, by Theorem 2.4.1. By Borel-Weil-Bott theorem ([3], also see [6, Th. 1.6.8, Ch. 1]), $H^s(Y; \mathbb{E}_{\gamma+jq\epsilon^*})$ is an irreducible finite dimensional K_0-representation with highest weight $w_Y(\gamma + jq\epsilon^* + \rho_\mathfrak{k}) - \rho_\mathfrak{k} = w_Y(\gamma_0) + (-t - jq - c)\epsilon^*$ since $w^0_\mathfrak{k}(\epsilon^*) = -\epsilon^*$, where $\sum_{\beta\in\Delta_2} \beta = c\epsilon^*$ for some $c \in \mathbb{N}$. Define $m_j := -t - jq - c$ for all $j \geq 0$. For $0 \leq p_r \leq \cdots \leq p_1 \leq m_j$, $E_{m_j\epsilon^* + p_1\gamma_1 + \cdots + p_r\gamma_r}$ is a subrepresentation of $\mathrm{Res}_\mathfrak{k}(V_{m_j\epsilon^*})$, in view of Lemma 7.1.2. So by Lemma 7.1.3, $E_{\gamma_0 - m_j\epsilon^* - p_1\gamma_r - \cdots - p_r\gamma_1}$ is a subrepresentation of $\mathrm{Res}_\mathfrak{k}(V_{w_Y(\gamma_0)+m_j\epsilon^*})$ since $w_Y(\gamma_j) = -\gamma_{r+1-j}$, for all $1 \leq j \leq r$ by Remark 2.5.4(i). Now $H^s(Y; \mathbb{E}_{\gamma+jq\epsilon^*})$ is isomorphic to $V_{w_Y(\gamma_0)+m_j\epsilon^*}$. So, for $0 \leq p_r \leq \cdots \leq p_1 \leq m_j$, $E_{\gamma_0 - m_j\epsilon^* - p_1\gamma_r - \cdots - p_r\gamma_1}$ is an L_0-submodule of $H^s(Y; \mathbb{E}_{\gamma+jq\epsilon^*})$.

Fix $a_1 \geq \cdots \geq a_r \geq 0$, where $a_1, \ldots, a_r \in \mathbb{Z}$. In view of §4.1 and Lemma 4.3.2, q is odd when c is odd. Let $\mathbb{N}' = \{j \in \mathbb{N} | (jq + c) \text{is even}\}$. There exists $j_0 \in \mathbb{N}$ such that for all $j \in \mathbb{N}'$ with $j \geq j_0$, $-(jq + c)/2 \geq a_1$. Define $p_{r+1-i} := -(jq + c)/2 - a_i, \quad 1 \leq i \leq r$. Then $0 \leq p_r \leq \cdots \leq p_1 < m_j$.

Now $\sum_{1\leq i\leq r} p_i\gamma_{r+1-i} = \sum_{1\leq i\leq r} p_{r+1-i}\gamma_i = \sum_{1\leq i\leq r}(-a_i - (jq + c)/2)\gamma_i = (jq + c)\epsilon^* - \sum_{1\leq i\leq r} a_i\gamma_i$ in view of Proposition 2.5.2(i), since $w^0_\mathfrak{k}(\epsilon) = -\epsilon$ by hypothesis. It follows that $\gamma_0 - m_j\epsilon^* - \sum_{1\leq i\leq r} p_i\gamma_{r+1-i} = \gamma + \sum_{1\leq i\leq r} a_i\gamma_i$. So for all $j \in \mathbb{N}'$ with $j \geq j_0$, $E_{\gamma+a_1\gamma_1+\cdots+a_r\gamma_r}$ is an L_0-submodule of $H^s(Y; \mathbb{E}_{\gamma+jq\epsilon^*})$. That is, for all integers $a_1 \geq \cdots \geq a_r \geq 0$, the L_0-type $E_{\gamma+a_1\gamma_1+\cdots+a_r\gamma_r}$ occurs in $\pi_{\gamma+\rho_\mathfrak{g}}$ with infinite multiplicity.

In particular, if $\gamma = t\nu^*$, each L_0-type in $\pi_{\gamma+\rho_\mathfrak{k}}$ occurs in $\pi_{\gamma+\rho_\mathfrak{g}}$ with infinite multiplicity. This completes the proof. □

There are three major obstacles in obtaining complete result in the non-quaternionic case. The first is the decomposition of $S^m(\mathfrak{u}_{-1})$ into L_0-types E_λ. Secondly, one has the problem of decomposing of the tensor product $E_\gamma \otimes E_\lambda$ into irreducible L_0-representations E_κ. Finally, one has the restriction problem of decomposing the irreducible K_0-representation $H^s(K_0/L_0; \mathbb{E}_\kappa)$ into L_0- subrepresentations. The latter two problems can, in principle, be solved using the work of Littelmann [17]. The problem of detecting occurrence of an infinite family of common L_0-types in the general case appears to be intractable.

We conclude this this thesis with the following questions:

Questions: Suppose that there exist infinitely many common L_0-types between a Borel-de Siebenthal discrete series representation $\pi_{\gamma+\rho_\mathfrak{g}}$ of G_0 and the holomorphic discrete series representation $\pi_{\gamma+\rho_\mathfrak{k}}$ of K_0^*. Then (i) Does there exist a one dimensional L_0-subrepresentation in $S^m(\mathfrak{u}_{-1})$? (ii) Is it true that $w^0_\mathfrak{k}(\Delta_0) = \Delta_0$?

Bibliography

[1] Borel, A.; Hirzebruch, F. Characteristic classes and homogeneous spaces. I. Amer. J. Math. **80** (1958) 458–538.

[2] Borel, A.; J.de Siebenthal Les sous-groupes fermés de rang maximum des groupes de Lie clos. Comment. Math. Helv. **23** (1949), 200–221.

[3] Bott, R. Homogeneous vector bundles, Ann. Math. (2), **66** (1957), 203–248.

[4] Bourbaki, N. Groupes et algébres de Lie. Chapitre IV-VII. Hermann, Paris 1968.

[5] Dupont Johan L. *Differential Geometry.* Lecture Notes Series, **62**. Aarhus Universitet, Matematisk Institut, Aarhus, 1993. iv+177 pp.

[6] Fels, Gregor; Huckleberry, Alan; Wolf, Joseph A. *Cycle spaces of flag domains. A complex geometric view point.* Progress in Mathematics, **245**. Birkhäuser Boston, Inc., Boston, MA, 2006. xx+339 pp.

[7] Harish-Chandra, Integrable and square-integrable representations of a semisimple Lie group. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 314–317.

[8] Harish-Chandra, Representations of semisimple Lie groups. VI. Integrable and square-integrable representations. Amer. J. Math. **78** (1956), 564–628.

[9] Harish-Chandra, Discrete series for semisimple Lie groups. II. Explicit determination of the characters. Acta Math. **116** (1966) 1–111.

[10] Helgason, Sigurdur *Differential Geometry, Lie Groups, and Symmetric Spaces.* Corrected reprint of the 1978 original. Graduate Studies in Mathematics, **34**. American Mathematical Society, Providence, RI, 2001.

[11] Hotta, R.; Parthasarathy, R. Multiplicity formulae for discrete series, Invent. Math. **26** (1974) 133–178.

[12] Humphreys, J. E. *Introductions to Lie Algebras and Representation Theory.* Second printing, revised. Graduate Studies in Mathematics, **9**. Springer-Verlag, New York-Berlin, 1978. xi+171 pp.

[13] Knapp, Anthony W. *Representation theory of semisimple groups. An overview based on examples.* Reprint of the 1986 original. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 2001.

[14] Knapp, Anthony W. *Lie groups beyond an introduction.* Second edition. Progress in Mathematics, **140**. Birkhäuser Boston, Inc., Boston, MA, 2002.

[15] Kobayashi, Toshiyuki Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups and its applications. Invent. Math. **117** (1994) 181–205.

[16] Kobayashi, Toshiyuki Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups II: Microlocaly analysis and asymptotic K-support. Annals of Mathematics Vol **147**, No. 3 (May 1998)

[17] Littelmann, Peter A Littlewood-Richardson rule for symmetrizable Kac-Moody algebras. Invent. Math. **116** (1994), 329–346.

[18] Littelmann, Peter Paths and root operators in representation theory. Ann. of Math. (2) **142** (1995), 499–525.

[19] Ørsted, Bent; Wolf, Joseph A. Geometry of the Borel-de Siebenthal discrete series. J. Lie Theory **20** (2010), no. 1, 175–212.

[20] Parthasarathy, R. An algebraic construction of a class of representations of a semi-simple Lie algebra. Math. Ann. **226** (1977), 1–52.

[21] Repka, Joe Tensor products of holomorphic discrete series representations. Canad. J. Math. **31** (1979), 836–844.

[22] Sato, M.; Kimura, T. A classification of irreducible prehomogeneous vector spaces and their relative invariants. Nagoya Math. J. **65** (1977), 1–155.

[23] Schmid, Wilfried Die Randwerte holomorpher Funktionen auf hermitesche symmetrischen Räumen. Invent. Math. **9** (1969/1970) 61–80.

[24] Warner, Frank W. *Foundations of Differentiable Manifolds and Lie Groups.* Corrected reprint of the 1971 edition. Graduate Studies in Mathematics, **94**. Springer-Verlag, New York-Berlin, 1983. ix+272 pp.

[25] Warner, Garth *Harmonic Analysis on Semi-Simple Lie Groups I.* Springer-Verlag, Berlin, Heidelberg, New York, 1972. xvi+529 pp.