

# On bases for local Weyl modules in type $A$

*By*

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## List of publications arising from the thesis

### Journal

1. “Stability of the Chari-Pressley-Loktev bases for local Weyl modules of  $\mathfrak{sl}_2[t]$ ”, K. N. Raghavan, B. Ravinder, S. Viswanath, *Algebr. Represent. Theory* **18** (2015), no. 3, 613–632.

### Preprints

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**Dedicated to my parents**

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# Abstract

This thesis is a study of the Chari-Pressley-Loktev (CPL) bases [5, 7] for local Weyl modules of the current algebra  $\mathfrak{sl}_{r+1}[t]$ . As convenient parametrizing sets of these bases, we introduce the notion of partition overlaid patterns (POPs), which play a role analogous to that played by (Gelfand-Tsetlin) patterns in the representation theory of the special linear Lie algebra.

The notion of a POP leads naturally to the notion of area of a pattern. We observe that there is a unique pattern of maximal area among all those with a given bounding sequence and given weight. We give a combinatorial proof of this and discuss its representation theoretic relevance.

We prove the “stability”, i.e., compatibility in the long range, of CPL bases with respect to inclusions of local Weyl modules in the case  $r = 1$  and state it as a conjecture for  $r > 1$ . In order to state the conjecture, we establish a certain bijection between colored partitions and POPs, which is of interest in itself.

Irreducible representations of the special linear Lie algebra occur as grade zero pieces of the corresponding local Weyl modules. The CPL basis being homogeneous, those basis elements that are of grade zero form a basis for the irreducible representation space. We prove a triangular relationship between this basis and the classical Gelfand-Tsetlin basis.

# Synopsis

Let  $\mathfrak{g}$  be a finite-dimensional complex simple Lie algebra. Let  $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$  denote the corresponding current algebra, namely, the extension by scalars of  $\mathfrak{g}$  to the polynomial ring  $\mathbb{C}[t]$ . We think of  $\mathfrak{g}[t]$  as a Lie algebra over the complex numbers, graded by  $t$ , and are interested in the representation theory of its graded finite-dimensional modules. Local Weyl modules, introduced by Chari and Pressley in [7], are interesting graded finite-dimensional representations of  $\mathfrak{g}[t]$ . Let us fix  $\mathfrak{h}$  a Cartan subalgebra of  $\mathfrak{g}$  and  $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$  a Borel subalgebra of  $\mathfrak{g}$  containing  $\mathfrak{h}$ .

Corresponding to every dominant integral weight  $\lambda$  of  $\mathfrak{g}$ , there is one local Weyl module denoted by  $W(\lambda)$ . Recall that an element  $u \neq 0$  of a  $\mathfrak{g}[t]$ -module is said to be a *highest weight vector* of weight  $\lambda$  if

$$(\mathfrak{n}^+ \otimes \mathbb{C}[t]) u = 0, \quad (\mathfrak{h} \otimes t\mathbb{C}[t]) u = 0, \quad \text{and} \quad H u = \langle \lambda, H \rangle u, \quad \forall H \in \mathfrak{h}.$$

The  $W(\lambda)$  is universal among finite-dimensional  $\mathfrak{g}[t]$ -modules generated by a highest weight vector of weight  $\lambda$ , in the sense that any such module is a quotient of  $W(\lambda)$  [7]. The grade zero piece of the local Weyl module  $W(\lambda)$  is isomorphic to the finite-dimensional irreducible representation  $V(\lambda)$  of  $\mathfrak{g}$ , and the  $\mathfrak{h}$ -weights of  $W(\lambda)$  are precisely those of  $V(\lambda)$ .

Let us now specialize to the type  $A$  case, i.e., when  $\mathfrak{g}$  is the special Linear Lie algebra  $\mathfrak{sl}_{r+1}$ . Let  $\mathfrak{h}$  and  $\mathfrak{b}$  be respectively the diagonal and upper triangular subalgebras of

$\mathfrak{g} = \mathfrak{sl}_{r+1}$ . In [7], Chari and Pressley produced nice monomial bases for local Weyl modules in the case  $\mathfrak{g} = \mathfrak{sl}_2$ . Chari and Loktev in [5] clarified and extended the construction of these bases to the case  $\mathfrak{g} = \mathfrak{sl}_{r+1}$ , and used it to prove the conjecture [6] about the dimension of local Weyl modules and to show that local Weyl modules are in fact isomorphic to Demazure modules of certain representations of the affine Lie algebra  $\widehat{\mathfrak{g}}$ . In this thesis, we undertake a deeper study of these bases.

This thesis consists of six chapters, which we briefly describe below:

- In Chapter 1, we recall some basic concepts and preliminaries that will help present our results in this thesis.
- In Chapter 2, we present our results on the “stability” of the Chari-Pressley bases for local Weyl modules of  $\mathfrak{sl}_2[t]$ .
- In Chapter 3, we introduce the notion of a *partition overlaid pattern* (POP). We also introduce the notion of *area* of a Gelfand-Tsetlin pattern and describe our results on area maximizing Gelfand-Tsetlin patterns.
- In chapter 4, we give a bijection between colored partitions and POPs.
- In chapter 5, we state a conjecture about the “stability” of the Chari-Loktev bases for  $\mathfrak{sl}_3$  and beyond, using the bijection in chapter 4.
- In chapter 6, we present our results on triangularity of Gelfand-Tsetlin and Chari-Loktev bases for representations of  $\mathfrak{sl}_{r+1}$ .

The main results in this thesis are explained in the following sections.

# Stability of the Chari-Pressley-Loktev bases for local Weyl modules of $\mathfrak{sl}_2[t]$

In chapter 2 of the thesis, we investigate further the Chari-Pressley bases for local Weyl modules of  $\mathfrak{g} = \mathfrak{sl}_2$ , taking into account the perspective gained from the later work of Chari and Loktev [5]. The dominant integral weights for  $\mathfrak{g} = \mathfrak{sl}_2$  being parametrized by the non-negative integers, there is one local Weyl module  $W(n)$  for every integer  $n \geq 0$ . Let us restrict ourselves here, for the sake of simplicity, to the case when  $n$  is even. The local Weyl modules then get identified with Demazure modules of the basic representation  $L(\Lambda_0)$  of the affine Lie algebra  $\widehat{\mathfrak{sl}}_2$ . As such, they are related by a chain of inclusions:

$$(1) \quad W(0) \hookrightarrow W(2) \hookrightarrow W(4) \hookrightarrow \cdots \hookrightarrow W(n) \hookrightarrow W(n+2) \hookrightarrow \cdots \hookrightarrow L(\Lambda_0).$$

First, we give the corresponding chain of inclusions of indexing sets of the Chari-Pressley bases for these modules. Once the inclusion of indexing sets is established, it is natural to ask if the Chari-Pressley bases respect these inclusions. In the first part of this thesis, we focus on this question.

To state a little more precisely what we do, let  $\mathfrak{P}(n)$  denote the parametrizing set of the Chari-Pressley basis for  $W(n)$ : the elements of  $\mathfrak{P}(n)$  are pairs  $(k, \lambda)$  where  $k$  is an integer with  $0 \leq k \leq n$ , and  $\lambda$  is a partition whose Young diagram fits into an  $(n-k) \times k$  box. We assign a weight for an element of  $\mathfrak{P}(n)$  as the weight of the corresponding Chari-Pressley basis element of  $W(n)$  in  $L(\Lambda_0)$ . We first define a weight preserving embedding  $\psi$  of  $\mathfrak{P}(n)$  into  $\mathfrak{P}(n+2)$  for each  $n$ , thereby obtaining a chain  $\mathfrak{P}(0) \hookrightarrow \mathfrak{P}(2) \hookrightarrow \mathfrak{P}(4) \hookrightarrow \cdots$ . We study the compatibility of the Chari-Pressley bases with respect to this chain of embeddings.

As a first step, we define a normalized version of the Chari-Pressley bases by replacing the powers in the monomials by divided powers and introducing a sign factor. These



normalized bases, which we refer to throughout as the CPL (short for Chari-Pressley-Loktev) bases. For  $\xi \in \mathfrak{P}(n)$ , denote  $\mathbf{c}(\xi)$  as the corresponding CPL basis element of  $W(n)$ . For  $\xi \in \mathfrak{P}(n)$ , the CPL basis elements  $\mathbf{c}(\xi) \in W(n)$  and  $\mathbf{c}(\psi(\xi)) \in W(n+2)$  lie in the same weight space of  $L(\Lambda_0)$ . However, it is not true in general that  $\mathbf{c}(\xi)$  and  $\mathbf{c}(\psi(\xi))$  are equal as elements of  $L(\Lambda_0)$ , as the following example shows.

**Example 0.0.1.** *Let  $\lambda$  be the partition  $2 + 1$ , i.e.,  $\lambda = (2, 1, 0, 0, \dots)$ . Let  $\xi = (2, \lambda) \in \mathfrak{P}(4)$ , then  $\psi(\xi) = (3, \lambda) \in \mathfrak{P}(6)$ . Using the commutation relations in  $\widehat{\mathfrak{sl}}_2$ , it is easy to compute:*

$$\begin{aligned}\mathbf{c}(\xi) &= \frac{1}{3} (h \otimes t^{-3} - (h \otimes t^{-1})^3) v_{\Lambda_0}, \\ \mathbf{c}(\psi(\xi)) &= (h \otimes t^{-3} + (h \otimes t^{-2})(h \otimes t^{-1})) v_{\Lambda_0},\end{aligned}$$

where  $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  is the element of the standard Cartan subalgebra of  $\mathfrak{sl}_2$  and  $v_{\Lambda_0}$  is an element of  $L(\Lambda_0)$  of weight  $\Lambda_0$ . Both these vectors have weight  $\Lambda_0 - 3\delta$ . It is well known that the vectors  $(h \otimes t^{-3}) v_{\Lambda_0}$ ,  $(h \otimes t^{-2})(h \otimes t^{-1}) v_{\Lambda_0}$ ,  $(h \otimes t^{-1})^3 v_{\Lambda_0}$  form a basis for the weight space  $L(\Lambda_0)_{\Lambda_0 - 3\delta}$ . Thus, we conclude  $\mathbf{c}(\xi) \neq \mathbf{c}(\psi(\xi))$ .

However, in our main result (Theorem 0.0.2) we show that  $\mathbf{c}(\xi) = \mathbf{c}(\psi(\xi))$  for all “stable”  $\xi$ . More precisely, let

$$\mathfrak{P}^{\text{stab}}(n) := \{(k, \lambda) \in \mathfrak{P}(n) : |\lambda| \leq \min(n - k, k)\},$$

where  $|\lambda|$  is the sum of parts of the partition  $\lambda$ . We note that  $\xi \in \mathfrak{P}^{\text{stab}}(n)$  implies  $\psi(\xi) \in \mathfrak{P}^{\text{stab}}(n+2)$ . The main theorem of this part of thesis is the following (see [20, Theorem 6] for the journal version):

**Theorem 0.0.2.** *For every  $\xi = (k, \lambda) \in \mathfrak{P}^{\text{stab}}(n)$ , we have*

$$\mathbf{c}(\xi) = \mathbf{c}(\psi(\xi)),$$

*i.e., they are equal as elements of  $L(\Lambda_0)$ .*

The proof of Theorem 0.0.2 uses certain translation operators introduced by Frenkel and Kac [11] for completely different purposes.

As a consequence of Theorem 0.0.2, we obtain a basis for  $L(\Lambda_0)$  consisting of the stable CPL basis elements. More precisely, given an element  $\xi$  of  $\mathfrak{P}(n)$ , let  $\xi_k$  be its image in  $\mathfrak{P}(n+2k)$  (where  $k$  is a non-negative integer), and let  $\mathfrak{c}(\xi_k)$  be the corresponding CPL basis element. Consider the sequence  $\mathfrak{c}(\xi_k)$ ,  $k = 0, 1, 2, \dots$ , of elements in  $L(\Lambda_0)$ . Our main result (Theorem 0.0.2) implies that this sequence stabilizes for large  $k$ . In fact, it says that  $\mathfrak{c}(\xi_k)$  equals the stable value as soon as  $k$  is such that the weight space of  $W(n+2k)$  corresponding to the weight of  $\xi$  equals that of  $L(\Lambda_0)$ . Passing to the direct limit, we obtain a basis for  $L(\Lambda_0)$  consisting of the stable CPL basis elements. Moreover, we obtain an explicit description of the stable CPL basis in terms of elements of the Fock space of the homogeneous Heisenberg subalgebra of  $\widehat{\mathfrak{sl}}_2$ .

## On area maximizing Gelfand-Tsetlin patterns

One of our tasks in this thesis is to reinterpret the parametrizing set of the Chari-Loktev basis for local Weyl modules of  $\mathfrak{sl}_{r+1}[t]$ . Towards this end we introduce the notion of a *partition overlaid pattern* or *POP*. To define a POP, we first recall the notion of a Gelfand-Tsetlin pattern. A *Gelfand-Tsetlin pattern* (or just *pattern*)  $\mathcal{P}$  is an array of

integral row vectors  $\underline{\lambda}^j = (\lambda_1^j, \dots, \lambda_j^j)$ ,  $1 \leq j \leq r+1$  :

$$\begin{array}{cccc}
 & & & \lambda_1^1 \\
 & & & \lambda_1^2 \quad \lambda_2^2 \\
 & & \dots & \dots \quad \dots \\
 & \lambda_1^r & & \dots \quad \lambda_r^r \\
 \lambda_1^{r+1} & \lambda_2^{r+1} & & \dots \quad \lambda_{r+1}^{r+1}
 \end{array}$$

subject to the following conditions:

$$\lambda_i^{j+1} \geq \lambda_i^j \geq \lambda_{i+1}^{j+1}, \quad \forall 1 \leq i \leq j \leq r.$$

We call the last sequence  $\underline{\lambda}^{r+1}$  of the pattern  $\mathcal{P}$  is its *bounding sequence*. For example, the pattern consisting of row vectors 5; 7, 4; and 7, 5, 3 is written ( $r = 2$  here):

$$(2) \quad \begin{array}{ccc}
 & & 5 \\
 & 7 & 4 \\
 7 & 5 & 3
 \end{array}$$

A *partition overlaid pattern* (POP for short) consists of a GT pattern  $\mathcal{P} : \underline{\lambda}^1, \dots, \underline{\lambda}^{r+1}$ , and for every ordered pair  $(j, i)$  of integers with  $1 \leq j \leq r$  and  $1 \leq i \leq j$ , a partition  $\underline{\pi}(j)^i$  that fits into the rectangle  $(\lambda_i^{j+1} - \lambda_i^j, \lambda_i^j - \lambda_{i+1}^{j+1})$ . Here, a partition *fits into a rectangle*  $(a, b)$ , where  $a$  and  $b$  are non-negative integers, means the number of (non-zero) parts is at most  $a$  and the largest part is at most  $b$ . Example: a partition overlay on the pattern displayed in (2) consists of three partitions  $\underline{\pi}(1)^1$ ,  $\underline{\pi}(2)^1$ , and  $\underline{\pi}(2)^2$  that fit respectively into the rectangles  $(2, 1)$ ,  $(0, 2)$ , and  $(1, 1)$ .

Dominant integral weights for  $\mathfrak{g} = \mathfrak{sl}_{r+1}$  may be identified with non-increasing sequences of non-negative integers of length  $r+1$  with the last element of the sequence being

0. Patterns with bounding sequence (corresponding to)  $\lambda$  parametrize the Gelfand-Tsetlin (GT) basis for  $V(\lambda)$  [13]. Analogously, POPs with bounding sequence  $\lambda$  parametrize the Chari-Loktev (CL) basis for  $W(\lambda)$ . The weight of a pattern is the sequence of differences of successive row sums; this gives the  $\mathfrak{h}$ -weight of the corresponding GT basis element of  $V(\lambda)$ . The weight of the underlying pattern of a POP equals the  $\mathfrak{h}$ -weight of the corresponding CL basis element of  $W(\lambda)$ , and further the number of boxes in the partition overlay is its grade.

The notion of a POP leads naturally to the notion of the *area* of a pattern: For a pattern  $\mathcal{P}$ :  $\underline{\lambda}^1, \dots, \underline{\lambda}^{r+1}$ , we define the number  $\sum_{1 \leq i \leq j \leq r} (\lambda_i^{j+1} - \lambda_i^j)(\lambda_i^j - \lambda_{i+1}^{j+1})$  as its *area*. Example: the area of the pattern displayed in (2) is  $2 + 0 + 1 (= 3)$ . For a weight  $\mu$  of  $V(\lambda)$ , it turns out that the piece of highest grade in the  $\mu$ -weight space of the local Weyl module  $W(\lambda)$  is one dimensional. We give a representation theoretic proof of this fact. This suggests—even proves albeit circuitously—that there must be a unique pattern of highest area among all those with bounding sequence  $\lambda$  and weight  $\mu$ . We give a direct, elementary, and purely combinatorial proof of this.

To state our result more precisely, we recall the notion of majorization. For an element  $\underline{x} = (x_1, \dots, x_n)$  in  $\mathbb{R}^{r+1}$ , let  $\underline{x}^\downarrow = (x_1^\downarrow, \dots, x_n^\downarrow)$  be the vector whose co-ordinates are obtained by rearranging the  $x_j$  in weakly decreasing order. For elements  $\underline{x}$  and  $\underline{y}$  in  $\mathbb{R}^{r+1}$ , we say that  $\underline{x}$  *majorizes*  $\underline{y}$  and write  $\underline{x} \succ_m \underline{y}$  if

$$x_1^\downarrow + \dots + x_k^\downarrow \geq y_1^\downarrow + \dots + y_k^\downarrow, \quad \text{for all } 1 \leq k < r + 1, \quad \text{and} \quad x_1 + \dots + x_{r+1} = y_1 + \dots + y_{r+1}.$$

We are now ready to state our main result in this direction (see [21]). In the following theorem we will allow patterns to have real entries.

**Theorem 0.0.3.** *Let  $\lambda = \lambda_1 \geq \dots \geq \lambda_n$  be a non-increasing sequence of real numbers and  $\mu = (\mu_1, \dots, \mu_{r+1})$  an element of  $\mathbb{R}^{r+1}$  that is majorized by  $\lambda$ :  $\lambda \succ_m \mu$ . Then there is a unique pattern  $\mathcal{P}$  of maximum area among all those with bounding sequence  $\lambda$  and*

weight  $\mu$ . More over, if  $\lambda$  and  $\mu$  are integral, then the pattern  $\mathcal{P}$  has integer entries.

The proof of Theorem 0.0.3 is purely combinatorial. In the course of the proof, we also prove that the pattern  $\mathcal{P} : \underline{\lambda}^1, \dots, \underline{\lambda}^{r+1}$ , has the following properties:

- For any  $j$ ,  $1 \leq j \leq r + 1$ , its  $j^{\text{th}}$  row  $\underline{\lambda}^j$  majorizes the  $j^{\text{th}}$  row  $\underline{\kappa}^j$  of any pattern with bounding sequence  $\lambda$  and weight  $\mu$ :  $\underline{\lambda}^j \succ_m \underline{\kappa}^j$ .
- Its area equals  $\frac{1}{2}(\|\lambda\|^2 - \|\mu\|^2)$ , where  $\|\cdot\|$  is the Euclidean norm on  $\mathbb{R}^{r+1}$ .

## A bijection and the stability conjecture

In chapter 6 of the thesis, we state a conjecture about the “stability” of the Chari-Loktev bases for  $\mathfrak{sl}_3$  and beyond. Note that the conjecture is proved for  $\mathfrak{sl}_2$  in Chapter 2. To describe what is meant by stability, let  $\theta$  be the highest root of  $\mathfrak{g} = \mathfrak{sl}_{r+1}$ . We then have natural inclusions of local Weyl modules:

$$(3) \quad W(\lambda) \hookrightarrow W(\lambda + \theta) \hookrightarrow W(\lambda + 2\theta) \hookrightarrow \dots,$$

for a dominant integral weight  $\lambda$  of  $\mathfrak{g}$ . For  $\mathfrak{g} = \mathfrak{sl}_2$ , the above chain of inclusions is just (1). We may ask if there are corresponding natural inclusions of indexing sets of the Chari-Loktev bases for these modules. To prove that this is indeed the case, we first establish a combinatorial bijection, which is a generalization of the construction of Durfee squares [1]. Loosely speaking, the bijection identifies POPs with  $r$ -colored partitions. We use this bijection to give inclusions of indexing sets of the Chari-Loktev bases. Once the inclusion of indexing sets is established, we may ask if the Chari-Loktev bases respect the inclusions. We believe that they do have this *stability* property, and in fact formally commit ourselves to this effect (Conjecture 0.0.4).

We first describe below, a motivation for the bijection which identifies POPs with

$r$ -colored partitions. Given a dominant integral weight  $\lambda$  of  $\mathfrak{g} = \mathfrak{sl}_{r+1}$ , the local Weyl modules  $W(\lambda + k\theta)$ , for every non-negative integer  $k$ , are identified as Demazure modules of a fundamental representation  $L(\Lambda)$  of  $\widehat{\mathfrak{g}}$ . Hence, we have the chain (3) of Demazure submodules of  $L(\Lambda)$ , whose union is  $L(\Lambda)$ . Given a weight of  $L(\Lambda)$  (which corresponds to a non-negative integer  $d$ ), there exists large  $k$  such that the corresponding weight space of  $W(\lambda + k\theta)$  equals that of  $L(\Lambda)$ . Hence, we have a bijection from certain POPs onto the set of  $r$ -colored partitions of  $d$ . Indeed, the dimension of weight space of  $L(\Lambda)$  of the weight corresponding to  $d$  is the number of  $r$ -colored partitions of  $d$  [16, §12.13], and the Chari-Loktev basis for  $W(\lambda + k\theta)$  parametrized by POPs. We prove this by constructing an explicit bijection, and use this construction to state a conjecture about the “stability” of the Chari-Loktev bases.

We explain below what we do little more precisely. For the sake of simplicity, we consider here the weight  $\Lambda - d\delta$  of  $L(\Lambda)$  corresponding to a non-negative integer  $d$ . Let  $L(\Lambda)_{\Lambda - d\delta}$  (resp.  $W(\lambda + k\theta)_{\Lambda - d\delta}$ ) denote the weight space of  $L(\Lambda)$  (resp.  $W(\lambda + k\theta)$ ) of weight  $\Lambda - d\delta$ . We find a positive integer  $K_0$ , and construct a bijection  $\Omega_k$ , for  $k \geq K_0$ , from the set of  $r$ -colored partitions of  $d$  onto the set  $\mathbb{P}_k$  of POPs whose corresponding Chari-Loktev basis elements lie in  $W(\lambda + k\theta)_{\Lambda - d\delta}$ . We use the bijection  $\Psi_k := \Omega_{k+1} \circ \Omega_k^{-1}$ , for  $k \geq K_0$ , from  $\mathbb{P}_k$  to  $\mathbb{P}_{k+1}$ , to state a conjecture about “stability” of the Chari-Loktev bases.

As a first step, we slightly modify the definition of these bases as in [20, §3.2], by normalizing the generators of the local Weyl modules and replacing the powers in the monomials by divided powers. We call these modified bases as the CPL bases. For  $\xi \in \mathbb{P}_k$ , denote  $\mathfrak{c}(\xi)$  as the corresponding CPL basis element of  $W(\lambda + k\theta)_{\Lambda - d\delta}$ . For  $k \geq K_0$ , for  $\xi \in \mathbb{P}_k$  and  $\Psi_k(\xi) \in \mathbb{P}_{k+1}$ , the corresponding CPL bases elements respectively  $\mathfrak{c}(\xi) \in W(\lambda + k\theta)$  and  $\mathfrak{c}(\Psi_k(\xi)) \in W(\lambda + (k+1)\theta)$ , lie in the same weight space  $L(\Lambda)_{\Lambda - d\delta}$  of  $L(\Lambda)$ . We conjecture below that they are in fact equal as elements of  $L(\Lambda)$  up to a sign factor (see [21]).

**Conjecture 0.0.4.** *Let  $\xi \in \mathbb{P}_k$ . For  $k \geq K_0$ , we have*

$$\mathbf{c}(\xi) = \pm \mathbf{c}(\Psi_k(\xi)),$$

*i.e., they are equal as elements of  $L(\Lambda)$  up to a sign factor.*

Theorem 0.0.2 establishes the  $\mathfrak{g} = \mathfrak{sl}_2$  case of this conjecture. However, generalizing our methods to  $\mathfrak{g} = \mathfrak{sl}_3$  and beyond presents formidable technical difficulties.

## Triangularity of Gelfand-Tsetlin and Chari-Loktev bases for representations of $\mathfrak{sl}_{r+1}$

We recall that Chari and Loktev [5], in their study of local Weyl modules of the current algebra  $\mathfrak{sl}_{r+1}[t]$ , constructed a nice monomial basis for these modules consisting of homogeneous elements. Since the grade zero piece of the local Weyl module  $W(\lambda)$  is just the finite-dimensional irreducible representation  $V(\lambda)$  of  $\mathfrak{sl}_{r+1}$ , the Chari-Loktev basis elements of degree zero give a monomial basis for  $V(\lambda)$  [5, Corollary 2.1.3]. We shall call this the Chari-Loktev (CL) basis for  $V(\lambda)$ . Recall that for  $V(\lambda)$ , we also have the Gelfand-Tsetlin (GT) basis [13], which is parametrized by the set of patterns with bounding sequence  $\lambda$ .

In the last part of this thesis, we compare the Gelfand-Tsetlin and the Chari-Loktev bases for the irreducible representation  $V(\lambda)$  of  $\mathfrak{sl}_{r+1}$ . In order to compare them, we will assume that our bases are normalized such that the same highest weight vector belongs to both bases. Note that Chari-Loktev basis elements of the grade zero piece of  $W(\lambda)$  are corresponding to POPs with empty partition overlays or in other words simply to patterns. Thus, the CL and GT bases for  $V(\lambda)$  are both parametrized by the set of patterns with bounding sequence  $\lambda$ . We now ask a very natural question: is it true that

one basis is upper triangular with respect to the other, relative to some partial order on the set of patterns? We answer this question in the affirmative, and show that this holds with respect to the *row-wise dominance* partial order on patterns. We also compute the diagonal elements of the transition matrix. The *row-wise dominance* partial order  $\geq$  on patterns is defined by  $\mathcal{P} \geq \mathcal{Q}$  if for every  $j$ ,  $1 \leq j \leq r+1$ , the  $j^{\text{th}}$  row  $\underline{\lambda}^j$  of  $\mathcal{P}$  succeeds the  $j^{\text{th}}$  row  $\underline{\kappa}^j$  of  $\mathcal{Q}$  in the dominance order on partitions, i.e.,

$$\lambda_1^j + \cdots + \lambda_i^j \geq \kappa_1^j + \cdots + \kappa_i^j, \quad \forall 1 \leq i \leq j.$$

To state our result more precisely, let  $\text{GT}(\lambda)$  denote the set of patterns with bounding sequence  $\lambda$ . For  $\mathcal{P} \in \text{GT}(\lambda)$ , let  $\zeta_{\mathcal{P}}$  (resp.  $\text{CL}(\mathcal{P})$ ) denote the corresponding GT basis (resp. CL basis) element of  $V(\lambda)$ . We then prove (see [21]):

**Theorem 0.0.5.** *Let  $\lambda$  be a dominant integral weight of  $\mathfrak{sl}_{r+1}$ . Let  $\mathcal{P} \in \text{GT}(\lambda)$  with the array of integral row vectors  $\underline{\lambda}^j = (\lambda_1^j, \dots, \lambda_j^j)$ , for  $1 \leq j \leq r+1$ . Then we have the following:*

$$(4) \quad \text{CL}(\mathcal{P}) = \sum_{\substack{\mathcal{Q} \in \text{GT}(\lambda) \\ \mathcal{Q} \geq \mathcal{P}}} c_{\mathcal{Q}} \zeta_{\mathcal{Q}}, \quad \text{for some } c_{\mathcal{Q}} \in \mathbb{C},$$

where the co-efficient  $c_{\mathcal{P}}$  of  $\zeta_{\mathcal{P}}$  in (4) is equal to

$$\prod_{1 \leq i < j \leq r+1} \prod_{d_{ji}=0}^{\lambda_i^j - \lambda_i^{j-1} - 1} \prod_{j'=i+1}^{j-1} \frac{1}{(\lambda_i^{j-1} - \lambda_{j'}^{j-1} + d_{ji}) + (j' - i + 1)}.$$



# Chapter 1

## Preliminaries

In this chapter, we recall certain well-known definitions and results which will be used in this thesis. We begin with brief history.

The theory of semisimple Lie algebras and their representations lies at the heart of modern mathematics. The finite-dimensional simple Lie algebras over the field of complex numbers were classified in the works of Elie Cartan and Wilhelm Killing in the 1930's. There are four infinite series  $A_r$  ( $r \geq 1$ );  $B_r, C_r$  ( $r \geq 2$ );  $D_r$  ( $r \geq 4$ ) which are called the classical Lie algebras, and five exceptional Lie algebras  $E_6, E_7, E_8, F_4, G_2$ . The Lie algebras of type  $A, D$ , and  $E$  are called of type simply laced. The structure of these Lie algebras is uniformly described in terms of certain finite sets of vectors in a Euclidean space called root systems. The theory of finite-dimensional representations of semisimple Lie algebras is largely reduced to the study of their irreducible representations, due to Weyl's complete reducibility theorem. The irreducibles are parametrized by their highest weights.

In the late 1960's, Victor Kac and Robert Moody built on this work and independently defined and studied a class of Lie algebras, now called the Kac-Moody Lie algebras. These are generalizations of the finite-dimensional simple Lie algebras. The theory of Kac-

Moody Lie algebras and their representations has numerous connections with other areas of mathematics and physics. The reader is referred to, e.g., the books of Bourbaki [2], Carter [3], Dixmier [8], Humphreys [14], or Kac [16] for a detailed exposition of the theory.

Throughout the thesis,  $\mathbb{C}$  denotes the field of complex numbers,  $\mathbb{Z}$  the set of integers,  $\mathbb{Z}_{\geq 0}$  the set of non-negative integers,  $\mathbb{N}$  the set of positive integers,  $\mathbb{C}[t]$  the polynomial ring in an indeterminate  $t$ ,  $\mathbb{C}[t, t^{-1}]$  the ring of Laurent polynomials, and  $\mathbf{U}(\mathfrak{a})$  the universal enveloping algebra corresponding to a complex Lie algebra  $\mathfrak{a}$ .

## 1.1 The simple Lie algebra $\mathfrak{g}$

Let  $\mathfrak{g}$  be a finite-dimensional simple Lie algebra over  $\mathbb{C}$  of rank  $r$ , with Cartan subalgebra  $\mathfrak{h}$ . Set  $I = \{1, 2, \dots, r\}$ . Let  $R$  (resp.  $R^+$ ) be the set of roots (resp. positive roots) of  $\mathfrak{g}$  with respect to  $\mathfrak{h}$  and let  $\theta \in R^+$  be the highest root in  $R$ . Let  $(\cdot | \cdot)$  be a non-degenerate, symmetric, invariant bilinear form on  $\mathfrak{h}^*$  normalized so that the square length of a long root is two. For  $\alpha \in R$ , let  $\alpha^\vee \in \mathfrak{h}$  be the corresponding co-root and let  $\mathfrak{g}_\alpha$  be the corresponding root space of  $\mathfrak{g}$ . It is well-known that  $\dim \mathfrak{g}_\alpha = 1, \forall \alpha \in R$ . For each  $\alpha \in R^+$ , we fix non-zero elements  $x_\alpha^\pm \in \mathfrak{g}_{\pm\alpha}$  such that  $[x_\alpha^+, x_\alpha^-] = \alpha^\vee$ . We set  $\mathfrak{n}^\pm = \bigoplus_{\alpha \in R^+} \mathfrak{g}_{\pm\alpha}$ .

The weight lattice  $P$  (resp. the set of dominant weights  $P^+$ ) is the  $\mathbb{Z}$ -span (resp.  $\mathbb{Z}_{\geq 0}$ -span) of the fundamental weights  $\varpi_i, i \in I$  of  $\mathfrak{g}$ . The root lattice  $Q$  is the  $\mathbb{Z}$ -span of the simple roots  $\alpha_i, i \in I$  of  $\mathfrak{g}$ . The dominant root lattice  $Q^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$ . Set  $d_i = 2/(\alpha_i | \alpha_i), \forall i \in I$ . We define  $L = \sum_{i \in I} \mathbb{Z} d_i \varpi_i$ , a sub lattice of  $P$ , and  $M = \sum_{i \in I} \mathbb{Z} d_i \alpha_i$ , a sub lattice of  $Q$ . We note that  $L$  and  $M$  are the images of the co-weight and co-root lattices respectively under the identification of  $\mathfrak{h}$  and  $\mathfrak{h}^*$  induced by the form  $(\cdot | \cdot)$ .

### 1.1.1 The Weyl group of $\mathfrak{g}$

For each  $i \in I$ , the fundamental reflection  $s_{\alpha_i}$  (or  $s_i$ ) is given by

$$s_{\alpha_i}(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i, \quad \forall \lambda \in \mathfrak{h}^*.$$

The subgroup  $W$  of  $GL(\mathfrak{h}^*)$  generated by all fundamental reflections  $s_i$ ,  $i \in I$  is called the Weyl group of  $\mathfrak{g}$ . Given  $w \in W$ , let  $\ell(w)$  be the length of a reduced expression for  $w$ . Let  $w_0$  be the longest element in  $W$ .

### 1.1.2 The finite-dimensional irreducible $\mathfrak{g}$ -modules

It is well-known that the finite-dimensional irreducible  $\mathfrak{g}$ -modules (up to isomorphism) are indexed by the elements of  $P^+$ . For  $\lambda \in P^+$ , the corresponding finite-dimensional irreducible  $\mathfrak{g}$ -module  $V(\lambda)$  is the cyclic  $\mathfrak{g}$ -module generated by an element  $v_\lambda$  with the following defining relations:

$$x_\alpha^+ v_\lambda = 0, \quad h v_\lambda = \langle \lambda, h \rangle v_\lambda, \quad (x_\alpha^-)^{\langle \lambda, \alpha^\vee \rangle + 1} v_\lambda = 0, \quad \forall \alpha \in R^+, h \in \mathfrak{h}.$$

## 1.2 The affine Lie algebra $\widehat{\mathfrak{g}}$

Let  $\mathfrak{g}$  be a finite-dimensional simple Lie algebra over  $\mathbb{C}$  as in §1.1. Let  $\widehat{\mathfrak{g}}$  be the corresponding (untwisted) affine Lie algebra defined by

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

where  $c$  is central and the other Lie brackets are given by

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + m\delta_{m,-n}(x|y)c,$$

$$[d, x \otimes t^m] = m(x \otimes t^m),$$

for all  $x, y \in \mathfrak{g}$  and integers  $m, n$ . The Lie subalgebras  $\widehat{\mathfrak{h}}$ ,  $\widehat{\mathfrak{n}}^+$ , and  $\widehat{\mathfrak{b}}$  of  $\widehat{\mathfrak{g}}$  are defined as follows:

$$\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d, \quad \widehat{\mathfrak{n}}^+ = \mathfrak{n}^+ \otimes \mathbb{C}[t] \oplus (\mathfrak{n}^- \oplus \mathfrak{h}) \otimes t\mathbb{C}[t], \quad \widehat{\mathfrak{b}} = \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}^+.$$

We regard  $\mathfrak{h}^*$  as a subspace of  $\widehat{\mathfrak{h}}^*$  by setting  $\langle \lambda, c \rangle = \langle \lambda, d \rangle = 0$  for  $\lambda \in \mathfrak{h}^*$ . For  $\xi \in \widehat{\mathfrak{h}}^*$ , let  $\xi|_{\mathfrak{h}}$  be the element of  $\mathfrak{h}^*$  obtained by restricting  $\xi$  to  $\mathfrak{h}$ . Let  $\delta, \Lambda_0 \in \widehat{\mathfrak{h}}^*$  be given by

$$\langle \delta, \mathfrak{h} + \mathbb{C}c \rangle = 0, \quad \langle \delta, d \rangle = 1, \quad \langle \Lambda_0, \mathfrak{h} + \mathbb{C}d \rangle = 0, \quad \langle \Lambda_0, c \rangle = 1.$$

Extend the non-degenerate form on  $\mathfrak{h}^*$  to a non-degenerate symmetric bilinear form on  $\widehat{\mathfrak{h}}^*$  by setting,

$$(\mathfrak{h}^*|\mathbb{C}\delta + \mathbb{C}\Lambda_0) = (\delta|\delta) = (\Lambda_0|\Lambda_0) = 0 \quad \text{and} \quad (\delta|\Lambda_0) = 1.$$

Set  $\widehat{I} = I \cup \{0\}$ . The elements  $\alpha_i, i \in \widehat{I}$  where  $\alpha_0 = \delta - \theta$  are the set of simple roots of  $\widehat{\mathfrak{g}}$ , and the elements  $\alpha_i^\vee, i \in \widehat{I}$  where  $\alpha_0^\vee = c - \theta^\vee$  are the corresponding co-roots. The Chevalley generators  $e_i$  and  $f_i$  ( $i \in \widehat{I}$ ) of  $\widehat{\mathfrak{g}}$  are given by following:

$$e_0 = x_\theta^- \otimes t, \quad f_0 = x_\theta^+ \otimes t^{-1}, \quad e_i = x_{\alpha_i}^+ \otimes 1, \quad f_i = x_{\alpha_i}^- \otimes 1, \quad (i \in I).$$

Let  $\widehat{R}^+$  be the set of positive roots,

$$\widehat{R}^+ = \{\alpha + n\delta : \alpha \in R, n \in \mathbb{N}\} \cup R^+ \cup \{n\delta : n \in \mathbb{N}\},$$

and  $\widehat{R}^-$  be the set of negative roots,

$$\widehat{R}^- = \{\alpha + n\delta : \alpha \in R, n \in -\mathbb{N}\} \cup R^- \cup \{n\delta : n \in -\mathbb{N}\}.$$

Let  $\widehat{R}_{re} = \{\alpha + n\delta : \alpha \in R, n \in \mathbb{Z}\}$  be the set of real roots, and  $\widehat{R}_{im} = \{n\delta : n \in \mathbb{Z} \setminus \{0\}\}$  be the set of imaginary roots. The set of roots  $\widehat{R}$  of  $\widehat{\mathfrak{g}}$  is given by  $\widehat{R} = \widehat{R}_{re} \cup \widehat{R}_{im} = \widehat{R}^- \cup \widehat{R}^+$ . The root space decomposition of  $\widehat{\mathfrak{g}}$  is given by

$$\mathfrak{g} = \bigoplus_{\gamma \in \widehat{R}} \mathfrak{g}_\gamma \oplus \widehat{\mathfrak{h}},$$

where  $\mathfrak{g}_\gamma = \{x \in \widehat{\mathfrak{g}} : [h, x] = \langle \gamma, h \rangle x, \forall h \in \widehat{\mathfrak{h}}\}$ . It is well-known that  $\dim \widehat{\mathfrak{g}}_\gamma = 1, \forall \gamma \in \widehat{R}_{re}$ . For each real root  $\alpha + n\delta$ , we have the Lie subalgebra of  $\widehat{\mathfrak{g}}$  generated by  $\{x_\alpha^+ \otimes t^n, x_\alpha^- \otimes t^{-n}\}$  which is isomorphic to  $\mathfrak{sl}_2$ . Let  $\widehat{Q} = \sum_{i \in \widehat{I}} \mathbb{Z}\alpha_i$  be the root lattice, and  $\widehat{Q}^+ = \sum_{i \in \widehat{I}} \mathbb{Z}_{\geq 0}\alpha_i$ . The weight lattice (resp. the set of dominant integral weights) is defined by

$$\widehat{P} \text{ (resp. } \widehat{P}^+) = \{\lambda \in \widehat{\mathfrak{h}}^* : \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z} \text{ (resp. } \mathbb{Z}_{\geq 0}), \forall i \in \widehat{I}\}.$$

For an element  $\lambda \in \widehat{P}$ , the integer  $\langle \lambda, c \rangle$  is called the *level* of  $\lambda$ .

### 1.2.1 The Weyl group of $\widehat{\mathfrak{g}}$

For each  $i \in \widehat{I}$ , the fundamental reflection  $s_{\alpha_i}$  (or  $s_i$ ) is given by

$$s_{\alpha_i}(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i, \quad \forall \lambda \in \widehat{\mathfrak{h}}^*.$$

The subgroup  $\widehat{W}$  of  $GL(\widehat{\mathfrak{h}}^*)$  generated by all fundamental reflections  $s_i, i \in \widehat{I}$  is called the affine Weyl group or the Weyl group of  $\widehat{\mathfrak{g}}$ . We regard  $W$  naturally as a subgroup of  $\widehat{W}$ . Given  $\alpha \in \widehat{\mathfrak{h}}^*$ , let  $t_\alpha \in GL(\widehat{\mathfrak{h}}^*)$  be defined by

$$t_\alpha(\lambda) = \lambda + (\lambda|\delta)\alpha - (\lambda|\alpha)\delta - \frac{1}{2}(\lambda|\delta)(\alpha|\alpha)\delta \quad \text{for } \lambda \in \widehat{\mathfrak{h}}^*.$$

The translation subgroup  $T_M$  of  $\widehat{W}$  is defined by  $T_M = \{t_\alpha \in GL(\widehat{\mathfrak{h}}^*) : \alpha \in M\}$  (where, you may recall the definition of  $M$  from §1.1).

The following proposition gives the relation between  $W$  and  $\widehat{W}$ . It is well-known and may be found in [16].

**Proposition 1.2.1.** [16, Proposition 6.5]  $\widehat{W} = W \times T_M$ .

The extended affine Weyl group  $\widetilde{W}$  is the semi-direct product

$$\widetilde{W} = W \rtimes T_L,$$

where  $T_L = \{t_\alpha \in GL(\widehat{\mathfrak{h}}^*) : \alpha \in L\}$ . Let  $\widehat{C} = \{\Lambda \in \widehat{\mathfrak{h}}^* : \langle \Lambda, \alpha_i^\vee \rangle \geq 0 \forall i \in \widehat{I}\}$  be the fundamental Weyl chamber for  $\widehat{\mathfrak{g}}$ . Let  $\Sigma = \{\sigma \in \widetilde{W} : \sigma(\widehat{C}) = \widehat{C}\}$ ; it is a subgroup of the group of diagram automorphisms of  $\widehat{\mathfrak{g}}$ . Then  $\Sigma$  provides a complete system of coset representatives of  $\widetilde{W}/\widehat{W}$  and we have  $\widetilde{W} = \widehat{W} \rtimes \Sigma$  (see [10], as also [2]).

Given  $w \in \widehat{W}$ , let  $\ell(w)$  be the length of a reduced expression for  $w$ . The length function  $\ell$  is extended to  $\widetilde{W}$  by setting

$$(1.1) \quad \ell(w\sigma) = \ell(w),$$

for  $w \in \widehat{W}$  and  $\sigma \in \Sigma$ .

## 1.2.2 The category $\mathcal{O}$

A  $\widehat{\mathfrak{g}}$ -module  $V$  is called  $\widehat{\mathfrak{h}}$ -diagonalizable if it admits a weight space decomposition

$$V = \bigoplus_{\mu \in \widehat{\mathfrak{h}}^*} V_\mu,$$

where  $V_\mu = \{v \in V : hv = \langle \mu, h \rangle v, \forall h \in \widehat{\mathfrak{h}}\}$ . A non-zero vector of  $V_\mu$  is called a weight vector of weight  $\mu$ . Let  $P(V) := \{\mu \in \widehat{\mathfrak{h}}^* : V_\mu \neq 0\}$  denote the set of weights of  $V$ . For  $\Lambda \in \widehat{\mathfrak{h}}^*$ , let us denote  $D(\Lambda) := \{\mu \in \widehat{\mathfrak{h}}^* : \mu \leq \Lambda\}$ . Recall that the partial order  $\leq$  on  $\widehat{\mathfrak{h}}^*$  is defined by  $\mu \leq \Lambda$  iff  $\Lambda - \mu \in \widehat{Q}^+$ .

**Definition 1.2.2.** A  $\widehat{\mathfrak{g}}$ -module  $V$  is said to be in category  $\mathcal{O}$  if

1. It is  $\widehat{\mathfrak{h}}$ -diagonalizable with finite-dimensional weight spaces, and
2. There exist finitely many elements  $\Lambda_1, \dots, \Lambda_m \in \widehat{\mathfrak{h}}^*$  such that  $P(V) \subset \cup_{i=1}^m D(\Lambda_i)$ .

The morphisms in  $\mathcal{O}$  are homomorphisms of  $\widehat{\mathfrak{g}}$ -modules. The category  $\mathcal{O}$  is abelian.

### 1.2.3 Highest-weight modules

Highest-weight modules are important examples of objects from the category  $\mathcal{O}$ .

**Definition 1.2.3.** A  $\widehat{\mathfrak{g}}$ -module  $V$  is said to be a highest-weight module with highest weight  $\Lambda \in \widehat{\mathfrak{h}}^*$  if there exists a non-zero vector  $v_\Lambda$  such that

$$(1.2) \quad \widehat{\mathfrak{n}}^+ v_\Lambda = 0, \quad h v_\Lambda = \langle \Lambda, h \rangle v_\Lambda, \quad \forall h \in \widehat{\mathfrak{h}}, \quad \text{and} \quad \mathbf{U}(\widehat{\mathfrak{g}}) v_\Lambda = V.$$

**Remark 1.2.4.** By condition (1.2) it is easy to see that  $\mathbf{U}(\widehat{\mathfrak{n}}^-) v_\Lambda = V$ , and we have  $V = \bigoplus_{\mu \leq \Lambda} V_\mu$ ,  $V_\Lambda = \mathbb{C}v_\Lambda$ ,  $\dim V_\mu < \infty \quad \forall \mu \in \widehat{\mathfrak{h}}^*$ . Therefore, a highest-weight module is an object of category  $\mathcal{O}$ .

Now, we recall an important family of highest-weight modules known as Verma modules.

**Definition 1.2.5.** A  $\widehat{\mathfrak{g}}$ -module  $M(\Lambda)$  with highest weight  $\Lambda$  is called a Verma module if every  $\widehat{\mathfrak{g}}$ -module with highest weight  $\Lambda$  is a quotient of  $M(\Lambda)$ .

The following proposition justifies the importance of Verma modules.

**Proposition 1.2.6.** [16, Proposition 9.2]

1. For every  $\Lambda \in \widehat{\mathfrak{h}}^*$  there exists a unique (up to isomorphism) Verma module  $M(\Lambda)$ .

2. Viewed as a  $U(\widehat{\mathfrak{n}}^-)$ -module,  $M(\Lambda)$  is a free module of rank 1 generated by the highest weight vector.
3.  $M(\Lambda)$  contains a unique proper maximal submodule  $M'(\Lambda)$ .

It follows from part 3 of the above proposition that for  $\Lambda \in \widehat{\mathfrak{h}}^*$ , there is a unique irreducible module of highest weight  $\Lambda$  which we denote by  $L(\Lambda) := M(\Lambda)/M'(\Lambda)$ . The  $\widehat{\mathfrak{g}}$ -modules  $L(\Lambda)$ , for  $\Lambda \in \widehat{\mathfrak{h}}^*$ , exhaust all irreducible modules of the category  $\mathcal{O}$  [16, Proposition 9.3].

## 1.2.4 Integrable modules

**Definition 1.2.7.** A  $\widehat{\mathfrak{g}}$ -module  $V$  is said to be integrable if the following holds:

- It is  $\widehat{\mathfrak{h}}$ -diagonalizable with finite-dimensional weight spaces.
- The Chevalley generators  $e_i$  and  $f_i$  ( $i \in \widehat{I}$ ) are locally nilpotent on  $V$ . i.e., given any  $v \in V$ , there exists  $n \geq 0$  such that  $e_i^n v = 0 = f_i^n v$ .

We will further restrict our attention to the category  $\mathcal{O}^{\text{int}}(\widehat{\mathfrak{g}})$  of integrable modules in category  $\mathcal{O}$ . We record the following fact from [16].

**Proposition 1.2.8.** [16, Lemma 10.1] The  $\widehat{\mathfrak{g}}$ -module  $L(\Lambda)$  is integrable if and only if  $\Lambda \in \widehat{P}^+$ .

The following Proposition gives the defining relations for the modules  $L(\Lambda)$ ,  $\Lambda \in \widehat{P}^+$ .

**Proposition 1.2.9.** [16, Corollary 10.4] Let  $\Lambda \in \widehat{P}^+$ . The  $\widehat{\mathfrak{g}}$ -module  $L(\Lambda)$  is the cyclic module generated by  $v_\Lambda$ , with defining relations

$$\begin{aligned} h v_\Lambda &= \langle \Lambda, h \rangle v_\Lambda \quad \forall h \in \widehat{\mathfrak{h}}, \\ e_i v_\Lambda &= 0 \quad (i \in \widehat{I}), \\ f_i^{\langle \Lambda, \alpha_i^\vee \rangle + 1} v_\Lambda &= 0 \quad (i \in \widehat{I}). \end{aligned}$$



In particular, an integrable highest-weight module of  $\widehat{\mathfrak{g}}$  is automatically irreducible.

The  $\widehat{\mathfrak{g}}$ -modules  $L(\Lambda)$ , for  $\Lambda \in \widehat{P}^+$ , exhaust all irreducible integrable modules of the category  $\mathcal{O}$  [16, Corollary 10.7]. Denote by  $P(\Lambda)$  the set of weights of  $L(\Lambda)$ .

The following proposition may be found in [16]. For  $\mathfrak{g} = \mathfrak{sl}_2$ , see also [3, Proposition 20.22].

**Proposition 1.2.10.** [16, Lemma 12.6, Proposition 12.13] *Assume that  $\mathfrak{g}$  is simply laced of rank  $r$ . Let  $\Lambda \in \widehat{P}^+$  be of level 1. Then*

1.  $P(\Lambda) = \{t_\mu(\Lambda) - n\delta : \mu \in Q, n \in \mathbb{Z}_{\geq 0}\}$ ,
2. For  $\mu \in Q$ ,  $n \in \mathbb{Z}_{\geq 0}$ , we have

$$\dim L(\Lambda)_{t_\mu(\Lambda) - n\delta} = \text{the number of partitions of } n \text{ into } r \text{ colors.}$$

### 1.2.5 Demazure modules

Let  $L(\Lambda)$  be the irreducible integrable highest-weight module of  $\widehat{\mathfrak{g}}$  corresponding to a dominant integral weight  $\Lambda$ . Given an element  $w$  of  $\widehat{W}$ , define a  $\widehat{\mathfrak{b}}$ -submodule  $V_w(\Lambda)$  of  $L(\Lambda)$  by

$$V_w(\Lambda) = \mathbf{U}(\widehat{\mathfrak{b}})(L(\Lambda)_{w\Lambda}).$$

We call the  $\widehat{\mathfrak{b}}$ -module  $V_w(\Lambda)$  as a Demazure module. Since  $f_i L(\Lambda)_{w\Lambda} = 0$  holds if and only if  $\langle w\Lambda, \alpha_i^\vee \rangle \leq 0$ , we see that  $V_w(\Lambda)$  is  $\mathfrak{g}$ -stable if and only if  $\langle w\Lambda, \alpha_i^\vee \rangle \leq 0, \forall i \in I$ . The notion of Demazure module associated to an element of  $\widetilde{W}$  is defined by setting

$$V_{w\sigma}(\Lambda) = V_w(\sigma\Lambda),$$

for  $\sigma \in \Sigma$  and  $w \in \widehat{W}$ .

### 1.2.6 Inclusions of Demazure modules

We first recall the notion of Bruhat order on  $\widehat{W}$ . Let  $w_1, w_2 \in \widehat{W}$ . Bruhat order is the partial order relation  $\leq$  on  $\widehat{W}$  defined by  $w_1 \leq w_2$  if given a reduced expression  $s_1 s_2 \cdots s_r$  for  $w_2$ ,  $w_1$  can be obtained as a subexpression of this reduced expression. i.e.,  $w_1 = s_{i_1} s_{i_2} \cdots s_{i_q}$  for some  $1 \leq i_1 < i_2 < \cdots < i_q \leq r$  (see [15, §5.10]).

Let  $\Lambda \in \widehat{P}^+$  and  $\widehat{W}_\Lambda := \{w \in \widehat{W} : w\Lambda = \Lambda\}$ . We now recall the notion of Bruhat order on  $\widehat{W}/\widehat{W}_\Lambda$ . For elements  $w_1, w_2$  of  $\widehat{W}/\widehat{W}_\Lambda$ , let  $w'_1$  (resp.  $w'_2$ ) is a minimal length element of  $\widehat{W}$  in the coset  $w_1$  (resp.  $w_2$ ). Bruhat order  $\leq$  on  $\widehat{W}/\widehat{W}_\Lambda$  is defined by  $w_1 \leq w_2$  if  $w'_1 \leq w'_2$  in the usual Bruhat order on  $\widehat{W}$ .

For elements  $w_1 \leq w_2$  of  $\widehat{W}/\widehat{W}_\Lambda$ , where  $\leq$  denotes the Bruhat order on  $\widehat{W}/\widehat{W}_\Lambda$ , the Demazure module  $V_{w_1}(\Lambda)$  is included in  $V_{w_2}(\Lambda)$  (as submodules of  $L(\Lambda)$ ) (see [17, Proposition 3.2.4]).

## 1.3 The current algebra $\mathfrak{g}[t]$

The current algebra  $\mathfrak{g}[t]$  associated to  $\mathfrak{g}$  is defined as  $\mathfrak{g} \otimes \mathbb{C}[t]$ , with the Lie bracket

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} \quad \forall x, y \in \mathfrak{g}, m, n \in \mathbb{Z}_{\geq 0}.$$

The degree grading on  $\mathbb{C}[t]$  gives a natural  $\mathbb{Z}_{\geq 0}$ -grading on  $\mathbf{U}(\mathfrak{g}[t])$ : the element  $(a_1 \otimes t^{r_1}) \cdots (a_k \otimes t^{r_k})$ , for  $a_i \in \mathfrak{g}, r_i \in \mathbb{Z}_{\geq 0}$ , has grade  $r_1 + \cdots + r_k$ . A graded  $\mathfrak{g}[t]$ -module is a  $\mathbb{Z}$ -graded vector space  $V = \bigoplus_{n \in \mathbb{Z}} V[n]$  such that

$$(\mathfrak{g} \otimes t^m)V[n] \subset V[n+m], \quad \forall m \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z}.$$

Let  $\text{ev}_0 : \mathfrak{g}[t] \rightarrow \mathfrak{g}$  be the morphism of Lie algebras given by setting  $t = 0$ . The pull back of any  $\mathfrak{g}$ -module  $V$  by  $\text{ev}_0$  defines a graded  $\mathfrak{g}[t]$ -module structure on  $V$ , and we denote

this module by  $\text{ev}_0^* V$ . We define the morphism of graded  $\mathfrak{g}[t]$ -modules as a degree zero morphism of  $\mathfrak{g}[t]$ -modules. For  $m \in \mathbb{Z}$  and a graded  $\mathfrak{g}[t]$ -module  $V$ , we let  $\tau_m V$  be the  $m$ -th graded shift of  $V$ , defined by setting  $(\tau_m V)[n] = V[n - m]$ .

### 1.3.1 The local Weyl modules of $\mathfrak{g}[t]$

In [7], Chari and Pressley introduced the notion of local Weyl modules for the loop algebra  $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ . In [9], a more general case was considered by replacing the Laurent polynomial ring with the co-ordinate ring of an algebraic variety. Later in [4], a functorial approach is used to study local Weyl modules associated with the Lie algebra  $\mathfrak{g} \otimes A$ , where  $A$  is a commutative  $\mathbb{C}$ -algebra with unit.

**Definition 1.3.1.** *Given  $\lambda \in P^+$ , the local Weyl module  $W(\lambda)$  is the cyclic  $\mathfrak{g}[t]$ -module generated by an element  $w_\lambda$ , with following defining relations:*

$$(1.3) \quad \begin{aligned} (\mathfrak{n}^+ \otimes \mathbb{C}[t]) w_\lambda = 0, \quad (\mathfrak{h} \otimes t\mathbb{C}[t]) w_\lambda = 0, \quad \text{and} \quad h w_\lambda = \langle \lambda, h \rangle w_\lambda, \quad \forall h \in \mathfrak{h}, \\ (x_\alpha^- \otimes 1)^{\langle \lambda, \alpha^\vee \rangle + 1} w_\lambda = 0, \quad \forall \alpha \in R^+. \end{aligned}$$

We set the grade of  $w_\lambda$  to be zero. Since the defining relations of  $W(\lambda)$  are graded, it inherits a  $\mathbb{Z}_{\geq 0}$ -grading from the grading on  $\mathbf{U}(\mathfrak{g}[t])$ . For  $s \in \mathbb{N}$ , the subspace of grade  $s$  is given by

$$W(\lambda)[s] = \text{span} \{ (a_1 \otimes t^{r_1}) \cdots (a_k \otimes t^{r_k}) w_\lambda : k \geq 1, a_i \in \mathfrak{g}, r_i \in \mathbb{Z}_{\geq 0}, \sum r_i = s \},$$

and the subspace of grade zero is given by

$$W(\lambda)[0] = \mathbf{U}(\mathfrak{g}) w_\lambda.$$

The following proposition is well-known and the proof is analogous to that in [7, §§1-2].

**Proposition 1.3.2.** [7] For  $\lambda \in P^+$ , we have the following:

1.  $W(\lambda)$  has a unique finite-dimensional graded irreducible quotient, which is isomorphic to  $\text{ev}_0^*V(\lambda)$ . In particular,  $W(\lambda) \neq \{0\}$ .
2. The zeroth graded piece  $W(\lambda)[0]$  of  $W(\lambda)$  is isomorphic to  $V(\lambda)$ .
3.  $W(\lambda)$  is finite-dimensional. Moreover, any finite-dimensional  $\mathfrak{g}[t]$ -module  $V$  generated by an element  $v \in V$  satisfying the relations

$$(1.4) \quad (\mathfrak{n}^+ \otimes \mathbb{C}[t])v = 0, \quad (\mathfrak{h} \otimes t\mathbb{C}[t])v = 0, \quad \text{and} \quad hv = \langle \lambda, h \rangle v, \quad \forall h \in \mathfrak{h},$$

is a quotient of  $W(\lambda)$ .

**Definition 1.3.3.** For  $\lambda \in P^+$ , an element  $v \neq 0$  of a  $\mathfrak{g}[t]$ -module satisfying the relations (1.4) is said to be a highest weight vector of weight  $\lambda$ .

### 1.3.2 The graded character of local Weyl modules

For  $s \geq 0$ , the subspace  $W(\lambda)[s]$  of grade  $s$  of the local Weyl module  $W(\lambda)$  is a  $\mathfrak{g}$ -submodule, and we have the following weight space decomposition for  $W(\lambda)$ :

$$W(\lambda) = \bigoplus_{(\mu, s) \in P \times \mathbb{Z}_{\geq 0}} W(\lambda)_{\mu, s},$$

where  $W(\lambda)_{\mu, s} := \{w \in W(\lambda)[s] : hw = \langle \mu, h \rangle w, \forall h \in \mathfrak{h}\}$ . For  $\mu \in P$ , let

$$W(\lambda)_\mu := \bigoplus_{s \geq 0} W(\lambda)_{\mu, s} = \{w \in W(\lambda) : hw = \langle \mu, h \rangle w, \forall h \in \mathfrak{h}\}.$$

The  $\mu$  for which  $W(\lambda)_\mu \neq 0$  are the *weights* of  $W(\lambda)$ . The graded character  $\text{ch}_q W(\lambda)$  of  $W(\lambda)$  is defined as,

$$(1.5) \quad \text{ch}_q W(\lambda) := \sum_{(\mu,s) \in P \times \mathbb{Z}_{\geq 0}} \dim W(\lambda)_{\mu,s} q^s e^\mu \in \mathbb{Z}[P][q].$$

### 1.3.3 Local Weyl modules as level one Demazure modules

The following theorem gives the connection of local Weyl modules with Demazure modules. For  $\mathfrak{g} = \mathfrak{sl}_2$ , it follows from a result in [7]. For  $\mathfrak{g} = \mathfrak{sl}_{r+1}$ , it is proved in [5] by using the result in [7]. For  $\mathfrak{g}$  simply laced, it is proved in [10] also by using the result in [7].

**Theorem 1.3.4.** [10, Theorem 7] *Assume that  $\mathfrak{g}$  is simply laced. Given  $\lambda \in P^+$ , let  $w \in \widehat{W}$ ,  $\sigma \in \Sigma$  and  $\Lambda \in \widehat{P}^+$  such that*

$$w\sigma\Lambda \equiv w_0\lambda + \Lambda_0 \pmod{\mathbb{Z}\delta}.$$

*Then we have the following isomorphism of  $\mathfrak{g}[t]$ -modules,*

$$W(\lambda) \cong V_{w\sigma}(\Lambda).$$

## 1.4 The Lie algebra $\mathfrak{g} = \mathfrak{sl}_{r+1}$

Throughout this section  $\mathfrak{g} = \mathfrak{sl}_{r+1}$ , the Lie algebra of  $(r+1) \times (r+1)$  trace zero matrices over the field  $\mathbb{C}$  of complex numbers.

### 1.4.1 Notation

Let  $\mathfrak{h}$  be the *standard* Cartan subalgebra of  $\mathfrak{g}$  consisting of trace zero diagonal matrices. For  $1 \leq i \leq r+1$ , let  $\varepsilon_i \in \mathfrak{h}^*$  be the projection to the  $i^{\text{th}}$  co-ordinate. Let  $\varpi_i =$

$\varepsilon_1 + \cdots + \varepsilon_i, 1 \leq i \leq r$ , be the set of fundamental weights of  $\mathfrak{g}$ . Let  $\alpha_i = \varepsilon_i - \varepsilon_{i+1}, 1 \leq i \leq r$ , be a set of simple roots of  $\mathfrak{g}$  with respect to  $\mathfrak{h}$ , and  $\alpha_{i,j} = \alpha_i + \cdots + \alpha_j, 1 \leq i \leq j \leq r$ , be the set positive roots of  $\mathfrak{g}$ . For  $1 \leq i, j \leq r+1$ , let  $E_{i,j}$  be the  $(r+1) \times (r+1)$  matrix with 1 in the  $(i, j)^{th}$  position and 0 elsewhere. Define subalgebras  $\mathfrak{n}^\pm$  of  $\mathfrak{g}$  by

$$\mathfrak{n}^\pm = \bigoplus_{1 \leq i \leq j \leq r} \mathbb{C}x_{i,j}^\pm,$$

where  $x_{i,j}^+ = E_{i,j+1}$  and  $x_{i,j}^- = E_{j+1,i}$ . Now we have the following decomposition for  $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

## 1.4.2 The Gelfand-Tsetlin bases for irreducible representations of $\mathfrak{sl}_{r+1}$

**Definition 1.4.1.** A Gelfand-Tsetlin pattern (or just pattern)  $\mathcal{P}$  is an array of integral row vectors  $\underline{\lambda}^j = (\lambda_1^j, \dots, \lambda_j^j), 1 \leq j \leq r+1$ :

$$\begin{array}{ccccccc} & & & & & & \lambda_1^1 \\ & & & & & & \\ & & & & & & \lambda_1^2 & \lambda_2^2 \\ & & & & & & \dots & \dots & \dots \\ & & & & & & \lambda_1^r & \dots & \lambda_r^r \\ & & & & & & \lambda_1^{r+1} & \lambda_2^{r+1} & \dots & \lambda_{r+1}^{r+1} \end{array}$$

subject to the following conditions:

$$\lambda_i^{j+1} \geq \lambda_i^j \geq \lambda_{i+1}^{j+1}, \quad \forall 1 \leq i \leq j \leq r.$$

We call the last sequence  $\underline{\lambda}^{r+1}$  of the pattern  $\mathcal{P}$  is its *bounding sequence*. For a

sequence  $\underline{\lambda}$  of non-increasing integers, we let  $\text{GT}(\underline{\lambda})$  denote the set of patterns with bounding sequence  $\underline{\lambda}$ .

The following theorem is originally given in [13] (see [19, Theorem 2.3] for the current formulation).

**Theorem 1.4.2.** [13] *Given  $\lambda = \sum_{i=1}^r m_i \varpi_i \in P^+$ , set  $\lambda_i = m_1 + \cdots + m_i \forall 1 \leq i \leq r$ , and consider the non-increasing sequence  $\underline{\lambda} : \lambda_1 \geq \cdots \geq \lambda_r \geq 0$ . Then there exists a basis  $\{\zeta_{\mathcal{P}}\}$  for the irreducible representation  $V(\lambda)$  of  $\mathfrak{g}$  parametrized by the set of patterns  $\mathcal{P} \in \text{GT}(\underline{\lambda})$  such that the action of generators of  $\mathfrak{g}$  is given by the formulas*

$$\begin{aligned} (E_{k,k} - E_{k+1,k+1}) \zeta_{\mathcal{P}} &= \left( 2 \sum_{i=1}^k \lambda_i^k - \sum_{i=1}^{k-1} \lambda_i^{k-1} - \sum_{i=1}^{k+1} \lambda_i^{k+1} \right) \zeta_{\mathcal{P}}, \\ E_{k,k+1} \zeta_{\mathcal{P}} &= - \sum_{i=1}^k \frac{(l_{k,i} - l_{k+1,1}) \cdots (l_{k,i} - l_{k+1,k+1})}{(l_{k,i} - l_{k,1}) \cdots (l_{k,i} - l_{k,i-1})(l_{k,i} - l_{k,i+1}) \cdots (l_{k,i} - l_{k,k})} \zeta_{\mathcal{P} + \delta_{k,i}}, \\ E_{k+1,k} \zeta_{\mathcal{P}} &= \sum_{i=1}^k \frac{(l_{k,i} - l_{k-1,1}) \cdots (l_{k,i} - l_{k-1,k-1})}{(l_{k,i} - l_{k,1}) \cdots (l_{k,i} - l_{k,i-1})(l_{k,i} - l_{k,i+1}) \cdots (l_{k,i} - l_{k,k})} \zeta_{\mathcal{P} - \delta_{k,i}}, \end{aligned}$$

where  $l_{k,i} = \lambda_i^k - i + 1$ , and the arrays  $\mathcal{P} \pm \delta_{k,i}$  are obtained from  $\mathcal{P}$  by replacing  $\lambda_i^k$  by  $\lambda_i^k \pm 1$ . It is assumed that  $\zeta_{\mathcal{P}} = 0$  if the array  $\mathcal{P}$  is not a pattern.

We know  $E_{j,i} = [E_{j,j-1}, [E_{j-1,j-2}, [\cdots, [E_{i+2,i+1}, E_{i+1,i}] \cdots]]]$ , for  $i \neq j$ . Hence for  $j > i$ , the action  $E_{j,i} \zeta_{\mathcal{P}}$  of  $E_{j,i}$  on  $\zeta_{\mathcal{P}}$  is a linear combination of elements  $\zeta_{\kappa}$ , where  $\kappa$  is a pattern obtained from  $\mathcal{P}$  by decreasing each of indices  $\lambda_{r_m}^m$ , for  $i \leq m \leq j-1$ , by 1 while leaving all other indices unchanged. The action of  $E_{j,i}$  on  $\zeta_{\mathcal{P}}$ , for  $j < i$ , is defined in a similar way.

We call the set  $\{\zeta_{\mathcal{P}} : \mathcal{P} \in \text{GT}(\underline{\lambda})\}$  as the Gelfand-Tsetlin (GT) basis for  $V(\lambda)$ . The *weight* of a pattern is defined as the sequence of differences of successive row sums; this gives the  $\mathfrak{h}$ -weight of the corresponding GT basis element of  $V(\lambda)$ .

### 1.4.3 The Chari-Pressley-Loktev bases for local Weyl modules of $\mathfrak{sl}_{r+1}[t]$

In this subsection, we recall some results of Chari-Pressley [7] and Chari-Loktev [5]. We begin by introducing some notation.

Given a non-negative integer  $l$ , for  $\mathbf{s} = (s(l) \geq \dots \geq s(1)) \in \mathbb{Z}_{\geq 0}^l$  and  $1 \leq i \leq j \leq r$ , let  $\mathbf{x}_{i,j}^-(l, \mathbf{s})$  be the element of  $\mathbf{U}(\mathfrak{n}^- \otimes \mathbb{C}[t])$  defined by

$$(1.6) \quad \mathbf{x}_{i,j}^-(l, \mathbf{s}) = (x_{i,j}^- \otimes t^{s(1)}) \cdots (x_{i,j}^- \otimes t^{s(l)}),$$

if  $l > 0$  and  $\mathbf{x}_{i,j}^-(0, \emptyset) = 1$ .

Fix  $\lambda = \sum_{i=1}^r m_i \varpi_i \in P^+$ . The set  $\mathfrak{C}(\lambda)$  consists of elements  $(l_{i,j}, \mathbf{s}_{i,j})_{1 \leq i \leq j \leq r}$  with  $l_{i,j} \in \mathbb{Z}_{\geq 0}$ ,  $\mathbf{s}_{i,j} = (s_{i,j}(l_{i,j}) \geq \dots \geq s_{i,j}(1)) \in \mathbb{Z}_{\geq 0}^{l_{i,j}}$  such that

$$\text{either } l_{i,j} = 0 \text{ or } l_{i,j} > 0 \text{ and } s_{i,j}(l_{i,j}) \leq m_i + \sum_{s=j+1}^r l_{i+1,s} - \sum_{s=j}^r l_{i,s}.$$

The set  $\mathfrak{B}(\lambda)$  consist of elements

$$(1.7) \quad \mathbf{x}_{1,1}^-(l_{1,1}, \mathbf{s}_{1,1}) \mathbf{x}_{1,2}^-(l_{1,2}, \mathbf{s}_{1,2}) \mathbf{x}_{2,2}^-(l_{2,2}, \mathbf{s}_{2,2}) \mathbf{x}_{1,3}^-(l_{1,3}, \mathbf{s}_{1,3}) \cdots \mathbf{x}_{r,r}^-(l_{r,r}, \mathbf{s}_{r,r}) w_\lambda,$$

in  $W(\lambda)$  with  $(l_{i,j}, \mathbf{s}_{i,j})_{1 \leq i \leq j \leq r} \in \mathfrak{C}(\lambda)$ .

The following theorem gives bases for the local Weyl modules of  $\mathfrak{sl}_{r+1}[t]$ . For  $\mathfrak{g} = \mathfrak{sl}_2$  it is proved by Chari and Pressley in [7]. Later, in [5], Chari and Loktev proved it for  $\mathfrak{g} = \mathfrak{sl}_{r+1}$  using the result in [7].

**Theorem 1.4.3.** [5, 7] *For  $\lambda \in P^+$ , the set  $\mathfrak{B}(\lambda)$  is a basis for the local Weyl module  $W(\lambda)$ .*



# Chapter 2

## Stability of the

## Chari-Pressley-Loktev bases for local

## Weyl modules of $\mathfrak{sl}_2[t]$

The results of this chapter have appeared in [20].

### 2.1 Notation and Preliminaries

In this section, we recall results from chapter 1 which will be used in this chapter, for  $\mathfrak{g} = \mathfrak{sl}_2$ .

### 2.1.1 The affine Lie algebra $\widehat{\mathfrak{sl}}_2$

Let  $\mathfrak{sl}_2$  be the Lie algebra of  $2 \times 2$  trace zero matrices over the field  $\mathbb{C}$  of complex numbers with *standard basis*

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Let  $\mathfrak{h} = \mathbb{C}h$  be the *standard* Cartan subalgebra and  $(A, B) \mapsto \text{trace}(AB)$  the normalized invariant bilinear form on  $\mathfrak{sl}_2$ .

Let  $\widehat{\mathfrak{sl}}_2$  be the affine Lie algebra defined by

$$\widehat{\mathfrak{sl}}_2 = \mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

where  $c$  is central and the other Lie brackets are given by

$$(2.1) \quad [At^m, Bt^n] = [A, B]t^{m+n} + m\delta_{m,-n}(A, B)c,$$

$$(2.2) \quad [d, At^m] = m(At^m),$$

for all  $A, B \in \mathfrak{sl}_2$  and integers  $m, n$ : here, as throughout this chapter,  $At^s$  is shorthand for  $A \otimes t^s$ . We let  $\widehat{\mathfrak{h}} = \mathbb{C}h \oplus \mathbb{C}c \oplus \mathbb{C}d$ , and regard  $\mathfrak{h}^*$  as a subspace of  $\widehat{\mathfrak{h}}^*$  by setting  $\langle \lambda, c \rangle = \langle \lambda, d \rangle = 0$  for  $\lambda \in \mathfrak{h}^*$ .

Let  $\alpha_0, \alpha_1$  denote the simple roots of  $\widehat{\mathfrak{sl}}_2$  and let  $\alpha_0^\vee = c - h, \alpha_1^\vee = h$  be the corresponding coroots. Let  $e_i, f_i$  ( $i = 0, 1$ ) denote the Chevalley generators of  $\widehat{\mathfrak{sl}}_2$ ; these are given by

$$e_1 = x, \quad f_1 = y, \quad e_0 = yt, \quad f_0 = xt^{-1}.$$

We have

$$\langle \alpha_1, h \rangle = 2, \quad \langle \alpha_1, c \rangle = 0, \quad \langle \alpha_1, d \rangle = 0 \quad \text{and} \quad \langle \alpha_0, h \rangle = -2, \quad \langle \alpha_0, c \rangle = 0, \quad \langle \alpha_0, d \rangle = 1.$$

Let  $\delta = \alpha_0 + \alpha_1$  denote the null root,  $\widehat{Q} = \mathbb{Z}\alpha_0 + \mathbb{Z}\alpha_1$  the root lattice, and  $\widehat{Q}^+$  the non-negative integer span of  $\alpha_0, \alpha_1$ . The weight lattice (resp. the set of dominant weights) is defined by

$$\widehat{P} \text{ (resp. } \widehat{P}^+) = \{\lambda \in \widehat{\mathfrak{h}}^* : \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z} \text{ (resp. } \mathbb{Z}_{\geq 0}), i = 0, 1\}.$$

We define  $\Lambda_0 \in \widehat{P}^+$  by  $\langle \Lambda_0, h \rangle = 0, \langle \Lambda_0, c \rangle = 1, \langle \Lambda_0, d \rangle = 0$ .

The Weyl group  $\widehat{W}$  of  $\widehat{\mathfrak{sl}}_2$  is the subgroup of  $GL(\widehat{\mathfrak{h}}^*)$  generated by the simple reflections  $s_0, s_1$ . These are defined by  $s_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i$  for  $\lambda \in \widehat{\mathfrak{h}}^*$ , and  $i = 0, 1$ . There is a non-degenerate, symmetric, bilinear  $\widehat{W}$ -invariant form  $(\cdot|\cdot)$  on  $\widehat{\mathfrak{h}}^*$ , given by requiring that  $\mathbb{C}\alpha_1$  be orthogonal to  $\mathbb{C}\delta + \mathbb{C}\Lambda_0$ , together with the relations  $(\alpha_1|\alpha_1) = 2, (\delta|\delta) = (\Lambda_0|\Lambda_0) = 0, (\delta|\Lambda_0) = 1$ .

Given  $\alpha \in \mathfrak{h}^*$ , we define  $t_\alpha \in GL(\widehat{\mathfrak{h}}^*)$  by

$$(2.3) \quad t_\alpha(\lambda) = \lambda + (\lambda|\delta)\alpha - (\lambda|\alpha)\delta - \frac{1}{2}(\lambda|\delta)(\alpha|\alpha)\delta \quad \text{for } \lambda \in \widehat{\mathfrak{h}}^*.$$

Now let  $\varpi_1 = \alpha_1/2$ ; then  $Q = \mathbb{Z}\alpha_1$  and  $P = \mathbb{Z}\varpi_1$  are the root and weight lattices of the underlying  $\mathfrak{sl}_2$ . We also let  $P^+ = \mathbb{Z}_{\geq 0}\varpi_1$  be the set of dominant weights of the underlying finite-type diagram. The *translation subgroup*  $T_Q$  of  $\widehat{W}$  is defined by  $T_Q = \{t_{j\alpha_1} : j \in \mathbb{Z}\}$ . We have  $\widehat{W} = W \rtimes T_Q$ , where  $W = \{1, s_1\}$  is the underlying finite Weyl group.

The extended affine Weyl group  $\widetilde{W}$  is the semi-direct product

$$\widetilde{W} = W \rtimes T_P,$$

where  $T_P = \{t_{j\varpi_1} : j \in \mathbb{Z}\}$ . Now consider the element  $\sigma = s_1 t_{-\varpi_1} \in \widetilde{W}$ . This induces the

diagram automorphism of the Dynkin diagram of  $\widehat{\mathfrak{sl}}_2$ ; we have

$$\sigma\alpha_0 = \alpha_1, \sigma\alpha_1 = \alpha_0, \sigma\rho = \rho.$$

Here,  $\rho \in \widehat{\mathfrak{h}}^*$  is the Weyl vector, defined by  $\langle \rho, \alpha_i^\vee \rangle = 1$  for  $i = 0, 1$  and  $\langle \rho, d \rangle = 0$ . We also have  $\widetilde{W} = \widehat{W} \rtimes \Sigma$ , where  $\Sigma = \{1, \sigma\}$  is the subgroup generated by  $\sigma$ .

### 2.1.2 The basic representation $L(\Lambda_0)$ of $\widehat{\mathfrak{sl}}_2$

Given  $\Lambda \in \widehat{P}^+$ , let  $L(\Lambda)$  be the irreducible  $\widehat{\mathfrak{sl}}_2$ -module with highest weight  $\Lambda$ . It is the cyclic  $\widehat{\mathfrak{sl}}_2$ -module generated by  $v_\Lambda$ , with defining relations

$$(2.4) \quad h v_\Lambda = \langle \Lambda, h \rangle v_\Lambda \quad \forall h \in \widehat{\mathfrak{h}},$$

$$(2.5) \quad e_i v_\Lambda = 0 \quad (i = 0, 1),$$

$$(2.6) \quad f_i^{\langle \Lambda, \alpha_i^\vee \rangle + 1} v_\Lambda = 0 \quad (i = 0, 1).$$

It has weight space decomposition  $L(\Lambda) = \bigoplus_{\mu \in \widehat{\mathfrak{h}}^*} L(\Lambda)_\mu$ . The  $\mu$  for which  $L(\Lambda)_\mu \neq 0$  are the *weights of*  $L(\Lambda)$ . The module  $L(\Lambda_0)$  is particularly well-understood; the following well-known proposition describes the weight spaces of  $L(\Lambda_0)$  (see Proposition 1.2.10).

**Proposition 2.1.1.** [16]

1. The set of weights of  $L(\Lambda_0)$  is  $\{t_{j\alpha_1}(\Lambda_0) - d\delta \mid j \in \mathbb{Z}, d \in \mathbb{Z}_{\geq 0}\}$ .
2.  $\dim(L(\Lambda_0)_{t_{j\alpha_1}(\Lambda_0) - d\delta}) = p(d)$ , the number of partitions of  $d$ .

We let  $\Lambda_1 = \sigma\Lambda_0$ . Then,  $\Lambda_0, \Lambda_1$  are (a choice of) fundamental weights corresponding to the coroots  $\alpha_0^\vee, \alpha_1^\vee$ , i.e.,  $\langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$  for  $i, j \in \{0, 1\}$ . We let  $v_{\Lambda_i}$  denote a highest weight vector of  $L(\Lambda_i)$  for  $i = 0, 1$ .

### 2.1.3 The current algebra and its local Weyl modules

The current algebra  $\mathfrak{sl}_2[t] = \mathfrak{sl}_2 \otimes \mathbb{C}[t]$  is a Lie algebra with Lie bracket is obtained from that of  $\mathfrak{sl}_2$  by extension of scalars to  $\mathbb{C}[t]$ :  $[At^m, Bt^n] = [A, B]t^{m+n}$  for all  $A, B$  in  $\mathfrak{sl}_2$  and non-negative integers  $m, n$ . As such, it is a subalgebra of  $\widehat{\mathfrak{sl}_2}$ .

**Definition 2.1.2.** (see §1.3.1) Given  $n \in \mathbb{Z}_{\geq 0}$ , the *local Weyl module*  $W(n)$  is the cyclic  $\mathfrak{sl}_2[t]$ -module with generator  $w_n$  and relations:

$$(2.7) \quad (xt^s)w_n = 0, \quad (ht^{s+1})w_n = 0, \quad hw_n = nw_n, \quad y^{n+1}w_n = 0 \quad \text{for all } s \geq 0.$$

### 2.1.4 Local Weyl modules as Demazure modules

We recall that the *standard* Borel subalgebra of  $\widehat{\mathfrak{sl}_2}$  is

$$\widehat{\mathfrak{b}} = \mathfrak{sl}_2 \otimes t\mathbb{C}[t] \oplus \mathbb{C}x \oplus \widehat{\mathfrak{h}}.$$

Let  $w$  be in  $\widehat{W}$  and  $\Lambda$  in  $\widehat{P}^+$ . The weight space  $L(\Lambda)_{w\Lambda}$  of  $L(\Lambda)$  has dimension one (since two weights that are Weyl group conjugates have the same multiplicities).

Define  $V_w(\Lambda) := \mathbf{U}(\widehat{\mathfrak{b}})(L(\Lambda)_{w\Lambda})$ . Then,  $V_w(\Lambda)$  is a  $\mathbf{U}(\widehat{\mathfrak{b}})$ -submodule of  $L(\Lambda)$ , called the *Demazure module* of  $L(\Lambda)$  associated to  $w$ . More generally, given an element  $w$  of the extended affine Weyl group  $\widetilde{W}$ , we write  $w = u\tau$  with  $u \in \widehat{W}, \tau \in \Sigma$  and define the associated Demazure module by  $V_w(\Lambda) := V_u(\tau(\Lambda))$  (see §1.2.5).

We will consider the modules  $V_{t_\lambda}(\Lambda_0)$  for  $\lambda \in P$ . It is convenient to use the notation of [10] and set

$$D(1, \lambda) := V_{t_{-\lambda}}(\Lambda_0).$$

Since  $\Sigma = \{1, \sigma\}$ , the  $D(1, \lambda)$  are Demazure modules for  $L(\Lambda_0)$  (when  $\lambda \in Q$ ) or  $L(\Lambda_1)$  (when  $\lambda \notin Q$ ). Further,  $D(1, \lambda)$  is  $\mathfrak{sl}_2[t]$ -stable (not just  $\widehat{\mathfrak{b}}$ -stable) if, and only if,  $\lambda \in P^+$ .

The following theorem identifies the  $\mathfrak{sl}_2[t]$ -stable Demazure modules with the local Weyl modules of the current algebra (see Theorem 1.3.4):

**Theorem 2.1.3.** [5] *The local Weyl module  $W(n)$  is isomorphic to the Demazure module  $D(1, n\varpi_1)$ , as modules of the current algebra  $\mathfrak{sl}_2[t]$ .*

The isomorphism maps the generator  $w_n$  of  $W(n)$  to a vector of  $L(\Lambda_{\bar{n}})$ , which we will also denote  $w_n$ . Here  $\bar{n}$  is 0 if  $n$  is even and 1 if  $n$  is odd. By [5, Corollary 1.5.1] (see also [10, Corollary 1]), the weight  $\gamma$  of the vector  $w_n \in L(\Lambda_{\bar{n}})$  is a Weyl conjugate of  $\Lambda_{\bar{n}}$ . Further, we must have  $\langle \gamma, h \rangle = n$ . It follows from (2.3) that  $\gamma = t_{n\alpha_1/2}(\Lambda_0)$  (respectively  $t_{(n-1)\alpha_1/2}(\Lambda_1)$ ) if  $n$  is even (respectively, if  $n$  is odd).

Since the  $\gamma$ -weight space of  $L(\Lambda_{\bar{n}})$  is one-dimensional, this isomorphism identifying the local Weyl module as a Demazure module is unique up to scaling. We will fix the following choice of  $w_n$  for the rest of this chapter:

$$(2.8) \quad w_n := \begin{cases} (xt^{-\frac{n}{2}})^{\binom{n}{2}} v_{\Lambda_0} & \text{if } n \text{ is even,} \\ (xt^{-\frac{n+1}{2}})^{\binom{n-1}{2}} v_{\Lambda_1} & \text{if } n \text{ is odd.} \end{cases}$$

Here we have used the ‘‘divided power notation’’:  $X^{(p)} := X^p/p!$ . It is clear that  $w_n$  has weight  $\gamma$ ; the fact that  $w_n \neq 0$  will follow from Proposition 2.3.8(1) for  $n$  even, and from the arguments of §2.3.3 for  $n$  odd. We will henceforth identify  $W(n)$  with  $D(1, n\varpi_1)$  by the isomorphism defined by this choice of  $w_n$ , and think of  $W(n)$  as a subspace of  $L(\Lambda_{\bar{n}})$ .

### 2.1.5 Inclusions of local Weyl modules

Let  $\Lambda \in \widehat{P}^+$  and  $\widehat{W}_\Lambda := \{w \in \widehat{W} \mid w\Lambda = \Lambda\}$ . For elements  $w_1 \leq w_2$  of  $\widehat{W}/\widehat{W}_\Lambda$ , where  $\leq$  denotes the Bruhat order on  $\widehat{W}/\widehat{W}_\Lambda$ , the Demazure module  $V_{w_1}(\Lambda)$  is included in  $V_{w_2}(\Lambda)$

(as submodules of  $L(\Lambda)$ ) (see §1.2.6). Specializing to our case, we have, for  $n$  even,

$$(2.9) \quad W(n) = V_{t_{-n\varpi_1}}(\Lambda_0) \subseteq V_{t_{-(n+2)\varpi_1}}(\Lambda_0) = W(n+2),$$

since  $t_{-n\varpi_1} \leq t_{-(n+2)\varpi_1} = s_1 s_0 t_{-n\varpi_1}$ . For  $n$  odd, we have  $W(n) = V_{t_{-(n-1)\varpi_1} s_1}(\Lambda_1)$ , since  $t_{-\varpi_1} = s_1 \sigma$ . A similar argument to the above establishes  $W(n) \subset W(n+2)$  in this case as well. We thus have the following chains of embeddings:

$$(2.10) \quad W(0) \hookrightarrow W(2) \hookrightarrow \dots \hookrightarrow W(2n) \hookrightarrow W(2n+2) \hookrightarrow \dots \hookrightarrow L(\Lambda_0).$$

$$(2.11) \quad W(1) \hookrightarrow W(3) \hookrightarrow \dots \hookrightarrow W(2n+1) \hookrightarrow W(2n+3) \hookrightarrow \dots \hookrightarrow L(\Lambda_1).$$

## 2.2 The main results

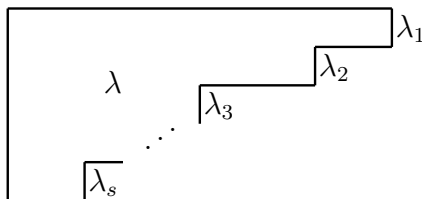
### 2.2.1 Bases for local Weyl modules

We first recall some results of [7] (see §1.4.3) which give a basis for the local Weyl module  $W(n)$ . We begin by introducing some notation. Let  $\mathcal{Y}$  denote the set of all integer partitions. Elements of  $\mathcal{Y}$  are infinite sequences  $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$  of non-negative integers such that (i)  $\lambda_i \geq \lambda_{i+1}$  for all  $i \geq 1$  and (ii)  $\lambda_j = 0$  for all sufficiently large  $j$ . We let  $|\lambda| = \sum_i \lambda_i$ , and write  $\lambda \vdash r$  to mean  $\lambda \in \mathcal{Y}$  with  $|\lambda| = r$ . Let  $\text{supp } \lambda = \min\{j \geq 0 : \lambda_{j+1} = 0\}$ . Given non-negative integers  $a, b$ , let

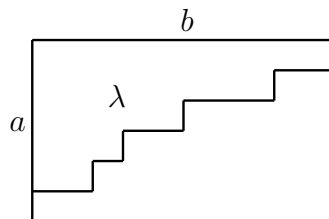
$$(2.12) \quad \mathcal{Y}(a, b) := \{\lambda \in \mathcal{Y} : \lambda_1 \leq b \text{ and } \text{supp } \lambda \leq a\}.$$

We identify partitions with *Young diagrams* in the standard way: the Young diagram corresponding to a partition  $\lambda$  is also denoted  $\lambda$  and consists of an arrangement of square boxes, all of the same size (the sides are of unit length), numbering  $|\lambda|$  in all, arranged left-and top-justified,  $\lambda_1$  on the first row,  $\lambda_2$  on the second row (which is below the first

row), and so on:



where  $s = \text{supp } \lambda$ . In this language,  $\mathcal{Y}(a, b)$  is the set of partitions whose Young diagrams fit into a rectangular  $a \times b$  box:



Next, we define the set which will parametrize bases for local Weyl modules:

$$(2.13) \quad \mathfrak{P} := \{(m, k, \lambda) : m, k \in \mathbb{Z} \text{ with } m \geq k \geq 0, \text{ and } \lambda \in \mathcal{Y}(m - k, k)\}.$$

In light of [5] a triple  $(m, k, \lambda) \in \mathfrak{P}$  should be thought of as the pair  $(\text{GT}_{m,k}, \lambda)$  where

$$\text{GT}_{m,k} = \begin{pmatrix} & k \\ m & 0 \end{pmatrix}$$

is a Gelfand-Tsetlin pattern for  $\mathfrak{sl}_2$  (see Definition 1.4.1). Associated to this pattern is a box of size  $(m - k) \times (k - 0)$ , and the condition in (2.13) says that the Young diagram of  $\lambda$  should fit into this box.

For each non-negative integer  $n$ , we also define

$$\mathfrak{P}(n) := \{(m, k, \lambda) \in \mathfrak{P} : m = n\}.$$

Given  $\xi = (n, k, \lambda) \in \mathfrak{P}(n)$  with  $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ , define the following element of



$W(n)$ :

$$(2.14) \quad B(\xi) := \left( \prod_{i=1}^{n-k} yt^{\lambda_i} \right) w_n.$$

We note that since  $[yt^j, yt^k] = 0$  for all  $j, k \geq 0$ , the order of terms in the product in equation (2.14) is irrelevant. We now have the following important theorem due to Chari and Pressley [7] (see Theorem 1.4.3):

**Theorem 2.2.1.** [7] *Let  $n \geq 0$ . Then  $\{B(\xi) \mid \xi \in \mathfrak{P}(n)\}$  is a basis for the local Weyl module  $W(n)$ .*

### 2.2.2 The CPL basis elements $\mathfrak{c}(\xi)$

Our primary goal in this chapter is to study the compatibility of the Chari-Pressley bases for  $W(n)$  with the chain of embeddings in equations (2.10) and (2.11). As a first step, we slightly modify the definition of these bases, introducing normalization factors and parametrizing them by the complements of partitions in  $\mathcal{Y}(n-k, k)$ , rather than by the partitions themselves. More precisely, given  $\xi = (n, k, \lambda) \in \mathfrak{P}(n)$ , define

$$(2.15) \quad \mathfrak{c}(\xi) := z(\xi) \left( \prod_{i=1}^{n-k} yt^{k-\lambda_i} \right) w_n.$$

where  $z(\xi)$  is a normalization factor. To specify  $z(\xi)$ , we first let  $m_j := \#\{i : \lambda_i = j\}$  denote the multiplicity of the part  $j$  in  $\lambda$  for each  $j \geq 1$ , and let  $m_0 := n - k - \text{supp } \lambda$ . Then, we have  $\prod_{i=1}^{n-k} yt^{k-\lambda_i} = \prod_{j=0}^k (yt^{k-j})^{m_j}$ . The normalization factor is given by

$$z(\xi) := \frac{(-1)^{\lfloor \frac{n}{4} \rfloor - \lfloor \frac{n-k}{2} \rfloor}}{\prod_{j=0}^k m_j!}.$$

Here,  $[x]$  denotes the greatest integer less than or equal to  $x$ . We may also rewrite (2.15) in terms of divided powers; we have

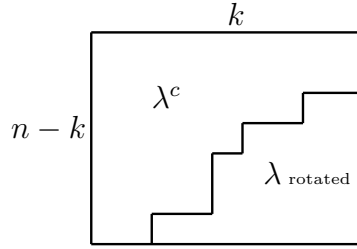
$$\mathbf{c}(\xi) = \text{sgn}(\xi) y^{(m_k)} (yt^1)^{(m_{k-1})} \dots (yt^k)^{(m_0)} w_n,$$

where  $\text{sgn}(\xi) = (-1)^{[\frac{n}{4}] - [\frac{n-k}{2}]}$ .

Given  $\xi = (n, k, \lambda) \in \mathfrak{P}(n)$ , with  $s = \text{supp } \lambda$ , define  $\lambda^c \in \mathcal{Y}(n-k, k)$  by

$$\lambda^c := (k, k, \dots, k, k - \lambda_s, k - \lambda_{s-1}, \dots, k - \lambda_1, 0, 0, \dots),$$

where the initial string of  $k$ 's is of length  $n - k - s$ . The Young diagrams of  $\lambda^c$  and  $\lambda$ , the latter rotated by  $180^\circ$  and appropriately translated, are complements of each other in the  $(n - k) \times k$  box:



Letting  $\xi^c = (n, k, \lambda^c)$ , it is clear that  $\xi^c \in \mathfrak{P}(n)$  and  $\mathbf{c}(\xi) = z(\xi) B(\xi^c)$ . This of course implies that the set

$$\mathcal{C}(n) := \{\mathbf{c}(\xi) : \xi \in \mathfrak{P}(n)\}$$

is also a basis for  $W(n)$ . We call this the *CPL basis* of  $W(n)$ .

We now view  $W(n)$  as a subspace of  $L(\Lambda_{\bar{n}})$  as in equations (2.10) and (2.11). The weight of  $\mathbf{c}(\xi)$  in  $L(\Lambda_{\bar{n}})$  is given by the following lemma.

**Lemma 2.2.2.** *Let  $\xi = (n, k, \lambda) \in \mathfrak{P}$ . Then*

1. *Weight of  $\mathbf{c}(\xi) = t_{(k-n)\alpha_1}$  (weight of  $w_n$ )  $- |\lambda|\delta$ .*
2. *If  $n$  is even, the weight of  $\mathbf{c}(\xi)$  in  $L(\Lambda_0)$  is  $t_{(k-\frac{n}{2})\alpha_1}(\Lambda_0) - |\lambda|\delta$ .*

3. If  $n$  is odd, the weight of  $\mathbf{c}(\xi)$  in  $L(\Lambda_1)$  is  $t_{(k-\frac{n+1}{2})\alpha_1}(\Lambda_1) - |\lambda|\delta$ .

*Proof.* From (2.15), we have

$$\begin{aligned} \text{wt}(\mathbf{c}(\xi)) &= \text{wt}(w_n) - (n-k)\alpha_1 + \delta \sum_{i=1}^{n-k} (k - \lambda_i) \\ &= \text{wt}(w_n) + (k-n)\alpha_1 + k(n-k)\delta - |\lambda|\delta. \end{aligned}$$

Let  $\beta = \frac{n}{2}\alpha_1$  if  $n$  is even, and  $\frac{n-1}{2}\alpha_1$  if  $n$  is odd. Then  $\text{wt}(w_n) = t_\beta(\Lambda_{\bar{n}})$ . Since  $t_{(k-n)\alpha_1}$  and  $t_\beta$  commute, the first part of the lemma is implied by the following identity, which can be verified directly using (2.3):

$$t_{(k-n)\alpha_1}(\Lambda_{\bar{n}}) = \Lambda_{\bar{n}} + (k-n)t_{-\beta}(\alpha_1) + k(n-k)\delta.$$

Assertions (2) and (3) are obvious from (1). □

### 2.2.3 The main theorem: stability of the CPL bases

We wish to study the compatibility of the bases  $\mathcal{C}(n)$  and  $\mathcal{C}(n+2)$  with respect to the embedding  $W(n) \hookrightarrow W(n+2)$ . As a first step, we define a weight preserving embedding at the level of the parametrizing sets of these bases. Define the map  $\psi : \mathfrak{P} \rightarrow \mathfrak{P}$  by

$$\psi(n, k, \lambda) = (n+2, k+1, \lambda).$$

This is well defined, since  $\mathcal{Y}(n-k, k)$  is a subset of  $\mathcal{Y}(n-k+1, k+1)$ . Further,  $\psi$  is injective, and maps  $\mathfrak{P}(n)$  to  $\mathfrak{P}(n+2)$  for all  $n$ . Now, the following is immediate from Lemma 2.2.2.

**Lemma 2.2.3.** *Let  $\xi \in \mathfrak{P}(n)$ . Then the basis vectors  $\mathbf{c}(\xi) \in W(n)$  and  $\mathbf{c}(\psi(\xi)) \in W(n+2)$  lie in the same weight space of  $L(\Lambda_{\bar{n}})$ .*

However, it is not true in general that  $\mathbf{c}(\xi)$  and  $\mathbf{c}(\psi(\xi))$  are equal as elements of  $L(\Lambda_{\bar{n}})$ , as the following example shows.

**Example 2.2.4.** Let  $\lambda$  be the partition  $2 + 1$ , i.e.,  $\lambda = (2, 1, 0, 0, \dots)$ . Let  $\xi = (4, 2, \lambda)$ . Then  $\xi \in \mathfrak{P}(4)$ , and  $\psi(\xi) = (6, 3, \lambda)$ . Using (2.15), (2.8) and the commutation relations in  $\widehat{\mathfrak{sl}}_2$ , it is easy to compute:

$$\begin{aligned}\mathbf{c}(\xi) &= \frac{1}{3} (ht^{-3} - (ht^{-1})^3) v_{\Lambda_0}, \\ \mathbf{c}(\psi(\xi)) &= (ht^{-3} + ht^{-2}ht^{-1}) v_{\Lambda_0}.\end{aligned}$$

Both these vectors have weight  $\Lambda_0 - 3\delta$ . It is well known that the vectors  $ht^{-3}v_{\Lambda_0}$ ,  $ht^{-2}ht^{-1}v_{\Lambda_0}$ ,  $(ht^{-1})^3v_{\Lambda_0}$  form a basis for the weight space  $L(\Lambda_0)_{\Lambda_0 - 3\delta}$ . Thus, we conclude  $\mathbf{c}(\xi) \neq \mathbf{c}(\psi(\xi))$ .  $\square$

We will however see below that  $\mathbf{c}(\xi) = \mathbf{c}(\psi(\xi))$  for all *stable*  $\xi$ . More precisely, let

$$(2.16) \quad \mathfrak{P}^{\text{stab}}(n) := \begin{cases} \{(n, k, \lambda) \in \mathfrak{P}(n) : |\lambda| \leq \min(n - k, k)\} & \text{if } n \text{ is even,} \\ \{(n, k, \lambda) \in \mathfrak{P}(n) : |\lambda| \leq \min(n - k, k - 1)\} & \text{if } n \text{ is odd,} \end{cases}$$

and  $\mathfrak{P}^{\text{stab}} = \bigsqcup_{n \geq 0} \mathfrak{P}^{\text{stab}}(n)$ .

We note that  $\xi \in \mathfrak{P}^{\text{stab}}(n)$  implies  $\psi(\xi) \in \mathfrak{P}^{\text{stab}}(n + 2)$ . The following is the main result of this chapter.

**Theorem 2.2.5.** *Let  $n$  be a non-negative integer and  $\xi = (n, k, \lambda) \in \mathfrak{P}^{\text{stab}}$ . Then*

$$\mathbf{c}(\xi) = \mathbf{c}(\psi(\xi)),$$

*i.e., they are equal as elements of  $L(\Lambda_{\bar{n}})$ .*

This theorem is proved in §§2.3.1-2.3.3.

## 2.2.4 Passage to the direct limit: a basis for $L(\Lambda_0)$

Theorem 2.2.5 allows us to construct a basis for  $L(\Lambda_p)$  ( $p = 0, 1$ ) by taking the direct limit of the  $\mathcal{C}(n)$  (for  $n \equiv p \pmod{2}$ ). We explain this below for  $p = 0$ , the case  $p = 1$  being similar. Consider  $L(\Lambda_0)$ , and let  $\mu = t_{j\alpha_1}(\Lambda_0) - d\delta$  ( $j \in \mathbb{Z}, d \in \mathbb{Z}_{\geq 0}$ ) be a weight of this module. Define

$$(2.17) \quad \mathfrak{P}_\mu := \{(n, k, \lambda) \in \mathfrak{P} : k - \frac{n}{2} = j \text{ and } |\lambda| = d\}.$$

We note that  $\xi = (n, k, \lambda) \in \mathfrak{P}_\mu$  forces  $n$  to be even; further, it is clear from Lemma 2.2.2 that  $\mathfrak{c}(\xi)$  has weight  $\mu$  iff  $\xi \in \mathfrak{P}_\mu$ .

Now, let  $\mathfrak{P}_\mu(n) = \mathfrak{P}_\mu \cap \mathfrak{P}(n)$ . This set parametrizes the basis elements of  $W(n)$  of weight  $\mu$ . By (2.17), the cardinality of  $\mathfrak{P}_\mu(n)$  is the number of partitions of  $d$  which fit into a  $(\frac{n}{2} - j) \times (\frac{n}{2} + j)$  box. Thus, for large enough  $n$ ,  $\mathfrak{P}_\mu(n)$  contains exactly  $p(d)$  (the number of partitions of  $d$ ) elements; in particular this implies that  $\psi$  induces a bijection of the sets  $\mathfrak{P}_\mu(n)$  and  $\mathfrak{P}_\mu(n+2)$ . Further, it is also clear that for large  $n$ , every  $\xi \in \mathfrak{P}_\mu(n)$  is stable. More precisely, we have

$$(2.18) \quad |\mathfrak{P}_\mu(n)| = p(d) \text{ and } \mathfrak{P}_\mu(n) \subset \mathfrak{P}^{\text{stab}} \text{ for all even } n \geq 2(d + |j|).$$

Choosing any such  $n$ , say  $n = 2(d + |j|)$ , we define the following (linearly independent) subset of  $L(\Lambda_0)_\mu$ :

$$\mathcal{B}_\mu := \{\mathfrak{c}(\xi) : \xi \in \mathfrak{P}_\mu(n)\}.$$

By Theorem 2.2.5 and the remarks above, this is independent of the choice of  $n$ . Since by Proposition 2.1.1, the dimension of  $L(\Lambda_0)_\mu$  is also  $p(d)$ , we conclude that  $\mathcal{B}_\mu$  is a basis for the weight space  $L(\Lambda_0)_\mu$ . Finally, to obtain a basis for  $L(\Lambda_0)$ , we take the disjoint union over the weights of  $L(\Lambda_0)$ :

$$\mathcal{B} := \bigsqcup_{\mu} \mathcal{B}_\mu.$$

We may view  $\mathcal{B}$  as a direct limit of the CPL bases  $\mathcal{C}(n)$  ( $n$  even) for the Demazure modules (=local Weyl modules) of  $L(\Lambda_0)$ .

### 2.2.5 A variation on the theme

We note that the generator  $w_n$  of  $W(n) = D(1, n\varpi_1)$  is not a lowest weight vector of the Demazure module  $D(1, n\varpi_1)$ ; while the lowest weight in  $D(1, n\varpi_1)$  is  $t_{-n\varpi_1}(\Lambda_0)$ , the weight of  $w_n$  is in fact  $t_{n\varpi_1}(\Lambda_0)$ . From the basis  $B(\xi)$  of equation (2.14), it is easy to construct a basis consisting of monomials in the raising operators of the current algebra acting on a lowest weight vector  $v_n$  of the Demazure module. Given  $\xi = (n, k, \lambda) \in \mathfrak{P}(n)$ , with  $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ , define the following element of  $W(n)$ :

$$(2.19) \quad \overline{B}(\xi) := \left( \prod_{i=1}^{n-k} xt^{\lambda_i} \right) v_n.$$

We now have:

**Proposition 2.2.6.** *The set  $\{\overline{B}(\xi) \mid \xi \in \mathfrak{P}(n)\}$  is a basis for the local Weyl module  $W(n)$ .*

The proof appears in §2.3.3. This basis also admits a normalized version which exhibits similar stabilization behavior as the CPL basis.

## 2.3 Proof of the main theorem

### 2.3.1 The key special case

In this subsection we prove Theorem 2.2.5 in the special case that  $\xi = (n, k, \lambda) \in \mathfrak{P}^{\text{stab}}$  with  $n$  even and  $k = n/2$ . In this case, the weight of  $\mathfrak{c}(\xi)$  in  $L(\Lambda_0)$  is  $\Lambda_0 - |\lambda|\delta$ . From

equations (2.15) and (2.8), we have

$$(2.20) \quad \mathbf{c}(\xi) := z(\xi) \left( \prod_{i=1}^k y t^{k-\lambda_i} \right) (x t^{-k})^{(k)} v_{\Lambda_0}.$$

Now, let  $\mathfrak{t} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} h t^n \oplus \mathbb{C} c$  denote the homogeneous Heisenberg subalgebra of  $\widehat{\mathfrak{sl}}_2$ . Recall that the subspace  $\bigoplus_{p \geq 0} L(\Lambda_0)_{\Lambda_0 - p\delta}$  is invariant under  $\mathfrak{t}$ , and is isomorphic to the canonical commutation relations representation (Fock space) of  $\mathfrak{t}$ . Thus, each element of this subspace can be uniquely expressed as a polynomial in (the infinitely many variables)  $h t^{-1}, h t^{-2}, \dots$ , acting on  $v_{\Lambda_0}$  [11]. In particular, there is a unique polynomial  $f_\xi(h t^{-1}, h t^{-2}, \dots)$  such that

$$\mathbf{c}(\xi) = f_\xi(h t^{-1}, h t^{-2}, \dots) v_{\Lambda_0}.$$

Our first goal is to determine  $f_\xi$  explicitly by applying the *straightening rules* in  $\mathbf{U}(\widehat{\mathfrak{sl}}_2)$  to equation (2.20). We will then show that  $f_\xi = f_{\psi(\xi)}$  for  $\xi \in \mathfrak{P}^{\text{stab}}$ , thereby establishing Theorem 2.2.5 in this case.

For  $r \geq 1$ , we let  $[r] := \{1, 2, \dots, r\}$ . Let  $\pi \in \mathcal{Y}$  be a partition such that  $|\pi| = r$  and  $\text{supp } \pi = s$ . A *set partition* of  $[r]$  of *type*  $\pi$  is a collection  $B = \{B_1, B_2, \dots, B_s\}$  of pairwise disjoint subsets of  $[r]$  such that  $\bigcup_{i=1}^s B_i = [r]$  and  $|B_i| = \pi_i$  for all  $i \in [s]$ . We let  $\mathcal{Y}(\pi)$  denote the set of all set partitions of  $[r]$  of type  $\pi$ .

Now, let  $B = \{B_1, B_2, \dots, B_s\} \in \mathcal{Y}(\pi)$ ; given  $\sigma \in S_r$  (the symmetric group on  $r$  letters),  $p = (p_1, p_2, \dots, p_r) \in \mathbb{N}^r$  and  $q = (q_1, q_2, \dots, q_r) \in \mathbb{N}^r$ , define the following element of  $\mathbf{U}(\mathfrak{t})$ :

$$(2.21) \quad W(B, \sigma; p, q) := \prod_{j=1}^s h t^{\sum_{i \in B_j} (p_i - q_{\sigma(i)})}.$$

We also define

$$(2.22) \quad \mathcal{H}(\pi; p, q) := \frac{1}{\pi_1! \dots \pi_s!} \sum_{\substack{B \in \mathcal{Y}(\pi) \\ \sigma \in \mathcal{S}_r}} W(B, \sigma; p, q).$$

With these notations we can state the following theorem.

**Theorem 2.3.1.** *Let  $r \geq 1$ . For every triple  $(p, q, v)$  with  $p = (p_1, p_2, \dots, p_r) \in \mathbb{N}^r$ ,  $q = (q_1, q_2, \dots, q_r) \in \mathbb{N}^r$  and  $v \in L(\Lambda_0)$ , satisfying*

1.  $p_i < q_j$  for all  $i, j \in [r]$ ,
2.  $\sum_{i \in A} p_i \geq \sum_{j \in B} q_j$  for all subsets  $A, B$  of  $[r]$  such that  $|A| = |B| + 1$ ,
3.  $yt^{(\sum_{i \in A} p_i - \sum_{j \in B} q_j)} v = 0$  for all subsets  $A, B$  of  $[r]$  such that  $|A| = |B| + 1$ ,

we have

$$(2.23) \quad \left( \prod_{i=1}^r yt^{p_i} \right) \left( \prod_{j=1}^r xt^{-q_j} \right) v = (-1)^r \sum_{\pi \vdash r} C(\pi) \mathcal{H}(\pi; p, q) v,$$

where for  $\pi = (\pi_1, \pi_2, \dots)$ ,  $C(\pi) = \prod_{i=1}^{\text{supp } \pi} \pi_i! (\pi_i - 1)!$ .

*Proof.* We proceed by induction on  $r$ . First, for  $r = 1$ , consider  $yt^{p_1} xt^{-q_1} v$ . Since  $yt^{p_1} v = 0$  and  $p_1 \neq q_1$ , we have

$$yt^{p_1} xt^{-q_1} v = [yt^{p_1}, xt^{-q_1}] v = -ht^{p_1 - q_1} v,$$

as required. Now let  $r \geq 2$ , and assume the result for  $r-1$ . Consider  $(\prod_{i=1}^r yt^{p_i})(\prod_{j=1}^r xt^{-q_j}) v$ .

Since  $yt^{p_r} v = 0$  and  $p_r \neq q_j$  for all  $j$ , we may replace  $yt^{p_r} (\prod_{j=1}^r xt^{-q_j}) v$  by

$$[yt^{p_r}, \prod_{j=1}^r xt^{-q_j}] v = (-1) \sum_{l=1}^r \left( \prod_{j=l+1}^r xt^{-q_j} \right) ht^{p_r - q_l} \left( \prod_{j=1}^{l-1} xt^{-q_j} \right) v.$$



Next, using  $[ht^{p_r - q_i}, xt^{-q_j}] = 2xt^{-q_j - q_i + p_r}$ , we can commute the  $ht^{p_r - q_i}$  term past the  $(\prod_{j=1}^{l-1} xt^{-q_j})$ . This yields

$$(2.24) \quad \begin{aligned} (-1) \prod_{i=1}^r yt^{p_i} \prod_{j=1}^r xt^{-q_j} v &= \sum_{l=1}^r \prod_{i=1}^{r-1} yt^{p_i} \prod_{\substack{j=1 \\ j \neq l}}^r xt^{-q_j} (ht^{p_r - q_l} v) \\ &+ 2 \sum_{\substack{l, m=1 \\ m < l}}^r \prod_{i=1}^{r-1} yt^{p_i} \prod_{\substack{j=1 \\ j \neq l, m}}^r xt^{-q_j} (xt^{-q_m - q_l + p_r} v). \end{aligned}$$

We now consider the first sum in equation (2.24). Fix  $l \in [r]$  and let  $p'$  and  $q'$  denote the  $r - 1$  tuples obtained by deleting  $p_r$  from  $p$  and  $q_l$  from  $q$  respectively. We also let  $v' = ht^{p_r - q_l} v$ . Then, we claim that the triple  $(p', q', v')$  satisfies the hypotheses (1)-(3) of the theorem. The first two hypotheses are clear; now given  $A \subset [r - 1]$  and  $B \subset [r] \setminus \{l\}$  with  $|A| = |B| + 1$ , we have

$$(2.25) \quad \begin{aligned} yt^{(\sum_{i \in A} p_i - \sum_{j \in B} q_j)} v' &= \left[ yt^{(\sum_{i \in A} p_i - \sum_{j \in B} q_j)}, ht^{p_r - q_l} \right] v \\ &= 2yt^{(\sum_{i \in A \cup \{r\}} p_i - \sum_{j \in B \cup \{l\}} q_j)} v = 0, \end{aligned}$$

thereby verifying hypothesis (3). By the induction hypothesis, we obtain

$$(2.26) \quad \prod_{i=1}^{r-1} yt^{p_i} \prod_{\substack{j=1 \\ j \neq l}}^r xt^{-q_j} (ht^{p_r - q_l} v) = (-1)^{r-1} \sum_{\pi' \vdash r-1} C(\pi') \mathcal{H}(\pi'; p', q') ht^{p_r - q_l} v.$$

The second sum in equation (2.24) is treated analogously. Fix  $l, m \in [r]$  with  $m < l$  and let  $q''$  denote the  $r - 1$  tuple obtained from  $q$  by deleting  $q_l, q_m$  and appending  $q_l + q_m - p_r$ . We also let  $p'' = (p_1, p_2, \dots, p_{r-1})$  and  $v'' = v$ . The triple  $(p'', q'', v'')$  evidently satisfies the hypotheses of the theorem. Again, the induction hypothesis implies

$$(2.27) \quad \prod_{i=1}^{r-1} yt^{p_i} \prod_{\substack{j=1 \\ j \neq l, m}}^r xt^{-q_j} (xt^{-q_m - q_l + p_r} v) = (-1)^{r-1} \sum_{\pi'' \vdash r-1} C(\pi'') \mathcal{H}(\pi''; p'', q'') v.$$

Fix a partition  $\pi \vdash r$ , with  $\pi = (\pi_1, \pi_2, \dots)$  and  $s = \text{supp } \pi$ . We can now find the coefficient  $C(\pi)$  that occurs in equation (2.23). Since the  $yt^{p_i}$  commute pairwise and likewise the  $xt^{-q_j}$ , it is clear that the expression for  $(\prod_{i=1}^r yt^{p_i})(\prod_{j=1}^r xt^{-q_j})v$  is invariant under the  $S_r \times S_r$  action that permutes the  $p_i$  and  $-q_j$  among themselves. Thus, to find  $C(\pi)$  it is enough to find the coefficient of the canonical word

$$(2.28) \quad ht^{\sum_{i=1}^{\pi_1} (p_i - q_i)} ht^{\sum_{i=\pi_1+1}^{\pi_1+\pi_2} (p_i - q_i)} \dots ht^{\sum_{i=\pi_1+\dots+\pi_{s-1}+1}^r (p_i - q_i)} v$$

in the RHS of (2.24).

We consider two cases (a)  $\pi_s = 1$ , and (b)  $\pi_s \geq 2$ . In case (a), it is clear from equations (2.24), (2.26) and (2.27) that the canonical word above occurs only in  $\prod_{i=1}^{r-1} yt^{p_i} \prod_{j=1}^{r-1} xt^{-q_j} (ht^{p_r - q_r} v)$ , and with coefficient  $C(\pi')$  where  $\pi' = (\pi_1, \pi_2, \dots, \pi_{s-1}) \vdash r - 1$ . Thus,

$$(2.29) \quad C(\pi) = C(\pi') = \prod_{i=1}^{s-1} \pi_i! (\pi_i - 1)! = \prod_{i=1}^s \pi_i! (\pi_i - 1)!,$$

since  $\pi_s = 1$ .

In case (b), we have  $\pi_s \geq 2$ . Again, examining equations (2.24), (2.26) and (2.27), it follows that the canonical word in this case occurs only in

$$\prod_{i=1}^{r-1} yt^{p_i} \prod_{\substack{j=1 \\ j \neq l, m}}^r xt^{-q_j} (xt^{-q_m - q_l + p_r}) v,$$

for all  $l, m$  such that

$$\pi_1 + \dots + \pi_{s-1} + 1 \leq m < l \leq r.$$

Each such pair  $(l, m)$  contributes a coefficient  $C(\pi'')$  where  $\pi'' = (\pi_1, \pi_2, \dots, \pi_{s-1}, \pi_s - 1) \vdash r - 1$ . Since  $r - \sum_{i=1}^{s-1} \pi_i = \pi_s$ , we get

$$C(\pi) = \binom{\pi_s}{2} 2C(\pi'') = \pi_s (\pi_s - 1) \left( \prod_{i=1}^{s-1} \pi_i! (\pi_i - 1)! \right) (\pi_s - 1)! (\pi_s - 2)! = \prod_{i=1}^s \pi_i! (\pi_i - 1)!,$$

as required. This proves Theorem 2.3.1.  $\square$

Let  $\lambda = (\lambda_1, \lambda_2, \dots)$  be a partition with  $\text{supp } \lambda = r \geq 1$ . Let  $\pi \vdash r$  with  $\text{supp } \pi = s$ , and let  $B = \{B_1, B_2, \dots, B_s\}$  be an element of  $\mathcal{Y}(\pi)$ . Define the following elements of  $\mathbf{U}(h \otimes t^{-1}\mathbb{C}[t^{-1}])$ :

$$(2.30) \quad W(B, \lambda) := \prod_{p=1}^s ht^{-\sum_{j \in B_p} \lambda_j}, \text{ and}$$

$$(2.31) \quad \mathcal{H}(\pi, \lambda) := \sum_{B \in \mathcal{Y}(\pi)} W(B, \lambda).$$

**Example 2.3.2.**  $\mathcal{H}(\pi = (3), \lambda = (\lambda_1, \lambda_2, \lambda_3)) = ht^{-(\lambda_1 + \lambda_2 + \lambda_3)}$ .

$\mathcal{H}(\pi = (2, 1), \lambda = (\lambda_1, \lambda_2, \lambda_3)) = ht^{-(\lambda_1 + \lambda_2)}ht^{-\lambda_3} + ht^{-(\lambda_1 + \lambda_3)}ht^{-\lambda_2} + ht^{-(\lambda_2 + \lambda_3)}ht^{-\lambda_1}$ .

$\mathcal{H}(\pi = (1, 1, 1), \lambda = (\lambda_1, \lambda_2, \lambda_3)) = ht^{-\lambda_1}ht^{-\lambda_2}ht^{-\lambda_3}$ .  $\square$

We now have the following important corollary to Theorem 2.3.1:

**Corollary 2.3.3.** *Let  $r \geq 1$ . Fix a partition  $\lambda = (\lambda_1, \lambda_2, \dots)$  with  $\text{supp } \lambda = r$ . Then, for all  $k \geq |\lambda|$ , we have*

$$(2.32) \quad \left( \prod_{i=1}^r yt^{k-\lambda_i} \right) (xt^{-k})^{(r)} v_{\Lambda_0} = (-1)^r \sum_{\pi \vdash r} C'(\pi) \mathcal{H}(\pi, \lambda) v_{\Lambda_0}.$$

Here, for  $\pi = (\pi_1, \pi_2, \dots)$ ,  $C'(\pi)$  is given by

$$C'(\pi) = \prod_{i=1}^{\text{supp } \pi} (\pi_i - 1)!.$$

*Proof.* Consider  $p = (p_1, p_2, \dots, p_r)$  and  $q = (q_1, q_2, \dots, q_r)$  with  $p_i = k - \lambda_i$  and  $q_i = k$  for all  $i \in [r]$ . We claim that the triple  $(p, q, v_{\Lambda_0})$  satisfies the hypotheses of Theorem 2.3.1. To see this, observe first that  $p_i < q_j$  for all  $i, j \in [r]$ . Further, if  $A$  is a non-empty

subset of  $[r]$ , we have

$$\sum_{i \in A} p_i = \sum_{i \in A} (k - \lambda_i) \geq |A|k - |\lambda| \geq (|A| - 1)k.$$

Finally, the highest weight vector  $v_{\Lambda_0} \in L(\Lambda_0)$  clearly satisfies  $yt^p v_{\Lambda_0} = 0 \forall p \geq 0$ . Thus, by Theorem 2.3.1, we obtain

$$(2.33) \quad \left( \prod_{i=1}^r yt^{k-\lambda_i} \right) (xt^{-k})^r v_{\Lambda_0} = (-1)^r \sum_{\pi \vdash r} C(\pi) \mathcal{H}(\pi; p, q).$$

with  $C(\pi) = \prod_{i=1}^{\text{supp } \pi} \pi_i! (\pi_i - 1)!$ . Now since  $q_j = k$  for all  $j$ , it is clear from equations (2.22) and (2.31) that

$$(2.34) \quad \mathcal{H}(\pi; p, q) = \frac{r!}{\pi_1! \pi_2! \cdots \pi_s!} \mathcal{H}(\pi, \lambda).$$

Equations (2.33) and (2.34) complete the proof.  $\square$

We observe that while the expression on the left hand side of equation (2.32) depends on  $k$ , the one on the right hand side is independent of it. The fact that these two expressions are equal for  $k \geq |\lambda|$  is precisely what leads to the stability properties of interest.

The following lemma collects together the straightening rules in  $L(\Lambda_0)$  that are used in the course of proving Theorem 2.2.5. In principle, these can all be proved directly by working in the vertex operator realization of  $L(\Lambda_0)$  [11]. The proofs below are simpler, and are included here for the sake of completeness.

**Lemma 2.3.4.** *Let  $v_{\Lambda_0}$  denote a highest weight vector of  $L(\Lambda_0)$ . Then*

$$1. (yt^m)^{(l)} (xt^{-m})^{(m)} v_{\Lambda_0} = (xt^{-m})^{(m-l)} v_{\Lambda_0} \quad \forall 1 \leq l \leq m.$$

$$2. \prod_{i=1}^r xt^{2i-1} \prod_{i=1}^r yt^{-(2i-1)} v_{\Lambda_0} = v_{\Lambda_0} \quad \forall r \in \mathbb{N}.$$

$$3. \prod_{i=1}^r yt^{2i-1} \prod_{i=1}^r xt^{-(2i-1)} v_{\Lambda_0} = v_{\Lambda_0} \quad \forall r \in \mathbb{N}.$$

4. Let  $p > q \geq 0$  and let  $v \in L(\Lambda_0)$  satisfy  $yt^p v = ht^{p-q} v = 0$ . Then

$$yt^p (xt^{-q})^{(s)} v = -(xt^{-q})^{(s-2)} xt^{p-2q} v \quad \forall s \geq 2.$$

5. For  $r \in 2\mathbb{N}$  and  $0 \leq j \leq \frac{r}{2}$ , we have

$$\left( \prod_{i=1}^{\frac{r}{2}+j} yt^{2i-1} \right) (xt^{-r})^{(2j)} \left( \prod_{i=1}^{\frac{r}{2}-j} xt^{-(2i-1)} \right) v_{\Lambda_0} = (-1)^j v_{\Lambda_0}.$$

6. For  $r \in 2\mathbb{N} - 1$  and  $0 \leq j \leq \frac{r-1}{2}$ , we have

$$\left( \prod_{i=1}^{\frac{r+1}{2}+j} yt^{2i-1} \right) (xt^{-r})^{(2j+1)} \left( \prod_{i=1}^{\frac{r-1}{2}-j} xt^{-(2i-1)} \right) v_{\Lambda_0} = (-1)^j v_{\Lambda_0}.$$

$$7. \left( \prod_{i=1}^r yt^{2i-1} \right) (xt^{-r})^{(r)} v_{\Lambda_0} = (-1)^{\lfloor \frac{r}{2} \rfloor} v_{\Lambda_0} \quad \forall r \in \mathbb{N}.$$

$$8. (xt^{-r})^{(r)} v_{\Lambda_0} = (-1)^{\lfloor \frac{r}{2} \rfloor} T_{r\alpha_1}(v_{\Lambda_0}) \neq 0 \quad \forall r \in \mathbb{N}.$$

*Proof.* (1) Consider the Lie subalgebra of  $\widehat{\mathfrak{sl}}_2$  spanned by  $E := yt^m, F := xt^{-m}$  and  $H := -h + mc$ . This is isomorphic to  $\mathfrak{sl}_2$ . Further,  $E, F$  act locally nilpotently on  $L(\Lambda_0)$ , and we have  $Hv_{\Lambda_0} = mv_{\Lambda_0}, Ev_{\Lambda_0} = 0$ . The standard  $\mathfrak{sl}_2$  calculation now shows  $E^{(l)}F^{(m)}v_{\Lambda_0} = F^{(m-l)}v_{\Lambda_0}$ .

(2) Using Proposition 2.3.7, it is easy to see that this is just a restatement of the identity  $T_{-r\alpha_1}T_{r\alpha_1}v_{\Lambda_0} = v_{\Lambda_0}$ .

(3) As in (2), this is now the identity  $T_{r\alpha_1}T_{-r\alpha_1}v_{\Lambda_0} = v_{\Lambda_0}$ .

(4) With the given hypotheses, we compute

$$yt^p(xt^{-q})^s v = [yt^p, (xt^{-q})^s] v = - \sum_{i=0}^{s-1} (xt^{-q})^i ht^{p-q} (xt^{-q})^{s-1-i} v.$$

We also have  $[ht^{p-q}, (xt^{-q})^u] = 2u(xt^{-q})^{u-1} xt^{p-2q}$  for all  $u \geq 1$ . Applying this to the above equation completes the proof.

(5) For  $j = 0$ , this is just the statement of (3). For  $1 \leq j \leq \frac{r}{2}$ , define  $v_j := \prod_{i=1}^{\frac{r}{2}-j} xt^{-(2i-1)} v_{\Lambda_0}$ . From weight considerations, it can be easily seen that  $v_j$  satisfies  $yt^{r+2j-1} v_j = 0 = ht^{2j-1} v_j$ . Thus, by (4), we obtain

$$yt^{r+2j-1}(xt^{-r})^{(2j)} v_j = -(xt^{-r})^{(2j-2)} xt^{-(r-2j+1)} v_j = -(xt^{-r})^{(2j-2)} v_{j-1}.$$

The result now follows by induction on  $j$ .

(6) This is analogous to (5).

(7) For  $r$  even, put  $j = \frac{r}{2}$  in (5) to obtain

$$\prod_{i=1}^r yt^{2i-1}(xt^{-r})^{(r)} v_{\Lambda_0} = (-1)^{\frac{r}{2}} v_{\Lambda_0}.$$

Similarly, for  $r$  odd, put  $j = \frac{r-1}{2}$  in (6):

$$\prod_{i=1}^r yt^{2i-1}(xt^{-r})^{(r)} v_{\Lambda_0} = (-1)^{\frac{r-1}{2}} v_{\Lambda_0}.$$

The equations above give us the desired result for all  $r \in \mathbb{N}$ .

(8) Let  $r \in \mathbb{N}$ . Then  $(xt^{-r})^{(r)} v_{\Lambda_0}$  and  $T_{r\alpha_1}(v_{\Lambda_0}) = \prod_{i=1}^r xt^{-(2i-1)} v_{\Lambda_0}$  belong to the 1-dimensional space  $L(\Lambda_0)_{\Lambda_0+r\alpha_1-r^2\delta}$ , and so we must have

$$(xt^{-r})^{(r)} v_{\Lambda_0} = a \prod_{i=1}^r xt^{-(2i-1)} v_{\Lambda_0},$$

for some  $a \in \mathbb{C}$ . But by (3) and (7), it follows that  $a = (-1)^{\lfloor \frac{r}{2} \rfloor}$  and that these vectors are non-zero.  $\square$

We can now deduce the key special case of Theorem 2.2.5 that we are after, namely for  $\xi$  of the form  $(n, n/2, \lambda)$  with  $n$  even. Firstly, given a partition  $\lambda \in \mathcal{Y}$ , let  $r = \text{supp } \lambda$  and  $m_j(\lambda) = \#\{i : \lambda_i = j\}$  denote the multiplicity of the part  $j$  in  $\lambda$  for each  $j \geq 1$ . If  $r \geq 1$ , define the following element of  $\mathbf{U}(h \otimes t^{-1}\mathbb{C}[t^{-1}])$ :

$$(2.35) \quad f_\lambda(ht^{-1}, ht^{-2}, \dots) := \frac{(-1)^r}{\prod_{j \geq 1} m_j(\lambda)!} \sum_{\pi \vdash r} C'(\pi) \mathcal{H}(\pi, \lambda),$$

where  $C'(\pi) = \prod_{i=1}^{\text{supp } \pi} (\pi_i - 1)!$  as in Corollary 2.3.3. If  $r = 0$ , i.e.,  $\lambda$  is the empty partition, we let  $f_\lambda := 1$ .

Now, let  $\xi = (n, \frac{n}{2}, \lambda) \in \mathfrak{P}^{\text{stab}}$  with  $n$  even. As mentioned before, the weight of  $\mathbf{c}(\xi)$  in this case is  $\Lambda_0 - |\lambda|\delta$ . The expression of  $\mathbf{c}(\xi)$  as a polynomial in  $ht^{-1}, ht^{-2}, \dots$  acting on  $v_{\Lambda_0}$  is given by the following theorem.

**Theorem 2.3.5.** *Let  $n$  be even and let  $\xi = (n, n/2, \lambda) \in \mathfrak{P}^{\text{stab}}$ . Then*

$$(2.36) \quad \mathbf{c}(\xi) = f_\lambda(ht^{-1}, ht^{-2}, \dots) v_{\Lambda_0}.$$

*Proof.* Let  $r = \text{supp } \lambda$  and  $k = n/2$ . If  $r = 0$ , then  $\mathbf{c}(\xi) = (yt^k)^{(n-k)} w_n = (yt^k)^{(k)} (xt^{-k})^{(k)} v_{\Lambda_0} = v_{\Lambda_0}$ , by Lemma 2.3.4(1). Now, for  $r \geq 1$ ,

$$\left( \prod_{i=1}^r yt^{k-\lambda_i} \right) (yt^k)^{(n-k-r)} w_n = \left( \prod_{i=1}^r yt^{k-\lambda_i} \right) (yt^k)^{(k-r)} (xt^{-k})^{(k)} v_{\Lambda_0} = \left( \prod_{i=1}^r yt^{k-\lambda_i} \right) (xt^{-k})^{(r)} v_{\Lambda_0},$$

again by Lemma 2.3.4(1). The theorem now follows from this and equations (2.32), (2.15) and (2.35).  $\square$

We now observe that  $f_\lambda$  depends only on  $\lambda$  and not on  $n$ , thereby proving Theorem 2.2.5 when  $\xi$  is of the form  $(n, n/2, \lambda)$ :

**Corollary 2.3.6.** *Let  $n$  be even and let  $\xi = (n, n/2, \lambda) \in \mathfrak{P}^{\text{stab}}$ . Then  $\mathfrak{c}(\xi) = \mathfrak{c}(\psi(\xi))$ .*

### 2.3.2 The general case when $n$ is even

We now turn to the remaining cases of Theorem 2.2.5 for even  $n$ , i.e.,  $\xi = (n, k, \lambda) \in \mathfrak{P}^{\text{stab}}$  with  $n$  even and  $k \neq n/2$ . We will now show how to reduce these to the case  $k = n/2$  using the translation operators of Frenkel and Kac. We recall the necessary facts from [11], stated for our context.

Let  $\Delta := \{\alpha_1, -\alpha_1\}$  be the set of all roots of  $\mathfrak{sl}_2$ , and set  $E_{\alpha_1} := x$  and  $E_{-\alpha_1} := y$ . Let  $(V, \pi)$  be an integrable representation of  $\widehat{\mathfrak{sl}}_2$  with weight space decomposition  $V = \bigoplus_{\mu \in \widehat{\mathfrak{h}}^*} V_\mu$ . For a real root  $\alpha = \gamma + k\delta$  ( $\gamma \in \Delta, k \in \mathbb{Z}$ ) of  $\widehat{\mathfrak{sl}}_2$  we define

$$(2.37) \quad r_\alpha^\pi := e^{-\pi(E_\alpha)} e^{\pi(E_{-\alpha})} e^{-\pi(E_\alpha)}.$$

where  $E_\alpha := E_\gamma t^k$ . The operator  $r_\alpha^\pi$  is a linear automorphism of  $V$  such that  $r_\alpha^\pi(V_\mu) = V_{s_\alpha(\mu)}$ , where  $s_\alpha \in \widehat{W}$  is the reflection defined by  $\alpha$ .

Next, we introduce the translation operators  $T_\beta^\pi$  on  $V$  for each  $\beta \in Q = \mathbb{Z}\Delta$ . For  $\gamma \in \Delta$ , define

$$(2.38) \quad T_\gamma^\pi := r_{\delta-\gamma}^\pi r_\gamma^\pi.$$

and let  $T_{p\gamma}^\pi := (T_\gamma^\pi)^p \forall p \in \mathbb{Z}_{\geq 0}$ . These operators satisfy  $T_\beta^\pi(V_\mu) = V_{t_\beta(\mu)}$  for all  $\mu \in \widehat{\mathfrak{h}}^*$ ,  $\beta \in Q$ .

We will only need these operators in two cases, namely when  $(V, \pi)$  is either the adjoint representation or the basic representation of  $\widehat{\mathfrak{sl}}_2$ . We note that  $T_\beta^{\text{ad}}$  is in fact a



Lie algebra automorphism of  $\widehat{\mathfrak{sl}}_2$ . For ease of notation, we will denote the translation operators corresponding to the basic representation simply by  $T_\beta$ , suppressing the  $\pi$  in the superscript.

The key properties of the translation operators are given by Propositions 1.2 and 2.3 of [11]. We summarize them for our context below:

**Proposition 2.3.7.** (Frenkel-Kac)

1.  $T_{p\alpha_1}^{\text{ad}}(xt^k) = xt^{k-2p} \forall p, k \in \mathbb{Z}$ .
2.  $T_{p\alpha_1}^{\text{ad}}(yt^k) = yt^{k+2p} \forall p, k \in \mathbb{Z}$ .
3.  $T_{p\alpha_1}T_{q\alpha_1} = T_{(p+q)\alpha_1} \forall p, q \in \mathbb{Z}$ .
4.  $T_{p\alpha_1}AT_{-p\alpha_1}(v) = T_{p\alpha_1}^{\text{ad}}(A)v \forall A \in \widehat{\mathfrak{sl}}_2, v \in L(\Lambda_0), p \in \mathbb{Z}$ .
5.  $T_{p\alpha_1}(v_{\Lambda_0}) = \prod_{i=1}^p xt^{-(2i-1)} v_{\Lambda_0} \forall p \geq 0$ .
6.  $T_{p\alpha_1}(v_{\Lambda_0}) = \prod_{i=1}^{-p} yt^{-(2i-1)} v_{\Lambda_0} \forall p \leq 0$ .

The following is the key proposition that allows us to carry out a reduction to the case  $k = n/2$ .

**Proposition 2.3.8.** *Let  $n$  be even. Then, we have:*

1.  $w_n = (-1)^{\lfloor \frac{n}{4} \rfloor} T_{n\alpha_1/2}(v_{\Lambda_0})$ .
2. *Given  $0 \leq k \leq n$ , let  $\gamma = (k - n/2)\alpha_1$ . Then*

$$(2.39) \quad (-1)^{\lfloor \frac{n}{4} \rfloor} w_n = (-1)^{\lfloor \frac{n-k}{2} \rfloor} T_\gamma(w_{2(n-k)}).$$

3. *Given  $\xi = (n, k, \lambda) \in \mathfrak{P}^{\text{stab}}$ , let  $\xi^\dagger = (2(n-k), n-k, \lambda)$  and  $\gamma(\xi) = (k - n/2)\alpha_1$ . Then  $\xi^\dagger \in \mathfrak{P}^{\text{stab}}$ , and*

$$(2.40) \quad \mathbf{c}(\xi) = T_{\gamma(\xi)}(\mathbf{c}(\xi^\dagger)).$$

*Proof.* The proof of (1) will be given in the appendix (see Lemma 2.3.4(8)). Equation (2.39) follows easily from (1) and Proposition 2.3.7(3). To prove (3), we start with equation (2.15) and use Proposition 2.3.7 again to obtain

$$(2.41) \quad T_{-\gamma(\xi)}(\mathbf{c}(\xi)) = z(\xi) \left( \prod_{i=1}^{n-k} T_{-\gamma(\xi)}^{\text{ad}}(yt^{k-\lambda_i}) \right) (T_{-\gamma(\xi)} w_n).$$

Now,  $T_{-\gamma(\xi)}^{\text{ad}}(yt^{k-\lambda_i}) = yt^{n-k-\lambda_i}$ . Further, it is clear from definition that  $z(\xi) = (-1)^{\lfloor \frac{n}{4} \rfloor - \lfloor \frac{n-k}{2} \rfloor} z(\xi^\dagger)$ .

Plugging these and (2.39) into (2.41), we obtain (2.40).  $\square$

We can now complete the proof of Theorem 2.2.5 for  $n$  even. Given  $\xi = (n, k, \lambda) \in \mathfrak{P}^{\text{stab}}$ , recall that  $\psi(\xi) = (n+2, k+1, \lambda)$ . It is now immediate from the definitions that

$$\gamma(\xi) = \gamma(\psi(\xi)) \text{ and } \psi(\xi^\dagger) = \psi(\xi)^\dagger.$$

Proposition 2.3.8 and Corollary 2.3.6 now imply Theorem 2.2.5 for the case that  $n$  is even.  $\square$

### 2.3.3 The proof for odd $n$

In this subsection, we show how to reduce the case of  $n$  odd to that of  $n$  even, using automorphisms of  $\widehat{\mathfrak{sl}}_2$ .

Let  $\tau$  be an automorphism of  $\widehat{\mathfrak{sl}}_2$  such that  $\tau\widehat{\mathfrak{h}} = \widehat{\mathfrak{h}}$ . We have the induced action of  $\tau$  on  $\widehat{\mathfrak{h}}^*$  by  $\langle \tau\lambda, h \rangle = \langle \lambda, \tau^{-1}h \rangle$ . Given an  $\widehat{\mathfrak{sl}}_2$ -module  $V$ , let  $V^\tau$  denote the module with the twisted action

$$g \circ v = \tau^{-1}(g)v \text{ for } g \in \widehat{\mathfrak{sl}}_2, v \in V.$$

Observe that for automorphisms  $\tau_1, \tau_2$ , we have  $V^{\tau_1\tau_2} \simeq (V^{\tau_2})^{\tau_1}$ .

We now study the twisted actions on  $L(\Lambda_0)$  by two specific automorphisms  $\tilde{\sigma}, \tilde{\phi}$  of  $\widehat{\mathfrak{sl}}_2$ . First, recall from §2.1 that  $\sigma = s_1 t_{-\varpi_1} \in \widehat{W}_{ex}$  is an automorphism of the Dynkin diagram of  $\widehat{\mathfrak{sl}}_2$ ; it swaps  $\alpha_0, \alpha_1$  and fixes  $\rho$ . Consider the Lie algebra automorphism  $\tilde{\sigma}$  of  $\widehat{\mathfrak{sl}}_2$  given by the relations

$$\tilde{\sigma}(e_i) = e_{1-i}, \quad \tilde{\sigma}(f_i) = f_{1-i}, \quad \tilde{\sigma}(\alpha_i^\vee) = \alpha_{1-i}^\vee \quad (i = 0, 1) \quad \text{and} \quad \tilde{\sigma}(\rho^\vee) = \rho^\vee.$$

Here  $\rho^\vee \in \widehat{\mathfrak{h}}$  is the unique element for which  $\langle \alpha_0, \rho^\vee \rangle = 1, \langle \alpha_1, \rho^\vee \rangle = 1$  and  $\langle \Lambda_0, \rho^\vee \rangle = 0$ . Clearly  $\tilde{\sigma}$  is an involution, and

$$\tilde{\sigma}(yt^m) = xt^{m-1}, \quad \tilde{\sigma}(xt^m) = yt^{m+1}, \quad \tilde{\sigma}(ht^m) = -ht^m + \delta_{m,0} c \quad \forall m \in \mathbb{Z}.$$

Further,  $\tilde{\sigma}$  leaves  $\widehat{\mathfrak{h}}$  invariant, and its induced action on  $\widehat{\mathfrak{h}}^*$  coincides with  $\sigma$ .

To define the second automorphism  $\tilde{\phi}$ , we employ the following simple lemma, which follows directly from the Lie bracket relations (2.1), (2.2).

**Lemma 2.3.9.** *Let  $\phi$  be an automorphism of  $\mathfrak{sl}_2$ , which preserves the Killing form. Then  $\phi$  can be extended to an automorphism  $\tilde{\phi}$  of  $\widehat{\mathfrak{sl}}_2$  by defining  $\tilde{\phi}(c) = c, \tilde{\phi}(d) = d$  and  $\tilde{\phi}(At^m) = \phi(A) t^m \quad \forall A \in \mathfrak{sl}_2, m \in \mathbb{Z}$ .*

Now, consider the involution  $\phi$  of  $\mathfrak{sl}_2$  defined by

$$(2.42) \quad \phi(x) = y, \quad \phi(y) = x, \quad \phi(h) = -h.$$

This preserves the Killing form, so by Lemma 2.3.9, it extends to an automorphism (in fact, an involution)  $\tilde{\phi}$  of  $\widehat{\mathfrak{sl}}_2$ . It is again clear that (i)  $\tilde{\phi}$  preserves  $\widehat{\mathfrak{h}}$ , and (ii) the induced action of  $\tilde{\phi}$  on  $\widehat{\mathfrak{h}}^*$  coincides with the simple reflection  $s_1$ .

**Proposition 2.3.10.** *With notation as above, we have (i)  $L(\Lambda_0)^{\tilde{\sigma}} \simeq L(\Lambda_1)$ , and (ii)  $L(\Lambda_0)^{\tilde{\phi}} \simeq L(\Lambda_0)$ .*

*Proof.* To prove (i), consider the  $\mathbf{U}(\widehat{\mathfrak{sl}}_2)$ -linear map  $L(\Lambda_1) \rightarrow L(\Lambda_0)^{\tilde{\sigma}}$  which sends  $v_{\Lambda_1}$  to  $v_{\Lambda_0}$ . To show this is well defined, we only need to check that  $v_{\Lambda_0} \in L(\Lambda_0)^{\tilde{\sigma}}$  satisfies the relations (2.4)-(2.6) for  $\Lambda = \Lambda_1$ . Since  $\tilde{\sigma}$  interchanges each pair  $(e_0, e_1)$ ,  $(f_0, f_1)$  and acts as  $\sigma$  on  $\widehat{\mathfrak{h}}^*$ , all three relations follow. Now, this map is a surjection, since  $v_{\Lambda_0}$  generates  $L(\Lambda_0)^{\tilde{\sigma}}$ . Since  $L(\Lambda_1)$  is irreducible, it must be an isomorphism.

A similar argument establishes (ii). We map  $L(\Lambda_0) \rightarrow L(\Lambda_0)^{\tilde{\phi}}$  by sending  $v_{\Lambda_0}$  to  $v_{\Lambda_0}$ . To show that this extends to a well-defined  $\mathbf{U}(\widehat{\mathfrak{sl}}_2)$ -linear map on all of  $L(\Lambda_0)$ , we verify that  $v_{\Lambda_0} \in L(\Lambda_0)^{\tilde{\phi}}$  satisfies (2.4)-(2.6) for  $\Lambda = \Lambda_0$ . As above, (2.4) holds since the action of  $\tilde{\phi}$  on  $\widehat{\mathfrak{h}}^*$  coincides with  $s_1$ , and  $s_1\Lambda_0 = \Lambda_0$ . Further, in  $L(\Lambda_0)$ , we have  $\tilde{\phi}^{-1}(e_0)v_{\Lambda_0} = xt v_{\Lambda_0} = 0$  and  $\tilde{\phi}^{-1}(e_1)v_{\Lambda_0} = yv_{\Lambda_0} = 0$ . This establishes (2.5). Finally, for (2.6), we compute in  $L(\Lambda_0)$ :  $\tilde{\phi}^{-1}(f_1)v_{\Lambda_0} = xv_{\Lambda_0} = 0$ , and  $\tilde{\phi}^{-1}(f_0)^2v_{\Lambda_0} = (yt^{-1})^2v_{\Lambda_0}$ . Since  $yt^{-1}$  is in a real root space of  $\widehat{\mathfrak{sl}}_2$ , it is easy to see that this last term is also zero by a standard  $\mathfrak{sl}_2$  argument (using the  $\mathfrak{sl}_2$  spanned by  $xt, yt^{-1}$  and  $h + c$ ). The fact that it is an isomorphism follows as in (i).  $\square$

Let  $\tau = \tilde{\sigma}\tilde{\phi}$ . Then Proposition 2.3.10 implies

$$L(\Lambda_1) \simeq L(\Lambda_0)^{\tilde{\sigma}} \simeq \left(L(\Lambda_0)^{\tilde{\phi}}\right)^{\tilde{\sigma}} \simeq L(\Lambda_0)^\tau.$$

The isomorphism  $F : L(\Lambda_1) \rightarrow L(\Lambda_0)^\tau$  maps  $v_{\Lambda_1} \mapsto v_{\Lambda_0}$ . It is then determined on all of  $L(\Lambda_1)$  by  $\widehat{\mathfrak{sl}}_2$ -linearity, i.e., by the relation

$$F(Xv) = \tau^{-1}(X)F(v) \quad \forall X \in \widehat{\mathfrak{sl}}_2, \quad v \in L(\Lambda_1).$$

We now prove Theorem 2.2.5 for  $\xi = (n, k, \lambda) \in \mathfrak{P}^{\text{stab}}$  with  $n$  odd. From (2.15) and (2.8), we have

$$\mathbf{c}(\xi) = z(\xi) \left( \prod_{i=1}^{n-k} yt^{k-\lambda_i} \right) (xt^{-\frac{n+1}{2}})^{\binom{n-1}{2}} v_{\Lambda_1}.$$

Applying the isomorphism  $F$ , we obtain

$$F(\mathbf{c}(\xi)) = z(\xi) \left( \prod_{i=1}^{n-k} yt^{k-\lambda_i-1} \right) (xt^{-\frac{n-1}{2}})^{\binom{n-1}{2}} v_{\Lambda_0} = \mathbf{c}(n-1, k-1, \lambda),$$

since  $(-1)^{\lfloor \frac{n}{4} \rfloor} = (-1)^{\lfloor \frac{n-1}{4} \rfloor}$  for  $n$  odd. Observe by (2.16) that  $(n, k, \lambda) \in \mathfrak{P}^{\text{stab}}$  for  $n$  odd, implies that  $(n-1, k-1, \lambda)$  is also in  $\mathfrak{P}^{\text{stab}}$ . Theorem 2.2.5 now follows for  $\xi$  since we have already proved it for all even  $n$ . This completes the proof of that theorem in all cases.  $\square$

Finally, we observe that the above ideas also give us a proof of Proposition 2.2.6. With notation as in that proposition, first let  $n$  be even. If  $G : L(\Lambda_0) \rightarrow L(\Lambda_0)^{\tilde{\phi}}$  is the isomorphism constructed in the proof of Proposition 2.3.10, observe that  $G(w_n) = (yt^{-\frac{n}{2}})^{\binom{n}{2}} v_{\Lambda_0} = v_n$ , say, is a lowest weight vector of  $D(1, n\varpi_1)$ . Further, for  $\xi \in \mathfrak{P}(n)$ , we have  $G(B(\xi)) = \overline{B}(\xi)$ , thereby proving Proposition 2.2.6 in this case. The stable basis elements in this set up are simply the images of the  $\mathbf{c}(\xi)$ ,  $\xi \in \mathfrak{P}^{\text{stab}}$ , under the appropriate isomorphism  $G$ . The case of odd  $n$  is analogous, via the isomorphism  $G' : L(\Lambda_1) \rightarrow L(\Lambda_1)^{\tilde{\sigma}\tilde{\phi}\tilde{\sigma}^{-1}}$ .

# Chapter 3

## On area maximizing Gelfand-Tsetlin patterns

The results of this chapter will appear in [21].

### 3.1 Notation

This section establishes notation and terminology. The notion of a partition overlaid pattern, or POP, is introduced in §3.1.8. POPs parametrize Chari-Loktev bases for local Weyl modules of  $\mathfrak{sl}_{r+1}$  (just as patterns parametrize the Gelfand-Tsetlin bases for irreducible representations). The notion of area (triangular and trapezoidal) of a pattern introduced in §3.1.3 plays an important role in what follows.

#### 3.1.1 Interlacing condition on sequences

We will be dealing with finite non-increasing sequences of real numbers, like so:  $\lambda_1 \geq \dots \geq \lambda_n$ . It is convenient to fix terminology and notation as follows:

- A sequence such as  $\lambda_1 \geq \dots \geq \lambda_n$  will be denoted for short by  $\underline{\lambda}$ .
- The sequence  $\lambda_1 \geq \dots \geq \lambda_n$  is *integral* if the  $\lambda_j$  are all integers. It is *non-negative integral* if the  $\lambda_j$  are all non-negative integers.
- The number of elements in the sequence is the *length* of the sequence.
- Let  $\underline{\lambda}$  and  $\underline{\mu}$  be sequences of lengths  $n$  and  $n - 1$  respectively. We say that they *interlace* and write  $\underline{\lambda} \gtrsim \underline{\mu}$  if

$$(3.1) \quad \lambda_1 \geq \mu_1 \geq \lambda_2, \quad \lambda_2 \geq \mu_2 \geq \lambda_3, \quad \dots, \quad \text{and} \quad \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n.$$

The interlacing condition may be remembered easily if  $\underline{\lambda}$  and  $\underline{\mu}$  are arranged like so:

$$(3.2) \quad \begin{array}{ccccccccccc} & \mu_1 & & \mu_2 & & \cdots & & \cdots & & \cdots & & \mu_{n-2} & & \mu_{n-1} \\ \lambda_1 & & \lambda_2 & & \lambda_3 & & \cdots & & \cdots & & \lambda_{n-2} & & \lambda_{n-1} & & \lambda_n \end{array}$$

If we now imagine  $\geq$  relations among numbers as we move in the north-easterly or the south-easterly direction, that is precisely the condition for interlacing.

- Given sequences  $\underline{\lambda}$  and  $\underline{\mu}$  of lengths  $n$  and  $n - 1$  respectively, we will feel free to use several alternative expressions to express the condition that they interlace:

$$\underline{\mu} \gtrsim \underline{\lambda}; \quad \underline{\mu} \text{ interlaces } \underline{\lambda}; \quad \underline{\lambda} \text{ interlaces } \underline{\mu}; \quad \underline{\lambda} \text{ and } \underline{\mu} \text{ are interlaced}; \quad \text{etc.}$$

### Weak interlacing

Let  $\underline{\lambda} : \lambda_1 \geq \dots \geq \lambda_n$  be a non-increasing sequence of real numbers and  $\underline{\mu} = (\mu_1, \dots, \mu_{n-1})$  be an element of  $\mathbb{R}^{n-1}$ . We say that  $\underline{\lambda}$  *weakly interlaces*  $\underline{\mu}$  and write  $\underline{\lambda} \gtrsim_w \underline{\mu}$  if for every  $j$ ,  $1 \leq j \leq n - 1$ , and every sequence  $1 \leq i_1 < \dots < i_j \leq n - 1$ , we have:

$$(3.3) \quad \lambda_1 + \dots + \lambda_j \geq \mu_{i_1} + \dots + \mu_{i_j} \geq \lambda_{n-j+1} + \dots + \lambda_n$$





## Rows of a pattern

Let  $\mathcal{P}$  be the pattern  $\underline{\lambda}^1, \dots, \underline{\lambda}^n$ . The entries of  $\underline{\lambda}^k$  are sometimes referred to as the *entries in row  $k$*  of  $\mathcal{P}$ . The lone entry in the first row of the pattern (3.4) is 5; the entries in its second row are 7, 4; and those in third are 7, 5, and 3.

## The weight of a pattern

The *weight* of a pattern with bounding sequence of length  $n$  is the  $n$ -tuple  $(a_1, \dots, a_n)$ , where  $a_j$  is the difference of the sum of the entries in row  $j$  and the sum of the entries in row  $j - 1$ . It is understood that the sum of the entries in the zeroth row is zero. The weight of the pattern in (3.4), for instance, is  $(5, 6, 4)$ .

### 3.1.3 Trapezoidal Area and (Triangular) Area of a pattern

Let  $\underline{\lambda}$  and  $\underline{\mu}$  be two sequences of lengths  $n$  and  $n - 1$  respectively that are interlaced. The *triangular area* or just *area* of the pair  $(\underline{\lambda}, \underline{\mu})$  is defined by:

$$(3.5) \quad \Delta(\underline{\lambda}, \underline{\mu}) := \sum_{i=1}^{n-1} (\lambda_i - \mu_i)(\mu_i - \lambda_{i+1})$$

And the *trapezoidal area* of the pair  $(\underline{\lambda}, \underline{\mu})$  is defined by:

$$(3.6) \quad \square(\underline{\lambda}, \underline{\mu}) := \sum_{1 \leq i \leq j \leq n-1} (\lambda_i - \mu_i)(\mu_j - \lambda_{j+1})$$

The above definitions make sense even when  $n = 1$ :  $\underline{\mu}$  is empty and both areas vanish (since they are empty sums).

The (*triangular*) *area* of a pattern  $\mathcal{P}$  with rows  $\underline{\lambda}^1, \dots, \underline{\lambda}^n$  is defined by:

$$(3.7) \quad \Delta(\mathcal{P}) := \Delta(\underline{\lambda}^n, \underline{\lambda}^{n-1}) + \Delta(\underline{\lambda}^{n-1}, \underline{\lambda}^{n-2}) + \dots + \Delta(\underline{\lambda}^2, \underline{\lambda}^1) + \Delta(\underline{\lambda}^1, \underline{\lambda}^0)$$

Its *trapezoidal area* is defined by:

$$(3.8) \quad \square(\mathcal{P}) := \square(\underline{\lambda}^n, \underline{\lambda}^{n-1}) + \square(\underline{\lambda}^{n-2}, \underline{\lambda}^{n-2}) + \cdots + \square(\underline{\lambda}^2, \underline{\lambda}^1) + \square(\underline{\lambda}^1, \underline{\lambda}^0)$$

Observe that both areas are zero for a pattern with a single row (with only one entry).

### 3.1.4 Majorization

For an element  $\underline{x} = (x_1, \dots, x_n)$  in  $\mathbb{R}^n$ , let  $\underline{x}^\downarrow = (x_1^\downarrow, \dots, x_n^\downarrow)$  be the vector whose coordinates are obtained by rearranging the  $x_j$  in weakly decreasing order. For elements  $\underline{x}$  and  $\underline{y}$  in  $\mathbb{R}^n$ , we say that  $\underline{x}$  *weakly majorizes*  $\underline{y}$  and write  $\underline{x} \succ_{\text{wm}} \underline{y}$  if

$$(3.9) \quad x_1^\downarrow + \cdots + x_k^\downarrow \geq y_1^\downarrow + \cdots + y_k^\downarrow \quad \text{for all } 1 \leq k \leq n$$

The right hand side in the above equation is evidently the largest possible value of  $\sum_{i=1}^k y_{j_i}$  over all sequences  $1 \leq j_1 < \dots < j_k \leq n$ . Thus (3.9) is equivalent to the a priori stronger condition:

$$(3.10) \quad x_1^\downarrow + \cdots + x_k^\downarrow \geq y_{j_1} + \cdots + y_{j_k} \quad \text{for all } 1 \leq k \leq n \quad \text{and for all } 1 \leq j_1 < \dots < j_k \leq n$$

We say that  $\underline{x}$  *majorizes*  $\underline{y}$  and write  $\underline{x} \succ_{\text{m}} \underline{y}$  if  $\underline{x} \succ_{\text{wm}} \underline{y}$  and  $x_1 + \dots + x_n = y_1 + \dots + y_n$ .

Observe the following: for real  $n$ -tuples  $\underline{x}$  and  $\underline{y}$  with  $\underline{x} \succ_{\text{m}} \underline{y}$ , given any  $k$ ,  $1 \leq k \leq n$ , and any sequence  $1 \leq i_1 < \dots < i_k \leq n$ , we have

$$(3.11) \quad y_{i_1} + \cdots + y_{i_k} \geq x_{n-k+1}^\downarrow + \cdots + x_n^\downarrow$$

Indeed, let  $\{i_{k+1}, \dots, i_n\} := \{1, \dots, n\} \setminus \{i_1, \dots, i_k\}$ . Then, on the one hand, when  $y_{i_{k+1}} + \cdots + y_{i_n}$  is added to the left hand side and  $x_1^\downarrow + \cdots + x_{n-k}^\downarrow$  to the right hand side the resulting quantities are equal, and, on the other,  $y_{i_{k+1}} + \cdots + y_{i_n} \leq y_1^\downarrow + \cdots + y_{n-k}^\downarrow \leq$

$$x_1^\downarrow + \cdots + x_{n-k}^\downarrow.$$

### 3.1.5 Majorization and weak interlacing

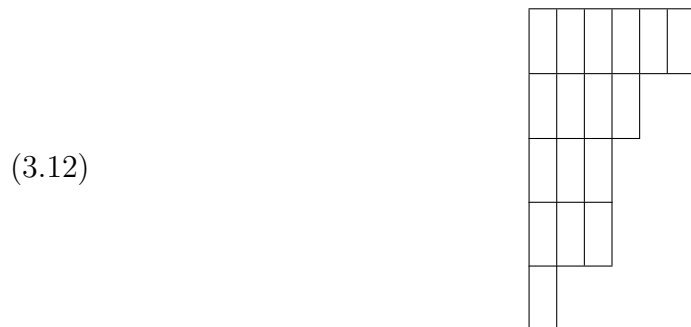
Let  $\underline{\lambda} : \lambda_1 \geq \dots \geq \lambda_n$  be a non-increasing sequence of real numbers and  $\underline{\mu} = (\mu_1, \dots, \mu_n) \in \mathbb{R}^n$  such that  $\underline{\lambda} \succ_m \underline{\mu}$ . Then  $\underline{\lambda} \succeq_w (\mu_1, \dots, \mu_{n-1})$ . (Proof: (3.10) and (3.11).)

### 3.1.6 Partitions

A *partition* consists of a finite sequence of non-negative integers, possibly with repetitions, arranged in non-increasing order. Example: 6, 4, 4, 3, 1, 0, 0. The non-zero elements of the sequence are called the *parts* of the partition. If the sum of the parts of a partition  $\underline{\pi} : \pi_1 \geq \pi_2 \geq \dots$  is  $n$ , the partition is said to be a *partition of  $n$* , and we write  $|\underline{\pi}| = n$ . The example above is a partition of 18 with 5 parts.

The trailing zeros in a partition are non-significant. Thus 6, 4, 3, 3, 1, 0, 0 is the same partition as 6, 4, 3, 3, 1. We allow the empty sequence to be a partition: it is the only partition of 0.

Each partition has an associated *shape*. Given a partition  $\underline{\pi} : \pi_1 \geq \pi_2 \geq \dots$  of  $n$ , its associated shape consists of a grid of  $n$  squares, all of the same size, arranged top- and left-justified, with  $\pi_1$  squares in the first row,  $\pi_2$  squares in the second, and so on (the rows are counted from the top downwards). The shape corresponding to the partition 6, 4, 3, 3, 1, 0, 0, for example, is this:



We say that a partition *fits into a rectangle*  $(a, b)$ , where  $a$  and  $b$  are non-negative integers, if the number of parts is at most  $a$  and the largest part (if it exists) is at most  $b$ . The terminology should make sense if we think of the shape associated to a partition. The partition whose shape is displayed above fits into the rectangle  $(a, b)$  if and only if  $a \geq 5$  and  $b \geq 6$ .

### Complementary partitions

Let  $\underline{\pi} : \pi_1 \geq \pi_2 \geq \dots$  be a partition that fits into the rectangle  $(a, b)$ —in other words,  $b \geq \pi_1$  and  $\pi_j = 0$  for  $j > a$ . The *complement to  $\underline{\pi}$  in the rectangle  $(a, b)$*  is the partition  $\underline{\pi}^c$  defined as follows:  $\pi_j^c = b - \pi_{a+1-j}$  for  $1 \leq j \leq a$  and  $\pi_j^c = 0$  for  $j > a$ . For example, the complement of the partition 6, 4, 3, 3, 1 in the rectangle  $(7, 6)$  is 6, 6, 5, 3, 3, 2.

### 3.1.7 Colored partitions

Let  $r$  be a positive integer. An  *$r$ -colored partition* or a *partition into  $r$  colors* is a partition in which each part is assigned an integer between 1 and  $r$ . The number assigned to a part is its *color*. We may think of an  $r$ -colored partition as just an ordered  $r$ -tuple of  $(\underline{\pi}^1, \dots, \underline{\pi}^r)$  of partitions: the partition  $\underline{\pi}^j$  consists of all parts of color  $j$  of the  $r$ -colored partition. An  *$r$ -colored partition of  $n$*  is an  $r$ -colored partition with  $|\underline{\pi}^1| + \dots + |\underline{\pi}^r| = n$ .

### 3.1.8 Partition overlaid patterns

A *partition overlaid pattern* (*POP* for short) consists of an integral GT pattern  $\underline{\lambda}^1, \dots, \underline{\lambda}^n$ , and, for every ordered pair  $(j, i)$  of integers with  $1 \leq j < n$  and  $1 \leq i \leq j$ , a partition  $\underline{\pi}(j)^i$  that fits into the rectangle  $(\lambda_i^{j+1} - \lambda_i^j, \lambda_i^j - \lambda_{i+1}^{j+1})$ . Example: a partition overlay on the pattern displayed in (3.4) consists of three partitions  $\underline{\pi}(2)^1$ ,  $\underline{\pi}(2)^2$ , and  $\underline{\pi}(1)^1$  that fit respectively into the rectangles  $(0, 2)$ ,  $(1, 1)$ , and  $(2, 1)$ .

POPs parametrize bases of local Weyl modules of current algebras of type  $A$  (as proved by Chari-Loktev [5] and recalled in §3.3.6) just as integral GT patterns parametrize bases of irreducible representations of simple Lie algebras of type  $A$  (as proved by Gelfand-Tsetlin and is well known).

The *bounding sequence*, (*triangular*) *area*, *trapezoidal area*, *weight*, etc. of a POP are just the corresponding notions attached to the underlying pattern. The *number of boxes* in a POP  $\mathfrak{P}$  is the sum  $\sum_{(j,i)} |\underline{\pi(j)}^i|$  of the number of boxes in each of its constituent partitions. It is denoted by  $|\mathfrak{P}|$ . Among POPs with a fixed underlying pattern, the maximum possible value of the number of boxes is evidently the (triangular) area of the pattern. The *depth* of a POP  $\mathfrak{P}$  is defined by  $\text{depth } \mathfrak{P} := \square(\mathcal{P}) - |\mathfrak{P}|$ , where  $\mathcal{P}$  is the underlying pattern of  $\mathfrak{P}$ .

### 3.1.9 Weights identified as tuples

Let  $\mathfrak{g} = \mathfrak{sl}_{r+1}$  be the simple Lie algebra consisting of  $(r+1) \times (r+1)$  complex traceless matrices ( $r \geq 1$ ). Let  $\mathfrak{h}$  and  $\mathfrak{b}$  be respectively the diagonal and upper triangular subalgebras of  $\mathfrak{g}$ . Linear functionals on  $\mathfrak{h}$  are called *weights*.

Let  $\epsilon_i$ ,  $1 \leq i \leq r+1$ , be the weight that maps a diagonal matrix to its entry in position  $(i, i)$ . Observe that  $\epsilon_1 + \cdots + \epsilon_{r+1} = 0$ . Every weight may be expressed as  $a_1\epsilon_1 + \cdots + a_{r+1}\epsilon_{r+1}$ , with  $\underline{a} \in \mathbb{C}^{r+1}$ . Two elements in  $\mathbb{C}^{r+1}$  are said to be *equivalent* if their difference is a multiple of  $\underline{\mathbf{1}} := (1, \dots, 1)$ , so that weights are identified with equivalence classes in  $\mathbb{C}^{r+1}$ .

We will use this identification often tacitly. For a weight  $\eta$ , we denote by  $\underline{\eta}$  an element in the corresponding equivalence class in  $\mathbb{C}^{r+1}$ . Depending upon the context, this  $\underline{\eta}$  may denote a particular representative: we will see two instances of this below.

A weight is *integral* if there exists a tuple  $\underline{a}$  in  $\mathbb{C}^{r+1}$  consisting of integers that corresponds to it; it is *dominant* if  $a_1 \geq \dots \geq a_{r+1}$ . These notions correspond to the respective

notions in the representation theory of  $\mathfrak{g}$ . Dominant integral weights are thus in bijection with integer tuples of the form  $\lambda_1 \geq \dots \geq \lambda_r \geq \lambda^{r+1} = 0$ . As an example, consider the highest root  $\theta$  of  $\mathfrak{g}$ . The corresponding element of  $\mathbb{C}^{r+1}$  is  $\underline{\theta} = (2, 1, \dots, 1, 0)$ .

A weight  $\mu$  is a weight of the irreducible representation  $V(\lambda)$  with highest weight a dominant integral weight  $\lambda$  if and only if  $\underline{\mu} \preceq_m \underline{\lambda}$ , where the tuple  $\underline{\mu} = (\mu_1, \dots, \mu_{r+1})$  representing  $\mu$  is so chosen that  $\lambda_1 + \dots + \lambda_{r+1} = \mu_1 + \dots + \mu_{r+1}$ . (See also Proposition 3.2.1 and Theorem 3.2.2 in §3.2 below.)

Fix an invariant form  $(\cdot | \cdot)$  on  $\mathfrak{h}^*$  such that for every root  $\alpha$  we have  $(\alpha | \alpha) = 2$ . Given  $\lambda \in \mathfrak{h}^*$ , how do we compute  $(\lambda | \lambda)$  in terms of the corresponding tuple  $\underline{\lambda}$ ? We have  $(\lambda | \lambda) = \|\underline{\lambda}\|^2 := \lambda_1^2 + \dots + \lambda_{r+1}^2$  provided that  $\underline{\lambda}$  is so chosen that  $\lambda_1 + \dots + \lambda_{r+1} = 0$ . We will have occasion to compute  $(\lambda | \lambda) - (\mu | \mu)$  for  $\lambda, \mu$  in  $\mathfrak{h}^*$ . We observe that it equals  $\|\underline{\lambda}\|^2 - \|\underline{\mu}\|^2$  provided that  $\underline{\lambda}$  and  $\underline{\mu}$  satisfy  $\lambda_1 + \dots + \lambda_{r+1} = \mu_1 + \dots + \mu_{r+1}$ .

## 3.2 On area maximizing Gelfand-Tsetlin patterns

This section is elementary and combinatorial. Its purpose is to prove Theorem 3.2.2 below. The representation theoretic relevance of the theorem is discussed in §3.3. For an  $n$ -tuple  $\underline{x} := (x_1, \dots, x_n)$  of real numbers, the norm is defined as usual:  $\|\underline{x}\| := \sqrt{x_1^2 + \dots + x_n^2}$ .

We begin with a proposition which should be well known. We state and prove it in order to put things in context and in the interest of completeness.

**Proposition 3.2.1.** *Let  $\underline{\lambda}^1, \dots, \underline{\lambda}^n$  be a GT pattern with weight  $\underline{\mu}$ . Then  $\underline{\lambda}^n \succ_m \underline{\mu}$ .*

*Proof.* Proceed by induction on  $n$ . In case  $n = 1$ , we have  $\underline{\mu} = \underline{\lambda}^1$ , and the result is obvious.

Now suppose that  $n \geq 2$ . By the induction hypothesis,  $\underline{\lambda}^{n-1} \succ_m \underline{\mu}^{n-1}$ , where  $\underline{\mu}^{n-1}$ :

$\mu_1, \dots, \mu_{n-1}$  is the weight of  $\underline{\lambda}^1, \dots, \underline{\lambda}^{n-1}$ . Since  $\underline{\lambda}^{n-1} = \underline{\lambda}^{n-1\downarrow}$ , this means the following: for any  $k$ ,  $1 \leq k \leq n-1$ , and any sequence  $1 \leq i_1 < \dots < i_k \leq n-1$ , we have:

$$(3.13) \quad \lambda_1^{n-1} + \lambda_2^{n-1} + \dots + \lambda_k^{n-1} \geq \mu_{i_1} + \dots + \mu_{i_k}$$

Since  $\underline{\lambda}^n = \underline{\lambda}^{n\downarrow}$ , in order to show  $\underline{\lambda}^n \succ_m \underline{\mu}$ , we need to prove the following two sets of inequalities: for any  $k$ ,  $0 \leq k \leq n-1$ , and any sequence  $1 \leq i_1 < \dots < i_k \leq n-1$ :

$$(3.14) \quad \lambda_1^n + \lambda_2^n + \dots + \lambda_k^n \geq \mu_{i_1} + \dots + \mu_{i_k}$$

$$(3.15) \quad \lambda_1^n + \lambda_2^n + \dots + \lambda_{k+1}^n \geq \mu_{i_1} + \dots + \mu_{i_k} + \mu_n$$

Equation (3.14) follows by combining (3.13) with the inequalities  $\lambda_1^n \geq \lambda_1^{n-1}, \dots, \lambda_k^n \geq \lambda_k^{n-1}$  (which hold since  $\underline{\lambda}^{n-1} \geq \underline{\lambda}^n$ ). As to (3.15), since  $\mu_n = (\lambda_1^n + \dots + \lambda_n^n) - (\lambda_1^{n-1} + \dots + \lambda_{n-1}^{n-1})$ , it is equivalent to:

$$(3.16) \quad \lambda_1^{n-1} + \lambda_2^{n-1} + \dots + \lambda_k^{n-1} + (\lambda_{k+1}^{n-1} - \lambda_{k+2}^n) + \dots + (\lambda_{n-1}^{n-1} - \lambda_n^n) \geq \mu_{i_1} + \dots + \mu_{i_k}$$

But each of  $(\lambda_{k+1}^{n-1} - \lambda_{k+2}^n), \dots, (\lambda_{n-1}^{n-1} - \lambda_n^n)$  is non-negative (because  $\underline{\lambda}^{n-1} \geq \underline{\lambda}^n$ ), and thus (3.16) too follows from (3.13).  $\square$

Here is the main result of this section:

**Theorem 3.2.2.** *Let  $\underline{\lambda} = \lambda_1 \geq \dots \geq \lambda_n$  be a non-increasing sequence of real numbers and  $\underline{\mu} = (\mu_1, \dots, \mu_n)$  an element of  $\mathbb{R}^n$  that is majorized by  $\underline{\lambda}$ :  $\underline{\lambda} \succ_m \underline{\mu}$ . Then there is a unique GT pattern  $\mathcal{P} : \underline{\lambda}^1, \dots, \underline{\lambda}^n$  with bounding sequence  $\underline{\lambda}^n = \underline{\lambda}$ , weight  $\underline{\mu}$ , and satisfying*

the following:

(3.17)

For any  $j$ ,  $1 \leq j \leq n$ , its  $j^{\text{th}}$  row  $\underline{\lambda}^j$  majorizes the  $j^{\text{th}}$  row  $\underline{\kappa}^j$  of any pattern with bounding sequence  $\underline{\lambda}$  and weight  $\underline{\mu}$ :  $\underline{\lambda}^j \succ_m \underline{\kappa}^j$ .

This unique pattern  $\mathcal{P}$  has the following properties:

(A) It is integral if  $\underline{\lambda}$  and  $\underline{\mu}$  are integral.

(B) Its triangular area equals  $\frac{1}{2}(\|\underline{\lambda}\|^2 - \|\underline{\mu}\|^2)$ , which is strictly more than the area of any other pattern with bounding sequence  $\underline{\lambda}$  and weight  $\underline{\mu}$ .

For the proof of the theorem, which appears in §3.2.1 below, we now make preparations.

**Proposition 3.2.3.** *Let  $\underline{\lambda}$  and  $\underline{\lambda}'$  be sequences of lengths  $n$  and  $n - 1$  respectively that are interlaced:  $\underline{\lambda} \geq \underline{\lambda}'$ . Then the trapezoidal area  $\square(\underline{\lambda}, \underline{\lambda}')$  is given by*

$$(3.18) \quad 2\square(\underline{\lambda}, \underline{\lambda}') = \|\underline{\lambda}\|^2 - \|\underline{\lambda}'\|^2 - ((\lambda_1 + \cdots + \lambda_n) - (\lambda'_1 + \cdots + \lambda'_{n-1}))^2$$

*Proof.* Proceed by induction on  $n$ . In the case  $n = 1$ , both sides vanish (when correctly interpreted). Now suppose that  $n \geq 2$ . Let  $\underline{\kappa}$  and  $\underline{\kappa}'$  be the sequences of length  $n - 1$  and  $n - 2$  obtained by deleting respectively  $\lambda_n$  from  $\underline{\lambda}$  and  $\lambda'_{n-1}$  from  $\underline{\lambda}'$ : the sequence  $\underline{\kappa}'$  is empty in case  $n = 2$ . Also, let

$$(3.19) \quad T(n) := (\lambda_n + \sum_{j=1}^{n-1} (\lambda_j - \lambda'_j)) \quad T(n-1) := (\lambda_{n-1} + \sum_{j=1}^{n-2} (\lambda_j - \lambda'_j))$$

so that the last term in the desired equation (3.18) is  $-T(n)^2$ .

Then, firstly, by induction:

$$(3.20) \quad 2\square(\underline{\kappa}, \underline{\kappa}') = \|\underline{\kappa}\|^2 - \|\underline{\kappa}'\|^2 - T(n-1)^2$$



Secondly, as is easily seen:

$$(3.21) \quad \square(\underline{\lambda}, \underline{\lambda}') = \square(\underline{\kappa}, \underline{\kappa}') + (\lambda'_{n-1} - \lambda_n) \sum_{j=1}^{n-1} (\lambda_j - \lambda'_j)$$

$$(3.22) \quad = \square(\underline{\kappa}, \underline{\kappa}') + (\lambda'_{n-1} - \lambda_n)(T(n-1) - \lambda'_{n-1})$$

Finally, since  $T(n) = T(n-1) - (\lambda'_{n-1} - \lambda_n)$ :

$$(3.23) \quad T(n)^2 = T(n-1)^2 + (\lambda'_{n-1} - \lambda_n)^2 - 2(\lambda'_{n-1} - \lambda_n)T(n-1)$$

Adding twice of (3.22) with (3.20) and (3.23), we get

$$\begin{aligned} 2\square(\underline{\lambda}, \underline{\lambda}') + T(n)^2 &= \|\underline{\kappa}\|^2 - \|\underline{\kappa}'\|^2 - (\lambda'_{n-1} - \lambda_n)(\lambda'_{n-1} + \lambda_n) \\ &= (\|\underline{\kappa}\|^2 + \lambda_n^2) - (\|\underline{\kappa}'\|^2 + \lambda'_{n-1}{}^2) \\ &= \|\underline{\lambda}\|^2 - \|\underline{\lambda}'\|^2 \end{aligned}$$

and the proposition is proved. □

**Corollary 3.2.4.** *The trapezoidal area of a pattern  $\mathcal{P}: \underline{\lambda}^1, \underline{\lambda}^2, \dots, \underline{\lambda}^n$  is given by*

$$(3.24) \quad \square(\mathcal{P}) = \frac{1}{2}(\|\underline{\lambda}\|^2 - \|\underline{\mu}\|^2)$$

where  $\underline{\lambda} = \underline{\lambda}^n$  is the bounding sequence and  $\underline{\mu}$  the weight of  $\mathcal{P}$ .

*Proof.* By the proposition:

$$(3.25) \quad \square(\underline{\lambda}^j, \underline{\lambda}^{j-1}) = \frac{1}{2}(\|\underline{\lambda}^j\|^2 - \|\underline{\lambda}^{j-1}\|^2 - \mu_j^2) \quad \text{for } j = n, n-1, \dots, 1$$

Adding these  $n$  equations gives us the desired result. □

**Lemma 3.2.5.** *Let  $\underline{\lambda}^n : \lambda_1^n \geq \dots \geq \lambda_n^n$  and  $\underline{\lambda}^{n-1} : \lambda_1^{n-1} \geq \dots \geq \lambda_{n-1}^{n-1}$  be non-increasing sequences of real numbers that are interlaced:  $\underline{\lambda}^n \geq \underline{\lambda}^{n-1}$ . Suppose that*

1.  $\underline{\lambda}^n$  is integral;
2.  $\lambda_1^{n-1} + \cdots + \lambda_{n-1}^{n-1}$  is an integer; and
3.  $\square(\underline{\lambda}^n, \underline{\lambda}^{n-1}) = \triangle(\underline{\lambda}^n, \underline{\lambda}^{n-1})$ .

Then  $\underline{\lambda}^{n-1}$  is integral.

*Proof.* Let  $k$  be the largest integer,  $1 \leq k \leq n-1$ , such that  $\lambda_j^{n-1} = \lambda_j^n$  for all  $j < k$ . From (3) it follows that  $\lambda_j^{n-1} = \lambda_{j+1}^n$  for all  $j > k$ . From (1) it follows that  $\lambda_j^{n-1}$  is an integer for  $j \neq k$ . From (2) it follows that  $\lambda_k^{n-1}$  is also an integer.  $\square$

**Corollary 3.2.6.** *Let  $\underline{\lambda} : \lambda_1 \geq \dots \geq \lambda_n$  be a non-increasing sequence of integers. Let  $\underline{\mu}$  be in  $\mathbb{Z}^n$  such that  $\underline{\lambda} \succ_m \underline{\mu}$ . Let  $\mathcal{P}$  be a GT pattern with bounding sequence  $\underline{\lambda}$ , weight  $\underline{\mu}$ , and triangular area  $\frac{1}{2}(\|\underline{\lambda}\|^2 - \|\underline{\mu}\|^2)$ . Then  $\mathcal{P}$  is integral.*

*Proof.* Let  $\underline{\lambda}^j$  denote the  $j^{\text{th}}$  row of  $\mathcal{P}$ . By Corollary 3.2.4,  $\square(\mathcal{P}) = \triangle(\mathcal{P})$ , so  $\square(\underline{\lambda}^j, \underline{\lambda}^{j-1}) = \triangle(\underline{\lambda}^j, \underline{\lambda}^{j-1})$  for all  $j$ ,  $n \geq j \geq 1$ . Since  $\mu_j = (\lambda_1^j + \cdots + \lambda_j^j) - (\lambda_1^{j-1} + \cdots + \lambda_{j-1}^{j-1})$ , it follows (by an easy decreasing induction) that  $\lambda_1^j + \cdots + \lambda_j^j$  is an integer for all  $j$ ,  $n \geq j \geq 1$ . By applying Lemma 3.2.5 repeatedly, we see successively that  $\underline{\lambda}^{n-1}, \dots, \underline{\lambda}^1$  are all integral.  $\square$

### 3.2.1 Proof of Theorem 3.2.2

**Lemma 3.2.7.** *Let  $\underline{\lambda}^n : \lambda_1^n \geq \dots \geq \lambda_n^n$  be a non-increasing sequence of length  $n$  of real numbers and  $\underline{\mu} = (\mu_1, \dots, \mu_n)$  an element of  $\mathbb{R}^n$  such that  $\underline{\lambda}^n \succ_m \underline{\mu}$ . Then there exists a unique non-increasing sequence  $\underline{\lambda}^{n-1} : \lambda_1^{n-1} \geq \dots \geq \lambda_{n-1}^{n-1}$  of length  $n-1$  of real numbers such that the following hold:*

1.  $\underline{\lambda}^{n-1} \succ \underline{\lambda}^n$
2.  $\lambda_1^{n-1} + \cdots + \lambda_{n-1}^{n-1} = \mu_1 + \cdots + \mu_{n-1}$

3. Let  $\underline{\kappa}^n : \kappa_1^n \geq \dots \geq \kappa_n^n$  be a non-increasing sequence of length  $n$  of real numbers and  $\underline{\kappa}^{n-1} = (\kappa_1^{n-1}, \dots, \kappa_{n-1}^{n-1})$  an element of  $\mathbb{R}^{n-1}$  such that:

- (i)  $\underline{\lambda}^n \succ_m \underline{\kappa}^n$ , (ii)  $\underline{\kappa}^n \succeq_w \underline{\kappa}^{n-1}$ , and  
 (iii)  $\kappa_1^{n-1} + \dots + \kappa_{n-1}^{n-1} = \mu_1 + \dots + \mu_{n-1}$ .

Then  $\underline{\lambda}^{n-1} \succ_m \underline{\kappa}^{n-1}$ .

Moreover, the unique sequence  $\underline{\lambda}^{n-1}$  has the following properties:

(a)  $\underline{\lambda}^{n-1}$  is integral if  $\underline{\lambda}^n$  and  $\underline{\mu}$  are integral.

(b)  $\square(\underline{\lambda}^n, \underline{\lambda}^{n-1}) = \Delta(\underline{\lambda}^n, \underline{\lambda}^{n-1})$ .

(c) Let  $\underline{\kappa}^{n-1} : \kappa_1^{n-1} \geq \dots \geq \kappa_{n-1}^{n-1}$  be a non-increasing sequence of real numbers such that:

- (i')  $\underline{\kappa}^{n-1} \succeq \underline{\lambda}^n$ , (ii')  $\kappa_1^{n-1} + \dots + \kappa_{n-1}^{n-1} = \mu_1 + \dots + \mu_{n-1}$ , and  
 (iii')  $\underline{\lambda}^{n-1} \neq \underline{\kappa}^{n-1}$ .

Then  $\square(\underline{\lambda}^n, \underline{\kappa}^{n-1}) \succeq \Delta(\underline{\lambda}^n, \underline{\kappa}^{n-1})$ .

*Proof.* The uniqueness of  $\underline{\lambda}^{n-1}$  is easy to see. Indeed, if  $\underline{\eta}^{n-1}$  be a another sequence with properties (1)–(3), then by applying (3) with  $\underline{\kappa}^n = \underline{\lambda}$  and  $\underline{\kappa}^{n-1} = \underline{\eta}^{n-1}$ , we see that  $\underline{\lambda}^{n-1} \succ_m \underline{\eta}^{n-1}$ . By the same argument with the roles of  $\underline{\eta}^{n-1}$  and  $\underline{\lambda}^{n-1}$  switched, we see that  $\underline{\eta}^{n-1} \succ_m \underline{\lambda}^{n-1}$ . It follows from the definition of  $\succ_m$  that it is a partial order, so we conclude  $\underline{\eta}^{n-1} = \underline{\lambda}^{n-1}$ .

We now turn to the existence of  $\underline{\lambda}^{n-1}$ . Consider the auxiliary non-decreasing sequence of terms  $\underline{\tilde{\lambda}}^n : \tilde{\lambda}_1^n \leq \dots \leq \tilde{\lambda}_n^n$ , where

$$(3.26) \quad \tilde{\lambda}_k^n := (\lambda_1^n + \dots + \lambda_n^n) - \lambda_k^n$$

Since  $\underline{\lambda}^{n\downarrow} = \underline{\lambda}^n$ , it follows from equations (3.11) and (3.10) that

$$(3.27) \quad \tilde{\lambda}_1^n \leq \mu_1 + \cdots + \mu_{n-1} \leq \tilde{\lambda}_n^n$$

Fix a  $j_0$ ,  $1 \leq j_0 \leq n-1$ , such that

$$(3.28) \quad \tilde{\lambda}_{j_0}^n \leq \mu_1 + \cdots + \mu_{n-1} \leq \tilde{\lambda}_{j_0+1}^n$$

In fact, there is a unique such  $j_0$  except when  $\mu_1 + \cdots + \mu_{n-1} = \tilde{\lambda}_j^n$  for some  $j$ .

Set

$$(3.29) \quad \lambda_j^{n-1} := \begin{cases} \lambda_j^n & \text{for } j < j_0 \\ \lambda_{j+1}^n & \text{for } j > j_0 \end{cases}$$

$$(3.30) \quad \lambda_{j_0}^{n-1} := (\mu_1 + \cdots + \mu_{n-1}) - (\lambda_1^n + \cdots + \lambda_{j_0-1}^n + \lambda_{j_0+2}^n + \cdots + \lambda_n^n)$$

From (3.30) and (3.28), we see that

$$(3.31) \quad \lambda_{j_0}^n - \lambda_{j_0}^{n-1} = \tilde{\lambda}_{j_0+1}^n - (\mu_1 + \cdots + \mu_{n-1}) \geq 0 \quad \& \quad \lambda_{j_0}^{n-1} - \lambda_{j_0+1}^n = (\mu_1 + \cdots + \mu_{n-1}) - \tilde{\lambda}_{j_0}^n \geq 0$$

To observe that  $\underline{\lambda}^{n-1}$  is a non-decreasing sequence, first note that, from (3.29) and the fact that  $\underline{\lambda}^n$  is non-decreasing, we have:

$$(3.32) \quad \lambda_1^{n-1} \geq \cdots \geq \lambda_{j_0-1}^{n-1} \quad \lambda_{j_0+1}^{n-1} \geq \cdots \geq \lambda_{n-1}^{n-1}$$

And, combining (3.29), (3.31), and the non-decreasing property of  $\underline{\lambda}^n$ , we have:

$$(3.33) \quad \lambda_{j_0-1}^{n-1} = \lambda_{j_0-1}^n \geq \lambda_{j_0}^n \geq \lambda_{j_0}^{n-1} \quad \lambda_{j_0}^{n-1} \geq \lambda_{j_0+1}^n \geq \lambda_{j_0+2}^n = \lambda_{j_0+1}^{n-1}$$

Item (1) follows from (3.29) and (3.31), item (2) from (3.29) and (3.30). As to item (3), we must show that for any  $j$ ,  $1 \leq j \leq n-1$ , and any choice  $1 \leq i_1 < \dots < i_j \leq n-1$ , the following holds:

$$(3.34) \quad \lambda_1^{n-1} + \dots + \lambda_j^{n-1} \geq \kappa_{i_1}^{n-1} + \dots + \kappa_{i_j}^{n-1}$$

For  $j < j_0$ , we have, by successively using (3.29), (i), and (ii):

$$(3.35) \quad \lambda_1^{n-1} + \dots + \lambda_j^{n-1} = \lambda_1^n + \dots + \lambda_j^n \geq \kappa_1^n + \dots + \kappa_j^n \geq \kappa_{i_1}^{n-1} + \dots + \kappa_{i_j}^{n-1}$$

For  $j \geq j_0$ , substituting for  $\lambda_i^{n-1}$  from (3.29) and (3.30), the left hand side of (3.34) becomes:

$$\lambda_1^{n-1} + \dots + \lambda_j^{n-1} = (\mu_1 + \dots + \mu_{n-1}) - (\lambda_{j+2}^n + \dots + \lambda_n^n)$$

Using (iii), we get

$$\lambda_1^{n-1} + \dots + \lambda_j^{n-1} = (\kappa_1^{n-1} + \dots + \kappa_{n-1}^{n-1}) - (\lambda_{j+2}^n + \dots + \lambda_n^n)$$

Letting  $\{i_{j+1}, \dots, i_{n-1}\}$  denote the complement  $\{1, \dots, n-1\} \setminus \{i_1, \dots, i_j\}$ , the right hand side of the last equation may be rewritten as

$$(\kappa_{i_1}^{n-1} + \dots + \kappa_{i_j}^{n-1}) + \left( (\kappa_{i_{j+1}}^{n-1} + \dots + \kappa_{i_{n-1}}^{n-1}) - (\lambda_{j+2}^n + \dots + \lambda_n^n) \right)$$

The second parenthetical term here is non-negative, for by (ii) and (i)

$$\kappa_{i_{j+1}}^{n-1} + \dots + \kappa_{i_{n-1}}^{n-1} \geq \kappa_{j+2}^n + \dots + \kappa_n^n \geq \lambda_{j+2}^n + \dots + \lambda_n^n$$

and the proof of item (3) is complete.

Assertion (a) is immediate from the definitions (3.29) and (3.30) of  $\underline{\lambda}^{n-1}$ . As to

(b), it is clear from definitions (3.5) and (3.6) that it holds, since either  $\lambda_j^n = \lambda_j^{n-1}$  or  $\lambda_j^{n-1} = \lambda_{j+1}^n$  for every  $j \neq j_0$ . Towards the proof of (c), let  $\ell$  and  $\rho$  be respectively the smallest and largest  $j$ ,  $1 \leq j \leq n-1$ , such that  $\lambda_j^{n-1} \neq \kappa_j^{n-1}$ . Taking  $\underline{\kappa}^n$  to be  $\underline{\lambda}^n$  in (3), we obtain  $\underline{\lambda}^{n-1} \succ_m \underline{\kappa}^{n-1}$ , which implies that  $\lambda_\ell^{n-1} > \kappa_\ell^{n-1}$  and  $\kappa_\rho^{n-1} > \lambda_\rho^{n-1}$ . In particular,  $\ell < \rho$ . From (1), we have  $\lambda_\ell^n \geq \lambda_\ell^{n-1} > \kappa_\ell^{n-1}$  and  $\kappa_\rho^{n-1} > \lambda_\rho^{n-1} \geq \lambda_{\rho+1}^n$ , so that  $(\lambda_\ell^n - \kappa_\ell^{n-1})(\kappa_\rho^{n-1} - \lambda_{\rho+1}^n)$  is a non-trivial contribution to  $\square(\underline{\lambda}^n, \underline{\kappa}^{n-1}) - \triangle(\underline{\lambda}^n, \underline{\kappa}^{n-1})$ .  $\square$

**Corollary 3.2.8.** *With hypothesis and notation as in the lemma,  $\underline{\lambda}^{n-1} \succ_m (\mu_1, \dots, \mu_{n-1})$ .*

*Proof.* Put  $\underline{\kappa}^n = \underline{\lambda}^n$  and  $\underline{\kappa}^{n-1} = (\mu_1, \dots, \mu_{n-1})$  in item (3). Hypothesis (ii) holds by the observation in §3.1.5.  $\square$

### Proof of Theorem 3.2.2

The uniqueness of the pattern  $\mathcal{P}$  being obvious, it is enough to prove its existence. Apply Lemma 3.2.7 to the given pair  $\underline{\lambda}$  and  $\underline{\mu}$  (by taking  $\underline{\lambda}^n$  in the statement of the lemma to be  $\underline{\lambda}$ ). The  $\underline{\lambda}^{n-1}$  we obtain as a result is such that  $\underline{\lambda}^{n-1} \succ_m (\mu_1, \dots, \mu_{n-1})$  (Corollary 3.2.8) so we can apply the lemma again, this time to the pair  $\underline{\lambda}^{n-1}$  and  $(\mu_1, \dots, \mu_{n-1})$ . Continuing thus, we obtain, by items (1) and (2) of the lemma, a GT pattern—let us denote it  $\mathcal{P}$ —with bounding sequence  $\underline{\lambda}$  and weight  $\underline{\mu}$ .

We claim that the pattern  $\mathcal{P}$  satisfies (3.17). To prove this, proceed by reverse induction on  $j$ . For  $j = n$ , we have  $\underline{\lambda}^n = \underline{\kappa}^n = \underline{\lambda}$ , so the statement is evident. For the induction step, suppose we have proved that  $\underline{\lambda}^j \succ_m \underline{\kappa}^j$ . Note that  $\underline{\lambda}^{j-1}$  is constructed by applying Lemma 3.2.7 with  $\underline{\lambda}^j$  in place of  $\underline{\lambda}^n$  (in the notation of the lemma). Assertion (3.17) follows from item (3) of the lemma, by substituting respectively  $\underline{\kappa}^j$ ,  $\underline{\kappa}^{j-1}$ , and  $(\mu_1, \dots, \mu_{j-1})$  for  $\underline{\kappa}^n$ ,  $\underline{\kappa}^{n-1}$ , and  $(\mu_1, \dots, \mu_{n-1})$ .

The pattern  $\mathcal{P}$  is integral if  $\underline{\lambda}$  and  $\underline{\mu}$  are so (Lemma 3.2.7 (a)), so (A) is clear.

Finally we prove (B). Let  $\mathcal{P}'$  be a pattern distinct from  $\mathcal{P}$  with bounding sequence  $\underline{\lambda}$

and weight  $\underline{\mu}$ . By Corollary 3.2.4,  $\square(\mathcal{P}) = \square(\mathcal{P}') = (||\underline{\lambda}||^2 - ||\underline{\mu}||^2)/2$ . By Lemma 3.2.7 (b),  $\square(\mathcal{P}) = \triangle(\mathcal{P})$ . Let  $j$  be largest,  $1 \leq j \leq n$ , such that the  $j^{\text{th}}$  row  $\underline{\kappa}^j$  of  $\mathcal{P}'$  is distinct from the  $j^{\text{th}}$  row of  $\mathcal{P}$ . Then  $j < n$  and, by Lemma 3.2.7 (c),  $\square(\underline{\kappa}^{j+1}, \underline{\kappa}^j) \succeq \triangle(\underline{\kappa}^{j+1}, \underline{\kappa}^j)$ , so  $\square(\mathcal{P}') \succeq \triangle(\mathcal{P}')$ , and (B) is proved.  $\square$

### 3.3 Relevance of the main theorem to the theory of Local Weyl modules

In this section we discuss the relevance of our main theorem (Theorem 3.2.2) of §3.2 to representation theory. We do this by means of giving a representation theoretic proof of a version of the theorem: see §3.3.7 below. The proof is based on the theory of local Weyl modules for current algebras. We first recall the required results from Chapter 1.

#### 3.3.1 The current algebra $\mathfrak{g}[t]$ and the affine algebra $\widehat{\mathfrak{g}}$

Let  $\mathfrak{g}$  be a complex simple finite dimensional Lie algebra. The corresponding *current algebra*, denoted  $\mathfrak{g}[t]$ , is merely the extension of scalars to the polynomial ring  $\mathbb{C}[t]$  of  $\mathfrak{g}$ . There is a natural grading on  $\mathfrak{g}[t]$  given by the degree in  $t$ : thus  $X \otimes t^s$  has degree  $s$  (here  $X$  is in  $\mathfrak{g}$  and  $s$  is a non-negative integer). There is an induced grading by non-negative integers on the universal enveloping algebra  $U(\mathfrak{g}[t])$ . We can talk about graded modules (graded by integers) of  $\mathfrak{g}[t]$ .

Let  $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$  be the affine algebra corresponding to  $\mathfrak{g}$ . The current algebra  $\mathfrak{g}[t]$  is evidently a subalgebra of  $\widehat{\mathfrak{g}}$ .

### 3.3.2 Fixing notation

Fix a Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  and a Borel subalgebra  $\mathfrak{b}$  containing  $\mathfrak{h}$  of  $\mathfrak{g}$ . Let  $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$  be the triangular decomposition of  $\mathfrak{g}$  with  $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ . Let  $(\cdot | \cdot)$  denote the invariant form on  $\mathfrak{h}^*$  such that  $(\alpha | \alpha) = 2$  for all long roots  $\alpha$ .

Put  $\widehat{\mathfrak{h}} := \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$  and  $\widehat{\mathfrak{b}} := \mathfrak{g} \otimes t\mathbb{C}[t] \oplus \mathfrak{b} \oplus \mathbb{C}c \oplus \mathbb{C}d$ . Denote by  $\Lambda_0$  and  $\delta$  elements of  $\widehat{\mathfrak{h}}^*$  such that  $\langle \Lambda_0, c \rangle = \langle \delta, d \rangle = 1$  and  $\langle \Lambda_0, \mathfrak{h} \rangle = \langle \Lambda_0, d \rangle = \langle \delta, \mathfrak{h} \rangle = \langle \delta, c \rangle = 0$ . Extend  $(\cdot | \cdot)$  to  $\widehat{\mathfrak{h}}^*$  by setting  $(\mathfrak{h}^* | \Lambda_0) = (\mathfrak{h}^* | \delta) = (\Lambda_0 | \Lambda_0) = (\delta | \delta) = 0$ ,  $(\Lambda_0 | \delta) = 1$ , where  $\mathfrak{h}^*$  is identified as the subspace of  $\widehat{\mathfrak{h}}^*$  that kills  $c$  and  $d$ .

Fix a dominant integral weight  $\lambda$  of  $\mathfrak{g}$  (with respect to  $\mathfrak{h}$  and  $\mathfrak{b}$ ).

### 3.3.3 The local Weyl module $W(\lambda)$

An element  $w_\lambda$  of a  $\mathfrak{g}[t]$ -module is of *highest weight*  $\lambda$  if:

$$(3.36) \quad (\mathfrak{n}^+ \otimes \mathbb{C}[t]) w_\lambda = 0, \quad (\mathfrak{h} \otimes t\mathbb{C}[t]) w_\lambda = 0, \quad H w_\lambda = \langle \lambda, H \rangle w_\lambda \quad \text{for } H \in \mathfrak{h}$$

The *local Weyl module*  $W(\lambda)$  corresponding to  $\lambda$  is the cyclic  $\mathfrak{g}[t]$ -module generated by an element  $w_\lambda$  of highest weight  $\lambda$  (in other words, subject to the relations (3.36)) and further satisfying:

$$(3.37) \quad \mathfrak{g}_{-\alpha}^{\langle \lambda, \alpha^\vee \rangle + 1} w_\lambda = 0 \text{ for every simple root } \alpha \text{ of } \mathfrak{g}$$

where  $\alpha^\vee$  the co-root corresponding to  $\alpha$  and  $\mathfrak{g}_{-\alpha}$  is the root space in  $\mathfrak{g}$  corresponding to  $-\alpha$ . It is evident that  $W(\lambda)$  is graded (since the relations in (3.36) and (3.37) are all homogeneous). We let the generator  $w_\lambda$  have grade 0, so that  $W(\lambda) = U(\mathfrak{n}^- \otimes \mathbb{C}[t]) w_\lambda$  is graded by the non-negative integers. It is well known—the proofs are analogous to those in [7, §2]—that  $W(\lambda)$  is finite dimensional and moreover that it is maximal among finite



dimensional modules generated by an element of highest weight  $\lambda$  (which means, more precisely, that for any finite dimensional cyclic  $\mathfrak{g}[t]$ -module  $W$  generated by an element  $w$  of highest weight  $\lambda$ , there exists a unique  $\mathfrak{g}[t]$ -module map from  $W(\lambda)$  onto  $W$  mapping  $u$  to  $w$ ).

### 3.3.4 Local Weyl modules as Demazure modules

Let  $w_0$  be the longest element of the Weyl group  $W$  of  $\mathfrak{g}$ . Let  $\Lambda$  be the dominant integral weight of  $\widehat{\mathfrak{g}}$  and  $w$  an element of the affine Weyl group such that

$$(3.38) \quad w\Lambda = t_{w_0\lambda}\Lambda_0 \quad (w \text{ in the affine Weyl group, } \Lambda \text{ dominant})$$

where (see [16, (6.5.2)]):

$$(3.39) \quad t_\gamma\zeta := \zeta + \langle \zeta, c \rangle \gamma - ((\zeta|\gamma) + \frac{1}{2}(\gamma|\gamma)\langle \zeta, c \rangle)\delta \quad \text{for } \gamma \text{ in } \mathfrak{h}^* \text{ and } \zeta \text{ in } \widehat{\mathfrak{h}}^*$$

From the above two equations, we obtain that  $\Lambda$  has level 1, or, in other words  $\langle \Lambda, c \rangle = 1$ :

$$(3.40) \quad \langle \Lambda, c \rangle = (\Lambda|\nu(c)) = (\Lambda|\delta) = (w\Lambda|w\delta) = (w\Lambda|\delta) = (t_{w_0\lambda}\Lambda_0|\delta) = (\Lambda_0|\delta) = 1$$

where  $\nu : \widehat{\mathfrak{h}} \rightarrow \widehat{\mathfrak{h}}^*$  is the isomorphism as in [16, §2.1]:  $(w\Lambda|w\delta) = (w\Lambda|\delta)$  because of the invariance of the form under the action of the affine Weyl group and the fact that  $\delta$  is fixed by the affine Weyl group; the penultimate equality holds because of (3.39).

Let  $L(\Lambda)$  be the integrable  $\widehat{\mathfrak{g}}$ -module with highest weight  $\Lambda$ . With  $w$  as above, denote by  $V_w(\Lambda)$  the Demazure submodule  $U(\widehat{\mathfrak{b}})(L(\Lambda)_{w\Lambda})$  of  $L(\Lambda)$ : here  $L(\Lambda)_{w\Lambda}$  denotes the (one-dimensional)  $\widehat{\mathfrak{h}}$ -weight space of  $L(\Lambda)$  of weight  $w\Lambda$ .

We recall the identification of local Weyl modules as Demazure modules (see Theorem 1.3.4):

**Theorem 3.3.1.** ([5, 1.5.1 Corollary] for type  $A$  and [10, Theorem 7] in general) *Let  $\mathfrak{g}$  be simply laced and  $\lambda$  be a dominant integral weight of  $\mathfrak{g}$ . With  $w$  and  $\Lambda$  as in (3.38), let  $v$  be a non-zero element of the one-dimensional  $\widehat{\mathfrak{h}}$ -weight space of the Demazure module  $V_w(\Lambda)$  of weight  $w_0w\Lambda = t_\lambda w_0\Lambda_0 = t_\lambda\Lambda_0$ . Then  $v$  satisfies the relations (3.36) and therefore, since  $V_w(\Lambda)$  is finite dimensional, there exists a unique  $\mathfrak{g}[t]$ -map from the local Weyl module  $W(\lambda)$  onto  $V_w(\Lambda)$  mapping the generator to  $v$ . This map is an isomorphism.*

### 3.3.5 The key proposition

The following lemma will be useful in proving the main proposition of this section. It may be well known, we give a proof here for completeness.

**Lemma 3.3.2.** *Assume that  $\mathfrak{g}$  is simply laced and  $\lambda$  be a dominant integral weight of  $\mathfrak{g}$ . Let  $\Lambda \in \widehat{P}^+$  is of level 1 such that the local Weyl module  $W(\lambda)$  is isomorphic to a  $\mathfrak{g}[t]$ -submodule of  $L(\Lambda)$ . Let  $\mu \in \widehat{\mathfrak{h}}^*$  is a weight of  $W(\lambda)$ . If  $\mu + \delta$  is not a weight of  $W(\lambda)$ , then it is also not a weight of  $L(\Lambda)$ .*

*Proof.* We know that any weight of  $L(\Lambda)$  is of the form  $t_\gamma(\Lambda) - d\delta$ , where  $\gamma \in Q$  and  $d \in \mathbb{Z}_{\geq 0}$  [16, §12.6]. Therefore, we take  $\mu = t_\gamma(\Lambda) - d\delta$ , for some  $\gamma \in Q$  and  $d \in \mathbb{Z}_{\geq 0}$ . Let  $v_{t_\gamma(\Lambda)}$  be a non-zero element of  $L(\Lambda)$  of weight  $t_\gamma(\Lambda)$ . Let  $v_\mu$  be a non-zero element of  $W(\lambda)$  of weight  $\mu$ . Consider the Lie subalgebra  $\mathfrak{a} = \bigoplus_{m \in \mathbb{Z}} (\mathfrak{h} \otimes t^m)$  of  $\widehat{\mathfrak{g}}$ , and also its representation  $V = \bigoplus_{n \in \mathbb{Z}} L(\Lambda)_{t_\gamma(\Lambda) - n\delta}$ . It is well-known that  $V$  is an irreducible  $\mathfrak{a}$ -module [16, Proposition 9.13]. This gives that  $V$  is the cyclic  $\mathfrak{a}$ -module generated by  $v_{t_\gamma(\Lambda)}$ . Since  $v_\mu \in V$  and  $(\mathfrak{h} \otimes t^m)v_{t_\gamma(\Lambda)} = 0 \forall m \in \mathbb{N}$ , we get

$$v_\mu = (h_1 \otimes t^{-m_1}) \cdots (h_p \otimes t^{-m_p}) v_{t_\gamma(\Lambda)},$$

for some  $h_i \in \mathfrak{h}$  and  $m_i \in \mathbb{Z}_{\geq 0}$ . Thus, we have

$$(h_p \otimes t^{m_p}) \cdots (h_1 \otimes t^{m_1}) v_\mu = (h_p \otimes t^{m_p}) \cdots (h_1 \otimes t^{m_1}) (h_1 \otimes t^{-m_1}) \cdots (h_p \otimes t^{-m_p}) v_{t_\gamma(\Lambda)},$$

which is a non-zero scalar multiple of  $v_{t_\gamma(\Lambda)}$ . Hence, we get that  $v_{t_\gamma(\Lambda)} \in W(\lambda)$ , i.e.,  $t_\gamma(\Lambda)$  is a weight of  $W(\lambda)$ . Suppose  $\mu + \delta$  is not a weight of  $W(\lambda)$ , then we get that  $\mu = t_\gamma(\Lambda)$  and therefore  $\mu + \delta$  is not a weight of  $L(\Lambda)$ . This completes the proof.  $\square$

We now state and prove the main proposition of this section. Observe that the  $\mathfrak{g}$ -weights of the local Weyl module  $W(\lambda)$  are precisely the weights of the irreducible  $\mathfrak{g}$ -module  $V(\lambda)$  with highest weight  $\lambda$ : indeed this follows from the finite dimensionality of  $W(\lambda)$  and the fact that  $W(\lambda) = U(\mathfrak{n}^-[t])w$ , where  $w$  is the generator.

**Proposition 3.3.3.** *Let  $\lambda$  be a dominant integral weights of  $\mathfrak{g}$  and  $\mu$  a weight of the irreducible  $\mathfrak{g}$ -module  $V(\lambda)$ . Let  $M$  be the maximal integer such that the  $M^{\text{th}}$ -graded piece  $W(\lambda)_\mu[M]$  of the  $\mu$ -weight space of the local Weyl module  $W(\lambda)$  is non-zero. Then, under the assumption that  $\mathfrak{g}$  is simple of simply laced type, we have:*

1.  $M = \frac{1}{2} \cdot ((\lambda|\lambda) - (\mu|\mu))$
2.  $W(\lambda)_\mu[M]$  has dimension 1

*Proof.* Fix notation as in §3.3.4. Identify  $W(\lambda)$  with the Demazure module  $V_w(\Lambda)$  as explained there. Since  $W(\lambda) = U(\mathfrak{n}^-[t]) \cdot w$  and the generator  $w$  is identified with an element of  $V_w(\Lambda)_{t_\lambda\Lambda_0}$ , the  $\widehat{\mathfrak{h}}$ -weights of  $W(\lambda)$  are of the form

$$(3.41) \quad t_\lambda\Lambda_0 - \eta + d\delta$$

where  $\eta$  is a positive integral linear combination of the simple roots of  $\mathfrak{g}$ , and  $d$  is a non-negative integer. From (3.39) we have:

$$(3.42) \quad t_\lambda\Lambda_0 = \Lambda_0 + \lambda - \frac{1}{2}(\lambda|\lambda)\delta$$

so we may rewrite (3.41) as:

$$(3.43) \quad \lambda - \eta + \Lambda_0 - \frac{1}{2}(\lambda|\lambda)\delta + d\delta$$

Observe that this weight acts on  $\mathfrak{h}$  as  $\lambda - \eta$ .

Let  $\eta$  be such that  $\lambda - \eta = \mu$ . Then by the hypothesis of maximality of  $M$ , we have:

- $\kappa := t_\lambda \Lambda_0 + (\mu - \lambda) + M\delta$  is a weight of  $W(\lambda)$
- but  $\kappa_1 := t_\lambda \Lambda_0 + (\mu - \lambda) + M'\delta$  is not (a weight of  $W(\lambda)$ ) for  $M' > M$ .

Then  $\kappa_1$  is not a weight of  $L(\Lambda)$  either (see Lemma 3.3.2). Thus  $\kappa$  is a *maximal weight* of  $L(\Lambda)$  in the sense of [16, §12.6]. Since  $\Lambda$  is of level one (3.40), there exists, by [16, Lemma 12.6], an element  $\gamma$  of the root lattice of  $\mathfrak{g}$  such that  $\kappa = t_\gamma \Lambda$ . In particular,  $\kappa$  is a Weyl group translate of the highest weight  $\Lambda$  of  $L(\Lambda)$ , and so the multiplicity of the  $\kappa$ -weight space in the Demazure module  $V_w(\Lambda)$  (and so also in  $W(\lambda)$ ) cannot exceed 1. This proves (2).

We now prove (1), by equating two expressions for  $\kappa$ . On the one hand, by the definition of  $\kappa$  and (3.42), we get

$$(3.44) \quad \kappa = \Lambda_0 + \mu - \frac{1}{2}(\lambda|\lambda)\delta + M\delta$$

On the other hand we have  $\kappa = t_\gamma \Lambda$ . Since  $\Lambda$  is of level 1, we obtain using [16, (6.5.3)] that

$$(3.45) \quad \kappa = t_\gamma \Lambda = \Lambda_0 + (\bar{\Lambda} + \gamma) + \frac{1}{2}((\Lambda|\Lambda) - (\bar{\Lambda} + \gamma|\bar{\Lambda} + \gamma))\delta$$

where  $\bar{\Lambda}$  denotes the restriction to  $\mathfrak{h}$  of  $\Lambda$ . We have  $(\Lambda|\Lambda) = (w\Lambda|w\Lambda) = (t_{w_0\lambda}\Lambda_0|t_{w_0\lambda}\Lambda_0) = 0$ , where the last equality follows from (3.42).

Equating the  $\mathfrak{h}^*$  components on the right hand sides of (3.44) and (3.45), we get  $\mu = \bar{\Lambda} + \gamma$ ; now equating the coefficients of  $\delta$ , we get (1).  $\square$

We now derive a consequence of the proposition above (Corollary 3.3.5) by combining it with a result of Kodera-Naoi [18], which we first recall. While Corollary 3.3.5 may be well known to experts, including a proof of it here is not out of place, particularly since we use it crucially later in the later sections. For a dominant integral weight  $\nu$  of  $\mathfrak{g}$ , let  $b_\nu$  denote the minimum value of  $(\chi|\chi)$  for a weight  $\chi$  of the irreducible representation  $V(\nu)$ . It is clear from item (1) of the proposition that the maximum grade in which the local Weyl module  $W(\nu)$  lives is  $((\nu|\nu) - b_\nu)/2$ .

Recall that the socle of a module is by definition the largest semisimple submodule.

**Theorem 3.3.4** ([18, §3]). *For a dominant integral weight  $\nu$  of a simply laced simple Lie algebra  $\mathfrak{g}$ , the socle (as a  $\mathfrak{g}[t]$ -module) of the local Weyl module  $W(\nu)$  is its homogeneous piece of largest possible grade, namely  $((\nu|\nu) - b_\nu)/2$ . Moreover, the socle is simple.*

**Corollary 3.3.5.** *Let  $\mathfrak{g}$  be a simple algebra of simply laced type. Let  $\lambda$  and  $\mu$  be dominant integral weights of  $\mathfrak{g}$  such that  $\mu$  is a weight of the irreducible representation  $V(\lambda)$  of  $\mathfrak{g}$  with highest weight  $\lambda$ . Then the space  $\text{Hom}_{\mathfrak{g}[t]}(W(\mu), W(\lambda))$  of  $\mathfrak{g}[t]$ -homomorphisms from the local Weyl module  $W(\mu)$  to the local Weyl module  $W(\lambda)$  is one dimensional. Any non-zero element of this space is an injection.*

*Proof.* Let  $\varphi$  be an element of  $\text{Hom}_{\mathfrak{g}[t]}(W(\mu), W(\lambda))$ . Write  $\varphi w_\mu = v_0 + v_1 + \dots$ , where  $w_\mu$  is the generator of  $W(\mu)$ , and  $v_j$  are homogeneous elements of weight  $\mu$  and pairwise different grades, with  $v_0$  being of maximum possible grade  $M := ((\lambda|\lambda) - (\mu|\mu))/2$ . Then each  $v_j$  is a highest weight vector in the sense of (3.36), so there exists a  $\mathfrak{g}[t]$ -homomorphism  $\varphi_j : W(\mu) \rightarrow W(\lambda)$  defined by  $\varphi(w_\mu) = v_j$ . The image of  $\varphi_j$  is homogeneous and the maximum possible grade in it is at most  $M + ((\mu|\mu) - b_\mu)/2 = ((\lambda|\lambda) - b_\lambda)/2$  (since  $b_\mu = b_\lambda$  from the hypothesis). Thus the image of  $\varphi_j$ , for  $j > 0$ , does not meet the socle of  $W(\lambda)$  (by the theorem) and hence is zero. This proves that the  $v_j$  are all zero (for  $j > 0$ ).

Thus  $\varphi w_\mu = v_0$ . By (2) of Proposition 3.3.3, the  $M^{\text{th}}$  graded piece  $W(\lambda)_\mu[M]$  of the weight space  $W(\lambda)_\mu$  is 1-dimensional. This proves that  $\text{Hom}_{\mathfrak{g}[t]}(W(\mu), W(\lambda))$  has dimension at most 1.

On the other hand, it is clear that  $\mathfrak{h} \otimes \mathbb{C}[t]$  kills  $W(\lambda)_\mu[M]$  since  $M$  is maximal. Since  $\mu$  is dominant,  $(\mu|\alpha) \geq 0$  for every positive root  $\alpha$ , so that  $(\mu + \alpha|\mu + \alpha) > (\mu|\mu)$ . By item (1) of the proposition,  $W(\lambda)_{\mu+\alpha}[M] = 0$ , so  $\mathfrak{n}^+ \otimes \mathbb{C}[t]$  kills  $W(\lambda)_\mu[M]$ . Thus the space  $W(\lambda)_\mu[M]$  consists of highest weight vectors of weight  $\mu$  in the sense of (3.36). We thus obtain  $\dim \text{Hom}_{\mathfrak{g}[t]}(W(\mu), W(\lambda)) = \dim W(\lambda)_\mu[M] = 1$ .

Let  $\varphi \neq 0$  be a  $\mathfrak{g}[t]$ -homomorphism from  $W(\mu)$  to  $W(\lambda)$ . To show that it is injective, it is enough to show that its restriction  $\varphi'$  to the socle of  $W(\mu)$  is injective (because every non-zero submodule meets the socle). Since  $\varphi$  is homogeneous of degree  $((\lambda|\lambda) - (\mu|\mu))/2$  (that is, it shifts grades up by that amount), and the socles are the pieces of highest grade (by the theorem), it follows that  $\text{Im } \varphi' = \text{Im } \varphi \cap \text{socle } W(\lambda)$ . But  $\text{Im } \varphi$  meets the socle (since  $\varphi \neq 0$ ), so it follows that  $\text{Im } \varphi' \neq 0$ , so  $\varphi' \neq 0$ , and so  $\varphi'$  is injective (by the simplicity of the socle of  $W(\mu)$ ).  $\square$

### 3.3.6 The Chari-Loktev bases for local Weyl modules in type $A$

We now recall, in the language of partition overlaid patterns or POPs (see §3.1.8), the result about bases of local Weyl modules in type  $A$  proved by Chari-Loktev [5] (see §1.4.3).

We tacitly use the correspondence as in §3.1.9 between weights and tuples. Fix a dominant integral weight  $\lambda$  with corresponding tuple  $\underline{\lambda} : \lambda_1 \geq \dots \geq \lambda_r \geq \lambda_{r+1} = 0$ . The local Weyl module  $W(\lambda)$  has a homogeneous weight basis indexed by POPs with bounding sequence  $\underline{\lambda}$ . The  $\mathfrak{h}$ -weight of a basis element indexed by a POP equals the weight of the pattern underlying the POP; the grade of the basis element equals the number of boxes in the POP. Thus we obtain a formula for the graded character of  $W(\lambda)$

(see §1.3.2):

$$(3.46) \quad \boxed{\text{ch}_q W(\lambda) = \sum_{\underline{\lambda}} e^{\text{weight}(\mathfrak{P})} q^{|\mathfrak{P}|}}$$

where the subscript  $\underline{\lambda}$  in the summation indicates that the sum is over all POPs with bounding sequence  $\underline{\lambda}$ .

### Chari-Loktev (CL) monomials and basis elements

Let  $x_{ij}^-$ , for  $1 \leq i \leq j \leq r$ , denote the  $(r+1) \times (r+1)$  complex matrix all of whose entries are zero except the one in position  $(j+1, i)$  which is 1. Let  $a, b$  be non-negative integers and  $\underline{\pi}$  a partition that fits into the rectangle  $(a, b)$ . For  $k$  an integer with  $1 \leq k \leq b$ , let  $\mathbf{m}(a, b, \underline{\pi}, k)$  denote the number of parts of  $\underline{\pi}$  that equal  $k$ . Let  $\mathbf{m}(a, b, \underline{\pi}, 0)$  denote  $a - \sum_{k=1}^b \mathbf{m}(a, b, \underline{\pi}, k)$ . Let  $x_{ij}^-(a, b, \underline{\pi})$  denote

$$(x_{ij}^- \otimes 1)^{\mathbf{m}(a, b, \underline{\pi}, 0)} \cdot (x_{ij}^- \otimes t^1)^{\mathbf{m}(a, b, \underline{\pi}, 1)} \cdot \dots \cdot (x_{ij}^- \otimes t^b)^{\mathbf{m}(a, b, \underline{\pi}, b)} = \prod_{i=0}^b (x_{ij}^- \otimes t^i)^{\mathbf{m}(a, b, \underline{\pi}, i)},$$

where  $X^{(p)}$  denotes the divided power  $X^p/p!$ . The order of factors in the above product is immaterial since they commute with each other.

Let  $\mathfrak{P}$  be a POP with bounding sequence  $\underline{\lambda}$ . Let  $\underline{\lambda}^1, \dots, \underline{\lambda}^r, \underline{\lambda}^{r+1} = \underline{\lambda}$  be the rows of the underlying pattern of  $\mathfrak{P}$ , and  $\underline{\pi}(j)^i$ ,  $1 \leq i \leq j \leq r$ , be the partition overlay. For  $1 \leq j \leq r$ , denote by  $\rho_{\mathfrak{P}}^j$  the monomial

$$\prod_{i=1}^j x_{ij}^-(\lambda_i^{j+1} - \lambda_i^j, \lambda_i^j - \lambda_{i+1}^{j+1}, \underline{\pi}(j)^i)$$

where again the order is immaterial in the product. We finally let

$$(3.47) \quad \rho_{\mathfrak{P}} := \rho_{\mathfrak{P}}^1 \cdot \rho_{\mathfrak{P}}^2 \cdot \dots \cdot \rho_{\mathfrak{P}}^{r-1} \cdot \rho_{\mathfrak{P}}^r$$

where now the order of the factors matters. The monomial (3.47) is called the *Chari-Loktev (or CL) monomial* corresponding to  $\mathfrak{P}$ .

**Theorem 3.3.6.** [5, Theorem 2.1.3] *The elements  $v_{\mathfrak{P}} := \rho_{\mathfrak{P}} w_{\lambda}$ , as  $\mathfrak{P}$  varies over all POPs with bounding sequence  $\underline{\lambda}$ , form a basis for the local Weyl module  $W(\lambda)$ .*

We call  $v_{\mathfrak{P}}$  the *Chari-Loktev (or CL) basis element* corresponding to  $\mathfrak{P}$ .

### 3.3.7 Representation theoretic proof of Theorem 3.2.2

We are now ready to give a representation theoretic proof of the following version of our main theorem (Theorem 3.2.2):

*Let  $\underline{\lambda} : \lambda_1 \geq \dots \geq \lambda_n$  be a non-increasing sequence of integers. Let  $\underline{\mu} \in \mathbb{Z}^n$  such that  $\underline{\lambda} \succ_m \underline{\mu}$ . Then there is a unique pattern (with real entries) with bounding sequence  $\underline{\lambda}$ , weight  $\underline{\mu}$ , and (triangular) area  $\frac{1}{2}(\|\underline{\lambda}\|^2 - \|\underline{\mu}\|^2)$ . This pattern is integral. Any other pattern (with real entries) with bounding sequence  $\underline{\lambda}$  and weight  $\underline{\mu}$  has (triangular) area strictly less than  $\frac{1}{2}(\|\underline{\lambda}\|^2 - \|\underline{\mu}\|^2)$ .*

Subtracting  $\lambda_n$  from all entries of a pattern sets up an area and integrability preserving bijection between patterns with bounding sequence  $\underline{\lambda}$  and weight  $\underline{\mu}$  on the one hand and those with bounding sequence  $\lambda_1 - \lambda_n \geq \dots \geq \lambda_{n-1} - \lambda_n \geq 0$  and weight  $\underline{\mu} - (\lambda_n, \dots, \lambda_n)$  on the other. Moreover passing from  $\underline{\lambda}$  and  $\underline{\mu}$  to  $\lambda_1 - \lambda_n \geq \dots \lambda_{n-1} - \lambda_n \geq 0$  and  $\underline{\mu} - (\lambda_n, \dots, \lambda_n)$  does not affect the hypothesis  $\underline{\lambda} \succ_m \underline{\mu}$ . We may therefore assume without loss of generality that  $\lambda_n = 0$ .

Let  $M$  be the maximal (triangular) area attained among all integral patterns with bounding sequence  $\underline{\lambda}$  and weight  $\underline{\mu}$ . Then by the theorem of Chari-Loktev recalled above in §3.3.6, it follows that  $M$  is the maximal such that  $W(\underline{\lambda})_{\underline{\mu}}[M] \neq 0$  and moreover that the dimension of  $W(\underline{\lambda})_{\underline{\mu}}[M]$  equals the number of integral patterns with bounding sequence



$\underline{\lambda}$ , weight  $\underline{\mu}$ , and (triangular) area  $M$ . It now follows from Proposition 3.3.3 that the number of such patterns is 1 and that  $M = \frac{1}{2}(\|\underline{\lambda}\|^2 - \|\underline{\mu}\|^2)$ .

Now suppose that we have a pattern with real entries with bounding sequence  $\underline{\lambda}$  and weight  $\underline{\mu}$ . Then its trapezoidal area is  $\frac{1}{2}(\|\underline{\lambda}\|^2 - \|\underline{\mu}\|^2)$  (Corollary 3.2.4). So its triangular area is at most this number. Moreover, if its triangular area equals this number, then it is integral (Corollary 3.2.6).

# Chapter 4

## A bijection between colored partitions and POPs

The results of this chapter will appear in [21]. This chapter is entirely combinatorial and may be read independently of the rest of the thesis. Its goal is Theorem 4.5.2, which gives a certain bijection between colored partitions of a number and POPs of a certain kind. Quite apart from any interest this bijection may have in its own right, we use it in the next section to state the conjectural stability of the Chari-Loktev bases. The stability property expresses compatibility of the bases with inclusions of local Weyl modules, and in order to make sense of this there must be in the first place an identification of the indexing set of the basis of the included module as a subset of the indexing set of the basis of the ambient module. The combinatorial bijection of this section establishes the desired identification.

### 4.1 Breaking up a partition

In this section, we describe a procedure to break up a partition into smaller partitions depending upon some input. This can be viewed as a generalization of the construction

of Durfee squares. We first treat the case when the input is a single integer. We then treat the general case when the input is a non-decreasing sequence of integers.

#### 4.1.1 The case when a single integer is given

First suppose that we are given:

- a partition  $\underline{\pi} : \pi_1 \geq \pi_2 \geq \dots$ , and
- an integer  $c$ .

It is convenient to put  $\pi_0 = \infty$ . Consider the function  $m \mapsto \pi_m + c - m$  on non-negative integers. It is decreasing, takes value  $\infty$  at 0, is non-negative at  $c$  if  $c$  is non-negative, and is negative for large  $m$ . Let  $a$  be the largest non-negative integer such that  $\pi_a \geq a - c$ . Note that  $a \geq c$ .

Put  $b := a - c$ . Consider the partitions  $\underline{\pi}^1 : \pi_1 - b \geq \dots \geq \pi_a - b$  and  $\underline{\pi}^2 : \pi_{a+1} \geq \pi_{a+2} \geq \dots$ . The former has at most  $a$  parts; the latter has largest part at most  $b$  (since  $\pi_{a+1} < (a + 1) - c$  by choice of  $a$ ). The last assertion can be stated as follows:

$$(4.1) \quad \pi_d \leq b \text{ for } d > a$$

It is easily seen that

$$(4.2) \quad |\underline{\pi}| = ab + |\underline{\pi}^1| + |\underline{\pi}^2|$$

Consider the association  $(c, \underline{\pi}) \mapsto (a, b; \underline{\pi}^1, \underline{\pi}^2)$ . Since  $(c, \underline{\pi})$  can be recovered from  $(a, b; \underline{\pi}^1, \underline{\pi}^2)$ , the association is one-to-one. Its image, as  $c$  varies over all integers and  $\underline{\pi}$  over all partitions, consists of all  $(a, b; \underline{\pi}^1, \underline{\pi}^2)$  such that  $a, b$  are non-negative integers,  $\underline{\pi}^1$  is a partition with at most  $a$  parts, and  $\underline{\pi}^2$  is a partition with largest part at most  $b$ .

Figure 4.1 illustrates the procedure just described. Note that the  $c = 0$  case is the Durfee square construction.

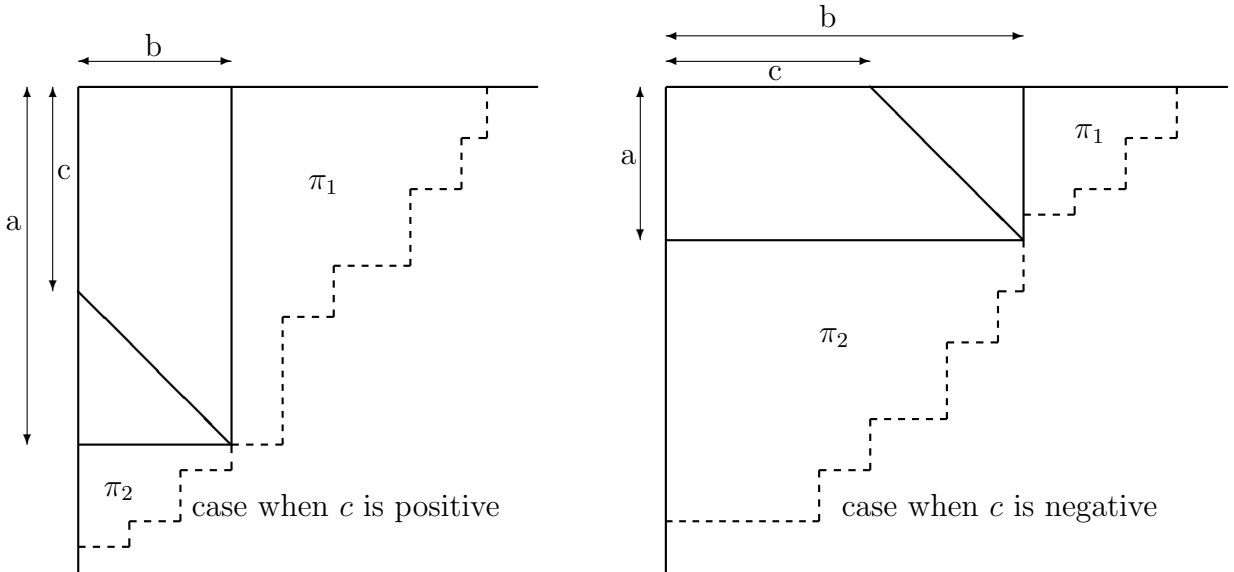


Figure 4.1: Illustration of the procedure in §4.1.1

#### 4.1.2 The case when a non-decreasing sequence of integers is given

Now suppose that we are given, for  $t \geq 2$  an integer, the following:

- a partition  $\underline{\pi} : \pi_1 \geq \pi_2 \geq \dots$ , and
- a sequence  $\underline{c} : c_1 \leq \dots \leq c_{t-1}$  of integers.

As before, it is convenient to set  $\pi_0 = \infty$ . For  $1 \leq j \leq t-1$ , let  $a_j$  be the largest non-negative integer such that  $\pi_{a_j} \geq a_j - c_j$ . Since the  $c_j$  are non-decreasing, it is clear that the  $a_j$  are also non-decreasing:  $a_1 \leq \dots \leq a_{t-1}$ . Set  $b_j := a_j - c_j$ .

**Proposition 4.1.1.** *The  $b_j$  thus defined are non-increasing:  $b_1 \geq \dots \geq b_{t-1}$ .*

*Proof.* Fix  $j$  such that  $1 \leq j \leq t-2$  (there is nothing to prove in case  $t = 2$ ). If  $a_{j+1} = a_j$ , then  $b_{j+1} = a_{j+1} - c_{j+1} = a_j - c_{j+1} \leq a_j - c_j = b_j$ . If  $a_{j+1} > a_j$ , then, on the

one hand,  $\pi_{a_{j+1}} \leq b_j$  by (4.1); and, on the other,  $b_{j+1} \leq \pi_{a_{j+1}}$  (by the definitions of  $a_{j+1}$  and  $b_{j+1}$ ).  $\square$

We define  $t$  partitions  $\underline{\pi}^1, \underline{\pi}^2, \dots, \underline{\pi}^t$  as follows. Set  $a_0 = 0, a_t = \infty; b_0 = \infty, b_t = 0$ ; and for  $j, 1 \leq j \leq t$ :

$$(4.3) \quad \underline{\pi}^j : \pi_{a_{j-1}+1} - b_j \geq \pi_{a_{j-1}+2} - b_j \geq \dots \geq \pi_{a_j} - b_j$$

The above equation can be rewritten as follows:

$$(4.4) \quad \pi_{k-a_{j-1}}^j := \pi_k - b_j \quad \text{for } k \text{ such that } a_{j-1} < k \leq a_j$$

Note that  $\underline{\pi}^j$  fits into the rectangle  $(a_j - a_{j-1}, b_{j-1} - b_j)$  in the sense of §3.1.6 (since  $\pi_{a_{j-1}+1} \leq b_{j-1}$  by (4.1)). We have

$$(4.5) \quad |\underline{\pi}| = |\underline{\pi}^1| + \dots + |\underline{\pi}^t| + \sum_{j=1}^{t-1} (a_j - a_{j-1})b_j = |\underline{\pi}^1| + \dots + |\underline{\pi}^t| + \sum_{j=1}^{t-1} a_j(b_j - b_{j+1})$$

Consider the association  $(\underline{c}, \underline{\pi}) \mapsto (\underline{a}, \underline{b}; \underline{\pi}^1, \dots, \underline{\pi}^t)$ , where  $\underline{a}, \underline{b}$  refer respectively to the sequences  $a_1 \leq \dots \leq a_{t-1}$  and  $b_1 \geq \dots \geq b_{t-1}$ . Since  $(\underline{c}, \underline{\pi})$  can be recovered from  $(\underline{a}, \underline{b}; \underline{\pi}^1, \dots, \underline{\pi}^t)$ , the association is one-to-one. Its image, as  $\underline{c}$  varies over all non-decreasing integer sequences of length  $t - 1$  and  $\underline{\pi}$  over all partitions, consists of all  $(\underline{a}, \underline{b}; \underline{\pi}^1, \dots, \underline{\pi}^t)$  such that  $\underline{a}, \underline{b}$  are non-negative integer sequences of length  $t - 1$  with  $\underline{a}$  non-decreasing and  $\underline{b}$  non-increasing, and for every  $j, 1 \leq j \leq t, \underline{\pi}^j$  is a partition that fits into the rectangle  $(a_j - a_{j-1}, b_{j-1} - b_j)$ .

The picture in Figure 4.2 describes the procedure just described.

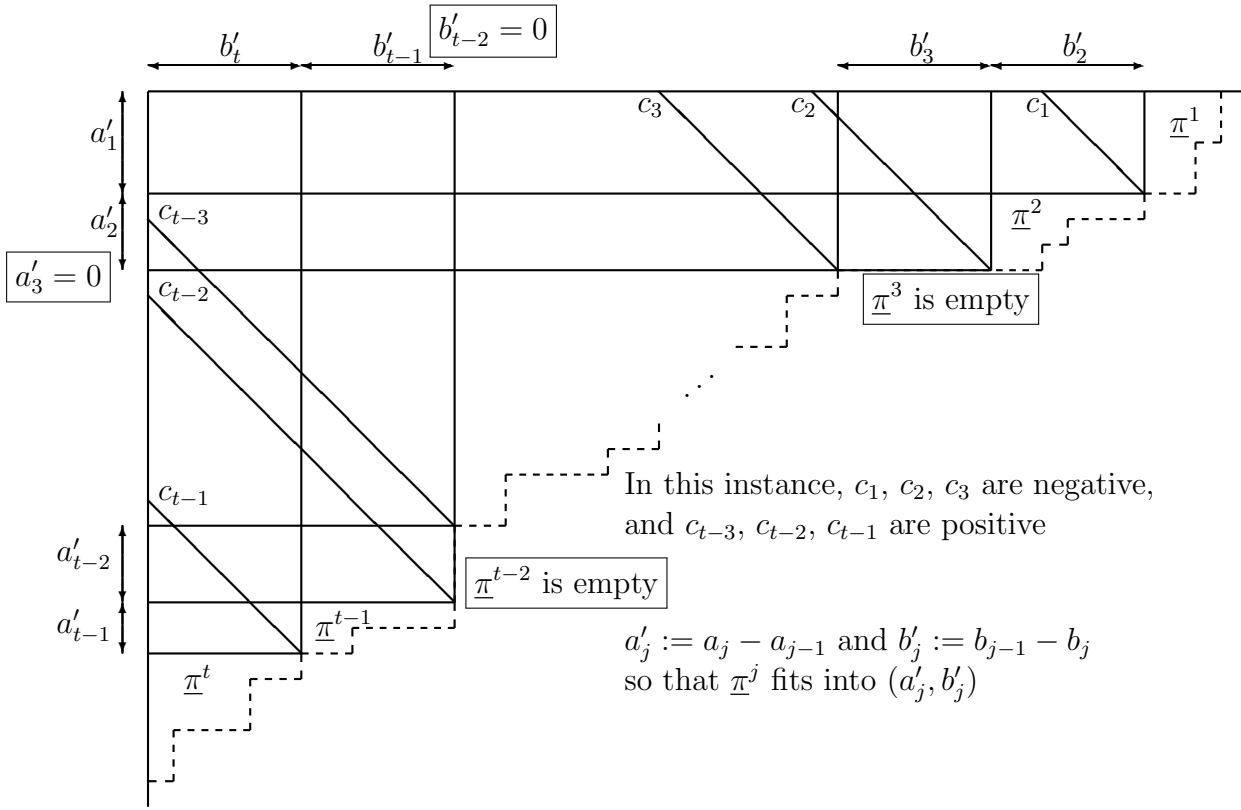


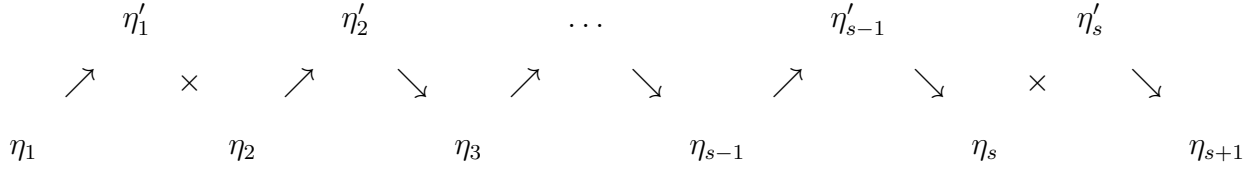
Figure 4.2: Illustration of the procedure in §4.1.2

## 4.2 Producing nearly interlacing sequences with approximate partition overlays

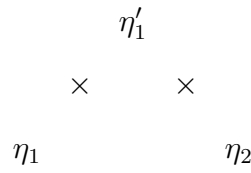
Fix an integer  $s \geq 1$  and an integer sequence  $\underline{\eta}: \eta_1, \dots, \eta_{s+1}$  with  $\eta_2 \geq \dots \geq \eta_s$ . An integer sequence  $\underline{\eta}': \eta'_1, \dots, \eta'_s$  with  $\eta'_2 \geq \dots \geq \eta'_s$  is said to *nearly interlace*  $\underline{\eta}$  if either  $s = 1$  (in which case no further condition is imposed) or  $s \geq 2$  and

$$(4.6) \quad \eta_1 \geq \eta'_1, \quad \eta'_s \geq \eta_{s+1}, \quad \text{and } \eta'_2 \geq \dots \geq \eta'_{s-1} \text{ interlaces } \eta_2 \geq \dots \geq \eta_s.$$

The following is a pictorial depiction of this definition (where  $x \longrightarrow y$  means  $x \geq y$ , and  $\times$  indicates the absence of any relation):



In case  $s = 1$ , the definition imposes no further constraint on  $\underline{\eta}'$  and so the pictorial depiction is:



We define the *proper trapezoidal area* of the nearly interlacing sequences  $\underline{\eta}, \underline{\eta}'$  as above by:

$$(4.7) \quad \square^{\text{prop}}(\underline{\eta}, \underline{\eta}') := \sum_{1 \leq i < j \leq s} (\eta_i - \eta'_i)(\eta'_j - \eta_{j+1})$$

Given sequences  $\underline{\eta}, \underline{\eta}'$  as above that nearly interlace, a sequence  $\underline{\pi}^1, \dots, \underline{\pi}^s$  of partitions is said to *approximately overlay*  $\underline{\eta}, \underline{\eta}'$ , if either  $s = 1$  (in which case no further condition is imposed) or  $s \geq 2$  and

$$(4.8) \quad \left\{ \begin{array}{l} \underline{\pi}^1 \text{ has at most } \eta_1 - \eta'_1 \text{ parts, } \underline{\pi}^s \text{ has largest part at most } \eta'_s - \eta_{s+1}, \text{ and} \\ \text{for } j = 2, \dots, s-1, \text{ the partition } \underline{\pi}^j \text{ fits into the rectangle } (\eta_j - \eta'_j, \eta'_j - \eta_{j+1}). \end{array} \right.$$

Now suppose that we are given:

- a partition  $\underline{\pi} : \pi_1 \geq \pi_2 \geq \dots$ , and
- an integer  $\mu$ .

Our goal first of all in this section is to associate to the data  $(\mu, \underline{\pi})$  a nearly interlacing

sequence with an approximate partition overlay ( $s$  and  $\underline{\eta}$  are fixed once and for all). The map is denoted by  $\Xi_{\underline{\eta}}$ . We then investigate the nature of  $\Xi_{\underline{\eta}}$  (Lemma 4.2.2).

Put  $c_1 := \mu - \eta_2, \dots, c_{s-1} := \mu - \eta_s$ . Then  $c_1 \leq \dots \leq c_{s-1}$ . We apply the procedure of §4.1.2 with  $t = s$  and  $\underline{c}$  as above to obtain  $(\underline{a}, \underline{b}, \underline{\pi}^1, \dots, \underline{\pi}^s)$ . In case  $s = 1$ , we take  $\underline{a}$  and  $\underline{b}$  to be empty and set  $\underline{\pi}^1 := \underline{\pi}$ . In case  $s = 2$ , the procedure of §4.1.2 reduces to that of §4.1.1.

We now define the sequence  $\underline{\eta}'$ . In case  $s = 1$ , set

$$(4.9) \quad \eta'_1 := \eta_1 + \eta_2 - \mu \quad (\text{case } s = 1)$$

Now suppose  $s \geq 2$ . As before, it is convenient to set  $a_0 = b_s = 0$  and  $a_s = b_0 = \infty$ .

Define

$$(4.10) \quad \eta'_j := \eta_j - (a_j - a_{j-1}) \quad \text{for } j = 1, \dots, s-1 \quad \text{and} \quad \eta'_s := \eta_{s+1} + b_{s-1} \quad (\text{case } s \geq 2)$$

**Proposition 4.2.1.** *Suppose that  $s \geq 2$ . Then:*

1. For  $j = 2, \dots, s$ , we have  $\eta'_j = \eta_{j+1} + (b_{j-1} - b_j)$ .
2.  $\underline{\eta}$  and  $\underline{\eta}'$  are nearly interlaced.
3. The sequence  $\underline{\pi}^1, \dots, \underline{\pi}^s$  of partitions approximately overlays  $\underline{\eta}, \underline{\eta}'$ .
4.  $(\eta_1 + \dots + \eta_{s+1}) - (\eta'_1 + \dots + \eta'_s) = \mu$
5.  $|\square^{\text{prop}}(\underline{\eta}, \underline{\eta}')| + (|\underline{\pi}^1| + \dots + |\underline{\pi}^s|) = |\underline{\pi}|$
6.  $\sum_{i=1}^j (\eta_i - \eta'_i) = a_j$  for  $0 \leq j < s$  and  $\sum_{i=j+1}^s (\eta'_i - \eta_{i+1}) = b_j$  for  $1 \leq j \leq s$ .



*Proof.* For  $j = s$ , (1) is just the definition of  $\eta'_s$ . Fix  $j$  in the range  $2, \dots, s-1$ . We have:

$$\begin{aligned}
\eta'_j &= \eta_j - (a_j - a_{j-1}) && \text{definition of } \eta'_j \\
&= \eta_j - ((c_j + b_j) - (c_{j-1} + b_{j-1})) && \text{definition of } \underline{b} \\
&= \eta_j - ((\mu - \eta_{j+1} + b_j) - (\mu - \eta_j + b_{j-1})) && \text{definition of } \underline{c} \\
&= \eta_{j+1} + (b_{j-1} - b_j)
\end{aligned}$$

This proves (1). For (2), we observe:

- For  $1 \leq j \leq s-1$ , we have  $\eta_j \geq \eta_j - (a_j - a_{j-1}) = \eta'_j$  since  $\underline{a}$  is a non-decreasing sequence.
- For  $2 \leq j \leq s$ , we have  $\eta'_j = \eta_{j+1} + (b_{j-1} - b_j) \geq \eta_{j+1}$  by (1) and the fact that  $\underline{b}$  is a non-increasing sequence (Proposition 4.1.1).

Assertion (3) follows since  $\underline{\pi}^j$  fits into  $(a_j - a_{j-1}, b_{j-1} - b_j)$  by construction.

Assertion (4) is just the definition (4.9) in case  $s = 1$ . Now suppose  $s \geq 2$ . By the definition (4.10) of  $\eta'_j$ , we have

$$\begin{aligned}
\eta'_1 + \dots + \eta'_s &= (\eta_1 - (a_1 - a_0)) + \dots + (\eta_{s-1} - (a_{s-1} - a_{s-2})) + (\eta_{s+1} + b_{s-1}) \\
&= \eta_1 + \dots + \eta_{s-1} + \eta_{s+1} - a_{s-1} + b_{s-1} \\
&= \eta_1 + \dots + \eta_{s-1} + \eta_{s+1} - c_{s-1} \quad (\text{since } b_{s-1} = a_{s-1} - c_{s-1} \text{ by definition}) \\
&= \eta_1 + \dots + \eta_{s-1} + \eta_s + \eta_{s+1} - \mu \quad (\text{since } c_{s-1} = \mu - \eta_s \text{ by definition})
\end{aligned}$$

For (5), first rewrite the definition (4.7) to get:

$$\Box^{\text{PROP}}(\underline{\eta}, \underline{\eta}') = \sum_{1 \leq i \leq s-1} (\eta_i - \eta'_i) \left( \sum_{i+1 \leq j \leq s} (\eta'_j - \eta_{j+1}) \right)$$

Substituting from (4.10) and item (1) into the right hand side above, we get

$$\square^{\text{prop}}(\underline{\eta}, \underline{\eta}') = \sum_{1 \leq i \leq s-1} (a_i - a_{i-1}) \left( \sum_{i+1 \leq j \leq s} (b_{j-1} - b_j) \right) = \sum_{1 \leq i \leq s-1} (a_i - a_{i-1}) b_i$$

Assertion (5) now follows from (4.5).

For (6), using (4.10) we obtain  $\sum_{i=1}^j (\eta_i - \eta'_i) = \sum_{i=1}^j (a_i - a_{i-1}) = a_j$ . Similarly, using (1) we obtain  $\sum_{i=j+1}^s (\eta'_i - \eta_{i+1}) = \sum_{i=j+1}^s (b_{i-1} - b_i) = b_j$ .  $\square$

**Lemma 4.2.2.** *Fix an integer  $s \geq 1$  and an integer sequence  $\underline{\eta}$ :  $\eta_1, \dots, \eta_{s+1}$  with  $\eta_2 \geq \dots \geq \eta_s$ . Let  $\Xi_{\underline{\eta}}$  denote the association described above:*

- *from the set of all pairs  $(\mu, \underline{\pi})$ , where  $\mu$  is an integer and  $\underline{\pi}$  a partition*
- *to the set of all tuples  $(\underline{\eta}', \underline{\pi}^1, \dots, \underline{\pi}^s)$ , where  $\eta'$ :  $\eta'_1, \dots, \eta'_s$  is an integer sequence nearly interlacing  $\underline{\eta}$ , and  $\underline{\pi}^1, \dots, \underline{\pi}^s$  a sequence of partitions approximately overlaying  $\underline{\eta}, \underline{\eta}'$*

*The association  $\Xi_{\underline{\eta}}$  is a bijection. More precisely, the association  $\Xi'_{\underline{\eta}}$  in the other direction to be defined below (in the course of the proof of this lemma) is the two-sided inverse of  $\Xi_{\underline{\eta}}$ .*

*Proof.* We define  $\Xi'_{\underline{\eta}}$ . Let  $\underline{\eta}'$  and  $\underline{\pi}^1, \dots, \underline{\pi}^s$  with the specified properties be given. The image of  $(\underline{\eta}', \underline{\pi}^1, \dots, \underline{\pi}^s)$  under  $\Xi'_{\underline{\eta}}$  is defined to be  $(\mu^*, \underline{\pi}^*)$  where  $\mu^*$  and  $\underline{\pi}^*$  are as defined below. Set

$$(4.11) \quad \mu^* := (\eta_1 + \dots + \eta_{s+1}) - (\eta'_1 + \dots + \eta'_s)$$

To define  $\underline{\pi}^*$ , we take  $k$  to be a positive integer and define  $\pi_k^*$ . For  $j$ ,  $0 \leq j \leq s-1$ , set  $a_j^* := \sum_{i=1}^j (\eta_i - \eta'_i)$ . Put  $a_s^* := \infty$ . We have  $0 = a_0^* \leq a_1^* \leq \dots \leq a_{s-1}^* < a_s^* = \infty$ . Thus

there exists unique  $j$ ,  $1 \leq j \leq s$ , such that  $a_{j-1}^* < k \leq a_j^*$ . Set

$$(4.12) \quad \pi_k^* := \pi_{k-a_{j-1}^*}^j + b_j^* \quad \text{where } b_j^* := \sum_{i=j+1}^s (\eta'_i - \eta_{i+1})$$

Since  $\eta'_i \geq \eta_{i+1}$  for all  $i \geq 2$ , it is clear that  $b_j^*$  and hence also  $\pi_k^*$  is non-negative.

Let us verify that  $\pi_k^* \geq \pi_{k+1}^*$  for all  $k$ . If  $a_{j-1}^* < k < a_j^*$ , then  $a_{j-1}^* < k+1 \leq a_j^*$ , so that, from (4.12),  $\pi_k^* - \pi_{k+1}^* = \pi_{k-a_{j-1}^*}^j - \pi_{k+1-a_{j-1}^*}^j \geq 0$  (since  $\underline{\pi}^j$  is a partition). Now suppose that  $k = a_j^*$ . Then, from (4.12),

$$\pi_k^* - \pi_{k+1}^* = (\pi_{k-a_{j-1}^*}^j + b_j^*) - (\pi_1^{j+1} + b_{j+1}^*) = \pi_{a_j^*-a_{j-1}^*}^j + ((\eta'_{j+1} - \eta_{j+2}) - \pi_1^{j+1}) \geq 0$$

where the last inequality holds since  $\underline{\pi}^{j+1}$  has largest part at most  $\eta'_{j+1} - \eta_{j+2}$  by hypothesis. This proves that  $\underline{\pi}^*$  is a partition and finishes the definition of  $\Xi'_\eta$ .

We now verify that  $\Xi'_\eta \circ \Xi_\eta$  is the identity. Suppose we first apply  $\Xi_\eta$  to  $(\mu, \underline{\pi})$  to get  $(\underline{\eta}', \underline{\pi}^1, \dots, \underline{\pi}^s)$  to which in turn we apply  $\Xi'_\eta$  to get  $(\mu^*, \underline{\pi}^*)$ . From (4) of Proposition 4.2.1 and (4.11) it follows that  $\mu^* = \mu$ . It follows from the definitions of  $\underline{a}^*$  and  $\underline{b}^*$  above and (6) of Proposition 4.2.1 that  $\underline{a}^* = \underline{a}$  and  $\underline{b}^* = \underline{b}$ . It now follows from the definitions (4.4) and (4.12) respectively of  $\underline{\pi}^j$  and  $\underline{\pi}^*$  that  $\underline{\pi}^* = \underline{\pi}$ .

Finally we verify that  $\Xi_\eta \circ \Xi'_\eta$  is the identity. Let  $(\mu^*, \underline{\pi}^*)$  be the result of application of  $\Xi'_\eta$  to  $(\underline{\eta}', \underline{\pi}^1, \dots, \underline{\pi}^s)$ . To calculate the action of  $\Xi_\eta$  on  $(\mu^*, \underline{\pi}^*)$ , we must compute  $\underline{c}$ ,  $\underline{a}$ , and  $\underline{b}$  as in §4.1.2. From the definition of  $\underline{c}$  at the beginning of this section and those of  $\mu^*$ ,  $\underline{a}^*$ , and  $\underline{b}^*$  above, we have:

$$(4.13) \quad c_j = \mu^* - \eta_{j+1} = \left( \sum_{i=1}^j (\eta_i - \eta'_i) \right) - \left( \sum_{i=j+1}^s (\eta'_i - \eta_{i+1}) \right) = a_j^* - b_j^*$$

We claim that  $\underline{a}^* = \underline{a}$ . Assuming this claim, it follows from (4.13) and the definition of  $\underline{b}$  in §4.1.2 that  $\underline{b} = \underline{b}^*$ . From (4.10) and the definition of  $\underline{a}^*$  above, it follows that the nearly interlacing sequence part of the image of  $\Xi_\eta$  is  $\underline{\eta}'$ . From (4.4), (4.12), and the equalities

$\underline{a} = \underline{a}^*$ ,  $\underline{b} = \underline{b}^*$ , it now follows that the image under  $\Xi_\eta$  of  $(\mu^*, \underline{\pi}^*)$  equals  $(\underline{\eta}', \underline{\pi}^1, \dots, \underline{\pi}^s)$ .

It remains only to prove the claim above that  $\underline{a} = \underline{a}^*$ , or in other words that for  $j$ ,  $1 \leq j \leq s-1$ ,  $a_j^*$  is the largest integer such that  $\pi_{a_j^*}^* \geq a_j^* - c_j$ . From (4.12) and (4.13), we have

$$\pi_{a_j^*}^* = \pi_{a_j^* - a_{j-1}^*}^j + b_j^* \geq b_j^* = a_j^* - c_j$$

We now show that  $\pi_{a_j^*+1}^* < a_j^* + 1 - c_j$ . Fix  $\ell$ ,  $j+1 \leq \ell \leq s$ , such that  $a_j^* = a_{\ell-1}^* < a_j^* + 1 \leq a_\ell^*$ . From (4.12) and (4.13), we get

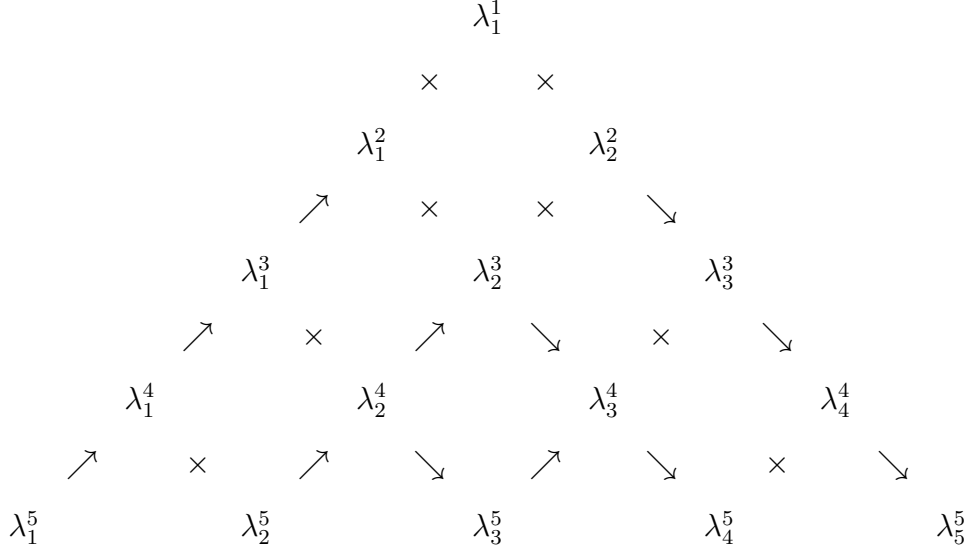
$$\begin{aligned} 1 + a_j^* - c_j - \pi_{a_j^*+1}^* &= 1 + (a_j^* - c_j) - \pi_1^\ell - b_\ell^* \\ &= 1 + (b_j^* - b_\ell^*) - \pi_1^\ell \\ &= 1 + (\eta'_{j+1} - \eta_{j+2}) + \dots + (\eta'_{\ell-1} - \eta_\ell) + ((\eta'_\ell - \eta_{\ell+1}) - \pi_1^\ell) \end{aligned}$$

Since  $(\eta'_i - \eta_{i+1}) \geq 0$  for  $2 \leq i \leq s$  and the largest part  $\pi_1^\ell$  of  $\underline{\pi}^\ell$  is at most  $(\eta'_\ell - \eta_{\ell+1})$ , the right hand side in the last line of the above display is positive.  $\square$

### 4.3 Near patterns with approximate partition overlays

Fix an integer  $r \geq 1$ . Suppose that, for every  $j$ ,  $1 \leq j \leq r+1$ , we have an integer sequence  $\underline{\lambda}^j$ :  $\lambda_1^j, \dots, \lambda_j^j$  of length  $j$  with  $\lambda_2^j \geq \dots \geq \lambda_{j-1}^j$ . We say that this collection of sequences forms a *near pattern* if  $\underline{\lambda}^j$  nearly interlaces  $\underline{\lambda}^{j+1}$  for every  $j$ ,  $1 \leq j \leq r$ . The last sequence  $\underline{\lambda}^{r+1}$  is called the *bounding sequence* of the near pattern. The following is a pictorial depiction of this definition for  $r = 4$  (where  $x \longrightarrow y$  means  $x \geq y$ , and  $\times$

indicates the absence of any relation):



The *proper trapezoidal area* of a near pattern  $\mathcal{P} = \{\underline{\lambda}^j \mid 1 \leq j \leq r + 1\}$  is defined by:

$$(4.14) \quad \square^{\text{prop}}(\mathcal{P}) := \sum_{j=2}^r \square^{\text{prop}}(\underline{\lambda}^{j+1}, \underline{\lambda}^j) = \sum_{j=2}^r \sum_{1 \leq i < h \leq j} (\lambda_i^{j+1} - \lambda_i^j)(\lambda_h^j - \lambda_{h+1}^{j+1})$$

The *weight* of a near pattern  $\mathcal{P}$  as above is the tuple  $(\mu_1, \dots, \mu_{r+1})$ , where

$$\mu_{j+1} := \sum_{i=1}^{j+1} \lambda_i^{j+1} - \sum_{i=1}^j \lambda_i^j \quad \text{for } 1 \leq j \leq r \quad \text{and} \quad \mu_1^1 := \lambda_1^1$$

Let  $\mathcal{P} = \{\underline{\lambda}^j \mid 1 \leq j \leq r + 1\}$  be a near pattern. Suppose that we are given partitions  $\underline{\pi}(j)^i$ , for  $1 \leq j \leq r$  and  $1 \leq i \leq j$ . We say that this collection of partitions *approximately overlays* the near pattern  $\mathcal{P}$  if:

- for  $2 \leq j \leq r$ , the partition  $\underline{\pi}(j)^1$  has at most  $\lambda_1^{j+1} - \lambda_1^j$  parts
- for  $2 \leq j \leq r$ , the partition  $\underline{\pi}(j)^j$  has largest part at most  $\lambda_j^j - \lambda_{j+1}^{j+1}$
- for  $3 \leq j \leq r$  and  $2 \leq i \leq j - 1$ , the partition  $\underline{\pi}(j)^i$  fits into the rectangle  $(\lambda_i^{j+1} - \lambda_i^j, \lambda_i^j - \lambda_{i+1}^{j+1})$

The above conditions can also be expressed by saying that for every  $j$ ,  $1 \leq j \leq r$ , the sequence  $\underline{\pi(j)^i}$ ,  $1 \leq i \leq j$ , of partitions approximately overlays the nearly interlacing sequences  $\underline{\lambda}^{j+1}$ ,  $\underline{\lambda}^j$  in the sense of (4.8).

The *number of boxes* in an approximate partition overlay as above of a near pattern is defined to be  $\sum_{1 \leq i \leq j \leq r} |\underline{\pi(j)^i}|$ . The terminology is justified by thinking of the partitions in terms of their shapes.

We sometimes use the the term *approximately overlaid near pattern*, *AONP* for short, for a near pattern with an approximate partition overlay.

### 4.3.1 A bijection on AONPs

Fix an integer  $s \geq 1$  and an integer sequence  $\underline{\lambda}^{r+1}$ :  $\lambda_1^{r+1}, \dots, \lambda_{r+1}^{r+1}$  with  $\lambda_2^{r+1} \geq \dots \geq \lambda_r^{r+1}$ . Let  $\mathcal{M}$  denote the set of tuples  $(\underline{\mu}; \underline{\pi(1)}, \dots, \underline{\pi(r)})$ , where  $\underline{\mu}$ :  $\mu_2, \dots, \mu_{r+1}$  is a sequence of  $r$  integers and  $\underline{\pi(1)}, \dots, \underline{\pi(r)}$  is a sequence of  $r$  partitions (the reason for the indexing of  $\mu_j$  starting with 2 will become clear presently). Let  $\mathcal{N}_{\underline{\lambda}^{r+1}}$  denote the set of all AONPs with bounding sequence  $\underline{\lambda}^{r+1}$ .

Given an element of  $(\underline{\mu}; \underline{\pi(1)}, \dots, \underline{\pi(r)})$  in  $\mathcal{M}$ , set  $(\underline{\lambda}^r, \underline{\pi(r)^1}, \dots, \underline{\pi(r)^r}) := \Xi_{\underline{\lambda}^{r+1}}(\mu_{r+1}, \underline{\pi(r)})$ , where  $\Xi_{\underline{\lambda}^{r+1}}$  is as defined in §4.2. By (2) of Proposition 4.2.1, it follows that  $\underline{\lambda}^r$  is such that  $\lambda_2^r \geq \dots \geq \lambda_{r-1}^r$ . We may thus inductively define:

$$(4.15) \quad (\underline{\lambda}^j, \underline{\pi(j)^1}, \dots, \underline{\pi(j)^j}) := \Xi_{\underline{\lambda}^{j+1}}(\mu_{j+1}, \underline{\pi(j)})$$

From (2) and (3) of Proposition 4.2.1, it follows that the sequences  $\underline{\lambda}^j$  ( $1 \leq j \leq r+1$ ) and partitions  $\underline{\pi(j)^i}$  ( $1 \leq j \leq r$ ,  $1 \leq i \leq j$ ) form an AONP with bounding sequence  $\underline{\lambda}^{r+1}$ . Thus we have defined a map from  $\mathcal{M}$  to  $\mathcal{N}_{\underline{\lambda}^{r+1}}$ , which too we denote by  $\Xi_{\underline{\lambda}^{r+1}}$  by abuse of notation.

**Lemma 4.3.1.** *The map  $\Xi_{\underline{\lambda}^{r+1}}$  from  $\mathcal{M}$  to  $\mathcal{N}_{\underline{\lambda}^{r+1}}$  just defined is a bijection.*

*Proof.* We construct a two sided inverse. For an element  $\{\underline{\lambda}^j; \underline{\pi(j)}^i \mid 1 \leq j \leq r, 1 \leq i \leq j\}$  of  $\mathcal{N}_{\underline{\lambda}^{r+1}}$ , set

$$(\mu_{j+1}^*, \underline{\pi(j)}^*) := \Xi'_{\underline{\lambda}^{j+1}}(\underline{\lambda}^j, \underline{\pi(j)}^1, \dots, \underline{\pi(j)}^j)$$

where  $\Xi'_{\underline{\lambda}^{j+1}}$  is as defined in the proof of Lemma 4.2.2. Since  $\Xi'_{\underline{\lambda}^{j+1}}$  is the two sided inverse of  $\Xi_{\underline{\lambda}^{j+1}}$  (by the assertion of that lemma), it follows that the map

$$\{\underline{\lambda}^j; \underline{\pi(j)}^j\} \mapsto (\mu_2^*, \dots, \mu_{r+1}^*; \underline{\pi(1)}^*, \dots, \underline{\pi(r)}^*)$$

is the required two sided inverse. By abusing notation again, we denote it by  $\Xi'_{\underline{\lambda}^{r+1}}$ .  $\square$

**Proposition 4.3.2.** *The underlying near pattern  $\mathcal{P}$  in the image of  $(\mu_2, \dots, \mu_{r+1}; \underline{\pi(1)}, \dots, \underline{\pi(r)})$  under  $\Xi_{\underline{\lambda}^{r+1}}$  has bounding sequence  $\underline{\lambda}^{r+1}$  and weight  $(\mu_1, \mu_2, \dots, \mu_{r+1})$  where  $\mu_1 := (\sum_{j=1}^{r+1} \lambda_j^{r+1}) - (\sum_{j=2}^{r+1} \mu_j)$ . The number  $n$  of boxes in the approximate partition overlay satisfies:*

$$(4.16) \quad \square^{\text{PROP}}(\mathcal{P}) + n = |\underline{\pi(1)}| + \dots + |\underline{\pi(r)}|$$

*Proof.* The first assertion is immediate from the definition. From the definition of  $\Xi_{\underline{\lambda}^{r+1}}$  and Proposition 4.2.1 (4) and (5), we obtain, for  $j$  such that  $1 \leq j \leq r$ ,  $(\sum_{i=1}^{j+1} \lambda_i^{j+1}) - (\sum_{i=1}^j \lambda_i^j) = \mu_{j+1}$  and  $\square^{\text{PROP}}(\underline{\lambda}^{j+1}, \underline{\lambda}^j) + \sum_{i=1}^j |\underline{\pi(j)}^i| = |\underline{\pi(j)}|$ . Adding the first set of equations, we get  $\lambda_1^1 = (\sum_{j=1}^{r+1} \lambda_j^{r+1}) - (\sum_{j=2}^{r+1} \mu_j) = \mu_1$ . Adding the second, we get (4.16).  $\square$

## 4.4 Shift by $k$

Let an integer  $k$  be fixed. Given a sequence  $\underline{\eta}$ :  $\eta_1, \dots, \eta_{s+1}$ , where ( $s \geq 0$  is an integer) we denote by  $\tilde{\underline{\eta}}$  the sequence  $\tilde{\eta}_1, \dots, \tilde{\eta}_{s+1}$ , where

$$(4.17) \quad \begin{cases} \tilde{\eta}_1 := \eta_1 + 2k & \tilde{\eta}_j := \eta_j + k \quad \text{for } 2 \leq j \leq s & \text{and } \tilde{\eta}_{s+1} := \eta_{s+1} & \text{(if } s \geq 1) \\ \tilde{\eta}_1 := \eta_1 + k & & & \text{(if } s = 0) \end{cases}$$

We refer to  $\tilde{\underline{\eta}}$  as the *shift by  $k$*  (or just *shift* if  $k$  is clear from the context) of  $\underline{\eta}$ . The suppression of the dependence on  $k$  in the notation  $\tilde{\underline{\eta}}$  should cause no confusion.

The shift  $\tilde{\underline{\eta}}, \tilde{\underline{\eta}'}$  of a pair  $\underline{\eta}, \underline{\eta}'$  of nearly interlacing sequences is also nearly interlacing. A sequence of partitions approximately overlays  $\underline{\eta}, \underline{\eta}'$  if and only if it approximately overlays  $\tilde{\underline{\eta}}, \tilde{\underline{\eta}'}$ .

The *shift of a near pattern* consists of the shifts of the constituent sequences of the pattern. It is also a near pattern. A collection of partitions approximately overlays a near pattern if and only if it approximately overlays the shifted pattern.

To shift a near interlacing sequence or near pattern with an approximate partition overlay, we just shift the constituent sequences. The partitions in the overlay stay as they are.

Shift preserves proper trapezoidal area (of a near interlacing sequence or near pattern). If a near pattern has weight  $(\mu_1, \dots, \mu_{s+1})$ , the near pattern shifted by  $k$  has weight  $(\mu_1 + k, \dots, \mu_{s+1} + k)$ . Positive shifts of interlacing sequences (respectively patterns) continue to be interlacing (respectively patterns).

**Proposition 4.4.1.** *Let an integer  $s \geq 1$  and an integer sequence  $\underline{\eta}: \eta_1, \dots, \eta_{s+1}$  with  $\eta_2 \geq \dots \geq \eta_s$  be fixed. For  $\underline{\mu}$  an integer and  $\underline{\pi}$  a partition, if  $\Xi_{\underline{\eta}}(\underline{\mu}, \underline{\pi}) = (\underline{\eta}', \underline{\pi}^1, \dots, \underline{\pi}^s)$ , then  $\Xi_{\tilde{\underline{\eta}}}(\underline{\mu} + k, \underline{\pi}) = (\tilde{\underline{\eta}'}, \underline{\pi}^1, \dots, \underline{\pi}^s)$ .*

*Proof.* Indeed, the sequences  $\underline{c}$ ,  $\underline{a}$ , and  $\underline{b}$  involved in the calculation of  $\Xi_{\underline{\eta}}(\underline{\mu}, \underline{\pi})$  (see §4.1.2) are the same as the corresponding ones involved in the calculation of  $\Xi_{\tilde{\underline{\eta}}}(\underline{\mu} + k, \underline{\pi})$ . The result now follows from the definition (4.3) of  $\underline{\pi}^j$  and (4.9), (4.10) of  $\underline{\eta}'$ .  $\square$

**Lemma 4.4.2.** *Let  $\underline{\eta}: \eta_1, \dots, \eta_{s+1}$  and  $\underline{\eta}': \eta'_1, \dots, \eta'_s$  be integer sequences (for some  $s \geq 1$ ), let  $\underline{\mu} = (\mu_1, \dots, \mu_{s+1})$  be a tuple of integers, and  $\underline{\pi}^1, \dots, \underline{\pi}^s$  a sequence of partitions. Assume that:*

1.  $\underline{\eta}$  is non-increasing:  $\eta_1 \geq \dots \geq \eta_{s+1}$



2.  $\underline{\eta}, \underline{\eta}'$  nearly interlace and  $\underline{\pi}^1, \dots, \underline{\pi}^s$  is an approximate partition overlay on  $\underline{\eta}, \underline{\eta}'$
3.  $\underline{\mu} \preceq_m \underline{\eta}$
4.  $(\eta_1 + \dots + \eta_{s+1}) - (\eta'_1 + \dots + \eta'_s) = \mu_{s+1}$

Let  $k$  be an integer and  $\tilde{\eta}, \tilde{\eta}'$  be the shifts by  $k$  of  $\underline{\eta}, \underline{\eta}'$ . If  $k \geq \square^{\text{prop}}(\underline{\eta}, \underline{\eta}') + |\underline{\pi}^1| + |\underline{\pi}^s|$ , then

- (a)  $\tilde{\eta}, \tilde{\eta}'$  interlace (which in particular implies that  $\tilde{\eta}'$  is non-increasing:  $\tilde{\eta}'_1 \geq \dots \geq \tilde{\eta}'_s$ )
- (b)  $\underline{\pi}^1, \dots, \underline{\pi}^s$  overlays  $\tilde{\eta}, \tilde{\eta}'$  (i.e.,  $\underline{\pi}^j$  fits into the rectangle  $(\tilde{\eta}_j - \tilde{\eta}'_j, \tilde{\eta}'_j - \tilde{\eta}_{j+1})$  for  $1 \leq j \leq s$ )
- (c)  $(\mu_1 + k, \dots, \mu_s + k) \preceq_m \tilde{\eta}'$

*Proof.* For (b), since  $\underline{\pi}^1, \dots, \underline{\pi}^s$  approximately overlays  $\underline{\eta}, \underline{\eta}'$ , it is enough to show:

$$(4.18) \quad \underline{\pi}^1 \text{ has largest part at most } \tilde{\eta}'_1 - \tilde{\eta}_2 \quad \text{and} \quad \underline{\pi}^s \text{ has at most } \tilde{\eta}_s - \tilde{\eta}'_s \text{ parts}$$

These assertions imply in particular that  $\tilde{\eta}'_1 \geq \tilde{\eta}_2$  and  $\tilde{\eta}_s \geq \tilde{\eta}'_s$ , from which (a) follows (since  $\tilde{\eta}, \tilde{\eta}'$  nearly interlace).

To prove (4.18), we first show that  $\tilde{\eta}'_1 - \tilde{\eta}_2 \geq \pi_1^1$ . Putting  $\tilde{\eta}'_1 = \eta'_1 + 2k$  and  $\tilde{\eta}_2 = \eta_2 + k$ , we see that the desired inequality is equivalent to

$$(4.19) \quad (\eta_1 - \eta_2) + k \geq (\eta_1 - \eta'_1) + \pi_1^1$$

We consider two cases. First suppose that  $\eta'_j - \eta_{j+1} \geq 1$  for some  $2 \leq j \leq s$ . Then, since

$$k \geq \square^{\text{prop}}(\underline{\eta}, \underline{\eta}') + |\underline{\pi}^1| \geq (\eta_1 - \eta'_1)(\eta'_j - \eta_{j+1}) + \pi_1^1 \geq (\eta_1 - \eta'_1) + \pi_1^1$$

and  $(\eta_1 - \eta_2) \geq 0$  by hypothesis (1), we obtain (4.19). In the second case, we have  $\eta'_j = \eta_{j+1}$  for all  $j \geq 2$ . Then, by hypothesis (4), we get  $\eta_1 - \eta'_1 = \mu_{s+1} - \eta_2$ . Substituting

this into (4.19), we get

$$(4.20) \quad (\eta_1 - \mu_{s+1}) + k \geq \pi_1^1$$

But since  $k \geq |\underline{\pi}^1| \geq \pi_1^1$  and  $\eta_1 \geq \mu_{s+1}$  by hypothesis (3), we obtain (4.20).

The proof of the latter half of (4.18), namely that  $\tilde{\eta}'_s - \tilde{\eta}'_{s+1} \geq p$  (where  $p$  denotes the number of parts in  $\underline{\pi}^s$ ), is analogous to that of the former half above. Putting  $\tilde{\eta}_2 = \eta_s + k$   $\tilde{\eta}'_s = \eta'_s$ , we see that the desired inequality is equivalent to

$$(4.21) \quad (\eta_s - \eta_{s+1}) + k \geq (\eta'_s - \eta_{s+1}) + p$$

We consider two cases. First suppose that  $\eta_j - \eta'_j \geq 1$  for some  $1 \leq j < s$ . Then, since

$$k \geq \square^{\text{prop}}(\underline{\eta}, \underline{\eta}') + |\underline{\pi}^s| \geq (\eta_j - \eta'_j)(\eta'_s - \eta_{s+1}) + p \geq (\eta'_s - \eta_{s+1}) + p$$

and  $(\eta_s - \eta_{s+1}) \geq 0$  by hypothesis (1), we obtain (4.21). In the second case, we have  $\eta_j = \eta'_j$  for all  $j < s$ . Then, by hypothesis (4), we get  $\eta'_s - \eta_{s+1} = \eta_s - \mu_{s+1}$ . Substituting this into (4.21), we get

$$(4.22) \quad (\mu_{s+1} - \eta_{s+1}) + k \geq p$$

But since  $k \geq |\underline{\pi}^s| \geq p$  and  $\mu_{s+1} \geq \eta_{s+1}$  by hypothesis (3), we obtain (4.22).

We now turn to (c). That  $\tilde{\eta}'_1 + \cdots + \tilde{\eta}'_s = (\mu_1 + k) + \cdots + (\mu_s + k)$  follows from the definition of  $\tilde{\eta}'$ , hypothesis (4), and the fact implied by hypothesis (3) that  $\eta_1 + \cdots + \eta_{s+1} = \mu_1 + \cdots + \mu_{s+1}$ . It follows from (a) that  $\tilde{\eta}'_1 \geq \cdots \geq \tilde{\eta}'_s$ . It only remains to show that for any  $j < s$  and  $1 \leq i_1 < \cdots < i_j \leq s$  we have

$$(4.23) \quad \tilde{\eta}'_1 + \cdots + \tilde{\eta}'_j \geq (\mu_{i_1} + k) + \cdots + (\mu_{i_j} + k)$$

Substituting the definitions  $\tilde{\eta}'_1 = \eta'_1 + 2k$  and  $\tilde{\eta}'_i = \eta'_i + k$  for  $1 < i \leq j$ , we may rewrite (4.23) equivalently as

$$(4.24) \quad (\eta_1 + \cdots + \eta_j) - (\mu_{i_1} + \cdots + \mu_{i_j}) + k \geq (\eta_1 - \eta'_1) + \cdots + (\eta_j - \eta'_j)$$

We consider two cases. First suppose that  $\eta'_i - \eta_{i+1} \geq 1$  for some  $j < i \leq s$ . Then, since

$$k \geq \square^{\text{PROP}}(\underline{\eta}, \underline{\eta}') \geq ((\eta_1 - \eta'_1) + \cdots + (\eta_j - \eta'_j)) (\eta'_i - \eta_{i+1}) \geq (\eta_1 - \eta'_1) + \cdots + (\eta_j - \eta'_j)$$

and  $(\eta_1 + \cdots + \eta_j) - (\mu_{i_1} + \cdots + \mu_{i_j}) \geq 0$  by hypothesis (3), we obtain (4.24). In

the second case, we have  $\eta'_i = \eta_{i+1}$  for all  $i > j$ . Then, by hypothesis (4), we get

$$(\eta_1 - \eta'_1) + \cdots + (\eta_j - \eta'_j) = \mu_{s+1} - \eta_{j+1}.$$

Substituting this into (4.24), we get

$$(4.25) \quad (\eta_1 + \cdots + \eta_{j+1}) - (\mu_{i_1} + \cdots + \mu_{i_j} + \mu_{s+1}) + k \geq 0$$

But since  $k \geq 0$  and  $(\eta_1 + \cdots + \eta_{j+1}) \geq (\mu_{i_1} + \cdots + \mu_{i_j} + \mu_{s+1})$  by hypothesis (3), we are done.  $\square$

**Corollary 4.4.3.** *Let  $\underline{\lambda} : \lambda_1 \geq \dots \geq \lambda_{r+1}$  be a non-increasing integer sequence (where  $r \geq 1$  is an integer). Let  $\underline{\mu} \in \mathbb{Z}^{r+1}$  be such that  $\underline{\lambda} \succ_m \underline{\mu}$ . Let  $\mathcal{P}$  be a near pattern with bounding sequence  $\underline{\lambda}$  and weight  $\underline{\mu}$ . Let  $\{\underline{\pi}(j)^i \mid 1 \leq i \leq j \leq r\}$  be an approximate partition overlay on  $\mathcal{P}$ . Then the shift by  $\square^{\text{PROP}}(\mathcal{P}) + \sum_{1 \leq j \leq r} (|\underline{\pi}(j)^1| + |\underline{\pi}(j)^j|)$  of the AONP  $(\mathcal{P}, \{\underline{\pi}(j)^i\})$  is a POP.*

*Proof.* We proceed by induction on  $r$ . Let  $\underline{\lambda}^j, 1 \leq j \leq r+1$ , be the constituent sequences of  $\mathcal{P}$ . We have  $\underline{\lambda} = \underline{\lambda}^{r+1}$ . Consider the shift  $\tilde{\mathcal{P}}$  of  $\mathcal{P}$  by  $k$ , where

$$k = \square^{\text{PROP}}(\underline{\lambda}^{r+1}, \underline{\lambda}^r) + |\underline{\pi}(r)^1| + |\underline{\pi}(r)^r|.$$

By the lemma, we obtain:

(a)  $\tilde{\lambda}^{r+1}, \tilde{\lambda}^r$  interlace

(which in particular implies that  $\tilde{\lambda}^r$  is non-increasing:  $\tilde{\lambda}_1^r \geq \dots \geq \tilde{\lambda}_r^r$ )

(b)  $\underline{\pi(r)}^1, \dots, \underline{\pi(r)}^s$  overlays  $\tilde{\lambda}^{r+1}, \tilde{\lambda}^r$

(c)  $(\mu_1 + k, \dots, \mu_r + k) \preccurlyeq_m \tilde{\lambda}^r$

In particular, this gives a proof in the base case  $r = 1$  of the induction.

Now suppose  $r \geq 2$ . Let  $\mathcal{P}_1$  denote the pattern obtained from  $\mathcal{P}$  by omitting its last row. We may apply the induction hypothesis to  $\tilde{\lambda}^r, (\mu_1 + k, \dots, \mu_r + k)$ , and the AONP  $(\tilde{\mathcal{P}}_1, \{\underline{\pi(j)}^i \mid 1 \leq i \leq j \leq r-1\})$ , to conclude that the shift by  $\square^{\text{prop}}(\tilde{\mathcal{P}}_1) + \sum_{1 \leq j < r} (|\underline{\pi(j)}^1| + |\underline{\pi(j)}^j|)$  of this AONP is a POP.

Note the following:

- $\square^{\text{prop}}$  is preserved under shifts
- $\square^{\text{prop}}(\mathcal{P}) = \square^{\text{prop}}(\underline{\lambda}^{r+1}, \underline{\lambda}^r) + \square^{\text{prop}}(\mathcal{P}_1)$
- upward shifts of  $\tilde{\lambda}^{r+1}, \tilde{\lambda}^r$  do not affect (a) and (b)
- shift by  $\square^{\text{prop}}(\mathcal{P}_1) + \sum_{1 \leq j < r} (|\underline{\pi(j)}^1| + |\underline{\pi(j)}^j|)$  of  $\tilde{\mathcal{P}}_1$  (respectively  $\tilde{\lambda}^{r+1}, \tilde{\lambda}^r$ ) equals shift by  $\square^{\text{prop}}(\mathcal{P}) + \sum_{1 \leq j \leq r} (|\underline{\pi(j)}^1| + |\underline{\pi(j)}^j|)$  of  $\mathcal{P}_1$  (respectively  $\underline{\lambda}^{r+1}, \underline{\lambda}^r$ )

The result follows. □

## 4.5 Bijection between $r$ -colored partitions and POPs

We are at last ready to state and prove the desired theorem. Fix integers  $r \geq 1$  and  $d \geq 0$ . Let  $\mathcal{P}_r(d)$  denote the set of all  $r$ -colored partitions of  $d$ .

Fix  $\underline{\lambda}^{r+1} = \underline{\lambda} : \lambda_1 \geq \dots \geq \lambda_{r+1}$  a non-increasing sequence of integers, and  $\underline{\mu} = (\mu_1, \dots, \mu_{r+1}) \in \mathbb{Z}^{r+1}$  such that  $\underline{\lambda} \succcurlyeq_m \underline{\mu}$ . Let us denote a general AONP with bounding

sequence of length  $r + 1$  by  $(\mathcal{P}, \{\underline{\pi(j)}^i \mid 1 \leq i \leq j \leq r\})$ , where  $\mathcal{P}$  denotes the underlying near pattern and  $\underline{\pi(j)}^i$  the partitions in the approximate overlay. Let  $\mathcal{N}_{\underline{\lambda}, \underline{\mu}}(d)$  denote the set of all AONPs where  $\mathcal{P}$  has bounding sequence  $\underline{\lambda}^{r+1}$ , weight  $(\mu_1, \dots, \mu_{r+1})$ , and satisfies the following condition:

$$(4.26) \quad \square^{\text{PROP}}(\mathcal{P}) + \sum_{1 \leq i \leq j \leq r} |\underline{\pi(j)}^i| = d$$

Let  $\Phi_{\underline{\lambda}, \underline{\mu}}$  be the map from  $\mathcal{P}_r(d)$  to the set of AONPs given by

$$\underline{\pi(1)}, \dots, \underline{\pi(r)} \quad \mapsto \quad \Xi_{\underline{\lambda}^{r+1}}(\mu_2, \dots, \mu_{r+1}; \underline{\pi(1)}, \dots, \underline{\pi(r)})$$

where  $\Xi_{\underline{\lambda}^{r+1}}$  is as defined in §4.3.1.

**Proposition 4.5.1.** *The map  $\Phi_{\underline{\lambda}, \underline{\mu}}$  is a bijection from  $\mathcal{P}_r(d)$  to  $\mathcal{N}_{\underline{\lambda}, \underline{\mu}}(d)$ .*

*Proof.* It follows from Proposition 4.3.2 that the image of  $\Phi_{\underline{\lambda}, \underline{\mu}}$  lies in  $\mathcal{N}_{\underline{\lambda}, \underline{\mu}}(d)$ . Since  $\Xi_{\underline{\lambda}^{r+1}}$  is a bijection (Lemma 4.3.1), it follows that  $\Phi_{\underline{\lambda}, \underline{\mu}}$  is an injection. We now show that it is also onto  $\mathcal{N}_{\underline{\lambda}, \underline{\mu}}(d)$ . Given an element of  $\mathcal{N}_{\underline{\lambda}, \underline{\mu}}(d)$ , its image under  $\Xi'_{\underline{\lambda}^{r+1}}$  maps to that element under  $\Xi_{\underline{\lambda}^{r+1}}$  (see the proof of Lemma 4.3.1), so it is of the form  $(\mu_2, \dots, \mu_{r+1}; \underline{\pi(1)}, \dots, \underline{\pi(r)})$ , where  $\underline{\pi(1)}, \dots, \underline{\pi(r)}$  is an  $r$ -colored partition of  $d$  (Proposition 4.3.2).  $\square$

For an integer  $k$ , let  $\mathcal{S}^k$  denote the “shift by  $k$ ” operator (§4.4). Let  $\mathcal{N}_{\underline{\lambda}, \underline{\mu}}^k(d)$  (respectively  $\mathbb{P}_{\underline{\lambda}, \underline{\mu}}^k(d)$ ) denote the set of all AONPs (respectively POPs) where  $\mathcal{P}$  has bounding sequence  $\tilde{\underline{\lambda}}$  (the shift by  $k$  of  $\underline{\lambda}$ ), weight  $(\mu_1 + k, \dots, \mu_{r+1} + k)$ , and (4.26) is satisfied.

**Theorem 4.5.2.** *For  $k \geq d$ , the composition  $\mathcal{S}^k \circ \Phi_{\underline{\lambda}, \underline{\mu}}$  defines a bijection from  $\mathcal{P}_r(d)$  to  $\mathbb{P}_{\underline{\lambda}, \underline{\mu}}^k(d)$ .*

*Proof.* The operator  $\mathcal{S}^k$  is evidently a bijection from  $\mathcal{N}_{\underline{\lambda}, \underline{\mu}}(d)$  to  $\mathcal{N}_{\underline{\lambda}, \underline{\mu}}^k(d)$  (see §4.4). By Corollary 4.4.3, its image lies in  $\mathbb{P}_{\underline{\lambda}, \underline{\mu}}^k(d)$ , so  $\mathcal{N}_{\underline{\lambda}, \underline{\mu}}^k(d) = \mathbb{P}_{\underline{\lambda}, \underline{\mu}}^k(d)$ . The result follows from Proposition 4.5.1.  $\square$

## 4.6 The complementation involution $\mathcal{C}$

For  $\mathfrak{P} = (\mathcal{P}, \underline{\pi(j)^i} \mid 1 \leq i \leq j \leq r)$  a POP with  $\underline{\eta^j}$  being the  $j^{\text{th}}$  row of the pattern  $\mathcal{P}$ , let  $\overline{\mathfrak{P}}$  denote the POP  $(\mathcal{P}, \underline{\pi^c(j)^i} \mid 1 \leq i \leq j \leq r)$ , where  $\underline{\pi^c(j)^i}$  denotes the complement of  $\underline{\pi(j)^i}$  in the rectangle  $(\eta_i^{j+1} - \eta_i^j, \eta_i^j - \eta_{i+1}^{j+1})$  (see §3.1.6). For  $\mathfrak{P}$  in  $\mathbb{P}_{\underline{\lambda}, \underline{\mu}}^k(d)$ , we have

$$\sum_{1 \leq i \leq j \leq r} |\underline{\pi^c(j)^i}| = \Delta(\mathcal{P}) - \sum_{1 \leq i \leq j \leq r} |\underline{\pi(j)^i}| = \Delta(\mathcal{P}) + \square^{\text{prop}}(\mathcal{P}) - d = \square(\mathcal{P}) - d$$

The association  $\mathcal{C} : \mathfrak{P} \mapsto \overline{\mathfrak{P}}$  is evidently reversible. The above calculation shows that  $\mathcal{C}$  defines a bijection from  $\mathbb{P}_{\underline{\lambda}, \underline{\mu}}^k(d)$  onto  $\mathbb{P}_{\underline{\tilde{\lambda}}, \underline{\mu}}^k[d]$ , where  $\mathbb{P}_{\underline{\tilde{\lambda}}, \underline{\mu}}^k[d]$  denotes the set of POPs with bounding sequence  $\underline{\tilde{\lambda}}$ , weight  $(\mu_1 + k, \dots, \mu_{r+1} + k)$ , and depth  $d$  (see §3.1.8 for the definition of depth).

Precomposing the bijection of Theorem 4.5.2 with  $\mathcal{C}$  continues to be a bijection:

**Corollary 4.6.1.** *For  $k \geq d$ , the composite map  $\mathcal{C} \circ \mathcal{S}^k \circ \Phi_{\underline{\lambda}, \underline{\mu}}$  is a bijection from  $\mathcal{P}_r(d)$  to  $\mathbb{P}_{\underline{\tilde{\lambda}}, \underline{\mu}}^k[d]$ .*

**Proposition 4.6.2.** *The bijections of Theorem 4.5.2 and Corollary 4.6.1 are compatible. More precisely, for  $j \geq 0$ , we have:*

$$(4.27) \quad \mathcal{S}^{j+k} \circ \Phi_{\underline{\lambda}, \underline{\mu}} = \mathcal{S}^j \circ \Phi_{\underline{\tilde{\lambda}}, (\mu_1+k, \dots, \mu_{r+1}+k)} \quad \text{in the theorem}$$

$$(4.28) \quad \mathcal{C} \circ \mathcal{S}^{j+k} \circ \Phi_{\underline{\lambda}, \underline{\mu}} = \mathcal{C} \circ \mathcal{S}^j \circ \Phi_{\underline{\tilde{\lambda}}, (\mu_1+k, \dots, \mu_{r+1}+k)} \quad \text{in the corollary}$$

*Proof.* The left hand side of (4.28) may be written as  $\mathcal{C} \circ \mathcal{S}^j \circ (\mathcal{S}^k \circ \Phi_{\underline{\lambda}, \underline{\mu}})$ . Now, by Proposition 4.4.1,  $\mathcal{S}^k \circ \Phi_{\underline{\lambda}, \underline{\mu}} = \Phi_{\underline{\tilde{\lambda}}, (\mu_1+k, \dots, \mu_{r+1}+k)}$ . The proof of (4.27) is similar.  $\square$

**Corollary 4.6.3.** *For  $k \geq d$  and  $j \geq 0$ , the map  $\mathcal{C} \circ \mathcal{S}^j \circ \mathcal{C} : \mathbb{P}_{\underline{\lambda}, \underline{\mu}}^k[d] \rightarrow \mathbb{P}_{\underline{\lambda}, \underline{\mu}}^{k+j}[d]$  is a bijection.*

*Proof.*  $\mathcal{C} \circ \mathcal{S}^j \circ \mathcal{C}$  is equal to the composition of two bijections: the inverse of  $\mathcal{C} \circ \mathcal{S}^k \circ \Phi_{\underline{\lambda}, \underline{\mu}}$  followed by  $\mathcal{C} \circ \mathcal{S}^{j+k} \circ \Phi_{\underline{\lambda}, \underline{\mu}}$ .  $\square$

# Chapter 5

## The stability conjecture for $\mathfrak{sl}_3$ and beyond

The results of this chapter will appear in [21].

### 5.1 The stability conjecture: first version

In this section and the next, we state the conjecture about stability of Chari-Loktev bases under a chain of inclusions of local Weyl modules (for  $\mathfrak{g} = \mathfrak{sl}_{r+1}$ ). The conjecture has been proved in Chapter 2 in the case  $r = 1$ . We begin by recalling details about the chain of inclusions. The theorem of the previous section (§4) gives us an identification of the indexing set of the basis for an included module as a subset of the indexing set of the basis of the larger module. It then makes sense to ask whether the bases are well behaved with respect to inclusions. The stability conjecture says that this is so in the stable range.

We state two versions of the conjecture. The first version has the advantage that it can be stated quickly (assuming only the result in Corollary 3.3.5, and without any

further reference to the details in §3.3 about the identification of local Weyl modules as Demazure modules). The second version, stated in the next section, is more involved but provides also motivation.

### 5.1.1 The set up

Let  $\lambda$  be a dominant integral weight and  $\theta$  the highest root of  $\mathfrak{g} = \mathfrak{sl}_{r+1}$ . We identify  $\lambda$  and  $\theta$  with  $(r+1)$ -tuples of integers  $\underline{\lambda}$  and  $\underline{\theta}$  respectively as in §3.1.9:  $\underline{\lambda}: \lambda_1 \geq \cdots \lambda_r \geq \lambda_{r+1} = 0$  and  $\underline{\theta} = (2, 1, \dots, 1, 0)$ .

### 5.1.2 Chains of inclusions

Consider the chain of  $\mathfrak{g}[t]$ -module inclusions (see Corollary 3.3.5), each of which is defined uniquely up to scaling:

$$(5.1) \quad W(\lambda) \hookrightarrow W(\lambda + \theta) \hookrightarrow W(\lambda + 2\theta) \hookrightarrow \dots$$

The Chari-Loktev basis for  $W(\lambda + k\theta)$  is indexed by the set  $\mathbb{P}_{\underline{\lambda}}^k$  of POPs with bounding sequence  $\underline{\lambda} + k\underline{\theta}$  (§3.3.6). Mirroring the above chain of inclusions of local Weyl modules, we have a chain of inclusions of these indexing sets:

$$\mathbb{P}_{\underline{\lambda}}^0 \hookrightarrow \mathbb{P}_{\underline{\lambda}}^2 \hookrightarrow \mathbb{P}_{\underline{\lambda}}^1 \hookrightarrow \dots$$

Indeed, the map  $\mathcal{C}\mathcal{S}^k\mathcal{C}$ , where  $\mathcal{C}$  is the complementation (§4.6) and  $\mathcal{S}^k$  is the shift by  $k$  (§4.4), defines an injection  $\mathbb{P}_{\underline{\lambda}}^j \hookrightarrow \mathbb{P}_{\underline{\lambda}}^{j+k}$ . These injections are compatible as  $j$  and  $k$  vary (since  $\mathcal{C}$  is an involution and  $\mathcal{S}^k\mathcal{S}^{k'} = \mathcal{S}^{k+k'}$ ).



### 5.1.3 Fixing the inclusions

For a POP  $\mathfrak{P}$ , let  $v_{\mathfrak{P}}$  and  $\rho_{\mathfrak{P}}$  denote respectively the Chari-Loktev basis element and the Chari-Loktev monomial corresponding to  $\mathfrak{P}$  (see §3.3.6). We fix the chain of inclusions (5.1) such that for every  $j \geq 0$ , let the generator  $w_{\lambda+j\theta}$  of  $W(\lambda + j\theta)$  be mapped to:

(5.2)

$$w_{\lambda+j\theta} \mapsto v_{\mathfrak{P}_j^1} = \rho_{\mathfrak{P}_j^1} w_{\lambda+(j+1)\theta} \quad \text{under the inclusion } W(\lambda + j\theta) \hookrightarrow W(\lambda + (j+1)\theta)$$

where  $\mathfrak{P}_j$  denotes the unique element of  $\mathbb{P}_{\underline{\lambda}}^j$  of weight  $\underline{\lambda} + j\underline{\theta}$  (corresponding to the generator of  $W(\lambda + j\theta)$ ), and  $\mathfrak{P}^k$  denotes the image of  $\mathfrak{P}$  under the inclusion  $\mathbb{P}_{\underline{\lambda}}^j \hookrightarrow \mathbb{P}_{\underline{\lambda}}^{j+k}$  (for  $\mathfrak{P}$  in  $\mathbb{P}_{\underline{\lambda}}^j$ ,  $k \geq 0$ ).

### 5.1.4 The stability conjecture: first version

It is now natural to ask whether, for all  $j \geq 0$ , for all  $k \geq 0$ , and for all  $\mathfrak{P} \in \mathbb{P}_{\underline{\lambda}}^j$ , we have

$$(5.3) \quad v_{\mathfrak{P}} \mapsto \pm v_{\mathfrak{P}^k} \quad \text{under the inclusion } W(\lambda + j\theta) \hookrightarrow W(\lambda + (j+k)\theta)$$

Simple instances (see Example 2.2.4) show that (5.3) is too much to expect in general. We do however conjecture that it holds in the “stable range”:

**Conjecture 5.1.1** (STABILITY of Chari-Loktev bases). *With notation as above, let  $\mathfrak{P}$  be a POP in  $\mathbb{P}_{\underline{\lambda}}^j$ . Let  $\underline{\mu}$  be the weight of  $\mathfrak{P}$  and  $d$  its depth. Note that  $\underline{\mu} \preceq_{\mathfrak{m}} \underline{\lambda} + j\underline{\theta}$  and in particular  $\sum_{i=1}^{r+1} \mu_i = (\sum_{i=1}^{r+1} \lambda_i) + j(r+1)$ . The assertion (5.3) holds if  $j \geq \ell + d$ , where  $\ell$  be the least non-negative integer such that  $\underline{\mu} - (j - \ell)\underline{\mathbf{1}} \preceq_{\mathfrak{m}} \underline{\lambda} + \ell\underline{\theta}$ . Here  $\underline{\mathbf{1}}$  stands for the element  $(1, \dots, 1) \in \mathbb{R}^{r+1}$ .*

## 5.2 The stability conjecture: second version

In this section, we state the second version of the conjecture about stability of Chari-Loktev bases under a chain of inclusions of local Weyl modules (for  $\mathfrak{g} = \mathfrak{sl}_{r+1}$ ). For a brief description of what the conjecture is about, see the introduction to §5.1, where the first version is stated. The conjecture has been proved in Chapter 2 in the case  $r = 1$ .

### 5.2.1 The set up

Let  $\theta$  be the highest root and  $\lambda$  a dominant integral weight of  $\mathfrak{g} = \mathfrak{sl}_{r+1}$ .

### 5.2.2 Local Weyl modules as Demazure modules

Adopt the notation of §3.3. There exists a unique dominant integral weight  $\Lambda$  of  $\widehat{\mathfrak{g}}$  such that  $t_{w_0(\lambda)}(\Lambda_0) = w\Lambda$ , with  $w$  in the affine Weyl group. By results of Chari-Loktev and Fourier-Littelmann recalled in Theorem 3.3.1 above, for  $k$  a non-negative integer, the local Weyl module  $W(\lambda + k\theta)$  is isomorphic as a  $\mathfrak{g}[t]$ -module to the Demazure submodule  $V_{t_{w_0(\lambda+k\theta)}}(\Lambda_0) = V_{t_{w_0(k\theta)}w}(\Lambda)$  of  $L(\Lambda)$ . The isomorphism is unique up to scaling. (At the moment we choose the scaling arbitrarily. Later on, we will fix the scaling, so that the isomorphism is fixed.) The isomorphism maps the generator  $w_{\lambda+k\theta}$  of  $W(\lambda + k\theta)$  to a vector of  $L(\Lambda)$  of weight  $t_{\lambda+k\theta}(\Lambda_0)$ . We denote the image of  $w_{\lambda+k\theta}$  also by the same symbol.

### 5.2.3 A chain of local Weyl modules

Since  $\lambda$  and  $\theta$  are both dominant,  $\ell(t_{w_0(\lambda+k\theta)}) = \ell(t_{w_0\lambda}) + k \cdot \ell(t_{w_0\theta})$ , so  $t_{w_0(\lambda+k\theta)} \leq t_{w_0(\lambda+m\theta)}$  in the Bruhat order (on the extended affine Weyl group) for  $k \leq m$ . Consider the following

chain:

$$t_{w_0(\lambda)} \leq t_{w_0(\lambda+\theta)} \leq \cdots \leq t_{w_0(\lambda+k\theta)} \leq t_{w_0(\lambda+(k+1)\theta)} \leq \cdots$$

and the corresponding chain of Demazure submodules of  $L(\Lambda)$ :

$$(5.4) \quad W(\lambda) \hookrightarrow W(\lambda + \theta) \hookrightarrow \cdots \hookrightarrow W(\lambda + k\theta) \hookrightarrow W(\lambda + (k + 1)\theta) \hookrightarrow \cdots \quad (\hookrightarrow L(\Lambda))$$

The union of the modules in the above chain equals  $L(\Lambda)$  (because  $t_\theta = s_0 s_\theta$  and every simple reflection of  $\mathfrak{g}$  occurs in any reduced expression for  $s_\theta$ ).

### 5.2.4 Weights of local Weyl modules

It is well known (see Theorem 1.2.10) that any weight of  $L(\Lambda)$  is of the form  $t_\gamma(\Lambda) - d\delta$ , where  $\gamma$  is an integral linear combination of the simple roots of  $\mathfrak{g}$ , and  $d$  is a non-negative integer ( $\Lambda$  is of level 1 as observed in (3.40)). Fix such a weight. The dimension of the corresponding weight space is given by (see Theorem 1.2.10):

$$(5.5) \quad \dim L(\Lambda)_{t_\gamma(\Lambda) - d\delta} = \text{the number of partitions of } d \text{ into } r \text{ colors.}$$

Since the union of the local Weyl modules in (5.4) is  $L(\Lambda)$ , there exists  $K_0$  such that, for all  $k \geq K_0$ , we have the equality  $W(\lambda + k\theta)_{t_\gamma(\Lambda) - d\delta} = L(\Lambda)_{t_\gamma(\Lambda) - d\delta}$  of weight spaces. In particular, for  $k \geq K_0$ , the dimension of  $W(\lambda + k\theta)_{t_\gamma(\Lambda) - d\delta}$  equals the number of  $r$  colored partitions of  $d$ .

### 5.2.5 Parametrizing set of the Chari-Loktev basis for the weight space $W(\lambda + k\theta)_{t_\gamma(\Lambda) - d\delta}$

Following §3.1.9, identify  $\lambda$  with a non-decreasing sequence  $\underline{\lambda} : \lambda_1 \geq \cdots \geq \lambda_r \geq \lambda_{r+1} = 0$ , and the highest root  $\theta$  with  $\underline{\theta}$ . Let  $\mu$  denote the restriction to  $\mathfrak{h}$  of  $t_\gamma(\Lambda) - d\delta$ . Identify  $\mu$

with the tuple  $\underline{\mu} = (\mu_1, \dots, \mu_{r+1})$  in its equivalence class such that  $\sum_{i=1}^{r+1} \lambda_i = \sum_{i=1}^{r+1} \mu_i$ .

**Proposition 5.2.1.** *The CL basis of  $W(\lambda + k\theta)_{t_\gamma(\Lambda)-d\delta}$  is parametrized by the set  $\mathbb{P}_{\underline{\lambda}, \underline{\mu}}^k[d]$  of POPs with bounding sequence  $\underline{\lambda} + k\underline{\theta}$ , weight  $\underline{\mu} + k\underline{\mathbf{1}}$  (where  $\underline{\mathbf{1}}$  denotes  $(1, \dots, 1)$ ), and depth  $d$ .*

*Proof.* By Corollary 3.2.4, the number  $n$  of boxes in any POP in  $\mathbb{P}_{\underline{\lambda}, \underline{\mu}}^k[d]$  is constant and given by:

$$n = \frac{1}{2}(\|\underline{\lambda} + k\underline{\theta}\|^2 - \|\underline{\mu} + k\underline{\mathbf{1}}\|^2) - d$$

Now, by the result of Chari-Loktev recalled in §3.3.6, the set  $\mathbb{P}_{\underline{\lambda}, \underline{\mu}}^k[d]$  parametrizes the CL basis for  $W(\lambda + k\theta)_\mu[n]$ , the  $n$ -graded piece of the  $\mu$ -weight space of  $W(\lambda + k\theta)$ . Thus it is enough to observe that the weight space  $W(\lambda + k\theta)_{t_\gamma(\Lambda)-d\delta}$  has grade  $n$ , or, equivalently, that the weight space  $W(\lambda + k\theta)_{t_\gamma\Lambda}$  has grade  $n + d$ .

We now argue that in order to prove the last claim we may assume  $k$  to be large. Suppose we know that  $W(\lambda + (k + \ell)\theta)_{t_\gamma\Lambda}$  has grade  $(\|\underline{\lambda} + (k + \ell)\underline{\theta}\|^2 - \|\underline{\mu} + (k + \ell)\underline{\mathbf{1}}\|^2)/2$  for some  $\ell \geq 0$ . Since the inclusion  $W(\lambda + k\theta) \hookrightarrow W(\lambda + (k + \ell)\theta)$  increases degree by  $(\|\underline{\lambda} + (k + \ell)\underline{\theta}\|^2 - \|\underline{\lambda} + k\underline{\theta} + \ell\underline{\mathbf{1}}\|^2)/2$ —see Corollary 3.3.5—we conclude that the grade of  $W(\lambda + k\theta)_{t_\gamma\Lambda}$  is given by  $(\|\underline{\lambda} + k\underline{\theta} + \ell\underline{\mathbf{1}}\|^2 - \|\underline{\mu} + (k + \ell)\underline{\mathbf{1}}\|^2)/2$ , which as an easy calculation shows is equal to  $(\|\underline{\lambda} + k\underline{\theta}\|^2 - \|\underline{\mu} + k\underline{\mathbf{1}}\|^2)/2$ .

Since the union of the chain (5.4) of local Weyl modules is  $L(\Lambda)$ , it follows that  $W(\lambda + k\theta)_{t_\gamma\Lambda} = L(\Lambda)_{t_\gamma\Lambda}$  for all large  $k$ . Since  $L(\Lambda)_{t_\gamma\Lambda}$  is the highest grade piece of the  $\mu$ -weight space of  $L(\Lambda)$ , it follows that  $W(\lambda + k\theta)_{t_\gamma\Lambda}$  is the highest grade piece of the  $\mu$ -weight space of  $W(\lambda + k\theta)$ . It now follows from Proposition 3.3.3 (1) that the grade of  $W(\lambda + k\theta)_{t_\gamma\Lambda}$  is  $n + d$ .  $\square$

### 5.2.6 An explicit value for the bound $K_0$ in §5.2.4

Let  $\ell$  be large enough so that  $\underline{\mu}$  is a weight of  $\lambda + \ell\theta$  (it is easy to see that  $\lambda$  and  $\underline{\mu}$  are equal modulo the root lattice of  $\mathfrak{g}$ , so that such an  $\ell$  exists). Then  $\underline{\lambda} + \ell\underline{\theta} \succ_m \underline{\mu} + \ell\underline{\mathbf{1}}$ , and we may apply the results of §4.5. From Corollary 4.6.1, we conclude that, for  $k \geq d$ , the map  $\mathcal{C} \circ \mathcal{S}^k \circ \Phi_{\underline{\lambda} + \ell\underline{\theta}, \underline{\mu} + \ell\underline{\mathbf{1}}}$  is a bijection from  $\mathcal{P}_r(d)$  to the set  $\mathbb{P}_{\underline{\lambda}, \underline{\mu}}^{\ell+k}[d]$  of POPs bounding sequence  $\underline{\lambda} + (\ell + k)\underline{\theta}$ , weight  $\underline{\mu} + (\ell + k)\underline{\mathbf{1}}$ , and depth  $d$ . Thus we may take  $K_0 = \ell + d$ .

### 5.2.7 Fixing the scaling

With notation as in §5.2.2, fix arbitrarily an embedding of  $W(\lambda)$  in  $L(\Lambda)$ , and identify  $W(\lambda)$  with its image in  $L(\Lambda)$ . For  $k \geq 1$ , fix the embedding of  $W_{\lambda+k\theta}$  in  $L(\Lambda)$  such that the generator  $w_{\lambda+k\theta}$  is identified with

$$(x_\theta^+ \otimes t^{-((\lambda|\theta)+k)})^{(k)} w_\lambda$$

where  $X^{(p)}$  denotes the divided power  $X^p/p!$ . It is clear that  $w_{\lambda+k\theta}$  has weight  $t_{\lambda+k\theta}(\Lambda_0)$ ; that  $w_{\lambda+k\theta} \neq 0$  follows from the following observation:

$$(5.6) \quad (x_\theta^- \otimes t^{((\lambda|\theta)+k)})^{(k)} (x_\theta^+ \otimes t^{-((\lambda|\theta)+k)})^{(k)} w_\lambda = w_\lambda.$$

The proof of (5.6) follows by standard  $\mathfrak{sl}_2$  calculations, using the following relations in  $W(\lambda)$ :

$$(x_\theta^- \otimes t^{((\lambda|\theta)+k)}) w_\lambda = 0, \quad \text{for all } k \in \mathbb{N}$$

We will identify  $W(\lambda + k\theta)$  with its image  $V_{t_{w_0(\lambda+k\theta)}}(\Lambda_0)$  in  $L(\Lambda)$  via the isomorphism fixed as above.

### 5.2.8 The stability conjecture: second version

We now state the stability conjecture. Let  $\lambda$  be a dominant integral weight and  $\theta$  the highest root of  $\mathfrak{g} = \mathfrak{sl}_{r+1}$ . Identify  $\lambda$  with  $\underline{\lambda}$ :  $\lambda_1 \geq \dots \geq \lambda_r \geq \lambda_{r+1} = 0$  and  $\theta$  with  $\underline{\theta} = (2, 1, \dots, 1, 0)$  as in §3.1.9. Let  $\gamma$  be an element of the root lattice of  $\mathfrak{g}$  and  $d \geq 0$  a non-negative integer. Let  $\mu$  denote the restriction to the diagonal subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  of the weight  $t_\gamma \Lambda - d\delta$  of the integrable highest weight module  $L(\Lambda)$  of the affine Lie algebra  $\widehat{\mathfrak{g}}$ , where  $\Lambda$  is as in §5.2.2. Choose the tuple  $\underline{\mu}$  corresponding to  $\mu$  so that  $\sum_{i=1}^{r+1} \mu_i = \sum_{i=1}^{r+1} \lambda_i$ .

For any integer  $k \geq 0$ , the set  $\mathbb{P}_{\underline{\lambda}, \underline{\mu}}^k[d]$  (of POPs with bounding sequence  $\underline{\lambda} + k\underline{\theta}$ , weight  $\underline{\mu} + k\underline{1}$ , and depth  $d$ ) indexes the CL basis of  $W(\lambda + k\theta)_{t_\gamma(\Lambda) - d\delta}$ . For  $\mathfrak{P}$  in  $\mathbb{P}_{\underline{\lambda}, \underline{\mu}}^k[d]$  and  $j \geq 0$  an integer, let  $\mathfrak{P}^j$  denote the image of  $\mathfrak{P}$  in  $\mathbb{P}_{\underline{\lambda}, \underline{\mu}}^{k+j}[d]$  under the map  $\mathcal{C} \circ \mathcal{S}^j \circ \mathcal{C} : \mathbb{P}_{\underline{\lambda}, \underline{\mu}}^k[d] \rightarrow \mathbb{P}_{\underline{\lambda}, \underline{\mu}}^{k+j}[d]$  (§4.4, 4.6).

**Conjecture 5.2.2** (STABILITY of CL basis; second version). *Let  $\ell$  be the least non-negative integer such that  $\underline{\mu} + \ell\underline{1} \preceq_m \underline{\lambda} + \ell\underline{\theta}$ . Then for all  $k \geq \ell + d$ , for all  $j \geq 0$ , and for all  $\mathfrak{P} \in \mathbb{P}_{\underline{\lambda}, \underline{\mu}}^k[d]$ , we have the equality  $v_{\mathfrak{P}} = \pm v_{\mathfrak{P}^j}$  of CL basis elements up to sign in  $L(\Lambda)$  under the identification of local Weyl modules  $W(\lambda + m\theta)$  with their images in  $L(\Lambda)$  under isomorphisms fixed as in §5.2.7.*

The conjecture mirrors the fact that  $\mathcal{C} \circ \mathcal{S}^j \circ \mathcal{C}$  is an isomorphism for  $k \geq \ell + d$  (Corollary 4.6.3).

# Chapter 6

## Triangularity of Gelfand-Tsetlin and Chari-Loktev bases for representations of $\mathfrak{sl}_{r+1}$

The results of this chapter will appear in [21]. Throughout this chapter, we will assume that the simple Lie algebra  $\mathfrak{g} = \mathfrak{sl}_{r+1}$ . We adopt the notation of §1.4.

### 6.1 The main result

We first recall Chari-Loktev bases for irreducible representations of  $\mathfrak{g} = \mathfrak{sl}_{r+1}$ . Given  $\lambda = \sum_{i=1}^r m_i \varpi_i \in P^+$ , set  $\lambda_i = m_1 + \cdots + m_i \forall 1 \leq i \leq r$ , and consider the non-increasing sequence  $\underline{\lambda} : \lambda_1 \geq \cdots \geq \lambda_r \geq \lambda_{r+1} = 0$ . For a pattern  $\mathcal{P} : \underline{\lambda}^1, \dots, \underline{\lambda}^r, \underline{\lambda}^{r+1} = \underline{\lambda}$  in the set  $\text{GT}(\underline{\lambda})$  of GT patterns with bounding sequence  $\underline{\lambda}$ , define an element  $\text{CL}(\mathcal{P})$  of  $V(\lambda)$  as follows:

$$\text{CL}(\mathcal{P}) := (x_{1,1}^-)^{\lambda_1^2 - \lambda_1} (x_{1,2}^-)^{\lambda_1^3 - \lambda_1^2} (x_{2,2}^-)^{\lambda_2^3 - \lambda_2^2} (x_{1,3}^-)^{\lambda_1^4 - \lambda_1^3} (x_{2,3}^-)^{\lambda_2^4 - \lambda_2^3} \cdots (x_{r,r}^-)^{\lambda_r^{r+1} - \lambda_r^r} v_\lambda.$$

The following theorem is proved in [5] (see also §3.3.6 for the current formulation).

**Theorem 6.1.1.** [5, Corollary 2.1.3] *The set  $\{\text{CL}(\mathcal{P}) : \mathcal{P} \in \text{GT}(\underline{\lambda})\}$  forms a basis for  $V(\lambda)$ .*

The Gelfand-Tsetlin (GT) basis for  $V(\lambda)$  is given in §1.4.2: corresponding to pattern  $\mathcal{P}$  the GT basis vector is denoted by  $\zeta_{\mathcal{P}}$ . The pattern  $\Lambda$  with the  $j^{\text{th}}$  row  $\underline{\Lambda}^j = (\lambda_1, \dots, \lambda_j)$ ,  $\forall 1 \leq j \leq r+1$ , is the unique pattern in  $\text{GT}(\underline{\lambda})$  whose corresponding GT basis element  $\zeta_{\Lambda}$  has weight  $\lambda$ . We assume  $\zeta_{\Lambda} = v_{\lambda}$ , with out loss of generality, by suitably normalizing the GT basis for  $V(\lambda)$ .

The *row-wise dominance* partial order  $\geq$  on the set  $\text{GT}(\underline{\lambda})$  is defined by  $\mathcal{P} \geq \mathcal{Q}$  if for every  $j$ ,  $1 \leq j \leq r+1$ , the  $j^{\text{th}}$  row  $\underline{\lambda}^j$  of  $\mathcal{P}$  succeeds the  $j^{\text{th}}$  row  $\underline{\kappa}^j$  of  $\mathcal{Q}$  in the dominance order on partitions, i.e.,

$$\lambda_1^j + \dots + \lambda_i^j \geq \kappa_1^j + \dots + \kappa_i^j, \quad \forall 1 \leq i \leq j.$$

If  $\mathcal{P} \geq \mathcal{Q}$ , then we say that  $\mathcal{P}$  *dominates*  $\mathcal{Q}$ . If  $\mathcal{P} \geq \mathcal{Q}$  and  $\mathcal{P} \neq \mathcal{Q}$ , then we say that  $\mathcal{P}$  *strictly dominates*  $\mathcal{Q}$  and write  $\mathcal{P} \succ \mathcal{Q}$ .

We are now in a position to state our main result of this chapter.

**Theorem 6.1.2.** *Given  $\lambda = \sum_{i=1}^r m_i \varpi_i \in P^+$ , set  $\lambda_i = m_1 + \dots + m_i \forall 1 \leq i \leq r$ , and consider the non-increasing sequence  $\underline{\lambda} : \lambda_1 \geq \dots \geq \lambda_r \geq 0$ . For a pattern  $\mathcal{P} : \underline{\lambda}^1, \dots, \underline{\lambda}^r, \underline{\lambda}^{r+1} = \underline{\lambda}$  in  $\text{GT}(\underline{\lambda})$  we have the following:*

$$(6.1) \quad \text{CL}(\mathcal{P}) = \sum_{\substack{\mathcal{Q} \in \text{GT}(\underline{\lambda}) \\ \mathcal{Q} \geq \mathcal{P}}} c_{\mathcal{Q}} \zeta_{\mathcal{Q}}, \quad \text{for some } c_{\mathcal{Q}} \in \mathbb{C},$$

where the co-efficient  $c_{\mathcal{P}}$  of  $\zeta_{\mathcal{P}}$  in (6.1), is equal to,

$$(6.2) \quad \prod_{1 \leq i < j \leq r+1} \prod_{d_{ji}=0}^{\lambda_i^j - \lambda_i^{j-1} - 1} \prod_{j'=i+1}^{j-1} \frac{1}{(\lambda_i^{j-1} - \lambda_{j'}^{j-1} + d_{ji}) + (j' - i + 1)}.$$



## 6.2 Proof of the main result

We first, define the notion of length of a pattern. For a pattern  $\mathcal{P} : \underline{\lambda}^1, \dots, \underline{\lambda}^r, \underline{\lambda}^{r+1}$ , we define the *length*  $\ell(\mathcal{P})$  of  $\mathcal{P}$  as follows:

$$\ell(\mathcal{P}) := \sum_{1 \leq i \leq j \leq r} (\lambda_i^{j+1} - \lambda_i^j).$$

We now prove Theorem 6.1.2, proceeding by induction on length  $\ell(\mathcal{P})$  of  $\mathcal{P}$ . If  $\ell(\mathcal{P}) = 0$ , then  $\mathcal{P} = \Lambda$  and nothing to prove. Let  $\ell(\mathcal{P}) \geq 1$ . We assume Theorem 6.1.2 for all  $\mathcal{P}' \in \text{GT}(\underline{\lambda})$  with  $\ell(\mathcal{P}') < \ell(\mathcal{P})$ , and prove it for  $\mathcal{P}$ . Let  $j_0 = \min\{j : \lambda_i^j - \lambda_i^{j-1} \neq 0 \text{ for some } i\}$  and  $i_0 = \min\{i : \lambda_i^{j_0} - \lambda_i^{j_0-1} \neq 0\}$ .

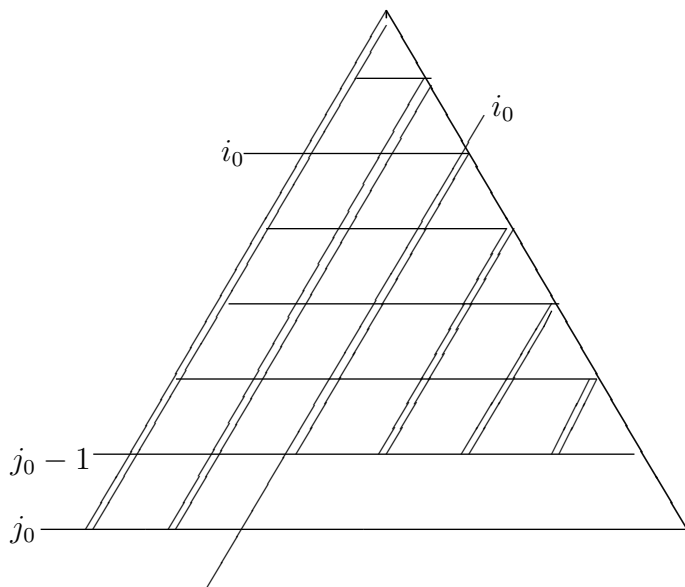


Figure 6.1: The figure shows the first  $j_0$  rows of the pattern of  $\mathcal{P}$ . The double lines indicate equalities, while the single lines are  $\geq$  relations.

Now we define  $\tilde{\mathcal{P}} \in \text{GT}(\underline{\lambda})$  as follows:

$$(6.3) \quad \tilde{\mathcal{P}} : \tilde{\lambda}^1, \dots, \tilde{\lambda}^{r+1}, \quad \text{with} \quad \tilde{\lambda}_i^j := \begin{cases} \lambda_i^j + 1, & \text{if } i = i_0 \text{ and } i_0 \leq j \leq j_0 - 1, \\ \lambda_i^j, & \text{otherwise.} \end{cases}$$

It is easy to observe that

$$(6.4) \quad \tilde{\mathcal{P}} \geq \mathcal{P}, \quad \ell(\tilde{\mathcal{P}}) = \ell(\mathcal{P}) - 1, \quad \text{and} \quad \text{CL}(\mathcal{P}) = x_{i_0, j_0-1}^- \text{CL}(\tilde{\mathcal{P}}) = E_{j_0, i_0} \text{CL}(\tilde{\mathcal{P}}).$$

By induction hypothesis, we have

$$(6.5) \quad \text{CL}(\tilde{\mathcal{P}}) = \sum_{\substack{\mathcal{R} \in \text{GT}(\underline{\lambda}) \\ \mathcal{R} \geq \tilde{\mathcal{P}}}} c'_{\mathcal{R}} \zeta_{\mathcal{R}}, \quad \text{for some } c'_{\mathcal{R}} \in \mathbb{C},$$

and the co-efficient  $c'_{\tilde{\mathcal{P}}}$  of  $\zeta_{\tilde{\mathcal{P}}}$  in (6.5), is equal to,

$$(6.6) \quad \prod_{1 \leq i < j \leq r+1} \prod_{d_{ji}=0}^{\tilde{\lambda}_i^j - \tilde{\lambda}_i^{j-1} - 1} \prod_{j'=i+1}^{j-1} \frac{1}{(\tilde{\lambda}_i^{j-1} - \tilde{\lambda}_{j'}^{j-1} + d_{ji}) + (j' - i + 1)}.$$

Using (6.4) and (6.5), we get

$$(6.7) \quad \text{CL}(\mathcal{P}) = \sum_{\substack{\mathcal{R} \in \text{GT}(\underline{\lambda}) \\ \mathcal{R} \geq \tilde{\mathcal{P}}}} c'_{\mathcal{R}} E_{j_0, i_0} \zeta_{\mathcal{R}}.$$

We now prove Theorem 6.1.2 for  $\mathcal{P}$  starting with proving (6.1). To prove (6.1), it is enough to show by (6.7) that

$$(6.8) \quad E_{j_0, i_0} \zeta_{\mathcal{R}} = \sum_{\substack{\mathcal{Q} \in \text{GT}(\underline{\lambda}) \\ \mathcal{Q} \geq \tilde{\mathcal{P}}}} c''_{\mathcal{Q}} \zeta_{\mathcal{Q}}, \quad \text{for some } c''_{\mathcal{Q}} \in \mathbb{C}, \quad \forall \mathcal{R} \geq \tilde{\mathcal{P}}.$$

We now prove (6.8). Let  $\mathcal{R} : \underline{\tau}^1, \dots, \underline{\tau}^{r+1}$  be such that  $\mathcal{R} \geq \tilde{\mathcal{P}}$ . Let  $\mathcal{Q} : \underline{\kappa}^1, \dots, \underline{\kappa}^{r+1}$  be

a pattern obtained from  $\mathcal{R}$  such that

$$\kappa_i^j := \begin{cases} \tau_i^j - 1, & i = r_j \text{ and } i_0 \leq j \leq j_0 - 1, \\ \tau_i^j, & \text{otherwise,} \end{cases}$$

for some  $1 \leq r_j \leq j$ . We only need to show that  $\mathcal{Q} \geq \mathcal{P}$  (see §1.4.2 for the action of  $E_{j_0, i_0}$  on  $\zeta_{\mathcal{R}}$ ). For  $1 \leq j < i_0$  and  $j_0 \leq j \leq r + 1$ , we have  $\kappa^j = \underline{\tau}^j \geq \tilde{\lambda}^j = \underline{\lambda}^j$ . Hence we need to show the following:

$$(6.9) \quad \kappa_1^j + \cdots + \kappa_m^j \geq \lambda_1^j + \cdots + \lambda_m^j, \quad \forall i_0 \leq j \leq j_0 - 1, \quad \forall 1 \leq m \leq j.$$

We prove (6.9) by splitting  $1 \leq m \leq j$  into three parts; (i)  $1 \leq m < r_j$  (ii)  $r_j \leq m < i_0$  (possibly empty), and (iii)  $i_0 \leq m \leq j$ . For  $1 \leq m < r_j$ , (6.9) follows from the observation that  $\underline{\tau}^j \geq \underline{\lambda}^j$ . Similarly for  $i_0 \leq m \leq j$ , it follows from the observation that  $\underline{\tau}^j \geq \tilde{\lambda}^j$ . We now show (6.9) for  $r_j \leq m < i_0$ . Since  $\mathcal{Q}$  is a pattern and  $m < i_0 \leq j$ , we have  $\kappa_i^j \geq \kappa_i^m = \tau_i^m$ ,  $\forall 1 \leq i \leq m$ . This gives that

$$(6.10) \quad \kappa_1^j + \cdots + \kappa_m^j \geq \tau_1^m + \cdots + \tau_m^m.$$

Now  $\underline{\tau}^m \geq \tilde{\lambda}^m$  and  $m < i_0 \leq j \leq j_0 - 1$  gives,

$$(6.11) \quad \tau_1^m + \cdots + \tau_m^m \geq \tilde{\lambda}_1^m + \cdots + \tilde{\lambda}_m^m = \lambda_1^m + \cdots + \lambda_m^m = \lambda_1^j + \cdots + \lambda_m^j.$$

Combining (6.10) and (6.11), we get (6.9) in this case. This completes the proof of (6.1) for  $\mathcal{P}$ . We now prove (6.2) for  $\mathcal{P}$ . We observe that  $\zeta_{\mathcal{P}}$  occurs in  $E_{j_0, i_0} \zeta_{\tilde{\mathcal{P}}}$  and it does not occur in  $E_{j_0, i_0} \zeta_{\mathcal{R}}$ ,  $\forall \mathcal{R} \succeq \tilde{\mathcal{P}}$ . Indeed, the patterns getting by adding 1 to an element of  $j^{\text{th}}$  row  $\underline{\lambda}^j$  of  $\mathcal{P}$ ,  $\forall i_0 \leq j \leq j_0 - 1$ , and dominates  $\tilde{\mathcal{P}}$  are the only  $\tilde{\mathcal{P}}$ . Hence we get by

(6.7) that

(6.12)

the co-efficient of  $\zeta_{\mathcal{P}}$  in  $\text{CL}(\mathcal{P})$ , is equal to,  $(c'_{\tilde{\mathcal{P}}})$ (the co-efficient of  $\zeta_{\mathcal{P}}$  in  $E_{j_0, i_0} \zeta_{\tilde{\mathcal{P}}}$ ).

It is easy to observe that

the co-efficient of  $\zeta_{\mathcal{P}}$  in  $E_{j_0, i_0} \zeta_{\tilde{\mathcal{P}}}$ , is equal to,

the co-efficient of  $\zeta_{\mathcal{P}}$  in  $E_{j_0, j_0-1} E_{j_0-1, j_0-2} \cdots E_{i_0+1, i_0} \zeta_{\tilde{\mathcal{P}}}$ ,

which is also equal to

$$(6.13) \quad \prod_{j=i_0}^{j_0-1} \left( \text{the co-efficient of } \zeta_{\tilde{\mathcal{P}}-\delta_{i_0, i_0}-\delta_{i_0+1, i_0}-\cdots-\delta_{j, i_0}} \text{ in } E_{j+1, j} \zeta_{\tilde{\mathcal{P}}-\delta_{i_0, i_0}-\delta_{i_0+1, i_0}-\cdots-\delta_{j-1, i_0}} \right)$$

$$= (1) \left( \prod_{j=i_0+1}^{j_0-1} \frac{1}{l_{j, i_0} - l_{j, j}} \right) = \prod_{j=i_0+1}^{j_0-1} \frac{1}{(\lambda_{i_0}^j - \lambda_j^j) + (j - i_0 + 1)} = \prod_{j=i_0+1}^{j_0-1} \frac{1}{(\lambda_{i_0}^{j_0-1} - \lambda_j^{j_0-1}) + (j - i_0 + 1)},$$

(see Theorem 1.4.2 for the action of  $E_{j+1, j}$ ). Now the proof of (6.2) for  $\mathcal{P}$  follows from (6.12) by using (6.3), (6.6), and (6.13). This completes the proof of Theorem 6.1.2.

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