Positive cones of cycles and Seshadri constants on certain projective varieties

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As members of the Viva Voce Committee, we certify that we have read the dissertation prepared by Rupam Karmakar entitled "Positive cones of cycles and Seshadri constants on certain projective varieties" and recommend that it may be accepted as fulfilling the thesis requirement for the award of Degree of Doctor of Philosophy.

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Chapter 1

Introduction

This chapter is divided into three sections. In each section we begin with a brief discussion of the historical advancement of the central topics in this thesis and describe our main results at the end.

Throughout this thesis, a variety will be assumed to be reduced and irreducible. A divisor on a projective variety will mean a Cartier divisor. The canonical isomorphism between the Picard group and the divisor class group of a projective variety will be used whenever necessary.

1.1 Nef cones of divisors

Let X be a complex projective variety. The nef cone $\mathrm{Nef}^1(X) \subseteq N^1(X)$ and the ample cone $\mathrm{Amp}(X) \subseteq N^1(X)$ of divisors on X have long stood in the centre of the theory of positivity in algebraic geometry. Over the last few decades these cones have been studied extensively by various authors(see [Laz1] (Section 1.5), [Miy], [F], [BP], [MOH] etc.). We discuss some of these important results about the positive cones of divisors according to the relevance of our treatise here.

Yoichi Miyaoka in 1987 started studying nef and effective divisors on a projective bundle $\mathbb{P}(E)$ over a smooth curve C, where E is a vector bundle of rank r over C. In his paper [Miy], he had shown that semi-stability of E is characterised by the nefness of the *normalized hyperplane class* γ_E. We examine the result precisely in the theorem below.

Theorem 1.1.1 ([Miy], Theorem 3.1)**.** *Let E be a vector bundle on C and $\pi : \mathbb{P}(E) \longrightarrow C$ be the associated projective bundle. $\mathcal{O}_{\mathbb{P}(E)}(1)$ denotes the tautological line bundle and γ_E is the numerical class of $c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) - \pi^*\mu(E)$, where $\mu(E) = \frac{c_1(E)}{\mathrm{rank}(E)}$. Then the following conditions are equivalent:*

(i) E is semi-stable.

(ii) γ_E is nef.

(iii) $\overline{\mathrm{NE}}(\mathbb{P}(E)) = \mathbb{R}_+\gamma_E^{r-1} + \mathbb{R}_+\gamma_E^{r-2}\pi^*d$ *where d is a positive generator of $N^1(C)_{\mathbb{Z}}$*

(iv) $\mathrm{Nef}(\mathbb{P}(E)) = \mathbb{R}_+\gamma_E + \mathbb{R}_+\pi^*d$.

(v) Every effective divisor on $\mathbb{P}(E)$ if numerically effective i.e. nef.

Later in 2006, U. Bruzzo and D. Hernández in [BH] proved that the slope-semistability of a vector bundle E with $\Delta(E) = 0$ on a projective variety X is equivalent to the nefness of the normalized hyperplane class in $\mathbb{P}(E)$. This result shows that nefness of a divisor plays an important role in determining the geometry of a vector bundle on a projective variety.

Theorem 1.1.2 ([BH] Theorem 1.3)**.** *Let E be a vector bundle of rank r on a projective variety X. Then the following conditions are equivalent:*

(i) The normalized hyperplane class $c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) - \frac{1}{\mathrm{rank}(E)}\pi^*(c_1(E))$ is nef.

(ii) E is semistable and $\Delta(E) = 0$.

Where $\mu(E)$ is the slope of E, $\pi : \mathbb{P}(E) \longrightarrow X$ is the projection map and $\Delta(E)$

is the characteristic class of E defined as

$$\Delta(E) = c_2(E) - \frac{r-1}{2r}c_1(E)^2$$

In 2011, Mihai Fulger generalized Miyaoka's results and showed that pseudo-effective cones of cycles $\overline{\mathrm{Eff}}_i(\mathbb{P}(E))$ are determined by all the numerical data in the Harder-Narasimhan filtration of E. We draw up this particular result in more details in Section 1.3.

I. Biswas and A. J. Parameswaran in 2014 studied the nef cones of the Grassmann bundle $Gr_r(E)$, parameterizing r dimensional quotients of E.

Let C be a smooth complex projective curve. Let E be a vector bundle over C and $\phi : Gr_r(E) \longrightarrow C$ be the corresponding Grassmann bundle, parameterizing all r dimensional quotients in the fibres of E for $1 \leq r \leq \mathrm{rank}(E) - 1$ and let $\mathcal{O}_{Gr_r(E)}(1)$ be the tautological line bundle on $Gr_r(E)$. Let

$$0 = E_0 \subset E_1 \subset ... \subset E_l = E$$

be the Harder-Narasimhan filtration of E. Let $t \in [1, l]$ be the unique largest integer such that

$$\sum_{i=t}^{l} \mathrm{rank}(E_i/E_{i-1}) \geq r.$$

Define

$$\theta_{E,r} := (r - \mathrm{rank}(E/E_t))\mu(E_t/E_{t-1}) + \deg(E/E_t).$$

Theorem 1.1.3 ([BP]). *(i)* $\theta_{E,r} > 0 \implies \mathcal{O}_{Gr_r(E)}(1)$ *is ample.*

(ii) $\theta_{E,r} < 0 \implies \mathcal{O}_{Gr_r(E)}(1)$ *is not nef.*

(iii) $\theta_{E,r} = 0 \implies \mathcal{O}_{Gr_r(E)}(1)$ *is nef but not ample.*

(iv) The boundary of the nef cone in $N^1(Gr_r(E))$ *is given by* L *and* $\mathcal{O}_{Gr_r(E)}(1) \otimes L^{-\theta_{E,r}}$*, where* L *is the class of a fibre of the projection map* $\phi : Gr_r(E) \longrightarrow C$*.*

The above theorem in [BP] is proved for a curve C over an algebraically closed field of arbitrary characteristic. Since we don't work with the fields of positive characteristic here, we only stated the theorem in characteristic zero.

Let E_1 and E_2 be two vector bundles over a complex curve C and let $X = \mathbb{P}(E_1) \times_C \mathbb{P}(E_2)$ be the fibre product over C. Consider the following diagram:

$$\begin{array}{ccc} X = \mathbb{P}(E_1) \times_C \mathbb{P}(E_2) & \xrightarrow{p_2} & \mathbb{P}(E_2) \\ \big\downarrow{\scriptstyle p_1} & & \big\downarrow{\scriptstyle \pi_2} \\ \mathbb{P}(E_1) & \xrightarrow{\quad \pi_1 \quad} & C \end{array}$$

Let,

$$\eta_1 = \left[\mathcal{O}_{\mathbb{P}(E_1)}(1)\right] \in N^1(\mathbb{P}(E_1)) \quad , \quad \eta_2 = \left[\mathcal{O}_{\mathbb{P}(E_2)}(1)\right] \in N^1(\mathbb{P}(E_2)),$$

$$\zeta_1 = p_1^*(\eta_1),\ \zeta_2 = p_2^*(\eta_2),\ F \text{ is the fibre of } \pi_1 \circ p_1.$$

Motivated by [F] and [BP] in this thesis we compute the nef cone of nef line bundles on X.

Theorem 1.1.4. *Let* E_1 *and* E_2 *be two vector bundles on a smooth complex projective curve* C *and let* $X = \mathbb{P}(E_1) \times_C \mathbb{P}(E_2)$ *as discussed earlier. Then,*

$$\mathrm{Nef}(\mathbb{P}(E_1) \times_C \mathbb{P}(E_2)) = \left\{a\tau_1 + b\tau_2 + cF \mid a, b, c \in \mathbb{R}_{\geq 0}\right\}.$$

where $\tau_1 = \zeta_1 - \mu_{11}F$ *and* $\tau_2 = \zeta_2 - \mu_{21}F$ *and* μ_{11} *and* μ_{21} *are the smallest slopes of any torsion-free quotients of* E_1 *and* E_2 *respectively.*

1.2 Seshadri constants

The Seshadri constant measures the local positivity of a line bundle on a projective variety around a given point. It was introduced by Demailly in 1992 and gradually it grew on its own as an important invariant in algebraic geometry. Several expositions on this topic can be found in the literature of the last two decades(see [Ba], [BDHKKSS], [CN], [Gar], [BHNN] etc.). We talk about some of these works relevant to our study.

A *ruled surface* X is a $\mathbb{P}^1$-bundle over a smooth curve C. It corresponds to a rank 2 normalized(in terms of Hartshorne [H]) vector bundle E over C and is denoted by $\pi : (X := \mathbb{P}(E)) \longrightarrow C$ with fibre f. Let X_0 be the section of minimum self-intersection and let $\bigwedge^2 E \cong \mathcal{O}_X(\mathbf{e})$ and $e = -\deg(\mathbf{e})$. The Néron-Severi group $N^1(X)$ is generated by X_0 and f.

In 2005, L. F. Garcia(see [Gar]) studied Seshadri constants of nef line bundles on ruled surfaces. We summarize his main results in following two theorems. But before going into that we recall an important definition.

Definition 1.2.1. *An irreducible curve C_0 on X passing through a point $x \in X$ with multiplicity $m \geq 1$ and $C^2 < m^2$ is said to be a Seshadri exceptional curve based at x.*

For a Seshadri exceptional curve based at x we define a continuous function

$$q_{C_0} : \mathrm{Nef}(X) \longrightarrow R, \quad q_{C_0}(L, x) = \frac{L \cdot C}{\mathrm{mult}_x C_0}.$$

Theorem 1.2.1 ([Gar], Theorem 4.14)**.** *Let X be a ruled surface with invariant $e > 0$. Let $L \equiv aX_0 + bf$ be a nef divisor on X. Then*

(i) If $x \in X_0$, then the Seshadri constant $\varepsilon(X, L, x) = \min\{q_f(L.x), q_{X_0}(L, x)\} = \min\{a, b - ae\}$.

(ii) If $x \notin X_0$, then $\varepsilon(X, L, x) = q_f(L, x) = a$.

Theorem 1.2.2 ([Gar], Theorem 4.16)**.** *Let X be a ruled surface with invariant $e \leq 0$. Let $L \equiv aX_0 + bf$ be a nef divisor on X. Then*

(i) If $e = 0$ and x lies on a curve numerically equivalent to X_0, then

$$\varepsilon(X, L, x) = \min\{q_f(L.x), q_{X_0}(L, x)\} = \min\{a, b\}.$$

(ii) In other case:

(a) If $b - \frac{1}{2}ae \geq \frac{1}{2}a$, then

$$\varepsilon(X, L, x) = a.$$

(b) If $0 \leq b - \frac{1}{2}ae \leq \frac{1}{2}a$, then

$$2\big(b - \tfrac{1}{2}ae\big) \leq \varepsilon(X, L, x) \leq \sqrt[2]{L} = \sqrt{2a\big(b - \tfrac{1}{2}ae\big)}.$$

More recently in 2018, I.Biswas, K. Hanumanthu, D.S. Nagaraj and P. E. Newstead studied Seshadri constants on Grassmann bundles over smooth curves under the assumption that the corresponding vector bundle is unstable. This generalizes L. F. Garcia's(see [Gar]) result and explores much more.

Assume that E is an unstable vector bundle over C. Let $Gr_r(E)$ be the corresponding Grassmann bundle over C. We follow the notations used in Section 1.1. Let us fix an integer $1 \leq m \leq l$ and define

$$s := \operatorname{rank}(E/E_{m-1}), \quad d := \deg(E/E_{m-1})$$

The closed cone of curves $\overline{\text{NE}}(Gr_r(E))$ is generated by classes of Γ_l and Γ_t, where Γ_l is the class of a line in the fibre $\mathcal{L}$ of the projection map $\phi : Gr_r(E) \longrightarrow C$ and

Γ_t denote the image of the section

$$t : C \longrightarrow Gr_r(E)$$

corresponding to the s-dimensional quotients $E \longrightarrow E/E_{m-1}$.

Theorem 1.2.3 ([BHNN], Theorem 3.1). *Assume that there exists an integer $c < l$ such that* $\text{rank}(E_c) = s$ *and that* $\deg(E_c)$ *is an integral multiple of s. Let $L \equiv a\mathcal{O}_{Gr_r(E)}(1) + b\mathcal{L}$ be an ample line bundle on $Gr_r(E)$. Then the Seshadri constants of L are given by the following:*

(1) $\varepsilon(X, L, x) \geq \min\{a, b\}$ *for all $x \in Gr_r(E)$.*

(2) *If $b \leq a$, then $\varepsilon(X, L, x) = b$ for all $x \in Gr_r(E)$.*

(3) *If $a < b$, then:*

(*i*) *if x does not belong to the base locus of the linear system $\mid \mathcal{O}_{Gr_r(E)}(1) \mid$, then $\varepsilon(X, L, x) = b$;*

(*ii*) *if x belongs to the base locus of $\mid \mathcal{O}_{Gr_r(E)}(1) \mid$, then $a \leq \varepsilon(X, L, x) \leq b$;*

(*iii*) *if $x \in \gamma_t$,then $\varepsilon(X, L, x) = a$.*

Theorem 1.2.4 ([BHNN], Theorem 3.3). *Assume that $\mu(E_m/E_{m-1}) - \mu(E_{m-1}/E_{m-2}) \leq \theta$. With the same hypothesis as of the previous theorem, the Seshadri constants of L is given by the following:*

$$\varepsilon(X, L, x) = \begin{cases} b & if \;\; b \leq a \;\; or \;\; x \notin \Gamma_t \\ a & if \;\; a < b \;\; and \;\; x \in \Gamma_t \end{cases}$$

Let E_1 and E_2 be two vector bundle over a complex smooth projective curve and let $X = \mathbb{P}(E_1) \times_C \mathbb{P}(E_2)$ as discussed earlier in Section 1.1. Motivated by [Gar] and [BHNN] in this thesis we study the Seshadri constants of nef line bundles on X.

Theorem 1.2.5. *Let E_1 and E_2 be two vector bundles on a smooth curve C with μ_{11} and μ_{21} being the smallest slopes of any torsion-free quotient of E_1 and E_2 respectively and let $X = \mathbb{P}(E_1) \times_C \mathbb{P}(E_2)$. Let L be an ample line bundle on X numerically equivalent to $a\tau_1 + b\tau_2 + cF \in N^1(X)$. Then, the Seshadri constants of L,*

$$\varepsilon(X, L, x) \geq \min\{a, b, c\}, \ \ \forall x \in X.$$

Moreover,

(4.1.1) *if $a = \min\{a, b, c\}$, then $\varepsilon(X, L, x) = a, \ \ \forall x \in X$*

(4.1.2) *if $b = \min\{a, b, c\}$, then $\varepsilon(X, L, x) = b, \ \ \forall x \in X$.*

In the above theorem when $c = \min\{a, b, c\}$ more can be said about the Seshadri constants of ample line bundles on X if E_1 and E_2 are unstable vector bundles over C.

Theorem 1.2.6. *Let E_1 and E_2 be two unstable vector bundles over a smooth curve C of rank r_1 and r_2 respectively and $X = \mathbb{P}(E_1) \times_C \mathbb{P}(E_2)$. Let L be an ample line bundle on X numerically equivalent to $a\tau_1+b\tau_2+cF \in N^1(X)$. When $c = \min\{a, b, c\}$ the Seshadri constants of L have the following properties.*

(i) *Assume $c \leq a \leq b$, $\operatorname{rank}(E_1) = 2$ and E_1 is normalised.*

If x is a point outside $\mathbf{B}_{-}(\zeta_1)$, then $\varepsilon(X, L, x) = a$.

If x belongs to $\mathbf{B}_{-}(\zeta_1)$, then $c \leq \varepsilon(X, L, x) \leq a$.

(ii) *Assume $c \leq b \leq a$, $\operatorname{rank}(E_2) = 2$ and E_2 is normalised.*

If x is a point outside $\mathbf{B}_{-}(\zeta_2)$, then $\varepsilon(X, L, x) = b$.

If x belongs to $\mathbf{B}_{-}(\zeta_2)$, the $c \leq \varepsilon(X, L, x) \leq b$.

(iii) If x is on some curve whose class is proportional to $\overline{\delta_3}$, then $\varepsilon(X, L, x) =$

c, where $\overline{\delta_3} = \delta_3 + \mu_{11}\delta_1 + \mu_{21}\delta_2$.

1.3 Effective cones of cycles

For the past few years various positive cones of higher co-dimension cycles have gained much attention amongst fellow geometers. At the fag end of the last decade Claire Voisin in [V] used the cones of effective cycles to study the generalized Hodge conjecture for coniveau 2 complete intersections and stated a conjecture about the the cones of effective cycles in intermediate dimensions. She showed that generalized Hodge conjecture for coniveau 2 complete intersections follows from this effectiveness conjecture. Around the same time Thomas Peternell in [P] worked with cones of effective cycles to study Robin Hartshorne's conjecture, which says that

For a projective manifold Z and submanifolds X and Y with ample normal bundles, if $\dim X + \dim Y \geq \dim Z$, then $X \cap Y \neq \phi$.

Later there has been significant progress in the theoretical understanding of such cycles due to [FL1] and [FL2] and others. Although similar in nature, these cycles do not share all the important properties of divisors or curves(see [DELV], [F], [DJV], [CC], [FL1], [FL2]) for details.

Let E be a vector bundle over a complex smooth projective curve C and let $\pi : \mathbb{P}(E) \longrightarrow C$ be the projective bundle associated to it. Let ξ be the class of the tautological bundle $\mathcal{O}(1)$ on $\mathbb{P}(E)$ and let f be the class of a fibre of the projection π. E admits the unique Harder-Narasimhan filtration

$$E = E_0 \supset E_1 \supset ...E_l = 0.$$

Denote $Q_i := E_{i-1}/E_i$, $r_i := \operatorname{rank} Q_i, d_i := \deg Q_i$ and $\mu_i = \mu(Q_i) := \frac{d_i}{r_i}$.

In 2011 M. Fulger([F]) computed effective cones of cycles $\overline{\mathrm{Eff}}^i(\mathbb{P}(E))$ on $\mathbb{P}(E)$.

We describe the results below.

Theorem 1.3.1 ([F], Lemma 2.2)**.** *If E is semistable of rank n and slope μ, then for all $i \in \{1, 2, ..., n-1\}$*

$$\overline{\mathrm{Eff}}^i(\mathbb{P}(E)) = \langle (\xi - \mu f)^i, \xi^{i-1} f \rangle$$

Assume that E is an unstable vector bundle over C. From Harder-Narasimhan filtration of E we get the following exact sequence

$$0 \longrightarrow E_1 \longrightarrow E \longrightarrow Q_1 \longrightarrow 0$$

Let $i : \mathbb{P}(Q_1) \longrightarrow \mathbb{P}(E)$ be the canonical inclusion. Then

Theorem 1.3.2 ([F], Lemma 2.3)**.** *For all $i \in \{1, 2, ..., r_1\}$*

$$\overline{\mathrm{Eff}}_i(\mathbb{P}(E)) = \left\langle [\mathbb{P}(Q_1)](\xi - \mu f)^{r_1 - i}, \xi^{n-i-1} f \right\rangle$$

In particular i_* induces an isomorphism $\overline{\mathrm{Eff}}_i(\mathbb{P}(Q_1)) \cong \overline{\mathrm{Eff}}_i(\mathbb{P}(E))$ for $i < r_1$.

Now the projection map

$$q : \mathbb{P}(E) \setminus \mathbb{P}(Q_1) \longrightarrow \mathbb{P}(E_1)$$

can be seen as a rational map whose indeterminacies are resolved by blowing up $\mathbb{P}(Q_1)$. As a result of the blow up we have the following commutative diagram:

$$\begin{array}{ccc} Bl_{\mathbb{P}(Q_1)}\mathbb{P}(E) & \xrightarrow{\eta} & \mathbb{P}(E_1) \\ \downarrow{\scriptstyle B} & & \downarrow{\scriptstyle p} \\ \mathbb{P}(E) & \xrightarrow{\pi} & C \end{array}$$

The next theorem shows that pseudo-effective cycles of dimension bigger than rank(Q_1) can be tied down to pseudo-effective cycles of $\mathbb{P}(E_1)$.

Theorem 1.3.3 ([F], Lemma 2.7). *The map* $B_*\eta^*|_{\overline{\mathrm{Eff}}^i(\mathbb{P}(E_1))}$ *is an isomorphism onto* $\overline{\mathrm{Eff}}^i(\mathbb{P}(E_1))$ *for* $i < n - r_1$.

There are few other examples known in literature where people have studied pseudo effective cones of higher co-dimension cycles. In 2011, O. debarre, L. Ein, R. Lazarsfeld and C. Voisin studied nef and pseudo-effective cones of cycles of higher co-dimension on the self products of elliptic curves with complex multiplication and on the product of a very general abelian surface with itself([DELV]). In 2016, I. Coskun, J. Lesieutre and J. C. Ottem studied pseudo-effective cones of higher co-dimension cycles on point blow-ups of projective spaces. They provided bounds on the number of points blown up for which these cones are linearly generated and for which these cones are finitely generated([CLO]). In 2018, N. Pintye and A. Prendergast-Smith([PP]) studied pseudo-effective cones of cycles on some linear blow-ups of projective spaces.

Let E_1 and E_2 be two vector bundles over a smooth complex projective curve C. Let $X = \mathbb{P}(E_1) \times_C \mathbb{P}(E_2)$ as discussed earlier in Section 1.1. Motivated by [F] we compute the pseudo-effective cones of cycles on X.

Theorem 1.3.4. *Let* E_1 *and* E_2 *be two semistable vector bundles over* C *of rank* r_1 *and* r_2 *respectively with* $r_1 \leq r_2$ *and* $X = \mathbb{P}(E_1) \times_C \mathbb{P}(E_2)$. *Then for all* $k \in \{1, 2, ..., r_1 + r_2 - 1\}$

$\overline{\mathrm{Eff}}^k(X)$ *is generated by complete intersections of nef divisor classes in* X.

Let $E_i (i = 1, 2)$ be two unstable bundles on X, having the following H-N filtrations.

$$E_i = E_{i0} \supset E_{i1} \supset ... \supset E_{il_i} = 0$$

for $i = 1, 2$. Write $Q_{11} = E_1/E_{11}, Q_{21} = E_2/E_{21}, n_{11} = \mathrm{rank}(Q_{11})$ and $n_{21} =$

rank(Q_{21}).

Theorem 1.3.5. *Let E_1 and E_2 be two unstable bundle of rank r_1 and r_2 and degree d_1 and d_2 respectively over a smooth curve C and $X = \mathbb{P}(E_1) \times_C \mathbb{P}(E_2)$. Then for all $k \in \{1, 2, ..., \mathbf{n}\}$ ($\mathbf{n} := n_{11} + n_{21} - 1$)*

$\overline{\mathrm{Eff}}_k([\mathbb{P}(Q_{11}) \times_C \mathbb{P}(Q_{21})])$ *is isomorphic to* $\overline{\mathrm{Eff}}_k(X)$ *for* $k \leq \mathbf{n}$.

Theorem 1.3.6. $\overline{\mathrm{Eff}}^k(X) \cong \overline{\mathrm{Eff}}^k(Z')$ *and* $\overline{\mathrm{Eff}}^k(Z') \cong \overline{\mathrm{Eff}}^k(Z'')$. *So,* $\overline{\mathrm{Eff}}^k(X) \cong \overline{\mathrm{Eff}}^k(Z'')$ *for* $k < r_1 + r_2 - 1 - \mathbf{n}$

where $Z' = \mathbb{P}(E_{11}) \times_C \mathbb{P}(E_2)$ *and* $Z'' = \mathbb{P}(E_{11}) \times_C \mathbb{P}(E_{21})$.

Contents

Notations

- $\mathbb{Z}$ The ring of integers
- $\mathbb{Q}$ The field of rational numbers
- $\mathbb{R}$ The field of real numbers
- $\mathbb{R}_{>0}$ The set of positive real numbers
- $\mathbb{R}_{\geq 0}$ The set of all non-negetive real numbers
- $\mathbb{C}$ The field of complex numbers
- $\mathbb{P}^n_k$ Projective n-space over an algebraically closed field k
- $c_i(E)$ i-th Chern class of a vector bundle E
- $\mathrm{Div}(X)$ The set of all divisors on a variety X
- $[D]$ Numerical equivalence class of a divisor D on X.
- $\rho(X)$ The Picard rank of X
- $\mathrm{Pic}(X)$ The Picard group of X
- $\mathrm{mult}_x C$ Multiplicity at the point x of a curve C passing through x
- $N^1(X)$ Real Néron-Severi group of X.

Summary

Let E_1 and E_2 be two vector bundles over a smooth complex projective curve C and let $\mathbb{P}(E_1) \times_C \mathbb{P}(E_2)$ be the fibre product of projective bundles associated with them. In this thesis, we study various positive cones of cycles and Seshadri constants of nef line bundles on $\mathbb{P}(E_1) \times_C \mathbb{P}(E_2)$.

(I) The nef cone $\mathrm{Nef}(X) \subseteq N^1(X)$ of divisors on a projective variety X is an important invariant which gives useful information about the projective embeddings of X. The nef cone of various smooth irreducible projective varieties has been studied by many authors in the last few decades (See [Laz1] (Section 1.5), [Miy], [F], [BP], [MOH], [KMR] for more details). In this thesis we compute $\mathrm{Nef}(\mathbb{P}_C(E_1) \times_C \mathbb{P}_C(E_2))$ without restriction on the rank or semistability of E_1 and E_2.

(II) Let X be a smooth complex projective variety and let L be a nef line bundle on X. The Seshadri constant of L at $x \in X$ is defined as

$$\varepsilon(X, L, x) := \inf_{x \in C} \left\{ \frac{L \cdot C}{\mathrm{mult}_x C} \right\}$$

where the infimum is taken over all the closed curves in X passing through x having the multiplicity $\mathrm{mult}_x C$ at x.

Seshadri constants on ruled surfaces $\mathbb{P}_C(E)$ ($\mathrm{rank}(E) = 2$) over a smooth curve C have been studied by many authors (see [Gar], [HM] etc.). More generally, [BHNN] computes the Seshadri constants of ample line bundles on the Grassmann bundle

$Gr_r(E)$ over a smooth curve C under the assumption that E is an unstable bundle on C. In particular, under some suitable conditions on the Harder-Narasimhan filtration of E, [BHNN] computes the Seshadri constants of ample line bundles on $\mathbb{P}_C(E)$, whenever E is an unstable vector bundle over a smooth curve C.

Motivated by this, in this thesis we study the Seshadri constants of ample line bundles on $\mathbb{P}(E_1) \times_C \mathbb{P}(E_2)$, where E_1 and E_2 are vector bundles over a smooth irreducible curve C of rank r_1 and r_2 respectively, under some assumptions on E_1 and E_2, and have given bounds in some other cases.

(III) Recently the theory of cones of cycles of higher dimension has been the subject of increasing interest(see [F], [DELV], [DJV], [CC] etc). Lately, there has been significant progress in the theoretical understanding of such cycles, due to [FL1], [FL2] and others. But the the number of examples where the cone of effective cycles have been explicitly computed is relatively small till date(see [F], [CLO] etc).

Let E_1 and E_2 be two vector bundles over a smooth curve C and consider the fibre product $\mathbb{P}(E_1)\times_C\mathbb{P}(E_2)$. In this thesis, we compute the cones of effective cycles on $\mathbb{P}(E_1) \times_C \mathbb{P}(E_2)$ in the following cases.

(i) When both E_1 and E_2 are semistable vector bundles of rank r_1 and r_2 respectively over C.

(ii) When Neither E_1 nor E_2 is semistable vector bundles of rank r_1 and r_2 respectively over C.

Summary

Let E_1 and E_2 be two vector bundles over a smooth complex projective curve C and let $\mathbb{P}(E_1) \times_C \mathbb{P}(E_2)$ be the fibre product of projective bundles associated with them. In this thesis, we study various positive cones of cycles and Seshadri constants of nef line bundles on $\mathbb{P}(E_1) \times_C \mathbb{P}(E_2)$.

(I) The nef cone $\mathrm{Nef}(X) \subseteq N^1(X)$ of divisors on a projective variety X is an important invariant which gives useful information about the projective embeddings of X. The nef cone of various smooth irreducible projective varieties has been studied by many authors in the last few decades (See [Laz1] (Section 1.5), [Miy], [F], [BP], [MOH], [KMR] for more details). In this thesis we compute $\mathrm{Nef}(\mathbb{P}_C(E_1) \times_C \mathbb{P}_C(E_2))$ without restriction on the rank or semistability of E_1 and E_2.

(II) Let X be a smooth complex projective variety and let L be a nef line bundle on X. The Seshadri constant of L at $x \in X$ is defined as

$$\varepsilon(X, L, x) := \inf_{x \in C} \left\{ \frac{L \cdot C}{\mathrm{mult}_x C} \right\}$$

where the infimum is taken over all the closed curves in X passing through x having the multiplicity $\mathrm{mult}_x C$ at x.

Seshadri constants on ruled surfaces $\mathbb{P}_C(E)$ ($\mathrm{rank}(E) = 2$) over a smooth curve C have been studied by many authors (see [Gar], [HM] etc.). More generally, [BHNN] computes the Seshadri constants of ample line bundles on the Grassmann bundle

$Gr_r(E)$ over a smooth curve C under the assumption that E is an unstable bundle on C. In particular, under some suitable conditions on the Harder-Narasimhan filtration of E, [BHNN] computes the Seshadri constants of ample line bundles on $\mathbb{P}_C(E)$, whenever E is an unstable vector bundle over a smooth curve C.

Motivated by this, in this thesis we study the Seshadri constants of ample line bundles on $\mathbb{P}(E_1) \times_C \mathbb{P}(E_2)$, where E_1 and E_2 are vector bundles over a smooth irreducible curve C of rank r_1 and r_2 respectively, under some assumptions on E_1 and E_2, and have given bounds in some other cases.

(III) Recently the theory of cones of cycles of higher dimension has been the subject of increasing interest(see [F], [DELV], [DJV], [CC] etc). Lately, there has been significant progress in the theoretical understanding of such cycles, due to [FL1], [FL2] and others. But the the number of examples where the cone of effective cycles have been explicitly computed is relatively small till date(see [F], [CLO] etc).

Let E_1 and E_2 be two vector bundles over a smooth curve C and consider the fibre product $\mathbb{P}(E_1) \times_C \mathbb{P}(E_2)$. In this thesis, we compute the cones of effective cycles on $\mathbb{P}(E_1) \times_C \mathbb{P}(E_2)$ in the following cases.

(i) When both E_1 and E_2 are semistable vector bundles of rank r_1 and r_2 respectively over C.

(ii) When Neither E_1 nor E_2 is semistable vector bundles of rank r_1 and r_2 respectively over C.

Chapter 2

Preliminaries

2.1 Nef and ample line bundles

Let X be an irreducible projective variety over the field of complex numbers $\mathbb{C}$. L and D denote a line bundle and a Cartier divisor on X respectively. We denote by $\mathrm{Div}(X)$ the group of all cartier divisors on X.

Definition 2.1.1. *L is said to be very ample if there exists a closed embedding of X into some projective space $\mathbb{P}^N$ such that*

$$L = \mathcal{O}_X(1) := \mathcal{O}_{\mathbb{P}^N}(1)_{|X}$$

Definition 2.1.2. *L is said to be ample if there exists an integer $\tilde{n}(F)$ such that for every coherent sheaf F and for every $n \geq \tilde{n}(F)$, the sheaf $F \otimes L^{\otimes n}$ is generated by its global sections.*

A Cartier divisor D is said to be very ample or ample if the associated line bundle $\mathcal{O}_X(D)$ is so.

The relation between ample and very ample line bundles on X is given by the following theorem.

Theorem 2.1.1 (see [H], Chapter II Theorem 7.6)**.** *Let X be a scheme of finite type over a noetherian ring A and L be a line bundle on X. Then L is ample if and only if $L^{\otimes n}$ is very ample over SpecA for some $n > 0$.*

Definition 2.1.3. {*Numerical equivalence*} *Two Cartier divisors D_1 and D_2 are numerically equivalent if $D_1 \cdot C = D_2 \cdot C$ for all irreducible curves $C \subseteq X$.*

Numerical equivalence of line bundles is defined analogously. A divisor is *numerically trivial* if it is numerically equivalent to zero divisor and the subgroup of all numerically trivial divisors is named $\mathrm{Num}(X)$. The group $N^1(X)_{\mathbb{Z}} := \mathrm{Div}(X)/\,\mathrm{Num}(X)$ of numerical equivalence classes of divisors is called the Néron-Severi group. It is a free abelian group of finite rank and we call the rank of $N^1(X)_{\mathbb{Z}}$ the *Picard number* of X, $\rho(X)$.

Next we recall the notion of $\mathbb{Q}$-divisors and $\mathbb{R}$- divisors on a variety.

Definition 2.1.4. *Let X be an irreducible projective variety over $\mathbb{C}$. An $\mathbb{Q}$- divisor on X is an element of the vector space*

$$\mathrm{Div}_{\mathbb{Q}}(X) := \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

An $\mathbb{Q}$-divisor D can be written as a finite sum

$$D = \sum c_i B_i,$$

where $c_i \in \mathbb{Q}$ and $B_i \in \mathrm{Div}(X)$.

Similarly a $\mathbb{R}$-divisor on X is an element of the real vector space

$$\mathrm{Div}_{\mathbb{R}}(X) := \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$$

A $\mathbb{R}$*-divisor* D *can be written as a finite sum*

$$D = \sum d_j B_j,$$

where $d_j \in \mathbb{R}$ *and* $B_j \in \mathrm{Div}(X)$

Equivalences and operations on $\mathbb{Q}$-divisors and $\mathbb{R}$- divisors are mostly analogous to the integral counterpart. We are not going to discuss those in details. See ([Laz1]) for more details.

Let D be an $\mathbb{Q}$-divisor on X. The *stable base locus* of D is

$$\mathbf{B}(D) := \bigcap_{m \in \mathbb{N}} \mathrm{Bs}(\mid mD \mid)_{red},$$

where the intersection is taken over all m such that mD is an integral divisor and the *base locus* $\mathrm{Bs}(|D|)$ of a complete linear system $|D|$ of Cartier divisors on X is the set of common zeros of all sections of the associated line bundle $L(D)$. This is an interesting invariant but some well known pathologies associated to linear series made it's study not so fruitful and discouraging in many cases. For example it can happen that the stable base locus $\mathbf{B}(D)$ does not depend only on the numerical equivalence class of D. However this problem can be avoided by considering the following approximation of $\mathbf{B}(D)$.

The *restricted base locus* of a $\mathbb{R}$-divisor D on X is defined to be

$$\mathbf{B}_-(D) := \bigcup_A \mathbf{B}(D + A),$$

where the union is taken over all ample divisors A such that $D + A$ is a $\mathbb{Q}$-divisor.

It follows easily from the definition that $\mathbf{B}_-(D)$ depends only on the numerical equivalence class of D. See [ELMNP] for further details.

Let $N^1(X)$ be the real vector space of numerical equivalence classes of $\mathbb{R}$- divisors. Then there is an isomorphism

$$N^1(X) = N^1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$$

The following two theorems state two very important phenomena about the amplitude of a divisor(or a line bundle). One says that the amplitude can be detected cohomologically and the other deals with the numerical characterisation of ample line bundles.

Theorem 2.1.2 (see [Laz1], Theorem 1.2.6)**.** *Let L be a line bundle on a complete scheme X. Then the following conditions are equivalent:*
(i) L is ample
(ii) For every coherent sheaf F there exists an integer $m_0(F)$ such that for all $i > 0$ and all $m \geq m_0(F)$, $H^i(X, F \otimes L^{\otimes m}) = 0$.

The statement (ii) of the above theorem is often called Serre's vanishing theorem.

Theorem 2.1.3 (Nakai- Moishezon - Kleiman criterion, see [Laz1] Theorem 1.2.23)**.** *Let L be a line bundle on a projective variety X. Then L is ample if and only if*

$$\int_V c_1(L)^{\dim(V)} > 0$$

for every positive-dimensional irreducible subvariety of X.

An analogous statement of the above theorem can be stated for $\mathbb{R}$- divisors and the credit for that goes to Campana and Peternell. We recall the statement below.

Theorem 2.1.4 (Nakai criterion for $\mathbb{R}$- divisors, see [Laz1] Theorem 2.3.18)**.** *Let X be a projective scheme and δ be a class of $\mathbb{R}$- divisors in $N^1(X)_{\mathbb{R}}$. Then δ is ample* *if and only if*

$$\left(\delta^{dimV} \cdot V\right) > 0$$

for every irreducible subvariety of X of positive dimension.

We now move into another very important notion in the theory of positivity of line bundles, the *nefness* or the *numerical effectiveness* of line bundles. Let us begin with the definition.

Definition 2.1.5. *Let X be a complete variety. A line bundle L on X(or a Cartier divisor with $\mathbb{Z}$ or $\mathbb{R}$ coefficients) is nef if*

$$\int_C c_1(L) \geq 0 \quad \big(or \quad (D \cdot C) \geq 0\big)$$

for all irreducible curves C in X.

The introduction of some important cones in $N^1(X)$ was pioneered by Kleiman. Let X be a complete complex variety. Let V be a finite-dimensional vector space. A *cone* in V is a set $W \subseteq V$ stable under the multiplication by positive scalars. The *ample cone* $\mathrm{Amp}(X) \subset N^1(X)$ is the cone of ample $\mathbb{R}$- divisor classes on X. The *nef cone* $\mathrm{Nef}(X) \subset N^1(X)$ is the convex cone of all nef $\mathbb{R}$-divisor classes on X.

Theorem 2.1.5 (Kleiman, see [Laz1] Theorem 1.4.23)**.** *Let X be a projective variety. The nef cone is the closure of the ample cone and the ample cone is the interior of the nef cone.*

$$\mathrm{Nef}(X) = \overline{\mathrm{Amp}(X)}, \quad \mathrm{Amp}(X) = int\big(\mathrm{Nef}(X)\big)$$

Let X be a complete variety. We denote by $Z_1(X)$ the real vector space of *real one-cycles* on X, consisting of $\mathbb{R}$-linear combinations of all irreducible curves on X.

So, an element $\gamma \in Z_1(X)$ can be written as

$$\gamma = \sum b_i C_i$$

where $b_i \in \mathbb{R}$ and C_i's are irreducible curves on X. Two one-cycles $\gamma_1, \gamma_2 \in Z_1(X)$ are said to be *numerically equivalent* if

$$\big(D \cdot \gamma_1\big) = \big(D \cdot \gamma_2\big)$$

for all $D \in \mathrm{Div}_{\mathbb{R}}(X)$. The real vector space of all numerical equivalence classes of one-cycles is denoted by $N_1(X)$ and we have the following perfect pairing

$$N^1(X) \times N_1(X) \longrightarrow \mathbb{R}, \quad (\delta, \gamma) \mapsto (\delta \cdot \gamma)$$

Definition 2.1.6. *The cone of curves* $\mathrm{NE}(X)$ *is the cone spanned by all real effective one-cycles on* X*. More precisely*

$$\mathrm{NE}(X) = \Big\{ \sum b_i [C_i] | C_i \subset X \textit{ is an irreducible curve and } b_i \in \mathbb{R}_{\geq 0} \Big\}$$

The closure $\overline{\mathrm{NE}}(X) \subset N^1(X)_{\mathbb{R}}$ *is called the closed cone of curves.*

It's a fact that $\overline{\mathrm{NE}}(X)$ is dual to $\mathrm{Nef}(X)$ which will be discussed in Section 2.3 in a more general set up.

2.2 Seshadri constants

The Seshadri constant measures the local positivity of a line bundle. It was first introduced by Demailly , mainly to use with the aim to solve the Fujita conjecture. Gradually it became clear that they are considerably rich and fascinating invariants in their own rights. before going into the definition of Seshadri constants we recall the Seshadri criterion for ampleness where it all began.

Theorem 2.2.1. *Let X be a projective variety and D be a divisor on X. Then D is ample if and only if there is a positive number ε such that for every point $x \in X$ and for every irreducible curve $C \subset X$ passing through x*

$$D \cdot C \geq \varepsilon \operatorname{mult}_x C$$

What comes naturally from the above theorem is the optimality question of ε. That leads us to the following definition of Demailly.

Definition 2.2.1 (Seshadri constants at a point)**.** *Let X be a smooth projective variety and L be a line bundle on X. the Seshadri constants of L at a point $x \in X$ if defined as*

$$\varepsilon(X, L, x) := \inf_{x \in C} \left\{ \frac{L \cdot C}{\operatorname{mult}_x C} \right\}$$

where the infimum is taken over all irreducible curves $C \subset X$ passing through X.

Definition 2.2.2. *We say that a curve C computes the Seshadri constant $\varepsilon(X, L, x)$ if*

$$\varepsilon(X, L, x) = \frac{L \cdot C}{\operatorname{mult}_x C}$$

In that case the curve C is called a Seshadri curve.

It is unknown if a Seshadri curve exists in general. We have an alternate definition of the Seshadri constants as expressed below.

Proposition 2.2.1. *Let X be an irreducible projective variety and*

$$\mu : X' = Bl_x X \longrightarrow X$$

be the blow up of X at a point $x \in X$ with the exceptional divisor $E \subseteq X'$. Also assume L to be a nef divisor on X. Then the Seshadri constant of L at x is defined as

$$\varepsilon(X, L, x) := max\{\alpha \geq 0 \quad |\mu^* L - \alpha.E \text{ is nef}\}$$

It can be proved easily that the two definitions of Seshadri constants mentioned above are actually same. See ([Laz1]), Theorem 5.1.5.

Now we list some of the important properties of the Seshadri constants.

Proposition 2.2.2 (see [Laz1])**.** (*i*) *The Seshadri constant $\varepsilon(X, L, x)$ depends only on the numerical equivalence class of L.*

(*ii*) *If L is very ample, then $\varepsilon(X, L, x) \geq 1$ for every $x \in X$.*

(*iii*) *If $W \subseteq X$ is an irreducible subvariety of positive dimension containing $x \in X$, then*

$$\varepsilon(X, L, x) \leq \Big(\frac{L^{\dim W} \cdot W}{\operatorname{mult}_x W}\Big)^{\frac{1}{\dim W}}$$

In particular, one has the trivial bound

$$\varepsilon(X, L, x) \leq \sqrt[n]{\frac{L^n}{\operatorname{mult}_x X}}$$

when n is the dimension of X.

2.3 Positive cones of higher-dimension cycles

In Section 2.1 we have discussed various positive cones of divisors(Co-dimension one cycles) on a projective variety. In this section we will talk about some positive cones of higher co-dimension cycles on a projective variety.

Let X be a smooth projective variety. For a closed subscheme $V \subseteq X$ we define its *fundamental integral cycle* as in [[Ful],1.5]. A *k-cycle* is a finite formal sum

$$\sum n_i[V_i]$$

where V_i's are k-dimensional subvarieties of X and $n_i \in \mathbb{Z}$. The group of integral k-cycles on X denoted by $Z_k(X)$ is a free abelian group of k-dimensional subvarieties on X. Similarly, we can use the denominations *rational* or *real* when the coefficients are $\mathbb{Q}$ or $\mathbb{R}$ respectively. Many equivalence relations have been introduced on $Z_k(X)$ (or $Z_k(X)_{\mathbb{Q}}$ or $Z_k(X)_{\mathbb{R}}$) to study the cycles on X. Here we focus on the *numerical equivalence* of cycles which suits our need.

Let E be a vector bundle on X. Then for any integar i, W. Fulton in ([Ful]) constructs the Chern class $c_i(X)$ of E which maps a class $\alpha \in A_k(X)$ to $c_i(X) \cap \alpha \in A_{k-i}X$. Since the operation is commutative and associative, we can define $P(E_I) \cap [Z]$ for any finite collection of vector bundles $E_{ii\in I}$ and any weighted homogenous polynomial $P(E_I)$ of weight i on the Chern classes of these vector bundles.

Definition 2.3.1. *A k-cycle Z on X is said to be numerically trivial if*

$$\deg(P(E_I) \cap Z) = 0$$

where $P(E_I)$ is a weighted homogenous polynomial of weight k in Chern classes of a finite set of vector bundles on X.

The set of *numerically trivial* k-cycles form a group called $\mathrm{Num}_k(X)$. See ([Ful], Chapter 19) and ([FL1]) for more details.

We denote $N_k(X)_{\mathbb{Z}}$ to be the quotient of $Z_k(X)$ by numerically trivial cycles. $N_k(X)_{\mathbb{Z}}$ is a lattice inside $N_k(X)_{\mathbb{Q}} := N_k(X)_{\mathbb{Z}} \otimes \mathbb{Q}$ and $N_k(X) := N_k(X)_{\mathbb{Z}} \otimes \mathbb{R}$. $N_k(X)$ is a free abelian group of finite rank and called the *numerical group*. It's abstract dual $N_k(X)$ is called the *numerical dual group*. We have the formal identification

$$N^k(X) = \frac{\text{homogenous Chern polynomials with real coefficients with weight}\, k}{\text{Chern polynomials}\, P\, \text{such that}\, P \cap \alpha = 0\, \text{for all}\, \alpha \in N_k(X)_{\mathbb{R}}}$$

However if X is a non-singular variety of dimension n, we set

$$N^k(X)_{\mathbb{Z}} = N_{n-k}(X)_{\mathbb{Z}}, \quad N^k(X)_{\mathbb{Q}} = N_{n-k}(X)_{\mathbb{Q}}, \quad N^k(X) = N_{n-k}(X)$$

Definition 2.3.2. *A class $\alpha \in N_k(X)$ is said to be effective if there are subvarieties $V_1, V_2, ..., V_m$ and non-negetive real numbers $n_1, n_2, ..., n_m$ such that α can be written as $\alpha = \sum n_i[V_i]$. The pseudo-effective cone $\overline{\mathrm{Eff}}_k(X) \subseteq N_k(X)$ is the closure of the cone generated by classes of effective cycles. The pseudo-effective dual classes also form a closed cone in $N^k(X)$ that we denote by $\overline{\mathrm{Eff}}^k(X)$.*

We list some of the basic properties of the pseudo-effective cone in the following proposition.

Proposition 2.3.1 (see [FL1], [FL2])**.** *(i) $\overline{\mathrm{Eff}}_k(X)$ span $N_k(X)$ and do not contain lines.*

(ii) For a morphism $f : X \longrightarrow Y$ of projective varieties $f_*(\overline{\mathrm{Eff}}_k(X)) \subseteq \overline{\mathrm{Eff}}_k(Y)$. Equality holds, if π is surjective.

(iii) $f^*(\overline{\mathrm{Eff}}_k(Y) \subset \overline{\mathrm{Eff}}_{k+r}(X)$ if f is flat of relative dimension r and f^* exists.

(iv) Suppose $t_1, t_2, ..., t_k$ are ample classes in $N^1(X)$. Then $t_1 \cdot t_2 \cdot ... \cdot t_k \cap [X]$ belongs to the interior of $\overline{\mathrm{Eff}}^k(X)$.

Definition 2.3.3. *The nef cone $\mathrm{Nef}^k(X)$ is the dual cone of $\mathrm{Eff}_k(X)$ in $N^k(X)$.*

More precisely

$$\mathrm{Nef}^k(X) = \Big\{ \alpha \in N^k(X) \quad | \alpha \cdot \beta \geq 0 \quad \forall \beta \in \overline{\mathrm{Eff}}_k(X) \Big\}$$

Proposition 2.3.2 (see [FL1], [FL2])**.** (*i*) $\mathrm{Nef}^k(X)$ *generates* $N^k(X)$ *and only contain half lines.*

(*ii*) *For a dominant morphism* $g : X \longrightarrow Y$ *of projective varieties,* $g^*\beta \in \mathrm{Nef}^k(Y)$ *implies* $\beta \in \mathrm{Nef}^k(X)$.

(*iii*) *If* $t_1, t_2, ..., t_k$ *are ample classes in* $N^1(X)$*, then* $t_1 \cdot t_2 \cdot ... \cdot t_k$ *belongs to the interior of* $\mathrm{Nef}^k(X)$.

Chapter 3

Nef cone of ample line bundles

3.1 Geometry of products of projective bundles over curves

Let E_1 and E_2 be two vector bundles over a smooth complex projective curve C of rank r_1, r_2 and degrees d_1, d_2 respectively. Let $\mathbb{P}(E_1) = \mathbf{Proj}\ (\oplus_{d\geq 0} Sym^d(E_1))$ and $\mathbb{P}(E_2) = \mathbf{Proj}\ (\oplus_{d\geq 0} Sym^d(E_2))$ be the associated projective bundles together with the projection morphisms $\pi_1 : \mathbb{P}(E_1) \longrightarrow C$ and $\pi_2 : \mathbb{P}(E_2) \longrightarrow C$ respectively. Let $X = \mathbb{P}(E_1) \times_C \mathbb{P}(E_2)$ be the fibre product over C. Consider the following commutative diagram:

$$\begin{array}{ccc} X = \mathbb{P}(E_1) \times_C \mathbb{P}(E_2) & \xrightarrow{p_2} & \mathbb{P}(E_2) \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \pi_2} \\ \mathbb{P}(E_1) & \xrightarrow{\pi_1} & C \end{array}$$

Let f_1, f_2 and F denote the numerical equivalence classes of the fibres of the maps π_1, π_2 and $\pi_1 \circ p_1 = \pi_2 \circ p_2$ respectively. Note that $X \cong \mathbb{P}(\pi_1^*(E_2)) \cong \mathbb{P}(\pi_2^*(E_1))$. We first fix the following notations for the numerical equivalence classes in $N^1(X)$,

$$\eta_1 = \left[\mathcal{O}_{\mathbb{P}(E_1)}(1)\right] \in N^1(\mathbb{P}(E_1)) \quad , \quad \eta_2 = \left[\mathcal{O}_{\mathbb{P}(E_2)}(1)\right] \in N^1(\mathbb{P}(E_2)),$$
$$\zeta_1 = p_1^*(\eta_1),\, \zeta_2 = p_2^*(\eta_2)$$

We here summarise some results that have been discussed in [KMR] (See Section 3 in [KMR] for more details) :

$$F = p_1^*(f_1) = p_2^*(f_2) \quad , \quad F^2 = 0 \ , \ N^1(X) = \mathbb{R}(\zeta_1) \oplus \mathbb{R}(\zeta_2) \oplus \mathbb{R}F \ ,$$

$$\zeta_2^{r_2} \cdot F = 0 \ , \ \zeta_2^{r_2+1} = 0 \ , \ \zeta_1^{r_1} \cdot F = 0 \ , \ \zeta_1^{r_1+1} = 0 \ ,$$

$$\zeta_1^{r_1} = (\deg(E_1))F \cdot \zeta_1^{r_1-1} \quad , \quad \zeta_2^{r_2} = (\deg(E_2))F \cdot \zeta_2^{r_2-1} \quad ,$$

$$\zeta_1^{r_1} \cdot \zeta_2^{r_2-1} = \deg(E_1) \quad , \quad \zeta_2^{r_2} \cdot \zeta_1^{r_1-1} = \deg(E_2) \ .$$

Also, the dual basis of $N_1(X)$ is $\{\delta_1, \delta_2, \delta_3\}$ where,

$$\delta_1 = F \cdot \zeta_1^{r_1-2} \cdot \zeta_2^{r_2-1} \quad , \quad \delta_2 = F \cdot \zeta_1^{r_1-1} \cdot \zeta_2^{r_2-2},$$

$$\delta_3 = \zeta_1^{r_1-1} \cdot \zeta_2^{r_2-1} - \deg(E_1)F \cdot \zeta_1^{r_1-2} \cdot \zeta_2^{r_2-1} - \deg(E_2)F \cdot \zeta_1^{r_1-1} \cdot \zeta_1^{r_2-2}.$$

Let C be a smooth curve over the field of complex numbers $\mathbb{C}$ and let E be a vector bundle over C. The slope of E is defined as

$$\mu(E) := \tfrac{\deg E}{r} \in \mathbb{Q}$$

A vector bundle E over C is said to be semistable if $\mu(F) \leq \mu(E)$ for all non-zero subbundles $F \subseteq E$. For every vector bundle E, there is a unique filtration

$$E = E_0 \supset E_1 \supset \cdots \supset E_{l-1} \supset E_l = 0$$

called the Harder-Narasimhan filtration, such that E_i/E_{i+1} is semistable for each $i \in \{0, 1, \cdots, l-1\}$ and $\mu(E_i/E_{i+1}) > \mu(E_{i-1}/E_i)$ for all $i \in \{1, 2, \cdots, l-1\}$.

Let E_1 and E_2 be two vector bundles of rank r_1 and r_2 and degree d_1 and d_2 respectively over a smooth curve C.

Let E_1 admits the unique Harder-Narasimhan filtration

$$E_1 = E_{10} \supset E_{11} \supset ... \supset E_{1l_1} = 0$$

with $Q_{1i} := E_{1(i-1)}/E_{1i}$ being semistable for all $i \in [1, l_1 - 1]$. Denote $n_{1i} = \text{rank}(Q_{1i})$,
$d_{1i} = \deg(Q_{1i})$ and $\mu_{1i} = \mu(Q_{1i}) := \frac{d_{1i}}{n_{1i}}$ for all i.

Similarly, let E_2 also admits the unique Harder-Narasimhan filtration

$$E_2 = E_{20} \supset E_{21} \supset ... \supset E_{2l_2} = 0$$

with $Q_{2i} := E_{2(i-1)}/E_{2i}$ being semistable for $i \in [1, l_2 - 1]$. Denote $n_{2i} = \text{rank}(Q_{2i})$, $d_{2i} = \deg(Q_{2i})$ and $\mu_{2i} = \mu(Q_{2i}) := \frac{d_{2i}}{n_{2i}}$ for all i.

Theorem 3.1.1. *Let E_1 and E_2 be two vector bundles on a smooth complex projective curve C and let $X = \mathbb{P}(E_1) \times_C \mathbb{P}(E_2)$ as discussed earlier. Then,*

$$\text{Nef}(\mathbb{P}(E_1) \times_C \mathbb{P}(E_2)) = \left\{a\tau_1 + b\tau_2 + cF \mid a, b, c \in \mathbb{R}_{\geq 0}\right\}.$$

where $\tau_1 = \zeta_1 - \mu_{11}F$ and $\tau_2 = \zeta_2 - \mu_{21}F$ and μ_{11} and μ_{21} are the smallest slopes of any torsion-free quotients of E_1 and E_2 respectively, with the same notation as above.

3.2 Proof of Theorem 3.1.1

Proof. By the result of [F], $\text{Nef}(\mathbb{P}(E_i)) = \left\{a_i(\eta_i - \mu_{i1}f_i) + b_i f_i \mid a_i, b_i \in \mathbb{R}_{\geq 0}\right\}$ for $i = 1, 2$. Since pullback of nef line bundles are nef, we get , $\tau_1 = \zeta_1 - \mu_{11}F$,

$\tau_2 = \zeta_2 - \mu_{21}F$ and F are nef.

Now, from the Harder-Narasimhan filtration of E_i's $(i = 1, 2)$ as described above, we get the following short exact sequences

$$0 \longrightarrow E_{i1} \longrightarrow E_i \longrightarrow Q_{i1} \longrightarrow 0$$

for $i = 1, 2$.

Let $j_i : \mathbb{P}(Q_{i1}) \longrightarrow \mathbb{P}(E_i)$ denote the canonical embeddings for $i = 1, 2$.

We now proceed along the lines of [Section 2, [F]]. The result in [Example 3.2.17, [Ful]] adjusted to bundles of quotients over curves shows that

$$\big[\mathbb{P}(Q_{11})\big] = \eta_1^{r_1 - n_{11}} + (d_{11} - d_1)\eta_1^{r_1 - n_{11} - 1} f_1 \in N_{n_{11}}(\mathbb{P}(E_1))$$

and

$$\big[\mathbb{P}(Q_{21})\big] = \eta_2^{r_2 - n_{21}} + (d_{21} - d_2)\eta_2^{r_2 - n_{21} - 1} f_2 \in N_{n_{21}}(\mathbb{P}(E_2))$$

where $n_{11} = \operatorname{rank}(Q_{11})$, $n_{21} = \operatorname{rank}(Q_{21}), d_{11} = \deg(Q_{11})$ and $d_{21} = \deg(Q_{21})$.

As $(\eta_1 - \mu_{11} f_1)$ and $(\eta_2 - \mu_{21} f_2)$ are both nef divisors, we have

$\theta_{11} := \big[\mathbb{P}(Q_{11})\big] \cdot \big(\eta_1 - \mu_{11} f_1\big)^{n_{11} - 1}$

$= \big\{\eta_1^{r_1 - n_{11}} + (d_{11} - d_1)\eta_1^{r_1 - n_{11} - 1} f_1\big\} \cdot \big(\eta_1 - \mu_{11} f_1\big)^{n_{11} - 1} \in \overline{\mathrm{Eff}}_1(\mathbb{P}(E_1))$

and

$\theta_{21} := \big[\mathbb{P}(Q_{21})\big] \cdot \big(\eta_2 - \mu_{21} f_2\big)^{n_{21} - 1}$

$= \big\{\eta_2^{r_2 - n_{21}} + (d_{21} - d_2)\eta_2^{r_2 - n_{21} - 1} f_2\big\} \cdot \big(\eta_2 - \mu_{21} f_2\big)^{n_{21} - 1} \in \overline{\mathrm{Eff}}_1(\mathbb{P}(E_2))$.

Note that, p_1 and p_2 are proper, flat morphisms, and as the base space is smooth,

p_1, p_2 are also smooth. Hence, numerical pullbacks of cycles are well defined and the flatness of p_1 and p_2 ensure that pullbacks of numerical classes preserve the pseudo-effectivity. We consider $D := p_1^*(\theta_{11}) \cdot p_2^*(\theta_{21})$, which is equal to

$$p_1^*\big[\mathbb{P}(Q_{11})\big] \cdot p_2^*\big[\mathbb{P}(Q_{21})\big] \cdot \big\{\zeta_1 - \mu_{11}F\big\}^{n_{11}-1} \cdot \big\{\zeta_2 - \mu_{21}F\big\}^{n_{21}-1}$$

By using the above descriptions of θ_{11} and θ_{21}, D can be written as

$$D = \big\{\zeta_1^{r_1-n_{11}} + (d_{11}-d_1)F \cdot \zeta_1^{r_1-n_{11}-1}\big\} \cdot \big\{\zeta_2^{r_2-n_{21}} + (d_{21}-d_2)F \cdot \zeta_2^{r_2-n_{21}-1}\big\} \cdot \big\{\zeta_1 - \mu_{11}F\big\}^{n_{11}-1} \cdot \big\{\zeta_2 - \mu_{21}F\big\}^{n_{21}-1}$$

$$= \big\{\zeta_1^{r_1-1} + (\mu_{11}-d_1)F \cdot \zeta_1^{r_1-2}\big\} \cdot \big\{\zeta_2^{r_2-1} + (\mu_{21}-d_2)F \cdot \zeta_2^{r_2-2}\big\}$$

$$= \zeta_1^{r_1-1} \cdot \zeta_2^{r_2-1} + (\mu_{11}-d_1)F \cdot \zeta_1^{r_1-2} \cdot \zeta_2^{r_2-1} + (\mu_{21}-d_2)F \cdot \zeta_2^{r_2-2} \cdot \zeta_1^{r_1-1}$$

which is clearly a 1-cycle in X. Now, $p_1^*\big[\mathbb{P}(Q_{11})\big] \cdot p_2^*\big[\mathbb{P}(Q_{21})\big] = \big[\mathbb{P}(Q_{11}) \times_C \mathbb{P}(Q_{21})\big]$ is an effective cycle in X , and $\zeta_1 - \mu_{11}F, \zeta_2 - \mu_{21}F$ are nef divisors in X. Hence, $D \in \overline{\mathrm{Eff}}_1(X)$.

Since $\tau_1 \cdot D = \big\{\zeta_1 - \mu_{11}F\big\} \cdot D = 0, \tau_2 \cdot D = \big\{\zeta_2 - \mu_{21}F\big\} \cdot D = 0$ and $F^2 = 0$, τ_1, τ_2, F are in the boundary of $\mathrm{Nef}(X)$.

If $a\tau_1 + b\tau_2 + cF$ is any element in $\mathrm{Nef}(X)$, then $(a\tau_1 + b\tau_2 + cF) \cdot D \geq 0$, which implies that $c \geq 0$. Also, $F \cdot \tau_1^{r_1-2} \cdot \tau_2^{r_2-1}$ and $F \cdot \tau_1^{r_1-1} \cdot \tau_2^{r_2-2}$ are intersections of nef divisors. Now

$$\begin{aligned}(a\tau_1 + b\tau_2 + cF) \cdot (F \cdot \tau_1^{r_1-2} \cdot \tau_2^{r_2-1}) &= aF \cdot \tau_1^{r_1-1} \cdot \tau_2^{r_2-1} + bF \cdot \tau_1^{r_1-2} \cdot \tau_2^{r_2} + cF^2 \cdot \tau_1^{r_1-2} \cdot \tau_2^{r_2-1} \\ &= aF \cdot \zeta_1^{r_1-1} \cdot \zeta_2^{r_2-1} + bF \cdot \zeta_1^{r_1-2} \cdot \zeta_2^{r_2} + 0 \\ &= a + 0 + 0 \\ &= a\end{aligned}$$

and

$$
\begin{aligned}
(a\tau_1 + b\tau_2 + cF) \cdot (F \cdot \tau_1^{r_1-1} \cdot \tau_2^{r_2-2}) &= aF \cdot \tau_1^{r_1} \cdot \tau_2^{r_2-2} + bF \cdot \tau_1^{r_1-1} \cdot \tau_2^{r_2-1} + cF^2 \cdot \tau_1^{r_1-1} \cdot \tau_2^{r_2-2} \\
&= b + 0 + 0 \\
&= b
\end{aligned}
$$

Since, $a\tau_1 + b\tau_2 + cF \in \mathrm{Nef}(X)$, we have $a \geq 0, b \geq 0$. This completes the proof. □

Corollary 3.2.1. *Assume that the hypotheses of Theorem 3.1.1 holds. Then, the closed cone of curves of X is given by*

$$
\overline{\mathrm{NE}}(X) = \left\{ p\delta_1 + q\delta_2 + r(\delta_3 + \mu_{11}\delta_1 + \mu_{21}\delta_2) \mid p, q, r \in \mathbb{R}_{\geq 0} \right\}.
$$

Remark 3.2.1. *If E_1 and E_2 both are semistable bundles in Theorem 3.1.1 , then for each $i \in \{1, 2\}$, $\mathbb{P}(Q_{i1}) \subset \mathbb{P}(E_i)$ becomes an equality and by putting μ_1 and $\mu_2, (\mu_i = \mu(E_i),\ i = 1, 2)$ in place of μ_{11} and μ_{21} in the description above, we recover an earlier result in [KMR] (see Theorem 4.1 in [KMR]). Similar alterations can be made if one of the vector bundles is semistable and the other is unstable.*

Chapter 4

Seshadri constants

4.1 Seshadri constants of ample line bundles on $X = \mathbb{P}(E_1) \times_C \mathbb{P}(E_2)$

In this section, we will compute the Seshadri constants of ample line bundles on $X = \mathbb{P}(E_1) \times_C \mathbb{P}(E_2)$ in certain cases and will give bounds in some other cases. See the introduction for the definition of Seshadri constant.

Theorem 4.1.1. *Let E_1 and E_2 be two vector bundles on a smooth curve C with μ_{11} and μ_{21} being the smallest slopes of any torsion-free quotient of E_1 and E_2 respectively and let $X = \mathbb{P}(E_1) \times_C \mathbb{P}(E_2)$. Let L be an ample line bundle on X which is numerically equivalent to $a\tau_1 + b\tau_2 + cF \in N^1(X)$. Then, the Seshadri constants of L satisfy,*

$$\varepsilon(X, L, x) \geq \min\{a, b, c\}, \;\; \forall x \in X.$$

Moreover,

(*i*) *if $a = \min\{a, b, c\}$, then $\varepsilon(X, L, x) = a, \;\; \forall x \in X$*

(ii) if $b = \min\{a, b, c\}$*, then* $\varepsilon(X, L, x) = b$, $\forall x \in X$.

Theorem 4.1.2. *Let* E_1 *and* E_2 *be two unstable vector bundles over a smooth curve* C *of rank* r_1 *and* r_2 *respectively and* $X = \mathbb{P}(E_1) \times_C \mathbb{P}(E_2)$*. Let* L *be an ample line bundle on* X *numerically equivalent to* $a\tau_1 + b\tau_2 + cF \in N^1(X)$*. When* $c = \min\{a, b, c\}$ *the Seshadri constants of* L *have the following properties.*

(i) Assume $c \leq a \leq b$, $\mathrm{rank}(E_1) = 2$ *and* E_1 *is normalised.*

If x *is a point outside* $\mathbf{B}_-(\zeta_1)$*, then* $\varepsilon(X, L, x) = a$.

If x *belongs to* $\mathbf{B}_-(\zeta_1)$*, then* $c \leq \varepsilon(X, L, x) \leq a$.

(ii) Assume $c \leq b \leq a$, $\mathrm{rank}(E_2) = 2$ *and* E_2 *is normalised.*

If x *is a point outside* $\mathbf{B}_-(\zeta_2)$*, then* $\varepsilon(X, L, x) = b$.

If x *belongs to* $\mathbf{B}_-(\zeta_2)$*, the* $c \leq \varepsilon(X, L, x) \leq b$.

(iii) If x *is on some curve whose class is proportional to* $\overline{\delta_3}$*, then* $\varepsilon(X, L, x) = c$*, where* $\overline{\delta_3} = \delta_3 + \mu_{11}\delta_1 + \mu_{21}\delta_2$.

4.2 Proof of Theorem 4.1.1

Before going into the proof of the Theorem 4.1.1, we will prove the following useful lemma.

Lemma 4.2.1. *Let* L *be an* $\mathbb{R}$*-divisor of type* (a, b) *on* $\mathbb{P}^n \times \mathbb{P}^m$*, with* $a, b \in \mathbb{R}_{\geq 0}$*. Then,*

$$\varepsilon(\mathbb{P}^n \times \mathbb{P}^m, L, p) = \min\{a, b\} \ \forall p \in \mathbb{P}^n \times \mathbb{P}^m$$

Proof. Let B be an irreducible curve in $\mathbb{P}^n \times \mathbb{P}^m$. Then, B can be written as $B = x(1, 0)^{n-1} + y(0, 1)^{m-1}$ for some $x, y \in \mathbb{R}_{\geq 0}$. Also, for any $p \in \mathbb{P}^n \times \mathbb{P}^m$, we have

$\deg B \geq \operatorname{mult}_p B$. Hence,

$$\frac{L \cdot B}{\operatorname{mult}_p B} = \frac{ay + bx}{\operatorname{mult}_p B} \geq \min\{a, b\} \cdot \frac{y + x}{\operatorname{mult}_p B} \geq \min\{a, b\}$$

Now, for any point $p \in \mathbb{P}^n \times \mathbb{P}^m$, write $p = (p_1, p_2)$, with $p_1 \in \mathbb{P}^n$ and $p_2 \in \mathbb{P}^m$.

Then, $p \in p_1 \times l_2$ and $p \in l_1 \times p_2$, where l_1 and l_2 are classes of lines in $\mathbb{P}^n$ and $\mathbb{P}^m$ respectively. This gives us

$$\varepsilon(\mathbb{P}^n \times \mathbb{P}^m, L, p) \leq \frac{L \cdot (p_1 \times l_2)}{1} = a \quad \textit{and} \quad \varepsilon(\mathbb{P}^n \times \mathbb{P}^m, L, p) \leq \frac{L \cdot (l_1 \times p_2)}{1} = b$$

which implies $\varepsilon(\mathbb{P}^n \times \mathbb{P}^m, L, p) \leq \min\{a, b\}$. This proves the lemma. □

Proof. of Theorem 4.1.1 : By Theorem 3.1.1 and Corollary 3.2.1,

$$\operatorname{Nef}(X) = \left\{a\tau_1 + b\tau_2 + cF \mid a, b, c \in \mathbb{R}_{\geq 0}\right\}$$

and

$$\overline{\mathrm{NE}}(X) = \left\{p\delta_1 + q\delta_2 + r\overline{\delta_3} \in N_1(X) \mid p, q, r \in \mathbb{R}_{\geq 0}\right\},$$

where $\overline{\delta_3} = \delta_3 + \mu_{11}\delta_1 + \mu_{21}\delta_2$.

Let B be a reduced and irreducible curve passing through $x \in X$ with multiplicity m at $x \in X$. Then B can be written as $B = p\delta_1 + q\delta_2 + r\overline{\delta_3} \in \overline{\mathrm{NE}}(X) \subseteq N_1(X)$. Two cases can occur :

Case I

Assume that B is not contained in any fibre of the map $(\pi_1 \circ p_2)$ over the curve C. Hence, by Bézout's Theorem :

$$F \cdot B \geq \operatorname{mult}_x B = m \tag{4.1}$$

This implies, $r \geq m$. Since L is ample, $a, b, c > 0$. Hence,

$$\frac{L \cdot B}{\operatorname{mult}_x B} = \frac{L \cdot B}{m} = \frac{(a\tau_1 + b\tau_2 + cF) \cdot (p\delta_1 + q\delta_2 + r\overline{\delta_3})}{m}$$

$$= \frac{ap + bq}{m} + c \cdot \frac{r}{m} \geq c \cdot \frac{r}{m} \geq c.$$

Case II

Assume that B is contained in some fibre F of the map $(\pi_1 \circ p_1)$ over the curve C. Hence, $F \cdot B = 0$ which implies $r = 0$. We know that the fibres of the map $(\pi_1 \circ p_1)$ are isomorphic to $\mathbb{P}^{r_1-1} \times \mathbb{P}^{r_2-1}$. Since B is curve in $\mathbb{P}^{r_1-1} \times \mathbb{P}^{r_2-1}$ passing through x of multiplicity m, then from Lemma 4.2.1 , $\frac{L \cdot B}{\operatorname{mult}_x B} \geq \min\{a, b\}$.

Combining both cases, we have, $\varepsilon(X, L, x) := \inf\limits_{x \in C} \{\frac{L \cdot C}{\operatorname{mult}_x C}\} \geq \min\{a, b, c\}$, $\forall x \in X$.

Now, a point $x \in X$ can be written as $x = (x_1, x_2)$, where $x_1 \in \mathbb{P}(E_1), x_2 \in \mathbb{P}(E_2)$. Take the class of a line l_2 in the fibre f_2 of π_2 passing through x_2. Then, $x \in x_1 \times l_2 = \delta_1\{= F \cdot \zeta_1^{r_1-2} \cdot \zeta_2^{r_2-1}\}$ in $N_1(X)$. So,

$$\varepsilon(X, L, x) \leq \frac{L \cdot \delta_1}{1} = a.$$

When $a = \min\{a, b, c\}$, using the above inequality and the fact that $\varepsilon(X, L, x) \geq \min\{a, b, c\}$, we conclude that $\varepsilon(X, L, x) = a$.

Similarly, take the class of a line l_1 in the fibre f_1 of π_1 passing through x_1. Then, $x \in l_1 \times x_2 = \delta_2\{= F \cdot \zeta_1^{r_1-1} \cdot \zeta_2^{r_2-2}\}$ in $N_1(X)$. So,

$$\varepsilon(X, L, x) \leq \frac{L \cdot \delta_2}{1} = b.$$

So, if $b = \min\{a, b, c\}$, the above inequality and $\varepsilon(X, L, x) \geq \min\{a, b, c\}$ implies that $\varepsilon(X, L, x) = b$. This proves (i) and (ii). □

4.3 Proof of Theorem 4.1.2

Proof. Let $B \subseteq X$ be a reduced and irreducible curve passing through $x \in X$ and m be the multiplicity of B at x. Let $B = p\delta_1 + q\delta_2 + r\overline{\delta_3} \in \overline{\mathrm{NE}}(X) \subseteq N_1(X)$, where p, q, r are in $\mathbb{R}_{\geq 0}$ and $\overline{\delta_3} = \delta_3 + \mu_{11}\delta_1 + \mu_{21}\delta_2$.

First, assume that $c \leq a \leq b$. Let x be a point outside of $\mathbf{B}_{-}(\zeta_1)$. Then, B is also not contained in $\mathbf{B}_{-}(\zeta_1)$. Hence, $p_2^*\eta_1 \cdot B \geq 0$ i.e,

$$\zeta_1 \cdot (p\delta_1 + q\delta_2 + r\overline{\delta_3}) \geq 0.$$

which implies, $p + r\mu_{11} \geq 0$.

Now if B is not contained in the fibre, then by Case(I) in the proof of Theorem 4.1.1, we get , $r \geq m$. Hence,

$$\varepsilon(X, L, x) = \frac{ap + bq + cr}{m} \geq \frac{r}{m}(c - a\mu_{11}) + \frac{bq}{m} \geq \frac{r}{m}(c - a\mu_{11}) \geq (c - a\mu_{11}) \geq -a\mu_{11} \geq a.$$

(since $\mathrm{rank}(E_1) = 2$ and E_1 is normalised, $\mu(Q_{11}) = \mu_{11} = \deg(Q_{11}) \leq -1$).

And if B is contained in the fibre, then by Case (II) in the proof of Theorem 4.1.1, we get, $(p+q) \geq m$. Hence,

$$\varepsilon(X, L, x) = \frac{ap+bq}{m} \geq \frac{a(p+q)}{m} \geq a$$

as our assumption is $b \geq a \geq c$. We already know that $\varepsilon(X, L, x) \leq a$ from the proof of (4.1.1). So, $\varepsilon(X, L, x) = a$. If x belongs to $\mathbf{B}_{-}(\zeta_1)$, then it is obvious that $c \leq \varepsilon(X, L, x) \leq a$. This completes the proof of (i). A similar kind of argument will prove (ii).

To prove (iii), observe that $L \cdot \overline{\delta_3} = c$. So,

$$\varepsilon(X, L, x) \leq \frac{L \cdot \overline{\delta_3}}{\text{mult}_x \overline{\delta_3}} \leq \frac{c}{\text{mult}_x \overline{\delta_3}} \leq c.$$

Therefore, by the above inequality and first part of theorem 4.1.1, we get, $\varepsilon(X, L, x) = c$. ☐

Corollary 4.3.1. *Assume the hypotheses of Theorem 4.1.2 holds and let L be an ample line bundle on X numerically equivalent to $a\tau_1 + b\tau_2 + cF \in N^1(X)$. Then, we have,*

(i) $\varepsilon(X, L) = \min\{a, b, c\}$.

(ii) $\varepsilon(X, L, 1) \leq \min\{a, b\}$.

Proof. Since $\varepsilon(X, L, x) \geq \min\{a, b, c\}$, for all $x \in X$, we have,

$$\varepsilon(X, L) = \inf_{x \in X} \varepsilon(X, L, x) \geq \min\{a, b, c\}.$$

Now, if $\min\{a, b, c\} = a$, $\varepsilon(X, L, x) = \varepsilon(X, L) = \min\{a, b, c\} = a, \forall x \in X$. Similarly, if $\min\{a, b, c\} = b, \varepsilon(X, L, x) = \varepsilon(X, L) = \min\{a, b, c\} = b, \forall x \in X$. Also, when $\min\{a, b, c\} = c$, then, $\varepsilon(X, L, x) = \min\{a, b, c\} = c \geq \varepsilon(X, L)$, if x is on some curve of class proportional to $\overline{\delta_3}$. Therefore, combining all three cases, we have, $\varepsilon(X, L) = \min\{a, b, c\}$.

In the proof of Theorem 4.1.1 we have shown that for all $x \in X$, $\varepsilon(X, L, x) \leq \min\{a, b\}$. So, this implies that $\varepsilon(X, L, 1) \leq \min\{a, b\}$. □

Theorem 4.3.1. *Let E_1 be a semistable vector bundle of rank r_1 and E_2 be an unstable vector bundle of rank r_2 over a smooth curve C and let $X = \mathbb{P}(E_1) \times_C \mathbb{P}(E_2)$. Let L be an ample bundle on X numerically equivalent to $a\tau_1 + b\tau_2 + cF \in N^1(X)$. When $c = \min\{a, b, c\}$ the Seshadri constants of L have the following properties.*

Assume that $c \leq b \leq a$, $\mathrm{rank}(E_2) = 2$ and E_2 is normalised.

(i) if x is a point outside $\mathbf{B}_-(\zeta_2)$, then $\varepsilon(X, L, x) = b$.

(ii) if x belongs to $\mathbf{B}_-(\zeta_2)$, then $c \leq \varepsilon(X, L, x) \leq b$.

Proof. The proof is similar to the proof of the Theorem 4.1.2. □

Chapter 5

Pseudo-effective cone of cycles

5.1 Pseudo-effective cone of cycles on $X = \mathbb{P}(E_1) \times_C \mathbb{P}(E_2)$

In this section we compute the pseudo-effective cone of cycles on $X = \mathbb{P}(E_1) \times_C \mathbb{P}(E_2)$, where E_1 and E_2 are two vector bundles over a smooth curve C.

Theorem 5.1.1. *Let $r_1 = \text{rank}(E_1)$ and $r_2 = \text{rank}(E_2)$ and without loss of generality assume that $r_1 \leq r_2$. Then the bases of $N^k(X)$ are given by*

$$N^k(X) = \begin{cases} \left(\{\zeta_1^i \cdot \zeta_2^{k-i}\}_{i=0}^{k}, \{F \cdot \zeta_1^j \cdot \zeta_2^{k-j-1}\}_{j=0}^{k-1}\right) & if \quad k < r_1 \\ \\ \left(\{\zeta_1^i \cdot \zeta_2^{k-i}\}_{i=0}^{r_1-1}, \{F \cdot \zeta_1^j \cdot \zeta_2^{k-j-1}\}_{j=0}^{r_1-1}\right) & if \quad r_1 \leq k < r_2 \\ \\ \left(\{\zeta_1^i \cdot \zeta_2^{k-i}\}_{i=t+1}^{r_1-1}, \{F \cdot \zeta_1^j \cdot \zeta_2^{k-j-1}\}_{j=t}^{r_1-1}\right) & if \quad k = r_2 + t \quad where \quad t \in \{0, 1, 2, ..., r_1 - 2\}. \end{cases}$$

Proof. To begin with consider the case where $k < r_1$. We know that $X \cong \mathbb{P}(\pi_2^* E_1)$

and the natural morphism $\mathbb{P}(\pi_2^*E_1) \longrightarrow \mathbb{P}(E_2)$ can be identified with p_2. With the above identifications in place the chow group of X has the following isomorphism [see Theorem 3.3 , page 64 [Ful]]

$$A(X) \cong \bigoplus_{i=0}^{r_1-1} \zeta_1^i A(\mathbb{P}(E_2)) \tag{5.1}$$

Choose i_1, i_2 such that $0 \leq i_1 < i_2 \leq k$. Consider the k- cycle $\alpha := F \cdot \zeta_1^{r_1-i_1-1} \cdot \zeta_2^{r_2+i_1-k-1}$.

Then $\zeta_1^{i_1} \cdot \zeta_2^{k-i_1} \cdot \alpha = 1$ but $\zeta_1^{i_2} \cdot \zeta_2^{k-i_2} \cdot \alpha = 0$. So, $\{\zeta_1^{i_1} \cdot \zeta_2^{k-i_1}\}$ and $\{\zeta_1^{i_2} \cdot \zeta_2^{k-i_2}\}$ can not be numerically equivalent.

Similarly, take j_1, j_2 such that $0 \leq j_1 < j_2 \leq k$ and consider the k-cycle
$\beta := \zeta_1^{r_1-j_1-1} \cdot \zeta_2^{r_2+j_1-k}$.

Then as before it happens that $F \cdot \zeta_1^{j_1} \cdot \zeta_2^{k-j_1-1} \cdot \beta = 1$ but $F \cdot \zeta_1^{j_2} \cdot \zeta_2^{k-j_2-1} \cdot \beta = 0$. So $\{F \cdot \zeta_1^{j_1} \cdot \zeta_2^{k-j_1-1}\}$ and $\{F \cdot \zeta_1^{j_2} \cdot \zeta_2^{k-j_2-1}\}$ can not be numerically equivalent.

For the remaining case let's assume $0 \leq i \leq j \leq k$ and consider the k-cycle $\gamma := F \cdot \zeta_1^{r_1-i-1} \cdot \zeta_2^{r_2+i-1-k}$.

Then $\{\zeta_1^i \cdot \zeta_2^{k-i}\} \cdot \gamma = 1$ and $\{F \cdot \zeta_1^j \cdot \zeta_2^{k-j-1}\} \cdot \gamma = 0$. So, they can not be numerically equivalent. From these observations and 5.1 we obtain a basis of $N^k(X)$ which is given by

$$N^k(X) = \left(\{\zeta_1^i \cdot \zeta_2^{k-i}\}_{i=0}^k, \{F \cdot \zeta_1^j \cdot \zeta_2^{k-j-1}\}_{j=0}^{k-1} \right)$$

For the case $r_1 \leq k < r_2$ observe that $\zeta_1^{r_1+1} = 0$, $F \cdot \zeta_1^{r_1} = 0$ and $\zeta_1^{r_2} = deg(E_1) F \cdot \zeta_1^{r_1-1}$.

When $k \geq r_2$ we write as $k = r_2 + t$ where t ranges from 0 to $r_1 - 1$. In that case the observations like $\zeta_2^{r_2+1} = 0$, $F \cdot \zeta_2^{r_2} = 0$ and $\zeta_2^{r_2} = \deg(E_2) F \cdot \zeta_2^{r_2-1}$ proves our case. □

Now we are ready to treat the case where both E_1 and E_2 are semistable vector bundles over C.

Theorem 5.1.2. *Let E_1 and E_2 be two semistable vector bundles over C of rank r_1 and r_2 respectively with $r_1 \leq r_2$ and $X = \mathbb{P}(E_1) \times_C \mathbb{P}(E_2)$. Then for all $k \in \{1, 2, ..., r_1 + r_2 - 1\}$*

$$\overline{\mathrm{Eff}}^k(X) = \begin{cases} \left\langle \left\{(\zeta_1 - \mu_1 F)^i (\zeta_2 - \mu_2 F)^{k-i}\right\}_{i=0}^{k}, \left\{F \cdot \zeta_1^j \cdot \zeta_2^{k-j-1}\right\}_{j=0}^{k-1} \right\rangle & if \quad k < r_1 \\ \left\langle \left\{(\zeta_1 - \mu_1 F)^i (\zeta_2 - \mu_2 F)^{k-i}\right\}_{i=0}^{r_1-1}, \left\{F \cdot \zeta_1^j \cdot \zeta_2^{k-j-1}\right\}_{j=0}^{r_1-1} \right\rangle & if \quad r_1 \leq k < r_2 \\ \left\langle \left\{(\zeta_1 - \mu_1 F)^i (\zeta_2 - \mu_2 F)^{k-i}\right\}_{i=t+1}^{r_1-1}, \left\{F \cdot \zeta_1^j \cdot \zeta_2^{k-j-1}\right\}_{j=t}^{r_1-1} \right\rangle \\ if \quad k = r_2 + t, \quad t = 0, ..., r_1 - 1. \end{cases}$$

where $\mu_1 = \mu(E_1)$ and $\mu_2 = \mu(E_2)$.

5.2 Proof of Theorem 5.1.2

Proof. Firstly, $(\zeta_1 - \mu_1 F)^i \cdot (\zeta_2 - \mu_2 F)^{k-i}$ and $F \cdot \zeta_1^i \cdot \zeta_2^{k-j-1}[= F \cdot (\zeta_1 - \mu_1 F)^i \cdot (\zeta_2 - \mu_2 F)^{k-j-1}]$ are intersections of nef divisors. So, they are pseudo-effective for all $i \in \{0, 1, 2, ..., k\}$. conversely, when $k < r_1$ notice that we can write any element C of $\overline{\mathrm{Eff}}^k(X)$ as

$$C = \sum_{i=0}^{k} a_i (\zeta_1 - \mu_1 F)^i \cdot (\zeta_2 - \mu_2 F)^{k-i} + \sum_{j=0}^{k} b_j F \cdot \zeta_1^j \cdot \zeta_2^{k-j-1}$$

where $a_i, b_i \in \mathbb{R}$.

For a fixed i_1 intersect C with $D_{i_1} := F \cdot (\zeta_1 - \mu_1 F)^{r_1-i_1-1} \cdot (\zeta_2 - \mu_2 F)^{r_2-k+i_1-1}$ and for a fixed j_1 intersect C with $D_{j_1} := (\zeta_1 - \mu_1 F)^{r_1-j_1-1} \cdot (\zeta_2 - \mu_2 F)^{r_2+j_1-k}$. These intersections lead us to

$$C \cdot D_{i_1} = a_{i_1} \quad and \quad C \cdot D_{j_1} = b_{j_1}$$

Since $C \in \overline{\mathrm{Eff}}^k(X)$ and D_{i_1}, D_{j_1} are intersection of nef divisors, a_{i_1} and b_{j_1} are non-negetive. Now running i_1 andj_1 through $\{0, 1, 2, ..., k\}$ we get all the a_i's and b_i's are non-negetive and that proves our result for $k < r_1$. The cases where $r_1 \leq k < r_2$ and $k \geq r_2$ can be proved very similarly after the intersection products involving ζ_1 and ζ_2 in Section 1 of Chapter 3 are taken into count. □

Next we study the more interesting case where E_1 and E_2 are two unstable vector bundles of rank r_1 and r_2 and degree d_1 and d_2 respectively over a smooth curve C.

Let E_1 be the unique Harder-Narasimhan filtration

$$E_1 = E_{10} \supset E_{11} \supset ... \supset E_{1l_1} = 0$$

with $Q_{1i} := E_{1(i-1)}/E_{1i}$ being semistable for all $i \in [1, l_1 - 1]$. Denote $n_{1i} = \mathrm{rank}(Q_{1i})$,
$d_{1i} = \deg(Q_{1i})$ and $\mu_{1i} = \mu(Q_{1i}) := \frac{d_{1i}}{n_{1i}}$ for all i.

Similarly, E_2 also admits the unique Harder-Narasimhan filtration

$$E_2 = E_{20} \supset E_{21} \supset ... \supset E_{2l_2} = 0$$

with $Q_{2i} := E_{2(i-1)}/E_{2i}$ being semistable for $i \in [1, l_2 - 1]$. Denote $n_{2i} = \mathrm{rank}(Q_{2i})$, $d_{2i} = \deg(Q_{2i})$ and $\mu_{2i} = \mu(Q_{2i}) := \frac{d_{2i}}{n_{2i}}$ for all i.

Consider the natural inclusion $\bar{i} = i_1 \times i_2 : \mathbb{P}(Q_{11}) \times_C \mathbb{P}(Q_{21}) \longrightarrow \mathbb{P}(E_1) \times_C \mathbb{P}(E_2)$, which is induced by natural inclusions $i_1 : \mathbb{P}(Q_{11}) \longrightarrow \mathbb{P}(E_1)$ and $i_2 : \mathbb{P}(Q_{21}) \longrightarrow \mathbb{P}(E_2)$. In the next theorem we will see that the cycles of $\mathbb{P}(E_1) \times_C \mathbb{P}(E_2)$ of dimension at most $n_{11} + n_{21} - 1$ can be tied down to cycles of $\mathbb{P}(Q_{11}) \times_C \mathbb{P}(Q_{21})$ via $\bar{i}$.

Theorem 5.2.1. *Let E_1 and E_2 be two unstable bundle of rank r_1 and r_2 and degree d_1 and d_2 respectively over a smooth curve C and $r_1 \leq r_2$ without loss of generality and $X = \mathbb{P}(E_1) \times_C \mathbb{P}(E_2)$.*

Then for all $k \in \{1, 2, ..., \mathbf{n}\}$ ($\mathbf{n} := n_{11} + n_{21} - 1$)

Case(1): $\quad n_{11} \leq n_{21}$

$$\overline{\mathrm{Eff}}_k(X) = \begin{cases} \Big\langle \Big\{ [\mathbb{P}(Q_{11}) \times_C \mathbb{P}(Q_{21})](\zeta_1 - \mu_{11}F)^i(\zeta_2 - \mu_{21}F)^{\mathbf{n}-k-i} \Big\}_{i=t+1}^{n_{11}-1}, \\ \qquad \Big\{ F \cdot \zeta_1^{r_1 - n_{11} + j} \cdot \zeta_2^{r_2 + n_{11} - k - j - 2} \Big\}_{j=t}^{n_{11}-1} \Big\rangle \\ \qquad if \quad k < n_{11} \quad and \quad t = 0, 1, 2, ..., n_{11} - 2 \\ \Big\langle \Big\{ [\mathbb{P}(Q_{11}) \times_C \mathbb{P}(Q_{21})](\zeta_1 - \mu_{11}F)^i(\zeta_2 - \mu_{21}F)^{\mathbf{n}-k-i} \Big\}_{i=0}^{n_{11}-1}, \\ \qquad \Big\{ F \cdot \zeta_1^{r_1 - n_{11} + j} \cdot \zeta_2^{r_2 + n_{11} - k - j - 2} \Big\}_{j=0}^{n_{11}-1} \Big\rangle \\ \qquad if \quad n_{11} \leq k < n_{21}. \\ \Big\langle \Big\{ [\mathbb{P}(Q_{11}) \times_C \mathbb{P}(Q_{21})](\zeta_1 - \mu_{11}F)^i(\zeta_2 - \mu_{21}F)^{\mathbf{n}-k-i} \Big\}_{i=0}^{\mathbf{n}-k}, \\ \qquad \Big\{ F \cdot \zeta_1^{r_1 - n_{11} + j} \cdot \zeta_2^{r_2 + n_{11} - k - j - 2} \Big\}_{j=0}^{\mathbf{n}-k} \Big\rangle \\ \qquad if \quad k \geq n_{21}. \end{cases}$$

$Case(2)$: $n_{21} \le n_{11}$

$$\overline{\mathrm{Eff}}_k(X) = \begin{cases} \Big\langle \Big\{ [\mathbb{P}(Q_{11}) \times_C \mathbb{P}(Q_{21})](\zeta_2 - \mu_{21}F)^i(\zeta_1 - \mu_{11}F)^{\mathbf{n}-k-i} \Big\}_{i=t+1}^{n_{21}-1}, \\ \qquad \Big\{ F \cdot \zeta_2^{r_2-n_{21}+j} \cdot \zeta_1^{r_1+n_{21}-k-j-2} \Big\}_{j=t}^{n_{21}-1} \Big\rangle \\ \qquad if \quad k < n_{21} \quad and \quad t = 0, 1, 2, ..., n_{21}-2 \\ \Big\langle \Big\{ [\mathbb{P}(Q_{11}) \times_C \mathbb{P}(Q_{21})](\zeta_2 - \mu_{21}F)^i(\zeta_1 - \mu_{11}F)^{\mathbf{n}-k-i} \Big\}_{i=0}^{n_{21}-1}, \\ \qquad \Big\{ F \cdot \zeta_2^{r_2-n_{21}+j} \cdot \zeta_1^{r_1+n_{21}-k-j-2} \Big\}_{j=0}^{n_{21}-1} \Big\rangle \\ \qquad if \quad n_{21} \le k < n_{11}. \\ \Big\langle \Big\{ [\mathbb{P}(Q_{11}) \times_C \mathbb{P}(Q_{21})](\zeta_2 - \mu_{21}F)^i(\zeta_1 - \mu_{11}F)^{\mathbf{n}-k-i} \Big\}_{i=0}^{\mathbf{n}-k}, \\ \qquad \Big\{ F \cdot \zeta_2^{r_2-n_{21}+j} \cdot \zeta_1^{r_1+n_{21}-k-j-2} \Big\}_{j=0}^{\mathbf{n}-k} \Big\rangle \\ \qquad if \quad k \ge n_{11}. \end{cases}$$

Thus in both cases $\bar{i}_*$ *induces an isomorphism between* $\overline{\mathrm{Eff}}_k([\mathbb{P}(Q_{11}) \times_C \mathbb{P}(Q_{21})])$ *and* $\overline{\mathrm{Eff}}_k(X)$ *for* $k \le \mathbf{n}$.

5.3 Proof of Theorem 5.2.1

Proof. to begin with consider $Case(1)$ and then take $k \ge n_{21}$. Since $(\zeta_1 - \mu_{11}F)$ and $(\zeta_2 - \mu_{21}F)$ are nef

$$\phi_i := [\mathbb{P}(Q_{11}) \times_C \mathbb{P}(Q_{21})](\zeta_1 - \mu_{11}F)^i(\zeta_2 - \mu_{21}F)^{\mathbf{n}-k-i} \in \overline{\mathrm{Eff}}_k(X).$$

for all $i \in \{0, 1, 2, ..., \mathbf{n} - k\}$.

Now The result in [Example 3.2.17, [Ful]] adjusted to bundles of quotients over curves shows that

$$[\mathbb{P}(Q_{11})] = \eta_1^{r_1 - n_{11}} + (d_{11} - d_1)\eta_1^{r_1 - n_{11} - 1} f_1$$

and

$$[\mathbb{P}(Q_{21})] = \eta_2^{r_2 - n_{21}} + (d_{21} - d_2)\eta_2^{r_2 - n_{21} - 1} f_2$$

Also, $p_1^*[\mathbb{P}(Q_{11})] \cdot p_2^*[\mathbb{P}(Q_{21})] = [\mathbb{P}(Q_{11}) \times_C \mathbb{P}(Q_{21})]$. With little calculations it can be shown that

$\phi_i \cdot (\zeta_1 - \mu_{11}F)^{n_{11}-i} \cdot (\zeta_2 - \mu_{21}F)^{k+i+1-n_{11}}$

$= (\zeta_1^{r_1-n_{11}} + (d_{11} - d_1)F \cdot \zeta_1^{r_1-n_{11}-1})(\zeta_2^{r_2-n_{21}} + (d_{21} - d_2)F \cdot \zeta_2^{r_2-n_{21}-1})(\zeta_1 - \mu_{11}F)^{n_11-i} \cdot (\zeta_2 - \mu_{21}F)^{k+i+1-n_{11}}$

$= (\zeta_1^{r_1} - d_1F \cdot \zeta_1^{r_1-1})(\zeta_2^{r_2} - d_2F \cdot \zeta_2^{r_2-1}) = 0.$

So, ϕ_i 's are in the boundary of $\overline{\text{Eff}}_k(X)$ for all $i \in \{0, 1, ..., \mathbf{n} - k\}$. The fact that $F \cdot \zeta_1^{r_1-n_{11}+j} \cdot \zeta_2^{r_2+n_{11}-k-j-2}$'s are in the boundary of $\overline{\text{Eff}}_k(X)$ for all $i \in \{0, 1, ..., \mathbf{n} - k\}$ can be deduced from the proof of Theorem 5.1.2. The other cases can be proved similarly.

The proof of $Case(2)$ is similar to the proof of $Case(1)$.

Now, to show the isomorphism between pseudo-effective cones induced by $\bar{i}_*$ observe that Q_{11} and Q_{21} are semi-stable bundles over C. So, Theorem 5.1.2 gives the expressions for $\overline{\text{Eff}}_k([\mathbb{P}(Q_{11}) \times_C \mathbb{P}(Q_{21})])$. Let $\zeta_{11} = \mathcal{O}_{\mathbb{P}(\tilde{\pi}_2^*(Q_{11}))}(1) = \tilde{p}_1^*(\mathcal{O}_{\mathbb{P}(Q_{11})}(1))$ and $\zeta_{21} = \mathcal{O}_{\mathbb{P}(\tilde{\pi}_1^*(Q_{21})}(1) = \tilde{p}_2^*(\mathcal{O}_{\mathbb{P}(Q_{21})}(1))$, where $\tilde{\pi}_2 = \pi_2|_{\mathbb{P}(Q_{21})}$, $\tilde{\pi}_1 = \pi_1|_{\mathbb{P}(Q_{11})}$ and $\tilde{p}_1 : \mathbb{P}(Q_{11}) \times_C \mathbb{P}(Q_{21}) \longrightarrow \mathbb{P}(Q_{11})$, $\tilde{p}_2 : \mathbb{P}(Q_{11}) \times_C \mathbb{P}(Q_{21}) \longrightarrow \mathbb{P}(Q_{21})$ are the projection maps. Also notice that $\bar{i}^*\zeta_1 = \zeta_{11}$ and $\bar{i}^*\zeta_1 = \zeta_{21}$.

Using the above relations and projection formula the isomorphism between $\overline{\text{Eff}}_k([\mathbb{P}(Q_{11})\times_C \mathbb{P}(Q_{21})])$ and $\overline{\text{Eff}}_k(X)$ for $k \leq \mathbf{n}$ can be proved easily.

□

Next we want to show that higher dimension pseudo effective cycles on X can be related to the pseudo effective cycles on $\mathbb{P}(E_{11}) \times_C \mathbb{P}(E_{21})$. More precisely there is a isomorphism between $\overline{\text{Eff}}^k(X)$ and $\overline{\text{Eff}}^k([\mathbb{P}(E_{11}) \times_C \mathbb{P}(E_{21})])$ for $k < r_1 + r_2 - 1 - \mathbf{n}$. Useing the coning construction as in [F] we show this in two steps, first we establish an isomorphism between $\overline{\text{Eff}}^k([\mathbb{P}(E_1) \times_C \mathbb{P}(E_2)])$ and $\overline{\text{Eff}}^k([\mathbb{P}(E_{11}) \times_C \mathbb{P}(E_2)])$ and then an isomorphism between $\overline{\text{Eff}}^k([\mathbb{P}(E_{11}) \times_C \mathbb{P}(E_2)])$ and $\overline{\text{Eff}}^k([\mathbb{P}(E_{11}) \times_C \mathbb{P}(E_{21})])$ in similar fashion. But before proceeding any further we need to explore some more facts.

Let E be an unstable vector bundle over a non-singular projective variety V. There is a unique filtration

$$E = E^0 \supset E^1 \supset E^2 \supset ... \supset E^l = 0$$

which is called the Harder-Narasimhan filtration of E with $Q^i := E^{i-1}/E^i$ being semistable for $i \in [1, l-1]$. Now the following short-exact sequence

$$0 \longrightarrow E^1 \longrightarrow E \longrightarrow Q^1 \longrightarrow 0$$

induced by the Harder-Narasimhan filtration of E gives us the natural inclusion $j : \mathbb{P}(Q^1) \hookrightarrow \mathbb{P}(E)$. Considering $\mathbb{P}(Q^1)$ as a subscheme of $\mathbb{P}(E)$ we obtain the commutative diagram below by blowing up $\mathbb{P}(Q^1)$.

$$
\begin{array}{ccc}
\tilde{Y} = Bl_{\mathbb{P}(Q^1)}\mathbb{P}(E) & \xrightarrow{\Phi} & \mathbb{P}(E^1) = Z \\
\downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle q} \\
Y = \mathbb{P}(E) & \xrightarrow{\quad p \quad} & V
\end{array}
\tag{5.2}
$$

where Ψ is blow-down map.

The following theorem is reverberation of a similar result in [F]. We provide the details of the proof to cover every aspect of the result.

Theorem 5.3.1. *With the above notation, there exists a locally free sheaf G on Z such that $\tilde{Y} \simeq \mathbb{P}_Z(G)$ and $\nu : \mathbb{P}_Z(G) \longrightarrow Z$ it's corresponding bundle map.*

In particular if we place $V = \mathbb{P}(E_2)$, $E = \pi_2^* E_1$, $E^1 = \pi_2^* E_{11}$ and $Q^1 = \pi_2^* Q_{11}$ then the above commutative diagram becomes

$$
\begin{array}{ccc}
\tilde{Y'} = Bl_{\mathbb{P}(\pi_2^* Q_{11})}\mathbb{P}(\pi_2^* E_1) & \xrightarrow{\Phi'} & \mathbb{P}(\pi_2^* E_{11}) = Z' \\
\downarrow{\scriptstyle \Psi'} & & \downarrow{\scriptstyle \overline{p}_2} \\
Y' = \mathbb{P}(\pi_2^* E_1) & \xrightarrow{\quad p_2 \quad} & \mathbb{P}(E_2)
\end{array}
\tag{5.3}
$$

where $p_2 : \mathbb{P}(\pi_2^* E_1) \longrightarrow \mathbb{P}(E_2)$ and $\overline{p}_2 : \mathbb{P}(\pi_2^* E_{11}) \longrightarrow \mathbb{P}(E_2)$ are projection maps.

and there exists a locally free sheaf G' on Z' such that $\tilde{Y'} \simeq \mathbb{P}_{Z'}(G')$ and $\nu' : \mathbb{P}_{Z'}(G') \longrightarrow Z'$ it's bundle map.

Now let $\zeta_{Z'} = \mathcal{O}_{Z'}(1)$, $\gamma = \mathcal{O}_{\mathbb{P}_{Z'}(G')}(1)$, F the numerical equivalence class of a fibre of $\pi_2 \circ p_2$, F_1 the numerical equivalence class of a fibre of $\pi_2 \circ \overline{p}_2$, $\tilde{E}$ the class of the exceptional divisor of Ψ' and $\zeta_1 = p_1^*(\eta_1) = \mathcal{O}_{\mathbb{P}(\pi_2^* E_1)}(1)$. Then we have the following relations:

$$(5.4) \qquad \gamma = (\Psi')^* \zeta_1, \quad (\Phi')^* \zeta_{Z'} = (\Psi')^* \zeta_1 - \tilde{E}, \quad (\Phi')^* F_1 = (\Psi')^* F$$

$$(5.5) \qquad \tilde{E} \cdot (\Psi')^* (\zeta_1 - \mu_{11} F)^{n_{11}} = 0$$

Additionally, if we also denote the support of the exceptional divisor of $\tilde{Y'}$ *by* $\tilde{E}$ *, then* $\tilde{E} \cdot N(\tilde{Y'}) = (j_{\tilde{E}})_* N(\tilde{E})$*, where* $j_{\tilde{E}} : \tilde{E} \longrightarrow \tilde{Y'}$ *is the canonical inclusion.*

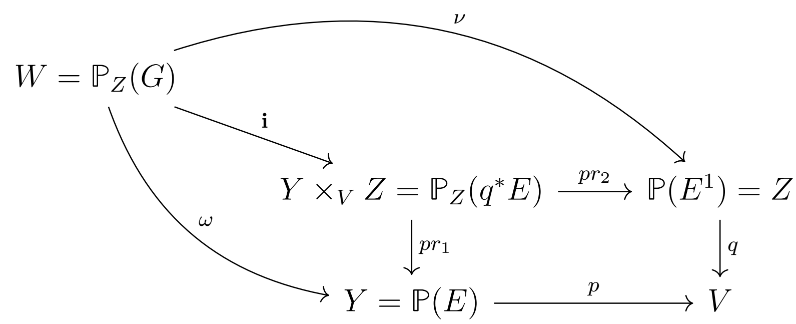
5.4 Proof of Theorem 5.3.1

Proof. With the above hypothesis the following commutative diagram is formed:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & q^*E^1 & \longrightarrow & q^*E & \longrightarrow & q^*Q^1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}(E^1)}(1) & \longrightarrow & G & \longrightarrow & q^*Q^1 & \longrightarrow & 0 \end{array}$$

where G is the push-out of morphisms $q^*E^1 \longrightarrow q^*E$ and $q^*E^1 \longrightarrow \mathcal{O}_{\mathbb{P}(E^1)}(1)$ and the first vertical map is the natural surjection. Now let $W = \mathbb{P}_Z(G)$ and $\nu : W \longrightarrow Z$ be it's bundle map. So there is a canonical surjection $\nu^*G \longrightarrow \mathcal{O}_{\mathbb{P}_Z(G)}(1)$. Also note that $q^*E \longrightarrow G$ is surjective by snake lemma. Combining these two we obtain a surjective morphism $\nu^*q^*E \longrightarrow \mathcal{O}_{\mathbb{P}_Z(G)}(1)$ which determines $\omega : W \longrightarrow Y$. We claim that we can identify $(\tilde{Y}, \Phi, \Psi)$ and (W, ν, ω). Now Consider the following commutative diagram:

(5.6)



where $\mathbf{i}$ is induced by the universal property of the fiber product. Since $\mathbf{i}$ can also be obtained from the surjective morphism $q^*E \longrightarrow G$ it is a closed immersion. Let $\mathcal{T}$ be the $\mathcal{O}_Y$ algebra $\mathcal{O}_Y \oplus \mathcal{I} \oplus \mathcal{I}^2 \oplus ...$, where $\mathcal{I}$ is the ideal sheaf of $\mathbb{P}(Q^1)$ in Y. We have an induced map of $\mathcal{O}_Y$- algebras $Sym(p^*E^1) \longrightarrow \mathcal{T} * \mathcal{O}_Y(1)$ which is onto because the image of the composition $p^*E^1 \longrightarrow p^*E \longrightarrow \mathcal{O}_Y(1)$ is $\mathcal{T} \otimes \mathcal{O}_Y(1)$. This induces a closed immersion

$$\mathbf{i'} : \tilde{Y} = Proj(\mathcal{T} * \mathcal{O}_Y(1)) \longrightarrow Proj(Sym(p^*E^1) = Y \times_V Z.$$

$\mathbf{i'}$ fits to a similar commutative diagram as (5) and as a result Φ and Ψ factor through pr_2 and pr_1. Both W and $\tilde{Y}$ lie inside $Y \times_V Z$ and ω and Ψ factor through pr_1 and ν and Φ factor through pr_2. So to prove the identification between $(\tilde{Y}, \Phi, \Psi)$ and (W, ν, ω) , it is enough to show that $\tilde{Y} \cong W$. This can be checked locally. So, after choosing a suitable open cover for V it is enough to prove $\tilde{Y} \cong W$ restricted to each of these open sets. Also we know that $p^{-1}(U) \cong \mathbb{P}_U^{rk(E)-1}$ when $E_{|U}$ is trivial and $\mathbb{P}_U^n = \mathbb{P}_{\mathbb{C}}^n \times U$. Now the the isomorphism follows from [proposition 9.11, [EH]] after adjusting the the definition of projectivization in terms of [H].

We now turn our attention to the diagram (5.2). observe that if we fix the notations $W' = \mathbb{P}'_Z(G')$ with $\omega' : W' \longrightarrow Y'$ as discussed above then we have an identification between $(\tilde{Y}', \Phi', \Psi')$ and (W', ν', ω').

$\omega' : W' \longrightarrow Y'$ comes with $(\omega')^* \mathcal{O}_{Y'}(1) = \mathcal{O}_{\mathbb{P}_{Z'}(G')}(1)$. So, $\gamma = (\Psi')^* \zeta_1$ is achieved. $(\Phi')^* F_1 = (\Psi')^* F$ follows from the commutativity of the diagram (5.2).

The closed immersion $\mathbf{i}'$ induces a relation between the $\mathcal{O}(1)$ sheaves of $Y \times_V Z$ and $\tilde{Y}$. For $Y \times_V Z$ the $\mathcal{O}(1)$ sheaf is $pr_2^* \mathcal{O}_Z(1)$ and for $Proj(\mathcal{T} * \mathcal{O}_Y(1)$ the $\mathcal{O}(1)$ sheaf is $\mathcal{O}_{\tilde{Y}}(-\tilde{E}) \otimes (\Psi)^* \mathcal{O}_Y(1)$. Since Φ factors through pr_2, $(\Phi)^* \mathcal{O}_Z(1) = \mathcal{O}_{\tilde{Y}}(-\tilde{E}) \otimes (\Psi)^* \mathcal{O}_Y(1)$. In the particular case (see diagram (5.2)) $(\Phi')^* \mathcal{O}_{Z'}(1) = \mathcal{O}_{\tilde{Y}'}(-\tilde{E}) \otimes (\Psi')^* \mathcal{O}_{Y'}(1)$ i. e. $(\Phi')^* \zeta_{Z'} = (\Psi')^* \zeta_1 - \tilde{E}$.

Next consider the short exact sequence:

$$0 \longrightarrow \mathcal{O}_{Z'}(1) \longrightarrow G' \longrightarrow \overline{p}_2^* \pi_2^* Q_{11} \longrightarrow 0$$

We wish to calculate below the total Chern class of G' through the Chern class relation obtained from the above short exact sequence.

$$c(G') = c(\mathcal{O}_{Z'}(1)) \cdot c(\overline{p}_2^* \pi_2^* Q_{11}) = (1 + \zeta_{Z'}) \cdot \overline{p}_2^* \pi_2^* (1 + d_{11}[pt]) = (1 + \xi_{Z'})(1 + d_{11} F_1)$$

From the Grothendieck relation for G' we have

$$\gamma^{n_{11}+1} - \Phi'^*(\zeta_{Z'} + d_{11} F_1) \cdot \gamma^{n_{11}} + \Phi'^*(d_{11} F_1 \cdot \zeta_{Z'}) \cdot \gamma^{n_{11}-1} = 0$$
$$\Rightarrow \gamma^{n_{11}+1} - (\Psi'^* \zeta_1 - \tilde{E}) + d_{11} \Psi'^* F) \cdot \gamma^{n_{11}} + d_{11}(\Psi'^* \zeta_1 - \tilde{E}) \cdot \Psi'^* F) \cdot \gamma^{n_{11}-1} = 0$$
$$\Rightarrow \tilde{E} \cdot \gamma^{n_{11}} - d_{11} \tilde{E} \cdot \Psi'^* F \cdot \gamma^{n_{11}-1} = 0$$
$$\Rightarrow \tilde{E} \cdot \Psi'^* (\zeta_1 - \mu_{11} F)^{n_{11}} = 0$$

For the last part note that $\tilde{E} = \mathbb{P}(\pi_2^* Q_{11}) \times_{\mathbb{P}(E_2)} Z'$. Also $N(\tilde{Y}')$ and $N(\tilde{E})$ are free $N(Z')$-module. Using these informations and projection formula, the identity $\tilde{E} \cdot N(\tilde{Y}') = (j_{\tilde{E}})_* N(\tilde{E})$ is obtained easily.

□

Now we are in a position to prove the next theorem.

Theorem 5.4.1. $\overline{\mathrm{Eff}}^k(X) \cong \overline{\mathrm{Eff}}^k(Y') \cong \overline{\mathrm{Eff}}^k(Z')$ *and* $\overline{\mathrm{Eff}}^k(Z') \cong \overline{\mathrm{Eff}}^k(Z'')$. *So,* $\overline{\mathrm{Eff}}^k(X) \cong \overline{\mathrm{Eff}}^k(Z'')$ *for* $k < r_1 + r_2 - 1 - \mathbf{n}$

where $Z' = \mathbb{P}(E_{11}) \times_C \mathbb{P}(E_2)$ *and* $Z'' = \mathbb{P}(E_{11}) \times_C \mathbb{P}(E_{21})$

5.5 Proof of Theorem 5.4.1

Proof. Since $Y' = \mathbb{P}(\pi_2^* E_1) \cong \mathbb{P}(E_1) \times_C \mathbb{P}(E_2) = X$, $\overline{\mathrm{Eff}}^k(X) \cong \overline{\mathrm{Eff}}^k(Y')$ is followed at once. To prove that $\overline{\mathrm{Eff}}^k(X) \cong \overline{\mathrm{Eff}}^k(Z')$ we first define the the map: $\theta_k : N^k(X) \longrightarrow N^k(Z')$ by

$$\zeta_1^i \cdot \zeta_2^{k-i} \mapsto \bar{\zeta_1}^i \cdot \bar{\zeta_2}^{k-i}, \quad F \cdot \zeta_1^j \cdot \zeta_2^{k-j-1} \mapsto F_1 \cdot \bar{\zeta}_1^j \cdot \bar{\zeta}_2^{k-j-1}$$

where $\bar{\zeta_1} = \overline{p}_1^*(\mathcal{O}_{\mathbb{P}(E_{11})}(1))$ and $\bar{\zeta_2} = \overline{p}_2^*(\mathcal{O}_{\mathbb{P}(E_2)}(1))$. $\overline{p}_1 : \mathbb{P}(E_{11}) \times_C \mathbb{P}(E_2) \longrightarrow \mathbb{P}(E_{11})$ and $\overline{p}_2 : \mathbb{P}(E_{11}) \times_C \mathbb{P}(E_2) \longrightarrow \mathbb{P}(E_2)$ are respective projection maps.

It is evident that the above map is in isomorphism of abstract groups. We claim that this induces an isomorphism between $\overline{\mathrm{Eff}}^k(X)$ and $\overline{\mathrm{Eff}}(Z')$. First we construct an inverse for θ_k. Define $\Omega_k : N^k(Z') \longrightarrow N^k(X)$ by

$$\Omega_k(l) = {\Psi'}_* {\Phi'}^*(l)$$

Ω_k is well defined since Φ' is flat and Ψ' is birational. Ω_k is also pseudo-effective. Now we need to show that Ω_k is the inverse of θ_k.

$$\begin{aligned}\Omega_k(\bar{\zeta_1}^{\,i}\cdot\bar{\zeta_2}^{\,k-i}) &= \Psi'_*((\Phi'^*\bar{\zeta}_1)^i\cdot(\Phi'^*\bar{\zeta}_2)^{k-i})\\ &= \Psi'_*((\Phi'^*\zeta_{Z'})^i\cdot(\Phi'^*\bar{\zeta}_2)^{k-i})\\ &= \Psi'_*((\Psi'^*\zeta_1-\tilde{E})^i\cdot(\Psi'^*\zeta_2)^{k-i})\\ &= \Psi'_*((\sum_{0\le c\le i}(-1)^i\tilde{E}^c(\Psi'^*\zeta_1)^{i-c})\cdot(\Psi'^*\zeta_2)^{k-i})\end{aligned}$$

Similarly,

$$\Omega_k(F_1\cdot\bar{\zeta_1}^{\,j}\cdot\bar{\zeta_2}^{\,k-j-1}) = \Psi'_*((\sum_{0\le d\le j}(-1)^j\tilde{E}^d(\Psi'^*\zeta_1)^{j-d})\cdot(\Psi'^*\zeta_2)^{k-j-1})$$

So,

$$\Omega_k\Big(\textstyle\sum_i a_i\,\bar{\zeta_1}^{\,i}\cdot\bar{\zeta_2}^{\,k-i}+\sum_j b_j\,F_1\cdot\bar{\zeta_1}^{\,j}\cdot\bar{\zeta_2}^{\,k-j-1}\Big)$$

$$=\Big(\sum_i a_i\,{\zeta_1}^i\cdot{\zeta_2}^{k-i}+\sum_j b_j\,F\cdot{\zeta_1}^j\cdot{\zeta_2}^{k-j-1}\Big)+\Psi'_*\Big(\sum_i\sum_{1\le c\le i}\tilde{E}^c\Psi'^*(\alpha_{i,c})+\sum_j\sum_{1\le d\le j}\tilde{E}^d\Psi'^*(\beta_{j,d})\Big)$$

for some cycles $\alpha_{i,c},\beta_{j,d}\in N(X)$. But, $\Psi'^*(\tilde{E}^t)=0$ for all $1\le t\le i\le r_1+r_2-1-\mathbf{n}$ for dimensional reasons. Hence, the second part in the right hand side of the above equation vanishes and we make the conclusion that $\Omega_k=\theta_k^{-1}$.

Next we seek an inverse of Ω_k which is pseudo-effective and meet our demand of being equal to θ_k. Define $\eta_k : N^k(X)\longrightarrow N^k(Z')$ by

$$\eta_k(s)=\Phi'_*(\delta\cdot\Psi'^*s)$$

where $\delta=\Psi'^*(\xi_2-\mu_{11}F)^{n_{11}}$.

By the relations (5.4) and (5.5) , $\Psi'^*((\zeta_1^i\cdot\zeta_2^{k-i})$ is $\Phi'^*(\bar{\zeta_1}^{\,i}\cdot\bar{\zeta_2}^{\,k-i})$ modulo $\tilde{E}$ and

$\delta \cdot \tilde{E} = 0$. Also $\phi'_*\delta = [Z']$ which is derived from the fact that $\Phi'_*\gamma^{n_{11}} = [Z']$ and the relations (5.4) and (5.5). Therefore

$$\eta_k(\zeta_1^i \cdot \zeta_2^{k-i}) = \Phi'_*(\delta \cdot \Phi'^*(\bar{\zeta_1}^i \cdot \bar{\zeta_2}^{k-i})) = (\bar{\zeta_1}^i \cdot \bar{\zeta_2}^{k-i}) \cdot [Z'] = \bar{\zeta_1}^i \cdot \bar{\zeta_2}^{k-i}$$

In a similar way, $\Psi'^*(F \cdot \zeta_1^j \cdot \zeta_2^{k-j-1})$ is $\Phi'^*(F_1 \cdot \bar{\zeta_1}^j \cdot \bar{\zeta_2}^{k-j-1})$ modulo $\tilde{\mathbf{E}}$ and as a result of this

$$\eta_k(F \cdot \zeta_1^j \cdot \zeta_2^{k-j-1}) = F_1 \cdot \bar{\zeta_1}^j \cdot \bar{\zeta_2}^{k-j-1}$$

So, $\eta_k = \theta_k$.

Next we need to show that η_k is a pseudo- effective map. Notice that $\Psi'^*s = \bar{s} + \mathbf{j}_*s'$ for any effective cycle s on X, where $\bar{s}$ is the strict transform under Ψ' and hence effective. Now δ is intersection of nef classes. So, $\delta \cdot \bar{s}$ is pseudo-effective. Also $\delta \cdot \mathbf{j}_*s' = 0$ from theorem 5.3.1 and Φ'_* is pseudo-effective. Therefore η_k is pseudo-effective and first part of the theorem is proved. We will sketch the prove for the second part i.e. $\overline{\mathrm{Eff}}^k(Z') \cong \overline{\mathrm{Eff}}^k(Z'')$ which is similar to the proof of the first part. Consider the following diagram:

(5.7)
$$\begin{array}{ccc} Z'' = \mathbb{P}(E_{11}) \times_C \mathbb{P}(E_{21}) & \xrightarrow{\hat{p}_2} & \mathbb{P}(E_{21}) \\ \downarrow{\hat{p}_1} & & \downarrow{\hat{\pi}_2} \\ \mathbb{P}(E_{11}) & \xrightarrow{\hat{\pi}_1} & C \end{array}$$

Define $\hat{\theta}_k : N^k(Z') \longrightarrow N^k(Z'')$ by

$$\bar{\zeta_1}^i \cdot \bar{\zeta_2}^{k-i} \mapsto \hat{\zeta_1}^i \cdot \hat{\zeta_2}^{k-i}, \quad F \cdot \bar{\zeta_1}^j \cdot \bar{\zeta_2}^{k-j-1} \mapsto F_2 \cdot \hat{\zeta_1}^j \cdot \hat{\zeta_2}^{k-j-1}$$

where $\hat{\zeta}_1 = \hat{p_1}^*(\mathcal{O}_{\mathbb{P}(E_{11})}(1)), \hat{\zeta}_2 = \hat{p_2}^*(\mathcal{O}_{\mathbb{P}(E_{21})}(1))$ and F_2 is the class of a fibre of $\hat{\pi}_1 \circ \hat{p}_1$.

This is a isomorphism of abstract groups and behaves exactly the same as θ_k. The methods applied to get the result for θ_k can also be applied successfully here. $\square$

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Thesis Highlight

Name of the Student: Rupam Karmakar
Name of the CI/OCC: Dr. Sanoli Gun Enrolment No.: MATH10201404005
Title: Positive cones of cycles and Seshadri constants on certain projective varieties
Discipline: Mathematical Sciences Subarea of Discipline: Mathematics
Date of viva voce: 05/10/2020

In this thesis, we study various positive cones of cycles and Seshadri constants of nef line bundles on products of projective bundles X over smooth complex projective curves.

The nef cone of divisors of a projective variety X is an important invariant which gives useful information about the projective embeddings of X. The nef cone of various smooth irreducible projective varieties has been studied by many authors in the last few decades.
In this thesis, we compute the nef cones of nef divisors on products of projective bundles without any restriction on the rank or semistability of the associated vector bundles.
The Seshadri constant measures the local positivity of a line bundle on a projective variety around a given point. It was introduced by Demailly in 1992 and gradually it grew on its own as an important invariant in algebraic geometry. Several expositions on this topic can be found in the literature of the last two decades.
In this thesis, we study the Seshadri constants of nef line bundles on products of projective bundles over smooth complex projective curves. We compute the Seshadri constants under some assumptions on the associated vector bundles and provide bounds in some other cases.
For the past few years various positive cones of higher co-dimension cycles have gained much attention amongst fellow geometers with significant progress in the theoretical understanding of such cycles. Although similar in nature, these cycles do not share all the important properties of divisors or curves.
In this thesis, we compute the pseudo-effective cones of higher-codimension cycles on products of projective bundles over smooth complex projective curves, when both the vector bundles are semistable and when both the vector bundles are unstable.