Quasi-isometries of $\mathbb{Z}^n$ and twisted conjugacy in certain linear groups

By

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A thesis submitted to the

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Recommendations of the Viva Voce Committee

As members of the Viva Voce Committee, we certify that we have read the dissertation prepared by Oorna Mitra entitled "Quasi-isometries of $\mathbb{Z}^n$ and twisted conjugacy in certain linear groups " and recommend that it may be accepted as fulfilling the thesis requirement for the award of Degree of Doctor of Philosophy.

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Oorna Mitra

Oorna Mitra

DECLARATION

I hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Oorna Mitra

LIST OF PUBLICATIONS ARISING FROM THE THESIS

Journal

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Submitted

1. *Twisted conjugacy in general linear groups over polynomial algebras over finite fields* Joint with Parameswaran Sankaran, axXiv:1912.10184, https://arxiv.org/abs/1912.10184.

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Summary

The thesis addresses two problems, which are independent of each other.

In the first part, we study $\mathcal{QI}(\mathbb{Z}^n)$, the quasi-isometry group of the finitely generated abelian group $\mathbb{Z}^n$ for $n \geq 2$. We show that certain groups of diffeomorphisms embed into it and therefore, conclude that $\mathcal{QI}(\mathbb{Z}^n)$ is "large". See 2.

Every finitely generated group Γ can be made into a metric space with respect to the word metric. Any two finite generating sets give rise to equivalent word metrics. One can naturally associate a group, namely *the quasi-isometry group*, $\mathcal{QI}(\Gamma)$ to the metric space Γ whose elements are *coarse* equivalence classes of certain self-maps. $\mathcal{QI}(\Gamma)$ is a *quasi-isometry invariant*, which is an important object of study in geometric group theory.

We have the following result concerning $\mathcal{QI}(\mathbb{Z}^n)$ for $n \geq 2$.

Theorem 0.0.1. *Let $n \geq 2$. The following groups can be embedded in $\mathcal{QI}(\mathbb{Z}^n)$.*

(i) $\mathrm{Bilip}(\mathbb{S}^{n-1})$, *in particular,* $\mathrm{Diff}^r(\mathbb{S}^{n-1}), 1 \leq r \leq \infty$,

(ii) $\mathrm{Bilip}(\mathbb{D}^n, \mathbb{S}^{n-1})$ embeds in $\mathcal{QI}(\mathbb{Z}^n)$; in particular $\mathrm{Diff}^r(\mathbb{D}^n, \mathbb{S}^{n-1}), 1 \leq r \leq \infty$, *where $\mathbb{D}^n$ denotes the disk $\{v \in \mathbb{R}^n \mid ||v|| \leq 1\}$,*

(iii) $\mathrm{Diff}^r_\kappa(\mathbb{R}^n), 1 \leq r \leq \infty$ embeds in $\mathcal{QI}(\mathbb{Z}^n)$.

Theorem 0.0.2. *Let $n \geq 2$ and $\mathcal{R} := \{f = (f_{ij}) \in Map(\mathbb{R}_{>0}, SO(n))$: for any*

$i, j \leq n$, $f_{ij} : \mathbb{R}_{>0} \to \mathbb{R}$ is C^1 and $|f'_{ij}(t)| < c/t \ \forall t$ for some constant $c = c(f)\}$. The map $\Phi : \mathcal{R} \to \mathcal{QI}(\mathbb{R}^n)$ defined as $f \mapsto [\phi_f]$ is a homomorphism such that $K \subset \ker(\Phi) \subset N$, where $\phi_f(v) = f(||v||)(v), v \neq 0$ and $\phi_f(0) = 0$, $K = \{f \in \mathcal{R} \mid \exists N = N(f)$ such that $f(t) = I_n \ \forall t \geq N\}$ and $N = \{f \in \mathcal{R} \mid \lim_{t\to\infty} f(t) = I_n\}$.

In the second part, which is the major part of the thesis, we study twisted conjugacy in general and special linear groups G over polynomial and Laurent polynomial rings over subfields of the algebraic closure of finite fields. See 3.

Given an automorphism $\phi : \Gamma \to \Gamma$ of an infinite group Γ, one has an action of Γ on itself given by $g.x = gx\phi(g^{-1})$. The orbits of this action are the ϕ-twisted conjugacy classes. $R(\phi)$ denotes the Reidemeister number of ϕ, which is the number of ϕ-twisted conjugacy classes in Γ if finite, otherwise $R(\phi) := \infty$. Γ is said to have the R_∞-property if $R(\phi) = \infty, \forall \phi \in \mathrm{Aut}(\Gamma)$.

The notion of twisted conjugacy in groups originated in Nielsen-Reidemeister fixed point theory. The problem of determining which class of groups have the R_∞ -property has been an active area of research in the study of twisted conjugacy classes in infinite groups.

We address this question for the class of general and special linear groups over polynomial and Laurent polynomial algebras over a field $F \subset \bar{\mathbb{F}}_p$.

We have the following results.

Theorem 0.0.3. *Let F be a subfield of $\bar{\mathbb{F}}_p$ and let $G = GL_n(R), SL_n(R)$ where $R = F[t]$ or $F[t, t^{-1}]$. Then G has the R_∞-property in the following cases: (i) $n \geq 3$, (ii) $GL_2(F[t]), SL_2(F[t])$, (iii) $GL_2(\mathbb{F}_q[t, t^{-1}])$.*

Theorem 0.0.4. *Let $R = F[t]$ or $F[t, t^{-1}]$ where F is a field contained in $\bar{\mathbb{F}}_p$. Let $n \geq 3$. Suppose that H is a subgroup of $G = GL_n(R)$ that contains $SL_n(R)$. Then H has the R_∞-property.*

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Chapter 1

Introduction

The thesis consists of two parts, which are independent of each other.

In the first part, we study $\mathcal{QI}(\mathbb{Z}^n)$, the quasi-isometry group of the finitely generated abelian group $\mathbb{Z}^n$ for $n \geq 2$. We show that certain groups of diffeomorphisms embed into it and therefore, conclude that $\mathcal{QI}(\mathbb{Z}^n)$ is "large". See 2.

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1.1 Quasi-isometry

Let $(X, d_X), (Y, d_Y)$ be metric spaces and let $\lambda \geq 1, \epsilon \geq 0$. A set-map $f : X \to Y$ is called a *quasi-isometric embedding* if, for any $x_0, x_1 \in X$, the following inequality holds:

$$-\epsilon + (1/\lambda)d_X(x_0, x_1) \leq d_Y(f(x_0), f(x_1)) \leq \lambda d_X(x_0, x_1) + \epsilon. \tag{1}$$

If, in addition, there exists a constant C such that, for any $y \in Y$, there exists an $x \in X$ such that $d_Y(f(x), y) < C$, we say that f is a *quasi-isometry* and if such an f exists, X and Y are said to be $\mathcal{QI}$ *equivalent* spaces.

Quasi-isometries capture large scale features of metric spaces. It ignores local features and focuses only on how the space appears from far off. In geometric group theory, it is very important to understand if two finitely generated groups with word metrics are $\mathcal{QI}$ equivalent or not and for this reason, it is important to study quasi-isometry invariants associated with finitely generated groups. Some of the invariants are growth rate, spaces of ends, hyperbolicity of a group.

If Γ is a finitely generated group with word metric d, then the set of all self-quasi-isometry classes of Γ forms a group, namely the quasi-isometry group of Γ, and is denoted $\mathcal{QI}(\Gamma)$. If two finitely generated groups with the word metrics are quasi-isometrically equivalent, then their quasi-isometry groups are isomorphic, that is $\mathcal{QI}(\Gamma)$ is a quasi-isometry invariant. For more details on quasi-isometry and quasi-isometry groups, see 2.1. For this reason, these groups are interesting to be understood.

In general, the groups $\mathcal{QI}(\Gamma)$ are difficult to determine and it appears that there are very few families of groups Γ for which $\mathcal{QI}(\Gamma)$ has an explicit description. Some of them are irreducible lattices in semisimple Lie groups (see [F] and the references therein), solvable Baumslag-Solitar groups $BS(1, n)$ [FM, Theorem 7.1], the groups $BS(m, n), 1 < m < n$ [Wh, Theorem 4.3], the group $B_2(\mathbb{Z}[1/m])$ of 2×2-upper triangular matrices over $\mathbb{Z}[1/m]$ [T], $B_n(\mathbb{Z}[1/p])$ for p a prime and $n > 2$ [Wo], the lamplighter groups $G \wr \mathbb{Z}$ with G finite [EFW].

Even for the very simple case, when $\Gamma = \mathbb{Z}^n$, the groups $\mathcal{QI}(\mathbb{Z}^n)$ remain largely unexplored and are poorly understood, especially when $n > 1$. We note here that the inclusion $\mathbb{Z}^n \hookrightarrow \mathbb{R}^n$ is a quasi-isometry (where the metric on $\mathbb{R}^n$ is understood to be the Euclidean and $\mathbb{Z}^n$ is equipped with word metric) and so we have a natural isomorphism $\mathcal{QI}(\mathbb{Z}^n) \to \mathcal{QI}(\mathbb{R}^n)$.

When $n = 1$, Gromov and Pansu [GP, §3.3.B] noted that $\text{Bilip}(\mathbb{R}) \to \mathcal{QI}(\mathbb{R})$ is surjective and that $\mathcal{QI}(\mathbb{R})$ is infinite dimensional. Sankaran [S] showed that $\mathcal{QI}(\mathbb{Z}) \cong \mathcal{QI}(\mathbb{R})$ contains, for example, the free group of rank the continuum and a copy of the group of all *compactly supported* diffeomorphisms of $\mathbb{R}$. The proof techniques used in [S] heavily relied on the one-dimensionality of the real line and do not extend to higher dimensions.

We show that, as in the case of $\mathbb{Z}$, the group $\mathcal{QI}(\mathbb{Z}^n) \cong \mathcal{QI}(\mathbb{R}^n), n \geq 2$, is 'large' and contains many diffeomorphism groups. Before stating our main results, we fix some notations below.

Let M be a smooth Riemannian manifold with metric d and boundary ∂M which is possibly empty.

- $\text{Bilip}(M) := \{f : M \to M : f$ is a homeomorphism and there exists a constant $\lambda \geq 1$ such that $(1/\lambda)d(x, y) \leq d(f(x), f(y)) \leq \lambda d(x, y)$ for all $x, y \in M\}$.

- $\text{Diff}^r(M) :=$ the group of all C^r-diffeomorphisms of M where $1 \leq r \leq \infty$.

- $\text{Bilip}(M, \partial M) := \{f \in \text{Bilip}(M) : f(x) = x$ for all $x \in \partial M\}$.

- $\text{Diff}^r(M, \partial M) := \{f \in \text{Diff}^r(M) : f(x) = x$ for all $x \in \partial M\}$.

- $\text{Diff}^r_\kappa(M) :=$ the group of all compactly supported C^r-diffeomorphisms of M.

Theorem 1.1.1. *Let $n \geq 2$. The following groups can be embedded in $\mathcal{QI}(\mathbb{Z}^n)$.*

(i) $\text{Bilip}(\mathbb{S}^{n-1})$, *in particular,* $\text{Diff}^r(\mathbb{S}^{n-1}), 1 \leq r \leq \infty$,

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Theorem 1.1.2. *Let $n \geq 2$ and $\mathcal{R} := \{f = (f_{ij}) \in Map(\mathbb{R}_{>0}, SO(n))$: for any*

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1.2 Twisted conjugacy

Let Γ be a group and $\phi : \Gamma \to \Gamma$ be an automorphism. Consider the action $g.x = gx\phi(g^{-1})$ of Γ on itself. The orbits of this action are called ϕ-twisted conjugacy classes or Reidemeister classes of ϕ. The cardinality of the set of all ϕ-twisted conjugacy classes is called the Reidemeister number of ϕ and is denoted $R(\phi)$. When the orbit space is infinite, we write $R(\phi) = \infty$. Γ is said to have the R_∞ -property if there are infinitely many ϕ-twisted conjugacy classes for every automorphism ϕ of Γ. If Γ has the R_∞-property, we shall call Γ an R_∞ group.

1.2.1 Motivation

The problem of determining which class of groups have the R_∞ -property was initiated by Fel'shtyn and Hill. This problem is one of the principal problems in the theory of twisted conjugacy classes in infinite groups. The notion of twisted conjugacy in groups originated in Nielsen-Reidemeister fixed point theory. We explain below the connection to the fixed point theory.

Let X be a compact connected polyhedron and let $f : X \to X$ be a continuous map. Denote the fixed point set of f by $\mathrm{Fix}(f) = \{x \in X | f(x) = x\}$. The nonvanishing of the classical Lefschetz number $L(f)$ guarantees the existence of fixed points. Unfortunately, $L(f)$ yields no information about the size of the set of fixed points of f . However, the Nielsen number, recalled below., $N(f)$, a more subtle homotopy invariant, provides a lower bound on the size of this set.

Let $p : \tilde{X} \to X$ be the universal covering of X and $\tilde{f} : \tilde{X} \to \tilde{X}$ a lifting of f, ie. $p \circ \tilde{f} = f \circ p$. Two liftings $\tilde{f}$ and $\tilde{f}'$ are called conjugate if there is a $\gamma \in \Gamma = \pi_1(X)$ such that $\tilde{f}' = \gamma \circ \tilde{f} \circ \gamma^{-1}$. This is an equivalence relation in the set of all liftings of f. It can be seen that either $p(\mathrm{Fix}(\tilde{f})) \cap p(\mathrm{Fix}(\tilde{f}')) = \phi$ or $p(\mathrm{Fix}(\tilde{f})) = p(\mathrm{Fix}(\tilde{f}'))$ iff $\tilde{f}$,$\tilde{f}'$ are conjugates. The subset $p(\mathrm{Fix}(\tilde{f})) \subset \mathrm{Fix}(f)$ is called a fixed point class of f determined by the lifting class $[\tilde{f}]$. A fixed point class is called essential if its 'index' is nonzero. The number of essential fixed point classes is called the Nielsen number of f , denoted by $N(f)$. The Nielsen number is always finite, $N(f) \leq \mid \mathrm{Fix}(f) \mid$ and $N(f) = N(g)$ if f is homotopic to g. In early 1940's, Wecken proved that for a compact triangulable manifold X of dimension $n \geq 3$ and for any self map f of X, $N(f)$ is actually equal to $\mid \mathrm{Fix}(g) \mid$ for some g homotopic to f.

Fixing a lift $\tilde{f} : \tilde{X} \to \tilde{X}$ of f, every lifting of f can be written uniquely as $\alpha \circ \tilde{f}$, with $\alpha \in \Gamma$. Now for every $\alpha \in \Gamma$, $\tilde{f} \circ \alpha$ is a lifting of f so there is a unique $\alpha' \in \Gamma$ such that $\alpha' \circ \tilde{f} = \tilde{f} \circ \alpha$. This gives a homomorphism $\phi : \Gamma \mapsto \Gamma$ such that $\tilde{f} \circ \alpha = \phi(\alpha) \circ \tilde{f}$ for each $\alpha \in \Gamma$. ϕ can be identified with the induced homomorphism $f_* : \Gamma \mapsto \Gamma$. Furthermore, $[\alpha \circ \tilde{f}] = [\alpha' \circ \tilde{f}]$ iff $\alpha' = \gamma \alpha f_*(\gamma^{-1})$ for some $\gamma \in \Gamma$. This shows that lifting classes of f are in 1-1 correspondence with f_*-twisted conjugacy classes in Γ and since a lifting class $p(\mathrm{Fix}(\tilde{f}))$ might be empty, we have |lifting classes of $f| \leq R(f_*)$, which implies $N(f) \leq R(f_*)$. The number $R(f_*)$ need not be finite. For the so-called Jiang spaces, either $N(f) = R(f_*)$ provided $R(f_*)$ is finite or $N(f) = 0$ when $R(f_*)$ is infinity. In some cases, for such spaces, if the fundamental group $\pi_1(X)$ is an R_∞ group, then every homeomorphism of X can be deformed to a fixed point free map.

1.2.2 Known results and our problem

There is no definitive method to determine if a group is an R_∞ group or not. The techniques depend on the structures and properties of the groups under consideration which makes this problem non-trivial and interesting. It is known that the R_∞-property is not inherited by finite index subgroups in general. This also shows that this property is not geometric, i.e, not invariant under quasi-isometry. For example, the infinite dihedral group, which contains the infinite cyclic group as an index 2 subgroup, has the R_∞-property (whereas $R(-Id_{\mathbb{Z}}) = 2$). The R_∞-property is not preserved under quotients. For example, any finitely generated free group has this property while for example, their commutator quotient is a free abelian group that does not have this property.

We list below some important classes of groups which are known to have (or not have) the R_∞ -property:

- Non-elementary hyperbolic groups and non-elementary relatively hyperbolic groups have R_∞-property [LL], [Fe].

- Baumslag-Solitar groups $BS(m,n) = \langle a, b \mid ab^m a^{-1} = b^n \rangle$ have the R_∞-property [FG] except when $mn = 0$, in which case the group is virtually infinite cyclic, and when $mn = 1$, in this case the group is isomorphic to $\mathbb{Z}^2$.

We address this question for the class of general and special linear groups over polynomial and Laurent polynomial algebras over a field F which is a subfield of the algebraic closure of $\mathbb{F}_p$.

The same question for linear groups over fields of characteristic zero was considered by Nasybullov [Na1], [Na2] and Felshtyn and Nasybullov [FN] culminating in the result that a Chevalley group of classical type over an algebraically closed field F of characteristic zero has the R_∞-property if and only if F has finite transcendence degree over $\mathbb{Q}$. Lang [L] has shown that if $\rho : G \to G$ a Frobenius endomorphism

$g \mapsto g^q$ where $q = p^e$, then the Lang map $L : G \to G$ defined as $L(z) = z^{-1}.\rho(z)$ is surjective, where G is a connected algebraic group over an algebraically closed field F of characteristic $p > 0$ which proves that $R(\rho) = 1$. Hence, any connected algebraic group over an algebraically closed field of characteristic $p > 0$ fails to have R_∞-property. We look at groups over *rings* of characteristic $p > 0$.

We note that the group $SL_n(\mathbb{F}_q[t])$ is a lattice in $SL_n(\mathbb{F}_q((t^{-1})))$ for all $n \geq 2$. Also, the group $SL_n(\mathbb{F}_q[t, t^{-1}])$ is a lattice in $SL_n(\mathbb{F}_q((t^{-1}))) \times SL_n(\mathbb{F}_q((t)))$ where $SL_n(\mathbb{F}_q[t, t^{-1}])$ is embedded diagonally in the latter group for all $n \geq 2$. It was shown in [**?**] that any irreducible lattice in a connected non-compact semisimple real Lie group with finite centre has the R_∞-property. The present work may be viewed as a first step in classifying, according to the R_∞-property, lattices in semisimple connected algebraic groups over local fields of positive characteristics.

1.2.3 Main results

The main results are listed below.

Theorem 1.2.1. *Let F be a subfield of $\bar{\mathbb{F}}_p$ and let $G = GL_n(R), SL_n(R)$ where $R = F[t]$ or $F[t, t^{-1}]$. Then G has the R_∞-property in the following cases: (i) $n \geq 3$, (ii) $GL_2(F[t]), SL_2(F[t])$, (iii) $GL_2(\mathbb{F}_q[t, t^{-1}])$.*

Theorem 1.2.2. *Let $R = F[t]$ or $F[t, t^{-1}]$ where F is a field contained in $\bar{\mathbb{F}}_p$. Let $n \geq 3$. Suppose that H is a subgroup of $G = GL_n(R)$ that contains $SL_n(R)$. Then H has the R_∞-property.*

Chapter 2

Quasi-isometry group of Euclidean space

2.1 Quasi-isometry and Quasi-isometry group

We begin by recalling the basic notion of quasi-isometry in the study of coarse geometry.

Let $(X, d_X), (Y, d_Y)$ be a metric space and let $\lambda \geq 1, \epsilon \geq 0$. A set-map $f : X \to Y$ is called a (λ, ϵ)-*quasi-isometric embedding* if, for any $x_0, x_1 \in X$, the following inequality holds:

$$-\epsilon + (1/\lambda)d_X(x_0, x_1) \leq d_Y(f(x_0), f(x_1)) \leq \lambda d_X(x_0, x_1) + \epsilon. \qquad (1)$$

If, in addition, the image of f is C-*dense*, that is, if there exists a constant C such that, for any $y \in Y$, there exists an $x \in X$ such that $d_Y(f(x), y) < C$, we say that f is a (λ, ϵ, C)-*quasi-isometry*, or more briefly a *quasi-isometry*. In general a quasi-isometry is neither one-one nor onto. More importantly, a quasi-isometry need not even be continuous. For example, the map $\mathbb{R} \to \mathbb{Z}$, defined as $x \mapsto \lfloor x \rfloor$ is a quasi-isometry when we take $\lambda = 1, \epsilon = 1, C = 1/2$. The precise values of the constants λ, ϵ, C are often not so important as their existence.

Since a quasi-isometry $f : X \to Y$ in general is neither one-to-one nor onto, it would not be possible to find an inverse for f. However, there exists another quasi-isometry $g : Y \to X$, called *a quasi-inverse of* f, with possibly a different set of implied constants, such that $f \circ g \sim id_Y$ and $g \circ f \sim id_X$. Here we say that two set-maps $f_0, f_1 : X \to Y$ are equivalent—and we write $f_0 \sim f_1$—if $\sup_{x \in X} d_Y(f_0(x), f_1(x)) < \infty$. This is an equivalence relation on the set of all maps from X to Y; the equivalence class of f_0 is denoted $[f_0]$ and is referred to as *the quasi-isometry class* of f_0. It is easy to see that $f_0, f_1 : X \to Y$ and $g_0, g_1 : Y \to Z$ are set maps such that $f_0 \sim f_1, g_0 \sim g_1$, then $g_0 \circ f_0 \sim g_1 \circ f_1$. It follows that set of all self-quasi-isometry classes of a metric space X is a group under the multiplication rule $[f].[g] := [f \circ g]$ and is denoted $\mathcal{QI}(X)$. Any quasi-isometry $f : X \to Y$ induces an isomorphism $\mathcal{QI}(X) \to \mathcal{QI}(Y)$ defined as $[h] \mapsto [f \circ h \circ g]$ where $g : X \to Y$ is a quasi-inverse of f.

If Γ is a finitely generated group with a finite generating set A, then one has word metric d_A on Γ, defined in terms of the length function l_A as follows: $d_A(\gamma_0, \gamma_1) = l_A(\gamma_0^{-1}\gamma_1), \forall \gamma_0, \gamma_1 \in \Gamma$. If $d_{A'}$ is the word metric associated to another finite generating set A' then the identity map of Γ is a quasi-isometry between (Γ, d_A) and $(\Gamma, d_{A'})$. Hence the isomorphism type of $\mathcal{QI}(\Gamma)$ is intrinsic to the group Γ and is independent of the choice of the finite generating set used in its definition. Taking A to be the standard basis of $\mathbb{Z}^n$, the inclusion $\mathbb{Z}^n \hookrightarrow \mathbb{R}^n$ is a quasi-isometry (where the metric on $\mathbb{R}^n$ is understood to be the Euclidean) and so we have a natural isomorphism $\mathcal{QI}(\mathbb{Z}^n) \to \mathcal{QI}(\mathbb{R}^n)$. A more general fact is true, which is known as the Švarc-Milnor Lemma. It states that if Γ acts properly and cocompactly via isometries on a complete geodesic metric space X in which closed balls of finite radii are compact, then, for any choice of $x_0 \in X$ the orbit map

$\gamma \mapsto \gamma.x_0$ is a quasi-isometry $\Gamma \to X$ (where the metric on Γ is always understood to be a word metric) and so $\mathcal{QI}(\Gamma) \cong \mathcal{QI}(X)$. See [BH, Proposition 8.19, Chapter I.8] for a more general formulation and proof. Examples of metric spaces X satisfying the conditions are complete Riemannian manifolds.

It is easily seen that if two groups Γ_0, Γ_1 are (abstractly) commensurable, that is, if there exists a group G which is isomorphic to a finite index subgroup of $\Gamma_i, i = 0, 1$, then Γ_0, Γ_1 and G are quasi-isometric. It follows from the fact that any automorphism of finitely generated groups is a quasi-isometry and the inclusion map from a finite index subgroup to the whole group is quasi-isometry. Also, if $K \subset \Gamma$ is a finite normal subgroup, then the natural projection $\Gamma \to \Gamma/K$ is a quasi-isometry. In general, if Γ_0 and Γ_1 are quasi-isometric, it is not true that they are commensurable. However, it turns out that any finitely generated group quasi-isometric to $\mathbb{Z}^n$ is virtually free abelian of rank n. See [BH, Theorem 8.40, Chapter I]. As a consequence, $\mathbb{Z}^m$ and $\mathbb{Z}^n$ are not quasi-isometric to each other if $m \neq n$. This fact can also be seen using the notion of growth rate.

2.2 Bi-Lipschitz homeomorphisms of $\mathbb{S}^{n-1}$ and $\mathbb{D}^n$

Recall that a map $f : X \to Y$ between metric spaces is *bi-Lipschitz* if there exists a constant $\lambda \geq 1$ such that $(1/\lambda)d(x, y) \leq d(f(x), f(y)) \leq \lambda d(x, y)$. Thus f satisfies Equation (1) with $\epsilon = 0$. Any such λ will be called a bi-Lipschitz constant for f. Any bi-Lipschitz map is continuous and one-to-one but not necessarily onto. If $f : X \to Y$ is bi-Lipschitz homeomorphism then its inverse is also bi-Lipschitz (with the same bi-Lipschitz constant as for f). The group of all bi-Lipschitz self-homeomorphisms of X is denoted by $\mathrm{Bilip}(X, d)$. Evidently, it contains the group $\mathrm{Iso}(X)$ of isometries of X.

Recall that two metrics d and δ on X are bi-Lipschitz equivalent if there exists a

$\lambda \geq 1$ such that $(1/\lambda)d(x,y) \leq \delta(x,y) \leq \lambda d(x,y)$ for all $x, y \in X$. Suppose that d and δ are bi-Lipschitz equivalent. Then $\text{Bilip}(X,d) = \text{Bilip}(X,\delta)$. For example, the Euclidean metric and the l_1-metric are bi-Lipschitz equivalent. Also the identity map is a quasi-isometry $(X,d) \to (X,\delta)$ and the quasi-isometry group $\mathcal{QI}(X)$ with respect to d or δ is the same. If d is clear from the context, we shall abbreviate $\text{Bilip}(X,d)$ to $\text{Bilip}(X)$.

Since any bi-Lipschitz homeomorphism is evidently a quasi-isometry, we obtain a natural homomorphism $\eta : \text{Bilip}(X) \to \mathcal{QI}(X)$. The kernel of η consists precisely of those bi-Lipschitz homeomorphisms that are a bounded distance away from *id*. That is, $f \in \ker(\eta)$ if and only if $\sup_{x \in X} d(f(x), x) < \infty$.

2.2.1 Embedding of Bilip($\mathbb{S}^{n-1}$) into Bilip($\mathbb{R}^n$)

Denote by $\mathbb{R}^n_0$ the punctured Euclidean space $\mathbb{R}^n \setminus \{0\}$. In the case of $\mathbb{R}^n_0 \subset \mathbb{R}^n$, the metric induced by the Riemannian metric is the *same* as the restriction of the Euclidean metric, which we shall use to define the group $\text{Bilip}(\mathbb{R}^n_0)$. Note that if $f : \mathbb{R}^n_0 \to \mathbb{R}^n_0$ is a bi-Lipschitz homeomorphism, then f extends to a bi-Lipschitz homeomorphism of $\mathbb{R}^n$ fixing 0. We shall identify $\text{Bilip}(\mathbb{R}^n_0)$ with the subgroup of $\text{Bilip}(\mathbb{R}^n)$ that fixes the origin.

We regard $\mathbb{R}^n_0$ as a product $\mathbb{S}^{n-1} \times \mathbb{R}_{>0}$ using polar coordinates. This allows for any continuous self-map $\phi : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ to be extended radially to $\tilde{\phi} : \mathbb{R}^n \to \mathbb{R}^n$ that fixes the origin. Explicitly

$$\tilde{\phi}(v) = \begin{cases} ||v|| f(v/||v||), & \text{if } v \neq 0 \\ 0, & \text{if } v = 0. \end{cases}$$

Lemma 2.2.1. *We keep the above notations. Let $\phi : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ be bi-Lipschitz. Then so is $\widetilde{\phi} : \mathbb{R}^n \to \mathbb{R}^n$. Moreover, $\phi \mapsto \widetilde{\phi}$ is a monomorphism*

$\rho : \mathrm{Bilip}(\mathbb{S}^{n-1}) \to \mathrm{Bilip}(\mathbb{R}^n)$ *and* $||\widetilde{\phi} - id|| = \infty$ *if* ϕ *is nontrivial.*

Proof. Let $\lambda^{-1}||x-y|| \leq ||\phi(x) - \phi(y)|| \leq \lambda||x-y||$ for all $x, y \in \mathbb{S}^{n-1}$. Let $v, w \in \mathbb{R}^n_0$ where $||v|| = ||w|| =: r$. Then $||\widetilde{\phi}(v) - \widetilde{\phi}(w)|| = ||r\phi(v/r) - r\phi(w/r)|| = r||\phi(v/r) - \phi(w/r)|| \leq r\lambda||v/r - w/r|| = \lambda||v-w||$. Similarly $||\widetilde{\phi}(v) - \widetilde{\phi}(w)|| \geq \lambda^{-1}||v-w||$.

Suppose that $||v|| = s||w||, s > 1$. Set $v' := v/s$ so that $||\widetilde{\phi}(v') - \widetilde{\phi}(w)|| \leq \lambda||v' - w||$. Note that $||\widetilde{\phi}(v) - \widetilde{\phi}(v')|| = ||v - v'|| \leq ||v - w||$ since v, v' are on the same ray issuing from the origin; the last inequality holds because v' is the point closest to v on the sphere $S(0, ||w||)$. Now $||\widetilde{\phi}(v) - \widetilde{\phi}(w)|| \leq ||\widetilde{\phi}(v) - \widetilde{\phi}(v')|| + ||\widetilde{\phi}(v') - \widetilde{\phi}(w)|| \leq ||v - v'|| + \lambda||v' - w|| \leq ||v - w|| + \lambda||v - w|| = (\lambda + 1)||v - w||$, where the last inequality holds since in the triangle with vertices v', v, w, the angle at v' is obtuse. An entirely similar argument applies for $\widetilde{\phi^{-1}} = \widetilde{\phi}^{-1}$ and so $\widetilde{\phi}|_{\mathbb{R}^n_0}$ is bi-Lipschitz with bi-Lipschitz constant $(1 + \lambda)$. As observed already, this implies that $\widetilde{\phi}$ is bi-Lipschitz.

It is readily verified that ρ is a homomorphism of groups. Since $\widetilde{\phi}|_{\mathbb{S}^{n-1}} = \phi$ we see that it is a monomorphism. As for the last statement, choose $x \in \mathbb{S}^{n-1}$ such that $\phi(x) \neq x$. Then $||\widetilde{\phi}(rx) - rx|| = r||\phi(x) - x|| \to \infty$ as $r \to \infty$ and so $||\widetilde{\phi} - id|| = \infty$. □

2.2.2 Embeddings of Bilip($\mathbb{D}^n, \mathbb{S}^{n-1}$) into Bilip($\mathbb{R}^n$)

For any integer $k \geq 0$, let $G_k = \mathrm{Homeo}(D_k, \partial D_k)$ where $D_k \subset \mathbb{R}^n$ is the closed unit disk centred at ke_1. Thus $D_0 = \mathbb{D}^n$ and $G_0 = \mathrm{Homeo}(\mathbb{D}^n, \mathbb{S}^{n-1})$. The group G_k will be regarded as a subgroup of $\mathrm{Homeo}_\kappa(\mathbb{R}^n)$ whenever it is convenient to do so. Note that $\Phi_k : G_0 \to G_k$, defined as $f \mapsto \tau_k f \tau_k^{-1}$ where τ_k is the translation $v \mapsto v + ke_1$ is an isomorphism of groups. Also, if $h_j \in G_{2j}, j \geq 0$, then $h_k \circ h_l = h_l \circ h_k$ whenever $k \neq l$ and the infinite composition $h_0 \circ h_1 \circ h_2 \circ \cdots$ is a well-defined homeomorphism of $\mathbb{R}^n$ whose support is contained in $\cup_{j\geq 0} D_{2j}$. This element will be denoted more briefly as $\prod_{j\geq 0} h_j$. Explicitly,

$$\prod_{j\geq 0} h_j(v) = \begin{cases} h_j(v), & v \in D_{2j}, \\ v, & v \notin \cup_{j\geq 0} D_{2j}. \end{cases}$$

One has an embedding $\Phi : G_0^\omega \cong \prod_{k\geq 0} G_{2k} \hookrightarrow \mathrm{Homeo}(\mathbb{R}^n)$ defined as

$$\Phi((g_j)) = \prod_{j\geq 0} \Phi_{2j}(g_j).$$

for $(g_j) \in G_0^\omega$.

Let $\delta : G_0 \to G_0^\omega$ be the diagonal embedding. We note that if $H \subset G_0$ is a group of bi-Lipschitz homeomorphisms, then $\Phi(\delta(H)) \subset \mathrm{Bilip}(\mathbb{R}^n)$. If $H \subset G_0 \cap \mathrm{Diff}^r_\kappa(\mathbb{B}^n)$, then $\Phi(\delta(H)) \subset \mathrm{Diff}^r(\mathbb{R}^n)$.

In general $\Phi((g_j))$ is not bi-Lipschitz even if $g_j \in G_j$ is bi-Lipschitz for every j. However, if the $g_j, j \geq 0$, are *uniformly bi-Lipschitz*, that is, if there exists a $\lambda \geq 1$ such that $\lambda^{-1}||x-y|| \leq ||g_j(x) - g_j(y)|| \leq \lambda||x-y||$ $\forall x, y \in \mathbb{R}^n$ for *every* $j \geq 0$, then $\Phi((g_j))$ is bi-Lipschitz.

For any $(g_j) \in G_0^\omega$, the homeomorphism $\Phi((g_j))$ is quasi-isometrically equivalent to the identity since $\mathrm{Fix}(\Phi((g_j)))$ is 1-dense in $\mathbb{R}^n$. So we modify
$\Phi : G_0^\omega \to \mathrm{Homeo}(\mathbb{R}^n)$.

Let $\rho_j : \mathbb{R}^n \to \mathbb{R}^n$ be defined as $\rho_j(v) = 4^j e_1 + 2^j v, j \geq 1$. Then
$\rho_j(\mathbb{D}^n) = D(4^j e_1, 2^j) =: C_j$. It is convenient to set $C_0 := \mathbb{D}^n$ and $\rho_0 = id_{\mathbb{R}^n}$. We note that $C_i \cap C_j = \emptyset$ if $i > j \geq 0$. If $g \in G_0$, then $\rho_j g \rho_j^{-1}$ has support in C_j and so $\rho_i g \rho_i^{-1}$ and $\rho_j g \rho_j^{-1}$ commute if $i \neq j$. Therefore $g \mapsto \prod_{j\geq 0} \rho_j g \rho_j^{-1}$ is a well-defined homomorphism $\Psi : G_0 \to \mathrm{Homeo}(\mathbb{R}^n)$. It is evident that Ψ is a monomorphism since $\Psi(g)|_{\mathbb{D}^n} = g$.

Lemma 2.2.2. *With the above notations, let $g \in G_0$. Then the following statements hold:*

(i) if $g \in G_0$ is non-trivial, then $||\Psi(g) - id|| = \infty$,

(ii) if $g \in \mathrm{Bilip}(\mathbb{D}^n, \mathbb{S}^{n-1})$, then $\Psi(g) \in \mathrm{Bilip}(\mathbb{R}^n)$,

Proof. (i) Choose $x_0 \in \mathbb{B}$ such that $g(x_0) \neq x_0$. If $N \geq 1$ is any integer, choose an integer k so large that $2^k||g(x_0) - x_0|| > N$. Then
$||\Psi(g)(2^k x_0 + 4^k e_1) - 2^k x_0 - 4^k e_1|| = ||\rho_k g(x_0) - 2^k x_0 - 4^k e_1|| = ||2^k g(x_0) - 2^k x_0|| > N$.
This proves (i).

(ii) Suppose that $\lambda \geq 1$ be a bi-Lipschitz constant for g. We claim that λ is also a bi-Lipschitz constant for $\psi(g)$. To see this, first let $x, y \in D(4^j e_1, 2^j)$. Then
$x_0 := \rho_j^{-1}(x) = (x - 4^j e_1)/2^j \in \mathbb{D}$ and so $g(x_0) \in \mathbb{D}$. Similarly $g(y_0) \in \mathbb{D}$ where
$y_0 := \rho_j^{-1}(y) = (y - 4^j e_1)/2^j$. Therefore
$\Psi(g)(x) - \psi(g)(y) = \rho_j g(x_0) - \rho_j g(y_0) = 2^j g(x_0) - 2^j g(y_0) = 2^j(g(x_0) - g(y_0))$.
Since $\lambda^{-1}||x_0 - y_0|| \leq ||g(x_0) - g(y_0)|| \leq \lambda||x_0 - y_0||$, multiplying throughout by 2^j we see that
$\lambda^{-1}||x - y|| = 2^j\lambda^{-1}||x_0 - y_0|| \leq ||\Psi(g)(x) - \Psi(g)(y)|| \leq \lambda 2^j||x_0 - y_0|| = \lambda||x - y||$.

If $x_1, x_2 \in \mathbb{R}^n$ are fixed by $\Psi(g)$, then, trivially $||\Psi(g)(x_1) - \Psi(g)(x_2)|| = ||x_0 - y_0||$.
Suppose that $x_0 \in C_j, y_0 \in C_k, j \neq k$. The straight line segment joining x_0, y_0 meets ∂C_j and ∂C_k at unique points x_1, x_2 respectively and we have
$\psi(g)(x_i) = x_i, i = 1, 2$. So, $||\Psi(g)(x_0) - \Psi(g)(y_0)|| \leq$
$||\Psi(g)(x_0) - \Psi(g)(x_1)|| + ||\Psi(g)(x_1) - \Psi(g)(x_2)|| + ||\Psi(g)(x_2) - \Psi(g)(y_0)|| \leq$
$\lambda||x_0 - x_1|| + ||x_1 - x_2|| + \lambda||x_2 - y_0|| \leq \lambda(||x_0 - x_1|| + ||x_1 - x_2|| + ||x_2 - y_0||) = \lambda||x_0 - y_0||$.
Similarly, $\lambda^{-1}||x_0 - y_0|| \leq ||\Psi(g)(x_0) - \Psi(g)(y_0)||$. □

2.2.3 Proof of Theorem 1.1.1

We denote by $\mathrm{Diff}^r(M)$ the group of all C^r-diffeomorphisms of a smooth manifold M where $1 \leq r \leq \infty$. The group of compactly supported homeomorphisms of M will be denoted $\mathrm{Homeo}_\kappa(M)$. If $G(M)$ is a group of homeomorphisms of M, the subgroup $\mathrm{Homeo}_\kappa(M) \cap G(M) \subset G(M)$ will be denoted $G_\kappa(M)$. If M is a manifold with boundary, $\mathrm{Diff}^r(M, \partial M)$ denotes the subgroup of $\mathrm{Diff}^r(M)$ that fixes the boundary ∂M pointwise. If M is a Riemannian manifold, $\mathrm{Diff}^r_{\mathrm{b}}(M)$ denotes the subgroup of $\mathrm{Diff}^r(M)$ consisting of those diffeomorphisms $\phi : M \to M$ such that the norms $||T_x\phi||, ||T_x\phi^{-1}||$ of the differentials $T_x\phi : T_xM \to T_{\phi(x)}M$ are uniformly bounded in the following sense: There exists a real number $\lambda = \lambda(\phi) > 1$ such that $\lambda^{-1} \leq ||T_x\phi||, ||T_x\phi^{-1}|| \leq \lambda$ for all $x \in M$. Note that $\mathrm{Diff}^r_{\mathrm{b}}(M)$ contains $\mathrm{Diff}^r_\kappa(M)$, the group of all compactly supported diffeomorphisms of M. If $f : M \to M$ is a C^r-self-map of a Riemannian manifold, we define $||f||$ to be $||f|| = \sup_{x\in M} ||T_xf||$ if it is finite, otherwise we set $||f|| = \infty$. If both $||f||$ and $||f^{-1}||$ are finite, then, it is easy to see that both f and f^{-1} are bi-Lipschitz. Hence, for a compact Riemannian manifold M with boundary ∂M, possibly empty, we have $\mathrm{Diff}^r(M) \subset \mathrm{Bilip}(M)$ and $\mathrm{Diff}^r(M, \partial M) \subset \mathrm{Bilip}(M, \partial M)$.

We are now ready to prove Theorems 1.1.1.

Proof of Theorem 1.1.1: Since $\mathcal{QI}(\mathbb{Z}^n) \cong \mathcal{QI}(\mathbb{R}^n)$, we need only obtain embeddings into $\mathcal{QI}(\mathbb{R}^n)$.

(i) One has a well-defined homomorphism $\eta : \mathrm{Bilip}(\mathbb{R}^n) \to \mathcal{QI}(\mathbb{R}^n)$ defined as $f \mapsto [f]$. The kernel of this homomorphism is the subgroup $\{f \in \mathrm{Bilip}(\mathbb{R}^n) \mid ||f - id|| < \infty\}$ By Lemma 2.2.1, we have an embedding $\rho : \mathrm{Bilip}(\mathbb{S}^{n-1}) \to \mathrm{Bilip}(\mathbb{R}^n)$ defined as $\phi \mapsto \widetilde{\phi}$ where $||\widetilde{\phi} - id|| = \infty$ if $\phi \neq id$. Hence $\eta \circ \rho$ is a monomorphism. Since elements of $\mathrm{Diff}^r(\mathbb{S}^{n-1}), 1 \leq r \leq \infty$, are bi-Lipschitz as mentioned above, it follows that $\mathrm{Diff}^r(\mathbb{S}^{n-1})$ are subgroups of

Bilip($\mathbb{S}^{n-1}$). This proves (i).

(ii) By Lemma 2.2.2, one has an embedding $\Psi : \text{Bilip}(\mathbb{D}^n, \mathbb{S}^{n-1}) \to \text{Bilip}(\mathbb{R}^n)$. Since $||\Psi(g) - id|| = \infty$ for $g \neq id$, the composition $\eta \circ \Psi : \text{Bilip}(\mathbb{D}^n, \mathbb{S}^{n-1}) \to \mathcal{QI}(\mathbb{R}^n)$ is a monomorphism. Since $\mathbb{D}^n$ is convex, the Riemannian metric on it induced from the Euclidean metric on $\mathbb{R}^n$ is the same as the restriction of the Euclidean metric. In particular, any $f \in \text{Diff}^r(\mathbb{D}^n, \mathbb{S}^{n-1})$ is bi-Lipschitz. That is, $\text{Diff}^r(\mathbb{D}^n, \mathbb{S}^{n-1})$ is a subgroup of $\text{Bilip}(\mathbb{D}^n, \mathbb{S}^{n-1})$ and our assertion follows. □

(iii) Since $\text{Diff}^r_\kappa(\mathbb{R}^n) \cong \text{Diff}^r_\kappa(\mathbb{B}^n)$ and since the latter group is naturally a subgroup of $\text{Diff}^r(\mathbb{D}^n, \mathbb{S}^{n-1})$ the assertion follows from (ii). □

2.3 Spiral group

Let $f : \mathbb{R}_{>0} \to \text{SO}(n)$ be a continuous map. We write $f(t) = (f_{ij}(t))$. Define $\phi = \phi_f : \mathbb{R}^n_0 \to \mathbb{R}^n_0$, as $\phi(x) = f(||x||)(x)$. It is readily verified that ϕ is a homeomorphism which maps each sphere centred at the origin to itself. It extends to a homeomorphism, again denoted ϕ_f, of $\mathbb{R}^n$ that fixes the origin. Also $f \mapsto \phi_f$ is a homomorphism of groups $\phi : \text{Maps}(\mathbb{R}_{>0}, \text{SO}(n)) \to \text{Homeo}(\mathbb{R}^n)$ where the group structure on $\text{Maps}(\mathbb{R}_{>0}, \text{SO}(n))$ is obtained by pointwise operations. It is in fact a monomorphism since f can be recovered from ϕ_f by restricting ϕ_f to the positive ray $\mathbb{R}_{>0}e_1$. We shall refer to the image of $\text{Maps}(\mathbb{R}_{>0}, \text{SO}(n))$ as the *spiral group* of $\mathbb{R}^n$ and denote it by Spiral($\mathbb{R}^n$).

2.3.1 Proof of Theorem 1.1.2

We will use the following norms for $n \times n$-matrices over $\mathbb{R}$. If $A = (a_{ij})$, define $||A||_E := (\sum a_{ij}^2)^{1/2}$. As usual, denoting the operator norm of A by $||A|| = \sup_{||x||=1} ||Ax||$, we have $||A|| \leq ||A||_E \leq \sqrt{n}||A||$. Also

$||Ax|| \leq ||A||.||x|| \leq ||A||_E.||x||$.

Lemma 2.3.1. *With notations as above, suppose that $f : \mathbb{R}_{>0} \to SO(n)$ is C^1 and satisfies the following condition: there exist a constant $C = C(f) \geq 0$ such that, for all $1 \leq i, j \leq n$, $|f'_{ij}(t)| \leq C/t\ \forall t > 0$. Then $\phi_f : \mathbb{R}^n \to \mathbb{R}^n$ is bi-Lipschitz.*

Proof. Let $x, y \in \mathbb{R}^n$. Suppose that $r := ||x|| = ||y||$. Then
$||\phi(x) - \phi(y)|| = ||f(r)x - f(r)y|| = ||x - y||$.

Suppose that $s := ||y|| > ||x|| = r$, $x \neq 0 \neq y$. Set $A := f(r), B := f(s)$. Then
$||\phi(x) - \phi(y)|| = ||Ax - By|| \leq ||Ax - Bx|| + ||Bx - By|| \leq ||A - B||.||x|| + ||x - y||$.

We have $B - A = (f_{ij}(s) - f_{ij}(r)) = (f'_{ij}(t_{ij})).(s - r)$ for some $t_{ij} \in (r, s)$ by the mean value theorem. By our hypothesis, $|f'_{ij}(t_{ij})| \leq C/t_{ij} \leq C/r$. So, using the norm $||X||_E = \sqrt{\sum_{i,j} |x_{ij}|^2}$ for $X = (x_{ij}) \in M_n(\mathbb{R})$, we obtain that
$||B - A||_E^2 = ||(f'_{ij}(t_{ij}))||_E^2(s - r)^2 \leq n^2C^2/t_{ij}^2 \cdot (s - r)^2 \leq n^2C^2(s - r)^2/r^2$. This implies that $||Bx - Ax|| \leq nC(s - r)||x||/r = nC(||y|| - ||x||) \leq nC||y - x||$. So
$||\phi(x) - \phi(y)|| \leq (nC + 1)||x - y||\ \forall x, y \in \mathbb{R}^n$.

Let $g(t) = f(t)^{-1} \in SO(n)$. Then $g_{ij} = f_{ji}, 1 \leq i, j \leq n$, and so g also satisfies the condition $|g'_{ij}(t)| \leq C/t\ \forall t > 0$. Note that $\psi := (\phi_f)^{-1} = \phi_g$ and so applying the above consideration to $\psi = \phi_g$ we obtain $||\psi(a) - \psi(b)|| \leq (nC + 1)||a - b||$; equivalently $||x - y||/(nC + 1) \leq ||\phi(x) - \phi(y)||$ for all $x, y \in \mathbb{R}^n$. □

Straightforward verification shows that the space of all C^1-maps $f : \mathbb{R}_{>0} \to \mathrm{SO}(n)$ satisfying the hypothesis of the above lemma forms a group under pointwise operations. We shall denote this group by $\mathcal{R}$. Let
$N := \{f \in \mathcal{R} \mid \lim_{r\to\infty} f(r) = I_n\}$ and let
$K = \{f \in \mathcal{R} \mid f(t) = I_n\ \forall t \geq b, \text{ for some } b > 0\}$. Then N and K are subgroups of $\mathcal{R}$. Indeed it can be seen that both N and K are normal subgroups of $\mathcal{R}$.

In view of the above lemma, one has a well-defined map $\Phi : \mathcal{R} \to \mathcal{QI}(\mathbb{R}^n)$ where

$\Phi(f) := [\phi_f]$. A routine verification shows that Φ is a homomorphism of groups. An element $f \in \mathcal{R}$ is in the kernel of Φ if and only if there exists a constant $M = M(f) > 1$ such that, for all $r \geq M$, we have $||f(r)(v) - v|| < M$ for all v in $S_r \subset \mathbb{R}^n$, the sphere of radius r centred at the origin. The following lemma allows us express the inequality in terms of the eigenvalues of $f(r) \in \mathrm{SO}(n)$.

Lemma 2.3.2. *Let $r \geq M > 1$. Let $A \in \mathrm{SO}(n)$. Then: $||Av - v|| < M$ for all $v \in S_r$ if and only if $|1 - \lambda| < M/r$ for any eigenvalue λ of A.*

Proof. If -1 is an eigenvalue of A, then choosing an eigenvector $v \in S_r$ corresponding to -1 we have $||Av - v|| = 2r > M$. On the other hand, $|1 - \lambda| < M/r \leq 1$ implies that, if $\lambda = \exp(i\theta)$, then $\theta \in (-\pi/3, \pi/3)$ and in particular $\lambda \neq -1$. So we assume that -1 is not an eigenvalue of A.

Set $V_0 := \mathrm{Fix}(A)$ and let $V = V_0^{\perp} \subset \mathbb{R}^n$. Since -1 is not an eigenvalue of A, we see that $\dim V$ is even. Let $\dim V = 2m$ and let $V_1, \cdots, V_m$ be two-dimensional pairwise orthogonal subspaces of V such that $A(V_j) = V_j$ and the eigenvalues of $A|_{V_j}$ are $\lambda_j, \bar{\lambda}_j \neq 1$. Thus, writing $\lambda_j = \exp(i\theta_j)$ with $\theta_j \in (-\pi, \pi)$, given any $v \in V_j$, Av is got by rotating v by angle θ_j about the origin in the plane V_j. So $||Av - v|| = 2|\sin(\theta_j/2)|.||v|| = |1 - \lambda_j|.||v||$.

Now, given any $v \in S_r$, write $v = v_0 + v_1 + \cdots + v_m, v_j \in V_j$. Then $||Av - v||^2 = \sum_{1 \leq j \leq m} ||Av_j - v_j||^2 = \sum_{1 \leq j \leq m} |1 - \lambda_j|^2 ||v_j||^2 \leq |1 - \lambda_p|^2.r^2$ where p is such that $|1 - \lambda_j| \leq |1 - \lambda_p|\ \forall j$. The upper bound $|1 - \lambda_p|^2 r^2$ is attained when $|1 - \lambda_j| = |1 - \lambda_p|$ for all j and $v \in V_j$. Therefore $||Av - v|| < M\ \forall v \in S_r$ if and only if $|1 - \lambda_j| < M/r\ \forall j$. □

We are ready to prove Theorem 1.1.2.

Proof of Theorem 1.1.2: Let $\mathcal{K} \subset \mathcal{R}$ be the set of all $f \in \mathcal{R}$, satisfying the following condition: there exists a constant $M = M(f) > 1$ such that for any $r \geq M$ and for any eigenvalue λ of $f(r) \in \mathrm{SO}(n)$, we have $|1-\lambda| < M/r$. By the above lemma, $\mathcal{K}$ equals the kernel of Φ. It is clear that $K \subset \mathcal{K}$. Also, if $f \in \mathcal{K}$, the existence of a constant M such that the eigenvalues λ of $f(r)$ satisfy the inequality $|1-\lambda| < M/r$ for all $r \geq M$ implies that the eigenvalues of $f(r)$ tend to 1 as r goes to ∞. In view of the compactness of $\mathrm{SO}(n)$, we must have $\lim_{r\to\infty} f(r) = I$. So $\mathcal{K} \subset N$. $\square$

Chapter 3

Twisted conjugacy in general linear groups over $\mathbb{F}_q[t]$ and $\mathbb{F}_q[t, t^{-1}]$.

We begin this chapter by fixing some notations. Throughout this chapter (unless mentioned otherwise), F denotes any subfield of algebraic closure $\bar{\mathbb{F}_p}$ of $\mathbb{F}_p$, where p is a prime, $R = F[t]$ or $F[t, t^{-1}]$ and $A = \mathbb{F}_q[t]$ or $\mathbb{F}_q[t, t^{-1}]$, where $q = p^e, e > 1$.

We discuss several properties such as generators, commutator subgroups, structures, automorphisms of the groups $\mathrm{GL}_n(R)$ or $\mathrm{SL}_n(R)$ for $n \geq 2$ in the first few sections of this chapter. The groups $\mathrm{GL}_n(R)$ or $\mathrm{SL}_n(R)$ behave very differently depending on whether $n \geq 3$ or $n = 2$. We consider the $n \geq 3$ case in 3.1. In case of $n = 2$, the behaviour of the groups under consideration depend heavily on R. We discuss the case of $n = 2$ and $R = F[t]$ in 3.2 and the case of $n = 2$ and $R = \mathbb{F}_q[t, t^{-1}]$ in 3.3.

Finally in 3.4, we first state some basic facts about twisted conjugacy in 3.4.1 and then prove theorem 1.2.1 and theorem 1.2.2. The proofs are divided into several cases and given in several subsections of §3.4.

3.1 The groups $\mathbf{GL}_n(R), \mathbf{SL}_n(R)$ when $n \geq 3$

In this section, we discuss about generators 3.1.1, commutator subgroup 3.1.2 and automorphisms 3.2.3 of the groups $\mathrm{GL}_n(R), \mathrm{SL}_n(R)$ for $n \geq 3$. The description of the automorphisms of these groups is due to O. T. O'Meara [O].

3.1.1 A generating set

Let $E_n(R) \subset \mathrm{GL}_n(R)$ denote the subgroup generated by the elementary matrices $e_{ij}(\lambda), \lambda \in R, 1 \leq i, j \leq n, i \neq j$. By definition, $e_{ij}(\lambda)$ is the matrix whose diagonal entries are 1, the (i, j)-th entry is λ and all other entries are zero. These matrices are also known as *transvection matrices*.

One has the obvious inclusions $E_n(R) \subset \mathrm{SL}_n(R) \subset \mathrm{GL}_n(R)$. As R is a Euclidean domain, we have $E_n(R) = \mathrm{SL}_n(R)$ for $n \geq 2$; see [Ro, Theorem 2.3.2].

From now on, we will only focus on the case when $n \geq 3$.

We note that the commutator $[e_{ik}(x), e_{kj}(y)] = e_{ij}(xy)$ if i, j, k are all distinct and $e_{ij}(x)e_{ij}(y) = e_{ij}(x + y)\ \forall x, y \in R$. It follows that, since $n \geq 3$,
$\mathrm{SL}_n(F[t]) = \langle e_{ij}(t), e_{ij}(a) : a \in F, 1 \leq i, j \leq n, i \neq j\rangle$ and
$\mathrm{SL}_n(F[t, t^{-1}]) = \langle e_{ij}(t), e_{ij}(t^{-1}), e_{ij}(a) : a \in F, 1 \leq i, j \leq n, i \neq j\rangle$. In particular, $\mathrm{SL}_n(R)$ is finitely generated if F is finite. We also observe that
$\mathrm{GL}_n(F[t]) = \langle e_{ij}(t), e_{ij}(a), \mathrm{diag}(1, \ldots, 1, b) : a \in F, b \in F^\times, 1 \leq i, j \leq n, i \neq j\rangle$ and
$\mathrm{GL}_n(F[t, t^{-1}]) = \langle e_{ij}(t), e_{ij}(t^{-1}), e_{ij}(a), \mathrm{diag}(1, \ldots, 1, b), \mathrm{diag}(1, \ldots, 1, t) : a \in F, b \in F^\times, 1 \leq i, j \leq n, i \neq j\rangle$, which is true for any field F. In particular, $\mathrm{GL}_n(R)$ is finitely generated if F is finite.

3.1.2 Commutator subgroup

We begin by recalling our assumption $n \geq 3$. Then we have the commutator relation $[e_{ik}(x), e_{kj}(1)] = e_{ij}(x)$ if i, j, k are all distinct and $\forall x \in R$, which implies that $E_n(R) = [E_n(R), E_n(R)]$. Hence, $\mathrm{SL}_n(R)$ is perfect and $[\mathrm{GL}_n(R), \mathrm{GL}_n(R)] = \mathrm{SL}_n(R)$. This holds true for any integral domain R and $n \geq 3$.

3.1.3 Automorphisms

Let $n \geq 3$ and let $G = \mathrm{SL}_n(R)$ or $\mathrm{GL}_n(R)$. An automorphism $\phi : G \to G$ is called *standard* if it is in the subgroup generated by the following four types of automorphisms:

(i) Conjugation by $g \in \mathrm{GL}_n(R)$, denoted $\iota_g : G \to G$, defined as $x \mapsto gxg^{-1}$. It is inner if $g \in G$.

(ii) Automorphisms of G induced by automorphisms of the ring R. We make no distinction in the notation between the automorphism of the ring R and the induced automorphism of G.

(iii) Automorphisms which are homotheties. Recall that a homomoprhism $\mu = \mu_\chi : G \to G$ is a *homothety* if there is a character $\chi : G \to R^\times$ ($R^\times$ is the group of units of R) such that $\mu_\chi(x) = \chi(x)x$. Since $x, \chi(x)x \in G$ it follows that $\chi(x)I_n \in G$. Being a scalar matrix, $\chi(x)I_n$ belongs to the centre of G. A homothety μ_χ fails to be injective if and only if there exists a central element $z = zI_n \in G$ other than I_n such that $\chi(z) = z^{-1}$.

(iv) Contragradient automorphism $\epsilon : G \to G$ defined as $x \mapsto {}^t x^{-1}$, $\forall x \in G$.

O. T. O'Meara [O] has shown that any automorphism of G is standard.

Theorem 3.1.1. (O'Meara [O]) *Suppose F denotes any subfield of algebraic closure $\bar{\mathbb{F}}_p$ of $\mathbb{F}_p$, where p is a prime, $R = F[t]$ or $F[t, t^{-1}]$ and $G = GL_n(R)$ or $SL_n(R)$ where $n \geq 3$. Then any automorphism $\phi : G \to G$ can be expressed as* *follows:*

$$\phi = \mu_\chi \circ \rho \circ \iota_g \quad or \quad \phi = \mu_\chi \circ \rho \circ \iota_g \circ \epsilon$$

where μ_χ is a homothety automorphism corresponding to a character $\chi : G \to R^\times, g \in GL_n(R), \rho : G \to G$ is induced by a ring automorphism denoted by the same symbol $\rho : R \to R$, and, ϵ is the contragradient $x \mapsto {}^t x^{-1}$.

Remark 3.1.2. *O'Meara's theorem holds for a much more general setup. It describes automorphisms of $GL_n(R)$ and $SL_n(R)$, where R is a commutative integral domain that is not a field and $n \geq 3$. In this general setup, the element g in ι_g comes from $GL_n(F)$, where F is the fraction field of R. From 5.5 Theorem B, 5.6 Theorem C in [O] and the fact that $R = F[t]$ or $F[t, t^{-1}]$ is a PID, it follows that in our case g in ι_g can be taken to vary in $GL_n(R)$.*

Commutation relations.

We have the following commutation relations:

(i) $\rho \circ \iota_g = \iota_{\rho(g)} \circ \rho$,

(ii) $\epsilon \circ \rho = \rho \circ \epsilon$,

(iii) $\epsilon \circ \iota_g = \iota_{\epsilon(g)} \circ \epsilon$,

(iv) $\mu_\chi \circ \rho = \rho \circ \mu_\eta$ where $\eta = \rho_0^{-1} \circ \chi \circ \rho : G \to R^\times$. Here $\rho_0 : R^\times \to R^\times$ is defined by $\rho : R \to R$.

(v) $\mu_{-\chi} \circ \epsilon = \epsilon \circ \mu_{\chi \circ \epsilon}$ where $-\chi(g) = \chi(g)^{-1}$ $\forall g \in G$.

Ring automorphisms.

We shall assume that R is one of the rings $F[t], F[t, t^{-1}]$ where F is a subfield of $\bar{\mathbb{F}}_p$. An automorphisms $\rho : R \to R$ restricts to an automorphism of F. Note that for any $x \in F$, $\rho(x) = x^{p^r}$ for some $r \geq 1$ where the value of r may depend on x. The group $\mathrm{Aut}(F)$ is a finite cyclic group generated by $x \to x^p$ if F is finite. When

$F \subset \bar{\mathbb{F}}_p$ is infinite, it is the inverse limit of $\{\mathrm{Aut}(E)\}$ where E varies over finite subfields of F.

When $R = F[t]$, $\rho(t) = at + b$ where $a, b \in \mathbb{F}_q \subset F$ with $q = p^e$ for some $e \geq 1$, $a \neq 0$. It is readily seen that $\mathrm{Aut}(R) = \mathrm{Aff}(R) \rtimes \Phi$ where $\Phi = \mathrm{Aut}(F)$ and $\mathrm{Aff}(R)$ denotes the group of F-algebra automorphism of R defined by $t \mapsto at + b, a \in F^\times, b \in F$.

When $R = F[t, t^{-1}]$, the automorphisms can be factored as $\rho_0 \circ \rho_1$ where $\rho_1(t) = t$ and ρ_0 is an F-algebra automorphism such that $\rho_0(t) = \lambda t^\varepsilon$ where $\lambda \in F^\times, \varepsilon \in \{1, -1\}$. It is readily seen that $\mathrm{Aut}(R) \cong (F^\times \rtimes \Phi) \times \mathbb{Z}/2\mathbb{Z}$ where $\Phi = \mathrm{Aut}(F)$.

Let $\rho : R \to R$ be any automorphism. Suppose that $\rho(t) = at^\varepsilon + b$ where $\varepsilon \in \{1, -1\}, a \in F^\times, b \in F$. It is understood that $b = 0$ when $R = F[t, t^{-1}]$ and that $\varepsilon = 1$ when $R = F[t]$. Since a Frobenius automorphism stabilizes finite subfields of F, it follows that $\mathbb{F}_q[t]$ (resp. $\mathbb{F}_q[t, t^{-1}]$) is mapped to itself by ρ provided $a, b \in \mathbb{F}_q \subset F$. Let $s \in R$ be the product $\prod_{\alpha \in \mathbb{F}_q^\times, \beta \in \mathbb{F}_q} (\alpha t + \beta) = \prod_{\gamma \in \mathbb{F}_q} (t + \gamma)^{q-1}$ when $R = F[t]$ and $s = t^{q-1} + t^{1-q}$ when $R = F[t, t^{-1}]$. Then $\rho(s) = s$. Also, $\mathbb{F}_p \subset \mathbb{F}_q$ is fixed under all Frobenius automorphisms. It follows that the polynomial algebra $\mathbb{F}_p[s] \subset R$ is fixed under ρ in each of the cases $R = F[t], F[t, t^{-1}]$.

By our choice of q, if $\mathbb{F}_q \subset \mathbb{F}_\ell \subset F$, we see that $R_\ell := \mathbb{F}_\ell[t] \subset F[t]$ (resp. $R_\ell := \mathbb{F}_\ell[t, t^{-1}] \subset F[t, t^{-1}]$) is stable by ρ. Consequently, the groups $G_\ell := \mathrm{GL}_n(R_\ell), \mathrm{SL}_n(R_\ell)$ are also stable by ρ. We note that G is the union of the groups G_ℓ.

Homothety.

We note that $\mathrm{SL}_n(R)$ is perfect (§3.1.2) and so it has no non-trivial character. Consequently, any homothety automorphism of $\mathrm{SL}_n(R)$ is trivial.

Now consider $\mathrm{GL}_n(R), n \geq 3$, where $R = F[t, t^{-1}]$ and $\mu = \mu_\chi$ a homothety automorphism of $\mathrm{GL}_n(R)$ correspoding to a character $\chi : \mathrm{GL}_n(R) \to R^\times = F^\times \times \langle t \rangle$. *We claim that* $Im(\chi) \subset F^\times$. To see this, we suppose that $\chi(a) = \lambda t^k$ for some $a \in \mathrm{GL}_n(R)$ and $k \geq 1$. Note that $\mathrm{GL}_n(R) = \mathrm{SL}_n(R).H$ where the subgroup $H \cong R^\times$ consisting of diagonal matrices $h(\alpha) = \mathrm{diag}(1, \ldots, 1, \alpha)$ with $\alpha \in R^\times$. Since $\mathrm{SL}_n(R)$ is perfect as $n \geq 3$, we see that χ is trivial on $\mathrm{SL}_n(R)$. So, we may assume that $a = h(\alpha) \in H$ with $\alpha = \alpha_0.t^c, c \neq 0$. This implies that $\chi(h(t)) = \lambda t^m$ for some $m \neq 0$ and $\lambda \in F^\times$. Since $\mathrm{SL}_n(R)$ is characteristic in $\mathrm{GL}_n(R)$, we have the following commuting diagram with exact rows in which $\mu', \bar{\mu}$ are automorphisms defined by μ:

$$\begin{array}{ccccccccc} 1 & \to & \mathrm{SL}_n(R) & \to & \mathrm{GL}_n(R) & \stackrel{\det}{\to} & R^\times & \to & 1 \\ & & \mu' \downarrow & & \downarrow \mu & & \downarrow \bar{\mu} & & \\ 1 & \to & \mathrm{SL}_n(R) & \to & \mathrm{GL}_n(R) & \stackrel{\det}{\to} & R^\times & \to & 1 \end{array}$$

We have $\bar{\mu}\det(h(t)) = \bar{\mu}(t) = \gamma t^{\pm 1}$ for some $\gamma \in F^\times$. The commutativity of the above diagram leads to $\gamma t^{\pm 1} = \bar{\mu}(\det(h(t))) = \det(\mu(h(t))) = \det(\chi(h(t)).h(t)) = \det(\lambda t^m I_n).\det(h(t)) = \lambda^n.t^{mn+1}$. It follows that $\gamma = \lambda^n$ and $mn + 1 = \pm 1$. Since $n \geq 3$, we conclude that $\mathrm{Im}(\chi) \subset F^\times$.

As a consequence, we have the following

Corollary 3.1.3. *Let* $n \geq 3$ *and* $R = F[t], F[t, t^{-1}]$ *where* $F \subset \bar{\mathbb{F}}_p$.

(i) Any outer automorphism of $SL_n(R)$ *is represented by one of the following automorphisms:*

(a) $\iota_{h(\alpha)} \circ \rho, \iota_{h(\alpha)} \circ \rho \circ \epsilon, \alpha \in R^\times$ *where* $h(\alpha) = diag(1, \ldots, 1, \alpha) \in GL_n(R)$. *It is of finite order when* $\alpha \in F^\times$.

(ii) Any outer automorphism of $GL_n(R)$ *is represented by one of the following automorphisms:*

(a) $\mu_\chi \circ \rho, \mu_\chi \circ \rho \circ \epsilon$, *where* χ *is a suitable character* $\chi : GL_n(R) \to F^\times$. □

3.2 The groups $\mathbf{GL}_2(F[t])$ and $\mathbf{SL}_2(F[t])$

In this section, we denote $R := F[t]$ and F is a subfield of the algebraic closure of $\mathbb{F}_q$. We describe the amalgamated free product structure of $\mathrm{GL}_2(F[t])$ and $\mathrm{SL}_2(F[t])$ in 3.2.1 due to Nagao [N]. We prove that these groups are infinitely generated 3.2.2 and then discuss about commutator subgroups 3.2.3 and automorphisms 3.2.4 of these groups. As for the automorphism group of $\mathrm{GL}_2(F[t])$, Reiner [R] described a set of generators of $\mathrm{Aut}(\mathrm{GL}_2(F[t]))$. We partially extend his results to the case of $\mathrm{SL}_2(F[t])$ and obtain a set of representatives for the elements of its outer automorphisms.

3.2.1 Structure

The groups $\mathrm{GL}_2(F[t])$ and $\mathrm{SL}_2(F[t])$ both have amalgamated free product structures. See [N] and also [Se], Chapter II, §1.6, Theorem 6.

Let $B \subset G = \mathrm{GL}_2(F[t])$ denote the subgroup of all upper triangular matrices and let $G_0 = \mathrm{GL}_2(F)$ and set $B_0 := B \cap G_0$. Nagao has shown that $G = G_0 *_{B_0} B$. It follows that, writing $G_1 := \mathrm{SL}_2(F), B_1 = B_0 \cap G_1, B_1(F[t]) = B \cap \mathrm{SL}_2(F[t])$, we have $\mathrm{SL}_2(F[t]) = G_1 *_{B_1} B_1(F[t])$.

3.2.2 Infinite generation

In this section, we prove that $\mathrm{GL}_2(F[t])$ and $\mathrm{SL}_2(F[t])$ are not finitely generated. This is a theorem by Nagao (see [N]) and the proof given here is almost the same as that of Nagao's.

We follow the notations of 3.2.1. We first note that any element of any generating set of G, due to the amalgamated free product structure of G, can be expressed as a word in letters from G_0 and U, the subgroup of the unipotent upper triangular matrices in G. Now each element of the abelian group U can be expressed as a product of matrices from G_0 and matrices of the form $e_{12}(t^k), k \in \mathbb{N}$. If G is finitely generated, then G is generated by G_0 and $e_{12}(t^i), i \in \{1, \ldots, n-1\}$. Since $e_{12}(t^n)$ has to be a product of the matrices from G_0 and matrices of the form $e_{12}(t^i), i \in \{1, \ldots, n-1\}$, we have the following relation :

$g_1 e_{12}(f_1) g_2 e_{12}(f_2) \ldots g_k e_{12}(f_k) c = e_{12}(t^n)$, where $f_i \in F[t]$ with $f_i(0) = 0$, $g_i, c \in G_0$, $1 \leq i \leq k, deg(f_i) < n, g_i \neq Id$ for $i \neq 1$ and possibly $c \neq 1$. It implies that $e_{12}(-t^n) g_1 e_{12}(f_1) g_2 e_{12}(f_2) \ldots g_k e_{12}(f_k) c = Id$. But the left hand side is either a reduced expression or leads to a reduced expression with the leading term $e_{12}(f_1 - t^n) \notin G_0$ as $deg(f_1) < n$. This is a contradiction.

A similar proof also shows that $\mathrm{SL}_2(F[t])$ is infinitely generated.

3.2.3 Commutator subgroup

We first note that when $F = \mathbb{F}_2$, $\mathrm{GL}_2(F[t]) = \mathrm{SL}_2(F[t])$.

Let $F \neq \mathbb{F}_2$. Then F contains $\mathbb{F}_q$ where $q \geq 3$. We have we have $[\mathrm{diag}(\lambda, 1), e_{12}(x)] = e_{12}((\lambda - 1)x) \in [\mathrm{GL}_2(F), \mathrm{GL}_2(F)]$. Since $q \geq 3$, we can choose λ of $\mathbb{F}_q^\times$ so that $u := \lambda - 1 \neq 0$. Replacing x in the above by $x_1 := ux$, we see that $e_{12}(ux_1) = e_{12}(u^2 x) \in [\mathrm{GL}_2(R), \mathrm{GL}_2(R)]$. Since u has finite order, repeating this process leads to $e_{12}(x) \in [\mathrm{GL}_2(R), \mathrm{GL}_2(R)]\ \forall x \in R$. Similarly $e_{21}(x) \in [\mathrm{GL}_2(R), \mathrm{GL}_2(R)]$. Therefore $E_2(R) \subset [\mathrm{GL}_2(R), \mathrm{GL}_2(R)]$. Since $E_2(R) = \mathrm{SL}_2(R)$ and $[\mathrm{GL}_2(R), \mathrm{GL}_2(R)] \subset \mathrm{SL}_2(R)$, we have .$[\mathrm{GL}_2(R), \mathrm{GL}_2(R)] = \mathrm{SL}_2(R)$ when F contains $\mathbb{F}_q$ where $q \geq 3$.

Therefore, $\mathrm{SL}_2(F[t])$ is a characteristic subgroup of $\mathrm{GL}_2(F[t])$ for any subfield F of $\bar{\mathbb{F}}_p$.

3.2.4 Automorphisms

Let $F \subset \bar{\mathbb{F}}_p$ be a subfield. In the case $G = \mathrm{GL}_2(F[t])$, Reiner [R] showed that there are *non-standard* automorphisms. Specifically, let $U \subset G$ be the subgroup consisting of all unipotent upper triangular matrices over $F[t]$. Thus U is naturally isomorphic to the $\mathbb{F}_p$-vector space $F[t]$ where $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \in U$ corresponds to $x \in F[t]$. Suppose that ν is any F-vector space isomorphism of $F[t]$ which restricts to identity on degree-zero polynomials $F.1$. Then Reiner [R] showed that ν, regarded as an automorphism of U, extends to a unique automorphism, also denoted $\nu : \mathrm{GL}_2(F[t]) \to \mathrm{GL}_2(F[t])$ which restricts to the identity on $\mathrm{GL}_2(F) \subset \mathrm{GL}_2(F[t])$. We call such an automorphism a *Reiner automorphisms.* It turns out that the standard automorphisms, together with the Reiner automorphisms, generate the full automorphism group of G. See [R, §3]. This result follows essentially from Lemma 3.2.1 below and Lemmas 3.2.2 and 3.2.3. Our proof uses the result of Nagao that G splits as an amalgamated free product (See [N] and also [Se], Chapter II, §1.6, Theorem 6.). We will also obtain a set of generators for the automorphism group of $\mathrm{SL}_2(F[t])$.

Lemma 3.2.1. *With the notations as §3.2.1, suppose that $\phi : G \to G$ is an automorphism where $G = GL_2(F[t])$ or $SL_2(F[t])$. Then there exists a $g \in G$ such that*
(i) $\phi(GL_2(F))$ (resp. $\phi(SL_2(F))$) equals ${}^gGL_2(F)$ (resp. ${}^gSL_2(F)$) ,
(ii) $\phi(B) = {}^hB$ (resp. $\phi(B_1) = {}^hB_1$) for some $h \in {}^gG_0$ (resp. $h \in {}^gG_1$).

Proof. . We shall consider only the case of $G = \mathrm{GL}_2(F[t])$, the proof for $\mathrm{SL}_2(F[t])$ being analogous.
(i) For any finite subfield $\mathbb{F}_q$ of F, we see that $\phi(\mathrm{GL}_2(\mathbb{F}_q))$ is contained in a conjugate of one of the two factors $\mathrm{GL}_2(F)$ or B by [Se], Chapter I, §1.3. It is easily seen that $\phi(GL_2(\mathbb{F}_q))$ does not embed in B and so we obtain that $\mathrm{GL}_2(\mathbb{F}_q)$ embeds in ${}^g\mathrm{GL}_2(F)$ for some $g \in G$ that possibly depends on q. If F itself is finite,

then we may take $\mathbb{F}_q$ to be F and we are done. So assume that $F \subset \bar{\mathbb{F}}_p$ is not finite. Consider $\mathbb{F}_\ell \subset F$, an extension field of $\mathbb{F}_q$. We need only show that $\phi(\mathrm{GL}_2(\mathbb{F}_\ell))$ is contained in the *same* conjugate ${}^gG_0$ of G_0. Equivalently, we assume that $\phi(\mathrm{GL}_2(\mathbb{F}_q)) \subset G_0$ and show that $\phi(\mathrm{GL}_2(\mathbb{F}_\ell)) \subset G_0$. Suppose that $\phi(\mathrm{GL}_2(\mathbb{F}_\ell)) \subset {}^hG_0$, where $h \notin G_0$. Now ${}^hG_0 \cap G_0$ contains $\phi(\mathrm{GL}_2(\mathbb{F}_q))$. We shall presently show that ${}^hG_0 \cap G_0$ is contained in a conjugate of B_0. This is a contradiction since $\mathrm{GL}_2(\mathbb{F}_q)$ does not embed in B_0.

It remains to show that ${}^hG_0 \cap G_0$ is contained in a conjugate of B_0. Let $x \in G_0 \cap {}^hG_0$ and write $x = hx_0h^{-1}$, for some $x_0 \in G_0$. If $x_0 \in B_0$, then $x \in {}^hB_0$. So assume that for some $x \in G_0 \cap {}^hG_0$, we have $x = hx_0h^{-1}$ with $x_0 \in G_0 \setminus B_0$. Since $x, x_0 \in G_0 \setminus B_0$ and since $h \notin G_0$, by the normal form theorem [LS, Theorem 2.6, Chapter IV] we see that the element $x^{-1}hx_0h^{-1}$ is not the trivial element, a contradiction, which establishes our claim.

(ii) Replacing ϕ by $\iota_g \circ \phi$ with $g \in G$ as in (i), we may (and do) assume that $\phi(G_0) = G_0$. Moreover, since $\phi(e_{12}(1))$ is an element of order p, it is conjugate in G_0 to an element of $U_0 := U \cap G_0$. Again we may assume that $v := \phi(e_{12}(1)) \in U_0$. Set $u := \phi(e_{12}(t))$. Now $Z_G(u) = \phi(Z_G(e_{12}(t))) = \phi(Z_G(e_{12}(1))) = Z_G(v) = ZU$ where $Z_G(x)$ denotes the centralizer of x in G and Z denotes the centre of G. On the other hand $Z_G(u) = \phi(Z_G(e_{12}(t))) = \phi(ZU)$. Therefore $\phi(ZU) = ZU$. It follows that $\phi(U) = U$ and, since $B = N_G(U)$ the normalizer of U, we see that $\phi(B) = B$. □

Let $R = F[t]$. We begin by describing the automorphism groups of $B \subset \mathrm{GL}_2(R)$ and $B_1(R) = B \cap \mathrm{SL}_2(R)$. Let H denote the diagonal subgroup of B and H_1 that of $B_1(R)$. Then $B = HU, B_1(R) = H_1U$. Under the identification $U \cong R$, the conjugation action of H on U corresponds to the action of H on R given by $h.x = \lambda\mu^{-1}x$ where $h = \mathrm{diag}(\lambda, \mu) \in H$. The action of H_1 on R is obtained by restricting the H action.

Let α be an automorphism of the abelian group R. Then α is an $\mathbb{F}_p$-vector space automorphism. It extends to an automorphism of B which is identity on H if and only if, for all $x \in R$, we have $\alpha(h.x) = h.\alpha(x)$, i.e, $\alpha(\lambda\mu^{-1}x) = \lambda\mu^{-1}\alpha(x)$. Taking $\mu = 1$ we see that the latter condition is equivalent to α being F-linear.

Similarly α extends to an automorphism of $B_1(R)$ fixing H_1 element-wise if and only if $\alpha(\lambda^2 x) = \lambda^2\alpha(x)\ \forall x \in F[t], \forall \lambda \in F$. When $p = 2$, the square map $\lambda \mapsto \lambda^2$ is an automorphism of $F^\times$ and so the condition on α is equivalent to the F-linearity of α.

Suppose that p is an odd prime. Let $F' \subset F$ be the $\mathbb{F}_p$-subalgebra generated by $S = \{\lambda^2 \mid \lambda \in F^\times\}$; thus α is F'-linear. We claim that $F' = F$, that is, every element of F is a sum of square elements of F. To see this, note that, since any finite subfield $\mathbb{F}_\ell \subset F$ has at least $\ell/2$ square elements (including 0), we see that the subfield $F' \cap \mathbb{F}_\ell$ of $\mathbb{F}_\ell$ has at least $\ell/2$ elements. Since p is odd we conclude that $\mathbb{F}_\ell \subset F'$. Since $F' \subset F \subset \bar{\mathbb{F}}_p$, we conclude that $F' = F$.

The following lemma is an immediate consequence.

Lemma 3.2.2. *Let $R = F[t]$. We keep the above notations. An automorphism α of the abelian group $U \cong F[t]$ extends to an automorphism of B (resp. $B_1(R)$) that restricts to the identity on H (resp. H_1) if and only if α is F-linear.* □

Suppose that $\alpha : B \to B$ is automorphism as in the above lemma. Then it restricts to the identity on B_0 if $\alpha(\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)) = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. Similarly, if $\alpha : B_1(R) \to B_1(R)$ is an automorphism as in the above lemma, then it restricts to the identity on B_1 if and only if $\alpha(\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)) = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. Any automorphism of B that restricts to the identity on B_0 extends to a unique automorphism of G which is identity on G_0, in view of the amalgamated free product structure $G = G_0 *_{B_0} B$. These are the Reiner automorphisms of G. Analogous statement for automorphisms of $B_1(R)$ holds and we refer to the resulting automorphisms of $\mathrm{SL}_2(R)$ as the *Reiner automorphisms* of

$\mathrm{SL}_2(R)$.

Lemma 3.2.3. *Let $R = F[t]$ and let $G = GL_2(R)$ or $SL_2(R)$. Let $\phi : G \to G$ be any automorphism. Then there exists a $y \in G$ such that the automorphism $\psi := \iota_{y^{-1}} \circ \phi$ has the following properties:*
(i) $\psi(G_0) = G_0$ (resp. $\psi(G_1) = G_1$),
(ii) $\psi(B) = B$ (resp. $\psi(B_1(R)) = B_1(R)$),
(iii) $\psi(B_0) = B_0$ (resp. $\psi(B_1) = B_1$).

Proof. This is an immediate corollary of Lemma 3.2.1. In fact, the proof already established (i) and (ii). The third assertion follows since $B_0 = G_0 \cap B_1$ in the case of $\mathrm{GL}_2(F[t])$ and $B_1 = G_1 \cap B_1(F[t])$. □

3.3 The group $\mathbf{GL}_2(\mathbb{F}_q[t, t^{-1}])$

In this section, we focus on the group $G := \mathrm{GL}_2(\mathbb{F}_q[t, t^{-1}])$. Throughout this section, we denote $\mathbb{F}_q[t, t^{-1}]$ by A. First, we describe the amalgamated free product structure of a certain characteristic subgroup of G which will help us to give a description of the automorphisms of G 3.3.4. Then we discuss about a set of generators 3.3.2 and commutator subgroup 3.1.2 of G. Finally, in 3.3.4, we describe the automorphisms of G. We show that $\mathrm{Out}(G)$ is a finite group and every outer automorphism class can be represented by a finite order automorphism of G.

3.3.1 Structure

Recall from [Se, Chapter II, §1.4] the action of $\mathrm{GL}_2(K)$ on a tree X with fundamental domain an edge where K is a local field with a discrete valuation ν. The valuation ν gives rise to a metric d on K, defined as $d(x, y) := |x - y|_\nu$, where $|x|_\nu := c^{-\nu(x)}$ and $c > 1$. The vertices of the tree are equivalence classes of

$\mathcal{O}$-lattices in $V = K^2$ where $\mathcal{O} = \mathcal{O}_\nu$ is the local ring $\{x \in K \mid \nu(x) \geq 0\}$ and two lattices $L, L' \in V$ are equivalent if there exists an $x \in K^\times$ such that $L' = xL$. A pair of vertices Λ, Λ' form an edge $\Lambda\Lambda'$ of X if there are $\mathcal{O}$-lattices L, L' representing Λ, Λ' such that $L' \subset L$ and $L/L' \cong \mathcal{O}/\pi\mathcal{O}$, where $\pi \in \mathcal{O}$ is a uniformizer. (That is, π is a generator of the maximal ideal $\{x \in \mathcal{O} \mid \nu(x) > 0\}$.)

In our context, we have $K = \mathbb{F}_q(t)$ and ν to be
$\nu(a/b) = \mathrm{ord}_0(a) - \mathrm{ord}_0(b), a, b \in \mathbb{F}_q[t^{-1}]$ where ord_0 denotes the order of vanishing of t^{-1} at 0. For example, if $a = t^{-3}$ and $b = t^{-1} + 1$, then $\nu(a/b) = 3$. With respect to the metric arising from the above valuation ν, $A \subset K$ is dense in K. We consider the restricted action of $G = \mathrm{GL}_2(A)$ on X. The $G = \mathrm{GL}_2(A)$ action on X has fundamental domain $e = vv'$ where v, v' may be taken to be the classes of L, L' respectively where $L = \mathcal{O}e_1 + \mathcal{O}e_2$ and $L' = \mathcal{O}e_1 + \mathcal{O}te_1$. Then conjugation by $h := \left(\begin{smallmatrix} 0 & 1 \\ t & 0 \end{smallmatrix}\right) \in \mathrm{GL}_2(A)$ interchangess v and v' so that $h.e = e$ although $h.v = v',\ h.v' = v$. However, when restricted to the subgroup $\mathrm{SL}_2(A)$, the action on X is *without inversion*—an element $g \in \mathrm{SL}_2(A)$ stabilizes an edge of X if and only if its vertices are fixed. The edge $e = vv'$ is also the fundamental domain for the $\mathrm{SL}_2(A)$ action on X as the centre Z of $\mathrm{GL}_2(A)$ acts trivially on X. This leads to the description of $\mathrm{SL}_2(A)$ as an amalgamated free product:
$\mathrm{SL}_2(A) = \mathrm{SL}_2(\mathbb{F}_q[t]) *_{\mathcal{B}'} {}^\tau\mathrm{SL}_2(\mathbb{F}_q[t])$ where $\tau = \delta(1, t)$ and
$\mathcal{B}' = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{F}_q[t]) \mid\ t|c\}$.

We also need to consider the subgroup $G^0 \subset G$ defined as the kernel of the homomorphism $G \to \mathbb{Z}$ defined as $g \mapsto \nu(\det(g))$. Note that $\mathrm{GL}_2(\mathbb{F}_q[t]), \mathrm{SL}_2(A) \subset G^0$. The group also G^0 acts on X without inversion and the stabilizers of v, v' are the equal to the stabilizers of L, L' respectively by [Se, Chapter II, §1.4, Theorem 2]. So $G^0_v = \mathrm{GL}_2(\mathbb{F}_q[t])$ and $G^0_{v'} = {}^\tau G^0_v$. This leads to the description $G^0 = G^0_v *_{\mathcal{B}^0} G^0_{v'}$ where $\mathcal{B}^0 = G^0_v \cap {}^\tau G^0_{v'}$. See [Se, Ch. II, §1.4, Theorem 3].

We note that since $\mathrm{SL}_2(A)$ is characteristic in $\mathrm{GL}_2(A)$, and since $\mathbb{F}_q^\times$ is characteristic in $A^\times \cong \mathbb{F}_q^\times \times \mathbb{Z}$, it follows that the group G^0 is characteristic in G.

Also, as had already been noted in 3.2.1, $\mathrm{GL}_2(\mathbb{F}_q[t]) = \mathrm{GL}_2(\mathbb{F}_q) *_{B_0} B_1$ where $B_1 = B(\mathbb{F}_q[t])$ is the group of upper triangular matrices in $G_1 := \mathrm{GL}_2(\mathbb{F}_q[t])$ and $B_0 = B_1 \cap \mathrm{GL}_2(\mathbb{F}_q)$. We set $G_0 = \mathrm{GL}_2(\mathbb{F}_q)$ and $B_0 = B_1 \cap G_0$. We shall denote by T the group of all diagonal matrices in G, by U the group of all unipotent upper triangular matrices in G and by B the group of all upper triangular matrices in G. We set $T_0 := T \cap G_0, U_0 := U \cap G_0$. We have $B = T.U$. Set $u := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in G_0$. We note that $B^- := {}^uB$ (resp. $U^- = {}^uU$) is the group of all lower triangular matrices (resp. unipotent lower triangular matrices) in G and ${}^uT = T$. Moreover the Weyl group $W(G,T) = N_G(T)/T$ is isomorphic to $\mathbb{Z}_2$, generated by the image of $u \in N_G(T)$ and we have $W(G_0, T_0) = W(G,T)$.

3.3.2 A generating set

We claim that G is finitely generated. In fact, it is generated by $\mathrm{GL}_2(\mathbb{F}_q) \cup \{\delta(t,1)\}$. Let $H = gp\langle \mathrm{GL}_2(\mathbb{F}_q), \delta(t,1)\rangle \subset G$. Since $u = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in G_0 = \mathrm{GL}_2(\mathbb{F}_q)$ we see that $u\delta(t,1)u = \delta(1,t) \in H$. So $T \subset H$. Since $\delta(\lambda t^k, 1)e_{12}(1)\delta(\lambda^{-1}t^{-k}, 1) = e_{12}(\lambda t^k)$, we see that $e_{12}(\lambda t^k) \in H \ \forall \lambda \in \mathbb{F}_q^\times, \forall k \in \mathbb{Z}$. As U is isomorphic to the additive group of A, in view of the relation $e_{12}(x)e_{12}(y) = e_{12}(x+y)$ we see that $U \subset H$. Also, $U^- = uUu \subset H$, where U^- is the set of all lower triangular matrices in G. Since A is a Euclidean domain, G equals the group generated by the diagonal matrices and the elementary matrices $e_{ij}(a), i \neq j, a \in A$. (See [Ro, Theorem 2.3.2].) The observation that H contains T and all the elementary matrices implies that $H = G$.

3.3.3 Commutator subgroup

In this section, we show that $\mathrm{SL}_2(A)$ is a characteristic subgroup of $\mathrm{GL}_2(A)$, where $A = \mathbb{F}_q[t, t^{-1}]$. The case where $q = 2$ and $q \geq 3$ are handled separately.

Let $A = \mathbb{F}_2[t, t^{-1}]$. Denote by N the subgroup $[\mathrm{GL}_2(A), \mathrm{GL}_2(A)] \subset \mathrm{SL}_2(A)$. We shall write $g \sim h$ to mean that $gN = hN$ in $\mathrm{SL}_2(A)/N$. Let $\eta : A \to \mathbb{F}_2$ be algebra homomorphism defined by $t \mapsto 1$. Since the induced homomorphism $\mathrm{SL}_2(A) \to \mathrm{SL}_2(\mathbb{F}_2)$ is a surjection, we see that $\eta(N)$ equals the commutator subgroup of $\mathrm{SL}_2(\mathbb{F}_2) \cong S_3$. (Here S_3 is the permutation group.) Hence $u := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is not in N, moreover $u \sim b$ for any element $b \in \mathrm{SL}_2(\mathbb{F}_2)$ of order 2. In particular $u \sim e_{12}(1)$.

We also see that $[\delta(t, 1), u] = \delta(t, t^{-1}) \in N$, $e_{12}(x) \sim ue_{12}(x)u^{-1} = e_{21}(x)\ \forall x \in A$. So $\mathrm{SL}_2(A)/N = U(A)/N \cap U(A)$ where $U(A) \cong A$ stands for 2×2-unipotent upper triangular matrices over A. Since $[\delta(t, 1), e_{12}(x)] = e_{12}((1 + t)x)\ \forall x \in A$ we see that $\mathrm{SL}_2(A)/N$ is a quotient of the abelian group $\bar{A} = A/J \cong \mathbb{F}_2$ where J is the ideal $A(1 + t)$. We have a surjective homomorphism $\mathrm{SL}_2(A) \to \mathrm{SL}_2(\bar{A}) \cong S_3$ induced by the canonical quotient map $A \to \bar{A} = \mathbb{F}_2$. Since $S_3/[S_3, S_3] \cong \mathbb{Z}/2\mathbb{Z}$, we must have $\mathrm{SL}_2(A)/N = U(\bar{A}) \cong \bar{A} \cong \mathbb{Z}/2\mathbb{Z}$. Since $\mathrm{SL}_2(A)$ is the kernel of the determinant $\mathrm{GL}_2(A) \to A^\times = \langle t \rangle \cong \mathbb{Z}$, we have $\mathrm{GL}_2(A)/[\mathrm{GL}_2(A), \mathrm{GL}_2(A)] \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. So we see that $\mathrm{SL}_2(A)$ is inverse image of the torsion subgroup by the natural quotient map $\mathrm{GL}_2(A) \to \mathrm{GL}_2(A)/[\mathrm{GL}_2(A), \mathrm{GL}_2(A)]$. Hence $\mathrm{SL}_2(A)$ is characteristic in $\mathrm{GL}_2(A)$.

Now we consider $A = \mathbb{F}_q[t, t^{-1}]$ where $q \geq 3$. We have $[\delta(\lambda, 1), e_{12}(x)] = e_{12}((\lambda - 1)x) \in [\mathrm{GL}_2(A), \mathrm{GL}_2(A)]$. Since $q \geq 3$, we can choose λ of $\mathbb{F}_q^\times$ so that $c := \lambda - 1 \neq 0$. Replacing x in the above by $x_1 := cx$, we see that $e_{12}(cx_1) = e_{12}(c^2x) \in [\mathrm{GL}_2(A), \mathrm{GL}_2(A)]$. Since c has finite order, repeating this process leads to $e_{12}(x) \in [\mathrm{GL}_2(A), \mathrm{GL}_2(A)]\ \forall x \in A$. Similarly

$e_{21}(x) \in [\mathrm{GL}_2(A), \mathrm{GL}_2(A)]$. Therefore $E_2(A) \subset [\mathrm{GL}_2(A), \mathrm{GL}_2(A)]$. Since $E_2(A) = \mathrm{SL}_2(A)$ and $[\mathrm{GL}_2(A), \mathrm{GL}_2(A)] \subset \mathrm{SL}_2(A)$, we have .$[\mathrm{GL}_2(A), \mathrm{GL}_2(A)] = \mathrm{SL}_2(A)$ for $q \geq 3$.

Therefore, $\mathrm{SL}_2(\mathbb{F}_q[t, t^{-1}])$ is a characteristic subgroup of $\mathrm{GL}_2(\mathbb{F}_q[t, t^{-1}])$ for any $q \geq 2$.

3.3.4 Automorphisms

We keep the notations of §3.3.1. From §3.3.2 we see that any automorphism of $G = \mathrm{GL}_2(A)$ is determined by its values on $G_0 \cup \{\delta(t, 1)\}$ where $G_0 = \mathrm{GL}_2(\mathbb{F}_q)$. We shall show that the outer automorphism group of G is isomorphic to certain subgroup of $T_0 \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathrm{Out}(G_0))$, where T_0 is the diagonal subgroup of G_0.

Let $\Phi_0 \subset \mathrm{Aut}(G_0)$ denote the group of all Frobenius automorphisms of G_0. Elements of Φ_0 are induced by the automorphisms of $\mathbb{F}_q$ and we have $\Phi_0 \cong \mathbb{Z}_e$ where $q = p^e$. It is well-known (see [B, p. 13]) that any outer automorphism of G_0 is represented by a Frobenius automorphism $\rho \in \Phi_0$ or by $\rho \circ \epsilon$ where ϵ is the contragradient automorphism. More precisely, the natural quotient $\mathrm{Aut}(G_0) \to \mathrm{Out}(G_0)$ restricted to $\Phi_0 \times \mathbb{Z}_2$ where the $\mathbb{Z}_2$ factor is generated by ϵ, is an isomorphism. We shall denote by $\widetilde{\Phi}_0$ the subgroup of $\mathrm{Aut}(G_0)$ generated by $\Phi_0 \cup \{\varepsilon\}$. Similarly, $\Phi, \widetilde{\Phi} \subset \mathrm{Aut}(G)$ are the group of Frobenius automorphisms of G and the subgroup generated by $\Phi \cup \{\varepsilon\}$ respectively. Of course $\Phi \cong \Phi_0, \widetilde{\Phi} \cong \widetilde{\Phi}_0$ via restriction to G_0.

We begin by showing that any outer automorphism is represented by an automorphism that stabilises G_0.

Lemma 3.3.1. *Let $\theta : G \to G$ be any automorphism. Then there exists a $g \in G$ such that $\phi = \iota_g \circ \theta$ stabilises G_0.*

Proof. Let $H = \theta(G_0)$.

Since G^0 (recall that $G^0 \subset G$ is the kernel of the homomorphism $G \to \mathbb{Z}$ defined as $g \mapsto \nu(\det(g))$) is characteristic in G and since $G_0 = \mathrm{GL}_2(\mathbb{F}_q) \subset G^0$, it follows that $\theta(G_0)$ is contained in G^0. Since G_0 is finite, it follows that $\theta(G_0)$ is conjugate in G^0 to a subgroup in one of the two factors $G^0_v = \mathrm{GL}_2(\mathbb{F}_q[t])$ or $G^0_{v'} = {}^{\tau}\mathrm{GL}_2(\mathbb{F}_q[t])$. Since $\tau \in G$, we see that the two factors are conjugates in G. Therefore we have an $x \in G$ such that $\iota_x \circ \theta(G_0)$ is contained in $G^0_v = \mathrm{GL}_2(\mathbb{F}_q[t])$. We set $\psi := \iota_x \circ \theta$.

As noted in 3.2.1, $\mathrm{GL}_2(\mathbb{F}_q[t])$ is an amalgamated free product $G_0 *_{B_0} B_1$; see [Se, Chapter II, §1.6, Theorem 6]. Any finite subgroup in $\mathrm{GL}_2(\mathbb{F}_q[t])$ is contained in a conjugate of either $\mathrm{GL}_2(\mathbb{F}_q)$ or B_1. So there exists a $y \in \mathrm{GL}_2(\mathbb{F}_q[t])$ such that $\psi(G_0) \subset {}^{y}G_0$ or $\psi(G_0) \subset {}^{y}B_1$. But $B_1 \cong (\mathbb{F}_q^\times)^2 \ltimes A$ has no subgroup of order $q(q-1)^2(q+1) = o(G_0)$ except when $q = 3$. When $q = 3$, B_1 has no element of order 4 whereas $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in G_0$ has order 4. So B_1 does not contain a copy of G_0 and we must have, $\psi(G_0) = {}^{y}G_0$. Taking $g = y^{-1}x$ and $\phi := \iota_{y^{-1}} \circ \psi = \iota_g \circ \theta$, we see that the assertion holds. □

Next we show that an automorphism ϕ of G as in Lemma 3.3.1 can be composed with ι_g for some $g \in G_0$ so that resulting automorphism stabilises T.

Lemma 3.3.2. *Suppose that $\phi : G \to G$ is an automorphism that stabilises G_0. Then there exists an $h \in G_0$ such that $\psi(T) = T$ where $\psi = \iota_h \circ \phi$. Moreover $\psi(U^\pm) = U^\pm$ or $\psi(U^\pm) = U^\mp$ according as $\psi_0 := \psi|_{G_0} = \rho \in \Phi_0$ or $\psi_0 = \rho \circ \epsilon$, where $U^+ = U$ (resp. U^-) denotes the subgroup of all unipotent upper (resp. lower) triangular matrices in G.*

Proof. Denote by $\phi_0 \in \mathrm{Aut}(G_0)$ defined by ϕ. It is known that any outer automorphism of G_0 is represented by a Frobenius automorphism ρ or $\rho \circ \epsilon$ where ϵ is the contragradient $g \mapsto {}^{t}g^{-1}$; see [B, p. 13]. So we can find an $h \in G_0$ such that, with $\psi := \iota_h \circ \phi$, we have $\psi_0 = \rho$ or $\psi_0 = \rho \circ \epsilon$.

We claim that ψ stabilises T. Let $U_0 \subset G_0$ be the subgroup of all unipotent upper triangular matrices. When $\psi_0 = \rho \in \Phi_0$, we have $\phi(U_0) = U_0$. Since the centraliser of U_0 in G equals ZU, we must have $\psi(ZU) = ZU$. Here, and in what follows, Z denotes the centre of G. Since $U \cong A$ consists precisely of all elements of ZU of order p, it follows that $\psi(U) = U$. Similarly, we have $\psi(U^-) = U^-$. Since the normaliser of U (resp. U^-) in G equals B, the subgroup of upper triangular matrices in G (resp. B^-, the subgroup of lower triangular matrices in G), it follows that $\psi(B) = B$ and $\psi(B^-) = B^-$. When $\psi_0 = \rho \circ \epsilon$, we have $\psi(U_0) = {U_0}^-$. By similar arguments as before, we have $\psi(B) = B^-$ and $\psi(B^-) = B$. It follows that in both the cases we have, $T = B \cap B^- = \psi(B) \cap \psi(B^-) = \psi(B \cap B^-) = \psi(T)$. □

Let $\phi \in \mathrm{Aut}(G)$. Since $\phi(Z) = Z$, we must have $\delta(t,t) = \delta(\lambda t^{\varepsilon}, \lambda t^{\varepsilon})$ with $\lambda \in \mathbb{F}_q^{\times}, \varepsilon \in \{1, -1\}$. Suppose that $\phi(T) = T$. Then $\phi(T_0) = T_0$. Let $\phi(\delta(t,1)) = h\delta(t^a, t^b)$ where $h = \delta(\alpha, \beta) \in T_0$ and $a, b \in \mathbb{Z}$. We claim that:
(i) exactly one of a, b is zero and that the non-zero one equals ε,
(ii) $\alpha\beta = \lambda$.
Since $\mathrm{SL}_2(A)$ is characteristic in $\mathrm{GL}_2(A)$, we see that ϕ induces an automorphism $\bar{\phi}$ on the quotient $\mathrm{GL}_2(A)/\mathrm{SL}_2(A) \cong \mathbb{F}_q^{\times} \times \langle t \rangle$ and we have $\bar{\phi}(\det(g)) = \det(\phi(g))$. Therefore $\det(\delta(t,1)) = t$ implies that $a + b \in \{1, -1\}$. Also since ϕ induces an automorphism of T/Z we must have $a - b \in \{1, -1\}$. Thus we have $a = 0, b \in \{1, -1\}$ or $b = 0, a \in \{1, -1\}$.

Since $u\delta(t,1)u = \delta(1,t)$ where $u = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, we have $\delta(t,t) = u\delta(t,1)u\delta(t,1)$. Using the fact u normalises T and that $u^2 = 1$, we see that $\phi(u) = \left(\begin{smallmatrix} 0 & c \\ 1/c & 0 \end{smallmatrix}\right) \in G$ for some $c \in A^{\times}$. So $\phi(\delta(1,t)) = \delta(\beta t^b, \alpha t^a)$.

It follows that $(\lambda t^{\varepsilon}, \lambda t^{\varepsilon}) = \phi(\delta(t,t)) = \phi(\delta(1,t)\delta(t,1)) = \alpha\beta t^{a+b} I$. Hence $a + b = \varepsilon, \alpha\beta = \lambda$. Since $a - b = \pm 1$, our claim follows.

Definition 3.3.3. *Let $\phi : G \to G$ be an automorphism such that ϕ stabilises T. Let $h \in T_0, \varepsilon \in \{1, -1\}$. We say that ϕ is of* type $(h, \varepsilon, 1)$ *(resp. (h, ε, u)) if*

$\phi(\delta(t,1)) = h\delta(t^{\varepsilon},1)$ *(resp.* $\phi(\delta(t,1)) = h.\delta(1,t^{\varepsilon}) = h.^{u}\delta(t^{\varepsilon},1)$*).*

Since $G_0 \cup \{\delta(t,1)\}$ generates G, it is clear that there is at most one automorphism ϕ of a given type $(h,\varepsilon,u^i), i \in \{0,1\}$ and prescribed $\phi_0 := \phi|_{G_0}$.

Lemma 3.3.4. *Every outer automorphism of* $GL_2(A)$ *is represented by a unique automorphism* ϕ *such that*
(i) $\phi(T) = T, \phi(G_0) = G_0$*,*
(ii) $\phi_0 = \phi|_{G_0} \in \widetilde{\Phi}_0$*, the subgroup of automorphisms of* G_0 *generated by the contragradient* ϵ *and the Frobenius automorphisms* ρ*,*
(iii) ϕ *is of type* (h,ε,u^i) *for some* $h \in T_0, \varepsilon \in \{1,-1\}, i \in \{0,1\}$*, and,*
(iv) $\phi \in Aut(G)$ *has finite order.*

Proof. By Lemma 3.3.1 and Lemma 3.3.2, given any automorphism θ, there exists an element $x \in G$ such that $\phi := \iota_x \circ \theta$ stabilises G_0, T.

Since $\mathrm{Out}(G_0) \cong \widetilde{\Phi}_0$, replacing ϕ by $\iota_g \circ \phi$ with a suitable $g \in G_0$ if necessary, we may (and do) assume that $\phi_0 := \phi|_{G_0} \in \widetilde{\Phi}_0$. Thus, we have ϕ_0 equals ρ or $\rho \circ \epsilon$ for a suitable Frobenius automorphism ρ. This proves (i) and (ii).

Since $\phi(\delta(t,1)) \in T$, we have $\phi(\delta(t,1)) = \delta(\alpha t^{\varepsilon},\beta) = h\delta(t^{\varepsilon},1)$ or $\delta(\alpha,\beta t^{\varepsilon}) = h\delta(1,t^{\varepsilon})$ for some $h = (\alpha,\beta) \in T_0, \varepsilon \in \{1,-1\}$. Thus, ϕ is of type $(h,\varepsilon,1)$ in the former case while it is of type (h,ε,u) in the latter. This proves (iii).

The uniqueness of ϕ follows from the fact that G is generated by $G_0 \cup \{\delta(t,1)\}$.

It remains to show that $o(\phi)$ is finite. Note that the type of ϕ^2 is $(v,1,1)$ for some $v \in T_0$. Since ϕ is of finite order if and only if ϕ^2 is, we assume, without loss of generality, that the type of ϕ is $(v,1,1)$. Thus $\phi(\delta(t,1)) = v\delta(t,1)$ where $v \in T_0$.

First we show that $\phi^m(\delta(t,1)) = \delta(t,1)$ for some m. Set $v_0 := v$ and $v_j := \phi^j(v_0) = \phi(v_{j-1}) \in T_0, j \geq 1$. Let r be the order of $\phi|_{T_0}$. Setting $\nu_0 := \prod_{0 \leq j < r} v_j$ we obtain that $\phi(\nu_0) = \nu_0$.

A simple calculation leads to $\phi^r(\delta(t,1)) = (\prod_{0\leq j<r} v_j)\delta(t,1) = \nu_0\delta(t,1)$. Applying ϕ once again we obtain that $\phi^{2r}(\delta(t,1)) = \phi(\nu_0\delta(t,1)) = \nu_0^2\delta(t,1))$. By induction, $\phi^{rs}(\delta(t,1)) = \nu_0^s\delta(t,1)$. Taking $m = r(q-1)$ we obtain that
$\phi^m(\delta(t,1)) = \nu_0^{q-1}\delta(t,1) = \delta(t,1)$.

Let $o(\phi_0) = k$ where $\phi_0 = \phi|_{G_0}$. Let $n = km$ where m is such that
$\phi^m(\delta(t,1)) = \delta(t,1)$. Fix generators $x_1,\ldots,x_r$ of G_0. Since $G_0 \cup \{\delta(t,1)\}$ generates G, any element $g \in G$ has an expression as a word $g = \omega(x_1,\cdots,x_r,\delta(t,1))$.
Applying ϕ^n, we see that $\phi^n(g) = \phi^n(\omega(x_1,\cdots,x_r,\delta(t,1))) =$
$\omega(\phi^n(x_1),\cdots,\phi^n(x_r),\phi^n(\delta(t,1))) = \omega(x_1,\cdots,x_r,\delta(t,1)) = g$. Hence $\phi^n = id$. $\square$

Let $\Gamma \subset \mathrm{Aut}(G)$ be the set of all automorphisms of ϕ of G which stabilise G_0 and T and further satisfying the condition that $\phi_0 = \phi|_{G_0} \in \widetilde{\Phi}$.

We shall now show that $\mathrm{Out}(G)$ is a subgroup of an explicitly described finite group in terms of type and $\mathrm{Out}(G_0)$.

Suppose that ϕ,ψ are automorphisms of G which stabilise G_0 and T. Suppose that $(h,\varepsilon,u^i),(g,\eta,u^j)$ are the types of ϕ and ψ respectively. Let $\phi_0 = \phi|_{G_0}$ and $\psi_0 = \psi|_{G_0}$. Then $\psi\circ\phi$ stabilises G_0,T and $(\psi\circ\phi)_0 = \psi_0\circ\phi_0$. Also, $\psi\circ\phi$ is of type $(x,\epsilon\eta,u^{i+j})$ where $x \in T_0$ is seen to be as follows by a straightforward computation:

$$x = \begin{cases} g\psi_0(h) & \text{if } (\varepsilon,i) = (1,0), \\ g^{-1}\psi_0(h) & \text{if } (\varepsilon,i) = (-1,0), \\ {}^u g\psi_0(h) & \text{if } (\varepsilon,i) = (1,1), \\ {}^u g^{-1}\psi_0(h) & \text{if } (\varepsilon,i) = (-1,1). \end{cases} \tag{1}$$

We consider the action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on T_0 where the action of the first factor is given inversion and the second by conjugation by u. Also, the group $\mathrm{Out}(G_0)$, identified to the group generated by the contragradient and the Frobenius automorphisms, acts on T_0. These two actions commute and yield an action of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathrm{Out}(G_0)$ on T_0. We define Γ to be the resulting semi direct product $T_0 \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathrm{Out}(G_0))$. Explicitly,

$$(g, \eta, u^j; \psi_0).(h, \varepsilon, u^i; \phi_0) = (x, \eta\varepsilon, u^{i+j}; \psi_0 \circ \phi_0) \tag{2}$$

where x is given by Equation (1).

Theorem 3.3.5. *The group $Out(GL_2(A))$ is isomorphic to a subgroup of $\Gamma = T_0 \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2 \times Out(GL_2(\mathbb{F}_q))$.*

Proof. As observed in Lemma 3.3.1, any subgroup of G isomorphic to G_0 is conjugate to G_0. Also, by the above lemma, every outer automorphism of G is represented by an automorphism that maps G_0 to itself. Suppose that $\phi, \psi \in \mathrm{Aut}(G)$ stabilise G_0 and that $\psi = \iota_g \circ \phi$ with $g \in G$. Then $\iota_g(G_0) = G_0$ and $\psi_0 = \iota_g \circ \phi_0$. Although g may not be in G_0, we claim that $\psi_0 = \iota_h \circ \phi_0$ for some $h \in G_0$. To see this, consider the projection $r : G \to G_0$ induced by the $\mathbb{F}_q$-algebra homomorphism $A \to \mathbb{F}_q$ defined by $t \mapsto 1$. Then r restricts to the identity on G_0. So, setting $h = r(g)$ we have $\psi_0 = \iota_h \circ \phi_0$. We have a well-defined map $\Psi_1 : \mathrm{Out}(G) \to \mathrm{Out}(G_0)$ obtained as $[\phi] \mapsto [\phi_0]$ where ϕ is as in Lemma 3.3.4. It is clear that Ψ_1 is a homomorphism.

Next let $\Psi_0 : \mathrm{Out}(G) \to T_0 \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ be the map $[\phi] \mapsto (h, \varepsilon, u^i)$ where ϕ as in Lemma 3.3.4 and is of type (h, ε, u^i). The uniqueness part of the lemma implies that this is a well-defined map. By Equation (2), $\Psi : \mathrm{Out}(G) \to \Gamma$, defined as $[\phi] \mapsto (\Psi_0([\phi]), \Psi_1([\phi]))$, is a well-defined homomorphism of groups.

Since $G_0 \cup \{\delta(t, 1)\}$ generates G and since $[\phi_0] \in \mathrm{Out}(G_0)$ uniquely determines ϕ_0 in the subgroup $\widetilde{\Phi}_0$ of $\mathrm{Aut}(G_0)$, there is at most one outer automorphism of G that maps to a given element of Γ. This shows that Ψ is a monomorphism. □

Remark 3.3.6. *It is not clear to us whether Ψ is surjective. When $h = (\alpha, 1) \in T_0$*

and $\phi_0 = \rho$ *(resp.* $\rho \circ \epsilon$*) and* $h = \delta(\lambda, 1)$ *with* $\lambda \in \mathbb{F}_q^\times$*, then we may take* ϕ *to be induced by an* $\mathbb{F}_p$*-algebra automorphism which is the Frobenius automorphism* ρ *on* $\mathbb{F}_q$ *and sends* t *to* λt^ε*. Then we see that* $\Psi([\phi]) = (h, \varepsilon, 1, [\phi_0]) \in \Gamma$*.*

3.4 Proof of Theorem 1.2.1 and Theorem 1.2.2

In this section, we prove our main theorems 1.2.1 and 1.2.2. We begin by recalling some basic results about twisted conjugacy 3.4.1. We first prove R_∞-property for groups $\mathrm{GL}_n(R), \mathrm{SL}_n(R)$ for F a finite field and $n \geq 3$, which is actually a special case of theorem 1.2.1, (i). Then we prove it for $\mathrm{GL}_n(R), \mathrm{SL}_n(R)$ for F a subfield of $\bar{\mathbb{F}}_p$ and $n \geq 3$, which is the theorem 1.2.1, (i). Next, in 3.4.4, we prove theorem 1.2.2. Finally we focus on $n = 2$ case and prove theorem 1.2.1, (ii) and 1.2.1, (iii) in 3.4.5 and 3.4.6 respectively.

3.4.1 Basic results on twisted conjugacy

In this section, we recall some definitions and basic results about twisted conjugacy.

Given an endomorphism $\phi : G \to G$ of a group G and $x \in G$, we shall denote by $[x]_\phi$ the ϕ-twisted conjugacy class of x. We write $x \sim_\phi y$ if $[x]_\phi = [y]_\phi$.

We collect here some basic results concerning twisted conjugacy and the R_∞-property which are relevant for our purposes. Let G be an infinite group (not necessarily finitely generated) and let $K \subset G$ be a normal subgroup. Let $\eta : G \to H$ be the canonical quotient map where $H = G/K$. Suppose that $\phi : G \to G$ is an automorphism such that $\phi(K) = K$ so that we have the following diagram in which the rows are exact and isomorphisms $\phi', \bar{\phi}$ are defined by ϕ:

$$\begin{array}{ccccccccc} 1 & \to & K & \to & G & \to & H & \to & 1 \\ & & \downarrow \phi' & & \downarrow \phi & & \downarrow \bar{\phi} & & \\ 1 & \to & K & \to & G & \to & H & \to & 1 \end{array} \tag{1}$$

Any ϕ-twisted conjugacy class in G maps into a $\bar{\phi}$-twisted conjugacy class in H. Also, any ϕ'-twisted conjugacy class in K is contained in a ϕ-twisted conjugacy class of G. Moreover, if H is finite, then any ϕ-twisted conjugacy class contains at most $o(H)$ many distinct ϕ'-twisted conjugacy classes of K. So, if $R(\phi') = \infty$, then $R(\phi) = \infty$. See [MS1, Lemma 2.2] and its proof. We summarise these results as a lemma.

We recall that a subgroup K of G is *characteristic* in G if every automorphism of G restricts to an automorphism of K.

Lemma 3.4.1. *Suppose that $\phi : G \to G$ is an automorphism of an infinite group such that the rows in (1) are exact and the homomorphisms $\phi', \bar{\phi}$ are isomorphisms. Then:*
(i) If $R(\bar{\phi}) = \infty$, then $R(\phi) = \infty$. In particular, if K is characteristic in G, then G has the R_∞-property if H does.
(ii) Suppose that H is finite. Then $R(\phi) = \infty$ if $R(\phi') = \infty$. In particular, if K is characteristic and has finite index in G, then G has the R_∞-property if K does. □

The following lemma is well-known and can be found, for example, in [GS1, §3].

Let $g \in G$ and let $\iota_g : G \to G$ denote the inner automorphism $x \mapsto gxg^{-1}$. If $\phi : G \to G$ is any automorphism, then there exists a bijection $\mathcal{R}(\iota_g \circ \phi) \to \mathcal{R}(\phi)$ defined as $[x]_{\iota_g \circ \phi} \mapsto [xg]_\phi$. Therefore $R(\phi) = \infty$ if and only if $R(\iota_g \circ \phi) = \infty$.

Lemma 3.4.2. *Suppose that G has infinitely many distinct (usual) conjugacy classes. Then G has the R_∞-property if $R(\phi) = \infty$ for all $\phi \in S$ where $S \subset Aut(G)$*

is a transversal for the quotient $Aut(G) \to Out(G)$, the outer automorphism group of G. □

Remark 3.4.3. Examples of groups which satisfy the hypothesis of the lemma are the infinite groups which are residually finite. (Cf. [MS1, Proposition 2.4].) So, by [LS], the lemma is applicable if G is an infinite, finitely generated subgroup of a linear group over a field. For any infinite ring R and $n \geq 2$, the groups $\mathrm{SL}_n(R), \mathrm{GL}_n(R)$ are readily seen to have infinitely many conjugacy classes. For example, for any $x \in R$, the block diagonal matrix $B(x) := \mathrm{diag}(A(x), I_{n-2})$ where $A(x) = \left(\begin{smallmatrix} 1+x & x \\ 1 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(A)$ has trace $2 + x$ and so the $B(x), x \in R$ are in pairwise distinct conjugacy classes. The same is true of any subgroup $H \subset \mathrm{GL}_n(R)$ that contains $\mathrm{SL}_n(R)$.

The following lemma can be proved along the same lines as [GS1, Lemma 3.4] or [GS2, Lemma 2.3].

Lemma 3.4.4. *Let $\theta : G \to G$ be a finite order automorphism. Let $r = o(\theta)$. (i) Suppose that $[x]_\theta = [y]_\theta$. Then $\prod_{0 \leq j < r} \theta^j(x)$ and $\prod_{0 \leq j < r} \theta^j(y)$ are conjugates in G. (ii) Suppose that $x, y \in Fix(\theta)$ and that $[x]_\theta = [y]_\theta$. Then x^r and y^r are conjugates in G.* □

3.4.2 For the groups $\mathbf{GL}_n(A), \mathbf{SL}_n(A)$ when $n \geq 3$

Recall that $A = \mathbb{F}_q[t]$ or $\mathbb{F}_q[t, t^{-1}]$, where $q = p^e, e > 1$.

We note that since the groups $G = G(A) = \mathrm{GL}_n(A), \mathrm{SL}_n(A)$ have infinitely many (usual) conjugacy classes, it suffices to show that $R(\phi) = \infty$ for a set $\mathcal{S}$ of representatives of the outer automorphisms of G. We take $\mathcal{S}$ to be as in Corollary 3.1.3.

Consider the automorphism $\rho : G \to G$ induced by a ring automorphism

$\rho : A \to A$. Let $S = \mathbb{F}_p[s] \subset A$ be as in 3.1.3. Then S is contained in the subring $\mathrm{Fix}(\rho) \subset A$ and the group $G(S) \subset G(A)$ is element-wise fixed by ρ.

Set $x_m = e_{12}(s^m)e_{21}(-s^m) \in \mathrm{SL}_2(S)$ so that $x_m = \left(\begin{smallmatrix} 1-s^{2m} & s^m \\ -s^m & 1 \end{smallmatrix}\right)$. We observe that $\rho(x_m) = x_m = e_{12}(s^m)\epsilon(e_{12}(s^m))$ and that the x_m satisfy the polynomial $X^2 + (s^{2m} - 2)X + I_2 = 0$. We regard x_m as also as an element of $SL_n(S)$ by identifying it with the block diagonal matrix $\delta(x_m, I_{n-2})$. These elements will be shown to be in pairwise distinct ϕ-twisted conjugacy classes for many automorphisms of G. The following lemma will play a crucial role in our proof.

Lemma 3.4.5. *Let $A = \mathbb{F}_q[t], \mathbb{F}_q[t, t^{-1}]$. Fix $r \geq 1$. The elements $x_m = e_{12}(s^m)e_{21}(-s^m), \in SL_n(A), m \geq 1$, are such that $tr(x_m^r)$ are pairwise distinct.*

Proof. Set $x := \left(\begin{smallmatrix} 1-u^2 & u \\ -u & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{F}_p[u])$. We see that $\mathrm{tr}(x) = 2 - u^2, \mathrm{tr}(x^2) = 2 - 4u^2 + u^4$. As the characteristic polynomial of x is $X^2 - (2 - u^2)X + 1$, we obtain the relation $\mathrm{tr}(x^r) = (2 - u^2)\mathrm{tr}(x^{r-1}) - \mathrm{tr}(x^{r-2})$ for any $r \geq 3$. It follows by induction that $\mathrm{tr}(x^r)$ is a polynomial in u of degree $2r$ with leading coefficient $(-1)^r \in \mathbb{F}_p$.

The last assertion still holds when x is viewed as an element of $\mathrm{GL}_n(\mathbb{F}_p[u]), n \geq 3$. Applying this to the elements $x_m \in \mathrm{GL}_n(A)$ defined above, $\mathrm{tr}(x_m^r) \in S = \mathbb{F}_p[s] \subset A$ is a polynomial in s of degree $2rm$. Hence $\mathrm{tr}(x_m^r), m \geq 1$, are pairwise distinct. □

We are now ready to prove R_∞-property for $\mathrm{GL}_n(F[t]), \mathrm{SL}_n(F[t]), \mathrm{GL}_n(F[t, t^{-1}]), \mathrm{SL}_n(F[t, t^{-1}])$ for F a finite field and $n \geq 3$.

Proof. The proof will depend on the type of automorphism as listed in Corollary 3.1.3. We consider both $G = \mathrm{GL}_n(A), \mathrm{SL}_n(A)$ with $A = \mathbb{F}_q[t], \mathbb{F}_q[t, t^{-1}]$ simultaneously whenever feasible. The symbol ρ will always denote an automorphism of G induced by a ring automorphism of A, ϵ the contragradient, ι_g the conjugation induced by a $g \in \mathrm{GL}_n(A)$, etc.

Type ρ: Note that $x_m \in \mathrm{Fix}(\rho)$ and ρ is of finite order. Taking $r = o(\rho)$ in Lemma 3.4.5, we see, by Lemma 3.4.4, that the $x_m^r, m \geq 1$, are in pairwise distinct ρ-twisted conjugacy classes and so $R(\rho) = \infty$.

We consider, next, an automorphism $\theta = \epsilon$ or $\rho \circ \epsilon$. We shall show that $e_{12}(s^m), e_{12}(s^k)$ are not in the same θ-twisted conjugacy class if $k \neq m$ by using Lemma 3.4.4.

Type ϵ: When $\theta = \epsilon$, it has order 2. We have $x_m = e_{12}(s^m).\epsilon(e_{12}(s^m))$, and since $\mathrm{tr}(x_m) \neq \mathrm{tr}(x_k)$, applying Lemma 3.4.4 we see that $[e_{12}(s^m)]_\theta \neq [e_{12}(s^k)]_\theta$. Hence $R(\epsilon) = \infty$.

Type $\rho \circ \epsilon$: When $\theta = \rho \circ \epsilon$, it has order $2r$ where $r = o(\rho)$. Since ρ and ϵ commute, $\theta^j(e_{12}(s^m)) = \epsilon^j \rho^j(e_{12}(s^m)) = \epsilon^j(e_{12}(s^m))$ equals $e_{12}(s^m)$ or $e_{21}(-s^m)$ depending on if j is even or odd respectively. It follows that $\prod_{0 \leq j < 2r} \theta^j(e_{12}(s^m)) = (e_{12}(s^m)e_{21}(-s^m))^r = x_m^r$. Since $\mathrm{tr}(x_m^r) \neq \mathrm{tr}(x_k^r)$ when $k \neq m$, we see that $[e_{12}(s^m)]_\theta \neq [e_{12}(s^k)]_\theta$ if $m \neq k$. So $R(\rho \circ \epsilon) = \infty$.

Type $\theta = \mu_\chi \circ \rho, \mu_\chi \circ \rho \circ \epsilon$: These types are relevant only for $G = \mathrm{GL}_n(A)$.

When $A = \mathbb{F}_q[t]$ we set $K := \mathrm{SL}_n(A)$, which is of finite index in $\mathrm{GL}_n(\mathbb{F}_q[t])$. Note that, in view of Lemma 3.4.1 and the fact that $\mu_\chi = Id$ on K, we have, $R(\mu_\chi \circ \rho) = R(\mu_\chi \circ \rho|_K) = R(\rho|_K) = R(\rho) = \infty$.

Now assume that $A = \mathbb{F}_q[t, t^{-1}]$. Let $\theta = \mu_\chi \circ \rho$ and let $r = o(\rho)$. We claim that the elements $x_m^r \in \mathrm{SL}_n(A), m \geq 1$, are in pairwise distinct θ-twisted conjugacy classes. To see this, note that since μ_χ restricts to the identity on $\mathrm{SL}_n(A)$ as observed in Remark **??**, $x_m \in \mathrm{Fix}(\theta)$. Suppose that $x_k = z x_m \theta(z^{-1})$ for some $z \in \mathrm{GL}_n(A)$. Note that $\theta(z^{-1}) = \rho(z^{-1}).u$ where $u := \chi(\rho(z^{-1}))I_n$. Since $Im(\chi) \subset \mathbb{F}_q^\times$, we see that u is a scalar matrix in $\mathrm{GL}_n(\mathbb{F}_q)$.

Thus $x_k = z x_m \rho(z^{-1}).u$. Since u is in the centre of $\mathrm{GL}_n(A)$ we obtain that for any $j \geq 1$, applying θ repeatedly, we obtain that for any $j \geq 0$,

$x_k = \theta^j(x_k) = \rho^j(z)x_m\rho^{j+1}(z^{-1}).u_j$ for a suitable scalar matrix u_j in $\mathrm{GL}_n(\mathbb{F}_q)$. Setting $r := o(\rho)$, we are lead to the equation $x_k^r = z{x_m}^r z^{-1}.v$ for some scalar matrix $v \in \mathrm{GL}_n(\mathbb{F}_q)$. Taking trace on both sides we obtain that $\mathrm{tr}(x_k^r) = v.\mathrm{tr}(x_m^r) \in \mathbb{F}_q[s]$. This is a contradiction to Lemma 3.4.5 as the degree of $\mathrm{tr}(x_j^r)$ as a polynomial in s equals $2jr$. This shows that $R(\mu_\chi \circ \rho) = \infty$.

The proof that $R(\mu_\chi \circ \rho \circ \epsilon) = \infty$ uses $e_{12}(s^m) \in \mathbb{F}_p[s]$ and is similar to the proof for the type $\rho \circ \epsilon$, just as the above proof for $\mu_\chi \circ \rho$ parallels the proof for type ρ.

This completes the proof that $\mathrm{GL}_n(A)$ has the R_∞-property for $n \geq 3$.

It remains to consider the case of automorphisms of $\mathrm{SL}_n(A)$ as in Corollary (i)(b).

Type $\iota_h \circ \rho$: We consider an automorphism ϕ of $\mathrm{SL}_n(A)$ of the form $\phi = \iota_h \circ \rho$ with $h = h(a) \in H$ with $a \in A^\times$ as in Corollary 3.1.3(ii). Suppose that k and m are distinct but $x_k = zx_m\phi(z^{-1}) = zx_m h\rho(z^{-1})h^{-1}$. So $x_k h$ and $x_m h$ are ρ-twisted conjugates. We apply Lemma 3.4.4(i) to ρ. Setting $r = o(\rho)$ we obtain, $\prod_{0\leq j<r} \rho^j(x_k h)$ and $\prod_{0\leq j<r} \rho^j(x_m h)$ are conjugates. Since x_k and $\rho^j(h) = h(\rho^j(a))$ commute (as $n \geq 3$), we obtain that $\prod_{0\leq j<r} \rho^j(x_k h) = x_k^r.h(\prod \rho^j(a))$ and the same holds when k is replaced by m. Taking trace leads to $\mathrm{tr}(x_k^r) = \mathrm{tr}(x_m^r)$, contradicting Lemma 3.4.5.

Type $\iota_h \circ \rho \circ \epsilon$: Finally, it remains to consider automorphisms of $\mathrm{SL}_n(A)$ of the form $\psi := \iota_h \circ \rho \circ \epsilon$ with $h = h(a), a \in A^\times$, as in Corollary 3.4.4. We assert that $e_{12}(s^m), m \geq 1$, are in pairwise distinct ψ-twisted conjugacy classes. Suppose not. Then there exist distinct positive integers k, m such that $e_{12}(s^m) = ze_{12}(s^k)\psi(z^{-1})$. Therefore $e_{12}(s^k)h$ and $e_{12}(s^m)h$ are $\rho \circ \epsilon$-twisted conjugates. This implies, by Lemma 3.4.4, that $v_k := \prod_{0\leq j<2r} \theta^j(e_{12}(s^k)h)$ and $v_m := \prod_{0\leq j<2r} \theta^j(e_{12}(s^m)h)$ are conjugates, where $\theta = \rho \circ \epsilon$, $r = o(\rho)$. As in the case $\rho \circ \epsilon$, using $\epsilon(h) = h^{-1}$, and $e_{12}(u), h, \rho(h)$, pairwise commute, we obtain $v_k = x_k^r.\prod_{0\leq j<r} \rho^{2j}(h) \prod_{0<j\leq r} \rho^{2j-1}(h^{-1}) = x_k^r h(b)$ where $b \in A^\times$ depends only on

$r = o(\rho)$ and not on k—actually, $b \in \mathbb{F}_q^\times$, but this is not relevant here. Since v_k, v_m are conjugates, we get $\mathrm{tr}(v_k) = \mathrm{tr}(v_m)$ which implies that $\mathrm{tr}(x_m^r) = \mathrm{tr}(x_k^r)$, contradicting Lemma 3.4.5. This shows that $R(\psi) = \infty$, completing the proof.

□

3.4.3 Proof of Theorem 1.2.1 (i)

Let $R = F[t]$ or $F[t, t^{-1}]$, where F is an infinite subfield of $\bar{\mathbb{F}}_p$ and let G denote one of the groups $\mathrm{GL}_n(R), \mathrm{SL}_n(R), n \geq 3$. The proof of Theorem 1.2.1(i) is similar to the respective cases when the ring is $\mathbb{F}_q[t], \mathbb{F}_q[t, t^{-1}]$ and in fact may be reduced to these cases. For this reason we shall omit the details. Since $n \geq 3$, O'Meara's theorem is applicable, and so we have that any outer automorphism is represented by an automorphism ϕ as follows:
(i) When $G = \mathrm{GL}_n(R)$, $\phi = \mu_\chi \circ \rho, \mu_\chi \circ \rho \circ \epsilon$ where ρ is an automorphism induced by an $\mathbb{F}_p$-algebra automorphism of R, ϵ is the contragradient automorphism and μ_χ is the homothety $g \mapsto \chi(g).g$ associated to a character $\chi : G \to R^\times$.
(ii) When $G = \mathrm{SL}_n(R)$, $\phi = \iota_h \circ \rho, \iota_h \circ \rho \circ \epsilon$ where $h \in \mathrm{GL}_n(R)$ with ρ, ϵ as in (i). In fact we need only take h to be a diagonal matrix $h(a) := \delta(I_{n-1}, a), a \in R^\times$.

Any automorphism ρ of R is an $\mathbb{F}_p$-algebra automorphism. Since $R^\times = F^\times$ or $F^\times \times \langle t \rangle$, according as $R = F[t]$ or $F[t, t^{-1}]$, it is readily seen that ρ restricts to an automorphism of F and sends t to $\alpha t^\varepsilon + \beta$ for some $\varepsilon \in \{1, -1\}, \alpha \in F^\times, \beta \in F$ where $\beta = 0$ when $R = F[t, t^{-1}]$ and $\varepsilon = 1$ when $R = F[t]$. We choose $q = p^e$ such that $\alpha, \beta \in \mathbb{F}_q \subset F$.

As in 3.1.3, $\rho(s) = s$ where $s = \prod_{\gamma \in \mathbb{F}_q}(t + \gamma)^{q-1}$ when $R = F[t]$ and $s = t^{q-1} + t^{1-q}$ when $R = F[t, t^{-1}]$. Hence $\mathbb{F}_p[s] \subset \mathrm{Fix}(\rho) \subset R$.

Note that since $F \subset \bar{\mathbb{F}}_p$, any subfield of F is a normal extension of $\mathbb{F}_p$. So any automorphism of F stabilises all its subfields. By our choice of q, if $\mathbb{F}_q \subset \mathbb{F}_\ell \subset F$, we see that $R_\ell := \mathbb{F}_\ell[t] \subset F[t]$ (resp. $R_\ell := \mathbb{F}_\ell[t, t^{-1}] \subset F[t, t^{-1}]$) is stable by ρ. Consequently, the groups $G_\ell := \mathrm{GL}_n(R_\ell), \mathrm{SL}_n(R_\ell)$ are also stable by ρ. We note that G is the union of the groups G_ℓ.

Since $n \geq 3$, the group $\mathrm{SL}_n(R)$ is perfect, it follows that any character $\chi : \mathrm{GL}_n(R) \to R^\times$ is determined by its restriction to the subgroup H consisting of diagonal matrices $h(a) = \delta(I_{n-1}, a), a \in R^\times$. In particular, if $a \in \mathbb{F}_q^\times$, then $\chi(h(a)) \in \mathbb{F}_q^\times$. If χ is a character that yields a homothety automorphism μ_χ, then we have already observed in §?? that $\mathrm{Im}(\chi) \subset F^\times \subset R^\times$ even when $R = F[t, t^{-1}]$.

Let $x_m = e_{12}(s^m).e_{21}(-s^m) \in G_q = \mathrm{GL}_n(R_q), m \geq 1$, be as in Lemma 3.4.5 where $R_q = \mathbb{F}_q[t], \mathbb{F}_q[t, t^{-1}]$ according as $R = F[t], F[t, t^{-1}]$ respectively. Then $x_m \in \mathrm{Fix}(\rho)$. Suppose that there exists an element $z \in G$ such that $x_k = z x_m \rho(z^{-1})$ with $k \neq m$. There exists $\ell = q^d = p^{de}$ a sufficiently large power of q such that $\mathbb{F}_\ell \subset F$ and $x_k, x_m, z \in G_\ell$. Then $\rho^{de}|_{G_\ell} = id$. This implies, by Lemma 3.4.4 that x_m^{de} and x_k^{de} are conjugates in G_ℓ. This contradicts Lemma 3.4.5 and we conclude that $R(\rho) = \infty$. The proof for $\rho \circ \epsilon$ is similar to the proof of the corresponding type of automorphism for the case when F is a finite field given in §3.4.2.

Now let $\phi = \mu_\chi \circ \rho$. Note that μ_χ restricts to the identity on $\mathrm{SL}_n(R)$. In particular, $\phi(x_m) = \rho(x_m) = x_m$ where $x_m = e_{12}(s^m).e_{21}(-s^m) \in \mathrm{Fix}(\rho) \cap G_q$. Suppose that $x_k \sim_\phi x_m$ for some $k \neq m$. Let $z \in G$ such that $x_k = z x_m \phi(z^{-1}) = z x_m \rho(z^{-1}).uI_n$ where $u = \chi(\rho(z^{-1})) \in F^\times$. Then there exists an $\ell = q^d = p^{de}$ for a sufficiently large d so that $z \in G_\ell$. Then $\rho^{de}|_{G_\ell} = id$. Applying ϕ repeatedly to both sides of this equation and using $\phi(x_k) = \rho(x_k) = x_k, \phi(x_m) = x_m$ we obtain $x_k = \rho^j(z) x_m \rho^{j+1}(z^{-1}).u_j I_n$ for $j \geq 1$ for suitable $u_j \in F^\times$. Multiplying these equations in order for $0 \leq j < de$ and using the fact that $\rho^{de}(z^{-1}) = z^{-1}$ we obtain that $x_k^{de} = z x_m^{de} z^{-1}.vI_n$ for some $v \in F^\times$. Taking trace on both sides we get $\mathrm{tr}(x_k^{de}) = v.\mathrm{tr}(x_m^{de})$. This is a contradiction since $v \in F$ and the degrees of traces of x_k^{de}, x_m^{de} as polynomials in s are $2kde, 2mde$ respectively which are unequal as

$k \neq m$. Hence we conclude that $R(\phi) = \infty$ in this case.

The proof is similar when $\phi = \mu_\chi \circ \rho \circ \epsilon$. This completes the proof when $G = \mathrm{GL}_n(R)$.

When $G = \mathrm{SL}_n(R)$, we need to show that $R(\phi) = \infty$ when $\phi = \iota_h \circ \rho, \iota_h \circ \rho \circ \epsilon$ where $h = h(a) = \delta(I_{n-1}, a), a \in R^\times$. We choose $q = p^e$ so that ρ restricts to G_q and hence to G_ℓ for all $\ell = q^d$. The rest of the proof is as in the proof of the case when F is a finite field for the automorphisms of $\mathrm{SL}_n(R_\ell)$ of the corresponding types, given in 3.4.2. The details are left to the reader. □

3.4.4 Proof of Theorem 1.2.2

Let $R = F[t]$ or $F[t, t^{-1}]$ where F is an infinite subfield of $\bar{\mathbb{F}}_p$. There is a bijective correspondence between subgroups of $R^\times$ and subgroups of $G := \mathrm{GL}_n(R)$ that contain $\mathrm{SL}_n(R)$ where $D \subset R^\times$ corresponds to $H = H(D) = \{g \in G \mid \det(g) \in D\}$. We shall denote by D_0 the torsion subgroup of $D \subset R^\times$ and by $H_0 \subset H = H(D)$ the subgroup that corresponds to D_0. Note that H_0 is characteristic in H. When $R = F[t]$, $H = H_0$, and, when $R = F[t, t^{-1}]$, $H/H_0 \cong D/D_0$ is a subgroup of $\langle \alpha t \rangle \cong \mathbb{Z}$ for some $\alpha \in F$.

Note that $D_0 \subset F^\times$ is finite when F is. Assume that $F \subset \bar{\mathbb{F}}_p$ is infinite. For each finite subfield $\mathbb{F}_{p^r} \subset F$, we set $D_{0,r} = D_0 \cap \mathbb{F}_{p^r}^\times$. Then $D_0 = \cup D_{0,r}$, and, setting $H_{0,r} = H(D_{0,r})$, we have $H_0 = \cup H_{0,r}$. The group $\mathrm{SL}_n(R)$ has finite index in $H_{0,r}$ and so it follows that $H_{0,r}$ has the R_∞-property. Although each $H_{0,r} \subset H_0$ is characteristic, its index in H_0 is in general is not finite. As a first step towards the proof of Theorem 1.2.2, we establish the following.

Lemma 3.4.6. *We keep the above notations. Let $R = F[t], F[t, t^{-1}]$ where $F \subset \bar{\mathbb{F}}_p$ is any subfield. Then H_0 has the R_∞-property.*

Proof. If $F = \mathbb{F}_q$, then D_0 is finite and so $\mathrm{SL}_n(R)$ is a finite index characteristic subgroup of H_0. Hence H_0 has the R_∞-property by Theorem 3.4.1.

Suppose that $F \subset \bar{\mathbb{F}}_p$ is infinite. Let $\theta : H_0 \to H_0$ be any automorphism and let θ', its restriction to $\mathrm{SL}_n(R)$. We may replace θ by $\iota_g \circ \theta$ for a suitable $g \in \mathrm{SL}_n(R)$ if necessary so that θ' is one of the following types: $\iota_h \circ \rho, \iota_h \circ \rho \circ \epsilon$ where $h = h(a) = \det(I_{n-1}, a)$ for a suitable $a = a_0 t^l, a_0 \in F$ where $l = 0$ when $R = F[t]$ and ρ is induced by a ring automorphism $\rho : R \to R$. Suppose that $\rho(t) = \alpha t^\varepsilon + \beta$ where $\alpha, \beta \in F, \alpha \neq 0$ and $\varepsilon \in \{1, -1\}$. It is understood that $\varepsilon = 1$ when $R = F[t]$ and $\beta = 0$ when $R = F[t, t^{-1}]$. By choosing $q = p^e$ to be a large enough power of p, we may assume that $\alpha, \beta, a_0 \in \mathbb{F}_q$ so that θ' stabilizes $\mathrm{SL}_n(R_q)$ where $R_q = \mathbb{F}_q[t], \mathbb{F}_q[t, t^{-1}]$ according as $R = F[t], F[t, t^{-1}]$ respectively. We proceed as in the proof of Theorem 1.2.1 using the elements $x_k, e_{12}(s) \in \mathrm{SL}_n(R_q), k \geq 1$, with entries in $\mathbb{F}_p[s] \subset \mathrm{Fix}(\rho)$ to deduce that $R(\theta) = \infty$. We sketch a proof in the case when $\theta' = \iota_h \circ \rho$.

Suppose that $x_k = z x_m \theta(z^{-1})$ for some $z \in H_0$. Write $z = y.h(c)$ where $y \in \mathrm{SL}_n(R)$ and $c = \det(z) \in F^\times$. Since $\mathrm{SL}_{n-1}(F) \subset \mathrm{SL}_n(F)$ is stable by $\theta' = \iota_h \circ \rho$, and since $h(c)$ is in the centralizer of $\mathrm{SL}_{n-1}(F)$, we see that $\theta(h(c))$ is also in the centralizer of $\mathrm{SL}_{n-1}(F)$. It follows that for each $j \geq 1$, we have $\theta^j(h(c)) = h(c_j)$ for some $c_j \in R^\times$. Since $h(c_j) \in H_0$, $c_j \in F^\times$. Choose $d \geq 1$ so that $c \in \mathbb{F}_{q^d}^\times$. It follows that $c_j \in \mathbb{F}_{q^d}$ for all j. We shall repeatedly use the fact that the $h(c_j)$ commutes with x_k and x_m for all j to show that $x_k^N h(u) = y(x_m^N h(v))y^{-1}$ for suitable elements$u, v \in \mathbb{F}_{q^d}$ where $N := de$ is positive integer such that ρ^N restricts to the identify on $\mathrm{SL}_n(R_{q^d})$.

Substituting $z = yh(c)$ and using $\theta' = \iota_h.\rho$ on $\mathrm{SL}_n(F)$ in the equation $x_k = z x_m \theta(z^{-1})$ leads to $x_k = y x_m h(c) h(c_1^{-1}) \theta'(y^{-1}) = y x_m h(c) h(c_1^{-1}) h \rho(y) h^{-1}$. Hence $x_k h = y x_m h.h(c) h(c_1^{-1}) \rho(y^{-1}) = y(x_m h) h_0 \rho(y^{-1})$, where $h_0 = h(c c_1^{-1})$.

Applying ρ^j we obtain

$$x_k\rho^j(h) = \rho^j(y)(x_m\rho^j(h))\rho^j(h_0)\rho^{j+1}(y^{-1}).$$

Now, using the fact that $\rho^N = id$ on $\mathrm{SL}_n(R_{q^d})$ where $N = 2de$ and taking product (in order) of these equations for $0 \leq j < N$ we obtain

$$x_k^N.(\prod_{0\leq j<N} \rho^j(h)) = y(x_m. \prod_{0\leq j<N} \rho^j(h.h_0))y^{-1}.$$

Thus, we obtain $x_k^N.h(u) = yx_mh(v)y^{-1}$ for appropriate elements $u, v \in \mathbb{F}_{q^d}^\times$. Taking trace on both sides we obtain

$$\mathrm{tr}(x_k^N) + (n-3) + u = \mathrm{tr}(x_m^N) + (n-3) + v.$$

This contradicts Lemma 3.4.5 as $k \neq m$ by assumption. Hence $R(\theta) = \infty$, completing the proof. □

We shall now prove Theorem 1.2.2.

Proof of Theorem 1.2.2. Let $D = \{\det(g) \in R^\times \mid g \in H\}$ so that $H = H(D)$. By the above lemma, $H_0 := H(D_0)$ has the R_∞-property where D_0 is the torsion subgroup of D. So we may assume that $D \neq D_0$. This implies that $R = F[t, t^{-1}]$ and that $\bar{D} := D/D_0 \cong \mathbb{Z}$ generated by an element αt^l for some $l \geq 1$ and $\alpha \in \mathbb{F}_q^\times \subset F$ for some $q = p^e$.

Since H_0 is characteristic in H, any automorphisms $\phi : H \to H$ induces an automorphism $\bar{\phi} : \bar{D} \to \bar{D}$. If $\bar{\phi}(\alpha t^l) = \alpha t^l$, then $R(\bar{\phi}) = \infty$ which implies that $R(\phi) = \infty$.

So suppose that $\bar{\phi}(\alpha t^l) = \alpha^{-1}t^{-l}$. Let ϕ_0 denote the restriction of ϕ to H_0 and let ϕ' be the restriction to $\mathrm{SL}_n(R)$. Replacing ϕ by $\iota_g \circ \phi$ for a suitable $g \in \mathrm{SL}_n(R)$ if necessary, we may (and do) assume that ϕ' is one of the following types: $\iota_h \circ \rho, \iota_h \circ \rho \circ \epsilon$ where $h = h(a) = \delta(I_{n-1}, a)$ for some $a \in R^\times$. There are two cases to consider depending on the type of ϕ'. In each case the proof is similar to the proof of Theorem 1.2.1 (i). We will sketch a proof in the case $\phi' = \iota_h \circ \rho$.

Consider the elements $x_k = e_{12}(s^k)e_{21}(-s^k) \in \mathrm{SL}_n(\mathbb{F}_q[t, t^{-1}]), k \geq 1$, where $s = t^{(q-1)} + t^{(1-q)}$. Suppose that, for some $k \neq m$, we have $x_k = z x_m \phi(z^{-1})$ for some $z \in H$. Taking determinant on both sides we obtain that $\det(z) = \det(\phi(z))$. Since $\bar{\phi}$ on $\bar{D}$ is not the identity map, we must have $\det(z) \in D_0$ and so $z \in H_0$. This is a contradiction since we showed in the proof of Lemma 3.4.6 that the x_j's are in pairwise distict ϕ_0-twisted conjugacy classes. □

3.4.5 Proof of Theorem 1.2.1 (ii)

We begin by recalling the normal form for elements of an amalgamated free product $G = \Gamma_0 *_\Lambda \Gamma_1$. This involves choices of right coset representatives S_0, S_1 for the canonical projections $\Gamma_i \to \Gamma_i/\Lambda$, where we assume that the trivial element $1 \in S_i, i = 0, 1$. The normal form theorem asserts that every element $\gamma \in \Gamma$ has a unique expression $\gamma = u x_1 \cdots x_k$ where $u \in \Lambda$, $x_i \in S_j \setminus \{1\}$, $\forall i$ with $j \in \{0, 1\}$ and two successive factors x_i, x_{i+1} do not belong to the same S_j. [1] However, for our purposes, a weaker version of the normal form will suffice. It says that any element $\gamma \in G \setminus \Lambda$ can be expressed as $\gamma = x_1 \cdots x_k$ with each $x_i \in \Gamma_j \setminus \Lambda$ for some $j = 0, 1$, and no two successive factors $x_i, x_{i+1}, 1 \leq i < k$, belong the the same Γ_j. Any two such expressions of the same element γ differs by replacing x_i, x_{i+1} by $x_i h_i, h_i^{-1} x_{i+1}$ where $h_i \in \Lambda, 1 \leq i < k$. We will call any such expression *weakly reduced*. If $\gamma \in \Lambda$, $\gamma \neq 1$, then its (weakly) reduced expression is $\gamma = \gamma$. Any two weakly reduced expressions have the same length. The length $l(\gamma)$ of an element $\gamma \in G$ is the

[1] To be precise, the normal form of an element is a *sequence* $(u, x_1, \ldots, x_k)$ rather than an expression. We will not make this distinction as the sequence that engenders the expression will be clear from the context.

length of any of its weakly reduced expression and we set $l(1) := 0$. Note that $l(ab) \leq l(a) + l(b)$ and equality holds if in some weakly reduced expressions of a, b, the last term of a and the first term of b do not belong to the same Γ_j. Also $l(ac) = l(ca) = l(a)$ if $c \in \Lambda, a \notin \Lambda$. We refer the reader to [LS],[Se] for a detailed exposition on the normal form.

We need the following lemma.

Lemma 3.4.7. *Let $G = \Gamma_0 *_\Lambda \Gamma_1$ where we assume that $\Lambda \neq \Gamma_i, i = 0, 1$. Let $k, m \geq 2$, with m even. Suppose that $z = u_1 \cdots u_k, w = v_k \cdots v_1, x = a_1 \cdots a_m$ are weakly reduced expressions where $u_k, v_k \in \Gamma_0$. Then either $l(zxw) = m$ or $l(zxw)$ is odd.*

Proof. Set $\Gamma'_i = \Gamma_i \setminus \Lambda, i = 0, 1$. Our hypotheses that $k \geq 2$ and u_k, v_k belong to the same factor Γ_0 implies that implies that $u_j, v_j \in \Gamma'_0$ if $j \equiv k \mod 2$ and $u_j, v_j \in \Gamma'_1$ if $j \equiv k + 1 \mod 2$. Also, since m is even, exactly one of a_1, a_m belongs to Γ'_0; we assume that $a_m \in \Gamma'_0$, the proof in the other case being similar. Then $y_1 := a_m v_k \in \Gamma_0$. If $y_1 \in \Gamma'_0$, then $xw = a_1 \cdots a_{m-1} y_1 v_{k-1} \cdots v_1$ is a weakly reduced expression of length $m - 1 + k$. Since $u_k \in \Gamma'_0, a_1 \in \Gamma'_1$, we have $l(zxw) = l(z) + l(xw) = 2k + m - 1$. This proves our assertion in this case.

So assume that $y_1 \in \Lambda$. Then the subexpression $y_2 := a_{m-1} y_1 v_{k-1} \in \Gamma_1$. If $y_2 \notin \Lambda$, then $l(z a_1 \cdots a_{m-2} y_2) = k + (m - 1)$ and so $l(zxw) = k + m - 1 + (k - 2)$ which is odd, proving our assertion in this case.

Let $\ell = \min\{k, m\}$. In general, we define $y_j, 1 \leq j \leq \max\{\ell, k\}$ inductively to be a subexpression of $zxw = u_1 \ldots u_k a_1 \ldots a_m v_k \ldots v_1$ as follows (where $a_0 = 1 = v_0$):

$$y_{j+1} = \begin{cases} a_{m-j} y_j v_{k-j}, & 1 \leq j \leq \ell - 1, \\ a_{m-k} y_k, & \text{if } j = k \leq m, \\ u_{k-j+m} y_j v_{k-j}, & \text{if } m \leq j \leq k. \end{cases}$$

By our assumption $y_1 \in \Lambda$. We let $i \leq \max\{k, l\}$ be the largest integer such that $y_j \in \Lambda \ \forall j \leq i$.

First suppose that $i < \ell$. Then $y_{i+1} \in \Gamma'_r, v_{k-i-1} \in \Gamma'_{1-r}$ for some r. So $za_1 \ldots a_{m-i-1}y_{i+1}$ is a weakly reduced expression of length $k + (m - i)$. Since $zxw = za_1 \ldots a_{m-i-1}y_{i+1}v_{k-i-1} \ldots v_1$ and since $v_{k-i-1} \notin \Gamma_r$, we have $l(zxw) = k + (m - i) + (k - i - 1) = 2k + m - 2i + 1$ which is odd.

Next let $i = k = \ell$. If $k < m$, then $y_{k+1} = a_{m-k}y_k \in \Gamma'_0$ and we have $zxw = za_1 \ldots a_{m-k-1}y_{k+1}$ is a weakly reduced expression of length $k + m - k = m$. If $k = m$, then we have $zxw = u_1 \ldots u_{k-1}(u_k y_k)$ has length $k = m$.

Finally, let $\ell = m < i \leq k$. Since m is even, $u_{k-i+m}, v_{k-i} \in \Gamma'_r$ for a suitable r and $y_{i+1} = u_{k-i+m}y_i v_{k-i} \in \Gamma'_r$. So $zxw = u_1 \ldots u_{k-i+m-1}y_{i+1}v_{k-i-1} \ldots v_1$ is a reduced expression of length $2k - 2i + m - 1$ which is odd, completing the proof. □

Corollary 3.4.8. *Let G be as in Lemma 3.4.7. If $\psi : G \to G$ is an automorphism such that $\psi(\Gamma_i) = \Gamma_i$, $i = 0, 1$. Then $R(\psi) = \infty$.*

Proof. Our hypothesis on ψ yields that $\psi(\Lambda) = \Lambda$. Choose elements $g_i \in \Gamma'_i$, $i = 0, 1$. Let $x_r = (g_0 g_1)^r$. Then $l(x_r) = 2r$.

We claim that $x_r, r \geq 1$, are in pairwise distinct ψ-twisted conjugacy classes. Suppose that $x_r \sim_\psi x_s$ for some $r \neq s$ and we shall arrive at a contradiction.

Since $x_r \sim_\psi x_s$, there exists an element $z \in G$ such that $x_s = zx_r\psi(z^{-1})$. Suppose that $z \in \Gamma_i$ for some i. First consider the case when $z \in \Lambda$. Then $\psi(z^{-1}) \in \Lambda$. Then $2s = l(x_s) = l(zx\psi(z^{-1})) = l(zx) = l(x) = 2r$ a contradiction since $s \neq r$. So assume that $z \in \Gamma_i \setminus \Lambda$. We suppose that $i = 0$, the case $i = 1$ is handled analogously. Now $l(zx_r\psi(z^{-1}) = 2r + 1$ when $zg_0 \in \Gamma_0 \setminus \Lambda$ or $l(zx_r\psi(z^{-1})) = 2r$ when $zg_0 \in \Lambda$. In either case $zx_r\psi(z^{-1}) \neq x_s$ as their lengths are not equal.

Suppose that $z \notin \Lambda$. Then $k := l(z) \geq 2$. Let $z = z_1 \cdots z_k$ be a weakly reduced expression. In view of our hypothesis on ψ we have $\psi(\Gamma'_j) = \Gamma'_j$ and so $w := \psi(z^{-1}) = w_k w_{k-1} \cdots w_1$ is a weakly reduced expression where $w_j := \psi(z_j^{-1})$. Also, for the same reason, $z_1, w_1 \in \Gamma'_j$ for some j. Therefore, by Lemma 3.4.7, we conclude that $l(zx_r w)$ is odd. Since $l(x_s) = 2s$ is even we again get a contradiction. This establishes our claim and we conclude that $R(\psi) = \infty$. □

We shall now prove Theorem 1.2.1 $\mathrm{GL}_2(F[t])$, and $\mathrm{SL}_2(F[t])$

Proof of Theorem 1.2.1 for $GL_2(F[t], SL_2(F[t])$. Let $R = F[t]$ and let $G = \mathrm{GL}_2(R)$ or $\mathrm{SL}_2(R)$. Then G is an amalgamated free product $\Gamma_0 *_\Lambda \Gamma_1$ where, using the notation of §**??**, we have $\Gamma_0 = G_0 = \mathrm{GL}_2(F), \Gamma_1 = B, \Lambda = B_0$ when $G = \mathrm{GL}_2(R)$ and $\Gamma_0 = \mathrm{SL}_2(F)$, $\Gamma_1 = B_1(R)$, $\Lambda = B_1$, when $G = SL_2(R)$.

The groups $\mathrm{GL}_2(R), \mathrm{SL}_2(R)$ have infinitely many usual conjugacy classes as noted in Remark 3.4.3. Therefore to show the R_∞-property for G, it suffices to show that $R(\psi) = \infty$ for a complete set of coset representatives of $\mathrm{Out}(G)$ by Lemma 3.4.2. Lemma 3.2.3 shows that given any automorphism ϕ of G, its class $[\phi] \in Out(G)$ is represented by an automorphism ψ such that $\psi(\Gamma_i) = \Gamma_i$. Thus the hypotheses of Lemma 3.4.7 hold and so applying Corollary 3.4.8 to ψ we see that $R(\psi) = \infty$. This completes the proof. □

3.4.6 Proof of Theorem 1.2.1 (iii)

A fixed subgroup of automorphisms of $\mathrm{GL}_2(A)$ stabilising G_0 and T

Suppose that $\phi \in \mathrm{Aut}(G)$ stabilises G_0 and T. Furthermore, assume that $\phi_0 = \phi|_{G_0} = \rho$, a Frobenius automorphism. We will show that $\mathrm{GL}_2(\mathbb{F}_p[s])$ is contained in $\mathrm{Fix}(\phi)$ where $s = t^{q-1} + t^{1-q}$.

We will only give the details when ϕ is of type (h, ε, u^i) with $\varepsilon = -1$. The arguments are similar in the case $\varepsilon = 1$.

We begin with the elementary observation that $\mathrm{GL}_2(\mathbb{F}_p)$ is fixed by $\phi_0 = \rho$. In particular, $e_{12}(1)$ and $e_{21}(1)$ are fixed by ϕ irrespective of the type of ϕ.

Case (i) : Suppose that ϕ is of type $(h, -1, 1)$.

Since for $x \in T_0$, $x^{q-1} = I$ we have
$\phi(\delta(t^{(q-1)k}, 1)) = h^{(q-1)k}\delta(t^{(1-q)k}, 1) = \delta(t^{(1-q)k}, 1)$ for all $k \in \mathbb{Z}$. Setting
$e_k := \delta(t^{(q-1)k}, 1).e_{12}(1).\delta(t^{(1-q)k}, 1) = e_{12}(t^{(q-1)k})$ for $k \in \mathbb{Z}$, we obtain that

$$\phi(e_k) = \phi(\delta(t^{(q-1)k}, 1).e_{12}(1).\phi(\delta(t^{(1-q)k}, 1)) = e_{12}(t^{(1-q)k}) = e_{-k}.$$

So $e_k e_{-k} = e_{12}(t^{(q-1)k} + t^{(1-q)k}) \in \mathrm{Fix}(\phi)$ for all $k \in \mathbb{Z}$. Set $s = t^{q-1} + t^{1-q}$. As $\mathbb{F}_p[s]$ equals the $\mathbb{F}_p$-span of $1, t^{(q-1)k} + t^{(1-q)k}, k \geq 1$, we see that $U(\mathbb{F}_p[s])$ is contained in $\mathrm{Fix}(\phi)$. Since, as observed already, $\mathrm{GL}_2(\mathbb{F}_p) \subset \mathrm{Fix}(\phi)$, it follows by Nagao's theorem (recalled in 3.2.1) that $\mathrm{GL}_2(\mathbb{F}_p[s]) \subset \mathrm{Fix}(\phi)$.

Case (ii) : Suppose that ϕ is of type $(h, -1, u)$. We have
$\phi(\delta(t^{(q-1)k}, 1)) = \delta(1, t^{(1-q)k})$ and $\phi(\delta(t^{(1-q)k}, 1)) = \delta(1, t^{(q-1)k})$ for all $k \in \mathbb{Z}$. Hence, for any $k \in \mathbb{Z}$, we obtain that

$$\phi(e_k) = \phi(\delta(t^{(q-1)k}, 1)).e_{12}(1).\phi(\delta(t^{(1-q)k}, 1)) = e_{12}(t^{(q-1)k}) = e_k.$$

As in Case (i), we see the that $\mathrm{GL}_2(\mathbb{F}_p[s]) \subset \mathrm{Fix}(\phi)$ where $s = t^{q-1} + t^{1-q}$. In fact in this case we obtain that the groups $\mathrm{GL}_2(\mathbb{F}_p[t^{q-1}]), \mathrm{GL}_2(\mathbb{F}_p[t^{1-q}])$ and hence the subgroup generated by them, is contained in $\mathrm{Fix}(\phi)$.

Remark 3.4.9. Suppose that $\phi_0 = \rho \circ \epsilon$. Then the fixed subgroup of ϕ does not contain any non-diagonal upper (or lower) triangular matices. In particular, it does not contain a subgroup of the form $\mathrm{SL}_2(R)$ for any subring $R \subset A$. Neverthless, the following observation will be useful in showing that $R(\phi) = \infty$.

We have for all $k \in \mathbb{Z}$,

$$\phi(\delta(t^{k(q-1)}, 1)) = \begin{cases} \delta(t^{k(1-q)}, 1) & \text{if } \phi \text{ is of type } (h, -1, 1), \\ \delta(1, t^{k(1-q)}) & \text{if } \phi \text{ is of type } (h, -1, u). \end{cases}$$

Since $\phi(e_{12}(1)) = e_{21}(-1)$ and $\phi(e_{21}(1)) = e_{12}(-1)$, we obtain that $\phi(e_k)$ equals $(e'_{-k})^{-1}$ or $(e'_k)^{-1}$ according as ϕ is of type $(h, -1, 1)$ or $(h, -1, u)$ respectively. So, in both case, $\phi(e_k e_{-k}) = (e'_k e'_{-k})^{-1}$. Similarly $\phi(e'_k e'_{-k}) = (e_k e_{-k})^{-1}$. Since $k \in \mathbb{Z}$ is arbitrary, this implies that $\phi(e_{12}(s^k)) = e_{21}(s^k)^{-1} = e_{21}(-s^k)$ and
$\phi(e_{21}(s^k)) = e_{12}(-s^k)$ for all $k \geq 1$, where $s = t^{q-1} + t^{1-q}$.

Proof of Theorem 1.2.1 for $\mathbf{GL}_2(\mathbb{F}_q[t, t^{-1}])$

We are now ready to prove Theorem 1.2.1 for $G = \mathrm{GL}_2(A)$ where $A = \mathbb{F}_q[t, t^{-1}]$.

Proof. Let $\psi : G \to G$ be any automorphism of $G = \mathrm{GL}_2(A)$ where $A = \mathbb{F}_q[t, t^{-1}]$. Since $R(\psi) = R(\phi)$ if ϕ and ψ represent the same outer automorphism of G, we need only show that $R(\phi) = \infty$ where ϕ is an automorphism as in Lemma 3.3.4.

Suppose that ϕ is of type (h, ε, u^i), $\varepsilon \in \{1, -1\}, i \in \{0, 1\}, h \in T_0$. There are two cases to consider depending on the value of ε.

Let $\varepsilon = 1$. The subgroup $K := \{g \in G \mid \det(g) \in \mathbb{F}_q^\times\} = gp\langle \mathrm{SL}_2(A), \lambda I \mid \lambda \in \mathbb{F}_q^\times \rangle$ is characteristic in $\mathrm{GL}_2(A)$ and $G/K \cong \mathbb{Z}$ generated by $\delta(t, 1)K$. Denote by $\bar{\phi}$ the automorphism of G/K induced by ϕ. Then $\bar{\phi}$ induces the identity on G/K. So $R(\bar{\phi}) = \infty$ and therefore $R(\phi) = \infty$ by Lemma 3.4.1.

Suppose that $\varepsilon = -1$. There are two cases to consider depending on whether ϕ_0 equals ρ or $\rho \circ \epsilon$. We use the notations of §3.4.6.

Case (i) : $\phi_0 = \rho$, a Frobenius automorphism.

With notations as in §3.4.6, set $x_m := e_{12}(s^m)e_{21}(-s^m) \in \mathrm{Fix}(\phi)$. Then by Lemma

3.4.5, we see that for a fixed $k \geq 1$, the x_m^k are in pairwise distinct conjugacy classes in G. Taking $k = o(\phi)$, it follows from Lemma 3.4.4 that the x_m are in pairwise distinct ϕ-twisted conjugacy classes of G.

Case (ii) : $\phi_0 = \rho \circ \epsilon$.

We claim that the elements $e_{12}(s^m), m \geq 1$, are in pairwise distinct ϕ-twisted conjugacy classes. The proof of the claim is exactly as in Case $\rho \circ \epsilon$ in 3.4.2. Let $r = o(\phi)$. As seen in Remark 3.4.9, $\prod_{0 \leq j < 2r} \phi^j(e_{12}(s^m)) = (e_{12}(s^m).e_{21}(-s^m))^r$. Setting $x_m := e_{12}(s^m).e_{21}(-s^m)$, it was shown in Lemma 3.4.5 that $\{\mathrm{tr}(x_m^r)\}_{m \geq 1}$ are pariwise distinct. Hence, by Lemma 3.4.4(i), the $e_{12}(s^m), m \geq 1$, are in pairwise distinct ϕ-twisted conjugacy classes. □

3.5 Some related open problems

The following problems come up naturally from the already done work.

(1) Let $R = F[t]$, or $F[t, t^{-1}]$ where $F \subset \bar{\mathbb{F}}_p$ and let I be a nonzero ideal in R. Determine whether the *congruent* subgroup $\mathrm{GL}_n(R, I)$ (resp. $\mathrm{SL}_n(R, I)$), $n \geq 2$, defined as the kernel of the natural homomorphism $\mathrm{GL}_n(R) \to \mathrm{GL}_n(R/I)$ (resp. $\mathrm{SL}_n(R) \to \mathrm{SL}_n(R/I)$) has the R_∞-property.

(2) Let $R = F[t]$, or $F[t, t^{-1}]$ where $F \subset \bar{\mathbb{F}}_p$. Determine whether the finite index subgroups of $\mathrm{GL}_n(R)$ and $\mathrm{SL}_n(R), n \geq 2$, have R_∞-property.

(3) Let A be a the ring obtained from R (as in (1)) by inverting finitely many irreducible polynomials. Determine whether $\mathrm{GL}_n(A)$, resp. $\mathrm{SL}_n(A)$, $n \geq 2$, has the R_∞-property.

(4) Fix $k \in \mathbb{N}$. Let $R = F[t_1, .., t_k]$ (F as in (i)). Determine whether $\mathrm{GL}_n(R)$ and $\mathrm{SL}_n(R), n \geq 3$, have R_∞-property. We note here that for any associative free algebra A over a commutative field F on at most countably many free generators

$GL_2(A)$ is isomorphic to $GL_2(F[t])$ (see [**?**]).

Quasi-isometries of Z^n and twisted conjugacy in certain linear groups

Abstract

The thesis consists of two parts, which are independent of each other.

In the first part, we study the quasi-isometry group of the finitely generated abelian group of rank ≥ 2. We show that certain groups of diffeomorphisms embed into it and therefore, conclude that this group is "large".

In the second part, which is the major part of the thesis, we study twisted conjugacy in general and special linear groups over polynomial and Laurent polynomial rings over subfields of the algebraic closure of finite fields.

LIST OF PUBLICATIONS ARISING FROM THE THESIS

Journal

1. *Embedding certain diffeomorphism groups into the quasi-isometry groups of Euclidean* spaces Joint with Parameswaran Sankaran, Topology and its Applications, Vol 265 (2019). DOI: 10.1016/j.topol.2019.106833

Submitted

1. *Twisted conjugacy in general linear groups over polynomial algebras over finite fields* Joint with Parameswaran Sankaran, axXiv:1912.10184.

Thesis Title :
Quasi-isometries of Z^n and twisted conjugacy in certain linear groups

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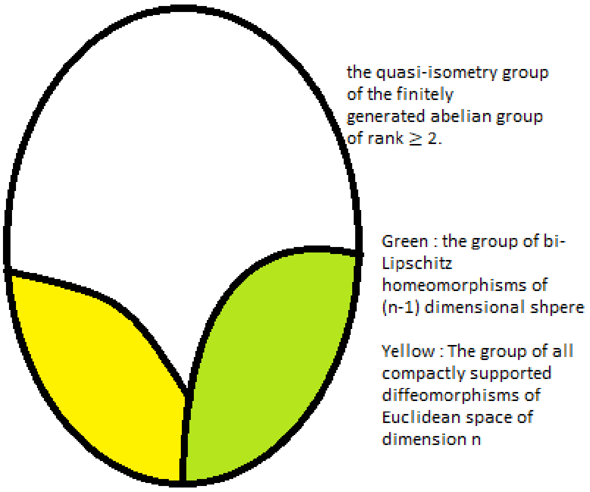
In the first part, we study the quasi-isometry group of the finitely generated abelian group of rank ≥ 2. We show that certain groups of diffeomorphisms embed into it and therefore, conclude that the group is “large”.



In the second part, which is the major part of the thesis, we study twisted conjugacy in general and special linear groups G over polynomial and Laurent polynomial rings over subfields of the algebraic closure of finite fields.
We have the following results.
(i) The general and special linear groups of rank ≥ 2 over polynomial and Laurent polynomial rings over subfields of the algebraic closure of finite fields have infinitely many twisted conjugacy classes
for every automorphism.
(ii) The general and special linear groups of rank 1 over polynomial ring over subfields of the algebraic closure of finite
fields have infinitely many twisted conjugacy classes for every automorphism.
(iii) The general linear groups of rank 1 over Laurent
polynomial ring over finite fields have infinitely many twisted conjugacy classes
for every automorphism.
(iv) Any subgroup of the general linear group of rank ≥ 2 over polynomial and Laurent polynomial rings over subfields of the algebraic closure of finite fields that contains the special linear group of rank ≥ 2 over the same rings have infinitely many twisted conjugacy classes for every automorphism.