

Spectral Statistics for Anderson Model with Decaying  
Randomness and Singular Potentials

By

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## DECLARATION

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# Synopsis

In the study of materials for their electrical properties one of the important models is the Anderson model, named after the Physicist P W Anderson. It is an approximation for the Schrödinger equation of one electron moving in the potential of a collection of orderly placed atoms, viz., a crystal.

The approximation, called the tight binding approximation, is based on constructing wave function to the Schrödinger equation, which are superposition of those of isolated atoms located at the atomic sites. Such a model can be identified to be a discrete Schrödinger operator on  $\ell^2(\mathbb{Z}^d)$ . P. W. Anderson studied the tight binding approximation for large disorder and showed that such models produced insulators when the disorder is large, a theory for which he shared the Nobel Prize in 1977 along with N. F. Mott and J. H. van Vleck.

Mathematically rigorous studies of the Anderson Model and other models of random operators started in the 1970's. In 1977 Goldsheid-Molchanov-Pastur [27] provided the proof of *localization* for Anderson related model in one dimension. Several years later Kunz-Souillard [45] gave the proof of *localization* for the actual Anderson model, also for dimension one. Since then the study of random operators has become an important field of mathematical physics, which has led to a tremendous amount of research activity and many mathematical results.

The proof of Anderson transition and existence of extended states are still a challenge, but a rigorous understanding of Anderson localization has been achieved. Several powerful methods have been found to prove Anderson localization. In contrast to one-dimensional models multi-dimensional models require large disorder for localization.

There are two well known methods available for the study of Anderson localization in arbitrary dimension, the method of *multiscale analysis* developed in 1983 by Fröhlich-Spencer [23] which immediately gave those proofs of localization for large disorder, by Simon-Wolff [58], Fröhlich-Martinelli-Scoppola-Spencer [24], Delyon-Levy-Souillard [17],

the other is the *fractional moments method* introduced by Aizenman-Molchanov [5] in 1993.

In this thesis we study the Spectral statistics of Anderson Model on  $\mathbb{Z}^d$  with two different types of potentials, one decaying random potential and the other i.i.d random potential with Hölder continuous single site distribution. We mainly study the limits of eigenvalue statistics of finite dimensional restrictions of the model.

The study of eigenvalue statistics for the Anderson model was done by Molchanov [48] in one-dimension and by Minami [47] in higher dimension (see also Germinet-Klopp [25]) in the localized regime of the spectrum. Eigenfunction statistics was studied by Nakano [50] for continuous model and Killip-Nakano [40] for lattice model with single site distribution having bounded density.

For decaying potentials the spectral statistics was studied for different cases. Killip-Stoiciu [45] studied the CMV matrices whose matrix elements decay like  $n^{-\alpha}$ . They showed that, for (i)  $\alpha > 1/2$  statistics is the clock, (ii)  $\alpha = 1/2$  limiting process is circular  $\beta$ -ensemble, (iii)  $0 < \alpha < 1/2$  the statistics is Poisson. Analogues of Killip-Stoiciu [45] was done by Kotani-Nakano [41] for the one-dimensional Schrödinger operator with decaying potentials in the continuum model and obtained the same statistics for  $\alpha > 1/2$  and  $\alpha = 1/2$ . Krichevski-Valkó-Virág [43] studied the one-dimensional discrete Schrödinger operator with the random potential decaying like  $n^{-1/2}$  and obtained the Sine $\beta$ -process.

The thesis is divided into six chapters. In the first chapter we present the basic probability theory, Point processes. In the second chapters we give the mathematical formulation of Anderson Model on  $\mathbb{Z}^d$  and its spectral theory. The third chapter deals with some mathematical tools required to identify the spectrum of an operator and the Aizenman-Molchanov method to show localization for large disorder. In the fourth chapter we show that the local spectral statistics of Anderson Model (with decaying random potentials) inside the a.c spectrum (extended states) is the same as that of the free part. In the fifth chapter we consider the model with i.i.d random potential whose single site distribution

is  $\alpha$ -Hölder ( $0 < \alpha \leq 1$ ) continuous and show that in the localized regime the eigenvalue statistics is Poisson. In the last chapter we estimate the integrated density of states of the Anderson model with decaying random potentials. We also show that the average spacing of eigenvalues of the finite dimensional approximation are different for the continuous and the pure point parts of the spectrum. The last three chapters contain the original work of the thesis. A brief description of the thesis, chapterwise is given below.

**(1) Preliminaries:**

In this chapter we present basic probability theory needed to prove the results in the last three chapters, including Borel-Cantelli lemma, Martingale convergence theorem and a basic theory of Poisson point processes. The materials here comes from [60], [6], [33] and [20].

**(2) Random Operators:**

In this chapter we study the different forms of spectral theorem for a self-adjoint operator on a separable Hilbert space and discuss the proof of Weyl's Criterion to show existence of essential spectrum. We then discuss the notion of measurable family of random operators ( $\omega \rightarrow H^\omega$ ) from a measurable space  $(\Omega, \mathcal{B}_\Omega)$  to  $\mathcal{B}(\mathcal{H})$  [10]. We give the definition of ergodic family  $\{H^\omega\}_{\omega \in \Omega}$  of operators affiliated to dynamical system  $(\Omega, \mathcal{B}_\Omega, \mathbb{P}, G)$  [16]. We show that the spectrum of ergodic family of operators is almost surely a constant set, the proof is recalled from [51]. Finally define the Anderson Model on lattice as:

$$H^\omega = \Delta + V^\omega, \quad \omega \in \mathbb{R}^{\mathbb{Z}^d} \quad \text{on} \quad \ell^2(\mathbb{Z}^d),$$

where  $\Delta$  is the discrete Laplacian on  $\ell^2(\mathbb{Z}^d)$  and  $V^\omega(n) = \omega_n$   $n \in \mathbb{Z}^d$  are i.i.d random variables with distribution  $\mu$  on  $\mathbb{R}$ . The random operators  $H^\omega$  as in above is ergodic family of random operators affiliated to the dynamical system  $(\mathbb{R}^{\mathbb{Z}^d}, \mathcal{B}_{\mathbb{R}^{\mathbb{Z}^d}}, \otimes \mu, \mathbb{Z}^d)$ . We discuss the existence of Integrated density of states (IDS) for the above model.

### (3) Criterion for Identifying the Spectrum:

This chapter is devoted to the study of different methods to identify the spectrum of a self-adjoint operator of our interest. All most all these methods use the properties of a measure and it's transformation via the spectral theorem. Let  $\nu$  be a positive measure on  $\mathbb{R}$  which integrates the function  $\frac{1}{1+x^2}$  then the Borel transformation  $F_\nu$  of  $\nu$  is given by

$$F_\nu(z) = \int_{\mathbb{R}} \left( \frac{1}{x-z} - \frac{x}{1+x^2} \right) d\nu(x), \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

We study the method how we can recover the different Lebesgue components of  $\nu$  namely the atomic part  $\nu_{pp}$ , the singular continuous  $\nu_{sc}$  and the absolutely continuous part  $\nu_{ac}$  from  $F_\nu$ . Then using the Borel transformation method we discuss the Aronszajn-Donoghue ([3] and [15]) theory of rank one perturbations of self-adjoint operators. This theory and its improvements were used to solve several spectral problems.

### (4) Level Repulsion for a class of Decaying Random Potentials:

In this chapter we study the spectral statistics of Anderson Model with decaying random potentials inside the absolutely continuous spectrum. Here we take the Model as follows

$$H^\omega = \Delta + V^\omega \text{ on } \ell^2(\mathbb{Z}^d),$$

where  $V^\omega(n) = a_n q_n(\omega)$ ,  $\{q_n\}_{n \in \mathbb{Z}^d}$  are i.i.d random variables with an absolutely continuous distribution  $\mu$  and  $\{a_n\}_{n \in \mathbb{Z}^d}$  is a sequence of positive real numbers such that  $a_n \rightarrow 0$  as  $|n| \rightarrow \infty$ . We show that if  $a_n = O(|n|^{-\alpha})$ ,  $\alpha > 2$  then for certain class of  $\mu$  the local spectral statistics inside the a.c spectrum  $([-2d, 2d])$  is the same as that of the free part  $\Delta$ . This is one of the new results.

### (5) Poisson Statistics for Anderson Model with Singular Randomness:

Here we consider the above model with  $a_n = 1, \forall n$  and  $\mu$  to be  $\alpha$ -Hölder ( $0 < \alpha \leq 1$ )

continuous. We study the point process given by

$$\xi_L^\omega(\cdot) = \sum_{\lambda \in \sigma(H_L^\omega)} \delta_{\beta_L(\lambda - E)}(\cdot), \quad E \in \mathbb{R}, \quad \beta_L = (2L + 1)^{d/\alpha},$$

where  $H_L^\omega$  is the restriction of  $H^\omega$  to  $\Lambda_L \subset \mathbb{Z}^d$ , the cube with side length  $2L + 1$  center at 0. We show that if  $E$  is in localized regime then for bounded symmetric interval  $I$  the limit points, convergence being in distribution, of the sequence of random variables  $\{\xi_L^\omega(I)\}_L$  are Poisson random variables. This result is new for the cases  $0 < \alpha < 1$ .

### (6) Integrated Density of States for Decaying Random Potentials:

We consider  $H^\omega$  with decaying random potentials as described above in Chapter 4. We show that if  $a_n = O(|n|^{-\alpha}), \alpha > 0$  and  $\frac{d\mu}{dx}(x) = O(|x|^{-\delta}), \delta > 1$  as  $|x| \rightarrow \infty$  then the average spacing of eigenvalues of  $H_{\Lambda_L}^\omega$ , the finite dimensional approximation of  $H^\omega$  are different for the continuous and the pure point parts of the spectrum. In pure point regime (i.e outside  $[-2d, 2d]$ ) it is of order  $(2L + 1)^{-(d - \alpha(\delta - 1))}$  and inside continuous spectrum (i.e inside  $[-2d, 2d]$ ) it is of order  $(2L + 1)^{-d}$ . We also provide some bounds for the Integrated density of states of this model. These bounds are new.

# Chapter 1

## Preliminaries

## 1.1 Basics of Probability Theory

In this section we enumerate basic facts about random variables, their independence and distribution. Let  $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$  be a **probability space**, where  $\mathcal{B}_\Omega$  is a  $\sigma$ -algebra of subsets of  $\Omega$  and  $\mathbb{P}$  is a measure on  $\mathcal{B}_\Omega$  with  $\mathbb{P}(\Omega) = 1$ .  $\Omega$  is known as **sample space** and elements of  $\mathcal{B}_\Omega$  are called **events**. First, we formally define a random variable.

**Definition 1.1.1.** *A random variable  $X$  on a probability space  $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$  is a complex valued  $\mathcal{B}_\Omega$ -measurable function on  $\Omega$ , i.e.,  $X^{-1}(A) \in \mathcal{B}_\Omega$ , for every  $A \in \mathcal{B}_\mathbb{C}$ , where  $\mathcal{B}_\mathbb{C}$  is the collection of Borel sets of  $\mathbb{C}$ .*

After we have defined random variables, we define **distributions** of random variables.

**Definition 1.1.2.** *Let  $X$  be a real valued random variable on a probability space  $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$ . Then, the distribution of  $X$  is a measure  $P_X$  on  $(\mathbb{R}, \mathcal{B}_\mathbb{R})$  defined as follows:*

$$P_X(A) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \in A\}) = \mathbb{P}(X^{-1}(A)), \quad \forall A \in \mathcal{B}_\mathbb{R}.$$

Let  $X$  and  $Y$  be two random variables on  $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$ . We say  $X$  and  $Y$  are identically distributed if their distributions agree. Now, we define **distribution functions** of random variables.

**Definition 1.1.3.** *Let  $X$  be a real random variable on a probability space  $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$ . Let  $F_X$  be a function over  $\mathbb{R}$  defined as follows:*

$$F_X(x) = P_X(-\infty, x) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \leq x\}).$$

*Then,  $F_X(\cdot)$  is called the distribution function of  $X$ .*

Let  $X$  be a real random variable on  $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$ . *Expectation* of  $X$  is denoted by  $\mathbb{E}^\omega(X)$  and defined by

$$\mathbb{E}^\omega(X) = \int_{\Omega} X d\mathbb{P}.$$

Now, we define the notion of **independence** of events.



**Definition 1.1.4.** Let  $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$  be a probability space. Let  $B = \{B_\alpha : \alpha \in I\} \subset \mathcal{B}_\Omega$  be a collection of events over an index set  $I$ .  $B$  is called independent if for every finite sub-collection  $\{\alpha_1, \alpha_2, \dots, \alpha_k\} \subset I$ , where  $1 \leq k < \infty$ , the following holds:

$$\mathbb{P}\left(\bigcap_{i=1}^k B_{\alpha_i}\right) = \prod_{i=1}^k \mathbb{P}(B_{\alpha_i}).$$

The notion of independence can be extended to random variables as follows.

**Definition 1.1.5.** A collection of real random variables  $\{X_\alpha : \alpha \in I\}$  (over an index set  $I$ ) on a probability space  $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$  is called independent if for any  $1 \leq k < \infty$ , for any  $\{\alpha_1, \alpha_2, \dots, \alpha_k\} \subset I$  and  $\{A_i \in \mathcal{B}_\mathbb{R} : 1 \leq i \leq k\}$ , the following holds:

$$\mathbb{P}\left(\{\omega : X_{\alpha_1}(\omega) \in A_1, X_{\alpha_2}(\omega) \in A_2, \dots, X_{\alpha_k}(\omega) \in A_k\}\right) = \prod_{i=1}^k \mathbb{P}\left(\{\omega : X_{\alpha_i}(\omega) \in A_i\}\right)$$

If the collection of random variables  $\{X_\alpha : \alpha \in I\}$  is independent and identically distributed (i.i.d.) with common distribution  $P_0$  then,

$$\mathbb{P}\left(\{\omega : X_{\alpha_1}(\omega) \in A_1, X_{\alpha_2}(\omega) \in A_2, \dots, X_{\alpha_k}(\omega) \in A_k\}\right) = \prod_{i=1}^k P_0(A_i).$$

**Definition 1.1.6.** Let  $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$  be a probability space and  $\{A_n\}_{n \geq 1}$  be sequences of events in  $\mathcal{B}_\Omega$ . Then we refer to the event

$$A_n \text{ occurs infinitely often} = \overline{\lim} A_n := \bigcap_{k=1}^{\infty} \left( \bigcup_{n \geq k} A_n \right)$$

as “ $A_n$  occurs infinitely often”.

Before we move to the next section we present the statement of Borel-Cantelli lemma.

The interested read may refer to [4, Theorem 2.2.2] for the proof.

**Lemma 1.1.7 (Borel-Cantelli lemma).** Let  $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$  be a probability space and  $\{A_n\}_{n \geq 1}$  be a sequence of events in  $\mathcal{B}_\Omega$ . Then

1. If  $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$ , then  $\mathbb{P}(\overline{\lim} A_n) = 0$ .
2. If  $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$  and  $\{A_n\}_{n \geq 1}$  are independent, then  $\mathbb{P}(\overline{\lim} A_n) = 1$ .

The above lemma is very useful in showing the existence of spectrum for Anderson Model with decaying potentials, considered in the fourth and sixth of the thesis.

### 1.1.1 Martingale Convergence Theorems

In this section we define Conditional Expectation and Martingales and present theorems related to convergence of Martingales. In order to define conditional expectation, we need the following theorem:

**Theorem 1.1.8.** *Let  $X$  be a random variable on a probability space  $(\Omega, \mathcal{B}, \mathbb{P})$ , with  $\mathbb{E}(|X|) < \infty$  and let  $\mathcal{G}$  be a sub  $\sigma$ -algebra of  $\mathcal{B}$ . Then, there exists an essentially unique  $\mathcal{G}$  measurable random variable  $Z$  such that*

$$\int_A X(\omega) d\mathbb{P}(\omega) = \int_A Z(\omega) d\mathbb{P}(\omega) \quad \forall A \in \mathcal{G};$$

*i.e.,*  $\mathbb{E}(\chi_A X) = \mathbb{E}(\chi_A Z) \quad \forall A \in \mathcal{G}.$

This random variable  $Z$  is called the conditional expectation of  $X$  given  $\mathcal{G}$  and is denoted by  $E(X|\mathcal{G})$ . When  $\mathcal{G} = \langle Y \rangle$ , the  $\sigma$ -algebra generated by a random variable  $Y$ ,  $E(X|\mathcal{G})$  reduces to  $E(X|Y)$ , the conditional expectation of  $X$  given  $Y$ . The proof of Theorem 1.1.8 can be found in [6, Theorem 3.4.1]. Now, we are ready to define Martingales.

**Definition 1.1.9. (Martingales)** *Let  $(\Omega, \mathcal{B}, \mathbb{P})$  be a probability space. A martingale sequence of length  $n$  is a chain  $X_1, X_2, \dots, X_n$  of random variables and corresponding sub  $\sigma$ -algebras  $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n$  that satisfy the following relations:*

1. Each  $X_i$  is measurable with respect to the corresponding  $\mathcal{G}_i$ , and  $\mathbb{E}(|X_i|) < \infty$ , for every  $i = 1, 2, \dots, n$ .

2. The  $\sigma$ -algebras are increasing, i.e  $\mathcal{G}_i \subset \mathcal{G}_{i+1}$  for every  $i = 1, 2, \dots, n - 1$ .

3.  $E(X_{i+1}|\mathcal{G}_i) = X_i$  for  $i = 1, 2, \dots, n - 1$ .

We can have an infinite martingale sequence  $\{(X_i, \mathcal{G}_i) : i \geq 1\}$ , if for every  $n \geq 1$ ,  $\{(X_i, \mathcal{G}_i) : 1 \leq i \leq n\}$  is a martingale sequence of length  $n$ .

Let  $\{(X_n, \mathcal{G}_n)\}$  be a sequence of martingales. Without loss of generality, we can assume the full  $\sigma$ -algebra  $\mathcal{B}$  to be the smallest  $\sigma$ -algebra generated by  $\bigcup_n \mathcal{G}_n$ . The following theorem talks about convergence of Martingales.

Before we state the theorem we define  $L^p(\Omega, \mathcal{B}, \mathbb{P})$  to be the following set,

$$L^p(\Omega, \mathcal{B}, \mathbb{P}) = \{[f] : \int_{\Omega} |f|^p d\mathbb{P} < \infty\}, \quad 1 \leq p < \infty,$$

$$[f] = \{g : \Omega \rightarrow \mathbb{C}, g \text{ is } \mathcal{B} \text{ measurable s.t } g(\omega) = f(\omega) \text{ a.e } \omega (\mathbb{P})\}.$$

For each  $[f] \in L^p(\Omega, \mathcal{B}, \mathbb{P})$  we define *norm* to be

$$\|[f]\| = \left( \int_{\Omega} |f|^p d\mathbb{P} \right)^{1/p},$$

or simply  $\|f\|_p = \|[f]\|$  for any representative  $f \in [f]$ .

**Theorem 1.1.10.** *If for some  $p \geq 1$ ,  $X \in L^p(\Omega, \mathcal{B}, \mathbb{P})$  and  $X_n = E(X|\mathcal{G}_n)$  then  $X_n$  is a martingale and  $\lim_{n \rightarrow \infty} \|X_n - X\|_p = 0$ .*

Suppose  $\{(X_n, \mathcal{G}_n) : n \geq 1\}$  is a martingale on a probability space  $(\Omega, \mathcal{B}, \mathbb{P})$ . The following is an interesting question to ask: Can we find a random variable  $X \in L^p(\Omega, \mathcal{B}, \mathbb{P})$  such that  $X_n = E(X|\mathcal{G}_n)$ ? The answer is provided by the following theorem.

**Theorem 1.1.11.** *If for some  $p > 1$ ,  $\{(X_n, \mathcal{G}_n) : n \geq 1\}$  a martingale is bounded in  $L^p(\Omega, \mathcal{B}, \mathbb{P})$  in the sense that  $\sup_n \|X_n\|_p < \infty$  then, there exist a random variable  $X$  in  $L^p(\Omega, \mathcal{B}, \mathbb{P})$  such that  $X_n = E(X|\mathcal{G}_n)$  and  $\lim_{n \rightarrow \infty} \|X_n - X\|_p = 0$ .*

The convergence in the above theorem is valid almost everywhere.

**Theorem 1.1.12.** *Let  $X \in L^p(\Omega, \mathcal{B}, \mathbb{P})$  for some  $p \geq 1$ . Then the martingale  $X_n = E(X|\mathcal{G}_n)$  converges to  $X$  for almost all  $\omega$  w.r.t  $\mathbb{P}$ .*

The proofs of the above theorems can be found in [60, 5.2 Martingale Convergence Theorems].

## 1.2 Point Process

A measure  $\mu$  on  $\mathbb{R}$  is called *boundedly-finite* if  $\mu(A) < \infty$ , for every bounded set  $A \in \mathcal{B}_{\mathbb{R}}$  and is called *integer-valued* if  $\mu(A) \in \mathbb{N} \cup \{\infty\}$  for any  $A \in \mathcal{B}_{\mathbb{R}}$ . Let  $\mathcal{N}_{\mathbb{N}}(\mathbb{R})$  denote the set of all integer-valued boundedly-finite Radon measures on  $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ . We proceed to define a **point process**.

**Definition 1.2.1.** A point process  $\xi^\omega$  with a state space  $\mathbb{R}$  is a measurable mapping from a probability space  $(\Omega, \mathcal{B}, \mathbb{P})$  into  $\mathcal{N}_{\mathbb{N}}(\mathbb{R})$ , i.e., for each  $A \in \mathcal{B}_{\mathbb{R}}$ ,  $\xi^\omega(A)$ ,  $(\omega \rightarrow \xi^\omega(A))$  is a random variable on  $(\Omega, \mathcal{B}, \mathbb{P})$ .

Given a topological space  $\mathcal{X}$  (*state space*) which is locally compact, second countable and Hausdorff, let  $\mathcal{M}(\mathcal{X})$  be the set of all boundedly-finite Borel measures on  $\mathcal{X}$  and let  $\mathcal{B}_{\mathcal{X}}$  be the Borel algebra of  $\mathcal{X}$ . In general, we can define random measures on  $\mathcal{X}$ .

**Definition 1.2.2.** A random measure  $\xi^\omega$  with a state space  $\mathcal{X}$  is a measurable mapping from a probability space  $(\Omega, \mathcal{B}, \mathbb{P})$  into  $\mathcal{M}(\mathcal{X})$ , i.e., for each  $A \in \mathcal{B}_{\mathcal{X}}$ ,  $\xi^\omega(A)$ ,  $(\omega \rightarrow \xi^\omega(A))$  is a random variable on  $(\Omega, \mathcal{B}, \mathbb{P})$ .

$$\begin{array}{ccc}
 \Omega & \xrightarrow{\omega \rightarrow \xi^\omega} & \mathcal{M}(\mathcal{X}) \\
 \searrow & & \swarrow \\
 \omega \rightarrow \xi^\omega(A) & & \xi^\omega \rightarrow \xi^\omega(A) \\
 & \searrow & \swarrow \\
 & \mathbb{R}^+ & 
 \end{array} \tag{1.2.1}$$

One of the characteristics of a random measure is its **intensity**, which is defined as follows.

**Definition 1.2.3.** The intensity of the random measure  $\xi^\omega$  is given by,

$$\nu(A) = \mathbb{E}^\omega(\xi^\omega(A)), \quad \forall A \in \mathcal{B}_{\mathcal{X}}.$$

Note that, a point process is a special case of a random measure ( $\mathcal{X} = \mathbb{R}$ , integer-valued). A collection of random measures  $\{\xi_\alpha^\omega : \alpha \in I\}$  on same probability space  $(\Omega, \mathcal{B}, \mathbb{P})$  is said to be independent and identically distributed if for any  $A \in \mathcal{B}_{\mathbb{R}}$  the collection of random

variables  $\{\xi_\alpha^\omega(A) : \alpha \in I\}$  is i.i.d.

The following definition gives the notion of infinite divisibility of a random measure.

**Definition 1.2.4.** *A point process (or random measure) is infinitely divisible if, for every  $k$ , it can be represented as the superposition of  $k$  independent, identically distributed, point process (or random measure) components.*

In symbols, a point process  $\xi^\omega$  is infinitely divisible if, for every  $k$ , we can write

$$\xi^\omega = \xi_{1,k}^\omega + \xi_{2,k}^\omega + \cdots + \xi_{k,k}^\omega,$$

where  $\xi_{i,k}^\omega$  ( $i = 1, \dots, k$ ) are i.i.d. components. Now, we define probability distributions associated with random measures.

**Definition 1.2.5.** *The finite-dimensional distributions (fidi distributions, for short) of a random measure  $\xi^\omega$  are the joint distributions, for all finite families of bounded Borel sets  $A_1, A_2, \dots, A_k$  of the random variables  $\xi^\omega(A_1), \xi^\omega(A_2), \dots, \xi^\omega(A_k)$ . The family of proper distribution functions are given by*

$$F_k(A_1, A_2, \dots, A_k; x_1, x_2, \dots, x_k) = \mathbb{P}(\{\omega : \xi^\omega(A_i) \leq x_i, i = 1, 2, \dots, k\}). \quad (1.2.2)$$

A random measure is characterized completely by its associated fidi distribution.

**Proposition 1.2.6.** *The distribution of a random measure is completely determined by the fidi distributions (1.2.2) for all finite families  $(A_1, A_2, \dots, A_k)$  of disjoint sets from a semiring  $\mathcal{A}$  of bounded sets generating  $\mathcal{B}_X$ .*

For the proof of above Proposition we refer to [20, Proposition 9.2.III]. Now, we formally define the convergence of random measures in the sense of their fidi distributions.

**Definition 1.2.7.** *The sequence  $\{\xi_n^\omega\}$  of random measures converges in the sense of convergence of fidi distributions to a random measure  $\xi$ , if for every finite family  $\{A_1, A_2, \dots, A_k\}$*

of bounded continuity sets  $A_i \in \mathcal{B}_{\mathcal{X}}$ , the joint distributions of  $\{\xi_n^\omega(A_1), \xi_n^\omega(A_2), \dots, \xi_n^\omega(A_k)\}$  converge weakly in  $\mathcal{B}(\mathbb{R}^k)$  to the joint distribution of  $\{\xi(A_1), \xi(A_2), \dots, \xi(A_k)\}$  i.e.,

$$\lim_{n \rightarrow \infty} F_{k,n}(A_1, A_2, \dots, A_k; x_1, x_2, \dots, x_k) = F_k(A_1, A_2, \dots, A_k; x_1, x_2, \dots, x_k),$$

for all  $(x_1, x_2, \dots, x_k) \in \mathbb{R}^k$  at which  $F_k$  is continuous.

$A \in \mathbb{B}_{\mathcal{X}}$  is a bounded continuity set if  $E^\omega(\xi^\omega(\partial A)) = 0$ , where  $\partial A$  denotes the boundary of the set  $A$ .  $F_{k,n}, F_k$  be the fidi distribution functions of  $\xi_n^\omega, \xi^\omega$  respectively as in (1.2.2). Now we define the Laplace Transformation of a random measure.

**Definition 1.2.8. (Laplace Transformation)** *The Laplace Transformation of a random measure  $\xi^\omega$  is defined by*

$$L_\xi(f) = \mathbb{E}^\omega[\exp(-\xi^\omega(f))], \quad f \in BM_+(\mathcal{X}), \quad (1.2.3)$$

where  $BM_+(\mathcal{X})$  denotes the set of all bounded measurable positive function on  $\mathcal{X}$  and  $\xi^\omega(f)$  is given by

$$\xi^\omega(f) = \int_{\mathcal{X}} f(x) d\xi^\omega(x). \quad (1.2.4)$$

It turns out that the weak convergence of random measures is equivalent to convergence of their respective Laplace transformations.

**Definition 1.2.9.** *A sequence of point processes  $\{\xi_n^\omega\}$  is said to converge weakly to a point process  $\xi^\omega$  if*

$$\lim_{n \rightarrow \infty} \mathbb{E}^\omega[\exp(-\xi_n^\omega(f))] = \mathbb{E}^\omega[\exp(-\xi^\omega(f))], \quad \forall f \in C_c^+(\mathbb{R}),$$

where  $C_c^+(\mathbb{R})$  is the set of all non-negative continuous functions on  $\mathbb{R}$  with compact support.

We end this section with the definition of Poisson point process.

**Definition 1.2.10. (Poisson Point Process)** A point process  $\xi^\omega$  is said to be a Poisson point process with intensity measure  $\nu$  if the following two conditions are satisfied:

(i) If  $A$  is a bounded Borel set of  $\mathbb{R}$  then,  $\xi^\omega(A)$  should follow Poisson distribution with parameter  $\nu(A)$ . Thus,

$$\mathbb{P}(\omega : \xi^\omega(A) = k) = e^{-\nu(A)} \frac{\nu(A)^k}{k!}, \quad k = 0, 1, 2, \dots .$$

(ii) If  $A_1, A_2, \dots, A_n$  are disjoint Borel sets of  $\mathbb{R}$ , then  $\xi^\omega(A_1), \xi^\omega(A_2), \dots, \xi^\omega(A_n)$  are independent random variables.

### 1.2.1 Limit Theorems for Superpositions

The formal setting for studying the sum or superposition of a large number of point processes or random measures is a triangular array  $\{\xi_{ni}^\omega : i = 1, 2, \dots, m_n; n = 1, 2, \dots\}$  and its associated row sums

$$\xi_n^\omega = \sum_{i=1}^{m_n} \xi_{ni}^\omega, \quad n = 1, 2, \dots .$$

If for each  $n$ , the processes  $\{\xi_{ni}^\omega : i = 1, 2, \dots, m_n\}$  are mutually independent then, it is said to be an independent triangular array.

**Definition 1.2.11.** An array  $\{\xi_{ni}^\omega : i = 1, 2, \dots, m_n; n = 1, 2, \dots\}$  is said to be uniformly asymptotically negligible (u.a.n) if for all  $\epsilon > 0$  and all bounded  $A \in \mathcal{B}_X$

$$\limsup_{n \rightarrow \infty} \mathbb{P}(\omega : \xi_{ni}^\omega(A) > \epsilon) = 0. \quad (1.2.5)$$

The following proposition gives the criterion for the infinite divisibility of a point process.

**Proposition 1.2.12.** A point process is infinitely divisible if and only if it can be represented as the limit of the row sums of a u.a.n. array.

The proof is given in [20, Proposition 11.2.IV]. The following theorem enumerates the

conditions for a row-sum of a u.a.n. array to converge weakly to a Poisson point process.

**Theorem 1.2.13.** *The triangular u.a.n. array  $\{\xi_{ni}^\omega : i = 1, 2, \dots, m_n; n = 1, 2, \dots\}$  converges weakly to a Poisson process with intensity measure  $\mu$  if and only if for all bounded Borel sets  $A$  with  $\mu(\partial A) = 0$ ,*

$$\sum_{i=1}^{m_n} \mathbb{P}(\{\omega : \xi_{ni}^\omega(A) \geq 2\}) \longrightarrow 0 \quad (n \rightarrow \infty) \quad (1.2.6)$$

and

$$\sum_{i=1}^{m_n} \mathbb{P}(\{\omega : \xi_{ni}^\omega(A) \geq 1\}) \longrightarrow \mu(A) \quad (n \rightarrow \infty). \quad (1.2.7)$$

We refer to [20, Theorem 11.2.V] for the proof of above theorem.



## Chapter 2

# Random Operators

In this chapter, we discuss the properties of a self-adjoint operator on Hilbert space. We present the spectral theorem of a self-adjoint operator. The spectral theorem is very important in the theory of operators on a Hilbert space. It provides a complete structure of self-adjoint operator. There are many equivalent forms of the spectral theorem. We give some criteria to identify the different components of the spectrum of self-adjoint operator. In the last two sections we give the mathematical formulation of Anderson model on  $\ell^2(\mathbb{Z}^d)$  and describe its spectrum. Here we consider the model as a family of operator of the form  $H = \Delta + V$ , where  $\Delta$  is the kinetic energy (deterministic) and  $V$  is the potentials given by a multiplication operator. Here we assume the potentials is random. We end this Chapter with the definition of Density of states. The density of states count the number of eigenvalue of  $H$  per unit volume in some sense.

## 2.1 Self-Adjoint Operator and Spectral Theorem

In this section we provide some basic properties of self-adjoint operator on Hilbert space and their spectral theorem. Let  $\mathcal{H}$  be a complex separable Hilbert space with inner product denoted by  $\langle \cdot, \cdot \rangle$  which is linear w.r.t the second variable. A linear operator on  $\mathcal{H}$  is a pair  $(\text{dom}(T), T)$ , where  $\text{dom}(T)$  is a vector subspace of  $\mathcal{H}$  and  $T : \text{dom}(T) \rightarrow \mathcal{H}$  is a linear transformation. The operator  $T$  is called *bounded* if  $\text{dom}(T) = \mathcal{H}$  and

$$\|T\| := \sup_{\|\psi\|=1} \|T\psi\| < \infty.$$

The vector space of all bounded operators on  $\mathcal{H}$  is denoted by  $\mathcal{B}(\mathcal{H})$  and it is a Banach space under the operator norm defined above.

An operator  $P \in \mathcal{B}(\mathcal{H})$  is called an *orthogonal projection* if

$$P^2 = P \text{ and } \langle P\varphi, \psi \rangle = \langle \varphi, P\psi \rangle, \forall \varphi, \psi \in \mathcal{H}.$$

An operator  $U \in \mathcal{B}(\mathcal{H})$  is called *unitary* if it is onto and satisfies the following condition:

$$\langle U\varphi, U\psi \rangle = \langle \varphi, \psi \rangle \quad \forall \varphi, \psi \in \mathcal{H}.$$

In order to define notion of closability of an operator, we need the following definition.

**Definition 2.1.1.** *The graph of a linear operator  $(\text{dom}(T), T)$  is the set of all pairs given by*

$$\Gamma(T) = \{(\varphi, T\varphi) : \varphi \in \text{dom}(T)\}.$$

Thus,  $\Gamma(T)$  is subset of  $\mathcal{H} \times \mathcal{H}$ , which is a Hilbert space with inner product given by

$$\langle (\varphi_1, \psi_1), (\varphi_2, \psi_2) \rangle = \langle \varphi_1, \varphi_2 \rangle + \langle \psi_1, \psi_2 \rangle.$$

Let  $T_1$  and  $T$  be two operators on  $\mathcal{H}$ .  $T_1$  is said to be an extension of  $T$  iff  $\text{dom}(T) \subset \text{dom}(T_1)$  and  $T_1\varphi = T\varphi$ ,  $\forall \varphi \in \text{dom}(T)$ . We write  $T \subset T_1$  to denote  $T_1$  is an extension of  $T$ .

An operator  $T$  is said to be *densely defined* if  $\text{dom}(T)$  is dense in  $\mathcal{H}$ . If  $T$  is densely defined and there is a constant  $C > 0$ , such that  $\|T\psi\| \leq C\|\psi\|$ ,  $\forall \psi \in \text{dom}(T)$ , then,  $T$  has a unique extension to a bounded operator on  $\mathcal{H}$ . Now, we can formally define a closed operator.

**Definition 2.1.2.** *An operator  $T$  is called closed if it's graph  $\Gamma(T)$  is closed in  $\mathcal{H} \times \mathcal{H}$ . An operator is closable if it has a closed extension.*

In other words,  $T$  is closed iff for any sequence  $\{\varphi_n\}_n \in \text{dom}(T)$  such that  $\varphi_n \rightarrow \varphi$  and  $T\varphi_n \rightarrow \psi$ . Here, we have  $\varphi \in \text{dom}(T)$  and  $T\varphi = \psi$ . If  $T$  is closable then, it's smallest closed extension  $\bar{T}$  is an extension of  $T$  s.t  $\bar{T} \subset T_1$  for any closed extension  $T_1$  of  $T$ . Then  $\bar{T}$  is called the closure of  $T$ . The following proposition says that the closure of  $\Gamma(T)$  must be a graph of some operator if  $T$  is closable.

**Proposition 2.1.3.** *If  $T$  is closable, then  $\Gamma(\bar{T}) = \overline{\Gamma(T)}$ .*

For the proof we refer to [53, Proposition, page 250].

In the following, we define symmetric and self-adjoint operators on a Hilbert space.

**Definition 2.1.4.** *Let  $T$  be a densely defined operator on a Hilbert space  $\mathcal{H}$ . Then  $T$  is called symmetric if  $\langle T\psi, \varphi \rangle = \langle \psi, T\varphi \rangle$ ,  $\forall \varphi, \psi \in \text{dom}(T)$ .*

**Definition 2.1.5.** *Let  $T$  be a densely defined operator on a Hilbert space  $\mathcal{H}$ . Then its adjoint operator  $(\text{dom}(T^*), T^*)$  is defined as follows:*

$$\text{dom}(T^*) = \{\varphi \in \mathcal{H} : |\langle T\psi, \varphi \rangle| \leq C_\varphi \|\psi\|, C_\varphi > 0 \forall \psi \in \text{dom}(T)\}.$$

*If  $\varphi \in \text{dom}(T^*)$  then by Riesz lemma [53, Theorem II.4] there exist a unique  $\eta \in \mathcal{H}$  such that  $\langle T\psi, \varphi \rangle = \langle \psi, \eta \rangle$ . Now define  $T^*\varphi = \eta$ .*

The above two definitions imply that if  $T$  is symmetric then  $T \subset T^*$ .

**Definition 2.1.6.** *A symmetric operator  $T$  is said to be self-adjoint if  $T = T^*$ , i.e  $\text{dom}(T) = \text{dom}(T^*)$  and  $\langle T\psi, \varphi \rangle = \langle \psi, T^*\varphi \rangle$ ,  $\forall \varphi, \psi \in \text{dom}(T) = \text{dom}(T^*)$ .*

The following theorem enumerates the properties of  $T^*$ , when  $T$  is a closable operator. For the proof we refer to [53, Theorem VIII.1].

**Theorem 2.1.7.** *Let  $T$  be a densely defined operator on a Hilbert space  $\mathcal{H}$ . Then*

1.  $T^*$  is closed.
2.  $T$  is closable iff  $\text{dom}(T^*)$  is dense, in which case  $\overline{T} = (T^*)^*$ .
3. If  $T$  is closable, then  $\overline{T}^* = T^*$ .

The following theorem states the criteria for a symmetric operator to be self-adjoint operator. For the proof of the theorem we refer to [53, Theorem VIII.3].

**Theorem 2.1.8.** *Let  $T$  be a symmetric operator on  $\mathcal{H}$ . Then the following are equivalent:*

1.  $T$  is self-adjoint.

2.  $T$  is closed,  $\text{Ker}(T^* - z) = \{0\}$  and  $\text{Ker}(T^* - \bar{z}) = \{0\}$  with  $\text{Im}z \neq 0$ .

3.  $\text{Ran}(T - z) = \mathcal{H}$  and  $\text{Ran}(T - \bar{z}) = \mathcal{H}$ .

Now, we define **spectrum** of an operator.

**Definition 2.1.9.** Let  $T$  be a linear operator on  $\mathcal{H}$ , then  $\rho(T)$ , the resolvent set of  $T$  defined by

$$\rho(T) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is boundedly invertible}\}.$$

The spectrum of  $T$  is the set  $\sigma(T) = \mathbb{C} \setminus \rho(T)$ .

The following proposition describes the properties of spectrum of a self-adjoint operator.

We refer to [53, Theorem VI.8] for the proof of the proposition.

**Proposition 2.1.10.** Let  $T$  be a self-adjoint operator then

1.  $\sigma(T) \subseteq \mathbb{R}$  and if  $z \in \mathbb{C} \setminus \mathbb{R}$  then  $\|(T - z)^{-1}\| \leq |\text{Im}z|^{-1}$ .

2. If  $\lambda_1, \lambda_2$  are two distinct eigenvalues of  $T$  and  $\varphi_1, \varphi_2$  are the corresponding eigenfunctions then  $\langle \varphi_1, \varphi_2 \rangle = 0$ .

Now, we define spectral measure.

**Definition 2.1.11.** If  $\Omega$  is a set,  $\mathcal{B}_\Omega$  is a  $\sigma$ -algebra of subsets of  $\Omega$ , and  $\mathcal{H}$  is a Hilbert space, a spectral measure for  $(\Omega, \mathcal{B}_\Omega, \mathcal{H})$  is a function  $E : \mathcal{B}_\Omega \rightarrow \mathcal{B}(\mathcal{H})$  such that

1. For each  $\Delta$  in  $\mathcal{B}_\Omega$ ,  $E(\Delta)$  is an orthogonal projection.

2.  $E(\emptyset) = 0$  and  $E(\Omega) = 1$ .

3.  $E(\Delta_1 \cap \Delta_2) = E(\Delta_1)E(\Delta_2)$  for  $\Delta_1$  and  $\Delta_2$  in  $\mathcal{B}_\Omega$  and

4. If  $\{\Delta_n\}_{n=1}^\infty$  are pairwise disjoint sets from  $\mathcal{B}_\Omega$  then,

$$E\left(\bigcup_{n=1}^{\infty} \Delta_n\right)\varphi = \lim_{N \rightarrow \infty} \sum_{n=1}^N E(\Delta_n)\varphi, \quad \forall \varphi \in \mathcal{H}.$$

For  $\varphi \in \mathcal{H}$ ,  $\mu_\varphi(\cdot) = \langle \varphi, E(\cdot)\varphi \rangle$  is a countably additive positive measure on  $\Omega$ . For  $\varphi, \psi \in \mathcal{H}$ ,  $\mu_{\varphi, \psi}(\cdot) = \langle \varphi, E(\cdot)\psi \rangle$  is a complex measure on  $\Omega$  with total variation bounded by  $\|\varphi\|\|\psi\|$ . Now, suppose  $g$  is a complex-valued measurable function on  $\Omega$  and  $E$  is a spectral measure on  $\mathcal{B}_\Omega$ . Also, let

$$D_g = \left\{ \varphi \in \mathcal{H} : \int_{\mathbb{R}} |g(\lambda)|^2 d\mu_\varphi(\lambda) < \infty \right\}.$$

Then,  $\text{dom}(g(T)) = D_g$  is dense in  $\mathcal{H}$  and the operator  $(D_g, g(T))$  is defined as follows:

$$\langle \varphi, g(T)\varphi \rangle = \int_{\mathbb{R}} g(\lambda) d\mu_\varphi(\lambda). \quad (2.1.1)$$

Symbolically, we write

$$g(T) = \int_{\mathbb{R}} g(\lambda) dE(\lambda).$$

In particular, if we take  $(\mathbb{R}, \mathcal{B}_\mathbb{R})$  and  $g(\lambda) = \lambda$  then,  $g(T) = T$  and for  $\varphi, \psi \in \text{dom}(T)$ ,

$$\langle \varphi, T\psi \rangle = \int_{\mathbb{R}} \lambda d\mu_{\varphi, \psi}(\lambda).$$

We write

$$T = \int_{\mathbb{R}} \lambda dE(\lambda).$$

Now, we are in a position to state the spectral theorem. For the proof, we refer to The Spectral Theorem 4.11 of Chapter X in [9].

**Theorem 2.1.12. (Spectral Theorem).** *If  $T$  is a self-adjoint operator on a separable Hilbert space  $\mathcal{H}$  then, there is a unique spectral measure  $E$  on  $(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mathcal{H})$  such that*

1.  $T = \int_{\mathbb{R}} \lambda dE(\lambda)$ ,
2.  $E(\Delta) = 0$  if  $\Delta \cap \sigma(T) = \emptyset$  and
3. if  $U$  is an open subset of  $\mathbb{R}$  and  $U \cap \sigma(T) \neq \emptyset$  then,  $E(U) \neq 0$ .

From the above theorem, it is easy to see that if  $E_T(\cdot)$  is the spectral measure for a self-adjoint operator  $T$  on a separable Hilbert space  $\mathcal{H}$  then,

$$\sigma(T) = \{\lambda \in \mathbb{R} : E_T(\lambda - \epsilon, \lambda + \epsilon) \neq 0 \ \forall \epsilon > 0\}. \quad (2.1.2)$$

With this definition of spectrum, we can decompose  $\sigma(T)$  as

$$\sigma(T) = \sigma_{ess}(T) \cup \sigma_{dis}(T),$$

where  $\sigma_{ess}(T)$ ,  $\sigma_{dis}(T)$  are called essential and discrete spectrum of  $T$  respectively. They are given by

$$\sigma_{ess}(T) = \{\lambda \in \mathbb{R} : \dim \text{Ran} E_T(\lambda - \epsilon, \lambda + \epsilon) = \infty \ \forall \epsilon > 0\} \text{ and} \quad (2.1.3)$$

$$\sigma_{dis}(T) = \{\lambda \in \mathbb{R} : \dim \text{Ran} E_T(\lambda - \epsilon, \lambda + \epsilon) < \infty \text{ for some } \epsilon > 0\}. \quad (2.1.4)$$

The range of  $E_T(\lambda - \epsilon, \lambda + \epsilon)$  is a closed subspace of  $\mathcal{H}$  and  $\dim \text{Ran} E_T(\lambda - \epsilon, \lambda + \epsilon)$  denotes the dimension of this subspace.

It is a fact that  $\sigma_{ess}(T)$  is always closed, whereas  $\sigma_{dis}(T)$  is not necessarily closed.  $\sigma_{dis}(T)$  consists of all isolated eigenvalues of finite multiplicity. The following theorem is extremely useful to identify the essential spectrum of a self-adjoint operator.

**Theorem 2.1.13. (Weyl's Criterion).** *Let  $T$  be a self-adjoint operator on a separable Hilbert space. Then  $\lambda \in \sigma_{ess}(T)$  if and only if there exist a sequence  $\{\psi_n\}_n \in \text{dom}(T)$  with  $\|\psi_n\| = 1$  and  $\langle \psi_n, \psi_m \rangle = 0$  for  $n \neq m$  such that  $\lim_{n \rightarrow \infty} \|(T - \lambda)\psi_n\| = 0$ .*

We refer to [53, Theorem VII.12] for the proof.

We define the following subspaces of a Hilbert space  $\mathcal{H}$ :

$$\begin{aligned} \mathcal{H}_{pp} &= \{\psi \in \mathcal{H} : \mu_\psi \text{ is atomic}\} \\ \mathcal{H}_{ac} &= \{\psi \in \mathcal{H} : \mu_\psi \text{ is absolutely continuous}\} \text{ and} \\ \mathcal{H}_{sing} &= \{\psi \in \mathcal{H} : \mu_\psi \text{ is singular continuous}\}. \end{aligned}$$

Using these subspaces, we can have the following theorem. The proof can be found in [53, Theorem VII.4].

**Theorem 2.1.14.**  $\mathcal{H} = \mathcal{H}_{pp} \oplus \mathcal{H}_{ac} \oplus \mathcal{H}_{sing}$ . Each of these subspace is invariant under  $T$ .  $T \upharpoonright \mathcal{H}_{pp}$  has complete set of eigenvectors,  $T \upharpoonright \mathcal{H}_{ac}$  has only absolutely continuous spectral measures and  $T \upharpoonright \mathcal{H}_{sing}$  has only continuous singular spectral measure.

From the above theorem, it follows that  $\sigma_{pp}(T) = \overline{\sigma(T \upharpoonright \mathcal{H}_{pp})}$ . That is,  $\sigma_{pp}(T)$  is the closure of set of all eigenvalues of  $T$ . Also,  $\sigma_{ac}(T) = \sigma(T \upharpoonright \mathcal{H}_{ac})$  and  $\sigma_{sing}(T) = \sigma(T \upharpoonright \mathcal{H}_{sing})$ . Therefore,  $\sigma(T) = \sigma_{pp}(T) \cup \sigma_{cont}(T)$ , where  $\sigma_{cont}(T) = \sigma_{ac}(T) \cup \sigma_{sing}(T)$ .

It can be proved that a self-adjoint operator is nothing but component-wise multiplication operator.

**Theorem 2.1.15.** Let  $T$  be a self-adjoint operator on separable Hilbert space  $\mathcal{H}$ . Then, there exists a set of orthonormal vectors  $\{\psi_n\}_{n=1}^N$ , the closure of whose span is  $\mathcal{H}$  and, also, there exists a unitary operator  $U : \mathcal{H} \longrightarrow \bigoplus_{n=1}^N L^2(\sigma(T), d\mu_{\psi_n})$  ( $N = 1, 2, \dots$  or  $\infty$ ) such that

$$(UTU^{-1}f)_n(x) = xf_n(x).$$

In the above we represent an element  $f \in \bigoplus_{n=1}^N L^2(\sigma(T), d\mu_{\psi_n})$  as an  $N$ -tuple  $(f_1(x), f_2(x), \dots, f_N(x))$  with  $f_n \in L^2(\sigma(T), d\mu_{\psi_n})$  such that  $\sum_{n=1}^N \int_{\sigma(T)} |f_n(x)|^2 d\mu_{\psi_n}(x) < \infty$ .

Now, if  $\mu = \sum_{n=1}^{\infty} \frac{1}{2^n} \mu_{\psi_n}$  (called a total spectral measure) then, we have  $\sigma_{pp}(T) = \text{supp } \mu_{pp}$ ,  $\sigma_{ac}(T) = \text{supp } \mu_{ac}$  and  $\sigma_{sing}(T) = \text{supp } \mu_{sing}$ .

### Example of Self-adjoint operator:

Let  $(\Omega, \mathcal{B}_\Omega, \mu)$  be a  $\sigma$ -finite measure space and  $\varphi : \Omega \longrightarrow \mathbb{R}$  be measurable. Define the operator  $M_\varphi : L^2(\Omega, d\mu) \longrightarrow L^2(\Omega, d\mu)$  as follows:

$$\text{dom}(M_\varphi) := \{f \in L^2(\Omega, d\mu) : \varphi f \in L^2(\Omega, d\mu)\}, \quad M_\varphi f = \varphi f.$$



Then:

1.  $M_\varphi$  is a self-adjoint operator,
2. If  $\varphi \in L^\infty(\Omega, d\mu)$  then  $M_\varphi$  is bounded and  $\|M_\varphi\| = \|\varphi\|_\infty$ ,
3. The spectrum of  $M_\varphi$  is given by  $\sigma(M_\varphi) = \text{ess} - \text{ran}(\varphi)$ , where

$$\text{ess} - \text{ran}(\varphi) := \{\lambda \in \mathbb{R} : \mu(\varphi^{-1}(\lambda - \epsilon, \lambda + \epsilon)) > 0 \forall \epsilon > 0\} \text{ and}$$

4. If  $E$  is the spectral measure for  $M_\varphi$  then, for every Borel set  $\Delta$  of  $\sigma(M_\varphi)$ ,

$$E(\Delta) = M_{\chi_{\varphi^{-1}(\Delta)}}, \text{ where } \chi_A \text{ denotes the indicator function of } A.$$

## 2.2 Measurability Concepts

Let  $\Omega$  be a topological space with the Borel  $\sigma$ -algebra  $\mathcal{B}_\Omega$ . For a separable Hilbert space  $\mathcal{H}$  it's Borel  $\sigma$ -algebra  $\mathcal{B}_\mathcal{H}$  is equal to the smallest  $\sigma$ -algebra, for which all functions  $\mathcal{H} \ni f \rightarrow \langle f, g \rangle \in \mathbb{C}$ , for  $g \in \mathcal{H}$ , are measurable. Hence, a function  $\Omega \ni \omega \rightarrow f(\omega) \in \mathcal{H}$  is measurable (for the Borel  $\sigma$ -algebra  $\mathcal{B}_\mathcal{H}$ ) if and only if all the functions  $\Omega \ni \omega \rightarrow \langle f(\omega), g \rangle \in \mathbb{C}$  are measurable. For details we refer to [10, V.1 Measurability Concepts].

**Definition 2.2.1.** *We say that a function  $\Omega \ni \omega \rightarrow T(\omega) \in \mathcal{B}(\mathcal{H})$  is measurable if  $\forall f, g \in \mathcal{H}$  the function  $\Omega \ni \omega \rightarrow \langle f, T(\omega)g \rangle \in \mathbb{C}$  is measurable.*

The following proposition characterizes the measurability of a family of self-adjoint operators. For the proof of the proposition, we refer to Proposition V.1.2 of [10].

**Proposition 2.2.2.** *If  $H^\omega$  is a self-adjoint operator on  $\mathcal{H}$ , for each  $\omega \in \Omega$ , and  $\{E_{H^\omega}(A) : A \in \mathcal{B}_\mathbb{R}\}$  denotes the family of spectral projections of  $H^\omega$  then the following three properties are equivalent:*

1.  $\Omega \ni \omega \rightarrow E_{H^\omega}(A) \in \mathcal{B}(\mathcal{H})$  is measurable  $\forall A \in \mathcal{B}_\mathbb{R}$ ,

2.  $\Omega \ni \omega \rightarrow e^{itH^\omega} \in \mathcal{B}(\mathcal{H})$  is measurable  $\forall t \in \mathbb{R}$ ,

3.  $\Omega \ni \omega \rightarrow (H^\omega - zI)^{-1} \in \mathcal{B}(\mathcal{H})$  is measurable  $\forall z \in \mathbb{C} \setminus \mathbb{R}$ .

Now, we formally define a measurable family of self-adjoint operators on a Hilbert space.

**Definition 2.2.3.** *A function  $\Omega \ni \omega \rightarrow H^\omega$  with values in the set of self-adjoint operators on  $\mathcal{H}$  is said to be measurable if any of the three equivalent properties of the above proposition is satisfied.*

## 2.3 Spectra of Ergodic Families

In this section, we discuss some random self-adjoint operators and their spectral properties. To begin with, we consider a probability space  $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$  and a group  $G$  that acts on  $\Omega$ . Now, for each  $g$  in  $G$  there is an automorphism  $T_g$  on  $\Omega$  such that  $\{T_g : g \in G\}$  is a family of measure preserving ergodic transformations on  $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$ . We also assume that  $\mathcal{H}$  is a separable Hilbert space on which  $G$  has a unitary representation and this family of unitary operators is denoted by  $\{U_g : g \in G\}$ . We make the following assumptions on  $G$ .

**Hypothesis 2.3.1.** *There is a total set  $S$  of vectors in  $\mathcal{H}$  such that for each  $f \in S$ , there is sequence  $g_n(f)$  of elements of  $G$  such that  $\{U_{g_n} f\}_n$  is a collection of orthonormal vectors.*

Now, we consider a family of measurable self-adjoint operators  $H^\omega$ ,  $\omega \in \Omega$ , on  $\mathcal{H}$  such that they have a common domain.

**Definition 2.3.2.** *Let  $(\Omega, \mathcal{B}_\Omega, \mathbb{P}, G)$ ,  $\{T_g, U_g, g \in G\}$  and  $\mathcal{H}$  be defined as above. Then, we say that  $\{H^\omega : \omega \in \Omega\}$  is an ergodic family of operators affiliated to  $(\Omega, \mathcal{B}_\Omega, \mathbb{P}, G)$  if it satisfies the following condition:*

$$U_g H^\omega U_g^* = H^{T_g \omega}, \quad \forall g \in G. \quad (2.3.1)$$

The following theorems are originally due to Pastur [51]. For the proofs we refer to [10, Lemma V.2.1] and [10, Proposition V.2.4].

**Theorem 2.3.3.** *If  $H^\omega$  is an ergodic family of self-adjoint operators affiliated to  $(\Omega, \mathcal{B}_\Omega, \mathbb{P}, G)$  then, for almost all  $\omega$ , dimension of  $\text{ran}(E_{H^\omega}(A))$  is either 0 or  $\infty$ , for every  $A \in \mathcal{B}_\mathbb{R}$ .*

**Theorem 2.3.4.** *If  $H^\omega$  is an ergodic family of self-adjoint operators affiliated to  $(\Omega, \mathcal{B}_\Omega, \mathbb{P}, G)$ , then there is a (closed, non-random) subset  $\Sigma$  of  $\mathbb{R}$  such that*

$$\sigma(H^\omega) = \Sigma \quad \text{a.e. } \omega.$$

Moreover, there are non-random sets  $\Sigma_{ac}, \Sigma_{sc}, \Sigma_{pp}$  such that

$$\sigma_{ac}(H^\omega) = \Sigma_{ac}, \sigma_{sc}(H^\omega) = \Sigma_{sc}, \sigma_{pp}(H^\omega) = \Sigma_{pp} \quad \text{a.e. } \omega.$$

## 2.4 The Anderson Model

Let  $\omega = \{\omega_n\}_{n \in \mathbb{Z}^d}$  be a sequence of real-valued i.i.d. random variables with distribution  $\mu$ . In this case the probability space is given by the product  $\Omega = \mathbb{R}^{\mathbb{Z}^d}$  and usual product  $\sigma$ -algebra  $\mathcal{B}_\Omega$ . The probability measure  $\mathbb{P}$  on  $\Omega$  is given by  $\mathbb{P} = \bigotimes_{n \in \mathbb{Z}^d} \mu$ , constructed via Kolmogorov consistency theorem. We refer to this probability space as  $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$ . The Anderson Model is a random Hamiltonian  $H^\omega$  on  $\ell^2(\mathbb{Z}^d)$ , defined for  $\omega \in \Omega$  by

$$(H^\omega u)(n) = (\Delta u)(n) + \omega_n u(n), \quad \forall u \in \ell^2(\mathbb{Z}^d), \quad n \in \mathbb{Z}^d, \quad (2.4.1)$$

where  $\Delta$  is the discrete Laplacian on  $\ell^2(\mathbb{Z}^d)$  given by

$$(\Delta \varphi)(n) = \sum_{|k|=1} \varphi(n+k), \quad \varphi \in \ell^2(\mathbb{Z}^d). \quad (2.4.2)$$

Defining the random potential  $V^\omega : \mathbb{Z}^d \rightarrow \mathbb{R}$  as  $V^\omega(n) = \omega_n$ , we may also write

$$H^\omega = \Delta + V^\omega. \quad (2.4.3)$$

We note that  $H^\omega$  is not a single operator, but an operator-valued function on a probability space  $(\Omega, \mathcal{B}_\omega, \mathbb{P})$ . Generally its operator theoretic properties depend on  $\omega$ . Our goal is to find out the properties of  $H^\omega$  holds almost everywhere.

One may think of the Anderson model  $H^\omega$  as the Hamiltonian governing the quantum mechanical motion of a single electron in a discretized alloy-type random medium. In this view the random potential  $V^\omega(n) = \omega_n$ ,  $n \in \mathbb{Z}^d$ , represents a solid formed by atoms located at the sites  $n$  of the lattice  $\mathbb{Z}^d$  and carrying random electrical charges  $\omega_n$ . It is easy to see  $H^\omega$  is self-adjoint, the dynamics of the electron is given through the unitary group  $e^{-itH^\omega}$ , defined via the spectral theorem, which provides the solution  $\psi(t) = e^{-itH^\omega}\psi_0$ ,  $\psi(0) = \psi_0$  of the Schrödinger equation  $H^\omega\psi(t) = i\psi'(t)$ ,  $\psi(0) = \psi_0$ . The possible energies of the electron are given by the  $\sigma(H^\omega)$ .

For each  $i \in \mathbb{Z}^d$ , define  $T_i : \Omega \rightarrow \Omega$  by  $(T_i\omega)(n) = \omega_{n-i}$ . So, the group  $G = \mathbb{Z}^d$  acts on  $\Omega$  by translation. Also,  $G$  has unitary representation in  $\ell^2(\mathbb{Z}^d)$  given by

$$(U_i\varphi)(n) = \varphi(n-i), \quad \varphi \in \ell^2(\mathbb{Z}^d), \quad \forall i \in G = \mathbb{Z}^d.$$

If we take the orthonormal set  $S = \{\delta_k : k \in \mathbb{Z}^d\}$ , where  $\delta_k(n) = 1$ , for  $n = k$  and 0 otherwise then, Hypothesis 2.3.1 is satisfied by  $G = \mathbb{Z}^d$ . We also have  $U_i H^\omega U_i^* = H^{T_i\omega}$ , for every  $i \in \mathbb{Z}^d$ . In the case when  $\mu$  has compact support, we see that  $\omega \rightarrow \langle \varphi, H^\omega \psi \rangle$  are measurable functions of  $\omega \forall \varphi, \psi \in \mathcal{H}$ . If the support of  $\mu$  is unbounded then  $\omega \rightarrow \langle \varphi, (H^\omega - z)^{-1} \psi \rangle$  are measurable  $\forall z \in \mathbb{C}^+$ ,  $\varphi, \psi \in \mathcal{H}$ . The family  $\{H^\omega\}$  of self-adjoint operators with these properties is said to be weakly measurable ergodic family of self-adjoint operators affiliated to  $(\Omega, \mathcal{B}_\Omega, \mathbb{P}, G = \mathbb{Z}^d)$ .

We end this section with the following theorem, which says that  $\sigma(H^\omega)$  is almost surely

deterministic, i.e. there exists a closed subset  $\Sigma$  of  $\mathbb{R}$  such that  $\sigma(H^\omega) = \Sigma$  a.e  $\omega$ . We refer [57, Theorem 2] for the proof of the theorem.

**Theorem 2.4.1.** *The spectrum of  $H^\omega$  given by*

$$\sigma(H^\omega) = [-2d, 2d] + \text{supp } \mu, \quad \text{a.e } \omega, \quad (\sigma(\Delta) = [-2d, 2d]).$$

## 2.5 The Density of States

This section is devoted to describing the so-called integrated density of states which is a quantity of fundamental importance for models in condensed matter physics. The spectrum of  $H^\omega$  is very often uncountable, so we cannot simply count eigenvalues within the interval  $(a, b)$  which is same as the dimension of  $\text{Rang} E_{H^\omega}(a, b)$ . In fact, we have seen in Theorem 2.3.3 that dimension of any spectral projection is either 0 or  $\infty$  a.e  $\omega$ . Therefore, a counting function has to be obtained differently. In order to do this, we restrict  $H^\omega$  to a finite cube  $\Lambda$  of  $\mathbb{Z}^d$ , then we count the number of eigenvalues (which are the energy levels of the system) in a given interval  $[a, b]$ . We look for a volume independent quantity to give an account of very large system. So we divide the count by  $|\Lambda|$ , the number of points in  $|\Lambda|$ . Finally, we compute  $\Lambda \uparrow \mathbb{Z}^d$ . The limit gives an average number of the energy levels per unit volume. Such a quantity is experimentally measured for crystals. This limit procedure is called the **thermodynamic limit**. For the proof of the following proposition we refer to [37, Proposition 5.2].

**Proposition 2.5.1.** *If  $\varphi$  is a bounded measurable function on  $\mathbb{R}$ ,  $\Lambda$  finite cubes centered at the origin, then*

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \text{Tr}(\varphi(H^\omega) \chi_\Lambda) = \mathbb{E}^\omega(\langle \delta_0, \varphi(H^\omega) \delta_0 \rangle) \quad \text{a.e } \omega.$$

In the following, we define the integrated density of states.

**Definition 2.5.2.** *The measure  $\nu$  defined by*

$$\nu(A) = \mathbb{E}^\omega(\langle \delta_0, E_{H^\omega}(A)\delta_0 \rangle), \text{ for any Borel set } A$$

*is called density of state measure. The distribution function  $N$  of  $\nu$  defined by  $N(x) = \nu(-\infty, x]$  is known as the integrated density of states. Since  $N$  is a non-decreasing function on  $\mathbb{R}$ , it is differentiable almost everywhere (w.r.t Lebesgue measure) and if its derivative  $n(E) = \frac{dN(x)}{dx} |_{x=E}$  exists at  $E \in \mathbb{R}$ , we call it the density of states at the energy  $E$ .*

## Chapter 3

# Criteria for Identifying the Spectrum

In this chapter we present Borel transformation of a measure  $\mu$  on  $\mathbb{R}$  and discuss methods to recover the different components of  $\mu$ , namely  $\mu_{pp}$ ,  $\mu_{ac}$  and  $\mu_{sing}$  from its Borel transform. If  $A$  is a self-adjoint operator on a separable Hilbert space  $\mathcal{H}$  and  $\varphi \in \mathcal{H}$ . Then we see, using the spectral theorem, that the function  $z \rightarrow \langle \varphi, (A - z)^{-1} \varphi \rangle$  are Borel transformation of the spectral measure  $\langle \varphi, E_A(\cdot) \varphi \rangle$  associate with the self-adjoint operator  $A$  and the vector  $\varphi$ . So we use it to identify the spectral types of self-adjoint operators. We wrap up the chapter by presenting the fractional moment criterion of Aizenman-Molchanov to show the localization for Anderson Model. In [5] Aizenman-Molchanov showed the exponential decay of Green's function for large disorder. Once we have the decay for Green's function then Simon-Wolff criterion immediately imply the point spectrum for this model.

Note that we use the word *measure* for a positive measure and use the adjectives *signed* and *complex* for a non-positive measure.

### 3.1 Borel Transformation

We begin by defining Borel transformation for a class of measures.

**Definition 3.1.1.** *Let  $\mu$  be a measure on  $\mathbb{R}$  satisfying the following condition:*

$$\int_{\mathbb{R}} \frac{1}{1+x^2} d\mu(x) < \infty. \quad (3.1.1)$$

*Then, the Borel transform  $F_\mu$  of  $\mu$  is given by*

$$F_\mu(z) = \int_{\mathbb{R}} \left( \frac{1}{x-z} - \frac{x}{1+x^2} \right) d\mu(x), \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (3.1.2)$$

$F_\mu(z)$  is an analytic function in  $\mathbb{C}^+ \cup \mathbb{C}^-$ , and it maps these two components to themselves. Conversely, we have the well-known *Herglotz representation* theorem for analytic functions on upper half plane and it is given by the following theorem. Its proof can be found in [16, Theorem 1.4.2].



**Theorem 3.1.2.** *Let  $F : \mathbb{C}^+ \rightarrow \mathbb{C}^+$  be an analytic function. Then, there exists a non-negative number  $C$ , a real number  $D$  and a measure  $\mu$  satisfying the condition 3.1.1, such that*

$$F(z) = Cz + D + \int_{\mathbb{R}} \left( \frac{1}{x-z} - \frac{x}{1+x^2} \right) d\mu(x), \quad z \in \mathbb{C}^+.$$

*The values  $C$ ,  $D$  and  $\mu$  are uniquely associated with  $F$ .*

It has been shown that boundary values exist for these analytic functions. The proof of the theorem is given in [16, Theorem 1.4.6]

**Theorem 3.1.3.** *Let  $F : \mathbb{C}^+ \rightarrow \mathbb{C}^+$  be an analytic function. Then, the boundary values  $\lim_{\epsilon \rightarrow 0} F(x + i\epsilon)$  exist and are finite for almost every  $x$ .*

This theorem has two corollaries related to Borel transformation of a measure, which are stated below. Note that if  $\mu$  is a finite complex measure then, by the linearity of the integral, we can extend the notion of Borel transformation to such measures using the formula  $\mu = \mu_+^R - \mu_-^R + i(\mu_+^I - \mu_-^I)$ , where  $\mu_{\pm}^R$  and  $\mu_{\pm}^I$  are positive and negative components of the real and imaginary part of  $\mu$ , respectively.

**Corollary 3.1.4.** *Let  $\mu$  be a non-zero finite complex measure with finite total variation  $|\mu|$  and  $F_{\mu}$  be it's Borel transformation. Then,  $F_{\mu}(x + i0) = \lim_{\epsilon \rightarrow 0} F_{\mu}(x + i\epsilon)$  exist and  $0 < |F_{\mu}(x + i0)| < \infty$ , for a.e.  $x$  (w.r.t Lebesgue Measure).*

**Corollary 3.1.5.** *Let  $\mu_1$  and  $\mu_2$  be two probability measures. If  $F_{\mu_1}(x + i0) = F_{\mu_2}(x + i0)$ , for every  $x$  in a set of positive Lebesgue measure, then  $F_{\mu_1} = F_{\mu_2}$  on  $\mathbb{C}^+$ , which means  $\mu_1 = \mu_2$ .*

The details of above corollaries are given in Corollary 1.4.8 and Corollary 1.4.9 of [16].

The following theorem gives the method of recovering the different components of a measure  $\mu$ , namely  $\mu_{ac}$ ,  $\mu_{sing}$  and  $\mu_{pp}$ , from it's Borel transformation  $F_{\mu}$ .

We refer to [16, Corollary 1.4.5], [16, Theorem 1.4.16] and [56, Theorem 11.6] for the proof of the theorem.

**Theorem 3.1.6.** *Let  $\mu$  be a measure on  $\mathbb{R}$  satisfying the condition 3.1.1 and let  $F_\mu$  be its Borel transformation. Then,*

1.  $\frac{1}{\pi} \operatorname{Im}(F_\mu(x + i\epsilon))dx \xrightarrow{\text{weakly}} d\mu(x)$  as  $\epsilon \rightarrow 0$ , in the sense that

$$\frac{1}{\pi} \int_{\mathbb{R}} f(x) \operatorname{Im}(F_\mu(x + i\epsilon))dx \longrightarrow \int_{\mathbb{R}} f(x) d\mu(x) \quad \text{as } \epsilon \rightarrow 0,$$

for all bounded continuous function  $f$ .

2.  $\frac{1}{\pi} \operatorname{Im}(F_\mu(x + i0)) = \frac{d\mu_{ac}(x)}{dx}$  a.e.  $x$ .

3. The singular part  $\mu_{sing}$  of  $\mu$  is supported on the set

$$\{x : \lim_{\epsilon \rightarrow 0} \operatorname{Im}(F_\mu(x + i\epsilon)) = \infty\}.$$

4. If  $\mu(\{x\}) \neq 0$  then,

$$\mu(\{x\}) = \lim_{\epsilon \rightarrow 0} \epsilon \operatorname{Im}(F_\mu(x + i\epsilon)).$$

## 3.2 Rank One Perturbations

In this section we recall Aronszajn-Donoghue theory of rank one perturbations [56, Chapter 11]. Consider a self-adjoint operator  $H_0$  on  $\mathcal{H}$  and let  $\phi$  be a unit vector in  $\mathcal{H}$ . Let  $P_\phi$  be the orthogonal projection onto the span of  $\phi$ . Now define the following operators:

$$H_\lambda = H_0 + \lambda P_\phi, \quad \lambda \in \mathbb{R}. \quad (3.2.1)$$

Let  $\mu_\lambda$  be the spectral measure of  $H_\lambda$  associated with the vector  $\phi$ . Also, let  $\mu_0$  be the corresponding measure of  $H_0$ . Let  $F_\lambda$  and  $F_0$  be the Borel transformations of  $\mu_\lambda$  and  $\mu_0$ , respectively. Thus, formally

$$\mu_\lambda(\cdot) = \langle \phi, E_{H_\lambda}(\cdot)\phi \rangle, \quad \mu_0(\cdot) = \langle \phi, E_{H_0}(\cdot)\phi \rangle.$$

$$F_\lambda(z) = \langle \phi, (H_\lambda - z)^{-1} \phi \rangle, \quad F_0(z) = \langle \phi, (H_0 - z)^{-1} \phi \rangle, \quad z \in \mathbb{C}^+.$$

We observe the relations

$$F_\lambda(z) = \frac{F_0(z)}{1 + \lambda F_0(z)}, \quad \text{Im}(F_\lambda(z)) = \frac{\text{Im}(F_0(z))}{|1 + \lambda F_0(z)|^2},$$

which are derived using the resolvent equation

$$(H_\lambda - z)^{-1} = (H_0 - z)^{-1} - \lambda(H_\lambda - z)^{-1} P_\phi (H_0 - z)^{-1}.$$

Let us define the following sets for  $\lambda \neq 0$ .

$$\begin{aligned} S_{\lambda,0} &= \{x \in \mathbb{R} : (DF_0)(x) < \infty, F_0(x + i0) = -\lambda^{-1}\}, \\ S_{\lambda,\infty} &= \{x \in \mathbb{R} : (DF_0)(x) = \infty, F_0(x + i0) = -\lambda^{-1}\}, \\ L_0 &= \{x \in \mathbb{R} : 0 < \text{Im}(F_0(x + i0)) < \infty\}, \end{aligned}$$

where

$$(DF_0)(x) = \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}} \frac{1}{(x - y)^2 + \epsilon^2} d\mu_0(y).$$

The following theorem was proved by Aronszajn [3] for Sturm-Liouville operators which was then extended to rank one perturbations by Donoghue [15]. The theorem gives a few properties of Lebesgue decomposition of  $\mu_\lambda$  in terms of the measure  $\mu_0$ . For the proof of the theorem we refer to [16, Theorem 3.1.3] or [56, Theorem 12.2].

**Theorem 3.2.1 (Aronszajn-Donoghue).** *Let  $H_\lambda$  and  $\phi$  be as in the equation (3.2.1) then,*

1. *The pure point part  $\mu_{\lambda,pp}$  of  $\mu_\lambda$  is supported on the set  $S_{\lambda,0}$  and is given by*

$$\mu_{\lambda,pp}(\{x\}) = \sum_{y \in S_{\lambda,0}} \frac{1}{\lambda^2 (DF_0)(y)} \delta(x - y).$$

2. *The singular part  $\mu_{\lambda,sing}$  is supported in the set  $S_{\lambda,\infty}$ .*

3. The absolute continuous part  $\mu_{\lambda,ac}$  is supported in the set  $L_0$ , for all  $\lambda$ .

The following formula, known as the spectral averaging formula, states that  $\int \mu_\lambda d\lambda$  is same as the Lebesgue measure. The proof of the proposition is given in [16, Proposition 3.1.4] or [56, Theorem 11.8].

**Proposition 3.2.2.** *Let  $\mu_\lambda(\cdot) = \langle \phi, E_{H_\lambda}(\cdot)\phi \rangle$ ,  $\lambda \in \mathbb{R}$ , be the family of probability measures, associated with the operators  $H_\lambda$  as in equation 3.2.1. Then,*

$$dx = \int_{\mathbb{R}} \mu_\lambda(x) d\lambda,$$

in the sense that if  $f \in L^1(\mathbb{R}, dx)$  then,  $f \in L^1(\mathbb{R}, d\mu_\lambda)$ , for a.e.  $\lambda$ , and  $\int f(x) d\mu_\lambda(x) \in L^1(\mathbb{R}, d\lambda)$  and

$$\int \left( \int f(x) d\mu_\lambda(x) \right) d\lambda = \int f(x) dx.$$

The following beautiful criterion to identify the point spectrum for rank one perturbations is due to Simon and Wolff [58]. The proof of this theorem uses Theorem 3.2.1 and Proposition 3.2.2 and can be found in [16, Theorem 3.1.7].

**Theorem 3.2.3. (Simon-Wolff criterion)** *Let  $H_\lambda$  and  $\phi$  be as in equation (3.2.1) and consider the family of measures  $\mu_\lambda$ ,  $\lambda \in \mathbb{R}$ . Suppose  $\mu_\lambda[a, b] \neq 0$  for almost every  $\lambda$ . Then the following are equivalent.*

1. For almost all  $\lambda$ ,  $\mu_\lambda$  is pure point in  $[a, b]$ .
2. For almost every  $x$  in  $[a, b]$  (w.r.t Lebesgue measure)  $(DF_0)(x) < \infty$ .

There is another interesting theorem we include here which is related to rank one perturbations and is due to Jaksic and Last [35]. Before we state the theorem we define the following.

$$\begin{aligned} \mathcal{H}_{\lambda,\phi} &= \overline{\text{span}\{H_\lambda^j \phi : j = 0, 1, \dots\}} \quad \phi \in \mathcal{H}, \\ \mathcal{H}_{\lambda,\psi} &= \overline{\text{span}\{H_\lambda^j \psi : j = 0, 1, \dots\}} \quad \psi \in \mathcal{H}, \\ \mu_{\lambda,\phi}(\cdot) &= \langle \phi, E_{H_\lambda}(\cdot)\phi \rangle, \quad \mu_{\lambda,\psi}(\cdot) = \langle \psi, E_{H_\lambda}(\cdot)\psi \rangle. \end{aligned}$$

The proof of the theorem can be found in [35, Theorem 2.4] and [16, Theorem 3.1.9].

**Theorem 3.2.4. (Jaksic-Last)** *Consider a self-adjoint operator  $H_0$  and a pair of vectors  $\phi, \psi \in \mathcal{H}$ . Suppose  $H_\lambda = H_0 + \lambda P_\phi$ . Further, suppose  $\mathcal{H}_{\lambda,\phi}$  is not orthogonal to  $\mathcal{H}_{\lambda,\psi}$ , for some  $\lambda$ . Then, for almost every  $\lambda$ ,  $\mu_{\lambda,\phi}$  is absolutely continuous with respect to  $\mu_{\lambda,\psi}$ .*

### 3.3 Aizenman-Molchanov Method

In this section we present the ideas of proof of localization (i.e  $\sigma(H^\omega) = \emptyset$ , for  $H^\omega$  given below) using Aizenman-Molchanov Method. In this method, one can show that for the operators of the form  $H^\omega = \Delta + V^\omega$  on  $\ell^2(\mathbb{Z}^d)$  (as in section 2.4), the Green's function  $\langle \delta_n, (H^\omega - E - i\epsilon)^{-1} \delta_m \rangle$  have a good decay in  $|n - m|$ . This is done by showing that the averages  $\mathbb{E}^\omega(|\langle \delta_n, (H^\omega - E - i\epsilon)^{-1} \delta_m \rangle|^s)$  have good decay in  $|n - m|$  for some  $0 < s < 1$ . This type of estimate together with Simon-Wolff criterion give that the spectrum of  $H^\omega$  is pure point and we say  $H^\omega$  exhibits Anderson localization. This method is also known as fractional moment method and is due to Aizenman-Molchanov [5]. Localization in the Anderson Model is of great interest as it shows at what energies the materials described by the model become insulators and do not conduct electricity.

**Definition 3.3.1.** *We say that a random Schrödinger operator  $H^\omega$  has spectral localization in an interval  $I \subset \mathbb{R}$  if  $H^\omega$  almost surely has pure point spectrum in  $I$ . That is, if*

$$\sigma(H^\omega) \cap I \subset \sigma_{pp}(H^\omega) \quad \text{a.e. } \omega.$$

We say that the localization is exponential if eigenfunctions corresponding to eigenvalue in  $I$  decay exponentially.

**Remark 3.3.2.** *Thus, exponential spectral localization holds in  $I$  if for almost every  $\omega$ , each eigenfunction  $\psi_\omega$  corresponding to an eigenvalue in  $I$  obeys*

$$|\psi_\omega(x)| \leq C_\omega e^{-\eta|x-x_\omega|}$$

with  $\eta > 0$  and some finite  $C_\omega$ . Then,  $x_\omega$  is called the localization center.

We say that  $H^\omega$  exhibits *dynamical localization* in  $I$  if there exist constants  $C < \infty$  and  $\eta > 0$  such that

$$\mathbb{E}^\omega \left( \sup_{t \in \mathbb{R}} |\langle \delta_j, e^{-itH^\omega} E_{H^\omega}(I) \delta_k \rangle| \right) \leq C e^{-\eta|j-k|}, \quad \forall j, k \in \mathbb{Z}^d. \quad (3.3.1)$$

Now, we describe the model of random Schrödinger operators for which the localization (described above) is physically expected. In order to do this, we introduce additional disorder parameter  $\lambda > 0$  in (2.4.3) and define

$$H_\lambda^\omega = \Delta + \lambda V^\omega \quad (3.3.2)$$

In fact, the above model fits into (2.4.3) with  $V^\omega(n) = \lambda \omega_n$ , i.i.d. random variables, whose common distribution  $\mu_\lambda$  is given by the re-scaled  $\mu$  (distribution of  $\omega_n$ ) i.e.,

$$\mu_\lambda(B) := \mu(B/\lambda) \quad \forall B \in \mathcal{B}_\mathbb{R}.$$

First, we give a definition for class of measures which will be considered to prove the localization of  $H_\lambda^\omega$ .

**Definition 3.3.3.** A probability measure  $\mu$  is said to be  $\tau$ -regular for  $0 < \tau \leq 1$  if for any  $x \in \mathbb{R}$ ,

$$\mu([x - a, x + a]) \leq C|a|^\tau, \quad \text{for any } a > 0,$$

with  $C$  independent of  $x$ . The smallest number  $C$  with this property is denoted by  $C_{\mu,\tau}$ .

The following lemma, which is known as “decoupling lemma”, is very crucial for this theory. Its proof can be found in [16, Lemma 5.1.14].

**Lemma 3.3.4.** Suppose  $\mu$  is an absolutely continuous probability measure that is also

$\tau$ -regular. Then, for any  $0 < s < \tau \leq 1$  and any  $\beta \in \mathbb{C}$  we have

$$\inf_{\alpha \in \mathbb{C}} \int_{\mathbb{R}} \frac{|x - \alpha|^s}{|x - \beta|^s} d\mu(x) \geq C_{s,\tau}(\mu) \int_{\mathbb{R}} \frac{1}{|x - \beta|^s} d\mu(x),$$

where  $C_{s,\tau}(\mu) \rightarrow 0$  as  $s \rightarrow \tau$ .

Now, we introduce the *Green's functions* as the matrix-elements of the resolvent of  $H_\lambda^\omega$ ,

$$G_\lambda^\omega(z; n, m) := \langle \delta_n, (H_\lambda^\omega - E - i\epsilon)^{-1} \delta_m \rangle, \quad \text{Im}z \neq 0.$$

The next lemma gives an estimate for the average of Green's functions. The proof of the lemma is given in [16, Lemma 5.1.16]. Assuming that  $\mu$  has compact support leads to an easier proof which is given in [57, Lemma 4.1].

**Lemma 3.3.5.** *Consider the random operators  $H_\lambda^\omega$  described in equation (3.3.2) with  $\tau$ -regular  $\mu$ . Then, there exists a constant  $C(\lambda, s)$ , independent of  $E \in \mathbb{R}$  and  $\epsilon > 0$ , such that the following holds:*

$$\mathbb{E}^\omega (|G_\lambda^\omega(E + i\epsilon; n, m)|^s) \leq C(\lambda, s) < \infty, \quad \text{for some } 0 < s < \tau \leq 1, \quad \forall n, m \in \mathbb{Z}^d. \quad (3.3.3)$$

These results put together lead to the following theorem whose proof can be found in [5, Theorem 4.3] with  $U_0(\cdot) = 0$ .

**Theorem 3.3.6.** *Let  $0 < s < \tau \leq 1$ . Let  $H_\lambda^\omega$  be as in equation (3.3.2) and let  $\mu$  satisfy the condition in the Lemma 3.3.5. Then, there exists  $\lambda_0 > 0$  such that for  $\lambda > \lambda_0$  there are constants  $C < \infty$  and  $r > 0$  with*

$$\mathbb{E}^\omega (|G_\lambda^\omega(E + i\epsilon; n, m)|^s) \leq C e^{-r|n-m|} \quad \forall n, m \in \mathbb{Z}^d \text{ and } E \in \mathbb{R}, \quad \epsilon > 0, \quad (3.3.4)$$

and the spectrum of  $H_\lambda^\omega$  is pure point a.e.  $\omega$ , for  $\lambda > \lambda_0$ .

This theorem immediately implies that

$$\int_a^b \left( \sum_{m \in \mathbb{Z}^d} \mathbb{E}^\omega (|G_\lambda^\omega(E + i\epsilon; n, m)|^s) \right) dE < \infty, \quad \text{for } \lambda > \lambda_0.$$

which, in turn, implies

$$\sum_{m \in \mathbb{Z}^d} |G_\lambda^\omega(E + i0; n, m)|^s < \infty, \quad \text{a.e. } (\omega, E) \in \Omega \times [a, b],$$

and hence

$$\sum_{m \in \mathbb{Z}^d} |G_\lambda^\omega(E + i0; n, m)|^2 < \infty \quad \text{a.e. } \omega, \quad \text{a.e. } E \in [a, b].$$

The last statement verifies the Simon-Wolff criterion for the pure point spectrum in  $[a, b]$  for large enough  $\lambda$ .



## Chapter 4

# Level Repulsion for a Class of Decaying Random Potentials

## 4.1 Introduction

In this chapter, we consider the Anderson model with decaying randomness and show that statistics near the band edges in the absolutely continuous spectrum in dimensions  $d \geq 3$  is independent of the randomness and agrees with that of the free part  $\Delta$ . We also consider the operators at small coupling and identify the length scales at which the statistics agrees with the free one in the limit when the coupling constant goes to zero.

The studies of statistics of eigenvalues of random Schrödinger operators was done in one dimension by Molchanov [48] and in the Anderson model at high disorder by Minami [47] initially. Both these works formalized the rigorous procedure for exhibiting Poisson statistics in these random models. They show that the eigenvalue statistics near an energy  $E$  in the spectrum follows a Poisson random measure with intensity being  $n(E)$  times the Lebesgue measure, where  $n(E)$  is the density of states at  $E$ .

Subsequently Poisson statistics was shown for the trees by Aizenman-Warzel [7]. An elegant proof of the Minami estimate needed in showing Poisson statistics was obtained by Combes-Germinet-Klein who also showed Poisson statistics in the continuum models in [14]. In the paper [25] Germinet-Klopp gave a proof not only of the Poisson statistics but also showed that the level spacing distribution follows the exponential law.

In one dimension for a class of decaying random potentials the eigenvalue statistics was shown to follow the beta-ensemble by Kotani-Nakano [41]. Our goal is to look at the models of decaying random potentials in  $d$  dimension where a sharp mobility edge exists, as shown in Kirsch-Krishna-Obermeit [42] and Jacksic-Last [35], and find out if there is a sharp transition in the local statistics.

We are concerned about the statistics in the absolutely continuous spectral regime. We consider two cases, one where the random potential is decaying and other where the random potential has small coupling. In the former case we identify the rate of decay of the potential and the dimension in which the statistics agrees with that of the free operator  $\Delta$ . In the latter case we identify the lengths of cubes for which the statistics

agrees with the one for the cases  $\Delta = 0$ .

The model we consider is given by

$$H^\omega = \Delta + V^\omega, \quad (\Delta u)(n) = \sum_{|m-n|=1} u(m), \quad (V^\omega u)(m) = V^\omega(m)u(m), \quad (4.1.1)$$

for  $u \in \ell^2(\mathbb{Z}^d)$  where  $\{V(n)\}$  is a collection of independent real valued random variables on  $\Omega = \mathbb{R}^{\mathbb{Z}^d}$ . We denote the standard basis of  $\ell^2(\mathbb{Z}^d)$  by  $\{\delta_n, n \in \mathbb{Z}^d\}$ . The spectrum  $\sigma(\Delta)$  of the operator  $\Delta$  is well known to be purely absolutely continuous and is given by the interval  $[-2d, 2d]$ . We consider a cube of side length  $2L$  centered at the origin in  $\mathbb{Z}^d$  namely,

$$\Lambda_L = \{n = (n_1, n_2, \dots, n_d) \in \mathbb{Z}^d : |n_i| \leq L, i = 1, 2, \dots, d\} \quad (4.1.2)$$

and take  $\chi_{\Lambda_L}$  as the orthogonal projection on to  $\ell^2(\Lambda_L)$ . We define  $(2L+1)^d$  dimensional matrices  $\Delta_L, \Delta_{L,E}$  associated with an  $E \in (-2d, 2d)$  by

$$\Delta_L = \chi_{\Lambda_L} \Delta \chi_{\Lambda_L}, \quad \Delta_{L,E} = (L+1) \chi_{\Lambda_L} (\Delta - E) \chi_{\Lambda_L}.$$

We also consider the matrices

$$H_{L,E}^\omega = (L+1) \chi_{\Lambda_L} (H^\omega - E) \chi_{\Lambda_L}, \quad E \in (-2d, 2d).$$

It is known [38], [35], [42] that the spectrum of  $H^\omega$  is purely absolutely continuous in  $(-2d, 2d)$  when the variance of  $V^\omega(n)$  is finite and the sequence  $a_n$  satisfies  $a_n \approx |n|^{-2-\epsilon}$  as  $|n| \rightarrow \infty$ .

In the next section we study the measures

$$\mu_{L,E}^0 = \frac{1}{(2L+1)^{d-1}} \text{Tr}(E_{\Delta_{L,E}}()), \quad \mu_{L,E}^\omega = \frac{1}{(2L+1)^{d-1}} \text{Tr}(E_{H_{L,E}^\omega}()) \quad (4.1.3)$$

where we have notationally denoted the (projection valued) spectral measure of a self-adjoint operator  $A$  by  $E_A()$ .

## 4.2 Decaying randomness : Statistics

In this section, we consider perturbations of  $\Delta$  by independent random single site potentials with either a short range rate of decay at  $\infty$  or having a disorder parameter which is small.

**Hypothesis 4.2.1.** *Let  $V^\omega(n) = a_n q_n(\omega)$  where  $q_n(\omega)$  are independent random variables distributed according to a probability measure  $\nu$  such that  $\int |x| d\nu(x) < \infty$ .*

*We assume that :*

(1) *the sequence  $a_n$  satisfies  $a_n > 0, n \in \mathbb{Z}^d$  and  $a_n(1 + |n|)^{2+\epsilon}$  is bounded.*

*or*

(2)  *$a_n = \eta, n \in \mathbb{Z}^d, \eta > 0$ .*

We consider the operators  $H^\omega$  as given in the equation (4.1.1) and the measures  $\mu_{L,E}^0, \mu_{L,E}^\omega$  given in equation (4.1.3) associated with the compressions of the operators  $\Delta, H^\omega$  to  $\Lambda_L$ .

**Theorem 4.2.2.** *Consider the self-adjoint operators  $H^\omega$  with  $V^\omega$  satisfying Hypothesis (4.2.1 (1)) with the measures  $\mu_{L,E}^\omega$  and  $\mu_{L,E}^0$  defined in equation (4.1.3) associated with  $E \in (-2d, 2d)$ . Then for  $d \geq 3$  we have*

$$\int_{\mathbb{R}} f(x) d\mu_L^\omega(x) - \int_{\mathbb{R}} f(x) d\mu_L^0(x) \xrightarrow{L \rightarrow \infty} 0 \quad a.e \omega \quad \forall f \in C_0^\infty(\mathbb{R}), \quad (4.2.1)$$

where  $C_0^\infty(\mathbb{R})$  denote the set of all infinitely differentiable functions on  $\mathbb{R}$  vanishing at  $\infty$ .

*Proof.* For simplicity we fix  $E \in (-2d, 2d)$  and drop the subscript  $E$  from the measures  $\mu_{L,E}^\omega, \mu_{L,E}^0$  below. Let  $f \in C_0^\infty(\mathbb{R})$  and consider the difference

$$\int_{\mathbb{R}} f(x) d\mu_L^\omega(x) - \int_{\mathbb{R}} f(x) d\mu_L^0(x).$$

Using the spectral theorem and the definitions of the measures  $\mu_L^0, \mu_L^\omega$  we can write the

above difference as

$$\begin{aligned} & \int_{\mathbb{R}} f(x) d\mu_L^0(x) - \int_{\mathbb{R}} f(x) d\mu_L^\omega(x) \\ &= \int \widehat{f}(\xi) \frac{1}{(2L+1)^{d-1}} \text{Tr} \left( e^{iH_L^0 \xi} - e^{iH_L^\omega \xi} \right) d\xi. \end{aligned} \quad (4.2.2)$$

We compute

$$\begin{aligned} & \text{Tr} \left( e^{iH_L^0 \xi} - e^{iH_L^\omega \xi} \right) \\ &= \text{Tr} \left( \chi_{\Lambda_L} \left( e^{iH_L^0 \xi} - e^{iH_L^\omega \xi} \right) \right) \\ &= \int_0^\xi \sum_{n \in \Lambda_L} \langle \delta_n, e^{iH_L^\omega(\xi-\eta)} i(H_L^\omega - H_L^0) e^{iH_L^0 \eta} \delta_n \rangle d\eta \\ &= \int_0^\xi \sum_{n, k \in \Lambda_L} \langle \delta_n, e^{iH_L^\omega(\xi-\eta)} \delta_k \rangle i((L+1)V^\omega(k)) \langle \delta_k, e^{iH_L^0 \eta} \delta_n \rangle d\eta. \end{aligned} \quad (4.2.3)$$

Therefore combining the above two equations, we estimate using Cauchy-Schwarz

$$\begin{aligned} & \left| \int_{\mathbb{R}} f(x) d\mu_L^0(x) - \int_{\mathbb{R}} f(x) d\mu_L^\omega(x) \right| \\ & \leq \frac{(L+1)}{(2L+1)^{d-1}} \int d\xi |(i+\xi)\widehat{f}(\xi)| \\ & \quad \times \frac{1}{|i+\xi|} \int_0^\xi d\eta \sum_{k \in \Lambda_L} |V^\omega(k)| \|e^{i(\xi-\eta)H_L^\omega} \delta_k\| \|e^{i\eta H_L^0} \delta_k\| \\ & \leq \frac{1}{(2L+1)^{d-2}} \sum_{n \in \Lambda_L} |V^\omega(n)| \int |(i+\xi)\widehat{f}| d\xi. \end{aligned} \quad (4.2.4)$$

We set

$$X_L(\omega, f) = \int_{\mathbb{R}} f(x) d\mu_L^\omega(x) - \int_{\mathbb{R}} f(x) d\mu_L^0(x). \quad (4.2.5)$$

Then from the above inequality we get the bound

$$|X_L(\omega, f)| \leq \|(i+\xi)\widehat{f}\|_1 \frac{1}{(2L+1)^{d-2}} \sum_{n \in \Lambda_L} a_n |q_n(\omega)|.$$

This estimate, the decay condition on  $a_n$  assumed in the hypothesis 4.2.1 and the fact

that for  $n$  in  $\Lambda_L$  we have  $(2L + 1)^{-d+2} \leq (1 + |n|)^{-d+2}$  together imply the estimates

$$\begin{aligned}
|X_L(\omega, f)| &= \|(i + \xi)\widehat{f}\|_1 \frac{1}{(2L + 1)^{(d-2)}} \sum_{n \in \Lambda_L} a_n |q_n(\omega)| \\
&\leq C(2L + 1)^{-\frac{\epsilon}{2}} \sum_{n \in \Lambda_L} (1 + |n|)^{-d-\frac{\epsilon}{2}} |q_n(\omega)| \\
&\leq C(2L + 1)^{-\frac{\epsilon}{2}} \sum_{n \in \Lambda_L} \frac{|q_n(\omega)| - \gamma}{(1 + |n|)^{d+\frac{\epsilon}{2}}} + C(2L + 1)^{-\frac{\epsilon}{2}} \sum_{n \in \mathbb{Z}^d} \frac{\gamma}{(1 + |n|)^{d+\frac{\epsilon}{2}}} \quad (4.2.6)
\end{aligned}$$

for each fixed  $L$  and almost every  $\omega$ . We define the random variables

$$M_L(\omega) = \sum_{n \in \Lambda_L} (1 + |n|)^{-d-\epsilon/2} (|q_n(\omega)| - \gamma), \quad \text{where } \gamma = \mathbb{E}|q_n(\omega)| = \int |x| d\nu(x).$$

Since  $|q_n(\omega)| - \gamma$  are i.i.d random variables with mean zero by hypothesis 4.2.1, we find that the conditional expectation of  $M_L$  given  $M_i, i = 1, \dots, L - 1$ , satisfies

$$\mathbb{E}(M_L(\omega) | M_0(\omega), \dots, M_{L-1}(\omega)) = M_{L-1}(\omega) + \mathbb{E} \left( \sum_{|n|=L} (|q_n(\omega)| - \gamma) \right) = M_{L-1}(\omega),$$

showing that  $M_L(\omega)$  is a martingale. Since

$$\sup_L \mathbb{E}(M_L(\omega)) < \infty,$$

the martingale convergence theorem (Theorem 5.7, Varadhan [60]) shows that  $M_L(\omega)$  converges almost everywhere to a random variable which is finite almost everywhere which implies that

$$L^{-\epsilon/2} M_L(\omega)$$

converges to zero almost everywhere. Using this fact in the estimate (4.2.6) we find that

$$|X_L(\omega, f)| \rightarrow 0 \quad \text{as } L \rightarrow \infty \text{ a.e } \omega. \quad (4.2.7)$$

The above is valid for any  $f \in C_0^\infty(\mathbb{R})$ , since for functions  $f$  in this class  $\|(i + \xi)\widehat{f}\|_1$  is

finite. Now (4.2.7) together with (4.2.5) give (4.2.1).  $\square$

We now consider the case of weakly coupled random potentials and find the scales on which the statistics is similar to that of the free part as the coupling constant goes to zero. Let  $\epsilon(\eta)$  be a function of  $\eta$  such that

$$\epsilon(\eta) \rightarrow \infty \text{ if } \eta \rightarrow 0 \text{ and } \lim_{\eta \rightarrow 0} \epsilon(\eta)^2 \eta = 0.$$

**Theorem 4.2.3.** *Consider the self-adjoint operators  $H^\omega$  with  $V^\omega$  satisfying Hypothesis (4.2.1(2)), with coupling constant  $\eta$ . Consider the measures  $\mu_{L,E}^\omega$  and  $\mu_{L,E}^0$  defined in equation (4.1.3) associated with  $E \in (-2d, 2d)$ . Then for  $d \geq 1$ , the sequences of measures  $\{\mu_{\epsilon(\eta),E}^\omega\}$  and  $\{\mu_{\epsilon(\eta),E}^0\}$  have the same limit points almost everywhere in the sense of distributions as  $\eta \rightarrow 0$ .*

*Proof.* The proof is essentially the same as the proof of Theorem 4.2.2. In the present case, the first step in the inequality (4.2.6) becomes,

$$\begin{aligned} |X_{\epsilon(\eta)}(\omega, f)| &\leq \|(1 + \xi)\widehat{f}\|_1 \epsilon(\eta)^{-d+2} \sum_{n \in \Lambda_{\epsilon(\eta)}} \eta |q_n(\omega)| \\ &\leq \|(1 + \xi)\widehat{f}\|_1 \epsilon(\eta)^2 \eta \left( \epsilon(\eta)^{-d} \sum_{n \in \Lambda_{\epsilon(\eta)}} |q_n(\omega)| \right), \end{aligned} \tag{4.2.8}$$

after which the proof is similar to the one given in the proof of Theorem (4.2.2) making use of the fact that  $\epsilon(\eta)^2 \eta \rightarrow 0$  as  $\eta \rightarrow 0$ .  $\square$

### 4.3 Eigenvalues and eigenvectors of $\Delta_L$

In this section, we study the eigenvalues of  $\Delta_L$  and show that for energies at the edges of the band  $(-2d, 2d)$  there are limit points for the distributions  $\Psi_{L,E}^0$  associated with the measures  $\mu_{L,E}^0$ .

The eigenvalues  $\lambda_{j_1, \dots, j_d}^L$  and the (un-normalized) eigenfunctions  $\Psi_{j_1, \dots, j_d, L}$  of  $\Delta_L$  are given

by (with the superscript for  $\lambda$  denoting an index and not a power)

$$\begin{aligned}\lambda_{j_1, \dots, j_d}^L &= 2 \sum_{\ell=1}^d \cos(\theta_{j_\ell, L}), \quad \theta_{j, L} = \frac{j\pi}{2(L+1)}, \\ \Psi_{j_1, \dots, j_d, L}(n) &= \prod_{\ell=1}^d \phi_{j_\ell, L}(n_\ell), \quad n = (n_1, \dots, n_d) \in \Lambda_L, \\ \phi_{j, L}(m) &= \begin{cases} \cos(\theta_{j, L} m), & \text{if } j \text{ is odd,} \\ \sin(\theta_{j, L} m), & \text{if } j \text{ is even,} \end{cases} \quad , \quad m \in \{-L, \dots, L\},\end{aligned}\tag{4.3.1}$$

where  $j_\ell \in \{1, 2, \dots, 2L+1\}$ ,  $\ell = 1, \dots, d$ .

The eigenvalues of  $\Delta_{L, E}$  are correspondingly  $\{\lambda_{j_1, \dots, j_d}^L - E\}$  for  $E \in [-2d, 2d]$ .

We start with a lemma on the multiplicities of the eigenvalues.

**Lemma 4.3.1.** *Let  $E_{\Delta_L}$  denote the projection valued measure associated with  $\Delta_L$ . Then for any  $\lambda \in \mathbb{R}$ ,*

$$\text{Tr}(E_{\Delta_L}(\{\lambda\})) \leq d(2L+1)^{d-1}.$$

*Proof.* If  $\lambda$  is not an eigenvalue of  $\Delta_L$ ,  $E_{\Delta_L}(\{\lambda\}) = 0$  and the bound is trivial. so we assume that  $\lambda \in \sigma(\Delta_L)$ . The statement in the lemma follows if we show that the eigenvalues of  $\Delta_L$  have multiplicity at most the bound given in the lemma. Let

$$S = \left\{ 2 \cos\left(\frac{k\pi}{2(L+1)}\right) : k \in \{1, \dots, 2L+1\} \right\}.$$

The points of  $S$  are distinct and so  $S$  has cardinality  $2L+1$  and the map

$$f(x_1, \dots, x_d) = x_1 + x_2 + \dots + x_d$$

from  $S^d$  to  $[-2d, 2d]$  gives precisely all the eigenvalues of  $\Delta_L$ . Clearly the equation  $f(x_1, \dots, x_d) = \lambda$  allows the free choice of at most  $d-1$  of the variables  $x_j$ . If we fix  $x_1$  then the number of choices of the remaining variables is at most  $(2L+1)^{d-1}$ . Since we can fix any one of the  $d$  variables  $x_j$  the bound stated in the lemma follows.  $\square$



**Remark 4.3.2.** Since scaling the matrix  $\Delta_L$  or adding a constant multiple of the identity matrix to it does not change the multiplicities of eigenvalues, the above lemma implies that

$$\text{Tr}(E_{L(\Delta_L - E)}(\{\lambda\})) \leq d(2L + 1)^{d-1}.$$

for any  $\lambda \in \mathbb{R}$ .

**Lemma 4.3.3.** Let  $d \geq 1$  and  $E \in (-2d, 2d)$ ,  $2d - 2 < |E| < 2d$ , then for any  $f \in C_0^\infty(\mathbb{R})$ , we have

$$\sup_{L \in \mathbb{N}} \int f(x) d\mu_{L,E}^0(x) < \infty.$$

*Proof.* We give the proof only for the case  $2d - 2 < E < 2d$ , the proof for the  $-2d < E < -2d + 2$  is similar. Let  $f \in C_0^\infty(\mathbb{R})$  have support in  $[-K, K]$ . Let  $\Lambda_L^r$  be a cube of side length  $L$  in  $\mathbb{Z}^r$ ,  $r \in \{0, 1, 2, \dots, d-1\}$ , take  $\Delta_L^0 = 0$  and set

$$(\Delta^r u)(n) = \sum_{|n-i|=1} u(n+i), \quad u \in \ell^2(\mathbb{Z}^r), \quad \Delta_L^r = \chi_{\Lambda_{L,r}} \Delta_r \chi_{\Lambda_{L,r}}.$$

Then

$$\begin{aligned} & \int f(x) d\mu_{L,E}^0(x) \\ &= \frac{1}{(2L+1)^{(d-1)}} \sum_{k=1}^{2L+1} \sum_{\lambda \in \sigma(\Delta_L^{d-1})} f((L+1)(2 \cos(\theta_{k,L}) + \lambda - E)). \end{aligned} \quad (4.3.2)$$

The support of  $f$  is in  $[-K, K]$ , so the above sum is only over  $k$  such that  $(L+1)(2 \cos(\theta_{k,L}) + \lambda - E) \in [-K, K]$ . Therefore setting

$$J_{\lambda,E,L} = \left[ \frac{E-\lambda}{2} - \frac{K}{2(L+1)}, \frac{E-\lambda}{2} + \frac{K}{2(L+1)} \right], \quad V_{L,r} = (2L+1)^{-r} \quad (4.3.3)$$

we have

$$\begin{aligned}
& \left| \int f(x) d\mu_{L,E}^0(x) \right| \\
& \leq \|f\|_\infty V_{L,d-1} \sum_{\lambda \in \sigma(\Delta_L^{d-1})} \# \left\{ k \in \frac{2(L+1)}{\pi} \arccos(J_{\lambda,E,L} \cap [-1,1]) \right\}, \tag{4.3.4}
\end{aligned}$$

where

$$\arccos(S) = \{\arccos(x) : x \in S\}.$$

Letting  $|(a, b)| = (b - a)$ , noting that the number of integers in  $(a, b)$  is at most  $(b - a) + 1$  and using the monotonicity of  $\arccos$  in  $[-1, 1]$ , the inequality (4.3.4) becomes

$$\begin{aligned}
& \left| \int f(x) d\mu_{L,E}^0(x) \right| \\
& \leq \|f\|_\infty V_{L,d-1} \sum_{\lambda \in \sigma(\Delta_L^{d-1}), |\frac{E-\lambda}{2} \pm \frac{K}{2(L+1)}| \leq 1} 1 + \\
& \frac{2(L+1)}{\pi} \left( \arccos\left(\frac{E-\lambda}{2} - \frac{K}{2(L+1)}\right) - \arccos\left(\frac{E-\lambda}{2} + \frac{K}{2(L+1)}\right) \right) \tag{4.3.5} \\
& \leq \|f\|_\infty + \|f\|_\infty V_{L,d-1} \\
& \times \sum_{\lambda \in \sigma(\Delta_L^{d-1}), |\frac{E-\lambda}{2} \pm \frac{K}{2(L+1)}| \leq 1} \frac{2(L+1)}{\pi} \left( \frac{2K}{2(L+1)} \frac{1}{\sqrt{1 - \left(\frac{E-\lambda}{2} + x_L\right)^2}} \right),
\end{aligned}$$

where we have used the mean value theorem in the last step for writing the differences of the  $\arccos$  terms, which is justified since  $|\frac{E-\lambda}{2} \pm \frac{K}{2(L+1)}| \leq 1$ . By the mean value theorem it also follows that  $|x_L| < \frac{K}{2(L+1)}$  and  $|\frac{E-\lambda}{2} + x_L| < 1$ . If  $d = 1$ , the proof is over at this stage since for large  $L$ , the right hand side is bounded for any  $|E| < 1$ . Therefore from now on we assume that  $d \geq 2$ . Simplifying the above inequality by majorizing it by twice the second term, which we can do, otherwise the proof would be complete, we get

$$\begin{aligned}
& \left| \int f(x) d\mu_{L,E}^0(x) \right| \\
& \leq \|f\|_\infty + \|f\|_\infty V_{L,d-1} \sum_{\lambda \in \sigma(\Delta_L^{d-1}), |\frac{E-\lambda}{2} \pm \frac{K}{2(L+1)}| \leq 1} \frac{2K}{\pi} \left( \frac{1}{\sqrt{1 - \left(\frac{E-\lambda}{2} + x_L\right)^2}} \right), \tag{4.3.6}
\end{aligned}$$

The above term is uniformly bounded in  $L$  if  $(\frac{E-\lambda}{2} + x_L)^2 \leq \frac{1}{2}$ . So we assume that  $(\frac{E-\lambda}{2} + x_L)^2 \geq \frac{1}{2}$  and in that case the sum over  $\lambda$  splits into two parts, according as  $\pm(\frac{E-\lambda}{2} + x_L) > \frac{1}{2}$ . Therefore we set

$$I_{\pm} = \|f\|_{\infty} V_{L,d-1} \sum_{\lambda \in \sigma(\Delta_L^{d-1}), \pm \frac{E-\lambda}{2} \pm \frac{K}{2(L+1)} \geq \frac{1}{2}} \frac{2K}{\pi} \left( \frac{1}{\sqrt{1 - (\frac{E-\lambda}{2} + x_L)^2}} \right).$$

We continue with the proof for  $I_+$  the proof of the other case is similar. We have

$$\frac{1}{\sqrt{1 - (\frac{E-\lambda}{2} + x_L)^2}} \leq \frac{1}{\sqrt{1 - \frac{E-\lambda}{2} - x_L} \sqrt{1 + \frac{E-\lambda}{2} + x_L}} \leq \frac{1}{\sqrt{1 - \frac{E-\lambda}{2} - x_L}}.$$

Using this bound we find

$$I_+ \leq \|f\|_{\infty} V_{L,d-1} \sum_{\lambda \in \sigma(\Delta_L^{d-1}), \frac{E-\lambda}{2} + \frac{K}{2(L+1)} \geq \frac{1}{2}} \frac{2K}{\pi} \left( \frac{1}{\sqrt{1 - \frac{E-\lambda}{2} - x_L}} \right), \quad (4.3.7)$$

We now use the fact that  $\lambda \in \sigma(\Delta_L^{d-1})$  can be split into  $\lambda = \lambda_1 + \lambda_2$ , where  $\lambda_2 \in \sigma(\Delta_L^1)$  and  $\lambda_1 \in \sigma(\Delta_L^{d-2})$ . Then the above inequality becomes

$$\begin{aligned} I_+ &\leq \frac{2K \|f\|_{\infty}}{\pi} V_{L,d-2} \sum_{\lambda_1 \in \sigma(\Delta_L^{d-2})} \\ &\times \frac{1}{(2L+1)} \sum_{\lambda_2 \in \sigma(\Delta_L^1), E - \lambda_1 - \lambda_2 - 2x_L > 0} \left( \frac{1}{\sqrt{1 - \frac{E - \lambda_1 - \lambda_2}{2} - x_L}} \right), \end{aligned} \quad (4.3.8)$$

We claim that the sum

$$I(\gamma) = \frac{1}{(2L+1)} \sum_{\lambda_2 \in \sigma(\Delta_L^1), \lambda_2 < 2\gamma} \left( \frac{1}{\sqrt{\gamma + \frac{\lambda_2}{2}}} \right)$$

where  $\gamma = 1 - \frac{E-\lambda_1}{2}$ , is uniformly bounded in  $\gamma$  and  $L$ . If the claim is true then we get

the bound

$$I_+ \leq \frac{2K\|f\|_\infty}{\pi} V_{L,d-2} \sum_{\lambda_1 \in \sigma(\Delta_L^{d-2})} C < \frac{2K\|f\|_\infty}{\pi} C, \quad (4.3.9)$$

giving the lemma. We therefore prove the claim. Using the explicit expressions for the points in  $\sigma(\Delta_L^1)$  we computed earlier in equation (4.3.1), we get

$$I(\gamma) = \frac{1}{(2L+1)} \sum_{k=1, \gamma > \cos(\frac{k\pi}{2(L+1)})}^{2L+1} \left( \frac{1}{\sqrt{\gamma - \cos \frac{k\pi}{2(L+1)}}} \right)$$

Since the function

$$g(x) = \frac{1}{\sqrt{\gamma - \cos(x\pi)}}$$

is monotonically decreasing in  $0 \leq x \leq 1$  we bound the sum above by the integral

$$I(\gamma) \leq \delta_L + \left( \frac{2(L+1)}{2L+1} \right) \int_0^1 g(x) \chi_{(([-1,\gamma])}(\cos(x\pi)) dx.$$

where  $\delta_L$  is a small error that is uniformly bounded in  $L$ . Changing variables  $y = \cos(x\pi)$  gives the bound

$$\begin{aligned} I(\gamma) &\leq \delta_L + \frac{2}{\pi} \int_{-1}^{\gamma} g(y) \frac{1}{\sqrt{1-y^2}} dy \\ &\leq \delta_L + \frac{2}{\pi} \int_{-1}^{\gamma} \frac{1}{\sqrt{\gamma-y}} \frac{1}{\sqrt{1-y^2}} dy \\ &\leq \delta_L + \frac{1}{\sqrt{1-\gamma}} \int_{-1}^{\gamma} \frac{1}{\sqrt{\gamma-y}} \frac{1}{\sqrt{1+y}} dy. \end{aligned} \quad (4.3.10)$$

The condition on  $E$  assures us that  $\gamma < 0$ , therefore the factor  $\frac{1}{\sqrt{1-\gamma}}$  is bounded by 1, on the other hand a bound by splitting the integral into two pieces up to and from the midpoint  $(1 + |\gamma|)/2$  yields

$$\int_{|\gamma|}^1 \frac{1}{\sqrt{(y-|\gamma|)(1-y)}} dy \leq 2.$$

Note: In case  $|\gamma| = 1$  we define  $I(\gamma)$  to be

$$I(\gamma) = \lim_{\epsilon \downarrow} I(\gamma, \epsilon)$$

where

$$I(\gamma, \epsilon) = \frac{1}{(2L+1)} \sum_{\lambda_2 \in \sigma(\Delta_L^1), |\lambda_2| < 2-\epsilon} \left( \frac{1}{\sqrt{\gamma + \frac{\lambda_2}{2}}} \right)$$

and bound  $I(\gamma, \epsilon)$  for each  $\epsilon > 0$ , which we can do since all the terms are finite for each  $\epsilon > 0$ . This avoids the logarithmic singularity in the integral when we replace the sum defining  $I(\gamma)$  by an integral.  $\square$

**Proposition 4.3.4.** *The measures  $\mu_{L,E}^0$  have limit points in the vague sense when  $2d-2 < |E| < 2d$ .*

*Proof.* By the lemma above the measures  $\mu_{L,E}^0$  are uniformly bounded on the space of continuous functions of compact support. Hence by Helly's selection theorem they have limit points in the vague sense (by a diagonal argument if necessary). To show that there is at least one non-zero limit point we show that for some positive function of compact support,

$$\liminf_{L \in \mathbb{N}} \int f(x) d\mu_{L,E}^0(x) > 0.$$

To this end consider a  $K > 1$  fixed and let  $0 \leq f \leq 1$  be a continuous function with  $f(x) = 1, -K \leq x \leq K$ . Then we see from equations (4.3.2), (4.3.3) that

$$\int f(x) d\mu_{L,E}^0(x) \geq \sum_{\lambda \in \sigma(\Delta_L^{d-1})} \# \left\{ k \in \frac{2(L+1)}{\pi} \arccos(J_{\lambda,E,L} \cap [-1, 1]) \right\}.$$

As estimated in equation (4.3.4) we estimate the number of integers  $k$  by the Lebesgue measure of the interval, but now taking a smaller interval  $[\frac{E-\lambda}{2} - \frac{K}{4(L+1)}, \frac{E-\lambda}{2} + \frac{K}{4(L+1)}]$  to

get as in equation (4.3.5) (now for lower bound)

$$\begin{aligned} & \int f(x) d\mu_{L,E}^0(x) \\ & \geq \sum_{\lambda \in \sigma(\Delta_L^{d-1})} \frac{2(L+1)}{\pi} \left( \arccos\left(\frac{E-\lambda}{2} - \frac{K}{4(L+1)}\right) - \arccos\left(\frac{E-\lambda}{2} + \frac{K}{4(L+1)}\right) \right) \end{aligned} \quad (4.3.11)$$

using the monotonicity of arccos in  $[-1, 1]$ . For some  $0 < \delta < 1$ , we take  $L$  large so that  $\frac{K}{4(L+1)} < \delta/4$ , hence using the mean value theorem we get the lower bound

$$\begin{aligned} & \frac{2(L+1)}{\pi} \left( \arccos\left(\frac{E-\lambda}{2} - \frac{K}{4(L+1)}\right) - \arccos\left(\frac{E-\lambda}{2} + \frac{K}{4(L+1)}\right) \right) \\ & = \frac{2(L+1)}{\pi} \frac{K}{2(L+1)} \frac{1}{\sqrt{1 - \left(\frac{E-\lambda}{2} + x_L\right)^2}} = \frac{K}{\pi} \frac{1}{\sqrt{1 - \left(\frac{E-\lambda}{2} + x_L\right)^2}}. \end{aligned} \quad (4.3.12)$$

Therefore from equation (4.3.11) and the above we get since  $|x_L| < \delta/4$  for large enough  $L$ ,

$$\int f(x) d\mu_{L,E}^0(x) \geq \frac{K}{\pi(2L+1)^{d-1}} \sum_{\lambda \in \sigma(\Delta_L^{d-1})} \sum_{\frac{|E-\lambda|}{2} \leq 1-\delta/2} \sqrt{\frac{1}{2}}.$$

The right hand side clearly has a limit in terms of the density of states of  $\Delta^{d-1}$  namely

$$\frac{K}{\pi\sqrt{2}} \mathcal{N}_{d-1}((E-2+\delta, E+2-\delta))$$

where  $\mathcal{N}_r$  is the density of states of  $\Delta^r$ . For  $|E| \in (2d-2, 2d)$ ,  $(E-2+\delta, E+2-\delta) \cap (-2d+2, 2d-2) \neq \emptyset$  for small enough  $\delta$  showing the positivity of the right hand side.  $\square$

We end this chapter with a conjecture,

**Conjecture 4.3.5.** *If  $d \geq 4$ ,  $E \in (-2d, 2d)$ , the limit points of  $\mu_{L,E}^0$  are given by*

$$\sum_{k \in \mathbb{Z}} \int \sin(\theta) n_{d-1}(E - 2 \cos(\theta)) \delta_{\pi k \sin(\theta)}(\cdot) d\theta,$$

where  $n_d$  is density of states of  $\Delta$  in  $d$  dimensions.

## Chapter 5

### Poisson Statistics for Anderson

### Model with Singular Randomness

## 5.1 Introduction

In this chapter we consider the Anderson model on the  $d$ -dimensional lattice with the single site potential having singular distribution, mainly  $\alpha$ -Hölder continuous ones and show that the eigenvalue statistics is Poisson in the region of exponential localization.

We consider the Anderson model, namely the operators

$$H^\omega = H_0 + \sum_{j \in \mathbb{Z}^d} \omega_j P_j, \quad \omega \in \Omega, \quad (5.1.1)$$

on  $\ell^2(\mathbb{Z}^d)$  with  $P_j$  the orthogonal projection onto  $\ell^2(\{j\})$ . We take  $H_0$  to commute with translations on  $\mathbb{Z}^d$ . Typically we have  $H_0 = \Delta$ , which is the discrete Laplacian with diagonal part dropped.  $H_0 = 0$  is also included in the model.

We consider a cube  $\Lambda_L$  of side length  $L$  and cover the cube with smaller disjoint cubes  $C_p$  of side length  $l_L$ , so that  $\Lambda_L = \cup_{p=1}^{N_L^d} C_p$ . Given these we consider the matrices  $H_{\Lambda_L}^\omega$  and  $H_{C_p}^\omega$  obtained by compressing the operator  $H^\omega$  to the finite dimensional subspaces  $\ell^2(\Lambda_L)$  and  $\ell^2(C_p)$  respectively.

An explicit collection of such cubes  $C_p$  is obtained by dividing  $(-L-1, L]^d$  into  $N_L^d$  equal cubes  $C_p^*$  of the form  $(c, d]^d$  for  $p = 1, 2, \dots, N_L^d$ , with side length  $\frac{2L+1}{N_L}$  and defining

$$C_p = C_p^* \cap \mathbb{Z}^d, \quad \text{int}(C_p) = \{x \in C_p : \text{dist}(x, \partial C_p) > l_L\}, \quad (5.1.2)$$

where  $\{N_L\}_L$  and  $\{l_L\}_L$  are both increasing sequences of integers, which will be specified later. For any cube  $B \subset \mathbb{Z}^d$  the boundary of  $B$  is denoted by  $\partial B$  and is defined by

$$\partial B = \{x \in B : \exists x' \in B^c \text{ such that } |x - x'| = 1\}.$$

**Hypothesis 5.1.1.** *We assume that the single site distribution  $\mu$  is uniformly  $\alpha$ -Hölder continuous for  $0 < \alpha \leq 1$ .*

We look in the region of exponential localization to study eigenvalue statistics. In this



context by *exponential localization* we mean that some appropriate fractional moments of the Green functions associated with the operators are exponentially decaying as shown by Aizenman-Molchanov [5] (see also [8]) in the strong disorder case, namely when  $\mu$  in Hypothesis 5.1.1 has large support and large variation. For simplicity we assume complete localization and state their results as a hypothesis for the present work.

**Hypothesis 5.1.2.** *Let  $\Lambda \subseteq \mathbb{Z}^d$  be any large cube, then the inequality*

$$\sup_{z \in \mathbb{C}^+, \operatorname{Re}(z) \in [a, b]} \mathbb{E} \left( |\langle \delta_n, (H_\Lambda^\omega - z)^{-1} \delta_m \rangle|^s \right) \leq C e^{-\gamma |n-m|},$$

*is valid for some  $0 < s < \alpha \leq 1$  and  $0 < \gamma < \infty$ .*

Henceforth the energy  $E \in \sigma(H^\omega)$  appearing in all the quantities below is assumed to lie in  $[a, b]$  occurring in the above hypothesis.

Using the constant  $\gamma$  in the above hypothesis, we specify the numbers  $N_L, l_L$  of equation (5.1.2). We fix an  $\epsilon$ ,  $0 < \epsilon < 1$  and choose

$$N_L = O(L^{1-\epsilon}), \quad l_L = \frac{5d}{\alpha\gamma} \ln(2L + 1). \quad (5.1.3)$$

Given the operators in equation (5.1.1) satisfying the above hypotheses and an  $\alpha$ ,  $0 < \alpha \leq 1$ , we define the following random measures on  $\mathbb{R}$ , where we set  $\beta_L = (2L + 1)^{\frac{d}{\alpha}}$ .

$$\begin{aligned} \xi_{L,E}^\omega(I) &= \operatorname{Tr}(\chi_{\Lambda_L} E_{\beta_L(H_L^\omega - E)}(I)) \\ \eta_{p,E}^\omega(I) &= \operatorname{Tr}(\chi_{\Lambda_L} E_{\beta_L(H_{C_p}^\omega - E)}(I)), \quad p = 1, \dots, N_L^d \\ \zeta_{L,E}^\omega(I) &= \operatorname{Tr}(\chi_{\Lambda_L} E_{\beta_L(H^\omega - E)}(I)). \end{aligned} \quad (5.1.4)$$

The eigenvalue statistics was studied for random Schrödinger operators by Molchanov [48] followed by Minami [47], who obtained an estimate for ensuring that the Lévy measure of the eigenvalue point process is degenerate. Eigenvalue statistics was studied by Germinet-Klopp [25], Aizenman-Warzel [7] for the canopy graph. Recently Poisson statistics for regular graphs was obtained by Geisinger [26]. Combes-Germinet-Klein [13] obtained the

Minami estimate in a more transparent form while extending the original estimate to more general single site distributions and obtaining estimates on probabilities associated with existence of multiple eigenvalues in an interval. The statistics results were extended to include localization centers by Nakano-Killip [40] and for the Schrödinger case Nakano [50] showed infinite divisibility of the limiting point processes. In the case of Anderson model with higher rank random potentials Tautenhan-Veselić [59] obtained the Minami estimate leading to the Poisson statistics. Recently Hislop-Krishna [30] considered higher rank random potentials for Anderson models and showed the eigenvalue statistics to be compound Poisson. They also showed that in all the cases of random Schrödinger and Anderson models with higher rank i.i.d random potentials, the Wegner estimate and complete localization leads to a compound Poisson eigenvalue statistics in general.

Level repulsion was proved for a class of Anderson models with decaying randomness by Dolai-Krishna [18], for a class of Schrödinger operators in one dimension Kotani-Nakano [41] obtained  $\beta$ -ensemble governing the statistics and for localization centers level repulsion was shown in the Anderson model by Nakano [49].

All these works, except that of Combes-Germinet-Klein [13], assumed that the single site distributions have an absolutely continuous bounded density, which amounts to taking  $\alpha = 1$ .

We show here that by changing the scale appropriately we can include more singular single site distributions. However this comes at a price. In view of the subtleties involved with singular measures, in particular the absence of a de la Vallée Poussin type theorem, the results become weaker.

It is well known that once we have the Wegner estimate, limit points of the point processes in equation (5.1.4) are also point processes. We state this fact as a theorem below. The proof involves showing tightness of the family of measures  $\xi_{L,E}^\omega$  and proof is given as part of Hislop-Krishna [30, Proposition 4.1].

**Theorem 5.1.3.** *Consider  $H^\omega$  as in equation (5.1.1) satisfying Hypothesis 5.1.1 with  $0 < \alpha \leq 1$ . Then every limit point, in the sense of distributions, of  $\xi_{L,E}^\omega$ , is a point*

process.

The main question is then to determine the nature of the limiting point processes which we do below for  $\alpha$ -Hölder continuous measures.

To the best of our knowledge ours is the first instance where singular single site distributions are allowed to obtain eigenvalue statistics.

We use the same symbol  $\mathcal{N}$  for the IDS and the measure associated with it, and it should be clear from the context the object used.

We define the  $\alpha$ -derivative and the  $\alpha$ -upper derivatives of the integrated density of states  $\mathcal{N}$  of our model, by

$$d_{\mathcal{N}}^{\alpha}(E) = \lim_{\epsilon \rightarrow 0} \frac{\mathcal{N}((E - \epsilon, E + \epsilon))}{(2\epsilon)^{\alpha}}, D_{\mathcal{N}}^{\alpha}(E) = \limsup_{\epsilon \rightarrow 0} \frac{\mathcal{N}((E - \epsilon, E + \epsilon))}{(2\epsilon)^{\alpha}}.$$

We define the measures  $\mathcal{L}_{\alpha}$  as

$$\mathcal{L}_{\alpha}(I) = \alpha 2^{\alpha-1} \int_I |y|^{\alpha-1} dy \tag{5.1.5}$$

for any bounded Borel subset  $I \subset \mathbb{R}$ .

Our main theorem is that the limiting point processes give Poisson distributions for a class of intervals. For technical reasons, that have to do with the fact that we are dealing with singular measures, we consider only intervals symmetric about the origin below to obtain the parameters of the limiting Poisson distributions. In view of the fact that for singular measures  $\nu$ ,  $d_{\nu}^{\alpha}(x)$  may not exist for almost all  $x$  w.r.t.  $\nu$ , we have to deal with upper derivatives in which case we can only talk about limit points of the random measures we considered above.

**Theorem 5.1.4.** *Consider  $H^{\omega}$  as in equation (5.1.1) satisfying Hypotheses 5.1.1 with  $0 < \alpha \leq 1$  and Hypothesis 5.1.2 with  $E$  in the region of localization. For any bounded*

open interval  $I$ , suppose  $\gamma_{E,I}$  is non-zero such that

$$\gamma_{E,I} = \lim_{n \rightarrow \infty} \mathbb{E}(\xi_{L_n(I),E}^\omega(I)).$$

Then the random variables  $\xi_{L_n(I),E}^\omega(I)$  converge in distribution to the Poisson random variable with parameter  $\gamma_{E,I}$ .

This theorem implies the following.

**Corollary 5.1.5.** *With the assumptions of Theorem 5.1.4, if  $0 < D_{\mathcal{N}}^\alpha(E) < \infty$ , then for each bounded open interval  $I = -I$ , there is a subsequence  $L_n(I)$  such that  $\xi_{L_n(I),E}^\omega(I)$  converges in distribution to a Poisson random variable with parameter  $D_{\mathcal{N}}^\alpha(E)\mathcal{L}_\alpha(I)$ .*

It is interesting to note that the measures  $\mathcal{L}_\alpha$  occur in the theorem of Jensen-Krishna, [16, Theorem 1.3.2] dealing with continuous wavelet transforms of measures, where the constants  $c_\alpha$  are integrals of the function  $\psi$ , the function that generates the “continuous wavelet”, with respect to  $\mathcal{L}_\alpha$ .

Since the limsup of a sequence is always a limit point of the sequence, the above theorem shows that when the upper derivative  $D_{\mathcal{N}}^\alpha(E)$  is positive, there is at least one subsequence of  $\xi_{L,E}^\omega(I)$  that converges in distribution to a Poisson random variable, the parameter of the corresponding Poisson distribution is then  $D_{\mathcal{N}}^\alpha(E)\mathcal{L}_\alpha(I)$ .

It was shown by Krishna [39, Corollary 1.7], Combes-Germinet-Klein [13], (and by Combes-Hislop-Klopp [12], Stollmann [55] for continuous models), that when the single site distribution is uniformly  $\alpha$ -Hölder continuous (Hypothesis 5.1.1), the IDS,  $\mathcal{N}$  is uniformly  $\alpha$ -Hölder continuous. When  $\mathcal{N}$  is  $\alpha$ -Hölder continuous we can use the decomposition of Theorem 69, Rogers [54], with  $h(x) = |x|^\alpha$ , to obtain for any bounded Borel subset  $E \subset \mathbb{R}$ ,

$$\mathcal{N}(E) = \int_E f(t) d\mu^h(x) + J(E),$$

with  $J$  being strongly  $h$  continuous. Accordingly if this  $f$  is non-zero a.e.  $\mu^h$  then we will have non-zero  $D_{\mathcal{N}}^\alpha(E)$  for those  $E$  for which  $f(E)$  is non-zero.

However since the theorems of [39],[12], may not be optimal, in the sense that the  $\mathcal{N}$  may have better modulus of continuity in some part of its support, (as seen in Kaminaga-Nakamura-Krishna [34], where the IDS is analytic in some region of the spectrum even for measures  $\mu$  with singular component in them) even when  $\mu$  is only  $\alpha$ -Hölder continuous, we cannot be sure that the  $f$  is indeed non-zero for the given  $E$ .

We finally note that the subsequences of Theorem 5.1.3 and those in Corollary 5.1.5 may be different.

## 5.2 Ideas of Proofs

The strategy of proof Theorem 5.1.4 is to first follow the procedure adopted by Minami [47] where one first shows that three classes of random measures considered above are asymptotically essentially the same in the sense that if one of the limits below exists then it does for all and they are all the same. A similar statement holds for any subsequence also.

**Proposition 5.2.1.** *Consider the processes defined in equation (5.1.4) associated with the operators  $H^\omega$  satisfying Hypotheses 5.1.1,5.1.2. Then for any bounded interval  $I \subset \mathbb{R}$  we have*

$$\lim_{L \rightarrow \infty} \zeta_{L,E}^\omega(I) = \lim_{L \rightarrow \infty} \xi_{L,E}^\omega(I) = \lim_{L \rightarrow \infty} \sum_p^{N_L^d} \eta_{p,E}^\omega(I). \quad (5.2.1)$$

The limits above are in the sense of convergence in distributions and the Proposition follows from Theorem 5.3.3 as in Minami's paper [47] and we omit the proof. An immediate Corollary of the above Proposition and Lemma 5.3.1 is the following.

**Corollary 5.2.2.** *For any bounded interval  $I \subset \mathbb{R}$  we have*

$$\lim_{L \rightarrow \infty} \mathbb{E}(\zeta_{L,E}^\omega(I)) = \lim_{L \rightarrow \infty} \mathbb{E}(\xi_{L,E}^\omega(I)) = \lim_{L \rightarrow \infty} \mathbb{E}\left(\sum_p^{N_L^d} \eta_{p,E}^\omega(I)\right). \quad (5.2.2)$$

Once these results are established, our strategy is to use the the celebrated Lévy-Khintchine

representation theorem for measures. The Lévy-Khintchine theorem (see Theorem 1.2.1, Applebaum [1]) says that a measure  $\nu$  is infinitely divisible if and only if its characteristic function  $\widehat{\nu}(t)$  is of the form

$$e^{iat+bt^2+\int_{|x|\leq c}(e^{itx}-1-itx)dM(x)}$$

for some  $\sigma$ -finite measure  $M$ , which is called the Lévy measure associated with  $\nu$ . In the case the measure  $M$  is finite we can absorb the linear term into the number  $a$  and rewrite this expression in the form

$$e^{iat+bt^2+\int_{|x|\leq c}(e^{itx}-1)dM(x)}$$

It turns out that a distribution is Poisson iff  $a = b = 0$  and  $M$  is supported on  $\{1\}$  (notationally such a measure is written as a positive multiple of  $\delta(x - 1)dx$  by some authors). The parameter of the Poisson distribution is then the number  $M(\{1\})$ .

We emphasize here that to show Proposition 5.2.1 it is sufficient to have exponential localization (in the sense of Aizenman-Molchanov [5]) and the Wegner estimate for the IDS  $\mathcal{N}$ . The result that  $M(\mathbb{R} \setminus \{1\}) = 0$  uses the Minami estimate.

Therefore the idea is to compute the Fourier transforms of the random variables

$$\sum_p \eta_{p,E}^\omega(I)$$

which are a sum of i.i.d random variables and show that the limit of the Fourier transform has the desired form.

In view of the Corollary 5.2.2, the value of the parameter of the Poisson distribution is computed using the fact that the parameter is the expectation of the Poisson distribution which in this case is obtained as the limit of  $\mathbb{E}(\zeta_{L,E}^\omega(I))$  either for the whole sequence or if the limit does not exist for some subsequences for which it does.

In the context of absolutely continuous single site distributions  $\mu$  these limits exist at points in the spectrum where the density of states exists. In our context where we are

dealing with singular single site distributions which have no density with respect to the Lebesgue measure we need to consider derivatives or upper derivatives with respect to Hausdorff measures to obtain these limit points.

### 5.3 Wegner and Minami Estimates

It was shown by Krishna [39, Corollary 1.7] that if the single site distribution  $\mu$  is uniformly  $\alpha$ -Hölder,  $0 < \alpha \leq 1$  continuous, then the integrated density of states (IDS) is also uniformly  $\alpha$ -Hölder continuous.

We state this fact in the form given by Combes-Germinet-Klein [13]. Given a probability measure  $\mu$  let  $S_\mu(s) := \sup_{a \in \mathbb{R}} \mu[a, a + s]$ . Define

$$Q_\mu(s) := \begin{cases} \|\rho\|_\infty s & \text{if } \mu \text{ has bounded density } \rho, \\ 8S_\mu(s) & \text{otherwise.} \end{cases}$$

If  $\mu$  is Hölder continuous with exponent  $\alpha \in (0, 1]$  then  $Q_\mu(s) \leq Us^\alpha$  for small  $s > 0$ , for some constant  $U$ .

In [13] Combes-Germinet-Klein prove the Wegner estimate and the Minami estimate for more general measure  $\mu$  (single site distribution). We collect their results in the following lemma which immediately gives the following corollary. The inequality (5.3.1) is [13, inequality (2.2)], the inequality (5.3.2) is [13, Theorem 2.3] and the inequality (5.3.3) is [13, Theorem 2.1], so we omit the proofs.

**Lemma 5.3.1.** *For all bounded interval  $I \subset \mathbb{R}$  and any finite volume  $\Lambda \subset \mathbb{Z}^d$ , we have*

$$\mathbb{E}(\langle \delta_n, E_{H^\omega}(I) \delta_n \rangle) \leq Q_\mu(|I|), \quad (5.3.1)$$

$$\mathbb{E}(Tr(E_{H_\Lambda^\omega}(I))) \leq Q_\mu(|I|) |\Lambda|, \quad (5.3.2)$$

$$\mathbb{E}\left(Tr(E_{H_\Lambda^\omega}(I))(Tr(E_{H_\Lambda^\omega}(I)) - 1)\right) \leq \left(Q_\mu(|I|) |\Lambda|\right)^2. \quad (5.3.3)$$

**Corollary 5.3.2.** Consider  $\mathcal{N}$ , the IDS of the operators  $H^\omega$  satisfying Hypothesis 5.1.1. Then for any  $\psi \in C_c(\mathbb{R})$  and  $n \in \mathbb{Z}^d$ , we have

$$\int_{\mathbb{R}} \psi(x) d\mathcal{N}(x) = \mathbb{E}(\langle \delta_n, \psi(H^\omega) \delta_n \rangle) \leq \|\psi\|_\infty Q_\mu(|s_\psi|), \quad s_\psi = \text{supp } \psi. \quad (5.3.4)$$

$$\mathbb{E}(\text{Tr}(\psi(H_\Lambda^\omega))) \leq \|\psi\|_\infty Q_\mu(|s_\psi|) |\Lambda|. \quad (5.3.5)$$

Given any measure  $\nu$  we denote notationally  $\nu(f) = \int f(x) d\nu(x)$  below, where again the limits are to be understood as in the sense stated for Proposition 5.2.1.

**Theorem 5.3.3.** Let  $H^\omega$  satisfy the Hypotheses 5.1.1, 5.1.2 and let  $E \in \sigma_p(H^\omega)$ . Then for each  $\psi \in C_c(\mathbb{R})$ , we have

$$\lim_{L \rightarrow \infty} \zeta_{L,E}^\omega(\psi) = \lim_{L \rightarrow \infty} \xi_{L,E}^\omega(\psi) = \lim_{L \rightarrow \infty} \sum_p^{N_L^d} \eta_{p,E}^\omega(\psi), \quad (5.3.6)$$

with convergence in the sense of distributions.

**Proof:** By general theory (see Kallenberg [32, Theorem 4.5]), the theorem follows if we show that

$$\lim_{L \rightarrow \infty} \mathbb{E} \left| e^{-\zeta_{L,E}^\omega(\psi)} - e^{-\sum_p^{N_L^d} \eta_{p,E}^\omega(\psi)} \right| = 0, \quad (5.3.7)$$

$$\lim_{L \rightarrow \infty} \mathbb{E} \left| e^{-\xi_{L,E}^\omega(\psi)} - e^{-\sum_p^{N_L^d} \eta_{p,E}^\omega(\psi)} \right| = 0. \quad (5.3.8)$$

Since the set of function  $\phi_z(x) = \text{Im} \frac{1}{x-z}$ ,  $z \in \mathbb{C}^+$  are dense in  $C_c(\mathbb{R})$  it is sufficient to verify (5.3.7) for such functions, for more details we refer [31, Appendix: The Stone-Weierstrass Gavotte].

For  $n \in \text{int}(C_p)$  and  $z \in \mathbb{C}^+$  we have the well known perturbation formula, using the resolvent estimate,

$$G^{\Lambda_L}(z; n, n) = G^{C_p}(z; n, n) + \sum_{(m,k) \in \partial C_p} G^{C_p}(z; n, m) G^{\Lambda_L}(z; k, n) \quad (5.3.9)$$

where  $(m, k) \in \partial C_p$  means  $m \in C_p$ ,  $k \in \mathbb{Z}^d \setminus C_p$  such that  $|m - k| = 1$ .



Denote  $z_L = E + \beta_L^{-1}z$  then we have, proceeding as in the proof by Minami [47],

$$\begin{aligned}
& \left| \xi_{L,E}^\omega(\phi_z) - \sum_p^{N_L^d} \eta_{p,E}^\omega(\phi_z) \right| \tag{5.3.10} \\
&= \frac{1}{\beta_L} \left| \text{Tr} \text{Im} G^{\Lambda_L}(z_L) - \sum_p \text{Tr} \text{Im} G^{C_p}(z_L) \right| \\
&\leq \frac{1}{\beta_L} \sum_p \sum_{n \in C_p \setminus \text{int}(C_p)} \{ \text{Im} G^{C_p}(z_L; n, n) + \text{Im} G^{\Lambda_L}(z_L; n, n) \} \\
&\quad + \frac{1}{\beta_L} \sum_p \sum_{n \in \text{int}(C_p)} \sum_{(m,k) \in \partial C_p} |G^{C_p}(z_L; n, m)| |G^{\Lambda_L}(z_L; k, n)| \\
&= A_L + B_L.
\end{aligned}$$

From Combes-Germinet-Klein [13, A.9] we have for given  $k > 0$

$$\text{Im} z \mathbb{E}(G^\Lambda(z; n, n)) \leq \pi \left( 1 + \frac{k}{2} \right) S_\mu \left( \frac{2 \text{Im} z}{k} \right). \tag{5.3.11}$$

Since  $\text{Im} z_L = \beta_L^{-1} \text{Im} z$  with  $\text{Im} z > 0$  so using the  $\alpha$ -Hölder continuity of  $\mu$  we get

$$\begin{aligned}
\frac{1}{\beta_L} \mathbb{E} \left( G^\Lambda(z_L; n, n) \right) &\leq \frac{1}{\text{Im} z} \pi \left( 1 + \frac{k}{2} \right) S_\mu \left( \frac{2 \text{Im} z_L}{k} \right), \quad \Lambda = C_p, \quad \Lambda_L \tag{5.3.12} \\
&\leq C \left( \frac{2 \beta_L^{-1} \text{Im} z}{k} \right)^\alpha \\
&\leq C (2L + 1)^{-d}, \quad (\text{since } \beta_L = (2L + 1)^{d/\alpha}).
\end{aligned}$$

From the inequality (5.3.10) and above we get

$$\mathbb{E}(A_L) \leq C (2L + 1)^{-d} N_L^d \left( \frac{2L + 1}{N_L} \right)^{d-1} l_L = O(L^{-\epsilon} \ln(L)), \tag{5.3.13}$$

in view of our choices for  $N_L, l_L$  in equation (5.1.3).

On the other hand the term  $B_L$  is split as

$$\begin{aligned} B_L &= \frac{1}{\beta_L} \sum_p \sum_{n \in \text{int}(C_p)} \sum_{(m,k) \in \partial C_p} |G^{C_p}(z_L; n, m)| |G^{\Lambda_L}(z_L; k, n)| \\ &= \frac{1}{\beta_L} \sum_p \sum_{n \in \text{int}(C_p)} \sum_{(m,k) \in \partial C_p} |G^{\Lambda_L}(z_L; k, n)|^s |G^{\Lambda_L}(z_L; k, n)|^{1-s} |G^{C_p}(z_L; n, m)| \end{aligned} \quad (5.3.14)$$

Then using the fact that  $(m, k) \in \partial C_p$  and  $n \in \text{int}(C_p)$  so that  $|n - k| > l_L$  for large enough  $L$ , the Hypothesis 5.1.2 with the number  $s$  chosen from there, to estimate,

$$|G^{\Lambda_L}(z_L; k, n)|^{1-s} \leq \frac{1}{|Imz_L|^{1-s}} \text{ and } |G^{C_p}(z_L; n, m)| \leq \frac{1}{|Imz_L|},$$

we obtain the following bound from taking expectation in the equality (5.3.14).

$$\mathbb{E}(B_L) \leq \frac{1}{\beta_L |Imz_L|^{2-s}} N_L^d \left( \frac{2L+1}{N_L} \right)^d \left( \frac{2L+1}{N_L} \right)^{d-1} l_L e^{-\gamma l_L}. \quad (5.3.15)$$

We simplify the right hand side of the above inequality to get

$$\mathbb{E}^\omega(B_L) \leq \frac{1}{\beta_L |Imz_L|^{2-s}} N_L^d \left( \frac{2L+1}{N_L} \right)^d \left( \frac{2L+1}{N_L} \right)^{d-1} l_L e^{-\gamma l_L} \quad (5.3.16)$$

$$= O(L^{(1-s)\frac{d}{\alpha}} N_L^d \left( \frac{L}{N_L} \right)^{2d-1} e^{-\gamma \frac{5d}{\gamma\alpha} \ln(2L+1)} \ln(L)) \quad (5.3.17)$$

$$= O(L^{(\frac{d}{\alpha}(1-s)+(2d-1)+(d-1)(1-\epsilon)-\frac{5d}{\alpha})} \ln(L)) \quad (5.3.18)$$

$$= O(L^{-1}), \quad (5.3.19)$$

since  $\alpha \leq 1$ . In particular we have from (5.3.13) and (5.3.16)

$$\mathbb{E}(A_L + B_L) \rightarrow 0 \text{ as } L \rightarrow \infty. \quad (5.3.20)$$

Finally the inequality  $|e^{-x} - e^{-y}| \leq |x - y|$  for  $x, y > 0$  together with the bound (5.3.10) and above convergence gives the required vanishing of the limits equation (5.3.7).

Again using the resolvent equation for  $G(z; n, n) = \langle \delta_n, (H^\omega - z)^{-1} \delta_n \rangle$  and the equality

$$\langle \delta_n, (\oplus H_{C_p}^\omega \oplus H_{\Lambda_L^c}^\omega - z)^{-1} \delta_n \rangle = \langle \delta_n, (\oplus H_{C_p}^\omega - z)^{-1} \delta_n \rangle$$

valid for each  $n \in \Lambda_L$ , gives us the relation

$$G(z; n, n) = G^{C_p}(z; n, n) + \sum_{(m,k) \in \partial C_p} G^{C_p}(z; n, m) G(z; k, n), \quad n \in C_p,$$

for each  $p = 1, \dots, N_L^d$ . The convergence in equation (5.3.8) is then obtained by essentially repeating the argument above.  $\square$

## 5.4 Proof of the main Theorem

We first prove the Theorem 5.1.4. In the argument below we consider a subsequence  $L_n$  which converges to the limsup in equation (5.4.11) and use Proposition 5.2.1 to only consider  $\xi_{L,E}$  instead of  $\zeta_{L,E}$ . We will show that

$$\lim_{n \rightarrow \infty} \mathbb{E}(e^{it\xi_{L_n,E}^\omega(I)}) = e^{(e^{it}-1)\gamma_{E,I}}.$$

This will then show, by Lévy-Khintchine theorem that  $\xi_{L_n,E}^\omega(I)$  converge in distribution to the Poisson random variable with parameter  $\gamma_{E,I}$ . Since the convergence in distribution for a sequence of random variables is equivalent to the convergence of their Fourier transforms point wise combined with Theorem 5.3.3, it is enough to look at the limit with  $\xi_{L_n,E}^\omega(I)$  replaced by  $\sum_{p=1}^{N_{L_n}^d} \eta_{L_n,E}^\omega(I)$ .

We first note that from equation (5.2.2) we have,

$$\lim_{n \rightarrow \infty} \mathbb{E} \left( \sum_p^{N_{L_n}^d} \eta_{p,E}^\omega(I) \right) = \lim_{n \rightarrow \infty} \mathbb{E}(\zeta_{L_n,E}^\omega(I)) = \gamma_{E,I} \tag{5.4.1}$$

We now compute the limits of Fourier transforms

$$\begin{aligned}
\lim_{n \rightarrow \infty} \mathbb{E}(e^{it\xi_{L_n, E}^\omega(I)}) &= \lim_{n \rightarrow \infty} \mathbb{E}\left(e^{it \sum_{p=1}^{N_{L_n}^d} \eta_{p, E}^\omega(I)}\right) \\
&= \lim_{n \rightarrow \infty} \prod_{p=1}^{N_{L_n}^d} \mathbb{E}(e^{it\eta_{p, E}^\omega(I)}) \\
&= \lim_{n \rightarrow \infty} \left[ \mathbb{E}(e^{it\eta_{p, E}^\omega(I)}) \right]^{N_{L_n}^d}.
\end{aligned} \tag{5.4.2}$$

Now for  $p = 1, \dots, N_{L_n}^d$ ,

$$\begin{aligned}
\mathbb{E}(e^{it\eta_{p, E}^\omega(I)}) &= \sum_{m=0}^{\infty} e^{itm} \mathbb{P}(\eta_{p, E}^\omega(I) = m) \\
&= 1 + \mathbb{E}(\eta_{p, E}^\omega(I)) [e^{it} - 1] + R_{L_n}
\end{aligned} \tag{5.4.3}$$

where  $R_{L_n}$  is given by

$$\begin{aligned}
R_{L_n} &= \sum_{m=0}^{\infty} e^{itm} \mathbb{P}(\eta_{p, E}^\omega(I) = m) - 1 - \mathbb{E}(\eta_{p, E}^\omega(I)) [e^{it} - 1] \\
&= \sum_{m=0}^{\infty} e^{itm} \mathbb{P}(\eta_{p, E}^\omega(I) = m) - \sum_{m=0}^{\infty} \mathbb{P}(\eta_{p, E}^\omega(I) = m) \\
&\quad - [e^{it} - 1] \sum_{m=0}^{\infty} m \mathbb{P}(\eta_{p, E}^\omega(I) = m) \\
&= \sum_{m=2}^{\infty} (e^{itm} - m e^{it} + m - 1) \mathbb{P}(\eta_{p, E}^\omega(I) = m).
\end{aligned}$$

Then using the inequality

$$|e^{itm} - m e^{it} + m - 1| \leq (m + 1) + (m - 1) \leq 2m, \quad \text{when } m \geq 2$$

and setting  $J_{L,E} = E + \beta_L^{-1}I$  we get,

$$\begin{aligned}
|R_{L_n}| &\leq \sum_{m=2}^{\infty} (|e^{itm} - me^{it}| + (m-1)) \mathbb{P}(\eta_{p,E}^{\omega}(I) = m) \\
&\leq \sum_{m=2}^{\infty} ((m+1) + (m-1)) \mathbb{P}(\eta_{p,E}^{\omega}(I) = m) \\
&\leq 2 \sum_{m=2}^{\infty} m \mathbb{P}(\eta_{p,E}^{\omega}(I) = m) \\
&\leq 2 \sum_{m=2}^{\infty} m(m-1) \mathbb{P}(\eta_{p,E}^{\omega}(I) = m) \\
&\leq 2\mathbb{E}\left(\text{Tr}(E_{H_{C_p}^{\omega}}(J_{L_n,E}))(\text{Tr}(E_{H_{C_p}^{\omega}}(J_{L_n,E})) - 1)\right),
\end{aligned}$$

Now from the Minami estimate (5.3.3) of Lemma 5.3.1 we have

$$\begin{aligned}
&N_{L_n}^d \mathbb{E}\left(\text{Tr}(E_{H_{C_p}^{\omega}}(J_{L_n,E}))(\text{Tr}(E_{H_{C_p}^{\omega}}(J_{L_n,E})) - 1)\right) \\
&\leq N_{L_n}^d \left(Q_{\mu}(|J_{L_n,E}|) |C_p|\right)^2 \\
&\leq N_{L_n}^d (|J_{L_n,E}|^{\alpha} |C_p|)^2 \\
&= O\left(\beta_{L_n}^{-2\alpha} N_{L_n}^d \left(\frac{2L_n + 1}{N_{L_n}}\right)^{2d}\right).
\end{aligned} \tag{5.4.4}$$

The above calculation together with the estimate (5.4.4) will give

$$N_{L_n}^d R_{L_n} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

From the above computation we get

$$N_{L_n}^d \left[ \mathbb{E}(\eta_{p,E}^{\omega}(I))[e^{it} - 1] + R_{L_n} \right] \xrightarrow{n \rightarrow \infty} \gamma_{E,I}[e^{it} - 1].$$

We use the equations (5.4.2) and (5.4.3) to obtain the equality

$$\mathbb{E}(e^{it\xi_{L_n,E}^\omega}) = \left[ 1 + \frac{N_{L_n}^d [\mathbb{E}(\eta_{p,E}^\omega(I)) [e^{it} - 1] + R_{L_n}]}{N_{L_n}^d} \right]^{N_{L_n}^d} \quad (5.4.5)$$

which combined with the convergence of  $(1 + \frac{z_n}{n})^n$  to  $e^z$ , whenever  $z_n \rightarrow z$  as  $n \rightarrow \infty$  gives us finally the limit

$$\mathbb{E}(e^{it\xi_{L_n,E}^\omega}) \xrightarrow{n \rightarrow \infty} e^{\gamma_{E,I}(e^{it}-1)}.$$

□

**Proof of Corollary 5.1.5:**

We first note that,  $D_\mu^\alpha(E) < \infty$  iff  $\limsup \frac{\mathcal{N}(E+\epsilon I)}{(2\epsilon)^\alpha} < \infty$ , for all bounded symmetric intervals  $I$ . We will show that

$$\limsup \frac{(\beta_L^\alpha \mathbb{E}(\langle \delta_0, E_{H^\omega}(E + \beta_L^{-1}I)\delta_0 \rangle))}{|I|^\alpha} \geq \frac{1}{2^d} D_{\mathcal{N}}^\alpha(E).$$

Then by the assumption of theorem the right hand side is positive, so a limit point of  $\mathbb{E}(\zeta_L(I))$  is positive. We recall that  $\beta_L^\alpha = (2L+1)^d$ . Let  $I$  be a bounded open symmetric interval and choose  $\epsilon \in (\beta_{L+1}^{-1}, \beta_L^{-1}]$ . Then we have

$$E + \beta_L^{-1}I \supseteq E + \epsilon I \quad \text{and} \quad \mathcal{N}(E + \beta_L^{-1}I) \geq \mathcal{N}(E + \epsilon I).$$

Therefore we have, since  $\beta_{L+1}^\alpha \epsilon^\alpha \geq 1$ ,

$$\begin{aligned} \frac{\beta_L^\alpha \mathcal{N}(E + \beta_L^{-1}I)}{|I|^\alpha} &\geq \left( \frac{\beta_L}{\beta_{L+1}} \right)^\alpha \frac{\mathcal{N}(E + \epsilon I)}{(\epsilon|I|)^\alpha} \\ &\geq \left( \frac{\beta_L}{\beta_{L+1}} \right)^\alpha \frac{\mathcal{N}(E - c\epsilon, E + c\epsilon)}{(\epsilon|I|)^\alpha}, \end{aligned} \quad (5.4.6)$$

These inequalities imply that

$$\sup_{L \geq M} \frac{\beta_L^\alpha \mathcal{N}(E + \beta_L^{-1}I)}{|I|^\alpha} \geq \left( \frac{1}{1 + \frac{2}{2M+1}} \right)^d \sup_{\epsilon \in (\beta_{L+1}^{-1}, \beta_L^{-1}], L \geq M} \frac{\mathcal{N}(E + \epsilon I)}{(\epsilon|I|)^\alpha} \quad (5.4.7)$$

$$\geq \left( \frac{1}{1 + \frac{2}{2M+1}} \right)^d \sup_{\epsilon \in (0, \beta_M^{-1}]} \frac{\mathcal{N}(E + \epsilon I)}{(\epsilon|I|)^\alpha}, \quad (5.4.8)$$

where we used the facts that

$$\bigcup_{L \geq M} (\beta_{L+1}^{-1}, \beta_L^{-1}] = (0, \beta_M^{-1}] \quad \text{and} \quad \left( \frac{\beta_L}{\beta_{L+1}} \right)^\alpha \geq \left( \frac{1}{1 + \frac{2}{2M+1}} \right)^d, \quad \text{for } L \geq M.$$

We now let  $M \rightarrow \infty$  in the inequality (5.4.8). Then from the definition of limsup we get

$$\overline{\lim}_{L \rightarrow \infty} \frac{\beta_L^\alpha \mathcal{N}(E + \beta_L^{-1}I)}{|I|^\alpha} \geq D_{\mathcal{N}}^\alpha(E). \quad (5.4.9)$$

Similarly starting with  $\epsilon \in (\beta_{L+1}^{-1}, \beta_L^{-1}]$  we get the inequality

$$\frac{\beta_{L+1}^\alpha \mathcal{N}(E + \beta_{L+1}^{-1}I)}{|I|^\alpha} \leq \left( \frac{\beta_{L+1}}{\beta_L} \right)^\alpha \frac{\mathcal{N}(E + \epsilon I)}{(\epsilon|I|)^\alpha}$$

and proceed as in the above argument, with upper bounds now, to get

$$\overline{\lim}_{L \rightarrow \infty} \frac{\beta_L^\alpha \mathcal{N}(E + \beta_L^{-1}I)}{|I|^\alpha} \leq D_{\mathcal{N}}^\alpha(E). \quad (5.4.10)$$

Putting the inequalities (5.4.9) and (5.4.10) we get

$$\overline{\lim}_{L \rightarrow \infty} \frac{\beta_L^\alpha \mathcal{N}(E + \beta_L^{-1}I)}{|I|^\alpha} = D_{\mathcal{N}}^\alpha(E).$$

The above inequality shows that, noting again that  $\beta_L^\alpha = (2L + 1)^d$ ,

$$\begin{aligned}
\gamma_{E,I} &= \overline{\lim}_{L \rightarrow \infty} \mathbb{E}(\zeta_{L,E}^\omega(I)) & (5.4.11) \\
&= \overline{\lim}_{L \rightarrow \infty} \mathbb{E}\left(\sum_{n \in \Lambda_L} \langle \delta_n, E_{H^\omega}(I) \delta_n \rangle\right) \\
&= \overline{\lim}_{L \rightarrow \infty} \beta_L^\alpha \mathcal{N}(E + \beta_L^{-1} I) \\
&= D_{\mathcal{N}}^\alpha(E) |I|^\alpha = D_{\mathcal{N}}^\alpha(E) \mathcal{L}_\alpha(I),
\end{aligned}$$

where to pass to the third line we used the fact that  $\mathbb{E}(\langle \delta_n, E_{H^\omega}(I) \delta_n \rangle)$  does not depend on  $n$ . Since the limsup above is a limit point of the sequence considered, we have the corollary.  $\square$

**Examples 5.4.1.** *We now give an example of random operators that have singular density of states and for which the local eigenvalue statistics is Poisson. We note while this example may appear trivial, it is one for which none of the existing theorems can show Poisson eigenvalue statistics.*

*Consider the operator*

$$H^\omega = \sum_{n \in \mathbb{Z}^d} \omega_n P_n$$

*$P_n$  is projection onto  $\ell^2(\{n\})$  as in the model (5.1.1) with  $\{\omega_n\}$  i.i.d random variable with distribution  $\mu$ . Then the IDS agrees with the distribution of the measure  $\mu$ , so if we choose a singular  $\alpha$ -continuous measure  $\mu$  (such as the Cantor measure, for which  $\alpha = \log(2)/\log(3)$ ), then the conditions of our theorem are valid for  $H_0 = 0$  (which is in some sense infinite disorder limit of the large disorder Anderson model).*

*Therefore Poisson eigenvalue statistics holds for points in the spectrum.*



## Chapter 6

# Integrated Density of States for Decaying Random Potentials

## 6.1 Introduction

There is a large body of work on the integrated density of states for ergodic potentials, however the integrated density of states is not known for each non-stationary random potentials. In this chapter, we investigate some bounds for the integrated density of states in the pure point regime for the case of decaying model as define below.

The random Schrödinger operator with decaying randomness is a random Hamiltonian  $H^\omega$  on  $\ell^2(\mathbb{Z}^d)$  given by

$$H^\omega = -\Delta + V^\omega, \quad \omega \in \Omega. \quad (6.1.1)$$

$\Delta$  is the adjacency operator defined by (2.4.2). The random potential  $V^\omega$  which is the multiplication operator on  $\ell^2(\mathbb{Z}^d)$  by the sequence  $\{a_n q_n(\omega)\}_{n \in \mathbb{Z}^d}$  defined by

$$V^\omega = \sum_{n \in \mathbb{Z}^d} a_n q_n(\omega) |\delta_n\rangle \langle \delta_n|, \quad (6.1.2)$$

where  $\{\delta_n\}_{n \in \mathbb{Z}^d}$  is the standard basis for  $\ell^2(\mathbb{Z}^d)$ . Here  $\{a_n\}_{n \in \mathbb{Z}^d}$  is a sequence of positive real numbers such that  $a_n \rightarrow 0$  as  $|n| \rightarrow \infty$  and  $\{q_n\}_{n \in \mathbb{Z}^d}$  are real valued i.i.d. random variables with an absolutely continuous probability distribution  $\mu$  which has bounded density. We realize  $q_n$  as  $\omega(n)$  on  $(\mathbb{R}^{\mathbb{Z}^d}, \mathcal{B}_{\mathbb{R}^{\mathbb{Z}^d}}, \mathbb{P})$ ,  $\mathbb{P} = \bigotimes \mu$  constructed via Kolmogorov theorem. We refer to this probability space as  $(\Omega, \mathcal{B}, \mathbb{P})$  henceforth. For any  $B \subset \mathbb{Z}^d$  we consider the orthogonal projection  $\chi_B$  onto  $\ell^2(B)$  and define the matrices

$$H_B^\omega = (\langle \delta_n, H^\omega \delta_m \rangle)_{n, m \in B}, \quad G^B(z; n, m) = \langle \delta_n, (H_B^\omega - z)^{-1} \delta_m \rangle, \quad G^B(z) = (H_B^\omega - z)^{-1}. \quad (6.1.3)$$

$$G(z) = (H^\omega - z)^{-1}, \quad G(z; n, m) = \langle \delta_n, (H^\omega - z)^{-1} \delta_m \rangle, \quad z \in \mathbb{C}^+.$$

We note that  $H_B^\omega$  is the matrix

$$H_B^\omega = \chi_B H^\omega \chi_B : \ell^2(B) \longrightarrow \ell^2(B), \quad a.e. \omega.$$

We note that by assumptions on  $V^\omega$ , the operators  $H^\omega$  are self-adjoint a.e.  $\omega$  and have a common core domain consisting of vectors of finite support. Let  $\Lambda_L \subset \mathbb{Z}^d$  be the  $d$ -dimensional cube centered at origin as given in (4.1.2). We then assume that,

**Hypothesis 6.1.1.** 1. The measure  $\mu$  is absolute continuous with density  $\rho$  that satisfies

$$\rho(x) = \begin{cases} 0 & \text{if } |x| < 1 \\ \frac{1}{|x|^\delta} & \text{if } |x| \geq 1, \text{ for some } \delta > 1. \end{cases} \quad (6.1.4)$$

2. The sequence  $a_n$  satisfy  $0 < a_n \simeq |n|^{-\alpha}$ ,  $\alpha > 0$ .

3. The pair  $(\alpha, \delta)$  is chosen such that  $d - \alpha(\delta - 1) > 0$  holds. This implies that  $\beta_L \rightarrow \infty$  as  $L \rightarrow \infty$ , where  $\beta_L$  is given by

$$\begin{aligned} \beta_L &= \sum_{n \in \Lambda_L} a_n^{(\delta-1)} \\ &\simeq \sum_{n \in \Lambda_L} |n|^{-\alpha(\delta-1)} \simeq O\left((2L+1)^{d-\alpha(\delta-1)}\right). \end{aligned} \quad (6.1.5)$$

**Remark 6.1.2.** We have taken an explicit  $\rho(x)$  in (6.1.4) in order to simplify the calculations in the proofs of our results. Our results also hold for  $\rho(x) = O\left(\frac{1}{|x|^\delta}\right)$ ,  $\delta > 1$  as  $|x| \rightarrow \infty$ .

In [42], Kirsch-Krishna-Obermeit consider  $H^\omega = -\Delta + V^\omega$  on  $\ell^2(\mathbb{Z}^d)$  with the same  $V^\omega$  as defined in (6.1.2). They showed that  $\sigma(H^\omega) = \mathbb{R}$  and  $\sigma_c(H^\omega) \subseteq [-2d, 2d]$  a.e.  $\omega$ , under some conditions on  $\{a_n\}_{n \in \mathbb{Z}^d}$  and  $\mu$  (The density of  $\mu$  should not decay too fast at infinity and  $a_n$  should not decay too fast). For the precise condition on  $a_n$ 's and  $\mu$  we recall Definition 2.1 from [42], which is given as follows.

**Definition 6.1.3.** Let  $\{a_n\}$  be a bounded, positive sequence on  $\mathbb{R}$ . Then,  $\{a_n\}$  - supp  $\mu$  is defined by

$$\{a_n\} - \text{supp } \mu := \left\{ x \in \mathbb{R} : \sum_n \mu(a_n^{-1}(x - \epsilon, x + \epsilon)) = \infty \forall \epsilon > 0 \right\}. \quad (6.1.6)$$

We call a probability measure  $\mu$  asymptotically large with respect to  $a_n$  if

$$\{a_{kn}\} - \text{supp } \mu = \mathbb{R}, \text{ for all } k \in \mathbb{Z}^+.$$

To show the existence of point spectrum outside  $[-2d, 2d]$  they verified Simon-Wolf criterion [56, Theorem 12.5] by showing exponential decay of the fractional moment of the Green function [42, Lemma 3.2]. The decay is valid for  $|n - m| > 2R$  with energy  $E \in \mathbb{R} \setminus [-2d, 2d]$  and is given by

$$\mathbb{E}^\omega(|G^{\Lambda_L}(E + i\epsilon : n, m)|^s) \leq D_{P(n,m)} e^{-c\left(\frac{|n-m|}{2}\right)}, \quad E \in \mathbb{R} \setminus [-2d, 2d], \quad (6.1.7)$$

where  $\epsilon > 0$ ,  $0 < s < 1$ ,  $c$  is a positive constant and  $R \in \mathbb{Z}^+$ . Here,  $D_{P(n,m)}$  is a constant independent of  $E$  and  $\epsilon$ , but polynomially bounded in  $|n|$  and  $|m|$ .

Jaksic-Last showed in [35, Theorem 1.2] that for  $d \geq 3$ , if  $a_n \simeq |n|^{-\alpha}$   $\alpha > 1$  then there is no singular spectrum inside  $(-2d, 2d)$  of  $H^\omega$ .

In the second section of this chapter we describe the the spectrum of  $H^\omega$  using [42, Theorem 2.7]. We then show that the average spacing of eigenvalues of  $H_{\Lambda_L}^\omega$  close to the energy  $E \in \mathbb{R} \setminus [-2d, 2d]$  is of order  $\beta_L^{-1}$ , whereas those close to  $E \in [-2d, 2d]$  have average spacing of the order  $\frac{1}{(2L+1)^d}$ . This shows that the eigenvalues of  $H_{\Lambda_L}^\omega$  are more densely distributed inside  $[-2d, 2d]$ , the continuous spectrum of  $H^\omega$ , than in the pure point regime i.e., outside  $[-2d, 2d]$ .

As we move to state our main results of this chapter, we define the following:

$$N_L^\omega(E) = \#\{j : E_j \leq E, E_j \in \sigma(H_{\Lambda_L}^\omega)\}, \quad (6.1.8)$$

$$\tilde{N}_L^\omega(E) = \#\{j : E_j \geq E, E_j \in \sigma(H_{\Lambda_L}^\omega)\}, \quad (6.1.9)$$

$$\gamma_L(\cdot) = \frac{1}{\beta_L} \sum_{n \in \Lambda_L} \mathbb{E}^\omega(\langle \delta_n, E_{H_{\Lambda_L}^\omega}(\cdot) \delta_n \rangle). \quad (6.1.10)$$

Our main results are as follows:

**Theorem 6.1.4.** *If  $E < -2d$  and  $\epsilon = -2d - E > 0$  then, we have*

$$\frac{1}{(\delta - 1)(4d + \epsilon)^{(\delta-1)}} \leq \underline{\lim}_{L \rightarrow \infty} \frac{1}{\beta_L} \mathbb{E}^\omega(N_L^\omega(E)) \leq \overline{\lim}_{L \rightarrow \infty} \frac{1}{\beta_L} \mathbb{E}^\omega(N_L^\omega(E)) \leq \frac{1}{(\delta - 1)\epsilon^{(\delta-1)}}.$$

For  $E = 2d + \epsilon > 2d$  we have

$$\frac{1}{(\delta - 1)(4d + \epsilon)^{(\delta-1)}} \leq \underline{\lim}_{L \rightarrow \infty} \frac{1}{\beta_L} \mathbb{E}^\omega(\tilde{N}_L^\omega(E)) \leq \overline{\lim}_{L \rightarrow \infty} \frac{1}{\beta_L} \mathbb{E}^\omega(\tilde{N}_L^\omega(E)) \leq \frac{1}{(\delta - 1)\epsilon^{(\delta-1)}}.$$

Now we investigate the average number of eigenvalues of  $H_{\Lambda_L}^\omega$  inside  $[-2d, 2d]$ , which can be given as follows:

**Corollary 6.1.5.** *For any interval  $(M_1, M_2) \not\supseteq [-2d, 2d]$  we have*

$$\lim_{L \rightarrow \infty} \frac{1}{(2L + 1)^d} \mathbb{E}^\omega(\#\{\sigma(H_{\Lambda_L}^\omega) \cap (M_1, M_2)\}) = 1. \quad (6.1.11)$$

**Corollary 6.1.6.** *If  $M_1 < -2d$  and  $M_2 > 2d$  then, we have*

$$\overline{\lim}_{L \rightarrow \infty} \gamma_L((-\infty, M_1] \cup [M_2, \infty)) \leq \frac{1}{\delta - 1} \left[ \frac{1}{(-2d - M_1)^{(\delta-1)}} + \frac{1}{(M_2 - 2d)^{(\delta-1)}} \right] \quad (6.1.12)$$

For any interval  $I \subseteq \mathbb{R} \setminus [-2d, 2d]$  with length  $|I| > 4d$  there is a constant  $C_I > 0$  such that

$$\underline{\lim}_{L \rightarrow \infty} \gamma_L(I) \geq C_I > 0. \quad (6.1.13)$$

**Corollary 6.1.7.** *Let  $M_1 < -2d$  and  $M_2 > 2d$  and  $\gamma_L \upharpoonright_{(M_1, M_2)^c}$  denote the restriction of  $\gamma_L$  to  $\mathbb{R} \setminus (M_1, M_2)$ . The sequence of measure  $\{\gamma_L \upharpoonright_{(M_1, M_2)^c}\}_L$  admits a subsequence which converges vaguely to a non-trivial measure, say  $\gamma$ .*

The above theorem gives estimates for the average of  $N_L^\omega(E)$  and  $\tilde{N}_L^\omega(E)$ , but we can also get a point-wise estimate of the above quantities which is given by following theorem.

**Theorem 6.1.8.** *If  $d \geq 2$ ,  $0 < \alpha < \frac{1}{2}$  and  $1 < \delta < \frac{1}{2\alpha}$  then for almost all  $\omega$*

$$\frac{1}{(\delta - 1)(2d - E)^{(\delta-1)}} \leq \liminf_{L \rightarrow \infty} \frac{1}{\beta_L} N_L^\omega(E) \leq \overline{\lim}_{L \rightarrow \infty} \frac{1}{\beta_L} N_L^\omega(E) \leq \frac{1}{(\delta - 1)(-2d - E)^{(\delta-1)}} \quad \text{for } E < -2d,$$

$$\frac{1}{(\delta - 1)(2d + E)^{(\delta-1)}} \leq \liminf_{L \rightarrow \infty} \frac{1}{\beta_L} \tilde{N}_L^\omega(E) \leq \overline{\lim}_{L \rightarrow \infty} \frac{1}{\beta_L} \tilde{N}_L^\omega(E) \leq \frac{1}{(\delta - 1)(E - 2d)^{(\delta-1)}} \quad \text{for } E > 2d.$$

In [22], Figotin-Germinet-Klein-Müller studied the Anderson Model on  $L^2(\mathbb{R}^d)$  with decaying random potentials given by

$$H^\omega = -\Delta + \lambda \gamma_\alpha V^\omega \quad \text{on } L^2(\mathbb{R}^d),$$

where  $\lambda > 0$  is the disorder parameter and  $\gamma_\alpha$  is the envelope function

$$\gamma_\alpha(x) := (1 + |x|^2)^{-\frac{\alpha}{2}}, \quad \alpha \geq 0.$$

They assumed that the density of the single site distribution is in  $L^\infty(\mathbb{R}^d)$  and has compact support. They showed that if  $\alpha \in (0, 2)$  then  $H^\omega$  has infinitely many eigenvalues in  $(-\infty, 0)$  a.e.  $\omega$ . In [22, Theorem 3], they gave the bound for  $N^\omega(E)$ ,  $E < 0$  (number of eigenvalues of  $H^\omega$  below  $E$ ) in terms of density of states for the stationary (i.i.d. case) Model.

In [28], Gordon-Jaksić-Molchanov-Simon studied the Model given by

$$H^\omega = -\Delta + \sum_{n \in \mathbb{Z}^d} (1 + |n|^\alpha) q_n(\omega), \quad \alpha > 0 \quad \text{on } \ell^2(\mathbb{Z}^d),$$

where  $\{q_n\}$  are i.i.d. random variables uniformly distributed on  $[0, 1]$ . They showed that if  $\alpha > d$  then  $H^\omega$  has discrete spectrum a.e.  $\omega$ . For the case when  $\alpha \leq d$  they construct a strictly decreasing sequence  $\{a_k\}_{k \in \mathbb{N}}$  of positive numbers such that if  $\frac{d}{k} \geq \alpha > \frac{d}{k+1}$  then for a.e.  $\omega$  we have the following:

(i)  $\sigma(H^\omega) = \sigma_{pp}(H^\omega)$  and eigenfunctions of  $H^\omega$  decay exponentially,

(ii)  $\sigma_{ess}(H^\omega) = [a_k, \infty)$  and

(iii)  $\#\sigma_{disc}(H^\omega) < \infty$ .

They also showed that

(a) If  $\frac{d}{k} > \alpha > \frac{d}{k+1}$  and  $E \in (a_j, a_{j-1})$ ,  $1 \leq j \leq k$ , then

$$\lim_{L \rightarrow \infty} \frac{N_L^\omega(E)}{L^{d-j\alpha}} = N_j(E)$$

exists for a.e.  $\omega$  and is a non random function.

(b) If  $\alpha = \frac{d}{k}$  and  $E \in (a_j, a_{j-1})$ ,  $1 \leq j < k$  the above is valid. If  $E \in (a_k, a_{k-1})$  then

$$\lim_{L \rightarrow \infty} \frac{N_L^\omega(E)}{\ln L} = N_k(E)$$

exists for a.e.  $\omega$  and is a non random function.

In this chapter, we essentially show that for decaying potentials the confinement length is  $(2L+1)^d$  inside  $[-2d, 2d]$  and  $\beta_L$  outside  $[-2d, 2d]$ . On the other hand, for the growing potentials (as in [28]), the confinement length is a function of energy.

## 6.2 On the pure point and continuous spectrum

In this section, we work out the spectrum of  $H^\omega$  under the Hypothesis 6.1.1. Let  $x < 0$  and  $\epsilon > 0$  such that  $x + \epsilon < 0$  then, for large enough  $|n| \geq M$  we have  $a_n^{-1}(x + \epsilon) \leq -1$  since  $a_n^{-1} \rightarrow \infty$  as  $|n| \rightarrow \infty$ . Therefore, we have, for  $|n| \geq M$

$$\begin{aligned} \mu\left(\frac{1}{a_n}(x - \epsilon, x + \epsilon)\right) &= \int_{a_n^{-1}(x-\epsilon)}^{a_n^{-1}(x+\epsilon)} \rho(t) dt \\ &= a_n^{(\delta-1)} \int_{x-\epsilon}^{x+\epsilon} \frac{1}{|t|^\delta} dt. \end{aligned}$$

Hence,

$$\sum_{n \in \mathbb{Z}^d} \mu\left(\frac{1}{a_n}(x - \epsilon, x + \epsilon)\right) \geq \int_{x-\epsilon}^{x+\epsilon} \frac{1}{|t|^\delta} dt \sum_{|n| \geq M} a_n^{(\delta-1)} = \infty, \quad (6.2.1)$$

since  $\beta_L = \sum_{n \in \Lambda_L} a_n^{(\delta-1)} \rightarrow \infty$  as  $L \rightarrow \infty$  (using 6.1.5).

For  $x > 0$ , a similar calculation as above will give

$$\sum_{n \in \mathbb{Z}^d} \mu \left( \frac{1}{a_n} (x - \epsilon, x + \epsilon) \right) = \infty, \quad \epsilon > 0. \quad (6.2.2)$$

Now let  $\epsilon > 0$ . Since  $a_n^{-1} \rightarrow \infty$  as  $|n| \rightarrow \infty$ , there exists an  $M$  such that  $a_n^{-1}\epsilon > 1$  for  $|n| \geq M$ . So, we have

$$\begin{aligned} \sum_{n \in \mathbb{Z}^d} \mu \left( \frac{1}{a_n} (-\epsilon, \epsilon) \right) &\geq \sum_{|n| \geq M} \mu(-a_n^{-1}\epsilon, a_n^{-1}\epsilon) \\ &= 2 \sum_{|n| \geq M} \int_1^{a_n^{-1}\epsilon} \frac{1}{t^\delta} dt \\ &= \frac{2}{\delta-1} \sum_{|n| \geq M} (1 - \epsilon^{1-\delta} a_n^{\delta-1}). \end{aligned}$$

Since,  $\frac{2}{\delta-1} \sum_{n \in \Lambda_L} (1 - \epsilon^{1-\delta} a_n^{\delta-1}) \approx \frac{2}{\delta-1} [(2L+1)^d - (2L+1)^{d-\alpha(\delta-1)}] \rightarrow \infty$  as  $L \rightarrow \infty$ , it follows that

$$\sum_{n \in \mathbb{Z}^d} \mu \left( \frac{1}{a_n} (-\epsilon, \epsilon) \right) = \infty. \quad (6.2.3)$$

If  $0 < \epsilon_1 < \epsilon_2$  then, we have

$$\mu \left( a_n^{-1} (x - \epsilon_1, x + \epsilon_1) \right) \leq \mu \left( a_n^{-1} (x - \epsilon_2, x + \epsilon_2) \right) \quad \forall x \in \mathbb{R}.$$

Now, using the above inequality together with (6.2.1), (6.2.2) and (6.2.3) we have, for all  $\epsilon > 0$ ,

$$\sum_{n \in \mathbb{Z}^d} \mu \left( a_n^{-1} (x - \epsilon, x + \epsilon) \right) = \infty, \quad \text{for all } x \in \mathbb{R}. \quad (6.2.4)$$

Then, using (6.2.4) from [42, Definition 2.1] we see that

$$M = \bigcap_{k \in \mathbb{Z}^+} (a_{kn} - \text{supp } \mu) = \mathbb{R}.$$



Therefore, [42, Corollary 2.5] and [42, Theorem 2.3] will give the following description about the spectrum of  $H^\omega$ .

$$\sigma_{ess}(H^\omega) = [-2d, 2d] + \mathbb{R} = \mathbb{R} \text{ and } \sigma_c(H^\omega) \subseteq [-2d, 2d] \text{ a.e } \omega.$$

### 6.3 Proof of main results

#### Proof of Theorem 6.1.4.

Define

$$A_{L,\pm}^\omega = \pm 2d + \sum_{n \in \Lambda_L} a_n q_n(\omega) P_{\delta_n}.$$

and

$$N_{\pm,L}^\omega(E) = \#\{j; E_j \leq E, E_j \in \sigma(A_{L,\pm}^\omega)\}, \quad N_L^\omega(E) = \#\{j : E_j \leq E, E_j \in \sigma(H_{\Lambda_L}^\omega)\}.$$

Since  $\sigma(\Delta) = [-2d, 2d]$ , following operator inequality

$$A_{L,-}^\omega \leq H_{\Lambda_L}^\omega \leq A_{L,+}^\omega. \tag{6.3.1}$$

is obvious, with

$$H_{\Lambda_L}^\omega = \chi_{\Lambda_L} \Delta \chi_{\Lambda_L} + \sum_{n \in \Lambda_L} a_n q_n(\omega) P_{\delta_n}.$$

A simple application of the min-max principle [36, Theorem 6.44] shows that

$$N_{+,L}^\omega(E) \leq N_L^\omega(E) \leq N_{-,L}^\omega(E). \tag{6.3.2}$$

Now, the spectrum  $\sigma(A_{L,\pm}^\omega)$  of  $A_{L,\pm}^\omega$  consists of only eigenvalues and is given by

$$\sigma(A_{L,\pm}^\omega) = \{n \in \Lambda_L : \pm 2d + a_n q_n(\omega)\}.$$

Let  $E < -2d$  with  $E = -2d - \epsilon$ , for some  $\epsilon > 0$ . Then,

$$\begin{aligned}
N_{-,L}^\omega(E) &= \#\{n \in \Lambda_L : -2d + a_n q_n(\omega) \leq -2d - \epsilon\} \\
&= \#\{n \in \Lambda_L : q_n(\omega) \in (-\infty, -a_n^{-1}\epsilon]\} \\
&= \sum_{n \in \Lambda_L} \chi_{\{q_n(\omega) \in (-\infty, -a_n^{-1}\epsilon]\}}.
\end{aligned} \tag{6.3.3}$$

Since  $q_n$  are i.i.d, if we take expectation of both sides of (6.3.3) we get

$$\begin{aligned}
\mathbb{E}^\omega(N_{-,L}^\omega(E)) &= \sum_{n \in \Lambda_L} \mu(-\infty, -a_n^{-1}\epsilon] \\
&= \sum_{n \in \Lambda_L} \int_{-\infty}^{-a_n^{-1}\epsilon} \rho(x) dx.
\end{aligned} \tag{6.3.4}$$

Since  $a_n^{-1} \rightarrow \infty$  as  $|n| \rightarrow \infty$  and  $\epsilon > 0$ , there exists an  $M \in \mathbb{N}$  such that

$$a_n^{-1}\epsilon > 1, \quad -a_n^{-1}\epsilon < -1 \quad \forall |n| > M.$$

Therefore, from (6.3.3) we get, for large enough  $L$ ,

$$\mathbb{E}^\omega(N_{-,L}^\omega(E)) = \sum_{n \in \Lambda_L} \int_{-\infty}^{-a_n^{-1}\epsilon} \rho(x) dx \tag{6.3.5}$$

$$= \sum_{n \in \Lambda_L, |n| > M} \int_{-\infty}^{-a_n^{-1}\epsilon} \rho(x) dx + \sum_{n \in \Lambda_L, |n| \leq M} \int_{-\infty}^{-1} \rho(x) dx. \tag{6.3.6}$$

Since  $\#\{n \in \mathbb{Z}^d : |n| \leq M\} \leq (2M + 1)^d$ , we have

$$\begin{aligned}
\sum_{n \in \Lambda_L, |n| \leq M} \int_{-\infty}^{-1} \rho(x) dx &\leq (2M + 1)^d \int_{-\infty}^{-1} \rho(x) dx \\
&= (2M + 1)^d \int_{-\infty}^{-1} \frac{1}{|x|^\delta} dx \\
&= \frac{(2M + 1)^d}{(\delta - 1)}, \quad \delta > 1 \text{ is given.}
\end{aligned} \tag{6.3.7}$$

If we take  $\beta_L = \sum_{n \in \Lambda_L} a_n^{(\delta-1)}$  then,  $\beta_L \rightarrow \infty$  as  $L \rightarrow \infty$  and we have, from (6.3.7),

$$\lim_{L \rightarrow \infty} \frac{1}{\beta_L} \sum_{n \in \Lambda_L, |n| \leq M} \int_{-\infty}^{-1} \rho(x) dx = 0. \quad (6.3.8)$$

Now,

$$\begin{aligned} \sum_{n \in \Lambda_L, |n| > M} \int_{-\infty}^{-a_n^{-1}\epsilon} \rho(x) dx &= \sum_{n \in \Lambda_L, |n| > M} a_n^{-1} \int_{-\infty}^{-\epsilon} \rho(a_n^{-1}t) dt \\ &= \sum_{n \in \Lambda_L, |n| > M} a_n^{(\delta-1)} \int_{-\infty}^{-\epsilon} \frac{1}{|t|^\delta} dt \\ &= \frac{\epsilon^{1-\delta}}{\delta-1} \sum_{n \in \Lambda_L, |n| > M} a_n^{(\delta-1)}, \quad \delta > 1. \end{aligned} \quad (6.3.9)$$

This equality gives,

$$\lim_{L \rightarrow \infty} \frac{1}{\beta_L} \sum_{n \in \Lambda_L, |n| > M} \int_{-\infty}^{-a_n^{-1}\epsilon} \rho(x) dx = \frac{\epsilon^{1-\delta}}{\delta-1}. \quad (6.3.10)$$

Then, using (6.3.8) and (6.3.10) in (6.3.5), we get

$$\lim_{L \rightarrow \infty} \frac{1}{\beta_L} \mathbb{E}^\omega(N_{-,L}^\omega(E)) = \frac{\epsilon^{1-\delta}}{\delta-1} = \frac{1}{(\delta-1)\epsilon^{(\delta-1)}} > 0. \quad (6.3.11)$$

A similar calculation with  $\mathbb{E}^\omega(N_{+,L}^\omega(E))$  gives,

$$\lim_{L \rightarrow \infty} \frac{1}{\beta_L} \mathbb{E}^\omega(N_{+,L}^\omega(E)) = \frac{(4d+\epsilon)^{1-\delta}}{\delta-1} = \frac{1}{(\delta-1)(4d+\epsilon)^{(\delta-1)}} > 0. \quad (6.3.12)$$

Now, using (6.3.11) and (6.3.12) from (6.3.2), we conclude the inequality

$$\frac{1}{(\delta-1)(4d+\epsilon)^{(\delta-1)}} \leq \liminf_{L \rightarrow \infty} \frac{1}{\beta_L} \mathbb{E}^\omega(N_L^\omega(E)) \leq \overline{\lim}_{L \rightarrow \infty} \frac{1}{\beta_L} \mathbb{E}^\omega(N_L^\omega(E)) \leq \frac{1}{(\delta-1)\epsilon^{(\delta-1)}}. \quad (6.3.13)$$

If we define

$$\tilde{N}_{\pm,L}^{\omega}(E) = \#\{j : E_j \geq E, E_j \in \sigma(A_{L\pm}^{\omega})\}, \quad \tilde{N}_L^{\omega}(E) = \#\{j : E_j \geq E, E_j \in \sigma(H_{\Lambda_L}^{\omega})\} \quad (6.3.14)$$

then the Min-max theorem and (6.3.1) together give

$$\tilde{N}_{-,L}^{\omega}(E) \leq \tilde{N}_L^{\omega}(E) \leq \tilde{N}_{+,L}^{\omega}(E). \quad (6.3.15)$$

If  $E = 2d + \epsilon > 2d$ , for some  $\epsilon > 0$ , a similar calculation results in

$$\frac{1}{(\delta - 1)(4d + \epsilon)^{(\delta-1)}} \leq \liminf_{L \rightarrow \infty} \frac{1}{\beta_L} \mathbb{E}^{\omega}(\tilde{N}_L^{\omega}(E)) \leq \overline{\lim}_{L \rightarrow \infty} \frac{1}{\beta_L} \mathbb{E}^{\omega}(\tilde{N}_L^{\omega}(E)) \leq \frac{1}{(\delta - 1)\epsilon^{(\delta-1)}}. \quad (6.3.16)$$

The inequalities (6.3.13) and (6.3.16) together prove the Theorem 6.1.4.  $\square$

**Proof of Corollary 6.1.5:**

Since  $H_{\Lambda_L}^{\omega}$  is a matrix of order  $(2L + 1)^d$ , we have  $\#\sigma(H_{\Lambda_L}^{\omega}) = (2L + 1)^d$ . If  $M_1 < -2d$  and  $M_2 > 2d$  then,

$$\#\left\{\sigma(H_{\Lambda_L}^{\omega}) \cap (-\infty, M_1]\right\} + \#\left\{\sigma(H_{\Lambda_L}^{\omega}) \cap (M_1, M_2)\right\} + \#\left\{\sigma(H_{\Lambda_L}^{\omega}) \cap [M_2, \infty)\right\} = (2L+1)^d. \quad (6.3.17)$$

Since

$$\frac{1}{(2L + 1)^d} \mathbb{E}^{\omega}\left\{\sigma(H_{\Lambda_L}^{\omega}) \cap (-\infty, M_1]\right\} = \frac{\beta_L}{(2L + 1)^d} \frac{1}{\beta_L} \mathbb{E}^{\omega}(N_L^{\omega}(M_1)), \quad (6.3.18)$$

and from (6.3.13) and Hypothesis 6.1.1 we have

$$\overline{\lim}_{L \rightarrow \infty} \frac{1}{\beta_L} \mathbb{E}^{\omega}(N_L^{\omega}(M_1)) < \infty, \quad \text{and} \quad \lim_{L \rightarrow \infty} \frac{\beta_L}{(2L + 1)^d} = 0,$$

the following limit holds

$$\lim_{L \rightarrow \infty} \frac{1}{(2L+1)^d} \mathbb{E}^\omega \{ \sigma(H_{\Lambda_L}^\omega) \cap (-\infty, M_1] \} = 0. \quad (6.3.19)$$

Similarly, using (6.3.16) we get

$$\lim_{L \rightarrow \infty} \frac{1}{(2L+1)^d} \mathbb{E}^\omega \{ \sigma(H_{\Lambda_L}^\omega) \cap [M_2, \infty) \} = 0. \quad (6.3.20)$$

Using the inequalities (6.3.17), (6.3.19) and (6.3.20), we see that for any interval  $(M_1, M_2)$  containing  $[-2d, 2d]$

$$\lim_{L \rightarrow \infty} \frac{1}{(2L+1)^d} \mathbb{E}^\omega (\# \{ \sigma(H_{\Lambda_L}^\omega) \cap (M_1, M_2) \}) = 1.$$

□

**Corollary 6.1.6:**

If  $M_1 < -2d$  then from (6.1.10) we have

$$\begin{aligned} \gamma_L(-\infty, M_1] &= \frac{1}{\beta_L} \mathbb{E}^\omega \left( \text{Tr}(E_{H_{\Lambda_L}^\omega}(-\infty, M_1]) \right) \\ &= \frac{1}{\beta_L} \mathbb{E}^\omega (N_L^\omega(M_1)). \quad (\text{using (6.1.8)}). \end{aligned} \quad (6.3.21)$$

This equality together with (6.3.13) gives

$$\overline{\lim}_{L \rightarrow \infty} \gamma_L(-\infty, M_1] \leq \frac{1}{(\delta-1)(-2d-M_1)^{\delta-1}} \quad (\text{using } \epsilon = -2d - M_1). \quad (6.3.22)$$

Similarly, for  $M_2 > 2d$ , using (6.3.16), we get

$$\overline{\lim}_{L \rightarrow \infty} \gamma_L[M_2, \infty) \leq \frac{1}{(\delta-1)(M_2-2d)^{\delta-1}} \quad (\text{using } \epsilon = M_2 - 2d). \quad (6.3.23)$$

Now, (6.3.22) and (6.3.23) together prove (6.1.12).

Let  $J = [E_1, E_2] \subset (-\infty, -2d)$  with  $|J| > 4d$ , set  $E_1 = -2d - \epsilon_1$ ,  $E_2 = -2d - \epsilon_2$  such that  $\epsilon_1 - \epsilon_2 > 4d$ . Then,

$$\begin{aligned}\gamma_L(J) &= \frac{1}{\beta_L} \mathbb{E}^\omega(N_L^\omega(E_2)) - \frac{1}{\beta_L} \mathbb{E}^\omega(N_L^\omega(E_1)) \\ &\geq \frac{1}{\beta_L} \mathbb{E}^\omega(N_{+,L}^\omega(E_2)) - \frac{1}{\beta_L} \mathbb{E}^\omega(N_{-,L}^\omega(E_1)) \quad (\text{using (6.3.2)}).\end{aligned}\tag{6.3.24}$$

Therefore, (6.3.12) and (6.3.11) give (6.1.13), namely

$$\liminf_{L \rightarrow \infty} \gamma_L(J) \geq \frac{1}{(\delta - 1)} \left[ \frac{1}{(4d + \epsilon_2)^{(\delta-1)}} - \frac{1}{\epsilon_1^{(\delta-1)}} \right] > 0.$$

Similar result holds even when  $J \subset (2d, \infty)$  with  $|J| > 4d$ . □

### Proof of Corollary 6.1.7:

From (6.1.12) we have

$$\sup_L \gamma_L((-\infty, M_1] \cup [M_2, \infty)) < \infty.\tag{6.3.25}$$

We write  $\mathbb{R} \setminus (M_1, M_2) = \bigcup_n A_n$ , countable union of compact sets. Now,  $\gamma_L \upharpoonright_{A_n}$  (restriction of  $\gamma_L$  to  $A_n$ ) admits a weakly convergence subsequence by Banach-Alaoglu Theorem. Then, by a diagonal argument we select a subsequence of  $\{\gamma_L\}$  which converges vaguely to a non-trivial measure, say  $\gamma$  on  $\mathbb{R} \setminus (M_1, M_2)$ .

The non-triviality of  $\gamma$  is given by the fact that if  $J \subset \mathbb{R} \setminus (M_1, M_2)$  is an interval such that  $4d < |J| < \infty$  then from (6.1.13) we get

$$\inf_L \gamma_L(J) > 0.$$

□

Before we proceed to the proof of Theorem 6.1.8, we state the following lemma.

**Lemma 6.3.1.** *Let  $\{X_n\}$  be sequence of random variables on a probability space  $(\Omega, \mathcal{B}, \mathbb{P})$*

satisfying

$$\sum_{n=1}^{\infty} \mathbb{P}(\omega : |X_n(\omega) - X(\omega)| > \epsilon) < \infty, \quad \epsilon > 0.$$

Then  $X_n \xrightarrow{n \rightarrow \infty} X$  a.e.  $\omega$ .

The proof follows from the Borel-Cantelli lemma [1.1.7](#).

**Proof of Theorem [6.1.8](#):**

Let  $E = -2d - \epsilon$  for some  $\epsilon > 0$  and define

$$X_n(\omega) := \chi_{\{\omega: q_n(\omega) \leq -a_n^{-1}\epsilon\}}. \quad (6.3.26)$$

Since  $\{q_n\}_n$  are i.i.d.,  $\{X_n\}$  is a sequence of independent random variables. Now, from [\(6.3.3\)](#) we have

$$N_{-,L}^\omega(E) = \sum_{n \in \Lambda_L} X_n(\omega). \quad (6.3.27)$$

We want to prove the following:

$$\lim_{L \rightarrow \infty} \frac{N_{-,L}^\omega(E) - \mathbb{E}^\omega(N_{-,L}^\omega(E))}{\beta_L} = 0 \quad a.e \ \omega. \quad (6.3.28)$$

In view of Lemma [6.3.1](#), in order to prove the above equation, it is enough to show that

$$\sum_{L=1}^{\infty} \mathbb{P}\left(\omega : \frac{|N_{-,L}^\omega(E) - \mathbb{E}^\omega(N_{-,L}^\omega(E))|}{\beta_L} > \eta\right) < \infty \quad \forall \ \eta > 0. \quad (6.3.29)$$

Now, using Chebyshev's inequality we get

$$\sum_{L=1}^{\infty} \mathbb{P}\left(\omega : \frac{|N_{-,L}^\omega(E) - \mathbb{E}^\omega(N_{-,L}^\omega(E))|}{\beta_L} > \eta\right) \leq \sum_{L=1}^{\infty} \frac{1}{\eta^2 \beta_L^2} \mathbb{E}^\omega \left( N_{-,L}^\omega(E) - \mathbb{E}^\omega(N_{-,L}^\omega(E)) \right)^2. \quad (6.3.30)$$

We proceed to estimate the RHS of the above inequality.

$$\begin{aligned}
\mathbb{E}^\omega \left( N_{-,L}^\omega(E) - \mathbb{E}^\omega(N_{-,L}^\omega(E)) \right)^2 &= \mathbb{E}^\omega \left( \sum_{n \in \Lambda_L} (X_n(\omega) - \mathbb{E}^\omega(X_n(\omega))) \right)^2 \\
&= \sum_{n \in \Lambda_L} \mathbb{E}^\omega \left( X_n(\omega) - \mathbb{E}^\omega(X_n(\omega)) \right)^2 \quad (X_n \text{ are independent}) \\
&= \sum_{n \in \Lambda_L} \left[ \mathbb{E}^\omega(X_n^2) - (\mathbb{E}^\omega(X_n))^2 \right] \\
&\leq \sum_{n \in \Lambda_L} \mathbb{E}^\omega(X_n^2) \\
&= \sum_{n \in \Lambda_L} \mathbb{E}^\omega(X_n) \quad (X_n^2 = X_n) \\
&= \mathbb{E}^\omega(N_{-,L}^\omega(E)) \quad (\text{using (6.3.27)}).
\end{aligned}$$

Now using the above estimate in (6.3.30) we get,

$$\begin{aligned}
\sum_{L=1}^{\infty} \mathbb{P} \left( \omega : \frac{|N_{-,L}^\omega(E) - \mathbb{E}^\omega(N_{-,L}^\omega(E))|}{\beta_L} > \eta \right) &\leq \frac{1}{\eta^2} \sum_{L=1}^{\infty} \frac{1}{\beta_L^2} \mathbb{E}^\omega(N_{-,L}^\omega(E)) \quad (6.3.31) \\
&= \frac{1}{\eta^2} \sum_{L=1}^{\infty} \frac{1}{\beta_L} \frac{1}{\beta_L} \mathbb{E}^\omega(N_{-,L}^\omega(E)) \\
&\leq \frac{C}{\eta^2} \sum_{L=1}^{\infty} \frac{1}{\beta_L} \quad (\text{using (6.3.11)}) \\
&\simeq \sum_{L=1}^{\infty} \frac{1}{L^{d-\alpha(\delta-1)}} \quad (\text{using (6.1.5)}).
\end{aligned}$$

As we have assumed in the theorem that  $0 < \alpha < \frac{1}{2}$ ,  $1 < \delta < \frac{1}{2\alpha}$  and  $d \geq 2$ , we have  $d - \alpha(\delta - 1) > 1$ . Thus, (6.3.29) follows from (6.3.31).

Therefore, from (6.3.28), for a.e.  $\omega$ , we have

$$\begin{aligned}
\lim_{L \rightarrow \infty} \frac{1}{\beta_L} N_{-,L}^\omega(E) &= \lim_{L \rightarrow \infty} \frac{1}{\beta_L} \mathbb{E}^\omega(N_{-,L}^\omega(E)) \quad (6.3.32) \\
&= \frac{1}{(\delta - 1)\epsilon^{(\delta-1)}} \quad (\text{using (6.3.11)}) \\
&= \frac{1}{(\delta - 1)(-2d - E)^{(\delta-1)}} \quad (E = -2d - \epsilon).
\end{aligned}$$



A similar calculation gives, for a.e.  $\omega$ ,

$$\begin{aligned} \lim_{L \rightarrow \infty} \frac{1}{\beta_L} N_{+,L}^\omega(E) &= \lim_{L \rightarrow \infty} \frac{1}{\beta_L} \mathbb{E}^\omega(N_{+,L}^\omega(E)) \\ &= \frac{1}{(\delta-1)(4d+\epsilon)^{(\delta-1)}} \quad (\text{using (6.3.12)}) \\ &= \frac{1}{(\delta-1)(2d-E)^{(\delta-1)}} \quad (E = -2d - \epsilon). \end{aligned} \tag{6.3.33}$$

The inequalities (6.3.32), (6.3.33) together with (6.3.2) give, for  $E < -2d$  for a.e.  $\omega$ ,

$$\frac{1}{(\delta-1)(2d-E)^{(\delta-1)}} \leq \liminf_{L \rightarrow \infty} \frac{1}{\beta_L} N_L^\omega(E) \leq \overline{\lim}_{L \rightarrow \infty} \frac{1}{\beta_L} N_L^\omega(E) \leq \frac{1}{(\delta-1)(-2d-E)^{(\delta-1)}}. \tag{6.3.34}$$

For  $E > 2d$  we compute  $\tilde{N}_{\pm,L}^\omega(E)$  (as in (6.3.14)) exactly in the same way as give above.

Thus, we can prove that, for a.e.  $\omega$ ,

$$\begin{aligned} \lim_{L \rightarrow \infty} \frac{1}{\beta_L} \tilde{N}_{+,L}^\omega(E) &= \lim_{L \rightarrow \infty} \frac{1}{\beta_L} \mathbb{E}^\omega(\tilde{N}_{+,L}^\omega(E)) \\ &= \frac{1}{(\delta-1)(E-2d)^{(\delta-1)}} \end{aligned}$$

and

$$\begin{aligned} \lim_{L \rightarrow \infty} \frac{1}{\beta_L} \tilde{N}_{-,L}^\omega(E) &= \lim_{L \rightarrow \infty} \frac{1}{\beta_L} \mathbb{E}^\omega(\tilde{N}_{-,L}^\omega(E)) \\ &= \frac{1}{(\delta-1)(2d+E)^{(\delta-1)}}. \end{aligned}$$

These equalities, together with (6.3.15) give the following. For  $E > 2d$ , a.e.  $\omega$ ,

$$\frac{1}{(\delta-1)(2d+E)^{(\delta-1)}} \leq \liminf_{L \rightarrow \infty} \frac{1}{\beta_L} \tilde{N}_L^\omega(E) \leq \overline{\lim}_{L \rightarrow \infty} \frac{1}{\beta_L} \tilde{N}_L^\omega(E) \leq \frac{1}{(\delta-1)(E-2d)^{(\delta-1)}}. \tag{6.3.35}$$

□

We conclude the chapter with the following comment:

It will be interesting to investigate the non-randomness of  $\frac{1}{L^{d-\alpha(\delta-1)}} N_L^\omega(\omega)$ ,  $E > 2d$

(or  $\frac{1}{L^{d-\alpha(\delta-1)}}\tilde{N}_L^\omega$ ,  $E < -2d$ ) as  $L \rightarrow \infty$ . Once we have the existence of this limit it will be easy to investigate the eigenvalue statistics in the pure point regime for the decaying model.

# Chapter 7

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