Eigenvalue Statistics of Higher Rank Anderson Tight Binding Model Over The Canopy Tree

By

Narayanan P. A.

MATH10201304004

The Institute of Mathematical Sciences, Chennai

A thesis submitted to the

Board of Studies in Mathematical Sciences

In partial fulfillment of requirements

for the Degree of

DOCTOR OF PHILOSOPHY

of

HOMI BHABHA NATIONAL INSTITUTE

February, 2021

Homi Bhabha National Institute

Recommendations of the Viva Voce Committee

As members of the Viva Voce Committee, we certify that we have read the dissertation prepared by Narayanan P. A. entitled "Eigenvalue Statistics of Higher Rank Anderson Tight Binding Model Over The Canopy Tree" and recommend that it may be accepted as fulfilling the thesis requirement for the award of Degree of Doctor of Philosophy.

K. Srini. ____________________ Date: February 25, 2021

Chairman - K. Srinivas

K. Vijay. ____________________ Date: February 25, 2021

Guide/Convenor - Vijay Kodiyalam

S. Warzel ____________________ Date: February 25, 2021

Examiner - Simone Warzel

Anirban Mukhopadhyay ____________________ Date: February 25, 2021

Member 1 - Anirban Mukhopadhyay

Amritanshu Prasad ____________________ Date: February 25, 2021

Member 2 - Amritanshu Prasad

Raghavan KN ____________________ Date: February 25, 2021

Member 3 - K. N. Raghavan

Final approval and acceptance of this thesis is contingent upon the candidate's submission of the final copies of the thesis to HBNI.

I hereby certify that I have read this thesis prepared under my direction and recommend that it may be accepted as fulfilling the thesis requirement.

Date: February 25, 2021

K. Vijay.

Place: Chennai **Signature**

Guide

STATEMENT BY AUTHOR

This dissertation has been submitted in partial fulfillment of requirements for an advanced degree at Homi Bhabha National Institute (HBNI) and is deposited in the Library to be made available to borrowers under rules of the HBNI.

Brief quotations from this dissertation are allowable without special permission, provided that accurate acknowledgement of source is made. Requests for permission for extended quotation from or reproduction of this manuscript in whole or in part may be granted by the Competent Authority of HBNI when in his or her judgement the proposed use of the material is in the interests of scholarship. In all other instances, however, permission must be obtained from the author.

Narayanan P. A.

DECLARATION

I hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Narayanan P. A.

LIST OF PUBLICATIONS ARISING FROM THE THESIS

Journal

1. "On multiplicity of spectrum for Anderson type operators with higher rank perturbations", Anish Mallick and Narayanan P. A, Journal: Operators and Matrices, 2019, (Volume 13, Issue 3, pp. 733-744)

Narayanan P. A.

DEDICATIONS

I dedicate this work to my parents P. K. Achutha Pisharody and P. K. Vanaja Pisharasiar.

ACKNOWLEDGEMENTS

Ph. D. is a turning point in one's career. It is made possible with the support and the effort of a lot of people. To thank all of them enough is never possible. But, here I attempt to acknowledge some of them, as my memory permits.

I thank my Ph.D. supervisor Prof. Krishna Maddaly for putting in his valuable time in my regard, for his patience and for all the numerous discussions and helps he has done during the long tenure of my Ph. D. life. I thank Prof. Vijay Kodiyalam for taking the responsibility of a formal guide after the retirement of Prof. Krishna Maddaly from I.M.Sc., despite his numerous roles and responsibilities at I.M.Sc. and beyond. I will be for ever grateful to him for this.

I thank my doctoral committee members Prof. K. Srinivas, Prof. Vijay Kodiyalam, Prof. Anirban Mukhopadhay, Prof. Amritanshu Prasad and Prof. K. N. Raghavan agreeing to take up this responsibility, for putting in their valuable time in this cause and for their patience. I also take this opportunity to thank from my heart Prof. V. S. Sunder and Prof. D. S. Nagaraj who were in the doctoral committee before their retirement.

I fondly remember my teachers from Nursery School and Lower Primary (St. Mary's LP School Edathirinji) Treesa Teacher, Kathrina teacher and Baby teacher. I thank teachers from Upper Primary, High School and Higher Secondary: Rajalakshmi teacher, Rama teacher, Jeeja teacher, Rekha teacher, Maya teacher, Haridasan master, Susheelan master, Pramodini teacher, Radha teacher, Mini teacher, Sreedevi teacher, Narendran master, Sunanda teacher, Ushaja teacher, Rosy teacher, Jyothi teacher, Jayasri teacher, Seena teacher and Uma teacher who have all helped me find the student in me and encouraged my interest in Mathematics.

I thank my teachers in Christ College, Irinjalakuda where I did Bachelor of Science in Mathematics. I especially thank Prof. N. J. Thomas who taught me Real Analysis and has been encouraging as a teacher can ever be, ever since. I thank Prof. Tintu and Prof. Seena who encouraged me to study at Cochin University of Science and Technology. I have benefited a lot from discussions with them. The teachers and the atmosphere at Christ College has helped me a lot to nurture my interest in Mathematics.

I thank Prof. M. N. Narayanan Nampoothiri who taught me at the Cochin university of Science and Technology. Thank you sir for helping me find belief in me, especially in the context of Mathematics. I thank Prof. A. Vijayakumar who taught me at Cochin University of Science and Technology for the wonderful teacher he has been and for being a well wisher. He was crucial in developing interest in me to join I.M.Sc.

I thank the teachers during my life at I.M.Sc., Prof. V. S. Sunder, from whom I had the good fortune of taking several courses, Prof. D. S. Nagaraj, Prof. P. Shankaran, Prof. S. Keshavan, Prof. R. Balasubramanyam, Prof. Anilesh Mohari, Prof. K. N. Raghavan, Prof. Amritanshu Prasad, Prof. Partha Sarathi Chakroborty, Prof. Anirban Mukhopadhay, Prof. S. Viswanath, Prof. K. Srinivas, and Prof. Shanoli Gun. I must especially thank Prof. Gun for numerous helps, especially at the beginning of my tenure at I.M.Sc. Every student who meets her will remember that she always had some kind words for them. I have benefited a lot from the discussions with all of them; if I have become a bit more matured mathematics student, it is due to them.

The excellent library at I.M.Sc. has been a place of calm and focused study for me through out my stay there. I fondly recall the pleasant and professional behaviour from the library staff. I am especially grateful to Dr. Ushadevi P. for all the helps I have availed regarding the library.

No academic work at I.M.Sc. would have been possible without active help from the office staff and other staffs. The Registrar Mr. Vishnu Prasad has always been accessible. I am also grateful to Ms. Indra R and Ms. Prema P. for all the many occasions I had sought help from them. Mr. G. Srinivasan has been especially helpful and forthcoming for all the numerous computer related matters. I thank all the staff and non-staff members at I.M.Sc. They were all forthcoming and truly professional beyond their call of duty.

I thank Kiran Kumar V. B., Didimos and Tijo. I have learned a lot of Mathematics from discussions with them at CUSAT, and there after.

I thank my good friends Krishnadas, Akhil, Viby (who is unfortunately no longer with us; may his soul rest in peace), Anoop and Bipin with whom I had the good fortune of studying Mathematics. I acknowledge my friends and batchmates in I.M.Sc. not in any particular order: Nidhish Unnikrishnan, Arunkumar G., Keshab Chandra Bakshi, Raghu Venkata Tej Panthangi, Priyamvad Srivastava, Sohan Lal Saini, Ekata Saha, Biswajyoti Saha and Purna Pushkala. We had a great time together. I, also, acknowledge my friends Tiju Cheriyan, Sumesh K., Akhilesh P., Divakaran Divakaran, Rathul, Anvil, Jilmy, Uday, Anupam, Sruthy and Ranjan.

This acknowledgement will never be complete without thanking my friends and colleagues Dr. Dhriti Ranjan Dolai and Dr. Anish Mallick. Dhriti da has always tended the care of an elder brother through out the time of the Ph.D. and afterwards. Anish bhaiyya gave help, support and guidance at all the crucial times of this work. He was always approachable and ever ready to help. Although no words are enough to describe my gratitude to Dhriti da and Anish bhaiyya, I am taking this opportunity to thank them.

I am always thankful and in amazement to my parents for bringing me up despite what they have gone through in life.

Contents

1 Introduction **25**

2 Preliminaries **33**

2.1 Basics from Functional Analysis and Probability 33

2.2 Measurable Operator Families 41

2.2.1 Bethe lattice operators 43

2.2.2 Random Operators on the Canopy Tree 45

2.2.3 Measurability of the random operators on the Bethe lattice and the Canopy tree 50

2.3 Localisation 53

2.4 Wegner and Minami Estimate 58

3 Multiplicity of the Canopy Hamiltonian **63**

4 Eigenvalue statistics for Bethe lattice **73**

4.1 Main Theorems 73

4.2 Proofs of Main Theorems 76

4.2.1 Proof of Theorem 4.1.1 76

4.2.2 Infinite divisibility and Compound Poisson Variables 79

5 Conclusion **87**

Bibliography **89**

List of Figures

2.1 Bethe lattice with K=3 43

2.2 First few recursion steps for the Canopy tree for $K = 2$. 46

4.1 Boxes corresponding to the projections 76

Summary

In this thesis, we study the higher rank Anderson tight-binding model on the Bethe lattice.

Anderson tight binding Hamiltonian over Bethe lattice, is a highly studied and important model, and it is one of the few models where the existence of both the absolute continuous ([4, 27, 41]) as well as the pure point ([2], [3], [29]) spectrum are proven.

Aizenman-Warzel ([5]) has shown that, studying the eigenvalue statistics for the Anderson model on the Bethe lattice, actually amounts to studying the eigenvalue statistics not of the Bethe lattice model, but of the Anderson model on a related graph called the canopy tree graph. They have also shown that, unlike the case in Bethe lattice, the canopy tree Hamiltonian has only pure point spectrum. Aizenman-Warzel ([5]) studied the rank-one case, which means that the single site potentials are independent of each other. But, we allow the single site potentials on a collection of vertices to be the same. The main focus of this manuscript is to study the local eigenvalue statistics for the Anderson tight binding model over the canopy tree, when single site potential affects a collection of vertices of the tree.

Hislop-Krishna ([34]) studied the higher rank Anderson model on the lattice, $\mathbb{Z}^d$, and showed that its local eigenvalue statistics is compound Poissonian. Whereas Aizenman-Warzel showed the eigenvalue statistics for the rank-one case to be Poissonian, in this thesis, for rank > 2 case, following Hislop-Krishna ([34]), the statistics is shown to be strictly compound Poissonian.

Next, we give a brief summary of our results. Bethe lattice is a connected regular graph with a root, i.e., every vertex of the graph has K neighbours, for some $K \in \mathbb{N}$, and there is a special vertex called the root, usually denoted as 0 (zero). We consider the higher rank Anderson type operator on the Bethe lattice $\mathcal{B}$, defined as

$$H_\lambda^\omega := \Delta + \lambda \sum_{p \in \mathcal{J}} \omega_p P_p,$$

where, Δ is the graph Laplacian, $\lambda > 0$ is the disorder parameter, $\{\omega_p\}$ are i.i.d. real valued random variables following an absolutely continuous distribution which has a compactly supported, essentially bounded derivative ρ. The projections P_p are projections on to the subspace of $\ell^2(\mathcal{B})$, defined by the subgraph with vertex set $\Lambda_d(p) := \{x \in \mathcal{B} : d(p,x) \leq d, p \prec_{\mathcal{B}} x\}$, where the partial order $\prec_{\mathcal{B}}$ is defined as $x \prec_{\mathcal{B}} y$ if and only if $d(0,y) = d(0,x) + d(x,y)$; in other words, x lies in the (unique) path from the root to the vertex y.

The canopy graph is an infinite tree. It is constructed as follows. Its vertex set $\mathcal{V}_C$ is subdivided into different layers. We start with a layer which contains infinitely many vertices. This is designated as the boundary layer of the graph. Divide this vertex set into disjoint subsets, each of cardinality K, for some $K \in \mathbb{N}$. Corresponding to each such subset, we designate a new vertex to which all the vertices in the subset are connected by edges. The vertices thus obtained form the second layer of the canopy graph. The same process is repeated on the second layer to get a third layer and so on, ad infinitum. Similar to the Bethe lattice Hamiltonian, we define a random operator on the canopy tree:

$$H_{C,\lambda}^\omega := \Delta + \sum_{p \in \mathcal{J}} \omega_p P_p.$$

Here, P_p will be projections on the the subspace of $\ell^2(C)$, defined by the subgraph of C having the vertex set $\Lambda_{C,d}$, where $\Lambda_{C,d} := \{x \in C : d(p,x) \leq d, p \prec_C x\}$. Here, $d(p,x)$ is the minimum of lengths of all the paths connecting p and x in C, the partial order $\prec_C$ is defined by $x \prec_C y$ if $d(x, \partial C) = d(x,y) + d(y, \partial C)$ (in other words y is in a path that start from x and goes to the boundary).

In order to study the eigenvalue statistics for the Canopy model at an energy level $E_0 \in \mathbb{R}$, we consider the following random point process.

$$\mu^{\omega,\lambda}_{E_0,L}(f) := \mathrm{Tr}(f(|\Lambda_L|(H^{\omega}_{\lambda,L} - E_0))), \forall f \text{ in } C_c(\mathbb{R}), \tag{0.0.1}$$

where Λ_L is the subgraph of the Canopy tree with the following vertex set: $\mathcal{V}(\Lambda_L) := \{x \in \mathcal{V}(C) : d(0,x) \le L\}$, $H^{\omega}_{\Lambda_L}$ is the random operator obtained by restricting $H^{\omega}_{C,\lambda}$ to Λ_L, i.e.,

$$H^{\omega}_{\lambda,L} := \chi_{\Lambda_L} H^{\omega}_{C,\lambda} \chi_{\Lambda_L}.$$

Now, we can give our main results.

Theorem 0.0.1. *Consider the random operators H^{ω} and $H^{\omega}_{\lambda,L}$ defined in* (2.2.12) *satisfying the assumptions (2.2.6). Let $E_0 \in \Sigma_{K-1}$, $\lambda > 0$ large and a bounded interval $I \subset \mathbb{R}$. Then the limit points of the sequence of random variables $\{\mu^{\omega,\lambda}_{E_0,L}(I)\}_{L\in\mathbb{N}}$, defined in* (1.0.2)*, are compound Poisson distributed with the associated Lévy measure having at least one point from the set $\{K-1, K, K+1, \cdots, M_0\}$ in its support.*

As shown by Hislop-Krishna ([34]), higher rank perturbations can result in, not necessarily a Poisson process, but a compound Poisson process. However, it is not clear, in general, if Anderson model with higher rank perturbations would result in a non-trivial compound Poisson statistics. It happens that, our particular model with higher rank perturbations has eigenvalues with arbitrarily high multiplicities (which are proportional to the size of the perturbing projections), which is the content of the next theorem. This, together with showing compound Poisson statistics, results in a strictly compound Poisson statistics for our model.

The following multiplicity result is from the joint work with Anish Mallick [8].

Theorem 0.0.2. *For $K > 2$, let C denote the Canopy tree of degree $K+1$ and on the Hilbert space $\ell^2(C)$, define the random operator H^{ω}_C by* (2.2.11)*, for some $m_0 \ge 2$. Set the random variables $\{\omega_x\}_{x\in\mathcal{N}}$ to be independent and identically distributed following* *an absolutely continuous distribution μ. Then*

$$\sigma(\Delta_{m_0-1}) + Supp(\mu) \subset \sigma_{pp}(H_C^\omega) \qquad a.s,$$

and the maximum multiplicity of point spectrum in $\sigma(\Delta_{m_0-1}) + Supp(\mu)$ is at least $K-1$.

As shown by Aizenman-Warzel ([5]), and also reflected in our work, is the fact that, the current notion of eigenvalue statistics does not distinguish between pure point and absolutely continuous part of the spectrum. Therefore, it is natural to ask, whether we can modify the definition of the eigenvalue statistics to reflect different types of spectra. As shown in the paper Aizenman-Warzel ([5]), trees with a backbone structure can be treated similar to the canopy tree. So, we believe that, extending our results to models on those trees will not be that difficult.

We conclude by saying that studying Anderson model on various graphs, especially, the Bethe lattice, is a highly interesting area, which has scope for much further work.

Summary

In this thesis, we study the higher rank Anderson tight-binding model on the Bethe lattice.

Anderson tight binding Hamiltonian over Bethe lattice, is a highly studied and important model, and it is one of the few models where the existence of both the absolute continuous ([4, 27, 41]) as well as the pure point ([2], [3], [29]) spectrum are proven.

Aizenman-Warzel ([5]) has shown that, studying the eigenvalue statistics for the Anderson model on the Bethe lattice, actually amounts to studying the eigenvalue statistics not of the Bethe lattice model, but of the Anderson model on a related graph called the canopy tree graph. They have also shown that, unlike the case in Bethe lattice, the canopy tree Hamiltonian has only pure point spectrum. Aizenman-Warzel ([5]) studied the rank-one case, which means that the single site potentials are independent of each other. But, we allow the single site potentials on a collection of vertices to be the same. The main focus of this manuscript is to study the local eigenvalue statistics for the Anderson tight binding model over the canopy tree, when single site potential affects a collection of vertices of the tree.

Hislop-Krishna ([34]) studied the higher rank Anderson model on the lattice, $\mathbb{Z}^d$, and showed that its local eigenvalue statistics is compound Poissonian. Whereas Aizenman-Warzel showed the eigenvalue statistics for the rank-one case to be Poissonian, in this thesis, for rank > 2 case, following Hislop-Krishna ([34]), the statistics is shown to be strictly compound Poissonian.

Next, we give a brief summary of our results. Bethe lattice is a connected regular

graph with a root, i.e., every vertex of the graph has K neighbours, for some $K \in \mathbb{N}$, and there is a special vertex called the root, usually denoted as 0 (zero). We consider the higher rank Anderson type operator on the Bethe lattice $\mathcal{B}$, defined as

$$H_\lambda^\omega := \Delta + \lambda \sum_{p \in \mathcal{J}} \omega_p P_p,$$

where, Δ is the graph Laplacian, $\lambda > 0$ is the disorder parameter, $\{\omega_p\}$ are i.i.d. real valued random variables following an absolutely continuous distribution which has a compactly supported, essentially bounded derivative ρ. The projections P_p are projections on to the subspace of $\ell^2(\mathcal{B})$, defined by the subgraph with vertex set $\Lambda_d(p) := \{x \in \mathcal{B} : d(p,x) \leq d, p \prec_{\mathcal{B}} x\}$, where the partial order $\prec_{\mathcal{B}}$ is defined as $x \prec_{\mathcal{B}} y$ if and only if $d(0,y) = d(0,x) + d(x,y)$; in other words, x lies in the (unique) path from the root to the vertex y.

The canopy graph is an infinite tree. It is constructed as follows. Its vertex set $\mathcal{V}_C$ is subdivided into different layers. We start with a layer which contains infinitely many vertices. This is designated as the boundary layer of the graph. Divide this vertex set into disjoint subsets, each of cardinality K, for some $K \in \mathbb{N}$. Corresponding to each such subset, we designate a new vertex to which all the vertices in the subset are connected by edges. The vertices thus obtained form the second layer of the canopy graph. The same process is repeated on the second layer to get a third layer and so on, ad infinitum. Similar to the Bethe lattice Hamiltonian, we define a random operator on the canopy tree:

$$H_{C,\lambda}^\omega := \Delta + \sum_{p \in \mathcal{J}} \omega_p P_p.$$

Here, P_p will be projections on the the subspace of $\ell^2(C)$, defined by the subgraph of C having the vertex set $\Lambda_{C,d}$, where $\Lambda_{C,d} := \{x \in C : d(p,x) \leq d, p \prec_C x\}$. Here, $d(p,x)$ is the minimum of lengths of all the paths connecting p and x in C, the partial order $\prec_C$ is defined by $x \prec_C y$ if $d(x, \partial C) = d(x,y) + d(y, \partial C)$ (in other words y is in a path that start from x and goes to the boundary).

In order to study the eigenvalue statistics for the Canopy model at an energy level $E_0 \in \mathbb{R}$, we consider the following random point process.

$$\mu^{\omega,\lambda}_{E_0,L}(f) := \text{Tr}(f(|\Lambda_L|(H^{\omega}_{\lambda,L} - E_0))), \forall f \text{ in } C_c(\mathbb{R}), \tag{0.0.1}$$

where Λ_L is the subgraph of the Canopy tree with the following vertex set: $\mathcal{V}(\Lambda_L) := \{x \in \mathcal{V}(C) : d(0,x) \le L\}$, $H^{\omega}_{\Lambda_L}$ is the random operator obtained by restricting $H^{\omega}_{C,\lambda}$ to Λ_L, i.e.,

$$H^{\omega}_{\lambda,L} := \chi_{\Lambda_L} H^{\omega}_{C,\lambda} \chi_{\Lambda_L}.$$

Now, we can give our main results.

Theorem 0.0.1. *Consider the random operators H^{ω} and $H^{\omega}_{\lambda,L}$ defined in* (2.2.12) *satisfying the assumptions (2.2.6). Let $E_0 \in \Sigma_{K-1}$, $\lambda > 0$ large and a bounded interval $I \subset \mathbb{R}$. Then the limit points of the sequence of random variables $\{\mu^{\omega,\lambda}_{E_0,L}(I)\}_{L\in\mathbb{N}}$, defined in* (1.0.2)*, are compound Poisson distributed with the associated Lévy measure having at least one point from the set $\{K-1, K, K+1, \cdots, M_0\}$ in its support.*

As shown by Hislop-Krishna ([34]), higher rank perturbations can result in, not necessarily a Poisson process, but a compound Poisson process. However, it is not clear, in general, if Anderson model with higher rank perturbations would result in a non-trivial compound Poisson statistics. It happens that, our particular model with higher rank perturbations has eigenvalues with arbitrarily high multiplicities (which are proportional to the size of the perturbing projections), which is the content of the next theorem. This, together with showing compound Poisson statistics, results in a strictly compound Poisson statistics for our model.

The following multiplicity result is from the joint work with Anish Mallick [8].

Theorem 0.0.2. *For $K > 2$, let C denote the Canopy tree of degree $K+1$ and on the Hilbert space $\ell^2(C)$, define the random operator H^{ω}_C by* (2.2.11)*, for some $m_0 \ge 2$. Set the random variables $\{\omega_x\}_{x\in\mathcal{N}}$ to be independent and identically distributed following* *an absolutely continuous distribution* μ. *Then*

$$\sigma(\Delta_{m_0-1}) + Supp(\mu) \subset \sigma_{pp}(H_C^\omega) \qquad a.s,$$

and the maximum multiplicity of point spectrum in $\sigma(\Delta_{m_0-1}) + Supp(\mu)$ *is at least* $K-1$.

As shown by Aizenman-Warzel ([5]), and also reflected in our work, is the fact that, the current notion of eigenvalue statistics does not distinguish between pure point and absolutely continuous part of the spectrum. Therefore, it is natural to ask, whether we can modify the definition of the eigenvalue statistics to reflect different types of spectra. As shown in the paper Aizenman-Warzel ([5]), trees with a backbone structure can be treated similar to the canopy tree. So, we believe that, extending our results to models on those trees will not be that difficult.

We conclude by saying that studying Anderson model on various graphs, especially, the Bethe lattice, is a highly interesting area, which has scope for much further work.

Chapter 1

Introduction

In 1958, P.W. Anderson [63] introduced a model of disordered systems such as alloys, which is later named after him as the Anderson model. One of the phenomenon Anderson explained using the model (with convincing, but not rigorous arguments) was that of localisation (which has come to be known as the Anderson localisation). Physically, it means that electrons in disordered media are trapped in a small region of space, in the presence of large disorder. In 1977, Anderson shared the Physics Nobel prize, along with Mott and van Vleck for their work in this area.

In the 1970s mathematically rigorous results started to appear; the first result being that of Goldsheid, Molchanov, and Pastur in 1977 [33]. They showed Anderson localisation for a related one-dimensional model. Pastur [54] gave the fundamental theorem that, for ergodic random operators, almost surely, the spectrum is constant. The first mathematically rigorous proof of Anderson localisation for the Anderson model was due to the work of Kunz and Souillard [46]. Their work, also, was for the one-dimensional model. The spectral localisation (that is, the existence of dense pure point spectrum) for models in arbitrary dimensions was proved, independently, by Fröhlich, Martinelli, Scoppola, and Spencer [29]; Simon and Wolff [61]; and Delyon, Lévy, and Souillard [23], based on the path-breaking paper of Fröhlich and Spencer [28] in 1983 which introduced a method called the 'multiscale analysis'.

Two general methods to show the Anderson localisation for arbitrary dimensions are available, the first one is the multiscale analysis and the other is the fractional moment method due to Aizenman and Molchanov [2], in 1993. The latter method is easier, but not as widely applicable as the first method. For the analogue of the Anderson model in $L^2(\mathbb{R}^d)$, the fractional moment method was extended by Aizenman, Elgart, Naboko, Schenker, and Stolz [1] in 2005.

The phenomenon of Anderson localisation is relatively well understood but the other important question of the existence of de-localised states, or equivalently, that of absolutely continuous part of the spectrum, is still a major open problem for the higher dimensional Anderson model. It is generally believed that, for dimensions more than two, ac spectrum is present for the models with low disorder. For the integer lattice ($\mathbb{Z}^d$) models with decaying potentials existence of the ac-spectrum has been proved in Krishna [43], Krishna [44], Kirsch [39] and Jakšić-Last [36].

The other important question is about the local structure of the spectrum, the structure of the spectrum in the neighbourhoods of a point in the spectrum, a study going by the name of 'eigenvalue statistics'. The eigenvalue statistics in one dimension was studied by Molchanov [52], and later for higher dimensions by Minami [51]. In the region of localisation where fractional moment bounds are valid (where (4.1.1) holds), they showed that the eigenvalues follow Poisson statistics. Subsequently, Poisson statistics was shown for the trees by Aizenman-Warzel [5], and by Geisinger [30] for regular graphs. In some recent results, Germinet-Klopp [31] extended the results of Killipp-Nakano [38]. These works are focused on eigenfunction statistics in the regime of pure point spectrum. There are also works in the region of the absolutely continuous spectrum, like Kotani-Nakano [42], Avila-Last-Simon [11], and Dolai-Mallick [24]. Mallick-Dolai [7] studied the eigenfunction statistics of the Anderson model with singular randomness where the single-site distribution is Holder continuous. Dolai-Krishna [25] showed Poisson statistics for an Anderson model with singular randomness. Dolai-Mallick [24] studied the spectral statistics of random Schrödinger operators where the potential is unbounded. There are a few results for spectral statistics for non-rank one Anderson models, for example, Hislop-Krishna [34] and Combes-Germinet-Klein [16].

Anderson tight-binding Hamiltonian over the Bethe lattice is a well studied and important model and it is one of the few models where the existence of both the absolutely continuous [4, 27, 41] as well as the pure point [2, 3, 29] spectrum are shown to exist for small disorder. However, the eigenvalue statistics as defined by Minami [51] does not carry over to the Bethe lattice, but extends to the Canopy tree (as explained by Aizenman-Warzel [5]). The main focus of this manuscript is to study the local eigenvalue statistics for the Anderson tight binding model over the Canopy tree when the single-site potential is constant on a collection of vertices of the tree (as defined in (2.2.11)). Although, to define the local statistics we look at the cut-off operator on the Bethe lattice.

There is a qualitative difference between the Poisson local eigenvalue statistics obtained my Minami [51], Aizenman-Warzel [5] and others in various contexts and the compound Poisson local statistics obtained by Hislop-Krishna [34]. In the Poisson case, the sequence of random measures considered do converge to a Poisson random measure, while in the compound Poisson case one can only consider a sequence of random variables associated with an interval and the sequence of random measures and show that the limit points of these random variables have a compound Poisson distribution. One is not able to prove a theorem at the level of random measures. The reason is the lack of a proof of the independence of the limit points as the intervals vary. (More precisely it is not clear that the limit points X_I^ω, $X_{I_1}^\omega$ of the sequences $\mu_{E_0,L}^{\omega,\lambda}(I)$, $\mu_{E_0,L}^{\omega,\lambda}(I_1)$ given in equation (1.0.2) are independent random variables, for $I \cap I_1 = \emptyset$)

Thus, we consider the higher rank Anderson type operator on the Bethe lattice $\mathcal{B}$ defined as

$$H_\lambda^\omega := \Delta + \lambda \sum_{y \in J} \omega_y P_y, \tag{1.0.1}$$

where Δ is the graph Laplacian on the Bethe lattice, $\lambda > 0$ is the disorder parameter, $\{\omega_y\}_{y\in J}$ are independent identically distributed real random variables following an absolutely continuous distribution $\rho(x)dx$ where $\rho \in L^\infty(\mathbb{R})$ and $supp(\rho)$ compact. The P_y's are certain projections on the Hilbert space associated with the Bethe lattice, defined in [2.2.3].

To study the local eigenvalue statistics at $E_0 \in \mathbb{R}$, we look at the limit points of the random variables $\{\mu^{\omega,\lambda}_{E_0,L}(I)\}_{L\in\mathbb{N}}$, associated with an interval $I \subset \mathbb{R}$ defined by

$$\mu^{\omega,\lambda}_{E_0,L}(I) = \mathrm{Tr}(\chi_I(|\Lambda_L|(H^\omega_{\lambda,L} - E_0))), \tag{1.0.2}$$

where $H^\omega_{\lambda,L}$ are suitable cut-off operators (defined in 2.2.9) and χ_I is the indicator function of the interval I and $|\Lambda_L|$ is the number of vertices in the subtree Λ_L defined in [2.2.4]. As stated earlier, this method of defining a limiting random variable does not provide the local eigenvalue statistics over the Bethe lattice, but on the Canopy tree. In this thesis, we show that the above sequence of random variables has limit points that are compound Poisson random variables.

In the work [5], the authors concluded simple Poisson point process as the eigenvalue statistics for the Anderson tight-binding model over the Canopy tree. One of the important points they raised is the fact that the infinite divisibility of the eigenvalue process cannot be taken similar to the $\mathbb{Z}^d$ case. This is because $\frac{|\partial\Lambda_L|}{|\Lambda_L|}$ does not converges to zero as $L \to \infty$. However, because of the exponential nature of the growth of the surface area, and the fact that we can achieve any rate of decay in Theorem 4.1.1, we can get the infinite divisibility needed for Poisson process by dividing the trees into subtrees of height $\approx \alpha L$ (for $0 < \alpha < \frac{1}{2}$). Usually, this would fail to produce the correct decay needed to establish the infinite divisibility; but in this case, this rate of decay suffices.

The local statistics for the higher rank Anderson model on the Canopy tree that we consider in this thesis, is an extension of the work in Aizenman-Warzel [5] and we show that the local statistics is Compound Poisson as done in Hislop-Krishna [35]. The significant part of our thesis is to show that the Compound Poisson statistics is non-trivial in the sense that it is not Poisson, but has higher multiplicity.

There are three main components involved in determining local statistics for random operators. These are the Wegner estimate, originally shown by Wegner [64], the Minami estimate obtained by Minami [51] and fractional moment bounds originally obtained by Aizenman-Molchanov [2]. It was shown in [35] that the Wegner estimate and the fractional moment bounds are sufficient to conclude that the statistics is Compound Poisson for a wide class of random operators. There, the authors also generalize the Minami estimate to higher rank coefficients for the i.i.d random potentials using the method of Combes et.al. [15]. It is this form of the Minami estimate that we use in this thesis. One departure, which is the significant part of this thesis, that gives us a non-trivial example of Compound Poisson statistics, is to show that the random operators we consider have spectral multiplicity higher than one for a subset of energies.

This thesis consists of five chapters. The first chapter is the current one where we informally discuss the problem we address and introduce the content of the future chapters.

The second chapter has, by now well established, preliminaries about the random operator families and their spectral properties. In this chapter, we introduce the Canopy tree as the correct object to consider for discussing the local statistics on the Bethe lattice as done by Aizenman and Warzel in [5]. After briefly discussing the definitions of the Anderson-model operator on the Bethe lattice and the eigenvalue statistics there, we introduce the operators on the Canopy tree, similar to equation (1.0.1). These are the adjacency operator on the Canopy tree perturbed by a random operator coming from a countable collection of i.i.d real-valued random variables and a countable collection of finite rank projections $\{P_n\}$ which are of the same rank, on the Hilbert space of square-summable sequences on the Canopy tree. These finite rank projections are supported on disjoint parts of the Canopy tree and form a partition of unity on the said Hilbert space. Since our purpose is to study the eigenvalue statistics, we then take the finite compressions of these operators by restricting them to finite-dimensional Hilbert spaces associated with finite subtrees of the Canopy tree, considered as matrices. These matrices are unitarily equivalent to the matrices which are similar compressions of the operators on the Bethe lattice corresponding to some subtrees there. Therefore we can and will talk about these matrices as being on the Bethe lattice or the Canopy tree interchangeably. When there is confusion, we always understand that the matrices are on the Canopy tree, its subtrees and the associated Hilbert spaces, but keeping in mind the origins are from the Bethe lattice.

We then consider the matrix-valued kernels of the resolvents of these random operators, expressions such as $P_n(H^\omega - z)^{-1}P_m$ and discuss the case when

$$\mathbb{E}(\|P_n(H^\omega - z)^{-1}P_m\|^s)$$

where $0 < s < 1$, decay exponentially when the distance between the supports of P_n and P_m increases to infinity and the disorder parameter λ is large, independent of L. This decay has spectral consequences and the intervals of energies (Re(z)) where this decay holds contain only pure point spectrum. This fact is the well known Simon-Wolff [61] criterion for the case of rank one operators (that is, when P_ns are of rank one). The Simon-Wolff criterion can be used for the higher rank case, but we give an alternative argument by Graf [49] to conclude the pure point spectrum in such intervals. We present the essential estimate to complete Graf's proof in this Chapter. We note here that the fractional moment bound is sufficient for later arguments on the local statistics and the spectral conclusion is not necessary.

The Wegner estimate is presented later in Lemma 2.4.2, which gives the absolute continuity of the averaged spectral measures of the random operators we consider. This estimate is sufficient to conclude the Compound Poisson nature of the limiting random variables that we investigate when we have the fractional moment bounds on the resolvent kernels. The (generalized) Minami estimate is presented following the Wegner estimate in Lemma 2.4.3. The Minami estimate was crucial, in the case when P_n had rank one, in showing that the limiting infinitely divisible measure has the same mean and variance, for any finite interval I, enabling one to conclude that the limiting statistics is Poisson. In the present case, it is not as crucial, since the generalized Minami estimate only shows that the Compound Poisson random variables have finite multiplicity.

The following chapter has one of the original results proved by the author in a joint work with Mallick [8]. Here we show that, in the Canopy graph context and for a choice of the $\{P_n\}$s, some part of the spectrum has multiplicity bigger than one [Theorem 3.0.1], if the rank of P_n is larger than one. This is done by explicitly constructing more than one mutually orthogonal eigenfunctions associated with eigenvalues of the random operator that we consider.

In their work, Aizenman and Warzel in [5] showed that the Laplacian on the Canopy graph has infinitely many degenerate eigenvalues. They explicitly construct the eigenfunctions, by using the symmetries of the Laplacian. It is not immediately clear how to use this result to say anything about the multiplicity of the Canopy Hamiltonian in the rank one case, since the symmetries of the Laplacian do not carry over to the perturbed operator.

But in the higher rank case, the structure of the Canopy tree comes in handy. There are local symmetries of the tree that we can choose based on the higher rank perturbation we have, that induce unitaries on the Hilbert space, which in turn preserve the operator, under conjugation. This is a mechanism that is special to some graphs, which is explained in more detail in Chapter 3 and also elsewhere [8].

As an application of the result on multiplicity we obtain a lower bound on the probabilities of some events in Proposition [3.0.4], that will imply that the Levy measures associated with the limiting random variables that we obtain, have supports away from $\{1\}$.

Chapter 4 has the main theorem [4.1.2] on the local statistics and proofs of the different components needed there. The main idea is to show that a limiting random variable associated with a sequence given in equation (1.0.2) is a Compound Poisson random variable. For doing this we follow the arguments of [35], by computing the limits of averages of the Fourier transforms of the distributions of these random variables. The structure of the limit shows the infinite divisibility and the Compound Poisson nature of the limiting random variable, by an application of the Lévy-Khintchine Theorem [9, Theorem 1.2.14] (given in Chapter 2, Theorem 2.1.15). The Lévy-Khintchine Theorem gives a calculable method to identify if a random variable is Poisson or Compound Poisson, by looking at the Fourier transform of its distribution.

We then use Proposition [3.0.4], to show that the Levy measure associated with the limiting random variable has support away from $\{1\}$, completing the arguments needed to prove our main result.

In the fifth and concluding chapter, we discuss open problems for further enquiry that came up during our study into the problems that this thesis discusses.

Chapter 2

Preliminaries

2.1 Basics from Functional Analysis and Probability

In this section, we recall some basic concepts and theorems of functional analysis such as the spectral theorem, the functional-calculus and the classification of the spectrum of a self-adjoint operator into various types and some necessary preliminaries from probability. We assume familiarity of functional analysis, say, at the level of topics covered in [55]. For probability part, topics covered in [59] will be fairly sufficient. The material of this section is standard and can be found in various books on functional analysis; e.g., [20, 22, 50, 53, 55, 60, 65, 66] and measure and probability theory such as [9, 10, 12, 13, 26, 37, 47, 56, 57].

We recall, now, the spectral theorem for self-adjoint operators. We refer to [55, p. 263] for a reference for the following discussion about projection valued measures.

Definition 2.1.1 (Projection Valued Measure)**.** *Let $(X, \mathcal{B})$ be a measurable space, and let $\mathcal{H}$ be Hilbert space. A projection valued measure on $(X, \mathcal{B})$ on to $\mathcal{H}$ is a map*

$$E : \mathcal{B} \to \mathcal{P}(\mathcal{H}), \tag{2.1.1}$$

where $\mathcal{P}(\mathcal{H})$ is the set of all orthogonal projections on $\mathcal{H}$, which satisfy the following *conditions.*

1. *Let $B_n \in \mathcal{B}$ be a disjoint collection. Then, $E(\cup_n B_n) = \sum E(B_n)$, where the summation is defined point wise and the sum converges strongly.*

2. $E(\emptyset) = 0f$ *and* $E(X) = I$.

3. $E(B_1 \cap B_2) = E(B_1)E(B_2)$.

Projection valued measures on the real line and self-adjoint operators are related. For projection valued measures on the real line, we want to distinguish between two cases. If for some $a \in \mathbb{R}$ we have $E((-a,a)) = I$, then we say E has bounded support, otherwise it is said to have unbounded support. As the quoted theorems below would show, the projection valued measures with bounded support are associated with the bounded self-adjoint operators and those with unbounded support are associated with the unbounded self-adjoint operators. Given a projection valued measure E on $\mathcal{H}$, and two vectors ψ and η in $\mathcal{H}$, we can associate a finite complex measure $E_{\psi,\eta}(\cdot)$ on $\mathbb{R}$ as follows.

(2.1.2) $$E_{\psi,\eta}(B) := \langle \psi, E(B)\eta \rangle,$$

for $B \in \mathcal{B}$, the sigma algebra of all Borel subsets of the real line.

Theorem 2.1.2 (Functional Calculus)**.** *Let H be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Then there is a unique mapping Θ from the collection of all bounded Borel measurable functions on $\mathbb{R}$ to the set of all linear operators on the Hilbert space, $\mathcal{L}(\mathcal{H})$, with the following properties.*

- $\Theta(\varphi_1 + \varphi_2) = \Theta(\varphi_1) + \Theta(\varphi_2)$,

- $\Theta(\varphi_1\varphi_2) = \Theta(\varphi_1)\Theta(\varphi_2)$,

- $\Theta(\overline{\varphi}) = \Theta(\varphi)^*$, *where $\overline{\varphi}$ is defined by $\overline{\varphi}(x) := \overline{\varphi(x)}$, the complex conjugate of $\varphi(x)$.*

*A mapping with these properties is called an algebraic *-homomorphism between the *-algebras the set of all bounded Borel functions on the real line and the set of all linear operators,* $\mathcal{L}(\mathcal{H})$. *For definitions and more details on algebras, we refer to [58].*

- Θ *is continuous with respect to the norms. In fact,* $\|\Theta((h)\|_{\mathcal{L}(\mathcal{H})} \leq \|h\|_\infty$.
- *Let* $f_n(x)$ *be a sequence of bounded Borel measurable functions such that* $\lim_{n\to\infty} f_n(x) = x$ *for each* $x \in \mathbb{R}$ *and* $|f_n(x)| \leq |x|$ *for all* x *and* n*. Then for any* $\psi \in Dom(H)$,
$$\lim_{n\to\infty} \Theta(f_n)\psi = H\psi.$$
- *If the sequence* $\{f_n\}$ *converges to* f *pointwise and if the sequence* $\{\|f_n\|_\infty\}$ *is bounded, then* $\lim_{n\to\infty}\Theta(f_n) = \Theta(f)$*, strongly.*
- *If* $H\psi = \lambda\psi$*, then* $\Theta(f)\psi = f(\lambda)\psi$.
- *If* $f \geq 0$*, then* $\Theta(f) \geq 0$.
- *If* $HT = TH$*, then* $\Theta(f)T = T\Theta(f)$.

The projection valued measure has the following very important property. We refer to [55, p. 263, Theorem VIII.6]. Note that our Theorems 2.1.3 and 2.1.4 together is called the spectral theorem there ([55, p.263,Theorem VIII.6]).

Theorem 2.1.3. *Let* E *be a projection valued measure on* $(\mathbb{R}, \mathcal{B})$ *on to* $\mathcal{H}$*. Then we define a self-adjoint operator* H *on* $\mathcal{H}$ *as follows:*

$$\mathrm{Dom(H)} := \{\psi \in \mathcal{H} : \left| \int x \ dE_{\phi,\psi}(x) \right| \leq C_\psi \|\phi\|, \ for \ all \ \phi \in \mathcal{H}\}$$

and H *is defined by,* $\langle \phi, H\psi \rangle := \int x dE_{\phi,\psi}(x)$ *on all* $\psi \in \mathrm{Dom}(H), \phi \in \mathcal{H}$*. In this case, we write* $H = \int x \ dE(x)$.

In the case when E has bounded support it is clear that $\mathrm{Dom}(H) = \mathcal{H}$ and so we get a bounded self-adjoint operator associated with it. The spectral theorem gives a converse of the above theorem for self-adjoint operators (See [55, p. 263, VIII.6]).

Theorem 2.1.4 (Spectral Theorem)**.** *Let H be a self-adjoint operator on a Hilbert space* $\mathcal{H}$ *with domain Dom(H). Then there is a unique projection valued measure E, on* $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ *(which is called the spectral measure of* H*) such that we have the following equalities.*

$$H = \int x \, dE(x), \ \mathrm{Dom}(H) = \{\psi \in \mathcal{H} : \left| \int_{\mathbb{R}} \lambda \mathrm{d}E_{\phi,\psi}(\lambda) \right| \leq C_\psi \|\phi\|, \ \forall \phi \in \mathcal{H}\},$$

where the expression $\int x \, dE(x)$ defines an operator as in the previous theorem.

For bounded measurable functions, the following theorem allows us to take a self-adjoint operator as an input to the functions. It is known as the functional calculus. It is equivalent to the spectral theorem, e.g., [55, p. 262, Theorem VIII.5].

The following lemma makes the important observation that we can recover the unique spectral measure corresponding to a self-adjoint operator using functional-calculus. A proof of this can be found in [55] in the discussion after Theorem VIII.5.

Lemma 2.1.5. *Let H be a self-adjoint operator on a Hilbert space* $\mathcal{H}$*. Then the unique spectral measure of H is given by*

$$\mathcal{B}(\sigma(H)) \ni B \to \chi_B(H) \in \mathcal{P}(\mathcal{H}).$$

Using spectral projections we can classify the spectrum into various components. The discrete spectrum consists of those points in the spectrum which have a neighbourhood such that the corresponding spectral projection is finite dimensional. The essential spectrum consists of those spectral points where the spectral projection of every neighbourhood of these points is infinite-dimensional. The Lebesgue decomposition theorem, (e.g., [59, p. 121, Theorem 6.10]) tells us that any measure μ on $\mathbb{R}$ has a unique decomposition as

$\mu = \mu_{pp} + \mu_{ac} + \mu_{sc}$ as the sum of pure point, absolutely continuous and singular continuous parts with respect to the Lebesgue measure.

Now, we define the following subspaces. See [48, Definition 2.6.8] for a reference.

Definition 2.1.6. *Let H be a self-adjoint operator on the Hilbert space $\mathcal{H}$. Let E_H be the spectral measure for H. For $\psi \in \mathcal{H}$, let $E_{\psi,\psi} = \langle \psi, E_H()\psi \rangle$. Then define the subspaces of $\mathcal{H}$*

(2.1.3) $$\mathcal{H}_{pp} := \{\psi \in \mathcal{H} : E_{\psi,\psi}(\cdot) \text{is pure point}\}$$

(2.1.4) $$\mathcal{H}_{ac} := \{\psi \in \mathcal{H} : E_{\psi,\psi}(\cdot) \text{is absolutely continuous}\}$$

(2.1.5) $$\mathcal{H}_{sc} := \{\psi \in \mathcal{H} : E_{\psi,\psi}(\cdot) \text{is singular continuous}\}$$

In terms of the above subspaces, we have the following decomposition for $\mathcal{H}$. ([48, Remark 2.6.9, p.48].)

Theorem 2.1.7. *Let H be a self-adjoint operator, then*

$$\mathcal{H} = \mathcal{H}_{pp} \bigoplus \mathcal{H}_{ac} \bigoplus \mathcal{H}_{sc}.$$

Also, these subspaces are invariant under H.

We denote by $E_{p,H}$ the spectral projection of H on to the subspace $\mathcal{H}_{pp}$ and by $E_{c,H}$ the spectral projection on to the subspace $\mathcal{H}_{ac} \bigoplus \mathcal{H}_{sc}$ for later use.

We can now state an important decomposition of the spectrum of a self-adjoint operator. [See [48, 2.6.4,2.6.5]] and the discussion following Definition [2.6.8] there.

Definition 2.1.8. *We denote the restrictions of H to a subspace V by $H \restriction V$.*

(2.1.6) $$\sigma_{pp}(H) := \overline{\{\lambda : \lambda \text{ is an eigenvalue of} H\}}.$$

(2.1.7) $$\sigma_{ac}(H) := \sigma(H \restriction \mathcal{H}_{ac}).$$

(2.1.8) $$\sigma_{sc}(H) := \sigma(H \restriction \mathcal{H}_{sc}).$$

The following is a very useful theorem connecting the spectral projections of a self-adjoint operator to the resolvents of the operator. We refer to [48, Theorem 2.4.2, p. 41].

Theorem 2.1.9 (Stone's Formula)**.** *Let A be a self-adjoint operator. Suppose $a, b \in \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$, $a < b$, then*

(2.1.9)
$$E_A([a,b]) + E_A((a,b)) = \text{s}-\lim_{\epsilon\to 0+} \frac{1}{\pi i} \int_a^b ((A-(t+i\epsilon)I)^{-1} - (A-(t-i\epsilon)I)^{-1})dt,$$

where $E_A(\cdot)$ is the spectral projection for A, and the integrals above converge in the strong operator topology.

We now turn to some concepts and theorems from probability theory that are used in this thesis.

Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability space. The distribution of a random variable is the probability measure $\mathbb{P} \circ X^{-1}$ on $\mathbb{R}$. The distribution function of the random variable X (sometimes, also called the cumulative distribution function in the literature) is the non-decreasing positive function $F_X(x) = \mathbb{P} \circ X^{-1}((-\infty, x))$.

A finite or countable collection of random variables $\{X_1, \ldots, X_N, \ldots\}$ is called i.i.d (independent and identically distributed) if all the random variables are pairwise independent and their distributions are the same. The strong law of large numbers and the central limit theorem are well known (these can be found in many of the references given in the introduction to this section, e.g., see [56, Chapter 8] for a discussion on these theorems), but we need and use a concept of divisibility of random variables coming from the divisibility of measures, which we define below.

The following definition can be found in [9, Section 1.2.1, p.21].

Definition 2.1.10. *A convolution $\mu * \nu$ of two probability measures μ, ν on $\mathbb{R}$ is defined by*

$$\mu * \nu(A) = \int \chi_A(x+y) d\mu(x) d\mu(y),$$

where A is any Borel set on $\mathbb{R}$.

Definition 2.1.11. *A probability measure μ is said to be divisible if there is another measure ν and a positive integer k such that $\mu = \underbrace{\nu * \nu * \cdots * \nu}_{k \ \ times}$, where $*$ denotes convolution of measures. The measure μ is said to be infinitely divisible if it is divisible for all $k \in \mathbb{N}$, where the ν may be different as k varies. We will say a random variable is infinitely divisible if its distribution is infinitely divisible.*

For a discussion on infinite divisibility of probability measures, see [9, Proposition 1.2.6, p.25]. We also need the concept of convergence of a sequence of random variables. For the following, we refer to [9, Section 1.1.5, p. 14].

Definition 2.1.12. *A sequence X_n of real valued random variables is said to converge to X in distribution : if $F_{X_n}(x) \to F_X(x)$ for all points x which are points of continuity of the distribution function F_X.*

Here is an equivalent condition that enables us to verify the convergence of a sequence of random variables. We denote by $\mathbb{E}(X) = \int X(\omega) \ d\mathbb{P}(\omega)$ for a real valued random variable on $(\Omega, \mathcal{B}, \mathbb{P})$. For a proof of the following handy theorem, we refer to [13, p. 349, Theorem 26.3].

Theorem 2.1.13. *A sequence $\{X_n\}$ of random variables converges in distribution to a random variable X iff $\mathbb{E}(e^{itX_n}) \to \mathbb{E}(e^{itX})$ for each $t \in \mathbb{R}$.*

The following can be found in [9, Examples 1.2.9,1.2.10].

Definition 2.1.14 (Poisson and Compound Poisson distributions)**.** *A random variable X is* *said to have the Poisson distribution with parameter λ if its distribution $\mathbb{P}\circ X^{-1}$ is of the form*

$$\mathbb{P}\circ X^{-1}(\{k\}) = \frac{\lambda^k}{k!}e^{-\lambda}, \ \ k \in \{0\}\cup\mathbb{N},$$

and zero on any interval that does not contain a positive integer.

If Y is a Poisson distributed random variable and $X_1,\ldots,X_N,\ldots$ are i.i.d random variables all of which are independent of Y, then the distribution of the random variable $\sum_{j=1}^{Y} X_i$ is called the compound Poisson distribution.

We add the Lévy-Khintchine theorem which is an important characterisation of infinitely divisible random variables based on their characteristic functions, which is crucial in the proof of our result showing the compound Poisson nature of the limiting random variable of the random process on the Canopy tree. Let ν be a Borel measure defined on $\mathbb{R}\setminus\{0\}$. It is a Lévy measure if $\int_{\mathbb{R}\setminus\{0\}}\left(|y|^2\wedge 1\right)\nu(dy) < \infty$, where $'\wedge'$ is the operation of taking minimum. Note that any finite measure on $\mathbb{R}\setminus\{0\}$ is a Lévy measure.

The Following is Lévy-Khintchine theorem (See [9, Theorem 1.2.14] and the notes following the theorem).

Theorem 2.1.15 (Lévy-Khintchine)**.** *Let X be a random variable. X is infinitely divisible if and only if there exists $a > 0$, $b \in \mathbb{R}$ such that the characteristic function of X has the following form.*

$$\mathbb{E}(e^{itX}) = e^{ibt-\frac{1}{2}at^2} + \int_{\mathbb{R}\setminus\{0\}}\left(e^{ist} - 1 - ist\chi_{\mathbb{B}(0)}(s)\right)\mathrm{d}\nu(s), \tag{2.1.10}$$

where $\mathbb{B}(0)$ is the unit ball. The triplet (a,b,ν) helps us to identify the distribution of the random variable, as follows. The random variable is:

- Gaussian: If $\nu = 0$ and $b = 0$.
- Poisson: If $a = b = 0$ and $\nu = c\delta_1$ for some positive constant c .

- Compound Poisson: If $a=0$, $b=c\int_{\mathbb{B}(0)} x\,\mathrm{d}\mu(x)$, $\nu=c\mu$ for some constant c and a probability measure μ.

2.2 Measurable Operator Families

In this section, we consider the random operator-valued functions that this thesis studies, namely the Anderson-model random operator on the Bethe lattice and on the Canopy tree. We start with the definition of measurability of bounded operator-valued functions. After introducing the classical theorem by Pastur on the almost sure constancy of the spectrum of the ergodic family of operators, we state the definitions of the operator families that we consider in this thesis, and the definition of the eigenvalue statistics for them. Then we briefly explain the observation by Aizenman-Warzel in [5] about the connection between the eigenvalue statistics over the Bethe lattice and the Canopy tree. We show, subsequently, that the two main operator-valued functions that we consider are measurable. We refer to [14, Chapter V] for a broad treatment of this subject.

Let $(\Omega,\mathcal{F})$ be a measurable space. Consider a function

$$\Omega \ni \omega \to H^{\omega} \in \mathcal{S}(\mathcal{H}), \tag{2.2.1}$$

where $\mathcal{S}(\mathcal{H})$ is the set of all bounded self-adjoint operators on the Hilbert space $\mathcal{H}$.

It is more convenient to think of the above function as an indexed family, $\{H^{\omega}\}_{\omega\in\Omega}$, which we do henceforth. This family is said to be a measurable family, if for all ψ and η in $\mathcal{H}$, the map

$$\Omega \ni \omega \to \langle \psi, H^{\omega}\eta\rangle \in \mathbb{C} \tag{2.2.2}$$

is a measurable map. [We refer to [62, p. 12, Def. 1.2.1]].

Definition 2.2.1. *Let* $(\Omega,\mathcal{F},\mathbb{P})$ *be a probability space and let* $\{H^\omega\}_{\omega\in\Omega}$ *be a measurable family of self-adjoint operators on a Hilbert space* $\mathcal{H}$ *. The family is called an ergodic family if there is an ergodic family of measure preserving transformations,* $(T_i)_{i\in I}$*, on* $(\Omega,\mathcal{F},\mathbb{P})$ *and a family,* $(U_i)_{i\in I}$*, of unitary operators in* $\mathcal{H}$ *so that the following covariance condition is satisfied*:

$$H^{T_i(\omega)} = U_i^* H^\omega U_i. \tag{2.2.3}$$

We also note the following fact: If H^ω is an ergodic family, then the family of spectral projections $(E_{H^\omega}(B))_{\omega\in\Omega}$, for each $B\in\mathcal{F}$ is also an ergodic family for the same $(T_i)_{i\in I}$ and $(U_i)_{i\in I}$, see [[62, p. 13, Def. 1.2.3] and the remark that follows there.

Ergodic families of self-adjoint operators have invariant spectra, a theorem originally proved by Pastur [54]. The version of the theorem stated here appears in [62, p. 13, Theorem 1.2.5] and a complete proof of the theorem can be found in [14, Proposition V.2.4, p.250].

Theorem 2.2.2. *Let* $(\Omega,\mathcal{F},\mathbb{P})$ *be a complete probability space. Let* $(H^\omega)_{\omega\in\Omega}$ *be a measurable ergodic family of self-adjoint operators on a Hilbert space* $\mathcal{H}$*. Then, there is a closed subset* Σ *of* $\mathbb{R}$ *such that,*

$$\sigma(H^\omega) = \Sigma, \quad a.e.\ \omega. \tag{2.2.4}$$

There are also sets Σ_{ac},Σ_{sc} *and* Σ_p *such that*

$$\Sigma_{ac} = \sigma_{ac}(H^\omega),\ \ \Sigma_{sc} = \sigma_{sc}(H^\omega),\ \ \Sigma_p = \sigma_p(H^\omega),$$

for almost every ω*.*

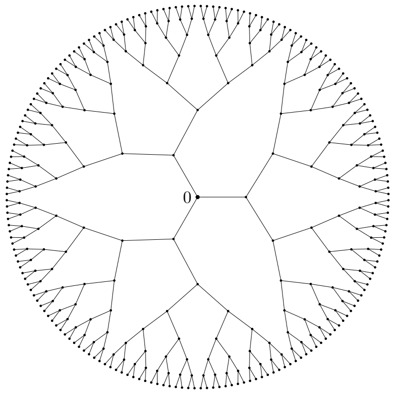

Figure 2.1: Bethe lattice with K=3

2.2.1 Bethe lattice operators

The main operator families we consider are defined on the Bethe lattice and the Canopy tree. In this subsection, we consider the random operator on the Bethe lattice.

Let $\mathcal{B} = (V_{\mathcal{B}}, E_{\mathcal{B}})$ denote the infinite rooted tree with the root $0 \in V_{\mathcal{B}}$, where all the vertices have $K+1$ neighbours (in the figure to the right, K is 2). On the Hilbert space $\ell^2(\mathcal{B})$, we have the graph Laplacian Δ defined by

$$(\Delta\psi)(x) := \sum_{d(x,y)=1} \psi(y), \ \forall x \in V_{\mathcal{B}}, \psi \in \ell^2(\mathcal{B}).$$

Here, $d(x,y)$ is the number of edges in the shortest path between the vertices x and y.

Define the set $J_B \subset V_{\mathcal{B}}$ as,

$$J_B := \{x \in V_{\mathcal{B}} : d(0,x) \in (m_0+1)\mathbb{N} \cup \{0\}\}. \tag{2.2.5}$$

On the Bethe lattice, we define a subtree, $\Lambda_l(x)$ for each $l \in \mathbb{N}$ and any vertex $x \in V_{\mathcal{B}}$ as follows. The vertex set of the subtree $\Lambda_l(x)$ is given by

$$V_{\Lambda_l(x)} := \{y \in V_{\mathcal{B}} : d(x,y) \le l \text{ and } x \prec y\}. \tag{2.2.6}$$

Here, $x \prec y$ means that the vertex y is such that $\mathrm{d}(0,y) = \mathrm{d}(0,x) + \mathrm{d}(x,y)$; i.e., x lies between 0 and y (0, of course, is the special vertex, root, that we have chosen on the Bethe lattice). The edges of the subtree $\Lambda_l(x)$ are inherited from the Bethe lattice; i.e., there is an edge between two vertices y_1 and y_2 of $\Lambda_l(x)$ if and only if there is an edge between y_1 and y_2 as vertices of the Bethe lattice.

Also, for $U \subseteq V_{\mathcal{B}}$ we have the orthogonal projections on $\ell^2(\mathcal{B})$ given by

$$(\chi_U \psi)(x) = \begin{cases} \psi(x), & x \in U, \\ 0, & \text{otherwise}, \end{cases} \qquad \forall \psi \in \ell^2(\mathcal{B})$$

Now, we define a collection of orthogonal projections

$$(P_y \psi)(x) := \chi_{\Lambda_{m_0}(y)}(x) \psi(x), \tag{2.2.7}$$

for $y \in J_B$. Here, m_0 is a fixed positive integer independent of the y's. Note that, $\mathrm{rank}(P_y) = \frac{K^{m_0+1}-1}{K-1}$ which we will denote by M_0. We note that, by the definition, $\{P_y, y \in J_{\mathcal{B}}\}$ forms a partition of the identity on $\ell^2(\mathcal{B})$.

We need a few assumptions to define the random operator on the Bethe lattice.

Assumption 2.2.3. *1. We assume that the random variables $\{\omega_x, x \in \mathcal{J}_{\mathcal{B}}\}$ are i.i.d with a common absolutely continuous distribution μ having density ρ supported in the bounded set $[a,b]$. In addition ρ is continuous in (a,b) and is strictly positive there.*

2. *We consider a collection of finite rank projections $\{P_y,\ y \in \mathcal{J}_{\mathcal{B}}\}$, given in equation 2.2.7, of equal rank M_0. The collection $\{P_y\}$ forms a partition of unity on the Hilbert space $\ell^2(\mathcal{B})$.*

Given $\{(\omega_x, P_x), x \in \mathcal{J}_{\mathcal{B}}\}$ as in the above assumption, the higher rank Anderson operator on the Bethe lattice $\mathcal{B}$ is defined as

$$H_\lambda^\omega := \Delta + \lambda \sum_{y \in J_B} \omega_y P_y, \tag{2.2.8}$$

where $\lambda > 0$ is the disorder parameter.

Now we define the cut-off operators of the full random operator on the Bethe lattice.

$$H_{\lambda,L}^\omega := \chi_{\Lambda_L(0)} H_\lambda^\omega \chi_{\Lambda_L(0)}, \tag{2.2.9}$$

acting on $\ell^2(\Lambda_L(0))$ to itself. These are the matrices that we consider to define the eigenvalue statistics on the Bethe lattice. Henceforth, for convenience, we will denote $\Lambda_L(0)$ by Λ_L. Since we will deal with the subtree Λ_L a lot, we recall its definition, once again, for the convenience of the reader.

Definition 2.2.4. *The subtree Λ_L of the Bethe lattice is defined as follows. The vertex set of the subtree Λ_L is given by*

$$V_{\Lambda_L} := \{y \in V_{\mathcal{B}} : \mathrm{d}(0,y) \le L\}.$$

The edge relations of the subtree Λ_L is just the restriction of the edge relations of the Bethe lattice to the vertex set V_{Λ_L}.

2.2.2 Random Operators on the Canopy Tree

Now, we consider the Anderson model on the Canopy tree. See [5, p. 6].

Definition 2.2.5. *A Canopy tree C of degree $K+1$ is given by the pair $(\mathcal{V},\mathcal{E})$, where the vertex set is $\mathcal{V} = \mathbb{Z} \times (\mathbb{N} \cup \{0\})$ and the edge set is*

$$\mathcal{E} = \left\{ \left\{ (x,n), \left(\left\lfloor \frac{x}{K} \right\rfloor, n+1 \right) \right\} : x \in \mathbb{Z} \}, n \in \mathbb{N} \cup \{0\} \right\}.$$

We will denote the boundary of the tree by the set of vertices

$$\partial C := \{(y,0) : y \in \mathbb{Z}\}.$$

Note that, every vertex on the Canopy tree, except those on the boundary, has degree $K+1$. The vertices on the boundary have degree one. On $\mathcal{V}$, we denote by d the usual metric of the graph, that is, for any two vertices v,w in $\mathcal{V}$, $d(v,w)$ is the length of the shortest path connecting v and w. We will also need a binary relation $\prec$ on $\mathcal{V}$ which is defined by

$$v \prec w \;\Leftrightarrow\; d(v,\partial C) \le d(w,\partial C) \;\&\; d(v,w) = d(w,\partial C) - d(v,\partial C),$$

where $d(v,\partial C)$ is the distance of v from the boundary. Thus, $v \prec w$ means that v lies in the shortest path between w and the boundary ∂C. For $w \in \mathcal{V}$, the forward neighbor set is defined by

$$N_w = \{v \in \mathcal{V} : v \prec w \;\&\; d(v,w) = 1\}.$$

We note that N_w is empty for $w \in \partial C$, but for any other vertex it has cardinality K. Finally, for $w \in \mathcal{V}$ and $l \in \mathbb{N}$, just as in the case of the Bethe lattice, we consider a finite subtree, $\tilde{\Lambda}_l(w)$ of C, whose vertices are

$$V_{\tilde{\Lambda}_l(w)} := \{v \in \mathcal{V} : v \prec w, d(v,w) \le l\}, \tag{2.2.10}$$

and whose edges are obtained by restricting the edges of C to $V_{\tilde{\Lambda}_l(w)}$. Sometimes, we will denote the vertex set of the subtree also by $\tilde{\Lambda}_l(w)$. The context will make it clear which is meant.

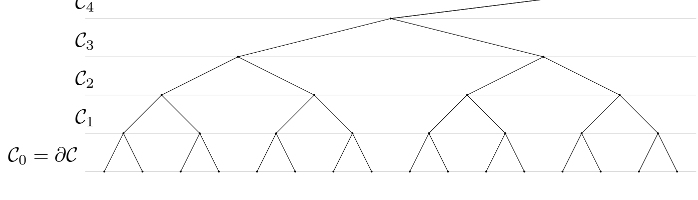

Figure 2.2: First few recursion steps for the Canopy tree for $K=2$.

Let

$$J_C := \{y \in V_C : \mathrm{d}(\partial C, y) = m_0 + (m_0+1)k, \text{ for some } k \in \mathbb{N} \cup \{0\}\},$$

Analogous to the assumptions 2.2.3 on the Bethe lattice, on the Canopy tree, we have the following assumptions.

Assumption 2.2.6. 1. *We assume that the random variables $\{\omega_x, x \in \mathcal{J}_C\}$ are i.i.d with a common absolutely continuous distribution μ having density ρ supported in the bounded set $[a,b]$. In addition ρ is continuous in (a,b) and is strictly positive there.*

2. *We consider a collection of finite rank projections $\{P_y,\ y \in \mathcal{J}_C\}$, given by $P_y := \chi_{\tilde{\Lambda}_{m_0}(y)}$ for $y \in J_C$, of equal rank M_0. The collection $\{P_y\}$ forms a partition of unity on the Hilbert space $\ell^2(C)$.*

We then have the random operator associated with the collection $\{(\omega_x, P_x) : x \in \mathcal{J}_C\}$.

$$H^{\omega}_{C,\lambda} = \Delta_C + \lambda \sum_{y \in J_C} \omega_y P_y, \tag{2.2.11}$$

$\lambda > 0$ is the disorder parameter. Similar to how we defined the cut-off operators for the Bethe lattice operators, we define the cut-offs for the Canopy tree operators below.

Let $\tilde{\Lambda}_L(x_L)$ denote the subtree of the Canopy tree, whose root, which we have denoted by x_L, is the vertex $(0,L)$ of the Canopy tree. Note that, it is at a distance L from the boundary. Therefore, we see that the boundary of the subtree $\tilde{\Lambda}_L(x_L)$ is a subset of the boundary of the Canopy tree. Define

$$H^{\omega}_{C,\lambda,L} := \chi_{\tilde{\Lambda}_L(x_L)} H^{\omega}_{C,\lambda} \chi_{\tilde{\Lambda}_L(x_L)}. \tag{2.2.12}$$

As noted by Aizenman and Warzel [5, p. 3], it is the random operator on the Canopy tree that captures the eigenvalue statistics for the Anderson model on the Bethe lattice. Below, we give the details of this mechanism. For this discussion, we will denote $\tilde{\Lambda}_L(x_L)$ by $\tilde{\Lambda}_L$.

Also, ω will denote the elements of $\mathbb{R}^{V_{\Lambda_L}}$ and $\tilde{\omega}$ will denote the elements of $\mathbb{R}^{V_{\tilde{\Lambda}_L}}$, where V_{Λ_L} is the set of all vertices of the subtree Λ_L and $V_{\tilde{\Lambda}_L}$ is the set of all vertices of the subtree $\tilde{\Lambda}_L$.

The connection between the cut-off operators $H^{\omega}_{\lambda,L}$ and $H^{\tilde{\omega}}_{C,\lambda,L}$

Given any $L \in \mathbb{N}$, the subtree Λ_L of the Bethe lattice can be isometrically embedded into the Canopy tree. In particular, we have a natural isometric isomorphism of the metric graph Λ_L on to the metric subtree $\tilde{\Lambda}_L$ of the Canopy tree. Note that, $|\Lambda_L| = |\tilde{\Lambda}_L|$. The isomorphism is described in the following way.

The root of the subtree Λ_L is mapped to the vertex $(0,L)$ of the Canopy tree. The K vertices adjacent to the root of Λ_L are mapped to the K adjacent vertices of $(0,L)$ in the forward direction (i.e., between the vertex $(0,L)$ and the boundary), so these vertices are in the $(L-1)$th layer, C_{L-1}, of the Canopy tree. The vertices which are at a distance two from the root of Λ_L are mapped to the vertices of $\tilde{\Lambda}_L$, which are at a distance two from $(0,L)$, in the $(L-2)$-th layer of the Canopy tree, in such a way that the adjacency relations are preserved. Proceeding like this, we can map the entire Λ_L on to $\tilde{\Lambda}_L$. It is clear from the construction that this map is a bijection and a graph isomorphism. Since our metric on both the graphs is the usual shortest-distance metric, the distances are preserved under a graph isomorphism. Let us call this isomorphism between Λ_L and $\tilde{\Lambda}_L$, as Φ_L.

The graph-isomorphism Φ_L induces a unitary map from $\ell^2(\Lambda_L)$ to $\ell^2(\tilde{\Lambda}_L)$ by mapping δ_x to $\delta_{\Phi_L(x)}$, where δ_x denotes the unit vector in $\ell^2(\Lambda_L)$ or in $\ell^2(\tilde{\Lambda}_L)$ supported at the vertex x. We denote these unitary operators by U_L.

Intuitively, from the perspective of the chosen root of the Bethe lattice, the subtree Λ_L describes the Bethe lattice as $L \to \infty$; while from the perspective of the vertices near the boundary of Λ_L, this limit describes the Canopy tree.

The Eigenvalue Statistics

To obtain the eigenvalue statistics on the Bethe lattice, we have to study a sequence of random measures $\mu^{\omega,\lambda}_{E_0,L}$, associated with a point E_0 in the spectrum of H^{ω}_{λ}, defined by,

$$\mu^{\omega,\lambda}_{E_0,L}(f) = \mathrm{Tr}(f(|\Lambda_L|(H^{\omega}_{\lambda,L} - E_0))),\ \forall f \in C_c(\mathbb{R}). \tag{2.2.13}$$

We study the distributional limit of the random variables $\mu^{\omega,\lambda}_{E_0,L}(\chi_I)$, for each fixed bounded interval I. To study this limit, it is enough to calculate the limit of the Fourier transform of the associated distributions [See [13, p. 349, Theorem 26.3]], i.e., $\lim_{L\to\infty}\mathbb{E}_{\omega}\left[e^{\iota t \mu^{\omega,\lambda}_{E_0,L}(\chi_I)}\right]$.

On the Canopy side, we have the analogous definition of the random measures,

$$\tilde{\mu}^{\omega,\lambda}_{E_0,L}(f) = \mathrm{Tr}(f(|\Lambda_L|(H^{\omega}_{\lambda,C,L} - E_0))),\ \forall f \in C_c(\mathbb{R}). \tag{2.2.14}$$

We will show, in Theorem 4.1.2,

$$\lim_{L\to\infty} \mathbb{E}_{\omega}\left[e^{\iota t \tilde{\mu}^{\omega,\lambda}_{E_0,L}(\chi_I)}\right] = e^{\sum_{k=1}^{M_0}(e^{\iota t k}-1)p_k(I)}.$$

We claim that studying $\mathbb{E}_{\omega}\left[e^{\iota t \tilde{\mu}^{\omega,\lambda}_{E_0,L}(\chi_I)}\right]$ is the same as studying $\mathbb{E}_{\omega}\left[e^{\iota t \mu^{\omega,\lambda}_{E_0,L}(\chi_I)}\right]$. This is because of the following reasons.

As we already saw, given an ω and an L, we can associate a cut-off operator on the Bethe lattice, namely, $H^{\omega}_{\lambda,L}$. Similarly, we have associated a cut-off operator on the Canopy-tree, $H^{\tilde{\omega}}_{C,\lambda,L}$.

We can define a map from $\mathbb{R}^{V_{\Lambda_L}}$ onto $\mathbb{R}^{V_{\tilde{\Lambda}_L}}$ by sending $\omega = (\omega_x)_{x\in V_{\Lambda_L}}$ to $\tilde{\omega} = (\omega_{\Phi_L(x)})_{\Phi_L(x)\in V_{\tilde{\Lambda}_L}}$. This is just a permutation map (and hence a unitary). Let us call this map, also, Φ_L, for obvious reasons. Then, this is a measure preserving map when each of the spaces are equipped with the same product measure, which is the product of μ, $|\Lambda_L|$ $(=|\tilde{\Lambda}_L|)$ times. We can see that for all ω in $\mathbb{R}^{V_{\Lambda_L}}$, the $\tilde{\omega} \in \mathbb{R}^{V_{\tilde{\Lambda}_L}}$ thus obtained is such that the matrices $H^{\omega}_{\lambda,L}$ and $H^{\tilde{\omega}}_{C,\lambda,L}$ are unitarily equivalent under the natural unitary maps U_L, defined above. Also, the traces are invariant under unitary transformations. Therefore, for the pair ω and $\tilde{\omega}$, we have

$$\mu^{\omega,\lambda}_{E_0,L}(f) = \tilde{\mu}^{\tilde{\omega},\lambda}_{E_0,L}(f), \text{ for all } f \in C_c(\mathbb{R}).$$

This discussion shows that for fixed L and I, the random variables $\mu^{\omega,\lambda}_{E_0,L}(\chi_I)$ and $\tilde{\mu}^{\tilde{\omega},\lambda}_{E_0,L}(\chi_I)$ are identically distributed: For any $n \in \mathbb{N} \cup \{0\}$, the set $\{\omega : \mu^{\omega,\lambda}_{E_0,L}(\chi_I) = n\}$ is taken to the set $\{\tilde{\omega} : \tilde{\mu}^{\tilde{\omega},\lambda}_{E_0,L}(\chi_I) = n\}$, under the measure preserving map Φ_L. Therefore, they have the same measure. Therefore, studying the eigenvalue statistics over the Bethe lattice is the same as studying the statistics over the Canopy tree.

From now onwards, for convenience of notation, we denote $\tilde{\Lambda}_L(x_L)$ by Λ_L. Also, the cut-off operators $H^{\omega}_{C,\lambda,L}$ will be denoted as $H_{\lambda,L}$, henceforth. (Although we have used the notation Λ_L for similar subtrees of the Bethe lattice [2.2.4], it should not cause any confusion. One, from now onwards, unless otherwise stated, Λ_L will be the subtree of the Canopy tree that we defined here, secondly, the context will make it clear which is meant and thirdly, these two subtrees are isomorphic trees, as explained in the previous paragraphs. Similar considerations are behind adopting the notation $H_{\lambda,L}$ (from the Bethe lattice notation), therefore, from now on, $H_{\lambda,L}$ will denote the cut-off operators of the Canopy tree operators, unless otherwise specified.)

2.2.3 Measurability of the random operators on the Bethe lattice and the Canopy tree

Now, we turn to show that the operator families we defined are measurable families of bounded self-adjoint operators. Consider the operator family $H^{\omega}_{\lambda,G} := H_0 + \lambda V^{\omega}$, where G is either the Bethe lattice or the Canopy tree, H_0 is the adjacency matrix on the respective graphs, and V^ω is defined as

$$V^\omega \psi := \sum_{y \in \mathcal{J}} \omega_y P_y \psi,$$

where $\mathcal{J}$ and P_y are as in the definitions of the respective operators [2.2.8 and 2.2.11]. Before showing the measurability, we show that the operators in this family are bounded and self-adjoint; more precisely, for a.e. $\omega \in \Omega$, $H^\omega_{\lambda,G}$ is bounded; and for all $\omega \in \Omega$ these operators are self-adjoint.

Claim: For all ω, The operator $H^\omega_{\lambda,G}$ is bounded.

Proof. First, we show that H_0, which is the graph Laplacian on either the Bethe lattice or the Canopy tree is a bounded operator. Let's denote for $x \in G$, $N(x)$ the set of neighbours of the vertex x. (Here, we have done a small abuse of notation by using the letter G for both the graph and the vertex set of the graph. It can be understood from the context which is meant.) So,

$$(H_0\psi)(x) := \sum_{y \in N(x)} \psi(y).$$

Let y_x be such that $|\psi(x)| \leq |\psi(y_x)|$ for all $y \in N(x)$. This implies,

$$|(H_0\psi)(x)| \leq (K+1)|\psi(y_x)|.$$

We have,

$$\begin{aligned} ||H_0\psi||^2 &= \sum_{x \in G} |(H_0\psi)(x)|^2 \\ &\leq (K+1)^2 \sum_{x \in G} |\psi(y_x)|^2. \end{aligned}$$

Note that the same y_x can be in the neighborhood of at most $K+1$ points in G. Therefore,

$$||H_0\psi||^2 \leq (K+1)^4 ||\psi||^2.$$

This shows that H_0 is bounded.

The assumption on $\{P_y\}$, that it forms a partition of unity and the fact that ω_y is bounded by the assumption on the support of ρ shows that

$$||V^{\omega}\psi|| \leq C||\psi||, \ C = \sup\{|r| : r \in supp(\rho)\}.$$

So H^{ω} is a bounded operator for each $\omega \in \Omega$.

Also, we can directly show that $\forall \psi, \eta \in \ell^2(G), \langle H_0\psi, \eta\rangle = \langle \psi, H_0\eta\rangle$; which shows that H_0 is a symmetric operator. Similarly, considering that P_y are orthogonal projections by definition and therefore symmetric, we can directly show, in this case as well that, $\forall \psi, \eta \in \ell^2(G), \langle V^{\omega}\psi, \eta\rangle = \langle \psi, V^{\omega}\eta\rangle$, $\forall \omega \in \Omega$. Hence, V^{ω} is also symmetric. Now, for almost every ω, $H^{\omega}_{\lambda,G}$ being the sum of two symmetric bounded operators, is symmetric and bounded; and hence self-adjoint. □

Claim: The random operator family $\{H^{\omega}_{\lambda,G}\}$ is a measurable family.

Proof. To show the measurability, we have to show that (since H^{ω} is a family of bounded operators), for all ψ and η in $\mathcal{H}$ (which is $\ell^2(G)$), the map $\Omega \ni \omega \longrightarrow \langle \psi, H^{\omega}\eta\rangle \in \mathbb{C}$ is measurable. Now,

$$\begin{aligned}\langle \psi, H^{\omega}\eta\rangle &= \langle \psi, (H_0 + V^{\omega})\eta\rangle \\ &= \langle \psi, H_0\eta\rangle + \langle \psi, V^{\omega}\eta\rangle.\end{aligned}$$

It is clear that $\sum_y \alpha_y \omega_y$ is measurable for any collection of finitely many complex numbers α_y. Since for any $\psi, \eta \in \ell^2(G)$, the sum

$$\sum_y \langle \psi, P_y \eta\rangle$$

converges absolutely and since

$$\langle \psi, V^\omega \eta \rangle = \sum_y \omega_y \langle \psi, P_y \eta \rangle,$$

the measurability of $\langle \psi, V^\omega \eta \rangle$ is clear from the fact that the limit of measurable functions is measurable. □

2.3 Localisation

In this section, we present a proof of the localisation implied by the fractional moment bounds on the operators we consider. The often-used Simon-Wolff [61] criterion applies, since the fractional moment bounds imply that the matrix $\Im\left(P_x(H^\omega - (\Re z) - i0)^{-1} P_x\right)$ is zero, therefore $P_x(H^\omega - (\Re z) - i0)^{-1} P_x$ is self-adjoint and using its spectral decomposition we can reduce the matrix-valued equation to a numerical equation of the form in Simon-Wolff [61]. However, the proof of the localisation given by Graf [49] using RAGE theorem, in the region of energies where the fractional moment bound holds, goes through, so we present it here. A crucial fact used in the proof of Graf, is that the expectation of ℓ^2-norm of $(\Im z)^{\frac{1}{2}}(H^\omega - z)^{-1} P_x \phi$ is finite in the spectrum, (while the expectation of the ℓ^2-norm of $(H^\omega - z)^{-1} P_x \phi$ is always infinite, there) which he bounds by the expectation of ℓ^s-norm of $(H^\omega - z)^{-1} P_x \phi$ for some $0 < s < 1$ and for any unit vector ϕ in the range of P_x. That this is finite is implied by the fractional moment bounds 2.3.1.

For the following theorem we denote by H^ω_λ, the operators $H^\omega_{\lambda,G}$ with G being the Bethe lattice or the Canopy tree. See 2.2.8, 2.2.3, 2.2.11 and 2.2.6 for the definitions of the respective operators and the assumptions involved about the potentials.

Theorem 2.3.1. *Consider the operators H^ω_λ defined above . Whenever the fractional moment bound*

$$\mathbb{E}\|P_x(H^\omega_\lambda - z)^{-1} P_y\|^s \leq C_s e^{-\gamma|x-y|}, \ \gamma > \ln(K+1), \tag{2.3.1}$$

holds for all $x,y \in C$ and some $0 < s < 1$, uniformly in $(\Re z) \in I$, the operators H^ω have no continuous spectrum in I, for almost every ω.

Proof. In the following, we write $|y-x| = \mathrm{d}(x,y)$ for ease of writing. Let $A^{\omega,\kappa} = H^\omega_\lambda + \kappa P_x$ and correspondingly let for $\Im z > 0$,

$$G_{\omega,\kappa}(x,y,z) = P_x(A^{\omega,\kappa} - z)^{-1} P_y$$
(2.3.2)
$$G_\omega(x,y,z) = P_x(H^\omega_\lambda - z)^{-1} P_y,$$

for any x,y. Using the resolvent identity, we have

(2.3.3)
$$G_{\omega,\kappa}(x,y,z) = \left(\kappa + G_\omega(x,x;z)^{-1}\right)^{-1}.$$

Since $\{P_y\}$ is a collection of projections which forms a partition of identity, taking an orthonormal basis $\{\phi_j\}$ in the range of P_x, recalling that M_0 is the dimension of the range of P_x and bounding the norm of a matrix by its trace norm, we have for any R,

(2.3.4)
$$\begin{aligned}
(\Im z) \sum_{|y-x| \geq R} \|G_{\omega,\kappa}(x,y,z)\|^2 &= (\Im z) \sum_{|y-x| \geq 0} \|G_{\omega,\kappa}(x,y,z)^* G_{\omega,\kappa}(x,y,z)\| \\
&\leq (\Im z) \sum_{j=1}^{M_0} \sum_{|y-x| \geq 0} \langle \phi_j, G_{\omega,\kappa}(x,y,z)^* G_{\omega,\kappa}(x,y,z) \phi_j \rangle \\
&= (\Im z) \sum_{j=1}^{M_0} \langle \phi_j, P_x (A^{\omega,\kappa} - \overline{z})^{-1} (A^{\omega,\kappa} - z)^{-1} P_x \phi_j \rangle \\
&= \frac{1}{2} \sum_{j=1}^{M_0} \langle \phi_j, P_x (A^{\omega,\kappa} - z)^{-1} P_x \phi_j \rangle - \langle \phi_j, P_x (A^{\omega,\kappa} - \overline{z})^{-1} P_x \phi_j \rangle \\
&\leq M_0 \|P_x (A^{\omega,\kappa} - z)^{-1} P_x\| \leq M_0 \left\| \left(\kappa + G_\omega(x,x,z)^{-1}\right)^{-1} \right\|,
\end{aligned}$$

using the equation (2.3.3) and the fact that $\|B\| = \|B^*\|$ for any matrix B. We also get the following inequality, since $(\Im z)\|G_{\omega,\kappa}(x,y,z)\| \leq 1$,

(2.3.5)
$$|\Im(z)| \sum_{|y-x| \geq R} \|G_{\omega,\kappa}(x,y,z)\|^2 \leq \sum_{|y-x| \geq R} \|G_\omega(x,y,z)\|,$$

where the right hand side is a finite quantity for any z with $\Im z>0$ by using Combes-Thomas bound (for example [40, Theorem 11.2]). Therefore, interpolating between these two inequalities we get, for any $0<s<1$,

$$\begin{aligned}(2.3.6)\qquad &(\Im z)\sum_{|y-x|\geq R}\|G_{\omega,\kappa}(x,y;z)\|^2\\ &\leq M_0^{1-s}\|\left(\kappa+G_\omega(x,x,z)^{-1}\right)^{-1}\|^{1-s}\left(\sum_{|y-x|\geq R}\|G_\omega(x,y,z)\|\right)^s\\ &\leq M_0^{1-s}\|\left(\kappa+G_\omega(x,x,z)^{-1}\right)^{-1}\|^{1-s}\sum_{|y-x|\geq R}\|G_\omega(x,y,z)\|^s,\end{aligned}$$

since $(\sum a_i)^s\leq\sum a_i^s$, for positive a_is; also, for $\Im z>0$, the sums in the above inequality converge for any $s>0$ by Combes-Thomas bound again. Again bounding the norm of a matrix by its trace norm and taking $\{\lambda_i\}$ to be the eigenvalues of $G_\omega(x,x,z)^{-1}$, we have

$$\begin{aligned}(2.3.7)\qquad \|\left(\kappa+G_\omega(x,x,z)^{-1}\right)^{-1}\|^{1-s}&\leq M_0\max_j|\kappa+\lambda_j|^{-1+s}\\ &\leq M_0|\kappa+\lambda|^{-1+s},\end{aligned}$$

for some $\lambda\in\{\lambda_1,\ldots,\lambda_{M_0}\}$. Using this bound and integrating κ over a unit interval, we get

$$\begin{aligned}(2.3.8)\qquad &\int_{-\frac{1}{2}}^{\frac{1}{2}}d\kappa(\Im z)\sum_{|y-x|\geq R}\|G_{\omega,\kappa}(x,y;z)\|^2\\ &\leq M_0^{2-s}\left(\int_1^2 d\kappa|\kappa+\lambda|^{-1+s}\right)\sum_{|y-x|\geq R}\|G_\omega(x,y,z)\|^s\\ &\leq C_sM_0^{2-s}\sum_{|y-x|\geq R}\|G_\omega(x,y,z)\|^s,\end{aligned}$$

with C_s independent of λ and hence z, as the κ integral is independent of λ by our assumption on s (that it is positive and smaller than one). Then the inequality (2.3.8) and the fractional moment bound (2.3.1) valid for any $E\in I$, together imply using the Lemma

2.3.2 that for any unit vector $\phi \in Ran(P_x)$,

$$
\begin{aligned}
(2.3.9) \qquad & \int_{-\frac{1}{2}}^{\frac{1}{2}} d\kappa \ \mathbb{E}\|E_{A^{\omega,\kappa},c}(I)\phi\|^2 \\
& \leq \varlimsup_{R\to\infty} \varliminf_{\Im z \downarrow 0} \frac{\Im z}{\pi} \int_I dE \sum_{|y-x|>R} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\kappa \ \mathbb{E}\left(\|G_{\omega,\kappa}(x,y;z)\|^2 \right) \\
& \leq \frac{C|I|}{\pi} \varlimsup_{R\to\infty} \sum_{|x-y|>R} e^{-\gamma|x-y|} = 0.
\end{aligned}
$$

This inequality is valid for any unit vector $\phi \in Ran(P_x)$ and any x. Therefore, the continuous spectrum of $A^{\omega,\kappa}$ is empty for almost every (ω,κ) in the interval I. If we take κ in a small enough interval, then the range of $\omega_x + \kappa$ as ω_x varies in the support of ρ contains the support of ρ; therefore, complete localisation for $H_\lambda^\omega + \kappa P_x$ almost every (ω,κ), in particular, implies complete localisation for H_λ^ω for almost every ω, completing the proof. □

We give the lemma used in the last part of the above theorem which is quite general and uses the RAGE theorem [21]. Let $\mathcal{H}$ be a separable Hilbert space and A any self-adjoint operator on it. Let $\{\psi_n, n \in \mathbb{N}\}$ be an orthonormal basis for $\mathcal{H}$. Denote by $E_{c,A}$ and $E_{p,A}$ the orthogonal projections onto the continuous and point spectral subspaces of A, respectively. For $N \in \mathbb{N}$, let P_N denote the orthogonal projection onto the subspace generated by $\{\psi_n, n \geq N\}$ and $P_N^\perp$ its orthogonal complement. Note that for any finite N, $P_N^\perp$ is a finite rank and hence a compact operator.

Lemma 2.3.2. *Consider $\mathcal{H}$ and A as above with A a bounded self-adjoint operator, its spectrum contained in the interval I. Then for any $f \in \mathcal{H}$, with $\|f\| = 1$,*

$$
(2.3.10) \qquad \|E_{c,A} f\|^2 = \varliminf_{N\to\infty} \varliminf_{\epsilon\to 0} \frac{\epsilon}{\pi} \int_I \|P_N (A - E - i\epsilon)^{-1} f\|^2 \, dE.
$$

Proof: We start by showing

$$
(2.3.11) \qquad \|E_{c,A} f\|^2 = \varliminf_{N\to\infty} \lim_{T\to\infty} \frac{1}{T} \int_0^T \|P_N e^{-iAt} E_{c,A} f\|^2 \, dt, \ \forall f \in \mathcal{H}.
$$

The relation follows since $P_N = I - P_N^\perp$, so replacing P_N with this sum, we get

$$(2.3.12) \qquad \|E_{c,A}f\|^2 = \varliminf_{N\to\infty} \lim_{T\to\infty}\frac{1}{T}\int_0^T \left(\|P_N e^{-iAt}f\|^2 + \|P_N^\perp e^{-iAt}f\|^2\right) dt, \ \forall f \in \mathcal{H}.$$

The additional term

$$(2.3.13) \qquad \varliminf_{N\to\infty} \lim_{T\to\infty}\frac{1}{T}\int_0^T \|P_N^\perp e^{-iAt}f\|^2 \, dt, \ \forall f \in \mathcal{H}$$

is zero by RAGE Theorem, [21, Theorem 5.8(a)] since $P_N^\perp$ is compact, thus showing the relation (2.3.11).

The above proof also shows that the limits

$$(2.3.14) \qquad \lim_{T\to\infty}\frac{1}{T}\int_0^T \|P_N e^{-iAt}E_{c,A}f\|^2 \, dt$$

exist.

Now for any bounded continuous function g on $\mathbb{R}$, we have the relations

$$(2.3.15) \qquad \lim_{T\to\infty}\frac{1}{T}\int_0^T g(t)dt = \lim_{s\to 0} 2s\int_0^\infty e^{-2st}g(t)dt$$

by [32, Theorem 2]. Plugging in $g(t) = \|P_N e^{-iAt}E_{c,A}f\|^2$ in the above equation, we get

$$(2.3.16) \qquad \|E_{c,A}f\|^2 = \varliminf_{N\to\infty} \lim_{\epsilon\to 0} 2\epsilon\int_0^\infty e^{-2\epsilon t}\|P_N e^{-iAt}E_{c,A}f\|^2 \, dt.$$

We now note that if f is an eigenvector of A, then $\|P_N e^{-iAt}f\| = \|P_N f\|$, so

$$(2.3.17) \qquad \lim_{N\to\infty}\lim_{\epsilon\to 0} 2\epsilon\int_0^\infty e^{-2\epsilon t}\|P_N e^{-iAt}f\|dt = \lim_{N\to\infty}\lim_{\epsilon\to 0}\|P_N f\|^2 = 0,$$

since P_N goes to zero strongly. Therefore using the fact that finite linear combinations of eigenvectors of A are dense in the subspace $E_{p,A}\mathcal{H}$ and the triangle inequality, we see that

$$\underline{\lim}_{N\to\infty} \lim_{\epsilon\to 0} \|E_{c,A}f\|^2 = 2\epsilon \int_0^\infty e^{-2\epsilon t}\|P_N e^{-iAt}E_{c,A}f\|^2\, dt. \tag{2.3.18}$$

$$= \underline{\lim}_{N\to\infty} \lim_{\epsilon\to 0} 2\epsilon \int_0^\infty e^{-2\epsilon t}\|P_N e^{-iAt} f\|^2\, dt. \tag{2.3.19}$$

Now writing e^{-itA} in the spectral representation of A, an application of the Parseval identity and noting that the integral over the energy E outside the interval I converges to zero, we get the required relation (2.3.10) of the Lemma, as in the proof of Graf's Theorem [49]. □

2.4 Wegner and Minami Estimate

In this section, we give two very important estimates regarding the expected number of eigenvalues of the cut-offs of the random operator, namely, the Wegner estimate and the Minami estimate. These estimates depend on a more general fact of averaging the perturbed self-adjoint operators over the perturbation known as the spectral averaging. We recall, first, the spectral averaging result from Krishna-Stollmann [45]. We denote by E_D the spectral measure of the self-adjoint operator D as usual.

Theorem 2.4.1. *Let A be a self-adjoint operator and B a bounded non-negative operator. Then we have for any unit vector* $\phi \in Ran(B)$,

$$\int_0^1 dt\ \langle \phi, E_{A+tB}(I)\phi\rangle \leq C|I|,$$

with the constant independent of A,B*, where Ran(B) is the range of the operator B.*

Proof. The proof is the direct application of the Corollary [45, Corollary 2.5] taking $h(t) = \chi_{[0,1]}(t)$ there. The independence of the constant can be seen from the explicit bound given in [35, Item 3.2]. □

The Wegner estimate was originally proved by Wegner [64] and found wide use in the theory of random operators. See for example [15, 18, 19]. In the case of projection valued perturbations, the needed spectral averaging is done using matrix-valued Herglotz functions. In the following, the matrices $H^{\omega}_{\Lambda,L}$ are defined on the Bethe lattice and $H^{\omega}_{\tilde{\Lambda},L}$ on the Canopy tree. In the following theorem, the inequality (2.4.3) is traditionally referred to as the Wegner estimate, but we give other variants, inequalities (2.4.1, 2.4.2), which we need in its proof.

Lemma 2.4.2. (Wegner Estimate) *For any bounded interval $I \subset \mathbb{R}$, we have for any unit vector ϕ such that $P_p\phi = \phi$, (the projections P_p are defined in 2.2.3),*

$$\mathbb{E}_{\omega}\left[\left\langle \phi, E_{H^{\omega}_{\lambda,L}}(I)\phi \right\rangle\right] \leq C|I|, \tag{2.4.1}$$

$$\mathbb{E}_{\omega}[\mathrm{Tr}(P_y E_{H^{\omega}_{\lambda,L}}(I))P_y] \leq C|I|M_0, \tag{2.4.2}$$

$$\mathbb{E}_{\omega}[\mathrm{Tr}(E_{H^{\omega}_{\lambda,L}}(I))] \leq C|I||\Lambda_L|. \tag{2.4.3}$$

The same bounds hold if we replace $H^{\omega}_{\Lambda,L}$ by $H^{\omega}_{\tilde{\Lambda},L}$.

Proof. The first statement of the Lemma is a direct application of the spectral averaging theorem 2.4.1. For the second and the third inequalities we take an orthonormal basis $\{\phi_j\}$ for $\ell^2(\Lambda_L)$ which is subordinate to $\{P_y,\ y \in \Lambda_L\}$, (i.e. for each ϕ_j there is a $P_y,\ y \in \Lambda_L$ such that $P_y\phi_j = \phi_j$) and show the inequality (2.4.3), the proof of the inequality (2.4.2) is similar. We have with this choice of the basis,

$$\begin{aligned}
&\mathbb{E}_{\omega}[\mathrm{Tr}(E_{H^{\omega}_{\lambda,L}}(I))] \\
&= \sum_{j=1}^{|\Lambda_L|} \mathbb{E}_{\omega}\left[\left\langle \phi_j, E_{H^{\omega}_{\lambda,L}}(I)\phi_j \right\rangle\right] \\
&= \sum_{j=1}^{|\Lambda_L|} \mathbb{E}_{\omega_y^{\perp}}\left[\int \left\langle \phi_j, E_{H^{\omega_y^{\perp}}_{\lambda,L} + \omega_y P_y}(I)\phi_j \right\rangle\ \rho(\omega_y) d\omega_y\right] \\
&\leq \|\rho\|_{\infty} \sum_{j=1}^{|\Lambda_L|} \mathbb{E}_{\omega_y^{\perp}}\left[\int \left\langle \phi_j, E_{H^{\omega_y^{\perp}}_{\lambda,L} + \omega_y P_y}(I)\phi_j \right\rangle\ d\omega_y\right] \\
&\leq C\|\rho\|_{\infty}|\Lambda_L||\ I\ |.
\end{aligned} \tag{2.4.4}$$

In the third line of the above inequality we choose for each j, the y such that $P_y\phi_j = \phi_j$ and use the spectral averaging theorem 2.4.1 for the last inequality. □

Minami showed Poisson statistics in the Anderson model at large disorder for almost all points in the spectrum where the density of states is positive in [51]. He uses an estimate, now called by his name, which essentially shows that the limiting process has the same mean and variance, a characteristic of the Poisson random variables and processes. A proof, different from that given by Minami, of this estimate, has been given by Combes-Germinet-Klein [17]. These proofs of the Minami estimate require the independent random variables constituting the potential to be of rank one (where we think of the term $\omega_x P_x$ as a projection valued random variable). In the case of rank higher than one, Hislop-Krishna [34] generalised the Minami estimate.

Since we have operators where the potential consists of higher-rank independent random variables, we will not be able to show a Minami estimate for our model, but we can show the generalised Minami estimate as in [34, p. 6].

Lemma 2.4.3. (Extended Minami Estimate) *For any bounded interval $I \subset \mathbb{R}$, we have,*

$$\sum_{m>M_0} \mathbb{P}[\mathrm{Tr}(E_{H^{\omega}_{\lambda,L}}(I)) > m] \le (\pi \|\rho\|_\infty |\Lambda_L||I|)^2, \tag{2.4.5}$$

where M_0 is the common rank of the perturbing projections.

Proof. Since a perturbation of rank M_0 can change at most M_0 eigenvalues of any self-adjoint matrix in a fixed interval, we have for any b,

$$|\mathrm{Tr}(E_{H^{\omega}_{\lambda,L}}(I)) - \mathrm{Tr}(E_{H^{\omega}_{\lambda,L}+bP_y}(I))| \le M_0.$$

Therefore we get,

$$|\mathrm{Tr}(E_{H^{\omega}_{\lambda,L}}(I)) - M_0| \le \mathrm{Tr}(E_{H^{\omega_y^\perp}_{\lambda,L}+bP_y}(I)),$$

and in particular when $tr(E_{H^\omega_{\lambda,L}}(I)) \geq M_0$, writing $\omega = (\omega_y^\perp, \omega_y)$ for each y and any b,

$$\mathrm{Tr}(E_{H^\omega_{\lambda,L}}(I)) - M_0 \leq \int_0^1 \mathrm{Tr}(E_{H^{\omega_y^\perp}_{\lambda,L}+bP_y}(I))db.$$

So, we can use the above and get,

$$\begin{aligned}
&\sum_{m>M_0} \mathbb{P}[\mathrm{Tr}(E_{H^\omega_{\lambda,L}}(I)) > m] \\
&= \sum_{m>M_0} \int_0^1 db \;\; \mathbb{P}[\mathrm{Tr}(E_{H^\omega_{\lambda,L}}(I)) > m] \\
&\leq \int_0^1 db \;\; \mathbb{E}_\omega\Big[\mathrm{Tr}(E_{H^\omega_{\lambda,L}}(I))(\mathrm{Tr}(E_{H^\omega_{\lambda,L}}(I)) - M_0)\chi(\mathrm{Tr}(E_{H^\omega_{\lambda,L}}(I)) > M_0)\Big] \\
&= \int_0^1 db \;\; \mathbb{E}_\omega\left[\sum_{j=1}^{|\Lambda_L|}\Big\langle \phi_j, E_{H^\omega_{\lambda,L}}(I)\phi_j\Big\rangle(\mathrm{Tr}(E_{H^\omega_{\lambda,L}}(I)) - M_0)\chi(\mathrm{Tr}(E_{H^\omega_{\lambda,L}}(I)) > M_0)\right] \\
&= \int_0^1 db \;\; \mathbb{E}_\omega\left[\sum_{j=1}^{|\Lambda_L|}\Big\langle \phi_j, E_{H^\omega_{\lambda,L}}(I)\phi_j\Big\rangle(\mathrm{Tr}(E_{H^{\omega_y^\perp}_{\lambda,L}+bP_y}(I))\right] \\
&\leq \int_0^1 db \;\; \sum_{j=1}^{|\Lambda_L|}\mathbb{E}_{\omega_y^\perp}\left[(\mathrm{Tr}(E_{H^{\omega_y^\perp}_{\lambda,L}+bP_y}(I))\int\Big\langle \phi_j, E_{H^\omega_{\lambda,L}}(I)\phi_j\Big\rangle\, \rho(\omega_y)\; dy\right] \\
&\leq C\|\rho\|_\infty \; |I| \sum_j^{\Lambda_L} \int_0^1 db\, \mathbb{E}_{\omega_y^\perp}\left[(\mathrm{Tr}(E_{H^{\omega_y^\perp}_{\lambda,L}+bP_y}(I))\right].
\end{aligned}$$

The last integral is of the form $\mathbb{E}_{\tilde{\omega}}\mathrm{Tr}(H^{\tilde{\omega}}_{\lambda,L}(I))$ for each y, with ω_y replaced by b and $\rho(\omega_y)$ replaced by $\chi_{[0,1]}(b)$. Therefore for each j the summand has an upper bound of $C\|\rho\|_\infty \; \||\Lambda_L|$ by the Wegner estimate, inequality (2.4.3). Therefore we get an overall bound of $C(\|\rho\|_\infty \; |\Lambda_L||I|)^2$.

$\square$

Chapter 3

Multiplicity of the Canopy Hamiltonian

In this short chapter, we discuss the multiplicity of the Anderson tight binding model Hamiltonian over the Canopy tree. We show that this Hamiltonian has eigenvalues of multiplicity at least $K-1$. Depending on the size of the perturbing projections this multiplicity can be made arbitrarily high. Not only that, we can find intervals I with $I \cap \sigma_{pp}(H^{\omega}_{C,\lambda}) \neq \emptyset$ such that this collection of eigenvalues which have multiplicity at least $K-1$ is dense in $I \cap \sigma_{pp}(H^{\omega}_{C,\lambda})$. This statement is very important in showing that the eigenvalue statistics of this model is a non-trivial compound Poisson process. Without this, we could only show that the statistics is possibly a compound Poisson process without ruling out the possibility that it could be a pure Poisson process.

Finding out the multiplicity of operators (let alone random operators) is generally difficult. To the best of our knowledge, there are only a few Anderson type models where we know the multiplicity of the random Hamiltonian. This makes the Canopy model a good place for further studies in this regard.

The main theorem of this chapter, Theorem 3.0.1, is from the joint paper with Anish Mallick [8]. In the paper [5], Aizenman and Warzel have shown that the spectrum of the adjacency operator of the Canopy tree consists of degenerate eigenvalues (i.e., eigenvalues of multiplicity more than one). But, this does not carry over to the multiplicity of the

Hamiltonian on the Canopy tree, when the perturbing projections are rank-one. Since we have a Minami estimate for the Anderson model over the Canopy tree (in the rank-one case), we know that the operators should have non-degenerate eigenvalues for almost every ω.

By a symmetry of a graph, we mean an automorphism of the graph. By a symmetry of an operator, we mean a unitary which fixes the operator under conjugation. A symmetry of a graph induces a unitary on the Hilbert space over the graph, as explained below. Some of these unitaries can turn out to be symmetries for the operator on the graph. It is the symmetries of the Canopy tree that cause the degeneracy of the eigenvalues of the adjacency operator on the Canopy tree. (In general, a symmetry of an operator need not imply the degeneracy for the eigenvalues of the operator; but, in the case of the Canopy tree, there are symmetries of the operator that cause the degeneracy of the eigenvalues.)

For the operator obtained by perturbing with higher-rank projections, not all the symmetries are lost; some of the unitaries induced by the automorphisms of the Canopy tree are, still, symmetries of this perturbed operator. This causes the operators to have eigenvalues of multiplicity (more than one).

In general, about the multiplicity, we can observe the following. Let U be a symmetry for an operator A on a Hilbert space. If E is an eigenvalue of A with the corresponding eigenfunction ψ, then $U\psi$ is an eigenfunction for UAU^* for the same λ. So, if $UAU^* = A$, then E is an eigenvalue of A with multiplicity at least two, provided $U\psi$ and ψ are independent.

The proof of the multiplicity of the Canopy Hamiltonian depends upon two properties of the Canopy graph. First, for each $l \in \mathbb{N}$, let Λ_l be the subtree defined in 2.2.10; then, there is an automorphism Γ_l on the Canopy tree whose restriction to the subtree Λ_l is also an automorphism of Λ_l and Γ_l fixes the vertices outside of the Λ_l. Similarly, if we have an automorphism of Λ_l, then we can extend that to an automorphism of the whole Canopy tree, trivially, by defining its action outside Λ_l as identity. This has the consequence that the Canopy tree has a lot of symmetries. Now, more importantly, we have the second observation, see [5, A2,p. 33], that there are eigenfunctions for the adjacency operator of the Canopy tree which are supported only on these Λ_l's. These two observations will result in the eigenvalues of higher multiplicity for the Canopy operators, as can be seen in the proof of theorem 3.0.1.

Let ψ_l be an eigenfunction of the Canopy adjacency operator whose support is Λ_l. Let E be the corresponding eigenvalue. Now, if we perturb the adjacency operator of the Canopy tree with a projection whose support is all of Λ_l, i.e., if we consider the operator $\Delta+\omega_x P_x$ where Δ is the adjacency operator of the Canopy tree and x is the root of the subtree Λ_l and the support of P_x is Λ_l, then it is easy to see that ψ_l is an eigenfunction for this operator $\Delta+\omega_x P_x$ corresponding to the eigenvalue $E+\omega_x$. Then, the first observation would imply that $E+\omega_x$ is an eigenvalue of multiplicity more than one, provided $\Gamma_l\psi_l$ and ψ_l are independent. Indeed, in such a case, $E+\omega_x$ will be an eigenvalue of multiplicity greater than one for the full perturbed operator on the Canopy tree; not just for $\Delta+\omega_x P_x$.

In short, if we have an eigenvalue E of the Canopy adjacency operator with an eigenfunction ψ_l whose support is contained in Λ_l (= the support of P_x) and with the property that $\Gamma_l\psi_l$ and ψ_l are independent, then we have $E+\omega_x$ as an eigenvalue of the canopy Hamiltonian, with multiplicity greater than one.

The essential idea in the proof of our main theorem in this chapter, Theorem 3.0.1, is that if P_x is such that the subtree $\Lambda_l(x)$ where P_x is supported shares its boundary with that of the whole Canopy tree, then any $E\in\sigma(\Delta_{l-1})$ can serve as such an eigenvalue. This is, essentially, the main argument in the proof of the second observation, above, in [5].

In particular we can show that $E=0$ is an eigenvalue of multiplicity at least $K-1$, for the adjacency matrix $\Delta_{m_0}, m_0\geq 1$ using the same arguments. This observation is used in the proof of the subsequent theorem, **??**.

Now, if we consider the Hamiltonians on the Canopy tree (that we have defined in 2.2.11), i.e., the random operator

$$H^{\omega}_{C,\lambda}=\Delta_C+\lambda\sum_{y\in J_C}\omega_y P_y, \tag{3.0.1}$$

the above observations allow us to conclude that each of the ω_y, where $y \in J_C$ is such that y's are vertices of the Canopy tree at a distance m_0 from the boundary (we have defined m_0 such that it is the "depth" of the subtrees which are the support of the projections P_y's), is an eigenvalue of multiplicity more than one (in fact, of multiplicity at least $K-1$, where the K is the connectivity of the Canopy graph).

Now, we recall some notations from the previous chapter. We will denote the boundary of the tree by

$$\partial C = \{(y,0) : y \in \mathbb{Z}\},$$

and for any $i \in \mathbb{N} \cup \{0\}$, the set of vertices which are i distance away from the boundary by

$$C_i = \{(y,i) : y \in \mathbb{Z}\}.$$

Note that C_0 is then the boundary of the Canopy tree. On $\mathcal{V}$, we denote by $d(\cdot,\cdot)$ the usual metric of the graph. That is, for any two vertices v,w in $\mathcal{V}$, $d(v,w)$ is the length of the shortest path connecting v and w. We will also need a binary relation $\prec$ on $\mathcal{V}$ which is defined by

$$v \prec w \iff d(v,\partial C) \leq d(w,\partial C) \;\&\; d(v,w) = d(w,\partial C) - d(v,\partial C),$$

where $d(v,\partial C)$ is the distance of v from the boundary. Thus, $v \prec w$ means that v lies in the shortest path between w and the boundary ∂C. For $w \in \mathcal{V}$, the forward neighbour set is defined by

$$N_w = \{v \in \mathcal{V} : v \prec w \;\&\; d(v,w) = 1\}. \tag{3.0.2}$$

Note that N_w is empty for $w \in \partial C$, but for any other vertex it has cardinality K.

Finally, for $w \in \mathcal{V}$ and $L \in \mathbb{N}$, recall that for any vertex w, the subtree Λ_L is

$$\Lambda_L(w) := \{v \in \mathcal{V} : v \prec w, d(v,w) \leq L\},$$

where the edges are obtained by restricting the edges of C to $\Lambda_L(w)$. From now on, we will exclusively consider the eigenvalue statistics of the Canopy tree, although we want to study the eigenvalue statistics over the Bethe lattice. As explained before [See 2.2.2], this suffices. The random operator of interest on the Hilbert space $\ell^2(C)$ and the assumptions involved are defined in 2.2.11 and in 2.2.6 in the previous chapter. We will denote Δ_n to be the adjacency matrix for the tree $\Lambda_n(x)$, for $x \in \mathcal{V}_n$, for $n \in \mathbb{N}$ (since all of these trees are isomorphic, we do not need to specify the root other than the distance from the boundary).

The following theorem shows that the operator $H^\omega_{C,\lambda}$ have non-trivial multiplicity over certain parts of the spectrum. This can be viewed as extending the result of Theorem 1.6 of Aizenman-Warzel [5] to Anderson type operator. Instead of infinite degeneracy, as in the case of [6, Theorem 1.6], we only get finite degeneracy due to the presence of randomness.

Theorem 3.0.1. *For $K > 2$, let C denote the Canopy tree of degree $K+1$ and on the Hilbert space $\ell^2(C)$, define the random operator H^ω_C by* (2.2.11)*, for some $m_0 \geq 2$. Set the random variables $\{\omega_x\}_{x\in\mathcal{N}}$ to be independent and identically distributed following an absolutely continuous distribution μ. Then*

$$\sigma(\Delta_{m_0-1}) + \mathrm{Supp}(\mu) \subset \sigma_{pp}(H^\omega_C) \qquad a.s,$$

and the maximum multiplicity of point spectrum in $\sigma(\Delta_{m_0-1}) + \mathrm{Supp}(\mu)$ is at least $K-1$.

Proof. Note that all the $\Lambda_{m_0-1}(x)$ are identical for any $x \in C_{m_0-1}$. Let $\mathcal{T}_{m_0-1}$ denote a tree with root e (for some vetex e), which is isomorphic to the tree $\Lambda_{m_0-1}(x)$, for $x \in C_{m_0-1}$. We will denote by ϕ_x the isomorphism

$$\phi_x : \Lambda_{m_0-1}(x) \to \mathcal{T}_{m_0-1}.$$

We will view Δ_{m_0-1} as the adjacency matrix for the graph $\mathcal{T}_{m_0-1}$. Finally, for $E \in \sigma(\Delta_{m_0-1})$ consider a normalized eigenvector ψ corresponding to the eigenvalue E.

Claim: *For any* $x \in C_{m_0} \subset \mathcal{N}$, $E + \omega_x$ *is an eigenvalue of the operator* H_C^ω *with multiplicity at least* $K-1$.

To show this, we are going to define the $K-1$ orthonormal eigenvectors for $E+\omega_x$. Let N_x be the set of neighbours of x, as defined in 3.0.2. Let $\alpha := (\alpha_y), y \in N_x$ be an element in $\mathbb{R}^{n-1}$, satisfying the following conditions

(3.0.3)
$$\sum_y \alpha_y = 0 \ \ \& \ \ \sum_y |\alpha_y|^2 = 1.$$

For each such α, define the vector $\Psi^{(\alpha)} \in \ell^2(C)$ by

$$\Psi^{(\alpha)}(p) = \begin{cases} \alpha_y \psi(\phi_y(p)), & \text{if } p \prec y \text{ for some } y \in N_x \\ 0, & \text{if } p \notin \cup_{y \in N_x} \Lambda_{m_0-1}(y) \end{cases} \qquad \forall p \in C,$$

Observe that $\Psi^{(\alpha)}$ satisfies

$$[(H_C^\omega - (E+\omega_x))\Psi^{(\alpha)}](p) = 0 \qquad \forall p \in \mathcal{V} \setminus \Lambda_{m_0}(x)$$

trivially, because all the entries that show up are defined to be zero. For any $p \in \Lambda_{m_0-1}(y)$ where $y \in N_x$, we have

$$[(H_C^\omega - (E+\omega_x))\Psi^{(\alpha)}](p) = \alpha_y [\Delta_{C_{m_0}} \psi](\phi_y(p)) - E\psi(\phi_y(p)) = 0.$$

Here we are using the fact that $\Psi^{(\alpha)}(x) = 0$, hence $[\Delta_C \Psi^{(\alpha)}](p) = [\Delta_{C_{m_0}} \psi](\phi_y(p))$. Finally, at x we have

$$\begin{aligned} &[(H_C^\omega - (E+\omega_x))\Psi^{(\alpha)}](x) \\ &\quad = \sum_{y \in N_x} \Psi^{(\alpha)}(y) = \psi(e) \sum_{y \in N_x} \alpha_y = 0 \end{aligned}$$

by definition of (α_y). We also have, for any $(\alpha_y)_y$ and $(\beta_y)_y$ that satisfies (3.0.3),

$$\left\langle \Psi^{(\alpha)}, \Psi^{(\beta)} \right\rangle_{\ell^2(C)} = \sum_{y \in N_x} \alpha_y \beta_y.$$

Hence there are $K-1$ orthonormal vectors $\Psi^{(\alpha)}$ which are eigenvectors for H_C^ω for the eigenvalue $E+\omega_x$.

Now using the fact that $\{\omega_x\}_{x \in \mathcal{V}_{m_0}}$ are i.i.d, we have

$$\overline{\{E+\omega_x : x \in \mathcal{V}_{m_0}\}} = E + \mathrm{Supp}(\mu) \qquad a.s,$$

which completes the proof of the theorem by using the above claim. □

Remark 3.0.2. *In particular, using the same arguments, we can see that zero is an eigenvalue of multiplicity at least $K-1$ for the adjacency operators Δ_{m_0}, $m_0 \geq 2$. To see this, observe that for $m_0 = 1$, the adjacency operator Δ_{m_0} has zero as an eigenvalue, by the simple direct calculation. This implies, following the arguments of the proof, that zero is an eigenvalue of multiplicity at least $K-1$ for the adjacency operator Δ_{m_0} for $m_0 \geq 2$.*

Remark 3.0.3. *The above remark, together with Theorem 3.0.1, implies that* Supp(μ) *is contained in the part of the pure point spectrum of the canopy Hamiltonian where the multiplicity is at least $K-1$. This observation is important in the proof of Theorem 3.0.4 where we need to consider degenerate points in the spectrum such that ρ is strictly positive there.*

The important matter, here, is that $E+\omega_x$ is an eigenvalue of multiplicity at least $K-1$ of the Canopy Hamiltonian, for all ω in the canonical probability space over the Canopy tree, vertices x on the boundary of the Canopy tree and eigenvalues E of the graph Laplacians over the finite subtrees Λ_{m_0-1}. (m_0 is already fixed by the definition of the Canopy random Hamiltonian by way of $\frac{K^{m_0+1}-1}{K-1}$ being the rank of the perturbing projections.) This has the consequence that there are parts of the spectrum of the Canopy Hamiltonian where any neighbourhood of any point contains a countable number of eigenvalues of the above-mentioned type.

In the following we consider as before the vertex set $\mathcal{V}_C$ of the Canopy tree C and take the finite subtree Λ_L rooted at the vertex $(0,L) \in \mathcal{V}_C$. Associated with this subtree and the operators $H^\omega_{C,\lambda}$ given in equation 2.2.11, we define the matrices $H^\omega_{\lambda,L}$ as in 2.2.12. We define $\Lambda_{L,x}$ to be the subtree of Λ_L rooted at $x \in \Lambda_L$. That is, $\Lambda_{L,x} := \Lambda_{L-\mathrm{d}(x_L,x)}(x)$. We assume that the common distribution μ of the random variables $\{\omega_x, x \in C\}$ has support $[a,b]$. Let $\Sigma_{K-1} = \sigma(\Delta_{m_0-1}) + (a,b)$.

Apart from the intrinsic interest in the question of multiplicity of operators, the following theorem is the reason why we need the previous theorem. This is the crucial fact needed to show that the compound Poisson random variables obtained as the limit points associated with the eigenvalue statistics are, actually, non-trivial compound Poisson.

Theorem 3.0.4. *Consider the i.i.d random variables $\{\omega_x, x \in C\}$ and assume that their common distribution is given by $d\mu(x) = \rho(x)dx$ with $\rho(x) > 0, x \in (a,b)$. Given $E \in \Sigma_{K-1}$ let $E_0 \in \sigma(\Delta_{m_0-1})$ be such that $E - E_0 \in (a,b)$. Then for any interval I, there is a L_0 large, independent of I, such that whenever $L \geq L_0$, we have*

$$(3.0.4) \qquad \sum_{x \in \Lambda_L, d(x,\partial C) = L - l_L} \mathrm{Prob}\left(\mathrm{Tr}(E_{H^\omega_{\Lambda_{l_L,x}}}(E + I|\Lambda_L|^{-1})) \geq K-1 \right) \geq \frac{1}{2}\left(1 - e^{-\frac{1}{2}\rho(E-E_0)}\right) > 0.$$

In the following discussion, wherever we have to consider $|\Lambda_L| = \frac{K^{L+1}-1}{K-1}$, we have replaced that with the quantity K^L. This simplified the notations, and should not affect any of the bounds or limits, since $\frac{K^{L+1}-1}{K-1} \approx K^L$ for large enough L.

By the assumption that $E - E_0 \in (a,b)$, we see that for any I, there is a L_0 large so that $E - E_0 + IK^{-L_0} \subset (a,b)$ and hence $E - E_0 + IK^{-L} \subset (a,b)$ for all $L \geq L_0$. If $\omega_y \in E - E_0 + IK^{-L} \subset (a,b)$ for some $y \in \partial\Lambda_{L,x} \cap \partial C$, then $E_0 + \omega_y \in E_0 + (a,b)$ and so it is an eigenvalue of multiplicity at least $K-1$ by the multiplicity theorem. But $\omega_y \in E - E_0 + IK^{-L}$ implies that $E_0 + \omega_y \in E_0 + E - E_0 + IK^{-L} = E + IK^{-L}$; therefore,

$\mathrm{Tr}((E_{H_{L,x}}(E+IK^{-L}))\geq K-1$. Hence we have the inclusion,

$$\left\{\omega\colon \mathrm{Tr}(E_{H_{L,x}}(E-E_0+IK^{-L})\geq K-1)\right\}\supset\left\{\omega:\omega_y\in E-E_0+IK^{-L}, \text{ for some } y\in\partial\Lambda_{L,x}\right\}.$$

Therefore, taking probabilities we have

(3.0.5)

$$\begin{aligned}
&\sum_{x\in\Lambda_L,d(x,\partial C)=L-l_L}\mathrm{Prob}\Big(\mathrm{Tr}(E_{H_{L,x}}(E-E_0+IK^{-L})\geq K-1\Big)\\
&\geq \mathrm{Prob}\Big(\left\{\omega:\mathrm{Tr}(E_{H_{L,x}}(IK^{-L})\geq K-1 \text{ for some } x\in\Lambda_L,d(x,\partial C)=L-l_L\right\}\Big)\\
&\geq \mathrm{Prob}\{\omega:\omega_y\in E-E_0+IK^{-L}, \text{ for some } y\in\partial\Lambda_{L,x} \text{ and for some } x\in\Lambda_L,d(x,\partial C)=L-l_L\}.
\end{aligned}$$

Define the sets, for some x such that $\mathrm{d}(x,\partial C)=L-l_L$ and $y\in\partial\Lambda_{L,x}$: $\Omega_{y,L}:=\{\omega:\omega_y\in E-E_0+IK^{-L}\}$. Let $\Omega_L:=\{\omega:\omega_y\in E-E_0+IK^{-L}$, for some $y\in\partial\Lambda_{L,x}\cap\partial C$ and for some $x\in\Lambda_L,d(x,\partial C)=L-l_L\}$. Therefore,

$$\Omega_L=\cup_{y\in\partial\Lambda_L}\Omega_{y,L}.$$

Since ω_ys are independent and identically distributed, we have $\mathbb{P}(\Omega_{y,L})=\mu(E-E_0+IK^{-L})=c_L$, say. Then by the inclusion exclusion principle [56, Proposition 4.4,p. 30] applied to independent and identically distributed events, we have (note that, K^L is the size of the boundary, $\partial\Lambda_L$)

(3.0.6)

$$\begin{aligned}
\mathbb{P}(\cup_{y\in\partial\Lambda_L}\Omega_{y,L})&=\sum_y\mathbb{P}(\Omega_{y,L})-\sum_{y_1\neq y_2}\mathbb{P}(\Omega_{y_1,L}\cap\Omega_{y_2,L})+\sum_{y_1\neq y_2\neq y_3}\mathbb{P}(\Omega_{y_1,L}\cap\Omega_{y_2,L}\cap\Omega_{y_3,L})-\cdots\\
&\quad+(-1)^{K^L}\mathbb{P}(\cap_{y\in\partial\Lambda_L}\Omega_{y,L})\\
&=\sum_y c_L-\sum_{y_1\neq y_2}c_L^2+\sum_{y_1\neq y_2\neq y_3}c_L^3-\cdots+(-1)^{K^L}c_L^{K^L}\\
&=K^Lc_L-\binom{K^L}{2}c_L^2+\binom{K^L}{3}c_L^3-\cdots+(-1)^{K^L}c_L^{K^L}.
\end{aligned}$$

The last expression is equal to $1-(1-c_L)^{K^L}$. Recall that $c_L=\mathbb{P}(\Omega_{y,L})=\mathbb{P}(\omega:\omega_y\in$

$E-E_0+IK^{-L})$. Since the density ρ of μ is continuous in its support, we have by the mean value theorem, $c_L = \mathbb{P}(\omega\colon\ \omega_y \in E-E_0+IK^{-L}) = \rho(e)|I|K^{-L}$,for some $e \in E-E_0+IK^{-L}$. By assumption on E we have $E-E_0+IK^{-L} \subset (a,b)$ for L large, so by the continuity of ρ in (a,b), we also have, for large enough L, $|\rho(e)-\rho(E-E_0)| < \frac{1}{2}\rho(E-E_0)$, or $\rho(e) \geq \frac{1}{2}\rho(E-E_0)$ in $E-E_0+IK^{-L}$. Let L_0 be large so that when $L \geq L_0$, both of $E-E_0+IK^{-L} \subset (a,b)$ and $\rho(e) \geq \frac{1}{2}\rho(E-E_0)$ in $E-E_0+IK^{-L}$ hold. Then, we have, for large $L \geq L_0$,

$$(1-c_L)^{K^L} \leq (1-\frac{1}{2}\rho(E-E_0)|I|K^{-L})^{K^L},$$

Since $(1-x/n)^n$ converges to e^{-x}, the right hand side of the above inequality converges to $e^{-\frac{1}{2}\rho(E-E_0)|I|}$, which is strictly smaller than 1 by the positivity of $\rho(E-E_0)$ and $|I|$. Therefore, increasing L_0 if necessary, we have for all $L \geq L_0$,

$$\mathbb{P}(\cup_{y\in\partial\Lambda_L}\Omega_{y,L}) = 1-(1-c_L)^{K^L} \geq \frac{1}{2}(1-e^{-\frac{1}{2}\rho(E-E_0)|I|}).$$

This shows 3.0.4 for $L \geq L_0$.

Chapter 4

Eigenvalue statistics for Bethe lattice

In this chapter, we give our main results on the local eigenvalue statistics of the higher rank Anderson model we consider in this thesis. Here, we will work only with the Canopy tree, its subtree, the associated Hilbert spaces, matrices and operators, though we talk of them as if they are on the Bethe lattice for reasons explained earlier (See. 2.2.2). We show that the eigenvalues in an interval around a point in a part of the point spectrum have a compound Poisson distribution, which is not Poisson. In the first section, we state the main theorem of the thesis and some other theorems needed for proving it. The proofs of all these theorems will be given in the next section (4.2).

4.1 Main Theorems

Our main theorem of this thesis is that the eigenvalue statistics, of the random operator we consider, is compound Poisson in a part of the point spectrum. The proof of this fact uses properties of the spectrum including its multiplicity which was given in the last chapter and a fractional moment bound on the resolvent kernels of the random operator that allows us to prove the infinite divisibility of a sequence of random variables we need to consider. These together imply that the 'Levy measure' of the limiting random variables is non-trivial as explained in the introductory chapter.

We note that in the following theorem, the exponent of decay of the averages of fractional powers of the resolvent kernels has to be large enough so that the averages of the matrix-valued elements have norms that are square-integrable in y. We recall that the operators and matrices occurring in the following theorems are defined in equations (2.2.11, 2.2.10) and satisfy the assumptions in (2.2.6)

The idea behind showing the local statistics to be compound Poisson is the following. We take the subtree $\Lambda_L(x_L)$ of the Canopy tree, consider its subtrees $\{\Lambda_{L-\ell_L}(x)\}$, with $\ell_L \approx \alpha L$, $0<\alpha<1$ and show first that the random variables $Tr(E_{H^\omega_{\lambda,L}}(E+I|\Lambda_L|^{-1})$ and a finite, but increasing with L, sum of i.i.d random variables have the same limit points as $L\to\infty$. For this proof, we need the fractional moment bounds that show exponential decay of fractional moments of some matrix valued resolvent kernels in Theorem 4.1.1. Once this is done, an explicit form of the characteristic function of the distribution of limiting random variables is obtained; which show, by an application of the Levy-Khintchine theorem, that the limits are compound Poisson. We show that the limit points are non-trivial compound Poisson random variables by computing the Levy measure associated with the random variables and show that they are non-trivial on the set $\{K-1,...,M_0\}$. We also compute the expectation of the limit points as an aside. We note here that the strategy followed originally by Minami and other authors including Hislop-Krishna, to show that the limiting random variables are non-trivial is different from ours. They need to consider another sequence of random variables and compute their limit points and show they agree with the two we already computed, but in our case the explicit bound on the Levy measure we have, lets us omit this step.

Theorem 4.1.1. *Let $H^\omega_{\lambda,L}$ be defined as in* (2.2.12). *Then, for any $0<s<1$, $E_0\in\mathbb{R}$, and $\gamma>\ln(K+1)$, there exist $\lambda_{\gamma,s}>0$ and $C>0$ such that,*

$$\sup_{\epsilon>0}\mathbb{E}_{\omega}\left[\left\|P_x\left(H^\omega_{\lambda,L}-E-i\epsilon\right)^{-1}P_y\right\|^s\right]\le Ce^{-\gamma \mathrm{d}(x,y)} \tag{4.1.1}$$

for all $\lambda > \lambda_{\gamma,s}$ and L large enough so that $x, y \in \Lambda_L$.

The above theorem describes the exponential decay of the Green's function. What is more important is the fact that any rate of decay is achievable by changing the disorder parameter.

We aim to prove the following theorem, where the number M_0 is the rank of the projections P_y. We recall that we are working with the subtrees Λ_L rooted at $x_L = (0, L)$ and the corresponding random variables

$$\mu^{\omega,\lambda}_{E_0,L}(I) = \mathrm{Tr}(E_{|\Lambda_L|(H^{\omega}_{\lambda,L} - E_0)}(I)),$$

associated with a point E_0 and an interval I, and also recall that $\Sigma_{K-1} = \sigma(\Delta_{m_0-1}) + (a,b)$, where $\mathrm{Supp}(\mu) = [a,b]$, μ is the common distribution of the random variables ω_y,for $y \in J_C$.

Theorem 4.1.2. *Consider the random operators H^{ω} and $H^{\omega}_{\lambda,L}$ defined in* (2.2.12) *satisfying the assumptions (2.2.6). Let $E_0 \in \Sigma_{K-1}$, $\lambda > 0$ large and a bounded interval $I \subset \mathbb{R}$. Then the limit points of the sequence of random variables $\{\mu^{\omega,\lambda}_{E_0,L}(I)\}_{L\in\mathbb{N}}$, defined in* (1.0.2)*, are compound Poisson distributed with the associated Lévy measure having at least one point from the set $\{K-1, K, K+1, \cdots, M_0\}$ in its support.*

Remark 4.1.3. *The fact that the Lévy measure is supported at a point other than $\{1\}$ shows that the compound Poisson distribution is not Poisson. The different*

limit points are distinguished by the different Lévy measures of their distribution.

4.2 Proofs of Main Theorems

4.2.1 Proof of Theorem 4.1.1

We donote by Λ_L the subtree $\tilde{\Lambda}_L(x_L)$ of the Canopy tree, defined in page 47. For $y \in \Lambda_L$, we will denote

$$\Lambda_l(y) := \{x \in \Lambda_L : \mathrm{d}(x,y) \le l \;\&\; x \prec y\}, \tag{4.2.1}$$

for $l \in \mathbb{N}$. For any $p \in J_C$, the projections are defined as $P_p = \chi_{\Lambda_{m_0}(p)}$ (see the definition in Assumptions 2.2.6). See [figure 4.1] below.

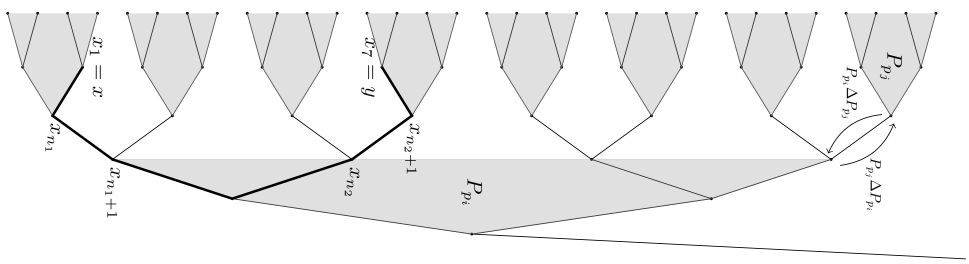

Figure 4.1: Boxes corresponding to the projections

Using the resolvent equation between $H^{\omega}_{\lambda,L}$ and

$$\tilde{H}^{\omega}_L := (I-P_p)H^{\omega}_{\lambda,L}(I-P_p) + P_p H^{\omega}_{\lambda,L} P_p$$

we have,

$$\begin{aligned}&\left\langle \delta_x, (H^{\omega}_{\lambda,L}-z)^{-1}\delta_y \right\rangle = \\ &-\Big\langle \delta_x, \big[P_p\Delta P_p + (\omega_p - z)P_p - P_p\Delta(\chi_{\Lambda_L}-P_p)(\tilde{H}^{\omega}_L - z)^{-1}(\chi_{\Lambda_L}-P_p)\Delta P_p\big]^{-1} \\ &\qquad P_p\Delta(\chi_{\Lambda_L}-P_p)(\tilde{H}^{\omega}_L - z)^{-1}\delta_y \Big\rangle\end{aligned} \tag{4.2.2}$$

for $x\in\Lambda_{m_0}(p)$ and $y\in\Lambda_L\setminus\Lambda_{m_0}(p)$. By taking $y\in\Lambda_{m_0}(p)$, we can also show that,

$$
\text{(4.2.3)}\qquad \begin{aligned} &P_p(H^{\omega}_{\lambda,L}-z)^{-1}P_p=\\ &\qquad\left[P_p\Delta P_p+(\omega_p-z)P_p-P_p\Delta(\chi_{\Lambda_L}-P_p)(\tilde{H}^{\omega}_L-z)^{-1}(\chi_{\Lambda_L}-P_p)\Delta P_p\right]^{-1}.\end{aligned}
$$

Using the fact that there is a unique path from x to y, (in the sense that if we remove any edge within this path, then x and y will lie in different components) say, $x=x_1,\ldots,x_n=y$, and taking $n_1<n$ so that $x_{n_1}\in\Lambda_{m_0}(p)$ and $x_{n_1+1}\in\Lambda_L\setminus\Lambda_{m_0}(p)$, the expression (4.2.2) gives us

$$
\text{(4.2.4)}\qquad \left\langle\delta_x,(H^{\omega}_{\lambda,L}-z)^{-1}\delta_y\right\rangle=\Gamma_{x_1,x_{n_1}}\left\langle\delta_{x_{n_1+1}},(\tilde{H}^{\omega}_L-z)^{-1}\delta_y\right\rangle,
$$

where $\Gamma_{a,b}$ is:

$$
\text{(4.2.5)}\qquad \left\langle\delta_a,\left[P_p\Delta P_p+(\omega_p-z)P_p-P_p\Delta(\chi_{\Lambda_L}-P_p)(\tilde{H}^{\omega}_L-z)^{-1}(\chi_{\Lambda_L}-P_p)\Delta P_p\right]^{-1}\delta_b\right\rangle
$$

for $a,b\in\Lambda_{m_0}(p)$. Repeating this procedure inductively, we have,

$$
\text{(4.2.6)}\qquad \left\langle\delta_x,(H^{\omega}_{\lambda,L}-z)^{-1}\delta_y\right\rangle=\prod_{i=0}^{m}\Gamma_{x_{n_i+1},x_{n_{i+1}}},
$$

with $x=x_1,\ldots,x_n=y$ the shortest path between x and y, and $\{n_i\}_{i=1}^m$ with the property that for each i there exists $p_i\in J$ such that $x_{n_{i-1}+1},x_{n_i}\in\Lambda_{m_0}(p_i)$, $n_0=0$, and $n_{m+1}=n$. Finally,

$$
\text{(4.2.7)}\qquad \begin{aligned}\Gamma_{x_{n_i+1},x_{n_{i+1}}}=\Bigg\langle\delta_{x_{n_i+1}},\Bigg[&P_{p_i}\Delta P_{p_i}+(\omega_{p_i}-z)P_{p_i}\\ &-P_{p_i}\Delta\left(\chi_{\Lambda_L}-\sum_{j=1}^{i}P_{p_j}\right)(\tilde{H}^{\omega}_{i,L}-z)^{-1}\left(\chi_{\Lambda_L}-\sum_{j=1}^{i}P_{p_j}\right)\Delta P_{p_i}\Bigg]^{-1}\delta_{x_{n_{i+1}}}\Bigg\rangle,\end{aligned}
$$

where

$$
\tilde{H}^{\omega}_{i,L}:=(\chi_{\Lambda_L}-P_{p_i})H^{\omega}_{i-1,L}(\chi_{\Lambda_L}-P_{p_i})+P_{p_i}\Delta P_{p_i}+\omega_{p_i}P_{p_i}
$$

with $\tilde{H}^{\omega}_{0,L} := H^{\omega}_{\lambda,L}$. Observe that,

$$P_p \Delta\big(\chi_{\Lambda_L} - P_p\big)(\tilde{H}^{\omega}_L - z)^{-1}\big(\chi_{\Lambda_L} - P_p\big)\Delta P_p$$

is the multiplication operator over the boundary of the subtree $\Lambda_{m_0}(p)$. After removing the subtree $\Lambda_{m_0}(p)$, we are left with disjoint trees, and $\tilde{H}^{\omega}_L$ restricted to each of these subtrees are independent of each other. For $y \in \Lambda_{m_0}(p)$, define

$$N_y = \{x \in \Lambda_L : \mathrm{d}(x,y) = 1 \ \& \ x \notin \Lambda_{m_0}(p)\},$$

which is the set of neighbours of the vertex y, which lie outside $\Lambda'_{m_0(p)}$. We have,

$$\begin{aligned} &P_p \Delta\big(\chi_{\Lambda_L} - P_p\big)(\tilde{H}^{\omega}_L - z)^{-1}\big(\chi_{\Lambda_L} - P_p\big)\Delta P_p \\ &\quad = \sum_{y \in \Lambda_{m_0}(p)} \big|\delta_y\big\rangle\big\langle\delta_y\big| \sum_{x \in N_y} \big\langle \delta_x, (\tilde{H}^{\omega}_L - z)^{-1}\delta_x\big\rangle \end{aligned}$$

and the independence of $\tilde{H}^{\omega}_L$ on each of the subtree implies the independence of $\big\{\big\langle \delta_x, (\tilde{H}^{\omega}_L - z)^{-1}\delta_x\big\rangle\big\}_x$ for each $x \in \cup_{y \in \Lambda_{m_0}(p)} N_y$.

To prove the theorem we will use the expression (4.2.6). Notice that, in that expression, $\Gamma_{x_{n_i+1}, x_{n_{i+1}}}$ is independent of the random variables $\{\omega_{p_j}\}_{j=1}^{i-1}$. So we have,

$$\begin{aligned} &\mathop{\mathbb{E}}_{\omega}\Big[\big|\big\langle \delta_x, (H^{\omega}_{\lambda,L} - z)^{-1}\delta_y\big\rangle\big|^s\Big] = \\ &\qquad \mathop{\mathbb{E}}_{\omega^{\perp}_{p_0}, \cdots, \omega^{\perp}_{p_m}}\left[\mathop{\mathbb{E}}_{\omega_{p_0}}\left[|\Gamma_{x_{n_0+1}, x_{n_1}}|^s \mathop{\mathbb{E}}_{\omega_{p_1}}\left[\cdots \mathop{\mathbb{E}}_{\omega_{p_m}}\left[|\Gamma_{x_{n_m+1}, x_{n_{m+1}}}|^s\right]\right]\right]\right]. \end{aligned}$$

Therefore, all we need to do is to estimate $\mathbb{E}_{\omega_{p_i}}\left[|\Gamma_{x_{n_i+1}, x_{n_{i+1}}}|^s\right]$ independent of $\{\omega_n\}_{n \neq p_i}$. Let $\{E^{\omega}_j(z)\}_{j=1}^{rank(P_{p_i})}$, counted with multiplicity, denote the eigenvalues of

$$P_{p_i}\Delta P_{p_i} - P_{p_i}\Delta\left(\chi_{\Lambda_L} - \sum_{j=1}^{i} P_{p_j}\right)(\tilde{H}^{\omega}_{i,L} - z)^{-1}\left(\chi_{\Lambda_L} - \sum_{j=1}^{i} P_{p_j}\right)\Delta P_{p_i}.$$

Then, by the definition of Γ (see (4.2.7)), we have,

$$|\Gamma_{x_{n_i+1},x_{n_{i+1}}}| \leq \sum_{j=1}^{rank(P_{p_i})} \frac{1}{|E_i^\omega(z) - \lambda\omega_{p_i} - z|}.$$

Hence,

$$\mathop{\mathbb{E}}_{\omega_{p_i}}\left[|\Gamma_{x_{n_i+1},x_{n_{i+1}}}|^s\right] \leq \mathop{\mathbb{E}}_{\omega_{p_i}}\left[\sum_{j=1}^{rank(P_{p_i})} \frac{1}{|E_i^\omega(z) - \lambda\omega_{p_i} - z|^s}\right]$$
$$\leq C\frac{|\Lambda_{m_0}(p_i)|}{\lambda^s}.$$

Therefore, for large enough λ, $C\frac{|\Lambda_{m_0}(p_i)|}{\lambda^s} < 1$. So, using

$$\mathop{\mathbb{E}}_{\omega}\left[\left|\left\langle \delta_x, (H_{\lambda,L}^\omega - z)^{-1}\delta_y\right\rangle\right|^s\right] \leq \left(C\frac{|\Lambda_{m_0}(p_i)|}{\lambda^s}\right)^m, \tag{4.2.8}$$

we get the estimate (4.1.1), proving the theorem.

4.2.2 Infinite divisibility and Compound Poisson Variables

To show the infinite divisibility of the sequence of measures $\{\mu_{E_0,L}^{\omega,\lambda}\}_L$, first, define the measures $\eta_{E_0,L,x}^{\omega,\lambda}$ for $x \in \Lambda_L$ as

$$\eta_{E_0,L,x}^{\omega,\lambda}(f) := \mathrm{Tr}(f(|\Lambda_L|(\chi_{\Lambda_{L-l_L}(x)} H_{\lambda,L}^\omega \chi_{\Lambda_{L-l_L}(x)} - E_0))),\ f \in C_c(\mathbb{R}), \tag{4.2.9}$$

where $l_L = m_0\lfloor\frac{\alpha L}{m_0}\rfloor$, $0 < \alpha < \frac{1}{2}$.

The following lemma says that the random variables $\mu_{E_0,L}^{\omega,\lambda}(I)$ and $\sum_{x:\mathrm{d}(x_L,x)=l_L}\eta_{E_0,L,x}^{\omega,\lambda}(I)$ have the same set of limit points in the topology of distributional convergence. We have used the Fourier transform characterization of the distributional convergence [13, p. 349, Theorem 26.3].

Lemma 4.2.1. *Let $I \subset \mathbb{R}$ be a bounded interval, then*

(4.2.10)
$$\lim_{L\to\infty} \mathbb{E}_{\omega}\left[\left| e^{\iota \mu^{\omega,\lambda}_{E_0,L}(I)} - e^{\iota \sum_{x:\mathrm{d}(x_L,x)=l_L} \eta^{\omega,\lambda}_{E_0,L,x}(I)} \right|\right] = 0.$$

Proof. Given any positive measure ν, let $F_\nu(z) = \int \frac{1}{w-z} d\nu(w), \Im z > 0$. Then the sequence of measures $\frac{1}{\pi}\Im F_\nu(z) d\Re(z)$ converges to ν in distribution [we refer to [48, p. 20, Corollary 1.4.5]]. Therefore showing the convergence in equation (4.2.10) reduces to showing

(4.2.11)
$$\lim_{L\to\infty}\lim_{\Im z\to 0} \mathbb{E}_{\omega}\left[\left| \mu^{\omega,\lambda}_{E_0,L}(\Im(\cdot - z)^{-1}) - \sum_{x:\mathrm{d}(x_L,x)=l_L} \eta^{\omega,\lambda}_{E_0,L,x}(\Im(\cdot - z)^{-1}) \right|\right] = 0.$$

Hence, (denote $H^{\omega}_{\lambda,L,x} := \chi_{\Lambda_{L-l_L}(x)} H^{\omega}_{\lambda,L} \chi_{\Lambda_{L-l_L}(x)}$)

$$\begin{aligned}
&\left| \mu^{\omega,\lambda}_{E_0,L}(\Im(\cdot - z)^{-1}) - \sum_{x:\mathrm{d}(x_L,x)=l_L} \eta^{\omega,\lambda}_{E_0,L,x}(\Im(\cdot - z)^{-1}) \right| \\
&= \frac{1}{|\Lambda_L|}\Bigg| \sum_{y\in\Lambda_L} \Im\left\langle \delta_y, (H^{\omega}_{\lambda,L} - E_0 - |\Lambda_L|^{-1} z)^{-1} \delta_y \right\rangle \\
&\qquad - \sum_{x:\mathrm{d}(x_L,x)=l_L} \sum_{y\in\Lambda_{L-l_L}(x)} \Im\left\langle \delta_y, (H^{\omega}_{\lambda,L,x} - E_0 - |\Lambda_L|^{-1} z)^{-1} \delta_y \right\rangle \Bigg| \\
&\le \frac{1}{|\Lambda_L|} \sum_{\mathrm{d}(x_L,x)<l_L} \Im G_L(y,y;z_L) \\
&\quad + \frac{1}{|\Lambda_L|} \sum_{x:\mathrm{d}(x_L,x)=l_L} \sum_{\substack{y\in\Lambda_{L-l_L}(x) \\ \mathrm{d}(x,y)<l_L}} \Im G_L(y,y;z_L) + \Im G_{L,x}(y,y;z_L) \\
&\quad + \frac{1}{|\Lambda_L|} \sum_{x:\mathrm{d}(x_L,x)=l_L} \sum_{\substack{y\in\Lambda_{L-l_L}(x) \\ \mathrm{d}(x,y)\ge l_L}} |G_L(y,y;z_L) - G_{L,x}(y,y;z_L)|,
\end{aligned}$$

where $G_L(x,y;z_L) = \left\langle \delta_x, (H^{\omega}_{\lambda,L} - E_0 - |\Lambda_L|^{-1} z)^{-1} \delta_y \right\rangle$ and $G_{L,x}(x,y;z_L) = \langle \delta_x, (H^{\omega}_{\lambda,L,x} - E_0 - |\Lambda_L|^{-1} z)^{-1} \delta_y \rangle$. Using (2.4.2) for the first and the second sums, we get

$$\begin{aligned}
\frac{1}{|\Lambda_L|} \sum_{\mathrm{d}(x_L,x)<l_L} \mathbb{E}_{\omega}[\Im G_L(y,y;z_L)] &\le \pi \|\rho\|_\infty \frac{1 + \frac{K+1}{K-1}(K^{l_L} - 1)}{1 + \frac{K+1}{K-1}(K^{L+1} - 1)} \\
&= O(K^{-(1-\alpha)L}) \xrightarrow{L\to\infty} 0.
\end{aligned}$$

$$\frac{1}{|\Lambda_L|}\sum_{x:\mathrm{d}(x_L,x)=l_L}\sum_{\substack{y\in\Lambda_{L-l_L}(x)\\ \mathrm{d}(x,y)<l_L}}\mathbb{E}_{\omega}\big[\Im G_L(y,y;z_L)+\Im G_{L,x}(y,y;z_L)\big]$$

$$\leq 2\pi\|\rho\|_\infty\frac{(K+1)K^{l_L}\frac{K^{l_L}-1}{K-1}}{1+\frac{K+1}{K-1}(K^{L+1}-1)}=O(K^{(1-2\alpha)L})\xrightarrow{L\to\infty}0.$$

For the third term, we use the resolvent equation, and get (Px denotes the neighbouring vertex such that $\mathrm{d}(x_L,x)=\mathrm{d}(x_L,Px)+1$; i.e., the vertex previous to x.)

$$\frac{1}{|\Lambda_L|}\sum_{x:\mathrm{d}(x_L,x)=l_L}\sum_{\substack{y\in\Lambda_{L-l_L}(x)\\ \mathrm{d}(x,y)\geq l_L}}\mathbb{E}[|G_L(y,y;z_L)-G_{L,x}(y,y;z_L)|]$$

$$=\frac{1}{|\Lambda_L|}\sum_{x:\mathrm{d}(x_L,x)=l_L}\sum_{\substack{y\in\Lambda_{L-l_L}(x)\\ \mathrm{d}(x,y)\geq l_L}}\mathbb{E}[|G_L(y,Px;z_L)G_{L,x}(x,y;z_L)|]$$

$$\leq\frac{1}{|\Lambda_L|}\sum_{x:\mathrm{d}(x_L,x)=l_L}\sum_{\substack{y\in\Lambda_{L-l_L}(x)\\ \mathrm{d}(x,y)\geq l_L}}\frac{1}{(|\Lambda_L|^{-1}\Im z)^{2-s}}\mathbb{E}[|G_{L,x}(x,y;z_L)|^s]$$

$$\leq\frac{|\Lambda_L|^{1-s}(K+1)K^{l_L-1}}{(\Im z)^{2-s}}\sum_{n=l_L}^{\infty}K^n e^{n(\tilde{C}-s\frac{1}{m_0}\ln\lambda)}$$

$$=O\Big(e^{(1-s+2\alpha)\ln K L+\alpha L(\tilde{C}-s\frac{1}{m_0}\ln\lambda)}\Big),\tag{4.2.12}$$

where we used the fact that $|G_L(x,y;z_L)|\leq\frac{1}{|\Lambda_L|^{-1}\Im z}$, and the last expression comes from the proof of Theorem 4.1.1 (see (4.2.8)). For

$$\frac{m_0((1-s+2\alpha)\ln K+\alpha\tilde{C})}{s\alpha}<\ln\lambda,$$

observe that (4.2.12) goes to zero as $L\to\infty$. This completes the proof of (4.2.11), and hence the lemma. □

Before attempting to prove the main result, we need to establish some results on the limit of the process.

Lemma 4.2.2. *Let $L_n = m_0 n$ for $n\in\mathbb{N}$. Then, for any bounded interval I, there exists a sub-sequence $\{\tilde{L}_n\}_n$ of $\{L_n\}_n$, such that for all $k\in\{K-1,\dots,M_0\}$ $\{\sum_{\mathrm{d}(x_{\tilde{L}_m},x)=l_{\tilde{L}_m}}\mathbb{P}[\eta^{\omega,\lambda}_{E_0,\tilde{L}_m,x}(\chi_I)=$*

$k]\}_{m\in\mathbb{N}}$ *converges and it is strictly positive. Here* l_L *is the sequence defined in Lemma 4.2.1.*

Proof. Using the Lemma 4.2.1, we have

$$0 \le \sum_{\mathrm{d}(x_{L_m},x)=l_L} \mathbb{P}[\eta^{\omega,\lambda}_{E_0,L,x}(\chi_I)=k] \le \frac{1}{k}\sum_{\mathrm{d}(L_m,x)=l_L} \mathbb{E}[\eta^{\omega,\lambda}_{E_0,L,x}(\chi_I)] \le \frac{1}{k}K^{l_L}|\Lambda_{L-l_L}|\frac{1}{|\Lambda_L|} \le C,$$

where we have used the Wegner estimate 2.4.3 in the last step. Therefore, we can find a sub-sequence $\{\tilde{L}_m\}_m$ of $\{nm_0\}_{n\in\mathbb{N}}$, such that

$$\left\{ \sum_{\mathrm{d}(x_{\tilde{L}_m},x)=l_{\tilde{L}_m}} \mathbb{P}[\eta^{\omega,\lambda}_{E_0,\tilde{L}_m,x}(\chi_I)=k] \right\}$$

converges. Since we are concerned with only finitely many sequences

$$\left\{ \sum_{\mathrm{d}(x_{\tilde{L}_m},x)=l_{\tilde{L}_m}} \mathbb{P}\{\eta^{\omega,\lambda}_{E_0,\tilde{L}_m,x}(\chi_I)=k \right\}_{k=1,2,3,\dots,M_0},$$

we can find a common sequence such that the sequence $\{\sum_{\mathrm{d}(x_L,x)=l_{\tilde{L}_m}} \mathbb{P}\{\eta^{\omega,\lambda}_{E_0,\tilde{L}_m,x}(\chi_I)=k\}$ converges for all $k=1,2,3,\dots,M_0$.

Now, by Theorem 3.0.4, we know that these limits are strictly positive. Hence the lemma. □

Proof of Theorem 4.1.2

In the following, we are following Hislop-Krishna [34] quite closely, except in the calculation of the Lévy measure. To prove the theorem, all we need to do is to compute

$$\lim_{L\to\infty\omega} \mathbb{E}\left[e^{\iota t\mu^{\omega,\lambda}_{E_0,L}(\chi_I)}\right],$$

where χ_I is the characteristic function of a bounded interval $I\subset\mathbb{R}$. Notice that, for χ_I, the random variables $\mu^{\omega,\lambda}_{E_0,L}(\chi_I)$ and $\{\eta^{\omega,\lambda}_{E_0,L,x}(\chi_I)\}_x$ are integer valued.

Using the Lemma 4.2.1, we have

$$\begin{aligned}\lim_{L\to\infty}\mathbb{E}_{\omega}\left[e^{\iota t\mu^{\omega,\lambda}_{E_0,L}(\chi_I)}\right] &= \lim_{L\to\infty}\mathbb{E}_{\omega}\left[e^{\sum_{\mathrm{d}(x_L,x)=l_L}\eta^{\omega,\lambda}_{E_0,L,x}(\chi_I)}\right]\\ &= \lim_{L\to\infty}\prod_{\mathrm{d}(x_L,x)=l_L}\left(\mathbb{E}_{\omega}\left[e^{\iota t\eta^{\omega,\lambda}_{E_0,L,x}(\chi_I)}\right]\right)\\ &= \lim_{L\to\infty} e^{\sum_{\mathrm{d}(x_L,x)=l_L}\ln\left(\mathbb{E}_{\omega}\left[e^{\iota t\eta^{\omega,\lambda}_{E_0,L,x}(\chi_I)}-1\right]+1\right)}. \end{aligned}\tag{4.2.13}$$

The second line follows because of the independence of $\{\eta^{\omega,\lambda}_{E_0,L,x}\}_{\mathrm{d}(x_L,x)=l_L}$ (this is where l_L is a multiple of m_0 is used so that all of the the projections P_{p_j} have support at at most one $\{\Lambda_{L-l_L}(x)\}$). Using $|e^{\iota x}-1|\le|x|$ for real x, we have

$$\left|\mathbb{E}_{\omega}\left[e^{\iota t\eta^{\omega,\lambda}_{E_0,L,x}(\chi_I)}-1\right]\right|\le\mathbb{E}_{\omega}\left[\left|e^{\iota t\eta^{\omega,\lambda}_{E_0,L,x}(\chi_I)}-1\right|\right]\le|t|\mathbb{E}_{\omega}\left[\eta^{\omega,\lambda}_{E_0,L,x}(\chi_I)\right],\tag{4.2.14}$$

and using the Wegner estimate (2.4.3) over the operator $\chi_{\Lambda_{L-l_L}(x)}H^{\omega}_{\lambda,L}\chi_{\Lambda_{L-l_L}(x)}$, (the measure $\eta^{\omega,\lambda}_{E_0,L,x}$ is defined using this, see (4.2.9)), we get,

$$\left|\mathbb{E}_{\omega}\left[e^{\iota t\eta^{\omega,\lambda}_{E_0,L,x}(\chi_I)}-1\right]\right|\le|t|\mathbb{E}_{\omega}\left[\eta^{\omega,\lambda}_{E_0,L,x}(\chi_I)\right]\le C|t||I|\frac{|\Lambda_{L-l_L}(x)|}{|\Lambda_L|}\xrightarrow{L\to\infty}0.\tag{4.2.15}$$

From the expressions (4.2.14), (4.2.15), and the fact that $|\ln(1+x)-x|\le|x|^2$ for $|x|\ll1$, we have

$$\begin{aligned}&\ln\left(\mathbb{E}_{\omega}\left[e^{\iota t\eta^{\omega,\lambda}_{E_0,L,x}(\chi_I)}-1\right]+1\right)\\ &\quad=\mathbb{E}_{\omega}\left[e^{\iota t\eta^{\omega,\lambda}_{E_0,L,x}(\chi_I)}-1\right]+O\left(\left|\mathbb{E}_{\omega}\left[e^{\iota t\eta^{\omega,\lambda}_{E_0,L,x}(\chi_I)}-1\right]\right|^2\right)\\ &\quad=\mathbb{E}_{\omega}\left[e^{\iota t\eta^{\omega,\lambda}_{E_0,L,x}(\chi_I)}-1\right]+O\left(|t|^2\left(\mathbb{E}_{\omega}\left[\eta^{\omega,\lambda}_{E_0,L,x}(\chi_I)\right]\right)^2\right)\\ &\quad=\mathbb{E}_{\omega}\left[e^{\iota t\eta^{\omega,\lambda}_{E_0,L,x}(\chi_I)}-1\right]+O\left(|t|^2|I|^2\left(\frac{|\Lambda_{L-l_L}(x)|}{|\Lambda_L|}\right)^2\right).\end{aligned}\tag{4.2.16}$$

Using (4.2.16) and the fact that $|\Lambda_{L-l_L}(x_L)|\Big(\frac{|\Lambda_{L-l_L}(x)|}{|\Lambda_L|}\Big)^2 \xrightarrow{L\to\infty} 0$ on (4.2.13), we get,

$$
\begin{aligned}
\lim_{L\to\infty}\mathop{\mathbb{E}}_{\omega}\Big[e^{\iota t\mu^{\omega,\lambda}_{E_0,L}(\chi_I)}\Big] &= \lim_{L\to\infty} e^{\sum_{\mathrm{d}(x_L,x)=l_L}\ln\Big(\mathbb{E}_\omega\Big[e^{\iota t\eta^{\omega,\lambda}_{E_0,L,x}(\chi_I)}-1\Big]+1\Big)} \\
&= \lim_{L\to\infty} e^{\sum_{\mathrm{d}(x_L,x)=l_L}\mathbb{E}_\omega\Big[e^{\iota t\eta^{\omega,\lambda}_{E_0,L,x}(\chi_I)}-1\Big]}.
\end{aligned}
\tag{4.2.17}
$$

Focusing on the exponent of the last equation, we have

$$
\begin{aligned}
&\sum_{\mathrm{d}(x_L,x)=l_L}\mathop{\mathbb{E}}_{\omega}\Big[e^{\iota t\eta^{\omega,\lambda}_{E_0,L,x}(\chi_I)}-1\Big] \\
&\quad=\sum_{\mathrm{d}(x_L,x)=l_L}\sum_{k=1}^{\infty}(e^{\iota tk}-1)\mathbb{P}[\eta^{\omega,\lambda}_{E_0,L,x}(\chi_I)=k] \\
&\quad=\sum_{k=1}^{M_0}(e^{\iota tk}-1)\Big(\sum_{\mathrm{d}(x_L,x)=l_L}\mathbb{P}[\eta^{\omega,\lambda}_{E_0,L,x}(\chi_I)=k]\Big)+R(L),
\end{aligned}
\tag{4.2.18}
$$

where

$$
\begin{aligned}
R(L)&=\sum_{\mathrm{d}(x_L,x)=l_L}\sum_{k=M_0+1}^{\infty}(e^{\iota tk}-1)\mathbb{P}[\eta^{\omega,\lambda}_{E_0,L,x}(\chi_I)=k] \\
|R(L)|&\leq 2\sum_{\mathrm{d}(x_L,x)=l_L}\sum_{k=M_0+1}^{\infty}\mathbb{P}[\eta^{\omega,\lambda}_{E_0,L,x}(\chi_I)=k] \\
&\leq(\pi\|\rho\|_\infty|I|)^2|\Lambda_{L-l_L}(x_L)|\left(\frac{|\Lambda_{L-l_L}(x)|}{|\Lambda_L|}\right)^2\xrightarrow{L\to\infty}0.
\end{aligned}
\tag{4.2.19}
$$

The last line follows from the Minami estimate (2.4.5). From Lemma 4.2.2, we have a sequence $\{L_n\}_{n\in\mathbb{N}}$, such that, for $k\leq M_0$

$$
\lim_{n\to\infty}\sum_{\mathrm{d}(x_L,x)=l_{L_n}}\mathbb{P}[\eta^{\omega,\lambda}_{E_0,L_n,x}(\chi_I)=k]=p_k(I).
\tag{4.2.20}
$$

Now Theorem (4.2.20), (4.2.19), and (4.2.17) give

$$
\lim_{n\to\infty}\mathop{\mathbb{E}}_{\omega}\Big[e^{\iota t\mu^{\omega,\lambda}_{E_0,L_n}(\chi_I)}\Big]=e^{\sum_{k=1}^{M_0}(e^{\iota tk}-1)p_k(I)},
\tag{4.2.21}
$$

where one of the $p_k > 0$ for $k \in \{K-1,...,M_0\}$, due to Theorem 3.0.4. This completes the proof of the theorem.

Remark 4.2.3. *Note that if X is a random variable, then* $\mathbb{E}X = -i\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}(e^{itX})\,|_{t=0}$. *Therefore, the intensity measure for the limit of the sequence of random variables* $\{\mu^{\omega,\lambda}_{E_0,L_n}(I)\}$ *is given by,*

$$-i\frac{\mathrm{d}}{\mathrm{d}t}e^{\sum_{k=1}^{M_0}(e^{itk}-1)p_k(I)}\,|_{t=0} = -ie^{\sum_{k=1}^{M_0}(e^{itk}-1)p_k(I)}\sum_{k=1}^{M_0} ik(e^{itk})p_k(I)\,|_{t=0} = \sum_{k=1}^{M_0} kp_k(I).$$

Chapter 5

Conclusion

Here, we summarize the thesis and give some questions that can be considered for future studies, in this area.

Anderson tight binding model on the Bethe lattice is one of the important and highly studied models, both by mathematicians and physicists. It is one of the few models where the phenomena of localisation and delocalisation both have been established. Although the existence of the Poisson/ Compound Poisson statistics is usually expected to occur in the region of localisation, the Poisson statistics has been previously observed for the Bethe lattice Anderson model, even in the regime of the absolutely continuous spectrum. Aizenman and Warzel [5] explained this phenomenon by showing that, actually, it is the Canopy operator which captures the limit of the finite volume restrictions of the Bethe lattice operator; and that the Canopy operator has only completely localised energies for all disorders. As shown by Hislop-Krishna, the higher rank perturbations can result in, not necessarily a Poisson process, but a compound Poisson process.

It is generally not true that for the Anderson model with higher-rank perturbations, we will get a compound Poisson process. If we can show that the model has sufficiently many eigenvalues with multiplicity at least two, then we can show that the statistics is a strictly compound Poisson process. Our model happens to have arbitrarily high multiplicities for some of its eigenvalues (which are countably infinite), proportional to the size of the perturbing projections. This results in a non-trivial compound Poisson statistics for our model. Studying the multiplicity of operators is generally a difficult thing to do. To the best of our knowledge, there are no other Anderson type models where we can study the multiplicity in such a transparent manner as in the Canopy model. This adds the importance of the Canopy Hamiltonian in regards to studying multiplicity.

As shown by Aizenman and Warzel, and also reflected in this thesis, is the fact that the current notion of eigenvalue statistics does not distinguish between pure point and absolutely continuous part of the spectrum. Therefore, it is natural to ask, whether we can modify the definition to reflect different types of spectra. As shown in the paper Aizenman-Warzel [5], trees with a backbone structure can be treated similar to the Canopy tree. So, we believe that extending our results to models on those trees will not be that difficult.

Bibliography

[1] Michael Aizenman, Alexander Elgart, Serguei Naboko, H. Jeffrey Schenker, and Gunter Stolz. Moment analysis for localization in random schrödinger operators. *Inventiones mathematicae*, 163(2):343–413, 2005.

[2] Michael Aizenman and Stanislav Molchanov. Localization at large disorder and at extreme energies: An elementary derivation. *Communications in Mathematical Physics*, 157(2):245–278, 1993.

[3] Michael Aizenman, Jeffrey H Schenker, Roland M Friedrich, and Dirk Hundertmark. Finite-volume fractional-moment criteria for Anderson localization. *Communications in Mathematical Physics*, 224(1):219–253, 2001.

[4] Michael Aizenman, Robert Sims, and Simone Warzel. Stability of the absolutely continuous spectrum of random Schrödinger operators on tree graphs. *Probability Theory and Related Fields*, 136(3):363–394, 2006.

[5] Michael Aizenman and Simone Warzel. The canopy graph and level statistics for random operators on trees. *Mathematical Physics, Analysis and Geometry*, 9(4):291–333, 2006.

[6] Michael Aizenman and Simone Warzel. The canopy graph and level statistics for random operators on trees. *Math. Phys. Anal. Geom.*, 9(4):291–333 (2007), 2006.

[7] M Anish and Dhriti Ranjan Dolai. Eigenfunction statistics for anderson model with holder continuous single site potential. *Proc. Indian Acad.Sci.Math.Sci.*, 126(4):577–589, 2016.

[8] Narayanan P. A. Anish Mallick. On multiplicity of spectrum for anderson type operators with higher rank perturbations. *Operators and Matrices*, 13:733–744, 2018.

[9] D. Applebaum, B. Bollobas, University of Cambridge, W. Fulton, A. Katok, F. Kirwan, P. Sarnak, B. Simon, and B. Totaro. *Lévy Processes and Stochastic Calculus*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2004.

[10] S. Athreya and V.S. Sunder. *Measure and Probability*. Universities Press, 2009.

[11] Artur Avila, Yoram Last, and Barry Simon. Bulk universality and clock spacing of zeros for ergodic Jacobi matrices with absolutely continuous spectrum. *Analysis & PDE*, 3(1):81–108, 2010.

[12] Rabi Bhattacharya and Edward C Waymire. *A basic course in probability theory*. Springer Science & Business Media, 2007.

[13] P. Billingsley. *Probability and Measure*. Wiley Series in Probability and Statistics. Wiley, 1995.

[14] René Carmona and Jean Lacroix. *Spectral theory of random Schrödinger operators*. Springer Science & Business Media, 2012.

[15] Jean-Michel Combes, François Germinet, and Abel Klein. Generalized eigenvalue-counting estimates for the Anderson model. *Journal of Statistical Physics*, 135(2):201, 2009.

[16] Jean-Michel Combes, François Germinet, and Abel Klein. Poisson statistics for eigenvalues of continuum random Schrödinger operators. *Analysis & PDE*, 3(1):49–80, 2010.

[17] Jean-Michel Combes, François Germinet, and Abel Klein. Poisson statistics for eigenvalues of continuum random Schrödinger operators. *Analysis & PDE*, 3(1):49–80, 2010.

[18] Jean-Michel Combes, Peter D. Hislop, and Frédéric Klopp. An optimal Wegner estimate and its application to the global continuity of the integrated density of states for random Schrödinger operators. *Duke Math. J.*, 140(3):469–498, 12 2007.

[19] J.M. Combes and P.D. Hislop. Localization for some continuous, random Hamiltonians in d-dimensions. *Journal of Functional Analysis*, 124(1):149 – 180, 1994.

[20] J.B. Conway. *A Course in Functional Analysis*. Graduate texts in mathematics. Springer-Verlag, 1985.

[21] H.L. Cycon, R.G. Froese, B. Simon, and W. Kirsch. *Schrödinger Operators: With Applications to Quantum Mechanics and Global Geometry*. Springer study edition. Springer, 1987.

[22] H Garth Dales. *Introduction to Banach algebras, operators, and harmonic analysis*, volume 57. Cambridge University Press, 2003.

[23] F. Delyon, Y. Lévy, and B. Souillard. Anderson localization for multi-dimensional systems at large disorder or large energy. *Commun. Math. Phys.*, 100:463–470, 1985.

[24] Dhriti Ranjan Dolai and Mallick Anish. Spectral statistics for one-dimensional anderson model with unbounded but decaying potentials. *Infinite Dimensional Analysis, Quantum Probability and related Topics*, 22(2), 2019.

[25] Dhriti Ranjan Dolai and M Krishna. Poisson statistics for Anderson model with singular randomness. *J. Ramanujan Math. Soc.*, 30(3):251–266, 2015.

[26] Gerald B Folland. *Real analysis: modern techniques and their applications*. John Wiley & Sons, 2013.

[27] Richard Froese, David Hasler, and Wolfgang Spitzer. Absolutely continuous spectrum for the Anderson model on a tree: A geometric proof of Klein's theorem. *Communications in Mathematical Physics*, 269(1):239–257, 2007.

[28] J. Fröhlich and T. Spencer. Absence of diffusion in the Anderson tight binding model for large disorder or low energy. *Comm. Math. Phys*, 88:151–184, 1983.

[29] Jürg Fröhlich, Fabio Martinelli, Elisabetta Scoppola, and Thomas Spencer. Constructive proof of localization in the Anderson tight binding model. *Communications in Mathematical Physics*, 101(1):21–46, 1985.

[30] Leander Geisinger. Poisson eigenvalue statistics for random Schrödinger operators on regular graphs. In *Annales Henri Poincaré*, volume 16, pages 1779–1806. Springer, 2015.

[31] François Germinet and Frédéric Klopp. Spectral statistics for the discrete Anderson model in the localized regime. In N. Ueki N. Minami, editor, *Spectra of Random Operators and Related Fields*, RIMS Kôkyûroku Bessatsu, B27, pages 11–24. Université de Kyoto, Kyoto, Japon, 2011.

[32] Emanuel Gluskin and Shmuel Miller. On the recovery of the time average of continuous and discrete time functions from their laplace and z-transforms. *International Journal of Circuit Theory and Applications*, 41(9):988–997, 2013. https://arxiv.org/abs/1109.3356.

[33] I. Ya. Goldsheid, S. A. Molchanov, and L. Pastur. A random homogeneous Schrödinger operator has a pure point spectrum. *Funkcional. Anal. i Prilozen*, 11:96, 1977.

[34] Peter D. Hislop and M. Krishna. Eigenvalue statistics for random Schrödinger operators with non rank one perturbations. *Communications in Mathematical Physics*, 340(1):125–143, 2015.

[35] Peter D. Hislop and M. Krishna. Eigenvalue statistics for random Schrödinger operators with non rank one perturbations. *Comm. Math. Phys.*, 340(1):125–143, 2015.

[36] Vojkan Jakšić and Yoram Last. Corrugated surfaces and ac spectrum. *Reviews in Mathematical Physics*, 12(11):1465–1503, 2000.

[37] Olav Kallenberg. *Lectures on random measures*. University of Goteberg, 1974.

[38] Rowan Killip and Fumihiko Nakano. Eigenfunction statistics in the localized Anderson model. In *Annales Henri Poincaré*, volume 8, pages 27–36. Springer, 2007.

[39] W. Kirsch. Scattering theory for sparse random potentials. *Random Oper. Stoch. Equ.*, 10:329–334, 2002.

[40] Werner Kirsch. An invitation to random Schrödinger operators. In *Random Schrödinger operators*, volume 25 of *Panor. Synthèses*, pages 1–119. Soc. Math. France, Paris, 2008. With an appendix by Frédéric Klopp.

[41] Abel Klein. Extended states in the Anderson model on the Bethe lattice. *Advances in Mathematics*, 133(1):163–184, 1998.

[42] Shinichi Kotani and Fumihiko Nakano. Poisson statistics for 1d Schrödinger operators with random decaying potentials. *Electron. J. Probab.*, 22:Paper No. 69, 31, 2017.

[43] M. Krishna. Anderson models with decaying randomness: Existence of extended states. *Proc. Indian Acad. Sci. Math.*, 100:285–294, 1990.

[44] M. Krishna. Absolutely continuous spectrum for sparse potentials. *Proc. Indian Acad. Sci. Math.*, 103:333–339, 1993.

[45] M. Krishna and Peter Stollmann. Direct integrals and spectral averaging. *J. Operator Theory*, 69(1):279–285, 2013.

[46] H. Kunz and B. Souillard. Sur le spectre des opèrateurs aux diffèrences finies alèatoires. *Comm. Math. Phys.*, 78:201–246, 1980/81.

[47] John W. Lamperti. *Probability*. John Wiley & Sons, 2011.

[48] Demuth M and Krishna M. *Determining Spectra in Quantum Theory*. Birkhäuser Basel, 2005.

[49] Graf. G. M. Anderson localisation and the space-time characteristic of continuum states. *Jour. Stat. Phys*, 75, 1994.

[50] Hazewinkel Michiel, editor. *Encyclopedia of Mathematics*, volume 4. Springer, 1995.

[51] Nariyuki Minami. Local fluctuation of the spectrum of a multidimensional Anderson tight binding model. *Comm. Math. Phys.*, 177(3):709–725, 1996.

[52] SA Molčanov. The local structure of the spectrum of the one-dimensional Schrödinger operator. *Communications in Mathematical Physics*, 78(3):429–446, 1981.

[53] Mahendra Ganpatrao Nadkarni. *Spectral theory of dynamical systems*. Springer Science & Business Media, 1998.

[54] L. A. Pastur. Spectral properties of disordered systems in the one-body approximation. *Comm. Math. Phys.*, pages 179–196, 1980.

[55] Michael Reed and Barry Simon. *Methods of modern mathematical physics. vol. 1. Functional analysis*. Academic, 1980.

[56] S.M. Ross. *A First Course in Probability*. Pearson Education, Incorporated, 2014.

[57] W Rudin. *Real and complex analysis*. London [etc.]:[sn], 1970.

[58] W. Rudin. *Functional Analysis*. International series in pure and applied mathematics. McGraw-Hill, 2006.

[59] Walter Rudin. *Real and Complex Analysis*. McGraw Hill International Editions, 1987.

[60] Barry Simon. *Trace ideals and their applications*, volume 35. Cambridge University Press Cambridge, 1979.

[61] T. Simon, B. Wolff. Singular continuum spectrum under rank one perturbations and localisation for random Hamiltonians. *Comm. Pure. App. Math.*, 39:75–90, 1986.

[62] Peter Stollmann. *Caught by disorder: bound states in random media*, volume 20. Springer Science & Business Media, 2012.

[63] Anderson P. W. Absence of diffusion in certain random lattices. *Phys. Rev*, 109:1492–1505, 1958.

[64] F. Wegner. Bounds on the density of states in disordered systems. *Zeit. fur Phy*, B44:9–15, 1981.

[65] Joachim Weidmann. *Linear operators in Hilbert spaces*, volume 68. Springer Science & Business Media, 2012.

[66] Kosaku Yosida. Functional analysis. reprint of the sixth (1980) edition. classics in mathematics. *Springer-Verlag, Berlin*, 11:14, 1995.

In this thesis, we consider the Anderson tight binding model with non-rank-one (i.e., higher rank) perturbations. We consider the operators on the Bethe lattice. The Bethe lattice is an infinite rooted, regular tree where each vertex has K-many neighbours in the outward direction. So, except the special vertex, root (which has K-many neighbours), every vertex would have K+1-many neighbours. The Anderson model we consider is the following one.

$$H_\lambda^\omega := \Delta + \lambda \sum_{p \in \mathcal{J}} \omega_p P_p,$$

where P_p are non-rank-one projections of the same rank. Our objective is to study the eigenvalue statistics over the Bethe lattice. It is known in the literature that the eigenvalue statistics of the Anderson tight-binding model over the Bethe lattice is captured by another, related graph called the Canopy tree. Therefore, we look at the Anderson tight-binding model over the Canopy tree where the randomness is non-rank-one.

The Canopy tree is an infinite tree which can be described as follows. We start with a countable collection of vertices, thought of as the zero-th layer of the Canopy tree. Partition this layer into subsets of cardinality K. Edges are drawn between the vertices in the zero-th layer to those of the first layer in such a way that all the K-vertices in each subset of the zero-th layer are connected to a single vertex sitting in the first layer. Similar connections are drawn from the vertices of the first layer to the vertices in the second layer. Continue this process ad infinitum. See the picture below.

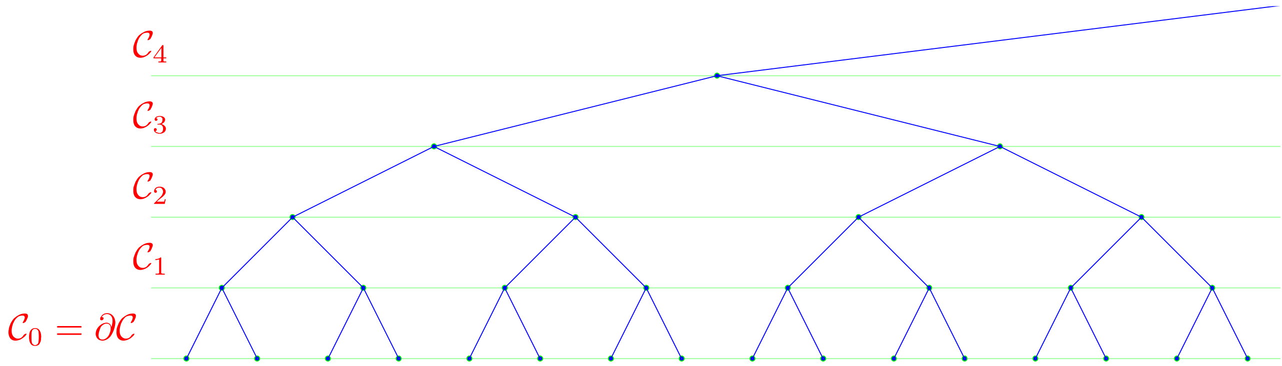

Figure 1:Canopy tree with K=2

We look at the following Anderson tight-binding model Hamiltonians on the Canopy tree, where the projections are of higher rank.

$$H_{\mathscr{C},\lambda}^\omega := \Delta + \sum_{p \in \mathcal{J}} \omega_p P_p .$$

To study the eigenvalue statistics, we consider the following process.

$$\mu_{E_0,L}^{\omega,\lambda}(f) := \mathrm{Tr}\left(f\left(\left|\Lambda_L\right| \left(H_{\lambda,L}^\omega - E_0 \right) \right) \right), \forall f \in C_c(R),$$

where $H_{\lambda,L}^\omega$ are the cut-off operators, obtained by restricting $H_{\mathscr{C},\lambda}^\omega$ to the subtrees Λ_L (which are suitable subtrees of the Canopy tree). We show that for any fixed bounded interval I, the sequence of random variables $\mu_{E_0,L}^{\omega,\lambda}(\chi_I)$ converges to a compound Poisson random variable. Then, we obtain a positive lower bound for the Levy measure associated with the limiting random variable, at points $k \in \{K-1, K, \ldots M_0\}$, where M_0 is the common rank of the perturbing projections P_p. This shows that the limiting random variable is a non-trivial compound Poisson random variable. For the last step we studied the multiplicity of the model.