Pi-systems of Symmetrizable Kac-Moody Algebras

By

Krishanu Roy

MATH10201305003

The Institute of Mathematical Sciences, Chennai

A thesis submitted to the

Board of Studies in Mathematical Sciences

In partial fulfillment of requirements

for the Degree of

DOCTOR OF PHILOSOPHY

of

HOMI BHABHA NATIONAL INSTITUTE

April, 2020

Homi Bhabha National Institute

Recommendations of the Viva Voce Committee

As members of the Viva Voce Committee, we certify that we have read the dissertation prepared by Krishanu Roy entitled "Pi-Systems of Symmetrizable Kac-Moody Algebras" and recommend that it may be accepted as fulfilling the thesis requirement for the award of Degree of Doctor of Philosophy.

Chairman - Parameswaran Sankaran — Date: April 21, 2020

Guide/Convenor - K N Raghavan — Date: April 21, 2020

Co-Guide - Sankaran Viswanath — Date: April 21, 2020

Nagaraj. D.S.
Member 1 - D S Nagaraj — Date: April 21, 2020

K. Vijay.
Member 2 - Vijay Kodiyalam — Date: April 21, 2020

Tanusree Khandai
Examiner - Tanusree Khandai — Date: April 21, 2020

Final approval and acceptance of this thesis is contingent upon the candidate's submission of the final copies of the thesis to HBNI.

I hereby certify that I have read this thesis prepared under my direction and recommend that it may be accepted as fulfilling the thesis requirement.

Date: April 21, 2020

Place: Chennai

Guide

STATEMENT BY AUTHOR

This dissertation has been submitted in partial fulfillment of requirements for an advanced degree at Homi Bhabha National Institute (HBNI) and is deposited in the Library to be made available to borrowers under rules of the HBNI.

Brief quotations from this dissertation are allowable without special permission, provided that accurate acknowledgement of source is made. Requests for permission for extended quotation from or reproduction of this manuscript in whole or in part may be granted by the Competent Authority of HBNI when in his or her judgement the proposed use of the material is in the interests of scholarship. In all other instances, however, permission must be obtained from the author.

Krishanu Roy

Krishanu Roy

DECLARATION

I hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Krishanu Roy

Krishanu Roy

LIST OF PUBLICATIONS ARISING FROM THE THESIS

Journal

1. Maximal closed subroot systems of real affine root systems., Krishanu Roy and R Venkatesh, Transformation groups, 2019, 24, 1261-1308.

Krishanu Roy

Krishanu Roy

Contents

List of Figures

List of Tables

Summary

In this thesis, we undertake a systematic study of π-systems of symmetrizable Kac-Moody algebras and regular subalgebras of affine Kac-Moody algebras. A π-system Σ is a finite subset of the real roots of a Kac-Moody algebra $\mathfrak{g}$ satisfying the property that pairwise differences of elements of Σ are not roots of $\mathfrak{g}$. As part of his classification of regular semisimple subalgebras of semisimple Lie algebras, Dynkin introduced and studied the notion of π-systems. These precisely form the simple systems of such subalalgebras. We generalize the definition of π-systems and regular subalgebras and establish their fundamental properties. We show that π-systems, regular subalgebras and closed subroot systems of affine Kac-Moody algebras are in one-to-one correspondence. We completely classify and give explicit descriptions of the maximal closed subroot systems (or maximal π-systems in other words) of affine Kac-Moody algebras. As an application we describe a procedure to get the classification of all regular subalgebras of affine Kac Moody algebras in terms of their root systems. We also study the orbits of the Weyl group action on π-systems of symmetrizable Kac-Moody algebras, showing that for many π-systems of interest in physics, the action is transitive (up to negation). Finally, we formulate general principles for constructing π-systems and criteria for the non-existence of π-systems of certain types and use these to determine the set of maximal hyperbolic diagrams in ranks 3-10 relative to the partial order of admitting a π-system.

Summary

In this thesis, we undertake a systematic study of π-systems of symmetrizable Kac-Moody algebras and regular subalgebras of affine Kac-Moody algebras. A π-system Σ is a finite subset of the real roots of a Kac-Moody algebra $\mathfrak{g}$ satisfying the property that pairwise differences of elements of Σ are not roots of $\mathfrak{g}$. As part of his classification of regular semisimple subalgebras of semisimple Lie algebras, Dynkin introduced and studied the notion of π-systems. These precisely form the simple systems of such subalalgebras. We generalize the definition of π-systems and regular subalgebras and establish their fundamental properties. We show that π-systems, regular subalgebras and closed subroot systems of affine Kac-Moody algebras are in one-to-one correspondence. We completely classify and give explicit descriptions of the maximal closed subroot systems (or maximal π-systems in other words) of affine Kac-Moody algebras. As an application we describe a procedure to get the classification of all regular subalgebras of affine Kac Moody algebras in terms of their root systems. We also study the orbits of the Weyl group action on π-systems of symmetrizable Kac-Moody algebras, showing that for many π-systems of interest in physics, the action is transitive (up to negation). Finally, we formulate general principles for constructing π-systems and criteria for the non-existence of π-systems of certain types and use these to determine the set of maximal hyperbolic diagrams in ranks 3-10 relative to the partial order of admitting a π-system.

Chapter 1

Maximal Closed subroot systems of real affine root systems

1.1 Preliminaries

We denote the set of complex numbers by $\mathbb{C}$ and, respectively, the set of integers, non–negative integers, and positive integers by $\mathbb{Z}$, $\mathbb{Z}_+$, and $\mathbb{N}$.

We refer to [15] for the general theory of affine Lie algebras and we refer to [1, 19] for the general theory of affine root systems. Throughout, A will denote an indecomposable affine Cartan matrix, and S will denote the corresponding Dynkin diagram with the labeling of vertices as in Table Aff2 from [15, pg.54–55]. Let $\mathring{S}$ be the Dynkin diagram obtained from S by dropping the zero node and let $\mathring{A}$ be the Cartan matrix, whose Dynkin diagram is $\mathring{S}$.

Let $\mathfrak{g}$ and $\mathring{\mathfrak{g}}$ be the affine Lie algebra and the finite–dimensional simple Lie algebra associated to A and $\mathring{A}$ over $\mathbb{C}$, respectively. We shall realize $\mathring{\mathfrak{g}}$ as a subalgebra of $\mathfrak{g}$.

We fix $\mathring{\mathfrak{h}} \subseteq \mathfrak{h}$ Cartan subalgebras of $\mathring{\mathfrak{g}}$ and respectively $\mathfrak{g}$. Then we have

$$\mathfrak{h} = \mathring{\mathfrak{h}} \oplus \mathbb{C}K \oplus \mathbb{C}d.$$

where K is the canonical central element, and d is the derivation. Consider $\mathring{\mathfrak{h}}^*$ as a subspace of $\mathfrak{h}^*$ by setting $\lambda(K) = \lambda(d) = 0$ for all $\lambda \in \mathring{\mathfrak{h}}^*$. Let $\delta \in \mathfrak{h}$ be given by $\delta(d) = a_0$, where a_0 is 2 if $\mathfrak{g}$ is of type $\mathtt{A}_{\mathtt{2n}}^{(2)}$ and 1 otherwise, and $\delta(\mathring{\mathfrak{h}} \oplus \mathbb{C}K) = 0$. Let $(\ ,\ )$ be a standard symmetric non-degenerate invariant bilinear form on $\mathfrak{h}^*$.

1.1.1 Affine root system

We denote by $\Delta(\mathfrak{g})$ the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$, and the set of real roots of $\mathfrak{g}$ by $\Delta_{\mathrm{re}}(\mathfrak{g}) =: \Phi$ and the set of imaginary roots of $\mathfrak{g}$ by $\Delta_{\mathrm{im}}(\mathfrak{g})$. *We call Φ as affine root systems here. By abuse of notations, we say that Φ is of affine type X (resp. untwisted or twisted) if and only if $\Delta(\mathfrak{g})$ is of affine type X (resp. untwisted or twisted).* The set of roots of $\mathring{\mathfrak{g}}$ with respect to $\mathring{\mathfrak{h}}$ is denoted by $\mathring{\Phi}$ and note that $\mathring{\Phi}$ can be identified as a subroot system of Φ. Let Φ_ℓ and Φ_s (resp. $\mathring{\Phi}_\ell$ and $\mathring{\Phi}_s$) denote respectively the subsets of Φ (resp. $\mathring{\Phi}$) consisting of the long and short roots. We set

$$m = \begin{cases} 1, & \text{if } \Phi \text{ is of untwisted type} \\ 2, & \text{if } \Phi \text{ is of type } \mathtt{A}_{\mathtt{2n}}^{(2)}\ (\mathtt{n} \geq \mathtt{1}), \mathtt{A}_{\mathtt{2n-1}}^{(2)}\ (\mathtt{n} \geq \mathtt{3}), \mathtt{D}_{\mathtt{n+1}}^{(2)}\ (\mathtt{n} \geq \mathtt{2}) \text{ or } \mathtt{E}_{\mathtt{6}}^{(2)} \\ 3, & \text{if } \Phi \text{ is of type } \mathtt{D}_{\mathtt{4}}^{(3)}. \end{cases}$$

We have (see [15, Page no. 83]) $\Phi = \{\alpha + r\delta : \alpha \in \mathring{\Phi}, r \in \mathbb{Z}\}$ if $m = 1$ and

$$\Phi = \{\alpha + r\delta : \alpha \in \mathring{\Phi}_s, r \in \mathbb{Z}\} \cup \{\alpha + mr\delta : \alpha \in \mathring{\Phi}_\ell, r \in \mathbb{Z}\}$$

if $m = 2$ or 3, but Φ is not of type $\mathtt{A}_{\mathtt{2n}}^{(2)}$ and else

$$\Phi = \{\tfrac{1}{2}(\alpha+(2r-1)\delta : \alpha \in \mathring{\Phi}_\ell, r \in \mathbb{Z}\} \cup \{\alpha+r\delta : \alpha \in \mathring{\Phi}_s, r \in \mathbb{Z}\} \cup \{\alpha+2r\delta : \alpha \in \mathring{\Phi}_\ell, r \in \mathbb{Z}\}.$$

1.1.2 Weyl group

Given $\alpha \in \Phi$, we denote by $\alpha^\vee \in \mathfrak{h}$ the coroot associated to α. Then we set $\langle \beta, \alpha^\vee \rangle := \beta(\alpha^\vee) = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$. Define reflections $\mathbf{s}_\alpha \colon \mathfrak{h}^* \to \mathfrak{h}^*$ for $\alpha \in \Phi$ as follows:

$$\mathbf{s}_\alpha(\beta) = \beta - \langle \beta, \alpha^\vee \rangle \alpha$$

where $\beta \in \mathfrak{h}^*$. For $\alpha \in \mathring{\Phi}$, $\mathbf{s}_\alpha$ restricts to the reflection in α on $\mathring{\mathfrak{h}}^*$. We let $W := \langle \mathbf{s}_\alpha : \alpha \in \Phi \rangle$ denote the Weyl group of $\mathfrak{g}$ and denote by $\mathring{W} := \langle \mathbf{s}_\alpha : \alpha \in \mathring{\Phi} \rangle$ the Weyl group of $\mathring{\mathfrak{g}}$ generated by those reflections.

1.1.3 Definitions

In this section, we recall some general definitions and facts about finite and affine root systems.

Definition 1.1.1. A proper non–empty subset Ψ of Φ (resp., $\mathring{\Phi}$) is called

1. a subroot system of Φ (resp., $\mathring{\Phi}$), if $\mathbf{s}_\alpha(\beta) \in \Psi$ for all $\alpha, \beta \in \Psi$;
2. closed in Φ (resp., $\mathring{\Phi}$), if $\alpha, \beta \in \Psi$ and $\alpha + \beta \in \Phi$ (resp., $\mathring{\Phi}$) implies $\alpha + \beta \in \Psi$;
3. closed subroot system of Φ (resp., $\mathring{\Phi}$), if it is both subroot system and closed.

Definition 1.1.2. A proper closed subroot system Ψ of Φ (resp., $\mathring{\Phi}$) is said to be a maximal closed subroot system of Φ (resp., $\mathring{\Phi}$) if $\Psi \subseteq \Delta \subsetneq \Phi$ (resp., $\mathring{\Phi}$) implies $\Delta = \Psi$ for any closed subroot system Δ of Φ (resp., $\mathring{\Phi}$).

Definition 1.1.3. Let $\Psi \leq \Phi$ be a subroot system. The gradient root system associated with Ψ is defined to be

$$\mathrm{Gr}(\Psi) := \left\{ (\alpha + r\delta)|_{\mathring{\mathfrak{h}}} : \alpha + r\delta \in \Psi \right\},$$

where recall that $\mathring{\mathfrak{h}}$ is the Cartan subalgebra of $\mathring{\mathfrak{g}}$. Since $\delta|_{\mathring{\mathfrak{h}}} = 0$, we have $(\alpha + r\delta)|_{\mathring{\mathfrak{h}}} = \alpha|_{\mathring{\mathfrak{h}}} = \alpha$ for $\alpha + r\delta \in \Psi$. In particular we have

$$\mathrm{Gr}(\Phi) = \begin{cases} \mathring{\Phi} \cup \frac{1}{2}\mathring{\Phi}_\ell & \text{if } \widehat{\mathfrak{g}} \text{ is of type } \mathtt{A}_{\mathtt{2n}}^{(2)} \\ \mathring{\Phi} & \text{otherwise.} \end{cases}$$

The definition of $\mathrm{Gr}(\Psi)$ is dependent on the ambient root system Φ. But we do not want to put Φ as an additional parameter in the notation. Note that $\mathrm{Gr}(\Psi)$ does not need be a reduced root system in general. For example, $\mathrm{Gr}(\Phi)$ is non-reduced finite root system of type $\mathtt{BC_n}$ when $\mathfrak{g}$ is of type $\mathtt{A}_{\mathtt{2n}}^{(2)}$. It is easy to see that the gradient root system associated with Ψ is a subroot system of $\mathrm{Gr}(\Phi)$ in the sense of Definition 1.1.1(1). We say $\mathrm{Gr}(\Psi)$ is reduced if $\mathrm{Gr}(\Psi)$ does not contain a subroot system of type $\mathtt{BC_r}$ for any $r \geq 1$. The Weyl group of $\mathrm{Gr}(\Psi)$ generated by $\{\mathbf{s}_\alpha : \alpha \in \mathrm{Gr}(\Psi)\}$ is denoted by $W_{\mathrm{Gr}(\Psi)}$.

Definition 1.1.4. Let $\Psi \leq \mathrm{Gr}(\Phi)$ be a subroot system. The lift of Ψ in Φ is defined to be

$$\widehat{\Psi} := \bigcup_{\alpha \in \Psi} \{\alpha + r\delta : \text{ for all } r \text{ such that } \alpha + r\delta \in \Phi\}$$

It is easy to see that the lift $\widehat{\Psi}$ of Ψ is a subroot system of Φ.

Definition 1.1.5. Let Ψ be an irreducible subroot system of Φ. We say that Ψ is of type $\mathtt{X}_{\mathtt{n}}^{(\mathtt{r})}$ if there exists a vector space isomorphism $\varphi : \mathbb{R}\Psi \to \mathbb{R}\Gamma$ such that

$$\varphi(\Psi) = \Gamma \quad \text{and} \quad \langle \beta, \alpha^\vee \rangle = \langle \varphi(\beta), \varphi(\alpha^\vee) \rangle \quad \text{for all} \quad \alpha, \beta \in \Psi,$$

where Γ is a real root system of type $\mathtt{X}_{\mathtt{n}}^{(\mathtt{r})}$ and $\mathbb{R}\Psi$ (resp., $\mathbb{R}\Gamma$) denotes the vector space spanned by Ψ (resp., Γ) over $\mathbb{R}$.

Let Ψ be a reducible subroot system of Φ. We say that Ψ is of type $\mathtt{X}_{\mathtt{n}_1}^{(\mathtt{r}_1)} \oplus \mathtt{X}_{\mathtt{n}_2}^{(\mathtt{r}_2)} \oplus \cdots \oplus \mathtt{X}_{\mathtt{n}_k}^{(\mathtt{r}_k)}$ if $\Psi = \Psi_1 \oplus \Psi_2 \oplus \cdots \oplus \Psi_k$ such that Ψ_i is irreducible for all $1 \leq i \leq k$, Ψ_i are mutually orthogonal and Ψ_i is of type $\mathtt{X}_{\mathtt{n}_i}^{(\mathtt{r}_i)}$ for all $1 \leq i \leq k$.

Remark 1.1.6. Notice that the vector space sum of the irreducible components of a reducible root system need not be direct. For example, consider the affine root system Δ of type $\mathtt{G}_2^{(1)}$ and its real roots $\Phi = \{\alpha + n\delta : \alpha \in \mathring{\Phi}, n \in \mathbb{Z}\}$ where $\mathring{\Phi}$ is of type $\mathtt{G}_2$. Let $\{\alpha_1, \alpha_2\}$ be the simple system of $\mathring{\Phi}$, such that α_2 is a short root. Then define

$$\Psi = \{\pm\alpha_2 + n\delta : n \in \mathbb{Z}\} \cup \{\pm\theta + n\delta : n \in \mathbb{Z}\},$$

where θ is the long root of $\mathring{\Phi}$. Clearly, Ψ is a closed subroot system of type $\mathtt{A}_1^{(1)} \oplus \mathtt{A}_1^{(1)}$ but the sum of vector spaces spanned by each component is not direct.

The following Lemma is immediate from the above definitions.

Lemma 1.1.7. *Let Φ be an irreducible affine root system and let* $\mathrm{Gr}(\Phi)$ *be its corresponding gradient root system. If Ψ is a closed subroot system of* $\mathrm{Gr}(\Phi)$ *then the lift $\widehat{\Psi}$ is also a closed subroot system of Φ.*

1.2 Maximal closed subroot systems of an irreducible finite crystallographic root system

We make the following conventions throughout this chapter $\mathtt{B}_1 = \mathtt{C}_1 = \mathtt{D}_1 = \mathtt{A}_1$, $\mathtt{B}_2 = \mathtt{C}_2$, $\mathtt{D}_2 = \mathtt{A}_1 \oplus \mathtt{A}_1$, $\mathtt{D}_3 = \mathtt{A}_3$, $\mathtt{A}_1^{(1)} = \mathtt{B}_1^{(1)} = \mathtt{C}_1^{(1)}$, $\mathtt{B}_2^{(1)} = \mathtt{C}_2^{(1)}$, $\mathtt{D}_2^{(1)} = \mathtt{A}_1^{(1)} \oplus \mathtt{A}_1^{(1)}$, $\mathtt{A}_3^{(1)} = \mathtt{D}_3^{(1)}$, $\mathtt{A}_1^{(2)} = \mathtt{A}_1^{(1)}$ and $\mathtt{A}_3^{(2)} = \mathtt{D}_3^{(2)}$. Below we list all maximal closed subroot systems of an irreducible finite crystallographic root system of rank n from [16, Page 136].

Table 1.1: Types of maximal closed subroot systems of irreducible finite root systems

Type	Reducible	Irreducible
$\mathtt{A}_\mathtt{n}$	$\mathtt{A}_\mathtt{r} \oplus \mathtt{A}_{\mathtt{n-r-1}}$ $(0 \le \mathtt{r} \le \mathtt{n}-2)$	$\mathtt{A}_{\mathtt{n}-1}$
$\mathtt{B}_\mathtt{n}$	$\mathtt{B}_\mathtt{r} \oplus \mathtt{D}_{\mathtt{n-r}}$ $(1 \le \mathtt{r} \le \mathtt{n}-2)$	$\mathtt{B}_{\mathtt{n}-1}$, $\mathtt{D}_\mathtt{n}$
$\mathtt{C}_\mathtt{n}$	$\mathtt{C}_\mathtt{r} \oplus \mathtt{C}_{\mathtt{n-r}}$ $(1 \le \mathtt{r} \le \mathtt{n}-1)$	$\mathtt{A}_{\mathtt{n}-1}$
$\mathtt{D}_\mathtt{n}$	$\mathtt{D}_\mathtt{r} \oplus \mathtt{D}_{\mathtt{n-r}}$ $(2 \le \mathtt{r} \le \mathtt{n}-2)$	$\mathtt{A}_{\mathtt{n}-1}$, $\mathtt{D}_{\mathtt{n}-1}$
$\mathtt{E}_6$	$\mathtt{A}_5 \oplus \mathtt{A}_1$, $\mathtt{A}_2 \oplus \mathtt{A}_2 \oplus \mathtt{A}_2$	$\mathtt{D}_5$
$\mathtt{E}_7$	$\mathtt{A}_5 \oplus \mathtt{A}_2$, $\mathtt{A}_1 \oplus \mathtt{D}_6$	$\mathtt{E}_6$, $\mathtt{A}_7$
$\mathtt{E}_8$	$\mathtt{A}_1 \oplus \mathtt{E}_7$, $\mathtt{E}_6 \oplus \mathtt{A}_2$, $\mathtt{A}_4 \oplus \mathtt{A}_4$	$\mathtt{D}_8$, $\mathtt{A}_8$
$\mathtt{F}_4$	$\mathtt{A}_2 \oplus \mathtt{A}_2$, $\mathtt{C}_3 \oplus \mathtt{A}_1$	$\mathtt{B}_4$
$\mathtt{G}_2$	$\mathtt{A}_1 \oplus \mathtt{A}_1$	$\mathtt{A}_2$

1.2.1 Characterization of closed subroot systems

We will closely follow the arguments in [7] (see also [6]) to complete the classification of maximal closed subroot systems of affine root systems. The authors of [7] considered only the untwisted affine root systems or more generally considered real root systems of loop algebras of Kac–Moody algebras in [7]. Here in this chapter we will deal with both untwisted and twisted affine root systems. We leave out the proofs of most of the results presented in this section as it closely follows the arguments of [7].

Recall that Φ is the set of real roots of the irreducible affine Kac–Moody Lie algebra $\mathfrak{g}$. Let Ψ be a subroot system of Φ. Define

$$Z_\alpha(\Psi) = \{r : \alpha + r\delta \in \Psi\}, \text{ for } \alpha \in \mathrm{Gr}(\Psi).$$

It is easy to see that $\Psi = \{\alpha + r\delta : \alpha \in \mathrm{Gr}(\Psi), r \in Z_\alpha(\Psi)\}$. We immediately have (see, Lemma 8 in [7])

$$(1.2.1) \qquad Z_\beta(\Psi) - \langle \beta, \alpha^\vee \rangle Z_\alpha(\Psi) \subseteq Z_{\mathbf{s}_\alpha(\beta)}(\Psi), \text{ for all } \alpha, \beta \in \mathrm{Gr}(\Psi).$$

Lemma 1.2.1 ([7], Lemma 13)**.** *Let Φ be an irreducible affine root system and let Ψ be a subroot system of Φ and assume that $\mathrm{Gr}(\Psi)$ is reduced. Let Γ be a simple system of $\mathrm{Gr}(\Psi)$ and let $p : \Gamma \to \mathbb{Z}$ be an arbitrary function. Then there exists a unique $\mathbb{Z}$–linear extension p to $\mathrm{Gr}(\Psi)$, which we denote again by p for simplicity, $p : \mathrm{Gr}(\Psi) \to \mathbb{Z}$ given by $\alpha \mapsto p_\alpha$ satisfying*

$$(1.2.2) \qquad p_\beta - \langle \beta, \alpha^\vee \rangle p_\alpha = p_{\boldsymbol{s}_\alpha(\beta)}$$

for all $\alpha, \beta \in \mathrm{Gr}(\Psi)$.

The following proposition is very crucial.

Proposition 1.2.2. *Let Φ be an irreducible affine root system and let Ψ be a subroot system of Φ. Then there exists a function $p^\Psi : \mathrm{Gr}(\Psi) \to \mathbb{Z}, \alpha \mapsto p^\Psi_\alpha$, and non-negative integers n^Ψ_α for each $\alpha \in \mathrm{Gr}(\Psi)$ such that $Z_\alpha(\Psi) = p^\Psi_\alpha + n^\Psi_\alpha \mathbb{Z}$. Moreover the function p^Ψ is $\mathbb{Z}-$linear if $\mathrm{Gr}(\Psi)$ is reduced.*

Proof. We will first assume that $\mathrm{Gr}(\Psi)$ is reduced. Let Γ be a simple system of $\mathrm{Gr}(\Psi)$ and choose arbitrary elements $p^\Psi_\alpha \in Z_\alpha(\Psi)$ for each $\alpha \in \Gamma$. Define a function

$p^\Psi : \Gamma \to \mathbb{Z}$ given by $\alpha \mapsto p^\Psi_\alpha$. Now, fix the unique $\mathbb{Z}$–linear extension of p^Ψ to $\mathrm{Gr}(\Psi)$ as in Lemma 1.2.1. Define

$$Z'_\alpha(\Psi) = Z_\alpha(\Psi) - p^\Psi_\alpha = \{r - p^\Psi_\alpha : r \in Z_\alpha(\Psi)\} \text{ for } \alpha \in \mathrm{Gr}(\Psi).$$

Since each root of $\mathrm{Gr}(\Psi)$ is conjugate to some simple root by an element in $W_{\mathrm{Gr}(\Psi)}$, we get $p^\Psi_\alpha \in Z_\alpha(\Psi)$, for all $\alpha \in \mathrm{Gr}(\Psi)$ and

$$Z'_\beta(\Psi) - \langle \beta, \alpha^\vee \rangle Z'_\alpha(\Psi) \subseteq Z'_{\mathbf{s}_\alpha(\beta)}(\Psi), \text{ for all } \alpha, \beta \in \mathrm{Gr}(\Psi),$$

using the equation (1.2.1) and (1.2.2). One can easily see that $Z'_\alpha(\Psi)$ are subgroups for all $\alpha \in \mathrm{Gr}(\Psi)$, since $0 \in Z'_\alpha(\Psi)$, $Z'_\alpha(\Psi) = Z'_{-\alpha}(\Psi)$ and $Z'_\alpha(\Psi) + 2Z'_\alpha(\Psi) = Z'_\alpha(\Psi)$ for all $\alpha \in \mathrm{Gr}(\Psi)$ (proof of this fact is same as the proof of Lemma 22 in [7]). Hence there exists $n^\Psi_\alpha \in \mathbb{Z}_+$ for each $\alpha \in \mathrm{Gr}(\Psi)$ such that $Z'_\alpha(\Psi) = n^\Psi_\alpha \mathbb{Z}$. This completes the proof in this case.

We are now left with the case $\mathrm{Gr}(\Psi)$ is non-reduced. Since the sets $Z_\alpha(\Psi)$ depends only on the individual irreducible components of Ψ, we can assume that Ψ is irreducible. In particular, $\mathrm{Gr}(\Psi)$ is of type $\mathtt{BC_r}$ for some $r \geq 1$. So, we have

$$\mathrm{Gr}(\Psi) = \big\{ \pm \epsilon_i, \pm 2\epsilon_i, \pm\epsilon_i \pm \epsilon_j : 1 \leq i \neq j \leq r \big\}$$

if $r \geq 2$ or $\mathrm{Gr}(\Psi) = \{\pm\epsilon_1, \pm 2\epsilon_1\}$ if $r = 1$ (see [4, Page no. 547]). Write $\mathrm{Gr}(\Psi)_s = \{\pm\epsilon_i : 1 \leq i \leq r\}$, $\mathrm{Gr}(\Psi)_{\mathrm{im}} = \{\pm\epsilon_i \pm \epsilon_j : 1 \leq i \neq j \leq r\}$ and $\mathrm{Gr}(\Psi)_\ell = \{\pm 2\epsilon_i : 1 \leq i \leq r\}$. By convention, we have $\mathrm{Gr}(\Psi)_{\mathrm{im}} = \emptyset$ if $r = 1$. Let $\Gamma = \{\alpha_1 = \epsilon_1 - \epsilon_2, \cdots, \alpha_{r-1} = \epsilon_{r-1} - \epsilon_r, \alpha_r = \epsilon_r\}$ be the simple system of $\mathrm{Gr}(\Psi)$ and here by convention we have $\Gamma = \{\epsilon_1\}$ when $r = 1$. Choose arbitrary elements $p^\Psi_\alpha \in Z_\alpha(\Psi)$ for each $\alpha \in \Gamma$ and define the function $p^\Psi : \Gamma \to \frac{1}{2}\mathbb{Z}$, $\alpha \mapsto p^\Psi_\alpha$ as before. Fix the unique $\mathbb{Z}-$linear extension of $\overline{p^\Psi}$ to $\mathrm{Gr}(\Psi)$ as in Lemma 1.2.1. Since the long roots of $\mathrm{Gr}(\Psi)$ are not Weyl group conjugate to simple roots, we will not have

$\overline{p_\alpha^\Psi} \in Z_\alpha(\Psi)$ for all long roots $\alpha \in \mathrm{Gr}(\Psi)_\ell$ as before in reduced case. But this is the only obstruction that we have in this case. To overcome this issue, first fix a $\mathbb{Z}-$linear extension of $p^\Psi : \Gamma \to \frac{1}{2}\mathbb{Z}$ to $p^\Psi : \mathrm{Gr}(\Psi)_s \cup \mathrm{Gr}(\Psi)_{\mathrm{im}} \to \frac{1}{2}\mathbb{Z}$ and choose $p_\alpha^\Psi \in Z_\alpha(\Psi)$ arbitrarily for the positive roots of $\mathrm{Gr}(\Psi)_\ell$. Then we see that $-p_\alpha^\Psi \in Z_{-\alpha}(\Psi)$ for $\alpha \in \mathrm{Gr}(\Psi)_\ell$. So, we take $p_{-\alpha}^\Psi := -p_\alpha^\Psi$ for the negative roots of $\mathrm{Gr}(\Psi)_\ell$ and define a natural extension

$$p^\Psi : \mathrm{Gr}(\Psi) \to \frac{1}{2}\mathbb{Z} \text{ of } p^\Psi : \mathrm{Gr}(\Psi)_s \cup \mathrm{Gr}(\Psi)_{\mathrm{im}} \to \frac{1}{2}\mathbb{Z}$$

by assigning these arbitrarily chosen p_α^Ψ to α for each long root α. Now, note that this new extension $p^\Psi : \mathrm{Gr}(\Psi) \to \frac{1}{2}\mathbb{Z}$ is no longer $\mathbb{Z}-$linear map. As before, we define $Z'_\alpha(\Psi) = Z_\alpha(\Psi) - p_\alpha^\Psi$ for all $\alpha \in \mathrm{Gr}(\Psi)$. Then by definition of $Z'_\alpha(\Psi)$, we have $0 \in Z'_\alpha(\Psi)$ for all $\alpha \in \mathrm{Gr}(\Psi)$. Note that $Z_\alpha(\Psi)$ satisfies the equation (1.2.1), which implies that

$$Z'_\beta(\Psi) - \langle \beta, \alpha^\vee \rangle Z'_\alpha(\Psi) \subseteq Z'_{\mathbf{s}_\alpha(\beta)}(\Psi) + (p^\Psi_{\mathbf{s}_\alpha(\beta)} - (p_\beta^\Psi - \langle \beta, \alpha^\vee \rangle p_\alpha^\Psi)), \text{ for all } \alpha, \beta \in \mathrm{Gr}(\Psi).$$

Since $p_\alpha^\Psi = -p_\alpha^\Psi$ for all $\alpha \in \mathrm{Gr}(\Psi)$, we get $Z'_\alpha(\Psi) - 2Z'_\alpha(\Psi) \subseteq Z'_{-\alpha}(\Psi)$ for all $\alpha \in \mathrm{Gr}(\Psi)$. This implies $Z'_{-\alpha}(\Psi) = Z'_\alpha(\Psi)$ and $Z'_\alpha(\Psi) + 2Z'_\alpha(\Psi) = Z'_\alpha(\Psi)$ for all $\alpha \in \mathrm{Gr}(\Psi)$. Precisely this fact and $0 \in Z'_\alpha(\Psi), \alpha \in \mathrm{Gr}(\Psi)$ used in the proof of [7, Lemma 22] to prove that $Z'_\alpha(\Psi)$ is a subgroup of $\mathbb{Z}$ for all $\alpha \in \mathrm{Gr}(\Psi)$. Note that for $\alpha, \beta \in \mathrm{Gr}(\Psi)$ we have $\mathbf{s}_{\alpha + p_\alpha^\Psi \delta}(\beta + p_\beta^\Psi \delta) = \mathbf{s}_\alpha(\beta) + (p_\beta^\Psi - \langle \beta, \alpha^\vee \rangle p_\alpha^\Psi)\delta$, which implies that $p_\beta^\Psi - \langle \beta, \alpha^\vee \rangle p_\alpha^\Psi \in Z_{\mathbf{s}_\alpha(\beta)}(\Psi)$. Hence $(p^\Psi_{\mathbf{s}_\alpha(\beta)} - (p_\beta^\Psi - \langle \beta, \alpha^\vee \rangle p_\alpha^\Psi))$ must be in $Z'_{\mathbf{s}_\alpha(\beta)}(\Psi)$ for all $\alpha, \beta \in \mathrm{Gr}(\Psi)$. Thus, we have

$$Z'_\beta(\Psi) - \langle \beta, \alpha^\vee \rangle Z'_\alpha(\Psi) \subseteq Z'_{\mathbf{s}_\alpha(\beta)}(\Psi) \text{ for all } \alpha, \beta \in \mathrm{Gr}(\Psi).$$

as before. Since the sets $Z'_\alpha(\Psi)$ are subgroups of $\mathbb{Z}$, there exists $n_\alpha^\Psi \in \mathbb{Z}_+$ such that $Z_\alpha(\Psi) = p_\alpha^\Psi + n_\alpha^\Psi \mathbb{Z}$ for all $\alpha \in \mathrm{Gr}(\Psi)$. This completes the proof in this case. $\square$

From the Proposition 1.2.2, it is clear that a subroot system Ψ of Φ is completely determined by the gradient subroot system $\mathrm{Gr}(\Psi)$ and the cosets $Z_\alpha(\Psi) = p_\alpha^\Psi + n_\alpha^\Psi\mathbb{Z}$, $\alpha \in \mathrm{Gr}(\Psi)$. Naturally if Ψ is closed in Φ, then the "closedness property of Ψ in Φ" will give us some more restrictions on the gradient subroot systems and the cosets $Z_\alpha(\Psi)$. We will completely characterize these restrictions on the gradient subroot systems $\mathrm{Gr}(\Psi)$ and the cosets $Z_\alpha(\Psi)$ corresponding to "closedness property of Ψ in Φ" in Proposition 1.2.6, 1.2.7, 1.2.8, 1.3.1, 1.4.2 and use this information to determine all possible maximal closed subroot systems Ψ of Φ. The following lemma tells us about the relationships between the integers n_α^Ψ. Proof of this lemma closely follows the arguments of [7, Lemma 14] and only uses the fact that

$$Z'_\beta(\Psi) - \langle \beta, \alpha^\vee \rangle Z'_\alpha(\Psi) \subseteq Z'_{\mathbf{s}_\alpha(\beta)}(\Psi) \text{ for all } \alpha, \beta \in \mathrm{Gr}(\Phi),$$

so we will omit the proof.

Lemma 1.2.3. *[Lemma 14, [7]] Let Ψ be a subroot system of Φ and let n_α^Ψ be defined as above. We have $\langle \beta, \alpha^\vee \rangle n_\alpha^\Psi \mathbb{Z} \subseteq n_\beta^\Psi \mathbb{Z}$ for all $\alpha, \beta \in \mathrm{Gr}(\Psi)$, and $n_\alpha^\Psi = n_\beta^\Psi$ for all $\alpha, \beta \in \mathrm{Gr}(\Psi)$ with $\beta \in W_{\mathrm{Gr}(\Psi)}\alpha$. In particular if $n_\alpha^\Psi = 0$ for some $\alpha \in \mathrm{Gr}(\Psi)$ then $n_\beta^\Psi = 0$ for all $\beta \in W_{\mathrm{Gr}(\Psi)}\alpha$.*

Note that when $n_\beta^\Psi \neq 0$, we have $\langle \beta, \alpha^\vee \rangle n_\alpha^\Psi \mathbb{Z} \subseteq n_\beta^\Psi \mathbb{Z}$ if and only if n_β^Ψ divides $\langle \beta, \alpha^\vee \rangle n_\alpha^\Psi$.

1.2.2 Reducible gradient

Suppose $\mathrm{Gr}(\Psi)$ is reducible say $\mathrm{Gr}(\Psi) = \Psi_1 \oplus \cdots \oplus \Psi_k$, then by Lemma 1.2.3 for each $1 \leq i \leq k$ we have $n_\alpha^\Psi = n_\beta^\Psi$ for all $\alpha, \beta \in (\Psi_i)_\ell$ (resp. for all $\alpha, \beta \in (\Psi_i)_s$ and for all $\alpha, \beta \in (\Psi_i)_{\mathrm{im}}$), denote this unique number by $n_\ell^{\Psi_i}(\Psi)$ (resp. $n_s^{\Psi_i}(\Psi)$ and $n_{\mathrm{im}}^{\Psi_i}(\Psi)$). We drop Ψ in $n_\ell^{\Psi_i}(\Psi)$ (resp. in $n_s^{\Psi_i}(\Psi)$ and in $n_{\mathrm{im}}^{\Psi_i}(\Psi)$) and simply denote it by $n_\ell^{\Psi_i}$ (resp. $n_s^{\Psi_i}$ and $n_{\mathrm{im}}^{\Psi_i}$) if the underlying subroot system Ψ is understood. Note that long roots (or short roots or intermediate roots) of $\mathrm{Gr}(\Psi)$ from the different components are not conjugate under the action of $W_{\mathrm{Gr}(\Psi)}$. In particular $n_\ell^{\Psi_1}, \cdots, n_\ell^{\Psi_k}$ (resp. $n_s^{\Psi_1}, \cdots, n_s^{\Psi_k}$ or $n_{\mathrm{im}}^{\Psi_1}, \cdots, n_{\mathrm{im}}^{\Psi_k}$) may not be equal. If $\mathrm{Gr}(\Psi)$ is irreducible, we denote $n_\ell^{\mathrm{Gr}(\Psi)}$ and (resp. $n_{\mathrm{im}}^{\mathrm{Gr}(\Psi)}$) by n_ℓ^Ψ and n_s^Ψ (resp. n_{im}^Ψ) or simply by n_ℓ and n_s (resp. n_{im}) if the underlying subroot system Ψ is understood. By convention, we have $n_s = n_{\mathrm{im}}$ in case $\mathrm{Gr}(\Psi)$ is of type $\mathtt{BC_1}$. Sometimes we will denote n_s as n_Ψ to emphasize its importance. We also simply denote $Z_\alpha(\Psi)$, p_α^Ψ and n_α^Ψ by Z_α, p_α, n_α if the underlying subroot system Ψ is understood.

1.2.3 Properties of n_α

The following Lemma compares the cosets Z_α of two subroot systems of Φ.

Lemma 1.2.4. *Let $\Psi \subseteq \Delta \subseteq \Phi$ be two subroot systems of Φ.*

1. *Then we have $\mathrm{Gr}(\Psi) \subseteq \mathrm{Gr}(\Delta)$.*

2. *The cosets satisfy $Z_\alpha(\Psi) \subseteq Z_\alpha(\Delta)$ for all $\alpha \in \mathrm{Gr}(\Psi)$ and in particular $n_\alpha^\Psi \mathbb{Z} \subseteq n_\alpha^\Delta \mathbb{Z}$ for all $\alpha \in \mathrm{Gr}(\Psi)$.*

3. *If $\mathrm{Gr}(\Psi) = \mathrm{Gr}(\Delta)$ and $n_\alpha^\Delta = n_\alpha^\Psi$ for all $\alpha \in \mathrm{Gr}(\Delta)$, then we have $\Psi = \Delta$.*

Proof. By the definition of gradient, we have $\mathrm{Gr}(\Psi) \subseteq \mathrm{Gr}(\Delta)$ and by the definition of $Z_\alpha(\Psi)$, we have $Z_\alpha(\Psi) \subseteq Z_\alpha(\Delta)$ for all $\alpha \in \mathrm{Gr}(\Psi)$. In particular, we have

$$p_\alpha^\Psi + n_\alpha^\Psi \mathbb{Z} \subseteq p_\alpha^\Delta + n_\alpha^\Delta \mathbb{Z}, \text{ for } \alpha \in \mathrm{Gr}(\Psi).$$

This implies $(p_\alpha^\Delta - p_\alpha^\Psi) \in n_\alpha^\Delta \mathbb{Z}$ and $n_\alpha^\Psi \mathbb{Z} \subseteq (p_\alpha^\Delta - p_\alpha^\Psi) + n_\alpha^\Delta \mathbb{Z} = n_\alpha^\Delta \mathbb{Z}$. This proves the Statement (2). Finally for the last part, assume that $\mathrm{Gr}(\Psi) = \mathrm{Gr}(\Delta)$ and $n_\alpha^\Delta = n_\alpha^\Psi$

for all $\alpha \in \mathrm{Gr}(\Delta)$. For $\alpha \in \mathrm{Gr}(\Delta)$, we have $(p_\alpha^\Delta - p_\alpha^\Psi) \in n_\alpha^\Delta \mathbb{Z}$, and hence

$$p_\alpha^\Psi + n_\alpha^\Delta \mathbb{Z} = p_\alpha^\Delta + n_\alpha^\Delta \mathbb{Z}.$$

This implies that $Z_\alpha(\Psi) = Z_\alpha(\Delta)$ for all $\alpha \in \mathrm{Gr}(\Delta)$ since $n_\alpha^\Delta = n_\alpha^\Psi$. Thus, we have $\Psi = \Delta$ since $\Psi = \{\alpha + r\delta : \alpha \in \mathrm{Gr}(\Psi), r \in Z_\alpha(\Psi)\}$ and $\Delta = \{\alpha + r\delta : \alpha \in \mathrm{Gr}(\Delta), r \in Z_\alpha(\Delta)\}$. This completes the proof. □

We record the following lemma for the future use.

Lemma 1.2.5. *Let Φ be an irreducible affine root system and let $\Psi \leq \Phi$ be a closed subroot system with an irreducible gradient subroot system $\mathrm{Gr}(\Psi)$. Then $n_\alpha^\Psi = 0$ for some $\alpha \in \mathrm{Gr}(\Psi)$ implies that $n_\beta^\Psi = 0$ for all $\beta \in \mathrm{Gr}(\Psi)$*

Proof. Suppose $n_\alpha^\Psi = 0$ for some $\alpha \in \mathrm{Gr}(\Phi)$. Then, since $\mathrm{Gr}(\Psi)$ is irreducible, given any $\beta \in \mathrm{Gr}(\Psi)$ there exists a finite sequence of roots $\beta_1 = \alpha, \cdots, \beta_r = \beta$ such that $(\beta_i, \beta_{i+1}) \neq 0$ for all $1 \leq i \leq r-1$. Then by Lemma 1.2.3, we have $\langle \beta_i, \beta_{i+1}^\vee \rangle n_{\beta_{i+1}}^\Psi \mathbb{Z} \subseteq n_{\beta_i}^\Psi \mathbb{Z}$ for all $1 \leq i \leq r-1$. From this it is clear that $n_{\beta_1}^\Psi = 0 \implies n_{\beta_2}^\Psi = 0 \implies \cdots \implies n_{\beta_r}^\Psi = 0$. Thus, we have $n_\beta^\Psi = 0$ for all $\beta \in \mathrm{Gr}(\Phi)$. This completes the proof.

□

1.2.4 Closed subroot systems of untwisted affine root systems

The following proposition determines the integers n_α for the closed subroot systems of untwisted affine root systems.

Proposition 1.2.6. *Let Φ be an irreducible untwisted affine root system.*

1. *Suppose Ψ is a closed subroot system of Φ with an irreducible gradient subroot system* $\mathrm{Gr}(\Psi)$*, then $n_\alpha = n_\beta$ for all $\alpha, \beta \in \mathrm{Gr}(\Psi)$. Denote this unique number by n_Ψ.*

2. *Suppose Ψ is a maximal closed subroot system of Φ with* $\mathrm{Gr}(\Psi) = \mathring{\Phi}$*, then n_Ψ must be a prime number.*

Proof. Suppose $n_\alpha = 0$ for some $\alpha \in \mathrm{Gr}(\Psi)$, then by Lemma 1.2.5, we have $n_\beta = 0$ for all $\beta \in \mathrm{Gr}(\Psi)$. Hence, the Statement (1) is clear in this case. So, assume that $n_\alpha \neq 0$ for all $\alpha \in \mathrm{Gr}(\Psi)$. Suppose $\mathrm{Gr}(\Psi)$ is simply laced, then we have $n_\alpha = n_\beta$ for all $\alpha, \beta \in \mathrm{Gr}(\Psi)$ by Lemma 1.2.3. Hence, the Statement (1) is immediate in this case. So, we assume that $\mathrm{Gr}(\Psi)$ is non simply-laced irreducible root system. We can choose two short roots α_1 and α_2 in $\mathrm{Gr}(\Psi)$ such that their sum $\alpha_1 + \alpha_2$ is a long root in $\mathrm{Gr}(\Psi)$. Then from Lemma 1.2.3, we have $n_{\alpha_1} = n_{\alpha_2} = n_s$ and $n_\ell = n_{\alpha_1+\alpha_2}$. As Ψ is closed we have

$$Z_{\alpha_1} + Z_{\alpha_2} = Z_{\alpha_1+\alpha_2}.$$

Since $p_{\alpha_1+\alpha_2} = p_{\alpha_1} + p_{\alpha_2}$, we get $Z'_{\alpha_1} + Z'_{\alpha_2} = Z'_{\alpha_1+\alpha_2}$, which implies that $n_\ell \mid n_s$. On the other hand $\langle \beta, \alpha^\vee \rangle = \pm 1$ for short root β and long root α, see [14, Page no. 45]. Using this and by Lemma 1.2.3, we get $n_s \mid n_\ell$ and hence $n_\ell = n_s$. This completes the proof of Statement (1).

For the second part, it follows from 1.2.2 and Statement (1) that there exists p_α such that $Z_\alpha = p_\alpha + n_\Psi \mathbb{Z}$ for all $\alpha \in \mathring{\Phi}$. Suppose $n_\Psi = 0$, then we have

$$\Psi = \{\alpha + p_\alpha^\Psi \delta : \alpha \in \mathrm{Gr}(\Phi)\} \subsetneq \Delta,$$

where Δ is a proper closed subroot system of Φ given by $\Delta = \{\alpha + (p_\alpha^\Psi + 2r)\delta : \alpha \in \mathrm{Gr}(\Phi), r \in \mathbb{Z}\}$. This is a contradiction to our assumption that Ψ is maximal closed subroot system in Φ, so we must have $n_\Psi \neq 0$. Suppose $n_\Psi = 1$, then it is immediate that $Z_\alpha = \mathbb{Z}$ for all $\alpha \in \mathring{\Phi}$. Hence, $\Psi = \Phi$ which is again a contradiction. So, we must have $n_\Psi \neq 1$. Suppose n_Ψ is not a prime number and let $n_\Psi = uv$ be a nontrivial factorization of n_Ψ, then we have

$$\Omega := \left\{\alpha + (p_\alpha + ur)\delta : \alpha \in \mathring{\Phi}, r \in \mathbb{Z}\right\}$$

is a closed subroot system of Φ since the function $\alpha \mapsto p_\alpha$ is $\mathbb{Z}-$linear and satisfies the Equation (1.2.2) and

$$\mathbf{s}_{\alpha+(p_\alpha+ur)\delta}(\beta + (p_\beta + ur')\delta) = \mathbf{s}_\alpha(\beta) + (p_{\mathbf{s}_\alpha(\beta)} + ur' - ur\langle\beta, \alpha^\vee\rangle)\delta$$

for $\alpha, \beta \in \mathring{\Phi}$ and $r, r' \in \mathbb{Z}$. But $\Psi \subsetneqq \Omega \subsetneqq \Phi$, which contradicts the fact that Ψ is maximal closed subroot system in Φ. This completes the proof of Statement (2). $\square$

1.2.5 Closed subroot systems of twisted affine root systems not of type $\mathtt{A}_{2n}^{(2)}$

We have the following proposition which is similar to Proposition 1.2.6 for twisted affine root systems not of type $\mathtt{A}_{2n}^{(2)}$. Recall the definition of m from Section 1.1.1.

Proposition 1.2.7. *Let Φ be an irreducible twisted affine root system not of type $\mathtt{A}_{2n}^{(2)}$ and let $\Psi \leq \Phi$ be a subroot system with an irreducible gradient subroot system $\mathrm{Gr}(\Psi)$. Let n_ℓ and n_s be defined as in Section 1.2.2.*

1. *Suppose Ψ is a closed subroot system of Φ such that $\mathrm{Gr}(\Psi)$ is simply laced, then we get $n_\alpha = n_\beta$ for all $\alpha, \beta \in \mathrm{Gr}(\Psi)$. Denote this unique number by n_Ψ.*

2. *Suppose Ψ is a closed subroot system of Φ such that $\mathrm{Gr}(\Psi)$ is non*

simply-laced, then we get $n_\ell = n_s$ if $m|n_s$ and we get $n_\ell = mn_s$ if $m \not| n_s$. Denote n_s by n_Ψ.

3. *Suppose Ψ is a maximal closed subroot system of Φ with $\mathrm{Gr}(\Psi) = \mathring{\Phi}$, then n_Ψ is a prime number.*

Proof. Suppose $n_\alpha = 0$ for some $\alpha \in \mathrm{Gr}(\Psi)$, then by Lemma 1.2.5, we have $n_\beta = 0$ for all $\beta \in \mathrm{Gr}(\Psi)$. Hence, the Statements (1) and (2) are clear in this case. So, we assume that $n_\alpha \neq 0$ for all $\alpha \in \mathrm{Gr}(\Psi)$. Suppose $\mathrm{Gr}(\Psi)$ is simply laced, then we have $n_\alpha = n_\beta$ for all $\alpha, \beta \in \mathrm{Gr}(\Psi)$ by Lemma 1.2.3. Hence, the Statement (1) follows. So, we assume that $\mathrm{Gr}(\Phi)$ is non-simply laced and irreducible to prove the Statement (2). Since $\mathrm{Gr}(\Psi)$ is irreducible and non simply-laced, we can choose two short roots α_1 and α_2 in $\mathrm{Gr}(\Psi)$ such that their sum $\alpha_1 + \alpha_2$ is a long root in $\mathrm{Gr}(\Psi)$. Again using Lemma 1.2.3, we have $n_{\alpha_1} = n_{\alpha_2} = n_s$ and $n_\ell = n_{\alpha_1+\alpha_2}$. As Ψ is closed, we have

$$(Z_{\alpha_1} + Z_{\alpha_2}) \cap m\mathbb{Z} = Z_{\alpha_1+\alpha_2}.$$

Since $p_{\alpha_1+\alpha_2} = p_{\alpha_1} + p_{\alpha_2}$ and $p_{\alpha_1+\alpha_2} \in m\mathbb{Z}$, we get $\left(Z'_{\alpha_1} + Z'_{\alpha_2}\right) \bigcap m\mathbb{Z} = Z'_{\alpha_1+\alpha_2}$, which implies that

$$n_s\mathbb{Z} \cap m\mathbb{Z} = n_\ell\mathbb{Z}.$$

Thus, we get $n_\ell = n_s$ if $m|n_s$ and $n_\ell = mn_s$ if $m \not| n_s$. This proves the Statement (2) of the proposition.

For the last part, observe that $\Psi < \Phi$ is properly contained in Φ since Ψ is a maximal closed subroot system of Φ. We know that there exists p_α such that $Z_\alpha = p_\alpha + n_\alpha\mathbb{Z}$ for all $\alpha \in \mathring{\Phi}$. Suppose $n_\Psi = 0$, then we have

$$\Psi = \{\alpha + p_\alpha^\Psi\delta : \alpha \in \mathrm{Gr}(\Phi)\} \subsetneq \Delta,$$

where Δ is a proper closed subroot system of Φ given by

$\Delta = \{\alpha + (p_\alpha^\Psi + mr)\delta : \alpha \in \mathrm{Gr}(\Phi), r \in \mathbb{Z}\}$. This is a contradiction to our assumption that Ψ is maximal closed subroot system in Φ, so we must have $n_\Psi \neq 0$. If $n_\Psi = 1$, then it is immediate that $n_\ell = m$ and $n_s = 1$. This implies that $Z_\alpha = \mathbb{Z}$ for short roots α and $Z_\alpha = m\mathbb{Z}$ for long roots α. Hence, we get $\Psi = \Phi$ since $\mathrm{Gr}(\Psi) = \mathring{\Phi}$, again a contradiction. Suppose n_Ψ is not a prime number, then let $n_\Psi = uv$ be a nontrivial factorization of n_Ψ such that $m|u$ if $m|n_\Psi$. Let

$$\Omega = \{\alpha + (p_\alpha + ur)\delta : \alpha \in \mathring{\Phi}, r \in \mathbb{Z}\}$$

if $\mathrm{Gr}(\Psi)$ is simply laced or $m \mid n_\Psi$. Otherwise let

$$\Omega = \{\alpha + (p_\alpha + mur)\delta, \beta + (p_\beta + ur)\delta : \alpha \in \mathring{\Phi}_\ell, \beta \in \mathring{\Phi}_s, r \in \mathbb{Z}\}.$$

We claim that Ω is a closed subroot system of Φ. Note that the function $\alpha \mapsto p_\alpha$ is $\mathbb{Z}-$linear and satisfies the Equation (1.2.2). Let $\alpha \in \mathring{\Phi}_\ell, \beta \in \mathring{\Phi}_s$, then for $r, r' \in \mathbb{Z}$ we have

$$\mathbf{s}_{\beta+(p_\beta+ur)\delta}(\alpha + (p_\alpha + mur')\delta) = \mathbf{s}_\beta(\alpha) + ((p_\alpha + mur') - (p_\beta + ur)\langle \alpha, \beta^\vee \rangle)\delta.$$

Since $p_{\mathbf{s}_\beta(\alpha)} = p_\alpha - \langle \alpha, \beta^\vee \rangle p_\beta$, we have

$$\mathbf{s}_{\beta+(p_\beta+ur)\delta}(\alpha + (p_\alpha + mur')\delta) = \mathbf{s}_\beta(\alpha) + (p_{\mathbf{s}_\beta(\alpha)} + mur' - ur\langle \alpha, \beta^\vee \rangle)\delta.$$

Now, since $\langle \alpha, \beta^\vee \rangle = \langle \beta, \alpha^\vee \rangle m$ and $\mathbf{s}_\beta(\alpha)$ is a long root, we have $\mathbf{s}_{\beta+(p_\beta+ur)\delta}(\alpha + (p_\alpha + mur')\delta) \in \Omega$. Similarly, for $\alpha \in \mathring{\Phi}_\ell, \beta \in \mathring{\Phi}_s$ and $r, r' \in \mathbb{Z}$ we have

$$\mathbf{s}_{\alpha+(p_\alpha+mur')\delta}(\beta + (p_\beta + ur)\delta) = \mathbf{s}_\alpha(\beta) + (p_{\mathbf{s}_\alpha(\beta)} + ur - mur'\langle \beta, \alpha^\vee \rangle)\delta \in \Omega$$

since $\mathbf{s}_\alpha(\beta)$ is a short root. Remaining cases are similarly done, so it proves that Ω

is a subroot system. Since sum of a short root and long root from $\mathring{\Phi}$ can not be a long root again, we get Ω is closed subroot system in Φ. But $\Psi \subsetneqq \Omega \subsetneqq \Phi$, which contradicts the fact that Ψ is a maximal closed subroot system in Φ. This completes the proof of Statement (3). $\square$

1.2.6 Closed subroot systems of $\mathtt{A}_{\mathtt{2n}}^{(2)}$

We have the following result which is analogues to the Propositions 1.2.6 and 1.2.7 in the $\mathtt{A}_{\mathtt{2n}}^{(2)}$ setting.

Proposition 1.2.8. *Let Φ be an irreducible twisted affine root system of type $\mathtt{A}_{\mathtt{2n}}^{(2)}$ and let $\Psi \leq \Phi$ be a subroot system with an irreducible gradient subroot system $\mathrm{Gr}(\Psi)$. Let n_ℓ, n_{im} and n_s be defined as in Section 1.2.2.*

1. *Suppose Ψ is a closed subroot system of Φ such that $\mathrm{Gr}(\Psi)$ is simply laced, then we get $n_\alpha = n_\beta$ for all $\alpha, \beta \in \mathrm{Gr}(\Psi)$.*
2. *Suppose Ψ is a closed subroot system of Φ such that $\mathrm{Gr}(\Psi)$ is non-simply laced and does not contain any short root, then we get $n_\ell = n_{\mathrm{im}}$ if $2|n_{\mathrm{im}}$ and we get $n_\ell = 2n_{\mathrm{im}}$ if $2 \not| n_{\mathrm{im}}$.*
3. *Suppose Ψ is a closed subroot system of Φ such that $\mathrm{Gr}(\Psi)$ is non-simply laced and does not contain any long root, then we get $n_s = n_{\mathrm{im}}$.*
4. *Suppose Ψ is a closed subroot system of Φ with $\mathrm{Gr}(\Psi)$ containing short, intermediate and long roots, then $n_s = n_{\mathrm{im}}$, $n_\ell = 2n_s$ and n_s is an odd number. Denote n_s by n_Ψ.*
5. *Suppose Ψ is a maximal closed subroot system of Φ with $\mathrm{Gr}(\Psi) = \mathrm{Gr}(\Phi)$, then n_Ψ must be a prime number.*

Proof. Suppose $n_\alpha = 0$ for some $\alpha \in \mathrm{Gr}(\Psi)$, then by Lemma 1.2.5, we have $n_\beta = 0$ for all $\beta \in \mathrm{Gr}(\Psi)$. Hence, the Statements (1), (2), (3) and (4) are clear in this case. So, we assume that $n_\alpha \neq 0$ for all $\alpha \in \mathrm{Gr}(\Psi)$. Suppose $\mathrm{Gr}(\Psi)$ is simply laced, then we have $n_\alpha = n_\beta$ for all $\alpha, \beta \in \mathrm{Gr}(\Psi)$ by Lemma 1.2.3. Hence, the Statement (1) follows. Suppose Ψ is a closed subroot system of Φ such that $\mathrm{Gr}(\Psi)$ does not contain any short root, then Ψ is a closed subroot system of $\mathtt{A}^{(2)}_{\mathtt{2n-1}}$. Hence, the Statement (2) follows from Proposition 1.2.7. Suppose Ψ is a closed subroot system of Φ such that $\mathrm{Gr}(\Psi)$ does not contain any long root. By Lemma 1.2.3, $n_s \mid n_{\mathrm{im}}$ and $n_{\mathrm{im}} \mid 2n_s$. Then by Proposition 1.2.2 and Lemma 1.2.5, we have $n_\alpha \in \mathbb{N}$ and $p_\alpha \in Z_\alpha(\Psi)$ such that $Z_\alpha(\Psi) = p_\alpha + n_\alpha\mathbb{Z}$ for all $\alpha \in \mathrm{Gr}(\Phi)$. If there is only one short root in $\mathrm{Gr}(\Psi)$, then we have $n_s = n_{\mathrm{im}}$ by convention. So assume that we can choose two short roots $\alpha, \beta \in \mathrm{Gr}(\Psi)$ such that $\alpha + \beta$ is an intermediate root. Then since Ψ is closed, we have

$$(p_\alpha + n_s\mathbb{Z}) + (p_\beta + n_s\mathbb{Z}) = p_{\alpha+\beta} + n_s\mathbb{Z} \subseteq p_{\alpha+\beta} + n_{\mathrm{im}}\mathbb{Z},$$

which implies that $n_s\mathbb{Z} \subseteq n_{\mathrm{im}}\mathbb{Z}$ and $n_{\mathrm{im}} \mid n_s$ and hence $n_s = n_{\mathrm{im}}$. This completes proof of Statement (3).

Suppose Ψ is a closed subroot system of Φ such that $\mathrm{Gr}(\Psi)$ contains short, intermediate and long roots, then $n_s = n_{\mathrm{im}}$ as before. By Lemma 1.2.3, $n_{\mathrm{im}} \mid n_\ell$ and $n_\ell \mid 2n_{\mathrm{im}}$. Then by Proposition 1.2.2 and Lemma 1.2.5, we have $n_\alpha \in \mathbb{N}$ and $p_\alpha \in Z_\alpha(\Psi)$ such that $Z_\alpha(\Psi) = p_\alpha + n_\alpha\mathbb{Z}$ for all $\alpha \in \mathrm{Gr}(\Phi)$. Since Ψ is closed, we have $Z_{2\alpha}(\Psi) - Z_\alpha(\Psi) \subseteq Z_\alpha(\Psi)$ for a short root $\alpha \in \mathrm{Gr}(\Psi)$. This implies that

$$(p_{2\alpha} - p_\alpha) + n_\ell\mathbb{Z} \subseteq p_\alpha + n_s\mathbb{Z} \text{ and hence } p_{2\alpha} + n_\ell\mathbb{Z} \subseteq (2p_\alpha + n_s\mathbb{Z}) \cap 2\mathbb{Z},$$

since $p_{2\alpha} + n_\ell\mathbb{Z} \subseteq 2\mathbb{Z}$. From this, we conclude that n_s must be odd since $2p_\alpha$ is odd. Since $(2p_\alpha + n_s\mathbb{Z}) \cap 2\mathbb{Z} \subseteq Z_{2\alpha}(\Psi) = p_{2\alpha} + n_\ell\mathbb{Z}$ we have

$$p_{2\alpha} + n_\ell\mathbb{Z} = (2p_\alpha + n_s\mathbb{Z}) \cap 2\mathbb{Z} = (2p_\alpha + n_s) + 2n_s\mathbb{Z}.$$

This implies, we must have $n_\ell = 2n_s$. This completes proof of Statement (4).

Suppose Ψ is a maximal closed subroot system with $\mathrm{Gr}(\Psi) = \mathrm{Gr}(\Phi)$ and $n_\alpha = 0$ for some $\alpha \in \mathrm{Gr}(\Phi)$, then by Lemma 1.2.5, we have $n_\beta = 0$ for all $\beta \in \mathrm{Gr}(\Phi)$. This implies that $\Psi = \{\alpha + p_\alpha^\Psi\delta : \alpha \in \mathrm{Gr}(\Phi)\} \subsetneq \Delta$, where Δ is a proper closed subroot system of Φ given by

$$\Delta = \{\alpha+(p_\alpha^\Psi+3r)\delta : \alpha \in \mathrm{Gr}(\Phi)_s \cup \mathrm{Gr}(\Phi)_{\mathrm{im}}, r \in \mathbb{Z}\} \cup \{\alpha+(p_\alpha^\Psi+6r)\delta : \alpha \in \mathrm{Gr}(\Phi)_\ell, r \in \mathbb{Z}\}.$$

Then Ψ can not be maximal closed subroot system in Φ, a contradiction to our assumption. Hence, $n_\alpha \neq 0$ for all $\alpha \in \mathrm{Gr}(\Phi)$. Suppose $n_\Psi = 1$, then we get $\Psi = \Phi$ from Statement (4), a contradiction. So, $n_\Psi \neq 1$. Now suppose n_Ψ is a composite number and $n_\Psi = pq$. Since n_Ψ is an odd integer, without loss of generality we can assume that p is an odd integer. Then $\Psi \subsetneq \Delta$, where Δ is a proper closed subroot system of Φ given by

$$\Delta = \{\alpha+(p_\alpha^\Psi+pr)\delta : \alpha \in \mathrm{Gr}(\Phi)_s \cup \mathrm{Gr}(\Phi)_{\mathrm{im}}, r \in \mathbb{Z}\} \cup \{\alpha+(p_\alpha^\Psi+2pr)\delta : \alpha \in \mathrm{Gr}(\Phi)_\ell, r \in \mathbb{Z}\}.$$

Hence, n_Ψ must be a prime number. This completes the proof. $\square$

1.3 Untwisted Case

Throughout this section we assume that Φ is an irreducible untwisted affine root system. Note that $\mathrm{Gr}(\Phi) = \mathring{\Phi}$ and $\widehat{\mathring{\Phi}} = \Phi$.

We need the following simple result to complete the classification of maximal closed subroot systems in this case. The Statement (2) of the following proposition already appears in the proof of [11, Lemma 4.1].

Proposition 1.3.1. *Let Φ be an irreducible untwisted affine root system and let $\Psi \leq \Phi$ be a subroot system.*

1. *If $\Psi \leq \Phi$ is a closed subroot system, then $\mathrm{Gr}(\Psi) \leq \mathring{\Phi}$ is a closed subroot system.*

2. *If $\Psi \leq \Phi$ is a maximal closed subroot system, then either $\mathrm{Gr}(\Psi) = \mathring{\Phi}$ or $\mathrm{Gr}(\Psi) \subsetneq \mathring{\Phi}$ is a maximal closed subroot system. In particular we get $\Psi = \widehat{\mathrm{Gr}(\Psi)}$ when $\mathrm{Gr}(\Psi) \subsetneq \mathring{\Phi}$.*

Proof. Statement (1) is immediate from the definition. Now, suppose $\mathrm{Gr}(\Psi) \neq \mathring{\Phi}$, then we claim that $\mathrm{Gr}(\Psi) \subsetneq \mathring{\Phi}$ is a maximal closed subroot system. Otherwise, there exist a closed subroot system Ω such that $\mathrm{Gr}(\Psi) \subsetneq \Omega \subsetneq \mathring{\Phi}$ which immediately implies that $\Psi \subsetneq \widehat{\Omega} \subsetneq \Phi$. This leads to a contradiction as $\widehat{\Omega}$ is closed in Φ by Lemma 1.1.7. Since $\widehat{\mathrm{Gr}(\Psi)}$ is a proper closed subroot system which contains Ψ, we must have $\Psi = \widehat{\mathrm{Gr}(\Psi)}$. This completes the proof of Statement (2). □

1.3.1 Main theorem for untwisted case

Now, we are ready to state our main theorem for untwisted case.

Theorem 1.3.2. *Let Ψ be a maximal closed subroot system of Φ.*

1. *If $\mathrm{Gr}(\Psi) = \mathring{\Phi}$, then there exists a $\mathbb{Z}$–linear function $p : \mathrm{Gr}(\Psi) \to \mathbb{Z}$ satisfying (1.2.2) and a prime number n_Ψ such that*

$$\Psi = \{\alpha + (p_\alpha + rn_\Psi)\delta : \alpha \in \mathrm{Gr}(\Psi), r \in \mathbb{Z}\}.$$

Conversely, given a $\mathbb{Z}$–linear function $p : \mathring{\Phi} \to \mathbb{Z}$ satisfying (1.2.2) and a prime number n_Ψ the subroot system Ψ defined above gives a maximal subroot system of Φ. The affine type of Ψ is same as affine type of Φ.

2. *If $\mathrm{Gr}(\Psi) \subsetneq \mathring{\Phi}$ is a maximal closed subroot system, then*

$$\Psi = \{\alpha + r\delta : \alpha \in \mathrm{Gr}(\Psi), r \in \mathbb{Z}\}.$$

Conversely, if $\mathring{\Psi}$ is a proper maximal subroot system of $\mathring{\Phi}$ then the lift $\widehat{\mathring{\Psi}}$ is a maximal subroot system of Φ. The affine type of $\widehat{\mathring{\Psi}}$ is $\mathtt{X}_\mathtt{n}^{(1)}$ if $\mathring{\Psi}$ is of finite type $\mathtt{X}_\mathtt{n}$.

Proof. Forward part of Statement (1) follows from the Proposition 1.2.6. For the converse part let $\Psi = \{\alpha + (p_\alpha + rn_\Psi)\delta : \alpha \in \mathrm{Gr}(\Psi), r \in \mathbb{Z}\}$, where the function $p : \mathrm{Gr}(\Psi) \to \mathbb{Z}$ is $\mathbb{Z}$–linear and satisfying (1.2.2) and n_Ψ is a prime number. It is easy to verify that Ψ is a closed subroot system of Φ since p is $\mathbb{Z}$–linear and satisfying (1.2.2). Now, suppose $\Psi \subsetneq \Delta \subseteq \Phi$, then $\mathrm{Gr}(\Delta) = \mathring{\Phi}$ since $\mathrm{Gr}(\Psi) = \mathring{\Phi}$. Now, by part (2) of Lemma 1.2.4, we have n_Δ divides n_Ψ. This implies $n_\Delta = 1$ or $n_\Delta = n_\Psi$ since n_Ψ is a prime number. If $n_\Delta = n_\Psi$, then by part (3) of Lemma 1.2.4, we get $\Psi = \Delta$, a contradiction. So, we must get $n_\Delta = 1$, this implies that $\Delta = \Phi$. This completes the proof of Statement (1).

Forward part of Statement (2) follows from the part (2) of Proposition 1.3.1 and the converse part is straightforward from the part (2) of Proposition 1.3.1 and the Lemma 1.2.4. □

Remark 1.3.3. Our main classification theorem for the untwisted case is indeed an immediate corollary of the results of [7], see also [6]. Essentially all the machineries were developed in [7] to complete the classification of maximal closed subroot system of untwisted affine root system. Since the purpose of their paper is to classify all the subroot systems in terms of the admissible subgroups of the coweight lattice of a root system Ψ, and the scaling functions on Ψ, the authors do not write Theorem 1.3.2 as a corollary of their results. The main purpose of this chapter is to get a similar classification theorem of maximal subroot systems for the twisted affine root system case as well.

We end this section by listing out all possible types of maximal closed subroot systems of irreducible untwisted affine root systems and give few examples to demonstrate how one gets the this list from Theorem 1.3.2 and Table 1.1.

Example 1.3.4. Let $\Phi = \mathtt{B}_{\mathtt{n}}^{(1)}$. Then $\mathring{\Phi} = \mathtt{B}_{\mathtt{n}} = \{\pm\epsilon_i, \pm\epsilon_i \pm \epsilon_j : 1 \leq i \neq j \leq n\}$. The root system $\mathtt{B}_{\mathtt{n}}$ has a maximal closed subroot system Δ of type $\mathtt{B}_{\mathtt{n-1}}$ with a simple system $\{\epsilon_2 - \epsilon_3, \epsilon_3 - \epsilon_4, \cdots, \epsilon_{n-1} - \epsilon_n, \epsilon_n\}$ (see [16, Page 136]). By Theorem 1.3.2, $\widehat{\Delta}$ is a maximal closed subroot system of Φ and by Definition 1.1.5, the type of $\widehat{\Delta}$ is $\mathtt{B}_{\mathtt{n-1}}^{(1)}$.

Example 1.3.5. Let $\Phi = \mathtt{G}_2^{(1)}$. Then
$\mathring{\Phi} = \mathtt{G}_2 = \{\epsilon_i - \epsilon_j, \pm(\epsilon_i + \epsilon_j - 2\epsilon_k) : 1 \leq i, j, k \leq 3, i \neq j\}$. The root system $\mathtt{G}_2$ has a maximal closed subroot system Δ of type $\mathtt{A}_1 \oplus \mathtt{A}_1$ with a simple system $\{\epsilon_1 - \epsilon_2, \epsilon_1 + \epsilon_2 - 2\epsilon_3\}$ (see [16, Page 136]). By Theorem 1.3.2, $\widehat{\Delta}$ is a maximal closed subroot system of Φ and the type of $\widehat{\Delta}$ is $\mathtt{A}_1^{(1)} \oplus \mathtt{A}_1^{(1)}$.

Example 1.3.6. Let $\Phi = \mathtt{D}_{\mathtt{n}}^{(1)}$. Then $\mathring{\Phi} = \mathtt{D}_{\mathtt{n}} = \{\pm\epsilon_i \pm \epsilon_j : 1 \leq i \neq j \leq n\}$. The root system $\mathtt{D}_{\mathtt{n}}$ has a maximal closed subroot system Δ of type $\mathtt{D}_{\mathtt{n-1}}$ with a simple system $\{\epsilon_2 - \epsilon_3, \epsilon_3 - \epsilon_4, \cdots, \epsilon_{n-1} - \epsilon_n, \epsilon_{n-1} + \epsilon_n\}$ (see [16, Page 136]). By Theorem 1.3.2, $\widehat{\Delta}$ is a maximal closed subroot system of Φ and the type of $\widehat{\Delta}$ is $\mathtt{D}_{\mathtt{n-1}}^{(1)}$.

The following table is immediate from Theorem 1.3.2 and Table 1.1.

Remark 1.3.7. The Table 1.2 has already appeared in [11] and note that the authors of [11] have omitted the possibility of a maximal closed subroot system $\mathtt{D}_{\mathtt{n-1}}^{(1)} \subset \mathtt{D}_{\mathtt{n}}^{(1)}$ in their list.

Table 1.2: Types of maximal closed subroot systems of irreducible untwisted affine root systems

Type	Reducible	Irreducible
$\mathtt{A}_{\mathtt{n}}^{(1)}$	$\mathtt{A}_{\mathtt{r}}^{(1)} \oplus \mathtt{A}_{\mathtt{n-r-1}}^{(1)}$ $(0 \le \mathtt{r} \le \mathtt{n}-1)$	$\mathtt{A}_{\mathtt{n}}^{(1)}$
$\mathtt{B}_{\mathtt{n}}^{(1)}$	$\mathtt{B}_{\mathtt{r}}^{(1)} \oplus \mathtt{D}_{\mathtt{n-r}}^{(1)}$ $(1 \le \mathtt{r} \le \mathtt{n}-2)$	$\mathtt{B}_{\mathtt{n-1}}^{(1)}$, $\mathtt{D}_{\mathtt{n}}^{(1)}$, $\mathtt{B}_{\mathtt{n}}^{(1)}$
$\mathtt{C}_{\mathtt{n}}^{(1)}$	$\mathtt{C}_{\mathtt{r}}^{(1)} \oplus \mathtt{C}_{\mathtt{n-r}}^{(1)}$ $(1 \le \mathtt{r} \le \mathtt{n}-1)$	$\mathtt{A}_{\mathtt{n-1}}^{(1)}$, $\mathtt{C}_{\mathtt{n}}^{(1)}$
$\mathtt{D}_{\mathtt{n}}^{(1)}$	$\mathtt{D}_{\mathtt{r}}^{(1)} \oplus \mathtt{D}_{\mathtt{n-r}}^{(1)}$ $(2 \le \mathtt{r} \le \mathtt{n}-2)$	$\mathtt{A}_{\mathtt{n-1}}^{(1)}$, $\mathtt{D}_{\mathtt{n-1}}^{(1)}$, $\mathtt{D}_{\mathtt{n}}^{(1)}$
$\mathtt{E}_{6}^{(1)}$	$\mathtt{A}_{5}^{(1)} \oplus \mathtt{A}_{1}^{(1)}$, $\mathtt{A}_{2}^{(1)} \oplus \mathtt{A}_{2}^{(1)} \oplus \mathtt{A}_{2}^{(1)}$	$\mathtt{D}_{5}^{(1)}$, $\mathtt{E}_{6}^{(1)}$
$\mathtt{E}_{7}^{(1)}$	$\mathtt{A}_{5}^{(1)} \oplus \mathtt{A}_{2}^{(1)}$, $\mathtt{A}_{1}^{(1)} \oplus \mathtt{D}_{6}^{(1)}$	$\mathtt{E}_{6}^{(1)}$, $\mathtt{A}_{7}^{(1)}$, $\mathtt{E}_{7}^{(1)}$
$\mathtt{E}_{8}^{(1)}$	$\mathtt{A}_{1}^{(1)} \oplus \mathtt{E}_{7}^{(1)}$, $\mathtt{E}_{6}^{(1)} \oplus \mathtt{A}_{2}^{(1)}$, $\mathtt{A}_{4}^{(1)} \oplus \mathtt{A}_{4}^{(1)}$	$\mathtt{D}_{8}^{(1)}$, $\mathtt{A}_{8}^{(1)}$, $\mathtt{E}_{8}^{(1)}$
$\mathtt{F}_{4}^{(1)}$	$\mathtt{A}_{2}^{(1)} \oplus \mathtt{A}_{2}^{(1)}$, $\mathtt{A}_{1}^{(1)} \oplus \mathtt{C}_{3}^{(1)}$	$\mathtt{B}_{4}^{(1)}$, $F_{4}^{(1)}$
$\mathtt{G}_{2}^{(1)}$	$\mathtt{A}_{1}^{(1)} \oplus \mathtt{A}_{1}^{(1)}$	$\mathtt{A}_{2}^{(1)}$, $\mathtt{G}_{2}^{(1)}$

1.4 Twisted Case not of type $\mathtt{A}_{2\mathtt{n}}^{(2)}$

Throughout this section we assume that Φ is an irreducible twisted affine root system which is not of type $\mathtt{A}_{2\mathtt{n}}^{(2)}$. Let $\Psi \le \Phi$ be a closed subroot system. Unlike in untwisted case, we have three choices for $\mathrm{Gr}(\Psi)$ in this case. Indeed because of this fact, the classification of maximal closed subroot systems of twisted affine root systems becomes more technical. We begin with the definition of the third possible case.

Definition 1.4.1. A subroot system $\mathring{\Psi}$ of $\mathring{\Phi}$ is said to be *semi-closed* if

1. $\mathring{\Psi}$ is not closed in $\mathring{\Phi}$ and
2. if $\alpha, \beta \in \mathring{\Psi}$ such that $\alpha + \beta \in \mathring{\Phi}\backslash\mathring{\Psi}$, then α and β must be short roots and $\alpha + \beta$ must be a long root.

The condition (1) in Definition 1.4.1 implies that there must exist two roots $\alpha, \beta \in \mathring{\Psi}$ such that $\alpha + \beta \in \mathring{\Phi} \setminus \mathring{\Psi}$ and the condition (2) ensures that α and β are short roots and $\alpha + \beta$ is a long root. Thus, if $\mathring{\Psi}$ is semi-closed in $\mathring{\Phi}$, then there exists short roots α and β such that their sum $\alpha+\beta$ is a long root and $\alpha+\beta \in \mathring{\Phi} \backslash \mathring{\Psi}$.

Proposition 1.4.2. *Let Φ be an irreducible twisted affine root system not of type $\mathtt{A}_{\mathtt{2n}}^{(2)}$ and let $\Psi \leq \Phi$ be a subroot system. If $\Psi \leq \Phi$ is a closed subroot system, then either*

1. $\mathrm{Gr}(\Psi) = \mathrm{Gr}(\Phi)$ *or*

2. $\mathrm{Gr}(\Psi)$ *is a proper closed subroot system of* $\mathrm{Gr}(\Phi)$ *or*

3. $\mathrm{Gr}(\Psi)$ *is a proper semi-closed subroot system of* $\mathrm{Gr}(\Phi)$.

Proof. Let $\mathrm{Gr}(\Psi)$ neither be equal to $\mathrm{Gr}(\Phi)$ nor be a proper closed subroot system of $\mathrm{Gr}(\Phi)$. Then there must exist two roots $\alpha_1, \alpha_2 \in \mathrm{Gr}(\Psi)$ such that $\alpha_1 + \alpha_2 \in \mathrm{Gr}(\Phi) \setminus \mathrm{Gr}(\Psi)$. We claim that the roots α_1, α_2 must be short roots and their sum $\alpha_1 + \alpha_2$ must be a long root. Since $\alpha_1, \alpha_2 \in \mathrm{Gr}(\Psi)$, there exists $u, v \in \mathbb{Z}$ such that $\alpha_1 + u\delta$, $\alpha_2 + v\delta \in \Psi$. As Ψ is closed and $\alpha_1 + \alpha_2 \in \mathrm{Gr}(\Phi) \setminus \mathrm{Gr}(\Psi)$, we have $\alpha_1 + \alpha_2 + (u+v)\delta \notin \Phi$. This implies that $\alpha_1 + \alpha_2$ is a long root.

Suppose that both α_1 and α_2 are long roots. Then both u and v are integer multiples of m, and hence so is $u+v$, which contradicts the fact that $\alpha_1 + \alpha_2 + (u+v)\delta \notin \Phi$. So, they can not be both long. Since a sum of a short root and a long root can not be a long root, we have both α_1 and α_2 are short roots. This proves that $\mathrm{Gr}(\Psi)$ must be a proper semi-closed subroot system of $\mathrm{Gr}(\Phi)$. $\square$

1.4.1 Main theorem for twisted case not of type $\mathtt{A}_{\mathtt{2n}}^{(2)}$

Now, we assume that $\Psi \leq \Phi$ is a maximal closed subroot system. Then by Proposition 1.4.2, we have three choices for $\mathrm{Gr}(\Psi)$. First two cases of Proposition 1.4.2 are easier to study and they are similar to the untwisted affine root systems.

The case (3) of Proposition 1.4.2 requires a case-by-case analysis. In this section, we study the easier cases (1) and (2) and in Sections 1.5.3, 1.6, 1.7 and 1.8 we will treat the case (3) for all affine root systems Φ distinct from $\mathtt{A}_{\mathtt{2n}}^{(2)}$. Root systems of type $\mathtt{A}_{\mathtt{2n}}^{(2)}$ will be considered separately in Section 1.9 for $n \geq 2$ and in Section 1.10 for $n = 1$.

Proposition 1.4.3. *Let Φ, Ψ as before. If $\Psi \leq \Phi$ is a maximal closed subroot system and $\mathrm{Gr}(\Psi)$ is a proper closed subroot system of $\mathrm{Gr}(\Phi)$, then $\mathrm{Gr}(\Psi) < \mathrm{Gr}(\Phi)$ is a maximal closed subroot system such that it contains at least one short root. In this case, we have $\Psi = \widehat{\mathrm{Gr}(\Psi)}$.*

Proof. The proof that $\mathrm{Gr}(\Psi) < \mathrm{Gr}(\Phi)$ is a maximal closed subroot system follows immediately from the part (2) of Proposition 1.3.1. Now, suppose that $\mathrm{Gr}(\Psi)$ contains only long roots. Then it is easy to see that $\Psi \leq \widehat{\mathring{\Phi}_\ell}$. But
$\Psi \leq \widehat{\mathring{\Phi}_\ell} \subsetneq \Omega = \left\{\alpha + mr\delta : \alpha \in \mathring{\Phi},\ r \in \mathbb{Z}\right\}$ and Ω is a closed subroot system of Φ, which is a contradiction to the fact that Ψ is maximal closed. □

Now, we present our main classification theorem for the maximal closed subroot systems of twisted affine root system Φ (which is not of type $\mathtt{A}_{\mathtt{2n}}^{(2)}$) whose gradient subroot system is equal to $\mathring{\Phi}$ or is a proper closed subroot system of $\mathring{\Phi}$.

Theorem 1.4.4. *Let Φ be an irreducible twisted affine root system which is not of type $\mathtt{A}_{\mathtt{2n}}^{(2)}$ and let Ψ be a maximal closed subroot system of Φ.*

1. *If $\mathrm{Gr}(\Psi) = \mathring{\Phi}$, then there exists a $\mathbb{Z}$–linear function $p : \mathrm{Gr}(\Psi) \to \mathbb{Z}$ and a prime number n_Ψ such that p satisfies the condition (1.2.2), $p_\alpha \in m\mathbb{Z}$ for long roots α and*

$$\Psi = \begin{cases} \{\alpha + (p_\alpha + rn_\Psi)\delta, \beta + (p_\beta + mrn_\Psi)\delta : \alpha \in \mathring{\Phi}_s, \beta \in \mathring{\Phi}_\ell, r \in \mathbb{Z}\} & \text{if } m \neq n_\Psi, \\ \{\alpha + (p_\alpha + rn_\Psi)\delta : \alpha \in \mathring{\Phi},\ r \in \mathbb{Z}\} & \text{if } m = n_\Psi. \end{cases}$$

Conversely, given a prime number n_Ψ and a $\mathbb{Z}$–linear function $p : \mathring{\Phi} \to \mathbb{Z}$ satisfying $p_\alpha \in m\mathbb{Z}$ for long roots $\alpha \in \mathring{\Phi}_\ell$ and (1.2.2), the subroot system Ψ defined above gives us a maximal closed subroot system of Φ.

2. *If $\mathrm{Gr}(\Psi) \subsetneq \mathring{\Phi}$ is a proper closed subroot system, then $\mathrm{Gr}(\Psi) < \mathring{\Phi}$ is a maximal closed subroot system such that it contains at least one short root and in this case $\Psi = \widehat{\mathrm{Gr}(\Psi)}$. Conversely, if $\mathring{\Psi} \subsetneq \mathring{\Phi}$ is a maximal closed subroot system with a short root then $\widehat{\mathring{\Psi}}$ is a maximal closed subroot system.*

Remark 1.4.5. For Case (1) i.e., $\mathrm{Gr}(\Psi) = \mathring{\Phi}$, the type of Ψ is $\mathtt{X}_\mathtt{n}^{(2)}$ if the type of $\mathring{\Phi}$ is $\mathtt{X}_\mathtt{n}$ and $m \neq n_\Psi$ and the type of Ψ is $\mathtt{X}_\mathtt{n}^{(1)}$ if the type of $\mathring{\Phi}$ is $\mathtt{X}_\mathtt{n}$ and $m = n_\Psi$. For Case (2), the type of Ψ is $\mathtt{X}_{\mathtt{n}_1}^{(\mathtt{r}_1)} \oplus \cdots \oplus \mathtt{X}_{\mathtt{n}_\mathtt{s}}^{(\mathtt{r}_\mathtt{s})}$, where $\mathtt{X}_{\mathtt{n}_\mathtt{i}}$'s are irreducible components of $\mathrm{Gr}(\Psi)$ and $r_i = 1$ if $\mathtt{X}_{\mathtt{n}_\mathtt{i}}$ is simply-laced else it is 2.

Proof of Statement (1). The forward part of Statement (1) is clear from the parts (2) and (3) of Proposition 1.2.7. Converse part of Statement (1) will be proved case by case.

Case (1.1). First assume that n_Ψ is a prime number such that $n_\Psi \neq m$ and $\Psi = \{\alpha + (p_\alpha + rn_\Psi)\delta, \beta + (p_\beta + mrn_\Psi)\delta : \alpha \in \mathring{\Phi}_s, \beta \in \mathring{\Phi}_\ell, r \in \mathbb{Z}\}$ where p_α satisfies the condition (1.2.2) and $p_\alpha \in m\mathbb{Z}$ for long roots α. It is easy to verify that Ψ is a closed subroot system of Φ. By the definition of Ψ, we have

$$Z_\alpha(\Psi) = p_\alpha + n_\Psi\mathbb{Z} \text{ for } \alpha \in \mathring{\Phi}_s \text{ and } Z_\alpha(\Psi) = p_\alpha + mn_\Psi\mathbb{Z} \text{ for } \alpha \in \mathring{\Phi}_\ell.$$

Now, we will prove that Ψ is a maximal closed subroot system of Φ. Suppose $\Psi \subsetneq \Delta \subseteq \Phi$ for some closed subroot system Δ of Φ. Then we claim that Δ must be equal to Φ. Since $\Psi \subseteq \Delta$, we have $\mathrm{Gr}(\Psi) = \mathrm{Gr}(\Delta) = \mathring{\Phi}$. By part (2) of Proposition 1.2.7, n_Δ determines the subgroups $Z'_\alpha(\Delta)$ and hence the cosets $Z_\alpha(\Delta)$. But by part (2) Lemma 1.2.4, we get n_Δ divides n_Ψ. This implies that either $n_\Delta = 1$ or $n_\Delta = n_\Psi$ since n_Ψ is a prime number. Assume first that $n_\Delta = n_\Psi$, then we get

$n_\alpha^\Psi = n_\alpha^\Delta$ for all $\alpha \in \mathring{\Phi}$ by part (2) of Proposition 1.2.7. Since $\mathrm{Gr}(\Psi) = \mathrm{Gr}(\Delta)$ and $n_\alpha^\Psi = n_\alpha^\Delta$ for all $\alpha \in \mathrm{Gr}(\Delta)$, we have $\Psi = \Delta$ using part (3) of Lemma 1.2.4, a contradiction. So, n_Δ must be equal to 1. In this case, we get $n_\alpha^\Delta = n_\alpha^\Phi$ for all $\alpha \in \mathring{\Phi}$ again using the part (2) of Proposition 1.2.7. This immediately implies that $\Delta = \Phi$ by part (3) of Lemma 1.2.4, since $\mathrm{Gr}(\Delta) = \mathrm{Gr}(\Phi)$.

Case (1.2). Now assume that $n_\Psi = m$ and $\Psi = \{\alpha + (p_\alpha + rm)\delta : \alpha \in \mathring{\Phi},\ r \in \mathbb{Z}\}$. One easily sees that Ψ is a closed subroot system of Φ. So, it remains to show that Ψ is a maximal closed subroot system of Φ. Suppose $\Psi \subsetneq \Delta \subseteq \Phi$ for some closed subroot system Δ of Φ. Then we need to prove that Δ must be equal to Φ. Since $\Psi \subseteq \Delta$, we get $\mathrm{Gr}(\Delta) = \mathring{\Phi}$ and by part (2) of Lemma 1.2.4, we get $n_\Delta = m$ or $n_\Delta = 1$. If $n_\Delta = m$, then by part (2) of Proposition 1.2.7, we get $n_\alpha^\Delta = n_\alpha^\Psi$ for all $\alpha \in \mathring{\Phi}$. This forces $\Delta = \Psi$, a contradiction. So, this case does not arise. Hence we must have $n_\Delta = 1$ which implies that $\Delta = \Phi$ as before in Case 1.1. This completes the proof of Statement (1).

Proof of Statement (2). The forward part of Statement (2) is clear from the Proposition 1.4.3. Conversely, suppose $\mathring{\Psi}$ is a maximal closed subroot system in $\mathring{\Phi}$ such that it contains at least one short root, say $\beta \in \mathring{\Psi}$, then we claim that the lift $\widehat{\mathring{\Psi}}$ in Φ must be a maximal closed subroot system. Let Δ be a closed subroot system in Φ such that

$$\widehat{\mathring{\Psi}} \subsetneq \Delta \subseteq \Phi.$$

Then we need to prove that Δ must be equal to Φ. We observe the following facts first.

1. By considering respective gradients, we have $\mathring{\Psi} \subseteq \mathrm{Gr}(\Delta) \subseteq \mathring{\Phi}$. This implies that $\mathrm{rank}(\mathring{\Psi}) \leq \mathrm{rank}(\mathrm{Gr}(\Delta)) \leq \mathrm{rank}(\mathring{\Phi})$.

2. By Proposition 1.4.2, we know that $\mathrm{Gr}(\Delta)$ is either closed in $\mathring{\Phi}$ or semi-closed in $\mathring{\Phi}$.

3. Since $\mathring{\Psi}$ contains the short root β, we have $\beta + r\delta \in \widehat{\mathring{\Psi}} \subseteq \Delta$ for all $r \in \mathbb{Z}$.

4. $\mathring{\Psi}$ can be both irreducible and reducible subroot system of $\mathring{\Phi}$ (see Table 1.1 and [16, Page 136]).

Now, we will deal with all possible cases of Δ. We begin with the easiest case.

Case (2.1). Assume that $\mathrm{Gr}(\Delta)$ is closed in $\mathring{\Phi}$. Then we claim that $\mathrm{Gr}(\Delta) = \mathring{\Phi}$. Since $\mathring{\Psi}$ is maximal closed in $\mathring{\Phi}$ and $\mathring{\Psi} \subseteq \mathrm{Gr}(\Delta) \subseteq \mathring{\Phi}$, we must have either $\mathrm{Gr}(\Delta) = \mathring{\Psi}$ or $\mathrm{Gr}(\Delta) = \mathring{\Phi}$. If $\mathrm{Gr}(\Delta) = \mathring{\Psi}$, then we have $\Delta \subseteq \widehat{\mathring{\Psi}}$, a contradiction. So, we must have $\mathrm{Gr}(\Delta) = \mathring{\Phi}$. Since $n_\Delta = 1$, we get $n_\ell^\Delta = m$ by part (2) Proposition 1.2.7. Hence, we get $\Delta = \Phi$ by part (3) of Lemma 1.2.4.

Case (2.2). Now, we are left with the case that $\mathrm{Gr}(\Delta)$ is not closed but semi-closed in $\mathring{\Phi}$. We will prove that this case also can not arise. Let $\mathrm{Gr}(\Delta)$ be not closed but semi-closed in $\mathring{\Phi}$. By Proposition 1.4.2, there exists short roots $\alpha_1, \alpha_2 \in \mathrm{Gr}(\Delta)$ such that $\alpha_1 + \alpha_2$ is a long root and $\alpha_1 + \alpha_2 \in \mathring{\Phi} \backslash \mathrm{Gr}(\Delta)$, fix these short roots α_1 and $\alpha_2 \in \mathrm{Gr}(\Delta)$. First we observe that $\mathrm{Gr}(\Delta)$ can not be irreducible. Otherwise, $\mathrm{Gr}(\Delta)$ is irreducible and $\beta + r\delta \in \Delta$ for all $r \in \mathbb{Z}$ would imply $n_\Delta = 1$ and hence $n_\ell^\Delta = m$ by part (2) of Proposition 1.2.7. Since we have $\alpha_1 + r\delta, \alpha_2 + r\delta \in \Delta$ for all $r \in \mathbb{Z}$, which implies that $(\alpha_1 + \alpha_2) + m\delta = (\alpha_1 + (m-1)\delta) + (\alpha_2 + \delta) \in \Delta$, a contradiction to the fact that $\alpha_1 + \alpha_2 \notin \mathrm{Gr}(\Delta)$. So, $\mathrm{Gr}(\Delta)$ must be reducible. Let $\mathrm{Gr}(\Delta) = \Delta_1 \oplus \cdots \oplus \Delta_k$ be the decomposition of $\mathrm{Gr}(\Delta)$ into irreducible components. Then it is immediate that $\mathrm{rank}(\mathrm{Gr}(\Delta)) = \mathrm{rank}(\Delta_1) + \cdots + \mathrm{rank}(\Delta_k)$.

Case (2.2.1). We now consider the case when $\mathring{\Psi}$ is irreducible. Since $\mathring{\Psi}$ is irreducible, it must be contained in one of components of $\mathrm{Gr}(\Delta)$. Without loss of generality we can assume that $\mathring{\Psi} \subseteq \Delta_1$. We have either

$$\mathrm{rank}(\mathring{\Psi}) = \mathrm{rank}(\mathring{\Phi}) \text{ or } \mathrm{rank}(\mathring{\Psi}) = \mathrm{rank}(\mathring{\Phi}) - 1$$

since $\mathring{\Psi}$ is irreducible maximal closed subroot system of $\mathring{\Phi}$ (see Table 1.1 and [16, Page 136]). If $\operatorname{rank}(\mathring{\Psi}) = \operatorname{rank}(\mathring{\Phi})$, then we get $\operatorname{rank}(\Delta_i) = 0,$ for all $i = 2, \cdots, k$ which is a contradiction to the fact that $\operatorname{Gr}(\Delta)$ is reducible. So, we get $\operatorname{rank}(\mathring{\Psi}) = \operatorname{rank}(\mathring{\Phi}) - 1$. Since

$$\operatorname{rank}(\Delta_2) + \cdots + \operatorname{rank}(\Delta_k) \leq \operatorname{rank}(\mathring{\Phi}) - \operatorname{rank}(\mathring{\Psi}) = 1,$$

we must have $k = 2$ and $\operatorname{rank}(\Delta_2) = 1$. This implies that $\operatorname{Gr}(\Delta) = \Delta_1 \oplus A_1$ with $\mathring{\Psi} \subseteq \Delta_1$. Since $\beta + r\delta \in \Delta$ for all $r \in \mathbb{Z}$ and Δ_1 is irreducible, we have $n_s^{\Delta_1}(\Delta) = 1$. In particular $\alpha + r\delta \in \Delta$ for all the short roots $\alpha \in \Delta_1$ and $r \in \mathbb{Z}$. Clearly, one of the short roots α_j, $j = 1, 2$ must be in Δ_1, say $\alpha_1 \in \Delta_1$. Since $\alpha_2 \in \operatorname{Gr}(\Delta)$, there exists $r \in \mathbb{Z}$ such that $\alpha_2 + r\delta \in \Delta$. Now,
$(\alpha_1 + \alpha_2) + m\delta = (\alpha_2 + r\delta) + (\alpha_1 + (m - r)\delta) \in \Delta$ since Δ is closed and $(\alpha_1 + (m - r)\delta) \in \Delta$ because $n_s^{\Delta_1} = 1$. This is again contradicting the fact that $\alpha_1 + \alpha_2 \notin \operatorname{Gr}(\Delta)$.

Case (2.2.2). We are now left with the case $\mathring{\Psi}$ is reducible. Recall that $\mathring{\Phi}$ is non simply-laced irreducible finite crystallographic root system. So, by the classification of maximal closed subroot systems of the finite root systems (see Table 1.1 and [16, Page 136]), we know that we must have $\operatorname{rank}(\mathring{\Psi}) = \operatorname{rank}(\mathring{\Phi})$ and $\mathring{\Psi} = \Psi_1 \oplus \Psi_2$, where Ψ_1, Ψ_2 are irreducible components of $\mathring{\Psi}$ except in the case that when $\mathring{\Phi} = \mathtt{B_n}$ and $(\Psi_1, \Psi_2) = (\mathtt{B_{n-2}}, \mathtt{A_1} \oplus \mathtt{A_1})$. We will treat the cases $\mathring{\Phi} = \mathtt{B_n}$ and $(\Psi_1, \Psi_2) = (\mathtt{B_{n-2}}, \mathtt{A_1} \oplus \mathtt{A_1})$ separately. Since $\operatorname{rank}(\mathring{\Psi}) = \operatorname{rank}(\operatorname{Gr}(\Delta)) = \operatorname{rank}(\mathring{\Phi})$ and $\operatorname{Gr}(\Delta)$ is reducible, $\mathring{\Psi}$ can not be contained in one single irreducible component of $\operatorname{Gr}(\Delta)$.

Subcase 1. Assume that Ψ_1, Ψ_2 are irreducible, i.e., $(\Psi_1, \Psi_2) \neq (\mathtt{B_{n-2}}, \mathtt{A_1} \oplus \mathtt{A_1})$. Since $\mathring{\Psi}$ can not be contained in one single irreducible component of $\operatorname{Gr}(\Delta)$ and

Ψ_1, Ψ_2 are irreducible, we may assume that $\Psi_1 \subseteq \Delta_1$, $\Psi_2 \subseteq \Delta_2$. Then

$$\operatorname{rank}(\mathring{\Phi}) = \operatorname{rank}(\Psi_1) + \operatorname{rank}(\Psi_2) \leq \operatorname{rank}(\Delta_1) + \cdots + \operatorname{rank}(\Delta_k) \leq \operatorname{rank}(\mathring{\Phi})$$

implies that $k = 2$ and $\operatorname{rank}(\Psi_1) = \operatorname{rank}(\Delta_1)$, $\operatorname{rank}(\Psi_2) = \operatorname{rank}(\Delta_2)$. Since $\beta \in \mathring{\Psi}$, it must be either in Ψ_1 or in Ψ_2. Assume that $\beta \in \Psi_1$, then as before in the Case 2.2.1 we get $n_s^{\Delta_1}(\Delta) = 1$. Hence, by previous arguments which appear in the Case 2.2.1, we observe that Δ_2 must contain those short roots α_1 and α_2. Now, since Δ_2 contains the short roots α_1 and α_2 we observe that Ψ_2 must contain only long roots. Otherwise, we will get $n_s^{\Delta_2}(\Delta) = 1$ (since $\widehat{\mathring{\Psi}} \subseteq \Delta$) which will again lead to the contradiction $\alpha_1 + \alpha_2 \in \operatorname{Gr}(\Delta)$. Hence, Δ_2 must be non simply-laced. Again by the classification, see Table 1.1 and [16, Page no. 136], we can have only the following possibilities of $(\mathring{\Phi}, \mathring{\Psi})$ such that $\mathring{\Psi} = \Psi_1 \oplus \Psi_2$ with simply laced Ψ_2:

$$(\mathtt{B_n}, \mathtt{B_{n-1}} \oplus \mathtt{A_1}), (\mathtt{B_n}, \mathtt{B_{n-i}} \oplus \mathtt{D_i}),\ 3 \leq \mathtt{i} \leq \mathtt{n}-2,\ \ (\mathtt{F_4}, \mathtt{C_3} \oplus \mathtt{A_1}),\ (\mathtt{F_4}, \mathtt{A_2} \oplus \mathtt{A_2}),\ \ (\mathtt{G_2}, \mathtt{A_1} \oplus \mathtt{A_1}).$$

We will prove that these possibilities can not occur. Hence, the case "$\mathring{\Psi}$ is reducible" is not possible and hence the case $\operatorname{Gr}(\Delta)$ is semi-closed in $\mathring{\Phi}$ is not possible. Recall that $\Psi_2 \subseteq \Delta_2$ satisfying the following properties:

- $\operatorname{rank}(\Psi_2) = \operatorname{rank}(\Delta_2)$, Ψ_2 is simply laced and Ψ_2 contains only long roots
- Δ_2 is non simply-laced
- Δ_2 contains the short roots α_1 and α_2 whose sum $\alpha_1 + \alpha_2$ is a long root in $\mathring{\Phi}$.

This immediately implies that the cases $(\mathring{\Phi}, \mathring{\Psi}) = (\mathtt{B_n}, \mathtt{B_{n-1}} \oplus \mathtt{A_1}), (\mathtt{F_4}, \mathtt{C_3} \oplus \mathtt{A_1})$, and $(\mathtt{G_2}, \mathtt{A_1} \oplus \mathtt{A_1})$ are not possible. If $(\mathring{\Phi}, \mathring{\Psi}) = (\mathtt{F_4}, \mathtt{A_2} \oplus \mathtt{A_2})$, then Δ_2 must contain $\mathtt{A_2}$ properly which implies that Δ_2 must be $\mathtt{G_2}$. But $\mathtt{G_2}$ can not be a subroot system of $\mathtt{F_4}$, so this case also does not occur.

Now, consider the case $(\mathring{\Phi}, \mathring{\Psi}) = (\mathtt{B_n}, \mathtt{B_{n-i}} \oplus \mathtt{D_i})$ with $3 \leq i \leq n-2$. Then we have $\Psi_1 = \mathtt{B_{n-i}}$ and $\Psi_2 = D_i$. Since Δ_2 is non simply-laced irreducible finite root system, the only possibilities of Δ_2 are $\mathtt{B_i}, \mathtt{C_i}, \mathtt{F_4}$ and $\mathtt{G_2}$. We will directly prove that these possibilities can not occur. By counting the number of short roots in $\mathtt{B_{n-4}} \oplus \mathtt{F_4}$ and $\mathtt{B_n}$, one can easily see that $\mathtt{B_{n-4}} \oplus \mathtt{F_4}$ can not occur as a subroot system of $\mathtt{B_n}$. Similarly, $\mathtt{B_{n-2}} \oplus \mathtt{G_2}$ does not occur as a subroot system of $\mathtt{B_n}$. Since $\mathtt{B_{n-4}} \oplus \mathtt{F_4}$ and $\mathtt{B_{n-2}} \oplus \mathtt{G_2}$ can not occur as subroot systems of $\mathtt{B_n}$, we can not have $\Delta_2 = \mathtt{G_2}$ or $\mathtt{F_4}$. So, we are left with the cases $\Delta_2 = \mathtt{B_i}$ or $\mathtt{C_i}$. The $\mathtt{D_i}$ can not occur as subroot system of $\mathtt{C_i}$ with only consisting of long roots, hence Δ_2 can not be $\mathtt{C_i}$. Thus $\Delta_2 = \mathtt{B_i}$ is the only case remaining, in this case $\mathtt{D_i}$ must be the subroot system of $\mathtt{B_i}$ consisting of all long roots of $\mathtt{B_i}$. Since $\Delta_2 = \mathtt{B_i}$ and $\alpha_1 + \alpha_2$ is a long root in $\mathring{\Phi}$, we have $\alpha_1 + \alpha_2 \in \mathtt{D_i} \subseteq \mathrm{Gr}(\Delta)$, a contradiction.

Subcase 2. Finally we are left with the case $\mathring{\Phi} = \mathtt{B_n}$ and $(\Psi_1, \Psi_2) = (\mathtt{B_{n-2}}, \mathtt{A_1} \oplus \mathtt{A_1})$. Since $\mathring{\Psi}$ can not be contained in one single irreducible component of $\mathrm{Gr}(\Delta)$, we may have two cases.

(i) $\Psi_1 = \mathtt{B_{n-2}} \subseteq \Delta_1$ and $\Psi_2 = \mathtt{A_1} \oplus \mathtt{A_1} \subseteq \Delta_2$. In this case, $k = 2$, $\mathrm{rank}(\Delta_1) = n-2$ and $\mathrm{rank}(\Delta_2) = 2$. Since $\beta \in \mathring{\Psi}$, we have either $\beta \in \Psi_1$ or $\beta \in \Psi_2$. Let $\beta \in \Psi_1$. This implies that $n_s^{\Delta_1}(\Delta) = 1$ which implies that $\alpha_1, \alpha_2 \notin \Delta_1$. Hence, $\alpha_1, \alpha_2 \in \Delta_2$ and Ψ_2 can not have short roots and must contain only long roots. Thus, $\Delta_2 = \mathtt{B_2}$ or $\mathtt{G_2}$, not possible like in Subcase 1. So, $\beta \in \Psi_2 \implies n_s^{\Delta_2}(\Delta) = 1 \implies \alpha_1, \alpha_2 \in \Delta_1$. This implies that Ψ_1 can not contain short roots and only contain long roots and Δ_1 must be non simply-laced. But $\Psi_1 = \mathtt{B_{n-2}}$ is non simply-laced for $n \geq 4$, so it contains a short root of $\mathring{\Phi}$. If $n = 3$ then $\mathrm{rank}(\Delta_1) = 1$, which implies that $\Psi_1 = \Delta_1$. So Δ_1 can not be non simply-laced in this case, again a contradiction. So this case is not possible.

(ii) $\Psi_1 = \mathtt{B_{n-2}} \subseteq \Delta_1$ and $\Psi_2 \subseteq \Delta_2 \oplus \Delta_3$. In this case, $k = 3$, $\mathrm{rank}(\Delta_1) = n-2$

and $\Delta_2 = \Delta_3 = \mathtt{A_1}$. Since sum of two roots from $\Delta_2 \oplus \Delta_3 = \mathtt{A_1} \oplus \mathtt{A_1}$ can not be a root again, we must have one of the α_j, $j = 1, 2$ in Δ_1. So, β can not be in Ψ_1. Thus, $\beta \in \Psi_2$. But this can not happen like in the case (i).

This completes the proof. $\square$

1.4.2 Examples and Table

In this section we list out all possible types of maximal closed subroot systems of irreducible twisted affine root systems which has closed gradient subroot systems and we demonstrate how to get this list from the Theorem 1.4.4 by a few examples.

Example 1.4.6. Let $\Phi = \mathtt{D}^{(2)}_{\mathtt{n+1}}$. Then $\mathring{\Phi} = \mathtt{B_n} = \{\pm\epsilon_i, \pm\epsilon_i \pm \epsilon_j; 1 \le i \ne j \le n\}$. The root system $\mathtt{B_n}$ has a maximal closed subroot system Δ of type $\mathtt{B_{n-1}}$ with a simple system $\{\epsilon_2 - \epsilon_3, \epsilon_3 - \epsilon_4, \cdots \epsilon_{n-1} - \epsilon_n, \epsilon_n\}$ (see [16, Page 136]). Note that Δ contains short roots. By Theorem 1.4.4, $\widehat{\Delta}$ is a maximal closed subroot system of Φ and the type of $\widehat{\Delta}$ is $\mathtt{D}^{(2)}_{\mathtt{n}}$.

Example 1.4.7. Let $\Phi = \mathtt{E}^{(2)}_{6}$. Then
$\mathring{\Phi} = \mathtt{F_4} = \big\{ \pm \epsilon_i, \pm\epsilon_i \pm \epsilon_j, \frac{1}{2}(\lambda_1\epsilon_1 + \lambda_2\epsilon_2 + \lambda_3\epsilon_3 + \lambda_4\epsilon_4) : \lambda_i = \pm 1, 1 \le i \ne j \le 4\big\}$.
The root system $\mathtt{F_4}$ has maximal closed subroot system Δ_1 of type $\mathtt{A_2} \oplus \mathtt{A_2}$ with a simple system $\{\epsilon_1 + \epsilon_2, \epsilon_2 - \epsilon_3, \epsilon_4, \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)\}$ and Δ_2 of type $\mathtt{B_4}$ with a simple system $\{\epsilon_1 + \epsilon_2, \epsilon_2 - \epsilon_3, \epsilon_3 - \epsilon_4, \epsilon_4\}$ (see [16, Page 136]). Note that Δ_1, Δ_2 both contains short roots. By Theorem 1.4.4, $\widehat{\Delta_1}$ is a maximal closed subroot system of Φ of type $\mathtt{A}^{(1)}_{2} \oplus \mathtt{A}^{(1)}_{2}$ and $\widehat{\Delta_2}$ is a maximal closed subroot system of Φ of type $\mathtt{D}^{(2)}_{5}$.

The following table is immediate from the Theorem 1.4.4 and Table 1.1.

Remark 1.4.8. Note that the Table 1.3 gives us only the part of the classification. The list in the Table 1.3 has already appeared in [11] (see [11, Table 1 & 2]) and

Table 1.3: Types of maximal closed subroot systems of irreducible twisted affine root systems (not of type $\mathtt{A}_{\mathtt{2n}}^{(2)}$) with closed gradient subroot systems

Type	Reducible	Irreducible
$\mathtt{D}_{\mathtt{n+1}}^{(2)}$	$\mathtt{D}_{\mathtt{r+1}}^{(2)} \oplus \mathtt{D}_{\mathtt{n-r}}^{(1)}$ $(2 \leq \mathtt{r} \leq \mathtt{n}-2)$	$\mathtt{B}_{\mathtt{n}}^{(1)}$, $\mathtt{D}_{\mathtt{n+1}}^{(2)}$, $D_n^{(2)}$
$\mathtt{A}_{\mathtt{2n-1}}^{(2)}$	$\mathtt{A}_{\mathtt{2r-1}}^{(2)} \oplus \mathtt{A}_{\mathtt{2n-2r-1}}^{(2)}$ $(1 \leq \mathtt{r} \leq \mathtt{n}-1)$	$\mathtt{A}_{\mathtt{2n-1}}^{(2)}$, $\mathtt{C}_{\mathtt{n}}^{(1)}$, $\mathtt{A}_{\mathtt{n-1}}^{(1)}$
$\mathtt{E}_{6}^{(2)}$	$\mathtt{A}_{1}^{(1)} \oplus \mathtt{A}_{5}^{(2)}$, $\mathtt{A}_{2}^{(1)} \oplus \mathtt{A}_{2}^{(1)}$	$\mathtt{E}_{6}^{(2)}$, $\mathtt{F}_{4}^{(1)}$, $D_5^{(2)}$
$\mathtt{D}_{4}^{(3)}$	$\mathtt{A}_{1}^{(1)} \oplus \mathtt{A}_{1}^{(1)}$	$\mathtt{D}_{4}^{(3)}$, $\mathtt{G}_{2}^{(1)}$, $\mathtt{A}_{2}^{(1)}$

note that the authors of [11] have omitted the possibility of a maximal closed subroot system $\mathtt{A}_{2}^{(1)} \oplus \mathtt{A}_{2}^{(1)} \subset \mathtt{E}_{6}^{(2)}$ and $\mathtt{D}_{5}^{(2)} \subset \mathtt{E}_{6}^{(2)}$ in their list.

We are now left with the case (3) *of Proposition 1.4.2 (in twisted affine root systems which is not of type* $\mathtt{A}_{\mathtt{2n}}^{(2)}$*) and the type* $\mathtt{A}_{\mathtt{2n}}^{(2)}$ *in completing the classification theorem. The aim of the remaining part of this chapter is to consider the case* (3) *of Proposition 1.4.2 and the type* $\mathtt{A}_{\mathtt{2n}}^{(2)}$*. The case* (3) *of Proposition 1.4.2 requires a type by type analysis so, in Section 1.5, 1.6, 1.7, 1.8 we consider the types* $\mathtt{D}_{\mathtt{n+1}}^{(2)}$*,* $\mathtt{A}_{\mathtt{2n-1}}^{(2)}$*,*$\mathtt{D}_{4}^{(3)}$ *and* $\mathtt{E}_{6}^{(2)}$ *separately. Finally we will deal the types* $\mathtt{A}_{\mathtt{2n}}^{(2)}, \mathtt{n} \neq 1$ *and* $\mathtt{A}_{2}^{(2)}$ *in Sections 1.9 and 1.10. We will denote* $I_n = \{1, \cdots, n\}$ *in what follows next.*

1.5 The case $\mathtt{D}_{\mathtt{n+1}}^{(2)}$

Throughout this section we assume that Φ is of type $\mathtt{D}_{\mathtt{n+1}}^{(2)}$. In particular, the gradient root system of $\mathtt{D}_{\mathtt{n+1}}^{(2)}$ is of type $\mathtt{B}_{\mathtt{n}}$. We have the following explicit description of $\mathtt{D}_{\mathtt{n+1}}^{(2)}$, see [4, Page no. 545, 579]:

$$\Phi = \{\pm\epsilon_i + r\delta, \pm\epsilon_i \pm \epsilon_j + 2r\delta : r \in \mathbb{Z}, 1 \leq i \neq j \leq n\}$$

and

$$\mathring{\Phi} = \{\pm\epsilon_i, \pm\epsilon_i \pm \epsilon_j : 1 \leq i \neq j \leq n\}.$$

We need the following definition.

Definition 1.5.1. For a subset $I \subseteq I_n$, we define

$$\Psi_I(\mathtt{D}^{(2)}_{\mathtt{n+1}}) = \Big\{ \pm \epsilon_s + 2r\delta : s \in I, r \in \mathbb{Z}\Big\} \cup \Big\{ \pm \epsilon_s + (2r+1)\delta : s \notin I, r \in \mathbb{Z}\Big\}$$
$$\cup \Big\{ \pm \epsilon_s \pm \epsilon_t + 2r\delta : s \neq t, \quad s,t \in I \text{ or } s,t \notin I, \; r \in \mathbb{Z}\Big\}.$$

Lemma 1.5.2. *$\Psi_I(\mathtt{D}^{(2)}_{\mathtt{n+1}})$ is a closed subroot system of Φ for any subset $I \subseteq I_n$.*

Proof. Set $J = I_n \backslash I$. Write $\Psi_I^{\text{even}} = \Big\{ \pm \epsilon_s + 2r\delta : s \in I, r \in \mathbb{Z}\Big\}$, $\Psi_J^{\text{odd}} = \Big\{ \pm \epsilon_s + (2r+1)\delta : s \notin I, r \in \mathbb{Z}\Big\}$ and $\Psi_{I\times J}^{\text{even}} = \Big\{ \pm \epsilon_s \pm \epsilon_t + 2r\delta : s \neq t, \quad s,t \in I \text{ or } s,t \notin I, \; r \in \mathbb{Z}\Big\}$. Since the integers appear in the δ part of elements of Ψ_I^{even} and Ψ_J^{odd} have different parities, their sum can not be a root in Φ again. It is clear that if the sum of two roots $\alpha, \beta \in \Psi_I^{\text{even}}$ (or $\in \Psi_J^{\text{odd}}$) is again a root in Φ then $\alpha + \beta$ must be in $\Psi_{I\times J}^{\text{even}}$. Similarly, if $\alpha \in \Psi_{I\times J}^{\text{even}}$, $\beta \in \Psi_I^{\text{even}}$ (resp. $\beta \in \Psi_J^{\text{odd}}$) and $\alpha + \beta \in \mathtt{D}^{(2)}_{\mathtt{n+1}}$ then we must have $\alpha+\beta \in \Psi_I^{\text{even}}$ (resp. $\alpha + \beta \in \Psi_J^{\text{odd}}$).

Finally consider the case $\alpha, \beta \in \Psi_{I\times J}^{\text{even}}$. Write $\alpha = \pm\epsilon_s \pm \epsilon_t + 2r\delta$ and $\beta = \pm\epsilon_u \pm \epsilon_v + 2r'\delta$. Suppose $\alpha + \beta \in \mathtt{D}^{(2)}_{\mathtt{n+1}}$, then we must have $|\{s,t\} \cap \{u,v\}| = 1$ and in this case the sign of this common element in α and β must be opposite. Since either both $s,t \in I$ or both $s,t \in J$ (and it is true for u,v as well), we must have $\alpha + \beta \in \Psi_{I\times J}^{\text{even}}$. □

Proposition 1.5.3. *Suppose Φ is of type $\mathtt{D}^{(2)}_{\mathtt{n+1}}$ and $\Psi \leq \Phi$ is a maximal closed subroot system with proper semi-closed gradient subroot system $\mathrm{Gr}(\Psi) < \mathring{\Phi}$, then there exist a set $I \subsetneq I_n$ such that $\Psi = \Psi_I(\mathtt{D}^{(2)}_{\mathtt{n+1}})$.*

Proof. Since $\mathrm{Gr}(\Psi)$ is a semi-closed subroot system, there exist $i, j \in I_n$ such that $\epsilon_i, \epsilon_j \in \mathrm{Gr}(\Psi)$ but $\epsilon_i + \epsilon_j \notin \mathrm{Gr}(\Psi)$. We claim that elements of $Z_{\epsilon_i}(\Psi)$ and $Z_{\epsilon_j}(\Psi)$ can not have same parities. Suppose $Z_{\epsilon_i}(\Psi)$ and $Z_{\epsilon_j}(\Psi)$ contain same parity elements, say $2r+1 \in Z_{\epsilon_i}(\Psi), 2s+1 \in Z_{\epsilon_j}(\Psi)$. Then we have $\epsilon_i + \epsilon_j + 2(r+s+1)\delta \in \Psi$ since Ψ is closed, a contradiction to the choices of i, j. Proof is same for even integers. Hence, without loss of generality we can assume that $Z_{\epsilon_i}(\Psi) \subseteq 2\mathbb{Z}$ and $Z_{\epsilon_j}(\Psi) \subseteq 2\mathbb{Z}+1$.

Now, we claim that for each $\epsilon_k \in \mathrm{Gr}(\Psi)$ either $Z_{\epsilon_k}(\Psi) \subseteq 2\mathbb{Z}$ or $Z_{\epsilon_k}(\Psi) \subseteq 2\mathbb{Z}+1$. Suppose there exists $s, r \in \mathbb{Z}$ such that $\epsilon_k + 2s\delta, \epsilon_k + (2r+1)\delta \in \Psi$ with $k \neq i, j$. Then one immediately sees that $\epsilon_k + \epsilon_i, \epsilon_j - \epsilon_k \in \mathrm{Gr}(\Psi)$ since Ψ is closed and $Z_{\epsilon_i}(\Psi) \subseteq 2\mathbb{Z}$ and $Z_{\epsilon_j}(\Psi) \subseteq 2\mathbb{Z}+1$. This implies that $\epsilon_i + \epsilon_j \in \mathrm{Gr}(\Psi)$, a contradiction. Hence, either $Z_{\epsilon_k}(\Psi) \subseteq 2\mathbb{Z}$ or $Z_{\epsilon_k}(\Psi) \subseteq 2\mathbb{Z}+1$ for each $\epsilon_k \in \mathrm{Gr}(\Psi)$. Define

$$I = \{k \in I_n : Z_{\epsilon_k}(\Psi) \subseteq 2\mathbb{Z}\}.$$

Since $j \notin I$, we have $\Psi_I(\mathtt{D}^{(2)}_{\mathtt{n+1}}) \subsetneq \Phi$. We claim that $\Psi \subseteq \Psi_I(\mathtt{D}^{(2)}_{\mathtt{n+1}})$. Suppose, we have $\pm\epsilon_s \pm \epsilon_t + 2r\delta \in \Psi$ with $s \in I$ and $t \notin I$. Since $s \in I$, we have $\mp\epsilon_s + 2r'\delta \in \Psi$ for some $r' \in \mathbb{Z}$. Then we get

$$(\pm\epsilon_s \pm \epsilon_t + 2r\delta) + (\mp\epsilon_s + 2r'\delta) \in \Phi \text{ implies that } \pm\epsilon_t + 2(r+r')\delta \in \Psi$$

since Ψ is closed. This implies that $2(r+r') \in Z_{\epsilon_t}(\Psi)$, a contradiction to the choice of t. Since Ψ is maximal closed subroot system, we have $\Psi = \Psi_I(\mathtt{D}^{(2)}_{\mathtt{n+1}})$. This completes the proof.

$\square$

Conversely, given a proper subset $I \subsetneq I_n$, we will show that $\Psi_I(\mathtt{D}^{(2)}_{\mathtt{n+1}})$ defined above in the Definition 1.5.1 must be a maximal closed subroot system of Φ.

Proposition 1.5.4. *Suppose Φ is of type $\mathtt{D}^{(2)}_{\mathtt{n+1}}$. For $I \subsetneq I_n$, we have $\Psi_I(\mathtt{D}^{(2)}_{\mathtt{n+1}})$ is a maximal closed subroot system of Φ. The type of $\Psi_I(\mathtt{D}^{(2)}_{\mathtt{n+1}})$ is $\mathtt{B}^{(1)}_{\mathtt{r}} \oplus \mathtt{B}^{(1)}_{\mathtt{n-r}}$, where $|I| = r$.*

Proof. We have already seen in Lemma 1.5.2 that $\Psi_I(\mathtt{D}^{(2)}_{\mathtt{n+1}})$ is a closed subroot system of Φ. So, it only remains to prove that $\Psi_I(\mathtt{D}^{(2)}_{\mathtt{n+1}})$ is maximal closed in Φ. Suppose Ω is a closed subroot system of Φ such that $\Psi_I(\mathtt{D}^{(2)}_{\mathtt{n+1}}) \subsetneq \Omega \subseteq \Phi$, then we claim that $\Omega = \Phi$. Since $\Psi_I(\mathtt{D}^{(2)}_{\mathtt{n+1}}) \subsetneq \Omega$, there are three possibilities for elements of $\Omega \backslash \Psi_I(\mathtt{D}^{(2)}_{\mathtt{n+1}})$. We have either

1. $\epsilon_s + (2r+1)\delta \in \Omega$ for some $r \in \mathbb{Z}$ and $s \in I$ or
2. $\epsilon_s + 2r\delta \in \Omega$ for some $r \in \mathbb{Z}$ and $s \notin I$ or
3. $\epsilon_s \pm \epsilon_t + 2r\delta \in \Omega$ for some $r \in \mathbb{Z}$, $s \in I$ and $t \notin I$.

In each of the cases, we repeatedly use the fact that Ω is closed in Φ and $\Psi_I(\mathtt{D}^{(2)}_{\mathtt{n+1}}) \subseteq \Omega$ and prove that $\Omega = \Phi$.

Case (1). Suppose there exists $\epsilon_s + (2r+1)\delta \in \Omega$ for some $r \in \mathbb{Z}$ and $s \in I$. By adding

$$\epsilon_s + (2r+1)\delta \text{ with } \epsilon_t + (2\mathbb{Z}+1)\delta \text{ for } t \notin I, \text{ we get } \epsilon_s + \epsilon_t + 2\mathbb{Z}\delta \subseteq \Omega \text{ for all } t \notin I.$$

And by adding $-\epsilon_s - 2r\delta \in \Omega$ with $\epsilon_s + \epsilon_t + 2\mathbb{Z}\delta$ for $t \notin I$, we get $\epsilon_t + 2\mathbb{Z}\delta \subseteq \Omega$ for all $t \notin I$ which implies that $\epsilon_t + \mathbb{Z}\delta \subseteq \Omega$ for all $t \notin I$. Similarly, by adding $-\epsilon_s - (2r+1)\delta \in \Omega$ with $\epsilon_s + \epsilon_t + 2\mathbb{Z}\delta \subseteq \Omega$ for $t \in I$, where $s \neq t$, we get $\epsilon_t + (2\mathbb{Z}+1)\delta \subseteq \Omega$ for all $t \in I$ with $s \neq t$. Now, fix $t \notin I$ and by adding $-\epsilon_t - (2r+1)\delta \in \Omega$ with $\epsilon_s + \epsilon_t + 2\mathbb{Z}\delta \subseteq \Omega$, we get $\epsilon_s + (2\mathbb{Z}+1)\delta \subseteq \Omega$. This implies that $\epsilon_t + \mathbb{Z}\delta \subseteq \Omega$ for all $t \in I$. Thus, we have $\epsilon_t + \mathbb{Z}\delta \subseteq \Omega$ for all $t \in I_n$. Since Ω is closed subroot system, this immediately implies that $\Omega = \Phi$.

Case (2). Suppose there exists $\epsilon_s + 2r\delta \in \Omega$ for some $r \in \mathbb{Z}$ and $s \notin I$. By adding $\epsilon_s + 2r\delta$ with $\epsilon_t + 2\mathbb{Z}\delta$ for $t \in I$, we get $\epsilon_s + \epsilon_t + 2\mathbb{Z}\delta \subseteq \Omega$ for all $t \in I$. And by adding $-\epsilon_s - (2r+1)\delta \in \Omega$ with $\epsilon_s + \epsilon_t + 2\mathbb{Z}\delta$ for $t \in I$, we get $\epsilon_t + (2\mathbb{Z}+1)\delta \subseteq \Omega$

for all $t \in I$. This implies that $\epsilon_t + \mathbb{Z}\delta \subseteq \Omega$ for all $t \in I$. Similarly, by adding $-\epsilon_s - 2r\delta \in \Omega$ with $\epsilon_s + \epsilon_t + 2\mathbb{Z}\delta \subseteq \Omega$ for $t \notin I$, where $s \neq t$, we get $\epsilon_t + 2\mathbb{Z}\delta \subseteq \Omega$ for all $t \notin I$ with $s \neq t$. Now, fix $t \in I$ such that $t \neq s$ and by adding $-\epsilon_t - 2r\delta \in \Omega$ with $\epsilon_s + \epsilon_t + 2\mathbb{Z}\delta \subseteq \Omega$ we get $\epsilon_s + 2\mathbb{Z}\delta \subseteq \Omega$. This implies that $\epsilon_t + \mathbb{Z}\delta \subseteq \Omega$ for all $t \notin I$. Thus, we proved $\epsilon_t + \mathbb{Z}\delta \subseteq \Omega$ for all $t \in I_n$. Since Ω is closed subroot system, we immediately get $\Omega = \Phi$.

Case (3). Finally assume that $\epsilon_s \pm \epsilon_t + 2r\delta \in \Omega$ for some $r \in \mathbb{Z}$, $s \in I$ and $t \notin I$. Add $\mp\epsilon_t - (2r+1)\delta \in \Omega$ with $\epsilon_s \pm \epsilon_t + 2r\delta \in \Omega$ then we get $\epsilon_s + \delta \in \Omega$. Thus, we are back to the Case (1). This completes the proof. □

Remark 1.5.5. The authors of [11] have omitted the possibility of a maximal closed subroot system $\mathtt{B}_{\mathtt{r}}^{(1)} \oplus \mathtt{B}_{\mathtt{n-r}}^{(1)} \subset \mathtt{D}_{\mathtt{n+1}}^{(2)}$ in their classification list, see [11, Table 1 & 2].

1.6 The case $\mathtt{A}_{\mathtt{2n-1}}^{(2)}$

Throughout this section we assume that Φ is of type $\mathtt{A}_{\mathtt{2n-1}}^{(2)}$. In particular, the gradient root system of $\mathtt{A}_{\mathtt{2n-1}}^{(2)}$ is of type $\mathtt{C}_{\mathtt{n}}$. We have the following explicit description of $\mathtt{A}_{\mathtt{2n-1}}^{(2)}$, see [4, Page no. 547, 573]:

$$\Phi = \{\pm 2\epsilon_i + 2r\delta, \pm\epsilon_i \pm \epsilon_j + r\delta : r \in \mathbb{Z}, 1 \leq i \neq j \leq n\}$$

and $\mathring{\Phi} = \{\pm 2\epsilon_i, \pm\epsilon_i \pm \epsilon_j : 1 \leq i \neq j \leq n\}$.

Consider $\mathring{\Phi}_s = \Big\{ \pm\epsilon_i \pm \epsilon_j : i, j \in I_n, i \neq j \Big\} =: \mathcal{D}_n$. Clearly, the short roots $\mathring{\Phi}_s$ for a root system of type $\mathtt{D}_{\mathtt{n}}$ (see [4, Page no. 146]) and

$$\Gamma_n = \{\alpha_1 = \epsilon_1 - \epsilon_2, \cdots, \alpha_{n-1} = \epsilon_{n-1} - \epsilon_n, \alpha_n = \epsilon_{n-1} + \epsilon_n\}$$

is a simple root system of $\mathcal{D}_n$. It is easy to see that $\epsilon_s - \epsilon_t = \alpha_s + \cdots + \alpha_{t-1}$ and

$$\epsilon_s + \epsilon_t = \begin{cases} \alpha_s + \cdots + \alpha_{t-2} + \alpha_t & \text{if } t = n, \\ \alpha_s + \cdots + \alpha_{t-1} + 2(\alpha_t + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n & \text{if } t < n. \end{cases}$$

Let $p : \Gamma_n \to \{0, 1\}$ be a function such that $p_{\alpha_{n-1}}$ and p_{α_n} have different parity and let $p : \mathcal{D}_n \to \mathbb{Z}$ be its $\mathbb{Z}$–linear extension given by $\pm\epsilon_s \pm \epsilon_t \mapsto p_{\pm\epsilon_s\pm\epsilon_t}$. Since the map p is $\mathbb{Z}$–linear, we have

$$p_{\epsilon_s+\epsilon_t} = \begin{cases} p_{\epsilon_s-\epsilon_t} - p_{\alpha_{t-1}} + p_{\alpha_t} & \text{if } t = n, \\ p_{\epsilon_s-\epsilon_t} + 2(p_{\alpha_t} + \cdots + p_{\alpha_{n-2}}) + p_{\alpha_{n-1}} + p_{\alpha_n} & \text{if } t < n. \end{cases}$$

This implies $p_{\epsilon_s-\epsilon_t}$ and $p_{\epsilon_s+\epsilon_t}$ have different parity for $s < t$. Since $p_{\epsilon_s-\epsilon_t} = -p_{\epsilon_t-\epsilon_s}$, we conclude that $p_{\epsilon_s-\epsilon_t}$ and $p_{\epsilon_s+\epsilon_t}$ also have different parity for $s > t$. Now, define

$$\Psi_p(\mathtt{A}^{(2)}_{\mathtt{2n-1}}) := \big\{ \pm\epsilon_s \pm \epsilon_t + (p_{\pm\epsilon_s\pm\epsilon_t} + 2r)\delta : 1 \le t \ne s \le n, r \in \mathbb{Z} \big\}.$$

Lemma 1.6.1. *Let $p : \mathcal{D}_n \to \mathbb{Z}$ be a $\mathbb{Z}-$linear function such that $p_{\epsilon_s-\epsilon_t}$ and $p_{\epsilon_s+\epsilon_t}$ have different parity for each $1 \le s \ne t \le n$. Then $\Psi_p(\mathtt{A}^{(2)}_{\mathtt{2n-1}})$ is a maximal closed subroot system of Φ.*

Proof. Since p is $\mathbb{Z}$–linear, we have

$$\mathbf{s}_{\alpha+(p_\alpha+2r)\delta}(\beta + (p_\beta + 2r')\delta) = \mathbf{s}_\alpha(\beta) + (p_{\mathbf{s}_\alpha(\beta)} + 2(r' - r\langle\beta, \alpha^\vee\rangle))\delta$$

for $\alpha, \beta \in \mathcal{D}_n$ and $r, r' \in \mathbb{Z}$, where $\mathbf{s}_\alpha$ is the reflection with respect to α defined in Section 1.1.2. This implies that $\Psi_p(\mathtt{A}^{(2)}_{\mathtt{2n-1}})$ is a subroot system of Φ. For $s, t \in I_n$, $t \ne s$, we can not have $2\epsilon_s + (p_{\epsilon_s-\epsilon_t} + p_{\epsilon_s+\epsilon_t} + 2r)\delta \in \Phi$ for any $r \in \mathbb{Z}$, since $p_{\epsilon_s-\epsilon_t}$ and $p_{\epsilon_s+\epsilon_t}$ have different parity. This implies that $\Psi_p(\mathtt{A}^{(2)}_{\mathtt{2n-1}})$ is a closed subroot system of Φ. Now, suppose there is a closed subroot system Δ of Φ such that

$\Psi_p(\mathtt{A}^{(2)}_{\mathtt{2n-1}}) \subsetneq \Delta \subseteq \Phi$. Then we claim that $\Delta = \Phi$. Since $\Psi_p(\mathtt{A}^{(2)}_{\mathtt{2n-1}}) \subsetneq \Delta$, we have two possibilities for elements of $\Delta \backslash \Psi_p(\mathtt{A}^{(2)}_{\mathtt{2n-1}})$. We have either

1. $2\epsilon_s + 2r\delta \in \Delta$ for some $s \in I_n$ and $r \in \mathbb{Z}$ or
2. $\epsilon_s \pm \epsilon_t + (p_{\epsilon_s \pm \epsilon_t} + 2r + 1)\delta \in \Delta$ for some $s \neq t \in I_n$ and $r \in \mathbb{Z}$.

Case (1). Suppose there exists $s \in I_n$ such that $2\epsilon_s + 2r\delta \in \Delta$ for some $r \in \mathbb{Z}$. Then since $\epsilon_t - \epsilon_s + (p_{\epsilon_t - \epsilon_s} + 2\mathbb{Z})\delta \subseteq \Delta$ for any $t \neq s$, we have

$$\epsilon_t + \epsilon_s + (p_{\epsilon_t - \epsilon_s} + 2\mathbb{Z})\delta = (2\epsilon_s + 2r\delta) + \epsilon_t - \epsilon_s + (p_{\epsilon_t - \epsilon_s} + 2\mathbb{Z})\delta \subseteq \Delta.$$

for all $t \in I_n$ with $t \neq s$. As $\epsilon_t + \epsilon_s + (p_{\epsilon_t + \epsilon_s} + 2\mathbb{Z})\delta \subseteq \Psi_p(\mathtt{A}^{(2)}_{\mathtt{2n-1}})$ and $p_{\epsilon_t + \epsilon_s}$ and $p_{\epsilon_t - \epsilon_s}$ have different parity, we get $(\epsilon_t + \epsilon_s) + \mathbb{Z}\delta \subseteq \Delta$ for all $t \neq s$. This in turn implies that

$$(\epsilon_t + \epsilon_s + p_{\epsilon_t - \epsilon_s}\delta) + \epsilon_t - \epsilon_s + (p_{\epsilon_t - \epsilon_s} + 2\mathbb{Z})\delta = 2\epsilon_t + 2\mathbb{Z}\delta \subseteq \Delta$$

for all $t \in I_n$ with $t \neq s$. Now, $\epsilon_t - \epsilon_s + \mathbb{Z}\delta = (2\epsilon_t + 2\mathbb{Z}\delta) - (\epsilon_t + \epsilon_s + \mathbb{Z}\delta) \subseteq \Delta$ for all $t \neq s$. So far we have proved that $2\epsilon_s + 2r\delta \in \Delta$ implies that $\pm\epsilon_t \pm \epsilon_s + \mathbb{Z}\delta$, $\pm 2\epsilon_t + 2\mathbb{Z}\delta \subseteq \Delta$ for all $t \in I_n$ such that $t \neq s$. By repeating the earlier arguments with all possible $t \in I_n$ such that $t \neq s$, we see that $\Delta = \Phi$.

Case (2). Now, assume that there exists $s, t \in I_n$ such that $\epsilon_s \pm \epsilon_t + (p_{\epsilon_s \pm \epsilon_t} + 2r + 1)\delta \in \Delta$ for some $r \in \mathbb{Z}$. Since $\epsilon_s \mp \epsilon_t + (p_{\epsilon_s \mp \epsilon_t} + 2r')\delta \in \Delta$ for all $r' \in \mathbb{Z}$ and $p_{\epsilon_s \pm \epsilon_t}$, $p_{\epsilon_s \mp \epsilon_t}$ have different parity, we have $2\epsilon_s + 2r\delta \in \Delta$. So, we are back to the Case (1) and hence $\Delta = \Phi$. This completes the proof.

$\square$

Proposition 1.6.2. *Let Φ be an irreducible affine root system of type $\mathtt{A}^{(2)}_{\mathtt{2n-1}}$. Then $\Psi \leq \Phi$ is a maximal closed subroot system with a proper semi-closed gradient* *subroot system* $\mathrm{Gr}(\Psi) < \mathring{\Phi}$ *if and only if there exist* $\mathbb{Z}-$*linear function* $p : \mathcal{D}_n \to \mathbb{Z}$ *such that* $p_{\epsilon_s - \epsilon_t}$ *and* $p_{\epsilon_s + \epsilon_t}$ *have different parity for each* $1 \le s \ne t \le n$ *and*

$$\Psi = \Psi_p(\mathtt{A}^{(2)}_{\mathtt{2n-1}}) = \big\{ \pm\epsilon_s \pm \epsilon_t + (p_{\pm\epsilon_s \pm \epsilon_t} + 2r)\delta : 1 \le t \ne s \le n, r \in \mathbb{Z} \big\}.$$

The affine type of $\Psi_p(\mathtt{A}^{(2)}_{\mathtt{2n-1}})$ *is* $\mathtt{D}^{(1)}_{\mathtt{n}}$.

Proof. Let $\Psi \le \Phi$ be a maximal closed subroot system with a proper semi-closed gradient subroot system $\mathrm{Gr}(\Psi) < \mathring{\Phi}$. By Proposition 1.4.2, there exist $s, t \in I_n$ such that $\epsilon_s + \epsilon_t, \epsilon_s - \epsilon_t \in \mathrm{Gr}(\Psi)$ but $2\epsilon_s \notin \mathrm{Gr}(\Psi)$. Define

$$I = \{i \in I_n : 2\epsilon_i \in \mathrm{Gr}(\Psi)\}$$

Then it is immediate that $I \subsetneq I_n$ by previous observation. Suppose that $I \ne \emptyset$. Then we will prove that $\Psi \subseteq \Psi_I \subsetneq \Phi$, where

$$\Psi_I = \{\pm 2\epsilon_i + 2r\delta, \pm\epsilon_k \pm \epsilon_\ell + r\delta, \pm\epsilon_{k'} \pm \epsilon_{\ell'} + r\delta : i \in I_n, k \ne \ell \in I,\ k' \ne \ell' \notin I, r \in \mathbb{Z}\}.$$

It is easy to see that Ψ_I is the lift of the closed subroot system

$$\{\pm 2\epsilon_i, \pm\epsilon_k \pm \epsilon_\ell, \pm\epsilon_{k'} \pm \epsilon_{\ell'} : i \in I_n,\ k, \ell \in I, k \ne \ell,\ \ k', \ell' \notin I, k' \ne \ell'\}$$

of $\mathring{\Phi}$. So, Ψ_I is a closed subroot system of Φ by Lemma 1.1.7 and since $I \subsetneq I_n$, it is proper if $I \ne \emptyset$. Suppose that $\epsilon_i \pm \epsilon_j + r\delta \in \Psi$, for some $i \in I, j \notin I, r \in \mathbb{Z}$. Then since $i \in I$, we have $2\epsilon_i + 2r'\delta \in \Psi$ for some $r' \in \mathbb{Z}$. Since Ψ is closed, we have

$$\epsilon_i \mp \epsilon_j + (2r' - r)\delta = 2\epsilon_i + 2r'\delta - (\epsilon_i \pm \epsilon_j + r\delta) \in \Psi.$$

This implies that that
$\pm(2\epsilon_j + 2(r - r')\delta) = (\epsilon_i \pm \epsilon_j + r\delta) - (\epsilon_i \mp \epsilon_j + (2r' - r)\delta) \in \Psi$, a contradiction to

the fact that $j \notin I$. So, we have $\Psi \subseteq \Psi_I$. Since Ψ_I is a closed subroot system, we must have $\Psi = \Psi_I$ which is absurd as the gradient root system of Ψ_I is closed. So, we must have $I = \emptyset$.

Since $2\epsilon_i \notin \mathrm{Gr}(\Psi)$ for all $i \in I_n$, the elements in $Z_{\epsilon_i+\epsilon_j}(\Psi)$ and $Z_{\epsilon_i-\epsilon_j}(\Psi)$ must have different parity for all $1 \leq i \neq j \leq n$. Otherwise, we will get $2\epsilon_i + (r+r')\delta = (\epsilon_i + \epsilon_j + r\delta) + (\epsilon_i - \epsilon_j + r'\delta) \in \Psi$ for some $r, r' \in \mathbb{Z}$ such that $r \equiv r' \bmod 2$. This is contradicting the fact that $2\epsilon_i \notin \mathrm{Gr}(\Psi)$ for all $i \in I_n$. Hence, by 1.2.2, there exists $\mathbb{Z}-$linear function $p^\Psi : \mathcal{D}_n \to \mathbb{Z}$ such that for each $1 \leq i \neq j \leq n$, we have $Z_{\epsilon_i+\epsilon_j}(\Psi) \subseteq p^\Psi_{\epsilon_i-\epsilon_j} + 2\mathbb{Z}$ and $Z_{\epsilon_i-\epsilon_j}(\Psi) \subseteq p^\Psi_{\epsilon_i+\epsilon_j} + 2\mathbb{Z}$ with $p^\Psi_{\epsilon_i-\epsilon_j} \not\equiv p^\Psi_{\epsilon_i+\epsilon_j} (\mathrm{mod}\ 2)$ and

$$\Psi \subseteq \Psi_{p^\Psi}(\mathtt{A}^{(2)}_{\mathtt{2n-1}}) = \big\{\pm\epsilon_i \pm \epsilon_j + (p^\Psi_{\pm\epsilon_i\pm\epsilon_j} + 2r)\delta : 1 \leq i, j \leq n, i \neq j, r \in \mathbb{Z}\big\}$$

Since $\Psi_{p^\Psi}(\mathtt{A}^{(2)}_{\mathtt{2n-1}})$ is a closed subroot system in Φ by Lemma 1.6.1, we have the equality $\Psi = \Psi_{p^\Psi}(\mathtt{A}^{(2)}_{\mathtt{2n-1}})$. Converse part is immediate from the Lemma 1.6.1. This completes the proof. $\square$

Remark 1.6.3. The authors of [11] have omitted the possibility of a maximal closed subroot system $\mathtt{D}^{(1)}_{\mathtt{n}} \subset \mathtt{A}^{(2)}_{\mathtt{2n-1}}$ in their classification list, see [11, Table 1 & 2].

1.7 The case $\mathtt{D}_4^{(3)}$

Throughout this section we assume that Φ is of type $\mathtt{D}_4^{(3)}$. In particular, the gradient root system of Φ is of type $\mathtt{G_2}$. We have the following explicit description of $\mathtt{D}_4^{(3)}$, see [4, Page no. 559, 608]:

$$\Phi = \big\{\epsilon_i - \epsilon_j + r\delta, \pm(\epsilon_i + \epsilon_j - 2\epsilon_k) + 3r\delta : i, j, k \in I_3, i \neq j,\ r \in \mathbb{Z}\big\}$$

and $\mathring{\Phi} = \big\{\epsilon_i - \epsilon_j, \pm(\epsilon_i + \epsilon_j - 2\epsilon_k) : i, j, k \in I_3, i \neq j\big\}$.

Lemma 1.7.1. *Suppose Φ is of type $\mathtt{D}_4^{(3)}$ and $\Psi \le \Phi$ is a maximal closed subroot system with a proper semi-closed gradient subroot system, then* $\mathrm{Gr}(\Psi) = \mathring{\Phi}_s$.

Proof. Since $\mathrm{Gr}(\Psi)$ is semi-closed, then by Proposition 1.4.2 there exists two short roots $\alpha, \beta \in \mathrm{Gr}(\Psi)$ such that $\alpha + \beta \notin \mathrm{Gr}(\Psi)$. Since $\mathbf{s}_\alpha(\beta) \in \mathrm{Gr}(\Psi)$ and is another short root different from α and β, we have $\mathring{\Phi}_s \subseteq \mathrm{Gr}(\Psi)$. Since $\mathring{\Phi}_s$ is a maximal subroot system of G_2 and $\mathrm{Gr}(\Psi) \neq \mathring{\Phi}$, we get $\mathrm{Gr}(\Psi) = \mathring{\Phi}_s$. $\square$

Let $\{i, j, k\}$ be a permutation of $I_3 = \{1, 2, 3\}$ and $\ell \in \mathbb{Z}$. Define

$$\Psi^+(i, j, k; \ell) := \left\{\epsilon_i - \epsilon_j + 3r\delta, \epsilon_j - \epsilon_k + (3r + \ell)\delta, \epsilon_i - \epsilon_k + (3r + \ell)\delta : r \in \mathbb{Z}\right\}$$

and $\Psi(i, j, k; \ell) := \Psi^+(i, j, k; \ell) \cup (-\Psi^+(i, j, k; \ell))$.

Lemma 1.7.2. *$\Psi(i, j, k; \ell)$ is a subroot system of Φ for any permutation $\{i, j, k\}$ of I_3 and $\ell \in \mathbb{Z}$.*

Proof. Write $\alpha_1 = \epsilon_i - \epsilon_j$, $\alpha_2 = \epsilon_j - \epsilon_k$ and $\alpha_3 = \epsilon_i - \epsilon_k$. Then $(\alpha_1, \alpha_2) = -1$ and $(\alpha_1, \alpha_3) = (\alpha_2, \alpha_3) = 1$. This implies that
$\mathbf{s}_{\alpha_1+3r\delta}(\alpha_2 + (3r' + \ell)\delta) = \alpha_3 + (3(r + r') + \ell)\delta$,
$\mathbf{s}_{\alpha_1+3r\delta}(\alpha_3 + (3r' + \ell)\delta) = \alpha_2 + (3(r' - r) + \ell)\delta$ and
$\mathbf{s}_{\alpha_2+(3r+\ell)\delta}(\alpha_3 + (3r' + \ell)\delta) = \alpha_1 + 3(r' - r)\delta$ are in $\Psi(i, j, k; \ell)$. Similarly, we see that $\mathbf{s}_\alpha(\beta) \in \Psi(i, j, k; \ell)$ for all $\alpha, \beta \in \Psi(i, j, k; \ell)$. This proves that $\Psi(i, j, k; \ell)$ is a subroot system of Φ. $\square$

Proposition 1.7.3. *$\Psi(i, j, k; \ell)$ is a maximal closed subroot system of Ψ for any permutation $\{i, j, k\}$ of I_3 and $\ell \in \mathbb{Z}$ such that $\ell \equiv 1$ or $2 \pmod 3$.*

Proof. Lemma 1.7.2 implies that $\Psi(i, j, k; \ell)$ is a subroot system of Φ. Since $\ell \equiv 1$ or $2 \pmod 3$, we have

$$(\epsilon_j - \epsilon_k + (3r + \ell)\delta) + (\epsilon_i - \epsilon_k + (3r' + \ell)\delta) = (\epsilon_i + \epsilon_j - 2\epsilon_k + (3(r + r') + 2\ell)\delta) \notin \Phi.$$

It is easy to check that $\alpha+\beta \in \Phi$ for $\alpha, \beta \in \Psi(i,j,k;\ell)$ implies that $\alpha+\beta \in \Psi(i,j,k;\ell)$ in remaining cases. This proves that $\Psi(i,j,k;\ell)$ is a closed subroot system of Φ when $\ell \equiv 1$ or $2 \pmod 3$. So, it remains to prove that it is maximal closed subroot system in Φ. Let Δ be a closed subroot system of Φ such that $\Psi(i,j,k;\ell) \subsetneq \Delta \subseteq \Phi$. Observe that $\Delta \backslash \Psi(i,j,k;\ell)$ may contain a short root or a long root. There are three possibilities for short roots of $\Delta \backslash \Psi(i,j,k;\ell)$ and it will be considered in the Cases (1), (2) and (3). The possibility of $\Delta \backslash \Psi(i,j,k;\ell)$ containing a long root is considered in Case (4).

Case (1). Let $\epsilon_i - \epsilon_j + (3r+r')\delta \in \Delta$ for some $r, r' \in \mathbb{Z}$ such that $r' \not\equiv 0 \pmod 3$. This implies that

$$(\epsilon_i - \epsilon_j + (3r+r')\delta) + (\epsilon_j - \epsilon_k + (\ell + 3\mathbb{Z})\delta) = \epsilon_i - \epsilon_k + (\ell + r' + 3\mathbb{Z})\delta \subseteq \Delta.$$

So, $(\epsilon_i - \epsilon_k + (\ell + r' + 3\mathbb{Z})\delta) + (\epsilon_k - \epsilon_j + (-\ell + 3\mathbb{Z})\delta) = \epsilon_i - \epsilon_j + (r' + 3\mathbb{Z})\delta \subseteq \Delta$, and

$$(\epsilon_j - \epsilon_i + 3\mathbb{Z}\delta) + (\epsilon_i - \epsilon_k + (\ell + r' + 3\mathbb{Z})\delta) = \epsilon_j - \epsilon_k + (\ell + r' + 3\mathbb{Z})\delta \subseteq \Delta.$$

Summing these two we have $\epsilon_i - \epsilon_k + (\ell + 2r' + 3\mathbb{Z})\delta \subseteq \Delta$. This implies that $\epsilon_i - \epsilon_k + \mathbb{Z}\delta \subseteq \Delta$ and using this we get $\alpha + r\delta \in \Delta$ for all short roots α and $r \in \mathbb{Z}$. Since any long root of G_2 can be written as sum of two short roots, we have $\Delta = \Phi$.

Case (2). Let $\epsilon_j - \epsilon_k + (3r + r' + \ell)\delta \in \Delta$ for some $r, r' \in \mathbb{Z}$ such that $r' \not\equiv 0 \pmod 3$. Then

$$(\epsilon_i - \epsilon_k + \ell\delta) + (\epsilon_k - \epsilon_j - (3r + r' + \ell)\delta) = \epsilon_i - \epsilon_j + (-3r - r')\delta \in \Delta.$$

So, we are back to Case (1). Thus, we get $\Delta = \Phi$.

Case (3). Let $\epsilon_i - \epsilon_k + (3r + r' + \ell)\delta \in \Delta$ for some $r, r' \in \mathbb{Z}$ such that

$r' \not\equiv 0 \pmod 3$. Then

$$(\epsilon_i - \epsilon_k + (3r + r' + \ell)\delta) + (\epsilon_k - \epsilon_j - \ell\delta) = \epsilon_i - \epsilon_j + (3r + r')\delta \in \Delta.$$

Again we are back to Case (1). Thus, we get $\Delta = \Phi$.

Case (4). Finally assume that Δ contains a long root and let $\epsilon_s + \epsilon_t - 2\epsilon_u + 3r\delta \in \Delta$ for some $r \in \mathbb{Z}$ and a permutation $\{s, t, u\}$ of I_3. Then subtracting a suitable short root from $\epsilon_s + \epsilon_t - 2\epsilon_u + 3r\delta$ will bring us back to one of the three previous cases and we get $\Delta = \Phi$.

Hence, $\Psi(i, j, k; \ell)$ is a maximal closed subroot system of Φ. $\square$

Conversely, we prove that any maximal closed subroot system Ψ of Φ must be of the form $\Psi = \Psi(i, j, k; \ell)$ for some permutation $\{i, j, k\}$ of I_3 and $\ell \in \mathbb{Z}$ satisfying $\ell \equiv 1$ or $2 \pmod 3$.

Proposition 1.7.4. *Let Φ be the affine root system of type $\mathtt{D}_4^{(3)}$. Then $\Psi \leq \Phi$ is a maximal closed subroot system with a proper semi-closed gradient subroot system* $\mathrm{Gr}(\Psi)$ *if and only if* $\mathrm{Gr}(\Psi) = \mathring{\Phi}_s$ *and $\Psi = \Psi(i, j, k; \ell)$ for some permutation $\{i, j, k\}$ of I_3 and $\ell \in \mathbb{Z}$ satisfying $\ell \equiv 1$ or $2 \pmod 3$. The type of $\Psi(i, j, k; \ell)$ is $\mathtt{A}_2^{(1)}$.*

Proof. Let Ψ be a maximal closed subroot system of Φ. Then by Lemma 1.7.1, we get $\mathrm{Gr}(\Psi) = \mathring{\Phi}_s$ and it is irreducible. This also implies that Ψ can not contain any long root of Φ. From the Proposition 1.2.2, we see that Ψ must contain the roots

$$\big\{\epsilon_1 - \epsilon_2 + (p_1 + n_s r)\delta, \epsilon_2 - \epsilon_3 + (p_2 + n_s r)\delta, \epsilon_1 - \epsilon_3 + (p_3 + n_s r)\delta : r \in \mathbb{Z}\big\}$$

for some $p_1, p_2, p_3 \in \mathbb{Z}$ and $n_s \in \mathbb{Z}$. Since Ψ is closed and does not contain any long roots, we get $p_1 - p_2 \not\equiv 0 \pmod 3$ as $\epsilon_1 + \epsilon_3 - 2\epsilon_2 + (p_1 - p_2)\delta = (\epsilon_1 - \epsilon_2 + p_1\delta) + (\epsilon_3 - \epsilon_2 - p_2\delta) \notin \Psi$. Similarly, we get

$p_2 + p_3 \not\equiv 0 \pmod 3$ and $p_1 + p_3 \not\equiv 0 \pmod 3$. This implies that $p_1 \pmod 3$, $p_2 \pmod 3$ and $-p_3 \pmod 3$ are distinct elements. Hence, one of the p_i must be $\equiv 0 \pmod 3$. We claim that there exists a permutation $\{i, j, k\}$ of I_3 such that

$$\Psi = \big\{\pm(\epsilon_i - \epsilon_j + (q_1 + n_s r)\delta), \pm(\epsilon_j - \epsilon_k + (q_2 + n_s r)\delta), \pm(\epsilon_i - \epsilon_k + (q_3 + n_s r)\delta) : r \in \mathbb{Z}\big\},$$

where q_1, q_2 and q_3 satisfy $q_1 \equiv 0 \pmod 3$, $q_2 \equiv q_3 \pmod 3$ and $q_3 \not\equiv 0 \pmod 3$. If $p_1 \equiv 0 \pmod 3$, then take $(q_1, q_2, q_3) = (p_1, p_2, p_3)$ and take the permutation to be identity. If $p_2 \equiv 0 \pmod 3$, then take $(q_1, q_2, q_3) = (p_2, -p_3, -p_1)$ and take the permutation to be they cycle $(1\ 2\ 3)$ and if $p_3 \equiv 0 \pmod 3$, then take $(q_1, q_2, q_3) = (p_3, -p_2, p_1)$ and take the permutation to be the cycle $(2\ 3)$.

Now, we claim that $n_s \equiv 0 \pmod 3$. Suppose not, then there exists $r \in \mathbb{Z}$ such that $rn_s \equiv q_2 \pmod 3$ which implies that $\epsilon_i + \epsilon_k - 2\epsilon_j + (q_1 + rn_s - q_2)\delta = (\epsilon_i - \epsilon_j + (q_1 + rn_s)\delta) + (\epsilon_k - \epsilon_j - q_2\delta) \in \Psi$, a contradiction. Thus, there exists a permutation $\{i, j, k\}$ of I_3 and $\ell \equiv 1$ or $2 \pmod 3$ such that $\Psi \subseteq \Psi^+(i, j, k; \ell)$. Since $\Psi(i, j, k; \ell)$ is closed, we get that $\Psi = \Psi(i, j, k; \ell)$. This proves the forward part. The converse is clear from the Proposition 1.7.3. $\square$

1.8 The case $\mathtt{E}_6^{(2)}$

Throughout this section we assume that Φ is of type $\mathtt{E}_6^{(2)}$. In particular, the gradient root system $\mathring{\Phi}$ of $\mathtt{E}_6^{(2)}$ is of type $\mathtt{F}_4$. We have the following explicit description of $\mathtt{E}_6^{(2)}$, see [4, Page no. 557, 604]:

$$\Phi = \big\{\pm\epsilon_i + r\delta, \pm\epsilon_i \pm \epsilon_j + 2r\delta, \tfrac{1}{2}(\lambda_1\epsilon_1 + \lambda_2\epsilon_2 + \lambda_3\epsilon_3 + \lambda_4\epsilon_4) + r\delta, : \lambda_i = \pm 1, 1 \le i \ne j \le 4, r \in \mathbb{Z}\big\}$$

The short roots of $\mathring{\Phi}$ form a root system of type $\mathtt{D}_4$ ([4, Page no. 147]). We set $\mathcal{D}_4 := \mathring{\Phi}_s = \big\{ \pm\epsilon_i, \frac{1}{2}(\lambda_1\epsilon_1 + \lambda_2\epsilon_2 + \lambda_3\epsilon_3 + \lambda_4\epsilon_4) : i \in I_4, \lambda_j = \pm 1, \forall\ j \in I_4\big\}$ and $\Gamma_4 = \{\epsilon_2, \epsilon_3, \epsilon_4, \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)\}$ is a simple root system of $\mathcal{D}_4$.

Let $p : \Gamma_4 \to \mathbb{Z}$ be a function and let $p : \mathcal{D}_4 \to \mathbb{Z}$ be its $\mathbb{Z}$–linear extension, such that exactly two p_{ϵ_i} are even and the rest two are odd. Define

$$\Psi_p(\mathtt{E}_6^{(2)}) := \big\{\alpha + (p_\alpha + 2r)\delta : \alpha \in \mathcal{D}_4, r \in \mathbb{Z}\big\} \cup \big\{\pm\epsilon_i \pm \epsilon_j + 2r\delta : p_{\epsilon_i} + p_{\epsilon_j} \in 2\mathbb{Z}, r \in \mathbb{Z}\big\}.$$

Note that $p_{-\epsilon_i} = -p_{\epsilon_i}$ and $p_{\epsilon_i} + p_{\epsilon_j} \in 2\mathbb{Z}$ if and only if $p_{\epsilon_i}, p_{\epsilon_j}$ have the same parity.

Lemma 1.8.1. $\Psi_p(\mathtt{E}_6^{(2)})$ *is a closed subroot system of* Φ.

Proof. First we prove that $\Psi_p(\mathtt{E}_6^{(2)})$ is a subroot system of Φ. Since p is $\mathbb{Z}$–linear and satisfies the equation 1.2.2, we have

$$\mathbf{s}_{\alpha+(p_\alpha+2r)\delta}(\beta + (p_\beta + 2r')\delta) = \mathbf{s}_\alpha(\beta) + (p_{\mathbf{s}_\alpha(\beta)} + 2(r' - r\langle\beta, \alpha^\vee\rangle))\delta \in \Psi_p(\mathtt{E}_6^{(2)}).$$

Suppose $\pm\epsilon_i \pm \epsilon_j \in \Psi_p(\mathtt{E}_6^{(2)})$, we have p_{ϵ_i} and p_{ϵ_j} have the same parity since $p_{\epsilon_i} + p_{\epsilon_j} \in 2\mathbb{Z}$. This implies p_{ϵ_k} and p_{ϵ_ℓ} also have the same parity by our choice of p, where $\{k, \ell\} = I_4 \backslash \{i, j\}$. So, $p_{\epsilon_k} + p_{\epsilon_\ell} \in 2\mathbb{Z}$, and hence $\pm\epsilon_k \pm \epsilon_\ell + 2r\delta \in \Psi_p(\mathtt{E}_6^{(2)})$ for all $r \in \mathbb{Z}$. We have

$$\mathbf{s}_{\alpha+(p_\alpha+2r)\delta}(\pm\epsilon_i \pm \epsilon_j + 2r'\delta) = \mathbf{s}_\alpha(\pm\epsilon_i \pm \epsilon_j) + 2(r' - (\pm\epsilon_i \pm \epsilon_j, \alpha)(p_\alpha + 2r))\delta,$$

for $\alpha \in \mathcal{D}_4$ and $r, r' \in \mathbb{Z}$. Now, since

$$\mathbf{s}_\alpha(\pm\epsilon_i \pm \epsilon_j) \text{ is a root of the form } \begin{cases} \pm\epsilon_i \pm \epsilon_j \text{ if } \alpha = \pm\epsilon_k \\ \pm\epsilon_i \pm \epsilon_j \text{ or } \ \pm \epsilon_k \pm \epsilon_\ell \text{ if } \alpha = \sum\limits_{r=1}^{4} \lambda_r\epsilon_r, \end{cases}$$

where $\{k, \ell\} = I_4 \backslash \{i, j\}$, we have $\mathbf{s}_{\alpha+(p_\alpha+2r)\delta}(\pm\epsilon_i \pm \epsilon_j + 2r'\delta) \in \Psi_p(\mathtt{E}_6^{(2)})$. It is easy to see that,

$$\mathbf{s}_{\pm\epsilon_k\pm\epsilon_\ell+2r\delta}(\epsilon_i \pm \epsilon_j + 2r'\delta) = \pm\epsilon_i \pm \epsilon_j + 2r'\delta \in \Psi_p(\mathtt{E}_6^{(2)}).$$

Since $p_{\pm\epsilon_i}$ and $p_{\pm\epsilon_j}$ have the same parity, we have

$$p_{\alpha-(\alpha,\pm\epsilon_i\pm\epsilon_j)(\pm\epsilon_i\pm\epsilon_j)} = p_\alpha - (\alpha, \pm\epsilon_i \pm \epsilon_j)(p_{\pm\epsilon_i} + p_{\pm\epsilon_j}) \equiv p_\alpha \pmod 2.$$

This implies that

$$\mathbf{s}_{\pm\epsilon_i\pm\epsilon_j+2r\delta}(\alpha+(p_\alpha+2r')\delta) = (\alpha-(\alpha,\pm\epsilon_i\pm\epsilon_j)(\pm\epsilon_i\pm\epsilon_j))+(p_\alpha+2(r'-(\alpha,\pm\epsilon_i\pm\epsilon_j)r))\delta \in \Psi_p(\mathtt{E}_6^{(2)})$$

for $\alpha \in \mathcal{D}_4$ and $r, r' \in \mathbb{Z}$ since $(\alpha - (\alpha, \pm\epsilon_i \pm \epsilon_j)(\pm\epsilon_i \pm \epsilon_j)) \in \mathcal{D}_4$ for $\alpha \in \mathcal{D}_4$. This proves that $\Psi_p(\mathtt{E}_6^{(2)})$ is a subroot system of Φ. Now, we prove that $\Psi_p(\mathtt{E}_6^{(2)})$ is closed in Φ. We have the following cases.

Case (1). Let $x = (\alpha + (p_\alpha + 2r)\delta) + (\beta + (p_\beta + 2r')\delta) \in \Phi$ for some $\alpha, \beta \in \mathcal{D}_4$. If $\alpha + \beta \in \mathcal{D}_4$, then it is easy to see that $x = (\alpha + \beta) + (p_{\alpha+\beta} + 2(r + r'))\delta \in \Psi_p(\mathtt{E}_6^{(2)})$. If $\alpha + \beta \notin \mathcal{D}_4$, then p_α and p_β are of the same parity. We have the following possibilities when $\alpha + \beta \notin \mathcal{D}_4$:

- if $\alpha = \pm\epsilon_i, \beta = \pm\epsilon_j \in \mathcal{D}_4$, then
  $x = (\pm\epsilon_i \pm \epsilon_j) + (p_\alpha + p_\beta + 2(r + r'))\delta \in \Psi_p(\mathtt{E}_6^{(2)})$ since $p_\alpha \equiv p_{\epsilon_i} \pmod 2$ and $p_\beta \equiv p_{\epsilon_j} \pmod 2$ have the same parity.

- if $\alpha = \frac{1}{2}(\lambda_i\epsilon_i + \lambda_j\epsilon_j) + \frac{1}{2}(\lambda_k\epsilon_k + \lambda_\ell\epsilon_\ell)$ and $\beta = \frac{1}{2}(\lambda_i\epsilon_i + \lambda_j\epsilon_j) - \frac{1}{2}(\lambda_k\epsilon_k + \lambda_\ell\epsilon_\ell)$, then we have $\alpha - (\lambda_k\epsilon_k + \lambda_\ell\epsilon_\ell) = \beta$ which implies that
  $p_\alpha - (\lambda_k p_{\epsilon_k} + \lambda_\ell p_{\epsilon_\ell}) = p_\beta$. Since $p_\alpha \equiv p_\beta \pmod 2$, we must have
  $p_{\epsilon_k} \equiv p_{\epsilon_\ell} \pmod 2$. Hence, p_{ϵ_i} and p_{ϵ_j} are of the same parity by our choice of the function p. This implies
  $x = (\lambda_i\epsilon_i + \lambda_j\epsilon_j) + (p_\alpha + p_\beta + 2(r + r'))\delta \in \Psi_p(\mathtt{E}_6^{(2)})$.

Case (2). Let $x = (\alpha + (p_\alpha + 2r)\delta) + (\pm\epsilon_i \pm \epsilon_j + 2r'\delta) \in \Phi$ for some $\alpha \in \mathcal{D}_4$ and $(\pm\epsilon_i \pm \epsilon_j + 2r'\delta) \in \Psi_p(\mathtt{E}_6^{(2)})$. Since $\alpha + (\pm\epsilon_i \pm \epsilon_j) \in \mathrm{Gr}(\Phi)$, we have $\alpha + (\pm\epsilon_i \pm \epsilon_j) \in \mathcal{D}_4$. Since p_{ϵ_i} and p_{ϵ_j} have the same parity, we have $p_{\alpha-(\pm\epsilon_i\pm\epsilon_j)} = p_\alpha - (p_{\pm\epsilon_i} + p_{\pm\epsilon_j}) \equiv p_\alpha \pmod 2$. This implies that $x = \alpha + (\pm\epsilon_i \pm \epsilon_j) + (p_\alpha + 2(r + r'))\delta \in \Phi$, since $\alpha + (\pm\epsilon_i \pm \epsilon_j) \in \mathcal{D}_4$.

Case (3). Let $x = (\alpha + 2r\delta) + (\beta + 2r'\delta) \in \Phi$ for some $\alpha + 2r\delta, \beta + 2r'\delta \in \Psi_p(\mathtt{E}_6^{(2)})$ with $\alpha, \beta \notin \mathcal{D}_4$. Then we must have $\alpha = \pm\epsilon_i \pm \epsilon_j$ and $\beta = \mp\epsilon_j \pm \epsilon_k$ for some $i \neq j, j \neq k \in I_4$. Since p_{ϵ_i} and p_{ϵ_j} have the same parity and p_{ϵ_j} and p_{ϵ_k} have the same parity, we have $i = k$ by our choice of the function p. In this case, x can not be in Φ, so this case is not possible. This completes the proof. $\square$

Note that $Z_\alpha(\Psi_p(\mathtt{E}_6^{(2)})) = p_\alpha + 2\mathbb{Z}$ for all $\alpha \in \mathcal{D}_4$ and $Z_{\pm\epsilon_i\pm\epsilon_j}(\Psi_p(\mathtt{E}_6^{(2)})) = 2\mathbb{Z}$ for $\pm\epsilon_i \pm \epsilon_j \in \mathrm{Gr}(\Psi_p(\mathtt{E}_6^{(2)}))$, in particular $Z_\alpha(\Psi_p(\mathtt{E}_6^{(2)})) = 2\mathbb{Z}$ or $1 + 2\mathbb{Z}$ depending on p_α being even or odd.

Lemma 1.8.2. *$\Psi_p(\mathtt{E}_6^{(2)})$ is a maximal closed subroot system of Φ.*

Proof. Suppose there is a closed subroot system Δ of Φ such that $\Psi_p(\mathtt{E}_6^{(2)}) \subsetneq \Delta \subseteq \Phi$. This implies that that $\mathrm{Gr}(\Psi_p(\mathtt{E}_6^{(2)})) \subseteq \mathrm{Gr}(\Delta)$ and $Z_\alpha(\Psi_p(\mathtt{E}_6^{(2)})) \subseteq Z_\alpha(\Delta)$. Note that $Z_\alpha(\Psi_p(\mathtt{E}_6^{(2)})) = Z_\alpha(\Delta) = 2\mathbb{Z}$ for $\alpha = \pm\epsilon_i \pm \epsilon_j \in \mathrm{Gr}(\Psi_p(\mathtt{E}_6^{(2)}))$.

So, there are three possibilities for elements of $\Delta\backslash\Psi_p(\mathtt{E}_6^{(2)})$.

Case (1). Suppose $Z_\alpha(\Psi_p(\mathtt{E}_6^{(2)})) \subsetneq Z_\alpha(\Delta)$ for some $\alpha = \epsilon_s$. Then there exists $r_1, r_2 \in \mathbb{Z}$ such that $\epsilon_s + r_1\delta, \epsilon_s + r_2\delta \in \Delta$ and r_1, r_2 have different parity. Then either $(p_{\epsilon_t} + r_1) \in 2\mathbb{Z}$ or $(p_{\epsilon_t} + r_2) \in 2\mathbb{Z}$ for each $t \in I_4$ with $t \neq s$. Hence, $\epsilon_t + \epsilon_s + 2\mathbb{Z}\delta = \epsilon_t + (p_{\epsilon_t} + 2\mathbb{Z})\delta + \epsilon_s + r_i\delta \subseteq \Delta$ for $i = 1$ or 2. Similarly, one sees that

$$\pm\epsilon_t \pm \epsilon_s + 2\mathbb{Z}\delta \subseteq \Delta \text{ for all } t \in I_4, t \neq s.$$

Choose $t \in I_4$ such that p_{ϵ_t} and p_{ϵ_s} have different parity. Then $\epsilon_s - (p_{\epsilon_t} + 2\mathbb{Z})\delta = \epsilon_t + \epsilon_s + 2\mathbb{Z}\delta - (\epsilon_t + p_{\epsilon_t}\delta) \subseteq \Delta$. This implies that $\epsilon_s + \mathbb{Z}\delta \subseteq \Delta$. This implies $\epsilon_t + \mathbb{Z}\delta = (\epsilon_t + \epsilon_s + 2\mathbb{Z}\delta) + (-\epsilon_s + \mathbb{Z}\delta) \subseteq \Delta$ for all $t \neq s$. Hence, we have $\epsilon_t + \mathbb{Z}\delta \subseteq \Delta$ for all $t \in I_4$. From this it is easy to see that $\Delta = \Phi$.

Case (2). Suppose $Z_\alpha(\Psi_p(\mathtt{E}_6^{(2)})) \subsetneq Z_\alpha(\Delta)$ for some $\alpha = \sum_{i=1}^4 \lambda_i\epsilon_i$. Then there exists $r_1, r_2 \in \mathbb{Z}$ with different parity such that $\frac{1}{2}\big(\sum_{i=1}^4 \lambda_i\epsilon_i\big) + r_1\delta$, $\frac{1}{2}\big(\sum_{i=1}^4 \lambda_i\epsilon_i\big) + r_2\delta \in \Delta$. So, we have

$$\lambda_1\epsilon_1 + (r_k + s)\delta = \frac{1}{2}\big(\sum_{i=1}^{4} \lambda_i\epsilon_i\big) + r_k\delta + \frac{1}{2}\big(\lambda_1\epsilon_1 + \sum_{i=2}^{4}(-\lambda_i)\epsilon_i\big) + s\delta \in \Delta$$

for $k = 1, 2$ and $s = p_{\frac{1}{2}\left(\lambda_1\epsilon_1 + \sum\limits_{i=2}^{4}(-\lambda_i)\epsilon_i\right)}$. Since $(r_1 + s)$ and $(r_2 + s)$ have different parity, we are back to Case (1) and hence $\Delta = \Phi$.

Case (3). Suppose $\mathrm{Gr}(\Psi_p(\mathtt{E}_6^{(2)})) \subsetneq \mathrm{Gr}(\Delta)$. Then there exists $i, j \in I_4, i \neq j$, such that p_{ϵ_i} and p_{ϵ_j} have different parity and $\epsilon_i \pm \epsilon_j + 2r\delta \in \Delta$ for some $r \in \mathbb{Z}$. Since $\mp\epsilon_j + p_{\mp\epsilon_j}\delta \in \Psi_p(\mathtt{E}_6^{(2)})$, we get $\epsilon_i + (p_{\mp\epsilon_j} + 2r)\delta \in \Delta$. Since p_{ϵ_i}, $p_{\mp\epsilon_j} + 2r$ have different parity and $\epsilon_i + p_{\epsilon_i}\delta \in \Delta$, we are back to the Case (1) again and hence $\Delta = \Phi$. This completes the proof.

□

Proposition 1.8.3. *Suppose Φ is of type $\mathtt{E}_6^{(2)}$. Then $\Psi \leq \Phi$ is a maximal closed subroot system with a proper semi-closed gradient subroot system if and only if there exists a $\mathbb{Z}-$linear function $p : \mathcal{D}_4 \to \mathbb{Z}$ such that $\Psi = \Psi_p(\mathtt{E}_6^{(2)})$ and exactly two of p_{ϵ_i} are even. The type of $\Psi_p(\mathtt{E}_6^{(2)})$ is $\mathtt{C}_4^{(1)}$.*

Proof. Since Ψ is a maximal closed subroot system in Φ and not contained in the proper closed subroot system Ψ_0 of Φ, where $\Psi_0 := \{\pm\epsilon_i + r\delta, \pm\epsilon_i \pm \epsilon_j + 2r\delta, : 1 \leq i \neq j \leq 4, r \in \mathbb{Z}\}$, there is a short root of the form $\frac{1}{2}(\sum_{j=1}^4 \nu_j\epsilon_j)$ in $\mathrm{Gr}(\Psi)$, fix this short root in $\mathrm{Gr}(\Psi)$. Now, define

$I := \{i \in I_4 : \epsilon_i \in \mathrm{Gr}(\Psi)\}$.

First, we prove that I must be non-empty subset of I_4. Assume that $I = \emptyset$. Since $\mathrm{Gr}(\Psi)$ is semi-closed, there exist short roots α_1 and α_2 such that $\alpha_1 + \alpha_2$ is a long root and $\alpha_1 + \alpha_2 \in \mathring{\Phi} \backslash \mathrm{Gr}(\Psi)$. Since $I = \emptyset$, we can take $\alpha_1 = \frac{1}{2}(\lambda_{i_1}\epsilon_{i_1} + \lambda_{i_2}\epsilon_{i_2} + \lambda_{i_3}\epsilon_{i_3} + \lambda_{i_4}\epsilon_{i_4})$ and $\alpha_2 = \frac{1}{2}(-\lambda_{i_1}\epsilon_{i_1} - \lambda_{i_2}\epsilon_{i_2} + \lambda_{i_3}\epsilon_{i_3} + \lambda_{i_4}\epsilon_{i_4})$. Since $\epsilon_i \notin \mathrm{Gr}(\Psi)$ for all $i \in I_4$ and Ψ is a closed subroot system, the only short roots that $\mathrm{Gr}(\Psi)$ can contain are α_1, α_2, $\alpha_3 = \frac{1}{2}(-\lambda_{i_1}\epsilon_{i_1} + \lambda_{i_2}\epsilon_{i_2} - \lambda_{i_3}\epsilon_{i_3} + \lambda_{i_4}\epsilon_{i_4})$ and $\alpha_4 = \frac{1}{2}(-\lambda_{i_1}\epsilon_{i_1} + \lambda_{i_2}\epsilon_{i_2} + \lambda_{i_3}\epsilon_{i_3} - \lambda_{i_4}\epsilon_{i_4})$ along with their negatives. For example, if $\beta = \frac{1}{2}(-\lambda_{i_1}\epsilon_{i_1} + \lambda_{i_2}\epsilon_{i_2} + \lambda_{i_3}\epsilon_{i_3} + \lambda_{i_4}\epsilon_{i_4}) \in \mathrm{Gr}(\Psi)$, then $\alpha_1 + (-\beta) = \lambda_{i_1}\epsilon_{i_1} \in \mathrm{Gr}(\Psi)$ since $\alpha_1 + (-\beta)$ is a short root and $\mathrm{Gr}(\Psi)$ is semi-closed. This is clearly a contradiction to our assumption that $I = \emptyset$. So, $\mathrm{Gr}(\Psi) \subseteq \Delta := \{\pm\alpha_i : i \in I_4\} \cup \mathring{\Phi}_\ell$. But Δ is a closed subroot system of $\mathring{\Phi}$ and hence $\widehat{\Delta}$ is a closed subroot system in Φ. Since $\Psi \subseteq \widehat{\Delta}$, we must have $\Psi = \widehat{\Delta}$ and $\mathrm{Gr}(\Psi) = \Delta$, a contradiction to the fact that $\mathrm{Gr}(\Psi)$ is a proper semi-closed subroot system of $\mathring{\Phi}$. This proves that I must be non-empty. Indeed we will prove that $|I|$ must be 4, hence $I = I_4$. We will rule out all other possibilities one by one.

Case (1). We claim that we must have $|I| \geq 2$, hence $|I| \neq 1$. Let $i \in I$. As before, since $\mathrm{Gr}(\Psi)$ is semi-closed there exist short roots α and β such that $\alpha + \beta$ is a long root and $\alpha + \beta \in \mathring{\Phi} \backslash \mathrm{Gr}(\Psi)$. Now, both these short roots must lie in $\left\{\frac{1}{2}\sum_{j=1}^4 \lambda_j\epsilon_j : \lambda_j = \pm 1\right\}$, otherwise we are done. So, without loss of generality we assume that $\alpha = \frac{1}{2}(\lambda_{i_1}\epsilon_{i_1} + \lambda_{i_2}\epsilon_{i_2} + \lambda_{i_3}\epsilon_{i_3} + \lambda_{i_4}\epsilon_{i_4})$ and $\beta = \frac{1}{2}(-\lambda_{i_1}\epsilon_{i_1} - \lambda_{i_2}\epsilon_{i_2} + \lambda_{i_3}\epsilon_{i_3} + \lambda_{i_4}\epsilon_{i_4})$. If $i_3 = i$, then $\mathbf{s}_{\epsilon_i}(\alpha) = \alpha - \lambda_{i_3}\epsilon_i = \frac{1}{2}(\lambda_{i_1}\epsilon_{i_1} + \lambda_{i_2}\epsilon_{i_2} - \lambda_{i_3}\epsilon_{i_3} + \lambda_{i_4}\epsilon_{i_4})$. Since $\mathrm{Gr}(\Psi)$ is semi-closed, we have $\beta + \mathbf{s}_{\epsilon_i}(\alpha) = \lambda_{i_4}\epsilon_{i_4} \in \mathrm{Gr}(\Psi)$. Similarly, if $i_4 = i$, then we get $\lambda_{i_3}\epsilon_{i_3} \in \mathrm{Gr}(\Psi)$. Now, if $i_1 = i$, then $\mathbf{s}_{\epsilon_i}(\alpha) + (-\beta) = \lambda_{i_2}\epsilon_{i_2} \in \mathrm{Gr}(\Psi)$. Similarly, if $i_2 = i$, then we get $\lambda_{i_1}\epsilon_{i_1} \in \mathrm{Gr}(\Psi)$. This proves that we must have $|I| \geq 2$ in all cases.

Case (2). Now, we claim that $|I| \neq 3$. Suppose that $|I| = 3$ and $I = I_4 \backslash \{k\}$ for some $k \in I_4$. Recall that we have a short root of the form $\frac{1}{2}(\sum_{j=1}^{4} \nu_j \epsilon_j)$ in $\mathrm{Gr}(\Psi)$. For $j \in I_4$ such that $j \neq k$, there exists $r_j \in \mathbb{Z}$ such that $\nu_j \epsilon_j + r_j \delta \in \Psi$ since $\nu_j \epsilon_j \in \mathrm{Gr}(\Psi)$. Since $|I| = 3$, there exists $j_1, j_2 \in I$ such that $r_{j_1} + r_{j_2} \in 2\mathbb{Z}$. This implies that $\sum_{\ell=1}^{2}(\nu_{j_\ell}\epsilon_{j_\ell} + r_{j_\ell}\delta) \in \Psi$ since Ψ is closed in Φ. Now, since $\frac{1}{2}(\sum_{j=1}^{4} \nu_j \epsilon_j) + r\delta \in \Psi$ for some $r \in \mathbb{Z}$ and Ψ is closed, we have $\frac{1}{2}(-\nu_{j_1}\epsilon_{j_1} - \nu_{j_2}\epsilon_{j_2} + \nu_{j_3}\epsilon_{j_3} + \nu_k\epsilon_k) + \left(r - \sum\limits_{\ell=1}^{2} r_{j_\ell}\right)\delta =$ $\frac{1}{2}\left(\sum\limits_{j=1}^{4} \nu_j \epsilon_j\right) + r\delta - \sum\limits_{\ell=1}^{2}(\nu_{j_\ell}\epsilon_{j_\ell} + r_{j_\ell}\delta) \in \Psi$. Adding $\frac{1}{2}(-\nu_{j_1}\epsilon_{j_1} - \nu_{j_2}\epsilon_{j_2} + \nu_{j_3}\epsilon_{j_3} + \nu_k\epsilon_k) + (r - \sum_{\ell=1}^{2} r_{j_\ell})\delta$ and $\frac{1}{2}(\sum_{j=1}^{4} \nu_j\epsilon_j) + r\delta \in \Psi$ we get $(\nu_{j_3}\epsilon_{j_3} + \nu_k\epsilon_k) + (2r - \sum\limits_{\ell=1}^{2} r_{j_\ell})\delta \in \Psi$. Again adding $-\nu_{j_3}\epsilon_{j_3} - r_{j_3}\delta$ with $(\nu_{j_3}\epsilon_{j_3} + \nu_k\epsilon_k) + (2r - \sum\limits_{\ell=1}^{2} r_{j_\ell})\delta \in \Psi$, we get $\nu_k\epsilon_k + (2r - \sum\limits_{k=1}^{3} r_{j_k})\delta \in \Psi$ which contradicts the assumption that $k \notin I$. This proves that $|I| \neq 3$. So, we proved that $|I| = 2$ or 4 are the only possibilities.

Case (3). Now, assume that $|I| = 4$, hence $I = I_4$. In this case, we claim that there exists a $\mathbb{Z}-$linear function $p : \mathcal{D}_4 \to \mathbb{Z}$ with the property that exactly two p_{ϵ_i} are even and the rest two are odd such that $\Psi = \Psi_p(\mathtt{E}_6^{(2)})$. Since $\mathrm{Gr}(\Psi)$ contains $\pm\epsilon_i, 1 \leq i \leq 4$, and a short root of the form $\frac{1}{2}\sum_{j=1}^{4} \nu_j\epsilon_j$, $\mathrm{Gr}(\Psi)$ must contain all the short roots of F_4. We now claim that for each short root $\alpha \in \mathrm{Gr}(\Psi)$, $Z_\alpha(\Psi)$ contains either only odd integers or even integers, i.e., it can not contain integers with different parity. We will do this case by case.

- Suppose there is $i \in I_4$ such that $2r_1, 2r_2 + 1 \in Z_{\epsilon_i}(\Psi)$ for some $r_1, r_2 \in \mathbb{Z}$. Using this, one easily sees that there exist $s_{ij}^{\pm} \in \mathbb{Z}$ such that $\epsilon_i \pm \epsilon_j + 2s_{ij}^{\pm}\delta \in \Psi$ for all $j \neq i$, since $I = I_4$ and Ψ is closed. Hence,

$$\pm\epsilon_j \mp \epsilon_k + (2s_{ij}^{\pm} - 2s_{ik}^{\pm})\delta = \epsilon_i \pm \epsilon_j + 2s_{ij}^{\pm}\delta - (\epsilon_i \pm \epsilon_k + 2s_{ik}^{\pm}\delta) \in \Psi$$

for all $j \neq k$ contradicting our assumption on $\mathrm{Gr}(\Psi)$ that it is semi-closed.

This proves that $Z_{\epsilon_i}(\Psi)$ contains either only odd integers or only even integers.

- Now, assume that $Z_\alpha(\Psi)$ contains both odd and even integers for some $\alpha = \frac{1}{2}(\sum_{j=1}^{4} \mu_j\epsilon_j)$, i.e. $\exists\ r_1, r_2 \in \mathbb{Z}$ with different parity such that $\frac{1}{2}(\sum_{j=1}^{4} \mu_j\epsilon_j) + r_1\delta, \frac{1}{2}(\sum_{j=1}^{4} \mu_j\epsilon_j) + r_2\delta \in \Psi$. Then this implies that $\frac{1}{2}(\mu_1\epsilon_1 - \mu_2\epsilon_2 + \mu_3\epsilon_3 + \mu_4\epsilon_4) + (r_1 - k_2)\delta = \frac{1}{2}(\sum\limits_{j=1}^{4} \mu_j\epsilon_j) + r_1\delta - (\mu_2\epsilon_2 + k_2\delta) \in \Psi$, where $k_2 \in Z_{\epsilon_2}(\Psi)$. Similarly, we get $\frac{1}{2}(\mu_1\epsilon_1 - \mu_2\epsilon_2 - \mu_3\epsilon_3 - \mu_4\epsilon_4) + (r_1 - k_2 - k_3 - k_4)\delta \in \Psi$, where $k_j \in Z_{\epsilon_j}(\Psi)$. Which in turn implies that

$$(\alpha + r_1\delta) + \left(\tfrac{1}{2}(\mu_1\epsilon_1 - \sum_{j=2}^{4} \mu_j\epsilon_j) + (r_1 - \sum_{j=2}^{4} k_j)\delta\right) = \mu_1\epsilon_1 + (2r_1 - \sum_{j=2}^{4} k_j)\delta \in \Psi,$$

since Ψ is closed in Φ. Similarly, we have $\mu_1\epsilon_1 + (r_1 + r_2 - \sum\limits_{j=2}^{4} k_j)\delta \in \Psi$. Which means $Z_{\epsilon_1}(\Psi)$ contains integers of different parity which by Case(1) is impossible. This proves our claim.

Let p a function $p : \Gamma_4 \to \mathbb{Z}$ such that $p_\beta \in Z_\beta(\Psi)$ for each β in Γ_4, where Γ_4 is a simple root system of $\mathcal{D}_4$ defined in 1.8. Extend the function p to $\mathcal{D}_4$ $\mathbb{Z}-$linearly, denote this extension again by p. We now claim that exactly two p_{ϵ_i} are even. Suppose all p_{ϵ_i} have the same parity, then "Ψ is closed in Φ" would imply that $\mathrm{Gr}(\Psi) = \mathring{\Phi}$. This is a contradiction to our assumption that $\mathrm{Gr}(\Psi)$ is semi-closed. So, all p_{ϵ_i} can not have the same parity. Now, assume that there exists $k \in I_4$ such that p_{ϵ_i} have the same parity for all $i \neq k$, and p_{ϵ_k} has different parity. Let $\beta_1 = \frac{1}{2}(\sum_{i\neq k} \epsilon_i + \epsilon_k) + r\delta \in \Psi$ for some $r \in \mathbb{Z}$. Since Ψ is closed, we have

$$\beta_2 = \frac{1}{2}(\sum_{i\neq k}(-\epsilon_i) + \epsilon_k) + (r - \sum_{i\neq k} p_{\epsilon_i})\delta \in \Psi$$

and hence we get $\beta_1 + \beta_2 = \epsilon_k + (2r - \sum\limits_{i\neq k} p_{\epsilon_i})\delta \in \Psi$. This implies that p_{ϵ_k} and

$(2r-\sum_{i\neq k}p_{\epsilon_i})$ are in $Z_{\epsilon_k}(\Psi)$. But p_{ϵ_k} and $(2r-\sum_{i\neq k}p_{\epsilon_i})$ have different parity, which is a contradiction to our previous observation that $Z_\alpha(\Psi)$ contains only either odd integers or even integers. Thus, we proved that exactly two p_{ϵ_i} are even and the rest are odd. Now, using the arguments in the proof of Case (3) in Lemma 1.8.2, we see that there is no $i,j\in I_4$ with $i\neq j$ such that p_{ϵ_i} and p_{ϵ_j} have different parity and $\pm\epsilon_i\pm\epsilon_j\in \mathrm{Gr}(\Psi)$. This implies that $\Psi\subseteq\Psi_p(\mathtt{E}_6^{(2)})$. Since Ψ is maximal closed, we have $\Psi=\Psi_p(\mathtt{E}_6^{(2)})$.

Case (4). Finally assume that $|I|=2$ and $I=\{i,j\}$. Since $\mathrm{Gr}(\Psi)$ is semi-closed, then we claim that we have

$$\mathrm{Gr}(\Psi)\cap\mathcal{D}_4=\big\{\pm\epsilon_i,\pm\epsilon_j,\pm\tfrac{1}{2}(\sum_{r=1}^{4}\mu_r\epsilon_r):\mu_r=\nu_r, r\neq i,j\big\}.$$

Since $\alpha=\frac{1}{2}(\sum_{r=1}^4\nu_r\epsilon_r)\in\mathrm{Gr}(\Psi)$, we have $\mathbf{s}_{\epsilon_i}(\alpha)=\alpha-\nu_i\epsilon_i\in\mathrm{Gr}(\Psi)$ and $\mathbf{s}_{\epsilon_j}(\alpha)=\alpha-\nu_i\epsilon_j\in\mathrm{Gr}(\Psi)$. This proves that $\mathrm{Gr}(\Psi)\cap\mathcal{D}_4\supseteq\big\{\pm\epsilon_i,\pm\epsilon_j,\pm\frac{1}{2}(\sum_{r=1}^4\mu_r\epsilon_r):\mu_r=\nu_r, r\neq i,j\big\}$. Suppose $\beta=\frac{1}{2}(\sum_{r=1}^4\mu_r\epsilon_r)\in\mathrm{Gr}(\Psi)$ such that $\mu_k\neq\nu_k$ for some $k\neq i,j$. Let $\ell\in I_4\backslash\{i,j,k\}$. If $\mu_\ell\neq\nu_\ell$, then $-\beta$ satisfies the required condition, i.e., $-\mu_k=\nu_k$ and $-\mu_\ell=\nu_\ell$. So, assume that $\mu_\ell=\nu_\ell$, then $\frac{1}{2}(-\mu_i\epsilon_i-\mu_j\epsilon_j-\mu_k\epsilon_k+\mu_\ell\epsilon_\ell)\in\mathrm{Gr}(\Psi)$ and Ψ is closed, so we have $\epsilon_\ell\in\mathrm{Gr}(\Psi)$. This is clearly a contradiction to our assumption that $I=\{i,j\}$. This proves that

$$\mathrm{Gr}(\Psi)\cap\mathcal{D}_4=\big\{\pm\epsilon_i,\pm\epsilon_j,\pm\tfrac{1}{2}(\sum_{r=1}^{4}\mu_r\epsilon_r):\mu_r=\nu_r, r\neq i,j\big\}.$$

From this one easily sees that the only long roots $\mathrm{Gr}(\Psi)$ can contain are $\pm\epsilon_i\pm\epsilon_j$ and $\pm\epsilon_k\pm\epsilon_\ell$, where $\{k,\ell\}=I_4\backslash\{i,j\}$. Note that $\pm\epsilon_k$ and $\pm\epsilon_\ell$ can not be written as sum of elements from $\mathrm{Gr}(\Psi)\cap\mathcal{D}_4$. We now claim that $Z_\alpha(\Psi)$ does not contain elements of different parity for each short root α in $\mathrm{Gr}(\Psi)$. Assume this claim for time being. Then for each $\alpha\in\mathrm{Gr}(\Psi)\cap\mathcal{D}_4$, we have $Z_\alpha(\Psi)\subseteq p_\alpha+2\mathbb{Z}$ for some

$p_\alpha \in Z_\alpha(\Psi)$. Note that p_α is determined by Ψ for $\alpha \in \mathrm{Gr}(\Psi) \cap \mathcal{D}_4$. Now we extend this function $p : \mathrm{Gr}(\Psi) \cap \mathcal{D}_4 \to \mathbb{Z}$ to entire $\mathcal{D}_4$, $\mathbb{Z}$–linearly by defining p_{ϵ_k} in the following way:

- If both p_{ϵ_i} and p_{ϵ_j} have the same parity, then define p_{ϵ_k} to be an integer with different parity than p_{ϵ_i}.
- If p_{ϵ_i} and p_{ϵ_j} have different parity, then define p_{ϵ_k} arbitrarily.

The extended function $p : \mathcal{D}_4 \to \mathbb{Z}$, then satisfies the conditions that (1) exactly two p_{ϵ_r} are even and the rest two p_{ϵ_r} are odd and (2) it takes the same values p_α which was determined by Ψ for $\alpha \in \mathrm{Gr}(\Psi) \cap \mathcal{D}_4$. Note that the parity of p_{ϵ_ℓ} is completely determined by the parity of p_{ϵ_i}, p_{ϵ_j}, p_{ϵ_k} and $p_{\frac{1}{2}\sum_{r=1}^{4}\nu_r\epsilon_r)}$. By the choice of p, we have $\Psi \subsetneq \Psi_p(\mathtt{E}_6^{(2)})$. This proves that Ψ can not be maximal closed subroot system in Φ. Hence, the case $|I| = 2$ is not possible.

Proof of the claim: Now, we will complete the proof of the claim that $Z_\alpha(\Psi)$ does not contain elements of different parity for each short root α in $\mathrm{Gr}(\Psi)$. Let α_1, α_2 be two short roots in $\mathrm{Gr}(\Psi)$ such that $\alpha_1 + \alpha_2$ is a long root and $\alpha_1 + \alpha_2 \in \mathring{\Phi}\backslash\mathrm{Gr}(\Psi)$. We now prove that if $Z_\beta(\Psi)$ contains elements of different parity for some short root β in $\mathrm{Gr}(\Psi)$, then $Z_\alpha(\Psi)$ must contain elements of different parity for all short roots α in $\mathrm{Gr}(\Psi)$. This will contradict the fact that $\alpha_1 + \alpha_2 \in \mathring{\Phi}\backslash\mathrm{Gr}(\Psi)$, hence the claim follows.

- Assume that $Z_{\epsilon_i}(\Psi)$ contains elements of different parity, then we have $\pm\epsilon_i \pm \epsilon_j \in \mathrm{Gr}(\Psi)$ as Ψ is closed. This implies that $Z_{\epsilon_j}(\Psi)$ also contains elements of different parity. Let $\alpha = \frac{1}{2}(\sum\limits_{r=1}^{4}\mu_r\epsilon_r) \in \mathrm{Gr}(\Psi)$. We have $\frac{1}{2}(\sum_{r\neq s}\mu_r\epsilon_r - \mu_s\epsilon_s) \in \mathrm{Gr}(\Psi)$ for $s = i, j$. Since for $s = i, j$,

$$\frac{1}{2}(\sum_{r\neq s}\mu_r\epsilon_r - \mu_s\epsilon_s) + r_1\delta + \mu_s\epsilon_s + r_2\delta = \alpha + (r_1 + r_2)\delta$$

and $Z_{\epsilon_s}(\Psi)$ contains elements of different parity, we have $Z_\alpha(\Psi)$ also contains elements of different parity.

- Now, assume that $Z_\alpha(\Psi)$ contains elements of different parity for $\alpha = \frac{1}{2}(\sum_{r=1}^{4} \mu_r\epsilon_r)$ with $\mu_r = \nu_r, r \neq i, j$. Since we have $\frac{1}{2}(\sum\limits_{r\neq i} \mu_r\epsilon_r - \mu_i\epsilon_i) \in \mathrm{Gr}(\Psi)$, we get $Z_{\epsilon_i}(\Psi)$ contains elements of different parity. So, we are back to previous case.

This completes the proof. $\square$

1.9 The case $\mathtt{A}_{\mathtt{2n}}^{(2)}$

Throughout this section we assume that Φ is of type $\mathtt{A}_{\mathtt{2n}}^{(2)}$ and $n \geq 2$. In particular, the gradient root system $\mathrm{Gr}(\Phi)$ of $\mathtt{A}_{\mathtt{2n}}^{(2)}$ is of type $\mathtt{BC_n}$. We have the following explicit description of $\mathtt{A}_{\mathtt{2n}}^{(2)}$, see [4, Page no. 547, 583]:

$$\Phi = \big\{ \pm \epsilon_i + (r + \tfrac{1}{2})\delta, \pm 2\epsilon_i + 2r\delta, \pm\epsilon_i \pm \epsilon_j + r\delta, : 1 \leq i \neq j \leq n, r \in \mathbb{Z} \big\}$$

and $\mathrm{Gr}(\Phi) = \big\{ \pm \epsilon_i, \pm 2\epsilon_i, \pm\epsilon_i \pm \epsilon_j : 1 \leq i \neq j \leq n \big\} = \mathring{\Phi} \cup \frac{1}{2}\mathring{\Phi}_\ell$. In particular, we have three root lengths in $\mathrm{Gr}(\Phi)$ and we denote the short, intermediate and long roots of $\mathrm{Gr}(\Phi)$ by $\mathrm{Gr}(\Phi)_s$, $\mathrm{Gr}(\Phi)_{\mathrm{im}}$ and $\mathrm{Gr}(\Phi)_\ell$ respectively. Let $\Gamma = \{\alpha_1 = \epsilon_1 - \epsilon_2, \cdots, \alpha_{n-1} = \epsilon_{n-1} - \epsilon_n, \alpha_n = \epsilon_n\}$ be the simple system for $\mathrm{Gr}(\Phi)$.

Before we proceed further we fix some notations. For $I \subseteq I_n$, we set

$\Psi^+(I, \frac{1}{2}) := \big\{\epsilon_i + (2r + \frac{1}{2})\delta, (\epsilon_k + \epsilon_\ell) + (2r+1)\delta, (\epsilon_k - \epsilon_\ell) + 2r\delta : i, k, \ell \in I, k \neq \ell,\ r \in \mathbb{Z}\big\}$,

$\Psi^+(I, \frac{3}{2}) := \big\{\epsilon_i + (2r + \frac{3}{2})\delta, (\epsilon_k + \epsilon_\ell) + (2r+1)\delta, (\epsilon_k - \epsilon_\ell) + 2r\delta : i, k, \ell \in I, k \neq \ell,\ r \in \mathbb{Z}\big\}$ and

$\Psi^+(I, 0, 1) := \big\{(\epsilon_k + \epsilon_\ell) + 2r\delta, (\epsilon_k - \epsilon_\ell) + (2r+1)\delta : k \in I, \ell \in I_n \backslash I, r \in \mathbb{Z}\big\}$.

Now, define

$\Psi_I(\mathtt{A}_{\mathtt{2n}}^{(2)}) :=$

$\Psi^+\big(I,\frac{1}{2}\big)\cup(-\Psi^+\big(I,\frac{1}{2}\big))\cup\Psi^+(I,0,1)\cup(-\Psi^+(I,0,1))\cup\Psi^+\big(I_n\backslash I,\frac{3}{2}\big)\cup(-\Psi^+\big(I_n\backslash I,\frac{3}{2}\big)).$

Note that $\mathrm{Gr}(\Psi_I(\mathtt{A}_{\mathtt{2n}}^{(2)})) = \{\pm\epsilon_i, \pm\epsilon_k \pm \epsilon_\ell : i,k,\ell \in I_n, k \neq \ell\}$ for a root system of type B_n.

Proposition 1.9.1. *For $I \subseteq I_n$, $\Psi_I(\mathtt{A}_{\mathtt{2n}}^{(2)})$ is a maximal closed subroot system of Φ.*

Proof. It is easy to check that $\Psi_I(\mathtt{A}_{\mathtt{2n}}^{(2)})$ is a closed subroot system of Φ. We prove that it is a maximal closed subroot system in Φ. Let Δ be a closed subroot system of Φ such that $\Psi_I(\mathtt{A}_{\mathtt{2n}}^{(2)}) \subsetneq \Delta \subseteq \Phi$. The following are the possibilities for elements of $\Delta\backslash\Psi_I(\mathtt{A}_{\mathtt{2n}}^{(2)})$: if $\alpha \in \Delta\backslash\Psi_I(\mathtt{A}_{\mathtt{2n}}^{(2)})$, then α must be equal to either

- $\epsilon_i + (2r+\frac{3}{2})\delta \in \Delta$ or $2\epsilon_i + 2r\delta$, where $i \in I,\ r \in \mathbb{Z}$
- $\epsilon_i + (2r+\frac{1}{2})\delta \in \Delta$ or $2\epsilon_i + 2r\delta \in \Delta$, where $i \notin I,\ r \in \mathbb{Z}$
- $(\epsilon_k+\epsilon_\ell) + 2r\delta \in \Delta$ or $(\epsilon_k-\epsilon_\ell) + (2r+1)\delta \in \Delta$, where $k,\ell \in I$ and $r \in \mathbb{Z}$
- $(\epsilon_k+\epsilon_\ell) + 2r\delta \in \Delta$ or $(\epsilon_k-\epsilon_\ell) + (2r+1)\delta \in \Delta$, where $k,\ell \notin I$ and $r \in \mathbb{Z}$
- $(\epsilon_k+\epsilon_\ell) + (2r+1)\delta \in \Delta$ or $(\epsilon_k-\epsilon_\ell) + 2r\delta \in \Delta$, where $k \in I$, $\ell \notin I$ and $r \in \mathbb{Z}$

Suppose there exists $i \in I$ such that $\epsilon_i + (2r+\frac{3}{2})\delta \in \Delta$ for some $r \in \mathbb{Z}$. Then since $\epsilon_i + (2\mathbb{Z}+\frac{1}{2})\delta \subseteq \Delta$, we have $(\epsilon_i + (2\mathbb{Z}+\frac{1}{2})\delta) + (\epsilon_i + (2r+\frac{3}{2})\delta = 2\epsilon_i + 2\mathbb{Z}\delta \subseteq \Delta$. This implies that $(2\epsilon_i + 2\mathbb{Z}\delta) - (\epsilon_i + (2r+\frac{3}{2})\delta) = \epsilon_i + (\mathbb{Z}+\frac{1}{2})\delta \subseteq \Delta$. For $j \in I$, $\mathbf{s}_{\epsilon_i-\epsilon_j}(2\epsilon_i + 2\mathbb{Z}\delta) = 2\epsilon_j + 2\mathbb{Z}\delta \subseteq \Delta$. Similarly, for $j \notin I$ we have $\mathbf{s}_{\epsilon_i+\epsilon_j}(2\epsilon_i + 2\mathbb{Z}\delta) = -2\epsilon_j + 2\mathbb{Z}\delta \subseteq \Delta$. As before this implies that $\epsilon_j + \frac{2\mathbb{Z}+1}{2}\delta \in \Delta$ for all $j \in I_n$. Hence, $\Delta = \Phi$. Suppose there exists $i \in I$ such that $2\epsilon_i + 2r\delta \in \Delta$ for some $r \in \mathbb{Z}$. Then $\epsilon_i + \frac{3}{2}\delta = (2\epsilon_i + 2r\delta) + (-\epsilon_i - (2(r-1)+\frac{1}{2})\delta) \in \Delta$, so we are back to the first case. Hence, $\Delta = \Phi$.

All the remaining cases are done similarly. For example, if $(\epsilon_k+\epsilon_\ell) + 2r\delta \in \Delta$ for some $r \in \mathbb{Z}$ and $k,\ell \in I$, then we have

$\epsilon_k + \frac{3}{2}\delta = (\epsilon_k + \epsilon_\ell) + 2r\delta + (-\epsilon_\ell - (2(r-1) + \frac{1}{2})\delta) \in \Delta$, so we are back to first case. This completes the proof.

$\square$

1.9.1 Certain type of maximal closed subroot systems of $\mathtt{A}_{\mathtt{2n}}^{(2)}$

We now see another type of maximal closed subroot systems of Φ. For $J \subsetneq I_n$, define

$$A_J := \{\pm 2\epsilon_i, \pm\epsilon_s \pm \epsilon_t : i \in I_n \backslash J,\ s \neq t \in I_n \backslash J\} \cup \{\pm 2\epsilon_j, \pm\epsilon_j, \pm\epsilon_k \pm \epsilon_\ell : j \in J,\ k \neq \ell \in J\}$$

and denote by $\widehat{A_J}$ the lift of A_J in Φ. Here we make the convention that

$$A_J = \begin{cases} \{\pm 2\epsilon_i, \pm\epsilon_s \pm \epsilon_t : i \in I_n,\ s \neq t \in I_n\} \text{ if } J = \emptyset \\ \{\pm 2\epsilon_i, \pm\epsilon_j : i \in I_n, j \in J\} \text{ if } |J| = 1 \text{ and } n = 2 \\ \{\pm 2\epsilon_i, \pm\epsilon_j, \pm\epsilon_s \pm \epsilon_t : i \in I_n,\ s \neq t \in I_n \backslash J\} \text{ if } J = \{j\} \text{ and } n > 2 \\ \{\pm 2\epsilon_i, \pm\epsilon_j, \pm\epsilon_k \pm \epsilon_\ell : i \in I_n,\ j \in J,\ k \neq \ell \in J\} \text{ if } |I_n \backslash J| = 1 \end{cases}$$

Note that A_J is a proper closed subroot system of BC_n for any $J \subsetneq I_n$ and it is of type $C_{n-r} \oplus BC_r$ if $|J| = r$. Hence, the lift $\widehat{A_J}$ of A_J is a closed subroot system in Φ. We have,

Proposition 1.9.2. *The lift $\widehat{A_J}$ of A_J in Φ is a maximal closed subroot system Φ for $J \subsetneq I_n$.*

Proof. Let Δ be a closed subroot system of Φ such that $\widehat{A_J} \subsetneq \Delta \subseteq \Phi$. Then there are three possibilities for elements of $\Delta \backslash \widehat{A_J}$.

Case(1). Suppose $\epsilon_i + (r + \frac{1}{2})\delta \in \Delta$ for some $i \notin J$ and $r \in \mathbb{Z}$. Then since

$\pm\epsilon_i \pm \epsilon_s + \mathbb{Z}\delta \subseteq \Delta$ for all $s \notin J$ with $i \neq s$, we have

$$(\epsilon_i + (r + \tfrac{1}{2})\delta) + (-\epsilon_i \pm \epsilon_s + \mathbb{Z}\delta) = \pm\epsilon_s + (\mathbb{Z} + \tfrac{1}{2})\delta \subseteq \Delta \text{ for all } s \notin J,\ i \neq s.$$

If $J = \emptyset$, then we get $\pm\epsilon_i + (\mathbb{Z} + \frac{1}{2})\delta \subseteq \Delta$ by repeating the earlier argument with the choice of $s \in I_n$ which is different from i. If $J \neq \emptyset$, then $\epsilon_j + (\mathbb{Z} + \frac{1}{2})\delta \subseteq \Delta$ for all $j \in J$. Fix $j \in J$. Then we have
$(\epsilon_i + (r + \frac{1}{2})\delta) + (\epsilon_j + (\mathbb{Z} + \frac{1}{2})\delta) = \epsilon_i + \epsilon_j + \mathbb{Z}\delta \subseteq \Delta$. Now,

$$\epsilon_i + (\mathbb{Z} + \tfrac{1}{2})\delta = (-\epsilon_j + (\mathbb{Z} + \tfrac{1}{2})\delta) + (\epsilon_i + \epsilon_j + \mathbb{Z}\delta) \subseteq \Delta.$$

This proves that $\epsilon_s + (\mathbb{Z} + \frac{1}{2})\delta \subseteq \Delta$ for all $s \notin J$. Hence, we have
$\pm\epsilon_s + (\mathbb{Z} + \frac{1}{2})\delta \subseteq \Delta$ for all $s \in I_n$. This implies that $\Delta = \Phi$.

Case(2). Suppose $\epsilon_j + \epsilon_k + r\delta \in \Delta$ for some $j \in J, k \notin J$ and $r \in \mathbb{Z}$. Then since $-\epsilon_j + (\mathbb{Z} + \frac{1}{2})\delta \subseteq \Delta$, we have

$$\epsilon_k + (\mathbb{Z} + \tfrac{1}{2})\delta = (\epsilon_j + \epsilon_k + r\delta) + (-\epsilon_j + (\mathbb{Z} + \tfrac{1}{2})\delta) \subseteq \Delta.$$

So, we are back to the Case (1), hence $\Delta = \Phi$.

Case(3). Suppose $\epsilon_k - \epsilon_j + r\delta \in \Delta$ for some $j \in J, k \notin J$ and $r \in \mathbb{Z}$. Then since $\epsilon_j + (\mathbb{Z} + \frac{1}{2})\delta \subseteq \Delta$, we have

$$\epsilon_k + (\mathbb{Z} + \tfrac{1}{2})\delta = (\epsilon_k - \epsilon_j + r\delta) + (\epsilon_j + (\mathbb{Z} + \tfrac{1}{2})\delta) \subseteq \Delta.$$

So, we are back to the Case (1), hence $\Delta = \Phi$. $\square$

1.9.2 Main theorem for $\mathtt{A}_{\mathtt{2n}}^{(2)}$

Let $\Psi \leq \Phi$ be a maximal subroot system. Now, we are ready to state our final classification theorem for the case $\mathtt{A}_{\mathtt{2n}}^{(2)}$.

Theorem 1.9.3. *Suppose Φ is of type $\mathtt{A}_{\mathtt{2n}}^{(2)}$ and $\Psi \leq \Phi$ is a maximal closed subroot system. Then*

(i) $\Psi =$ the lift of A_J for some $J \subsetneq I_n = \widehat{A_J}$ or

(ii) $\Psi = \Psi_I(\mathtt{A}_{\mathtt{2n}}^{(2)})$ for some $I \subseteq I_n$ or

(iii) there exist an odd prime number n_s and a $\mathbb{Z}$–linear function $p : \mathrm{Gr}(\Phi)_s \cup \mathrm{Gr}(\Phi)_{\mathrm{im}} \to \frac{1}{2}\mathbb{Z}$ satisfying the equation (1.2.2) such that

$$\begin{aligned}\Psi(p, n_s) :=&\big\{ \pm \epsilon_i \pm (p_{\epsilon_i} + rn_s)\delta, \pm 2\epsilon_i \pm (2p_{\epsilon_i} + n_s + 2rn_s)\delta : i \in I_n,\ r \in \mathbb{Z}\big\} \\ &\cup \big\{ \pm \epsilon_i \pm \epsilon_j + (\pm p_{\epsilon_i} \pm p_{\epsilon_j} + rn_s)\delta : i, j \in I_n, i \neq j,\ r \in \mathbb{Z}\big\}.\end{aligned}$$

Conversely, all the subroot systems defined above are maximal closed subroot systems of Φ.

Proof. Define $J = \{i \in I_n : \epsilon_i \in \mathrm{Gr}(\Psi)\}$. Now, two cases are possible: $J \subsetneq I_n$ or $J = I_n$.

Case (1). First consider the case $J \subsetneq I_n$. In this case, we claim that $\Psi = \widehat{A_J}$. This is immediate if we prove that $\mathrm{Gr}(\Psi) \subseteq A_J$. Suppose $\mathrm{Gr}(\Psi) \nsubseteq A_J$, then there must exist $k \in J$ and $\ell \notin J$ such that $\epsilon_k \pm \epsilon_\ell \in \mathrm{Gr}(\Psi)$. This means that there exists $r, r' \in \mathbb{Z}$ such that $\epsilon_k \pm \epsilon_\ell + r\delta \in \Psi, \epsilon_k + (r' + \frac{1}{2})\delta \in \Psi$. Since Ψ is closed in Φ, we get

$$(\epsilon_k \pm \epsilon_\ell + r\delta) + (-\epsilon_k - (r' + \tfrac{1}{2}))\delta = \pm \epsilon_\ell + (r - r' - \tfrac{1}{2})\delta \in \Psi,$$

which contradicts the fact that $\ell \notin J$. So, $\mathrm{Gr}(\Psi) \subseteq A_J$ and hence $\Psi \subseteq \widehat{A_J}$. Since $\widehat{A_J}$ is closed in Φ, we have $\Psi = \widehat{A_J}$.

Case (2). Now, consider the case $J = I_n$. Since Ψ is closed, we have $\pm\epsilon_i \pm \epsilon_j \in \mathrm{Gr}(\Psi)$ for all $1 \leq i \neq j \leq n$. It is easy to see that if $\mathrm{Gr}(\Psi)$ contains $2\epsilon_i$ for some $i \in I_n$, then it contains $\pm 2\epsilon_j$ for all $j \in I_n$ as $\mathbf{s}_{\epsilon_i - \epsilon_j}(2\epsilon_i) = 2\epsilon_j$. So, we get either $\mathrm{Gr}(\Psi) = \{\pm\epsilon_i, \pm\epsilon_i \pm \epsilon_j : i, j \in I_n, i \neq j\}$ or $\mathrm{Gr}(\Psi) = \mathrm{Gr}(\Phi)$.

Case (2.1). Suppose $\mathrm{Gr}(\Psi) = \{\pm\epsilon_i, \pm\epsilon_i \pm \epsilon_j : i, j \in I_n, i \neq j\}$, then we claim that $\Psi = \Psi_I(\mathtt{A}^{(2)}_{\mathtt{2n}})$ for some $I \subseteq I_n$. By Proposition 1.2.2, we have

$$\exists \;\; k_i \in \mathbb{Z} \text{ such that } Z_{\epsilon_i}(\Psi) = (k_i + \tfrac{1}{2}) + n_s\mathbb{Z}, \text{ for each } i \in I_n.$$

Since $Z_{\epsilon_i}(\Psi) + Z_{\epsilon_i}(\Psi) = (2k_i + 1) + n_s\mathbb{Z}$ and $2\epsilon_i \notin \mathrm{Gr}(\Psi)$, we must have $n_s \in 2\mathbb{Z}$. Set $I = \{i \in I_n : k_i \in 2\mathbb{Z}\}$, then we immediately get $\Psi \subseteq \Psi_I(\mathtt{A}^{(2)}_{\mathtt{2n}})$. Since $\Psi_I(\mathtt{A}^{(2)}_{\mathtt{2n}})$ is closed, we have $\Psi = \Psi_I(\mathtt{A}^{(2)}_{\mathtt{2n}})$.

Case (2.2). Finally assume that $J = I_n$ and $\mathrm{Gr}(\Psi) = \mathrm{Gr}(\Phi)$. Then by Proposition 1.2.2, we have $n_\alpha \in \mathbb{N}$ and $p_\alpha \in Z_\alpha(\Psi)$ such that $Z_\alpha(\Psi) = p_\alpha + n_\alpha\mathbb{Z}$ for all $\alpha \in \mathrm{Gr}(\Phi)$. By Proposition 1.2.8, we have $n_s = n_{\mathrm{im}}$, $n_\ell = 2n_s$ and n_s is an odd prime number. Conversely, let n_s be a given odd prime number and $p : \mathrm{Gr}(\Phi)_s \cup \mathrm{Gr}(\Phi)_{\mathrm{im}} \to \frac{1}{2}\mathbb{Z}$ be a given $\mathbb{Z}-$linear map satisfying the condition 1.2.2. It is a straightforward checking that $\Psi(p, n_s)$ is a closed subroot system of Φ. Now, we prove that $\Psi(p, n_s)$ must be a maximal closed subroot system in Φ. Suppose there is a maximal subroot system Δ such that $\Psi(p, n_s) \subseteq \Delta \subsetneq \Phi$. Then since $\mathrm{Gr}(\Delta) = \mathrm{Gr}(\Phi)$ (by earlier arguments) Δ must be of the form $\Psi(p', n_s')$ for some function $p' : \mathrm{Gr}(\Phi)_s \cup \mathrm{Gr}(\Phi)_{\mathrm{im}} \to \frac{1}{2}\mathbb{Z}$ and odd prime number n_s'. Now,

$$Z_\alpha(\Psi) \subseteq Z_\alpha(\Delta), \alpha \in \mathrm{Gr}(\Phi)$$

implies that $n_s = n'_s$ and $p_\alpha \equiv p'_\alpha (\mathrm{mod}\ n_s)$ for all $\alpha \in \mathrm{Gr}(\Phi)$. Hence, $\Psi(p, n_s) = \Delta$. This proves that $\Psi(p, n_s)$ is a maximal subroot system of Φ. This completes the proof.

$\square$

Remark 1.9.4. One can easily check that the type of $\widehat{A_J}$ is $\mathtt{A}_{\mathtt{2n-1}}^{(2)}$ if $J = \emptyset$ else $\mathtt{A}_{\mathtt{2r}}^{(2)} \oplus \mathtt{A}_{\mathtt{2n-2r-1}}^{(2)}$, where $|J| = r$, the type of $\Psi_I(\mathtt{A}_{\mathtt{2n}}^{(2)})$ is $\mathtt{B}_{\mathtt{n}}^{(1)}$ and the type of $\Psi(p, n_s)$ is $\mathtt{A}_{\mathtt{2n}}^{(2)}$. Clearly, the root systems of type $\mathtt{D}_{\mathtt{r}}^{(1)} \oplus \mathtt{A}_{\mathtt{2n-2r}}^{(2)}$ do not occur as a maximal closed subroot system of $\mathtt{A}_{\mathtt{2n}}^{(2)}$ as it is stated in [11, Table 1 & 2]. In [11], the authors do not give any description of the closed subroot systems of type $\mathtt{D}_{\mathtt{r}}^{(1)} \oplus \mathtt{A}_{\mathtt{2n-2r}}^{(2)}$ of $\mathtt{A}_{\mathtt{2n}}^{(2)}$. But we presume that it must be the lift $\widehat{\Delta}$ of

$$\Delta = \Big\{ \pm \epsilon_k \pm \epsilon_\ell : 1 \leq k \neq \ell \leq r \Big\} \cup \Big\{ \pm \epsilon_i, \pm 2\epsilon_i, \pm\epsilon_i \pm \epsilon_j : r+1 \leq i \neq j \leq n \Big\}.$$

It is easy to see that Δ is a closed subroot system of $\mathtt{BC_n}$ of type $\mathtt{D_r} \oplus \mathtt{BC_{n-r}}$. Hence, $\widehat{\Delta}$ is a closed subroot system of $\mathtt{A}_{\mathtt{2n}}^{(2)}$ of type $\mathtt{D}_{\mathtt{r}}^{(1)} \oplus \mathtt{A}_{\mathtt{2n-2r}}^{(2)}$. But this is not maximal as $\Delta \subsetneq \widehat{A_J}$ for $J = \{r+1, \cdots, n\}$.

1.10 The case $\mathtt{A}_2^{(2)}$

Throughout this section we assume that Φ is of type $\mathtt{A}_2^{(2)}$. We have the following explicit description of $\mathtt{A}_2^{(2)}$, see [4, Page no. 565]:

$$\Phi = \Big\{ \pm \epsilon_1 + (r + \tfrac{1}{2})\delta, \pm 2\epsilon_1 + 2r\delta : r \in \mathbb{Z} \Big\}$$

and $\mathrm{Gr}(\Phi) = \{\pm\epsilon_1, \pm 2\epsilon_1\}$.

We have the following classification theorem for the case $\mathtt{A}_2^{(2)}$.

Theorem 1.10.1. *Suppose Φ is of type $\mathtt{A}_2^{(2)}$ and Ψ is a maximal closed subroot* *system of Φ. Then one of the following holds:*

1. $\Psi = \Psi(k,q) := \big\{ \pm\epsilon_1 \pm (k+\frac{1}{2}+rq)\delta, \pm 2\epsilon_1 \pm (2k+1+(2r+1)q)\delta : r \in \mathbb{Z}\big\}$ *for some* $k \in \mathbb{Z}_+$ *and odd prime number* q *and* $\mathrm{Gr}(\Psi) = \{\pm\epsilon_1, \pm 2\epsilon_1\}$.

2. $\Psi = \{\pm(\epsilon_1 + (2r+\frac{1}{2})\delta) : r \in \mathbb{Z}\}$ *or* $\{\pm(\epsilon_1 + (2r+\frac{3}{2})\delta) : r \in \mathbb{Z}\}$ *and* $\mathrm{Gr}(\Psi) = \{\pm\epsilon_1\}$

3. $\Psi = \{\pm(2\epsilon_1 + 2r\delta) : r \in \mathbb{Z}\}$ *and* $\mathrm{Gr}(\Psi) = \{\pm 2\epsilon_1\}$.

If $\Psi = \Psi(k,q)$*, then the type of* Ψ *is* $\mathtt{A}_2^{(2)}$*, otherwise it is* $\mathtt{A}_1^{(1)}$*.*

Proof. Let Ψ be a maximal closed subroot system. Then we have three possibilities for $\mathrm{Gr}(\Psi)$: either $\mathrm{Gr}(\Psi) = \{\pm\epsilon_1\}$ or $\mathrm{Gr}(\Psi) = \{\pm 2\epsilon_1\}$ or $\mathrm{Gr}(\Psi) = \{\pm\epsilon_1, \pm 2\epsilon_1\}$.

Case (1). First let $\mathrm{Gr}(\Psi) = \{\pm\epsilon_1, \pm 2\epsilon_1\}$. Then by Proposition 1.2.2, we have

$$Z_{\pm\epsilon_1}(\Psi) = \pm p_{\epsilon_1} + n_s\mathbb{Z} \subseteq \tfrac{1}{2} + \mathbb{Z} \text{ and } Z_{\pm 2\epsilon_1}(\Psi) = \pm p_{2\epsilon_1} + n_\ell\mathbb{Z} \subseteq 2\mathbb{Z}.$$

for some $p_{\epsilon_1} \in \frac{1}{2} + \mathbb{Z}$ and $p_{2\epsilon_1} \in 2\mathbb{Z}$. As Ψ is closed and $p_{2\epsilon_1} + n_\ell\mathbb{Z} \subseteq 2\mathbb{Z}$, we have

$$(p_{2\epsilon_1} - p_{\epsilon_1}) + n_\ell\mathbb{Z} \subseteq p_{\epsilon_1} + n_s\mathbb{Z} \text{ and hence } p_{2\epsilon_1} + n_\ell\mathbb{Z} \subseteq (2p_{\epsilon_1} + n_s\mathbb{Z}) \cap 2\mathbb{Z}.$$

From this we conclude that n_s must be an odd integer since $2p_{\epsilon_1}$ is an odd integer. Since for all $r \in \mathbb{Z}$ such that $2p_{\epsilon_1} + n_s r \in 2\mathbb{Z}$, we have $2p_{\epsilon_1} + n_s r \in Z_{2\epsilon_1}(\Psi)$. This implies

$$p_{2\epsilon_1} + n_\ell\mathbb{Z} = (2p_{\epsilon_1} + n_s\mathbb{Z}) \cap 2\mathbb{Z} = (2p_{\epsilon_1} + n_s) + 2n_s\mathbb{Z}.$$

This implies, we must have $n_\ell = 2n_s$. So, Ψ must be equal to $\Psi(k, n_s)$, where $k = p_{\epsilon_1} - \frac{1}{2} \in \mathbb{Z}_+$ and n_s is an odd integer. One can easily see that $\Psi(k, n_s)$ is maximal if and only if n_s is an odd prime number.

Case (2). Now, let $\mathrm{Gr}(\Psi) = \{\pm\epsilon_1\}$. Then we claim that $\Psi = \{\pm(\epsilon_1 + (2r + \frac{1}{2})\delta) : r \in \mathbb{Z}\}$ or $\{\pm(\epsilon_1 + (2r + \frac{3}{2})\delta) : r \in \mathbb{Z}\}$. Suppose $\pm(\epsilon_1 + (r + \frac{1}{2})\delta), \pm(\epsilon_1 + (s + \frac{1}{2})\delta) \in \Psi$ for some $r, s \in \mathbb{Z}$, then we claim that r and s are of the same parity. If they have different parity, then $(r + s + 1) \in 2\mathbb{Z}$ which implies that $\pm 2\epsilon_1 \in \mathrm{Gr}(\Psi)$, a contradiction. This proves that

$$\text{either } \Psi \subseteq \{\pm(\epsilon_1 + (2r + \tfrac{1}{2})\delta) : r \in \mathbb{Z}\} \text{ or } \Psi \subseteq \{\pm(\epsilon_1 + (2r + \tfrac{3}{2})\delta) : r \in \mathbb{Z}\}.$$

Since both sets on the right hand side are closed in Φ, we get the equality. Now, we prove that both sets $\{\pm(\epsilon_1 + (2r + \frac{1}{2})\delta) : r \in \mathbb{Z}\}$ and $\{\pm(\epsilon_1 + (2r + \frac{3}{2})\delta) : r \in \mathbb{Z}\}$ are maximal closed in Φ. Let $\Delta \leq \Phi$ be a closed subroot system such that either

$$\{\pm(\epsilon_1 + (2r + \tfrac{1}{2})\delta) : r \in \mathbb{Z}\} \subsetneq \Delta \text{ or } \{\pm(\epsilon_1 + (2r + \tfrac{3}{2})\delta) : r \in \mathbb{Z}\} \subsetneq \Delta.$$

This implies that $\{\pm\epsilon_1\} \subseteq \mathrm{Gr}(\Delta)$ and hence either $\mathrm{Gr}(\Delta) = \{\pm\epsilon_1\}$ or $\mathrm{Gr}(\Delta) = \{\pm\epsilon_1, \pm 2\epsilon_1\}$. If $\mathrm{Gr}(\Delta) = \{\pm\epsilon_1\}$, then by previous argument, we get

$$\text{either } \Delta \subseteq \{\pm(\epsilon_1 + (2r + \tfrac{1}{2})\delta) : r \in \mathbb{Z}\} \text{ or } \Delta \subseteq \{\pm(\epsilon_1 + (2r + \tfrac{3}{2})\delta) : r \in \mathbb{Z}\},$$

which is not possible. So, we must have $\mathrm{Gr}(\Delta) = \{\pm\epsilon_1, \pm 2\epsilon_1\}$. Then from the proof of Case (1) we get $\Delta = \Psi(k, q)$ for some $k \in \mathbb{Z}_+$ and an odd integer $q \in \mathbb{Z}$. But since

$$\{\pm(\epsilon_1 + (2r + \tfrac{1}{2})\delta) : r \in \mathbb{Z}\} \subsetneq \Delta \text{ or } \{\pm(\epsilon_1 + (2r + \tfrac{3}{2})\delta) : r \in \mathbb{Z}\} \subsetneq \Delta.$$

we have either $\frac{1}{2} + 2\mathbb{Z} \subseteq k + \frac{1}{2} + q\mathbb{Z}$ or $\frac{3}{2} + 2\mathbb{Z} \subseteq k + \frac{1}{2} + q\mathbb{Z}$ which implies that $2\mathbb{Z} \subseteq q\mathbb{Z}$. This implies that $q = 1$ and $\Delta = \Phi$.

Case (3). Finally assume that $\mathrm{Gr}(\Psi) = \{\pm 2\epsilon_1\}$. Then it is clear that $\Psi \subseteq \{\pm(2\epsilon_1 + 2r\delta) : r \in \mathbb{Z}\}$. Since $\{\pm(2\epsilon_1 + 2r\delta) : r \in \mathbb{Z}\}$ is closed, we have

$\Psi = \{\pm(2\epsilon_1 + 2r\delta) : r \in \mathbb{Z}\}$. Conversely, $\{\pm(2\epsilon_1 + 2r\delta) : r \in \mathbb{Z}\}$ must be closed in Φ. Let Δ be a closed subroot system of Φ such that $\{\pm(2\epsilon_1 + 2r\delta) : r \in \mathbb{Z}\} \subsetneq \Delta$. Then we have $\{\pm 2\epsilon_1\} \subseteq \mathrm{Gr}(\Delta)$ and it immediately implies that $\mathrm{Gr}(\Delta) = \{\pm\epsilon_1, \pm 2\epsilon_1\}$ as $\{\pm(2\epsilon_1 + 2r\delta) : r \in \mathbb{Z}\} \subsetneq \Delta$. Then from the proof of Case (1) we get $\Delta = \Psi(k, q)$ for some $k \in \mathbb{Z}_+$ and an odd integer $q \in \mathbb{Z}$. This implies that $2\mathbb{Z} \subseteq 2k + 1 + q + 2q\mathbb{Z}$ which implies that $2\mathbb{Z} \subseteq 2q\mathbb{Z}$. Since q is an odd integer, we get $q = 1$ and $\Delta = \Phi$. This completes the proof.

$\square$

1.11 Final table

Now, we are ready to state our final classification theorem for irreducible twisted affine root systems.

Table 1.4: Types of maximal subroot system of irreducible twisted affine root systems

Type	With closed gradient	With semi-closed gradient
$\mathtt{A}_2^{(2)}$	$\mathtt{A}_2^{(2)}$	$A_1^{(1)}$
$\mathtt{A}_{2n}^{(2)}$	$\mathtt{A}_{2r}^{(2)} \oplus \mathtt{A}_{2n-2r-1}^{(2)}$ $(1 \le r \le n-1)$, $\mathtt{A}_{2n}^{(2)}$, $\mathtt{A}_{2n-1}^{(2)}$	$\mathtt{B}_n^{(1)}$
$\mathtt{D}_{n+1}^{(2)}$	$\mathtt{D}_{r+1}^{(2)} \oplus \mathtt{D}_{n-r}^{(1)}$ $(1 \le r \le n-2)$, $\mathtt{B}_n^{(1)}$, $\mathtt{D}_{n+1}^{(2)}$, $\mathtt{D}_n^{(2)}$	$\mathtt{B}_r^{(1)} \oplus \mathtt{B}_{n-r}^{(1)}$ $(2 \le r \le n-2)$
$\mathtt{A}_{2n-1}^{(2)}$	$\mathtt{A}_{2r-1}^{(2)} \oplus \mathtt{A}_{2n-2r-1}^{(2)}$ $(1 \le r \le n-1)$, $\mathtt{A}_{2n-1}^{(2)}$, $\mathtt{C}_n^{(1)}$, $\mathtt{A}_{n-1}^{(1)}$	$\mathtt{D}_n^{(1)}$
$\mathtt{E}_6^{(2)}$	$\mathtt{A}_1^{(1)} \oplus \mathtt{A}_5^{(2)}$, $\mathtt{A}_2^{(1)} \oplus \mathtt{A}_2^{(1)}$, $\mathtt{E}_6^{(2)}$, $\mathtt{F}_4^{(1)}$, $\mathtt{D}_5^{(2)}$	$\mathtt{C}_4^{(1)}$
$\mathtt{D}_4^{(3)}$	$\mathtt{A}_1^{(1)} \oplus \mathtt{A}_1^{(1)}$, $\mathtt{D}_4^{(3)}$, $\mathtt{G}_2^{(1)}$, $\mathtt{A}_2^{(1)}$	$\mathtt{A}_2^{(1)}$

We end this section with the following remark.

Remark 1.11.1. As we pointed out in the introduction the authors of [11] have omitted a few possible cases in their classification list for the twisted case. We list out all the differences between our classification list and their classification list. The following possible cases are omitted in twisted case, see [11, Table 1, Table 2, Theorem 5.8]:

- $\mathtt{A}_2^{(1)} \oplus \mathtt{A}_2^{(1)} \subset \mathtt{E}_6^{(2)}$
- $\mathtt{D}_5^{(2)} \subset \mathtt{E}_6^{(2)}$
- $\mathtt{B}_\mathtt{r}^{(1)} \oplus \mathtt{B}_{\mathtt{n-r}}^{(1)} \subset \mathtt{D}_{\mathtt{n+1}}^{(2)}$
- $\mathtt{D}_\mathtt{n}^{(1)} \subset \mathtt{A}_{\mathtt{2n-1}}^{(2)}$

The root systems of type $\mathtt{D}_\mathtt{r}^{(1)} \oplus \mathtt{A}_{\mathtt{2n-2r}}^{(2)}$ does not occur as a maximal closed subroot system in $\mathtt{A}_{\mathtt{2n}}^{(2)}$, in contrast to what is stated in [11, Table 2].

1.12 Closed subroot systems and Regular subalgebras

In this section we will describe a procedure to classify all the regular subalgebras of affine Kac–Moody subalgebras both in untwisted and twisted case. We follow the same notations as in the preliminary section.

Recall that Φ denotes the set of real roots of the affine Lie algebra $\mathfrak{g}$ and $\Delta(\mathfrak{g})$ denotes the roots of $\mathfrak{g}$. We will record the following fact from [18, Remark 3.1]. It is fairly standard, but we give a proof for this fact for completeness.

Lemma 1.12.1. *Let Ψ be a closed subset of Φ such that $\Psi = -\Psi$ and $\mathbf{s}_\alpha(\beta) \in \Psi$ for all $\alpha, \beta \in \Psi$ with $\beta \pm \alpha \in \Delta_{\mathrm{im}}(\mathfrak{g})$ or $\beta \pm 2\alpha \in \Delta_{\mathrm{im}}(\mathfrak{g})$. Then Ψ must be a closed subroot system of Φ.*

Proof. We only need to prove that Ψ is a subroot system. Note that all root strings in Φ are unbroken. Let $\alpha, \beta \in \Psi$ such that $\langle \beta, \alpha^\vee \rangle \in \mathbb{Z}_+$. If $\beta - s\alpha \in \Delta_{\mathrm{im}}(\mathfrak{g})$ for some $s \in \mathbb{Z}_+$ we must have $s \in \{1, 2\}$ and hence $\mathbf{s}_\alpha(\beta) \in \Psi$.

Otherwise $\beta - s\alpha \in \Phi$ for all $0 \leq s \leq \langle \beta, \alpha^\vee \rangle$. Since $-\alpha \in \Psi$ we get by the closedness of Ψ that $\beta - s\alpha \in \Psi$. Thus $\mathbf{s}_\alpha(\beta) \in \Psi$. The case $-\langle \beta, \alpha^\vee \rangle \in \mathbb{Z}_+$ works similarly and we omit the details. $\square$

Lemma 1.12.2. *Let $\mathfrak{g}'$ be a $\mathfrak{h}$–invariant subalgebra of $\mathfrak{g}$ and let $\Delta(\mathfrak{g}') \subseteq \Delta(\mathfrak{g})$ be the set of roots of $\mathfrak{g}'$ with respect to $\mathfrak{h}$. Let $\Psi(\mathfrak{g}') = \Delta(\mathfrak{g}') \cap \Phi$ be the set of real roots of $\mathfrak{g}'$. Suppose $\Delta(\mathfrak{g}') = -\Delta(\mathfrak{g}')$, then $\Psi(\mathfrak{g}')$ must be a closed subroot system of Φ.*

Proof. First recall that $\dim(\mathfrak{g}_\alpha) = 1$ for all $\alpha \in \Phi$. Since $\Phi = -\Phi$, we have $\Psi(\mathfrak{g}') = -\Psi(\mathfrak{g}')$. Suppose $\alpha, \beta \in \Psi(\mathfrak{g}')$ and $\alpha + \beta \in \Phi$ then it is immediate that $\alpha + \beta \in \Psi(\mathfrak{g}')$, since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. This implies $\Psi(\mathfrak{g}')$ is closed in Φ. So, by Lemma 1.12.1 it remains to prove that, $\mathbf{s}_\alpha(\beta) \in \Psi$ for all $\alpha, \beta \in \Psi(\mathfrak{g}')$ with $\beta \pm \alpha \in \Delta_{\mathrm{im}}(\mathfrak{g})$ or $\beta \pm 2\alpha \in \Delta_{\mathrm{im}}(\mathfrak{g})$.

Case (1). Assume that $\mathfrak{g}$ is not of type $\mathtt{A}_{\mathtt{2n}}^{(2)}$. Let $\alpha, \beta \in \Psi(\mathfrak{g}')$ such that $\beta = \alpha + r\delta$ for some $r \in \mathbb{Z}$. We have $\mathbf{s}_\alpha(\beta) = -\alpha + r\delta$. The finite dimensional subspace

$$V = \mathfrak{g}_{\alpha+r\delta} \oplus \mathfrak{g}_{r\delta} \oplus \mathfrak{g}_{-\alpha+r\delta} \subseteq \mathfrak{g} \text{ is a } \mathfrak{sl}_2 = \mathfrak{g}_\alpha \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \oplus \mathfrak{g}_{-\alpha}\text{–module}$$

since $[\mathfrak{g}_\alpha, \mathfrak{g}_{\alpha+r\delta}] = 0$ and $[\mathfrak{g}_{-\alpha}, \mathfrak{g}_{-\alpha+r\delta}] = 0$ and it decomposes as $V \cong_{\mathfrak{sl}_2} V(2) \oplus V(0)^{\oplus k}$, where $V(\lambda)$ denotes the finite dimensional irreducible $\mathfrak{sl}_2$–module corresponding to the non–negative integer $\lambda \in \mathbb{Z}_+$ and $k = \dim(\mathfrak{g}_{r\delta}) - 1$. In particular, we have $[\mathfrak{g}_\beta, \mathfrak{g}_{-\alpha}] \neq 0$ and $[[\mathfrak{g}_\beta, \mathfrak{g}_{-\alpha}], \mathfrak{g}_{-\alpha}] = \mathfrak{g}_{-\alpha+r\delta} = \mathfrak{g}_{\mathbf{s}_\alpha(\beta)}$, since $\dim(\mathfrak{g}_{\mathbf{s}_\alpha(\beta)}) = 1$. Since $\mathfrak{g}_\beta, \mathfrak{g}_{-\alpha} \subseteq \mathfrak{g}'$, we have $\mathfrak{g}_{\mathbf{s}_\alpha(\beta)} \subseteq \mathfrak{g}'$. This implies $\mathbf{s}_\alpha(\beta) \in \Psi(\mathfrak{g}')$. Similarly we get $\mathbf{s}_\alpha(\beta) \in \Psi(\mathfrak{g}')$ if $\beta = -\alpha + r\delta$.

Case (2). Assume that $\mathfrak{g}$ is of type $\mathtt{A}_{\mathtt{2n}}^{(2)}$. Let $\alpha, \beta \in \Psi(\mathfrak{g}')$ such that $\beta = 2\alpha + r\delta$ for some $r \in \mathbb{Z}$. We have $\mathbf{s}_\alpha(\beta) = -2\alpha + r\delta$. The finite dimensional subspace

$$V = \mathfrak{g}_{2\alpha+r\delta} \oplus \mathfrak{g}_{\alpha+r\delta} \oplus \mathfrak{g}_{r\delta} \oplus \mathfrak{g}_{-\alpha+r\delta} \oplus \mathfrak{g}_{-2\alpha+r\delta} \subseteq \mathfrak{g} \text{ is a } \mathfrak{sl}_2 = \mathfrak{g}_\alpha \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \oplus \mathfrak{g}_{-\alpha}\text{–module}$$

and it decomposes as $V \cong_{\mathfrak{sl}_2} V(4) \oplus V(0)^{\oplus k}$, where $k = \dim(\mathfrak{g}_{r\delta}) - 1$. In particular, we have $[\mathfrak{g}_\beta, \mathfrak{g}_{-\alpha}] = \mathfrak{g}_{\alpha+r\delta} \subseteq \mathfrak{g}'$ and $\mathfrak{g}' \supseteq [[\mathfrak{g}_\beta, \mathfrak{g}_{-\alpha}], \mathfrak{g}_{-\alpha}] = [\mathfrak{g}_{\alpha+r\delta}, \mathfrak{g}_{-\alpha}] \neq 0$ and

$$\mathfrak{g}' \supseteq [[[\mathfrak{g}_\beta, \mathfrak{g}_{-\alpha}], \mathfrak{g}_{-\alpha}], \mathfrak{g}_{-\alpha}] = [[\mathfrak{g}_{\alpha+r\delta}, \mathfrak{g}_{-\alpha}], \mathfrak{g}_{-\alpha}] = \mathfrak{g}_{-\alpha+r\delta},$$

since $\dim(\mathfrak{g}_{-\alpha+r\delta}) = 1$ and $\mathfrak{g}_\beta, \mathfrak{g}_{-\alpha} \subseteq \mathfrak{g}'$. This immediately implies that $\mathfrak{g}_{\mathbf{s}_\alpha(\beta)} = [\mathfrak{g}_{-\alpha+r\delta}, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{g}'$. Hence we have $\mathbf{s}_\alpha(\beta) \in \Psi(\mathfrak{g}')$. The cases $\beta = \pm\alpha + r\delta$ or $-2\alpha + r\delta$ and $\mathfrak{g}$ is of type $\mathtt{A}_{\mathtt{2n}}^{(2)}$ follows using similar ideas, so we omit the details.

□

In [8], E. B. Dynkin introduced a notion of regular semi-simple subalgebras in order to classify all the semi-simple subalgebras of finite dimensional complex semi-simple Lie algebras. As a natural generalization of Dynkin's definition, one can give a constructive definition of regular subalgebras in the context of affine Kac–Moody algebras as well (see for example [11]).

Definition 1.12.3. Let Ψ be a closed subroot system of Φ. The subalgebra $\mathfrak{g}(\Psi)$ of $\mathfrak{g}$ generated by $\mathfrak{g}_\alpha$, for $\alpha \in \Psi$, is called the regular subalgebra associated with Ψ.

One can easily see that the definition of regular subalgebras works well for all Kac–Moody algebras. Clearly $\mathfrak{g}(\Psi)$ is invariant under the adjoint action of $\mathfrak{h}$ (the Cartan subalgebra of $\mathfrak{g}$). Moreover we have,

$$\mathfrak{g}(\Psi) = \mathfrak{h}(\Psi) \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{g})} (\mathfrak{g}_\alpha \cap \mathfrak{g}(\Psi)),$$

where $\mathfrak{h}(\Psi) = \mathbb{C}-$span of $\{\alpha^\vee : \alpha \in \Psi\}$. Denote the roots of $\mathfrak{g}(\Psi)$ with respect to $\mathfrak{h}$ by $\Delta(\Psi) := \{\alpha \in \Delta(\mathfrak{g}) : \mathfrak{g}_\alpha \cap \mathfrak{g}(\Psi) \neq 0\}$. Then it is immediate that $\Psi \subseteq \Delta(\Psi) \cap \Phi$. Note that for real roots α, we have $\mathfrak{g}_\alpha \cap \mathfrak{g}(\Psi) = \mathfrak{g}_\alpha$, but for imaginary roots we may not necessarily have equality. As we have mentioned in the introduction, we have a bijective correspondence between regular subalgebras and closed subroot systems of Φ. We need the following proposition in order to prove this bijective correspondence.

Proposition 1.12.4. *Let Ψ be a closed subroot system of Φ and let $\Psi = \Psi_1 \oplus \cdots \oplus \Psi_k$ be its direct sum decomposition of irreducible components. Let $\beta \in \Delta(\Psi)$, then there exists $\beta_1, \cdots, \beta_r \in \Psi$ such that the following holds:*

(1) $\beta = \beta_1 + \cdots + \beta_r$ and we have $\beta_1 + \cdots + \beta_i \in \Delta(\Psi)$, for each $1 \leq i \leq r$.

(2) There exists $1 \leq i_0 \leq k$ such that $\beta_1, \cdots, \beta_r \in \Psi_{i_0}$.

(3) Suppose $\beta_1 + \cdots + \beta_i \in \Delta(\Psi) \cap \Phi$ for some $1 \leq i \leq r$, then we get $\beta_1 + \cdots + \beta_i \in \Psi_{i_0}$.

Proof. Since $\mathfrak{g}_\alpha$, $\alpha \in \Psi$ generates $\mathfrak{g}(\Psi)$, it is easy to see that the right normed Lie words

$$\{[x_{\beta_r}, [x_{\beta_{r-1}}, [\cdots, [x_{\beta_2}, x_{\beta_1}]] \in \mathfrak{g}(\Psi) : \beta = \beta_1 + \cdots + \beta_r, \beta_i \in \Psi, 1 \leq i \leq r, r \in \mathbb{N}\}$$

spans $\mathfrak{g}(\Psi)_\beta$. Thus if $\beta \in \Delta(\Psi)$, then there exists $r \in \mathbb{N}$ and $\beta_i \in \Psi, 1 \leq i \leq r$, such that $\beta = \beta_1 + \cdots + \beta_r$ and the right normed Lie word $[x_{\beta_r}, [x_{\beta_{r-1}}, [\cdots, [x_{\beta_2}, x_{\beta_1}]] \neq 0$ for some $x_{\beta_i} \in \mathfrak{g}_{\beta_i}$, $1 \leq i \leq r$. Fix these x_{β_i}'s. Now it is easy to see that $[x_{\beta_r}, [x_{\beta_{r-1}}, [\cdots, [x_{\beta_2}, x_{\beta_1}]] \neq 0$ only if

$$x_{\beta_1} \neq 0 \text{ and } [x_{\beta_i}, [\cdots, [x_{\beta_2}, x_{\beta_1}]] \neq 0 \text{ for all } 2 \leq i \leq r$$

and hence we have $\beta_1 + \cdots + \beta_i \in \Delta(\Psi), 1 \leq i \leq r$. This completes the proof of Statement (1).

To prove Statement (2) and (3), first observe that the irreducible components $\Psi_1, \cdots, \Psi_k$ of Ψ are closed in Φ.

Case (1). Suppose $\beta_1 + \cdots + \beta_i \in \Phi$ for all $1 \le i \le r$, then the Statement (2) and (3) follows from induction and the fact that $\alpha + \beta \notin \Phi$ if $\alpha \in \Psi_p$ and $\beta \in \Psi_q$ for $1 \le p \ne q \le k$. In this case we have, $\beta_1 \in \Psi_{i_0} \implies \beta_1, \cdots, \beta_r \in \Psi_{i_0}$ and $\beta_1 + \cdots + \beta_i \in \Psi_{i_0}$ for all $1 \le i \le r$.

Case (2). Suppose $\beta_1 + \cdots + \beta_i \notin \Phi$ for some $2 \le i \le r$. Let $i \in \{1, \cdots r\}$ be the minimum such that $\beta_1 + \cdots + \beta_i \notin \Phi$, in particular we have $\beta_1 + \cdots + \beta_j \in \Phi$ for all $1 \le j < i$. Then by previous argument, there exists $i_0 \in \{1, \cdots, k\}$ such that $\beta_1, \cdots, \beta_{i-1} \in \Psi_{i_0}$ and $\beta_1 + \cdots + \beta_j \in \Psi_{i_0}$ for all $1 \le j < i$. Write $\beta_1 + \cdots + \beta_{i-1} = \alpha + s\delta \in \Psi_{i_0}$, where $\alpha \in \mathrm{Gr}(\Psi_{i_0})$. Since $\beta_1 + \cdots + \beta_i \notin \Phi$, we must have $\beta_i = -\alpha + s'\delta$. Observe that $(\beta_1 + \cdots + \beta_{i-1}, \beta_i) = -(\alpha, \alpha) \ne 0$. So we immediately get $\beta_i = -\alpha + s'\delta \in \Psi_{i_0}$ and $\beta_1 + \cdots + \beta_{i-1} + \beta_i = (s + s')\delta$. Suppose $\beta_{i+1} = \beta + s''\delta \notin \Psi_{i_0}$ then we get $[x_{\beta+s''\delta}, x_{\alpha+s\delta}] = 0$ and $[x_{\beta+s''\delta}, x_{-\alpha+s'\delta}] = 0$ as $(\beta + s''\delta) + (\alpha + s\delta) \notin \Delta(\Psi)$ and $(\beta + s''\delta) + (-\alpha + s'\delta) \notin \Delta(\Psi)$. This immediately implies that

$$[x_{\beta_{i+1}}, [x_{\beta_i}, [x_{\beta_{i-1}}, [\cdots, [x_{\beta_2}, x_{\beta_1}]] = [x_{\beta+s''\delta}, [x_{\alpha+s\delta}, x_{-\alpha+s'\delta}]] = 0$$

which is a contradiction to our choice of $x_{\beta_1}, \cdots, x_{\beta_{i+1}}$. Thus we must have $\beta_{i+1} = \beta + s''\delta \in \Psi_{i_0}$. Now induction completes the proof of Statement (2).

We only need to prove that $\beta_1 + \cdots + \beta_i + \beta_{i+1} \in \Psi_{i_0}$ in order to complete the proof of Statement (3). First recall from the Proposition 1.2.2 that there exists $n_\alpha \in \mathbb{Z}$ for $\alpha \in \mathrm{Gr}(\Psi)$ such that $Z_\alpha(\Psi_{i_0}) = p_\alpha + n_\alpha \mathbb{Z}$.

Case (2.1). Suppose $n_\alpha = 0$ for some $\alpha \in \mathrm{Gr}(\Psi_{i_0})$, then $n_\beta = 0$ for all $\beta \in \mathrm{Gr}(\Psi_{i_0})$ by Lemma 1.2.5. Then we have $\beta_1 + \cdots + \beta_j \in \Phi$ for all $1 \le j \le r$ in this case, so the Statement (3) is immediate in this case.

Case (2.2). So assume that $n_\alpha \ne 0$ for all $\alpha \in \mathrm{Gr}(\Psi_{i_0})$. Write

$\beta_1 + \cdots + \beta_{i-1} = \alpha + (p_\alpha + n_\alpha k_\alpha)\delta,\ \beta_i = -\alpha + (-p_\alpha + n_\alpha k'_\alpha)\delta$ and $\beta_{i+1} = \beta + (p_\beta + n_\beta k_\beta)\delta$ Then we have $\beta_1 + \cdots + \beta_i = n_\alpha(k_\alpha + k'_\alpha)\delta$. We need to prove that $\beta_1 + \cdots + \beta_i + \beta_{i+1} = \beta + (p_\beta + n_\beta k_\beta + n_\alpha(k_\alpha + k'_\alpha))\delta$ must be in Ψ_{i_0}.

Case (2.2.1). Assume that Φ is not of type $\mathtt{A}_{\mathtt{2n}}^{(2)}$. Suppose both α and β are long or short then we have $n_\alpha = n_\beta$ by Lemma 1.2.3, hence $\beta_1 + \cdots + \beta_i + \beta_{i+1} \in \Psi_{i_0}$, since $Z_\beta(\Psi_{i_0}) = p_\beta + n_\alpha\mathbb{Z}$. If β is short and α is long then we have $n_\beta = n_\alpha$ or $n_\alpha = mn_\beta$ by Statement (2) of Proposition 1.2.7, hence we have $\beta_1 + \cdots + \beta_i + \beta_{i+1} \in \Psi_{i_0}$. Now assume that α is short and β is long then we have $n_\beta = n_\alpha$ if $m | n_\alpha$ and $n_\beta = mn_\alpha$ if $m \nmid n_\alpha$. Again the claim follows easily when $n_\beta = n_\alpha$. So we are left with case $n_\beta = mn_\alpha$. Recall that $m = 2$ or 3 in this case, so it is prime number. Now note that $p_\beta + n_\beta k_\beta \equiv 0\ (mod\ m)$ and $(p_\beta + n_\beta k_\beta + n_\alpha(k_\alpha + k'_\alpha)) \equiv 0\ (mod\ m)$ together implies, $n_\alpha(k_\alpha + k'_\alpha) \equiv 0\ (mod\ m)$. Since $m \nmid n_\alpha$, we get $k_\alpha + k'_\alpha \equiv 0\ (mod\ m)$. This implies we have $n_\alpha(k_\alpha + k'_\alpha) \equiv 0\ (mod\ n_\beta)$ and hence we have $\beta_1 + \cdots + \beta_i + \beta_{i+1} \in \Psi_{i_0}$. This completes the proof of Statement (3) in this case.

Case (2.2.2). Assume that Φ is of type $\mathtt{A}_{\mathtt{2n}}^{(2)}$. Suppose both α and β are long or short or intermediate then we have $n_\alpha = n_\beta$ by Lemma 1.2.3, hence $\beta_1 + \cdots + \beta_i + \beta_{i+1} \in \Psi_{i_0}$, since $Z_\beta(\Psi_{i_0}) = p_\beta + n_\alpha\mathbb{Z}$. If β is short (resp. intermediate) and α is intermediate (resp. short) then we have $n_\beta = n_\alpha$ by Proposition 1.2.8, hence we have $\beta_1 + \cdots + \beta_i + \beta_{i+1} \in \Psi_{i_0}$. If β is short or intermediate and α is long then we have $n_\beta = n_\alpha$ or $n_\alpha = 2n_\beta$ by Proposition 1.2.7, hence we have $\beta_1 + \cdots + \beta_i + \beta_{i+1} \in \Psi_{i_0}$. Now assume that α is short or intermediate and β is long then we have $n_\beta = n_\alpha$ if $2 | n_\alpha$ and $n_\beta = 2n_\alpha$ if $m \nmid n_\alpha$. Again the claim follows easily when $n_\beta = n_\alpha$. So we are left with case $n_\beta = 2n_\alpha$. Now note that $p_\beta + n_\beta k_\beta \equiv 0\ (mod\ 2)$ and $(p_\beta + n_\beta k_\beta + n_\alpha(k_\alpha + k'_\alpha)) \equiv 0\ (mod\ 2)$ together implies, $n_\alpha(k_\alpha + k'_\alpha) \equiv 0\ (mod\ 2)$. Since $2 \nmid n_\alpha$, we get $k_\alpha + k'_\alpha \equiv 0\ (mod\ 2)$. This implies we have $n_\alpha(k_\alpha + k'_\alpha) \equiv 0\ (mod\ n_\beta)$ and hence we have $\beta_1 + \cdots + \beta_i + \beta_{i+1} \in \Psi_{i_0}$. This completes the proof of Statement (3) in this case.

$\square$

Corollary 1.12.5. *Let Ψ be a closed subroot system of Φ and let $\Delta(\Psi)$ be the set of roots of $\mathfrak{g}(\Psi)$ with respect to $\mathfrak{h}$. Then we have $\Psi = \Delta(\Psi) \cap \Phi$. Thus the map $\Psi \mapsto \mathfrak{g}(\Psi)$ is a one-to-one correspondence between the set of closed subroot systems of Φ and the set of regular subalgebras of $\mathfrak{g}$.*

Proof. Immediate from Proposition 1.12.4. $\square$

1.12.1 Connection to π-system

E. B. Dynkin showed that linearly independent π-systems arise precisely as simple systems of regular subalgebras of finite dimensional semi-simple algebras. So it is natural to expect to define regular subalgebras in terms of $\pi-$systems in our context. Now we give equivalent definition of regular subalgebras in terms of $\pi-$systems. A $\pi-$system Σ is a finite subset of Φ satisfying the property that for each $\alpha, \beta \in \Sigma$, we have $\alpha - \beta$ is not a root (i.e., $\alpha - \beta \notin \Delta(\mathfrak{g})$). Note that we do not demand Σ to be linearly independent in the definition of $\pi-$systems. Let $\mathfrak{g}(\Sigma)$ be the subalgebra of $\mathfrak{g}$ generated by $\{\mathfrak{g}_\alpha : \alpha \in \Sigma \cup (-\Sigma)\}$ and let $\Delta(\Sigma)$ be the set of roots of $\mathfrak{g}(\Sigma)$ with respect to $\mathfrak{h}$. Denote by W_Σ the Weyl group generated by the reflections $\{\mathbf{s}_\alpha : \alpha \in \Sigma\}$. We refer to [3] for more details and historical remarks about $\pi-$systems. We have a natural choice of $\pi-$system for each closed subroot system of Φ.

Lemma 1.12.6. *Let Ψ be a closed subroot system of Φ and let $\Psi = \Psi_1 \oplus \cdots \oplus \Psi_k$ be its direct sum decomposition of irreducible components. Let Σ_i be a simple system of Ψ_i for each $1 \leq i \leq k$. Then $\Sigma = \bigcup_{i=i}^{k} \Sigma_i$ is a $\pi-$system.*

Proof. Let $\alpha \in \Sigma_i$ and $\beta \in \Sigma_j$, we need show that $\alpha - \beta$ is not a root of $\mathfrak{g}$. If $i = j$, then clearly $\alpha - \beta$ is not a root of $\mathfrak{g}$. Assume that $i \neq j$ and $\alpha - \beta$ is a root. Since $(\alpha, \beta) = 0$, we have $(\alpha - \beta, \alpha - \beta) > 0$. So, $\alpha - \beta$ is a real root of $\mathfrak{g}$. Since Ψ is closed in Φ, we have $\alpha - \beta \in \Psi$. But we have $(\alpha - \beta, \alpha) > 0$ and $(\alpha - \beta, \beta) < 0$, which demands $\alpha - \beta \in \Sigma_i \cap \Sigma_j$. This is clearly a contradiction and it completes the proof. $\square$

Suppose Σ is a $\pi-$system then $\Sigma \cup -\Sigma$ is closed under multiplication by -1. So it motivates us to define symmetric subsets of real roots. More precisely, a subset Σ_s of Φ is said to be symmetric if $\Sigma_s = -\Sigma_s$. Let $\mathfrak{g}(\Sigma_s)$ be the subalgebra of $\mathfrak{g}$ generated by $\{\mathfrak{g}_\alpha : \alpha \in \Sigma_s\}$. We are now ready to state our equivalent definitions of regular subalgebras of $\mathfrak{g}$.

Theorem 1.12.7. *Let $\mathfrak{g}$ an affine Kac-Moody algebra and let $\mathfrak{g}'$ be its subalgebra. Then the following definitions are equivalent:*

1. *there exists a closed subroot system Ψ of Φ such that $\mathfrak{g}' = \mathfrak{g}(\Psi)$,*
2. *there exists a $\pi-$system Σ of Φ such that $\mathfrak{g}' = \mathfrak{g}(\Sigma)$,*
3. *there exists a symmetric subset Σ_s of Φ such that $\mathfrak{g}' = \mathfrak{g}(\Sigma_s)$.*

Proof. First assume that $\mathfrak{g}' = \mathfrak{g}(\Psi)$ for some closed subroot system Ψ of Φ. Then by Lemma 1.12.6, we have the $\pi-$system Σ which is a union of simple systems of corresponding irreducible components of Ψ. Since Ψ is reduced, we have $\Psi = W_\Sigma(\Sigma)$. Since $\Delta(\Sigma) = -\Delta(\Sigma)$ and $\mathfrak{g}(\Sigma)$ is $\mathfrak{h}$–invariant, we have $\Psi = W_\Sigma(\Sigma) \subseteq \Delta(\Sigma)$ by Lemma 1.12.2. This implies that $\mathfrak{g}_\alpha \subseteq \mathfrak{g}(\Sigma)$ for all $\alpha \in \Psi$, hence we have $\mathfrak{g}(\Psi) \subseteq \mathfrak{g}(\Sigma)$. Since $\Sigma \subseteq \Psi$, we have $\mathfrak{g}(\Sigma) \subseteq \mathfrak{g}(\Psi)$. So, we have the equality $\mathfrak{g}(\Psi) = \mathfrak{g}(\Sigma)$. This also implies that $\Delta(\Psi) = \Delta(\Sigma)$ and we have $\Delta(\Sigma) \cap \Phi = \Psi$ from Corollary 1.12.5. This proves (1) implies (2). The fact (2) implies (3) follows immediately if we take $\Sigma_s = \Sigma \cup -\Sigma$.

Now we prove (3) implies (1). Suppose $\mathfrak{g}' = \mathfrak{g}(\Sigma_s)$ for some symmetric subset Σ_s of Φ. It is easy to see that $\Delta(\Sigma_s) = -\Delta(\Sigma_s)$. Let $\Psi = \Delta(\Sigma_s) \cap \Phi$. Again by Lemma 1.12.2, Ψ is a closed subroot system of Φ. Clearly $\mathfrak{g}(\Psi) \subseteq \mathfrak{g}(\Sigma_s)$ since $\mathfrak{g}_\alpha \subseteq \mathfrak{g}(\Sigma_s)$ for all $\alpha \in \Psi$. Since $\Sigma_s \subseteq \Delta(\Sigma_s) \cap \Phi = \Psi$, we have $\mathfrak{g}(\Sigma_s) \subseteq \mathfrak{g}(\Psi)$. So, we have the equality $\mathfrak{g}(\Psi) = \mathfrak{g}(\Sigma_s)$. This completes the proof. $\square$

Corollary 1.12.8. *The association $\Sigma \mapsto \Delta(\Sigma) \cap \Phi$ gives a bijective correspondence between the set of $\pi-$systems of Φ and the closed subroot systems of Φ.*

Remark 1.12.9. One can easily see that our definition of regular subalgebras is little different from the regular subalgebras which appears in [20, Section 2], see [21] for its generalization. Suppose the closed subroot system has a simple system (i.e., the corresponding $\pi-$system is linearly independent) then our definition of regular subalgebra matches up with the definition of Naito's, see [21], indeed in this case our regular subalgebra is the derived subalgebra of Naito's regular subalgebra which is a Kac–Moody algebra by definition. Note that the closed subroot systems of an affine root system does not need to have simple systems in general. For example, consider the affine root system $\Delta = \mathtt{G}_2^{(1)}$ and $\Delta_{\mathrm{re}} = \{\alpha + n\delta : \alpha \in \mathtt{G}_2, n \in \mathbb{Z}\}$. Let $\{\alpha_1, \alpha_2\}$ be the simple system of $\mathtt{G}_2$, such that α_2 is a short root. Then define

$$\Psi = \{\pm\alpha_2 + n\delta : n \in \mathbb{Z}\} \cup \{\pm\theta + n\delta : n \in \mathbb{Z}\},$$

where θ is the long root of $\mathtt{G}_2$. Clearly, Ψ is a closed subroot system of type $\mathtt{A}_1^{(1)} \oplus \mathtt{A}_1^{(1)}$ which has no linearly independent simple system by rank comparison. So, here in this chapter we are dealing with a much bigger class of subalgebras of affine Kac–Moody algebras.

1.12.2 Existence of finite chain

We have the following explicit description for the closed subroot systems of untwisted affine root systems.

Proposition 1.12.10. *Let Φ be an untwisted affine root system. We have, Ψ (does not need to be of affine type) is a closed subroot system of Φ if and only if there exists*

- *mutually orthogonal irreducible closed subroot systems $\Psi_1, \cdots, \Psi_k$ of $\mathring{\Phi}$ and*
- *$n_i \in \mathbb{Z}$ and $\mathbb{Z}$–linear function $p^i : \Psi_i \to \mathbb{Z},\ \alpha \mapsto p^i_\alpha$, satisfying the equation 1.2.2, for each $1 \leq i \leq k$*

such that

$$\Psi = \widehat{\Psi_1} \oplus \cdots \oplus \widehat{\Psi_k} \tag{1.12.1}$$

where $\widehat{\Psi_i} = \{\alpha + (p^i_\alpha + rn_i)\delta \in \Psi : \alpha \in \Psi_i, r \in \mathbb{Z}\}$, $1 \leq i \leq k$. The subroot system $\widehat{\Psi_i}$ is of finite type if and only if the integer n_i associated to $\widehat{\Psi_i}$ is zero.

Proof. Let Ψ be a closed subroot system of untwisted affine root system Φ. Then by Proposition 1.3.1, we know that $\mathrm{Gr}(\Psi)$ is a closed subroot system of $\mathring{\Phi}$. Let

$$\mathrm{Gr}(\Psi) = \Psi_1 \oplus \cdots \oplus \Psi_k$$

be the decomposition of $\mathrm{Gr}(\Psi)$ into irreducible components. Then each Ψ_i is an irreducible finite subroot system of $\mathring{\Phi}$. Since $\mathrm{Gr}(\Psi)$ is closed in Φ, we see that each Ψ_i is closed in $\mathring{\Phi}$. Let $\widehat{\Psi_i}$ denote the lift of Ψ_i in Ψ. Then for each $1 \leq i \leq k$, by Proposition 1.2.6, there exists $n_i \in \mathbb{Z}$ and a $\mathbb{Z}$–linear function $p^i : \Psi_i \to \mathbb{Z},\ \alpha \mapsto p^i_\alpha$, satisfying the equation 1.2.2 such that for each $\alpha \in \Psi_i$, $Z_\alpha(\widehat{\Psi_i}) = p^i_\alpha + n_i\mathbb{Z}$. This implies that $\widehat{\Psi_i} = \{\alpha + (p^i_\alpha + rn_i)\delta \in \Psi : \alpha \in \Psi_i, r \in \mathbb{Z}\}$, $1 \le i \le k$. Notice that if $n_i = 0$, then the lift of Ψ_i must be of finite type and the types of $\widehat{\Psi_i}$ and Ψ_i are same. Converse part is straightforward. This completes the proof.

□

Let Φ be an affine root system and Ψ be a closed subroot system of Φ as before. Write $\Psi = \Psi_a \oplus \Psi_f$ where Ψ_a (resp. Ψ_f) is the affine (resp. finite) part of Ψ. Since Ψ is closed, we have the subroot systems Ψ_a and Ψ_f are closed in Φ and $\mathring{\Phi}$ respectively. Since we know the classification of all the closed subroot systems in the finite type (see [1, 8]), we only need to classify all the closed subroot systems of Φ which are of affine type. It can be done using the following theorem and the information about maximal closed subroot systems which appears in previous sections.

Theorem 1.12.11. *Let Φ be an affine root system and Ψ be a closed subroot system in Φ of affine type. Then there exists a finite chain of closed subroot systems in Φ, $\Phi = \Phi_0 \supseteq \Phi_1 \supseteq \cdots \supseteq \Phi_k = \Psi$ such that Φ_i is maximal closed in Φ_{i-1} for $1 \le i \le k$.*

First we fix a notation. For a closed subroot system Δ of Φ with decomposition into indecomposable components $\mathrm{Gr}(\Delta) = \Delta_1 \oplus \cdots \oplus \Delta_k$, we denote by

$$\mathrm{ht}(\Delta) = \sum_{i=1}^{k} n_s^{\Delta_i}(\Delta) + \sum_{i=1}^{k} n_{\mathrm{im}}^{\Delta_i}(\Delta) + \sum_{i=1}^{k} n_\ell^{\Delta_i}(\Delta)$$

Here it is understood that $n_{\mathrm{im}}^{\Delta_i}(\Delta) = 0$ if there is no intermediate roots and so on. We need the following lemma to prove the theorem 1.12.11.

Lemma 1.12.12. *Let Φ be an affine root system and $\Psi \subsetneq \Delta \subseteq \Phi$ be closed subroot systems of Φ of affine type. Let $\mathrm{Gr}(\Psi) = \Psi_1 \oplus \cdots \oplus \Psi_\ell$ be the decomposition of $\mathrm{Gr}(\Psi)$ into irreducible components. Then we have*

(i) *either* $\mathrm{Gr}(\Psi) \subsetneq \mathrm{Gr}(\Delta)$ *or*

(ii) $\mathrm{Gr}(\Psi) = \mathrm{Gr}(\Delta)$ *and* $\mathrm{ht}(\Delta) < \mathrm{ht}(\Psi)$.

Proof. Suppose $\mathrm{Gr}(\Psi) \subsetneq \mathrm{Gr}(\Delta)$, then there is nothing to prove. So, assume that $\mathrm{Gr}(\Psi) = \mathrm{Gr}(\Delta)$. It is easy to see that for each $1 \leq i \leq \ell$ we have $n_s^{\Psi_i}(\Delta)$ is a divisor of $n_s^{\Psi_i}(\Psi)$, in particular $n_s^{\Psi_i}(\Delta) \leq n_s^{\Psi_i}(\Psi)$. Similarly, we have $n_{\mathrm{im}}^{\Psi_i}(\Delta) \leq n_{\mathrm{im}}^{\Psi_i}(\Psi)$ and $n_\ell^{\Psi_i}(\Delta) \leq n_\ell^{\Psi_i}(\Psi)$ for all $1 \leq i \leq \ell$. This immediately implies that $\mathrm{ht}(\Delta) \leq \mathrm{ht}(\Psi)$.

If $\mathrm{ht}(\Delta) = \mathrm{ht}(\Psi)$, then we must have $n_s^{\Psi_i}(\Delta) = n_s^{\Psi_i}(\Psi)$, $n_{\mathrm{im}}^{\Psi_i}(\Delta) = n_{\mathrm{im}}^{\Psi_i}(\Psi)$ and $n_\ell^{\Psi_i}(\Delta) = n_\ell^{\Psi_i}(\Psi)$ for all $1 \leq i \leq \ell$. This implies that $p_\alpha^\Psi + n_\alpha^\Psi\mathbb{Z} \subseteq p_\alpha^\Delta + n_\alpha^\Delta\mathbb{Z}$ for all $\alpha \in \mathrm{Gr}(\Psi)$. Since $n_\alpha^\Psi = n_\alpha^\Delta$, we get $p_\alpha^\Psi \equiv p_\alpha^\Delta (\mathrm{mod}\ n_\alpha^\Psi)$ for all $\alpha \in \mathrm{Gr}(\Psi)$. This immediately implies that Ψ must be equal to Δ which is a contradiction to the assumption.

□

Theorem 1.12.11 is an immediate corollary of the following proposition.

Proposition 1.12.13. *Let Φ be an affine root system and Ψ be a closed subroot system in Φ of affine type. Then there is no infinite chain of closed subroot systems in Φ, such that*

$$\Psi = \Phi_0 \subsetneq \Phi_1 \subsetneq \cdots \subsetneq \Phi_k \subsetneq \Phi_{k+1} \subsetneq \cdots \subseteq \Phi.$$

Proof. We prove this result by contradiction. Assume that there is an infinite chain of closed subroot systems in Φ, such that

$$\Psi = \Phi_0 \subsetneq \Phi_1 \subsetneq \cdots \subsetneq \Phi_k \subsetneq \Phi_{k+1} \subsetneq \cdots \subseteq \Phi.$$

Then we have $\mathrm{Gr}(\Psi) = \mathrm{Gr}(\Phi_0) \subseteq \mathrm{Gr}(\Phi_1) \subseteq \cdots \subseteq \mathrm{Gr}(\Phi_k) \subseteq \cdots \subseteq \mathrm{Gr}(\Phi)$. Since

$\mathrm{Gr}(\Phi)$ is finite, there must exists a $k \in \mathbb{Z}$ such that $\mathrm{Gr}(\Phi_k) = \mathrm{Gr}(\Phi_i)$ for all $i \geq k$. Since $\Phi_k \subsetneq \Phi_j \subsetneq \Phi_i$, by lemma 1.12.12, we have

$$\mathrm{ht}(\Phi_i) < \mathrm{ht}(\Phi_j) < \mathrm{ht}(\Phi_k) \quad \text{for all } k < j < i$$

which is absurd. This completes the proof. $\square$

Chapter 2

Weyl group action on π-system

2.1 preliminaries

An integer matrix $A = (a_{ij})$ of size $n \times n$, where n is a positive integer, is called a *generalized Cartan matrix*, *GCM* for short, if the following conditions are satisfied:

1. $a_{ii} = 2$ for all $1 \leq i \leq n$
2. $a_{ij} \leq 0$ whenever $1 \leq i \neq j \leq n$
3. $a_{ij} = 0$ if $a_{ji} = 0$ for $1 \leq i, j \leq n$

Given a GCM A of size n, we let $\mathfrak{g}(A)$ denote the *Kac-Moody Lie algebra* associated to A [15, §1.3], with Cartan subalgebra $\mathfrak{h}(A)$ and Chevalley generators e_i, f_i for $1 \leq i \leq n$. Let $\mathfrak{g}'(A)$ denote the derived subalgebra $[\mathfrak{g}(A), \mathfrak{g}(A)]$ of $\mathfrak{g}(A)$. Let $\alpha_i(A), 1 \leq i \leq n$ denote the simple roots of $\mathfrak{g}(A)$ and let $Q(A)$ be its root lattice, i.e., the free abelian group generated by the $\alpha_i(A)$. Both $\mathfrak{g}(A)$ and $\mathfrak{g}'(A)$ are $Q(A)$-graded Lie algebras, with $\deg e_i = \alpha_i(A) = -\deg f_i$ and $\deg h = 0$ for all $h \in \mathfrak{h}(A)$ [15, Chapter 1]. We let $\Delta, \Delta^{re}, \Delta^{im}$ denote the sets of roots, real roots and imaginary roots respectively. For a root α, we let $\mathfrak{g}(A)_\alpha$ denote the corresponding root space. Each real root α defines a reflection s_α of $\mathfrak{h}^*$ by $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \, \alpha$ where $\alpha^\vee \in \mathfrak{h}(A)$ is the coroot corresponding to α. The Weyl group $W(A)$ is the subgroup of $GL(\mathfrak{h}^*)$ generated by the s_α, $\alpha \in \Delta^{re}$. We use terminology and notation as in the early chapters of [15] without any further comment.

2.1.1 Multisets of real roots

Let A be a GCM, and let $\Sigma = \{\beta_1, \beta_2, \cdots, \beta_m\}$ be a collection of real roots of $\mathfrak{g}(A)$ (possibly with repetitions). We define the $m \times m$ matrix

$$M(\Sigma) := [\langle \beta_j, \beta_i^\vee \rangle]_{i,j=1}^m$$

We note that this is not a GCM in general. We let $\Sigma^\vee := \{\beta_1^\vee, \beta_2^\vee, \cdots, \beta_m^\vee\}$ be the corresponding multiset of coroots. Viewing these as real roots of $\mathfrak{g}(A^T)$, we observe $M(\Sigma^\vee) = M(\Sigma)^T$.

A reordering of the elements of Σ corresponds to a simultaneous permutation of the rows and columns of the matrix: $M(\Sigma) \mapsto P\, M(\Sigma)\, P^T$ for some $m \times m$ permutation matrix P. We will most often identify two such matrices without explicit mention.

2.1.2 π-systems

Definition 2.1.1. Let A be a GCM. A *π-system* in A is a finite collection of distinct real roots $\{\beta_i\}_{i=1}^m$ of $\mathfrak{g}(A)$ such that $\beta_i - \beta_j$ is not a root for any $1 \le i \ne j \le m$.

This definition is essentially due to Dynkin [8] (for A of finite type) and Morita [20] (in general), both of whom require that the $\{\beta_i\}_{i=1}^m$ be linearly independent;

Morita calls such sets *fundamental subsets of roots.* The following proposition is stated in Morita (for the linearly independent case) without proof (see also Naito [21]). We supply the easy details.

Proposition 2.1.2. *Let A be a GCM, and $\Sigma = \{\beta_i\}_{i=1}^m$ be a π-system in A. Then the matrix $M(\Sigma)$ is a GCM.*

PROOF: For any real root β we have $\langle \beta^\vee, \beta\rangle = 2$. Indeed, letting $\beta = w\alpha$ for a simple root α and w an element of the Weyl group, we have $\beta^\vee = w(\alpha^\vee)$, and $\langle \beta^\vee, \beta\rangle = \langle w(\alpha^\vee), w\alpha\rangle = \langle \alpha^\vee, \alpha\rangle = 2$. Suppose β and γ are distinct real roots such that $\gamma - \beta$ is not a root. Consider $\{\gamma - p\beta, \ldots, \gamma + q\beta\}$ the "β-string through γ" ([15, Prop. 5.1]). Clearly $p = 0$ and $\langle \beta^\vee, \gamma\rangle = p - q \leq 0$.

With β and γ as in the previous paragraph, if $\langle \beta^\vee, \gamma\rangle = 0$, then $q = 0$, so that $\beta + \gamma$ is not a root, so the γ-string $\{\beta - p'\gamma, \ldots, \beta + q'\gamma\}$ through β consists only of β, and so $\langle \gamma^\vee, \beta\rangle = p' - q' = 0$. $\square$

We call $B := M(\Sigma)$ the *type* of Σ, and refer to Σ as a *π-system of type B in A.*

In Dynkin and Morita's original definitions, a π-system was required to be linearly independent. Dynkin does however mention π-systems of finite-dimensional simple Lie algebras with this condition relaxed [8] . In the symmetrizable Kac-Moody context, Morita [20] and Naito [21] obtained the key initial results. A decade later, Feingold-Nicolai [10] rediscovered the definition of π-systems , but imposed the restriction that all roots of a π-system be positive. They did not require linear independence, but as was pointed out by Henneaux et al [[12], x4.3], their main theorem on embeddings arising out of π-systems is false unless this condition is imposed. Our Theorem 2.1.3 is the corrected statement, in the more general setting of π-systems that are not necessarily subsets of the positive real roots. Our Theorem 2.2.1 serves as a link between the definitions of Morita and

Feingold-Nicolai.

2.1.3 Symmetrizable GCMs and π-systems

An $n \times n$ GCM A is *symmetrizable* if there exists a diagonal $n \times n$ matrix D with positive rational diagonal entries such that DA is symmetric. Let $\Sigma = \{\beta_i : 1 \leq i \leq m\}$ be a π-system of type B in A. We note that if A is a symmetrizable GCM, then so is B. Fix a choice of diagonal matrix D which symmetrizes A, and let $(\cdot \mid \cdot)$ denote the corresponding symmetric bilinear form on $Q(A) \otimes_{\mathbb{Z}} \mathbb{C}$, defined by:

$$(\alpha_i(A) \mid \alpha_j(A)) = D_{ii}\, a_{ij} \tag{2.1.1}$$

Since the β_i are real roots of $\mathfrak{g}(A)$, we know by [Kac, Chapter 5] that:

$$b_{ij} = \langle \beta_i^\vee, \beta_j \rangle = \frac{2\,(\beta_i \mid \beta_j)}{(\beta_i \mid \beta_i)}$$

Thus, $D' = \mathrm{diag}((\beta_i \mid \beta_i)\,/2)$ is a diagonal matrix with positive rational entries that symmetrizes B. This choice of symmetrization defines a symmetric bilinear form on $Q(B) \otimes_{\mathbb{Z}} \mathbb{C}$. As in equation (2.1.1) above, this is given by $(\alpha_i(B) \mid \alpha_j(B)) = D'_{ii}\, b_{ij} = (\beta_i \mid \beta_j)$. In other words, given the compatible choices of symmetrizations (D, D') as above, the $\mathbb{C}$-linear map

$$q_{\Sigma} : Q(B) \otimes_{\mathbb{Z}} \mathbb{C} \to Q(A) \otimes_{\mathbb{Z}} \mathbb{C}, \quad \alpha_i(B) \mapsto \beta_i \text{ for } 1 \leq i \leq m \tag{2.1.2}$$

is form preserving. Given $\alpha \in Q(A) \otimes_{\mathbb{Z}} \mathbb{C}$ with $(\alpha \mid \alpha) \neq 0$, the corresponding reflection s_α is given by:

$$s_\alpha(\gamma) = \gamma - \frac{2\,(\gamma \mid \alpha)}{(\alpha \mid \alpha)}\,\alpha$$

for $\gamma \in Q(A) \otimes_{\mathbb{Z}} \mathbb{C}$. We note that $q_{\Sigma}(s_\alpha(\beta)) = s_{\alpha'}(\beta')$ where $\alpha, \beta \in Q(B) \otimes_{\mathbb{Z}} \mathbb{C}$ and α', β' are their images under q_{Σ}.

Theorem 2.1.3. *Let A be an $n \times n$ symmetrizable GCM and $\Sigma = \{\beta_i\}_{i=1}^m$ a π-system of type B in A. Let e_{β_i}, $e_{-\beta_i}$ be non-zero elements in the root spaces $\mathfrak{g}(A)_{\beta_i}$ and $\mathfrak{g}(A)_{-\beta_i}$ respectively, such that $[e_{\beta_i}, e_{-\beta_i}] = \beta_i^\vee$. Then there exists a unique Lie algebra homomorphism $i_{\Sigma} : \mathfrak{g}'(B) \to \mathfrak{g}'(A)$ such that $e_i \mapsto e_{\beta_i}$, $f_i \mapsto e_{-\beta_i}$, $h_i \mapsto \beta_i^\vee$.*

Proof: Since A is symmetrizable, so is B, and $\mathfrak{g}'(B)$ is generated by e_i, f_i, h_i, $1 \le i \le m$ subject to the relations [15, Theorem 9.11]:

$$[h_i, e_j] = b_{ij}\, e_j \qquad [h_i, f_j] = -b_{ij}\, f_j \tag{2.1.3}$$

$$[h_i, h_j] = 0 \tag{2.1.4}$$

$$[e_i, f_j] = \delta_{ij}\, h_i \qquad \text{and} \tag{2.1.5}$$

$$(\operatorname{ad} e_i)^{1-b_{ij}} e_j = (\operatorname{ad} f_i)^{1-b_{ij}} f_j = 0 \tag{2.1.6}$$

Any Lie algebra homomorphism from $\mathfrak{g}'(B)$ is thus determined by the images of e_i, f_i and h_i $(1 \le i \le m)$. Thus there is at most one Lie algebra homomorphism with the requisite properties.

To show that there exists such a homomorphism, we need only verify that the relations in (2.1.3) through (2.1.6) are satisfied. Relations (2.1.3) and (2.1.4) are clearly satisfied. As for (2.1.5) we consider two cases: if $j = i$, then it follows since $[e_{\beta_i}, e_{-\beta_i}] = \beta_i^\vee$; if $j \ne i$, then it follows since $\beta_i - \beta_j$ is not a root of $\mathfrak{g}(A)$ by the definition of π-system . As for (2.1.6), it follows from the fact [15, Prop. 5.1] that the β_i-string through β_j consists of β_j, $\beta_j + \beta_i$, $\ldots$, $\beta_j + k\beta_i$, where $k = \langle \beta_i^\vee, \beta_j \rangle$. $\square$

The following proposition is equivalent to that of Naito [21, Theorem 3.6], though his proof is different (without using the Serre relations). In the interest of completeness, we give an (slightly simpler) argument.

Proposition 2.1.4. *With notation as in the above theorem, if Σ is linearly independent (in $Q(A) \otimes_{\mathbb{Z}} \mathbb{C}$), one can extend the map i_Σ to a map from $\mathfrak{g}(B)$ to $\mathfrak{g}(A)$. Further, this map is injective.*

Proof. Suppose that $\{\mathfrak{h}; \alpha_1^\vee, \ldots, \alpha_n^\vee; \alpha_1, \ldots, \alpha_n\}$ is a realization of A [15, Chapter 1]. Let $\mathfrak{k}$ be any subspace of $\mathfrak{h}$ of smallest possible dimension such that (i) $\mathfrak{k}$ contains $\beta_1^\vee$, ..., $\beta_m^\vee$, and (ii) the restrictions of β_1, ..., β_m to $\mathfrak{k}$ are linearly independent as elements of $\mathfrak{k}^*$ (this is possible since we are given that the β_i are linearly independent). Then

1. $(\mathfrak{k}, \beta_1^\vee, \ldots, \beta_m^\vee; \beta_1|_{\mathfrak{k}}, \ldots, \beta_m|_{\mathfrak{k}})$ is a realization of B.
2. $\operatorname{rank} B \geq \operatorname{rank} A - 2(n-m)$.

Assertion (1) follows easily from the definition of realization. As for assertion (2), observe that $\{\beta_i^\vee\}_{i=1}^m$ is in the span of $\{\alpha_i^\vee\}_{i=1}^n$: this follows from the definition of $\beta^\vee$ for a real root β as $w(\alpha_i^\vee)$ where w is an element of the Weyl group such that $\beta = w(\alpha_i)$. We have $B = YAX$, where $X = (x_{ij})$ is the $n \times m$ matrix such that $\beta_j = \sum_{i=1}^n x_{ij}\alpha_i$ and $Y = (y_{ij})$ is the $m \times n$ matrix such that $\beta_j^\vee = \sum_{i=1}^n y_{ji}\alpha_i^\vee$. The matrices X and Y are both of rank m. The assertion now follows easily from elementary linear algebra.

Now, $\mathfrak{g}(B)$ is generated by $\mathfrak{k}$, e_i, f_i subject to the relations specified in the proof of Theorem 2.1.3 together with the following:

$$[k, e_i] = \beta_i(k)e_i \qquad [k, f_i] = -\beta_i(k)f_i \qquad [k_1, k_2] = 0 \qquad \text{for } k, k_1, k_2 \text{ in } \mathfrak{k}$$

We map $\mathfrak{k}$ to $\mathfrak{h}$ via the natural inclusion; e_i, f_i are mapped to $e_{\beta_i}, e_{-\beta_i}$ as before. We only need to check that the additional relations above hold. But these are obvious.

Finally, we show that the homomorphism is an embedding. The kernel of the homomorphism being an ideal of $\mathfrak{g}(B)$, it either contains the derived algebra $\mathfrak{g}'(B)$ or is contained in the center [15, §1.7(b)]. Since $e_i \mapsto e_{\beta_i}$ (and e_i is contained in $\mathfrak{g}'(B)$ by (2.1.3)) the first possibility is ruled out. Thus the kernel is contained in the center. But the center is contained in the subspace $\mathfrak{k}$ ([15, Prop. 1.6]) and on $\mathfrak{k}$ the homomorphism is an inclusion. Thus the kernel is zero.

□

Remark 2.1.5. The following easy observations are often useful:

1. If Σ is linearly independent, then q_{Σ} is an injection.

2. If $\det B \neq 0$, then Σ is linearly independent.

Example 2.1.6. (i) Let A be a GCM of finite type. Dynkin [8] showed that if $\mathfrak{m}$ is a regular semisimple subalgebra of $\mathfrak{g}(A)$, then there exists a GCM B of finite type and a π-system Σ of type B in A such that $\mathfrak{m} = i_{\Sigma}(\mathfrak{g}(B))$.

(ii) Let us take $A = [2]$, so that $\mathfrak{g}(A) = \mathfrak{g}'(A) = \mathfrak{sl}_2\mathbb{C}$. Let $\Sigma = \{\alpha_1, -\alpha_1\} = \Delta(A)$. This is clearly a π-system in A, of type $B = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$. The corresponding Kac-Moody algebra $\mathfrak{g}(B)$ is the affine Lie algebra $\widehat{\mathfrak{sl}_2\mathbb{C}}$. We then have [15, Chapter 7], $\mathfrak{g}'(B) = \mathfrak{sl}_2\mathbb{C} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$, the universal central extension of the loop algebra of $\mathfrak{sl}_2$. The generators of $\mathfrak{g}'(B)$ are $e_1 = X, f_1 = Y, e_2 = Y \otimes t, f_2 = X \otimes t^{-1}$, where $X = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $Y = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ are the standard generators of $\mathfrak{sl}_2\mathbb{C}$.

The map defined in Theorem 2.1.3 is thus:

$$e_1 \mapsto X,\ f_1 \mapsto Y,\ e_2 \mapsto Y,\ f_2 \mapsto X$$

(iii) More generally, let A be any finite type GCM and $\mathfrak{g}(A)$ the corresponding finite dimensional simple Lie algebra, with highest root θ. Consider the π-system Σ consisting of the simple roots of $\mathfrak{g}(A)$ together with $-\theta$. This has type B, the GCM of the untwisted affinization of $\mathfrak{g}(A)$. The map defined by Theorem 2.1.3 coincides with the *evaluation map* at $t = 1$:

$$\mathfrak{g}'(B) = \mathfrak{g}(A) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \to \mathfrak{g}(A)$$

$$c \mapsto 0 \text{ and } \zeta \otimes f(t) \mapsto f(1)\,\zeta \text{ for all } \zeta \in \mathfrak{g}(A),\ f \in \mathbb{C}[t, t^{-1}]$$

□

Lemma 2.1.7. *Let A be an $n \times n$ GCM. Let I be an ideal of $\mathfrak{g}'(A)$ that does not contain any simple root vectors, i.e., $e_i, f_i \notin I$ for all i. Then I does not contain any root vectors, i.e., $\mathfrak{g}'(A)_\alpha \cap I = (0)$ for all roots α.*

Proof. Suppose α is a positive, non-simple root. Assume $e_\alpha \in I$ for some nonzero $e_\alpha \in \mathfrak{g}'(A)_\alpha$. By [Kac, Lemma 1.5], there exists i_1 such that $[f_{i_1}, e_\alpha] \neq 0$. If $\alpha - \alpha_i$ is not a simple root, find i_2 such that $[f_{i_2}, [f_{i_1}, e_\alpha]] \neq 0$. Proceeding this way, after finitely many steps we get $[f_{i_k}[\cdots f_{i_2}, [f_{i_1}, e_\alpha]] \cdots] = e_i \in I$, which contradicts the hypothesis on I. If α were a negative root to begin with, the proof is analogous. □

Remark 2.1.8. 1. Let I be an ideal of $\mathfrak{g}'(A)$. We observe that if I contains one of $e_i, f_i, \alpha_i^\vee$, then it contains all three.

2. If A is an indecomposable GCM, then any proper ideal of $\mathfrak{g}'(A)$ satisfies the hypothesis of lemma 2.1.7. To see this, suppose e_i is in I. Then, so are f_i and $\alpha_i^\vee$. Since A is indecomposable, for each fixed j, there exist $i_1, i_2, \cdots i_s$ such that $a_{ii_1} a_{i_1 i_2} \cdots a_{i_s j} \neq 0$. Since, $[\alpha_i^\vee, e_{i_1}] = a_{ii_1} e_{i_1}$, we conclude I contains e_{i_1}, and hence also $f_{i_1}, \alpha_{i_1}^\vee$. Proceeding in this manner, we get $e_j, f_j, \alpha_j^\vee \in I$.

Since this holds for all j, we obtain $I = \mathfrak{g}'(A)$, a contradiction.

While the map i_Σ of Theorem 2.1.3 need not be injective when Σ is linearly dependent, we nevertheless have the following useful result which states that it is injective on each root space.

Corollary 2.1.9. *The map $i_\Sigma : \mathfrak{g}'(B) \to \mathfrak{g}'(A)$ defined in Theorem 2.1.3 is injective when restricted to $\mathfrak{g}'(B)_\alpha$ for $\alpha \in \Delta(B)$. Further, the image of $\mathfrak{g}'(B)_\alpha$ is contained in $\mathfrak{g}'(A)_{q_\Sigma(\alpha)}$.*

Corollary 2.1.10. *1. $q_\Sigma(\Delta^{re}(B)) \subset \Delta^{re}(A)$ and $q_\Sigma(\Delta^{im}(B)) \subset \Delta^{im}(A) \cup \{0\}$.*

2. If further Σ is linearly independent, then $q_\Sigma(\Delta^{im}(B)) \subset \Delta^{im}(A)$.

Proof. Corollary 2.1.9 implies that if α is a root of $\mathfrak{g}'(B)$, then $q_\Sigma(\alpha)$ is either 0 or a root of $\mathfrak{g}'(A)$. Further, since $(\alpha \mid \alpha) = (q_\Sigma(\alpha) \mid q_\Sigma(\alpha))$, real roots map to real roots and imaginary roots to imaginary roots or 0; since real roots are precisely those roots of positive norm. The second part is obvious from the linear independence assumption, since an imaginary root is nonzero, it cannot map to zero. $\square$

The above corollary, for linearly independent Σ was first obtained by Naito [21, Theorem 3.8]. Next, we have the converse to Theorem 2.1.3:

Proposition 2.1.11. *Let $A_{n\times n}, B_{m\times m}$ be symmetrizable GCMs. Suppose $\phi : \mathfrak{g}'(B) \to \mathfrak{g}'(A)$ is a Lie algebra homomorphism satisfying $0 \neq \phi(e_i) \in \mathfrak{g}'(A)_{\beta_i}$, $0 \neq \phi(f_i) \in \mathfrak{g}'(A)_{-\beta_i}$ for all $1 \leq i \leq m$, for some real roots $\{\beta_i\}_{i=1}^m$ of $\mathfrak{g}'(A)$. Then, the set $\Sigma = \{\beta_i\}_{i=1}^m$ is a π-system of type B in A.*

Proof. Given a real root β and any root γ of $\mathfrak{g}'(A)$, it follows from elementary $\mathfrak{sl}_2$ theory (applied to the β-string through γ) that

$$[\mathfrak{g}'(A)_\beta, \mathfrak{g}'(A)_\gamma] \neq 0 \text{ iff } \beta + \gamma \text{ is a root of } \mathfrak{g}'(A) \tag{2.1.7}$$

Now, since $[e_i, f_j] = 0$ for $1 \le i \ne j \le m$, we apply ϕ to conclude that $[\mathfrak{g}'(A)_{\beta_i}, \mathfrak{g}'(A)_{-\beta_j}] = 0$. Hence $\beta_i - \beta_j$ is not a root of $\mathfrak{g}'(A)$, and Σ is thus a π-system .

Next, we show that the type of this π-system is exactly B. Note that $|\langle \beta_i^\vee, \beta_j \rangle|$ is the largest integer k for which $\beta_j + k'\beta_i$ is a root of $\mathfrak{g}'(A)$ for $0 \le k' \le k$. Let $\alpha_i(B)$ denote the simple roots of $\mathfrak{g}'(B)$; their images under q_Σ are the β_i. We have $\ell = |b_{ij}|$ is the largest integer for which $\alpha_j(B) + \ell'\alpha_i(B)$ is a root of $\mathfrak{g}'(B)$ for $0 \le \ell' \le \ell$. In fact $\gamma = \alpha_j(B) + \ell\alpha_i(B) \in \Delta^{re}(B)$, and by corollary 2.1.10, $q_\Sigma(\gamma) \in \Delta^{re}(A)$. Thus, $k \ge \ell$.

By (2.1.7) above, $[\mathfrak{g}'(B)_{\alpha_i(B)}, \, \mathfrak{g}'(B)_\gamma] = 0$, and since these two real root spaces map isomorphically to the corresponding real root spaces of $\mathfrak{g}'(A)$, we conclude $[\mathfrak{g}'(A)_{\beta_i}, \, \mathfrak{g}'(B)_{q_\Sigma(\gamma)}] = 0$. By (2.1.7) again, $\beta_i + q_\Sigma(\gamma) = \beta_j + \,(\ell+1)\beta_i$ is not a root of $\mathfrak{g}'(A)$. Hence $k \le \ell$, and we obtain $\langle \beta_i^\vee, \beta_j \rangle = b_{ij}$ as required. $\square$

Corollary 2.1.12. *If A has a π-system of type B and B has a π-system of type C, then A has a π-system of type C.*

PROOF: Theorem 2.1.3 gives us Lie algebra morphisms $\mathfrak{g}'(C) \to \mathfrak{g}'(B) \to \mathfrak{g}'(A)$. By corollary 2.1.9, both these maps are injective on real root spaces. The generators e_i, f_i of $\mathfrak{g}'(C)$ map to real root vectors of $\mathfrak{g}'(B)$. Thus, under the composition of these two morphisms, e_i, f_i map to non-zero real root vectors of $\mathfrak{g}'(A)$. The corresponding roots are clearly negatives of each other. Proposition 2.1.11 now completes the proof. $\square$

If Σ_1, Σ_2 denote the π-systems of the above corollary, of types B and C respectively, then the π-system of type C in A that one obtains from the proof above is just $q_{\Sigma_1}(\Sigma_2)$.

As mentioned in the introduction, π-systems were first defined by Dynkin in his study of regular semisimple subalgebras of semisimple Lie algebras. In this setting, any set of simple roots of a closed subroot system of the root system (of a semisimple Lie algebra) is a π-system . The converse is also true, as can be seen from theorem 2.1.3.

In the infinite dimensional setting, Naito [21] defined a *regular subalgebra* of a Kac-Moody algebra $\mathfrak{g}(A)$ to be any subalgebra of the form $i_\Sigma(\mathfrak{g}(B))$ for Σ a linearly independent π-system of type B in A, where B varies over all GCMs (cf. Proposition 2.1.4).

2.2 Weyl group action on π-systems

Let A be a symmetrizable GCM. Let $W(A)$ denote the Weyl group of A. It acts on the set of roots of A, preserving each of the subsets of real and imaginary roots. Further this action preserves the bilinear invariant form. Thus, there is an induced action of $W(A)$ on the set of all π-systems in A of a given type B.

When A is of finite type, it is easy to see that every linearly independent π-system in A is $W(A)$-conjugate to a π-system contained in the set of *positive roots* of A. To see this, take an element $\gamma \in \mathfrak{h}^*_{\mathbb{R}}$ which has positive inner product with the elements of the π-system . The element $w \in W(A)$ which maps γ into the dominant Weyl chamber will clearly also map the π-system to a subset of the positive roots.

This proof fails in the general case; such w does not exist unless γ is in the Tits cone. For instance, the negative simple roots of A form a π-system of type A in A. This set cannot be $W(A)$-conjugated to a subset of positive roots if A is not of finite type; this can be seen using for instance [15, Theorem 3.12c]. The next theorem shows that this is essentially the only obstruction.

Theorem 2.2.1. *Let A, B be symmetrizable GCMs and Σ a linearly independent π-system of type B in A. If B is* indecomposable*, then:*

1. *There exists $w \in W(A)$ such that $w\Sigma \subset \Delta_+^{re}(A)$ or $w\Sigma \subset \Delta_-^{re}(A)$.*

2. *There exist $w_1, w_2 \in W(A)$ such that $w_1\Sigma \subset \Delta_+^{re}(A)$ and $w_2\Sigma \subset \Delta_-^{re}(A)$ if and only if B is of finite type.*

The proof occupies the next subsection.

2.2.1 Proof of theorem

The proof of theorem 2.2.1 closely follows that of [15, Proposition 5.9]. The first part of this theorem, in the special case $|\Sigma| = 2$ was proved by Naito in [21]. We first recall some relevant facts about the roots of a Kac-Moody algebra. Let B be an *indecomposable* GCM, and let $\mathfrak{g}(B)$ denote the corresponding Kac-Moody algebra. Let $Q(B)$ denote its root lattice. We use the notation introduced already for the sets of roots, real roots, positive roots etc. Let $\mathbb{R}_+$ denote the set of non-negative reals. Define:

$$C^{\text{im}} = \bigcup_{\alpha \in \Delta_+^{im}(B)} \mathbb{R}_+\alpha, \quad C^{\text{re}} = \bigcup_{\alpha \in \Delta_+^{re}(B)} \mathbb{R}_+\alpha.$$

We then have the following result due to Kac [15, §5.8]:

Proposition 2.2.2. *(Kac) In the metric topology on the real span of $Q(B)$, $\overline{C^{\text{im}}}$ is the convex hull of the set of limit points of C^{re}. In particular, it is a convex cone.*

Now suppose $Q(B) \subset \mathbb{E}$ for some real vector space $\mathbb{E}$. Let $\{\epsilon_i\}_{i=1}^n$ be a basis of $\mathbb{E}$. Define $\mathbb{E}_+$ to be the $\mathbb{R}_+$ span of the ϵ_i, and let $\mathbb{E}_- = -\mathbb{E}_+$.

Lemma 2.2.3. *If $\Delta(B) \subset \mathbb{E}_+ \cup \mathbb{E}_-$, then $\Delta_+^{im}(B) \subset \mathbb{E}_+$ or $\Delta_+^{im}(B) \subset \mathbb{E}_-$.*

Proof. Consider the set $\overline{C^{\mathrm{im}}}$; it has the following properties: (i) It is convex, by Proposition 2.2.2. (ii) It is contained in $\mathbb{E}_+ \cup \mathbb{E}_-$, by the given hypothesis. (iii) It does not contain a line (i.e., for nonzero $x \in \mathbb{E}$, both x and $-x$ cannot belong to this set), because $\overline{C^{\mathrm{im}}} \subset \mathbb{R}_+ (\Delta_+(B))$.

Suppose there exists $\alpha \in \Delta_+^{im}(B) \cap \mathbb{E}_-$ and $\beta \in \Delta_+^{im}(B) \cap \mathbb{E}_+$. Then there exists a point on the line joining α and β which does not belong to $\mathbb{E}_-$ or $\mathbb{E}_+$ because of property (iii). This point belongs to $\overline{C^{\mathrm{im}}}$ by property (i) but this is not possible beacuse of property (ii). Hense $\overline{C^{\mathrm{im}}}$ must be entirely contained either in $\mathbb{E}_+$ or in $\mathbb{E}_-$. □

Under the same hypothesis as lemma 2.2.3, we have:

Lemma 2.2.4. *If $\Delta_+^{im}(B) \subset \mathbb{E}_+$, then all but finitely many real roots of B lie in $\mathbb{E}_+$.*

Proof. First, we define an inner product on $\mathbb{E}$ by requiring the ϵ_i to be an orthonormal basis. This defines the standard metric topology on $\mathbb{E}$, and thereby on the $\mathbb{R}$-span of $Q(B)$.

Let $M := \Delta_+^{re}(B) \cap \mathbb{E}_-$, and $\widehat{M} := \{\alpha/\|\alpha\| : \alpha \in M\}$. Here, the norm is that of the Euclidean space $\mathbb{E}$. Observe that $\widehat{M}$ is a subset of $C^{\mathrm{re}} \cap \mathbb{E}_- \cap S$, where S is the unit sphere in $\mathbb{E}$. If $\widehat{M}$ is an infinite set, then, it has a limit point, say ζ. Now $\zeta \in \mathbb{E}_- \cap S$, and by Proposition 2.2.2, $\zeta \in \overline{C^{\mathrm{im}}}$. But $\overline{C^{\mathrm{im}}} \subset \mathbb{E}_+$ by hypothesis. This contradiction establishes the lemma. □

Proposition 2.2.5. *Let $\Delta(B) \subset \mathbb{E}_+ \cup \mathbb{E}_-$. There exists $w \in W(B)$ such that $w\Delta_+(B) \subset \mathbb{E}_+$ or $w\Delta_+(B) \subset \mathbb{E}_-$.*

Proof. By lemma 2.2.3, the positive imaginary roots are all contained in $\mathbb{E}_+$ or in $\mathbb{E}_-$; we may suppose (replacing the ϵ_i with their negatives if need be) that $\Delta_+^{im}(B) \subset \mathbb{E}_+$. Consider $F := \Delta_+^{re}(B) \cap \mathbb{E}_-$; this is finite by lemma 2.2.4. If this set is non-empty, it contains some simple root α of $\mathfrak{g}(B)$. Since the simple reflection s_α defines a bijective self-map of $\Delta_+^{re}(B)\backslash\{\alpha\}$, it is clear that $F' := s_\alpha\left(\Delta_+^{re}(B)\right) \cap \mathbb{E}_-$ contains one fewer element than F. Iterating this procedure, we can find w, a product of simple reflections, such that $w\,\Delta_+^{re}(B) \cap \mathbb{E}_-$ is empty, as required. $\square$

Finally, we are in a position to prove theorem 2.2.1. With notation as in the theorem, observe that the linear independence of Σ implies that $q_{\Sigma} : Q(B) \to Q(A)$ is injective. By corollary 2.1.10, $q_{\Sigma}(\Delta(B)) \subset \Delta(A) = \Delta_+(A) \cup \Delta_-(A)$. We define $\mathbb{E}$ to be the $\mathbb{R}$-span of $\Delta(A)$ and take $\{\epsilon_i\}$ to be the basis of simple roots of $\mathfrak{g}(A)$. Then, clearly, $q_{\Sigma}(\Delta(B)) \subset \mathbb{E}_+ \cup \mathbb{E}_-$. Identifying $\Delta(B)$ with its image under q_{Σ}, and appealing to proposition 2.2.5 completes the proof of part (1).

To prove part (2), since $w_1\Sigma \subset \Delta_+^{re}(A)$, we have $w_1(q_{\Sigma}(\Delta_+(B))) \subset \Delta_+(A)$. Consider the set $R := q_{\Sigma}(\Delta_+^{im}(B))$. We have (i) $R \subset \Delta^{im}(A)$, by corollary 2.1.10, and (ii) $w_1R \subset \Delta_+(A)$. Since the sets $\Delta_\pm^{im}(A)$ are both $W(A)$-invariant, this implies $R \subset \Delta_+^{im}(A)$. Similarly, from $w_2\Sigma \subset \Delta_-^{re}(A)$, we conclude $R \subset \Delta_-^{im}(A)$. This means R is empty, or in other words, that B is of finite type.

Conversely, if B is of finite type, then $\Delta_+(B)$ is finite. Hence its intersections with $\Delta_+(A)$ and $\Delta_-(A)$ are both finite sets. The proof of Proposition 2.2.5 shows that there exist elements of $W(A)$ which map $\Delta_+(B)$ to subsets of $\Delta_\pm(A)$. $\square$

As is evident from Example 2.1.6(ii), the conclusion of theorem 2.2.1 is false if Σ is not assumed to be linearly independent, even when A is of finite type.

Let A, B be symmetrizable GCMs. A π-system Σ of type B in A is said to be *positive* (resp. *negative*) if it is $W(A)$-conjugate to a π-system all of whose elements are positive (respectively negative) roots. Theorem 2.2.1 implies that if Σ is linearly independent and B is indecomposable and not of finite type, then Σ is either positive or negative. We record below a simple criterion to determine the sign that was obtained in the course of the proof of theorem 2.2.1.

Proposition 2.2.6. *Let A, B be symmetrizable GCMs, with B indecomposable and not of finite type. Let Σ be a linearly independent π-system of type B in A. Then the following are equivalent:*

1. *Σ is positive (resp. negative).*

2. *$q_{\Sigma}(\alpha) \in \Delta_{+}^{im}(A)$ (resp. $\Delta_{-}^{im}(A)$) for* every *$\alpha \in \Delta_{+}^{im}(B)$.*

3. *$q_{\Sigma}(\alpha) \in \Delta_{+}^{im}(A)$ (resp. $\Delta_{-}^{im}(A)$) for some $\alpha \in \Delta_{+}^{im}(B)$.*

$\square$

Let $\mathfrak{m}(B, A)$ denote the number of $W(A)$-orbits of π-systems of type B in A (this could be infinity in general). When A, B are of finite type, Borel-de Siebenthal and Dynkin determined the pairs for which $\mathfrak{m}(B, A) > 0$. Dynkin went further, and also determined the values of $\mathfrak{m}(B, A)$; these turn out to be 1 for almost all cases, except for a few where it is 2 [8, Tables 9-11]

2.3 π-systems of affine type

Let $S(A)$ denote the Dynkin diagram associated to the GCM A [15]. Any subset of the vertices of $S(A)$ together with the edges between them will be called a *subdiagram* of $S(A)$ (and we will use $\subseteq$ to denote the relation of being a subdiagram). Given $\alpha = \sum_{i=1}^{n} c_i \alpha_i$, we define $\operatorname{supp} \alpha$ to be the set $\{i : c_i \neq 0\}$ and view it as a subset of the vertices of $S(A)$. Given a subdiagram Y of $S(A)$, we say α is supported in Y if $\operatorname{supp} \alpha$ is contained in the set of vertices of Y. We also let $Y^{\perp}$ denote the set of vertices of $S(A)$ that are not connected by an edge to any vertex of Y.

Lemma 2.3.1. *Let A be a symmetrizable GCM and Y a subdiagram of $S(A)$ of affine type. Let δ_Y denote the null root of Y. If $\beta \in \Delta(A)$ is such that*

$(\beta \mid \delta_Y) = 0$, *then* $\operatorname{supp}\beta \subset Y \sqcup Y^\perp$.

Proof. We write $\beta = \sum_{p \in S(A)} c_p \alpha_p$, where all the coefficients are non-negative, or all non-positive. Let $\operatorname{supp}\beta$ denote the set of p for which c_p is nonzero. Now, $(\alpha_p \mid \delta_Y)$ is 0 for $p \in Y$, and ≤ 0 when $p \notin Y$. Since all coefficients are of the same sign, every $p \in \operatorname{supp}\beta$ must be either in Y or in $Y^\perp$. □

Theorem 2.3.2. *Let A be a symmetrizable GCM and B be a GCM of affine type. Suppose Σ is a linearly independent π-system of type B in A. Then,*

1. *There exists an affine subdiagram Y of $S(A)$ and $w \in W(A)$ such that every element of $w\Sigma$ is supported in Y.*
2. *Suppose (Y', w') is another such pair, i.e., with Y' a subdiagram of affine type, $w' \in W(A)$ such that $w'\Sigma$ is supported in Y'. Then $Y = Y'$ and $w'w^{-1} \in W(Y \sqcup Y^\perp)$.*
3. *$\mathfrak{m}(B, A) = \infty$.*

Proof. Let $\Sigma = \{\beta_i\}_1^m$. Let $\{\alpha_i(B)\}_1^m$ denote the simple roots of $\mathfrak{g}(B)$ and let δ_B denote its null root. Let $\delta_\Sigma = q_\Sigma(\delta_B)$. By corollary 2.1.10(2) and the fact that q_Σ preserves forms, we obtain that δ_Σ is an isotropic root of $\mathfrak{g}(A)$. By [15, Proposition 5.7], there exists $w \in W(A)$ such that $w(\delta_\Sigma)$ is supported on an affine subdiagram Y of $S(A)$ and $w(\delta_\Sigma) = k\delta_Y$ for some nonzero integer k, where δ_Y is the null root of Y.

Now, $0 = (\alpha_i(B) \mid \delta_B) = (\beta_i \mid \delta_\Sigma) = k\,(w\beta_i \mid \delta_Y)$ for all $i = 1, \cdots, m$. We conclude $\operatorname{supp} w\beta_i \subset Y \sqcup Y^\perp$, by lemma 2.3.1. Since $w\beta_i$ is a root, its support is connected, and hence contained entirely in Y or entirely in $Y^\perp$. However, $w\Sigma$ is a π-system of type B, an indecomposable GCM. So, $w\Sigma$ cannot be written as a disjoint union of two mutually orthogonal subsets. This means that either $\operatorname{supp} w\beta_i \subset Y$ for all i, or

$\operatorname{supp} w\beta_i \subset Y^\perp$ for all i. The latter is impossible since $k\delta_Y = w(\delta_\Sigma)$ is a positive integral combination of the $w\beta_i$. This proves part (1).

Now, if (Y', w') is another such pair, then since the only isotropic roots of $\mathfrak{g}(A)$ supported on subsets of Y' are the multiples of $\delta_{Y'}$, we obtain $w'(\delta_\Sigma) = k'\delta_{Y'}$ for $k' \neq 0$. Define $\sigma = w'w^{-1}$, so $\sigma(k\delta_Y) = k'\,\delta_{Y'}$. Since δ_Y is a positive imaginary root of $\mathfrak{g}(A)$, so is $\sigma\delta_Y$; thus k and k' have the same sign. We may suppose $k, k' > 0$. Now $k\delta_Y$ and $k'\delta_{Y'}$ are antidominant weights (i.e., their negatives are dominant weights) of $\mathfrak{g}(A)$, which are $W(A)$-conjugate. By [15, Proposition 5.2b], we get $k\delta_Y = k'\delta_{Y'}$. Thus, $Y = Y'$, $k = k'$ and $\sigma\delta_Y = \delta_Y$.

Since δ_Y is antidominant, the simple reflections that fix δ_Y generate the stabilizer of δ_Y. By lemma 2.3.1, this stabilizer is just $W(Y \sqcup Y^\perp)$. Thus $\sigma \in W(Y \sqcup Y^\perp)$, proving part (2).

Finally, let $\Sigma = w\Sigma$ denote the π-system of part (1). Now Y is of affine type, untwisted or twisted. In either case, from the description of the real roots of an affine Kac-Moody algebra [15, Chap 6], the following holds:
$\Delta^{re}(Y) + 6p\,\delta_Y \subset \Delta^{re}(Y)$ for all $p \in \mathbb{Z}$. Consider

$$\Sigma_p := \{\alpha + 6p\,\delta_Y : \alpha \in \Sigma\} \text{ for } p \in \mathbb{Z}.$$

Since δ_Y is orthogonal to every root of $\mathfrak{g}(Y)$, it is clear that Σ_p is a linearly independent π-system of type B in A, supported in Y. From the proof of part (1), we know $q_{\Sigma}(\delta_B) = k\delta_Y$ for some nonzero integer k. From the definition of Σ_p, we obtain

$$q_{\Sigma_p}(\delta_B) = (k + 6ph)\delta_Y \tag{2.3.1}$$

where h is the Coxeter number of the affine Kac-Moody algebra $\mathfrak{g}(B)$. We claim that the Σ_p are pairwise $W(A)$-inequivalent. Suppose Σ_m and Σ_n are in the same

$W(A)$-orbit. Then, from part (2), we obtain $\Sigma_m = \sigma(\Sigma_n)$ for some $\sigma \in W(Y \sqcup Y^{\perp})$. In particular, this means $q_{\Sigma_m}(\delta_B) = \sigma(q_{\Sigma_n}(\delta_B))$. Since σ fixes δ_Y, equation (2.3.1) implies $m = n$. This completes the proof of part (3).

□

Corollary 2.3.3. *Let A be a symmetrizable GCM such that $S(A)$ has no subdiagrams of affine type. Then A contains no linearly independent π-systems of affine type.*

This follows immediately from the proposition. We remark that Figure 2.4.2 contains examples of such $S(A)$.

Remark 2.3.1.

1. The conclusion of theorem 2.3.2 is false without the linear independence assumption, as in Example 2.1.6 (ii), (iii).

2. Let A, B be symmetrizable GCMs, with B of affine type. Suppose A contains a linearly independent π-system of type B. Theorem 2.3.2 implies that some affine type subdiagram Y of $S(A)$ also contains a linearly independent π-system of type B. This allows us to determine the possible set of such B in two steps: (i) find all affine subdiagrams Y of $S(A)$, and (ii) for each such Y, list out all the B's which occur as GCMs of linearly independent π-systems of Y.

3. We note that step (ii) above can in-principle be carried out using the results of [23] (see also [21, 11, 7]).

2.4 Hyperbolics and Overextensions

Let A be a symmetrizable GCM and $X = S(A)$ be its Dynkin diagram. If A is symmetric, we will call X *simply-laced*.

Definition 2.4.1. Let Z be a simply-laced Dynkin diagram. We say that Z is an *overextension* or of Ext type if there exists a vertex p in Z such that the subdiagram $Y = Z\backslash\{p\}$ is of affine type and $(\delta_Y \mid \alpha_p) = -1$.

We let Ext denote the set of overextensions. It is easy to see that the following is the complete list of overextensions, up to isomorphism:

$$A_n^{++}\ (n \geq 1),\ \ D_n^{++}\ (n \geq 4),\ E_n^{++}\ (n = 6, 7, 8)$$

(see Figure 2.4.1). Here, X_n^{++} has $n+2$ vertices. We remark that the corresponding GCMs are all nonsingular; hence a π-system of Ext type is necessarily linearly independent.

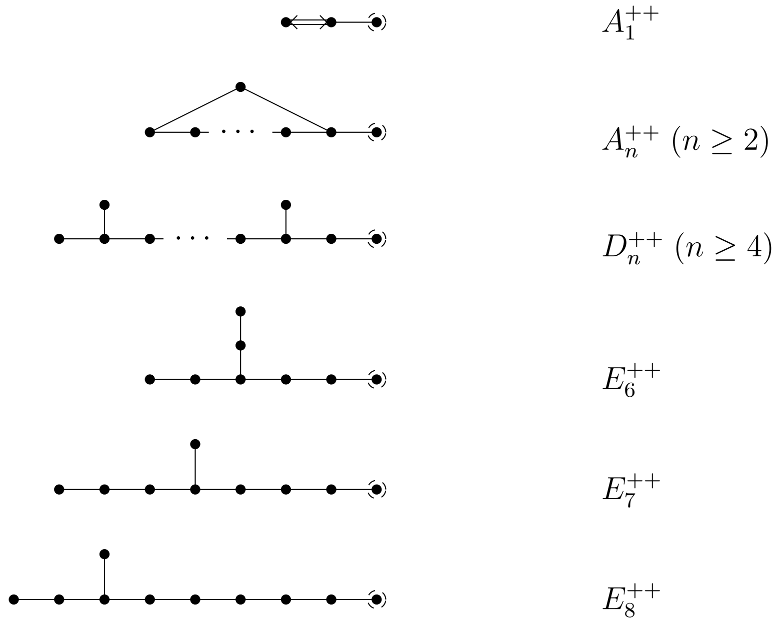


Figure 2.4.1: Ext type diagrams

2.4.1 Finite and affine part of overextension

From figure 2.4.1, one makes the important observation (via case-by-case check) that if Z is an overextension, then the vertex p satisfying the condition in definition 2.4.1 is unique. This vertex is marked by a dashed circle in figure 2.4.1.

We will call p the *overextended vertex* of Z, and Y the *affine part* of Z.

We had $(\delta_Y \mid \alpha_p) = -1$. Let $\delta_Y = \sum_{q \in Y} c_q \alpha_q$ with $c_q \in \mathbb{Z}_+$ for all q. Observing that $c_q\,(\alpha_q \mid \alpha_p) \leq 0$ for all q, it follows that: (i) There is a unique vertex q of Y such that $(\alpha_q \mid \alpha_p) \neq 0$, (ii) For this vertex, we have $c_q = 1$ and $(\alpha_q \mid \alpha_p) = -1$, (iii) In particular, this means q is a *special vertex* of the affine diagram Y (in the terminology of Kac, Chapter 6). Let Z° denote the finite type diagram obtained from Y by deleting q. We will call it the *finite part* of Z. We note that:

$$\delta_Y = \alpha_q + \theta_{Z^\circ}$$

where θ_{Z° denotes the highest root of Z°. It will be convenient to denote Y by $\widehat{Z^\circ}$.

The following trivial observation is useful: let X be a simply-laced Dynkin diagram and Z a diagram of Ext type. Suppose there exists π, a π-system of type Z in X; we let $\pi^\circ, \widehat{\pi^\circ}$ denote the subsets of π corresponding to the finite and affine parts of Z respectively. For any $w \in W(X)$, $w\pi$ is a π-system of type Z in X and $(w\pi)^\circ = w(\pi^\circ)$, $\widehat{w\pi^\circ} = w(\widehat{\pi^\circ})$.

2.4.2 Hyperbolics

We recall that an indecomposable, *symmetrizable* GCM A is said to be of *Hyberbolic type* if it is not of finite or affine type and every proper principal submatrix of A is a direct sum of finite or affine type GCMs.

There are finitely many GCMs of hyperbolic type in ranks 3-10 and infinitely many in rank 2. The former were enumerated, to varying degrees of completeness and detail, in [24, 5, 17]. More recently, this list was organized and independently verified in [2]. We will use this latter reference as our primary source for the Dynkin diagrams of hyperbolic type. Note that [2] does not require symmetrizability in the definition of a hyperbolic type GCM, so it contains 142 symmetrizable and 96 non-symmetrizable ones. We let Hyp denote the set of all symmetrizable GCMs of hyperbolic type of rank ≥ 3.

We recall from §2.3 the *subdiagram partial order* on the set of symmetrizable GCMs. We write $B \subseteq A$ if the Dynkin diagram $S(B)$ is a subdiagram of $S(A)$; equivalently B is a principal submatrix of A, possibly after a simultaneous permutation of its rows and columns. This is clearly a partial order, once we identify the matrices $\{PAP^T : P \text{ is a permutation matrix}\}$ with each other.

2.4.3 Simply laced hyperbolics

We now isolate the *symmetric* GCMs of hyperbolic type. By checking the classification case-by-case (see for instance [25, Tables 1,2] or [2]), one finds that these are either (i) of Ext type:

$$A_n^{++},\ (1 \leq n \leq 7),\quad D_n^{++},\ (4 \leq n \leq 8),\quad E_n^{++},\ (6 \leq n \leq 8) \tag{2.4.1}$$

or (ii) one of the diagrams in Figure 2.4.2, or (iii) one of the rank 2 symmetric GCMs $\begin{bmatrix} 2 & -a \\ -a & 2 \end{bmatrix}$ for $a \geq 3$. We observe by inspection of figure 2.4.1 that the diagrams in (ii) and (iii) do not contain a subdiagram of Ext type.

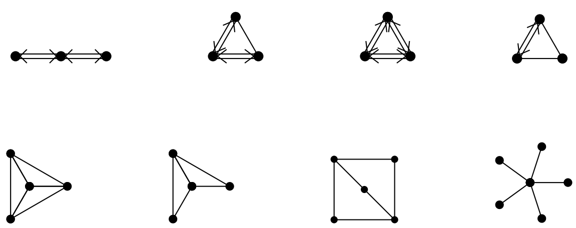


Figure 2.4.2: Simply-laced hyperbolics (ranks 3-10) that are not of Ext type.

The next lemma underscores the special role played by the hyperbolic overextensions. These are precisely the minimal elements of the set of overextensions relative to the partial order $\subseteq$.

Lemma 2.4.1.

$$\min(\mathrm{Ext}, \subseteq) = \mathrm{Ext} \cap \mathrm{Hyp}$$

Proof. Observe that $E_7^{++} \subseteq A_n^{++}$ for $n \geq 8$ and $E_8^{++} \subseteq D_n^{++}$ for $n \geq 9$. We are thus left with the diagrams of equation (2.4.1) as possible candidates for minimal elements. Now, each of these diagrams except D_8^{++} contains a unique subdiagram of affine type, obtained by removing a single vertex. So these diagrams cannot contain a proper subdiagram of Ext type. As for the diagram $Z = D_8^{++}$, it contains two subdiagrams of affine type, $Y_1 = E_8^{(1)}$ and $Y_2 = D_8^{(1)}$, obtained by deleting appropriate vertices p_1, p_2, but only the former satisfies $(\delta_Y \mid \alpha_p) = -1$ (this is -2 for the latter). Thus, D_8^{++} is also minimal. □

2.5 Weyl group orbits of π-systems of type A_1^{++}

In this section, we focus on the diagram A_1^{++}. The corresponding Kac-Moody algebra was first studied by Feingold and Frenkel [9].

We consider the problem of determining $\mathfrak{m}(A_1^{++}, X)$ for a simply-laced Dynkin diagram X. This is an important special case of the more general result of the next section. The latter result will be obtained by arguments similar to the ones used here, albeit with more notational complexity.

We begin with the following lemma which asserts that every Dynkin diagram of Ext type has a "canonical" π-system of type A_1^{++}.

Lemma 2.5.1. *Given a Dynkin diagram Z of* Ext *type, define:*

$$\pi(Z) := \{\theta_{Z^\circ}, \delta_Y - \theta_{Z^\circ}, \alpha_p\}$$

(notations Z°, Y, p, θ_{Z° are as defined in §2.4.1). Then $\pi(Z)$ is a linearly independent, positive π-system of type A_1^{++}.

Proof. We only need to show that the type of $\pi(Z)$ is A_1^{++}, the other assertions following from the observation that the three roots in $\pi(Z)$ are real, positive and have disjoint supports (cf. §2.4.1). Since Z is simply-laced, we normalize the form such that all real roots have norm 2. Thus

$$(\theta_{Z^\circ} \mid \delta_Y - \theta_{Z^\circ}) = -(\theta_{Z^\circ} \mid \theta_{Z^\circ}) = -2$$

It is clear from §2.4.1 that $(\theta_{Z^\circ} \mid \alpha_p) = 0$ and $(\delta_Y \mid \alpha_p) = -1$. This completes the verification. □

Theorem 2.5.2. *Let X be a simply-laced Dynkin diagram. Then:*

1. *X has a π-system of type A_1^{++} if and only if it contains a subdiagram of* Ext *type.*
2. *The number of $W(X)$-orbits of π-systems of type A_1^{++} in X is twice the number of such subdiagrams (and is, in particular, finite).*

Proof. In light of Theorem 2.2.1, any π-system of type A_1^{++} in X is $W(X)$-equivalent to a positive or a negative π-system , but not both. Thus, to prove the above theorem, it is sufficient to construct a bijection from the set of Ext type subdiagrams of X to $W(X)$-equivalence classes of *positive* π-systems of type A_1^{++} in X. We claim that the following map defines such a bijection:

$$Z \mapsto [\pi(Z)]$$

We will first establish the injectivity. Suppose Z_1, Z_2 are Ext type subdiagrams of X, with affine parts Y_1, Y_2 and overextended vertices p_1, p_2 respectively. Suppose

$\pi(Z_1) \sim \pi(Z_2)$ i.e., there exists $\sigma \in W(X)$ such that $\sigma(\pi(Z_1)) = \pi(Z_2)$. Consider the π-systems :

$$\pi_j = \{\theta_{Z_j^\circ}, \delta_{Y_j} - \theta_{Z_j^\circ}\}, \quad j = 1, 2.$$

We note that:

1. π_j is of type $A_1^{(1)}$.

2. π_j is supported in the affine subdiagram Y_j of X.

3. $\sigma(\pi_1) = \pi_2$.

Now, it follows from part (2) of theorem 2.3.2 that $Y_1 = Y_2$ and $\sigma \in W(Y_1 \sqcup Y_1^\perp)$. Since $p_1 \notin Y_1 \sqcup Y_1^\perp$, we can only have $\sigma\alpha_{p_1} = \alpha_{p_2}$ if $p_1 = p_2$. Thus, $Z_1 = Z_2$ as required.

Next, we turn to the surjectivity of this map. Let $\{\beta_{-1}, \beta_0, \beta_1\}$ be a positive π-system of X of type A_1^{++}. Since $\{\beta_0, \beta_1\}$ form a π-system of type $A_1^{(1)}$, which is affine, it follows from theorem 2.3.2 that there is a unique affine type subdiagram Y of X and an element $w \in W(X)$ such that $w\beta_i$ is supported in Y for $i = 0, 1$. Further (as in the proof of theorem 2.3.2), since $w(\beta_0 + \beta_1)$ is an isotropic root of $\mathfrak{g}(X)$, we must have $w(\beta_0 + \beta_1) = k\delta_Y$ for some *nonzero* integer k. Since $(\beta_0 + \beta_1 \mid \beta_{-1}) = -1$, we conclude $k = \pm 1$. But $\beta_0 + \beta_1 \in Q_+(X)$ by proposition 2.2.6, and $w^{-1}(\delta_Y) \in \Delta_+^{im}$ since δ_Y is a positive imaginary root. This implies $k = 1$.

Let $\beta_i' = w\beta_i$; thus β_0', β_1' are supported in Y, their sum equals δ_Y and $\left(\delta_Y \mid \beta_{-1}'\right) = -1$. We now need the following lemma:

Lemma 2.5.3. *Let X be a simply-laced Dynkin diagram, Y an affine subdiagram of X and β a real root of X satisfying $(\delta_Y \mid \beta) = -1$. Then there exists $\sigma \in W(Y \sqcup Y^\perp)$ such that $\sigma\beta$ is a simple root of X.*

We defer the proof of this lemma to the next subsection. Here, we use it to complete the proof of Theorem 2.5.2. We take $\beta = \beta'_{-1}$ in lemma 2.5.3. We obtain $\sigma \in W(Y \sqcup Y^{\perp})$ such that $\sigma\beta'_{-1} = \alpha_p$ for some vertex p of X. Define $Z := Y \cup \{p\}$. Since σ stabilizes δ_Y, we have $(\delta_Y \mid \alpha_p) = -1$; thus Z is of Ext type.

Since β'_0, β'_1 are supported in Y, so are $\sigma\beta'_0, \sigma\beta'_1$; further $\sigma\beta'_0 + \sigma\beta'_1 = \delta_Y$. Now

$$(\sigma\beta'_1, \alpha_p) = (\sigma\beta'_1, \sigma\beta'_{-1}) = 0.$$

This implies that $\sigma\beta'_1$ is supported in Z°. Since Z° is a simply-laced finite type diagram, all its real roots are conjugate under its Weyl group. Thus, there exists $\tau \in W(Z^\circ)$ such that $\tau\sigma\beta'_1 = \theta_{Z^\circ}$. Since τ stabilizes both δ_Y and α_p, we conclude that $\{\tau\sigma\beta'_i : i = -1, 0, 1\} = \pi(Z)$, as required. $\square$

2.5.1 Proof of lemma

We now turn to the proof of Lemma 2.5.3. We use the notations of the lemma. Since δ_Y is an antidominant weight of X, β must be a positive root. Further it is clear from $(\delta_Y \mid \beta) = -1$ that β must have the form:

$$\beta = \alpha_p + \sum_{q \in Y \sqcup Y^{\perp}} c_q(\beta)\alpha_q \tag{2.5.1}$$

where p is a vertex of X such that $(\delta_Y \mid \alpha_p) = -1$, and $c_q(\beta)$ are non-negative integers. Consider the $W(Y \sqcup Y^{\perp})$-orbit of β. Since the coefficient of α_p remains the same, any element γ of this orbit is a positive root that has the same form as the right hand side of (2.5.1) for some non-negative coefficients $c_q(\gamma)$. Let γ be a minimal height element of this orbit, i.e., one for which $\sum_q c_q(\gamma)$ is minimal. Then, we have: (i) $(\gamma \mid \alpha_q) \leq 0$ for all $q \in Y \sqcup Y^{\perp}$, since otherwise $s_q\gamma$ would have strictly smaller height, (ii) $(\gamma \mid \gamma) = (\alpha_p \mid \alpha_p)$ since all real roots have the same norm (X is simply-laced). We compute:

$$0 = (\gamma + \alpha_p \mid \gamma - \alpha_p) = \sum_{q \in Y \sqcup Y^\perp} c_q(\gamma)\ (\gamma + \alpha_p \mid \alpha_q)$$

Since $(\alpha_p \mid \alpha_q) \leq 0$, we conclude from (i) above that either $c_q(\gamma) = 0$ or $(\gamma \mid \alpha_q) = (\alpha_p \mid \alpha_q) = 0$ for each $q \in Y \sqcup Y^\perp$. If some $c_q(\gamma) \neq 0$, it would imply that γ has disconnected support, which is impossible since γ is a root. Thus, $\gamma = \alpha_p$ and the proof of the lemma is complete. □

2.5.2 Generalisation of lemma

We note that the key step in the proof above was showing that the set of all real roots β which have the form of equation (2.5.1) forms a single orbit under the standard parabolic subgroup $W(Y \sqcup Y^\perp)$ of W. In fact, those very same arguments prove a strengthened assertion. We formulate this below.

Given a Dynkin diagram X with simple roots α_i and given any α in its root lattice, we define the coefficients $c_i(\alpha)$ by:

$$\alpha = \sum_{i \in X} c_i(\alpha)\, \alpha_i$$

If J is a subdiagram of X, we define $\alpha_J = \sum_{i \in J} c_i(\alpha)\, \alpha_i$ and $\alpha_J^\dagger = \sum_{i \notin J} c_i(\alpha)\, \alpha_i$.

Proposition 2.5.4. *Let X be a symmetrizable Dynkin diagram with invariant bilinear form $(\cdot \mid \cdot)$ and simple roots α_i. Let J be a subdiagram of X, and fix a* nonzero *element $\zeta = \sum_{i \notin J} b_i\, \alpha_i$ of the root lattice of $X \backslash J$. Consider the set*

$$\mathcal{O} = \{\beta \in \Delta^{re}(X) : \beta_J^\dagger = \zeta \ \text{ and } \ (\beta \mid \beta) = (\zeta \mid \zeta)\}$$

Then:

1. *If ζ is a root of $\mathfrak{g}(X\backslash J)$, then $\mathcal{O} = W_J\,\zeta$ where W_J is the standard parabolic subgroup $\langle s_j : j \in J\rangle$ of W.*

2. *If ζ is not a root of $\mathfrak{g}(X\backslash J)$, then $\mathcal{O}$ is empty.*

Proof. Suppose $\mathcal{O}$ is non-empty, then ζ or $-\zeta$ lies in $Q_+(X\backslash J)$. We may assume the former case holds, so in fact $\mathcal{O} \subset \Delta_+^{re}(X)$. Since $\mathcal{O}$ is W_J-stable, it decomposes into W_J-orbits. Let $\mathcal{O}'$ denote one such orbit. let β denote an element of minimal height in $\mathcal{O}'$; as in the proof of Lemma 2.5.3, this implies $(\beta \mid \alpha_j) \leq 0$ for all $j \in J$; hence $(\beta \mid \alpha) \leq 0$ for all elements $\alpha \in Q_+(J)$. We now have $0 = (\beta + \zeta \mid \beta - \zeta) = \left(\beta + \beta_J^\dagger \mid \beta_J\right)$. But as observed already, $(\beta \mid \beta_J) \leq 0$; further $\left(\beta_J^\dagger \mid \beta_J\right) \leq 0$ since these elements have disjoint supports. This implies $(\beta \mid \beta_J) = \left(\beta_J^\dagger \mid \beta_J\right) = 0$. Suppose β_J is nonzero, the latter implies that $\beta = \beta_J + \beta_J^\dagger$ has disconnected support. Hence it cannot be a root. This contradiction shows $\beta_J = 0$, i.e., $\beta = \beta_J^\dagger = \zeta$. In particular, ζ is a root, and belongs to any W_J orbit in $\mathcal{O}$. Hence $\mathcal{O} = W_J\,\zeta$. □

Remark 2.5.1.
1. If X is simply-laced and J is a singleton, say $J = \{p\}$, and $\zeta = \alpha_p$, then $\mathcal{O}$ consists precisely of those real roots β of X which have the form of equation (2.5.1).

2. If X is of finite type and ζ is a root of $X\backslash J$, then Proposition 2.5.4 is a consequence of *Oshima's lemma* [22, Lemma 4.3], [7, Lemma 1.2].

2.5.3 Corollaries

We now have the following corollary of Theorem 2.5.2.

Corollary 2.5.5. *Let X be a Dynkin diagram of* Ext *type. Then:*

1. *If $X \in$* Hyp*, then there are exactly two π-systems of type A_1^{++} in X, up to*

$W(X)$-equivalence. In other words:

$$\mathfrak{m}(A_1^{++}, X) = 2 \text{ for } X = A_n^{++}\,(1 \leq n \leq 7),\ D_n^{++}\,(4 \leq n \leq 8),\ E_n^{++}\,(n = 6, 7, 8).$$

2. $\mathfrak{m}(A_1^{++}, A_8^{++}) = 6$, $\mathfrak{m}(A_1^{++}, A_n^{++}) = 10$ *for* $n \geq 9$.

3. $\mathfrak{m}(A_1^{++}, D_9^{++}) = 6$, $\mathfrak{m}(A_1^{++}, D_n^{++}) = 4$ *for* $n \geq 10$.

Proof: The first part follows from Lemma 2.4.1 and Theorem 2.5.2. For parts (2), (3), we need to count the number of subdiagrams of the ambient diagram which are of Ext type. We list these out in each case, leaving the easy verification to the reader.

1. A_8^{++}: one subdiagram of type A_8^{++} and two of type E_7^{++}.
2. A_n^{++} ($n \geq 9$): one subdiagram of type A_n^{++} and two each of types E_7^{++} and E_8^{++}.
3. D_9^{++}: one subdiagram of type D_9^{++} and two of type E_8^{++}.
4. D_n^{++} ($n \geq 10$): one subdiagram of type D_n^{++} and one of type E_8^{++}.

□

We also have the following result concerning the simply-laced hyperbolic diagrams not included in the previous corollary.

Corollary 2.5.6. *Let X be a simply-laced hyperbolic Dynkin diagram. If $X \notin \mathrm{Ext}$, then X does not contain a π-system of type A_1^{++}.*

Proof: This follows from the observation made in 2.4.3 that such diagrams do not contain subdiagrams of Ext type. □

Finally, we remark that Theorem 2.5.2 can be applied just as easily even when X is neither in Ext nor Hyp. For example, the diagram $X = E_{11}$, obtained by further extension of E_8^{++} [13] contains a unique subdiagram of Ext type, namely E_8^{++}. Thus, $\mathfrak{m}(A_1^{++}, E_{11}) = 2$.

2.6 The general case

Theorem 2.6.1. *Let X be a simply-laced Dynkin diagram and let K be a diagram of* Ext *type. Then:*

1. *There exists a π-system in X of type K if and only if there exists an* Ext *type subdiagram Z of X such that Z° has a π-system of type K°.*
2. *The number of $W(X)$ orbits of π-systems of type K in X is given by:*

$$\mathfrak{m}\left(K, X\right) = 2 \sum_{\substack{Z \subseteq X \\ Z \in \mathrm{Ext}}} \mathfrak{m}(K^\circ, Z^\circ) \tag{2.6.1}$$

where K°, Z° denote their finite parts.

We remark that equation (2.6.1) reduces the computation of the multiplicity of K in X to a sum of multiplicities involving only finite type diagrams. The latter, as mentioned earlier, are completely known [8]. Observe also that for $K = A_1^{++}$, K° is of type A_1. Since any Z° occurring on the right hand side of (2.6.1) is simply-laced, we have $\mathfrak{m}(K^\circ, Z^\circ) = 1$. So this reduces exactly to Theorem 2.5.2 in this case.

Corollary 2.6.2. *Let K be a Dynkin diagram of* Ext *type. Then,*

1. *$\mathfrak{m}(K, X)$ is finite for all simply-laced diagrams X.*

2. $\mathfrak{m}(K, X) = 2\,\mathfrak{m}(K^\circ, X^\circ)$ *for all* $X \in \mathrm{Hyp} \cap \mathrm{Ext}$.

We now prove theorem 2.6.1.

Proof. It is enough to prove the second part of the theorem. Now, by Theorem 2.2.1, any π-system in X of type K is either positive or negative, but not both. Consider the sets:

- $\mathcal{A}$: the set of $W(X)$-orbits of positive π-systems of type K in X;
- $\widehat{\mathcal{B}}$: the set of all pairs (Z, Σ) where Z is an Ext type subdiagram of X and Σ is a positive π-system of type K° in Z°.
- $\mathcal{B} = \widehat{\mathcal{B}}/\!\sim$, the equivalence classes of $\widehat{\mathcal{B}}$ under the equivalence relation defined by: $(Z, \Sigma) \sim (Z', \Sigma')$ iff $Z = Z'$ and Σ' is in the $W(Z^\circ)$-orbit of Σ.

Since $2|\mathcal{A}|$ and $2|\mathcal{B}|$ are the two sides of equation (2.6.1), it is sufficient to construct a bijection from the set $\mathcal{B}$ to $\mathcal{A}$. We first define a map from $\widehat{\mathcal{B}}$ to $\mathcal{A}$. Let $(Z, \Sigma) \in \widehat{\mathcal{B}}$. Let Z° and $\widehat{Z^\circ}$ denote the finite and affine parts of Z, and let p denote its overextended vertex. Since Σ is a π-system of type K° in Z°, we identify $\Delta(K^\circ)$ with a subset of $\Delta(Z^\circ)$ via corollary 2.1.10. Let θ_Σ denote the highest root in $\Delta(K^\circ)$ (identified with its image in $\Delta(Z^\circ) \subset Q(Z)$). Consider the set

$$\pi(Z, \Sigma) = \{\alpha_p, \delta_{\widehat{Z^\circ}} - \theta_\Sigma\} \cup \Sigma$$

It is straighforward to see that this is a π-system . Further, it is of type K.

We now claim that the map: $\widehat{\mathcal{B}} \to \mathcal{A}, \ \ (Z, \Sigma) \mapsto [\pi(Z, \Sigma)]$ factors through $\mathcal{B}$ and defines a bijection between $\mathcal{B}$ and $\mathcal{A}$.

Firstly, suppose $(Z, \Sigma) \sim (Z, \Sigma')$, i.e., $w\Sigma = \Sigma'$ for some $w \in W(Z^\circ)$. Since clearly $w\alpha_p = \alpha_p$, $w\delta_{\widehat{Z^\circ}} = \delta_{\widehat{Z^\circ}}$ and $\theta_{\Sigma'} = w\theta_\Sigma$, we conclude that $\pi(Z, \Sigma') = w\,\pi(Z, \Sigma)$. So the map does indeed factor through $\mathcal{B}$. We will now show it is an injection.

Suppose $(Z_i, \Sigma_i) \in \widehat{\mathcal{B}}$, $i = 1, 2$ are such that $[\pi(Z_1, \Sigma_1)] = [\pi(Z_2, \Sigma_2)]$, i.e., there exists $\sigma \in W(X)$ such that $\sigma(\pi(Z_1, \Sigma_1)) = \pi(Z_2, \Sigma_2)$. Let p_i denote the overextended vertex of Z_i.

Consider the π-systems :

$$\pi_j = \{\delta_{\widehat{Z_j^\circ}} - \theta_{\Sigma_j}\} \cup \Sigma_j, \quad j = 1, 2.$$

We note that: (i) π_j is of type $\widehat{K^\circ}$, (ii) π_j is supported in the affine subdiagram $\widehat{Z_j^\circ}$ of X, and (iii) $\sigma(\pi_1) = \pi_2$.

Now, it follows from part (2) of theorem 2.3.2 that $\widehat{Z_1^\circ} = \widehat{Z_2^\circ}$ and $\sigma \in W(\widehat{Z_1^\circ} \sqcup \widehat{Z_1^\circ}^\perp)$. Since $p_1 \notin \widehat{Z_1^\circ} \sqcup \widehat{Z_1^\circ}^\perp$, we can only have $\sigma\alpha_{p_1} = \alpha_{p_2}$ if $p_1 = p_2$. Thus, $Z_1 = Z_2$. We write $\sigma = \tau\tau'$ with $\tau \in W(\widehat{Z_1^\circ})$ and $\tau' \in W(\widehat{Z_1^\circ}^\perp)$.

Since $\sigma\pi_1 = \pi_2$, we obtain $\tau\,\Sigma_1 = \Sigma_2$ (in fact, $\tau\,\pi_1 = \pi_2$) since τ' fixes each element of π_1 pointwise. Further, $\sigma\alpha_{p_1} = \alpha_{p_1}$ implies that $\sigma \in W(\{p_1\}^\perp)$. In particular, $\tau \in W(\widehat{Z_1^\circ}) \cap W(\{p_1\}^\perp) = W(Z_1^\circ)$. Hence we obtain $(Z_1, \Sigma_1) \sim (Z_2, \Sigma_2)$, in other words, the map defined above is injective on $\mathcal{B}$.

Next, we show surjectivity of the map. Let π be a positive π-system in X of type K; we will show that $[\pi]$ is in the image of the map. Let $\pi^\circ, \widehat{\pi^\circ}$ be the subsets of π corresponding to the finite and affine parts of K respectively. Now, $\widehat{\pi^\circ}$ is a positive π-system of type $\widehat{K^\circ}$ in X. By theorem 2.3.2, there is an affine type subdiagram Y of X, and an element $w \in W(X)$ such that every element of (the positive π-system) $w(\widehat{\pi^\circ}) = \widehat{(w\pi)^\circ}$ is supported in Y. Since $[\pi] = [w\pi]$, let us replace π with $w\pi$ in what follows. Thus, π is a positive π-system of type K such that $\widehat{\pi^\circ}$ is supported in Y. Let $\beta \in \pi$ correspond to the overextended vertex of K, and let $\delta_{\widehat{\pi^\circ}}$ denote the null root of $\widehat{K^\circ}$, identified with its image in $\Delta(\widehat{\pi^\circ}) \subset \Delta(X)$. Thus $\delta_{\widehat{\pi^\circ}}$ (i) is a positive imaginary root of X (by corollary 2.1.10), (ii) is supported in Y, and (iii) satisfies $(\delta_{\widehat{\pi^\circ}} \mid \beta) = -1$. The first two conditions imply $\delta_{\widehat{\pi^\circ}} = r\delta_Y$ for some $r \geq 1$, while the third implies $r = 1$.

As in the proof of Theorem 2.5.2, we now appeal to Lemma 2.5.3 to find an element $\sigma \in W(Y \sqcup Y^\perp)$ such that $\sigma\beta = \alpha_p$ for some vertex p of X. Define $Z = Y \cup \{p\}$; this is clearly an Ext type subdiagram of X. Consider the positive π-system $\xi = \sigma\pi$ of type K. We have:

(a) $\alpha_p \in \xi$, (b) $\widehat{\xi^\circ}$ is supported in Y and (c) $\delta_{\widehat{\xi^\circ}} = \delta_Y$.

Further, $(\alpha \mid \beta) = 0$ for all $\alpha \in \pi^\circ$ gives us $(\sigma\alpha \mid \sigma\beta) = 0$, i.e., $(\alpha' \mid \alpha_p) = 0$ for all $\alpha' \in \xi^\circ$. This in turn implies that: (d) ξ° is supported in Z°.

From (a), (c) and (d) we conclude $\xi = \pi(Z, \xi^\circ)$. Since $[\pi] = [\xi]$ and ξ° is of type K°, the proof is complete.

$\square$

Chapter 3

The partial order $\preceq$

Let A, B be GCMs. We define $B \preceq A$ if there is a linearly independent π-system of type B in A. We now show that $\preceq$ defines a partial order on the set of symmetrizable hyperbolic GCMs (where we identify two GCMs that differ only by a simultaneous permutation of rows and columns). Clearly this relation is reflexive. By corollary 2.1.12 this relation is transitive. We now prove that this relation is anti-symmetric.

Lemma 3.0.1. *Let A be an $n \times n$ GCM (not necessarily symmetrizable). Let $\{\alpha_i\}_{i=1}^n$ be the simple roots of $\mathfrak{g}(A)$. Let $\{\beta_i\}_{i=1}^n$ be* any set of real roots *of $\mathfrak{g}(A)$. Let $\alpha_i^\vee, \beta_i^\vee$ denote the corresponding coroots. Consider the integer matrix: $B = [\langle \beta_i^\vee, \beta_j \rangle]_{ij}$. Then:*

1. $\det A$ *divides* $\det B$.

2. *Further if A, B are invertible with $|\det A| = |\det B|$, then $\{\beta_i\}_{i=1}^n$ and $\{\beta_i^\vee\}_{i=1}^n$ form $\mathbb{Z}$-bases of $Q(A)$ and $Q^\vee(A)$ respectively.*

PROOF: We write:

$$\beta_i^\vee = \sum_{k=1}^n u_{ik}\,\alpha_k^\vee$$
$$\beta_j = \sum_{\ell=1}^n v_{j\ell}\,\alpha_\ell$$

where $u_{ik}, v_{j\ell}$ are integers. Using the equations above, we compute:

$$B = UAV^T$$

where $U = [u_{ij}]$ and $V = [v_{ij}]$ are integer matrices. Taking determinants, we obtain $\det B = \det U \det V \det A$, proving the first assertion. For the second assertion, the given condition implies $|\det U| = |\det V| = 1$, i.e., U and V are in $\mathrm{GL}_n(\mathbb{Z})$. This is clearly equivalent to what needs to be shown. $\square$

Proposition 3.0.2. *Let A, B be $n \times n$ symmetrizable GCMs of hyperbolic type, with $\det A = \det B$. Suppose $\Sigma = \{\beta_i\}_{i=1}^n$ is a π-system of type B in A. Then Σ is $W(A)$-conjugate to $\Pi(A)$ or $-\Pi(A)$, where $\Pi(A)$ is the set of simple roots of $\mathfrak{g}(A)$. In particular, A and B are equal up to a simultaneous permutation of rows and columns.*

PROOF: Consider the map $q_{\Sigma} : Q(B) \to Q(A)$ of equation (2.1.2), defined by $\alpha_i(B) \mapsto \beta_i$ for all i, where $\Pi(B) = \{\alpha_i(B) : 1 \leq i \leq n\}$ is the set of simple roots of $\mathfrak{g}(B)$. We assume for convenience that the symmetric bilinear forms on $Q(A)$ and $Q(B)$ are chosen compatibly as in §2.1.3, so that q_{Σ} is form preserving (the arguments below will still work for any choices of standard invariant forms, since they only differ by scaling by positive rationals).

Using the given hypothesis and the fact that hyperbolic GCMs are necessarily invertible, we obtain from the second part of lemma 3.0.1 that: (i) Σ is a $\mathbb{Z}$-basis of $Q(A)$ and (ii) $\Sigma^\vee = \{\beta_i^\vee\}_{i=1}^n$ is a $\mathbb{Z}$-basis of $Q^\vee(A)$.

We observe from (i) above that q_{Σ} is a form preserving lattice isomorphism of $Q(B)$ onto $Q(A)$. We now claim that $q_{\Sigma}(\Delta(B)) = \Delta(A)$. Corollary 2.1.10 implies that $q_{\Sigma}(\Delta(B)) \subset \Delta(A)$. We only need to prove the reverse inclusion. Towards this end, we recall the following description of the set of roots of a symmetrizable Kac-Moody algebra $\mathfrak{g}(C)$ of Finite, Affine or Hyperbolic type [15, Prop 5.10]:

(3.0.1)

$$\Delta^{re}(C) = \{\alpha = \sum_j k_j\, \alpha_j(C)\, \in Q(C) : |\alpha|^2 > 0 \text{ and } k_j\, |\alpha_j(C)|^2/|\alpha|^2 \in \mathbb{Z} \text{ for all } j\}$$

(3.0.2)

$$\Delta^{im}(C) = \{\alpha \in Q(C)\backslash\{0\} : |\alpha|^2 \leq 0\}$$

forms where $\alpha_j(C)$ are the simple roots, $Q(C)$ is the root lattice, and we fix any standard invariant form on $\mathfrak{g}(C)$. We apply this when $C = A, B$ below.

Since $|q_{\Sigma}(\alpha)|^2 = |\alpha|^2$ for all $\alpha \in Q(B)$, it is clear from equation (3.0.2) that $q_{\Sigma}(\Delta^{im}(B)) = \Delta^{im}(A)$. Now let $\beta \in \Delta^{re}(A)$ and define $\alpha = q_{\Sigma}^{-1}(\beta)$. We need to prove that $\alpha \in \Delta^{re}(B)$. Let $\beta = \sum_j k_j \beta_j$ for some integers k_j; thus $\alpha = \sum_j k_j\, \alpha_j(B)$. Since β is a real root, $|\alpha|^2 = |\beta|^2 > 0$. Define

$$c_j = k_j\, |\alpha_j(B)|^2/|\alpha|^2 = k_j\, |\beta_j|^2/|\beta|^2$$

Equation (3.0.1) implies that α is a real root of $\mathfrak{g}(B)$ iff $c_j \in \mathbb{Z}$ for all j. Consider $\beta^\vee \in Q^\vee(A)$; by (ii) above, we know that $\Sigma^\vee$ forms a $\mathbb{Z}$-basis of the coroot lattice $Q^\vee(A)$. Now $\gamma^\vee = 2\nu^{-1}(\gamma)/|\gamma|^2$ for any real root γ of $\mathfrak{g}(A)$ [15, Prop. 5.1], where ν is the linear isomorphism from the Cartan subalgebra of $\mathfrak{g}(A)$ to its dual induced by the form. A simple computation now shows :

$$\beta^\vee = \sum_j c_j \, \beta_j^\vee$$

This proves the integrality of the c_j, and hence our claim.

Thus, $q_{\Sigma}(\Delta(B)) = \Delta(A)$. Since $q_{\Sigma}(\Pi(B)) = \Sigma$, this means that Σ is a *root basis* of $\Delta(A)$ [15, §5.9], i.e., Σ is a $\mathbb{Z}$-basis of $Q(A)$ such that every element of $\Delta(A)$ can be expressed as an integral linear combination of Σ with all coefficients of the same sign. By [15, Proposition 5.9], we conclude that Σ is $W(A)$-conjugate to $\pm\Pi(A)$.

Finally, since $\Pi(A)$ is a π-system of type A in A, we conclude that $A = B$, up to a simultaneous permutation of rows and columns. $\square$

Proposition 3.0.3. *Let $A_{n\times n}$ and $B_{m\times m}$ be symmetrizable GCMs of hyperbolic type such that $A \preceq B$ and $B \preceq A$. Then, $m = n$, and there exists a permutation matrix P such that $PAP^T = B$.*

Proof: Since $A \preceq B$, there exists a linearly independent π-system of type A in B; in particular, this implies $n \leq m$. Similarly, $m \leq n$, so we obtain $m = n$. Applying lemma 3.0.1, we conclude that $\det A \mid \det B$ and $\det B \mid \det A$, so in fact $\det A = \pm \det B$. Since hyperbolic GCMs have strictly negative determinant, we must have $\det A = \det B$. Proposition 3.0.2 completes the proof. $\square$

In other words, $\preceq$ is a partial order on the set of equivalence classes of hyperbolic GCMs, where we identify GCMs that differ by a simultaneous reordering of rows and columns. We restrict ourselves to the set Hyp comprising hyperbolic GCMs of rank ≥ 3. Then, we will determine the maximal elements of Hyp with respect to this partial order (up to equivalence).

3.1 Construction of π-systems

In this section we will develop some principles for constructing π-systems in a given Dynkin diagram. These are generalizations of the principles developed in [25] for simply-laced diagrams.

In fact, all our principles below are instances of the following simple, but powerful method of constructing π-systems .

General principle: Let X be the Dynkin diagram of a symmetrizable GCM. Let Λ denote a proper subdiagram of X and let Λ' be the subdiagram formed by the vertices not in Λ. Let Σ, Σ' be π-systems in Λ, Λ' respectively, *consisting of positive real roots*. Then $\Sigma \cup \Sigma'$ is a π-system in X.

This principle follows from the observations that (i) the (real) roots of a subdiagram are precisely the (real) roots of the ambient diagram that are supported on the subdiagram, (ii) the difference of two positive roots with disjoint supports will have coefficients of mixed sign, and can therefore not be a root. In all our applications below, we will always take Σ' to consist of the set of all simple roots of Λ'.

Observe that the GCM of $\Sigma \cup \Sigma'$ is of the form

(3.1.1)
$$\begin{bmatrix} B & * \\ * & B' \end{bmatrix}$$

where B, B' are the GCMs of Σ, Σ' respectively. The terms denoted $*$ are of the form $2(\beta_1 \mid \beta_2)/(\beta_2 \mid \beta_2)$ where $\beta_1 \in \Lambda, \beta_2 \in \Lambda'$ or vice versa. We now isolate some special instances of this general principle, which will be used repeatedly in the sequel.

3.1.1 Principle A:

Let Y be an affine Dynkin diagram, twisted or untwisted, but $Y \neq A_{2l}^{(2)}$. Let $\{\alpha_0, \cdots, \alpha_n\}$ denote the simple roots of Y. Let $\overline{Y}$ denote the underlying finite type diagram, obtained from Y by deleting the node corresponding to α_0.

Let X be the diagram obtained by adding an extra vertex to Y, which is connected only to α_0, and by a single edge. Since Y is symmetrizable, so is X. We denote the simple root corresponding to this vertex α_{-1}. Let $A = (a_{ij})$ denote the GCM of X; thus $a_{ij} = 2\,(\alpha_i \mid \alpha_j)/(\alpha_i \mid \alpha_i)$ for $-1 \leq i, j \leq n$.

We note in passing that when Y is simply-laced, X is of Ext type. Let δ_Y denote the null root of Y, so $\delta_Y = \sum_{i=0}^{n} a_i \alpha_i$ with $a_i \in \mathbb{N}$. We let s_i denote the reflection corresponding to the simple root α_i.

Since Y is an affine diagram other than $A_{2l}^{(2)}$, we have $a_0 = 1$ [15, Chapter 4, Tables Aff 1-3]. In the general principle, we take the subdiagram $\Lambda = Y$ and Λ' to be the singleton set containing the vertex (-1). Define Σ to be the π-system in Y of type Y comprising the roots $\{s_0\,\gamma_i : 0 \leq i \leq n\}$ where the γ_i are given by:

$$\gamma_0 = \alpha_0 + \delta_Y, \quad \gamma_j = \alpha_j \ (j \geq 1)$$

We note that when Y is twisted, α_0 is a short root and hence $\alpha_0 + \delta_Y$ is a root in this case; it is of course a root when Y is untwisted. Define $\Sigma' = \{\alpha_{-1}\}$; this is clearly of finite type A_1.

We let $\Sigma \cup \Sigma' = \{\beta_i : -1 \leq i \leq n\}$ with $\beta_{-1} = \alpha_{-1}$ and $\beta_i = s_0\,\gamma_i$ for $i \geq 1$. All the hypotheses of the general principle are satisfied. As observed in equation (3.1.1), to find the type of $\Sigma \cup \Sigma'$, it only remains to compute the numbers $b_{ij} = 2\,(\beta_i \mid \beta_j)/(\beta_i \mid \beta_i)$ where $i = -1, j \geq 0$ or vice-versa.

Now: (i) $(\beta_{-1} \mid \beta_j) = (s_0\,\beta_{-1} \mid \gamma_j) = (\alpha_0 \mid \alpha_j)$ for $j \geq 1$, since $s_0\,\alpha_{-1} = \alpha_0 + \alpha_{-1}$

and α_{-1} is orthogonal to all roots of $\overline{Y}$. (ii) $|\beta_{-1}|^2 = |\alpha_{-1}|^2 = |\alpha_0|^2$. This gives us: $b_{j,-1} = a_{j0}$ and $b_{-1,j} = a_{0j}$ for $j \geq 1$.

Finally, we compute: $(\beta_{-1} \mid \beta_0) = (\alpha_{-1} \mid s_0(\alpha_0 + \delta_Y))$. But $s_0(\alpha_0 + \delta_Y) = -\alpha_0 + \delta_Y = \theta$, where θ is the highest long (respectively short) root of $\overline{Y}$ if Y is untwisted (respectively twisted). But $(\alpha_{-1} \mid \theta) = 0$ since as before α_{-1} is orthogonal to all roots of $\overline{Y}$. In other words $b_{0,-1} = b_{-1,0} = 0$.

The Dynkin diagram $S(B)$ is thus obtained from $X = S(A)$ by removing the edge between vertices 0 and -1, and instead connecting the vertex -1 to every neighbour of 0 with the same edge labels, i.e., such that $b_{j,-1} = a_{j0}$ and $b_{-1,j} = a_{0j}$.

3.1.2 Principle B:

Let X be the Dynkin diagram of a symmetrizable GCM A and let Y denote a subset of its vertices such that Y forms a subdiagram of affine type. We set $r = 1$ if Y is untwisted, $r = 3$ if Y if of type $D_4^{(3)}$ and $r = 2$ for all other twisted types. Let δ_Y denote the null root of the diagram Y. In the general principle, we choose $\Lambda = Y$. For each $p \in Y$, fix a non-negative integer k_p; if α_p is a long root of Y, we require further that $r | k_p$ (for Y of type $A_{2n}^{(2)}$ this requirement only applies to the longest root length). Let $\beta_p = \alpha_p + k_p\,\delta_Y$ and define $\Sigma = \{\beta_p : p \in Y\}$; this is a π-system of type Y in Y. For $q \notin Y$, let $\beta_q = \alpha_q$ and define $\Sigma' = \{\beta_q : q \notin Y\}$. Then, by the general principle, $\Sigma \cup \Sigma'$ is a π-system in X. Let $B = (b_{ij})_{i,j \in X}$ denote its type. As above, $b_{ij} = a_{ij}$ whenever i, j are both in Y or both not in Y. To compute b_{pq} and b_{qp} for $p \in Y, q \notin Y$, we have:

$$(\beta_p, \beta_q) = (\alpha_p, \alpha_q) + k_p(\delta_Y, \alpha_q)$$

Hence $b_{pq} = a_{pq} + k_p \frac{2(\delta_Y, \alpha_q)}{(\alpha_p, \alpha_p)}$ and $b_{qp} = a_{qp} + k_p \frac{2(\delta_Y, \alpha_q)}{(\alpha_q, \alpha_q)}$. These can be explicitly computed in each case of interest.

While we will have occassion to use this principle in its full generality, we give below some special instances of it which occur often. Since Y is affine, we assume that the vertices of Y have the standard labelling $0, 1, \cdots, n$ as in [15, Chapter 4]. Suppose $X \backslash Y$ contains only a single vertex (labelled -1) which is connected by a single edge to the vertex 0 of Y.

(i) First let us suppose that Y is untwisted. Fix p such that $1 \leq p \leq n$. Choose $k_p = 1$ and $k_s = 0$ for all $0 \leq s \leq n$, $s \neq p$. We only need to compute b_{ij} for $i = -1, j \geq 0$ or vice-versa. Now, clearly $b_{-1,j} = a_{-1,j}$ and $b_{j,-1} = a_{j,-1}$ for $j \geq 0$, $j \neq p$. Further,

$$(\beta_p, \beta_{-1}) = (\alpha_p, \alpha_{-1}) + (\delta_Y, \alpha_{-1}) = (\alpha_0, \alpha_{-1}) = -\frac{|\alpha_0|^2}{2}$$

Since $|\beta_i|^2 = |\alpha_i|^2$ for all i, we conclude that $b_{-1,p} = -|\alpha_0|^2/|\alpha_{-1}|^2 = -1$ and $b_{p,-1} = -|\alpha_0|^2/|\alpha_p|^2$. Now since α_0 is a long root of Y, we obtain

$$b_{p,-1} = \begin{cases} -1 & \text{if } \alpha_p \text{ is a long root of } Y \\ -2 & \text{if } Y \neq G_2^{(1)}, \text{ and } \alpha_p \text{ is a short root of } Y \\ -3 & \text{if } Y = G_2^{(1)}, \text{ and } \alpha_p \text{ is a short root of } Y \end{cases}$$

In terms of Dynkin diagrams, the diagram $S(B)$ coincides with $S(A)$ except that there is a single, double or triple edge joining vertices -1 and p (with an arrow pointing towards p) depending on the three cases above.

(ii) If Y is twisted, fix a vertex $1 \leq p \leq n$ and define (i) $k_s = 0$ for $0 \leq s \leq n$, $s \neq p$ (ii) $k_p = r$ if α_p is a long root (longest root in case of $A_{2n}^{(2)}$) and $k_p = 1$ otherwise. As above we have: (a) $b_{ij} = a_{ij}$ for $i, j \neq p$, (b) $b_{ij} = a_{ij}$ for $i, j \neq -1$, (c) $b_{p,-1} = -1$ and (d) $b_{-1,p} = -|\alpha_p|^2/|\alpha_0|^2$. Since α_0 is a short root of Y, we have:

$$b_{-1,p} = \begin{cases} -1 & \text{if } \alpha_p \text{ is not a long root of } Y \\ -2 & \text{if } Y \neq D_4^{(3)}, \text{ and } \alpha_p \text{ is a long root of } Y \\ -3 & \text{if } Y = D_4^{(3)}, \text{ and } \alpha_p \text{ is a long root of } Y \end{cases}$$

As before, this implies that the diagram $S(B)$ coincides with $S(A)$ except that there is a single, double or triple edge joining vertices -1 and p (with an arrow pointing away from p) depending on the three cases above.

(iii) If instead of $1 \leq p \leq n$, we choose the vertex $p = 0$ in (i) or (ii) above, we obtain $b_{0,-1} = b_{-1,0} = -2$, and $b_{ij} = a_{ij}$ for all other pairs (i, j). In the Dynkin diagram $S(B)$, this would be denoted by a double edge between vertices 0 and -1, marked with two arrows, one pointing toward each vertex.

For principles **C, D, E**, we let X denote the Dynkin diagram of any symmetrizable GCM.

Principle C: (Shrinking) Suppose I is a subset of the vertices of X such that I forms a (connected) subdiagram of Finite type. It is well known that $\beta_\bullet = \sum_{i \in I} \alpha_i$ is a root of $\mathfrak{g}(I)$. Since I is of finite type, this root is real. In the general principle, we choose the subset $\Lambda = I$ and the π-system $\Sigma = \{\beta_\bullet\}$. Let $\Sigma' = \{\alpha_j : j \notin I\}$. Let B denote the GCM of $\Sigma \cup \Sigma'$. We have for $j \notin I$,

$$\frac{(\beta_\bullet, \alpha_j)}{(\alpha_j, \alpha_j)} = \sum_{i \in I} \frac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$$

Further, letting $k_i = |\alpha_i|^2 / |\beta_\bullet|^2$ for $i \in I$, we have

$$\frac{(\beta_\bullet, \alpha_j)}{(\beta_\bullet, \beta_\bullet)} = \sum_{i \in I} k_i \frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$$

Thus,

$$b_{j\bullet} = \sum_{i \in I} a_{ji}, \quad b_{\bullet j} = \sum_{i \in I} k_i \, a_{ij} \tag{3.1.2}$$

We note that k_i is the ratio of root lengths in a finite type diagram, and is therefore one of $\frac{1}{3}, \frac{1}{2}, 1, 2, 3$. If no two vertices of I have a common neighbour $j \notin I$, then the Dynkin diagram $S(B)$ may be thought of as being obtained from X by contracting the vertices of I to a single "fat" vertex $\bullet$. The edges in X between $i \in I$ and $j \notin I$ are now drawn between $\bullet$ and j in $S(B)$ (with possibly new edge weights). The rest of the diagram X is carried over unchanged.

Principle D: (Deletion) If we delete any subset of vertices from the vertex set of X and define Σ to be the set of remaining $\{\alpha_i\}$, then Σ is a π-system in X. Its Dynkin diagram is clearly a subdiagram of X.

Principle E:

(i) Let the vertices of X be labelled $1, 2, \cdots, n$. Suppose X contains a subdiagram of finite type B_2, i.e., there are vertices p, q in X joined by a double bond directed (say) towards p. In other words, $a_{pq} = -2, a_{qp} = -1$. In the general principle, we take Λ to be this subdiagram of type B_2 and define $\Sigma = \{\beta_p, \beta_q\}$ to be the π-system of type $A_1 \times A_1$ in Λ given by:

$$\beta_p = s_p(\alpha_q) = \alpha_q + 2\alpha_p, \quad \beta_q = \alpha_q.$$

Define $\beta_j = \alpha_j$ for $1 \leq j \leq n$, $j \neq p, q$ and let Σ' be the set of these β_j. Let B denote the GCM of $\Sigma \cup \Sigma' = \{\beta_i : 1 \leq i \leq n\}$; clearly $b_{ij} = a_{ij}$ for $i, j \neq p$. Now,

$$\frac{(\beta_p, \beta_j)}{(\beta_j, \beta_j)} = \frac{(\alpha_q, \alpha_j)}{(\alpha_j, \alpha_j)} + 2\frac{(\alpha_p, \alpha_j)}{(\alpha_j, \alpha_j)}, \text{ i.e., } b_{jp} = a_{jq} + 2a_{jp}$$

Since $|\alpha_q|^2 = 2|\alpha_p|^2$, we have

$$\frac{(\beta_p, \beta_j)}{(\beta_p, \beta_p)} = \frac{(\alpha_q, \alpha_j)}{(\alpha_q, \alpha_q)} + 2\frac{(\alpha_p, \alpha_j)}{2(\alpha_p, \alpha_p)} \text{ i.e, } b_{pj} = a_{qj} + a_{pj}$$

Note in particular that since Σ has type $A_1 \times A_1$, we have $b_{pq} = b_{qp} = 0$, i.e., the double edge between p, q in X has been removed in $S(B)$.

(ii) Now suppose the Dynkin diagram X has a subdiagram of finite type G_2, i.e., there are vertices p, q in X joined by a triple bond directed towards p. As above, choose Λ to be this subdiagram of type G_2 and define $\Sigma = \{\beta_p, \beta_q\}$ to be the π-system of type A_2 in Λ given by:

$$\beta_p = s_p(\alpha_q) = \alpha_q + 3\alpha_p, \quad \beta_q = \alpha_q.$$

Choose Σ' as above, to consist of all the simple roots α_i of X other than $i = p, q$. A similar computation establishes that $b_{jp} = a_{jq} + 3a_{jp}$, $b_{pj} = a_{qj} + a_{pj}$ and $b_{ij} = a_{ij}$ for all other pairs (i, j). Note in particular that since Σ is of type A_2, one has $b_{pq} = b_{qp} = -1$, i.e., the triple edge between p, q in X has now been replaced by a single edge in $S(B)$.

(iii) Suppose X contains a subdiagram of type $A_2^{(2)}$, i.e., there are vertices p, q in X with $a_{pq} = -4, a_{qp} = -1$ (depicted in the Dynkin diagram by four bonds directed towards p). We choose $\Sigma = \{\beta_p, \beta_q\}$ to be the π-system of type $A_1^{(1)}$ in Λ given by:

$$\beta_p = s_p(\alpha_q) = \alpha_q + 4\alpha_p, \quad \beta_q = \alpha_q.$$

Reasoning as before, we deduce $b_{jp} = a_{jq} + 4a_{jp}$, $b_{pj} = a_{qj} + a_{pj}$ and $b_{ij} = a_{ij}$ for all other pairs (i, j). Here, since Σ has type $A_1^{(1)}$, the quadruple edge from q to p has been replaced by a two-way double edge.

3.2 Non-Maximal Hyperbolic Diagrams

In Tables 3.1-3.10, we have listed the 142 symmetrizable hyperbolic Dynkin diagrams in ranks 3-10. We will denote by Γ_k the hyperbolic Dynkin diagram occurring with serial number k in these tables. These diagrams are taken from Tables 1–23 of [2] which contain the full list of 238 hyperbolic diagrams without the assumption of symmetrizability. The diagram Γ_k occurs as item number k in Tables 1–23 of [2]. Since we only consider the 142 symmetrizable hyperbolic diagrams rather than all 238 of them, there are "gaps" in the serial numbers that occur in our tables.

The entries in our tables contain the following information: for each serial number k, the second column is the corresponding Dynkin diagram, the third column is another serial number, say ℓ such that $\Gamma_k \preceq \Gamma_\ell$ and the fourth column indicates the principle(s) used to construct a π-system of type Γ_k in Γ_ℓ. We note that ℓ is not unique in general, but since our primary goal is to identify the maximal diagrams relative to $\preceq$, we will be content with finding one value of ℓ.

The diagrams Γ_k for which we are unable to find a suitable ℓ using any of our principles are candidates for maximal elements. We show in §3.4 that each of these diagrams is indeed maximal. The entries corresponding to these diagrams are indicated by 'Max' in the third column while the fourth column contains the value of the determinant of the GCM of the diagram.

In this section we give a few examples to illustrate the Principles A-E developed in the previous section. The other entries of the table may be verified by similar arguments.

Principle A: Taking $X = \Gamma_{219}$ and $Y = F_4^{(1)}$ in principle A, we obtain a π-system of type Γ_{207} in Γ_{219}. Similarly, choosing $X = \Gamma_{159}$ and $Y = G_2^{(1)}$, we obtain $\Gamma_{150} \preceq \Gamma_{159}$.

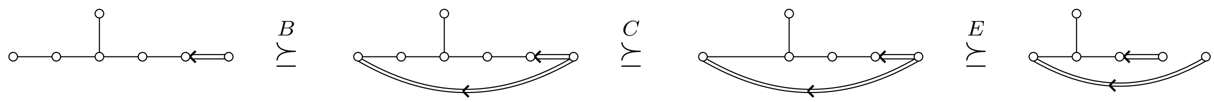
Principle B: Let $X = \Gamma_{159}$, $Y = G_2^{(1)}$ and α_p be the long simple root of G_2. Applying principle B allows us to construct a π-system of type Γ_{129} in Γ_{159}. Similarly, taking $X = \Gamma_{160}$, Y to be the twisted affine diagram $D_4^{(3)}$ and α_p to be the short simple root of G_2, we conclude that $\Gamma_{130} \preceq \Gamma_{160}$.

Principle C: Principle C allows us to shrink diagrams in a specified manner. For instance, one readily obtains from this principle that: $\Gamma_{222} \preceq \Gamma_{226} \preceq \Gamma_{231} \preceq \Gamma_{236}$.

Principle D: Typically the deletion principle D is used in conjunction with one of the other principles. For instance, first applying principle B to $X = \Gamma_{163}$, $Y = D_3^{(2)}$ and $p = 0$ (i.e., the affine simple root of Y) one obtains the rank 4 diagram obtained from Γ_{163} by replacing its single edge by the two-way double edge $\Longleftrightarrow$. Now applying principle D to delete the node at the other end gives us Γ_{106}.

Principle E: This principle only applies when the ambient diagram has a double, triple or quadruple edge. For example, an application of this principle shows there exists a π-system of type Γ_{220} in Γ_{218} and one of type Γ_{161} in Γ_{160}.

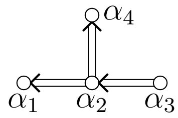
We close this subsection with the example of $\Gamma_{223} \succeq \Gamma_{212}$ which requires a sequential application of the three principles B, C and E:



3.2.1 The exceptions : principle (*)

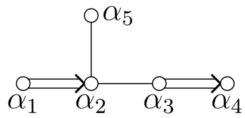
As mentioned above, for each non-maximal diagram Γ_k, Principles A-E can typically be used to exhibit a diagram Γ_ℓ such that $\Gamma_k \preceq \Gamma_\ell$. However, there are four non-maximal diagrams which are not directly amenable to any of these principles. We give below special constructions in these cases.

(i) $\Gamma_{91} \preceq \Gamma_{157}$: Consider the Dynkin diagram Γ_{157}:



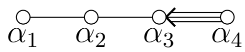
The π-system $\Sigma = \{\alpha_1 + \alpha_2, \alpha_3, \alpha_1 + \alpha_2 + 2\alpha_4\}$ is of type Γ_{91}.

(ii) $\Gamma_{158} \preceq \Gamma_{191}$: Consider the Dynkin diagram Γ_{191}:



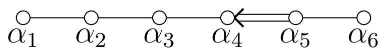
The π-system $\Sigma = \{\alpha_1, \alpha_1 + 2\alpha_2, \alpha_5 + \alpha_2 + \alpha_3, \alpha_4\}$ is of type Γ_{158}.

(iii) $\Gamma_{172} \preceq \Gamma_{160}$: Consider the Dynkin diagram Γ_{160}:



The π-system $\Sigma = \{\alpha_1 + \alpha_2 + \alpha_3,\ \alpha_4,\ \alpha_4 + 3\alpha_3,\ \alpha_2\}$ is of type Γ_{172}.

(iv) $\Gamma_{214} \preceq \Gamma_{218}$: Consider the Dynkin diagram Γ_{218}:



The π-system $\Sigma = \{\alpha_1, \alpha_2, \alpha_5 + 2\alpha_4 + 2\alpha_3, \alpha_6, \alpha_5, \alpha_4\}$ is of type Γ_{214}.

3.3 Non-existence of π-systems

In this section, we give a few simple criteria that can be used to demonstrate the non-existence of π-systems of certain types in an ambient Lie algebra.

The following is an immediate corollary of the discussion of §2.1.3, together with the fact that a real root is Weyl conjugate to some simple root, and therefore has the same length.

Lemma 3.3.1. (Root length criterion) *Let A, B be indecomposable symmetrizable GCMs such that $B \preceq A$. For each pair of simple roots of B, the ratio of their lengths equals that of some pair of simple roots of A (with respect to any choices of standard invariant forms on $\mathfrak{g}(A)$ and $\mathfrak{g}(B)$).*

For instance, this implies that there doesn't exist a π-system of type G_2 in any other finite type GCM.

The next result follows directly from lemma 3.0.1, proposition 3.0.2 and the fact that hyperbolic GCMs have strictly negative determinant. It has been extracted here as a separate statement on account of its wider applicability.

Lemma 3.3.2. (Determinant criterion) *Let A, B be symmetrizable hyperbolic GCMs of the same size. If $B \preceq A$ and $B \neq A$ (up to simultaneous reordering of rows and columns), then* $\det B = k \det A$ *for some* $k \geq 2$.

Let X be the Dynkin diagram of a symmetrizable Kac-Moody algebra and let W denote its Weyl group. We define X_{short} to be the subdiagram formed by the simple roots of shortest length, i.e,

$$X_{\text{short}} = \{p \in X : |\alpha_p| = \min_{i \in X} |\alpha_i|\}$$

Similarly X_{long} is the subdiagram formed by the simple roots of longest length. We also let

$$\Delta^{re}_{\text{short}}(X) = \{\alpha \in \Delta^{re}(X) : |\alpha| = \min_{i \in X} |\alpha_i|\} = W \cdot X_{\text{short}}$$

and $\Delta^{re}_{\text{long}}(X) = W \cdot X_{\text{long}}$.

We say X is *doubly-laced* if X contains only single or double edges (with arrows) and *triply-laced* if it contains only single and triple edges (with arrows). The next lemma is a direct consequence of these definitions.

Lemma 3.3.3. *Let X be a doubly- or triply-laced Dynkin diagram (we set $d = 2$ in the former case, $d = 3$ in the latter). Then:*

1. $d \mid \langle \alpha_i^\vee, \alpha_j \rangle$ *for all* $i \in X_{\text{short}}$, $j \in X \backslash X_{\text{short}}$.

2. $d \mid \langle \alpha_j^\vee, \alpha_i \rangle$ *for all* $i \in X_{\text{long}}$, $j \in X \backslash X_{\text{long}}$.

Now consider π-systems Σ in X such that $\Sigma \subset \Delta^{re}_{\text{short}}$ or $\Sigma \subset \Delta^{re}_{\text{long}}$. We seek to understand the possible types of such Σ. The proposition of the next subsection is the important result that will enable us to answer this question. This proposition is vastly more general and can be applied to a wide variety of settings.

Proposition 3.3.4. *Let X be the Dynkin diagram of a symmetrizable Kac-Moody algebra, Y a subdiagram of X and $d \geq 2$ an integer. Suppose that either:*

$$d \mid \langle \alpha_j^\vee, \alpha_i \rangle \text{ for all } i \in Y,\ j \in X \backslash Y, \text{ or} \tag{3.3.1}$$

$$d \mid \langle \alpha_i^\vee, \alpha_j \rangle \text{ for all } i \in Y,\ j \in X \backslash Y. \tag{3.3.2}$$

Let $\Sigma = \{\beta_i : 1 \leq i \leq m\}$ be a multiset with $\beta_i \in W \cdot \Delta^{re}(Y)$. Then, there exists a multiset $\overline{\Sigma} = \{\overline{\beta}_i : 1 \leq i \leq m\}$ with $\overline{\beta}_i \in \Delta^{re}(Y)$ such that

$$M(\Sigma) \equiv M(\overline{\Sigma}) \pmod d$$

Proof: Let s_i denote the simple reflection corresponding to the vertex $i \in X$ and let $W(Y)$ be the (standard parabolic) subgroup of W generated by the $\{s_i : i \in Y\}$. The given hypothesis implies by [15, Prop 3.13] that for each $i \in Y$, $j \in X \backslash Y$, $(s_i s_j)^{m_{ij}} = 1$ where $m_{ij} = 2, 4, 6$ or ∞. Since these are even (or

∞), it follows that the map $W \to W(Y)$ defined on the generators by:

$$s_i \mapsto \begin{cases} s_i & i \in Y \\ 1 & i \in X\backslash Y \end{cases}$$

extends to a group homomorphism. We denote it $w \mapsto \overline{w}$.

Let $Q(X)$, $Q^\vee(X)$ denote the root and coroot lattices of X. We define sublattices $R, R^\vee$ as follows. If (3.3.1) holds, then $R := d\,Q(X)$, and

$$R^\vee := d\,Q^\vee(Y) \oplus Q^\vee(X\backslash Y) = \bigoplus_{i\in Y} \mathbb{Z}\,(d\alpha_i^\vee) \;\oplus\; \bigoplus_{j\notin Y} \mathbb{Z}\alpha_j^\vee$$

If (3.3.2) holds, then

$$R := d\,Q(Y) \oplus Q(X\backslash Y) \quad \text{and} \quad R^\vee = d\,Q^\vee(X)$$

The given hypotheses readily imply that R and $R^\vee$ are W-invariant. We now make the following important observation:

(3.3.3)

Given $(w,\alpha) \in W \times \Delta^{re}(Y)$, we have $w\alpha \in \overline{w}\alpha + R$ and $w(\alpha^\vee) \in \overline{w}(\alpha^\vee) + R^\vee$

It is enough to prove this on the generators $w = s_k$ of W. This is obvious when $k \in Y$ and follows from equations (3.3.1), (3.3.2) when $k \in X\backslash Y$.

Now, given $\beta \in W \cdot \Delta^{re}(Y)$, say $\beta = \sigma\alpha$ for some $(\sigma,\alpha) \in W \times \Delta^{re}(Y)$, we define $\overline{\beta} := \overline{\sigma}\alpha$. This is a real root of Y, and in view of (3.3.3) above, the association $\beta \mapsto \overline{\beta}$ is well-defined modulo R. Further, if $\gamma = \tau\alpha'$ is another root in the W-orbit of $\Delta^{re}(Y)$, then

$$\langle\, \overline{\beta}^\vee, \overline{\gamma} \,\rangle = \langle\, \overline{\sigma}(\alpha^\vee),\, \overline{\tau}\alpha' \,\rangle \equiv \langle\, \sigma(\alpha^\vee),\, \tau\alpha' \,\rangle \pmod{d} \tag{3.3.4}$$

The congruence modulo d in this equation is an easy consequence of equation (3.3.3), together with the observations that

$$\langle\, Q^\vee(X)\,,\, R\,\rangle \equiv \langle\, R^\vee\,,\, Q(Y)\,\rangle \equiv 0 \pmod{d} \qquad \text{if equation (3.3.1) holds.}$$

$$\langle\, R^\vee\,,\, Q(X)\,\rangle \equiv \langle\, Q^\vee(Y)\,,\, R\,\rangle \equiv 0 \pmod{d} \qquad \text{if equation (3.3.2) holds.}$$

Finally, if $\Sigma = \{\beta_i : 1 \leq i \leq m\}$ is a multi-subset of $W \cdot \Delta^{re}(Y)$, define $\overline{\Sigma} = \{\overline{\beta}_i : 1 \leq i \leq m\}$. Equation (3.3.4) now implies $M(\Sigma) \equiv M(\overline{\Sigma}) \pmod{d}$ as required. □

We obtain several useful corollaries.

Corollary 3.3.5. *Let X be a doubly-laced Dynkin diagram. Suppose that X_{short} (respectively X_{long}) is of type A_1, i.e., is a single vertex, then there is no π-system of type A_2 in X contained wholly in $\Delta^{re}_{\text{short}}(X)$ (respectively $\Delta^{re}_{\text{long}}(X)$).*

Corollary 3.3.6. *Let X be a doubly-laced Dynkin diagram. Suppose that X_{short} (respectively X_{long}) is of type A_2, then there is no π-system of type $A_2 \times A_1$ in X contained wholly in $\Delta^{re}_{\text{short}}(X)$ (respectively $\Delta^{re}_{\text{long}}(X)$).*

Corollary 3.3.7. *Let X be a triply-laced Dynkin diagram. Suppose that X_{short} (respectively X_{long}) is of type A_1, then there is no π-system of type $A_1 \times A_1$ in X contained wholly in $\Delta^{re}_{\text{short}}(X)$ (respectively $\Delta^{re}_{\text{long}}(X)$).*

We indicate how to prove Corollary 3.3.6, the others being similar. Lemma 3.3.3 allows us to apply Proposition 3.3.4 with $Y = X_{\text{short}}$ (or X_{long}) and $d = 2$. The set of shortest (or longest) real roots of X is nothing but $W \cdot \Delta^{re}(Y)$. Given any π-system (in fact any multiset of real roots) Σ of X contained wholly in the Weyl group orbit of $\Delta^{re}(Y)$, we obtain the multisubset $\overline{\Sigma}$ of $\Delta^{re}(Y)$ such that $M(\Sigma)$ coincides with $M(\overline{\Sigma})$ modulo $d = 2$. For Y of type A_2, it only remains to verify that no such multisubset exists if we take $M(\Sigma)$ to be the GCM of type $A_2 \times A_1$, i.e., the matrix

$$M = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

So let $\overline{\Sigma} = \{\overline{\beta}_1, \overline{\beta}_2, \overline{\beta}_3\}$ be such that $M(\overline{\Sigma})$ is congruent to M mod 2. We observe that the root system of type A_2 has the property that given two (real) roots α, β, we have $\langle\, \alpha^\vee, \beta \,\rangle$ is even iff $\beta = \pm\alpha$. Since the third row and column of M is zero mod 2, we conclude that $\overline{\beta}_1$ and $\overline{\beta}_2$ must both be of the form $\pm\overline{\beta}_3$. But this would imply $\langle\, \beta_1^\vee, \beta_2 \,\rangle$ is also even, which is a contradiction. $\square$

The following two lemmas are more restrictive in scope, in that they only apply when the ambient Lie algebra is of finite, affine or (symmetrizable) hyperbolic type, and only to the case of shortest roots.

Lemma 3.3.8. *Suppose X is a triply-laced Dynkin diagram of finite, affine or hyperbolic type. Suppose X_{short} is of type A_1, then there is no π-system of type A_2 in X contained wholly in $\Delta^{re}_{\text{short}}(X)$.*

Proof. Let p denote the vertex of X such that $X_{\text{short}} = \{p\}$. We normalize the standard invariant form on X such that $|\alpha_p|^2 = \min_{j \in X} |\alpha_j|^2 = 1$. Since X is triply-laced, $|\alpha_j|^2$ is a nonzero power of 3 for all $j \neq p$. Now suppose $\Sigma = \{\beta_1, \beta_2\}$ is a π-system of type A_2 in X such that $\Sigma \subset \Delta^{re}_{\text{short}}(X) = W\alpha_p$ (the Weyl group orbit of α_p). Applying an element of W if necessary, we can assume $\beta_1 = \alpha_p$. By the arguments used in the proof of Proposition 3.3.4, specifically equation (3.3.3), we obtain:

$$\beta_2 = \pm\alpha_p + \gamma$$

for some $\gamma \in R$, where $R = \mathbb{Z}\,(3\alpha_p) \oplus \bigoplus_{j \neq p} \mathbb{Z}\alpha_j$. Thus

$$\langle \beta_2, \beta_1^\vee \rangle = \pm\langle \alpha_p, \alpha_p^\vee \rangle + \langle \gamma, \alpha_p^\vee \rangle \in \pm 2 + 3\,\mathbb{Z}$$

since $\langle \alpha_j, \alpha_p^\vee \rangle = 0$ or -3 for all $j \neq p$. Now Σ has type A_2, so $\langle \beta_2, \beta_1^\vee \rangle = -1$. We must have $\beta_2 = \alpha_p + \gamma$, with $\langle \gamma, \alpha_p^\vee \rangle = -3$. We compute:

$$|\beta_2|^2 = |\alpha_p|^2 + |\gamma|^2 + 2\,(\alpha_p \mid \gamma) = |\alpha_p|^2 + |\gamma|^2 + \langle\, \gamma, \alpha_p^\vee \,\rangle$$

since $|\alpha_p|^2 = 1$. Since β_2 is W-conjugate to α_p, their norms coincide, and we obtain $|\gamma|^2 = -\langle\, \gamma, \alpha_p^\vee \,\rangle = 3$. We write

$$\gamma = 3k_p\, \alpha_p + \sum_{j \neq p} k_j\, \alpha_j$$

where the $k_\bullet$ are integers. We observe that $\frac{3k_p|\alpha_p|^2}{|\gamma|^2} = k_p \in \mathbb{Z}$. For $j \neq p$, $\frac{k_j|\alpha_j|^2}{|\gamma|^2} = \frac{k_j|\alpha_j|^2}{3} \in \mathbb{Z}$ since 3 divides $|\alpha_j|^2$. Since X is of finite, affine or hyperbolic type, we use equation (3.0.1) to conclude that γ is a real root of X. But $\gamma = \beta_2 - \beta_1$, which contradicts the fact that Σ is a π-system . □

Lemma 3.3.9. *Suppose X is a doubly-laced Dynkin diagram of finite, affine or hyperbolic type. Suppose X_{short} is of type A_2, then there is no π-system of type $A_1 \times A_1$ in X contained wholly in $\Delta^{re}_{\text{short}}(X)$.*

Proof. Let $X_{\text{short}} = \{p, q\}$ and let $\{\beta_1, \beta_2\}$ be two elements in the W-orbit of $\{\alpha_p, \alpha_q\}$ which form a π-system of type $A_1 \times A_1$. Applying an element of W and interchanging p, q if necessary, we can assume $\beta_1 = \alpha_p$. By the arguments used in the proof of Proposition 3.3.4, we obtain:

$$\beta_2 = \alpha + \gamma$$

for some $\alpha \in \Delta^{re}(X_{\text{short}})$ and $\gamma \in R$ where $R = 2\,Q(X_{\text{short}}) \oplus Q(X \backslash X_{\text{short}})$. We have

$$0 = \langle\, \beta_2, \beta_1^\vee \,\rangle = \langle\, \alpha, \alpha_p^\vee \,\rangle + \langle\, \gamma, \alpha_p^\vee \,\rangle \in \langle\, \alpha, \alpha_p^\vee \,\rangle + 2\mathbb{Z}$$

As in the proof of Corollary 3.3.6, we note that $\langle\, \alpha, \alpha_p^\vee \,\rangle$ is even iff $\alpha = \pm\alpha_p$. Since

$\alpha_p \equiv -\alpha_p \pmod{R}$, we may assume $\beta_2 = \alpha_p + \gamma$. We conclude $\langle \gamma, \alpha_p^\vee \rangle = -2$. Normalizing the standard invariant form such that $|\alpha_p|^2 = |\alpha_q|^2 = 1$, we compute: $|\beta_2|^2 = |\alpha_p|^2 + |\gamma|^2 + \langle \gamma, \alpha_p^\vee \rangle$. As before, this implies $|\gamma|^2 = -\langle \gamma, \alpha_p^\vee \rangle = 2$. Letting:

$$\gamma = 2k_p\,\alpha_p + 2k_q\,\alpha_q + \sum_{j \neq p,q} k_j\,\alpha_j$$

we obtain: (i) $\frac{2k_p|\alpha_p|^2}{|\gamma|^2} = k_p \in \mathbb{Z}$, (ii) $\frac{2k_q|\alpha_q|^2}{|\gamma|^2} = k_q \in \mathbb{Z}$, and (iii) $\frac{k_j|\alpha_j|^2}{|\gamma|^2} = \frac{k_j|\alpha_j|^2}{2} \in \mathbb{Z}$ for each $j \neq p, q$, since in this case $|\alpha_j|^2$ is a nonzero power of 2. Equation (3.0.1) implies γ is a real root of X, contradicting the fact that $\{\beta_1, \beta_2\}$ was a π-system to begin with. $\square$

3.3.1 Remark

We note that both the above lemmas do not hold if 'short' is replaced by 'long'. For example:

1. If $X = G_2$, then X_{long} is of type A_1. But the set of all long roots forms a *closed subroot system* isomorphic to A_2; a π-system of type A_2 in G_2 consisting entirely of long roots is $\{\alpha_1, \alpha_1 + 3\alpha_2\}$ where α_1, α_2 are respectively the long and short simple roots of G_2.

2. If $X = B_3$, then $X_{\text{long}} = \{p, q\}$ (say) is of type A_2. Consider $\Sigma = \{-\theta\} \cup \{\alpha_p, \alpha_q\}$ where θ is the highest root of X. This forms a π-system consisting entirely of long roots; it has type A_3, and hence contains a subsystem of type $A_1 \times A_1$.

3.4 Maximal Hyperbolic diagrams

In this section, we consider the 22 symmetrizable hyperbolic diagrams Γ_k which cannot be exhibited as π-systems of other diagrams using Principles A-E. Such diagrams only exist in ranks 3, 4, 6 and 10 and there are 5, 9, 5 and 3 such diagrams (respectively) in those ranks. We will prove that these are all in fact maximal diagrams relative to the partial order $\preceq$. As mentioned in §3.2, the entries corresponding to these diagrams are labelled 'Max' in the third column and contain the determinant of their GCMs in the fourth.

3.4.1 Rank 10

Since $\det \Gamma_{238} = -1$, it is maximal by the determinant criterion (lemma 3.3.2). The same lemma shows that Γ_{236} and Γ_{237} are not $\preceq$ comparable. Both these latter diagrams have two root lengths, while Γ_{238} has only one, so the root length criterion (lemma 3.3.1) shows that neither of them can be $\preceq \Gamma_{238}$. Thus all three are maximal diagrams of rank 10.

3.4.2 Rank 6

Since Γ_{218} and Γ_{219} have determinant -1, they are both maximal among rank 6 diagrams by the determinant criterion. The root length criterion ensures that neither of these is $\preceq \Gamma_{238}$, so to show maximality of these two diagrams, it only remains to prove that neither of them can be realized as π-systems of Γ_{236} or Γ_{237}. But this follows readily from corollary 3.3.5.

Diagrams Γ_{216} and Γ_{217} have three root lengths. By the root length criterion they cannot be realized as π-systems of any of the rank 10 maximal diagrams or of the other candidate diagrams Γ_k ($k = 215, 218, 219$) in rank 6. Since each of these two diagrams have determinant -2, they are mutually incomparable by the determinant criterion. This establishes maximality of Γ_{216} and Γ_{217}.

Finally to show maximality of Γ_{215}, we observe that it cannot be realized as a π-system of: (i) Γ_k for $k = 236, 237$ by corollary 3.3.5 (ii) Γ_{238} by the root length criterion (iii) Γ_k for $k = 216, 217$ by the determinant criterion (iv) Γ_{218} by corollary 3.3.6 (v) Γ_{219} by lemma 3.3.9.

3.4.3 Rank 4

Since $\det \Gamma_{159} = \det \Gamma_{160} = -1$, they are maximal amongst rank 4 diagrams. Since both these diagrams are triply laced, they contain a pair of simple roots α_i, α_j such that $|\alpha_i|^2/|\alpha_j|^2 = 3$. However none of the maximal diagrams in rank 6 or 10 have triple edges, so the root length criterion ensures that neither of $\Gamma_{159}, \Gamma_{160}$ occur as π-systems of those diagrams. Hence Γ_{159} and Γ_{160} are maximal.

The root length criterion shows that Γ_{173} is maximal since it contains 4 root lengths. It also shows that none of the Γ_k for $166 \leq k \leq 170$ can be realized as π-systems of Γ_{159} or Γ_{160} or of any of the maximal diagrams of ranks 6 or 10. Since $\det \Gamma_k = -2$ or -3 for $166 \leq k \leq 170$, the determinant criterion implies they are pairwise incomparable. This establishes their maximality.

Finally to show maximality of Γ_{171}, we observe that it cannot be realized as a π-system of: (i) any of the maximal diagrams of rank 6 or 10, by the root length criterion (ii) Γ_k for $166 \leq k \leq 170$, by the determinant criterion (iii) Γ_{160} by corollary 3.3.7 (iv) Γ_{159} by lemma 3.3.8.

3.4.4 Rank 3

The determinant criterion ensures that Γ_k, $117 \leq k \leq 121$ are pairwise incomparable. By the root length criterion, these diagrams cannot be realized as π-systems of any diagram of rank ≥ 4. Thus, they are all maximal.

3.4.5 Remarks

This completes the verification that all 22 candidate diagrams in ranks 3-10 are in fact maximal. We make the following interesting observation:

$$\Gamma \text{ is a maximal hyperbolic diagram } \not\Rightarrow \Gamma^T \text{ is maximal}$$

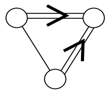
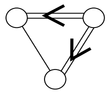
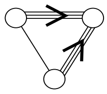
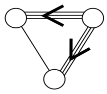
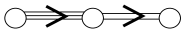
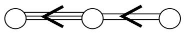
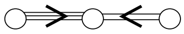
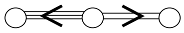
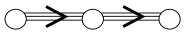
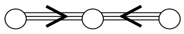
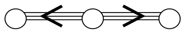
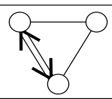
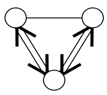
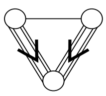
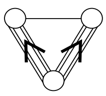
where Γ^T is the dual diagram, obtained by reversing all the arrows in Γ (corresponds to taking the transpose of the GCM). Examples (in fact the only ones) of such diagrams are:

1. $\Gamma = \Gamma_{215}$ is maximal, while $\Gamma^T = \Gamma_{214} \preceq \Gamma_{218}$.

2. $\Gamma = \Gamma_{171}$ is maximal, while $\Gamma^T = \Gamma_{172} \preceq \Gamma_{160}$.

We note that the proof of maximality of these two diagrams involves lemmas 3.3.8 and 3.3.9, neither of which holds when "dualized" (as remarked in §3.3.1). In particular, the above examples show that the operation of taking duals is not an automorphism of the partial order $\preceq$, i.e., if A, B are GCMs such that $B \preceq A$, then it is not necessarily true that $B^T \preceq A^T$.

Table 3.1: Rank 3 diagrams

S. No	Dynkin Diagram	$\preceq$	Principle
3		134	C
4		135	C
10		140	C
11		140	C
25		166	C
26		167	C
27		168	C
28		169	C
29		170	C
30		171	C
31		172	C
32		103	B
40		103	B
49		164	B , C
50		165	B , C

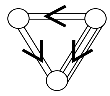
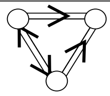
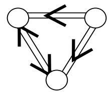
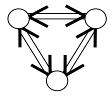
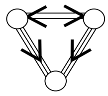
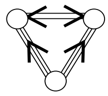
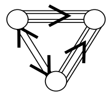
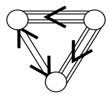
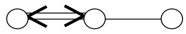
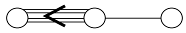
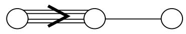
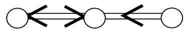
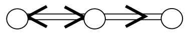
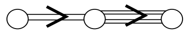
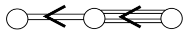
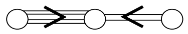
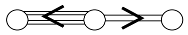
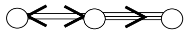
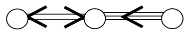
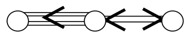
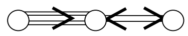
S. No	Dynkin Diagram	$\preceq$	Principle
54		157	B
55		162	B , C
56		163	B , C
80.		103	B
83		113	B
84.		114	B
90		123	E
91		157	*
103		126	B , D
104		164	C
105		165	C
106		163	B , D
107		162	B , D
108		173	C
109		173	C

Table 3.2: Rank 3 diagrams (continued)

S.No	Dynkin Diagram	$\preceq$	Principle
110		174	C
111		175	C
112		103	B
113		159	B , C
114		160	B , C
115		158	D , E
116		157	C

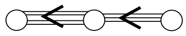
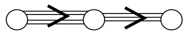
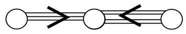
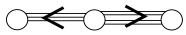
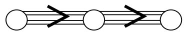
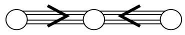
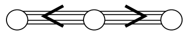
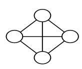
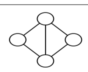
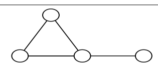
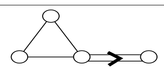
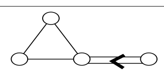
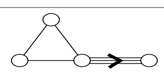
S.No	Dynkin Diagram	$\preceq$	Principle
117		Max	det=-6
118		Max	det=-6
119		Max	det=-6
120		Max	det=-6
121		Max	det=-8
122		158	C
123		157	D , E

Table 3.3: Rank 4 diagrams

S. No	Dynkin Diagram	$\preceq$	Principle
124		126	B
125		126	B
126		177	C
127		178	C
128		179	C
129		159	B

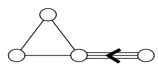
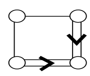
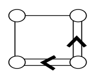
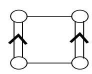
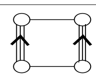
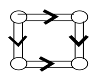
S. No	Dynkin Diagram	$\preceq$	Principle
130		160	B
134		162	B
135		163	B
136		180	C
140		171	B
146		174	B

Table 3.4: Rank 4 diagrams (continued)

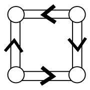
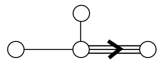
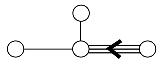
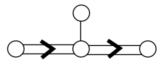
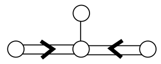
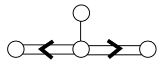
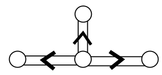
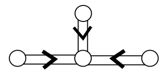
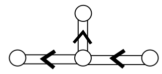
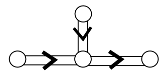
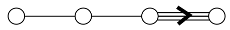
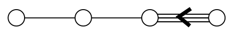
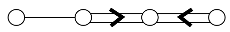
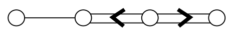
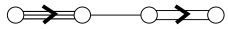
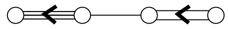
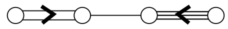
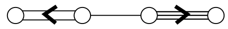
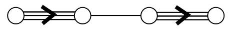
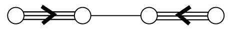
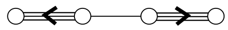
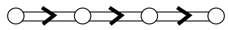
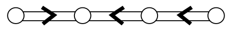
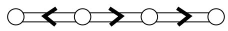
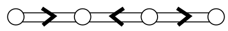
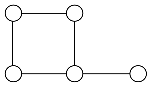
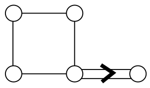
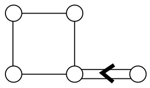
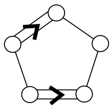
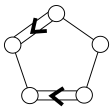
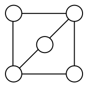
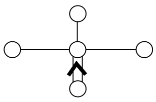
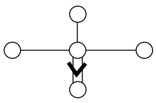
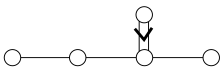
S. No	Dynkin Diagram	$\preceq$	principle
148		176	B
150		159	A
151		160	A
152		191	C
153		189	C
154		190	C
155		163	A
156		162	A
157		173	E
158		191	*
159		Max	det=-1
160		Max	det=-1
161		160	E
162		197	C
163		198	C
164		195	C
165		196	C
166		Max	det=-2
167		Max	det=-2
168		Max	det=-2
169		Max	det=-2
170		Max	det=-3
171		Max	det=-3
172		160	*
173		Max	det=-4
174		217	B, D
175		216	B, D
176		214	B, C, D

Table 3.5: Rank 5 diagrams

S. No	Dynkin Diagram	$\preceq$	Principle
177		199	C
178		188	B
179		187	B
180		200	C
181		200	C
184		177	A
185		206	C
186		205	C
187		209	C

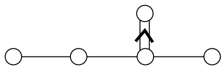
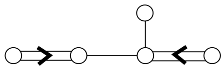
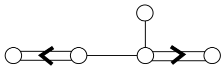
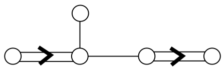
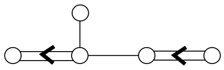
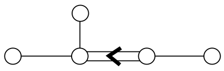
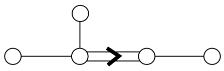
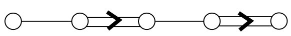
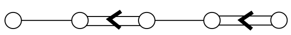
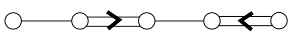
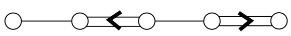
S. No	Dynkin Diagram	$\preceq$	Principle
188		210	C
189		212	C
190		211	C
191		213	C
192		213	C
193		208	C
194		207	C
195		217	C
196		216	C
197		215	C
198		214	C

Table 3.6: Rank 6 diagrams

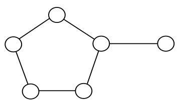
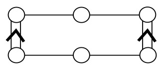
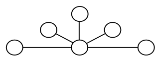
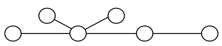
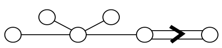
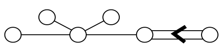
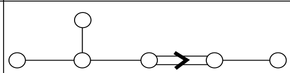
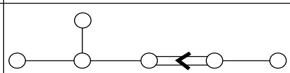
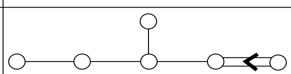
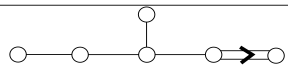
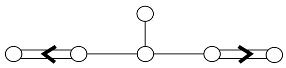
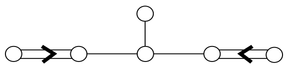
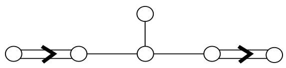
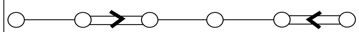
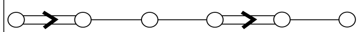
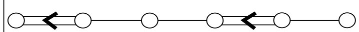
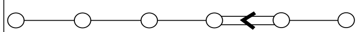
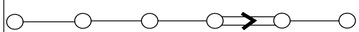
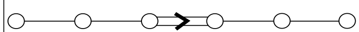
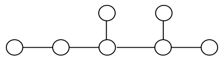
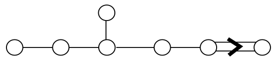
S.No	Dynkin Diagram	$\preceq$	Principle
199		224	C
200		214	B
203		221	A , C
204		221	C
205		210	A
206		209	A
207		219	A
208		218	A
209		223	C
210		222	C
211		214	E
212		223	B ,C, E
213		216	E
214		218	*
215		Max	det=-2
216		Max	det=-2
217		Max	det=-2
218		Max	det=-1
219		Max	det=-1
220		218	E

Table 3.7: Rank 7 diagrams

S. No	Dynkin Diagram	$\preceq$	Principle
221		225	C
222		226	C

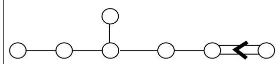
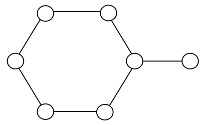
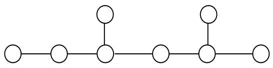
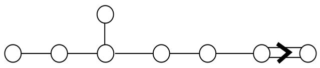
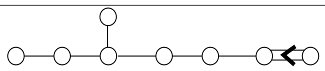
S. No	Dynkin Diagram	$\preceq$	Principle
223		227	C
224		228	C

Table 3.8: Rank 8 diagrams

S. No	Dynkin Diagram	$\preceq$	Principle
225		230	C
226		231	C
227		232	C

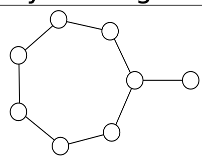
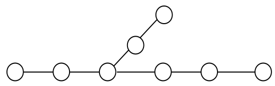
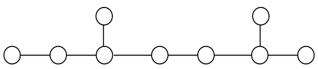
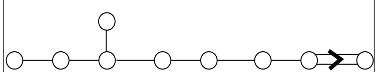
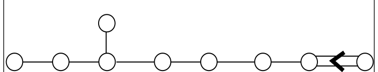
S. No	Dynkin Diagram	$\preceq$	Principle
228		233	C
229		238	B , D

Table 3.9: Rank 9 diagrams

S. No	Dynkin Diagram	$\preceq$	Principle
230		235	C
231		236	C
232		237	C

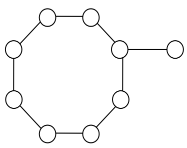
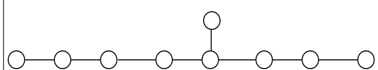
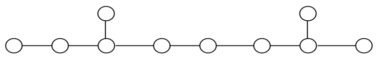
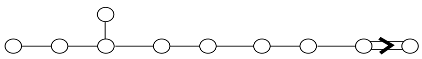
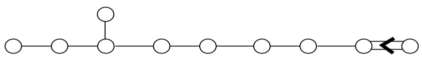
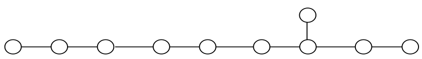
S. No	Dynkin Diagram	$\preceq$	Principle
233		238	B , C
234		238	B , D

Table 3.10: Rank 10 diagrams

S.No	Dynkin Diagram	$\preceq$	Principle
235		238	A
236		Max	det=-2
237		Max	det=-2
238		Max	det=-1

Pi-systems of symmetrizable Kac-Moody algebras

Abstract

In this thesis, we undertake a systematic study of π-systems of symmetrizable Kac-Moody algebras and regular subalgebras of affine Kac-Moody algebras. A π-system Σ is a finite subset of the real roots of a Kac-Moody algebra $\mathfrak{g}$ satisfying the property that pairwise differences of elements of Σ are not roots of $\mathfrak{g}$.

As part of his classification of regular semisimple subalgebras of semisimple Lie algebras, Dynkin introduced and studied the notion of π-systems. These precisely form the simple systems of such subalalgebras. We generalize the definition of π-systems and regular subalgebras and establish their fundamental properties. We show that π-systems, regular subalgebras and closed subroot systems of affine Kac-Moody algebras are in one-to-one correspondence. We completely classify and give explicit descriptions of the maximal closed subroot systems (or maximal π-systems in other words) of affine Kac-Moody algebras. As an application we describe a procedure to get the classification of all regular subalgebras of affine Kac Moody algebras in terms of their root systems.

We also study the orbits of the Weyl group action on π-systems of symmetrizable Kac-Moody algebras, showing that for many π-systems of interest in physics, the action is transitive. The main results of this thesis are follows:

- We give explicit descriptions of the maximal closed subroot systems of affine root systems.
- We address the Weyl group action on π-systems.
- We formulate general principles for constructing π-systems and criteria for the non-existence of π-systems of certain types and use these to determine the set of maximal hyperbolic diagrams in ranks 3-10 relative to the partial order of admitting a π-system.

List of publications arising from the thesis

Journal Papers.

1. Maximal closed subroot systems of real affine root systems.
   Krishanu Roy and R.Venkatesh.
   Transformation groups, 2019, Volume 24, Pages 1261-1308.

<u>Thesis Highlight</u>

Name of the Student: Krishanu Roy

Name of the CI/OCC: Dr. K N Raghavan **Enrolment No.:** MATH10201305003

Thesis Title: Pi-systems of symmetrizable Kac-Moody algebras.

Discipline: Mathematical Science **Subarea of Discipline:** Lie algebras.

Date of viva voce: 21/04/2020

In this thesis, we undertake a systematic study of pi-systems of symmetrizable Kac-Moody algebras and regular subalgebras of affine Kac-Moody algebras. A pi-system is a finite subset of the real roots of a Kac-Moody algebra satisfying the property that pairwise differences of elements of this subset are not roots of the Kac-Moody algebras.

As part of his classification of regular semisimple subalgebras of semisimple Lie algebras, Dynkin introduced and studied the notion of pi-systems. These precisely form the simple systems of such subalalgebras. We generalize the definition of pi-systems and regular subalgebras and establish their fundamental properties. We show that pi-systems, regular subalgebras and closed subroot systems of affine Kac-Moody algebras are in one-to-one correspondence. We completely classify and give explicit descriptions of the maximal closed subroot systems (or maximal pi-systems in other words) of affine Kac-Moody algebras. As an application we describe a procedure to get the classification of all regular subalgebras of affine Kac Moody algebras in terms of their root systems.

We also study the orbits of the Weyl group action on pi-systems of symmetrizable Kac-Moody algebras, showing that for many pi-systems of interest in physics, the action is transitive. The main results of this thesis are follows:

1) We give explicit descriptions of the maximal closed subroot systems of affine root systems.

2) We address the Weyl group action on pi-systems.

3) We formulate general principles for constructing pi-systems and criteria for the non-existence of pi-systems of certain types and use these to determine the set of maximal hyperbolic diagrams in ranks 3-10 relative to the partial order of admitting a pi-system.