A study of Kostant-Kumar modules via Littelmann paths

By

Mrigendra Singh Kushwaha

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The Institute of Mathematical Sciences, Chennai

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Recommendations of the Viva Voce Committee

As members of the Viva Voce Committee, we certify that we have read the dissertation prepared by Mrigendra Singh Kushwaha entitled "A study of Kostant-Kumar modules via Littelmann paths" and recommend that it may be accepted as fulfilling the thesis requirement for the award of Degree of Doctor of Philosophy.

K. Vijay.
Chairman - Vijay Kodiyalam — Date: July 22, 2021

Sviswanath
Guide/Convenor - Sankaran Viswanath — Date: July 22, 2021

Raghavan KN
Co-Guide - K N Raghavan — Date: July 22, 2021

Examiner - C.S. Rajan — Date: July 22, 2021

Amritanshu Prasad
Member 1 - Amritanshu Prasad — Date: July 22, 2021

Anirban Mukhopadhyay
Member 2 - Anirban Mukhopadyay — Date: July 22, 2021

Member 3 - Sushmita Venugopalan — Date: July 22, 2021

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Date: July 22, 2021

Place: Chennai

Raghavan KN
Co-guide

Sviswanath
Guide

Chapter 1

Introduction

1.1 Kac-Moody Lie algebras

We recall some basic definitions and results concerning Kac-Moody algebras and their representations, following Kac's book [7].

1.1.1 Generalized Cartan matrix (GCM)

An $n \times n$ complex matrix A is called a *generalized Cartan matrix* if it satisfies the following conditions:

1. $a_{ii} = 2$ for $i = 1, 2, ..., n$.

2. a_{ij} are nonpositive integers for $i \neq j$.

3. $a_{ij} = 0$ implies $a_{ji} = 0$.

A generalized Cartan matrix $A = (a_{ij})_{i,j=1}^{n}$ is called *symmetrizable* if $A = DB$ for some invertible diagonal matrix $D = diag\{d_1, d_2, ..., d_n\}$ and a symmetric matrix B.

Let $\mathfrak{h}$ be a complex vector space such that $dim\ \mathfrak{h} - n = n - \ell$, where ℓ is the rank of A. Let $\Pi = \{\alpha_1, \alpha_2, ..., \alpha_n\} \subset\ \mathfrak{h}^*$, and $\Pi^\vee = \{\alpha_1^\vee, \alpha_2^\vee, ..., \alpha_n^\vee\} \subset \mathfrak{h}$ be linearly independent such that $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$ for $i, j \in \{1, 2, ..., n\}$, where $\langle , \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}$ denotes the pairing $\langle \alpha, h \rangle = \alpha(h)$.

Let $A = (a_{ij})_{i,j=1}^n$ be an $n \times n$ symmetrizable generalized Cartan matrix and $(\mathfrak{h}, \Pi, \Pi^\vee)$ as above. The symmetrizable Kac-Moody algebra $\mathfrak{g}(A)$ is a Lie algebra generated by $\{e_i, f_i | i = 1, 2, ..., n\}$, $\mathfrak{h}$ subject to the following relations:

1. $[e_i, f_i] = \delta_{ij} \alpha_i^\vee$, for all $i, j = 1, 2, ..., n$.
2. $[h, h'] = 0$, for all $h, h' \in \mathfrak{h}$.
3. $[h, e_i] = \langle \alpha_i, h \rangle e_i$ for $i = 1, 2, ..., n; h \in \mathfrak{h}$.
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The subalgebra $\mathfrak{h}$ of $\mathfrak{g}(A)$ is the called *Cartan subalgebra*. Π is called the root basis and its elements are called *simple roots*. $\Pi^\vee$ is called the coroot basis and its elements are called *simple coroots* following the same terminology as in the theory of finite dimensional simple Lie algebras.

Theorem 1.1.1. *The symmetrizable Kac-Moody Lie alebra $\mathfrak{g}(A)$ has the following* triangular decomposition*:*

$$\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

where $\mathfrak{n}_+$(resp $\mathfrak{n}_-$) denotes the subalgebra of $\mathfrak{g}(A)$, which is generated by $\{e_i | i = 1, 2, ..., n\}$ (resp $\{f_i | i = 1, 2, ..., n\}$).

1.1.2 The Weyl group

Consider the map $s_i : \mathfrak{h}^* \to \mathfrak{h}^*$ for all $i = 1, 2, ..., n$ defined by:

$$s_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i$$

We can immediately observe that $s_i(\alpha_i) = -\alpha_i$; $s_i^2 = 1$, this shows that s_i is a reflection along hyperplane $\{h \in \mathfrak{h}^* | \langle \lambda, \alpha_i^\vee \rangle = 0\}$. Reflections s_i are called *simple reflections.*

The Weyl group W is the group of automorphisms of $\mathfrak{h}^*$ generated by simple reflections $\{s_1, s_2, ..., s_n\}$.

1.1.3 Integrable module

A $\mathfrak{g}(A)$-module V is called $\mathfrak{h}$-*diagonalizable* if:

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$$

where $V_\lambda = \{u \in V | h(v) = \langle \lambda, h \rangle v; \forall h \in \mathfrak{h}\}$. We say V_λ is a weight space, and if $V_\lambda \neq 0$ then λ is called a *weight*, and dim V_λ is called the multiplicity of λ and is denoted by mult V_λ.

Definition 1.1.2. *An $\mathfrak{h}$-diagonalizable module V over a Kac-Moody Lie algebra $\mathfrak{g}(A)$ is called integrable if all e_i and f_i for $i = 1, 2, ..., n$ are locally nilpotent on V.*

Note that the adjoint module of a Kac-Moody Lie algebra is an integrable module.

1.1.4 Highest weight module

Let $U(\mathfrak{g}(A))$ denote the universal enveloping algebra of $\mathfrak{g}(A)$.

Definition 1.1.3. *A* $\mathfrak{g}(A)$*-module* V *is called a* highest weight module *with* highest weight $\Lambda \in \mathfrak{h}^*$ *if there exist a non-zero vector* $v_\Lambda \in V$ *such that following holds:*

$$\mathfrak{n}_+(v_\Lambda) = 0;\ h(v_\Lambda) = \Lambda(h)v_\Lambda \ for\ h \in \mathfrak{h}$$

$$U(\mathfrak{g}(A))(v_\Lambda) = V$$

The vector v_Λ *is called a* highest weight vector *of* V.

Proposition 1.1.4. *Let* V *be a highest weight module with highest weight vector* v_Λ *and highest weight* $\Lambda \in \mathfrak{h}^*$*, then:*

$$V = \bigoplus_{\lambda \leq \Lambda} V_\lambda;\ V_\Lambda = \mathbb{C}v_\Lambda$$

and also $dim V_\lambda < \infty$. *Where* $\lambda \leq \Lambda$ *means* $\Lambda - \lambda$ *is a non-negative linear combination of simple roots.*

We will mention some examples of highest weight modules.

Verma module:- Let $\lambda \in \mathfrak{h}^*$ and K_λ be the left ideal of $U(\mathfrak{g}(A))$ generated by $\mathfrak{n}_+$ and $h - \lambda(h)$ for all $h \in \mathfrak{h}$. Thus

$$K_\lambda = U(\mathfrak{g}(A))\mathfrak{n}_+ + \sum_{h \in \mathfrak{h}^*} U(\mathfrak{g}(A))(h - \lambda(h))$$

The module $M(\lambda) = U(\mathfrak{g}(A))/K_\lambda$ is a highest weight $U(\mathfrak{g}(A))$-module called the *Verma module* with highest weight λ.

Irreducible highest weight module:- The Verma module $M(\lambda)$ has a unique maximal submodule $J(\lambda)$, the module is $V(\lambda) = M(\lambda)/J(\lambda)$ is an irreducible highest weight module with highest weight $\lambda \in \mathfrak{h}^*$.

Formal Character:- Let V be a highest weight $\mathfrak{g}(A)$-module. We define the formal character of V by:

$$char(V) = \sum_{\lambda \in P(V)} dim(V_\lambda) e^\lambda$$

where $P(V)$ is the set of weights of V and e^λ are formal exponentials, satisfying the following rules:

$$e^\lambda \cdot e^{\lambda'} = e^{\lambda+\lambda'} \ (\lambda, \lambda' \in \mathfrak{h}^*) \ ; \ e^0 = 1$$

1.1.5 Integrable highest weight modules

Let A be an $n \times n$ symmetrizable GCM. Let V be a highest weight $\mathfrak{g}(A)$-module.

An element $\lambda \in \mathfrak{h}^*$ is called an *integral weight* if $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}$ for all $i = 1, 2, ..., n$. Let P denote the additive group of all integral weights. An element $\lambda \in P$ called a *dominant integral weight* if $\langle \lambda, \alpha_i^\vee \rangle \geq 0$ for all $i = 1, 2, ..., n$. Let P^+ denote the set of all dominant integral weights.

Proposition 1.1.5. *The irreducible highest weight $\mathfrak{g}(A)$-module $V(\Lambda)$ is integrable if and only if $\Lambda \in P^+$.*

Proposition 1.1.6. *The tensor product of a finite number of integrable highest weight modules is a direct sum of modules $V(\Lambda)$ with $\Lambda \in P^+$.*

1.1.6 Demazure module

Let $\Lambda \in P^+$ and $V(\Lambda)$ be the irreducible highest weight integrable module.

Lemma 1.1.7. *Let λ be a weight of $V(\Lambda)$, and w be an element in the Weyl group W. Then $w(\lambda)$ also is a weight of $V(\Lambda)$ and $dimV(\Lambda)_{w(\lambda)} = dimV(\Lambda)_\lambda$.*

Let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$ be the Borel subalgebra contaning $\mathfrak{h}$, and $U(\mathfrak{b})$ the universal enveloping algebra of $\mathfrak{b}$. Fix a Weyl group element w and $\Lambda \in P^+$. Let

$v_{w(\Lambda)} \in V(\Lambda)$ be a non zero weight vector of weight $w(\Lambda)$. The Demazure module $\mathcal{D}_w(\Lambda)$ is defined by:

$$\mathcal{D}_w(\Lambda) := U(\mathfrak{b})(v_{w(\Lambda)})$$

1.2 Lakshmibai-Seshadri paths (LS paths)

Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra, with Cartan subalgebra $\mathfrak{h}$ and let P denote its weight lattice. A path is piecewise linear map $\pi : [0,1] \to P \otimes_{\mathbb{Z}} \mathbb{R}$ such that $\pi(0) = 0$.

Let P^+ denote the dominant weight lattice and W the Weyl group. For $\lambda \in P^+$, stabilizer of λ denoted by W_λ. Let "$\geq$" be the Bruhat order on W/W_λ.

a-chain

Let $w > w'$ be two elements of W/W_λ and let $0 < a < 1$ be a rational number. By an *a-chain* for the pair (w, w') we mean a sequence of cosets in W/W_λ:

$$u_0 := w > u_1 := s_{\beta_1} w > u_2 := s_{\beta_2} s_{\beta_1} w > ... > u_s := s_{\beta_s} ... s_{\beta_2} s_{\beta_1} w = w'$$

where $\beta_1, \beta_2, ..., \beta_s$ are positive real roots such that for all $i = 1, 2, ..., s$:

$$\ell(u_i) = \ell(u_{i-1}) - 1; \text{ and } a\langle u_{i-1}(\lambda), \beta_i^\vee \rangle \in \mathbb{Z}.$$

Definition 1.2.1. *Let $\lambda \in P^+$. A sequence*

$$\pi = (w_1 > w_2 > ... > w_r; 0 < a_1 < a_2 < ... < a_r = 1)$$

where $w_i \in W/W_\lambda$ and $0 < a_i < 1$ are rational numbers for all $i = 1, 2, ..., r$. We say π is a LS path of shape λ if, for all $i = 1, 2, ..., r-1$ there exists an a_i-chain *for the pair* (w_i, w_{i+1}).

We identify this with the path $\pi : [0,1] \to P \otimes_{\mathbb{Z}} \mathbb{R}$ given by:

$$\pi(t) := \sum_{i=1}^{j-1} (a_i - a_{i-1}) w_i(\lambda) + (t - a_{j-1}) w_j(\lambda); \ for\ t \in [a_{j-1}, a_j]$$

By the above definition we can see that the end point of path $\pi(1) \in P$.

We let $\mathcal{P}_\lambda$ denote the set of all LS paths of shape λ.

Theorem 1.2.2. *[14] Let* $\mathfrak{g}$ *be a symmetrizable Kac-Moody Lie algebra and* $V(\lambda)$ *be the integrable highest weight* $\mathfrak{g}$*-module with highest weight* $\lambda \in P^+$*. Then:*

$$char(V(\lambda)) = \sum_{\pi \in \mathcal{P}_\lambda} e^{\pi(1)}$$

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Summary

Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra. For each dominant integral weight λ of $\mathfrak{g}$, let V_λ denote the corresponding irreducible integrable highest weight $\mathfrak{g}$-module and let v_λ be a highest weight vector in V_λ. Given dominant integral weights λ, μ and an element w of the Weyl group of $\mathfrak{g}$, the *Kostant-Kumar* (KK) module $K(\lambda, w, \mu)$ is the cyclic $\mathfrak{g}$-submodule of $V_\lambda \otimes V_\mu$ generated by $v_\lambda \otimes v_{w\mu}$, where $v_{w\mu}$ is a nonzero vector in the one-dimensional weight space of weight $w\mu$ in V_μ.

Littelmann has given a path model for the tensor product $V_\lambda \otimes V_\mu$. We give, in the spirit of Littelmann, a path model for Kostant-Kumar modules in terms of Lakshmibai-Seshadri (LS) paths. Littelmann's path model gives a generalized Littlewood-Richardson rule for decomposing tensor products into irreducibles. An analogous rule for Kostant-Kumar modules was given by Joseph under the hypothesis that the Kac-Moody algebra is symmetric. We extend Joseph result to finite type Lie algebras and use this rule to study Parthasarathy-Ranga Rao-Varadarajan (PRV) components and generalized PRV components in Kostant-Kumar modules.

At the end, we discuss Kostant-Kumar modules for the finite dimensional Lie algebras $\mathfrak{g}$ of type A. In this case, it is well known that the semistandard Young tableaux are very useful to study representations theory. We gave a procedure to associate a permutation $w(T)$ to semistandard Young tableau T. Permuatation

$w(T)$ corresponds to the *right key* of T introduced by Lascoux-Schützenberger.

It is well known that Littlewood-Richardson (LR) tableaux count multiplicities of irreducible modules in the tensor product. Given a LR tableaux S of type μ, we can easily associate a semi standard Young tableau T of shape μ. We associate a permutation $w(S)$ to LR tableau S, by simply defining $w(S) := w(T)$. Then Littlewood-Richardson tableaux S such that $w(S) \leq w$ count multiplicities of irreducible modules in the KK module $K(\lambda, w, \mu)$.

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Consider the map $s_i : \mathfrak{h}^* \to \mathfrak{h}^*$ for all $i = 1, 2, ..., n$ defined by:

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We will mention some examples of highest weight modules.

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The module $M(\lambda) = U(\mathfrak{g}(A))/K_\lambda$ is a highest weight $U(\mathfrak{g}(A))$-module called the *Verma module* with highest weight λ.

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$v_{w(\Lambda)} \in V(\Lambda)$ be a non zero weight vector of weight $w(\Lambda)$. The Demazure module $\mathcal{D}_w(\Lambda)$ is defined by:

$$\mathcal{D}_w(\Lambda) := U(\mathfrak{b})(v_{w(\Lambda)})$$

1.2 Lakshmibai-Seshadri paths (LS paths)

Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra, with Cartan subalgebra $\mathfrak{h}$ and let P denote its weight lattice. A path is piecewise linear map $\pi : [0,1] \to P \otimes_{\mathbb{Z}} \mathbb{R}$ such that $\pi(0) = 0$.

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where $\beta_1, \beta_2, ..., \beta_s$ are positive real roots such that for all $i = 1, 2, ..., s$:

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$$\pi(t) := \sum_{i=1}^{j-1}(a_i - a_{i-1})w_i(\lambda) + (t - a_{j-1})w_j(\lambda); \ \text{for } t \in [a_{j-1}, a_j]$$

By the above definition we can see that the end point of path $\pi(1) \in P$.

We let $\mathcal{P}_\lambda$ denote the set of all LS paths of shape λ.

Theorem 1.2.2. *[14] Let* $\mathfrak{g}$ *be a symmetrizable Kac-Moody Lie algebra and* $V(\lambda)$ *be the integrable highest weight* $\mathfrak{g}$*-module with highest weight* $\lambda \in P^+$*. Then:*

$$char(V(\lambda)) = \sum_{\pi \in \mathcal{P}_\lambda} e^{\pi(1)}$$

Chapter 2

Some results on extremal elements in Coxeter groups

2.1 Generalities on extremal elements in Coxeter groups

The purpose of this section is to formulate and prove the required result about these minimal elements—Corollary 2.1.19 below—in the more natural context of Coxeter groups. The arguments leading up to the result are all elementary. The reader willing to accept it at face value without proof may want to skip this section at a first pass.

The results in §2.1.3, 2.1.4, and 2.1.5 are well known (e.g., from [2, 12] as specifically indicated in a few places below), but we have included them because we need them and it is easier to prove them ab initio in our set up than to refer to sources. It not only makes the thesis more self-contained but also more readable with these results stated and proved rather than just quoted.

2.1.1 Notation for this section

Let (W, S) be a Coxeter system. Let $\leq$ denote the (strong) Bruhat order on W. For a subset K of the Coxeter group W, we let $\mathbf{min}\, K$ and $\mathbf{max}\, K$ denote respectively the unique minimal and unique maximal elements of K in the Bruhat order (if they do exist). For u in W and s in S, the elements u and su (respectively u and us) are comparable. Thus $\mathbf{min}\,\{u, su\}$, $\mathbf{max}\,\{u, su\}$, $\mathbf{min}\,\{u, us\}$, and $\mathbf{max}\,\{u, us\}$ make sense. We denote these respectively by $u \wedge su$, $u \vee su$, $u \wedge us$, and $u \vee us$.

2.1.2 The results

The following basic fact is repeatedly applied in this section:

(*)

For elements $u \leq v$ in W and s in S, we have $u \wedge su \leq v \wedge sv$ and $u \vee su \leq v \vee sv$.

The "right analogue" of the above fact asserts: $u \wedge us \leq v \wedge vs$ and $u \vee us \leq v \vee vs$ (under the same hypothesis). Only the left analogues of the "one sided" results below are explicitly stated. Their right analogues hold good too.

Remark 2.1.1. Suppose that $u \leq v$ is a covering relation in W (that is, $\text{length}(u) = \text{length}(v) - 1$ and $u = tv$ for some reflection t in W). Then, if for s in S, we have $sv < v$ and $u < su$, then $t = s$. Indeed, it follows from (*) that $u \leq sv$, but then equality is forced since u and sv have the same length.

A simple application of (*) gives:

Proposition 2.1.2. *Suppose that a subset K of the Coxeter group W has a unique minimal element u under $\leq$. Then, for any s in S, the subset $K \cup sK$ also has a unique minimal element under $\leq$, namely, $u \wedge su$. Analogously, if K admits a* *unique maximal element v under $\leq$, then $v \vee sv$ is the unique maximal element of $K \cup sK$.*

Corollary 2.1.3. *Let $\mathfrak{s}$: s_1, s_2, $\ldots$ be a (possibly infinite) sequence of simple reflections (elements of S). For $\mathfrak{s}'$: s_{i_1}, s_{i_2}, $\ldots$, s_{i_m} a finite subsequence of $\mathfrak{s}$, let $w(\mathfrak{s}')$ denote the element $s_{i_m}s_{i_{m-1}}\cdots s_{i_1}$ of the Coxeter group (note the order reversal). Let K be a subset of W with a unique minimal element u with respect to $\leq$. Then $\cup_{\mathfrak{s}'}w(\mathfrak{s}')K$, where the union runs over all finite subsequences $\mathfrak{s}'$ of $\mathfrak{s}$, has a unique minimal element u_∞, the stable value of u_j as $j \to \infty$, where u_j is recursively defined: $u_0 = u$, and $u_{j+1} = u_j \wedge s_{j+1}u_j$ for $j \geq 0$.*

PROOF: For j a non-negative integer, let $\mathfrak{s}_j$ denote the subsequence s_1, s_2, $\ldots$, s_j of $\mathfrak{s}$. By a repeated application of Proposition 2.1.2, we see that u_j is the unique minimal element of $K_j := \cup_{\mathfrak{s}'}w(\mathfrak{s}')K$, where the union runs over subsequences of $\mathfrak{s}_j$. Since the subsets K_j increase with j, it follows that $u_{j+1} \leq u_j$. Since any decreasing sequence in the Bruhat order stabilizes, we conclude that u_j is constant for j sufficiently large. □

Remark 2.1.4. What about the maximal analogue of Corollary 2.1.3? Let K be a subset of W that has a unique maximal element v. With notation as in the proof just above, we conclude analogously that v_k is the unique maximal element of K_k, where v_k is defined recursively as follows: $v_0 = v$, and $v_{i+1} = v \vee s_{i+1}v$ for $0 \leq i$. Since the K_k increase with k, we have $v_{k+1} \geq v_k$. If the v_k stabilize to a stable value v_∞ as $k \to \infty$ (which in general need not happen), then v_∞ is the unique maximal element of $\cup_{k\geq 0}K_k$. In particular, the maximal analogue holds if the sequence $\mathfrak{s}$ is finite.

We now apply Corollary 2.1.3 (and its right analogue) in two special cases. First, let σ be an element of W and, with notation as in the corollary, choose the sequence $\mathfrak{s}$: $s_1, s_2, \ldots, s_m$ of elements of S to be such that $s_m s_{m-1} \cdots s_1$ is a reduced expression for σ. Then $\{w(\mathfrak{s}') \mid \mathfrak{s}' \text{ is a subsequence of } \mathfrak{s}\}$ equals $I(\sigma) := \{\sigma' \in W \mid \sigma' \leq \sigma\}$. We conclude that $I(\sigma)K$ has a unique minimal element and further that this element is the unique minimal element in $I(\sigma)u$. Now applying the right analogue of this argument, we obtain:

Corollary 2.1.5. *Let K be a subset of the group W that admits a unique minimal element u. Then, for any two elements σ_1 and σ_2 of W, the set $I(\sigma_1)KI(\sigma_2)$ has a unique minimal element, and this element is the unique minimal element in $I(\sigma_1)uI(\sigma_2)$.*

The special case of the above result (as also its maximal analogue, namely, Corollary 2.1.8) when K is a singleton and σ_2 is the identity element appears in [12, Lemma 11 (i)].

Corollary 2.1.6. *Let σ and φ be elements of W, and s an element of S. Suppose that $\sigma s < \sigma$. Then $\mathbf{min}\, I(\sigma)\varphi$ equals either $\mathbf{min}\, I(\sigma)s\varphi$ or $\mathbf{min}\, I(\sigma s)\varphi$ accordingly as $s\varphi < \varphi$ or $\varphi < s\varphi$.*

PROOF: Choose a sequence $\mathfrak{s}$: $s = s_1, s_2, \ldots, s_m$ such that $s_m s_{m-1} \cdots s_1$ is a reduced expression for σ. Let φ_i, φ_i', and φ_i'' be sequences defined recursively as follows:

- $\varphi_0 = \varphi$, and $\varphi_{i+1} = \varphi_i \wedge s_{i+1}\varphi_i$ for $0 \leq i < m$
- $\varphi_0' = s\varphi$, and $\varphi_{i+1}' = \varphi_i' \wedge s_{i+1}\varphi_i'$ for $0 \leq i < m$
- $\varphi_1'' = \varphi$, and $\varphi_{i+1}'' = \varphi_i'' \wedge s_{i+1}\varphi_i''$ for $1 \leq i < m$

By Corollary 2.1.3, $\mathbf{min}\, I(\sigma)\varphi$, $\mathbf{min}\, I(\sigma)s\varphi$, and $\mathbf{min}\, I(\sigma s)\varphi$ are equal respectively to φ_m, φ_m', and φ_m''.

First suppose that $s\varphi < \varphi$. Then $\varphi_1 = \varphi_1'$: indeed, $\varphi_1 = \varphi \wedge s\varphi = s\varphi$, and $\varphi_1' = s\varphi \wedge s(s\varphi)) = s\varphi$. Thus $\varphi_i = \varphi_i'$ for all $1 \leq i$, and in particular for $i = m$.

Now suppose that $\varphi < s\varphi$. Then $\varphi_1 = \varphi_1''$: indeed, $\varphi_1 = \varphi \wedge s\varphi = \varphi$, and $\varphi_1'' = \varphi$ by definition. Thus $\varphi_i = \varphi_i''$ for all $1 \leq i$, and in particular for $i = m$. $\square$

Towards a second application of Corollary 2.1.3, let S_1 be a subset of S and W_1 the subgroup of W generated by S_1. Recall that such a subgroup of W is called a *standard parabolic subgroup.* With notation as in Corollary 2.1.3, choose the sequence $\mathfrak{s}$: s_1, s_2, $\ldots$ to consist of elements of S_1 and such that every element of W_1 arises as $w(\mathfrak{s}')$ for some finite subsequence $\mathfrak{s}'$ of $\mathfrak{s}$. Then $\{w(\mathfrak{s}') \,|\, \mathfrak{s}' \text{ is a subsequence of } \mathfrak{s}\}$ equals W_1. We conclude that W_1K has a unique minimal element. Now applying the right analogue of this argument, we obtain:

Corollary 2.1.7. *Let K be a subset of the group W that admits a unique minimal element u. Then, for any two standard parabolic subgroups W_1 and W_2 of W, the set W_1KW_2 has a unique minimal element, and this element is the unique minimal element in W_1uW_2. In particular, any double coset of a pair of standard parabolic subgroups has a unique minimal element.*

Of course, when the subgroups W_1 and W_2 are finite, Corollary 2.1.7 is a special case of Corollary 2.1.5. Indeed, letting w_1 and w_2 be the unique maximal elements of W_1 and W_2 respectively, we have $W_1 = I(w_1)$ and $W_2 = I(w_2)$.

As the maximal analogues of the above two corollaries, we have:

Corollary 2.1.8. *Let K be a subset of W having a unique maximal element v. Then, for any two elements σ_1 and σ_2 of W, the set $I(\sigma_1)KI(\sigma_2)$ has a unique maximal element, namely, the unique such element in $I(\sigma_1)vI(\sigma_2)$. In particular, for any two finite standard parabolic subgroups W_1 and W_2 of W, the union W_1KW_2 of double cosets has a unique maximal element, namely, the unique such* *element in* $W_1 v W_2$.

Remark 2.1.9. (Relation to Deodhar's $\star$ operation.) In [2, Lemma 2.4], Deodhar states: there exists a unique associative binary operation $\star$ on W such that $w \star \mathrm{id} = w$ and $w \star s = w \vee ws$ for all $w \in W$ and $s \in S$. The uniqueness is clear. For the proof of the existence, we define $w \star x := \mathbf{max}\, I(w)I(x)$ for all w and x in W ($\mathbf{max}\, I(w)I(x)$ exists by Corollary 2.1.8). It is easy to verify, using Corollary 2.1.8, that this operation has the requisite properties: $\mathbf{max}\, I(w)I(\mathrm{id}) = \mathbf{max}\, I(w) = w$; $\mathbf{max}\, I(w)I(s) = \mathbf{max}\, wI(s) = w \vee ws$; and

$$\begin{aligned} \mathbf{max}\, I(w)I(\mathbf{max}\, I(x)I(y)) &= \mathbf{max}\, I(w)\mathbf{max}\,(I(x)I(y)) = \mathbf{max}\, I(w)I(x)I(y) \\ &= \mathbf{max}\,(\mathbf{max}\, I(w)I(x))I(y) = \mathbf{max}\, I(\mathbf{max}\,(I(w)I(x)))I(y), \end{aligned}$$

so associativity holds.

We have:

- The unique maximal element of $I(\sigma_1)KI(\sigma_2)$ in Corollary 2.1.8 is $\sigma_1 \star v \star \sigma_2$.
- $I(w)I(x) = I(w \star x)$ for all w and x in W.
- Let K be a subset of W with a unique maximal element v. For any collection $\sigma_1, \ldots, \sigma_s, \tau_1, \ldots, \tau_t$ of elements in W, the set $I(\sigma_1)\cdots I(\sigma_s)KI(\tau_1)\cdots I(\tau_t)$ equals $I(\sigma_1 \star \cdots \star \sigma_s)KI(\tau_1 \star \cdots \star \tau_t)$ and admits a unique maximal element, namely, $\sigma_1 \star \cdots \star \sigma_s \star v \star \tau_1 \star \cdots \star \tau_t$.

Remark 2.1.10. Consider the specialized Hecke algebra $\mathcal{H}$ defined as the associative algebra with identity (over say a field k) generated by variables T_s, $s \in S$, and subject to the relations $T_s^2 = T_s$ (for all s in S) and the braid relations. For $w \in W$, let T_w be the element $T_{s_{i_1}} T_{s_{i_2}} \cdots T_{s_{i_r}}$ of $\mathcal{H}$ where $s_{i_1} s_{i_2} \cdots s_{i_r}$ is a reduced expression for w: this definition does not depend on the choice of reduced expression because the braid relations are satisfied. The algebra $\mathcal{H}$ is just the semigroup algebra of the semigroup W with respect to the $\star$ operation as in Remark 2.1.9: $\{T_w \mid w \in W\}$ is a basis for $\mathcal{H}$ and $T_w T_x = T_{w\star x}$ for all w and x in W.

Let kW be the free k-vector space with elements of W as a basis. We can make kW to be $\mathcal{H}$-$\mathcal{H}$ bimodule as follows. For s in S, let $\wedge_s$ denote the (left) operator on W defined by $\wedge_s w := sw \wedge w$ and $s\wedge$ the right operator on W defined by $ws\wedge := w \wedge ws$ (for w in W). The linear extensions of the operators $s\wedge$ and $\wedge_s$ to kW are denoted by the same symbols. We have, for s, t in S and w in W:

- $\wedge_s(\wedge_s w) = \wedge_s w$ and $(ws\wedge)s\wedge = ws\wedge$.
- $(\wedge_s w)_t\wedge = \wedge_s(w_t\wedge)$.
- Let $s_1, \ldots, s_m$ be a sequence of elements of S such that $s_m s_{m-1} \cdots s_1$ is a reduced expression for an element σ of W. Then $\wedge_m \cdots \wedge_1 (w) = \mathbf{min}\, I(\sigma)w$ (see the paragraph preceding Corollary 2.1.5) and analogously $w \wedge_1 \cdots \wedge_m = \mathbf{min}\, wI(\sigma^{-1})$, where $\wedge_j$ and $j\wedge$ stand for $\wedge_{s_j}$ and $s_j\wedge$ respectively. Thus the operators $s\wedge$ (respectively $\wedge_s$), $s \in S$, satisfy the braid relations.

Thus, letting T_s, $s \in S$, act on kW on the left by $\wedge_s$ and on the right by $s\wedge$, we get a bimodule structure on kW.

2.1.3 Bruhat order on double coset spaces

Let W_1 and W_2 be standard parabolic subgroups of W. It is convenient to identify the coset space $W_1\backslash W/W_2$ as a subset of W via the association $W_1uW_2 \mapsto \mathbf{min}\, W_1uW_2$. The Bruhat order on $W_1\backslash W/W_2$ is the restriction to this subset of the Bruhat order on W.

Corollary 2.1.11. *Given two elements W_1uW_2, W_1vW_2 in $W_1\backslash W/W_2$, we have $W_1uW_2 \leq W_1vW_2$ in Bruhat order if and only if there exist u' in W_1uW_2 and v'*

in W_1vW_2 such that $u' \leq v'$. In particular, if $\mathbf{min}\, W_1uW_2 \leq v'$ for some v' in W_1vW_2, then $\mathbf{min}\, W_1uW_2 \leq v''$ for every $v'' \in W_1vW_2$.

Proof: For the only if part, just take $u' = \mathbf{min}\, W_1uW_2$ and $v' = \mathbf{min}\, W_1vW_2$. For the if part, apply Corollary 2.1.7 with $K = \{u', v'\}$. Since $u' \leq v'$, we conclude that $\mathbf{min}\, W_1u'W_2 = \mathbf{min}\, W_1KW_2$. But $\mathbf{min}\, W_1u'W_2 = \mathbf{min}\, W_1uW_2$ and $\mathbf{min}\, W_1KW_2 \leq \mathbf{min}\, W_1v'W_2 = \mathbf{min}\, W_1vW_2$ since $v' \in K$. $\square$

Corollary 2.1.12. *For u an element of W and s an element of S, suppose that $suW_1 \lneq uW_1$. Then for every v in uW_1, we have $sv < v$. Conversely, if $su < u$ for the unique minimal element u in uW_1, then $suW_1 \lneq uW_1$, and su is the unique minimal element in suW_1.*

Proof: For the first statement, observe the following: if $v < sv$, then, by Corollary 2.1.11, $uW_1 = vW_1 \leq svW_1 = suW_1$, a contradiction. For the first part of the converse, observe that $suW_1 \leq uW_1$ by Corollary 2.1.11, and that equality cannot hold (if su were to belong to uW_1 the minimality of u in uW_1 would be contradicted). For the second part of the converse, suppose that $x \in suW_1$. Then $sx \in uW_1$, and so $u \leq sx$, which means $su = u \wedge su \leq sx \wedge x \leq x$. $\square$

Remark 2.1.13. For u in W and s in S, it is possible that $usW_1 \lneq uW_1$ but there exists v in uW_1 with $vs > v$. For example, let

$$W = \langle s_1, s_2, s_3 \,|\, s_1^2 = s_2^2 = s_3^2 = 1,\ s_1s_2s_1 = s_2s_1s_2,\ s_2s_3s_2 = s_3s_2s_3,\ s_1s_3 = s_3s_1\rangle$$

$W_1 = \langle s_1, s_3\rangle$, $u = s_1s_2$, and $s = s_2$. Then $uW_1 = \{s_1s_2, s_1s_2s_1, s_1s_2s_3, s_1s_2s_1s_3\}$, the minimal element in uW_1 is u, and $usW_1 = W_1 \lneq uW_1$, but $vs > v$ for $v = s_1s_2s_3$ in uW_1.

Proposition 2.1.14. *Suppose that $W_1uW_2 \leq W_1vW_2$. Then, given u' in W_1uW_2, there exists v' in W_1vW_2 with $u' \leq v'$.*

Proof: Proceed by induction on the length of u'. Let u_0 be the minimal element in W_1uW_2. We have $u_0 \leq u'$. If $u' = u_0$, then $u' \leq v'$ for any v' in W_1vW_2 (since $u_0 \leq v_0$ by definition, where v_0 is the minimal element of W_1vW_2). Now suppose that $u_0 \lneq u'$. Then there exists either $s \in S \cap W_1$ such that $su' < u'$, or $t \in S \cap W_2$ such that $u't < u'$. Let us suppose that the former condition holds (the case when the latter holds is handled analogously). Observe that su' belongs to W_1uW_2. By induction, there exists v' in W_1vW_2 such that $su' \leq v'$. By (*) at the beginning of this section, we have $u' \leq su' \vee u' \leq v' \vee sv'$. But $v' \vee sv'$ belongs to W_1vW_2. $\square$

2.1.4 Deodhar's Lemma

Let W_1 be a standard parabolic subgroup of W and let σ, w be elements of W. Set $J_{\sigma W_1}(w) := \{v \in \sigma W_1 \mid w \leq v\}$.

Proposition 2.1.15. *Let s be in S.*

1. $J_{\sigma W_1}(w) \supseteq J_{\sigma W_1}(w')$ *for* $w \leq w'$.
2. $J_{\sigma W_1}(w) \subseteq sJ_{s\sigma W_1}(w \wedge sw)$
3. $J_{\sigma W_1}(w)$ *is non-empty if and only if* $wW_1 \leq \sigma W_1$.

Proof: Statement (1) is immediate. For (2), just observe that $w \leq x$ implies $w \wedge sw \leq x \wedge sx \leq sx$. As for (3), the only if part is trivial, and the if part follows from Proposition 2.1.14. $\square$

Lemma 2.1.16. *Suppose that σ is the minimal element of σW_1, and let $s \in S$ be such that $s\sigma < \sigma$. Then*

1. *$sx < x$ for any x in $J_{\sigma W_1}(w)$; and $v < sv$ for any v in $J_{s\sigma W_1}(w \wedge sw)$.*
2. *$J_{\sigma W_1}(w) = sJ_{s\sigma W_1}(w \wedge sw)$. In particular,*
   $J_{\sigma W_1}(w) = J_{\sigma W_1}(w \wedge sw) = J_{\sigma W_1}(w \vee sw)$.
3. *If either $J_{\sigma W_1}(w)$ or $J_{s\sigma W_1}(w \wedge sw)$ has a unique minimal element u, then so does the other and su is that unique minimal element.*

Proof: (1): We have $s\sigma W_1 \lneq \sigma W_1$ by the second part of Corollary 2.1.12. From the first part of that corollary, it follows that $sx < x$ for any x in σW_1. Since $J_{\sigma W_1}(w) \subseteq \sigma W_1$ by definition, the first statement follows. The second statement too follows from the first assertion in Corollary 2.1.12.

(2): For the first assertion, given item (2) of Proposition 2.1.15, it is enough to show that $J_{\sigma W_1}(w) \supseteq sJ_{s\sigma W_1}(w \wedge sw)$. Suppose x belongs to $J_{s\sigma W_1}(w \wedge sw)$. Then evidently sx belongs to σW_1. By item (1) (of the present lemma), we have $sx = x \vee sx$. Since $x \geq w \wedge sw$ by hypothesis, we have $sx = x \vee sx \geq (w \wedge sw) \vee s(w \wedge sw) \geq w$.

To see that $J_{\sigma W_1}(w) = J_{\sigma W_1}(w \wedge sw)$ (respectively, $J_{\sigma W_1}(w) = J_{\sigma W_1}(w \vee sw)$), put $v = w \wedge sw$ (respectively, $v = w \vee sw$). Then $v \wedge sv = w \wedge sw$, and so, by the first assertion, $J_{\sigma W_1}(v) = sJ_{s\sigma W_1}(v \wedge sv) = sJ_{s\sigma W_1}(w \wedge sw) = J_{\sigma W_1}(w)$.

(3) Suppose u is the unique minimal element in $J_{s\sigma W_1}(w \wedge sw)$. Then su belongs to $J_{\sigma W_1}(w)$ by (2). Let x be any element in $J_{\sigma W_1}(w)$. We have $x \vee sx = x$ by (1), and $sx \in J_{s\sigma W_1}(w \wedge sw)$ by (2), so $u \leq sx$. Thus $su \leq u \vee su \leq sx \vee s(sx) = sx \vee x = x$.

Suppose u is the unique minimal element in $J_{\sigma W_1}(w)$. Then su belongs to $J_{s\sigma W_1}(w \wedge sw)$ by (2). Let y be any element in $J_{s\sigma W_1}(w \wedge sw)$. We have $u \wedge su = su$ by (1), and $sy \in J_{\sigma W_1}(w)$ by (2), so $u \leq sy$. Thus

$su = u \wedge su \leq sy \wedge s(sy) = sy \wedge y \leq y.$ $\square$

Proposition 2.1.17. ("Deodhar's lemma", see e.g. [13, Lemma 5.8]) *Suppose that $J_{\sigma W_1}(w)$ is not empty. Then it contains a unique minimal element. Moreover, this element can be constructed recursively as follows: let σ be the minimal element in its coset σW_1 and let $s_1, \ldots, s_m$ be a sequence of elements of S such that $s_1 \cdots s_m$ is a reduced expression for σ; put $v_0 = w$ and $v_j = v_{j-1} \wedge s_j v_{j-1}$ for $1 \leq j \leq m$; then v_m belongs W_1, and the minimal element of $J_{\sigma W_1}(w)$ is just σv_m.*

Proof: By repeated application of Lemma 2.1.16 (2), we have $J_{\sigma W_1}(w) = \sigma J_{W_1}(v_m)$. Since $J_{\sigma W_1}(w)$ is non-empty, it follows that $J_{W_1}(v_m)$ is non-empty as well, which means v_m belongs to W_1 and is the unique minimal element in $J_{W_1}(v_m)$. That σv_m is the unique minimal element of $J_{\sigma W_1}(w)$ follows by a repeated application of Lemma 2.1.16 (3). $\square$

Remark 2.1.18. We make a few remarks regarding the construction in Proposition 2.1.17.

1. The element v_m in the statement of Proposition 2.1.17 is $\mathbf{min}\, I(\sigma^{-1})w$ (see the third item in the list in Remark 2.1.10). Thus $J_{\sigma W_1}(w)$ is non-empty if and only if $\mathbf{min}\, I(\sigma^{-1})w$ belongs to W_1 and in this case its unique minimal element is $\sigma(\mathbf{min}\, I(\sigma^{-1})w)$.

2. Given a double coset $W_1 \sigma W_2$, where W_2 is also a standard parabolic subgroup, there need not be a unique minimal element among those in the double coset that are $\geq w$. Consider for instance the following simple example. Let W be the Weyl group of type A_2: $S = \{s_1, s_2\}$ and $W = \langle s_1, s_2 \,|\, s_1^2 = s_2^2 = 1, s_1 s_2 s_1 = s_2 s_1 s_2 \rangle$. Put $W_1 = W_2 = \langle s_1 \rangle = \{1, s_1\}$, $\sigma = s_2$, and $w = s_1$. Observe that among the elements in

$W_1 s_2 W_2 = \{s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1\}$ that are $\geq s_1$, there are two minimal ones, namely, $s_1 s_2$ and $s_2 s_1$.

3. Let w' be an element of w such that $w \leq w'$. Then, evidently, $\mathbf{min}\, J_{\sigma W_1}(w) \leq \mathbf{min}\, J_{\sigma W_1}(w')$ (assuming that both sets are non-empty).

4. Let σ' be an element of W such that $\sigma W_1 \leq \sigma' W_1$. Suppose that $J_{\sigma W_1}(w)$ is non-empty. Then $J_{\sigma' W_1}(w)$ is non-empty too: for any u in σW_1 (and in particular for any u in $J_{\sigma W_1}(w)$), there exists, by Proposition 2.1.14, u' in $\sigma' W_1$ such that $u \leq u'$, and evidently u' belongs to $J_{\sigma W_1}(w')$ when u belongs to $J_{\sigma W_1}(w)$. However, it need not be true that $\mathbf{min}\, J_{\sigma W_1}(w) \leq \mathbf{min}\, J_{\sigma' W_1}(w)$, as the following simple example shows. Let W be the Weyl group of type A_3:

$$\langle s_1, s_2, s_3 \,|\, s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^3 = (s_2 s_3)^3 = (s_1 s_3)^2 = 1 \rangle$$

   Let W_1 be the parabolic subgroup $\langle s_2 \rangle = \{1, s_2\}$, $w = s_2$, $\sigma = s_1 s_3$, and $\sigma' = s_1 s_2 s_3$. Then $J_{\sigma W_1}(w)$ is non-empty, and $\mathbf{min}\, J_{\sigma W_1}(w) = s_1 s_3 s_2 \not\leq \mathbf{min}\, J_{\sigma' W_1}(w) = s_1 s_2 s_3$.

5. (Deodhar's functions f and g [2]) Suppose that σ is the least element in the coset σW_1. Let σ' be an element of the Weyl group that is the least in its coset $\sigma' W_1$ and suppose that $\sigma W_1 \leq \sigma' W_1$ (equivalently $\sigma \leq \sigma'$).

   Given x in W_1, there exists x' in W_1 such that $\sigma x \leq \sigma' x'$ (see Proposition 2.1.14). Deodhar in [2, Lemma 2.2] states that there is a function $f' : W_1 \to W_1$ (depending upon σ and σ') such that, for x and x' in W_1, $\sigma x \leq \sigma' x'$ if and only if $f'x \leq x'$. (Deodhar writes g for f'.) In the notation of Remark 2.1.10, this function is given by $f'x = \wedge_{\sigma'^{-1}} \sigma x$. Indeed it follows from Proposition 2.1.17 that $\wedge_{\sigma'^{-1}} \sigma x$ has the required property.

   Deodhar also asserts the existence of a function $f : W_1 \to W_1$ (depending upon σ and σ') such that, for x and x' in W_1, $\sigma x \leq \sigma' x'$ if and only if $x \leq fx'$. To describe this function, let $s_1, \ldots, s_m$ be a sequence of elements of S such that $s_1 \cdots s_m$ is a reduced expression for σ'. Put $v_0 = \sigma$, and $v_{i+1} := v_i \wedge s_i v_i$ inductively for $1 \leq i \leq m$. Let p be least, $0 \leq p \leq m$, such that $v_p = \mathrm{id}$: such a p exists because $\sigma \leq \sigma'$. Let $s_{i_1}, \ldots, s_{i_q}$ be the subsequence of $s_{p+1}, \ldots, s_m$ consisting precisely of those elements that belong to W_1. Putting $z := s_{i_1} \star \cdots \star s_{i_q}$, Deodhar's function f is given by $fx' = z \star w'$. We omit the justification (which is not difficult) since we have no use in what follows for this function f.

6. (see [12, Lemma 11 (ii)]) Put $K_{\sigma W_1}(w) := \{v \in \sigma W_1 \,|\, v \leq w\}$ $(= I(w) \cap \sigma W_1)$. If $K_{\sigma W_1}(w)$ is non-empty (or, equivalently, $\sigma \leq w$) then it has a unique maximal element. Indeed, writing w as $\sigma' x'$ where σ' is the minimal element in wW_1 and x' is in W_1, this unique maximal element is $\sigma f x'$, where f is Deodhar's function of the previous item.

Let W be the group of permutations of $[n]$ with S as the set of simple transpositions $(1, 2)$, $\ldots$, $(n-1, n)$. For r, $1 \leq r < n$, let W_r be the standard maximal parabolic subgroup generated by all simple transpositions except $(r, r+1)$. Let σ be a permutation of $[n]$ with one-line notation $\sigma_1\sigma_2 \ldots \sigma_n$. For σ to be of minimal length in its coset σW_r, it is necessary and sufficient that the sequences $\sigma_1 \ldots \sigma_r$ and $\sigma_{r+1} \ldots \sigma_n$ are both increasing. Suppose that this is the case.

Let w be another permutation. For $J_{\sigma W_r}(w)$ to be non-empty, it is necessary and sufficient that $w_1^r \leq \sigma_1$, $\ldots$, $w_r^r \leq \sigma_r$, where $w_1^r < \ldots < w_r^r$ are the elements w_1, $\ldots$, w_r arranged in increasing order. Suppose that this is the case.

Let us suppose further that the $w_1, \ldots, w_r$ were themselves in increasing order, so that $w_1^r = w_1$, $\ldots$, $w_r^r = w_r$. In this case, $\tau := \mathbf{min}\, J_{\sigma W_r}(w)$ is determined as follows. Put $\tau_j = \sigma_j$ for $1 \leq j \leq r$. For $j > r$, an induction on j determines τ_j as follows. Let $\tau_1^{j-1} < \ldots < \tau_{j-1}^{j-1}$ be $\tau_1, \ldots, \tau_{j-1}$ in increasing order and $w_1^j < \ldots < w_j^j$ be $w_1, \ldots, w_j$ in increasing order. Then $\tau_j = w_k^j$, where k is the largest, $1 \leq k \leq j$, such that $\tau_{k-1}^{j-1} < w_k^j$ (we put $\tau_0^{j-1} = -\infty$).

As an example, let $n = 6$, $r = 3$, $w = 145362$, and $\sigma = 246135$. Then, by the recipe above, $\tau = 246153$.

Justification for the recipe appears later: see Example 5.2.2 in §5.2.2. $\square$

2.1.5 Standard tuples and standard lifts

Let $W_1, \ldots, W_m$ be a sequence of standard parabolic subgroups of W and let $\theta = (\tau_1, \ldots, \tau_m) \in W/W_1 \times \cdots \times W/W_m$. We call θ *standard* if there exists a chain $\tilde{\tau}_1 \geq \ldots \geq \tilde{\tau}_m$ of elements in W such that $\tilde{\tau}_j W_j = \tau_j$ for $1 \leq j \leq m$. Such a chain is called a *standard lift* of θ.

Fix a standard tuple θ of cosets and a standard lift of it as above. Put $\sigma_m = \mathbf{min}\, \tau_m$. Observe that $\tilde{\tau}_{m-1} \geq \tilde{\tau}_m \geq \sigma_m$. This means that $J_{\tau_{m-1}}(\sigma_m)$ is not empty, and so it has a unique minimal element by Proposition 2.1.17. Put $\sigma_{m-1} = \mathbf{min}\, J_{\tau_{m-1}}(\sigma_m)$. Proceeding this way, choose inductively $\sigma_j = \mathbf{min}\, J_{\tau_j}(\sigma_{j+1})$ for j equal to $m-2$, $m-3$, $\ldots$, 1. We call the chain $\sigma_1 \geq \ldots \geq \sigma_m$ the *minimal standard lift* of θ. We denote by $\mathfrak{w}(\theta)$ the initial element σ_1 of the minimal standard lift of θ.

Let $\sigma_1 \geq \ldots \geq \sigma_n$ be the minimal standard lift of θ (for some standard tuple θ of cosets). As is easily observed (by a downward induction on j), $\sigma_j \leq \tilde{\tau}_j$ for $1 \leq j \leq m$ for any standard lift $\tilde{\tau}_1 \geq \ldots \geq \tilde{\tau}_m$ of θ. Furthermore, this property characterises the minimal standard lift.

2.1.6 The takeaway from this section

Finally, we isolate the takeaway from this section in its (admittedly strange and whimsical) specific form that will be invoked later.

Corollary 2.1.19. *Let (W,S) be a Coxeter system, τ, φ be elements of W, and W_1, W_2 be standard parabolic subgroups of W. Let $I(\tau^{-1})$ be the Bruhat interval $\{w \in W \mid w \leq \tau^{-1}\}$. Then:*

1. *If τ', φ' are elements of W such that $\tau W_1 = \tau' W_1$ and $\varphi W_2 = \varphi' W_2$, then:*

$$W_1 I({\tau'}^{-1})\varphi' W_2 = W_1 I(\tau^{-1})\varphi W_2$$

2. *There exists a unique minimal element in $W_1 I(\tau^{-1})\varphi W_2$, denoted $\mathbf{min}\, W_1 I(\tau^{-1})\varphi W_2$.*

3. *Let s be an element of S such that $s\tau < \tau$ and $s\varphi W_2 \geq \varphi W_2$. Then*

$$\mathbf{min}\, W_1 I(\tau^{-1})\varphi W_2 = \mathbf{min}\, W_1 I(\tau^{-1}s)\varphi W_2$$

4. *Let s be an element of S such that $s\varphi < \varphi$ and $s\tau W_1 \leq \tau W_1$. Then*

$$\mathbf{min}\, W_1 I(\tau^{-1})\varphi W_2 = \mathbf{min}\, W_1 I(\tau^{-1})s\varphi W_2$$

Proof: For (1), it being evident that $W_1 I(\tau^{-1})\varphi W_2 = W_1 I(\tau^{-1})\varphi' W_2$, it is enough to show that $W_1 I(\tau^{-1}) = W_1 I({\tau'}^{-1})$. By Corollary 2.1.7, $W_1\tau^{-1} = W_1{\tau'}^{-1}$ has a unique minimal element, say σ^{-1}. It is enough to show that $W_1 I(\sigma^{-1}) = W_1 I(\tau^{-1})$. Since $\sigma^{-1} \leq \tau^{-1}$, it follows that $I(\sigma^{-1}) \subseteq I(\tau^{-1})$ and so $W_1 I(\sigma^{-1}) \subseteq W_1 I(\tau^{-1})$. To prove the other way containment, write $\tau^{-1} = u^{-1}\sigma^{-1}$ with $u \in W_1$ and $\ell(\tau) = \ell(u) + \ell(\sigma)$, where ℓ stands for "length". Suppose that

$\rho^{-1} \leq \tau^{-1}$. Then $\rho^{-1} = v^{-1}\rho'^{-1}$ with $v \leq u$ (hence $v \in W_1$) and $\rho' \leq \sigma$. Thus $W_1\rho^{-1} = W_1 v^{-1}\rho'^{-1} = W_1\rho'^{-1}$, and we are done.

Assertion (2) follows from Corollaries 2.1.5 and 2.1.7.

Proof of (3): By Corollary 2.1.5, $\mathbf{min}\, I(\tau^{-1})\varphi$ exists. First suppose that $s\varphi > \varphi$. Then, by Corollary 2.1.6, $\mathbf{min}\, I(\tau^{-1})\varphi = \mathbf{min}\, I(\tau^{-1}s)\varphi$, and, by Corollary 2.1.7, the desired equality follows. Next suppose that $s\varphi < \varphi$. Then, by Corollary 2.1.11, $s\varphi W_2 \leq \varphi W_2$ and, given our hypothesis that $s\varphi W_2 \geq \varphi W_2$, we conclude that $s\varphi W_2 = \varphi W_2$. Put $\varphi' := s\varphi$. Then $\varphi = s\varphi' > \varphi' = s\varphi$, and, by the first case, we have $\mathbf{min}\, W_1 I(\tau^{-1})\varphi' W_2 = \mathbf{min}\, W_1 I(\tau^{-1}s)s\varphi' W_2$. But, since $\varphi' W_2 = \varphi W_2$, this is exactly the desired equality.

Proof of (4): This is analogous to the proof of (3). By Corollary 2.1.5, $\mathbf{min}\, I(\tau^{-1})\varphi$ exists. First suppose that $s\tau < \tau$. Then, by Corollary 2.1.6, $\mathbf{min}\, I(\tau^{-1})\varphi = \mathbf{min}\, I(\tau^{-1})s\varphi$, and, by Corollary 2.1.7, the desired equality follows. Next suppose that $s\tau > \tau$. Then, by Corollary 2.1.11, $s\tau W_1 \geq \tau W_1$ and, given our hypothesis that $s\tau W_1 \leq \tau W_1$, we conclude that $s\tau W_1 = \tau W_1$. Put $\tau' := s\tau$. Then $s\tau' = \tau < s\tau = \tau'$, and, by the first case, we have $\mathbf{min}\, W_1 I(\tau'^{-1})\varphi W_2 = \mathbf{min}\, W_1 I(\tau'^{-1})s\varphi W_2$. But, since $\tau' W_1 = \tau W_1$, we have, by part (1), $W_1 I(\tau'^{-1}) = W_1 I(\tau^{-1})$, and the desired equality follows. $\square$

Chapter 3

The Kostant-Kumar filtration

3.1 The KK filtration on concatenated LS paths

In this section, we introduce the two key elements that underpin this entire thesis, namely:

- the definition of Kostant-Kumar (KK) sets of concatenated Lakshmibai-Seshadri (LS) paths (Equation (3.1.2)).
- the result that such a KK set is invariant under the action of root operators (Proposition 3.1.3)

The following notation remains fixed throughout this thesis: $\mathfrak{g}$ denotes a symmetrizable Kac-Moody algebra; λ and μ are fixed dominant integral weights; W is the Weyl group and W_λ, W_μ are respectively the stabilizers in W of λ, μ.

We assume familiarity with the basic notions and results of Littelmann's theory [14, 15] of paths. Let $\mathcal{P}_\lambda$, $\mathcal{P}_\mu$ be respectively the sets of Lakshmibai-Seshadri (LS) paths of shape λ, μ. Let $\mathcal{P}_\lambda \star \mathcal{P}_\mu := \{\pi \star \pi' \,|\, \pi \in \mathcal{P}_\lambda,\ \pi' \in \mathcal{P}_\mu\}$, where $\star$ denotes concatenation.

Recall that a path π in $\mathcal{P}_\lambda$ consists of a sequence $\tau_1 > \tau_2 > \ldots > \tau_r$ of elements in W/W_λ and a sequence $0 = a_0 < a_1 < \ldots < a_{r-1} < a_r = 1$ of rational numbers (subject to some integrality conditions as in [14, §2], the details of which are not so relevant for the moment). We call τ_1 the *initial direction* and τ_r the *final direction* of π.

3.1.1 Definition of a KK set in $\mathcal{P}_\lambda \star \mathcal{P}_\mu$

Given a path $\pi \star \pi'$ in $\mathcal{P}_\lambda \star \mathcal{P}_\mu$, we define, using Corollary 2.1.19, part (2), its associated Weyl group element $\mathfrak{w}(\pi \star \pi')$ by:

$$\mathfrak{w}(\pi \star \pi') := \mathbf{min}\, W_\lambda I(\tau^{-1})\varphi W_\mu \tag{3.1.1}$$

where τ and φ are lifts in W respectively of the final direction of π and the initial direction of π'. Part (1) of Corollary 2.1.19 says that $\mathfrak{w}(\pi \star \pi')$ is independent of the choice of the lifts τ and φ.

Given an element $\varphi = W_\lambda w W_\mu$ of the double coset space $W_\lambda \backslash W / W_\mu$, we define the associated *KK set* by:

$$\mathcal{P}(\lambda, \varphi, \mu) := \{\pi \star \pi' \in \mathcal{P}_\lambda \star \mathcal{P}_\mu \,|\, \mathfrak{w}(\pi \star \pi') \leq w\} \tag{3.1.2}$$

The choice of the lift w in W of φ does not matter (see Corollary 2.1.11), and we often write $\mathcal{P}(\lambda, w, \mu)$ in place of $\mathcal{P}(\lambda, \varphi, \mu)$.

Clearly $\mathcal{P}(\lambda, \varphi, \mu) \subseteq \mathcal{P}(\lambda, \varphi', \mu)$ if $\varphi \leq \varphi'$. Thus the KK sets form an increasing filtration of the space $\mathcal{P}_\lambda \star \mathcal{P}_\lambda$ of concatenated LS paths, with underlying poset being the double coset space $W_\lambda \backslash W / W_\mu$ with its Bruhat order. We call this the *KK filtration* on paths.

Remark 3.1.1. For this remark alone, we suppose that W is finite. Let w_0 be the longest element of W. For w and x in W, let $w \circledast x := (ww_0 \star xw_0)w_0$, where $\star$ is Deodhar's operation on the Weyl group discussed in §2.1.9. Let π in $\mathcal{P}_\lambda$ comprise the sequence $\tau_1 > \tau_2 > \ldots > \tau_r$ of elements in W/W_λ and the sequence $0 = a_0 < a_1 < \ldots < a_{r-1} < a_r = 1$ of rational numbers. Let $\pi^\dagger$ be the path in $\mathcal{P}_\lambda$ comprising $w_0\tau_r > \ldots > w_0\tau_1$ and $0 = 1 - a_r < 1 - a_{r-1} < \ldots < 1 - a_1 < 1 = 1 - a_0$. Then

$$\mathfrak{w}(\pi \star \pi') = \phi(\pi^\dagger)^{-1} \circledast \phi(\pi')$$

where $\phi(\eta)$ for an LS path η is the minimal lift in W of the initial direction of η.

3.1.2 Stability of KK sets under root operators

For α a simple root, let e_α and f_α be the root operators on paths as defined in [15]. Although this definition differs from the earlier one in [14], it is "backwards-compatible": as explained in [15, Corollary 2 on page 512], the results of [14] are unaffected and we can freely quote them.

Let π be a path in $\mathcal{P}_\lambda$. Recall from [14]:

1. The straight line path π_λ from the origin to λ belongs to $\mathcal{P}_\lambda$.
2. π is piece-wise linear and its end point $\pi(1)$ is an integral weight.
3. For a simple root α, if $e_\alpha(\pi)$ (respectively $f_\alpha(\pi)$) does not vanish, then it belongs to $\mathcal{P}_\lambda$, its end point is $\pi(1) + \alpha$ (respectively $\pi(1) - \alpha$), and $f_\alpha(e_\alpha(\pi))$ (respectively $e_\alpha(f_\alpha(\pi))$) equals π.
4. Let α be a simple root and τ the initial (respectively, final) direction of π. If $e_\alpha\pi$ does not vanish, then its initial (respectively, final) direction is either τ or $s_\alpha\tau$. The same holds for f_α in place of e_α.

5. π is obtained from π_λ by applying a suitable finite sequence of the root operators f_α, as α varies. In particular, the end point of π is of the form $\lambda - \kappa$ where κ is a non-negative integral linear combination of the simple roots.

6. Every value that is a local minimum of the function $h^\pi_\alpha(t) := \langle \pi(t), \alpha^\vee \rangle$ on $t \in [0, 1]$ is an integer, for every simple root α. (A value $h^\pi_\alpha(t_0)$ is called a *local minimum* if $h^\pi_\alpha(t_0) \leq h^\pi_\alpha(t)$ for $0 \leq |t - t_0| < \epsilon$ for some $\epsilon > 0$.) This follows from the proof of [15, Lemma 4.5, part (d)] although the definition of local minimum there is less inclusive.

7. If $e_\alpha(\pi)$ vanishes for every simple root α, then $\pi = \pi_\lambda$ [14, Corollary in §3.5]. In particular, if (the image of) π lies entirely in the dominant Weyl chamber, then $\pi = \pi_\lambda$.

Lemma 3.1.2. *Let $\pi \star \pi'$ be a path in $\mathcal{P}_\lambda \star \mathcal{P}_\mu$ and α a simple root. Then:*

1. *Every local minimum value of the function $h^{\pi\star\pi'}_\alpha(t) := \langle (\pi \star \pi')(t), \alpha^\vee \rangle$ is an integer.*

Suppose that $e_\alpha(\pi \star \pi')$ does not vanish. Then:

2. *$e_\alpha(\pi \star \pi')$ equals either $e_\alpha \pi \star \pi'$ or $\pi \star e_\alpha \pi'$.*

3. *$\mathfrak{w}(\pi \star \pi') = \mathfrak{w}(e_\alpha(\pi \star \pi'))$.*

PROOF: Statement (1) holds because $\pi(1)$ is an integral weight (item 2 above) and local minima of both functions h^π_α and $h^{\pi'}_\alpha$ are integers (item 6 above).

Item (2) appears as [15, Lemma 2.7]. At any rate, it follows readily from the definition of e_α once we know that the absolute minima of the functions h^π_α and $h^{\pi'}_\alpha$ are both integers, which is guaranteed by item 6 above.

To prove (3), let τ be the final direction of π and φ the initial direction of π'. First suppose that $e_\alpha(\pi \star \pi') = e_\alpha \pi \star \pi'$. By item 4 above, the final direction of $e_\alpha \pi$ is either τ or $s_\alpha\tau$. If it is τ, then there is nothing for us to do. In case it is $s_\alpha\tau$, then, from the definition of e_α and properties of π and π', it follows that $s_\alpha\tau < \tau$ and $s_\alpha\varphi \geq \varphi$. The assertion now follows from part 3 of Corollary 2.1.19.

Now suppose that $e_\alpha(\pi \star \pi') = \pi \star e_\alpha\pi'$. By item 4 above, the initial direction of $e_\alpha\pi'$ is either φ or $s_\alpha\varphi$. If it is φ, then there is nothing for us to do. In case it is $s_\alpha\varphi$, then, from the definition of e_α and properties of π and π', it follows that $s_\alpha\varphi < \varphi$ and $s_\alpha\tau \leq \tau$. The assertion now follows from part 4 of Corollary 2.1.19. □

The equivalence relation on $\mathcal{P}_\lambda \star \mathcal{P}_\mu$ defined by root operators

Given $\pi \star \pi'$ and $\sigma \star \sigma'$ paths in $\mathcal{P}_\lambda \star \mathcal{P}_\mu$, let us say $\pi \star \pi'$ *is related to* $\sigma \star \sigma'$ if $\pi \star \pi'$ equals either $e_\alpha(\sigma \star \sigma')$ or $f_\alpha(\sigma \star \sigma')$ for some simple root α. This relation is symmetric since $\pi \star \pi' = e_\alpha(\sigma \star \sigma')$ if and only if $f_\alpha(\pi \star \pi') = \sigma \star \sigma'$. Denote by $\sim$ the reflexive and transitive closure of this relation (as we vary over all simple roots).

As an immediate consequence of the item (3) of Lemma 3.1.2, we have:

Proposition 3.1.3. *The association $\pi \star \pi' \mapsto \mathfrak{w}(\pi \star \pi')$ is constant on equivalence classes of the equivalence relation $\sim$. In particular, for any $\varphi \in W_\lambda \backslash W / W_\mu$, the KK set $\mathcal{P}(\lambda, \varphi, \mu)$ is a union of such equivalence classes.*

In other words, each KK set is stable under the root operators. We will show in §4.3 that a KK set provides a *path model* for the corresponding KK module.

3.2 More Preliminaries

Notation is fixed as in §3.1: $\mathfrak{g}$ is a symmetrizable Kac-Moody algebra; λ, μ are dominant integral weights; W is the Weyl group; and W_λ, W_μ are the stabilisers in W of λ, μ respectively.

3.2.1 Geometric interpretation of minimal representatives in $W_\lambda \backslash W / W_\mu$

We now give a geometric interpretation of the unique minimal element in a given double coset in $W_\lambda \backslash W / W_\mu$. The association $\overline{w} \leftrightarrow w\mu$ (for $w \in W$) gives a bijection of the coset space W/W_μ with the set $W\mu$ of W-conjugates of μ. We identify the sets W/W_μ and $W\mu$ via this bijection. The double coset space $W_\lambda \backslash W / W_\mu$ may then be identified with the set of W_λ-orbits of $W\mu$.

Proposition 3.2.1. *Every W_λ-orbit of the set $W\mu$ of W-conjugates of μ contains a unique element $w\mu$ such that $\lambda + tw\mu$ is dominant for some real number $t > 0$.*

PROOF: Each such orbit contains a unique $w\mu$ that is W_λ-dominant. The W_λ-dominance means precisely that $\langle w\mu, \alpha^\vee \rangle \geq 0$ for every simple root α in W_λ. It is easily verified that $w\mu$ has the desired property. Conversely, if $w\mu$ is not W_λ-dominant, then $\langle w\mu, \alpha^\vee \rangle < 0$ for some simple root α in W_λ, and so $\langle \lambda + tw\mu, \alpha^\vee \rangle = t\langle w\mu, \alpha^\vee \rangle < 0$ for $t > 0$. □

The double coset space $W_\lambda \backslash W / W_\mu$ may thus be identified with the set of those Weyl conjugates $w\mu$ of μ such that $\lambda + tw\mu$ is dominant for some positive t. We illustrate this by means of an example. Let $\mathfrak{g}$ be of type B_2. Let e_1 and e_2 be the standard basis vectors in $\mathbb{R}^2$ with its standard inner product. We may take $\alpha_1 := e_1$ and $\alpha_2 := e_2 - e_1$ to be the simple roots. Then the set of all positive roots is $\{\alpha_2, \alpha_1, \alpha_2 + \alpha_1, \alpha_2 + 2\alpha_1\}$, and the fundamental weights are $\varpi_1 = \frac{1}{2}(\epsilon_1 + \epsilon_2)$ and $\varpi_2 = \epsilon_2$. The Weyl group consists of 8 elements:

$$W = \{1, s_1, s_2, s_1 s_2, s_2 s_1, s_2 s_1 s_2, s_1 s_2 s_1, s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1\}$$

where s_1 and s_2 are the reflections in the hyperplanes perpendicular to α_1 and α_2 respectively. The shaded portion in the figure is the dominant Weyl chamber.

Take $\lambda = 2\omega_1$ and $\mu = 2\omega_1 + \omega_2$. The stabilizers of λ and μ are respectively: $W_\lambda = \{1, s_2\}$ and $W_\mu = \{1\}$. The set of double cosets $W_\lambda \backslash W / W_\mu$ is:

$$\{\{1, s_2\}, \{s_1, s_2 s_1\}, \{s_1 s_2, s_2 s_1 s_2\}, \{s_1 s_2 s_1, s_2 s_1 s_2 s_1\}\}$$

As is clear from Figure 3.2.1, μ, $s_1\mu$, $s_1 s_2 \mu$, and $s_1 s_2 s_1 \mu$ are all the conjugates of μ for which the line segment joining λ to the conjugate lies for some positive distance in the dominant Weyl chamber. $\square$

Proposition 3.2.2. *Given a double coset in $W_\lambda \backslash W / W_\mu$, let w be the unique minimal element in it with respect to the Bruhat order (as guaranteed by Corollary 2.1.7). Then $w\mu$ is such that $\lambda + tw\mu$ is dominant for all small positive t.*

PROOF: From the proof of Proposition 3.2.1, it is enough to show that $w\mu$ is W_λ-dominant. Suppose that this is not so. Then there exists simple root α with s_α in W_λ such that $\langle w\mu, \alpha^\vee \rangle < 0$. We then have $s_\alpha w < w$, which contradicts the hypothesis that w is the minimal element in its double coset. $\square$

3.2.2 Two key propositions

An LS path π of shape μ is said to be λ-*dominant* if $\lambda + \pi(t)$ belongs to the dominant Weyl chamber for every $t \in [0, 1]$. The set of λ-dominant paths of

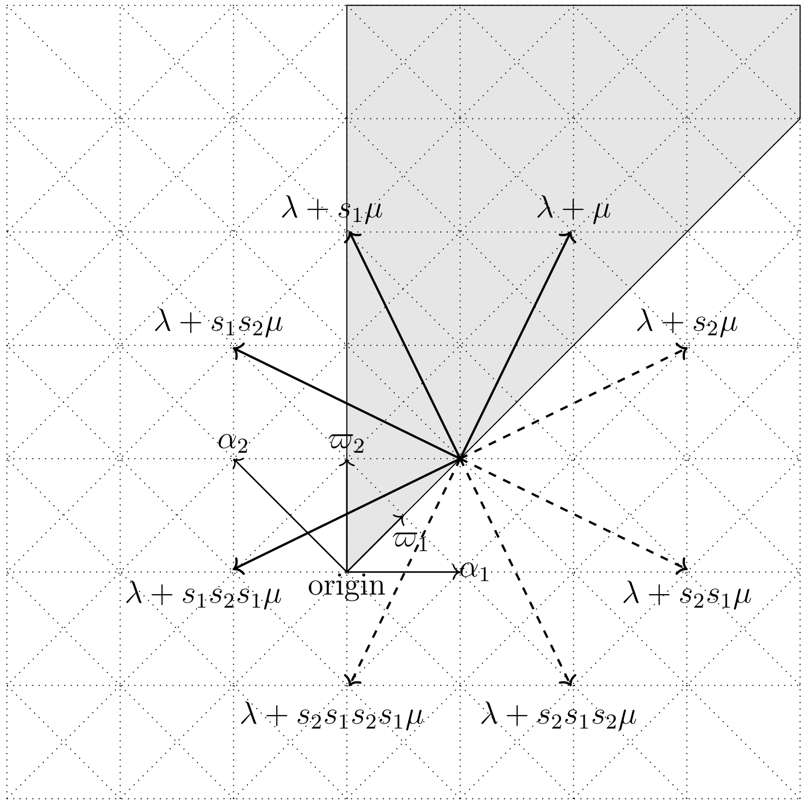


Figure 3.2.1: Illustration of Propositions 3.2.1, 3.2.2; see Example 3.2.1

shape μ is denoted by $\mathcal{P}_\mu^\lambda$. For w an element of the Weyl group, $\mathcal{P}_\mu^\lambda(w)$ denotes the elements of $\mathcal{P}_\mu^\lambda$ whose initial direction is $\leq wW_\mu$.

Proposition 3.2.3. *Let θ be a path in $\mathcal{P}_\lambda \star \mathcal{P}_\mu$. Then there exists a unique path η in the equivalence class (of the relation $\sim$ defined in §3.1.2) containing θ such that $e_\alpha\eta$ vanishes on η for all simple roots α. Moreover, η has the following properties:*

1. *η lies entirely in the dominant Weyl chamber.*

2. *$\eta = \pi_\lambda \star \pi$ for some π in $\mathcal{P}_\mu^\lambda$.*

3. *$\mathfrak{w}(\theta) = \mathfrak{w}(\eta) = v$ where v is minimal in the Weyl group such that $v\mu$ is the initial direction of π. In particular, if $\theta \in \mathcal{P}(\lambda, w, \mu)$ (for some w in W), then $\pi \in \mathcal{P}_\mu^\lambda(w)$.*

Proof: For the existence of η, there is the following standard argument. Construct by induction a sequence θ_0, θ_1, … of elements in the equivalence class of θ as follows. Choose θ_0 to be θ. Given θ_i, if $e_\alpha\theta_i$ vanishes for all simple roots α, then set $\eta = \theta_i$ and we are done. If not, then choose α simple root arbitrarily such that $e_\alpha\theta_i$ does not vanish and put $\theta_{i+1} = e_\alpha\theta_i$. By induction θ_{i+1} belongs to the equivalence class of θ. We will eventually find an η this way, for this process must terminate at some point. In fact, the length of the sequence is bounded by the sum of the coefficients of κ where κ is the non-negative integral linear combination of the simple roots such that the end point of θ equals $\lambda + \mu - \kappa$.

Since $e_\alpha\eta$ vanishes for all simple α and since the absolute minimum of the function $h^\eta_\alpha(t)$ is an integer for every simple α (see item (1) in Lemma 3.1.2 above), it follows from the definition of e_α that η lies entirely in the dominant Weyl chamber.

The uniqueness of η now follows from [15, Corollary 1 (b) of §7].

Write $\eta = \zeta \star \pi$ with $\zeta \in \mathcal{P}_\lambda$ and $\pi \in \mathcal{P}_\mu$. Since η lies entirely in the dominant Weyl chamber, clearly so does ζ. Thus $\zeta = \pi_\lambda$ by item 7 in §3.1.2 above, and π belongs to $\mathcal{P}^\lambda_\mu$.

The equality $\mathfrak{w}(\theta) = \mathfrak{w}(\eta)$ follows from Proposition 3.1.3. Since η lies entirely in the dominant Weyl chamber, it follows that $\lambda + tv\mu$ is dominant for sufficiently small $t \geq 0$. By Proposition 3.2.2, the unique minimal element of $W_\lambda v W_\mu$ lies in vW_μ and hence equals v. But $w(\eta) = \mathbf{min}\, W_\lambda v W_\mu$ by its definition. □

Proposition 3.2.4. *With notation as in Proposition 3.2.3, write $\theta = \pi_1 \star \pi_2$. Then the following conditions are equivalent:*

1. $\eta = \pi_\lambda \star \pi_\mu$ *(that is, $\pi = \pi_\mu$)*
2. $\mathfrak{w}(\theta) = \text{identity}$
3. *there exist $\tilde{\tau}$ and $\tilde{\varphi}$ in W such that $\tilde{\tau} \geq \tilde{\varphi}$, $\tilde{\tau}W_\lambda$ is the final direction of π_1, and $\tilde{\varphi}W_\mu$ is the initial direction of π_2.*

Proof: Let v be as defined in item (3) of Proposition 3.2.3. Observe that condition (1) is equivalent to saying that v is identity. Since $\mathfrak{w}(\theta) = v$ by Proposition 3.2.3 (3), we have (1)$\Leftrightarrow$(2).

(2)$\Rightarrow$(3): Let τ and φ be arbitrary elements in W such that τW_λ is the final direction of π_1 and φW_μ is the initial direction of π_2. Condition (2) says that $\mathbf{min}\, W_\lambda I(\tau^{-1})\varphi W_\mu$ equals identity. Let $u \in W_\lambda$, $\sigma \leq \tau$ in W, and $v \in W_\mu$ be such that $u^{-1}\sigma^{-1}\varphi v =$ identity, or $\varphi v = \sigma u$. We have $\sigma W_\lambda \leq \tau W_\lambda$ (by Corollary 2.1.11). By Proposition 2.1.14, there exists $\tilde{\tau}$ in τW_λ such that $\sigma u \leq \tilde{\tau}$. Taking $\tilde{\varphi} = \varphi v = \sigma u$, (3) is proved.

(3)$\Rightarrow$(2): Since $\tilde{\varphi} \leq \tilde{\tau}$, it follows that $\tilde{\varphi}^{-1}$ belongs to $I(\tilde{\tau}^{-1})$. This implies that $\mathbf{min}\, W_\lambda I(\tilde{\tau}^{-1})\tilde{\varphi} W_\mu$ equals identity. But $\mathfrak{w}(\theta) = \mathbf{min}\, W_\lambda I(\tilde{\tau}^{-1})\tilde{\varphi} W_\mu$ by definition.
$\square$

3.2.3 Extremal paths

Let θ be a path in $\mathcal{P}_\lambda \star \mathcal{P}_\mu$ and let η be as in Proposition 3.2.3 above. Following Montagard [20], we call θ *extremal* if the dominant Weyl conjugate $\overline{\theta(1)}$ of the end point of θ equals the end point $\lambda + \pi(1)$ of η.

The following observation [20, Theorem 2.2 (i)] applied to the path $\pi_\lambda \star \pi_{u\mu}$ is already used in Littelmann's proof [14, §7] of the PRV conjecture (here u denotes an element of W and $\pi_{u\mu}$ the straight line path to the extremal weight $u\mu$ in V_μ):

Proposition 3.2.5. *If a path $\theta \in \mathcal{P}_\lambda \star \mathcal{P}_\mu$ lies entirely in the dominant Weyl chamber except perhaps for a portion of its last straight line segment, then θ is extremal in the above sense.*

Chapter 4

A path model for Kostant-Kumar modules

4.1 The KK (sub)modules of $V_\lambda \otimes V_\mu$

In this section we recall the definition of Kostant-Kumar (KK) modules and two basic results about them (Propositions 4.1.1 and 4.1.2).

Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra. Let λ, μ be dominant integral weights. Let V_λ, V_μ be the irreducible integrable $\mathfrak{g}$-modules with respective highest weights λ, μ. Let W_λ, W_μ be the respective stabilizers of λ, μ in the Weyl group W.

4.1.1 Filtration by KK modules of $V_\lambda \otimes V_\mu$

Fix an element w of the Weyl group. Let v_λ be a highest weight vector in V_λ. Let $v_{w\mu}$ be a non-zero vector in the (one-dimensional) weight space $V_{w\mu}$ of weight $w\mu$ in V_μ. The *Kostant-Kumar module*, or simply *KK module*, $K(\lambda, w, \mu)$ is defined to be the cyclic submodule of the tensor product $V_\lambda \otimes V_\mu$ generated by $v_\lambda \otimes v_{w\mu}$:

$$K(\lambda, w, \mu) := U\mathfrak{g}\,(v_\lambda \otimes v_{w\mu}) \tag{4.1.1}$$

where $U\mathfrak{g}$ denotes the universal enveloping algebra of $\mathfrak{g}$.

Proposition 4.1.1. *Let u and w be elements of the Weyl group such that $W_\lambda u W_\mu = W_\lambda w W_\mu$. Then $K(\lambda, u, \mu) = K(\lambda, w, \mu)$.*

PROOF: For the proof, we will first recall a basic result from [7, §3.8]. Let V be an integrable representation of $\mathfrak{g}$. For a simple reflection s, there is a corresponding linear automorphism r^V of V (defined in [7, Lemma 3.8]) such that:

1. For $v \in V$ a weight vector of weight η, $r^V(v)$ is a weight vector of weight $s(\eta)$.
2. $r^{V\otimes V'} = r^V \otimes r^{V'}$ for V' an integrable representation.
3. For $v \in V$, there exists $x_v \in U\mathfrak{g}$ such that $r^V(v) = x_v(v)$.

It suffices to show that $K(\lambda, u, \mu) \subseteq K(\lambda, w, \mu)$, for then the other containment also holds by interchanging the roles of u and w. Write $u = \tau w \varphi$ with $\tau \in W_\lambda$ and $\varphi \in W_\mu$. We have $u\mu = \tau w \varphi \mu = \tau w \mu$. Let $\tau = s_{i_1} \cdots s_{i_k}$ be a reduced expression for τ. Note that all s_{i_j} belong to W_λ. Consider the operator $r_{i_1} \cdots r_{i_k}$ on $V_\lambda \otimes V_\mu$ where $r_{i_j} = r_{i_j}^{V_\lambda \otimes V_\mu}$ is the linear automorphism corresponding to s_{i_j} (as recalled above).

On the one hand, by properties (1) and (2) above, we have:

$$r_{i_k}(v_\lambda \otimes v_{w\mu}) = r_{i_k}^{V_\lambda}(v_\lambda) \otimes r_{i_k}^{V_\mu}(v_{w\mu}) = c \cdot v_{s_{i_k}\lambda} \otimes v_{s_{i_k} w\mu} = c \cdot v_\lambda \otimes v_{s_{i_k} w\mu}$$

where c is a non-zero scalar. By a chain of similar calculations, we get

$$r_{i_1} \cdots r_{i_k}(v_\lambda \otimes v_{w\mu}) = c' \cdot v_\lambda \otimes v_{\tau w\mu} = c' \cdot v_\lambda \otimes v_{u\mu} \tag{4.1.2}$$

where c' is a non-zero scalar.

On the other hand, by property (3), there exist elements $x_{i_1}, \ldots, x_{i_k}$ of $U\mathfrak{g}$ such that

$$r_{i_1} \cdots r_{i_k}(v_\lambda \otimes v_{w\mu}) = x_{i_1} \cdots x_{i_k}(v_\lambda \otimes v_{w\mu}) \tag{4.1.3}$$

From (4.1.2) and (4.1.3) we get

$$v_\lambda \otimes v_{u\mu} = c'^{-1} \cdot r_{i_1} \cdots r_{i_k}(v_\lambda \otimes v_{w\mu}) = c'^{-1} \cdot x_{i_1} \cdots x_{i_k}(v_\lambda \otimes v_{w\mu})$$

and thus $K(\lambda, u, \mu) = U\mathfrak{g}(v_\lambda \otimes v_{u\mu}) \subseteq U\mathfrak{g}(v_\lambda \otimes v_{w\mu}) = K(\lambda, w, \mu)$. □

Proposition 4.1.2. *For elements u and w of the Weyl group W such that $W_\lambda u W_\mu \leq W_\lambda w W_\mu$ in the Bruhat order on $W_\lambda \backslash W / W_\mu$ (see §2.1.3), we have $K(\lambda, u, \mu) \subseteq K(\lambda, w, \mu)$.*

Proof: By Proposition 4.1.1, we may assume $u = \mathbf{min}\, W_\lambda u W_\mu$ and $w = \mathbf{min}\, W_\lambda w W_\mu$, so that $u \leq w$. Let $U\mathfrak{b}(v_{w\mu})$ be the Demazure module generated by $v_{w\mu}$. Since $u \leq w$, we have $v_{u\mu} \in U\mathfrak{b}(v_{w\mu})$. Thus $v_\lambda \otimes v_{u\mu} \in U\mathfrak{b}(v_\lambda \otimes v_{w\mu}) \subseteq U\mathfrak{g}(v_\lambda \otimes v_{w\mu})$, and $U\mathfrak{g}(v_\lambda \otimes v_{u\mu}) \subseteq U\mathfrak{g}(v_\lambda \otimes v_{w\mu})$. □

Remark 4.1.3. The KK module $K(\lambda, 1, \mu) = U\mathfrak{g}(v_\lambda \otimes v_\mu)$ corresponding to the identity element 1 of the Weyl group is the copy of the irreducible representation $V_{\lambda+\mu}$ in $V_\lambda \otimes V_\mu$. When $\mathfrak{g}$ is of finite type, the KK module $K(\lambda, w_0, \mu)$ corresponding to the longest element w_0 of the Weyl group is the whole tensor product $V_\lambda \otimes V_\mu$. Indeed, letting $\mathfrak{b}_+$ and $\mathfrak{b}_-$ denote respectively the positive and negative Borel subalgebras, we have

$$
\begin{aligned}
U\mathfrak{g}(v_\lambda \otimes v_{w_0\mu}) = U\mathfrak{b}_- \cdot U\mathfrak{b}_+(v_\lambda \otimes v_{w_0\mu}) &= U\mathfrak{b}_-(v_\lambda \otimes U\mathfrak{b}_+ v_{w_0\mu}) \\
&= U\mathfrak{b}_-(v_\lambda \otimes V_\mu) = (U\mathfrak{b}_- v_\lambda) \otimes V_\mu = V_\lambda \otimes V_\mu
\end{aligned}
$$

4.2 Recall of a decomposition rule for Kostant-Kumar (KK) modules

The decomposition rule (Theorem 4.2.1 below) that gives the break up of a KK module into a direct sum of irreducibles is well known. For example, at least in the case when $\mathfrak{g}$ is symmetric (i.e., has a symmetric generalized Cartan matrix), it follows immediately from Joseph's results [6, Theorems 5.25, 5.22]. Our purpose in this section is to state the theorem and also give, for the sake of readability and completeness, a proof in the case when $\mathfrak{g}$ is of finite type. The restrictive hypothesis on $\mathfrak{g}$ (namely that it be of finite type or symmetric) is imposed only due to the use of a positivity result of Lusztig [16, 22.1.7] by Joseph in [6] and is possibly not required: see [6, §1.4].

Theorem 4.2.1. (Joseph [6]) *Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra that is either of finite type or symmetric. Let λ, μ be dominant integral weights and w an element of the Weyl group. Then the decomposition of the KK module $K(\lambda, w, \mu)$ as a direct sum of irreducible $\mathfrak{g}$-modules is given by*

$$
K(\lambda, w, \mu) = \bigoplus V_{\lambda+\pi(1)} \quad \text{where the sum is over } \pi \in \mathcal{P}^\lambda_\mu(w) \tag{4.2.1}
$$

where $\mathcal{P}^\lambda_\mu(w)$ denotes the set of λ-dominant LS paths of shape μ with initial direction $\leq wW_\mu$ (λ-dominance of a path is defined in §3.2.2).

In the case when $\mathfrak{g}$ is symmetric, the theorem follows from Joseph's results as already indicated (see also Naoi [22, Remark 2.12]). In the finite type case, a proof is recorded below (§4.2.2). For this proof, we need a result of Lakshmibai, Littelmann, and Magyar [12], which is a combinatorial analogue of the existence of “excellent filtrations”, a la Joseph [3], Mathieu [18, 19], Polo [24], and van der Kallen [25]. We first recall this result.

4.2.1 A result of Lakshmibai-Littelmann-Magyar

In order to state the result, we introduce some notation. The term *path* in this section means a piecewise linear path whose endpoint lies in the weight lattice (for instance, a concatenation of LS paths of various shapes). Let $\mathcal{P}$ be a set of paths. We define its *character*, denoted $\mathbf{char}\,\mathcal{P}$, by: $\mathbf{char}\,\mathcal{P} := \sum_{\eta\in\mathcal{P}} e^{\eta(1)}$. If π is any path, we let $\pi \star \mathcal{P}$ denote the set of paths $\{\pi \star \eta : \eta \in \mathcal{P}\}$. Suppose π is a path such that $\pi(t)$ belongs to the dominant Weyl chamber for all $t \in [0, 1]$. Fix a reduced word $w = s_{\beta_1} s_{\beta_2} \cdots s_{\beta_k}$: here β_i are simple roots. Define

$$C(\pi, w) := \{f_{\beta_1}^{n_1} f_{\beta_2}^{n_2} \cdots f_{\beta_k}^{n_k}\, \pi : \; n_i \geq 0 \text{ for all } i\}$$

This set is independent of the reduced word chosen, and has character:

$$\mathbf{char}\, C(\pi, w) = \Lambda_w(e^{\pi(1)})$$

where Λ_w is the Demazure operator corresponding to w (see, e.g., [14, §5.1]). Further, when π is the straight line path π_μ, we have $C(\pi_\mu, w) = \mathcal{P}_\mu(w)$, the set of LS paths of shape μ with initial direction $\leq wW_\mu$. The following key result appears in [12, Proposition 12] (see also [5, Theorem in §2.11, also §3.5]).

Proposition 4.2.2. *With notation as above, there exists a Weyl group valued* *function* $\pi \mapsto w(\pi)$ *on* $\mathcal{P}_\mu^\lambda(w)$ *such that*

$$\pi_\lambda \star C(\pi_\mu, w) = \bigsqcup_{\pi \in \mathcal{P}_\mu^\lambda(w)} C(\pi_\lambda \star \pi, w(\pi)) \tag{4.2.2}$$

(The precise form of the function $\pi \mapsto w(\pi)$ *is immaterial for our purposes.)*

Computing characters of both sides in (4.2.2), we obtain:

$$\mathbf{char}\, \pi_\lambda \star C(\pi_\mu, w) = \sum_{\pi \in \mathcal{P}_\mu^\lambda(w)} \Lambda_{w(\pi)}(e^{\lambda+\pi(1)}) \tag{4.2.3}$$

4.2.2 Proof of Theorem 4.2.1 for $\mathfrak{g}$ of finite type

By a result of Kumar [9, Theorem 2.14], the character of the KK module $K(\lambda, w, \mu)$ is given by

$$\mathbf{char}\, K(\lambda, w, \mu) = \Lambda_{w_0}(e^\lambda \cdot \Lambda_w(e^\mu)) \tag{4.2.4}$$

where w_0 is the longest Weyl group element. Since $\Lambda_w(e^\mu)$ is the character of $C(\pi_\mu, w)$, we obtain

$$\mathbf{char}\, K(\lambda, w, \mu) = \Lambda_{w_0}(\mathbf{char}\, \pi_\lambda \star C(\pi_\mu, w)) \tag{4.2.5}$$

Substituting from (4.2.3) into (4.2.4), we obtain:

$$\mathbf{char}\, K(\lambda, w, \mu) = \sum_{\pi \in \mathcal{P}_\mu^\lambda(w)} \Lambda_{w_0}\Lambda_{w(\pi)}(e^{\lambda+\pi(1)}) = \sum_{\pi \in \mathcal{P}_\mu^\lambda(w)} \Lambda_{w_0}(e^{\lambda+\pi(1)})$$

since $\Lambda_{w_0}\Lambda_\sigma = \Lambda_{w_0}$ for all $\sigma \in W$. This latter fact follows from the following well-known property of the Demazure operators: if α is a simple root, then $\Lambda_{s_\alpha}\Lambda_w$ equals $\Lambda_{s_\alpha w}$ or Λ_w according as $s_\alpha w$ is $> w$ or $< w$.

But now $\Lambda_{w_0}(e^{\lambda+\pi(1)})$ is the character of the $\mathfrak{g}$-module $V_{\lambda+\pi(1)}$ (by the Demazure character formula applied to w_0). Thus the modules on both sides of (4.2.1) have the same character, and the proof is complete. $\square$

Example 4.2.3. Consider the situation of Example 3.2.1 and Figure 3.2.1. The λ-dominant paths of shape μ, colour coded by their initial directions, are all listed below and depicted in Figure 4.2.1:

Initial direction	Colour Coding	Path data
identity	violet	$1; \quad 0 < 1$
s_1	red	$s_1; \quad 0 < 1$
s_1	red	$s_1 > 1; \quad 0 < 1/2 < 1$
$s_1 s_2$	cyan	$s_1 s_2 > s_2; \quad 0 < 1/4 < 1$
$s_1 s_2$	cyan	$s_1 s_2 > s_2; \quad 0 < 1/2 < 1$
$s_1 s_2 s_1$	orange	$s_1 s_2 s_1 > s_2 s_1 > s_1 > 1; \quad 0 < 1/4 < 1/3 < 1/2 < 1$
$s_1 s_2 s_1$	orange	$s_1 s_2 s_1 > s_2 s_1 > s_1; \quad 0 < 1/4 < 1/3 < 1$
$s_1 s_2 s_1$	orange	$s_1 s_2 s_1 > s_2 s_1 > s_1; \quad 0 < 1/2 < 2/3 < 1$
$s_1 s_2 s_1$	orange	$s_1 s_2 s_1 > s_2; \quad 0 < 1/2 < 1$

Thus the KK modules decompose as follows:

- $K(\lambda, 1, \mu) = V_{\lambda+\mu}$.

- $K(\lambda, s_1, \mu) = V_{\lambda+\mu} \oplus V_{2\varpi_1+2\varpi_2} \oplus V_{3\varpi_2}$.

- $K(\lambda, s_1 s_2, \mu) = V_{\lambda+\mu} \oplus V_{2\varpi_1+2\varpi_2} \oplus V_{3\varpi_2} \oplus V_{4\varpi_1} \oplus V_{2\varpi_1+\varpi_2}$.

- $K(\lambda, s_1 s_2 s_1, \mu) = V_{\lambda+\mu} \oplus V_{2\varpi_1+2\varpi_2} \oplus V_{3\varpi_2} \oplus V_{4\varpi_1} \oplus V_{2\varpi_1+\varpi_2}^{\oplus 2} \oplus V_{2\varpi_1} \oplus V_{\varpi_2} \oplus V_{2\varpi_2}$.

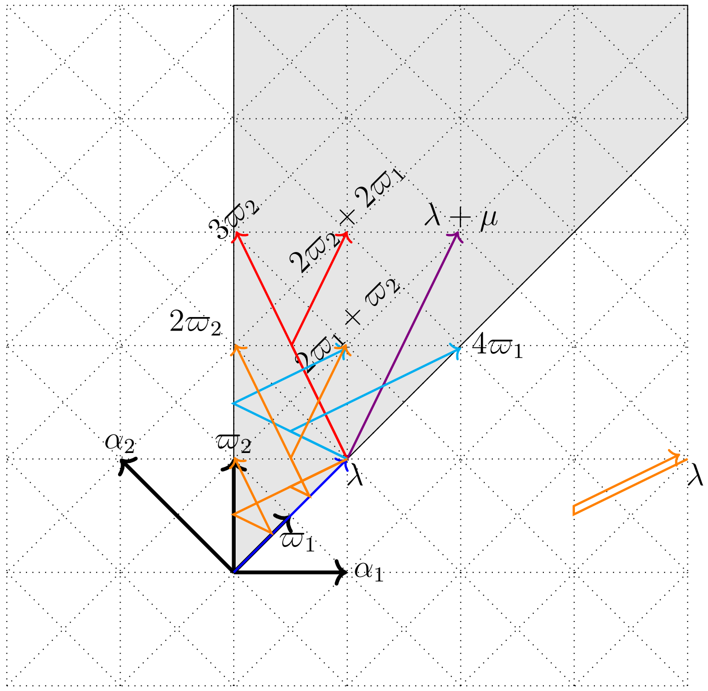


Figure 4.2.1: Decomposition of KK modules of $V_{2\varpi_1} \otimes V_{2\varpi_1+\varpi_2}$ for $\mathfrak{g}$ of type B_2: see Example 4.2.3. The orange coloured path ending at $\lambda = 2\varpi_1$ is shown separately for clarity.

4.3 A path model for Kostant-Kumar (KK) modules

We deduce a path model for KK modules by combining the decomposition rule (Theorem 4.2.1) with the invariance under the root operators (Proposition 3.1.3) of the association (3.1.1) of the Weyl group element $\mathfrak{w}(\pi \star \pi')$ to a concatenation $\pi \star \pi'$ of two LS paths. The restriction on $\mathfrak{g}$ (namely, that it be of finite type or symmetric) in the theorem is only because of the use of the decomposition rule, and is possibly not required.

Theorem 4.3.1. *Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra that is either of finite type or symmetric (as in Theorem 4.2.1). Let λ, μ be dominant integral weights and w an element of the Weyl group. Let $\mathcal{P}_\lambda$ and $\mathcal{P}_\mu$ respectively be the sets of Lakshmibai-Seshadri paths of shapes λ and μ. For $\pi \in \mathcal{P}_\lambda$ and $\pi \in \mathcal{P}_\mu$, let $\mathfrak{w}(\pi \star \pi')$ be the Weyl group element associated as in* (3.1.1) *in §3.1.1 to the* *concatenated path $\pi \star \pi'$. Then the KK set*

$$\mathcal{P}(\lambda, w, \mu) = \{\pi \star \pi' \mid \pi \in \mathcal{P}_\lambda, \pi' \in \mathcal{P}_\mu, \mathfrak{w}(\pi \star \pi') \leq w\}$$

is a path model *for the KK module $K(\lambda, w, \mu)$ in the sense that*

$$\mathbf{char}\, K(\lambda, w, \mu) = \sum_{\eta \in \mathcal{P}(\lambda, w, \mu)} \exp \eta(1) \tag{4.3.1}$$

Proof: From Theorem 4.2.1, we have:

$$\mathbf{char}\, K(\lambda, w, \mu) = \sum_{\pi \in \mathcal{P}_\mu^\lambda(w)} \mathbf{char}\, V_{\lambda+\pi(1)} \tag{4.3.2}$$

where $\mathcal{P}_\mu^\lambda(w)$ is the set of λ-dominant LS paths of shape μ with initial direction $\leq w$. For $\pi \in \mathcal{P}_\mu^\lambda(w)$, let $\mathcal{P}_{\pi_\lambda \star \pi}$ be the equivalence class in $\mathcal{P}_\lambda \star \mathcal{P}_\mu$ containing $\pi_\lambda \star \pi$ (under the equivalence relation $\sim$ defined by the root operators—see §3.1.2), where π_λ denotes the straight line path from the origin to λ. Since $\pi_\lambda \star \pi$ lies entirely in the dominant Weyl chamber (this is what it means for π to be λ-dominant), it follows from the "Isomorphism Theorem" in [15, Theorem 7.1] that

$$\sum_{\sigma \in \mathcal{P}_{\pi_\lambda \star \pi}} \exp \sigma(1) \;=\; \sum_{\sigma \in \mathcal{P}_{\lambda+\pi(1)}} \exp \sigma(1) \tag{4.3.3}$$

(where of course $\mathcal{P}_{\lambda+\pi(1)}$ denotes the set of LS paths of shape $\lambda + \pi(1)$). By the "Character formula" [14, page 330], the right hand side of (4.3.3) equals $\mathbf{char}\, V_{\lambda+\pi(1)}$, so putting together (4.3.2) and (4.3.3) gives

$$\mathbf{char}\, K(\lambda, w, \mu) \;=\; \sum_{\pi \in \mathcal{P}_\mu^\lambda(w)} \sum_{\sigma \in \mathcal{P}_{\pi_\lambda \star \pi}} \exp \sigma(1) \tag{4.3.4}$$

Thus, for the proof of the theorem, it suffices to show the following:

$$(4.3.5) \qquad \mathcal{P}(\lambda, w, \mu) \;=\; \sqcup_{\pi\in\mathcal{P}_\mu^\lambda(w)}\mathcal{P}_{\pi_\lambda\star\pi} \qquad \text{(disjoint union)}$$

To prove (4.3.5), first let $\pi \in \mathcal{P}_\mu^\lambda(w)$. Let u be an element of the Weyl group such that $u\mu$ is the initial direction of π. From our assumption that $uW_\mu \leq wW_\mu$, it follows that $W_\lambda uW_\mu \leq W_\lambda wW_\mu$ (see Corollary 2.1.11) and $\mathfrak{w}(\pi_\lambda \star \pi) \leq w$ (evidently $\mathfrak{w}(\pi_\lambda \star \pi)$ is the minimal element in $W_\lambda uW_\mu$). By Proposition 3.1.3, it follows that the Weyl group elements associated via $\mathfrak{w}$ to elements of $\mathcal{P}_{\pi_\lambda\star\pi}$ are all the same. This proves $\mathcal{P}(\lambda, w, \mu) \supseteq \mathcal{P}_{\pi_\lambda\star\pi}$.

Now let φ be an element in $\mathcal{P}(\lambda, w, \mu)$. Apply Proposition 3.2.3 to φ and let η be as in the conclusion. Then $\eta = \pi_\lambda \star \pi$ for some $\pi \in \mathcal{P}_\mu^\lambda(w)$, and the containment $\subseteq$ is proved.

That the union on the right hand side of (4.3.5) is disjoint follows from the uniqueness of η in Proposition 3.2.3 (which in turn rests on [15, Corollary 1 (b) of §7]): $\pi_\lambda \star \pi$ is the unique path in $\mathcal{P}_{\pi_\lambda\star\pi}$ on which e_α vanishes for all simple roots α. $\square$

4.4 PRV components and generalised PRV components in KK modules

We show how the decomposition rule (Theorem 4.2.1) leads easily to results about the existence of PRV components (Theorem 4.4.1) and generalised PRV components (Theorem 4.4.3) in KK modules. The arguments are well known: see e.g. those by Joseph in [4, §2.7]. In fact, Theorem 4.4.1 for the finite case follows from items (i) and (iii) of the theorem in [4, §2.7].

Theorem 4.4.1 is at once a generalisation of two results: the so called refined PRV and KPRV theorems:

- Its special case when $\mathfrak{g}$ is of finite type and $w = w_0$ (the longest element of the Weyl group) is due to Kumar [10, Theorem 1.2], who refers to his result as "a refinement of the PRV conjecture" and says that it was conjectured by D.-N. Verma.

- The special case when $\sigma = w$ was proved by Kumar [9, page 117] and independently Mathieu [17, Corollaire 3]. Kumar calls it "the strengthened PRV conjecture (due to Kostant)". We have called it "KPRV" following Khare [8].

Theorem 4.4.3 is a KK version of Montagard's result [20, Theorem 3.1] about generalised PRV components.

4.4.1 The map Φ

Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra, and fix dominant integral weights λ and μ. Let W_λ and W_μ denote respectively the stabilizers in the Weyl group W of λ and μ.

Consider the map from the Weyl group W to the set Λ^+ of dominant integral weights given by $\sigma \mapsto \overline{\lambda + \sigma\mu}$, where $\overline{\lambda + \sigma\mu}$ denotes the dominant Weyl conjugate of the weight $\lambda + \sigma\mu$. This map factors through the natural quotient map from W to $W_\lambda \backslash W / W_\mu$. We denote by Φ the map $W_\lambda \backslash W / W_\mu \to \Lambda^+$ given by $W_\lambda \sigma W_\mu \mapsto \overline{\lambda + \sigma\mu}$.

4.4.2 PRV components in KK modules

The restrictive hypothesis on $\mathfrak{g}$ in the following theorem (as also in Theorem 4.4.3), namely that it be either of finite type or symmetric, is inherited from the decomposition rule (Theorem 4.2.1) and is possibly not required.

Theorem 4.4.1. *Let $\mathfrak{g}$ be a symmetrizable Kac Moody algebra that is either of finite type or symmetric (as in Theorem 4.2.1). Let λ, μ be dominant integral weights and w, σ be elements of the Weyl group. Let ν be the dominant Weyl conjugate of the weight $\lambda + \sigma\mu$. Then the irreducible $\mathfrak{g}$-module V_ν occurs in the decomposition into irreducible $\mathfrak{g}$-modules of the KK module $K(\lambda, w, \mu)$ at least as many times as there are elements $\tau \in W_\lambda \backslash W / W_\mu$ such that $\tau \leq W_\lambda w W_\mu$ and $\Phi(\tau) = \nu$.*

Proof: We describe a map $\tilde{\Phi}$ from $W_\lambda \backslash W / W_\mu$ to the set $\mathcal{P}_\mu^\lambda$ of λ-dominant LS paths of shape μ. Given $\tau \in W_\lambda \backslash W / W_\mu$, let v be the unique minimal element in τ. Consider the path $\varphi = \pi_\lambda \star \pi_{v\mu}$ in $\mathcal{P} \star \mathcal{P}_\mu$, where π_λ and $\pi_{v\mu}$ are the straight line paths from the origin to λ and $v\mu$ respectively. Note that $\mathfrak{w}(\varphi) = v$. Apply Proposition 3.2.3 to φ and let η be as in its conclusion. Then $\eta = \pi_\lambda \star \pi$ for some $\pi \in \mathcal{P}_\mu^\lambda$ and $\mathfrak{w}(\eta) = v$. We define $\tilde{\Phi}(\tau) := \pi$. Since π determines η from which we can recover v and in turn τ, it follows that $\tilde{\Phi}$ is injective.

It follows from Proposition 3.2.5 that φ as above is extremal, which means that $\eta(1) = \overline{\varphi(1)} = \overline{\lambda + v\mu} = \Phi(\tau)$. Thus $\tilde{\Phi}$ is a "lift" to $\mathcal{P}_\mu^\lambda$ of Φ, meaning that $\Phi(\tau)$ is the end point of $\tilde{\Phi}(\tau)$ shifted by λ for any τ in $W_\lambda \backslash W / W_\mu$. Combining this fact and the injectivity of $\tilde{\Phi}$ with the decomposition rule (Theorem 4.2.1), we immediately obtain the theorem. □

4.4.3 KPRV recovered

It follows immediately from the theorem that V_ν occurs at least once in $K(\lambda,\sigma,\mu)$. We now observe that it occurs at most once by repeating the following elementary argument from [9, §2.7]. Indeed in any $\mathfrak{g}$-homomorphism from $K(\lambda,\sigma,\mu)$ module to V_ν, the vector $v_\lambda \otimes v_{\sigma\mu}$ has to map to an element of weight $\lambda+\sigma\mu$. But the dimension of the $\lambda+\sigma\mu$-weight space in V_ν is clearly one, since $\lambda+\sigma\mu$ is a Weyl conjugate of ν. Thus the space of $\mathfrak{g}$-homomorphisms from $K(\lambda,\sigma,\mu)$ to V_ν is one dimensional, and we have:

Corollary 4.4.2. *Let $\mathfrak{g}$, λ, μ, σ, and ν be as in Theorem 4.4.1. Then the irreducible $\mathfrak{g}$-module V_ν occurs exactly once in the decomposition into irreducible $\mathfrak{g}$-modules of the KK module $K(\lambda,\sigma,\mu)$.*

Put $\lambda=\mu$. Let w be an element of the Weyl group such that $W_\lambda w W_\lambda \neq W_\lambda w^{-1} W_\lambda$ (e.g., when λ is regular and $w \neq w^{-1}$). Since $\lambda + w\lambda$ and $\lambda + w^{-1}\lambda$ are Weyl conjugates, it follows that V_ν where $\nu = \overline{\lambda + w\lambda}$ appears uniquely in $K(\lambda, w, \lambda)$ and in $K(\lambda, w^{-1}, \lambda)$ (by Corollary 4.4.2) and at least twice in the tensor product $V_\lambda \otimes V_\lambda$ (by Theorem 4.4.1). □

4.4.4 Generalised PRV components in KK modules

Importing to our context a result of Montagard [20, Theorem 3.1], we prove the following:

Theorem 4.4.3. *Let $\mathfrak{g}$, λ, and μ be as in Theorem 4.4.1. Let v, u be elements in the Weyl group and β a positive root such that either $v^{-1}\beta$ or $u^{-1}\beta$ is a simple root. Let k be an integer such that $0 \leq k \leq \min\{\langle v\lambda, \beta^\vee\rangle, \langle u\mu, \beta^\vee\rangle\}$ and the integral weight $\nu = v\lambda + u\mu - k\beta$ is dominant. Then the irreducible $\mathfrak{g}$-module V_ν occurs in the decomposition of the KK module $K(\lambda, w, \mu)$ into irreducibles where $w = v^{-1}s_\beta u$.*

PROOF: First suppose that $v^{-1}\beta$ is simple. Since $k \leq \langle v^{-1}u\mu, v^{-1}\beta^\vee\rangle$, it follows that $f^k_{v^{-1}\beta}\pi_{v^{-1}u\mu}$ does not vanish. Consider the path $\varphi := \pi_\lambda \star f^k_{v^{-1}\beta}\pi_{v^{-1}u\mu}$ in $\mathcal{P}_\lambda \star \mathcal{P}_\mu$. As is easily verified, the dominant Weyl conjugate of $\varphi(1)$ is ν and $\mathfrak{w}(\varphi)$ is either $\mathbf{min}\, W_\lambda v^{-1}s_\beta u W_\mu$ or $\mathbf{min}\, W_\lambda v^{-1}uW_\mu$ depending upon whether $k > 0$ or $k = 0$. An easy verification (using the hypothesis that $\langle u\mu, \beta^\vee\rangle \geq 0$) shows that $w = v^{-1}s_\beta u \geq v^{-1}u$. Thus $w \geq \mathfrak{w}(\varphi)$ in either case, and $\varphi \in \mathcal{P}(\lambda, w, \mu)$.

Apply Proposition 3.2.3 to the path φ and let η be as in its conclusion. Then η is of the form $\pi_\lambda \star \pi$ with $\pi \in \mathcal{P}^\lambda_\mu(w)$. By the decomposition rule (Theorem 4.2.1), $V_{\eta(1)}$ occurs in $K(\lambda, w, \mu)$. But Montagard [20, Proof of Theorem 3.1] shows that φ is extremal, which means that $\nu = \overline{\varphi(1)} = \eta(1)$, and the proof is done in this case.

Now suppose that $u^{-1}\beta$ is simple. Then, applying the result in the previous case, we conclude that V_ν occurs in the KK submodule $K(\mu, w^{-1}, \lambda)$ of $V_\mu \otimes V_\lambda$. But under the $\mathfrak{g}$-isomorphism $a \otimes b \leftrightarrow b \otimes a$ of $V_\mu \otimes V_\lambda$ with $V_\lambda \otimes V_\mu$, the submodules $K(\mu, w^{-1}, \lambda)$ and $K(\lambda, w, \mu)$ map isomorphically to each other. $\square$

Remark 4.4.4.

1. In the set up of the theorem, let $\mathfrak{g}$ be of finite type. In place of the hypothesis that either $v^{-1}\beta$ or $u^{-1}\beta$ is simple, let us assume that β is simple. In this case too [20, Theorem 3.1] says that V_ν occurs in the full tensor product $V_\lambda \otimes V_\mu$. We have not handled this case.

2. Suppose that β is a negative root such that the hypothesis is satisfied: either $v^{-1}\beta$ or $u^{-1}\beta$ is simple, $0 \leq k \leq \min\{\langle v\lambda, \beta^\vee\rangle, \langle u\mu, \beta^\vee\rangle\}$, and $\nu = v\lambda + u\mu - k\beta$ is dominant. Then $\langle \nu, \beta^\vee\rangle = 0$ (by the dominance of ν) and the hypothesis is also satisfied if we put $s_\beta\beta = -\beta$, $s_\beta v$, $s_\beta u$, k, and ν in place respectively of β, v, u, k, and ν.

3. In the proof of the first case (when $v^{-1}\beta$ is simple), the hypothesis that $k \leq \langle v\lambda, \beta^\vee\rangle$ is not explicitly used, but it is implicitly used in the invocation of Montagard's criterion for a path to be extremal.

In response to a question of the referee, we identify certain cases in which the multiplicity of the generalised PRV component in Theorem 4.4.3 is precisely one:

Corollary 4.4.5. *Fix notation and hypothesis as in Theorem 4.4.3.*

1. *Suppose that $k = 0$. Then V_ν occurs precisely once in $K(\lambda, v^{-1}u, \mu)$.*

2. *Suppose that k is equal either to $\langle v\lambda, \beta^\vee\rangle$ or $\langle u\mu, \beta^\vee\rangle$. Then V_ν occurs precisely once in $K(\lambda, w, \mu)$.*

3. *Suppose that β is simple. Then V_ν occurs precisely once in $K(\lambda, w, \mu)$.*

PROOF: (1) In this case, we have $\nu = v\lambda + u\mu = \overline{\lambda + v^{-1}u\mu}$, and so the result follows from KPRV (Corollary 4.4.2).
(2) Suppose that $\langle v\lambda, \beta^\vee\rangle = k$ (the case when $\langle u\mu, \beta^\vee\rangle = k$ is similar). Then $\nu = v\lambda + u\mu - k\beta = \overline{\lambda + w\mu}$, and the result follows once again from Corollary 4.4.2.
(3) The proof in this case is similar to that of Corollary 4.4.2. By Theorem 4.4.3, V_ν occurs at least once in $K(\lambda, w, \mu)$. It is therefore enough to show that the space of $\mathfrak{g}$-homomorphisms from $K(\lambda, w, \mu)$ to V_ν is at most one dimensional. The generator $v_\lambda \otimes v_{w\mu}$ of the $U\mathfrak{g}$-module $K(\lambda, w, \mu)$ has weight $\lambda + w\mu$. Since weight is preserved under a $\mathfrak{g}$-homomorphism, it is enough to show that $\lambda + w\mu$ has multiplicity at most one in V_ν. A small calculation shows that $\lambda + w\mu = v^{-1}(\nu - (\langle u\mu, \beta^\vee\rangle - k)\beta)$. Since β is assumed to be simple, the multiplicity of $\nu - (\langle u\mu, \beta^\vee\rangle - k)\beta$ (and so also of its v^{-1} translate) in V_ν is at most one, and we are done. $\square$

We illustrate the result of Theorem 4.4.3 and also the idea behind its proof by means of an example borrowed from Montagard [20]. Let the root system of $\mathfrak{g}$ be G_2. Let e_1 and e_2 be the standard basis vectors of $\mathbb{R}^2$ with its standard inner product. We may take $\alpha_1 := e_1$ and $\alpha_2 := -\frac{3}{2}e_1 + \frac{\sqrt{3}}{2}e_2$ to be the simple roots. The

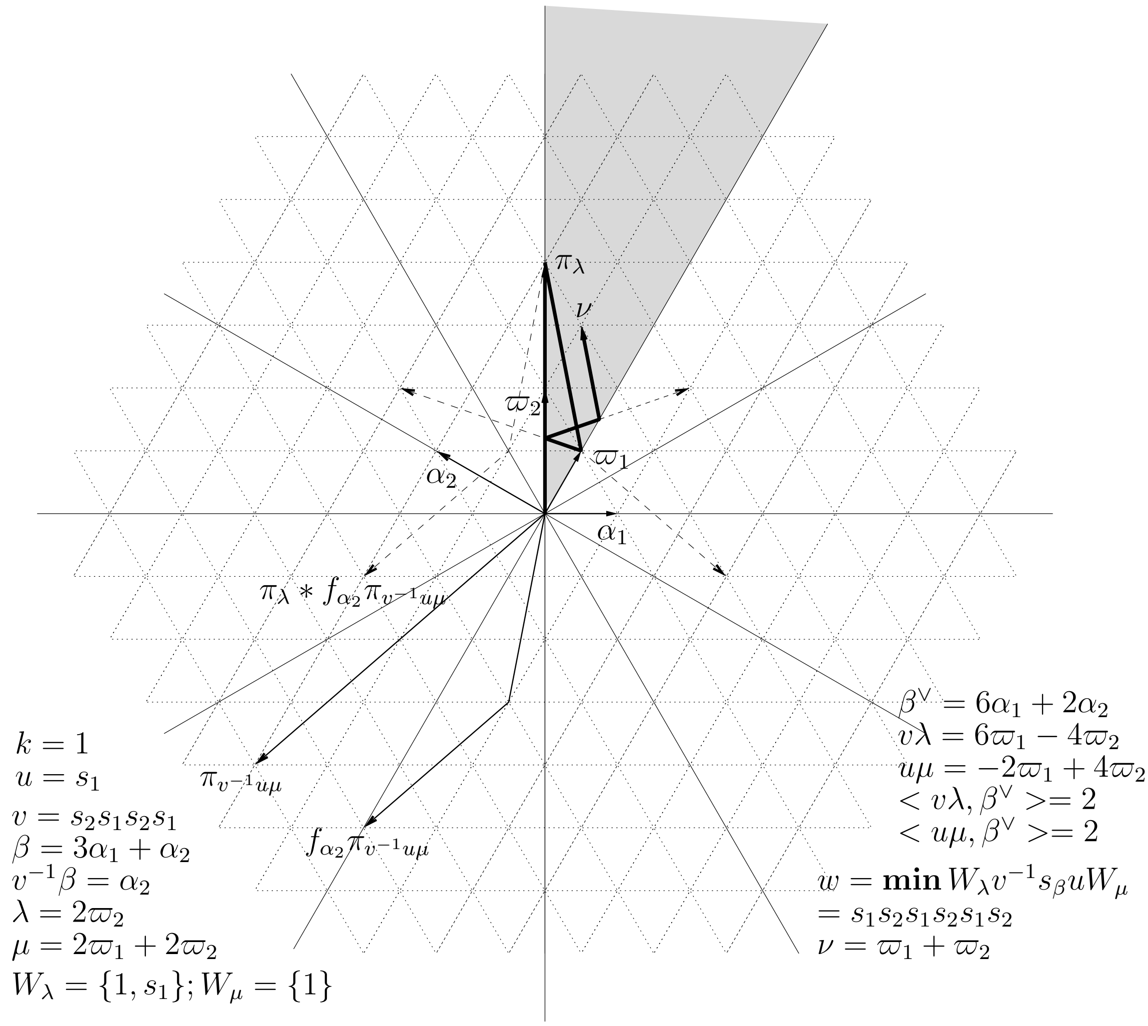


Figure 4.4.1: $V_\nu \in K(\lambda, w, \mu)$: see Example 4.4.4

set of all positive roots is:

$$\{\alpha_2, \alpha_1, \alpha_2 + \alpha_1, \alpha_2 + 2\alpha_1, \alpha_2 + 3\alpha_1, 2\alpha_2 + 3\alpha_1\}$$

Let $s_1 = s_{\alpha_1}$, $s_2 = s_{\alpha_2}$ be the simple reflections and ϖ_1, ϖ_2 the fundamental weights. The dominant integral weights λ, μ, ν, the Weyl group elements u, v, w, the root β, and the integer k are all as shown in Figure 4.4.1. The path η ending at ν appears in bold. $\square$ Consider the situation of Example 4.2.3 and Figure 4.2.1. The decompositions of the KK modules $K(\lambda, w, \mu)$ are given there. The PRV and generalised PRV components, identified by their highest weights, are listed below against the smallest KK modules to which they belong.

KK modules	PRV Components	Generalised PRV components
$K(\lambda, 1, \mu)$	$\lambda + \mu = 4\varpi_1 + \varpi_2$	
$K(\lambda, s_1, \mu)$	$3\varpi_2$	$2\varpi_1 + 2\varpi_2$
$K(\lambda, s_1 s_2, \mu)$	$2\varpi_1 + \varpi_2$	$4\varpi_1$
$K(\lambda, s_1 s_2 s_1, \mu)$	ϖ_2	$2\varpi_2, \quad 2\varpi_1$

The only component that is not listed above is the second copy of $V_{2\varpi_1+\varpi_2}$, the one which belongs to the full tensor product but not to any smaller KK module. It is not captured by the theorems in this section: it is not a PRV or generalised PRV component.

The multiplicities of the listed components are all 1 in the respective smallest KK modules to which they belong. For the PRV components, this follows from Corollary 4.4.2. For $2\varpi_1 + 2\varpi_2$ $(\lambda + \mu - \alpha_1)$ and $4\varpi_1$ $(\lambda + s_2\mu - \alpha_1)$, this follows from Corollary 4.4.5 (3), but not for $2\varpi_1$ $(s_1\lambda + s_2\mu - (\alpha_1 + \alpha_2))$ or $2\varpi_2$ $(s_2\lambda + s_1\mu - (\alpha_1 + \alpha_2))$. $\square$ Let $\mathfrak{g}$ be of type A_2. Let α_1, α_2 be the simple roots and $\varpi_1 = \epsilon_1$, $\varpi_2 = \epsilon_1 + \epsilon_2$ the corresponding fundamental weights. Let $\lambda = \varpi_1 + 2\varpi_2$ and $\mu = 2\varpi_1 + \varpi_2$. The decomposition into irreducibles of the tensor product $V_\lambda \otimes V_\mu$ is depicted in Figure **??**. Except for one copy of $V_{2\varpi_1+2\varpi_2}$ which belongs to $K(\lambda, s_2 s_1, \mu)$, every component occurs as a PRV or generalised PRV component. These, identified by their highest weights, are listed below against the smallest KK modules to which they belong.

KK modules	PRV Components	Generalised PRV components
$K(\lambda, 1, \mu)$	$\lambda + \mu = 3\varpi_1 + 3\varpi_2$	
$K(\lambda, s_1, \mu)$	$\varpi_1 + 4\varpi_2$	
$K(\lambda, s_2, \mu)$	$4\varpi_1 + \varpi_2$	
$K(\lambda, s_1 s_2, \mu)$	$2\varpi_1 + 2\varpi_2$	
$K(\lambda, s_2 s_1, \mu)$	$\varpi_1 + \varpi_2$	$3\varpi_1, \quad 3\varpi_2$
$K(\lambda, s_1 s_2 s_1, \mu)$	0	$\varpi_1 + \varpi_2$

Except for the generalised PRV component $\varpi_1 + \varpi_2$ (which is contained in the full tensor product but not in any smaller KK module), the other components occur with multiplicity 1 in the respective smallest KK modules to which they belong. For the PRV components, this follows from Corollary 4.4.2. For $3\varpi_1$ $(= s_2\lambda + \mu - \alpha_1)$ and $3\varpi_2$ $(= \lambda + s_1\mu - \alpha_2)$, this follows from Corollary 4.4.5 (3). The generalised PRV component $\varpi_1 + \varpi_2$ $(= s_1\lambda + s_2\mu - (\alpha_1 + \alpha_2))$ shows that the hypothesis of simplicity of β in Corollary 4.4.5 (3) cannot be omitted. $\square$

Chapter 5

A Tableau Decomposition rule for Kostant-Kumar modules

5.1 Tableau decomposition rule for Kostant-Kumar (KK) modules

Fix an integer $d \geq 2$. Let $\mathfrak{g} = \mathfrak{sl}_d$, the simple Lie algebra of traceless complex $d \times d$ matrices. There is, in this special case, the classical Littlewood-Richardson (LR for short) rule (see e.g. [26, 27]) that gives, in terms of tableaux, the decomposition into irreducibles of the tensor product of two finite dimensional irreducible representations of $\mathfrak{g}$. The multiplicities of the irreducibles in this rule are called "LR coefficients" and they count certain "LR tableaux". Our purpose in this section is to deduce, from the general decomposition rule (Theorem 4.2.1), a version of this classical rule, which we call the "refined LR rule", for decomposing as a direct sum of irreducibles any KK submodule of the tensor product: see §5.1.4 for the statement. We call the multiplicities of the irreducibles in this refined rule the "refined LR coefficients".

The refined LR coefficients also count certain LR tableaux. The identification of the set of LR tableaux to be counted is based upon the association of a permutation to each LR tableau (§5.1.6). There is a very natural association of a semi-standard Young tableau (SSYT) to an LR tableau (§5.1.6) and the permutation is just the initial element of the minimal standard lift when the SSYT is interpreted as a standard concatenation of LS paths (§6.1.2).

In the light of this last mentioned fact, it is noteworthy that the procedure we give for determining the permutation (§5.1.5) from the SSYT is not a repeated application of Deodhar's lemma (Proposition 2.1.17): it seems to be more efficient than that. Lascoux and Schützenberger [28] associate to each SSYT a "right key" (which by definition is another SSYT) from which the permutation can be read off. Willis [29] gives an alternative method–"the scanning method"—for finding the right key of an SSYT. Our procedure is different from those in [28, 29].

5.1.1 Preliminaries

The choices involved (Cartan subalgebra $\mathfrak{h}$, Borel subalgebra $\mathfrak{b}$, etc.) are fixed as usual: the subalgebra of diagonal (respectively, upper triangular) traceless complex $d \times d$ matrices is taken to be $\mathfrak{h}$ (respectively, $\mathfrak{b}$). We denote by ϵ_j the linear functional on $\mathfrak{h}$ that maps a matrix to its entry in position (j, j).

Recall that a *partition* is a weakly decreasing sequence $\lambda_1 \geq \lambda_2 \geq \ldots$ (sometimes also written $\lambda_1 + \lambda_2 + \cdots$) of non-negative integers that is eventually zero. The non-zero elements of the sequence are called the *parts*. We tacitly identify partitions with their (Young) shapes. To a partition $\lambda : \lambda_1 \geq \ldots \geq \lambda_d \geq 0 \geq \ldots$ with at most d parts, we attach the dominant integral weight $\lambda_1\epsilon_1 + \cdots + \lambda_d\epsilon_d$. A second such partition $\lambda'_1 \geq \ldots \geq \lambda'_d \geq 0 \geq \ldots$ corresponds to the same weight as λ if and only if $\lambda_1 - \lambda'_1 = \cdots = \lambda_d - \lambda'_d$ (since $\epsilon_1 + \ldots + \epsilon_d = 0$ is evidently the only linear dependence relation up to scaling on $\epsilon_1, \ldots, \epsilon_d$). Thus partitions with less than d parts are in one-to-one correspondence with dominant integral weights. We will abuse notation and use the same symbol for both a partition with less than d parts and the corresponding dominant integral weight.

Let $[j] := \{1, \ldots, j\}$ for any integer $j \geq 1$. The Weyl group is identified with the group of permutations of the set $[d]$. The one line notation for a permutation w of $[d]$ is $w_1 \ldots w_d$, where $w_j := w(j)$ (for $1 \leq j \leq d$).

5.1.2 Semi-standard Skew tableaux (SSST for short)

Let ν and λ be two partitions with the shape of ν containing the shape of λ. A *(semi-standard) skew tableau*, *SSST* for short, of shape ν/λ is a filling up by positive integers of those boxes that are in the shape of ν but not in the shape of λ such that the entries in each row are weakly increasing rightward and those in each column are strictly increasing downward. Here are two examples with $\nu = 7 + 5 + 4 + 3 + 1$ and $\lambda = 4 + 4 + 1 + 1$:

$$\begin{array}{|c|c|c|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot & 2 & 2 & 3 \\ \hline \cdot & \cdot & \cdot & \cdot & 7 \\ \cline{1-5} \cdot & 3 & 3 & 5 \\ \cline{1-4} \cdot & 4 & 6 \\ \cline{1-3} 1 \\ \cline{1-1} \end{array} \qquad\qquad \begin{array}{|c|c|c|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \\ \hline \cdot & \cdot & \cdot & \cdot & 2 \\ \cline{1-5} \cdot & 1 & 2 & 3 \\ \cline{1-4} \cdot & 3 & 4 \\ \cline{1-3} 1 \\ \cline{1-1} \end{array} \tag{5.1.1}$$

Reverse reading words and ballot sequences

Let T be a SSST of shape ν/λ. Its *reverse reading word*, denoted $\mathbf{w}_{\text{row}}(T)$, is defined as follows: read the entries of T from right to left in every row, scanning the rows from top to bottom. For the two SSSTs in the display above, the reverse reading words respectively are:

(5.1.2) $\quad 3\ 2\ 2\ 7\ 5\ 3\ 3\ 6\ 4\ 1 \qquad\qquad 1\ 1\ 1\ 2\ 3\ 2\ 1\ 4\ 3\ 1$

The word $\mathbf{w}_{\text{row}}(T)$ (or, more generally, any word in the positive integers) is said to be a *ballot sequence* if for any integer $j \geq 1$ the number of times j occurs up to any point in the word (while scanning it from left to right) is at least the number of times $j+1$ occurs up to that point. In (5.1.2), the word on the left is not a ballot sequence but the one on the right is.

Type and weight of a word and of a SSST

The *type* of any word $\mathbf{w}$ in the positive integers is the sequence μ: μ_1, μ_2, $\ldots$, where μ_j denotes the number of occurrences of j in $\mathbf{w}$. The type of the word on the left in (5.1.2) is 1, 2, 3, 1, 1, 1, 1, 0, 0, $\ldots$. Evidently, permuting the letters of a word does not change its type. If $\mathbf{w}$ is a word in $[d]$, then we may further associate to it the integral weight $\mu_1\epsilon_1 + \cdots + \mu_d\epsilon_d$ of $\mathfrak{g} = \mathfrak{sl}_d$. This is called the *weight* of the word and denoted $\mathrm{wt}(\mathbf{w})$.

The *type* and *weight* of a SSST T are defined respectively to be the type and weight of its reverse reading word $\mathbf{w}_{\text{row}}(T)$.[1]

If the word $\mathbf{w}$ is a ballot sequence, then its *type* is a partition: $\mu_1 \geq \mu_2 \geq \ldots$, and in this case we use the notation for partitions to denote types. For example, the type of the word on the right in (5.1.2) is $\mu = 5+2+2+1$. The weight of such a word in $[d]$ is dominant.

[1]Later on we will introduce the "column word" $\mathbf{w}_{\text{col}}(T)$ of T, which being a permutation of $\mathbf{w}_{\text{row}}(T)$ shares its type and weight.

5.1.3 Littlewood-Richardson (LR for short) tableaux and coefficients

An *LR tableau* (LR is short for *Littlewood-Richardson*) is a SSST T whose reverse reading word $\mathbf{w}_{\text{row}}(T)$ is a ballot sequence. Let λ and μ be partitions. Let $\mathcal{T}_\mu^\lambda$ denote the set of LR tableau of shape ν/λ and type μ—here ν is allowed to vary. If T in $\mathcal{T}_\mu^\lambda$ has shape ν/λ, we write $\nu(T)$ for ν. As is well-known, $\mathcal{T}_\mu^\lambda$ has representation theoretic and geometric significance. For example (see e.g. [26, 27]) $s_\lambda s_\mu = \sum_{T\in\mathcal{T}_\mu^\lambda} s_{\nu(T)}$, where s_τ denotes the Schur function associated to a partition τ.

For a fixed partition ν, the number of T in $\mathcal{T}_\mu^\lambda$ with $\nu(T)=\nu$ is usually denoted $c_{\lambda\mu}^\nu$. The numbers $c_{\lambda\mu}^\nu$ are called *LR coefficients*. In terms of these, we may write the the above rule for multiplication of Schur functions as $s_\lambda s_\mu = \sum_\nu c_{\lambda\mu}^\nu s_\nu$.

Bruhat order on permutations

Any permutation u of $[j]$ (for some integer $j \geq 1$) can naturally be considered as a permutation of $[k]$, for any integer $k \geq j$. Given two permutations u and u' (of $[j]$ and $[j']$ respectively), we write $u \leq u'$ if that is so in the Bruhat order on permutations of $[k]$ for some $k \geq$ both j and j'. If $u \leq u'$ for one such k, then it is so for all such k.

Refined Littlewood-Richardson coefficients: their definition

In §5.1.6 below, we specify a procedure that assigns a permutation u to a given SSST T.[2] Fix a permutation w and let $\mathcal{T}_\mu^\lambda(w)$ denote the subset of $\mathcal{T}_\mu^\lambda$ consisting

[2]It is easy to associate to T a SSYT S of shape μ—see §5.1.6. Interpreting S as a standard concatenation of LS paths in the sense of Proposition 6.1.3 in the appendix, the associated permutation u is just the initial element of the minimal standard lift of S, as will be proved in §5.2. Observe that, if as in §5.1.4 the number of parts in ν is at most d, then the entries in S and the

of those elements for which the associated permutation u satisfies $u \leq w$ (in the Bruhat order as defined in §5.1.3 above). The result (5.1.3) below ascribes representation theoretic meaning to $\mathcal{T}_\mu^\lambda(w)$.

For a fixed partition ν, we denote by $c_{\lambda\mu}^\nu(w)$ the number of T in $\mathcal{T}_\mu^\lambda(w)$ with $\nu(T) = \nu$. We call the numbers $c_{\lambda\mu}^\nu(w)$ *refined LR coefficients*.

5.1.4 Tableau decomposition rule for KK modules

Suppose that λ, μ are partitions with less than d parts (or, equivalently, dominant integral weights for $\mathfrak{g} = \mathfrak{sl}_d$) and that w is a permutation of $[d]$ (or, equivalently, an element of the Weyl group). Then the decomposition of the Kostant-Kumar module $K(\lambda, w, \mu)$ (defined in §4.1.1) as a direct sum of irreducible $\mathfrak{g}$-modules is given by:

$$\boxed{K(\lambda, w, \mu) = \bigoplus_{T \in \mathcal{T}_\mu^\lambda(w)} V_{\nu(T)}} \tag{5.1.3}$$

where $V_{\nu(T)}$ is interpreted to be zero in case $\nu(T)$ has more than d parts. (Recall from §5.1.1 that to any partition with at most d parts there is associated a dominant integral weight of $\mathfrak{g}$.)

Here is an alternative way to express the above decomposition rule:

$$\boxed{K(\lambda, w, \mu) = \bigoplus_{\overline{\nu}} V_{\overline{\nu}}^{\oplus c_{\lambda\mu}^\nu(w)}} \tag{5.1.4}$$

where the sum runs over all partitions $\overline{\nu}$ with less than d parts, and ν depending

number of parts in μ are also bounded above by d, so that the interpretation of S as a concatenation of LS paths associated to $\mathfrak{g} = \mathfrak{sl}_d$ is possible, and u is a permutation of $[d]$.

on $\overline{\nu}$ denotes the unique partition with at most d parts (if it exists) such that

$$\overline{\nu}_j = \nu_j - \nu_d \text{ for } 1 \leq j < d \quad \text{and} \quad \nu_1 + \cdots + \nu_d = (\lambda_1 + \cdots + \lambda_{d-1}) + (\mu_1 + \cdots + \mu_{d-1})$$

The proof of (5.1.3) will be given below in §5.1.9.

The statement for polynomial representations of $GL_d(\mathbb{C})$

For convenience of reference, we now state, without proof, a version of the decomposition rule (5.1.3) for polynomial representations of the general linear group $GL_d(\mathbb{C})$. Suppose that λ, μ are partitions with at most d parts and V_λ, V_μ the corresponding irreducible polynomial representations. Let w be a permutation of $[d]$. Then the decomposition of the Kostant-Kumar module $K(\lambda, w, \mu)$ (defined similarly as in §4.1.1) as a direct sum of irreducible polynomial representations is given by:

$$\boxed{K(\lambda, w, \mu) = \bigoplus_\nu V_\nu^{\oplus c^\nu_{\lambda\mu}(w)}} \tag{5.1.5}$$

where the sum runs over all partitions ν with at most d parts.

An example

Here is a simple example illustrating the rules (5.1.3) and (5.1.4). Let $d = 3$, $\lambda = 2 + 1$, and $\mu = 3 + 1$. As the reader can readily verify, there are 7 elements T in $\mathcal{T}^\lambda_\mu$ with $\nu(T)$ having at most 3 parts. These are listed below along with the permutations of [3] attached to them (as in §5.1.6):

$$\begin{array}{|c|c|c|c|c|}\hline \cdot & \cdot & 1 & 1 & 1 \\ \hline \cdot & 2 \\ \cline{1-2}\end{array} \leftrightarrow 123 \qquad \begin{array}{|c|c|c|c|}\hline \cdot & \cdot & 1 & 1 \\ \hline \cdot & 1 & 2 \\ \cline{1-3}\end{array} \leftrightarrow 213 \qquad \begin{array}{|c|c|c|c|c|}\hline \cdot & \cdot & 1 & 1 & 1 \\ \hline \cdot \\ \cline{1-1} 2 \\ \cline{1-1}\end{array} \leftrightarrow 132$$

$$\begin{array}{|c|c|c|c|}\hline \cdot & \cdot & 1 & 1 \\ \hline \cdot & 1 \\ \cline{1-2} 2 \\ \cline{1-1}\end{array} \leftrightarrow 231 \qquad \begin{array}{|c|c|c|c|}\hline \cdot & \cdot & 1 & 1 \\ \hline \cdot & 2 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array} \leftrightarrow 312 \qquad \begin{array}{|c|c|c|}\hline \cdot & \cdot & 1 \\ \hline \cdot & 1 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} \leftrightarrow 312 \qquad \begin{array}{|c|c|c|}\hline \cdot & \cdot & 1 \\ \hline \cdot & 1 \\ \cline{1-2} 1 & 2 \\ \cline{1-2}\end{array} \leftrightarrow 321$$

And so we have:

$$K(\lambda, 123, \mu) = V_{5+2} \qquad K(\lambda, 213, \mu) = V_{4+3} \oplus V_{5+2} \qquad K(\lambda, 132, \mu) = V_4 \oplus V_{5+2}$$

$$K(\lambda, 231, \mu) = V_{3+1} \oplus V_4 \oplus V_{4+3} \oplus V_{5+2} \quad K(\lambda, 312, \mu) = V_{2+2} \oplus V_{3+1} \oplus V_4 \oplus V_{4+3} \oplus V_{5+2}$$

$$K(\lambda, 321, \mu) = V_1 \oplus V_{2+2} \oplus V_{3+1} \oplus V_{3+1} \oplus V_4 \oplus V_{4+3} \oplus V_{5+2}$$

5.1.5 SSYT and permutations attached to them

Let μ be a partition. A *semi-standard Young tableau*, *SSYT* for short, *of shape* μ is just a (semi-standard) skew tableau of shape μ/empty in the sense of §5.1.2. Here is an example of a SSYT of shape $\mu = 4 + 2 + 1$:

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & 6 & 8 \\ \hline 2 & 4 \\ \cline{1-2} 7 \\ \cline{1-1}\end{array} \tag{5.1.6}$$

Associating a permutation to a given SSYT

Let S be a SSYT of shape μ and let m be the largest entry of S. We associate to S a permutation u of $[m]$, as follows. Let μ'_1 be the number of parts in μ. Observe that $m \geq \mu'_1$ since the entries in every column of S are strictly increasing downwards.

Let $u_1 u_2 \ldots u_m$ be the one-line notation for u. We will describe below an inductive procedure to produce the sequence $u_1, \ldots, u_{\mu'_1}$. As for $u_{\mu'_1+1}, \ldots, u_m$, we take these to be just the elements of $[m] \setminus \{u_1, \ldots, u_{\mu'_1}\}$ arranged in increasing order.

It is easy to produce u_1: it is just the largest (right most) entry in the first row of S. Suppose that $u_1, u_2, \ldots, u_{p-1}$ have been produced (with $1 < p \leq \mu'_1$). We now describe a procedure to determine u_p.

Let $\mathbf{b}$ be a box in S. Suppose that a box $\mathbf{b}'$ in S is weakly to the Northeast of $\mathbf{b}$ and has an entry that is less than that of $\mathbf{b}$. Then we write $\mathbf{b} \succ \mathbf{b}'$. For example, in the SSYT of (5.1.6), if $\mathbf{b}$ is the one with entry 7, then $\mathbf{b}'$ could be any of those containing 1, 2, 3, 4, or 6; if $\mathbf{b}$ is the one with entry 4, then $\mathbf{b}'$ could only be the one containing 3.

The $\mathbf{b}$-*depth* of such a box $\mathbf{b}'$ is defined to be the largest δ such that there is a chain $\mathbf{b} \succ \mathbf{b}_1 \succ \mathbf{b}_2 \succ \ldots \succ \mathbf{b}_\delta = \mathbf{b}'$. The $\mathbf{b}$-depth of $\mathbf{b}$ itself is defined to be 0.

Let $\mathbf{b}^p$ denote the right most box in row p. We write *p-depth* for $\mathbf{b}^p$-depth. For $1 \leq j \leq p$, we let y_j be maximal possible entry in a box whose p-depth is $p - j$. (The box in row j in the same column as $\mathbf{b}^p$ has p-depth $p - j$, so y_j exists.) By definition, y_p is the entry in the box $\mathbf{b}^p$. As is easily seen, $y_1 < \ldots < y_p$. We call this the *p-depth sequence of S*.

Let $a_1 < \ldots < a_{p-1}$ be the elements $u_1, \ldots, u_{p-1}$ arranged in increasing order. Let k, $1 \leq k \leq p$, be the largest such that $a_{k-1} < y_k$ ($a_0 = -\infty$ by convention).

Take u_p to be y_k.

Proposition 5.1.1. *With notation as above, we evidently have:*

- $a_1 < \ldots < a_{k-1} < y_k < a_k < \ldots < a_{p-1}$.
- u_p *is distinct from* $u_1, \ldots, u_{p-1}$. □

Remark 5.1.2. The element y_j in the p-depth sequence of S is just the entry in the right most box of p-depth $p-j$: "right most box" means box in the right most column; since no two boxes in the same column have the same p-depth, this is well defined. Indeed let $\mathbf{b}$ be that box and e its entry. Clearly $e \leq y_j$. To show $y_j \leq e$, first observe that no column to the right of the one containing $\mathbf{b}$ has a box of p-depth $p-j$ (by choice of $\mathbf{b}$); secondly that e dominates the entry in any box that is weakly to the Northwest of $\mathbf{b}$ (since S is a SSYT); and finally that any box of p-depth $p-j$ strictly South and weakly West of $\mathbf{b}$ can only have an entry that is at most e (for otherwise the p-depth of $\mathbf{b}$ would exceed $p-j$).

Illustration of the procedure above

Let S be the SSYT in (5.1.6). The permutation associated to it is 83612457 in one-line notation. Evidently $u_1 = 8$ and $\mu'_1 = 3$; the 2-depth sequence is $3 < 4$ and $u_2 = 3$; the 3-depth sequence is $3 < 6 < 7$ and $u_3 = 6$.

A technical result that will be used later

The following lemma will be invoked later on, in Example 5.2.2.

Lemma 5.1.3. *Let S be a SSYT and q the number of boxes in its right most column. Let S' be the SSYT obtained from S by deleting its last column. Fix $p > q$. If in the procedure for producing u_p (where u is the permutation associated to S),* *we use the p-sequence of S' in place of that of S, it makes no difference (that is, we still get the same u_p).*

PROOF: Let $y_1 < \ldots < y_p$ and $y'_1 < \ldots < y'_p$ be the p-depth sequences of S and S' respectively, and suppose that $u_p = y_k$. Since the entries in the last column of S all belong to $\{u_1, \ldots, u_{p-1}\}$ but, by Proposition 5.1.1, y_k does not belong to that set, it follows that any box of S with y_k as its entry belongs to S'. Thus $y_k = y'_k$.

On the other hand, $y'_j \le y_j \le a_{j-1}$ for all $j > k$ (where $a_1 < \ldots < a_{p-1}$ is the arrangement in increasing order of $u_1, \ldots, u_{p-1}$), so k is the largest such that $a_{k-1} < y'_k$. □

5.1.6 Association of permutations to LR tableaux

Recall that the definition in §5.1.3 of refined LR coefficients refers to a certain association of permutations to LR tableaux. We describe this association now, after first associating SSYTs to LR tableaux.

Let T be an LR tableau of shape ν/λ and type μ. If ν has at most d parts, then so has μ, for each entry on row j of T is at most j (for all $j \ge 1$).

The SSYT associated to T

We associate to T a SSYT S of shape μ as follows. The entries in row j of S from left to right are just the row numbers of T in which the entry j appears, counted with multiplicity and arranged in weakly increasing order. That the entries in every column of S are strictly increasing downward follows readily from the assumption that the reverse reading word of T is a ballot sequence: indeed, for integers $k \ge 1$ and $j \ge 2$, if the k^{th} appearance of j (as we read the reverse reading word from left to right) is in row r, then the k^{th} appearance of $j-1$ is in some row strictly above the r^{th}.

The permutation associated to T

Consider the permutation u associated as in §5.1.5 to the SSYT S. We associate u to T itself. For example, for the skew tableau on the right in (5.1.1), the associated SSYT is the one shown below and the associated permutation is 51324:

(5.1.7)
$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 3 & 5 \\ \hline 2 & 3 \\ \cline{1-2} 3 & 4 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}$$

5.1.7 p-dominance of words

Let p: $p_1 \geq p_2 \geq \ldots$ be a partition. We denote by $\mathbf{w}(p)$ the word (in the positive integers) that has p_1 ones, p_2 twos, $\ldots$ in succession: this is just the reverse reading word of the SSYT of shape p all of whose entries in row j are j (for all j). Note that $\mathbf{w}(p)$ is a ballot sequence.

A word (in the positive integers) is said to be *p-dominant* if when preceded by $\mathbf{w}(p)$ the resulting word is a ballot sequence.

Proposition 5.1.4. *For a given word $\mathbf{w}$ there is a unique smallest partition $p_{\mathbf{w}}$ such that $\mathbf{w}$ is $p_{\mathbf{w}}$-dominant ($p_{\mathbf{w}}$ is the smallest in the sense that its shape is contained in the shape of any partition p for which $\mathbf{w}$ is p-dominant).*

PROOF: A letter $e > 1$ of the given word $\mathbf{w}$ is said to be a *violator* if the number of $e-1$ occurring before it does not exceed the number of e occurring before it. For j a positive integer, let p_j be the number of violators in $\mathbf{w}$ that exceed j. (For example, p_1 is the total number of violators.) It is elementary to see that the partition $p_1 \geq p_2 \geq \ldots$ is the unique smallest one for which $\mathbf{w}$ is p-dominant. $\square$

Weights of words in $[d]$

Let w be a word in $[d]$. The *weight of* w, denoted $\mathrm{wt}(w)$, is defined to be the weight

The words $\mathbf{w}_{\text{row}}$ and $\mathbf{w}_{\text{col}}$ attached to a SSST

Let T be a SSST. We have already defined its reverse reading word $\mathbf{w}_{\text{row}}(T)$ in §5.1.2. We now define its *reverse column word*, denoted $\mathbf{w}_{\text{col}}(T)$, as follows: we read the entries top to bottom in every column beginning with the right most column and ending with the left most. For the SSST in (5.1.1), the reverse column words respectively are 3227536341 and 1112324131.

For the SSST on the left in (5.1.1), the partitions $p_{\mathbf{w}}$ attached (as in Proposition 5.1.4) to its words $\mathbf{w}_{\text{row}}$ and $\mathbf{w}_{\text{col}}$ turn out to be the same, namely $5+3+2+2+1+1$. For the SSST on the right in (5.1.1), both $\mathbf{w}_{\text{row}}$ and $\mathbf{w}_{\text{col}}$ are ballot sequences (so $p_{\mathbf{w}}$ is empty for both). Indeed we have:

Proposition 5.1.5. *Let T be a SSST and p a partition. Then $\mathbf{w}_{\text{row}}(T)$ is p-dominant if and only if $\mathbf{w}_{\text{col}}(T)$ is so.*

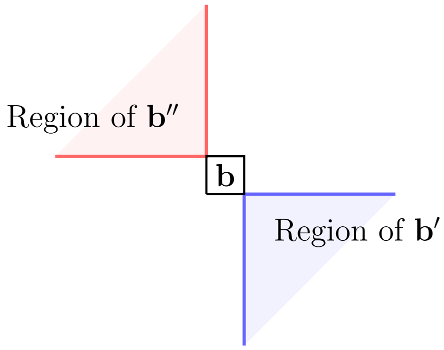
Remark 5.1.6. This statement is well known at least in the case of a SSYT (see, e.g., [30, Exercise 5.2.4]). A proof from first principles is given below for the sake of completeness.

PROOF: For boxes $\mathbf{b}_1$ and $\mathbf{b}_2$ of T, the phrase $\mathbf{b}_1$ "occurs before" $\mathbf{b}_2$ in $\mathbf{w}_{\text{col}}(T)$ (respectively $\mathbf{w}_{\text{row}}(T)$) has the obvious meaning. We let $\mathbf{b}$ be an arbitrary box in T. Its position is denoted by (r, c) and entry by e.

1. Let $\mathbf{b}'$ be a box that occurs before $\mathbf{b}$ in $\mathbf{w}_{\text{col}}(T)$ but not in $\mathbf{w}_{\text{row}}(T)$. Let its position be denoted by (r', c') and entry by e'. Then $r < r'$, $c < c'$ and, since T is semi-standard, $e < e'$.

2. Let $\mathbf{b}''$ be a box that occurs before $\mathbf{b}$ in $\mathbf{w}_{\text{row}}(T)$ but not in $\mathbf{w}_{\text{col}}(T)$. Let its position be denoted by (r'', c'') and entry by e''. Then $r'' < r$, $c'' < c$ and, since T is semi-standard, $e'' < e$.

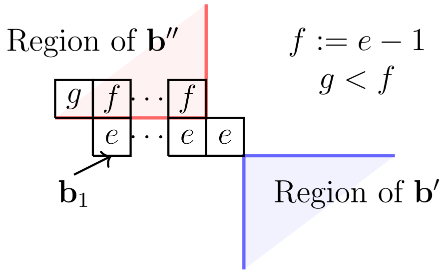
The following figure depicts the situation:



Suppose first that $\mathbf{w}_{\text{col}}(T)$ is p-dominant. Consider the contributions to the words $\mathbf{w}_{\text{row}}(T)$ and $\mathbf{w}_{\text{col}}(T)$ of an arbitrarily fixed box $\mathbf{b}$ in T. With notation as above, observe that no box $\mathbf{b}'$ has $e - 1$ as an entry and no box $\mathbf{b}''$ has e as an entry. Thus, letting m_r and n_r (respectively m_c and n_c) denote respectively the number of occurrences of e and $e - 1$ (strictly) before $\mathbf{b}$ in $\mathbf{w}_{\text{row}}(T)$ (respectively $\mathbf{w}_{\text{col}}(T)$), we have $m_r \leq m_c$ and $n_c \leq n_r$. Since $m_c \leq n_c$ by p-dominance of $\mathbf{w}_{\text{col}}(T)$, we have $m_r \leq m_c \leq n_c \leq n_r$, so $\mathbf{w}_{\text{row}}(T)$ is p-dominant too.

Now suppose that $\mathbf{w}_{\text{row}}(T)$ is p-dominant. By way of contradiction, suppose that $\mathbf{w}_{\text{col}}(T)$ is not p-dominant. Choose a box $\mathbf{b}$ in T which "violates" the p-dominance of $\mathbf{w}_{\text{col}}(T)$, meaning that (with notation as above) $n_c < m_c$. Since no box of type $\mathbf{b}'$ or $\mathbf{b}''$ can have an entry equal to e—we have $e'' < e < e'$—it follows that $m_c = m_r$.

Consider a box of type $\mathbf{b}''$ with entry equal to $e - 1$. Let us denote by $\mathbf{b}''_1$ any such box and suppose that there are k such boxes. Then $n_r = n_c + k$, since $e' > e$. The entry in the box just below a box $\mathbf{b}_1''$ must be e (since such a box is weakly North and strictly West of $\mathbf{b}$ on the one hand, but on the other hand its entry must be strictly larger than $e-1$). Thus all the k boxes $\mathbf{b}_1''$ must occur in row $r-1$, and T looks like:



Now let $\mathbf{b}_1$ be the box in T that is k boxes to the left of $\mathbf{b}$. Let us count the number of entries equal to e (respectively $e-1$) that occur before $\mathbf{b}_1$ in $\mathbf{w}_{\mathrm{row}}(T)$. This count equals $m_r + k = m_c + k$ (respectively, $n_r = n_c + k$). We have $n_c + k < m_c + k$ (since $n_c < m_c$ by choice of $\mathbf{b}$). But this means that the box $\mathbf{b}_1$ violates the p-dominance of $\mathbf{w}_{\mathrm{row}}(T)$, a contradiction. □

5.1.8 Deconstructing a SSST

Let T be a SSST of shape ν/λ. As before, we think of λ as being fixed and ν as varying. For k a positive integer:

- Let $n_r(k)$ denote the number of times k appears in row r.
- Consider the boxes of T belonging to λ and those with entries not exceeding k. Together they form a Young shape. Denote by λ^k this shape as well as the corresponding partition. It is convenient to set $\lambda^0 = \lambda$. Observe that

$$\lambda = \lambda^0 \subseteq \lambda^1 \subseteq \lambda^2 \subseteq \dots \tag{5.1.8}$$

where $\subseteq$ between shapes means that the former is contained in the latter. We have $\lambda^k_r - \lambda^{k-1}_r = n_r(k)$.

- Denote by $\mathbf{w}_k(T)$ the word comprising the row numbers of T in which k appears, listed with multiplicity and in weakly decreasing order. In terms of the integers $n_r(k)$, we have $\mathbf{w}_k(T) = \ldots 2^{n_2(k)}1^{n_1(k)}$.

The hypothesis that T is semistandard puts a constraint on the sequence of shapes that can possibly arise as (5.1.8). Indeed, the fact that an of entry of T is strictly larger than the one vertically just above it (if the latter happens to exist) means precisely that no two boxes in $\lambda^k \setminus \lambda^{k-1}$ are in the same column, or, in other words:

(5.1.9)
$$\lambda^k_r \leq \lambda^{k-1}_{r-1} \qquad \forall r > 1 \quad \forall k \geq 1$$

In terms of λ and $n_r(k)$, this can also be expressed as the following set of conditions:

(5.1.10)
$$\lambda_r + n_r(1) + \cdots + n_r(k) \leq \lambda_{r-1} + n_{r-1}(1) + \cdots + n_{r-1}(k-1) \qquad \forall r > 1 \quad \forall k \geq 1$$

The position word $\mathbf{w}_{\mathsf{pos}}$ and its λ-dominance

To see what (5.1.10) translates to in terms of the words $\mathbf{w}_k(T)$, let us define the *position word* of T, denoted $\mathbf{w}_{\mathrm{pos}}(T)$, to be the concatenation $w_1(k)w_2(k)\ldots$. For example, the position words of the SSST in (5.1.1) are, respectively, 5111334342 and 5311132434. It is readily seen that (5.1.10) is equivalent to the λ-dominance of the word $\mathbf{w}_{\mathrm{pos}}(T)$ (in the sense of §5.1.7).

Recovering the SSST T

Evidently the SSST T can be recovered from the collection of integers $n_r(k)$ (presuming knowledge of the fixed partition λ). Thus it can be recovered either from the sequence (5.1.8) of increasing shapes or from the sequence $\mathbf{w}_1(T)$, $\mathbf{w}_2(T)$, $\ldots$ of words. Moreover, if either the sequence (5.1.8) satisfies the constraint (5.1.9) or, equivalently, if the sequence $\mathbf{w}_1(T)$, $\mathbf{w}_2(T)$, $\ldots$ is such that $\mathbf{w}_{\mathrm{pos}}(T)$ is λ-dominant, then there exists a corresponding T.

Bijection between $\mathcal{T}_\mu^\lambda(w)[d]$ and $\mathcal{S}_\mu^\lambda(w)[d]$

As preparation for the proof in §5.1.9 below of the tableau version of the decomposition rule (5.1.3) of KK modules, we apply the observations above to the case when T is LR.

Fix notation as in §5.1.4. Let $\mathcal{T}_\mu^\lambda[d]$ denote the subset of $\mathcal{T}_\mu^\lambda$ consisting of those elements T such that $\nu(T)$ has at most d parts. Let $\mathcal{S}_\mu^\lambda$ denote those SSYT of shape μ whose column word is λ-dominant (in the sense of §5.1.7), and let $\mathcal{S}_\mu^\lambda(w)$ be the subset of those elements of $\mathcal{S}_\mu^\lambda$ for which the associated permutation u (as in §5.1.5) satisfies $u \leq w$. Put:

$$\mathcal{S}_\mu^\lambda[d] := \{S \in \mathcal{S}_\mu^\lambda \mid \text{no entry of } S \text{ exceeds } d\} \qquad \mathcal{S}_\mu^\lambda(w)[d] := \mathcal{S}_\mu^\lambda(w) \cap \mathcal{S}_\mu^\lambda[d]$$

The weight of a SSYT with entries from $[d]$ is its weight thought of as a SSST (see 5.1.2).

Proposition 5.1.7. *Let T be an element of $\mathcal{T}_\mu^\lambda$ and S the SSYT attached to T as in §5.1.6. The association $T \mapsto S$ gives a bijection between $\mathcal{T}_\mu^\lambda$ and $\mathcal{S}_\mu^\lambda$, under which $\nu(T) = \lambda + \mathrm{wt}(S)$, and which also restricts to a bijection between the pairs $\mathcal{T}_\mu^\lambda(w)$, $\mathcal{S}_\mu^\lambda(w)$ and $\mathcal{T}_\mu^\lambda(w)[d]$, $\mathcal{S}_\mu^\lambda(w)[d]$.*

Proof: We first show that $T \mapsto S$ gives a bijection between $\mathcal{T}_\mu^\lambda$ and $\mathcal{S}_\mu^\lambda$. From Proposition 5.1.5 it follows that the λ-dominance of $\mathbf{w}_{\mathrm{row}}(S)$ and $\mathbf{w}_{\mathrm{col}}(S)$ are equivalent, so

$$\mathcal{S}_\mu^\lambda = \{S \text{ is a SSYT of shape } \mu \,|\, \mathbf{w}_{\mathrm{row}}(S) \text{ is } \lambda\text{-dominant}\}$$

It is easy to see from their definitions that the words $\mathbf{w}_{\mathrm{row}}(S)$ and $\mathbf{w}_{\mathrm{pos}}(T)$ are the same. Thus, from §5.1.8, we conclude:

- $\mathbf{w}_{\mathrm{pos}}(T) = \mathbf{w}_{\mathrm{row}}(S)$ is λ-dominant, so S belongs to $\mathcal{S}_\mu^\lambda$.
- The sequence $\mathbf{w}_1(T)$, $\mathbf{w}_2(T)$, $\ldots$ defined in §5.1.8 and hence T itself can be recovered readily from S by reading the entries in every row of S from right to left. This shows that $T \mapsto S$ is one-to-one.
- Given S' in $\mathcal{S}_\mu^\lambda$, the λ-dominance of $\mathbf{w}_{\mathrm{row}}(S')$ means that there exists a skew tableau T' of shape ν/λ (for some ν) that corresponds to it (in the sense of §5.1.8). The fact that the entries along any column of S' are strictly increasing downwards translates to the fact that the corresponding T' as above is LR, so T' belongs to $\mathcal{T}_\mu^\lambda$ and $T' \mapsto S'$. This shows that $T \mapsto S$ is surjective.

This finishes the proof that $T \mapsto S$ gives a bijection from $\mathcal{T}_\mu^\lambda$ to $\mathcal{S}_\mu^\lambda$.

It is clear from the description of the association $T \mapsto S$ that S has type μ and that $\lambda + \nu(T) = \mathrm{wt}(S)$.

The association of a permutation to an LR tableau proceeds via the SSYT attached to it, so it immediately follows that $T \mapsto S$ gives a bijection from $\mathcal{T}_\mu^\lambda(w)$ to $\mathcal{S}_\mu^\lambda(w)$.

Finally, the number of parts of $\nu(T)$ on the one hand and the maximum value of an entry in S on the other are upper bounds for each other under $T \mapsto S$, so we get a bijection between $\mathcal{T}_\mu^\lambda(w)[d]$ and $\mathcal{S}_\mu^\lambda(w)[d]$. $\square$

5.1.9 Proof of the tableau KK decomposition rule of §5.1.4

The decomposition rule (5.1.3) in terms of tableaux can be derived, as we now show, from the general decomposition rule (4.2.1) for KK-modules in §4.2. The derivation consists of stringing together three bijections that preserve invariants.

The first of these is the bijection between $\mathcal{T}_\mu^\lambda[d]$ and $\mathcal{S}_\mu^\lambda[d]$ of Proposition 5.1.7. The second and third bijections are from the appendix: by Corollary 6.1.4, we may identify $\mathcal{S}_\mu[d]$, the set of SSYT of shape μ with entries from $[d]$, with $\mathcal{P}_{\text{std}}$, the set of standard concatenations of LS paths as in §6.1.2; and, finally, there is the crystal isomorphism Γ of §6.1.4 between the set $\mathcal{P}_\mu$ of LS paths of shape μ and $\mathcal{P}_{\text{std}}$.

In the subsection below, the good properties required of the second bijection are established. For the first bijection, this was done in Proposition 5.1.7. As for the crystal isomorphism Γ, it preserves end points and λ-dominance as shown in Proposition 6.1.6; and the minimal element in the initial direction of π in $\mathcal{P}_\mu$ is the initial element of the minimal standard lift of $\Gamma\pi$ as shown in Proposition 6.1.7.

The final upshot is a bijection $T \leftrightarrow \pi$ between $\mathcal{T}_\mu^\lambda[d]$ on the one hand and $\mathcal{P}_\mu^\lambda$ on the other such that (a) $\nu(T)$ equals the end point $\pi(1)$ and (b) the permutation u attached to T as in §5.1.6 equals the minimal element in the initial direction of π. This will finish the proof of the tableau decomposition rule (5.1.3).

Good properties of the bijection of Corollary 6.1.4

Proposition 5.1.8. *Under the identification between $\mathcal{S}_\mu[d]$ and $\mathcal{P}_{\text{std}}$ of Corollary 6.1.4, let S in $\mathcal{S}_\mu[d]$ correspond to θ in $\mathcal{P}_{\text{std}}$. Then:*

1. *The weight* $\mathrm{wt}(S)$ *of* S *equals the end point* $\theta(1)$ *of the path* θ.

2. *The permutation* u *associated to* S *by the procedure of §5.1.5 equals the initial element of the minimal standard lift of* θ.

3. *The column word* $\mathbf{w}_{\mathrm{col}}(S)$ *of* S *is* λ*-dominant (in the sense of §5.1.7) if and only if the path* θ *is* λ*-dominant.*

PROOF: Item (1) is immediate from the definitions. As for item (2), the whole of §5.2 is devoted to its proof.

Turning to item (3), we first prove the "only if part". Let c denote the number of columns in the shape of S, let r_j denote the number of boxes in column j of S (for $1 \leq j \leq c$), and let $\mathbf{w}'_j$ denote the word $s_{1j} \ldots s_{r_j j}$ (where, as for a matrix, s_{ab} denotes the entry of S in row a and column b). The word $\mathbf{w}_{\mathrm{col}}(S)$ is, by definition, $\mathbf{w}'_c\mathbf{w}'_{c-1}\cdots\mathbf{w}'_1$. Its λ-dominance clearly implies that of any left subword of it, in particular that of the subwords $\mathbf{w}'_c$, $\mathbf{w}'_c\mathbf{w}'_{c-1}$, $\ldots$, $\mathbf{w}'_c\mathbf{w}'_{c-1}\cdots\mathbf{w}'_2$, and $\mathbf{w}'_c\mathbf{w}'_{c-1}\cdots\mathbf{w}'_2\mathbf{w}'_1 = \mathbf{w}_{\mathrm{col}}(S)$. This in turn implies that the weights $\lambda + \mathrm{wt}(\mathbf{w}'_c)$, $\lambda + \mathrm{wt}(\mathbf{w}'_c\mathbf{w}'_{c-1})$, $\ldots$, $\lambda + \mathrm{wt}(\mathbf{w}'_c\mathbf{w}'_{c-1}\cdots\mathbf{w}'_2)$, and $\lambda + \mathrm{wt}(\mathbf{w}'_c\mathbf{w}'_{c-1}\cdots\mathbf{w}'_2\mathbf{w}'_1) = \lambda + \mathrm{wt}(\mathbf{w}_{\mathrm{col}}(S))$ are all dominant. But the dominance of these c weights is, as is readily seen, precisely equivalent to the λ-dominance of θ.

For the "if part", we first make an observation (whose elementary proof we skip). Suppose that a word $\mathbf{w}$ in $[d]$ is a concatenation $\mathbf{w}_1\mathbf{w}_2$ of words $\mathbf{w}_1$ and $\mathbf{w}_2$ such that $\mathbf{w}_1$ is λ-dominant, $\mathbf{w}_2$ is weakly increasing (left to right), and $\lambda + \mathrm{wt}(\mathbf{w})$ is dominant. Then $\mathbf{w}$ is λ-dominant.

The λ-dominance of θ implies that $\lambda + \mathrm{wt}(\mathbf{w}'_c)$, $\lambda + \mathrm{wt}(\mathbf{w}'_c\mathbf{w}'_{c-1})$, $\ldots$, $\lambda + \mathrm{wt}(\mathbf{w}'_c\mathbf{w}'_{c-1}\cdots\mathbf{w}'_2)$, and $\lambda + \mathrm{wt}(\mathbf{w}'_c\mathbf{w}'_{c-1}\cdots\mathbf{w}'_2\mathbf{w}'_1) = \lambda + \mathrm{wt}(\mathbf{w}_{\mathrm{col}}(S))$ are all dominant. Since each $\mathbf{w}'_j$ is strictly increasing, we conclude using the observation that $\mathbf{w}_{\mathrm{col}}(S)$ is λ-dominant. $\square$

5.2 An important property of the procedure of §5.1.5

Let S be an SSYT (see §5.1.5) none of whose entries exceeds d, and u the permutation of $[d]$ obtained by application to S of the procedure of §5.1.5. As explained in §6.1.2 (see, in particular, Corollary 6.1.4) such SSYTs may be identified as certain standard concatenations of LS paths whose shapes are fundamental weights (for $\mathfrak{g} = \mathfrak{sl}_d$). In what follows, we will use the notation for an SSYT to denote also the corresponding standard concatenation of paths. Let v be the initial element of the minimal standard lift of S (§2.1.5).

The purpose of this section is to show that $u = v$. The proof is given in §5.2.5 and §5.2.7 after preparations in the earlier subsections.

The procedure of §5.1.5 seems to be quite different from and more efficient than a repeated application of Deodhar's lemma (Example 2.1.4) to compute the initial element of the minimal standard lift v. Besides, the justification we give in Example 5.2.2 of the recipe of Example 2.1.4 is itself based on the result of this section (that $u = v$).

5.2.1 Notation relating to permutations

Let x be a permutation and let $x_1x_2\ldots$ denote its one-line notation.

We call $\{i \,|\, x_i > x_{i+1}\}$ the *descent set* of x. We say that x has *only r significant elements* if its descent set is contained in $[r]$, or, in other words, if the sequence $x_{r+1}x_{r+2}\ldots$ is increasing. E.g., the only permutation that has zero significant elements is the identity.

For s an integer, let $\underline{x}^s$ denote the sequence $x_1^s < \ldots < x_s^s$ of the first s elements of x (namely $x_1, \ldots, x_s$) arranged in increasing order.

On the tableau criterion for Bruhat order

Recall the following "tableau criterion" for comparability in Bruhat order of two permutations: $x \leq z$ if and only if $\underline{x}^s \leq \underline{z}^s$ for all s, where $\underline{x}^s \leq \underline{z}^s$ is short hand for $x_j^s \leq z_j^s$ for all $1 \leq j \leq s$.

Lemma 5.2.1. ([1, Corollary (5)]) *For $x \leq z$, it suffices that $\underline{x}^s \leq \underline{z}^s$ holds for either (a) all s in the descent set of x, or (b) all s not in the descent set of z.*

5.2.2 An example

For x a permutation, we denote by ${}_{(r)}x$ the permutation obtained from x by rearranging the first r elements in its one-line notation in increasing order. In other words, ${}_{(r)}x$ is the permutation whose one-line notation is $x_1^r \ldots x_r^r x_{r+1} x_{r+2} \ldots$.

Lemma 5.2.2. ${}_{(s)}x \leq {}_{(r)}x \leq {}_{(1)}x = x$ *for* $s \geq r \geq 1$.

Proof: Put $y = {}_{(s)}x$ and $z = {}_{(r)}x$. The descent set of y is contained in $\{s, s+1, \ldots\}$. For any $t \geq s$, we have $\underline{y}^t = \underline{z}^t$, so it follows from Lemma 5.2.1 that $y \leq z$. Observe that ${}_{(1)}x = x$. $\square$

Given a permutation x of $[n]$ and an integer $r \leq n$, we let $S(r, x)$ denote the SSYT constructed as follows: it has $n - r + 1$ columns; column j (counting from the left) has $n + 1 - j$ boxes and its entries are the first $n + 1 - j$ entries of x arranged in increasing order. E.g., if x is the permutation of [5] with one-line notation 45312, then $S(x, 3)$ is:

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 3 & 4 \\ \hline 3 & 4 & 5 \\ \hline 4 & 5 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}$$

Lemma 5.2.3. *The initial element of the minimal standard lift of $S(r, x)$ is ${}_{(r)}x$.*

Proof: Let a be this initial element. By an induction argument, we may assume that ${}_{(s)}x$ is the initial element of the minimal standard lift of $S(s, x)$ for $s > r$. Thus $z := {}_{(r+1)}x \leq a$. Since the first r elements of a match the respective ones of $y := {}_{(r)}x$, it follows in particular that $\underline{y}^r \leq \underline{a}^r$. Since the descent set of y is contained in $\{r, r+1, \ldots\}$, and $\underline{y}^s = \underline{z}^s \leq \underline{a}^s$ for $s > r$, it follows from Lemma 5.2.1 that $y \leq a$.

On the other hand, evidently ${}_{(n)}x \leq {}_{(n-1)}x \leq \ldots \leq {}_{(r)}x$ is a standard lift of $S(r, x)$, so $a \leq {}_{(r)}x = y$. □

Let notation be fixed as in Example 2.1.4. We described there a procedure for determining $\tau := J_{\sigma W_r}(w)$ without however providing a justification for it. We now provide such a justification as an application of the main result of this section $(u = v)$.

Let S' denote the SSYT $S(r, w)$ and S the SSYT obtained by attaching to S' on the right a column with r boxes whose entries from top to bottom are $\sigma_1, \ldots, \sigma_r$. For the values $n = 6$, $r = 3$, $\sigma = 246135$, and $w = 145362$ used as an illustration in

Example 2.1.4, S is:

(5.2.1)

1	1	1	1	2
1	3	3	4	4
3	4	4	5	6
4	5	5		
5	6			
6				

By Lemma 5.2.3, the initial element of the minimal standard lift of S' is ${}_{(r)}w = w$, so the initial element v of the minimal standard lift of S is the least element having the following two properties: $w \leq v$ and the first r elements of v (in its one-line notation) are $\sigma_1, \ldots, \sigma_r$, in that order.

Now, τ is the least element having the two properties: $w \leq \tau$ and $\tau W_r = \sigma W_r$. Evidently ${}_{(r)}\tau W_r = \tau W_r = \sigma W_r$, and, by Lemma 5.2.2, $w = {}_{(r)}w \leq {}_{(r)}\tau \leq \tau$. So ${}_{(r)}\tau = \tau$ and the first r elements of τ are $\sigma_1, \ldots, \sigma_r$, in that order. This means $\tau = v$.

Thus, by the main result of this section, the element u obtained by applying the procedure of §5.1.5 to S equals τ. It is easily seen that $u_j = \sigma_j$, for $1 \leq j \leq r$. For $j > r$, to determine u_j, we may use, by Lemma 5.1.3, the j-depth sequence $y'_1 < \ldots < y'_j$ of S' instead of that of S. The entries in the column with j boxes of S' being $w^j_1 < \ldots < w^j_j$, it is clear that $w^j_i \leq y'_i$ for $1 \leq i \leq j$. On the other hand, since each y'_i must be an entry in one of the columns of S' with at most j boxes, it follows that $w^j_i = y'_i$ for every i.

This completes the justification of the recipe of Exercise 2.1.4 to compute $J_{\sigma W_r}(w)$. $\square$

5.2.3 Truncations of permutations and SSYTs

For r an integer, let $x^{(r)}$ denote the permutation obtained from x by rearranging its elements in position $r+1$ and beyond in increasing order. We call $x^{(r)}$ the *r-truncation of x*. Evidently $x^{(r)}$ has only r significant elements. As an easy consequence of Lemma 5.2.1, we have:

Lemma 5.2.4. *Suppose that $x \leq z$. Then $x^{(r)} \leq z^{(r)} \leq z$.*

For r an integer, let $S^{(r)}$ denote the SSYT obtained by taking the first r rows of S: if S has at most r rows, then $S^{(r)}$ is all of S. We call $S^{(r)}$ the *r-truncation of S*. Let $v(r)$ denote the initial element of the minimal standard lift of $S^{(r)}$.

Proposition 5.2.5. *Every permutation in the minimal standard lift of $S^{(r)}$ has only r significant elements. In particular, if S has at most r rows, then v has only r significant elements: $v^{(r)} = v$.*

PROOF: We use Lemma 5.2.4 to observe that the r-truncation of any standard lift of $S^{(r)}$ continues to be a standard lift, and moreover that the r-truncation of the minimal standard lift is itself. $\square$

Proposition 5.2.6. *$v(r) = v^{(r)}$.*

PROOF: Using Lemma 5.2.4 again, we observe that the r-truncation of any standard lift of S gives a standard lift of $S^{(r)}$. Thus $v(r) \leq v^{(r)}$.

Let $\sigma_1 \leq \ldots \leq \sigma_k = v(r)$ be the minimal standard lift of $S^{(r)}$. By Proposition 5.2.5, there are only r significant elements in every σ_j. We will construct a standard lift $\tilde{\sigma}_1 \leq \ldots \leq \tilde{\sigma}_k$ of S whose r-truncation is $\sigma_1 \leq \ldots \leq \sigma_k$. It will then follow that $v \leq \tilde{\sigma}_k$, and so, by Lemma 5.2.4, $v^{(r)} \leq \tilde{\sigma}_k^{(r)} = \sigma_k = v(r)$.

To construct $\tilde{\sigma}_j$, proceed as follows. The first r elements of $\tilde{\sigma}_j$ are the same as those of σ_j. The first entries of $\tilde{\sigma}_j$ also match the entries top-downwards in column j of S

(until the latter entries are exhausted). The remaining entries of $\tilde{\sigma}_j$ are arranged in decreasing order. Criterion (b) of Lemma 5.2.1 is useful to verify $\tilde{\sigma}_j \le \tilde{\sigma}_{j+1}$. $\square$

5.2.4 The main part of the proof (that $u = v$)

Let k be the number of columns in S. For i, $1 \le i \le k$, let $S[i]$ denote the SSYT consisting only of the first i columns (from the left) of S, and $u[i]$ the permutation obtained by running the procedure of §5.1.5 on $S[i]$. By the description of the procedure, it is clear that running the procedure on $S[i]^{(p)}$ yields $u[i]^{(p)}$.

Let p denote a positive integer. We proceed by induction on p to show the following three assertions.[3]

a. $u[1]^{(p)} \le u[2]^{(p)} \le \ldots \le u[k]^{(p)} = u^{(p)}$.

   Consider a rectangular grid of boxes with p boxes in every column and k boxes in every row. Suppose we fill the boxes in column i of this grid by the first p entries of $u[i]$ in increasing order. It is clear from item (a) above that we then get a SSYT (see Lemma 5.2.1). Let S_p be the SSYT whose first p rows are this rectangular SSYT and whose rows $p+1$ and beyond are the same as the corresponding ones of S. For i, $1 \le i \le k$, we let $S_p[i]$ denote the SSYT consisting of only the first i columns of S_p.

   For example, on the left in the following display is shown S_2 for S as in (5.1.6);

[3] It is only assertion (a) that we are really interested in. Once we have it, it follows rather easily that $v \le u$ (see §5.2.5). The other two assertions are technical devices that facilitate the proof of (a).

and on the right is shown S_4 for S as in (5.2.1):

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & 3 & 3 \\ \hline 2 & 4 & 6 & 8 \\ \hline 7 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 2 \\ \hline 2 & 3 & 3 & 3 & 3 \\ \hline 3 & 4 & 4 & 4 & 4 \\ \hline 4 & 5 & 5 & 5 & 6 \\ \hline 5 & 6 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array}$$

b. Fix i, $1 \leq i \leq k$. For any $r > p$, the r-depth of sequence of $S[i]$ equals the r-depth sequence of $S_p[i]$.

c. Fix i, $1 \leq i \leq k$. Let $a_1 < \ldots < a_p$ be the the first p entries arranged in increasing order of $u[i]$.

Let $y_1 < \ldots < y_{p+1}$ be the $(p+1)$-depth sequence of $S[i]$. Let s, $1 \leq s \leq p+1$, be such that $a_1 < \ldots < a_{s-1} < y_s < a_s < \ldots < a_p$ is the arrangement in increasing order of the first $p+1$ elements of $u[i]$ (see Proposition 5.1.1 and the sentence preceding it).

Then $y_1 = a_1, \ldots, y_{s-1} = a_{s-1}$, and y_s occurs in row s of $S_p[i]$ and in a column weakly to the right of that in which $\mathbf{b}^{p+1}$ occurs ($\mathbf{b}^{p+1}$ is the right most box in row $p+1$ of S).

Base case of the induction

The assertions are easily verified in case $p = 1$. Indeed, for every i, $1 \leq i \leq k$, $u[i]^{(1)}$ has only one significant element and its first element is the entry in the first row in column i of S. This proves (a). Assertion (b) is immediate since $S_1 = S$. Assertion (c) is vacuous in case $s = 1$. In case $s = 2$, we have $a_1 < y_2$, where a_1 is the entry in the first row and column i of $S[i]$ and y_2 is the right most entry in row 2 of $S[i]$. It follows that the box in row 1 and column i has 2-depth 2, so $a_1 \le y_1$. Since a_1 is the largest entry in the first row and y_1 occurs as an entry in the first row, it follows that $y_1 \le a_1$. Thus $y_1 = a_1$.

Proof of assertion (a)

To simplify notation, write g and h for $u[i]$ and $u[i+1]$ respectively. We need to prove that $\underline{g}^j \le \underline{h}^j$ for all $j \le p$ (Lemma 5.2.1). By the induction hypothesis, we know this to be true for $j < p$, so it remains to be proved only for $j = p$. Let us write $a_1 < \ldots < a_{p-1}$ for $\underline{g}^{p-1}$ and $b_1 < \ldots < b_{p-1}$ for $\underline{h}^{p-1}$.

Let $e_1 < \ldots < e_p$ and $f_1 < \ldots < f_p$ be the p-depth sequences of $S[i]$ and $S[i+1]$ respectively. We have, evidently, $e_j \le f_j$. Let s and t, $1 \le s, t \le p$, be such that

$$a_1 < \ldots < a_{s-1} < e_s < a_s < \ldots < a_{p-1} \quad \text{and} \quad b_1 < \ldots < b_{t-1} < f_t < b_t < \ldots < b_{p-1}$$

are the sequences $\underline{g}^p$ and $\underline{h}^p$.

In the case $s \le t$,[4] the desired conclusion $\underline{g}^p \le \underline{h}^p$ follows rather easily from $\underline{g}^{p-1} \le \underline{h}^{p-1}$. Indeed we have, in the case $s < t$:

$$\begin{array}{ll}
g^p_j = a_j \le b_j = h^p_j & \text{for } 1 \le j \le s-1 \\
g^p_s = e_s < a_s \le b_s = h^p_s & \\
g^p_j = a_{j-1} \le b_{j-1} = h^p_{j-1} < h^p_j & \text{for } s+1 \le j < t \\
g^p_t = a_{t-1} \le b_{t-1} < f_t = h^p_t & \\
g^p_j = a_{j-1} \le b_{j-1} = h^p_j & \text{for } t < j \le p
\end{array}$$

For $s = t$, the three middle lines in the display above should be replaced by $g^p_s = e_s \le f_s = h^p_s$.

So let us assume $s > t$. The cases $1 \le j < t$ and $s < j \le p$, are similar respectively

[4]The case $s < t$ never actually occurs, but that does not concern us here.

to the first and last cases above. It is for j in the range $t \leq j \leq s$ that we need some care. The key observation here is that $a_j = e_j$ for $j < s$. This follows from assertion (c) with p replaced by $p-1$ (which we may assume to be true by induction). Indeed, using this, we are done as follows:

$$\begin{aligned} g_j^p &= a_j = e_j \leq f_j \leq b_{j-1} = h_j^p \quad \text{for } t < j \leq s \\ g_t^p &= a_{t-1} = e_t < f_t = h_t^p \qquad \square \end{aligned}$$

Proof of assertion (b)

Fix $r > p$. By induction, we know the statement for $p-1$ in place p, so the r-depth sequences of $S[i]$ and $S_{p-1}[i]$ are the same. It is therefore enough to prove that the r-depth sequences of $S_{p-1}[i]$ and $S_p[i]$ are the same. It is convenient to omit the "$[i]$" and just write S_{p-1} and S_p for $S_{p-1}[i]$ and $S_p[i]$ respectively. Assertion (b) follows immediately from Corollaries 5.2.9 and 5.2.11 below.

We denote by i_0 the column number in which the right most box $\mathbf{b}^p$ in row p of S_{p-1} (equivalently S) occurs. The entries in any column of S_{p-1} are also entries in that same column of S_p. For every box $\mathbf{b}$ of S_{p-1}, we denote by $\mathbf{b}'$ the (unique) box of S_p in the same column as $\mathbf{b}$ and having the same entry. The association $\mathbf{b} \mapsto \mathbf{b}'$ is evidently one-to-one. Either $\mathbf{b}'$ is in the same row as $\mathbf{b}$ or in the next lower row.

We classify boxes of S_p as follows:

- *Old* boxes are those that are in the image of the above map $\mathbf{b} \mapsto \mathbf{b}'$. *New* boxes are those that are not old.

- An *unmoved* box is an old box $\mathbf{b}'$ that is in the same row as its preimage $\mathbf{b}$. We write $\mathbf{b}' = \mathbf{b}$ in this case. A *moved* box is an old one that is not unmoved, or, in other words, an old one that is in a row one lower than its preimage.

A box $\mathbf{b}$ of S_{p-1} is *moved* or *unmoved* accordingly as its image $\mathbf{b}'$ in S_p is so.

As an illustration, shown on the left in the display below is S_3 and on the right is S_4 in a particular case: $p = 3$ and $i_0 = 2$. The entries in all the new boxes are in bold and underlined; those in unmoved boxes are in red; those in moved boxes are in blue. The 4-depth sequences for $S_3[3]$ and $S_3[4]$ respectively are 7, 5, 2, 1 and 7, 5, 4, 2; those for $S_3[l]$ for $l \geq 5$ are all 7, 6, 4, 3.

(5.2.2)
$$
\begin{array}{cccccccc}
1 & 1 & 1 & 2 & 3 & 3 & 4 & 7 \\
2 & 2 & 2 & 4 & 6 & 7 & 8 & 8 \\
3 & 3 & 5 & 5 & 8 & 8 & 9 & 9 \\
6 & 7 & & & & & & \\
7 & & & & & & & \\
9 & & & & & & &
\end{array}
\qquad
\begin{array}{cccccccc}
1 & 1 & 1 & 2 & 3 & 3 & \underline{\mathbf{3}} & \underline{\mathbf{3}} \\
2 & 2 & 2 & 4 & \underline{\mathbf{4}} & \underline{\mathbf{4}} & 4 & 7 \\
3 & 3 & 5 & 5 & 6 & 7 & 8 & 8 \\
6 & 7 & \underline{\mathbf{7}} & \underline{\mathbf{7}} & 8 & 8 & 9 & 9 \\
7 & & & & & & & \\
9 & & & & & & &
\end{array}
$$

Proposition 5.2.7. 1. *In any column i of S_{p-1} (respectively S_p) with $i \leq i_0$, all boxes are unmoved (respectively old and unmoved). In particular, the right most box $\mathbf{b}^r$ in row r (with $r > p$) of S_p is old and unmoved.*

2. *Let $\mathbf{n}$ be a new box. Then, to the left of $\mathbf{n}$ and in the same row, in a column with number $i \geq i_0$, there is an old box carrying the same entry as $\mathbf{n}$.*[5]

PROOF: Item (1) is clear. Indeed S_{p-1} and S_p are identical in columns $i \leq i_0$.

To prove item (2), suppose that $\mathbf{n}$ occurs in column c of S_p. Then $S_{p-1}[c]$ has $p-1$ boxes in its last column. Let $a_1 < \ldots < a_{p-1}$ be the entries in that column (top to bottom). Let $y_1 < \ldots < y_p$ be the p-depth sequence of $S[c]$ (or, what amounts to the same by the induction hypothesis, of $S_{p-1}[c]$) and let s, $1 \leq s \leq p$, be such that $a_1 < \ldots < a_{s-1} < y_s < a_s < \ldots < a_{p-1}$ are the entries in the last column of $S_p[c]$. The box with y_s as its entry is $\mathbf{n}$, and $\mathbf{n}$ occurs in row s of $S_p[c]$. We may assume

[5]Any such box is actually unmoved, but we don't need that bit of detail.

by induction that assertion (c) of §5.2.4 is true with $p-1$ in place of p, and conclude that y_s appears as an entry in row s of $S_{p-1}[c]$ in a column with number $i \geq i_0$. $\square$

Proposition 5.2.8. *Let* $\mathbf{b}_1$ *and* $\mathbf{b}_2$ *be boxes of* S_{p-1}*. Then* $\mathbf{b}_1 \succ \mathbf{b}_2$ *if and only if* $\mathbf{b}_1' \succ \mathbf{b}_2'$*.*

PROOF: Neither the entry nor the column number changes on passage from $\mathbf{b}$ to $\mathbf{b}'$. While the row number could increase by at most 1 on this passage, consider the facts that both S_{p-1} and S_p are SSYTs and that $\mathbf{b}_2$ (respectively $\mathbf{b}_2'$) is weakly to the East of $\mathbf{b}_1$ (respectively $\mathbf{b}_1'$) and carries an entry which is strictly less. Together these imply that $\mathbf{b}_2$ (respectively $\mathbf{b}_2'$) occurs in a higher row than $\mathbf{b}_1$ (respectively $\mathbf{b}_1'$). $\square$

Corollary 5.2.9. *Fix* $r > p$*. Suppose that* $\mathbf{b}^r = \mathbf{b}_0 \succ \ldots \succ \mathbf{b}_\delta$ *is a chain of boxes in* S_{p-1}*. Then* $\mathbf{b}^r = \mathbf{b}_0' \succ \ldots \succ \mathbf{b}_\delta'$ *is a chain of boxes in* S_p*. In particular, the* r*-depth sequence of* S_{p-1} *is term for term dominated by the* r*-depth sequence of* S_p*.*

PROOF: That we get a chain on passing from $\mathbf{b}$ to $\mathbf{b}'$ is clear from Proposition 5.2.8. That $\mathbf{b}^{r\prime} = \mathbf{b}^r$ follows from Proposition 5.2.7 (1). It is clear from the description of the association $\mathbf{b} \mapsto \mathbf{b}'$ that the entries of $\mathbf{b}_\delta$ and $\mathbf{b}_\delta'$ are the same. $\square$

Proposition 5.2.10. *Fix* $r > p$*. Given a chain* $\mathbf{b}^r = \tilde{\mathbf{b}}_0 \succ \ldots \succ \tilde{\mathbf{b}}_\delta$ *of boxes in* S_p*, there exists a chain* $\mathbf{b}^r = \mathbf{b}_0 \succ \ldots \succ \mathbf{b}_\delta$ *of boxes in* S_{p-1} *with* $\mathbf{b}_\delta$ *having the same entry as* $\tilde{\mathbf{b}}_\delta$ *and being weakly to the Northwest of it (meaning, the row and column numbers of* $\mathbf{b}_\delta$ *each is at most that of the corresponding number of* $\tilde{\mathbf{b}}_\delta$*).*

PROOF: Proceed by induction on δ. For $\delta = 0$, the statement is easily seen to be true since $\mathbf{b}^r = \mathbf{b}^{r\prime}$ is old and unmoved (Proposition 5.2.7 (1)). Suppose that $\delta \geq 1$.

First suppose that $\tilde{\mathbf{b}}_\delta$ is an old box. Let $\mathbf{b}_\delta$ be the unique box in S_{p-1} such that $\mathbf{b}'_\delta = \tilde{\mathbf{b}}_\delta$. Note that $\mathbf{b}_\delta$ shares its entry and column number with $\tilde{\mathbf{b}}_\delta$ and is weakly to the North of it. By induction, choose $\mathbf{b}^r = \mathbf{b}_0 \succ \ldots \succ \mathbf{b}_{\delta-1}$ with $\mathbf{b}_{\delta-1}$ being weakly to the Northwest of $\tilde{\mathbf{b}}_{\delta-1}$ and having the same entry. Since $\mathbf{b}_{\delta-1}$ is weakly to the West of $\mathbf{b}_\delta$ with a strictly larger entry, it follows that it is on a strictly lower row, and so $\mathbf{b}_{\delta-1} \succ \mathbf{b}_\delta$.

Now suppose that $\tilde{\mathbf{b}}_\delta$ is a new box. Using Proposition 5.2.7 (2), replace it by an old and unmoved box having the same entry and being to the left in the same row. Suppose that the new $\tilde{\mathbf{b}}_\delta$ is in column c. If any $\tilde{\mathbf{b}}_j$ for $j < \delta$ has a column number higher than c, replace it by the one in the same row in column number c. We now get a chain with $\tilde{\mathbf{b}}_\delta$ being old, so we are reduced to the case settled in the previous paragraph. □

Corollary 5.2.11. *Fix $r > p$. The r-depth sequence of S_{p-1} dominates term for term the r-depth sequence of S_p.*

Proof of assertion (c)

By assertion (b), we may take $y_1 < \ldots < y_{p+1}$ to be the $(p+1)$-depth sequence of $S_p[i]$. Fix $j < s$. We would like to show that $y_j = a_j$. In what follows, we write just "depth" to mean "$(p+1)$-depth". Recall that, by definition, y_j is the maximal entry in a box of depth $p+1-j$; and a_j occurs as the entry in row j and column i of $S_p[i]$. Any box of depth $p+1-j$ occurs in row j or above, and a_j dominates all the entries in those rows. Thus it is enough to show that the box in row j and column i of $S_p[i]$ has depth $p+1-j$. Further, since any box in row j has depth at most $p+1-j$, it is enough to show that the depth of that box is at least $p+1-j$.

Further, it is enough to show this for $j = s - 1$, since it follows then for the other $j < s$ as well.

By definition, y_s occurs as an entry in a box $\mathbf{b}$ of $S_p[i]$ of depth $p + 1 - s$. Such a $\mathbf{b}$ can only appear in row s or above. But since $a_{s-1} < y_s$, it follows that $\mathbf{b}$ cannot occur in row $s - 1$ or above. So it appears in row s, and so $\mathbf{b} \succ \mathbf{c}$ where $\mathbf{c}$ is the box in row $s - 1$ and column i of $S_p[i]$, which means that $\mathbf{c}$ has depth $p + 2 - s$. □

5.2.5 Proof that $v \leq u$

It follows from assertion (a) that $u[1]^{(p)} \leq \ldots \leq u[k]^{(p)}$ is a standard lift of $S^{(p)}$. Since $v^{(p)} = v(p)$ is the initial element of the minimal lift of $S^{(p)}$ (by Proposition 5.2.6), it follows that $v^{(p)} \leq u[k]^{(p)} = u^{(p)}$. Since $v = v^{(p)}$ and $u = u^{(p)}$ for large p, it follows that $v \leq u$. □

5.2.6 A technical lemma (that is invoked in §5.2.7)

Lemma 5.2.12. *Let $\sigma_1 \leq \ldots \leq \sigma_k$ be a standard lift of S. Consider any box of $\mathbf{b}^p$-depth δ in S, for some positive integer p. (Recall that $\mathbf{b}^p$ denotes the right most box in row p of S.) Let y be the entry in that box and c be the serial number (from the left) of the column in which that box appears. Then, among the first p elements of σ_c (in its one-line notation), there are at least $\delta + 1$ that are at least y.*

Proof: Proceed by induction on δ. Suppose first that $\delta = 0$. The only box with $\mathbf{b}^p$-depth 0 is the box $\mathbf{b}^p$ itself. Since $\mathbf{b}^p$ occurs on row p, the conclusion is easily verified to be true.

Now suppose that $\delta \geq 1$. From the hypothesis (and the definition of $\mathbf{b}^p$-depth), there exists, for some $c' < c$, a box in the column c' of S with entry $y' > y$ and of $\mathbf{b}^p$-depth $\delta - 1$. By the induction hypothesis, there exist, among the first p

elements of $\sigma_{c'}$, δ that are at least y'. Since $\sigma_{c'} \leq \sigma_c$, the same assertion holds with $\sigma_{c'}$ replaced by σ_c. Now, y too occurs in the first p elements of σ_c. Thus there are at least $\delta + 1$ among the first p elements of σ_c that are at least y. $\square$

Corollary 5.2.13. *Let $y_1 < \ldots < y_p$ be the p-depth sequence of S (this was used in the procedure in §5.1.5 to determine u_p). Then, for every j, $1 \leq j \leq p$, among the first p elements of v, there occur at least $p+1-j$ elements that are at least y_j.*

Proof: By definition, y_j occurs as an entry in some box of S of $\mathbf{b}^p$-depth $p - j$. Suppose c is the column number in which such a box appears. Choose the standard lift in the lemma above to be the minimal one. Then, by the lemma, among the first p elements of σ_c, there occur at least $p + 1 - j$ that are at least y_j. Since $\sigma_c \leq v$, the same assertion holds with v in place of σ_c. $\square$

5.2.7 Proof that $u \leq v$

For p a positive integer, we prove, by induction on p, that $u^{(p)} \leq v^{(p)}$. Since $u^{(p)} = u$ and $v^{(p)} = v$ for large p, it will follow that $u \leq v$. First consider the case $p = 1$. Let the right most entry in the first row of S be a. From the description of the procedure to produce u in §5.1.5, it is clear that $u_1 = a$. On the other hand, evidently, the initial element of any standard lift of S has a as its first element (in its one-line notation), so in particular $v_1 = a$. This proves $u^{(1)} = v^{(1)}$.

Now let $p > 1$. By the induction hypothesis, we have $u^{(p-1)} \leq v^{(p-1)}$. It is enough therefore to prove that $u_p \leq v_p$.

Since we have proved that $v \leq u$ (§5.2.5), it follows that that $v^{(p-1)} = u^{(p-1)}$. Let $y_1 < \ldots < y_p$ be the p-depth sequence of S and let j, $1 \leq j \leq p$, be such that $u_p = y_j$. Then there are exactly $p - j$ elements among $u_1, \ldots, u_{p-1}$ that are at least y_j. Corollary 5.2.13 guarantees that among the first p elements of v, there are at least $p+1-j$ that are at least y_j. Since $u_j = v_j$ for $j \leq p-1$, it follows that $u_p = y_j \leq v_p$, and we are done. $\square$

Chapter 6

On the concatenation of LS paths

6.1 Multiple concatenations of LS paths

The immediate provocation for this appendix comes from the need to quote its results (Propositions 6.1.2 and 6.1.7) in the proof of the tableau decomposition rule for KK modules (§5.1.9). These results are part of folklore. They are already hinted at by Littelmann in [14]: see the "precise combinatorial criterion" alluded to in the paragraph preceding the theorem in §8.1 of that paper. They are also later stated in [31, §11] with a sketch of proofs. However, we could not find a suitable reference with complete proofs. This appendix aims to provide precisely such a reference, presupposing knowledge of (a) Littelmann's basic definitions and results on paths as in [15] and (b) the results recalled and proved from scratch in §2.1 above.

6.1.1 Standard concatenations

Let $\mathfrak{g}$ be symmetrizable Kac-Moody algebra. Let $\lambda_1, \ldots, \lambda_n$ be dominant integral weights. For j, $1 \leq j \leq n$, let $\mathcal{P}_j$ denote the set of LS paths of shape λ_j. Consider the set $\mathcal{P} := \mathcal{P}_1 \star \cdots \star \mathcal{P}_n := \{\pi_1 \star \cdots \star \pi_n \,|\, \pi_j \in \mathcal{P}_j \text{ for } 1 \leq j \leq n\}$ of paths. For paths θ and θ' in $\mathcal{P}$, let us write $\theta \sim \theta'$ if either $e_\alpha\theta = \theta'$ or $f_\alpha\theta = \theta'$ for some simple root α. This is a symmetric relation. Let us continue to denote by $\sim$ the reflexive and transitive closure of this relation on $\mathcal{P}$.

The path $\eta(\theta)$

Fix a $\theta = \pi_1 \star \cdots \star \pi_n$ in $\mathcal{P}$. As in Proposition 3.2.3, which is the special case $n = 2$ of the present set up, it follows that:

- In the equivalence class of $\mathcal{P}$ containing θ, there exists a unique path $\eta(\theta)$ that is killed by e_α for every simple root α.
- The $\eta(\theta)$ as above lies entirely in the dominant chamber.

Standard concatenations

We want to characterize those θ for which $\eta(\theta) = \pi_{\lambda_1} \star \cdots \star \pi_{\lambda_n}$, where as usual π_{λ_j} denotes the straight line path from the origin to λ_j. Towards this, put $W_j := W_{\lambda_j}$, the stabiliser of λ_j in the Weyl group W, and let $\tau_{1,j} > \ldots > \tau_{r_j,j}$ be the chain of elements in W/W_j forming the LS path π_j (for $1 \leq j \leq n$). Consider the tuple

$$(6.1.1) \qquad \big(\tau_{1,1}, \ldots \tau_{r_1,1}, \quad \ldots \quad , \tau_{1,j}, \ldots, \tau_{r_j,j}, \quad \ldots \quad , \tau_{1,n}, \ldots, \tau_{r_n,n}\big)$$

which is an element of

$$(6.1.2) \qquad (W/W_1)^{\times\ r_1 \text{ times}} \times \cdots \times (W/W_j)^{\times\ r_j \text{ times}} \times \cdots \times (W/W_n)^{\times\ r_n \text{ times}}$$

We call the path θ *standard* if the tuple (6.1.1) is standard in the sense of §2.1.5. A standard lift (respectively, a minimal standard lift) in the sense of §2.1.5 of the tuple (6.1.1) is called a *standard lift* (respectively, *minimal standard lift*) of θ. We denote by $\mathfrak{w}(\theta)$ the initial element of the minimal standard lift of θ.

We denote by $\mathcal{P}_{\text{std}}$ the subset of $\mathcal{P}$ consisting of standard paths.

Example 6.1.1. The path $\pi_{\lambda_1} \star \cdots \star \pi_{\lambda_n}$ is standard, for (identity, ..., identity) is its minimal standard lift. Moreover, it is the only standard path in $\mathcal{P}$ with identity as the initial element of its minimal standard lift. Thus:

$$\{\theta \in \mathcal{P}_{\text{std}} \,|\, \mathfrak{w}(\theta) \leq \text{identity}\} = \{\pi_{\lambda_1} \star \cdots \star \pi_{\lambda_n}\} \tag{6.1.3}$$

Here is a characterization of the paths θ in $\mathcal{P}$ for which $\eta(\theta) = \pi_{\lambda_1} \star \cdots \star \pi_{\lambda_n}$:

Proposition 6.1.2. (see [14, §8.1]) $\eta(\theta) = \pi_{\lambda_1} \star \cdots \star \pi_{\lambda_n}$ *if and only if* θ *is standard.*

The proof of this proposition is given in §6.1.3.

6.1.2 Specializing to a classical case: the case of the special linear Lie algebra

Preserve the notation of the previous subsection and specialize to the situation of §5.1: an integer $d \geq 2$ is fixed, $\mathfrak{g} = \mathfrak{sl}_d$, etc. Let μ be a dominant integral weight, or, equivalently a partition with less than d parts. Write μ as $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{d-1} \geq 0 \geq \ldots$. Let $\varpi_1 = \epsilon_1$, $\varpi_2 = \epsilon_1 + \epsilon_2$, ..., $\varpi_{d-1} = \epsilon_1 + \cdots + \epsilon_{d-1}$ be the fundamental weights. Let W_{ϖ_j}, $1 \leq j < d$, denote the stabiliser in W of ϖ_j.

Put $m_1 = \mu_1 - \mu_2$, ..., $m_{d-2} = \mu_{d-2} - \mu_{d-1}$, $m_{d-1} = \mu_{d-1} - \mu_d = \mu_{d-1}$, and

$n = m_1 + m_2 + \cdots + m_{d-1}$ (note that $n = \mu_1$). Let $\lambda_1, \ldots, \lambda_n$ be:

$$\underbrace{\varpi_1, \ldots, \varpi_1}_{m_1 \text{ times}}, \ldots, \underbrace{\varpi_j, \ldots, \varpi_j}_{m_j \text{ times}}, \ldots, \underbrace{\varpi_\ell, \ldots, \varpi_\ell}_{m_\ell \text{ times}}$$

so that $\mu = \lambda_1 + \cdots + \lambda_n$.

The elements of W/W_{ϖ_j} are parametrized by subsets of cardinality j of $[d]$. Each such subset is written as $\{1 \le i_1 < \ldots < i_j \le d\}$. Given two such subsets $\underline{i} = \{1 \le i_1 < \ldots < i_j \le d\}$ and $\underline{i}' = \{1 \le i'_1 < \ldots < i'_j \le d\}$, we have $\underline{i} \le \underline{i}'$ in the Bruhat order on W/W_{ϖ_j} if and only if $i_1 \le i'_1, \ldots, i_{j-1} \le i'_{j-1}$, and $i_j \le i'_j$. For a permutation σ of $[d]$ whose one line notation is $\sigma_1 \ldots \sigma_d$, the coset σW_{ϖ_j} corresponds to $\{1 \le i_1 < \ldots < i_j \le d\}$, where $i_1, \ldots, i_j$ are the elements $\sigma_1, \ldots, \sigma_j$ arranged in increasing order.

For permutations σ and τ of $[d]$ with respective one-line notations $\sigma_1 \ldots \sigma_d$ and $\tau_1 \ldots \tau_d$, we have $\sigma \le \tau$ in the Bruhat order if and only if $\sigma W_{\varpi_j} \le \tau W_{\varpi_j}$ for every j, $1 \le j < d$: see, for example, [1].

The LS paths of shape ϖ_j are all straight lines, so they too are parametrized by elements of W/W_{ϖ_j}. Thus a path in $\mathcal{P}$ can be represented by a "tableau", where a *tableau* consists of $m_{d-1} + \ldots + m_1$ top-justified columns of boxes, where each of the first m_{d-1} columns (from the left) has $d-1$ boxes, each of the next m_{d-2} columns has $d-2$ boxes, and so on; the boxes are filled with numbers between 1 and d, the entries in each column being strictly increasing downwards.[1]

Let, for example, $d = 5$ and $\mu = 6 + 3 + 3 + 2$. Then $m_1 = 3$, $m_2 = 0$, $m_3 = 1$, $m_4 = 2$; $n = 6$, and $\lambda_1, \ldots, \lambda_6$ equals $\varpi_1, \varpi_1, \varpi_1, \varpi_3, \varpi_4, \varpi_4$. And the paths in $\mathcal{P}$ can be identified with tableaux consisting of 6 top-justified columns of boxes, the first two columns having 4 boxes each, the next column having 3 boxes, and the

[1]The reversal of order, which is admittedly annoying, is necessary to preserve entrenched conventions.

last three columns having 1 box each. Here are two examples of such tableaux:

$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 2 & 1 & 4 & 3 \\ \hline 3 & 2 & 4 \\ \cline{1-3} 4 & 3 & 5 \\ \cline{1-3} 5 & 5 \\ \cline{1-2} \end{array} \qquad\qquad \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 2 & 3 & 4 & 5 \\ \hline 2 & 3 & 4 \\ \cline{1-3} 3 & 4 & 5 \\ \cline{1-3} 5 & 5 \\ \cline{1-2} \end{array} \tag{6.1.4}$$

For $\theta = \tau_1 \star \cdots \star \tau_6$ in $\mathcal{P}$, the entries in the first column of the corresponding tableau define τ_6 (which is a path of shape $\lambda_5 = \varpi_4$), the entries in the second column define τ_5, and so on, until the entries in the last column define τ_1.

Proposition 6.1.3. *A path θ in $\mathcal{P}$ is standard as defined earlier in this section (§6.1.1) if and only if the entries in the tableau corresponding to it are weakly increasing in every row from left to right.*

Proof: Proposition 2.1.15 (3) is relevant here. In particular, we could use it prove the if part, but instead we directly construct an explicit standard lift. Let j be such that $1 \leq j < d$, and let $\underline{h} = \{1 \leq h_1 < \ldots < h_j \leq d\}$ be an element in W/W_{ϖ_j}. Denote by $\tilde{\underline{h}}$ the permutation whose one line notation is $h_1 \ldots h_j h'_1 \ldots h'_{d-j}$, where $h'_1, \ldots, h'_{d-j}$ are the elements of $[d] \setminus \{h_1, \ldots, h_j\}$ arranged in decreasing order. Clearly, $\tilde{\underline{h}} W_{\varpi_j} = \underline{h}$. Let $k \leq j$ and let $\underline{i} = \{1 \leq i_1 < \ldots < i_k \leq d\}$ be an element of W/W_{ϖ_k} such that $h_1 \leq i_1, \ldots, h_{k-1} \leq i_{k-1}$, and $h_k \leq i_k$. Let $\tilde{\underline{i}}$ be defined from $\underline{i}$ (as $\tilde{\underline{h}}$ is from $\underline{h}$). Then, as is not to hard to see, $\tilde{\underline{h}} \leq \tilde{\underline{i}}$. This proves the if part.

Let j, k be integers such that $1 \leq k \leq j < d$. Let σ, τ be permutations of $[d]$ with respective one line notations $\sigma_1 \ldots \sigma_d$ and $\tau_1 \ldots \tau_d$. Then
$\sigma W_{\varpi_j} = \{h_1 < \ldots < h_j\} = \underline{h}$ where $h_1, \ldots, h_j$ are just $\sigma_1, \ldots, \sigma_j$ arranged in increasing order, and $\tau W_{\varpi_k} = \{i_1 < \ldots < i_k\} = \underline{i}$ where $i_1, \ldots, i_k$ are just $\tau_1, \ldots, \tau_k$ arranged in increasing order. Suppose that $\sigma \leq \tau$. Then $h'_1 \leq i_1, \ldots, h'_k \leq i_k$, where $h'_1, \ldots, h'_k$ are $\sigma_1, \ldots, \sigma_k$ arranged in increasing order. It follows that
$h_1 \leq i_1, \ldots, h_k \leq i_k$, since evidently $h_1 \leq h'_1, \ldots, h_k \leq h'_k$. This proves the only if

part of the assertion. □

Corollary 6.1.4. *The set $\mathcal{P}_{\text{std}}$ of standard paths in $\mathcal{P}$ may be identified with the set $\mathcal{S}_\mu[d]$ of SSYT of shape μ (in the sense of §5.1.5) with entries from $[d]$.*

The path represented by the tableau on the left in (6.1.4) is not standard whereas the one represented by the tableau on the right is standard: the tableau on the left is not a SSYT whereas the tableau on the right is.

6.1.3 Proof of Proposition 6.1.2

Towards the proof, we first prove a lemma.

Lemma 6.1.5. *If θ is standard, then so is every element in the equivalence class of $\mathcal{P}$ containing θ.*

Proof: Let θ be standard and α be a simple root. We will presently show that $f_\alpha\theta$ is standard in case it does not vanish. The proof that $e_\alpha\theta$ is also standard, which we omit, is analogous. This will suffice to prove the lemma. Let us write $W/W_{\lambda_{i_1}} \times \cdots \times W/W_{\lambda_{i_m}}$ for the Cartesian product (6.1.2), and denote by $(\tau_1, \ldots, \tau_m)$ the tuple (6.1.1).

Suppose that $f_\alpha\theta$ does not vanish. From the definition of f_α, it follows that, by increasing m and replacing τ_j by τ_j, τ_j for some choices of j, $1 \leq j \leq m$, as necessary, we may assume that the tuple as in (6.1.1) corresponding to $f_\alpha\theta$ is $(\tau_1', \ldots, \tau_m')$ where for every j, $1 \leq j \leq m$, we have

$$\text{(6.1.5)} \qquad \tau_j' \text{ is either } \tau_j \text{ or } s_\alpha\tau_j, \text{ depending upon certain conditions.}$$

To exploit these conditions, it is useful to introduce the following terminology. Let

j be an integer, $1 \leq j \leq m$. We call j

$$
(6.1.6) \qquad \begin{array}{cl}
\textit{changing} & \text{if } \tau'_j = s_\alpha \tau_j \neq \tau_j \\
\textit{changeable (but not changing)} & \text{if } \tau'_j = \tau_j < s_\alpha \tau_j \\
\textit{resisting} & \text{if } s_\alpha \tau_j < \tau_j \\
\textit{flat} & \text{if } s_\alpha \tau_j = \tau_j
\end{array}
$$

Using this terminology, we record some simple observations ((6.1.7)), (6.1.8), and (6.1.11) below) that we need for the proof. All of these follow readily from the definition of f_α as in [15]. To begin with:

(6.1.7) j is changing only if $\tau_j < s_\alpha \tau_j$,
so the cases in (6.1.6) are exhaustive and mutually exclusive.

In particular this means that $\tau'_j = \tau_j$ if j is resisting. So we have:

(6.1.8) If j is resisting or flat or changeable (but not changing), then $\tau'_j = \tau_j$.

We call j, $1 \leq j \leq m$, *unobstructed* if there exists k, $j \leq k \leq m$, such that k is changing and there does not exist j' with j' resisting and $j \leq j' < k$. We call j *obstructed* if it is not unobstructed. Evidently:

(6.1.9) j is unobstructed if it is changing, and j is obstructed if it is resisting.

so, from (6.1.8):

(6.1.10) If j is obstructed, then $\tau'_j = \tau_j$.

We also have (from the definition of the operator f_α):

(6.1.11) If j is changeable (but not changing), then j is obstructed.

Let now $\tilde{\tau}_1 \geq \ldots \geq \tilde{\tau}_m$ be a standard lift of θ. For j, $1 \leq j \leq m$, define $\tilde{\tau}'_j$ by:

$$\tilde{\tau}'_j = \begin{cases} s_\alpha \tilde{\tau}_j & \text{if } j \text{ is changing} \\ \tilde{\tau}_j & \text{if } j \text{ is obstructed} \end{cases} \tag{6.1.12}$$

and, when j is flat and unobstructed, by a downward induction as required:

(6.1.13) $\tilde{\tau}'_j :=$ the smaller of $\tilde{\tau}_j$ and $s_\alpha \tilde{\tau}_j$ that is larger than or equal to $\tilde{\tau}'_{j+1}$

Since $\tilde{\tau}'_{j+1}$ is either $\tilde{\tau}_{j+1}$ or $s_\alpha \tilde{\tau}_{j+1}$ (by downward induction), it follows (by an application of the basic observation (*) in §2.1.2 applied to the hypothesis that $\tilde{\tau}_j \geq \tilde{\tau}_{j+1}$) that

$$\tilde{\tau}_j \vee s_\alpha \tilde{\tau}_j \geq \tilde{\tau}'_{j+1} \tag{6.1.14}$$

so at least one of $\tilde{\tau}_j$ and $s_\alpha \tilde{\tau}_j$ is larger than or equal to $\tilde{\tau}'_{j+1}$ and (6.1.13) makes sense.

We now argue that $\tilde{\tau}'_j \geq \tilde{\tau}'_{j+1}$ for all $1 \leq j < m$. If j is flat and unobstructed, then this follows from the definition (6.1.13) of $\tilde{\tau}'_j$. If j is either changing or resisting, then $\tilde{\tau}'_j = \tilde{\tau}_j \vee s_\alpha \tilde{\tau}_j$ (from (6.1.6), (6.1.7), and (6.1.8)), so it follows from (6.1.14) that $\tilde{\tau}'_j \geq \tilde{\tau}'_{j+1}$. By the mutual exclusivity of the cases in (6.1.6) (which follows from (6.1.7) as already remarked) and (6.1.11), we may assume that j is obstructed but not resisting. But then $j+1$ is also obstructed, and so $\tilde{\tau}'_{j+1} = \tilde{\tau}_{j+1}$ by (6.1.12), and $\tilde{\tau}'_j = \tilde{\tau}_j \geq \tilde{\tau}_{j+1} = \tilde{\tau}'_{j+1}$.

We claim that $\tilde{\tau}'_1 \geq \ldots \geq \tilde{\tau}'_m$ is a standard lift of $f_\alpha \theta$. It remains only to verify that

$\tilde{\tau}'_j W_{\lambda_{i_j}} = \tau'_j$ for every j, $1 \leq j \leq m$. This is easily done, as follows:

- j changing: $\tilde{\tau}'_j = s_\alpha \tilde{\tau}_j$ by (6.1.12), so $\tilde{\tau}'_j W_{\lambda_{i_j}} = s_\alpha \tilde{\tau} W_{\lambda_{i_j}} = s_\alpha \tau_j$. But $s_\alpha \tau_j = \tau'_j$ by (6.1.6).

- j obstructed: $\tilde{\tau}'_j = \tilde{\tau}_j$ by (6.1.12), so $\tilde{\tau}'_j W_{\lambda_{i_j}} = \tilde{\tau} W_{\lambda_{i_j}} = \tau_j$. But $\tau_j = \tau'_j$ by (6.1.10).

- j flat: $\tilde{\tau}'_j$ is either $\tilde{\tau}_j$ or $s_\alpha \tilde{\tau}_j$, so $\tilde{\tau}'_j W_{\lambda_{i_j}}$ is either τ_j or $s_\alpha \tau_j$. But $\tau_j = s_\alpha \tau_j = \tau'_j$ by (6.1.8). $\square$

Proposition 6.1.2 Write η for $\eta(\theta)$. If $\eta = \pi_{\lambda_1} \star \cdots \star \pi_{\lambda_n}$, then η is standard and so θ is standard by the previous lemma. Now suppose that θ is standard. Then so is η by the lemma. Let us write $W/W_{\lambda_{i_1}} \times \cdots \times W/W_{\lambda_{i_m}}$ for the Cartesian product (6.1.2), and denote by $(\sigma_1, \ldots, \sigma_m)$ the minimal standard lift of η.

Let α be any simple root. We claim that there cannot exist k, $1 \leq k \leq m$, such that:

$$(6.1.15) \qquad s_\alpha \sigma_k < \sigma_k \qquad \text{and} \qquad s_\alpha \sigma_j W_{\lambda_{i_j}} = \sigma_j W_{\lambda_{i_j}} \text{ for all } 1 \leq j < k$$

To prove the claim, we suppose such a k exists and arrive at a contradiction. We have $s_\alpha \sigma_k W_{\lambda_{i_k}} \leq \sigma_k W_{\lambda_{i_k}}$. If strict inequality holds here, then $e_\alpha \eta$ does not vanish, a contradiction, so equality holds. If $s_\alpha \sigma_{k+1} > \sigma_{k+1}$, then $s_\alpha \sigma_k = \sigma_k \wedge s_\alpha \sigma_k > \sigma_{k+1} \wedge s_\alpha \sigma_{k+1} = \sigma_{k+1}$, a contradiction to the hypothesis that $(\sigma_1, \ldots, \sigma_m)$ is a minimal standard lift of η (because then $s_\alpha \sigma_k$ would work as a lift in place of σ_k). Thus (6.1.15) holds with k replaced by $k+1$. Repeating these arguments sufficiently many times, we conclude that $s_\alpha \sigma_m < \sigma_m$ and $s_\alpha \sigma_j W_{\lambda_{i_j}} = \sigma_j W_{\lambda_{i_j}}$ for all $1 \leq j \leq m$. But then σ_m is not the minimal element in the coset $\sigma_m W_{\lambda_{i_m}}$, which contradicts the hypothesis that $(\sigma_1, \ldots, \sigma_m)$ is the minimal standard lift of η.

To show that $\eta = \pi_{\lambda_1} \star \cdots \star \pi_{\lambda_n}$, it suffices to show that σ_1 is the identity element of the Weyl group W. If σ_1 is not the identity element, let α be a simple root such that $s_\alpha \sigma_1 < \sigma_1$. Then (6.1.15) holds with $k = 1$, a contradiction.

6.1.4 The crystal isomorphism

Fix notation as in the beginning of §6.1.1. Let $\mathcal{P}_{\text{std}}$ denote the set of all standard paths in $\mathcal{P}$. By Proposition 6.1.2, $\mathcal{P}_{\text{std}}$ is precisely the set of paths θ in $\mathcal{P}$ for which $\eta(\theta) = \pi_{\lambda_1} \star \cdots \star \pi_{\lambda_n}$. Thus, by [15, Theorem 7.1], there is a (unique) crystal isomorphism[2] $\Gamma : \mathcal{P}_\lambda \to \mathcal{P}_{\text{std}}$, where $\mathcal{P}_\lambda$ denotes the set of LS paths of shape $\lambda = \lambda_1 + \cdots + \lambda_n$.

Proposition 6.1.6. *The isomorphism Γ has the following properties:*

- *The straight line path π_λ (from the origin to λ) is mapped under Γ to $\pi_{\lambda_1} \star \cdots \star \pi_{\lambda_n}$.*
- *The end point of π in $\mathcal{P}_\lambda$ is the same as that of its image $\Gamma\pi$.*
- *π is λ-dominant if and only $\Gamma\pi$ is so.*

Proof: The first item is because π_λ (respectively $\pi_{\lambda_1} \star \cdots \star \pi_{\lambda_n}$) is the unique path in $\mathcal{P}_\lambda$ (respectively $\mathcal{P}_{\text{std}}$) on which e_α vanishes for every simple α. The second is because (a) π_λ and $\pi_{\lambda_1} \star \cdots \star \pi_{\lambda_n}$ both have λ as end point, (b) every path in $\mathcal{P}_\lambda$ (respectively $\mathcal{P}_{\text{std}}$) can be obtained by acting a sequence of f_α operators on π_λ (respectively $\pi_{\lambda_1} \star \cdots \star \pi_{\lambda_n}$), (c) the first item, and finally (d) if f_α does not vanish on any path σ (in either $\mathcal{P}_\lambda$ or $\mathcal{P}_{\text{std}}$) then $f_\alpha\sigma(1) = \sigma(1) - \alpha$. As for the third item, we make two observations from which it follows that Γ preserves λ-dominance:

[2] "Crystal isomorphism" just means a bijection that commutes with the action of the root operators f_α and e_α.

- a path σ in $\mathcal{P}_\lambda \cup \mathcal{P}_{\text{std}}$ is λ-dominant if and only if $e_\alpha(\pi_\lambda \star \sigma)$ vanishes for all simple roots α.

- For paths π_1 and π_2 in $\mathcal{P}_\lambda \cup \mathcal{P}_{\text{std}}$, $e_\alpha(\pi_1 \star \pi_2)$ equals either $e_\alpha\pi_1 \star \pi_2$ or $\pi_1 \star e_\alpha\pi_2$ depending precisely upon whether or not $i \geq j$ where i (respectively j) is the maximum non-negative integer k such that $f_\alpha^k\pi_1$ (respectively $e_\alpha^k\pi_2$) does not vanish. $\square$

Proposition 6.1.7. *For an LS path π of shape λ, the minimal element in the initial direction of π equals the initial element $\mathfrak{w}(\Gamma\pi)$ of the standard minimal lift of $\Gamma\pi$.*

The proposition follows by combining Corollary 6.1.10 with Lemma 6.1.11.

A useful observation (Corollary 6.1.10)

Let μ be a dominant integral weight. For a Weyl group valued function $\mathfrak{F} : \mathcal{P}_\mu \to W$ on the set $\mathcal{P}_\mu$ of LS paths of shape μ, and v an element of W, put $\mathcal{P}_{\mu,v}(\mathfrak{F}) := \{\pi \in \mathcal{P}_\mu \,|\, \mathfrak{F}(\pi) \leq v\}$.

Lemma 6.1.8. *Suppose that the following conditions hold for π an arbitrary path in $\mathcal{P}_\mu$ and $w := \mathfrak{F}(\pi)$:*

1. *If α a simple root with $s_\alpha w < w$, then $e_\alpha\pi$ does not vanish.*

2. *Suppose $f_\alpha\pi$ does not vanish. Then either* (a) *$\mathfrak{F}(f_\alpha\pi) = w$ or* (b) *$\mathfrak{F}(f_\alpha\pi) = s_\alpha w > w$ and $e_\alpha\pi$ vanishes.*

Then, for v in W and β simple such that $s_\beta v < v$:

$$\mathcal{P}_{\mu,v}(\mathfrak{F}) = \{f_\beta^k \pi \mid \pi \in \mathcal{P}_{\mu,s_\beta v}(\mathfrak{F}),\ k \geq 0,\ f_\beta^k \pi \text{ does not vanish}\}$$

PROOF: Since $\mathcal{P}_{\mu,s_\beta v}(\mathfrak{F}) \subseteq \mathcal{P}_{\mu,v}(\mathfrak{F})$ and, by (2), $\mathcal{P}_{\mu,v}(\mathfrak{F})$ is closed under the action of f_α, it follows that the right hand side is contained in $\mathcal{P}_{\mu,v}(\mathfrak{F})$. To prove the other containment, let σ be in $\mathcal{P}_{\mu,v}(\mathfrak{F})$. Let $k \geq 0$ be maximal such that $e_\beta^k \sigma$ does not vanish, and put $\pi := e_\beta^k \sigma$. Then $f_\beta^k \pi = \sigma$, so it is enough to show that π is in $\mathcal{P}_{\mu,s_\beta v}(\mathfrak{F})$.

Put $w := \mathfrak{F}(\pi)$. On the one hand, since $f_\beta^k \pi = \sigma$, it follows from (2) that $\mathfrak{F}(\sigma)$ equals either w or $s_\beta w$, so that $w \leq w \vee s_\beta w = \mathfrak{F}(\sigma) \vee s_\beta \mathfrak{F}(\sigma) \leq v \vee s_\beta v = v$. But, on the other, if $s_\beta w < w$, then $e_\beta \pi$ does not vanish by (1), a contradiction to the maximality of k. Thus we have $w < s_\beta w$ and $w = w \wedge s_\beta w \leq v \wedge s_\beta v = s_\beta v$. $\square$

Corollary 6.1.9. *Let $\iota : \mathcal{P}_\mu \to W$ be the function that maps each path to the minimal element in its initial direction. Then, for v in W and β simple such that $s_\beta v < v$:*

$$\mathcal{P}_{\mu,v}(\iota) = \{f_\beta^k \pi \mid \pi \in \mathcal{P}_{\mu,s_\beta v}(\iota), k \geq 0, f_\beta^k \pi \text{ does not vanish}\}$$

PROOF: It follows easily from the definition of the operators e_α and f_α that the hypothesis of the lemma are satisfied for the function ι. (See also [14, Lemma in §5.3].) $\square$

Corollary 6.1.10. *Suppose in addition to the conditions* (1) *and* (2) *of Lemma 6.1.8 the function $\mathfrak{F}$ satisfies the following: $\mathcal{P}_{\mu,\mathrm{identity}}(\mathfrak{F}) = \{\pi_\lambda\}$. Then $\mathfrak{F} = \iota$.*

PROOF: We proceed by induction on v to show that $\mathcal{P}_{\mu,v}(\mathfrak{F}) = \mathcal{P}_{\mu,v}(\iota)$. This will suffice. If $v =$ identity, then both sets are equal to $\{\pi_\lambda\}$ and the result holds. So suppose that $v >$ identity. Choose a simple root α such that $w := s_\alpha v < v$. By the induction hypothesis, $\mathcal{P}_{\mu,w}(\mathfrak{F}) = \mathcal{P}_{\mu,w}(\iota)$. But, by Lemma 6.1.8, we have

$$\mathcal{P}_{\mu,v}(\mathfrak{F}) = \{f_\alpha^k \pi \mid \pi \in \mathcal{P}_{\mu,w}(\mathfrak{F}),\ k \geq 0,\ f_\alpha^k \pi \text{ does not vanish}\}$$

and, by Corollary 6.1.9, we have

$$\mathcal{P}_{\mu,v}(\iota) = \{f_\alpha^k \pi \mid \pi \in \mathcal{P}_{\mu,w}(\iota),\ k \geq 0,\ f_\alpha^k \pi \text{ does not vanish}\},$$

so it is clear that $\mathcal{P}_{\mu,v}(\mathfrak{F}) = \mathcal{P}_{\mu,v}(\iota)$. $\square$

The above corollary together with the following lemma proves Proposition 6.1.7. The proof of the lemma occupies §6.1.5

Lemma 6.1.11. *Fix notation as in the first paragraph of §6.1.4. Let $\mathfrak{F} : \mathcal{P}_\lambda \to W$ be the Weyl group valued function on $\mathcal{P}_\lambda$ given by $\mathfrak{F}(\pi) := \mathfrak{w}(\Gamma(\pi))$. Then $\mathcal{P}_{\lambda,\text{identity}}(\mathfrak{F}) = \{\pi_\lambda\}$ and $\mathfrak{F}$ satisfies the conditions* (1) *and* (2) *of Lemma 6.1.8.*

6.1.5 Proof of Proposition 6.1.7: and Lemma 6.1.11

We first prove:

Lemma 6.1.12. *With notation as in the statement and proof of Lemma 6.1.5, suppose that $\tilde{\tau}_1 \geq \ldots \geq \tilde{\tau}_m$ be the minimal standard lift of θ. Then*

1. *Suppose that $\tilde{\tau}_p > s_\alpha \tilde{\tau}_p$. Then p is either resisting or flat. If p is flat, then there exists r, $p < r \leq m$, with r resisting and every q such that $p < q < r$ is flat.*

2. $\tilde{\tau}'_j \geq \tilde{\tau}_j$ *for all* j, $1 \leq j \leq m$.

3. Suppose that j *is changing and* $j < m$. *Then* $\tilde{\tau}'_{j+1} \wedge s_\alpha \tilde{\tau}'_{j+1} = \tilde{\tau}_{j+1}$.

4. $\tilde{\tau}'_1 \geq \ldots \geq \tilde{\tau}'_m$ *is the minimal standard lift of* $f_\alpha \theta$

Proof: (1) If p is changing or changeable (but not changing), then $\tau_p < s_\alpha \tau_p$, so it would mean that $\tilde{\tau}_p < s_\alpha \tilde{\tau}_p$ (Corollary 2.1.12). This proves that p can only be either flat or resisting. Suppose now that p is flat. Let r be the least integer, $p < r \leq m$, (if it exists) such that r is not flat. If such an r doesn't exist, put $r = m + 1$. For all q, $p \leq q < r$, put $\sigma_q := \tilde{\tau}_q \wedge s_\alpha \tilde{\tau}_q$. Then $\sigma_p = s_\alpha \tilde{\tau}_p$. We have $\sigma_p \geq \ldots \geq \sigma_{r-1}$ (by the basic fact (*) in §2.1.2). If $r < m$ and r is not resisting, then $\tilde{\tau}_r = \tilde{\tau}_r \wedge s_\alpha \tilde{\tau}_r$, so that $\sigma_{r-1} \geq \tilde{\tau}_r$. Thus $\sigma_p \geq \ldots \sigma_{r-1} \geq \tilde{\tau}_r \geq \ldots \geq \tilde{\tau}_m$ would be a standard lift of $(\tau_p, \ldots, \tau_m)$, which we could complete to a standard lift of θ. But then $\sigma_p = s_\alpha \tilde{\tau}_p < \tilde{\tau}_p$, which contradicts the hypothesis that $\tilde{\tau}_1 \geq \ldots \geq \tilde{\tau}_m$ is the minimal standard lift.

(2) Proceed by downward induction on j. Since $\tilde{\tau}'_j = \tilde{\tau}_j$ in case j is obstructed, and $\tilde{\tau}'_j = s_\alpha \tilde{\tau}_j > \tilde{\tau}_j$ in case j is changing, we may assume that j is flat and unobstructed, so $j < m$ and $\tau'_j = \tau_j$. We have, by the induction hypothesis, $\tilde{\tau}'_{j+1} \geq \tilde{\tau}_{j+1}$, and so by (3) of Remark 2.1.18:

$$\tilde{\tau}'_j \geq \mathbf{min}\, J_{\tau'_j}(\tilde{\tau}'_{j+1}) = \mathbf{min}\, J_{\tau_j}(\tilde{\tau}'_{j+1}) \geq \mathbf{min}\, J_{\tau_j}(\tilde{\tau}_{j+1}) = \tilde{\tau}_j.$$

(3) Since by definition $\tilde{\tau}'_{j+1}$ is either $\tilde{\tau}_{j+1}$ or $s_\alpha \tilde{\tau}_{j+1}$, and $\tilde{\tau}'_{j+1} \geq \tilde{\tau}_{j+1}$ by item (2), it is enough to show that $\tilde{\tau}'_{j+1} < s_\alpha \tilde{\tau}'_{j+1}$. If not, then, by item (1), there exists r such that $j < r \leq m$ with r resisting and every q such that $j < q < r$ is flat. But this cannot happen since j is changing, by the definition of the operator f_α.

(4) Let $\tilde{\tau}''_1 \geq \ldots \geq \tilde{\tau}''_m$ be another standard lift of $f_\alpha \theta$. It suffices to show that $\tilde{\tau}''_j \geq \tilde{\tau}'_j$ for every j, $1 \leq j \leq m$. Proceed by downward induction on j. It is

convenient to put $\tilde{\tau}_{m+1} = \tilde{\tau}'_{m+1} = \tilde{\tau}''_{m+1} =$ identity. By the induction hypothesis, $\tilde{\tau}''_{j+1} \geq \tilde{\tau}'_{j+1}$.

In case j is obstructed, $\tau'_j = \tau_j$ by (6.1.10), and we have

$$\tilde{\tau}''_j \geq \mathbf{min}\, J_{\tau'_j}(\tilde{\tau}''_{j+1}) = \mathbf{min}\, J_{\tau_j}(\tilde{\tau}''_{j+1}) \geq \mathbf{min}\, J_{\tau_j}(\tilde{\tau}_{j+1}) = \tilde{\tau}_j = \tilde{\tau}'_j$$

Suppose now that j is changing. Then $\tau_j < s_\alpha \tau_j = \tau'_j$ by (6.1.6) and (6.1.7). By (3) of Remark 2.1.18, Corollary 2.1.12 and Lemma 2.1.16 (2), and item (3) above:

$$\begin{aligned}\tilde{\tau}''_j \geq \mathbf{min}\, J_{s_\alpha \tau_j}(\tilde{\tau}''_{j+1}) \geq \mathbf{min}\, J_{s_\alpha \tau_j}(\tilde{\tau}'_{j+1}) &= s_\alpha\, \mathbf{min}\, J_{\tau_j}(\tilde{\tau}'_{j+1} \wedge s_\alpha \tau'_{j+1}) \\ &= s_\alpha\, \mathbf{min}\, J_{\tau_j}(\tilde{\tau}_{j+1}) = s_\alpha \tilde{\tau}_j = \tilde{\tau}'_j\end{aligned}$$

The only remaining case is when j is flat and unobstructed. We then have $j < m$ and $\tau'_j = \tau_j$. By the induction hypothesis and item (2) above, we have $\tilde{\tau}''_{j+1} \geq \tilde{\tau}'_{j+1} \geq \tilde{\tau}_{j+1}$, so by (3) of Remark 2.1.18:

$$\tilde{\tau}''_j \geq \mathbf{min}\, J_{\tau'_j}(\tilde{\tau}''_{j+1}) = \mathbf{min}\, J_{\tau_j}(\tilde{\tau}''_{j+1}) \geq \mathbf{min}\, J_{\tau_j}(\tilde{\tau}'_{j+1}) \geq \mathbf{min}\, J_{\tau_j}(\tilde{\tau}_{j+1}) = \tilde{\tau}_j$$

This means we would be done in case $\tilde{\tau}_j \geq \tilde{\tau}'_j$ (which by item (2) is equivalent to $\tilde{\tau}_j = \tilde{\tau}'_j$). But, $\tilde{\tau}'_j$ is by definition the smaller of $\tilde{\tau}_j$ and $s_\alpha \tilde{\tau}_j$ that is larger than $\tilde{\tau}'_{j+1}$. So it only remains to consider the case when $\tilde{\tau}_j < s_\alpha \tilde{\tau}_j$ and $\tilde{\tau}_j \not\geq \tilde{\tau}'_{j+1}$. In this situation, $\tilde{\tau}_{j+1} < s_\alpha \tilde{\tau}_{j+1} = \tilde{\tau}'_{j+1}$ (for $\tilde{\tau}'_{j+1}$ is by definition either $s_\alpha \tilde{\tau}_{j+1}$ or $\tilde{\tau}_{j+1}$, and $\tilde{\tau}_j \geq \tilde{\tau}_{j+1}$). This implies by item (1) that $j+1$ is obstructed and therefore j is also obstructed, a contradiction. $\square$

Lemma 6.1.11 That $\mathcal{P}_{\lambda,\text{identity}}(\mathfrak{F}) = \pi_\lambda$ follows from (6.1.3).

Now Put $\theta = \Gamma\pi$ and $\mathfrak{w}(\theta) = w$. Let $\theta = (\tau_1, \ldots, \tau_m)$ and let $(\tilde{\tau}_1, \ldots, \tilde{\tau}_m)$ be the minimal standard lift of θ (so that $w = \tilde{\tau}_1$).

Proof of condition (1) of Lemma 6.1.8: Let α be a simple root such that $s_\alpha w < w$. To show that $e_\alpha \pi$ does not vanish, it is enough to show that $e_\alpha \theta$ does not vanish, and for this it is enough to show that there exists r, $1 \leq r \leq m$, such that $s_\alpha \tau_r < \tau_r$, and $s_\alpha \tau_j = \tau_j$ for all j, $1 \leq j < r$. By way of contradiction, suppose that $s_\alpha \tau_r > \tau_r$ for the least r such that $s_\alpha \tau_r \neq \tau_r$ (the case when $s_\alpha \tau_j = \tau_j$ for all $1 \leq j \leq m$ is included in the consideration: we put $r = m+1$ in this case). For j, $1 \leq j < r$, set $\sigma'_j := \tilde{\tau}_j \wedge s_\alpha \tilde{\tau}_j$. Observe that $(\sigma'_1, \ldots, \sigma'_{r-1}, \tilde{\tau}_r, \ldots, \tilde{\tau}_m)$ is also a standard lift of θ. But then $\sigma'_1 = s_\alpha w < w = \tilde{\tau}_1$, which contradicts the choice of $(\tilde{\tau}_1, \ldots, \tilde{\tau}_m)$ as the minimal standard lift of θ.

Proof of condition (2) of Lemma 6.1.8: Suppose that $f_\alpha \pi$ does not vanish. Then $f_\alpha \theta$ does not vanish either. By Lemma 6.1.5, $f_\alpha \theta$ is standard. Moreover, by Lemma 6.1.12 $(\tilde{\tau}'_1, \ldots, \tilde{\tau}'_m)$ is the minimal standard lift of θ. Since $\tilde{\tau}'_1$ is either $\tilde{\tau}_1$ or $s_\alpha \tilde{\tau}_1$ by its definition, it follows that $\mathfrak{F}(f_\alpha \pi)$ is either w or $s_\alpha w$. Suppose that $\mathfrak{F}(f_\alpha \pi) \neq \tilde{\tau}_1 = w$. Then, since $\tilde{\tau}'_1 \geq \tau_1$ by item (2) of Lemma 6.1.12, it follows that $\mathfrak{F}(f_\alpha \pi) = s_\alpha w > w$. Moreover, this happens only if 1 is unobstructed, which means that minimum is 0 of the function $t \mapsto \langle \pi(t), \alpha^\vee \rangle$ on the interval $[0,1]$, and so $e_\alpha \pi$ vanishes.

Thesis Title :

A Study of Kostant -Kumar modules via Littelmann paths.

In this thesis we study about Kostant-Kumar modules (KK module for short). KK modules are the certain cyclic submodules of the tensor product of two integrable irriducible modules of symmetrizable Kac-Moody Lie algebras.

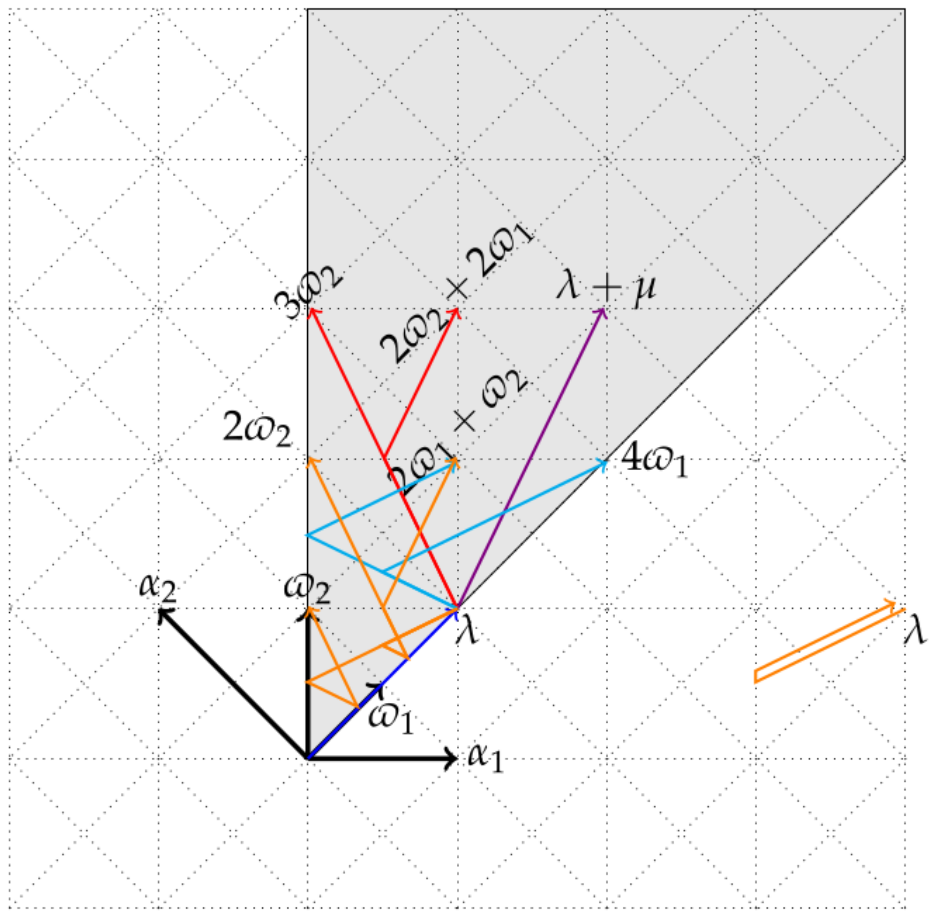


Figure 1: Decomposition KK modules of tensor product in Lie algebra of type B2

We extends Joseph decomposition rules of KK modules for finite type Lie algebra. Above figure is graphic of decomposition of KK modules in Lie algebra of type B2. We also give, in the spirit of Littelmann, a path model for KK modules for symmetrizable Kac-Moody Lie algebras provided it is symmetric or of finite type.