On intermediate subfactors

By

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Keshab Chandra Bakshi

List of publications arising from the thesis

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   arXiv link: https://arxiv.org/abs/1710.00285

Preprint

4. Relative position between interemdiate subfactors, with Sayan Das, Zhengwei Liu and Yunxiang Ren, 2017.

Keshab Chandra Bakshi

DEDICATED TO

Professor V. S. Sunder

"A band of minstrels suddenly appears, dances, and sings, and it departs in the same sudden manner. They come and they return, but none recognizes them."

– Sri Ramakrishna.

"He who despises Euclidean Geometry is like a man who, returning from foreign parts, disparages his home"

–H. G. Forder

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Contents

Synopsis

This thesis primarily deals with the intermediate subfactors of a subfactor, say $N \subset M$, of type II_1. A factor M is a von Neumann algebra (that is, a 'self adjoint subalgebra' of $\mathcal{L}(H)$ such that $M'' = M$) with trivial center. We say a factor M is of type II_1 if it is infinite dimensional, and has a normal state (in which case it is automatically faithful and tracial) denoted by τ. A subfactor of a type II_1 factor M is a subalgebra $N \subset M$ which is itself a type II_1 factor and contains the identity element of M. Interest in subfactors began when V. Jones introduced index, denoted by $[M : N]$, (See [22]) of the subfactor $N \subset M$ of type II_1. He has shown that $[M : N]$ lies in the set $\{4\cos^2(\frac{\pi}{n}) : n \geq 3\} \cup [4, +\infty]$. The central object in the subfactor theory is the standard invariant of a given subfactor. A deep theorem by Popa says the standard invariant completely determines strongly amenable subfactors (see [40]). (Ocneanu had earlier proved the important special case of Popa's theorem when the subfactor had 'finite depth'.) Moreover, Popa introduced standard λ-lattice as an axiomatization of the standard invariant (see [43]) which is equivalent to Ocneanu's paragroup axiomatisation for subfactors of finite depth. Jones subsequently in [23] introduced subfactor planar algebras, an axiomatization, with a topological flavour, of the standard invariant of subfactors.

Pimsner and Popa have shown (in [36]) that for an inclusion $N \subset M$ of II_1 factors, M is a finitely generated projective module over N if and only if $[M : N]$ is finite by constructing a certain kind of family $\{m_j : 1 \leq j \leq n+1\}$ of elements in M, with $n = \lfloor [M : N] \rfloor$, which they called "orthonormal basis" for the pair $N \subset M$. In the first chapter of this thesis we consider (as in [25]) more general bases for factors of type II_1 which are not necessarily orthonormal and showed that this can also be done for connected inclusion of finite dimensional von Neumann algebras (in the sense that the Bratteli diagram is connected). For both these cases we obtain a characterization of 'Jones' basic construction' (see [22]) in terms of bases and prove the phenomenon of 'multistep basic construction' (see [37]). The first chapter is the content of [1].

In the second chapter, we consider the subfactor planar algebra $P^{(N \subset Q)}$ for an intermediate subfactor $N \subset Q \subset M$ of an irreducible subfactor $N \subset M$ of finite index. In [5] authors showed that this can be described in terms of the subfactor planar algebra $P^{(N \subset M)}$. We give an alternative proof of this fact by showing that if T is any planar tangle, the associated operator $Z_T^{(N \subset Q)}$ can be read off from $Z_T^{(N \subset M)}$ by a procedure involving the so-called *biprojection* corresponding to the intermediate subfactor $N \subset Q \subset M$ (for definition of biprojection see [6], [29]) and a scalar $\alpha(T)$ carefully chosen so as to ensure that the formula defining $Z_T^{(N \subset Q)}$ is multiplicative with respect to composition of tangles. The difference in proofs here and in [5] stems from the two ways that a planar algebra P can be described, respectively, (i) as in [27] (where one says what the underlying spaces P_n are, and explicitly describes the multilinear operator Z_T^P associated to a planar tangle T, and then verifies that these tangle maps satisfy the necessary compatibility conditions, as in Theorem 2.3.3), and (ii) by specifying a non-degenerate scalar-valued partition function Z on 0-tangles (labelled by $S = \coprod_{k \in \ Col} S_k$) which is invariant under planar isotopy and multiplicative on connected components (as in [23]). Thus, one may say that a 'bonus' in our approach is that we know how any planar tangle acts on a vector in its domain. We also apply our result to one example involving crossed product by groups. The second chapter consists of results from [2].

We denote by $\mathcal{L}(N \subset M)$ the set of all intermediate von Neumann sub-algebras for the subfactor $N \subset M$. The set $\mathcal{L}(N \subset M)$ forms a lattice under the two operations $P \wedge Q = P \cap Q$ and $P \vee Q = (P \cup Q)''$. If $N \subset M$ is irreducible, that is $N' \cap M = \mathbb{C}$, then $\mathcal{L}(N \subset M)$ is exactly the lattice of intermediate subfactors. Watatani in [49] has shown that in this case the lattice is a finite set. In the third chapter we improve existing upper bounds for the cardinality of this set. For this we have introduced a natural notion of ‘angle’ involving biprojections and investigated various properties of the same. In the final section of the third chapter we investigate the intermediate subfactors for general finite-index case. We show if the norm difference between two biprojections is less than half then the corresponding intermediate subfactors are actually isomorphic. This final chapter is the content of [3] and [4].

We describe our results in more detail below.

On Pimsner Popa bases

We write $(N \subseteq M, tr)$ to denote a unital inclusion of finite von Neumann algebras, with ‘tr’ a faithful normal tracial state, and write $N \subset M \overset{e_1}{\subset} M_1$ for Jones' resulting basic construction. The trace tr is called a *Markov trace of modulus* τ if it extends to a positive trace $Tr : M_1 \longmapsto \mathbb{C}$ such that $Tr(xe_n) = \tau tr(x)$ for $x \in M$. We are interested in the following two cases:

$\mathcal{C}ase(1)$: N and M are II_1 factors with finite index $[M : N]$, so there exists unique Markov trace tr on M of modulus τ where $\tau = [M : N]^{-1}$.

$\mathcal{C}ase(2)$: Let $N \subseteq M$ be a connected inclusion of finite dimensional C^*- algebras and hence there exists unique Markov trace tr on M of modulus τ where $\tau = \|G\|^{-2}$ where G is the inclusion matrix for $N \subseteq M$.

Then it is known that in both cases ((1) and (2)) there exists a unique Markov trace on M_1 (in fact, for each M_n, all of the same modulus), and we can iterate the basic construction to obtain a tower,

$$M_1 \subseteq M_2 \subseteq .. \subseteq M_n \subseteq M_{n+1}....$$

where $M_{n+1} = \langle M_n, e_{n+1} \rangle$ is the result of applying the basic construction for the pair

$M_{n-1} \subseteq M_n$ and e_{n+1} is the projection implementing the tr_{M_n} preserving conditional expectation of M_n onto M_{n-1}. As mentioned in the introduction, Pimsner and Popa in [36] have introduced a notion of 'basis' for M over N which is orthogonal in nature. In a similar manner, we find a slightly less restrictive notion of basis in [25]. In this thesis (in section 2) we see that this notion of basis in [25] can also be carried out in our case (2) of connected inclusions of finite dimensional C^* algebras. Further in Chapter 1 we characterize bases, in both cases (1) and (2), by three equivalent conditions as stated below:

Theorem 0.0.1. *Let N and M be as in Case(1) or in Case(2). Then the following notions for a finite set $\{\lambda_i : i \in I = 1, 2, ...n\} \subseteq M$, are equivalent:*

1. *Let E_N be the tr- preserving conditional expectation of M onto N and define the matrix $Q \in M_n(N)$ whose (i, j) entry is given by $q_{ij} = E_N(\lambda_i {\lambda_j}^*)$. Then Q is a projection in $M_n(N)$ such that $tr_{M_n(N)}(Q) = \tau^{-1}/n$.*
2. *$\sum_{i=1}^{n} {\lambda_i}^* e_1 \lambda_i = 1$, where e_1 is the Jones projection.*
3. *For any $x \in M$, $x = \sum_{i=1}^{n} E_N(x{\lambda_i}^*)\lambda_i$.*

We have succeeded in obtaining a simple characterization of Jones' basic construction in terms of bases, in both the cases:

Lemma 0.0.2. *Let $N \subseteq M$ be as in Case(1) or Case(2). Assume $\{\lambda_i : i \in \{1, 2, ..n\}$ is a basis for M/N(which exists in both the cases). Let P be a II_1 factor in Case(1) or a finite dimensional C^*-algebra in Case (2) such that P contains M and also contains a projection f such that $\sum_{i=1}^{n} {\lambda_i}^* f \lambda_i = 1$ and satisfies further the following properties :*

1. *$fxf = E_N(x)f$ for all $x \in M$*
2. *$\{\tau^{-1/2} f \lambda_i\}$ is a basis for P/M.*

3. $n \longmapsto nf$ is an injective map from N into P. (Condition (3) needs to be assumed in Case(2), unlike Case(1) where it is automatically met.)

Then there exists an isomorphism from $M_1 = \langle M, e_1 \rangle$ onto P which maps e_1 to f and fixes M.

In this situation we say that P is an instance of basic construction applied to the inclusion $N \subseteq M$ with a choice of projection implementing the conditional expectation being given by f.

As a pleasant application we deduce the multistep basic construction for both the cases:

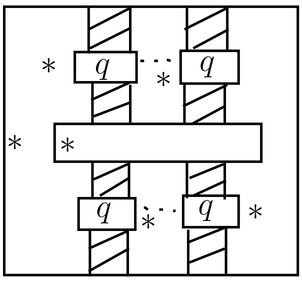
Theorem 0.0.3. *Let $N \subseteq M$ be a pair of von Neumann algebras as in Case(1) or (2) and $N \subseteq M \subseteq M_1 \subseteq \cdots$ be the tower of II_1 factors (resp., finite dimensional C* algebras) in Case(1) (resp., in Case(2)) obtained by iterating the basic construction. Let $e_i \in M_i$ be the Jones' projections. Then for $m \geq 0, k \geq -1$, where $M_{-1} = N$, it is true that $M_k \subseteq M_{k+m} \subseteq M_{k+2m}$ is an instance of basic construction with a choice of projection implementing the conditional expectation of M_{k+m} onto M_k being given by*

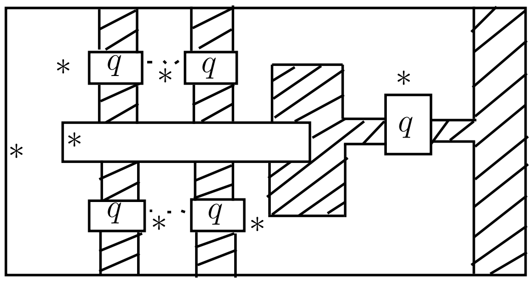
$$
\begin{aligned}
&e_{[k,k+m]} \\
&\quad = \tau^{-m(m-1)/2}(e_{k+m+1}e_{k+m}\cdots e_{k+2})(e_{k+m+2}e_{k+m+1}\cdots e_{k+3}) \\
&\qquad \cdots (e_{k+2m}e_{k+2m-1}\cdots e_{k+m+1})
\end{aligned}
$$

Intermediate planar algebras

It is well-known from [6], [29] and [9] that in case $N' \cap M = \mathbb{C}$, there is a bijective correspondence between biprojections q (corresponding to the Jones projection of $L^2(M)$ onto $L^2(Q)$) and the intermediate subfactor Q, where $N \subset Q \subset M$. Denote by $N \subset Q \subset Q_1 \subset Q_2 \subset \ldots$ the Jones tower of $N \subset Q$. (As we will be discussing several planar algebras here - those corresponding to $N \subset M, N \subset Q, Q \subset M$, etc. - we shall reserve the symbol P for $P^{(N \subset M)}$, so $P_n = N' \cap M_{n-1}$.)

Definition 0.0.4. *Denote the following tangle by E_n (pretending that there are no labels, namely q's):*





according as n is even or odd respectively. We shall use these to define a map $T \mapsto F(T)$ from the class of n-tangles to the class of partially labelled *n-tangles with $(n+1)$ internal discs all but the last of which are 2-boxes labelled with a q, with the tangle T inserted in the last disc of colour n. Thus, $F(T) = E_n \circ_{(D_1, D_2, \cdots, D_n, D_{n+1})}$ $(q, q, \cdots, q, T)$.*

If it is clear from the context then we write E instead of E_n.

Define functions $F_n : P_n \mapsto P_n$ by $F_n(x) = Z_{E_n}(q \otimes q \otimes \cdots \otimes q \otimes x)$ for $x \in P_n$. We often write $F(x)$ instead of $F_n(x)$ if there is no confusion.

Definition 0.0.5. *Let T be a k-tangle with $b \geq 1$ internal discs $D_1, \ldots D_b$ of colours $k_1, \ldots k_b$. Then define $\alpha(T) = [M : Q]^{\frac{1}{2}c(T)}$, where*

$$c(T) = (\lceil k/2 \rceil + \lfloor k_1/2 \rfloor + \cdots + \lfloor k_b/2 \rfloor) - l(T)$$

with $l(T)$ being the number of closed loops after capping the black intervals of the external disc of T and cupping the black intervals of all internal discs of T. (Note the use of both ceiling ($\lceil\cdot\rceil$) and floor ($\lfloor\cdot\rfloor$) functions.)

We shall show that:

Theorem 0.0.6. *If $P'_k = ran(F(I^k_k))$ and $Z'_T = \alpha(T) Z_{F(T)|_{\otimes P'_{k_i(T)}}}$, then $(P', T \mapsto Z'_T|_{\otimes P'_{k_i(T)}})$ is a subfactor planar algebra which is isomorphic to $P^{(N\subset Q)}$.*

The proof of this theorem has two main ingredients: (a) the verification that P' is a planar algebra; and (b) the verification that this is isomorphic to the planar algebra of $N \subset Q$. The proof of (b) is an application of a theorem of Jones (see [27] Theorem 2.1,which is the formulation that we shall use) while the only really non-trivial part of proving (a) is in the verification of compatibility of the partition function to gluing of tangles. In order to verify that the operation of tangles (in P') is compatible with composition of tangles, we will need to verify that

$$\alpha(T \circ \tilde{T}) Z_{F(T\circ\tilde{T})} = \alpha(T)\alpha(\tilde{T}) Z_{F(T)\circ F(\tilde{T})} ,$$

which is seen to translate to:

$$Z_{F(T\circ F(\tilde{T}))} = [M:Q]^{-1/2(\tilde{k}_0 - l(T) - l(\tilde{T}) + l(T\circ\tilde{T})} Z_{F(T\circ\tilde{T})} , \tag{1}$$

With our formulation of intermediate planar algebra, as an application we have recovered the result of [30] which establishes an one-one correspondence between a planar subalgebra P^Θ of the 'group planar algebra' which is naturally associated with a group Θ of automorphisms of the given group G and the planar agebra corresponding to the 'subgroup-subfactor' associated with the group inclusion $\Theta \subset (G \rtimes \Theta)$. It seems that there is a slight problem with the constant in the defining isomorphism β_k of [30]. We clarify this point.

Angle between intermediate subfactors

The study of the lattice structure of von Neumann subalgebras was first studied by Murray and von Neumann in [33]. For a general finite-index subfactor $N \subset M$ the intermediate subfactor lattice $\mathcal{L}(N \subset M)$ may not be finite. Even in the case when $N' \cap M$ is abelian the set of intermediate subfactors may be infinite as shown in [48] (Theorem 5.4). Our starting point is the following remarkable result of Watatani:

Theorem 0.0.7. *[49] Let $N \subset M$ be an irreducible subfactor of type II_1 such that $[M : N] < \infty$. Then the set $\mathcal{L}(N \subset M)$ is finite.*

R. Longo in [31] provided an explicit bound depending only on $[M : N]$ for the cardinality of $\mathcal{L}(N \subset M)$ for irreducible subfactor $N \subset M$ of type II_1 with $[M : N] < \infty$. He showed that $|\mathcal{L}(N \subset M)| \leq l^l$, where $l = [M : N]^2$. We have improved Longo's bound using planar algebraic machinery and as a consequence provide another proof of Theorem 0.0.7. For this we define 'angle' between intermediate subfactors and exploit the fact that the 'angle' has certain rigidity. We also prove various properties of 'angle'.

In the first section we introduce a notion of 'angle' between intermediate subfactors of $N \subset M$ and describe few properties of 'angle'.

Definition 0.0.8. *Let $N \subseteq M$ be a subfactor of type II_1 with finite index. Consider two proper intermediate subfactors P and Q. Let e_P and e_Q be the biprojections corresponding to P and Q respectively. Define **angle**, denoted by $\alpha^N_M(P, Q)$, between P and Q as follows:*

$$\cos(\alpha^N_M(P, Q)) = \frac{\langle (e_P - e_1), (e_Q - e_1) \rangle_2}{\|e_P - e_1\|_2 \|e_Q - e_1\|_2}$$

where, $\langle x, y \rangle_2 = tr(y^* x)$ and hence $\|x\|_2 = (tr(x^* x))^{1/2}$ for $x, y \in P_2^{N \subseteq M}$.

Definition 0.0.9. *We define exterior angle, denoted by $\beta^N_M(P, Q)$, between P and Q as $\alpha^M_{M_1}(P_1, Q_1)$. Here $P_1 = \langle M, e_P \rangle$ and $Q_1 = \langle M, e_Q \rangle$ are Jones' basic construction.*

Let $\{\lambda_i\}$(resp. $\{\mu_j\}$) be (right) basis for P/N (resp. Q/N). Then, after expressing $\alpha_M^N(P,Q)$ in terms of $\{\lambda_i\}$ and $\{\mu_j\}$ it becomes evident that the $\alpha_M^N(P,Q)$ does not depend on M. More explicitly,

Fact 0.0.10. *1. Consider subfactors $N \subset P, Q \subset M, S$. Then $\alpha_M^N(P,Q) = \alpha_S^N(P,Q)$.*

2. For subfactors $R, N \subset P, Q \subset M$, $\beta_M^N(P,Q) = \beta_M^R(P,Q)$.

In general, $\alpha_M^N(P,Q)$ and $\beta_M^N(P,Q)$ are not equal. We investigate a special case when $\alpha_M^N(P,Q)$ equals $\pi/2$. We denote by (N,P,Q,M) the intermediate subfactors $N \subset P, Q \subset M$. Then we say (N,P,Q,M) is a commuting square if $E_P^M E_Q^M = E_N^M$ holds and is called a co-commuting square if the quadruple (M', Q', P', N') is a commuting square. We show (N,P,Q,M) is a commuting square iff $\alpha_M^N(P,Q) = \pi/2$. Thus (N,P,Q,M) is a co-commuting square iff $\beta_M^N(P,Q) = \pi/2$.

Below we characterize $\alpha_M^N(P,Q) = \pi/2$ in terms of bases:

Theorem 0.0.11. *For a quadruple (N,P,Q,M) the following are equivalent:*

1. *(N,P,Q,M) is a commuting square, that is $\alpha(P,Q) = \pi/2$.*

2. *If $\{\lambda_i\}$ (resp. $\{\mu_j\}$) is any basis for P/N (resp. Q/N), define $p = \sum_{i,j} \lambda_i \mu_j e_1 {\mu_j}^* {\lambda_i}^*$. Then,*
$$p = \bigvee \{v e_Q v^* : v \in \mathcal{U}(P)\}.$$

3. *If $\{\lambda_i\}$ (resp. $\{\mu_j\}$) is any basis for P/N (resp. Q/N), define $q = \sum_{i,j} \mu_j \lambda_i e_1 {\lambda_i}^* {\mu_j}^*$. Then,*
$$q = \bigvee \{u e_P u^* : u \in \mathcal{U}(Q)\}.$$

Then, we investigate when $\alpha(P,Q) = \pi/2 = \beta(P,Q)$. Explicitly we characterize non- degenerate commuting squares into various equivalent conditions in the following theorem:

Theorem 0.0.12. *Consider a quadruple (N, P, Q, M) such that $\alpha(P, Q) = \pi/2$. Let as before $\{\lambda_i\}$ (resp. $\{\mu_j\}$) be a basis for P/N (resp. Q/N). Then the following are equivalent:*

1. *$\beta(P, Q) = \pi/2$, that is (N, P, Q, M) is a co-commuting square.*

2. *$\{\lambda_i \mu_j\}$ is a basis for M/N.*

3. *$\{\mu_j \lambda_i\}$ is a basis for M/N.*

4. *(N, P, Q, M) is a non-degenerate commuting square (See [40]).*

In the process, we recover the well-known characterization of non-degenerate commuting square (with slight modification) due to Popa as in [40].

Put as usual $\delta = \sqrt{[M : N]}$ and $\tau = [M : N]^{-1}$. Firstly, we show that there is a certain rigidity in the possible values of angle. More precisely, we have the following theorem:

Theorem 0.0.13. *If P and Q are two distinct minimal intermediate subfactors of an irreducible subfactor $N \subset M$, then $\alpha_M^N(P, Q) > \frac{\pi}{3}$.*

Then we estimate the cardinality of the set of intermediate subfactors of an irreducible subfactor and thus in particular we give a purely planar algebraic proof of Theorem 0.0.7.

Theorem 0.0.14. *Suppose $N \subset M$ is an irreducible subfactor of type II_1 of finite index. Then,*

1. *The number of minimal intermediate subfactors is at most $3^{[M:N]} - 1$.*

2. *The number of intermediate subfactors is at most $9^{[M:N]}$.*

In the final section we try to investigate the intermediate subfactors in the general case. The following is the well-known fact about the unitary equivalence of close projections in a Hilbert space.

Theorem 0.0.15. *[34][Lemma 6.2.2] Let p and q be two projections on a Hilbert space H such that $\|p-q\| < 1$,then there exists a unitary $u \in C^*(p, q, 1_H)$ such that $\|1_H - u\| \leq \sqrt{2}\|p-q\|$ and $upu^* = q$.*

Motivated by above and Eric Christensen's perturbation techniques (see [12],[11]) we obtain the following result:

Theorem 0.0.16. *Let $N \subseteq M$ be a subfactor with $[M : N] < \infty$. Let P and Q be two intermediate subfactors of $N \subseteq M$. If $\|e_P - e_Q\| < 1/2$, then there exists an $*$-isomorphism $\Phi : P \longmapsto Q$ such that $\Phi|_N = id$. In particular, the subfactors $N \subset P$ and $N \subset Q$ are isomorphic.*

List of Figures

Chapter 1

Pimsner-Popa basis

As stated in the Synopsis, we assume $N \subseteq M$ is a unital inclusion of finite von Neumann algebras of one of the following two types.

$\mathcal{C}ase(1)$: N and M are II_1 factors with finite index $[M : N]$ and hence there exists unique Markov trace tr on M of modulus τ where $\tau = [M : N]^{-1}$.

$\mathcal{C}ase(2)$: Let $N \subseteq M$ be a connected inclusion of finite dimensional C^* algebras and hence there exists unique Markov trace tr on M of modulus τ where $\tau = \|G\|^{-2}$ where G is the inclusion matrix for $N \subseteq M$.

Pimsner and Popa have shown (in [36]) that for an inclusion $N \subset M$ of II_1 factors, M is a finitely generated projective module over N if and only if $[M : N]$ is finite by constructing a family $\{m_j : 1 \leq j \leq n+1\}$ of elements in M, with n equal to the integer part of $[M : N]$, which they called “orthonormal basis” for the pair $N \subseteq M$. In a similar manner, we find a slightly less restrictive notion of basis in [25]. In the first section of this chapter we quickly recall few basic results that we will often use throughout the thesis. In the next section of this chapter we characterize bases, in both cases (1) and (2), by three equivalent conditions. We extend this more general notion of basis (as in [25]) in the case (2) of connected inclusions of finite dimensional C^*- algebras. One advantage of this characterization is a transparent proof of Corollary (1.2.6). This result has been mentioned for the case of II_1 factors in [25] (Lemma(4.3.4 (i)), but the proof there seems incomplete. Our characterization of bases now clarifies this point, and also shows that bases behave in a nice way with respect to the Jones' tower.

As an application we show (in 3.1) how the use of bases leads to a natural proof of existence, in case (2), (see [10](Theorem 2.1)) of a unique extension of an automorphism on M which leaves N globally invariant, to an automorphism on the hyperfinite II_1 factor M_∞ which is compatible with the tower in the sense of fixing the Jones projections. It has been also proved that the initial automorphism will be automatically trace-preserving.

In [37] (Proposition 1.2) Pimsner and Popa have characterized basic construction for II_1 factor inclusion in two equivalent ways. See also [21] (Section 5). In this thesis we have characterized basic construction in terms of basis we introduced (see Lemma 1.3.6). We have succeeded to obtain a simple characterization of M_1 for finite demensional C^*- algebra case also. In [37] (Theorem 2.6) Pimsner and Popa have used their characterization of basic construction to describe the k-th step of the basic construction. In the Section 3.2 we have also given another proof of this construction using our characterization of basic construction and have also done the same for connected inclusion of finite dimensional C^* -algebras.

1.1 Preliminary

Suppose $N \subset M$ is an inclusion of finite von Neumann algebras. Fix some faithful normal tracial state tr on M, and consider the M-M-bimodule $\mathcal{H} = L^2(M, tr)$, with its distinguished cyclic trace vector Ω. Denote the trace preserving conditional expectation of M onto N by E. Then the orthogonal projection e_N of $\mathcal{H}$ onto the subspace $L^2(N, tr)$ is called the Jones' Projection, which satisfies :

1. $e_N \in N'$ and $N = M \cap \{e_N\}'$.
2. $e_N(x\Omega) = E_N^M(x)\Omega$ for all $x \in M$.
3. $e_N x e_N = E_N^M(x) e_N$ for all $x \in M$.
4. $J e_N = e_N J$ where J denotes the modular conjugation operator on $\mathcal{H}$.
5. $\langle M, e_N \rangle = J N' J$.

Then it is known that in both cases ((1) and (2)), as mentioned above, there exists a unique Markov trace on M (see [25]), and we can iterate the basic construction (see [37]) to obtain a tower,

$$M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \subseteq M_{n+1} \cdots$$

where $M_{n+1} = \langle M_n, e_{n+1} \rangle$ is the result of applying the basic construction for the pair $M_{n-1} \subseteq M_n$ and e_{n+1} is the projection implementing the tr_{M_n} preserving conditional expectation of M_n onto M_{n-1}. We then obtain a II_1 factor M_∞ in both the cases, which is hyperfinite in Case (2). Let $\{e_n : 1 \leq n < \infty\}$ be the sequence of projections in the II_1 factor M_∞ and write tr for the unique trace on it. Below we mention few basic facts about those (Jones') projections from [25] (Proposition 3.3.2) which will be often useful for this thesis.

1. The number $\tau = tr(e_n)$ is independent of n; further, τ is equal to $[M : N]^{-1}$ (respectively $\|G\|^{-2}$) for Case(1) (respectively Case(2));
2. (Markov Property) $tr(x e_n) = \tau tr(x)$ for all $x \in M_{n-1}, n \geq 1$;
3. $e_n \in {M'}_{n-2} \cap M_n$ for all $n \geq 1$, and in particular,

$$e_n e_m = e_m e_n \text{ if } |m - n| > 1;$$

4. $e_{n+1}e_ne_{n+1} = \tau e_{n+1}\ \forall\ n \geq 1$;

5. $e_ne_{n+1}e_n = \tau e_n\ \forall\ n \geq 1$.

For more details or a crash course in subfactor theory the reader is referred to [25],[17] and [22].

If $N \subset M$ are type II_1 factors as in Case (1), Pimsner and Popa in [36] have shown that 'M is a finitely generated projective module over N iff the index $[M : N]$ is finite' by constructing a so-called 'orthonormal basis of M over N' (see [36] (Proposition 1.3). This notion of 'orthonormal basis' has been extended to Case(2) in [25] (Lemma 5.7.3). Moreover, a general notion of bases (not necessarily orthogonal) has been introduced in [25] (Section 4.3). Motivated by Pimsner-Popa basis of subfactors Watatani has introduced (in the memoir [50]) what he calls 'quasi-basis for conditional expectation E' in a purely algebraic setting. Assuming the existence of quasi-basis he developed index for a conditional expectation of index-finite type (we call this 'Watatani Index' and denote it by $Index_wE$) which he shows to be independent of the choice of quasi-basis. He then investigated Jones' index theory in C^*-algebra setting. This 'quasi-bases' can be viewed as an example of a module frame of Hilbert C^*-module (see [15]).

1.2 Bases

For both the Cases the following easy but very useful Lemma holds whose proof can be found in [36] (Lemma 1.2) and for Case(2) see [25] (Remark 4.3.2(a)).

Lemma 1.2.1. *If $x_1 \in M_1$, then there exists unique element $x_0 \in M$ such that $x_1e_1 = x_0e_1$, this element is given by $x_0 = \tau^{-1}E_M(x_1e_1)$.*

In the following theorem we give three equivalent descriptions of basis, not necessarily orthonormal in the sense of Pimsner-Popa.

Theorem 1.2.2. *Let N and M be as in Case(1) or in Case(2). Then for a finite set $\{\lambda_i : i \in I = 1, 2, ...n\} \subseteq M$, the following are equivalent:*

1. *Let E_N be the tr- preserving conditional expectation of M onto N and define a matrix Q whose (i, j) entry is given by $q_{ij} = E_N(\lambda_i \lambda_j{}^*)$. Then Q is a projection in $M_n(N)$ such that $tr_{M_n(N)}(Q) = \tau^{-1}/n$.*

2. *$\sum_{i=1}^n \lambda_i{}^* e_1 \lambda_i = 1$, where e_1 is the Jones projection.*

3. *For any $x \in M$, $x = \sum_{i=1}^n E_N(x\lambda_i{}^*)\lambda_i$.*

Proof. (1) $\Longrightarrow$ (2) : This proof is mainly inspired by [36]. Assume (1) holds. Since tr on M is Markov, it extends to a unique trace on M_1, namely tr_{M_1}. Put $v_i = e_1\lambda_i$ and

$$v = \begin{bmatrix} v_1 & 0 & \dots & 0 \\ v_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ v_n & 0 & \dots & 0 \end{bmatrix}.$$

Then, $v_i v_j{}^* = e_1 \lambda_i \lambda_j{}^* e_1 = E_N(\lambda_i \lambda_j{}^*)e_1 = q_{ij}e_1$. Thus,

$$vv^* = \begin{bmatrix} q_{11}e_1 & q_{12}e_1 & \dots & q_{1n}e_1 \\ q_{21}e_1 & q_{22}e_1 & \dots & q_{2n}e_1 \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1}e_1 & q_{n2}e_1 & \dots & q_{nn}e_1 \end{bmatrix} = QE$$

where

$$E = \begin{bmatrix} e_1 & 0 & \dots & 0 \\ 0 & e_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e_1 \end{bmatrix}.$$

Thus by a property of Jones projection [22] (Proposition 3.1.4), $vv^* = QE = EQ$ and hence v is a partial isometry. Thus v^*v is a projection; i.e., $\sum_i v_i^* v_i$ is a projection

f (say) in $\langle M, e_1 \rangle = M_1$. But $f = \sum_i {\lambda_i}^* e_1 \lambda_i$ satisfies the following equations :

$$\begin{aligned} tr_{M_1}(f) &= n\ tr_{M_n(M_1)}\ (vv^*) \\ &= n\ tr_{M_n(M_1)}(QE) \\ &= n\ (1/n) \sum_i tr_{M_1}(q_{ii} e_1) \\ &= \sum_i \tau\ tr(q_{ii}) \ \ \text{(Markov property)} \\ &= \tau\ n\ tr_{M_n(N)}(Q) \\ &= 1 \ \ \text{(by (1))}. \end{aligned}$$

Thus $(1 - f) \geq 0$ and $tr_{M_1}(f) = 1$. Then faithfulness of tr_{M_1} implies $f = 1$. So, $\sum_i {\lambda_i}^* e_1 \lambda_i = 1$. Thus (1) implies (2).

(2) $\Longrightarrow$ (3): We assume that (2) holds. Let $x^* \in M$, then

$$\begin{aligned} x^* e_1 &= (\sum_i {\lambda_i}^* e_1 \lambda_i) x^* e_1 \\ &= \sum_i {\lambda_i}^* E_N(\lambda_i x^*) e_1 \\ &= (\sum_i {\lambda_i}^* E_N(\lambda_i x^*)) e_1. \end{aligned}$$

Again applying Lemma 1.2.1 and then taking adjoint we get (3).

(3) $\Longrightarrow$ (2) : We assume (3). Let x and y be two arbitrary elements of M. Then,

$$\begin{aligned} (\sum_i {\lambda_i}^* e_1 \lambda_i)(x e_1 y) &= \sum_i {\lambda_i}^* e_1 \lambda_i x e_1 y \\ &= \sum_i {\lambda_i}^* E_N(\lambda_i x) e_1 y \end{aligned}$$

$$= (xe_1y) \quad (\text{by } (3)).$$

Similarly,

$$\begin{aligned}(xe_1y)(\sum_i \lambda_i{}^* e_1 \lambda_i) &= \sum_i xe_1y\lambda_i{}^* e_1\lambda_i \\ &= \sum_i xe_1 E_N(y\lambda_i{}^*)\lambda_i \\ &= (xe_1y) \quad (\text{by } (3)).\end{aligned}$$

Then we know the space Me_1M, which is the linear span of $\{xe_1y : x, y \in M\}$, is a strongly dense *-subalgebra of M_1, see for instance [17] (Proposition 3.6.1(vii)). And since multiplication is separately strongly continuous it follows that $\sum_i \lambda_i{}^* e_1 \lambda_i = 1$.

(2) $\Longrightarrow$ (1) : Suppose (2) is true. Then,

$$\begin{aligned}e_1(\sum_k q_{ik}q_{kj}) &= e_1(\sum_k E_N(\lambda_i\lambda_k{}^*)E_N(\lambda_k\lambda_j{}^*)) \\ &= e_1(\sum_k E_N(\lambda_i\lambda_k{}^* E_N(\lambda_k\lambda_j{}^*))) \\ &= \sum_k e_1\lambda_i\lambda_k{}^* E_N(\lambda_k\lambda_j{}^*)e_1 \\ &= \sum_k e_1\lambda_i\lambda_k{}^* e_1\lambda_k\lambda_j{}^* e_1 \\ &= e_1\lambda_i(\sum_k \lambda_k{}^* e_1\lambda_k)\lambda_j{}^* e_1 \\ &= e_1\lambda_i\lambda_j{}^* e_1 \quad (\text{by } (2)) \\ &= e_1 E_N(\lambda_i\lambda_j{}^*) \\ &= e_1 q_{ij}.\end{aligned}$$

Thus applying Lemma 1.2.1 we get $Q^2 = Q$. Clearly $Q^* = Q$. Hence Q is a projection in $M_n(N)$. Now

$$\begin{aligned}
tr_{M_n(N)}(Q) &= (1/n)\sum_i tr(q_{ii}) \\
&= (1/n)\sum_i tr(E_N(\lambda_i \lambda_i^*)) \\
&= (1/n)\sum_i tr(\lambda_i \lambda_i^*) \\
&= (\tau^{-1}/n)\sum_i tr(e_1 \lambda_i \lambda_i^*) \quad \text{(Markov Property)} \\
&= (\tau^{-1}/n)\sum_i tr(\lambda_i^* e_1 \lambda_i) \\
&= (\tau^{-1}/n).
\end{aligned}$$

Hence (2) implies (1). □

Remark 1.2.3. *Taking adjoints in (3) it follows that the above three are also equivalent to* $x = \sum_{i=1}^{n} \lambda_i^* E_N(\lambda_i x)$, *for all* $x \in M$.

Definition 1.2.4. *A finite set* $\{\lambda_i : i \in I\} \subset M$ *satisfying any one of the equivalent conditions (i)-(iii) of Theorem 1.2.2 will simply be called a (left)* **basis** *for* M/N *(viewing* M *as a left* N*-module). On the other hand, the set of adjoints of a 'left-basis' would be called a 'right basis'(that is, in this case* $\sum_i \lambda_i e_1 \lambda_i^* = 1$ *or equivalently* $x = \sum_{i=1}^{n} E_N(x\lambda_i)\lambda_i^* = \sum_{i=1}^{n} \lambda_i E_N(\lambda_i^* x)$ *for all* $x \in M$*). A 'left basis' will be called a 'two-sided basis' if it is simultaneously a 'right basis'.*

Existence of bases: Comparing [36] (Proposition 1.3(c)(2)) and Theorem 1.2.2 we remark that any Pimsner-Popa basis for II_1 factor inclusions is automatically a basis according to our notion. For Case(1) an explicit construction has been given in [36](Proposition 1.3) while for Case (2) see [25] (Lemma 5.7.3), and [24] (Proposition 2.5). Explicitly, using path algebra due to Ocneanu and Sunder, a basis $\{\lambda_i : i \in I\}$ for the connected inclusion of finite dimensional C^*- algebra $A_0 \subseteq B_0$ has been explicitly constructed in the proof of Lemma 5.7.3 of [25] (see Equation 5.7.9). For

Case(2) see also [14](section 9.4). Observe that, Theorem 1.2.2 (1) now says that $Index_w\ E$ is same as Jones' index for Case (1) and equals to $\|G\|^2$ for Case (2).

Remark 1.2.5. *The row vector* $[E_N(x{\lambda_1}^*), .., E_N(x{\lambda_n}^*)] \in M_{1\times n}(N)Q$ *and conversely if* $[x_1, .., x_n] \in M_{1\times n}(N)Q$ *satisfies* $x = \sum_{i=1}^n x_i\lambda_i$ *then* $x_j = E_N(x{\lambda_j}^*)$ *for all* $j \in I$.

Exactly the same proof as in [25] (Proposition 4.3.3(b)(ii)) works.

Corollary 1.2.6. *Let* $N \subseteq M \subseteq P$ *be a tower of* II_1 *factors with* $[P : N] < \infty$ *(or a tower of finite dimensional* C^**-algebras where the two inclusions are connected with inclusion matrices* G *and* H *respectively). In either case, let* $\{\lambda_i : 1 \leq i \leq m\}$ *be a basis for* M/N *and* $\{\mu_j : 1 \leq j \leq n\}$ *be a basis for* P/M*, then* $\{\lambda_i\mu_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ *is a basis for* P/N.

Proof. Let $x \in P$ and as $\{\mu_j\}$ is a basis for P/M, we get, $x = \sum_{j=1}^n E_M(x{\mu_j}^*)\mu_j$. Now note $E_M(x{\mu_j}^*) \in M$ and $\{\lambda_i\}$ is a basis for M/N. Now item (3) of the Theorem 1.2.2 yields,

$$E_M(x{\mu_j}^*) = \sum_{i=1}^m E_N\{E_M(x{\mu_j}^*){\lambda_i}^*\}\lambda_i.$$

Thus we get,

$$\begin{aligned} x &= \sum_{j=1}^n [\sum_{i=1}^m E_N\{E_M(x{\mu_j}^*){\lambda_i}^*\}\lambda_i]\mu_j \\ &= \sum_{i=1}^m \sum_{j=1}^n E_N\{E_M(x{\mu_j}^*{\lambda_i}^*)\}\lambda_i\mu_j. \end{aligned}$$

Thus applying Theorem 1.2.2 (3) again we see that $\{\lambda_i\mu_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ is a basis for P/N. □

Corollary 1.2.7. *If* $\{\lambda_i : i \in I = \{1, 2, ...n\}\}$ *is a basis for* M/N*, then* $\{\tau^{-1/2}e_1\lambda_i\}$ *is a basis for* M_1/M*, where* $N \subseteq M$ *is an in inclusion as in Case (1) or Case (2).*

Proof. In Case (1), M_1 is a II_1 factor such that $[M_1 : M] = [M : N] < \infty$. In Case (2), inclusion matrix for $M \subseteq M_1$ is G^t and hence is a connected inclusion. In both the cases let e_2 be the Jones projection for the inclusion $M \subseteq M_1$. Then,

$$\begin{aligned}\sum_{i=1}^{n}\{\tau^{-1/2}e_1\lambda_i\}^* e_2\{\tau^{-1/2}e_1\lambda_i\} &= \tau^{-1}\sum_{i=1}^{n}\lambda_i{}^* e_1e_2e_1\lambda_i\\ &= \tau^{-1}\tau\sum_{i=1}^{n}\lambda_i{}^* e_1\lambda_i\\ &= 1 \quad (\text{since } \ \{\lambda_i\} \ \text{ is a basis}).\end{aligned}$$

Now (2) of the Theorem 1.2.2 yields the result. □

Remark 1.2.8. *From Corollary 1.2.7 every element of M_1 is expressible in the form $\sum_{i=1}^n x_ie_1y_i$ for some $x_i, y_i \in M$ (in fact, $x = \tau^{-1}\sum_{i=1}^n E_M(x\lambda_i^*e_1)e_1\lambda_i$); however this does not allow us to define a $*$-homomorphism on M_1 by merely specifying the image of an element of the form xe_1y, as we will need to verify that such a 'definition' is unambiguous; but we may define the above canonical decomposition to unambiguously define maps on M_1 once we know where to map elements of N, the basis vectors λ_i and e_1. This problem of ambiguity was part of the reason for us the study this notion of bases. The reader need only compare the crisp clarity of the proofs of unambiguity in the definition of α_1 in Theorem 1.3.3 and of ϕ in Lemma 1.3.6 with the corresponding proofs of Theorem 2.1 in [10] (actually only to be found in the arXiv version) and of Proposition 1.2 in [37], to appreciate this remark.*

Corollary 1.2.9. *Let $N \subseteq M$ as in Case (1) or (2) and $\{\lambda_i : i \in I\}$ be a basis for M/N. For $\widehat{i(k)} = (i_1, i_2, \cdots i_k) \in I^k, k \geqslant 1$ define*

$$\lambda_{\widehat{i(k)}} = \tau^{-k(k-1)/4}\lambda_{i_1}e_1\lambda_{i_2}e_2e_1\lambda_{i_3}\cdots\lambda_{i_{k-1}}e_{k-1}\cdots e_1\lambda_{i_k}.$$

Then $\{\lambda_{\widehat{i(k)}} : \widehat{i(k)} \in I^k\}$ is a basis for M_{k-1}/N.

Proof. Clearly the statement is true for $k = 1$ with the understanding that $M_0 = M$. Suppose the statement is true for k. Now applying Corollary 1.2.7 recursively we get M_k/M_{k-1} has basis $\{\tau^{-k/2}e_k e_{k-1}..e_1\lambda_{i_{k+1}} : i_{k+1} \in I\}$. Then applying Corollary 1.2.6 we see that M_k/N has basis,

$$\{\tau^{-k(k-1)/4}\tau^{-k/2}\lambda_{i_1}e_1\lambda_{i_2}e_2e_1\lambda_{i_3}..\lambda_{i_{k-1}}e_{k-1}..e_1\lambda_{i_k}e_ke_{k-1}..e_1\lambda_{i_{k+1}}\}$$

which is equal to $\{\lambda_{\widehat{i(k+1)}} : \widehat{i(k+1)} \in I^{k+1}\}$; and the proof of the inductive step is complete. $\square$

1.3 Applications

1.3.1 Compatible automorphisms of the Hyperfinite II_1 factor

Consider an inclusion as in Case (2). Then, we have a unique Markov trace tr on M. Next consider the Jones tower $N \subseteq M \subseteq M_1 \subseteq M_2 \subseteq$ ..... and let R be the hyperfinite II_1 factor arising from this tower [25]. Suppose further we have an automorphism α_0 on M such that $\alpha_0(N) = N$. In this present section we shall show using our concept of basis how we can costruct a unique extension of α_0 to an automorphism α of the hyperfinite II_1 factor R which is compatible with respect to the tower in the sense of fixing all the Jones projections and leaving M_i invariant.

This can be thought of as the finite dimensional C^*-algebraic version of [32] (Lemma 5.1). In that paper Loi studied automorphisms for a pair of factors using standard form of von Neumann algebras, whereas our treatment is based on basis for the corresponding inclusion. In the similar direction in [26], Kawahigashi dealt with automorphisms commuting with a faithful normal conditional expectation for a pair of σ-finite von Neumann algebras and related this with an action of a locally compact abelian group. See also [45], where the author was more concerned with commuting squares.

Lemma 1.3.2. *Let N, M, tr, α_0 be as above.Then α_0 is automatically trace preserving, that is $tr \circ \alpha_0 = tr$.*

Proof. Let the minimal central projections in N be $\{p_1, p_2, \cdots, p_m\}$ and those in M be $\{q_1, q_2, \cdots, q_n\}$. Then the inclusion matrix G is an $m \times n$ matrix. Observe α_0 permutes p_i's and q_j's. Say $p_i \mapsto p_{\tau(i)}$ and $q_j \mapsto q_{\sigma(j)}$ for $\tau \in \Sigma_m$ and $\sigma \in \Sigma_n$. As α_0 is an automorphism, $G(i,j) = G(\tau(i), \sigma(j))$. Equivalently, $G = TGS$ for permutation matrices T and S of sizes m and n respectively. Let $\vec{t}$ be the trace vector corresponding to tr for M. Then it is the unique positive Perron-Frobenius eigenvector of G^tG, hence also of $S^{-1}G^tGS$. But that implies $S\vec{t}$ is a positive eigenvector of G^tG with the same eigenvalue as of $\vec{t}$ and by uniqueness of Perron -Frobenius theory (see Chapter XIII[16]) we get $S\vec{t} = \vec{t}$. Hence $tr \circ \alpha_0 = tr$. □

Theorem 1.3.3. *Let α_0 be an automorphism of M such that $\alpha_0(N) = N$. Then there is a unique (trace preserving) automorphism α_1 of M_1 such that $\alpha_1(e_1) = e_1$, $\alpha_1(M) = M$ and the restriction of α_1 to M is α_0.*

Proof. We know there is a basis for M/N. Fix such a basis $\{\lambda_i : i \in I\}$. Take Q as in Theorem 1.2.2 (1). Firstly, we show $\{\alpha_0(\lambda_i) : i \in I\}$ is also a basis for M/N. Let Q_1 be the matrix with (i,j) entry given by $q_1(i,j) = E_N\{\alpha_0(\lambda_i {\lambda_j}^*)\}$. We claim $E_N(\alpha_0(x)) = \alpha_0(E_N(x))$ for all $x \in M$. Take any $y \in N$. Then $\alpha_0(N) = N$ implies there exists $s \in N$ such that $\alpha_0(s) = y$. Now,

$$\begin{aligned} tr(\alpha_0(E_N(x))y) &= tr(\alpha_0(E_N(x)s)) \\ &= tr(E_N(x)s) \quad (as\ tr \circ \alpha_0 = tr) \\ &= tr(E_N(xs)) \quad (as\ s \in N) \end{aligned}$$

$$
\begin{aligned}
&= tr(xs) \\
&= tr(\alpha_0(xs)) \quad (as\ tr \circ \alpha_0 = tr) \\
&= tr(\alpha_0(x)y) \quad (\text{ for all } x \in M).
\end{aligned}
$$

Thanks to the uniqueness of the trace preserving conditional expectation, we get $E_N(\alpha_0(x)) = \alpha_0(E_N(x))$. Thus Theorem 1.2.2 implies that Q_1 is a projection in $M_n(N)$ since Q is so. Now,

$$
\begin{aligned}
tr_{M_n(N)}(Q_1) &= (1/n)\sum_{i \in I} tr(q_1(i,i)) \\
&= (1/n)\sum_{i \in I} tr[E_N\{\alpha_0(\lambda_i {\lambda_j}^*)\}] \\
&= (1/n)\sum_{i \in I} tr(\lambda_i {\lambda_j}^*) \quad (by\ Lemma\ (1.3.2)) \\
&= tr_{M_n(N)} Q.
\end{aligned}
$$

Thus it follows from Theorem 1.2.2 that $\{\alpha_0(\lambda_i)\}$ is a basis for M/N. Let $x \in M_1$. Corollary 1.2.7 then implies

$$
x = \sum_{i \in I} \tau^{-1} E_M(x {\lambda_i}^* e_1) e_1 \lambda_i.
$$

Then define,

$$
\alpha_1(x) = \tau^{-1} \sum_{i \in I} \alpha_0(E_M(x {\lambda_i}^* e_1)) e_1 \alpha_0(\lambda_i).
$$

There is clearly no ambiguity in the definition of α_1.

Next we show that α_1 is a homomorphism. Consider $y \in M_1$. Now using the properties of Jones' projection and the fact that α_0 is a homomorphism we get the following series of equations:

$$
\begin{aligned}
&\alpha_1(x)\alpha_1(y) \\
&\quad = \tau^{-2} \sum_{i,j} \alpha_0[E_M(x {\lambda_i}^* e_1)] e_1 \alpha_0(\lambda_i) \alpha_0[E_M(y {\lambda_j}^* e_1)] e_1 \alpha_0(\lambda_j)
\end{aligned}
$$

$$= \tau^{-2} \sum_{i,j} \alpha_0[E_M(x\lambda_i{}^* e_1)] E_N(\alpha_0[\lambda_i E_M(y\lambda_j{}^* e_1)]) e_1 \alpha_0(\lambda_j)$$

$$= \tau^{-2} \sum_{i,j} \alpha_0[E_M(x\lambda_i{}^* e_1) E_N(\lambda_i E_M(y\lambda_j{}^* e_1))] e_1 \alpha_0(\lambda_j)$$

(since α_0 and E_N commute)

$$= \tau^{-2} \sum_{i,j} \alpha_0[E_M\{x\lambda_i{}^* e_1 E_N(\lambda_i E_M(y\lambda_j{}^* e_1))\}] e_1 \alpha_0(\lambda_j)$$

$$= \tau^{-2} \sum_{j} \alpha_0[E_M\{x E_M(y\lambda_j{}^* e_1) e_1\}] e_1 \alpha_0(\lambda_j) \qquad (1)$$

(since $\sum_i \lambda_i{}^* e_1 \lambda_i = 1$).

Similarly,

$$\alpha_1(xy)$$

$$= \tau^{-1} \sum_{i} \alpha_0[E_M(xy\lambda_i{}^* e_1)] e_1 \alpha_0(\lambda_i)$$

$$= \tau^{-1} \sum_{i} \alpha_0[E_M\{x(\sum_j \tau^{-1} E_M(y\lambda_j{}^* e_1) e_1 \lambda_j) \lambda_i{}^* e_1\}] e_1 \alpha_0(\lambda_i)$$

$$= \tau^{-2} \sum_{i,j} \alpha_0[E_M\{x E_M(y\lambda_j{}^* e_1) E_N(\lambda_j \lambda_i{}^*) e_1\}] e_1 \alpha_0(\lambda_i)$$

$$= \tau^{-2} \sum_{i,j} \alpha_0[E_M\{x E_M(y\lambda_j{}^* e_1 E_N(\lambda_j \lambda_i{}^*)) e_1\}] e_1 \alpha_0(\lambda_i)$$

$$= \tau^{-2} \sum_{i} \alpha_0[E_M\{x E_M(y\lambda_i{}^* e_1) e_1\}] e_1 \alpha_0(\lambda_i) \qquad (2)$$

(since $\sum_j \lambda_j{}^* e_1 \lambda_j = 1$).

Now comparing equations (1) and (2) we conclude that α_1 is indeed a homomorphism.

Next we show α_1 fixes e_1. Observe,

$$e_1 = \tau^{-1} \sum_i E_M(e_1 \lambda_i{}^* e_1) e_1 \lambda_i.$$

Now using our definition of α_1 and property of Jones' projection it is easy to see that,

$$\begin{aligned}
\alpha_1(e_1) &= \tau^{-1}\sum_i \alpha_0[E_M\{E_N(\lambda_i{}^*)e_1\}]e_1\alpha_0(\lambda_i) \\
&= \sum_i \alpha_0(E_N(\lambda_i{}^*))e_1\alpha_0(\lambda_i) \\
&= \sum_i E_N(\alpha_0(\lambda_i{}^*))e_1\alpha_0(\lambda_i) \quad \text{(as } E_N \text{ and } \alpha_0 \text{ commute)} \\
&= \sum_i e_1\alpha_0(\lambda_i)^* e_1\alpha_0(\lambda_i) \\
&= e_1.
\end{aligned}$$

In the last equation we have used the fact that $\{\alpha_0(\lambda_i)\}$ is a basis for M/N.

Next we will show that α_1 agrees with α_0 when it is restricted to M. Now, since α_0 is an automorphism, for $x \in M$, we find that,

$$\begin{aligned}
\alpha_1(x) &= \tau^{-1}\sum_i \alpha_0\{E_M(x\lambda_i{}^* e_1)\}e_1\alpha_0(\lambda_i) \\
&= \tau^{-1}\sum_i \alpha_0(x\lambda_i{}^*)E_M(e_1)e_1\alpha_0(\lambda_i) \\
&= \sum_i \alpha_0(x)\alpha_0(\lambda_i)^* e_1\alpha_0(\lambda_i) \qquad \text{(since } E_M(e_1) = \tau) \\
&= \alpha_0(x) \qquad\qquad \text{(as } \{\alpha_0(\lambda_i)\} \text{ is a basis for } M/N).
\end{aligned}$$

Now we want to show that α_1 is onto. Let $y \in M_1$. Then, $y = \sum_i y_i e_1 \alpha_0(\lambda_i)$, for some $y_i \in M$, since $\alpha_0(\lambda_i)$ is a basis for M/N. As, α_0 is an automorphism there is a unique $x_i \in M$ such that $\alpha_0(x_i) = y_i$. Put $x = \sum_i x_i e_1 \lambda_i$. Then x belongs to M_1. Now as we have already proved that α_1 is a homomrphism which preserves e_1 and agree with α_0 when restricted to M it follows trivially that $\alpha_1(x) = y$. Thus α_1 is onto.

Lastly we show α_1 is one-one. Observe, α_1 is $*$-preserving, since if $x = \sum_i x_i e_1 \lambda_i$ we find, exactly as above, that

$$\alpha_1(x^*) = \sum_i \alpha_1(\lambda_i)^* e_1 \alpha_1(x_i)^* = \{\sum_i \alpha_1(x_i) e_1 \alpha_1(\lambda_i)\}^* = \alpha_1(x)^*.$$

Now,

$$\begin{aligned}
tr(\alpha_1(x)) &= tr(\sum_i \alpha_0(x_i) e_1 \alpha_0(\lambda_i)) \\
&= \sum_i tr\{e_1 \alpha_0(\lambda_i) \alpha_0(x_i)\} \\
&= \tau \sum_i tr\{\alpha_0(\lambda_i x_i)\} \quad \text{(Markov property)} \\
&= \tau \sum_i tr(\lambda_i x_i) \quad \text{(by Lemma 1.3.2)} \\
&= \sum_i tr(x_i e_1 \lambda_i) \quad \text{(Markov property)} \\
&= tr(x).
\end{aligned}$$

So α_1 is tr-preserving and hence one-one. The uniqueness assertion is obvious since M and e_1 generate M_1. Thus α_1 satisfies all the properties mentioned in the theorem. □

Corollary 1.3.4. *Let α_0 be as in the previous theorem. Then there is a unique (trace preserving) automorphism α of the hyperfinite* II_1 *factor R such that $\alpha(e_i) = e_i, \alpha(M_i) = M_i$ for all $i \geq 1$ and $\alpha|M = \alpha_0$.*

Proof. Apply Theorem 1.3.3 recursively for the tower of basic construction to get a unique (trace preserving) automorphism α_i on M_i which leaves M_j invariant and fixes all e_j such that $1 \leq j \leq i$ and $\alpha_i|M_j = \alpha_j$. Thus we can define an automorphism (compatible with respect to the tower) α_∞ on $\cup_i M_i$ by, $\alpha_\infty(x) = \alpha_j(x)$ for $x \in M_j$. Now as α_∞ is bounded it extends to trace preserving automorphism α (say) on R.

Also since M_1 and e_i s generate R, uniqueness is straightforward. □

1.3.5 Iterating basic construction

The following gives a characterization of basic construction using bases, for both Case(1) and Case(2). This would be needed for our proof of the assertion regarding k-th step basic constructions.

Lemma 1.3.6. *Let $N \subseteq M$ be as in Case(1) or Case(2). Assume $\{\lambda_i : i \in \{1, 2, \cdots, n\}\}$ is a basis for M/N(which exist in both cases). Let P be a II_1 factor in Case(1) or a finite dimensional C^*-algebra in Case (2) such that P contains M and also contains a projection f such that $\sum_{i=1}^{n} {\lambda_i}^* f \lambda_i = 1$ and satisfies further the following three properties :*

1. *$fxf = E_N(x)f$ for all $x \in M$;*
2. *$\{\tau^{-1/2} f \lambda_i\}$ is a basis for P/M;*
3. *$n \longmapsto nf$ is an injective map from N into P (Condition (3) needs to be assumed in Case(2), unlike Case(1) where it is automatically met).*

Then there exists an isomorphism from $M_1 = \langle M, e_1 \rangle$ onto P which maps e_1 to f and fixes M .

In this situation we say that P is an instance of basic construction applied to the inclusion $N \subseteq M$ with a choice of projection implementing the conditional expectation being given by f.

Proof. $\mathcal{C}ase1$: Let $x \in M_1$. Now from Corollary 1.2.7 it follows that

$$x = \sum_i \tau^{-1/2} E_M(x\tau^{-1/2} {\lambda_i}^* e_1) e_1 \lambda_i.$$

Put $a_i = \tau^{-1}E_M(x\lambda_i{}^*e_1)$, then define a map $\phi : M_1 \mapsto P$ by $\phi(x) = \sum_i a_i f\lambda_i$, which is clearly well-defined. Note, if $y = \sum_i b_i e_1\lambda_i$ such that $[b_1, b_2, \cdots, b_n] \in M_{1\times n}(M)Q$, then by Remark 1.2.5 we conclude that

$$\phi(y) = \sum_i b_i f\lambda_i. \tag{1.1}$$

Let Q_1 be the matrix whose (i, j)-th entry is given by

$$q_1(i, j) = E_M((\tau^{-1/2}e_1\lambda_i)(\tau^{-1/2}e_1\lambda_j)^*).$$

Then $q_1(i, j) = E_N(\lambda_i\lambda_j{}^*) = q_{ij}$. Now, let x be as above and let $y \in M_1$. Put $y = \sum_i b_i e_1\lambda_i$ where $b_i = \tau^{-1}E_M(y\lambda_i{}^*e_1)$. Then the following equations follow from properties of Jones' projection,

$$\begin{aligned}
\phi(xy) &= \phi\{\sum_{i,j} a_i E_N(\lambda_i b_j)e_1\lambda_j\} \\
&= \phi\{\sum_{i,j} \tau^{-1}E_M(x\lambda_i{}^*e_1)E_N(\lambda_i b_j)e_1\lambda_j\} \\
&= \phi\{\sum_{i,j} \tau^{-1}E_M(x\lambda_i{}^*e_1E_N(\lambda_i b_j))e_1\lambda_j\} \\
&= \phi\{\sum_{j} \tau^{-1}E_M(xb_je_1)e_1\lambda_j\} \quad (\text{since } \sum_i \lambda_i{}^*e_1\lambda_i = 1) \\
&= \phi\{\sum_{j} \tau^{1}E_M[x\tau^{-1}E_M(y\lambda_j{}^*e_1)e_1]e_1\lambda_j\}.
\end{aligned}$$

Now it can be easily checked that the row vector:

$$[\tau^{-1}E_M\{x\tau^{-1}E_M(y\lambda_1{}^*e_1)e_1\}, \cdots, \tau^{-1}E_M\{x\tau^{-1}E_M(y\lambda_n{}^*e_1)e_1\}] \in M_{1\times n}(M)Q.$$

Thus, it follows from Equation(1.1) that,

$$\begin{aligned}\phi(xy) &= \sum_j \tau^{-1} E_M[x\tau^{-1}E_M(y\lambda_j{}^* e_1)e_1] f\lambda_j \\ &= \sum_{i,j} \tau^{-1} E_M(a_i e_1 \lambda_i b_j e_1) f\lambda_j \\ &= \sum_{i,j} \tau^{-1} a_i E_M(E_N(\lambda_i b_j)e_1) f\lambda_j \\ &= \sum_{i,j} a_i E_N(\lambda_i b_j) f\lambda_j \\ &= \sum_{i,j} a_i f\lambda_i b_j f\lambda_j \qquad \text{(by assumption (1))} \\ &= \phi(x)\phi(y).\end{aligned}$$

Also, we have,

$$\begin{aligned}\phi(e_1) &= \tau^{-1} \sum_i E_M(e_1 \lambda_i{}^* e_1) f\lambda_i \\ &= \sum_i E_N(\lambda_i{}^*) f\lambda_i \qquad \text{(since } E_M(e_1) = \tau) \\ &= \sum_i f\lambda_i{}^* f\lambda_i \qquad \text{(by assunption (1))} \\ &= f \qquad \text{(since, } \sum_i \lambda_i{}^* f\lambda_i = 1).\end{aligned}$$

Thus ϕ is a nonzero homomorphism. Now assume, $x \in M$, then,

$$\begin{aligned}\phi(x) &= \sum_i \tau^{-1} E_M(x\lambda_i{}^* e_1) f\lambda_i \\ &= \sum_i x\lambda_i{}^* f\lambda_i \qquad \text{(since } x\lambda_i{}^* \in M) \\ &= x.\end{aligned}$$

ϕ is also *-preserving, as, if $x = \sum_i a_i e_1 \lambda_i$ is any element of M_1, then the following identities hold:

$$\begin{aligned}\phi(\{\sum_i a_i e_1 \lambda_i\}^*) &= \sum_i {\lambda_i}^* f {a_i}^* \quad \text{(since } \phi(e_1) = f \text{ and } \phi|_M = id)\\ &= \{\sum_i a_i f \lambda_i\}^*\\ &= \{\phi(\sum_i a_i e_1 \lambda_i)\}^*.\end{aligned}$$

Thus $\phi(x^*) = \phi(x)^*$.

Since M_1 is a factor, ϕ is automatically injective.

Finally we show ϕ is onto. For this purpose assume $z \in P$, assumption (2) then implies $z = \sum_i c_i f \lambda_i$ for some $c_i \in M$. Put, $y = \sum_i c_i e_1 \lambda_i$ which belongs to M_1 and since ϕ is a homomorphism sending e_1 to f and whose restriction to M is identity, we clearly get $\phi(y) = z$, proving onto. Thus ϕ is an isomorphism satisfying all the conditions stated in the Lemma.

$\mathcal{C}ase2$: Note assumption(2) implies $P = MfM$. Also this together with assumption(1) imply that $PfP = MfM$. Thus $P = PfP$ which forces $Z_P(f) = 1$. Now just applying Corollary 5.3.2 in [25] we get the result.

This completes the Lemma.

□

Now we give another proof of k-th step basic construction for an inclusion of II_1 factors using basis and also we show it can be done for Case(2).

Theorem 1.3.7. *Let $N \subseteq M$ be a pair of von Neumann algebras as in Case(1) or (2) and $N \subseteq M \subseteq M_1 \subseteq \cdots$ be the tower of II_1 factors (or finite dimensional C^*-algebras) in Case(1) (or in Case(2) respectively) which can be obtained by iterating the basic construction. Let $e_i \in M_i$ be the Jones' projections. Then for*

$m \geq 0, k \geq -1$, $M_k \subseteq M_{k+m} \subseteq M_{k+2m}$ is an instance of basic construction with a choice of projection implementing the conditional expectation of M_{k+m} onto M_k is given by

$$\begin{aligned} e_{[k,k+m]} & \\ &= \tau^{-m(m-1)/2}(e_{k+m+1}e_{k+m}\cdots e_{k+2})(e_{k+m+2}e_{k+m+1}\cdots e_{k+3}) \\ &\quad \cdots(e_{k+2m}e_{k+2m-1}\cdots e_{k+m+1}). \end{aligned}$$

Proof. Without loss of generality we shall prove that $M_{-1} \subseteq M_n \subseteq M_{2n+1}$ is an instance of basic construction with $e_{[-1,n]}$ is the required projection. Assume $\{\lambda_i : i \in 1, 2, ..n\}$ is a basis for M/N(which exists in both the Cases). Now from Corollary 1.2.9 we know that $\{\lambda_{\widehat{i(n+1)}}\}$ is a basis for M_n/N.

Now applying Corollary 1.2.6 and Corollary 1.2.7 repeatedly we get M_{2n+1}/M_n has basis,

$$\tau^{-1/2\{(n+1)+(n+2)+\cdots+(2n+1)\}}(e_{n+1}\cdots e_1)\lambda_{i_1}(e_{n+2}\cdots e_1)\lambda_{i_2}..(e_{2n}..e_1)\lambda_{i_n}(e_{2n+1}..e_1)\lambda_{i_{n+1}}.$$

Observe that,

$$\begin{aligned}(e_{n+1}\cdots e_1)\lambda_{i_1}(e_{n+2}\cdots e_1)\lambda_{i_2}\cdots(e_{2n+1}\cdots e_1)\lambda_{i_{n+1}} &= (e_{n+1}\cdots e_1)(e_{n+2}\cdots e_2)(e_{n+3}\cdots e_3) \\ &\quad \cdots(e_{2n+1}\cdots e_{n+1})\lambda_{i_1}e_1\lambda_{i_2}e_2e_1\lambda_{i_3} \\ &\quad \cdots\lambda_{i_n}e_n\cdots e_1\lambda_{i_{n+1}}.\end{aligned}$$

In other words it shows that, M_{2n+1}/M_n has basis as, $\{\tau^{-(n+1)/2}e_{[-1,n]}\lambda_{\widehat{i(n+1)}}\}$.

Note, $[M_n : N] = [M : N]^{(n+1)} = \tau^{-(n+1)}$. Thus condition (2) of the Lemma 1.3.6 holds for Case(1).

To do the same for finite dimensional C^*-algebras we break this into two Cases.

$\mathcal{S}ubcase1$: Suppose n is odd. Then the inclusion matrix for $N \subseteq M$ would be $(GG^t)^k$ where $n = (2k-1)$. But it is easy to see that $\|(GG^t)^k\| = \|G\|^{2k} = \|G\|^{(n+1)} = \tau^{-(n+1)/2}$. Thus condition (2) of the Lemma 1.3.6 holds in this Case.

$\mathcal{S}ubcase2$: Here n is even, $n = 2m$ (say). Then the inclusion matrix for $N \subseteq M$ would be $G(G^tG)^m$. Then we see, $\|G^tG(G^tG)^m\| \leq \|G^t\|\|G(G^tG)^m\| = \|G\|\|G(G^tG)^m\|$. Now applying the Case(1) in left hand side, we get that $\|G\|^{2m+1} \leq \|G(G^tG)^m\|$. The opposite inequality is obvious. Thus condition (2) of Lemma 1.3.6 holds in this Case also.

We need to show that, for all $k \geq 1$, (for both Cases),

$$\sum_{i_1, i_2, \cdots, i_k} \lambda^*_{\widehat{i(k)}} e_{[-1,k-1]} \lambda_{\widehat{i(k)}} = 1. \tag{1.2}$$

We prove it by induction over $k \geq 1$. It is easy to see that, $\lambda_{\widehat{i(n)}}(\tau^{-n/2} e_n..e_1 \lambda_{i_{n+1}}) = \lambda_{\widehat{i(n+1)}}$ and hence, $(\tau^{-n/2} \lambda^*_{i_{n+1}} e_1 \cdots e_n) \lambda^*_{\widehat{i(n)}} = \lambda^*_{\widehat{i(n+1)}}$. Suppose, as induction hypothesis, for $n \geq 1$,

$$\sum_{i_1, i_2, \cdots, i_n} \lambda^*_{\widehat{i(n)}} e_{[-1,n-1]} \lambda_{(\widehat{i(n)}} = 1. \tag{1.3}$$

Since $\sum_i {\lambda_i}^* e_1 \lambda_i = 1$, we see that Equation (1.2) holds for $k = 1$.

Also we know, for $n \geq 1$,

$$e_{[-1,n]} = \tau^{-n}(e_{n+1}e_{n+2}...e_{2n+1}) e_{[-1,n-1]} (e_{2n}e_{2n-1}..e_{n+1}).$$

Thus,

$$\sum_{i_1, i_2, .., i_{n+1}} \lambda^*_{\widehat{i(n+1)}} e_{[-1,n]} \lambda_{\widehat{i(n+1)}}$$
$$= \sum_{i_1, i_2, .. i_{n+1}} \tau^{-2n} \lambda^*_{i_{n+1}} (e_1 e_2 ... e_n) \lambda^*_{\widehat{i(n)}} (e_{n+1}e_{n+2}..e_{2n+1}) e_{[-1,n-1]}$$

$$
\begin{aligned}
&(e_{2n}..e_{n+1})\lambda_{\widehat{i(n)}}(e_ne_{n-1}..e_1)\lambda_{i_{n+1}} \\
&= \sum_{i_1,i_2,,..i_{n+1}} \tau^{-2n}\lambda^*_{i_{n+1}}(e_1e_2..e_n)(e_{n+1}..e_{2n+1})\lambda^*_{\widehat{i(n)}}e_{[-1,n-1]}\lambda_{\widehat{i(n)}} \\
&(e_{2n}e_{2n-1}..e_{n+1})(e_ne_{n-1}..e_1)\lambda_{i_{n+1}} \\
&= \tau^{-2n}\sum_{i_{n+1}}\lambda^*_{i_{n+1}}(e_1e_2....e_{2n+1})(e_{2n}e_{2n-1}...e_1)\lambda_{i_{n+1}} \\
&[\text{by Equation } (1.3)] \\
&= \sum_{i_{n+1}}\lambda^*_{i_{n+1}}e_1\lambda_{i_{n+1}} \\
&[\text{since } (e_1e_2..e_{2n+1})(e_{2n}e_{2n-1}..e_1) = \tau^{2n}e_1] \\
&= 1.
\end{aligned}
$$

Here, the second equation holds as $\lambda_{\widehat{i(n)}} \in M_{n-1}$ and $(e_{n+1}e_{n+2}..e_{2n+1})$, $(e_{2n}e_{2n-1}..e_{n+1})$ both commute with M_{n-1}.

Hence the induction is complete.

Now we show property(1) of the Lemma 1.3.6. As induction hypothesis, suppose, for $n \geq 0$,

$$e_{[-1,n]}x_ne_{[-1,n]} = E_N(x_n)e_{[-1,n]} \text{ for } x_n \in M_n.$$

It trivially holds for $n = 0$. Then, for $n \geq 0$, and for $x_{n+1} \in M_{n+1}$, we get the following array of equations,

$$
\begin{aligned}
&e_{[-1,n+1]}x_{n+1}e_{[-1,n+1]} \\
&\quad = \tau^{-2(n+1)}(e_{n+2}..e_{2n+3})e_{[-1,n]}(e_{2n+2}..e_{n+2})x_{n+1}(e_{n+2}..e_{2n+3})e_{[-1,n]}(e_{2n+2}..e_{n+2}) \\
&\quad = \tau^{-2(n+1)}(e_{n+2}..e_{2n+3})e_{[-1,n]}(e_{2n+2}..e_{n+3})E_{M_n}(x_{n+1})(e_{n+2}..e_{2n+3})e_{[-1,n]}(e_{2n+2}..e_{n+2}) \\
&\quad = \tau^{-2(n+1)}(e_{n+2}..e_{2n+3})e_{[-1,n]}(e_{2n+2}..e_{n+3})(e_{n+2}..e_{2n+3})E_{M_n}(x_{n+1})e_{[-1,n]}(e_{2n+2}..e_{n+2}) \\
&\quad = \tau^{-2(n+1)}(e_{n+2}..e_{2n+3})e_{[-1,n]}(\tau^ne_{2n+2}e_{2n+3})E_{M_n}(x_{n+1})e_{[-1,n]}(e_{2n+2}..e_{n+2}) \\
&\quad = \tau^n\tau^{-2(n+1)}(e_{n+2}..e_{2n+2})e_{[-1,n]}(e_{2n+3}e_{2n+2}e_{2n+3})E_{M_n}(x_{n+1})e_{[-1,n]}(e_{2n+2}..e_{n+2})
\end{aligned}
$$

$$= \tau^{-(n+1)}(e_{n+2}..e_{2n+3})e_{[-1,n]}E_{M_n}(x_{n+1})e_{[-1,n]}(e_{2n+2}..e_{n+2})$$

$$= \tau^{-(n+1)}(e_{n+2}..e_{2n+3})E_N(x_{n+1})e_{[-1,n]}(e_{2n+2}..e_{n+2}) \qquad \text{[Induction hypothesis]}$$

$$= E_N(x_{n+1})e_{[-1,n+1]}.$$

The fourth equation holds because of the almost trivial fact that

$$(e_{2n+2}..e_{n+3})(e_{n+2}..e_{2n+3}) = \tau^n e_{2n+2}e_{2n+3}. \tag{1.4}$$

It should be mentioned that throughout we have used the fact that, for $n \geq 0$,

$$e_{[-1,n+1]} = \tau^{-(n+1)}(e_{n+2}..e_{2n+3})e_{[-1,n]}(e_{2n+2}..e_{n+2}).$$

This completes the induction.

Now using Lemma 1.3.6 we get the desired result for II_1 factor Case.

For finite dimensional C^*- algebra the only remaining thing is to prove that the map $x \longmapsto xe_{[-1,n]}$ for $x \in N$ is injective. From Lemma 1.2.1 it follows that $xe_1 = 0$ implies $x = 0$ for $x \in N$, proving the above fact for $n = 0$. Suppose the statement is true for $(n-1)$, that is for $x \in N, xe_{[-1,n-1]} = 0$ implies $x = 0$. Let for $x \in N, xe_{[-1,n]} = 0$. Thus, $(\|xe_{[-1,n]}\|_2)^2 = tr(xe_{[-1,n]}x^*) = 0$. Note,

$$0 = tr(e_{[-1,n]}x^*x)$$

$$= tr((e_{n+1}e_{n+2}..e_{2n+1})e_{[-1,n-1]}(e_{2n}..e_{n+1})x^*x)$$

$$= tr(e_{[-1,n-1]}(e_{2n}..e_{n+1})(e_{n+1}..e_{2n+1})x^*x) \qquad (\text{since } x^*x \in N)$$

$$= tr(e_{[-1,n-1]}(\tau^{n-1}e_{2n}e_{2n+1})x^*x). \qquad (\text{by Equation (1.4)})$$

But as we know 'tr' is Markov, we conclude from the last equation $tr(e_{[-1,n-1]}x^*x) = 0$, that is $tr(xe_{[-1,n-1]}x^*) = 0$. In other words,$xe_{[-1,n-1]} = 0$ and now from induction hypothesis we conclude $x = 0$. Hence the induction is complete.

This completes the proof for both the Cases. $\square$

Chapter 2

Intermediate planar algebra revisited

Jones initiated the modern subfactor theory by defining the index of a subfactor and exploring various properties of the same in [22]. Later, standard invariant has been developed and axiomatized by Ocneanu's Paragroup ([35]), Popa's λ-lattices ([43]) and Jones' planar algebras ([23]) as a complete invariant for a 'good' (and an important) class of subfactors. The notion of planar algebras has been evolving since Jones introduced it. In the sequel we will mainly follow the notation of planar algebra as in [27]. After recalling the notation and basic facts in the first section of this chapter we consider the situation in which an irreducible subfactor $N \subset M$ has an intermediate subfactor Q; and describe the tower of iterated basic construction of $N \subseteq Q$ in terms of the the corresponding tower of $N \subseteq M$. This is the crucial step in obtaining standard invariant of $N \subset Q$ in terms of $N \subset M$. The Jones' tower of $N \subset Q$ in terms of $N \subset M$ was described in [5]. Here we give another proof using the characterization of the basic construction as in Lemma 1.3.6, in terms of Pimsner-Popa bases. It should be mentioned that D. Bisch gave a partial description of standard invariant of $N \subset Q$ in [7] giving the standard invariant of the inclusion $N \subset Q_1$, where Q_1 is the first step basic construction for $N \subset Q$.

In the third section we work out a reformulation of the proof of the fact that the planar algebra $P^{(N \subseteq Q)}$ may be derived from $P^{(N \subseteq M)}$ (see [5] and [28]) by requiring that the action of a planar tangle T is given by Equation (2.1) below. It is well-known from earlier work of Bisch [6] - and reformulated in [9] and [29] in the planar algebraic terms that we will actually use here - that such intermediate subfactors are in bijective correspondence with so-called *biprojections*, say, $q \in P_2^{(N \subset M)}$. We wish here to describe (in Theorem 2.3.3) the planar algebra of $N \subset Q$ in terms of the planar algebra of $N \subset M$, while the planar algebra of $Q \subset M$ can be obtained by applying these results to $M \subset M_1$. The biprojection q corresponding to the intermediate subfactor Q gives rise naturally to a mapping $F = \{F_m\}$ from tangles of any color (say m) to partially labelled tangles of the same color (see Definition 2.1.12), and a scalar-valued function α defined on the collection of all tangles (see Definition 2.3.1), such that $P^{(N \subset Q)}$ may be identified with a planar algebra, call it P', with $P'_n = range(Z_{F(I_n^n)}^{(N \subset M)})$,where I_n^n is the identity tangle of colour n and the multilinear map $Z^{(N \subset Q)}$ associated to a tangle $T_{k_1, \cdots, k_b}^{k_0}$ is given by

$$Z_T^{(N \subset Q)} = \alpha(T) Z_{F(T)}^{(N \subset M)}, \tag{2.1}$$

where both sides are thought of as acting on $\otimes_{i=1}^b P'_{k_i}$. The slightly involved proof of the above assertion takes some work - see Theorem 2.3.4 and Theorem 2.3.3. The difference in proofs here and in [5] stems from the two ways that a planar algebra P can be described, respectively, (i) as in [27] (where one says what the underlying spaces P_n are, and explicitly describes the multilinear operator Z_T^P associated to a planar tangle T, and then verifying that these tangle maps satisfy the necessary compatibility conditions, as in Theorem 2.3.3), and (ii) by specifying a non-degenerate scalar-valued partition function Z on 0-tangles (labelled by $S = \coprod_{k \in \ Col} S_k$) which is invariant under planar isotopy and multiplicative on connected components (as in

[23]). Thus, one may say that a 'bonus' in our approach is that we know how any planar tangle acts on a vector in its domain.

With our formulation of intermediate planar algebra, as an application, in Section 2.4.5 we have recovered the result of [30] which establishes an one-one correspondence between a planar subalgebra P^{Θ} of the 'group planar algebra' which is naturally associated with a group Θ of automorphisms of the given group G and the planar agebra corresponding to the 'subgroup-subfactor' associated with the inclusion $\Theta \subset (G \rtimes \Theta)$. It seems that there is a slight inaccuracy with the constant in the defining isomorphism β_k of [30], while the corrected constant may be found in Definition 2.4.14.

2.1 Notation and some basic facts

In this chapter, all factors will be of type II_1, and all subfactors $N \subset M$ will be of finite index $[M : N]$. By tr_M we will mean the unique normal faithful trace defined on M. E_N^M will denote the trace preserving conditional expectation from M onto N; we shall often omit M and write E_N when doing so is unambiguous.

Following Bisch ([6]), we denote the 'Jones towers' built from the basic construction for $N \subset Q \subset M$ as:

$$N \subset Q \subset M \subset P_1 \subset M_1 \subset P_2 \subset M_2 \subset P_3 \subset \ldots \subset M_{2n+1} \ .$$

We write $e_{\epsilon,i}, \epsilon \in \{0, 1\}, i \geq 1$ for the projections:

$$e_{0,i} : L^2(M_{i-1}) \to L^2(P_{i-1}) \qquad \text{and} \qquad e_{1,i} : L^2(M_{i-1}) \to L^2(M_{i-2})$$

so that $P_i = \langle M_{i-1}, e_{0,i} \rangle$ and $M_i = \langle M_{i-1}, e_{1,i} \rangle$ (here we set $M_0 = M$, $M_{-1} = N$, $P_0 = Q$).

The description of the algebras generated by $e_{0,i}$ and $e_{1,i}$ are given in [8]. We will use the following relations appearing in [8]. In what follows, as usual, $[a, b] = 0$ means that a and b commute, and $[a, B] = 0$ means that a commutes with all elements of the set B.

Fact 2.1.1. *The following relations hold:*

1. $e_{0,i}e_{1,i} = e_{1,i}$,
2. $[e_{a,i}, e_{b,j}] = 0$ *for* $|i - j| \geq 2$,
3. $[e_{0,i}, e_{0,i\pm1}] = 0$,
4. $[e_{0,i},\ P_{i-1}] = [e_{1,i},\ M_{i-2}] = 0$,
5. *for* i *even,* $e_{0,i}e_{1,i\pm1}e_{0,i} = [Q : N]^{-1}e_{0,i}e_{0,i\pm1}$, *and* $e_{1,i}e_{0,i\pm1}e_{1,i} = [M : Q]^{-1}e_{1,i}$,
6. *for* i *odd,* $e_{0,i}e_{1,i\pm1}e_{0,i} = [M : Q]^{-1}e_{0,i}e_{0,i\pm1}$, *and* $e_{1,i}e_{0,i\pm1}e_{1,i} = [Q : N]^{-1}e_{1,i}$,
7. $e_{1,i}e_{1,i\pm1}e_{1,i} = [M : N]^{-1}e_{1,i}$.

Proof. For a proof look at Proposition 5.1 of [8] . □

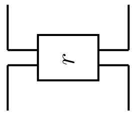
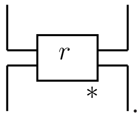
V. Jones introduced his theory of **planar algebras** in [23] as an axiomatization of the standard invariant of subfactors. A summary of planar algebra terminology is given in [29] and also a crash course on planar algebras is given in [27]. We will mainly follow the notation for planar algebras from [27](Section 2). Thus, we write P_k for the k-box space $N' \cap M_{k-1}$, $\delta = [M : N]^{-1/2}$ and write Z_T for the multilinear operator corresponding to a 'planar tangle' T. Consider the set $Col = \{0, 1, 2, \cdots\} \times \{\pm1\}$, elements of which we refer to as colours. We will typically write a color as (k, ϵ) where ϵ is either $+$ or $-$ and stands for $+1$ or 1. A planar tangle is defined on $\mathbb{R}^2$ which is consisting of the following data:

(1) An output disc and in the interior there are finitely many input discs.

(2) Each disc has an even number (possibly 0) of points marked on its boundary circles.

(3) Finitely many smooth strands in the interior of the output disc and the complement of the input discs who meets the boundaries of discs transversally.

(4) Each boundary is partitioned into finitely many intervals and a $*$ is assigned to a distinguished interval to indicate the relative position.

(5) The connected components of $\mathbb{R}^2$ (of the complement of the curves) in the output disc are called regions. The regions admit a checkerboard shading such that across any curve, the shading toggles. For each disc, one of its boundary arcs is distinguished and marked with a $*$ placed near it (whereas in [27],$*$ was marked to a distinguished point). As is usual, we will normally draw the discs as boxes with their $*$ arcs unmarked and assumed to contain their north-west corner (and in exceptional cases when it has been necessary to use a '2-click rotation, as in the following figure, for instance, the $*$-interval will be explicitly marked); typically, when a 2-box has a q in it, the $*$-arc has to be in a white arc, and for biprojections, it is immaterial which white arc has the $*$, and we may omit indicating the $*$. Similarly, we shall sometimes omit drawing the external disc. (If from the context the shading is clear we will omit that also.) If $r \in P_2$ sometimes we also write  for .
Lastly the tangles are defined only up to a planar isotopy preserving the $*$-arcs, the shading and the numbering of the internal discs.

Definition 2.1.2. *A planar algebra is a collection* $\{P_{(k,\epsilon)} : (k,\epsilon) \in Col\}$ *of vector spaces over* $\mathbb{C}$ *such that given any tangle* $T = T^{(k_0,\epsilon_0)}_{(k_1,\epsilon_1),(k_2,\epsilon_2),\cdots,(k_b,\epsilon_b)}$, *there is an associated linear map* Z_T

$$\begin{cases} P_{(k_1,\epsilon_1)} \otimes P_{(k_2,\epsilon_2)} \otimes \cdots \otimes P_{(k_b,\epsilon_b)} \longrightarrow P_{(k_0,\epsilon_0)} \text{ if } b > 0, \\ \mathbb{C} \longrightarrow P_{(k_0,\epsilon_0)} \text{ if } b = 0, \end{cases}$$

which need to satisfy Compatibility with Renumbering, Compatibility with Composition and Non-degeneracy axiom (see [27] for details).

In the Figure 2.1 below, we give examples of some important tangles.

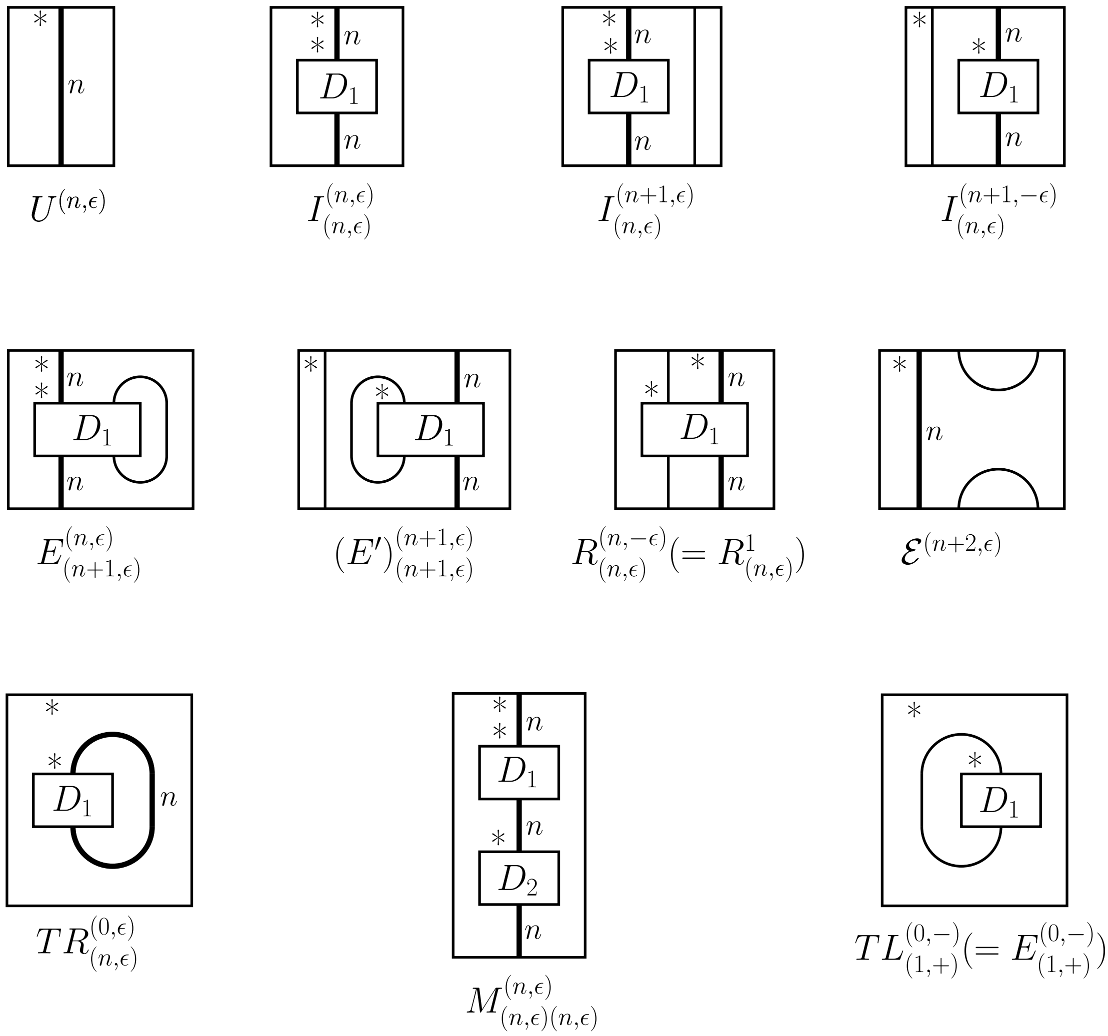


Figure 2.1: Important tangles

We now define morphism of planar algebras.

Definition 2.1.3. *If P, Q are planar algebras, a morphism from P to Q is a collection $\{\phi_{(k,\epsilon)} : P_{(k,\epsilon)} \longrightarrow Q_{(k,\epsilon)}\}_{(k,\epsilon)\in Col}$ of linear maps such that given any tangle*

$T = T^{(k_0,\epsilon_0)}_{(k_1,\epsilon_1),(k_2,\epsilon_2),\cdots,(k_b,\epsilon_b)}$, *the following diagram commutes:*

$$\begin{array}{ccc} \otimes_{j=1}^{b} P_{(k_j,\epsilon_j)} & \xrightarrow{Z_T^P} & P_{(k_0,\epsilon_0)} \\ {\scriptstyle \otimes_{j=1}^{b}\phi_{(k_j,\epsilon_j)}} \downarrow & & \downarrow {\scriptstyle \phi_{(k_0,\epsilon_0)}} \\ \otimes_{j=1}^{b} Q_{(k_j,\epsilon_j)} & \xrightarrow[Z_T^Q]{} & Q_{(k_0,\epsilon_0)} \end{array}$$

Figure 2.2: Planar algebra morphism

Definition 2.1.4. *A planar algebra P is said to be* ***connected*** *if $dimP_{(0,\pm)} = 1$. In this case, there are canonical identifications $P_{(0,\pm)} \cong \mathbb{C}$. Suppose P is a connected planar algebra and T is a 0-tangle (by which we shall mean a both $+$ and $-$ tangle). Thus the corresponding multilinear operator Z_T assigns a scalar to each 'labelled 0-tangles' and is referred to as the partition function associated to the planar algebra (see [28]).*

Definition 2.1.5. *A connected planar algebra P is said to have* ***modulus*** *δ if there is a scalar δ such that $Z^P_{T^{0,\pm}} = \delta Id_{\mathbb{C}}$ where $T^{0,+}$ (resp., $T^{0,-}$) denotes the $(0,+)$ (resp., $(0,-)$) tangle with no internal disc and a single closed loop.*

Following the notation of [27] we denote the so-called generating tangles of [23] by $\{\mathcal{E}^k : k \geq 2\}, \{(E')^k_k : k \geq 1\}$ and $\{E^k_{k+1}, M_k, I^{k+1}_k : k \in Col\}$ which are called Jones Projection tangles, left conditional expectation tangles,(right) conditional expectation tangles, multiplication tangles and inclusion tangles respectively (see Figure 2.1). The most interesting planar algebras are 'subfactor planar algebras' which will be mostly used in this thesis. This first appeared in [23].

Definition 2.1.6. *We shall say that P is a subfactor planar algebra if:*

(1) P is connected, finite-dimensional(in the sense that $dimP_{(k,\epsilon)} < \infty$, for all $(k,\epsilon) \in Col.$), spherical (in the sense that if its partition function assigns the same value to any two 0-tangles which are isotopic as tangles on the 2-sphere), and has positive modulus;

(2) each P_k is a C^*-algebra in such a way that, if T is a k_o-tangle with external disc D_0 and b internal discs D_i of colors k_i, and if $x_i \in P_{k_i}, 1 \leq i \leq b$ then $Z_T(x_1 \otimes \cdots \otimes x_b)^* = Z_{T^*}(x_1{}^* \otimes \cdots \otimes x_b{}^*)$; and

(3) if we define the 'pictorial trace' on P by $tr_{k+1}(x)1_+ = \delta^{-(k+1)} Z_{E_1^{0,+}} Z_{E_2^1} \cdots Z_{E_{k+1}^k}(x)$ for $x \in P_{k+1}$, then tr_m is a faithful positive trace on P_m for all $m \geq 1$.

One has the following striking correspondence between subfactor planar algebras and subfactors.

Theorem 2.1.7. *[23] Let $N \subset M(= M_0) \subset^{e_1} M_1 \subset \cdots \subset^{e_k} M_k \subset^{e_{k+1}} \cdots$ be the tower of the basic construction associated to an extremal subfactor with $[M : N] = \delta^2 < \infty$. Then there exists a unique subfactor planar algebra $P = P^{N \subseteq M}$ of modulus δ satisfying the following conditions:*

1. *$P_k^{N \subseteq M} = N' \cap M_{k-1} \;\; \forall \;\; k \geq 1$ -where this is regarded as an equality of $*$-algebras which is consistent with the inclusions on the two sides;*
2. *$Z_{\mathcal{E}^{k+1}}(1) = \delta e_k \;\; \forall \;\; k \geq 1$;*
3. *$Z_{(E')_k^k}(x) = \delta E_{M' \cap M_{k-1}} \forall \;\; x \in N' \cap M_{k-1}, \;\; \forall \;\; k \geq 1$;*
4. *$Z_{E_{k+1}^k}(x) = \delta E_{N' \cap M_{k-1}}(x) \;\; \forall \;\; x \in N' \cap M_k$; and this is required to hold for all k in Col, where for $k = 0_{\mp}$, the equation is interpreted as $Z_{E_1^{\mp}}(x) = \delta tr_M(x) \;\; \forall \;\; x \in N' \cap M$.*

Conversely (Theorem 4.3.1 of [23]), given any subfactor planar algebra P, there exists an extremal subfactor $N \subset M$ such that P is equivalent to $P^{N \subset M}$ as planar algebras. We will say that a planar algebra P and a subfactor $N \subset M$ are associated to one another if P is equivalent to $P^{N \subset M}$.

The following theorem on generating tangles will be often useful in the sequel.

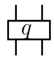
Theorem 2.1.8. *[27][Theorem 3.3] Let $\mathcal{T}$ denote the set of all coloured tangles, and suppose $\mathcal{T}_1$ is a subclass of $\mathcal{T}$ which satisfies:*

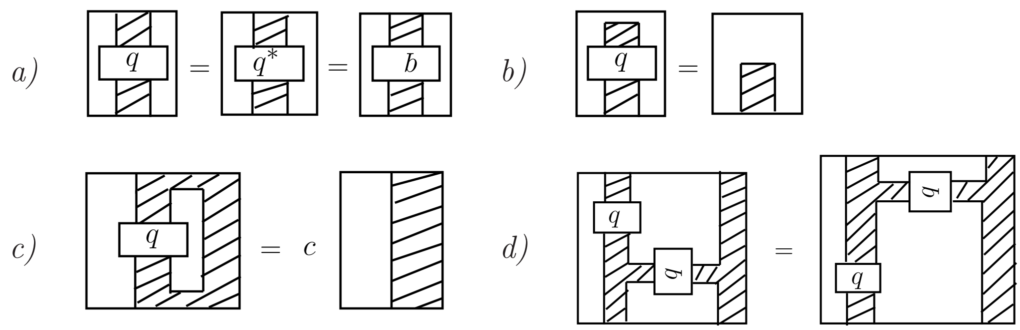
1. $\{1^{0_+}, 1^{0_-}\} \cup \{\mathcal{E}^k : k \geq 2\} \cup \{(E')^k_k : k \geq 1\} \cup \{E^k_{k+1}, M_k, I^{k+1}_k : k \in Col\} \subset \mathcal{T}_1$

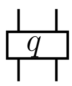
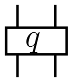
2. *$\mathcal{T}_1$ is closed under composition, when it makes sense; i.e., $T, S \in \mathcal{T}_1, k_0(S) = k_i(T)$ implies $T \circ_{D_i(T)} S \in \mathcal{T}_1$. Then $\mathcal{T}_1 = T$.*

Remark 2.1.9. *There is an older version of planar algebra, which we will actually use in this thesis. In this case, the set of colors is the subset $\{0_+, 0_-, 1, 2, 3, \cdots\}$. All discs (except the (0_-)-disc) have $*$-arcs touching the white regions. But as shown in [44][Remark 3.6] this definition is equivalent to the definition mentioned earlier.*

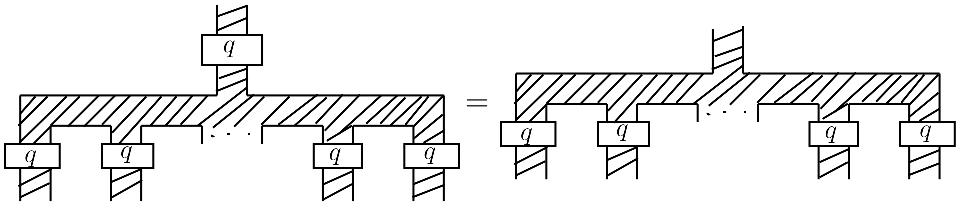
It is well-known from [6], [29] and [9] that in case $N' \cap M = \mathbb{C}$, there is a bijective correspondence between biprojections q (corresponding to the Jones projection of $L^2(M)$ onto $L^2(Q)$) and the intermediate subfactor Q, where $N \subset Q \subset M$. More precisely, we have the following (reformulation of) Theorem 3.2 of [6].

Theorem 2.1.10. *([6] [29] [9]) Let $N \subset M$ be an extremal II_1 subfactor. Let $P^{(N\subset M)}$ be the planar algebra of $N \subset M$, and let $\Phi_{N\subset M}$ be the presenting map of $P^{(N\subset M)}$ on itself, i.e. $\Phi_{N\subset M} : \mathcal{P}(L) \to P^{(N\subset M)}$ with $L = \coprod P_k^{(N\subset M)}$. Suppose there exists an intermediate subfactor Q, $N \subset Q \subset M$. If we let*  *denote the biprojection corresponding to Q, (which is necessarily a projection) we have*



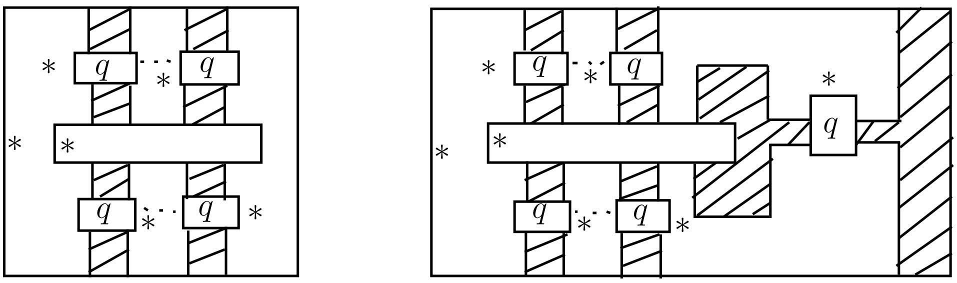
with $c = [M : N]^{1/2}[M : Q]^{-1}$. Furthermore, in the case $N' \cap M = \mathbb{C}$, the converse is also true. Namely a 2-box  *satisfying a)-d) above implies the existence of an* *intermediate subfactor Q, $N \subset Q \subset M$ corresponding to*  .

Corollary 2.1.11. *The following exchange relation holds:*



Denote by $N \subset Q \subset Q_1 \subset Q_2 \subset \ldots$ the Jones tower of $N \subset Q$.

Definition 2.1.12. *Denote the following (unlabelled) tangle by E_n (pretending that there are no labels, namely q's):*



according as n is even or odd respectively. We shall use these to define a map $T \mapsto F(T)$ from the class of k-tangles to the class of partially labelled *k-tangles with $(k+1)$ internal discs all but the last of which are 2-boxes labelled with a q, with the tangle T inserted in the last disc of colour k. Thus, $F(T) = E_k \circ_{(D_1, D_2, \cdots, D_k, D_{k+1})}$ $(q, q, \cdots, q, T)$.*

If it is clear from the context then we write E instead of E_n.

Define functions $F_n : P_n \mapsto P_n$ by $F_n(x) = Z_{E_n}(q \otimes q \otimes \cdots \otimes q \otimes x)$ for $x \in P_n$. We often write $F(x)$ instead of $F_n(x)$ if there is no confusion.

Following [5] define the natural inclusion map

$$i : F_n(P_n) \to F_{n+1}(P_{n+1})$$

given by (via the inclusion $P_n \subset P_{n+1}$)

$$t \mapsto F_{n+1}(t).$$

We denote this inclusion by $\subset_i$. Our starting point is the following result from [5]:

Theorem 2.1.13. *The lattice of algebras:*

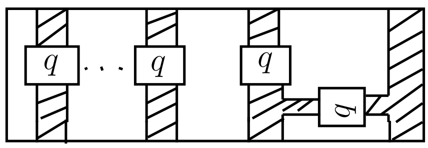
$$\begin{array}{ccccccccccc} F_0(P_0) & \subset_i & F_1(P_1) & \subset_i & F_2(P_2) & \subset_i & \dots & \subset_i & F_n(P_n) & \subset_i & \dots \\ & & \cup & & \cup & & & & \cup & & \\ & & F_1(P_{1,1}) & \subset_i & F_2(P_{1,2}) & \subset_i & \dots & \subset_i & F_n(P_{1,n}) & \subset_i & \dots \end{array}$$

is isomorphic to the standard invariant of $N \subset Q$ (here, $P_{1,k} = M' \cap M_{k-1}$ for all $k \geq 1$):

$$\begin{array}{ccccccccccc} N' \cap N & \subset & N' \cap Q & \subset & N' \cap Q_1 & \subset & \dots & \subset & N' \cap Q_{n-1} & \subset & \dots \\ & & \cup & & \cup & & & & \cup & & \\ & & Q' \cap Q & \subset & Q' \cap Q_1 & \subset & \dots & \subset & Q' \cap Q_{n-1} & \subset & \dots \end{array}$$

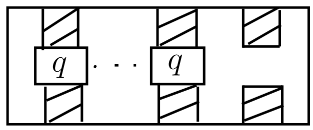
The Jones projections are

$$F_{2n+1}(P_{2n+1}) \ni e^Q_{2n} = [M : Q]^{1/2}[Q : N]^{-1/2}$$



and

$$F_{2n+2}(P_{2n+2}) \ni e^Q_{2n+1} = [M : N]^{-1/2}$$



for $n \geq 1$. The trace on $F_{2n}(P_{2n})$ and $F_{2n+1}(P_{2n+1}))$ is given by $tr_{N \subset Q}(x) = [M : Q]^n tr_{N \subset M}(x)$.

2.2 Jones' tower of intermediate subfactors

As promised before, in this section we describe the tower of iterated basic construction of $N \subseteq Q$ in terms of the corresponding tower of $N \subseteq M$. This is the key result in proving Theorem 2.1.13 and has been proved in [5]. Here we give another proof using Lemma 1.3.6 of the first chapter.

The following well known fact is often useful (see [5]):

Fact 2.2.1. *Given a* II_1 *factor* $A \subset \mathcal{B}(\mathcal{H})$ *and projections* $r \in A$ *and* $s \in (rAr)'$, *we have*

$$tr_{rArs}(rzrs) = (tr_A(r))^{-1} tr_A(rzr) \tag{2.2}$$

for all $z \in A$.

Notation 2.2.2. *Put* $p_{[0,2n-1]} = e_{0,1}e_{0,3}\cdots e_{0,2n-1}$;

and $A_n = \{Np_{[0,2n-1]}, p_{[0,2n-1]}Mp_{[0,2n-1]}, \cdots, p_{[0,2n-1]}M_{2n-1}p_{[0,2n-1]}\}$.

Proposition 2.2.3. *Each member of the set* A_n *is a* II_1 *factor.*

Proof. We prove this by Induction over n.

Let us prove for $n = 1$. Thus we need to prove that

(2.3) All elements of $A_1 = \{Np_{0,1}, p_{0,1}Mp_{0,1}, p_{0,1}M_1p_{0,1}\}$ are II_1 factors.

Since $p_{0,1} = e_{0,1} \in N'$, $Np_{[0,1]}$ is a II_1 factor. By definition of the Jones' projection $e_{0,1}$, we have $e_{0,1}Me_{0,1} = Qe_{0,1}$ and $e_{0,1}$ commutes with the II_1 factor Q. This proves that $e_{0,1}Me_{0,1}$ is a II_1 factor. Finally, as $e_{0,1} \in M_1$ it is clear that $p_{[0,1]}M_1p_{[0,1]}$ is a II_1 factor. Thus each member of A_1 is indeed a II_1 factor.

Let us do one more step for clarity, that is, $n = 2$; and we need to prove that

(2.4)

Elements of $A_2 = \{Np_{[0,3]}, p_{[0,3]}Mp_{[0,3]}, p_{[0,3]}M_1p_{[0,3]}, p_{[0,3]}M_2p_{[0,3]}, p_{[0,3]}M_3p_{[0,3]}\}$ are factors.

Now observe that $e_{0,3}$ commutes with every member of A_1. Thus, it follows from the already established (2.3) that each member of the set $\{Np_{[0,3]}, p_{[0,3]}Mp_{[0,3]}, p_{[0,3]}M_1p_{[0,3]}\}$ is a II_1 factor. Also by definition of the Jones' projection $e_{0,3}$, we have $e_{0,3}M_2e_{0,3} = P_2e_{0,3}$ and $e_{0,3}$ commutes with P_2 and $e_{0,1}$. Thus $p_{[0,3]}M_2p_{[0,3]} = e_{0,1}e_{0,3}M_2e_{0,3}e_{0,1} = e_{0,1}P_2e_{0,1}e_{0,3} = P_2p_{[0,3]}$ is a II_1 factor (since, $e_{0,1}P_2e_{0,1}$ is a II_1 factor as $e_{0,1} \in P_2$). Finally $p_{[0,3]}M_3p_{[0,3]}$ is a II_1 factor since $p_{[0,3]} \in M_3$. Thus each member of A_2 is indeed a II_1 factor.

Assume, as the induction hypothesis, that the statement in the claim is valid for $n = k$. We prove it for $n = k + 1$. In other words, we assume that

(2.5)

All the members in $A_k =: \{p_{[0,2k-1]}M_ip_{[0,2k-1]} : -1 \leq i \leq 2k - 1\}$ are II_1 factors.

Since $e_{0,2k+1}$ commutes with each member of A_k each element of the set $\{p_{[0,2n+1]}M_ip_{[0,2k+1]} : -1 \leq i \leq 2k - 1\}$ is a II_1 factor (by Induction hypothesis). It only remains to show that $p_{[0,2k+1]}M_ip_{[0,2k+1]}$ is a II_1 factor for $i = 2k, 2k + 1$.Recall that $e_{0,2k+1}$ is the Jones' projection corresponding to the basic construction $P_{2k} \subseteq M_{2k} \subseteq P_{2k+1}$. Thus, $e_{0,2k+1}M_{2k}e_{0,2k+1} = P_{2k}e_{0,2k+1}$. Since $e_{0,2k+1}$ commutes with P_{2k} as well as with all the projections $e_{0,2i-1}$ for $1 \leq i \leq k$ we see that $p_{[0,2k+1]}M_{2k}p_{[0,2k+1]}$ is a II_1 factor. Finally, it is clear that $p_{[0,2k+1]}M_{2k+1}p_{[0,2k+1]}$ is a II_1 factor since $p_{[0,2k+1]} \in M_{2k+1}$. In conclusion, each member of the set A_{k+1} is indeed a II_1 factor.

This completes the proof of the Proposition. $\square$

Theorem 2.2.4. *Given $N \subset Q \subset M$ and the notation introduced above, set $p = e_{0,1}\ e_{0,3}\ e_{0,5} \cdots e_{0,2n-1}$. Then the chain*

$$Np \subset pMp \subset pM_1p \subset pM_2p \subset \ldots \subset pM_{2n-1}p$$

is isomorphic to the first $2n-1$ steps of the basic construction of $N \subset Q$. The Jones projections are given by $e_{0,2i}p : L^2(pM_{2i-1}p) \to L^2(pM_{2i-2}p)$ and $e_{1,2i+1}p : L^2(pM_{2i}p) \to L^2(pM_{2i-1}p)$. The unique normalized trace on the chain, denoted $tr_{N\subset Q}$, is given by $tr_{N\subset Q}(x) = [M : Q]^n tr_{N\subset M}(x)$.

Proof. We put, $p_{[0,2n-1]} = e_{0,1}\ e_{0,3}\ e_{0,5} \cdots e_{0,2n-1}$. The final trace assertion is immediate from the fact $tr_{N\subset M}(p_{[0,2n-1]}) = [M : Q]^{-n}$ (this fact follows easily from Lemma 4.2 of [5]). It suffices to show that the above chain is a basic construction and that the inclusion $Np_{[0,2n-1]} \subset p_{[0,2n-1]}Mp_{[0,2n-1]}$ is isomorphic to $N \subset Q$. We do this in several steps.

Step 1 : Firstly we show, $Ne_{0,1} \subseteq e_{0,1}Me_{0,1} \subseteq e_{0,1}M_1e_{0,1}$ is isomorphic to the (first step) basic construction of $N \subseteq Q$, where the corresponding Jones' projection is given by $e_{1,1}e_{0,1}(= e_{1,1})$. We prove this using Lemma 1.3.6. Note that $e_{0,1} = q$.

Since $e_{0,1}$ belongs to Q' (and hence belongs to N'), $Qe_{0,1}$ (and hence $Ne_{0,1}$) is a von Neumann algebra. Furthermore, as $e_{0,1}Me_{0,1} = Qe_{0,1}$ it is clear that $(Ne_{0,1} \subseteq e_{0,1}Me_{0,1}) \cong (N \subseteq Q)$ via the map $x \longmapsto xe_{0,1}$ for $x \in Q$. The map $x \longmapsto xe_{0,1}$ is injective by Lemma 1.2.1. In particular, $[Qe_{0,1} : Ne_{0,1}] = [Q : N]$. Let $\{\lambda_i\}$ be a (left) basis for Q/N, which always exists by [36], then since $E^{Qe_{0,1}}_{Ne_{0,1}}(xe_{0,1}) = E^Q_N(x)e_{0,1}, \{\lambda_i e_{0,1}\}$ is a (left) basis for $Qe_{0,1}/Ne_{0,1}$. Now, $e_{1,1} \in e_{0,1}M_1e_{0,1}$ since $e_{1,1} \leq e_{0,1}$ and observe,

$$\sum (\lambda_i e_{0,1})^* e_{1,1}(\lambda_i e_{0,1})$$
$$= \sum \lambda_i{}^* e_{0,1}e_{1,1}e_{0,1}\lambda_i \quad [\text{since } e_{0,1} \in Q']$$

$$= \sum {\lambda_i}^* e_{1,1} \lambda_i \ \ [\text{since } e_{1,1} = e_{1,1}e_{0,1} = e_{0,1}e_{1,1}]$$
$$= \sum {\lambda_i}^* e_N^Q e_{0,1} \lambda_i \ \ [\text{since } e_N^Q e_{0,1} = e_{1,1}]$$
$$= \sum ({\lambda_i}^* e_N^Q \lambda_i) e_{0,1}$$
$$= e_{0,1} \ \ [\text{since} \sum {\lambda_i}^* e_N^Q \lambda_i = 1].$$

Also, for all $m \in M$ the following two equations hold true:

$$e_{1,1}(e_{0,1}me_{0,1})e_{1,1} = e_{1,1}me_{1,1} = E_N^M(m)e_{1,1} \tag{2.6}$$

and,

(2.7)

$$E_{Ne_{0,1}}^{Qe_{0,1}}(e_{0,1}me_{0,1})e_{1,1}\{= E_{Ne_{0,1}}^{Qe_{0,1}}(E_Q^M(m)e_{0,1})e_{1,1} = E_N^Q(E_Q^M(m))e_{0,1}e_{1,1}\} = E_N^M(m)e_{1,1}.$$

Equations (2.6) and (2.7) imply that $e_{1,1}(e_{0,1}me_{0,1})e_{1,1} = E_{Ne_{0,1}}^{Qe_{0,1}}(e_{0,1}me_{0,1})e_{1,1}$. We next show for all $x \in e_{0,1}M_1e_{0,1}$ the following is true:

$$E_{Qe_{0,1}}^{e_{0,1}M_1e_{0,1}}(x) = [M : Q]E_Q^{M_1}(x)e_{0,1}. \tag{2.8}$$

This follows from the following array of equations which hold true $\forall y \in Q$:

$$tr_{N\subseteq Q}([M : Q]E_Q^{M_1}(x)e_{0,1}.ye_{0,1})$$
$$= [M : Q]^2 tr_{N\subseteq M}(E_Q^{M_1}(x)ye_{0,1})$$
$$= [M : Q]^2 tr_{N\subseteq M}(E_Q^{M_1}(xy)e_{0,1})$$
$$= [M : Q] tr_{N\subseteq M}(E_Q^{M_1}(xy)) \ \ [\text{since } tr_{N\subseteq M}(e_{0,1}) = [M : Q]^{-1}]$$
$$= [M : Q] tr_{N\subseteq M}(xy)$$
$$= tr_{N\subseteq Q}(e_{0,1}xy) \ \ [\text{since } e_{0,1}x = x]$$
$$= tr_{N\subseteq Q}(x.ye_{0,1}).$$

Then we show, $\{\sqrt{[Q:N]}e_{1,1}\lambda_i e_{0,1}\}$ is a (left) basis for $e_{0,1}M_1e_{0,1}/e_{0,1}Me_{0,1}$. To prove this firstly note that,by Jones' local index formula (see [22] or sections 2.2-2.3 of [25]) and extremality (see [40] page 176) the following equation holds

$$[e_{0,1}M_1e_{0,1}:Qe_{0,1}] = tr(e_{0,1})^2[M_1:Q] = [Q:N] = [Qe_{0,1}:Ne_{0,1}] \tag{2.9}$$

To obtain the first equality we have used the fact that $[e_{0,1}M_1e_{0,1} : Qe_{0,1}] = tr_{Q'}(e_{0,1})tr_{M_1}(e_{0,1}[M_1 : Q]$. Next we show that the following array of equations hold:

$$
\begin{aligned}
&E^{e_{0,1}M_1e_{0,1}}_{Qe_{0,1}}[(\sqrt{[Q:N]}e_{1,1}\lambda_i e_{0,1})(\sqrt{[Q:N]}e_{1,1}\lambda_j e_{0,1})^*]\\
&\quad= [Q:N]E^{e_{0,1}M_1e_{0,1}}_{Qe_{0,1}}(e_{1,1}e_{0,1}\lambda_i\lambda_j^*e_{1,1}) \quad [\text{since } \ e_{0,1}\in Q']\\
&\quad= [Q:N]E^{e_{0,1}M_1e_{0,1}}_{Qe_{0,1}}(e_{0,1}e_{1,1}\lambda_i\lambda_j^*e_{1,1}e_{0,1}) \quad [\text{since } e_{1,1}=e_{1,1}e_{0,1}=e_{0,1}e_{1,1}]\\
&\quad= [Q:N][M:Q]E^{M_1}_{Q}(e_{0,1}e_{1,1}\lambda_i\lambda_j^*e_{1,1}e_{0,1})e_{0,1} \quad [\text{by Equation(2.8)}]\\
&\quad= [M:N]E^{M_1}_{Q}(e_{1,1}\lambda_i\lambda_j^*e_{1,1})e_{0,1}\\
&\quad= [M:N]E^{M_1}_{Q}[E^{M}_{N}(\lambda_i\lambda_j^*)e_{1,1}]e_{0,1}\\
&\quad= [M:N]E^{Q}_{N}(\lambda_i\lambda_j^*)E^{M_1}_{Q}(e_{1,1})e_{0,1}\\
&\quad= [M:N]E^{Q}_{N}(\lambda_i\lambda_j^*)E^{M}_{Q}(E^{M_1}_{M}(e_{1,1}))e_{0,1}\\
&\quad= E^{Q}_{N}(\lambda_i\lambda_j^*)e_{0,1}\\
&\quad= E^{Qe_{0,1}}_{Ne_{0,1}}(\lambda_i\lambda_j^*e_{0,1})\\
&\quad= E^{Qe_{0,1}}_{Ne_{0,1}}(\lambda_i e_{0,1}.e_{0,1}\lambda_j^*) \quad [\text{since } \ e_{0,1}\in Q']
\end{aligned}
$$

Now as $\{\lambda_i e_{0,1}\}$ is a (left) basis for $Qe_{0,1}/Ne_{0,1}$ the last equation in the above array of equations together with Equation (2.9) tells that $\{\sqrt{[Q:N]}e_{1,1}\lambda_i e_{0,1}\}$ is a (left) basis for $e_{0,1}M_1e_{0,1}/e_{0,1}Me_{0,1}$. Here we have used Theorem 1.2.2. Now applying Lemma 1.3.6 we get the desired result.

Step 2 : Here again we have $N \subseteq Q \subseteq M$, with biprojection $e_{0,1}$. We claim

$(p_{[0,3]}Mp_{[0,3]} \subseteq p_{[0,3]}M_1p_{[0,3]} \subseteq p_{[0,3]}M_2p_{[0,3]}) \cong (Q \subseteq Q_1 \subseteq Q_2)$. Here Jones' projection is given by $e_{0,2}p_{[0,3]}$. We will again apply Lemma 1.3.6. We prove it in various steps.

(a) Firstly, as $e_{0,3}$ commutes with $e_{0,1}$ and every element of M_1, it follows from Step 1 that $\{\sqrt{[Q:N]}e_{1,1}\lambda_i p_{[0,3]}\}$ is a (left) basis for $p_{[0,3]}M_1p_{[0,3]}/p_{[0,3]}Mp_{[0,3]}$.

(b) Next we claim the following:

Claim: For $m_1 \in M_1$,

$$(e_{0,2}p_{[0,3]})(p_{[0,3]}m_1p_{[0,3]})(e_{0,2}p_{[0,3]}) = E^{p_{[0,3]}M_1p_{[0,3]}}_{p_{[0,3]}Mp_{[0,3]}}(p_{[0,3]}m_1p_{[0,3]})e_{0,2}p_{[0,3]}. \tag{2.10}$$

Proof: As $e_{0,3}$ commutes with $e_{0,1}$ and every element of M_1, it follows that,

$$\begin{aligned}
&E^{p_{[0,3]}M_1p_{[0,3]}}_{p_{[0,3]}Mp_{[0,3]}}(p_{[0,3]}m_1p_{[0,3]})\\
&\quad = E^{e_{0,1}M_1e_{0,1}e_{0,3}}_{Qe_{0,1}e_{0,3}}(e_{0,1}m_1e_{0,1}e_{0,3})\\
&\quad = E^{e_{0,1}M_1e_{0,1}}_{Qe_{0,1}}(e_{0,1}m_1e_{0,1})e_{0,3} \quad \text{[by Fact 2.2.1]}\\
&\quad = [M:Q]E^{M_1}_{Q}(e_{0,1}m_1e_{0,1})p_{[0,3]} \quad \text{[by Equation (2.8)]}
\end{aligned}$$

Since $e_{0,1} \in P_1$ it is easy to see that, $E^{M_1}_{P_1}(e_{0,1}m_1e_{0,1}) = E^{M_1}_{P_1}(e_{0,1}m_1e_{0,1})e_{0,1} = me_{0,1}$ for some unique $m \in M$, which exists by Lemma 1.2.1. That implies,

$$E^{M_1}_{M}(E^{M_1}_{P_1}(e_{0,1}m_1e_{0,1})) = mE^{M_1}_{M}(e_{0,1}) = mE^{P_1}_{M}(e_{0,1}) = m[M:Q]^{-1}.$$

In other words, $E^{M_1}_{M}(e_{0,1}m_1e_{0,1}) = m[M:Q]^{-1}$. Therefore $m = [M:Q]E^{M_1}_{M}(e_{0,1}m_1e_{0,1})$. Thus,

$$E^{M_1}_{P_1}(e_{0,1}m_1e_{0,1}) = [M:Q]E^{M_1}_{M}(e_{0,1}m_1e_{0,1})e_{0,1} \tag{2.11}$$

Thus,

$$
\begin{aligned}
&(e_{0,2}p_{[0,3]})(p_{[0,3]}m_1p_{[0,3]})(e_{0,2}p_{[0,3]})\\
&\quad = e_{0,2}p_{[0,3]}m_1p_{[0,3]}e_{0,2}p_{[0,3]}\\
&\quad = e_{0,3}e_{0,2}e_{0,1}m_1e_{0,1}e_{0,2}e_{0,3} \quad \text{[by Fact 2.1.1(3)]}\\
&\quad = e_{0,1}E^{M_1}_{P_1}(e_{0,1}m_1e_{0,1})e_{0,2}e_{0,3} \quad \text{[by definition of } e_{0,2}\text{]}\\
&\quad = [M:Q]e_{0,1}E^{M_1}_{M}(e_{0,1}m_1e_{0,1})e_{0,1}e_{0,2}e_{0,3} \quad \text{[by Equation (2.11)]}\\
&\quad = [M:Q]E^{M}_{Q}(E^{M_1}_{M}(e_{0,1}m_1e_{0,1}))e_{0,1}e_{0,2}e_{0,3}\\
&\quad = [M:Q]E^{M_1}_{Q}(e_{0,1}m_1e_{0,1})p_{[0,3]}e_{0,2}p_{[0,3]} \quad \text{[by Fact 2.1.1(3)]}\\
&\quad = E^{p_{[0,3]}M_1p_{[0,3]}}_{p_{[0,3]}Mp_{[0,3]}}(p_{[0,3]}m_1p_{[0,3]})e_{0,2}p_{[0,3]}
\end{aligned}
$$

This justifies the claim. ■

(c) We prove $[p_{[0,3]}M_2p_{[0,3]} : p_{[0,3]}M_1p_{[0,3]}] = [Q:N] = [p_{[0,3]}M_1p_{[0,3]} : p_{[0,3]}Mp_{[0,3]}]$. Firstly note,

$$
\begin{aligned}
p_{[0,3]}M_2p_{[0,3]} &= e_{0,1}e_{0,3}M_2e_{0,3}e_{0,1}\\
&= e_{0,1}P_2e_{0,3}e_{0,1} \quad \text{[by definition of } e_{0,3}\text{]}\\
&= p_{[0,3]}P_2p_{[0,3]} \quad \text{[since } e_{0,3}\in P_2'\text{]}
\end{aligned}
$$

It is trivial to see that, $[P_2 : M_1] = [M_1 : P_1] = \frac{[M_1:M]}{[P_1:M]} = \frac{[M:N]}{[M:Q]} = [Q:N]$. Thus,

$$
\begin{aligned}
[Q:N] &= [P_2 : M_1]\\
&= [e_{0,1}P_2e_{0,1} : e_{0,1}M_1e_{0,1}] \quad [\ as\ e_{0,1}\in M_1\subseteq P_2]\\
&= [p_{[0,3]}P_2p_{[0,3]} : p_{[0,3]}M_1p_{[0,3]}]\\
&= [p_{[0,3]}M_2p_{[0,3]} : p_{[0,3]}M_1p_{[0,3]}]
\end{aligned}
$$

Now observe, $[p_{[0,3]}M_2p_{[0,3]} : p_{[0,3]}M_1p_{[0,3]}] = [Q:N] = [Qp_{[0,1]} : Np_{[0,1]}]$. Also, by Equation (2.9), $[Qp_{[0,1]} : Np_{[0,1]}] = [p_{[0,1]}M_1p_{[0,1]} : p_{[0,1]}Mp_{[0,1]}] = [p_{[0,3]}M_1p_{[0,3]} : p_{[0,3]}Mp_{[0,3]}]$. This proves (c).

(d) Claim: $\sum_i \{(\sqrt{[Q:N]}e_{1,1}\lambda_i p_{[0,3]})^* e_{0,2}p_{[0,3]}(\sqrt{[Q:N]}e_{1,1}\lambda_i p_{[0,3]})\} = p_{[0,3]}$.

Proof: The following equations hold true:

$$\begin{aligned}
&\sum (e_{1,1}\lambda_i p_{[0,3]})^* e_{0,2}p_{[0,3]}(e_{1,1}\lambda_i p_{[0,3]}) \\
&\quad = \sum p_{[0,3]}\lambda_i^* e_{1,1}e_{0,2}e_{1,1}\lambda_i p_{[0,3]} \\
&\quad = [Q:N]^{-1}\sum p_{[0,3]}\lambda_i^* e_{1,1}\lambda_i p_{[0,3]} \quad \text{[Fact 2.1.1(6)]} \\
&\quad = [Q:N]^{-1}\sum p_{[0,3]}\lambda_i^* e_N^Q e_{0,1}\lambda_i p_{[0,3]} \\
&\quad = [Q:N]^{-1}p_{[0,3]}.
\end{aligned}$$

The last equation follows from the fact that $\{\lambda_i\}$ is a (left) basis for Q/N and hence $\sum \lambda_i^* e_N^Q \lambda_i = 1$. ■

(e) Firstly it is easy to check that,

$$E_{M_1}^{P_2}(e_{0,2}) = [M_1 : P_1]^{-1} = [Q:N]^{-1}. \tag{2.12}$$

Next we show that,

$$E_{P_1}^{M_1}(e_{1,1}) = [Q:N]^{-1}e_{0,1} \tag{2.13}$$

To see this, note that by [36] there exists unique $m_0 \in M$ such that $E_{P_1}^{M_1}(e_{1,1}) = E_{P_1}^{M_1}(e_{1,1})e_{0,1} = m_0 e_{0,1}$. Thus, $E_M^{M_1}(e_{1,1}) = m_0 E_M^{M_1}(e_{0,1})$. Hence, $m_0 = [Q:N]^{-1}$. This proves Equation 2.13.

Claim: $\{[Q:N]e_{0,2}p_{[0,3]}e_{1,1}\lambda_i p_{[0,3]}\}$ is a (left) basis for $p_{[0,3]}M_2p_{[0,3]}/p_{[0,3]}M_1p_{[0,3]}$.

Proof: By Fact 2.2.1 it is trivial to check that for all $m_2 \in M_2$:

$$E^{p_{[0,3]}M_2p_{[0,3]}}_{p_{[0,3]}M_1p_{[0,3]}}(p_{[0,3]}m_2p_{[0,3]}) = p_{[0,3]}E^{M_2}_{M_1}(m_2)p_{[0,3]} \tag{2.14}$$

Then the following array of equations hold:

$$\begin{aligned}
&E^{p_{[0,3]}M_2p_{[0,3]}}_{p_{[0,3]}M_1p_{[0,3]}}[(e_{0,2}e_{1,1}\lambda_i p_{[0,3]})(e_{0,2}e_{1,1}\lambda_j p_{[0,3]})^*] \\
&\qquad = E^{p_{[0,3]}M_2p_{[0,3]}}_{p_{[0,3]}M_1p_{[0,3]}}(p_{[0,3]}e_{0,2}e_{1,1}\lambda_i\lambda_j^*e_{1,1}e_{0,2}p_{[0,3]}) \\
&\qquad = p_{[0,3]}E^{M_2}_{M_1}(e_{0,2}e_{1,1}\lambda_i\lambda_j^*e_{1,1}e_{0,2})p_{[0,3]} \quad \text{[by Equation (2.14)]} \\
&\qquad = p_{[0,3]}E^{M_2}_{M_1}(E^{M_1}_{P_1}(e_{1,1}\lambda_i\lambda_j^*e_{1,1})e_{0,2})p_{[0,3]} \\
&\qquad = p_{[0,3]}E^{M_1}_{P_1}(e_{1,1}\lambda_i\lambda_j^*e_{1,1})E^{M_2}_{M_1}(e_{0,2})p_{[0,3]} \\
&\qquad = p_{[0,3]}E^{Q}_{N}(\lambda_i\lambda_j^*)E^{M_1}_{P_1}(e_{1,1})E^{P_2}_{M_1}(e_{0,2})p_{[0,3]} \\
&\qquad = p_{[0,3]}E^{Q}_{N}(\lambda_i\lambda_j^*)[Q:N]^{-1}E^{M_1}_{P_1}(e_{1,1})p_{[0,3]} \quad \text{[by Equation (2.12)]} \\
&\qquad = p_{[0,3]}E^{Q}_{N}(\lambda_i\lambda_j^*)[Q:N]^{-1}[Q:N]^{-1}e_{0,1}p_{[0,3]} \quad \text{[by Equation (2.13)]} \\
&\qquad = [Q:N]^{-2}p_{[0,3]}E^{Q}_{N}(\lambda_i\lambda_j^*)p_{[0,3]} \\
&\qquad = [Q:N]^{-2}E^{Qp_{[0,3]}}_{Np_{[0,3]}}(\lambda_i p_{[0,3]}p_{[0,3]}\lambda_j^*).
\end{aligned}$$

Since, $\{\lambda_i p_{[0,3]}\}$ is a (left)basis for $Qp_{[0,3]}/Np_{[0,3]}$ it follows from Theorem 1.2.2 that $\{[Q:N]e_{0,2}p_{[0,3]}e_{1,1}\lambda_i p_{0,3}\}$ (in other words, $\{[Q:N]e_{0,2}e_{1,1}\lambda_i p_{[0,3]}\}$) is a (left) basis for $p_{[0,3]}M_2p_{[0,3]}/p_{[0,3]}M_1p_{[0,3]}$. ■

Thus combining (a),(b),(c),(d) and (e) and applying Lemma 1.3.6 we complete the proof of Step 2.

Step 3 : In general, apply **Step 1** to the following subfactors for $2n-1 \geq i \geq 3$ and i odd: $p_{[0,2n-i]}M_{2n-i}p_{[0,2n-i]} \subseteq p_{[0,2n-i]}P_{2n-i+1}p_{[0,2n-i]} \subseteq p_{[0,2n-i]}M_{2n-i+1}p_{[0,2n-i]}$. Here biprojection is given by $p_{[0,2n-i]}e_{0,2n-i+2}$. We get $p_{[0,2n-i+2]}M_{2n-i}p_{[0,2n-i+2]} \subseteq p_{[0,2n-i+2]}M_{2n-i+1}p_{[0,2n-i+2]} \subseteq p_{[0,2n-i+2]}M_{2n-i+2}p_{[0,2n-i+2]}$ is a Jones' tower with corresponding Jones' projection is given by $e_{1,2n-i+2}p_{[0,2n-i+2]}$. But simply observe that for $i \neq 3$,

$$e_{0,2n-1}e_{0,2n-3}\cdots e_{0,2n-i+4} \in M'_{2n-i}, M'_{2n-i+1} \text{ and } M'_{2n-i+2}.$$

So, using Fact 2.1.1 (3) we have

$$p_{[0,2n-1]}M_{2n-i}p_{[0,2n-1]} \subseteq p_{[0,2n-1]}M_{2n-i+1}p_{[0,2n-1]} \subseteq p_{[0,2n-1]}M_{2n-i+2}p_{[0,2n-i+2]}$$

is a Jones' tower with the Jones' projection $e_{1,2n-i+2}p_{[0,2n-1]}$. For $i = 3$ this is obvious.

Apply **Step 2** to the following subfactors for $2n-1 \geq j \geq 3$ and j odd: $p_{[0,2n-j-2]}M_{2n-j-2}p_{[0,2n-j-2]} \subseteq p_{[0,2n-j-2]}P_{2n-j-1}p_{[0,2n-j-2]} \subseteq p_{[0,2n-j-2]}M_{2n-j-1}p_{[0,2n-j-2]}$. Here biprojection is given by $p_{[0,2n-j-2]}e_{0,2n-j}$. Then we get $p_{[0,2n-j+2]}M_{2n-j-1}p_{[0,2n-j+2]} \subseteq p_{[0,2n-j+2]}M_{2n-j}p_{[0,2n-j+2]} \subseteq p_{[0,2n-j+2]}M_{2n-j+1}p_{[0,2n-j+2]}$ is a Jones' tower where the corresponding Jones' projection is given by $e_{0,2n-j+1}p_{[0,2n-j+2]}$. Let us elaborate this a little. For an intermediate subfactor $N \subset Q \subset M$, with the corresponding biprojection $e_{0,1}$ we proved before $(p_{[0,3]}Mp_{[0,3]} \subseteq p_{[0,3]}M_1p_{[0,3]} \subseteq p_{[0,3]}M_2p_{[0,3]}) \cong (Q \subseteq Q_1 \subseteq Q_2)$. Here Jones' projection is given by $e_{0,2}p_{[0,3]}$. Now, for $2n-1 \geq j \geq 3$ and j odd, we have a situation where $N = p_{[0,2n-j-2]}M_{2n-j-2}p_{[0,2n-j-2]}$, $Q = p_{[0,2n-j-2]}P_{2n-j-1}p_{[0,2n-j-2]}$ and $M = p_{[0,2n-j-2]}M_{2n-j-1}p_{[0,2n-j-2]}$. Here biprojection is given by $p_{[0,2n-j-2]}e_{0,2n-j}$. Now we will apply Step 2 to obtain the following Jones' tower: $p_{[0,2n-j+2]}M_{2n-j-1}p_{[0,2n-j+2]} \subseteq p_{[0,2n-j+2]}M_{2n-j}p_{[0,2n-j+2]} \subseteq p_{[0,2n-j+2]}M_{2n-j+1}p_{[0,2n-j+2]}$, with the corresponding Jones' projection is given by $e_{0,2n-j+1}p_{[0,2n-j+2]}$.

Again note that for $j \neq 3$, $e_{0,2n-1}e_{0,2n-3}\cdots e_{0,2n-j+4} \in M'_{2n-j-1}, M'_{2n-j}$ and M'_{2n-j+1}. Thus it follows from Fact 2.1.1 (3) that

$$p_{[0,2n-1]}M_{2n-j-1}p_{[0,2n-1]} \subseteq p_{[0,2n-1]}M_{2n-j}p_{[0,2n-1]} \subseteq p_{[0,2n-1]}M_{2n-j+1}p_{[0,2n-1]}$$

is a Jones's tower with the corresponding Jones' projection is given by $e_{0,2n-j+1}p_{[0,2n-1]}$. For $j = 3$ this is trivially satisfied.

Lastly note that, from **Step 1** and Fact 2.1.1 it follows that

$$Np_{[0,2n-1]} \subseteq p_{[0,2n-1]}Mp_{[0,2n-1]} \subseteq p_{[0,2n-1]}M_1p_{[0,2n-1]}$$

is a Jones' tower with the corresponding Jones' projection $e_{1,1}p_{[0,2n-1]}$.

Combining all the facts mentioned above we prove the result as stated in the theorem. □

2.3 The Intermediate Planar Algebra

This section is devoted to our reformulation of the proof of the fact that the planar algebra $P^{(N\subseteq Q)}$ may be derived from $P^{(N\subseteq M)}$ (see [5]) by requiring that the action of a planar tangle T is given by Equation (2.1) of the Introduction.

Definition 2.3.1. *Let T be a k_0-tangle with $b \geq 1$ internal discs $D_1, \cdots, D_b$ of colours $k_1, \cdots, k_b$. Then define $\alpha(T) = [M : Q]^{\frac{1}{2}c(T)}$, where*

$$c(T) = (\lceil k_0/2 \rceil + \lfloor k_1/2 \rfloor + \cdots + \lfloor k_b/2 \rfloor) - l(T),$$

with $l(T)$ being the total number of closed loops after capping the black intervals of the external disc of T and cupping the black intervals of all internal discs of T.

Proposition 2.3.2. *If $T = T^{k_0}_{k_1,\cdots,k_b}$ and $\tilde{T} = \tilde{T}^{\tilde{k}_0}_{\tilde{k}_1,\cdots,\tilde{k}_{\tilde{b}}}$ are tangles with discs of indicated colours such that $\tilde{k}_0 = k_i$ for some $1 \leq i \leq b$, then*

$$\frac{\alpha(T)\alpha(\tilde{T})}{\alpha(T \circ_i \tilde{T})} = [M : Q]^{\frac{1}{2}(\tilde{k_0} - l(T) - l(\tilde{T}) + l(T\circ_i\tilde{T}))}.$$

Proof. This is simple arithmetic:

$$
\begin{aligned}
c(T) &= (\lceil k_0/2\rceil + \lfloor k_1/2\rfloor + \cdots + \lfloor k_b/2\rfloor) - l(T) \\
c(\tilde{T}) &= (\lceil \tilde{k}_0/2\rceil + \lfloor \tilde{k}_1/2\rfloor + \cdots + \lfloor \tilde{k}_{\tilde{b}}/2\rfloor) - l(\tilde{T}) \\
c(T \circ_i \tilde{T}) &= \lceil k_0/2\rceil + \lfloor k_1/2\rfloor + \cdots + \lfloor k_b/2\rfloor - \lfloor \tilde{k}_0/2\rfloor + \lfloor \tilde{k}_1/2\rfloor + \cdots + \lfloor \tilde{k}_{\tilde{b}}/2\rfloor - l(T \circ_i \tilde{T})
\end{aligned}
$$

Hence, after all the cancellation, we find that

$$
\begin{aligned}
\frac{\alpha(T)\alpha(\tilde{T})}{\alpha(T \circ_i \tilde{T})} &= [M:Q]^{1/2\left(c(T)+c(\tilde{T})-c(T\circ_i\tilde{T})\right)} \\
&= [M:Q]^{1/2\left(\tilde{k}_0-l(T)-l(\tilde{T})+l(T\circ_i\tilde{T})\right)},
\end{aligned}
$$

since $\lceil n/2\rceil + \lfloor n/2\rfloor = n$ for all integral n. □

Thus, we assume that P is an irreducible subfactor planar algebra and $q \in P_2$ be a biprojection. Let $T = T^{k_0}_{k_1,\cdots,k_b}$ and $\tilde{T} = \tilde{T}^{\tilde{k}_0}_{\tilde{k}_1,\cdots,\tilde{k}_{\tilde{b}}}$ be tangles with $k_i = \tilde{k}_0$. By $F(T)$ we will denote the partially labelled tangle obtained from T by 'surrounding it with q's'. It has the same number and colors of discs as T does. Let P'_k be the range of the action of the tangle $F(I^k_k)$ (that is, $P'_k = range(Z^{(N\subset M)}_{F(I^k_k)})$).

We shall show that:

Theorem 2.3.3. *If* $P'_n = range(Z^{(N\subset M)}_{F(I^n_n)})$ *and* $Z'_T = \alpha(T) Z_{F(T)|_{\otimes P'_{k_i(T)}}}$*, then* $(P', T \mapsto Z'_T|_{\otimes P'_{k_i(T)}})$ *is a subfactor planar algebra which is isomorphic to* $P^{(N\subset Q)}$.

The proof of this theorem has two main ingredients: (a) the verification that P' is a planar algebra; and (b) the verification that this is isomorphic to the planar algebra of $N \subset Q$. The proof of (b) is an application of Theorem 2.1.7 while the only really non-trivial part of proving (a) is in the verification of compatibility of the partition function to gluing of tangles. In order to verify that the operation of tangles (in P') is compatible with composition of tangles, we will need to verify that

$$\alpha(T \circ \tilde{T}) Z_{F(T \circ \tilde{T})} = \alpha(T)\alpha(\tilde{T}) Z_{F(T) \circ F(\tilde{T})} ,$$

which, in view of Proposition 2.3.2, is seen to translate to:

$$Z_{F(T) \circ F(\tilde{T})} = [M : Q]^{-1/2(\tilde{k}_0 - l(T) - l(\tilde{T}_0) + l(T \circ \tilde{T})} Z_{F(T \circ \tilde{T})} , \tag{2.15}$$

which is what the next few pages are devoted to. We start on part (b) in the few lines after the proof of "compatibility under substitution".

Theorem 2.3.4. *The equation*

$$Z_{F(T) \circ_i F(\tilde{T})} = \tau(q)^{\frac{1}{2}(k_i + l(T \circ_i \tilde{T}) - l(T) - l(\tilde{T}))} Z_{F(T \circ_i \tilde{T})},$$

holds for inputs coming from P'. Here, τ is the unique Markov trace on M_1. Thus in particular, $\tau(q) = [M : Q]^{-1}$.

Proof. Here, for any tangle T, $l(T)$ is the number of loops obtained after black-cupping the internal discs of T and black-capping the external disc of T.

The proof of Theorem 2.3.4 proceeds by a series of reductions to easier and easier cases until the result is obvious. There are 4 main steps.

Step 0: Step showing that without loss of generality we can always assume that T and $\widetilde{T}$ are never 0_- tangles:

1. If $\widetilde{T}$ is a 0_- tangle it is clear that $l(T \circ \widetilde{T}) = l(T) + l(\widetilde{T})$. Also $F(\widetilde{T}) = \widetilde{T}$. Thus the equation in Theorem 2.3.4 is automatially satisfied. By same argument this is actually also true whenever $\widetilde{T}$ is a 0_+-tangle.

2. We show below that in Theorem 2.3.4 one can assume without loss of generality

that if T has no points in the outside boundary it must be a 0_+ tangle. Consider the 0_+-tangle W as in Figure 2.3 below.

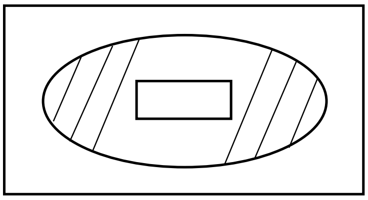


Figure 2.3: Tangle W

Assume that the equation in Theorem 2.3.4 is true whenever T is a 0_+-tangle. Let T be a 0_--tangle. Now $W \circ T$ is a 0_+-tangle with $l(W \circ T) = l(T) + l(W) = l(T) + 1$ and $l((W \circ T) \circ \widetilde{T}) = l(T \circ \widetilde{T}) + 1$. By assumption the following equation holds (for inputs coming from P'):

$$Z_{(W \circ T) \circ F(\widetilde{T})} = \tau(q)^{\frac{1}{2}[\widetilde{k_0} + l((W \circ T) \circ \widetilde{T}) - l(W \circ T) - l(\widetilde{T})]} Z_{(W \circ T) \circ \widetilde{T}} \tag{2.16}$$

By composition law of tangle action we obtain left hand side of the above euation is equal to $\sqrt{[M : N]} Z_{T \circ F(\widetilde{T})}$ and similarly the right hand side is equal to $\sqrt{[M : N]} Z_{T \circ \widetilde{T}}$. Thus from Equation 2.16 we conclude that the following equation

$$Z_{T \circ F(\widetilde{T})} = \tau(q)^{\frac{1}{2}[\widetilde{k_0} + l(T \circ \widetilde{T}) - l(T) - l(\widetilde{T})]} Z_{T \circ \widetilde{T}}$$

holds true. Thus it is justifiable to assume without loss of generality T is never a 0_--tangle.

Step 1: Reduction to the case T is a 0_+-tangle : Let S be the 0_+-tangle as in Figure 2.4 below and $\tilde{S} = \tilde{T}$. We claim that the truth of the equation for S and $\tilde{S}$ implies it for T and $\tilde{T}$. The new disc of S is the last numbered one.

Observe that, by definition, $l(S) = l(T), l(\tilde{S}) = l(\tilde{T})$ and $l(S \circ_i \tilde{S}) = l(T \circ_i \tilde{T})$.

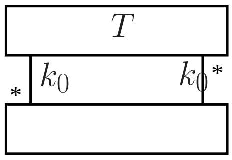


Figure 2.4: Step 1

To prove Theorem 2.3.4, it suffices to trace both sides against an arbitrary element $x \in P_{k_0}$ and verify that the results are the same.

Now, $\delta^{k_0}\tau(Z_{F(T)\circ_i F(\tilde{T})}(\cdots)x) = Z_{F(S)\circ_i F(\tilde{S})}(\cdots, F(x))$ and

$\delta^{k_0}\tau(Z_{F(T\circ_i\tilde{T})}(\cdots)x) = Z_{F(S\circ_i\tilde{S})}(\cdots, F(x))$.

Also, we are given that

$$Z_{F(S)\circ_i F(\tilde{S})} = \tau(q)^{\frac{1}{2}(k_i+l(S\circ_i\tilde{S})-l(S)-l(\tilde{S}))} Z_{F(S\circ_i\tilde{S})},$$

holds when all inputs come from P'.

Thus the $\delta^{k_0}\tau(Z_{F(T)\circ_i F(\tilde{T})}(\cdots)x)$ above equals

$$\tau(q)^{\frac{1}{2}(k_i+l(S\circ_i\tilde{S})-l(S)-l(\tilde{S}))} Z_{F(S\circ_i\tilde{S})}(\cdots, F(x)) =$$
$$\delta^{k_0}\tau(q)^{\frac{1}{2}(k_i+l(T\circ_i\tilde{T})-l(T)-l(\tilde{T}))}\tau(Z_{F(T\circ_i\tilde{T})}(\cdots)x).$$

The desired reduction follows. This reduction having been made, we will henceforth assume that T is a 0_+-tangle and therefore the equation that must be seen to hold on P' is:

$$Z_{T\circ_i F(\tilde{T})} = \tau(q)^{\frac{1}{2}(k_i+l(T\circ_i\tilde{T})-l(T)-l(\tilde{T}))} Z_{T\circ_i\tilde{T}}. \qquad (*)$$

(since $F(T) = T$ for a 0_+-tangle T).

Step 2: Reduction to the case T is of the form in Figure 3.11 where $\hat{T}$ is some

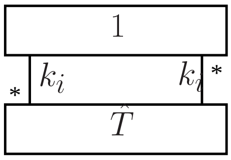
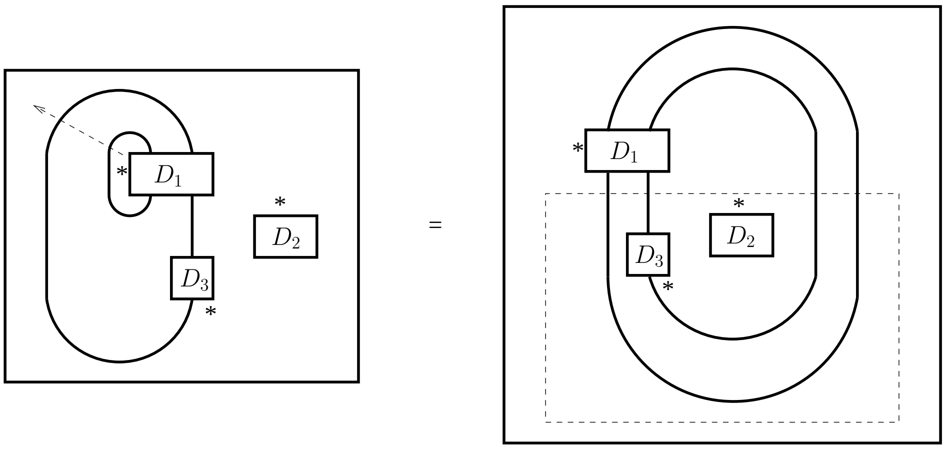


Figure 2.5: Step 2

k_i-tangle and $i = 1$: This follows from sphericality as follows:

Consider the 0_+-tangle T. Now from the $*$-arc of the internal disc D_1 draw an arrow upto points of infinity. This arrow will intersect finitely many strings of T (possibly zero). Now move the string one-by-one into the other side using sphericality so that the $*$-region of D_1 and the $*$-region of the external disc are same. Afterthat put all the other discs except D_1 and the loops, if they exists, inside a big rectangle. Call the tangle inside the rectangle (rectangle will be the external boundary of a new tangle) $\hat{T}$ and put an appropriate $*$ so that it exactly looks like Figure 2.4 (page 73). The example below will explain everything:



Step 3: Reduction to the case $\tilde{T}$ is Temperley-Lieb: This is handled in two different ways according as k_i is even or odd.

Subcase 3.1: Suppose that k_i is even. Let $U = U^{k_1}_{2k_1,k_1}$ and $\tilde{S} = \tilde{S}^{2k_1}$ be the following tangles in Figure 2.6, and $S = T \circ_1 (U \circ_2 \tilde{T})$. It is then clear that $S \circ_1 \tilde{S} = T \circ_1 \tilde{T}$.

We claim that the validity of Equation (*) holding for the pair $(S, \tilde{S})$ implies its

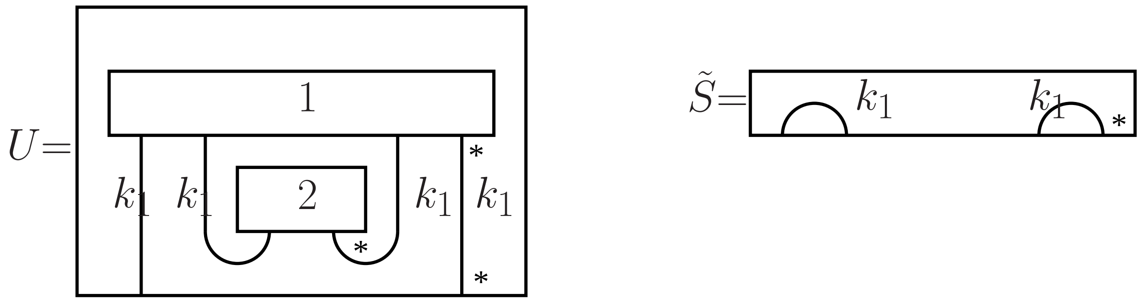


Figure 2.6: Step 3a

validity for the pair $(T, \tilde{T})$. To see this, assume that

$$Z_{S \circ_1 F(\tilde{S})} = \tau(q)^{\frac{1}{2}(2k_1 + l(S \circ_1 \tilde{S}) - l(S) - l(\tilde{S}))} Z_{S \circ_1 \tilde{S}}.$$

Now observe that $Z_{S \circ_1 \tilde{S}} = Z_{T \circ_1 \tilde{T}}$ and $Z_{S \circ_1 F(\tilde{S})} = Z_{T \circ_1 F(\tilde{T})}$ since k_1 is even and using that $q^2 = q$ several times. Also, note that $l(\tilde{S}) = k_1$ and $l(S) = l(\tilde{T}) + l(\hat{T}) = l(\tilde{T}) + l(T)$. Substituting all this in the previous equation and simplifying, we get the desired Equation (*).

Subcase 3.2: Suppose that k_1 is odd. Now let $U = U^{k_1}_{2(k_1+1),k_1}$ and $\tilde{S} = \tilde{S}^{2(k_1+1)}$ be the following tangles in Figure 2.7 and let $S = T \circ_1 (U \circ_2 \tilde{T})$. It is then clear that

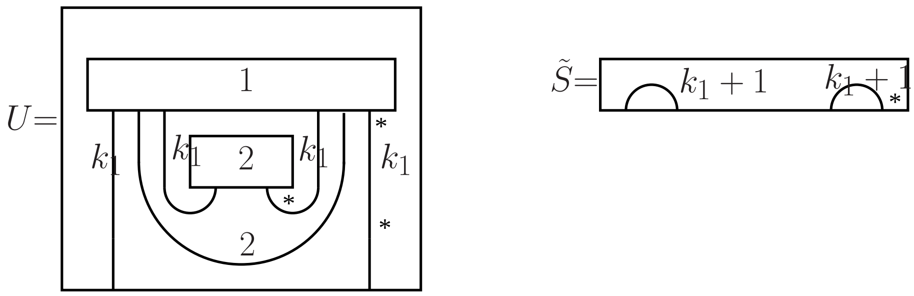


Figure 2.7: Step 3b

$S \circ_1 \tilde{S}$ differs from $T \circ_1 \tilde{T}$ in having one extra floating loop.

We again claim that the validity of Equation (*) holding for the pair $(S, \tilde{S})$ implies its validity for the pair $(T, \tilde{T})$. To see this, assume that

$$Z_{S \circ_1 F(\tilde{S})} = \tau(q)^{\frac{1}{2}(2(k_1+1) + l(S \circ_1 \tilde{S}) - l(S) - l(\tilde{S}))} Z_{S \circ_1 \tilde{S}}.$$

Now observe that $Z_{S \circ_1 \tilde{S}} = \delta Z_{T \circ_1 \tilde{T}}$. Also note that $l(\tilde{S}) = k_1 + 1$, $l(S) = l(\tilde{T}) + l(\hat{T}) = l(\tilde{T}) + l(T)$, and $l(S \circ_1 \tilde{S}) = l(T \circ_1 \tilde{T}) + 1$.

To finish the proof, it suffices to see that $Z_{S \circ_1 F(\tilde{S})} = \delta\tau(q) Z_{T \circ_1 F(\tilde{T})}$. We will first do this in the case $k_1 = 5$. The tangle $S \circ_1 F(\tilde{S})$ is depicted in Figure 2.8. With a

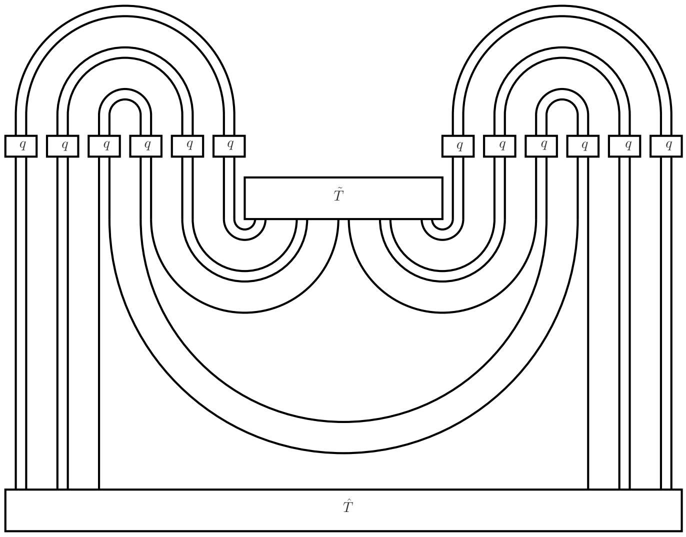


Figure 2.8: Step 3c

little bit of manipulation, this reduces to Figure 2.9.

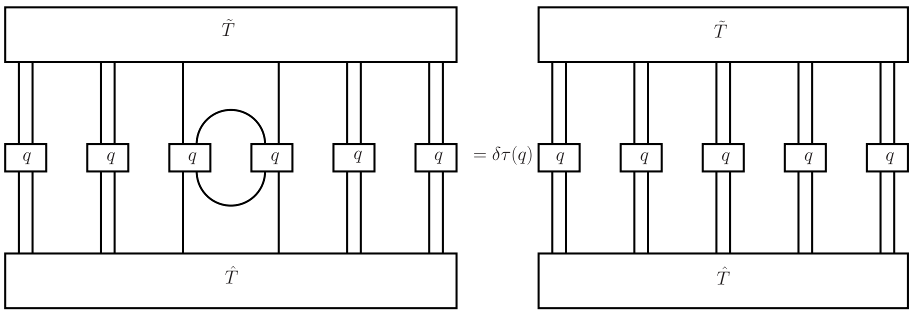


Figure 2.9: Step 3d

Since the last picture is clearly $T \circ_1 F(\tilde{T})$, we're done. It should be clear that a similar proof works whenever k_1 is odd.

Step 4: Resolution of the case $\tilde{T}$ is Temperley-Lieb in three different subcases by induction on k_1. In each of the subcases, we will show that the statement for a suitably chosen S and $\tilde{S}$ with $k_0(\tilde{S}) < k_0(\tilde{T})$, implies it for T and $\tilde{T}$.

Subcase 4.1: Suppose that in $\tilde{T}$ some $2i-1$ and $2i$ are joined by a string so that $\tilde{T}$ has the form in Figure 2.10 for some Temperley-Lieb tangle $\tilde{S}$ of colour $k_1 - 1$. In

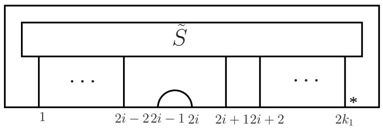


Figure 2.10: Step 4a

this case, let S be the tangle in Figure 2.11.

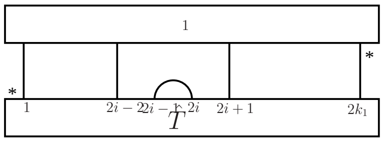


Figure 2.11: Step 4b

Note that $T \circ_1 \tilde{T} = S \circ_1 \tilde{S}$. It follows that $l(T \circ_1 \tilde{T}) = l(S \circ_1 \tilde{S})$ and it is easy to see that $l(S) = l(T)$ $(= l(\hat{T}))$ and that $l(\tilde{S}) = l(\tilde{T}) - 1$. To show that the statement for the pair $S, \tilde{S}$ implies that for the pair $T, \tilde{T}$, it therefore suffices now to see that $T \circ_1 F(\tilde{T}) = S \circ_1 F(\tilde{S})$. This follows easily from the fact that 'q capped on top can be replaced by the identity' (see Theorem 2.1.10).

Subcase 4.2: Suppose that in $\tilde{T}$ some $2i$ and $2i+1$ are joined by a string so that $\tilde{T}$ has the form in Figure 2.12

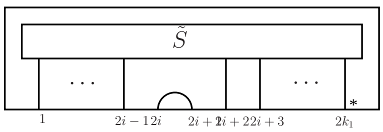


Figure 2.12: Step 4c

for some Temperley-Lieb tangle $\tilde{S}$ of colour $k_1 - 1$. Note that $l(\tilde{S}) = l(\tilde{T})$. Here there are two further subcases.

Subcase 4.2(a): The black intervals $[2i-1, 2i]$ and $[2i+1, 2i+2]$ are part of distinct black regions in $\hat{T}$. In this case, let S be the tangle Figure 2.13

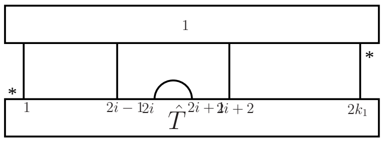


Figure 2.13: Step 4d

Here too $T \circ_1 \tilde{T} = S \circ_1 \tilde{S}$ and it follows that $l(T \circ_1 \tilde{T}) = l(S \circ_1 \tilde{S})$. Recall that $l(\tilde{S}) = l(\tilde{T})$. The pictures for computing $l(T)$ and $l(S)$ are shown in Figure 2.14. (The picture for $l(T)$ is above the one for $l(S)$).

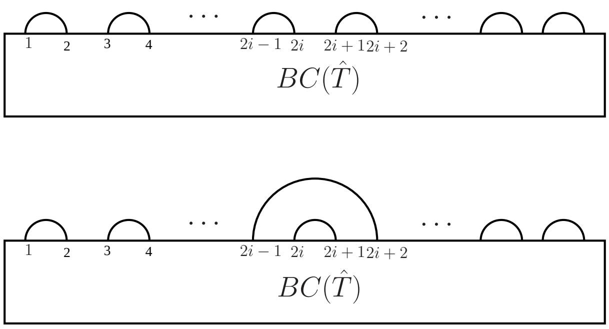


Figure 2.14: Step 4e

Here $BC(\hat{T})$ is the tangle obtained by 'black cupping' the insides of all boxes of $\hat{T}$. We need to compare the number of loops in the top and bottom pictures. Observe that the black regions of $\hat{T}$ and that of $BC(\hat{T})$ are in natural bijective correspondence and therefore the black intervals $[2i-1, 2i]$ and $[2i+1, 2i+2]$ are part of distinct black regions in $BC(\hat{T})$. Thus the loops containing $2i-1$ (and $2i$) and $2i+1$ (and $2i+2$) are different in the first picture while these two loops are cut and spliced into a single loop in the second picture. It follows that $l(S) = l(T) - 1$.

Now suppose that we know that

$$Z_{S \circ_1 F(\tilde{S})} = \tau(q)^{\frac{1}{2}(k_1 - 1 + l(S \circ_1 \tilde{S}) - l(S) - l(\tilde{S}))} Z_{S \circ_1 \tilde{S}}.$$

It follows that

$$Z_{S\circ_1 F(\tilde{S})} = \tau(q)^{\frac{1}{2}(k_1+l(T\circ_1\tilde{T})-l(T)-l(\tilde{T}))} Z_{T\circ_1\tilde{T}},$$

and so to complete the proof it suffices to see that $Z_{S\circ_1 F(\tilde{S})} = Z_{T\circ_1 F(\tilde{T})}$.

To see this, first note that the 'antipode symmetry' of the Theorem 2.1.10(a) implies that $T \circ_1 F(\tilde{T}) = V \circ_1 F(\hat{T})$ where V is the tangle in Figure 2.15.

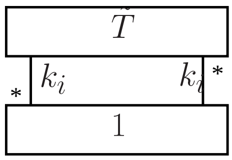


Figure 2.15: Step 4f

Thus $T \circ_1 F(\tilde{T})$ is given by the picture on the left in Figure 2.16 which equals the one on the right using properties of q.

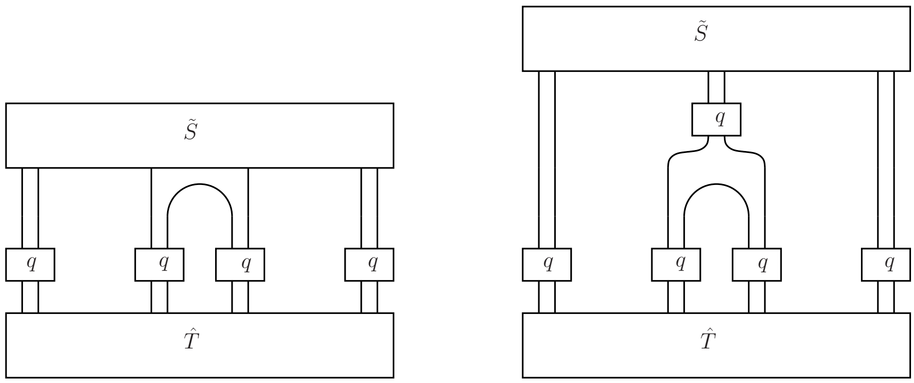


Figure 2.16: Step 4g

The two middle q's in the picture on the right may be deleted using one application of Lemma 2.3.5 to $\hat{T}$ and then what is left is clearly $S \circ_1 F(\tilde{S})$.

Lemma 2.3.5. *Let T be a k-tangle and $[k] = \{1, 2, \cdots, k\}$ be regarded as the set of black external boundary arcs of T, enumerated, say, in clockwise direction starting from the one immediately next (counterclockwise) to the $*$ arc. Let $A \subseteq [k]$ be such that any black region of T intersects at most one element of A. Surround T with q's* *in all positions except those given by A and call this partially labelled tangle $F_A(T)$. Then $Z_{F_A(T)} = Z_{F(T)}$ on P'.*

Proof. Consider the external boundary of any black region of T that intersects an external boundary arc. Say it looks like something in Figure 2.17. Here the dark

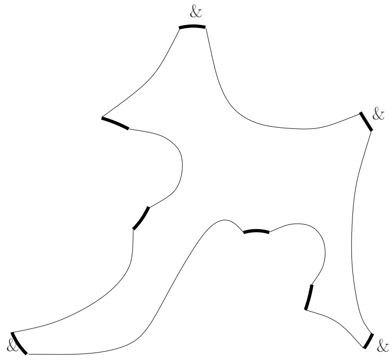


Figure 2.17: Step 4h

portions represent boundary arcs of discs of T while the light portions represent strings. Say the portions marked & are boundary arcs of the external disc of T while the rest are boundary arcs of various internal discs of T. By assumption, at most one of the portions marked & is in A.

Now while calculating $F(T)$, this black region looks as in Figure 2.18, where every 2-box has a q in it. The external portions have a q by definition of $F(T)$,

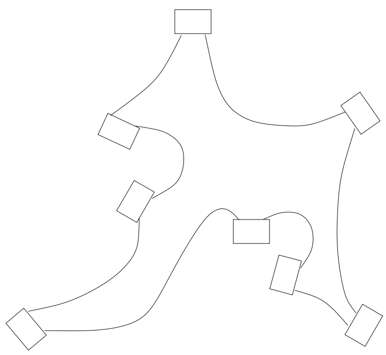


Figure 2.18: Step 4i

while the internal portions have a q because we're only interested in the values of the tangle when inputs come from P'. Now observe that in calculating $F_A(T)$, at most one of the q's is missing - which does not matter because of the exchange relation that q satisfies (see Corollary 2.1.11). □

Subcase 4.2(b): The black intervals $[2i-1, 2i]$ and $[2i+1, 2i+2]$ are part of the same black region in $\hat{T}$.

Draw a dotted line from the midpoint of the interval $[2i-1, 2i]$ to the midpoint of the interval $[2i+1, 2i+2]$ in $\hat{T}$ that lies entirely in the black region that these are both part of. This line does not intersect any string of $\hat{T}$ (by definition of a region) and so the part of $\hat{T}$ that lies inside this dotted line is a 1-box that joins the points $2i$ and $2i+1$. By irreducibility we may replace this one box by a scalar times a string and thus assume that in $\hat{T}$ too, the points $2i$ and $2i+1$ are joined together. Thus $\hat{T}$ is of the form in Figure 2.19

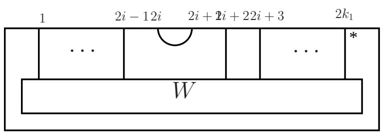


Figure 2.19: Step 4j

for some tangle W of colour $k_1 - 1$. Set S to be the tangle in Figure 2.20.

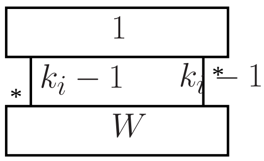


Figure 2.20: Step 4k

Again we claim that the truth of the statement of S and $\tilde{S}$ implies that of the statement for T and $\tilde{T}$. So suppose that

$$Z_{S \circ_1 F(\tilde{S})} = \tau(q)^{\frac{1}{2}(k_1 - 1 + l(S \circ_1 \tilde{S}) - l(S) - l(\tilde{S}))} Z_{S \circ_1 \tilde{S}}.$$

Note that $T \circ_1 \tilde{T} = S \circ_1 \tilde{S}$ with one extra floating loop and therefore $Z_{T \circ_1 \tilde{T}} = \delta Z_{S \circ_1 \tilde{S}}$

and $l(T \circ_1 \tilde{T}) = l(S \circ_1 \tilde{S}) + 1$. Also $l(T) = l(\hat{T}) = l(W) = l(S)$ and we recall that $l(\tilde{S}) = l(\tilde{T})$.

It remains to compare $S \circ_1 F(\tilde{S})$ and $T \circ_1 F(\tilde{T})$. Observe that $T \circ_1 F(\tilde{T})$ equals the picture on the left in Figure 2.21 which equals $\delta\tau(q)$ times picture on the right using properties of q - which is clearly $S \circ_1 F(\tilde{S})$.

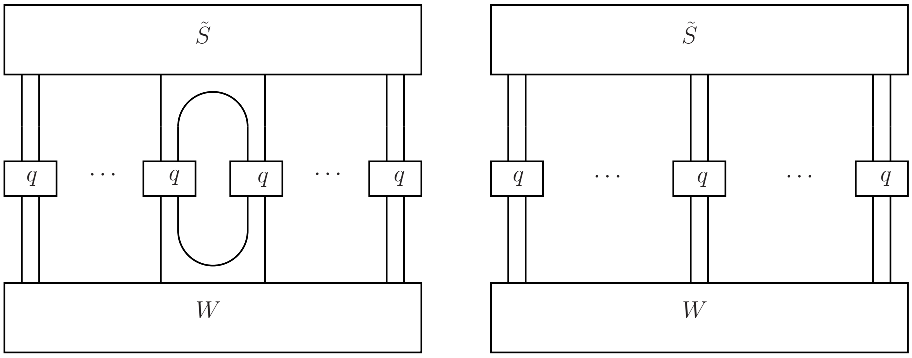


Figure 2.21: Step 4L

Therefore $Z_{T \circ_1 F(\tilde{T})} = \delta\tau(q) Z_{S \circ_1 F(\tilde{S})}$. It now follows that

$$Z_{T \circ_1 F(\tilde{T})} = \tau(q)^{\frac{1}{2}(k_1 + l(T \circ_1 \tilde{T}) - l(T) - l(\tilde{T}))} Z_{T \circ_1 \tilde{T}}.$$

This completes the proof of Theorem 2.3.4. $\square$

We proceed to verify that our prescription for the tangle action does indeed specify various compatibility requirements that must be satisfied in order to define a planar algebra.

(1) Compatibility with renumbering

Let, $\sigma \in \Sigma_b$ (the group of permutations on the set $\{1, 2, \cdots, b\}$). Consider the tangle $\sigma(T)$ which as a subset of $\mathbb{R}^2$ is the same as T except that its $\sigma(i)$-th disc is the i-th disc of T. We have to show the following diagram commutes:

$$\begin{array}{ccc} P'_{k_1} \otimes P'_{k_2} \otimes \cdots \otimes P'_{k_b} & \xrightarrow{U_\sigma} & P'_{k_{\sigma^{-1}(1)}} \otimes P'_{k_{\sigma^{-1}(2)}} \otimes \cdots \otimes P'_{k_{\sigma^{-1}(b)}} \\ \downarrow Z'_T & \swarrow Z'_{\sigma(T)} & \\ P'_{k_0} & & \end{array} .$$

Where,

$$U_\sigma(x_1 \otimes \cdots \otimes x_b) = x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(b)}$$

for x_i belonging to $P'_i \subseteq P_i$. Now,

$$\begin{aligned} & Z'_{\sigma(T)} \circ U_\sigma(x_1 \otimes \cdots \otimes x_b) \\ & = Z'_{\sigma(T)}(x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(b)}) \\ & = \alpha(\sigma(T))F_{k_0}(Z_{\sigma(T)}(x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(b)})) \\ & = \alpha(T)F_{k_0}(Z_{\sigma(T)} \circ U_\sigma(x_1 \otimes \cdots \otimes x_b)) \quad [\text{since } \ \alpha(\sigma(T)) = \alpha(T)] \\ & = \alpha(T)F_{k_0}(Z_T(x_1 \otimes \cdots \otimes x_b)) \quad [\text{ renumbering axiom for Z}] \\ & = Z'_T(x_1 \otimes \cdots \otimes x_b) \quad [\text{by definition}]. \end{aligned}$$

(2) Non-degeneracy

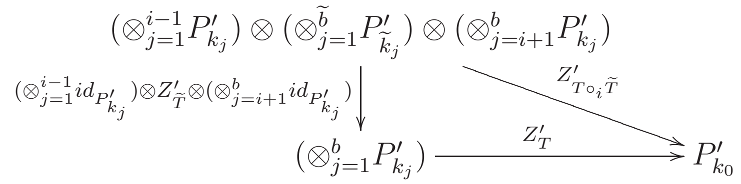
We have to show, $Z'_{I^k_k} = id_{P_k'}$. Now for $x \in P'_k$,

$$\begin{aligned} & Z'_{I^k_k}(x) \\ & = \alpha(I^k_k)F_k(Z_{I^k_k}(x)) \ [\text{by definition}] \\ & = \alpha(I^k_k)F_k(x) \ [\text{non-degeneracy of Z}] \\ & = F_k(x) \qquad [\text{since } \alpha(I^k_k) = 1] \\ & = x. \end{aligned}$$

(3) Compatibility with respect to substitution

Let $T = T^{k_0}_{k_1,\cdots,k_b}$ and $\widetilde{T} = T^{\widetilde{k_0}}_{\widetilde{k_1},\cdots,\widetilde{k_{\tilde{b}}}}$ with $\tilde{k_0} = k_i$ for some $i \in \{1, \cdots, b\}$. We need to check that the following diagram commutes:

When $\tilde{b} > 0$:



Let $(\otimes_{j=1}^{i-1} x_j) \otimes (\otimes_{j=1}^{\tilde{b}} \widetilde{x}_j) \otimes (\otimes_{j=i+1}^{b} x_j)$ belongs to $(\otimes_{j=1}^{i-1} P'_{k_j}) \otimes (\otimes_{j=1}^{\tilde{b}} P'_{\widetilde{k_j}}) \otimes (\otimes_{j=i+1}^{b} P'_{k_j})$. Then,

$$
\begin{aligned}
&Z'_T \circ (id \otimes Z'_{\widetilde{T}} \otimes id)[(\otimes_{j=1}^{i-1} x_j) \otimes (\otimes_{j=1}^{\tilde{b}} \widetilde{x}_j) \otimes (\otimes_{j=i+1}^{b} x_j)] \\
&\quad = Z'_T[(\otimes_{j=1}^{i-1} x_j) \otimes Z'_{\widetilde{T}}(\otimes_{j=1}^{\tilde{b}} \widetilde{x}_j) \otimes (\otimes_{j=i+1}^{b} x_j)] \\
&\quad = Z'_T[(\otimes_{j=1}^{i-1} x_j) \otimes \alpha(\widetilde{T}) Z_{E \circ \widetilde{T}}((\otimes^{\widetilde{k_0}} q) \otimes (\otimes_{j=1}^{\tilde{b}} \widetilde{x}_j)) \otimes (\otimes_{j=i+1}^{b} x_j)] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by definition } of Z'_T) \\
&\quad = \alpha(T)\alpha(\widetilde{T}) Z_{E \circ T}[(\otimes^{k_0} q) \otimes (\otimes_{j=1}^{i-1} x_j) \otimes Z_{E \circ \widetilde{T}}((\otimes^{\widetilde{k_0}} q) \otimes (\otimes_{j=1}^{\tilde{b}} \widetilde{x}_j)) \otimes (\otimes_{j=i+1}^{b} x_j)] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by definition } of Z'_T) \\
&\quad = \alpha(T)\alpha(\widetilde{T}) Z_{E \circ (T \circ_i (E \circ \widetilde{T}))}[(\otimes^{k_0} q) \otimes (\otimes_{j=1}^{i-1} x_j) \otimes (\otimes^{\widetilde{k_0}} q) \otimes (\otimes_{j=1}^{\tilde{b}} \widetilde{x}_j) \otimes (\otimes_{j=i+1}^{b} x_j)] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(since } Z \text{ is associative)} \\
&\quad = \alpha(T)\alpha(\widetilde{T}) \frac{\alpha(T \circ_i \widetilde{T})}{\alpha(T)\alpha(\widetilde{T})} Z_{E \circ (T \circ_i \widetilde{T})}[(\otimes^{k_0} q) \otimes (\otimes_{j=1}^{i-1} x_j) \otimes (\otimes_{j=1}^{\tilde{b}} \widetilde{x}_j) \otimes (\otimes_{j=i+1}^{b} x_j)] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Theorem 2.3.4)} \\
&\quad = Z'_{T \circ_i \widetilde{T}}[(\otimes_{j=1}^{i-1} x_j) \otimes (\otimes_{j=1}^{\tilde{b}} \widetilde{x}_j) \otimes (\otimes_{j=i+1}^{b} x_j)] \qquad \text{(by definition).}
\end{aligned}
$$

When $\tilde{b} = 0$: We need to check that the following diagram commutes:

$$\begin{array}{ccc} (\otimes_{j=1}^{i-1} P'_{k_j}) \otimes \mathbb{C} \otimes (\otimes_{j=i+1}^{b} P'_{k_j}) & \xrightarrow{\cong} & \otimes_{j=1,\, j\neq i}^{b} P'_{k_j} \\ {\scriptstyle (\otimes_{j=1}^{i-1} id_{P'_{k_j}}) \otimes Z'_{\widetilde{T}} \otimes (\otimes_{j=i+1}^{b} id_{P'_{k_j}})} \downarrow & & {\scriptstyle Z'_{T\circ_i \widetilde{T}}} \downarrow \\ \otimes_{j=1}^{b} P'_{k_j} & \xrightarrow{Z'_T} & P'_{k_0} \end{array}$$

The proof is as above.

Thus $T \mapsto Z'_T$ is compatible with substitution.

In conclusion, the collection $P' = \{P'_k : k \in Col\}$ of vector spaces, equipped with the assignment $T \mapsto Z'_T$ of multilinear maps, is a planar algebra. Now, using the fact that a planar algebra is spherical if and only if the 'left' and 'right' traces are same, it follows that (P', Z'_T) is indeed a spherical planar algebra.

Proof. (of part (b) in the notation of the paragraph following the statement of Theorem 2.3.4.)

Here we use Theorem 2.1.7. That $P'_k = (P^{(N \subseteq Q)})_k$ and consistency under inclusions of two sides follows from definition. That (P', Z') has modulus $\sqrt{[Q:N]}$ follows from definition of α. We need, further, to show the following:

Fact 2.3.6. *$(i) Z'_{\mathcal{E}^{2k+1}}(1) = \sqrt{[Q:N]} e^Q_{2k}$ and $(ii) Z'_{\mathcal{E}^{2k}}(1) = \sqrt{[Q:N]} e^Q_{2k-1}$.*

See the figure 2.22. Left one is for case (ii) and right one is for case (i).

Justification of (i): $\alpha(\mathcal{E}^{2k+1}) = \sqrt{[M:Q]}$.

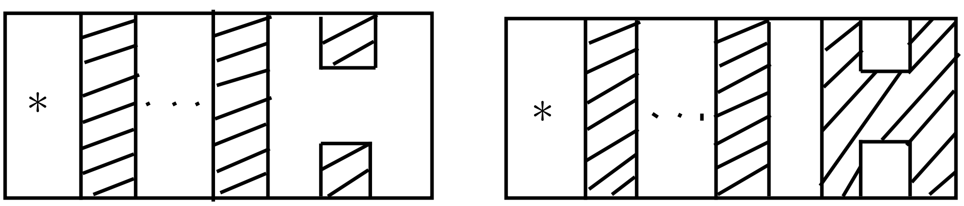
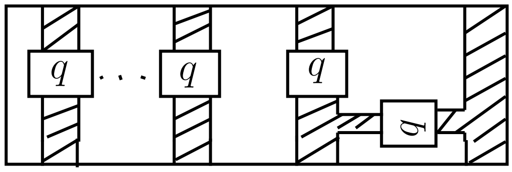


Figure 2.22: Jones Projection

By definition,

$Z'_{\mathcal{E}^{2k+1}}(1)$

$= \sqrt{[M:Q]} F_{2n+1}(Z^{N \subseteq M}_{\mathcal{E}^{2k+1}}(1))$

$= \sqrt{[M:Q]}$ 

$= \sqrt{[Q:N]} e^{Q}_{2k}$ [by Theorem 2.1.13].

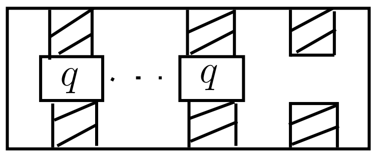
This justifies the fact(i).

Justification of (ii): $\alpha(\mathcal{E}^{2k}) = [M:Q]^{-\frac{1}{2}}$.

By definition,

$Z'_{\mathcal{E}^{2k}}(1)$

$= [M:Q]^{-\frac{1}{2}} F_{2k}(Z^{N \subseteq M}_{\mathcal{E}^{2k}}(1))$

$= [M:Q]^{-\frac{1}{2}}$ 

$= \sqrt{[Q:N]} e^{Q}_{2k-1}$ [by Theorem 2.1.13].

This justifies the fact(ii).

Fact 2.3.7. $Z'_{(E')^n_n}(x) = \sqrt{[Q:N]} E_{Q' \cap Q_{n-1}}(x)$ *for all* x *belongs to* $N' \cap Q_{n-1}$ *and* $n \geq 1$*, where the corresponding trace of* $E^{N' \cap Q_{n-1}}_{Q' \cap Q_{n-1}}$ *is* $tr_{N \subseteq Q}$.

Justification: The tangles $(E')^n_n$ are as in figure 2.23 according as n is odd (on the left) or even (on the right).

Consider the case when n is odd. Now, for all $y \in M' \cap M_{n-1}$ and $x \in {P_n}'$,

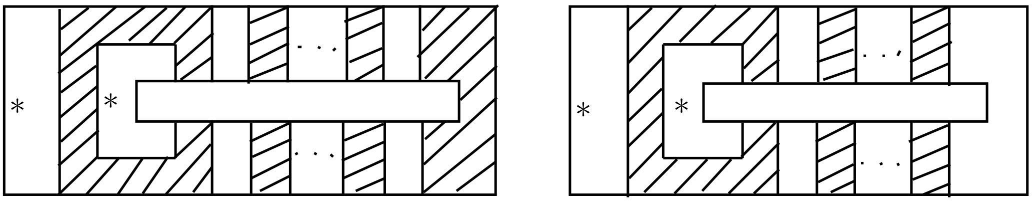
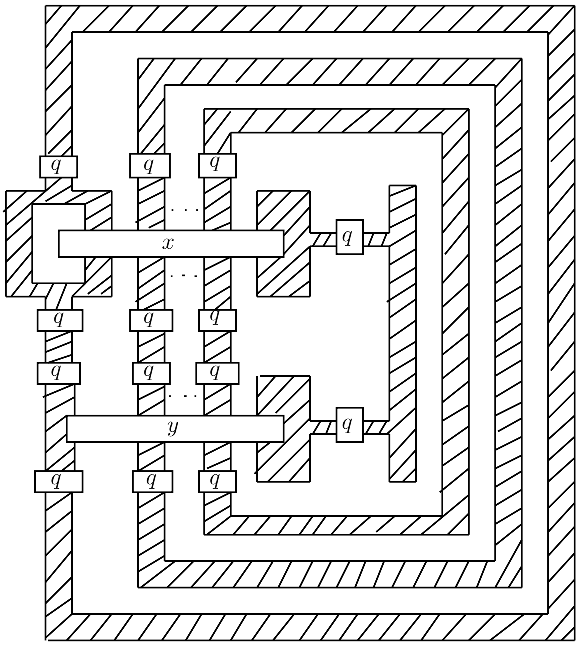


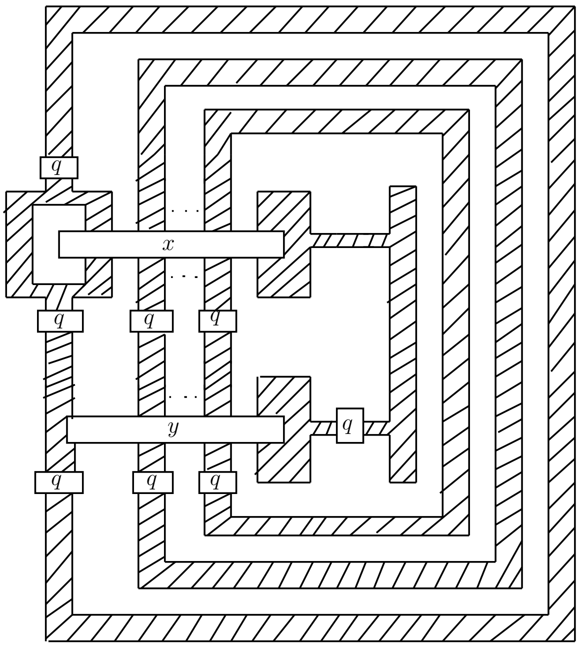
Figure 2.23: Left Conditional Expectation

$tr([M:Q]F(E^{N'\cap M_{n-1}}_{M'\cap M_{n-1}}(x))F(y))$ is equal to

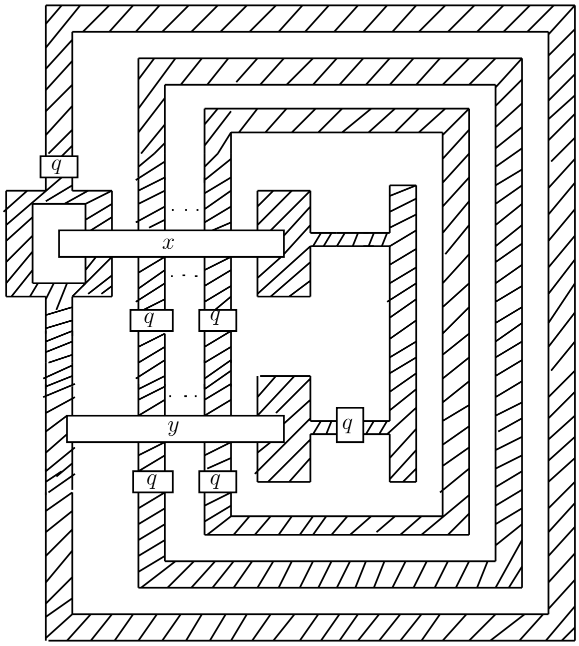
$$\delta^{-n}\frac{[M:Q]}{\sqrt{[M:N]}}$$



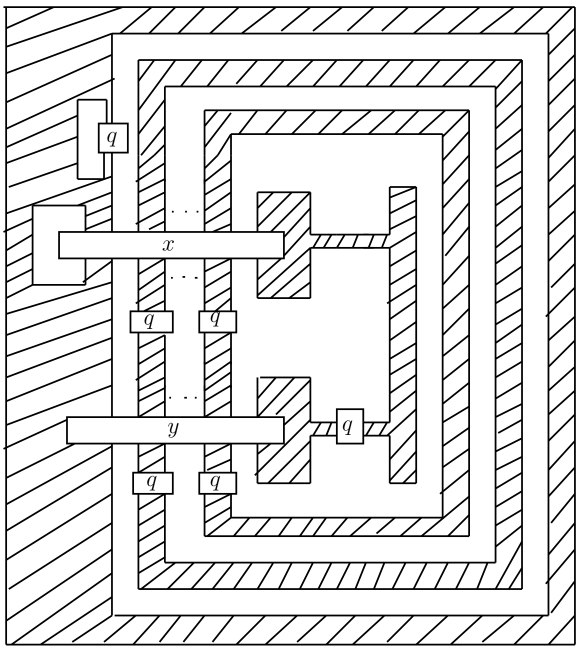
$$=\delta^{-n}\frac{[M:Q]}{\sqrt{[M:N]}}$$



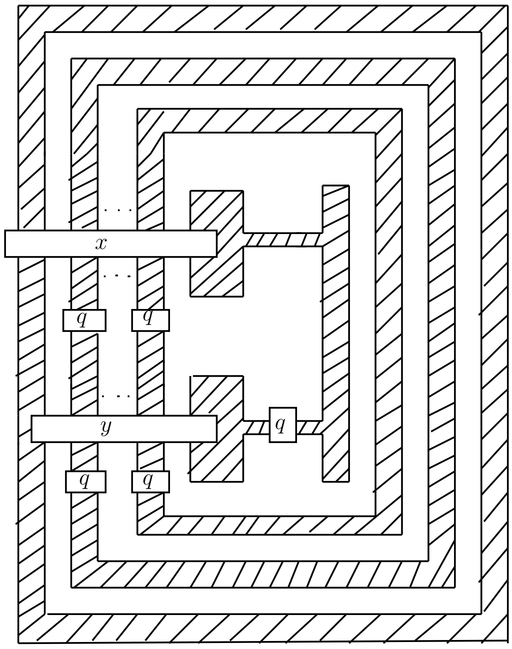
[*by Theorem* 2.1.10 (*a*)]

$= \delta^{-n} \frac{[M:Q]}{\sqrt{[M:N]}}$ 

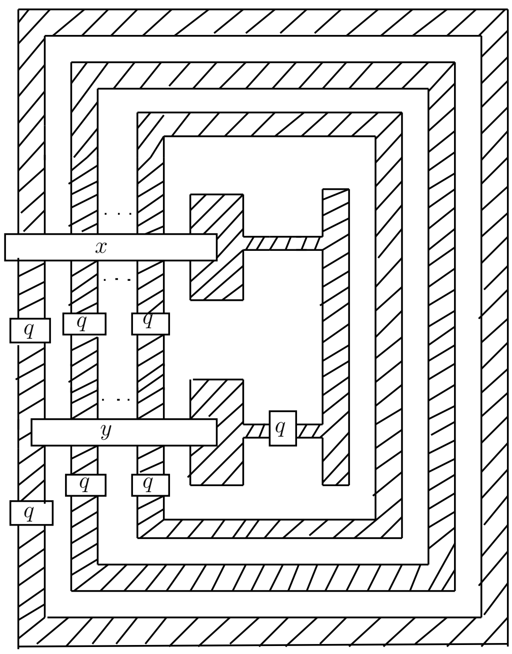
[*since* $x \in F_n(P_n)$ and by exchange relation]

$= \delta^{-n} \frac{[M:Q]}{\sqrt{[M:N]}}$ 

[by extremality]

$= \delta^{-n}$ 

[by Theorem 2.1.10 (c) and extremality]

$= \delta^{-n}$ 

[as x belongs to $F_n(P_n)$]

$$= tr(xF(y)).$$

Thus it follows that

$$E^{F(N'\cap M_{n-1})}_{F(M'\cap M_{n-1})}(x) = [M:Q]F(E^{N'\cap M_{n-1}}_{M'\cap M_{n-1}}(x)). \tag{2.17}$$

Now,

$$\begin{aligned}
&Z'_{(E')^n_n}(x)\\
&\quad = \alpha((E')^n_n)F(Z_{(E')^n_n}(x)) \ \ [by\ definition]\\
&\quad = [M:Q]^{\frac{1}{2}}[M:N]^{\frac{1}{2}}F(E^{N'\cap M_{n-1}}_{M'\cap M_{n-1}}(x)) \ \ [by\ definition\ of\ \alpha]\\
&\quad = [Q:N]^{\frac{1}{2}}E^{F(N'\cap M_{n-1})}_{F(M'\cap M_{n-1})}(x) \ \ [\text{by Equation (2.17)}].
\end{aligned}$$

This completes the proof for odd case. Even case is exactly similar, so we omit it.

Fact 2.3.8. *$Z'_{E^n_{n+1}}(x) = \sqrt{[Q:N]}E_{N'\cap Q_{n-1}}(x)$ for all x belongs to $N'\cap Q_n$ and this is required to hold for all n in Col, where for $n = 0_{\pm}$, the equation is interpreted as $Z'_{E^{0+}_1}(x) = \sqrt{[Q:N]}tr_{N\subseteq Q}(x)$ for all x belongs to $N'\cap Q$. Here again the trace corresponding to the conditional expectation is given by $tr_{N\subseteq Q}$.*

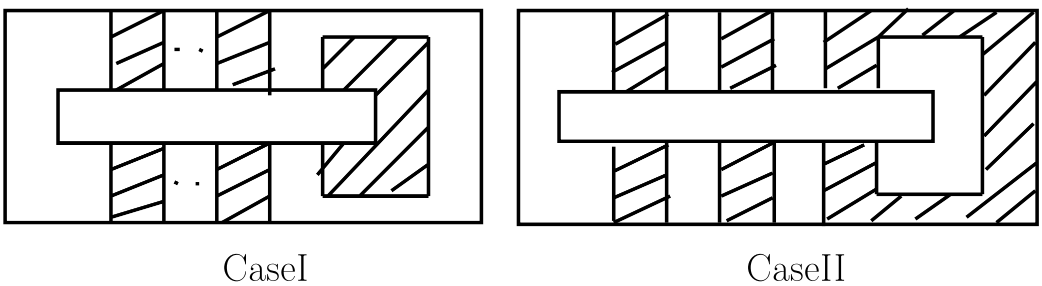
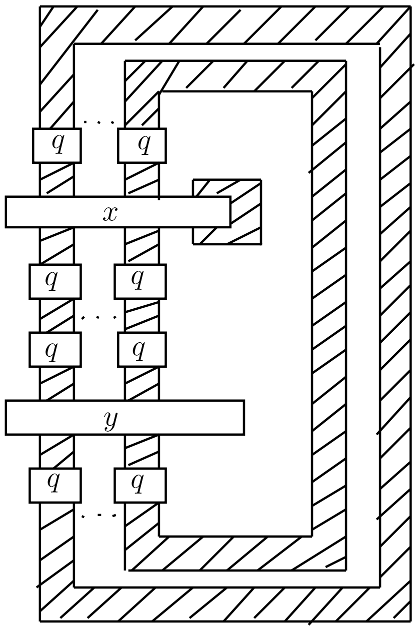


Figure 2.24: Conditional Expectation

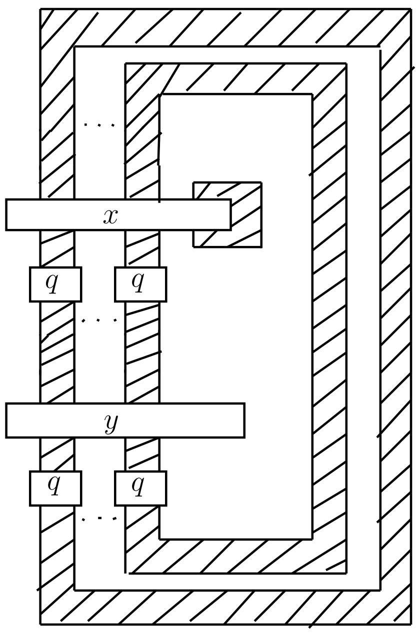
Justification: Consider the conditional expectation tangle as in Figure 2.24. For Case I (that is, n is even) we give a diagramatic proof and for Case II (that is, n is odd) we give analytic proof.

$\mathcal{C}aseI(E^{2n}_{2n+1})$: For all $y \in F(N' \cap M_{2n-1})$, $tr(F(E^{N' \cap M_{2n}}_{N' \cap M_{2n-1}}(x))F(y))$ is equal to

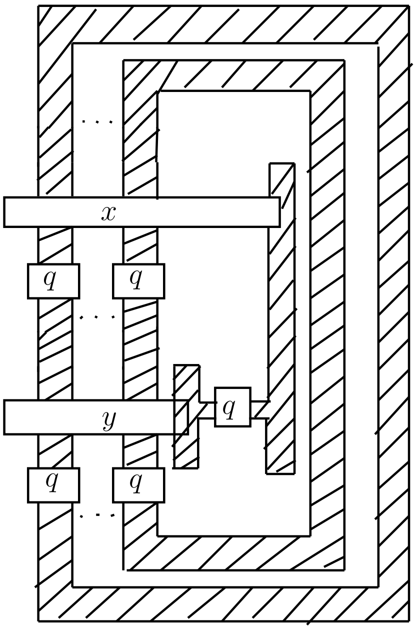
δ^{-2n}



$= \delta^{-2n}$



[by Theorem 2.1.10(a)]

$= \delta^{-2n}$ 

[since, $y \in P_{2n} \subseteq P_{2n+1}$ and use $Theorem$ 2.1.10(b)]

$= tr(xF(y))$.

Thus,

$$E^{F(N' \cap M_{2n})}_{F(N' \cap M_{2n-1})}(x) = F(E^{N' \cap M_{2n}}_{N' \cap M_{2n-1}}(x)). \tag{2.18}$$

Then the following equations hold:

$Z'_{E^{2n}_{2n+1}}(x)$

$= \alpha(E^{2n}_{2n+1})F(Z_{E^{2n}_{2n+1}}(x))$ [by definition]

$$= [M:Q]^{-\frac{1}{2}} F(Z_{E^{2n}_{2n+1}}(x)) \ \ \text{[by definition of } \alpha]$$

$$= [M:Q]^{-\frac{1}{2}} [M:N]^{\frac{1}{2}} F(E^{N'\cap M_{2n}}_{N'\cap M_{2n-1}}(x))$$

$$= [Q:N]^{\frac{1}{2}} E^{F(N'\cap M_{2n})}_{F(N'\cap M_{2n-1})}(x) \ \ \text{[by Equation (2.18)]}$$

This completes the proof of Case I.

$\mathcal{C}aseII(E^{2n-1}_{2n})$: By definition, $\alpha(E^{2n-1}_{2n}) = \sqrt{[M:Q]}$. Then, for $x \in P'_{2n} = F_{2n}(P_{2n})$,

(2.19)

$$Z'_{E^{2n-1}_{2n}}(x) = \sqrt{[M:Q]} F(Z^{N\subseteq M}_{E^{2n-1}_{2n}}(x)) = \sqrt{[M:Q]}\sqrt{[M:N]} F(E^{N'\cap M_{2n-1}}_{N'\cap M_{2n-2}}(x))$$

Then we claim,

$$E^{F(N'\cap M_{2n-1})}_{F(N'\cap M_{2n-2})}(x) = [M:Q] F(E^{N'\cap M_{2n-1}}_{N'\cap M_{2n-2}}(x))$$

Justification: Firstly observe (see [5]),

$$F_{2n}(P_{2n}) = p_{[0,2n-1]}(N'\cap M_{2n-1})p_{[0,2n-1]},$$

where as before $p_{[0,2n-1]} = e_{0,1}e_{0,3}, \cdots, e_{0,2n-1}$. Put, $x = p_{[0,2n-1]}m_{2n-1}p_{[0,2n-1]}$ for $m_{2n-1} \in N'\cap M_{2n-1}$. Then the following self-explanatory array of equations hold for any $m_{2n-2} \in N'\cap M_{2n-2}$(using Fact 2.1.1 repeatedly):

$$tr_{N\subseteq Q}([M:Q]p_{[0,2n-1]}E^{N'\cap M_{2n-1}}_{N'\cap M_{2n-2}}(p_{[0,2n-1]}m_{2n-1}p_{[0,2n-1]})p_{[0,2n-1]}p_{[0,2n-1]}m_{2n-2}p_{[0,2n-1]})$$

$$= [M:Q]^{n+1} tr(p_{[0,2n-1]}E^{N'\cap M_{2n-1}}_{N'\cap M_{2n-2}}(p_{[0,2n-1]}m_{2n-1}p_{[0,2n-1]})p_{[0,2n-1]}m_{2n-2}p_{[0,2n-1]})$$

$$= [M:Q]^{n+1} tr(E^{N'\cap M_{2n-1}}_{N'\cap M_{2n-2}}(p_{[0,2n-1]}m_{2n-1}p_{[0,2n-1]})p_{[0,2n-1]}m_{2n-2}p_{[0,2n-1]})$$

$$= [M:Q]^{n+1} tr(E^{N'\cap M_{2n-1}}_{N'\cap M_{2n-2}}(p_{[0,2n-1]}m_{2n-1}p_{[0,2n-1]})p_{[0,2n-3]}E^{N'\cap M_{2n-2}}_{N'\cap P_{2n-2}}(m_{2n-2})p_{[0,2n-3]}e_{0,2n-1})$$

$$= [M:Q]^{n+1} tr(E^{N'\cap M_{2n-1}}_{N'\cap M_{2n-2}}(p_{[0,2n-1]}m_{2n-1}p_{[0,2n-1]}E^{N'\cap M_{2n-2}}_{N'\cap P_{2n-2}}(m_{2n-2})p_{[0,2n-3]})e_{0,2n-1})$$

$$= [M:Q]^{n} tr(p_{[0,2n-1]}m_{2n-1}p_{[0,2n-1]}E^{N'\cap M_{2n-2}}_{N'\cap P_{2n-2}}(m_{2n-2})p_{[0,2n-3]}) \qquad \text{[Markov Property]}$$

$$= [M:Q]^{n} tr(p_{[0,2n-1]}m_{2n-1}p_{[0,2n-1]}p_{[0,2n-1]}m_{2n-2}p_{[0,2n-1]})$$

$$= tr_{N\subseteq Q}(p_{[0,2n-1]}m_{2n-1}p_{[0,2n-1]}p_{[0,2n-1]}m_{2n-2}p_{[0,2n-1]})$$

Thus from definition of trace preserving conditional expectation we conclude that the claim is justified. Hence from Equation (2.19) it follows that,

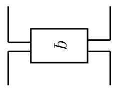
$$Z'_{E_{2n}^{2n-1}}(x) = \frac{\sqrt{[M:Q]}\sqrt{[M:N]}}{[M:Q]} E^{F(N'\cap M_{2n-1})}_{F(N'\cap M_{2n-2})}(x) = \sqrt{[Q:N]}E^{P'_{2n}}_{P'_{2n-1}}(x)$$

This is what we wanted to show. □

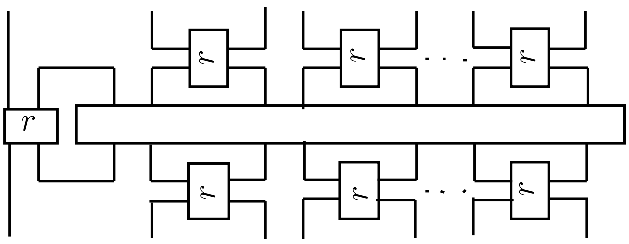
Thus the proof of Theorem 2.3.3 is complete.

2.4 Examples

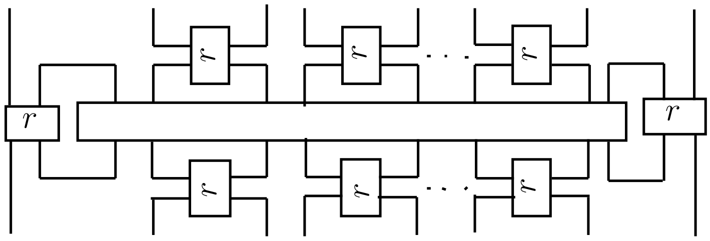
2.4.1 Dual Intermediate Planar algebra

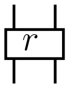
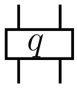
We now consider the other intermediate subfactor $Q \subset M$. We can describe its planar algebra using Theorem 2.3.3 and the fact that it is the dual subfactor to $M \subset P_1$. Namely, apply Theorem 2.3.3 to the planar algebra $(P_{1,n}(L))_n$ of $M \subset M_1$, with respect to the projection $[M:Q]^{1/2}[Q:N]^{-1/2}$ . We obtain the planar algebra $(P_n^{(M\subset P_1)})_n$. The planar algebra of $Q \subset M$ is its dual, $(P_{1,n}^{(M\subset P_1)})_n$. If we carry out this process, we obtain the following planar algebra:

Definition 2.4.2. *Denote the following (unlabelled) tangle by E'_n (pretending that there are no labels, namely r's):*



,



(where  $= \sqrt{\frac{[M:Q]}{[Q:N]}}$  *) according as n is odd or even respectively. We shall use these to define a map $T \mapsto G(T)$ from the class of k-tangles to the class of* partially labelled *k-tangles with $(k+1)$ internal discs all but the last of which are 2-boxes labelled with a r, with the tangle T inserted in the last disc of colour k. Thus, $G(T) = E'_k \circ_{(D_1, D_2, \cdots, D_k, D_{k+1})} (r, r, \cdots, r, T)$.*

If it is clear from the context then we write E' instead of E'_n.

Define functions $G_n : P_n \mapsto P_n$ by $G_n(x) = Z_{E'_n}(r \otimes r \otimes \cdots \otimes r \otimes x)$ for $x \in P_n$. We often write $G(x)$ instead of $G_n(x)$ if there is no confusion.

Definition 2.4.3. *Let T be a k-tangle with $b \geq 1$ internal discs $D_1, \ldots D_b$ of colors $k_1, \ldots k_b$. Then define $\widetilde{\alpha}(T) = [Q : N]^{\frac{1}{2}\widetilde{c}(T)}$, where*

$$\widetilde{c}(T) = (\lceil k_0/2 \rceil + \lfloor k_1/2 \rfloor + \cdots + \lfloor k_b/2 \rfloor) - \widetilde{l}(T)$$

with $\widetilde{l}(T)$ being the number of closed loops after capping the white intervals of the external disc of T and cupping the white intervals of all internal discs of T.

It is straightforward to verify the following corollary:

Corollary 2.4.4. *If $P''_k = range(Z^{(N \subset M)}_{G(I^k_k)})$ and $Z''_T = \widetilde{\alpha}(T) Z_{G(T)|_{\otimes P''_{k_i(T)}}}$, then $(P'', T \mapsto Z''_T|_{\otimes P''_{k_i(T)}})$ is a subfactor planar algebra which is isomorphic to $P^{(Q \subset M)}$.*

2.4.5 Crossed Product Example

Landau described the planar algebra $P(G)$ of the group subfactor $R^G \subset R$ (corresponding to the fixed-points of an outer action of a finite group G on R) which has a presentation with generators given by $L_2 = G$ and $L_k = \phi$ for $k \neq 2$, and the relation that a simple closed loop of either color be the scalar $\sqrt{|G|}$ and the additional six relations labelled $00, 0, 1, 2, 3, 4$ as in [29]. Denote by e, the identity of G.

We have another group Θ and an action $\alpha : \Theta \mapsto Aut(G)$ as in [30]. Without loss of generality we can assume α is $1-1$. Denote by f, the identity of Θ. The map that replaces the label of each 2-box with the label's image under $\theta \in \Theta$ defines an automorphism of $P(G)$ (we will denote this also by θ). Then the set P^{Θ} of invariants for this action is a sub-planar algebra of P, and the set of Θ-invariant k-boxes of $P(G)$ constitutes precisely the set of k-boxes of P^{Θ}.

Remark 2.4.6. *We follow the same notation as in [30] [Remark* 3.3.1.(*b*)*] to denote an orthonormal basis of* $P(G)_k$ *(with respect to the inner product given by the natural trace): define* $S(\bar{g})$ *(where* $\bar{g} \in G^{k-1}$*) to be the labelled* k*-tangle* ($k > 2$) *given by the following two Figures 2.25 and 2.26 for* k *odd and even respectively.*

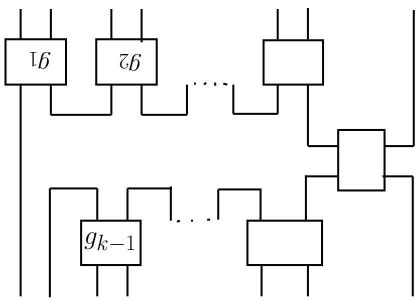
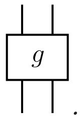


Figure 2.25: k Odd

Also,

$S(g) =$ 

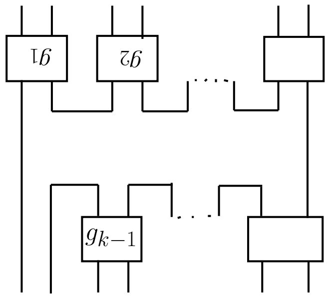


Figure 2.26: k Even

We use Latin alphabets to denote the elements of G*, whereas we use Greek symbols to write the elements of* Θ*. As usual we write the elements of* $G \rtimes \Theta$ *as ordered pairs* (g, θ) *with the ususal multiplication* $(g_1, \theta_1)(g_2, \theta_2) = (g_1\theta_1(g_2), \theta_1\theta_2)$*. Also for each integer* $k \geq 1$ *and* $\theta \in \Theta$*, we simply write* $\theta(g_1, \cdots, g_k)$ *to denote the map* $\alpha_\theta^{(k)} \in Aut(G^k)$ *defined by* $\alpha_\theta^{(k)}(g_1, g_2, \cdots, g_k) = (\alpha_\theta(g_1), \alpha_\theta(g_2), \cdots, \alpha_\theta(g_k))$*. Lastly by* $\bar{\delta}_n$ *we denote the n-tuple* $(\delta_1, \cdots, \delta_n)$*. If from the context it is obvious what is n,we simply write* $\bar{\delta}$*. For convenience we denote by* $\bar{\delta}_{[k,n]}$*(respectively,* $\bar{\delta}_{(k,n]}$*) the tuple* $(\delta_k, \cdots, \delta_n)$*(respectively,* $(\delta_{k+1}, \cdots, \delta_n)$*).*

We prove the following theorem ([30]):

Theorem 2.4.7. *Let* G, Θ *be as above, and let* $G \rtimes \Theta$ *denote the semi-direct product,and let* $N = R^{G\rtimes\Theta} \subset R^\Theta = M$ *denote the corresponding subgroup-subfactor. Then,*

$$P^\Theta \simeq P^{(N\subset M)}.$$

We prove this theorem in various steps.

Fact 2.4.8. *For* $k \geq 3$*, using exchange relation repeatedly and other relations labelled* $0, 1, 2$ *as stated in [29] we get,*

$$S(g_1, g_2, \cdots, g_{k-1})S(h_1, h_2, \cdots, h_{k-1})$$
$$= (\sqrt{|G|})^{(\lceil k/2\rceil-1)}(\prod_{i=2}^{\lceil k/2\rceil} \delta(h_1g_{k+1-i}, h_i))S(h_1g_1, h_1g_2, \cdots, h_1g_{\lceil k/2\rceil}, h_{\lceil k/2\rceil+1}, \cdots, h_{k-1}).$$

Then, for $k = 2$, simply observe, $S(g_1)S(g_2) = S(g_1 g_2)$.

Fact 2.4.9. *Define $\Theta S(\bar{g}) = \sum_{\theta\in\Theta} S(\theta(\bar{g}))$ for $\bar{g} \in G^{k-1}$. Then as stated in [30] $\{\Theta S(\bar{g}) : [\bar{g}] \in G^{k-1}/\Theta\}$ is an orthogonal basis for P_k^{Θ}. A simple calculation shows the following:*

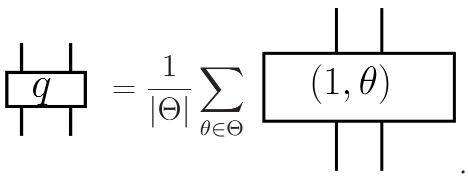
$$\begin{aligned}
&\Theta S(g_1, g_2, \cdots, g_{k-1})\Theta S(h_1, h_2, \cdots, h_{k-1}) \\
&\quad = (\sqrt{|G|})^{(\lceil k/2\rceil - 1)} \sum_{\theta''\in\Theta} \big(\prod_{i=2}^{\lceil k/2\rceil} \delta(h_1\theta''(g_{k+1-i}), h_i)\big) \\
&\quad \Theta S(h_1\theta''(g_1), h_1\theta''(g_2), \cdots, h_1\theta''(g_{\lceil k/2\rceil}), h_{\lceil k/2\rceil+1}, h_{\lceil k/2\rceil+2}, \cdots, h_{k-1}).
\end{aligned}$$

For $k = 2$ the above is being interpreted as

$$\Theta S(g)\Theta S(h) = \sum_{\theta''\in\Theta} \Theta S(g\theta''(h)).$$

Remark 2.4.10. *Note that there is a slight correction in constant in Facts 2.4.8 and 2.4.9 as compared to [30] Remark 3.3.1 (f) and (g), respectively.*

Fact 2.4.11. *Let q be the biprojection corresponding to the intermediate subfactor R^{Θ} such that $R^{G\rtimes\Theta} \subset R^{\Theta} \subset R$. In other words,*



Then using exchange relation we easily get the following result as mentioned in [30]:

$$\begin{aligned}
&F_n(S((g_1, \theta_1), (g_2, \theta_2), \cdots, (g_{n-1}, \theta_{n-1}))) \\
&\quad = \frac{1}{|\Theta|^n} \sum_{\substack{\theta\in\Theta \\ \bar{\gamma}\in\Theta^{n-1}}} S((\theta(g_1), \gamma_1), (\theta(g_2), \gamma_2), \cdots, (\theta(g_{n-1}), \gamma_{n-1})).
\end{aligned}$$

Remark 2.4.12. *Observe that the formula in Fact 2.4.11 depends only on the orbit of* $(g_1, g_2, \cdots, g_{n-1})$ *under* Θ. *Following [30] we put,*

$$U(g_1, g_2, \cdots, g_{k-1}) = \sum_{\substack{\theta \in \Theta \\ \bar{\gamma} \in \Theta^{k-1}}} S((\theta(g_1), \gamma_1), (\theta(g_2), \gamma_2), \cdots, (\theta(g_{k-1}), \gamma_{k-1})).$$

Then it is simple to verify that $\{U(\bar{g}) : [\bar{g}] \in G^{k-1}/\Theta\}$ *is an orthogonal basis for* $F_k(P^{(R^{G \rtimes \Theta} \subset R)})$.

Lemma 2.4.13.

$$\begin{aligned} &U(g_1, g_2, \cdots, g_{k-1}) U(h_1, h_2, \cdots, h_{k-1}) \\ &\quad = (\sqrt{|G|})^{(\lceil k/2 \rceil - 1)} (\sqrt{|\Theta|})^{(\lceil k/2 \rceil - 1)} (|\Theta|)^{\lfloor k/2 \rfloor} \sum_{\theta'' \in \Theta} (\prod_{i=2}^{\lceil k/2 \rceil} \delta(h_1 \theta''(g_{k+1-i}), h_i)) \\ &\qquad U(h_1 \theta''(g_1), h_1 \theta''(g_2), \cdots, h_1 \theta''(g_{\lceil k/2 \rceil}), h_{\lceil k/2 \rceil + 1}, h_{\lceil k/2 \rceil + 2}, \cdots, h_{k-1}). \end{aligned}$$

For $k = 2$, *the above is being interpreted as*

$$U(g)U(h) = |\Theta| \sum_{\theta'' \in \Theta} U(g\theta''(h)).$$

Note that there is a slight inaccuracy in the corresponding equation (Equation(8)) in [30].

Proof.

$$\begin{aligned} &U(g_1, g_2, \cdots, g_{k-1}) U(h_1, h_2, \cdots, h_{k-1}) \\ &\quad = \sum_{\substack{\theta \in \Theta \\ \bar{\gamma} \in \Theta^{k-1}}} S((\theta(g_1), \gamma_1), (\theta(g_2), \gamma_2), \cdots, (\theta(g_{k-1}), \gamma_{k-1})) \\ &\qquad \sum_{\substack{\phi \in \Theta \\ \bar{\sigma} \in \Theta^{k-1}}} S((\phi(h_1), \sigma_1), (\phi(h_2), \sigma_2), \cdots, (\phi(h_{k-1}), \sigma_{k-1})) \end{aligned}$$

$$= (\sqrt{G})^{(\lceil k/2\rceil -1)}(\sqrt{|\Theta|})^{(\lceil k/2\rceil -1)} \sum_{\theta,\phi,\bar{\gamma},\bar{\sigma}} (\prod_{i=2}^{\lceil k/2\rceil} \delta((\phi(h_1)\sigma_1\theta(g_{k+1-i}),\sigma_1\gamma_{k+1-i}),(\phi(h_i),\sigma_i)))$$

$$S((\phi(h_1)\sigma_1\theta(g_1),\sigma_1\gamma_1),(\phi(h_1)\sigma_1\theta(g_2),\sigma_1\gamma_2),\cdots,(\phi(h_1)\sigma_1\theta(g_{\lceil k/2\rceil}),\sigma_1\gamma_{\lceil k/2\rceil}),$$

$$(\phi(h_{\lceil k/2\rceil+1}),\sigma_{\lceil k/2\rceil+1}),\cdots,(\phi(h_{k-1}),\sigma_{k-1}))) \qquad \text{[Using Fact 2.4.8]}$$

$$= (\sqrt{G})^{(\lceil k/2\rceil -1)}(\sqrt{|\Theta|})^{(\lceil k/2\rceil -1)} \sum_{\phi,\theta} \sum_{\substack{\sigma_1\in\Theta \\ \bar{\delta}\in\Theta^{k-1} \\ \bar{\gamma}_{(\lceil k/2\rceil,k-1]}}} (\prod_{i=2}^{\lceil k/2\rceil} \delta(\phi(h_1\phi^{-1}\sigma_1\theta(g_{k+1-i})),\phi(h_i))$$

$$S((\phi(h_1\phi^{-1}\sigma_1\theta(g_1)),\delta_1),(\phi(h_1\phi^{-1}\sigma_1\theta(g_2)),\delta_2),\cdots,(\phi(h_1\phi^{-1}\sigma_1\theta(g_{\lceil k/2\rceil})),\delta_{\lceil k/2\rceil}),$$

$$(\phi(h_{\lceil k/2\rceil+1}),\delta_{\lceil k/2\rceil+1}),\cdots,(\phi(h_{k-1}),\delta_{k-1}))) \quad \text{[Putting } \delta_i = \begin{cases} \sigma_1\gamma_i, & \text{if } 1\le i\le \lceil k/2\rceil \\ \sigma_i, & \text{else]} \end{cases}$$

$$= (\sqrt{G})^{(\lceil k/2\rceil -1)}(\sqrt{|\Theta|})^{(\lceil k/2\rceil -1)}(|\Theta|)^{(\lfloor k/2\rfloor -1)}|\Theta| \sum_{\phi,\bar{\delta}} \sum_{\theta''\in\Theta} (\prod_{i=2}^{\lceil k/2\rceil} \delta(h_1\theta''(g_{k+1-i}),h_i)$$

$$S((\phi(h_1\theta''(g_1)),\delta_1),(\phi(h_1\theta''(g_2)),\delta_2),\cdots,(\phi(h_1\theta''(g_{\lceil k/2\rceil})),\delta_{\lceil k/2\rceil}),$$

$$(\phi(h_{\lceil k/2\rceil+1}),\delta_{\lceil k/2\rceil+1}),\cdots,(\phi(h_{k-1}),\delta_{k-1}))) \qquad \text{[Putting } \phi^{-1}\sigma_1\theta = \theta''\text{]}$$

This completes the proof. □

Definition 2.4.14. *Define linear maps $\Phi_k : P_k^{\Theta} \longmapsto F_k(P_k(G \ltimes \Theta))$ by,*

$$\Phi_k(\Theta S(\bar{g})) = |\Theta|^{-\lfloor k/2\rfloor}(\sqrt{|\Theta|})^{(1-\lceil k/2\rceil)}U(\bar{g}),$$

here $[\bar{g}] \in G^{k-1}/\Theta$.

Remark 2.4.15. *What we have called Φ_k Landau and Sunder have termed as β_k in [30]. It seems that there is a mistake in the definition of β_k in [30].*

To prove Theorem 2.4.7 we need to check that the following equation holds:

$$\Phi_{k_0}(Z_T(x_1\otimes\cdots\otimes x_b)) = Z'_T(\Phi_{k_1}(x_1)\otimes\cdots\otimes\Phi_{k_b}(x_b))\} \tag{2.20}$$

for any tangle $T(=T^{k_0}_{k_1,\cdots,k_b})$. Here Z'_T is defined as in Theorem 2.3.3.

In view of Theorem 2.1.8, it suffices to prove

Theorem 2.4.16. *The collection $\mathscr{T}$ of those tangles T which satisfy Equation (2.20) contains a class of 'generating tangles' namely $\mathscr{T} \supset \{1^{0+}, 1^{0-}\} \cup \{\mathcal{E}^k : k \geq 2\} \cup \{(E')^k_k : k \geq 1\} \cup \{E^k_{k+1}, M_k, I^{k+1}_k : k \in Col\}$*

By 'compatibility with composition' axiom of tangle action of a planar algebra it is apparent that the collection $\mathscr{T}$ is closed under composition. We prove in detail that $\mathscr{T}$ contains the multiplication tangles and the right conditional expectation tangles. In other cases we just sketch the proofs.

Lemma 2.4.17. *$M_k \in \mathscr{T}$.*

Proof. Firstly note,

$$\Phi_k(Z_{M_k}(\Theta S(g_1, g_2, \cdots, g_{k-1}) \otimes \Theta S(h_1, h_2, \cdots, h_{k-1})))$$
$$= \Phi_k(\Theta S(g_1, g_2, \cdots, g_{k-1}) \Theta S(h_1, h_2, \cdots, h_{k-1}))$$
$$= (\sqrt{|G|})^{(\lceil k/2 \rceil - 1)} \sum_{\theta'' \in \Theta} (\prod_{i=2}^{\lceil k/2 \rceil} \delta(h_1\theta''(g_{k+1-i}), h_i)) \times$$
$$\Phi_k(\Theta S(h_1\theta''(g_1), h_1\theta''(g_2), \cdots, h_1\theta''(g_{\lceil k/2 \rceil}), h_{\lceil k/2 \rceil+1}, h_{\lceil k/2 \rceil+2}, \cdots, h_{k-1}))$$

[Using Fact 2.4.9]

$$= (\sqrt{|\Theta|})^{1-\lceil k/2 \rceil} (|\Theta|)^{-\lfloor k/2 \rfloor} (\sqrt{|G|})^{(\lceil k/2 \rceil - 1)} \sum_{\theta''} (\prod_{i=2}^{\lceil k/2 \rceil} \delta(h_1\theta''(g_{k+1-i}), h_i)) \times$$
$$U(h_1\theta''(g_1), h_1\theta''(g_2), \cdots, h_1\theta''(g_{\lceil k/2 \rceil}), h_{\lceil k/2 \rceil+1}, h_{\lceil k/2 \rceil+2}, \cdots, h_{k-1})$$

[Definition 2.4.14].

On the other hand, since $\alpha(M_k) = 1$ the following equations hold:

$$Z'_{M_k}(\Phi_k(\Theta S(g_1, g_2, \cdots, g_{k-1})) \otimes \Phi_k(\Theta S(h_1, h_2, \cdots, h_{k-1})))$$
$$= \alpha(M_k) F_k(Z_{M_k}((|\Theta|)^{-2\lfloor k/2 \rfloor} (\sqrt{|\Theta|})^{2(1-\lceil k/2 \rceil)} U(g_1, g_2, \cdots, g_{k-1}) \otimes U(h_1, h_2, \cdots, h_{k-1})))$$

[Definition 2.4.14]

$$= (\sqrt{|G|})^{(\lceil k/2\rceil-1)}(\sqrt{|\Theta|})^{(\lceil k/2\rceil-1)}(|\Theta|)^{\lfloor k/2\rfloor}(|\Theta|)^{-2\lfloor k/2\rfloor}(\sqrt{|\Theta|})^{2(1-\lceil k/2\rceil)}\sum_{\theta''\in\Theta}(\prod_{i=2}^{\lceil k/2\rceil}\delta(h_1\theta''(g_{k+1-i}),h_i))$$

$$U(h_1\theta''(g_1),h_1\theta''(g_2),\cdots,h_1\theta''(g_{\lceil k/2\rceil}),h_{\lceil k/2\rceil+1},h_{\lceil k/2\rceil+2},\cdots,h_{k-1}$$

[by Lemma 2.4.13]

$$= (\sqrt{|G|})^{(\lceil k/2\rceil-1)}(|\Theta|)^{-\lfloor k/2\rfloor}(\sqrt{|\Theta|})^{1-\lceil k/2\rceil)}\sum_{\theta''}(\prod_{i=2}^{\lceil k/2\rceil}\delta(h_1\theta''(g_{k+1-i}),h_i))$$

$$U(h_1\theta''(g_1),h_1\theta''(g_2),\cdots,h_1\theta''(g_{\lceil k/2\rceil}),h_{\lceil k/2\rceil+1},h_{\lceil k/2\rceil+2},\cdots,h_{k-1}).$$

This completes the proof. □

Lemma 2.4.18. $E^k_{k+1}\in\mathscr{T}$.

Proof. **Case I**: $k=2n$. Put $T=E^k_{k+1}$. For $n=1$, use relation 1 to get $Z_T(S(g_1,g_2))=S({g_1}^{-1})$. If $n\geq 2$, we again using relation 1 get the following result easily:

$$Z_T(S(g_1,g_2,\cdots,g_k))=S(g_1,g_2,\cdots,g_{(k/2)},g_{(\frac{k}{2}+2)},\cdots,g_k). \tag{2.21}$$

Also observe, $\alpha(T)=|\Theta|^{-1/2}$.

We show for $n\geq 2, T\in\mathscr{T}$ ($n=1$ is exactly similar).

$$\begin{aligned}
&\Phi_k(Z_T(\Theta S(g_1,g_2,\cdots,g_k)))\\
&\quad=\Phi_k(Z_T(\sum_{\theta\in\Theta}S(\theta(g_1),\theta(g_2),\cdots,\theta(g_k))))\\
&\quad=\Phi_k(\sum_{\theta}Z_T(S(\theta(g_1),\theta(g_2),\cdots,\theta(g_k))))\\
&\quad=\Phi_k(\sum_{\theta}S(\theta(g_1),\theta(g_2),\cdots,\theta(g_{k/2}),\theta(g_{(\frac{k}{2}+2)}),\cdots,\theta(g_k)))\ \text{[by Equation (2.21)]}\\
&\quad=\Phi_k(\Theta S(g_1,g_2,\cdots,g_{(k/2)},g_{(\frac{k}{2}+2)},\cdots,g_k))\\
&\quad=(\sqrt{|\Theta|})^{(1-\lceil k/2\rceil)}(|\Theta|)^{-\lfloor k/2\rfloor}U(g_1,g_2,\cdots,g_{(k/2)},g_{(\frac{k}{2}+2)},\cdots,g_k).
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
&Z'_T(\Phi_{k+1}(\Theta S(g_1, g_2, \cdots, g_k))) \\
&\quad = \alpha(T) F_k(Z_T((\sqrt{|\Theta|})^{(1-\lceil \frac{k+1}{2} \rceil)}(|\Theta|)^{-\lfloor \frac{k+1}{2} \rfloor} U(g_1, g_2, \cdots, g_k))) \\
&\quad = (|\Theta|)^{-1/2}(\sqrt{|\Theta|})^{(1-\lceil \frac{k+1}{2} \rceil)}(|\Theta|)^{-\lfloor \frac{k+1}{2} \rfloor} F_k(Z_T(\sum_{\substack{\theta \in \Theta \\ \bar{\gamma} \in \Theta^k}} (\theta(g_1), \gamma_1), (\theta(g_2), \gamma_2), \cdots, (\theta(g_k), \gamma_k)))) \\
&\quad = (|\Theta|)^{-1/2}(\sqrt{|\Theta|})^{(1-\lceil \frac{k+1}{2} \rceil)}(|\Theta|)^{-\lfloor \frac{k+1}{2} \rfloor} F_k(\sum_{\theta, \bar{\gamma}} (\theta(g_1), \gamma_1), (\theta(g_2), \gamma_2), \cdots \\
&\quad \cdots, (\theta(g_{(k/2)}), \gamma_{(k/2)}), (\theta(g_{(\frac{k}{2}+2)}), \gamma_{(\frac{k}{2}+2)}), \cdots, (\theta(g_k), \gamma_k))) \\
&\quad = (|\Theta|)^{-1/2}(\sqrt{|\Theta|})^{(1-\lceil \frac{k+1}{2} \rceil)}(|\Theta|)^{-\lfloor \frac{k+1}{2} \rfloor} |\Theta| F_k(U(g_1, g_2, \cdots, g_{(k/2)}, g_{(\frac{k}{2}+2)}, \cdots, g_k)). \\
&\quad = (\sqrt{|\Theta|})^{(2-\lceil \frac{k+1}{2} \rceil)}(|\Theta|)^{-\lfloor \frac{k+1}{2} \rfloor} U(g_1, g_2, \cdots, g_{(k/2)}, g_{(\frac{k}{2}+2)}, \cdots, g_k).
\end{aligned}
$$

Simple algebraic calculation tells us, $\lceil (k+1)/2 \rceil = \lceil (k/2) \rceil + 1$, and $\lfloor (k+1)/2 \rfloor = \lfloor k/2 \rfloor$. Thus we have proved,

$$\Phi_k(Z_T(\Theta S(g_1, g_2, \cdots, g_k))) = Z'_T(\Phi_{k+1}(\Theta S(g_1, g_2, \cdots, g_k))).$$

In other words, $E^k_{k+1} \in \mathscr{T}$.

Case II: $k = 2n - 1$. Put $T = E^k_{k+1}$. The case $n = 1$ is trivial. Using relation 2 as in [30] and exchange relation we easily get:

$$Z_T(S(g_1, g_2, g_3)) = \sqrt{|G|}\delta(g_2, g_3) S(g_1, g_2).$$

More generally, for $n > 2$ it follows that

(2.22)

$$Z_T(S(g_1, g_2, \cdots, g_k)) = \sqrt{|G|}\delta(g_{(\lceil k/2 \rceil)}, g_{(\lceil k/2 \rceil + 1)}) S(g_1, g_2, \cdots, g_{(\lceil k/2 \rceil)}, g_{(\lceil k/2 \rceil + 2)}, \cdots, g_k).$$

Then the following equations are easy to check:

$$\begin{aligned}
&\Phi_k(Z_T(\Theta S(g_1, g_2, \cdots, g_k))) \\
&\quad = \Phi_k(Z_T(\sum_{\theta\in\Theta} S(\theta(g_1), \theta(g_2), \cdots, \theta(g_k)))) \\
&\quad = \Phi_k(\sum_{\theta} \sqrt{|G|} \quad \delta(g_{(\lceil k/2\rceil)}, g_{(\lceil k/2\rceil+1)}) S(\theta(g_1), \theta(g_2), \cdots, \theta(g_{(\lceil k/2\rceil)}), \theta(g_{(\lceil k/2\rceil+2)}), \cdots, \theta(g_k)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{[by Equation 2.22]} \\
&\quad = \Phi_k(\sqrt{|G|} \quad \delta(g_{(\lceil k/2\rceil)}, g_{(\lceil k/2\rceil+1)}) \Theta S(g_1, g_2, \cdots, g_{(\lceil k/2\rceil)}, g_{(\lceil k/2\rceil+2)}, \cdots, g_k)) \\
&\quad = \sqrt{|G|} \quad \delta(g_{(\lceil k/2\rceil)}, g_{(\lceil k/2\rceil+1)}) (\sqrt{|\Theta|})^{(1-\lceil k/2\rceil)} (|\Theta|)^{-\lfloor k/2\rfloor} U(g_1, g_2, \cdots, g_{(\lceil k/2\rceil)}, g_{(\lceil k/2\rceil+2)}, \cdots, g_k).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&Z'_T(\Phi_{k+1}(\Theta S(g_1, g_2, \cdots, g_k))) \\
&\quad = (|\Theta|)^{1/2} F_k(Z_T((\sqrt{|\Theta|})^{(1-\lceil \frac{k+1}{2}\rceil)} (|\Theta|)^{-\lfloor \frac{k+1}{2}\rfloor} U(g_1, g_2, \cdots, g_k))) \text{ [since } \alpha(T) = |\Theta|^{1/2}]. \\
&\quad = (|\Theta|)^{1/2} (\sqrt{|\Theta|})^{(1-\lceil \frac{k+1}{2}\rceil)} (|\Theta|)^{-\lfloor \frac{k+1}{2}\rfloor} F_k(Z_T(\sum_{\substack{\theta\in\Theta \\ \bar{\gamma}\in\Theta^k}} S((\theta(g_1), \gamma_1), (\theta(g_2), \gamma_2), \cdots, (\theta(g_k), \gamma_k)))) \\
&\quad = (|\Theta|)^{1/2} (\sqrt{|\Theta|})^{(1-\lceil \frac{k+1}{2}\rceil)} (|\Theta|)^{-\lfloor \frac{k+1}{2}\rfloor} (\sqrt{|G|}\sqrt{|\Theta|}) \delta(g_{(\lceil k/2\rceil)}, g_{(\lceil k/2\rceil+1)}) \\
&\qquad F_k(U(g_1, g_2, \cdots, g_{(\lceil k/2\rceil)}, g_{(\lceil k/2\rceil+2)}, \cdots, g_k)) \qquad\qquad \text{[by Equation 2.22]} \\
&\quad = \sqrt{|G|} (\sqrt{|\Theta|})^{(1-\lceil \frac{k+1}{2}\rceil)} (|\Theta|)^{(1-\lfloor \frac{k+1}{2}\rfloor)} \delta(g_{(\lceil k/2\rceil)}, g_{(\lceil k/2\rceil+1)}) U(g_1, g_2, \cdots, g_{(\lceil k/2\rceil)}, g_{(\lceil k/2\rceil+2)}, \cdots, g_k).
\end{aligned}$$

In this case observe that, $\lceil k/2\rceil = \lceil (k+1)/2\rceil$ and $\lfloor (k+1)/2\rfloor = \lfloor k/2\rfloor + 1$. Thus,

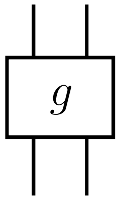
$$\Phi_k(Z_T(\Theta S(g_1, g_2, \cdots, g_k))) = Z'_T(\Phi_{k+1}(\Theta S(g_1, g_2, \cdots, g_k))).$$

In other words, $E^k_{k+1} \in \mathscr{T}$. $\square$

Lemma 2.4.19. $I^{k+1}_k \in \mathscr{T}$.

Proof. Put $T = I_k^{k+1}$. Clearly, $\alpha(T) = 1$. Put

$$h = \frac{1}{\sqrt{|G|}} \sum_{g \in G}$$



.

Clearly $\theta(h) = h$ for all $\theta \in \Theta$.

Case I [$k = 2n$ even]:

It is easy to check that $Z_T(S(g)) = S(g^{-1}, h)$. In general, for $n > 2$ we have

$$Z_T(S(g_1, g_2, \cdots, g_{k-1})) = S(g_1, g_2, \cdots, g_{(k/2)}, h, g_{(k/2+1)}, \cdots, g_{k-1}).$$

Case II [$k = 2n - 1$ odd]:

$k = 1$ is a trivial case. For $k = 3$ we get $Z_T(S(g_1, g_2)) = S(g_1, g_2, g_2)$. In general,

$$Z_T(S(g_1, g_2, \cdots, g_{k-1})) = S(g_1, g_2, \cdots, g_{(\lceil k/2 \rceil)}, g_{(\lceil k/2 \rceil)}, g_{(\lceil k/2 \rceil + 1)}, \cdots, g_{k-1}),$$

for $k > 3$.

The proof of the above equations is routine, and omitted. □

Lemma 2.4.20. $(E')_k^k \in \mathscr{T}$.

Proof. We omit the details. Put $T = (E')_k^k$. In this case $\alpha(T) = (|\Theta|)^{1/2}$. The case $k = 1$ is trivial. In general, simply observe:

$$Z_T(S(g_1, g_2, \cdots, g_{k-1})) = \sqrt{|G|}\delta(g_1, e)S(e, g_2, \cdots, g_{k-1}).$$

□

This complets the proof of Theorem 2.4.16.

Lastly we apply Theorem 2.3.3 to conclude that the proof of Theorem 2.4.7 is now complete.

Chapter 3

Angle between intermediate subfactors

First recall the concept of a 'commuting square', which is a quadruple of (finite) von Neumann algebras with inclusions as in the diagram:

$$\begin{array}{ccc} Q & \subset & M \\ \cup & & \cup \\ N & \subset & P \end{array}$$

(we always denote the above quadruple by (N, P, Q, M)) such that $E^M_P E^M_Q = E^M_Q E^M_P = E^M_N$. Commuting squares were first introduced by S. Popa in [38] and were later found to be a very useful tool in subfactor theory (see for example [25], [17], [40], [41], [42] to name a few). In this thesis we only work with commuting squares of type II_1 factors ; and all subfactors will be of finite index.

We denote by $\mathcal{L}(N \subset M)$ the set of all intermediate von Neumann subalgebras for the subfactor $N \subset M$. The set $\mathcal{L}(N \subset M)$ forms a lattice under the two operations $P \wedge Q = P \cap Q$ and $P \vee Q = (P \cup Q)''$. The study of the lattice structure of von Neumann subalgebras was initiated by Murray and von Neumann in [33].

If $N \subset M$ is irreducible, that is $N' \cap M = \mathbb{C}$, then $\mathcal{L}(N \subset M)$ is exactly the lattice of intermediate subfactors. In this direction Watatani in [49] has obtained the following remarkable result:

Theorem 3.0.21. *[49] Let $N \subset M$ be an irreducible subfactor of type* II_1 *such that $[M : N] < \infty$. Then the set $\mathcal{L}(N \subset M)$ is finite.*

R. Longo in [31] provided an explicit bound depending only on $[M : N]$ for the cardinality of $\mathcal{L}(N \subset M)$ for irreducible subfactor $N \subset M$ of type II_1 with $[M : N] < \infty$. He showed that $|\mathcal{L}(N \subset M)| \leq l^l$, where $l = [M : N]^2$.

Intermediate subfactors may be thought of as generalization of subgroups for the following reasons. Let G be a finite group with an outer action on the II_1 factor M. Then the intermediate subfactors of $M \subset M \rtimes G$ are given by $M \rtimes H$, where H is a subgroup of G. Thus 'bounding the cardinality of $\mathcal{L}(N \subset M)$ for the irreducible subfactor $N \subset M$' is related to the famous Wall's conjecture(1961)(It should be mentioned that during the June 2012 AIM workshop, "Cohomology bounds and growth rates" a counterexample was found to this conjecture):

Conjecture 3.0.22 (Wall's conjecture)**.** *For a finite group G, let* $\max(G)$ *denote the number of maximal proper subgroups of G. Then we have*

$$\max(G) \leq |G|.$$

We have attacked this problem of finding upper bound for the cardinality of $\mathcal{L}(N \subset M)$ by introducing a notion of **angle** (see Definition 3.1.1), denoted by $\alpha_M^N(P, Q)$, between intermediate subfactors P and Q of any finite-index subfactor $N \subset M$ and exploit the fact that the angle has certain rigidity in the case $N' \cap M = \mathbb{C}$. More precisely,

Theorem 3.0.23. *If P and Q are two distinct minimal intermediate subfactors of the irreducible subfactor $N \subset M$ with $[M : N] < \infty$, then $\alpha_M^N(P, Q) > \frac{\pi}{3}$.*

This rigidity result of 'angle' leads us to give an estimation on the size of the lattice of intermediate subfactors in the following theorem:

Theorem 3.0.24. *Suppose $N \subset M$ is an irreducible subfactor, then we have upper bound of the size of the lattice of intermediate subfactors as below:*

$$|\mathcal{L}(N, M)| \leq 9^{[M:N]}$$

This improves existing upper bounds for the cardinality of this set and, in particular provides another proof of Theorem 3.0.21 using planar algebraic machinery.

Motivated by relative position of two subspaces (see [20] for example) Sano and Watatani has introduced in [47] angle between two von Neumann subalgebras of a finite von Neumann algebra. This has been further investigated in [19] and [18]. This angle contains spectral data. Our notion of 'angle' is a very natural one and it actually measures how far intermediate subfactors should be to form a commuting square: we show $\alpha_M^N(P, Q) = \pi/2$ if and only if the quadruple (N, P, Q, M) forms a commuting square. Motivated by [47] and as is usual in geometry, we consider the 'exterior angle', denoted by $\beta_M^N(P, Q)$, between P and Q (see Definition 3.1.3) and it follows that if $\beta_M^N(P, Q) = \pi/2$ if and only if (M', Q', P', N') forms a commuting square (here all the commutants have been taken in $\mathcal{B}(L^2(M))$), that is, (N, P, Q, M) forms a co-commuting square. We further derive various equivalent conditions for a quadruple to become a commuting square (see Theorem 3.1.15). This we do after expressing angle in terms of Pimsner-Popa bases. Moreover, we examine when interior and exterior angles are equal. We succeed in answering this in the special (and important) case if angle equals ninety degree (see Theorem 3.1.17) and as a pleasant consequence recover the various equivalent conditions of 'non-degenerate commuting square' by S. Popa ([40]) in terms of bases (see Corollary 3.1.19).

For a general finite-index subfactor $N \subseteq M$ the set of all intermediate subfactors may not be finite. Even in the case when $N' \cap M$ is abelian the set of intermediate subfactors may be infinite as shown in [48] (Theorem 5.4). In this direction we have the following result:

Theorem 3.0.25. *Let $N \subseteq M$ be a subfactor with $[M : N] < \infty$. Let P and Q be two intermediate subfactors of $N \subseteq M$. If $\|e_P - e_Q\| < 1/2$, then there exists an $*$-isomorphism $\Phi : P \longmapsto Q$ such that $\Phi|_N = id$.*

As an easy corollary we immediately recover the recent cute result of F. Xu (see Corollary 3.4.7).

3.1 Angle and commuting square

Definition 3.1.1. *Let $N \subseteq M$ be a subfactor with finite index. Consider two intermediate subfactors P and Q. Let e_P and e_Q be the biprojections corresponding to P and Q respectively. Put, $v_P = \frac{(e_P - e_1)}{\|e_P - e_1\|_2} \in N' \cap M_1$. Define **angle**, denoted by $\alpha_M^N(P, Q)$, between P and Q as follows:*

$$\cos(\alpha_M^N(P, Q)) = \langle v_P, v_Q \rangle_2.$$

where, $\langle x, y \rangle_2 = tr(y^* x)$ and hence $\|x\|_2 = (tr(x^* x))^{1/2}$ for $x, y \in P_2^{N \subseteq M}$. As usual, angle takes only the principal value, that is: $0 \leq \alpha_M^N(P, Q) \leq \pi$. If from the context it is clear what N and M are, we may omit them from $\alpha_M^N(P, Q)$.

Lemma 3.1.2. $0 \leq \cos(\alpha(P, Q)) \leq 1$ *and* $\alpha(P, Q) = 0$ *iff* $e_P = e_Q$. *In particular,* $\alpha(P, Q) \in [0, \pi/2]$.

Proof. That $0 \leq \cos(\alpha(P, Q)) \leq 1$ follows from Cauchy-Schwarz inequality. Then we get $\alpha(P, Q) = 0$ iff v_P is a multiple of v_Q. Since both v_P and v_Q are positive and $\|v_P\| = \|v_Q\| = 1$ it follows that $v_P = v_Q$. Thus $(e_P - e_1) = \gamma(e_Q - e_1)$ for some constant γ. As $(e_P - e_1)$ and $(e_Q - e_1)$ are both projections we conclude $\gamma = 1$ proving $e_P = e_Q$. $\square$

Definition 3.1.3. *We define exterior angle, denoted by $\beta_M^N(P,Q)$, between P and Q as $\alpha_{M_1}^M(P_1, Q_1)$. Here $P_1 = \langle M, e_P\rangle$ and $Q_1 = \langle M, e_Q\rangle$ are Jones' basic construction. This is similar to [47]. As before, if from the context it is clear what N and M are, we may omit them from $\beta_M^N(P,Q)$.*

Notation: Put as usual $\delta = \sqrt{[M:N]}$ and $\tau = [M:N]^{-1}$. Note, $tr(e_1) = \tau$.

Proposition 3.1.4. *The quadruple (N, P, Q, M) forms a commuting square iff $\alpha(P,Q) = \frac{\pi}{2}$.*

Proof. Suppose (N, P, Q, M) forms a commuting square. Then $e_P e_Q = e_1$. Thus $\alpha(P,Q) = \pi/2$.

Conversely, suppose $\alpha(P,Q) = \pi/2$. Thus $tr(e_P e_Q) = \tau$. It follows that, $tr(e_P e_Q e_P) = \tau$ and hence $tr(e_P e_Q e_P - e_1) = 0$. Now observe, $(e_P e_Q e_P - e_1) = (e_P e_Q - e_1)(e_P e_Q - e_1)^*$. Then faithfulness of trace yields $e_P e_Q = e_1$. In other words, (N, P, Q, M) is a commuting square. This completes the proof. $\square$

Recall the following definition ([47]):

Definition 3.1.5. *A quadruple (N, P, Q, M) is called a co-commuting square if the quadruple (M', P', Q', N') is a commuting square (here commutants have been taken in $\mathcal{B}(L^2(M))$). Equivalently, the quadruple (M, P_1, Q_1, M_1) is a commuting square, where P_1(resp. Q_1) is the the basic construction $\langle M, e_P\rangle$ (resp. $\langle M, e_Q\rangle$).*

In general, it is not true that $\alpha_M^N(P,Q) = \beta_M^N(P,Q)$ (See for instance Fact 3.1.8). Motivated by [47] we try to investigate a special case when $\alpha_M^N(P,Q) = \pi/2$. For this we first express angle in terms of Pimsner-Popa basis.

Remark 3.1.6. *In this chapter we will consider what we called in the first chapter 'right basis'. Thus the condition for* $\{\lambda_i\}$ *to be a right basis would be* $\sum_{i=1}^{n} \lambda_i e_1 \lambda_i{}^* = 1$ *or equivalently,* $x = \sum_{i=1}^{n} E_N(x\lambda_i)\lambda_i{}^* = \sum_{i=1}^{n} \lambda_i E_N(\lambda_i{}^* x)$ *for all* $x \in M$.

Proposition 3.1.7. *Consider intermediate subfactors* P *and* Q *of* $N \subset M$*. Let* $\{\lambda_i\}$*(resp.* $\{\mu_j\}$*) be (right) basis for* P/N *(resp* Q/N*). Then,*

$$\cos(\alpha(P,Q)) = \frac{\sum_{i,j} tr(E_N^M(\lambda_i{}^*\mu_j)\mu_j{}^*\lambda_i) - 1}{\sqrt{[P:N]-1}\sqrt{[Q:N]-1}} \tag{3.1}$$

Proof. Firstly observe that for any intermediate subfactor, say P, of $N \subset M$ and basis $\{\lambda_i\}$ we have $e_P^M = \sum_i \lambda_i e_1 \lambda_i{}^*$. This follows trivially from the following array of equations and is well-known: for any $x \in M$,

$$\begin{aligned}
&(\sum_i \lambda_i e_1 \lambda_i{}^*)(x\Omega) \\
&\quad = (\sum_i \lambda_i (E_N^M(\lambda_i{}^* x)))\Omega \\
&\quad = (\sum_i \lambda_i E_N^P(\lambda_i{}^* E_P^M(x)))\Omega \\
&\quad = E_P^M(x)\Omega \\
&\quad = e_P^M(x\Omega).
\end{aligned}$$

In the above Ω denotes the cyclic vector for the standard space $L^2(M)$.

Thus with our notation $e_Q^M = \sum_j \mu_j e_1 \mu_j{}^*$. Then it follows from Definition 3.1.1:

$$\begin{aligned}
&\cos(\alpha(P,Q)) \\
&\quad = \frac{tr(e_P e_Q) - \tau}{\sqrt{tr(e_P) - \tau}\sqrt{tr(e_Q) - \tau}} \\
&\quad = \frac{tr(\sum_{i,j} \lambda_i e_1 \lambda_i{}^* \mu_j e_1 \mu_j{}^*) - \tau}{\sqrt{tr(\sum_i \lambda_i e_1 \lambda_i{}^*) - \tau}\sqrt{tr(\sum_j \mu_j e_1 \mu_j{}^*) - \tau}} \\
&\quad = \frac{\sum_{i,j} tr(e_1 E_N^M(\lambda_i{}^*\mu_j)\mu_j{}^*\lambda_i) - \tau}{\sqrt{\sum_i tr(e_1 \lambda_i{}^*\lambda_i) - \tau}\sqrt{\sum_j tr(e_1 \mu_j{}^*\mu_j) - \tau}}
\end{aligned}$$

$$= \frac{\sum_{i,j} tr(E_N^M({\lambda_i}^* \mu_j){\mu_j}^* \lambda_i) - 1}{\sqrt{[P:N]-1}\sqrt{[Q:N]-1}}$$

This completes the proof. □

Fact 3.1.8. *Consider intermediate subfactors P and Q such that $N \subset P \subset Q \subset M$.Then the following two two equations hold (as is seen from the definitions):*

$$\cos(\alpha(P,Q)) = \sqrt{\frac{[P:N]-1}{[Q:N]-1}}.$$

and

$$\cos(\beta(P,Q)) = \sqrt{\frac{[M:Q]-1}{[M:P]-1}}.$$

This shows that $\alpha(P,Q)$ and $\beta(P,Q)$ may not be equal. For example, consider subfactors $N \subset P \subset Q \subset M$ such that $[P:N] = 2, [M:Q] = 3, [Q:P] = 5$. Then by the above two formulas we get $\cos(\alpha(P,Q)) = \frac{1}{3}$ and $\cos(\beta(P,Q)) = \frac{1}{\sqrt{7}}$.

Proposition 3.1.9. *Consider factors of type II_1 such that $R, N \subset P, Q \subset M, S$. Then $\alpha_M^N(P,Q) = \alpha_S^N(P,Q)$ and $\beta_M^N(P,Q) = \beta_M^R(P,Q)$.*

Proof. This follows from Proposition 3.1.7. □

Proposition 3.1.10. *Consider again $N \subset P, Q \subset M$ and let $\{\lambda_i\}$(resp. $\{\mu_j\}$) be basis for P/N (resp. Q/N). Then the following are equivalent:*

1. *$\alpha(P,Q) = \pi/2$*
2. *$q := \sum_{i,j} \mu_j \lambda_i e_1 {\lambda_i}^* {\mu_j}^*$ is a projection such that $q \geq e_P$.*
3. *$p := \sum_{i,j} \lambda_i \mu_j e_1 {\mu_j}^* {\lambda_i}^*$ is a projection such that $p \geq e_Q$.*

Proof. (1) $\Rightarrow$ (2)

That q is a projection is easy and was observed in [47]. We prove it for sake of completeness. By the second line of the proof of Proposition 3.1.7, $q = \sum_i \mu_i e_P \mu_i{}^*$ and hence $q = q^*$.

Then,

$$\begin{aligned} q^2 &= \sum_{i,j} \mu_i e_P \mu_i{}^* \mu_j e_P \mu_j{}^* \\ &= \sum_{i,j} \mu_i E_P^M(\mu_i{}^* \mu_j) e_P \mu_j{}^* \\ &= \sum_{i,j} \mu_i E_P^M E_Q^M(\mu_i{}^* \mu_j) e_P \mu_j{}^* \\ &= \sum_{i,j} \mu_i E_N^M(\mu_i{}^* \mu_j) e_P \mu_j{}^* \text{ [applying Proposition 3.1.4]} \\ &= \sum_{j} \mu_j e_P \mu_j{}^* \text{ [since} \{\mu_j\} \text{is a basis for } Q/N] \\ &= q. \end{aligned}$$

Now we show that $(e_P)q = e_P$.

$$\begin{aligned} (e_P)q &= \sum_j e_P \mu_j e_P \mu_j{}^* \\ &= \sum_j e_P E_P^M(\mu_j) \mu_j{}^* \\ &= \sum_j e_P (E_P^M E_Q^M(\mu_j)) \mu_j{}^* \\ &= \sum_j e_P E_N^M(\mu_j) \mu_j{}^* \text{ [applying Proposition 3.1.4]} \\ &= e_P \text{ [since } \{\mu_j\} \text{ is a basis for } Q/N] \end{aligned}$$

Thus q is projection such that $q \geq e_P$. This completes the proof of (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1)

$(e_P)q = e_P$ implies $\sum_j e_P E_P^M(\mu_j){\mu_j}^* = e_P$. Taking trace to both sides we get,

$$\sum_j tr(E_P^M(\mu_j){\mu_j}^*) = 1. \tag{3.2}$$

Then from the definition of angle it follows easily that

$$\cos(\alpha(P,Q)) = \frac{tr(e_P e_Q) - \tau}{\sqrt{tr(e_P) - \tau}\sqrt{tr(e_Q) - \tau}} \tag{3.3}$$

Put $r = \sum_j {\mu_j}^* e_P \mu_j$. Thus,

$$\begin{aligned} tr(re_1) &= tr(\sum_j {\mu_j}^* e_P \mu_j e_1) \\ &= tr(e_P \sum_j \mu_j e_1 {\mu_j}^*) \\ &= tr(e_P e_Q) \ \ [\text{since } \sum_j \mu_j e_1 {\mu_j}^* = e_Q]. \end{aligned}$$

Thus it follows from Equation 3.3 that:

$$\cos(\alpha(P,Q)) = \frac{tr(re_1) - \tau}{\sqrt{tr(e_P) - \tau}\sqrt{tr(e_Q) - \tau}} \tag{3.4}$$

But,

$$\begin{aligned} re_1 &= \sum_j {\mu_j}^* e_P \mu_j e_1 \\ &= \sum_j {\mu_j}^* e_P \mu_j e_P e_1 \quad [\text{since } \ e_P e_1 = e_1] \\ &= \sum_j {\mu_j}^* E_P^M(\mu_j) e_1 \end{aligned}$$

Thus $tr(re_1) = \tau tr({\mu_j}^* E_P^M(\mu_j)) = \tau tr(E_P^M(\mu_j){\mu_j}^*)$. Then Equation (3.2) implies that $tr(re_1) = \tau$. Thus by Equation(3.4) $\alpha(P,Q) = \pi/2$.

This completes the proof of (2) $\Rightarrow$ (1).

(1) $\Leftrightarrow$ (3)

Simply observe, $\alpha(P,Q) = \alpha(Q,P)$. The rest follows from above two implications. This completes the proof. $\square$

Remark 3.1.11. *(1) In the above proposition, $q = e_P$ if and only if $Q = N$.*

Proof. Firstly observe, by Markov property of trace,

$$tr(q) = tr(\sum_j \mu_j e_P {\mu_j}^*) = \frac{\sum_j tr(\mu_j {\mu_j}^*)}{[M:P]}.$$

But as $\{\mu_j\}$ is a basis for Q/N, $\sum_j \mu_j {\mu_j}^* = [Q:N]$. Thus

$$tr(q) = \frac{[M:N]}{[M:P][M:Q]}. \tag{3.5}$$

Suppose $q = e_P$. After taking trace on both sides we get $[M:N] = [M:Q]$ implying $Q = N$.

Conversely $Q = N$ implies $tr(q) = tr(e_P)$ (See Equation (3.5)). Since by Proposition 3.1.10 $q \geq e_P$, it follows $q = e_P$. $\square$

(2) Similarly, $p = e_Q$ if and only if $P = N$.

As an easy corollary we immediately get the following well-known result (see [47]):

Corollary 3.1.12. $\alpha(P,Q) = \pi/2$ *implies* $[M:Q] \geq [P:N]$ *and hence* $[M:P] \geq [Q:N]$.

Proof. Simply observe q is a projection and apply Equation (3.5). $\square$

Proposition 3.1.13. *Let $\alpha(P, Q) = \pi/2$ and p, q be as in Theorem 3.1.10. Then,*

$$\bigvee\{ve_Qv^* : v \in \mathcal{U}(P)\} = p$$

and

$$\bigvee\{ue_Pu^* : u \in \mathcal{U}(Q)\} = q$$

Proof. First note that as observed in Proposition 3.1.10 for any basis $\{\mu_j\}$ of Q/N, $q = \sum_j \mu_j e_P \mu_j{}^*$ is a projection such that $q \geq e_P$. Consider an arbitrary unitary element $u \in \mathcal{U}(Q)$. Then it is trivial to see that $\{u^*\mu_j\}$ is a basis for Q/N. Thus $u^*qu \geq e_P$ and hence $ue_Pu^* \leq q$. Therefore, $\bigvee\{ue_Pu^* : u \in \mathcal{U}(Q)\} \leq q$. Observe that, since $q = \sum_j \mu_j e_P \mu_j{}^*$,

$$\begin{aligned} range(q) &\subset [\mu L^2(P) : \mu \in Q] \\ &= [uL^2(P) : u \in \mathcal{U}(Q)] \\ &= [range(\{ue_Pu^* : u \in \mathcal{U}(Q)\})]. \end{aligned}$$

Thus, $\bigvee\{ue_Pu^* : u \in \mathcal{U}(Q)\} \geq q$. In conclusion, $\bigvee\{ue_Pu^* : u \in \mathcal{U}(Q)\} = q$. Proof for p is exactly similar. □

Remark 3.1.14. *Let $\alpha = \pi/2$ and p, q be as in Theorem 3.1.10. Then it is not hard to show that $p, q \geq e_P \vee e_Q$. In general, it is not true that $e_P \vee e_Q = e_{P\vee Q}$, although $e_P \vee e_Q \leq p, q \leq e_{P\vee Q}$.*

Below we give a characterization of commuting square in terms of basis:

Theorem 3.1.15. *For a quadruple (N, P, Q, M) the following are equivalent:*

1. *(N, P, Q, M) is a commuting square, that is $\alpha(P, Q) = \pi/2$.*

2. *Let p be as in Theorem 3.1.10. Then,*

$$p = \bigvee\{ve_Qv^* : v \in \mathcal{U}(P)\}.$$

3. *Let q be as in Theorem 3.1.10. Then,*

$$q = \bigvee\{ue_Pu^* : u \in \mathcal{U}(Q)\}.$$

Proof. (1) $\Rightarrow$ (2)

This is Proposition 3.1.13.

(2) $\Rightarrow$ (1)

Clearly $\bigvee\{ve_Qv^* : v \in \mathcal{U}(P)\} \geq e_Q$. Hence $p \geq e_Q$. Again applying Proposition 3.1.10 we get $\alpha(P,Q) = \pi/2$.

Thus (1) and (2) are equivalent.

By symmetry, (1) and (3) are equivalent. This completes the proof. □

Notation 3.1.16. *Let (N, P, Q, M) be a quadruple of type II_1 factors. Put $b_1 = [M : P], b_2 = [M : Q]$ and $b = [M : N]$.*

Below we investigate when $\alpha(P,Q) = \pi/2 = \beta(P,Q)$. Explicitly, we characterize simultaneously commuting and co-commuting squares in terms of various equivalent conditions.

Theorem 3.1.17. *Consider a quadruple (N, P, Q, M). Then the following are equivalent:*

1. *$\alpha(P,Q) = \beta(P,Q) = \pi/2$, that is (N, P, Q, M) is a commuting and co-commuting square.*

2. *If* $\{\lambda_i\}$*(resp.* $\{\mu_j\}$*) is any basis for* P/N *(resp.* Q/N*), then* $\{\lambda_i\mu_j\}$ *is a basis for* M/N*. In other words,* $p = 1$*.*

3. *If* $\{\lambda_i\}$*(resp.* $\{\mu_j\}$*) is any basis for* P/N *(resp.* Q/N*),then* $\{\mu_j\lambda_i\}$ *is a basis for* M/N*. In other words,* $q = 1$*.*

4. *Any basis (not necessarily orthonormal) for* P/N *is a basis for* M/Q*.*

5. *Any basis (not necessarily orthonormal) for* Q/N *is a basis for* M/P*.*

Proof. Let $\{\lambda_i\}$ be any basis for P/N. Fix a basis $\{\mu_j\}$ for Q/N. Thus, (2) implies $\{\lambda_i\mu_j\}$ is a basis for M/N. Hence, $\sum_{i,j}\lambda_i\mu_j e_1 {\mu_j}^* {\lambda_i}^* = 1$. Thus, $\sum_i \lambda_i e_Q {\lambda_i}^* = 1$ (since we know $\sum_j \mu_j e_1 {\mu_j}^* = e_Q$). Again applying Theorem 1.2.2 we obtain $\{\lambda_i\}$ is a basis for M/Q. This proves (2) $\Rightarrow$ (4).

Simply use Corollary 1.2.6 to show that (4) $\Rightarrow$ (2).

Thus, (4) $\Leftrightarrow$ (2). Similarly, (5) $\Leftrightarrow$ (3).

Suppose (1) holds true. Then, since $\alpha(P,Q) = \pi/2$ using Corollary 3.1.12 we obtain $[M : P] \geq [Q : N]$. Again applying Corollary 3.1.12 and using the fact that $\beta(P,Q) = \pi/2$ we similarly obtain $[M_1 : Q_1] \geq [P_1 : M]$. But observe that $[M_1 : Q_1] = \frac{[M_1:M]}{[Q_1:M]} = \frac{[M:N]}{[M:Q]} = [Q : N]$. Also, $[P_1 : M] = [M : P]$. Thus we obtain $[Q : N] \geq [M : P]$. In conclusion, $[M : P] = [Q : N]$. Thus by Equation (3.5), $tr(q) = 1$. Now, since $\alpha(P,Q) = \pi/2$, from Proposition 3.1.10 it follows that q is a projection implying $q = 1$. Thus, by Theorem 1.2.2, $\{\mu_j\lambda_i\}$ is a basis for M/N. Therefore, we have proved that (1) $\Rightarrow$ (3).

Suppose (3) holds true, that is (fix as before $\{\lambda_i\}$(resp. $\{\mu_j\}$) as any basis for P/N (resp. Q/N)) $q = 1$ (see Theorem 1.2.2). Hence, $b_1 b_2 = b$. Thus applying Proposition 3.1.10 we immediately get $\alpha(P,Q) = \pi/2$ implying p is also a projection. From Equation(3.5) it is obvious that $tr(p) = tr(q)$. Thus $p = 1$. Therefore, (5) $\Leftrightarrow$ (3) $\Rightarrow$ (2) $\Leftrightarrow$ (4). Thus, $\{\lambda_i\}$ is also a basis for M/Q and $\{\mu_j\}$ is also a basis for M/P.

Hence, $\{\sqrt{b_2}\lambda_i e_Q\}$ (resp. $\{\sqrt{b_1}\mu_j e_P\}$) is a basis for Q_1/M (resp. P_1/M). Denote by $\widetilde{p}$ the operator corresponding to p, as in Proposition 3.1.10, for the quadruple (M, Q_1, P_1, M_1). Similarly, define $\widetilde{q}$. Now we see that the following equations hold true:

$$\begin{aligned}
\widetilde{p} \quad &= \sum_{i,j} b_1 b_2 \lambda_i e_Q \mu_j e_P e_2 e_P {\mu_j}^* e_Q {\lambda_i}^* \\
&= \sum_{i,j} b_1 b_2 \lambda_i \mu_j e_Q e_P e_2 e_P e_Q {\mu_j}^* {\lambda_i}^* \\
&= \sum_{i,j} b_1 b_2 \lambda_i \mu_j e_1 e_2 e_1 {\mu_j}^* {\lambda_i}^* \quad [\text{since} \quad \alpha(P,Q) = \pi/2] \\
&= \sum_{i,j} \lambda_i \mu_j e_1 {\mu_j}^* {\lambda_i}^* \quad [\text{since} \quad b_1 b_2 = b] \\
&= p.
\end{aligned}$$

Thus $\widetilde{p} = p = 1$. Again applying Proposition 3.1.10 we get $\beta(P,Q) = \pi/2$. In other words, (3) $\Rightarrow$ (1).

By symmetry (that is $\beta(P,Q) = \beta(Q,P)$), (1) $\Leftrightarrow$ (2).

This completes the proof of all the equivalent statements. □

Definition 3.1.18. *A commuting square*

$$\begin{array}{ccc}
Q & \subset & M \\
\cup & & \cup \\
N & \subset & P
\end{array}$$

is said to be non-degenerate if $\overline{spPQ} = M$*, that is* P *generates* M *as a right* Q*-module.*

Now, the following corollary follows easily. This is the characterization of non-degenerate commuting square due to S. Popa (see [40])(with slight modification):

Corollary 3.1.19. *[40] For a commuting square (N, P, Q, M) of II_1-factors with all inclusions of finite index, the following statements are equivalent:*

1. *(N, P, Q, M) is a co-commuting square, that is $\beta_M^N(P, Q) = \pi/2$.*

2. *$\bigvee\{ve_Qv^* : v \in \mathcal{U}(P)\} = 1$*

3. *$\bigvee\{ue_Pu^* : u \in \mathcal{U}(Q)\} = 1$*

4. *Any basis (not necessarily orthonormal) for P/N is a basis for M/Q.*

5. *Any basis (not necessarily orthonormal) for Q/N is a basis for M/P.*

6. *$PQ := span\{\sum_{i=1}^n x_iy_i : x_i \in P, y_i \in Q\} = M$, in particular, (N, P, Q, M) is non-degenerate.*

7. *$QP = M$, in particular, (N, Q, P, M) is non-degenerate*

Proof. Suppose $\{\lambda_i\}, \{\mu_j\}, p$ and q are as before.

By Theorem 3.1.17 and Proposition 3.1.13, it is trivial to see that conditions (1), (2) and (4) all are equivalent to satisfy the equation $p = 1$. Similarly, (1),(3) and (5) are equivalent to the equation $q = 1$.

Suppose (3) holds true. Thus, by Theorem 3.1.15, $\{\mu_j\lambda_i\}$ is a basis for M/N and hence $M = QP$, implying (7). Conversely, suppose (7) holds true. Thus any $x \in M$ can be written as $x = \sum_k b_ka_k$, where $b_k \in Q$ and $a_k \in P$. Then it is easy to check that for any $x \in M$:

$$\begin{aligned} q(x\Omega) =& q((\sum_k b_ka_k)\Omega) \\ &= \sum_{j,k} \mu_j e_P({\mu_j}^* b_ka_k\Omega) \\ &= \sum_{j,k} \mu_j E_P^M({\mu_j}^* b_k)a_k\Omega \end{aligned}$$

$$= \sum_{j,k} \mu_j E_P^M E_Q^M (\mu_j{}^* b_k) a_k \Omega$$

$$= \sum_{j,k} \mu_j E_N^Q (\mu_j{}^* b_k) a_k \Omega \quad \text{[by commuting square condition]}$$

$$= \sum_{k} b_k a_k \Omega \quad \text{[since } \{\mu_j\} \text{ is a basis for } Q/N]$$

$$= x\Omega.$$

Thus $q = 1$.

That (6) is equivalent to $p = 1$ is exactly similar.

This completes the proof. □

Remark 3.1.20. *It is worth mentioning that Popa has shown that if (4) of Theorem 3.1.17 holds for a quadruple* (N, P, Q, M)*, then* $\overline{spPQ} = M$ *with the additional assumption that the quadruple is a commuting square; whereas we have shown in Theorem 3.1.17 that if (4) holds, then automatically the quadruple will be a non-degenerate commuting square.*

Corollary 3.1.21. *Let* (N, P, Q, M) *be a quadruple. If for some basis* $\{\mu_j\}$ *for* Q/N *it happens that* $\{\mu_j{}^*\}$ *is a basis for* M/P*, then* $\alpha_M^N(P,Q) = \beta_M^N(P,Q) = \pi/2$*. Similar statement holds for* $\{\lambda_i\}$*.*

Proof. Put as before $r = \sum_j \mu_j{}^* e_P \mu_j$. By assumption $r = 1$. Thus, by Equation (3.4), $\alpha(P,Q) = \pi/2$. By property of basis $\sum_j \mu_j \mu_j{}^* = [Q : N]$ and $\sum_j \mu_j{}^* \mu_j = [M : P]$. Thus $[M : P] = [Q : N]$ and hence $b_1 b_2 = b$ and therefore by Theorem **??** $\beta(P,Q) = \pi/2$. □

Corollary 3.1.22. *If* P/N *and* Q/N *both have two sided basis, then* $\alpha(P,Q) = \beta(P,Q) = \pi/2$ *implies that* M/N *has two sided basis.*

Proof. Just use the fact (2) ⇔ (3) of Theorem 3.1.17. □

Corollary 3.1.23. *Consider the the intermediate subfactor P such that $N \subset P \subset M$. Suppose, $\{\lambda_i\}$(respectively, $\{\gamma_j\}$) is a two-sided basis for P/N(resp. for M/P). If there exists another intermediate subfactor Q such that $\alpha(P,Q) = \beta(P,Q) = \pi/2$, then $\{\lambda_i\gamma_j\}$ is a two-sided basis for M/N.*

Proof. Firstly applying Theorem 3.1.19 we find that $\{\gamma_j\}$ is a two sided basis for Q/N. Then simply using Corollary 3.1.22 we immediately obtain the result. □

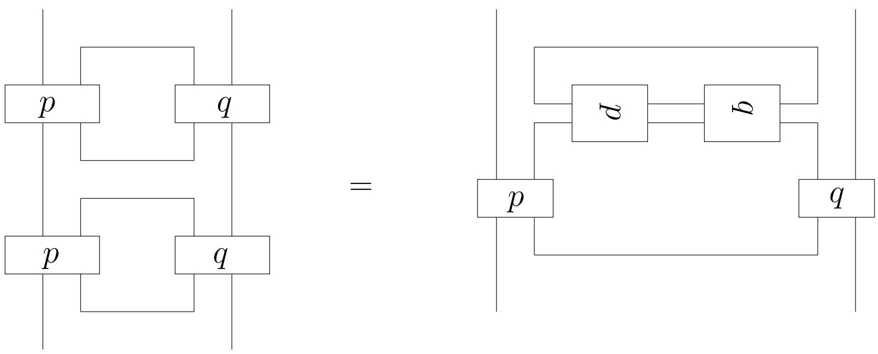
3.2 Boundedness of angle

Notation: In this section and the next section we will assume $N \subseteq M$ is an irreducible subfactor. In the irreducible case it is well known that the set of intermediate subfactors form a lattice under the operations $P \wedge Q = P \cap Q$ and $P \vee Q = (P \cup Q)''$.

Let us recall the definition of co-product of two elements a and b of $N' \cap M_1$, denoted by $a \star b$, as in [19][Definition 3.6]. Note, they have denoted the co-product by $a \circ b$.

Lemma 3.2.1. *Suppose e_P and e_Q are two biprojections, then $e_P \vee e_Q$ is a subprojection of $\frac{1}{\delta tr(e_P e_Q)} e_P \star e_Q$.*

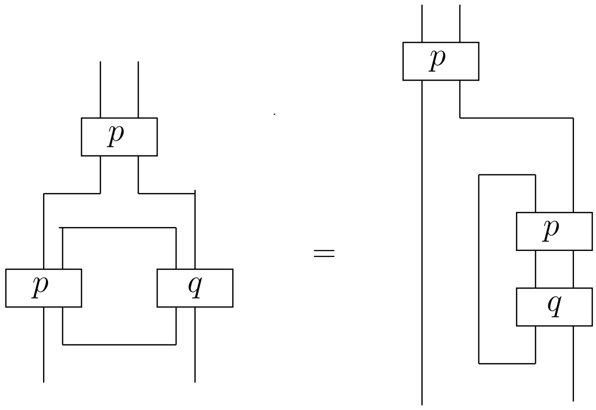
Proof. Let us as usual denote by p (resp. q) the biprojection e_P(resp. e_Q). Using exchange relation twice we get the following:



$$= \delta tr(pq)(p \star q)$$

The above implies $\frac{1}{\delta tr(pq)}(p \star q)$ is a projection.

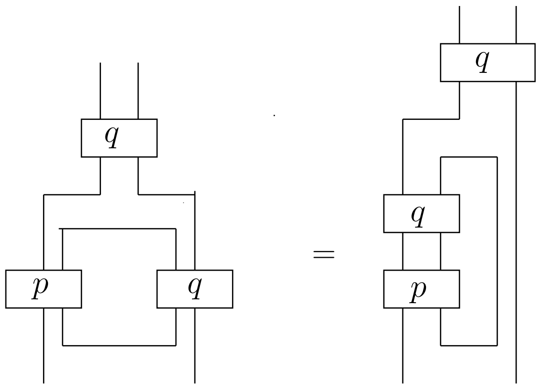
Now, using exchange relation for the biprojection p we get the equations as in Figure 3.1.



$$= \delta tr(pq)p.$$

Figure 3.1: e_P is a subprojection of $\frac{1}{\delta tr(e_P e_Q)} e_P \star e_Q$

Using exchange relation for the biprojection q we get the equations as in Figure 3.2.



$$= \delta tr(pq)q$$

Figure 3.2: e_Q is a subprojection of $\frac{1}{\delta tr(e_P e_Q)} e_P \star e_Q$

Thus from the above discussions we conclude $e_P \vee e_Q$ is a subprojection of $\frac{1}{\delta tr(e_P e_Q)} e_P \star e_Q$ finishing the proof. □

Definition 3.2.2. *Let $N \subseteq M$ be a subfactor. Then Q is called a maximal (respectively, minimal) intermediate subfactor of $N \subseteq M$ if whenever there exists an intermediate subfactor P such that if $N \subseteq Q \subseteq P \subseteq M$ (respectively, $N \subseteq P \subseteq Q \subseteq M$) holds, then P equals either Q or M (respectively, P equals either N or Q). We exclude N and M from the definition of maximal (or minimal) intermediate subfactor for obvious reason. Note that, maximal intermediate subfactors in $N \subseteq M$ correspond to minimal intermediate subfactors in $M \subseteq M_1$.*

Proposition 3.2.3. *Suppose P, Q are minimal intermediate subfactors, then*

$$\frac{tr(e_P)tr(e_Q)}{tr(e_P e_Q)} \geq tr(e_P) + tr(e_Q) - \tau \tag{3.6}$$

Proof. Since P and Q are minimal intermediate subfactors, $P \cap Q$ is N. Thus, $e_P \wedge e_Q = e_1$. Now by Lemma 3.2.1, we have

$$\frac{1}{\delta tr(e_P e_Q)} e_P \star e_Q \geq e_P \vee e_Q$$

Computing the trace of both sides and observing $tr(e_P \star e_Q) = \delta tr(e_P)tr(e_Q)$, we get

$$\begin{aligned} \frac{tr(e_P)tr(e_Q)}{tr(e_P e_Q)} &\geq tr(e_P \vee e_Q) \\ &= tr(e_P) + tr(e_Q) - tr(e_P \wedge e_Q) \\ &= tr(e_P) + tr(e_Q) - tr(e_1) \\ &= tr(e_P) + tr(e_Q) - \tau \end{aligned}$$

□

Theorem 3.2.4. *Suppose P, Q are two distinct minimal intermediate subfactors, then $\alpha(P, Q) > \frac{\pi}{3}$.*

Proof. Firstly observe, $(tr(e_P) + tr(e_Q) - \tau)$ is non-zero. From Equation (3.6), we have

$$tr(e_P e_Q) \leq \frac{tr(e_P)tr(e_Q)}{tr(e_P)+tr(e_Q)-\tau} \tag{3.7}$$

$$tr(e_P e_Q) - \tau \leq \frac{tr(e_P)tr(e_Q)}{tr(e_P)+tr(e_Q)-\tau} - \tau \tag{3.8}$$

$$= \frac{tr(e_P)tr(e_Q) - \tau(tr(e_P)+tr(e_Q)) + \tau^2}{tr(e_P)+tr(e_Q)-\tau} \tag{3.9}$$

$$= \frac{(tr(e_P)-\tau)(tr(e_Q)-\tau)}{tr(e_P)+tr(e_Q)-\tau} \tag{3.10}$$

Therefore,

$$\begin{aligned}
\cos(\alpha(P,Q)) &= \frac{tr((e_P-e_1)(e_Q-e_1))}{tr(e_P-e_1)^{1/2} tr(e_Q-e_1)^{1/2}} \\
&= \frac{tr(e_P e_Q)-\tau}{(tr(e_P)-\tau)^{1/2}(tr(e_Q)-\tau)^{1/2}} \quad \text{(since, } e_P e_1 = e_1 e_Q = e_1\text{)} \\
&\leq \frac{(tr(e_P)-\tau)^{1/2}(tr(e_Q)-\tau)^{1/2}}{tr(e_P)+tr(e_Q)-\tau} \quad \text{(by Equation (3.10))} \\
&< \frac{(tr(e_P)-\tau)^{1/2}(tr(e_Q)-\tau)^{1/2}}{tr(e_P)-\tau+tr(e_Q)-\tau} \\
&\leq 1/2
\end{aligned}$$

Therefore, $\alpha(P,Q) > \frac{\pi}{3}$. □

Corollary 3.2.5. *If P, Q are two distinct minimal intermediate subfactors of $N \subseteq M$, then $\|v_P - v_Q\|_2 > 1$*

3.3 Number of intermediate subfactors

Theorem 3.3.1. *Suppose $N \subset M$ is an irreducible subfactor of type II_1 of finite index. The number of minimal intermediate subfactors is at most $3^{[M:N]} - 1$.*

Proof. Let $\mathcal{L}_m(N,M)$ be the set of all minimal intermediate subfactors of $N \subset M$ and d be the dimension of the higher relative commutant $N' \cap M_1$. Consider the real inner product space $(N' \cap M_1)_{s.a}$ which is also of (real) dimension d. The vector $v_P = (e_P - e_1)/||e_P - e_1||_2$ is a unit vector in this space.

Then by Theorem 3.2.4, we see that $\{v_P : P \in \mathcal{L}_m(N, M)\}$ is a set of unit vectors in $(N' \cap M_1)_{s.a}$ and by Corollary 3.2.5 $\|v_P - v_Q\|_2 > 1$ for distinct P and Q. We now estimate the cardinality of this set.

Consider the d dimensional ball B_P with center at each v_P and radius 1/2. It follows that for any $P, Q \in \mathcal{L}_m(N, M)$, B_P and B_Q are disjoint. Furthermore, we have

$$B_P \subset B(3/2) \cap (B(1/2))^c \ \ \forall\ P \in \mathcal{L}_m(N, M).$$

Here, $B(r)$ stands for the d dimensional ball with center at origin and radius r. Thus,

$$\begin{aligned} |\mathcal{L}_m(N, M)| &\leq \frac{Vol(B(3/2)) - Vol(B(1/2))}{Vol(B(1/2))} \\ &= \frac{(3/2)^d - (1/2)^d}{(1/2)^d} \\ &= 3^d - 1. \end{aligned}$$

From subfactor theory, we know that $\dim(N' \cap M_1) \leq [M : N]$ (see [46] [Lemma 2.1]). This completes the proof. □

Definition 3.3.2. *Suppose δ^2 is a real number greater than or equal to 2, we define*

$$I(\delta^2) = \sup_{N \subset M} \{|\mathcal{L}(N, M)| : N \subset M \textit{ is a subfactor with } [M : N] \leq \delta^2\}$$
$$m(\delta^2) = \sup_{N \subset M} \{|\mathcal{L}_m(N, M)| : N \subset M \textit{ is a subfactor with } [M : N] \leq \delta^2\}$$

Corollary 3.3.3. *Let δ^2 be a real number greater than or equal to 2, we have*

$$m(\delta^2) \leq 3^{\delta^2}.$$

Lemma 3.3.4. *Suppose $\delta^2 \geq 4$, then we have*

$$I(\delta^2) \leq m(\delta^2) I(\delta^2/2).$$

Proof. Note that the subfactor $R \subset R \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ is of index 4 and $\mathbb{Z}_2 \times \mathbb{Z}_2$ has two non-trivial proper subgroups. Thus, $m(\delta^2) \geq 2$ when $\delta^2 \geq 4$.

To prove the lemma, we need to show for an arbitrary subfactor $N \subset M$ with $[M : N] \leq \delta^2$,

$$|\mathcal{L}(N, M)| \leq m(\delta^2) I(\delta^2/2).$$

Case 1: Suppose $|\mathcal{L}_m(N, M)| = 0$, then $|\mathcal{L}(N, M)| = 2$. (Since $\mathcal{L}(N, M) = \{N, M\}$.) Note that $m(\delta^2) \geq 2$ and $I(\delta^2) \geq 2$, the lemma follows directly.

Case 2: Suppose $|\mathcal{L}_m(N, M)| = 1$. Let P be the minimal intermediate subfactor, then we have

$$\mathcal{L}(N, M) = \mathcal{L}(P, M) \cup \{N\}.$$

Thus,

$$\begin{aligned} |\mathcal{L}(N, M)| &= |\mathcal{L}(P, M)| + 1 \\ &\leq I([M : P]) + 1. \end{aligned}$$

Since $[M : P] = [M : N]/[P : N]$ and $[P : N] \geq 2$, we have $[M : P] \leq [M : N]/2 \leq \delta^2/2$. Therefore,

$$\begin{aligned} |\mathcal{L}(N, M)| &\leq I(\delta^2/2) + 1 \\ &\leq 2I(\delta^2/2) \\ &\leq m(\delta^2) I(\delta^2/2). \end{aligned}$$

Case 3: Suppose $|\mathcal{L}_m(N, M)| \geq 2$. It follows that

$$\mathcal{L}(N, M)\backslash\{N, M\} \subset \bigcup_{P \in \mathcal{L}_m(N,M)} (\mathcal{L}(P, M)\backslash M).$$

Therefore,

$$\begin{aligned}
|\mathcal{L}(N, M)| &\leq \sum_{P \in \mathcal{L}_m(N,M)} (|\mathcal{L}(P, M)| - 1) + 2 \\
&\leq \sum_{P \in \mathcal{L}_m(N,M)} (I([M : P]) - 1) + 2 \\
&\leq \sum_{P \in \mathcal{L}_m(N,M)} (I(\delta^2/2) - 1) + 2 \\
&\leq |\mathcal{L}_m(N, M)|I(\delta^2/2) - |\mathcal{L}_m(N, M)| + 2 \\
&\leq |\mathcal{L}_m(N, M)|I(\delta^2/2) \\
&\leq m(\delta^2)I(\delta^2/2).
\end{aligned}$$

□

Theorem 3.3.5. *Suppose $N \subset M$ is an irreducible subfactor of type II_1 of finite index. The number of intermediate subfactors is at most $9^{[M:N]}$.*

Proof. First note that if we have $2 \leq [M : N] < 4$, then there there are no non-trivial intermediate subfactors for $N \subset M$. Therefore,

$$|\mathcal{L}(N, M)| = 2 < 9^2 \leq 9^{[M:N]}.$$

Suppose $\delta^2 = [M : N] \geq 4$, by Lemma 3.3.4, we have

$$\begin{aligned}
|\mathcal{L}(N, M)| &\leq I(\delta^2) \\
&\leq m(\delta^2)I(\delta^2/2) \\
&\leq m(\delta^2)m(\delta^2/2)I(\delta^2/2^2)
\end{aligned}$$

$$\leq m(\delta^2)m(\delta^2/2)m(\delta^2/4)\cdots m(\delta^2/2^k)I(\delta^2/2^{k+1}),$$

where k is the smallest integer such that $2 \leq \delta^2/2^{k+1} < 4$.

By Theorem 3.3.1, we have

$$\begin{aligned}
|\mathcal{L}(N,M)| &\leq I(\delta^2/2^{k+1})\prod_{j=0}^{k} 3^{\delta^2/2^j} \\
&\leq \prod_{j=0}^{k+1} 3^{\delta^2/2^j} \quad (\text{since } I(\delta^2/2^{k+1}) = 2 < 3^{\delta^2/2^{k+1}}) \\
&\leq \prod_{j=0}^{+\infty} 3^{\delta^2/2^j} \\
&\leq 3^{2\delta^2} = 9^{\delta^2}.
\end{aligned}$$

This completes the proof. □

The following section will appear in a preprint [4], which is work in progress.

3.4 General Case

As mentioned in the introduction, a non-irreducible finite-index subfactor $N \subseteq M$ may have infinitely many intermediate subfactors. Even in the case $N' \cap M$ abelian the set of intermediate subfactors may be infinite as shown in [48] (Theorem 5.4). We show in Theorem 3.4.5 that for a general finite-index subfactor if operator norm of difference of two biprojections is less than 1/2, then they are isomorphic.

First recall the following standard facts which will be very useful for this section.

Lemma 3.4.1. *[34][Lemma 6.2.2] Let A be a unital C^*-algebra. Suppose p and q are two projections in A with $\|p-q\| < 1$. Then, there exists a unitary $w \in A$ such that*

$$wpw^* = q \text{ and } \|w - I\| \leq \sqrt{2}\|p - q\|.$$

Lemma 3.4.2. *Let $\mathcal{H}$ be a Hilbert space and $p \in \mathcal{B}(\mathcal{H})$ be a projection. If $r \in \mathcal{B}(\mathcal{H})$ is a self-adjoint element satisfying $\|p - r\| \leq \delta$ for some $\delta < 1/2$, there exists a projection $q \in C^*\{r\}$ such that $\|p - q\| \leq 2\delta$.*

Proof. This is a standard calculation. See for instance [13][Lemma 3.1.6]. We prove it for sake of completeness.

Claim: Let $a \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator with discrete spectrum $\{\lambda_j\}$. Then for $\lambda \notin \sigma(a)$, $\|(a - \lambda I)^{-1}\| = \sup_j \frac{1}{|\lambda - \lambda_j|}$.

Proof of claim: Firstly since, $\lambda \notin \sigma(a)$ we can define a continuous function $f \in C(\sigma(a))$ defined by $f(t) = (t - \lambda)^{-1}$. Thus applying continuous funtional calculas for the function $(t-\lambda)(t-\lambda)^{-1}$ we readily get $f(a) = (a-\lambda)^{-1}$. Hence, $\|(a-\lambda)^{-1}\| = |f|_\infty = \sup_j \frac{1}{|\lambda - \lambda_j|}$. This justifies the claim.

For any $\lambda \in \mathbb{C}$ such that $d(\lambda, \{0, 1\}) > \delta$ we have by above claim $\|(p - \lambda)^{-1}\| \leq \frac{1}{d(\lambda,\{0,1\})}$. Thus,

$$\|(p - \lambda)^{-1}(r - \lambda) - 1\| = \|(p - \lambda)^{-1}(r - p)\| < 1.$$

Thus $(r - \lambda)$ is invertible and hence $\lambda \notin \sigma(r)$. Thus the characteristic function $\mathcal{X}_{[1-\delta,1+\delta]}$ is continuous on the spectrum of r. By continuous functional calculals $q := \mathcal{X}_{[1-\delta,1+\delta]}(r) \in C^*\{r\}$ is a projection such that $\|p - r\| \leq \delta$. The rest follows from triangle inequality. This completes the proof of the Lemma. □

Let us also mention below the so-called 'relative Dixmier property' which would be needed in the sequel.

Definition 3.4.3. *Let $N \subseteq M$ be an inclusion of von Neumann algebras. For any $x \in M$, we will denote by $C_N(x)$ the norm closure of the convex hull of $\{uxu^* : u$ is* *a unitary in* $N(=\mathcal{U}(N))\}$. *We say* $N \subset M$ *has relative Dixmier property if for any* $x \in M, C_N(x) \cap (N' \cap M) \neq \phi$.

Popa in [39] showed the following theorem.

Theorem 3.4.4. *Let* $N \subseteq M$ *be an inclusion of von Neumann algebras with a conditional expectation* $E : M \longrightarrow N$ *of finite probabilistic index. Then,* $N \subseteq M$ *has relative Dixmier property.*

Motivated by Lemma 3.4.1 and Eric Christensen's perturbation techniques (see [12],[11]) we obtain the following result:

Theorem 3.4.5. *Let* $N \subseteq M$ *be a subfactor with* $[M : N] < \infty$. *Let* P *and* Q *be two intermediate subfactors of* $N \subseteq M$. *If* $\|e_P - e_Q\| < 1/2$, *then there exists a* $*$*-isomorphism* $\Phi : Q \longmapsto P$ *such that* $\Phi|_N = id$. *In other words, the subfactors* $N \subset P$ *and* $N \subset Q$ *are isomorphic.*

Proof. The proof is inspired by Christensen ([12], [11]). Put $\|e_P - e_Q\| = \gamma$. Let $Q \subseteq M \subseteq Q_1$ be the Jones' basic construction, that is, $\langle M, e_Q\rangle = Q_1$. Similarly, let $\langle M, e_P\rangle = P_1$. Then, for $\lambda_i \geq 0$ with $\sum_i \lambda_i = 1$ and u_i unitaries in P, we get:

$$\begin{aligned}\|\sum_i \lambda_i u_i e_Q u_i{}^* - e_P\| &= \|\sum_i \lambda_i u_i e_Q u_i{}^* - \sum_i \lambda_i u_i e_P u_i{}^*\| \\ &= \|\sum_i \lambda_i u_i (e_Q - e_P) u_i{}^*\| \\ &\leq \sum_i \lambda_i \|u_i\| \|e_Q - e_P\| \|u_i{}^*\| \\ &= \gamma.\end{aligned}$$

We saw above that

$$\|z - e_P\| \leq \gamma \quad \forall z \quad \in C_P(e_Q). \tag{3.11}$$

Now, since by Theorem 3.4.4, $P \subset Q_1$ has relative Dixmier property, there exists $s \in P' \cap Q_1$ such that $s \in C_P(e_Q)$. By Equ. (3.11), $\|s - e_P\| \leq \gamma$. But, $\|\frac{(s+s^*)}{2} - e_P\| \leq \gamma$. Thus we obtain a self-adjoint element $t (= \frac{(s+s^*)}{2})$ in $C_P(e_Q) \cap (P' \cap Q_1)$ such that $\|t - e_P\| \leq \gamma$. Then applying Lemma 3.4.2 we get a projection $\tilde{p}$ satisfying $\|\tilde{p} - e_P\| \leq 2\gamma \lneq 1$. Put $q = J\tilde{p}J$. Then, $q \in Q' \cap P_1 \subseteq N' \cap P_1$. Also, $Je_PJ = e_P \in P' \cap P_1 \subseteq N' \cap P_1$. Thus, $\|q - e_P\| = \|\tilde{p} - e_P\| \leq 2\gamma < 1$. Then, by Lemma 3.4.1, there exists a unitary $w \in N' \cap P_1$ such that

$$we_Pw^* = q. \tag{3.12}$$

For $x \in Q$, since $q \in Q'$, from Equation 3.12, it follows easily that, $(w^*xw)e_P = w^*xqw = w^*qxw = e_P(w^*xw) = e_P(w^*xw)e_P \in e_PP_1e_P$. Thus one can define $\widetilde{\Phi} \colon Q \longmapsto e_PP_1e_P = Pe_P$ by $\widetilde{\Phi}(x) = (w^*xw)e_P$. We now show that $\widetilde{\Phi}$ is a $*$-homomorphism.

$$\begin{aligned}
\widetilde{\Phi}(x)\widetilde{\Phi}(y) \quad &= (w^*xwe_P)(w^*ywe_P) \\
&= w^*xqywe_P = w^*qxywe_P \\
&= e_Pw^*xywe_P = \widetilde{\Phi}(xy).
\end{aligned}$$

Since $w \in N' \cap P_1$ it follows that:

$$\widetilde{\Phi}(n) = w^*nwe_P = ne_P \quad \forall n \in N.$$

We know that there exists a surjective $*$-isomorphism α from Pe_P onto P such that $\alpha(ne_P) = n$ for all $n \in N$. Define $\Phi = \alpha \circ \widetilde{\Phi}$. Then clearly Φ is a normal $*$-homomorphism from Q to P which is identity on N. Also since Q is a factor, Φ is injective. So we need only to check that Φ is surjective.

Since $\|e_P - e_Q\| < 1/2 < 1$ there exists a unitary $u \in M_1$ such that $e_P = ue_Qu^*$. So, $tr_{M_1}(e_P) = tr_{M_1}(e_Q)$ which implies that $[M : P] = [M : Q]$ and thus

$[P : N] = [Q : N]$. Note that $\Phi(Q)$ is a subfactor of P containing N and we have $[\Phi(Q) : \Phi(N)] = [\Phi(Q) : N] = [Q : N]$ (since $\Phi(Q)$ is isomorphic to Q with $\Phi(N) = N$). Thus,

$$[P : N] = [P : \Phi(Q)][\Phi(Q) : N] = [P : \Phi(Q)][Q : N] = [P : \Phi(Q)][P : N]$$

From the above equation we conclude that $[P : \Phi(Q)] = 1$ implying $\Phi(Q) = P$.

Hence Φ is surjective, and the proof is complete. □

Corollary 3.4.6. *The Galois group $Gal(N \subset M)$ for an inclusion $N \subset M$ of factors is defined by $Gal(N \subset M) = \{\alpha \in Aut(M) : \alpha|_N = id_N\}$. Then with the same notation as in above theorem we have $Gal(N \subset P) \cong Gal(N \subset Q)$.*

Proof. Let $\theta \in Gal(N \subset P)$. Then $\Phi^{-1} \circ \theta \circ \Phi \in Gal(N \subset Q)$ and clearly this defines a group isomorphism. Details are routine and hence omitted. □

Corollary 3.4.7. *[51](Theorem 2.8) Let $N \subseteq P, Q \subseteq M$ be inclusions of subfactors such that $N \subseteq M$ is irreducible of finite index. If $\|e_P - e_Q\| < 1/2$, then $P = Q$.*

Proof. This follows easily from the proof of Theorem 3.4.5. Firstly observe that for any $x \in N' \cap P_1$ it is trivial to check that (see for instance [36]) $xe_P = [M : P]E_M(xe_P)e_P$. But as $N \subset M$ is irreducible it follows that $E_M(xe_P) \in \mathbb{C}$ and hence $xe_P \in \mathbb{C}e_P$. In particular $we_P \in \mathbb{C}e_P$. and hence $we_Pw^* \in \mathbb{C}e_P$. Now by Equation (3.12) it follows that $e_P \in Q' \cap P_1$ as $q \in Q' \cap P_1$. Since, $P = \{e_P\}' \cap M$ it follows that $Q \subset P$. Similarly, $P \subset Q$. This completes the proof. □

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