Cohomology of locally symmetric spaces

By

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List of Publications arising from the thesis

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Contents

Synopsis

Introduction

Our object of study has been the topology of locally symmetric spaces of non-compact type. Let us fix some notations. Let X be a (globally) symmetric space of non-compact type. Let G be the identity component of its isometry group. Then G is a connected semisimple Lie group with trivial centre and no compact factors. G acts transitively on X. Fix a point o in X. Let K be the isotropy subgroup at o. Then K is a maximal compact subgroup of G and $X = G/K$. Let Γ be a torsion free lattice in G. Then $\Gamma\backslash X = \Gamma\backslash G/K$ is a locally symmetric space. We denote the Lie algebra of G by $\mathfrak{g}_0$. One has the Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$, where $\mathfrak{k}_0$ is the Lie algebra of K. Let $\mathfrak{g}$ denote the complexification of $\mathfrak{g}_0$ and $\mathfrak{p}$ that of $\mathfrak{p}_0$. We can identify $\mathfrak{p}$ with the complexified tangent space of X at o. If X is a Hermitian symmetric space then $\mathfrak{p}$ breaks up into a direct sum $\mathfrak{p} = \mathfrak{p}^+ + \mathfrak{p}^-$, where $\mathfrak{p}^+$ is the holomorphic part and $\mathfrak{p}^-$ is the anti-holomorphic part.

The thesis consists of two parts. The first part addresses homotopy classification of maps between higher rank irreducible locally symmetric spaces including possible degrees in terms of the lattices involved. In particular we have addressed the question of when the degree can be negative.

The second part involves construction of cohomology classes of a family of compact locally symmetric spaces associated to $SO^*(2n)$. These classes are Poincaré duals of certain totally geodesic submanifolds. Using this, we detect occurrence of certain irreducible unitary representations associated to θ-stable parabolic subalgebras of $\mathfrak{g}$, in the direct Hilbert sum decomposition of $L^2(\Gamma\backslash SO^*(2n))$.

We now outline the main results of the thesis in more detail.

Degrees of maps between locally symmetric spaces

It is important in topology to know when there exist maps of non-zero degree between two members of a class of connected manifolds of same dimension. We consider here the class of higher rank irreducible locally symmetric spaces.

Theorem 0.0.1. *Let G, H be connected semisimple Lie groups with trivial centre and without compact factors and let K, L be maximal compact subgroups of G and H respectively. Suppose that the (real) rank of G is at least 2 and that $\dim G/K \geq \dim H/L$. Let Γ be an irreducible torsion-free lattice in G and let Λ be any torsion-free lattice in H. Then there exists a non-negative integer $\delta = \delta(\Gamma, \Lambda)$ such that the following hold: Any continuous map $f : \Gamma\backslash G/K \to \Lambda\backslash H/L$ is either null-homotopic or is homotopic to a proper map g such that $\deg(g) = \pm\delta$.*

Straightforward arguments using Margulis' normal subgroup theorem, Mostow-Margulis-Prasad rigidity theorem and a result of Prasad in [27] show that if a non null homotopic map f does exist then essentially $G = H$ and $\Gamma < \Lambda$. Using a result in [11], we show that the number $\delta(\Gamma, \Lambda)$ is *independent* of the continuous map f.

As a consequence of the above theorem we get the following result.

Theorem 0.0.2. *Let $X = \Gamma\backslash G/K, Y = \Lambda\backslash H/L$ where G, H, Γ, Λ satisfy the hypotheses of Theorem 0.0.1. Then the set $[X, Y]$ of all (free) homotopy classes of maps from X to Y is finite.*

Assuming there exist orientation preserving maps of positive degree, we obtain information about when the degree can also be negative. This is related to the question whether or not a globally symmetric space of non-compact type admits an orientation reversing isometry. For irreducible symmetric spaces of type IV we have the following result.

Theorem 0.0.3. *An irreducible globally symmetric space G/K of type IV admits an orientation reversing isometry if and only if either $\dim_{\mathbb{C}} G = \dim K$ is odd, or, K is locally isomorphic to one of the groups $SU(4n+3), n \geq 0$, and $SO(4m), m \geq 1$.*

Aiding our analysis is the observation that a symmetric space of non-compact type admits an orientation reversing isometry if and only if its compact dual does. We have completely settled the question of existence of orientation reversing isometry for all irreducible symmetric spaces of type I (which are compact duals of those of type III) associated to the classical groups as well as some exceptional groups, by either showing that some Pontrjagin number is non-zero or explicitly producing an orientation reversing isometry. Table 1 summarizes our results. There OR indicates the existence of an orientation reversing isometry and OP indicates that *every* isometry is orientation preserving.

We contrast the situation in Theorem 0.0.2 with that of cardinality of free homotopy classes of maps from a locally symmetric space rank 1 to any locally symmetric space, by showing that in many cases it is infinite. In fact we prove

Proposition 0.0.4. *Let X be any connected CW complex with positive first Betti number. Let Y be an Eilenberg-MacLane complex $K(\Lambda, 1)$ where Λ is any group that has infinitely many conjugacy classes. Then the set $[X, Y]$ of (free) homotopy classes of maps from X to Y is infinite.*

It has been proved in many cases that a locally symmetric space of rank one must have non-vanishing first Betti number. Thus taking X to be a locally symmetric

Type	U/K	parameter	OP/OR
A I	$SU(n)/SO(n)$	$n \equiv 0,2,3 \mod 4$	OR
		$n \equiv 1 \mod 4$	OP
A II	$SU(2n)/Sp(n)$	$2\|n$	OR
		$2\|(n-1)$	OP
A III	$\mathbb{C}G_{p+q,p}$	$2\|pq$	OP
		$pq \equiv 1 \mod 2$	OR
BD I	$\widetilde{G}_{p+q,p}$	$2\|p, 2\|q, 8\|pq$	OP
		otherwise	OR
D III	$SO(2n)/U(n)$	$n \equiv 2,3 \mod 4$	OR
		$n \equiv 0,1 \mod 4$	OP
C I	$Sp(n)/U(n)$	$n \equiv 1,2 \mod 4$	OR
		$n \equiv 0,3 \mod 4$	OP
C II	$\mathbb{H}G_{p,q}$	$2\|pq$ or $p \neq q$	OP
		$p = q \equiv 1 \mod 2$	OR
E III	$\frac{E_6}{Spin(10)\times U(1)}$	–	OP
E VII	$\frac{E_7}{E_6\times U(1)}$	–	OR
F II	$F_4/SO(9)$	–	OP
G	$G_2/SO(4)$	–	OP

Table 1: Results for irreducible symmetric spaces of Type III.

space with non-vanishing first Betti number and Y to be any locally symmetric space, in Proposition 0.0.4, we see that $[X, Y]$ is infinite.

These results have been published in [23].

Cohomology of cocompact lattices in $SO^*(2n)$

In [22], Millson and Raghunathan construct pairs of complementary dimensional submanifolds of certain locally symmetric spaces associated to the Lie groups $G = SO_0(p,q), SU(p,q)$ and $Sp(p,q)$. They show that the cup product of the Poincaré duals of these pairs of submanifolds are non-zero and hence they represent non zero classes in cohomology. The G-invariant forms are obvious non-trivial cohomology classes. They show that every locally symmetric space for which this construction of non-trivial geometric cycles have been made, admits a finite cover such that the

Poincaré duals of the corresponding cycles in the cover are not G-invariant. The theoretical framework for construction of such *special cycles* has been put in place in [29] and has been applied to the case $G = SU^*(2n)$ by Schwermer and Waldner in [34], the case where G is the non-compact real form of the exceptional group G_2 by Waldner and the case where $G = SL(n,\mathbb{R})$ and $SL(n,\mathbb{C})$ by Schimpf.

We investigate the case of locally symmetric spaces associated to the group $G = SO^*(2n)$, which is the real simple Lie group of type DIII. This group can be described in terms of quaternions as $SO^*(2n) \cong SO(n,\mathbb{H}) := \{g \in SL(n,\mathbb{H}) : \tau_r(g)^\top g = Id\}$, where $\tau_r : \mathbb{H} \to \mathbb{H}$ takes $p + iq + jr + ks$ to $p + iq - jr + ks$ and on a matrix it acts entry-wise. Since $SO(n,\mathbb{H})$ is of higher rank, all lattices in it are arithmetic. Let F be a totally real number field. Except when $n = 4$, any F-structure of $SO(n,\mathbb{H})$ is of the form $SU(A,\tau_r,\mathbb{H}_F^{\alpha,\beta}) := \{g \in SL(n,\mathbb{H}_F^{\alpha,\beta}) : \tau_r(g)^\top Ag = A\}$, where $\alpha,\beta < 0$ and $A \in GL(n,\mathbb{H}_F^{\alpha,\beta})$ is τ_r-Hermitian, that is, it satisfies $\tau_r(A)^\top = A$. See [25, Section18.5]. When $n = 4$, where there are more F-structures due to triality. We ignore these exotic F-structures. Without loss of generality we assume that A is diagonal, since all τ_r-Hermitian matrices are diagonalizable. The numbers α,β and entries of the matrix A can be chosen such that an application of restriction of scalars to the F-group $SU(A,\tau_r,\mathbb{H}_F^{\alpha,\beta})$ will result in a $\mathbb{Q}$-group whose $\mathbb{R}$ points is a product of simple real Lie groups, where only one factor is isomorphic to $SO(n,\mathbb{H})$ and the rest are compact groups. Corresponding to such an F-structure, we get a commensurability class of lattices in $SO(n,\mathbb{H})$. If $F \neq \mathbb{Q}$, such lattices are automatically cocompact. If $F = \mathbb{Q}$, then such lattices are cocompact if and only if the τ_r-Hermitian form corresponding to A, is $\mathbb{H}_{\mathbb{Q}}^{\alpha,\beta}$-anisotropic. All these well behaved F-structures $SU(A,\tau_r,\mathbb{H}_F^{\alpha,\beta})$, resulting in uniform lattices, will be said to be *of type* $DIII_u$.

Proposition 0.0.5 below yeilds a class of F-structures of type $DIII_u$ which admit an F-rational involution which is a Cartan involution when seen as an automorphism of the $\mathbb{R}$-points. We call such an involution an *F-rational Cartan involution.*

Proposition 0.0.5. *Consider the F-group $SU(A, \tau_r, \mathbb{H}_F^{\alpha,\beta})$ of type $DIII_u$, where A is diagonal. If the reduced norm of each diagonal element of A belongs to the same class in $F^\times/(F^\times)^2$, then there exists an F-rational Cartan involution given by conjugation by a diagonal matrix $Y \in GL(n, \mathbb{H}_F^{\alpha,\beta})$.*

We fix this Cartan involution and denote it by θ. Let K denote the maximal compact subgroup of the $\mathbb{R}$-points that is fixed by θ.

Our method of construction of special cycles is same as in [34]. First we produce some F-rational involutions that commute with θ. Any such involution will fix K. Conjugation by any matrix D which satisfies the following two conditions is an F-rational involution.

$$D^2 = -\lambda I_n, \text{ for some } \lambda \in F^\times \tag{1}$$

$$\tau_r(D)^\top AD = \mu A, \text{ for some } \mu \in F^\times. \tag{2}$$

It turns out we must have $\lambda = \pm\mu$. Such an involution will be called *sign involution* if $\lambda < 0$, an *involution of even type* if $\lambda > 0$ and $\lambda = \mu$, and an *involution of odd type* if $\lambda > 0$ and $\lambda = -\mu$. Each of these involutions commute with θ. Let σ denote any of these involutions. If σ is a sign involution then $\theta\sigma$ is an involution of even type and vice versa. If σ is an involution of odd type then $\theta\sigma$ is again an involution of odd type.

There exists a lattice Γ in the commensurability class corresponding to each F-structure which is fixed by σ and $\theta\sigma$. Thus they induce a pair of involution of the corresponding locally symmetric space $\Gamma\backslash G/K$. By Theorem 4.11 in [29], there exists a finite index subgroup $\Gamma' < \Gamma$ such that the fixed point submanifolds of the pairs of involutive isometries of $\Gamma'\backslash G/K$ induced by σ and $\theta\sigma$ have non zero intersection numbers, provided a certain orientability condition 'Or' is satisfied. It is always satisfied if σ is a sign involution or an involution of even type. If σ is an involution of odd type then Or is satisfied if n is odd. In summary our first main result is as follows.

Theorem 0.0.6. *Let Λ be a torsion free lattice in a commensurability class corresponding to an F-structure of class $DIII_u$ satisfying the condition given in Proposition 0.0.5. Then there exists a cofinal family of finite index subgroups $\Gamma \subset \Lambda$ such that the following holds. There exists cohomology classes of $\Gamma\backslash SO^*(2n)/SU(n)$, which are not G-invariant, in dimensions $2k(n-k)$ and $n(n-1)-2k(n-k)$, where k varies between 1 and $[\frac{n}{2}]$, and in the dimension $\frac{1}{2}n(n-1)$ if n is odd.*

Remark 0.0.7. *1. Any locally symmetric space associated to $SO^*(2n)$ is Kähler. Hence its cohomology can be decomposed into Hodge types. A natural question is what is the type decomposition of the cohomology classes that we construct. The classes in dimension $2k(n-k)$ and $n(n-1)-2k(n-k)$ are Poincaré duals of complex analytic submanifolds. Hence they must be of pure type (p,p). See [9, p. 162-163]. This fact will be crucially used in the following.*

2. Another consequence of the Kähler property is the following. In dimensions $2k(n-k)$ and $n(n-1)-2k(n-k)$, we not only detect non-G-invariant classes in the cohomology of a cofinal family of finite index subgroups $\Gamma \subset \Lambda$, but in fact for every torsion free lattice in the commensurability class of Λ. This is because a complex analytic submanifold in a finite index cover projects down to an analytic subvariety and such subvarieties always represent non-trivial homology cycle in the Kähler manifold. See [9, p. 110].

A major motivation for construction of geometric cycles is detection of occurrence of irreducible unitary representations of G with non-zero cohomology in $L^2(\Gamma\backslash G)$. By [36], these are representations whose Harish-Chandra modules are isomorphic to one of the $(\mathfrak{g}, K)$-modules $A_{\mathfrak{q}}$, where $A_{\mathfrak{q}}$ is the representation of $\mathfrak{g}$ obtained by cohomological induction on the trivial representation of a θ-stable parabolic subalgebra

$\mathfrak{q}$ of $\mathfrak{g}$. Let $\mathfrak{Q} := \{A_\mathfrak{q} : \mathfrak{q} \text{ a } \theta\text{-stable parabolic}\}/\simeq$, where $\simeq$ is the unitary equivalence relation. The possibility of such a detection comes from the Matsushima's isomorphism which can be stated as

$$H^*(\Gamma\backslash G/K;\mathbb{C}) \cong \prod_{A_\mathfrak{q}\in\mathfrak{Q}} m(A_\mathfrak{q},\Gamma)\ H^*(\mathfrak{g},K,A_\mathfrak{q}) \tag{3}$$

where $A_\mathfrak{q}$ is understood to represent its unitary equivalence class and $m(A_\mathfrak{q},\Gamma)$ is the multiplicity with which the G-representation with Harish Chandra module $A_\mathfrak{q}$ occurs in $L^2(\Gamma\backslash G)$. Let $\mathfrak{u}$ be the nilpotent radical of $\mathfrak{q}$. Then the cohomology group $H^j(\mathfrak{g},K,A_\mathfrak{q})$ is non-trivial only if $R(\mathfrak{q}) \leq j \leq \dim(G/K) - R(\mathfrak{q})$, where $R(\mathfrak{q}) = \dim_\mathbb{C}(\mathfrak{u}\cap\mathfrak{p})$.

If G/K is a Hermitian symmetric space then we have an additional structure on both sides of (3), in the form of Hodge bi-grading for cohomology of Kähler manifolds on the left and a similar bi-grading on the right arising from a bi-gradation of its cochain complex $\mathrm{Hom}_K(\wedge^k\mathfrak{p},A_\mathfrak{q}) \cong \oplus_{p+q=k}\mathrm{Hom}_K(\wedge^p\mathfrak{p}^+\otimes\wedge^q\mathfrak{p}^-,A_\mathfrak{q})$. These two bi-gradations are compatible. For any k, there is exactly one pair (p,q) for which $p+q=k$ and $H^{p,q}(\mathfrak{g},K,A_\mathfrak{q})$ is non-zero. This pair is the one that satisfies $p-q = R(\mathfrak{q})^+ - R(\mathfrak{q})^-$, where $R(\mathfrak{q})^+ = \dim(\mathfrak{u}\cap\mathfrak{p}^+)$ and $R(\mathfrak{q})^- = \dim(\mathfrak{u}\cap\mathfrak{p}^-)$. See [36, Proposition 6.19].

In our case of $G = SO^*(2n)$, we obtain a parametrization of the set $\mathfrak{Q}$ by associating a certain combinatorial diagram to each member of the set. The numbers $R(\mathfrak{q})^+$ and $R(\mathfrak{q})^-$ can be read off directly from these diagrams. Using this data and Remark 1, we isolate an $A_\mathfrak{q}$ which is detected by the special cycle we constructed in dimension $2n-2$.

Theorem 0.0.8. *Let Γ be any torsion-free lattice in the commensurability class corresponding to an F-structure of class $DIII_u$ on $SO^*(2n)$ satisfying the condition given in Proposition 0.0.5. Assume $n > 8$. Then:*

(i) There exists a unique $A_\mathfrak{q}$ up to unitary equivalence satisfying $R(\mathfrak{q})^+ = R(\mathfrak{q})^- =$

$n-2$, such that $m(A_{\mathfrak{q}}, \Gamma) > 0$.

(ii) There are cohomology classes in $H^*(\Gamma\backslash SO^*(2n)/U(n);\mathbb{C})$ not G-invariant in all even dimensions between $2n-4$ and $n(n-1)-(2n-4)$.

Remark 0.0.9. *The question of which $A_{\mathfrak{q}}$'s occur in $L^2(\Gamma\backslash SO^*(2n))$, where Γ is a torsion free cocompact lattice in $SO^*(2n)$, has been investigated by others, for example, by Li in [18]. The $A_{\mathfrak{q}}$ that we detect for the restricted family of lattices, as stated in Theorem 0.0.8, does not seem to have been detected earlier.*

***

Layout of thesis: Chapter 1 contains our results on degrees of maps between locally symmetric spaces and (non-) existence of orientation reversing isometries of symmetric spaces. In particular, Theorems 0.0.1 and 0.0.2 of the Synopsis appear as Theorems 1.2.1 and 1.2.2 respectively. Chapter 2 is about construction of special cycles for a certain family of locally symmetric spaces associated to the group $SO^*(2n) \cong SO(n,\mathbb{H})$. Theorem 0.0.6 of the Synopsis is restated with more details as Theorem 2.5.1. Chapter 3 is about irreducible unitary representations of $SO(n,\mathbb{H})$, with non-zero cohomology, that may occur in $L^2(\Gamma\backslash SO(n,\mathbb{H}))$, where Γ is a uniform lattice in $SO(n,\mathbb{H})$. The contents of Theorem 0.0.8 are stated more precisely in Theorem 3.5.1 and Corollary 3.5.2.

Chapters 2 and 3 can be read independently of Chapter 1.

Throughout this thesis by a locally symmetric space we always mean a locally symmetric space of non-compact type.

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Chapter 1

Degrees of maps between locally symmetric spaces

Let X be a locally symmetric space $\Gamma\backslash G/K$ where G is a connected non-compact semisimple real Lie group with trivial centre, K is a maximal compact subgroup of G, and $\Gamma \subset G$ is a torsion-free irreducible lattice in G. Let $Y = \Lambda\backslash H/L$ be another such space having the same dimension as X. Suppose that real rank of G is at least 2. In this chapter we show that any $f : X \to Y$ is either null-homotopic or is homotopic to a covering projection of degree an integer that depends only on Γ and Λ. As a corollary we obtain that the set $[X, Y]$ of homotopy classes of maps from X to Y is finite. We also obtain results on the (non-) existence of orientation reversing diffeomorphisms on X. The chapter is organised as follows: In §1.1 we introduce the notions of $\mathcal{F}$-co-Hopficity and minimal index. The main results, Theorems 1.2.1 and 1.2.2 are stated and proved in §1.2. In §1.3 we consider the problem of classifying locally symmetric spaces which admit an orientation reversing isometry. In §1.4 we consider the case of rank-1 locally symmetric spaces.

1.1 $\mathcal{F}$-Co-Hopficity and minimal index

Recall that a group Γ is said to be *residually finite* if, given any $\gamma \in \Gamma$, $\gamma \neq 1$, there exists a finite index subgroup Λ such that $\gamma \notin \Lambda$. A group Γ is called *Hopfian* (resp. *co-Hopfian*) if any surjective (resp. injective) homomorphism $\Gamma \to \Gamma$ is an automorphism. Any finitely generated subgroup of a general linear group over a field is residually finite, and, any finitely generated residually finite group is Hopfian. The latter result is due to Mal'cev. See [19]. In particular, any lattice in a connected semisimple linear Lie group G, being finitely generated, is residually finite and hence Hopfian.

Lemma 1.1.1. *Let Γ be an infinite torsion-free group.*
(i) Suppose that any non-trivial normal subgroup of Γ has finite index in Γ. Let $\phi : \Gamma \to \Lambda$ be any surjective homomorphism where Λ is infinite. Then ϕ is an isomorphism. If Γ is also co-Hopfian, then any non-trivial endomorphism of Γ is an isomorphism.
(ii) (Cf. Hirshon [11]) Let $\phi : \Gamma \to \Gamma$ be an endomorphism where $Im(\phi) \subset \Gamma$ has finite index in Γ. If Γ is finitely generated, residually finite and co-Hopfian, then ϕ is an automorphism.

Proof. (i) Note that $\ker(\phi)$ has infinite index in Γ since Λ is infinite. By our hypothesis on Γ, it follows that $\ker(\phi)$ is trivial and so ϕ is an isomorphism.

Let Γ be co-Hopfian. If $\phi : \Gamma \to \Gamma$ is a non-trivial endomorphism, then $\phi(\Gamma)$ is infinite as Γ is torsion-free. It follows from what has been shown already that ϕ is a monomorphism. Since Γ is co-Hopfian, we must have ϕ is onto and so ϕ is an automorphism.

(ii) This is essentially due to Hirshon [11, Corollary 3] who showed, without the hypothesis of co-Hopficity property, that ϕ is a monomorphism. The co-Hopficity of Γ implies that ϕ is an automorphism. $\square$

Definition Let Γ, Λ be any two infinite groups. Let $\delta(\Gamma, \Lambda)$ be defined as

$$\delta(\Gamma, \Lambda) := \min[\Lambda : \Lambda']$$

where the infimum is taken over all finite index subgroups Λ' of Λ which are isomorphic to Γ. If there is no such subgroup, we set $\delta(\Gamma, \Lambda) := 0$. We call $\delta(\Gamma, \Lambda)$ the *minimal index* of Γ in Λ.

Let X be a finite CW complex which is a $K(\Lambda, 1)$-space. Note that if Λ' is any finite index subgroup of Λ, then the coving space X' of X corresponding to the subgroup Λ' is also a finite CW complex which is a $K(\Lambda', 1)$-space. Also the Euler characteristic $\chi(\Lambda) := \chi(X)$ is non-zero if and only if $\chi(\Lambda')$ is non-zero and $\chi(\Lambda)[\Lambda : \Lambda'] = \chi(\Lambda')$. It follows that in case $\chi(\Lambda) \neq 0$, then the minimal index equals the index: $\delta(\Lambda', \Lambda) = [\Lambda : \Lambda']$.

Definition We say that Λ is *$\mathcal{F}$-co-Hopfian* if $[\Lambda : \Lambda_1] = [\Lambda : \Lambda_2]$ for any two finite index subgroups $\Lambda_1, \Lambda_2 \subset \Lambda$ such that $\Lambda_1 \cong \Lambda_2$.

The group $\mathbb{Z}$ is not $\mathcal{F}$-co-Hopfian. A non-abelian free group of finite rank is $\mathcal{F}$-co-Hopfian but not co-Hopfian. More generally, we see from the discussion preceding the above definition that if there exists a $K(\Lambda, 1)$-space where X is a finite CW complex with $\chi(X) \neq 0$, then Λ is $\mathcal{F}$-co-Hopfian. Also if Λ admits no non-trivial finite quotients, then Λ is vacuously $\mathcal{F}$-co-Hopfian.

We recall now a natural metric on a locally symmetric space. Let $\Theta : G \to G$ be an involutive automorphism with fixed group a maximal compact subgroup K. Let $\theta : \mathfrak{g} \to \mathfrak{g}$ be its differential where $\mathfrak{g} := Lie(G)$ and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition where $\mathfrak{k} = \{V \in \mathfrak{g} \mid \theta(V) = V\} = Lie(K), \mathfrak{p} = \{V \in \mathfrak{g} \mid \theta(V) = -V\}$. The Killing form of $\mathfrak{g}$ restricted to $\mathfrak{p}$ is positive definite. Since the tangent bundle of G/K is obtained as $G \times_K \mathfrak{p} \to G/K$ (where K acts on $\mathfrak{p}$ via the adjoint), this yields a G-invariant Riemannian metric on G/K with respect to which it is a globally symmetric space. The canonical Riemannian metric on $\Gamma\backslash G/K$ is obtained by requiring the covering projection $G/K \to \Gamma\backslash G/K = X$ to be a local isometry. We refer to volume of X with respect to this metric the *canonical volume* and denote it by $\mathrm{Vol}(X)$. By the strong rigidity theorem of Mostow-Margulis-Prasad ([7]) the canonical volume of X is a homotopy invariant. We have the following observation.

Lemma 1.1.2. *Let Λ be an infinite torsion-free group. Suppose that any one of the following holds: (i) there exists a finite $K(\Lambda, 1)$ complex and $\chi(\Lambda) \neq 0$; (ii) Λ is an irreducible lattice in a semisimple Lie group H with trivial centre and having no (non-trivial) compact factors. Then Λ is $\mathcal{F}$-co-Hopfian.*

Proof. By the above discussion, it only remains to consider case (ii). As remarked already, the Mostow-Margulis-Prasad rigidity theorem implies that the canonical volume of a locally symmetric space is a homotopy invariant. Let $p_j : M_j \to M$ be the covering of M such that $\pi_1(M_j) =: \Lambda_j, j = 1, 2$. Then $\mathrm{Vol}(M_j) = \deg(p_j).\mathrm{Vol}(M)$. Since $\Lambda_1 \cong \Lambda_2$, we have $\mathrm{Vol}(M_1) = \mathrm{Vol}(M_2)$ and so $[\Lambda : \Lambda_1] = \deg(p_1) = \deg(p_2) = [\Lambda : \Lambda_2]$. □

Lemma 1.1.3. *Let $\phi : \Gamma \to \Lambda$ be a homomorphism such that $[\Lambda : \phi(\Gamma)] < \infty$ where Γ is an infinite group in which every proper normal subgroup has finite index in Γ and Λ is an infinite torsion-free $\mathcal{F}$-co-Hopfian group. Then $\delta(\Gamma, \Lambda) = [\Lambda : \phi(\Lambda)]$.*

Proof. Since Λ is torsion free, the image of ϕ is infinite. Hence $\ker(\phi)$ has infinite index and hence, by our hypothesis on Γ, ϕ is a monomorphism. If Λ_1 is any finite index subgroup of Λ, such that $\Lambda_1 \cong \Gamma$, then $\Lambda_1 \cong \phi(\Gamma)$. By the $\mathcal{F}$-co-Hopf property of Λ, we have $[\Lambda : \Lambda_1] = [\Lambda : \phi(\Gamma)]$. The lemma is now immediate from the definition of $\delta(\Gamma, \Lambda)$. □

1.2 Main results

Our first main result is the following. Recall that a lattice $\Gamma \subset G$ in a connected semisimple real Lie group G is called *irreducible* if its image in G/N under the projection $G \to G/N$ is dense for any non-compact normal subgroup N of G.

Let $f : M \to N$ be any continuous map between two oriented connected manifolds of the same dimension n. Recall that if M and N are compact, the $\deg(f) \in \mathbb{Z}$ is defined by $f_*(\mu_M) = \deg(f).\mu_N$ where $f_* : H_n(M;\mathbb{Z}) = \mathbb{Z}\mu_M \to \mathbb{Z}\mu_N = H_n(N;\mathbb{Z})$ and μ_M, μ_N are the fundamental classes of M and N respectively. If M and N are non-compact and if f is *proper*, then $\deg(f) \in \mathbb{Z}$ is defined in an analogous manner using compactly supported cohomology.

Theorem 1.2.1. *Let G, H be connected semisimple Lie groups with trivial centre and without compact factors and let K, L be maximal compact subgroups of G and H respectively. Suppose that the (real) rank of G is at least 2 and that $\dim G/K \geq \dim H/L$. Let Γ be an irreducible torsionless lattice in G and let Λ be any torsionless lattice in H. There exists a non-negative integer $\delta = \delta(\Gamma, \Lambda)$ such that the following hold: Any continuous map $f : \Gamma\backslash G/K \to \Lambda\backslash H/L$ is either null-homotopic or is homotopic to a proper map g such that $\deg(g) = \pm\delta$.*

Proof. Without loss of generality we assume that f preserves the base points, which are taken to be the identity double-cosets. Thus $\pi_1(X) = \Gamma, \pi_1(Y) = \Lambda$ (suppressing the base point in the notation).

Suppose that f is not null-homotopic. Then $f_* : \pi_1(X) = \Gamma \to \Lambda = \pi_1(Y)$ is non-trivial. Note that Γ and Λ are torsion-free since X and Y are aspherical manifolds.

Since $rank(G) \geq 2$ and Γ is an irreducible lattice, by Margulis' normal subgroup theorem ([38, Chapter 8]) $\ker(f_*)$ is finite or $Im(f_*)$ is finite. As Γ and Λ are torsionless and f_* is non-trivial, we must have $\ker(f_*)$ is trivial. Hence f_* is an isomorphism of Γ onto a subgroup of $\Lambda_1 \subset \Lambda$. Since $\Lambda_1 \subset H$ is discrete, by the main result of Prasad [27] we see that $\dim H/L \geq \dim G/K$. Since $\dim X \geq \dim Y$ by hypothesis, we must have equality and, again by the same theorem, Λ_1 is a lattice in H. Since Λ is a lattice of H, we conclude that $\Lambda_1 \subset \Lambda$ must have finite index in Λ. It follows that $f : X \to Y$ factors as $p \circ f_1$ where $f_1 : X \to Y_1$, and $p : Y_1 \to Y$ is a finite covering projection, where $Y_1 := \Lambda_1 \backslash H/L$. Since f_1 induces isomorphism in fundamental groups, it is a homotopy equivalence as X, Y_1 are aspherical manifolds. By the Mostow-Margulis rigidity theorem, it follows that f_1 is homotopic to an isometry h. We let $g := p \circ h$.

It remains to show that $\deg(g) = \pm\deg(p) = \pm[\Lambda : \Lambda_1]$ equals $\pm\delta(\Gamma, \Lambda)$, where the sign is positive if f_1 is orientation preserving and is negative otherwise. By Lemma 1.1.2, Λ is $\mathcal{F}$-co-Hopfian. Since $rank(G) \geq 2$, in view of Margulis' normal subgroup theorem, we see that the hypotheses of Lemma 1.1.3 hold and we have $\delta(\Gamma, \Lambda) = [\Lambda : \Lambda_1]$. This completes the proof. □

We obtain the following result as a corollary of Theorem 1.2.1.

Theorem 1.2.2. *Let $X = \Gamma \backslash G/K, Y = \Lambda \backslash H/L$ where G, H, Γ, Λ satisfy the hypotheses Theorem 1.2.1. Then the set $[X, Y]$ of all (free) homotopy classes of maps from X to Y is finite.*

Proof. First note that there are only finitely many subgroups of Λ having index $\delta(\Gamma, \Lambda)$. Corresponding to any such group $\Gamma \cong \Lambda_1 \subset \Lambda$, the deck transformation group of the covering projection $p : Y_1 \to Y$ is finite; here Y_1 corresponds to the subgroup Λ_1. Identifying Y_1 with X, the set of homotopy self-equivalences of X equals the group $Out(\Gamma)$ of all outer automorphisms of $\Gamma = \pi_1(X)$. It is a well-known fact that this latter group is finite. For example, using the Mostow-Margulis-Prasad rigidity theorem, one has a natural homomorphism $Out(\Gamma) \to Aut(G)/G$ with kernel N_Γ/Γ where N_Γ denotes the normalizer of Γ in G. Since $Aut(G)$ has only finitely many components, $Aut(G)/G$ is finite. Also N_Γ is a lattice in G by a result of Borel [30, Chapter V]. Hence $Out(\Gamma)$ is finite. It follows that $[X, Y]$ is finite. □

1.3 Existence of orientation reversing isometries

In this section we investigate the possibility of existence of a negative degree map between locally symmetric spaces satisfying the hypotheses of Theorem 1.2.1, given a map of positive degree already exists. Let f be this given positive degree map. The proof of Theorem 1.2.1 in fact shows that f is homotopic to a local isometry from X to Y. So we may assume that $G = H$ and $\Gamma \subset \Lambda$. The following lemma gives a description of all such local isometries.

Lemma 1.3.1. *Let $\widetilde{X}$ be a symmetric space of non-compact type. Let X and Y be two locally symmetric spaces having same universal cover $\widetilde{X}$, corresponding to the torsionless lattices Γ and Λ respectively. Assume $\Gamma \subset \Lambda$. Then the set of local isometries from X to Y is in one to one correspondence with $\Lambda\backslash N(\Gamma, \Lambda)$, where $N(\Gamma, \Lambda) := \{g \in Iso(\widetilde{X}) : g\Gamma g^{-1} \subset \Lambda\}$ and Λ acts on it via left multiplication. Here $Iso(\widetilde{X})$ is the full isometry group of $\widetilde{X}$.*

Proof. The above statement is a generalization of the statement that the set of isometries of X is in bijective correspondence with $\Gamma\backslash N(\Gamma)$, where $N(\Gamma)$ denote the normalizer of Γ in $\mathrm{Iso}(\widetilde{X})$. See [5, Proposition 8.6, chapter I.8]. The proof is also a straightforward generalization of the proof given in [5] in that case. We sketch the main arguments here. Given an element $g \in N(\Gamma, \Lambda)$ the induced map from X to Y is well defined. Suppose we are given a local isometry $\mu : X \to Y$. It can be lifted to an isometry $\widetilde{\mu}$ of $\widetilde{X}$. Now one can to show that for any $\gamma \in \Gamma$, $\widetilde{\mu}\gamma\widetilde{\mu}^{-1}$ is a deck transformation of the covering $\widetilde{X} \to Y$. Hence $\widetilde{\mu}\gamma\widetilde{\mu}^{-1} \in \Lambda$ and therefore $\widetilde{\mu} \in N(\Gamma, \Lambda)$. Thus we get a surjective map from $N(\Gamma, \Lambda)$ to the set of local isometries from X to Y. This map factors through $\Lambda \backslash N(\Gamma, \Lambda)$ and we get our required one to one correspondence. □

We immediately get the following result as corollary, which gives a theoretical answer to the question of existence of negative degree maps.

Corollary 1.3.2. *Let X and Y be as in Lemma 1.3.1. A negative degree map from X to Y exists if and only if $N(\Gamma, \Lambda)$ intersects one of the components of $Iso(\widetilde{X})$ that act in an orientation reversing manner on $\widetilde{X}$.*

So the next question we address is: *does $\widetilde{X} = G/K$ admit an orientation reversing isometry?* We first show that we can transfer the question to the realm of symmetric spaces of compact type. See Proposition 1.3.3 for a precise statement.

Let U/K be the simply connected compact dual of G/K. Suppose that $\sigma_u : U/K \to U/K$ is an isometry which we assume, without loss of generality, fixes the identity coset o. As $U/Z(U)$ is covered by the identity component of the group of isometries of U/K, σ_u induces an automorphism, again denoted σ_u, of the Lie algebra $Lie(U) =: \mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ that stabilizes $\mathfrak{k}$ and hence $\mathfrak{p}_* := i\mathfrak{p}$ as well. (Recall that $Lie(G) = \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.) Let σ be the complex linear extension of σ_u to $\mathfrak{u} \otimes \mathbb{C} =: \mathfrak{g}_{\mathbb{C}}$. Then $\sigma(\mathfrak{g}) = \mathfrak{g}$ and so $\sigma_0 := \sigma|_{\mathfrak{g}}$ is an autmorphism of $\mathfrak{g}$. We denote by the same symbol σ_0 the automorphism of G/K induced by $\sigma_0 \in Aut(\mathfrak{g})$. Conversely, starting with an isometry σ_0 of G/K that fixes the identity coset of G/K, which is again denoted o, we obtain an isometry of U/K that fixes $o \in U/K$ (using the assumption that U/K is simply connected).

We have the natural isomorphism of tangent spaces $T_oU/K = \mathfrak{u}/\mathfrak{k} \cong \mathfrak{p}_*$ and $T_oG/K = \mathfrak{g}/\mathfrak{k} \cong \mathfrak{p}$. Note that $d\sigma_u : T_oU/K = \mathfrak{p}_* \to \mathfrak{p}_* = T_oU/K$ and $d\sigma_o : T_oG/K =$

$\mathfrak{p} \to \mathfrak{p} = T_0G/K$ are restrictions of the same complex linear map $\sigma : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$. Hence σ_u is orientation preserving if and only if σ_0 is orientation preserving.

Proposition 1.3.3. *Suppose that U/K is the simply connected compact dual of G/K. The space G/K admits an orientation reversing isometry if and only if U/K admits an orientation reversing isometry. In particular, if U/K does not admit any orientation reversing isometry, neither does $X = \Gamma\backslash G/K$ for any torsionless discrete subgroup $\Gamma \subset G$.* □

In view of the above proposition, we need only consider simply connected symmetric spaces U/K of compact type to decide whether G/K admits an orientation reversing isometry. We shall assume that G is simple and settle this question completely for all symmetric spaces G/K where G is of classical type or G is a complex Lie group. We shall also address a few cases where G is exceptional.

Recall that if a smooth compact manifold admits an orientation reversing diffeomorphism, then the manifold represents either the trivial element or an element of order 2 in the oriented cobordism ring Ω_* and hence all its Pontrjagin numbers are zero. See [21, p. 186]. Suppose that some Pontrjagin number of U/K is non-zero, then it does not admit any orientation reversing diffeomorphism and the same is true of $X = \Gamma\backslash G/K$ as well for any torsionless discrete subgroup Γ. In the case when Γ is a cocompact lattice, by the Hirzebruch proportionality principle ([12], [15]) the corresponding Pontrjagin number of X is non-zero and X represents an element of infinite (additive) order in Ω_*.

1.3.1 Symmetric spaces of type IV

When G is a simply connected complex simple Lie group, we have $U = K \times K$, G/K is of type IV, and $U/K \cong K$ is of type II. Let ϕ be an automorphism of $\mathfrak{g} = \mathfrak{k} + i\mathfrak{k} =: \mathfrak{k}^{\mathbb{C}}$ which preserves $\mathfrak{k}$. Then either ϕ is the $\mathbb{C}$-linear extension $\theta := \psi \otimes \mathbb{C}$

of an automorphism ψ of $\mathfrak{k}$ or is of the form $\sigma_0 \circ \theta$ where σ_0 is the complex conjugation. Since σ_0 restricted to $\mathfrak{p} = i\mathfrak{k}$ equals $-id$, G/K admits an orientation reversing isometry if $\dim_{\mathbb{C}} \mathfrak{g} = \dim \mathfrak{k}$ is odd. If $\phi = \theta$, then $\phi | i\mathfrak{k}$ is orientation reversing if and only if ψ is. Suppose that $\mathfrak{t} \subset \mathfrak{k}$ is the Lie algebra of a maximal torus $T \subset K$. By composing with an inner automorphism of $\mathfrak{k}$ if necessary, we may assume that ψ stabilizes $\mathfrak{t}$. Let Δ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{t}^{\mathbb{C}}$ and let Σ be the set of simple roots for a positive system of roots Δ^+. Then Δ is also the set of roots of $\mathfrak{k}$ with respect to $\mathfrak{t}$. We may further assume, by composing with an inner automorphism of $\mathfrak{g}$ representing a suitable element of the Weyl group of $(\mathfrak{k}, \mathfrak{t})$, that ψ preserves Δ^+. See [10, Theorem 3.29, Ch. X]. Then ψ induces an automorphism of the Dynkin diagram of $(\mathfrak{k}, \mathfrak{t})$. *We claim that ψ is orientation preserving if and only if it induces an even permutation of the set of nodes of the Dynkin diagram, namely, Σ.* To see this, for any complex linear form γ on $\mathfrak{t}^{\mathbb{C}}$, let $H_\gamma \in \mathfrak{t}^{\mathbb{C}}$ be the unique element $\mathfrak{t}^{\mathbb{C}}$ such that $B(H, H_\gamma) = \gamma(H), \forall H \in \mathfrak{t}^{\mathbb{C}}$. (Here $B(.,.)$ is the Killing form.) Then $iH_\alpha \in \mathfrak{t}$ for all $\alpha \in \Delta$. Let $X_\alpha, \alpha \in \Delta$, be a Weyl basis (in the sense of [10, Definition, p.421]). Then, for any $\beta \in \Delta^+$, $X_\beta - X_{-\beta}, i(X_\beta + X_{-\beta})$ form an $\mathbb{R}$-basis for the real vector space $\mathfrak{k} \cap (\mathfrak{g}_\beta + \mathfrak{g}_{-\beta}) =: \mathfrak{k}_\beta$. The $\mathfrak{k}_\beta, \beta \in \Delta^+$, are the non-trivial irreducible T-submodules of $\mathfrak{k}$ and we have $\psi(\mathfrak{k}_\beta) = \mathfrak{k}_\gamma$ where $\gamma = \psi^*(\beta) = \beta \circ \psi \in \Delta^+$. In fact, the assumption that $\psi(\Delta^+) = \Delta^+$ implies that the matrix of $\psi : \mathfrak{k}_\beta \to \mathfrak{k}_\gamma$ with respect to the ordered bases of $\mathfrak{k}_\beta, \mathfrak{k}_\gamma$ as above is of the form $\left(\begin{smallmatrix} x_\beta & -y_\beta \\ y_\beta & x_\beta \end{smallmatrix}\right)$ with $x_\beta^2 + y_\beta^2 = 1$ (see [10, §5, Ch. IX]). It follows that the $\psi : \mathfrak{k} \to \mathfrak{k}$ is orientation preserving if and only if $\psi|\mathfrak{t}$ is orientation preserving. Since the basis $iH_\alpha, \alpha \in \Sigma$, of $\mathfrak{t}$ is permuted by ψ, $\psi|\mathfrak{t}$ is orientation preserving if and only if $\psi^* : \Sigma \to \Sigma$ is an even permutation. This proves our claim. An inspection of the Dynkin diagrams reveals that an orientation reversing automorphism of $\mathfrak{k}$ exists precisely when $\mathfrak{k} = \mathfrak{su}(n), n \equiv 0, 3 \mod 4$, or $\mathfrak{k} = \mathfrak{so}(2n), n \geq 4$. We have proved

Theorem 1.3.4. *An irreducible globally symmetric space G/K of type IV admits an orientation reversing isometry if and only if either* $\dim_{\mathbb{C}} G = \dim K$ *is odd, or,* *K is locally isomorphic to one of the groups* $SU(4n+3), n \geq 0$, *and* $SO(4m), m \geq 1$.

From now on, till the end of §4, we assume that G is a connected real simple Lie group which is not a complex Lie group. Thus G/K is of type III and U is simple. The results are tabulated in Table 1.1.

1.3.2 Hermitian symmetric spaces

Let U/K be simply connected compact irreducible Hermitian symmetric space U/K, where U is simply connected and simple. There are six families of such spaces: AIII, DIII, BDI(rank 2), CI, EIII, and EVII, using the standard notations (as in [10]).

There are exactly two invariant complex structures $\mathcal{C}$ and $\mathcal{C}'$ which are conjugate to one another in the sense that the complex structure on $\mathfrak{p}_* = T_oU/K$ induced by $\mathcal{C}$ and $\mathcal{C}'$ are complex conjugate of each other. These two complex structures are related by an automorphism σ_u of U that stabilizes K thereby inducing the complex conjugation on $\mathfrak{p}_*$ (with respect to $\mathcal{C}$, say). In particular, the isometry σ_u of U/K is orientation reversing if and only if $\dim_{\mathbb{C}} U/K$ is odd. See [3, Remark 2, §13]. In particular, $E_7/U(1) \times E_6$, which has dimension 27 (over $\mathbb{C}$), admits an orientation reversing isometry.

When $\dim_{\mathbb{C}} U/K = d$ is even, we shall show that an appropriate Pontrjagin number of U/K is non-zero.

The signature of a compact irreducible Hermitian symmetric space are known; see [13, p. 163]. In the case of the complex Grassmann manifold (Type AIII), $\mathbb{C}G_{p+q,p} = SU(p+q)/S(U(p) \times U(q))$, the signature equals $\binom{\lfloor (p+q)/2 \rfloor}{\lfloor p/2 \rfloor}$ when $d = pq$ is even. (Cf. Shanahan [35, p. 489].) The signature equals $4\lfloor p/2 \rfloor$ for the complex quadric (Type BD I, rank 2), $SO(2+p)/(SO(2) \times SO(p)), p > 2$. In the case of type E III, namely $E_6/(Spin(10) \times U(1))$, the signature equals 3. We shall presently show that when

$\dim_{\mathbb{C}} SO(2p)/U(p) = p(p-1)/2 = d$ (resp. $\dim_{\mathbb{C}} Sp(n)/U(n) = n(n+1)/2 = d$) is even, $p_1^{d/2} \neq 0$ where p_1 is the first Pontrjagin class.

We shall now compute $p_1 := p_1(SO(2p)/U(p))$. For this purpose we shall use the notation and the formula for the total Chern class of $SO(2p)/U(p)$ given in [3, §16.3] with respect to an $SO(2p)$-invariant complex structure compatible with the usual differentiable structure on $SO(2p)/U(p)$. Let $\sigma_j = \sigma_j(x_1, \ldots, x_p)$ denote the jth symmetric polynomial in the indeterminates $x_1, \ldots, x_p$. We set $\deg(x_j) = 2$ and consider the graded polynomial algebra $\mathbb{K}[\sigma_1, \ldots, \sigma_p]$. If $\mathbb{K}$ is any field of characteristic other than 2, the cohomology algebra $H^*(SO(2p)/U(p); \mathbb{K})$ is isomorphic to $\mathbb{K}[\sigma_1, \ldots, \sigma_p]/I$ where I is the ideal generated by the elements $\lambda_j := \sigma_j(x_1^2, x_2^2, \ldots, x_p^2), 1 \leq j < p$ and σ_p. We take $\mathbb{K}$ to be $\mathbb{Q}$.

We have $c(SO(2n)/U(n)) = \prod_{1 \leq i < j \leq p}(1+x_i+x_j)$ from which we obtain the following formula for the total Pontrjagin class:

$$1 - p_1 + p_2 - \cdots = \prod_{1 \leq i < j \leq p} (1 + x_i + x_j) \prod_{1 \leq i < j \leq p} (1 - x_i - x_j),$$

equivalently,

$$p(SO(2n)/U(n)) = \prod_{1 \leq i < j \leq p} (1 + (x_i + x_j)^2).$$

Therefore $p_1 = \sum_{1 \leq i < j \leq p}(x_i+x_j)^2 = (p-1) \sum_{1 \leq i \leq p} x_i^2 + 2 \sum_{1 \leq i < j \leq p} x_i x_j = (n-1)\lambda_1 + 2\sigma_2 = 2\sigma_2 = \sigma_1^2$, since $\lambda_1 \in I$ and $\sigma_1^2 = \lambda_1 + 2\sigma_2$. Since $SO(2n)/U(n)$ is Kähler, and since $H^2(SO(2n)/U(n); \mathbb{Q}) = \mathbb{Q}\sigma_1$, we have $\sigma_1^d \neq 0$ where $d = \dim_{\mathbb{C}} SO(2p)/U(p) = p(p-1)/2$. Thus we see that $p_1^{d/2}[SO(2p)/U(p)] \neq 0$ when d is even.

An entirely analogous computation shows that $p_1^{d/2}[Sp(p)/U(p)] \neq 0$ when $\dim_{\mathbb{C}} Sp(p)/U(p) = n(n+1)/2 =: d$ is even, using the formula for the total Chern class of $Sp(p)/U(p)$ as given in [3, §16.4]. In fact, in this case the computations can be carried out in the integral cohomology ring.

To summarise we have proved, in view of Proposition 1.3.3, the following.

Theorem 1.3.5. *An irreducible hermitian symmetric domain G/K (or its simply connected compact dual U/K) admits an orientation reversing isometry if and only if its complex dimension is odd.* $\square$

1.3.3 Oriented Grassmann manifolds

The oriented Grassmann manifold $\widetilde{G}_{m+n,n} = SO(m+n)/SO(m) \times SO(n)$ is dual to $SO_0(m,n)/SO(m)\times SO(n)$. We leave out the well-known case of sphere, $\min\{m,n\} = 1$, and also $\widetilde{G}_{4,2} \cong S^2 \times S^2$, by assuming $m, n > 1$ and $m + n > 4$. When the dimension mn is odd, the symmetric space $\widetilde{G}_{m+n,n}$ admits an orientation reversing isometry. So assume that mn is even.

Shanahan [35] has shown that the signature of $\widetilde{G}_{m+n,n}$ equals $\binom{\lfloor (m+n)/4 \rfloor}{\lfloor n/4 \rfloor}$ when both m, n are even, $mn \equiv 0 \mod 8$, and is zero otherwise. When $m \equiv n \equiv 2 \mod 4$ and $m \neq n$, it was shown that $p_1^{mn/4}[\widetilde{G}_{m+n,n}] \neq 0$ in the proof of [33, Theorem 3.2]. Consequently, $\widetilde{G}_{m+n,n}$ does not admit an orientation reversing diffeomorphism in these cases.

It remains to consider the cases $\widetilde{G}_{m+n,n} = U/K$, $U = SO(m+n), K = SO(m) \times SO(n)$ where at least one of the numbers m, n is odd, or, $m = n \equiv 2 \mod 4$. Suppose that m is odd (the case n odd being analogous). Consider the isometry $\sigma_u : \widetilde{G}_{m+n,n} \to \widetilde{G}_{m+n,n}$ defined as $\sigma_u(xK) = D^{-1}xDK$ where $D := D_{m+n} = diag(1, \dots, 1, -1) \in O(m+n)$. Note that $D^{-1}KD = K$. Also the differential of σ_u at $o \in U/K$ is the linear map of $\mathfrak{p}_* = \{\left(\begin{smallmatrix} 0 & B \\ -B^\top & 0 \end{smallmatrix}\right) \mid B \in M_{m\times n}(\mathbb{R})\} \cong M_{m\times n}(\mathbb{R})$ defined by $B \mapsto BD_n$. Since m is odd, we conclude that σ_u is orientation reversing.

Finally, let $n \equiv 2 \mod 4$ and consider $\sigma_u : \widetilde{G}_{2n,n} \to \widetilde{G}_{2n,n}$ defined as $uK \mapsto J^{-1}uJK = JuJK$ is an isometry where $J \in SO(2n)$ the matrix $J := \left(\begin{smallmatrix} 0 & I_n \\ I_n & 0 \end{smallmatrix}\right)$.

Note that $J^{-1}KJ = K$.

Then the differential of σ_u at $o \in U/K$ is the linear isomorphism of $\mathfrak{p}_* \cong M_n(\mathbb{R})$ given by $B \mapsto -B^\top$, which is orientation reversing if and only if $\binom{n}{2}$ is odd. Since $n \equiv 2 \mod 4$, we conclude that σ_u is orientation reversing. (Cf. [33, Lemma 3.5].)

1.3.4 Quaternionic Grassmann manifolds

Next consider the quaternionic Grassmann manifold $\mathbb{H}G_{p+q,p} = Sp(p+q)/Sp(p) \times Sp(q)$. S. Mong [24] has shown that the signature of $\mathbb{H}G_{p+q,p}$ is zero if and only if both p, q are odd. We claim that the Pontrjagin number $p_1^{pq}[\mathbb{H}G_{p+q,p}] \neq 0$ when $p \neq q$. When $p = 1 < q$, $\mathbb{H}G_{q+1,1} \cong \mathbb{H}P^q$, the quaternionic projective space and the result is due to Borel and Hirzebruch [3, §15.5]. The general case can be reduced the case of quaternionic projective space. To see this, we assume, without loss of generality, that $q > p > 1$ and use the natural 'inclusion' of the quaternionic projective space $j : \mathbb{H}P^{q-p-1} \subset \mathbb{H}G_{p+q,p}$ induced by the obvious inclusion of $H := \begin{pmatrix} I_{p-1} & 0 & 0 \\ 0 & Sp(q-p+2) & 0 \\ 0 & 0 & I_{p-1} \end{pmatrix} \cong Sp(q-p+2)$ into $Sp(p+q)$ so that $H \cap (Sp(p) \times Sp(q)) = Sp(1) \times Sp(q-p+1) \subset Sp(q-p+2)$. If we view $\mathbb{H}G_{p+q,p}$ as the space of all q-dimensional left $\mathbb{H}$-vector spaces in $\mathbb{H}^{p+q}$ and $\mathbb{H}P^{q-p+1} = \mathbb{H}G_{q-p+2,1}$ as the space of 1-dimensional $\mathbb{H}$-vector spaces in $\mathbb{H}e_p + \cdots + \mathbb{H}e_{q+1}$ then $j(L) = L + \mathbb{H}e_1 + \cdots + \mathbb{H}e_{p-1}$ for $L \in \mathbb{H}P^{q-p+1}$. (As usual, $\mathbb{H}$ stands for the skew field of quaternions and $e_1, \ldots, e_{p+q}$, the standard basis for $\mathbb{H}^{p+q}$.) Then the normal bundle to the imbedding j is trivial, by using, for example, the description of the tangent bundle of $\mathbb{H}G_{p+q,p}$ due to Lam [16]. So $j^*(p_1(\mathbb{H}G_{p+q,p})) = p_1(\mathbb{H}P^{q-p+1}) \neq 0$ since $q - p + 1 \geq 2$. This shows that $p_1(\mathbb{H}G_{p+q,p}) \neq 0$. Observing that the integral cohomology rings of $\mathbb{C}G_{p+q,q}$ and $\mathbb{H}G_{p+q,p}$ are isomorphic by an isomorphism that doubles the degree and using the fact that $c_1^{pq} \neq 0$ where $c_1 \in H^2(\mathbb{C}G_{p+q,p}; \mathbb{Z}) \cong \mathbb{Z}$ is a generator, we see that $p_1^{pq} \neq 0$ in $H^{4pq}(\mathbb{H}G_{p+q,p}; \mathbb{Z})$. Thus the Pontrjagin number $p_1^{pq}[\mathbb{H}G_{p+q,p}] \neq 0$. (See also [33, Theorems 3.2(iii) and 3.3(ii)].) Hence when $p \neq q$, $\mathbb{H}G_{p+q,p}$ does not admit an orientation reversing diffeomorphism.

It remains to consider the case $p = q \equiv 1 \mod 2$. In this case $\mathbb{H}G_{2p,p}$ admits an orientation reversing isometry as we shall now show. For this purpose, we shall use the description $\mathfrak{sp}(2p) = \{\left(\begin{smallmatrix} A & Z \\ -{}^{\top}\bar{Z} & B \end{smallmatrix}\right) \mid Z = \left(\begin{smallmatrix} Z_1 & Z_2 \\ -\bar{Z}_2 & \bar{Z}_1 \end{smallmatrix}\right), Z_1, Z_2 \in M_p(\mathbb{C}), A, B \in \mathfrak{sp}(p)\}$, where $\mathfrak{sp}(p) = \{\left(\begin{smallmatrix} X & Y \\ -\bar{Y} & \bar{X} \end{smallmatrix}\right) \mid X \in \mathfrak{u}(p), {}^{\top}Y = Y \in M_p(\mathbb{C})\}$. Thus $\mathfrak{p}_*$ consists of all matrices of the form $\left(\begin{smallmatrix} 0 & Z \\ -{}^{\top}\bar{Z} & 0 \end{smallmatrix}\right) \in \mathfrak{sp}(2p)$. Conjugation by $J = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$ is an automorphism of $\mathfrak{sp}(2p)$ which maps $\left(\begin{smallmatrix} A & Z \\ -{}^{\top}\bar{Z} & B \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} B & -{}^{\top}\bar{Z} \\ Z & A \end{smallmatrix}\right)$. Evidently it stabilizes $\mathfrak{k} = \mathfrak{sp}(p) \times \mathfrak{sp}(p)$ and, since p is odd, reverses the orientation on $\mathfrak{p}_*$.

1.3.5 Other symmetric spaces of classical type

Consider the space $SU(n)/SO(n), n > 2$ (type AI). Note that $\mathfrak{p}_* = iSym^0_n(\mathbb{R})$, consisting of trace 0 symmetric matrices with purely imaginary entries. Thus $\mathfrak{p}_* \cong Sym^0(\mathbb{R})$. Conjugation by $D := diag(1, \ldots, 1, -1)$ yields an isometry σ_u defined as $xK \mapsto DxDK$ which induces on $\mathfrak{p}_*$ the map $X \mapsto DXD$. It is readily seen to be orientation reversing if and only if n is even. When $\dim SU(n)/SO(n) = (n+2)(n-1)/2$ is odd, i.e., when $n \equiv 0, 3 \mod 4$, the involutive isometry $gK \to {}^{\top}g^{-1}K$ is orientation reversing. It remains to consider the case $n \equiv 1 \mod 4$. When $n \equiv 1 \mod 2$, the outer automorphism group of $SL(n, \mathbb{R})$ is generated by the Cartan involution $X \to {}^{\top}X^{-1}$; see [26, p. 132-133]. It follows that $SL(n, \mathbb{R})/SO(n)$ and $SU(n)/SO(n)$ does not admit any orientation reversing isometry if $n \equiv 1 \mod 4$.

Next consider the symmetric space $SU(2n)/Sp(n)$ (type AII), which is dual to $SU^*(2n)/Sp(n)$ where $SU^*(2n) \subset SL(2n, \mathbb{C})$ consists of matrices Z which commute with the transformation ${}^{\top}(z_1, \ldots, z_{2n}) = {}^{\top}(\bar{z}_{n+1}, \ldots, \bar{z}_{2n}, -\bar{z}_1, \ldots, -\bar{z}_n), z \mapsto {}^{\top}(z_1, \ldots, z_{2n}) \in \mathbb{C}^{2n}$. Again the outer automorphism group of $SU^*(2n)$ is generated by the Cartan involution $X \to {}^{\top}X^{-1}$ by the work of Murakami [26, p. 131-132].

Hence $SU^*(2n)/Sp(n)$ (equivalently $SU(2n)/Sp(n)$) admits an orientation reversing isometry if and only if its dimension $(n-1)(2n+1)$ is odd, that is, if and only if n is even.

1.3.6 Symmetric spaces of types G and F II

Borel and Hirzebruch [3, §18,19] have computed the Pontrjagin numbers of $G_2/SO(4)$ (type G) and of the Cayley plane $F_4/Spin(9)$ (type F II). In particular they showed that $p_2[G_2/SO(4)] \neq 0$ and $p_4[F_4/Spin(9)] \neq 0$. Hence neither $G_2/SO(4)$ nor $F_4/Spin(9)$ admit any orientation reversing diffeomorphism. Also, by the formula of G. Hirsch, the Poincaré polynomials of $G_2/SO(4)$ and $F_4/Spin(9)$ are $1+t^4+t^8$ and $1+t^8+t^{16}$, it follows that their signatures are 1 (with suitable orientations).

We summarise the above results for type III symmetric spaces in Table 1.1. Here OR indicates the existence of an orientation reversing isometry and OP indicates that every isometry is orientation preserving.

1.4 Maps from rank 1 locally symmetric spaces

We will show that the restriction on rank of G in Theorem 1.2.1 cannot be weakened. We do this by proving Proposition 1.4.1 below. It has been conjectured that any locally symmetric space $X = \Gamma\backslash G/K$, where G is a connected semisimple (non-compact) linear Lie group of real rank 1, admits a finite cover whose first Betti number is non-zero. In group theoretic terms, it translates into the assertion that the lattice Γ admits a finite index subgroup Γ_0 whose abelianization is infinite. Although this conjecture seems to be as yet unresolved, it is known to be true in most cases when Γ is arithmetic. See [20], [2] and [31].

Type	U/K	parameter	OP/OR
A I	$SU(n)/SO(n)$	$n \equiv 0, 2, 3 \mod 4$	OR
		$n \equiv 1 \mod 4$	OP
A II	$SU(2n)/Sp(n)$	$2\|n$	OR
		$2\|(n-1)$	OP
A III	$\mathbb{C}G_{p+q,p}$	$2\|pq$	OP
		$pq \equiv 1 \mod 2$	OR
BD I	$\widetilde{G}_{p+q,p}$	$2\|p, 2\|q, 8\|pq$	OP
		otherwise	OR
D III	$SO(2n)/U(n)$	$n \equiv 2, 3 \mod 4$	OR
		$n \equiv 0, 1 \mod 4$	OP
C I	$Sp(n)/U(n)$	$n \equiv 1, 2 \mod 4$	OR
		$n \equiv 0, 3 \mod 4$	OP
C II	$\mathbb{H}G_{p,q}$	$2\|pq$ or $p \neq q$	OP
		$p = q \equiv 1 \mod 2$	OR
E III	$\frac{E_6}{Spin(10)\times U(1)}$	–	OP
E VII	$\frac{E_7}{E_6 \times U(1)}$	–	OR
F II	$F_4/SO(9)$	–	OP
G	$G_2/SO(4)$	–	OP

Table 1.1: Results for irreducible symmetric spaces of Type III.

We want to show that if $Y = \Lambda\backslash H/L$ is any locally symmetric space where Λ is torsionless and H is any connected semisimple linear Lie group and X is any manifold with positive first Betti number, then the set $[X, Y]$ is infinite. It is easily seen that, when H is regarded as a subgroup of $GL(N, \mathbb{R})$ for some N, there exists a λ in Λ not all of whose eigenvalues are on the unit circle. It follows that λ^m and λ^n are not conjugates in $GL(N, \mathbb{R})$ if $|m| \neq |n|$. Hence Λ has infinitely many distinct conjugacy classes. In fact, it follows from the work of Brauer [6] (cf. [28]) that any infinite, residually finite group has infinitely many conjugacy classes. In particular, any finitely generated infinite subgroup of $GL(N, \mathbb{R})$ has infinitely many conjugacy classes. The following very general result shows that $[X, Y]$ is infinite.

Proposition 1.4.1. *Let X be any connected CW complex with positive first Betti number. Let Y be an Eilenberg-MacLane complex $K(\Lambda, 1)$ where Λ is any group that has infinitely many conjugacy classes. Then the set $[X, Y]$ of (free) homotopy classes of maps from X to Y is infinite.*

Proof. Let $\Gamma = \pi_1(X)$. Since the spaces involved are CW complexes, and since Y is a $K(\Lambda, 1)$-space, any homomorphism $\Gamma \to \Lambda$ is induced by a continuous map $X \to Y$. If two maps $f, g : X \to Y$ are (freely) homotopic, then there exist inner-automorphisms $\iota_a \in Aut(\Gamma)$ and $\iota_b \in Aut(\Lambda)$ such that $g_* = \iota_b \circ f_* \circ \iota_a : \Gamma \to \Lambda$. Let $\alpha : \Gamma \to \mathbb{Z}$ be a non-zero element of $H^1(X; \mathbb{Z})$. We may assume that α is surjective; choose a $\gamma_0 \in \Gamma$ such that $\alpha(\gamma_0) = 1$. For any $\lambda \in \Lambda$, let $f_\lambda : X \to Y$ be any continuous map which induces the homomorphism $\theta_\lambda := \epsilon_\lambda \circ \alpha$ where $\epsilon_\lambda : \mathbb{Z} \to \Lambda$ is defined by $1 \mapsto \lambda$. Suppose that f_λ and f_μ are (freely) homotopic. Then there exist $a \in \Gamma$ and $b \in \Lambda$, such that $\theta_\mu = \iota_b \circ \theta_\lambda \circ \iota_a$. Evaluating both sides at γ_0, we obtain $\mu = b\lambda b^{-1}$; thus λ and μ are conjugates in Λ. Since Λ has infinitely many conjugacy classes it follows that $[X, Y]$ is infinite. $\square$

Chapter 2

F-structures on $SO(n, \mathbb{H})$ and construction of special cycles

The notations in the rest of the thesis are different from those in Chapter 1.

In this chapter we will construct submanifolds of certain compact locally symmetric spaces associated to the group $G = SO(n, \mathbb{H})$, whose Poincaré duals cannot be represented by G-invariant forms on $X = SO(n, \mathbb{H})/U(n)$. In §2.1 we describe all uniform lattices in $SO(n, \mathbb{H})$ (except certain exotic ones occurring when $n = 4$ due to triality). All the results here follow from known facts, but may not be readily available in literature in the form we require them. In §2.2 we will discuss how to construct these submanifolds. This is standard material. These submanifolds occur as fixed points of certain involutive isometries which are induced by involutive automorphisms of G. We construct such involutive automorphisms in §2.3. The results in this section are new. §2.4 recalls certain properties of a compact Kähler manifold and infers how they strengthen our results. In §2.5 we state and prove our main result, Theorem 2.5.1.

2.1 Uniform lattices in $SO(n, \mathbb{H})$

In this section we describe uniform lattices in $SO(n, \mathbb{H})$. Since $SO(n, \mathbb{H})$ is of higher rank, all lattices in it are arithmetic. In the following we first describe all F-structures of $SO(n, \mathbb{H})$, where F is a number field. Applying restriction of scalars on these F-structures will yield $\mathbb{Q}$-groups whose $\mathbb{R}$-points will be product of simple Lie groups. We deduce which simple Lie groups may occur in this product. We then describe the F-structures for which this product has exactly one non-compact factor. For $F \neq \mathbb{Q}$, each such F-structure produces a family of uniform lattices. Finally we give a necessary and sufficient condition on a $\mathbb{Q}$-structure for the corresponding arithmetic subgroups to be cocompact.

2.1.1 F-structure of $SO(n, \mathbb{H})$

Let $\mathbb{D}$ be a division ring. We will define certain subgroups of the group $GL(n, \mathbb{D})$. First we define the special linear group over $\mathbb{D}$. Let F be the centre of $\mathbb{D}$. It is a field. There always exists a field extension L of F, for which there is an L-algebra isomorphism $\phi : \mathbb{D} \otimes_F L \to M(d, L)$. We say that $\mathbb{D}$ **splits** over L. The restriction of ϕ to $\mathbb{D} \otimes 1$ gives an algebra embedding of $\mathbb{D}$ in $M(d, L)$. This embedding induces an F-algebra embedding of $M(n, \mathbb{D})$ in $M(n, M(d, L)) \cong M(nd, L)$ by replacing each entry of a matrix in $M(n, \mathbb{D})$ by a $d \times d$ block matrix with entries in L. By abuse of notation we denote this embedding $M(n, \mathbb{D}) \hookrightarrow M(nd, L)$ again by ϕ. Given such an embedding ϕ, we can define the subgroup $SL(n, \mathbb{D})$ of $GL(n, \mathbb{D})$ as follows.

$$SL(n, \mathbb{D}) := \{g \in GL(n, \mathbb{D}) : \det(\phi(g)) = 1\}$$

This definition is independent of the choice of the map ϕ.

Now we will define analogs of the special unitary group in the context of a division ring $\mathbb{D}$ that admits an *anti-involution* τ. By anti-involution we mean an F-anti-isomorphism of order 2. We begin by defining the analog of a Hermitian form on a right $\mathbb{D}$-module E.

Definition 2.1.1. *A* τ***-Hermitian form*** *on* E *is an* F*-bilinear map* $h : E \times E \to \mathbb{D}$ *which satisfies the following conditions*

- $h(v_1a_1 + v_2a_2, w) = \tau(a_1)h(v_1, w) + \tau(a_2)h(v_2, w)$
- $h(v, w_1a_1 + w_2a_2) = h(v, w_1)a_1 + h(v, w_2)a_2$
- $h(w, v) = \tau(h(v, w))$

for all $a_1, a_2 \in \mathbb{D}$.

Definition 2.1.2. *A* τ*-Hermitian form* h *is called* ***non-degenerate*** *if for each* $v \in E$*, there exists* $w \in E$*, such that* $h(v, w) \neq 0$.

Definition 2.1.3. *A matrix* $A \in M(n, \mathbb{D})$ *is called* τ***-Hermitian****, if* $\tau(A)^\top = A$.

Given a $\mathbb{D}$-basis $\{e_i : 1 \leq i \leq n\}$ of E, there is a one-to-one correspondence between τ-Hermitian forms on E and τ-Hermitian matrices. On one side the map is given by h going to the Gram matrix $(h(e_i, e_j))_{i,j}$. On the other side the map is given by sending A to the map $(v, w) \mapsto \tau(v)^\top Aw$, where v and w also denote the column vector representations with respect to the chosen basis. A τ-Hermitian form is non-degenerate if and only if its Gram matrix is invertible.

Now we define the subgroup of $SL(n, \mathbb{D})$ which preserves such a τ-Hermitian form induced from an invertible τ-Hermitian matrix A.

$$SU(A, \tau, \mathbb{D}) := \{g \in SL(n, \mathbb{D}) : \tau(g)^\top Ag = A\}.$$

We know for a Hermitian matrix A over $\mathbb{C}$, there always exists $P \in GL(n, \mathbb{C})$, such that $P^*AP = D$, where D is a diagonal. A similar statement holds for τ-Hermitian matrices which we state below. The proof is via standard arguments and we omit it.

Proposition 2.1.4. *Given a τ-Hermitian matrix A, there exists $P \in GL(n, \mathbb{D})$, such that $\tau(P)^\top AP$ is diagonal.*

This shows that for each invertible τ-Hermitian matrix A, the group $SU(A, \tau, \mathbb{D})$ is conjugate to a group $SU(B, \tau, \mathbb{D})$, where B is diagonal.

Now we introduce a special class of division algebras that we will be particularly interested in.

Definition 2.1.5. *Let F be a field and fix $\alpha, \beta \in F \setminus \{0\}$. The* ***quaternion algebra*** *$\mathbb{H}_F^{\alpha,\beta}$ over F is the algebra which is generated as a vector space over F by the four elements $1, i, j$ and k, with the following relations:*

$$i^2 = \alpha, j^2 = \beta, ij = k = -ji$$

These relations determine the multiplication.

When $F = \mathbb{R}$ and $\alpha = -1 = \beta$, we get the usual quaternion algebra $\mathbb{H}$.

Any quaternion algebra $\mathbb{H}_F^{\alpha,\beta}$ admits two anti-involutions τ_c and τ_r. The anti-involution τ_c takes a general element $p + iq + jr + ks \in \mathbb{H}$ to $p - iq - jr - ks$ and τ_r takes $p + iq + jr + ks$ to $p + iq - jr + ks$. The two anti-involutions are related as $j\tau_r(x)j^{-1} = \tau_c(x)$ for all $x \in \mathbb{H}_F^{\alpha,\beta}$.

If $\alpha, \beta < 0$, then for the quaternion algebra $\mathbb{H}_\mathbb{R}^{\alpha,\beta}$ over $\mathbb{R}$, we have a refinement of Proposition 2.1.4.

Proposition 2.1.6. *Let $\alpha, \beta < 0$. Given an invertible τ_r-Hermitian matrix A, there exists $P \in GL(n, \mathbb{H}_\mathbb{R}^{\alpha,\beta})$, such that $\tau_r(P)^\top AP = I_n$.*

Proof. By Proposition 2.1.4, we already have an orthogonal basis for h, the τ_r-Hermitian form induced by A. All we need to show is that this basis can be further orthonormalized. For that it is enough to show that if $h(v,v) \neq 0$, for some $v \in \mathbb{H}^n$, then there exists $\lambda \in \mathbb{H}$, such that $h(v\lambda, v\lambda) = \tau_r(\lambda)h(v,v)\lambda = 1$. Notice that $\tau_r(h(v,v)) = h(v,v)$, by the third condition in the definition of a τ-Hermitian form. Hence $h(v,v)$ is of the form $a + ib + kd$, for some $a, b, d \in \mathbb{R}$. Thus it is enough to solve for $x = p + iq + jr + ks$ in the equation

$$\tau_r(x)x = a + ib + kd \tag{2.1}$$

and take this as value of λ^{-1}. Expanding the LHS of (2.1) and equating the coefficients of $1, i$ and k from both sides, we get

$$\begin{aligned} p^2 + \alpha q^2 - \beta r^2 - \alpha\beta s^2 &= a \\ 2pq + 2\beta rs &= b \\ 2ps + 2qr &= d \end{aligned}$$

Let us assign $p = 0$, then we have $s = \frac{b}{2\beta r}$ and $q = \frac{d}{2r}$ from the second and third equations. Substituting these values in the first equation we get $-\beta r^2 - \frac{\alpha b^2}{4\beta r^2} + \frac{\alpha d^2}{4r^2} = a$. If we vary r from 0 to ∞, the LHS varies from $-\infty$ to ∞. Thus there exists a solution for r. $\square$

Proposition 2.1.6 shows that for each τ_r-Hermitian matrix $A \in GL(n, \mathbb{H}_{\mathbb{R}}^{\alpha,\beta})$, the group $SU(A, \tau_r, \mathbb{H}_{\mathbb{R}}^{\alpha,\beta})$ is conjugate to the group $SU(I_n, \tau_r, \mathbb{H}_{\mathbb{R}}^{\alpha,\beta})$. Note that $SO(n, \mathbb{H})$ $= SU(I_n, \tau_r, \mathbb{H})$. By [25, Section 18.5], all F-structures of $SO(n, \mathbb{H})$, except when $n = 4$, are of the form $SU(A, \tau_r, \mathbb{H}_F^{\alpha,\beta})$, where A is an invertible τ_r-Hermitian. In fact by Proposition 2.1.4, without loss of generality, we may assume that A is diagonal. For $n = 4$, there are more F-structures arising from triality. We will ignore these exotic F-structures.

2.1.2 $\mathbb{R}$-points of the group $SU(A, \tau_r, \mathbb{H}_F^{\alpha,\beta})$

As mentioned in §2.1.1, given a finite dimensional central division algebra $\mathbb{D}$ over F there exists a field extension L of F, for which there is an isomorphism $\phi : \mathbb{D} \otimes_F L \to M(d, L)$. All division algebras split over $\overline{F}$, the algebraic closure of F. For us, F will always be a number field which is a subfield of $\mathbb{R}$. Since $\overline{F} = \overline{\mathbb{Q}} \subset \mathbb{C}$, therefore all division algebras of the form $\mathbb{H}_F^{\alpha,\beta}$ split over $\mathbb{C}$. Depending upon choice of α and β, $\mathbb{H}_F^{\alpha,\beta}$ may also split over $\mathbb{R}$. This is the subject of the following lemma.

Lemma 2.1.7. *If either α or β is positive then $\mathbb{H}_F^{\alpha,\beta}$ splits over $\mathbb{R}$.*

Proof. First note that without loss of generality we may assume that $\alpha > 0$. This is because we have an isomorphism from $\mathbb{H}_F^{\alpha,\beta}$ to $\mathbb{H}_F^{\beta,\alpha}$ by interchanging i and j. We may also assume that $\beta < 0$. This is because we have an isomorphism from $\mathbb{H}_F^{\alpha,\beta}$ to $\mathbb{H}_F^{\alpha,-\beta/\alpha}$ by sending i to i and j to k/α. For $\mathbb{H}_F^{\alpha,\beta}$, with $\alpha > 0$ and $\beta < 0$, we have an $\mathbb{R}$-algebra homomorphism $\phi : \mathbb{H}_{\mathbb{R}}^{\alpha,\beta} \to M(2, \mathbb{R})$, that sends i to $\begin{pmatrix} \sqrt{\alpha} & 0 \\ 0 & -\sqrt{\alpha} \end{pmatrix}$ and j to $\begin{pmatrix} 0 & \sqrt{|\beta|} \\ -\sqrt{|\beta|} & 0 \end{pmatrix}$. It can be checked that ϕ is a well defined isomorphism. $\square$

Lemma 2.1.8. *If α and β are both negative then $\mathbb{H}_{\mathbb{R}}^{\alpha,\beta} \cong \mathbb{H}$.*

This follows from a more general result.

Lemma 2.1.9. *For any non zero $u, v \in F$, $\mathbb{H}_F^{u^2\alpha, v^2\beta} \cong \mathbb{H}_F^{\alpha,\beta}$*

Proof. The map sending i to ui and j to vj gives an $\mathbb{R}$-algebra isomorphism. $\square$

Now we describe the $\mathbb{R}$-points of the group $SU(A, \tau_r, \mathbb{H}_F^{\alpha,\beta})$.

Proposition 2.1.10. *If A is a τ_r-Hermitian matrix in $GL(n, \mathbb{H}_F^{\alpha,\beta})$, then*

$$SU(A, \tau_r, \mathbb{H}_F^{\alpha,\beta} \otimes \mathbb{R}) \cong \begin{cases} SO(p, 2n-p) & \text{if } \mathbb{H}_{\mathbb{R}}^{\alpha,\beta} \cong M(2,\mathbb{R}) \\ SO(n, \mathbb{H}) & \text{if } \mathbb{H}_{\mathbb{R}}^{\alpha,\beta} \cong \mathbb{H} \end{cases},$$

for a suitable $0 \leq p \leq 2n$.

Proof. When $\mathbb{H}_{\mathbb{R}}^{\alpha,\beta} \cong \mathbb{H}$, we have $SU(A, \tau_r, \mathbb{H}_{\mathbb{R}}^{\alpha,\beta}) \cong SU(A, \tau_r, \mathbb{H}) \cong SO(n, \mathbb{H})$, using Proposition 2.1.6. For the case where $\mathbb{H}_F^{\alpha,\beta}$ splits over $\mathbb{R}$, we have an explicit $\mathbb{R}$-algebra isomorphism $\phi : \mathbb{H}_{\mathbb{R}}^{\alpha,\beta} \to M(2, \mathbb{R})$ as described in the proof of Lemma 2.1.7. (As before we assume without loss of generality that $\alpha > 0$ and $\beta < 0$). The map ϕ induces an $\mathbb{R}$-algebra isomorphism from $M(n, \mathbb{H}_{\mathbb{R}}^{\alpha,\beta})$ to $M(n, M(2, \mathbb{R})) \cong M(2n, \mathbb{R})$ which we again denote by ϕ. We wish to understand the image of $SU(A, \tau_r, \mathbb{H}_{\mathbb{R}}^{\alpha,\beta})$ in $M(2n, \mathbb{R})$ under ϕ. For any $X \in M(n, \mathbb{H}_{\mathbb{R}}^{\alpha,\beta})$, a simple calculation shows that $\phi(\tau_r(X)^\top) = \phi(X)^\top$. Thus $g \in SL(n, \mathbb{H})$ satisfies $\tau_r(g)^\top A g = A$ if and only if $\phi(g) \in SL(2n, \mathbb{R})$ satisfies $\phi(g)^\top \phi(A) \phi(g) = \phi(A)$. Let the signature of the symmetric matrix $\phi(A)$ be $(p, 2n-p)$. Then the group $SU(A, \tau_r, \mathbb{H}_{\mathbb{R}}^{\alpha,\beta})$ is isomorphic to $SO(p, 2n-p)$. □

2.1.3 Restriction of scalars for $SU(A, \tau_r, \mathbb{H}_F^{\alpha,\beta})$

Let F be a real number field and let V_∞ denote the Archimedean places of F. Then each $s \in V_\infty$, induces an isomorphism of division rings $\mathbb{H}_F^{\alpha,\beta} \to \mathbb{H}_{s(F)}^{s(\alpha),s(\beta)}$ which in turn induces a ring isomorphism $M(n, \mathbb{H}_F^{\alpha,\beta}) \to M(n, \mathbb{H}_{s(F)}^{s(\alpha),s(\beta)})$. By abuse of notation we will again denote each of these isomorphisms by s.

Let $V_\infty^{\mathbb{R}}$ denote the real places and $V_\infty^{\mathbb{C}}$ denote the pairs of conjugate complex places.

Let $\iota \in V_\infty^{\mathbb{R}}$ denote the inclusion of F in $\mathbb{R}$. Then

$$\begin{aligned} \mathrm{Res}_{F/\mathbb{Q}}(SU(A, \tau_r, \mathbb{H}_F^{\alpha,\beta}))(\mathbb{R}) = &\prod_{s \in V_\infty^{\mathbb{R}}} SU(s(A), \tau_r, \mathbb{H}_{\mathbb{R}}^{s(\alpha),s(\beta)}) \\ &\times \prod_{s \in V_\infty^{\mathbb{C}}} SU(s(A), \tau_r, \mathbb{H}_{\mathbb{C}}^{s(\alpha),s(\beta)}). \end{aligned}$$

We wish to find the conditions on the elements $\alpha, \beta \in F$ and the matrix A, such that all the factors in the above product, except the one corresponding to $s = \iota$, are compact, and the one corresponding to $s = \iota$ is isomorphic to $SO(n, \mathbb{H})$. To begin with none of the places of F can be complex, since the factors corresponding to complex places are complex linear Lie groups and such groups cannot be compact unless they are trivial. So from now on we will assume that F is totally real. By Proposition 2.1.10, for ι to be the only place where $SU(s(A), \tau_r, \mathbb{H}_{\mathbb{R}}^{s(\alpha),s(\beta)}) \cong SO(n, \mathbb{H})$, it is necessary and sufficient that both α and β be negative and at least one of $s(\alpha)$ and $s(\beta)$ is positive for all places $s \neq \iota$.

One way to achieve this is the following. By primitive element theorem there exists an irreducible polynomial $f \in \mathbb{Q}[x]$, such that $F \cong \mathbb{Q}[x]/(f)$. All the embeddings of F in $\mathbb{C}$ are obtained by substituting some root of f for x in $\mathbb{Q}[x]/(f)$. Since F is totally real, we must have that the roots are all real. Let the roots be $a_0 < \cdots < a_k$ and assume that $F = \mathbb{Q}[a_0]$. Now choose $b \in \mathbb{Q}$ between a_0 and a_1, and consider the element $\beta = a_0 - b \in F$. Note that $\beta < 0$, but $s(\beta) > 0$ for all $\iota \neq s \in V_\infty$.

Since Proposition 2.1.4 says that $SU(A, \tau_r, \mathbb{H}_F^{\alpha,\beta})$ is conjugate to a group of the form $SU(A', \tau_r, \mathbb{H}_F^{\alpha,\beta})$, where A' is diagonal, we will assume from now on that A is diagonal. We will also assume that α, β has been chosen so that $\mathbb{H}_{\mathbb{R}}^{s(\alpha),s(\beta)} \cong M(2, \mathbb{R})$, for each $s \neq \iota$. Then by Proposition 2.1.10, for each $s \neq \iota$, $SU(s(A), \tau_r, \mathbb{H}_{\mathbb{R}}^{s(\alpha),s(\beta)}) \cong SO(p, 2n-p)$ for some $0 \leq p \leq 2n$. For $SO(p, 2n-p)$ to be compact we must have either $p = 0$ or $p = 2n$. Now $(p, 2n-p)$ is the signature of the symmetric matrix $\phi(s(A))$, where ϕ is as defined in the proof of Proposition 2.1.10. Since A is diagonal

$\phi(s(A))$ is a 2×2 block diagonal matrix. The signature of $\phi(s(A))$ is the sum of signatures of the 2×2 symmetric matrices $\phi(s(a_i))$, where $A = \mathrm{diag}(a_1, \cdots, a_n)$. The following lemma tells us what these signatures will be. Cf. [25, Exercise 6.4.9].

Lemma 2.1.11. *Let $\alpha > 0, \beta < 0$. Let $N_{\alpha,\beta}$ denote the reduced norm in $\mathbb{H}_F^{\alpha,\beta}$. Let $\phi : \mathbb{H}_{\mathbb{R}}^{\alpha,\beta} \to M(2, \mathbb{R})$ be any $\mathbb{R}$-algebra isomorphism. Let $x = p + iq + jr + ks \in \mathbb{H}_{\mathbb{R}}^{\alpha,\beta}$ be an invertible element (equivalently $N_{\alpha,\beta}(x) \neq 0$) such that $\tau_r(x) = x$. Then the number of positive eigenvalues of $\phi(x)$ is*

$$\epsilon_{\alpha,\beta}(x) = \begin{cases} 1 & \text{if } N_{\alpha,\beta}(x) < 0 \\ 2 & \text{if } N_{\alpha,\beta}(x) > 0 \text{ and } p > 0 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Note that $\tau_r(x) = x$ implies x is in fact equal to $p+iq+ks$. By Skolem-Noether theorem, the trace and determinant of $\phi(x)$ are independent of the choice of ϕ. In this case the trace is $2p$ and the determinant is $N_{\alpha,\beta}$. Hence the characteristic polynomial of $\phi(x)$ is $t^2 - 2pt + N_{\alpha,\beta}(x)$. The roots of this polynomial are $p \pm \sqrt{p^2 - N_{\alpha,\beta}(x)}$. Now the statement of the lemma becomes clear. □

If either α or β is positive, then $\mathbb{H}_F^{\alpha,\beta}$ splits over $\mathbb{R}$. We have done all our calculations assuming $\alpha > 0, \beta < 0$. A parallel body of calculations could have been done assuming $\alpha < 0, \beta > 0$ or $\alpha, \beta > 0$. Cf. [25, Exercises 6.4.7 and 6.4.10]. In any case let $\epsilon_{\alpha,\beta}(x)$ denote the number of positive eigenvalues of the relevant 2×2 matrices that block-diagonally make up the symmetric matrix which is preserved by elements of $\phi(SU(A, \tau_r, \mathbb{H}_{\mathbb{R}}^{\alpha,\beta}))$.

Proposition 2.1.12. *Let $\iota : F \hookrightarrow \mathbb{R}$ be the inclusion of a totally real number field $F \neq \mathbb{Q}$. Let $\alpha, \beta \in F$ such that $\alpha, \beta < 0$ and for each place $\iota \neq s \in V_\infty$, at least one of $s(\alpha)$ and $s(\beta)$ is positive. Let $a_1, \cdots, a_n \in \mathbb{H}_F^{\alpha,\beta} \setminus \{0\}$ be chosen so that for all i, $\tau_r(a_i) = a_i$ and for each place $\iota \neq s \in V_\infty$, $\sum_{i=1}^n \epsilon_{s(\alpha),s(\beta)}(s(a_i)) \in \{0, 2n\}$.*

Let A be the diagonal τ_r-Hermitian matrix whose diagonal entries are $a_1, \cdots, a_n$. Then there is a Lie group homomorphism $Res_{F/\mathbb{Q}}(SU(A, \tau_r, \mathbb{H}_F^{\alpha,\beta}))(\mathbb{R}) \to SO(n, \mathbb{H})$, which has compact kernel. Any arithmetic subgroup of $SO(n, \mathbb{H})$ corresponding to this $\mathbb{Q}$-structure is cocompact.

2.1.4 Cocompactness criterion when $F = \mathbb{Q}$

Recall that Godement's compactness criterion says that an arithmetic lattice is cocompact if and only if it has no non-trivial unipotent element. Let Γ be an arithmetic lattice with respect to a $\mathbb{Q}$-structure $SU(\tau_r, A, \mathbb{H}_{\mathbb{Q}}^{\alpha,\beta})$ of $SU(\tau_r, A, \mathbb{H}_{\mathbb{R}}^{\alpha,\beta})$. Let ϕ be an embedding of $SU(\tau_r, A, \mathbb{H}_{\mathbb{Q}}^{\alpha,\beta})$ in the space of rational valued matrices. Let $\gamma \in \Gamma$ be such that $\phi(\gamma)$ is a non-trivial unipotent matrix. This implies that there exists an integer $m \geq 2$, such that as $\mathbb{Q}$-linear transformations $(\phi(\gamma) - id)^m = 0$ and $(\phi(\gamma) - id)^{m-1} \neq 0$. Since ϕ is an injective group homomorphism, therefore this is same as saying $(\gamma - id)^m = 0$ and $(\gamma - id)^{m-1} \neq 0$. This implies $0 \neq \mathrm{Im}(\gamma - id)^{m-1} \subset \mathrm{Ker}(\gamma - id)$. Let $0 \neq v \in \mathrm{Im}(\gamma - id)^{m-1}$. That is, there exists a $w' \in (\mathbb{H}_{\mathbb{Q}}^{\alpha,\beta})^n$, such that $v = (\gamma - id)^{m-1}(w')$. Define $w := (\gamma - id)^{m-2}(w')$. Then $v = (\gamma - id)w$. In short we found two non-zero vectors v and w in $(\mathbb{H}_{\mathbb{Q}}^{\alpha,\beta})^n$, such that $\gamma v = v$ and $\gamma w = v + w$. Now suppose we denote the τ_r-Hermitian form corresponding to the τ_r-Hermitian matix A by $\langle\ ,\ \rangle$. Then

$$\begin{aligned}\langle v, w\rangle &= \langle \gamma v, \gamma w\rangle \\ &= \langle v, v + w\rangle \\ &= \langle v, v\rangle + \langle v, w\rangle\end{aligned}$$

This implies $\langle v, v\rangle = 0$. Thus $\langle\ ,\ \rangle$ is not anisotropic over $\mathbb{Q}$. On the other hand suppose $\langle\ ,\ \rangle$ is not anisotropic over $\mathbb{Q}$, that is, there exists $0 \neq v_1 \in \mathbb{H}_{\mathbb{Q}}^{\alpha,\beta}$, such that $\langle v_1, v_1\rangle = 0$. (v_1) is a linearly independent set, complete it to form a basis of

$(\mathbb{H}_{\mathbb{Q}}^{\alpha,\beta})^n$. As seen in the proof of Proposition 2.1.4, this basis can be orthogonalized to construct a basis $(v_1, \cdots, v_n)$. Define $u : (\mathbb{H}_{\mathbb{Q}}^{\alpha,\beta})^n \to (\mathbb{H}_{\mathbb{Q}}^{\alpha,\beta})^n$ to be the linear transformation that takes v_1 to v_1 and v_i to $v_i + v_1$, for all $i \geq 2$. Note that $u \in SU(\tau_r, A, \mathbb{H}_{\mathbb{Q}}^{\alpha,\beta})$ and $\phi(u)$ is unipotent. There exists an arithmetic lattice with respect to the $\mathbb{Q}$-structure $SU(\tau_r, A, \mathbb{H}_{\mathbb{Q}}^{\alpha,\beta})$ which contains the unipotent element u. Hence by Godement's criterion any lattice in this commensurability class is not cocompact. The following result summarises the above discussion.

Proposition 2.1.13. *The arithmetic lattices in $SU(\tau_r, A, \mathbb{H}_{\mathbb{R}}^{\alpha,\beta})$ corresponding to the $\mathbb{Q}$-structure $SU(\tau_r, A, \mathbb{H}_{\mathbb{Q}}^{\alpha,\beta})$ are cocompact if and only if the τ_r-Hermitian form corresponding to A is $\mathbb{Q}$-anisotropic.*

Definition 2.1.14. *The F-groups described in Propositions 2.1.12 and 2.1.13 will be said to be **of type** $DIII_u$.*

2.2 Construction of special cycles

We are interested in constructing cohomology classes of compact locally symmetric spaces associated to the group $SO(n, \mathbb{H})$. For any compact locally symmetric space associated to a semisimple Lie group G, the G-invariant forms on G/K represent non-zero cohomology classes. We wish to construct cohomology classes that are not G-invariant. These classes will be Poincaré duals of certain homology classes represented by submanifolds that we will construct. We remark here that since any compact locally symmetric spaces associated to $SO(n, \mathbb{H})$ is a compact Kähler manifold, any complex analytic submanifold will represent a non-zero class in homology. But the Poincaré dual of the complex analytic submanifold may be G-invariant. So we follow the method of Millson and Raghunathan, who in [22], construct pairs of complementary dimensional totally geodesic submanifolds of certain arithmetic locally symmetric spaces, whose intersection numbers are non-zero. This proves that these submanifolds represent non-zero classes in homology. Moreover in [22, Theorem 2.1], it is proved that these submanifolds can be lifted to a finite cover such that their Poincaré duals are not G-invariant. We will make this statement more precise below. In this section we describe the general framework for construction of such geometric cycles following the work of Rohlfs and Schwermer in [29].

Let $\mathbf{G}$ be a semisimple algebraic group over $\mathbb{Q}$. Let its real points be $G = \mathbf{G}(\mathbb{R})$, which is a simply connected Lie group. Let Λ be any torsion free arithmetic subgroup of G with respect to this $\mathbb{Q}$ structure. Let σ and τ be two finite order $\mathbb{Q}$-automorphisms of $\mathbf{G}$ which commute, so that the group $\Theta = \langle \sigma, \tau \rangle$ generated by the two remains finite order. We shall denote by the same symbols σ and τ the induced automorphisms of G. We assume that Θ takes a maximal compact subgroup K of G to itself. Thus we get an induced pair of commuting isometries of $X := G/K$ again denoted by σ and τ. Let $\mathbf{G}(\mathbb{Q})$ denote the $\mathbb{Q}$-points of $\mathbf{G}$. Since Θ consists of $\mathbb{Q}$-automorphisms, therefore $\mathbf{G}(\mathbb{Q})$ is taken to itself. Hence there exists a finite index torsion free subgroup Γ of Λ, which is taken to itself. Thus we get an induced finite order group of isometries of $\Gamma\backslash X$. Let $G(\sigma), \Gamma(\sigma)$ and $G(\tau), \Gamma(\tau)$ denote the fixed point sets in the appropriate groups under the action of σ and τ respectively. Let $X(\sigma)$ and $X(\tau)$ denote the fixed point sets in X under the actions of σ and τ respectively. Then $X(\sigma)$ and $X(\tau)$ are isometric to the symmetric spaces associated to the linear reductive Lie groups $G(\sigma)$ and $G(\tau)$ respectively. Let $C(\sigma, \Gamma)$ and $C(\tau, \Gamma)$ denote the connected components of the fixed point submanifolds of σ and τ respectively which are images of $\Gamma(\sigma)\backslash X(\sigma)$ and $\Gamma(\tau)\backslash X(\tau)$ under the natural inclusions. These are called *special cycles*. In this setting Rohlfs and Schwermer proves a result ([29, Theorem 4.11]) which we paraphrase below.

Theorem 2.2.1. ***(Rohlfs and Schwermer)*** *With the above notations suppose*

1. *the real Lie groups G, $G(\sigma)$ and $G(\tau)$ act in an orientation preserving manner on their respective spaces X, $X(\sigma)$ and $X(\tau)$,*

2. *the spaces* $X(\sigma)$ *and* $X(\tau)$ *intersect at exactly one point with positive intersection number,*

3. $\dim X(\sigma) + \dim X(\tau) = \dim X$.

Then there exists a $\langle \sigma, \tau \rangle$*-stable normal subgroup* $\Gamma' \subset \Gamma$ *of finite index such that the intersection number of* $C(\sigma, \Gamma')$ *and* $C(\tau, \Gamma')$ *is not equal to zero.*

With this notation the non-G-invariance statement ([22, Theorem 2.1]) of Millson and Raghunathan can be stated as below.

Theorem 2.2.2. ***(Millson and Raghunathan)*** *There exists a finite index subgroup* Γ'' *of* Γ' *such that the Poincaré duals of* $C(\sigma, \Gamma'')$ *and* $C(\tau, \Gamma'')$ *cannot be represented by* G*-invariant forms on* X*.*

Let us provide certain conditions under which the criteria of Theorem 2.2.1 are met. First we give a sufficient condition for an automorphism to take a given maximal compact subgroup K to itself.

Lemma 2.2.3. *Suppose* σ *is an automorphism of* G *and* θ *is a global Cartan involution such that* $\sigma \circ \theta = \theta \circ \sigma$*. Let* K *be the maximal compact subgroup that is point wise fixed by* θ*. Then* σ *keeps* K *invariant.*

The proof follows from standard arguments and we omit it.

If we can find an automorphism σ as in the above lemma, which is of finite order, then immediately $\sigma \circ \theta$ presents itself as another candidate satisfying the same condition and commuting with σ. The next lemma shows that they satisfy condition 2. of Theorem 2.2.1. The proof is elementary and we omit it.

Lemma 2.2.4. *Let* σ *be an automorphism of* G *which commutes with a global Cartan involution* θ*. Let* $o \in X = G/K$ *denote the trivial left coset of* K *in* G*. Then the* *fixed point submanifolds $X(\sigma)$ and $X(\sigma \circ \theta)$ of the actions of σ and $\sigma \circ \theta$ on X, intersect exactly at the point o.*

Lastly let us prove that $X(\sigma)$ and $X(\sigma \circ \theta)$ satisfy condition 3.

Lemma 2.2.5. *Suppose σ is an involution of G that commutes with a global Cartan involution θ. Then* $\dim(X(\sigma)) + \dim(X(\sigma \circ \theta)) = \dim X$.

Proof. It is enough to show that $T_oX = T_o(X(\sigma)) \oplus T_o(X(\sigma \circ \theta))$. But this follows from the observations that

$$T_oX = (+1 \text{ eigenspace of } d\sigma_o) \oplus (-1 \text{ eigenspace of } d\sigma_o)$$

and that the -1 eigenspace of $d\sigma_o$ is the $+1$ eigenspace of $d(\sigma \circ \theta)_o$, since $d\theta_o$ acts as $-Id$ on T_oX. □

Remark 2.2.6. *Condition* 1. *of Theorem 2.2.1 is called the **Or** condition. It has to be checked case by case.*

2.3 F-rational involutions

In §2.2 we described a way to produce non-G-invariant cohomology classes of arithmetic locally symmetric spaces by taking Poincaré duals of submanifolds which are fixed point subsets of isometries induced by $\mathbb{Q}$-involutions of G. In particular we considered pairs of $\mathbb{Q}$-involutions $(\sigma, \theta\sigma)$, where θ is a global Cartan involution and σ commutes with θ. If the $\mathbb{Q}$-structure is obtained by restriction of scalars applied to an F-group $\mathbf{G}$, then any F-morphism of $\mathbf{G}$ induces a $\mathbb{Q}$-morphism of $\mathrm{Res}_{F/\mathbb{Q}}(\mathbf{G})$. In §2.1.1 we have described all F-structures (except the exotic ones for $n = 4$) of $SO(n, \mathbb{H})$. In what follows we first produce some F-involutions σ for a restricted family of these F-structures. Note that $\sigma\theta$ is also an F-involution implies that θ itself must be F-rational. So next we show that for this restricted family an F-rational Cartan involution does exist. Finally we calculate the fixed point subgroups of the involutions we constructed and check whether they satisfy the condition Or. (See Remark 2.2.6 for definition of the Or condition.)

2.3.1 A fairly general way to create involutions

Let F be a subfield of $\mathbb{R}$. (We will only be concerned with the cases where F is either a totally real number field or $F = \mathbb{R}$.) Let α and β be two elements of F which are negative. We are given the F-group $SU(A, \tau_r, \mathbb{H}_F^{\alpha,\beta})$, where A is a non-singular τ_r-Hermitian diagonal matrix. We will produce some F-involutions of $SU(A, \tau_r, \mathbb{H}_F^{\alpha,\beta})$. These involutions will be conjugation by some diagonal matrices $D \in GL(n, \mathbb{H}_F^{\alpha,\beta})$. Conjugation by such a matrix D will be an involutive automorphism if it satisfies the following two criteria:

(1) $D^2 = -\lambda I_n,$ for some $\lambda \in F^\times$,

(2) $\tau_r(D)^\top AD = \mu A,$ for some $\mu \in F^\times$.

Note that D satisfies (1), if its elements satisfy either of the following conditions:

(a) All the diagonal elements of D are in F and their norms are all equal.

(b) All the diagonal elements of D are of the form $iq + jr + ks$, such that their norms are all equal.

In case (a), $-\lambda$ is the norm of all the diagonal elements in D, and, in case (b), λ is so. Suppose D is as in (a). Then clearly it also satisfies criterion 1 for $\mu = -\lambda$. Up to multiplication by an element of F, all such D are of the form $\text{diag}(\epsilon_1, \cdots, \epsilon_n)$, where $\epsilon_l = \pm 1$ for all l. Conjugation by such a D will be called *a sign involution*.

Now suppose D is as in case (b). Let the l^{th} diagonal element of D be denoted by x_l, and that of A be denoted by a_l. We have $N(x_l) = \lambda$ for all l. Applying the norm function to each diagonal element on both sides of the matrix equation (2), and noting that $N(\tau_r(z)) = N(z)$ for all $z \in \mathbb{H}_F^{\alpha,\beta}$, we get,

$$N(\tau_r(x_l)a_l x_l) = \mu^2 N(a_l)$$
$$\implies N(x_l)^2 = \mu^2$$
$$\implies N(x_l) = \lambda = \pm\mu \text{ for all } l.$$

Noting that $\tau_r(z) = j^{-1}\tau_c(z)j$ for all $z \in \mathbb{H}_F^{\alpha,\beta}$, (2) implies that, for each l,

$$\tau_r(x_l)a_l x_l = \pm N(x_l)a_l$$
$$\implies \tau_c(x_l)(ja_l)x_l = \pm N(x_l)(ja_l)$$
$$\implies (ja_l)x_l = \pm x_l(ja_l),$$

where the the sign is + if $\lambda = \mu$ and is $-$ if $\lambda = -\mu$. Thus each x_l either commutes or anti-commutes with ja_l. In the first case x_l must be in the F-span of ja_l and in the second case x_l must be perpendicular to ja_l in the vector space $iF + jF + kF$ equipped with the inner product corresponding to the quadratic form given by the restriction of the norm on $\mathbb{H}_F^{\alpha,\beta}$ to this subspace. If the diagonal matrix $D \in GL(n, \mathbb{H}_F^{\alpha,\beta})$ satisfy either of these conditions along with the condition that all its diagonal elements have the same norm, then conjugation by D is an involution. In the first case we will call this *an involution of even type* and in the second case *an involution of odd type.*

Lemma 2.3.1. *An involution of even type exists if and only if the norms of all the diagonal elements of A, belong to the same class in $F^\times/(F^\times)^2$.*

Proof. Suppose $N(a_1), \cdots, N(a_n)$ are in the same class in $F^\times/(F^\times)^2$. Then so are $N(ja_1), \cdots, N(ja_n)$. So there exists $t_1, \cdots, t_n \in F$ such that $t_1^2 N(ja_1) = \cdots = t_n^2 N(ja_n)$. Then conjugation by the diagonal matrix D, whose l^{th} diagonal element is $t_l j a_l$, is an involution of even type.

Conversely, suppose conjugation by a diagonal matrix D is an involution of even type. Then the l^{th} diagonal element of D is of the form $t_l j a_l$, where $t_l \in F$. Since the norms of all the diagonal elements of A are same, therefore $N(j)t_l^2 N(a_l) = \cdots = N(j)t_n^2 N(a_n)$. Hence $N(a_1), \cdots, N(a_n)$ all belong to the same class in $F^\times/(F^\times)^2$. □

The necessary and sufficient condition for existence of an involution of even type is also a sufficient condition for existence of involutions of odd type.

Lemma 2.3.2. *Suppose the diagonal entries of A all belong to the same class in $F^\times/(F^\times)^2$, then there exist involutions of odd type.*

Proof. Consider the vector space $iF + jF + kF$ equipped with the inner product corresponding to the quadratic form given by the restriction of the norm on $\mathbb{H}_F^{\alpha,\beta}$ to this subspace. Then the fact that $N(ja_1), \cdots, N(ja_n)$ all belong to the same class in $F^\times/(F^\times)^2$ implies that the one dimensional subspaces spanned by each ja_l are isometric. Now Witt Cancellation theorem ([17, Chapter XV, Section 10]) tells us that the two dimensional subspaces perpendicular to each of them are also isometric. Thus it is possible to pick an element x_l from the l^{th} of these perpendicular subspaces such that $N(x_1) = \cdots = N(x_n)$. Then conjugation by the matrix $D = \mathrm{diag}(x_1, \cdots, x_n)$ is an involution of odd type. □

Remark 2.3.3. *Sign involutions and involutions of odd and even type commute with involutions of even type.*

2.3.2 The F-rational Cartan involution

A Cartan involution for the group $SU(I_n, \tau_r, \mathbb{H}_{\mathbb{R}}^{\alpha,\beta})$, where both α and β are negative, is given by $g \mapsto (\tau_c(g)^\top)^{-1}$. Since $\tau_r(g)^\top g = I_n$ for each $g \in SO(n, \mathbb{H})$ and $\tau_r(z) = j\tau_c(z)j^{-1}$ for all $z \in \mathbb{H}$, we have that $(\tau_c(g)^\top)^{-1} = (jI_n)g(jI_n)^{-1}$. That is, the Cartan involution is just conjugation by jI_n. Now consider the group $SU(A, \tau_r, \mathbb{H}_{\mathbb{R}}^{\alpha,\beta})$ where $A \in GL(n, \mathbb{H}_{\mathbb{R}}^{\alpha,\beta})$ is a diagonal τ_r-Hermitian matrix. We have an isomorphism $SU(I_n, \tau_r, \mathbb{H}_{\mathbb{R}}^{\alpha,\beta}) \to SU(A, \tau_r, \mathbb{H}_{\mathbb{R}}^{\alpha,\beta})$, given by $g \mapsto PgP^{-1}$, where $\tau_r(P)^\top AP = I_n$. Note since A is diagonal, P can be (and will be) chosen to be diagonal too. A Cartan involution θ_A for $SU(A, \tau_r, \mathbb{H}_{\mathbb{R}}^{\alpha,\beta})$ can be described as one that makes the following diagram commute, where conjugation by any element X, is denoted by ι_X.

$$\begin{array}{ccc}
SU(A, \tau_r, \mathbb{H}_{\mathbb{R}}^{\alpha,\beta}) & \xrightarrow{\theta_A} & SU(A, \tau_r, \mathbb{H}_{\mathbb{R}}^{\alpha,\beta}) \\
\iota_P \uparrow & & \uparrow \iota_P \\
SU(I_n, \tau_r, \mathbb{H}_{\mathbb{R}}^{\alpha,\beta}) & \xrightarrow{\iota_{(jI_n)}} & SU(I_n, \tau_r, \mathbb{H}_{\mathbb{R}}^{\alpha,\beta})
\end{array}$$

Thus $\theta_A = \iota_{PjP^{-1}}$. Let p_l and a_l denote the l^{th} diagonal elements of P and A respectively, then the matrix equation $\tau_r(P)^\top AP = I_n$ is equivalent to saying that for all l, $\tau_r(p_l)a_l p_l = 1$. Then the l^{th} diagonal element of PjP^{-1} can be simplified as follows,

$$\begin{aligned}
p_l j p_l^{-1} &= (p_l^{-1})^{-1} j p_l^{-1} \\
&= \frac{1}{N(p_l^{-1})}\tau_c(p_l^{-1}) j p_l^{-1} \\
&= \frac{j}{N(p_l^{-1})}\tau_r(p_l^{-1}) p_l^{-1} \\
&= \frac{j a_l}{N(p_l^{-1})}
\end{aligned}$$

θ_A is an $\mathbb{R}$-involution for the $\mathbb{R}$-group $SU(A, \tau_r, \mathbb{H}_{\mathbb{R}}^{\alpha,\beta})$. We had discussed certain types of involutions in §2.3.1. From the above calculation it is clear that θ_A is an involution of even type. Now suppose α, β are negative elements of a totally real field F and $A \in GL(n, \mathbb{H}_F^{\alpha,\beta})$. Then Lemma 2.3.1 implies that if the norms of the diagonal elements of A belong to the same class in $F^\times/(F^\times)^2$, then θ_A is an F-rational Cartan involution.

Definition 2.3.4. *Let F be a totally real number field. Let $\alpha, \beta \in F$ such that $\alpha, \beta < 0$. Let $A \in GL(n, \mathbb{H}_F^{\alpha,\beta})$ be a τ_r-Hermitian diagonal matrix such that the norms of the diagonal elements of A belong to the same class in $F^\times/(F^\times)^2$. Then we say that the F-group $SU(A, \tau_r, \mathbb{H}_F^{\alpha,\beta})$ admits an F-rational Cartan involution* ***of diagonal type****.*

Let $SU(A, \tau_r, \mathbb{H}_F^{\alpha,\beta})$ be an F-group that admits an F-rational Cartan involution of diagonal type. Then θ_A as described above is an F-rational involution of even type. Let σ_A be a sign involution or an involution of even or odd type. Then by Remark 2.3.3, σ_A commutes with θ_A. Hence $\theta_A \sigma_A$ is an F-involution. Note that if σ_A is a sign involution then $\theta_A \sigma_A$ is an involution of even type and vice versa, whereas if σ_A is an involution of odd type then so is $\theta_A \sigma_A$.

2.3.3 Fixed point subgroups of involutions

We wish to calculate the subgroups of $SU(A, \tau_r, \mathbb{H}_\mathbb{R}^{\alpha,\beta})$, with $\alpha, \beta < 0$, which are point-wise fixed by the involutions described in §2.3.1. We will not do so directly but rather we will show that these fixed point subgroups are isomorphic to familiar Lie groups. As a first step we will transfer all the calculations to the setting of the group $SO(n, \mathbb{H})$, using the isomorphism

$$SU(A, \tau_r, \mathbb{H}_\mathbb{R}^{\alpha,\beta}) \xrightarrow{\iota_P} SU(I_n, \tau_r, \mathbb{H}_\mathbb{R}^{\alpha,\beta}) \xrightarrow{\Psi} SO(n, \mathbb{H})$$

where ι_P is as defined in §2.3.2 and Ψ is the $\mathbb{R}$-algebra isomorphism induced by the isomorphism $\psi : \mathbb{H}_\mathbb{R}^{\alpha,\beta} \to \mathbb{H}$ that sends i to $i\sqrt{|\alpha|}$ and j to $j\sqrt{|\beta|}$. Let us call the composition η. Given an involution σ_A of $SU(A, \tau_r, \mathbb{H}_{\mathbb{R}}^{\alpha,\beta})$, it induces an involution σ of $SO(n, \mathbb{H})$ which makes the following diagram commute.

$$\begin{array}{ccc} SU(A, \tau_r, \mathbb{H}_{\mathbb{R}}^{\alpha,\beta}) & \xrightarrow{\sigma_A} & SU(A, \tau_r, \mathbb{H}_{\mathbb{R}}^{\alpha,\beta}) \\ \eta \uparrow & & \uparrow \eta \\ SO(n, \mathbb{H}) & \xrightarrow{\sigma} & SO(n, \mathbb{H}) \end{array}$$

Now, it is clear that the fixed point subgroup of $SU(A, \tau_r, \mathbb{H}_{\mathbb{R}}^{\alpha,\beta})$ under σ_A is isomorphic to that of $SO(n, \mathbb{H})$ under σ via restriction of the isomorphism η. As a general strategy to simplify calculations, we will conjugate an involution σ by some automorphism of $SO(n, \mathbb{H})$, so that the the fixed point subgroup of the new involution is isomorphic to that of σ. For a specific type of involution, $SO(n, \mathbb{H})$ may not be best behaved group in terms of the fixed point subgroup being a familiar Lie group. In particular for involutions of even type we will work with a distinct isomorphic copy of $SO(n, \mathbb{H})$.

Note that if σ_A is a sign involution or an involution of even or odd type, then so is σ, respectively. This follows from the following general observations. Conjugation by any non-zero element in the division algebra $\mathbb{H}_{\mathbb{R}}^{\alpha,\beta}$ is an isometry of the real vector space $\mathbb{H}_{\mathbb{R}}^{\alpha,\beta}$ equipped with the inner product induced by the quadratic form coming from the reduced norm, and so is ψ. Now we do a case by case analysis.

We first consider a sign involution σ which is conjugation by a matrix of the form $D = \text{diag}(\epsilon_1, \cdots, \epsilon_n)$, where $\epsilon_i = \pm 1$. We can conjugate D by a permutation matrix to get a matrix of the form $I_{l,n-l} := \begin{pmatrix} I_l & \\ & -I_{n-l} \end{pmatrix}$. Note that conjugation by any permutation matrix is an automorphism of $SO(n, \mathbb{H})$. Hence it is enough to consider the case when $D = I_{l,n-l}$. An element of $SO(n, \mathbb{H})$ is fixed by σ if and only if it is block diagonal $\begin{pmatrix} X & \\ & Y \end{pmatrix}$, where $X \in SO(l, \mathbb{H}), Y \in SO(n-l, \mathbb{H})$. Thus the fixed point subgroup of $G = SO(n, \mathbb{H})$ under action of σ is $G(\sigma) \cong SO(l, \mathbb{H}) \times SO(n-l, \mathbb{H})$.

Next we turn to involutions of even type. An involution of even type is conjugation by a diagonal matrix D whose diagonal entries are real multiples of j and all having the same reduced norm. So we may assume $D = \mathrm{diag}(\epsilon_1 j, \cdots, \epsilon_n j)$. By the same argument that we used in the case of signed involutions, it is enough to consider the case when $D = jI_{l,n-l}$. Consider the group $G' := \{g \in SL(n, \mathbb{H}) : \tau_r(g)^\top I_{l,n-l} g = I_{l,n-l}\}$. We have an isomorphism $SO(n, \mathbb{H}) \to G'$ given by conjugation by the matrix $Q := \begin{pmatrix} I_l & \\ & iI_{n-l} \end{pmatrix}$. The corresponding involution σ' for G' is conjugation by the matrix $QDQ^{-1} = jI_n$. Now the fixed point subalgebra of $M(n, \mathbb{H})$ under conjugation by jI_n is $M(n, \mathbb{R} + j\mathbb{R})$. Hence the fixed point subgroup of G' under the action of σ' is $G'(\sigma') := \{g \in SL(n, \mathbb{R} + j\mathbb{R}) : \tau_r(g)^\top I_{l,n-l} g = I_{l,n-l}\}$. The determinant map on $M(n, \mathbb{R} + j\mathbb{R})$ is restriction of that on $M(n, \mathbb{H})$ and by $SL(n, \mathbb{R} + j\mathbb{R})$ we mean matrices whose determinants are 1, with this definition of determinant. Note that $\mathbb{R} + j\mathbb{R}$ is isomorphic to $\mathbb{C}$ as a ring and the operation τ_r restricted to $\mathbb{R} + j\mathbb{R}$ corresponds to complex conjugation on $\mathbb{C}$. Also note that with the above definition of determinant, matrices $g \in M(n, \mathbb{R}+j\mathbb{R})$ satisfying $\tau_r(g)^\top I_{l,n-l} g = I_{l,n-l}$ automatically belong to $SL(n, \mathbb{R} + j\mathbb{R})$. Hence we have $G'(\sigma') \cong U(l, n-l)$.

Now we consider an involution σ of odd type, which is conjugation by a matrix of the form $D = \mathrm{diag}(d_1, \cdots, d_n)$, where $d_l \in i\mathbb{R}+k\mathbb{R}$ and without loss of generality we may assume $N(d_l) = 1$ for all l. We will simplify the calculation by constructing a matrix $T \in GL(n, \mathbb{H})$ such that conjugation by T, denoted by ι_T, is an automorphism of $SO(n, \mathbb{H})$ and $TDT^{-1} = iI_n$. We construct T as follows. Note that $i \cdot d_l^{-1} \in \mathbb{R} + j\mathbb{R}$. Since $\mathbb{R} + j\mathbb{R}$ is a copy of $\mathbb{C}$ sitting inside $\mathbb{H}$, therefore all non-constant polynomials with coefficients in $\mathbb{R} + j\mathbb{R}$, have a root in $\mathbb{R} + j\mathbb{R}$. Let t_l be a solution of the polynomial equation $x^2 = i \cdot d_l^{-1}$. Since $N(d_l) = 1$ for all l, therefore $N(t_l) = 1$ and so $t_l^{-1} = \tau_c(t_l) = \tau_r(t_l)$. Hence we have $t_l d_l t_l^{-1} = t_l d_l \tau_c(t_l) = t_l^2 d_l = i$. Define

T to be $\text{diag}(t_1, \cdots, t_n)$. Then from the above calculations it is immediate that $TDT^{-1} = iI_n$. Also note that $\tau_r(T)^\top T = I_n$ and hence ι_T is an automorphism of $SO(n, \mathbb{H})$. Thus it is enough to consider the case when σ is conjugation by the matrix iI_n. The fixed point subalgebra of $M(n, \mathbb{H})$ under conjugation by iI_n is $M(n, \mathbb{R} + i\mathbb{R})$. Hence the fixed point subgroup of $G := SO(n, \mathbb{H})$ under the action of σ is $G(\sigma) := \{g \in SL(n, \mathbb{R} + i\mathbb{R}) : \tau_r(g)^\top g = I_n\}$, where $SL(n, \mathbb{R} + i\mathbb{R})$ has a similar meaning as that of $SL(n, \mathbb{R} + j\mathbb{R})$ above. Note that τ_r acts trivially on $\mathbb{R} + i\mathbb{R}$. Now by a similar argument as in the case of involutions of even type, we have $G(\sigma) \cong O(n, \mathbb{C})$.

Now we will check whether $G(\sigma)$ acts in an orientation preserving manner on $X(\sigma)$, that is, whether the *Or* condition is satisfied for the various involutions σ. When σ is a sign involution or an involution of even type then note that the fixed point subgroups are connected, hence they obviously satisfy the *Or* condition. For involutions of odd type we have the following lemma.

Lemma 2.3.5. *The action of the group $G(\sigma) \cong O(n, \mathbb{C})$ on $X(\sigma) \cong O(n, \mathbb{C})/O(n)$ preserves the orientation if and only if n is odd. Thus involutions of odd type satisfy condition* Or *precisely when n is odd.*

Proof. The Lie group $O(n, \mathbb{C})$ has exactly two components. The identity component must act in an orientation preserving manner. For the other component it is enough to check whether one particular element in it is orientation preserving. Note that $S = \text{diag}(-1, 1 \ldots, 1)$ is an element of the non-identity component. It is enough to check whether the determinant of the action of S on the tangent space of $SO(n, \mathbb{C})/SO(n)$ at the identity coset is positive or negative. The tangent space is isomorphic to the space of real skew symmetric matrices and the action is given by conjugation. The -1 eigenspace for the action of S has dimension $n - 1$. Now the lemma follows. □

The dimensions of the symmetric spaces $X(\sigma)$ associated to $G(\sigma)$ can be read off from [10, Chapter X, Tables IV and V]. The results of this subsection are summarized in the Table 2.1. The parameter l varies between 1 and $[n/2]$.

Type of involution σ	$G(\sigma)$ is isomorphic to	$\dim X(\sigma)$	*Or* condition
sign	$SO(l,\mathbb{H}) \times SO(n-l,\mathbb{H})$	$n(n-1) - 2l(n-l)$	satisfied
even	$U(l, n-l)$	$2l(n-l)$	satisfied
odd	$O(n,\mathbb{C})$	$\frac{1}{2}n(n-1)$	satisfied if and only if n is odd

Table 2.1: Summary of results of §2.3.3

2.4 Consequences of the Kähler property

As noted earlier any compact locally symmetric space associated to the group $SO(n,\mathbb{H})$ is a compact Kähler manifold. In this section we will recall some facts about compact Kähler manifolds and the consequences of these facts for complex analytic special cycles in cocompact Hermitian locally symmetric spaces.

For any complex analytic subvariety $V \hookrightarrow M$ of a compact Kähler manifold M, one has the notion of a fundamental homology class $\mu_V \in H_*(M;\mathbb{R})$, although V is not assumed to be smooth.

Fact: [9, p. 110] *Let M be a compact Kähler manifold and let $V \subset M$ be an analytic subvariety. Then the fundamental class μ_V of V is non-zero.*

Let us apply this fact in our context. Suppose that the symmetric space X associated to a simply connected semisimple Lie group G is Hermitian symmetric. Let Λ be a torsion free cocompact arithmetic subgroup associated to a certain $\mathbb{Q}$-structure of G. Recall from §2.2, if we are given a pair of commuting finite order $\mathbb{Q}$-automorphisms σ and τ of G satisfying the hypotheses of Theorem 2.2.1, then there exists a finite index subgroup Γ'' of Λ, such that the Poincaré duals of the special cycles $C(\sigma,\Gamma'')$

and $C(\tau, \Gamma'')$ are non-G-invariant cohomology classes of $\Gamma''\backslash X$. Now suppose σ and τ induce holomorphic automorphisms of X. (Equivalently $d\sigma_o : T_oX \to T_oX$ is $\mathbb{C}$-linear.) Then $\Lambda(\sigma)\backslash X(\sigma)$ and $\Lambda(\tau)\backslash X(\tau)$ are analytic subvarieties of $\Lambda\backslash X$. Hence by the above fact, they represent non-zero homology classes of $\Lambda\backslash X$. Moreover by naturality of Poincaré duality, the Poincaré duals of the submanifolds $\Lambda(\sigma)\backslash X(\sigma)$ and $\Lambda(\sigma)\backslash X(\sigma)$ are pull backs of those of $C(\sigma, \Gamma'')$ and $C(\tau, \Gamma'')$. Note that pull back classes represented by G-invariant forms of X are again of the same form. Thus if Poincaré duals of $\Lambda(\sigma)\backslash X(\sigma)$ and $\Lambda(\tau)\backslash X(\tau)$ are represented by G-invariant forms on X then so are $C(\sigma, \Gamma'')$ and $C(\tau, \Gamma'')$. This is a contradiction, hence we have the following corollary of the Theorem 2.2.1 and Theorem 2.2.2.

Corollary 2.4.1. *Let* $\mathbf{G}$ *be a linear algebraic group defined over* $\mathbb{Q}$ *such that its set of* $\mathbb{R}$*-points is a semisimple simply connected Lie group* G*. Let the symmetric space* X *associated to* G *be Hermitian symmetric. Let* σ *and* τ *be* $\mathbb{Q}$*-involutions of* $\mathbf{G}$ *such that the induced actions of* σ *and* τ *on* X *are holomorphic, satisfying*

1. *the real Lie groups* G*,* $G(\sigma)$ *and* $G(\tau)$ *act in an orientation preserving manner on their respective spaces* X*,* $X(\sigma)$ *and* $X(\tau)$*,*
2. *the spaces* $X(\sigma)$ *and* $X(\tau)$ *intersect at exactly one point with positive intersection number,*
3. $\dim X(\sigma) + \dim X(\tau) = \dim X$.

Let Λ *be a torsion free cocompact arithmetic subgroup associated to this* $\mathbb{Q}$*-group. Then the Poincaré duals of the fundamental classes of the analytic subvarieties* $\Lambda(\sigma)\backslash X(\sigma)$ *and* $\Lambda(\tau)\backslash X(\tau)$ *cannot be represented by* G*-invariant forms on* X*.*

The next fact is related to the Hodge decomposition of the cohomology of a compact Kähler manifold. A natural question is what is the type decomposition of the

Poincaré duals of the special cycles in the cohomology of a compact Hermitian locally symmetric space. We know that the Poincaré dual of the fundamental class of an analytic subvariety is always of pure type (p,p). See [9, p. 162-163].

Lemma 2.4.2. *The Poincaré dual of a complex analytic special cycle in a compact Hermitian locally symmetric space, is of pure type* (p,p)*.*

2.5 Main result

The main result of this chapter is the following.

Theorem 2.5.1. *Let F be a totally real number field. Consider the family of lattices in $G = SO(n,\mathbb{H})$ corresponding to an F-structure of class $DIII_u$ which admit an F-rational Cartan involution of diagonal type. Then:*

(i) For each torsion free lattice in the above family there exist cohomology classes of the associated locally symmetric spaces, that are not representable by G-invariant forms of $X = SO(n,\mathbb{H})/U(n)$, in dimensions

$$2l(n-l), n(n-1) - 2l(n-l), \text{ where } 1 \leq l \leq [n/2]$$

.

(ii) If n is odd, there exists a cofinal family of torsion free lattices in the above family such that, in the middle dimension

$$\frac{1}{2}n(n-1),$$

there exists a cohomology class of the associated locally symmetric spaces, that is not representable by G-invariant forms of $X = SO(n,\mathbb{H})/U(n)$.

Proof. The proof of (ii) is an application of Theorem 2.2.1 and Theorem 2.2.2. The proof of (i) is an application of Corollary 2.4.1, noting that the sign involutions and the involutions of even type induce holomorphic automorphisms of the Hermitian symmetric space $X = SO(n,\mathbb{H})/U(n)$. (See §3.2 for the explicit complex structure.) We take σ to be a sign involution or an involution of odd or even type and $\tau := \theta\sigma$, with θ being the F-rational Cartan involution of diagonal type. Cf. §2.3.2. That the pair $(\sigma, \theta\sigma)$ satisfies conditions 2. and 3. of Theorem 2.2.1 follows from Lemma 2.2.4 and Lemma 2.2.5. That condition 1. or the Or condition is also satisfied has been proved in §2.3.3. The dimensions at which the special cycles occur and hence the dimensions at which their Poincaré duals occur, have been listed in Table 2.1 at the end of §2.3.3. □

Remark 2.5.2. *The statement (i) of Theorem 2.5.1 will be strengthened in Corollary 3.5.2, whereby the list of given dimensions will be replaced by all even dimensions between $2n - 4$ and $n(n-1) - 2n - 4$. For this we will make crucial use of Lemma 2.4.2.*

Chapter 3

Special cycles and multiplicities of $A_{\mathfrak{q}}$ in $L^2(\Gamma\backslash SO(n,\mathbb{H}))$

In this chapter we will use the results of the Chapter 2 to detect the occurrence of a certain irreducible unitary representation in the decomposition of $L^2(\Gamma\backslash SO(n,\mathbb{H}))$, where Γ is any uniform lattice corresponding to an F-structure of type $DIII_u$ which admits an F-rational Cartan involution of diagonal type. See Theorem 3.5.1. This result also has non-trivial consequence for the cohomology of locally symmetric spaces associated to these lattices. See Corollary 3.5.2. The main tool that enables us to do this is the Matsushima's isomorphism, which we describe in §3.1. This is standard material. The irreducible unitary representations that can possibly be detected this way are associated to θ-stable parabolic subalgebras of $\mathfrak{g}_0 = \mathfrak{so}(n,\mathbb{H})$. In §3.2 we describe a root space decomposition of $\mathfrak{g} = \mathfrak{so}(2n,\mathbb{C})$, which is the complexification of $\mathfrak{so}(n,\mathbb{H})$. This will aid in the classification of θ-stable parabolic subalgebras of $\mathfrak{so}(n,\mathbb{H})$. We also fix a complex structure of $SO(n,\mathbb{H})/U(n)$ in this section. These are also known facts, but written in a way so that they are readily applicable to our problem. §3.3 is devoted to a description of the set of θ-stable parabolic subalgebras of $\mathfrak{so}(\mathfrak{n},\mathbb{H})$, up to an equivalence relation that mirrors the unitary equivalence in the set of irreducible unitary representations that are associated to these subalgebras. We do this by associating a certain combinatorial diagram, called *decorated staircase diagram*, to each equivalence class. This is new material. In §3.4 we describe the Hodge polynomials of relative Lie algebra cohomologies of $(\mathfrak{so}(2n,\mathbb{C}), U(n))$ with coefficients in the Harish-Chandra modules of the irreducible unitary representations associated to the θ-stable parabolic subalgebras. This follows from standard results but may not be directly available in literature. Finally in §3.5 we identify the irreducible unitary representation that is detected via the Matsushima isomorphism by construction of one of the complex analytic special cycles in Theorem 2.5.1. The consequence for cohomology of relevant locally symmetric spaces is also stated here. These are all new results.

3.1 Matsushima's isomorphism

Let G be a linear reductive group with finitely many components and let K be a maximal compact subgroup in it. Let $\mathfrak{g}_0, \mathfrak{k}_0$ denote the Lie algebras of G and K respectively. Let θ be the Cartan involution fixing $\mathfrak{k}_0$. Let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ be the Cartan decomposition. Let $\mathfrak{g}, \mathfrak{k}$ and $\mathfrak{p}$ denote the complexifications of $\mathfrak{g}_0, \mathfrak{k}_0$ and $\mathfrak{p}_0$ respectively. Let $(\pi, \mathcal{H}_\pi)$ denote any irreducible unitary representation of G. Let $\mathcal{H}^\infty_{\pi,K}$ denote the Harish-Chandra module of π. Consider the measure space $\Gamma\backslash G$, where Γ is a torsion free uniform lattice in G and the measure is the G-invariant measure associated to a Haar measure on G. Right translation by elements of G makes $L^2(\Gamma\backslash G)$ a unitary representation of G. By a theorem of Gelfand and Piatetsky-Shapiro, it is known that $L^2(\Gamma\backslash G)$ can be decomposed into a Hilbert direct sum of irreducible unitary representations, each occurring with finite multiplicities. See [8, Chapter 1, Section 2.3].

$$L^2(\Gamma\backslash G) = \hat{\bigoplus}_{\pi\in\hat{G}} m(\pi,\Gamma)\mathcal{H}_\pi$$

where $\hat{G}$ denotes the unitary dual of G and $m(\pi, \Gamma)$ is the multiplicity with which π occurs in $L^2(\Gamma\backslash G)$.

Then Matsushima's isomorphism (see [4, Chapter VII, Theorem 3.2]) states

$$H^*(\Gamma\backslash G/K; \mathbb{C}) \cong \bigoplus_{\pi \in \hat{G}} m(\pi, \Gamma) H^*(\mathfrak{g}, K; \mathcal{H}^\infty_{\pi,K})$$

where $H^*(\mathfrak{g}, K, \mathcal{H}^\infty_{\pi,K})$ is the relative Lie algebra cohomology with coefficients in the $(\mathfrak{g}, K)$-module $\mathcal{H}^\infty_{\pi,K}$. The only representations π, for which the $(\mathfrak{g}, K)$-cohomology do not vanish are those whose infinitesimal characters equal that of the trivial representation. By [36], their Harish-Chandra modules are isomorphic to one of the $(\mathfrak{g}, K)$-modules $A_\mathfrak{q}$, where $A_\mathfrak{q}$ is the representation of $\mathfrak{g}$ obtained by cohomological induction on the trivial representation of a θ-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$.

The cohomology groups $H^*(\mathfrak{g}, K; A_\mathfrak{q})$ can be explicitly computed, as we describe now. Let $\mathfrak{l}$ be the Levi factor and let $\mathfrak{u}$ be the nilpotent radical of $\mathfrak{q}$. Since $\mathfrak{q}$ is θ-stable (see Definition 3.3.1 below), $\mathfrak{l}_0 := \mathfrak{l} \cap \mathfrak{g}_0$ is a real form of $\mathfrak{l}$. Let L be the connected Lie subgroup of G, whose Lie algebra is $\mathfrak{l}_0$. Let $R(\mathfrak{q}) := \dim_\mathbb{C}(\mathfrak{p} \cap \mathfrak{u})$. Then we have an isomorphism

$$H^k(\mathfrak{g}, K; A_\mathfrak{q}) \cong H^{k-R(\mathfrak{q})}(\mathfrak{l}, L \cap K; \mathbb{C})$$

where $\mathbb{C}$ is the trivial $(\mathfrak{l}, L \cap K)$-module. The coboundary maps for the cochain complex $C^*(\mathfrak{l}, L \cap K; \mathbb{C}) := \mathrm{Hom}_{L\cap K}(\wedge^*(\mathfrak{l} \cap \mathfrak{p}), \mathbb{C})$ are all 0, hence the cohomology groups equal the groups in the cochain complex. Now $\mathrm{Hom}_{L\cap K}(\wedge^*(\mathfrak{l} \cap \mathfrak{p}), \mathbb{C})$ is the space of L-invariant forms on $L/(L \cap K)$. Hence it is isomorphic to the cohomology group of the compact dual of $L/(L \cap K)$, which we will denote by $Y_\mathfrak{q}$. Thus we have

$$H^k(\mathfrak{g}, K; A_\mathfrak{q}) \cong H^{k-R(\mathfrak{q})}(Y_\mathfrak{q}; \mathbb{C}). \tag{3.1}$$

The cohomology groups of symmetric spaces of compact types are well known.

From now on we will assume that G is semisimple with no compact factors and K has non-discrete centre. This implies that G/K is a Hermitian symmetric space. The Hodge decomposition of the cohomology of the Kähler manifold $\Gamma\backslash G/K$ gives a bigrading on the LHS of Matsushima's isomorphism. On the RHS we have a compatible bigradation for each Lie algebra cohomology, which we describe now. Let us fix an Ad K invariant complex structure J on $\mathfrak{p}_0$. Then we have a decomposition of K-modules, $\mathfrak{p} = \mathfrak{p}^+ + \mathfrak{p}^-$, into i and $-i$ eigenspaces $\mathfrak{p}^+$ and $\mathfrak{p}^-$ of J, respectively. Hence we have a decomposition of K-modules:

$$\wedge^k \mathfrak{p} \cong \bigoplus_{p+q=k} \wedge^p \mathfrak{p}^+ \otimes \wedge^q \mathfrak{p}^-.$$

For any $(\mathfrak{g}, K)$-module V, this induces a bigrading of the cohomology groups:

$$H^k(\mathfrak{g}, K; V) \cong \bigoplus_{p+q=k} H^{p,q}(\mathfrak{g}, K; V).$$

With this additional structure Matsushima's isomorphism tells us

$$H^{p,q}(\Gamma\backslash G/K; \mathbb{C}) \cong \bigoplus_{\pi\in\hat{G}} m(\pi,\Gamma) H^{p,q}(\mathfrak{g}, K; \mathcal{H}^{\infty}_{\pi,K}).$$

The $(\mathfrak{g}, K)$-module of interest to us is $A_{\mathfrak{q}}$. In equation (3.1) we had related the Lie algebra cohomology with coefficients in $A_{\mathfrak{q}}$ with the cohomology of a symmetric space of compact type $Y_{\mathfrak{q}}$. If G/K is Hermitian symmetric space, then so are all the $Y_{\mathfrak{q}}$. The Hodge polynomials $P_{u,v}(A_{\mathfrak{q}})$ and $P_{u,v}(Y_{\mathfrak{q}})$ of $H^*(\mathfrak{g}, K; A_{\mathfrak{q}})$ and $H^*(Y_{\mathfrak{q}}; \mathbb{C})$, respectively, are related as follows. (See [36, Proposition 6.19]). Define $R^+(\mathfrak{q}) = \dim(\mathfrak{u}\cap\mathfrak{p}^+)$ and $R^-(\mathfrak{q}) = \dim(\mathfrak{u}\cap\mathfrak{p}^-)$. Then

$$P_{u,v}(A_{\mathfrak{q}}) = u^{R^+(\mathfrak{q})} v^{R^-(\mathfrak{q})} P_{u,v}(Y_{\mathfrak{q}}).$$

3.2 Roots and complex structure when $\mathfrak{g}_0 = \mathfrak{so}(n, \mathbb{H})$

Recall that the group $SO(n, \mathbb{H})$ was defined as $\{g \in SL(n, \mathbb{H}) : \tau_r(g)^\top g = I_n\}$, where τ_r is an anti-involution on $\mathbb{H}$ that takes $p + iq + jc + kd$ to $p + iq - jc + kd$ and on a matrix it acts entry-wise. Recall too the other anti-involution τ_c on $\mathbb{H}$ that takes $p + iq + jc + kd$ to $p - iq - jc - kd$. Then the Lie algebra of $G := SO(n, \mathbb{H})$ is $\mathfrak{g}_0 := \{X \in M(n, \mathbb{H}) : \tau_r(X) + X = 0\}$ and the Cartan involution on $\mathfrak{g}_0$ is given by $X \mapsto -\tau_c(X)^\top$. The corresponding Cartan decomposition is given by $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$, where $\mathfrak{k}_0$ consists of matrices in $M(n, \mathbb{R} + j\mathbb{R})$ which are skew Hermitian as $\mathbb{R} + j\mathbb{R} \cong \mathbb{C}$, and $\mathfrak{p}_0$ consists of skew symmetric matrices with entries in $i\mathbb{R} + k\mathbb{R}$.

Define an $\mathbb{R}$-algebra embedding $\phi : \mathbb{H} \to M_2(\mathbb{C})$ given by sending $p + iq + jr + ks$ to $\begin{pmatrix} p + iq & r + is \\ -r + is & p - iq \end{pmatrix}$. This induces an embedding $\phi : M(n, \mathbb{H}) \to M(n, M(2, \mathbb{C})) \cong M(2n, \mathbb{C})$, that replaces each entry of the $n \times n$ matrix by its image under ϕ. Apply this embedding to $\mathfrak{g}_0$ and complexify it to get the complex simple Lie algebra $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$. Let $\mathfrak{k}$ and $\mathfrak{p}$ denote the complexifications of $\phi(\mathfrak{k}_0)$ and $\phi(\mathfrak{p}_0)$ respectively. Since $\mathfrak{so}(n, \mathbb{H})$ is equirank, the complexification of a maximal abelian subalgebra of $\phi(\mathfrak{k}_0)$ is a Cartan subalgebra for $\mathfrak{so}(2n, \mathbb{C})$. Following [14, Chapter II, Section 1, Examples 3 and 4], we describe a maximal abelian subalgebra of $\phi(\mathfrak{k}_0)$ and the roots and root spaces of $\mathfrak{so}(2n, \mathbb{C})$ with respect to its complexification. Let $\mathfrak{t}_0 \subset \phi(\mathfrak{k}_0)$ be the subalgebra consisting of diagonal matrices in $M(n, M(2, \mathbb{C}))$ whose diagonal elements are of the form $\begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}$, with $d \in \mathbb{R}$. It is a maximal abelian subalgebra of $\phi(\mathfrak{k}_0)$. The roots of $\mathfrak{so}(2n, \mathbb{C})$ with respect to its complexification $\mathfrak{t}$ are real on the real form $i\mathfrak{t}_0$. The roots can be described as follows. First let us denote an element of $i\mathfrak{t}_0$, whose l^{th} diagonal entry is $\begin{pmatrix} 0 & ix_l \\ -ix_l & 0 \end{pmatrix}$, by a sequence of real numbers $x = (x_1, \cdots, x_n)$. Let e_l be the functional that takes the x to x_l. Then $\Phi := \{\pm e_i \pm e_j : 1 \leq i \neq j \leq n\}$ is the set of roots for $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$ with respect to the Cartan subalgebra $\mathfrak{t}$. If $\gamma \in \Phi$, then the corresponding root space $\mathfrak{g}_\gamma$ is spanned by the vector E_γ, where E_γ is as follows. For $i < j$ and $\gamma = \pm e_i \pm e_j$, E_γ is a matrix in $M(n, M(2, \mathbb{C}))$, all whose entries except the $(i,j)^{th}$ and $(j,i)^{th}$ ones are zero. The $(i,j)^{th}$ and $(j,i)^{th}$ entries of E_γ are X_γ and $-X_\gamma^\top$, respectively, where $X_{e_i-e_j} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$, $X_{e_i+e_j} = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$, $X_{-e_i+e_j} = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$ and $X_{-e_i-e_j} = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$. The set of roots for the pair $(\mathfrak{k}, \mathfrak{t})$ is the subset $\Phi_c = \{e_i - e_j : i \neq j\}$ of Φ. These are the *compact roots*. The set $\Phi_n = \{\pm(e_i + e_j) : i \neq j\}$ consists of roots which are also the weights for the $\mathfrak{k}$-module $\mathfrak{p}$. These are the *non-compact roots*. We fix $\Phi_c^+ = \{e_i - e_j : i < j\}$ to be the set of positive roots for the pair $(\mathfrak{k}, \mathfrak{t})$.

We remark here that for any root γ, we have $\overline{\mathfrak{g}_\gamma} = \mathfrak{g}_{-\gamma}$, where $^-$ is the complex conjugation with respect to the real form $\mathfrak{g}_0$. Thus the sum of root spaces $\mathfrak{g}_\gamma + \mathfrak{g}_{-\gamma}$ is a complexification of $(\mathfrak{g}_\gamma + \mathfrak{g}_{-\gamma}) \cap \phi(\mathfrak{g}_0)$. For each root γ, we will identify the subspace V_γ of $\mathfrak{g}_0$, such that $\phi(V_\gamma) = (\mathfrak{g}_\gamma + \mathfrak{g}_{-\gamma}) \cap \phi(\mathfrak{g}_0)$. We need this information in §3.4. Let $M_{i,j}(n, \mathbb{H})$ denote the subspace of $M(n, \mathbb{H})$ consisting of matrices all whose entries, except $(i,j)^{th}$ and $(j,i)^{th}$ ones, are zero. Then $V_{e_i-e_j} = M_{i,j}(n, \mathbb{H}) \cap \mathfrak{k}_0$ and $V_{e_i+e_j} = M_{i,j}(n, \mathbb{H}) \cap \mathfrak{p}_0$.

Now we define an Ad K invariant complex structure on $\mathfrak{p}_0$. The complex structure J will in fact be Ad z, for some z in the centre $Z(K)$ of K. Here $K = SO(n, \mathbb{H}) \cap M(n, \mathbb{R} + j\mathbb{R}) \cong U(n)$. The centre consists of all scalar matrices in K. Take $z = ((1-j)/\sqrt{2})I_n$ and define $J = \text{Ad } z$. The i eigenspace of J is $\mathfrak{p}^+ := \bigoplus_{i<j} \mathfrak{g}_{e_i+e_j}$ and the $-i$ eigenspace is $\mathfrak{p}^- := \bigoplus_{i<j} \mathfrak{g}_{-e_i-e_j}$.

3.3 θ-stable parabolic subalgebras of $\mathfrak{so}(n, \mathbb{H})$

Let $\mathfrak{g}_0$ be any real semisimple Lie algebra and let θ be a Cartan involution of $\mathfrak{g}_0$. Let $\mathfrak{g}$ be the complexification of $\mathfrak{g}_0$. Our main interest is when $\mathfrak{g}_0 = \mathfrak{so}(n, \mathbb{H})$, but first we set up the general framework.

3.3.1 Description of θ-stable parabolic subalgebras

We begin by formally defining a θ-stable parabolic subalgebra of $\mathfrak{g}_0$.

Definition 3.3.1. *A parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ is called a **θ-stable parabolic subalgebra** of $\mathfrak{g}_0$ if the following conditions hold.*

1. $\theta(\mathfrak{q}) = \mathfrak{q}$,
2. $\mathfrak{q} \cap \overline{\mathfrak{q}} = \mathfrak{l}$,

where $\mathfrak{l}$ is the Levi factor of the parabolic subalgebra $\mathfrak{q}$ and $^{-}\colon \mathfrak{g} \to \mathfrak{g}$ is the complex conjugation with respect to the real form $\mathfrak{g}_0$.

All θ-stable parabolic subalgebras can be constructed in the following way. Start with a maximal toral subalgebra $\mathfrak{t}_0$ of $\mathfrak{k}_0$. Its complexification $\mathfrak{t}$ is a Cartan subalgbera for the reductive complex Lie algebra $\mathfrak{k}$. The real form of $\mathfrak{t}$, on which all the roots are real, is $i\mathfrak{t}_0$. Let Φ_c be the set of roots for $(\mathfrak{k}, \mathfrak{t})$. Thus we have

$$\mathfrak{k} = \mathfrak{t} + \sum_{\alpha \in \Phi_c} \mathfrak{k}_\alpha$$

Note that $\mathfrak{p}$ is a $\mathfrak{k}$-module, therefore we have a set of weights Φ_n of $\mathfrak{t}$ on $\mathfrak{p}$. Thus we have

$$\mathfrak{p} = \mathfrak{a} + \sum_{\beta \in \Phi_n} \mathfrak{p}_\beta$$

where $\mathfrak{a} = Z_{\mathfrak{p}}(\mathfrak{t})$. $\mathfrak{a}$ is the complexification of $\mathfrak{a}_0 = Z_{\mathfrak{p}_0}(\mathfrak{t}_0)$. Note that $\mathfrak{h} := \mathfrak{t} + \mathfrak{a}$ is a Cartan subalgebra for $\mathfrak{g}$ and $\mathfrak{h}_0 := i\mathfrak{t}_0 + \mathfrak{a}_0$ is its real form on which the roots are real. Let $\Phi = \Phi_c \cup \Phi_n$. For $\gamma \in \Phi$, define $\mathfrak{g}_\gamma$ to be $\mathfrak{k}_\gamma$ if $\gamma \in \Phi_c$ or $\mathfrak{p}_\gamma$ if $\gamma \in \Phi_c$. Now choose a positive system Φ_c^+ for Φ_c. Let $x \in i\mathfrak{t}_0$ be dominant with respect to Φ_c^+. Then we define the *θ-stable parabolic subalgebra associated to x* as

$$\mathfrak{q}_x := \mathfrak{h} + \sum_{\substack{\gamma \in \Phi \\ \gamma(x) \geq 0}} \mathfrak{g}_\gamma. \tag{3.2}$$

To see that $\mathfrak{q}_x$ is indeed a parabolic subalgebra, consider the restriction map $\mathfrak{h}_0^* \to i\mathfrak{t}_0^*$. We have fixed a notion of positivity on $i\mathfrak{t}_0$. Put a notion of positivity on $\mathfrak{h}_0$ such that positive elements are taken to non-negative elements under the restriction map. This shows that $\mathfrak{q}_x$ contains all positive root spaces of $\mathfrak{g}$ with respect to this notion of positivity. Hence $\mathfrak{q}_x$ is indeed a parabolic subalgebra. This discussion also shows that the Levi factor of $\mathfrak{q}_x$ is given by

$$\mathfrak{l}_x := \mathfrak{h} + \sum_{\substack{\gamma \in \Phi \\ \gamma(x) = 0}} \mathfrak{g}_\gamma$$

and the nilpotent radical is given by

$$\mathfrak{u}_x := \sum_{\substack{\gamma \in \Phi \\ \gamma(x) > 0}} \mathfrak{g}_\gamma.$$

Recall that $A_{\mathfrak{q}}$ is the representation of $\mathfrak{g}$ obtained by cohomological induction on the trivial representation of a θ-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$. Consider the family $\{A_{\mathfrak{q}} : \mathfrak{q} \text{ a } \theta\text{-stable parabolic subalgebra}\}$. The maximal compact subgroup K of G acts on the set of θ-stable parabolic subalgebras via the adjoint action. If $\mathfrak{q}$ and $\mathfrak{q}'$ belong to the same orbit under this action then $A_{\mathfrak{q}}$ and $A_{\mathfrak{q}'}$ are infinitesimally equivalent. Fix a maximal toral subalgebra $\mathfrak{t}_0$ and a positive system for Φ_c and let

$\mathfrak{Q}$ denote the set of θ-stable parabolic subalgebras of the form (3.2), where $x \in i\mathfrak{t}_0$ is dominant with respect to the fixed positive system.[1] Then each orbit in the set of θ-stable parabolic subalgebras under the K action has a representative in $\mathfrak{Q}$. Let $\mathfrak{q}, \mathfrak{q}' \in \mathfrak{Q}$, we say $\mathfrak{q} \sim \mathfrak{q}'$ if $\mathfrak{u} \cap \mathfrak{p} = \mathfrak{u}' \cap \mathfrak{p}$. This is an equivalence relation. By [32, Proposition 4.5], for $\mathfrak{q}, \mathfrak{q}' \in \mathfrak{Q}$, $A_{\mathfrak{q}}$ is infintesimally equivalent to $A_{\mathfrak{q}'}$ if and only if $\mathfrak{q} \sim \mathfrak{q}'$. Thus we have a bijection

$$\{A_{\mathfrak{q}} : \mathfrak{q} \text{ a } \theta\text{-stable parabolic subalgebra}\}/\simeq \quad \longleftrightarrow \quad \mathfrak{Q}/\sim$$

where $\simeq$ is infinitesimal equivalence relation.

From now on $\mathfrak{g}_0 = \mathfrak{so}(n, \mathbb{H})$. We retain all notations of §3.2. We had fixed a maximal toral subalgebra $\mathfrak{t}_0$ in the maximal compact subalgebra $\mathfrak{k}_0 = \mathfrak{so}(n, \mathbb{H}) \cap M(n, \mathbb{R} + j\mathbb{R})$ of $\mathfrak{g}_0$ and a positive system $\Phi_c^+ = \{e_i - e_j : 1 \le i < j \le n\}$ for $(\mathfrak{k}, \mathfrak{t})$. Observe that $\mathfrak{h} = \mathfrak{t}$ here. With this choice made, define the set $\mathfrak{Q}$ as above. Let $x \in it_0$ be dominant with respect to Φ_c^+. This is equivalent to saying that the tuple $x = (x_1, \cdots, x_n)$ satisfies $x_1 \ge \cdots \ge x_n$. Given such a decreasing sequence sequence x, we wish to determine the set of non-compact roots γ which satisfy $\gamma(x) \ge 0$. Table 3.1 gives this set.

$x_i + x_j$	weight of $\mathfrak{q}_x$
> 0	$e_i + e_j$
< 0	$-e_i - e_j$
$= 0$	$e_i + e_j, -e_i - e_j$

Table 3.1: Weights of $\mathfrak{q}_x$

Given a decreasing sequence x, the data whether $x_i + x_j$ is positive, negative or zero, for pairs (i, j) with $1 \le j < i \le n$, can be expressed by a combinatorial diagram called *decorated staircase diagram*, which we will define below. It will turn out that the set of these diagrams is in bijective correspondence with $\mathfrak{Q}/\sim$.

[1]The notation $\mathfrak{Q}$ was used to denote a different set only in the Synopsis.

3.3.2 Decorated staircase diagrams

We begin by defining a preliminary combinatorial object, which is illustrated in Figure 3.1, and setting up some terminology.

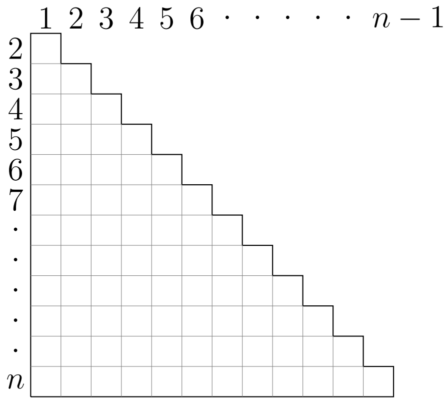

Figure 3.1: A staircase diagram

A *staircase diagram* $\mathcal{S}$ of height n is an arrangement of $\binom{n}{2}$ "boxes", which are identical squares, arranged in $n-1$ rows and columns, so that there are k boxes in the k^{th} row. The arrangement is left justified and bottom justified. The boxes are labelled by ordered pairs (i, j) where i, j denote the row and column indices respectively, $1 \le j < i \le n$. The vertices and edges of the boxes form a grid, which has the structure of a graph. They are also referred to as the vertices and edges of the staircase diagram.

The *boundary* of the staircase diagram is the boundary of the union of all the squares forming the staircase (thought of as subset of $\mathbb{R}^2$) and is a union of the *vertical*, the *horizontal*, and the *jagged* boundaries. The vertex at the intersection of the horizontal and the vertical boundary will be called the *origin* of the staircase diagram.

We say that two vertices p, q of a staircase diagram are *opposed* to each other if one of them is to the north and east of the other; we allow the possibility that they are on the same vertical or horizontal. We say that q is *above* p if q is to the north and east of p. In this case we also say that p is below q.

Any pair of distinct vertices p and q which are opposed to each other determine a rectangular region of the staircase diagram of which p, q are end points of a diagonal. In the degenerate case the rectangle has empty interior, when p, q are on the same vertical or horizontal.

A path (in the grid) joining p and q, which are opposed to each other, is said to be *monotone* if it is of shortest length, here the length refers to the number of edges in the path. A degenerate path consisting of a single vertex will also be said to be monotone.

Any vertex p of the staircase diagram determines a sub staircase diagram which consists of the collection of all boxes whose vertices are all to the north and east of p. It is empty if and only if p is on the jagged boundary.

We say that the $(i, j)^{th}$ box is *labelled by the vertex* p, if it contains p as its south west vertex.

To each decreasing sequence $x_1 \geq \cdots \geq x_n$ we associate a staircase diagram of height n which is *decorated* as follows. For $i > j$, if $x_i + x_j < 0$ then colour $(i, j)^{th}$ box black, if $x_i + x_j > 0$ colour it white and if $x_i + x_j = 0$ colour it grey. (In the illustrations we will use shading in place of the colour black.) The grey boxes will have a pattern which we first illustrate by an example. Arrange the terms of the sequence x as illustrated in Figure 3.2.

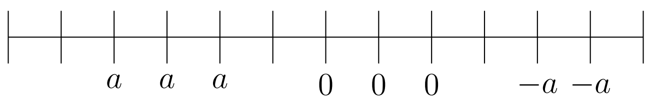

Figure 3.2: A typical decreasing sequence

Assume that all the other elements in the above diagram, other than the marked ones, have distinct modulus values, none of which is equal to a. Then the grey regions in the corresponding staircase diagram will have a pattern as shown in Figure 3.3.

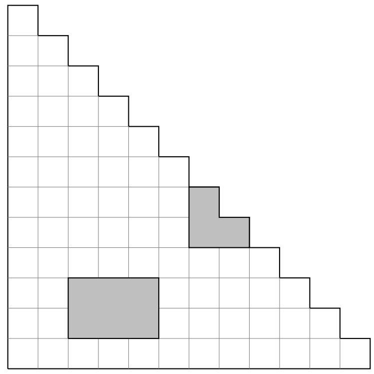

Figure 3.3: Pattern of grey regions

In general we may have several grey rectangles, where each will be positioned to the south west of the bottom left vertex of the previous one, if we see them from right to left. If there are k zeros in x, with $k \geq 2$, then the rightmost grey region will be a sub staircase diagram of height $k-1$.

Since the sequence x is decreasing we have the following two rules:

$$\text{If } k > l \text{ and } x_k + x_l \leq 0, \text{ then } x_i + x_j \leq 0, \text{ for all } i \geq k \text{ and } j \geq l. \tag{3.3}$$

$$\text{If } k > l \text{ and } x_k + x_l \geq 0, \text{ then } x_i + x_j \geq 0, \text{ for all } i \leq k \text{ and } j \leq l. \tag{3.4}$$

For the decoration this translates to the following pattern. The boxes which are to the left and above of any grey region must be white. The boxes to the right and below of any grey region must be black. Where there is no grey region to separate the black and white boxes, they are separated by monotone paths joining two grey areas or a grey area and the boundary. See Figure 3.4 for an illustration of such a decoration.

If two grey regions intersect at a point, we regard the point of intersection as a degenerate monotone path. Let $p_1, \cdots, p_k$, be the vertices of all such monotone paths arranged in an order so that p_{i+1} is below p_i, for each i. We note that these vertices completely determine the colouring. The grey boxes are those contained in the rectangles determined by two consecutive vertices or the sub staircase diagram

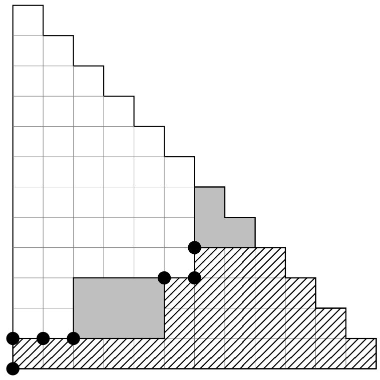

Figure 3.4: A decorated staircase diagram

diagram determined by p_1 (whenever these are not degenerate). The white boxes are the ones which are above or to the left of these rectangles. The black boxes are the ones which are below or to the right of them.

The point p_k must lie in the vertical or the horizontal boundary. It seems more natural to extend the monotone path with end point p_k, along the boundary, up to the origin. It makes no difference to the above discussion.

With this understanding we now give a formal definition.

Definition 3.3.2. *A* decorated staircase diagram of height n*, or* DS *in short, consists of a non-empty collection of distinct marked vertices* $p_1, \cdots, p_k$ *in a staircase diagram of height* n*, satisfying the following rules.*

(i) For all i*,* p_{i+1} *is below of* p_i*.*

(ii) The vertex p_2 *is not on the jagged boundary.*

(iii) The vertex p_k *is the origin of the staircase diagram.*

(iv) If the rectangle determined by p_i *and* p_{i+1} *is degenerate, then the distance between* p_i *and* p_{i+1} *is* 1*.*

Given a DS we can colour its boxes the way explained above. We will always think of a DS *with the colouring.*

Given a vertex p in a DS S, consider the sub staircase diagram determined by p, with colouring inherited from S. It is again a DS with the marked points determined by the colouring. We say it is the *sub-DS* determined by p and denote it by S_p.

Given a decreasing sequence x, it is clear from the two rules (3.3) and (3.4), that the decoration of a staircase diagram according to the sign of $x_i + x_j$ produces a DS. Now given a DS S we will explain an algorithm to construct a decreasing sequence x, such that the DS associated to x is S.

An algorithm to associate a decreasing sequence to a DS: Let S be a DS with marked vertices $p_1, \cdots, p_k$. Note that S_{p_i} is a sub-DS of $S_{p_{i+1}}$. We will inductively find sequences compatible with S_{p_i}, for all i, by concatenating to the left, right or both sides of the part that has already been constructed. At each step we add as many terms as the distance between p_i and p_{i+1} in the grid. As a visual aid, the reader may imagine one more layer of boxes on top of the jagged boundary, which are to be filled up by numbers in the sequence. See Figure 3.5 for an illustrative example.

In the following we will consider boxes labelled by the vertices p_i. Recall that we say the $(l, m)^{th}$ box in the staircase diagram is labelled by the vertex p, if it contains p as its south west vertex. But in our situation, the vertex p_1 may be on the jagged boundary and in that case there is no box in the staircase diagram labelled by it. Nevertheless we may speak of the $(l, m)^{th}$ box labelled by p_1, where l and m are the appropriate numbers.

If the $(l, m)^{th}$ box is labelled by p_1, then we put $x_j = 0$ for all $m \le j \le l$. (There may be no such j, but this can only happen if there is a rectangular grey region abutting the jagged boundary. This case is covered in the discussion below.) Suppose we have a compatible sequence for S_{p_k}. Let the box labelled by p_k be the $(l_1, m_1)^{th}$ one. Then we have already assigned values for x_j, where $m_1 \le j \le l_1$. Let the box labelled by

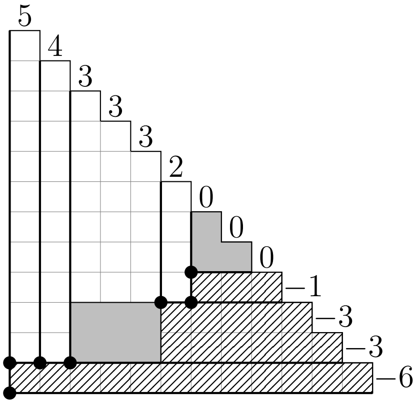

Figure 3.5: Algorithm for associating a decreasing sequence

p_{k+1} be the $(l_2, m_2)^{th}$ one. If it is white then $l_1 = l_2$ and $m_2 = m_1 - 1$. Choose x_{m_2} to be a number which is greater than $-x_{l_2}$, which has already been chosen. If it is black then $l_2 = l_1 + 1$ and $m_1 = m_2$. Choose x_{l_2} to be a number which is less than $-x_{m_2}$, which has already been chosen. If it is coloured grey then choose a number c which is greater than the modulus value of all the x_j, $m_1 \leq j \leq l_1$. Then put $x_j = c$ for all $m_2 \leq j < m_1$ and $x_i = -c$ for all $l_1 < i \leq l_2$. Note that this yields a compatible sequence for $S_{p_{k+1}}$ and finishes the induction step.

Remark 3.3.3. *In the above algorithm we may have consecutive marked vertices, such that the boxes labelled by them are white (or black). In this case the terms concatenated to the sequence corresponding to these two vertices may be taken to be the same. For instance in the example in Figure 3.5 we could have chosen the first number of the sequence to be* 4 *instead of* 5*. But we can avoid this situation by always choosing unequal numbers in such a case.*

Proposition 3.3.4. *For* $\mathfrak{g}_0 = \mathfrak{so}(n, \mathbb{H})$*, the set* $\mathfrak{Q}/\sim$ *is in one to one correspondence with the set of DS.*

Proof. By the above discussion we have a surjective map from $\mathfrak{Q}$ to the set of DS. Now we show that this map factors through $\mathfrak{Q}/\sim$. Suppose we have two sequences x and x' such that $\mathfrak{u}_x \cap \mathfrak{p} = \mathfrak{u}_{x'} \cap \mathfrak{p}$. The LHS is a sum of root spaces where the roots are $\{e_i + e_j : x_i + x_j > 0\} \cup \{-e_i - e_j : x_i + x_j < 0\}$ and the RHS is a sum of root spaces where the roots are $\{e_i + e_j : x'_i + x'_j > 0\} \cup \{-e_i - e_j : x'_i + x'_j < 0\}$. Thus these two set of roots must be equal. This means $x_i + x_j$ is positive or negative if and only if $x'_i + x'_j$ is positive or negative respectively. Also for the remaining pairs (i, j), $x_i + x_j = 0 = x'_i + x'_j$. Thus the two DS associated to $\mathfrak{q}_x$ and $\mathfrak{q}_{x'}$ must be same.

On the other hand a DS determines whether $x_i + x_j$ is positive, negative or zero for any pair (i, j). Hence $\mathfrak{q}_x \sim \mathfrak{q}_{x'}$ if and only if the corresponding DS are same. Hence we have a bijective map from $\mathfrak{Q}/\sim$ to the set of all DS. $\square$

Remark 3.3.5. *Let S be the DS corresponding to a class represented by a θ-stable parabolic subalgebra $\mathfrak{q}$. By the discussions in §3.2, $R^+(\mathfrak{q})$ equals the number of white boxes in S and $R^-(\mathfrak{q})$ equals the number of black boxes. Hence $R(\mathfrak{q})$ equals the number of white and black boxes. In the sequel the notations $R^+(S), R^-(S)$ and $R(S)$ will stand for the number of white boxes, the number of black boxes and the number of white and black boxes in S, respectively.*

3.4 Calculation of $H^*(\mathfrak{g}, K; A_\mathfrak{q})$ for $\mathfrak{g}_0 = \mathfrak{so}(n, \mathbb{H})$

Recall from §3.1 that $H^k(\mathfrak{g}, K; A_\mathfrak{q}) \cong H^{k-R(\mathfrak{q})}(Y_\mathfrak{q}; \mathbb{C})$. For each unitary equivalence class in $\{A_\mathfrak{q} : \mathfrak{q} \text{ a } \theta\text{-stable parabolic}\}$, we will determine $Y_\mathfrak{q}$ (which is independent of the choice of representative) and state its Hodge polynomial. The space $Y_\mathfrak{q}$ is the simply connected compact dual of the symmetric space associated to the non-compact semisimple Lie algebra $[\mathfrak{l}_0, \mathfrak{l}_0]$, where $\mathfrak{l}_0 := \mathfrak{l} \cap \mathfrak{g}_0$. So first we will determine $[\mathfrak{l}_0, \mathfrak{l}_0]$.

With notations as in §3.2 and §3.3, any θ-stable parabolic subalgebra in $\mathfrak{Q}$ is of the form $\mathfrak{q}_x$, where $x = (x_1, \cdots, x_n)$ is a decreasing sequence. Consider the equivalence relation $p \sim q$ if $|x_p| = |x_q|$ on the set $\{1 \leq j \leq n\}$. If $x_p = 0$ we denote the corresponding equivalence class $[p]$ by N_x. If $x_p \neq 0$, we define two subsets $I_{[p]} =: \{i \in [p] \mid x_i < 0\}, J_{[p]} = \{j \in [p] \mid x_j > 0\}$ of $[p]$. We denote the cardinalities of these sets by $|N_x|$, $|I_{[p]}|$ and $|J_{[p]}|$, respectively. Note that $I_{[p]}, J_{[p]}$ are disjoint sets of consecutive integers, at least one of which is non-empty and $I_{[p]} \cup J_{[p]} = [p]$. The sets $N_x, I_{[p]}, J_{[p]}, 1 \leq p \leq n, x_p \neq 0$, form a partition of the integers 1 up to n. If S is the DS associated to $\mathfrak{q}_x$, then the $(i,j)^{th}$ box is grey if either both i and j belong to N_x (these boxes constitute the grey region of S which is a sub staircase diagram), or if $i \in I_{[p]}$ and $j \in J_{[p]}$ (these boxes constitute one of the grey rectangles). By Remark 3.3.3, we may work with an x, such that whenever $I_{[p]}$ (respectively $J_{[p]}$) is not singleton, we have $J_{[p]} \neq \emptyset$ (respectively $I_{[p]} \neq \emptyset$). The advantage of working with such an x is that the members of $\Phi(\mathfrak{l}_x)$ can be read off directly from S, where $\Phi(\mathfrak{l}_x)$ denotes the set of roots of $(\mathfrak{l}_x, \mathfrak{t})$. More precisely, S already determines the non-compact roots contained in $\Phi(\mathfrak{l}_x)$, namely the roots $\pm(e_i + e_j)$ such that the $(i,j)^{th}$ box in the DS is grey. If moreover x is as above, then it also determines the compact roots contained in $\Phi(\mathfrak{l}_x)$. The roots $\pm(e_i - e_j) \in \Phi(\mathfrak{l}_x)$ if and only if either there exist k and l, such that the $(i,k)^{th}$ and $(j,l)^{th}$ boxes are in the same connected grey region or there exist k and l, such that the $(k,i)^{th}$ and $(l,j)^{th}$ boxes are in the same connected grey region. Henceforth it will always be assumed that x satisfies the property that, $|I_{[p]}| > 1$ implies $J_{[p]} \neq \emptyset$ and vice versa.

Now we will look at two special cases, which will enable us to understand the general case. The first case is where the sequence x is such that N_x contains atmost one element and there is only one equivalence class of $[p]$, with $x_p \neq 0$, such that $I_{[p]}$ and $J_{[p]}$ are both nonempty. That is, the DS corresponding to $\mathfrak{q}_x$ is of a form as shown in the left side diagram in the Figure 3.6. The right side diagram in Figure 3.6 should be thought of as an $n \times n$ matrix, where the boxes represent the positions of its entries. By the discussion in §3.2, $\mathfrak{l}_{x,0} \cap \mathfrak{p}_0$ consists of matrices in $\mathfrak{p}_0$ whose only non-zero entries are at the positions $(I_{[p]} \times J_{[p]}) \cup (J_{[p]} \times I_{[p]})$. In Figure 3.6, the corresponding positions in the matrix have been marked by a pattern

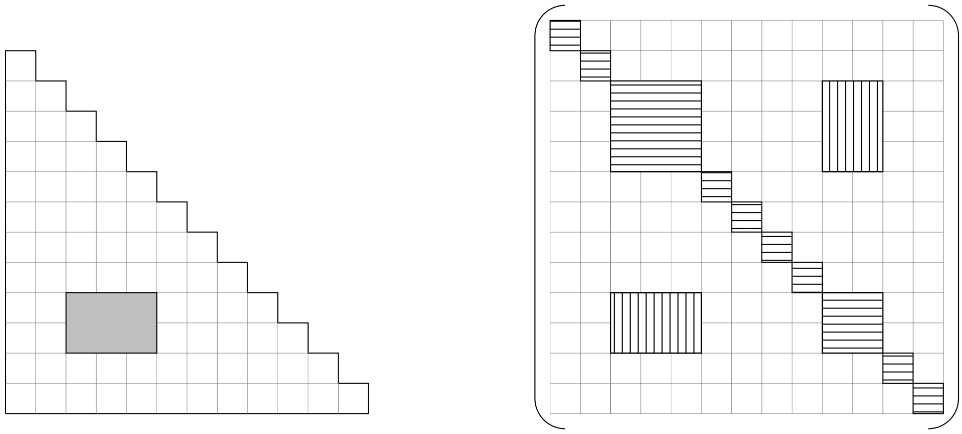

Figure 3.6: $\mathfrak{l}_{x,0}$ corresponding to a grey region

of vertical lines. Similarly $\mathfrak{l}_{x,0} \cap \mathfrak{k}_0$ consists of matrices in $\mathfrak{k}_0$ whose only non-zero entries are at the positions $(I_{[p]} \times I_{[p]}) \cup (J_{[p]} \times J_{[p]}) \cup \{(k,k) : 1 \leq k \leq n\}$. In Figure 3.6, the corresponding positions have been marked by a pattern of horizontal lines. From Figure 3.6, we observe that $\mathfrak{l}_{x,0}$ is a Lie algebra direct sum of some one dimensional subalgebras and its intersection with the set of matrices whose only non-zero entries are at the positions $[p] \times [p]$. This last summand is isomorphic to the fixed point set of $\mathfrak{so}(|I_{[p]}| + |J_{[p]}|, \mathbb{H})$ under conjugation by $jI_{|I_{[p]}|,|J_{[p]}|}$. Recall from §2.3.3 that conjugation by $jI_{|I_{[p]}|,|J_{[p]}|}$ is an involution of even type and we have already seen that the fixed point set in this case is isomorphic to $\mathfrak{u}(|I_{[p]}|, |J_{[p]}|)$. Hence $[\mathfrak{l}_{x,0}, \mathfrak{l}_{x,0}] = \mathfrak{su}(|I_{[p]}|, |J_{[p]}|)$.

Now we consider the next special case. This time assume that for all equivalence classes of the form $[p]$, with $x_p \neq 0$, either $I_{[p]} = \emptyset$ or $J_{[p]} = \emptyset$. Also assume $|N_x| > 1$. That is, the DS corresponding to $\mathfrak{q}_x$ is of a form as shown in the left side diagram in Figure 3.7. This time $\mathfrak{l}_{x,0} \cap \mathfrak{p}_0$ consists of matrices in $\mathfrak{p}_0$ whose only non-zero entries are the positions (i,j), with $i \neq j$ and $(i,j) \in N_x \times N_x$. On the other hand, $\mathfrak{l}_{x,0} \cap \mathfrak{k}_0$ consists of matrices in $\mathfrak{k}_0$ whose only non-zero entries are the positions $(N_x \times N_x) \cup \{(k,k) : 1 \leq k \leq n\}$. As before the corresponding entries have been marked by a pattern of vertical and horizontal lines respectively. We observe that $\mathfrak{l}_{x,0}$ is a Lie algebra direct sum of some one dimensional subalgebras and its intersection

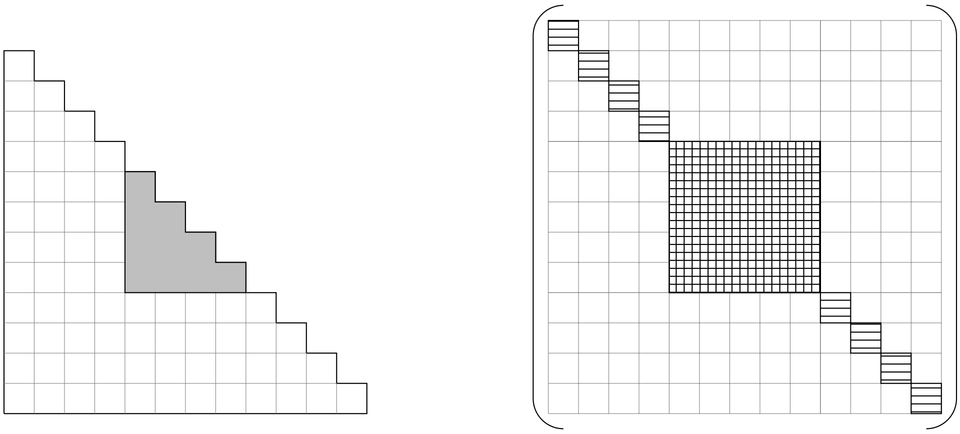

Figure 3.7: $\mathfrak{l}_{x,0}$ corresponding to a grey region

with the set of matrices whose only non-zero entries are at the positions $N_x \times N_x$. This last summand is isomorphic to $\mathfrak{so}(|N_x|, \mathbb{H})$. Hence $[\mathfrak{l}_{x,0}, \mathfrak{l}_{x,0}] = \mathfrak{so}(|N_x|, \mathbb{H})$.

The above arguments may be used to handle the general case. The crucial point here is that $I_{[p]}, J_{[p]}$ and N_x form a partition of $\{1, \cdots, n\}$. This makes the subalgebras corresponding to each $[p]$ an ideal and any two distinct such ideals are linearly disjoint. We state the general result below.

Proposition 3.4.1. *The derived algebra $[\mathfrak{l}_{x,0}, \mathfrak{l}_{x,0}]$ is a direct sum, over the parts of the partition $\{[p]\}$ omitting singletons, of the following simple Lie algebras:*

(i) $\mathfrak{su}(|I_{[p]}|, |J_{[p]}|)$ *if both $I_{[p]}, J_{[p]}$ are non-empty,*

(ii) $\mathfrak{so}(|N_x|, \mathbb{H})$.

Also,

$$R(\mathfrak{q}_x) = \dim_{\mathbb{C}}(\mathfrak{u} \cap \mathfrak{p}) = n(n-1)/2 - |N_x|(|N_x| - 1)/2 - \sum_{[p], x_p \neq 0} |I_{[p]}|.|J_{[p]}|.$$

From Proposition 3.4.1 it follows that

$$Y_{\mathfrak{q}} = (\prod_{[p], x_p \neq 0} \mathbb{C}G_{|[p]|, |I_{[p]}|}) \times SO(2|N_x|)/U(|N_x|),$$

where $\mathbb{C}G_{k,l}$ denotes the complex Grassmann manifold of l-planes in $\mathbb{C}^k$. It is understood that $U(0)$ is trivial and that the last factor in $Y_\mathfrak{q}$ is present only if $|N_x| \geq 2$.

The Poincaré polynomial $P_t(X)$ of a homogeneous manifold of the form $X = M/H$ where M is a compact connected Lie group and H is a connected subgroup having the same rank as M is given by the "formula of Hirsch". This is applicable to the complex Grassmann manifold $\mathbb{C}G_{k+l,k}$ and to $SO(2k)/U(k)$. See [21] and [1, Theorem 26.1]. From this and the observation that these manifolds are algebraic, the Hodge polynomials, with $z := uv$, are

$$P_{u,v}(\mathbb{C}G_{k+l,k}) = \frac{(1-z^{l+1})\cdots(1-z^{k+l})}{(1-z)\cdots(1-z^k)},$$
$$P_{u,v}(SO(2k)/U(k)) = (1+z)(1+z^2)\cdots(1+z^{k-1}).$$

Recall from §3.1, that the Hodge polynomials of $H^*(\mathfrak{g}, K; A_\mathfrak{q})$ and $H^*(Y_\mathfrak{q}; \mathbb{C})$ are related as $P_{u,v}(A_\mathfrak{q}) = u^{R^+(\mathfrak{q})}v^{R^-(\mathfrak{q})}P_{u,v}(Y_\mathfrak{q})$, where $R^+(\mathfrak{q}) = \dim(\mathfrak{u}\cap\mathfrak{p}^+)$ and $R^-(\mathfrak{q}) = \dim(\mathfrak{u}\cap\mathfrak{p}^-)$. Thus we have:

Proposition 3.4.2. *We keep the notations as above and set $\mathfrak{q} := \mathfrak{q}_x$. The Hodge polynomial $P_{u,v}(\mathfrak{q})$ of $H^*(\mathfrak{g}, K; A_\mathfrak{q})$ is given by*

$$P_{u,v}(A_\mathfrak{q}) = u^{R^+(\mathfrak{q})}v^{R^-(\mathfrak{q})}\Big(\prod_{[p],x_p\neq 0} P_{u,v}(\mathbb{C}G_{|[p]|,|I_{[p]}|})\Big) \times P_{u,v}(SO(2|N_x|)/U(|N_x|)).$$

3.5 The main result

In this section we first state the main theorem and its corollary. The proof of the theorem depends on a technical lemma which we prove at the end. Before that we prove the theorem and its corollary, assuming the statement of this lemma.

Theorem 3.5.1. *For $n \geq 9$, up to unitary equivalence there is a unique irreducible unitary representation of $SO(n, \mathbb{H})$, with Harish-Chandra module $A_\mathfrak{q}$, such* *that $R^+(\mathfrak{q}) = R^-(\mathfrak{q}) = n-2$. This representation occurs with non zero multiplicity in the decomposition of $L^2(\Gamma\backslash SO(n,\mathbb{H}))$, where Γ is any torsion free uniform lattice corresponding to an F-structure of type $DIII_u$ which admits an F-rational Cartan involution of diagonal type.*

Before stating the corollary let us define a partial order on the set $\mathbb{Z}[u,v]$ of two variable polynomials with integer coefficients. Let $P(u,v) = \sum_{i,j} p_{i,j}u^iv^j$ and $Q(u,v) = \sum_{i,j} q_{i,j}u^iv^j$ be two polynomials in $\mathbb{Z}[u,v]$. We say $P(u,v) \geq Q(u,v)$ if $p_{i,j} \geq q_{i,j}$, for all pairs (i,j).

Corollary 3.5.2. *Let $n \geq 9$. For each torsion free uniform lattice Γ, corresponding to an F-structure of type $DIII_u$ which admits an F-rational Cartan involution of diagonal type, we have the following inequality of Hodge polynomials.*

$$P_{u,v}(\Gamma\backslash SO(n,\mathbb{H})/U(n)) \geq u^{n-2}v^{n-2}(P_{u,v}(\mathbb{C}P^1) \times P_{u,v}(SO(2n-4)/U(n-2))) + P_{u,v}(SO(2n)/U(n)).$$

In particular there are cohomology classes of (p,p) type, which cannot be represented by $SO(n,\mathbb{H})$-invariant forms on $SO(n,\mathbb{H})/U(n)$, in each even dimension between $2n-4$ and $n(n-1)-(2n-4)$.

The first statement in Corollary 3.5.2 immediately follows from Proposition 3.4.2 applied to $A_\mathfrak{q} = A_\mathfrak{g}$ and $A_\mathfrak{q}$ equal to the one described in Theorem 3.5.1. The second statement follows from noting that $\mathbb{C}P^1 \times SO(2n-4)/U(n-2)$ had non-zero Betti numbers in all even dimensions.

Proof. of Theorem 3.5.1. We begin by discussing a general strategy for detection of unitary representations of a semisimple Lie group G whose Harish-Chandra module is one of the $A_\mathfrak{q}$. We will also assume that G/K is Hermitian symmetric. Then the results of §3.1 and §3.3 imply

$$H^{p,q}(\Gamma\backslash G/K;\mathbb{C}) \cong \prod_{\mathfrak{q}\in\mathfrak{Q}/\sim} m(A_{\mathfrak{q}},\Gamma)\ H^{p-R^{+}(\mathfrak{q}),q-R^{-}(\mathfrak{q})}(Y_{\mathfrak{q}},\mathbb{C}), \tag{3.5}$$

where $\mathfrak{q}$ is understood to represent its equivalence class in $\mathfrak{Q}/\sim$. When $\mathfrak{q}=\mathfrak{g}$, we have $R(\mathfrak{g})=0$, $m(\Gamma,\mathcal{A}_{\mathfrak{g}})=1$ and the image of $H^*(Y_{\mathfrak{g}};\mathbb{C})$ under the Matshushima isomorphism consists of cohomology classes that are represented by G-invariant forms on G/K. The construction of special cycles produce cohomology classes on the LHS of (3.5) that cannot be represented by G-invariant forms on G/K. Fix an integer c and consider the part of the cohomology that consists of (p,q) types where $p-q=c$. Suppose there exists a unique $\mathfrak{q}_1\in\mathfrak{Q}/\sim$ such that $R(\mathfrak{q}_1)$ is the least value in the set $\{R(\mathfrak{q}) : \mathfrak{g}\neq\mathfrak{q}\in\mathfrak{Q}/\sim, R^+(\mathfrak{q})-R^-(\mathfrak{q})=c\}$. Also suppose that a special cycle has been constructed whose Poincaré dual is of type (p_1,q_1), such that $p_1-q_1=c, p_1+q_1=d$ and $\{R(\mathfrak{q})\le d : \mathfrak{g}\neq\mathfrak{q}\in\mathfrak{Q}/\sim, R^+(\mathfrak{q})-R^-(\mathfrak{q})=c\}=\{R(\mathfrak{q}_1)\}$. Then it follows that $m(\Gamma,A_{\mathfrak{q}_1})\neq 0$.

In our case where $G=SO(n,\mathbb{H}), K=U(n)$ and Γ is as in the statement of the theorem, we will take $c=0$, that is, we will concentrate on cohomology classes of type (p,p). From Theorem 2.5.1, we know that there exists a cohomology class of $\Gamma\backslash SO(n,\mathbb{H})/U(n)$ of type $(n-1,n-1)$ arising from a special cycle. Lemma 3.5.4 below implies the existence of a unique $\mathfrak{q}\in\mathfrak{Q}/\sim$, which is not equal to $\mathfrak{g}$, satisfying $R^+(\mathfrak{q})=R^-(\mathfrak{q})\le n-1$, under the assumption that $n\ge 9$. This completes the proof. □

It remains to prove Lemma 3.5.4. We require a preliminary lemma first.

Lemma 3.5.3. *Suppose S is a DS of height n, all whose boxes are either grey or black. Then either there exists $1\le l\le n-1$, such that the first l rows of S consist of grey boxes and the remaining of black boxes, or there exists $1\le l\le n-1$ such that all boxes except the top l boxes of the first column are black.*

Proof. Recall from definition of DS that grey regions are either rectangular or in the shape of a sub staircase diagram. Observe that if p is the bottom left vertex of a grey region in the shape of a sub staircase diagram or the top right vertex of a rectangular grey region, then all boxes in the north west of p must be white. Let S be a DS satisfying the hypothesis of the lemma. If S has no rectangular grey region then there can only be a grey region in the shape of a sub staircase diagram. Now the above observation forces its bottom left vertex to be on the vertical boundary. Hence S must be of the first type described in the statement. On the other hand, if S has a rectangular grey region then the above observation forces it to be a subset of the first column. This also implies that it is the unique grey region in S. Hence S is of the second type described in the statement. □

Let S_0 be the DS which is decorated as follows. The box at the intersection of the first column and the last row is grey, rest of the boxes in the first column are white, rest of the boxes in the last row are black, and the remaining are grey. See Figure 3.8 for an illustration.

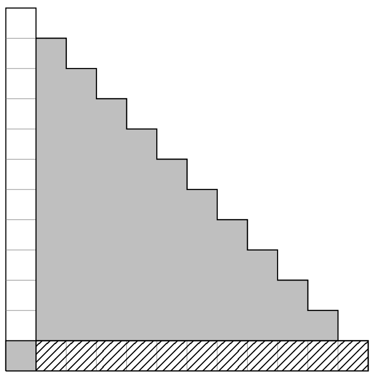

Figure 3.8: The DS S_0.

Lemma 3.5.4. *When $n \geq 9$, the only* DS S *of height n with $R^+(S) = R^-(S) \leq n-1$ is S_0.*

Proof. Let S be a DS of height n with $R^+(S) = R^-(S) \leq n - 1$. Suppose the $(n, 1)^{th}$ box is not grey. Then it is either white or black. If it is white then all the boxes in the first column are white. Let p be the right bottom vertex of the $(n,1)^{th}$ box. Then S_p is a DS as in Lemma 3.5.3. Suppose S_p is of the first type, then there cannot be more than one black row, since otherwise $R^-(S) = R^-(S_p) \geq 2n-3 > n-1$ for $n \geq 9$. On the other hand if S_p has only one black row, then $R^-(S) = n-2 < n-1 = R^+(S)$. This is a contradiction. If S_p is of the second type, then $R^-(S) = R^-(S_p) \geq \frac{1}{2}(n-2)(n-3)$ which is greater than $n-1$ if $n \geq 9$. So again we have a contradiction. An entirely analogous argument can be given to prove that the $(n,1)^{th}$ box cannot be black. Thus the $(n,1)^{th}$ box of S must be grey.

Suppose the grey rectangle containing the $(n,1)^{th}$ box is of dimension $a \times b$. Suppose $a = 1 = b$. The DS S_0 satisfies this condition. If $S \neq S_0$, let q be the top right vertex of the $(n,1)^{th}$ box. The conditions on S will force that S_q contains exactly one white box and exactly one black box. This configuration is not possible unless the height of S_q is 3 which will imply $n = 5$, a contradiction. Now suppose one of a, b is greater than 2. Counting the number of white boxes above this grey region and the number of black boxed to its right, we obtain, using $R^+(S) = R^-(S) \leq n-1$, that

$$n - 1 \geq \binom{a}{2} + a(n-a-b), \binom{b}{2} + b(n-a-b). \tag{3.6}$$

If $a \geq 2$, then counting the number of white boxes in the first two columns we obtain $2n - 2b - 3 \leq R^+ \leq n-1$. Hence $b \geq \lceil (n-2)/2 \rceil$. In particular $b \geq 2$, since $n \geq 9$. Thus there was no loss in generality in assuming $a \geq 2$. Also counting number of black boxes in the last two rows we get, $a \geq \lceil (n-2)/2 \rceil$. Since $a + b \leq n$, we get

$$\lceil (n-2)/2 \rceil \leq a, b \leq \lfloor (n+2)/2 \rfloor. \tag{3.7}$$

If $n = 2m$ is even, equation (3.7) simplifies as

$$m - 1 \leq a, b \leq m + 1 \tag{3.8}$$

Combining with equation (3.6) we get $2m-1 \geq (m-1)(m-2)/2$, which is equivalent to $m^2-7m+4 \leq 0$. Thus $m \leq 6$. Since $n \geq 9$, the only possibilities are $m = 5, 6$. If $m = 5$, then equation (3.8) implies, $4 \leq a, b \leq 6$. Since $\binom{6}{2}, \binom{5}{2} > 9$, equation (3.6) yields a contradiction to either a or b being equal to 5 or 6. Hence we must have $a = 4 = b$, but again equation (3.6) gives a contradiction. Similarly we can show that the case $m = 6$ cannot occur.

The case $n = 2m + 1$ is odd can be handled in a similar way. $\square$

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