Stability And Embedding Properties Of Some Projective Manifolds

by

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A thesis submitted to the
Board of Studies in Mathematical Sciences
In partial fulfillment of requirements
For the Degree of

DOCTOR OF PHILOSOPHY
of
HOMI BHABHA NATIONAL INSTITUTE

July, 2014

Homi Bhabha National Institute

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SESHADRI CHINTAPALLI

DECLARATION

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SESHADRI CHINTAPALLI

Abstract

This thesis is divided into two parts.

In the first part, we prove the semistability of logarithmic de Rham sheaves on a smooth projective variety (X, D), under suitable conditions. This is related to existence of Kähler -Einstein metric on the open variety. We investigate this problem when the Picard number is one. Fix a normal crossing divisor D on X and consider the logarithmic de Rham sheaf $\Omega_X(log\, D)$ on X. We prove semistability of this sheaf, when the log canonical sheaf $K_X + D$ is ample or trivial, or when $-K_X - D$ is ample i.e., when X is a log Fano n-fold of dimension $n \leq 6$. We also extend the semistability result for Kawamata coverings, and this gives examples whose Picard number can be greater than one.

In the second part, we investigate linear systems on hyperelliptic varieties. We prove analogues of well-known theorems on abelian varieties, like Lefschetz's embedding theorem and higher k-jet embedding theorems. Syzygy or N_p-properties are also deduced for appropriate powers of ample line bundles.

ACKNOWLEDGEMENT

Dedicated to my parents

Contents

Chapter 1

Introduction

We begin by summarizing the main results of the thesis. We divide this chapter into two sections. In section 1.1, we discus the problems related to semistability of logarithmic cotangent bundles on some projective manifolds. In section 1.2, we discus the problems related to the embedding properties of linear series on hyperelliptic varieties.

1.1 Semistability of logarithmic cotangent bundle on some projective manifolds.

The notion of stability of a vector bundle (in the sense of Mumford and Takemoto) play an important role in complex differential geometry and algebraic geometry. More general notion is the existence of Kähler-Einstein metric on compact Kähler manifold. Mumford [44] introduced stability for bundles on curves and later generalized to sheaves on higher dimensional varieties by Takemoto, Gieseker, Maruyama, and Simpson. The existence of a Kähler -Einstein metric implies the stability of the cotangent bundle is proved by Kobayashi [30] and Lübke [40].

Let X be a smooth projective variety over $\mathbb{C}$. Denote Ω_X, the cotangent bundle of X and K_X is the canonical line bundle on X. By the work of Aubin [1] and Yau [64], it is well known that Ω_X is stable whenever the canonical line bundle K_X is ample or trivial. The stability of Ω_X when $-K_X$ is ample i.e., when X is a Fano manifold, has attracted wide attention. By Tian [62] and Fahlaoui [13], we know that Ω_X is stable when X is a Del Pezzo surface except when X is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$ blown-up in a point. In the case of fano 3-folds with $b_2(X) = 1$, Steffens [60] in his thesis gave a complete answer to this problem. In [61], Subramanian proved the stability of cotangent bundle Ω_X when X is a smooth complete intersection in $\mathbb{P}^n$ of codimension l and multi-degree $(d_1, d_2, ..., d_l)$ with $d_1 \geq d_2 \geq ... \geq d_l$ and $d_l > \frac{(n+1-d_1-...-d_{l-1})}{2}$. Later Peternell- Wisniewski [56] gave complete answer for complete intersections in projective spaces. They also proved stability of Ω_X in the case of fano 4-folds with $b_2(X) = 1$ and stability of Ω_X of fano n-fold of large index with $b_2(X) = 1$. In [28], Hwang

proved the stability and semistability of Ω_X in the case of fano 5-folds and 6-folds respectively with picard number 1. Stability of Fano n-fold is still an open problem, for $n \geq 7$. In chapter 3 we will prove semistability for logarithmic cotangent bundle $\Omega_X(\log D)$.

Now we will briefly discuss the main results of Chapter 3.

Main results.

Let X be a smooth projective variety over $\mathbb{C}$ with picard number 1 and $D \subset X$ is a simple normal crossing divisor. Define $\Omega_X^a(\log D) := \bigwedge^a \Omega_X(\log D)$; these are logarithmic de Rham sheaves [12], whose local sections are meromorphic a-forms having at most a simple pole along D.

We prove the following theorem.

Theorem 1.1.1. *Suppose $(X, \mathcal{O}_X(1))$ is a smooth projective variety of dimension n over $\mathbb{C}$, with the Picard group $Pic(X) = \mathbb{Z}$. Let $D = \sum_{i=1}^{r} D_i$ be a simple normal crossing divisor on X and K_X denote the canonical class. If $K_X + \mathcal{O}_X(D)$ is ample or trivial, then $\Omega_X(\log D)$ is semistable.*

This statement can be extended to Kawamata coverings as follows.

Proposition 1.1.2. *Suppose $(Y, \mathcal{O}_Y(1))$ is a smooth projective variety of dimension n and $D = \sum_i D_i$ is a normal crossing divisor on Y. Assume $Pic(Y) = \mathbb{Z}$ and $K_Y + D$ is ample or trivial. Consider the Kawamata covering $\pi : X \to Y$, ramified along D and $D' = \pi^{-1}(D)_{red}$. Then the sheaf $\Omega_X(\log D')$ is semistable with respect to the ample class $\pi^* \mathcal{O}_Y(1)$.*

Note that the Picard number of X in the above proposition can be greater than one, which is relevant for other applications.

We next investigate log Fano manifolds (X, D), in small dimensions. In this situation the class $-K_X - D$ is ample. The classification of such pairs (X, D) is due to Maeda [42] and Fujita [14] in small dimensions. We have the following theorem:

Theorem 1.1.3. *Suppose (X, D) is a log Fano manifold of dimension n and $Pic(X) = \mathbb{Z}.\mathcal{O}_X(1)$. Let the canonical class $K_X = \mathcal{O}_X(-s)$ and D is in the linear system $|\mathcal{O}_X(k)|$, for $s, k > 0$.*

Assume one of the following holds:

a) $n = 2$ and $s = 3$,

b) $n = 3$ and $s \leq 4$

c) $n = 4$ and $s \leq 5$

d) $n = 5$ and $s \leq 6$ such that $s = 2, 5, 6$ or $(s, k) = (3, 2), (4, 3)$.

e) $n = 6$ and $s \leq 7$ such that $s \leq 4$, $s = 6, 7$, or $(s, k) = (5, 4), (5, 3)$.

If D is smooth and irreducible then the logarithmic cotangent bundle $\Omega_X(\log D)$ is semistable.

The proof involves a careful investigation of vanishing theorems within a certain range using residue sequences, and apply semistability of de Rham sheaves on Fano manifolds in small dimensions. The classification of Maeda yields complete statements for log Del Pezzo surfaces, log Fano threefolds and 4-folds.

We have remarked that $\Omega_{\mathbb{P}^2}(log\, D)$ is not semistable when $D = D_1 + D_2$, D_1 and D_2 are lines on $\mathbb{P}^2$.

1.2 Embedding properties of linear series on hyperelliptic varieties.

In this part of the thesis we prove results related to linear series. In particular, very ampleness, k-jet ampleness, projective normality, and higher syzygies of an ample line bundle on hyperelliptic varieties.

In recent years, the problems related to linear series have attracted considerable attention. The above questions are fairly well-understood on curves and we have some significant work done by Castelnuova, Fujita, and Green regarding N_p-property on curves. Indeed, Green [18] proved that if L is a line bundle on a curve C of genus g such that degree of L is at least $2g + 1 + p$ then L satisfies N_p-property, for $p \geq 0$. In the case of higher dimensional varieties Mukai conjectured that for any smooth polarized projective variety (X, L), $K_X \otimes L^{\otimes p+4}$ satisfies N_p-property, where K_X denotes the canonical line bundle on X. Mukai's conjecture has not yet been proved even for $p = 0$, but some significant work has been done in some special cases by Kempf [31], Y. Homma [27] Ein and Lazarsfeld [9]. In [15], [16] and [17], Gallego and Purnaprajna have done some nice work regarding syzygy properties on surfaces and three folds. There are still many open questions related to linear series on curves and surfaces. In the case of abelian varieties Lazarsfeld conjectured that if L is an ample line bundle on an abelian variety X then L^{p+3} satisfies N_p-property, for $p \geq 0$. Recently Pareschi [53], proved this conjecture. In [38], Lazarsfeld-Pareschi-Popa proved that L satisfies N_p if Seshadri constant of L is greater than $(p + 2)g$, where g is a dimension of an abelian variety X. Problems concerning about projective normal embedding of an ample line bundle on an abelian variety, have done by Iyer [29], Hwang-To [25], Koizumi [33] and Ohbuchi [51].

On the other hand, Fujita's conjecture on very ampleness of a line bundle has attracted attention in the past years. Indeed, if L is an ample line bundle on an algebraic variety X of dimension n, then $K_X \otimes L^{\otimes n+2}$ is very ample. Fujita's conjecture has been proven for algebraic surfaces but this problem is still open for higher dimensional varieties. The questions related to very ampleness and k-jet ampleness are completely known in the case of abelian varieties. Indeed, suppose X is an abelian variety and L is an ample line bundle on X then by the

theorem of Lefschetz $L^{\otimes 3}$ is very ample. In [52], Ohbuchi proved that $L^{\otimes 2}$ is very ample if L has no base divisor. In [2], Bauer-Szemberg proved that $L^{\otimes k+2}$ is k-jet ample and $L^{\otimes k+1}$ is k-jet ample if L has no base divisors. Recently G. Pareschi and M. Popa [54], [55] used an alternate approach called Mukai regularity to obtain most of the above results.

In the case of primitive line bundles L, i.e., line bundles of type $(1, d_2, ..., d_g)$, these problems are not much known. In the case of surfaces, L is of type $(1, d)$ is very ample iff $d \geq 5$ and there is no elliptic curve E on X with $(L.E) = 2$ (see [5], Theorem 10.4.1). For abelian varieties of dimension $g \geq 3$ not much is known. Recently Ein and Lazarsfeld [10] proved a theorem on global generation of adjoint line bundles. In [6], Birkenhake-Lange-Ramanan proved that very ampleness of a polarized abelian threefold of type $(1, 1, d)$.

In this chapter we are interested in powers of ample line bundles.

Let X be an hyperelliptic variety over $\mathbb{C}$. i.e., X is not isomorphic to an abelian variety but admitting an étale covering $A \to X$, where A is an abelian variety.

Now we will briefly discuss the main results of Chapter 4.

Main results.

We proved the following theorem which is an analogue of Lefschetz embedding theorem on abelian varieties.

Theorem 1.2.1. *Suppose X is a hyperelliptic variety of dimension n. Let L be an ample line bundle on X. Then we have*

1) L^k, for $k \geq 3$, is always very ample.

2) L^2 is very ample, if L has no base divisor.

We will generalize the above theorem, namely k-jet ampleness to hyperelliptic varieties as follows.

Theorem 1.2.2. *Suppose L is an ample line bundle on a hyperelliptic variety X. Then the following hold, for $k \geq 0$:*

1) L^{k+2} is k-jet ample

2) L^{k+1} is k-jet ample if L has no base divisor.

Regarding N_p-property, we show the analogue of Pareschi's theorem (Lazarsfeld's conjecture) on abelian varieties, extended to hyperelliptic varieties.

Theorem 1.2.3. *Suppose L is an ample line bundle on a hyperelliptic variety X. Then L^{p+k} satisfies N_p-property, for $k \geq 3$.*

The key point in the proofs is to note that a hyperelliptic variety X is realized as a finite group quotient A/G of an abelian variety A, for some finite group G acting freely on A [34,

Theorem 1.1, p.492]. Hence a line bundle on a hyperelliptic variety is regarded as a G-linearized line bundle on A. We introduce the notion of G-global generation of G-linearized sheaves and obtain a correspondence of the usual global generation on X with G-global generation on A. We then look at the notion of M-regularity of G-linearized sheaves and suitably extend the techniques used by Pareschi and Popa. The proofs are reduced to showing G-global generation of appropriate G-linearized coherent sheaves, obtained by applying the Fourier-Mukai functor. We employ different method, the 'averaging of sections' to obtain our results.

Chapter 2

Preliminaries

In this chapter we collect some basic definitions in algebraic geometry, which are essential for the rest of the thesis. We collect these definitions mainly from [23], [24], [39], [45], and [46].

2.0.1 Manifolds and Vector bundles.

Let $\mathbb{C}$ denotes the field of complex numbers.

Definition 2.0.4. *A complex manifold M of dimension n is a Hausdorff topological space with a countable basis $\mathcal{U} = \{U_\alpha\}$, and homeomorphisms $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha) \subset \mathbb{C}^n$ such that*

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta).$$

is holomorphic, for all α, β

We call the pair (U_α, ϕ_α) a coordinate chart of M. The collection $\mathcal{A}_M = \{(U_\alpha, \phi_\alpha)\}_\alpha$ is called an atlas for M.

Let M be a n-dimensional complex manifold and $p \in M$, and $z = (z_1, ..., z_n)$ a holomorphic coordinate system around p. We define tangent space, denoted by $T_p(M)$, the space of $\mathbb{C}$-linear derivations in the ring of holomorphic functions on M around p. *i.e.*,

$$T_p(M) = \mathbb{C}\{\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}\}.$$

Consider the set $T(M)$ formed by the disjoint union of all tangent spaces

$$T(M) = \bigsqcup_{p \in M} T_p(M).$$

Note that $T(M)$ is a complex manifold of dimension $2n$.

A complex-valued function f on open set $U \subset M$ is holomorphic if, for all α, $f \circ \phi_\alpha^{-1}$ is holomorphic on $\phi_\alpha(U_\alpha \cap U) \subset \mathbb{C}^n$.

Definition 2.0.5. *A continuous map $f : M \to N$ between two complex manifolds is holomorphic at a point $p \in M$ if there exists a chart (ϕ, U) near p and a chart (ψ, V) near $f(p) \in N$ such that*

$$\psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V)$$

defines a holomorphic map. This condition is independent of the choice of charts because overlap maps are biholomorphic. The map f is called holomorphic if it is holomorphic at every point of $p \in M$. An isomorphism between two complex manifolds M and N is a holomorphic map $f : M \to N$ such that f is bijective and f^{-1} is also a holomorphic map.

Examples 2.0.6. *(1) If V is a n-dimensional vector space over $\mathbb{C}$, then the projective space $\mathbb{P}(V) := \{$ the set of one dimensional subspaces of $V\}$ is a complex manifold of dimension $n-1$.*
  (2) The general linear group,

$$GL_n\mathbb{C} = \{A \in M(n, \mathbb{C}) | det A \neq 0\}$$

is a complex manifold of dimension n^2.

Definition 2.0.7. *Let M be a complex manifold. A complex vector bundle on M is a complex manifold E together with a holomorphic map $\pi : E \to M$ such that*
  • for each $p \in M$, the set $E_P = \pi^{-1}(p)$, is a complex vector space of finite dimension, (E_p is called fiber over p).
  • For every $p \in M$, there is a neighborhood U of p and a biholomorphic map

$$\phi_U : \pi^{-1}(U) \to U \times \mathbb{C}^n$$

such that $\phi_U(E_p) \subset \{p\} \times \mathbb{C}^n$, and ϕ_p, defined by the composition

$$\phi_p : E_p \to \{p\} \times \mathbb{C}^n \xrightarrow{p_2} \mathbb{C}^n$$

is a complex vector space isomorphism.
  The map ϕ_U is called a local trivialization of E over U. Here E is called the total space and M is called the base space. The dimension of the fibers E_p of E is called the rank of E. In particular, a vector bundle of rank 1 is called line bundle.

Note that for any pair of trivialization ϕ_U and ϕ_V the map

$$g_{UV} : U \cap V \to GL_n$$

given by

$$g_{UV}(x) = (\phi_U \circ \phi_V^{-1})|_{\{x\} \times \mathbb{C}^n}$$

is holomorphic. The maps g_{UV} are called transition functions for E relative to the trivializations ϕ_U, ϕ_V. The transition functions g_{UV} satisfy the following compatibility conditions:

$g_{UV}(x) \cdot g_{VU}(x) = I_n$ for all $x \in U \cap V$,

$g_{UV}(x) \cdot g_{VW}(x) \cdot g_{WU}(x) = I_n$ for all $x \in U \cap V \cap W$.

One can check that given an open cover $\mathcal{U} = \{U_\alpha\}$ of M and holomorphic maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_n$ satisfies above compatibility conditions, then there is a unique complex vector bundle $E \to M$ with transition functions $\{g_{\alpha\beta}\}$.

Example 2.0.8. *(Tangent bundle): Suppose M is a complex manifold, and $T_x(M)$ is the complex tangent space to M at x. Let*

$$T(M) = \bigsqcup_{x \in M} T_x(M).$$

and define

$$\pi : T(M) \to M$$

by

$$\pi(v) = x \text{ if } v \in T_x(M)$$

We can give a complex vector bundle structure to $T(M)$ on M as follows:
Let (U_α, ϕ_α) be an atlas for M. We have maps

$$\phi_\alpha : T_x(M) \to T_{\phi_\alpha(x)}(U_\alpha) \cong \mathbb{C}\{\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}\}.$$

for each $x \in U_\alpha$, hence a map

$$\phi_\alpha : \pi^{-1}(U_\alpha) = \bigcup_{x \in U_\alpha} T_x(M) \to U_\alpha \times \mathbb{C}^{2n}.$$

It is easy to verify that ϕ_α is bijective and fiber-preserving and more over that

$$\phi_\alpha{}^x : T_x(M) \to \{x\} \times \mathbb{C}^{2n} \xrightarrow{proj.} \mathbb{C}^{2n}$$

is a $\mathbb{C}$-linear isomorphism. The maps ϕ_α are biholomorphic follows from the complex manifold structure on $T(M)$. This vector bundle $T(M)$ is called the complex tangent bundle. We define transition functions

$$j_{\alpha,\beta} : U_\alpha \cap U_\beta \to GL(2n, \mathbb{C})$$

by setting

$$j_{\alpha,\beta}(x) = \phi_\alpha{}^x \circ (\phi_\beta{}^x)^{-1} : \mathbb{C}^{2n} \to \mathbb{C}^{2n}.$$

Definition 2.0.9. *A map between two complex vector bundles E, F over a complex manifold M is given by a holomorphic map $f : E \to F$ such that for every $p \in M$, one has $f(E_p) \subset F_p$ and $f_p = f|_{E_p} : E_p \to F_p$ is $\mathbb{C}$-linear. Two complex vector bundles E and F on M are isomorphic if there exists a map $f : E \to F$ such that $f : E_p \to F_p$ is an isomorphism for all $x \in M$.*

2.0.2 Sheaves.

Definition 2.0.10. *A presheaf $\mathcal{F}$ of abelian groups on a topological space X consists of the following data:*
- *for each open set U of X, an abelian group $\mathcal{F}(U)$,*
- *for each pair of open sets $U \subseteq V$, we have a restriction map $res_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$ such that*

(a) $res_{U,U} = id_{\mathcal{F}(U)}$ for every open set $U \subseteq X$,
(b) for $U \subseteq V \subseteq W$ open sets of X, we have $res_{W,U} = res_{V,U} \circ res_{W,V}$.

The elements of $\mathcal{F}(U)$ are called the sections of $\mathcal{F}$ over U, and the restriction maps $res_{V,U}$ are written as $s \mapsto s|_U$.

Definition 2.0.11. *The stalk of a presheaf $\mathcal{F}$ at a point $p \in X$, denoted by $\mathcal{F}_p$, defined as the direct limit of all $\mathcal{F}(U)$, for all open sets U containing p. Equivalently, we can define the stalk $\mathcal{F}_p$ to be the set $\{(s,U) : p \in U, s \in \mathcal{F}(U)\}$ modulo the the relation that $(s,U) \sim (t,V)$ if and only if there is some open neighborhood W of p with $W \subset U \cap V$ such that $s|_W = t|_W$.*

The elements of the stalk $\mathcal{F}_p$ are called as germs of $\mathcal{F}$ at p.

Definition 2.0.12. *The presheaf $\mathcal{F}$ is called a sheaf if it satisfies the following axiom:*
- *If $\{U_i\}_{i \in I}$ is a open cover of an open set U, and if we have elements $s_i \in \mathcal{F}(U_i)$ for each i, with the property that for each $i, j, s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there is a unique element $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$, for each i.*

Example 2.0.13. *Let X be a topological space and let $\mathcal{F}$ be a presheaf over X defined by*
(a) For any open subset $U \subseteq X$, $\mathcal{F}(U) := \{f : X \to \mathbb{R} | f$ is continuous$\}$.
(b) For $V \subseteq U$ open sets of X, and $f \in \mathcal{F}(U)$, $res_{U,V}(f) := f|_V$, the natural restriction map as a function. One can check that it is in fact a sheaf.

Definition 2.0.14. *A morphism $\phi : \mathcal{F} \to \mathcal{G}$ of presheaves $\mathcal{F}, \mathcal{G}$ on X is a family of homomorphisms $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ of abelian groups for each open subset U of X, such that for every pair $V \subseteq U$ of open sets in X, the diagram*

$$
\begin{array}{ccc}
\mathcal{F}(U) & \overset{\phi(U)}{\to} & \mathcal{G}(U) \\
\downarrow res_{U,V} & & \downarrow res_{U,V} \\
\mathcal{F}(V) & \overset{\phi(V)}{\to} & \mathcal{G}(V)
\end{array}
$$

is commutative.

Note that morphism between sheaves is nothing but morphism between presheaves. An isomorphism between sheaves is a morphism which has a two-sided inverse. We obtain the category of sheaves on the topological space X, which we denote by $(Sh(X))$. Also note that a morphism $\phi : \mathcal{F} \to \mathcal{G}$ of sheaves on X induces a morphism $\phi_p : \mathcal{F}_p \to \mathcal{G}_p$ on the stalks, for every point $p \in X$.

Theorem 2.0.15. *A morphism $\phi : \mathcal{F} \to \mathcal{G}$ of sheaves on X is an isomorphism if and only if the induced map on the stalk $\phi_p : \mathcal{F}_p \to \mathcal{G}_p$ is an isomorphism for every $p \in X$.*

Proof. See [24, Proposition 1.1, p.63]. □

For any morphism $\phi : \mathcal{F} \to \mathcal{G}$ of presheaves, we define presheaf kernel $ker\phi$ by $(ker\phi)(U) = ker\phi(U)$, which is a presheaf. If $\mathcal{F}, \mathcal{G}$ are sheaves then $ker\phi$ is also a sheaf. Similarly we can define subpresheaf, image presheaf, and quotient presheaf. But these are in general not sheaves. This leads us to the notion of a sheaf associated to a presheaf.

Definition-Proposition 2.0.16. *Given a presheaf $\mathcal{F}$, there is a sheaf $\mathcal{F}^+$ and a morphism $\theta : \mathcal{F} \to \mathcal{F}^+$, with the property that for any sheaf $\mathcal{G}$, and for any morphism $\varphi : \mathcal{F} \to \mathcal{G}$, there is a unique morphism $\psi : \mathcal{F}^+ \to \mathcal{G}$ such that $\varphi = \psi \circ \theta$. Furthermore the pair $(\mathcal{F}^+, \theta)$ is unique up to unique isomorphism. $\mathcal{F}^+$ is called the sheaf associated to the presheaf $\mathcal{F}$.*

Proof. See [24, Proposition 1.2, p.64]. □

2.0.3 Direct and inverse image of sheaves.

Suppose $f : X \to Y$ is a continuous map of topological spaces, and $\mathcal{F}$ is a presheaf on X. Then define a sheaf on Y, $f_*\mathcal{F}$ by $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}V)$, where V is an open subset of Y. Note that $f_*\mathcal{F}$ is a presheaf on Y, and is a sheaf if $\mathcal{F}$ is. This is called *direct image* sheaf of $\mathcal{F}$. For any sheaf $\mathcal{G}$ on Y, we define the inverse image sheaf $f^{-1}\mathcal{G}$ on X to be the sheaf associated the the presheaf $U \mapsto \lim_{V \supseteq f(U)} \mathcal{G}(V)$, where U is any open set in X, and the limit is taken over all open set V of Y containing $f(U)$.

2.1 Schemes.

2.1.1 Ringed Spaces.

Definition 2.1.1. *A ringed space is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X. A morphism of ringed spaces from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\sharp)$, where $f : X \to Y$ is a continuous map of topological spaces and a morphism $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings on Y.*

Note that we have a category $\mathcal{R}sp$, whose objects are ringed spaces, and whose morphisms are morphisms of ringed spaces.

Definition 2.1.2. *A ringed space $(X, \mathcal{O}_X)$ is called locally ringed space if for every point $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring. In that case, the maximal ideal in $\mathcal{O}_{X,x}$ is denoted by $\boldsymbol{m}_x$. The residue field $\mathcal{O}_{X,x}/\boldsymbol{m}_x$ of $\mathcal{O}_{X,x}$ is denoted by $k(x)$. A morphism of locally ringed spaces from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a morphism $(f, f^\sharp)$ of ringed spaces such that, for all $x \in X$, the induced map of local rings $f_x^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local homomorphism of local rings, i.e., $f_x^\sharp(\boldsymbol{m}_{f(x)}) \subset \boldsymbol{m}_x$.*

2.1.2 Affine Schemes

Let A be a ring and denote $Spec A$, the set of all prime ideals in A, called the prime spectrum or just spectrum of A. For any $f \in A$ we will denote $D(f)$, the set of all prime ideals $\mathbf{p} \in Spec A$ which does not contain f. Note that the family $\mathcal{B} = \{D(f)\}_{f \in A}$ is a basis for a topology of $X = Spec A$. Now we will define the sheaf of rings $\mathcal{O}_X$ on X as follows: for any open set $U \subset X$, $\mathcal{O}_X(U)$ is the set of all functions

$$s : U \to \coprod_{\mathbf{p} \in U} A_{\mathbf{p}},$$

such that $s(\mathbf{p}) \in A_{\mathbf{p}}$ for each $\mathbf{P}$ in U, which are locally represented by quotients. That is, for each $\mathbf{p} \in U$ there is a neighborhood V of $\mathbf{p}$, contained in U, and elements $a, f \in A$, such that for each $\mathbf{q} \in V, f \notin \mathbf{q}$, $s(\mathbf{q}) = a/f$ in $A_{\mathbf{q}}$. It is easy to check that $\mathcal{O}_X$ is a sheaf of rings, with the natural restriction maps.

Note that for any $\mathbf{p} \in X$, the stalk of $\mathcal{O}_X$ at $\mathbf{p}$ is $A_{\mathbf{p}}$, where $A_{\mathbf{p}}$ denotes the ring $S^{-1}A$, for $S = \{f \in A | f \notin \mathbf{p}\}$. Note that $A_{\mathbf{p}}$ is a local ring with the maximal ideal $\mathbf{p}A_{\mathbf{p}}$. Therefore, $(Spec A, \mathcal{O}_X)$ is a locally ringed space, we will denote it by $(Spec A, \mathcal{O}_{Spec A})$.

Proposition 2.1.3. *(a) If $\phi : A \to B$ is a homomorphism of rings, then ϕ induces a natural morphism of local ringed spaces*

$$(f, f^\sharp) : (Spec B, \mathcal{O}_{Spec B}) \to (Spec A, \mathcal{O}_{Spec A}).$$

(b) If A and B are rings, then any morphism of locally ringed spaces from $Spec B$ to $Spec A$ is induced by a homomorphism of rings $\phi : A \to B$ as in (a).

Proof. See [24, Proposition 2.3, p.73]. □

Definition 2.1.4. *An affine scheme is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic to the spectrum of some ring. A scheme is a locally ringed space $(X, \mathcal{O}_X)$ which admits an open covering $X = \bigcup_{i \in I} U_i$ such that $(U_i, \mathcal{O}_{X|_U})$ is an affine scheme for every i. We will denote it simply by X.*

A morphism of schemes is a morphism as locally ringed spaces. An isomorphism is a morphism with a two-sided inverse.

2.1.3 Projective scheme

Let $S = \bigoplus_{d \geq 0} S_d$ be a graded ring and denote $S_+ = \bigoplus_{d > 0} S_d$, ideal of S. Note that an ideal I is called homogeneous if it is generated by homogeneous elements. We let $ProjS$ denote the set of all homogeneous prime ideals $\mathbf{p}$ of S, which do not contain all of S_+. We set, for any homogeneous ideal $I(\neq S_+)$

$$V(I) = \{\mathbf{p} \in ProjS | I \subset \mathbf{p}\}.$$

We put the topology on $X := ProjS$, called the Zariski topology, by identifying $V(I)$ as closed sets. For any homogeneous element $f \in S_+$, set $D(f) = \{\mathbf{p} \in ProjS | f \notin \mathbf{p}\}$, then $D(f)$ is open in $ProjS$, these sets cover $ProjS$. Let $S_{(\mathbf{p})}$ denote the set of all elements of degree zero in the localization $T^{-1}S$, where T is the multiplicative system consisting of all homogeneous elements in S which are not in $\mathbf{p}$. Now we define the sheaf of rings $\mathcal{O}_X$ on X as follows: for any open set $U \subset X$, $\mathcal{O}_X(U)$ is the set of all functions

$$s : U \to \coprod_{\mathbf{p} \in U} S_{(\mathbf{p})},$$

such that $s(\mathbf{p}) \in S_{(\mathbf{p})}$ for each $\mathbf{P}$, which are locally represented by quotients. That is, for each $\mathbf{p} \in U$ there is a neighborhood V of $\mathbf{p}$, contained in U, and homogeneous elements $a, f \in S$ of same degree such that for each $\mathbf{q} \in V, f \notin \mathbf{q}$, $s(\mathbf{q}) = a/f$ in $S_{(\mathbf{q})}$. It is easy to check that $\mathcal{O}_X$ is a sheaf of rings, with the natural restriction maps.

The fact that $(X, \mathcal{O}_X)$ is a scheme is due to the following theorem.

Theorem 2.1.5. *Let S be a graded ring and set $X = ProjS$.*
(a) For every $\boldsymbol{p} \in X$, the stalk $\mathcal{O}_{X,\boldsymbol{p}}$ is isomorphic to the local ring $S_{(\boldsymbol{p})}$.
(b) For any homogeneous element $f \in S_+$ we have an isomorphism of locally ringed spaces $(D(f), \mathcal{O}_X | D(f)) \cong SpecS_{(f)}$, where $S_{(f)}$ consists of all elements of degree zero in the localization S_f.
(c) $X = ProjS$ is a scheme.

Proof. See [24, Theorem 2.5, p.76]. $\square$

Example 2.1.6. *Let $S = k[x_0, x_1, ..., x_n]$ be a polynomial ring over a field k, then $ProjS = \mathbb{P}^n_k$ is a Projective n-space over k.*

Definition 2.1.7. *An open subscheme of a scheme X is a scheme U, whose topological space is an open subset of X, and whose structure sheaf $\mathcal{O}_U$ is isomorphic to the restriction $\mathcal{O}_X | U$*

of the structure sheaf of X. An *open immersion* is a morphism $f : X \to Y$ of schemes which induces an isomorphism of X with an open subscheme of Y.

Definition 2.1.8. *A closed immersion is a morphism $f : Y \to X$ of schemes such that Y identifies as a closed subset of X and furthermore the induced map $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective. A closed subscheme of X is a closed subset Y of X endowed with the structure $(Y, \mathcal{O}_Y)$ of a scheme and with a closed immersion $j : Y \hookrightarrow X$.*

Note that a scheme X is called *irreducible* if its topological space is irreducible, i.e., the underlying topological space X cannot be expressed as the union of two proper closed subsets of X. A scheme is X called *connected* if its topological space is connected.

Definition 2.1.9. *(1) A scheme X is called reduced if for every open set $U \subseteq X$, the ring $\mathcal{O}_X(U)$ has no nilpotent elements. Equivalently, X is reduced if and only if the local rings $\mathcal{O}_p$, for all $p \in X$, have no nilpotent elements.*
(2) A scheme X is called integral if for every open set $U \subseteq X$, the ring $\mathcal{O}_X(U)$ is an integral domain.

Evidently, an integral scheme is always reduced. But the converse is not true in general. However, we have the following theorem:

Theorem 2.1.10. *A scheme is integral if and only if it is both reduced and irreducible.*

Proof. See [24, Theorem 3.1, p.82]. □

Definition 2.1.11. *The dimension of a topological space X, denoted $\dim X$ is the supremum of all integers n such that there exists a chain $Z_0 \subset Z_1 \subset ... \subset Z_n$ of distinct irreducible closed subsets of X. The dimension of a scheme X is the dimension of underlying topological space X.*

For any irreducible closed subset Z of X we define *codimension* of Z in X, denoted $codim(Z, X)$ is the supremum of all integers n such that there exists a chain $Z = Z_0 \subset Z_1 \subset ... \subset Z_n$ of distinct irreducible closed subsets of X, beginning with Z. If Y is any closed subset of X, we define

$$codim(Y, X) = \inf_{Z \subseteq Y} codim(Z, X)$$

Definition 2.1.12. *A scheme X is called noetherian if X can be covered by finite number of open affine subsets $Spec A_i$ with each A_i a noetherian ring.*

Definition-Proposition 2.1.13. *A morphism $f : X \to Y$ of schemes is called locally of finite type if one of the following equivalent conditions satisfies.*
(a) For each open affine subset $V = Spec B$ of Y, $f^{-1}(V)$ is covered by open affine subsets

$U_i = SpecA_i$, such that each A_i is a finitely generated B-algebra.

(b) There is an affine open covering $\{V_i = SpecB_i\}$ of Y, such that for each i, $f^{-1}(V_i)$ is covered by open affine subsets $U_{ij} = SpecA_{ij}$, where each A_{ij} is a finitely generated B_i-algebra.

The morphism f is called finite type if in addition f is quasi-compact, i.e., for every affine open subset $U \subseteq Y$, $f^{-1}(U)$ is quasi-compact.

Proof. See [46, Proposition 1, p.121]. □

Definition-Proposition 2.1.14. *A morphism $f : X \to Y$ of schemes is called finite morphism if one of the following equivalent conditions satisfies.*

(a) For each open affine subset $U = SpecB$ of Y, $f^{-1}(U)$ is affine, equal to $SpecA$, such that A is a finite B-module.

(b) There is an affine open covering $\{U_i = SpecB_i\}$ of Y such that for each i, $f^{-1}(U_i)$ is affine, equal to $SpecA_i$ with A_i is a finite B-module.

Proof. See [46, Proposition 5, p.124]. □

2.1.4 Fibered product

Let S be a scheme, and let X, Y be schemes over S, *i.e.*, schemes with morphisms to S. We define the *fibered product* of X, Y over S denoted $X \times_S Y$, to be a scheme over S, together with two morphisms $p_1 : X \times_S Y \to X$ and $p_2 : X \times_S Y \to Y$, satisfying the following universal property:

Let $f : Z \to X$, $g : Z \to Y$ be two morphisms of schemes over S. Then there exists a unique morphism $(f, g) : Z \to X \times_S Y$ of schemes making the following diagram commutative:

$$
\begin{array}{ccccc}
X & \overset{p_1}{\leftarrow} & X \times_S Y & \overset{p_2}{\to} & Y \\
& \nwarrow f & \uparrow (f,g) & \nearrow g & \\
& & Z & &
\end{array}
$$

Given any scheme S, fibered product exists in the category $\mathbb{S}ch(S)$ of schemes over S (See [24, Theorem 3.3, p.87]). For any scheme Y, we define $\mathbb{P}^n_Y = \mathbb{P}^n_\mathbb{Z} \times_{Spec\mathbb{Z}} Y$, and we call projective n-space over Y.

Definition 2.1.15. *A morphism $f : X \to Y$ of schemes is called separated if the image of the diagonal morphism $X \to X \times_Y X$ is closed. A morphism f is called proper if it is separated, of finite type, and universally closed. We say that f is unversally closed if it is a closed morphism, and for any base change $Z \to Y$, $X \times_Y Z \to Z$ is also a closed morphism.*

In this case we say that X is separated (resp. proper) over Y. A scheme X is separated(resp. proper) if it is separated (resp. proper) over $Spec\ \mathbb{Z}$. Note that any morphism of affine schemes is separated.

Definition 2.1.16. *A morphism f is called projective if it factors into a closed immersion $i : X \to \mathbb{P}_Y^n$, followed by the projection $p_2 : \mathbb{P}_Y^n \to Y$.*

The relation between projective and proper morphism is given in the following theorem:

Theorem 2.1.17. *A projective morphism of noetherian schemes is proper.*

Proof. See [24, Theorem 4.9, p.103]. □

2.1.5 Sheaves of Modules.

Definition 2.1.18. *Let $(X, \mathcal{O}_X)$ be a ringed space. A sheaf of $\mathcal{O}_X$-modules or an $\mathcal{O}_X$-module is a sheaf $\mathcal{F}$ on X, such that for each open set $U \subseteq X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, and for each inclusion of open sets $V \subseteq U$, the diagram:*

$$
\begin{array}{ccccc}
\mathcal{O}_X(U) & \times & \mathcal{F}(U) & \to & \mathcal{F}(U) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{O}_X(V) & \times & \mathcal{F}(V) & \to & \mathcal{F}(V).
\end{array}
$$

commutes.

A sheaf of ideals on X is a sheaf of $\mathcal{O}_X$-modules $\mathcal{I}$ which is a subsheaf of $\mathcal{O}_X$. In other words, for every open set U, $\mathcal{I}(U)$ is an ideal in $\mathcal{O}_X(U)$. A morphism $\mathcal{F} \to \mathcal{G}$ of sheaves of $\mathcal{O}_X$-modules is a morphism of sheaves, such that for each open set $U \subseteq X$, the map $\mathcal{F}(U) \to \mathcal{G}(U)$ is a homomorphism of $\mathcal{O}_X(U)$-modules. The category of $\mathcal{O}_X$-modules will be denoted by $\mathbb{M}od(X)$.

Note that using the usual operations on modules over a ring one can construct other $\mathcal{O}_X$-modules from the given $\mathcal{O}_X$-modules. Suppose $\mathcal{F}, \mathcal{G}$ are two $\mathcal{O}_X$-modules. We define the tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ to be the sheaf associated to the presheaf

$$
U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U).
$$

Which is an $\mathcal{O}_X$-module, and we denote it by $\mathcal{F} \otimes \mathcal{G}$. Similarly one can define $Hom_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ and the direct sum of a family of $\mathcal{O}_X$-modules. The kernel sheaf and cokernel sheaf of a homomorphism of $\mathcal{O}_X$-modules are $\mathcal{O}_X$-modules.

Definition 2.1.19. *Let X be a scheme. An $\mathcal{O}_X$-module $\mathcal{F}$ is called free if $\mathcal{F} \cong \mathcal{O}_X^r$, for some positive integer r. $\mathcal{F}$ is called locally free of rank r if there is an open covering $\{U_i\}$ of X such that*

$$
\mathcal{F}|_{U_i} \cong \mathcal{O}_X^r|_{U_i}.
$$

A locally free sheaf of rank 1 is called an invertible sheaf.

The set of all isomorphism classes of invertible sheaves on a scheme X form a group under the operation $\otimes$. For any invertible sheaf $\mathcal{L}$, we define inverse of $\mathcal{L}$ by $\mathcal{L}^{-1} = Hom(\mathcal{L}, \mathcal{O}_X)$. We denote it by Pic X, and called Picard group of X.

Remark 2.1.20. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. If $\mathcal{F}$ is sheaf of $\mathcal{O}_X$-module, then $f_*\mathcal{F}$ is sheaf of $f_*\mathcal{O}_Y$-module, and hence by the morphism $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$, $f_*\mathcal{F}$ has a natural structure of $\mathcal{O}_Y$-module. This $\mathcal{O}_Y$-module is called the direct image of $\mathcal{F}$ under f.*

Note that for any sheaf $\mathcal{G}$ of $\mathcal{O}_Y$-module on Y, $f^{-1}\mathcal{G}$ is $f^{-1}\mathcal{O}_Y$-module. Using the natural morphism $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ we can consider $\mathcal{O}_X$ as $f^{-1}\mathcal{O}_Y$. We define $f^*\mathcal{G}$ to be the tensor product

$$f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X.$$

Thus $f^*\mathcal{G}$ is an $\mathcal{O}_X$-module which we call inverse image of $\mathcal{G}$ under f. One can show that f^* and f_* are adjoint functors between the category of $\mathcal{O}_X$-modules and the category of $\mathcal{O}_Y$-modules, i.e., $Hom_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \cong Hom_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F})$.

Let A be a commutative ring and let $X = Spec A$ be an affine scheme. For any A-module M, we define an $\mathcal{O}_X$-module $\tilde{M}$ as follows. For each open subset $U \subseteq X$,

$$U \mapsto \{s : U \to \coprod_{\mathbf{p} \in U} M_{\mathbf{p}}\},$$

which are locally fractions (as we seen in the definition of affine scheme). This gives a sheaf on $Spec A$ with the obvious restriction maps. We denote it by $\tilde{M}$ and we call sheaf associated to M on $Spec A$.

We can easily check that for any principal open set $D(f)$ of X, $\tilde{M}(D(f)) = M_f$ and $\tilde{M}_{\mathbf{p}} = M_{\mathbf{p}}$ for every $\mathbf{p} \in Spec A$ and $\tilde{M}(X) = M$. It is clear that $\tilde{M}$ is an $\mathcal{O}_X$-module.

Definition-Proposition 2.1.21. *Let $(X, \mathcal{O}_X)$ be a scheme, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-module. Then the following are equivalent.*
(a) There is a affine open covering $\{U_i = Spec A_i\}$ of X such that for each i, there is an A_i-module M_i with $\mathcal{F}|_{U_i} \cong \tilde{M}_i$.
(b) for every affine open set $U = Spec A$ of X, there is an A-module M with $\mathcal{F}|_U \cong \tilde{M}$.

We say $\mathcal{F}$ quasi coherent if $\mathcal{F}$ satisfies above equivalent conditions. Assume X is noetherian, we say $\mathcal{F}$ is coherent if further more each M_i can be taken to be finitely generated A_i-module

Proof. See [24, Proposition 5.4, p.113]. $\square$

Definition 2.1.22. *Let Y be a closed subscheme of a scheme X, and let $i : Y \to X$ be the inclusion morphism. We define the ideal sheaf of Y, denoted $\mathcal{I}_Y$, to be the kernel of the morphism $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Y$. Note that $\mathcal{I}_Y$ is a quasi-coherent sheaf of ideals on X.*

One can check easily that any quasi-coherent sheaf of ideals on X is uniquely determined by a closed subscheme of X, (See, [24, Proposition 5.9, p.116]).

Now we will define quasi coherent sheaves on the $Proj$ of a graded ring. Let S be a graded ring and let M be a graded S-module. For any homogeneous prime $\mathbf{p} \in ProjS$, define T the set of all homogeneous elements not in $\mathbf{p}$. Denote $M_{(\mathbf{p})}$, the group of elements of degree 0 in the localization $T^{-1}M$. For each open subset $U \subseteq ProjS$,

$$U \mapsto \{s : U \to \coprod_{\mathbf{p} \in U} M_{(\mathbf{p})}\},$$

which are locally fractions (as we seen in the definition of projective scheme). This gives a sheaf on $ProjS$ with the obvious restriction maps. We denote it by $\tilde{M}$ and we call sheaf associated to M on $ProjS$. Note that $\tilde{M}$ is a quasi-coherent sheaf, and if S is noetherian then $\tilde{M}$ is coherent (See [24, Proposition 5.11, p.116]).

Definition 2.1.23. *Let S be a graded ring, and let $X = ProjS$. For any $n \in \mathbb{Z}$, we define the sheaf $\mathcal{O}_X(n)$ to be $\tilde{S(n)}$. We call $\mathcal{O}_X(1)$ the twisting sheaf of Serre. For any sheaf of $\mathcal{O}_X$-modules, $\mathcal{F}$, we denote by $\mathcal{F}(n)$ the twisted sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$, and we write $\mathcal{O}_X(0)$ simply as $\mathcal{O}_X$.*

Proposition 2.1.24. *Let S be a graded ring, and let $X = ProjS$. Assume that S is generated by S_1 as an S_0-algebra. Then*
(a) The sheaf $\mathcal{O}_X(n)$ is an invertible sheaf on X.
(b) $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \cong \mathcal{O}_X(n + m), \forall m, n \in \mathbb{Z}$.

Proof. See [24, Proposition 5.12, p.117]. □

2.2 Divisors.

Assume that X is a noetherian integral separated scheme which is regular in codimension one, here regular in codimension one means every local ring $\mathcal{O}_x$ of X of dimension one is regular.

Definition 2.2.1. *A prime divisor on X is a closed integral subscheme Y of codimension one. A Weil divisor on X is a formal $\mathbb{Z}$-linear combination of prime divisor of X. In other words, a Weil divisor is a finite sum $D = \sum n_i.Y_i$, where the Y_i are prime divisors.*

A Weil divisor $D = \sum n_i.Y_i$ is called *effective*, denoted $D \geq 0$, if all the $n_i \geq 0$. The set of all Weil divisors is a free abelian group generated by prime divisors. We will denote it by *Div X*. Suppose Y is a prime divisor and $y \in Y$ be the generic point of Y. Then the local

ring $\mathcal{O}_{X,y}$ is a discrete valuation ring with quotient field $K(X)$. Denote ϑ_Y, the corresponding valuation of Y. Note that since X is separated, Y is uniquely determined by its valuation. For any $f \in K(X)^*$, by [24, Lemma 6.1, p.131], $\vartheta_Y(f) = 0$ for all except finitely many prime divisors Y.

This implies, we can make the following definition.

Definition 2.2.2. *Given $f \in K(X)^*$, we define the divisor of f, denoted $div(f)$, by*

$$div(f) = \sum \vartheta_Y(f).Y,$$

where the sum is taken over all prime divisors of X.

$div(f)$ is called principal divisor of X. It is clear from the properties of discrete valuations that $div(\frac{f}{g}) = div(f) - div(g)$ and $div(fg) = div(f) + div(g)$, and hence the set of all principal divisors of X forms a subgroup of $Div(X)$.

Definition 2.2.3. *Two divisors D, D' on X are said to be linearly equivalent, written $D \sim D'$, if $D - D' = div(f)$, for some $f \in K(X)^*$.*

The quotient group $Div\ X$ divided by the subgroup of principal divisors is called the divisor class group of X, and it is denoted by $Cl\ X$. Note that, if A is an unique factorization domain then $Cl\ Spec A = 0$.

If $U \subset X$ is an open subset, we define the restriction map $Cl(X) \to Cl(U)$ by $\sum n_i Y_i \mapsto \sum n_i (Y_i \cap U)$.

We have the following use full exact sequence.

Theorem 2.2.4. *Let X be noetherian integral separated scheme which is regular in codimension one, and let $Z \subset X$ be a proper closed subset of X. Then*
(1) if $Z \subset X$ is a irreducible closed subscheme of codimension one, Then there is an exact sequence

$$\mathbb{Z} \xrightarrow{n \mapsto n[Z]} Cl(X) \longrightarrow Cl(X - Z) \longrightarrow 0.$$

(2) if $Z \subset X$ is of codimension at least 2. Then the canonical map

$$Cl(X) \longrightarrow Cl(X - Z)$$

is an isomorphism.

Proof. See [24, Proposition 6.5, p.133]. □

2.2.1 Cartier divisors.

Now we will define the notion of divisors on arbitrary schemes. Assume that X is a noetherian scheme. For any open open subset $U \subset X$, define $\mathcal{K}(U)$ be the total quotient ring $S(U)^{-1}\Gamma(U, \mathcal{O}_X)$ of $\Gamma(U, \mathcal{O}_X)$, where $S(U) = \{s \in \Gamma(U, \mathcal{O}_X) | s \text{ is not zero divisor in } \mathcal{O}_x, \forall x \in U\}$. By [45, Proposition 1, p.61], this gives a sheaf of $\mathcal{O}_X$-modules with the natural restriction maps, and it is unique. We will denote it by $\mathcal{K}_X$. The stalks $\mathcal{K}_x$ of $\mathcal{K}_X$ are just the total quotient rings of the stalks $\mathcal{O}_x$. Let $\mathcal{K}_X^*$ (respectively, $\mathcal{O}_X^*$) denotes the subgroup of $\mathcal{K}_X$ (respectively, $\mathcal{O}_X$) consisting of invertible elements.

Definition 2.2.5. *(1) A Cartier divisor on X is a global section of the sheaf $\mathcal{K}^*/\mathcal{O}^*$. More concretely, a Cartier divisor on X can be described by giving an open cover $\{U_i\}$ of X, and for each i an element $f_i \in \Gamma(U_i, \mathcal{K}^*)$, such that for each $i, j, f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}^*)$. (2) A Cartier divisor D on X, represented by $\{(U_i, f_i)\}$ is called effective if all the $f_i \in \Gamma(U_i, \mathcal{O}_{U_i})$.*

Note that the set of all Cartier divisors form a group. A Cartier divisor is called principal if it is in the image of the natural map $\Gamma(X, \mathcal{K}^*) \to \Gamma(X, \mathcal{K}^*/\mathcal{O}^*)$. Two Cartier divisors are linearly equivalent if their difference is principal. We define the group $CaCl\ X$ is the Cartier divisors modulo principal divisors. In general, Cartier divisors are not generalization of Weil divisor. But by [24, Proposition 6.11, p.141], $Cl\ X$ and $CaCl\ X$ are isomorphic, if X is integral, separated, and all of whose local rings are unique factorization domains.

Let D be an effective Cartier divisor on X, represented by $\{(U_i, f_i)\}$. Define the sheaf of ideals $\mathcal{I}$ on X which is locally generated by f_i. Let Y be the associated closed subscheme of codimension 1. This closed subscheme is called locally principal closed subscheme of X. By [24, Proposition 6.18, p.145], $\mathcal{I}_Y \cong \mathcal{O}_X(-D)$.

Now we will define a line bundle associated to a given Cartier divisor.

Definition 2.2.6. *Let D be a Cartier divisor on a noetherian scheme X, represented by $\{(U_i, f_i)\}$. We define a subsheaf $\mathcal{O}_X(D)$ of $\mathcal{K}_X$ by taking $\mathcal{O}_X(D)$ to be the sub-$\mathcal{O}_X$-module of $\mathcal{K}_X$ generated by f_i^{-1} on U_i. This is well defined, since f_i/f_j is invertible on $U_i \cap U_j$, so f_i^{-1} and f_j^{-1} generate the same $\mathcal{O}_X$-module.*

Note that $\mathcal{O}_X(D)$ can also be characterized by

$$\mathcal{O}_X(D)(U) = \{f \in \mathcal{K}_X^* | div(f) + D|_U \geq 0\} \cup \{0\}.$$

We have the following theorem.

Theorem 2.2.7. *(a) For any Cartier divisor D, $\mathcal{O}_X(D)$ is an invertible sheaf on X. The map $D \mapsto \mathcal{O}_X(D)$ gives a one-one correspondence between Cartier divisors on X and invertible*

subsheaves of $\mathcal{K}$.

(b) $\mathcal{O}_X(D_1 - D_2) \cong \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)^{-1}$.

(c) $D_1 \sim D_2$ *iff* $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$ *as abstract invertible sheaves.*

Proof. See [24, Proposition 6.13, p.144]. □

It is clear from the above theorem that the map $D \mapsto \mathcal{O}_X(D)$ gives an injective homomorphism from $CaCl\ X$ to $Pic(X)$. This map is surjective if X is integral scheme, [See [24, Proposition 6.15, p.145].

Finally, we note:

Theorem 2.2.8. *If* X *is integral, separated, and all of whose local rings are unique factorization domains, then there is a natural isomorphism* $Cl\ X \cong Pic(X)$.

Proof. See [24, Corollary 6.16, p.145]. □

2.2.2 The class group of projective space.

Let X be the smooth projective space $\mathbb{P}_k^n$ over a field k. Suppose $D = \sum n_i Y_i$ is a divisor on X. We define the degree of D by $deg(D) = \sum n_i.deg Y_i$, where $deg Y_i$ is the degree of the hyper surface Y_i.

Theorem 2.2.9. *Let* $H = \{x_0 = 0\}$ *be the hyperplane on* X. *Then:*

(a) If D *is any divisor of degree* d, *then* $D \sim dH$,

(b) for any $f \in K(X)^*$, $deg(div(f)) = 0$,

(c) the degree function gives an isomorphism $deg : Cl X \to \mathbb{Z}$.

Proof. See [24, Proposition 6.4, p.132]. □

It is clear from the above theorem that any line bundle on $\mathbb{P}_k^n$ is isomorphic to some $\mathcal{O}(m)$, where $m \in \mathbb{Z}$.

Now we define linear systems corresponding to line bundles.

2.2.3 Linear systems.

Let X be a non-singular projective variety over an algebraically closed field k. By Theorem 2.2.8, the notion of Weil divisors and Cartier divisors are equivalent. So we can talk about divisors instead of Weil divisors/Cartier divisors.

Definition 2.2.10. *Let* D *be a divisor on* X. *We define the complete linear system of* D, *denoted* $|D|$, *as*

$$|D| = \{D'|D' \geq 0, D' \sim D\}.$$

The base locus of $|D|$ is the intersection of all of the elements of $|D|$. We say $|D|$ is base point free if the base locus is empty.

Let D_0 be a divisor on X, and let $\mathcal{L} \cong \mathcal{O}(D_0)$ be the corresponding invertible sheaf on X. By [24, Theorem 5.19, p.122], $\Gamma(X, \mathcal{L})$ is a finite dimensional k-vector space. Let $\{U_i\}$ be an open cover of X, where $\mathcal{L}$ trivializes, and let $\phi : \mathcal{L}|_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i}$ be an isomorphism. In view of this, we can make the following definition.

Definition 2.2.11. *Let $s \in \Gamma(X, \mathcal{L})$ be a non zero section of $\mathcal{L}$. We define the divisor of zeros of s, denote $div(s)$, to be an effective Cartier divisor $\{(U_i, \phi(s))\}$ on X. It is clear that $div(s)$ is linearly equivalent to D_0. Note that ϕ is determined up to multiplication by an element of $\Gamma(U, \mathcal{O}_U{}^*)$, so $div(s)$ is well-defined Cartier divisor.*

Suppose $D \geq 0$ is a divisor linearly equivalent to D_0, then one can prove that $D = div(s)$ for some $s \in \Gamma(X, \mathcal{L})$. Finally, $s' = \lambda s$ for some $\lambda \in k^*$ and $s, s' \in \Gamma(X, \mathcal{L})$ if and only if $div(s) = div(s')$, (See [24, Proposition 7.7, p.157]). This implies, $|D_0|$ is one-to-one correspondence with the set $(\Gamma(X, \mathcal{L}) - \{0\})/k^*$. This gives $|D_0|$ a structure of the set of closed points of a projective space over k.

Suppose D is a divisor on X. Let n be the dimension of the vector space $\Gamma(X, \mathcal{O}_X(D))$, and $\mathbb{P}^{n-1} \simeq \mathbb{P}(\Gamma(X, \mathcal{O}_X(D))^*)$. Note that D defines a rational map:

$$\phi_D : X \dashrightarrow \mathbb{P}^{n-1}$$

Given a point $x \in X$, let

$$H_x = \{s \in \Gamma(X, \mathcal{O}_X(D)) | s(x) = 0\}.$$

Suppose x not in base locus of $|D|$, then H_x is a hyperplane in $\Gamma(X, \mathcal{O}_X(D))$, whence a point of $\phi_D(x) = [H_x] \in \mathbb{P}^{n-1}$. Note that the map ϕ_D is defined out side the base locus of $|D|$.

Definition 2.2.12. *A divisor D on X is called base point free or globally generated if the rational map $\phi_D : X \dashrightarrow \mathbb{P}(\Gamma(X, \mathcal{O}_X(D))^*)$ is a morphism. We say that D is very ample if ϕ_D defines an embedding of X. We say that D is ample if mD is very ample for some $m \in \mathbb{N}$.*

Note that a line bundle $\mathcal{L}$ is called very ample (respectively ample) if its corresponding divisor is very ample (respectively ample).

2.2.4 Differentials.

Now we will introduce the language of sheaf of relative differentials of one scheme over another.

Let $f : X \to Y$ be a separated morphism of schemes, and let $\Delta : X \to X \times_Y X$ be the diagonal morphism. Denote $\mathcal{I}$, ideal sheaf corresponds to the closed subscheme $\Delta(X) \subset X \times_Y X$.

Definition 2.2.13. *We define the sheaf of relative differentials of X over Y to be the quasi-coherent sheaf $\Omega_{X/Y} = \Delta^*(\mathcal{I}/\mathcal{I}^2)$.*

One can prove easily that, the sheaf $\Omega_{X/Y}$ is coherent if Y is noetherian and $f : X \to Y$ is of finite type. Suppose if $Y = Spec\, k$, k is field, then we write $\Omega_{X/Y}$ as Ω_X.

We have two use full exact sequences. The second exact sequence describes how differentials behave under a closed immersion.

Theorem 2.2.14. *(1) Given separated morphisms of schemes $X \xrightarrow{f} Y \xrightarrow{g} Z$, there is an exact sequence of sheaves on X:*

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0.$$

(2) Suppose $X \hookrightarrow Y$ closed subscheme of Y, with ideal sheaf $\mathcal{I}$, and $g : Y \to Z$ be a separated morphism of schemes. Then there is an exact sequence of sheaves on X:

$$\mathcal{I}/\mathcal{I}^2 \xrightarrow{\delta} \Omega_{Y/Z} \otimes \mathcal{O}_X \to \Omega_{X/Z} \to 0.$$

Proof. See [24, Theorem 8.11, 8.12, p.176]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We have the following exact sequence of differentials on a projective space.

Theorem 2.2.15. *(**The Euler exact sequence.**)*
Let A be a ring, let $Y = Spec\, A$, and let $X = \mathbb{P}_A^n$. Then there is an exact sequence of sheaves on X,

$$0 \to \Omega_{X/Y} \to \mathcal{O}_X(-1)^{n+1} \to \mathcal{O}_X \to 0.$$

Here, $\mathcal{O}_X(-1)^{n+1}$ means a direct sum of $n+1$ copies of $\mathcal{O}_X(-1)$.

Proof. See [24, Theorem 8.13, p.176]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following theorem will gives the connection between non-singular varieties over an algebraically closed field and differentials.

Theorem 2.2.16. *An n-dimensional scheme X over an algebraically closed field k is smooth if and only if Ω_X is a locally free sheaf of rank n.*

Proof. See [24, Theorem 8.15, p.177]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In view of above theorem, we can make the following definition.

Definition 2.2.17. *Let X be a n-dimensional nonsingular variety over k. We define the canonical sheaf of X to be $\omega_X = \bigwedge^n \Omega_X$, the nth exterior power of the sheaf of differentials. It is an invertible sheaf on X. We define the tangent sheaf of X to be $\mathcal{T}_X = \Omega_X^\vee = \mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)$.*

Now we study the behavior of the tangent sheaf and the canonical sheaf for a nonsingular subvariety of a variety X.

Let Y be a nonsingular subvariety of a nonsingular variety X over k. Let $\mathcal{I}$ be the ideal sheaf of Y on X.

Definition 2.2.18. *We define the conormal sheaf of Y in X to be a locally free sheaf $\mathcal{I}/\mathcal{I}^2$. Its dual $\mathcal{N}_{Y/X} = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y)$ is called the normal sheaf of Y in X. It is locally free of rank $r = codim(Y, X)$.*

Suppose Y is a nonsingular subvariety of codimension one, *i.e.*, Y is divisor on X. By [24, Proposition 8.20, p.182], the canonical sheaf $\omega_Y \cong \omega_X \otimes \mathcal{O}_X(D) \otimes \mathcal{O}_Y$.

2.3   Čech Cohomology.

Now we will define Čech cohomology groups for a sheaf of abelian groups on a topological space X with respect to an open cover of X.

Let X be a topological space, and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X. Let $\mathcal{F}$ be a sheaf of abelian groups on X. For any integer $p \geq 0$ and for any sequence of indices $i_0, i_1, ..., i_p$ in I with $i_0 < i_1 < ... < i_p$ we set

$$U_{i_0,i_1,...,i_p} = U_{i_0} \cap U_{i_1} \cap ... \cap U_{i_p}.$$

For each $p \geq 0$, set

$$\mathcal{C}^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < i_1 < ... < i_p} \mathcal{F}(U_{i_0,i_1,...,i_p}).$$

Any $\alpha \in \mathcal{C}^p(\mathcal{U}, \mathcal{F})$ is called a p-cochain of $\mathcal{U}$ in $\mathcal{F}$, and we denote by $\alpha_{i_0,i_1,...,i_p}$ the value of α in $\mathcal{F}(U_{i_0,i_1,...,i_p})$.

We define co-boundary map $d : \mathcal{C}^p(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^{p+1}(\mathcal{U}, \mathcal{F})$ by the formula

$$(d\alpha)_{i_0,i_1,...,i_p} = \sum_{k=0}^{p+1}(-1)^k \alpha_{i_0,i_1,...,\hat{i}_k,...,i_{p+1}}|_{U_{i_0,i_1,...,i_{p+1}}},$$

where $\hat{i}_k$ means that we remove the index i_k. One can check easily that $d^2 = 0$. Therefore, $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$ is a cochain complex of $\mathcal{F}$ with respect to the open cover $\mathcal{U}$.

Definition 2.3.1. *Let X be a topological space, and let $\mathcal{U}$ be an open cover of X. For any sheaf abelian groups $\mathcal{F}$ on X, we define pth Čech cohomology group of $\mathcal{F}$, with respect to the covering $\mathcal{U}$, to be*

$$H^p(\mathcal{U}, \mathcal{F}) = \frac{Ker\{\mathcal{C}^p(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^{p+1}(\mathcal{U}, \mathcal{F})\}}{Im\{\mathcal{C}^{p-1}(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^p(\mathcal{U}, \mathcal{F})\}}.$$

It follows immediately from the construction that $H^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$.

Definition 2.3.2. *Let X be a topological space, and let $\mathcal{F}$ be a sheaf on X. We set*

$$H^p(X, \mathcal{F}) = \varinjlim_{\mathcal{U}} H^p(\mathcal{U}, \mathcal{F})$$

where the direct limit is taken over the set of all open covers of X endowed with refinement of covering as the partial orderings.

Now we restrict our attention to cohomology of sheaves of $\mathcal{O}_X$-modules $\mathcal{F}$ on a scheme X. In this case we have nice properties of the Čech cohomology groups.

Theorem 2.3.3. *Let X be an affine scheme. Then for any quasi-coherent sheaf $\mathcal{F}$ on X and for any integer $p \geq 0$, we have $H^p(X, \mathcal{F}) = 0$*

Proof. See [39, Theorem 2.18, p.186]. □

For any open cover $\mathcal{U}$ of X, we have a canonical homomorphisms $H^p(\mathcal{U}, \mathcal{F}) \to H^p(X, \mathcal{F})$, for each integer $p \geq 0$. These homomorphisms need not be isomorphisms. On the other hand, we have the following.

Theorem 2.3.4. *Let X be a noetherian separated scheme, let $\mathcal{F}$ be a quasi-coherent sheaf on X, and $\mathcal{U}$ an affine covering of X. Then the canonical homomorphism*

$$H^p(\mathcal{U}, \mathcal{F}) \to H^p(X, \mathcal{F})$$

is an isomorphism for every $p \geq 0$.

Proof. See [39, Theorem 2.19, p.186]. □

The above results enables us to construct a long exact sequence of cohomology from a given short exact sequence of quasi-coherent sheaf on X, where X is a noetherian separated scheme.

Corollary 2.3.5. *For any short exact sequence of sheaves $0 \to \mathcal{F}'' \to \mathcal{F} \to \mathcal{F}' \to 0$ on X with $\mathcal{F}''$ quasi-coherent, we have a long exact sequence*

$$... \to H^{p-1}(X, \mathcal{F}') \xrightarrow{\partial} H^p(X, \mathcal{F}'') \to H^p(X, \mathcal{F}) \to H^p(X, \mathcal{F}') \xrightarrow{\partial} H^{p+1}(X, \mathcal{F}'') \to ....$$

Proof. See [39, Corollary 2.22, p.186]. □

Note that if $0 \to \mathcal{F}'' \to \mathcal{F} \to \mathcal{F}' \to 0$ is a short exact sequence of sheaves of abelian groups on a topological space X. In general we do not get a long exact sequence of Čech cohomology groups.

We can use Čech cohomology to determine when a noetherian scheme is affine. This is given in the following theorem.

Theorem 2.3.6. (*Serre*). *Let X be a noetherian scheme. Then the following are equivalent:*
(a) X is affine.
(b) $H^p(X, \mathcal{F}) = 0$, for all quasi-coherent sheaves $\mathcal{F}$ on X and for all $i \geq 1$.
(c) $H^1(X, \mathcal{F}) = 0$, for all coherent sheaf $\mathcal{F}$ on X

Proof. See [39, Theorem 2.23, p.187]. $\qquad\square$

2.3.1   Cohomology of projective schemes.

Let A be a noetherian ring, let $S = A[x_0, x_1, ..., x_r]$, and let $X = ProjS$ be the projective space $\mathbb{P}_A^r$ over A.

Theorem 2.3.7. *Let $X = \mathbb{P}_A^r$ then for any $n \in \mathbb{Z}$, we have:*
(a) $H^0(X, \mathcal{O}_X(n)) = S_n$ (if $n < 0$, we set $S_n = 0$ by convention).
(b) $H^p(X, \mathcal{O}_X(n)) = 0$ if $p \neq 0, r$.
(c) $H^r(X, \mathcal{O}_X(n)) \simeq H^0(X, \mathcal{O}_X(-n - r - 1))^\vee$ (where $\vee$ means dual as an A-module). In particular, $H^r(X, \mathcal{O}_X(n)) = 0$ if $n \geq -r$.

Proof. See [39, Theorem 3.1, p.195]. $\qquad\square$

We have the following fundamental theorem for the study of projective schemes.

Theorem 2.3.8. *Let X be a projective scheme over a noetherian ring A and let $\mathcal{F}$ be a coherent sheaf on X. Then we have the following properties.*
(a) For any integer $p \geq 0$, the A-module $H^p(X, \mathcal{F})$ is finitely generated.
(b) There exists an integer n_0 such that for every $n \geq n_0$ and for every $p \geq 1$, we have $H^p(X, \mathcal{F}(n)) = 0$.

Proof. See [39, Theorem 3.2, p.195]. $\qquad\square$

We have the following Serre duality theorem for the cohomology of coherent sheaves on a projective scheme. This will help us to reduce the computations of cohomology of coherent sheaves as we have seen Poincare duality in the case of cohomology of manifolds.

Theorem 2.3.9. *Let X be a smooth projective scheme over an algebraically closed filed k of dimensision n. Then for any locally free sheaf $\mathcal{F}$,*

$$H^i(X, \mathcal{F}) \simeq H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X)^\vee$$

Proof. See [24, Corollary 7.7, p.244]. $\qquad\square$

Definition 2.3.10. *Let $f : X \to Y$ be a morphism of schemes. An $\mathcal{O}_X$-module $\mathcal{F}$ is called flat over Y at a point $x \in X$ if the stalk $\mathcal{F}_x$ is a flat $\mathcal{O}_{y,Y}$-module, where $y = f(y)$ and we consider $\mathcal{F}_x$ as an $\mathcal{O}_{y,Y}$-module via the natural map $f^\sharp : \mathcal{O}_{y,Y} \to \mathcal{O}_{x,X}$. We say $\mathcal{F}$ is flat over Y if it is flat at every point of X. We say X is flat over Y if $\mathcal{O}_X$ is.*

2.3.2 Higher direct image sheaves.

Let $f : X \to Y$ be a separated morphism of schemes and let $\mathcal{F}$ be a quasi-coherent sheaf on X. We define higher direct image sheaves of $\mathcal{F}$ on Y as follows. For any open subset V of Y, we define the sheaf associated to the preashef,

$$V \mapsto H^p(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)})$$

on Y. We denote it by $R^p f_* \mathcal{F}$. These are quasi-coherent sheaves. For $p \geq 1$, $R^p f_* \mathcal{F}$ are called higher direct images of $\mathcal{F}$. Note that $R^0 f_* \mathcal{F} = f_* \mathcal{F}$.

Theorem 2.3.11. *Let $f : X \to Y$ be a separated and quasi-compact morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X, and $\mathcal{E}$ be a quasi-coherent sheaf which is flat over Y, then the canonical morphism of sheaves*

$$R^p f_* \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{E} \to R^p f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^* \mathcal{E})$$

is an isomorphism. The isomorphism is called the projection formula.

Proof. See [39, Theorem 2.32, p.190]. □

Theorem 2.3.12. *Let X be a noetherian scheme, and let $f : X \to Y$ be a morphism of X to an affine scheme $Y = Spec\ A$. Then for any quasi-coherent sheaf $\mathcal{F}$ on X, we have*

$$R^p(f_* \mathcal{F}) \cong H^p(X, \mathcal{F}).$$

Proof. See [24, Proposition 8.5, p.251]. □

The following theorem shows that cohomology commutes with flat base extension.

Theorem 2.3.13. *Let $f : X \to Y$ be a separated morphism of finite type of noetherian schemes, and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $u : Y' \to Y$ be a flat morphism of noetherian schemes.*

$$
\begin{array}{ccc}
X' & \xrightarrow{v} & X \\
\downarrow g & & \downarrow f \\
Y' & \xrightarrow{u} & Y
\end{array}
$$

Then for all $i \geq 0$ there are natural isomorphisms

$$u^* R^i f_*(\mathcal{F}) \cong R^i g_*(v^* \mathcal{F}).$$

Proof. See [24, Proposition 9.3, p.255]. □

Note that even if u is not flat, we have a natural map $u^* R^i f_*(\mathcal{F}) \rightarrow R^i g_*(v^* \mathcal{F})$.

Definition 2.3.14. *A morphism $f : X \rightarrow Y$ of schemes of finite type over k is etale if f is flat and unramified.*

We say f is unramified if for every $x \in X$, letting $y = f(x)$, we have $m_y \cdot \mathcal{O}_x = m_x$, and $k(x)$ is separable algebraic extension of $k(y)$.

Chapter 3

Semistability of logarithmic cotangent bundle on some projective manifolds

In the first section of this chapter we will recall the definitions of stability and some well known results related to stability of tangent bundle of low dimensional Fano manifolds. The main results Theorem 1.1.1, Proposition 1.1.2 and Theorem 1.1.3, in Chapter 1, will be proved in next sections.

Throughout this chapter, unless specified otherwise, a variety always mean a smooth projective variety over $\mathbb{C}$.

Let X be a n-dimensional smooth projective variety over $\mathbb{C}$ and K_X denotes the canonical line bundle on X. Fix an ample line bundle H on X.

3.1   Preliminaries.

For the definitions of this section we follow [26].

Definition 3.1.1. • *The dimension of a coherent sheaf $\mathcal{F}$ is the dimension of the closed set $Supp(\mathcal{F}) = \{x \in X | \mathcal{F}_x \neq 0\}$.*

• *$\mathcal{F}$ is called pure of dimension d if $dim(\mathcal{G}) = d$ for all non-trivial coherent subsheaves $\mathcal{G} \subset \mathcal{F}$.*

Definition 3.1.2. *The torsion filtration of a coherent sheaf $\mathcal{F}$ of dimension d, is the unique filtration*

$$0 \subset T_0(\mathcal{F}) \subset ... \subset T_d(\mathcal{F}) = \mathcal{F},$$

where $T_i(\mathcal{F})$ is the maximal subsheaf of $\mathcal{F}$ of dimension $\leq i$.

The torsion filtration of a coherent sheaf always exists. Note that $\mathcal{F}$ is pure sheaf of dimension d if and only if $T_{d-1}(\mathcal{F}) = 0$.

Suppose $\mathcal{F}$ is a locally free sheaf on X then it is torsion free and $dim(\mathcal{F}) = n$. Clearly $T_{n-1}(\mathcal{F}) = 0$ and hence it is pure.

Definition 3.1.3. *The slope of $\mathcal{F}$ with respect to the ample bundle H is defined by*

$$\mu(\mathcal{F}) := \frac{deg(\mathcal{F})}{rk(\mathcal{F})},$$

where $deg(\mathcal{F}) = c_1(\mathcal{F}).H^{n-1}$.

Definition 3.1.4. *A coherent sheaf $\mathcal{F}$ of dimension n on X is called stable in the sense of Mumford-Takemoto if $T_{n-2}(\mathcal{F}) = T_{n-1}(\mathcal{F})$ and $\mu(\mathcal{G}) < \mu(\mathcal{F})$ for all subsheaves $\mathcal{G} \subset \mathcal{F}$ with $0 < rk(\mathcal{G}) < rk(\mathcal{F})$.*

Similarly, $\mathcal{F}$ is semistable if $\mu(\mathcal{G}) \leq \mu(\mathcal{F})$.

The Euler characteristic of a coherent sheaf $\mathcal{F}$ is $\chi(\mathcal{F}) := \sum (-1)^i h^i(X, \mathcal{F})$, where $h^i(X, \mathcal{F}) = dim_{\mathbb{C}} H^i(X, \mathcal{F})$. The Hilbert polynomial $P(\mathcal{F})$ is defined by

$$m \mapsto \chi(\mathcal{F} \otimes \mathcal{O}_X(m)).$$

Suppose $\mathcal{F}$ is a coherent sheaf of dimension n. The reduced Hilbert polynomial $p(\mathcal{F})$ is defined by

$$p(\mathcal{F}, m) := \frac{P(\mathcal{F}, m)}{rk(\mathcal{F}).c_1(K_X)}.$$

The Harder-Narasimhan filtration.

Now we shall define the Harder-Narasimhan filtration.

Definition 3.1.5. *Let $\mathcal{F}$ be a non-trivial pure sheaf of dimension d. A Harder-Narasimhan filtration for $\mathcal{F}$ is an increasing filtration*

$$0 = HN_0(\mathcal{F}) \subset HN_1(\mathcal{F}) \subset ... \subset HN_l(\mathcal{F}) = \mathcal{F},$$

such that the factors $gr_i^{HN} = HN_i(\mathcal{F})/HN_{i-1}(\mathcal{F})$ for $i = 1, ..., l$, are semistable sheaves of dimension d with reduced Hilbert polynomials p_i satisfying

$$p_{max}(\mathcal{F}) := p_1 > ... > p_l =: p_{min}(\mathcal{F}).$$

Theorem 3.1.6. *Every pure sheaf $\mathcal{F}$ has a unique Harder-Narasimhan filtration.*

Proof. [26, Theorem 1.3.4, p17]. □

It is clear from the Definition 3.1.5 that $\mathcal{F}$ is semistable if and only if $\mathcal{F}$ is pure and $p_{max}(\mathcal{F}) = p_{min}(\mathcal{F})$.

Jordan-Hölder filtration.

Now we will define the Jordan-Hölder filtration for semi stable sheaves.

Definition 3.1.7. *Let $\mathcal{F}$ be a semistable sheaf of dimension d. A Jordan-Hölder filtration of $\mathcal{F}$ is a filtration*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset ... \subset \mathcal{F}_l = \mathcal{F},$$

such that the factors $gr_i(\mathcal{F}) = \mathcal{F}_i/\mathcal{F}_{i-1}$ are stable with reduced Hilbert polynomial $p(\mathcal{F})$.

Note that the sheaves $\mathcal{F}_i$, $i > 0$, are also semistable with Hilbert polynomial $p(\mathcal{F})$, and Jordan-Hölder filtration need not be unique.

We have the following theorem:

Theorem 3.1.8. *Jordan-Hölder filtration always exist. Up to isomorphism, the sheaf $gr(\mathcal{F}) := \bigoplus_i gr_i(\mathcal{F})$ does not depend on the choice of the Jordan-Hölder filtration.*

Proof. [26, Theorem 1.5.2, p23]. $\qquad\square$

Definition 3.1.9. *A semistable sheaf $\mathcal{F}$ is called polystable if $\mathcal{F}$ is the direct sum of stable sheaves.*

3.1.1 Stability and vanishing theorems.

Throughout this chapter, unless specified otherwise we assume that picard number of X is 1, and $H = \mathcal{O}_X(1)$ be the ample generator of $Pic(X)$. Let s be the index of X, *i.e.*, the canonical line bundle $K_X = \mathcal{O}_X(-s)$, $s \in \mathbb{Z}$. We remark that the stability of the cotangent bundle of X is implied by the vanishing of some Hodge cohomologies twisted by appropriate powers of the ample class $\mathcal{O}_X(1)$. This can be seen as follows. Suppose $S \subset \Omega_X^1$ is a coherent subsheaf of rank a and $\bigwedge^a S = \mathcal{O}_X(k)$, for some integer k. We have $\mathcal{O}_X(k) \subset \Omega_X^a$. The inclusion of sheaves gives a non trivial section of $\Omega_X^a \otimes \mathcal{O}_X(-k)$. The stability of the the cotangent bundle will hold if we have the following vanishing:

$H^0(X, \Omega_X^a \otimes \mathcal{O}_X(-k)) = 0$, for $0 < a < n$, and $k \geq a.\frac{-s}{n}$.

Assume that D is a smooth divisor from the linear system $|\mathcal{O}_X(d)|$. Consider the following exact sequences of sheaves on X and D respectively:

$$0 \to \Omega_X^q(t) \to \Omega_X^q(t+d) \to \Omega_{X|D}^q(t+d) \to 0$$

and

$$0 \to \Omega_D^q(t) \to \Omega_{X|D}^{q+1}(t+d) \to \Omega_D^{q+1}(t+d) \to 0.$$

We have the following key lemma which is useful further computations in this chapter.

Lemma 3.1.10. *The composition of appropriate maps on cohomology of the above sequences:*

$$H^{p-1}(X, \Omega_X^{q-1}) \to H^{p-1}(D, \Omega_{X|D}^{q-1}) \to H^{p-1}(D, \Omega_D^{q-1}) \to H^{p-1}(D, \Omega_{X|D}^q(d)) \to H^p(X, \Omega_X^q)$$

is cupping with $c_1(\mathcal{O}(d))$ (and thus is an isomorphism for $p + q < n + 1$).

Proof. [56, Lemma 1.2] $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

3.1.2 Stability of tangent bundle of a Fano manifold.

Now we shall recall the definition of Fano manifold and some well-known results related to stability of tangent bundle of a Fano manifold.

Definition 3.1.11. *A smooth projective variety X over $\mathbb{C}$ is called Fano if its anti canonical divisor $-K_X$ is ample.*

Theorem 3.1.12. *If X is a Del-Pezzo surface, then X has a stable tangent bundle TX, unless X is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, or $\mathbb{P}^2$ blown-up in a point.*

Proof. See [62] and [13]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Theorem 3.1.13. *Let X is a Fano 3-fold with $b_2 = 1$. Then the tangent bundle of X is stable.*

Proof. [60, Corollary 2.4, p.638]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Theorem 3.1.14. *Let X be a Fano 4-fold with $b_2 = 1$. Then the tangent bundle of X is stable.*

Proof. [56, Corollary 2.10, p.15]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

In the case of Fano 5-fold X, Peternell and Wisniewski proved stability of tangent bundle of Fano 5-fold except the case of index 2.

Theorem 3.1.15. *Let X be a Fano n-folds with Picard number 1. Then*
(a) TX is stable if $n = 5$.
(b) TX is semistable if $n = 6$.

Proof. [28, Theorem 2 and 3, p.605]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

3.1.3 Logarithmic De Rham sheaves.

In this subsection we shall define the logarithmic de Rham sheaves $\Omega_X(log\, D)$.

Let $D \subset X$ be a smooth irreducible divisor, that is, D does not contain multiple components.

Definition 3.1.16. *([58]) A meromorphic a-form α, $a \geq 0$, on X is called logarithmic a-form along a divisor D if both α and $d\alpha$ have at most simple poles along D.*

Meromorphic a-forms with logarithmic poles along D form a sheaf denoted by $\Omega_X^a(log\, D)$. More precisely, suppose $p \in X$ and $h_1, h_2, ..., h_n$ be local coordinates for X such that D is defined by $h_1 = 0$. We have

$$\Omega_{X,p}^1(log\, D) = \mathcal{O}_{X,p}\langle \frac{dh_1}{h_1}, dh_2, ..., dh_n \rangle$$

and $\Omega_X^a(log\, D) := \bigwedge^a \Omega_X^1(log\, D)$. In particular, $\Omega_X^0(log\, D) = \mathcal{O}_X$.

In the case of normal crossing divisors logarithmic forms are behaved well. Suppose $D = \sum_{i=1}^r D_i$ is a normal crossing divisor, i.e., D_i intersects D_j transversally, for $i \neq j$. Let $h_1, h_2, ..., h_n$ be local coordinates for X such that D_i is defined locally by $h_i = 0$, for $1 \leq i \leq r$, and $r \leq n$. In this case $\Omega_X^a(log\, D)$ is a locally free sheaf. More precisely, for any a-form $\alpha \in \Omega_X^a(log\, D)$ can be written locally as

$$\alpha = \sum_{1 \leq k_1 < ... < k_a \leq n} h_{k_1...k_a}.\delta_{k_1} \wedge ... \wedge \delta_{k_n}$$

where

$$\delta_i = \begin{cases} \frac{dh_i}{h_i} & i \leq r \\ dh_i & i > r \end{cases}$$

Consider the usual residue exact sequences [12, Properties 2.3.,p.13]:

$$0 \to \Omega_X \to \Omega_X(log\, D) \to \oplus_{i=1}^r \mathcal{O}_{D_i} \to 0, \tag{3.1.1}$$

and

$$0 \to \Omega_X^a(log\,(D - D_1)) \to \Omega_X^a(log\, D) \to \Omega_{D_1}^{a-1}(log\,(D - D_1)_{|D_1}) \to 0. \tag{3.1.2}$$

See [12, 2.2] for more details.

3.1.4 Slope of logarithmic De Rham sheaves.

Let $D_i \in |\mathcal{O}_X(k_i)|$, for some positive integers k_i, for $1 \leq i \leq r$. Consider the short exact sequence of sheaves on X

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0.$$

Since the first Chern class c_1 is additive over exact sequences, we have the equality:

$$c_1(\mathcal{O}_D) = -c_1(\mathcal{O}_X(-D)) = c_1(\mathcal{O}_X(D)). \tag{3.1.3}$$

Using the additivity of c_1 and from Equation (3.1.1), we have

$$
\begin{aligned}
c_1(\Omega_X(\log D)) &= c_1(\Omega_X) + c_1\left(\bigoplus_i \mathcal{O}_{D_i}\right) \\
&= c_1(\Omega_X) + \sum_i c_1(\mathcal{O}(D_i)), \text{ using (3.1.3)} \\
&= c_1(\Omega_X) + c_1\left(\mathcal{O}_X\left(\sum_i k_i\right)\right).
\end{aligned}
$$

The first Chern class modulo the rank, of the sheaf $\Omega_X^a(\log D)$ is

$$
\begin{aligned}
\frac{c_1(\Omega_X^a(\log D))}{\binom{n}{a}} &= \frac{\binom{n-1}{a-1}.c_1(\Omega_X(\log D))}{\binom{n}{a}} \\
&= \frac{\binom{n}{a} - \binom{n-1}{a}}{\binom{n}{a}}.c_1(\Omega_X(\log D)) \\
&= \frac{a}{n} c_1(\Omega_X(\log D)).
\end{aligned}
$$

Hence the slope is given as

$$
\begin{aligned}
\mu(\Omega_X^a(\log D)) &= \frac{a}{n} c_1(\Omega_X(\log D)).\mathcal{O}_X(1)^{n-1} \\
&= a.\frac{c_1(\Omega_X) + c_1(\mathcal{O}_X(\sum_i k_i))}{n}.\mathcal{O}_X(1)^{n-1} \\
&= \frac{a.(-s + \sum_{i=1}^{r} k_i)}{n}.\mathcal{O}_X(1)^n.
\end{aligned}
$$

As we have remarked in Subsection 3.1.1, the stability of $\Omega_X(\log D)$ is implied by the vanishing of some Hodge cohomologies. In particular, we have the following lemma.

Lemma 3.1.17. *The stability of $\Omega_X(\log D)$ is implied by the vanishing*

$$
H^0(X, \Omega_X^a(\log D)(-t)) = 0
$$

for $-t \le \frac{a.(s - \sum_{i=1}^{r} k_i)}{n}$ *and* $1 \le a < n$. *Similar assertion is true for semistability when we have strictly inequality in the slope inequality.*

Proof. Suppose there is a subsheaf $\mathcal{F} \subset \Omega_X(\log D)$ of rank $a < n$, destabilizing the sheaf. Then taking determinants, we get a nonzero morphism

$$
det(\mathcal{F}) \to \Omega_X^a(\log D).
$$

Let $det(\mathcal{F}) = \mathcal{O}_X(t)$, for some integer t. Hence the above morphism gives a nonzero section in $H^0(X, \Omega_X^a(\log D))$. The slope condition says that

$$
t > \frac{a.(-s + \sum_i k_i)}{n}
$$

Hence semistability is implied by the vanishing

$$
H^0(X, \Omega_X^a(\log D)(-t)) = 0, \text{ whenever } -t < \frac{a.(s - \sum_i k_i)}{n}.
$$

$\square$

3.2   Stability when the sheaf $K_X + \mathcal{O}_X(D)$ is non-negative.

In this section, we proceed to investigate the stability of $\Omega_X(\log D)$ under suitable assumptions on the canonical class with respect to the divisor D. More precisely, we prove Theorem (1.1) of the Chapter 1.

Theorem 3.2.1. *Suppose X is a smooth projective variety of dimension n over $\mathbb{C}$, with the Picard group $Pic(X) = \mathbb{Z}$. Let $D = \sum_{i=1}^{r} D_i$ be a simple normal crossing divisor on X, where $D_i \in |\mathcal{O}_X(k_i)|$, for some positive integers k_i, for $1 \le i \le r$. If $K_X + \mathcal{O}_X(\sum_{i=1}^{r} k_i)$ is ample or trivial, then $\Omega_X(\log D)$ is semistable.*

It suffices to prove vanishing of relevant cohomologies as indicated in Lemma 3.1.17.

We first prove the following vanishing. This is well-known and due to Norimatsu [50]. For the sake of completeness, we provide a simpler proof:

Lemma 3.2.2. *Suppose $(Y, \mathcal{O}_Y(1))$ is a smooth projective variety of dimension n. Let $D \subset Y$ be a normal crossing divisor and D is written as $\sum_{i=1}^{r} D_i$. Then for $t < 0$,*

$$H^0(Y, \Omega_Y^a(\log D)(t)) = 0.$$

Proof. We prove this by using induction on the number of components r of the divisor D.

We start with the case $r = 1$.

Consider the residue sequence

$$0 \to \Omega_Y^a \to \Omega_Y^a(\log D) \to \Omega_D^{a-1} \to 0.$$

Tensor with $\mathcal{O}(t)$, $t < 0$, and take the long exact cohomology sequence:

$$0 \to H^0(Y, \Omega_Y^a(t)) \to H^0(Y, \Omega_Y^a(\log D)(t)) \to H^0(D, \Omega_D^{a-1}(t)) \to \dots$$

Since $t < 0$, by Kodaira-Akizuki-Nakano theorem [12, 1.3,p.4], the first and the third cohomology groups vanish . This implies the middle cohomology $H^0(Y, \Omega_Y^a(\log D)(t))$ also vanishes.

Now assume that the lemma holds for divisors with at most $r-1$ components. Consider the residue sequence (3.1.2) and tensor with $\mathcal{O}(t)$, for $t < 0$. Now take the associated cohomology sequence

$$0 \to H^0(Y, \Omega_Y^a(\log(D-D_1))(t)) \to H^0(Y, \Omega_Y^a(\log D)(t)) \to H^0(D_1, \Omega_{D_1}^{a-1}(\log(D-D_1)_{|D_1})(t)) \to \dots$$

By induction hypothesis applied to $D-D_1$ on Y and D_1, we deduce the vanishing of the middle cohomology as required.

$\square$

The proof of Theorem 3.2.1 is a corollary of above lemma. Indeed, by Lemma 3.1.17, it suffices to check the vanishing

$$H^0(X, \Omega_X^a(\log D)(t)) = 0$$

for $t < \frac{a.(s - \sum_{i=1}^r k_i)}{n}$ and $1 \leq a < n$. Recall that $K_X = \mathcal{O}_X(-s)$, s is an integer.

The assumption on $K_X + D$ being ample or trivial implies that $s \leq \sum_{i=1}^r k_i$. Hence the slope condition $t < \frac{a.(s - \sum_{i=1}^r k_i)}{n}$ implies $t < 0$. Now by Lemma 3.2.2 we conclude the theorem.

3.3 Stability on Kawamata's finite coverings

In this section, we recall some details concerning branched finite coverings of a complex projective variety, and investigate stability of the logarithmic de Rham sheaves on the covering variety. Note that the Picard group of such coverings can be bigger than $\mathbb{Z}$. Hence it is of interest to look at such cases.

We begin by recalling Kawamata's covering construction:

Proposition 3.3.1. *Let $(Y, \mathcal{O}_Y(1))$ be a nonsingular projective variety of dimension n. Let $D = \sum_{i=1}^r D_i$ be a simple normal crossing divisor on Y and $D_i \in |\mathcal{O}_Y(k_i)|$, for some positive integers k_i. Then there is a smooth variety X together with a finite flat morphism $\pi : X \to Y$ such that $\pi^* D_i = k_i.D_i'$, for some divisors D_i' on X such that $D' = \sum_{i=1}^r D_i'$ is a normal crossing divisor on X. Furthermore, the canonical class $K_X = \pi^*(K_Y \otimes \mathcal{O}_Y(D))$.*

Proof. See [36, 4.1.6, 4.1.12]. □

Now we investigate the semistability of the logarithmic de Rham sheaves on the covering variety. More precisely, we prove Proposition (1.2) of the Chapter 1.

Proposition 3.3.2. *We keep notations as in Proposition 3.3.1 for the covering variety $\pi : X \to Y$. Assume that $Pic(Y) = \mathbb{Z}.\mathcal{O}_Y(1)$ and $k := \sum_{i=1}^r k_i$. If $K_Y + \mathcal{O}_Y(k)$ is ample or trivial then $\Omega_X(\log D')$ is semistable.*

Proof. Since $K_Y + \mathcal{O}(k)$ is ample or trivial, by Theorem 3.2.1, the sheaf $\Omega_Y(\log D)$ is semistable. By the generalized Hurwitz formula [12, Lemma 3.21, p.33], we have

$$\Omega_X(\log D') \simeq \pi^* \Omega_Y(\log D).$$

Now by [42, Lemma 1.17,p. 325], we deduce that the pullback sheaf $\Omega_X(\log D')$ is also semistable, with respect to the ample line bundle $\pi^* \mathcal{O}_Y(1)$. □

b) $n = 3$ and $s \leq 4$

c) $n = 4$ and $s \leq 5$ Now in the next section, we investigate the situation when the class $K_X + D$ is anti-ample.

3.4 Log Fano manifolds of small dimensions

The last section of this chapter we shall prove the Theorem (1.3) of the Chapter 1.

A pair (X, D) is a called a log Fano n-fold if the class $-K_X - D$ is ample. Here $D = \sum_i D_i$ is a normal crossing divisor, D_i are smooth irreducible divisors.

Assume that $Pic(X) = \mathbb{Z}.H$ and the anti canonical class is $-K_X = s.H$ and $D \in |k.H|$, for some $s, k > 0$. Hence the assumption on ampleness of $-K_X - D$ implies that $s > k$. In particular $s \geq 2$.

In this section we would like to discuss stability for possible cases, when n is small.

- $n = 2$

Here (X, D) is a Del Pezzo surface with $Pic(X) = \mathbb{Z}.H$. By Fujita's classification theorem [41, p.87], the following cases for (X, D) occur:

a) $(\mathbb{P}^2, H)$, where H is a line on $\mathbb{P}^2$.

b) $(\mathbb{P}^2, H_1 + H_2)$, where H_1, H_2 are lines on $\mathbb{P}^2$.

c) $(\mathbb{P}^2, Q)$, where Q is a conic in $\mathbb{P}^2$.

- $n = 3$

Here (X, D) is log Fano threefold with $Pic(X) = \mathbb{Z}.H$. By Maeda's classification [41, §6,p.95] according to the index s, the following cases occur:

a) **s = 4** and $X = \mathbb{P}^3$. Here D is equivalent to $H, 2H$ or $3H$. Hence we have,

1) $(\mathbb{P}^3, D)$, where D is a smooth cubic surface.

2) $(\mathbb{P}^3, D)$, where $D = D_1 + D_2$, and D_1 is a smooth quadric surface and D_2 is a plane.

3) $(\mathbb{P}^3, D)$, where $D = D_1 + D_2 + D_3$, and each D_i is a plane.

4) $(\mathbb{P}^3, D)$, where D is a smooth quadric surface.

5) $(\mathbb{P}^3, D)$, where $D = D_1 + D_2$, and each D_i is a plane.

6) $(\mathbb{P}^3, D)$, where D is a plane.

b) **s = 3** and $X = Q$, a smooth quadric threefold in $\mathbb{P}^4$. Here D is equivalent to H or $2H$. Hence we have,

1) (Q, D), where D is a smooth quartic surface in $\mathbb{P}^4$.

2) (Q, D), where $D = D_1 + D_2$ and each D_i is a smooth quadric surface.

3) (Q, D), where D is a smooth quadric surface.

c) **s = 2**. There are five different types of Fano 3-folds and D is a smooth irreducible divisor in the linear system $|H|$.

- $n = 4, 5, 6$ Here the possibilities are more and we refer to [14].

Note that the dual of a stable bundle is again stable, it suffices to prove that the cotangent bundle Ω_X^1 of the Fano manifold X is stable.

We can now state the main result of this section.

We will need the following result in the proof.

Lemma 3.4.1. *Suppose $(Y, \mathcal{O}(1))$ is a smooth projective variety of dimension n, and D is a smooth irreducible divisor in $|\mathcal{O}(k)|$. Fix $q < n - 1$. Then the restriction map*

$$H^0(Y, \Omega_Y^q(c)) \to H^0(D, \Omega_D^q(c))$$

is surjective, for all $c < k$.

Proof. See proof of [56, Lemma 2.9 a)]. □

Proposition 3.4.2. *Suppose (X, D) is a log Fano manifold of dimension n and $Pic(X) = \mathbb{Z}.H$. Let $K_X = \mathcal{O}_X(-s)$ and $D \in |\mathcal{O}_X(k)|$ such that $s, k > 0$.*

Assume one of the following holds:

a) $n = 2$ and $s = 3$,

b) $n = 3$ and $s \leq 4$

c) $n = 4$ and $s \leq 5$

d) $n = 5$ and $s \leq 6$ such that $s = 2, 5, 6$ or $(s, k) = (3, 2), (4, 3)$.

e) $n = 6$ and $s \leq 7$ such that $s \leq 4$, $s = 6, 7$, or $(s, k) = (5, 4), (5, 3)$.

If D is smooth and irreducible then the logarithmic cotangent bundle $\Omega_X(\log D)$ is semistable.

Proof. Suppose D is a smooth and irreducible divisor. Note that the ampleness of $-K_X - D$ implies that $s > k$.

From Lemma 3.1.17, the semistability of $\Omega_X(\log D)$ is implied by the vanishing

$$H^0(X, \Omega_X^a(\log D)(t)) = 0, \tag{3.4.4}$$

for $t < \frac{a.(s-k)}{n}$ and $1 \leq a < n$.

Recall the residue exact sequence;

$$0 \to \Omega_X^a(t) \to \Omega_X^a(\log D)(t) \to \Omega_D^{a-1}(t) \to 0.$$

Taking the global sections, we have the long exact sequence

$$0 \to H^0(X, \Omega_X^a(t)) \to H^0(X, \Omega_X^a(\log D)(t)) \to H^0(D, \Omega_D^{a-1}(t)) \to$$

$$H^1(X, \Omega_X^a(t)) \to H^1(X, \Omega_X^a(\log D)(t)) \to \dots$$

Then to prove the vanishing (3.4.4), it suffices to check that

$$H^0(X, \Omega_X^a(t)) = 0 \tag{3.4.5}$$

and the map

$$H^0(D, \Omega_D^{a-1}(t)) \to H^1(X, \Omega_X^a(t)) \tag{3.4.6}$$

is injective, whenever $t < \frac{a.(s-k)}{n}$ and $1 \leq a < n$.

We now look at the cases listed above, according to the dimension n.

a) $\mathbf{n = 2}$ and $\mathbf{s = 3}$.

By §3.4, the only possibility is $(X, D) = (\mathbb{P}^2, D)$, where D is a line or a conic in $\mathbb{P}^2$.

But for $X = \mathbb{P}^2$, $H^0(X, \Omega_X(t)) = 0$ for $t \leq 1$. Hence for $\frac{s-k}{2} = \frac{3-k}{2} \leq 1$, $k = 1, 2$, the vanishing (3.4.5) holds. When $t < 0$, then clearly $H^0(D, \mathcal{O}_D(t)) = 0$.

When $t = 0$, then by the hard Lefschetz theorem and the cupping map (for instance see Lemma 3.1.10) gives the injectivity of (3.4.6).

Hence $\Omega_{\mathbb{P}^2}(log\, D)$ is semistable.

b) $\mathbf{n = 3}$ and $\mathbf{s \leq 4}$.

Since X is a Fano 3-fold, by [60, 2.4, p.638], we have the stability of Ω_X. Therefore, by Maruyama's result [42, 2.6.1], Ω_X^a is semistable. Using the slope inequality in Lemma 3.1.17, we deduce that

$$H^0(X, \Omega_X^a(t)) = 0, \text{ for } t < \frac{a.s}{3}, \tag{3.4.7}$$

and when $a = 1$, the vanishing holds for $t \leq \frac{s}{3}$.

On the other hand, $\dfrac{a.(s-k)}{3} < \dfrac{a.s}{3}$ and this verifies (3.4.5).

Now we proceed to check (3.4.6) below.

Since $-K_D = \mathcal{O}_D(s-k)$ is ample, D is a Del Pezzo surface. But $Pic(D)$ can be greater than $\mathbb{Z}$, hence semistability of Ω_D does not always hold. Hence we argue as follows.

Since $0 < k < s \leq 4$ and $1 \leq a < 3$, the possible values for a are 1, 2 and the possible values for k and s are:

if $k = 1$, then $s = 2, 3, 4$.

if $k = 2$, then $s = 3, 4$.

if $k = 3$, then $s = 4$.

If $t < 0$, then the required vanishing of $H^0(D, \Omega_D^a(t))$, follows from Kodaira-Akizuki-Nakano theorem.

Suppose $a = 1$ and we have $0 \leq t < \dfrac{(s-k)}{3}$.

If we substitute the respective values of k and s in above range then the only possible value is $t = 0$. The required Hodge vanishing holds because both X and D are Fano manifolds.

Supose $a = 2$ and we have $0 \leq t < \dfrac{2 \cdot (s-k)}{3}$. In this case the only possible values are $t = 0, 1$ when $(k, s) = (1, 3)$ and $(2, 4)$. We deduce that it is sufficient to prove, when $t = 1$,

$$H^0(D, \Omega_D(1)) = 0$$

and when $t = 0$,

$$H^0(D, \mathcal{O}_D) \longrightarrow H^1(X, \Omega_X)$$

is injective.

Suppose $t = 1$.

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First consider the case when $(k,s) = (2,4)$. Then by §3.4 a) 4), $(X,D) = (\mathbb{P}^3, D)$, where D is a smooth quadric surface. Using Lemma 3.4.1, we deduce that the restriction map

$$H^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(1)) \to H^0(D, \Omega_D(1))$$

is surjective.

But we noticed in (3.4.7) or it also follows from [8], that $H^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}(1)) = 0$. Hence $H^0(D, \Omega_D(1)) = 0$.

When $(k,s) = (1,3)$, then by §3.4 b)3), $(X,D) = (Q,H)$, where Q is a smooth quadric three-fold and H is a hyperplane section. Hence D is again a quadric surface and $H^0(D, \Omega_D(1)) = 0$.

On the other hand for $t = 0$, by Lemma 3.1.10, the required injectivity follows.

Hence $\Omega_X(\log D)$ is semi-stable.

c) $\mathbf{n = 4}$ and $\mathbf{s \le 5}$.

Since X is a Fano 4-fold with $Pic(X) = \mathbb{Z}$, by [56, 2.10,p.15], Ω_X is stable. Therefore, by Maruyama's result the exterior powers are semistable and by Lemma 3.1.17, we have

$$H^0(X, \Omega_X^a(t)) = 0, \text{ for } t < \frac{a.s}{4}. \tag{3.4.8}$$

On the other hand, $\dfrac{a.(s-k)}{4} < \dfrac{a \cdot s}{4}$ and we have $H^0(X, \Omega_X^a(t)) = 0$ for $t < \dfrac{a \cdot (s-k)}{4}$.

Since $-K_D = \mathcal{O}_D(s-k)$ is ample, D is a Fano 3-fold with $Pic(D) = \mathbb{Z}.H_{|D}$ (by Lefschetz hyperplane section theorem). Hence Ω_D is stable [60, 2.4,p.638].

Therefore, again by Maruyama's result we have the semistability of its exterior powers and by Lemma 3.1.17,

$$H^0(D, \Omega_D^{a-1}(t)) = 0, \text{ for } t < \frac{(a-1).(s-k)}{3}.$$

Since $\dfrac{(a-1).(s-k)}{3} < \dfrac{a.(s-k)}{4}$, we only have to discuss the situation

$$\frac{(a-1).(s-k)}{3} \le t < \frac{a.(s-k)}{4}.$$

Since $0 < k < s \le 3$ and $1 \le a < 4$, the possible values for a are $1, 2, 3$ and the possible values for k and s are:

if $k = 1$, then $s = 2, 3$,

if $k = 2$, then $s = 3$.

If we substitute the respective values of k, s and a in $\dfrac{(a-1).(s-k)}{3} \le t < \dfrac{a.(s-k)}{4}$ then the only possible value which remains is $t = 0$ and when $a = 1$.

Therefore, as before injectivity of (3.4.6) follows from Lemma 3.1.10.

Suppose $\mathbf{s = 4}$, then we note that we need vanishing of only $H^0(D, \Omega_D^{a-1}(t))$, if $a = 2$ and $k = 1$. In this case X is a smooth quadric 4-fold and D is a smooth quadric threefold in $\mathbb{P}^3$. Hence we can apply Lemma 3.4.1, to get the desired vanishing.

Suppose $\mathbf{s = 5}$, then $X = \mathbb{P}^4$. We note that we need to check vanishing of $H^0(D, \Omega_D^{a-1}(t))$ only when $a = 2$, $k = 2, t = 1$ and when $a = 3$, $k = 2$, $t = 2$. In this case, D is a smooth quadric threefold. Both these vanishings follow from [59, Theorem (1), p.174].

Hence $\Omega_X(\log D)$ is semi-stable.

d) $\mathbf{n = 5}$ and $\mathbf{s \leq 6}$.

Since X is a Fano 5-fold, Ω_X is stable [28, Theorem 2,p.605]. Therefore, by Maruyama's result,
$$H^0(X, \Omega_X^a(t)) = 0, \text{ for } t < \frac{a.s}{5}.$$

Note that D is a Fano fourfold with $Pic(D) = \mathbb{Z}.H_{|D}$.

If $t \leq 0$ and except when $a = 1, t = 0$, then by Kodaira-Nakano vanishing theorem and by rational connectedness of D,
$$H^0(D, \Omega_D^{a-1}(t)) = 0.$$

So when $t \leq 0$, we have the desired vanishing $H^0(X, \Omega_X^a(\log D)(t)) = 0$.

Suppose $t = 0$ and $a = 1$ then by Lemma 3.1.10, the injectivity of (3.4.6) follows.

As in the previous case, we need to look at the case:
$$\frac{(a-1)(s-k)}{4} \leq t < \frac{a.(s-k)}{5}$$

to obtain vanishing of $H^0(D, \Omega_D^{a-1}(t))$.

We note that we need to check the following cases only:

1) $(s = 3, k = 1, a = 3, t = 1)$,
2) $(s = 4, k = 1, a = 2, t = 1)$,
3) $(s = 4, k = 2, a = 3, t = 1)$
4) $(s = 5, k = 1, a = 2, t = 1)$
5) $(s = 5, k = 1, a = 3, t = 2)$
6) $(s = 5, k = 1, a = 4, t = 3)$
7) $(s = 5, k = 2, a = 2, t = 1)$
8) $(s = 5, k = 3, a = 3, t = 1)$
9) $(s = 6, k = 2, a, t = a - 1)$
10) $(s = 6, k = 3, a = 2, t = 1)$
11) $(s = 6, k = 4, a = 3, t = 1)$.

We check that in 4),5),6), D is a smooth quadric hypersurface in $\mathbb{P}^5$. Hence the vanishing $H^0(D, \Omega_D^{a-1}(a-1)) = 0$ holds by [59, Theorem (1),p. 174]. Similarly, 9) also hold because D is a smooth quadric hypersurface in $\mathbb{P}^6$. We again use Snow's theorem and apply Lemma 3.4.1 to get the required vanishing on D, in case of 8), 10), and 11).

The remaining cases are not known to us.

Hence $\Omega_X(\log D)$ is semi-stable in the cases claimed.

e) $\mathbf{n = 6}$ and $\mathbf{s \leq 7}$.

Since X is a Fano 6-fold, by [28, Theorem 3,p.605], Ω_X is semi-stable. Therefore, by Maruyama's result

$$H^0(X, \Omega_X^a(t)) = 0 \text{ for } t < \frac{a.s}{6}.$$

On the other hand, $\dfrac{a.(s-k)}{6} < \dfrac{a.s}{6}$ we have

$$H^0(X, \Omega_X^a(t)) = 0, \text{ for } t < \frac{a.(s-k)}{6}.$$

Since $-K_D = \mathcal{O}_D(s-k)$ is ample, D is a Fano 5-fold with $Pic(D) = \mathbb{Z}.H_{|D}$ and hence Ω_D is stable [28, Theorem 2,p.605]. Therefore, again by Maruyama's result we have

$$H^0(D, \Omega_D^{a-1}(t)) = 0, \text{ for } t < \frac{(a-1).(s-k)}{5}.$$

Since $\dfrac{(a-1).(s-k)}{5} < \dfrac{a.(s-k)}{6}$, so we have only to discuss the situation

$$\frac{(a-1).(s-k)}{5} \le t < \frac{a.(s-k)}{6}.$$

Since $0 < k < s \le 4$ and $1 \le a < 6$, the possible values for a are 1, 2, 3, 4, 5 and the possible values for k and s are:

if $k = 1$, then $s = 2, 3, 4$,

if $k = 2$, then $s = 3, 4$,

if $k = 3$, then $s = 4$.

Suppose $a = 1$ we have $0 \le t < \dfrac{(s-k)}{6}$.

If we substitute the respective values of k and s in above range then the only possible value is $t = 0$. In this case it is enough to show $H^0(D, \mathcal{O}_D) \longrightarrow H^1(X, \Omega_X^1)$ is injective. But this follows from the cupping map Lemma 3.1.10.

Suppose $a = 2$ then we have $\dfrac{(s-k)}{5} \le t < \dfrac{2.(s-k)}{6}$. In this case t does not exist.

Suppose $a = 3, 4$, and if we substitute the respective values of k, s and a in

$$\frac{(a-1).(s-k)}{5} \le t < \frac{a.(s-k)}{6}.$$

Then the possible value is $t = 1$ when

- $a = 3$ and $(k, s) = (1, 4)$,
- $a = 4$ and $(k, s) = (1, 3), (2, 4)$.

In this case it is enough to show

$$H^0(D, \Omega_D^2(1)) = 0 \text{ when } (k, s) = (1, 4)$$

and

$$H^0(D, \Omega_D^3(1)) = 0 \text{ when } (k, s) = (1, 3), (2, 4).$$

But both the claims follows from stability of Ω_D.

Suppose $a = 5$ then we have $\dfrac{4.(s-k)}{5} \leq t < \dfrac{5 \cdot (s-k)}{6}$. If we substitute the respective values of k and s in above range then the only possible value is $t = 2$ when $(k, s) = (1, 4)$.

In this case it is enough to show

$$H^0(D, \Omega_D^4(2)) = 0 \text{ when } (k, s) = (1, 4).$$

But this follows from stability of Ω_D.

The following cases need only to be discussed:

s = 6:

1) $(s = 6, k = 1, a, t = a - 1)$

2) $(s = 6, k = 2, a = 2, t = 1)$

3) $(s = 6, k = 3, a = 2, t = 1)$

4) $(s = 6, k = 4, a = 3, t = 1)$.

s = 7:

5) $(s = 7, k = 2, a, t = a - 1)$

6) $(s = 7, k = 3, a = 2, t = 1)$.

In 1) and 5), we note that D is a smooth quadric hypersurface in $\mathbb{P}^6$ and by [59, Theorem 1, p.174], the vanishing $H^0(D, \Omega_D^{a-1}(a-1)) = 0$ holds. Again by Snow's theorem, and applying Lemma 3.4.1 we deduce the required vanishing in case 1)-4), 6).

Hence $\Omega_X(\log D)$ is semi-stable in the cases claimed.

$\square$

3.4.1 Counterexample when D is reducible

We now investigate the situation when D is reducible and when (X, D) is Del Pezzo surface.

Lemma 3.4.3. *Suppose $(X, D) = (\mathbb{P}^2, D_1 + D_2)$, where D_1, D_2 are lines on $\mathbb{P}^2$. Then $\Omega_{\mathbb{P}^2}(\log D)$ is not semistable.*

Proof. The semistability of $\Omega_{\mathbb{P}^2}(\log D)$ is equivalent to the vanishing (see (3.4.4)):

$$H^0(\mathbb{P}^2, \Omega_{\mathbb{P}^2}(\log D)(t)) = 0$$

for $t < \frac{(3-2)}{2}$, i.e. when $t \leq 0$.

When $t < 0$, this follows from Lemma 3.2.2.

When $t = 0$, we note that the injectivity of map

$$\bigoplus_{i=1,2} H^0(D_i, \mathcal{O}_{D_i}) \to H^1(\mathbb{P}^2, \Omega_{\mathbb{P}^2})$$

(see (3.1.1)) fails. Indeed, here $\bigoplus_{i=1,2} H^0(D_i, \mathcal{O}_{D_i})$ is of rank two and $H^1(\mathbb{P}^2, \Omega_{\mathbb{P}^2})$ is of rank one.

$\square$

Remark 3.4.4. *We suspect that in higher dimensional cases with several divisor components, the semistability may fail.*

Chapter 4

Embedding properties of linear series on hyperelliptic varieties

4.1 Preliminaries on linear systems.

Let X be a smooth projective variety defined over $\mathbb{C}$, and let L be an ample line bundle on X. Note that $H^0(X, L)$ is the finite dimensional vector space of global sections of the line bundle L. Let us recall (Section 2.2.3) the associated rational map:

$$\phi_L : X \to \mathbb{P}^n = P(H^0(X, L)^*),$$

given by

$$x \mapsto \{s \in H^0(X, L) | s(x) = 0\}.$$

One can ask when ϕ_L is a morphism (respectively, embedding). In other words, when ϕ_L is base point free (respectively, very ample).

More generally, we have the following notion of k-jet ampleness.

Definition 4.1.1. *A line bundle L is called k-jet ample, $k \geq 0$, if the restriction map*

$$H^0(L) \to H^0(L \otimes \mathcal{O}_X / m_{x_1}^{k_1} \otimes ... \otimes m_{x_p}^{k_p})$$

is surjective for distinct points $x_1, x_2, ..., x_p \in X$ such that $k_1 + k_2 + ... + k_p = k + 1$.

Note that 0-jet ample is same as global generation, and 1-jet ampleness is same as very ampleness.

Further questions on embedding were studied classically by the Italian school of Geometers. The study was with reference to the existence of trisecants or more generally multisecants to the given embedded variety.

Some geometric notions which evolved were:

Definition 4.1.2. *Let L be a very ample line bundle on X. We say ϕ_L is a projectively normal embedding (or, X is a projectively normal) if the multiplication map*

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) = Sym^r H^0(X, L) \to H^0(X, L^r)$$

is surjective, for each $r \geq 1$. We also called as N_0-property.

Definition 4.1.3. *We say that L satisfies N_1-property if it satisfies N_0-property and the ideal of the embedded variety is generated by quadrics.*

Mark Green ([18],[19]) unified the above concepts and introduced the N_p-property of L, also called the p-th syzygy property of L. For a projective variety X defined over an algebraically closed field k, and an ample line bundle L on X, consider the graded algebra $R_L = \oplus_{h=0}^{\infty} H^0(X, L^{\otimes h})$ over the polynomial ring $S_L = \oplus_{h=0}^{\infty} Sym^h H^0(X, L)$.

Now take a minimal resolution of R_L as a graded S_L-module:

$$0 \to \ldots \to E_p \to \ldots \to E_2 \to E_1 \to E_0 \to R_L \to 0$$

where $E_0 = S_L \oplus_j S_L(-a_{0j})$, $a_{0j} \geq 2$, and for $p \geq 1$, $E_p = \oplus_j S_L(-a_{pj})$, $a_{pj} \geq p + 1$.

Green introduced the following terminology:

• N_0-property of $L \iff E_0 = S_L$, i.e., the embedded variety X is projectively normal.

• N_1-property of $L \iff N_0$-property of L and $a_{1j} = 2$ for any j, i.e., the homogeneous ideal of the embedded variety X is generated by quadrics. In general,

• N_p-property of $L \iff N_{p-1}$-property of L and $a_{pj} = p + 1$ for any j, i.e., the first $(p - 1)$ maps of the resolution of the ideal of X are matrices with linear entries.

4.2   Known results on curves and surfaces.

The initial results on these questions included the case of curves and surfaces. Suppose C is a smooth projective curve of genus g and L is a line bundle on C.

Theorem 4.2.1. *a) (Castelnuova): If the degree of L is at least $2g + 1$ then L satisfies N_0-property.*
b) (Mattuck, Fujita, St.Donat): If the degree of L is at least $2g + 2$ then L satisfies N_1-property

The following theorem is due to Green [19], which generalizes the above results on curves.

Theorem 4.2.2. *Suppose C is a smooth projective curve of genus g and L is a line bundle on C. If the degree of L is at least $2g + 1 + p$ then L satisfies N_p-property.*

In the case of surfaces F. Gallego , B. Purnaprajna done some significant work.

Theorem 4.2.3. *[15] Let X be an Enriques surface over an algebraic closed field of charac-teristic 0. Let L be an ample base-point free line bundle. Then $L^{\otimes p+1}$ satisfies the property N_p, for all $p \geq 1$.*

Theorem 4.2.4. *[16] Let X be a K3 surface and let L be a base-point-free line bundle such that $L^2 \geq 4$. If $n \geq p+1$, then nL satisfies property N_p.*

Theorem 4.2.5. *[16] Let X be a K3 surface and let L be a base-point-free line bundle such that $L^2 \geq 8$ and the general member of $|L|$ is non-hyperelliptic, non-trigonal and not a plane quintic. Then rL satisfies property N_p for all $r \geq p$.*

For higher dimensions, Mukai proposed a generalization of the theorem 4.2.2.

4.2.1   Mukai conjecture for adjoint linear systems.

Conjecture 4.2.6. *Suppose (X, L) is a smooth polarized projective variety and K_X is the canonical class on X. Then $K_X + (p+4)L$ satisfies N_p-property.*

This conjecture is still unsolved. A progress on this conjecture was subsequently given:

Theorem 4.2.7. *[9] Suppose L is a very ample line bundle on a n-dimensional smooth pro-jective variety then $K_X + (n+1+p)L$ satisfies N_p-property.*

A stronger version of the Mukai conjecture in the case of Enriques surfaces and for the property N_0 is proved by Gallego and Purnaprajna. More precisely:

Theorem 4.2.8. *[21, Corollary 2.8, p.156] Let X be an Enriques surface and $L_1, ..., L_n$ ample line bundles on X. Let $L = K_X \otimes L_1 \otimes ... \otimes L_n$. If $n \geq 4$, then L satisfies property N_0.*

They also gave a partial answer in the case of abelian and bielliptic surfaces:

Theorem 4.2.9. *[21, Corollary 4.5, p.167] Let X be an Abelian or a bielliptic surface. Let M be an ample line and $L = K_X \otimes M^{\otimes n}$. If $n \geq 2p+2$ and $p \geq 1$, then L satisfies property N_p. In particular, if $n \geq 4$, L satisfies property N_1.*

4.3   Known results on abelian varieties.

An abelian variety A defined over $\mathbb{C}$ is a compact complex torus $\frac{\mathbb{C}^g}{\Gamma}$ (here $\Gamma \subset \mathbb{C}^g$ is a free abelian group on $2g$ generators) and there is an ample line bundle L on A.

Given a polarized abelian variety (A, L), Mumford([45]) associated certain groups: Let $t_a : A \to A$ be the translation map $x \mapsto a + x$.

• $K(L) = \{a \in A : L \simeq t_a^* L\}$ is called the fixed group of L.

• $\mathcal{G}(L) = \{(a, \phi) : L \simeq^\phi t_a^* L\}$ is called the theta group of L.

The theta group $\mathcal{G}(L)$ acts on $H^0(A, L)$ as follows: if $(a, \phi) \in \mathcal{G}(L)$ and $s \in H^0(A, L)$ then

$$(a, \phi).s = t_{-a}^* \phi(s).$$

Further, $H^0(A, L)$ is the unique irreducible $\mathcal{G}(L)$-module, upto isomorphisms, such that $\alpha \in \mathbb{C}^*$ acts as multiplication by itself ([45, Proposition 3]). These groups have been extensively used in the study of linear series on abelian varieties.

Denote $Pic^0(A)$, the group of line bundles on A whose first chern class is 0. The group $Pic^0(A)$ admits the structure of a complex torus (See [5, Proposition 2.2.1, p35]) in a natural way.

Definition 4.3.1. *[5] A holomorphic line bundle $\mathcal{P}$ on $A \times \hat{A}$ satisfying*
1) $\mathcal{P}|A \times L \simeq L$ for every $L \in \hat{A}$
2) $\mathcal{P}|0 \times \hat{A}$ is trivial
is called a Poincare bundle for A.

Some well-known results on linear series are:

Theorem 4.3.2. *Suppose (A, L) is a polarized abelian variety. Then*
1) (Lefschetz, [37]) nL, $n \geq 3$ is always very ample.
2) (Ohbuchi, [52]) L^2 is very ample if L has no base divisor.
3) (Bauer-Szemberg, [2]) L^{k+1} is k-jet ample for $k \geq 1$, and the same holds for L^{k+1}, $k \geq 1$ if L has no base divisor.
4) (Koizumi, [33]) nL satisfies N_0-property, for $n \geq 3$.
5) (Kempf, [31]) nL satisfies N_1-property, for $n \geq 4$.
6) (Pareschi, [53]) nL satisfies N_p-property, for $n \geq p + 3$.
7) (Iyer, [29]) Suppose (A, L) is a polarized g-dimensional simple abelian variety. If $\dim H^0(A, L) > 2g \cdot g!$, then L gives a projectively normal embedding, for all $g \geq 1$.

These results are not much known in the case of primitive line bundles L.

4.3.1 Primitive line bundles.

Let (A, L) be a polarized abelian variety of dimension g and $\hat{A}$ be its dual abelian variety. The polarization L induces an isogeny

$$\phi_L : A \to \hat{A}, x \mapsto t_x^* L \otimes L^{-1}.$$

The kernel of ϕ_L is of the form $(\oplus_{i=1}^g \mathbb{Z}/d_i\mathbb{Z})^2$ with positive integers $d_1, ..., d_g$ and $d_i | d_{i+1}$ for $i = 1, ..., g - 1$. The vector $(d_1, ..., d_g)$ is called the *type* of the polarization L.

Definition 4.3.3. *A line bundle L on an abelian variety A is said be primitive if L is of type $(1, d_2, ..., d_g)$. That is L is not of the form M^n for some $n \geq 2$ and an ample line bundle M on A.*

Some well-known results on linear series in the case of primitive line bundles are:

Theorem 4.3.4. *[5] Let L be an ample line bundle of type $(1, d)$ on A defining an irreducible polarization. Then L is globally generated if and only if $d \geq 3$ and is very ample if and only if $d \geq 5$ and there is no ellipite curve E on A with $(L.E) = 2$.*

In the case of abelian three fold Ein and Lazarsfeld proved a theorem on global generation of adjoint line bundles. More precisely,

Theorem 4.3.5. *[10] Let L be an ample line bundle of type $(1, d_2, d_3)$ on an abelian threefold A with $d_2.d_3 \geq 5$. Suppose there is no curve $C \subset A$ with $(L.C) \geq 29$ and there is no surface $S \subset A$ with $(L^2.C) \geq 16$. Then L is globally generated.*

The problem related to very ampleness in the case of abelian three folds proved by Birkenhake, Lange and Ramanan. More precisely,

Theorem 4.3.6. *[6] Let (A, L) be a general polarized abelian threefold of type $(1, 1, d)$, $d \geq 13$, $\neq 14$. Then the line bundle L is very ample.*

4.4 Mukai regularity and Continuous global generation:

The notion of Mukai regularity on abelian varieties, is based on Fourier-Mukai transform has been introduced by G.Pareschi and M.Popa ([54], [55]), to obtain the most of the above results.

4.4.1 Fourier-Mukai functor

Suppose A is an abelian variety of dimension g over $\mathbb{C}$ and $\hat{A}$ be its dual abelian variety (See [5, Section 2.4, p.34]). Denote $\mathcal{P}$, the normalized Poincaré line bundle on $A \times \hat{A}$.

Let us recall some facts from [43]:

Denote $\mathcal{C}oh(A)$ (respectively, $\mathcal{C}oh(\hat{A})$), the category of coherent sheaves on A (resp. on $\hat{A}$). Let

$$\hat{\mathcal{S}} : \mathcal{C}oh(A) \to \mathcal{C}oh(\hat{A})$$

be the functor defined as follows:

$$\hat{\mathcal{S}}\mathcal{F} := p_{2*}(p_1^*\mathcal{F} \otimes \mathcal{P}).$$

Similarly we can define the functor

$$\mathcal{S} : \mathcal{C}oh(\hat{A}) \to \mathcal{C}oh(A)$$

given as

$$\mathcal{S}\mathcal{G} := p_{1*}(p_2^*\mathcal{G} \otimes \mathcal{P}).$$

Denote $D(A)$ (respectively $D(\hat{A})$) the derived category of $\mathcal{C}oh(A)$ (respectively $\mathcal{C}oh(\hat{A})$). Then we have a derived functor ([43, Proposition 2.1, p.155]),

$$\mathcal{R}\hat{\mathcal{S}} : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\hat{\mathcal{A}})$$

given by

$$\mathcal{R}\hat{\mathcal{S}}\mathcal{F} = \mathcal{R}p_{2*}(p_1^*\mathcal{F} \otimes \mathcal{P}).$$

Similarly we obtain the derived functor

$$\mathcal{R}\mathcal{S} : \mathcal{D}(\hat{\mathcal{A}}) \to \mathcal{D}(\mathcal{A})$$

These derived functors are called the Fourier-Mukai functor.

4.4.2 Mukai-regularity

Now we recall the notion of of I.T (index theorem) and M-regularity from [43].

With notations as in previous subsection, denote $R^j\hat{\mathcal{S}}(\mathcal{F})$, the cohomologies of the derived complex $R\hat{\mathcal{S}}\mathcal{F}$. A coherent sheaf $\mathcal{F}$ on A satisfies W.I.T (the *weak index theorem*) with index i if $R^j\hat{\mathcal{S}}(\mathcal{F}) = 0$, for all $j \neq i$.

A stronger notion is as below.

Definition 4.4.1. *A coherent sheaf $\mathcal{F}$ on A is said to satisfy I.T (index theorem) with index i if $H^j(\mathcal{F} \otimes \alpha) = 0$, for all $\alpha \in \hat{A}$ and for all $j \neq i$.*

In this situation the sheaf $R^i\hat{\mathcal{S}}(\mathcal{F})$ is locally free. If $\mathcal{F}$ satisfies W.I.T or I.T. with index i, then the sheaf $R^i\hat{\mathcal{S}}(\mathcal{F})$ is denoted by $\hat{\mathcal{F}}$ and is called the *Fourier transform* of $\mathcal{F}$.

In particular, a sheaf $\mathcal{F}$ is said to satisfy index theorem (I.T) with index 0 if

$$H^i(\mathcal{F} \otimes \alpha) = 0, \forall \alpha \in \mathrm{Pic}^0(A), \forall i > 0.$$

Given a coherent sheaf $\mathcal{F}$ on A, we denote the support of the sheaf $R^i\hat{\mathcal{S}}(\mathcal{F})$ by

$$S^i(\mathcal{F}) := \mathrm{Supp}(R^i\hat{\mathcal{S}}(\mathcal{F})).$$

Definition 4.4.2. *A coherent sheaf $\mathcal{F}$ on A is called M-regular if*

$$codim\ S^i(\mathcal{F}) > i$$

for each $i = 1, ..., g$.

Remark 4.4.3. *1) Coherent sheaves on A which satisfy I.T with index 0, are examples of M-regular sheaves.*

2) Note that an ample line bundle H on A satisfies I.T with index 0 [54, Example 2.2, p.289].

Denote the cohomological support locus [22] by:

$$V^i(\mathcal{F}) := \{\eta \in \mathrm{Pic}^0(A) : h^i(\mathcal{F} \otimes \eta) \neq 0\} \subset \mathrm{Pic}^0(A).$$

Note that there is an inclusion $S^i(\mathcal{F}) \subset V^i(\mathcal{F})$.

Hence a sheaf is M-regular if

$$\mathrm{codim}(V^i(\mathcal{F})) > i \tag{4.4.1}$$

for any $i = 1, ..., g$.

The notion of M-regularity has significant geometric consequences via global generation of suitable sheaves. This will be illustrated in the next section.

The main result about M-regularity is the following:

Theorem 4.4.4. *[54, Theorem 2.4, p289]. Let $\mathcal{F}$ be a coherent sheaf and L an invertible sheaf supported on a subvariety Y of the abelian variety X (possibly X it self). If both $\mathcal{F}$ and L are M-regular as a sheaves on X, then $\mathcal{F} \otimes L$ is globally generated.*

To prove this theorem Pareschi and Popa introduced an intermediate notion, called continuous global generation.

Definition 4.4.5. *A coherent sheaf $\mathcal{F}$ on an irregular variety Y is called continuously globally generated if for any nonempty open set $U \subseteq Pic^0(Y)$ the sum of evaluation maps*

$$\bigoplus_{\alpha \in U} H^0(\mathcal{F} \otimes \alpha) \otimes \check{\alpha} \xrightarrow{\oplus ev} \mathcal{F}$$

is surjective.

Note that M-regularity implies continuous global generation, see [54, Theorem 2.13, p293].

We have the following intermediate result to prove Theorem 4.4.4.

Theorem 4.4.6. *[54, Theorem 2.4, p292]. Let Y be a subvariety of an irregular variety X, $\mathcal{F}$ a coherent sheaf and L a line bundle on Y, both continuously globally generated as sheaves on X. Then $\mathcal{F} \otimes L$ is globally generated.*

4.5 Main theorems on hyperelliptic varieties.

In this section, we will prove some of the above results on hyperelliptic varieties.

Definition 4.5.1. *[34] A smooth projective variety X is called a hyperelliptic variety if it is not isomorphic to an abelian variety but admitting an étale covering $A \to X$, where A is an abelian variety.*

Note that by [34, Theorem 1.1, p.492], there is a finite group G acting biholomorphically on A, without fixed points. In other words, we can write X as a group quotient $X = A/G$, with an étale quotient morphism
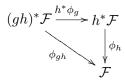
$$\pi : A \to X = A/G.$$

To investigate coherent sheaves on X, we note that their pullback on A under the morphism π, is equipped with an action of the group G. Hence to investigate line bundles and more generally coherent sheaves on X, it would suffice to investigate coherent sheaves on A with a G-action. To make this more precise, we recall the following facts.

4.5.1 G-linearized sheaves

Suppose A is an abelian variety and is equipped with an action by a finite group G. In this subsection, we recall G-linearized sheaves on an abelian variety A.

Definition 4.5.2. *[49, Definition 1.6, p.30]. A coherent sheaf $\mathcal{F}$ on A is called G-linearized (or a G-sheaf) if we have an isomorphism $\phi_g : g^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$, for all $g \in G$, and such that the following diagram of coherent sheaves on A*

$$(gh)^*\mathcal{F} \xrightarrow{h^*\phi_g} h^*\mathcal{F}$$
$$\phi_{gh} \searrow \quad \downarrow \phi_h$$
$$\mathcal{F}$$

is commutative, for any pair $g, h \in G$, i.e. $\phi_{gh} = \phi_h \circ h^*\phi_g$.

Assume that the action of the group G on A is free. We note that G-linearized sheaves are relevant to our situation, since it corresponds to coherent sheaves on the quotient variety A/G. In fact, we have:

Proposition 4.5.3. *Consider a pair (A, G) as above, and assume that the action of G on A is free. Then the functor $\mathcal{F} \mapsto \pi^*\mathcal{F}$ is an equivalence of category of coherent $\mathcal{O}_X$-modules on X and the category of coherent G-sheaves on A. The inverse functor is given by $\mathcal{G} \mapsto (\pi_*(\mathcal{G}))^G$ (the subsheaf of G-invariant sections of $\pi_*(\mathcal{G})$). Locally free sheaves correspond to locally free sheaves of the same rank.*

Proof. See [47, Proposition 2, p.70]. □

4.5.2 Mukai-regularity for G-linearized sheaves

Now we apply Fourier-Mukai functor $R\hat{\mathcal{S}}$ (See Section 4.4.1) on the G-linearized sheaves. More over we define the I.T (index theorem) and M-regularity in the case of G-linearized sheaves.

Definition 4.5.4. *1) A coherent G-sheaf $\mathcal{F}$ on A satisfies W.I.T (the weak index theorem) with index i if $R^j\hat{\mathcal{S}}(\mathcal{F}) = 0$, for all $j \neq i$.*
2) A coherent G-sheaf $\mathcal{F}$ on A is said to satisfy I.T (index theorem) with index i if $H^j(\mathcal{F} \otimes \alpha) = 0$, for all $\alpha \in \hat{A}$ and for all $j \neq i$.

Note that the sheaf $R^i\hat{\mathcal{S}}(\mathcal{F})$ is locally free.

Given a coherent G-sheaf $\mathcal{F}$ on A, we denote the support of the sheaf $R^i\hat{\mathcal{S}}(\mathcal{F})$ by

$$S^i(\mathcal{F}) := \mathrm{Supp}(R^i\hat{\mathcal{S}}(\mathcal{F})).$$

Now we recall the notion of M-regularity.

Definition 4.5.5. *A coherent G-sheaf $\mathcal{F}$ on A is called M-regular if*

$$codim\ S^i(\mathcal{F}) > i$$

for each $i = 1, ..., g$.

Remark 4.5.6. *1) Coherent G-sheaves on A which satisfy I.T with index 0, are examples of M-regular G-sheaves.*

2) We also note that an ample line bundle H satisfies I.T with index 0 [54, Example 2.2, p.289]. This will be relevant in our later sections.

Denote the cohomological support locus [22]:

$$V^i(\mathcal{F}) := \{\eta \in \mathrm{Pic}^0(A) : h^i(\mathcal{F} \otimes \eta) \neq 0\} \subset \mathrm{Pic}^0(A).$$

There is an inclusion $S^i(\mathcal{F}) \subset V^i(\mathcal{F})$.

Hence a G-sheaf is M-regular if

$$\mathrm{codim}(V^i(\mathcal{F})) > i \tag{4.5.2}$$

for any $i = 1, ..., g$.

4.6 G-global generation and global generation on hyperelliptic varieties

Suppose G be a finite group and $\mathcal{F}$ is a coherent G-sheaf on an abelian variety A. Consider the central extension of G by $\mathbb{C}^*$, the multiplicative group of nonzero complex numbers. In other words, there is an exact sequence:

$$1 \to \mathbb{C}^* \to \tilde{\mathcal{G}} \to G \to 0.$$

Here $\tilde{\mathcal{G}}$ consists of pairs$(g, \tilde{g})$, where g runs over G and $\tilde{g}$ is an automorphism of $\mathcal{F}$ covering g.

We assume that there is a splitting and let $\tilde{G} \subset \tilde{\mathcal{G}}$ denote the image of G under the splitting map. This is because, by definition, a G-linearized sheaf comes with a splitting as above. This will enable us to look at invariant sections of G-linearized coherent sheaves on A.

We note that $\tilde{G}$ acts on $H^0(A, \mathcal{F})$. Denote the subspace of $\tilde{G}$-invariants:

$$H^0(A, \mathcal{F})^{\tilde{G}} = \{s \in H^0(A, \mathcal{F}) : \tilde{g}s = s \ \forall \tilde{g} \in \tilde{G}\}.$$

Since our aim is to obtain global generation of coherent sheaves on the quotient variety $X = A/G$, we introduce the following corresponding notions for coherent G-sheaves on A as follows. In the next subsection, we will prove its equivalence with usual global generation on X.

4.6.1 G-global generation, G-very ampleness and G-k jet ampleness

We keep notations as above.

Definition 4.6.1. *A coherent G-sheaf $\mathcal{F}$ on A is called G-globally generated if the evaluation map*

$$ev : H^0(A, \mathcal{F})^{\tilde{G}} \otimes \mathcal{O}_A \to \mathcal{F}$$

is surjective. Here the map ev is evaluation of $\tilde{G}$-invariant sections at any point of A.

Now we formulate very ampleness for coherent G-sheaves as follows. For any $a \in A$, let $G.a := \{ga : g \in G\}$. Then this is the orbit of the point $a \in A$ under the action of G. Let $I_{G.a}$ denote the ideal sheaf of the orbit $G.a$ in A. Then this is a coherent G-sheaf on A.

Definition 4.6.2. *A G-line bundle L on A is called G-very ample if the coherent G- sheaf $L \otimes I_{G.a}$ is G-globally generated, for all $a \in A$.*

This notion can be extended to k-jet ampleness for G-line bundles as well.

Definition 4.6.3. *A G-line bundle L on A is G-k-jet ample if the coherent G sheaf*

$$L \otimes I_{G.a_1}^{k_1} \otimes ... \otimes I_{G.a_l}^{k_l}$$

is G-globally generated, for distinct points $a_1, a_2, ..., a_l \in A$ such that $k_1 + k_2 + ... + k_l = k$. In other words, the evaluation map given by $\tilde{G}$-invariant sections

$$H^0(A, L \otimes I_{G.a_1}^{k_1} \otimes ... \otimes I_{G.a_l}^{k_l})^{\tilde{G}} \to H^0(A, L \otimes I_{G.a_1}^{k_1} \otimes ... \otimes I_{G.a_l}^{k_l} \otimes \mathcal{O}_A/m_a)$$

is surjective, for each $a \in A$.

Note that G-0-jet ample is same as G-global generation and G-1-jet ampleness is same as G-very ampleness.

4.6.2 Equivalence of G-global generation and global generation on $X = A/G$

In this subsection, we note the relevance of G-global generation on the quotient variety X. We keep notations as in the previous subsection.

Then we have the following equivalence:

Lemma 4.6.4. *Suppose $\mathcal{F}$ is a coherent G-sheaf on A. Then $\mathcal{F}$ is G-globally generated if and only if the corresponding sheaf $(\pi_*(\mathcal{F}))^G$ is globally generated on the quotient variety $X = A/G$.*

Proof. We recall the one-one correspondence of coherent sheaves, as given in Proposition 4.5.3. Given a coherent sheaf $\mathcal{G}$ on the quotient variety $X = A/G$, consider its pullback $\pi^*\mathcal{G}$ on A, via the quotient morphism $\pi : A \to X = A/G$. Then $\pi^*\mathcal{G}$ is a coherent G-sheaf on A. It would suffice to prove that $\mathcal{G}$ is globally generated on X if and only if $\pi^*\mathcal{G}$ is G-globally generated on A, using Proposition 4.5.3. Firstly, we note the following decomposition [47, Remark 1, p.72]:

$$\pi_*\mathcal{O}_A = \bigoplus_{\chi \in \hat{G}} L_\chi,$$

if G is commutative. In any case, $\mathcal{O}_X$ is a direct summand of $\pi_*\mathcal{O}_A$. Here L_χ is a line bundle on X associated to the character χ on G. Using projection formula, we have:

$$\pi_*(\pi^*\mathcal{G}) = \bigoplus_{\chi \in \hat{G}} (\mathcal{G} \otimes L_\chi), \tag{4.6.3}$$

if G is commutative. But in any case, the sheaf $\mathcal{G}$ is a direct summand of $\pi_*(\pi^*\mathcal{G})$. This gives us an inclusion of the space of global sections:

$$\pi^* H^0(X, \mathcal{G}) \subset H^0(A, \pi^*\mathcal{G}).$$

In particular, the subspace of $\tilde{G}$-invariant sections of $H^0(A, \pi^*\mathcal{G})$ is given by the space $\pi^* H^0(X, \mathcal{G})$.

Suppose $\mathcal{G}$ is globally generated. This implies that the evaluation map:

$$H^0(X, \mathcal{G}) \otimes \mathcal{O}_X \to \mathcal{G}$$

is surjective. The pullback of this morphism of sheaves, via π, on A corresponds to the map

$$H^0(A, \pi^*\mathcal{G})^{\tilde{G}} \otimes \mathcal{O}_A \to \pi^*\mathcal{G}$$

and which is clearly surjective. This implies the G-global generation of $\pi^*\mathcal{G}$. Using the equivalence of categories in Proposition 4.5.3, we conclude the proof. $\qquad\square$

Corollary 4.6.5. *Suppose L is an ample G-line bundle on A and M be the corresponding line bundle on X (under the correspondence in Proposition 4.5.3). Then L is G-k jet ample if and only if M is k-jet ample on X.*

Proof. We need only to note that the ideal sheaf $I_{x_1}^{k_1} \otimes ... \otimes I_{x_l}^{k_l}$ of distinct points $x_1, ..., x_l \in X$ with multiplicities k_i, such that $\sum_i k_i = k$, corresponds to the ideal sheaf $I_{G.a_1}^{k_1} \otimes ... \otimes I_{G.a_l}^{k_l}$ on A, under the correspondence in Proposition 4.5.3. Here $G.a_i = \pi^{-1}(x_i)$, i.e. the inverse image of a point x_i is a G-orbit of a point $a_i \in A$, for $i = 1, ..., l$. Hence the coherent G-sheaf $L \otimes I_{G.a_1}^{k_1} \otimes ... \otimes I_{G.a_l}^{k_l}$ on A corresponds to the coherent sheaf $M \otimes I_{x_1}^{k_1} \otimes ... \otimes I_{x_l}^{k_l}$ on X. Now we apply Lemma 4.6.4, to conclude the proof. $\square$

4.7 G-global generation of G-linearized sheaves of weak index zero

In this section, we recall the notion of continuous global generation [54], adapted to coherent G- sheaves. Instead of the usual multiplication maps, we take the 'averaging' of sections, for the action of the group G. We note that the results of this section hold, for any action of the finite group, i.e., the action need not be free, except in Proposition 4.7.6.

Before proceeding to continuous global generation and its relevance to our set-up, recall the surjectivity statement for multiplication map of sections of ample line bundles [5, 7.3.3]. This is suitably generalized to higher rank sheaves, which are M-regular, by Pareschi and Popa [54]. We modify the multiplication maps by taking 'averaging' of sections, for the finite group G. In other words, we will consider multiplication maps for the $\tilde{G}$-invariant sections, suitably interpreted. This will be needed when we want to look at G-global generation of coherent G sheaves.

4.7.1 Surjectivity of 'Averaging' map

We keep the notations from the previous section.

Lemma 4.7.1. *Let $\mathcal{F}$ be M-regular coherent G-sheaf and H locally free G-sheaf satisfying I.T with index 0. Then for any Zariski open set $U \subseteq \hat{A}$, the map*

$$\bigoplus_{\alpha \in U} H^0(\mathcal{F} \otimes \alpha) \otimes H^0(H \otimes \check{\alpha}) \xrightarrow{\oplus Av} H^0(\mathcal{F} \otimes H)^{\tilde{G}}$$

is surjective. Here the 'averaging map' is given as

$$Av(s \otimes t) = \frac{1}{|G|} \sum_{\tilde{g} \in \tilde{G}} \tilde{g}(s \otimes t),$$

for $s \in H^0(\mathcal{F} \otimes \alpha)$ and $t \in H^0(H \otimes \check{\alpha})$.

Proof. Firstly, note that the map $\oplus Av$ factorizes as follows,

$$\bigoplus_{\alpha \in U} H^0(\mathcal{F} \otimes \alpha) \otimes H^0(H \otimes \check{\alpha}) \xrightarrow{\sum m_\alpha} H^0(\mathcal{F} \otimes H) \xrightarrow{h} H^0(\mathcal{F} \otimes H)^{\tilde{G}}.$$

where h is the averaging map. By [54, Theorem 2.5, p.290], the map $\sum m_\alpha$ is surjective. Clearly h is surjective, since h restricts to identity on $H^0(\mathcal{F} \otimes H)^{\tilde{G}} \subset H^0(\mathcal{F} \otimes H)$. Hence the composed map $\oplus Av = h \circ \sum m_\alpha$ is surjective.

<div style="text-align: right">□</div>

Corollary 4.7.2. *Let $\mathcal{F}$ be M-regular coherent G-sheaf and H locally free G-sheaf satisfying I.T with index 0. Then for any large positive integer N and for any subset $S \subset \hat{A}$ with $|S| = N$, the averaging map*

$$\bigoplus_{\alpha \in S} H^0(\mathcal{F} \otimes \alpha) \otimes H^0(H \otimes \check{\alpha}) \xrightarrow{\oplus Av} H^0(\mathcal{F} \otimes H)^{\tilde{G}}$$

is surjective

Proof. By above Lemma 4.7.1, the surjectivity of the averaging map

$$\bigoplus_{\alpha \in U} H^0(\mathcal{F} \otimes \alpha) \otimes H^0(H \otimes \check{\alpha}) \xrightarrow{\oplus Av} H^0(\mathcal{F} \otimes H)^{\tilde{G}}$$

implies that the family of linear subspaces $\{Im(Av_\alpha)\}_{\alpha \in U}$ spans the finite dimensional vector space $H^0(\mathcal{F} \otimes H)^{\tilde{G}}$. So for any large positive integer N, the images under Av of a finitely many N linear subspaces $H^0(\mathcal{F} \otimes \alpha) \otimes H^0(H \otimes \check{\alpha})$ span $H^0(\mathcal{F} \otimes H)^{\tilde{G}}$.

<div style="text-align: right">□</div>

4.7.2 G-Continuous Global Generation

In this subsection, we recall the notion of continuous global generation and its relevance to global generation [54]. We suitably modify this notion for coherent G-sheaves and show that it is related to G-global generation.

Definition 4.7.3. *A coherent G-sheaf $\mathcal{F}$ on A is called G-continuously globally generated if for any nonempty open set $U \subseteq \hat{A}$ the sum of average maps*

$$\bigoplus_{\alpha \in U} H^0(\mathcal{F} \otimes \alpha) \otimes \check{\alpha} \xrightarrow{\oplus Av} \mathcal{F}$$

is surjective. For $s \in H^0(A, \mathcal{F} \otimes \alpha)$ and a local section t of $\check{\alpha}$, we define locally on A:

$$Av(s \otimes t) = \frac{1}{|G|} \sum_{\tilde{g} \in \tilde{G}} \tilde{g}.(s \otimes t).$$

Note that locally $s \otimes t$ is a section of $\mathcal{F}$.

As earlier, we note that the sum could be taken over finite subsets of $\hat{A}$, of large cardinality.

Lemma 4.7.4. *Suppose $\mathcal{F}$ is a coherent G-sheaf and assume it is G-continuously globally generated. Then for any large positive integer N and for any subset $S \subset \hat{A}$ with $|S| = N$, the sum of average maps*

$$\bigoplus_{\alpha \in S} H^0(\mathcal{F} \otimes \alpha) \otimes \check{\alpha} \xrightarrow{\oplus Av} \mathcal{F}$$

is surjective.

Proof. This proof is similar to the argument given in Corollary 4.7.2. □

We now prove the following proposition relating tensor product of continuously G global generated sheaves and G-global generation.

Proposition 4.7.5. *Suppose $\mathcal{F}$ is a coherent G-sheaf and H is a G-line bundle on A. If both $\mathcal{F}$ and H are G-continuously globally generated then $\mathcal{F} \otimes H$ is G-globally generated.*

Proof. By Lemma 4.7.4, for any large positive integer N and for any subset $S \subset \hat{A}$ with $|S| = N$, the averaging map

$$\bigoplus_{\alpha \in S} H^0(\mathcal{F} \otimes \alpha) \otimes \check{\alpha} \xrightarrow{\oplus Av} \mathcal{F}$$

is surjective. Consider the following commutative diagram,

$$
\begin{array}{ccc}
\displaystyle\bigoplus_{\alpha \in S} H^0(\mathcal{F} \otimes \alpha) \otimes H^0(H \otimes \check{\alpha}) \otimes \mathcal{O}_A & \xrightarrow{\oplus Av} & H^0(\mathcal{F} \otimes H)^{\tilde{G}} \otimes \mathcal{O}_A \\
& \downarrow & \downarrow \\
\displaystyle\bigoplus_{\alpha \in S} H^0(\mathcal{F} \otimes \alpha) \otimes H \otimes \check{\alpha} = \bigoplus_{\alpha \in S} H^0(\mathcal{F} \otimes \alpha) \otimes \check{\alpha} \otimes H & \xrightarrow{Av \otimes id} & \mathcal{F} \otimes H.
\end{array}
$$

Then we have the surjectivity of the lower right map $Av \otimes id$.

We have to show surjectivity of the following evaluation map

$$ev : H^0(\mathcal{F} \otimes H)^{\tilde{G}} \otimes \mathcal{O}_A \to \mathcal{F} \otimes H.$$

We first show that

$$\mathrm{supp}(\mathrm{coker}(ev)) \subseteq \cap_{S \subset \hat{A}} \{\cup_{\alpha \in S} B(H \otimes \check{\alpha})\} =: Z.$$

Here the intersection varies over finite subsets S of $\hat{A}$ of large cardinality N and $B(H \otimes \check{\alpha})$ is the base locus of $H \otimes \check{\alpha}$. Let x be an element in $\mathrm{supp}(\mathrm{coker}(ev))$ such that x is not in Z. This implies, for some S and an $\alpha \in S$,

$$H^0(H \otimes \check{\alpha}) \otimes \mathcal{O}_A \to H \otimes \check{\alpha}$$

is surjective at x. Therefore, in the above commutative diagram, the evaluation map

$$ev : H^0(\mathcal{F} \otimes H)^{\tilde{G}} \otimes \mathcal{O}_A \to \mathcal{F} \otimes H.$$

is surjective at x. This gives a contradiction to x lying in $\operatorname{supp}(\operatorname{coker}(ev))$. Hence $\operatorname{supp}(\operatorname{coker}(ev))$ $\subseteq \cap_{S\subset\hat{A}}\{\cup_{\alpha\in S}B(H\otimes\check{\alpha})\}$. Since H is G- continuously globally generated, by the arguments in [54, Remark 2.11, Proposition 2.12, p.292], $\cap_S \cup_{\alpha\in S} B(H\otimes\check{\alpha})$ is empty, where $\cap$ runs over $S\subset\hat{A}$ of large cardinality. This implies $\operatorname{supp}(\operatorname{coker}(ev))$ is empty.

Hence the evaluation map,

$$ev : H^0(\mathcal{F}\otimes H)^{\tilde{G}}\otimes\mathcal{O}_A \to \mathcal{F}\otimes H$$

is surjective.

$\square$

The following proposition gives an analogue of [54, Proposition 2.13]. It shows that the M-regularity of a coherent G-sheaf implies G-continuous global generation. We assume that the group G acts freely on A.

Proposition 4.7.6. *If $\mathcal{F}$ is a M-regular coherent G-sheaf on A, then for any large positive integer N and for any subset S of $\hat{A}$ with cardinality N, the sum of average maps,*

$$\bigoplus_{\alpha\in S} H^0(\mathcal{F}\otimes\alpha)\otimes\check{\alpha} \overset{\oplus Av}{\to} \mathcal{F}$$

is surjective. In other words, $\mathcal{F}$ is G-continuously globally generated.

Proof. Let H be an ample G-line bundle such that $\mathcal{F}\otimes H$ is G-globally generated. Indeed, such a line bundle can be chosen, due to the correspondence in Proposition 4.5.3. We consider the sheaf $\mathcal{F}_X$ corresponding to $\mathcal{F}$, on $X = A/G$, and find an ample line bundle H_X on X such that $\mathcal{F}_X\otimes H_X$ is globally generated on X. Let H be the ample line bundle on A corresponding to H_X. By Lemma 4.6.4, the coherent G-sheaf $\mathcal{F}\otimes H$ is G- globally generated.

This implies that the evaluation map

$$H^0(\mathcal{F}\otimes H)^{\tilde{G}}\otimes\mathcal{O}_A \overset{ev}{\longrightarrow} \mathcal{F}\otimes H$$

is surjective. Since H is an ample G-line bundle, by Remark 4.5.6, H satisfies I.T with index 0. Therefore, by Corollary 4.7.2,

$$\bigoplus_{\alpha\in S} H^0(\mathcal{F}\otimes\alpha)\otimes H^0(H\otimes\check{\alpha})\otimes\mathcal{O}_A \overset{\oplus Av}{\longrightarrow} H^0(\mathcal{F}\otimes H)^{\tilde{G}}\otimes\mathcal{O}_A$$

is surjective. Now consider the following commutative diagram,

$$\begin{array}{ccc}
\bigoplus_{\alpha\in S} H^0(\mathcal{F}\otimes\alpha)\otimes H^0(H\otimes\check{\alpha})\otimes\mathcal{O}_A & \overset{\oplus Av}{\longrightarrow} & H^0(\mathcal{F}\otimes H)^{\tilde{G}}\otimes\mathcal{O}_A \\
\downarrow & & \downarrow \\
\bigoplus_{\alpha\in S} H^0(\mathcal{F}\otimes\alpha)\otimes H\otimes\check{\alpha} & \overset{Av\otimes id}{\longrightarrow} & \mathcal{F}\otimes H
\end{array}$$

where the sum varies over a finite subset S, of large cardinality. In the above commutative diagram, since $\oplus Av$ and the evaluation ev are surjective, it follows that the averaging map

$$\bigoplus_{\alpha \in S} H^0(\mathcal{F} \otimes \alpha) \otimes H \otimes \check{\alpha} \xrightarrow{Av \otimes id} \mathcal{F} \otimes H$$

is also surjective. Since H is a line bundle, we obtain the assertion on G-continuous global generation of the sheaf $\mathcal{F}$. $\qquad\square$

As a consequence of the above proposition, we obtain the main result of this section:

Corollary 4.7.7. *Suppose $\mathcal{F}$ is a coherent G-sheaf and H is a G-line bundle on A. If both $\mathcal{F}$ and H are M-regular sheaves on A, then the coherent G-sheaf $\mathcal{F} \otimes H$ is G-globally generated.*

Proof. By Proposition 4.7.6, $\mathcal{F}$ are H are G-continuously globally generated. By Proposition 4.7.5 $\mathcal{F} \otimes H$ is G-globally generated. $\qquad\square$

4.8 Embedding theorems on hyperelliptic varieties

In this section we prove analogues of very amplessnes results due to Ohbuchi and Lefschetz [54, Corollary 3.9], in the case of ample G-line bundle. By Corollary 4.6.5, we obtain similar embedding statements for the quotient variety $X = A/G$.

Lemma 4.8.1. *Let L_1 and L_2 be G-line bundles on A such that L_1 and $L_2 \otimes I_{Gx}$ are M-regular, for all $a \in A$. Then $L_1 \otimes L_2$ is G-very ample on A.*

Proof. By Corollary 4.7.7, $L_1 \otimes L_2 \otimes I_{G.a}$ is G-globally generated, for all $a \in A$. Hence $L_1 \otimes L_2$ is G-very ample. $\qquad\square$

Now we check M-regularity of G-line bundles which have no G-invariant base divisor. This will enable us to conclude very ampleness of powers of G-line bundles.

Proposition 4.8.2. *Suppose L be an ample G-line bundle and having no base divisor on an abelian variety A. Then $L \otimes I_{G.a}$ is M-regular on A.*

Proof. Firstly for any $a \in A$, consider the following exact sequence:

$$0 \to L \otimes I_{G.a} \to L \to L_{|G.a} \to 0.$$

Take the long exact cohomology sequence:

$$0 \to H^0(L \otimes I_{G.a}) \to H^0(L) \to \oplus_{g \in G} H^0(L \otimes \mathbb{C}(ga)) \to$$

$$H^1(L \otimes I_{G.a}) \to H^1(L) \to \oplus_{g \in G} H^1(L \otimes \mathbb{C}(ga)) \to \cdots.$$

Also note that since L is ample $H^i(A, L) = 0$, for all $i > 0$. Therefore the above long exact sequence reduces to

$$0 \to H^0(L \otimes I_{G.a}) \to H^0(L) \to (\oplus_{g \in G} H^0(L \otimes \mathbb{C}(ga)) \to H^1(L \otimes I_{G.a}) \to 0.$$

Now consider the cohomological support locus,

$$\text{Supp } V^i(L \otimes I_{G.a}) := \{\alpha \in \hat{A} : H^i(L \otimes I_{G.a} \otimes \alpha) \neq 0\}.$$

Note that

$$L \otimes I_{G.a} \otimes \alpha = \oplus_{g \in G}(L \otimes I_{ga} \otimes \alpha) \cong t_y^*(L \otimes I_{G.a-y}),$$

for some $y \in A$. The above exact sequences imply that, when $i > 1$, we have $\text{Supp} V^i(L \otimes I_{Gx})) = \emptyset$. This implies

$$\text{codim Supp} V^i(L \otimes I_{Gx}) > i$$

for all $i > 1$. When $i = 1$, $\text{Supp}(V^1(L \otimes I_{Gx}))$ is isomorphic to a base divisor of L. By hypothesis, L has no base divisor. Hence this implies codimension of $\text{Supp}(V^1(L \otimes I_{Gx}))$ is at least 2. Hence, using (4.5.2), $L \otimes I_{Gx}$ is M-regular.

$\square$

Now we consider powers of ample G-line bundles and apply the previous results to obtain embedding statements.

Theorem 4.8.3. *Suppose N is an ample line bundle on the quotient variety $X = A/G$. Then the following hold:*

a) N^2 is very ample, if N has no base divisor.

b) N^3 is always very ample.

Proof. Using Proposition 4.5.3, let L be the ample G-line bundle on A corresponding to the ample line bundle N on X.

To prove a), we assume that N has no base divisor. This implies that L has no G-invariant base divisor, in particular L has no base divisor. By Proposition 4.8.2, $L \otimes I_{Gx}$ is M-regular, for all $x \in X$. Furthermore since L is ample, L is M-regular by Remark 4.5.6. Hence by Corollary 4.7.7, $L \otimes L \otimes I_{Gx}$ is G-globally generated. Hence $L^{\otimes 2}$ is G-very ample. Now by Corollary 4.6.5, we conclude that N^2 is very ample on X.

To prove b), note that by Corollary 4.7.7, $L^{\otimes 2}$ is G-globally generated. This implies that $L^{\otimes 2}$ has no base divisor and hence by Theorem 4.8.2, $L^{\otimes 2} \otimes I_{Gx}$ is M-regular, for all $x \in X$. Hence, by Corollary 4.7.7, $L^{\otimes 2} \otimes I_{Gx}$ is G-continuously globally generated. This implies $L^{\otimes 3}$ is G-very ample and hence $N^{\otimes 3}$ is very ample on X. $\square$

To extend above results to k-jet ampleness on a hyperelliptic variety X, we note the below lemma for ample G-line bundles on an abelian variety A.

Lemma 4.8.4. *Suppose L is an ample G-line bundle on an abelian variety A. Then the following are equivalent:*

1) L is G-k-jet ample.

2) $L \otimes I_{G.a_1}^{k_1} \otimes ... \otimes I_{G.a_l}^{k_l}$ satisfies I.T. with index 0, for any l-distinct points $a_1, ..., a_l \in A$ such that $\sum k_i = k + 1$.

Proof. Using the correspondence in Proposition 4.5.3, it suffices to prove the equivalence for the corresponding line bundle $N := \pi_*(L)^G$ on X. Recall that $\pi : A \to X = A/G$ is the quotient morphism. Using (4.6.3), we note that

$$H^1(A, L) = \bigoplus_{\chi \in \hat{G}} H^1(X, N \otimes L_\chi).$$

Here L_χ denotes the line bundle on X associated to the character χ on G. Since L is ample we have the vanishing $H^1(A, L) = 0$. This implies the vanishing $H^1(X, N) = 0$. The rest of the proof is similar to [55, Lemma 3.3]. $\qquad\square$

Now we state the analogue of above theorem, for higher jet ampleness on a hyperelliptic variety X.

Proposition 4.8.5. *Suppose N is an ample line bundle on a hyperelliptic variety X. Then the following hold:*

1) N^{k+1} is k-jet ample if N has no base divisor, and for $k \geq 1$.

2) N^{k+2} is k-jet ample, and for $k \geq 0$.

Proof. The proof is similar to [55, Theorem 3.8] applied to the corresponding ample G-line bundle L on A. Indeed, by above Lemma 4.8.4, it suffices to check 3), i.e., the sheaf

$$L \otimes I_{G.a_1}^{k_1} \otimes ... \otimes I_{G.a_l}^{k_l}$$

is G-globally generated, for any l-distinct points $a_1, ..., a_l \in A$ such that $\sum k_i = k$.

We apply induction on k, and using the correspondence in Corollary 4.6.5, prove it for the ample G-line budle L on A.

Suppose $k = 1$. Then 1) holds, by Theorem 4.8.3.

Suppose the statement 1) holds for $k - 1$, i.e., L^k is G–$(k-1)$-jet ample. By Lemma 4.8.4, this implies for any l-distinct points $a_1, ..., a_l \in A$ such that $\sum_i k_i = k$, the sheaf $L^k \otimes I_{G.a_1}^{k_1} \otimes ... \otimes I_{G.a_l}^{k_l}$ satisfies I.T with index zero. By Remark 4.5.6 2), $L^k \otimes I_{G.a_1}^{k_1} \otimes ... \otimes I_{G.a_l}^{k_l}$ is M-regular. Hence, by Corollary 4.7.7, the sheaf $L \otimes L^k \otimes I_{G.a_1}^{k_1} \otimes ... \otimes I_{G.a_l}^{k_l}$ is G-globally generated, for l-distinct $a_1, ..., a_l \in A$, such that $\sum k_i = k$. Now by Lemma 4.8.4 3), L^{k+1} is G-k-jet ample.

The proof of 2) is similar, and we omit it. $\qquad\square$

4.9 Syzygy or N_p-property of line bundles on a hyperelliptic variety

In this section, we look at syzygy or N_p-properties defined by M. Green [19].

Suppose Z is a smooth projective variety defined over the complex numbers. An ample line bundle L on Z is said to satisfy N_p-property (See 4.1) if the first p-steps of the minimal graded free resolution of the algebra $R_L := \oplus_{n \geq 0} H^0(L^n)$ over the polynomial ring $S_L := \oplus_{n \geq 0} Sym^n H^0(L)$ are linear. In other words, a minimal resolution of R_L looks like:

$$S_L(-p-1)^{i_p} \to S_L(-p)^{\oplus i_{p-1}} \to ... \to S_L(-2)^{i_1} \to S_L \to R_L \to 0.$$

When $p = 0$, we say that L gives a projectively normal embedding. When $p = 1$, L satisfies N_0 and the ideal of the embedded variety is generated by quadrics.

More generally, even (see [55]), one can define properties measuring how far the first p steps of the resolution are from being linear. To do this, fix $p \geq 0$, and consider the first p steps of the minimal free resolution of R_L as an S_L-module.

$$E_p \to E_{p-1} \to \ \to E_1 \to E_0 \to R_L \to 0,$$

where $E_0 = S_L \oplus \bigoplus_j S_L(-a_{0j})$ with $a_{0j} \geq 2$ (since the linear series is complete), $E_1 = \bigoplus_j S_L(-a_{1j})$ with $a_{1j} \geq 2$ (since the embedding is non-degenerate) and so on, up to $E_p = \bigoplus_j S_L(-a_{pj})$ with $a_{pj} \geq p+1$. Then L is said to satisfy property N_p^r if $a_{pj} \leq p+1+r$. In particular, N_1^r means that $a_{1j} \leq 2+r$, i.e., the ideal $I_{X,L}$ is generated by forms of degree $\leq 2+r$, while property N_p^0 is the same as N_p.

4.9.1 Criterion for N_p^r-property

Usually, in practice, one looks at surjectivity of multiplication maps of sections of some natural bundles associated to L. We recall them below. Consider the exact sequence associated to a globally generated line bundle L, given by evaluation of its sections:

$$0 \to M_L \to H^0(L) \otimes \mathcal{O}_Z \to L \to 0.$$

Here M_L is a coherent sheaf and is the kernel of the evaluation map. In fact, it is a locally free sheaf.

Consider the exact sequence by taking the $p+1$-st exterior power of the above evaluation sequence:

$$0 \to \wedge^{p+1} M_L \otimes L^h \to \wedge^{p+1} H^0(L) \otimes L^h \to \wedge^p M_L \otimes L^{h+1} \to 0.$$

Then N_p^r-property holds if

$$H^1(\wedge^{p+1} M_L \otimes L^h) = 0, \text{ for all } h \geq r+1.$$

The converse is true if Z is an abelian variety, since $H^1(L^h) = 0$. See [53, p.660]. Moreover we have:

Lemma 4.9.1. *a)* If $H^1(M_L^{\otimes p+1} \otimes L^h) = 0$, for all $h \geq r+1$, then L satisfies N_p^r-property.

b) Assume that $H^1(M_L^{\otimes p+1} \otimes L^h) = 0$. Then $H^1(M_L^{\otimes p+1} \otimes L^h) = 0$ if and only if the multiplication map

$$H^0(L) \otimes H^0(M_L^{\otimes p} \otimes L^h) \to H^0(M_L^{\otimes p} \otimes L^{\otimes h+1})$$

is surjective.

Proof. See [55, Proposition 6.3]. □

4.9.2   Cohomology Vanishing on a hyperelliptic variety

Suppose X is a hyperelliptic variety of dimension n. As in earlier sections, we consider the quotient morphism $\pi : A \to X = A/G$. Here G is a finite group acting freely on A.

Suppose N is an ample line bundle on X. Assume it is globally generated. Consider the evaluation map on the sections of N:

$$0 \to M_N \to H^0(N) \otimes \mathcal{O}_X \to N \to 0.$$

Pullback of this exact sequence on A yields the exact sequence:

$$0 \to \pi^* M_N \to H^0(L)^{\tilde{G}} \otimes \mathcal{O}_A \to L \to 0.$$

Here $L := \pi^* N$ is the corresponding G-line bundle on A, and $H^0(L)^{\tilde{G}} \subset H^0(L)$ is the subspace of $\tilde{G}$-invariant sections. Denote $M_L^G := \pi^* M_N$. In particular, $\wedge^p M_L^G$ is a G-linearized bundle.

We first note the below vanishing, which we will need.

Lemma 4.9.2. *The cohomology vanishing*

$$H^1(A, \wedge^{p+1} M_L^G \otimes L^h) = 0$$

implies the cohomology vanishing

$$H^1(X, \wedge^{p+1} M_N \otimes N^h) = 0,$$

for each $h \geq r+1$ and $r \geq 0$.

Proof. Since the bundles $\wedge^{p+1} M_L^G$ and L^h are G-linearized bundles, the tensor product $\wedge^{p+1} M_L^G \otimes L^h$ is also a G-linearized bundle. In particular, the group $\tilde{G}$ acts on the cohomology groups $H^i(A, \wedge^{p+1} M_L^G \otimes L^h)$, for $i \geq 0$. The $\tilde{G}$-invariant subspace is precisely $H^i(A, \wedge^{p+1} M_L^G \otimes L^h)^{\tilde{G}}$. Now, we use projection formula as shown in Lemma 4.6.4, and using (4.6.3), we deduce that the $\tilde{G}$-invariant subspace is equal to the cohomology group $H^i(X, \wedge^{p+1} M_N \otimes N^h)$. This gives the assertion.

□

Lemma 4.9.3. *The cohomology vanishing*

$$H^1(A, \wedge^{p+1} M_L \otimes L^h) = 0$$

implies the cohomology vanishing

$$H^1(A, \wedge^{p+1} M_L^G \otimes L^h) = 0,$$

for each $h \geq r+1$ and $r \geq 0$.

Proof. Note that in the below exact sequence

$$0 \to M_L \to H^0(L) \otimes \mathcal{O}_A \to L \to 0$$

the group $\tilde{G}$ acts on $H^0(L)$ and on L equivariantly. Hence the inclusion of $\tilde{G}$-invariant sections $H^0(L)^{\tilde{G}} \subset H^0(L)$ provides an inclusion of bundles

$$M_L^G \subset M_L.$$

Moreover, since the averaging map of sections

$$H^0(L) \overset{Av}{\to} H^0(L)^{\tilde{G}}, \quad s \mapsto \frac{1}{|G|} \sum_{g \in \tilde{G}} g.s$$

is surjective, we deduce that the bundle M_L^G is a split summand of M_L.

Hence we have an inclusion of their exterior powers tensored with L^h:

$$\wedge^{p+1} M_L^G \otimes L^h \subset \wedge^{p+1} M_L \otimes L^h.$$

This is also a split summand and hence gives the inclusion of cohomologies:

$$H^1(A, \wedge^{p+1} M_L^G \otimes L^h) \subset H^1(A, \wedge^{p+1} M_L \otimes L^h).$$

We now deduce our assertion.

$\square$

Now, we apply above two lemmas to conclude our main consequence of this section.

Proposition 4.9.4. *Suppose M is an ample line bundle on a hyperelliptic variety X. Then M^{p+k} satisfies N_p-property, for any $k \geq 3$.*

Proof. Suppose M is an ample line bundle on X. By Theorem 4.8.3, we know that $N := M^k$, $k \geq 3$, is very ample. In particular, N is globally generated. Since $L = \pi^* N$ is an ample globally generated line bundle on A, by [53, Theorem 4.3, p. 663], we have the cohomology vanishing

$$H^1(A, \wedge^{p+1} M_L \otimes L^h) = 0$$

for any $h \geq 1$. Now apply Lemma 4.9.2 and Lemma 4.9.3, when $r = 0$, to conclude the cohomology vanishing

$$H^1(X, \wedge^{p+1} M_N \otimes N^h) = 0.$$

for any $h \geq 1$. This implies that $N^p = M^{p+k}$, $k \geq 3$, satisfies N_p-property.

$\square$

Theorem 4.9.5. *Suppose X is a hyperelliptic variety over $\mathbb{C}$, and N be an ample line bundle on X. If $(r+1)(n-1) > p+1$, then $N^{\otimes n}$ satisfies N_p^r.*

Proof. Note that $L := \pi^* N$ be an ample G-line bundle on A.

We have to show $H^1(X, \wedge^{p+1} M_{N^{\otimes n}} \otimes N^{\otimes nh}) = 0$ fo all $h \geq r+1$.

Since $(r+1)(n-1) > p+1$, $L^{\otimes n}$ satisfies N_p^r (By [53, Theorem 4.3, p.663]), i.e., $H^1(A, \wedge^{p+1} M_{L^{\otimes n}} \otimes L^{\otimes nh}) = 0$ fo all $h \geq r+1$. Now apply 4.9.2 and 4.9.3 to conclude the cohomology vanishing

$$H^1(X, \wedge^{p+1} M_{N^{\otimes n}} \otimes N^{\otimes nh}) = 0.$$

for any $h \geq r+1$.

$\square$

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