On the topology of nilpotent orbits in semisimple Lie algebras

By

Chandan Maity

MATH10201004002

The Institute of Mathematical Sciences, Chennai

A thesis submitted to the

Board of Studies in Mathematical Sciences

In partial fulfillment of requirements

for the Degree of

DOCTOR OF PHILOSOPHY

of

HOMI BHABHA NATIONAL INSTITUTE

May, 2017

Homi Bhabha National Institute

Recommendations of the Viva Voce Committee

As members of the Viva Voce Committee, we certify that we have read the dissertation prepared by Chandan Maity entitled " On the topology of nilpotent orbits in semisimple Lie algebras" and recommend that it may be accepted as fulfilling the thesis requirement for the award of Degree of Doctor of Philosophy.

________________________________ Date: 31/05/2017

Chairman - Parameswaran Sankaran

________________________________ Date: 31/05/2017

Guide/Convenor - Pralay Chatterjee

________________________________ Date: 31/05/2017

Examiner - Mahan Mj

________________________________ Date: 31/05/2017

Member 1 - D. S. Nagaraj

________________________________ Date: 31/05/2017

Member 2 - Anirban Mukhopadhyay

Final approval and acceptance of this thesis is contingent upon the candidate's submission of the final copies of the thesis to HBNI.

I hereby certify that I have read this thesis prepared under my direction and recommend that it may be accepted as fulfilling the thesis requirement.

Date: 31/05/2017

Place: Chennai Guide

STATEMENT BY AUTHOR

This dissertation has been submitted in partial fulfillment of requirements for an advanced degree at Homi Bhabha National Institute (HBNI) and is deposited in the Library to be made available to borrowers under rules of the HBNI.

Brief quotations from this dissertation are allowable without special permission, provided that accurate acknowledgement of source is made. Requests for permission for extended quotation from or reproduction of this manuscript in whole or in part may be granted by the Competent Authority of HBNI when in his or her judgement the proposed use of the material is in the interests of scholarship. In all other instances, however, permission must be obtained from the author.

Chandan Maity

DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Chandan Maity

LIST OF PUBLICATIONS ARISING FROM THE THESIS

Journal

1. “ On the second cohomology of nilpotent orbits in exceptional Lie algebras”, Pralay Chatterjee and Chandan Maity, *Bulletin des Sciences Mathématiques*, **141** (2017), no. 1, 10–24.

Preprint

1. “ The second cohomology of nilpotent orbits in classical Lie algebras”, Indranil Biswas, Pralay Chatterjee, and Chandan Maity. (https://arxiv.org/abs/1611.08369)

Chandan Maity

To

My Parents

ACKNOWLEDGEMENTS

I would like to take this opportunity to express my gratitude and thanks to my thesis supervisor Prof. Pralay Chatterjee for his inspiration, encouragement and invaluable guidance during my thesis work. I also thank him for being so friendly and helpful. I would like to thank Prof. Indranil Biswas for collaboration in one of the projects. I wish to thank Prof. Partha Sarathi Chakraborty, Prof. Anirban Mukhopadhyay, Prof. D. S. Nagaraj, Prof. P. Sankaran for encouragement and the courses taught by them during my course-work period at IMSc. I thank the IMSc office staff for handling some of the administrative formalities. Thanks are also due to my teachers at the Ramakrishna Mission Vivekananda University for the inspiring lectures they gave during my M.Sc. days.

I warmly thank Abhra, Anirbanda, Arghya, Jahanur, Kamalakshya, Krishanuda, Prateepda, Sandipanda, Sanjitda, Sarbeswarda, Satyajitda, Sumit for their friendship, company, and making my IMSc life enjoyable.

Last but far from least I wish to thank my parents and my sister Tapasi for their constant support, encouragement and unconditional love.

Contents

Synopsis

Introduction

Let G be a connected real semisimple Lie group with Lie algebra $\mathfrak{g}$. An element $X \in \mathfrak{g}$ is called *nilpotent* if $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ is a nilpotent operator. Let $\mathcal{O}_X$ denote the corresponding orbit $\{\mathrm{Ad}(g)X \mid g \in G\}$ under the adjoint action of G on $\mathfrak{g}$. These orbits form a rich class of homogeneous spaces which are extensively studied. Various topological aspects of these orbits have drawn attention over the years; see [CoMc] and references therein for an account. In this thesis, we describe the second cohomology groups of the nilpotent orbits in real classical non-compact Lie algebras which are non-complex. Considering the non-compact non-complex exceptional Lie algebras we also compute the dimensions of the second cohomology groups for most of the nilpotent orbits. For the rest of cases of nilpotent orbits in the exceptional Lie algebras, which are not covered in the above computations, we obtain upper bounds for the dimensions of the second cohomology groups. The methods involved above steered us to describe the first cohomology groups of the nilpotent orbits in all the simple real Lie algebras except $E_{6(-14)}$ and $E_{7(-25)}$. For the nilpotent orbits in $E_{6(-14)}$ and $E_{7(-25)}$ we give upper bounds for the dimensions of the first cohomology groups.

We next fix some notation. The *center* of a Lie algebra $\mathfrak{g}$ is denoted by $\mathfrak{z}(\mathfrak{g})$. We denote Lie groups by the capital letters, and unless mentioned otherwise, we denote their Lie algebras by the corresponding lower case German letters. Sometimes, for convenience, the Lie algebra of a Lie group G is also denoted by $\mathrm{Lie}(G)$. The connected component of a Lie group G containing the identity element is denoted by G°. For a subgroup H of G and a subset S of $\mathfrak{g}$, the subgroup of H that fixes S point wise is called the *centralizer* of S in H and is denoted by $\mathcal{Z}_H(S)$. Similarly, for a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a subset $S \subset \mathfrak{g}$, by $\mathfrak{z}_{\mathfrak{h}}(S)$ we will denote the subalgebra consisting elements of $\mathfrak{h}$ that commute with every element of S.

If G is a Lie group with Lie algebra $\mathfrak{g}$, then it is immediate that the coadjoint action of G° on the dual $\mathfrak{z}(\mathfrak{k})^*$ of $\mathfrak{z}(\mathfrak{k})$ is trivial; in particular, one obtains a natural action of G/G° on $\mathfrak{z}(\mathfrak{k})^*$. We denote by $[\mathfrak{z}(\mathfrak{g})^*]^{G/G^\circ}$ the space of fixed points of $\mathfrak{z}(\mathfrak{g})^*$ under the action of G/G°.

The second and first cohomologies of homogeneous spaces

We first formulate a convenient description of the second and first cohomology groups of a general homogeneous space of a connected Lie group. In [BC1, Theorem 3.3] the second cohomology groups of any homogeneous space of a connected Lie group are described under the additional restriction that all the maximal compact subgroups of the Lie group are semisimple. Our result holds under the mild restriction that the stabilizer of any point in the homogeneous space has finitely many connected components and hence generalizes [BC1, Theorem 3.3]. Thus the results are of independent interest in view of its applicability to a very large class of homogeneous spaces.

Theorem 0.0.1. *Let G be a connected Lie group, and let $H \subset G$ be a closed subgroup with finitely many connected components. Let K be a maximal compact subgroup of*

H, and M be a maximal compact subgroup of G containing K. Then

$$H^2(G/H,\, \mathbb{R}) \simeq \Omega^2\big(\frac{\mathfrak{m}}{[\mathfrak{m},\mathfrak{m}]+\mathfrak{k}}\big) \oplus [(\mathfrak{z}(\mathfrak{k}) \cap [\mathfrak{m},\mathfrak{m}])^*]^{K/K^\circ}$$

and

$$H^1(G/H,\mathbb{R}) \simeq \Omega^1(\frac{\mathfrak{m}}{[\mathfrak{m},\mathfrak{m}]+\mathfrak{k}}).$$

The next result, which follows from [CoMc, Lemma 3.7.3] and Theorem 0.0.1, is crucial to our computations of the second and first cohomology groups of the nilpotent orbits.

Theorem 0.0.2. *Let G be an algebraic group defined over $\mathbb{R}$ which is $\mathbb{R}$-simple. Let $X \in \operatorname{Lie} G(\mathbb{R}), X \neq 0$ be a nilpotent element and $\mathcal{O}_X$ be the orbit of X under the adjoint action of the identity component $G(\mathbb{R})^\circ$ on $\operatorname{Lie} G(\mathbb{R})$. Let $\{X,H,Y\}$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple in $\operatorname{Lie} G(\mathbb{R})$. Let K be a maximal compact subgroup in $\mathcal{Z}_{G(\mathbb{R})^\circ}(X,H,Y)$ and M be a maximal compact subgroup in $G(\mathbb{R})^\circ$ containing K. Then*

$$H^2(\mathcal{O}_X,\mathbb{R}) \simeq [(\mathfrak{z}(\mathfrak{k}) \cap [\mathfrak{m},\mathfrak{m}])^*]^{K/K^\circ}$$

and

$$\dim_{\mathbb{R}} H^1(\mathcal{O}_X,\mathbb{R}) = \begin{cases} 1 & \text{if } \ \mathfrak{k}+[\mathfrak{m},\mathfrak{m}] \subsetneqq \mathfrak{m} \\ 0 & \text{if } \ \mathfrak{k}+[\mathfrak{m},\mathfrak{m}] = \mathfrak{m}. \end{cases}$$

In particular, $\dim_{\mathbb{R}} H^1(\mathcal{O}_X,\mathbb{R}) \leq 1$.

Theorem 0.0.1 and Theorem 0.0.2 appear in the thesis and in [BCM].

Nilpotent orbits in non-compact non-complex real classical Lie algebras

We need the notion of Young diagrams and signed Young diagrams to state our results on the second and first cohomology groups of nilpotent orbits in real classical non-complex non-compact Lie algebras. Let n be a positive integer. By a *partition* of n we mean the symbol $[d_1^{t_{d_1}}, \ldots, d_s^{t_{d_s}}]$ where $t_{d_i}, d_i \in \mathbb{N}$, $1 \leq i \leq s$ satisfying $\sum_{i=1}^{s} t_{d_i} d_i = n$, $t_{d_i} \geq 1$ and $d_{i+1} > d_i > 0$ for all i. Let $\mathcal{P}(n)$ denote the *set of partitions of* n. For a partition $\mathbf{d} := [d_1^{t_{d_1}}, \ldots, d_s^{t_{d_s}}]$ we set $\mathbb{N}_\mathbf{d} := \{d_i \mid 1 \leq i \leq s\}$, $\mathbb{E}_\mathbf{d} := \{d \in \mathbb{N}_\mathbf{d} \mid d \text{ is even}\}$ and $\mathbb{O}_\mathbf{d} := \{d \in \mathbb{N}_\mathbf{d} \mid d \text{ is odd}\}$. The size of a rectangular array of empty boxes of height α and width β is denoted by $\alpha \times \beta$. A *Young diagram* corresponding to a partition $\mathbf{d} := [d_1^{t_{d_1}}, \ldots, d_s^{t_{d_s}}]$ of n is a disjoint union of rectangular arrays of empty boxes such that the sizes of the rectangular arrays are $t_{d_1} \times d_1, \ldots, t_{d_s} \times d_s$. Since there is an obvious correspondence between the set of Young diagrams of size n and the set $\mathcal{P}(n)$ of partitions of n, the *set of Young diagrams of size* n is also denoted by $\mathcal{P}(n)$. A *signed Young diagram* is a Young diagram together with appropriate signs $+1$ or -1 placed in the empty boxes in each rectangular array $t_{d_i} \times d_i$, for all $1 \leq i \leq s$. This is defined as follows. We consider the usual ordering of the signs, $-1 \leq +1$. In the first column of the rectangular array $t_{d_i} \times d_i$, the signs of 1 are non-increasing as we go down along the first column. It now remains to allot signs to the boxes in each row of the rectangular array $t_{d_i} \times d_i$. We divide into two cases according as $d_i \neq 3 \pmod 4$ or $d_i = 3 \pmod 4$. In the first case, when $d_i \neq 3 \pmod 4$, the signs of 1 alternate across each row. In the latter case, when $d_i = 3 \pmod 4$, for each row, the signs of 1 alternate across the row till the last but one box, and in the last box of the row the sign of the last but one box is repeated. For $d \in \mathbb{N}_\mathbf{d}$, let p_d (resp. q_d) be the number of $+1$ (resp. -1) in the 1st column of the rectangular array of size $t_d \times d$ in a singed Young diagram. The *signature of a signed Young diagram* is the ordered pair (p, q) where p (resp. q)

is the total number of boxes with the sign $+1$ (resp. -1).

The Young diagrams and the signed Young diagrams, with some more additional restrictions in some cases, parametrize the nilpotent orbits in all the classical Lie algebras. There is a natural surjection from the set of nilpotent orbits in $\mathfrak{sl}_n(\mathbb{R})$ to $\mathcal{P}(n)$ such that the fibers have cardinality either one or two. There is a natural bijection from the set of nilpotent orbits in $\mathfrak{sl}_n(\mathbb{H})$ to $\mathcal{P}(n)$. For a pair of integers (p, q) there is a natural bijection from the set of nilpotent orbits in $\mathfrak{su}(p, q)$ to the set of signed Young diagrams of signature (p, q). As before, for a pair of integers (p, q) there is a natural surjection from the set of nilpotent orbits in $\mathfrak{so}(p, q)$ to the set of signed Young diagrams of signature (p, q) such that rows of even length occur with even multiplicity and have their left most boxes labeled by $+1$. Furthermore, the fibers of the above surjection have cardinality either one or two or four. In the case of $\mathfrak{so}^*(2n)$ there is a natural bijection from the set of nilpotent orbits to the set of signed Young diagrams of size n in which rows of odd length have their leftmost boxes labeled by $+1$. There is a natural bijection from the set of nilpotent orbits in $\mathfrak{sp}(n, \mathbb{R})$ to the set of signed Young diagrams of size $2n$ in which rows of odd length occur with even multiplicity and have their left most boxes labeled by $+1$. For a pair of integers (p, q) there is a natural bijection from the set of nilpotent orbits in $\mathfrak{sp}(p, q)$ to the set of signed Young diagrams of signature (p, q) in which rows of even length have their leftmost boxes labeled by $+1$. The details of parametrization can be found in [CoMc, §9.3]. We also refer to Chapter 4 of the thesis and [BCM, §5] for an exposition of the above parametrizations and correction of certain errors in [CoMc, §9.3].

We use the notation as in the first paragraph of this section and the parametrizations of nilpotent orbits mentioned above to describe our main results.

Theorem 0.0.3. *Let $X \in \mathfrak{sl}_n(\mathbb{R})$ be a nilpotent element. Let $\mathbf{d} = [d_1^{t_{d_1}}, \dots, d_s^{t_{d_s}}] \in \mathcal{P}(n)$ be the partition associated to the orbit $\mathcal{O}_X$.*

1. *If* $n \geq 3$, $\#\mathbb{O}_{\mathbf{d}} = 1$ *and* $t_\theta = 2$ *for the* $\theta \in \mathbb{O}_{\mathbf{d}}$, *then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 1$.

2. *In all the other cases* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 0$.

Moreover, $\dim_{\mathbb{R}} H^1(\mathcal{O}_X, \mathbb{R}) = \begin{cases} 1 & \text{if } n = 2 \\ 0 & \text{if } n \geq 3. \end{cases}$

Theorem 0.0.4. *Let* X *be a nilpotent element in* $\mathfrak{sl}_n(\mathbb{H})$. *Then* $\dim_{\mathbb{R}} H^i(\mathcal{O}_X, \mathbb{R}) = 0$ *for* $i = 1, 2$.

Theorem 0.0.5. *Let* $X \in \mathfrak{su}(p, q)$ *be a nilpotent element. Recall that the orbit* $\mathcal{O}_X$ *corresponds to a signed Young diagram of signature* (p, q). *Let* $\mathbf{d} = [d_1^{t_{d_1}}, \ldots, d_s^{t_{d_s}}] \in \mathcal{P}(n)$ *be the partition associated to the corresponding Young diagram. Let* $l := \#\{d \mid d \in \mathbb{N}_{\mathbf{d}}, p_d \neq 0\} + \#\{d \mid d \in \mathbb{N}_{\mathbf{d}}, q_d \neq 0\}$.

1. *If* $\mathbb{N}_{\mathbf{d}} = \mathbb{E}_{\mathbf{d}}$, *then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = l - 1$ *and* $\dim_{\mathbb{R}} H^1(\mathcal{O}_X, \mathbb{R}) = 1$.

2. *If* $l = 1$ *and* $\mathbb{N}_{\mathbf{d}} = \mathbb{O}_{\mathbf{d}}$, *then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 0$ *and* $\dim_{\mathbb{R}} H^1(\mathcal{O}_X, \mathbb{R}) = 1$.

3. *If* $l \geq 2$ *and* $\#\mathbb{O}_{\mathbf{d}} \geq 1$, *then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = l - 2$ *and* $\dim_{\mathbb{R}} H^1(\mathcal{O}_X, \mathbb{R}) = 0$.

We next deal with $\mathfrak{so}(p, q)$. However, to avoid technical complications, we further assume the additional restrictions $p \neq 2, q \neq 2$ and $(p, q) \neq (1, 1)$. The complete results without these restrictions appear in the thesis and in [BCM].

Theorem 0.0.6. *Let* $p \neq 2, q \neq 2$ *and* $(p, q) \neq (1, 1)$. *Let* $X \in \mathfrak{so}(p, q)$ *be a nilpotent element. Recall that the orbit* $\mathcal{O}_X$ *corresponds to a signed Young diagram of signature* (p, q) *such that rows of even length occur with even multiplicity and have their left most boxes labeled by* $+1$. *Let* $\mathbf{d} = [d_1^{t_{d_1}}, \ldots, d_s^{t_{d_s}}] \in \mathcal{P}(n)$ *be the partition associated to the corresponding Young diagram. Let* $l := \#\mathbb{E}_{\mathbf{d}}$.

1. *If* $\#\{\theta \in \mathbb{O}_{\mathbf{d}} \mid p_\theta \neq 0\} = 1$, $\#\{\theta \in \mathbb{O}_{\mathbf{d}} \mid q_\theta \neq 0\} = 1$ *and* $p_{\theta_1} = q_{\theta_2} = 2$ *for the* $\theta_1 \in \{\theta \in \mathbb{O}_{\mathbf{d}} \mid p_\theta \neq 0\}$, $\theta_2 \in \{\theta \in \mathbb{O}_{\mathbf{d}} \mid q_\theta \neq 0\}$, *then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = l + 2$.

2. *Suppose either* $\#\{\theta \in \mathbb{O}_{\mathbf{d}} \mid p_\theta \neq 0\} = 1$ *and* $p_{\theta_1} = 2$ *for the* $\theta_1 \in \{\theta \in \mathbb{O}_{\mathbf{d}} \mid p_\theta \neq 0\}$, *or* $\#\{\theta \in \mathbb{O}_{\mathbf{d}} \mid q_\theta \neq 0\} = 1$ *and* $q_{\theta_2} = 2$ *for the* $\theta_2 \in \{\theta \in \mathbb{O}_{\mathbf{d}} \mid q_\theta \neq 0\}$. *Moreover, suppose that both the conditions do not hold simultaneously. Then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = l + 1$.

3. *In all other cases* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = l$.

Moreover, in all the above cases we have $\dim_{\mathbb{R}} H^1(\mathcal{O}_X, \mathbb{R}) = 0$.

Theorem 0.0.7. *Let* $X \in \mathfrak{so}^*(2n)$ *be a nilpotent element. Recall that the orbit* $\mathcal{O}_X$ *corresponds to a signed Young diagram of size* n *in which rows of odd length have their leftmost boxes labeled by* $+1$. *Let* $\mathbf{d} = [d_1^{t_{d_1}}, \ldots, d_s^{t_{d_s}}] \in \mathcal{P}(n)$ *be the partition associated to the corresponding Young diagram. Let* $l := \#\mathbb{O}_{\mathbf{d}}$. *Then*

$$\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = \begin{cases} 0 & \text{if } l = 0 \\ l - 1 & \text{if } l \geq 1 \end{cases}$$

and

$$\dim_{\mathbb{R}} H^1(\mathcal{O}_X, \mathbb{R}) = \begin{cases} 1 & \text{if } l = 0 \\ 0 & \text{if } l \geq 1. \end{cases}$$

Theorem 0.0.8. *Let* $X \in \mathfrak{sp}(n, \mathbb{R})$ *be a nilpotent element. Recall that the orbit* $\mathcal{O}_X$ *corresponds to a signed Young diagram of size* $2n$ *in which rows of odd length occur with even multiplicity and have their leftmost boxes labeled by* $+1$. *Let* $\mathbf{d} = [d_1^{t_{d_1}}, \ldots, d_s^{t_{d_s}}] \in \mathcal{P}(n)$ *be the partition associated to the corresponding Young diagram. Let* $l := \#\mathbb{O}_{\mathbf{d}}$. *Then*

$$\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = \begin{cases} 0 & \text{if } l = 0 \\ l - 1 & \text{if } l \geq 1 \end{cases}$$

and

$$\dim_{\mathbb{R}} H^1(\mathcal{O}_X, \mathbb{R}) = \begin{cases} 1 & \text{if } l = 0 \\ 0 & \text{if } l \geq 1. \end{cases}$$

Theorem 0.0.9. *Let $X \in \mathfrak{sp}(p,q)$ be a nilpotent element. Recall that the orbit $\mathcal{O}_X$ corresponds to a signed Young diagram of signature (p,q) in which rows of even length have their leftmost boxes labeled by $+1$. Let $\mathbf{d} := [d_1^{t_{d_1}}, \ldots, d_s^{t_{d_s}}] \in \mathcal{P}(n)$ be the partition associated to the corresponding Young diagram. Let $l := \#\mathbb{E}_{\mathbf{d}}$. Then*

$$\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = l \text{ and } \dim_{\mathbb{R}} H^1(\mathcal{O}_X, \mathbb{R}) = 0.$$

Theorems 0.0.3, 0.0.4, 0.0.5, 0.0.6, 0.0.7, 0.0.8, 0.0.9 appear in the thesis and in [BCM].

Nilpotent orbits in non-compact non-complex real exceptional Lie algebras

To describe our results we use the parametrizations of nilpotent orbits as given in [Dj1], [Dj2]. We consider the nilpotent orbits in $\mathfrak{g}$ under the action of $\operatorname{Int}\mathfrak{g}$, where $\mathfrak{g}$ is a non-compact non-complex real exceptional Lie algebra. We fix a semisimple algebraic group G defined over $\mathbb{R}$ such that $\mathfrak{g} = \operatorname{Lie}(G(\mathbb{R}))$. Here $G(\mathbb{R})$ denotes the associated real semisimple Lie group of the $\mathbb{R}$-points of G. Let $G(\mathbb{C})$ be the associated complex semisimple Lie group consisting of the $\mathbb{C}$-points of G. It is easy to see that the orbits in $\mathfrak{g}$ under the action of $\operatorname{Int}\mathfrak{g}$ are the same as the orbits in $\mathfrak{g}$ under the action of $G(\mathbb{R})^\circ$. In this case, for a nilpotent element $X \in \mathfrak{g}$, we set $\mathcal{O}_X := \{\operatorname{Ad}(g)X \mid g \in G(\mathbb{R})^\circ\}$. Let $\mathfrak{g} = \mathfrak{m} + \mathfrak{p}$ be a Cartan decomposition, and let θ be the corresponding Cartan involution. Let $\mathfrak{g}_{\mathbb{C}}$ be the Lie algebra of $G(\mathbb{C})$. Then $\mathfrak{g}_{\mathbb{C}}$ can be identified with the complexification of $\mathfrak{g}$. Let $\mathfrak{m}_{\mathbb{C}}$ and $\mathfrak{p}_{\mathbb{C}}$ be the $\mathbb{C}$-spans of $\mathfrak{m}$ and $\mathfrak{p}$ in $\mathfrak{g}_{\mathbb{C}}$, respectively. Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{m}_{\mathbb{C}} + \mathfrak{p}_{\mathbb{C}}$. Recall that, if $\mathfrak{g}$ is as above and $\mathfrak{g}$ is different from both $E_{6(-26)}$ and $E_{6(6)}$, then $\mathfrak{g}$ is of inner type, or equivalently, $\operatorname{rank} \mathfrak{m}_{\mathbb{C}} = \operatorname{rank} \mathfrak{g}_{\mathbb{C}}$. When $\mathfrak{g}$ is of inner type, the nilpotent orbits are parametrized by a finite sequence of integers of length l where $l := \operatorname{rank} \mathfrak{m}_{\mathbb{C}} = \operatorname{rank} \mathfrak{g}_{\mathbb{C}}$. When $\mathfrak{g}$ is not of inner type, that is, when $\mathfrak{g}$ is either $E_{6(-26)}$ or $E_{6(6)}$, then the nilpotent orbits are parametrized by a finite sequence of integers of length 4. We refer to [Dj1] and [Dj2] for the details of parametrizations of the nilpotent orbits in non-compact non-complex real exceptional Lie algebras.

Now we state the results on the second cohomology of the nilpotent orbits in non-compact non-complex exceptional Lie algebras in this set-up.

Recall that up to conjugation there is only one non-compact real form of G_2. We denote it by $G_{2(2)}$. There are only five nonzero nilpotent orbits in $G_{2(2)}$; see [Dj1, Table VI, p. 510].

Theorem 0.0.10. *Let the parametrization of the nilpotent orbits in $G_{2(2)}$ be as in [Dj1, Table VI, p. 510]. Let X be a nonzero nilpotent element in $G_{2(2)}$.*

1. *If the parametrization of the orbit $\mathcal{O}_X$ is given by either* 1 1 *or* 1 3, *then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 1$.
2. *If the parametrization of the orbit $\mathcal{O}_X$ is given by any of* 2 2, 0 4, 4 8, *then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 0$.

Recall that up to conjugation there are two non-compact real forms of F_4. They are denoted by $F_{4(4)}$ and $F_{4(-20)}$. There are 26 nonzero nilpotent orbits in $F_{4(4)}$; see [Dj1, Table VII, p. 510].

Theorem 0.0.11. *Let the parametrization of the nilpotent orbits in $F_{4(4)}$ be as in [Dj1, Table VII, p. 510]. Let X be a nonzero nilpotent element in $F_{4(4)}$.*

1. *Assume the parametrization of the orbit* $\mathcal{O}_X$ *is given by any of the sequences :* $001\ 1,\ 001\ 3,\ 110\ 2,\ 111\ 1,\ 131\ 3$. *Then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 1$.

2. *Assume the parametrization of the orbit* $\mathcal{O}_X$ *is given by any of the sequences :* $100\ 2,\ 200\ 0,\ 103\ 1,\ 111\ 3,\ 204\ 4$. *Then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) \leq 1$.

3. *If the parametrization of the orbit* $\mathcal{O}_X$ *is either* $101\ 1$ *or* $012\ 2$*, then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) \leq 2$.

4. *If* $\mathcal{O}_X$ *is not given by the parametrizations as in (1), (2), (3) above (# of such orbits are 14), then we have* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 0$.

There are two nonzero nilpotent orbits in $F_{4(-20)}$; see [Dj1, Table VIII, p. 511].

Theorem 0.0.12. *For every nilpotent element* $X \in F_{4(-20)}$, $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 0$.

Recall that up to conjugation there are four non-compact real forms of E_6. They are denoted by $E_{6(6)}$, $E_{6(2)}$, $E_{6(-14)}$ and $E_{6(-26)}$. There are 23 nonzero nilpotent orbits in $E_{6(6)}$; see [Dj2, Table VIII, p. 205].

Theorem 0.0.13. *Let the parametrization of the nilpotent orbits in* $E_{6(6)}$ *be as in [Dj2, Table VIII, p. 205]. Let* X *be a nonzero nilpotent element in* $E_{6(6)}$.

1. *If the parametrization of the orbit* $\mathcal{O}_X$ *is given by either* 1001 *or* 1101 *or* 1211*, then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 1$.

2. *Assume the parametrization of the orbit* $\mathcal{O}_X$ *is given by any of the sequences :* $0102, 0202, 1010$*,* $2002, 1011$. *Then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) \leq 1$.

3. *If* $\mathcal{O}_X$ *is not given by the parametrizations as in (1), (2) above (# of such orbits are 15), then we have* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 0$.

There are 37 nonzero nilpotent orbits in $E_{6(2)}$; see [Dj1, Table IX, p. 511].

Theorem 0.0.14. *Let the parametrization of the nilpotent orbits in $E_{6(2)}$ be as in [Dj1, Table IX, p. 511]. Let X be a nonzero nilpotent element in $E_{6(2)}$.*

1. *Assume the parametrization of the orbit $\mathcal{O}_X$ is given by any of the sequences :* 00000 4, 00200 2, 02020 0, 00400 8, 22222 2, 04040 4, 44044 4, 44444 8. *Then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 0$.

2. *Assume the parametrization of the orbit $\mathcal{O}_X$ is given by any of the sequences :* 10001 2, 10101 1, 21001 1, 10012 1, 11011 2, 01210 2, 10301 1, 11111 3, 22022 0. *Then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 2$.

3. *If the parametrization of the orbit $\mathcal{O}_X$ is given by either* 20002 0 *or* 00400 0 *or* 02020 4*, then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) \leq 2$.

4. *If the parametrization of the orbit $\mathcal{O}_X$ is given by* 20202 2*, then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) \leq 1$.

5. *If $\mathcal{O}_X$ is not given by the parametrizations as in (1), (2), (3), (4) above (# of such orbits are 16), then we have* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 1$.

There are 12 nonzero nilpotent orbits in $E_{6(-14)}$; see [Dj1, Table X, p. 512].

Theorem 0.0.15. *Let the parametrization of the nilpotent orbits in $E_{6(-14)}$ be as in [Dj1, Table X, p. 512]. Let X be a nonzero nilpotent element in $E_{6(-14)}$.*

1. *If the parametrization of the orbit $\mathcal{O}_X$ is given by* 40000 -2*, then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 0$.

2. *If $\mathcal{O}_X$ is not given by the above parametrization (# of such orbits are 11), then we have* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) \leq 1$.

There are two nonzero nilpotent orbits in $E_{6(-26)}$; see [Dj2, Table VII, p. 204].

Theorem 0.0.16. *For every nilpotent element $X \in E_{6(-26)}$,* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 0$.

Recall that up to conjugation there are three non-compact real forms of E_7. They are denoted by $E_{7(7)}$, $E_{7(-5)}$ and $E_{7(-25)}$. There are 94 nonzero nilpotent orbits in $E_{7(7)}$; see [Dj1, Table XI, pp. 513-514].

Theorem 0.0.17. *Let the parametrization of the nilpotent orbits in $E_{7(7)}$ be as in [Dj1, Table XI, pp. 513-514]. Let X be a nonzero nilpotent element in $E_{7(7)}$.*

1. *If the parametrization of the orbit $\mathcal{O}_X$ is given by* 1011101*, then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 3$.

2. *Assume the parametrization of the orbit $\mathcal{O}_X$ is given by any of the sequences:* 1001001, 1101011, 1111010, 0101111, 2200022, 3101021, 1201013, 1211121, 2204022. *Then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 2$.

3. *Assume the parametrization of the orbit $\mathcal{O}_X$ is given by any of the sequences :* 0100010, 1100100, 0010011, 3000100, 0010003, 0102010, 0200020, 2004002, 2103101, 1013012, 2020202, 1311111, 1111131, 1310301, 1030131, 2220222, 3013131, 1313103, 3113121, 1213113, 4220224, 3413131, 1313143, 4224224. *Then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 1$.

4. *Assume the parametrization of the orbit $\mathcal{O}_X$ is given by any of the sequences:* 2000002, 0101010, 2002002, 1110111, 2020020, 0200202, 1112111, 2022020, 0202202, 2202022, 0220220. *Then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) \leq 1$.

5. *Assume the parametrization of the orbit $\mathcal{O}_X$ is given by any of the sequences:* 2010001, 1000102, 0120101, 1010210, 1030010, 0100301, 3013010, 0103103. *Then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) \leq 2$.

6. *If the parametrization of the orbit $\mathcal{O}_X$ is given by either* 1010101 *or* 0020200*, then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) \leq 3$.

7. *If $\mathcal{O}_X$ is not given by the parametrizations as in (1), (2), (3), (4), (5), (6) above (# of such orbits are 39), then we have* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 0$.

There are 37 nonzero nilpotent orbits in $E_{7(-5)}$; see [Dj1, Table XII, p. 515].

Theorem 0.0.18. *Let the parametrization of the nilpotent orbits in $E_{7(-5)}$ be as in [Dj1, Table XII, p. 515]. Let X be a nonzero nilpotent element in $E_{7(-5)}$.*

1. *If the parametrization of the orbit $\mathcal{O}_X$ is given by either* 110001 1 *or* 000120 2*, then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 2$.
2. *Assume the parametrization of the orbit $\mathcal{O}_X$ is given by any of the sequences:* 000010 1, 010000 2, 000010 3, 010010 1, 200100 0, 010100 2, 000200 0, 010110 1, 010030 1, 010110 3, 201031 4, 010310 3. *Then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 1$.
3. *If the parametrization of the orbit $\mathcal{O}_X$ is given by either* 020200 0 *or* 111110 1*, then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) \leq 2$.
4. *Assume the parametrization of the orbit $\mathcal{O}_X$ is given by any of the sequences:* 020000 0, 201011 2, 040000 4, 040400 4. *Then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) \leq 1$.
5. *If $\mathcal{O}_X$ is not given by the parametrizations as in (1), (2), (3), (4) above (# of such orbits are 17), then we have* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 0$.

There are 22 nonzero nilpotent orbits in $E_{7(-25)}$; see [Dj1, Table XIII, p. 516].

Theorem 0.0.19. *Let the parametrization of the nilpotent orbits in $E_{7(-25)}$ be as in [Dj1, Table XIII, p. 516]. Let X be a nonzero nilpotent element in $E_{7(-25)}$.*

1. *Assume the parametrization of the orbit $\mathcal{O}_X$ is given by any of the sequences:* 000000 2, 000000 -2, 000002 -2, 200000 -2, 200002 -2, 400000 -2, 000004 -6, 200002 -6, 400004 -6, 400004 -10. *Then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 0$.
2. *If $\mathcal{O}_X$ is not given by any of the above parametrization (# of such orbits are 12), then we have* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) \leq 1$.

Recall that up to conjugation there are two non-compact real forms of E_8. They are denoted by $E_{8(8)}$ and $E_{8(-24)}$. There are 115 nonzero nilpotent orbits in $E_{8(8)}$; see [Dj1, Table XIV, pp. 517-519].

Theorem 0.0.20. *Let the parametrization of the nilpotent orbits in $E_{8(8)}$ be as in [Dj1, Table XIV, pp. 517-519]. Let X be a nonzero nilpotent element in $E_{8(8)}$.*

1. *Assume the parametrization of the orbit $\mathcal{O}_X$ is given by any of the sequences:* 10010011, 11110010, 10111011, 11110130. *Then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 2$.

2. *Assume the parametrization of the orbit $\mathcal{O}_X$ is given by any of the sequences:* 01000010, 10001000, 30000001, 10010001, 01010010, 01000110, 10100100, 00100003, 11001030, 10110100, 21010100, 01020110, 30001030, 11010101, 11101011, 11010111, 11111101, 21031031, 31010211, 12111111, 13111101, 13111141, 13103041, 31131211, 13131043, 34131341.
   Then $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 1$.

3. *If the parametrization of the orbit $\mathcal{O}_X$ is given* 00100101*, then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) \leq 3$.

4. *Assume the parametrization of the orbit $\mathcal{O}_X$ is given by any of the sequences:* 10001002, 10101001, 01200100, 02000200, 10101021, 10102100, 02020200, 01201031. *Then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) \leq 2$.

5. *Assume the parametrization of the orbit $\mathcal{O}_X$ is given by any of the sequences:* 11000001, 20010000, 01000100, 11001010, 20100011, 01010100, 02020000, 20002000, 20100031, 10101011, 00200022, 11110110, 01011101, 01003001, 11101101, 11101121, 10300130, 04020200, 02002022, 00400040, 11121121, 30130130, 02022022, 40040040. *Then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) \leq 1$.

6. *If $\mathcal{O}_X$ is not given by the parametrizations as in (1), (2), (3), (4), (5) above (# of such orbits are* 52*), then we have* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 0$.

There are 36 nonzero nilpotent orbits in $E_{8(-24)}$; see [Dj1, Table XV, p. 520].

Theorem 0.0.21. *Let the parametrization of the nilpotent orbits in $E_{8(-24)}$ be as in [Dj1, Table XV, p. 520]. Let X be a nonzero nilpotent element in $E_{8(-24)}$.*

1. *Assume the parametrization of the orbit $\mathcal{O}_X$ is given by any of the sequences:* $0000001\,1$, $1000000\,2$, $0000001\,3$, $1000001\,1$, $1100000\,1$, $1000010\,2$, $0000012\,2$, $1000011\,1$, $1000011\,3$, $1000003\,1$, $0110001\,2$, $1010011\,1$, $1000031\,3$.
   Then $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 1$.

2. *If the parametrization of the orbit $\mathcal{O}_X$ is given by either* $2000000\,0$ *or* $2000020\,0$, *then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) \leq 1$.

3. *If $\mathcal{O}_X$ is not given by the parametrizations as in (1), (2) above (# of such orbits are 21), then we have* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 0$.

Theorems 0.0.10, 0.0.11, 0.0.12, 0.0.13, 0.0.14, 0.0.15, 0.0.16, 0.0.17, 0.0.18, 0.0.19, 0.0.20, 0.0.21 appear in the thesis and in [CM].

Chapter 1

Introduction

The nilpotent orbits in the semisimple Lie algebras, under the adjoint action of the associated semisimple Lie groups, form a rich class of homogeneous spaces. Such orbits are studied at the interface of several disciplines in mathematics such as Lie theory, symplectic geometry, representation theory, algebraic geometry. Various topological aspects of these orbits have drawn attention over the years; see [CoMc], [Mc] and references therein for an account. In this thesis we contribute by studying two specific topological invariants, namely the second and the first de Rham cohomology groups, of such orbits in non-compact, non-complex simple Lie algebras. The results of this thesis appear in [BCM] and in [CM].

To put our work in proper perspective we first recall that all orbits in a semisimple Lie algebra under the adjoint action are equipped with the Kostant-Kirillov two form. For complex semisimple Lie groups, a criterion was given in [ABB, Theorem 1.2] for the exactness of the Kostant-Kirillov form on an adjoint orbit of a semisimple element. In [BC1, Proposition 1.2] the above criterion was generalized to orbits of arbitrary elements under the adjoint action of real semisimple Lie groups with semisimple maximal compact subgroups. This in turn led the authors to the natural question of describing the full second cohomology group of such orbits. Towards this, in [BC1], the second cohomology group of the nilpotent orbits in all the complex simple Lie algebras, under the adjoint action of the corresponding complex group, are computed; see [BC1, Theorems 5.4, 5.5, 5.6, 5.11, 5.12]. The computations in [BC1] naturally motivate us to continue the program for the remaining cases consisting of non-complex, non-compact simple Lie algebras.

In this thesis we carry forward what is initiated in [BC1], and compute the second cohomology groups of the nilpotent orbits in real classical Lie algebras which are non-complex and non-compact; see Theorems 6.1.1, 6.2.1, 6.3.4, 6.4.8, 6.4.9, 6.5.4, 6.6.5, 6.7.1. In this thesis we also compute the second cohomology groups of the nilpotent orbits for most of the nilpotent orbits in exceptional Lie algebras in this set-up, and for the rest of cases of the nilpotent orbits, which are not covered in the above computations, upper bounds for the dimensions of the second cohomology groups are obtained; see Theorems 7.1.1, 7.2.1, 7.2.2, 7.3.1, 7.3.2, 7.3.3, 7.3.4, 7.4.1, 7.4.2, 7.4.3, 7.5.1, 7.5.2. The methods involved above also steered us studying the first cohomology groups in all the simple real Lie algebras; see Theorems 8.1.1, 8.1.2, 8.1.3, 8.1.4, 8.1.5, 8.1.6, 8.1.7, 8.2.1, 8.2.2, 8.2.3.

In [BC1] to facilitate the computations in complex simple Lie algebras a suitable description of the second cohomology group of any homogeneous space of a connected Lie group was obtained in [BC1, Theorem 3.3] under the assumption that all the maximal compact subgroups of the Lie group are semisimple; see [BC2] for a relatively simple proof of [BC1, Theorem 3.3]. As only the complex simple Lie groups were considered in [BC1], this condition was not restrictive because the maximal compact subgroups in complex simple Lie groups are in fact simple Lie groups. However, in the present case of non-complex simple Lie groups, the maximal compact subgroups are not necessarily semisimple, and hence [BC1, Theorem 3.3] can not be applied anymore, in general, to do the computations. This necessitates a description of the second cohomology groups of homogeneous spaces of Lie groups without any imposed conditions on the maximal compact subgroups therein, and this is formulated in Theorem 5.1.3, generalizing [BC1, Theorem 3.3]. In Theorem 5.1.6 we also obtain a description of the first cohomology groups in the same general set-up as above.

We next briefly mention the strategy in our computations. As a preparatory step, we apply Theorem 5.1.3 to derive in Theorem 5.2.2 that the second and the first cohomology groups of nilpotent orbits in simple Lie algebras are closely related to the maximal compact subgroups of certain subgroups associated to a copy of $\mathfrak{sl}_2(\mathbb{R})$ containing the nilpotent element. Thus, in view of Theorem 5.2.2, the main goal in our computations of the second cohomology groups of the nilpotent orbits in real classical non-complex non-compact Lie algebras is to obtain the descriptions of how certain suitable maximal compact subgroups in the centralizers of the nilpotent elements are embedded in certain explicit maximal compact subgroups of the ambient simple Lie groups; see Remark 6.0.3 for some more details in this regard.

We now briefly outline the chapter-wise content of this thesis.

Chapter 2 is devoted to some notation, conventions, and background materials which will be used throughout the thesis.

In Chapter 3 we work out certain details on the structures of the nilpotent elements in the classical Lie algebras, and then combine them in Proposition 3.0.3 and Proposition 3.0.7. It should be noted that when $\mathbb{D}$ is either $\mathbb{R}$ or $\mathbb{C}$ the above propositions also follow from [SS]. However, the non-commutativity of $\mathbb{H}$ creates technical difficulties in extending the results to the case $\mathbb{D} = \mathbb{H}$ by directly implementing the proofs as in [SS]. Taking cues from [CoMc, §9.3, p.139] we take a different approach in the proofs by appealing to the Jacobson-Morozov theorem and the basic results on the structures of finite dimensional representations of $\mathfrak{sl}_2(\mathbb{R})$.

Chapter 4 deals with the parametrization of the nilpotent orbits. In §4.1 we elaborate in greater detail the parametrization of the nilpotent orbits in real non-complex non-compact classical Lie algebras which are given in [CoMc, §9.3]. Following [Dj1] and [Dj2], in §4.2 we describe the parametrization of the nilpotent orbits in real non-complex non-compact exceptional Lie algebras.

In Chapter 5 we give a convenient descriptions of the second and first cohomology groups of a homogeneous spaces of Lie group which are suitable for the purpose of computations in later chapters. Theorems 5.1.3 and 5.1.6 are the main results of this chapter, and they hold under the mild restriction that the stabilizer of any point in the homogeneous space has finitely many connected components. Although the general theories of cohomology groups of (compact) homogeneous spaces are widely studied in the past (see, for example, [Bo2], [CE], [Sp], [GHV]) we are unable to locate Theorems 5.1.3 and 5.1.6 in the existing literature to the best of our knowledge. The results are also of independent interest in view of its applicability to a very large class of homogeneous spaces. In the second section, we apply Theorems 5.1.3 and 5.1.6 to derive Theorem 5.2.2 which is key to our computations of the second and first cohomology groups of the nilpotent orbits done in the next chapters. Theorem 5.2.2 describes the second and the first cohomology groups of the nilpotent orbits in simple Lie algebras in terms of a maximal compact subgroup of the centralizer of a $\mathfrak{sl}_2(\mathbb{R})$-triple containing the nilpotent element and a maximal compact subgroup of the associated ambient Lie group containing the former one. As an interesting consequence of Theorem 5.2.2 it follows that the first cohomology group of any nilpotent orbit in a simple Lie algebra is at the most one dimensional.

In Chapter 6 the second cohomology groups of the nilpotent orbits in non-compact non-complex classical real Lie algebras are computed; see Theorems 6.1.1, 6.2.1, 6.3.4, 6.4.8, 6.4.9, 6.5.4, 6.6.5 and 6.7.1. The results are described in terms of the parametrizations given in §4.1. In particular, our computations yield that the second cohomology groups vanish for all the nilpotent orbits in $\mathfrak{sl}_n(\mathbb{H})$. In view of

Theorem 5.2.2, the main goal in our computations is to obtain the descriptions of how certain conjugates of suitable maximal compact subgroups in the centralizers of the nilpotent elements are embedded in certain explicit maximal compact subgroups of the ambient semisimple Lie groups; see Remark 6.0.3 for more explanations.

In Chapter 7 we consider non-compact non-complex exceptional Lie algebras and compute the dimensions of the second cohomology groups for most of the nilpotent orbits. For the rest of cases of the nilpotent orbits in the exceptional Lie algebras, which are not covered in the above computations, we obtain upper bounds for the dimensions of the second cohomology groups; see Theorems 7.1.1, 7.2.1, 7.2.2, 7.3.1, 7.3.2, 7.3.3, 7.3.4, 7.4.1, 7.4.2, 7.4.3, 7.5.1, 7.5.2. As in the classical case the computations here also use Theorem 5.2.2 crucially. In particular, we obtain that the second cohomology groups vanish for all the nilpotent orbits in $F_{4(-20)}$ and $E_{6(-26)}$.

The final chapter, namely Chapter 8, is devoted to computing the first cohomology groups of the nilpotent orbits. We begin this chapter by recording a simple observation that the first cohomology of all the nilpotent orbits vanish in the case of complex simple Lie algebras; see Theorem 8.0.1. The methods involved in Chapter 6 also led us describe the first cohomology groups of the nilpotent orbits in all the non-compact non-complex real classical Lie algebras; see Theorems 8.1.1, 8.1.2, 8.1.3, 8.1.4, 8.1.5, 8.1.6, 8.1.7. Our computations yield that the first cohomology groups vanish for all the nilpotent orbits in $\mathfrak{sl}_n(\mathbb{H})$, $\mathfrak{sp}(p,q)$, $\mathfrak{sl}_n(\mathbb{R})$ for $n \geq 3$. In Theorems 8.2.1, we prove that the first cohomology groups vanish for all the nilpotent orbits in a non-compact non-complex real exceptional Lie algebra $\mathfrak{g}$ where $\mathfrak{g} \not\simeq E_{6(-14)}$ and $\mathfrak{g} \not\simeq E_{7(-25)}$. The results in Theorem 8.2.2 and Theorem 8.2.3 give us either the exact dimensions or the bounds on the dimensions of the first cohomology groups of the nilpotent orbits in $E_{6(-14)}$ and $E_{7(-25)}$.

Chapter 2

Preliminaries

In this chapter we assemble some notation, conventions, and backgrounds that will be used throughout in this thesis. A few specialized notation are mentioned as and when they occur later. We also provide a proof of the well-known Jacobson-Morozov theorem.

2.1 Some general notation

Once and for all fix a square root of -1 and call it $\sqrt{-1}$. The *center* of a group G is denoted by $\mathcal{Z}(G)$ while the *center* of a Lie algebra $\mathfrak{g}$ is denoted by $\mathfrak{z}(\mathfrak{g})$. The Lie groups will be denoted by the capital letters, while the Lie algebra of a Lie group will be denoted by the corresponding lower case German letter, unless a different notation is explicitly mentioned. Sometimes, for notational convenience, the Lie algebra of a Lie group G is also denoted by $\mathrm{Lie}(G)$. The connected component of G containing the identity element is denoted by G°. For a subgroup H of G and a subset S of $\mathfrak{g}$, the subgroup of H that fixes S pointwise under the adjoint action is called the *centralizer* of S in H; the centralizer of S in H is denoted by $\mathcal{Z}_H(S)$. Similarly, for a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a subset $S \subset \mathfrak{g}$, by $\mathfrak{z}_{\mathfrak{h}}(S)$ we will denote the subalgebra of $\mathfrak{h}$ consisting of all the elements that commute with every element of S.

Let Γ be a group acting linearly on a vector space V. The subspace of V fixed pointwise by the action of Γ is denoted by V^Γ. If G is a Lie group with Lie algebra $\mathfrak{g}$, then it is immediate that the adjoint (respectively, coadjoint) action of G° on $\mathfrak{z}(\mathfrak{g})$ (respectively, $\mathfrak{z}(\mathfrak{g})^*$) is trivial; in particular, one obtains a natural action of G/G° on $\mathfrak{z}(\mathfrak{g})$ (respectively, $\mathfrak{z}(\mathfrak{g})^*$). We denote by $[\mathfrak{z}(\mathfrak{g})]^{G/G^\circ}$ (respectively, $[\mathfrak{z}(\mathfrak{g})^*]^{G/G^\circ}$) the space of points of $\mathfrak{z}(\mathfrak{g})$ (respectively, of $\mathfrak{z}(\mathfrak{g})^*$) fixed pointwise under the action of G/G°.

Let G be a semisimple Lie group with Lie algebra $\mathfrak{g}$. Consider the linear endomorphism

$$\mathrm{ad}_X \,:\, \mathfrak{g} \longrightarrow \mathfrak{g}\,, \quad Y \longmapsto [X, Y]\,.$$

An element $X \in \mathfrak{g}$ is called *semisimple* if the linear endomorphism ad_X is semisimple. An element $X \in \mathfrak{g}$ is called *nilpotent* if the linear endomorphism ad_X is nilpotent. The *set of nilpotent elements* in $\mathfrak{g}$ is denoted by $\mathcal{N}_\mathfrak{g}$. Consider the adjoint representation

$$\mathrm{Ad} \,:\, G \longrightarrow \mathrm{GL}(\mathfrak{g})$$

of G on $\mathfrak{g}$. The *adjoint orbit* of $X \in \mathfrak{g}$ is defined by $\mathcal{O}_X := \{\mathrm{Ad}(g)X \mid g \in G\}$. A *semisimple orbit* in $\mathfrak{g}$ is an adjoint orbit of a semisimple element X in $\mathfrak{g}$. A *nilpotent orbit* in $\mathfrak{g}$ is an adjoint orbit of a nilpotent element X in $\mathfrak{g}$. The *set of all nilpotent orbits in* $\mathfrak{g}$ *under the adjoint action of* G is denoted by $\mathcal{N}(G)$.

2.2 Partitions and (signed) Young diagrams

An *ordered set of order* n is a n-tuple $(v_1, \ldots, v_n)$, where $v_1, \ldots, v_n$ are elements of some set, such that $v_i \neq v_j$ if $i \neq j$. If $w \in \{v_1, \ldots, v_n\}$, then write

$w \in (v_1, \ldots, v_n)$. For two ordered sets $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_m)$, the ordered set $(v_1, \ldots, v_n, w_1, \ldots, w_m)$ will be denoted by $(v_1, \ldots, v_n) \vee (w_1, \ldots, w_m)$. Furthermore, for k-many ordered sets $(v^i_1, \ldots, v^i_{n_i})$, $1 \leq i \leq k$, define the ordered set $(v^1_1, \ldots, v^1_{n_1}) \vee \cdots \vee (v^k_1, \ldots, v^k_{n_k})$ to be the following juxtaposition of ordered sets $(v^i_1, \ldots, v^i_{n_i})$ with increasing i:

$$(v^1_1, \ldots, v^1_{n_1}) \vee \cdots \vee (v^k_1, \ldots, v^k_{n_k}) := (v^1_1, \ldots, v^1_{n_1}, \ldots, v^k_1, \ldots, v^k_{n_k}) .$$

By a *partition* of a positive integer n we mean the symbol $[d_1^{t_1}, \ldots, d_s^{t_s}]$, where $t_i, d_i \in \mathbb{N}$, $1 \leq i \leq s$, such that $\sum_{i=1}^{s} t_i d_i = n$, $t_i \geq 1$ and $d_{i+1} > d_i > 0$ for all i; see [CoMc, § 3.1, p. 30]. If $[d_1^{t_1}, \ldots, d_s^{t_s}]$ is a partition of n in the above sense then t_i is called the *multiplicity* of d_i. Henceforth, the multiplicity of d_i will be denoted by t_{d_i}; this is to avoid any ambiguity. Let $\mathcal{P}(n)$ denote the *set of all partitions of* n. For a partition $\mathbf{d} = [d_1^{t_{d_1}}, \ldots, d_s^{t_{d_s}}]$ of n, define

$$\mathbb{N}_{\mathbf{d}} := \{d_i \mid 1 \leq i \leq s\}, \quad \mathbb{E}_{\mathbf{d}} := \mathbb{N}_{\mathbf{d}} \cap (2\mathbb{N}), \quad \mathbb{O}_{\mathbf{d}} := \mathbb{N}_{\mathbf{d}} \setminus \mathbb{E}_{\mathbf{d}} . \tag{2.1}$$

Further define

$$\mathbb{O}^1_{\mathbf{d}} := \{d \mid d \in \mathbb{O}_{\mathbf{d}},\, d \equiv 1 \pmod 4\}, \; \mathbb{O}^3_{\mathbf{d}} := \{d \mid d \in \mathbb{O}_{\mathbf{d}},\, d \equiv 3 \pmod 4\}. \tag{2.2}$$

Following [CoMc, Theorem 9.3.3], a partition $\mathbf{d}$ of n will be called *even* if $\mathbb{N}_{\mathbf{d}} = \mathbb{E}_{\mathbf{d}}$. Let $\mathcal{P}_{\text{even}}(n)$ be the subset of $\mathcal{P}(n)$ consisting of all even partitions of n. We call a partition $\mathbf{d}$ of n to be *very even* if

- $\mathbf{d}$ is even, and
- t_η is even for all $\eta \in \mathbb{N}_{\mathbf{d}}$.

Let $\mathcal{P}_{\text{v.even}}(n)$ be the subset of $\mathcal{P}(n)$ consisting of all very even partitions of n. Now define

$$(2.3) \qquad \mathcal{P}_1(n) := \{\mathbf{d} \in \mathcal{P}(n) \mid t_\eta \text{ is even for all } \eta \in \mathbb{E}_{\mathbf{d}}\}$$

and

$$(2.4) \qquad \mathcal{P}_{-1}(n) := \{\mathbf{d} \in \mathcal{P}(n) \mid t_\theta \text{ is even for all } \theta \in \mathbb{O}_{\mathbf{d}}\}.$$

Clearly, we have $\mathcal{P}_{\text{v.even}}(n) \subset \mathcal{P}_1(n)$.

Following [CoMc, p. 140] we define a *Young diagram* to be a left-justified array of rows of empty boxes arranged so that no row is shorter than the one below it; the *size* of a Young diagram is the number of empty boxes appearing in it. There is an obvious correspondence between the set of Young diagrams of size n and the set $\mathcal{P}(n)$ of partitions of n. Hence the *set of Young diagrams of size* n is also denoted by $\mathcal{P}(n)$. A *signed Young diagram* is a Young diagram in which every box is labeled with $+1$ or -1 such that the sign of 1 alternate across rows except when the length of the row is of the form $3 \pmod 4$. In the latter case when the length of the row is of the form $3 \pmod 4$ we will alternate the sign of 1 till the last but one and repeat the sign of 1 in the last box as in the last but one box; see Remark 3.0.16 why the choices of signs in this case deviate from that in the previous cases. The sign of 1 need not alternate down columns. Two signed Young diagrams are equivalent if and only if each can be obtained from the other by permuting rows of equal length. The *signature of a signed Young diagram* is the ordered pair of integers (p, q) where p-many $+1$ and q-many -1 occur in it.

We next define certain sets using collections of matrices with entries comprising of signs ± 1, which are easily seen to be in bijection with sets of equivalence classes of various types of signed Young diagrams. These sets will be used in parametrizing the nilpotent orbits in the classical Lie algebras.

For a partition $\mathbf{d} \in \mathcal{P}(n)$ and $d \in \mathbb{N}_{\mathbf{d}}$, we define the subset $\mathbf{A}_d \subset \mathrm{M}_{t_d \times d}(\mathbb{C})$ of matrices (m^d_{ij}) with entries in the set $\{\pm 1\}$ as follows :

$$\mathbf{A}_d := \{(m^d_{ij}) \in \mathrm{M}_{t_d \times d}(\mathbb{C}) \mid (m^d_{ij}) \text{ satisfies } (\mathbf{Yd}.1) \text{ and } (\mathbf{Yd}.2)\}\,. \tag{2.5}$$

Yd.1 There is an integer $0 \leq p_d \leq t_d$ such that

$$m^d_{i1} := \begin{cases} +1 & \text{if } 1 \leq i \leq p_d \\ -1 & \text{if } p_d < i \leq t_d. \end{cases}$$

Yd.2

$$m^d_{ij} := (-1)^{j+1} m^d_{i1} \qquad \text{if } 1 < j \leq d,\ d \in \mathbb{E}_{\mathbf{d}} \cup \mathbb{O}^1_{\mathbf{d}};$$
$$m^d_{ij} := \begin{cases} (-1)^{j+1} m^d_{i1} & \text{if } 1 < j \leq d-1 \\ -m^d_{i1} & \text{if } j = d \end{cases},\ d \in \mathbb{O}^3_{\mathbf{d}}\,.$$

For any $(m^d_{ij}) \in \mathbf{A}_d$ set

$$\mathrm{sgn}_+(m^d_{ij}) := \#\{(i,j) \mid 1 \leq i \leq t_d,\ 1 \leq j \leq d,\ m^d_{ij} = +1\}$$

and

$$\mathrm{sgn}_-(m^d_{ij}) := \#\{(i,j) \mid 1 \leq i \leq t_d,\ 1 \leq j \leq d,\ m^d_{ij} = -1\}\,.$$

Remark 2.2.1. Form the above definitions of **Yd**.1 and **Yd**.2 the following observations are straightforward. For $d \in \mathbb{N}_{\mathbf{d}}$, let $M_d := (m^d_{ij}) \in \mathbf{A}_d$ (see (2.5) for the definition of $\mathbf{A}_d$).

1. If $d \in \mathbb{E}_{\mathbf{d}}$, then we have

$$\mathrm{sgn}_+(M_d) = t_d d/2 \quad \text{and} \quad \mathrm{sgn}_-(M_d) = t_d d/2\,.$$

2. If $d \in \mathbb{O}^1_{\mathbf{d}}$, then we have

$$\mathrm{sgn}_+(M_d) = (t_d d + p_d - q_d)/2 \quad \text{and} \quad \mathrm{sgn}_-(M_d) = (t_d d - p_d + q_d)/2\,.$$

3. If $d \in \mathbb{O}^3_{\mathbf{d}}$, then we have

$$\mathrm{sgn}_+(M_d) = (t_d d - p_d + q_d)/2 \quad \text{and} \quad \mathrm{sgn}_-(M_d) = (t_d d + p_d - q_d)/2\,.$$

□

Let $\mathcal{S}_{\mathbf{d}}(n) := \mathbf{A}_{d_1} \times \cdots \times \mathbf{A}_{d_s}$. Now define the subset $\mathcal{S}_{\mathbf{d}}(p,q) \subset \mathcal{S}_{\mathbf{d}}(n)$ by

$$\mathcal{S}_{\mathbf{d}}(p,q) := \Big\{(M_{d_1},\ldots,M_{d_s}) \in \mathcal{S}_{\mathbf{d}}(n) \mid \sum_{i=1}^{s} \mathrm{sgn}_+ M_{d_i} = p,\ \sum_{i=1}^{s} \mathrm{sgn}_- M_{d_i} = q\Big\} \tag{2.6}$$

where $p+q=n$. For a pair of non-negative integers (p,q) define

$$\mathcal{Y}(p,\, q) := \{(\mathbf{d},\, \mathbf{sgn}) \mid \mathbf{d} \in \mathcal{P}(n),\ \mathbf{sgn} \in \mathcal{S}_{\mathbf{d}}(p,\, q)\}. \tag{2.7}$$

It is easy to see that there is a natural bijection between the set $\mathcal{Y}(p,\, q)$ and the equivalence classes of signed Young diagrams of size $p+q$ with signature $(p,\, q)$. Hence, we will call $\mathcal{Y}(p,\, q)$ the *set of equivalence classes of signed Young diagrams of size $p+q$ with signature* $(p,\, q)$.

For any $\mathbf{d} \in \mathcal{P}(n)$ and $d \in \mathbb{N}_{\mathbf{d}}$, define the subset $\mathbf{A}_{d,1}$ of $\mathbf{A}_d$ by

$$\mathbf{A}_{d,1} := \{(m^d_{ij}) \in \mathbf{A}_d \mid m^d_{i1} = +1 \ \ \forall\ 1 \leq i \leq t_d\}\,.$$

Further define $\mathcal{S}^{\mathrm{even}}_{\mathbf{d}}(p,q) \subset \mathcal{S}_{\mathbf{d}}(p,q)$ and $\mathcal{S}^{\mathrm{odd}}_{\mathbf{d}}(n) \subset \mathcal{S}_{\mathbf{d}}(n)$ by

$$\mathcal{S}^{\mathrm{even}}_{\mathbf{d}}(p,\, q) := \{(M_{d_1},\, \ldots,\, M_{d_s}) \in \mathcal{S}_{\mathbf{d}}(p,\, q) \mid M_\eta \in \mathbf{A}_{\eta,1} \ \ \forall\ \ \eta \in \mathbb{E}_{\mathbf{d}}\} \tag{2.8}$$

and

$$\text{(2.9)} \qquad \mathcal{S}^{\text{odd}}_{\mathbf{d}}(n) := \{(M_{d_1}, \ldots, M_{d_s}) \in \mathcal{S}_{\mathbf{d}}(n) \mid M_\theta \in \mathbf{A}_{\theta,1} \ \ \forall \ \theta \in \mathbb{O}_{\mathbf{d}}\}.$$

For a pair (p,q) of non-negative integers we define the sets $\mathcal{Y}^{\text{even}}(p,q)$ and $\mathcal{Y}^{\text{even}}_1(p,q)$ by

$$\text{(2.10)} \qquad \mathcal{Y}^{\text{even}}(p,q) := \{(\mathbf{d}, \mathbf{sgn}) \mid \ \mathbf{d} \in \mathcal{P}(n), \ \mathbf{sgn} \in \mathcal{S}^{\text{even}}_{\mathbf{d}}(p,q)\},$$

$$\text{(2.11)} \qquad \mathcal{Y}^{\text{even}}_1(p,q) := \{(\mathbf{d}, \mathbf{sgn}) \mid \ \mathbf{d} \in \mathcal{P}_1(n), \ \mathbf{sgn} \in \mathcal{S}^{\text{even}}_{\mathbf{d}}(p,q)\}.$$

Similarly, for a non-negative integer n, set

$$\text{(2.12)} \qquad \mathcal{Y}^{\text{odd}}(n) := \{(\mathbf{d}, \mathbf{sgn}) \mid \ \mathbf{d} \in \mathcal{P}(n), \ \ \mathbf{sgn} \in \mathcal{S}^{\text{odd}}_{\mathbf{d}}(n)\},$$

$$\text{(2.13)} \qquad \mathcal{Y}^{\text{odd}}_{-1}(2n) := \{(\mathbf{d}, \mathbf{sgn}) \mid \mathbf{d} \in \mathcal{P}_{-1}(2n), \ \ \mathbf{sgn} \in \mathcal{S}^{\text{odd}}_{\mathbf{d}}(2n)\}.$$

Let $\mathbf{d} \in \mathcal{P}(n)$. For $\theta \in \mathbb{O}_{\mathbf{d}}$ and $M_\theta := (m^\theta_{rs}) \in \mathbf{A}_\theta$, define

$$l^+_{\theta,i}(M_\theta) := \#\{j \mid m^\theta_{ij} = +1\} \quad \text{and} \quad l^-_{\theta,i}(M_\theta) := \#\{j \mid m^\theta_{ij} = -1\}$$

for all $1 \le i \le t_\theta$; set

(2.14)

$$\mathcal{S}'_{\mathbf{d}}(p,q) := \left\{ (M_{d_1}, \ldots, M_{d_s}) \in \mathcal{S}^{\text{even}}_{\mathbf{d}}(p,q) \;\middle|\; \begin{array}{c} l^+_{\theta,i}(M_\theta) \text{ is even } \forall\, \theta \in \mathbb{O}_{\mathbf{d}},\ 1 \le i \le t_\theta \\ \text{or } l^-_{\theta,i}(M_\theta) \text{ is even } \forall\, \theta \in \mathbb{O}_{\mathbf{d}},\ 1 \le i \le t_\theta \end{array} \right\}.$$

2.3 Hermitian forms and associated groups

The notation $\mathbb{D}$ will stand for either $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$. We define the *usual conjugations* σ_c on $\mathbb{C}$ by $\sigma_c(x_1+\sqrt{-1}x_2) = x_1-\sqrt{-1}x_2$, and on $\mathbb{H}$ by $\sigma_c(x_1+\mathbf{i}x_2+\mathbf{j}x_3+\mathbf{k}x_4) = x_1-\mathbf{i}x_2-\mathbf{j}x_3-\mathbf{k}x_4$, $x_i \in \mathbb{R}$ for $i = 1,\dots,4$.

We now take a right vector space V defined over $\mathbb{D}$. Let $\mathrm{End}_{\mathbb{D}}(V)$ be the right $\mathbb{R}$-algebra of $\mathbb{D}$-*linear endomorphisms* of V, and let GL(V) be the *group of invertible elements* of $\mathrm{End}_{\mathbb{D}}(V)$. For a $\mathbb{D}$-linear endomorphism $T \in \mathrm{End}_{\mathbb{D}}(V)$ and an ordered $\mathbb{D}$-basis $\mathcal{B}$ of V, the *matrix of* T *with respect to* $\mathcal{B}$ is denoted by $[T]_{\mathcal{B}}$. Recall that if $\mathbb{D} = \mathbb{C}$ then $\mathrm{End}_{\mathbb{D}}(V)$ is also a (right) $\mathbb{C}$-algebra. When $\mathbb{D}$ is either $\mathbb{R}$ or $\mathbb{C}$, let

$$\mathrm{tr} : \mathrm{End}_{\mathbb{D}}(V) \longrightarrow \mathbb{D} \quad \text{and} \quad \det : \mathrm{End}_{\mathbb{D}}V \longrightarrow \mathbb{D}$$

respectively be the usual *trace* and *determinant* maps. Let A be a central simple $\mathbb{R}$-algebra. Let

$$\mathrm{Nrd}_A : A \longrightarrow \mathbb{R}$$

be the *reduced norm* on A, and let $\mathrm{Trd}_A : A \longrightarrow \mathbb{R}$ be the *reduced trace* on A.

Remark 2.3.1. Let $\mathbb{F}$ be a field and $\overline{\mathbb{F}}$ be the algebraic closure of $\mathbb{F}$. Recall that if A is a central simple algebra over $\mathbb{F}$, then there is an isomorphism $\phi\colon A\otimes_{\mathbb{F}}\overline{\mathbb{F}} \longrightarrow \mathrm{M}_n(\overline{\mathbb{F}})$ where $n^2 = \dim_{\mathbb{F}} A$. Thus we have

$$A \hookrightarrow \mathrm{M}_n(\overline{\mathbb{F}}), \quad a \longmapsto \phi(a\otimes 1).$$

Recall that the reduced norm Nrd_A and the reduced trace Trd_A on A are defined by, $\mathrm{Nrd}_A(a) := \det\phi(a\otimes 1)$ and $\mathrm{Trd}_A := \mathrm{tr}(\phi(a\otimes 1))$, respectively. By Skolem-Noether theorem the above definitions do not depend on the isomorphism ϕ. We now consider the specific case of the matrix algebra $A := \mathrm{M}_n(\mathbb{H})$ over $\mathbb{H}$. As the center of $\mathbb{H}$ is $\mathbb{R}$ it is easy to see that $\mathrm{M}_n(\mathbb{H})$ is a central simple algebra over $\mathbb{R}$.

Consider the $\mathbb{R}$-algebra embedding $\lambda : \mathrm{M}_n(\mathbb{H}) \to \mathrm{M}_{2n}(\mathbb{C})$ given by

$$\lambda(P) := \begin{pmatrix} P_1 & -\sigma_c(P_2) \\ P_2 & \sigma_c(P_1) \end{pmatrix},$$

where $P_1, P_2 \in \mathrm{M}_n(\mathbb{C})$ such that $P = P_1 + \mathbf{j}P_2$. It is easy to see that the above $\mathbb{R}$-algebra embedding λ induces a $\mathbb{C}$-algebra isomorphism between $\mathrm{M}_n(\mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathrm{M}_{2n}(\mathbb{C})$. Thus the reduced norm $\mathrm{Nrd}_{\mathrm{M}_n(\mathbb{H})}$ and reduced trace $\mathrm{Trd}_{\mathrm{M}_n(\mathbb{H})}$ on $\mathrm{M}_n(\mathbb{H})$ are given by

$$\mathrm{Nrd}_{\mathrm{M}_n(\mathbb{H})}(P_1 + \mathbf{j}P_2) := \det \begin{pmatrix} P_1 & -\sigma_c(P_2) \\ P_2 & \sigma_c(P_1) \end{pmatrix}$$

and

$$\mathrm{Trd}_{\mathrm{M}_n(\mathbb{H})}(P_1 + \mathbf{j}P_2) := \mathrm{tr} \begin{pmatrix} P_1 & -\sigma_c(P_2) \\ P_2 & \sigma_c(P_1) \end{pmatrix} = \mathrm{tr}(P_1 + \sigma_c(P_1)) = 2\mathrm{Re}(\mathrm{tr}(P)).$$

Thus it is immediate that $\mathrm{Trd}_{\mathrm{M}_n(\mathbb{H})}(\mathrm{M}_n(\mathbb{H})) \subset \mathbb{R}$. Now observe that, if $P_1, P_2 \in \mathrm{M}_n(\mathbb{C})$ then $\mathbf{j}(P_1+\mathbf{j}P_2)\mathbf{j}^{-1} = \sigma_c(P_1)+\mathbf{j}\sigma_c(P_2)$. Thus $\det(\lambda(P_1+\mathbf{j}P_2)) = \sigma_c(\det(\lambda(P_1+\mathbf{j}P_2)))$. This proves that $\mathrm{Nrd}_{\mathrm{M}_n(\mathbb{H})}(\mathrm{M}_n(\mathbb{H})) \subset \mathbb{R}$. These facts also follow from the generalities in the theory of central simple algebras. $\square$

We now define the associated groups. When $\mathbb{D} = \mathbb{R}$ or $\mathbb{C}$, define

$$\mathrm{SL}(V) := \{z \in \mathrm{GL}(V) \mid \det(z) = 1\} \text{ and } \mathfrak{sl}(V) := \{y \in \mathrm{End}_{\mathbb{D}}(V) \mid \mathrm{tr}(y) = 0\}.$$

If $\mathbb{D} = \mathbb{H}$ then recall that $\mathrm{End}_{\mathbb{D}}(V)$ is a central simple $\mathbb{R}$-algebra. In that case, define

$$\mathrm{SL}(V) := \{z \in \mathrm{GL}(V) \mid \mathrm{Nrd}_{\mathrm{End}_{\mathbb{D}} V}(z) = 1\}$$

and

$$\mathfrak{sl}(V) := \{y \in \mathrm{End}_{\mathbb{D}}(V) \mid \mathrm{Trd}_{\mathrm{End}_{\mathbb{D}}(V)}(y) = 0\}.$$

Let $\mathbb{D}$ be $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, as above. Let σ be either the identity map Id or an *involution* of $\mathbb{D}$, meaning σ is $\mathbb{R}$-linear with $\sigma^2 = \mathrm{Id}$ and $\sigma(xy) = \sigma(y)\sigma(x)$ for all $x,\, y \in \mathbb{D}$. Let $\epsilon = \pm 1$. Following [Bo1, § 23.8, p. 264] we call a map

$$\langle \cdot,\, \cdot \rangle :\, V \times V \longrightarrow \mathbb{D}$$

a ϵ-σ *Hermitian form* if

- $\langle \cdot,\, \cdot \rangle$ is additive in each argument,
- $\langle v,\, u \rangle = \epsilon\sigma(\langle u, v\rangle)$, and
- $\langle v\alpha,\, u \rangle = \sigma(\alpha)\langle u,\, v \rangle$ for all $u,\, v \in V$ and for all $\alpha \in \mathbb{D}$.

A ϵ-σ Hermitian form $\langle \cdot,\, \cdot \rangle$ is called *non-degenerate* if $\langle v,\, u \rangle = 0$ for all v if and only if $u = 0$. All ϵ-σ Hermitian forms considered here will be assumed to be non-degenerate.

We define

$$\mathrm{U}(V,\, \langle \cdot,\, \cdot \rangle) := \{T \in \mathrm{GL}(V) \mid \langle Tv,\, Tu \rangle = \langle v,\, u \rangle \;\; \forall \;\; v, u \in V\}$$

and

$$\mathfrak{u}(V,\, \langle \cdot,\, \cdot \rangle) := \{T \in \mathrm{End}_{\mathbb{D}}(V) \mid \langle Tv,\, u \rangle + \langle v,\, Tu \rangle = 0 \;\; \forall \;\; v, u \in V\}\,.$$

We next define

$$\text{(2.15)} \quad \mathrm{SU}(V, \langle \cdot, \cdot \rangle) := \mathrm{U}(V, \langle \cdot, \cdot \rangle) \cap \mathrm{SL}(V) \;\text{ and }\; \mathfrak{su}(V, \langle \cdot, \cdot \rangle) := \mathfrak{u}(V, \langle \cdot, \cdot \rangle) \cap \mathfrak{sl}(V)\,.$$

Recall that $\mathfrak{su}(V,\, \langle \cdot,\, \cdot \rangle)$ is a simple Lie algebra (cf. [Kn, Chapter I, Section 8]).

If $\mathbb{D} = \mathbb{C}$, then multiplication by $\sqrt{-1}$ sends the non-degenerate ϵ-σ Hermitian forms on V with $\epsilon = -1$, $\sigma \neq \mathrm{Id}$ to the non-degenerate ϵ-σ Hermitian forms with $\epsilon = 1$, $\sigma \neq \mathrm{Id}$, and this mapping is a bijection. Hence, when $\mathbb{D} = \mathbb{C}$ and $\sigma \neq \mathrm{Id}$, we consider only the case where $\epsilon = 1$. If $\mathbb{D} = \mathbb{H}$ and $\sigma = \mathrm{Id}$, then it can be easily seen that $\langle \cdot, \cdot \rangle \equiv 0$. As $\langle \cdot, \cdot \rangle$ is assumed to be non-degenerate, this, in particular, implies that there is no form $\langle \cdot, \cdot \rangle$ on V with $\mathbb{D} = \mathbb{H}$, $\sigma = \mathrm{Id}$.

Define $|z| := (z\sigma_c(z))^{1/2}$, for $z \in \mathbb{D}$. For $\alpha \in \mathbb{H}^*$ define

$$C_\alpha : \mathbb{H} \longrightarrow \mathbb{H}, \quad x \longmapsto \alpha x \alpha^{-1}.$$

Clearly C_α is a $\mathbb{R}$-algebra automorphism, and $C_\alpha = C_{\alpha/|\alpha|}$. When $\mathbb{D} = \mathbb{H}$, the following facts justify that it is enough to consider the involution σ_c instead of arbitrary involutions. The proof of the next lemma is a straightforward application of Skolem-Noether theorem which can be found in [Bo1, § 23.7, p. 262].

Lemma 2.3.2 (cf. [Bo1, § 23.7, p. 262])**.** *Let $\sigma : \mathbb{H} \longrightarrow \mathbb{H}$ be $\mathbb{R}$-linear with $\sigma(xy) = \sigma(y)\sigma(x)$ for all $x, y \in \mathbb{H}$. Then σ is an involution, meaning $\sigma^2 = \mathrm{Id}$, if and only if either $\sigma = \sigma_c$ or $\sigma = C_\alpha \circ \sigma_c$ for some α with $\alpha^2 = -1$.*

Lemma 2.3.3 (cf. [Bo1, § 23.8, p. 264])**.** *Let $\sigma : \mathbb{H} \longrightarrow \mathbb{H}$ be an involution, $\epsilon = \pm 1$ and*

$$\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{D}$$

a ϵ-σ Hermitian form. Let $\delta = \pm 1$ and $\alpha \in \mathbb{H}$ such that $|\alpha| = 1$ and $\alpha\sigma(\alpha)^{-1} = \delta$. Then $\alpha\langle \cdot, \cdot \rangle$ is a $\delta\epsilon$-$C_\alpha \circ \sigma$ Hermitian form.

As a consequence of Lemma 2.3.3 if $\sigma : \mathbb{H} \longrightarrow \mathbb{H}$ is an involution, $\epsilon = \pm 1$ and

$$\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{D}$$

a ϵ-σ Hermitian form, then $\alpha\langle \cdot, \cdot \rangle$ is a ϵ-σ_c Hermitian form with $\alpha \in \mathbb{H}$ being such that $\sigma = C_\alpha \circ \sigma_c$ and $\alpha^2 = -1$ (as in Lemma 2.3.2). In particular, an immediate consequence is that if σ, ϵ and $\langle\cdot,\,\cdot\rangle$ are as above, then there exists a ϵ-σ_c Hermitian form, say, $\langle\cdot,\,\cdot\rangle' : V \times V \longrightarrow \mathbb{D}$ such that $\mathrm{SU}(V,\,\langle\cdot,\,\cdot\rangle) = \mathrm{SU}(V,\,\langle\cdot,\,\cdot\rangle')$ and $\mathfrak{su}(V,\,\langle\cdot,\,\cdot\rangle) = \mathfrak{su}(V,\,\langle\cdot,\,\cdot\rangle')$. In view of the above observations, without loss of generality, we may only consider the involution σ_c. *From now on we will restrict to the involution* σ_c *instead of arbitrary involutions on* $\mathbb{D}$.

The case where $\mathbb{D} = \mathbb{C}$, $\sigma = \mathrm{Id}$ and $\epsilon = \pm 1$ is already investigated in [BC1]. Here the remaining three cases

1. $\mathbb{D} = \mathbb{R}$, $\sigma = \mathrm{Id}$ and $\epsilon = \pm 1$,

2. $\mathbb{D} = \mathbb{C}$, $\sigma = \sigma_c$ and $\epsilon = 1$, and

3. $\mathbb{D} = \mathbb{H}$, $\sigma = \sigma_c$ and $\epsilon = \pm 1$

will be investigated.

We next introduce certain standard nomenclature associated with the specific values of ϵ and σ. If $\sigma = \sigma_c$ and $\epsilon = 1$, then $\langle\cdot,\,\cdot\rangle$ is called a *Hermitian* form. When $\sigma = \sigma_c$ and $\epsilon = -1$, then $\langle\cdot,\,\cdot\rangle$ is called a *skew-Hermitian* form. The form $\langle\cdot,\,\cdot\rangle$ is called *symmetric* if $\sigma = \mathrm{Id}$ and $\epsilon = 1$. Lastly, if $\sigma = \mathrm{Id}$ and $\epsilon = -1$, then $\langle\cdot,\,\cdot\rangle$ is called a *symplectic* form. If $\langle\cdot,\,\cdot\rangle$ is a symmetric form on V, define

$$\mathrm{SO}(V,\,\langle\cdot,\,\cdot\rangle) := \mathrm{SU}(V,\,\langle\cdot,\,\cdot\rangle) \quad \text{and} \quad \mathfrak{so}(V,\,\langle\cdot,\,\cdot\rangle) := \mathfrak{su}(V,\,\langle\cdot,\,\cdot\rangle)\,. \tag{2.16}$$

Similarly, if $\langle\cdot,\,\cdot\rangle$ is a symplectic form on V, then define

$$\mathrm{Sp}(V,\,\langle\cdot,\,\cdot\rangle) := \mathrm{SU}(V,\,\langle\cdot,\,\cdot\rangle) \quad \text{and} \quad \mathfrak{sp}(V,\,\langle\cdot,\,\cdot\rangle) := \mathfrak{su}(V,\,\langle\cdot,\,\cdot\rangle)\,. \tag{2.17}$$

When $\mathbb{D} = \mathbb{H}$ and $\langle\cdot\,,\cdot\rangle$ is a skew-Hermitian form on V, define

$$\mathrm{SO}^*(V,\,\langle\cdot,\,\cdot\rangle) := \mathrm{SU}(V,\,\langle\cdot,\,\cdot\rangle) \quad \text{and} \quad \mathfrak{so}^*(V,\,\langle\cdot,\,\cdot\rangle) := \mathfrak{su}(V,\,\langle\cdot,\,\cdot\rangle)\,. \tag{2.18}$$

As before, V is a right vector space over $\mathbb{D}$. We now introduce some terminologies associated to certain types of $\mathbb{D}$-basis of V. When either $\mathbb{D} = \mathbb{R}, \sigma = \mathrm{Id}$ or $\mathbb{D} = \mathbb{C}, \sigma = \sigma_c$ or $\mathbb{D} = \mathbb{H}, \sigma = \sigma_c$, for a 1-$\sigma$ Hermitian form $\langle \cdot\,, \cdot \rangle$ on V, an orthogonal basis $\mathcal{A}$ of V is called *standard orthogonal* if $\langle v, v \rangle = \pm 1$ for all $v \in \mathcal{A}$. For a standard orthogonal basis $\mathcal{A}$ of V, set

$$p := \#\{v \in \mathcal{A} \mid \langle v, v \rangle = 1\} \quad \text{and} \quad q := \#\{v \in \mathcal{A} \mid \langle v, v \rangle = -1\}\,.$$

The pair (p, q), which is independent of the choice of the standard orthogonal basis $\mathcal{A}$, is called the *signature* of $\langle \cdot\,, \cdot \rangle$. When $\mathbb{D} = \mathbb{C}$ and $\sigma = \sigma_c$, if $\langle \cdot\,, \cdot \rangle$ is a skew-Hermitian form on V then $\sqrt{-1}\langle \cdot\,, \cdot \rangle$ is a Hermitian form on V; in this case the *signature of the skew-Hermitian form* $\langle \cdot\,, \cdot \rangle$ is defined to be the signature of the Hermitian form $\sqrt{-1}\langle \cdot\,, \cdot \rangle$.

In the case where $\mathbb{D} = \mathbb{R}$ or $\mathbb{C}$, $\sigma = \mathrm{Id}$ and $\epsilon = -1$, the dimension $\dim_{\mathbb{D}} V$ is an even number. Let $2n = \dim_{\mathbb{D}} V$. In this case an ordered basis $\mathcal{B} := (v_1, \ldots, v_n; v_{n+1}, \ldots, v_{2n})$ of V is said to be *symplectic* if $\langle v_i, v_{n+i} \rangle = 1$ for all $1 \leq i \leq n$ and $\langle v_i, v_j \rangle = 0$ for all $j \neq n+i$. The ordered set $(v_1, \ldots, v_n)$ is called the *positive part* of $\mathcal{B}$ and it is denoted by $\mathcal{B}_+$. Similarly, the ordered set $(v_{n+1}, \ldots, v_{2n})$ is called the *negative part* of $\mathcal{B}$, and it is denoted by $\mathcal{B}_-$. The *complex structure on V associated to the above symplectic basis $\mathcal{B}$* is defined to be the $\mathbb{R}$-linear map

$$J_{\mathcal{B}} : V \longrightarrow V\,, \quad v_i \longmapsto v_{n+i}\,, \quad v_{n+i} \longmapsto -v_i \;\; \forall \;\; 1 \leq i \leq n\,.$$

If $\mathbb{D} = \mathbb{H}$ and $\langle \cdot\,, \cdot \rangle$ is a skew-Hermitian form on V, an orthogonal $\mathbb{H}$-basis

$$\mathcal{B} := (v_1, \ldots, v_m)$$

of V ($m := \dim_{\mathbb{H}} V$) is said to be *standard orthogonal* if $\langle v_r, v_r \rangle = \mathbf{j}$ for all $1 \leq$

$r \leq m$ and $\langle v_r, v_s \rangle = 0$ for all $r \neq s$.

Take $P = (p_{ij}) \in \mathrm{M}_{r\times s}(\mathbb{D})$. Then P^t denotes the *transpose* of P. If $\mathbb{D} = \mathbb{C}$ or $\mathbb{H}$, then define $\overline{P} := (\sigma_c(p_{ij}))$. Let

$$\mathrm{I}_{p,q} := \begin{pmatrix} \mathrm{I}_p & \\ & -\mathrm{I}_q \end{pmatrix}, \quad \mathrm{J}_n := \begin{pmatrix} & -\mathrm{I}_n \\ \mathrm{I}_n & \end{pmatrix}. \tag{2.19}$$

The classical groups that we will be working with are:

$$\begin{aligned}
\mathrm{SL}_n(\mathbb{R}) &:= \{g \in \mathrm{GL}_n(\mathbb{R}) \mid \det(g) = 1\},\\
\mathrm{SL}_n(\mathbb{H}) &:= \{g \in \mathrm{GL}_n(\mathbb{H}) \mid \mathrm{Nrd}_{\mathrm{M}_n(\mathbb{H})}(g) = 1\},\\
\mathrm{SU}(p,q) &:= \{g \in \mathrm{SL}_{p+q}(\mathbb{C}) \mid \overline{g}^t \mathrm{I}_{p,q} g = \mathrm{I}_{p,q}\},\\
\mathrm{SO}(p,q) &:= \{g \in \mathrm{SL}_{p+q}(\mathbb{R}) \mid g^t \mathrm{I}_{p,q} g = \mathrm{I}_{p,q}\},\\
\mathrm{SO}^*(2n) &:= \{g \in \mathrm{SL}_n(\mathbb{H}) \mid \overline{g}^t \mathbf{j} \mathrm{I}_n g = \mathbf{j} \mathrm{I}_n\},\\
\mathrm{Sp}(n,\mathbb{R}) &:= \{g \in \mathrm{SL}_{2n}(\mathbb{R}) \mid g^t \mathrm{J}_n g = \mathrm{J}_n\},\\
\mathrm{Sp}(p,q) &:= \{g \in \mathrm{SL}_{p+q}(\mathbb{H}) \mid \overline{g}^t \mathrm{I}_{p,q} g = \mathrm{I}_{p,q}\}.
\end{aligned}$$

The corresponding Lie algebras are:

$$\begin{aligned}
\mathfrak{sl}_n(\mathbb{R}) &:= \{z \in \mathrm{M}_n(\mathbb{R}) \mid \mathrm{tr}(z) = 0\},\\
\mathfrak{sl}_n(\mathbb{H}) &:= \{z \in \mathrm{M}_n(\mathbb{H}) \mid \mathrm{Trd}_{\mathrm{M}_n(\mathbb{H})}(z) = 0\},\\
\mathfrak{su}(p,q) &:= \{z \in \mathfrak{sl}_{p+q}(\mathbb{C}) \mid \overline{z}^t \mathrm{I}_{p,q} + \mathrm{I}_{p,q} z = 0\},\\
\mathfrak{so}(p,q) &:= \{z \in \mathfrak{sl}_{p+q}(\mathbb{R}) \mid z^t \mathrm{I}_{p,q} + \mathrm{I}_{p,q} z = 0\},\\
\mathfrak{so}^*(2n) &:= \{z \in \mathfrak{sl}_n(\mathbb{H}) \mid \overline{z}^t \mathbf{j} \mathrm{I}_n + \mathbf{j} \mathrm{I}_n z = 0\},\\
\mathfrak{sp}(n,\mathbb{R}) &:= \{z \in \mathfrak{sl}_{2n}(\mathbb{R}) \mid z^t \mathrm{J}_n + \mathrm{J}_n z = 0\},\\
\mathfrak{sp}(p,q) &:= \{z \in \mathfrak{sl}_{p+q}(\mathbb{H}) \mid \overline{z}^t \mathrm{I}_{p,q} + \mathrm{I}_{p,q} z = 0\}.
\end{aligned}$$

For any group H, let H^n_Δ denote the diagonally embedded copy of H in the n-fold direct product H^n. Let V be a vector space over $\mathbb{D}$. Define $\mathfrak{d}_V : \mathrm{End}_{\mathbb{D}}(V) \longrightarrow \mathbb{D}^*$ to be $\mathfrak{d}_V := \det$ if $\mathbb{D} = \mathbb{C}$ or $\mathbb{R}$, and $\mathfrak{d}_V := \mathrm{Nrd}_{\mathrm{End}_{\mathbb{D}} V}$ if $\mathbb{D} = \mathbb{H}$. Let now V_i, $1 \leq i \leq m$, be right vector spaces over $\mathbb{D}$. As before, $\mathbb{D}$ is either $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$. For every $1 \leq i \leq m$, let $H_i \subset \mathrm{GL}(V_i)$ be a matrix subgroup. Now define the subgroup

$$S\big(\prod_i H_i\big) := \Big\{(h_1, \ldots, h_m) \in \prod_{i=1}^m H_i \;\Big|\; \prod_i \mathfrak{d}_{V_i}(h_i) = 1\Big\} \subset \prod_{i=1}^m H_i\,.$$

The following notation will allow us to write block-diagonal square matrices with many blocks in a convenient way. For r-many square matrices $A_i \in \mathrm{M}_{m_i}(\mathbb{D})$, $1 \leq i \leq r$, the block diagonal square matrix of size $\sum m_i \times \sum m_i$, with A_i as the i-th block in the diagonal, is denoted by $A_1 \oplus \cdots \oplus A_r$. This is also abbreviated as $\oplus_{i=1}^r A_i$. Furthermore, if $B \in \mathrm{M}_m(\mathbb{D})$ and s is a positive integer, then denote $B^s_\blacktriangle := \underbrace{B \oplus \cdots \oplus B}_{s\text{-many}}$.

The following lemma is a basic fact which readily follows from the Skolem-Noether theorem.

Lemma 2.3.4. *Let α, $\beta \in \mathbb{H}^*$ be such that* $\mathrm{Re}(\alpha) = \mathrm{Re}(\beta)$ *and* $|\alpha| = |\beta|$. *Then there exists an element $\lambda \in \mathbb{H}^*$ with $|\lambda| = 1$ such that $\alpha = \lambda\beta\lambda^{-1}$.*

Proof. It is enough to prove the lemma under the additional conditions $\mathrm{Re}(\alpha) = \mathrm{Re}(\beta) = 0$ and $|\alpha| = |\beta| = 1$. Note that $\alpha^2 = -1$ and $\mathbb{R}[\alpha]$ is a simple ring with unity (isomorphic to $\mathbb{C}$). Consider the $\mathbb{R}$-algebra homomorphisms

$$f\colon \mathbb{R}[\alpha] \longrightarrow \mathbb{H}\,, \quad \alpha \longmapsto \beta\,; \qquad \text{and} \qquad \iota\colon \mathbb{R}[\alpha] \hookrightarrow \mathbb{H}\,.$$

Then by Skolem-Noether theorem there exists a $\lambda \in \mathbb{H}^*$ such that $f(\alpha) = \lambda\,\iota(\alpha)\,\lambda^{-1}$, hence $\beta = \lambda\alpha\lambda^{-1}$. This completes the proof. □

Recall that $\mathrm{Nrd}_{\mathrm{M}_n(\mathbb{H})}$ is real valued on $\mathrm{M}_n(\mathbb{H})$; see Remark 2.3.1.

Lemma 2.3.5. *For $g \in \mathrm{GL}_n(\mathbb{H})$, $\mathrm{Nrd}_{\mathrm{M}_n(\mathbb{H})}(g)$ is a positive real number.*

Proof. First note that $g \in \mathrm{GL}_n(\mathbb{H})$ if and only if $\mathrm{Nrd}_{\mathrm{M}_n(\mathbb{H})}(g) \neq 0$. Thus we have a continuous group homomorphism $\mathrm{Nrd}_{\mathrm{M}_n(\mathbb{H})} \colon \mathrm{GL}_n(\mathbb{H}) \longrightarrow \mathbb{R}^*$. Since the group $\mathrm{GL}_n(\mathbb{H})$ is connected, $\mathrm{Nrd}_{\mathrm{M}_n(\mathbb{H})}(g) > 0$ for all $g \in \mathrm{GL}_n(\mathbb{H})$. $\square$

We next include the following basic result which will be used in Chapter 6.

Lemma 2.3.6. *Let G be a Lie group and H be a closed normal subgroup in G. Assume that both G and H have finitely many connected components. Let K be a maximal compact subgroup of G. Then $K \cap H$ is a maximal compact subgroup of H.*

Proof. Let M be a maximal compact subgroup in H. As G has finitely many connected components $g^{-1}Mg \subset K$ for some $g \in G$. As H is a normal subgroup of G, we have $g^{-1}Mg \subset H$. In particular, $g^{-1}Mg \subset K \cap H$. The conclusion now follows form the fact that $g^{-1}Mg$ is a maximal compact subgroup in H. $\square$

We will use the next lemma in Theorem 4.1.6. Let G be a group. For $\alpha \in G$, let $\mathcal{O}(G,\alpha) := \{g\alpha g^{-1} \mid g \in G\}$. Let $H \subset G$ be a normal subgroup. Then for any $x \in H$, $\mathcal{O}(G,x) \subset H$ and $h\mathcal{O}(G,x)h^{-1} = \mathcal{O}(G,x)$ for all $h \in H$.

Lemma 2.3.7. *Let $H \subset G$ be a normal subgroup with finite index. Let $S := \{\mathcal{O}(H,y) \mid y \in \mathcal{O}(G,x)\}$ where $x \in H$. Then*

$$\#(S) \;=\; \frac{\#(G/H)}{\#\big(\mathcal{Z}_G(x)/\mathcal{Z}_H(x)\big)}\,.$$

Proof. As $H \subset G$ is normal, for any $\alpha \in G$ we have $\mathcal{O}(H, g\alpha g^{-1}) = g\mathcal{O}(H,\alpha)g^{-1}$ for all $g \in G$. In particular, $g\mathcal{O}(H,y)g^{-1} \in S$ for all $y \in \mathcal{O}(G,x)$ and $g\mathcal{O}(H,x)g^{-1} = \mathcal{O}(H, gxg^{-1})$. Thus the action of G on S induced from the conjugation action is transitive.

$$\begin{aligned} \text{Stabilizer of } \mathcal{O}(H,x) &= \{g \in G \mid \mathcal{O}(H, gxg^{-1}) = \mathcal{O}(H,x)\} \\ &= \{g \in G \mid gxg^{-1} = hxh^{-1} \text{ for some } h \in H\} \\ &= \{g \in G \mid g \in \mathcal{Z}_G(x)H\} \\ &= \mathcal{Z}_G(x)H. \end{aligned}$$

Recall that $\mathcal{Z}_G(x)$ normalizes H. So $\mathcal{Z}_G(x)H$ is a group. Thus

$$S \simeq \frac{G}{\mathcal{Z}_G(x)H} \simeq \frac{G/H}{\mathcal{Z}_G(x)H/H} \simeq \frac{G/H}{\mathcal{Z}_G(x)/\mathcal{Z}_H(x)}.$$

□

2.4 The Jacobson-Morozov Theorem

We now give a brief exposition of the well-known Jacobson-Morozov theorem. For a Lie algebra $\mathfrak{g}$ over $\mathbb{R}$, a subset $\{X, H, Y\} \subset \mathfrak{g}$ is said to be a $\mathfrak{sl}_2(\mathbb{R})$-*triple* if $X \neq 0$, $[H, X] = 2X$, $[H, Y] = -2Y$ and $[X, Y] = H$. It is immediate that $\mathrm{Span}_{\mathbb{R}}\{X, H, Y\}$ for a $\mathfrak{sl}_2(\mathbb{R})$-triple $\{X, H, Y\} \subset \mathfrak{g}$ is a $\mathbb{R}$-subalgebra of $\mathfrak{g}$ which is isomorphic to the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$. We now state the well-known Jacobson-Morozov theorem.

Theorem 2.4.1 (Jacobson-Morozov, cf. [CoMc, Theorem 9.2.1])**.** *Let $X \in \mathfrak{g}$ be a non-zero nilpotent element in a real semisimple Lie algebra $\mathfrak{g}$. Then there exist $H, Y \in \mathfrak{g}$ such that $\{X, H, Y\} \subset \mathfrak{g}$ is a $\mathfrak{sl}_2(\mathbb{R})$-triple.*

Remark 2.4.2. When $\mathbb{D} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ the Jacobson-Morozov theorem for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{D})$ can be easily verified using the well-known Jordan canonical forms for nilpotent matrices. First let

$$(2.20) \qquad X_n := \begin{pmatrix} 0 & 1 & & \\ & \ddots & 1 \\ & & 0 \end{pmatrix}_{n\times n} \in \mathfrak{sl}_n(\mathbb{D}).$$

Clearly X_n is a non-zero nilpotent element in $\mathfrak{sl}_n(\mathbb{D})$. We now set

$$H_n := \operatorname{diag}(h_1, \ldots, h_n),$$

where $h_r = (n-1) - 2(r-1)$ for $1 \leq r \leq n$. We also set

$$Y_n := \begin{pmatrix} 0 & & & & \\ y_1 & 0 & & & \\ 0 & y_2 & 0 & & \\ \vdots & \ddots & & \ddots & \\ 0 & \cdots & 0 & y_{n-1} & 0 \end{pmatrix},$$

where $y_r = h_1 + \cdots + h_r$ for $1 \leq r \leq n-1$. Then it can be easily verified by a straightforward computation that $\{X_n, H_n, Y_n\}$ is a $\mathfrak{sl}_2(\mathbb{R})$-triple in $\mathfrak{sl}_n(\mathbb{D})$.

For $\mathbf{d} := [d_1^{t_{d_1}}, \ldots, d_s^{t_{d_s}}] \in \mathcal{P}(n)$ set

$$(2.21) \qquad X_{\mathbf{d}} := (X_{d_1})_{\blacktriangle}^{t_{d_1}} \oplus \cdots \oplus (X_{d_s})_{\blacktriangle}^{t_{d_s}},$$

where X_{d_r} is as in (2.20) and see §2.3 for the above notation. Set

$$(2.22) \qquad H_{\mathbf{d}} := (H_{d_1})_{\blacktriangle}^{t_{d_1}} \oplus \cdots \oplus (H_{d_s})_{\blacktriangle}^{t_{d_s}} \quad \text{and} \quad Y_{\mathbf{d}} := (Y_{d_1})_{\blacktriangle}^{t_{d_1}} \oplus \cdots \oplus (Y_{d_s})_{\blacktriangle}^{t_{d_s}}.$$

As $\{X_{d_i}, H_{d_i}, Y_{d_i}\}$ is a $\mathfrak{sl}_2(\mathbb{R})$-triple for all i it is clear that $\{X_{\mathbf{d}}, H_{\mathbf{d}}, Y_{\mathbf{d}}\}$ is also a $\mathfrak{sl}_2(\mathbb{R})$-triple in $\mathfrak{sl}_n(\mathbb{D})$. Now the Jacobson-Morozov theorem for $\mathfrak{sl}_n(\mathbb{D})$ follows from the fact that any nilpotent element in $\mathfrak{sl}_n(\mathbb{D})$ is conjugated by an element of $\mathrm{GL}_n(\mathbb{D})$ to $X_{\mathbf{d}}$ where $X_{\mathbf{d}}$ is as in (2.21). In case $\mathbb{D} = \mathbb{R}$ or $\mathbb{C}$, the fact that any nilpotent element in $\mathfrak{sl}_n(\mathbb{D})$ is conjugated by an element of $\mathrm{GL}_n(\mathbb{D})$ to some $X_{\mathbf{d}}$, where $X_{\mathbf{d}}$ is as in (2.21), follows from the basic results on Jordan canonical forms of matrices over fields; see [Her, §6.5] and more specifically [Her, Lemma 6.5.4]. We now observe that [Her, Lemma 6.5.4], which is crucial in proving the Jordan canonical forms of matrices over fields, remains valid when fields are replaced by division rings. Thus when $\mathbb{D} = \mathbb{H}$ the above fact still holds to be true. □

We next include a proof of the Theorem 2.4.1. The following lemma is a key fact required in the proof.

Lemma 2.4.3 (cf. [CoMc, Lemma 2.1.2])**.** *Let $\mathfrak{g}_{\mathbb{C}}$ be a reductive Lie algebra over $\mathbb{C}$ and $X \in \mathfrak{g}_{\mathbb{C}}$ be a semisimple element. Then $\mathfrak{z}_{\mathfrak{g}_{\mathbb{C}}}(X)$ is a reductive Lie algebra.*

Let $\mathfrak{g}_{\mathbb{C}}$ be a semisimple Lie algebra over $\mathbb{C}$. Let $X \in \mathfrak{g}_{\mathbb{C}}$. As the image of the linear map $\mathrm{ad}_X \colon \mathfrak{g}_{\mathbb{C}} \longrightarrow \mathfrak{g}_{\mathbb{C}}$ is $[X, \mathfrak{g}_{\mathbb{C}}]$ and $\ker \mathrm{ad}_X = \mathfrak{z}_{\mathfrak{g}_{\mathbb{C}}}(X)$ it follows that

$$\dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}} = \dim_{\mathbb{C}} [X, \mathfrak{g}_{\mathbb{C}}] + \dim_{\mathbb{C}} \mathfrak{z}_{\mathfrak{g}_{\mathbb{C}}}(X). \tag{2.23}$$

Let B be the Killing form of $\mathfrak{g}_{\mathbb{C}}$. Let $\mathfrak{z}_{\mathfrak{g}_{\mathbb{C}}}(X)^{\perp} := \{A \in \mathfrak{g}_{\mathbb{C}} \mid B(A, \mathfrak{z}_{\mathfrak{g}_{\mathbb{C}}}(X)) = 0\}$. As $\mathfrak{g}_{\mathbb{C}}$ is semisimple B is nondegenerate, and hence, we have

$$\dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}} = \dim_{\mathbb{C}} \mathfrak{z}_{\mathfrak{g}_{\mathbb{C}}}(X) + \dim_{\mathbb{C}} \mathfrak{z}_{\mathfrak{g}_{\mathbb{C}}}(X)^{\perp}. \tag{2.24}$$

For $A \in \mathfrak{g}_{\mathbb{C}}$ and $Z \in \mathfrak{z}_{\mathfrak{g}_{\mathbb{C}}}(X)$ we have $B(Z, [X, A]) = B([Z, X], A) = 0$. Thus $[X, \mathfrak{g}_{\mathbb{C}}] \subset \mathfrak{z}_{\mathfrak{g}_{\mathbb{C}}}(X)^{\perp}$. Now in view of (2.23) and (2.24), it follows that

$$[X, \mathfrak{g}_{\mathbb{C}}] = \mathfrak{z}_{\mathfrak{g}_{\mathbb{C}}}(X)^{\perp}. \tag{2.25}$$

We next prove the complex version of the Jacobson-Morozov theorem.

Theorem 2.4.4 (Jacobson-Morozov, cf. [CoMc, Theorem 3.3.1])**.** *Let $X \in \mathfrak{g}_{\mathbb{C}}$ be a non-zero nilpotent element in a complex semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Then there exist $H, Y \in \mathfrak{g}_{\mathbb{C}}$ such that $Span_{\mathbb{C}}\{X, H, Y\} \simeq \mathfrak{sl}_2(\mathbb{C})$.*

Proof. We will use induction on the dimension of $\mathfrak{g}_{\mathbb{C}}$. If $\dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}} = 3$, then $\mathfrak{g}_{\mathbb{C}}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ and we are done. Assume $\dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}} > 3$. If X is in a proper semisimple subalgebra of $\mathfrak{g}_{\mathbb{C}}$, then the conclusion follows from the induction hypothesis. So we assume that X does not lie in any proper semisimple subalgebra of $\mathfrak{g}_{\mathbb{C}}$.

Let B be the Killing form of $\mathfrak{g}_{\mathbb{C}}$. For $Z \in \mathfrak{z}_{\mathfrak{g}_{\mathbb{C}}}(X)$, we have $\mathrm{ad}_X \circ \mathrm{ad}_Z = \mathrm{ad}_Z \circ \mathrm{ad}_X$ and hence $\mathrm{ad}_X \circ \mathrm{ad}_Z$ is a nilpotent linear operator. Therefore $B(X, Z) = 0$. This implies that $B(X, \mathfrak{z}_{\mathfrak{g}_{\mathbb{C}}}(X)) = 0$ and thus by (2.25),

$$X \in \mathfrak{z}_{\mathfrak{g}_{\mathbb{C}}}(X)^{\perp} = [\mathfrak{g}_{\mathbb{C}}, X].$$

Thus there is a $H' \in \mathfrak{g}_{\mathbb{C}}$ such that $[H', X] = 2X$. Considering $\mathrm{ad}_{H'} \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}X)$, let $H' = H_s + H_n$ be the Jordan decomposition of H' in $\mathfrak{g}_{\mathbb{C}}$ where H_s is semisimple and H_n is nilpotent. Thus $[H_s, X] = 2X$ and $[H_n, X] = 0$. Hence we conclude that there exists a semisimple element $H \in \mathfrak{g}_{\mathbb{C}}$ such that $[H, X] = 2X$.

Claim: $H \in [\mathfrak{g}_{\mathbb{C}}, X]$.

On the contrary, assume that $H \notin [\mathfrak{g}_{\mathbb{C}}, X]$. Thus by (2.25), we have

$$B(H, \mathfrak{z}_{\mathfrak{g}_{\mathbb{C}}}(X)) \neq 0. \tag{2.26}$$

Using the Jacobi identity it follows that ad_H leaves $\mathfrak{z}_{\mathfrak{g}_{\mathbb{C}}}(X)$ invariant. As H is semisimple, we have the eigenspaces decomposition

$$\mathfrak{z}_{\mathfrak{g}_{\mathbb{C}}}(X) = (\mathfrak{z}_{\mathfrak{g}_{\mathbb{C}}}(X))_{\tau_1} \oplus \cdots \oplus (\mathfrak{z}_{\mathfrak{g}_{\mathbb{C}}}(X))_{\tau_r}$$

where $(\mathfrak{z}_{\mathfrak{g}_\mathbb{C}}(X))_{\tau_i} := \{Z \in \mathfrak{z}_{\mathfrak{g}_\mathbb{C}}(X) \mid [H, Z] = \tau_i Z\}$, $\tau_i \in \mathbb{C}$. If $Z \in (\mathfrak{z}_{\mathfrak{g}_\mathbb{C}}(X))_{\tau_i}$ is a non-zero element, with $\tau_i \neq 0$, then

$$\tau_i B(H, Z) = B(H, \tau_i Z) = B(H, [H, Z]) = B([H, H], Z) = 0.$$

This implies that $H \in (\mathfrak{z}_{\mathfrak{g}_\mathbb{C}}(X))_{\tau_i}^{\perp}$ for all $\tau_i \neq 0$. When $\tau_i = 0$, we have $(\mathfrak{z}_{\mathfrak{g}_\mathbb{C}}(X))_0 = \mathfrak{z}_{\mathfrak{z}_{\mathfrak{g}_\mathbb{C}}(X)}(H)$. Thus from (2.26), we conclude that there exits $Z \in \mathfrak{z}_{\mathfrak{z}_{\mathfrak{g}_\mathbb{C}}(X)}(H)$ such that $B(H, Z) \neq 0$. Let $Z = Z_s + Z_n$ be the Jordan decomposition of Z where Z_s is semisimple and Z_n is nilpotent. If Z is nilpotent (i.e., $Z_s = 0$) then we argue as before (see second paragraph of this proof) to conclude $B(H, Z) = 0$ which is a contradiction. Thus $Z_s \neq 0$. Using the Jordan decomposition, we moreover conclude that

(2.27) $\qquad Z_s$ is a non-zero semisimple element in $\mathfrak{z}_{\mathfrak{z}_{\mathfrak{g}_\mathbb{C}}(X)}(H)$.

By Lemma 2.4.3, $\mathfrak{z}_{\mathfrak{g}_\mathbb{C}}(Z_s)$ is a reductive subalgebra of $\mathfrak{g}_\mathbb{C}$, and hence $[\mathfrak{z}_{\mathfrak{g}_\mathbb{C}}(Z_s), \mathfrak{z}_{\mathfrak{g}_\mathbb{C}}(Z_s)]$ is a semisimple subalgebra of $\mathfrak{g}_\mathbb{C}$. Since $Z_s \neq 0$ and $\mathfrak{g}_\mathbb{C}$ is semisimple, it follows that $\mathfrak{z}_{\mathfrak{g}_\mathbb{C}}(Z_s)$ is a proper subalgebra of $\mathfrak{g}_\mathbb{C}$. By (2.27) we have $H, X \in \mathfrak{z}_{\mathfrak{g}_\mathbb{C}}(Z_s)$. Thus, $2X = [H, X] \in [\mathfrak{z}_{\mathfrak{g}_\mathbb{C}}(Z_s), \mathfrak{z}_{\mathfrak{g}_\mathbb{C}}(Z_s)]$. This contradicts the fact that X lies in a proper semisimple subalgebra of $\mathfrak{g}_\mathbb{C}$, proving the claim.

Let $Y \in \mathfrak{g}_\mathbb{C}$ be an element such that $H = [X, Y]$. As H semisimple, we have the decomposition of $\mathfrak{g}_\mathbb{C}$ into ad_H-eigenspaces as follows:

(2.28) $$\mathfrak{g}_\mathbb{C} = (\mathfrak{g}_\mathbb{C})_{\lambda_1} \oplus \cdots \oplus (\mathfrak{g}_\mathbb{C})_{\lambda_k},$$

where $(\mathfrak{g}_\mathbb{C})_{\lambda_i} := \{A \in \mathfrak{g}_\mathbb{C} \mid [H, A] = \lambda_i A\}$. Let $Y = Y_{\lambda_1} + \cdots + Y_{\lambda_k}$ with $Y_{\lambda_i} \in (\mathfrak{g}_\mathbb{C})_{\lambda_i}$.

Then $[X,\,(\mathfrak{g}_{\mathbb{C}})_{\lambda_i}] \subset (\mathfrak{g}_{\mathbb{C}})_{\lambda_i+2}$, for $1 \leq i \leq k$. Now,

$$\sum_{i=1}^{k}[X,\,Y_{\lambda_i}] \;=\; [X,\,Y] \;=\; H \in (\mathfrak{g}_{\mathbb{C}})_0.$$

Hence, in view of (2.28), we have $H \;=\; [X,\,Y_{-2}]$. Replacing Y by Y_{-2} we conclude that $[H,\,Y] \;=\; -2Y$ and $\text{Span}_{\mathbb{C}}\{X,\,H,\,Y\} \simeq \mathfrak{sl}_2(\mathbb{C})$. This completes the proof of the theorem. $\square$

We need the following lemma to prove the Jacobson-Morozov Theorem for real semisimple Lie algebra $\mathfrak{g}$.

Lemma 2.4.5 (Jacobson, cf. [CoMc, Lemma 9.2.2])**.** *Let $\mathfrak{g}$ be a real semisimple Lie algebra and $H,\,X,\,Y' \in \mathfrak{g}$ satisfy the relation $[H,\,X] \;=\; 2X$ and $[X,\,Y'] \;=\; H$. Then there exists $Y \in \mathfrak{g}$ such that $\{X,\,H,\,Y\}$ is a $\mathfrak{sl}_2(\mathbb{R})$-triple in $\mathfrak{g}$.*

Proof. Let $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C}$ be the complexification of $\mathfrak{g}$. We use Jacobi identity to conclude the following:

- ad_X maps the generalized λ-eigenspace of ad_H in $\mathfrak{g}_{\mathbb{C}}$ to the generalized $(\lambda+2)$-eigenspace, for any $\lambda \in \mathbb{C}$. This shows that X is nilpotent.
- $\text{ad}_H\,(\,\mathfrak{z}_{\mathfrak{g}}(X)\,) \;\subset\; \mathfrak{z}_{\mathfrak{g}}(X)$.
- $[\,X,\;[H,Y'] + 2Y'\,] \;=\; 0$.

Claim: All eigenvalues of $\text{ad}_H\colon \mathfrak{z}_{\mathfrak{g}}(X) \longrightarrow \mathfrak{z}_{\mathfrak{g}}(X)$ lie in $\mathbb{N}$.

Suppose the above claim is true. Then

$$\text{ad}_H + 2\,\text{Id} \;:\; \mathfrak{z}_{\mathfrak{g}}(X) \;\longrightarrow\; \mathfrak{z}_{\mathfrak{g}}(X), \quad A \longmapsto [H,\,A] + 2A$$

is non-singular, and hence $(\text{ad}_H + 2\,\text{Id})(Z) \;=\; -[H,\,Y'] - 2Y'$ for some $Z \in \mathfrak{z}_{\mathfrak{g}}(X)$. Replacing Y' by $Y := Y' + Z$, we see that $\{X,\,H,\,Y\}$ is a $\mathfrak{sl}_2(\mathbb{R})$-triple in $\mathfrak{g}$.

It now remains to prove the claim. For this let $\mathfrak{g}_i := \mathfrak{z}_{\mathfrak{g}}(X) \cap \mathrm{ad}_X^i(\mathfrak{g})$. Using induction on i, and the relation $\mathrm{ad}_H = \mathrm{ad}_X \circ \mathrm{ad}_{Y'} - \mathrm{ad}_{Y'} \circ \mathrm{ad}_X$, it follows that

$$\mathrm{ad}_H \circ \mathrm{ad}_X^i(W) = 2i\,\mathrm{ad}_X^i(W) + \mathrm{ad}_X^i \circ \mathrm{ad}_H(W),$$

(2.29)

$$\mathrm{ad}_X^{i+1} \circ \mathrm{ad}_{Y'}(W) = (i+1)\mathrm{ad}_H \circ \mathrm{ad}_X^i(W) - i(i+1)\mathrm{ad}_X^i(W) + \mathrm{ad}_{Y'} \circ \mathrm{ad}_X^{i+1}(W),$$

for $W \in \mathfrak{g}$. Set $Z := \mathrm{ad}_X^i(W) \in \mathfrak{g}_i$. Then again using induction on i and the above relations we conclude that

$$(i+1)\mathrm{ad}_H Z = i(i+1)\,Z + \mathrm{ad}_X^{i+1} \circ \mathrm{ad}_{Y'}(W)\,.$$

Using (2.29) it is easy to see that $\mathrm{ad}_X^{i+1} \circ \mathrm{ad}_{Y'}(W) \in \mathfrak{z}_{\mathfrak{g}}(X)$, for $\mathrm{ad}_X^i(W) \in \mathfrak{g}_i$. Thus, for all i, the operator ad_H acts on the vector space $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ by the scalar i. Since X is nilpotent, we have $\mathfrak{g}_i = 0$ for some large i, and hence all the eigenvalues ad_H on $\mathfrak{z}_{\mathfrak{g}}(X)$ lie in $\mathbb{N}$. □

Proof of Theorem 2.4.1. Let $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C}$ be the complexification of $\mathfrak{g}$. Then $X \in \mathfrak{g}_{\mathbb{C}}$ is a non-zero nilpotent element. Using Theorem 2.4.4, we have $\{X,\, H_{\mathbb{R}} + \sqrt{-1}H'_{\mathbb{R}},\, Y_{\mathbb{R}} + \sqrt{-1}Y'_{\mathbb{R}}\} \subset \mathfrak{g}_{\mathbb{C}}$ such that $\mathrm{Span}_{\mathbb{C}}\{X,\, H_{\mathbb{R}} + \sqrt{-1}H'_{\mathbb{R}},\, Y_{\mathbb{R}} + \sqrt{-1}Y'_{\mathbb{R}}\} \simeq \mathfrak{sl}_2(\mathbb{C})$ where $H_{\mathbb{R}}, H'_{\mathbb{R}}, Y_{\mathbb{R}}, Y'_{\mathbb{R}} \in \mathfrak{g}$. Note that $\{X,\, H_{\mathbb{R}}, Y_{\mathbb{R}}\} \subset \mathfrak{g}$ with $[H_{\mathbb{R}},\, X] = 2X$ and $[X,\, Y_{\mathbb{R}}] = H_{\mathbb{R}}$. Now the theorem follows from Lemma 2.4.5. □

The following result relates two $\mathfrak{sl}_2(\mathbb{R})$-triples with a pair of common elements.

Lemma 2.4.6 (cf. [CoMc, Lemma 3.4.4])**.** *Let X be a nilpotent element and let H be a semisimple element in a Lie algebra $\mathfrak{g}$ such that $\{X,\, H,\, Y_1\}$ and $\{X,\, H,\, Y_2\}$ are two $\mathfrak{sl}_2(\mathbb{R})$-triples in $\mathfrak{g}$. Then $Y_1 = Y_2$.*

Proof. Let $Y := Y_1 - Y_2$. Then we have $[X,\, Y] = 0$ and $[H,\, Y] = -2Y$. We fix the natural action of the $\mathbb{R}$-span of one of the $\mathfrak{sl}_2(\mathbb{R})$-triple, say $\{X,\, H,\, Y_1\}$, and consider $\mathfrak{g}$ as a module over the span. Decomposing $\mathfrak{g}$ into a direct sum of irreducible submodules we see that the above pair of relations forces $Y = 0$. □

We now record an immediate consequence of the above result.

Lemma 2.4.7. *Let $\{X, H, Y\}$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple in the Lie algebra $\mathfrak{g}$ of a Lie group G. Then $\mathcal{Z}_G(X, H) = \mathcal{Z}_G(X, H, Y)$.*

Proof. To prove the lemma it suffices to show that $\mathcal{Z}_G(X, H) \subset \mathcal{Z}_G(X, H, Y)$. Take any $g \in \mathcal{Z}_G(X, H)$. Then $\{\mathrm{Ad}(g)X, \mathrm{Ad}(g)H, \mathrm{Ad}(g)Y\} = \{X, H, \mathrm{Ad}(g)Y\}$ is another $\mathfrak{sl}_2(\mathbb{R})$-triple in $\mathfrak{g}$ containing X and H. Using Lemma 2.4.6 we have $\mathrm{Ad}(g)Y = Y$, implying that $g \in \mathcal{Z}_G(X, H, Y)$. □

We now state a result relating two $\mathfrak{sl}_2(\mathbb{R})$-triples with a common nilpotent element.

Theorem 2.4.8 (cf. [CoMc, Theorem 9.2.3])**.** *Let $X \in \mathfrak{g}$ be a non-zero nilpotent element in a real semisimple Lie algebra $\mathfrak{g}$ and G be the adjoint group of $\mathfrak{g}$. If $\{X, H, Y\}$ and $\{X, H', Y'\}$ are two $\mathfrak{sl}_2(\mathbb{R})$-triple in $\mathfrak{g}$ containing X, then they are conjugate under $\mathcal{Z}_G(X)$.*

Chapter 3

Basic results on nilpotent orbits

This chapter is devoted to working out certain details on the structures of the nilpotent elements in classical real semisimple Lie algebras. This is done in two steps. As suggested in [Mc, § 3.1–3.3, pp. 174-180] and in [CoMc, § 9.3, p. 139], considering a classical Lie algebra, we first apply the Jacobson-Morozov Theorem to assume that a given non-zero nilpotent element is a part of a $\mathfrak{sl}_2(\mathbb{R})$-triple of the classical Lie algebra. We then use the standard basic theory of finite dimensional $\mathfrak{sl}_2(\mathbb{R})$-representations to describe the structures of the $\mathfrak{sl}_2(\mathbb{R})$-isotypical components of the vector space of the underlying natural representation of the classical Lie algebra. When the corresponding classical groups are over $\mathbb{R}$ or $\mathbb{C}$, Proposition 3.0.3 and Proposition 3.0.7 follow from results [SS, p. 249, 1.6; p. 259, 2.19] due to Springer and Steinberg which they proved in a direct manner without using the standard theory of finite dimensional $\mathfrak{sl}_2(\mathbb{R})$-representations. It should be mentioned that the non-commutativity of $\mathbb{H}$ does not allow direct extensions of this approach to the classical groups over $\mathbb{H}$. The above two-step approach allows us to treat all the cases involving $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ in a uniform manner. We also detect an error in [CoMc, Lemma 9.3.1, p. 139] which we point out in Remark 3.0.16. This led us to modify the definition of signed Young diagrams as given in [CoMc, p. 140] and choose different signs in the last columns of the associated matrices, as done in **Yd**.2.

Given an endomorphism $T \in \mathrm{End}_{\mathbb{R}}(W)$, where W is a $\mathbb{R}$-vector space, and any $\lambda \in \mathbb{R}$, set

$$W_{T,\lambda} := \{w \in W \mid Tw = w\lambda\}.$$

Let V be a right vector space of dimension n over $\mathbb{D}$, where $\mathbb{D}$ is, as before, $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$. Let $\{X, H, Y\} \subset \mathfrak{sl}(V)$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple. Note that V is also a $\mathbb{R}$-vector space using the inclusion $\mathbb{R} \hookrightarrow \mathbb{D}$. Hence V is a module over $\mathrm{Span}_{\mathbb{R}}\{X, H, Y\} \simeq \mathfrak{sl}_2(\mathbb{R})$. For any positive integer d, let $M(d-1)$ denote the sum of all the $\mathbb{R}$-subspaces A of V such that

- $\dim_{\mathbb{R}} A = d$, and
- A is an irreducible $\mathrm{Span}_{\mathbb{R}}\{X, H, Y\}$-submodules of V.

Then $M(d-1)$ is the *isotypical component* of V containing all the irreducible submodules of V with highest weight $d-1$. Let

$$L(d-1) := V_{Y,0} \cap M(d-1). \tag{3.1}$$

As the endomorphisms X, H, Y of V are $\mathbb{D}$-linear, the $\mathbb{R}$-subspaces $M(d-1)$, $V_{Y,0}$ and $L(d-1)$ of V are also $\mathbb{D}$-subspaces. Let

$$t_d := \dim_{\mathbb{D}} L(d-1).$$

Remark 3.0.1. Let $\mathbf{d} := [d_1^{t_{d_1}}, \ldots, d_s^{t_{d_s}}] \in \mathcal{P}(n)$. Let $X_{\mathbf{d}} \in \mathrm{M}_n(\mathbb{D})$ be as in (2.21) and $H_{\mathbf{d}}, Y_{\mathbf{d}}$ be as in (2.22). We consider the space of column vectors $\mathbb{D}^n$ as a $\mathrm{Span}_{\mathbb{R}}\{X_{\mathbf{d}}, H_{\mathbf{d}}, Y_{\mathbf{d}}\}$-module (under the usual left multiplication of matrices from $\mathrm{M}_n(\mathbb{D})$ on the column vectors $\mathbb{D}^n$). Let $\mathbb{N}_{\mathbf{d}} := \{d_i \mid 1 \leq i \leq s\}$; see (2.1) for the definition. Then it is clear that, for $1 \le r \le s$

$$M(d_r - 1) = \mathrm{Span}_{\mathbb{D}}\{e_{t_1d_1+\cdots+t_{r-1}d_{r-1}+1}, \ldots, e_{t_1d_1+\cdots+t_rd_r}\}, \tag{3.2}$$

where $(e_1, \ldots, e_n)$ denotes the standard ordered basis of $\mathbb{D}^n$. □

The next lemma is an elementary application of the standard structure theory of irreducible $\mathfrak{sl}_2(\mathbb{R})$-modules.

Lemma 3.0.2. *Let V be a right $\mathbb{D}$-vector space, and let $\{X, H, Y\} \subset \mathfrak{sl}(V)$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple. Let d be a positive integer such that $M(d-1) \neq 0$. Let $\{w_1, w_2, \ldots, w_{t_d}\}$ be any $\mathbb{D}$-basis of $L(d-1)$. Then*

1. *$X^d w_j = 0$ and $H(X^l w_j) = X^l w_j(2l+1-d)$ for all $1 \le j \le t_d$;*
2. *the set $\{X^l w_j \mid 1 \le j \le t_d, \ 0 \le l \le d-1\}$ is a $\mathbb{D}$-basis of $M(d-1)$;*
3. *the $\mathbb{R}$-Span of $\{w_j, Xw_j, \ldots, X^{d-1}w_j\}$ is an irreducible $\mathrm{Span}_{\mathbb{R}}\{X, H, Y\}$-submodule of $M(d-1)$, and moreover, if W_j is the $\mathbb{D}$-Span of $\{w_j, Xw_j, \ldots, X^{d-1}w_j\}$, then*

$$\begin{aligned} M(d-1) &= W_1 \oplus W_2 \oplus \cdots \oplus W_{t_d} \\ &= L(d-1) \oplus XL(d-1) \oplus \cdots \oplus X^{d-1}L(d-1). \end{aligned} \tag{3.3}$$

Proof. As $M(d-1)$, $V_{Y,0}$ and $L(d-1)$ of are $\mathbb{D}$-subspaces of V, it suffices to prove the lemma for $\mathbb{D} = \mathbb{R}$. We have the following relations: for $1 \le j \le t_d$ and $0 \le l \le d-1$,

$$Hw_j = w_j(1-d), \quad H(X^l w_j) = w_j(2l+1-d). \tag{3.4}$$

Using induction on l, it follows from the relations $[H, X] = 2X$, $[H, Y] = -2Y$, $[X, Y] = H$, that $YX^l v = (X^{l-1}v)l(d-l)$ for all $v \in L(d-1)$ and $l > 0$. This in turn implies that

$$Y^l X^l w_j \;=\; w_j(l!)(d-1)(d-2)\cdots(d-l)\,. \tag{3.5}$$

Note that $X^d w_j \;=\; 0$ because $d-1$ is the highest weight. From (3.4) it follows that $X^l w_j$ and $X^k w_i$ are linearly independent if $l \,\neq\, k\,;\; 0 \,\leq\, l,\, k \,\leq\, d-1\,;\; 1 \,\leq\, j, i \,\leq\, t_d$. Furthermore, (3.5) implies that for each l with $0 \,\leq\, l \,<\, d$, the vectors $\{X^l w_j \mid 1 \,\leq\, j \,\leq\, t_d\}$ are $\mathbb{R}$-linearly independent. It is a basic fact that $\dim_{\mathbb{R}} M(d-1) \,=\, d\dim_{\mathbb{R}} L(d-1)$. Consequently, $\{X^l w_j \mid 1 \,\leq\, j \,\leq\, t_d,\; 0 \,\leq\, l \,\leq\, d-1\}$ is a $\mathbb{R}$-basis of $M(d-1)$. This proves (2). Part (3) follows immediately from (2). □

Consider the non-zero irreducible $\mathrm{Span}_{\mathbb{R}}\{X,\, H,\, Y\}$-submodules of V. Let $\{d_1,\, \ldots,\, d_s\}$, with $d_1 \,<\, \cdots \,<\, d_s$, be the integers that occur as $\mathbb{R}$-dimensions of such $\mathrm{Span}_{\mathbb{R}}\{X,\, H,\, Y\}$-modules. From Lemma 3.0.2(2) we have

$$\sum_{i=1}^{s} t_{d_i} d_i \;=\; \dim_{\mathbb{D}} V \;=\; n\,.$$

Thus

$$\mathbf{d} \;:=\; \left[d_1^{t_{d_1}},\, \ldots,\, d_s^{t_{d_s}}\right] \,\in\, \mathcal{P}(n)\,. \tag{3.6}$$

Consider $\mathbb{N}_{\mathbf{d}}$, $\mathbb{E}_{\mathbf{d}}$ and $\mathbb{O}_{\mathbf{d}}$ as defined in (2.1). Then we have

$$V = \bigoplus_{d\in\mathbb{N}_{\mathbf{d}}} M(d-1) \quad \text{and} \quad L(d-1) \;=\; V_{Y,0} \cap V_{H,1-d} \quad \text{for} \;\; d \,\geq\, 1\,. \tag{3.7}$$

When $\mathbb{D} = \mathbb{R}$ or $\mathbb{C}$ Proposition 3.0.3 follows from [SS, p. 249, 1.6].

Proposition 3.0.3. *Let $\{X, H, Y\} \subset \mathfrak{sl}(V)$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple, where V is a right $\mathbb{D}$-vector space. For all $d \,\in\, \mathbb{N}_{\mathbf{d}}$ and for any $\mathbb{D}$-basis of $L(d-1)$, say, $\{v_j^d \mid 1 \,\leq\, j \,\leq\, t_d \,:=\, \dim_{\mathbb{D}} L(d-1)\}$ the following two hold:*

1. $X^d v_j^d = 0$ *and* $H(X^l v_j^d) = X^l v_j^d(2l+1-d)$ *for* $1 \leq j \leq t_d$, $0 \leq l \leq d-1$, $d \in \mathbb{N}_{\mathbf{d}}$.

2. *For all* $d \in \mathbb{N}_{\mathbf{d}}$, *the set* $\{X^l v_j^d \mid 1 \leq j \leq t_d,\ 0 \leq l \leq d-1\}$ *is a* $\mathbb{D}$*-basis of* $M(d-1)$. *In particular,* $\{X^l v_j^d \mid 1 \leq j \leq t_d,\ 0 \leq l \leq d-1,\ d \in \mathbb{N}_{\mathbf{d}}\}$ *is a* $\mathbb{D}$*-basis of* V.

Proof. This follows from Lemma 3.0.2 and (3.7). □

Henceforth, $\sigma : \mathbb{D} \longrightarrow \mathbb{D}$ will denote either the identity map or σ_c (defined in Section 2.3) when $\mathbb{D}$ is $\mathbb{C}$ or $\mathbb{H}$. Let $\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{D}$ be a ϵ-σ Hermitian form. Let X be a non-zero nilpotent element in $\mathfrak{su}(V, \langle \cdot, \cdot \rangle)$; see (2.15) for the definition of $\mathfrak{su}(V, \langle \cdot, \cdot \rangle)$. Using Theorem 2.4.1, there exists $H, Y \in \mathfrak{su}(V, \langle \cdot, \cdot \rangle)$ such that $\mathrm{Span}_{\mathbb{R}}\{X, H, Y\}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. Thus, V becomes a $\mathrm{Span}_{\mathbb{R}}\{X, H, Y\}$-module.

We record the following straightforward but useful fact.

Lemma 3.0.4 (cf. [Mc, § 2.4, p. 171])**.** *Let* $\sigma : \mathbb{D} \longrightarrow \mathbb{D}$ *be either the identity map or* σ_c *when* $\mathbb{D}$ *is* $\mathbb{C}$ *or* $\mathbb{H}$. *Let* $\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{D}$ *be a* ϵ-σ *Hermitian form. Suppose* $A \in \mathrm{End}_{\mathbb{D}}(V)$ *such that* $\langle Ax, y \rangle + \langle x, Ay \rangle = 0$ *for all* $x, y \in V$. *Let* v *and* w *be two nonzero elements in* V *such that* $Av = v\lambda$ *and* $Aw = w\mu$ *for some* $\lambda, \mu \in \mathbb{R}$. *If* $\lambda + \mu \neq 0$, *then* $\langle v, w \rangle = 0$.

Proof. As $\langle Av, w \rangle + \langle v, Aw \rangle = 0$, it follows immediately that $\langle v, w \rangle(\lambda + \mu) = 0$. Now the lemma follows because $\lambda + \mu \neq 0$. □

Lemma 3.0.5 (cf. [Mc, § 3.2, p. 178])**.** *Let* V *be a right* $\mathbb{D}$*-vector space, and* $\epsilon = \pm 1$. *Let* σ *be as in Lemma 3.0.4, and let* $\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{D}$ *be a* ϵ-σ *Hermitian form. Let* $\{X, H, Y\} \subset \mathfrak{su}(V, \langle \cdot, \cdot \rangle)$ *be a* $\mathfrak{sl}_2(\mathbb{R})$*-triple. Let* $\mathbf{d}$ *be as in* (3.6), *and let* $d, d' \in \mathbb{N}_{\mathbf{d}}$ *be such that* $d \neq d'$. *Then* $M(d-1)$ *and* $M(d'-1)$ *are orthogonal with respect to* $\langle \cdot, \cdot \rangle$. *In particular, the Hermitian form* $\langle \cdot, \cdot \rangle$ *on* $M(d-1)$ *is non-degenerate for all* d.

Proof. We may assume that $d > d'$. Let $v \in L(d-1)$ and $u \in L(d'-1)$. By Lemma 3.0.4 we have that $\langle v, X^l u\rangle = 0$ when $0 \leq l \leq d'-1$. Moreover, $X^l u = 0$ if $l \geq d'$. Thus $\langle v,\, X^l u\rangle = 0$ for all $l \geq 0$. Hence, $\langle X^h v,\, X^l u\rangle = (-1)^h \langle v,\, X^{l+h} u\rangle = 0$. Now the lemma follows from (3.3) of Lemma 3.0.2. □

The next lemma, which further decomposes each isotypical component $M(d-1) \subset V$ into orthogonal subspaces, seems basic. However, as we are unable to locate it in the literature, we include a proof here.

Lemma 3.0.6. *Let V be a right $\mathbb{D}$-vector space, and $\epsilon = \pm 1$. Let $\sigma : \mathbb{D} \longrightarrow \mathbb{D}$ be either the identity map or σ_c when $\mathbb{D}$ is $\mathbb{C}$ or $\mathbb{H}$. Let $\langle \cdot, \cdot\rangle : V \times V \longrightarrow \mathbb{D}$ be a (non-degenerate) ϵ-σ Hermitian form. Let $\{X, H, Y\} \subset \mathfrak{su}(V, \langle \cdot, \cdot\rangle)$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple. Let $\mathbf{d}$ be as in* (3.6), *$d \in \mathbb{N}_{\mathbf{d}}$ and $t_d := \dim_{\mathbb{D}} L(d-1)$. Then there exists a $\mathbb{D}$-basis $\{w_1, \ldots, w_{t_d}\}$ of $L(d-1)$ such that the set*

$$\{X^l w_j \mid 1 \leq j \leq t_d,\ 0 \leq l \leq d-1\}$$

is a $\mathbb{D}$-basis of $M(d-1)$, and moreover, the value of $\langle \cdot, \cdot\rangle$ on a pair of these basis vector is 0, except in the following cases:

(1) *If $\sigma = \sigma_c$, then $\langle X^l w_j,\, X^{d-1-l} w_j \rangle \in \mathbb{D}^*$.*

(2) *If $\sigma = \mathrm{Id}$ and $\epsilon = 1$, then $\langle X^l w_j,\, X^{d-1-l} w_j\rangle \in \mathbb{D}^*$ for d odd, and*

$$\langle X^l w_j,\, X^{d-1-l} w_{j+1}\rangle \in \mathbb{D}^*$$

for d even and j odd.

(3) *If $\sigma = \mathrm{Id}$ and $\epsilon = -1$, then $\langle X^l w_j,\, X^{d-1-l} w_j\rangle \in \mathbb{D}^*$ for d even, and*

$$\langle X^l w_j,\, X^{d-1-l} w_{j+1}\rangle \in \mathbb{D}^*$$

for d odd and j odd.

Proof. We use induction on $\dim_{\mathbb{D}} V$. The proof is divided into two parts.

Part 1. In this part assume that one of the following three holds:

- $\mathbb{D} = \mathbb{R}$, $\sigma = \mathrm{Id}$, and $(-1)^{d-1}\epsilon = 1$;
- $\mathbb{D} = \mathbb{C}$, $\sigma = \sigma_c$ and $\epsilon = \pm 1$;
- $\mathbb{D} = \mathbb{H}$, $\sigma = \sigma_c$ and $\epsilon = \pm 1$.

We claim that there is an element $x_1 \in L(d-1)$ such that $\langle x_1, X^{d-1}x_1\rangle \neq 0$.

To prove the claim by contradiction, assume that $\langle x, X^{d-1}x\rangle = 0$ for all $x \in L(d-1)$. Lemma 3.0.4 implies that $\langle z_1, X^l z_2\rangle = 0$ for $l \neq d-1$ and $z_1, z_2 \in L(d-1)$. Fix a nonzero element $x \in L(d-1)$. Since $\langle \cdot, \cdot \rangle$ is non-degenerate on $M(d-1) \times M(d-1)$, there exists an element $y \in L(d-1)$ such that $\langle x, X^{d-1}y\rangle \neq 0$. As $x + y \in L(d-1)$, we also know that $\langle x+y,\ X^{d-1}(x+y)\rangle = 0$.

Now we will arrive at a contradiction considering the three cases separately.

First assume that $\mathbb{D} = \mathbb{R}$, $\sigma = \mathrm{Id}$ and $(-1)^{d-1}\epsilon = 1$. Then

$$0 = \langle x+y,\ X^{d-1}(x+y)\rangle = 2\langle x, X^{d-1}y\rangle\,.$$

This is evidently a contradiction.

Next assume that $\mathbb{D} = \mathbb{C}$, $\sigma = \sigma_c$ and $\epsilon = \pm 1$. Writing $\langle x, X^{d-1}y\rangle = a + \sqrt{-1}b$ and multiplying y by an appropriate scalar from $\mathbb{C}$ if required, we may assume that $a \neq 0$ as well as $b \neq 0$. Now the condition $\langle x+y, X^{d-1}(x+y)\rangle = 0$ implies that $(a + \sqrt{-1}b) + (-1)^{d-1}\epsilon(a - \sqrt{-1}b) = 0$. This contradicts the fact that both a and b are non-zero.

Finally, assume that $\mathbb{D} = \mathbb{H}$, $\sigma = \sigma_c$ and $\epsilon = \pm 1$. Writing $\langle x, X^{d-1}y\rangle = a_1 + \mathbf{i}b_1 + \mathbf{j}c_1 + \mathbf{k}d_1$ and multiplying y by an appropriate scalar from $\mathbb{H}$ if needed, we may assume that $a_1 \neq 0$ and $b_1 \neq 0$. Then,

$$\langle x\,,\ X^{d-1}y\rangle + (-1)^{d-1}\epsilon\sigma(\langle x,\ X^{d-1}y\rangle) \,=\, 0\,.$$

From this it follows that $(a_1 + \mathbf{i}b_1 + \mathbf{j}c_1 + \mathbf{k}d_1) + (-1)^{d-1}\epsilon(a_1 - \mathbf{i}b_1 - \mathbf{j}c_1 - \mathbf{k}d_1) = 0$. This gives a contradiction as both a_1 and b_1 are nonzero. This completes the proof of the claim.

Let W be the $\mathbb{D}$-Span of $\{X^l x_1 \mid 0 \leq l \leq d-1\}$, where x_1 is the element of $L(d-1)$ in the above claim. As the vectors $\{X^l x_1 \mid 0 \leq l \leq d-1\}$ are $\mathbb{D}$-linearly independent, and $\langle x_1, X^{d-1}x_1\rangle \neq 0$, it follows that $\langle\cdot,\cdot\rangle$ is non-degenerate on W. Hence,

$$V \,=\, W \oplus W^{\perp}\,,$$

where $W^{\perp} := \{v \in V \mid \langle v, W\rangle = 0\}$. As $\{X, H, Y\} \subset \mathfrak{su}(V, \langle\cdot\,,\cdot\rangle)$, it follows immediately that X, H, Y leave $W^{\perp}$ invariant. Let

$$X_1 := X|_{W^{\perp}}\,,\quad H_1 := H|_{W^{\perp}}\,,\quad Y_1 := Y|_{W^{\perp}}\,.$$

Let $\langle\cdot\,,\cdot\rangle'$ be the restriction of $\langle\cdot\,,\cdot\rangle$ to $W^{\perp}$. Then

$$\{X_1, H_1, Y_1\} \subset \mathfrak{su}(W^{\perp}, \langle\cdot\,,\cdot\rangle')$$

is a $\mathfrak{sl}_2(\mathbb{R})$-triple. Let $M_{W^{\perp}}(d-1)$ be the isotypical component of $W^{\perp}$ consisting of sum of all $\mathbb{R}$-subspaces B of $W^{\perp}$ with $\dim_{\mathbb{R}} B = d$ which are also irreducible $\mathrm{Span}_{\mathbb{R}}\{X_1, H_1, Y_1\}$-submodules of $W^{\perp}$. Then we have $M_{W^{\perp}}(d-1) = W^{\perp} \cap M(d-1)$ and $M(d-1) = W \oplus M_{W^{\perp}}(d-1)$. Since $\dim_{\mathbb{D}} W^{\perp} < \dim_{\mathbb{D}} V$, from the induction hypothesis, $M_{W^{\perp}}(d-1)$ has a $\mathbb{D}$-basis satisfying (1), (2), (3) of the lemma. This

$\mathbb{D}$-basis of $M_{W^\perp}(d-1)$ together with the $\mathbb{D}$-basis $\{X^l x_1 \mid 0 \leq l \leq d-1\}$ of W will give the required $\mathbb{D}$-basis of $M(d-1)$. This completes the proof using induction on $\dim_{\mathbb{D}} V$.

Part 2 : Here we deal with the remaining case where $\mathbb{D} = \mathbb{R}$, $\sigma = \mathrm{Id}$ and $(-1)^{d-1}\epsilon = -1$.

For all $x \in L(d-1)$, as

$$\langle x, X^{d-1}x\rangle = (-1)^{d-1}\epsilon\langle x, X^{d-1}x\rangle = -\langle x, X^{d-1}x\rangle ,$$

it is clear that $\langle x, X^{d-1}x\rangle = 0$. Lemma 3.0.4 gives that $\langle z_1, X^l z_2\rangle = 0$ for $l \neq d-1$, $z_1, z_2 \in L(d-1)$. Fix any nonzero $x_1 \in L(d-1)$. Since $\langle \cdot, \cdot\rangle$ is non-degenerate on $M(d-1) \times M(d-1)$, there exists $y_1 \in L(d-1) \setminus x_1\mathbb{D}$ such that $\langle x_1, X^{d-1}y_1\rangle \neq 0$. Let W' be the $\mathbb{D}$-Span of $\{X^l x_1, X^l y_1 \mid 0 \leq l \leq d-1\}$. As the vectors $\{X^l x_1, X^l y_1 \mid 0 \leq l \leq d-1\}$ are $\mathbb{D}$-linearly independent, and $\langle x_1, X^{d-1}y_1\rangle \neq 0$, it follows that $\langle \cdot, \cdot\rangle$ is non-degenerate on W'. As before, define $W'^{\perp} := \{v \in V \mid \langle v, W'\rangle = 0\}$. As $V = W' \oplus W'^{\perp}$, and $\dim_{\mathbb{D}} W'^{\perp} < \dim_{\mathbb{D}} V$, repeating the argument in part 1 the proof is completed. □

The next result is an analogue of Proposition 3.0.3 in the presence of a ϵ-σ Hermitian form. When $\mathbb{D} = \mathbb{R}$ or $\mathbb{C}$ Proposition 3.0.7 follows from [SS, p. 259, 2.19].

Proposition 3.0.7. *Let V be a right $\mathbb{D}$-vector space, $\epsilon = \pm 1$, $\sigma : \mathbb{D} \longrightarrow \mathbb{D}$ is either the identity map or it is σ_c when $\mathbb{D}$ is $\mathbb{C}$ or $\mathbb{H}$. Let $\langle \cdot, \cdot\rangle : V \times V \longrightarrow \mathbb{D}$ be a ϵ-σ Hermitian form. Let $\{X, H, Y\} \subset \mathfrak{su}(V, \langle \cdot, \cdot\rangle)$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple. Let $d \in \mathbb{N}_{\mathbf{d}}$ and $t_d := \dim_{\mathbb{D}} L(d-1)$. Then for all $d \in \mathbb{N}_{\mathbf{d}}$, there exists a $\mathbb{D}$-basis $\{v^d_j \mid 1 \leq j \leq t_d\}$ of $L(d-1)$ such that the following three hold:*

1. *$X^d v^d_j = 0$ and $H(X^l v^d_j) = X^l v^d_j(2l+1-d)$ for all $1 \leq j \leq t_d$, $0 \leq l \leq d-1$ and $d \in \mathbb{N}_{\mathbf{d}}$.*

2. *For all* $d \in \mathbb{N}_{\mathbf{d}}$*, the set* $\{X^l v_j^d \mid 1 \leq j \leq t_d,\ 0 \leq l \leq d-1\}$ *is a* $\mathbb{D}$*-basis of* $M(d-1)$*. In particular,* $\{X^l v_j^d \mid 1 \leq j \leq t_d,\ 0 \leq l \leq d-1,\ d \in \mathbb{N}_{\mathbf{d}}\}$ *is a* $\mathbb{D}$*-basis of* V*.*

3. *The value of* $\langle \cdot, \cdot \rangle$ *on any pair of the above basis vectors is* 0*, except in the following cases:*

   - *If* $\sigma = \sigma_c$*, then* $\langle X^l v_j^d, X^{d-1-l} v_j^d \rangle \in \mathbb{D}^*$*.*
   - *If* $\sigma = \mathrm{Id}$ *and* $\epsilon = 1$*, then* $\langle X^l v_j^d, X^{d-1-l} v_j^d \rangle \in \mathbb{D}^*$ *when* $d \in \mathbb{O}_{\mathbf{d}}$*, and*

     $$\langle X^l v_j^d, X^{d-1-l} v_{j+1}^d \rangle \in \mathbb{D}^*$$

     when $d \in \mathbb{E}_{\mathbf{d}}$ *and* j *is odd.*
   - *If* $\sigma = \mathrm{Id}$ *and* $\epsilon = -1$*, then* $\langle X^l v_j^d, X^{d-1-l} v_j^d \rangle \in \mathbb{D}^*$ *when* $d \in \mathbb{E}_{\mathbf{d}}$*, and*

     $$\langle X^l v_j^d, X^{d-1-l} v_{j+1}^d \rangle \in \mathbb{D}^*$$

     when $d \in \mathbb{O}_{\mathbf{d}}$ *and* j *is odd.*

Proof. Lemma 3.0.5 gives the orthogonal decomposition $V = \bigoplus_{d \in \mathbb{N}_{\mathbf{d}}} M(d-1)$ with respect to the non-degenerate form $\langle \cdot, \cdot \rangle$ on V. The proposition now follows from Lemma 3.0.6. □

Remark 3.0.8. We follow the notation of Proposition 3.0.7 in this remark. Set $V_j^d := \mathrm{Span}_{\mathbb{D}}\{X^l v_j^d \mid 0 \leq l \leq d-1\}$. The following observations are straightforward from Proposition 3.0.7.

1. When $\sigma = \sigma_c$ we have

   $$V = \bigoplus_{1 \leq j \leq t_d,\, d \in \mathbb{N}_{\mathbf{d}}} V_j^d$$

   where the above direct sum is an orthogonal direct sum with respect to $\langle \cdot, \cdot \rangle$.

2. When $\sigma = \mathrm{Id}$ and $\epsilon = 1$, we set $W_j^\eta := V_j^\eta + V_{j+1}^\eta$ where j is an odd integer, $1 \leq j \leq t_\eta, \eta \in \mathbb{E}_{\mathbf{d}}$. Then

$$V = (\bigoplus_{1\leq j\leq t_\theta,\, \theta\in\mathbb{O}_{\mathbf{d}}} V_j^\theta) \oplus (\bigoplus_{j \text{ is odd},\, 1\leq j\leq t_\eta,\, \eta\in\mathbb{E}_{\mathbf{d}}} W_j^\eta)$$

is an orthogonal direct sum with respect to $\langle\cdot\,,\cdot\rangle$.

3. When $\sigma = \mathrm{Id}$ and $\epsilon = -1$, we set $W_j^\theta := V_j^\theta + V_{j+1}^\theta$ where j is an odd integer, $1 \leq j \leq t_\theta, \theta \in \mathbb{O}_{\mathbf{d}}$. Then

$$V = (\bigoplus_{1\leq j\leq t_\eta,\, \eta\in\mathbb{E}_{\mathbf{d}}} V_j^\eta) \oplus (\bigoplus_{j \text{ is odd},\, 1\leq j\leq t_\theta,\, \theta\in\mathbb{O}_{\mathbf{d}}} W_j^\theta)$$

is an orthogonal direct sum with respect to $\langle\cdot\,,\cdot\rangle$.

□

Let $\langle\cdot\,,\cdot\rangle$ be a ϵ-σ Hermitian form on V. Define the form

$$(\cdot,\,\cdot)_d \,:\, L(d-1)\times L(d-1) \longrightarrow \mathbb{D}\,, \quad (v,\,u)_d \,:=\, \langle v\,,\, X^{d-1}u\rangle \tag{3.8}$$

as in [CoMc, p. 139].

Remark 3.0.9. In [CoMc, §9.3, p.139], starting with a nilpotent element X in $\mathfrak{su}(V, \langle\cdot\,,\cdot\rangle)$, the form in (3.8) is defined on the highest weight space of $M(d-1)$ involving the element Y of an $\mathfrak{sl}_2(\mathbb{R})$-triple $\{X,\, H,\, Y\}$. However, in Section 5.2 we work with a basis of $M(d-1)$ constructed using X (see Proposition 3.0.7 (2)). Hence for our convenience the form in (3.8) is defined using X. □

Remark 3.0.10. Observe that if $\{X,\, H,\, Y\}$ is a $\mathfrak{sl}_2(\mathbb{R})$-triple in the Lie algebra $\mathfrak{su}(V, \langle\cdot\,,\cdot\rangle)$, and $g \in \mathcal{Z}_{\mathrm{GL}(V)}(X, H, Y)$, then $(gx,\, gy)_d = (x,\, y)_d$ for all $x,\, y \in L(d-1)$ if and only if $\langle gv,\, gw\rangle = \langle v,\, w\rangle$ for all $v,\, w \in M(d-1)$. □

Remark 3.0.11. It is easy to see that when $\langle \cdot\,, \cdot \rangle$ is Hermitian, then the form $(\cdot,\, \cdot)_d$ is Hermitian (respectively, skew-Hermitian) if d is odd (respectively, even). When $\langle \cdot\,, \cdot \rangle$ is symmetric, then $(\cdot,\, \cdot)_d$ is symmetric (respectively, symplectic) if d is odd (respectively, even). When $\langle \cdot\,, \cdot \rangle$ is symplectic, then $(\cdot,\, \cdot)_d$ is symplectic (respectively, symmetric) if d is odd (respectively, even). Lastly, when $\langle \cdot\,, \cdot \rangle$ is skew-Hermitian, then $(\cdot,\, \cdot)_d$ is skew-Hermitian (respectively, Hermitian) if d is odd (respectively, even).

From Lemma 3.0.6 it follows that $(\cdot,\, \cdot)_d$ is non-degenerate. The $\mathbb{D}$-basis elements $\{v_j^d \mid 1 \leq j \leq t_d\}$ of $L(d-1)$ in Proposition 3.0.7 are modified as follows:

1. If $\mathbb{D} = \mathbb{R}$ and $\epsilon = 1$, by suitable rescaling each element of $\{v_j^d \mid 1 \leq j \leq t_d\}$ we may assume that

   - $\langle v_j^d,\, X^{d-1} v_j^d \rangle = \pm 1$ when $d \in \mathbb{O}_{\mathbf{d}}$, and
   - $\langle v_j^d,\, X^{d-1} v_{j+1}^d \rangle = 1$ when $d \in \mathbb{E}_{\mathbf{d}}$ and j is odd.

   In particular, $(v_1^d, \ldots, v_{t_d}^d)$ is an standard orthogonal basis of $L(d-1)$ with respect to $(\cdot,\, \cdot)_d$ for $d \in \mathbb{O}_{\mathbf{d}}$. If $\mathbb{D} = \mathbb{R}$ and $\epsilon = -1$, analogously we may assume that the elements of the $\mathbb{R}$-basis $\{v_j^d \mid 1 \leq j \leq t_d\}$ of $L(d-1)$ in Proposition 3.0.7 satisfy the condition that

   - $\langle v_j^d,\, X^{d-1} v_j^d \rangle = \pm 1$ when $d \in \mathbb{E}_{\mathbf{d}}$, and
   - $\langle v_j^d,\, X^{d-1} v_{t_d/2+j}^d \rangle = 1$ when $d \in \mathbb{O}_{\mathbf{d}}$ and $1 \leq j \leq t_d/2$.

   In particular, $(v_1^d, \ldots, v_{t_d}^d)$ is an orthogonal basis for $d \in \mathbb{E}_{\mathbf{d}}$, and

   $$(v_1^d, \ldots, v_{t_d/2}^d;\, v_{t_d/2+1}^d, \ldots, v_{t_d}^d)$$

   is a symplectic basis for $d \in \mathbb{O}_{\mathbf{d}}$ of $L(d-1)$ with respect to $(\cdot,\, \cdot)_d$.

2. If $\mathbb{D} = \mathbb{C}$, $\epsilon = 1$ and $\sigma = \sigma_c$, rescaling the elements of the $\mathbb{C}$-basis $\{v_j^d \mid 1 \leq j \leq t_d\}$ we may assume that

- $\langle v_j^d, X^{d-1} v_j^d \rangle = \pm 1$ when $d \in \mathbb{O}_{\mathbf{d}}$, and
- $\langle v_j^d, X^{d-1} v_j^d \rangle = \pm\sqrt{-1}$ when $d \in \mathbb{E}_{\mathbf{d}}$.

In particular, $(v_1^d, \ldots, v_{t_d}^d)$ is an orthogonal basis of $L(d-1)$ with respect to $(\cdot, \cdot)_d$ for $d \in \mathbb{N}_{\mathbf{d}}$.

3. If $\mathbb{D} = \mathbb{H}$, $\epsilon = 1$ and $\sigma = \sigma_c$, after rescaling and conjugating the elements of the $\mathbb{H}$-basis $\{v_j^d \mid 1 \leq j \leq t_d\}$ of $L(d-1)$ by suitable scalars (see Lemma 2.3.4) the elements of the $\mathbb{H}$-basis, we may assume that

    - $\langle v_j^d, X^{d-1} v_j^d \rangle = \pm 1$ when $d \in \mathbb{O}_{\mathbf{d}}$, and
    - $\langle v_j^d, X^{d-1} v_j^d \rangle = \mathbf{j}$ when $d \in \mathbb{E}_{\mathbf{d}}$.

    If $\mathbb{D} = \mathbb{H}$, $\epsilon = -1$ and $\sigma = \sigma_c$, analogously we may assume that the elements of the $\mathbb{H}$-basis $\{v_j^d \mid 1 \leq j \leq t_d\}$ of $L(d-1)$ satisfy

    - $\langle v_j^d, X^{d-1} v_j^d \rangle = \pm 1$ when $d \in \mathbb{E}_{\mathbf{d}}$, and
    - $\langle v_j^d, X^{d-1} v_j^d \rangle = \mathbf{j}$ when $d \in \mathbb{O}_{\mathbf{d}}$.

    In particular, $(v_1^d, \ldots, v_{t_d}^d)$ is an orthogonal basis of $L(d-1)$ with respect to $(\cdot, \cdot)_d$ for all $d \in \mathbb{N}_{\mathbf{d}}$.

□

Let $(v_1^d, \ldots, v_{t_d}^d)$ be an ordered $\mathbb{D}$-basis of $L(d-1)$ as in Proposition 3.0.7 satisfying the properties as Remark 3.0.11. The proofs of the following lemmas are straightforward and they are omitted.

Lemma 3.0.12. *Let $\mathbb{D} = \mathbb{R}$, $\sigma = \mathrm{Id}$ and $\epsilon = 1$. Fix $d \in \mathbb{N}_{\mathbf{d}}$ and $1 \leq j \leq t_d$.*

1. If $\eta \in \mathbb{E}_{\mathbf{d}}$ and j is odd, define

$$w_{jl}^{\eta} := \begin{cases} \left(X^l v_j^{\eta} + X^{\eta-1-l} v_{j+1}^{\eta}\right)\frac{1}{\sqrt{2}} & \text{if } 0 \leq l \leq \eta - 1 \\ \left(X^{2\eta-1-l} v_j^{\eta} - X^{l-\eta} v_{j+1}^{\eta}\right)\frac{1}{\sqrt{2}} & \text{if } \eta \leq l \leq 2\eta - 1. \end{cases}$$

Then $\langle w^\eta_{jl},\, w^\eta_{j(2\eta-1-l)}\rangle \,=\, 0$*, and* $\langle w^\eta_{jl},\, w^\eta_{jl}\rangle \,=\, (-1)^l\,\langle v^\eta_j,\, X^{\eta-1}v^\eta_{j+1}\rangle$.

2. *For* $\theta \,\in\, \mathbb{O}_{\mathbf{d}}$*, define*

$$w^\theta_{jl} \,:=\, \begin{cases} \left(X^l v^\theta_j + X^{\theta-1-l}v^\theta_j\right)\frac{1}{\sqrt{2}} & \text{if } \; 0 \,\leq\, l \,<\, (\theta-1)/2 \\ X^l v^\theta_j & \text{if } \; l \,=\, (\theta-1)/2 \\ \left(X^{\theta-1-l}v^\theta_j - X^l v^\theta_j\right)\frac{1}{\sqrt{2}} & \text{if } \; (\theta-1)/2 \,<\, l \,\leq\, \theta-1. \end{cases}$$

Then $\langle w^\theta_{jl},\, w^\theta_{j(\theta-1-l)}\rangle \,=\, 0$ *and*

$$\langle w^\theta_{jl},\, w^\theta_{jl}\rangle \,=\, \begin{cases} (-1)^l\langle v^\theta_j, X^{\theta-1}v^\theta_j\rangle & \text{if } \; 0 \,\leq\, l \,<\, (\theta-1)/2 \\ (-1)^l\langle v^\theta_j, X^{\theta-1}v^\theta_j\rangle & \text{if } \; l \,=\, (\theta-1)/2 \\ (-1)^{l+1}\langle v^\theta_j, X^{\theta-1}v^\theta_j\rangle & \text{if } \; (\theta-1)/2 \,<\, l \,\leq\, \theta-1. \end{cases}$$

Therefore, for any $\theta \,\in\, \mathbb{O}_{\mathbf{d}}$*,*

$$\langle w^\theta_{jl},\, w^\theta_{jl')}\rangle \,=\, 0$$

when $l \,\neq\, l'$ *and* $0 \,\leq\, l,\, l'\,\theta-1$.

Lemma 3.0.13. *Let* $\mathbb{D} \,=\, \mathbb{C}$*,* $\sigma \,=\, \sigma_c$ *and* $\epsilon \,=\, 1$*. Fix* $d \,\in\, \mathbb{N}_{\mathbf{d}}$ *and* $1 \,\leq\, j \,\leq\, t_d$*.*

1. *For* $\eta \,\in\, \mathbb{E}_{\mathbf{d}}$*, define*

$$\widetilde{w}^\eta_{jl} \,:=\, \begin{cases} \left(X^l v^\eta_j + X^{\eta-1-l}v^\eta_j\sqrt{-1}\right)\frac{1}{\sqrt{2}} & \text{if } \; 0 \,\leq\, l \,<\, \eta/2 \\ \left(X^{\eta-1-l}v^\eta_j - X^l v^\eta_j\sqrt{-1}\right)\frac{1}{\sqrt{2}} & \text{if } \; \eta/2 \,\leq\, l \,\leq\, \eta-1. \end{cases}$$

Then $\langle \widetilde{w}^\eta_{jl},\, \widetilde{w}^\eta_{j(\eta-1-l)}\rangle \,=\, 0$ *and* $\langle \widetilde{w}^\eta_{jl},\, \widetilde{w}^\eta_{jl}\rangle \,=\, (-1)^l\sqrt{-1}\langle v^\eta_j, X^{\eta-1}v^\eta_j\rangle$.

2. *For $\theta \in \mathbb{O}_{\mathbf{d}}$, define*

$$\widetilde{w}_{jl}^{\theta} := \begin{cases} \left(X^l v_j^{\theta} + X^{\theta-1-l} v_j^{\theta}\right)\frac{1}{\sqrt{2}} & \text{if } 0 \le l < (\theta-1)/2 \\ X^l v_j^{\theta} & \text{if } l = (\theta-1)/2 \\ \left(X^{\theta-1-l} v_j^{\theta} - X^l v_j^{\theta}\right)\frac{1}{\sqrt{2}} & \text{if } (\theta-1)/2 < l \le \theta-1. \end{cases}$$

Then $\langle \widetilde{w}_{jl}^{\theta}, \widetilde{w}_{j(\theta-1-l)}^{\theta} \rangle = 0$, and

$$\langle \widetilde{w}_{jl}^{\theta}, \widetilde{w}_{jl}^{\theta} \rangle = \begin{cases} (-1)^l \langle v_j^{\theta}, X^{\theta-1} v_j^{\theta} \rangle & \text{if } 0 \le l < (\theta-1)/2 \\ (-1)^l \langle v_j^{\theta}, X^{\theta-1} v_j^{\theta} \rangle & \text{if } l = (\theta-1)/2 \\ (-1)^{l+1} \langle v_j^{\theta}, X^{\theta-1} v_j^{\theta} \rangle & \text{if } (\theta-1)/2 < l \le \theta-1. \end{cases}$$

Therefore, for any $\theta \in \mathbb{O}_{\mathbf{d}}$,

$$\langle \widetilde{w}_{jl}^{\theta}, \widetilde{w}_{jl')}^{\theta} \rangle = 0$$

when $l \neq l'$ and $0 \le l, l' \le \theta - 1$.

Lemma 3.0.14. *Let $\mathbb{D} = \mathbb{H}$, $\sigma = \sigma_c$ and $\epsilon = 1$. Fix d and $1 \le j \le t_d$.*

1. *For $\eta \in \mathbb{E}_{\mathbf{d}}$, define*

$$\widehat{w}_{jl}^{\eta} := \begin{cases} \left(X^l v_j^{\eta} + X^{\eta-1-l} v_j^{\eta} \alpha_j\right)\frac{1}{\sqrt{2}} & \text{if } 0 \le l < \eta/2 \\ \left(X^{\eta-1-l} v_j^{\eta} - X^l v_j^{\eta} \alpha_j\right)\frac{1}{\sqrt{2}} & \text{if } \eta/2 \le l \le \eta-1 \end{cases}$$

where $\alpha_j = \langle v_j^{\eta}, X^{\eta-1} v_j^{\eta} \rangle$. Then

$$\langle \widehat{w}_{jl}^{\eta}, \widehat{w}_{j(\eta-1-l)}^{\eta} \rangle = 0 \quad \text{and} \quad \langle \widehat{w}_{jl}^{\eta}, \widehat{w}_{jl}^{\eta} \rangle = (-1)^{l+1} \mathrm{Nrd}(\langle v_j^{\eta}, X^{\eta-1} v_j^{\eta} \rangle).$$

2. *When* $\theta \in \mathbb{O}_{\mathbf{d}}$, *define*

$$\widehat{w}_{jl}^{\theta} := \begin{cases} \left(X^l v_j^{\theta} + X^{\theta-1-l} v_j^{\theta}\right)\frac{1}{\sqrt{2}} & \text{if } 0 \le l < (\theta-1)/2 \\ X^l v_j^{\theta} & \text{if } l = (\theta-1)/2 \\ \left(X^{\theta-1-l} v_j^{\theta} - X^l v_j^{\theta}\right)\frac{1}{\sqrt{2}} & \text{if } (\theta-1)/2 < l \le \theta - 1. \end{cases}$$

Then $\langle \widehat{w}_{jl}^{\theta}, \widehat{w}_{j(\theta-1-l)}^{\theta} \rangle = 0$, *and*

$$\langle \widehat{w}_{jl}^{\theta}, \widehat{w}_{jl}^{\theta} \rangle = \begin{cases} (-1)^l \langle v_j^{\theta}, X^{\theta-1} v_j^{\theta} \rangle & \text{if } 0 \le l < (\theta-1)/2 \\ (-1)^l \langle v_j^{\theta}, X^{\theta-1} v_j^{\theta} \rangle & \text{if } l = (\theta-1)/2 \\ (-1)^{l+1} \langle v_j^{\theta}, X^{\theta-1} v_j^{\theta} \rangle & \text{if } (\theta-1)/2 < l \le \theta - 1. \end{cases}$$

Therefore, for any $d \in \mathbb{N}_{\mathbf{d}}$,

$$\langle \widehat{w}_{jl}^{d}, \widehat{w}_{jl'}^{d} \rangle = 0$$

when $l \neq l'$ *and* $0 \le l, l' \le d - 1$.

The next corollary, which closely follows [CoMc, Lemma 9.3.1], gives a direct correspondence between the signature of $(\cdot, \cdot)_d$ on $L(d-1)$ and the signature of $\langle \cdot, \cdot \rangle$ on $M(d-1)$ when both $\langle \cdot, \cdot \rangle$ and $(\cdot, \cdot)_d$ have signatures. In part (3) of the corollary we record a correct version of a result in [CoMc, Lemma 9.3.1].

Corollary 3.0.15. *Let* $\langle \cdot, \cdot \rangle$ *be a* ϵ-σ *Hermitian form on* V. *Assume that* $\epsilon = 1$, *that is, the form* $\langle \cdot, \cdot \rangle$ *is symmetric or Hermitian.*

1. *If* $d \in \mathbb{E}_{\mathbf{d}}$ *then the signature of* $\langle \cdot, \cdot \rangle$ *on* $M(d-1)$ *is* $\left(\dim_{\mathbb{D}} M(d-1)/2, \dim_{\mathbb{D}} M(d-1)/2\right)$.

2. *If* $d \in \mathbb{O}_{\mathbf{d}}^1$, *and* (p_d, q_d) *is the signature of* $(\cdot, \cdot)_d$, *then the signature of* $\langle \cdot, \cdot \rangle$

on $M(d-1)$ *is*

$$((\dim_{\mathbb{D}} M(d-1) + p_d - q_d)/2,\ (\dim_{\mathbb{D}} M(d-1) + q_d - p_d)/2).$$

3. *If* $d \in \mathbb{O}^3_{\mathbf{d}}$, *and* (p_d, q_d) *is the signature of* $(\cdot, \cdot)_d$, *then the signature of* $\langle \cdot\, , \cdot \rangle$ *on* $M(d-1)$ *is*

$$((\dim_{\mathbb{D}} M(d-1) + q_d - p_d)/2,\ (\dim_{\mathbb{D}} M(d-1) + p_d - q_d)/2).$$

Proof. This follows directly from Lemmas 3.0.12, 3.0.13 and 3.0.14. □

Remark 3.0.16. We will now point out an error in [CoMc, p. 139, Lemma 9.3.1], and also explain why the definition of m^d_{id} in the case of $d \in \mathbb{O}^3_{\mathbf{d}}$ as in **Yd**.2 (in Section 2.2) is different from that in the case of $d \in \mathbb{E}_{\mathbf{d}} \cup \mathbb{O}^1_{\mathbf{d}}$. Let $\mathbb{D}$, V be as in Section 3, and let $\langle \cdot\, , \cdot \rangle$ be a Hermitian (respectively symmetric) form if $\mathbb{D} = \mathbb{H}, \mathbb{C}$ (respectively, $\mathbb{D} = \mathbb{R}$). Take a $\mathfrak{sl}_2(\mathbb{R})$-triple $\{X, H, Y\} \subset \mathfrak{su}(V, \langle \cdot\, , \cdot \rangle)$. Note that if $d \in \mathbb{O}^3_{\mathbf{d}}$, then the form $(\cdot, \cdot)_d$ in (3.8) is Hermitian (respectively, symmetric) when $\mathbb{D} = \mathbb{H}, \mathbb{C}$ (respectively, $\mathbb{D} = \mathbb{R}$). Let (p_d, q_d) be the signature of $(\cdot, \cdot)_d$ when $d \in \mathbb{O}^3_{\mathbf{d}}$. Corollary 3.0.15(3) says that the signature of the form $\langle \cdot\, , \cdot \rangle$ restricted to $M(d-1)$ is

$$((\dim_{\mathbb{D}} M(d-1) + q_d - p_d)/2,\ (\dim_{\mathbb{D}} M(d-1) + p_d - q_d)/2)$$

when $d \in \mathbb{O}^3_{\mathbf{d}}$. Set the signs in first column of the matrix (m^d_{ij}) as in **Yd**.1, and thus define $m^d_{i1} = +1$ when $1 \leq i \leq p_d$, and define $m^d_{i1} = -1$ when $p_d < i \leq t_d$.

However, in the case of $d \in \mathbb{O}^3_{\mathbf{d}}$, if we, following [CoMc, p. 139, Lemma 9.3.1], define $m^d_{ij} = (-1)^{j+1} m^d_{i1}$ for $1 < j \leq d$, then it can be easily verified that

$$\big(\mathrm{sgn}_+(m^d_{ij}),\ \mathrm{sgn}_-(m^d_{ij})\big) = \Big(\frac{\dim_{\mathbb{D}} M(d-1) + p_d - q_d}{2},\ \frac{\dim_{\mathbb{D}} M(d-1) + q_d - p_d}{2}\Big).$$

Thus, if $d \in \mathbb{O}^3_{\mathbf{d}}$ and $p_d \neq q_d$, then appealing to Corollary 3.0.15(3) we see that the signature of the form $\langle \cdot\,, \cdot \rangle$ restricted to $M(d-1)$ does not coincide with $(\mathrm{sgn}_+(m^d_{ij}), \mathrm{sgn}_-(m^d_{ij}))$. This shows that the second statement of [CoMc, p. 139, Lemma 9.3.1] is not true when $d \in \mathbb{O}^3_{\mathbf{d}}$ and $p_d \neq q_d$ (this means that $r \equiv 2 \pmod 4$ in the notation of [CoMc, p. 139, Lemma 9.3.1]). Recall that in **Yd**.2 (see Section 2.2), when $d \in \mathbb{O}^3_{\mathbf{d}}$ we have defined $m^d_{ij} = (-1)^{j+1} m^d_{i1}$ when $1 < j \leq d-1$ while $m^d_{id} := -m^d_{i1}$. Using the definitions of m^d_{i1} as above we have that

$$\big(\mathrm{sgn}_+(m^d_{ij}), \mathrm{sgn}_-(m^d_{ij})\big) = \Big(\frac{\dim_{\mathbb{D}} M(d-1) + q_d - p_d}{2}, \frac{\dim_{\mathbb{D}} M(d-1) + p_d - q_d}{2}\Big).$$

Thus, if we define m^d_{ij} as in **Yd**.1 and **Yd**.2, then the signature of $\langle \cdot\,, \cdot \rangle$ on $M(d-1)$ does coincide with $(\mathrm{sgn}_+(m^d_{ij}), \mathrm{sgn}_-(m^d_{ij}))$ for $d \in \mathbb{N}_{\mathbf{d}}$; see Remark 2.2.1. $\square$

Chapter 4

Parametrization of nilpotent orbits

In this chapter we describe certain parametrizations of the nilpotent orbits in non-compact non-complex simple real Lie algebras. We use the parametrizations to state the main results on the second and first cohomology groups of the nilpotent orbits in Chapters 6, 7, 8.

4.1 Nilpotent orbits in non-compact non-complex classical real Lie algebras

The results on the parametrizations of nilpotent orbits in non-compact non-complex classical real Lie algebras using Young diagrams and signed Young diagrams are well-known; e.g. see [CoMc, §9.3]. For the convenience of the readers, in this section we provide detailed proofs of these results.

4.1.1 Parametrization of nilpotent orbits in $\mathfrak{sl}_n(\mathbb{R})$

In this subsection we will recall a standard parametrization of $\mathcal{N}(\mathrm{SL}_n(\mathbb{R}))$, the set of all nilpotent orbits in $\mathfrak{sl}_n(\mathbb{R})$; see the last paragraph of §2.1 for notation. Let $X \in \mathcal{N}_{\mathfrak{sl}_n(\mathbb{R})}$ be a nilpotent element and $\mathcal{O}_X$ be the corresponding nilpotent orbit in $\mathfrak{sl}_n(\mathbb{R})$ under the adjoint action of $\mathrm{SL}_n(\mathbb{R})$. We first assume X to be non-zero. Let $\{X, H, Y\} \subset \mathfrak{sl}_n(\mathbb{R})$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple. Denoting $\mathbb{R}^n$, the right $\mathbb{R}$-vector space of column vectors, by V we recall that left multiplication by matrices in $\mathrm{M}_n(\mathbb{R})$ act as $\mathbb{R}$-linear transformations of $\mathbb{R}^n$. Let $\{d_1, \dots, d_s\}$ with $d_1 < \cdots < d_s$ be the finite set of natural numbers that occur as dimension of the non-zero irreducible $\mathrm{Span}_{\mathbb{R}}\{X, H, Y\}$-submodules of V. Recall that $M(d-1)$ is defined to be the isotypical component of V containing all irreducible submodules of V with highest weight $d-1$ and as in (3.1), we set $L(d-1) := V_{Y,0} \cap M(d-1)$. Let $t_{d_r} := \dim_{\mathbb{R}} L(d_r - 1)$ for $1 \leq r \leq s$. Then $\left[d_1^{t_{d_1}}, \dots, d_s^{t_{d_s}}\right] \in \mathcal{P}(n)$ (the set of partitions of n) because $\sum_{r=1}^{s} t_{d_r} d_r = n$. This induces a map, say,

$$\psi_{\mathfrak{sl}_n(\mathbb{R})} \colon \mathcal{N}_{\mathfrak{sl}_n(\mathbb{R})} \setminus \{0\} \longrightarrow \mathcal{P}(n)\,, \quad X \longmapsto \left[d_1^{t_{d_1}}, \dots, d_s^{t_{d_s}}\right]. \tag{4.1}$$

The above map $\psi_{\mathfrak{sl}_n(\mathbb{R})}$ has the following properties :

(4.2) $\psi_{\mathfrak{sl}_n(\mathbb{R})}(X) = \psi_{\mathfrak{sl}_n(\mathbb{R})}(hXh^{-1})$ for $h \in \mathrm{SL}_n(\mathbb{R})$.

(4.3) $\psi_{\mathfrak{sl}_n(\mathbb{R})}(X)$ does not depend on the $\mathfrak{sl}_2(\mathbb{R})$-triple $\{X, H, Y\}$ containing X.

First we will prove (4.2). Let $h \in \mathrm{SL}_n(\mathbb{R})$. Then it is easy to see that $\{hXh^{-1}$, hHh^{-1}, $hYh^{-1}\}$ is a $\mathfrak{sl}_2(\mathbb{R})$-triple in $\mathfrak{sl}_n(\mathbb{R})$. Considering V as $\mathrm{Span}_{\mathbb{R}}\{hXh^{-1}$, hHh^{-1}, $hYh^{-1}\}$-module, it follows that $hM(d-1)$ is the isotypical component of V containing all irreducible submodules V with highest weight $d-1$. Moreover, $hL(d_r-1) = V_{hYh^{-1},0} \cap hM(1-d_r)$. Therefore in both the cases we have the same partition $[d_1^{t_{d_1}}, \dots, d_s^{t_{d_s}}]$. This proves that $\psi_{\mathfrak{sl}_n(\mathbb{R})}(X) = \psi_{\mathfrak{sl}_n(\mathbb{R})}(hXh^{-1})$.

To prove (4.3), let $\{X, H', Y'\}$ be another $\mathfrak{sl}_2(\mathbb{R})$-triple in $\mathfrak{sl}_n(\mathbb{R})$ containing X. By Theorem 2.4.8, there exists $g \in \mathrm{SL}_n(\mathbb{R})$ such that $gXg^{-1} = X$, $gHg^{-1} = H'$, $gYg^{-1} = Y'$. Now (4.3) follows from (4.2).

It is easy to see that $\psi_{\mathfrak{sl}_n(\mathbb{R})}(X) \neq [1^n]$ when $X \neq 0$. We set $\psi_{\mathfrak{sl}_n(\mathbb{R})}(0) := [1^n]$. In view of (4.3) and (4.2), $\psi_{\mathfrak{sl}_n(\mathbb{R})}$ as in (4.1) induces a well-defined map

$$(4.4) \qquad \Psi_{\mathrm{SL}_n(\mathbb{R})} \colon \mathcal{N}(\mathrm{SL}_n(\mathbb{R})) \longrightarrow \mathcal{P}(n)\,, \quad \mathcal{O}_X \longmapsto \left[d_1^{t_{d_1}}, \ldots, d_s^{t_{d_s}}\right].$$

We next prove the well-known result which says that $\Psi_{\mathrm{SL}_n(\mathbb{R})}$ is "almost" a parametrization of the nilpotent orbits in $\mathfrak{sl}_n(\mathbb{R})$. Recall that $\mathcal{P}(n)$ denote the set of all partitions of n and $\mathcal{P}_{\mathrm{even}}(n)$ is the subset of $\mathcal{P}(n)$ consisting of all even partitions of n; see §2.2. We need the following lemma.

Lemma 4.1.1. *Let* $\mathbf{d} \in \mathcal{P}_{\mathrm{even}}(n)$*, and let* $X_{\mathbf{d}} \in \mathfrak{sl}_n(\mathbb{R})$ *be as in* (2.21)*. Let* $T \in \mathrm{GL}_n(\mathbb{R})$ *be such that* $TX_{\mathbf{d}} = X_{\mathbf{d}}T$*. Then* $\det T > 0$*.*

Proof. In this proof $\mathbb{D}$ stands for either $\mathbb{C}, \mathbb{R}$. Recall that $\mathbb{D}^n$ denotes the space of column vectors with entries in $\mathbb{D}$. We let the matrices in $\mathrm{M}_n(\mathbb{D})$ act on $\mathbb{D}^n$ by left multiplications. For $w = (x_1, \ldots, x_n)^t \in \mathbb{C}^n$ we set $\overline{w} = (\sigma_c(x_1), \ldots, \sigma_c(x_n))^t$ where 'σ_c' is the usual conjugation on $\mathbb{C}$.

Using the multiplicative Jordan decomposition it is enough to assume that T is a semisimple matrix in $\mathrm{GL}_n(\mathbb{R})$. For $\alpha \in \mathbb{C}$ we set $E_\alpha := \{v \in \mathbb{C}^n \mid Tv = \alpha v\}$. As $T \in \mathrm{GL}_n(\mathbb{R})$, if $\mu \in \mathbb{C}$ is an eigenvalue of T then so is $\sigma_c(\mu)$. Let $\lambda_1, \ldots, \lambda_r$; $\mu_1, \sigma_c(\mu_1), \ldots, \mu_s, \sigma_c(\mu_s)$ be all the eigenvalues of T where $\lambda_i \in \mathbb{R}$ for $1 \leq i \leq r$ and $\mu_j \in \mathbb{C} \setminus \mathbb{R}$ for $1 \leq j \leq s$. Then we have the decomposition

$$\mathbb{C}^n = (E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_r}) \bigoplus (E_{\mu_1} \oplus E_{\sigma_c(\mu_1)}) \oplus \cdots \oplus (E_{\mu_s} \oplus E_{\sigma_c(\mu_s)})$$

of $\mathbb{C}^n$ into eigenspaces of T. Set $E_{\lambda_i}(\mathbb{R}) := E_{\lambda_i} \cap \mathbb{R}^n$ and $F_{\mu_j} := \{w + \overline{w} \mid w \in E_{\mu_j}\}$.

Now it follows that

$$\mathbb{R}^n = \left(E_{\lambda_1}(\mathbb{R}) \oplus \cdots \oplus E_{\lambda_r}(\mathbb{R})\right) \bigoplus \left(F_{\mu_1} \oplus \cdots \oplus F_{\mu_s}\right). \tag{4.5}$$

Note that $\det(T|_{F_{\mu_j}}) > 0$. Thus to show $\det T > 0$, it is enough to prove $\dim_{\mathbb{R}} E_{\lambda_i}(\mathbb{R})$ is even for all $i = 1, \ldots, r$. Since $TX_{\mathbf{d}} = X_{\mathbf{d}}T$ and $X_{\mathbf{d}} \in \mathrm{M}_n(\mathbb{R})$, each direct summand in (4.5) remains invariant under $X_{\mathbf{d}}$. Let $X_{\lambda_i} := X_{\mathbf{d}}|_{E_{\lambda_i}}$ for $1 \leq i \leq r$ and $X_{\mu_j} := X_{\mathbf{d}}|_{F_{\mu_j}}$ for $1 \leq j \leq s$. Then X_{λ_i}, X_{μ_j} are nilpotent. For each $i = 1, \ldots, r$ and $j = 1, \ldots, s$ we define $H_{\lambda_i}, Y_{\lambda_i} \in \mathfrak{sl}(E_{\lambda_i})$ and $H_{\mu_j}, Y_{\mu_j} \in \mathfrak{sl}(F_{\mu_j})$ as in the following way:

- If $X_{\lambda_i} = 0$ we set $H_{\lambda_i} = Y_{\lambda_i} = 0$, and similarly if $X_{\mu_j} = 0$ we set $H_{\mu_j} = Y_{\mu_j} = 0$.
- If $X_{\lambda_i} \neq 0$ and $X_{\mu_j} \neq 0$ we let $\{X_{\lambda_i}, H_{\lambda_i}, Y_{\lambda_i}\}$, $\{X_{\mu_j}, H_{\mu_j}, Y_{\mu_j}\}$ be $\mathfrak{sl}_2(\mathbb{R})$-triples in $\mathfrak{sl}(E_{\lambda_i})$ and $\mathfrak{sl}(F_{\mu_j})$, respectively.

We now define

$$H := H_{\lambda_1} \oplus \cdots \oplus H_{\lambda_r} \oplus H_{\mu_1} \oplus \cdots \oplus H_{\mu_s} \text{ and } Y := Y_{\lambda_1} \oplus \cdots \oplus Y_{\lambda_r} \oplus Y_{\mu_1} \oplus \cdots \oplus Y_{\mu_s}.$$

Then clearly $\{X_{\mathbf{d}}, H, Y\}$ is a $\mathfrak{sl}_2(\mathbb{R})$-triple in $\mathfrak{sl}_n(\mathbb{R})$. Moreover, for all $i = 1, \ldots, r$ the spaces $E_{\lambda_i}(\mathbb{R})$ are $\{X_{\mathbf{d}}, H, Y\}$-submodules. Since $\mathbf{d} \in \mathcal{P}_{even}(n)$, each irreducible $\mathrm{Span}_{\mathbb{R}}\{X_{\mathbf{d}}, H, Y\}$-submodule of $\mathbb{R}^n$ has even dimension. Hence $\dim_{\mathbb{R}} E_{\lambda_i}(\mathbb{R})$ is even for $1 \leq i \leq r$. □

Theorem 4.1.2 ([CoMc, Theorem 9.3.3]). *For the map $\Psi_{\mathrm{SL}_n(\mathbb{R})}$ in (4.4),*

$$\#\Psi^{-1}_{\mathrm{SL}_n(\mathbb{R})}(\mathbf{d}) = \begin{cases} 1 & \text{for all } \mathbf{d} \in \mathcal{P}(n) \setminus \mathcal{P}_{\mathrm{even}}(n) \\ 2 & \text{for all } \mathbf{d} \in \mathcal{P}_{\mathrm{even}}(n). \end{cases}$$

Proof. First we will show that the map $\Psi_{\mathrm{SL}_n(\mathbb{R})}$ is surjective. This follows easily from Remark 2.4.2 and Remark 3.0.1 by applying them in the case $\mathbb{D} = \mathbb{R}$. Let $\mathbf{d} \in \mathcal{P}(n)$. Let $X_{\mathbf{d}} \in \mathrm{M}_n(\mathbb{D})$ be as in (2.21) and $H_{\mathbf{d}}, Y_{\mathbf{d}}$ be as in (2.22). We consider the space of column vectors $\mathbb{D}^n$ as a $\mathrm{Span}_{\mathbb{R}}\{X_{\mathbf{d}}, H_{\mathbf{d}}, Y_{\mathbf{d}}\}$-module (under the usual left multiplication of matrices from $\mathrm{M}_n(\mathbb{D})$ on the column vectors $\mathbb{D}^n$). Let $\mathbb{N}_{\mathbf{d}} := \{d_i \mid 1 \leq i \leq s\}$; see (2.1) for the definition. Then from (3.2) it follows immediately that $\Psi_{\mathrm{SL}_n(\mathbb{R})}(\mathcal{O}_{X_{\mathbf{d}}}) = \mathbf{d}$.

Next we compute the cardinality of the fiber of the map $\Psi_{\mathrm{SL}_n(\mathbb{R})}$. For $\mathbf{d} = [d_1^{t_{d_1}}, \dots, d_s^{t_{d_s}}] \in \mathcal{P}(n)$, let $X, Y \in \mathfrak{sl}_n(\mathbb{R})$ be such that $\Psi_{\mathrm{SL}_n(\mathbb{R})}(\mathcal{O}_X) = \Psi_{\mathrm{SL}_n(\mathbb{R})}(\mathcal{O}_Y) = \mathbf{d}$. Then $gXg^{-1} = Y$ for some $g \in \mathrm{GL}_n(\mathbb{R})$. Without loss of generality, we may assume $X = X_{\mathbf{d}}$, where $X_{\mathbf{d}}$ is as in (2.21), and $\det g = \pm 1$. If $\det g = 1$, then $\mathcal{O}_X = \mathcal{O}_Y$. Now we will show that if $\det g = -1$, then

- $\mathcal{O}_X = \mathcal{O}_Y$ when $\mathbf{d} \notin \mathcal{P}_{\mathrm{even}}(n)$,
- $\mathcal{O}_X \neq \mathcal{O}_Y$ when $\mathbf{d} \in \mathcal{P}_{\mathrm{even}}(n)$.

For $\mathbf{d} \in \mathcal{P}(n) \setminus \mathcal{P}_{\mathrm{even}}(n)$, we assume that d_r is odd for some $r\,(1 \leq r \leq s)$. Set

$$A := (\mathrm{I}_{d_1})_{\blacktriangle}^{t_{d_1}} \oplus \cdots \oplus (\mathrm{I}_{d_{r-1}})_{\blacktriangle}^{t_{d_{r-1}}} \oplus (-\mathrm{I}_{d_r}) \oplus (\mathrm{I}_{d_r})_{\blacktriangle}^{t_{d_r}-1} \oplus \cdots \oplus (\mathrm{I}_{d_s})_{\blacktriangle}^{t_{d_s}}.$$

Then $AX_{\mathbf{d}}A^{-1} = X_{\mathbf{d}}$, $\det A = -1$ and $gA \in \mathrm{SL}_n(\mathbb{R})$. Thus $\mathcal{O}_X = \mathcal{O}_Y$ as $Y = (gA)X_{\mathbf{d}}(gA)^{-1}$. When $\mathbf{d} \in \mathcal{P}_{\mathrm{even}}(n)$ and $\det g = -1$, we conclude $\mathcal{O}_X \neq \mathcal{O}_Y$ using Lemma 4.1.1. □

4.1.2 Parametrization of nilpotent orbits in $\mathfrak{sl}_n(\mathbb{H})$

In this subsection we will recall a standard parametrization of $\mathcal{N}(\mathrm{SL}_n(\mathbb{H}))$; see §2.1 for notation. Let $X \in \mathfrak{sl}_n(\mathbb{H})$ be a nilpotent element and $\mathcal{O}_X$ be the corresponding nilpotent orbit in $\mathfrak{sl}_n(\mathbb{H})$ under the adjoint action of $\mathrm{SL}_n(\mathbb{H})$. Let $X \neq 0$ and $\{X, H, Y\} \subset \mathfrak{sl}_n(\mathbb{H})$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple. Denoting $\mathbb{H}^n$, the right $\mathbb{H}$-vector space of column vectors, by V we recall that left multiplication by matrices in $\mathrm{M}_n(\mathbb{H})$ act as $\mathbb{H}$-linear transformations of $\mathbb{H}^n$. Let $\{d_1, \dots, d_s\}$ with $d_1 < \cdots < d_s$ be the integers that occur as $\mathbb{R}$-dimensions of non-zero irreducible $\mathrm{Span}_{\mathbb{R}}\{X, H, Y\}$-submodules of V. Recall that $M(d-1)$ is defined to be the isotypical component of V containing all irreducible $\mathrm{Span}_{\mathbb{R}}\{X, H, Y\}$-submodules of V with highest weight $(d-1)$, and as in (3.1), we set $L(d-1) := V_{Y,0} \cap M(d-1)$. Recall that the space $L(d_r - 1)$ is a $\mathbb{H}$-subspace for $r = 1, \dots, s$. Let $t_{d_r} := \dim_{\mathbb{H}} L(d_r - 1)$ for $1 \le r \le s$. Then as $\sum_{r=1}^{s} t_{d_r} d_r = n$ we see that $[d_1^{t_{d_1}}, \dots, d_s^{t_{d_s}}] \in \mathcal{P}(n)$, the set of all partitions of n. This gives a map, say,

$$\psi_{\mathfrak{sl}_n(\mathbb{H})} \colon \mathcal{N}_{\mathfrak{sl}_n(\mathbb{H})} \setminus \{0\} \longrightarrow \mathcal{P}(n) \,, \quad X \longmapsto \left[d_1^{t_{d_1}}, \dots, d_s^{t_{d_s}}\right]. \tag{4.6}$$

By an argument similar to the one given for the map $\psi_{\mathfrak{sl}_n(\mathbb{R})}$ in (4.1), it follows that the map $\psi_{\mathfrak{sl}_n(\mathbb{H})}$ satisfies the following properties :

(4.7) $\psi_{\mathfrak{sl}_n(\mathbb{H})}(X) = \psi_{\mathfrak{sl}_n(\mathbb{H})}(gXg^{-1})$ for $g \in \mathrm{SL}_n(\mathbb{H})$.

(4.8) $\psi_{\mathfrak{sl}_n(\mathbb{H})}(X)$ does not depend on the $\mathfrak{sl}_2(\mathbb{R})$-triple $\{X, H, Y\}$ containing X.

It follows easily that $\psi_{\mathfrak{sl}_n(\mathbb{H})}(X) \neq [1^n]$ when $X \neq 0$. By declaring $\psi_{\mathfrak{sl}_n(\mathbb{H})}(0) = [1^n]$, we have a well-defined map

$$\Psi_{\mathrm{SL}_n(\mathbb{H})} : \mathcal{N}(\mathrm{SL}_n(\mathbb{H})) \longrightarrow \mathcal{P}(n) \,, \quad \mathcal{O}_X \longmapsto \left[d_1^{t_{d_1}}, \dots, d_s^{t_{d_s}}\right].$$

The following well-known result says that $\Psi_{\mathrm{SL}_n(\mathbb{H})}$ parametrizes the nilpotent orbits in $\mathfrak{sl}_n(\mathbb{H})$.

Theorem 4.1.3 ([CoMc, Theorem 9.3.3]). *The map $\Psi_{\mathrm{SL}_n(\mathbb{H})} \colon \mathcal{N}(\mathrm{SL}_n(\mathbb{H})) \longrightarrow \mathcal{P}(n)$ is a bijection.*

Proof. First we will show that the map $\Psi_{\mathrm{SL}_n(\mathbb{H})}$ is surjective. This follows easily from Remark 2.4.2 and Remark 3.0.1 by applying them in the case $\mathbb{D} = \mathbb{H}$. Let $\mathbf{d} \in \mathcal{P}(n)$. Let $X_{\mathbf{d}} \in \mathrm{M}_n(\mathbb{D})$ be as in (2.21) and $H_{\mathbf{d}}, Y_{\mathbf{d}}$ be as in (2.22). We consider the space of column vectors $\mathbb{D}^n$ as a $\mathrm{Span}_{\mathbb{R}}\{X_{\mathbf{d}}, H_{\mathbf{d}}, Y_{\mathbf{d}}\}$ -module (under the usual left multiplication of matrices from $\mathrm{M}_n(\mathbb{D})$ on the column vectors $\mathbb{D}^n$). Let $\mathbb{N}_{\mathbf{d}} := \{d_i \mid 1 \leq i \leq s\}$; see (2.1) for the definition. Then from (3.2) it follows immediately that $\Psi_{\mathrm{SL}_n(\mathbb{H})}(\mathcal{O}_{X_{\mathbf{d}}}) = \mathbf{d}$.

Next we will show that the map $\Psi_{\mathrm{SL}_n(\mathbb{H})}$ is injective. Let $\mathbf{d} = [d_1^{t_{d_1}}, \ldots, d_s^{t_{d_s}}] \in \mathcal{P}(n)$, and let $X, N \in \mathfrak{sl}_n(\mathbb{H})$ be two non-zero nilpotent elements such that $\Psi_{\mathrm{SL}_n(\mathbb{H})}(\mathcal{O}_X) = \Psi_{\mathrm{SL}_n(\mathbb{H})}(\mathcal{O}_N) = \mathbf{d}$. Let $V := \mathbb{H}^n$ be the right $\mathbb{H}$-vector space of column vectors. Using Proposition 3.0.3, V has two $\mathbb{H}$-bases of the form $\{X^l v_r^d \mid 0 \leq l \leq d-1, 1 \leq r \leq s, d \in \mathbb{N}_{\mathbf{d}}\}$ and $\{N^l u_r^d \mid 0 \leq l \leq d-1, 1 \leq r \leq s, d \in \mathbb{N}_{\mathbf{d}}\}$. Let $g \in \mathrm{GL}_n(\mathbb{H})$ be such that $g(X^l v_r^d) = N^l u_r^d$ for all $0 \leq l \leq d-1, 1 \leq r \leq s, d \in \mathbb{N}_{\mathbf{d}}$. Then $gX(X^l v_r^d) = N^{l+1} u_r^d = Ng(X^l v_r^d)$ for all $0 \leq l \leq d-1, 1 \leq r \leq s, d \in \mathbb{N}_{\mathbf{d}}$. This in turn shows that $gXg^{-1} = N$. As the reduced norm $\mathrm{Nrd}_{\mathrm{M}_n(\mathbb{H})}(g)$ is a positive real number (see Lemma 2.3.5), multiplying g by a suitable positive real number we obtain $g' \in \mathrm{SL}_n(\mathbb{H})$ such that $g'X = Ng'$. Hence $\mathcal{O}_X = \mathcal{O}_N$. This completes the proof of the theorem. □

4.1.3 Parametrization of nilpotent orbits in $\mathfrak{su}(p, q)$

Let n be a positive integer and (p, q) be a pair of non-negative integers such that $p + q = n$. As we are dealing with non-compact groups, we will further assume that $p > 0$ and $q > 0$. For $x = (x_1, \ldots, x_n)^t \in \mathbb{C}^n$ we set $\overline{x} := (\sigma_c(x_1), \ldots, \sigma_c(x_n))^t$ where 'σ_c' is the usual conjugation on $\mathbb{C}$. Throughout this subsection $\langle \cdot, \cdot \rangle$ denotes the Hermitian form on $\mathbb{C}^n$ defined by $\langle x, y \rangle := \overline{x}^t \mathrm{I}_{p,q} y$, where $\mathrm{I}_{p,q}$ is as in (2.19).

We begin by recalling a standard parametrization of the set of nilpotent orbits $\mathcal{N}(\mathrm{SL}_n(\mathbb{C}))$; see the last paragraph of §2.1 for notation. Let $X' \in \mathcal{N}_{\mathfrak{sl}_n(\mathbb{C})}$ be a nilpotent element. We first assume $X' \neq 0$. Let $\{X', H', Y'\} \subset \mathfrak{sl}_n(\mathbb{C})$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple. Let $V := \mathbb{C}^n$ be the right $\mathbb{C}$-vector space of column vectors. Let $\{c_1, \ldots, c_l\}$ with $c_1 < \cdots < c_l$ be the finitely many integers that occur as $\mathbb{R}$-dimensions of non-zero irreducible $\mathrm{Span}_{\mathbb{R}}\{X', H', Y'\}$-submodules of V. Recall that $M(c-1)$ is defined to be the isotypical component of V containing all irreducible $\mathrm{Span}_{\mathbb{R}}\{X', H', Y'\}$-submodules of V with highest weight $(c-1)$, and as in (3.1), we set $L(c-1) := V_{Y',0} \cap M(c-1)$. Recall that the space $L(c_r-1)$ is a $\mathbb{C}$-subspace for $1 \leq r \leq l$. Let $t_{c_r} := \dim_{\mathbb{C}} L(c_r - 1)$ for $1 \leq r \leq l$. Then as $\sum_{r=1}^{l} t_{c_r} c_r = n$ we have $[c_1^{t_{c_1}}, \ldots, c_l^{t_{c_l}}] \in \mathcal{P}(n)$. This induces a map, say,

$$\psi_{\mathfrak{sl}_n(\mathbb{C})} : \mathcal{N}_{\mathfrak{sl}_n(\mathbb{C})} \setminus \{0\} \longrightarrow \mathcal{P}(n)\,, \quad X' \longmapsto \left[c_1^{t_{c_1}}, \ldots, c_l^{t_{c_l}}\right]. \tag{4.9}$$

By an argument similar to the one given for the map $\psi_{\mathfrak{sl}_n(\mathbb{R})}$ in (4.1), it follows that $\psi_{\mathfrak{sl}_n(\mathbb{C})}(X') = \psi_{\mathfrak{sl}_n(\mathbb{C})}(gX'g^{-1})$ for $g \in \mathrm{SL}_n(\mathbb{C})$. In particular, using Theorem 2.4.8 it follows that the map $\psi_{\mathfrak{sl}_n(\mathbb{C})}(X')$ does not depend on the $\mathfrak{sl}_2(\mathbb{R})$-triple $\{X', H', Y'\}$ containing X'. Note that $\psi_{\mathfrak{sl}_n(\mathbb{C})}(X') \neq [1^n]$ when $X' \neq 0$. We set $\psi_{\mathfrak{sl}_n(\mathbb{C})}(0) := [1^n]$. It is a basic fact (see [CoMc, Theorem 5.1.1, p. 69]) that the map $\psi_{\mathfrak{sl}_n(\mathbb{C})}$ induces a well-defined bijection

$$\Psi_{\mathrm{SL}_n(\mathbb{C})} : \mathcal{N}\left(\mathrm{SL}_n(\mathbb{C})\right) \longrightarrow \mathcal{P}(n)\,, \quad \mathcal{O}_{X'} \longmapsto \left[c_1^{t_{c_1}}, \ldots, c_l^{t_{c_l}}\right]. \tag{4.10}$$

As $\mathrm{SU}(p,q) \subset \mathrm{SL}_n(\mathbb{C})$ (and consequently as, the set of nilpotent elements $\mathcal{N}_{\mathfrak{su}(p,q)} \subset \mathcal{N}_{\mathfrak{sl}_n(\mathbb{C})}$) we have the inclusion map, say, $\vartheta_{\mathfrak{su}(p,q)} : \mathcal{N}_{\mathfrak{su}(p,q)} \longrightarrow \mathcal{N}_{\mathfrak{sl}_n(\mathbb{C})}$. Let

$$\psi'_{\mathfrak{su}(p,q)} := \psi_{\mathfrak{sl}_n(\mathbb{C})} \circ \vartheta_{\mathfrak{su}(p,q)} : \mathcal{N}_{\mathfrak{su}(p,q)} \longrightarrow \mathcal{P}(n)$$

be the composition.

Let now $X \in \mathfrak{su}(p,q)$ be a nilpotent element, and $\mathcal{O}_X$ be the corresponding nilpotent orbit of X in $\mathfrak{su}(p,q)$, under the adjoint action of $\mathrm{SU}(p,q)$. Assume $X \neq 0$, and let $\{X, H, Y\} \subset \mathfrak{su}(p,q)$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple. As before, we let $V := \mathbb{C}^n$, the right $\mathbb{C}$-vector space of column vector $\mathbb{C}^n$. Analogously as above, we also enumerate the finite set of natural numbers of the form $\dim_{\mathbb{R}} Q$ for all the non-isomorphic non-zero irreducible $\mathrm{Span}_{\mathbb{R}}\{X, H, Y\}$-submodules Q of V by $\{d_1, \ldots, d_s\}$ in such a way that the relation $d_1 < \cdots < d_s$ is satisfied. Let $t_{d_r} := \dim_{\mathbb{C}} L(d_r - 1)$ for $1 \leq r \leq s$. Then $\mathbf{d} := [d_1^{t_{d_1}}, \ldots, d_s^{t_{d_s}}] \in \mathcal{P}(n)$, and moreover, $\psi'_{\mathfrak{su}(p,q)}(X) = \mathbf{d}$.

We now consider $\mathcal{S}_{\mathbf{d}}(p,q)$ as defined in (2.6), and assign an element $\mathbf{sgn}_X \in \mathcal{S}_{\mathbf{d}}(p,q)$ to the element $X \in \mathcal{N}_{\mathfrak{su}(p,q)}$. Let $\mathbb{N}_{\mathbf{d}} := \{d_i \mid 1 \leq i \leq s\}$; see (2.1) for the definition. For all $d \in \mathbb{N}_{\mathbf{d}}$, we first define a $t_d \times d$ matrix, say $(m^d_{ij}(X))$, in $\mathbf{A}_d$; see (2.5) for the definition. Recall that the form $(\cdot,\cdot)_d \colon L(d-1) \times L(d-1) \longrightarrow \mathbb{C}$, as defined in (3.8), is Hermitian or skew-Hermitian according as d is odd or even. Let (p_d, q_d) be the signature of $(\cdot,\cdot)_d$; see §2.3 for the definition of the signature of a skew-Hermitian form. Consider $\mathbb{E}_{\mathbf{d}}$, $\mathbb{O}_{\mathbf{d}}$ as defined in (2.1) and $\mathbb{O}^1_{\mathbf{d}}$, $\mathbb{O}^3_{\mathbf{d}}$ as defined in (2.2). Define,

$$m^d_{i1}(X) := \begin{cases} +1 & \text{if } \ 1 \leq i \leq p_d \\ -1 & \text{if } \ p_d < i \leq t_d \end{cases} ; \ d \in \mathbb{N}_{\mathbf{d}},$$

and

$$m^d_{ij}(X) := (-1)^{j+1} m^d_{i1}(X) \qquad \text{if } 1 < j \leq d, \ d \in \mathbb{E}_{\mathbf{d}} \cup \mathbb{O}^1_{\mathbf{d}}; \tag{4.11}$$

$$m^\theta_{ij}(X) := \begin{cases} (-1)^{j+1} m^\theta_{i1}(X) & \text{if } 1 < j \leq \theta - 1 \\ -m^\theta_{i1}(X) & \text{if } j = \theta \end{cases} , \ \theta \in \mathbb{O}^3_{\mathbf{d}}. \tag{4.12}$$

The way the matrices $(m^d_{ij}(X))$ are defined, immediately implies that they verify (**Yd**.1) and (**Yd**.2). Set $\mathbf{sgn}_X := ((m^{d_1}_{ij}(X)), \ldots, (m^{d_s}_{ij}(X)))$. It then follows from

Remark 2.2.1 and Corollary 3.0.15 that

$$\sum_{k=1}^{s} \mathrm{sgn}_+(m_{ij}^{d_k}(X)) = p\,, \quad \sum_{k=1}^{s} \mathrm{sgn}_-(m_{ij}^{d_k}(X)) = q.$$

In particular, we have $\mathbf{sgn}_X \in \mathcal{S}_{\mathbf{d}}(p,q)$. We next show that $\mathbf{sgn}_X = \mathbf{sgn}_{gXg^{-1}}$ for all $g \in \mathrm{SU}(p,q)$. Clearly $\{gXg^{-1}, gHg^{-1},\, gYg^{-1}\}$ is a $\mathfrak{sl}_2(\mathbb{R})$-triple in $\mathfrak{su}(p,q)$. It is also clear that $gM(d-1)$ is the isotypical component of V containing all irreducible $\mathrm{Span}_{\mathbb{R}}\{gXg^{-1},\, gHg^{-1},\, gYg^{-1}\}$-submodules of V with highest weight $d-1$. Moreover, $gL(d-1) = V_{gYg^{-1},0} \cap gM(d-1)$. As in (3.8), let $(\cdot,\cdot)'_d \colon gL(d-1) \times gL(d-1) \longrightarrow \mathbb{C}$ be defined by $(v,u)'_d := \langle v\,,\, (gXg^{-1})^{d-1}u\rangle$ for all $v,u \in gL(d-1)$. As $g \in \mathrm{SU}(p,q)$, for all $u,v \in L(d-1)$ we have

$$(u,v)_d = \langle u, X^{d-1}v\rangle = \langle gu, gX^{d-1}v\rangle = \langle gu, (gXg^{-1})^{d-1}gv\rangle = (gu,gv)'_d\,.$$

Hence the signatures of $(\cdot,\cdot)_d$ and $(\cdot,\cdot)'_d$ are the same for all $d \in \mathbb{N}_{\mathbf{d}}$. In particular, $\mathbf{sgn}_X = \mathbf{sgn}_{gXg^{-1}}$.

Thus we have a map

$$\psi_{\mathfrak{su}(p,\,q)} : \mathcal{N}_{\mathfrak{su}(p,q)} \longrightarrow \mathcal{Y}(p,\,q)\,, \quad X \longmapsto \big(\psi'_{\mathfrak{su}(p,q)}(X),\, \mathbf{sgn}_X\big)\,;$$

where $\mathcal{Y}(p,q)$ is as in (2.7). The map $\psi_{\mathfrak{su}(p,\,q)}$ satisfies the following properties :

(4.13) $\psi_{\mathfrak{su}(p,q)}(X) = \psi_{\mathfrak{su}(p,q)}(gXg^{-1})$ for all $g \in \mathrm{SU}(p,q)$.

(4.14) $\psi_{\mathfrak{su}(p,q)}(X)$ does not depend on the $\mathfrak{sl}_2(\mathbb{R})$-triple $\{X,H,Y\}$ containing X.

It is immediate from above that (4.13) holds. To prove (4.14), we let $\{X,H',Y'\}$ be another $\mathfrak{sl}_2(\mathbb{R})$-triple in $\mathfrak{su}(p,q)$ containing X. By Theorem 2.4.8, there exists $h \in \mathrm{SU}(p,q)$ such that $hXh^{-1} = X$, $hHh^{-1} = H'$, $hYh^{-1} = Y'$. Now (4.14) follows from (4.13).

Thus $\psi_{\mathfrak{su}(p,q)}$ induces a well-defined map

$$\Psi_{\mathrm{SU}(p,q)} : \mathcal{N}(\mathrm{SU}(p,q)) \longrightarrow \mathcal{Y}(p,q), \quad \mathcal{O}_X \longmapsto \big(\psi'_{\mathfrak{su}(p,q)}(X),\, \mathbf{sgn}_X\big)\,. \tag{4.15}$$

Using our terminologies we next state a standard result which says that the map above gives a parametrization of the nilpotent orbits in $\mathfrak{su}(p,q)$.

Theorem 4.1.4. *The map* $\Psi_{\mathrm{SU}(p,q)} \colon \mathcal{N}(\mathrm{SU}(p,q)) \longrightarrow \mathcal{Y}(p,q)$ *in* (4.15) *is a bijection.*

Remark 4.1.5. On account of the error in [CoMc, Lemma 9.3.1] mentioned in Remark 3.0.16, the parametrization in Theorem 4.1.4 is a modification of the one in [CoMc, Theorem 9.3.3]. □

Proof. We divide the proof in two steps.

Step 1 : In this step we prove that $\Psi_{\mathrm{SU}(p,q)}$ is injective. Let $X, N \in \mathfrak{su}(p,q)$ be two non-zero nilpotent elements such that $\Psi_{\mathrm{SU}(p,q)}(\mathcal{O}_X) = \Psi_{\mathrm{SU}(p,q)}(\mathcal{O}_N)$. Let $\mathbf{d} := \psi'_{\mathfrak{su}(p,q)}(X) = \psi'_{\mathfrak{su}(p,q)}(N)$. Let $\{X^l v^d_j \mid 0 \le l \le d-1, 1 \le j \le t_d, d \in \mathbb{N}_{\mathbf{d}}\}$ and $\{N^l w^d_j \mid 0 \le l \le d-1, 1 \le j \le t_d, d \in \mathbb{N}_{\mathbf{d}}\}$ be two $\mathbb{C}$-bases of $V = \mathbb{C}^n$, as in Proposition 3.0.7, which satisfy Remark 3.0.11 (2). We also have $\mathbf{sgn}_X = \mathbf{sgn}_N$. Thus, after reordering the ordered sets $(v^d_1, \dots, v^d_{t_d})$ and $(w^d_1, \dots, w^d_{t_d})$ for all $d \in \mathbb{N}_{\mathbf{d}}$, if necessary, we may assume that

$$\langle v^d_j, X^{d-1} v^d_j \rangle = \langle w^d_j, N^{d-1} w^d_j \rangle \ \text{ for all } \ 1 \le j \le t_d, d \in \mathbb{N}_{\mathbf{d}}. \tag{4.16}$$

Let $h \in \mathrm{GL}_n(\mathbb{C})$ be such that $h(X^l v^d_j) = N^l w^d_j$ for all $0 \le l \le d-1, 1 \le j \le t_d, d \in \mathbb{N}_{\mathbf{d}}$. Then

$$hX(X^l v^d_j) = hX^{l+1} v^d_j = N^{l+1} w^d_j = N(N^l w^d_j) = Nh(X^l v^d_j)$$

for all $0 \le l \le d-1, 1 \le j \le t_d, d \in \mathbb{N}_{\mathbf{d}}$. This in turn shows that $hXh^{-1} = N$. We next show that $h \in \mathrm{U}(p,q)$. Using the equalities in (4.16) above it follows that

$$\langle hX^l v_j^d, hX^{d-1-l}v_j^d\rangle = \langle N^l w_j^d, N^{d-1-l}w_j^d\rangle = (-1)^l\langle w_j^d, N^{d-1}w_j^d\rangle$$

$$= (-1)^l\langle v_j^d, X^{d-1}v_j^d\rangle = \langle X^l v_j^d, X^{d-1-l}v_j^d\rangle,$$

for all $0 \leq l \leq d-1, 1 \leq j \leq t_d, d \in \mathbb{N}_{\mathbf{d}}$. As $\{X^l v_j^d \mid 0 \leq l \leq d-1, 1 \leq j \leq t_d, d \in \mathbb{N}_{\mathbf{d}}\}$ is a $\mathbb{C}$-basis of V, it is now clear from the relations among the basis elements in Proposition 3.0.7(3) in the case of $\sigma = \sigma_c, \epsilon = 1, \mathbb{D} = \mathbb{C}$ that $h \in \mathrm{U}(p,q)$. Let $\alpha \in \mathbb{C}$ be such that $\alpha^n = \det h$, and let $h' = \alpha^{-1}h$. Then $h' \in \mathrm{SU}(p,q)$ and $h'Xh'^{-1} = gXg^{-1} = N$. Thus $\mathcal{O}_X = \mathcal{O}_N$ which proves the injectivity of the map $\Psi_{\mathrm{SU}(p,q)}$.

Step 2 : In this step we prove that $\Psi_{\mathrm{SU}(p,q)}$ is surjective. Let us fix a signed Young diagram $(\mathbf{d}, \mathbf{sgn}) \in \mathcal{Y}(p,q)$. We set $n = p+q$. Then $\mathbf{d} \in \mathcal{P}(n)$, and $\mathbf{sgn} \in \mathcal{S}_{\mathbf{d}}(p,q)$. Let $X \in \mathcal{N}_{\mathfrak{sl}_n(\mathbb{C})}$, and $\{X,H,Y\} \subset \mathfrak{sl}_n(\mathbb{C})$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple such that $\psi_{\mathfrak{sl}_n(\mathbb{C})}(X) = \mathbf{d}$; see (4.9) and (4.10). Our strategy is to obtain a $P \in \mathrm{GL}_n(\mathbb{C})$ such that $P^{-1}XP \in \mathfrak{su}(p,q)$ and $\mathbf{sgn}_{P^{-1}XP} = \mathbf{sgn}$.

We next construct a nondegenerate Hermitian form $\langle\cdot\,,\cdot\rangle_{\mathrm{new}}$ on $V = \mathbb{C}^n$ with signature (p,q) such that $\{X,H,Y\} \subset \mathfrak{su}(V, \langle\cdot\,,\cdot\rangle_{\mathrm{new}})$; see (2.15) for the definition of $\mathfrak{su}(V, \langle\cdot\,,\cdot\rangle_{\mathrm{new}})$. Let $\mathbf{d} := [d_1^{t_1}, \ldots, d_s^{t_s}]$. Using Proposition 3.0.3(2), $\mathbb{C}^n$ has a $\mathbb{C}$-basis of the form $\{X^l v_j^d \mid 0 \leq l \leq d-1, 1 \leq j \leq t_d, d \in \mathbb{N}_{\mathbf{d}}\}$. Let $\mathbf{sgn} := (M_{d_1}, \ldots, M_{d_s})$, and let p_d, q_d be the number of $+1$, -1, respectively, appearing in the 1st column of the matrix of M_d (of size $t_d \times d$) for all $d \in \mathbb{N}_{\mathbf{d}}$. For $d \in \mathbb{N}_{\mathbf{d}}$, $1 \leq j \leq t_d$ and for $0 \leq l, r \leq d-1$ we define $b(X^l v_j^d, X^r v_j^d) \in \mathbb{C}$ by

$$b(X^l v_j^d, X^r v_j^d) = 0 \text{ if } l + r \neq d-1$$

and

$$b(X^l v_j^d, X^{d-1-l} v_j^d) := \begin{cases} (-1)^l & \text{if } d \in \mathbb{O}_{\mathbf{d}},\ 1 \le j \le p_d \\ (-1)^{l+1} & \text{if } d \in \mathbb{O}_{\mathbf{d}},\ p_d < j \le t_d \\ \sqrt{-1}(-1)^{l+1} & \text{if } d \in \mathbb{E}_{\mathbf{d}},\ 1 \le j \le p_d \\ \sqrt{-1}(-1)^l & \text{if } d \in \mathbb{E}_{\mathbf{d}},\ p_d < j \le t_d. \end{cases} \tag{4.17}$$

It now follows that for $0 \le l, r \le d-1$

$$b(X^l v_j^d, X^r v_j^d) = \overline{b(X^r v_j^d, X^l v_j^d)}. \tag{4.18}$$

Recall that, for all $d \in \mathbb{N}_{\mathbf{d}}$, $1 \le j \le t_d$, the $\mathbb{R}$-Span of $\{v_j^d, Xv_j^d, \ldots, X^{d-1}v_j^d\}$ is an irreducible $\text{Span}_{\mathbb{R}}\{X, H, Y\}$-submodule of $\mathbb{C}^n$; see Lemma 3.0.2 (2). We set $V_j^d := \text{Span}_{\mathbb{C}}\{X^l v_j^d \mid 0 \le l \le d-1\}$. As $\{X^l v_j^d \mid 0 \le l \le d-1\}$ is a $\mathbb{C}$-basis for V_j^d the equalities in (4.18) allow us to define a Hermitian form $\langle \cdot\,, \cdot \rangle_{dj}$ on V_j^d such that

$$\langle X^l v_j^d, X^r v_j^d \rangle_{dj} \ = \ b(X^l v_j^d, X^r v_j^d) \quad \text{for } 0 \le l, r \le d-1. \tag{4.19}$$

From the definition it is clear that $\langle \cdot\,, \cdot \rangle_{dj}$ is nondegenerate on V_j^d, and moreover $\langle Xx, y \rangle_{dj} + \langle x, Xy \rangle_{dj} = 0$ for all $x, y \in V_j^d$. Recall that

$$\mathbb{C}^n = \bigoplus_{d \in \mathbb{N}_{\mathbf{d}}, 1 \le j \le t_d} V_j^d\,. \tag{4.20}$$

Let $\langle \cdot\,, \cdot \rangle_{\text{new}}$ be the new Hermitian form on $\mathbb{C}^n$ such that its restriction to V_j^d agrees with $\langle \cdot\,, \cdot \rangle_{dj}$, and so that (4.20) is an orthogonal direct sum with respect to $\langle \cdot\,, \cdot \rangle_{\text{new}}$. Then $\langle \cdot\,, \cdot \rangle_{\text{new}}$ is nondegenerate on $V \times V$. Clearly, $\langle Xx, y \rangle_{\text{new}} + \langle x, Xy \rangle_{\text{new}} = 0$ for all $x, y \in V$. Recall that in Proposition 3.0.3 (1) we have that $YX^l v_j^d \ = \ (X^{l-1}v_j^d) l(d-l)$ for $0 < l \le d-1$, $1 \le j \le t_d$, $d \in \mathbb{N}_{\mathbf{d}}$, and $Yv_j^d \ = \ 0$ for $1 \le j \le t_d$, $d \in \mathbb{N}_{\mathbf{d}}$. As $\{X^l v_j^d \mid 0 \le l \le d-1, 1 \le j \le t_d, d \in \mathbb{N}_{\mathbf{d}}\}$ is a basis of $\mathbb{C}^n$, using the above relations, (4.17) and (4.19), we conclude that $\langle Hx, y\rangle_{\text{new}} + \langle x, Hy\rangle_{\text{new}} = 0$ and $\langle Yx, y\rangle_{\text{new}} + \langle x, Yy\rangle_{\text{new}} = 0$ for all $x, y \in V$. Thus $\{X, H, Y\} \subset \mathfrak{su}(V, \langle\cdot\,,\cdot\rangle_{\text{new}})$.

We next show that the signature of $\langle\cdot\,,\cdot\rangle_{\text{new}}$ is (p, q). Let $d \in \mathbb{N}_{\mathbf{d}}$. Recall that $M(d-1)$ denotes the isotypical component of $\mathbb{C}^n$ containing all irreducible $\text{Span}_{\mathbb{R}}\{X, H, Y\}$-submodules of $\mathbb{C}^n$ with highest weight $(d-1)$, and $L(d-1) = V_{Y,0} \cap M(d-1)$; see (3.1). As in (3.8), let $(\cdot,\cdot)_{\text{new}_d} \colon L(d-1) \times L(d-1) \longrightarrow \mathbb{C}$ be defined by $(v, u)_{\text{new}_d} := \langle v, X^{d-1}u\rangle_{\text{new}}$ for all $v, u \in L(d-1)$. From the defining properties of $\langle\cdot\,,\cdot\rangle_{\text{new}}$ it follows that $M(d-1)$ is a direct sum of the subspaces $V^d_1, \ldots, V^d_{t_d}$ which are mutually orthogonal with respect to $\langle\cdot\,,\cdot\rangle_{\text{new}}$. In particular, $(v^d_1, \ldots, v^d_{t_d})$ is a orthogonal basis of $L(d-1)$ with respect to $(\cdot,\cdot)_{\text{new}_d}$. Using this orthogonal basis and putting $l = 0$, in (4.17), we obtain that the signature of $(\cdot,\cdot)_{\text{new}_d}$ is (p_d, q_d); see §2.3 for the definition of the signature of a skew-Hermitian form. Now from Remark 2.2.1 and Corollary 3.0.15 it follows that the signature of $\langle\cdot\,,\cdot\rangle_{\text{new}}$ on $M(d-1)$ is $(\text{sgn}_+ M_d, \text{sgn}_- M_d)$. Recall that, as $\mathbf{sgn} \in \mathcal{S}_{\mathbf{d}}(p, q)$, we have $\sum_{d \in \mathbb{N}_{\mathbf{d}}} \text{sgn}_+ M_d = p$ and $\sum_{d \in \mathbb{N}_{\mathbf{d}}} \text{sgn}_- M_d = q$. Thus the signature of $\langle\cdot\,,\cdot\rangle_{\text{new}}$ is (p, q).

Since the signatures of both the forms $\langle\cdot\,,\cdot\rangle_{\text{new}}$ and $\langle\cdot\,,\cdot\rangle$ coincide there is a $P \in \text{GL}_n(\mathbb{C})$ such that

$$\langle x, y\rangle = \langle Px, Py\rangle_{\text{new}} \quad \text{for all } x, y \in \mathbb{C}^n. \tag{4.21}$$

Clearly $\{P^{-1}XP,\, P^{-1}HP,\, P^{-1}YP\}$ is a $\mathfrak{sl}_2(\mathbb{R})$-triple in $\mathfrak{su}(p, q)$. Now we will show that $\mathbf{sgn}_{P^{-1}XP} = \mathbf{sgn}$. Note that $P^{-1}M(d-1)$ is the isotypical component of $\mathbb{C}^n$ containing all the irreducible $\text{Span}_{\mathbb{R}}\{P^{-1}XP,\, P^{-1}HP,\, P^{-1}YP\}$-submodules of $\mathbb{C}^n$ with highest weight $(d-1)$. Moreover, $P^{-1}L(d-1) = V_{P^{-1}YP,0} \cap P^{-1}M(d-1)$. As in (3.8), let $(\cdot,\cdot)''_d \colon P^{-1}L(d-1) \times P^{1-}L(d-1) \longrightarrow \mathbb{C}$ be defined by $(x, y)''_d := \langle x, (P^{-1}XP)^{d-1}y\rangle$ for all $x, y \in P^{-1}L(d-1)$. Using (4.21) it follows that

$$(u, v)''_d = (Pu, Pv)_{\text{new}_d} \quad \text{for } u, v \in P^{-1}L(d-1);\ d \in \mathbb{N}_{\mathbf{d}}.$$

Thus the signatures $(\cdot,\cdot)''_d$ and $(\cdot,\cdot)_{\mathrm{new}_d}$ are both equal, which, in particular, shows that signature of $(\cdot,\cdot)''_d$ is (p_d, q_d) for all $d \in \mathbb{N}_{\mathbf{d}}$. This proves that $\mathbf{sgn}_{P^{-1}XP} = \mathbf{sgn}$. Hence $\Psi_{\mathrm{SU}(p,q)}(\mathcal{O}_{P^{-1}XP}) = (\mathbf{d}, \mathbf{sgn})$. This completes the proof of the theorem. $\square$

4.1.4 Parametrization of nilpotent orbits in $\mathfrak{so}(p,q)$

Let n be a positive integer and (p,q) be a pair of non-negative integers such that $p+q=n$. We will further assume $p>0$ and $q>0$ as we deal with non-compact groups. Throughout this subsection $\langle\cdot\,,\cdot\rangle$ denotes the symmetric form on $\mathbb{R}^n$ defined by $\langle x,y\rangle := x^t \mathrm{I}_{p,q} y$, for $x,y \in \mathbb{R}^n$, where $\mathrm{I}_{p,q}$ is as in (2.19).

In this subsection we will describe a suitable parametrization of the nilpotent orbits in $\mathfrak{so}(p,q)$ under the adjoint action of $\mathrm{SO}(p,q)^\circ$. Let $\Psi_{\mathrm{SL}_n(\mathbb{R})} \colon \mathcal{N}(\mathrm{SL}_n(\mathbb{R})) \longrightarrow \mathcal{P}(n)$ be the parametrization of $\mathcal{N}(\mathrm{SL}_n(\mathbb{R}))$ as in Theorem 4.1.2. As $\mathrm{SO}(p,q) \subset \mathrm{SL}_n(\mathbb{R})$ (consequently as, the set of nilpotent elements $\mathcal{N}_{\mathfrak{so}(p,q)} \subset \mathcal{N}_{\mathfrak{sl}_n(\mathbb{R})}$) we have the inclusion map, say, $\vartheta_{\mathfrak{so}(p,q)} \colon \mathcal{N}_{\mathfrak{so}(p,q)} \longrightarrow \mathcal{N}_{\mathfrak{sl}_n(\mathbb{R})}$. Let

$$\psi'_{\mathfrak{so}(p,q)} := \psi_{\mathfrak{sl}_n(\mathbb{R})} \circ \vartheta_{\mathfrak{so}(p,q)} \,:\, \mathcal{N}_{\mathfrak{so}(p,q)} \longrightarrow \mathcal{P}(n)$$

be the composition map. Recall that $\psi'_{\mathfrak{so}(p,q)}(\mathcal{N}_{\mathfrak{so}(p,q)}) \subset \mathcal{P}_1(n)$ where $\mathcal{P}_1(n)$ is as in (2.3); this follows form the first paragraph of Remark 3.0.11. Let $X \in \mathfrak{so}(p,q)$ be a non-zero nilpotent element and $\mathcal{O}_X$ be the corresponding nilpotent orbit in $\mathfrak{so}(p,q)$ under the adjoint action of $\mathrm{SO}(p,q)^\circ$. Let $\{X,H,Y\} \subset \mathfrak{so}(p,q)$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple. Let $V := \mathbb{R}^n$ be the right $\mathbb{R}$-vector space of column vectors. Let $\{d_1,\dots,d_s\}$ with $d_1 < \cdots < d_s$ be the finitely many integers that occur as $\mathbb{R}$-dimensions of non-zero irreducible $\mathrm{Span}_{\mathbb{R}}\{X,H,Y\}$-submodules of V. Recall that $M(d-1)$ is defined to be the isotypical component of V containing all irreducible $\mathrm{Span}_{\mathbb{R}}\{X,H,Y\}$-submodules of V with highest weight $d-1$ and as in (3.1) we set $L(d-1) := V_{Y,0} \cap M(d-1)$. Let $t_{d_r} := \dim_{\mathbb{R}} L(d_r-1)$ for $1 \le r \le s$. Then $\mathbf{d} := [d_1^{t_{d_1}}, \dots, d_s^{t_{d_s}}] \in \mathcal{P}_1(n)$, and moreover, $\psi'_{\mathfrak{so}(p,q)}(X) = \mathbf{d}$.

We now consider $\mathcal{S}^{\text{even}}_{\mathbf{d}}(p,q)$ as defined in (2.8), and assign an element $\mathbf{sgn}_X \in \mathcal{S}^{\text{even}}_{\mathbf{d}}(p,q)$ to the element $X \in \mathcal{N}_{\mathfrak{so}(p,q)}$. Let $\mathbb{N}_{\mathbf{d}} := \{d_i \mid 1 \le i \le s\}$; see (2.1) for the definition. For all $d \in \mathbb{N}_{\mathbf{d}}$ we first define a $t_d \times d$ matrix, say $(m^d_{ij}(X))$, in $\mathbf{A}_d$; see (2.5) for the definition. Recall that the form $(\cdot,\cdot)_d \colon L(d-1) \times L(d-1) \longrightarrow \mathbb{R}$, as defined in (3.8), is symmetric or symplectic according as d is odd or even. Consider $\mathbb{E}_{\mathbf{d}}$, $\mathbb{O}_{\mathbf{d}}$ as defined in (2.1). Let (p_θ, q_θ) be the signature of $(\cdot,\cdot)_\theta$ when $\theta \in \mathbb{O}_{\mathbf{d}}$. Define,

$$m^\eta_{i1}(X) := +1 \qquad \text{if } \; 1 \le i \le t_\eta, \quad \eta \in \mathbb{E}_{\mathbf{d}}\,;$$

$$m^\theta_{i1}(X) := \begin{cases} +1 & \text{if } \; 1 \le i \le p_\theta \\ -1 & \text{if } \; p_\theta < i \le t_\theta \end{cases} ,\theta \in \mathbb{O}_{\mathbf{d}}\,;$$

and for $j > 1$ we define $(m^d_{ij}(X))$ as in (4.11) and (4.12). The way the matrices $(m^d_{ij}(X))$ are defined, immediately implies that they verify $(\mathbf{Yd}.1)$ and $(\mathbf{Yd}.2)$. Set $\mathbf{sgn}_X := ((m^{d_1}_{ij}(X)), \ldots, (m^{d_s}_{ij}(X)))$. It then follows from Remark 2.2.1 and Corollary 3.0.15 that

$$\sum_{k=1}^{s} \text{sgn}_+(m^{d_k}_{ij}(X)) = p\,, \quad \sum_{k=1}^{s} \text{sgn}_-(m^{d_k}_{ij}(X)) = q.$$

Now from the above definition of $m^\eta_{i1}(X)$ for $\eta \in \mathbb{E}_{\mathbf{d}}$ we have $\mathbf{sgn}_X \in \mathcal{S}^{\text{even}}_{\mathbf{d}}(p,q)$. We next show that $\mathbf{sgn}_X = \mathbf{sgn}_{gXg^{-1}}$ for all $g \in \mathrm{SO}(p,q)^\circ$. Clearly, $\{gXg^{-1}, gHg^{-1}, gYg^{-1}\}$ is a $\mathfrak{sl}_2(\mathbb{R})$-triple in $\mathfrak{so}(p,q)$. It is also straightforward that $gM(d-1)$ is the isotypical component of V containing all irreducible $\text{Span}_{\mathbb{R}}\{gXg^{-1}, gHg^{-1}, gYg^{-1}\}$-submodules of V with highest weight $d-1$. Moreover, $gL(d-1) = V_{gYg^{-1},0} \cap gM(d-1)$. As in (3.8) for $\theta \in \mathbb{O}_{\mathbf{d}}$, let $(\cdot,\cdot)'_\theta : gL(\theta-1) \times gL(\theta-1) \longrightarrow \mathbb{R}$ be defined by $(v,u)'_\theta := \langle v, (gXg^{-1})^{\theta-1}u \rangle$ for all $v, u \in gL(d-1)$. As $g \in \mathrm{SO}(p,q)^\circ$, for all $v, w \in L(\theta-1)$ we have

$$(v,w)_\theta = \langle v, X^{\theta-1}w \rangle = \langle gv, gX^{\theta-1}w \rangle = \langle gv, (gXg^{-1})^{\theta-1}gw \rangle = (gv, gw)'_\theta.$$

Hence, the signature of $(\cdot,\cdot)_\theta$ and $(\cdot,\cdot)'_\theta$ are same for all $\theta \in \mathbb{O}_{\mathbf{d}}$. In particular, $\mathbf{sgn}_X = \mathbf{sgn}_{gXg^{-1}} \in \mathcal{S}^{\text{even}}_{\mathbf{d}}(p,q)$.

Thus we have a map

$$\psi_{\mathfrak{so}(p,q)} : \mathcal{N}_{\mathfrak{so}(p,q)} \longrightarrow \mathcal{Y}^{\text{even}}_1(p,q), \quad X \longmapsto \big(\psi'_{\mathfrak{so}(p,q)}(X),\, \mathbf{sgn}_X\big)\,;$$

where $\mathcal{Y}^{\text{even}}_1(p,q)$ is as in (2.11). The map $\psi_{\mathfrak{so}(p,q)}$ satisfies the following properties:

(4.22) $\qquad \psi_{\mathfrak{so}(p,q)}(X) = \psi_{\mathfrak{so}(p,q)}(gXg^{-1})$ for all $g \in \mathrm{SO}(p,q)^\circ$.

(4.23) $\qquad \psi_{\mathfrak{so}(p,q)}(X)$ does not depend on the $\mathfrak{sl}_2(\mathbb{R})$-triple $\{X,H,Y\}$.

It is immediate from the above that (4.22) holds. To prove (4.23), let $\{X,H',Y'\}$ be another $\mathfrak{sl}_2(\mathbb{R})$-triple in $\mathfrak{so}(p,q)$ containing X. By Theorem 2.4.8, there exists $h \in \mathrm{SO}(p,q)^\circ$ such that $hXh^{-1} = X$, $hHh^{-1} = H'$, $hYh^{-1} = Y'$. Now (4.23) follows from (4.22).

Thus $\psi_{so(p,q)}$ induces a well-defined map

$$\Psi_{\mathrm{SO}(p,q)^\circ} : \mathcal{N}(\mathrm{SO}(p,q)^\circ) \longrightarrow \mathcal{Y}^{\text{even}}_1(p,q), \quad \mathcal{O}_X \longmapsto \big(\psi'_{\mathfrak{so}(p,q)}(X),\, \mathbf{sgn}_X\big). \tag{4.24}$$

Using our terminologies we next formulate a standard result which says that the map above "almost" parametrizes the set $\mathcal{N}(\mathrm{SO}(p,q)^\circ)$. Recall from §2.2 that $\mathcal{P}_{\text{v.even}}$ is the subset of $\mathcal{P}(n)$ consisting of all very even partitions of n, $\mathcal{P}_1(n)$ is as in (2.3) and $\mathcal{S}'_{\mathbf{d}}(p,q)$ is as in (2.14).

Theorem 4.1.6. *The map $\Psi_{\mathrm{SO}(p,q)^\circ}$ in (4.24) satisfies the property that*

$$\#\Psi^{-1}_{\mathrm{SO}(p,q)^\circ}(\mathbf{d},\mathbf{sgn}) = \begin{cases} 4 & \textit{for all } \mathbf{d} \in \mathcal{P}_{\text{v.even}}(n) \\ 2 & \textit{for all } \mathbf{d} \in \mathcal{P}_1(n) \setminus \mathcal{P}_{\text{v.even}}(n),\ \mathbf{sgn} \in \mathcal{S}'_{\mathbf{d}}(p,q) \\ 1 & \textit{otherwise}. \end{cases}$$

Remark 4.1.7. Taking into account the error in [CoMc, Lemma 9.3.1], as pointed out in Remark 3.0.16, the above parametrization in Theorem 4.1.6 is a modification of Theorem 9.3.4 in [CoMc]. □

Proof. We divide the proof in two steps.

Step 1 : In this step we prove that $\Psi_{\mathrm{SO}(p,q)^\circ}$ is surjective. Let us fix a signed Young diagram $(\mathbf{d}, \mathbf{sgn}) \in \mathcal{Y}_1^{\rm even}(p,q)$. Set $n = p+q$. Then $\mathbf{d} \in \mathcal{P}_1(n)$, and $\mathbf{sgn} \in \mathcal{S}_{\mathbf{d}}^{\rm even}(p,q)$. Let $X \in \mathcal{N}_{\mathfrak{sl}_n(\mathbb{R})}$, and $\{X,H,Y\} \subset \mathfrak{sl}_n(\mathbb{R})$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple such that $\psi_{\mathfrak{sl}_n(\mathbb{R})}(X) = \mathbf{d}$; see (4.1) and Theorem 4.1.2. Our strategy is to obtain a $P \in \mathrm{GL}_n(\mathbb{R})$ such that $P^{-1}XP \in \mathfrak{so}(p,q)$ and $\mathbf{sgn}_{P^{-1}XP} = \mathbf{sgn} \in \mathcal{S}_{\mathbf{d}}^{\rm even}(p,q)$.

We next construct a nondegenerate symmetric form $\langle \cdot\,, \cdot \rangle_{\rm new}$ on $V = \mathbb{R}^n$ with signature (p,q) such that $\{X,H,Y\} \subset \mathfrak{so}(V, \langle \cdot\,, \cdot \rangle_{\rm new})$; see (2.16) for the definition of $\mathfrak{so}(V, \langle \cdot\,, \cdot \rangle_{\rm new})$. Let $\mathbf{d} := [d_1^{t_{d_1}}, \dots, d_s^{t_{d_s}}]$. Using Proposition 3.0.3(2), $\mathbb{R}^n$ has a $\mathbb{R}$-basis of the form $\{X^l v_j^d \mid 0 \le l \le d-1, 1 \le j \le t_d, d \in \mathbb{N}_{\mathbf{d}}\}$. Let $\mathbf{sgn} := (M_{d_1}, \dots, M_{d_s})$, and let p_θ, q_θ be the number of $+1$, -1, respectively, appearing in the 1st column of the matrix of M_θ (of size $t_\theta \times \theta$) for all $\theta \in \mathbb{O}_{\mathbf{d}}$. For $d \in \mathbb{N}_{\mathbf{d}}$, $1 \le j \le t_d$ and $0 \le l, r \le d-1$ we define $b(X^l v_j^d, X^r v_j^d) \in \mathbb{R}$ by

$$\begin{aligned} b(X^l v_j^\theta, X^r v_j^\theta) &= 0 \quad \text{if } l+r \neq \theta - 1\,, \theta \in \mathbb{O}_{\mathbf{d}}\,; \\ b(X^l v_j^\eta, X^r v_j^\eta) &= 0 \quad \text{if } \eta \in \mathbb{E}_{\mathbf{d}}\,; \\ b(X^l v_j^\eta, X^r v_{j+1}^\eta) &= 0 \quad \text{if } l+r \neq \eta - 1\,, j \text{ odd}, \eta \in \mathbb{E}_{\mathbf{d}}\,; \\ b(X^l v_{j+1}^\eta, X^r v_j^\eta) &= 0 \quad \text{if } l+r \neq \eta - 1\,, j \text{ odd}, \eta \in \mathbb{E}_{\mathbf{d}}\,, \end{aligned}$$

and

$$b(X^l v_j^\theta, X^{\theta-1-l} v_j^\theta) := \begin{cases} (-1)^l & \text{when } 1 \le j \le p_\theta \\ (-1)^{l+1} & \text{when } p_\theta < j \le t_\theta \end{cases} ; \theta \in \mathbb{O}_{\mathbf{d}}, \tag{4.25}$$

$$
(4.26)\quad
\begin{aligned}
b(X^l v_j^\eta, X^{\eta-1-l} v_{j+1}^\eta) &:= (-1)^l && \text{when } j \text{ is odd, } \eta \in \mathbb{E}_{\mathbf{d}},\ 1 \le j \le t_\eta,\\
b(X^l v_{j+1}^\eta, X^{\eta-1-l} v_j^\eta) &:= (-1)^{l+1} && \text{when } j \text{ is odd, } \eta \in \mathbb{E}_{\mathbf{d}},\ 1 \le j \le t_\eta.
\end{aligned}
$$

It now follows that

(4.27)

$$
b(X^l v_j^\theta, X^r v_j^\theta) = b(X^r v_j^\theta, X^l v_j^\theta) \quad \text{for } \theta \in \mathbb{O}_{\mathbf{d}}, 0 \le l, r \le \theta - 1,
$$

(4.28)

$$
b(X^l v_{j'}^\eta, X^r v_{j''}^\eta) = b(X^r v_{j''}^\eta, X^l v_{j'}^\eta) \ \text{for } \eta \in \mathbb{E}_{\mathbf{d}}, 0 \le l, r \le \eta - 1,\ j \le j', j'' \le j+1\,.
$$

Recall that, for all $d \in \mathbb{N}_{\mathbf{d}}$, $1 \le j \le t_d$, the $\mathbb{R}$-Span of $\{v_j^d,\ X v_j^d,\ \ldots,\ X^{d-1} v_j^d\}$ is an irreducible $\mathrm{Span}_{\mathbb{R}}\{X, H, Y\}$-submodule of $\mathbb{R}^n$; see Lemma 3.0.2 (2). For $1 \le j \le t_\theta, \theta \in \mathbb{O}_{\mathbf{d}}$, we set $V_j^\theta := \mathrm{Span}_{\mathbb{R}}\{X^l v_j^\theta \mid 0 \le l \le \theta - 1\}$. For $\eta \in \mathbb{E}_{\mathbf{d}}$, and an odd integer j, $1 \le j \le t_\eta$, we set $V_j^\eta := \mathrm{Span}_{\mathbb{R}}\{X^l v_j^\eta, X^l v_{j+1}^\eta \mid 0 \le l \le \eta - 1\}$. As $\{X^l v_j^\theta \mid 0 \le l \le \theta - 1\}$ is a $\mathbb{R}$-basis for V_j^θ the equalities in (4.27) allow us to define a symmetric form $\langle\cdot\,,\cdot\rangle_{\theta j}$ on V_j^θ such that

$$
(4.29)\qquad \langle X^l v_j^\theta,\ X^r v_j^\theta\rangle_{\theta j} \ = \ b(X^l v_j^\theta,\ X^r v_j^\theta) \quad \text{for } 0 \le l, r \le \theta - 1.
$$

Similarly as $\{X^l v_j^\eta, X^l v_{j+1}^\eta \mid 0 \le l \le \eta - 1\}$ is a $\mathbb{R}$-basis for V_j^η the equalities in (4.28) allow us to define a symmetric form $\langle\cdot\,,\cdot\rangle_{\eta j}$ on V_j^η such that

$$
(4.30)\ \langle X^l v_{j'}^\eta,\ X^r v_{j''}^\eta\rangle_{\eta j} \ = \ b(X^l v_{j'}^\eta,\ X^r v_{j''}^\eta) \quad \text{for } 0 \le l, r \le \eta - 1,\ j \le j', j'' \le j+1\,.
$$

From the definition it is clear that for all $d \in \mathbb{N}_{\mathbf{d}}$, $\langle\cdot\,,\cdot\rangle_{dj}$ is nondegenerate on V_j^d and moreover, $\langle Xx, y\rangle_{dj} + \langle x, Xy\rangle_{dj} = 0$ for all $x, y \in V_j^d$. Recall that

$$
(4.31)\qquad \mathbb{R}^n = \Big(\bigoplus_{j \text{ odd}, 1 \le j \le t_\eta, \eta \in \mathbb{E}_{\mathbf{d}}} V_j^\eta\Big) \oplus \Big(\bigoplus_{1 \le j \le t_\theta, \theta \in \mathbb{O}_{\mathbf{d}}} V_j^\theta\Big)\,.
$$

Let $\langle\cdot\,,\cdot\rangle_{\text{new}}$ be the new symmetric form on $V = \mathbb{R}^n$ such that its restriction to

V_j^d agrees with $\langle \cdot\,,\cdot \rangle_{dj}$, and so that (4.31) is an orthogonal direct sum with respect to $\langle \cdot\,,\cdot \rangle_{\rm new}$. Then $\langle \cdot\,,\cdot \rangle_{\rm new}$ is non-degenerate on $V \times V$. Clearly, $\langle Xx, y \rangle_{\rm new} + \langle x, Xy \rangle_{\rm new} = 0$ for all $x, y \in V$. Recall that in Proposition 3.0.3 (1) we have that $YX^l v_j^d = (X^{l-1}v_j^d) l(d-l)$ for $0 < l \leq d-1$, $1 \leq j \leq t_d$, $d \in \mathbb{N}_{\mathbf{d}}$, and $Yv_j^d = 0$ for $1 \leq j \leq t_d$, $d \in \mathbb{N}_{\mathbf{d}}$. As $\{X^l v_j^d \mid 0 \leq l \leq d-1, 1 \leq j \leq t_d, d \in \mathbb{N}_{\mathbf{d}}\}$ is a basis of $\mathbb{R}^n$, using the above relations,(4.25), (4.26), (4.29) and (4.30), we conclude that $\langle Hx, y \rangle_{\rm new} + \langle x, Hy \rangle_{\rm new} = 0$ and $\langle Yx, y \rangle_{\rm new} + \langle x, Yy \rangle_{\rm new} = 0$ for all $x, y \in V$. Thus $\{X, H, Y\} \subset \mathfrak{so}(V, \langle \cdot\,,\cdot \rangle_{\rm new})$.

We next show that the signature of $\langle \cdot\,,\cdot \rangle_{\rm new}$ is (p, q). Let $d \in \mathbb{N}_{\mathbf{d}}$. Recall that $M(d-1)$ denotes the isotypical component of $\mathbb{R}^n$ containing all irreducible ${\rm Span}_{\mathbb{R}}\{X, H, Y\}$-submodules of $\mathbb{R}^n$ with highest weight $(d-1)$, and $L(d-1) = V_{Y,0} \cap M(d-1)$; see (3.1). As in (3.8), let $(\cdot, \cdot)_{{\rm new}_d} \colon L(d-1) \times L(d-1) \longrightarrow \mathbb{R}$ be defined by $(v, u)_{{\rm new}_d} := \langle v, X^{d-1}u \rangle_{\rm new}$ for all $v, u \in L(d-1)$. From the defining properties of $\langle \cdot\,,\cdot \rangle_{\rm new}$ it follows that $M(\theta - 1) = \bigoplus_{1 \leq j \leq t_\theta} V_j^\theta$ for $\theta \in \mathbb{O}_{\mathbf{d}}$ and $M(\eta - 1) = \bigoplus_{j \text{ odd}, 1 \leq j \leq t_\eta} V_j^\eta$ for $\eta \in \mathbb{E}_{\mathbf{d}}$ where both the direct sums are orthogonal with respect to $\langle \cdot\,,\cdot \rangle_{\rm new}$. In particular, $(v_1^\theta, \ldots, v_{t_\theta}^\theta)$ is a orthogonal basis of $L(\theta - 1)$ with respect to $(\cdot, \cdot)_{{\rm new}_\theta}$ for all $\theta \in \mathbb{O}_{\mathbf{d}}$. Using this orthogonal basis and putting $l = 0$, in (4.25), we obtain that the signature of $(\cdot, \cdot)_{{\rm new}_\theta}$ is (p_θ, q_θ). Now from Remark 2.2.1 and Corollary 3.0.15 it follows that the signature of $\langle \cdot\,,\cdot \rangle_{\rm new}$ on $M(d-1)$ is $({\rm sgn}_+ M_d,\, {\rm sgn}_- M_d)$. Recall that, as $\mathbf{sgn} \in \mathcal{S}_{\mathbf{d}}^{\rm even}(p, q)$, we have $\sum_{d \in \mathbb{N}_{\mathbf{d}}} {\rm sgn}_+ M_d = p$ and $\sum_{d \in \mathbb{N}_{\mathbf{d}}} {\rm sgn}_- M_d = q$. Thus the signature of $\langle \cdot\,,\cdot \rangle_{\rm new}$ is (p, q).

Since the signatures of both the forms $\langle \cdot\,,\cdot \rangle_{\rm new}$ and $\langle \cdot\,,\cdot \rangle$ coincide there is a $P \in {\rm GL}_n(\mathbb{R})$ such that

$$\langle x, y \rangle \;=\; \langle Px, Py \rangle_{\rm new} \quad \text{for all } x, y \in \mathbb{R}^n. \tag{4.32}$$

Clearly $\{P^{-1}XP,\, P^{-1}HP,\, P^{-1}YP\} \subset \mathfrak{so}(p, q)$ is a $\mathfrak{sl}_2(\mathbb{R})$-triple. Now we will show that $\mathbf{sgn}_{P^{-1}XP} = \mathbf{sgn} \in \mathcal{S}_{\mathbf{d}}^{\rm even}(p, q)$. Note that $P^{-1}M(d-1)$ is the isotypical component of $\mathbb{R}^n$ containing all the irreducible $\mathrm{Span}_{\mathbb{R}}\{P^{-1}XP,\, P^{-1}HP,\, P^{-1}YP\}$-submodules of $\mathbb{R}^n$ with highest weight $(d-1)$. Moreover, $P^{-1}L(d-1) = V_{P^{-1}YP,0} \cap P^{-1}M(d-1)$. As in (3.8) for $\theta \in \mathbb{O}_{\mathbf{d}}$, let $(\cdot,\cdot)''_\theta \colon P^{-1}L(\theta-1) \times P^{-1}L(\theta-1) \longrightarrow \mathbb{R}$ be defined by $(x,y)''_\theta := \langle x, (P^{-1}XP)^{\theta-1}y\rangle$ for all $x, y \in P^{-1}L(\theta-1)$. Using (4.32) it follows that

$$(u,v)''_\theta = (Pu, Pv)_{\mathrm{new}_\theta} \quad \text{ for } u, v \in P^{-1}L(d-1);\, \theta \in \mathbb{O}_{\mathbf{d}}.$$

Thus the signatures $(\cdot,\cdot)''_\theta$ and $(\cdot,\cdot)_{\mathrm{new}_\theta}$ are both equal, which, in particular, shows that signature of $(\cdot,\cdot)''_\theta$ is $(p_\theta,\, q_\theta)$ for all $\theta \in \mathbb{O}_{\mathbf{d}}$. This proves that $\mathbf{sgn}_{P^{-1}XP} = \mathbf{sgn} \in \mathcal{S}^{\mathrm{even}}_{\mathbf{d}}(p,q)$. Hence $\Psi_{\mathrm{SO}(p,q)^\circ}(\mathcal{O}_{P^{-1}XP}) = (\mathbf{d}, \mathbf{sgn})$.

Step 2 : In this step we will compute the cardinality of the fibers of the map in (4.24). To do this first we will prove that if $\Psi_{\mathrm{SO}(p,q)^\circ}(\mathcal{O}_X) = \Psi_{\mathrm{SO}(p,q)^\circ}(\mathcal{O}_N) = (\mathbf{d}, \mathbf{sgn})$ for some $X, N \in \mathcal{N}_{\mathfrak{so}(p,q)}$, then there exists $g \in \mathrm{O}(p,q)$ such that $X = gNg^{-1}$. Let $\mathbf{d} := \psi'_{\mathfrak{so}(p,q)}(X) = \psi'_{\mathfrak{so}(p,q)}(N)$. Let $\{X^l v^d_j \mid 0 \le l \le d-1, 1 \le j \le t_d, d \in \mathbb{N}_{\mathbf{d}}\}$ and $\{N^l w^d_j \mid 0 \le l \le d-1, 1 \le j \le t_d, d \in \mathbb{N}_{\mathbf{d}}\}$ be two $\mathbb{R}$-bases of $V = \mathbb{R}^n$, as in Proposition 3.0.7 which satisfy Remark 3.0.11 (1). We also have $\mathbf{sgn}_X = \mathbf{sgn}_N \in \mathcal{S}^{\mathrm{even}}_{\mathbf{d}}(p,q)$. Thus, after reordering the ordered sets $(v^d_1, \dots, v^d_{t_d})$ and $(w^d_1, \dots, w^d_{t_d})$ for all $d \in \mathbb{N}_{\mathbf{d}}$, if necessary, we may assume that

$$\begin{aligned} \langle v^\theta_j, X^{\theta-1} v^\theta_j\rangle &= \langle w^\theta_j, N^{\theta-1} w^\theta_j\rangle \qquad \text{for all } \theta \in \mathbb{O}_{\mathbf{d}},\ 1 \le j \le t_\theta; \\ \langle v^\eta_j, X^{\eta-1} v^\eta_{j+1}\rangle &= \langle w^\eta_j, N^{\eta-1} w^\eta_{j+1}\rangle \quad \text{ for all } \eta \in \mathbb{E}_{\mathbf{d}},\, 1 \le j \le t_\eta\,. \end{aligned} \tag{4.33}$$

Let $g \in \mathrm{GL}_n(\mathbb{R})$ be such that

$$g(X^l v^d_j) = N^l w^d_j \ \text{ for all } 0 \le l \le d-1,\, 1 \le j \le t_d,\, d \in \mathbb{N}_{\mathbf{d}}. \tag{4.34}$$

Then $gX(X^l v^d_j) = Ng(X^l v^d_j)$, which in turn implies $gX = Ng$. Using (4.33) and

(4.34) we observe that

$$\langle gX^l v_j^\theta,\, gX^{\theta-1-l}v_j^\theta\rangle = \langle X^l v_j^\theta,\, X^{\theta-1-l}v_j^\theta\rangle \quad \text{for } \theta\in\mathbb{O}_{\mathbf{d}},\, 1\le j\le t_\theta\,;$$
$$\langle gX^l v_j^\eta,\, gX^{\eta-1-l}v_{j+1}^\eta\rangle = \langle X^l v_j^\eta,\, X^{\eta-1-l}v_{j+1}^\eta\rangle \ \text{ for } \eta\in\mathbb{E}_{\mathbf{d}},\, 1\le j\le t_\eta\,.$$

As $\{X^l v_j^d \mid 0\le l\le d-1, 1\le j\le t_d, d\in\mathbb{N}_{\mathbf{d}}\}$ is a $\mathbb{R}$-basis of $\mathbb{R}^n$, it is now clear from the relations among the basis elements in Proposition 3.0.7(3) in the case of $\sigma=\mathrm{Id}, \epsilon=1, \mathbb{D}=\mathbb{R}$ that $g\in\mathrm{O}(p,q)$.

We appeal to Lemma 6.0.1 (4) and Proposition 6.4.5 to make the following observations.

1. If $\mathbf{d}\in\mathcal{P}_{\mathrm{v.even}}(n)$, then $\mathcal{Z}_{\mathrm{O}(p,q)}(X,H,Y)=\mathcal{Z}_{\mathrm{SO}(p,q)^\circ}(X,H,Y)$.
2. If $\mathbf{d}\in\mathcal{P}_1(n)\setminus\mathcal{P}_{\mathrm{v.even}}(n)$, $\mathbf{sgn}\in\mathcal{S}'_{\mathbf{d}}(p,q)$, then $\#(\mathcal{Z}_{\mathrm{O}(p,q)}(X,H,Y)/\mathcal{Z}_{\mathrm{SO}(p,q)^\circ}(X,H,Y))=2$.
3. In all other cases $\#(\mathcal{Z}_{\mathrm{O}(p,q)}(X,H,Y)/\mathcal{Z}_{\mathrm{SO}(p,q)^\circ}(X,H,Y))=4$.

As $\#\mathrm{O}(p,q)/\mathrm{SO}(p,q)^\circ=4$, in view of Lemma 2.3.7, the proof is completed. □

4.1.5 Parametrization of nilpotent orbits in $\mathfrak{so}^*(2n)$

Let n be a positive integer. In this subsection we describe a suitable parametrization of the nilpotent orbits in $\mathfrak{so}^*(2n)$. For $w=(x_1,\dots,x_n)^t\in\mathbb{H}^n$ we set $\overline{w}=(\sigma_c(x_1),\dots,\sigma_c(x_n))^t$ where σ_c is the conjugation on $\mathbb{H}$ as defined in §2.3. Throughout this subsection $\langle\cdot\,,\cdot\rangle$ denotes the skew-Hermitian form on $\mathbb{H}^n$ defined by $\langle x,y\rangle := \overline{x}^t\mathbf{j}\mathrm{I}_n y$, for $x,y\in\mathbb{H}^n$.

Let $\Psi_{\mathrm{SL}_n(\mathbb{H})}\colon\mathcal{N}(\mathrm{SL}_n(\mathbb{H}))\longrightarrow\mathcal{P}(n)$ be the parametrization as in Theorem 4.1.3. As $\mathrm{SO}^*(2n)\subset\mathrm{SL}_n(\mathbb{H})$ (consequently as, the set of nilpotent elements $\mathcal{N}_{\mathfrak{so}^*(2n)}\subset$

$\mathcal{N}_{\mathfrak{sl}_n(\mathbb{H})}$) we have the inclusion map, say, $\vartheta_{\mathfrak{so}^*(2n)} : \mathcal{N}_{\mathfrak{so}^*(2n)} \longrightarrow \mathcal{N}_{\mathfrak{sl}_n(\mathbb{H})}$. Let

$$\psi'_{\mathfrak{so}^*(2n)} := \psi_{\mathfrak{sl}_n(\mathbb{H})} \circ \vartheta_{\mathfrak{so}^*(2n)} : \mathcal{N}_{\mathfrak{so}^*(2n)} \longrightarrow \mathcal{P}(n).$$

be the composition. Let $X \in \mathfrak{so}^*(2n)$ be a nilpotent element and $\mathcal{O}_X$ be the corresponding nilpotent orbit in $\mathfrak{so}^*(2n)$. First assume that $X \neq 0$. Let $\{X, H, Y\} \subset \mathfrak{so}^*(2n)$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple. Let $V := \mathbb{H}^n$ be the right $\mathbb{H}$-vector space of column vectors. The left multiplication by matrices in $\mathrm{M}_n(\mathbb{H})$ act as $\mathbb{H}$-linear transformations of $\mathbb{H}^n$. Let $\{d_1, \dots, d_s\}$ with $d_1 < \cdots < d_s$ be the finitely many integers that occur as $\mathbb{R}$-dimensions of non-zero irreducible $\mathrm{Span}_{\mathbb{R}}\{X, H, Y\}$-submodules of V. Recall that $M(d-1)$ is defined to be the isotypical component of V containing all irreducible $\mathrm{Span}_{\mathbb{R}}\{X, H, Y\}$-submodules of V with highest weight $(d-1)$, and as in (3.1), we set $L(d-1) := V_{Y,0} \cap M(d-1)$. Recall that the space $L(d_r - 1)$ is a $\mathbb{H}$-subspace for $1 \leq r \leq s$. Let $t_{d_r} := \dim_{\mathbb{H}} L(d_r - 1)$ for $1 \leq r \leq s$. Then $\mathbf{d} := [d_1^{t_{d_1}}, \dots, d_s^{t_{d_s}}] \in \mathcal{P}(n)$, and moreover, $\psi'_{\mathfrak{so}^*(2n)}(X) = \mathbf{d}$.

We now consider $\mathcal{S}^{\mathrm{odd}}_{\mathbf{d}}(n)$ as defined in (2.9), and assign an element $\mathbf{sgn}_X \in \mathcal{S}^{\mathrm{odd}}_{\mathbf{d}}(n)$ to the element $X \in \mathcal{N}_{\mathfrak{so}^*(2n)}$. Let $\mathbb{N}_{\mathbf{d}} := \{d_i \mid 1 \leq i \leq s\}$; see (2.1) for the definition. For all $d \in \mathbb{N}_{\mathbf{d}}$ we first define a $t_d \times d$ matrix, say $(m^d_{ij}(X))$, in $\mathbf{A}_d$; see (2.5) for the definition. Recall that the form $(\cdot,\cdot)_d \colon L(d-1) \times L(d-1) \longrightarrow \mathbb{H}$, as defined in (3.8), is skew-Hermitian or Hermitian according as d is odd or even. Consider $\mathbb{E}_{\mathbf{d}}$, $\mathbb{O}_{\mathbf{d}}$ as defined in (2.1). Let (p_η, q_η) be the signature of $(\cdot,\cdot)_\eta$ when $\eta \in \mathbb{E}_{\mathbf{d}}$. Define,

$$m^\theta_{i1}(X) := +1 \qquad \text{if } \ 1 \leq i \leq t_\theta, \quad \theta \in \mathbb{O}_{\mathbf{d}};$$

$$m^\eta_{i1}(X) := \begin{cases} +1 & \text{if } \ 1 \leq i \leq p_\eta \\ -1 & \text{if } \ p_\eta < i \leq t_\eta \end{cases}, \ \eta \in \mathbb{E}_{\mathbf{d}}\,;$$

and for $j > 1$ we define $(m^d_{ij}(X))$ as in (4.11) and (4.12). The way the matrices

$(m_{ij}^{d}(X))$ are defined, immediately implies that they verify (**Yd**.1) and (**Yd**.2). Set $\mathbf{sgn}_X := ((m_{ij}^{d_1}(X)), \ldots, (m_{ij}^{d_s}(X)))$. It then follows from the above definitions of $m_{i1}^{\theta}(X)$, $\theta \in \mathbb{O}_{\mathbf{d}}$ that $\mathbf{sgn}_X \in \mathcal{S}_{\mathbf{d}}^{\rm odd}(n)$. We next show that $\mathbf{sgn}_X = \mathbf{sgn}_{gXg^{-1}} \in \mathcal{S}_{\mathbf{d}}^{\rm odd}(n)$ for all $g \in {\rm SO}^*(2n)$. Clearly, $\{gXg^{-1},\, gHg^{-1},\, gYg^{-1}\}$ is a $\mathfrak{sl}_2(\mathbb{R})$-triple in $\mathfrak{so}^*(2n)$. It is also clear that $gM(d-1)$ is the isotypical component of V containing all irreducible ${\rm Span}_{\mathbb{R}}\{gXg^{-1},\, gHg^{-1},\, gYg^{-1}\}$-submodules of V with highest weight $d-1$. Moreover, $gL(d-1) = V_{gYg^{-1},0} \cap gM(d-1)$. As in (3.8), let $(\cdot,\cdot)'_d \colon gL(d-1) \times gL(d-1) \longrightarrow \mathbb{H}$ be defined by $(v,u)'_d := \langle v\,,\, (gXg^{-1})^{d-1}u\rangle$ for all $v, u \in gL(d-1)$. As $g \in {\rm SO}^*(2n)$ for all $u, v \in L(d-1)$, we have

$$(u,v)_d = \langle u, X^{d-1}v\rangle = \langle gu, gX^{d-1}v\rangle = \langle gu, (gXg^{-1})^{d-1}gv\rangle = (gu,gv)'_d\,.$$

Hence the signatures of $(\cdot,\cdot)_\eta$ and $(\cdot,\cdot)'_\eta$ are the same for all $\eta \in \mathbb{E}_{\mathbf{d}}$. In particular, $\mathbf{sgn}_X = \mathbf{sgn}_{gXg^{-1}} \in \mathcal{S}_{\mathbf{d}}^{\rm odd}(n)$.

Thus we have a map

$$\text{(4.35)} \qquad \psi_{\mathfrak{so}^*(2n)} : \mathcal{N}_{\mathfrak{so}^*(2n)} \longrightarrow \mathcal{Y}^{\rm odd}(n), \quad X \longmapsto \left(\psi'_{\mathfrak{so}^*(2n)}(X),\, \mathbf{sgn}_X\right),$$

where $\mathcal{Y}^{\rm odd}(n)$ is as in (2.12). The map $\psi_{\mathfrak{so}^*(2n)}$ satisfies the following properties:

(4.36) $\psi_{\mathfrak{so}^*(2n)}(X) = \psi_{\mathfrak{so}^*(2n)}(gXg^{-1})$ for all $g \in {\rm SO}^*(2n)$.

(4.37) $\psi_{\mathfrak{so}^*(2n)}(X)$ does not depend on the $\mathfrak{sl}_2(\mathbb{R})$-triple $\{X, H, Y\}$ containing X.

It is immediate that (4.36) holds. To prove (4.37), let $\{X, H', Y'\} \subset \mathfrak{so}^*(2n)$ be another $\mathfrak{sl}_2(\mathbb{R})$-triple containing X. By Theorem 2.4.8, there exists $h \in {\rm SO}^*(2n)$ such that $hXh^{-1} = X$, $hHh^{-1} = H'$, $hYh^{-1} = Y'$. Now (4.37) follows from (4.36).

Thus $\psi_{\mathfrak{so}^*(2n)}$ induces a well-defined map

$$\text{(4.38)} \qquad \Psi_{\mathrm{SO}^*(2n)} \colon \mathcal{N}(\mathrm{SO}^*(2n)) \longrightarrow \mathcal{Y}^{\mathrm{odd}}(n), \quad \mathcal{O}_X \longmapsto \big(\psi'_{\mathfrak{so}^*(2n)}(X), \, \mathbf{sgn}_X\big).$$

Using our terminologies we next state a standard result which says that the map above gives a parametrization of the nilpotent orbits in $\mathfrak{so}^*(2n)$.

Theorem 4.1.8 ([CoMc, Theorem 9.3.4])**.** *The map* $\Psi_{\mathrm{SO}^*(2n)} \colon \mathcal{N}(\mathrm{SO}^*(2n)) \longrightarrow \mathcal{Y}^{\mathrm{odd}}(n)$ *is a bijection.*

Proof. We divide the proof in two steps.

Step 1 : In this step we prove that $\Psi_{\mathrm{SO}^*(2n)}$ is injective. Let $X, N \in \mathfrak{so}^*(2n)$ be two non-zero nilpotent elements such that $\Psi_{\mathrm{SO}^*(2n)}(\mathcal{O}_X) = \Psi_{\mathrm{SO}^*(2n)}(\mathcal{O}_N)$. Let $\mathbf{d} := \psi'_{\mathfrak{so}^*(2n)}(X) = \psi'_{\mathfrak{so}^*(2n)}(N)$. Let $\{X^l v^d_j \mid 0 \leq l \leq d-1, 1 \leq j \leq t_d, d \in \mathbb{N}_\mathbf{d}\}$ and $\{N^l w^d_j \mid 0 \leq l \leq d-1, 1 \leq j \leq t_d, d \in \mathbb{N}_\mathbf{d}\}$ be two $\mathbb{H}$-bases of $V = \mathbb{H}^n$, as in Proposition 3.0.7 which satisfy Remark 3.0.11 (3). We also have $\mathbf{sgn}_X = \mathbf{sgn}_N \in \mathcal{S}^{\mathrm{odd}}_\mathbf{d}(n)$. Thus, after reordering the ordered sets $(v^d_1, \ldots, v^d_{t_d})$ and $(w^d_1, \ldots, w^d_{t_d})$ for all $d \in \mathbb{N}_\mathbf{d}$, if necessary, we may assume that

$$\text{(4.39)} \qquad \langle v^d_j, X^{d-1} v^d_j \rangle = \langle w^d_j, N^{d-1} w^d_j \rangle \ \text{ for all } \ 1 \leq j \leq t_d, d \in \mathbb{N}_\mathbf{d}.$$

Let $g \in \mathrm{GL}_n(\mathbb{H})$ be such that

$$\text{(4.40)} \qquad g(X^l v^d_j) = N^l w^d_j \ \text{ for all } 0 \leq l \leq d-1, 1 \leq j \leq t_d, d \in \mathbb{N}_\mathbf{d}.$$

Then it follows that $gX(X^l v^d_j) = Ng(X^l v^d_j)$ for all $0 \leq l \leq d-1, 1 \leq j \leq t_d, d \in \mathbb{N}_\mathbf{d}$. Thus $gXg^{-1} = N$. Now we show that $g \in \mathrm{SO}^*(2n)$. Using (4.39) and (4.40) above it follows that

$$\langle gX^l v^d_j, gX^{d-1-l} v^d_j \rangle = \langle X^l v^d_j, X^{d-1-l} v^d_j \rangle,$$

for all $0 \leq l \leq d-1, 1 \leq j \leq t_d, d \in \mathbb{N}_\mathbf{d}$. As $\{X^l v^d_j \mid 0 \leq l \leq d-1, 1 \leq j \leq t_d, d \in \mathbb{N}_\mathbf{d}\}$

is a $\mathbb{H}$-basis of V, it is now clear from the relations among the basis elements in Proposition 3.0.7(3) in the case of $\sigma = \sigma_c, \epsilon = -1, \mathbb{D} = \mathbb{H}$ that $g \in \mathrm{SO}^*(2n)$. Thus $\mathcal{O}_X = \mathcal{O}_N$ which proves the injectivity of the map $\Psi_{\mathrm{SO}^*(2n)}$.

Step 2 : In this step we prove that $\Psi_{\mathrm{SO}^*(2n)}$ is surjective. Let us fix a signed Young diagram $(\mathbf{d}, \mathbf{sgn}) \in \mathcal{Y}^{\mathrm{odd}}(n)$. Then $\mathbf{d} \in \mathcal{P}(n)$ and $\mathbf{sgn} \in \mathcal{S}^{\mathrm{odd}}_{\mathbf{d}}(n)$. Let $X \in \mathcal{N}_{\mathfrak{sl}_n(\mathbb{H})}$, and $\{X, H, Y\} \subset \mathfrak{sl}_n(\mathbb{H})$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple such that $\psi_{\mathfrak{sl}_n(\mathbb{H})}(X) = \mathbf{d}$; see (4.6) and Theorem 4.1.3. Our strategy is to obtain a $P \in \mathrm{GL}_n(\mathbb{H})$ such that $P^{-1}XP \in \mathfrak{so}^*(2n)$ and $\mathbf{sgn}_{P^{-1}XP} = \mathbf{sgn} \in \mathcal{S}^{\mathrm{odd}}_{\mathbf{d}}(n)$.

We next construct a nondegenerate skew-Hermitian form $\langle \cdot\,, \cdot \rangle_{\mathrm{new}}$ on $V = \mathbb{H}^n$ such that $\{X, H, Y\} \subset \mathfrak{so}^*(V, \langle \cdot\,, \cdot \rangle_{\mathrm{new}})$; see (2.18) for the definition of $\mathfrak{so}^*(V, \langle \cdot\,, \cdot \rangle_{\mathrm{new}})$. Let $\mathbf{d} := [d_1^{t_{d_1}}, \dots, d_s^{t_{d_s}}]$. Using Proposition 3.0.3(2), $\mathbb{H}^n$ has a $\mathbb{H}$-basis of the form $\{X^l v^d_j \mid 0 \leq l \leq d-1, 1 \leq j \leq t_d, d \in \mathbb{N}_{\mathbf{d}}\}$. Let $\mathbf{sgn} := (M_{d_1}, \dots, M_{d_s})$, and let p_η, q_η be the number of $+1$, -1, respectively, appearing in the 1st column of the matrix of M_η (of size $t_\eta \times \eta$) for all $\eta \in \mathbb{E}_{\mathbf{d}}$. For $d \in \mathbb{N}_{\mathbf{d}}$, $1 \leq j \leq t_d$ and for $0 \leq l, r \leq d-1$ we define $b(X^l v^d_j, X^r v^d_j) \in \mathbb{H}$ by

$$b(X^l v^d_j, X^r v^d_j) = 0 \quad \text{if } l + r \neq d - 1 \tag{4.41}$$

and

$$b(X^l v^d_j, X^{d-1-l} v^d_j) := \begin{cases} (-1)^l \mathbf{j} & \text{if } d \in \mathbb{O}_{\mathbf{d}},\ 1 \leq j \leq t_d \\ (-1)^l & \text{if } d \in \mathbb{E}_{\mathbf{d}},\ 1 \leq j \leq p_d \\ (-1)^{l+1} & \text{if } d \in \mathbb{E}_{\mathbf{d}},\ p_d < j \leq t_d. \end{cases} \tag{4.42}$$

It now follows that for $0 \leq l, r \leq d-1$

$$b(X^l v^d_j, X^r v^d_j) = -\,\overline{b(X^r v^d_j, X^l v^d_j)}. \tag{4.43}$$

Recall that, for all $d \in \mathbb{N}_{\mathbf{d}}$, $1 \leq j \leq t_d$, the $\mathbb{R}$-Span of $\{v_j^d,\, Xv_j^d,\, \ldots,\, X^{d-1}v_j^d\}$ is an irreducible $\mathrm{Span}_{\mathbb{R}}\{X, H, Y\}$-submodule of $\mathbb{H}^n$; see Lemma 3.0.2 (2). We set $V_j^d := \mathrm{Span}_{\mathbb{H}}\{X^l v_j^d \mid 0 \leq l \leq d-1\}$. As $\{X^l v_j^d \mid 0 \leq l \leq d-1\}$ is a $\mathbb{H}$-basis for V_j^d the equalities in (4.43) allow us to define a skew-Hermitian form $\langle \cdot\,, \cdot \rangle_{dj}$ on V_j^d such that

$$\langle X^l v_j^d, X^r v_j^d \rangle_{dj} \;=\; b(X^l v_j^d, X^r v_j^d) \quad \text{for } 0 \leq l, r \leq d-1. \tag{4.44}$$

From the definition it is clear that $\langle \cdot\,, \cdot \rangle_{dj}$ is nondegenerate on V_j^d, and moreover $\langle Xx, y \rangle_{dj} + \langle x, Xy \rangle_{dj} = 0$ for all $x, y \in V_j^d$. Recall that

$$\mathbb{H}^n = \bigoplus_{d \in \mathbb{N}_{\mathbf{d}},\, 1 \leq j \leq t_d} V_j^d\,. \tag{4.45}$$

Let $\langle \cdot\,, \cdot \rangle_{\mathrm{new}}$ be the new skew-Hermitian form on V such that its restriction to V_j^d agrees with $\langle \cdot\,, \cdot \rangle_{dj}$, and so that (4.45) is an orthogonal direct sum with respect to $\langle \cdot\,, \cdot \rangle_{\mathrm{new}}$. Then $\langle \cdot\,, \cdot \rangle_{\mathrm{new}}$ is nondegenerate on $V \times V$. Clearly, $\langle Xx, y \rangle_{\mathrm{new}} + \langle x, Xy \rangle_{\mathrm{new}} = 0$ for all $x, y \in V$. Recall that in Proposition 3.0.3 (1) we have that $YX^l v_j^d = (X^{l-1}v_j^d) l(d-l)$ for $0 < l \leq d-1$, $1 \leq j \leq t_d$, $d \in \mathbb{N}_{\mathbf{d}}$, and $Yv_j^d = 0$ for $1 \leq j \leq t_d$, $d \in \mathbb{N}_{\mathbf{d}}$. As $\{X^l v_j^d \mid 0 \leq l \leq d-1, 1 \leq j \leq t_d, d \in \mathbb{N}_{\mathbf{d}}\}$ is a basis of $\mathbb{H}^n$, using the above relations, (4.42) and (4.44), we conclude that $\langle Hx, y \rangle_{\mathrm{new}} + \langle x, Hy \rangle_{\mathrm{new}} = 0$ and $\langle Yx, y \rangle_{\mathrm{new}} + \langle x, Yy \rangle_{\mathrm{new}} = 0$ for all $x, y \in V$. Thus $\{X, H, Y\} \subset \mathfrak{so}^*(V, \langle \cdot\,, \cdot \rangle_{\mathrm{new}})$.

Since both the forms $\langle \cdot\,, \cdot \rangle_{\mathrm{new}}$ and $\langle \cdot\,, \cdot \rangle$ are nondegenerate and skew-Hermitian on $V = \mathbb{H}^n$, there is a $P \in \mathrm{GL}_n(\mathbb{H})$ such that

$$\langle x, y \rangle \;=\; \langle Px, Py \rangle_{\mathrm{new}} \quad \text{for all } x, y \in V. \tag{4.46}$$

Clearly $\{P^{-1}XP,\, P^{-1}HP,\, P^{-1}YP\}$ is a $\mathfrak{sl}_2(\mathbb{R})$-triple in $\mathfrak{so}^*(2n)$. Now we will show that $\mathbf{sgn}_{P^{-1}XP} = \mathbf{sgn} \in \mathcal{S}_{\mathbf{d}}^{\mathrm{odd}}(n)$. Note that $P^{-1}M(d-1)$ is the isotypical component of $\mathbb{H}^n$ containing all the irreducible $\mathrm{Span}_{\mathbb{R}}\{P^{-1}XP,\, P^{-1}HP,\, P^{-1}YP\}$-submodules of $\mathbb{H}^n$ with highest weight $(d-1)$. Moreover, $P^{-1}L(d-1) = V_{P^{-1}YP,0} \cap P^{-1}M(d-1)$. As in (3.8) for all $\eta \in \mathbb{E}_{\mathbf{d}}$, let $(\cdot,\cdot)''_\eta \colon P^{-1}L(\eta-1) \times P^{-1}L(\eta-1) \longrightarrow \mathbb{H}$ be defined by $(x,y)''_\eta := \langle x, (P^{-1}XP)^{\eta-1}y\rangle$ for all $x, y \in P^{-1}L(\eta-1)$. Using (4.46) it follows that

$$(u,v)''_\eta = (Pu, Pv)_{\mathrm{new}_\eta} \quad \text{for } u, v \in P^{-1}L(\eta-1);\ \eta \in \mathbb{E}_{\mathbf{d}}.$$

Thus the signatures $(\cdot,\cdot)''_\eta$ and $(\cdot,\cdot)_{\mathrm{new}_\eta}$ are both equal, which, in particular, shows that signature of $(\cdot,\cdot)''_\eta$ is (p_η, q_η) for all $\eta \in \mathbb{E}_{\mathbf{d}}$. This proves that $\mathbf{sgn}_{P^{-1}XP} = \mathbf{sgn} \in \mathcal{S}^{\mathrm{odd}}_{\mathbf{d}}(n)$. Hence $\Psi_{\mathrm{SO}^*(2n)}(\mathcal{O}_{P^{-1}XP}) = (\mathbf{d}, \mathbf{sgn})$. This completes the proof of the theorem. □

4.1.6 Parametrization of nilpotent orbits in $\mathfrak{sp}(n,\mathbb{R})$

Let n be a positive integer. In this subsection we describe a suitable parametrization of the nilpotent orbits in $\mathfrak{sp}(n,\mathbb{R})$ under the adjoint action of $\mathrm{Sp}(n,\mathbb{R})$. Throughout this subsection $\langle\cdot,\cdot\rangle$ denotes the symplectic form on $\mathbb{R}^{2n}$ defined by $\langle x,y\rangle := x^t \mathrm{J}_n y$, for $x, y \in \mathbb{R}^{2n}$, where J_n is as in (2.19).

Let $\Psi_{\mathrm{SL}_{2n}(\mathbb{R})} \colon \mathcal{N}(\mathrm{SL}_{2n}(\mathbb{R})) \longrightarrow \mathcal{P}(2n)$ be the parametrization of nilpotent orbits in $\mathfrak{sl}_{2n}(\mathbb{R})$; see Theorem 4.1.2. As $\mathrm{Sp}(n,\mathbb{R}) \subset \mathrm{SL}_{2n}(\mathbb{R})$, (consequently as, the set of nilpotent elements $\mathcal{N}_{\mathfrak{sp}(n,\mathbb{R})} \subset \mathcal{N}_{\mathfrak{sl}_{2n}(\mathbb{R})}$) we have the inclusion map, say, $\vartheta_{\mathfrak{sp}(n,\mathbb{R})} : \mathcal{N}_{\mathfrak{sp}(n,\mathbb{R})} \longrightarrow \mathcal{N}_{\mathfrak{sl}_{2n}(\mathbb{R})}$. Let $\psi'_{\mathfrak{sp}(n,\mathbb{R})} := \psi_{\mathfrak{sl}_{2n}(\mathbb{R})} \circ \vartheta_{\mathfrak{sp}(n,\mathbb{R})} \colon \mathcal{N}_{\mathfrak{sp}(n,\mathbb{R})} \longrightarrow \mathcal{P}(2n)$ be the composition. Recall that $\psi'_{\mathfrak{sp}(n,\mathbb{R})}(\mathcal{N}_{\mathfrak{sp}(n,\mathbb{R})}) \subset \mathcal{P}_{-1}(2n)$ where $\mathcal{P}_{-1}(2n)$ is as in (2.4); this follows form the Remark 3.0.11 (1). Let $X \in \mathfrak{sp}(n,\mathbb{R})$ be a non-zero nilpotent element and $\mathcal{O}_X$ be the corresponding nilpotent orbit in $\mathfrak{sp}(n,\mathbb{R})$. Let $\{X,H,Y\} \subset \mathfrak{sp}(n,\mathbb{R})$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple. Let V be $\mathbb{R}^{2n}$, the right $\mathbb{R}$-vector space of column vectors. Let $\{d_1,\ldots,d_s\}$ with $d_1 < \cdots < d_s$ be the finitely many integers that occur as $\mathbb{R}$-dimensions of non-zero irreducible $\mathrm{Span}_{\mathbb{R}}\{X,H,Y\}$-submodules of V. Recall that $M(d-1)$ is defined to be the isotypical component of V containing all irreducible submodules of V with highest weight $d-1$ and as in (3.1), we set $L(d-1) := V_{Y,0} \cap M(d-1)$. Let $t_{d_r} := \dim_{\mathbb{R}} L(d_r - 1)$ for $1 \le r \le s$. Then $\mathbf{d} := [d_1^{t_{d_1}}, \ldots, d_s^{t_{d_s}}] \in \mathcal{P}_{-1}(2n)$, and moreover, $\psi'_{\mathfrak{sp}(n,\mathbb{R})}(X) = \mathbf{d}$.

We now consider $\mathcal{S}^{\rm odd}_{\mathbf{d}}(2n)$ as defined in (2.9), and assign an element $\mathbf{sgn}_X \in \mathcal{S}^{\rm odd}_{\mathbf{d}}(2n)$ to the element $X \in \mathcal{N}_{\mathfrak{sp}(n,\mathbb{R})}$. Let $\mathbb{N}_{\mathbf{d}} := \{d_i \mid 1 \le i \le s\}$; see (2.1) for the definition. For all $d \in \mathbb{N}_{\mathbf{d}}$ we first define a $t_d \times d$ matrix, say $(m^d_{ij}(X))$, in $\mathbf{A}_d$; see (2.5) for the definition. Recall that the form $(\cdot,\cdot)_d \colon L(d-1) \times L(d-1) \longrightarrow \mathbb{R}$, as defined in (3.8), is symmetric or symplectic according as d is even or odd. Consider $\mathbb{E}_{\mathbf{d}}$, $\mathbb{O}_{\mathbf{d}}$ as defined in (2.1). Let (p_η, q_η) be the signature of $(\cdot,\cdot)_\eta$ when $\eta \in \mathbb{E}_{\mathbf{d}}$. Define,

$$m^{\theta}_{i1}(X) := +1 \qquad \text{if } \; 1 \le i \le t_\theta, \quad \theta \in \mathbb{O}_{\mathbf{d}};$$

$$m^{\eta}_{i1}(X) := \begin{cases} +1 & \text{if } \; 1 \le i \le p_\eta \\ -1 & \text{if } \; p_\eta < i \le t_\eta \end{cases}, \; \eta \in \mathbb{E}_{\mathbf{d}}\,;$$

and for $j > 1$ we define $(m^d_{ij}(X))$ as in (4.11) and (4.12). The way the matrices $(m^d_{ij}(X))$ are defined, immediately implies that they verify (**Yd**.1) and (**Yd**.2). Set $\mathbf{sgn}_X := ((m^{d_1}_{ij}(X)), \ldots, (m^{d_s}_{ij}(X)))$. It now follows from the above definition of $m^\theta_{i1}(X)$ for $\theta \in \mathbb{O}_{\mathbf{d}}$ that $\mathbf{sgn}_X \in \mathcal{S}^{\rm odd}_{\mathbf{d}}(2n)$.

We next show that $\mathbf{sgn}_X = \mathbf{sgn}_{gXg^{-1}} \in \mathcal{S}^{\rm odd}_{\mathbf{d}}(2n)$ for all $g \in \mathrm{Sp}(n,\mathbb{R})$. Clearly, $\{gXg^{-1}, gHg^{-1}, gYg^{-1}\}$ is a $\mathfrak{sl}_2(\mathbb{R})$-triple in $\mathfrak{sp}(n,\mathbb{R})$. It also clear that $gM(d-1)$ is the isotypical component of V containing all irreducible $\mathrm{Span}_{\mathbb{R}}\{gXg^{-1}, gHg^{-1}, gYg^{-1}\}$-submodules of V with highest weight $d-1$. Moreover, $gL(d-1) = V_{gYg^{-1},0} \cap gM(d-1)$. Now as in (3.8) for all $\eta \in \mathbb{E}_{\mathbf{d}}$, let $(\cdot,\cdot)'_\eta \colon gL(\eta-1) \times gL(\eta-1) \longrightarrow \mathbb{R}$ be defined by $(v,u)'_\eta := \langle v, (gXg^{-1})^{\eta-1} u\rangle$ for all $v, u \in gL(\eta-1)$. As $g \in \mathrm{Sp}(n,\mathbb{R})$, for all $u, v \in L(d-1)$ we have

$$(u,v)_\eta = \langle u, X^{\eta-1} v\rangle = \langle gu, gX^{\eta-1} v\rangle = \langle gu, (gXg^{-1})^{\eta-1} gv\rangle = (gu, gv)'_\eta\,.$$

Hence the signatures of $(\cdot,\cdot)_\eta$ and $(\cdot,\cdot)'_\eta$ are the same for all $\eta \in \mathbb{E}_{\mathbf{d}}$. In particular, $\mathbf{sgn}_X = \mathbf{sgn}_{gXg^{-1}} \in \mathcal{S}^{\rm odd}_{\mathbf{d}}(2n)$.

Thus we have a map

$$(4.47) \qquad \psi_{\mathfrak{sp}(n,\mathbb{R})} : \mathcal{N}_{\mathfrak{sp}(n,\mathbb{R})} \longrightarrow \mathcal{Y}^{\rm odd}_{-1}(2n), \quad X \longmapsto \big(\psi'_{\mathfrak{sp}(n,\mathbb{R})}(X),\, \mathbf{sgn}_X\big)\,,$$

where $\mathcal{Y}^{\rm odd}_{-1}(2n)$ is as in (2.13). The map $\psi_{\mathfrak{sp}(n,\mathbb{R})}$ satisfies the following properties:

(4.48) $\psi_{\mathfrak{sp}(n,\mathbb{R})}(X) = \psi_{\mathfrak{sp}(n,\mathbb{R})}(gXg^{-1})$ for all $g \in \mathrm{Sp}(n,\mathbb{R})$.

(4.49) $\psi_{\mathfrak{sp}(n,\mathbb{R})}(X)$ does not depend on the $\mathfrak{sl}_2(\mathbb{R})$-triple $\{X,H,Y\}$ containing X.

It is immediate from above that (4.48) holds. To prove (4.49), let $\{X,H',Y'\}$ be another $\mathfrak{sl}_2(\mathbb{R})$-triple in $\mathfrak{sp}(n,\mathbb{R})$ containing X. By Theorem 2.4.8, there exists $h \in \mathrm{Sp}(n,\mathbb{R})$ such that $hXh^{-1} = X$, $hHh^{-1} = H'$, $hYh^{-1} = Y'$. Now (4.49) follows from (4.48).

Thus we have a well-defined map

$$(4.50) \qquad \Psi_{\mathrm{Sp}(n,\mathbb{R})} \colon \mathcal{N}(\mathrm{Sp}(n,\mathbb{R})) \longrightarrow \mathcal{Y}^{\rm odd}_{-1}(2n), \quad \mathcal{O}_X \longmapsto \big(\psi'_{\mathfrak{sp}(n,\mathbb{R})}(X),\, \mathbf{sgn}_X\big).$$

Using our terminologies we next state a standard result which says that the map above gives a parametrization of the nilpotent orbits in $\mathfrak{sp}(n,\mathbb{R})$.

Theorem 4.1.9 ([CoMc, Theorem 9.3.5])**.** *The map* $\Psi_{\mathrm{Sp}(n,\mathbb{R})} : \mathcal{N}(\mathrm{Sp}(n,\mathbb{R})) \longrightarrow \mathcal{Y}^{\rm odd}_{-1}(2n)$ *in* (4.50) *is a bijection.*

Proof. We divide the proof in two steps.

Step 1 : In this step we prove that $\Psi_{\mathrm{Sp}(n,\mathbb{R})}$ is injective. Let $X, N \in \mathcal{N}_{\mathfrak{sp}(n,\mathbb{R})}$ be two non-zero elements such that $\Psi_{\mathrm{Sp}(n,\mathbb{R})}(\mathcal{O}_X) = \Psi_{\mathrm{Sp}(n,\mathbb{R})}(\mathcal{O}_N)$. Let $\mathbf{d} := \psi'_{\mathfrak{sp}(n,\mathbb{R})}(X) = \psi'_{\mathfrak{sp}(n,\mathbb{R})}(N) \in \mathcal{P}_{-1}(2n)$. Let $\{X^l v^d_j \mid 0 \le l \le d-1, 1 \le j \le t_d, d \in \mathbb{N}_{\mathbf{d}}\}$ and $\{N^l w^d_j \mid$

$0 \leq l \leq d-1, 1 \leq j \leq t_d, d \in \mathbb{N}_{\mathbf{d}}\}$ be two $\mathbb{R}$-bases of $V = \mathbb{R}^{2n}$ as in Proposition 3.0.7 when $\sigma = \mathrm{Id}, \epsilon = -1, \mathbb{D} = \mathbb{R}$. We also have $\mathbf{sgn}_X = \mathbf{sgn}_N \in \mathcal{S}^{\mathrm{odd}}_{\mathbf{d}}(2n)$. After suitable rescaling each element of the ordered sets $(v^d_1, \ldots, v^d_{t_d})$ and $(w^d_1, \ldots, w^d_{t_d})$ for all $d \in \mathbb{N}_{\mathbf{d}}$, if necessary, we may assume that

$$
\begin{aligned}
\langle v^\eta_j, X^{\eta-1} v^\eta_j\rangle &= \langle w^\eta_j, N^{\eta-1} w^\eta_j\rangle \qquad \text{for all } \eta \in \mathbb{E}_{\mathbf{d}},\, 1 \leq j \leq t_\eta\,;\\
\langle v^\theta_j, X^{\theta-1} v^\theta_{j+1}\rangle &= \langle w^\theta_j, N^{\theta-1} w^\theta_{j+1}\rangle \quad \text{for all } \theta \in \mathbb{O}_{\mathbf{d}},\, 1 \leq j \leq t_\theta.
\end{aligned} \tag{4.51}
$$

Let $g \in \mathrm{GL}_{2n}(\mathbb{R})$ be such that $g(X^l v^d_j) = N^l w^d_j$ for all $0 \leq l \leq d-1, 1 \leq j \leq t_d, d \in \mathbb{N}_{\mathbf{d}}$. It is now straightforward that $gX(X^l v^d_j) = Ng(X^l v^d_j)$ for all $0 \leq l \leq d-1, 1 \leq j \leq t_d, d \in \mathbb{N}_{\mathbf{d}}$. Thus we have $gX = Ng$. Using the equalities in (4.51) and the definition of g as above it follows that

$$
\begin{aligned}
\langle gX^l v^\eta_j,\, gX^{\eta-1-l} v^\eta_j\rangle &= \langle X^l v^\eta_j,\, X^{\eta-1-l} v^\eta_j\rangle \quad \text{for all } \eta \in \mathbb{E}_{\mathbf{d}}\,, 1 \leq j \leq t_\eta\,;\\
\langle gX^l v^\theta_j,\, gX^{\theta-1-l} v^\theta_{j+1}\rangle &= \langle X^l v^\theta_j,\, X^{\theta-1-l} v^\theta_{j+1}\rangle \ \text{for all } \theta \in \mathbb{O}_{\mathbf{d}}\,, 1 \leq j \leq t_\theta\,.
\end{aligned}
$$

As $\{X^l v^d_j \mid 0 \leq l \leq d-1, 1 \leq j \leq t_d, d \in \mathbb{N}_{\mathbf{d}}\}$ is a $\mathbb{R}$-basis of $\mathbb{R}^{2n}$, we conclude from the relations among the basis elements in Proposition 3.0.7(3) in the case of $\sigma = \mathrm{Id}, \epsilon = -1, \mathbb{D} = \mathbb{R}$ that $g \in \mathrm{Sp}(n,\mathbb{R})$. Thus $\mathcal{O}_X = \mathcal{O}_N$ which proves the injectivity of the map $\Psi_{\mathrm{Sp}(n,\mathbb{R})}$.

Step 2 : In this step we prove that $\Psi_{\mathrm{Sp}(n,\mathbb{R})}$ is surjective. Let us fix a signed Young diagram $(\mathbf{d}, \mathbf{sgn}) \in \mathcal{Y}^{\mathrm{odd}}_{-1}(2n)$. Then $\mathbf{d} \in \mathcal{P}_{-1}(2n)$, and $\mathbf{sgn} \in \mathcal{S}^{\mathrm{odd}}_{\mathbf{d}}(2n)$. Let $X \in \mathcal{N}_{\mathfrak{sl}_{2n}(\mathbb{R})}$ and $\{X, H, Y\} \subset \mathfrak{sl}_{2n}(\mathbb{R})$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple such that $\psi_{\mathfrak{sl}_{2n}(\mathbb{R})}(X) = \mathbf{d}$; see (4.1) and Theorem 4.1.2. Our strategy is to obtain a $P \in \mathrm{GL}_{2n}(\mathbb{R})$ such that $P^{-1}XP \in \mathfrak{sp}(n,\mathbb{R})$ and $\mathbf{sgn}_{P^{-1}XP} = \mathbf{sgn}$.

We next construct a nondegenerate symplectic form $\langle\cdot\,,\cdot\rangle_{\mathrm{new}}$ on $V = \mathbb{R}^{2n}$ such that $\{X, H, Y\} \subset \mathfrak{sp}(V, \langle\cdot\,,\cdot\rangle_{\mathrm{new}})$; see (2.17) for the definition of $\mathfrak{sp}(V, \langle\cdot\,,\cdot\rangle_{\mathrm{new}})$. Let $\mathbf{d} := [d_1^{t_{d_1}}, \ldots, d_s^{t_{d_s}}]$. Using Proposition 3.0.3(2), V has a $\mathbb{R}$-basis of the form

$\{X^l v_j^d \mid 0 \le l \le d-1, 1 \le j \le t_d, d \in \mathbb{N}_{\mathbf{d}}\}$. Let $\mathbf{sgn} := (M_{d_1}, \ldots, M_{d_s})$, and let p_η, q_η be the number of $+1$, -1, respectively, appearing in the 1st column of the matrix of M_η (of size $t_\eta \times \eta$) for all $\eta \in \mathbb{E}_{\mathbf{d}}$. For $d \in \mathbb{N}_{\mathbf{d}}$, $1 \le j \le t_d$ and for $0 \le l, r \le d-1$ we define $b(X^l v_j^d, X^r v_j^d) \in \mathbb{R}$ by

$$
\begin{aligned}
b(X^l v_j^\eta, X^r v_j^\eta) &= 0 \quad \text{if } l + r \neq \eta - 1\,, \eta \in \mathbb{E}_{\mathbf{d}}\,, \\
b(X^l v_j^\theta, X^r v_j^\theta) &= 0 \quad \text{if } \theta \in \mathbb{O}_{\mathbf{d}}\,, \\
b(X^l v_j^\theta, X^r v_{j+1}^\theta) &= 0 \quad \text{if } l + r \neq \theta - 1, j \text{ is odd }, \theta \in \mathbb{O}_{\mathbf{d}} \\
b(X^l v_{j+1}^\theta, X^r v_j^\theta) &= 0 \quad \text{if } l + r \neq \theta - 1, j \text{ is odd }, \theta \in \mathbb{O}_{\mathbf{d}},
\end{aligned}
$$

and

$$
b(X^l v_j^\eta, X^{\eta-1-l} v_j^\eta) := \begin{cases} (-1)^l & \text{when } 1 \le j \le p_\eta \\ (-1)^{l+1} & \text{when } p_\eta < j \le t_\eta \end{cases} ; \eta \in \mathbb{E}_{\mathbf{d}}, \tag{4.52}
$$

$$
\begin{aligned}
b(X^l v_j^\theta, X^{\theta-1-l} v_{j+1}^\theta) : &= (-1)^l \qquad \text{when } j \text{ is odd}, \ \theta \in \mathbb{O}_{\mathbf{d}}, \ 1 \le j \le t_\theta, \\
b(X^l v_{j+1}^\theta, X^{\theta-1-l} v_j^\theta) : &= (-1)^{l+1} \quad \text{when } j \text{ is odd}, \ \theta \in \mathbb{O}_{\mathbf{d}}, \ 1 \le j \le t_\theta\,.
\end{aligned} \tag{4.53}
$$

It now follows that

(4.54)

$$
b(X^l v_j^\eta, X^r v_j^\eta) = -\, b(X^r v_j^\eta, X^l v_j^\eta) \quad \text{for } \eta \in \mathbb{E}_{\mathbf{d}}, 0 \le l, r \le \eta - 1,
$$

(4.55)

$$
b(X^l v_{j'}^\theta, X^r v_{j''}^\theta) = -\, b(X^r v_{j''}^\theta, X^l v_{j'}^\theta) \text{ for } \theta \in \mathbb{O}_{\mathbf{d}}, 0 \le l, r \le \theta - 1, j \le j', j'' \le j+1.
$$

Recall that, for all $d \in \mathbb{N}_{\mathbf{d}}$, $1 \le j \le t_d$, the $\mathbb{R}$-Span of $\{v_j^d, Xv_j^d, \ldots, X^{d-1}v_j^d\}$ is an irreducible $\mathrm{Span}_{\mathbb{R}}\{X, H, Y\}$-submodule of $\mathbb{R}^{2n}$; see Lemma 3.0.2 (2). For $1 \le j \le t_\eta, \eta \in \mathbb{E}_{\mathbf{d}}$, we set $V_j^\eta := \mathrm{Span}_{\mathbb{R}}\{X^l v_j^\eta \mid 0 \le l \le \eta - 1\}$. For $\theta \in \mathbb{O}_{\mathbf{d}}$, and an odd integer j, $1 \le j \le t_\theta$, we set $V_j^\theta := \mathrm{Span}_{\mathbb{R}}\{X^l v_j^\theta, X^l v_{j+1}^\theta \mid 0 \le l \le \theta - 1\}$. As

$\{X^l v_j^\eta \mid 0 \le l \le \eta - 1\}$ is a $\mathbb{R}$-basis for V_j^η the equalities in (4.54) allow us to define a symplectic form $\langle \cdot\,, \cdot \rangle_{\eta j}$ on V_j^η such that

$$\langle X^l v_j^\eta,\, X^r v_j^\eta \rangle_{\eta j} \;=\; b(X^l v_j^\eta,\, X^r v_j^\eta) \quad \text{for } 0 \le l, r \le \eta - 1. \tag{4.56}$$

Similarly as $\{X^l v_j^\theta, X^l v_{j+1}^\theta \mid 0 \le l \le \theta - 1\}$ is a $\mathbb{R}$-basis for V_j^θ the equalities in (4.55) allow us to define a symplectic form $\langle \cdot\,, \cdot \rangle_{\theta j}$ on V_j^θ such that

$$\langle X^l v_{j'}^\theta,\, X^r v_{j''}^\theta \rangle_{\theta j} \;=\; b(X^l v_{j'}^\theta,\, X^r v_{j''}^\theta) \quad \text{for } 0 \le l, r \le \theta - 1,\ j \le j', j'' \le j+1\,. \tag{4.57}$$

From the definition it is clear that for all $d \in \mathbb{N}_{\mathbf{d}}$, $\langle \cdot\,, \cdot \rangle_{dj}$ is nondegenerate on V_j^d and moreover, $\langle Xx, y \rangle_{dj} + \langle x, Xy \rangle_{dj} = 0$ for all $x, y \in V_j^d$. Recall that

$$\mathbb{R}^{2n} = \Big(\bigoplus_{j \text{ odd}, 1 \le j \le t_\theta, \theta \in \mathbb{O}_{\mathbf{d}}} V_j^\theta \Big) \oplus \Big(\bigoplus_{1 \le j \le t_\eta, \eta \in \mathbb{E}_{\mathbf{d}}} V_j^\eta \Big)\,. \tag{4.58}$$

Let $\langle \cdot\,, \cdot \rangle_{\rm new}$ be the new symplectic form on $V = \mathbb{R}^{2n}$ such that its restriction to V_j^d agrees with $\langle \cdot\,, \cdot \rangle_{dj}$, and so that (4.58) is an orthogonal direct sum with respect to $\langle \cdot\,, \cdot \rangle_{\rm new}$. Then $\langle \cdot\,, \cdot \rangle_{\rm new}$ is non-degenerate on $V \times V$. Clearly, $\langle Xx, y \rangle_{\rm new} + \langle x, Xy \rangle_{\rm new} = 0$ for all $x, y \in V$. Recall that in Proposition 3.0.3 (1) we have that $YX^l v_j^d = (X^{l-1} v_j^d) l(d-l)$ for $0 < l \le d-1$, $1 \le j \le t_d$, $d \in \mathbb{N}_{\mathbf{d}}$, and $Y v_j^d = 0$ for $1 \le j \le t_d$, $d \in \mathbb{N}_{\mathbf{d}}$. As $\{X^l v_j^d \mid 0 \le l \le d-1, 1 \le j \le t_d, d \in \mathbb{N}_{\mathbf{d}}\}$ is a basis of $\mathbb{R}^{2n}$, using the above relations,(4.52), (4.53), (4.56) and (4.57), we conclude that $\langle Hx, y \rangle_{\rm new} + \langle x, Hy \rangle_{\rm new} = 0$ and $\langle Yx, y \rangle_{\rm new} + \langle x, Yy \rangle_{\rm new} = 0$ for all $x, y \in V$. Thus $\{X, H, Y\} \subset \mathfrak{sp}(V, \langle \cdot\,, \cdot \rangle_{\rm new})$.

Since both the forms $\langle \cdot\,, \cdot \rangle_{\rm new}$ and $\langle \cdot\,, \cdot \rangle$ are nondegenerate and symplectic on $\mathbb{R}^{2n}$, there is a $P \in {\rm GL}_{2n}(\mathbb{R})$ such that

$$\langle x, y \rangle \;=\; \langle Px, Py \rangle_{\rm new} \quad \text{for all } x, y \in V. \tag{4.59}$$

Clearly $\{P^{-1}XP, P^{-1}HP, P^{-1}YP\} \subset \mathfrak{sp}(n, \mathbb{R})$ is a $\mathfrak{sl}_2(\mathbb{R})$-triple. Now we will show that $\mathbf{sgn}_{P^{-1}XP} = \mathbf{sgn} \in \mathcal{S}^{\rm odd}_{\mathbf{d}}(2n)$. Note that $P^{-1}M(d-1)$ is the isotypical component of $\mathbb{R}^{2n}$ containing all the irreducible $\mathrm{Span}_{\mathbb{R}}\{P^{-1}XP, P^{-1}HP, P^{-1}YP\}$-submodules of $\mathbb{R}^{2n}$ with highest weight $(d-1)$. Moreover, $P^{-1}L(d-1) = V_{P^{-1}YP,0} \cap P^{-1}M(d-1)$. As in (3.8) for all $\eta \in \mathbb{E}_{\mathbf{d}}$, let $(\cdot,\cdot)''_\eta \colon P^{-1}L(\eta-1) \times P^{-1}L(\eta-1) \longrightarrow \mathbb{R}$ be defined by $(x,y)''_\eta := \langle x, (P^{-1}XP)^{\eta-1}y\rangle$ for all $x, y \in P^{-1}L(\eta-1)$. Using (4.59) it follows that

$$(u,v)''_\eta = (Pu, Pv)_{\mathrm{new}_\eta} \quad \text{for } u, v \in P^{-1}L(\eta-1);\ \eta \in \mathbb{E}_{\mathbf{d}}.$$

Thus the signatures $(\cdot,\cdot)''_\eta$ and $(\cdot,\cdot)_{\mathrm{new}_\eta}$ are both equal, which, in particular, shows that signature of $(\cdot,\cdot)''_\eta$ is (p_η, q_η) for all $\eta \in \mathbb{E}_{\mathbf{d}}$. This proves that $\mathbf{sgn}_{P^{-1}XP} = \mathbf{sgn} \in \mathcal{S}^{\rm odd}_{\mathbf{d}}(2n)$. Hence $\Psi_{\mathrm{Sp}(n,\mathbb{R})}(\mathcal{O}_{P^{-1}XP}) = (\mathbf{d}, \mathbf{sgn})$. This completes the proof of the theorem. □

4.1.7 Parametrization of nilpotent orbits in $\mathfrak{sp}(p,q)$

Let n be a positive integer and (p,q) be a pair of non-negative integers such that $p+q=n$. As we deal with non-compact groups, we will further assume $p>0$ and $q>0$. In this subsection we describe a suitable parametrization of the nilpotent orbits in $\mathfrak{sp}(p,q)$ under the adjoint action of $\mathrm{Sp}(p,q)$. For $w = (x_1, \dots, x_n)^t \in \mathbb{H}^n$ we set $\overline{w} = (\sigma_c(x_1), \dots, \sigma_c(x_n))^t$ where σ_c is the conjugation on $\mathbb{H}$ as defined in §2.3. Throughout this subsection $\langle\cdot,\cdot\rangle$ denotes the Hermitian form on $\mathbb{H}^n$ defined by $\langle x, y\rangle := \overline{x}^t \mathrm{I}_{p,q} y$, for $x, y \in \mathbb{H}^n$, where $\mathrm{I}_{p,q}$ is as in (2.19).

Let $\Psi_{\mathrm{SL}_n(\mathbb{H})} : \mathcal{N}(\mathrm{SL}_n(\mathbb{H})) \longrightarrow \mathcal{P}(n)$ be the parametrization as in Theorem 4.1.3. As $\mathrm{Sp}(p,q) \subset \mathrm{SL}_n(\mathbb{H})$ (consequently as, the set of nilpotent elements $\mathcal{N}_{\mathfrak{sp}(p,q)} \subset$

$\mathcal{N}_{\mathfrak{sl}_n(\mathbb{H})}$) we have the inclusion map, say, $\vartheta_{\mathfrak{sp}(p,q)} : \mathcal{N}_{\mathfrak{sp}(p,q)} \longrightarrow \mathcal{N}_{\mathfrak{sl}_n(\mathbb{H})}$. Let

$$\psi'_{\mathfrak{sp}(p,q)} := \psi_{\mathfrak{sl}_n(\mathbb{H})} \circ \vartheta_{\mathfrak{sp}(p,q)} \colon \mathcal{N}_{\mathfrak{sp}(p,q)} \longrightarrow \mathcal{P}(n)$$

be the composition. Let $X \in \mathfrak{sp}(p,q)$ be a non-zero nilpotent element and $\mathcal{O}_X$ be the corresponding nilpotent orbit in $\mathfrak{sp}(p,q)$. Let $\{X,H,Y\} \subset \mathfrak{sp}(p,q)$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple. Let V be $\mathbb{H}^n$, the right $\mathbb{H}$-vector space of column vectors. The left multiplication by matrices in $\mathrm{M}_n(\mathbb{H})$ act as $\mathbb{H}$-linear transformations of $\mathbb{H}^n$. We enumerate the finite set of natural numbers of the form $\dim_{\mathbb{R}} Q$ for all the non-isomorphic non-zero irreducible $\mathrm{Span}_{\mathbb{R}}\{X,H,Y\}$-submodules Q of V by $\{d_1, \dots, d_s\}$ in such a way that the relation $d_1 < \cdots < d_s$ is satisfied. Recall that $M(d-1)$ is defined to be the isotypical component of V containing all irreducible $\mathrm{Span}_{\mathbb{R}}\{X,H,Y\}$-submodules of V with highest weight $(d-1)$, and as in (3.1), we set $L(d-1) := V_{Y,0} \cap M(d-1)$. Recall that the space $L(d_r - 1)$ is a $\mathbb{H}$-subspace for $1 \leq r \leq s$. Let $t_{d_r} := \dim_{\mathbb{H}} L(d_r - 1)$ for $1 \leq r \leq s$. Then $\mathbf{d} := [d_1^{t_{d_1}}, \dots, d_s^{t_{d_s}}] \in \mathcal{P}(n)$, and moreover, $\psi'_{\mathfrak{sp}(p,q)}(X) = \mathbf{d}$.

We now consider $\mathcal{S}^{\mathrm{even}}_{\mathbf{d}}(p,q)$ as defined in (2.8), and assign an element $\mathbf{sgn}_X \in \mathcal{S}^{\mathrm{even}}_{\mathbf{d}}(p,q)$ to the element $X \in \mathcal{N}_{\mathfrak{sp}(p,q)}$. Let $\mathbb{N}_{\mathbf{d}} := \{d_i \mid 1 \leq i \leq s\}$; see (2.1) for the definition. For all $d \in \mathbb{N}_{\mathbf{d}}$ we first define a $t_d \times d$ matrix, say $(m^d_{ij}(X))$, in $\mathbf{A}_d$; see (2.5) for the definition. Recall that the form $(\cdot,\cdot)_d \colon L(d-1) \times L(d-1) \longrightarrow \mathbb{H}$, which is defined in (3.8), is Hermitian or skew-Hermitian according as d is odd or even. Consider $\mathbb{E}_{\mathbf{d}}$, $\mathbb{O}_{\mathbf{d}}$ as defined in (2.1). Let (p_θ, q_θ) be the signature of $(\cdot,\cdot)_\theta$ when $\theta \in \mathbb{O}_{\mathbf{d}}$. Define,

$$m^\eta_{i1}(X) := +1 \qquad \text{if } \ 1 \leq i \leq t_\eta, \quad \eta \in \mathbb{E}_{\mathbf{d}}\,;$$

$$m^\theta_{i1}(X) := \begin{cases} +1 & \text{if } \ 1 \leq i \leq p_\theta \\ -1 & \text{if } \ p_\theta < i \leq t_\theta \end{cases}, \ \theta \in \mathbb{O}_{\mathbf{d}}\,;$$

and for $j > 1$ we define $(m^d_{ij}(X))$ as in (4.11) and (4.12). The way the matrices

$(m^d_{ij}(X))$ are defined, immediately implies that they verify (**Yd**.1) and (**Yd**.2). Set $\mathbf{sgn}_X := ((m^{d_1}_{ij}(X)), \ldots, (m^{d_s}_{ij}(X)))$. It then follows from Remark 2.2.1 and Corollary 3.0.15 that

$$\sum_{k=1}^{s} \mathrm{sgn}_+(m^{d_k}_{ij}(X)) = p\,, \quad \sum_{k=1}^{s} \mathrm{sgn}_-(m^{d_k}_{ij}(X)) = q.$$

Now form the above definition of $m^{\eta}_{i1}(X)$ for $\eta \in \mathbb{E}_{\mathbf{d}}$ we conclude that $\mathbf{sgn}_X \in \mathcal{S}^{\rm even}_{\mathbf{d}}(p,q)$. We next show that $\mathbf{sgn}_X = \mathbf{sgn}_{gXg^{-1}} \in \mathcal{S}^{\rm even}_{\mathbf{d}}(p,q)$ for all $g \in \mathrm{Sp}(p,q)$. Clearly, $\{gXg^{-1},\, gHg^{-1},\, gYg^{-1}\}$ is a $\mathfrak{sl}_2(\mathbb{R})$-triple in $\mathfrak{sp}(p,q)$. It also clear that $gM(d-1)$ is the isotypical component of V containing all irreducible $\mathrm{Span}_{\mathbb{R}}\{gXg^{-1},$ $gHg^{-1},\, gYg^{-1}\}$-submodules of V with highest weight $d-1$. Moreover, $gL(d-1) = V_{gYg{-1},0} \cap gM(d-1)$. As in (3.8) for all $\theta \in \mathbb{O}_{\mathbf{d}}$, let $(\cdot,\cdot)'_\theta \colon gL(\theta-1) \times gL(\theta-1) \longrightarrow \mathbb{H}$ be defined by $(v,u)'_\theta := \langle v\,,\,(gXg^{-1})^{\theta-1}u\rangle$ for all $v,u \in gL(\theta-1)$. As $g \in \mathrm{Sp}(p,q)$, for all $u,w \in L(\theta-1)$ we have

$$(u,w)_\theta = \langle u, X^{\theta-1}w\rangle = \langle gu, gX^{\theta-1}w\rangle = \langle gu, (gXg^{-1})^{\theta-1}gw\rangle = (gu,gw)'_\theta.$$

Hence, the signature of $(\cdot,\cdot)_\theta$ and $(\cdot,\cdot)'_\theta$ are same for all $\theta \in \mathbb{O}_{\mathbf{d}}$. In particular, $\mathbf{sgn}_X = \mathbf{sgn}_{gXg^{-1}} \in \mathcal{S}^{\rm even}_{\mathbf{d}}(p,q)$.

Thus we have a map

$$\psi_{\mathfrak{sp}(p,q)} \,:\, \mathcal{N}_{\mathrm{Sp}(p,q)} \longrightarrow \mathcal{Y}^{\rm even}(p,q)\,, \quad X \longmapsto \left(\psi'_{\mathfrak{sp}(p,q)}(X),\, \mathbf{sgn}_X\right),$$

where $\mathcal{Y}^{\rm even}(p,q)$ is as in (2.10). The map $\psi_{\mathfrak{sp}(p,q)}$ satisfies the following properties:

(4.60) $\psi_{\mathfrak{sp}(p,q)}(X) = \psi_{\mathfrak{sp}(p,q)}(gXg^{-1})$ for all $g \in \mathrm{Sp}(p,q)$.

(4.61) $\psi_{\mathfrak{sp}(p,q)}(X)$ does not depend on the $\mathfrak{sl}_2(\mathbb{R})$-triple $\{X,H,Y\}$ containing X.

It is immediate from above that (4.60) holds. To prove (4.61), let $\{X,H',Y'\}$ be another $\mathfrak{sl}_2(\mathbb{R})$-triple in $\mathfrak{sp}(p,q)$ containing X. By Theorem 2.4.8, there exists

$h \in \mathrm{Sp}(p,q)$ such that $hXh^{-1} = X$, $hHh^{-1} = H'$, $hYh^{-1} = Y'$. Now (4.61) follows from (4.60).

Thus $\psi_{\mathfrak{sp}(p,q)}$ induces a well-defined map

$$\Psi_{\mathrm{Sp}(p,q)} : \mathcal{N}(\mathrm{Sp}(p,q)) \longrightarrow \mathcal{Y}^{\mathrm{even}}(p,q), \quad \mathcal{O}_X \longmapsto \big(\psi'_{\mathfrak{sp}(p,q)}(X), \, \mathbf{sgn}_X\big). \tag{4.62}$$

Using our terminologies we next state a standard result which says that the map above gives a parametrization of the nilpotent orbits in $\mathfrak{sp}(p,q)$.

Theorem 4.1.10. *The map* $\Psi_{\mathrm{Sp}(p,q)}$ *in* (4.62) *is a bijection.*

Remark 4.1.11. On account of the error in [CoMc, Lemma 9.3.1], as mentioned in Remark 3.0.16, the above parametrization in Theorem 4.1.10 is a modification of the one in [CoMc, Theorem 9.3.5]. □

Proof. We divide the proof in two steps.

Step 1 : In this step we prove that $\Psi_{\mathrm{Sp}(p,q)}$ is injective. Let $X, N \in \mathfrak{sp}(p,q)$ be two non-zero nilpotent elements such that $\Psi_{\mathrm{Sp}(p,q)}(\mathcal{O}_X) = \Psi_{\mathrm{SU}(p,q)}(\mathcal{O}_N)$. Let $\mathbf{d} := \psi'_{\mathfrak{sp}(p,q)}(X) = \psi'_{\mathfrak{sp}(p,q)}(N)$. Let $\{X^l v^d_j \mid 0 \leq l \leq d-1, 1 \leq j \leq t_d, d \in \mathbb{N}_{\mathbf{d}}\}$ and $\{N^l w^d_j \mid 0 \leq l \leq d-1, 1 \leq j \leq t_d, d \in \mathbb{N}_{\mathbf{d}}\}$ be two $\mathbb{H}$-bases of $V = \mathbb{H}^n$, as in Proposition 3.0.7, which satisfy Remark 3.0.11 (3). We also have $\mathbf{sgn}_X = \mathbf{sgn}_N \in \mathcal{S}^{\mathrm{even}}_{\mathbf{d}}(p,q)$. Thus, after reordering the ordered sets $(v^d_1, \dots, v^d_{t_d})$ and $(w^d_1, \dots, w^d_{t_d})$ for all $d \in \mathbb{N}_{\mathbf{d}}$, if necessary, we may assume that

$$\langle v^d_j, X^{d-1} v^d_j \rangle = \langle w^d_j, N^{d-1} w^d_j \rangle \ \text{ for all } \ 1 \leq j \leq t_d, d \in \mathbb{N}_{\mathbf{d}}. \tag{4.63}$$

Let $h \in \mathrm{GL}_n(\mathbb{H})$ be such that $h(X^l v^d_j) = N^l w^d_j$ for all $0 \leq l \leq d-1, 1 \leq j \leq t_d, d \in \mathbb{N}_{\mathbf{d}}$. Then

$$hX(X^l v^d_j) = hX^{l+1} v^d_j = N^{l+1} w^d_j = N(N^l w^d_j) = Nh(X^l v^d_j)$$

for all $0 \leq l \leq d-1, 1 \leq j \leq t_d, d \in \mathbb{N}_{\mathbf{d}}$. This in turn shows that $hXh^{-1} = N$. We next show that $h \in \mathrm{Sp}(p,q)$. Using the equalities in (4.63) above it follows that

$$\langle hX^l v_j^d, hX^{d-1-l} v_j^d \rangle = \langle N^l w_j^d, N^{d-1-l} w_j^d \rangle = (-1)^l \langle w_j^d, N^{d-1} w_j^d \rangle$$

$$= (-1)^l \langle v_j^d, X^{d-1} v_j^d \rangle = \langle X^l v_j^d, X^{d-1-l} v_j^d \rangle,$$

for all $0 \leq l \leq d-1, 1 \leq j \leq t_d, d \in \mathbb{N}_{\mathbf{d}}$. As $\{X^l v_j^d \mid 0 \leq l \leq d-1, 1 \leq j \leq t_d, d \in \mathbb{N}_{\mathbf{d}}\}$ is a $\mathbb{H}$-basis of V, it is now clear from the relations among the basis elements in Proposition 3.0.7(3) in the case of $\sigma = \sigma_c, \epsilon = 1, \mathbb{D} = \mathbb{H}$ that $h \in \mathrm{Sp}(p,q)$. Thus $\mathcal{O}_X = \mathcal{O}_N$ which proves the injectivity of the map $\Psi_{\mathrm{Sp}(p,q)}$.

Step 2 : In this step we prove that $\Psi_{\mathrm{Sp}(p,q)}$ is surjective. Let us fix a signed Young diagram $(\mathbf{d}, \mathbf{sgn}) \in \mathcal{Y}^{\mathrm{even}}(p,q)$. Set $n = p+q$. Then $\mathbf{d} \in \mathcal{P}(n)$, and $\mathbf{sgn} \in \mathcal{S}^{\mathrm{even}}_{\mathbf{d}}(p,q)$. Let X be a nilpotent matrix in $\mathfrak{sl}_n(\mathbb{H})$, and $\{X, H, Y\} \subset \mathfrak{sl}_n(\mathbb{H})$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple such that $\psi_{\mathfrak{sl}_n(\mathbb{H})}(X) = \mathbf{d}$; see (4.6) and Theorem 4.1.3. Our strategy is to obtain a $P \in \mathrm{GL}_n(\mathbb{H})$ such that $P^{-1}XP \in \mathfrak{sp}(p,q)$ and $\mathbf{sgn}_{P^{-1}XP} = \mathbf{sgn} \in \mathcal{S}^{\mathrm{even}}_{\mathbf{d}}(p,q)$.

We next construct a nondegenerate Hermitian form $\langle \cdot, \cdot \rangle_{\mathrm{new}}$ on $V = \mathbb{H}^n$ such that $\{X, H, Y\} \subset \mathfrak{su}(V, \langle \cdot, \cdot \rangle_{\mathrm{new}})$; see (2.15) for the definition of $\mathfrak{su}(V, \langle \cdot, \cdot \rangle_{\mathrm{new}})$. Let $\mathbf{d} := [d_1^{t_{d_1}}, \ldots, d_s^{t_{d_s}}]$. Using Proposition 3.0.3(2), $\mathbb{H}^n$ has a $\mathbb{H}$-basis of the form $\{X^l v_j^d \mid 0 \leq l \leq d-1, 1 \leq j \leq t_d, d \in \mathbb{N}_{\mathbf{d}}\}$. Let $\mathbf{sgn} := (M_{d_1}, \ldots, M_{d_s})$, and let p_θ, q_θ be the number of $+1$, -1, respectively, appearing in the 1^{st} column of the matrix of M_θ (of size $t_\theta \times \theta$) for all $\theta \in \mathbb{O}_{\mathbf{d}}$. For $d \in \mathbb{N}_{\mathbf{d}}$, $1 \leq j \leq t_d$ and for $0 \leq l, r \leq d-1$ we define $b(X^l v_j^d, X^r v_j^d) \in \mathbb{H}$ by

$$b(X^l v_j^d, X^r v_j^d) = 0 \quad \text{if } l + r \neq d-1 \tag{4.64}$$

and

$$
\text{(4.65)} \qquad b(X^l v_j^d, X^{d-1-l} v_j^d) := \begin{cases} (-1)^l \mathbf{j} & \text{if } d \in \mathbb{E}_{\mathbf{d}},\ 1 \le j \le t_d \\ (-1)^l & \text{if } d \in \mathbb{O}_{\mathbf{d}},\ 1 \le j \le p_d \\ (-1)^{l+1} & \text{if } d \in \mathbb{O}_{\mathbf{d}},\ p_d < j \le t_d. \end{cases}
$$

It now follows that for $0 \le l, r \le d-1$

$$
\text{(4.66)} \qquad b(X^l v_j^d, X^r v_j^d) = \overline{b(X^r v_j^d, X^l v_j^d)}.
$$

Recall that, for all $d \in \mathbb{N}_{\mathbf{d}}$, $1 \le j \le t_d$, the $\mathbb{R}$-Span of $\{v_j^d,\, X v_j^d,\, \ldots,\, X^{d-1} v_j^d\}$ is an irreducible $\mathrm{Span}_{\mathbb{R}}\{X, H, Y\}$-submodule of $\mathbb{H}^n$; see Lemma 3.0.2 (2). We set $V_j^d := \mathrm{Span}_{\mathbb{H}}\{X^l v_j^d \mid 0 \le l \le d-1\}$. As $\{X^l v_j^d \mid 0 \le l \le d-1\}$ is a $\mathbb{H}$-basis for V_j^d the equalities in (4.66) allow us to define a Hermitian form $\langle \cdot\,, \cdot \rangle_{dj}$ on V_j^d such that

$$
\text{(4.67)} \qquad \langle X^l v_j^d, X^r v_j^d \rangle_{dj} \;=\; b(X^l v_j^d, X^r v_j^d) \quad \text{for } 0 \le l, r \le d-1.
$$

From the definition it is clear that $\langle \cdot\,, \cdot \rangle_{dj}$ is nondegenerate on V_j^d, and moreover $\langle Xx, y \rangle_{dj} + \langle x, Xy \rangle_{dj} = 0$ for all $x, y \in V_j^d$. Recall that

$$
\text{(4.68)} \qquad \mathbb{H}^n = \bigoplus_{d \in \mathbb{N}_{\mathbf{d}}, 1 \le j \le t_d} V_j^d\,.
$$

Let $\langle \cdot\,, \cdot \rangle_{\mathrm{new}}$ be the new Hermitian form on V such that its restriction to V_j^d agrees with $\langle \cdot\,, \cdot \rangle_{dj}$, and so that (4.68) is an orthogonal direct sum with respect to $\langle \cdot\,, \cdot \rangle_{\mathrm{new}}$. Then $\langle \cdot\,, \cdot \rangle_{\mathrm{new}}$ is nondegenerate on $V \times V$. Clearly, $\langle Xx, y \rangle_{\mathrm{new}} + \langle x, Xy \rangle_{\mathrm{new}} = 0$ for all $x, y \in V$. Recall that in Proposition 3.0.3 (1) we have that $Y X^l v_j^d = (X^{l-1} v_j^d) l(d-l)$ for $0 < l \le d-1$, $1 \le j \le t_d$, $d \in \mathbb{N}_{\mathbf{d}}$, and $Y v_j^d = 0$ for $1 \le j \le t_d$, $d \in \mathbb{N}_{\mathbf{d}}$. As $\{X^l v_j^d \mid 0 \le l \le d-1, 1 \le j \le t_d, d \in \mathbb{N}_{\mathbf{d}}\}$ is a basis of $\mathbb{H}^n$, using the above relations, (4.42) and (4.44), we conclude that $\langle Hx, y \rangle_{\mathrm{new}} + \langle x, Hy \rangle_{\mathrm{new}} = 0$ and

$\langle Yx, y\rangle_{\rm new} + \langle x, Yy\rangle_{\rm new} = 0$ for all $x, y \in V$. Thus $\{X, H, Y\} \subset \mathfrak{su}(V, \langle \cdot\,,\cdot \rangle_{\rm new})$.

We next show that the signature of $\langle \cdot\,,\cdot \rangle_{\rm new}$ is (p, q). Let $d \in \mathbb{N}_{\mathbf{d}}$. Recall that $M(d-1)$ denotes the isotypical component of $\mathbb{H}^n$ containing all irreducible $\text{Span}_{\mathbb{R}}\{X, H, Y\}$-submodules of $\mathbb{H}^n$ with highest weight $(d-1)$, and $L(d-1) = V_{Y,0} \cap M(d-1)$; see (3.1). As in (3.8), let $(\cdot, \cdot)_{{\rm new}_d} \colon L(d-1) \times L(d-1) \longrightarrow \mathbb{H}$ be defined by $(v, u)_{{\rm new}_d} := \langle v, X^{d-1}u\rangle_{\rm new}$ for all $v, u \in L(d-1)$. From the defining properties of $\langle \cdot\,,\cdot \rangle_{\rm new}$ it follows that $M(d-1)$ is a direct sum of the subspaces $V^d_1, \dots, V^d_{t_d}$ which are mutually orthogonal with respect to $\langle \cdot\,,\cdot \rangle_{\rm new}$. In particular, $(v^d_1, \dots, v^d_{t_d})$ is a orthogonal basis of $L(d-1)$ with respect to $(\cdot, \cdot)_{{\rm new}_d}$. Using this orthogonal basis and putting $l = 0$ when $\theta \in \mathbb{O}_{\mathbf{d}}$, in (4.65), we obtain that the signature of $(\cdot, \cdot)_{{\rm new}_\theta}$ is (p_θ, q_θ) for all $\theta \in \mathbb{O}_{\mathbf{d}}$. Now from Remark 2.2.1 and Corollary 3.0.15 it follows that the signature of $\langle \cdot\,,\cdot \rangle_{\rm new}$ on $M(d-1)$ is $(\text{sgn}_+ M_d,\, \text{sgn}_- M_d)$. Recall that, as $\mathbf{sgn} \in \mathcal{S}^{\rm even}_{\mathbf{d}}(p, q)$, we have $\sum_{d \in \mathbb{N}_{\mathbf{d}}} \text{sgn}_+ M_d = p$ and $\sum_{d \in \mathbb{N}_{\mathbf{d}}} \text{sgn}_- M_d = q$. Thus the signature of $\langle \cdot\,,\cdot \rangle_{\rm new}$ is (p, q).

Since the signatures of both the forms $\langle \cdot\,,\cdot \rangle_{\rm new}$ and $\langle \cdot\,,\cdot \rangle$ coincide there is a $P \in \text{GL}_n(\mathbb{H})$ such that

$$\langle x, y \rangle \;=\; \langle Px, Py\rangle_{\rm new} \quad \text{for all } x, y \in V. \tag{4.69}$$

Clearly $\{P^{-1}XP,\, P^{-1}HP,\, P^{-1}YP\} \subset \mathfrak{sp}(p, q)$ is a $\mathfrak{sl}_2(\mathbb{R})$-triple. Now we will show that $\mathbf{sgn}_{P^{-1}XP} = \mathbf{sgn} \in \mathcal{S}^{\rm even}_{\mathbf{d}}(p, q)$. Note that $P^{-1}M(d-1)$ is the isotypical component of $\mathbb{H}^n$ containing all the irreducible $\text{Span}_{\mathbb{R}}\{P^{-1}XP,\, P^{-1}HP,\, P^{-1}YP\}$-submodules of $\mathbb{H}^n$ with highest weight $(d-1)$. Moreover, $P^{-1}L(d-1) = V_{P^{-1}YP,0} \cap P^{-1}M(d-1)$. As in (3.8) for $\theta \in \mathbb{O}_{\mathbf{d}}$, let $(\cdot, \cdot)''_\theta \colon P^{-1}L(\theta-1) \times P^{-1}L(\theta-1) \longrightarrow \mathbb{H}$ be defined by $(x, y)''_\theta := \langle x, (P^{-1}XP)^{\theta-1}y\rangle$ for all $x, y \in P^{-1}L(\theta-1)$. Using (4.69) it follows that

$$(u, v)''_\theta = (Pu, Pv)_{{\rm new}_\theta} \quad \text{for } u, v \in P^{-1}L(d-1);\ \theta \in \mathbb{O}_{\mathbf{d}}.$$

Thus the signatures $(\cdot,\cdot)''_\theta$ and $(\cdot,\cdot)_{\mathrm{new}_\theta}$ are both equal, which, in particular, shows that signature of $(\cdot,\cdot)''_\theta$ is $(p_\theta,\, q_\theta)$ for all $\theta \in \mathbb{O}_{\mathbf{d}}$. This proves that $\mathbf{sgn}_{P^{-1}XP} = \mathbf{sgn} \in \mathcal{S}^{\mathrm{even}}_{\mathbf{d}}(p,q)$. Hence $\Psi_{\mathrm{Sp}(p,q)}(\mathcal{O}_{P^{-1}XP}) = (\mathbf{d}, \mathbf{sgn})$. This completes the proof of the theorem. □

4.2 Nilpotent orbits in non-compact non-complex real exceptional Lie algebras

We refer to [Dj1], [Dj2] and [CM, Chapter 9] for the generalities required in this section. We follow the parametrization of nilpotent orbits in non-compact non-complex exceptional Lie algebras as given in [Dj1, Tables VI-XV] and [Dj2, Tables VII-VIII]. We consider the nilpotent orbits in $\mathfrak{g}$ under the action of $\mathrm{Int}\,\mathfrak{g}$, where $\mathfrak{g}$ is a non-compact non-complex real exceptional Lie algebra. We fix a semisimple algebraic group G defined over $\mathbb{R}$ such that $\mathfrak{g} = \mathrm{Lie}(G(\mathbb{R}))$. Here $G(\mathbb{R})$ denotes the associated real semisimple Lie group of the $\mathbb{R}$-points of G. Let $G(\mathbb{C})$ be the associated complex semisimple Lie group consisting of the $\mathbb{C}$-points of G. It is easy to see that orbits in $\mathfrak{g}$ under the action of $\mathrm{Int}\,\mathfrak{g}$ are the same as the orbits in $\mathfrak{g}$ under the action of $G(\mathbb{R})^\circ$. Thus in this set-up, for a nilpotent element $X \in \mathfrak{g}$, we set $\mathcal{O}_X := \{\mathrm{Ad}(g)X \mid g \in G(\mathbb{R})^\circ\}$. Let $\mathfrak{g} = \mathfrak{m} + \mathfrak{p}$ be a Cartan decomposition and θ be the corresponding Cartan involution. Let $\mathfrak{g}_\mathbb{C}$ be the Lie algebra of $G(\mathbb{C})$. Then $\mathfrak{g}_\mathbb{C}$ can be identified with the complexification of $\mathfrak{g}$. Let $\mathfrak{m}_\mathbb{C}, \mathfrak{p}_\mathbb{C}$ be the $\mathbb{C}$-spans of $\mathfrak{m}$ and $\mathfrak{p}$ in $\mathfrak{g}_\mathbb{C}$, respectively. Then $\mathfrak{g}_\mathbb{C} = \mathfrak{m}_\mathbb{C} + \mathfrak{p}_\mathbb{C}$. Let $M_\mathbb{C}$ be the connected subgroup of $G(\mathbb{C})$ with Lie algebra $\mathfrak{m}_\mathbb{C}$. Recall that, if $\mathfrak{g}$ is as above and $\mathfrak{g}$ is different from both $E_{6(-26)}$ and $E_{6(6)}$, then $\mathfrak{g}$ is of inner type, or equivalently, $\mathrm{rank}\,\mathfrak{m}_\mathbb{C} = \mathrm{rank}\,\mathfrak{g}_\mathbb{C}$. When $\mathfrak{g}$ is of inner type, the nilpotent orbits are parametrized by a finite sequence of integers of length l where $l := \mathrm{rank}\,\mathfrak{m}_\mathbb{C} = \mathrm{rank}\,\mathfrak{g}_\mathbb{C}$. When $\mathfrak{g}$ is not of inner type, that is, when $\mathfrak{g}$ is either $E_{6(-26)}$ or $E_{6(6)}$, then the nilpotent orbits are parametrized by a finite sequence of integers of length 4.

Let $X' \in \mathfrak{g}$ be a nonzero nilpotent element, and $\{X', H', Y'\} \subset \mathfrak{g}$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple. Then $\{X', H', Y'\}$ is $G(\mathbb{R})$-conjugate to another $\mathfrak{sl}_2(\mathbb{R})$-triple $\{\widetilde{X}, \widetilde{H}, \widetilde{Y}\}$ in $\mathfrak{g}$ such that $\theta(\widetilde{H}) = -\widetilde{H}$, $\theta(\widetilde{X}) = -\widetilde{Y}$. Set $E := (\widetilde{H} - i(\widetilde{X} + \widetilde{Y}))/2$, $F := (\widetilde{H} + i(\widetilde{X} + \widetilde{Y}))/2$ and $H := i(\widetilde{X} - \widetilde{Y})$. Then $\{E, H, F\}$ is a $\mathfrak{sl}_2(\mathbb{R})$-triple and $E, F \in \mathfrak{p}_{\mathbb{C}}$ and $H \in \mathfrak{m}_{\mathbb{C}}$. The $\mathfrak{sl}_2(\mathbb{R})$-triple $\{E, H, F\}$ is then called a $\mathfrak{p}_{\mathbb{C}}$-*Cayley triple associated to* X'.

4.2.1 Parametrization of nilpotent orbits in exceptional Lie algebras of inner type

We now recall from [Dj1, Column 2, Tables VI-XV] the parametrization of non-zero nilpotent orbits in $\mathfrak{g}$ when $\mathfrak{g}$ is an exceptional Lie algebra of inner type. Let $\mathfrak{h}_{\mathbb{C}} \subset \mathfrak{m}_{\mathbb{C}}$ be a Cartan subalgebra of $\mathfrak{m}_{\mathbb{C}}$ such that $\mathfrak{h}_{\mathbb{C}} \cap \mathfrak{m}$ is a Cartan subalgebra of $\mathfrak{m}$. As $\mathfrak{g}$ is of inner type, $\mathfrak{h}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Set $\mathfrak{h} := \mathfrak{h}_{\mathbb{C}} \cap i\mathfrak{m}$. Let R, R_0 be the root systems of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}), (\mathfrak{m}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$, respectively. Let $B := \{\alpha_1, \ldots, \alpha_l\}$ be a basis of R. Let $B_e := B \cup \{\alpha_0\}$ where α_0 is the negative of the highest root of (R, B). Then there exists an unique basis of R_0, say B_0, such that $B_0 \subset B_e$. Let C_0 be the closed Weyl chamber of R_0 in $\mathfrak{h}$ corresponding to the basis B_0. Let l_0 be the rank of $[\mathfrak{m}_{\mathbb{C}}, \mathfrak{m}_{\mathbb{C}}]$. Then either $l_0 = l$ or $l_0 = l - 1$. If $l_0 = l$ we set $B_0' := B_0$. If $l_0 = l - 1$ (in this case we have $B_0 \subset B$) we set $B_0' := B$. Clearly, $\#B_0' = l$. We enumerate $B_0' := \{\beta_1, \ldots, \beta_l\}$ as in [Dj1, 7, p. 506 and Table IV]. Let $X \in \mathfrak{g}$ be a nonzero nilpotent element, and $\{E, H, F\}$ be a $\mathfrak{p}_{\mathbb{C}}$-Cayley triple (in $\mathfrak{g}_{\mathbb{C}}$) associated to X. Then $\mathrm{Ad}(M_{\mathbb{C}})H \cap C_0$ is a singleton set, say $\{H_0\}$. The element H_0 is called *the characteristic* of the orbit $\mathrm{Ad}(M_{\mathbb{C}})E$ as it determines the orbit $M_{\mathbb{C}} \cdot E$ uniquely. Consider the map from the set of nilpotent orbits in $\mathfrak{g}$ to the set of integer sequences of length l, which assigns the sequence $\beta_1(H_0), \ldots, \beta_l(H_0)$ to each nilpotent orbits $\mathcal{O}_X$. In view of the Kostant-Sekiguchi theorem (cf. [CM, Theorem 9.5.1]), this gives a bijection between the set of nilpotent orbits in $\mathfrak{g}$ and the set of finite sequences of the form $\beta_1(H_0), \ldots, \beta_l(H_0)$ as above. We use this parametrization while dealing with nilpotent orbits in exceptional Lie algebras of inner type.

4.2.2 Parametrization of nilpotent orbits in $E_{6(-26)}$ or $E_{6(6)}$

We now recall from [Dj2, Column 1, Tables VII-VIII] the parametrization of non-zero nilpotent orbits in $\mathfrak{g}$ when $\mathfrak{g}$ is either $E_{6(-26)}$ or $E_{6(6)}$. We need a piece of notation here : henceforth, for a Lie algebra $\mathfrak{a}$ over $\mathbb{C}$ and an automorphism $\sigma \in \mathrm{Aut}_{\mathbb{C}}\,\mathfrak{a}$, the Lie subalgebra consisting of the fixed points of σ in $\mathfrak{a}$, is denoted by $\mathfrak{a}^{\sigma}$. Let now $\mathfrak{h}_{\mathbb{C}}$ be a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ (we point out the difference of our notation with that in [Dj2]; $\mathfrak{g}$ and $\mathfrak{h}$ of [Dj2, §1] are denoted here by $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{C}}$, respectively).

Let $\mathfrak{g} = E_{6(-26)}$. Let τ be the involution of $\mathfrak{g}_{\mathbb{C}}$ as defined in [Dj2, p. 198] which keeps $\mathfrak{h}_{\mathbb{C}}$ invariant. Then the subalgebra $\mathfrak{g}_{\mathbb{C}}^{\tau}$ is of type F_4, and $\mathfrak{h}_{\mathbb{C}}^{\tau}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}^{\tau}$. Let $G(\mathbb{C})^{\tau}$ be the connected Lie subgroup of $G(\mathbb{C})$ with Lie algebra $\mathfrak{g}_{\mathbb{C}}^{\tau}$. Let $\{\beta_1, \beta_2, \beta_3, \beta_4\}$ be the simple roots of $(\mathfrak{g}_{\mathbb{C}}^{\tau}, \mathfrak{h}_{\mathbb{C}}^{\tau})$ as defined in [Dj2, (1), p. 198]. Let $X \in E_{6(-26)}$ be a nonzero nilpotent element. Let $\{E, H, F\}$ be a $\mathfrak{p}_{\mathbb{C}}$-Cayley triple (in $\mathfrak{g}_{\mathbb{C}}$) associated to X. Then $H \in \mathfrak{g}_{\mathbb{C}}^{\tau}$ and $E, F \in \mathfrak{g}_{\mathbb{C}}^{-\tau}$. We may further assume that $H \in \mathfrak{h}_{\mathbb{C}}^{\tau}$. Then the finite sequence of integers $\beta_1(H), \beta_2(H), \beta_3(H), \beta_4(H)$ determine the orbit $\mathrm{Ad}(G(\mathbb{C})^{\tau})\, E$ uniquely; see [Dj2, p. 204].

Let $\mathfrak{g} = E_{6(6)}$. Let τ' be the involution of $\mathfrak{g}_{\mathbb{C}}$ as defined in [Dj2, p. 199] which keeps $\mathfrak{h}_{\mathbb{C}}$ invariant. Then the subalgebra $\mathfrak{g}_{\mathbb{C}}^{\tau'}$ is of type C_4, and $\mathfrak{h}_{\mathbb{C}}^{\tau'}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}^{\tau'}$. Let $G(\mathbb{C})^{\tau'}$ be the connected Lie subgroup of $G(\mathbb{C})$ with Lie algebra $\mathfrak{g}_{\mathbb{C}}^{\tau'}$. Let $\{\beta_0, \beta_1, \beta_2, \beta_3\}$ be the simple roots of $(\mathfrak{g}_{\mathbb{C}}^{\tau'}, \mathfrak{h}_{\mathbb{C}}^{\tau'})$ as defined in [Dj2, p. 199]. Let $X \in E_{6(6)}$ be a nonzero nilpotent element. Let $\{E, H, F\}$ be a $\mathfrak{p}_{\mathbb{C}}$-Cayley triple (in $\mathfrak{g}_{\mathbb{C}}$) associated to X. Then $H \in \mathfrak{g}_{\mathbb{C}}^{\tau'}$ and $E, F \in \mathfrak{g}_{\mathbb{C}}^{-\tau'}$. We may further assume that $H \in \mathfrak{h}_{\mathbb{C}}^{\tau'}$. It then follows that the finite sequence of integers

$\beta_0(H), \beta_1(H), \beta_2(H), \beta_3(H)$ determine the orbit $\mathrm{Ad}(G(\mathbb{C})^{\tau'})\, E$ uniquely; see [Dj2, p. 204].

Chapter 5

First and second cohomologies of homogeneous spaces of Lie groups

In this chapter we first formulate a convenient description of the second and first de Rham cohomology groups of homogeneous spaces of general connected Lie groups. In the second section we use the above results to obtain a description of the second and first cohomology groups of the nilpotent orbits.

5.1 Description of first and second cohomology groups of homogeneous spaces

We begin this section by a well-known definition. Given a Lie algebra $\mathfrak{a}$ and an integer $n \geq 0$, let $\Omega^n(\mathfrak{a})$ denote the space of all n-forms on $\mathfrak{a}$. A n-form $\omega \in \Omega^n(\mathfrak{a})$ is said to *annihilate* a given subalgebra $\mathfrak{b} \subset \mathfrak{a}$ if $\omega(X_1, \ldots, X_n) = 0$ whenever $X_i \in \mathfrak{b}$ for some i. Let $\Omega^n(\mathfrak{a}/\mathfrak{b})$ denote the space of n-forms on $\mathfrak{a}$ which annihilate $\mathfrak{b}$.

Let L be a compact Lie group with Lie algebra $\mathfrak{l}$. Let $J \subset L$ be a closed subgroup with Lie algebra $\mathfrak{j} \subset \mathfrak{l}$. The space of J-invariant p-forms on $\mathfrak{l}$ will be denoted by

$\Omega^p(\mathfrak{l})^J$. Note that $\omega \in \Omega^p(\mathfrak{l})^{J^\circ}$ if and only if

$$\sum_{i=1}^{p} \omega(X_1, \ldots, [Y, X_i], \ldots, X_p) = 0 \tag{5.1}$$

for all $Y \in \mathfrak{j}$ and all $(X_1, \ldots, X_p) \in \mathfrak{l}^p$. For a continuous function

$$W : J \longrightarrow \Omega^p(\mathfrak{l})$$

and a Haar measure μ_J on J, define the integral $\int_J W(g)d\mu_J(g) \in \Omega^p(\mathfrak{l})$ as follows:

$$(\int_J W(g)d\mu_J(g))(X_1, \ldots, X_p) := \int_J W(g)(X_1, \ldots, X_p)d\mu_J(g), \quad (X_1, \ldots, X_p) \in \mathfrak{l}^p.$$

The above integral $\int_J W(g)d\mu_J(g)$ is also denoted by $\int_J W d\mu_J$. The following equations are straightforward.

$$d\int_J W d\mu_J = \int_J dW d\mu_J\,; \tag{5.2}$$

For any $a \in L$,

$$\mathrm{Ad}(a)^* \int_J W d\mu_J = \int_J \mathrm{Ad}(a)^* W d\mu_J\,. \tag{5.3}$$

For any $\omega \in \Omega^p(\mathfrak{l})$, from the left-invariance of the Haar measure μ_J on J it follows that

$$\int_J (\mathrm{Ad}(g)^*\omega)d\mu_J(g) \in \Omega^p(\mathfrak{l})^J\,.$$

Lemma 5.1.1. *Let L be a compact Lie group with Lie algebra $\mathfrak{l}$. Let $p \geq 1$ be an integer.*

1. *If $\omega \in \Omega^p(\mathfrak{l})$ is invariant then $d\omega = 0$.*

2. *Every element of $H^p(\mathfrak{l}, \mathbb{R})$ contains an unique invariant $\omega \in \Omega^p(\mathfrak{l})$.*

3. If $J \subset L$ is a closed subgroup, then

$$\Omega^p(\mathfrak{l})^J \cap d(\Omega^{p-1}(\mathfrak{l})) = d(\Omega^{p-1}(\mathfrak{l})^J).$$

4. If L is connected and $\omega \in \Omega^2(\mathfrak{l})$, then $\omega \in \Omega^2(\mathfrak{l})^L$ if and only if $\omega \in \Omega^2(\mathfrak{l}/[\mathfrak{l}, \mathfrak{l}])$.

Proof. Statement (1) is proved in [CE, p. 102, 12.3]. Statement (2) is proved in [CE, p. 102, Theorem 12.1].

To prove (3), note that it suffices to show that

$$\Omega^p(\mathfrak{l})^J \cap d(\Omega^{p-1}(\mathfrak{l})) \subset d(\Omega^{p-1}(\mathfrak{l})^J).$$

Let μ_J denote the Haar measure on J such that $\mu_J(J) = 1$. For any $\omega \in \Omega^p(\mathfrak{l})^J \cap d(\Omega^{p-1}(\mathfrak{l}))$, we have $\omega = d\nu$ for some $\nu \in \Omega^{p-1}(\mathfrak{l})$. Now as ω is J-invariant, it follows that

$$\omega = \mathrm{Ad}(g)^* d\nu = d\mathrm{Ad}(g)^*\nu \tag{5.4}$$

for all $g \in J$. In particular, from (5.4) we have

$$\omega = \int_J (d\mathrm{Ad}(g)^*\nu) d\mu_J(g) = d\int_J (\mathrm{Ad}(g)^*\nu) d\mu_J(g).$$

As μ_J is preserved by the left multiplication by elements of J, it now follows that

$$\int_J (\mathrm{Ad}(g)^*\nu) d\mu_J(g) \in \Omega^{p-1}(\mathfrak{l})^J.$$

This in turn implies that $\omega \in d(\Omega^{p-1}(\mathfrak{l})^J)$.

The proof of (4) is essentially contained in the proof of [Br, p. 309, Corollary 12.9]; we will give the details. Take any $\omega \in \Omega^2(\mathfrak{l})^L$. Lemma 5.1.1(1) says that

$d\omega = 0$. Thus, for all $x, y, z \in \mathfrak{l}$,

$$(5.5) \qquad d\omega(x, y, z) = -\omega([x, y], z) + \omega([x, z], y) - \omega([y, z], x) = 0\,.$$

As ω is L-invariant, we also have

$$(5.6) \qquad -\omega([x, y], z) + \omega([x, z], y) = -(\omega([x, y], z) + \omega(y, [x, z]) = 0\,.$$

From (5.5) and (5.6) it follows that $\omega([y, z], x) = 0$, therefore

$$\omega([\mathfrak{l}, \mathfrak{l}], \mathfrak{l}) = 0.$$

This is equivalent to saying that $\omega \in \Omega^2(\mathfrak{l}/[\mathfrak{l}, \mathfrak{l}])$.

Conversely, if $\omega([\mathfrak{l}, \mathfrak{l}], \mathfrak{l}) = 0$, then it is immediate that ω satisfies (5.1) for $p = 2$. In particular, as L is connected, we conclude that $\omega \in \Omega^2(\mathfrak{l})^L$. This completes the proof of (4). □

Theorem 5.1.2 ([Mo])**.** *Let G be a connected Lie group, and let $H \subset G$ be a closed subgroup with finitely many connected components. Let M be a maximal compact subgroup of G such that $M \cap H$ is a maximal compact subgroup of H. Then the image of the natural embedding $M/(M \cap H) \hookrightarrow G/H$ is a deformation retraction of G/H.*

Theorem 5.1.2 is proved in [Mo, p. 260, Theorem 3.1] under the assumption that H is connected. However, as mentioned in [BC1], using [Ho, p. 180, Theorem 3.1], the proof as in [Mo] goes through when H has finitely many connected components.

Let G, H, M be as in Theorem 5.1.2, and let $K := M \cap H$. As $M/K \hookrightarrow G/H$ is a deformation retraction by Theorem 5.1.2, we have

$$(5.7) \qquad H^i(G/H, \mathbb{R}) \simeq H^i(M/K, \mathbb{R}) \quad \text{for all } i\,.$$

Theorem 5.1.3. *Let G be a connected Lie group, and let $H \subset G$ be a closed subgroup with finitely many connected components. Let K be a maximal compact subgroup of H, and let M be a maximal compact subgroup of G containing K. Then,*

$$H^2(G/H,\, \mathbb{R}) \,\simeq\, \Omega^2\big(\frac{\mathfrak{m}}{[\mathfrak{m},\, \mathfrak{m}] + \mathfrak{k}}\big) \oplus [(\mathfrak{z}(\mathfrak{k}) \cap [\mathfrak{m},\, \mathfrak{m}])^*]^{K/K^\circ}\,.$$

Proof. In view of (5.7) it is enough to show that

$$H^2(M/K,\, \mathbb{R}) \,\simeq\, \Omega^2\big(\mathfrak{m}/([\mathfrak{m},\, \mathfrak{m}] + \mathfrak{k})\big) \oplus [(\mathfrak{z}(\mathfrak{k}) \cap [\mathfrak{m},\, \mathfrak{m}])^*]^{K/K^\circ}\,. \tag{5.8}$$

As M is compact and connected, from [Sp, p. 310, Theorem 30] and the formula given in [Sp, p. 313] we conclude that there are natural isomorphisms

$$H^i(M/K,\, \mathbb{R}) \,\simeq\, \frac{\mathrm{Ker}\,(d : \Omega^i(\mathfrak{m}/\mathfrak{k})^K \to \Omega^{i+1}(\mathfrak{m}/\mathfrak{k})^K)}{d(\Omega^{i-1}(\mathfrak{m}/\mathfrak{k})^K)} \quad \forall\ i\,. \tag{5.9}$$

Setting $i = 2$ in (5.9),

$$H^2(M/K,\, \mathbb{R}) \,\simeq\, \frac{\mathrm{Ker}\,(d : \Omega^2(\mathfrak{m}/\mathfrak{k})^K \to \Omega^3(\mathfrak{m}/\mathfrak{k})^K)}{d(\Omega^1(\mathfrak{m}/\mathfrak{k})^K)}\,. \tag{5.10}$$

The numerator and the denominator in (5.10) will be identified.

We claim that

$$\mathrm{Ker}\,(d : \Omega^2(\mathfrak{m}/\mathfrak{k})^K \to \Omega^3(\mathfrak{m}/\mathfrak{k})^K) \,=\, \Omega^2(\mathfrak{m}/\mathfrak{k})^M \oplus d(\Omega^1(\mathfrak{m})^K). \tag{5.11}$$

To prove the claim, first note that $d(\Omega^2(\mathfrak{m}/\mathfrak{k})^M) \,=\, 0$ by Lemma 5.1.1(1). Therefore, we have

$$\Omega^2(\mathfrak{m}/\mathfrak{k})^M + d(\Omega^1(\mathfrak{m})^K) \,\subset\, \mathrm{Ker}\,(d : \Omega^2(\mathfrak{m}/\mathfrak{k})^K \to \Omega^3(\mathfrak{m}/\mathfrak{k})^K)\,.$$

To prove the converse, take any $\omega \in \operatorname{Ker}(d : \Omega^2(\mathfrak{m}/\mathfrak{k})^K \to \Omega^3(\mathfrak{m}/\mathfrak{k})^K)$. Then by Lemma 5.1.1(2) there is an element $\widetilde{\omega} \in \Omega^2(\mathfrak{m})^M$ such that

$$\omega - \widetilde{\omega} \in d(\Omega^1(\mathfrak{m}))\,. \tag{5.12}$$

As $\omega \in \Omega^2(\mathfrak{m})^K$ and $\widetilde{\omega} \in \Omega^2(\mathfrak{m})^M$, it follows that $\omega - \widetilde{\omega} \in \Omega^2(\mathfrak{m})^K$. So (5.12) and Lemma 5.1.1(3) together imply that

$$\omega - \widetilde{\omega} \in d(\Omega^1(\mathfrak{m})^K)\,.$$

Take any $f \in \Omega^1(\mathfrak{m})^K$ such that $\omega - \widetilde{\omega} = df$. As $f \in \Omega(\mathfrak{m})^K$, it follows that $df \in \Omega^2(\mathfrak{m}/\mathfrak{k})^K$. Thus $\widetilde{\omega} \in \Omega^2(\mathfrak{m}/\mathfrak{k})^M$. This in turn implies that $\omega \in \Omega^2(\mathfrak{m}/\mathfrak{k})^M + d(\Omega^1(\mathfrak{m})^K)$. Therefore,

$$\Omega^2(\mathfrak{m}/\mathfrak{k})^M + d(\Omega^1(\mathfrak{m})^K) \supset \operatorname{Ker}(d : \Omega^2(\mathfrak{m}/\mathfrak{k})^K \to \Omega^3(\mathfrak{m}/\mathfrak{k})^K)\,.$$

To complete the proof of the claim, it now remains to show that

$$\Omega^2(\mathfrak{m}/\mathfrak{k})^M \cap d(\Omega^1(\mathfrak{m})^K) = 0\,. \tag{5.13}$$

To prove (5.13), take any $f_1 \in \Omega^1(\mathfrak{m})^K$ such that $df_1 \in \Omega^2(\mathfrak{m}/\mathfrak{k})^M$. From Lemma 5.1.1(3) it follows that $df_1 = df_2$ for some $f_2 \in \Omega^1(\mathfrak{m})^M$. But then from Lemma 5.1.1(1) it follows that $df_2 = 0$. Thus we have $df_1 = df_2 = 0$. This proves (5.13), and the proof of the claim is complete.

Combining (5.10) and (5.11),

$$H^2(M/K, \mathbb{R}) \simeq \Omega^2(\mathfrak{m}/\mathfrak{k})^M \oplus \frac{d(\Omega^1(\mathfrak{m})^K)}{d(\Omega^1(\mathfrak{m}/\mathfrak{k})^K)}\,. \tag{5.14}$$

Moreover, as M is connected, Lemma 5.1.1(4) implies that

$$\Omega^2(\mathfrak{m}/\mathfrak{k})^M \simeq \Omega^2(\frac{\mathfrak{m}}{[\mathfrak{m},\,\mathfrak{m}]+\mathfrak{k}})\,. \tag{5.15}$$

We have

$$\mathrm{Ker}(d:\Omega^1(\mathfrak{m})\to\Omega^2(\mathfrak{m})) \;=\; \Omega^1(\mathfrak{m}/[\mathfrak{m},\,\mathfrak{m}]).$$

In view of the above it is straightforward to check that

$$\frac{d(\Omega^1(\mathfrak{m})^K)}{d(\Omega^1(\mathfrak{m}/\mathfrak{k})^K)} \simeq \frac{\Omega^1(\mathfrak{m})^K}{\Omega^1(\mathfrak{m}/\mathfrak{k})^K+\Omega^1(\mathfrak{m}/[\mathfrak{m},\,\mathfrak{m}])^K}\,. \tag{5.16}$$

We will identify the right-hand side of (5.16).

Consider the adjoint action of K on $\mathfrak{m}$. As K is compact, there is a K-invariant inner-product $\langle\cdot\,,\cdot\rangle$ on the $\mathbb{R}$-vector space $\mathfrak{m}$. Now decompose $\mathfrak{m}$ as follows.

$$\begin{aligned}\mathfrak{m} =&([\mathfrak{m},\,\mathfrak{m}]+\mathfrak{k})+\mathfrak{z}(\mathfrak{m})\\ =&([\mathfrak{m},\,\mathfrak{m}]+\mathfrak{k})\oplus\big((([\mathfrak{m},\,\mathfrak{m}]+\mathfrak{k})\cap\mathfrak{z}(\mathfrak{m}))^{\perp}\cap\mathfrak{z}(\mathfrak{m})\big).\end{aligned} \tag{5.17}$$

We next decompose $[\mathfrak{m},\mathfrak{m}]+\mathfrak{k}$ as

$$[\mathfrak{m},\,\mathfrak{m}]+\mathfrak{k} \;=\; \big(([\mathfrak{m},\,\mathfrak{m}]\cap\mathfrak{k})^{\perp}\cap[\mathfrak{m},\,\mathfrak{m}]\big)\oplus([\mathfrak{m},\,\mathfrak{m}]\cap\mathfrak{k})\oplus\big(([\mathfrak{m},\,\mathfrak{m}]\cap\mathfrak{k})^{\perp}\cap\mathfrak{k}\big)\,. \tag{5.18}$$

Using (5.17) and (5.18) the decomposition of $\mathfrak{m}$ is further refined as follows:

$$\begin{aligned}\mathfrak{m} &= ([\mathfrak{m},\,\mathfrak{m}]+\mathfrak{k})\oplus\big((([\mathfrak{m},\,\mathfrak{m}]+\mathfrak{k})\cap\mathfrak{z}(\mathfrak{m}))^{\perp}\cap\mathfrak{z}(\mathfrak{m})\big)\\ &= \big(([\mathfrak{m},\,\mathfrak{m}]\cap\mathfrak{k})^{\perp}\cap[\mathfrak{m},\,\mathfrak{m}]\big)\oplus([\mathfrak{m},\,\mathfrak{m}]\cap\mathfrak{k})\oplus\big(([\mathfrak{m},\,\mathfrak{m}]\cap\mathfrak{k})^{\perp}\cap\mathfrak{k}\big)\\ &\quad\oplus\big((([\mathfrak{m},\,\mathfrak{m}]+\mathfrak{k})\cap\mathfrak{z}(\mathfrak{m}))^{\perp}\cap\mathfrak{z}(\mathfrak{m})\big)\,.\end{aligned} \tag{5.19}$$

It is clear that all the direct summands in (5.19) are K-invariant. For notational convenience, set $\mathfrak{a} := (([\mathfrak{m},\,\mathfrak{m}] + \mathfrak{k}) \cap \mathfrak{z}(\mathfrak{m}))^{\perp}$ and $\mathfrak{b} := ([\mathfrak{m},\,\mathfrak{m}] \cap \mathfrak{k})^{\perp}$. Let

$$\sigma \,:\, \Omega^1(\mathfrak{m})) = \mathfrak{m}^* \longrightarrow ([\mathfrak{m},\,\mathfrak{m}] \cap \mathfrak{b})^* \oplus ([\mathfrak{m},\,\mathfrak{m}] \cap \mathfrak{k})^* \oplus (\mathfrak{k} \cap \mathfrak{b})^* \oplus (\mathfrak{a} \cap \mathfrak{z}(\mathfrak{m}))^* \tag{5.20}$$

be the isomorphism defined by

$$f \longmapsto (f|_{[\mathfrak{m},\,\mathfrak{m}]\cap\mathfrak{b}},\, f|_{[\mathfrak{m},\,\mathfrak{m}]\cap\mathfrak{k}},\, f|_{\mathfrak{k}\cap\mathfrak{b}},\, f|_{\mathfrak{a}\cap\mathfrak{z}(\mathfrak{m})})\,.$$

As each of the subspaces of $\mathfrak{m}$ in (5.19) is $\mathrm{Ad}(K)$–invariant, the restriction of the isomorphism σ in (5.20) to $\Omega^1(\mathfrak{m})^K$ induces an isomorphism

(5.21)

$$\widetilde{\sigma} \,:\, \Omega^1(\mathfrak{m})^K \xrightarrow{\sim} (([\mathfrak{m},\,\mathfrak{m}] \cap \mathfrak{b})^*)^K \oplus (([\mathfrak{m},\,\mathfrak{m}] \cap \mathfrak{k})^*)^K \oplus ((\mathfrak{k} \cap \mathfrak{b})^*)^K \oplus ((\mathfrak{a} \cap \mathfrak{z}(\mathfrak{m}))^*)^K\,.$$

As $\mathfrak{k} = ([\mathfrak{m},\,\mathfrak{m}] \cap \mathfrak{k}) \oplus (\mathfrak{k} \cap \mathfrak{b})$ and $[\mathfrak{m},\,\mathfrak{m}] = ([\mathfrak{m},\,\mathfrak{m}] \cap \mathfrak{k}) \oplus ([\mathfrak{m},\,\mathfrak{m}] \cap \mathfrak{b})$, it follows that

(5.22)

$$\widetilde{\sigma}(\Omega^1(\mathfrak{m}/\mathfrak{k})^K + \Omega^1(\mathfrak{m}/[\mathfrak{m},\,\mathfrak{m}])^K) = (([\mathfrak{m},\,\mathfrak{m}] \cap \mathfrak{b})^*)^K \oplus ((\mathfrak{k} \cap \mathfrak{b})^*)^K \oplus ((\mathfrak{a} \cap \mathfrak{z}(\mathfrak{m}))^*)^K\,.$$

Thus from (5.21) and (5.22) it follows that

$$\begin{aligned}
&\frac{\Omega^1(\mathfrak{m})^K}{\Omega^1(\mathfrak{m}/\mathfrak{k})^K + \Omega^1(\mathfrak{m}/[\mathfrak{m},\,\mathfrak{m}])^K} \\
&\simeq \frac{(([\mathfrak{m},\,\mathfrak{m}] \cap \mathfrak{b})^*)^K \oplus (([\mathfrak{m},\,\mathfrak{m}] \cap \mathfrak{k})^*)^K \oplus ((\mathfrak{k} \cap \mathfrak{b})^*)^K \oplus ((\mathfrak{a} \cap \mathfrak{z}(\mathfrak{m}))^*)^K}{(([\mathfrak{m},\,\mathfrak{m}] \cap \mathfrak{b})^*)^K \oplus ((\mathfrak{k} \cap \mathfrak{b})^*)^K \oplus ((\mathfrak{a} \cap \mathfrak{z}(\mathfrak{m}))^*)^K} \\
&\simeq (([\mathfrak{m},\,\mathfrak{m}] \cap \mathfrak{k})^*)^K\,.
\end{aligned} \tag{5.23}$$

As $[\mathfrak{m},\,\mathfrak{m}] \cap \mathfrak{k} = [\mathfrak{k},\,\mathfrak{k}] \oplus (\mathfrak{z}(\mathfrak{k}) \cap [\mathfrak{m},\,\mathfrak{m}])$, it follows that

$$(([\mathfrak{m},\,\mathfrak{m}] \cap \mathfrak{k})^*)^K \simeq ([\mathfrak{k},\,\mathfrak{k}]^*)^K \oplus ((\mathfrak{z}(\mathfrak{k}) \cap [\mathfrak{m},\,\mathfrak{m}])^*)^K\,. \tag{5.24}$$

In [BC1, § 3, (3.13)] it is proved that

$$([\mathfrak{k},\mathfrak{k}]^*)^K = 0\,. \tag{5.25}$$

We recall the proof for the sake of completeness. To prove (5.25), take any $\mu \in ([\mathfrak{k},\mathfrak{k}]^*)^K$. Then $\mu \circ \mathrm{Ad}(g)(X) = \mu(X)$ for all $X \in [\mathfrak{k},\mathfrak{k}]$ and $g \in K$. By differentiating, one has that $\mu(\mathrm{ad}(Y)(X)) = 0$ for all $X \in [\mathfrak{k},\mathfrak{k}]$ and $Y \in \mathfrak{k}$. Thus $\mu([\mathfrak{k},[\mathfrak{k},\mathfrak{k}]]) = 0$. But, as $[\mathfrak{k},\mathfrak{k}]$ is semisimple,

$$[\mathfrak{k},[\mathfrak{k},\mathfrak{k}]] = [\,\mathfrak{z}(\mathfrak{k}) + [\mathfrak{k},\mathfrak{k}]\,,\,[\mathfrak{k},\mathfrak{k}]\,] = [\,[\mathfrak{k},\mathfrak{k}]\,,\,[\mathfrak{k},\mathfrak{k}]\,] = [\mathfrak{k},\mathfrak{k}]\,.$$

Therefore, $\mu([\mathfrak{k},\mathfrak{k}]) = 0$. This proves the claim in (5.25).

Thus from (5.23), (5.24) and (5.25) we have

$$\frac{\Omega^1(\mathfrak{m})^K}{\Omega^1(\mathfrak{m}/\mathfrak{k})^K + \Omega^1(\mathfrak{m}/[\mathfrak{m},\,\mathfrak{m}])^K} \simeq ((\mathfrak{z}(\mathfrak{k}) \cap [\mathfrak{m},\,\mathfrak{m}])^*)^K\,. \tag{5.26}$$

Combining (5.16) and (5.26),

$$\frac{d(\Omega^1(\mathfrak{m})^K)}{d(\Omega^1(\mathfrak{m}/\mathfrak{k})^K)} \simeq \frac{\Omega^1(\mathfrak{m})^K}{\Omega^1(\mathfrak{m}/\mathfrak{k})^K + \Omega^1(\mathfrak{m}/[\mathfrak{m},\,\mathfrak{m}])^K} \simeq ((\mathfrak{z}(\mathfrak{k}) \cap [\mathfrak{m},\,\mathfrak{m}])^*)^K\,. \tag{5.27}$$

Moreover, as K° acts trivially on $((\mathfrak{z}(\mathfrak{k}) \cap [\mathfrak{m},\,\mathfrak{m}])^*)$,

$$((\mathfrak{z}(\mathfrak{k}) \cap [\mathfrak{m},\,\mathfrak{m}])^*)^K \simeq ((\mathfrak{z}(\mathfrak{k}) \cap [\mathfrak{m},\,\mathfrak{m}])^*)^{K/K^\circ}\,. \tag{5.28}$$

Combining (5.27) and (5.28),

$$\frac{d(\Omega^1(\mathfrak{m})^K)}{d(\Omega^1(\mathfrak{m}/\mathfrak{k})^K)} \simeq ((\mathfrak{z}(\mathfrak{k}) \cap [\mathfrak{m},\,\mathfrak{m}])^*)^{K/K^\circ}\,.$$

This and (5.15) together imply that the right-hand side of (5.14) coincides with the right-hand side of (5.8). This completes the proof of the theorem. □

Corollary 5.1.4. *Let G, H, K and M be as in Theorem 5.1.3. If* $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{m}) = 1$, *then*

$$H^2(G/H, \mathbb{R}) \simeq [(\mathfrak{z}(\mathfrak{k}) \cap [\mathfrak{m}, \mathfrak{m}])^*]^{K/K^\circ}.$$

Proof. As M is a compact Lie group, we have $\mathfrak{m} = \mathfrak{z}(\mathfrak{m}) \oplus [\mathfrak{m}, \mathfrak{m}]$. Thus we have $\dim_{\mathbb{R}}(\mathfrak{m}/([\mathfrak{m}, \mathfrak{m}] + \mathfrak{k})) \leq 1$. Now the corollary follows from Theorem 5.1.3. □

Corollary 5.1.5. *Let G, H, K and M be as in Theorem 5.1.3. If K is semisimple, then*

$$H^2(G/H, \mathbb{R}) \simeq \Omega^2(\frac{\mathfrak{m}}{[\mathfrak{m}, \mathfrak{m}] + \mathfrak{k}}).$$

Proof. As K is semisimple, we have $\mathfrak{z}(\mathfrak{k}) = 0$, so it follows from Theorem 5.1.3. □

Theorem 5.1.6. *Let G, H, K and M be as in Theorem 5.1.3. Then*

$$H^1(G/H, \mathbb{R}) \simeq \Omega^1(\frac{\mathfrak{m}}{[\mathfrak{m}, \mathfrak{m}] + \mathfrak{k}}).$$

Proof. In view of (5.7) it is enough to show that $H^1(M/K, \mathbb{R}) \simeq \Omega^1(\mathfrak{m}/([\mathfrak{m}, \mathfrak{m}] + \mathfrak{k}))$. As M is compact and connected, from [Sp, p. 310, Theorem 30] and the formula given in [Sp, p. 313] it follows that there are natural isomorphisms

$$H^i(M/K, \mathbb{R}) \simeq \frac{\mathrm{Ker}\,(d : \Omega^i(\mathfrak{m}/\mathfrak{k})^K \to \Omega^{i+1}(\mathfrak{m}/\mathfrak{k})^K)}{d(\Omega^{i-1}(\mathfrak{m}/\mathfrak{k})^K)} \quad \forall\ i. \tag{5.29}$$

Setting $i = 1$ in (5.29),

$$H^1(M/K, \mathbb{R}) \simeq \mathrm{Ker}\,(d : \Omega^1(\mathfrak{m}/\mathfrak{k})^K \to \Omega^2(\mathfrak{m}/\mathfrak{k})^K). \tag{5.30}$$

We claim that

$$\text{Ker}\,(d : \Omega^1(\mathfrak{m}/\mathfrak{k})^K \to \Omega^2(\mathfrak{m}/\mathfrak{k})^K) \;=\; \Omega^1(\mathfrak{m}/\mathfrak{k})^M\,. \tag{5.31}$$

To prove (5.31), first note that $\Omega^1(\mathfrak{m}/\mathfrak{k})^M \subseteq \Omega^1(\mathfrak{m}/\mathfrak{k})^K$. From Lemma 5.1.1(1) it follows that $d\alpha = 0$ for any $\alpha \in \Omega^1(\mathfrak{m}/\mathfrak{k})^M$. Thus

$$\Omega^1(\mathfrak{m}/\mathfrak{k})^M \subseteq \text{Ker}\,(d : \Omega^1(\mathfrak{m}/\mathfrak{k})^K \to \Omega^2(\mathfrak{m}/\mathfrak{k})^K)\,.$$

To prove the other way inclusion, take any $\alpha \in \text{Ker}\,(d : \Omega^1(\mathfrak{m}/\mathfrak{k})^K \to \Omega^2(\mathfrak{m}/\mathfrak{k})^K)$. Then $d\alpha = 0$ which in turn implies that $\alpha([X, Y]) = 0$ for all $X, Y \in \mathfrak{m}$. As M is connected, using (5.1) it follows that α is M-invariant. This proves the claim in (5.31).

As $\Omega^1(\mathfrak{m})^M \simeq \Omega^1(\mathfrak{m}/[\mathfrak{m}, \mathfrak{m}])$, it follows that

$$\Omega^1(\mathfrak{m}/\mathfrak{k})^M \simeq \Omega^1(\frac{\mathfrak{m}}{[\mathfrak{m}, \mathfrak{m}] + \mathfrak{k}})\,. \tag{5.32}$$

Combining (5.30), (5.31), (5.32) we have

$$H^1(M/K, \mathbb{R}) \simeq \Omega^1(\frac{\mathfrak{m}}{[\mathfrak{m}, \mathfrak{m}] + \mathfrak{k}})\,.$$

As noted before, the theorem follows from it. □

Corollary 5.1.7. *Let G, H, K and M be as in Theorem 5.1.3. If M is semisimple, then*

$$\dim_{\mathbb{R}} H^1(G/H, \mathbb{R}) = 0\,.$$

Proof. As M is semisimple, we have $\mathfrak{m} = [\mathfrak{m}, \mathfrak{m}]$, and hence the corollary follows from Theorem 5.1.6. □

Recall that any maximal compact subgroup of a complex semisimple Lie group is semisimple. The following corollary now follows form (5.7) and Corollary 5.1.7.

Corollary 5.1.8. *Let G be a connected complex semisimple Lie group, and let $H \subset G$ be a closed subgroup with finitely many connected components. Then*

$$\dim_{\mathbb{R}} H^1(G/H,\, \mathbb{R}) \;=\; 0\,.$$

In the special case where G is a simple real Lie group, the following result is a stronger form of Theorem 5.1.3 and Theorem 5.1.6.

Theorem 5.1.9. *Let G be a connected simple real Lie group, and let $H \subset G$ be a closed subgroup with finitely many connected components. Let K be a maximal compact subgroup of H and M a maximal compact subgroup of G containing K. Then*

$$H^2(G/H, \mathbb{R}) \;\simeq\; [(\mathfrak{z}(\mathfrak{k}) \cap [\mathfrak{m},\, \mathfrak{m}])^*]^{K/K^\circ}$$

and

$$\dim_{\mathbb{R}} H^1(G/H, \mathbb{R}) = \begin{cases} 1 & \text{if } \; \mathfrak{k} + [\mathfrak{m},\, \mathfrak{m}] \subsetneqq \mathfrak{m} \\ 0 & \text{if } \; \mathfrak{k} + [\mathfrak{m},\, \mathfrak{m}] = \mathfrak{m}. \end{cases}$$

In particular, $\dim_{\mathbb{R}} H^1(G/H,\, \mathbb{R}) \;\leq\; 1$.

Proof. Since M is a maximal compact subgroup of a real simple Lie group, it follows from [He, Proposition 6.2, p. 382] that $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{m})$ is either 0 or 1. In both these cases we have $\Omega^2(\mathfrak{m}/([\mathfrak{m},\, \mathfrak{m}] + \mathfrak{k})) \;=\; 0$. In view of Theorem 5.1.3 and (5.7), it follows that

$$H^2(G/H,\, \mathbb{R}) \;\simeq\; [(\mathfrak{z}(\mathfrak{k}) \cap [\mathfrak{m},\, \mathfrak{m}])^*]^{K/K^\circ}\,.$$

As G is simple, we have $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{m}) \;\leq\; 1$. Thus, we have either

$$\mathfrak{k} + [\mathfrak{m},\, \mathfrak{m}] \subsetneqq \mathfrak{m} \quad \text{or} \quad \mathfrak{k} + [\mathfrak{m},\, \mathfrak{m}] \;=\; \mathfrak{m}\,.$$

Therefore, from Theorem 5.1.6 and (5.7) we conclude that

$$\dim_{\mathbb{R}} H^1(G/H, \mathbb{R}) = \begin{cases} 1 & \text{if } \ \mathfrak{k} + [\mathfrak{m}, \mathfrak{m}] \subsetneqq \mathfrak{m} \\ 0 & \text{if } \ \mathfrak{k} + [\mathfrak{m}, \mathfrak{m}] = \mathfrak{m}\,. \end{cases}$$

□

5.2 Description of first and second cohomology groups of nilpotent orbits

The main result in this section, Theorem 5.2.2, is crucial in our computations of the second and first cohomology groups of the nilpotent orbits.

Lemma 5.2.1. *Let G be a semisimple algebraic group defined over $\mathbb{R}$. Let $X \in \operatorname{Lie} G(\mathbb{R})$ be a non-zero nilpotent element and let $\{X, H, Y\}$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple in $\operatorname{Lie} G(\mathbb{R})$. Then $\mathcal{Z}_G(X, H, Y)$ is a (reductive) Levi subgroup of $\mathcal{Z}_G(X)$ which is defined over $\mathbb{R}$.*

Proof. The nontrivial fact that the group $\mathcal{Z}_G(X, H, Y)$ is a (reductive) Levi subgroup of $\mathcal{Z}_G(X)$ is proved in [CoMc, p. 50, Lemma 3.7.3]. Since $X, H, Y \in \operatorname{Lie} G(\mathbb{R})$, it is immediate that the group $\mathcal{Z}_G(X, H, Y)$ is defined over $\mathbb{R}$. □

Theorem 5.2.2. *Let G be an algebraic group defined over $\mathbb{R}$ such that G is $\mathbb{R}$-simple. Let $0 \neq X \in \operatorname{Lie} G(\mathbb{R})$ be a nilpotent element and $\mathcal{O}_X$ be the orbit of X under the adjoint action of the identity component $G(\mathbb{R})^\circ$ on $\operatorname{Lie} G(\mathbb{R})$. Let $\{X, H, Y\}$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple in $\operatorname{Lie} G(\mathbb{R})$. Let K be a maximal compact subgroup in $\mathcal{Z}_{G(\mathbb{R})^\circ}(X, H, Y)$ and M a maximal compact subgroup of $G(\mathbb{R})^\circ$ containing K. Then,*

$$H^2(\mathcal{O}_X, \mathbb{R}) \ \simeq \ [(\mathfrak{z}(\mathfrak{k}) \cap [\mathfrak{m}, \mathfrak{m}])^*]^{K/K^\circ}$$

and

$$\dim_{\mathbb{R}} H^1(\mathcal{O}_X,\, \mathbb{R}) \;=\; \begin{cases} 1 & \textit{if} \;\; \mathfrak{k} + [\mathfrak{m},\, \mathfrak{m}] \subsetneqq \mathfrak{m} \\ 0 & \textit{if} \;\; \mathfrak{k} + [\mathfrak{m},\, \mathfrak{m}] \;=\; \mathfrak{m}\,. \end{cases}$$

Proof. From Lemma 5.2.1 it follows that the group $\mathcal{Z}_G(X,H,Y)$ is a (reductive) Levi subgroup of $\mathcal{Z}_G(X)$. In particular, we have the semidirect product decomposition:

$$\mathcal{Z}_{G(\mathbb{R})^\circ}(X) \;=\; \mathcal{Z}_{G(\mathbb{R})^\circ}(X,H,Y) R_u(\mathcal{Z}_G(X))(\mathbb{R})\,,$$

where $R_u(\mathcal{Z}_G(X))$ is the unipotent radical of $\mathcal{Z}_G(X)$. As $R_u(\mathcal{Z}_G(X))(\mathbb{R})$ simply connected and nilpotent, this implies that any maximal compact subgroup in $\mathcal{Z}_{G(\mathbb{R})^\circ}(X,H,Y)$ is a maximal compact subgroup in $\mathcal{Z}_{G(\mathbb{R})^\circ}(X)$. Since $G(\mathbb{R})^\circ$ is a connected simple real Lie group, the theorem now follows from Theorem 5.1.9. □

Chapter 6

Second cohomology of nilpotent orbits in non-compact non-complex classical Lie algebras

In this chapter we will compute the second de Rham cohomology groups of the nilpotent orbits in non-compact non-complex classical real Lie algebras. At the outset we mention that the justification for some of the detailed computations done in this chapter is explained in Remark 6.0.3.

Let V be a right $\mathbb{D}$-vector space, $\epsilon = \pm 1$, $\sigma : \mathbb{D} \longrightarrow \mathbb{D}$ be either the identity map or the usual conjugation σ_c when $\mathbb{D}$ is $\mathbb{C}$ or $\mathbb{H}$, and let $\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{D}$ be a ϵ-σ Hermitian form. Let $\mathrm{SL}(V)$ and $\mathrm{SU}(V, \langle \cdot, \cdot \rangle)$ be the groups defined in Section 2.3. We now follow the notation established at the beginning of Section 3. Let $\{X, H, Y\}$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple in $\mathfrak{sl}(V)$, and let $\mathbf{d} := [d_1^{t_{d_1}}, \ldots, d_s^{t_{d_s}}]$ be as in (3.6). Let $(v_1^d, \ldots, v_{t_d}^d)$ be the ordered $\mathbb{D}$-basis in Proposition 3.0.7 for $d \in \mathbb{N}_{\mathbf{d}}$. Then it follows from Proposition 3.0.3 and Proposition 3.0.7 that

$$\mathcal{B}^l(d) := (X^l v_1^d, \ldots, X^l v_{t_d}^d) \tag{6.1}$$

is an ordered $\mathbb{D}$-basis of $X^l L(d-1)$ for $0 \le l \le d-1$ with $d \in \mathbb{N}_{\mathbf{d}}$. Define

$$(6.2) \quad \mathcal{B}(d) := \mathcal{B}^0(d) \vee \cdots \vee \mathcal{B}^{d-1}(d) \;\forall\, d \in \mathbb{N}_{\mathbf{d}}, \quad \text{and} \quad \mathcal{B} := \mathcal{B}(d_1) \vee \cdots \vee \mathcal{B}(d_s).$$

Let

$$\Lambda_{\mathcal{B}} : \mathrm{End}(V) \longrightarrow \mathrm{M}_n(\mathbb{D}) \tag{6.3}$$

be the isomorphism of $\mathbb{R}$-algebras with respect to the ordered basis $\mathcal{B}$. Next define the character

$$\chi_{\mathbf{d}} : \prod_{d \in \mathbb{N}_{\mathbf{d}}} \mathrm{GL}(L(d-1)) \longrightarrow \mathbb{D}^*$$

by

$$\chi_{\mathbf{d}}\big(A_{t_{d_1}}, \ldots, A_{t_{d_s}}\big) := \begin{cases} \prod_{i=1}^s \big(\det A_{t_{d_i}}\big)^{d_i} & \text{if } \mathbb{D} = \mathbb{R} \text{ or } \mathbb{C} \\ \prod_{i=1}^s \big(\mathrm{Nrd}_{\mathrm{End}_{\mathbb{H}}(L(d_i-1))} A_{t_{d_i}}\big)^{d_i} & \text{if } \mathbb{D} = \mathbb{H}. \end{cases}$$

Lemma 6.0.1.

1. *The following equality holds:*

$$\mathcal{Z}_{\mathrm{SL}(V)}(X,H,Y) = \left\{ g \in \mathrm{SL}(V) \;\middle|\; \begin{array}{c} g(X^l L(d-1)) \subset X^l L(d-1), \\ \big[g|_{X^l L(d-1)}\big]_{\mathcal{B}^l(d)} = \big[g|_{L(d-1)}\big]_{\mathcal{B}^0(d)}, \\ \text{for all } 0 \le l < d, d \in \mathbb{N}_{\mathbf{d}} \end{array} \right\}.$$

2. *In particular,* $\mathcal{Z}_{\mathrm{SL}(V)}(X,H,Y) \simeq \big\{ g \in \prod_{d \in \mathbb{N}_{\mathbf{d}}} \mathrm{GL}(L(d-1)) \mid \chi_{\mathbf{d}}(g) = 1 \big\}$.
3. *If* $\{X, H, Y\}$ *is a* $\mathfrak{sl}_2(\mathbb{R})$*-triple in* $\mathfrak{su}(V, \langle\cdot,\cdot\rangle)$*, then*

$$\mathcal{Z}_{\mathrm{SU}(V,\langle\cdot,\cdot\rangle)}(X,H,Y) = \left\{ g \in \mathrm{SL}(V) \;\middle|\; \begin{array}{c} g(X^l L(d-1)) \subset X^l L(d-1), [g|_{X^l L(d-1)}]_{\mathcal{B}^l(d)} \\ = [g|_{L(d-1)}]_{\mathcal{B}^0(d)}, (gx, gy)_d = (x,y)_d, \\ \forall\, d \in \mathbb{N}_{\mathbf{d}},\ 0 \le l \le d-1,\ x, y \in L(d-1) \end{array} \right\};$$

here $(\cdot, \cdot)_d$ *is the form on* $L(d-1)$ *defined in* (3.8)*.*

4. *In particular,*

   $\mathcal{Z}_{\mathrm{SU}(V,\langle\cdot,\cdot\rangle)}(X,H,Y) \simeq \big\{g \in \prod_{d\in\mathbb{N}_{\mathbf{d}}} \mathrm{U}\big(L(d-1),(\cdot,\cdot)_d\big) \mid \chi_{\mathbf{d}}(g) = 1\big\}.$

Proof. For notational convenience, denote

$$\mathcal{G} := \left\{ g \in \mathrm{SL}(V) \;\middle|\; \begin{array}{c} g(X^l L(d-1)) \subset X^l L(d-1); \\ \big[g|_{X^l L(d-1)}\big]_{\mathcal{B}^l(d)} = \big[g|_{L(d-1)}\big]_{\mathcal{B}^0(d)} \,\forall\, 0 \le l \le d-1, d \in \mathbb{N}_{\mathbf{d}} \end{array} \right\}.$$

Take any $g \in \mathcal{Z}_{\mathrm{SL}(V)}(X,H,Y)$. Then $g(L(d-1)) \subseteq L(d-1)$ by (3.7). In particular, it follows that $g(X^l L(d-1)) \subseteq X^l L(d-1)$ because g commutes with X. Let $B_d := [g|_{L(d-1)}]_{\mathcal{B}^0(d)}$ for all $d \in \mathbb{N}_{\mathbf{d}}$. As g commutes with X, it follows that $\big[g|_{X^l L(d-1)}\big]_{\mathcal{B}^l(d)} = B_d$ for $0 \le l \le d-1$. This proves that $\mathcal{Z}_{\mathrm{SL}(V)}(X,H,Y) \subset \mathcal{G}$.

Take any $h \in \mathcal{G}$. Then $h(X^l L(d-1)) \subset X^l L(d-1)$ for all $0 \le l \le d-1$ and $d \in \mathbb{N}_{\mathbf{d}}$. For every $d \in \mathbb{N}_{\mathbf{d}}$, let (a^d_{ij}) denote the matrix $\big[h|_{L(d-1)}\big]_{\mathcal{B}^0(d)} \in \mathrm{GL}_{t_d}(\mathbb{D})$. Then $(a^d_{ij}) = \big[h|_{X^l L(d-1)}\big]_{\mathcal{B}^l(d)}$ for all $0 \le l \le d-1$.

We will show that h commutes with X and H. From (6.2) it follows that $\mathcal{B}$ is a $\mathbb{D}$-basis of V. Hence to prove that $Xh = hX$ we need to show $Xh(X^l v^d_j) = hX(X^l v^d_j)$ for all $1 \le j \le t_d$ and $0 \le l \le d-1$ with $d \in \mathbb{N}_{\mathbf{d}}$. However this follows from the following straightforward computation:

$$hX(X^l v^d_j) = hX^{l+1} v^d_j = \sum_{i=1}^{t_d} X^{l+1} v^d_i a^d_{ij} = X\Big(\sum_{i=1}^{t_d} X^l v^d_i a^d_{ij}\Big) = Xh(X^l v^d_j)\,.$$

As H acts as multiplication by a scalar in $\mathbb{R}$ (in fact, by a scalar in $\mathbb{Z}$) on the $\mathbb{D}$-basis $\mathcal{B}^l(d)$ (of $X^l L(d-1)$) for all $0 \le l \le d-1$ with $d \in \mathbb{N}_{\mathbf{d}}$, it is immediate that h commutes with H. In view of Lemma 2.4.7, we conclude that h commutes with Y. This completes the proof of statement (1).

The third statement follows from statement (1) and Remark 3.0.10. □

Remark 6.0.2. When $\mathbb{D} = \mathbb{R}$ or $\mathbb{C}$, the isomorphisms (2) and (4) in Lemma 6.0.1 were proved in [SS, p. 251, 1.8] and [SS, p. 261, 2.25] using only the Jordan canonical forms. However, as the non-commutativity of $\mathbb{H}$ creates technical difficulties in extending these results of [SS] to the case of $\mathbb{D} = \mathbb{H}$, we take a different approach by appealing to the Jacobson-Morozov theorem and the basic results on the structures of finite dimensional representations of $\mathfrak{sl}_2(\mathbb{R})$. □

Remark 6.0.3. We follow the notations of Theorem 5.2.2 in this remark. Theorem 5.2.2 asserts that when M is semisimple, in order to compute $H^2(\mathcal{O}_X, \mathbb{R})$ it is enough to know the isomorphism class of K. However, when M is not semisimple, it is not enough to know the isomorphism classes of K and M, rather we also need to know how K is embedded in M; see Theorem 5.2.2. Although the isomorphism classes of M are well-known when G is $\mathbb{R}$-simple, and the isomorphism classes of K can be obtained immediately using (2) and (4) of Lemma 6.0.1, hardly anything can be concluded, from these isomorphism classes, on how K is embedded in M. We devote the major part in the next Sections 6.3, 6.4, 6.5 and 6.6 to find out how K is sitting inside M for the nilpotent orbits in $\mathfrak{g}$ for which M is not semisimple. □

6.1 Second cohomology of nilpotent orbits in $\mathfrak{sl}_n(\mathbb{R})$

We follow the notation and parametrization of the nilpotent orbits as in §4.1.1 in our next result.

Theorem 6.1.1. *Let $X \in \mathfrak{sl}_n(\mathbb{R})$ be a nilpotent element. Let $\mathbf{d} = [d_1^{t_{d_1}}, \ldots, d_s^{t_{d_s}}] \in \mathcal{P}(n)$ be the partition associated to the orbit $\mathcal{O}_X$ (i.e., $\Psi_{\mathrm{SL}_n(\mathbb{R})}(\mathcal{O}_X) = \mathbf{d}$ in the notation of Theorem 4.1.2). Then the following hold:*

1. *If $n \geq 3$, $\#\mathbb{O}_{\mathbf{d}} = 1$ and $t_\theta = 2$ for $\theta \in \mathbb{O}_{\mathbf{d}}$, then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 1$.

2. In all the other cases $\dim_{\mathbb{R}} H^2(\mathcal{O}_X,\, \mathbb{R}) \,=\, 0$.

Proof. This is obvious when $X \,=\, 0$, so assume that $X \,\neq\, 0$.

The notation in Lemma 6.0.1 and the paragraph preceding it will be employed. Let $\{X,\, H,\, Y\} \,\subset\, \mathfrak{sl}_n(\mathbb{R})$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple. Let K be a maximal compact subgroup of $\mathcal{Z}_{\mathrm{SL}_n(\mathbb{R})}(X,H,Y)$. Let M be a maximal compact subgroup of $\mathrm{SL}_n(\mathbb{R})$ containing K. As $M \,\simeq\, \mathrm{SO}_n$, it follows that $\mathfrak{z}(\mathfrak{k}) \cap [\mathfrak{m},\, \mathfrak{m}] \,=\, 0$ when $n = 2$, and $[\mathfrak{m},\, \mathfrak{m}] \,=\, \mathfrak{m}$ when $n \,\geq\, 3$. Thus using Theorem 5.2.2,

$$H^2(\mathcal{O}_X,\, \mathbb{R}) \,\simeq\, \begin{cases} 0 & \text{if } \ n \,=\, 2 \\ \big[\mathfrak{z}(\mathfrak{k})^*\big]^{K/K^\circ} & \text{if } \ n \,\geq\, 3\,. \end{cases}$$

Treating $\mathbb{R}^n$ as a $\mathrm{Span}_{\mathbb{R}}\{X,H,Y\}$-module through the standard action of $\mathfrak{sl}_n(\mathbb{R})$, construct a $\mathbb{R}$-basis $\mathcal{B}$ as in (6.2), and consider the $\mathbb{R}$-algebra isomorphism $\Lambda_{\mathcal{B}}$ in (6.3). It now follows from Lemma 6.0.1(2) that the restriction of $\Lambda_{\mathcal{B}}$ induces an isomorphism of Lie groups:

$$\text{(6.4)} \qquad \Lambda_{\mathcal{B}} :\, \mathcal{Z}_{\mathrm{SL}_n(\mathbb{R})}(X,H,Y) \,\xrightarrow{\sim}\, S\Big(\prod_{d\in\mathbb{N}_{\mathbf{d}}} \mathrm{GL}_{t_d}(\mathbb{R})^d_\Delta\Big)\,.$$

As $\prod_{d\in\mathbb{N}_{\mathbf{d}}} (\mathrm{O}_{t_d})^d_\Delta$ is a maximal compact subgroup of $\prod_{d\in\mathbb{N}_{\mathbf{d}}} \mathrm{GL}_{t_d}(\mathbb{R})^d_\Delta$, and $S(\prod_{d\in\mathbb{N}_{\mathbf{d}}} \mathrm{GL}_{t_d}(\mathbb{R})^d_\Delta)$ is normal in $\prod_{d\in\mathbb{N}_{\mathbf{d}}} \mathrm{GL}_{t_d}(\mathbb{R})^d_\Delta$, it follows using Lemma 2.3.6 that $S\big(\prod_{d\in\mathbb{N}_{\mathbf{d}}} (\mathrm{O}_{t_d})^d_\Delta\big)$ is a maximal compact subgroup of $S\big(\prod_{d\in\mathbb{N}_{\mathbf{d}}} \mathrm{GL}_{t_d}(\mathbb{R})^d_\Delta\big)$. In view of the above observations it is now clear that for $n \,\geq\, 3$,

$$\text{(6.5)} \qquad H^2(\mathcal{O}_X,\, \mathbb{R}) \,\simeq\, [\mathfrak{z}(\mathfrak{k})^*]^{K/K^\circ} \quad \text{where} \quad K \,\simeq\, \prod_{\eta\in\mathbb{E}_{\mathbf{d}}} \mathrm{O}_{t_\eta} \times S\Big(\prod_{\theta\in\mathbb{O}_{\mathbf{d}}} \mathrm{O}_{t_\theta}\Big)\,.$$

Consider the group $A \,:=\, S(\mathrm{O}_{n_1} \times \cdots \times \mathrm{O}_{n_r})$ for positive integers $n_1,\, \ldots\, , n_r$. Let $\mathfrak{a}$ be the Lie algebra of A. It is then easy to prove (see the proof of Case-2 in

[BC1, Theorem 5.6]) that

$$\dim_{\mathbb{R}}[\mathfrak{z}(\mathfrak{a})]^{A/A^\circ} = \begin{cases} 1 & \text{if } r = 1 \quad \text{and} \quad n_r = 2 \\ 0 & \text{otherwise.} \end{cases} \tag{6.6}$$

It is also immediate that if B_1, B_2 are Lie groups, $B_3 := B_1 \times B_2$, and $\mathfrak{b}_i$, $1 \leq i \leq 3$, is the Lie algebra of B_i, then

$$[\mathfrak{z}(\mathfrak{b}_3)]^{B_3/B_3^\circ} \simeq [\mathfrak{z}(\mathfrak{b}_1)]^{B_1/B_1^\circ} \oplus [\mathfrak{z}(\mathfrak{b}_2)]^{B_2/B_2^\circ} . \tag{6.7}$$

Now the theorem follows from (6.6), (6.7) and (6.5). □

6.2 Second cohomology of nilpotent orbits in $\mathfrak{sl}_n(\mathbb{H})$

Our next result, which we state using the parametrization as in Theorem 4.1.3, says that the second cohomology groups of all the nilpotent orbits in $\mathfrak{sl}_n(\mathbb{H})$ vanish. As the Lie algebra $\mathfrak{sl}_1(\mathbb{H})$ is isomorphic to $\mathfrak{su}(2)$ which is a compact Lie algebra, we will further assume that $n \geq 2$.

Theorem 6.2.1. *For every nilpotent element $X \in \mathfrak{sl}_n(\mathbb{H})$ when $n \geq 2$,*

$$\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 0 .$$

Proof. We assume that $X \neq 0$ because the theorem is obvious when $X = 0$.

Suppose that $\Psi_{\mathrm{SL}_n(\mathbb{H})}(\mathcal{O}_X) = \mathbf{d}$. Using the notation in Lemma 6.0.1 and the paragraph preceding it, let $\{X, H, Y\} \subset \mathfrak{sl}_n(\mathbb{H})$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple. Let K be a maximal compact subgroup in $\mathcal{Z}_{\mathrm{SL}_n(\mathbb{H})}(X, H, Y)$. As $\mathrm{Sp}(n)$ is a maximal compact subgroup of $\mathrm{SL}_n(\mathbb{H})$, it follows from Theorem 5.2.2 that

$$H^2(\mathcal{O}_X, \mathbb{R}) \simeq [\mathfrak{z}(\mathfrak{k})^*]^{K/K^\circ} \tag{6.8}$$

for all $X \neq 0$. Treating $\mathbb{H}^n$ as a $\mathrm{Span}_{\mathbb{R}}\{X, H, Y\}$-module via the standard action of $\mathfrak{sl}_n(\mathbb{H})$, we construct a $\mathbb{H}$-basis $\mathcal{B}$ as in (6.2), and consider the $\mathbb{R}$-algebra isomorphism $\Lambda_{\mathcal{B}}$ in (6.3). It now follows from Lemma 6.0.1(2) that the restriction of $\Lambda_{\mathcal{B}}$ induces an isomorphism of Lie groups

$$\Lambda_{\mathcal{B}} : \mathcal{Z}_{\mathrm{SL}_n(\mathbb{H})}(X, H, Y) \xrightarrow{\sim} S\big(\prod_{d \in \mathbb{N}_{\mathbf{d}}} \mathrm{GL}_{t_d}(\mathbb{H})^d_{\Delta} \big) .$$

As $\prod_{d \in \mathbb{N}_{\mathbf{d}}} \mathrm{Sp}(t_d)^d_{\Delta}$ is a maximal compact subgroup of $\prod_{d \in \mathbb{N}_{\mathbf{d}}} \mathrm{GL}_{t_d}(\mathbb{H})^d_{\Delta}$, and

$$\prod_{d \in \mathbb{N}_{\mathbf{d}}} \mathrm{Sp}(t_d)^d_{\Delta} \subset S\big(\prod_{d \in \mathbb{N}_{\mathbf{d}}} \mathrm{GL}_{t_d}(\mathbb{H})^d_{\Delta} \big) ,$$

it follows that $\prod_{d \in \mathbb{N}_{\mathbf{d}}} \mathrm{Sp}(t_d)^d_{\Delta}$ is a maximal compact subgroup of $S\big(\prod_{d \in \mathbb{N}_{\mathbf{d}}} \mathrm{GL}_{t_d}(\mathbb{H})^d_{\Delta} \big)$. In particular, we have

$$K \simeq \prod_{d \in \mathbb{N}_{\mathbf{d}}} \mathrm{Sp}(t_d) .$$

As $\mathfrak{z}(\mathfrak{k}) = 0$, it now follows from (6.8) that $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 0$. ☐

6.3 Second cohomology of nilpotent orbits in $\mathfrak{su}(p, q)$

Let n be a positive integer and (p, q) a pair of non-negative integers such that $p + q = n$. As we are dealing with non-compact groups, we will further assume that $p > 0$ and $q > 0$. In this section, we follow notation and parametrization of the nilpotent orbits in $\mathfrak{su}(p, q)$ as in §4.1.3; see Theorem 4.1.4. Here we compute the second cohomology groups of nilpotent orbits in $\mathfrak{su}(p,q)$ under the adjoint action of $\mathrm{SU}(p,q)$. As $\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q))$, being a maximal compact subgroup in $\mathrm{SU}(p,q)$, is not semisimple, in view of Remark 6.0.3, we need to work out how a conjugate of a maximal compact subgroup of $\mathcal{Z}_{\mathrm{SU}(p,q)}(X)$ is embedded in $\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q))$, for an arbitrary nilpotent element $X\in\mathfrak{su}(p,q)$. Throughout this section $\langle\cdot\,,\cdot\rangle$ denotes the Hermitian form on $\mathbb{C}^n$ defined by $\langle x,y\rangle := \overline{x}^t \mathrm{I}_{p,q} y$, where $\mathrm{I}_{p,q}$ is as in (2.19).

Let $0\neq X\in\mathcal{N}_{\mathfrak{su}(p,q)}$, and $\{X,H,Y\}$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple in $\mathfrak{su}(p,q)$. Let $\Psi_{\mathrm{SU}(p,q)}(\mathcal{O}_X)=\big(\mathbf{d},\,\mathbf{sgn}_{\mathcal{O}_X}\big)$. Then $\Psi'_{\mathrm{SU}(p,q)}(\mathcal{O}_X)=\mathbf{d}$. Recall that $\mathbf{sgn}_{\mathcal{O}_X}$ determines the signature of $(\cdot,\,\cdot)_d$ on $L(d-1)$ for every $d\in\mathbb{N}_{\mathbf{d}}$; let $(p_d,\,q_d)$ be the signature of $(\cdot,\,\cdot)_d$, for $d\in\mathbb{N}_{\mathbf{d}}$. Let $(v_1^d,\,\ldots,\,v_{t_d}^d)$ be an ordered $\mathbb{C}$-basis of $L(d-1)$ as in Proposition 3.0.7. It now follows from Proposition 3.0.7(3)(a) that $(v_1^d,\,\ldots,\,v_{t_d}^d)$ is an orthogonal basis for $(\cdot,\,\cdot)_d$. We also assume that the vectors in the ordered basis $(v_1^d,\,\ldots,\,v_{t_d}^d)$ satisfies the properties in Remark 3.0.11(2). In view of the signature of $(\cdot,\,\cdot)_d$ we may further assume that

$$\sqrt{-1}(v_j^\eta,v_j^\eta)_\eta=\begin{cases}+1 & \text{if } 1\leq j\leq p_\eta\\ -1 & \text{if } p_\eta<j\leq t_\eta\end{cases}\;;\ \text{when } \eta\in\mathbb{E}_{\mathbf{d}}, \tag{6.9}$$

$$(v_j^\theta,v_j^\theta)_\theta=\begin{cases}+1 & \text{if } 1\leq j\leq p_\theta\\ -1 & \text{if } p_\theta<j\leq t_\theta\end{cases}\;;\ \text{when } \theta\in\mathbb{O}_{\mathbf{d}}. \tag{6.10}$$

Let $\big\{\widetilde{w}_{jl}^d \mid 1\leq j\leq t_d,\ 0\leq l\leq d-1\big\}$ be the $\mathbb{C}$-basis of $M(d-1)$ constructed using $(v_1^d,\,\ldots,\,v_{t_d}^d)$ as done in Lemma 3.0.13. For each $d\in\mathbb{N}_{\mathbf{d}}$, $0\leq l\leq d-1$, set

$$V^l(d) := \ \mathrm{Span}_{\mathbb{C}}\{\widetilde{w}_{1l}^d,\,\ldots,\,\widetilde{w}_{t_dl}^d\}\,.$$

The ordered basis $\big(\widetilde{w}_{1l}^d,\,\ldots,\,\widetilde{w}_{t_dl}^d\big)$ of $V^l(d)$ will be denoted by $\mathcal{C}^l(d)$.

Lemma 6.3.1. *The following holds:*

$$\mathcal{Z}_{\mathrm{SU}(p,q)}(X,H,Y)=\left\{g\in \mathrm{SU}(p,q)\,\middle|\, \begin{array}{c} g(V^l(d))\subset V^l(d)\ \ and \\ \left[g|_{V^l(d)}\right]_{\mathcal{C}^l(d)}=\left[g|_{V^0(d)}\right]_{\mathcal{C}^0(d)}\forall\, d\in\mathbb{N}_{\mathbf{d}},0\le l<d \end{array}\right\}.$$

Proof. As $\mathcal{Z}_{\mathrm{SU}(p,q)}(X,H,Y) = \mathrm{SU}(p,q)\cap\mathcal{Z}_{\mathrm{SL}_n(\mathbb{C})}(X,H,Y)$, using Lemma 6.0.1(1) it follows that

$$\mathcal{Z}_{\mathrm{SU}(p,q)}(X,H,Y)=\left\{g\in \mathrm{SU}(p,q)\,\middle|\, \begin{array}{c} g\big(X^lL(d-1)\big)\subset X^lL(d-1)\ \ \text{and} \\ \left[g|_{X^lL(d-1)}\right]_{\mathcal{B}^l(d)}=\left[g|_{L(d-1)}\right]_{\mathcal{B}^0(d)}\forall\, d\in\mathbb{N}_{\mathbf{d}},0\le l<d \end{array}\right\}.$$

For fixed $d\in\mathbb{N}_{\mathbf{d}}$ we consider the $t_d\times 1$-column matrices $\left[\widetilde{w}^d_{j\,l}\right]_{1\le j\le t_d}$, $\left[\widetilde{w}^d_{j\,(d-1-l)}\right]_{1\le j\le t_d}$ and $\left[X^lv^d_j\right]_{1\le j\le t_d}$, $\left[X^{d-1-l}v^d_j\right]_{1\le j\le t_d}$. Rewriting the definitions in Lemma 3.0.13 when $\eta\in\mathbb{E}_{\mathbf{d}}$,

$$\left[\widetilde{w}^\eta_{jl}\right]=\Big(\left[X^lv^\eta_j\right]+\left[X^{\eta-1-l}v^\eta_j\right]\sqrt{-1}\Big)\frac{1}{\sqrt{2}};\ \left[\widetilde{w}^\eta_{j(\eta-1-l)}\right]=\Big(\left[X^lv^\eta_j\right]-\left[X^{\eta-1-l}v^\eta_j\right]\sqrt{-1}\Big)\frac{1}{\sqrt{2}}$$

for $0\le l<\eta/2$. Furthermore, when $1\le\theta\in\mathbb{O}_{\mathbf{d}}$,

$$\left[\widetilde{w}^\theta_{jl}\right]=\Big(\left[X^lv^\theta_j\right]+\left[X^{\theta-1-l}v^\theta_j\right]\Big)\frac{1}{\sqrt{2}};\ \left[\widetilde{w}^\theta_{j(\theta-1-l)}\right]=\Big(\left[X^lv^\theta_j\right]-\left[X^{\theta-1-l}v^\theta_j\right]\Big)\frac{1}{\sqrt{2}},$$

for all $0\le l<(\theta-1)/2$, while for $l=(\theta-1)/2$,

$$\left[\widetilde{w}^\theta_{j\,(\theta-1)/2}\right]=\left[X^{(\theta-1)/2}v^\theta_j\right].$$

When $\theta=1$, then $\left[\widetilde{w}^\theta_j\right]=\left[v^\theta_j\right]$.

In particular, if $d\in\mathbb{N}_{\mathbf{d}}$ is fixed, then for every $0\le l\le d-1$ the following holds:

$$g(X^lL(d-1))\subset X^lL(d-1)\ \text{ if and only if }\ g(V^l(d))\subset V^l(d)\,,$$

and moreover,

$$[g|_{X^l L(d-1)}]_{\mathcal{B}^l(d)} = [g|_{L(d-1)}]_{\mathcal{B}^0(d)} \quad \text{if and only if} \quad [g|_{V^l(d)}]_{\mathcal{C}^l(d)} = [g|_{V^0(d)}]_{\mathcal{C}^0(d)}\,.$$

In fact, for any g as above, $\big[g|_{L(d-1)}\big]_{\mathcal{B}^0(d)} = \big[g|_{V^0(d)}\big]_{\mathcal{C}^0(d)}$. □

For every $d \in \mathbb{N}_{\mathbf{d}}$ and $0 \le l \le d-1$, orderings on the sets $\{v \in \mathcal{C}^l(d) \mid \langle v, v\rangle > 0\}$, $\{v \in \mathcal{C}^l(d) \mid \langle v, v\rangle < 0\}$, will be constructed. These ordered sets will be denoted by $\mathcal{C}^l_+(d)$ and $\mathcal{C}^l_-(d)$ respectively. The construction will be done in three steps according as $d \in \mathbb{E}_{\mathbf{d}}$ or $d \in \mathbb{O}^1_{\mathbf{d}}$ or $d \in \mathbb{O}^3_{\mathbf{d}}$.

For each $\eta \in \mathbb{E}_{\mathbf{d}}$ and $0 \le l \le \eta - 1$, define

$$\mathcal{C}^l_+(\eta) := \begin{cases} \big(\,\widetilde{w}^{\eta}_{1\,l},\ \ldots,\ \widetilde{w}^{\eta}_{p_\eta\,l}\big) & \text{if } l \text{ is even} \\ \big(\,\widetilde{w}^{\eta}_{(p_\eta+1)\,l},\ \ldots,\ \widetilde{w}^{\eta}_{t_\eta\,l}\big) & \text{if } l \text{ is odd,} \end{cases}$$

$$\mathcal{C}^l_-(\eta) := \begin{cases} \big(\,\widetilde{w}^{\eta}_{(p_\eta+1)\,l},\ \ldots,\ \widetilde{w}^{\eta}_{t_\eta\,l}\big) & \text{if } l \text{ is even} \\ \big(\,\widetilde{w}^{\eta}_{1\,l},\ \ldots,\ \widetilde{w}^{\eta}_{p_\eta\,l}\big) & \text{if } l \text{ is odd.} \end{cases}$$

For each $\theta \in \mathbb{O}^1_{\mathbf{d}}$, define

$$(6.11)\quad \mathcal{C}^l_+(\theta) := \begin{cases} \big(\,\widetilde{w}^{\theta}_{1\,l},\ \ldots,\ \widetilde{w}^{\theta}_{p_\theta\,l}\big) & \text{if } l \text{ is even and } 0 \le l < (\theta-1)/2 \\ \big(\,\widetilde{w}^{\theta}_{(p_\theta+1)\,l},\ \ldots,\ \widetilde{w}^{\theta}_{t_\theta\,l}\big) & \text{if } l \text{ is odd and } 0 \le l < (\theta-1)/2 \\ \big(\,\widetilde{w}^{\theta}_{1\,l},\ \ldots,\ \widetilde{w}^{\theta}_{p_\theta\,l}\big) & \text{if } l = (\theta-1)/2 \\ \big(\,\widetilde{w}^{\theta}_{1\,l},\ \ldots,\ \widetilde{w}^{\theta}_{p_\theta\,l}\big) & \text{if } l \text{ is odd and } (\theta+1)/2 \le l \le (\theta-1) \\ \big(\,\widetilde{w}^{\theta}_{(p_\theta+1)\,l},\ \ldots,\ \widetilde{w}^{\theta}_{t_\theta\,l}\big) & \text{if } l \text{ is even and } (\theta+1)/2 \le l \le (\theta-1) \end{cases}$$

and

$$
(6.12)\quad \mathcal{C}^l_-(\theta) := \begin{cases} \left(\widetilde{w}^{\theta}_{(p_\theta+1)\,l},\ \ldots,\ \widetilde{w}^{\theta}_{t_\theta\,l}\right) & \text{if } l \text{ is even and } 0 \leq l < (\theta-1)/2 \\ \left(\widetilde{w}^{\theta}_{1\,l},\ \ldots,\ \widetilde{w}^{\theta}_{p_\theta\,l}\right) & \text{if } l \text{ is odd and } 0 \leq l < (\theta-1)/2 \\ \left(\widetilde{w}^{\theta}_{(p_\theta+1)\,l},\ \ldots,\ \widetilde{w}^{\theta}_{t_\theta\,l}\right) & \text{if } l = (\theta-1)/2 \\ \left(\widetilde{w}^{\theta}_{(p_\theta+1)\,l},\ \ldots,\ \widetilde{w}^{\theta}_{t_\theta\,l}\right) & \text{if } l \text{ is odd and } (\theta+1)/2 \leq l \leq (\theta-1) \\ \left(\widetilde{w}^{\theta}_{1\,l},\ \ldots,\ \widetilde{w}^{\theta}_{p_\theta\,l}\right) & \text{if } l \text{ is even and } (\theta+1)/2 \leq l \leq (\theta-1). \end{cases}
$$

Similarly, for each $\zeta \in \mathbb{O}^3_{\mathbf{d}}$, define

$$
(6.13)\quad \mathcal{C}^l_+(\zeta) := \begin{cases} \left(\widetilde{w}^{\zeta}_{1\,l},\ \ldots,\ \widetilde{w}^{\zeta}_{p_\zeta\,l}\right) & \text{if } l \text{ is even and } 0 \leq l < (\zeta-1)/2 \\ \left(\widetilde{w}^{\zeta}_{(p_\zeta+1)\,l},\ \ldots,\ \widetilde{w}^{\zeta}_{t_\zeta\,l}\right) & \text{if } l \text{ is odd and } 0 \leq l < (\zeta-1)/2 \\ \left(\widetilde{w}^{\zeta}_{(p_\zeta+1)\,l},\ \ldots,\ \widetilde{w}^{\zeta}_{t_\zeta\,l}\right) & \text{if } l = (\zeta-1)/2 \\ \left(\widetilde{w}^{\zeta}_{(p_\zeta+1)\,l},\ \ldots,\ \widetilde{w}^{\zeta}_{t_\zeta\,l}\right) & \text{if } l \text{ is even and } (\zeta+1)/2 \leq l \leq (\zeta-1) \\ \left(\widetilde{w}^{\zeta}_{1\,l},\ \ldots,\ \widetilde{w}^{\zeta}_{p_\zeta\,l}\right) & \text{if } l \text{ is odd and } (\zeta+1)/2 \leq l \leq (\zeta-1) \end{cases}
$$

and

$$
(6.14)\quad \mathcal{C}^l_-(\zeta) := \begin{cases} \left(\widetilde{w}^{\zeta}_{(p_\zeta+1)\,l},\ \ldots,\ \widetilde{w}^{\zeta}_{t_\zeta\,l}\right) & \text{if } l \text{ is even and } 0 \leq l < (\zeta-1)/2 \\ \left(\widetilde{w}^{\zeta}_{1\,l},\ \ldots,\ \widetilde{w}^{\zeta}_{p_\zeta\,l}\right) & \text{if } l \text{ is odd and } 0 \leq l < (\zeta-1)/2 \\ \left(\widetilde{w}^{\zeta}_{1\,l},\ \ldots,\ \widetilde{w}^{\zeta}_{p_\zeta\,l}\right) & \text{if } l = (\zeta-1)/2 \\ \left(\widetilde{w}^{\zeta}_{1\,l},\ \ldots,\ \widetilde{w}^{\zeta}_{p_\zeta\,l}\right) & \text{if } l \text{ is even and } (\zeta+1)/2 \leq l \leq (\zeta-1) \\ \left(\widetilde{w}^{\zeta}_{(p_\zeta+1)\,l},\ \ldots,\ \widetilde{w}^{\zeta}_{t_\zeta\,l}\right) & \text{if } l \text{ is odd and } (\zeta+1)/2 \leq l \leq (\zeta-1). \end{cases}
$$

For all $d \in \mathbb{N}_{\mathbf{d}}$ and $0 \leq l \leq d-1$, define

$$
V^l_+(d) := \operatorname{Span}_{\mathbb{C}}\{v \in \mathcal{C}^l(d) \mid \langle v, v\rangle > 0\},\ V^l_-(d) := \operatorname{Span}_{\mathbb{C}}\{v \in \mathcal{C}^l(d) \mid \langle v, v\rangle < 0\}.
$$

It can be verified using (6.9), (6.10) together with the orthogonality relations in Lemma 3.0.13 that $\mathcal{C}^l_+(d)$ (respectively, $\mathcal{C}^l_-(d)$) is indeed an ordered set based on the (unordered) set $\{v \in \mathcal{C}^l(d) \mid \langle v, v\rangle > 0\}$ (respectively, $\{v \in \mathcal{C}^l(d) \mid \langle v, v\rangle < 0\}$) for all $d \in \mathbb{N}_{\mathbf{d}}$ and $0 \leq l \leq d-1$. In particular, $\mathcal{C}^l_+(d)$ and $\mathcal{C}^l_-(d)$ are ordered bases of $V^l_+(d)$ and $V^l_-(d)$ respectively, for all $d \in \mathbb{N}_{\mathbf{d}}$, $0 \leq l \leq d-1$.

In the next lemma, we specify a maximal compact subgroup of $\mathcal{Z}_{\mathrm{SU}(p,q)}(X,H,Y)$ in terms of the subspaces $V^l_+(d)$ and $V^l_-(d)$ defined as above which will be used in Proposition 6.3.3. For notational convenience, we will use $(-1)^l$ to denote the sign '+' or the sign '−' depending on whether l is an even integer or an odd integer.

Lemma 6.3.2. *Let K be the subgroup of $\mathcal{Z}_{\mathrm{SU}(p,q)}(X,H,Y)$ consisting of all $g \in \mathcal{Z}_{\mathrm{SU}(p,q)}(X,H,Y)$ satisfying the following conditions:*

1. $g(V^l_+(d)) \subset V^l_+(d)$ *and* $g(V^l_-(d)) \subset V^l_-(d)$, *for all* $d \in \mathbb{N}_{\mathbf{d}}$ *and* $0 \leq l \leq d-1$.

2. *When* $\eta \in \mathbb{E}_{\mathbf{d}}$,

$$\begin{array}{l} \left[g|_{V^0_+(\eta)}\right]_{\mathcal{C}^0_+(\eta)} = \left[g|_{V^l_{(-1)^l}(\eta)}\right]_{\mathcal{C}^l_{(-1)^l}(\eta)} \\ \left[g|_{V^0_-(\eta)}\right]_{\mathcal{C}^0_-(\eta)} = \left[g|_{V^l_{(-1)^{l+1}}(\eta)}\right]_{\mathcal{C}^l_{(-1)^{l+1}}(\eta)} \end{array} \;;\ \text{for all } 0 \leq l \leq \eta - 1.$$

3. *When* $\theta \in \mathbb{O}^1_{\mathbf{d}}$,

$$\left[g|_{V^0_+(\theta)}\right]_{\mathcal{C}^0_+(\theta)} = \begin{cases} \left[g|_{V^l_{(-1)^l}(\theta)}\right]_{\mathcal{C}^l_{(-1)^l}(\theta)} & \text{for all } 0 \leq l < (\theta-1)/2 \\ \left[g|_{V^{(\theta-1)/2}_+(\theta)}\right]_{\mathcal{C}^{(\theta-1)/2}_+(\theta)} & \\ \left[g|_{V^l_{(-1)^{l+1}}(\theta)}\right]_{\mathcal{C}^l_{(-1)^{l+1}}(\theta)} & \text{for all } (\theta-1)/2 < l \leq \theta - 1, \end{cases}$$

$$\left[g|_{V_-^0(\theta)}\right]_{\mathcal{C}_-^0(\theta)} = \begin{cases} \left[g|_{V^l_{(-1)^{l+1}}(\theta)}\right]_{\mathcal{C}^l_{(-1)^{l+1}}(\theta)} & \textit{for all } 0 \le l < (\theta-1)/2 \\ \left[g|_{V_-^{(\theta-1)/2}(\theta)}\right]_{\mathcal{C}_-^{(\theta-1)/2}(\theta)} & \\ \left[g|_{V^l_{(-1)^{l}}(\theta)}\right]_{\mathcal{C}^l_{(-1)^{l}}(\theta)} & \textit{for all } (\theta-1)/2 < l \le \theta - 1. \end{cases}$$

4. *When* $\zeta \in \mathbb{O}^3_{\mathbf{d}}$,

$$\left[g|_{V_+^0(\zeta)}\right]_{\mathcal{C}_+^0(\zeta)} = \begin{cases} \left[g|_{V^l_{(-1)^{l}}(\zeta)}\right]_{\mathcal{C}^l_{(-1)^{l}}(\zeta)} & \textit{for all } 0 \le l < (\zeta-1)/2 \\ \left[g|_{V_-^{(\zeta-1)/2}(\zeta)}\right]_{\mathcal{C}_-^{(\zeta-1)/2}(\zeta)} & \\ \left[g|_{V^l_{(-1)^{l+1}}(\zeta)}\right]_{\mathcal{C}^l_{(-1)^{l+1}}(\zeta)} & \textit{for all } (\zeta-1)/2 < l \le \zeta - 1, \end{cases}$$

$$\left[g|_{V_-^0(\zeta)}\right]_{\mathcal{C}_-^0(\zeta)} = \begin{cases} \left[g|_{V^l_{(-1)^{l+1}}(\zeta)}\right]_{\mathcal{C}^l_{(-1)^{l+1}}(\zeta)} & \textit{for all } 0 \le l < (\zeta-1)/2 \\ \left[g|_{V_+^{(\zeta-1)/2}(\zeta)}\right]_{\mathcal{C}_+^{(\zeta-1)/2}(\zeta)} & \\ \left[g|_{V^l_{(-1)^{l}}(\zeta)}\right]_{\mathcal{C}^l_{(-1)^{l}}(\zeta)} & \textit{for all } (\zeta-1)/2 < l \le \zeta - 1. \end{cases}$$

Then K is a maximal compact subgroup of $\mathcal{Z}_{\mathrm{SU}(p,q)}(X,H,Y)$.

Proof. In view of the description of $\mathcal{Z}_{\mathrm{SU}(p,q)}(X,H,Y)$ in the Lemma 6.3.1 we see that its subgroup

$$K \; := \left\{ g \in \mathrm{SU}(p,q) \;\middle|\; \begin{array}{c} g(V^l_+(d)) \subset V^l_+(d)\,, \;\; g(V^l_-(d)) \subset V^l_-(d) \;\text{ and} \\ \left[g|_{V^l(d)}\right]_{\mathcal{C}^l(d)} = \left[g|_{V^0(d)}\right]_{\mathcal{C}^0(d)} \;\text{for all } d \in \mathbb{N}_{\mathbf{d}},\, 0 \le l < d \end{array} \right\}$$
$$\subset \; \mathcal{Z}_{\mathrm{SU}(p,q)}(X,H,Y)$$

is maximal compact. Thus it suffices show that if $g \in \mathrm{SU}(p,q)$ and $g(V^l_+(d)) \subset V^l_+(d)$, $g(V^l_-(d)) \subset V^l_-(d)$, then $\left[g|_{V^l(d)}\right]_{\mathcal{C}^l(d)} = \left[g|_{V^0(d)}\right]_{\mathcal{C}^0(d)}$ for all $0 \le l \le d-1$, $d \in \mathbb{N}_{\mathbf{d}}$ if and only if g satisfies the conditions (2), (3) and (4) in the statement of the lemma. To do this, we first record the following relations among the ordered sets $\mathcal{C}^l(d), \mathcal{C}^l_{(-1)^{l+1}}(d)$ and $\mathcal{C}^l_{(-1)^l}(d)$ for all $d \in \mathbb{N}_{\mathbf{d}}$: When $\eta \in \mathbb{E}_{\mathbf{d}}$,

$$\mathcal{C}^l(\eta) = \mathcal{C}^l_{(-1)^l}(\eta) \vee \mathcal{C}^l_{(-1)^{l+1}}(\eta) \quad \text{for } 0 \le l \le \eta - 1. \tag{6.15}$$

When $\theta \in \mathbb{O}^1_{\mathbf{d}}$,

$$\mathcal{C}^l(\theta) = \begin{cases} \mathcal{C}^l_{(-1)^l}(\theta) \vee \mathcal{C}^l_{(-1)^{l+1}}(\theta) & \text{for all } 0 \le l < (\theta-1)/2 \\ \mathcal{C}^{(\theta-1)/2}_{+1}(\theta) \vee \mathcal{C}^{(\theta-1)/2}_{-1}(\theta) & \text{for } l = (\theta-1)/2 \\ \mathcal{C}^l_{(-1)^{l+1}}(\theta) \vee \mathcal{C}^l_{(-1)^l}(\theta) & \text{for all } (\theta-1)/2 < l \le \theta - 1. \end{cases} \tag{6.16}$$

When $\zeta \in \mathbb{O}^3_{\mathbf{d}}$,

$$\mathcal{C}^l(\zeta) = \begin{cases} \mathcal{C}^l_{(-1)^l}(\zeta) \vee \mathcal{C}^l_{(-1)^{l+1}}(\zeta) & \text{for all } 0 \le l < (\zeta-1)/2 \\ \mathcal{C}^{(\zeta-1)/2}_{-1}(\zeta) \vee \mathcal{C}^{(\zeta-1)/2}_{+1}(\zeta) & \text{for } l = (\zeta-1)/2 \\ \mathcal{C}^l_{(-1)^{l+1}}(\zeta) \vee \mathcal{C}^l_{(-1)^l}(\zeta) & \text{for all } (\zeta-1)/2 < l \le \zeta - 1. \end{cases} \tag{6.17}$$

Assuming that $g \in \mathrm{SU}(p,q)$, $g(V^l_+(d)) \subset V^l_+(d)$, $g(V^l_-(d)) \subset V^l_-(d)$ and

$$\big[g|_{V^l(d)}\big]_{\mathcal{C}^l(d)} = \big[g|_{V^0(d)}\big]_{\mathcal{C}^0(d)}$$

for all $0 \le l \le d-1$, $d \in \mathbb{N}_{\mathbf{d}}$, we next show that g satisfies the conditions (2), (3) and (4) in the lemma.

In view of (6.15), for all $\eta \in \mathbb{E}_{\mathbf{d}}$,

$$\Big[g|_{V^l(\eta)}\Big]_{\mathcal{C}^l_{(-1)^l}(\eta) \vee \mathcal{C}^l_{(-1)^{l+1}}(\eta)} = \Big[g|_{V^l(\eta)}\Big]_{\mathcal{C}^l(\eta)} = \Big[g|_{V^0(\eta)}\Big]_{\mathcal{C}^0(\eta)}$$

$$= \begin{pmatrix} \big[g|_{V^0_{+1}(\eta)}\big]_{\mathcal{C}^0_{+1}(\eta)} & 0 \\ 0 & \big[g|_{V^0_{-1}(\eta)}\big]_{\mathcal{C}^0_{-1}(\eta)} \end{pmatrix}.$$

Thus for all $\eta \in \mathbb{E}_{\mathbf{d}}$ and $0 \leq l \leq \eta - 1$,

$$\left[g|_{V^l_{(-1)^l}(\eta)}\right]_{\mathcal{C}^l_{(-1)^l}(\eta)} = \left[g|_{V^0_{+1}(\eta)}\right]_{\mathcal{C}^0_{+1}(\eta)} \text{ and } \left[g|_{V^l_{(-1)^{l+1}}(\eta)}\right]_{\mathcal{C}^l_{(-1)^{l+1}}(\eta)} = \left[g|_{V^0_{-1}(\eta)}\right]_{\mathcal{C}^0_{-1}(\eta)}.$$

Hence, (2) of the lemma holds.

In view of (6.16), for all $\theta \in \mathbb{O}^1_{\mathbf{d}}$ and $0 \leq l < (\theta - 1)/2$,

$$\left[g|_{V^l(\theta)}\right]_{\mathcal{C}^l_{(-1)^l}(\theta) \vee \mathcal{C}^l_{(-1)^{l+1}}(\theta)} = \left[g|_{V^l(\theta)}\right]_{\mathcal{C}^l(\theta)} = \left[g|_{V^0(\theta)}\right]_{\mathcal{C}^0(\theta)}$$

$$= \begin{pmatrix} \left[g|_{V^0_{+1}(\theta)}\right]_{\mathcal{C}^0_{+1}(\theta)} & 0 \\ 0 & \left[g|_{V^0_{-1}(\theta)}\right]_{\mathcal{C}^0_{-1}(\theta)} \end{pmatrix}.$$

Therefore if $\theta \in \mathbb{O}^1_{\mathbf{d}}$, then for all $0 \leq l < (\theta - 1)/2$,

$$\left[g|_{V^l_{(-1)^l}(\theta)}\right]_{\mathcal{C}^l_{(-1)^l}(\theta)} = \left[g|_{V^0_{+1}(\theta)}\right]_{\mathcal{C}^0_{+1}(\theta)} \text{ and } \left[g|_{V^l_{(-1)^{l+1}}(\theta)}\right]_{\mathcal{C}^l_{(-1)^{l+1}}(\theta)} = \left[g|_{V^0_{-1}(\theta)}\right]_{\mathcal{C}^0_{-1}(\theta)}.$$

From (6.16), we have

$$\left[g|_{V^{(\theta-1)/2}(\theta)}\right]_{\mathcal{C}^{(\theta-1)/2}_{+}(\theta) \vee \mathcal{C}^{(\theta-1)/2}_{-}(\theta)} = \left[g|_{V^{(\theta-1)/2}(\theta)}\right]_{\mathcal{C}^{(\theta-1)/2}(\theta)} = \left[g|_{V^0(\theta)}\right]_{\mathcal{C}^0(\theta)}$$

$$= \begin{pmatrix} \left[g|_{V^0_{+1}(\theta)}\right]_{\mathcal{C}^0_{+1}(\theta)} & 0 \\ 0 & \left[g|_{V^0_{-1}(\theta)}\right]_{\mathcal{C}^0_{-1}(\theta)} \end{pmatrix}.$$

Thus,

$$\left[g|_{V^{(\theta-1)/2}_{+}(\theta)}\right]_{\mathcal{C}^{(\theta-1)/2}_{+}(\theta)} = \left[g|_{V^0_{+}(\theta)}\right]_{\mathcal{C}^0_{+}(\theta)}, \left[g|_{V^{(\theta-1)/2}_{-}(\theta)}\right]_{\mathcal{C}^{(\theta-1)/2}_{-}(\theta)} = \left[g|_{V^0_{-}(\theta)}\right]_{\mathcal{C}^0_{-}(\theta)}.$$

When $(\theta - 1)/2 < l \leq \theta - 1$, we have

$$\left[g|_{V^l(\theta)}\right]_{\mathcal{C}^l_{(-1)^{l+1}}(\theta) \vee \mathcal{C}^l_{(-1)^l}(\theta)} = \left[g|_{V^l(\theta)}\right]_{\mathcal{C}^l(\theta)} = \left[g|_{V^0(\theta)}\right]_{\mathcal{C}^0(\theta)}$$

$$= \begin{pmatrix} \left[g|_{V^0_{+1}(\theta)}\right]_{\mathcal{C}^0_{+1}(\theta)} & 0 \\ 0 & \left[g|_{V^0_{-1}(\theta)}\right]_{\mathcal{C}^0_{-1}(\theta)} \end{pmatrix}.$$

Thus if $\theta \in \mathbb{O}^1_{\mathbf{d}}$, then for all $(\theta-1)/2 < l \leq \theta - 1$,

$$\left[g|_{V^l_{(-1)^{l+1}}(\theta)}\right]_{\mathcal{C}^l_{(-1)^{l+1}}(\theta)} = \left[g|_{V^0_{+1}(\theta)}\right]_{\mathcal{C}^0_{+1}(\theta)} \text{ and } \left[g|_{V^l_{(-1)^l}(\theta)}\right]_{\mathcal{C}^l_{(-1)^l}(\theta)} = \left[g|_{V^0_{-1}(\theta)}\right]_{\mathcal{C}^0_{-1}(\theta)}.$$

Hence, (3) of the lemma holds.

When $g(V^l_+(\zeta)) \subset V^l_+(\zeta)$, $g(V^l_-(\zeta)) \subset V^l_-(\zeta)$ and $\left[g|_{V^l(\zeta)}\right]_{\mathcal{C}^l(\zeta)} = \left[g|_{V^0(\zeta)}\right]_{\mathcal{C}^0(\zeta)}$ for all $0 \leq l \leq \zeta - 1$, $\zeta \in \mathbb{O}^3_{\mathbf{d}}$, using (6.17) it follows, similarly as above, that (4) of the lemma holds.

To prove the opposite implication, we assume that g satisfies the conditions $g(V^l_+(d)) \subset V^l_+(d)$, $g(V^l_-(d)) \subset V^l_-(d)$ as well as the conditions (2), (3), (4) of the lemma. Using the relations (6.15), (6.16) and (6.17) it is now straightforward to check that $\left[g|_{V^l(d)}\right]_{\mathcal{C}^l(d)} = \left[g|_{V^0(d)}\right]_{\mathcal{C}^0(d)}$ for all $0 \leq l \leq d-1, d \in \mathbb{N}_{\mathbf{d}}$. This completes the proof of the lemma. □

We now introduce some notation which will be required to state Proposition 6.3.3. For $d \in \mathbb{N}_{\mathbf{d}}$, define

$$\mathcal{C}_+(d) := \mathcal{C}^0_+(d) \vee \cdots \vee \mathcal{C}^{d-1}_+(d) \text{ and } \mathcal{C}_-(d) := \mathcal{C}^0_-(d) \vee \cdots \vee \mathcal{C}^{d-1}_-(d).$$

Let $\alpha := \#\mathbb{E}_{\mathbf{d}}$, $\beta := \#\mathbb{O}^1_{\mathbf{d}}$ and $\gamma := \#\mathbb{O}^3_{\mathbf{d}}$. We enumerate

$$\mathbb{E}_{\mathbf{d}} = \{\eta_i \mid 1 \leq i \leq \alpha\}$$

such that $\eta_i < \eta_{i+1}$,

$$\mathbb{O}^1_{\mathbf{d}} = \{\theta_j \mid 1 \leq j \leq \beta\}$$

such that $\theta_j < \theta_{j+1}$ and similarly

$$\mathbb{O}^3_{\mathbf{d}} = \{\zeta_j \mid 1 \leq j \leq \gamma\}$$

such that $\zeta_j < \zeta_{j+1}$. Now define

$$\mathcal{E}_+ := \mathcal{C}_+(\eta_1)\vee\cdots\vee\mathcal{C}_+(\eta_\alpha)\,;\; \mathcal{O}^1_+ := \mathcal{C}_+(\theta_1)\vee\cdots\vee\mathcal{C}_+(\theta_\beta)\,;\; \mathcal{O}^3_+ := \mathcal{C}_+(\zeta_1)\vee\cdots\vee\mathcal{C}_+(\zeta_\gamma);$$

$$\mathcal{E}_- := \mathcal{C}_-(\eta_1)\vee\cdots\vee\mathcal{C}_-(\eta_\alpha)\,;\; \mathcal{O}^1_- := \mathcal{C}_-(\theta_1)\vee\cdots\vee\mathcal{C}_-(\theta_\beta)\,;\; \mathcal{O}^3_- := \mathcal{C}_-(\zeta_1)\vee\cdots\vee\mathcal{C}_-(\zeta_\gamma).$$

Finally we define

$$\mathcal{H}_+ := \mathcal{E}_+ \vee \mathcal{O}^1_+ \vee \mathcal{O}^3_+, \quad \mathcal{H}_- := \mathcal{E}_- \vee \mathcal{O}^1_- \vee \mathcal{O}^3_- \text{ and } \mathcal{H} := \mathcal{H}_+ \vee \mathcal{H}_-. \tag{6.18}$$

It is clear that $\mathcal{H}$ is a standard orthogonal basis with $\mathcal{H}_+ = \{v \in \mathcal{H} \mid \langle v, v\rangle = 1\}$ and $\mathcal{H}_- = \{v \in \mathcal{H} \mid \langle v, v\rangle = -1\}$. In particular, $\#\mathcal{H}_+ = p$ and $\#\mathcal{H}_- = q$. From the definition of the $\mathcal{H}_+$ and $\mathcal{H}_-$ we have the following relations:

$$\sum_{i=1}^{\alpha} \frac{\eta_i}{2} t_{\eta_i} + \sum_{j=1}^{\beta} \big(\frac{\theta_j+1}{2} p_{\theta_j} + \frac{\theta_j-1}{2} q_{\theta_j}\big) + \sum_{k=1}^{\gamma} \big(\frac{\zeta_k-1}{2} p_{\zeta_k} + \frac{\zeta_k+1}{2} q_{\zeta_k}\big) = p$$

and

$$\sum_{i=1}^{\alpha} \frac{\eta_i}{2} t_{\eta_i} + \sum_{j=1}^{\beta} \big(\frac{\theta_j-1}{2} p_{\theta_j} + \frac{\theta_j+1}{2} q_{\theta_j}\big) + \sum_{k=1}^{\gamma} \big(\frac{\zeta_k+1}{2} p_{\zeta_k} + \frac{\zeta_k-1}{2} q_{\zeta_k}\big) = q.$$

The $\mathbb{C}$-algebra

$$\prod_{i=1}^{\alpha} \big(\mathrm{M}_{p_{\eta_i}}(\mathbb{C}) \times \mathrm{M}_{q_{\eta_i}}(\mathbb{C})\big) \times \prod_{j=1}^{\beta} \big(\mathrm{M}_{p_{\theta_j}}(\mathbb{C}) \times \mathrm{M}_{q_{\theta_j}}(\mathbb{C})\big) \times \prod_{k=1}^{\gamma} \big(\mathrm{M}_{p_{\zeta_k}}(\mathbb{C}) \times \mathrm{M}_{q_{\zeta_k}}(\mathbb{C})\big)$$

is embedded into $\mathrm{M}_p(\mathbb{C})$ and $\mathrm{M}_q(\mathbb{C})$ in the following two ways:

$$\mathbf{D}_p\colon \prod_{i=1}^{\alpha}\big(\mathrm{M}_{p_{\eta_i}}(\mathbb{C})\times \mathrm{M}_{q_{\eta_i}}(\mathbb{C})\big)\times\prod_{j=1}^{\beta}\big(\mathrm{M}_{p_{\theta_j}}(\mathbb{C})\times \mathrm{M}_{q_{\theta_j}}(\mathbb{C})\big)\times\prod_{k=1}^{\gamma}\big(\mathrm{M}_{p_{\zeta_k}}(\mathbb{C})\times \mathrm{M}_{q_{\zeta_k}}(\mathbb{C})\big)$$

$$\longrightarrow \mathrm{M}_p(\mathbb{C})$$

is defined by

$$\begin{aligned}&\big(A_{\eta_1},B_{\eta_1},\ldots,A_{\eta_\alpha},B_{\eta_\alpha};C_{\theta_1},D_{\theta_1},\ldots,C_{\theta_\beta},D_{\theta_\beta};E_{\zeta_1},F_{\zeta_1},\ldots,E_{\zeta_\gamma},F_{\zeta_\gamma}\big)\\ &\quad\longmapsto \bigoplus_{i=1}^{\alpha}\big(A_{\eta_i}\oplus B_{\eta_i}\big)_{\blacktriangle}^{\eta_i/2}\oplus\bigoplus_{j=1}^{\beta}\Big(\big(C_{\theta_j}\oplus D_{\theta_j}\big)_{\blacktriangle}^{\frac{\theta_j-1}{4}}\oplus C_{\theta_j}\oplus\big(C_{\theta_j}\oplus D_{\theta_j}\big)_{\blacktriangle}^{\frac{\theta_j-1}{4}}\Big)\\ &\qquad\oplus\bigoplus_{k=1}^{\gamma}\Big(\big(E_{\zeta_k}\oplus F_{\zeta_k}\big)_{\blacktriangle}^{\frac{\zeta_k+1}{4}}\oplus\big(F_{\zeta_k}\oplus E_{\zeta_k}\big)_{\blacktriangle}^{\frac{\zeta_k-3}{4}}\oplus F_{\zeta_k}\Big),\end{aligned}$$

and

$$\mathbf{D}_q\colon \prod_{i=1}^{\alpha}\big(\mathrm{M}_{p_{\eta_i}}(\mathbb{C})\times \mathrm{M}_{q_{\eta_i}}(\mathbb{C})\big)\times\prod_{j=1}^{\beta}\big(\mathrm{M}_{p_{\theta_j}}(\mathbb{C})\times \mathrm{M}_{q_{\theta_j}}(\mathbb{C})\big)\times\prod_{k=1}^{\gamma}\big(\mathrm{M}_{p_{\zeta_k}}(\mathbb{C})\times \mathrm{M}_{q_{\zeta_k}}(\mathbb{C})\big)$$

$$\longrightarrow \mathrm{M}_q(\mathbb{C})$$

is defined by

$$\begin{aligned}&\big(A_{\eta_1},B_{\eta_1},\ldots,A_{\eta_\alpha},B_{\eta_\alpha};C_{\theta_1},D_{\theta_1},\ldots,C_{\theta_\beta},D_{\theta_\beta};E_{\zeta_1},F_{\zeta_1},\ldots,E_{\zeta_\gamma},F_{\zeta_\gamma}\big)\\ &\quad\longmapsto \bigoplus_{i=1}^{\alpha}\big(B_{\eta_i}\oplus A_{\eta_i}\big)_{\blacktriangle}^{\eta_i/2}\oplus\bigoplus_{j=1}^{\beta}\Big(\big(D_{\theta_j}\oplus C_{\theta_j}\big)_{\blacktriangle}^{\frac{\theta_j-1}{4}}\oplus D_{\theta_j}\oplus\big(D_{\theta_j}\oplus C_{\theta_j}\big)_{\blacktriangle}^{\frac{\theta_j-1}{4}}\Big)\\ &\qquad\oplus\bigoplus_{k=1}^{\gamma}\Big(\big(F_{\zeta_k}\oplus E_{\zeta_k}\big)_{\blacktriangle}^{\frac{\zeta_k+1}{4}}\oplus\big(E_{\zeta_k}\oplus F_{\zeta_k}\big)_{\blacktriangle}^{\frac{\zeta_k-3}{4}}\oplus E_{\zeta_k}\Big).\end{aligned}$$

Define the characters

$$\chi_p\colon \prod_{i=1}^{\alpha}\big(\mathrm{GL}_{p_{\eta_i}}(\mathbb{C})\times\mathrm{GL}_{q_{\eta_i}}(\mathbb{C})\big)\times\prod_{j=1}^{\beta}\big(\mathrm{GL}_{p_{\theta_j}}(\mathbb{C})\times\mathrm{GL}_{q_{\theta_j}}(\mathbb{C})\big)\times\prod_{k=1}^{\gamma}\big(\mathrm{GL}_{p_{\zeta_k}}(\mathbb{C})\times\mathrm{GL}_{q_{\zeta_k}}(\mathbb{C})\big)$$

$$\longrightarrow \mathbb{C}^*$$

$$\big(A_{\eta_1}, B_{\eta_1}, \dots, A_{\eta_\alpha}, B_{\eta_\alpha}; C_{\theta_1}, D_{\theta_1}, \dots, C_{\theta_\beta}, D_{\theta_\beta}; E_{\zeta_1}, F_{\zeta_1}, \dots, E_{\zeta_\gamma}, F_{\zeta_\gamma}\big)$$

$$\longmapsto \prod_{i=1}^{\alpha}(\det A_{\eta_i}^{\eta_i/2} \det B_{\eta_i}^{\eta_i/2}) \prod_{j=1}^{\beta}(\det C_{\theta_j}^{\frac{\theta_j+1}{2}} \det D_{\theta_j}^{\frac{\theta_j-1}{2}}) \prod_{k=1}^{\gamma}(\det E_{\zeta_k}^{\frac{\zeta_k-1}{2}} \det F_{\zeta_k}^{\frac{\zeta_k+1}{2}})$$

and

$$\chi_q \colon \prod_{i=1}^{\alpha}\big(\mathrm{GL}_{p_{\eta_i}}(\mathbb{C})\times \mathrm{GL}_{q_{\eta_i}}(\mathbb{C})\big)\times\prod_{j=1}^{\beta}\big(\mathrm{GL}_{p_{\theta_j}}(\mathbb{C})\times \mathrm{GL}_{q_{\theta_j}}(\mathbb{C})\big)\times\prod_{k=1}^{\gamma}\big(\mathrm{GL}_{p_{\zeta_k}}(\mathbb{C})\times \mathrm{GL}_{q_{\zeta_k}}(\mathbb{C})\big)$$

$$\longrightarrow \mathbb{C}^*$$

$$\big(A_{\eta_1}, B_{\eta_1}, \dots, A_{\eta_\alpha}, B_{\eta_\alpha}; C_{\theta_1}, D_{\theta_1}, \dots, C_{\theta_\beta}, D_{\theta_\beta}; E_{\zeta_1}, F_{\zeta_1}, \dots, E_{\zeta_\gamma}, F_{\zeta_\gamma}\big)$$

$$\longmapsto \prod_{i=1}^{\alpha}(\det A_{\eta_i}^{\eta_i/2} \det B_{\eta_i}^{\eta_i/2}) \prod_{j=1}^{\beta}(\det C_{\theta_j}^{\frac{\theta_j-1}{2}} \det D_{\theta_j}^{\frac{\theta_j+1}{2}}) \prod_{k=1}^{\gamma}(\det E_{\zeta_k}^{\frac{\zeta_k+1}{2}} \det F_{\zeta_k}^{\frac{\zeta_k-1}{2}}).$$

Let $\Lambda_{\mathcal{H}} : \mathrm{End}_{\mathbb{C}}\mathbb{C}^n \longrightarrow \mathrm{M}_n(\mathbb{C})$ be the isomorphism of $\mathbb{C}$-algebras induced by the ordered basis $\mathcal{H}$ defined in (6.18). Let M be the maximal compact subgroup of $\mathrm{SU}(p,q)$ which leaves invariant simultaneously the two subspace spanned by $\mathcal{H}_+$ and $\mathcal{H}_-$. Clearly, $\Lambda_{\mathcal{H}}(M) = \mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q))$. In the next result we obtain an explicit description of $\Lambda_{\mathcal{H}}(K)$ in $\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q))$ where $K \subset M$ is the suitable maximal compact subgroup in the centralizer of the nilpotent element X, as in Lemma 6.3.2.

Proposition 6.3.3. *Let $X \in \mathcal{N}_{\mathfrak{su}(p,q)}$, $\Psi_{\mathrm{SU}(p,q)}(\mathcal{O}_X) = (\mathbf{d}, \mathbf{sgn}_{\mathcal{O}_X})$. Let $\alpha := \#\mathbb{E}_{\mathbf{d}}$, $\beta := \#\mathbb{O}^1_{\mathbf{d}}$ and $\gamma := \#\mathbb{O}^3_{\mathbf{d}}$. Let $\{X,H,Y\}$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple in $\mathfrak{su}(p,q)$ and $(p_d,\, q_d)$ the signature of the form $(\cdot,\cdot)_d$, $d \in \mathbb{N}_{\mathbf{d}}$, as defined in* (3.8)*. Let K be the maximal compact subgroup of $\mathcal{Z}_{\mathrm{SU}(p,q)}(X,H,Y)$ as in Lemma 6.3.2. Then $\Lambda_{\mathcal{H}}(K) \subset \mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q))$ is given by*

$$\Lambda_{\mathcal{H}}(K) = \left\{ \mathbf{D}_p(g) \oplus \mathbf{D}_q(g) \;\middle|\; \begin{array}{c} g \in \prod_{i=1}^{\alpha}\big(\mathrm{U}(p_{\eta_i})\times\mathrm{U}(q_{\eta_i})\big)\times\prod_{j=1}^{\beta}\big(\mathrm{U}(p_{\theta_j})\times\mathrm{U}(q_{\theta_j})\big) \\ \times\prod_{k=1}^{\gamma}\big(\mathrm{U}(p_{\zeta_k})\times\mathrm{U}(q_{\zeta_k})\big),\ \textit{and}\quad \chi_p(g)\chi_q(g)=1 \end{array} \right\}.$$

Proof. This follows by writing the matrices of the elements of the maximal compact subgroup K in Lemma 6.3.2 with respect to the basis $\mathcal{H}$ as in (6.18). $\square$

Theorem 6.3.4. *Let $X \in \mathfrak{su}(p,q)$ be a nilpotent element. Let $(\mathbf{d}, \mathbf{sgn}_{\mathcal{O}_X}) \in \mathcal{Y}(p,q)$ be the signed Young diagram of the orbit $\mathcal{O}_X$ (that is, $\Psi_{\mathrm{SU}(p,q)}(\mathcal{O}_X) = (\mathbf{d}, \mathbf{sgn}_{\mathcal{O}_X})$ as in the notation of Theorem 4.1.4). Let*

$$l := \#\{d \in \mathbb{N}_{\mathbf{d}} \mid p_d \neq 0\} + \#\{d \in \mathbb{N}_{\mathbf{d}}, \mid q_d \neq 0\}\,.$$

Then the following hold:

1. *If $\mathbb{N}_{\mathbf{d}} = \mathbb{E}_{\mathbf{d}}$, then $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = l - 1$.*
2. *If $l = 1$ and $\mathbb{N}_{\mathbf{d}} = \mathbb{O}_{\mathbf{d}}$, then $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 0$.*
3. *If $l \geq 2$ and $\#\mathbb{O}_{\mathbf{d}} \geq 1$, then $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = l - 2$.*

Proof. This is clear when $X = 0$. So assume that $X \neq 0$.

Let $\{X, H, Y\} \subset \mathfrak{su}(p,q)$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple. Let K be the maximal compact subgroup of $\mathcal{Z}_{\mathrm{SU}(p,q)}(X,H,Y)$ as in Lemma 6.3.2, and let $\mathcal{H}$ be as in (6.18). Let M be the maximal compact subgroup of $\mathrm{SU}(p,q)$ which leaves invariant simultaneously the two subspace spanned by $\mathcal{H}_+$ and $\mathcal{H}_-$. Then M contains K. It follows either from Proposition 6.3.3 or from Lemma 6.0.1 (4) that

$$K \simeq K' := S\Big(\prod_{d \in \mathbb{N}_{\mathbf{d}}} (\mathrm{U}(p_d) \times \mathrm{U}(q_d))^d_\Delta\Big)\,.$$

This implies that $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}) = l - 1$. We now appeal to Proposition 6.3.3 to make the following observations :

1. If $\mathbb{N}_{\mathbf{d}} = \mathbb{E}_{\mathbf{d}}$, then $\mathfrak{k} \subset [\mathfrak{m}, \mathfrak{m}]$.
2. If $\#\mathbb{O}_{\mathbf{d}} \geq 1$ and $l \geq 2$, then $\mathfrak{k}, \not\subset [\mathfrak{m}, \mathfrak{m}]$.

Since K is not necessarily connected, we need to show that the adjoint action of K on $\mathfrak{z}(\mathfrak{k})$ is trivial. For this, first denote

$$\mathbf{L} := \prod_{d \in \mathbb{N}_{\mathbf{d}}} (\mathrm{U}(p_d) \times \mathrm{U}(q_d))^d_{\Delta}$$

and identify K with K'. Let $\mathfrak{l}$ be the Lie algebra of $\mathbf{L}$. Then

$$[\mathbf{L},\, \mathbf{L}] \;=\; \prod_{d \in \mathbb{N}_{\mathbf{d}}} (\mathrm{SU}(p_d) \times \mathrm{SU}(q_d))^d_{\Delta}\,.$$

In particular $[\mathbf{L},\, \mathbf{L}] \subset K \subset \mathbf{L}$. Thus $[\mathfrak{l},\, \mathfrak{l}] = [\mathfrak{k},\, \mathfrak{k}]$, and hence $\mathfrak{z}(\mathfrak{k}) = \mathfrak{k} \cap \mathfrak{z}(\mathfrak{l})$. Since $\mathbf{L}$ is connected, the adjoint action of $\mathbf{L}$ is trivial on $\mathfrak{z}(\mathfrak{l})$. So the adjoint action of K on $\mathfrak{z}(\mathfrak{k})$ is trivial.

Proof of (1): From the above observations it follow that $\mathfrak{k} \subset [\mathfrak{m},\, \mathfrak{m}]$ when $\mathbb{N}_{\mathbf{d}} = \mathbb{E}_{\mathbf{d}}$. As the adjoint action of K on $\mathfrak{z}(\mathfrak{k})$ is trivial, we have $\big[(\mathfrak{z}(\mathfrak{k}) \cap [\mathfrak{m},\, \mathfrak{m}])\big]^{K/K^\circ} = \mathfrak{z}(\mathfrak{k}) \cap [\mathfrak{m},\, \mathfrak{m}] = \mathfrak{z}(\mathfrak{k})$. In view of Theorem 5.2.2 we now have $\dim_{\mathbb{R}} H^2(\mathcal{O}_X,\, \mathbb{R}) = l-1$.

Proof of (2): Suppose $\mathbf{d} = [d^{t_d}]$ where $t_d d = p+q$. Since $l = 1$, it follows that either $p_d = t_d$ or $q_d = t_d$. In both cases we have $K \simeq S\big(\mathrm{U}(t_d)^d_{\Delta}\big)$. So $\mathfrak{z}(\mathfrak{k})$ is trivial. Hence, in view of Theorem 5.2.2 we have $\dim_{\mathbb{R}} H^2(\mathcal{O}_X,\, \mathbb{R}) = 0$.

Proof of (3): From the above observations we have $\mathfrak{z}(\mathfrak{k}) \not\subset [\mathfrak{m},\, \mathfrak{m}]$ when $\#\mathbb{O}_{\mathbf{d}} \geq 1$ and $l \geq 2$. Since $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{m}) = 1$, it follows that $\dim_{\mathbb{R}}(\mathfrak{z}(\mathfrak{k}) \cap [\mathfrak{m},\, \mathfrak{m}]) = \dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}) - 1$. By Theorem 5.2.2, and the fact that the adjoint action of K on $\mathfrak{z}(\mathfrak{k})$ is trivial, we conclude that

$$\dim_{\mathbb{R}} H^2(\mathcal{O}_X,\, \mathbb{R}) \;=\; \dim_{\mathbb{R}}[(\mathfrak{z}(\mathfrak{k}) \cap [\mathfrak{m},\, \mathfrak{m}])^*]^{K/K^\circ} \;=\; l-2\,.$$

This completes the proof of the theorem. $\square$

6.4 Second cohomology of nilpotent orbits in $\mathfrak{so}(p,q)$

In this section we compute the second cohomology groups of nilpotent orbits in $\mathfrak{so}(p,q)$ under the adjoint action of $\mathrm{SO}(p,q)^\circ$. We assume that $p,\, q > 0$ as we deal with non-compact groups. Set $n := p+q$. In this section, we follow notation and parametrization of nilpotent orbits in $\mathfrak{so}(p,q)$ as in §4.1.4; see Theorem 4.1.6. Throughout this section $\langle \cdot\,, \cdot \rangle$ denotes the symmetric form on $\mathbb{R}^n$ defined by $\langle x, y \rangle := x^t \mathrm{I}_{p,q} y$, for $x, y \in \mathbb{R}^n$, where $\mathrm{I}_{p,q}$ is as in (2.19).

Let $0 \neq X \in \mathcal{N}_{\mathfrak{so}(p,q)}$, and $\{X, H, Y\} \subset \mathfrak{so}(p,q)$ a $\mathfrak{sl}_2(\mathbb{R})$-triple. Let $\Psi_{\mathrm{SO}(p,q)^\circ}(\mathcal{O}_X) = \big(\mathbf{d},\, \mathbf{sgn}_{\mathcal{O}_X}\big)$. Then we have $\Psi'_{\mathrm{SO}(p,q)^\circ}(\mathcal{O}_X) = \mathbf{d}$. Recall that $\mathbf{sgn}_{\mathcal{O}_X}$ determines the signature of $(\cdot,\,\cdot)_\theta$ on $L(\theta-1)$, $\theta \in \mathbb{O}_\mathbf{d}$; let (p_θ, q_θ) be the signature of $(\cdot,\,\cdot)_\theta$.

First assume that $\mathbb{N}_\mathbf{d} = \mathbb{O}_\mathbf{d}$. Let $(v_1^\theta, \ldots, v_{t_\theta}^\theta)$ be an ordered $\mathbb{R}$-basis of $L(\theta-1)$ as in Proposition 3.0.7. It now follows from Proposition 3.0.7(3)(b) that $(v_1^\theta, \ldots, v_{t_\theta}^\theta)$ is an orthogonal basis for $(\cdot,\,\cdot)_\theta$ when $\theta \in \mathbb{O}_\mathbf{d}$. We also assume that the vectors in the ordered basis $(v_1^\theta, \ldots, v_{t_\theta}^\theta)$ satisfies the properties in Remark 3.0.11(1). In view of the signature of $(\cdot,\,\cdot)_\theta$, $\theta \in \mathbb{O}_\mathbf{d}$, we may further assume that

$$(v_j^\theta, v_j^\theta)_\theta = \begin{cases} +1 & \text{if } 1 \leq j \leq p_\theta \\ -1 & \text{if } p_\theta < j \leq t_\theta. \end{cases} \tag{6.19}$$

For $\theta \in \mathbb{O}_\mathbf{d}$, let $\big\{ w_{jl}^\theta \mid 1 \leq j \leq t_\theta,\, 0 \leq l \leq \theta - 1 \big\}$ be the $\mathbb{R}$-basis of $M(\theta-1)$ as in Lemma 3.0.12. For each $0 \leq l \leq \theta-1$, define

$$V^l(\theta) := \ \mathrm{Span}_\mathbb{R}\{w_{1l}^\theta, \ldots, w_{t_\theta l}^\theta\}\,.$$

The ordered basis $\big(w_{1l}^\theta, \ldots, w_{t_\theta l}^\theta\big)$ of $V^l(\theta)$ is denoted by $\mathcal{C}^l(\theta)$.

Lemma 6.4.1. *For* $\mathbb{N}_{\mathbf{d}} = \mathbb{O}_{\mathbf{d}}$,

$$\mathcal{Z}_{\mathrm{SO}(p,q)}(X,H,Y) = \left\{ g \in \mathrm{SO}(p,q) \;\middle|\; \begin{array}{c} g(V^l(\theta)) \subset V^l(\theta) \text{ and} \\ \big[g|_{V^l(\theta)}\big]_{\mathcal{C}^l(\theta)} = \big[g|_{V^0(\theta)}\big]_{\mathcal{C}^0(\theta)} \, \forall\, \theta \in \mathbb{O}_{\mathbf{d}}, 0 \leq l < \theta \end{array} \right\}.$$

Proof. We omit the proof as it is identical to that of Lemma 6.3.1. □

We next impose orderings on the sets $\{v \in \mathcal{C}^l(\theta) \mid \langle v,v\rangle > 0\}$, $\{v \in \mathcal{C}^l(\theta) \mid \langle v,v\rangle < 0\}$. Define the ordered sets by $\mathcal{C}^l_+(\theta)$, $\mathcal{C}^l_-(\theta)$, $\mathcal{C}^l_+(\zeta)$ and $\mathcal{C}^l_-(\zeta)$ as in (6.11), (6.12), (6.13), (6.14), respectively according as $\theta \in \mathbb{O}^1_{\mathbf{d}}$ or $\zeta \in \mathbb{O}^3_{\mathbf{d}}$. For all $\theta \in \mathbb{O}_{\mathbf{d}}$ and $0 \leq l \leq \theta - 1$, set

$$V^l_+(\theta) := \mathrm{Span}_{\mathbb{R}}\{v \mid v \in \mathcal{C}^l(\theta), \langle v,v\rangle > 0\},\ V^l_-(\theta) := \mathrm{Span}_{\mathbb{R}}\{v \mid v \in \mathcal{C}^l(\theta), \langle v,v\rangle < 0\}.$$

It is straightforward from (6.19), and the orthogonality relations in Lemma 3.0.12, that $\mathcal{C}^l_+(\theta)$ and $\mathcal{C}^l_-(\theta)$ are indeed ordered bases of $V^l_+(\theta)$ and $V^l_-(\theta)$, respectively.

In the next lemma we specify a maximal compact subgroup of $\mathcal{Z}_{\mathrm{SO}(p,q)}(X,H,Y)$ in terms of the subspaces $V^l_+(\theta)$ and $V^l_-(\theta)$ defined as above which will be used in Proposition 6.4.4. As before, the notation $(-1)^l$ stands for the sign '+' or the sign '−' according as l is an even or odd integer.

Lemma 6.4.2. *Suppose that* $\mathbb{N}_{\mathbf{d}} = \mathbb{O}_{\mathbf{d}}$. *Let* K *be the subgroup of* $\mathcal{Z}_{\mathrm{SO}(p,q)}(X,H,Y)$ *consisting of all* $g \in \mathcal{Z}_{\mathrm{SO}(p,q)}(X,H,Y)$ *such that the following hold:*

1. $g(V^l_+(\theta)) \subset V^l_+(\theta)$ *and* $g(V^l_-(\theta)) \subset V^l_-(\theta)$, *for all* $\theta \in \mathbb{O}_{\mathbf{d}}$ *and* $0 \leq l \leq \theta - 1$.

2. When $\theta \in \mathbb{O}^1_{\mathbf{d}}$,

$$\left[g|_{V^0_+(\theta)}\right]_{\mathcal{C}^0_+(\theta)} = \begin{cases} \left[g|_{V^l_{(-1)^l}(\theta)}\right]_{\mathcal{C}^l_{(-1)^l}(\theta)} & \text{for all } 0 \leq l < (\theta-1)/2 \\ \left[g|_{V^{(\theta-1)/2}_+(\theta)}\right]_{\mathcal{C}^{(\theta-1)/2}_+(\theta)} & \\ \left[g|_{V^l_{(-1)^{l+1}}(\theta)}\right]_{\mathcal{C}^l_{(-1)^{l+1}}(\theta)} & \text{for all } (\theta-1)/2 < l \leq \theta-1, \end{cases}$$

$$\left[g|_{V^0_-(\theta)}\right]_{\mathcal{C}^0_-(\theta)} = \begin{cases} \left[g|_{V^l_{(-1)^{l+1}}(\theta)}\right]_{\mathcal{C}^l_{(-1)^{l+1}}(\theta)} & \text{for all } 0 \leq l < (\theta-1)/2 \\ \left[g|_{V^{(\theta-1)/2}_-(\theta)}\right]_{\mathcal{C}^{(\theta-1)/2}_-(\theta)} & \\ \left[g|_{V^l_{(-1)^l}(\theta)}\right]_{\mathcal{C}^l_{(-1)^l}(\theta)} & \text{for all } (\theta-1)/2 < l \leq \theta-1. \end{cases}$$

3. When $\zeta \in \mathbb{O}^3_{\mathbf{d}}$,

$$\left[g|_{V^0_+(\zeta)}\right]_{\mathcal{C}^0_+(\zeta)} = \begin{cases} \left[g|_{V^l_{(-1)^l}(\zeta)}\right]_{\mathcal{C}^l_{(-1)^l}(\zeta)} & \text{for all } 0 \leq l < (\zeta-1)/2 \\ \left[g|_{V^{(\zeta-1)/2}_-(\zeta)}\right]_{\mathcal{C}^{(\zeta-1)/2}_-(\zeta)} & \\ \left[g|_{V^l_{(-1)^{l+1}}(\zeta)}\right]_{\mathcal{C}^l_{(-1)^{l+1}}(\zeta)} & \text{for all } (\zeta-1)/2 < l \leq \zeta-1, \end{cases}$$

$$\left[g|_{V^0_-(\zeta)}\right]_{\mathcal{C}^0_-(\zeta)} = \begin{cases} \left[g|_{V^l_{(-1)^{l+1}}(\zeta)}\right]_{\mathcal{C}^l_{(-1)^{l+1}}(\zeta)} & \text{for all } 0 \leq l < (\zeta-1)/2 \\ \left[g|_{V^{(\zeta-1)/2}_+(\zeta)}\right]_{\mathcal{C}^{(\zeta-1)/2}_+(\zeta)} & \\ \left[g|_{V^l_{(-1)^l}(\zeta)}\right]_{\mathcal{C}^l_{(-1)^l}(\zeta)} & \text{for all } (\zeta-1)/2 < l \leq \zeta-1. \end{cases}$$

Then K is a maximal compact subgroup of $\mathcal{Z}_{\mathrm{SO}(p,q)}(X,H,Y)$.

Proof. We omit the proof as it is identical to the proof of Lemma 6.3.2. □

The following lemma is required in the proof of Theorem 6.4.9 (2)(iv). This is treated separately as $\mathbb{O}_{\mathbf{d}} \subsetneq \mathbb{N}_{\mathbf{d}}$. Recall that $\mathcal{B}^0(d)$ is an ordered basis of $L(d-1)$ as in (6.1) with $l = 0$ and satisfying Remark 3.0.11 (1).

Lemma 6.4.3. *Suppose that* $\Psi_{\mathrm{SO}(p,2)^\circ}(\mathcal{O}_X) = ([1^{p-2}, 2^2], ((m^1_{ij}), (m^2_{ij})))$*, where* (m^1_{ij}) *and* (m^2_{ij}) *are* $(p-2)\times 1$ *and* 2×2 *matrices, respectively, satisfying* $m^1_{i1} = +1$ *with* $1 \le i \le p-2$*,* $m^2_{i1} = +1$ *with* $1 \le i \le 2$*, and* **Yd**.*2. Let* K *be the subgroup of* $\mathcal{Z}_{\mathrm{SO}(p,2)}(X,H,Y)$ *consisting of all* $g \in \mathcal{Z}_{\mathrm{SO}(p,2)}(X,H,Y)$ *such that the following hold:*

1. $g(L(1)) \subset L(1)$*,* $g(XL(1)) \subset XL(1)$*,* $\left[g|_{L(1)}\right]_{\mathcal{B}^0(2)} = \left[g|_{XL(1)}\right]_{\mathcal{B}^1(2)}$ *and*
$$\left[g|_{L(1)}\right]_{\mathcal{B}^0(2)} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left[g|_{L(1)}\right]_{\mathcal{B}^0(2)}.$$
2. $g(L(0)) \subset L(0)$.

Then K *is a maximal compact subgroup of* $\mathcal{Z}_{\mathrm{SO}(p,2)}(X,H,Y)$.

Proof. Note that the form $(\cdot,\, \cdot)_1$ defined as in (3.8) is symmetric on $L(1-1) \times L(1-1)$ with signature $(p-2, 0)$, and the form $(\cdot,\, \cdot)_2$ defined as in (3.8) is symplectic on $L(2-1) \times L(2-1)$. Moreover, it follows from Proposition 3.0.7 that $\mathcal{B}^0(2) = (v^2_1\,;\, v^2_2)$ is a symplectic basis of $L(2-1)$ for $(\cdot,\, \cdot)_2$. Now the lemma follows from Lemma 6.0.1(4) and Lemma 6.6.2(1). □

We next introduce some notation which will be needed in Proposition 6.4.4 and in Proposition 6.4.5. We assume that $\mathbb{N}_{\mathbf{d}} = \mathbb{O}_{\mathbf{d}}$. For $\theta \in \mathbb{O}_{\mathbf{d}}$, define

$$\mathcal{C}_+(\theta) := \mathcal{C}^0_+(\theta) \vee \cdots \vee \mathcal{C}^{\theta-1}_+(\theta) \quad \text{and} \quad \mathcal{C}_-(\theta) := \mathcal{C}^0_-(\theta) \vee \cdots \vee \mathcal{C}^{\theta-1}_-(\theta)\,.$$

Let $\beta := \#\mathbb{O}^1_{\mathbf{d}}$ and $\gamma := \#\mathbb{O}^3_{\mathbf{d}}$. We enumerate $\mathbb{O}^1_{\mathbf{d}} = \{\theta_j \mid 1 \le j \le \beta\}$ such that $\theta_j < \theta_{j+1}$ and similarly $\mathbb{O}^3_{\mathbf{d}} = \{\zeta_j \mid 1 \le j \le \gamma\}$ such that $\zeta_j < \zeta_{j+1}$. Set

$$\mathcal{O}^1_+ := \mathcal{C}_+(\theta_1) \vee \cdots \vee \mathcal{C}_+(\theta_\beta)\,; \quad \mathcal{O}^3_+ := \mathcal{C}_+(\zeta_1) \vee \cdots \vee \mathcal{C}_+(\zeta_\gamma)\,;$$

$$\mathcal{O}^1_- := \mathcal{C}_-(\theta_1) \vee \cdots \vee \mathcal{C}_-(\theta_\beta) \quad \text{and} \quad \mathcal{O}^3_- := \mathcal{C}_-(\zeta_1) \vee \cdots \vee \mathcal{C}_-(\zeta_\gamma).$$

Now define

$$\mathcal{H}_+ := \mathcal{O}_+^1 \vee \mathcal{O}_+^3, \quad \mathcal{H}_- := \mathcal{O}_-^1 \vee \mathcal{O}_-^3 \quad \text{and} \quad \mathcal{H} := \mathcal{H}_+ \vee \mathcal{H}_- . \tag{6.20}$$

It is clear that $\mathcal{H}$ is a standard orthogonal basis of V such that $\mathcal{H}_+ = \{v \in \mathcal{H} \mid \langle v, v \rangle = 1\}$ and $\mathcal{H}_- = \{v \in \mathcal{H} \mid \langle v, v \rangle = -1\}$. In particular, $\#\mathcal{H}_+ = p$ and $\#\mathcal{H}_- = q$. From the definition of $\mathcal{H}_+$ and $\mathcal{H}_-$ as given in (6.20) we have the following relations:

$$\sum_{j=1}^{\beta} \big(\frac{\theta_j + 1}{2} p_{\theta_j} + \frac{\theta_j - 1}{2} q_{\theta_j}\big) + \sum_{k=1}^{\gamma} \big(\frac{\zeta_k - 1}{2} p_{\zeta_k} + \frac{\zeta_k + 1}{2} q_{\zeta_k}\big) = p$$

and

$$\sum_{j=1}^{\beta} \big(\frac{\theta_j - 1}{2} p_{\theta_j} + \frac{\theta_j + 1}{2} q_{\theta_j}\big) + \sum_{k=1}^{\gamma} \big(\frac{\zeta_k + 1}{2} p_{\zeta_k} + \frac{\zeta_k - 1}{2} q_{\zeta_k}\big) = q.$$

The $\mathbb{R}$-algebra $\prod_{j=1}^{\beta} \big(\mathrm{M}_{p_{\theta_j}}(\mathbb{R}) \times \mathrm{M}_{q_{\theta_j}}(\mathbb{R})\big) \times \prod_{k=1}^{\gamma} \big(\mathrm{M}_{p_{\zeta_k}}(\mathbb{R}) \times \mathrm{M}_{q_{\zeta_k}}(\mathbb{R})\big)$ is embedded in $\mathrm{M}_p(\mathbb{R})$ and in $\mathrm{M}_q(\mathbb{R})$ as follows:

$$\mathbf{D}_p \colon \prod_{j=1}^{\beta} \big(\mathrm{M}_{p_{\theta_j}}(\mathbb{R}) \times \mathrm{M}_{q_{\theta_j}}(\mathbb{R})\big) \times \prod_{k=1}^{\gamma} \big(\mathrm{M}_{p_{\zeta_k}}(\mathbb{R}) \times \mathrm{M}_{q_{\zeta_k}}(\mathbb{R})\big) \longrightarrow \mathrm{M}_p(\mathbb{R})$$

$$\big(C_{\theta_1}, D_{\theta_1}, \dots, C_{\theta_\beta}, D_{\theta_\beta}; E_{\zeta_1}, F_{\zeta_1}, \dots, E_{\zeta_\gamma}, F_{\zeta_\gamma}\big) \longmapsto$$

$$\bigoplus_{j=1}^{\beta} \Big(\big(C_{\theta_j} \oplus D_{\theta_j}\big)_{\blacktriangle}^{\frac{\theta_j - 1}{4}} \oplus C_{\theta_j} \oplus \big(C_{\theta_j} \oplus D_{\theta_j}\big)_{\blacktriangle}^{\frac{\theta_j - 1}{4}}\Big) \oplus \bigoplus_{k=1}^{\gamma} \Big(\big(E_{\zeta_k} \oplus F_{\zeta_k}\big)_{\blacktriangle}^{\frac{\zeta_k + 1}{4}} \oplus \big(F_{\zeta_k} \oplus E_{\zeta_k}\big)_{\blacktriangle}^{\frac{\zeta_k - 3}{4}} \oplus F_{\zeta_k}\Big)$$

and

$$\mathbf{D}_q \colon \prod_{j=1}^{\beta} \big(\mathrm{M}_{p_{\theta_j}}(\mathbb{R}) \times \mathrm{M}_{q_{\theta_j}}(\mathbb{R})\big) \times \prod_{k=1}^{\gamma} \big(\mathrm{M}_{p_{\zeta_k}}(\mathbb{R}) \times \mathrm{M}_{q_{\zeta_k}}(\mathbb{R})\big) \longrightarrow \mathrm{M}_q(\mathbb{R})$$

$$\big(C_{\theta_1}, D_{\theta_1}, \dots, C_{\theta_\beta}, D_{\theta_\beta}; E_{\zeta_1}, F_{\zeta_1}, \dots, E_{\zeta_\gamma}, F_{\zeta_\gamma}\big) \longmapsto$$

$$\bigoplus_{j=1}^{\beta}\Big(\big(D_{\theta_j}\oplus C_{\theta_j}\big)_{\blacktriangle}^{\frac{\theta_j-1}{4}}\oplus D_{\theta_j}\oplus\big(D_{\theta_j}\oplus C_{\theta_j}\big)_{\blacktriangle}^{\frac{\theta_j-1}{4}}\Big)\oplus\bigoplus_{k=1}^{\gamma}\Big(\big(F_{\zeta_k}\oplus E_{\zeta_k}\big)_{\blacktriangle}^{\frac{\zeta_k+1}{4}}\oplus\big(E_{\zeta_k}\oplus F_{\zeta_k}\big)_{\blacktriangle}^{\frac{\zeta_k-3}{4}}\oplus E_{\zeta_k}\Big).$$

Define two characters

$$\chi_p\colon \prod_{j=1}^{\beta}\big(\mathrm{O}_{p_{\theta_j}}\times\mathrm{O}_{q_{\theta_j}}\big)\times\prod_{k=1}^{\gamma}\big(\mathrm{O}_{p_{\zeta_k}}\times\mathrm{O}_{q_{\zeta_k}}\big)\longrightarrow\mathbb{R}\setminus\{0\}$$

$$\big(C_{\theta_1},D_{\theta_1},\dots,C_{\theta_\beta},D_{\theta_\beta};\,E_{\zeta_1},F_{\zeta_1},\dots,E_{\zeta_\gamma},F_{\zeta_\gamma}\big)\longmapsto$$

$$\prod_{j=1}^{\beta}(\det C_{\theta_j}^{\frac{\theta_j+1}{2}}\,\det D_{\theta_j}^{\frac{\theta_j-1}{2}})\prod_{k=1}^{\gamma}(\det E_{\zeta_k}^{\frac{\zeta_k-1}{2}}\,\det F_{\zeta_k}^{\frac{\zeta_k+1}{2}})=\prod_{j=1}^{\beta}\det C_{\theta_j}\prod_{k=1}^{\gamma}\det E_{\zeta_k}$$

and

$$\chi_q\colon \prod_{j=1}^{\beta}\big(\mathrm{O}_{p_{\theta_j}}\times\mathrm{O}_{q_{\theta_j}}\big)\times\prod_{k=1}^{\gamma}\big(\mathrm{O}_{p_{\zeta_k}}\times\mathrm{O}_{q_{\zeta_k}}\big)\longrightarrow\mathbb{R}\setminus\{0\}$$

$$\big(C_{\theta_1},D_{\theta_1},\dots,C_{\theta_\beta},D_{\theta_\beta};\,E_{\zeta_1},F_{\zeta_1},\dots,E_{\zeta_\gamma},F_{\zeta_\gamma}\big)\longmapsto$$

$$\prod_{j=1}^{\beta}(\det C_{\theta_i}^{\frac{\theta_i-1}{2}}\,\det D_{\theta_i}^{\frac{\theta_i+1}{2}})\prod_{k=1}^{\gamma}(\det E_{\zeta_k}^{\frac{\zeta_k+1}{2}}\,\det F_{\zeta_k}^{\frac{\zeta_k-1}{2}})=\prod_{j=1}^{\beta}\det D_{\theta_j}\prod_{k=1}^{\gamma}\det F_{\zeta_k}.$$

Let $\Lambda_{\mathcal{H}}:\mathrm{End}_{\mathbb{R}}\mathbb{R}^n\longrightarrow\mathrm{M}_n(\mathbb{R})$ be the isomorphism of $\mathbb{R}$-algebras induced by the ordered basis $\mathcal{H}$ in (6.20). Let M be the maximal compact subgroup of $\mathrm{SO}(p,q)$ which leaves invariant simultaneously the two subspaces spanned by $\mathcal{H}_+$ and $\mathcal{H}_-$. Clearly, $\Lambda_{\mathcal{H}}(M)=\mathrm{S}(\mathrm{O}(p)\times\mathrm{O}(q))$. In the next result we obtain an explicit description of $\Lambda_{\mathcal{H}}(K)$ in $\mathrm{S}(\mathrm{O}(p)\times\mathrm{O}(q))$ where $K\subset M$ is the maximal compact subgroup in the centralizer of the nilpotent element X, as in Lemma 6.4.2.

Proposition 6.4.4. *Let* $X\in\mathcal{N}_{\mathfrak{so}(p,q)}$, $\Psi_{\mathrm{SO}(p,q)^\circ}(\mathcal{O}_X)=(\mathbf{d},\mathbf{sgn}_{\mathcal{O}_X})$. *Assume that* $\mathbb{N}_{\mathbf{d}}=\mathbb{O}_{\mathbf{d}}$. *Let* $\beta:=\#\mathbb{O}^1_{\mathbf{d}}$ *and* $\gamma:=\#\mathbb{O}^3_{\mathbf{d}}$. *Let* $\{X,\,H,\,Y\}\subset\mathfrak{so}(p,q)$ *be a* $\mathfrak{sl}_2(\mathbb{R})$-*triple, and let* $(p_\theta,\,q_\theta)$ *be the signature of the form* $(\cdot,\,\cdot)_\theta$ *for all* $\theta\in\mathbb{O}_{\mathbf{d}}$ *as defined in* (3.8). *Let* K *be the maximal compact subgroup of* $\mathcal{Z}_{\mathrm{SO}(p,q)}(X,H,Y)$ *as in Lemma*

6.4.2. Then $\Lambda_{\mathcal{H}}(K) \subset \mathrm{S}(\mathrm{O}(p) \times \mathrm{O}(q))$ *is given by*

$$\Lambda_{\mathcal{H}}(K) = \left\{ \mathbf{D}_p(g) \oplus \mathbf{D}_q(g) \;\middle|\; \begin{array}{c} g \in \prod_{j=1}^{\beta} \left(\mathrm{O}_{p_{\theta_j}} \times \mathrm{O}_{q_{\theta_j}}\right) \times \prod_{k=1}^{\gamma} \left(\mathrm{O}_{p_{\zeta_k}} \times \mathrm{O}_{q_{\zeta_k}}\right) \\ \text{and} \quad \chi_p(g)\chi_q(g) = 1 \end{array} \right\}.$$

Proof. This follows by writing the matrices of the elements of the maximal compact subgroup K in Lemma 6.4.2 with respect to the basis $\mathcal{H}$ as in (6.20). □

As the subgroup $\mathrm{SO}(p,q)^\circ$ is normal in $\mathrm{SO}(p,q)$, so is $\mathcal{Z}_{\mathrm{SO}(p,q)^\circ}(X,H,Y)$ in $\mathcal{Z}_{\mathrm{SO}(p,q)}(X,H,Y)$. As K is a maximal compact subgroup in $\mathcal{Z}_{\mathrm{SO}(p,q)}(X,H,Y)$, it follows using Lemma 2.3.6 that $K_{\mathbb{O}} := K \cap \mathcal{Z}_{\mathrm{SO}(p,q)^\circ}(X,H,Y) = K \cap \mathrm{SO}(p,q)^\circ$ is a maximal compact subgroup of $\mathcal{Z}_{\mathrm{SO}(p,q)^\circ}(X,H,Y)$. The next proposition gives an explicit description of $\Lambda_{\mathcal{H}}(K_{\mathbb{O}})$ in $\mathrm{SO}(p) \times \mathrm{SO}(q)$.

Proposition 6.4.5. *Let* $X \in \mathcal{N}_{\mathfrak{so}(p,q)}$, $\Psi_{\mathrm{SO}(p,q)^\circ}(\mathcal{O}_X) = (\mathbf{d}, \mathbf{sgn}_{\mathcal{O}_X})$. *We assume that* $\mathbb{N}_{\mathbf{d}} = \mathbb{O}_{\mathbf{d}}$. *Let* $\beta := \#\mathbb{O}^1_{\mathbf{d}}$ *and* $\gamma := \#\mathbb{O}^3_{\mathbf{d}}$. *Let* $\{X,H,Y\} \subset \mathfrak{so}(p,q)$ *be a* $\mathfrak{sl}_2(\mathbb{R})$*-triple, and let* (p_θ, q_θ) *be the signature of the form* $(\cdot,\cdot)_\theta$ *for all* $\theta \in \mathbb{O}_{\mathbf{d}}$ *as defined in* (3.8). *Let* $K_{\mathbb{O}}$ *be the maximal compact subgroup of* $\mathcal{Z}_{\mathrm{SO}(p,q)^\circ}(X,H,Y)$ *as in the preceding paragraph. Then* $\Lambda_{\mathcal{H}}(K_{\mathbb{O}}) \subset \mathrm{SO}(p) \times \mathrm{SO}(q)$ *is given by*

$$\left\{ \mathbf{D}_p(g) \oplus \mathbf{D}_q(g) \;\middle|\; \begin{array}{c} g \in \prod_{j=1}^{\beta} \left(\mathrm{O}_{p_{\theta_j}} \times \mathrm{O}_{q_{\theta_j}}\right) \times \prod_{k=1}^{\gamma} \left(\mathrm{O}_{p_{\zeta_k}} \times \mathrm{O}_{q_{\zeta_k}}\right) \\ \text{and} \quad \chi_p(g) = 1, \ \chi_q(g) = 1 \end{array} \right\}.$$

Moreover, the above group is isomorphic to

$$S\Big(\prod_{j=1}^{\beta} \mathrm{O}_{p_{\theta_j}} \times \prod_{k=1}^{\gamma} \mathrm{O}_{p_{\zeta_k}}\Big) \times S\Big(\prod_{j=1}^{\beta} \mathrm{O}_{q_{\theta_j}} \times \prod_{k=1}^{\gamma} \mathrm{O}_{q_{\zeta_k}}\Big).$$

Proof. Let V_+ and V_- be the $\mathbb{R}$-spans of $\mathcal{H}_+$ and $\mathcal{H}_-$ respectively. Let M be the maximal compact subgroup in $\mathrm{SO}(p,q)$ which simultaneously leaves the subspaces V_+ and V_- invariant. It is clear that M° is a maximal compact subgroup of $\mathrm{SO}(p,q)^\circ$.

Hence

$$M^\circ = \mathrm{SO}(p,q)^\circ \cap M = \{g \in \mathrm{SO}(p,q) \mid \det g|_{V_+} = 1,\ \det g|_{V_-} = 1\}.$$

As $K \subset M$, we have that $K \cap \mathrm{SO}(p,q)^\circ = K \cap M^\circ$. The proposition now follows. □

For the next results we set :

$\mathbb{O}^-_{\mathbf{d}} := \{\theta \in \mathbb{O}_{\mathbf{d}} \mid (\cdot,\cdot)_\theta \text{ is negative definite}\}$, $\mathbb{O}^+_{\mathbf{d}} := \{\theta \in \mathbb{O}_{\mathbf{d}} \mid (\cdot,\cdot)_\theta \text{ is positive definite}\}$.

Lemma 6.4.6. *Let $X \in \mathfrak{so}(p,q)$ be a nilpotent element. Let $(\mathbf{d}, \mathbf{sgn}_{\mathcal{O}_X}) \in \mathcal{Y}^{\mathrm{even}}_1(p,q)$ be the signed Young diagram of the orbit $\mathcal{O}_X$ (that is, $\Psi_{\mathrm{SO}(p,q)^\circ}(\mathcal{O}_X) = (\mathbf{d}, \mathbf{sgn}_{\mathcal{O}_X})$ as in the notation of Theorem 4.1.6). We moreover assume that $\mathbb{N}_{\mathbf{d}} = \mathbb{O}_{\mathbf{d}}$. Let $K_{\mathbb{O}}$ be the maximal compact subgroup of $\mathcal{Z}_{\mathrm{SO}(p,q)^\circ}(X,H,Y)$ as in Proposition 6.4.5. Let $\mathfrak{k}_{\mathbb{O}}$ be the Lie algebra of $K_{\mathbb{O}}$. Then the following hold:*

1. *If $\#(\mathbb{O}_{\mathbf{d}} \setminus \mathbb{O}^-_{\mathbf{d}}) = 1$, $\#(\mathbb{O}_{\mathbf{d}} \setminus \mathbb{O}^+_{\mathbf{d}}) = 1$ and $p_{\theta_1} = q_{\theta_2} = 2$ for $\theta_1 \in \mathbb{O}_{\mathbf{d}} \setminus \mathbb{O}^-_{\mathbf{d}}$, $\theta_2 \in \mathbb{O}_{\mathbf{d}} \setminus \mathbb{O}^+_{\mathbf{d}}$, then $\dim_{\mathbb{R}} \left[\mathfrak{z}(\mathfrak{k}_{\mathbb{O}})\right]^{K_{\mathbb{O}}/K^\circ_{\mathbb{O}}} = 2$.*

2. *Suppose that either $\#(\mathbb{O}_{\mathbf{d}} \setminus \mathbb{O}^-_{\mathbf{d}}) = 1$, $p_{\theta_1} = 2$ for $\theta_1 \in \mathbb{O}_{\mathbf{d}} \setminus \mathbb{O}^-_{\mathbf{d}}$, or $\#(\mathbb{O}_{\mathbf{d}} \setminus \mathbb{O}^+_{\mathbf{d}}) = 1$, $q_{\theta_2} = 2$ for $\theta_2 \in \mathbb{O}_{\mathbf{d}} \setminus \mathbb{O}^+_{\mathbf{d}}$. Moreover, suppose that both the conditions do not hold simultaneously. Then $\dim_{\mathbb{R}} \left[\mathfrak{z}(\mathfrak{k}_{\mathbb{O}})\right]^{K_{\mathbb{O}}/K^\circ_{\mathbb{O}}} = 1$.*

3. *In all other cases, $\dim_{\mathbb{R}} \left[\mathfrak{z}(\mathfrak{k}_{\mathbb{O}})\right]^{K_{\mathbb{O}}/K^\circ_{\mathbb{O}}} = 0$.*

Proof. In view of (6.6), (6.7) and Proposition 6.4.5, the lemma is clear. □

Lemma 6.4.7. *Let W be a finite dimensional vector space over $\mathbb{R}$, and let $\langle \cdot\,, \cdot \rangle'$ be a non-degenerate symmetric bilinear form on W. Let $W_1, W_2 \subset W$ be subspaces such that $W_1 \perp W_2$ and $W = W_1 \oplus W_2$. Let $\langle \cdot\,, \cdot \rangle'_2$ be the restriction of $\langle \cdot\,, \cdot \rangle'$ to W_2.*

Then

$$\mathrm{SO}(W, \langle\cdot\,,\cdot\rangle')^\circ \cap \{g \in \mathrm{SO}(W, \langle\cdot\,,\cdot\rangle') \mid g(W_1) \subset W_1, g(W_2) \subset W_2, g|_{W_1} = \mathrm{Id}_{W_1}\}$$
$$=\{g \in \mathrm{SO}(W, \langle\cdot\,,\cdot\rangle') \mid g(W_1) \subset W_1, g(W_2) \subset W_2, g|_{W_1} = \mathrm{Id}_{W_1}, g|_{W_2} \in \mathrm{SO}(W_2, \langle\cdot\,,\cdot\rangle_2')^\circ\}.$$

In particular,

$$\mathrm{SO}(W, \langle\cdot\,,\cdot\rangle')^\circ \cap \{g \in \mathrm{SO}(W, \langle\cdot\,,\cdot\rangle') \mid g(W_1) \subset W_1,\ g(W_2) \subset W_2,\ g|_{W_1} = \mathrm{Id}_{W_1}\}$$

is isomorphic to $\mathrm{SO}(W_2, \langle\cdot\,,\cdot\rangle_2')^\circ$.

Proof. Let (p_2, q_2) be the signature of $\langle\cdot\,,\cdot\rangle_2'$. If either $p_2 = 0$ or $q_2 = 0$, then as $\mathrm{SO}(W_2, \langle\cdot\,,\cdot\rangle_2') = \mathrm{SO}(W_2, \langle\cdot\,,\cdot\rangle_2')^\circ$ the lemma follows immediately.

Assumption that $p_2 > 0$ and $q_2 > 0$. In this case, considering an orthogonal basis of W_2 for the form $\langle\cdot\,,\cdot\rangle_2'$ we easily construct a linear map $A : W \longrightarrow W$ such that $A|_{W_1} = \mathrm{Id}_{W_1}$, $A(W_2) \subset W_2$, $(A|_{W_2})^2 = \mathrm{Id}_{W_2}$, and $A|_{W_2} \in \mathrm{SO}(W_2, \langle\cdot\,,\cdot\rangle_2') \setminus \mathrm{SO}(W_2, \langle\cdot\,,\cdot\rangle_2')^\circ$. It is then clear that

$$A \in \mathrm{SO}(W, \langle\cdot\,,\cdot\rangle') \setminus \mathrm{SO}(W, \langle\cdot\,,\cdot\rangle')^\circ\,.$$

Let $\Gamma \subset \mathrm{GL}(W)$ be the subgroup generated by A and $\Gamma' \subset \mathrm{GL}(W_2)$ the subgroup generated by $A|_{W_2}$. It then follows that $\mathrm{SO}(W, \langle\cdot\,,\cdot\rangle') = \Gamma\,\mathrm{SO}(W, \langle\cdot\,,\cdot\rangle')^\circ$ and $\mathrm{SO}(W_2, \langle\cdot\,,\cdot\rangle_2') = \Gamma'\,\mathrm{SO}(W_2, \langle\cdot\,,\cdot\rangle_2')^\circ$. Now the lemma follows. □

We now describe the second cohomology groups of nilpotent orbits in $\mathfrak{so}(p,q)$ when $p > 0$, $q > 0$. As we will consider only simple Lie algebras, to ensure simplicity of $\mathfrak{so}(p,q)$, in view of [Kn, Theorem 6.105, p. 421] and isomorphisms (iv), (v), (vi), (ix), (x) in [He, Chapter X, §6, pp. 519-520], we need the additional restriction that $(p,q) \notin \{(1,1),(2,2)\}$.

Theorem 6.4.8. *Let* $p \neq 2$, $q \neq 2$ *and* $(p,q) \neq (1,1)$. *Let* $X \in \mathfrak{so}(p,q)$ *be a* *nilpotent element. Let* $(\mathbf{d},\, \mathbf{sgn}_{\mathcal{O}_X}) \in \mathcal{Y}_1^{\rm even}(p,q)$ *be the signed Young diagram of the orbit* $\mathcal{O}_X$ *(that is,* $\Psi_{{\rm SO}(p,q)^\circ}(\mathcal{O}_X) = (\mathbf{d},\, \mathbf{sgn}_{\mathcal{O}_X})$ *as in the notation of Theorem 4.1.6). Then the following hold:*

1. *If* $\#(\mathbb{O}_\mathbf{d} \setminus \mathbb{O}_\mathbf{d}^-) = 1$, $\#(\mathbb{O}_\mathbf{d} \setminus \mathbb{O}_\mathbf{d}^+) = 1$ *and* $p_{\theta_1} = q_{\theta_2} = 2$ *when* $\theta_1 \in \mathbb{O}_\mathbf{d} \setminus \mathbb{O}_\mathbf{d}^-$ *and* $\theta_2 \in \mathbb{O}_\mathbf{d} \setminus \mathbb{O}_\mathbf{d}^+$, *then* $\dim_\mathbb{R} H^2(\mathcal{O}_X,\, \mathbb{R}) = (\#\mathbb{E}_\mathbf{d} + 2)$.

2. *Suppose that either* $\#(\mathbb{O}_\mathbf{d} \setminus \mathbb{O}_\mathbf{d}^-) = 1$ *and* $p_{\theta_1} = 2$ *for* $\theta_1 \in \mathbb{O}_\mathbf{d} \setminus \mathbb{O}_\mathbf{d}^-$, *or* $\#(\mathbb{O}_\mathbf{d} \setminus \mathbb{O}_\mathbf{d}^+) = 1$ *and* $q_{\theta_2} = 2$ *for* $\theta_2 \in \mathbb{O}_\mathbf{d} \setminus \mathbb{O}_\mathbf{d}^+$. *Moreover, suppose that the above two conditions do not hold simultaneously. Then* $\dim_\mathbb{R} H^2(\mathcal{O}_X,\, \mathbb{R}) = (\#\mathbb{E}_\mathbf{d} + 1)$.

3. *In all other cases* $\dim_\mathbb{R} H^2(\mathcal{O}_X,\, \mathbb{R}) = \#\mathbb{E}_\mathbf{d}$.

Proof. Let $p+q = n$. As the theorem is evident when $X = 0$, we assume that $X \neq 0$.

Let $\{X,\, H,\, Y\} \subset \mathfrak{so}(p,q)$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple. Let $V := \mathbb{R}^n$ be the right $\mathbb{R}$-vector space of column vectors. We consider V as a ${\rm Span}_\mathbb{R}\{X,H,Y\}$-module via its natural $\mathfrak{so}(p,q)$-module structure. Let

$$V_\mathbb{E} := \bigoplus_{\eta \in \mathbb{E}_\mathbf{d}} M(\eta - 1); \quad V_\mathbb{O} := \bigoplus_{\theta \in \mathbb{O}_\mathbf{d}} M(\theta - 1).$$

Using Lemma 3.0.5 it follows that $V = V_\mathbb{E} \oplus V_\mathbb{O}$ is an orthogonal decomposition of V with respect to $\langle\cdot\,,\cdot\rangle$. Let $\langle\cdot\,,\cdot\rangle_\mathbb{E} := \langle\cdot\,,\cdot\rangle|_{V_\mathbb{E}\times V_\mathbb{E}}$ and $\langle\cdot\,,\cdot\rangle_\mathbb{O} := \langle\cdot\,,\cdot\rangle|_{V_\mathbb{O}\times V_\mathbb{O}}$. Let $X_\mathbb{E} := X|_{V_\mathbb{E}}$, $X_\mathbb{O} := X|_{V_\mathbb{O}}$, $H_\mathbb{E} := H|_{V_\mathbb{E}}$, $H_\mathbb{O} := H|_{V_\mathbb{O}}$, $Y_\mathbb{E} := Y|_{V_\mathbb{E}}$ and $Y_\mathbb{O} := Y|_{V_\mathbb{O}}$. Then we have the following natural isomorphism

$$(6.21) \quad \mathcal{Z}_{{\rm SO}(p,q)}(X,H,Y) \simeq \mathcal{Z}_{{\rm SO}(V_\mathbb{E},\langle\cdot\,,\cdot\rangle_\mathbb{E})}(X_\mathbb{E},H_\mathbb{E},Y_\mathbb{E}) \times \mathcal{Z}_{{\rm SO}(V_\mathbb{O},\langle\cdot\,,\cdot\rangle_\mathbb{O})}(X_\mathbb{O},H_\mathbb{O},Y_\mathbb{O}).$$

As, the form $(\cdot,\cdot)_\eta$ on $L(\eta-1)$ is non-degenerate and symplectic for all $\eta \in \mathbb{E}_\mathbf{d}$, it follows from Lemma 6.0.1 (4) that

$$\mathcal{Z}_{\mathrm{SO}(V_{\mathbb{E}},\langle\cdot,\cdot\rangle_{\mathbb{E}})}(X_{\mathbb{E}},H_{\mathbb{E}},Y_{\mathbb{E}}) \simeq \prod_{\eta\in\mathbb{E}_{\mathbf{d}}} \mathrm{Sp}(t_\eta/2,\,\mathbb{R}). \tag{6.22}$$

In particular, $\mathcal{Z}_{\mathrm{SO}(V_{\mathbb{E}},\langle\cdot,\cdot\rangle_{\mathbb{E}})}(X_{\mathbb{E}},H_{\mathbb{E}},Y_{\mathbb{E}})$ is connected, and hence using Lemma 6.4.7, (6.21) and (6.22) it follows that

$$\mathcal{Z}_{\mathrm{SO}(p,q)^\circ}(X,H,Y) \simeq \mathcal{Z}_{\mathrm{SO}(V_{\mathbb{E}},\langle\cdot,\cdot\rangle_{\mathbb{E}})}(X_{\mathbb{E}},H_{\mathbb{E}},Y_{\mathbb{E}}) \times \mathcal{Z}_{\mathrm{SO}(V_{\mathbb{O}},\langle\cdot,\cdot\rangle_{\mathbb{O}})^\circ}(X_{\mathbb{O}},H_{\mathbb{O}},Y_{\mathbb{O}}). \tag{6.23}$$

Let $K_{\mathbb{E}}$ be a maximal compact subgroup of $\mathcal{Z}_{\mathrm{SO}(V_{\mathbb{E}},\langle\cdot,\cdot\rangle_{\mathbb{E}})}(X_{\mathbb{E}},H_{\mathbb{E}},Y_{\mathbb{E}}) \simeq \prod_{\eta\in\mathbb{E}_{\mathbf{d}}} \mathrm{Sp}(t_\eta/2,\mathbb{R})$. Setting $\#\mathbb{O}_{\mathbf{d}} := r$, enumerate $\mathbb{O}_{\mathbf{d}} = \{a_1,\,\dots,\,a_r\}$ such that $a_i < a_{i+1}$ for all i. We next set $\mathbf{d}_{\mathbb{O}} := [a_1^{t_{a_1}},\,\dots,\,a_r^{t_{a_r}}]$. As $\sum_{d\in\mathbb{O}_{\mathbf{d}}} t_d d = \dim_{\mathbb{R}} V_{\mathbb{O}}$, we have $\mathbf{d}_{\mathbb{O}} \in \mathcal{P}(\dim_{\mathbb{R}} V_{\mathbb{O}})$. We recall that $K_{\mathbb{O}} := K \cap \mathcal{Z}_{\mathrm{SO}(p,q)^\circ}(X,H,Y) = K \cap \mathrm{SO}(p,q)^\circ$ is a maximal compact subgroup of $\mathcal{Z}_{\mathrm{SO}(p,q)^\circ}(X,H,Y)$, where K is the maximal compact subgroup of $\mathcal{Z}_{\mathrm{SO}(p,q)}(X,H,Y)$ as in Lemma 6.4.2. Let $\widetilde{K}$ be the image of $K_{\mathbb{O}}\times K_{\mathbb{E}}$ under the isomorphism in (6.23). It is evident that $\widetilde{K}$ is a maximal compact subgroup of $\mathcal{Z}_{\mathrm{SO}(p,q)^\circ}(X,H,Y)$. Let $\widetilde{M}$ be a maximal compact subgroup of $\mathrm{SO}(p,q)^\circ$ containing $\widetilde{K}$. Let $\widetilde{\mathfrak{k}}$ and $\widetilde{\mathfrak{m}}$ be the Lie algebras of $\widetilde{K}$ and $\widetilde{M}$ respectively. As $p \neq 2,\ q \neq 2$, we have $\widetilde{\mathfrak{m}} = [\widetilde{\mathfrak{m}},\,\widetilde{\mathfrak{m}}]$. Then using Theorem 5.2.2 it follows that, for all $X \neq 0$,

$$H^2(\mathcal{O}_X,\,\mathbb{R}) \;\simeq\; [\mathfrak{z}(\widetilde{\mathfrak{k}})^*]^{\widetilde{K}/\widetilde{K}^\circ}\,.$$

Let $\mathfrak{k}_{\mathbb{E}}, \mathfrak{k}_{\mathbb{O}}$ be the Lie algebras of $K_{\mathbb{E}}, K_{\mathbb{O}}$ respectively. As $K_{\mathbb{E}}$ is connected, in view of (6.7) we conclude that

$$[\mathfrak{z}(\widetilde{\mathfrak{k}})^*]^{\widetilde{K}/\widetilde{K}^\circ} \;\simeq\; \mathfrak{z}(\mathfrak{k}_{\mathbb{E}}) \oplus [\mathfrak{z}(\mathfrak{k}_{\mathbb{O}})]^{K_{\mathbb{O}}/K_{\mathbb{O}}^\circ}\,.$$

From (6.22) we have $\mathfrak{k}_{\mathbb{E}} \simeq \bigoplus_{\eta\in\mathbb{E}_{\mathbf{d}}} \mathfrak{u}(t_\eta/2)$. In particular, $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}_{\mathbb{E}}) = \#\mathbb{E}_{\mathbf{d}}$. As $\mathbb{N}_{\mathbf{d}_{\mathbb{O}}} = \mathbb{O}_{\mathbf{d}_{\mathbb{O}}}$, we use Lemma 6.4.6 to compute the dimension of $[\mathfrak{z}(\mathfrak{k}_{\mathbb{O}})]^{K_{\mathbb{O}}/K_{\mathbb{O}}^\circ}$. This completes the proof. ☐

We will next consider the remaining cases which are not covered in Theorem 6.4.8. These cases are: $(p, q) \in \{(2,1), (1,2)\}$; $p > 2,\ q = 2$ and $p = 2, q > 2$. Recall the definition of $\mathcal{Y}_1^{\rm even}(p,q)$ given in (2.11). If $p > 2$, then we list below the set of signed Young diagrams $\mathcal{Y}_1^{\rm even}(p,2)$ which correspond to non-zero nilpotent orbits in $\mathfrak{so}(p,2)$.

a.1 $\big([1^{p-1}, 3^1],\ ((m^1_{ij}), (m^3_{ij}))\big)$, where (m^1_{ij}) and (m^3_{ij}) are $(p-1) \times 1$ and 1×3 matrices respectively, satisfying $m^1_{i1} = +1, 1 \le i \le p-1;\ m^3_{i1} = +1, i = 1$ and **Yd**.2.

a.2 $\big([1^{p-1}, 3^1],\ ((m^1_{ij}), (m^3_{ij}))\big)$, where (m^1_{ij}) and (m^3_{ij}) are $(p-1) \times 1$ and 1×3 matrices respectively, satisfying $m^1_{i1} = +1, 1 \le i \le p-2,\ m^1_{i1} = -1, i = p-1;\ m^3_{i1} = -1, i = 1$ and **Yd**.2.

a.3 $\big([1^{p-3}, 5^1],\ ((m^1_{ij}), (m^5_{ij}))\big)$, where (m^1_{ij}) and (m^5_{ij}) are $(p-3) \times 1$ and 1×5 matrices respectively, satisfying $m^1_{i1} = +1, 1 \le i \le p-3;\ m^5_{i1} = +1, i = 1$ and **Yd**.2.

a.4 $\big([1^{p-2}, 2^2],\ ((m^1_{ij}), (m^2_{ij}))\big)$, where (m^1_{ij}) and (m^2_{ij}) are $(p-2) \times 1$ and 2×2 matrices respectively, satisfying $m^1_{i1} = +1, 1 \le i \le p-2;\ m^2_{i1} = +1, 1 \le i \le 2$ and **Yd**.2.

Similarly as above, if $q > 2$, then set $\mathcal{Y}_1^{\rm even}(2,q)$ consists of four elements which correspond to non-zero nilpotent orbits in $\mathfrak{so}(2,q)$. These are listed below:

b.1 $\big([1^{q-1}, 3^1],\ ((m^1_{ij}), (m^3_{ij}))\big)$, where (m^1_{ij}) and (m^3_{ij}) are $(q-1) \times 1$ and 1×3 matrices respectively, satisfying $m^1_{i1} = -1, 1 \le i \le q-1;\ m^3_{i1} = -1, i = 1$ and **Yd**.2.

b.2 $\big([1^{q-1}, 3^1],\ ((m^1_{ij}), (m^3_{ij}))\big)$, where (m^1_{ij}) and (m^3_{ij}) are $(q-1) \times 1$ and 1×3

matrices respectively, satisfying $m^1_{i1} = +1, i = 1,\ m^1_{i1} = -1, 2 \le i \le q-1;\ m^3_{i1} = +1, i = 1$ and **Yd**.2.

b.3 $\big([1^{q-3}, 5^1], ((m^1_{ij}), (m^5_{ij}))\big)$, where (m^1_{ij}) and (m^5_{ij}) are $(q-3)\times 1$ and 1×5 matrices respectively, satisfying $m^1_{i1} = -1, 1 \le i \le q-3;\ m^5_{i1} = -1, i = 1$ and **Yd**.2.

b.4 $\big([1^{q-2}, 2^2], ((m^1_{ij}), (m^2_{ij}))\big)$, where (m^1_{ij}) and (m^2_{ij}) are $(q-2)\times 1$ and 2×2 matrices respectively, satisfying $m^1_{i1} = -1, 1 \le i \le q-2;\ m^2_{i1} = +1, 1 \le i \le 2$ and **Yd**.2.

Theorem 6.4.9. *Let $\Psi_{\mathrm{SO}(p,q)^\circ} : \mathcal{N}(\mathrm{SO}(p,q)^\circ) \longrightarrow \mathcal{Y}_1^{\mathrm{even}}(p,q)$ be the parametrization in Theorem 4.1.6. Let $\mathcal{O}_X \in \mathcal{N}(\mathrm{SO}(p,q)^\circ)$. Then the following hold:*

1. *Suppose $(p,\, q) \in \{(2,1),\, (1,2)\}$, then $H^2(\mathcal{O}_X,\, \mathbb{R}) = 0$.*
2. *Assume that $p > 2,\ q = 2$.*
   (i) If $\Psi_{\mathrm{SO}(p,2)^\circ}(\mathcal{O}_X)$ is as in $(\mathbf{a}.1)$, then $\dim_{\mathbb{R}} H^2(\mathcal{O}_X,\, \mathbb{R}) = 0$.
   (ii) If $\Psi_{\mathrm{SO}(p,2)^\circ}(\mathcal{O}_X)$ is as in $(\mathbf{a}.2)$, then $\dim_{\mathbb{R}} H^2(\mathcal{O}_X,\, \mathbb{R}) = \begin{cases} 1 & \text{if } p = 4 \\ 0 & \text{otherwise.} \end{cases}$
   (iii) If $\Psi_{\mathrm{SO}(p,2)^\circ}(\mathcal{O}_X)$ is as in $(\mathbf{a}.3)$, then $\dim_{\mathbb{R}} H^2(\mathcal{O}_X,\, \mathbb{R}) = 0$.
   (iv) If $\Psi_{\mathrm{SO}(p,2)^\circ}(\mathcal{O}_X)$ is as in $(\mathbf{a}.4)$, then $\dim_{\mathbb{R}} H^2(\mathcal{O}_X,\, \mathbb{R}) = \begin{cases} 1 & \text{if } p = 4 \\ 0 & \text{otherwise.} \end{cases}$
3. *Assume $p = 2$ and $q > 2$.*
   (i) If $\Psi_{\mathrm{SO}(2,q)^\circ}(\mathcal{O}_X)$ is as in $(\mathbf{b}.1)$, then $\dim_{\mathbb{R}} H^2(\mathcal{O}_X,\, \mathbb{R}) = 0$.
   (ii) If $\Psi_{\mathrm{SO}(2,q)^\circ}(\mathcal{O}_X)$ is as in $(\mathbf{b}.2)$, then $\dim_{\mathbb{R}} H^2(\mathcal{O}_X,\, \mathbb{R}) = \begin{cases} 1 & \text{if } q = 4 \\ 0 & \text{otherwise.} \end{cases}$
   (iii) If $\Psi_{\mathrm{SO}(2,q)^\circ}(\mathcal{O}_X)$ is as in $(\mathbf{b}.3)$, then $\dim_{\mathbb{R}} H^2(\mathcal{O}_X,\, \mathbb{R}) = 0$.
   (iv) If $\Psi_{\mathrm{SO}(2,q)^\circ}(\mathcal{O}_X)$ is as in $(\mathbf{b}.4)$, then $\dim_{\mathbb{R}} H^2(\mathcal{O}_X,\, \mathbb{R}) = \begin{cases} 1 & \text{if } q = 4 \\ 0 & \text{otherwise.} \end{cases}$

Proof. As $X \neq 0$, we may assume that X lies in a $\mathfrak{sl}_2(\mathbb{R})$-triple, say $\{X, H, Y\}$, in $\mathfrak{so}(p, q)$.

Proof of (1): Let $\mathfrak{m}$ be the Lie algebra of a maximal compact subgroup of $\mathrm{SO}(p, q)^\circ$. As $(p, q) \in \{(2, 1), (1, 2)\}$, we have $[\mathfrak{m}, \mathfrak{m}] = 0$. Thus using Theorem 5.2.2 it follows that $H^2(\mathcal{O}_X, \mathbb{R}) = 0$.

Proof of (2): As $\mathbb{N}_{\mathbf{d}} = \mathbb{O}_{\mathbf{d}}$ in each of the cases (i), (ii) and (iii), we will use Proposition 6.4.5. Let K be the maximal compact subgroup of $\mathcal{Z}_{\mathrm{SO}(p,2)}(X, H, Y)$ as given in Lemma 6.4.2. Let M be the maximal compact subgroup of $\mathrm{SO}(p, 2)$ which leaves invariant simultaneously the two subspaces spanned by $\mathcal{H}_+$ and $\mathcal{H}_-$, where $\mathcal{H}_+$ and $\mathcal{H}_-$ are as in (6.20) with $q = 2$. Then $M^\circ = M \cap \mathrm{SO}(p, 2)$ is a maximal compact subgroup of $\mathrm{SO}(p, 2)^\circ$. Recall that $K_{\mathbb{O}} := K \cap M^\circ = K \cap \mathrm{SO}(p, 2)^\circ$ is a maximal compact subgroup of $\mathcal{Z}_{\mathrm{SO}(p,2)^\circ}(X, H, Y)$. Then, in the notation of Proposition 6.4.5, $\Lambda_{\mathcal{H}}(K_{\mathbb{O}}) \subset \mathrm{SO}(p) \times \mathrm{SO}(2)$. Let $\mathfrak{k}_{\mathbb{O}}$ and $\mathfrak{m}$ be the Lie algebras of $K_{\mathbb{O}}$ and M° respectively.

We now prove (i) of (2). Suppose $\Psi_{\mathrm{SO}(p,2)^\circ}(\mathcal{O}_X)$ is as in ($\mathbf{a}$.1). Using Proposition 6.4.5 it follows that

$$
\begin{aligned}
\Lambda_{\mathcal{H}}(K_{\mathbb{O}}) &= \big\{\mathbf{D}_p(g) \oplus \mathbf{D}_q(g) \bigm| g \in \mathrm{O}_{p-1} \times \mathrm{O}_1,\ \chi_p(g) = 1, \chi_q(g) = 1\big\} \\
&= \big\{C \oplus E \bigoplus E \oplus E \ \bigm|\ C \in \mathrm{O}_{p-1}, E \in \mathrm{O}_1,\ \det C \det E = 1\big\}.
\end{aligned} \tag{6.24}
$$

Therefore, $\mathfrak{z}(\mathfrak{k}_{\mathbb{O}}) \cap [\mathfrak{m}, \mathfrak{m}] = \mathfrak{so}_2$ when $p = 3$, and $\mathfrak{z}(\mathfrak{k}_{\mathbb{O}}) = 0$ when $p > 3$. From (6.24) it follows that $K_{\mathbb{O}} \simeq S(\mathrm{O}_2 \times \mathrm{O}_1)$ when $p = 3$. Since $\mathrm{O}_2/\mathrm{SO}_2$ acts non-trivially on $\mathfrak{so}_2$, when $p = 3$ we have $\big[\mathfrak{z}(\mathfrak{k}_{\mathbb{O}}) \cap [\mathfrak{m}, \mathfrak{m}]\big]^{K/K^\circ} = 0$. Thus using Theorem 5.2.2,

$$H^2(\mathcal{O}_X, \mathbb{R}) = 0$$

for all $p > 2$.

We next give a proof of (ii) of (2). Assume that $\Psi_{\mathrm{SO}(p,2)^\circ}(\mathcal{O}_X)$ is as in (**a**.2). Using Proposition 6.4.5 and notation therein,

$$\Lambda_{\mathcal{H}}(K_{\mathbb{O}}) = \{\mathbf{D}_p(g) \oplus \mathbf{D}_q(g) \mid g \in \mathrm{O}_{p-2} \times \mathrm{O}_1 \times \mathrm{O}_1,\ \chi_p(g) = 1, \chi_q(g) = 1\} \tag{6.25}$$

$$= \{C \oplus F \oplus F \bigoplus D \oplus F \ \mid\ C \in \mathrm{O}_{p-2}; D, F \in \mathrm{O}_1;\ \det C = 1, \det D \det F = 1\}.$$

It is clear from above that $\mathfrak{z}(\mathfrak{k}_{\mathbb{O}}) \cap [\mathfrak{m}, \mathfrak{m}] = \mathfrak{so}_2$ when $p = 4$ and $\mathfrak{z}(\mathfrak{k}_{\mathbb{O}}) = 0$ when $p \neq 4$, $p > 2$. When $p = 4$, then $K_{\mathbb{O}} \simeq \mathrm{SO}_2 \times S(\mathrm{O}_1 \times \mathrm{O}_1)$ from (6.25). As SO_2 acts trivially on $\mathfrak{so}_2$, using Theorem 5.2.2 we conclude that

$$\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = \begin{cases} 1 & \text{if } p = 4 \\ 0 & \text{otherwise.} \end{cases}$$

We now give a proof of (iii) of (2). Assume that $\Psi_{\mathrm{SO}(p,2)^\circ}(\mathcal{O}_X)$ is as in (**a**.3). Using Proposition 6.4.5 and notation therein,

$$\Lambda_{\mathcal{H}}(K_{\mathbb{O}}) = \{\mathbf{D}_p(g) \oplus \mathbf{D}_q(g) \mid g \in \mathrm{O}_{p-3} \times \mathrm{O}_1,\ \chi_p(g) = 1, \chi_q(g) = 1\}$$

$$= \{C \oplus E \oplus E \oplus E \bigoplus E \oplus E \ \mid\ C \in \mathrm{O}_{p-3}, E \in \mathrm{O}_1,\ \det C \det E = 1\}. \tag{6.26}$$

Therefore, we have $\mathfrak{z}(\mathfrak{k}_{\mathbb{O}}) \cap [\mathfrak{m}, \mathfrak{m}] = \mathfrak{so}_2$ when $p = 5$, and $\mathfrak{z}(\mathfrak{k}_{\mathbb{O}}) = 0$ for $p > 2, p \neq 5$. It follows from (6.26) that $K_{\mathbb{O}} \simeq S(\mathrm{O}_2 \times \mathrm{O}_1)$ when $p = 5$. Since $\mathrm{O}_2/\mathrm{SO}_2$ acts non-trivially on $\mathfrak{so}_2$, in the case when $p = 5$ we have $[\mathfrak{z}(\mathfrak{k}_{\mathbb{O}}) \cap [\mathfrak{m}, \mathfrak{m}]]^{K/K^\circ} = 0$. Thus in view of Theorem 5.2.2,

$$H^2(\mathcal{O}_X, \mathbb{R}) = 0$$

for all $p > 2$.

We now give a proof of (iv) of (2). Let $n = p+2$. Suppose $\Psi_{\mathrm{SO}(p,2)^\circ}(\mathcal{O}_X)$ is as in (**a**.4). We need to construct a standard orthogonal basis as done before. We follow the notation as in Lemma 6.4.3. Define, $\mathcal{A}_+(2) := \big((v_1^2 + Xv_2^2)/\sqrt{2}, (v_2^2 - Xv_1^2)/\sqrt{2}\big)$ and $\mathcal{A}_-(2) := \big((v_1^2 - Xv_2^2)/\sqrt{2}, (v_2^2 + Xv_1^2)/\sqrt{2}\big)$. Finally set $\mathcal{H}_+ := \mathcal{B}^0(1) \vee \mathcal{A}_+(2)$, $\mathcal{H}_- := \mathcal{A}_-(2)$ and $\mathcal{H} := \mathcal{H}_+ \vee \mathcal{H}_-$. Then it is clear that $\mathcal{H}$ is a standard orthogonal basis of V such that $\mathcal{H}_+ = \{v \in \mathcal{H} \mid \langle v, v\rangle = 1\}$ and $\mathcal{H}_- = \{v \in \mathcal{H} \mid \langle v, v\rangle = -1\}$. In particular, $\#\mathcal{H}_+ = p$ and $\#\mathcal{H}_- = 2$. Let $V_+(2)$, $V_-(2)$ be the spans of $\mathcal{A}_+(2)$, $\mathcal{A}_-(2)$ respectively. Let K be the maximal compact subgroup of $\mathcal{Z}_{\mathrm{SO}(p,2)}(X, H, Y)$ as in Lemma 6.4.3. We observe that if $g \in K$, then $g(V_+(2)) \subset V_+(2)$, $g(V_-(2)) \subset V_-(2)$ and

$$\big[g|_{V_+(2)}\big]_{\mathcal{A}_+(2)} = \big[g|_{V_-(2)}\big]_{\mathcal{A}_-(2)} = \big[g|_{L(1)}\big]_{\mathcal{B}^0(2)}.$$

Let $\Lambda_{\mathcal{H}} : \mathrm{End}_{\mathbb{R}}\mathbb{R}^n \longrightarrow \mathrm{M}_n(\mathbb{R})$ be the isomorphism of $\mathbb{R}$-algebras induced by the above ordered basis $\mathcal{H}$. Let M be the maximal compact subgroup in $\mathrm{SO}(p, 2)$ which simultaneously leaves the subspaces spanned by $\mathcal{H}_+$ and $\mathcal{H}_-$ invariant. Then $M^\circ = M \cap \mathrm{SO}(p, 2)^\circ$ is a maximal compact subgroup of $\mathrm{SO}(p, 2)^\circ$, and $\widetilde{K} := K \cap M^\circ$ is a maximal compact subgroup of $\mathcal{Z}_{\mathrm{SO}(p,2)^\circ}(X, H, Y)$. We have the following explicit description of $\Lambda_{\mathcal{H}}(\widetilde{K}) \subset \mathrm{SO}(p) \times \mathrm{SO}(2)$:

(6.27)

$$\Lambda_{\mathcal{H}}(\widetilde{K}) = \big\{A \oplus B \bigoplus B \mid A \in \mathrm{O}_{p-2}, B \in \mathrm{O}_2;\ \det A \det B = 1 \text{ and } \det B = 1\big\}.$$

In particular, $\widetilde{K} \simeq \mathrm{SO}_{p-2} \times \mathrm{SO}_2$. Let $\widetilde{\mathfrak{k}}$ and $\mathfrak{m}$ be the Lie algebras of $\widetilde{K}$ and M° respectively. From (6.27),

$$\mathfrak{z}(\widetilde{\mathfrak{k}}) \cap [\mathfrak{m}, \mathfrak{m}] = \begin{cases} \mathfrak{so}_2 & \text{if } p = 4 \\ 0 & \text{otherwise.} \end{cases}$$

As $\widetilde{K}$ is connected, the conclusion follows from Theorem 5.2.2. This completes the proof of (2).

The proofs of (3)(i), (3)(ii), (3)(iii) and (3)(iv) are similar to those of (2)(i), (2)(ii), (2)(iii) and (2)(iv) respectively and hence the details are omitted. □

6.5 Second cohomology of nilpotent orbits in $\mathfrak{so}^*(2n)$

Let n be a positive integer. In this section, we follow notation and parametrization of the nilpotent orbits in $\mathfrak{so}^*(2n)$ as in §4.1.5; see Theorem 4.1.8. Here we compute the second cohomology groups of nilpotent orbits in $\mathfrak{so}^*(2n)$ under the adjoint action of $\mathrm{SO}^*(2n)$. As $\mathrm{U}(n)$, being a maximal compact subgroup in $\mathrm{SO}^*(2n)$, is not semisimple, in view of Remark 6.0.3, we need to work out how a conjugate of a maximal compact subgroup of $\mathcal{Z}_{\mathrm{SO}^*(2n)}(X)$ is embedded in $\mathrm{U}(n)$, for an arbitrary nilpotent element $X \in \mathfrak{so}^*(2n)$. Throughout this section $\langle \cdot\,, \cdot \rangle$ denotes the skew-Hermitian form on $\mathbb{H}^n$ defined by $\langle x, y \rangle := \overline{x}^t \mathbf{j} \mathrm{I}_n y$, for $x, y \in \mathbb{H}^n$.

Let $0 \neq X \in \mathcal{N}_{\mathfrak{so}^*(2n)}$ and $\{X, H, Y\}$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple in $\mathfrak{so}^*(2n)$. Let $\Psi_{\mathrm{SO}^*(2n)}(\mathcal{O}_X) = \big(\mathbf{d}, \mathbf{sgn}_{\mathcal{O}_X}\big)$. Then $\Psi'_{\mathrm{SO}^*(2n)}(\mathcal{O}_X) = \mathbf{d}$. Recall that $\mathbf{sgn}_{\mathcal{O}_X}$ determines the signature of $(\cdot, \cdot)_\eta$ on $L(\eta - 1)$ for all $\eta \in \mathbb{E}_\mathbf{d}$; let (p_η, q_η) be the signature of $(\cdot, \cdot)_\eta$ on $L(\eta-1)$. Let $(v_1^d, \dots, v_{t_d}^d)$ be an ordered $\mathbb{H}$-basis of $L(d-1)$ as in Proposition 3.0.7. It now follows from Proposition 3.0.7(3)(a) that $(v_1^d, \dots, v_{t_d}^d)$ is an orthogonal basis of $L(d-1)$ for the form $(\cdot, \cdot)_d$ for all $d \in \mathbb{N}_\mathbf{d}$. We also assume that the vectors in the ordered basis $(v_1^d, \dots, v_{t_d}^d)$ satisfy the properties in Remark 3.0.11(3). Since $(\cdot, \cdot)_\theta$ is skew-Hermitian for all $\theta \in \mathbb{O}_\mathbf{d}$, using Lemma 2.3.4, we may assume that $(v_1^\theta, \dots, v_{t_\theta}^\theta)$ is a standard orthogonal basis for all $\theta \in \mathbb{O}_\mathbf{d}$. Thus

$$(v_j^\theta, v_j^\theta)_\theta = \mathbf{j} \quad \text{for all } 1 \leq j \leq t_\theta,\, \theta \in \mathbb{O}_\mathbf{d}. \tag{6.28}$$

In view of the signature of $(\cdot, \cdot)_\eta$, $\eta \in \mathbb{E}_{\mathbf{d}}$, we may assume that

$$(6.29) \qquad (v_j^\eta, v_j^\eta)_\eta = \begin{cases} +1 & \text{if } 1 \le j \le p_\eta \\ -1 & \text{if } p_\eta < j \le t_\eta. \end{cases}$$

For $\eta \in \mathbb{E}_{\mathbf{d}},\ 1 \le r \le p_\eta$, define

$$(6.30) \qquad w_{rl}^\eta := \begin{cases} \big(X^l v_r^\eta + X^{\eta-1-l} v_r^\eta \mathbf{j}\big)/\sqrt{2} & \text{if } l \text{ is even, } 0 \le l \le \eta/2 - 1 \\ \big(X^l v_r^\eta + X^{\eta-1-l} v_r^\eta \mathbf{j}\big)\mathbf{i}/\sqrt{2} & \text{if } l \text{ is odd, } 0 \le l \le \eta/2 - 1 \\ \big(X^{\eta-1-l} v_r^\eta - X^l v_r^\eta \mathbf{j}\big)\mathbf{i}/\sqrt{2} & \text{if } l \text{ is odd, } \eta/2 \le l \le \eta - 1 \\ \big(X^{\eta-1-l} v_r^\eta - X^l v_r^\eta \mathbf{j}\big)/\sqrt{2} & \text{if } l \text{ is even, } \eta/2 \le l \le \eta - 1. \end{cases}$$

Similarly for $\eta \in \mathbb{E}_{\mathbf{d}},\ p_\eta < r \le t_\eta$, define

$$(6.31) \qquad w_{rl}^\eta := \begin{cases} \big(X^l v_r^\eta + X^{\eta-1-l} v_r^\eta \mathbf{j}\big)\mathbf{i}/\sqrt{2} & \text{if } l \text{ is even, } 0 \le l \le \eta/2 - 1 \\ \big(X^l v_r^\eta + X^{\eta-1-l} v_r^\eta \mathbf{j}\big)/\sqrt{2} & \text{if } l \text{ is odd, } 0 \le l \le \eta/2 - 1 \\ \big(X^{\eta-1-l} v_r^\eta - X^l v_r^\eta \mathbf{j}\big)/\sqrt{2} & \text{if } l \text{ is odd, } \eta/2 \le l \le \eta - 1 \\ \big(X^{\eta-1-l} v_r^\eta - X^l v_r^\eta \mathbf{j}\big)\mathbf{i}/\sqrt{2} & \text{if } l \text{ is even, } \eta/2 \le l \le \eta - 1. \end{cases}$$

Using (6.29) we observe that for all $\eta \in \mathbb{E}_{\mathbf{d}}$,

$$\{w_{rl}^\eta \mid 0 \le l \le \eta - 1,\ 1 \le r \le t_\eta\}$$

is an orthogonal basis of $M(\eta - 1)$ with respect to $\langle \cdot\,, \cdot \rangle$, where $\langle w_{rl}^\eta, w_{rl}^\eta \rangle = \mathbf{j}$ for $0 \le l \le \eta - 1$, $1 \le r \le t_\eta$. For $\eta \in \mathbb{E}_{\mathbf{d}},\ 0 \le l \le \eta/2 - 1$, set

$$(6.32) \qquad W^l(\eta) := \mathrm{Span}_{\mathbb{H}}\{w_{r\,l}^\eta, w_{r\,\eta-1-l}^\eta \mid 1 \le r \le t_\eta\}.$$

Moreover, we define a standard orthogonal basis $\mathcal{D}^l(\eta)$ of $W^l(\eta)$ with respect to $\langle \cdot\,, \cdot \rangle$

as follows:

(6.33)

$$\mathcal{D}^l(\eta) := \begin{cases} \left(w^\eta_{1\,l}, \ldots, w^\eta_{p_\eta\,l}\right) \vee \left(w^\eta_{1\,(\eta-1-l)}, \ldots, w^\eta_{p_\eta\,(\eta-1-l)}\right) & \\ \bigvee \left(w^\eta_{(p_\eta+1)\,(\eta-1-l)}, \ldots, w^\eta_{t_\eta\,(\eta-1-l)}\right) \vee \left(w^\eta_{(p_\eta+1)\,l}, \ldots, w^\eta_{t_\eta\,l}\right) & \text{if } l \text{ is even} \\ \left(w^\eta_{1\,(\eta-1-l)}, \ldots, w^\eta_{p_\eta\,(\eta-1-l)}\right) \vee \left(w^\eta_{1\,l}, \ldots, w^\eta_{p_\eta\,l}\right) & \\ \bigvee \left(w^\eta_{(p_\eta+1)\,l}, \ldots, w^\eta_{t_\eta\,l}\right) \vee \left(w^\eta_{(p_\eta+1)\,(\eta-1-l)}, \ldots, w^\eta_{t_\eta\,(\eta-1-l)}\right) & \text{if } l \text{ is odd.} \end{cases}$$

Now fixing $\theta \in \mathbb{O}^1_{\mathbf{d}}$, for all $1 \le r \le t_\theta$, define

$$w^\theta_{rl} := \begin{cases} \left(X^l v^\theta_r + X^{\theta-1-l} v^\theta_r\right)/\sqrt{2} & \text{if } l \text{ is even, } 0 \le l < (\theta-1)/2 \\ \left(X^l v^\theta_r + X^{\theta-1-l} v^\theta_r\right)\mathbf{i}/\sqrt{2} & \text{if } l \text{ is odd, } 0 \le l < (\theta-1)/2 \\ X^l v^\theta_r & \text{if } l = (\theta-1)/2 \\ \left(X^{\theta-1-l} v^\theta_r - X^l v^\theta_r\right)/\sqrt{2} & \text{if } l \text{ is odd, } (\theta+1)/2 \le l \le \theta-1 \\ \left(X^{\theta-1-l} v^\theta_r - X^l v^\theta_r\right)\mathbf{i}/\sqrt{2} & \text{if } l \text{ is even, } (\theta+1)/2 \le l \le \theta-1. \end{cases}$$

For all $\zeta \in \mathbb{O}^3_{\mathbf{d}}$ and $1 \le r \le t_\zeta$, define

$$w^\zeta_{rl} := \begin{cases} \left(X^l v^\zeta_r + X^{\zeta-1-l} v^\zeta_r\right)/\sqrt{2} & \text{if } l \text{ is even, } 0 \le l < (\zeta-1)/2 \\ \left(X^l v^\zeta_r + X^{\zeta-1-l} v^\zeta_r\right)\mathbf{i}/\sqrt{2} & \text{if } l \text{ is odd, } 0 \le l < (\zeta-1)/2 \\ X^l v^\zeta_r \mathbf{i} & \text{if } l = (\zeta-1)/2 \\ \left(X^{\zeta-1-l} v^\zeta_r - X^l v^\zeta_r\right)/\sqrt{2} & \text{if } l \text{ is odd, } (\zeta+1)/2 \le l \le \zeta-1 \\ \left(X^{\zeta-1-l} v^\zeta_r - X^l v^\zeta_r\right)\mathbf{i}/\sqrt{2} & \text{if } l \text{ is even, } (\zeta+1)/2 \le l \le \zeta-1. \end{cases}$$

Using (6.28) we observe that for all $\theta \in \mathbb{O}_{\mathbf{d}}$,

$$\{w^\theta_{rl} \mid 0 \le l \le \theta-1,\ 1 \le r \le t_\theta\}$$

is an orthogonal basis of $M(\theta-1)$ with respect to $\langle \cdot\,, \cdot \rangle$, where $\langle w^\theta_{rl}, w^\theta_{rl} \rangle = \mathbf{j}$ for

$0 \le l \le \theta - 1$, $1 \le r \le t_\theta$. For each $\theta \in \mathbb{O}_{\mathbf{d}}$, $0 \le l \le \theta - 1$, set

$$V^l(\theta) := \mathrm{Span}_{\mathbb{H}}\{w^\theta_{rl} \mid 1 \le r \le t_\theta\}. \tag{6.34}$$

The standard orthogonal ordered basis $(w^\theta_{1l}, \ldots, w^\theta_{t_\theta l})$ of $V^l(\theta)$ with respect to $\langle \cdot\,, \cdot \rangle$ is denoted by $\mathcal{C}^l(\theta)$.

Let W be a right $\mathbb{H}$-vector space and $\langle \cdot\,, \cdot \rangle'$ be a non-degenerate skew-Hermitian form on W. Let $\dim_{\mathbb{H}} W = m$, and let $\mathcal{B}' := (v_1, \ldots, v_m)$ be a standard orthogonal basis of W such that $\langle v_r, v_r \rangle' = \mathbf{j}$ for all $1 \le r \le m$. Define

$$\mathrm{J}_{\mathcal{B}'} : W \longrightarrow W\,, \quad \sum_r v_r z_r \longmapsto \sum_r v_r \mathbf{j} z_r$$

for all column vectors $(z_1, \ldots, z_m)^t \in \mathbb{H}^m$. In the next lemma we recall an explicit description of maximal compact subgroup in the group $\mathrm{SO}^*(W, \langle \cdot\,, \cdot \rangle')$. Set

$$K_{\mathcal{B}'} := \big\{ g \in \mathrm{SO}^*(W, \langle \cdot\,, \cdot \rangle') \bigm| g\mathrm{J}_{\mathcal{B}'} = \mathrm{J}_{\mathcal{B}'} g \big\}.$$

The following lemma is standard; its proof is omitted.

Lemma 6.5.1. *Let W, $\langle \cdot\,, \cdot \rangle'$ and $\mathcal{B}'$ be as above. Then the following hold:*

1. *$K_{\mathcal{B}'}$ is a maximal compact subgroup of* $\mathrm{SO}^*(W, \langle \cdot\,, \cdot \rangle')$.
2. $K_{\mathcal{B}'} = \big\{ g \in \mathrm{SL}(W) \bigm| [g]_{\mathcal{B}'} = A + \mathbf{j}B$ *with* $A, B \in \mathrm{M}_m(\mathbb{R}), A + \sqrt{-1}B \in \mathrm{U}(m) \big\}$.

Recall that $\big\{ x \in \mathrm{End}_{\mathbb{H}} W \mid x\mathrm{J}_{\mathcal{B}'} = \mathrm{J}_{\mathcal{B}'} x \big\} = \big\{ x \in \mathrm{End}_{\mathbb{H}} W \bigm| [x]_{\mathcal{B}'} \in \mathrm{M}_m(\mathbb{R}) + \mathbf{j}\mathrm{M}_m(\mathbb{R}) \big\}$. We now consider the $\mathbb{R}$-algebra isomorphism

$$\Lambda'_{\mathcal{B}'} : \big\{ x \in \mathrm{End}_{\mathbb{H}} W \mid x\mathrm{J}_{\mathcal{B}'} = \mathrm{J}_{\mathcal{B}'} x \big\} \longrightarrow \mathrm{M}_m(\mathbb{C})\,, \quad x \longmapsto A + \sqrt{-1}B\,, \tag{6.35}$$

where A, $B \in \mathrm{M}_m(\mathbb{R})$ are the unique elements such that $[x]_{\mathcal{B}} = A + \mathbf{j}B$. In view of the above lemma it is clear that $\Lambda'_{\mathcal{B}'}(K_{\mathcal{B}'}) = \mathrm{U}(m)$, and hence $\Lambda'_{\mathcal{B}'} : K_{\mathcal{B}'} \longrightarrow \mathrm{U}(m)$ is an isomorphism of Lie groups.

In the next lemma we specify a maximal compact subgroup of $\mathcal{Z}_{\mathrm{SO}^*(2n)}(X, H, Y)$ which will be used in Proposition 6.5.3. Recall that $\overline{Z} := (\sigma_c(z_{rl})) \in \mathrm{M}_m(\mathbb{H})$; see Section 2.3.

Lemma 6.5.2. *Let K be the subgroup of $\mathcal{Z}_{\mathrm{SO}^*(2n)}(X, H, Y)$ consisting of all elements $g \in \mathcal{Z}_{\mathrm{SO}^*(2n)}(X, H, Y)$ satisfying the following conditions:*

1. $g\big(V^l(\theta)\big) \subset V^l(\theta)$ *for all* $\theta \in \mathbb{O}_{\mathbf{d}}$ *and* $0 \leq l \leq \theta - 1$.
2. *For all $\theta \in \mathbb{O}^1_{\mathbf{d}}$, there exist $A_\theta, B_\theta \in \mathrm{M}_{t_\theta}(\mathbb{R})$ with $A_\theta + \sqrt{-1}B_\theta \in \mathrm{U}(t_\theta)$ such that*

$$[g|_{V^l(\theta)}]_{\mathcal{C}^l(\theta)} = \begin{cases} A_\theta + \mathbf{j}B_\theta & \text{if } l \text{ is even, } 0 \leq l < (\theta-1)/2 \\ A_\theta - \mathbf{j}B_\theta & \text{if } l \text{ is odd, } 0 \leq l < (\theta-1)/2 \\ A_\theta + \mathbf{j}B_\theta & \text{if } l = (\theta-1)/2 \\ A_\theta + \mathbf{j}B_\theta & \text{if } l \text{ is odd, } (\theta+1)/2 \leq l \leq \theta - 1 \\ A_\theta - \mathbf{j}B_\theta & \text{if } l \text{ is even, } (\theta+1)/2 \leq l \leq \theta - 1. \end{cases}$$

3. *For all $\zeta \in \mathbb{O}^3_{\mathbf{d}}$, there exist $A_\zeta, B_\zeta \in \mathrm{M}_{t_\zeta}(\mathbb{R})$ with $A_\zeta + \sqrt{-1}B_\zeta \in \mathrm{U}(t_\zeta)$ such that*

$$[g|_{V^l(\zeta)}]_{\mathcal{C}^l(\zeta)} = \begin{cases} A_\zeta + \mathbf{j}B_\zeta & \text{if } l \text{ is even, } 0 \leq l < (\zeta-1)/2 \\ A_\zeta - \mathbf{j}B_\zeta & \text{if } l \text{ is odd, } 0 \leq l < (\zeta-1)/2 \\ A_\zeta - \mathbf{j}B_\zeta & \text{if } l = (\zeta-1)/2 \\ A_\zeta + \mathbf{j}B_\zeta & \text{if } l \text{ is odd, } (\zeta+1)/2 \leq l \leq \zeta - 1 \\ A_\zeta - \mathbf{j}B_\zeta & \text{if } l \text{ is even, } (\zeta+1)/2 \leq l \leq \zeta - 1. \end{cases}$$

4. $g(W^l(\eta)) \subset W^l(\eta)$ *for all* $\eta \in \mathbb{E}_{\mathbf{d}}$ *and* $0 \le l \le \eta/2 - 1$.

5. For all $\eta \in \mathbb{E}_{\mathbf{d}}$*, there exist* $A_{p_\eta}, B_{p_\eta}, C_{p_\eta}, D_{p_\eta} \in \mathrm{M}_{p_\eta}(\mathbb{R})$, $A'_{q_\eta}, B'_{q_\eta}, C'_{q_\eta}, D'_{q_\eta} \in \mathrm{M}_{q_\eta}(\mathbb{R})$ *with* $A_{p_\eta} + \mathbf{j}B_{p_\eta} + \mathbf{i}(C_{p_\eta} + \mathbf{j}D_{p_\eta}) \in \mathrm{Sp}(p_\eta)$ *and* $A'_{q_\eta} + \mathbf{j}B'_{q_\eta} + \mathbf{i}(C'_{q_\eta} + \mathbf{j}D'_{q_\eta}) \in \mathrm{Sp}(q_\eta)$ *such that*

$$\big[g|_{W^l(\eta)}\big]_{\mathcal{D}^l(\eta)} = \begin{pmatrix} A_{p_\eta} + \mathbf{j}B_{p_\eta} & -C_{p_\eta} + \mathbf{j}D_{p_\eta} & & \\ C_{p_\eta} + \mathbf{j}D_{p_\eta} & A_{p_\eta} - \mathbf{j}B_{p_\eta} & & \\ & & A'_{q_\eta} + \mathbf{j}B'_{q_\eta} & -C'_{q_\eta} + \mathbf{j}D'_{q_\eta} \\ & & C'_{q_\eta} + \mathbf{j}D'_{q_\eta} & A'_{q_\eta} - \mathbf{j}B'_{q_\eta} \end{pmatrix}.$$

Then K *is a maximal compact subgroup of* $\mathcal{Z}_{\mathrm{SO}^*(2n)}(X, H, Y)$.

Proof. Let $\mathcal{B}^l(d) = (X^l v_1^d, \ldots, X^l v_{t_d}^d)$ be the ordered basis of $X^l L(d-1)$ for $0 \le l \le d-1$, $d \in \mathbb{N}_{\mathbf{d}}$, as in (6.1). Let K' be the subgroup consisting of all $g \in \mathcal{Z}_{\mathrm{SO}^*(2n)}(X, H, Y)$ satisfying the following properties:

(6.36) For $\theta \in \mathbb{O}_{\mathbf{d}}, 0 \le l \le \theta - 1,\ g\big(V^l(\theta)\big) \subset V^l(\theta)$,

(6.37) $g|_{V^0(\theta)}$ commutes with $\mathrm{J}_{\mathcal{C}^0(\theta)}$,

(6.38)

$$\text{For all } \theta \in \mathbb{O}^1_{\mathbf{d}},\ \big[g|_{V^l(\theta)}\big]_{\mathcal{C}^l(\theta)} = \begin{cases} \big[g|_{V^0(\theta)}\big]_{\mathcal{C}^0(\theta)} & \text{if } l \text{ is even, } 0 \le l < (\theta-1)/2 \\ \overline{\big[g|_{V^0(\theta)}\big]}_{\mathcal{C}^0(\theta)} & \text{if } l \text{ is odd, } 0 \le l < (\theta-1)/2 \\ \big[g|_{V^0(\theta)}\big]_{\mathcal{C}^0(\theta)} & \text{if } l = (\theta-1)/2 \\ \big[g|_{V^0(\theta)}\big]_{\mathcal{C}^0(\theta)} & \text{if } l \text{ is odd, } (\theta+1)/2 \le l \le \theta-1 \\ \overline{\big[g|_{V^0(\theta)}\big]}_{\mathcal{C}^0(\theta)} & \text{if } l \text{ is even, } (\theta+1)/2 \le l \le \theta-1, \end{cases}$$

(6.39)

$$\text{For all } \zeta \in \mathbb{O}^3_{\mathbf{d}},\ \big[g|_{V^l(\zeta)}\big]_{\mathcal{C}^l(\zeta)} = \begin{cases} \big[g|_{V^0(\zeta)}\big]_{\mathcal{C}^0(\zeta)} & \text{if } l \text{ is even, } 0 \le l < (\zeta-1)/2 \\ \overline{\big[g|_{V^0(\zeta)}\big]}_{\mathcal{C}^0(\zeta)} & \text{if } l \text{ is odd, } 0 \le l < (\zeta-1)/2 \\ \overline{\big[g|_{V^0(\zeta)}\big]}_{\mathcal{C}^0(\zeta)} & \text{if } l = (\zeta-1)/2 \\ \big[g|_{V^0(\zeta)}\big]_{\mathcal{C}^0(\zeta)} & \text{if } l \text{ is odd, } (\zeta+1)/2 \le l \le \zeta-1 \\ \overline{\big[g|_{V^0(\zeta)}\big]}_{\mathcal{C}^0(\zeta)} & \text{if } l \text{ is even, } (\zeta+1)/2 \le l \le \zeta-1, \end{cases}$$

(6.40)

$$g\big(X^l L(\eta-1)\big) \subset X^l L(\eta-1),\ \big[g|_{X^l L(\eta-1)}\big]_{\mathcal{B}^l(\eta)} = \big[g|_{L(\eta-1)}\big]_{\mathcal{B}^0(\eta)}$$

if $\eta \in \mathbb{E}_{\mathbf{d}}, 0 \le l \le \eta-1$;

(6.41)

$g(W^l(\eta)) \subset W^l(\eta)$ for $\eta \in \mathbb{E}_{\mathbf{d}}, 0 \le l \le \eta/2-1,$ and $g|_{W^0(\eta)}$ commutes with $\mathrm{J}_{\mathcal{D}^0(\eta)}$.

Using Lemma 6.5.1(1) it is evident that K' is a maximal compact subgroup of $\mathcal{Z}_{\mathrm{SO}^*(2n)}(X,H,Y)$. Hence to prove the lemma it suffices to show that $K = K'$. Let $g \in \mathrm{SO}^*(2n)$. From Lemma 6.5.1(2) it is straightforward that g satisfies (1), (2), (3) of Lemma 6.5.2 if and only if g satisfies (6.36), (6.38), (6.39) and (6.37). Now suppose that $g \in \mathrm{SO}^*(2n)$ and g satisfying (4), (5) of Lemma 6.5.2. It is clear that

(6.41) holds. We observe that

$$\left[g|_{L(\eta-1)}\right]_{\mathcal{B}^0(\eta)} = \begin{pmatrix} A_{p_\eta} + \mathbf{j}B_{p_\eta} + \mathbf{i}(C_{p_\eta} + \mathbf{j}D_{p_\eta}) & 0 \\ 0 & A'_{q_\eta} + \mathbf{j}B'_{q_\eta} + \mathbf{i}(C'_{q_\eta} + \mathbf{j}D'_{q_\eta}) \end{pmatrix}.$$

This proves that (6.40) holds.

Now we assume that g satisfies (6.40) and (6.41). Let $A := \left[g|_{L(\eta-1)}\right]_{\mathcal{B}^0(\eta)}$. Then $A = \left[g|_{X^l L(\eta-1)}\right]_{\mathcal{B}^l(\eta)}$ for $1 \le l \le \eta - 1$. We observe that

$$\left[\mathrm{J}_{\mathcal{D}^0(\eta)}\right]_{\mathcal{B}^0(\eta)\vee\mathcal{B}^{\eta-1}(\eta)} = \begin{pmatrix} & \mathrm{I}_{p_\eta,q_\eta} \\ -\mathrm{I}_{p_\eta,q_\eta} & \end{pmatrix} \quad\text{and}\quad \left[g|_{W^0(\eta)}\right]_{\mathcal{B}^0(\eta)\vee\mathcal{B}^{\eta-1}(\eta)} = \begin{pmatrix} A & \\ & A \end{pmatrix}.$$

From (6.41) it follows that the above two matrices commute, which in turn implies that A commutes with $\begin{pmatrix} \mathrm{I}_{p_\eta} & \\ & -\mathrm{I}_{q_\eta} \end{pmatrix}$. Thus A is of the form $A = \begin{pmatrix} E_{p_\eta} & 0 \\ 0 & F_{q_\eta} \end{pmatrix}$ for some matrices $E_{p_\eta} \in \mathrm{GL}_{p_\eta}(\mathbb{H})$ and $F_{q_\eta} \in \mathrm{GL}_{q_\eta}(\mathbb{H})$. Write $E_{p_\eta} = A_{p_\eta} + \mathbf{j}B_{p_\eta} + \mathbf{i}(C_{p_\eta} + \mathbf{j}D_{p_\eta})$ and $F_{q_\eta} = A'_{q_\eta} + \mathbf{j}B'_{q_\eta} + \mathbf{i}(C'_{q_\eta} + \mathbf{j}D'_{q_\eta})$ where $A_{p_\eta}, B_{p_\eta}, C_{p_\eta}, D_{p_\eta} \in \mathrm{M}_{p_\eta}(\mathbb{R})$, $A'_{q_\eta}, B'_{q_\eta}, C'_{q_\eta}, D'_{q_\eta} \in \mathrm{M}_{q_\eta}(\mathbb{R})$. We now observe that

$$\left[g|_{W^l(\eta)}\right]_{\mathcal{D}^l(\eta)} = \begin{pmatrix} A_{p_\eta} + \mathbf{j}B_{p_\eta} & -C_{p_\eta} + \mathbf{j}D_{p_\eta} & & \\ C_{p_\eta} + \mathbf{j}D_{p_\eta} & A_{p_\eta} - \mathbf{j}B_{p_\eta} & & \\ & & A'_{q_\eta} + \mathbf{j}B'_{q_\eta} & -C'_{q_\eta} + \mathbf{j}D'_{q_\eta} \\ & & C'_{q_\eta} + \mathbf{j}D'_{q_\eta} & A'_{q_\eta} - \mathbf{j}B'_{q_\eta} \end{pmatrix}$$

where $\mathcal{D}^l(\eta)$ is defined as in (6.33).

Recall that $M(\eta-1) = \bigoplus_{l=0}^{\eta/2} W^l(\eta)$ is an orthogonal decomposition of $M(\eta-1)$ with respect to $\langle\cdot\,,\cdot\rangle$; see (6.32) and the paragraph preceding it. As $\mathcal{D}^0(\eta)$ is a standard orthogonal basis of $W^0(\eta)$, and $g|_{W^0(\eta)}$ commutes with $\mathrm{J}_{\mathcal{D}^0(\eta)}$, it follows that $\Lambda'_{\mathcal{D}^0(\eta)}(g|_{W^0(\eta)}) \in \mathrm{U}(2t_\eta)$. In other words,

$$\begin{pmatrix} A_{p_\eta}+\sqrt{-1}B_{p_\eta} & -C_{p_\eta}+\sqrt{-1}D_{p_\eta} & & \\ C_{p_\eta}+\sqrt{-1}D_{p_\eta} & A_{p_\eta}-\sqrt{-1}B_{p_\eta} & & \\ & & A'_{q_\eta}+\sqrt{-1}B'_{q_\eta} & -C'_{q_\eta}+\sqrt{-1}D'_{q_\eta} \\ & & C'_{q_\eta}+\sqrt{-1}D'_{q_\eta} & A'_{q_\eta}-\sqrt{-1}B'_{q_\eta} \end{pmatrix} \in \mathrm{U}(2t_\eta).$$

This implies that $E_{p_\eta} \in \mathrm{Sp}(p_\eta)$ and $F_{q_\eta} \in \mathrm{Sp}(q_\eta)$ and (5) of lemma (6.5.2) holds. This completes the proof. $\square$

We now introduce some notation which will be required to state Proposition 6.5.3. For $\eta \in \mathbb{E}_{\mathbf{d}}$, set

$$\mathcal{D}(\eta) := \mathcal{D}^0(\eta) \vee \cdots \vee \mathcal{D}^{\eta/2-1}(\eta)\,,$$

and for $\theta \in \mathbb{O}_{\mathbf{d}}$, set

$$\mathcal{C}(\theta) := \mathcal{C}^0(\theta) \vee \cdots \vee \mathcal{C}^{\theta-1}(\theta).$$

Let $\alpha := \#\mathbb{E}_{\mathbf{d}}$, $\beta := \#\mathbb{O}^1_{\mathbf{d}}$ and $\gamma := \#\mathbb{O}^3_{\mathbf{d}}$. We enumerate $\mathbb{E}_{\mathbf{d}} = \{\eta_i \mid 1 \le i \le \alpha\}$ such that $\eta_i < \eta_{i+1}$, $\mathbb{O}^1_{\mathbf{d}} = \{\theta_j \mid 1 \le j \le \beta\}$ such that $\theta_j < \theta_{j+1}$ and similarly $\mathbb{O}^3_{\mathbf{d}} = \{\zeta_j \mid 1 \le j \le \gamma\}$ such that $\zeta_j < \zeta_{j+1}$. Now define

$$\mathcal{E} := \mathcal{D}(\eta_1) \vee \cdots \vee \mathcal{D}(\eta_\alpha);\ \mathcal{O}^1 := \mathcal{C}(\theta_1) \vee \cdots \vee \mathcal{C}(\theta_\beta);\ \text{and}\ \ \mathcal{O}^3 := \mathcal{C}(\zeta_1) \vee \cdots \vee \mathcal{C}(\zeta_\gamma).$$

Also define

$$\mathcal{H} = \mathcal{E} \vee \mathcal{O}^1 \vee \mathcal{O}^3. \tag{6.42}$$

For an integer m define the $\mathbb{R}$-algebra embedding

$$\wp_{m,\mathbb{H}} : \mathrm{M}_m(\mathbb{H}) \longrightarrow \mathrm{M}_{2m}(\mathbb{C}), \quad R \longmapsto \begin{pmatrix} S & -\overline{T} \\ T & \overline{S} \end{pmatrix}$$

where $S, T \in \mathrm{M}_m(\mathbb{C})$ are the unique elements such that $R = S + \mathbf{j}T$. The following map is an $\mathbb{R}$-algebra embedding of $\prod_{i=1}^{\alpha} \big(\mathrm{M}_{p_{\eta_i}}(\mathbb{H}) \times \mathrm{M}_{q_{\eta_i}}(\mathbb{H})\big) \times \prod_{j=1}^{\beta} \mathrm{M}_{t_{\theta_j}}(\mathbb{C}) \times \prod_{k=1}^{\gamma} \mathrm{M}_{t_{\zeta_k}}(\mathbb{C})$ into $\mathrm{M}_n(\mathbb{C})$. Define

$$\mathbf{D}\colon \prod_{i=1}^{\alpha} \big(\mathrm{M}_{p_{\eta_i}}(\mathbb{H}) \times \mathrm{M}_{q_{\eta_i}}(\mathbb{H})\big) \times \prod_{j=1}^{\beta} \mathrm{M}_{t_{\theta_j}}(\mathbb{C}) \times \prod_{k=1}^{\gamma} \mathrm{M}_{t_{\zeta_k}}(\mathbb{C}) \longrightarrow \mathrm{M}_n(\mathbb{C})$$

by

$$\begin{aligned} &\big(C_{\eta_1}, D_{\eta_1}, \ldots, C_{\eta_\alpha}, D_{\eta_\alpha};\ A_{\theta_1}, \ldots, A_{\theta_\beta};\ B_{\zeta_1}, \ldots, B_{\zeta_\gamma}\big) \\ &\qquad \longmapsto \bigoplus_{i=1}^{\alpha} \Big(\wp_{p_{\eta_i},\mathbb{H}}(C_{\eta_i}) \oplus \wp_{q_{\eta_i},\mathbb{H}}(D_{\eta_i})\Big)_{\blacktriangle}^{\frac{\eta}{2}} \\ &\qquad \oplus \bigoplus_{j=1}^{\beta} \Big(\big(A_{\theta_j} \oplus \overline{A}_{\theta_j}\big)_{\blacktriangle}^{\frac{\theta_j-1}{4}} \oplus A_{\theta_j} \oplus \big(A_{\theta_j} \oplus \overline{A}_{\theta_j}\big)_{\blacktriangle}^{\frac{\theta_j-1}{4}}\Big) \\ &\qquad \oplus \bigoplus_{k=1}^{\gamma} \Big(\big(B_{\zeta_k} \oplus \overline{B}_{\zeta_k}\big)_{\blacktriangle}^{\frac{\zeta_k+1}{4}} \oplus \big(B_{\zeta_k} \oplus \overline{B}_{\zeta_k}\big)_{\blacktriangle}^{\frac{\zeta_k-3}{4}} \oplus \overline{B}_{\zeta_k}\Big). \end{aligned}$$

It is clear that $\mathcal{H}$ in (6.42) is a standard orthogonal basis of V with respect to $\langle \cdot\,, \cdot \rangle$. Let

$$\Lambda'_{\mathcal{H}} : \{x \in \mathrm{End}_{\mathbb{H}}\mathbb{H}^n \mid x\mathrm{J}_{\mathcal{H}} = \mathrm{J}_{\mathcal{H}}x\} \longrightarrow \mathrm{M}_n(\mathbb{C})$$

be the isomorphism of $\mathbb{R}$-algebras induced by the above ordered basis $\mathcal{H}$. Recall that $\Lambda'_{\mathcal{H}} : K_{\mathcal{H}} \longrightarrow \mathrm{U}(n)$ is an isomorphism of Lie groups. In the next result we obtain an explicit description of $\Lambda'_{\mathcal{H}}(K)$ in $\mathrm{U}(n)$ where $K \subset K_{\mathcal{H}}$ is the maximal compact subgroup in the centralizer of the nilpotent element X as in Lemma 6.5.2.

Proposition 6.5.3. *Let* $X \in \mathcal{N}_{\mathfrak{so}^*(2n)}$, $\Psi_{\mathrm{SO}^*(2n)}(\mathcal{O}_X) = \big(\mathbf{d}, \mathbf{sgn}_{\mathcal{O}_X}\big)$. *Let* $\alpha := \#\mathbb{E}_{\mathbf{d}}$,

$\beta := \#\mathbb{O}^1_{\mathbf{d}}$ and $\gamma := \#\mathbb{O}^3_{\mathbf{d}}$. Let $\{X, H, Y\} \subset \mathfrak{so}^*(2n)$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple; let (p_η, q_η) be the signature of the form $(\cdot,\cdot)_\eta$, for $\eta \in \mathbb{E}_{\mathbf{d}}$, as defined in (3.8). Let K be the maximal compact subgroup of $\mathcal{Z}_{\mathrm{SO}^*(2n)}(X, H, Y)$ as in Lemma 6.5.2. Then $\Lambda'_{\mathcal{H}}(K) \subset \mathrm{U}(n)$ is given by

$$\Lambda'_{\mathcal{H}}(K) = \left\{ \mathbf{D}(g) \;\middle|\; g \in \prod_{i=1}^{\alpha} \big(\mathrm{Sp}(p_{\eta_i}) \times \mathrm{Sp}(q_{\eta_i})\big) \times \prod_{j=1}^{\beta} \mathrm{U}(t_{\theta_j}) \times \prod_{k=1}^{\gamma} \mathrm{U}(t_{\zeta_k}) \right\}.$$

Proof. This follows by writing the matrices of the elements of the maximal compact subgroup K with respect to the basis $\mathcal{H}$ in (6.42). □

As we only consider simple Lie algebras, to ensure simplicity of $\mathfrak{so}^*(2n)$, in view of [Kn, Theorem 6.105, p. 421] and the isomorphisms (vii), (xi) in [He, Chapter X, §6, pp.519-520], we will further need to assume that $n \geq 3$.

Theorem 6.5.4. *Let $X \in \mathfrak{so}^*(2n)$ be a nilpotent element when $n \geq 3$. Let $(\mathbf{d}, \mathbf{sgn}_{\mathcal{O}_X}) \in \mathcal{Y}^{\mathrm{odd}}(n)$ be the signed Young diagram of the orbit $\mathcal{O}_X$ (that is, $\Psi_{\mathrm{SO}^*(2n)}(\mathcal{O}_X) = (\mathbf{d}, \mathbf{sgn}_{\mathcal{O}_X})$ in the notation of Theorem 4.1.8). Then*

$$\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = \begin{cases} 0 & \text{if } \#\mathbb{O}_{\mathbf{d}} = 0 \\ \#\mathbb{O}_{\mathbf{d}} - 1 & \text{if } \#\mathbb{O}_{\mathbf{d}} \geq 1\,. \end{cases}$$

Proof. As the theorem is evident when $X = 0$, we assume that $X \neq 0$.

In the proof we will use the notation established above. Let $\{X, H, Y\} \subset \mathfrak{so}^*(2n)$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple. Let K be the maximal compact subgroup of $\mathcal{Z}_{\mathrm{SO}^*(2n)}(X, H, Y)$ as in Lemma 6.5.2. Let $\mathcal{H}$ be as in (6.42), and let $K_{\mathcal{H}}$ be the maximal compact subgroup of $\mathrm{SO}^*(2n)$ as in the Lemma 6.5.1(1). Then $K \subset K_{\mathcal{H}}$. It follows either from Proposition 6.5.3 or from Lemma 6.0.1(4) that

$$K \simeq \prod_{\eta \in \mathbb{E}_{\mathbf{d}}} \big(\mathrm{Sp}(p_\eta) \times \mathrm{Sp}(q_\eta)\big) \times \prod_{\theta \in \mathbb{O}_{\mathbf{d}}} \mathrm{U}(t_\theta).$$

In particular, K is connected and $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}) = \#\mathbb{O}_{\mathbf{d}}$. Let $\mathfrak{k}_{\mathcal{H}}$ be the Lie algebra of $K_{\mathcal{H}}$. We now appeal to Proposition 6.5.3 to conclude that $\mathfrak{z}(\mathfrak{k}) \subset [\mathfrak{k}_{\mathcal{H}}, \mathfrak{k}_{\mathcal{H}}]$ when $\#\mathbb{O}_{\mathbf{d}} = 0$, and $\mathfrak{z}(\mathfrak{k}) \not\subset [\mathfrak{k}_{\mathcal{H}}, \mathfrak{k}_{\mathcal{H}}]$ when $\#\mathbb{O}_{\mathbf{d}} > 0$. As $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}_{\mathcal{H}}) = 1$, in view of Theorem 5.2.2 we have that for all $X \neq 0$,

$$\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = \dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}) \cap [\mathfrak{k}_{\mathcal{H}}, \mathfrak{k}_{\mathcal{H}}] = \begin{cases} 0 & \text{if } \#\mathbb{O}_{\mathbf{d}} = 0 \\ \#\mathbb{O}_{\mathbf{d}} - 1 & \text{if } \#\mathbb{O}_{\mathbf{d}} \geq 1 . \end{cases}$$

This completes the proof of the theorem. $\square$

6.6 Second cohomology of nilpotent orbits in $\mathfrak{sp}(n, \mathbb{R})$

Let n be a positive integer. In this section, we follow notation and parametrization of the nilpotent orbits in $\mathfrak{sp}(n, \mathbb{R})$ as in §4.1.6; see Theorem 4.1.9. Here we compute the second cohomology groups of nilpotent orbits in $\mathfrak{sp}(n, \mathbb{R})$ under the adjoint action of $\mathrm{Sp}(n, \mathbb{R})$. As $\mathrm{U}(n)$, being a maximal compact subgroup in $\mathrm{Sp}(n, \mathbb{R})$, is not semisimple, in view of Remark 6.0.3, we need to work out how a conjugate of a maximal compact subgroup of $\mathcal{Z}_{\mathrm{Sp}(n,\mathbb{R})}(X)$ is embedded in $\mathrm{U}(n)$, for an arbitrary nilpotent element $X \in \mathfrak{sp}(n, \mathbb{R})$. Throughout this section $\langle \cdot\,, \cdot \rangle$ denotes the symplectic form on $\mathbb{R}^{2n}$ defined by $\langle x, y \rangle := x^t \mathrm{J}_n y$, $x, y \in \mathbb{R}^{2n}$, where J_n is as in (2.19).

Let $0 \neq X \in \mathcal{N}_{\mathfrak{sp}(n,\mathbb{R})}$ and $\{X, H, Y\}$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple in $\mathfrak{sp}(n, \mathbb{R})$. Let $\Psi_{\mathrm{Sp}(n,\mathbb{R})}(\mathcal{O}_X) = (\mathbf{d}, \mathbf{sgn}_{\mathcal{O}_X})$. Recall that $\mathbf{sgn}_{\mathcal{O}_X}$ determines the signature of $(\cdot, \cdot)_\eta$ on $L(\eta - 1)$ for all $\eta \in \mathbb{E}_{\mathbf{d}}$; let (p_η, q_η) be the signature of $(\cdot, \cdot)_\eta$ on $L(\eta - 1)$. Let $(v_1^d, \ldots, v_{t_d}^d)$ be a $\mathbb{R}$-basis of $L(d - 1)$ as in Proposition 3.0.7. It now follows from Proposition 3.0.7(3)(c) that $(v_1^\eta, \ldots, v_{t_\eta}^\eta)$ is an orthogonal basis of $L(\eta - 1)$ for the form $(\cdot, \cdot)_\eta$. We also assume that the vectors in the basis $(v_1^d, \ldots, v_{t_d}^d)$ satisfy properties in Remark 3.0.11(1). In view of the signature of $(\cdot,\,\cdot)_\eta$, we may further assume that

$$(6.43)\qquad (v_j^\eta, v_j^\eta)_\eta = \begin{cases} +1 & \text{if } 1 \le j \le p_\eta \\ -1 & \text{if } p_\eta < j \le t_\eta \end{cases} ;\ \eta \in \mathbb{E}_{\mathbf{d}}.$$

For all $\theta \in \mathbb{O}_{\mathbf{d}}$, as $(\cdot,\,\cdot)_\theta$ is a symplectic form, we may assume that $(v_1^\theta, \ldots, v_{t_\theta/2}^\theta ;\ v_{t_\theta/2+1}^\theta, \ldots, v_{t_\theta}^\theta)$ is a symplectic basis of $L(\theta-1)$; see Section 2.3 for the definition of a symplectic basis. This is equivalent to saying that, for all $\theta \in \mathbb{O}_{\mathbf{d}}$,

$$(6.44)\qquad (v_j^\theta, v_{t_\theta/2+j}^\theta)_\theta = 1 \text{ for } 1 \le j \le t_\theta/2 \text{ and } (v_j^\theta, v_i^\theta)_\theta = 0 \text{ for all } i \ne j + t_\theta/2.$$

Now fixing $\theta \in \mathbb{O}_{\mathbf{d}}$, for all $1 \le j \le t_\theta$, define

$$(6.45)\qquad w_{jl}^\theta := \begin{cases} \left(X^l v_j^\theta + X^{\theta-1-l} v_j^\theta\right)\frac{1}{\sqrt{2}} & \text{if } 0 \le l < (\theta-1)/2 \\ X^l v_j^\theta & \text{if } l = (\theta-1)/2 \\ \left(X^{\theta-1-l} v_j^\theta - X^l v_j^\theta\right)\frac{1}{\sqrt{2}} & \text{if } (\theta-1)/2 < l \le \theta - 1. \end{cases}$$

For $\theta \in \mathbb{O}_{\mathbf{d}}$, $0 \le l \le \theta - 1$, set

$$(6.46)\qquad V^l(\theta) := \mathrm{Span}_{\mathbb{R}}\{w_{jl}^\theta \mid 1 \le j \le t_\theta\}.$$

The ordered basis $\left(w_{1l}^\theta, \ldots, w_{t_\theta l}^\theta\right)$ of $V^l(\theta)$ is denoted by $\mathcal{A}^l(\theta)$. Let $\mathcal{B}^l(d) = (X^l v_1^d, \ldots, X^l v_{t_d}^d)$ be the ordered basis of $X^l L(d-1)$ for $0 \le l \le d-1$, $d \in \mathbb{N}_{\mathbf{d}}$ as in (6.1).

Lemma 6.6.1. *The following holds:*

$$\mathcal{Z}_{\mathrm{Sp}(n,\mathbb{R})}(X,H,Y)=\left\{ g\in \mathrm{Sp}(n,\mathbb{R}) \;\middle|\; \begin{array}{c} g(V^l(\theta))\subset\ V^l(\theta)\ \textit{and} \\ \left[g|_{V^l(\theta)}\right]_{\mathcal{A}^l(\theta)}=\left[g|_{V^0(\theta)}\right]_{\mathcal{A}^0(\theta)}\forall\,\theta\in\mathbb{O}_{\mathbf{d}},0\le l<\theta; \\ g(X^lL(\eta-1))\subset X^lL(\eta-1)\ \textit{and} \\ \left[g|_{X^lL(\eta-1)}\right]_{\mathcal{B}^l(\eta)}=\left[g|_{L(\eta-1)}\right]_{\mathcal{B}^0(\eta)}\forall\,\eta\in\mathbb{E}_{\mathbf{d}},0\le l<\eta \end{array}\right\}.$$

Proof. The proof is similar to that of the Lemma 6.3.1; the details are omitted.

□

Using (6.44) and (6.45) we observe that for each $\theta\in\mathbb{O}_{\mathbf{d}}$ the space $M(\theta-1)$ is a direct sum of the subspaces $V^l(\theta)$, $0\le l\le\theta-1$, which are mutually orthogonal with respect to $\langle\cdot\,,\cdot\rangle$. We now re-arrange the ordered basis $\mathcal{A}^l(\theta)$ of $V^l(\theta)$ to obtain a symplectic basis $\mathcal{C}^l(\theta)$ of $V^l(\theta)$ with respect to $\langle\cdot\,,\cdot\rangle$ as follows. For $\theta\in\mathbb{O}^1_{\mathbf{d}}$, define

$$\mathcal{C}^l(\theta):=\begin{cases} \left(w^\theta_{1\,l},\dots,w^\theta_{t_\theta/2\,l}\right)\vee\left(w^\theta_{(t_\theta/2+1)\,l},\dots,w^\theta_{t_\theta\,l}\right) & \text{if } l \text{ is even, } 0\le l<(\theta-1)/2\\ \left(w^\theta_{(t_\theta/2+1)\,l},\dots,w^\theta_{t_\theta\,l}\right)\vee\left(w^\theta_{1\,l},\dots,w^\theta_{t_\theta/2\,l}\right) & \text{if } l \text{ is odd, } 0\le l<(\theta-1)/2\\ \left(w^\theta_{1\,l},\dots,w^\theta_{t_\theta/2\,l}\right)\vee\left(w^\theta_{(t_\theta/2+1)\,l},\dots,w^\theta_{t_\theta\,l}\right) & \text{if } l=(\theta-1)/2\\ \left(w^\theta_{(t_\theta/2+1)\,l},\dots,w^\theta_{t_\theta\,l}\right)\vee\left(w^\theta_{1\,l},\dots,w^\theta_{t_\theta/2\,l}\right) & \text{if } l \text{ is even, } (\theta+1)/2\le l\le\theta-1\\ \left(w^\theta_{1\,l},\dots,w^\theta_{t_\theta/2\,l}\right)\vee\left(w^\theta_{(t_\theta/2+1)\,l},\dots,w^\theta_{t_\theta\,l}\right) & \text{if } l \text{ is odd, } (\theta+1)/2\le l\le\theta-1. \end{cases}$$

Similarly, for each $\zeta\in\mathbb{O}^3_{\mathbf{d}}$, define

$$\mathcal{C}^l(\zeta):=\begin{cases} \left(w^\zeta_{1\,l},\dots,w^\zeta_{t_\zeta/2\,l}\right)\vee\left(w^\zeta_{(t_\zeta/2+1)\,l},\dots,w^\zeta_{t_\zeta\,l}\right) & \text{if } l \text{ is even, } 0\le l<(\zeta-1)/2\\ \left(w^\zeta_{(t_\zeta/2+1)\,l},\dots,w^\zeta_{t_\zeta\,l}\right)\vee\left(w^\zeta_{1\,l},\dots,w^\zeta_{t_\zeta/2\,l}\right) & \text{if } l \text{ is odd, } 0\le l<(\zeta-1)/2\\ \left(w^\zeta_{(t_\zeta/2+1)\,l},\dots,w^\zeta_{t_\zeta\,l}\right)\vee\left(w^\zeta_{1\,l},\dots,w^\zeta_{t_\zeta/2\,l}\right) & \text{if } l=(\zeta-1)/2\\ \left(w^\zeta_{(t_\zeta/2+1)\,l},\dots,w^\zeta_{t_\zeta\,l}\right)\vee\left(w^\zeta_{1\,l},\dots,w^\zeta_{t_\zeta/2\,l}\right) & \text{if } l \text{ is even, } (\zeta+1)/2\le l\le\zeta-1\\ \left(w^\zeta_{1\,l},\dots,w^\zeta_{t_\zeta/2\,l}\right)\vee\left(w^\zeta_{(t_\zeta/2+1)\,l},\dots,w^\zeta_{t_\zeta\,l}\right) & \text{if } l \text{ is odd, } (\zeta+1)/2\le l\le\zeta-1. \end{cases}$$

For $\eta \in \mathbb{E}_{\mathbf{d}}$, $0 \leq l \leq \eta/2 - 1$, set

$$W^l(\eta) := X^l L(\eta - 1) + X^{\eta-1-l} L(\eta - 1)\,. \tag{6.47}$$

We moreover re-arrange the ordered basis $\mathcal{B}^l(\eta) \vee \mathcal{B}^{\eta-1-l}(\eta)$ of $W^l(\eta)$ and obtain new basis $\mathcal{D}^l(\eta)$ as follows:

(6.48)

$$\mathcal{D}^l(\eta) := \begin{cases} \big(X^l v_1, \dots, X^l v_{p_\eta}\big) \vee \big(X^{\eta-1-l} v_{p_\eta+1}, \dots, X^{\eta-1-l} v_{t_\eta}\big) & \\ \bigvee \big(X^{\eta-1-l} v_1, \dots, X^{\eta-1-l} v_{p_\eta}\big) \vee \big(X^l v_{p_\eta+1}, \dots, X^l v_{t_\eta}\big) & \text{if } l \text{ is even} \\ \big(X^{\eta-1-l} v_1, \dots, X^{\eta-1-l} v_{p_\eta}\big) \vee \big(X^l v_{p_\eta+1}, \dots, X^l v_{t_\eta}\big) & \\ \bigvee \big(X^l v_1, \dots, X^l v_{p_\eta}\big) \vee \big(X^{\eta-1-l} v_{p_\eta+1}, \dots, X^{\eta-1-l} v_{t_\eta}\big) & \text{if } l \text{ is odd.} \end{cases}$$

Using (6.43) it can be easily verified that $\mathcal{D}^l(\eta)$ is a symplectic basis with respect to $\langle \cdot\,,\cdot \rangle$.

Let $J_{\mathcal{C}^l(\theta)}$ be the complex structure on $V^l(\theta)$ associated to the basis $\mathcal{C}^l(\theta)$ for $\theta \in \mathbb{O}_{\mathbf{d}}$, $0 \leq l \leq \theta - 1$, and let $J_{\mathcal{D}^l(\eta)}$ be the complex structure on $W^l(\eta)$ associated to the basis $\mathcal{D}^l(\eta)$ for $\eta \in \mathbb{E}_{\mathbf{d}}, 0 \leq l \leq \eta - 1$; see Section 2.3 for the definition of such complex structures.

The next lemma is a standard fact where we recall, without a proof, an explicit description of a maximal compact subgroup in a symplectic group. Let V' be a $\mathbb{R}$-vector space, $\langle \cdot\,,\cdot \rangle'$ be a non-degenerate symplectic form on V' and $\mathcal{B}'$ be a symplectic basis of V'. Let $J_{\mathcal{B}'}$ be the complex structure on V' associated to $\mathcal{B}'$. Let $2m := \dim_{\mathbb{R}} V'$. We set

$$K_{\mathcal{B}'} := \{g \in \mathrm{Sp}(V', \langle \cdot\,,\cdot \rangle') \mid g J_{\mathcal{B}'} = J_{\mathcal{B}'} g\}.$$

Lemma 6.6.2. *Let $V', \langle \cdot\,,\cdot \rangle', \mathcal{B}'$ and $J_{\mathcal{B}'}$ be as above. Then*

1. *$K_{\mathcal{B}'}$ is a maximal compact subgroup in* $\mathrm{Sp}(V', \langle \cdot\,,\cdot \rangle')$.

2. $K_{\mathcal{B}'} = \left\{ g \in \mathrm{SL}(V') \;\middle|\; \big[g\big]_{\mathcal{B}'} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \text{ where } A + \sqrt{-1}B \in \mathrm{U}(m) \right\}$.

Define the $\mathbb{R}$-algebra isomorphism

$$\text{(6.49)} \qquad \widetilde{\Lambda}_{\mathcal{B}'} : \{x \in \mathrm{End}_{\mathbb{R}} V' \mid xJ_{\mathcal{B}'} = J_{\mathcal{B}'}x\} \longrightarrow \mathrm{M}_m(\mathbb{C}), \quad x \longmapsto A + \sqrt{-1}B$$

where $\big[x\big]_{\mathcal{B}'} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$. In view of Lemma 6.6.2 it is clear that $\widetilde{\Lambda}_{\mathcal{B}'}(K_{\mathcal{B}'}) = \mathrm{U}(m)$, and thus $\widetilde{\Lambda}_{\mathcal{B}'} \colon K_{\mathcal{B}'} \to \mathrm{U}(m)$ is an isomorphism of Lie groups.

In the next lemma we describe a suitable maximal compact subgroup of $\mathcal{Z}_{\mathrm{Sp}(n,\mathbb{R})}(X, H, Y)$ which will be used in Proposition 6.6.4. Define the $\mathbb{R}$-algebra embedding

$$\wp_{m,\mathbb{C}} : \mathrm{M}_m(\mathbb{C}) \longrightarrow \mathrm{M}_{2m}(\mathbb{R}), \quad R \longmapsto \begin{pmatrix} S & -T \\ T & S \end{pmatrix}$$

where $S, T \in \mathrm{M}_m(\mathbb{R})$ are the unique elements such that $R = S + \sqrt{-1}T$.

Lemma 6.6.3. *Let K be the subgroup of $\mathcal{Z}_{\mathrm{Sp}(n,\mathbb{R})}(X, H, Y)$ consisting of elements g in $\mathcal{Z}_{\mathrm{Sp}(n,\mathbb{R})}(X, H, Y)$ satisfying the following conditions:*

1. *For all $\theta \in \mathbb{O}_{\mathbf{d}}$ and $0 \leq l \leq \theta - 1$, $g(V^l(\theta)) \subset V^l(\theta)$.*

2. *For all $\theta \in \mathbb{O}^1_{\mathbf{d}}$, there exist $A_\theta, B_\theta \in \mathrm{M}_{t_\theta/2}(\mathbb{R})$ with $A_\theta + \sqrt{-1}B_\theta \in \mathrm{U}(t_\theta/2)$ such that*

$$\big[g|_{V^l(\theta)}\big]_{\mathcal{C}^l(\theta)} = \begin{cases} \wp_{t_\theta/2,\mathbb{C}}(A_\theta + \sqrt{-1}B_\theta) & \text{if } l \text{ is even, } 0 \leq l < (\theta-1)/2 \\ \wp_{t_\theta/2,\mathbb{C}}(A_\theta - \sqrt{-1}B_\theta) & \text{if } l \text{ is odd, } 0 \leq l < (\theta-1)/2 \\ \wp_{t_\theta/2,\mathbb{C}}(A_\theta + \sqrt{-1}B_\theta) & \text{if } l = (\theta-1)/2 \\ \wp_{t_\theta/2,\mathbb{C}}(A_\theta + \sqrt{-1}B_\theta) & \text{if } l \text{ is odd, } (\theta+1)/2 \leq l \leq \theta-1 \\ \wp_{t_\theta/2,\mathbb{C}}(A_\theta - \sqrt{-1}B_\theta) & \text{if } l \text{ is even, } (\theta+1)/2 \leq l \leq \theta-1. \end{cases}$$

3. For all $\zeta \in \mathbb{O}^3_{\mathbf{d}}$*, there exist* $A_\zeta, B_\zeta \in \mathrm{M}_{t_\zeta/2}(\mathbb{R})$ *with* $A_\zeta + \sqrt{-1}B_\zeta \in \mathrm{U}(t_\zeta/2)$ *such that*

$$[g|_{V^l(\zeta)}]_{\mathcal{C}^l(\zeta)} = \begin{cases} \wp_{t_\zeta/2,\mathbb{C}}(A_\zeta + \sqrt{-1}B_\zeta) & \text{if } l \text{ is even, } 0 \le l < (\zeta-1)/2 \\ \wp_{t_\zeta/2,\mathbb{C}}(A_\zeta - \sqrt{-1}B_\zeta) & \text{if } l \text{ is odd, } 0 \le l < (\zeta-1)/2 \\ \wp_{t_\zeta/2,\mathbb{C}}(A_\zeta - \sqrt{-1}B_\zeta) & \text{if } l = (\zeta-1)/2 \\ \wp_{t_\zeta/2,\mathbb{C}}(A_\zeta + \sqrt{-1}B_\zeta) & \text{if } l \text{ is odd, } (\zeta+1)/2 \le l \le \zeta - 1 \\ \wp_{t_\zeta/2,\mathbb{C}}(A_\zeta - \sqrt{-1}B_\zeta) & \text{if } l \text{ is even, } (\zeta+1)/2 \le l \le \zeta - 1. \end{cases}$$

4. For all $\eta \in \mathbb{E}_{\mathbf{d}}$ *and* $0 \le l \le \eta - 1$, $\quad g(X^l L(\eta-1)) \subset X^l L(\eta-1)$.

5. For all $\eta \in \mathbb{E}_{\mathbf{d}}$*, there exist* $C_\eta \in \mathrm{O}_{p_\eta}$ *and* $D_\eta \in \mathrm{O}_{q_\eta}$ *such that*
$$[g|_{X^l L(\eta-1)}]_{\mathcal{B}^l(\eta)} = \begin{pmatrix} C_\eta & 0 \\ 0 & D_\eta \end{pmatrix}.$$

Then K *is a maximal compact subgroup of* $\mathcal{Z}_{\mathrm{Sp}(n,\mathbb{R})}(X,H,Y)$*.*

Proof. For our convenience we begin by introducing a new notation. Let m be an integer. For a matrix Z in $\mathrm{M}_{2m}(\mathbb{R})$, define

$$Z^\dagger := \begin{pmatrix} 0 & \mathrm{I}_m \\ \mathrm{I}_m & 0 \end{pmatrix} Z \begin{pmatrix} 0 & \mathrm{I}_m \\ \mathrm{I}_m & 0 \end{pmatrix}^{-1}.$$

Note that $\begin{pmatrix} P & -R \\ R & P \end{pmatrix}^\dagger = \begin{pmatrix} P & R \\ -R & P \end{pmatrix}$ for matrices $P, R \in \mathrm{M}_m(\mathbb{R})$.

Let $K' \subset \mathcal{Z}_{\mathrm{Sp}(n,\mathbb{R})}(X,H,Y)$ be the subgroup consisting of all elements g satisfying the following conditions:

(6.50)

For all $\theta \in \mathbb{O}_{\mathbf{d}}, 0 \leq l \leq \theta - 1,\ g\big(V^l(\theta)\big) \subset\ V^l(\theta)$.

(6.51)

$$\text{For all } \theta \in \mathbb{O}^1_{\mathbf{d}},\ \big[g|_{V^l(\theta)}\big]_{\mathcal{C}^l(\theta)} = \begin{cases} \big[g|_{V^0(\theta)}\big]_{\mathcal{C}^0(\theta)} & \text{if } l \text{ is even, } 0 \leq l < (\theta-1)/2 \\ \big[g|_{V^0(\theta)}\big]^{\dagger}_{\mathcal{C}^0(\theta)} & \text{if } l \text{ is odd, } 0 \leq l < (\theta-1)/2 \\ \big[g|_{V^0(\theta)}\big]_{\mathcal{C}^0(\theta)} & \text{if } l = (\theta-1)/2 \\ \big[g|_{V^0(\theta)}\big]_{\mathcal{C}^0(\theta)} & \text{if } l \text{ is odd, } (\theta+1)/2 \leq l \leq \theta - 1 \\ \big[g|_{V^0(\theta)}\big]^{\dagger}_{\mathcal{C}^0(\theta)} & \text{if } l \text{ is even, } (\theta+1)/2 \leq l \leq \theta - 1, \end{cases}$$

(6.52)

$$\text{For all } \zeta \in \mathbb{O}^3_{\mathbf{d}},\ \big[g|_{V^l(\zeta)}\big]_{\mathcal{C}^l(\zeta)} = \begin{cases} \big[g|_{V^0(\zeta)}\big]_{\mathcal{C}^0(\zeta)} & \text{if } l \text{ is even, } 0 \leq l < (\zeta-1)/2 \\ \big[g|_{V^0(\zeta)}\big]^{\dagger}_{\mathcal{C}^0(\zeta)} & \text{if } l \text{ is odd, } 0 \leq l < (\zeta-1)/2 \\ \big[g|_{V^0(\zeta)}\big]^{\dagger}_{\mathcal{C}^0(\zeta)} & \text{if } l = (\zeta-1)/2 \\ \big[g|_{V^0(\zeta)}\big]_{\mathcal{C}^0(\zeta)} & \text{if } l \text{ is odd, } (\zeta+1)/2 \leq l \leq \zeta - 1 \\ \big[g|_{V^0(\zeta)}\big]^{\dagger}_{\mathcal{C}^0(\zeta)} & \text{if } l \text{ is even, } (\zeta+1)/2 \leq l \leq \zeta - 1, \end{cases}$$

(6.53)

$g|_{V^0(\theta)}$ commutes with $J_{\mathcal{C}^0(\theta)}$,

(6.54)

$g\big(X^l L(\eta-1)\big) \subset X^l L(\eta-1), \quad \big[g|_{X^l L(\eta-1)}\big]_{\mathcal{B}^l(\eta)} = \big[g|_{L(\eta-1)}\big]_{\mathcal{B}^0(\eta)}$ for all $\eta \in \mathbb{E}_{\mathbf{d}}$,

$0 \leq l \leq \eta - 1$,

(6.55)

$g|_{W^0(\eta)}$ commutes with $J_{\mathcal{D}^0(\eta)}$.

Using Lemma 6.6.1 and Lemma 6.6.2(1) it is clear that K' is a maximal compact subgroup of $\mathcal{Z}_{\mathrm{Sp}(n,\mathbb{R})}(X,H,Y)$. Hence to prove the lemma it suffices to show that $K = K'$. Let $g \in \mathrm{Sp}(n,\mathbb{R})$. Using Lemma 6.6.2 (2) it is straightforward to check that g satisfies (1), (2), (3) in the statement of the lemma if and only if g satisfies (6.50), (6.51), (6.52) and (6.53). Now suppose that $g \in \mathrm{Sp}(n,\mathbb{R})$ and g satisfies (4) and (5) in the statement of the lemma. It is clear that (6.54) holds. We observe that

$$\left[J_{\mathcal{D}^0(\eta)}\right]_{\mathcal{D}^0(\eta)} = \begin{pmatrix} 0 & -\mathrm{I}_{t_\eta} \\ \mathrm{I}_{t_\eta} & 0 \end{pmatrix} \quad \text{and} \quad \left[g|_{W^0(\eta)}\right]_{\mathcal{D}^0(\eta)} = \begin{pmatrix} C_\eta & & & \\ & D_\eta & & \\ & & C_\eta & \\ & & & D_\eta \end{pmatrix}$$

where $\mathcal{D}^0(\eta)$ is defined by setting $l = 0$ in (6.48). From the matrix representations as above, it is clear that $J_{\mathcal{D}^0(\eta)}$ and $g|_{W^0(\eta)}$ commute. This proves that (6.55) holds.

Now we assume that g satisfies (6.54) and (6.55). It is clear that (4) in the statement of the lemma holds. Note that $A := \left[g|_{L(\eta-1)}\right]_{\mathcal{B}^0(\eta)} = \left[g|_{X^l L(\eta-1)}\right]_{\mathcal{B}^l(\eta)}$ for $1 \leq l \leq \eta - 1$. We observe that

$$\left[J_{\mathcal{D}^0(\eta)}\right]_{\mathcal{B}^0(\eta)\vee\mathcal{B}^{\eta-1}(\eta)} = \begin{pmatrix} 0 & -\mathrm{I}_{p_\eta,q_\eta} \\ \mathrm{I}_{p_\eta,q_\eta} & 0 \end{pmatrix} \quad \text{and} \quad \left[g|_{W^0(\eta)}\right]_{\mathcal{B}^0(\eta)\vee\mathcal{B}^{\eta-1}(\eta)} = \begin{pmatrix} A & \\ & A \end{pmatrix}.$$

From (6.55) it follows that the above two matrices commute, which in turn implies that A commutes with $\begin{pmatrix} \mathrm{I}_{p_\eta} & \\ & -\mathrm{I}_{q_\eta} \end{pmatrix}$. Thus A is of the form $A = \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}$ for some matrices $C \in \mathrm{GL}_{p_\eta}(\mathbb{R})$ and $D \in \mathrm{GL}_{q_\eta}(\mathbb{R})$. Now observe that

$$\left[g|_{W^0(\eta)}\right]_{\mathcal{D}^0(\eta)} = \begin{pmatrix} C & & & \\ & D & & \\ & & C & \\ & & & D \end{pmatrix}.$$

As $g|_{W^0(\eta)}$ commutes with $J_{\mathcal{D}^0(\eta)}$, it follows that

$$\begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} + \sqrt{-1}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{U}(t_\eta).$$

Thus, $C \in \mathrm{O}_{p_\eta}$ and $D \in \mathrm{O}_{q_\eta}$ and (5) in the statement of the lemma holds. This completes the proof. □

We next introduce some notation which will be needed in Proposition 6.6.4. Recall that the positive parts of the symplectic basis $\mathcal{D}(\eta)$, $\mathcal{C}(\theta)$ are denoted by $\mathcal{D}_+(\eta)$, $\mathcal{C}_+(\theta)$ respectively; see Section 2.3. Similarly, the negative parts of $\mathcal{D}(\eta)$, $\mathcal{C}(\theta)$ are denoted by $\mathcal{D}_-(\eta)$, $\mathcal{C}_-(\theta)$ respectively. For $\eta \in \mathbb{E}_{\mathbf{d}}$, set

$$\mathcal{D}_+(\eta) := \mathcal{D}_+^0(\eta) \vee \cdots \vee \mathcal{D}_+^{\eta/2-1}(\eta) \text{ and } \mathcal{D}_-(\eta) := \mathcal{D}_-^0(\eta) \vee \cdots \vee \mathcal{D}_-^{\eta/2-1}(\eta).$$

For $\theta \in \mathbb{O}_{\mathbf{d}}$, set

$$\mathcal{C}_+(\theta) := \mathcal{C}_+^0(\theta) \vee \cdots \vee \mathcal{C}_+^{\theta-1}(\theta) \text{ and } \mathcal{C}_-(\theta) := \mathcal{C}_-^0(\theta) \vee \cdots \vee \mathcal{C}_-^{\theta-1}(\theta).$$

Let $\alpha := \#\mathbb{E}_{\mathbf{d}}$, $\beta := \#\mathbb{O}_{\mathbf{d}}^1$ and $\gamma := \#\mathbb{O}_{\mathbf{d}}^3$. We enumerate $\mathbb{E}_{\mathbf{d}} = \{\eta_i \mid 1 \le i \le \alpha\}$ such that $\eta_i < \eta_{i+1}$, and $\mathbb{O}_{\mathbf{d}}^1 = \{\theta_j \mid 1 \le j \le \beta\}$ such that $\theta_j < \theta_{j+1}$; similarly enumerate $\mathbb{O}_{\mathbf{d}}^3 = \{\zeta_j \mid 1 \le j \le \gamma\}$ such that $\zeta_j < \zeta_{j+1}$. Now define

$$\mathcal{E}_+ := \mathcal{D}_+(\eta_1)\vee\cdots\vee\mathcal{D}_+(\eta_\alpha)\,;\ \mathcal{O}_+^1 := \mathcal{C}_+(\theta_1)\vee\cdots\vee\mathcal{C}_+(\theta_\beta)\,;\ \mathcal{O}_+^3 := \mathcal{C}_+(\zeta_1)\vee\cdots\vee\mathcal{C}_+(\zeta_\gamma);$$

$\mathcal{E}_- := \mathcal{D}_-(\eta_1)\vee\cdots\vee\mathcal{D}_-(\eta_\alpha)$; $\mathcal{O}^1_- := \mathcal{C}_-(\theta_1)\vee\cdots\vee\mathcal{C}_-(\theta_\beta)$; $\mathcal{O}^3_- := \mathcal{C}_-(\zeta_1)\vee\cdots\vee\mathcal{C}_-(\zeta_\gamma)$.

Also we define

$$\mathcal{H}_+ := \mathcal{E}_+ \vee \mathcal{O}^1_+ \vee \mathcal{O}^3_+,\ \mathcal{H}_- := \mathcal{E}_- \vee \mathcal{O}^1_- \vee \mathcal{O}^3_- \ \text{ and } \ \mathcal{H} := \mathcal{H}_+ \vee \mathcal{H}_-. \tag{6.56}$$

As before, for a matrix $A = (a_{ij}) \in \mathrm{M}_r(\mathbb{C})$, define $\overline{A} := (\overline{a}_{ij}) \in \mathrm{M}_r(\mathbb{C})$. Let

$$\mathbf{D}\colon \prod_{i=1}^{\alpha} \big(\mathrm{M}_{p_{\eta_i}}(\mathbb{R}) \times \mathrm{M}_{q_{\eta_i}}(\mathbb{R})\big) \times \prod_{j=1}^{\beta} \mathrm{M}_{t_{\theta_j}/2}(\mathbb{C}) \times \prod_{k=1}^{\gamma} \mathrm{M}_{t_{\zeta_k}/2}(\mathbb{C}) \longrightarrow \mathrm{M}_n(\mathbb{C})$$

be the $\mathbb{R}$-algebra embedding defined by

$$\begin{aligned}
&\big(C_{\eta_1}, D_{\eta_1}, \ldots, C_{\eta_\alpha}, D_{\eta_\alpha}; A_{\theta_1}, \ldots, A_{\theta_\beta}; B_{\zeta_1}, \ldots, B_{\zeta_\gamma}\big) \\
&\qquad \longmapsto \bigoplus_{i=1}^{\alpha} \big(C_{\eta_i} \oplus D_{\eta_i}\big)^{\eta_i/2}_{\blacktriangle} \oplus \bigoplus_{j=1}^{\beta} \Big(\big(A_{\theta_j} \oplus \overline{A}_{\theta_j}\big)^{\frac{\theta_j-1}{4}}_{\blacktriangle} \oplus A_{\theta_j} \oplus \big(A_{\theta_j} \oplus \overline{A}_{\theta_j}\big)^{\frac{\theta_j-1}{4}}_{\blacktriangle}\Big) \\
&\qquad\quad \oplus \bigoplus_{k=1}^{\gamma} \Big(\big(B_{\zeta_k} \oplus \overline{B}_{\zeta_k}\big)^{\frac{\zeta_k+1}{4}}_{\blacktriangle} \oplus \big(B_{\zeta_k} \oplus \overline{B}_{\zeta_k}\big)^{\frac{\zeta_k-3}{4}}_{\blacktriangle} \oplus \overline{B}_{\zeta_k}\Big).
\end{aligned}$$

It is clear that the basis $\mathcal{H}$ in (6.56) is a symplectic basis of V with respect to $\langle\cdot\,,\cdot\rangle$. Let $\widetilde{\Lambda}_{\mathcal{H}} : \{x \in \mathrm{End}_{\mathbb{R}}\mathbb{R}^{2n} \mid xJ_{\mathcal{H}} = J_{\mathcal{H}}x\} \longrightarrow \mathrm{M}_n(\mathbb{C})$ be the isomorphism of $\mathbb{R}$-algebras induced by the above symplectic basis $\mathcal{H}$. Recall that $\widetilde{\Lambda}_{\mathcal{H}} : K_{\mathcal{H}} \longrightarrow \mathrm{U}(n)$ is an isomorphism of Lie groups. Using (6.53) and (6.55) we observe that the group K defined in Lemma 6.6.3 satisfies the condition $K \subset K_{\mathcal{H}}$. In the next result we obtain an explicit description of $\widetilde{\Lambda}_{\mathcal{H}}(K)$ in $\mathrm{U}(n)$.

Proposition 6.6.4. *Let $X \in \mathcal{N}_{\mathfrak{sp}(n,\mathbb{R})}$ and $\Psi_{\mathrm{Sp}(n,\mathbb{R})}(\mathcal{O}_X) = (\mathbf{d}, \mathbf{sgn}_{\mathcal{O}_X})$. Let $\alpha := \#\mathbb{E}_{\mathbf{d}}$, $\beta := \#\mathbb{O}^1_{\mathbf{d}}$ and $\gamma := \#\mathbb{O}^3_{\mathbf{d}}$. Let $\{X, H, Y\}$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple in $\mathfrak{sp}(n,\mathbb{R})$, and let $(p_\eta,\, q_\eta)$ be the signature of $(\cdot,\, \cdot)_\eta$, $\eta \in \mathbb{E}_{\mathbf{d}}$, as defined in* (3.8)*. Let K be the maximal compact subgroup of $\mathcal{Z}_{\mathrm{Sp}(n,\mathbb{R})}(X, H, Y)$ as in Lemma 6.6.3. Then $\widetilde{\Lambda}_{\mathcal{H}}(K) \subset \mathrm{U}(n)$ is* *given by*

$$\widetilde{\Lambda}_{\mathcal{H}}(K) = \Big\{ \mathbf{D}(g) \;\Big|\; g \in \prod_{i=1}^{\alpha} \big(\mathrm{O}_{p_{\eta_i}} \times \mathrm{O}_{q_{\eta_i}}\big) \times \prod_{j=1}^{\beta} \mathrm{U}(t_{\theta_j}/2) \times \prod_{k=1}^{\gamma} \mathrm{U}(t_{\zeta_k}/2) \Big\}.$$

Proof. This follows by writing the matrices of the elements of the maximal compact subgroup K in Lemma 6.6.3 with respect to the symplectic basis $\mathcal{H}$ in (6.56). □

Theorem 6.6.5. *Let* $X \in \mathfrak{sp}(n, \mathbb{R})$ *be a nilpotent element. Let* $(\mathbf{d}, \mathbf{sgn}_{\mathcal{O}_X}) \in \mathcal{Y}^{\rm odd}_{-1}(2n)$ *be the signed Young diagram of the orbit* $\mathcal{O}_X$ *(that is,* $\Psi_{\mathrm{Sp}(n,\mathbb{R})}(\mathcal{O}_X) = (\mathbf{d}, \mathbf{sgn}_{\mathcal{O}_X})$ *as in the notation of Theorem 4.1.9). Then*

$$\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = \begin{cases} 0 & \text{if } \#\mathbb{O}_{\mathbf{d}} = 0 \\ \#\mathbb{O}_{\mathbf{d}} - 1 & \text{if } \#\mathbb{O}_{\mathbf{d}} \geq 1. \end{cases}$$

Proof. As the theorem is evident when $X = 0$ we assume that $X \neq 0$.

Let $\{X, H, Y\} \subset \mathfrak{sp}(n, \mathbb{R})$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple. Let K be the maximal compact subgroup of $\mathcal{Z}_{\mathrm{Sp}(n,\mathbb{R})}(X, H, Y)$ as in Lemma 6.6.3. Let $\mathcal{H}$ be as in (6.56) and $K_{\mathcal{H}}$ the maximal compact subgroup of $\mathrm{Sp}(n, \mathbb{R})$ as in Lemma 6.6.2(1). Then $K \subset K_{\mathcal{H}}$. Let $\mathfrak{k}_{\mathcal{H}}$ be the Lie algebra of $K_{\mathcal{H}}$. Using Proposition 6.6.4 it follows that $\mathfrak{z}(\mathfrak{k}) \subset [\mathfrak{k}_{\mathcal{H}}, \mathfrak{k}_{\mathcal{H}}]$ when $\#\mathbb{O}_{\mathbf{d}} = 0$, and $\mathfrak{z}(\mathfrak{k}) \not\subset [\mathfrak{k}_{\mathcal{H}}, \mathfrak{k}_{\mathcal{H}}]$ when $\#\mathbb{O}_{\mathbf{d}} \geq 1$. As $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}_{\mathcal{H}}) = 1$, it follows that

$$\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}) \cap [\mathfrak{k}_{\mathcal{H}}, \mathfrak{k}_{\mathcal{H}}] = \dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}) - 1$$

when $\#\mathbb{O}_{\mathbf{d}} \geq 1$. The group $\mathrm{O}_2/\mathrm{SO}_2 = \mathbb{Z}/2\mathbb{Z}$ acts non-trivially on $\mathfrak{so}_2$ and the group $\mathrm{U}(m)$ acts trivially on $\mathfrak{z}(\mathfrak{u}(m))$. We next use the observation in (6.7) to conclude that

$$\dim_{\mathbb{R}} \big[\mathfrak{z}(\mathfrak{k}) \cap [\mathfrak{k}_{\mathcal{H}}, \mathfrak{k}_{\mathcal{H}}]\big]^{K/K^\circ} = \begin{cases} 0 & \text{if } \#\mathbb{O}_{\mathbf{d}} = 0 \\ \#\mathbb{O}_{\mathbf{d}} - 1 & \text{if } \#\mathbb{O}_{\mathbf{d}} \geq 1\,. \end{cases}$$

Now the theorem follows from Theorem 5.2.2. $\square$

6.7 Second cohomology of nilpotent orbits in $\mathfrak{sp}(p,q)$

Let n be a positive integer and (p,q) be a pair of non-negative integers such that $p+q=n$. As we deal with non-compact groups, we will further assume $p>0$ and $q>0$. In our next result, we compute the second cohomology groups of the nilpotent orbits in $\mathfrak{sp}(p,q)$ under the adjoint action of $\mathrm{Sp}(p,q)$. To state the result we use the parametrization of the nilpotent orbits as in Theorem 4.1.10. Throughout this subsection $\langle \cdot\,,\cdot\rangle$ denotes the Hermitian form on $\mathbb{H}^n$ defined by $\langle x,y\rangle := \overline{x}^t \mathrm{I}_{p,q} y$, for $x,y\in\mathbb{H}^n$, where $\mathrm{I}_{p,q}$ is as in (2.19).

Theorem 6.7.1. *Let $X \in \mathfrak{sp}(p,q)$ be a nilpotent element. Let $(\mathbf{d},\mathbf{sgn}_{\mathcal{O}_X}) \in \mathcal{Y}^{\mathrm{even}}(p,q)$ be the signed Young diagram of the orbit $\mathcal{O}_X$ (that is, $\Psi_{\mathrm{Sp}(p,q)}(\mathcal{O}_X) = (\mathbf{d},\mathbf{sgn}_{\mathcal{O}_X})$ in the notation of Theorem 4.1.10). Then*

$$\dim_{\mathbb{R}} H^2(\mathcal{O}_X,\mathbb{R}) \;=\; \#\mathbb{E}_{\mathbf{d}}.$$

Proof. Let $p+q=n$. As the theorem follows trivially when $X=0$ we assume that $X\neq 0$. Let $\{X,H,Y\}\subset\mathfrak{sp}(p,q)$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple. Let $V:=\mathbb{H}^n$, the right $\mathbb{H}$-vector space of column vectors. We consider V as a $\mathrm{Span}_{\mathbb{R}}\{X,H,Y\}$-module via its natural $\mathfrak{sp}(p,q)$-module structure. Let

$$V_{\mathbb{E}} := \bigoplus_{\eta\in\mathbb{E}_{\mathbf{d}}} M(\eta-1)\,; \qquad V_{\mathbb{O}} := \bigoplus_{\theta\in\mathbb{O}_{\mathbf{d}}} M(\theta-1).$$

Using Lemma 3.0.5, we see that $V=V_{\mathbb{E}}\oplus V_{\mathbb{O}}$ is an orthogonal decomposition of V with respect to $\langle\cdot\,,\cdot\rangle$. Let $\langle\cdot\,,\cdot\rangle_{\mathbb{E}} := \langle\cdot\,,\cdot\rangle|_{V_{\mathbb{E}}\times V_{\mathbb{E}}}$ and $\langle\cdot\,,\cdot\rangle_{\mathbb{O}} := \langle\cdot\,,\cdot\rangle|_{V_{\mathbb{O}}\times V_{\mathbb{O}}}$. Let

$X_{\mathbb{E}} := X|_{V_{\mathbb{E}}}$, $X_{\mathbb{O}} := X|_{V_{\mathbb{O}}}$, $H_{\mathbb{E}} := H|_{V_{\mathbb{E}}}$, $H_{\mathbb{O}} := H|_{V_{\mathbb{O}}}$, $Y_{\mathbb{E}} := Y|_{V_{\mathbb{E}}}$ and $Y_{\mathbb{O}} := Y|_{V_{\mathbb{O}}}$. Then we have the following natural isomorphism :

$$(6.57) \quad \mathcal{Z}_{\mathrm{Sp}(p,q)}(X,H,Y) \simeq \mathcal{Z}_{\mathrm{SU}(V_{\mathbb{E}},\langle\cdot,\cdot\rangle_{\mathbb{E}})}(X_{\mathbb{E}},H_{\mathbb{E}},Y_{\mathbb{E}}) \times \mathcal{Z}_{\mathrm{SU}(V_{\mathbb{O}},\langle\cdot,\cdot\rangle_{\mathbb{O}})}(X_{\mathbb{O}},H_{\mathbb{O}},Y_{\mathbb{O}}).$$

Recall that the non-degenerate form $(\cdot,\cdot)_d$ on $L(d-1)$ is skew-Hermitian for all $d \in \mathbb{E}_{\mathbf{d}}$ and Hermitian for all $d \in \mathbb{O}_{\mathbf{d}}$; see Remark 3.0.11. Moreover, for $\theta \in \mathbb{O}_{\mathbf{d}}$ the signature of $(\cdot,\cdot)_\theta$ is (p_θ, q_θ). It follows from Lemma 6.0.1 (4) that

$$\mathcal{Z}_{\mathrm{SU}(V_{\mathbb{E}},\langle\cdot,\cdot\rangle_{\mathbb{E}})}(X_{\mathbb{E}},H_{\mathbb{E}},Y_{\mathbb{E}}) \simeq \prod_{\eta\in\mathbb{E}_{\mathbf{d}}} \mathrm{SO}^*(2t_\eta)$$

and

$$\mathcal{Z}_{\mathrm{SU}(V_{\mathbb{O}},\langle\cdot,\cdot\rangle_{\mathbb{O}})}(X_{\mathbb{O}},H_{\mathbb{O}},Y_{\mathbb{O}}) \simeq \prod_{\theta\in\mathbb{O}_{\mathbf{d}}} \mathrm{Sp}(p_\theta,q_\theta).$$

In particular, $\mathcal{Z}_{\mathrm{SU}(V_{\mathbb{E}},\langle\cdot,\cdot\rangle_{\mathbb{E}})}(X_{\mathbb{E}}, H_{\mathbb{E}}, Y_{\mathbb{E}})$ and $\mathcal{Z}_{\mathrm{SU}(V_{\mathbb{O}},\langle\cdot,\cdot\rangle_{\mathbb{O}})}(X_{\mathbb{O}}, H_{\mathbb{O}}, Y_{\mathbb{O}})$ are both connected groups. Let $K_{\mathbb{E}}$ be a maximal compact subgroup of $\mathcal{Z}_{\mathrm{SU}(V_{\mathbb{E}},\langle\cdot,\cdot\rangle_{\mathbb{E}})}(X_{\mathbb{E}}, H_{\mathbb{E}}, Y_{\mathbb{E}}) \simeq \prod_{\eta\in\mathbb{E}_{\mathbf{d}}} \mathrm{SO}^*(2t_\eta)$ and $K_{\mathbb{O}}$ be a maximal compact subgroup of $\mathcal{Z}_{\mathrm{SU}(V_{\mathbb{O}},\langle\cdot,\cdot\rangle_{\mathbb{O}})}(X_{\mathbb{O}}, H_{\mathbb{O}}, Y_{\mathbb{O}}) \simeq \prod_{\theta\in\mathbb{O}_{\mathbf{d}}} \mathrm{Sp}(p_\theta, q_\theta)$. Let K be the image of $K_{\mathbb{E}} \times K_{\mathbb{O}}$ under the isomorphism as in (6.57). It is clear that K is a maximal compact subgroup of $\mathcal{Z}_{\mathrm{Sp}(p,q)}(X,H,Y)$. Let M be a maximal compact subgroup of $\mathrm{Sp}(p,q)$ containing K. As $M \simeq \mathrm{Sp}(p)\times\mathrm{Sp}(q)$ is semisimple and K is connected, using Theorem 5.2.2 we have that

$$H^2(\mathcal{O}_X,\mathbb{R}) \simeq \mathfrak{z}(\mathfrak{k}), \quad \text{for all } X \neq 0.$$

Let $\mathfrak{k}_{\mathbb{O}}$ and $\mathfrak{k}_{\mathbb{E}}$ be the Lie algebras of $K_{\mathbb{O}}$ and $K_{\mathbb{E}}$, respectively. As $K_{\mathbb{O}}$ is semisimple, we have $\mathfrak{z}(\mathfrak{k}_{\mathbb{O}}) = 0$. Hence, $\mathfrak{z}(\mathfrak{k}) \simeq \mathfrak{z}(\mathfrak{k}_{\mathbb{E}}) \oplus \mathfrak{z}(\mathfrak{k}_{\mathbb{O}}) = \mathfrak{z}(\mathfrak{k}_{\mathbb{E}})$. Since $\mathfrak{k}_{\mathbb{E}} \simeq \bigoplus_{\eta\in\mathbb{E}_{\mathbf{d}}} \mathfrak{u}(t_\eta)$, we have $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}_{\mathbb{E}}) = \#\mathbb{E}_{\mathbf{d}}$. This completes the proof. $\square$

Chapter 7

Second cohomology of nilpotent orbits in non-compact non-complex exceptional real Lie algebras

In this chapter we study the second de Rham cohomology groups of the nilpotent orbits in non-compact non-complex exceptional Lie algebras over $\mathbb{R}$. The results in this chapter depend on the results of [Dj1, Tables VI-XV], [Dj2, Tables VII-VIII] and [Ki, Tables 1-12].

For the sake of convenience of writing the proofs, it will be useful to divide the nilpotent orbits in the following three types. Let $X \in \mathfrak{g}$ be a nonzero nilpotent element, and $\{X, H, Y\}$ be a $\mathfrak{sl}_2(\mathbb{R})$-triple in $\mathfrak{g}$. Let G be as in the beginning of §4.2. Let K be a maximal compact subgroup in $\mathcal{Z}_{G(\mathbb{R})^\circ}(X, H, Y)$, and M be a maximal compact subgroup in $G(\mathbb{R})^\circ$ containing K. A nonzero nilpotent orbit $\mathcal{O}_X$ in $\mathfrak{g}$ is said to be of

1. *type I* if $\mathfrak{z}(\mathfrak{k}) \neq 0$, $K/K^\circ = \mathrm{Id}$ and $\mathfrak{m} = [\mathfrak{m}, \mathfrak{m}]$;

2. *type II* if either $\mathfrak{z}(\mathfrak{k}) \neq 0$, $K/K^\circ \neq \mathrm{Id}$, $\mathfrak{m} = [\mathfrak{m}, \mathfrak{m}]$; or $\mathfrak{z}(\mathfrak{k}) \neq 0$, $\mathfrak{m} \neq [\mathfrak{m}, \mathfrak{m}]$;

3. *type III* if $\mathfrak{z}(\mathfrak{k}) = 0$.

In what follows we will use the next result repeatedly.

Corollary 7.0.1. *Let $\mathfrak{g}$ be a real simple non-compact exceptional Lie algebra. Let $X \in \mathfrak{g}$ be a nonzero nilpotent element.*

1. *If the orbit $\mathcal{O}_X$ is of type I, then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = \dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k})$.

2. *If the orbit $\mathcal{O}_X$ is of type II, then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) \leq \dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k})$.

3. *If the orbit $\mathcal{O}_X$ is of type III, then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 0$.

Proof. The proof of the corollary follows immediately from Theorem 5.2.2. □

Let $\mathfrak{g}$ be as above. In the proofs of our results in the following subsections we use the description of a Levi factor of $\mathfrak{z}_{\mathfrak{g}}(X)$ for each nilpotent element X in $\mathfrak{g}$, as given in the last columns of [Dj1, Tables VI-XV] and [Dj2, Tables VII-VIII]. This enables us compute the dimensions $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k})$ easily. We also use [Ki, Column 4, Tables 1-12] for the component groups for each nilpotent orbits in $\mathfrak{g}$.

7.1 Nilpotent orbits in the non-compact real form of G_2

Recall that up to conjugation there is only one non-compact real form of G_2. We denote it by $G_{2(2)}$. There are only five nonzero nilpotent orbits in $G_{2(2)}$; see [Dj1, Table VI, p. 510]. Note that in this case we have $\mathfrak{m} = [\mathfrak{m}, \mathfrak{m}]$.

Theorem 7.1.1. *Let the parametrization of the nilpotent orbits be as in §4.2.1. Let X be a nonzero nilpotent element in $G_{2(2)}$.*

1. *If the parametrization of the orbit $\mathcal{O}_X$ is given by either* $1\ 1$ *or* $1\ 3$*, then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 1$.

2. *If the parametrization of the orbit $\mathcal{O}_X$ is given by any of* $2\ 2$, $0\ 4$, $4\ 8$*, then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 0$.

Proof. From [Dj1, Column 7, Table VI, p. 510] we have $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}) = 1$ and from [Ki, Column 4, Table 1, p. 247] we have $K/K^{\circ} = \mathrm{Id}$ for the nilpotent orbits as in (1). Thus these are of type I. We refer to [Dj1, Column 7, Table VI, p. 510] for the orbits as given in (2). These orbits are of type III as $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}) = 0$. In view of the Corollary 7.0.1 the conclusions follow. □

7.2 Nilpotent orbits in non-compact real forms of F_4

Recall that up to conjugation there are two non-compact real forms of F_4. They are denoted by $F_{4(4)}$ and $F_{4(-20)}$.

Nilpotent orbits in $F_{4(4)}$.

There are 26 nonzero nilpotent orbits in $F_{4(4)}$; see [Dj1, Table VII, p. 510]. Note that in this case we have $\mathfrak{m} = [\mathfrak{m}, \mathfrak{m}]$.

Theorem 7.2.1. *Let the parametrization of the nilpotent orbits be as in §4.2.1. Let X be a nonzero nilpotent element in $F_{4(4)}$.*

1. *Assume the parametrization of the orbit* $\mathcal{O}_X$ *is given by any of the sequences :* 001 1, 001 3, 110 2, 111 1, 131 3. *Then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 1$.

2. *Assume the parametrization of the orbit* $\mathcal{O}_X$ *is given by any of the sequences :* 100 2, 200 0, 103 1, 111 3, 204 4. *Then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) \leq 1$.

3. *If the parametrization of the orbit* $\mathcal{O}_X$ *is either* 101 1 *or* 012 2*, then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) \leq 2$.

4. *If* $\mathcal{O}_X$ *is not given by the parametrizations as in (1), (2), (3) above (# of such orbits are 14), then we have* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 0$.

Proof. For the Lie algebra $F_{4(4)}$, we can easily compute $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k})$ from the last column of [Dj1, Table VII, p. 510] and K/K° from [Ki, Column 4, Table 2, pp. 247-248].

For the orbits $\mathcal{O}_X$, as in (1), we have $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}) = 1$ and $K/K^\circ = \mathrm{Id}$. Hence these are of type I. For the orbits $\mathcal{O}_X$, as in (2), we have $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}) = 1$ and $K/K^\circ \neq \mathrm{Id}$; hence they are of type II. For the orbits $\mathcal{O}_X$, as in (3), we have $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}) = 2$ and $K/K^\circ \neq \mathrm{Id}$. Hence these are also of type II. The rest of the 14 orbits, which are not given by the parametrizations in (1), (2), (3), are of type III as $\mathfrak{z}(\mathfrak{k}) = 0$. Now the theorem follows from Corollary 7.0.1. □

Nilpotent orbits in $F_{4(-20)}$

There are two nonzero nilpotent orbits in $F_{4(-20)}$; see [Dj1, Table VIII, p. 511].

Theorem 7.2.2. *For every nilpotent element* $X \in F_{4(-20)}$, $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 0$.

Proof. As the theorem follows trivially when $X = 0$ we assume that $X \neq 0$. We follow the parametrization of nilpotent orbits as in §4.2.1. From the last column of [Dj1, Table VIII, p. 511] we conclude that $\mathfrak{z}(\mathfrak{k}) = 0$. Hence the nonzero nilpotent orbits are of type III. Using Corollary 7.0.1 (3) we have $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 0$. □

7.3 Nilpotent orbits in non-compact real forms of E_6

Recall that up to conjugation there are four non-compact real forms of E_6. They are denoted by $E_{6(6)}$, $E_{6(2)}$, $E_{6(-14)}$ and $E_{6(-26)}$.

Nilpotent orbits in $E_{6(6)}$

There are 23 nonzero nilpotent orbits in $E_{6(6)}$; see [Dj2, Table VIII, p. 205]. Note that in this case we have $\mathfrak{m} = [\mathfrak{m}, \mathfrak{m}]$.

Theorem 7.3.1. *Let the parametrization of the nilpotent orbits be as in §4.2.2. Let X be a nonzero nilpotent element in $E_{6(6)}$.*

1. *If the parametrization of the orbit $\mathcal{O}_X$ is given by either* 1001 *or* 1101 *or* 1211, *then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 1$.
2. *Assume the parametrization of the orbit $\mathcal{O}_X$ is given by any of the sequences :* $0102, 0202, 1010,$ $2002, 1011$. *Then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) \leq 1$.
3. *If $\mathcal{O}_X$ is not given by the parametrizations as in (1), (2) above (# of such orbits are 15), then we have* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 0$.

Proof. For the Lie algebra $E_{6(6)}$, we can easily compute $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k})$ from the last column of [Dj2, Table VIII, p. 205] and K/K° from [Ki, Column 4, Table 4, p.253]. As pointed out in the 1st paragraph of [Ki, p. 254], there is an error in row 5 of [Dj2, Table VIII, p. 205]. Thus when $\mathcal{O}_X$ is given by the parametrization 2000 it follows from [Ki, p. 254] that $\mathfrak{z}(\mathfrak{k}) = 0$.

We have $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}) = 1$ and $K/K^\circ = \mathrm{Id}$ for the orbits given in (1). Thus these orbits are of type I. For the orbits, as in (2), we have $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}) = 1$ and $K/K^\circ = \mathbb{Z}_2$.

Hence, the orbits in (2) are of type II. For rest of the 15 nonzero nilpotent orbits, which are not given by the parametrizations of (1), (2), are of type III as $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}) = 0$. Now the results follow from Corollary 7.0.1. $\square$

Nilpotent orbits in $E_{6(2)}$

There are 37 nonzero nilpotent orbits in $E_{6(2)}$; see [Dj1, Table IX, p. 511]. Note that in this case we have $\mathfrak{m} = [\mathfrak{m}, \mathfrak{m}]$.

Theorem 7.3.2. *Let the parametrization of the nilpotent orbits be as in §4.2.1. Let X be a nonzero nilpotent element in $E_{6(2)}$.*

1. *Assume the parametrization of the orbit $\mathcal{O}_X$ is given by any of the sequences :* 00000 4, 00200 2, 02020 0, 00400 8, 22222 2, 04040 4, 44044 4, 44444 8. *Then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 0$.

2. *Assume the parametrization of the orbit $\mathcal{O}_X$ is given by any of the sequences :* 10001 2, 10101 1, 21001 1, 10012 1, 11011 2, 01210 2, 10301 1, 11111 3, 22022 0. *Then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 2$.

3. *If the parametrization of the orbit $\mathcal{O}_X$ is given by either* 20002 0 *or* 00400 0 *or* 02020 4, *then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) \leq 2$.

4. *If the parametrization of the orbit $\mathcal{O}_X$ is given by* 20202 2, *then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) \leq 1$.

5. *If $\mathcal{O}_X$ is not given by the parametrizations as in (1), (2), (3), (4) above (# of such orbits are 16), then we have* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 1$.

Proof. For the Lie algebra $E_{6(2)}$, we can easily compute $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k})$ from the last column of [Dj1, Table IX, p. 511] and K/K° from [Ki, Column 4, Table 5, pp. 255-256].

We have $\mathfrak{z}(\mathfrak{k}) = 0$ for the orbits, as given in (1), and these orbits are of type III. For the orbits, as given in (2), we have $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}) = 2$ and $K/K^\circ = \mathrm{Id}$. Thus the orbits in (2) are of type I. For the orbits, as given in (3), we have $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}) = 2$ and $K/K^\circ \neq \mathrm{Id}$, hence are of type II. For the orbits, as given in (4), we have $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}) = 1$ and $K/K^\circ = \mathbb{Z}_2$. Thus this orbit is of type II. For the rest of 16 orbits, which are not given in any of (1), (2), (3), (4), we have $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}) = 1$ and $K/K^\circ = \mathrm{Id}$. Thus these orbits are of type I. Now the conclusions follow from Corollary 7.0.1. □

Nilpotent orbits in $E_{6(-14)}$

There are 12 nonzero nilpotent orbits in $E_{6(-14)}$; see [Dj1, Table X, p. 512]. Note that in this case $\mathfrak{m} \simeq \mathfrak{so}_{10} \oplus \mathbb{R}$, and hence $[\mathfrak{m}, \mathfrak{m}] \neq \mathfrak{m}$.

Theorem 7.3.3. *Let the parametrization of the nilpotent orbits be as in §4.2.1. Let X be a nonzero nilpotent element in $E_{6(-14)}$.*

1. *If the parametrization of the orbit $\mathcal{O}_X$ is given by* $40000 \;\; -2$*, then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 0$.
2. *If $\mathcal{O}_X$ is not given by the above parametrization (# of such orbits are 11), then we have* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) \leq 1$.

Proof. For the Lie algebra $E_{6(-14)}$, we can easily compute $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k})$ from the last column of [Dj1, Table X, p. 512]. The orbit in (1) is of type I as $\mathfrak{z}(\mathfrak{k}) = 0$, and hence $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 0$. The other 11 orbits are of type II as $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}) = 1$ and $\mathfrak{m} \neq [\mathfrak{m}, \mathfrak{m}]$. Hence $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) \leq 1$. □

Nilpotent orbits in $E_{6(-26)}$

There are two nonzero nilpotent orbits in $E_{6(-26)}$; see [Dj2, Table VII, p. 204].

Theorem 7.3.4. *For every nilpotent element $X \in E_{6(-26)}$,* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 0$.

Proof. As the theorem follows trivially when $X = 0$ we assume that $X \neq 0$. We follow the parametrization of the nilpotent orbits as given in §4.2.2. The two nonzero nilpotent orbits in $E_{6(-26)}$ are of type III as $\mathfrak{z}(\mathfrak{k}) = 0$; see last column of [Dj2, Table VII, p. 204]. Hence, by Corollary 7.0.1(3) we conclude that $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 0$. □

7.4 Nilpotent orbits in non-compact real forms of E_7

Recall that up to conjugation there are three non-compact real forms of E_7. They are denoted by $E_{7(7)}$, $E_{7(-5)}$ and $E_{7(-25)}$.

Nilpotent orbits in $E_{7(7)}$

There are 94 nonzero nilpotent orbits in $E_{7(7)}$; see [Dj1, Table XI, pp. 513-514]. Note that in this case we have $\mathfrak{m} = [\mathfrak{m}, \mathfrak{m}]$.

Theorem 7.4.1. *Let the parametrization of the nilpotent orbits be as in §4.2.1. Let X be a nonzero nilpotent element in $E_{7(7)}$.*

1. *If the parametrization of the orbit $\mathcal{O}_X$ is given by* 1011101*, then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 3$.
2. *Assume the parametrization of the orbit $\mathcal{O}_X$ is given by any of the sequences:* 1001001, 1101011, 1111010, 0101111, 2200022, 3101021, 1201013, 1211121, 2204022*. Then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 2$.
3. *Assume the parametrization of the orbit $\mathcal{O}_X$ is given by any of the sequences :* 0100010, 1100100, 0010011, 3000100, 0010003, 0102010, 0200020, 2004002, 2103101, 1013012, 2020202, 1311111, 1111131, 1310301, 1030131, 2220222,

3013131, 1313103, 3113121, 1213113, 4220224, 3413131, 1313143, 4224224. *Then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 1$.

4. *Assume the parametrization of the orbit* $\mathcal{O}_X$ *is given by any of the sequences:* 2000002, 0101010, 2002002, 1110111, 2020020, 0200202, 1112111, 2022020, 0202202, 2202022, 0220220. *Then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) \leq 1$.

5. *Assume the parametrization of the orbit* $\mathcal{O}_X$ *is given by any of the sequences:* 2010001, 1000102, 0120101, 1010210, 1030010, 0100301, 3013010, 0103103. *Then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) \leq 2$.

6. *If the parametrization of the orbit* $\mathcal{O}_X$ *is given by either* 1010101 *or* 0020200, *then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) \leq 3$.

7. *If* $\mathcal{O}_X$ *is not given by the parametrizations as in (1), (2), (3), (4), (5), (6) above (# of such orbits are 39), then we have* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 0$.

Proof. For the Lie algebra $E_{7(7)}$, we can easily compute $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k})$ from the last column of [Dj1, Table XI, pp. 513-514] and K/K° from [Ki, Column 4, Table 8, pp. 260-264].

The orbit $\mathcal{O}_X$, as given in (1), is of type I as $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}) = 3$ and $K/K^\circ = \text{Id}$. For the orbits, as given in (2), we have $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}) = 2$ and $K/K^\circ = \text{Id}$. Hence these are also of type I. For the orbits, as given in (3), we have $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}) = 1$ and $K/K^\circ = \text{Id}$; hence they are of type I. For the orbits, as given in (4), we have $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}) = 1$ and $K/K^\circ = \mathbb{Z}_2$. Thus these are of type II. For the orbits, as given in (5), we have $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}) = 2$ and $K/K^\circ = \mathbb{Z}_2$. Hence these are also of type II. For the orbits, as given in (6), we have $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}) = 3$ and $K/K^\circ \neq \text{Id}$, hence they are of type II. Rest of the 39 orbits, which are not given by the parametrizations in (1), (2), (3), (4), (5), (6), are of type III as $\mathfrak{z}(\mathfrak{k}) = 0$. Now the results follow from Corollary 7.0.1. □

Nilpotent orbits in $E_{7(-5)}$

There are 37 nonzero nilpotent orbits in $E_{7(-5)}$; see [Dj1, Table XII, p. 515]. Note that in this case $\mathfrak{m} = [\mathfrak{m}, \mathfrak{m}]$.

Theorem 7.4.2. *Let the parametrization of the nilpotent orbits be as in §4.2.1. Let X be a nonzero nilpotent element in $E_{7(-5)}$.*

1. *If the parametrization of the orbit $\mathcal{O}_X$ is given by either* 110001 1 *or* 000120 2*, then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 2$*.*

2. *Assume the parametrization of the orbit $\mathcal{O}_X$ is given by any of the sequences:* 000010 1, 010000 2, 000010 3, 010010 1, 200100 0, 010100 2, 000200 0, 010110 1, 010030 1, 010110 3, 201031 4, 010310 3.
Then $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 1$*.*

3. *If the parametrization of the orbit $\mathcal{O}_X$ is given by either* 020200 0 *or* 111110 1*, then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) \leq 2$*.*

4. *Assume the parametrization of the orbit $\mathcal{O}_X$ is given by any of the sequences:* 020000 0*,* 201011 2*,* 040000 4*,* 040400 4*. Then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) \leq 1$*.*

5. *If $\mathcal{O}_X$ is not given by the parametrizations as in (1), (2), (3), (4) above (# of such orbits are 17), then we have* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 0$*.*

Proof. For the Lie algebra $E_{7(-5)}$, we can easily compute $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k})$ from the last column of [Dj1, Table XII, pp. 515] and K/K° from [Ki, Column 4, Table 9, pp. 266-268].

For the orbit $\mathcal{O}_X$, as in (1), we have $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}) = 2$ and $K/K^\circ = \text{Id}$. Hence these orbits are of type I. For the orbit $\mathcal{O}_X$, as in (2), we have $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}) = 1$ and $K/K^\circ = \text{Id}$. Hence these orbits are also of type I. For the orbit $\mathcal{O}_X$, as in (3), we have $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}) = 2$ and $K/K^\circ = \mathbb{Z}_2$, hence are of type II. For the orbit $\mathcal{O}_X$, as in

(4), we have $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}) = 1$ and $K/K^\circ = \mathbb{Z}_2$. Hence these are also of type II. Rest of the 17 orbits, which are not given by the parametrizations in (1), (2), (3), (4), are of type III as $\mathfrak{z}(\mathfrak{k}) = 0$. Now the conclusions follow from Corollary 7.0.1. $\square$

Nilpotent orbits in $E_{7(-25)}$

There are 22 nonzero nilpotent orbits in $E_{7(-25)}$; see [Dj1, Table XIII, p. 516]. In this case we have $\mathfrak{m} \neq [\mathfrak{m}, \mathfrak{m}]$.

Theorem 7.4.3. *Let the parametrization of the nilpotent orbits be as in §4.2.1. Let X be a nonzero nilpotent element in $E_{7(-25)}$.*

1. *Assume the parametrization of the orbit $\mathcal{O}_X$ is given by any of the sequences:* $000000\,2$, $000000\ -2$, $000002\ -2$, $200000\ -2$, $200002\ -2$, $400000\ -2$, $000004-6$, $200002\ -6$, $400004\ -6$, $400004\ -10$. *Then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 0$.

2. *If $\mathcal{O}_X$ is not given by any of the above parametrization (# of such orbits are* 12*), then we have* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) \leq 1$.

Proof. Note that the parametrization of nilpotent orbits in $E_{7(-25)}$ as in [Ki, Table 10] is different from [Dj1, Table X III, p. 516]. As the component group for all orbits in $E_{7(-25)}$ is Id; see [Ki, Column 4, Table 10, pp. 269-270], it does not depend on the parametrization. We refer to the last column of [Dj1, Table X III] for the orbits as given in (1). These are type III as $\mathfrak{z}(\mathfrak{k}) = 0$. For rest of the 12 orbits we have $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}) = 1$; see last column of [Dj1, Table X III]. As $\mathfrak{m} \neq [\mathfrak{m}, \mathfrak{m}]$, these are of type II. Now the results follow from Corollary 7.0.1. $\square$

7.5 Nilpotent orbits in non-compact real forms of E_8

Recall that up to conjugation there are two non-compact real forms of E_8. They are denoted by $E_{8(8)}$ and $E_{8(-24)}$.

Nilpotent orbits in $E_{8(8)}$

There are 115 nonzero nilpotent orbits in $E_{8(8)}$; see [Dj1, Table XIV, pp. 517-519]. Note that in this case we have $\mathfrak{m} = [\mathfrak{m}, \mathfrak{m}]$.

Theorem 7.5.1. *Let the parametrization of the nilpotent orbits be as in §4.2.1. Let X be a nonzero nilpotent element in $E_{8(8)}$.*

1. *Assume the parametrization of the orbit $\mathcal{O}_X$ is given by any of the sequences:* 10010011, 11110010, 10111011, 11110130. *Then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 2$.

2. *Assume the parametrization of the orbit $\mathcal{O}_X$ is given by any of the sequences:* 01000010, 10001000, 30000001, 10010001, 01010010, 01000110, 10100100, 00100003, 11001030, 10110100, 21010100, 01020110, 30001030, 11010101, 11101011, 11010111, 11111101, 21031031, 31010211, 12111111, 13111101, 13111141, 13103041, 31131211, 13131043, 34131341.
Then $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 1$.

3. *If the parametrization of the orbit $\mathcal{O}_X$ is given* 00100101, *then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) \leq 3$.

4. *Assume the parametrization of the orbit $\mathcal{O}_X$ is given by any of the sequences:* 10001002, 10101001, 01200100, 02000200, 10101021, 10102100, 02020200, 01201031. *Then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) \leq 2$.

5. *Assume the parametrization of the orbit* $\mathcal{O}_X$ *is given by any of the sequences:* 11000001, 20010000, 01000100, 11001010, 20100011, 01010100, 02020000, 20002000, 20100031, 10101011, 00200022, 11110110, 01011101, 01003001, 11101101, 11101121, 10300130, 04020200, 02002022, 00400040, 11121121, 30130130, 02022022, 40040040. *Then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) \leq 1$.

6. *If* $\mathcal{O}_X$ *is not given by the parametrizations as in (1), (2), (3), (4), (5) above (# of such orbits are* 52*), then we have* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 0$.

Proof. For the Lie algebra $E_{8(8)}$, we can easily compute $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k})$ from the last column of [Dj1, Table XIV, pp. 517-519] and K/K° from [Ki, Column 4, Table 11, pp. 271-275].

For the orbits $\mathcal{O}_X$, as given in (1), we have $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}) = 2$ and $K/K^\circ = \mathrm{Id}$. Hence these orbits are of type I. For the orbits $\mathcal{O}_X$, as given in (2), we have $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}) = 1$ and $K/K^\circ = \mathrm{Id}$. Hence these orbits are also of type I. For the orbit $\mathcal{O}_X$, as given in (3), we have $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}) = 3$ and $K/K^\circ \neq \mathrm{Id}$; hence they are of type II. For the orbits $\mathcal{O}_X$, as given in (4), we have $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}) = 2$ and $K/K^\circ \neq \mathrm{Id}$. Thus these orbits are of type II. For the orbits $\mathcal{O}_X$, as given in (5), we have $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}) = 1$ and $K/K^\circ \neq \mathrm{Id}$. Hence these are of type II. Rest of the 52 orbits, which are not given by the parametrizations of (1), (2), (3), (4), (5), are of type III as $\mathfrak{z}(\mathfrak{k}) = 0$. Now the conclusions follow from Corollary 7.0.1. □

Nilpotent orbits in $E_{8(-24)}$

There are 36 nonzero nilpotent orbits in $E_{8(-24)}$; see [Dj1, Table XV, p. 520]. Note that in this case we have $\mathfrak{m} = [\mathfrak{m}, \mathfrak{m}]$.

Theorem 7.5.2. *Let the parametrization of the nilpotent orbits be as in §4.2.1. Let* X *be a nonzero nilpotent element in* $E_{8(-24)}$.

1. *Assume the parametrization of the orbit* $\mathcal{O}_X$ *is given by any of the sequences:* $0000001\,1$, $1000000\,2$, $0000001\,3$, $1000001\,1$, $1100000\,1$, $1000010\,2$, $0000012\,2$, $1000011\,1$, $1000011\,3$, $1000003\,1$, $0110001\,2$, $1010011\,1$, $1000031\,3$.
   Then $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 1$.

2. *If the parametrization of the orbit* $\mathcal{O}_X$ *is given by either* $2000000\,0$ *or* $2000020\,0$, *then* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) \leq 1$.

3. *If* $\mathcal{O}_X$ *is not given by the parametrizations as in (1), (2) above (# of such orbits are 21), then we have* $\dim_{\mathbb{R}} H^2(\mathcal{O}_X, \mathbb{R}) = 0$.

Proof. For the Lie algebra $E_{8(-24)}$, we can easily compute $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k})$ from the last column of [Dj1, Table XV, p. 520] and K/K° from [Ki, Column 4, Table 12, pp. 277-278].

For the orbits $\mathcal{O}_X$, as given in (1), we have $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}) = 1$ and $K/K^\circ = \text{Id}$, hence these are of type I. For the orbits $\mathcal{O}_X$, as given in (2), we have $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}) = 1$ and $K/K^\circ \neq \text{Id}$. Hence these orbits are of type II. Rest of the 21 orbits, which are not given by the parametrizations of (1), (2), are of type III as $\mathfrak{z}(\mathfrak{k}) = 0$. Now the conclusions follow from Corollary 7.0.1. $\square$

Chapter 8

First cohomology of nilpotent orbits in simple non-compact Lie algebras

In this chapter, we compute the first de Rham cohomology groups of the nilpotent orbits. We begin by observing that in the case of complex simple Lie algebras the first cohomology of all the nilpotent orbits vanish.

Theorem 8.0.1. *Let $\mathfrak{g}$ be a complex simple Lie algebra. Then $H^1(\mathcal{O}_X, \mathbb{R}) = 0$, for all nilpotent elements $X \in \mathfrak{g}$.*

Proof. Any maximal compact subgroup of a simple complex Lie group is simple. The conclusion follows from Corollary 5.1.8. $\square$

8.1 First cohomology of nilpotent orbits in non-compact non-complex real classical Lie algebras

In this section we apply the results of the Chapter 6 to compute the first cohomology groups of the nilpotent orbits in the non-compact non-complex real classical Lie algebras. We first show that the first cohomology of all the nilpotent orbits in $\mathfrak{sl}_n(\mathbb{H})$ and $\mathfrak{sp}(p,q)$ vanish.

Theorem 8.1.1. *Let $\mathfrak{g}$ be either $\mathfrak{sl}_n(\mathbb{H})$ or $\mathfrak{sp}(p,q)$. Then $H^1(\mathcal{O}_X, \mathbb{R}) = 0$, for all nilpotent elements $X \in \mathfrak{g}$.*

Proof. Let G be $\mathrm{SL}_n(\mathbb{H})$ or $\mathrm{Sp}(p,q)$ according as $\mathfrak{g}$ is $\mathfrak{sl}_n(\mathbb{H})$ or $\mathfrak{sp}(p,q)$. Then any maximal compact subgroup of G is simple. The proof now follows from Theorem 5.2.2. □

Theorem 8.1.2. *Let $X \in \mathfrak{sl}_n(\mathbb{R})$ be a non-zero nilpotent element. Then*

$$\dim_{\mathbb{R}} H^1(\mathcal{O}_X, \mathbb{R}) = \begin{cases} 1 & \text{if } n = 2 \\ 0 & \text{if } n \geq 3. \end{cases}$$

Proof. We follow the notations as in the proof of Theorem 6.1.1. When $n \geq 3$ it is clear that $\mathfrak{m} = [\mathfrak{m}, \mathfrak{m}]$. When $n = 2$ we have $\mathfrak{m} \simeq \mathfrak{so}_2$ and $\Psi_{\mathrm{SL}_n(\mathbb{R})}(\mathcal{O}_X) = [2^1]$. Thus, using (6.4) we see that $\mathfrak{k} = 0$. Now the proof follows from Theorem 5.2.2. □

Theorem 8.1.3. *Let $X \in \mathfrak{su}(p,q)$ be a nilpotent element. Let $(\mathbf{d}, \mathbf{sgn}_{\mathcal{O}_X}) \in \mathcal{Y}(p,q)$ be the signed Young diagram of the orbit $\mathcal{O}_X$ (that is, $\Psi_{\mathrm{SU}(p,q)}(\mathcal{O}_X) = (\mathbf{d}, \mathbf{sgn}_{\mathcal{O}_X})$ as in the notation of Theorem 4.1.4). Let $l := \#\{d \mid d \in \mathbb{N}_{\mathbf{d}}, p_d \neq 0\} + \#\{d \mid d \in \mathbb{N}_{\mathbf{d}}, q_d \neq 0\}$.*

1. *If $\mathbb{N}_{\mathbf{d}} = \mathbb{E}_{\mathbf{d}}$, then $\dim_{\mathbb{R}} H^1(\mathcal{O}_X, \mathbb{R}) = 1$.*

2. *If* $l = 1$ *and* $\mathbb{N}_{\mathbf{d}} = \mathbb{O}_{\mathbf{d}}$*, then* $\dim_{\mathbb{R}} H^1(\mathcal{O}_X, \mathbb{R}) = 1$.

3. *If* $l \geq 2$ *and* $\#\mathbb{O}_{\mathbf{d}} \geq 1$*, then* $\dim_{\mathbb{R}} H^1(\mathcal{O}_X, \mathbb{R}) = 0$.

Proof. We follow the notations as in the proof of Theorem 6.3.4. We now appeal to Proposition 6.3.3 to make the following observations :

1. If $\mathbb{N}_{\mathbf{d}} = \mathbb{E}_{\mathbf{d}}$, then $\mathfrak{k} \subset [\mathfrak{m}, \mathfrak{m}]$. Hence, $\mathfrak{k} + [\mathfrak{m}, \mathfrak{m}] \subsetneq \mathfrak{m}$.

2. If $\mathbf{d} = [d^{t_d}]$, then $\mathfrak{z}(\mathfrak{k}) = 0$. Hence, $\mathfrak{k} + [\mathfrak{m}, \mathfrak{m}] \subsetneq \mathfrak{m}$.

3. If $\#\mathbb{O}_{\mathbf{d}} \geq 1$ and $l \geq 2$, then $\mathfrak{k} + [\mathfrak{m}, \mathfrak{m}] = \mathfrak{m}$.

As $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{m}) = 1$, in view of the Theorem 5.2.2, the proof follows. □

We next describe the first cohomology groups of nilpotent orbits in the simple Lie algebra $\mathfrak{so}(p, q)$ when $p > 0$, $q > 0$. Recall that in view of [Kn, Theorem 6.105, p. 421] and isomorphisms (iv), (v), (vi), (ix), (x) in [He, Chapter X, §6, pp. 519-520], to ensure simplicity of $\mathfrak{so}(p, q)$, we further assume that $(p, q) \notin \{(1, 1), (2, 2)\}$; see §6.4 also.

Theorem 8.1.4. *Consider* $\mathfrak{so}(p, q)$*, and assume that* $p \neq 2$*,* $q \neq 2$ *and* $(p, q) \neq (1, 1)$*. Then* $H^1(\mathcal{O}_X, \mathbb{R}) = 0$ *for all nilpotent elements* X *in* $\mathfrak{so}(p, q)$*.*

Proof. Let $\mathfrak{m}$, $\mathfrak{k}$ be as in the proof of Theorem 6.4.8. Since $p \neq 2$, $q \neq 2$, we have $\mathfrak{m} = [\mathfrak{m}, \mathfrak{m}]$. Using Theorem 5.2.2 we conclude that $H^1(\mathcal{O}_X, \mathbb{R}) = 0$. □

We will now consider the remaining cases of $\mathfrak{so}(p, q)$ which are not covered in Theorem 8.1.4; they are: $p > 2, q = 2$; $p = 2, q > 2$ and $(p, q) \in \{(2, 1), (1, 2)\}$. In Section 6.4 it was observed that when $p > 2, q = 2$, the non-zero nilpotent orbits correspond to only four possible signed Young diagrams as given in (**a**.1), (**a**.2), (**a**.3), (**a**.4), and similarly, when $p = 2, q > 2$, the non-zero nilpotent orbits correspond to only four possible signed Young diagrams as given in (**b**.1), (**b**.2), (**b**.3), (**b**.4).

Theorem 8.1.5. *Let $\Psi_{\mathrm{SO}(p,q)^\circ}$ be the parametrization in Theorem 4.1.6. Let $\mathcal{O}_X \in \mathcal{N}(\mathrm{SO}(p,q)^\circ)$. Then the following hold:*

1. *Suppose $(p,q) \in \{(2,1),\,(1,2)\}$, then $H^1(\mathcal{O}_X,\,\mathbb{R}) = 1$.*

2. *Assume that $p > 2$ and $q = 2$.*

   (i) If $\Psi_{\mathrm{SO}(p,2)^\circ}(\mathcal{O}_X)$ is as in either $(\mathbf{a}.1)$ or $(\mathbf{a}.2)$ or $(\mathbf{a}.3)$, then $\dim_{\mathbb{R}} H^1(\mathcal{O}_X,\mathbb{R}) = 1$.

   (ii) If $\Psi_{\mathrm{SO}(p,2)^\circ}(\mathcal{O}_X)$ is as in $(\mathbf{a}.4)$, then $H^1(\mathcal{O}_X,\,\mathbb{R}) = 0$.

3. *Assume that $p = 2$ and $q > 2$.*

   (i) If $\Psi_{\mathrm{SO}(2,q)^\circ}(\mathcal{O}_X)$ is as in $(\mathbf{b}.1)$ or $(\mathbf{b}.2)$ or $(\mathbf{b}.3)$, then $\dim_{\mathbb{R}} H^1(\mathcal{O}_X,\,\mathbb{R}) = 1$.

   (ii) If $\Psi_{\mathrm{SO}(2,q)^\circ}(\mathcal{O}_X)$ is as in $(\mathbf{b}.4)$, then $H^1(\mathcal{O}_X,\,\mathbb{R}) = 0$.

Proof. As $X \neq 0$, it lies in a $\mathfrak{sl}_2(\mathbb{R})$-triple, say $\{X,\,H,\,Y\}$, in $\mathfrak{so}(p,q)$.

Proof of (1): Let K' be a maximal compact subgroup of $\mathcal{Z}_{\mathrm{SO}(p,q)^\circ}(X,H,Y)$. Let $\mathfrak{k}'$ be the Lie algebra of K' and $\mathfrak{m}$ the Lie algebra of a maximal compact subgroup of $\mathrm{SO}(p,q)^\circ$ which contains K'. When $(p,q) \in \{(2,1),(1,2)\}$, we have $\dim_{\mathbb{R}} \mathfrak{m} = 1$ and $\Psi'_{\mathrm{SO}(p,q)^\circ}(\mathcal{O}_X) = [3^1]$. In particular, $\dim_{\mathbb{R}} L(3-1) = 1$. Using Lemma 6.0.1 (4) we have $\mathfrak{k}' = 0$. Hence, using Theorem 5.2.2, we have $\dim_{\mathbb{R}} H^1(\mathcal{O}_X,\,\mathbb{R}) = 1$.

Proof of (2): We first prove (2)(i). Let $\Psi_{\mathrm{SO}(p,2)^\circ}(\mathcal{O}_X)$ be as in $(\mathbf{a}.1)$, $(\mathbf{a}.2)$ or $(\mathbf{a}.3)$. Let K and M be the maximal compact subgroups of $\mathcal{Z}_{\mathrm{SO}(p,2)}(X,H,Y)$ and $\mathrm{SO}(p,2)$ respectively, as defined in the first paragraph of the proof of Theorem 6.4.9(2). Recall that $K_{\mathbb{O}} := K \cap M^\circ = K \cap \mathrm{SO}(p,2)^\circ$ is a maximal compact subgroup of $\mathcal{Z}_{\mathrm{SO}(p,2)^\circ}(X,H,Y)$. Let $\mathfrak{k}_{\mathbb{O}}$ and $\mathfrak{m}$ be the Lie algebras of $K_{\mathbb{O}}$ and M° respectively. Using (6.24), (6.25), (6.26) for the signed Young diagrams $(\mathbf{a}.1)$, $(\mathbf{a}.2)$, $(\mathbf{a}.3)$ respectively, we observe that in all the cases $\mathfrak{k}_{\mathbb{O}} \subset [\mathfrak{m},\,\mathfrak{m}]$. Now (3)(i) follows from Theorem 5.2.2.

We next prove (2)(ii). Let $\widetilde{\mathfrak{k}}$ and $\mathfrak{m}$ be as in the proof of (2)(iv) of Theorem 6.4.9. Then using (6.27), we have $\widetilde{\mathfrak{k}} + [\mathfrak{m},\, \mathfrak{m}] \;=\; \mathfrak{m}$. The statement (2)(ii) now follows using Theorem 5.2.2.

The proofs of (3)(i) and (3)(ii) are similar to those of (2)(i) and (2)(ii) respectively. ☐

As we deal with nilpotent orbits in simple Lie algebras, to ensure simplicity of $\mathfrak{so}^*(2n)$, in our next result we further assume that $n \geq 3$; see §6.5 also.

Theorem 8.1.6. *Let* $X \in \mathfrak{so}^*(2n)$ *be a nilpotent element when* $n \geq 3$. *Let* $(\mathbf{d},\, \mathbf{sgn}_{\mathcal{O}_X}) \in \mathcal{Y}^{\rm odd}(n)$ *be the signed Young diagram of the orbit* $\mathcal{O}_X$ *(that is,* $\Psi_{{\rm SO}^*(2n)}(\mathcal{O}_X) = (\mathbf{d},\, \mathbf{sgn}_{\mathcal{O}_X})$ *in the notation of Theorem 4.1.8). Then*

$$\dim_{\mathbb{R}} H^1(\mathcal{O}_X,\, \mathbb{R}) = \begin{cases} 1 & \text{if } \#\mathbb{O}_{\mathbf{d}} = 0 \\ 0 & \text{if } \#\mathbb{O}_{\mathbf{d}} \geq 1\,. \end{cases}$$

Proof. We follow the notation of the proof of Theorem 6.5.4. Using Proposition 6.5.3 we have $\mathfrak{k} \subset [\mathfrak{k}_{\mathcal{H}}, \mathfrak{k}_{\mathcal{H}}]$ when $\#\mathbb{O}_{\mathbf{d}} = 0$, and $\mathfrak{k} + [\mathfrak{k}_{\mathcal{H}}, \mathfrak{k}_{\mathcal{H}}] = \mathfrak{k}_{\mathcal{H}}$ when $\#\mathbb{O}_{\mathbf{d}} \geq 1$. As $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}_{\mathcal{H}}) = 1$, the proof is completed by Theorem 5.2.2. ☐

Theorem 8.1.7. *Let* $X \in \mathfrak{sp}(n, \mathbb{R})$ *be a nilpotent element. Let* $(\mathbf{d},\, \mathbf{sgn}_{\mathcal{O}_X}) \in \mathcal{Y}^{\rm odd}_{-1}(2n)$ *be the signed Young diagram of the orbit* $\mathcal{O}_X$ *(that is,* $\Psi_{{\rm Sp}(n,\mathbb{R})}(\mathcal{O}_X) = (\mathbf{d},\, \mathbf{sgn}_{\mathcal{O}_X})$ *in the notation of Theorem 4.1.9). Then*

$$\dim_{\mathbb{R}} H^1(\mathcal{O}_X,\, \mathbb{R}) = \begin{cases} 1 & \text{if } \#\mathbb{O}_{\mathbf{d}} = 0 \\ 0 & \text{if } \#\mathbb{O}_{\mathbf{d}} \geq 1\,. \end{cases}$$

Proof. We follow the notation of the proof of Theorem 6.6.5. Using Proposition 6.6.4, we conclude that $\mathfrak{k} \subset [\mathfrak{k}_{\mathcal{H}},\, \mathfrak{k}_{\mathcal{H}}]$ when $\#\mathbb{O}_{\mathbf{d}} = 0$ and $\mathfrak{k} + [\mathfrak{k}_{\mathcal{H}},\, \mathfrak{k}_{\mathcal{H}}] = \mathfrak{k}_{\mathcal{H}}$ when $\#\mathbb{O}_{\mathbf{d}} \geq 1$. As $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{k}_{\mathcal{H}}) = 1$, the proof is completed by Theorem 5.2.2. ☐

8.2 First cohomology of nilpotent orbits in non-compact non-complex real exceptional Lie algebras

In this section we derive some results on the dimension of the first cohomology groups of the nilpotent orbits in non-compact non-complex real exceptional Lie algebras. We begin by observing that the first cohomology groups vanish for all the nilpotent orbits in non-compact non-complex real exceptional Lie algebra $\mathfrak{g}$ when $\mathfrak{g} \not\simeq E_{6(-14)}$ and $\mathfrak{g} \not\simeq E_{7(-25)}$.

Theorem 8.2.1. *Let $\mathfrak{g}$ be a non-compact non-complex real exceptional Lie algebra which is neither isomorphic to $E_{6(-14)}$ nor to $E_{7(-25)}$. Then $H^1(\mathcal{O}_X, \mathbb{R}) = 0$ for all nilpotent elements $X \in \mathfrak{g}$.*

Proof. Any maximal compact subgroup of $\operatorname{Int}\mathfrak{g}$ is semisimple. The conclusion follows from Theorem 5.2.2. □

We next consider the case when $\mathfrak{g}$ is either $E_{6(-14)}$ or $E_{7(-25)}$. Recall that there are 12 nonzero nilpotent orbits in $E_{6(-14)}$; see [Dj1, Table X, p. 512].

Theorem 8.2.2. *Let the parametrization of the nilpotent orbits be as in §4.2.1. Let X be a nonzero nilpotent element in $E_{6(-14)}$.*

1. *If the parametrization of the orbit $\mathcal{O}_X$ is given by* $40000-2$, *then* $\dim_{\mathbb{R}} H^1(\mathcal{O}_X, \mathbb{R}) = 1$.

2. *If $\mathcal{O}_X$ is not given by the above parametrization (# of such orbits are 11), then we have* $\dim_{\mathbb{R}} H^1(\mathcal{O}_X, \mathbb{R}) \leq 1$.

Proof. For the Lie algebra $E_{6(-14)}$, we have $\mathfrak{m} = \mathfrak{so}_{10} \oplus \mathbb{R}$. For the orbit $\mathcal{O}_X$, as given in (1), we have $\mathfrak{k} = [\mathfrak{k}, \mathfrak{k}]$ from the last column, row 9 of [Dj1, Table

X, p. 512]. Hence $\mathfrak{k} + [\mathfrak{m}, \mathfrak{m}] \subsetneq \mathfrak{m}$. In view of Theorem 5.2.2 we conclude that $\dim_{\mathbb{R}} H^1(\mathcal{O}_X, \mathbb{R}) = 1$. For rest of the 11 orbits we conclude $\dim_{\mathbb{R}} H^1(\mathcal{O}_X, \mathbb{R}) \leq 1$ using Theorem 5.2.2. $\square$

There are 22 nonzero nilpotent orbits in $E_{7(-25)}$; see [Dj1, Table XIII, p. 516].

Theorem 8.2.3. *Let the parametrization of the nilpotent orbits be as in §4.2.1. Let X be a nonzero nilpotent element in $E_{7(-25)}$.*

1. *Assume the parametrization of the orbit $\mathcal{O}_X$ is given by any of the sequences:* $000000\,2$, $000000\ -2$, $000002\ -2$, $200000\ -2$, $200002\ -2$, $400000\ -2$, $000004\ -6$, $200002\ -6$, $400004\ -6$, $400004\ -10$. *Then* $\dim_{\mathbb{R}} H^1(\mathcal{O}_X, \mathbb{R}) = 1$.
2. *If $\mathcal{O}_X$ is not given by any of the above parametrization (# of such orbits are* 12*), then we have* $\dim_{\mathbb{R}} H^1(\mathcal{O}_X, \mathbb{R}) \leq 1$.

Proof. For the Lie algebra $E_{7(-25)}$, we have $\mathfrak{m} \neq [\mathfrak{m}, \mathfrak{m}]$. We refer to the last column of [Dj1, Table XIII, p. 516] to get the Lie algebra $\mathfrak{k}$. For the orbit $\mathcal{O}_X$, as given in (1), we have $\mathfrak{k} = [\mathfrak{k}, \mathfrak{k}]$. Hence $\mathfrak{k} + [\mathfrak{m}, \mathfrak{m}] \subsetneq \mathfrak{m}$. In view of Theorem 5.2.2, we conclude that $\dim_{\mathbb{R}} H^1(\mathcal{O}_X, \mathbb{R}) = 1$. For rest of the 12 orbits we conclude $\dim_{\mathbb{R}} H^1(\mathcal{O}_X, \mathbb{R}) \leq 1$ using Theorem 5.2.2. $\square$

Bibliography

[ABB] H. Azad, E. van den Ban and I. Biswas, Symplectic geometry of semisimple orbits. *Indag. Math.* **19** (2008), no. 4, 507–533.

[Br] G. Bredon, *Topology and geometry*, Graduate Texts in Mathematics, **139**, Springer-Verlag, New York, 1997.

[Bo1] A. Borel, *Linear Algebraic Groups*, Second Enlarged Edition, Graduate Texts in Mathematics, **126**, Springer-Verlag, New York, 1991.

[Bo2] A. Borel, Sur la cohomologie des espaces fibrés principaux et des espaces homognès de groupes de Lie compacts, *Ann. of Math.* **57** (1953), 115–207.

[BC1] I. Biswas and P. Chatterjee, On the exactness of Kostant-Kirillov form and the second cohomology of nilpotent orbits, *Internat. J. Math.* **23** (2012), no. 8, 1250086, 25 pp.

[BC2] I. Biswas and P. Chatterjee, Second cohomology of compact homogeneous spaces, *Internat. J. Math.* **24** (2013), no. 9, 1350076, 3 pp.

[BCM] I. Biswas, P. Chatterjee and C. Maity, The Second cohomology of nilpotent orbits in classical Lie algebras, Preprint (https://arxiv.org/abs/1611.08369).

[CE] C. Chevalley and S. Eilenberg, Cohomology theory of Lie groups and Lie algebras, *Trans. Amer. Math. Soc.* **63** (1948). 85–124.

[CM] P. Chatterjee and C. Maity, On the second cohomology of nilpotent orbits in exceptional Lie algebras, *Bull. Sci. Math.* **141** (2017), no. 1, 10–24.

[CoMc] D. H. Collingwood and W. M. McGovern, *Nilpotent orbits in semisimple Lie algebras*, Van Nostrand Reinhold Mathematics Series, Van Nostrand Reinhold Co., New York, 1993.

[Dj1] D. Z. Djokovic, Classification of Nilpotent Elements in Simple Exceptional Real Lie Algebras of Inner Type and Description of Their Centralizers, *J. Alg.* **112** (1988), 503–524.

[Dj2] D. Z. Djokovic, Classification of Nilpotent Elements in Simple Exceptional Real Lie Algebras $E_{6(6)}$ and $E_{6(-26)}$ and Description of Their Centralizers, *J. Alg.* **116** (1988), 196–207.

[GHV] W. Greub, S. Halperin and R. Vanstone, *Connections, curvature, and cohomology, Volume III: Cohomology of principal bundles and homogeneous spaces*, Pure and Applied Mathematics, Vol. 47-III. Academic Press, New York-London, 1976.

[Ho] G. Hochschild, *The structure of Lie groups*, Holden-Day, Inc., San Francisco, 1965.

[He] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New York, 1978.

[Her] I. N. Herstein, *Topics in algebra*, John Wiley & Sons, Singapore, 2006.

[Ki] D. R. King, The component groups of nilpotents in exceptional simple real Lie algebras, *Comm. Algebra*, **20** (1), 219-284 (1992).

[Kn] A. W. Knapp, *Lie Groups Beyond an Introduction,* Second edition, Progress in Mathematics, **140**, Birkhäusar, Basel, 2002.

[Mc] W. M. McGovern, *The adjoint representation and the adjoint action*, in : Algebraic quotients. Torus actions and cohomology. The adjoint representation and the adjoint action, 159–238, Encyclopaedia Math. Sci., **131**, Springer, Berlin, 2002.

[Mo] G. D. Mostow, On covariant fiberings of Klein spaces, *Amer. Jour. Math.* **77** (1955), 247–278.

[Sp] M. Spivak, *A comprehensive introduction to differential geometry*, Vol. V, Third edition, Publish or Perish, Inc., Houston, Texas, 1999.

[SS] T. A. Springer and R. Steinberg, *Conjugacy Classes*, in : Lecture Notes in Mathematics **131**, Seminar on Algebraic Groups and Related Finite Groups, Springer-Verlag, Berlin-Heidelberg-New York, 1970.