Cohomology of Generalized Dold Manifolds

By

Manas Mandal

MATH10201804004

The Institute of Mathematical Sciences, Chennai

A thesis submitted to the

Board of Studies in Mathematical Sciences

In partial fulfillment of requirements

for the Degree of

DOCTOR OF PHILOSOPHY

of

HOMI BHABHA NATIONAL INSTITUTE

November, 2024

Homi Bhabha National Institute

Recommendations of the Viva Voce Committee

As members of the Viva Voce Committee, we certify that we have read the dissertation prepared by Manas Mandal entitled "Cohomology of Generalized Dold Manifolds" and recommend that it may be accepted as fulfilling the thesis requirement for the award of Degree of Doctor of Philosophy.

Chairman - Sankaran Viswanath — Date: 14/11/2024

Guide/Convenor - Pralay Chatterjee — Date: 14/11/2024

Co-guide - Parameswaran Sankaran — Date: 14/11/2024

Examiner - Mahender Singh — Date: 14/11/2024

Member 1 - Anirban Mukhopadhyay — Date: 14/11/2024

Member 2 - Sushmita Venugopalan — Date: 14/11/2024

Member 3 - V. Uma — Date: 14/11/2024

Final approval and acceptance of this thesis is contingent upon the candidate's submission of the final copies of the thesis to HBNI.

I hereby certify that I have read this thesis prepared under our direction and recommend that it may be accepted as fulfilling the thesis requirement.

Date: 14/11/2024

Place: Chennai

Pralay Chatterjee(Guide)   Parameswaran Sankaran(Co-guide)

STATEMENT BY AUTHOR

This dissertation has been submitted in partial fulfillment of requirements for an advanced degree at Homi Bhabha National Institute (HBNI) and is deposited in the Library to be made available to borrowers under rules of the HBNI.

Brief quotations from this dissertation are allowable without special permission, provided that accurate acknowledgement of source is made. Requests for permission for extended quotation from or reproduction of this manuscript in whole or in part may be granted by the Competent Authority of HBNI when in his or her judgement the proposed use of the material is in the interests of scholarship. In all other instances, however, permission must be obtained from the author.

Manas Mandal

DECLARATION

I hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Manas Mandal

CERTIFICATION ON ACADEMIC INTEGRITY

1. I Manas Mandal, HBNI Enrolment number: MATH10201804004, hereby declare that the thesis titled “Cohomology of generalized Dold manifolds” is prepared by me and is the original work undertaken by me.

2. I also hereby undertake that this document has been duly checked through a plagiarism detection tool and the document is found to be plagiarism free as per the guidelines of the Institute/UGC.

3. I am aware and undertake that if plagiarism is detected in my thesis at any stage in the future, suitable penalty will be imposed as applicable as per the guidelines of the Institute/UGC.

Date: December 24, 2024

Manas Mandal

Endorsement by the thesis supervisor

I certify that the thesis written by the researcher is plagiarism free as mentioned above by the student.

Date: December 24, 2024

Pralay Chatterjee
Professor
IMSc, Chennai.

LIST OF PUBLICATIONS ARISING FROM THE THESIS

Journal

1. *Cohomology of generalized Dold spaces*, Mandal, Manas; Sankaran, Parameswaran, Topology Appl.310(2022), Paper No. 108040, 16 pp.

Communicated

1.*Cohomology and K-theory of generalized Dold manifolds fibred by complex flag manifolds*, Mandal, Manas; Sankaran, Parameswaran

Manas Mandal

Dedicated to *Maa, Baba, Bhai*

ACKNOWLEDGEMENTS

First and foremost, I would like to express my deepest gratitude to my supervisor, Prof. Parameswaran Sankaran. His exceptional guidance and support have been instrumental throughout my research journey. His patience and understanding have been crucial in overcoming many challenging times. Prof. Sankaran's encouragement and wisdom have inspired me to push the boundaries of my knowledge and pursue my academic goals. I am incredibly fortunate for the opportunity to work with him.

I also extend my sincere thanks to Prof. Pralay Chatterjee for teaching me topology and being my official guide. I have greatly benefited from his topology course and his adept handling of official matters and support in various aspects at IMSc, which deserve special appreciation.

I would like to acknowledge all the Mathematics faculty members at IMSc. I appreciate the courses taught by Prof. S. Viswanath, Prof. Pralay Chatterjee, Prof. Amritanshu Prasad, Prof. Dishant M. Pancholi, and Prof. Sundar Sobers. Special thanks to Prof. Sushmita Venugopalan for assisting me in my research methodology and to Prof. Sanoli Gun for her help in finding a supervisor and recommending me to Prof. Sankaran. I am deeply thankful to all my teachers since childhood, especially Anindita Mam, Binoy Sir, Suprakash Sir, Umapada Sir, Surja da, Tanujit Sir, Nabin da, Asit da, Atasi Mam, Avishek Sir, Suparna Mam, Sunil Sir, and Sudip Kumar Acharya Sir, for their guidance and support.

I appreciate the efforts of the administration and canteen staff at IMSc. I am grateful to R. Indra for her prompt assistance in various situations, and to Vasan for helping with many computer-related issues.

My IMSc mates have played an essential role in my academic and personal life.

While it is hard to name everyone, a few who come to mind are Ankur, Sidhu, Bera, Susho, Apurba, Vinod, Saha, Sathish, Aritra, Sourav, Manav, Sunil, Jyoti, Prithijit, Amit Kumar, Dhananjaya, Mrityunjoy, Sushant, Arunabha, Subrat, Hitesh, Sengupta, Sumit, Roni, Ravi, Anupam da, Sabiar da, Sujoy da, Pranendu da, Ujjal da, Snehajit da, Rupam da, Digjoy da, Sahil da, Arpan da, Debu da, Mrinal da, Arghya da, and many others for their companionship and support. I also want to thank all my badminton partners for sharing countless enjoyable and refreshing moments on the court, especially Apurba, Susho, Saha, Jyoti, Sadhana, Pavitra, Anupam da, Amit Kumar, Amit Suthar, and Shashikant. I cherish the companionship and random chats during the "5 o'clock Chay Adda", as well as the enjoyable cooking experiences and the food we shared in the "Robi-ranna" gatherings.

My heartfelt thanks go to my friends from my M.Sc.—Sanu, Sidhu, Samir, Suman, Sutirtha, Subrata, Swarup, Gopal, Chandan, Sourav, Asim, Sugata, Saikat, Bijoy, Shubhankar, Arnab, Shushovan, and many others—as well as to the incredibly supportive seniors, Swaraj da, Priyanshu da, Mrinal da, Dibakar da, Joy da, Mirazul da, Jayanta da, Subrata da, and more. Their companionship and support made my time in Kolkata memorable and enriching. I also extend my gratitude to friends from B.Sc., Swarnendu, Rajkumar, Umapada, Arun, Subha, Ajay, Madhumita, Suprakash, Krisnarjun, Dipankar, and many more. I would also like to thank my school friends, Avishek, Suman, Rupam, Ajay, Sahadeb, Saroj, Debasis, Shubhajit, Bivash, Pranay, Pathik, Gayatri, Priya, Mukta, Bratati, Irani, Ardhendu, Papun, Shibsankar, Nishikanta, and many others for their friendship and support.

I am deeply indebted to my parents and family for their unwavering love and encouragement. Their support has been a constant source of strength and inspiration.

Lastly, I want to express my gratitude to every member of IMSc for their contribution to the welcoming environment at the institute.

Contents

Summary **i**

Notations **iii**

1 Introduction **1**

1.1 Cohomology groups of GDS . . . 3

1.2 Mod 2 cohomology algebras of some GDS . . . 5

1.2.1 Cohomology of $P(S, X)$, where X is a torus manifold . . . 6

1.2.2 Cohomology of $P(S, \mathbb{C}G(\nu))$ . . . 7

1.3 Cohomology of $P(\mathbb{S}^m, \mathbb{C}G(\nu))$ . . . 9

1.3.1 Integral (co)homology of $P(\mathbb{S}^m, \mathbb{C}G(\nu))$ . . . 9

1.3.2 The ring structure of $H^*(P(\mathbb{S}^m, \mathbb{C}G(\nu)); \mathbb{Z}[1/2])$ . . . 11

1.4 K-theory of $P(\mathbb{S}^m, \mathbb{C}G(\nu))$ . . . 12

1.5 Applications . . . 13

2 Preliminaries **15**

2.1 σ-conjugate vector bundles . . . 16

2.2 Functoriality . . . 18

2.3 The Stiefel-Whitney classes of $\hat{\omega}$ . . . 19

2.4 Torus manifolds . . . 21

2.5 Half-spin bundles over $\mathbb{S}^{2r}$ . . . 23

3 Cohomology groups of GDS **27**

3.1 Mod 2 cohomology of GDS . . . 27

3.2 Chomology of GDS in $\mathbb{Z}[1/2]$ coefficients . . . 31

4 Mod 2 cohomology algebra of some GDS **39**

4.1 Cohomology of generalized Klein's bottles . . . 39

4.2 Cohomology of $P(S, X)$, where X is a torus manifold . . . 42

4.3 Cohomology of $P(S, \mathbb{C}G(\nu))$ . . . 48

5 Cohomology of $P(\mathbb{S}^m, \mathbb{C}G(\nu))$ **57**

5.1 Schubert cell structure of $\mathbb{C}G(\nu)$ . . . 58

5.2 A cell structure on $P(\mathbb{S}^m, \mathbb{C}G(\nu))$ . . . 60

5.3 Integral (co)homology of $P(\mathbb{S}^m, \mathbb{C}G(\nu))$ . . . 62

5.4 The ring structure of $H^*(P(\mathbb{S}^m, \mathbb{C}G(\nu)); \mathbb{Z}[1/2])$ . . . 69

6 K-theory of $P(\mathbb{S}^m, \mathbb{C}G(\nu))$ **75**

6.1 The additive structure of $K^*(P(\mathbb{S}^m, \mathbb{C}G(\nu)))$ . . . 76

6.2 The ring structure of $K^0(P(\mathbb{S}^m, \mathbb{C}G(\nu)))$. . . . 79

7 Applications **91**

7.1 Equivariant cohomology of (X, σ) 91

7.2 Fixed-points of self maps of $P(\mathbb{S}^m, \mathbb{C}G_{n,k})$ 92

7.3 Degrees of maps between certain generalized Dold manifolds 95

Bibliography **101**

Summary

A Dold manifold is defined as the quotient space $\mathbb{S}^m \times \mathbb{C}P^n/\sim$, where $(s, L) \sim (-s, \bar{L})$. These manifolds were first introduced by Albrecht Dold in 1956 to construct generators in odd dimensions for Thom's unoriented cobordism ring (see [Dol56]). The above definition was generalized by Nath and Sankaran to a broader class of manifolds in order to study certain manifold-properties, which they termed *generalized Dold manifolds* (see [NS19]). We generalize this even further and call it *generalized Dold spaces* (GDS) in [MS22] to study cohomology of these spaces.

A GDS is defined as follows: Consider two paracompact Hausdorff topological spaces S, X with two involutions α, σ on S, X respectively. Assume α is fixed point free and σ has fixed points. Then the GDS, denoted by $P(S, X)$, is defined to be the quotient space of $S \times X$ by the free $\mathbb{Z}_2$ action generated by the involution $\theta := \alpha \times \sigma$.

When S, X are CW complexes satisfying certain additional hypotheses, we show that, as $\mathbb{Z}_2$-vector spaces, $H^*(P(S, X); \mathbb{Z}_2) \cong H^*(Y; \mathbb{Z}_2) \otimes H^*(X; \mathbb{Z}_2)$ where $Y = S/\mathbb{Z}_2$. We also obtain the same result when $H^*(X; \mathbb{Z}_2)$ is generated by the Stiefel-Whitney classes of finitely many σ-conjugate (complex) vector bundles. See Proposition 3.1.2. In our study, we consider the integral cohomology groups of $P(S, X)$ assuming that X has a CW structure with cells only in even dimensions and that each cell is stable by the involution σ and certain conditions on S.

We obtain a formula for Stiefel-Whitney classes of the real vector bundle over

$P(S,X)$ associated to a σ-conjugate vector bundle over X when $H^1(X;\mathbb{Z}_2) = 0$. This formula is applied to obtain the *ring* structure of $H^*(P(S,X);\mathbb{Z}_2)$ when X is a torus manifold whose torus quotient is a homology polytope, or is a complex flag manifold. See Theorem 4.2.4 and Theorem 4.3.4. As an application, we compute the equivariant cohomology ring $H^*_{\mathbb{Z}_2}(X;\mathbb{Z}_2)$ (see Theorem 7.1.1).

We determine the integral cohomology *groups* of $P(m,\nu) = P(\mathbb{S}^m, \mathbb{C}G(\nu))$, where $\mathbb{C}G(\nu)$ is a complex flag manifold of type $\nu = (n_1, n_2, \ldots, n_s)$. In fact the additive structure of $H^*(P(S,X);\mathbb{Z})$ is obtained for a much wider class of spaces.

A description of $H^*(P(m,\nu);R)$ as a quotient of a polynomial ring is given in Theorem 5.4.2, when R is any commutative ring with identity in which 2 is invertible.

We have presented the K-groups of $P(m,\nu) = P(\mathbb{S}^m, \mathbb{C}G(\nu))$ as:
(i) $K^0(P(m,\nu)) \cong \mathbb{Z}^{b_e} \oplus \mathbb{Z}_{2^{\lfloor m/2 \rfloor}} \oplus A_0$ where $o(A_0) = 2^t$ for some t, $0 \leq t \leq b'_e - \lfloor m/2 \rfloor$,
(ii) $K^1(P(m,\nu)) \cong \mathbb{Z}^{b_o} \oplus A_1$ where $o(A_1) = 2^t$ for some t, $0 \leq t \leq b'_o$,
where b_e, b_o, b'_e, b'_o are computed in terms of m and ν. See Proposition 6.1.1.

We construct certain canonical complex vector bundles over $P(m,\nu)$ and consider a subring $\mathcal{K}^0$ of $K^0(P(m,\nu))$ generated by the classes of those vector bundles and show that as an abelian group, $K^0(P(m,\nu))/\mathcal{K}^0$ is finite. See Theorem 6.2.11.

As an application of Theorem 5.4.2, we provide a criterion for the existence of fixed points of a self map on the generalized Dold manifolds $P(m,n,k) = P(\mathbb{S}^m, \mathbb{C}G_{n,k})$, where $\mathbb{C}G_{n,k}$ is a complex Grassmannian manifold. See Proposition 7.2.4.

We have another application: Let $f : P(m,n,k) \to P(r,s,t)$ be a continuous map between two distinct oriented same dimensional generalized Dold manifolds satisfying the conditions: (i) $\lfloor k/2 \rfloor < \lfloor t/2 \rfloor$ if $m = r$ is odd, $k < t$ if $m = r$ is even, and (ii) $(m, 2t(s-t)) \neq (2,2)$. Then degree of f is zero. See Proposition 7.3.1.

Notations

$\mathbb{Z}$	The set of integers.
$\mathbb{Q}$	The set of rational numbers.
$\mathbb{R}$	The set of real numbers.
$\mathbb{C}$	The set of complex numbers.
$\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$	The set of integers modulo m.
$[n]$	The set $\{1, 2, 3, \cdots, n\}$.
$\lfloor n \rfloor$	The greatest integer $\leq n$.
α	Typically a free involution on S.
	Antipodal map when S is a sphere.
σ	Typically an involution on X with fixed points.
	The involution induced from conjugation on $\mathbb{C}^n$ when $X = \mathbb{C}G(\nu)$.
θ	$\alpha \times \sigma$.
$P(S, X)$	The generalized Dold space $S \times X/\sim_\theta,$ where $\theta := \alpha \times \sigma$.
$\mathbb{S}^m$	The unit sphere in $\mathbb{R}^{m+1}$.
$\mathbb{R}P^m$	The real projective space of real dimension m.
$\mathbb{C}P^n$	The complex projective space of complex dimension n.
$\mathbb{C}G_{n,k}$	The complex Grassmannian manifold consisting of
	k-dimensional complex subspaces of $\mathbb{C}^n$.
$\mathbb{C}G(\nu)$	The complex flag manifold of type ν.

ν	A finite sequence $(n_1, n_2, \ldots, n_s)$ of positive numbers.
$\lvert\nu\rvert$	$n = \sum_{1\le j\le s} n_j$.
ν_o	The number of odd numbers in the sequence $\nu = (n_1, n_2, \ldots, n_s)$.
$\lfloor \nu/2 \rfloor$	The sequence $(n'_1, n'_2, \ldots, n'_s)$ where $\nu = (n_1, n_2, \ldots, n_s)$ and $n'_j = \lfloor n_j/2 \rfloor$ for $j = 1, 2, \ldots, s$.
$\binom{n}{\nu}$	The multinomial coefficient $n!/(n_1! \cdots n_s!)$.
S_n	The permutation group on $[n]$.
S_ν	$S_{n_1} \times \cdots \times S_{n_s} \subset S_n$ where $\nu = (n_1, n_2, \ldots, n_s)$.
$O(n)$	The orthogonal group of $\mathbb{R}^n$.
$SO(n)$	The special orthogonal group of $\mathbb{R}^n$.
$\mathrm{Pin}(m)$	The pin group, a double cover of $O(m)$.
$\mathrm{Spin}(m)$	The spin group, a double cover of $SO(m)$.
$U(n)$	The unitary group of $\mathbb{C}^n$.
$SU(n)$	The special unitary group of $\mathbb{C}^n$.
$U(\nu)$	$U(n_1) \times U(n_2) \times \cdots U(n_s) \subset U(n)$ where $\nu = (n_1, n_2, \ldots, n_s)$.
$\epsilon_{\mathbb{R}}$	The trivial real line bundle.
$\epsilon_{\mathbb{C}}$	The trivial complex line bundle.
ξ	A twisted line bundle.
$\omega_{\mathbb{C}}$	The complexified vector bundle of a vector bundle ω.
$\hat{\omega}$	A real vector bundle of rank $2r$ over $P(S, X)$ associated to a σ-conjugate complex vector bundle ω over X of rank r. See §2.1
$\gamma_{n,k}$	The canonical complex k-plane bundle over $\mathbb{C}G_{n,k}$.
$\beta_{n,k}$	The canonical complex $(n-k)$-plane bundle over $\mathbb{C}G_{n,k}$.
Δ	$\mathrm{Spin}(m)$ complex representation. See §2.5
Δ^+, Δ^-	Half-spin representations of $\mathrm{Spin}(2r)$. See §2.5
ξ^+, ξ^-	The complex vector bundles over $\mathbb{S}^m = \mathrm{Spin}(m+1)/\mathrm{Spin}(m)$ associated to representations Δ^+, Δ^- respectively. See §2.5

η^-	The vector bundle $\alpha^!(\xi^+)$ over $\mathbb{S}^{2r}$ where α is the antipodal map.
$\tilde{\xi}$	The vector bundle $\xi^+ \oplus \eta^-$ over $\mathbb{S}^m$.
$\tilde{\alpha}$	A bundle isomorphism of $\tilde{\xi}$ covering α such that $\tilde{\alpha} \circ \tilde{\alpha} = id_{\tilde{\xi}}$.
ξ^0	The complex vector bundle over $\mathbb{S}^{2r}$,
	whose total space is $E(\xi^0) := E(\tilde{\xi})/\langle\tilde{\alpha}\rangle$.
e, e_j	Closed cells in B.
$\mathring{e}$	The open cell corresponding to the cell e.
$e^{\pm}$	For a cell e in B, $e^{\pm}$ are closed cells in $\tilde{B}$
	such that $p(e^{\pm}) = e$ as oriented cells.
$\varepsilon^{\pm}$	$(e^+ \pm e^-)/2$.
$\mathbf{i}$	$(i_1, i_2, \ldots, i_k)$, where $1 \leq i_p < i_q \leq n$ if $p < q$.
$I(n;k)$	The set of all $\mathbf{i} = (i_1, i_2, \ldots, i_k), 1 \leq i_p \leq n$.
$P(m,\nu)$	$\mathbb{S}^m \times \mathbb{C}G(\nu)/\sim$, where $(x, \mathbf{L}) \sim (-x, \bar{\mathbf{L}})$.
$X(\mathbf{i})$	The Schubert variety (cell) corresponding to $\mathbf{i}$. See §5.1
$I(\nu)$	The indexing set of all Schubert cells in $\mathbb{C}G(\nu)$. See §5.1
$\ell(\mathbf{i})$	$\dim_{\mathbb{C}} X(\mathbf{i})$.
$I_e(\nu)$	$\{\mathbf{i} \in I(\nu) \mid \ell(\mathbf{i}) \equiv 0 \pmod 2\}$.
$I_o(\nu)$	$\{\mathbf{i} \in I(\nu) \mid \ell(\mathbf{i}) \equiv 1 \pmod 2\} = I(\nu) \setminus I_e(\nu)$.
$\mathcal{B}_q$	A $\mathbb{Z}$-basis of $H_q(P(m,\nu);\mathbb{Z})/\text{torsion}$. See §5.3
$\mathcal{B}'_q$	A $\mathbb{Z}_2$-basis of the torsion subgroup of $H_q(P(m,\nu);\mathbb{Z})$. See §5.3
$\mathcal{B}$	$\bigcup_{q\geq 0} \mathcal{B}_q$.
$\mathcal{B}'$	$\bigcup_{q\geq 0} \mathcal{B}'_q$.
$\mathcal{B}_e$	$\bigcup_{q\geq 0} \mathcal{B}_{2q}$.
$\mathcal{B}_o$	$\bigcup_{q\geq 0} \mathcal{B}_{2q+1}$.
$\mathcal{B}'_e$	$\bigcup_{q\geq 0} \mathcal{B}'_{2q}$.
$\mathcal{B}'_o$	$\bigcup_{q\geq 0} \mathcal{B}'_{2q+1} = \mathcal{B}' \setminus \mathcal{B}'_e$.

b_e — $|\mathcal{B}_e|$.

b_o — $|\mathcal{B}_o|$.

β_e — $|\mathcal{B}'_e|$.

β_o — $|\mathcal{B}'_o|$.

b'_e — β_o.

b'_o — β_e.

ℓ_e — $|I_e(\nu)|$.

ℓ_o — $|I_o(\nu)|$.

γ_j — The canonical complex n_j-plane bundle over $\mathbb{C}G(\nu)$ whose fibre over $\mathbf{L} = (L_1, \ldots, L_s)$ is L_j.

$c_j(\omega)$ — The j-th Chern class of ω.

$c(\omega, t)$ — The total Chern polynomial $1 + c_1(\omega)t + c_2(\omega)t^2 + \cdots + c_r(\omega)t^r$.

δ_m — $[\xi^+] - 2^{r-1} = 2^{r-1} - [\xi^-]$ if m is even, and 0 if m is odd.

$\tilde{\xi}(\omega)$ — $\xi^+ \otimes \omega \oplus \eta^- \otimes \bar{\omega}$ for any σ-conjugate vector bundle ω over $\mathbb{C}G(\nu)$.

$\tilde{\theta}$ — $E(\tilde{\xi}(\omega)) \to E(\tilde{\xi}(\omega))$ is a bundle involution that covers θ defined as: $\tilde{\theta}(e^+ \otimes u, e^- \otimes v) = (\tilde{\alpha}^-(e^-) \otimes \bar{v}), \tilde{\alpha}^+(e^+) \otimes \bar{u}))$.

$\xi^0(\omega)$ — The complex vector bundle over $P(m, \nu)$, whose total space is $E(\tilde{\xi}(\omega))/\langle\tilde{\theta}\rangle$.

$\mathcal{K}$ — The subring of $K^0(\mathbb{S}^m \times \mathbb{C}G(\nu))$ generated by the elements (i) $x + \bar{x}$ and, (ii) $\delta_m(x - \bar{x})$ when m is even, as x varies in $K^0(\mathbb{C}G(\nu))$.

$\mathcal{K}^0$ — The subring of $K^0(P(m, \nu))$ generated by the following elements: (i) $[\xi_{\mathbb{C}}]$, $[\hat{\omega}_{\mathbb{C}}]$, and, (ii) when m is even, $[\xi^0(\omega)]$, where ω varies over σ-conjugate complex vector bundles over $\mathbb{C}G(\nu)$.

Chapter 1

Introduction

The classical Dold manifold $P(m,n)$ is defined as the orbit space of the $\mathbb{Z}_2$-action on $\mathbb{S}^m \times \mathbb{C}P^n$ generated by the involution $(v,[z]) \mapsto (-v,[\bar{z}]), v \in \mathbb{S}^m, [z] \in \mathbb{C}P^n$. Here $[\bar{z}]$ denotes $[\bar{z}_0 : \cdots : \bar{z}_n]$ when $[z] = [z_0 : \cdots : z_n] \in \mathbb{C}P^n$. These manifolds were first introduced by Albrecht Dold in 1956 to give explicit generators in odd dimensions for Thom's unoriented cobordism ring. See [Dol56].

The generalized Dold manifold $P(m,X)$ was introduced in [NS19] as the quotient of $\mathbb{S}^m \times X$ under the identification $(v,x) \sim (-v,\sigma(x))$, where $\sigma : X \to X$ be a complex conjugation on an almost complex manifold (X,J), that is, σ is an involution with non-empty fixed point set such that, for any $x \in X$, the differential $T_x\sigma : T_xX \to T_{\sigma(x)}X$ satisfies the equation $J_{\sigma(x)} \circ T_x\sigma = -T_x\sigma \circ J_x$. See [CF64, §24]. The main focus in [NS19] was the study of manifold-properties of $P(m,X)$ such as the description of its tangent bundle, a formula for its total Stiefel-Whitney class, the (stable) parallelizability and related properties, and its cobordism class.

Sarkar and Zvengrowski [SZ22] have constructed smooth manifolds which are a simultaneous generalization of projective product spaces due to Davis [Dav10] and of Dold manifolds and call them *generalized projective product spaces*. These are $\mathbb{Z}_2$-quotients $P(M,N)$ of products $M \times N$ by the diagonal action where M, N are manifolds admitting $\mathbb{Z}_2$-actions where the action on M is assumed to be free. When the fixed point set for $\mathbb{Z}_2$-action on N is non-empty, they are also generalized Dold spaces in our sense. Sarkar and Zvengrowski also obtain results on $\mathbb{Z}_2$-cohomology in many special cases, including when M is a product of spheres and N is a quasi-toric manifold, besides results on manifold-properties of generalized projective product spaces.

Here, our aim is to study cohomology and complex K-theory of the generalized Dold manifolds. While studying the homotopical/homological properties, it is natural to do away with stringent requirements such as X to be an almost complex manifold. Also we replace $(\mathbb{S}^m, \text{antipode})$, by a pair (S, α) where S is, say, a paracompact Hausdorff topological space and $\alpha : S \to S$ a fixed point free involution. Likewise, X is any Hausdorff topological space with an involution $\sigma : X \to X$ having a non-empty fixed point set. Then $P(S, \alpha, X, \sigma)$ (or more briefly $P(S, X)$) is the space $S \times_{\mathbb{Z}_2} X = S \times X/\sim$ where $(v, x) \sim (\alpha(v), \sigma(x))$ which is the quotient of $S \times X$ by the free $\mathbb{Z}_2$-action generated by $\alpha \times \sigma$.

Denote by Y the space $S/\sim_\alpha$. Then a GDS, $P(S, X)$ is the total space of a fibre bundle $X \hookrightarrow P(S, X) \twoheadrightarrow Y$. Also, we have an embedding $Y \times Fix(\sigma) \hookrightarrow P(S, X)$.

The class of GDS covers a wide range of spaces including real and complex projective spaces, generalized Klein's bottles, certain compact complex manifold bundles over real projective spaces, and various other fibre bundles.

We highlight the main results of the thesis on:

1. Cohomology groups of $P(S, \alpha, X, \sigma)$ for a wide class of spaces (S, α) and (X, σ).

2. Mod 2 cohomology algebra of $P(S, X)$, where S is a paracompact topological space and X is (i) a torus manifold (see [MP06]) under certain mild assumptions, which are satisfied by quasi-toric manifolds and (ii) a complex flag manifold $\mathbb{C}G(\nu)$.

3. Integral cohomology groups of $P(\mathbb{S}^m, \mathbb{C}G(\nu))$.

4. The ring structure of $H^*(P(\mathbb{S}^m, \mathbb{C}G(\nu)); \mathbb{Z}[1/2])$.

5. K-groups of $P(\mathbb{S}^m, \mathbb{C}G(\nu))$ up to 2-torsions.

In the following sections, we present the major findings of the thesis.

1.1 Cohomology groups of GDS

This chapter focuses on studying cohomology of generalized Dold spaces $P(S, X)$ where S, X are topological spaces under mild assumptions.

Suppose that S, X are locally finite CW complexes. Under appropriate hypotheses on the cell structures on S and X, we obtain a nice CW structure on $P(S, X)$ which has the property that the mod 2 cellular boundary map vanishes. Then we have the following result.

Proposition 1.1.1 (See Proposition 3.1.1)**.** *Suppose that (S, α) is any $\mathbb{Z}_2$-equivariant CW complex such that the induced cell structure on $Y = S/\sim_\alpha$ is perfect mod 2. Suppose that X is a CW complex such that (i) each skeleton $X^{(k)}, k \geq 1$ is finite, (ii) the cell structure is perfect mod 2, and, (iii) each cell is mapped to itself by $\sigma : X \to X$. Then the CW structure on $P(S, X)$ induced by the product CW structure on $S \times X$ is perfect mod 2. In particular, one has an isomorphism $H^*(P(S, X); \mathbb{Z}_2) \cong H^*(Y; \mathbb{Z}_2) \otimes H^*(X; \mathbb{Z}_2)$ of $H^*(Y; \mathbb{Z}_2)$-modules.*

We consider another situation where the X-bundle $p : P(S, X) \to Y$ admits a $\mathbb{Z}_2$-cohomology extension of fibre. Since $Fix(\sigma) = X^\sigma \neq \emptyset$, we have a cross-section $Y \to P(S, X)$ of the X-bundle and so it follows that $p^* : H^*(Y; \mathbb{Z}_2) \to H^*(P(S, X); \mathbb{Z}_2)$ is a monomorphism. We have the following result.

Proposition 1.1.2 (See Proposition 3.1.2). *Let* $(\omega_j, \hat{\sigma}_j), 1 \leq j \leq r$, *be* σ*-conjugate complex vector bundles over* (X, σ) *such that the cohomology algebra* $H^*(X; \mathbb{Z}_2)$ *is generated by the mod 2 Chern classes* $c_q(\omega_j) \in H^{2q}(X; \mathbb{Z}_2), 1 \leq q \leq rank(\omega_j), 1 \leq j \leq r$. *Assume that* X *satisfies any of the following: (i)* $\dim_{\mathbb{Z}_2} H^*(X; \mathbb{Z}_2) < \infty$, *(ii)* X *is a CW complex with finite* k*-skeleton for each* $k \geq 1$ *where each cell is stable by* σ. *Then we have an isomorphism of* $\mathbb{Z}_2$*-vector spaces:*

$$H^r(P(S, X); \mathbb{Z}_2) \cong \bigoplus_{p+q=r} H^p(Y; \mathbb{Z}_2) \otimes H^q(X; \mathbb{Z}_2).$$

In particular, $H^*(P(S, X); \mathbb{Z}_2)$ *is a free* $H^*(Y; \mathbb{Z}_2)$*-module with basis* B *where* B *is a* $\mathbb{Z}_2$*-basis for* $H^*(X; \mathbb{Z}_2)$. □

Suppose that $p : \tilde{B} \to B$ is a double covering projection where B is a connected locally finite CW complex. We obtain a G-equivariant CW structure on $\tilde{B}$ by lifting the cell structure on B where $G \cong \mathbb{Z}_2$ is the deck transformation group. Denote by $\phi : \tilde{B} \to \tilde{B}$ the involution that generates G. Then we have the following lemma.

Lemma 1.1.3 (See Lemma 3.2.2). *We keep the above notations. Let* R *be any commutative ring in which 2 is invertible. Then* $p_* : H_*(\tilde{B}; R) \to H_*(B; R)$ *maps* $Fix(\phi_*)$ *isomorphically onto* $H_*(B; R)$. *Moreover* $p_*[z] = 0$ *if* $\phi_*([z]) = -[z]$.

As an immediate corollary of the above lemma, we obtain the following proposition.

Proposition 1.1.4 (See Proposition 3.2.3). *Let* $[z] \in H_r(\tilde{B}; \mathbb{Z})$. *If* $p_*([z]) = 0$ *and* $\phi_*([z]) = [z]$, *then* $2^k[z] = 0$ *for some* $k \geq 0$. *If* $\phi_*([z]) = -[z]$, *then* $2p_*([z]) = 0$. □

It is not necessarily true that any torsion in $H_1(B; \mathbb{Z})$ is of order 2. A $2n$-dunce hat provides an example. Assuming some additional hypotheses, any nontrivial torsion element in $H_1(B; \mathbb{Z})$ can be shown to be of order 2. We have the following result regarding this.

Denote by $H_p^{\pm}(S;R)$ the subspace of $H_p(S;R)$ on which α_* acts as ± 1 and denote its dimension by $b_p^{\pm}$. Thus the r-th Betti number $b_r(S)$ of S equals $b_r^+ + b_r^-$.

Proposition 1.1.5 (See Proposition 3.2.6)**.** *We keep the above notations. Suppose that Y is a connected locally finite CW complex and that $H_*(S;\mathbb{Z})$ is free abelian. Suppose that X is a connected locally finite CW complex with cells only in even dimensions. Suppose that $\sigma : X \to X$ is an involution that stabilizes each cell in X and that σ is orientation preserving on any $2k$-dimensional cell e if and only if k is even. Let R be a ring where 2 is invertible. Then: (i) We have an isomorphism*

$$(1.1)\quad H_r(P(S,X);R) \cong \bigoplus_{p+4t=r} (H_p^+(S;R) \otimes H_{4t}(X;R) \oplus H_{p-2}^-(S;R) \otimes H_{4t+2}(X;R))$$

In particular,

$$(1.2)\qquad \dim H_r(P(S,X);\mathbb{Q}) = \sum_{p+4q=r} (b_p^+ b_{4q}(X) + b_{p-2}^- b_{4q+2}(X)).$$

(ii) Suppose that $\pi_1(S)$ is abelian. Any torsion element of $H_r(P(S,X);\mathbb{Z})$ is a 2-torsion.

The cohomology version of the above propositions is valid and is equivalent to it by the universal coefficient theorem.

1.2 Mod 2 cohomology algebras of some GDS

In this chapter, we highlight our results on mod 2 cohomology ring of $P(S,X)$, where S is a paracompact Hausdorff topological space and X is (i) a torus manifold (see [MP06]) under certain mild assumptions, which are satisfied by quasi-toric manifolds and (ii) a complex flag manifold.

1.2.1 Cohomology of $P(S,X)$, where X is a torus manifold

We consider a restricted class of torus manifolds (see [MP06]), namely, those torus manifolds where the torus T-action is locally standard and the orbit space is a *homology polytope*. This restricted class itself is a generalization of the notion of quasi-toric manifolds due to Davis and Januskiewicz [DJ91], where the orbit space is a simple convex polytope. Let Q be the orbit space X/T, which is then an n-dimensional manifold with corners (i.e., is modeled on $\mathbb{R}^n_{\geq 0}$).

We shall denote the group of 1-parameter subgroups $\mathrm{Hom}(\mathbb{S}^1, T)$ by $\mathbf{N}$ and the group of characters $\mathrm{Hom}(T, \mathbb{S}^1)$ by $\mathbf{N}^\vee \cong \mathbb{Z}^n$. One has a natural pairing $\langle .,. \rangle$: $\mathbf{N}^\vee \times \mathbf{N} \to \mathbf{Z}$ defined by $u \circ v(z) = z^{\langle u,v \rangle}$.

An *omni-orientation* of X is a choice of an orientation on X and on each characteristic submanifold $X_i, 1 \leq i \leq m$. The orientations on X, X_i determine a unique orientation on the normal space to $T_xX_i \subset T_xX$ for any $x \in X_i$ which in turn leads to an orientation on S_i. This determines a unique *primitive* vector $v_i \in \mathbf{N}$ whose image is S_i.

Fix an omni-orientation on X and assume that Q is a homology polytope. Denote by $\mathcal{Q}$ the set of all facets of Q. We obtain a map $\Lambda : \mathcal{Q} \to \mathbf{N}$ defined as $\Lambda(Q_i) = v_i, 1 \leq i \leq m$. Local standardness of the T action implies that $\Lambda(Q_{i_1}), \ldots, \Lambda(Q_{i_n})$ is a $\mathbb{Z}$-basis of $\mathbf{N}$ whenever $Q_{i_1}, \ldots, Q_{i_n} \in \mathcal{Q}$ meet at a vertex of Q. The map Λ is called the *characteristic function* of X. It was shown in [MP06, Lemma 4.5] that the pair (Q, Λ) determines X up to equivariant homeomorphism assuming the vanishing of $H^2(Q;\mathbb{Z})$. We identify X with $X(Q,\Lambda)$. In fact, let $X(Q,\Lambda)$ denote the space $T \times Q/\!\sim$ where $(t,q) \sim (t',q')$ if and only if $q = q'$ and $t^{-1}t'$ belongs to the subgroup S_q of T generated by the one parameter subgroups $\Lambda(Q_{i_1}), \ldots, \Lambda(Q_{i_k})$ where q is in the interior of the face $Q_{i_1} \cap \cdots \cap Q_{i_k}$.

Let $\sigma : X(Q,\Lambda) \to X(Q,\Lambda)$ be the involution $[t,q] \mapsto [t^{-1},q]$. Then $Fix(\sigma) =$

$X^\sigma = \{[t,q] \mid t^2 \in S_q, \ \forall q \in Q\}$ is non empty. Assume S to be a paracompact Hausdorff topological space and α is a free involution on S. Recall $S/{\sim_\alpha}$ is denoted by Y. Now we have the following proposition.

Theorem 1.2.1 (See Theorem 4.2.4)**.** *Let $X = X(Q,\Lambda)$ be a T-torus manifold where $X/T = Q$ is a homology polytope with m facets. Let $\sigma : X \to X$ be the involution $[t,q] \mapsto [t^{-1},q]$. Then, with the above notations, $H^*(P(S,X);\mathbb{Z}_2)$ is isomorphic, as an $A = H^*(Y;\mathbb{Z}_2)$-algebra, to the quotient $R(Q,\Lambda) := A[\tilde{x}_1,\ldots,\tilde{x}_m]/I$ where the ideal $I = I(Q,\Lambda)$ is generated by the following two types of elements:*
(i) $\sum_{1\leq j\leq m}\langle u, v_j\rangle \tilde{x}_j, \ u \in \mathbf{N}^\vee$, and
(ii) $\prod_{1\leq q\leq r}\tilde{x}_{j_q}$ whenever $Q_{j_1}\cap\cdots\cap Q_{j_r} = \emptyset$.

1.2.2 Cohomology of $P(S,\mathbb{C}G(\nu))$

Let $\nu := (n_1,\ldots,n_s)$ be an increasing sequence of positive integers and let $n = \sum_{1\leq j\leq s} n_j$. Then denote by $\mathbb{C}G(\nu)$ the complex flag manifold whose elements are complex vector subspaces $\underline{U} := (U_1,\ldots,U_s)$ of $\mathbb{C}^n$ where $U_i \perp U_j$ for $i \neq j$ and $\dim U_j = n_j, 1 \leq j \leq s$. The complex conjugation in $\mathbb{C}^n$ induces a complex conjugation σ on $\mathbb{C}G(\nu)$, and σ stabilizes each Schubert variety. The fixed point set of σ is the real flag manifold $\mathbb{R}G(\nu)$.

In this section, we study the cohomology of $P(S,\mathbb{C}G(\nu))$, where S is a paracompact topological space equipped with a free involution α and the involution on $\mathbb{C}G(\nu)$ is σ.

We denote by $\gamma_{\nu,j}$ (or more briefly γ_j), the complex vector bundle over $\mathbb{C}G(\nu)$ of rank n_j whose fibre over $\underline{U}$ is the vector space U_j. Also one has a natural isomorphism of vector bundles $\oplus_{1\leq j\leq s}\gamma_j \cong n\epsilon_{\mathbb{C}}$ which respects σ-conjugation. The integral cohomology ring of $\mathbb{C}G(\nu)$ is generated by $c_{i,j} := c_i(\gamma_j), 1 \leq i \leq n_j, 1 \leq j \leq s$, where the only relations among the $c_{i,j}$ are generated by the following (inhomogeneous) relation: $\prod_{1\le j\le s} c(\gamma_j) = 1$. It follows that the Stiefel-Whitney classes $w_{2i,j} = w_{2i}(\gamma_j), 1 \le i \le n_j, 1 \le j \le s$, generate $H^*(\mathbb{C}G(\nu);\mathbb{Z}_2)$ and the relations among these generators are all consequences of $\prod_{1\le j\le s} w(\gamma_j, t) = 1$.

Suppose that (S, α) is a paracompact space where α is a fixed point free involution. Then we have some real vector bundles $\hat{\gamma}_j = P(S, \gamma_j)$ over $P(S, \mathbb{C}G(\nu))$ constructed from γ_j. One has the following isomorphism of real vector bundles:

$$\bigoplus_{1\le j\le s} \hat{\gamma}_j \cong n\xi_\alpha \oplus n\epsilon_{\mathbb{R}}$$

where ξ_α is the real line bundle associated to the double cover $S \times \mathbb{C}G(\nu) \to P(S, \mathbb{C}G(\nu))$. Let y denote the first Stiefel-Whitney class of ξ_α. Therefore we obtain

$$w(\hat{\gamma}_s, t) = (1 + yt)^n . \prod_{1\le j<s} w(\hat{\gamma}_j, t)^{-1}.$$

We obtain a regular sequence $a_{2s}, n_s < s \le n$, in the polynomial algebra $R_\nu = \mathbb{Z}_2[\hat{w}_{2i,j} \mid 1 \le i \le n_j, 1 \le j \le s]$ that correspond to the coefficient of t^{2s} in $(1 + ty)^n(\prod \hat{w}_j(t))$. Now we set $\mathcal{R}_\nu$ to the polynomial algebra over $H^*(Y;\mathbb{Z}_2)$ generated by 'indeterminates' $\hat{w}_{2i,j}, 1 \le i \le n_j, 1 \le j < s$, and let $\mathcal{I}_\nu$ be the ideal of $\mathcal{R}_\nu$ generated by the elements $a_{2i} = a_{2i}(y^2, \hat{w}_{2l,j}), n_s < i \le n$. Then we have the following theorem.

Theorem 1.2.2 (See Theorem 4.3.4)**.** *Suppose that (S, α) is a paracompact Hausdorff topological space with a fixed point free involution α and let $\nu = n_1, n_2, \ldots, n_s$ be a sequence of positive numbers. With notations as above, we have an isomorphism $\mathcal{R}_\nu/\mathcal{I}_\nu \to H^*(P(S, \mathbb{C}G(\nu));\mathbb{Z}_2)$ of $H^*(Y;\mathbb{Z}_2)$-algebras defined by $\hat{w}_{2i,j} \mapsto w_{2i}(\hat{\gamma}_j)$.* □

As an application of Theorems 4.2.4, 4.3.4 we obtain the $\mathbb{Z}_2$-equivariant cohomology $H^*_{\mathbb{Z}_2}(X;\mathbb{Z}_2)$ of (X, σ) when X is either a torus manifold whose torus quotient is a homology polytope, or, is a complex flag manifold $\mathbb{C}G(\nu)$.

1.3 Cohomology of $P(\mathbb{S}^m, \mathbb{C}G(\nu))$

In this chapter, we highlight the results on cohomology of $P(\mathbb{S}^m, \mathbb{C}G(\nu))$, where $\mathbb{S}^m$ is equipped with antipodal involution α and the involution σ on $\mathbb{C}G(\nu)$ is induced from complex conjugation on $\mathbb{C}^n$. We denote $P(\mathbb{S}^m, \mathbb{C}G(\nu))$ by $P(m,\nu)$ for short.

1.3.1 Integral (co)homology of $P(\mathbb{S}^m, \mathbb{C}G(\nu))$

A well-known CW structure on $\mathbb{C}G(\nu)$ is given by Schubert cells $X(\mathbf{i})$, which has the structure of a complex projective variety. Let $I(\nu)$ denote the indexing set of all Schubert cells $X(\mathbf{i})$ in $\mathbb{C}G(\nu)$. This indexing set $I(\nu)$ is in bijection with the coset space S_n/S_ν where $S_\nu := S_{n_1} \times \cdots \times S_{n_s}$ and S_n is the permutation group on $\{1, 2, \cdots, n\}$. Denote the dimension of $X(\mathbf{i})$, as a complex variety, by $\ell(\mathbf{i})$.

A cell decomposition of $\mathbb{S}^m \times \mathbb{C}G(\nu)$ which is equivariant with respect to $\theta = \alpha \times \sigma$ is obtained by taking the product cells $X^{\pm}(j, \mathbf{i}) := C_j^{\pm} \times X(\mathbf{i})$ as $(j, \mathbf{i})$ varies in $I(m,\nu) := \{j \mid 0 \le j \le m\} \times I(\nu)\}$. The cell $C_j^{\pm} \times X(\mathbf{i})$ is given the product orientation. We shall denote the image of the oriented cell $X^+(j, \mathbf{i})$ under the double covering projection $\pi : \mathbb{S}^m \times \mathbb{C}G(\nu) \to P(m,\nu)$ by $X(j,\mathbf{i})$ and put the induced orientation on it. The deck transformation group of π is generated by $\theta = \alpha \times \sigma$.

Depending on $\mathbf{i}$ and j, sometimes $X(j,\mathbf{i}) \in C_*(P(m,\nu);\mathbb{Z})$ is a cycle whose homology class we denote by $[X(j,\mathbf{i})]$. We define the sets

$$\begin{aligned}
I_e(\nu) &:= \{\mathbf{i} \in I(\nu) \mid \ell(\mathbf{i}) \equiv 0 \pmod 2\},\\
I_o(\nu) &:= \{\mathbf{i} \in I(\nu) \mid \ell(\mathbf{i}) \equiv 1 \pmod 2\} = I(\nu) \setminus I_e(\nu),\\
\mathcal{B}_{2q} &:= \{[X(0,\mathbf{i})] \mid q = \ell(\mathbf{i}), \mathbf{i} \in I_e(\nu)\} \cup \{[X(m,\mathbf{i})] \mid 2q = m + 2\ell(\mathbf{i}), \mathbf{i} \in I_o(\nu)\},\\
\mathcal{B}_{2q+1} &:= \{[X(m,\mathbf{i})] \mid 2q+1 = m + 2\ell(\mathbf{i}), \mathbf{i} \in I_e(\nu)\},\\
\mathcal{B}'_q &:= \{[X(j,\mathbf{i})] \mid q = j + 2\ell(\mathbf{i}), j + \ell(\mathbf{i}) \equiv 1 \pmod 2, 0 \le j < m,\ \mathbf{i} \in I(\nu)\}.
\end{aligned}$$

Now we have the following result where we leave out the known cases namely, $m = 0$ as $P(0, \nu) \cong \mathbb{C}G(\nu)$ and $\nu = (1)$ in which case $P(m, \nu) = \mathbb{S}^m$.

Theorem 1.3.1 (See Theorem 5.3.1). *We keep the above notations. Suppose that $m \geq 1$ and $n = |\nu| \geq 2$.*
(i) For any $q \geq 0$, $H_q(P(m, \nu); \mathbb{Z}) \cong \mathbb{Z}^r \oplus (\mathbb{Z}_2)^t$ for some $r, t \geq 0$ depending on q.
(ii) The set $\mathcal{B}_q$ is a basis for $H_q(P(m, \nu); \mathbb{Z})/\text{torsion}$ and $\mathcal{B}'_q$ is a $\mathbb{Z}_2$-basis for the torsion subgroup of $H_q(P(m, \nu); \mathbb{Z})$.

For $\nu = (n_1, \ldots, n_s), s \geq 2$, set $\nu_o := \{1 \leq j \leq s \mid n_j \equiv 1 \mod 2\}$ and $\lfloor \nu/2 \rfloor := (n'_1, n'_2, \ldots, n'_s)$, where $n'_j := \lfloor n_j/2 \rfloor, 1 \leq j \leq s$. Define $\ell_e := |I_e(\nu)|$ and $\ell_o := |I_o(\nu)|$.

When A is a finitely generated abelian group we shall refer to the dimension over $\mathbb{Z}_2$ of the subgroup $A_2 := \{x \in A \mid 2x = 0\}$ as the 2-rank of A. For our applications to K-theory, we need the ranks (resp. 2-ranks) of $H^{\text{ev}}(P(m, \nu); \mathbb{Z}) = \bigoplus_{q \geq 0} H^{2q}(P(m, \nu); \mathbb{Z})$ and $H^{\text{odd}}(P(m, \nu); \mathbb{Z}) = \bigoplus_{q \geq 1} H^{2q-1}(P(m, \nu); \mathbb{Z})$, which are denoted b_e, b_o (resp. b'_e, b'_o). Then we have the following result.

Theorem 1.3.2 (See Theorem 5.3.4). *Let $m \geq 1$ and $\nu = (n_1, \ldots, n_s)$ where $s \geq 2$. Let $\nu_o = |\{1 \leq j \leq s \mid n_j \equiv 1 \pmod 2\}$. Then $H^{\text{ev}}(P(m, \nu); \mathbb{Z}) \cong \mathbb{Z}^{b_e} \oplus \mathbb{Z}_2^{b'_e}$ and $H^{\text{odd}}(P(m, \nu); \mathbb{Z}) \cong \mathbb{Z}^{b_o} \oplus \mathbb{Z}_2^{b'_o}$ where*
(i) Suppose that m is odd. Then

$$b_e = b_o = \ell_e = \begin{cases} \binom{n}{\nu}/2, & \text{if } \nu_o \geq 2 \\ \binom{n}{\nu} + \binom{\lfloor n/2 \rfloor}{\lfloor \nu/2 \rfloor}, & \text{if } \nu_o \leq 1. \end{cases}$$

When m is even, $b_e = \binom{n}{\nu}$ and $b_o = 0$.
(ii) $b'_e = \beta_o = \lfloor m/2 \rfloor \cdot \ell_e, \quad b'_o = \beta_e = \lfloor (m+1)/2 \rfloor \cdot \ell_o$.
In particular the integral cohomology of $P(1, \nu)$ has no torsion. □

1.3.2 The ring structure of $H^*(P(\mathbb{S}^m, \mathbb{C}G(\nu)); \mathbb{Z}[1/2])$

Let R be a commutative ring in which 2 is a unit. From Theorem 5.3.1 we know that as an R-module, $H^q(P(m,\nu); R)$ is free of rank $|\mathcal{B}_q|$. We now turn to the ring structure of $H^*(P(m,\nu); R)$. We shall first describe it as a subring of

$$H^*(\mathbb{S}^m \times \mathbb{C}G(\nu); R) \cong R[u_m] \otimes R[c_{r,j}; 1 \le r \le n_j, 1 \le j \le s]/\langle u_m^2,\ f_p, 1 \le p \le n\rangle$$

via the homomorphism π^* induced by the double covering projection $\pi : \mathbb{S}^m \times \mathbb{C}G(\nu) \to P(m,\nu)$, where $c_{r,j}$ denotes the r-th Chern class of $\gamma_{\nu,j}$. Since 2 is a unit in R, π^* is a monomorphism; see Proposition 3.2.6. Then, in Theorem 5.4.2 we shall describe $H^*(P(m,\nu); R)$ as a quotient of a polynomial algebra. Denote by $u_m \in H^m(\mathbb{S}^m; \mathbb{Z})$ the positive generator concerning the standard orientation on the sphere.

Theorem 1.3.3 (See Theorem 5.4.1)**.** *Suppose that 2 is a unit in R. Then $H^*(P(m,\nu)$ is isomorphic to the subalgebra $Fix(\theta^*) \subset H^*(\mathbb{S}^m \times \mathbb{C}G(\nu); R)$ which is generated by the following elements generate $Fix(\theta^*)$:*

Case (1): m is even.

$$\begin{aligned} u_m c_{2p-1}(\gamma_r), &\quad 1 \le 2p-1 \le n_r,\ 1 \le r \le s; \\ c_{2j}(\gamma_r), &\quad 1 \le j \le n_r,\ 1 \le r \le s;\ \textit{and} \\ c_{2p-1}(\gamma_i)c_{2q-1}(\gamma_j), &\quad 1 \le 2p-1 \le n_i,\ 1 \le 2q-1 \le n_j,\ 1 \le i \le j \le s. \end{aligned}$$

Case (2): m is odd.

$$\begin{aligned} u_m;\ \ c_{2j}(\gamma_r), &\quad 1 \le j \le n_r,\ 1 \le r \le s;\ \textit{and} \\ c_{2p-1}(\gamma_i)c_{2q-1}(\gamma_j), &\quad 1 \le 2p-1 \le n_i,\ 1 \le 2q-1 \le n_j,\ 1 \le i \le j \le s. \ \square \end{aligned}$$

Next, we present $H^*(P(m,\nu); R)$ as a quotient of a polynomial algebra over R. Since the ideal involves intricate notations, we omit its statement here; it will be detailed in Theorem 5.4.2.

1.4 K-theory of $P(\mathbb{S}^m, \mathbb{C}G(\nu))$

Here we study complex K-theory of the GDS $P(\mathbb{S}^m, \mathbb{C}G(\nu))$. We denote $P(\mathbb{S}^m, \mathbb{C}G(\nu))$ by $P(m,\nu)$ for short. Using the results on cohomology of $P(\mathbb{S}^m, \mathbb{C}G(\nu))$, we study the additive structure of K-groups in the following theorem.

Theorem 1.4.1. *Let $m \geq 1$, $\nu = (n_1, \ldots, n_s), s \geq 2$. Let b_e, b_o, b'_e, b'_o be as in Theorem 1.3.2. Then:*
(i) $K^0(P(m,\nu)) \cong \mathbb{Z}^{b_e} \oplus A_0$ where A_0 is a finite abelian group of order 2^k for some k, with $0 \leq k \leq b'_e$. The group A_0 contains a summand $\mathbb{Z}_{2^{\lfloor m/2 \rfloor}}$, generated by $y = [\xi_{\mathbb{C}}]-1$.
(ii) $K^1(P(m,\nu)) \cong \mathbb{Z}^{b_o} \oplus A_1$, where A_1 is a finite abelian group of order 2^t for some $0 \leq t \leq b'_o$.
In particular, $K^1(P(m,\nu))$ is a torsion group if m is even, and $K^0(P(1,\nu))$ is a torsion-free group

Next, we construct some canonical complex vector bundles over $P(m,\nu)$ such that the classes of these vector bundles generate a subring $\mathcal{K}^0 \subset K^0(P(m,\nu))$ with $K^0(P(m,\nu))/\mathcal{K}^0$ is a finite abelian 2-group. In particular, considering $\mathcal{K}^0$ as sub $\mathbb{Z}$-module of $K^0(P(m,\nu))$, we have $\mathcal{K}^0 \otimes \mathbb{Q} = K^0(P(m,\nu)) \otimes \mathbb{Q}$.

Assume that $m = 2r$ is even. An explicit description of a complex vector bundle ξ over $\mathbb{S}^m$ such that $[\xi] - \mathrm{rank}(\xi)$ is a generator of $\tilde{K}(S^m) \cong \mathbb{Z}$ is well-known. See [AH61], [Bot69]. We take this vector bundle ξ and denote it by ξ^+ for future usage. Define η^- to be $\alpha^!(\xi^+)$, where α is the antipodal map on the sphere. Now consider the following definitions.

Definition 1.4.2. *1. $\tilde{\xi}(\omega) := \xi^+ \otimes \omega \oplus \eta^- \otimes \bar{\omega}$ for any complex vector bundle ω over $\mathbb{C}G(\nu)$, where η^- is defined to be $\alpha^!(\xi^+)$.*

2. $\tilde{\theta} : E(\tilde{\xi}(\omega)) \to E(\tilde{\xi}(\omega))$ *is a bundle involution that covers* θ*, defined as:*

$$\tilde{\theta}(e^+ \otimes u, e^- \otimes v) = (\tilde{\alpha}^-(e^-) \otimes \bar{v}), \tilde{\alpha}^+(e^+) \otimes \bar{u})).$$

3. $\xi^0(\omega)$ *is defined to be the complex vector bundle over* $P(m, \nu)$*, whose total space is* $E(\tilde{\xi}(\omega))/\langle\tilde{\theta}\rangle$.

4. $\delta_m := \begin{cases} [\xi^+] - 2^{r-1} = 2^{r-1} - [\xi^-] & \text{if } m \equiv 0 \pmod 2 \\ 0 & \text{if } m \equiv 1 \pmod 2. \end{cases}$

5. $\mathcal{K}$ *is the subring of* $K^0(\mathbb{S}^m \times \mathbb{C}G(\nu))$ *generated by the elements (i)* $x + \bar{x}$ *and, (ii)* $\delta_m(x - \bar{x})$ *when* m *is even, as* x *varies in* $K^0(\mathbb{C}G(\nu))$.

6. $\mathcal{K}^0$ *is the subring of* $K^0(P(m, \nu))$ *generated by the following elements:*
(i) $[\xi_{\mathbb{C}}]$, $[\hat{\omega}_{\mathbb{C}}]$*, and, (ii) when* m *is even,* $[\xi^0(\omega)]$,
where ω *varies over* σ*-conjugate complex vector bundles over* $\mathbb{C}G(\nu)$.

Now we have the following proposition.

Theorem 1.4.3 (See Thoerem 6.2.11)**.** *Let* $m \geq 1$ *and let* $r = \lfloor m/2 \rfloor$*. The homomorphism* $\pi^! : K^0(P(m, \nu)) \to K^0(\mathbb{S}^m \times \mathbb{C}G(\nu))$ *defines a surjective ring homomorphism* $\mathcal{K}^0 \to \mathcal{K}$*, again denoted* $\pi^!$*, whose kernel equals the torsion ideal* $\mathcal{T}^0 \subset \mathcal{K}^0$*. The quotient group* $K^0(P(m, \nu))/\mathcal{K}^0$ *is a finite abelian* 2*-group.*

1.5 Applications

We provide some applications of our study of cohomology of generalized Dold spaces.

As an application of Theorems 4.2.4, 4.3.4 we obtain the $\mathbb{Z}_2$-equivariant cohomology $H^*_{\mathbb{Z}_2}(X; \mathbb{Z}_2)$ of (X, σ) when X is either a torus manifold whose torus quotient is a homology polytope, or, is a complex flag manifold $\mathbb{C}G(\nu)$.

Theorem 1.5.1 (See Theorem 7.1.1). *We keep the above notations from §1.2.1.*

(i) Let $X = X(Q, \Lambda)$ be a T-torus manifold where $Q = X/T$ is a homology polytope. Then $H^*_{\mathbb{Z}_2}(X; \mathbb{Z}_2)$ is isomorphic to the A-algebra $R(Q, \Lambda)$ where $A = H^*(\mathbb{R}P^\infty; \mathbb{Z}_2) \cong \mathbb{Z}_2[y]$.

(ii) Let $\nu = n_1 \leq \cdots < n_s, n = \sum n_j$. Then $H^*_{\mathbb{Z}_2}(\mathbb{C}G(\nu); \mathbb{Z}_2)$ is isomorphic to $\mathcal{R}_\nu/\mathcal{I}_\nu$ where $\mathcal{R}_\nu = A[\hat{w}_{2i,j}; 1 \leq i \leq n_j, 1 \leq j < s]$ where $A = H^*(\mathbb{R}P^\infty; \mathbb{Z}_2) = \mathbb{Z}_2[y]$ and $\mathcal{I}_\nu \subset \mathcal{R}_\nu$ is the ideal generated by the $a_{2i} \in \mathcal{R}_\nu, n_s < i \leq n$. □

We present the following result as an application of Theorem 5.4.2. It provides specific conditions under which a self-map on a generalized Dold manifold $P(m, n, k) := P(\mathbb{S}^m, \mathbb{C}G_{n,k})$ will have a fixed point.

Proposition 1.5.2 (See Proposition 7.2.4). *Let m be odd and either (i) $k \leq 3$ and $n > 2k$, or (ii) $k > 3$ and $n > 2k^2 - 1$ hold. Then for any continuous map $f : P(m, n, k) \to P(m, n, k)$ such that the degree $\deg(p \circ f \circ s) \neq 1$, there exists a fixed point.*

The following result is an application of Theorem 5.4.1. It provides some conditions for a map between generalized Dold manifolds to have degree zero.

Proposition 1.5.3 (See Proposition 7.3.1). *Let $f : P(m, n, k) \to P(r, s, t)$ be a continuous map between two oriented and same-dimensional generalized Dold manifolds. Further assume that*

(i) $\lfloor k/2 \rfloor < \lfloor t/2 \rfloor$ if $m = r$ is odd, and $k < t$ if $m = r$ is even,

(ii) $\mathbb{C}G_{s,t} \neq \mathbb{C}G_{2,1} = \mathbb{C}P^1$ if $r \neq m = 2$.

Then the degree of f is zero.

Chapter 2

Preliminaries

In this chapter, we shall discuss some important definitions and observations which will be necessary to proceed further towards the main results.

The classical Dold manifold $P(m,n)$ is defined as the orbit space of the $\mathbb{Z}_2$-action on $\mathbb{S}^m \times \mathbb{C}P^n$ generated by the involution $(v,[z]) \mapsto (-v,[\bar{z}]), v \in \mathbb{S}^m, [z] \in \mathbb{C}P^n$. Here $[\bar{z}]$ denotes $[\bar{z}_0 : \cdots : \bar{z}_n]$ when $[z] = [z_0 : \cdots : z_n] \in \mathbb{C}P^n$. See [Dol56].

Let us consider two connected, locally path connected, paracompact, Hausdorff topological spaces S, X along with a fixed point free involution $\alpha : S \to S$ and another involution $\sigma : X \to X$ with non-empty fixed point set. We denote by Y the quotient space $S/\sim$ where $v \sim \alpha(v), v \in S$. The space $P(S,\alpha,X,\sigma)$ (or more briefly $P(S,X)$ when there is no risk of confusion) is defined to be the quotient space $S \times X/\sim$ where $(v,x) \sim (\alpha(v),\sigma(x))$ and is called the *generalized Dold space*. Sometimes, we use GDS as a shorthand for the *generalized Dold space*.

Note that $P(S,X)$ is a (smooth) manifold if S,X (and α,σ) are (smooth) manifolds. When $P(S,X)$ is a manifold, we may refer to it as *generalized Dold manifolds*. The quotient maps $p_1 : S \to Y$ and $\pi : S \times X \to P(S,X)$ are covering projections with deck transformation groups isomorphic to $\mathbb{Z}_2$ generated by α and

$\alpha \times \sigma$ respectively. We denote by $[v] \in Y$ the element $p_1(v) = \{v, \alpha(v)\}$ and similarly $[v, x] = \pi(v, x) = \{(v, x), (\alpha(v), \sigma(x))\} \in P(S, X)$. The first projection $S \times X \to Y$ induces a map $p : P(S, X) \to Y$ which is the projection of a locally trivial bundle with fibre space X. Denote by $X^\sigma \subset X$ the fixed points of σ. For each $x \in X^\sigma := \mathrm{Fix}(\sigma)$, we have a cross-section $s_x = s : Y \to P(S, X)$ where $s([v]) = [v, x]$. In fact, we have an embedding of $Y \times X^\sigma$ into $P(S, X)$ defined by $([v], x) \mapsto [v, x]$.

The construction of generalised Dold space is functorial in (S, α) and (X, σ) for maps $(S', \alpha') \to (S, \alpha)$ and $(X', \sigma') \to (X, \sigma)$ that are $\mathbb{Z}_2$-equivariant maps.

Some examples of generalized Dold spaces that we will consider in our study are:

1. $P(\mathbb{S}^m, \alpha, \mathbb{S}^n, \sigma)$, where α is antipodal map on $\mathbb{S}^m$ and σ is the reflection map on $\mathbb{S}^n$ that sends $e_{n+1} \to -e_{n+1}$ and pointwise fixes $\mathbb{S}^{n-1} \subset \mathbb{R}^n = \{e_{n+1}\}^\perp$. Note that $P(\mathbb{S}^1, \alpha, \mathbb{S}^1, \sigma)$ is homeomorphic to Klein's bottle. So, the GDS $P(\mathbb{S}^m, \alpha, \mathbb{S}^n, \sigma)$ are *generalized Klein's bottles.*

2. $P(\mathbb{S}^m, \alpha, \mathbb{C}G(\nu), \sigma)$, where α is antipodal map on $\mathbb{S}^m$ and σ is the involution on the complex flag manifold $\mathbb{C}G(\nu)$, induced from the complex conjugation on $\mathbb{C}^n$.

3. $P(\mathbb{S}^m, \alpha, X, \sigma)$ where α is antipodal map on $\mathbb{S}^m$ and X is a torus manifold (see [MP06]) whose torus orbit space is a homology polytope with σ being a certain involution (see §4.2) on X.

2.1 σ-conjugate vector bundles

Let $\sigma : X \to X$ be an involution on a path connected paracompact Hausdorff topological space. We assume that X^σ is a non-empty proper subset of X. Let ω be a complex vector bundle over X. A *σ-conjugation* on ω is an involutive bundle map

$\hat{\sigma} : E(\omega) \to E(\omega)$ on the total space of ω that covers σ and is conjugate complex linear on the fibres of the projection $\pi_\omega : E(\omega) \to X$. If such a bundle involution exists, we call $(\omega, \hat{\sigma})$ (or more briefly ω) a σ-conjugate bundle.

Let $(\omega, \hat{\sigma})$ be a σ-conjugate vector bundle over X with bundle projection $E(\omega) \to X$. The zero-cross section $X \to E(\omega)$ is σ-equivariant. In particular, $\mathrm{Fix}(\hat{\sigma})$ is non-empty. We obtain a *real* vector bundle $P(S, \omega)$, more briefly denoted $\hat{\omega}$ when there is no danger of confusion, over $P(S, X)$ with projection $P(S, E(\omega), \hat{\sigma}) \to P(S, X)$ defined as $[v, e] \mapsto [v, \pi_\omega(e)]$.

Note that $\hat{\sigma}$ is also a σ-conjugation on the conjugate complex vector bundle $\bar{\omega}$ and we have an isomorphism of real vector bundles $\hat{\omega} \cong \hat{\bar{\omega}}$.

The above construction of $\hat{\omega}$ over $P(S, X)$, as prolongation of a σ-conjugate complex vector bundle over X, can be extended to real vector bundles as follows. Let η be any *real* vector bundle over X with a bundle involution $\hat{\sigma}$ that covers σ. We denote by $\hat{\eta}$ the vector bundle over $P(S, X)$ with total space $P(S, E(\eta)) = P(S, E(\eta), \hat{\sigma})$ and bundle projection $\pi_{\hat{\eta}} : P(S, E(\eta)) \to P(S, X)$ defined as $[v, e] \to [v, \pi_\eta(e)]$ $\forall v \in S, e \in E(\eta)$.

We can generalize the construction of the vector bundle $\hat{\omega}$ over $P(S, X)$ for a σ conjugate vector bundle ω over X. Suppose that ω is a real vector bundle over a space B with projection $p_\omega : E(\omega) \to B$. Let $f : B \to B$ be a fixed point free involution that is covered by an involutive bundle automorphism $\hat{f} : E(\omega) \to E(\omega)$. We obtain a vector bundle $\hat{\omega}$ over $\bar{B} := B/\!\sim_f$ with total space $E(\hat{\omega}) = E(\omega)/\sim_{\hat{f}}$. The bundle projection $E(\hat{\omega}) \to \bar{B}$ sends $[v] = \{v, \hat{f}(v)\}$ to $[b]$ for all $v \in E_b(\omega)$. If ω is a complex vector bundle and $\hat{f}$ is a complex vector bundle morphism, then $\hat{\omega}$ is a complex vector bundle over $\bar{B}$. We emphasize that the isomorphism class of $\hat{\omega}$ depends not only on ω, but also on $\hat{f}$.

The construction of $\hat{\omega}$ behaves well with respect to Whitney sum, tensor products, taking exterior powers, etc., so long as all the bundles involved are σ-conjugate (complex) vector bundles, i.e., the respective bundle involutions cover the *same* $\sigma : X \to X$. (See [NS19].)

2.2 Functoriality

Suppose that $\alpha_1 : S_1 \to S_1$ is a fixed point free involution and that $f : S_1 \to S$ is $\mathbb{Z}_2$-equivariant, i.e., $f(\alpha_1(v_1)) = \alpha f(v_1), v_1 \in S$. By the functoriality, we obtain a map $F : P(S_1, X) \to P(S, X)$. Explicitly F is induced by the $\mathbb{Z}_2$-equivariant map $f \times id : S_1 \times X \to S \times X$. Also, we have a morphism of real vector bundles $P(S_1, \omega) \to P(S, \omega)$ where the map between the total spaces $\hat{F} : P(S_1, E(\omega), \hat{\sigma}) \to P(S, E(\omega), \hat{\sigma})$ is again got by functoriality. Explicitly, $\hat{F}([v_1, e]) = [f(v_1), e]\ \forall v_1 \in S_1, e \in E(\omega)$. We have the following commuting diagram:

$$\begin{array}{ccc} P(S_1, E(\omega)) & \xrightarrow{\hat{F}} & P(S, E(\omega)) \\ \downarrow & & \downarrow \\ P(S_1, X) & \xrightarrow{F} & P(S, X) \end{array}$$

It follows that

$$F^*(P(S, \omega)) = P(S_1, \omega). \tag{1}$$

We have the following isomorphism of real vector bundle for any σ-conjugate complex vector bundle $(\omega, \hat{\sigma})$ over X:

$$P(S, \omega) \cong \xi_\alpha \otimes P(S, \omega). \tag{2}$$

An explicit bundle isomorphism can be obtained as follows: Write $\hat{\omega} := P(S, \omega)$. The total space $E(\xi_\alpha \otimes \hat{\omega})$ of $\xi_\alpha \otimes \hat{\omega}$ is a subspace of the quotient of $S \times X \times E(\epsilon_{\mathbb{R}} \otimes \omega)$ under the identification $(v, x, t \otimes w) \sim (\alpha(v), \sigma(x), -t \otimes \hat{\sigma}(w))$. We write the equivalence class of $(v, x, t \otimes w)$ as $[v, x, t \otimes w]$. The space $E(\xi_\alpha \otimes \omega)$ consists of triples $[v, x, t \otimes w]$ where $\pi_{\epsilon_{\mathbb{R}} \otimes \omega}(t \otimes w) = x \in X$. A bundle map $f : E(\xi_\alpha \otimes \hat{\omega}) \to E(P(S, \omega))$ that covers the identity is obtained as $[v, x, t \otimes w] \mapsto [v, x, \sqrt{-1}tw]$. The routine verification is left to the reader. (Cf. [Ucc65, Proposition 1.4(iii)], [NS19, Lemma 2.7].)

2.3 The Stiefel-Whitney classes of $\hat{\omega}$

Assuming that $H^1(X; \mathbb{Z}_2) = 0$, a formula for Stiefel-Whitney classes of $\hat{\omega}$ was obtained in [NS19] in the case of generalized Dold manifolds $P(\mathbb{S}^m, X, \sigma)$. The proof used functorial properties of the Dold construction and the splitting principle for complex vector bundles over X. These properties hold for any space X as the manifold properties of X play no role in the proof of the Stiefel-Whitney class formula. Specifically, the following formula holds for σ-conjugate vector bundles over $P(\mathbb{S}^m, X)$ for any CW complex X with $H^1(X; \mathbb{Z}_2) = 0$.

There exist suitable cohomology classes $\tilde{c}_j(\omega) \in H^{2j}(P(\mathbb{S}^m, X); \mathbb{Z}_2)$ which restricts to the mod 2 Chern class $c_j(\omega) = w_{2j}(\omega) \in H^{2j}(X)$ along the fibres of $P(\mathbb{S}^m, X) \to \mathbb{R}P^m$. One has the following formula for the i-th Stiefel-Whitney class of $P(\mathbb{S}^m, \omega) = \hat{\omega}$ in terms of the $\tilde{c}_j(\omega)$, (cf. [NS19, Prop. 2.11]):

$$w_i(\hat{\omega}) = \sum_{0 \le j \le r} \binom{r-j}{i-2j} x^{i-2j} \tilde{c}_j(\omega) \tag{3}$$

where $x := w_1(\xi_\alpha) \in H^1(P(\mathbb{S}^m, X); \mathbb{Z}_2) \cong H^1(\mathbb{R}P^m; \mathbb{Z}_2) \cong \mathbb{Z}_2$ is non-zero and $r = \mathrm{rank}_{\mathbb{C}}\omega$. (It is understood that the binomial coefficient $\binom{a}{b} = 0$ if $b > a$ or if $a < 0$ or $b < 0$.) See [NS19].

We have $\tilde{c}_1(\omega) = w_2(\hat{\omega}) + \binom{r}{2} x^2$ and $w_1(\hat{\omega}) = rx$. It can be seen easily, using (2) and induction, that $\tilde{c}_j(\omega)$ is expressible as a polynomial

$$\tilde{c}_j(\omega) = Q_j(x^2, w_2(\hat{\omega}), w_4(\hat{\omega}), \ldots, w_{2j}(\hat{\omega})) \text{ for } 1 \leq j \leq r.$$

Therefore, for $0 \leq j < r$, $w_{2j+1}(\hat{\omega})$ can be expressed as a polynomial in x and even Stiefel-Whitney classes $w_{2i}(\hat{\omega}), 0 \leq i \leq j$:

$$w_{2j+1}(\hat{\omega}) = xP_j(x^2, w_2(\hat{\omega}), w_4(\hat{\omega}), \ldots, w_{2j}(\hat{\omega})) \tag{4}$$

for a suitable polynomial $P_j = P_j(x^2, w_2, w_4, \ldots, w_{2j})$ of total degree $2j$ where $\deg(w_i) = i$. As noted already, $P_0 = r \in \mathbb{Z}_2$. Equations (3) and (4) are still valid when X is any connected CW complex as long as $H^1(X; \mathbb{Z}_2) = 0$.

We need to extend Equation (4) to the more general context of the bundle $\hat{\omega} := P(S, \omega)$ over $P(S, \alpha, X, \sigma)$ under the assumption that Y (and hence S) and X are CW complexes (although one can relax these restrictions even further).

We proceed as follows: First, extend the validity of (3) to the case when $S = \mathbb{S}^\infty$ with antipodal involution $-id$ so that $Y = \mathbb{R}P^\infty$. Then, any double cover $S \to Y$ can be classified by a map $\bar{f} : Y \to \mathbb{R}P^\infty$ and so we have an $\mathbb{Z}_2$-equivariant map $f : (S, \alpha) \to (\mathbb{S}^\infty, -id)$. Hence we obtain formula (3) with $x = w_1(\xi_\alpha)$ for the bundle $P(S, \omega)$, using the naturality of Stiefel-Whitney classes and the isomorphism (1). So, it remains only to consider the case $(S, \alpha) = (\mathbb{S}^\infty, -id)$.

Note that $P(\mathbb{S}^\infty, X) = \cup_{m \geq 1} P(\mathbb{S}^m, X)$. Moreover, the inclusion $P(\mathbb{S}^m, X) \hookrightarrow P(\mathbb{S}^\infty, X)$ is an $(m-1)$-equivalence (as can be seen using the homotopy exact sequence). Since the spaces involved are CW complexes i induces an isomorphism in cohomology up to dimension $m-1$. Given any σ-conjugate complex vector bundle ω over X of rank r we merely choose $m > 2r$. It follows that the formula (3) holds for $P(\mathbb{S}^\infty, \omega)$.

Using (4), or directly, we see that $P(S, \omega)$ is orientable if and only if $rank_{\mathbb{C}}(\omega)$ is even.

2.4 Torus manifolds

The notion of torus manifolds is due to Hattori and Masuda [HM03], [Mas99] . We shall use the definition as given in Masuda and Panov [MP06]. In fact, we consider a restricted class of torus manifolds (in the sense of Masuda-Panov), namely, those torus manifolds where the torus action is locally standard and the orbit space is a homology polytope. This restricted class itself is a generalization of the notion of quasi-toric manifolds due to Davis and Januskiewicz [DJ91], where the orbit space is a simple convex polytope. We begin by recalling the basic definitions.

A *torus manifold* is an even dimensional smooth compact orientable manifold X on which an n-dimensional torus $T \cong U(1)^n$ acts smoothly and effectively with non-empty fixed point set where $\dim X = 2n$. One says that the T-action on X is *locally standard* if X is covered by open sets $\{U\}$ that are equivariantly diffeomorphic to an open subset of $\mathbb{C}^n$ with the standard $T \cong U(1)^n$-action. This means that, for each U, there exists an automorphism $\psi : T \to T$ and an embedding $f : U \to \mathbb{C}^n$ such that $f(t.u) = \psi(t) f(u)\ \forall u \in U, t \in T$. The orbit space $Q := X/T$ is then an n-dimensional manifold with corners (i.e., is modelled on $\mathbb{R}^n_{\geq 0}$). It turns out that set of T-fixed points of X is finite and their images are the vertices of Q. It is easy to see (using the irreducible characters of the isotropy representation) that each T-fixed point is a connected component of the intersection of exactly n distinct T-stable submanifolds each having codimension 2 in X. There are only finitely many T-stable codimension 2 submanifolds in X, say $X_i, 1 \leq i \leq m$, and each of these are fixed by certain circle subgroup S_i of T. These are the *characteristic submanifolds* of X. Their images, $Q_i := X_i/T$ in Q are the *facets* of Q, which have codimension

1 in Q. The characteristic submanifolds of X are all orientable and are again torus manifolds under the T/S_i-action with orbit space Q_i.

A *preface* of Q is a non-empty intersection of a collection of facets of Q. A *face* of Q is a connected component of a preface. We regard Q itself as (the improper) face; all the other faces are *proper*. Q is said to be *face-acyclic* if all its faces (including Q itself) are acyclic, i.e., have the integral homology of a point. If Q is face acyclic, then every face contains a vertex of Q. If Q is face-acyclic and if every preface is a face, then Q is called a *homology polytope*. For example, when X is a quasi-toric manifold, then $Q = X/T$ is a simple convex polytope, which is evidently a homology polytope. (A (compact) convex polytope of dimension n is *simple* if exactly n facets meet at each vertex.)

A characteristic submanifold $X_i \hookrightarrow X$ determines a circle subgroup $S_i \subset T$; namely, the subgroup of T that fixes each point of X_i. Choosing an isomorphism f_i : $\mathbb{S}^1 \cong S_i$ amounts to choosing a primitive vector $v_i \in \mathrm{Hom}(\mathbb{S}^1, T) \cong \mathbb{Z}^n$ whose image equals S_i. Note that v_i is determined up to sign corresponding to two isomorphisms $\mathbb{S}^1 \to S_i$ namely $f_i, f_i \circ \iota$ where $\iota(z) = z^{-1}\ \forall z \in \mathbb{S}^1$. The sign is determined once an orientation on S_i is fixed.

We shall denote the group of 1-parameter subgroups $\mathrm{Hom}(\mathbb{S}^1, T)$ by $\mathbf{N}$ and the group of characters $\mathrm{Hom}(T, \mathbb{S}^1)$ by $\mathbf{N}^\vee \cong \mathbb{Z}^n$. One has a natural pariing $\langle ., . \rangle$: $\mathbf{N}^\vee \times \mathbf{N} \to \mathbf{Z}$ defined by $u \circ v(z) = z^{\langle u,v \rangle}$.

An *omni-orientation* of X is a choice of an orientation on X and on each characteristic submanifold $X_i, 1 \leq i \leq m$. The orientations on X, X_i determine a unique orientation on the normal space to $T_xX_i \subset T_xX$ for any $x \in X_i$ which in turn leads to an orientation on S_i. This determines a unique *primitive* vector $v_i \in \mathbf{N}$ whose image is S_i.

Fix an omni-orientation on X and assume that Q is a homology polytope. Denote by $\mathcal{Q}$ the set of all facets of Q. We obtain a map $\Lambda : \mathcal{Q} \to \mathbf{N}$ defined as $\Lambda(Q_i) = v_i, 1 \leq i \leq m$. Local standardness of the T action implies that $\Lambda(Q_{i_1}), \ldots, \Lambda(Q_{i_n})$ is a $\mathbb{Z}$-basis of $\mathbf{N}$ whenever $Q_{i_1}, \ldots, Q_{i_n} \in \mathcal{Q}$ meet at a vertex of Q. The map Λ is called the *characteristic function* of X. The pair (Q, Λ) determines X up to equivariant homeomophism assuming the vanishing of $H^2(Q; \mathbb{Z})$. In fact, let $X(Q, \Lambda)$ denote the space $T \times Q/\!\sim$ where $(t, q) \sim (t', q')$ if and only if $q = q'$ and $t^{-1}t'$ belongs to the subgroup of T generated by the one parameter subgroups $\Lambda(Q_{i_1}), \ldots, \Lambda(Q_{i_k})$ where q is in the interior of the face $Q_{i_1} \cap \cdots \cap Q_{i_k}$. Then $X(Q, \Lambda)$ is a smooth manifold on which T acts on the left with orbit space Q. Further Q is naturally embedded in $X(Q, \Lambda)$ via the map $q \mapsto [1, q]$. We regard Q as a subspace of X. The quotient map $X(Q, \Lambda) \to Q$ is therefore a retraction. It was shown in [MP06, Lemma 4.5], assuming the vanishing of $H^2(Q; \mathbb{Z})$, that $X(Q, \Lambda)$ is equivariantly homeomorphic to X. We identify X with $X(Q, \Lambda)$.

2.5 Half-spin bundles over $\mathbb{S}^{2r}$

Assume that $m = 2r$ is even. An explicit description of a complex vector bundle ξ over $\mathbb{S}^m$ such that $[\xi] - \operatorname{rank}(\xi)$ is a generator of $\tilde{K}(S^m) \cong \mathbb{Z}$ is well-known. See [AH61], [Bot69]. For the sake of completeness, we give the details.

Explicitly, ξ may be taken to be associated to a half-spin representation of $\mathrm{Spin}(m)$ where $\mathbb{S}^m$ is viewed as the homogeneous space $\mathrm{Spin}(m+1)/\mathrm{Spin}(m)$. We shall explain this construction below, assuming familiarity with the basic properties of the spin groups, as given in [Hus94].

We consider $\mathbb{R}^{m+1}$ as the standard $SO(m+1)$-representation. It becomes a representation of $\mathrm{Spin}(m+1)$ via the projection $\mathrm{Spin}(m+1) \to SO(m+1)$. Regard $\mathrm{Spin}(m) \subset \mathrm{Spin}(m+1)$ is the subgroup that stabilizes the vector $e_1 \in \mathbb{R}^{m+1}$. Consider bundles ξ^+, ξ^- associated to the half-spin (complex) representations Δ^+, Δ^- of the spin group Spin(m). Then $[\xi^+] - 2^{r-1}$ is a generator of $\tilde{K}(\mathbb{S}^m)$. See [AH61], [Hus94, Theorem 13.3, Chapter 14]. (Note that the degrees of the representations $\Delta^{\pm}$ equal 2^{r-1}.) It turns out that $[\xi^+] + [\xi^-] = 2^r$ in $\tilde{K}(\mathbb{S}^m)$. In fact, the representation $\Delta^+ \oplus \Delta^-$ is the restriction of the spin representation Δ of Spin($m+1$). As the bundle associated to $\Delta|_{\mathrm{Spin}(m)}$ is trivial, we obtain that $\xi^+ \oplus \xi^- = 2^r \epsilon_{\mathbb{C}}$, whence

$$[\xi^+] - 2^{r-1} = -([\xi^-] - 2^{r-1}). \tag{2.1}$$

Theorem 2.5.1. *Let $m = 2r$ be even. The ring $K^0(\mathbb{S}^m)$ is isomorphic to the truncated polynomial ring $\mathbb{Z}[u_m]/\langle u_m^2 \rangle$ under an isomorphism that sends u_m to $[\xi^+] - 2^{r-1}$. In particular, $[\xi^{\pm}]^2 = 2^r[\xi^{\pm}] - 2^{2r-2}$, and, $[\xi^+][\xi^-] = 2^{2r-2}$. Moreover, $\alpha^! : K^0(\mathbb{S}^m) \to K^0(\mathbb{S}^m)$ sends $[\xi^+]$ to $[\xi^-]$.*

Proof. The description of the ring structure and the fact that u_m is represented by $[\xi^+] - 2^{r-1}$, are well-known. (See [AH61] and [Hus94].) The asserted quadratic relation follows from $([\xi^+] - 2^{r-1})^2 = u_m^2 = 0$. Since $[\xi^+] + [\xi^-] = 2^r$, the same relation is satisfied by $[\xi^-]$. Also, $[\xi^+][\xi^-] = [\xi^+](2^r - [\xi^+]) = 2^{2r-2}$.

It remains to show that $\alpha^!(u_m) = -u_m$. This follows from the following facts: $H^m(\alpha; \mathbb{Q})$ is $-id$, the Chern character $\mathrm{ch} : K(\mathbb{S}^m) \to H^*(\mathbb{S}^m; \mathbb{Q})$ is a monomorphism, and $\alpha^* \circ \mathrm{ch} = \mathrm{ch} \circ \alpha^!$. □

We have not been able to find a natural and explicit bundle isomorphism $\xi^- \to \xi^+$ which covers the antipodal map $\alpha : \mathbb{S}^m \to \mathbb{S}^m$. For this reason, we work with the pull-back bundle $\alpha^!(\xi^+)$, which will suffice for our purposes.

Denote by η^- the pull-back bundle $\alpha^!(\xi^+)$. When $m \geq 2$ is even, we have $2^{r-1} \geq m/2$ and any two vector bundles of rank 2^{r-1} which represent the same class in $K(\mathbb{S}^m)$ are isomorphic, using [Hus94, Theorem 1.5, Chapter 8].

For future reference, we have the following remark.

Remark 2.5.2. *From the above discussion, we have* $[\eta^-] = [\xi^-] = 2^r - [\xi^+]$ *in* $K(\mathbb{S}^m)$ *for all* $m \equiv 0 \pmod 2$.

We have a bundle isomorphism $\tilde{\alpha}^- : E(\eta^-) \subset \mathbb{S}^m \times E(\xi^+) \to E(\xi^+)$ given by the second projection that covers the antipodal map α. Explicitly, $\tilde{\alpha}^- : E(\eta^-) \to E(\xi^+)$ is the map that $(v, e) \in E_v(\eta^-)$ to $e \in E_{-v}(\xi^+)$. Then $\tilde{\alpha}^-$ is a bundle map that covers α. Similarly $\tilde{\alpha}^+ : E(\xi^+) \to E(\eta^-)$ is the bundle map that sends $e \in E_v(\xi^+)$ to $(-v, e) \in E_{-v}(\eta^-)$. Henceforth, we shall identify the fibre $E_v(\eta^-) = \{v\} \times E_{-v}(\xi^+)$ with $E_{-v}(\xi^+)$ so that the total space $E(\eta^-)$ is identified with $E(\xi^+)$. We have, for any $v \in \mathbb{S}^m$ and $e \in E_v(\xi^+)$, $\tilde{\alpha}^+(\tilde{\alpha}^-(e)) = e$, i.e., $\tilde{\alpha}^+ \circ \tilde{\alpha}^- = id_{\eta^-}$. Similarly $\tilde{\alpha}^- \circ \tilde{\alpha}^+ = id_{\xi^+}$.

Set $\tilde{\xi} := \xi^+ \oplus \eta^-$. A vector in the fibre $E_v(\tilde{\xi})$ over $v \in \mathbb{S}^m$ will be denoted by a triple $(v; e, e'), e \in E_v(\xi^+), e' \in E_{-v}(\xi^+) = E_v(\eta^-), v \in \mathbb{S}^m$. Define $\tilde{\alpha} : E(\xi^+ \oplus \eta^-) \to E(\xi^+ \oplus \eta^-)$ as $\tilde{\alpha}(v; e, e') = (-v; \tilde{\alpha}_v^-(e'), \tilde{\alpha}_v^+(e))$. Then $\tilde{\alpha}$ covers α and $\tilde{\alpha} \circ \tilde{\alpha} = id$.

We therefore obtain a complex vector bundle ξ^0 over $\mathbb{R}P^m$ where $E(\xi^0) := E(\tilde{\xi})/\langle \tilde{\alpha} \rangle$. A point of $E_{[v]}(\xi^0)$ is $[v; x, y] = \{(v; x, y), (-v; y, x)\}$ where $x \in E_v(\xi^+), y \in E_v(\eta^-) = E_{-v}(\xi^+)$.

Let us consider the bundle maps $\tilde{\beta}^+ : E(\eta^-) \subset \mathbb{S}^m \times E(\xi^+) \to E(\xi^+)$ defined by $(v, e) \mapsto ie \in E_{-v}(\xi^+)$ and similarly $\tilde{\beta}^- : E(\xi^+) \to E(\eta^-)$ which maps $e \in E_v(\xi^+)$ to $(-v, ie) \in E_{-v}(\eta^-)$. These two bundle maps cover α. It is easy to see that $\tilde{\beta}^+ \circ \tilde{\beta}^- = -id_{E(\eta^-)}$ and $\tilde{\beta}^- \circ \tilde{\beta}^+ = -id_{E(\xi^+)}$. Define $\tilde{\beta} : E(\tilde{\xi}) \to E(\tilde{\xi})$, $(v; e, e') \mapsto (-v; -\tilde{\beta}^-(e'), \tilde{\beta}^+(e))$ which is a bundle involution covering α. Now we can form a vector bundles ξ^1 over $\mathbb{R}P^m$ where $E(\xi^1) = E(\tilde{\xi})/\langle \tilde{\beta} \rangle$. A point of $E_{[v]}(\xi^1)$ is $[v; x, y] = \{(v; x, y), (-v; -iy, ix)\}$ where $x \in E_v(\xi^+), y \in E_v(\eta^-) = E_{-v}(\xi^+)$.

Remark 2.5.3. *We note that* ξ^0 *and* ξ^1 *are isomorphic. An explicit isomorphism can be defined by* $E(\xi^0) \ni [v; x, y] \mapsto [v; -ix, y] \in E(\xi^1)$.

Chapter 3

Cohomology groups of GDS

In this chapter, we study cohomology of $P(S,X)$ for arbitrary topological spaces S,X. We generally consider S,X to be paracompact and Hausdorff spaces. First, we study the mod 2 cohomology of $P(S,X)$ and then consider cohomology in $\mathbb{Z}[1/2]$-coefficients as well.

3.1 Mod 2 cohomology of GDS

Let S,X be locally finite CW complexes. Under appropriate hypotheses on the cell-structures on S and X, we obtain a nice CW structure on $P(S,X)$ which has the property that the mod 2 cellular boundary map vanishes. We shall see that there are many classes of smooth manifolds S,X where there are such CW structures.

Denote by $(C_*(A),\partial^A)$ the cellular chain complex of a CW complex A over $\mathbb{Z}_2$. Every cell e has a unique orientation mod 2 and so defines a basis element, again denoted by e, of $C_q(A;\mathbb{Z}_2)=H_q(A^{(q)},A^{(q-1)};\mathbb{Z}_2)$. When A is clear from the context we write ∂ instead of ∂^A. In fact, we shall denote by the same symbol ∂ for the differential in any chain complexes when there is no danger of confusion which one is meant.

Let X be a connected CW complex. Assume that $\sigma : X \to X$ is an involution which stabilizes *each* cell of X. In particular the zero cells are contained in $\text{Fix}(\sigma)$. Since σ stabilizes each cell of X, we have $\sigma_*(e) = \pm e = e$ for each cell e of X and so $\sigma_* : C_*(X; \mathbb{Z}_2) \to C_*(X; \mathbb{Z}_2)$ is the identity map.

Let S be a $\mathbb{Z}_2$-equivariant CW complex where $\alpha : S \to S$ is the generator of the $\mathbb{Z}_2$-action on S. Since α is fixed point free involution, no open cell is mapped to itself under α. The CW-structure on (S, α) yields a CW structure on Y consisting of one cell $q(e) = q(\alpha(e))$ for each pair of cells $e, \alpha(e)$ of S, where q is the quotient map $S \to Y$. On the other hand, any CW structure on Y lifts to an equivariant CW-structure on S.

Also we obtain the product CW structure on $S \times X$ which is $\mathbb{Z}_2$-equivariant where the $\mathbb{Z}_2$-action is generated by $\theta := \alpha \times \sigma$. It consists of cells which are products $e \times d$ where e, d are cells of S and X respectively. The induced CW structure on $P(S, X)$ consists of one cell (e, d) for each pair of cells $e \times d, \theta(e \times d) = \alpha(e) \times \sigma(d) = \alpha(e) \times d$.

Suppose that the cell structure on X is perfect mod 2, that is, the differential in the cellular chain complex of X with $\mathbb{Z}_2$-coefficients vanishes in all dimensions. Since $\partial^X = 0$ in all dimensions, we have $H_*(X; \mathbb{Z}_2) \cong C_*(X; \mathbb{Z}_2)$. Our assumptions are trivially valid when X has no odd dimensional cells. In this case the cellular boundary map ∂ vanishes even with $\mathbb{Z}$-coefficients.

Fix $r \geq 0$. We have $C_r(S \times X; \mathbb{Z}_2) \cong \oplus_{p+q=r} C_p(S; \mathbb{Z}_2) \otimes C_q(X; \mathbb{Z}_2)$. Since $\partial^X = 0$, we have $\partial(e \otimes d) = \partial(e) \otimes d$. Morevover $\theta_*(e \otimes d) = \alpha_*(e) \otimes \sigma_*(d) = \alpha(e) \otimes d$. Recall that π denotes the covering projection $S \times X \to P(S, X)$. Since $\pi \circ \theta = \pi$ we have the following commuting diagram of chain complexes:

$$\begin{array}{ccc} C_r(S \times X; \mathbb{Z}_2) & \xrightarrow{\theta_*} & C_r(S \times X, \mathbb{Z}_2) \\ & \pi_* \searrow \quad \swarrow \pi_* & \\ & C_r(P(S, X); \mathbb{Z}_2). & \end{array}$$

We denote by (e, d) the cell $\pi(e \times d) = \pi(\alpha(e) \times d)$ in $P(S, X)$. Let $\dim e = p, \dim d = q$. Suppose that $\partial^S(e) = \sum_{j \in J(e)} e_j \in C_{p-1}(S; \mathbb{Z}_2)$ where $J(e)$ is a suitable subset of the indexing set for cells of S. Then, $\partial(e \times d) = \sum e_j \otimes d$ and so

$$\partial((e, d)) = \sum_{j \in J(e)} \pi_*(e_j \otimes d) = \sum_{j \in J(e)} (e_j, d) \in C_{r-1}(P(S, X); \mathbb{Z}_2). \tag{5}$$

Since $q_*(\alpha(e_j)) = q_*(e_j) \in C_{p-1}(Y)$, we have $(e_j, d) = (\alpha(e_j), d)$ for any cell d of X. If $\alpha(J(e)) = J(e)$, then both (e_j, d) and $(\alpha(e_j), d)$ occur in the sum (5) and cancel each other out and we conclude that $\partial((e, d)) = 0$. (Note that $e_j, \alpha(e_j)$ are distinct cells of S.) We are ready to prove the following proposition. For the notion of cohomology extension of fibre we refer the reader to Leray-Hirsch Theorem [Spa82, Theorem 9, §7, Chapter 5].

Proposition 3.1.1. *Suppose that (S, α) is any $\mathbb{Z}_2$-equivariant CW complex such that the induced cell structure on Y is perfect mod 2. Suppose that X is a CW complex such that (i) each skeleton $X^{(k)}, k \geq 1$ is finite, (ii) the cell-structure is perfect mod 2, and, (iii) each cell is mapped to itself by $\sigma : X \to X$. Then the CW structure on $P(S, X)$ induced by the product CW structure on $S \times X$ is perfect mod 2. In particular, one has an isomorphism $H^*(P(S, X); \mathbb{Z}_2) \cong H^*(Y; \mathbb{Z}_2) \otimes H^*(X; \mathbb{Z}_2)$ of $H^*(Y; \mathbb{Z}_2)$-modules.*

Proof. Let $\partial^S(e) = \sum_{j \in J(e)} e_j$. Since the induced cell structure on Y is perfect, we have $0 = \partial^Y(q_*(e)) = q_*(\partial^S(e)) = q_* \sum_{j \in J(e)} e_j$. Since $q_*(\partial^S(e)) = 0$ we see that $J(e)$ is stable by α. Hence, from Equation (5), for any cell d of X, we have $\partial((e, d)) = 0$. Thus the induced cell-structure on $P(S, X)$ is perfect mod 2.

Assume that X is a finite CW complex. Then we have

$$\begin{aligned} H_k(P(S, X); \mathbb{Z}_2) \cong C_k(P(S, X); \mathbb{Z}_2) &\cong \oplus_{i+j=k} C_i(Y; \mathbb{Z}_2) \otimes C_j(X; \mathbb{Z}_2) \\ &\cong \oplus_{i+j=k} H_i(Y; \mathbb{Z}_2) \otimes H_j(X; \mathbb{Z}_2). \end{aligned}$$

Consequently $H_*(P(S,X);\mathbb{Z}_2) \cong H_*(Y;\mathbb{Z}_2) \otimes H_*(X;\mathbb{Z}_2)$. Also $H^*(P(S,X);\mathbb{Z}_2) \cong H^*(Y;\mathbb{Z}_2) \otimes H^*(X;\mathbb{Z}_2)$ is a free module over $H^*(Y;\mathbb{Z}_2)$ via $\pi^* : H^*(Y;\mathbb{Z}_2) \to H^*(P(S,X);\mathbb{Z}_2)$.

In the general case, the inclusion $X^{(k)} \hookrightarrow X$ induces an inclusion $P(S,X^{(k)}) \hookrightarrow P(S,X)$ which covers the identity map of Y. The proposition follows from the observation that, for any $n \geq 1$, the inclusion-induced homomorphism $H^n(P(S,X);\mathbb{Z}_2) \to H^n(P(S,X^{(k)});\mathbb{Z}_2)$ is an isomorphism for all $k > n$. □

We shall obtain another situation where the X-bundle $\pi : P(S,X) \to Y$ admits a $\mathbb{Z}_2$-cohomology extension of fibre. As observed already, since $X^\sigma \neq \emptyset$, we have a cross-section $Y \to P(S,X)$ of the X-bundle and so it follows that $p^* : H^*(Y;\mathbb{Z}_2) \to H^*(P(S,X);\mathbb{Z}_2)$ is a monomorphism. We have the following result. For the notion of cohomology extension of fibre we refer the reader to [Spa82, Theorem 9, §7, Chapter 5].

Proposition 3.1.2. *Let $(\omega_j, \hat{\sigma}_j), 1 \leq j \leq r$, be σ-conjugate complex vector bundles over (X,σ) such that the cohomology algebra $H^*(X;\mathbb{Z}_2)$ is generated by the mod 2 Chern classes $c_q(\omega_j) \in H^{2q}(X;\mathbb{Z}_2), 1 \leq q \leq rank(\omega_j), 1 \leq j \leq r$. Assume that X satisfies any of the following: (i) $\dim_{\mathbb{Z}_2} H^*(X;\mathbb{Z}_2) < \infty$, (ii) X is a CW complex with finite k-skeleton for each $k \geq 1$ where each cell is stable by σ. Then we have an isomorphism of $\mathbb{Z}_2$-vector spaces:*

$$H^r(P(S,X);\mathbb{Z}_2) \cong \bigoplus_{p+q=r} H^p(Y;\mathbb{Z}_2) \otimes H^q(X;\mathbb{Z}_2). \tag{6}$$

In particular, $H^*(P(S,X);\mathbb{Z}_2)$ is a free $H^*(Y;\mathbb{Z}_2)$-module with basis B where B is a $\mathbb{Z}_2$-basis for $H^*(X;\mathbb{Z}_2)$.

Proof. Let $e_0 \in S$. Set $\hat{\omega}_j := P(S,\omega_j), 1 \leq j \leq r$. Then $\iota^*(\hat{\omega}_j) \cong \omega_j$ where ι is the inclusion $\iota : X \to P(S,X)$ defined as $x \mapsto [e_0, x]$. So $w_{2i}(\hat{\omega}_j)$ restricts to $w_{2i}(\omega_j) = c_i(\omega_j) \in H^{2i}(X;\mathbb{Z}_2)$. Since the $c_i(\omega_j), 1 \leq i \leq \text{rank}(\omega_j), 1 \leq j \leq r$, generate the cohomology algebra $H^*(X;\mathbb{Z}_2)$, it follows that the homomorphism $H^*(P(S,X);\mathbb{Z}_2) \to H^*(X;\mathbb{Z}_2)$ is surjective. Thus, the X-bundle $(P(S,X),Y,\pi)$ admits $\mathbb{Z}_2$-cohomology extension of the fibre. If $\dim_{\mathbb{Z}_2} H^*(X;\mathbb{Z}_2)$ is finite, such as when X is a finite CW complex, by Leray-Hirsch theorem we conclude that $H^*(P(S,X);\mathbb{Z}_2) \cong H^*(Y;\mathbb{Z}_2) \otimes H^*(X;\mathbb{Z}_2)$ as $H^*(Y;\mathbb{Z}_2)$-modules. This proves (i).

Suppose that X is a CW complex such that for any $k \geq 1$ the k-skeleton $X^{(k)}$ of X is a finite CW complex. Moreover, σ restricts to $X^{(k)}$. It follows from part (i) of the proposition that $H^*(P(S,X^{(k)});\mathbb{Z}_2) \cong H^*(Y;\mathbb{Z}_2) \otimes H^*(X^{(k)};\mathbb{Z}_2)$, which establishes the isomorphism (6). The last assertion follows readily from (6). □

3.2 Chomology of GDS in $\mathbb{Z}[1/2]$ coefficients

Here we study cohomology groups of generalized Dold spaces in R coefficients where 2 is invertible in R. Let us fix some notations and conventions before proceeding further.

Let B be a (locally finite) CW complex. We shall denote the closed cells of B by e, e_j, etc., and the corresponding open cells by $\mathring{e}, \mathring{e}_j$, etc. An orientation on a (closed) k-cell e in B is the choice of a generator of $H_k(e,\partial e;\mathbb{Z}) \cong H_k(\mathbb{S}^k;\mathbb{Z})$. An orientation on e is equivalent to an orientation of the open cell $\mathring{e} \cong \mathbb{R}^k$.

First, we shall consider the homology of $P(S,X)$ where X is a connected locally finite CW complex that has only even dimensional cells. We assume that S is a connected locally finite CW complex. Further restrictions on S and X will be placed later as and when required. Some of the results here are established in the context of a double cover $p : \widetilde{B} \to B$ with appropriate hypotheses on $\widetilde{B}$.

Lemma 3.2.1. *Let X be a connected locally finite CW complex with only even* *dimensional cells. Suppose that $\sigma : X \to X$ is an involution that stabilizes each cell in X and that σ is orientation preserving on any $2k$-dimensional cell e if and only if k is even. Then σ_* acts by $(-1)^k$ on $H_{2k}(X;\mathbb{Z})$.* □

We omit the proof, which is straightforward.

Suppose that $p : \tilde{B} \to B$ is a double covering projection where B is a connected locally finite CW complex. We obtain a G-equivariant CW structure on $\tilde{B}$ by lifting the cell structure on B where $G \cong \mathbb{Z}_2$ is the deck transformation group. Denote by $\phi : \tilde{B} \to \tilde{B}$ the involution that generates G. For each (closed) cell e of B, we have a pair of cells e^+, e^- in $\tilde{B}$ such that $p(e^\pm) = e$ and $\phi(e^+) = e^-$ (as unoriented cells).

Let R be any commutative ring. (It is always assumed that $0 \neq 1 \in R$.) We denote by $(C_*(B;R), \partial)$ the cellular chain complex with R-coefficients. Recall that $C_r(B;R)$ is the free R-module with basis the set of all closed oriented r-cells of B modulo the relations $e + e' = 0$ where e' is the same underlying cell as e but with opposite orientation. (We denote by the same symbol e an oriented cell as well as the corresponding element in the cellular chain group $C_k(B;R)$.) It is clear that $C_r(B;R)$ is isomorphic to the free R-module with basis the set of *unoriented* r-cells of B.

For each oriented cell e in B, fix orientations on e^+ and e^- so that $p|_{e^\pm} : e^\pm \to e$ is orientation preserving; thus $p_*(e^\pm) = e$. Then $\phi|_{e^+} : e^+ \to e^-$ is orientation preserving since $p = p \circ \phi$.

Suppose that 2 is invertible in R. We set $\varepsilon^+ := (e^+ + e^-)/2$ and $\varepsilon^- := (e^+ - e^-)/2$ for each closed cell e of B. Then $C_r(\tilde{B};R) = C_r^+(\tilde{B};R) \oplus C_r^-(\tilde{B};R)$ as a direct sum of R-modules where $C_r^+(\tilde{B};R)$ is the free R-module with basis $\{\varepsilon_i^+\}_i$ as e_i varies over the set of r-cells of B. Similarly $\{\varepsilon_i^-\}_i$ is an R-basis for $C_r^-(\tilde{B};R)$. It is readily seen that $C_r^+(\tilde{B};R) = \mathrm{Fix}(\phi_*)$, the R-submodule of $C_r(\tilde{B};R)$ element-wise fixed by ϕ_* and that ϕ_* acts as multiplication by -1 on $C_r^-(\tilde{B};R)$. Clearly p_* maps

$C_r^+(\tilde{B};R)$ isomorphically onto $C_*(B;R)$ and vanishes identically on $C_r^-(\tilde{B};R)$. Note that $\partial : C_*(\tilde{B};R) \to C_*(\tilde{B};R)$ maps $C_*^{\pm}(\tilde{B};R)$ to itself since $\partial \circ \phi_* = \phi_* \circ \partial$. Hence $H_r(\tilde{B};R)$ breaks up as a direct sum of two R-modules on which ϕ_* acts by 1 and (-1) respectively.

Lemma 3.2.2. *We keep the above notations. Let R be any commutative ring in which 2 is invertible. Then $p_* : H_*(\tilde{B};R) \to H_*(B;R)$ maps Fix(ϕ_*) isomorphically onto $H_*(B;R)$. Moreover $p_*[z] = 0$ if $\phi_*([z]) = -[z]$.*

Proof. Since $p_* = p_* \circ \phi_*$ in homology, if $\phi_*([z]) = -[z]$, then $p_*[z] = -p_*[z]$. Since 2 is invertible in R, we have $p_*([z]) = 0$.

Suppose that $\phi_*[z] = [z]$. Write $z = z^+ + z^-$ where $z^{\pm} \in C_r^{\pm}(\tilde{B};R)$. Since $\partial z_i^{\pm} \in C_r^{\pm}(\tilde{B};R)$ (as observed above), we have $\partial z^{\pm} = 0$ and so $[z] = [z^+] + [z^-]$. Since $[z] = \phi_*([z]) = [z^+] - [z^-]$, we have $2[z^-] = 0$ which implies that $[z^-] = 0$ as 2 is a unit in R. So we may (and do) assume that $z = z^+$.

Suppose that $p_*[z] = 0$. Write $p_*(z) = \partial(\sum a_j e_j)$ and set $\tilde{u} := \sum a_j \varepsilon_j^+ \in C_{r+1}^+(\tilde{B})$. Then $p_*(\partial(\tilde{u})) = \partial p_*(\tilde{u}) = p_*(z)$. Since $p_* : C_*^+(\tilde{B};R) \to C_*(B;R)$ is a monomorphism, we have $z = \partial\tilde{u}$ and so $[z] = 0$. Thus $p_*|_{\mathrm{Fix}(\phi_*)}$ is a monomorphism.

Next suppose that $z = \sum a_j e_j$ is a cycle in B (with $a_j \in R$). Then $p_*(\tilde{z}) = z$ where $\tilde{z} = \sum a_j \varepsilon_j \in C_r^+(\tilde{B};R)$. Since $p_*(\partial\tilde{z}) = \partial z = 0$ and since $p_* : C_*^+(\tilde{B};R) \to C_*(B;R)$ is a monomorphism, we see that $\partial\tilde{z} = 0$. Clearly $p_*([\tilde{z}]) = [z]$ and so p_* is an epimorphism. □

As an immediate consequence, we obtain the following.

Proposition 3.2.3. *Let $[z] \in H_r(\tilde{B};\mathbb{Z})$. If $p_*([z]) = 0$ and $\phi_*([z]) = [z]$, then $2^m[z] = 0$ for some $m \geq 1$. If $\phi_*([z]) = -[z]$, then $2p_*([z]) = 0$.* □

Example 3.2.4. Let B be the $2n$-dunce hat, defined as the mapping cone of $f : \mathbb{S}^1 \to \mathbb{S}^1$ where $f(z) = z^{2n}$. Assume that $n > 1$. Then $\tilde{B}$ is a two dimensional complex, $\pi_1(\tilde{B}) \cong \mathbb{Z}_n \to \mathbb{Z}_{2n} \cong \pi_1(B)$ is the inclusion homomorphism. We have $H_k(\tilde{B}; \mathbb{Z}) = 0$ for $k \geq 2$. The deck transformation group induces id on $H_1(\tilde{B}; \mathbb{Z}) \cong \mathbb{Z}_n$ and $p_* : H_1(\tilde{B}; \mathbb{Z}) \to H_1(B; \mathbb{Z})$ corresponds to inclusion $\mathbb{Z}_n \hookrightarrow \mathbb{Z}_{2n}$. We see that $H_*(p; R)$ is an isomorphism if 2 is a unit in R. Although B has a CW complex structure which is perfect mod 2, taking $n = 2^k$, $k \geq 1$, the 2-torsion elements in $H_1(B; \mathbb{Z})$ need not be of order 2. $\square$

Suppose that $\pi_1(\tilde{B}) \cong \mathbb{Z}^n$. We have an exact sequence of groups where $C_2 \cong \mathbb{Z}_2$:

$$1 \to \pi_1(\tilde{B}) \to \pi_1(B) \to C_2 \to 1. \tag{3.1}$$

If $\pi_1(B)$ is abelian, then either $\pi_1(B) \cong \mathbb{Z}^n \oplus \mathbb{Z}_2$ or $\pi_1(B) \cong \mathbb{Z}^n$ according as whether the exact sequence splits or not. Hence $H_1(B; \mathbb{Z}) \cong \mathbb{Z}^n \oplus \mathbb{Z}_2$ or $\mathbb{Z}^n$.

Suppose that $\pi_1(B)$ is not abelian. We will use additive notation for $\pi_1(\tilde{B})$. Conjugation by $y \in \pi_1(B) \backslash \pi_1(\tilde{B})$ defines an automorphism $A : \pi_1(\tilde{B}) \to \pi_1(\tilde{B})$. Fixing a basis $x_1, \ldots, x_n$, we obtain a matrix $A = (a_{ij}) \in GL(n, \mathbb{Z})$ such that $A^2 = I_n$ and $yxy^{-1} = Ax \ \forall x \in \mathbb{Z}^n$. Since $\pi_1(B)$ is not abelian, $A \neq I_n$. We assume that y is chosen so that it has order 2 when the above exact sequence splits. The abelianization of $\pi_1(B)$, namely $H_1(B; \mathbb{Z})$, has the following presentation

$$H_1(B, \mathbb{Z}) = (\mathbb{Z}x_1 + \cdots + \mathbb{Z}x_n + \mathbb{Z}y)/H$$

where H is generated by the following elements $2y - \sum_{1 \leq i \leq n} c_i x_i, (A - I_n)x_j, 1 \leq j \leq n$, for some $c \in \mathbb{Z}^n$. If $c = 0$, then y generates a cyclic subgroup of order 2. Suppose that $c \neq 0$. Then the above exact sequence does not split by our hypothesis on y, and there exists a j so that c_j is odd. We may assume that all the non-zero c_i are 1. Relabeling the x_j, we may assume that $c_1 = 1$. Now replacing x_1 by $\sum c_i x_i$ in the basis $x_1, \ldots, x_n$ of $\pi_1(B)$ if necessary, we may (and do) assume that $2y = x_1$. Note that $yx_1y^{-1} = x_1$. It follows that the image of y is an element of infinite order in $H_1(B;\mathbb{Z})$ and that $H_1(B;\mathbb{Z})$ is generated by at most n elements.

Set $A_- := \{x \in \pi_1(\tilde{B}) \mid Ax = -x\} \subset \pi_1(B)$. Its image $\bar{A}_-$ in $H_1(B;\mathbb{Z})$ is isomorphic to $\mathbb{Z}_2^k$ where $k = \mathrm{rank}(A_-)$.

We claim that $H_1(B;\mathbb{Z})/\bar{A}_-$ is a free abelian group of rank $n-k$. To see this, let $A_+ = \{x \in \pi_1(\tilde{B}) \mid Ax = x\} \subset \pi_1(B)$. Since A is invertible and since $A_+ + A_-$ has the same rank n as $\pi_1(\tilde{B})$, it suffices to show that A_+ maps monomorphically into $H_1(B;\mathbb{Z})$. This is immediate from the above presentation of $H_1(B;\mathbb{Z})$ since $yxy^{-1} = Ax = x$. We summarise the result in the proposition below.

Proposition 3.2.5. *Let $p : \tilde{B} \to B$ be a double cover where B is connected and locally path connected. Suppose that $\pi_1(\tilde{B})$ is free abelian of finite rank, say n. Then $H_1(B;\mathbb{Z}) \cong \mathbb{Z}^l \oplus \mathbb{Z}_2^k$ for some $k, l \geq 0$. Moreover, $k + l = n$ except when the deck transformation group $\mathbb{Z}_2$ acts trivially on $\pi_1(\tilde{B})$, in which case $l = n, k = 1$.* □

We apply Lemma 3.2.2 to the special case of the double covering projection $\pi : S \times X \to P(S,X)$ satisfying further hypotheses. Denote by $H_p^{\pm}(S;R)$ the R-submodule of $H_p(S;R)$ on which α_* acts as ± 1 and denote its dimension by $b_p^{\pm}$. Thus the r-th Betti number $b_r(S)$ of S equals $b_r^+ + b_r^-$.

Proposition 3.2.6. *We keep the above notations. Suppose that Y is a connected locally finite CW complex and that $H_*(S;\mathbb{Z})$ is free abelian. Assume that X is a connected locally finite CW complex with cells only in even dimensions. Assume that the involution $\sigma : X \to X$ satisfies the hypothesis of Lemma 3.2.1. Let R be a ring where 2 is invertible. Then (i) $\pi_* : H_*(S \times X;R) \to H_*(P(S,X);R)$ is a monomorphism. In fact, we have an isomorphism*

$$(3.2)\quad H_r(P(S,X);R) \cong \bigoplus_{p+4t=r} (H_p^+(S;R) \otimes H_{4t}(X;R) \oplus H_{p-2}^-(S;R) \otimes H_{4t+2}(X;R))$$

In particular,

$$\dim H_r(P(S,X);\mathbb{Q}) = \sum_{p+4q=r} (b_p^+ b_{4q}(X) + b_{p-2}^- b_{4q+2}(X)). \tag{3.3}$$

(ii) Suppose that $\pi_1(S)$ is abelian. Then any torsion element of $H_r(P(S,X);\mathbb{Z})$ has order a power of 2.

Proof. (i). We shall apply Lemma 3.2.2. In view of our hypotheses and the Künneth theorem, we have $H_r(S \times X;\mathbb{Z}) = \bigoplus_{p+2q=r} H_p(S;\mathbb{Z}) \otimes H_{2q}(X;\mathbb{Z})$. Recall that $\pi = \pi \circ \theta$ where $\theta = \alpha \times \sigma$. So $\pi_* = \pi_* \circ \theta_*$ and so $\theta_* = \alpha_* \otimes \sigma_*$. Since σ_* acts on $H_{2q}(X;\mathbb{Z})$ by $(-1)^q$ we obtain that $H_r(P(S,X);R) \cong \mathrm{Fix}(H_r(\theta;R)) = \bigoplus_{p+4t=r}(H_p^+(S;R) \otimes H_{4t}(X;R)) \oplus (H_{p-2}^-(S;R) \otimes H_{4t+2}(X;R))$. Taking $R = \mathbb{Q}$ we obtain the asserted formula for the r-th Betti number of $P(S,X)$.

(ii) We only need to show that there is no odd torsion element in $H_r(P(S,X);\mathbb{Z})$. Our hypothesis on $\pi_1(S)$ and the freeness of $H_1(S;\mathbb{Z})$ implies that $\pi_1(S) \cong \mathbb{Z}^k$ for some $k \geq 0$. If $k = 0$, then $\pi_1(P(S,X)) \cong \mathbb{Z}_2$. In general, $\pi_1(P(S,X))$ has an index 2 subgroup $H := \pi_*(\pi_1(S \times X)) \cong \mathbb{Z}^k$. Then by Proposition 3.2.5, $H_1(P(S,X);\mathbb{Z}) \cong \mathbb{Z}^l \oplus \mathbb{Z}_2^t$ for some $l, t \geq 0$ and so has no odd torsion.

Since $H_*(S \times X;\mathbb{Z})$ has no torsion and since π_* maps $\mathrm{Fix}(H_*(\theta;\mathbb{Q}))$ isomorphically onto $H_*(P(S,X);\mathbb{Q})$, π induces a monomorphism $\mathrm{Fix}(H_*(\theta;\mathbb{Z})) \to H_*(P(S,X);\mathbb{Z})$. Suppose that $H_r(P(S,X);\mathbb{Z})$ has an element of order an odd prime p. We assume that r is the least positive integer for which this happens. So $r > 1$ and $\mathrm{Tor}(H_{r-1}(P(S,X);\mathbb{Z}),\mathbb{Z}_p) = 0$. By the universal coefficient theorem, $H_r(P(S,X);\mathbb{Z}_p) \cong H_r(P(S,X);\mathbb{Z}) \otimes \mathbb{Z}_p$. Since the r-th Betti number of $P(S,X)$ equals $\dim H_r(P(S,X);\mathbb{Z}_p)$ by (i), every non-zero element in $H_r(P(S,X);\mathbb{Z}_p)$ has to be the reduction mod p of an element of $H_r(P(S,X);\mathbb{Z})$ of *infinite* order. This implies that $H_r(P(S,X);\mathbb{Z})$ has no element of order p, contrary to our choice of r. Hence the proof. □

Remark 3.2.7. *(i) The cohomology version of the above proposition is valid and is* *equivalent to it by the universal coefficient theorem.*

(ii) Suppose that S, X *as in the above theorem are compact connected orientable manifolds (without boundary). Suppose that* $\dim S = m, \dim X = 2d$. *Then* σ *is orientation preserving if and only if d is even. It follows that* $P(S, X)$ *is orientable if and only if* α *is orientation preserving and d is even, or,* α *orientation reversing and d is odd. This follows from computation of* $H_{m+2d}(P(S, X); \mathbb{Q})$. *This is also seen from the fact that* θ *is orientation preserving if and only if both* α, σ *are simultaneously either orientation preserving or reversing.*

Chapter 4

Mod 2 cohomology algebra of some GDS

In this chapter, we obtain results on mod 2 cohomology ring of $P(S, X)$, where S is a paracompact Hausdorff topological space and X is (i) a sphere, (ii) a torus manifold (see [MP06]) under certain mild assumptions, which are satisfied by quasi-toric manifolds and (iii) a complex flag manifold.

4.1 Cohomology of generalized Klein's bottles

Let $X = \mathbb{S}^n, n \geq 1$, and let $\sigma : \mathbb{S}^n \to \mathbb{S}^n$ be the reflection map that sends $e_{n+1} \to -e_{n+1}$ and pointwise fixes $\mathbb{S}^{n-1} \subset \mathbb{R}^n = \{e_{n+1}\}^{\perp}$. The cell structure on $\mathbb{S}^n$ with one 0-cell $d_0 = \{e_1\}$ and one (closed) n-cell $d_n = \mathbb{S}^n$ is stable by σ. With (S, α) as in Proposition 3.1.1, we have $H^*(P(S, \mathbb{S}^n); \mathbb{Z}_2) \cong H^*(Y; \mathbb{Z}_2) \otimes H^*(\mathbb{S}^n; \mathbb{Z}_2)$ which is a free $A := H^*(Y; \mathbb{Z}_2)$-module of rank 2. We shall determine $H^*(P(S, \mathbb{S}^n); \mathbb{Z}_2)$ as an A-algebra. The two cases $n \geq 2$ and $n = 1$ need to be treated separately.

Proposition 4.1.1. *Let $n \geq 2$. Suppose that S satisfies the hypothesis of Proposition 3.1.1. Let $A = H^*(Y; \mathbb{Z}_2)$. Then $H^*(P(S, \mathbb{S}^n); \mathbb{Z}_2) \cong A[u]/\langle u^2 \rangle$ as an A-algebra* *where* $\deg u = n$.

Proof. We first prove the claim when $S = \mathbb{S}^m$, then extend it to the case $S = \mathbb{S}^\infty$. The general case will be shown to follow from the case $S = \mathbb{S}^\infty$.

Let $S = \mathbb{S}^m$, $\alpha = -id$ so that $Y = \mathbb{R}P^m$. The Proposition is obvious when $m < n$ and so we assume that $m \geq n$. We shall denote $P(\mathbb{S}^m, \mathbb{S}^n)$ by $P_{m,n}$. One has an inclusion $P_{k,n} \subset P_{m,n}$ for $0 \leq k \leq m$, where $k = 0$ corresponds to the fibre inclusion $\mathbb{S}^n \hookrightarrow P_{m,n}$.

Let y be the non-zero element in $H^1(P_{m,n}; \mathbb{Z}_2) \cong H^1(\mathbb{R}P^m; \mathbb{Z}_2) \cong \mathbb{Z}_2$. Then $A = H^*(\mathbb{R}P^m; \mathbb{Z}_2) = \mathbb{Z}_2[y]/\langle y^{m+1} \rangle$. Observe that y is the Poincaré dual to the submanifold $P_{m-1,n} \hookrightarrow P_{m,n}$. Also, one may obtain $P_{k,n}$ as the intersection of $(m-k)$-copies of $P_{m-1,n}$ in general position and so y^{m-k} is the Poincaré dual of $P_{k,n}$. In particular the Poincaré dual of the fibre $\mathbb{S}^n \hookrightarrow P_{m,n}$ equals y^m.

Any $x \in \mathbb{S}^{n-1} = \mathrm{Fix}(\sigma)$ defines a cross-section $s : \mathbb{R}P^m \to P_{m,n}$ of the sphere bundle projection $P_{m,n} \to \mathbb{R}P^m$. Any two such sections are isotopic since $\mathbb{S}^{n-1}$ is connected. (Here we use the hypothesis that $n \geq 2$.) *We set* $u = u_n$ *to be the Poincaré dual of the submanifold* $s : \mathbb{R}P^m \to P_{m,n}$. Since the intersection $s(\mathbb{R}P^m) \cap \mathbb{S}^n$ is transverse and is exactly one point, we see that $y^m u \in H^{m+n}(P_{m,n}; \mathbb{Z}_2) \cong \mathbb{Z}_2$ is non-zero. Also, taking two distinct fixed points $x, x' \in \mathbb{S}^{n-1}$, we obtain cross-sections s, s' where $s(\mathbb{R}P^m) \cap s'(\mathbb{R}P^m) = \emptyset$. This shows that $u_n^2 = 0$, completing the proof of the proposition in this case.

Next, let $S = \mathbb{S}^\infty$. Choose $m > 2n$ so that the inclusion $j : P(\mathbb{S}^m, \mathbb{S}^n) \hookrightarrow P(\mathbb{S}^\infty, \mathbb{S}^n)$ is an $(m-1)$-equivalence. Let $U_n \in H^n(P(\mathbb{S}^\infty, \mathbb{S}^n); \mathbb{Z}_2) \cong H^n(P(\mathbb{S}^m, \mathbb{S}^n); \mathbb{Z}_2)$ be the element that corresponds to u_n. Since $u_n^2 = 0$ and j^* is an isomorphism in dimension $2n$, we have $U_n^2 = 0$. Thus taking $u = U_n$, the claim holds for $S = \mathbb{S}^\infty$.

In the general case of (S, α), we have a classifying map $\bar{f} : Y \to \mathbb{R}P^\infty$ which classifies the double covering $S \to Y$. Let $f : S \to \mathbb{S}^\infty$ be a lift of $\bar{f}$. Then we obtain a morphism of $\mathbb{S}^n$-bundles $F : P(S,\mathbb{S}^n) \to P(\mathbb{S}^\infty,\mathbb{S}^n)$ that covers $\bar{f}$ by functoriality. Let $\mathbf{u}_n = F^*(U_n)$. Since $H^*(P(\mathbb{S}^\infty,\mathbb{S}^n);\mathbb{Z}_2) \to H^*(P(S,\mathbb{S}^n);\mathbb{Z}_2)$ is a ring homomorphism, we have $\mathbf{u}_n^2 = 0$. Setting $u := \mathbf{u}_n$, the Claim follows. □

We turn to the case $n = 1$. Note that when $m = 1$, $P_{m,1}$ is the Klein bottle. It can be seen that, as in the special case of the Klein bottle, $P_{m,1}$ is a connected sum $\mathbb{R}P^{m+1}\#\mathbb{R}P^{m+1}$. To see this, we let $J^+, J^- \subset \mathbb{S}^1 \subset \mathbb{R}^2$ to be the closed arcs with end points $e_2, -e_2$ where $e_1 \in J^+, -e_1 \in J^-$. Note that σ stabilizes J^+ and J^-. Then $P(\mathbb{S}^m, J^+), P(\mathbb{S}^m, J^-) \subset P_{m,n}$ are twisted I-bundles over $\mathbb{R}P^m$ with common boundary $\mathbb{S}^m$. The projections of the I-bundle restricted to $\mathbb{S}^m$ is the double cover $\mathbb{S}^m \to \mathbb{R}P^m$. Since $\mathbb{R}P^{m+1}$ is got by attaching an m-cell via the double cover $\mathbb{S}^m \to \mathbb{R}P^m$, it follows that $P_{m,1}$ is the connected sum $\mathbb{R}P^{m+1}\#\mathbb{R}P^{m+1}$. Hence $H^*(P_{m,1};\mathbb{Z}_2) \cong \mathbb{Z}_2[a,b]/\langle a^{m+2}, b^{m+2}, ab\rangle$. The projection of the circle bundle $p : P_{m,1} \to \mathbb{R}P^m$ induces $p^* : H^*(\mathbb{R}P^m;\mathbb{Z}_2) \to H^*(P_{m,1};\mathbb{Z}_2)$, defined by $y \mapsto a+b$, where y is the generator of $A = H^*(\mathbb{R}P^m;\mathbb{Z}_2)$. (Note that $y = a + b \in H^1(P_{m,n};\mathbb{Z}_2)$ is the *only* element that satisfies the conditions $y^{m+1} = 0, y^m \neq 0$.) As an A-algebra, $H^*(P_{m,1};\mathbb{Z}_2)$ is isomorphic to $A[a]/\langle a^2 - ay\rangle$. When $m = \infty$, A is a polynomial algebra in y and we have $H^*(P_{\infty,1};\mathbb{Z}_2) \cong A[a]/\langle a^2 - ay\rangle$.

When (S,α) is as in Proposition 3.1.1, $P(S,\mathbb{S}^1) = Y^+ \cup Y^-$ where $Y^\pm = P(S,J^\pm)$ are total spaces of twisted I-bundles $\gamma^\pm$ over Y where $Y^+ \cap Y^- = P(S,\mathbb{S}^0) \cong S$. We have a continuous surjection $\eta : P(S,\mathbb{S}^1) \to P(S,J^+)/S = T(\gamma^+)$ where $T(\gamma^+)$ stands for the Thom space of γ^+. We let $a \in H^1(P(S,\mathbb{S}^1);\mathbb{Z}_2)$ be the image of the Thom class $a \in H^1(T(\gamma^+);\mathbb{Z}_2) \cong \mathbb{Z}_2$ under the homomorphism induced by η. Arguing as in the proof of the above proposition, we obtain the following.

Proposition 4.1.2. *Let (S,α) be as in Proposition 3.1.1. Then $H^*(P(S,\mathbb{S}^1);\mathbb{Z}_2) = A[a]\langle a^2 - ay\rangle$ where $\deg a = 1 = \deg y$.* □

4.2 Cohomology of $P(S, X)$, where X is a torus manifold

In this section, we determine the cohomology ring of $P(S, X)$ when X is a torus manifold under certain mild assumptions, which are satisfied by quasi-toric manifolds and S is paracompact. Basic facts concerning torus manifolds that are required for our purposes are recalled in §2.4. Now let's begin with the following lemma which will be needed in the sequel.

Let X be an oriented compact smooth manifold on which the torus $T \cong (\mathbb{S}^1)^k$ acts smoothly and effectively. Suppose that $H \subset T$ is a circle subgroup such that $F := X^H = \{x \in X \mid t.x = x \ \forall t \in H\}$ is an oriented connected submanifold of codimension 2. Then T stabilizes F. Let ν be the normal bundle of $F \hookrightarrow X$. We put a Riemannian metric on X that is invariant under T. Then ν is the orthogonal complement of the tangent bundle τF in $\tau X|_F$. Moreover, ν is a T-equivariant bundle over F. Note that H acts on ν as bundle automorphisms since F is pointwise fixed by H. Since F and X are oriented, so is ν and we may (and do) regard ν as a complex line bundle.

Lemma 4.2.1. *With notations as above, the complex line bundle ν is the restriction to F of a T-equivariant complex line bundle ω over X. Moreover, one has a T-equivariant cross-section $s : X \to E(\omega)$ which vanishes precisely on F. We have $c_1(\omega) = [F] \in H^2(X; \mathbb{Z})$ where $[F]$ denotes the cohomology class dual to $F \hookrightarrow X$.*

Proof. Without the equivariance part, the existence of ω and s were proved in [San08] and the equality $c_1(\omega) = [F]$ was deduced. So we need only construct ω and s as T-equivariant objects. As in the discussion before the statement of the lemma, we put a Riemannian metric on X that is invariant under T.

Denote by ω_0 the pull-back of ν to $D(\nu)$ via π where $\pi : D(\nu) \to F$ is the projection of the unit disk bundle associated to ν. Then $\omega_0 := \pi^*(\nu)$ is a T-equivariant complex line bundle which admits an equivariant cross-section $s_0 : D(\nu) \to E(\omega_0)$ defined as follows: Recall that $E(\omega_0)$ is the fibre product $\{(v, w) \in D(\nu) \times E(\nu) \mid \pi(v) = \pi(w)\} \subset D(\nu) \times E(\nu)$ and that T acts on $E(\omega_0)$ diagonally: $t.(v, w) = (t.v, t.w)$. We have $s_0(v) := (v, v)\ \forall v \in D(\nu)$. Note that s_0 vanishes precisely when $v = 0$, i.e., on the zero-cross section $F \to D(\nu)$. Moreover, $s_0(t.v) = (t.v, t.v) = t.(v, v) = t.s_0(v)$ for all $v \in D(\nu), t \in T$, showing that s_0 is T-equivariant. Let $S(\nu) = \partial D(\nu)$. We have an isomorphism of complex line bundles $\phi : E(\omega_0|_{S(\nu)}) \to S(\nu) \times \mathbb{C}$ defined as $\phi(v, zv) = (v, z||v||)\ \forall v \in S(\nu), z \in \mathbb{C}$. The section $\phi \circ s_0$ corresponds to the constant function 1 (i.e., $\phi \circ s_0(v) = (v, 1)\ \forall v \in S(\nu)$). Since the Riemannian metric on X is T-invariant, we have $\phi(t.(v, zv)) = \phi(t.v, zt.v)) = (t.v, z||t.v||) = (t.v, z||v||) = t.(v, z) = t.\phi(v, zv)\ \forall (v, w) \in E(\omega_0), t \in T$. This shows that, with the trivial T-action on $\mathbb{C}$ understood, ϕ is T-equivariant.

Let $N \subset X$ be an equivariant tubular neighbourhood of $F \subset X$ which is T-equivariantly diffeomorphic to $D(\nu)$. (See [Bre72, §2, Chapter VI].) Identifying N with $D(\nu)$ via such a diffeomorphism, we obtain a T-equivariant complex line bundle, again denoted ω_0, on N which restricts to ν on F, and a T-equivariant cross-section $s_0 : N \to E(\omega_0)$ which vanishes precisely on F. Moreover, we have an isomorphism $\phi : E(\omega_0|_{\partial N}) \to \partial N \times \mathbb{C}$. We glue $E(\omega_0)$ and $(X \setminus \mathrm{int}(N)) \times \mathbb{C}$, the total space of the trivial line bundle $\varepsilon_{\mathbb{C}}$, along $S(\nu) \times \mathbb{C}$ using ϕ to obtain a T-equivariant complex line bundle ω on X. The section s_0 extends to a T-equivariant section $s : X \to E(\omega)$ such that $s|_{X \setminus \mathrm{int}(N)}$ corresponds to the constant function $x \mapsto 1 \in \mathbb{C}$. In particular, s vanishes precisely on F. $\square$

Now consider a torus manifold X with $T \cong U(1)^n$ action such that X/T is a homology polytope as discussed earlier in §2.4. We continue using the facts and notations from the earlier discussion.

Let $\sigma : X(Q, \Lambda) \to X(Q, \Lambda)$ be the involution $[t, q] \mapsto [t^{-1}, q]$. (It is readily verified that this definition is meaningful.) Then $X^\sigma = \{[t,q] \mid t^2 \in S_q,\ \forall q \in Q\}$, which is the analogue of a *small cover* in the context of quasi-toric manifolds; see [DJ91]. Suppose that X_i is a characteristic submanifold of X, fixed by a subgroup $S_i \cong \mathbb{S}^1$ of T. Let q_i be in the interior of Q_i (i.e., $q_i \in Q_i$ but not in any proper face of Q_i). Then S_i is the isotropy subgroup at q_i, and X_i is the closure of the T-orbit of Q_i. Since $\sigma(Q_i) = Q_i$, it follows that $\sigma(X_i) = X_i$.

Let $\tilde{T} := T \rtimes \langle\sigma\rangle$ be the semi-direct product where $\sigma.t = t^{-1}.\sigma$ in $\tilde{T}$. Then $\tilde{T}$ acts on X. We put a Riemannian metric on X which is invariant with respect to $\tilde{T}$. In particular, σ preserves the Euclidean metric on the normal bundle ν_i over X_i. Let N_i denote a tubular neighbourhood of X_i that is stably by $\tilde{T}$ and is $\tilde{T}$-equivariantly diffeomorphic to $D(\nu_i)$, the disk bundle of ν_i. We denote by $\theta_i : N_i \to D(\nu_i)$ such a diffeomorphism. We have that $T\sigma$ restricts to a bundle map of the unit disk bundle $\pi_i : D(\nu_i) \to X_i$ covering $\sigma|_{X_i}$ and $\sigma(N_i) = N_i$. As in the proof of Lemma 4.2.1, we denote by $\omega_{i,0}$ the bundle $\theta_i^*(\pi_i^*(\nu_i))$ over N_i. We shall abuse notation and write π_i for the projection $N_i \to X_i$. This yields an involution $\hat{\sigma}_{i,0}$ on $\omega_{i,0}$ defined as $(u,v) \mapsto (\sigma(u), T_{\pi_i(u)}\sigma(v))$ where $u \in N_i$, and $v \in D(\nu_i)$ is in the fibre of $D(\nu_i) \to X_i$ over $\pi_i(u)$.

Lemma 4.2.2. *(i) The bundle map $T\sigma|_{D(\nu_i)}$ is orientation reversing on ν_i.*
(ii) The involution $\hat{\sigma}_{i,0} : E(\omega_{i,0}) \to E(\omega_{i,0})$ is a complex conjugation that covers $\sigma|_{N_i}$.

Proof. (i) It suffices to show that σ is orientation reversing on the fibre of $N_i \cong D(\nu_i) \to X_i$ over a point $q_i \in Q_i$. Since any neigbourhood of Q_i meets the interior of Q, we see that $N_i \cap int(Q) \neq \emptyset$. Let $q \in N_i \cap int(Q)$. Since $\sigma(q) = q$, and since σ is fibre preserving on $N_i \to X_i$, it follows that q is in the fibre D_{q_i} over a $q_i \in Q_i$. Let $t_0 \in S_i$ be the unique order two element. Then $[t_0, q] \in D_{q_i}$ and $\sigma([t_0,q]) = [t_0,q] \neq [1,q] = q$ and, moreover, no other point in the H_i-orbit of q is fixed by σ. This shows that σ is the reflection of the disk D_{q_i} about the 'diameter' of the disk D_{q_i} through q. Hence σ is orientation reversing.

(ii) Fixing orientation on ν_i, we obtain a reduction of the structure group of ν_i to $SO(2) \cong U(1)$, making ν_i a complex line bundle. Since $T\sigma|_{E(\nu_i)}$ preserves the Euclidean metric on ν_i, and is an involution, we conclude that it is a complex conjugation covering $\sigma|_{X_i}$. The same argument applied to the pull-back of ν_i via the projection $\pi_i : N_i \to X_i$ of the disk bundle shows that $(p, v) \mapsto (\sigma(p), T_{\pi_i(p)}\sigma(v))$ is a complex conjugation of $\pi_i^*(\nu_i) = \omega_{i,0}$ covering $\sigma|_{N_i} : N_i \to N_i$. □

Taking $F = X_i \subset X$, a characteristic submanifold in Lemma 4.2.1, we obtain a $T \rtimes \langle\sigma\rangle$-equivariant complex line bundle ω_i over X that extends the normal bundle ν_i over X_i, and a T-equivariant cross-section $s_i : X \to E(\omega_i)$ which vanishes precisely on X_i. In fact $\omega_i|_{N_i} = \omega_{i,0}$ and $\omega_i|_{X \setminus int(N_i)} = \varepsilon_{\mathbb{C}}$, the trivial complex line bundle.

We claim that the complex conjugation $\hat{\sigma}_{i,0}$ on $\omega_{i,0}$ extends to a complex conjugation on ω_i that covers $\sigma : X \to X$. Explicitly, we let $\hat{\sigma}_i$ to be the standard complex conjugation on the trivial bundle over $(X \setminus int(N_i))$. It remains to show that, under the identification of $\omega_{i,0}|_{\partial N_i}$ with the trivial bundle via the cross-section $s_0 : N_i \to E(\omega_i)$, the restriction $\hat{\sigma}_{i,0}|_{\partial N_i \times \mathbb{C}}$ is the same as the standard complex conjugation on $E(\omega_{i,0}|_{\partial N_i})$. This is clear since $s_0|_{\partial N_i}$ corresponds to the constant function $x \mapsto 1$ as noted in the proof of Lemma 4.2.1. Thus we have proved the following.

Lemma 4.2.3. *With the above notations, one has a σ-conjugation $\hat{\sigma}_i : E(\omega_i) \to E(\omega_i)$ for each $1 \leq i \leq m$. Moreover, the Chern class $c_1(\omega_i) \in H^2(X;\mathbb{Z})$ equals the cohomology class Poincaré dual to $X_i \hookrightarrow X$, i.e., $c_1(\omega_i) = [X_i] \in H^2(X;\mathbb{Z})$.* □

In view of Lemma 4.2.3 and the fact that the cohomology algebra $H^*(X;\mathbb{Z}_2)$ is generated by the classes $[X_i], 1 \leq i \leq m$, the hypotheses of Proposition 3.1.2 are satisfied. This leads to a description of the cohomology $H^*(P(S,X);\mathbb{Z}_2)$ as a module over $H^*(Y;\mathbb{Z}_2)$ for any pair (S,α) where S is a paracompact space and

$Y = S/\mathbb{Z}_2$. Our aim is to describe $H^*(P(S,X);\mathbb{Z}_2)$ as an $H^*(Y;\mathbb{Z}_2)$-algebra in terms of generators and relations.

Recall from [MP06] that the *integral* cohomology ring $H^*(X;\mathbb{Z})$ is the quotient of the polynomial algebra $\mathbb{Z}[x_1,\ldots,x_m]$ modulo the ideal I generated by the following two types of elements:

$$(i)\ x_{j_1}\ldots x_{j_r} = 0 \ \text{ whenever } Q_{j_1}\cap\cdots\cap Q_{j_r} = \emptyset,$$

$$(ii)\ \sum_{1\le j\le m}\langle u, v_j\rangle x_j = 0\ \forall u\in \mathrm{Hom}(T,\mathbb{S}^1),$$

where $v_j = \Lambda(Q_j)\in \mathbf{N}$. The element x_j corresponds to $[X_j]\in H^2(X;\mathbb{Z})$. In particular, X satisfies the hypothesis of Proposition 3.1.2 and we have $H^*(P(S,X);\mathbb{Z}_2)\cong H^*(Y,\mathbb{Z}_2)\otimes H^*(X;\mathbb{Z}_2)$.

For $u\in\mathbf{N}^\vee$, consider the complex line bundle $\omega_u := \otimes_{1\le j\le m}\omega_j^{\otimes a_j}$ where $a_j = \langle u, v_j\rangle\in\mathbb{Z}$. Then ω_u is isomorphic to the trivial complex line bundle since $c_1(\omega_u) = \sum a_j c_1(\omega_j) = \sum a_j[X_j] = 0$ in view of the relation (ii) above. Using the fact that $\bar\eta\cong \mathrm{Hom}_{\mathbb{C}}(\eta,\epsilon_{\mathbb{C}})$ and $\eta\otimes\eta_1$ are σ-conjugate complex vector bundles when η,η_1 are σ-conjugate vector bundles yields that ω_u is a σ-conjugate line bundle. (See [NS19, Example 2.2(iv)].) In fact, in the case of $\bar\eta$, the σ-conjugation $\hat\sigma$ of η is also a σ-conjugation of $\bar\eta$.

We have a $T\rtimes\mathbb{Z}_2$-equivariant cross-section $s_j : X\to E(\omega_j)$ whose zero locus equals X_j. We write s_j^a to denote the corresponding cross-section of $\omega_j^{\otimes a}$ for $a\ge 1$; when $a<0$, we set $s_j^a := s_j^{|a|}$, regarded as a section of $\bar\omega_j^{\otimes|a|}$. (Note that $E(\omega) = E(\bar\omega)$). When $a=0$, ω_j^a is the trivial bundle and s_j^a corresponds to the constant map $X\to\mathbb{C}$ to $x\mapsto 1$.

Let $\tilde x_j := w_2(\hat\omega_j)\in H^*(P(S,X);\mathbb{Z}_2)$ where $\hat\omega_j := P(S,\omega_j)$. Then $\sum b_j\tilde x_j$ restricts to $\sum b_j x_j = w_2(\otimes\omega_j^{\otimes b_j})$ along the fibres of the X-bundle $P(S,X)\to Y$. Let $p\in$

Fix(σ). Denote by s_p the cross section $Y \to P(S, X)$ defined as $[v] \mapsto [v, p]$. Since $H^1(X; \mathbb{Z}) = 0$, we have $H^2(P(S, X); \mathbb{Z}_2) \cong H^2(Y; \mathbb{Z}_2) \oplus H^2(X; \mathbb{Z}_2)$ by Proposition 3.1.2.

Claim: $s_p^*(\tilde{x}_j) = 0$ in $H^2(Y; \mathbb{Z}_2)$.
To see this, note that $s_p : Y \to P(S, X)$ factors as follows: $Y \stackrel{\cong}{\to} P(S, p) \hookrightarrow P(S, X)$. Now $\hat{\omega}_j|_{P(S,p)} = \xi_\alpha \oplus \epsilon_{\mathbb{R}}$ since $\omega_j|_p \cong \epsilon_{\mathbb{C}} = \{x_0\} \times \mathbb{C}$ (with standard complex conjugation). So $s_p^*(\tilde{x}_j) = w_2(\xi_\alpha \oplus \epsilon_{\mathbb{R}}) = 0$, as claimed.

It follows from the above Claim that $w_2(\otimes \hat{\omega}_j^{\otimes b_j}) = \sum b_j \tilde{x}_j$. Taking $b_j = \langle u, v_j \rangle$ and using the isomorphism $P(S, \otimes \omega_j^{\otimes b_j}) \cong P(S, \omega_u) \cong P(S, \epsilon_{\mathbb{C}})$ where the trivial complex line bundle has the standard conjugation, we obtain that, for any $u \in \mathbf{N}^\vee$,

$$\sum_{1 \le j \le m} \langle u, v_j \rangle \tilde{x}_j = 0. \tag{7}$$

Next, suppose that $\cap_{1 \le q \le r} Q_{j_q} = \emptyset$. The Whitney sum $\omega := \oplus_{1 \le q \le r} \omega_{j_q}$ admits a cross-section $s : X \to E(\omega)$ given by $s(x) = (s_{j_1}(x), \ldots, s_{j_r}(x))$. Clearly s vanishes along $\cap_{1 \le q \le r} X_{j_q} = \emptyset$, that is, s is nowhere vanishing and so we obtain a splitting $\omega \cong \eta \oplus \epsilon_{\mathbb{C}}$. The σ-conjugations on each summand ω_{j_q} of ω put together yields a σ-conjugation on ω. Since s is $T \rtimes \mathbb{Z}_2$-equivariant, the σ-cojugation on ω restricts to σ-conjugations on η and $\epsilon_{\mathbb{C}}$, and, on the latter it is the standard conjugation. It follows that $\hat{\omega} = \oplus_{1 \le q \le r} \hat{\omega}_{j_q} = \hat{\eta} \oplus \epsilon_{\mathbb{R}} \oplus \xi_\alpha$. Hence the top Stiefel-Whitney class of $\hat{\omega}$ is zero. That is,

$$\prod_{1 \le q \le r} \tilde{x}_{j_q} = 0 \ \text{ whenever } Q_{j_1} \cap \cdots \cap Q_{j_r} = \emptyset \ . \tag{8}$$

Let $A = H^*(Y; \mathbb{Z}_2)$ and let $A[\tilde{x}_1, \ldots, \tilde{x}_m]$ denote the polynomial algebra in the *indeterminates* $\tilde{x}_1, \ldots, \tilde{x}_m$. As a consequence of (7) and (8) we obtain the following.

Theorem 4.2.4. *Let $X = X(Q, \Lambda)$ be a T-torus manifold where $X/T = Q$ is* *a homology polytope with m facets. Let $\sigma : X \to X$ be the involution $[t, q] \mapsto [t^{-1}, q]$. Then, with the above notations, $H^*(P(S, X); \mathbb{Z}_2)$ is isomorphic, as an $A = H^*(Y; \mathbb{Z}_2)$-algebra, to the quotient $R(Q, \Lambda) := A[\tilde{x}_1, \ldots, \tilde{x}_m]/I$ where the ideal $I = I(Q, \Lambda)$ is generated by the following two types of elements:*

(i) $\sum_{1 \le j \le m} \langle u, v_j \rangle \tilde{x}_j,\ u \in \mathbf{N}^\vee$, *and,*

(ii) $\prod_{1 \le q \le r} \tilde{x}_{j_q}$ *whenever* $Q_{j_1} \cap \cdots \cap Q_{j_r} = \emptyset$.

The isomorphism is given by $\tilde{x}_j \mapsto w_2(\hat{\omega}_j) \in H^2(P(S, X); \mathbb{Z}_2)$.

Proof. From the description of the cohomology ring of X and the definition of $R(Q, \Lambda)$ it is clear that one has an isomorphism $R(Q, \Lambda) \cong H^*(X; A) \cong A \otimes_{\mathbb{Z}_2} H^*(X; \mathbb{Z}_2)$ of graded A-modules (by the universal coefficient theorem). In particular $R(Q, \Lambda)$ is a free A-module of rank equal to $\dim_{\mathbb{Z}_2} H^*(X; \mathbb{Z}_2) < \infty$. In fact, any $\mathbb{Z}_2$-basis consisting of monomials in $w_2(\omega_j)$ lifts an A-basis for $R(Q, \Lambda)$ got by replacing $w_2(\omega_j)$ by $\tilde{x}_j$.

We have a well-defined A-algebra homomorphism $\theta : R(Q, \Lambda) \to H^*(P(S, X); \mathbb{Z}_2)$ defined by $\tilde{x}_j \mapsto w_2(\hat{\omega}_j), 1 \le j \le m$, in view of Equations (7) and (8). By Proposition 3.1.2, we see that θ is a surjective homomorphism of A-modules. By the observation made above, as an A-module homomorphism, θ maps an A-basis to an A-basis and hence is an isomorphism. □

4.3 Cohomology of $P(S, \mathbb{C}G(\nu))$

In this section, our aim is to describe the $\mathbb{Z}_2$- cohomology ring of generalized Dold manifold $P(\mathbb{S}^m, \mathbb{C}G(\nu))$, which is fibred by the complex flag manifold $\mathbb{C}G(\nu)$ of type $\nu = (n_1, \ldots, n_s)$. Before proceeding to the case of $P(\mathbb{S}^m, \mathbb{C}G(\nu))$, we study the case of $P(\mathbb{S}^m, \mathbb{C}G(\nu))$, where $\mathbb{C}G_{n,k} = G_k(\mathbb{C}^n)$ is a complex Grassmannian manifold consisting of complex k-dimensional subspaces of $\mathbb{C}^n$.

The fixed point set of the (usual) complex conjugation σ on $\mathbb{C}G_{n,k}$ is the real Grassmann manifold $\mathbb{R}G_{n,k}$. Let $\xi = \xi_\alpha$ denote the line bundle associated to the double cover $S \to Y$. We shall also denote by ξ the line bundle over $P(S, \mathbb{C}G_{n,k})$ obtained as the pull back $p^!(\xi)$ via the projection $p : P(S, X) \to Y$ of the $\mathbb{C}G_{n,k}$-bundle. Denote by $\gamma_{n,k}, \beta_{n,k}$ the tautological complex k-plane bundle and its orthogonal complement bundle, which is of rank $(n-k)$. Note that the complex conjugation on $\mathbb{C}^n$ yields σ-conjugations on $\gamma_{n,k}$ and on $\beta_{n,k}$. Moreover, we have an isomorphism

$$\gamma_{n,k} \oplus \beta_{n,k} \cong n\epsilon_{\mathbb{C}}.$$

This yields an isomorphism of real vector bundles: (cf. [NS19, Example 2.4(ii)])

$$\hat{\gamma}_{n,k} \oplus \hat{\beta}_{n,k} \cong n\xi_\alpha \oplus n\epsilon_{\mathbb{R}} \tag{9}$$

where $\hat{\omega} := P(S, \omega)$. Consequently, the following relation among the Stiefel-Whitney classes holds, where $y = w_1(\xi_\alpha)$:

$$w(\hat{\gamma}_{n,k}).w(\hat{\beta}_{n,k}) = (1+y)^n.$$

We rewrite the above relation in terms of Stiefel-Whitney polynomials:

$$w(\hat{\beta}_{n,k}, t) = (1+yt)^n.w(\hat{\gamma}_{n,k}, t)^{-1} = \sum_{j \geq 0} a_j t^j = a(t) \tag{10}$$

where $a_j := a_j(y, w_1(\hat{\gamma}_{n,k}), \ldots, w_j(\hat{\gamma}_{n,k}))$ is the homogeneous polynomial of (total) degree j in $(1+y)^n.w(\hat{\gamma}_{n,k})^{-1}$.

Since $H^1(P(S, \mathbb{C}G_{n,k}); \mathbb{Z}_2) \cong H^1(Y; \mathbb{Z}_2) = \mathbb{Z}_2 y$, it is clear that $w_1(\hat{\gamma}_{n,k}), w_1(\hat{\beta}_{n,k}) \in \mathbb{Z}_2 y$. Moreover, $w_1(\hat{\gamma}_{n,k}) = 0$ (resp. $w_1(\hat{\beta}_{n,k}) = 0$) if and only if k is even (resp. $n-k$ is even) as a consequence of Equation (4) (or by a direct argument). Hence we see that, $w_1(\hat{\beta}_{n,k}) = ny + w_1(\hat{\gamma}_{n,k}) \in \mathbb{Z}_2 y$. This also follows from the above equation. Using Equation (4) and induction, we see that the Stiefel-Withney class $w_j(\hat{\beta}_{n,k}) = a_j$ is expressible as a polynomial in $y, w_2(\hat{\gamma}_{n,k}), \ldots, w_{2i}(\hat{\gamma}_{n,k})$ for all j where $i = \lfloor j/2 \rfloor$. We note that, by degree considerations, a_j is divisible by y when j is odd.

We may view (10) as *defining* $w_j(\hat{\beta}_{n,k}), 1 \leq j \leq 2(n-k)$, as the polynomial a_{2j}. Since $w_j(\hat{\beta}_{n,k}) = 0$ for $j > 2n-2k$, Equation (10) leads to the relations

$$a_j = a_j(y, w_2(\hat{\gamma}_{n,k}), \ldots, w_{2k}(\hat{\gamma}_{n,k})) = 0, j > 2n-2k. \tag{11}$$

Let $I \subset \mathbb{Z}_2[y, w_{2i}(\hat{\gamma}_{n,k}); 1 \leq i \leq k]$ be the ideal generated by $a_j, j > 2n-2k$. Suppose that the height of y equals $N \in \mathbb{N}$. Then $(1+yt)^{-1} = \sum_{0 \leq j \leq N} y^j t^j$ and we have $a(t).(\sum_{0 \leq j \leq N} y^j t^j) w(\hat{\gamma}_{n,k}, t) = 1$. It follows that a_{N+2n+j} is in the ideal generated by $a_{2n-2k+i}, 1 \leq i \leq N+2k$, for all $j \geq 1$ and so I is generated by $a_{2n-2k+i}, 1 \leq i \leq N+2k$. Moreover, it is easily seen that a_{2n+i} is in the ideal generated by $y, a_j, 2n-2k+1 \leq j \leq 2n$ for all $i \geq 1$.

Consider the graded polynomial algebra $R := \mathbb{Z}_2[y, \hat{w}_{2j}; 1 \leq j \leq k]$ in the indeterminates $y, \hat{w}_{2j}, 1 \leq j \leq k$, where $\deg y = 1, \deg \hat{w}_{2j} = 2j$. We regard $a_{2j} = a_{2j}(y, \hat{w}_2, \ldots, \hat{w}_{2j})$ as elements of R.

Lemma 4.3.1. *The elements $y, a_{2j} \in R, n-k < j \leq n$, form a regular sequence in R and $R/\langle y, a_{2j}, n-k < j \leq n \rangle \cong H^*(\mathbb{C}G_{n,k}; \mathbb{Z}_2)$.*

Proof. To see this, it suffices to show that $\bar{a}_{2j}, n-k < j \leq n$ is a regular sequence in $\bar{R} := R/\langle y \rangle$ where $\bar{a}_{2j} := a_{2j} \mod \langle y \rangle \in \bar{R}$. Note that $\bar{a}_{2j} = h_{2j}(\hat{w})$ where $h_{2j}(\hat{w})$ denotes the complete symmetric polynomial of degree $2j$ in $\hat{w}_2, \ldots, \hat{w}_{2k}$, that is, h_{2j} equals the coefficient of t^{2j} in $(1 + \hat{w}_2 t^2 + \cdots + \hat{w}_{2k} t^{2k})^{-1}$. One can show, as in [BT82, §23], that $\bar{a}_{2j}, n-k < j \leq n$, is a regular sequence in $\bar{R}$ and that $\bar{R}/\langle a_{2j}, n-k < j \leq n \rangle \cong H^*(\mathbb{C}G_{n,k}; \mathbb{Z}_2)$. □

It follows from the above lemma that $y^{m+1}, a_{2j}, n-k < j \leq n$, is a regular sequence in R and that the quotient $R/\langle y^{m+1}, a_{2j}; n-k < j \leq n\rangle$ is isomorphic to $\mathbb{Z}_2[y]/\langle y^{m+1}\rangle \otimes H^*(\mathbb{C}G_{n,k};\mathbb{Z}_2) = H^*(\mathbb{R}P^m;\mathbb{Z}_2) \otimes H^*(\mathbb{C}G_{n,k};\mathbb{Z}_2)$ as graded $\mathbb{Z}_2$-vector spaces. We are ready to prove the following proposition. Recall that $\xi_\alpha = \xi$ denotes the pull-back of the Hopf line bundle over $\mathbb{R}P^m$ via the projection of the X-bundle $P(\mathbb{S}^m, X) \to \mathbb{R}P^m$.

Proposition 4.3.2. *We keep all the notations as above. Then the $\mathbb{Z}_2$-cohomology algebra $H^*(P(\mathbb{S}^m, \mathbb{C}G_{n,k});\mathbb{Z}_2)$ is isomorphic to R/I where I is the ideal generated by $y^{m+1}, a_{2j}(\hat{w}), n-k < j \leq n$, where $\hat{w}_{2j}$ corresponds to $w_{2j}(\hat{\gamma}_{n,k})$ and y to $w_1(\xi)$.*

Proof. Consider the homomorphism $\eta : R \to H^*(P(\mathbb{S}^m, \mathbb{C}G_{n,k});\mathbb{Z}_2)$ of rings defined as $\eta(\hat{w}_{2j}) = w_{2j}(\hat{\gamma}_{n,k})$ and $\eta(y) = w_1(\xi)$. By Equation (10) and Proposition 3.1.2, η is surjective. It follows from Equation (11) that η factors as $R \to R/I \xrightarrow{\bar{\eta}} H^*(P(\mathbb{S}^m;\mathbb{C}G_{n,k});\mathbb{Z}_2)$. By the discussion preceding the statement of the proposition, we see that $\bar{\eta}$ is an isomorphism since R/I and $H^*(P(\mathbb{S}^m, \mathbb{C}G_{n,k});\mathbb{Z}_2) = H^*(\mathbb{R}P^m;\mathbb{Z}_2) \otimes H^*(\mathbb{C}G_{n,k};\mathbb{Z}_2)$ have the same dimension. □

Next we prove the following theorem.

Theorem 4.3.3. *We keep the above notations. Suppose that S is paracompact. The cohomology algebra $H^*(P(S, \mathbb{C}G_{n,k});\mathbb{Z}_2)$ is isomorphic, as an $H^*(Y;\mathbb{Z}_2)$-algebra, to $H^*(Y;\mathbb{Z}_2)[\hat{w}_2, \ldots, \hat{w}_{2k}]/\mathcal{I}$, where $\mathcal{I}$ is generated by $a_{2j}, n-k < j \leq n$, under an isomorphism that maps $\hat{w}_{2j}$ to $w_{2j}(\hat{\gamma}_{n,k})$.*

Proof. Set $\mathcal{R} := H^*(Y;\mathbb{Z}_2)[\hat{w}_2, \ldots, \hat{w}_{2k}]$. In view of Equation (11), it is clear that we have a surjective $H^*(Y;\mathbb{Z}_2)$-algebra homomorphism $\mathcal{R}/\mathcal{I} \to H^*(P(S,X);\mathbb{Z}_2)$ where $\hat{w}_{2j}$ maps to $w_{2j}(\hat{\gamma}_{n,k})$. Therefore it suffices to show that $\mathcal{R}/\mathcal{I}$ is isomorphic to $H^*(P(S,X);\mathbb{Z}_2)$ as an $H^*(Y;\mathbb{Z}_2)$-module.

Let A denote the $\mathbb{Z}_2$-subalgebra of $H^*(Y;\mathbb{Z}_2)$ generated by $y = w_1(\xi_\alpha)$ and let $R = A[\hat{w}_{2j}; 1 \leq j \leq k]$. Then $\mathcal{R} = H^*(Y;\mathbb{Z}_2) \otimes_A R$. Let $I \subset R$ denote the ideal generated by $a_{2j}, n-k < j \le n$. Then $\mathcal{R}/\mathcal{I} \cong H^*(Y;\mathbb{Z}_2) \otimes_A (R/I)$.

Suppose that $(S,\alpha) = (\mathbb{S}^m, -id)$. By Proposition 4.3.2, we have R/I is isomorphic to $H^*(P(\mathbb{S}^m;\mathbb{C}G_{n,k});\mathbb{Z}_2) \cong A \otimes H^*(\mathbb{C}G_{n,k};\mathbb{Z}_2)$. So $\mathcal{R}/\mathcal{I} \cong H^*(Y;\mathbb{Z}_2) \otimes_A A \otimes H^*(\mathbb{C}G_{n,k};\mathbb{Z}_2) \cong H^*(Y;\mathbb{Z}_2) \otimes H^*(\mathbb{C}G_{n,k};\mathbb{Z}_2)$.

Next suppose that $(S,\alpha) = (\mathbb{S}^\infty, -id)$. The inclusion $\mathbb{S}^m \hookrightarrow \mathbb{S}^\infty$ defines an inclusion $j_m : P(\mathbb{S}^m, X) \hookrightarrow P(\mathbb{S}^\infty, X)$ which induces a $H^*(Y;\mathbb{Z}_2)$-algebra homomorphism in cohomology. Moreover, j_m^* is an isomorphism up to dimension $m-1$. From this observation it follows that the theorem holds for $(\mathbb{S}^\infty, -id)$. In the general case, the result follows from functoriality and the fact that the X-bundle $P(S,X)$ arises as a pull-back of the X-bundle $P(\mathbb{S}^\infty, X)$ via a classifying map $S \to \mathbb{R}P^\infty$. This completes the proof. □

Proposition 4.3.2 and Theorem 4.3.3 are valid when the Grassmann manifold is replaced by complex flag manifolds. More precisely, let $\nu := (n_1, \dots, n_s)$ be an increasing sequence of positive numbers and let $n = \sum_{1 \le j \le s} n_j$. Denote by $\mathbb{C}G(\nu)$ the complex flag manifold whose elements are complex vector subspaces $\underline{U} := (U_1, \dots, U_s)$ of $\mathbb{C}^n$ where $U_i \perp U_j$ for $i \ne j$ and $\dim U_j = n_j, 1 \le j \le s$. Then $\mathbb{C}G(\nu) \cong U(n)/(U(n_1) \times \cdots \times U(n_s))$ has a natural structure of a complex manifold given by the usual inclusion of $U(n) \subset GL_n(\mathbb{C})$ so that $\mathbb{C}G(\nu) \cong GL_n(\mathbb{C})/P_\nu$ where P_ν is the subgroup which is block upper-triangular, where the diagonal sizes are $n_1, \dots, n_s$. (Under this identification, $\underline{U}$ corresponds to the sequence $U_1 \subset U_1 + U_2 \subset \cdots \subset U_1 + \cdots + U_s = \mathbb{C}^n$.) It is well-known that $\mathbb{C}G(\nu)$ has a CW-structure given by *Schubert cells* which are all even dimensional. The Schubert cells are obtained as B-orbits of T-fixed points where $T \subset GL_n(\mathbb{C})$ is the diagonal subgroup and $B \subset P_\nu$ is the group of upper triangular matrices. Their closures are the *Schubert varieties* in $\mathbb{C}G(\nu)$. The complex conjugation in $\mathbb{C}^n$ induces a complex conjugation σ on $\mathbb{C}G(\nu)$, and, moreover, σ stabilizes each Schubert variety. The fixed points of σ is the real flag manifold $\mathbb{R}G(\nu) \cong O(n)/(O(n_1) \times \cdots \times O(n_r))$ consisting of $\underline{U}$ where $U_j \cap \mathbb{R}^n$ is

n_j-dimensional for all j.

We denote by $\gamma_{\nu,j}$ (or more briefly γ_j), the complex vector bundle over $\mathbb{C}G(\nu)$ of rank n_j whose fibre over $\underline{U}$ is the vector space U_j. This is the pull-back of the bundle γ_{n,n_j} on $\mathbb{C}G_{n,n_j}$ via the projection $\mathbb{C}G(\nu) \to \mathbb{C}G_{n,n_j}$ that sends $\underline{U}$ to U_j. The complex conjugation $\mathbb{C}^n$ leads to a σ-conjugation on $\hat{\sigma}_j$ of γ_j. Also one has a natural isomorphism of vector bundles

$$\bigoplus_{1 \leq j \leq s} \gamma_j \cong n\epsilon_{\mathbb{C}} \tag{12}$$

which respects σ-conjugation, as in the case of Grassmann manifolds. The integral cohomology ring of $\mathbb{C}G(\nu)$ is generated by $c_{i,j} := c_i(\gamma_j), 1 \leq i \leq n_j, 1 \leq j \leq s$, where the only relations among the $c_{i,j}$ are generated by the following (inhomogeneous) relation: $\prod_{1 \leq j \leq s} c(\gamma_j) = 1$. It follows that $w_{2i,j} = w_{2i}(\gamma_j), 1 \leq i \leq n_j, 1 \leq j \leq s$, generate $H^*(\mathbb{C}G(\nu); \mathbb{Z}_2)$ and the relations among these generators are all consequences of $\prod_{1 \leq j \leq n_s} w(\gamma_j, t) = 1$.

Suppose that (S, α) is a paracompact space where α is a fixed point free involution. Then we have the real vector bundles $\hat{\gamma}_j = P(S, \gamma_j)$ over the generalized Dold space $P(S, \mathbb{C}G(\nu))$. One has the following isomorphism of real vector bundles, resulting from the isomorphism (12):

$$\bigoplus_{1 \leq j \leq s} \hat{\gamma}_j \cong n\xi_\alpha \oplus n\epsilon_{\mathbb{R}}$$

where ξ_α is the real line bundle associated to the double cover $S \times \mathbb{C}G(\nu) \to P(S, \mathbb{C}G(\nu))$. Therefore we obtain

$$w(\hat{\gamma}_s, t) = (1 + yt)^n . \prod_{1 \leq j < s} w(\hat{\gamma}_j, t)^{-1}. \tag{13}$$

Then $H^*(P(S, \mathbb{C}G(\nu)); \mathbb{Z}_2)$ is isomorphic to $H^*(Y; \mathbb{Z}_2) \otimes H^*(\mathbb{C}G(\nu); \mathbb{Z}_2)$ as an $H^*(Y; \mathbb{Z}_2)$-

module, by Proposition 3.1.2. Arguing as in the proof of Theorem 4.3.3, we observe that $H^*(P(S;\mathbb{C}G(\nu));\mathbb{Z}_2)$ is generated by as $H^*(Y;\mathbb{Z}_2)$-algebra by $\hat{w}_{2i,j} := w_{2i}(\hat{\gamma}_j), 1 \leq i \leq n_j, 1 \leq j < s$. Using Equation (13), we obtain a regular sequence $a_{2r}, n_s < r \leq n$, in the polynomial algebra $R_\nu = \mathbb{Z}_2[\hat{w}_{2i,j} \mid 1 \leq i \leq n_j, 1 \leq j \leq s]$ that correspond to the coefficient of t^{2r} in $(1+ty)^n(\prod \hat{w}_j(t))$ (rewriting $\hat{w}_{2i+1}$ in terms of $y, \hat{w}_{2l,j}, l \leq i$, using Equation (4)).

We set $\mathcal{R}_\nu$ to the polynomial algebra over $H^*(Y;\mathbb{Z}_2)$ generated by 'indeterminates' $\hat{w}_{2i,j}, 1 \leq i \leq n_j, 1 \leq j < s$, and let $\mathcal{I}_\nu$ be the ideal of $\mathcal{R}_\nu$ generated by the elements $a_{2i} = a_{2i}(y^2, \hat{w}_{2l,j}), n_s < i \leq n$. Then $a_{2i}, n_s < i \leq n$, form a regular sequence in $\mathcal{R}_\nu$. The proof of the following theorem is analogous to that of Theorem 4.3.3.

Theorem 4.3.4. *Suppose that (S,α) is a paracompact space with a fixed point free involution α and let $\nu = (n_1, n_2, \ldots, n_s)$ be a sequence of positive numbers. With notations as above, we have an isomorphism $\mathcal{R}_\nu/\mathcal{I}_\nu \to H^*(P(S,\mathbb{C}G(\nu));\mathbb{Z}_2)$ of $H^*(Y;\mathbb{Z}_2)$-algebras defined by $\hat{w}_{2i,j} \mapsto w_{2i}(\hat{\gamma}_j)$.* □

We remark that, setting $k = n - n_s$, the projection $\pi : \mathbb{C}G(\nu) \to G_{n,k}$ defined as $\underline{U} \mapsto U_1 + \cdots + U_{s-1}$ is $\mathbb{Z}_2$-equivariant and pulls back $\gamma_{n,k}$ (resp. $\beta_{n,k}$) to $\oplus_{1\leq j<s}\gamma_j$ (resp. γ_s). Hence $P(S,\pi)^*(\hat{\gamma}_{n,k}) = \oplus_{1\leq j<s}\hat{\gamma}_j$. Moreover, using Proposition 3.1.2 and the fact that $\pi^* : H^*(\mathbb{C}G_{n,k};\mathbb{Z}_2) \to H^*(\mathbb{C}G(\nu);\mathbb{Z}_2)$ is a monomorphism, we see that $P(S,\pi) : P(S,\mathbb{C}G(\nu)) \to P(S,\mathbb{C}G_{n,k})$ induces a monomorphism in $\mathbb{Z}_2$-cohomology.

We conclude this section with the following remark.

Remark 4.3.5. Let $\sigma : S \to S$ be a fixed point free involution. Let $X \hookrightarrow \mathbb{C}G(\nu)$ be a Schubert variety in a complex flag manifold $\mathbb{C}G(\nu)$. As had already been commented, X is stable by complex conjugation σ on $\mathbb{C}G(\nu)$ and $X^\sigma = X \cap \mathbb{R}G(\nu)$ is non-empty. Moreover, X admits a cell-decomposition having cells only in even dimensions, where the cells are the Schubert cells of $\mathbb{C}G(\nu)$ contained in X. Hence the inclusion $X \hookrightarrow \mathbb{C}G(\nu)$ induces a surjection $H^*(\mathbb{C}G(\nu); \mathbb{Z}) \to H^*(X; \mathbb{Z})$. It follows that the (mod 2) cohomology of X is generated by Chern classes (mod 2) of complex vector bundles on X. So Proposition 3.1.2 is applicable to the X-bundle $P(S, X) \to Y$ and we have $H^*(P(S, X); \mathbb{Z}_2) \cong H^*(Y; \mathbb{Z}_2) \otimes H^*(X)$ as $H^*(Y; \mathbb{Z}_2)$-modules. As for the $H^*(Y; \mathbb{Z}_2)$-algebra structure, it is determined by the surjection $H^*(P(S, \mathbb{C}G(\nu)); \mathbb{Z}_2) \to H^*(P(S, X); \mathbb{Z}_2)$. We omit the details.

Chapter 5

Cohomology of $P(\mathbb{S}^m, \mathbb{C}G(\nu))$

In this chapter, we study the cohomology of the specific classes of manifolds $P(S, X)$ where (i) $(S, \alpha) = (\mathbb{S}^m,$ antipodal map), and,
(ii) X is a complex flag manifold and $\sigma : X \to X$ is the complex conjugation (induced by the complex conjugation $^- : \mathbb{C}^n \to \mathbb{C}^n$).

Recall that $\mathbb{C}G_{n,k} \cong U(n)/(U(k) \times U(n-k))$ is the space all k-dimensional complex vector subspaces of $\mathbb{C}^n$. Let $\sigma : \mathbb{C}G_{n,k} \to \mathbb{C}G_{n,k}$ be the involution that sends $L \in \mathbb{C}G_{n,k}$ to $\bar{L} = \{\bar{v} \in \mathbb{C}^n \mid v \in L\}$.

We put the standard hermitian inner product on $\mathbb{C}^n$. The complex flag manifold $\mathbb{C}G(\nu)$ of type $\nu := (n_1 \cdots, n_s)$ is the space of all flags $\mathbf{L} := (L_1, \cdots, L_s)$ where: (i) L_j is a vector subspace of $\mathbb{C}^n$ of dimension $n = |\nu| := \sum_{1 \le j \le s} n_j$, and, (ii) where $L_i \perp L_j$ if $i \ne j$; thus $L_1 + \cdots + L_s = \mathbb{C}^n$. The complex flag manifold $\mathbb{C}G_\nu$ is identified with the homogeneous space $U(n)/U(\nu)$ where $U(\nu) := (U(n_1) \times \cdots \times U(n_s)) \subset U(n)$ is the subgroup of block diagonal matrices with block sizes $n_1, \cdots, n_s$. When $s = 2$, $\mathbb{C}G(\nu)$ is the Grassmann manifold $\mathbb{C}G_{n,n_1}$.

5.1 Schubert cell structure of $\mathbb{C}G(\nu)$

One has the Schubert cell structure for the complex Grassmann manifold $\mathbb{C}G(\nu)$ (cf. [MS74, Chapter 6]) which has a generalization to the case of complex flag manifolds given by the Bruhat decomposition (see [Bor53b]). The closed cells are the Schubert varieties in $\mathbb{C}G(\nu)$ are in bijection with the set of T-fixed points of $\mathbb{C}G(\nu)$ where $T \subset U(n)$ is the diagonal subgroup, which is a maximal torus of $U(n)$. These T-fixed points are in bijection with the coset space S_n/S_ν where $S_\nu := S_{n_1} \times \cdots \times S_{n_s}$. Here S_n is the permutation group on $[n] = \{1, 2, \cdots, n\}$. In the case of $\mathbb{C}G_{n,k}$, $S_n/(S_k \times S_{n-k})$ is identified with the set $I(n; k)$ of all strictly increasing sequences $\mathbf{i} = (i_1, i_2, \ldots, i_k), 1 \leq i_p \leq n$. The Schubert variety $X(\mathbf{i}) \subset \mathbb{C}G_{n,k}$ is defined as

$$X(\mathbf{i}) := \{L \in \mathbb{C}G_{n,k} \mid \dim_{\mathbb{C}}(L \cap \mathbb{C}^{i_p}) \geq p \ \forall p \leq k\}.$$

The corresponding T-fixed point is $E_{\mathbf{i}}$, the $\mathbb{C}$-span of the standard basis vectors $e_{i_p}, 1 \leq p \leq k$. $X(\mathbf{i})$ has the structure of a complex projective variety. One has the following formula for the dimension of $X(\mathbf{i})$ as a complex variety:

$$\dim_{\mathbb{C}} X(\mathbf{i}) = \ell(\mathbf{i}) := \sum_{1 \leq p \leq k} (i_p - p).$$

The corresponding (open) Schubert cell is $\mathring{X}(\mathbf{i}) = \{L \in \mathbb{C}G_{n,k} \mid \dim(L \cap \mathbb{C}^{i_p}) = p, \dim(L \cap \mathbb{C}^{i_p - 1}) = p - 1 \ \forall p\} \cong \mathbb{C}^{\ell(\mathbf{i})}$. We shall denote by $I_q(n; k) \subset I(n; k)$ the set of all $\mathbf{i} \in I(n; k)$ with $\ell(\mathbf{i}) = q$.

In the case of $\mathbb{C}G(\nu)$, we identify S_n/S_ν with the subset $I(\nu) \subset S_n$ defined as follows: $I(\nu)$ consists of all permutations $\mathbf{i} = (i_1, \cdots, i_n) = (\mathbf{i}_1, \ldots, \mathbf{i}_s)$ of $[n]$ (in one-line notation) in which each 'block' $\mathbf{i}_p$ (of size n_p) is in $I(n; n_p), 1 \leq p \leq s$, so that $\mathbf{i}_{p+1} = i_{m_p+1} < \cdots < i_{m_{p+1}}$ where $m_p := n_1 + \cdots + n_p$ (with $m_0 = 0$), $0 \leq p < s$. With respect to the generating set $S \subset S_n$ consisting of transpositions

$s_i = (i, i+1), 1 \leq i < n$, one has the length function $\ell = \ell_S$ on the Coxeter group (S_n, S). The set $I(\nu)$ is the set of minimal length representatives of elements of S_n/S_ν. We have the formula

$$\ell(\mathbf{i}) = \sum_{1 \leq t < n} |\{r > t \mid i_r < i_t\}|$$

When $s = 2$, this is consistent with the notation $\ell(\mathbf{i})$ for $\mathbf{i} \in I(n;k)$. The above formula for $\ell(\mathbf{i}), \mathbf{i} \in I(\nu)$, for $s \geq 3$ can be proved by induction on s: If $\mathbf{i} \in I(\nu)$, write $\nu' = (n_1, \ldots, n_{s-2}, n_{s-1} + n_s)$ and let $\mathbf{i}' = (\mathbf{i}_1, \cdots, \mathbf{i}_{s-2}, \mathbf{i}'_{s-1}) \in I(\nu')$ where $\mathbf{i}'_{s-1} \in I(n; n_{s-1} + n_s)$ is the sequence $\mathbf{i}_{s-1}, \mathbf{i}_s$ rearranged in the increasing order. Then, it is easily seen that

$$\ell(\mathbf{i}) = \ell(\mathbf{i}') + \ell(i'_{s-1}).$$

As in the case of the Grassmannian Schubert varieties, the set of all Schubert varieties (with reference to the standard complete flag $\mathbb{C} \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^n$) in $\mathbb{C}G(\nu)$ and the set of T-fixed points of $\mathbb{C}G(\nu)$ are in bijective correspondence with $I(\nu)$. An element $\mathbf{i} \in I(\nu)$ corresponds to the T-fixed point $(E_{\mathbf{i}_1}, E_{\mathbf{i}_2}, \ldots, E_{\mathbf{i}_s}) \in \mathbb{C}G(\nu)$ and the corresponding Schubert variety is denoted $X(\mathbf{i})$. An element $V = (V_1, \ldots, V_s) \in \mathbb{C}G(\nu)$ belongs to $X(\mathbf{i})$ if and only if $V_1 + \cdots + V_p$ belongs to $X(\mathbf{j}_p) \subset \mathbb{C}G_{n,m_p}$ for all $p < s$ where $\mathbf{j}_p \in I(n; m_p)$ is the sequence $(\mathbf{i}_1, \ldots, \mathbf{i}_p)$ rearranged in increasing order. The Schubert variety $X(\mathbf{i}) \subset \mathbb{C}G(\nu)$ has (complex) dimension the length of $\mathbf{i}$. That is, $\dim_{\mathbb{C}} X(\mathbf{i}) = \ell(\mathbf{i})$. See [LR08, §3.3], [Bri05, §1]. The dimension formula may be established for $s \geq 3$ by induction using the Grassmann bundle $\mathbb{C}G_{n,k} \hookrightarrow \mathbb{C}G(\nu) \to \mathbb{C}G(\nu')$ (with ν' as above) where $(V_1, \ldots, V_s) \in \mathbb{C}G(\nu)$ projects to $(V_1, \ldots, V_{s-2}, V_{s-1} + V_s) \in \mathbb{C}G(\nu')$.

The Schubert cells have natural orientations arising from the fact that the complex vector spaces have canonical orientations. In fact the open cell $\mathring{X}(\mathbf{i})$ is isomorphic to the affine space $\mathbb{C}^{\ell(\mathbf{i})}$. We note that the conjugation map $\sigma : \mathbb{C}G(\nu) \to \mathbb{C}G(\nu)$

that sends $\mathbf{L} = (L_1, \ldots, L_s)$ to $\bar{\mathbf{L}} = (\bar{L}_1, \ldots, \bar{L}_s)$ stabilizes each Schubert cell. In fact, under the identification $\mathring{X}(\mathbf{i}) \cong \mathbb{C}^{\ell(\mathbf{i})}$, the σ corresponds to the complex conjugation on $\mathbb{C}^{\ell(\mathbf{i})}$. In particular, σ is orientation preserving (resp. reversing) on $X(\mathbf{i})$ when $\ell(\mathbf{i})$ is even (resp. odd).

Since the real dimension of each Schubert cell is even, it follows that the differential of the cellular chain complex of $\mathbb{C}G(\nu)$ vanishes identically. Consequently, $H_*(\mathbb{C}G(\nu); \mathbb{Z}) \cong C_*(\mathbb{C}G(\nu); \mathbb{Z})$ and $H^*(\mathbb{C}G(\nu); \mathbb{Z}) \cong C^*(\mathbb{C}G(\nu); \mathbb{Z})$. Thus $\mathbb{C}G(\nu)$ satisfies the hypotheses of Lemma 3.2.1. The rank of $H^{2q}(\mathbb{C}G(\nu); \mathbb{Z})$ equals the cardinality of $I_q(\nu) := \{\mathbf{i} \in I(\nu) \mid \ell(i) = q\}$.

The fixed set $\mathrm{Fix}(\sigma) \subset \mathbb{C}G(\nu)$ is identified with the real flag manifold $\mathbb{R}G(\nu) = O(n)/(O(n_1) \times \cdots \times O(n_s))$ via the embedding defined by the complexification $V \mapsto V \otimes \mathbb{C} \subset \mathbb{C}^n$ where V is a real vector space $V \subset \mathbb{R}^n$.

5.2 A cell structure on $P(\mathbb{S}^m, \mathbb{C}G(\nu))$

We put the standard $\mathbb{Z}_2$-equivariant cell structure $\{C_k^{\pm}\}_{0 \leq k \leq m}$ on the sphere $\mathbb{S}^m$. Here the equivariance is with respect to the cyclic group generated by the antipodal map $\alpha : \mathbb{S}^m \to \mathbb{S}^m$. The cell C_k^+ (resp. C_k^-) is the upper (resp. lower) hemisphere of $\mathbb{S}^k \subset \mathbb{S}^m$ where $\mathbb{S}^k \subset \mathbb{S}^m$ is the unit sphere in $\mathbb{R}^{k+1}$ spanned by $e_j, 1 \leq j \leq k+1$. The projection $p : \mathbb{S}^m \to \mathbb{R}P^m$ is cellular with respect to the standard cell structure on $\mathbb{R}P^m$. The unique closed k-cell in $\mathbb{R}P^m$ is $\mathbb{R}P^k$ corresponding the inclusion $\mathbb{R}^{k+1} \hookrightarrow \mathbb{R}^{m+1}$.

We put the standard orientation on C_j^+ for each j and put the orientation on C_j^- induced on it via α. The cells C_j in $\mathbb{R}P^m$ are given the orientation induced from C_j^+ via $p|_{C_j^+} : C_j^+ \to C_j$.

A cell decomposition of $\mathbb{S}^m \times \mathbb{C}G(\nu)$ which is equivariant with respect to $\theta = \alpha \times \sigma$

is obtained by taking the product cells $X^{\pm}(j,\mathbf{i}) := C_j^{\pm} \times X(\mathbf{i})$ as $(j,\mathbf{i})$ varies in $I(m,\nu) := \{j \mid 0 \leq j \leq m\} \times I(\nu)\}$. The cell $C_j^{\pm} \times X(\mathbf{i})$ is given the product orientation.

We shall denote the image of the oriented cell $X^+(j,\mathbf{i})$ under the double covering projection $\pi : \mathbb{S}^m \times \mathbb{C}G(\nu) \to P(m,\nu)$ by $X(j,\mathbf{i})$ and put the induced orientation on it. The deck transformation group of π is generated by $\theta = \alpha \times \sigma$.

We note that $\theta|_{X^+(j,\mathbf{i})} : X^+(j,\mathbf{i}) \to X^-(j,\mathbf{i})$ is orientation preserving if and only if $\sigma|_{X^+(\mathbf{i})} : X^+(\mathbf{i}) \to X^-(\mathbf{i})$ is, if and only if $\ell(\mathbf{i})$ is even.

Since the boundary map of $C_*(\mathbb{C}G(\nu);\mathbb{Z})$ vanishes, it is readily seen that the boundary map of $C_*(\mathbb{S}^m \times \mathbb{C}G(\nu))$ is determined by $\partial : C_*(\mathbb{S}^m;\mathbb{Z})$ and we obtain, for all $(j,\mathbf{i}) \in I(m,\nu)$, that

$$\partial(X^{\pm}(j,\mathbf{i})) = X^{\pm}(j-1,\mathbf{i}) + (-1)^j X^{\mp}(j-1,\mathbf{i}), \tag{5.1}$$

and

$$\theta_*(X^{\pm}(j,\mathbf{i})) = (-1)^{\ell(\mathbf{i})} X^{\mp}(j,\mathbf{i}). \tag{5.2}$$

Now $p_*(X^+(j,\mathbf{i})) = X(j,\mathbf{i})$ and
$p_*(X^-(j,\mathbf{i})) = p_*(\theta_*((-1)^{\ell(\mathbf{i})}X^+(j,\mathbf{i}))) = p_*((-1)^{\ell(\mathbf{i})}X^+(j,\mathbf{i})) = (-1)^{\ell(\mathbf{i})}X(j,\mathbf{i})$.
It follows from Equation 5.1 that

$$\partial X(j,\mathbf{i}) = p_*\partial(X^+(j,\mathbf{i})) = (1 + (-1)^{j+\ell(\mathbf{i})})X(j-1,\mathbf{i}) \ \forall (j,\mathbf{i}) \in I(m,\nu). \tag{5.3}$$

5.3 Integral (co)homology of $P(\mathbb{S}^m, \mathbb{C}G(\nu))$

We now turn to the determination of the integral homology of $P(m, \nu)$. It follows from Equation 5.3 that, for $0 < j \leq m$, $X(j, \mathbf{i})$ is a cycle in $P(m, \nu)$ if and only if $j + \ell(\mathbf{i})$ is odd. When $j + \ell(\mathbf{i})$ is odd and $j < m$, we have $\partial(X(j+1, \mathbf{i})) = 2X(j, \mathbf{i})$ and so the homology class $[X(j, \mathbf{i})]$ has order 2.

It is evident from Equation 5.3 that $X(0, \mathbf{i})$ is a cycle for all $\mathbf{i} \in I(\nu)$. When $\ell(\mathbf{i})$ is even, $[X(0, \mathbf{i})]$ is of infinite order. If $\ell(\mathbf{i})$ is odd, $X(0, \mathbf{i})$ is of order 2. Similarly, if m is odd and $\ell(\mathbf{i})$ is even, $[X(m, \mathbf{i})]$ is of infinite order. If both m and $\ell(\mathbf{i})$ are even, then $X(m, \mathbf{i})$ is not a cycle. If m is even and $\ell(\mathbf{i})$ is odd, then $[X(m, \mathbf{i})]$ is of infinite order.

Define the sets

$$
\begin{aligned}
I_e(\nu) &:= \{\mathbf{i} \in I(\nu) \mid \ell(\mathbf{i}) \equiv 0 \pmod 2\}, \\
I_o(\nu) &:= \{\mathbf{i} \in I(\nu) \mid \ell(\mathbf{i}) \equiv 1 \pmod 2\} = I(\nu) \setminus I_e(\nu), \\
\mathcal{B}_{2q} &:= \{[X(0, \mathbf{i})] \mid q = \ell(\mathbf{i}), \mathbf{i} \in I_e(\nu)\} \cup \{[X(m, \mathbf{i})] \mid 2q = m + 2\ell(\mathbf{i}), \mathbf{i} \in I_o(\nu)\}, \\
\mathcal{B}_{2q+1} &:= \{[X(m, \mathbf{i})] \mid 2q+1 = m + 2\ell(\mathbf{i}), \mathbf{i} \in I_e(\nu)\}, \\
\mathcal{B}'_q &:= \{[X(j, \mathbf{i})] \mid q = j + 2\ell(\mathbf{i}), j + \ell(\mathbf{i}) \equiv 1 \pmod 2, 0 \leq j < m,\ \mathbf{i} \in I(\nu)\}.
\end{aligned}
$$

We note that if m is even, then $\mathcal{B}_{2q+1} = \emptyset$, and if m is odd, then $\{[X(m, \mathbf{i})] \mid 2q = m + 2\ell(\mathbf{i}), \mathbf{i} \in I_e(\nu)\} = \emptyset$.

Observe that the cycle group $Z_q(P(m, \nu); \mathbb{Z})$ is the free abelian group generated by $\{X(j, \mathbf{i})\}$ where $[X(j, \mathbf{i})] \in \mathcal{B}_q \cup \mathcal{B}'_q$ for any $q \geq 0$. In particular $H_q(P(m, \nu); \mathbb{Z})$ is generated by $\mathcal{B}_q \cup \mathcal{B}'_q$.

We have the following result where we leave out the known cases namely, $m = 0$ as $P(0, \nu) \cong \mathbb{C}G(\nu)$ and $\nu = (1)$ in which case $P(m, \nu) = \mathbb{S}^m$.

Theorem 5.3.1. *We keep the above notations. Suppose that $m \geq 1$ and $n = |\nu| \geq 2$. Then there is no odd torsion in $H_*(P(m, \nu); \mathbb{Z})$ and every 2-torsion element is of* *order* 2*. The set* $\mathcal{B}_q$ *is a basis for* $H_q(P(m,\nu);\mathbb{Z})/\text{torsion}$ *and* $\mathcal{B}'_q$ *is a* $\mathbb{Z}_2$*-basis for the torsion subgroup of* $H_q(P(m,\nu);\mathbb{Z})$*.*

Proof. We first show that $\mathcal{B}_q$ is $\mathbb{Z}$-linearly independent. Suppose that $Z = \sum a_{j,\mathbf{i}} X(j,\mathbf{i})$ with $a_{j,\mathbf{i}} \in \mathbb{Z}$, is a boundary where the sum is over $(j,\mathbf{i}) \in I(m,\nu)$ such that $[X(j,\mathbf{i})] \in \mathcal{B}_q$, i.e. $Z = \partial C$ for some $C \in C_{q+1}(P(m,\nu))$. Suppose that some $a_{j,\mathbf{i}} \neq 0$. Then there has to be a cell $X(k,\mathbf{l})$ such that $X(j,\mathbf{i})$ occurs with non-zero coefficient in the expression of $\partial X(k,\mathbf{l})$ for some $X(k,\mathbf{l})$. However, Equation 5.3 shows that the only possibility is $k = j+1 \leq m, \mathbf{l} = \mathbf{i}$ and that $\partial X(j+1,\mathbf{i}) = (1+(-1)^{j+1+\ell(\mathbf{i})})X(j,\mathbf{i})$. As $[X(j,\mathbf{i})] \in \mathcal{B}_q$, we have $\partial X(j+1,\mathbf{i}) = 0$. This shows that there is no such C and so the $\mathcal{B}_q$ is $\mathbb{Z}$-linearly independent.

Note that $\mathcal{B}_q \subset \text{Fix}(\theta_*)$. In view of this one can also prove the linear independence of $\mathcal{B}_q$ using Proposition 3.2.3.

Since $\mathcal{B}_q \cup \mathcal{B}'_q$ generates $H_q(P(m,\nu);\mathbb{Z})$ as observed above, it follows that $\mathcal{B}_q$ generates the abelian group $H_q(P(m,\nu);\mathbb{Z})/\text{torsion}$. Therefore $\mathcal{B}_q$ is a $\mathbb{Z}$-basis for $H_q(P(m,\nu);\mathbb{Z})$ mod torsion. It is easily seen that $\mathcal{B}'_q$ is $\mathbb{Z}_2$-linearly independent.

Any torsion element of $H_q(P(m,\nu);\mathbb{Z})$ is of order 2, in view of the fact that $H_q(P(m,\nu);\mathbb{Z})$ is generated by $\mathcal{B}_q \cup \mathcal{B}'_q$. Indeed, $\mathcal{B}_q$ generates a free abelian subgroup of $H_q(P(m,\nu);\mathbb{Z})$ and $\mathcal{B}'_q$ generates an elementary abelian 2-group contained in $H_q(P(m,\nu);\mathbb{Z})$. Hence the torsion subgroup equals the subgroup generated by $\mathcal{B}'_q$ and our assertion follows. □

The cohomology groups of $P(m,\nu)$ can be read off from the above theorem using the universal coefficient theorem. Thus $H^q(P(m,\nu);\mathbb{Z}) \cong \mathbb{Z}^t \oplus \mathbb{Z}_2^p$ where the values of t and p depend on m,ν can be explicitly determined in terms of $|\mathcal{B}_q|$ and $|\mathcal{B}'_{q-1}|$. As a warm up, we compute $H^2(P(m,\nu);\mathbb{Z})$. Note that $\mathcal{B}_2 = \emptyset$ and $\mathcal{B}'_1 = \{[X(1,\mathbf{i}_0)]\}$ where $\mathbf{i}_0 = (1,2\ldots,n)$ is the identity permutation (which is the unique element of length 0) if $m > 1$ and $\mathcal{B}'_1$ is empty if $m = 1$. So the rank of $H_2(P(m,\nu);\mathbb{Z})$ is zero.

Also $H_1(P(m,\nu);\mathbb{Z}) \cong \mathbb{Z}_2$ if $m > 1$ and is isomorphic to $\mathbb{Z}$ if $m = 1$.

Proposition 5.3.2. *Let $m \geq 1$. With the above notations,*

$$H^2(P(m,\nu);\mathbb{Z}) \cong \begin{cases} \mathbb{Z}_2 & \text{if } m > 1 \\ 0 & \text{if } m = 1. \end{cases}$$

Also the Picard group $Pic(P(m,\nu)) \cong \mathbb{Z}_2$ generated by $\xi_{\mathbb{C}}$ if $m > 1$ and is trivial otherwise.

Proof. The first assertion follows from the universal coefficient theorem. The second assertion follows from the fact that the first Chern class map yields an isomorphism of the Picard group onto $H^2(P(m,\nu);\mathbb{Z})$. □

We now turn to the determination of the rank and the torsion subgroup of the integral cohomology groups of $P(m,\nu)$ explicitly in terms of m,ν.

Write $H^{\text{ev}}(P(m,\nu);\mathbb{Z}) = \bigoplus_{q\geq 0} H^{2q}(P(m,\nu);\mathbb{Z}) \cong \mathbb{Z}^{b_e} \oplus \mathbb{Z}_2^{b'_e}$ and $H^{\text{odd}}(P(m,\nu);\mathbb{Z}) = \bigoplus_{q\geq 1} H^{2q-1}(P(m,\nu);\mathbb{Z}) \cong \mathbb{Z}^{b_o} \oplus \mathbb{Z}_2^{b'_o}$. We shall obtain formulas for the numbers b_e, b'_e, b_o, b'_o in terms of m,ν.

Set $\mathcal{B} := \bigcup_{q\geq 0} \mathcal{B}_q, \mathcal{B}_e := \bigcup_{q\geq 0} \mathcal{B}_{2q}$ and let $\mathcal{B}_o = \mathcal{B} \setminus \mathcal{B}_e$. Likewise $\mathcal{B}' : \bigcup_{q\geq 1} \mathcal{B}_q$ and $\mathcal{B}'_e := \{[X(j,\mathbf{i})] \in \mathcal{B}' \mid j \equiv 0 \pmod 2\}$, and $\mathcal{B}'_o = \mathcal{B}' \setminus \mathcal{B}'_e$. Also, set $\beta_e := |\mathcal{B}'_e|, \beta_o := |\mathcal{B}'_o|$.

By the universal coefficient theorem, $b_e = |\mathcal{B}_e|, b_o = |\mathcal{B}_o|$, whereas $b'_e = \beta_o, b'_o = \beta_e$.

Set $\ell_e := |I_e(\nu)|$ and $\ell_o = |I_o(\nu)|$. Thus $\ell_e + \ell_o = \binom{n}{\nu} = \chi(\mathbb{C}G(\nu))$. We shall determine the values of ℓ_e, ℓ_o.

The Poincaré polynomial $P_t(\nu)$ of $\mathbb{C}G(\nu)$ is the q-Gaussian multinomial coefficient $Q_q(\nu) := \binom{n}{\nu}_q = [n]_q/([n_1]_q \cdots [n_s]_q)$ where $q := t^2$ and $[k]_q = \prod_{1\leq j\leq k}(1-q^j)$.

The right hand side of Q_q is also the Poincaré polynomial $P_q(\mathbb{R}G(\nu);\mathbb{Z}_2)$ of the *real* flag manifold $\mathbb{R}G(\nu)$ with $\mathbb{Z}_2$-coefficients. See [Bor53b, §26], [Bor53a]. Therefore

$$\ell_e - \ell_o = \chi(\mathbb{R}G(\nu)). \tag{5.4}$$

We conclude that

$$\ell_e = (\chi(\mathbb{C}G(\nu)) + \chi(\mathbb{R}G(\nu)))/2; \quad \ell_o = (\chi(\mathbb{C}G(\nu)) - \chi(\mathbb{R}G(\nu)))/2. \tag{5.5}$$

If $\chi(\mathbb{R}G(\nu)) = 0$, then $\ell_e = \ell_o = \binom{n}{\nu}/2$.

The Euler-Poincaré characteristic of $\mathbb{R}G(\nu)$ can be computed using standard arguments and the classical result that $\chi(G/T) = |W(G,T)|$, the order of the Weyl group $W(G,T)$ when G is a compact connected Lie group and T is a maximal torus. We include a proof for the sake of completeness as we could not find an explicit reference for the result.

Lemma 5.3.3. *Let $\nu = (n_1, \ldots, n_s), s \geq 2$. Set $\nu_o := \{1 \leq j \leq s \mid n_j \equiv 1 \mod 2\}$.*

Then $\chi(\mathbb{R}G(\nu)) = 0$ if $\nu_o \geq 2$. Suppose that $\nu_o \leq 1$. Set $n'_j := \lfloor n_j/2 \rfloor, 1 \leq j \leq s$, and $\lfloor \nu/2 \rfloor := (n'_1, n'_2, \ldots, n'_s)$. Then $\chi(\mathbb{R}G(\nu)) = \binom{\lfloor n/2 \rfloor}{\lfloor \nu/2 \rfloor}$.

Proof. Without loss of generality, assume that n_j is even for all $j > \nu_o$. When $\nu_o \geq 2$, $\mathbb{R}G(\nu)$ is fibred by the real Grassmann manifold $\mathbb{R}G(n_1, n_2)$ over $\mathbb{R}G(n_1 + n_2, n_3, \ldots, n_s)$. Since $\dim \mathbb{R}G(n_1, n_2) = n_1 n_2$ is odd, $\chi(\mathbb{R}G(\nu)) = 0$ and so the vanishing of $\chi(\mathbb{R}G(\nu))$ follows from the multiplicative property of the Euler-Poincaré characteristics for fibrations. See [Spa82, Theorem 1,§3, Chapter 9]

Let $\nu_o \leq 1$. Consider first the Euler-Poincaré characteristic of the *oriented* flag manifold $\widetilde{\mathbb{R}G}(\nu) := SO(n)/SO(\nu)$ where $SO(\nu) := SO(n_1) \times \cdots \times SO(n_s)$. As $\nu_o \leq 1$, $SO(\nu)$ has the same rank as $SO(n)$. We take $T = (SO(2))^{\lfloor n/2 \rfloor} \subset SO(\nu)$ to be the standard maximal torus of $SO(n)$ which sits in $SO(n)$ as 2×2 block diagonal with top diagonal entry being 1 if n is odd. Using the fibration $SO(n)/T \to SO(n)/SO(\nu)$ with fibre $SO(\nu)/T$, we obtain that $\chi(SO(n)/T) = \chi(SO(\nu)/T)).\chi(SO(n)/SO(\nu))$. It is a classical result that when G is a compact connected Lie group and $T \subset G$ is a maximal torus of G, then $\chi(G/T) = |W(G,T)|$, the order of the Weyl group of G with respect to T. See [MT91, Part-II]. It is known [Hus94] that $W(SO(n),T)$ has order $2^{\lfloor (n-1)/2 \rfloor} \cdot \lfloor n/2 \rfloor !$. This yields that $\chi(\widetilde{\mathbb{R}G}(\nu)) = 2^{s-1}\binom{\lfloor n/2 \rfloor}{\lfloor \nu/2 \rfloor}$. The value of $\chi(\mathbb{R}G(\nu))$ is then determined to be $\binom{\lfloor n/2 \rfloor}{\lfloor \nu/2 \rfloor}$ using the covering projection $\widetilde{\mathbb{R}G}(\nu) \to \mathbb{R}G(\nu)$, which has degree 2^{s-1}. □

With notations as in the above lemma, if $\nu_o \leq 1$, then $\ell_e = (\binom{n}{\nu} + \binom{\lfloor n/2 \rfloor}{\lfloor \nu/2 \rfloor})/2$ and $\ell_o = (\binom{n}{\nu} - \binom{\lfloor n/2 \rfloor}{\lfloor \nu/2 \rfloor})/2$.

We tabulate below the values of ℓ_e and ℓ_o.

ν	ℓ_e	ℓ_o
$\nu_o \geq 2$	$\binom{n}{\nu}/2$	$\binom{n}{\nu}/2$
$\nu_o \leq 1$	$(\binom{n}{\nu} + \binom{\lfloor n/2 \rfloor}{\lfloor \nu/2 \rfloor})/2$	$(\binom{n}{\nu} - \binom{\lfloor n/2 \rfloor}{\lfloor \nu/2 \rfloor})/2$

Table 1: The values of ℓ_e and ℓ_o.

From the definition of $\mathcal{B}'_e$ we see that if $\mathbf{i} \in I_o(\nu)$, then $[X(j,\mathbf{i})] \in \mathcal{B}'_e$ for $\lfloor (m+1)/2 \rfloor$ distinct values of j, namely, those which satisfy $j \equiv 0 \pmod 2, 0 \leq j < m$. Moreover, if $[X(j,\mathbf{i})] \in \mathcal{B}'_e$, then $\mathbf{i} \in I_o(\nu)$. Therefore $\beta_e = \lfloor (m+1)/2 \rfloor \cdot \ell_o$. Similarly $\beta_o = \lfloor m/2 \rfloor \cdot \ell_e$.

The values of b_e, b_o depend on ν and the parity of m. We proceed to determine them.

We have $b_e + b_o = \dim_{\mathbb{Q}}(H^*(P(m,\nu);\mathbb{Q}))$ and $b_e - b_o = \chi(P(m,\nu))$.

Case 1: m is even. Then $H^q(P(m,\nu);\mathbb{Q}) = 0$ if q is odd and so $b_o = 0$. It follows that $b_e = \chi(P(m,\nu)) = (1/2)\chi(\mathbb{S}^m \times \mathbb{C}G(\nu)) = \binom{n}{\nu} = n!/(n_1! \cdots n_s!)$.

Case 2: <u>m is odd.</u> We have and $b_e - b_o = \chi(P(m,\nu)) = 0$. Therefore $b_e = b_o$. Also $b_e = |\{\mathbf{i} \in I(\nu) \mid \ell(\mathbf{i}) \equiv 0 \mod 2\}| = \ell_e$.

We summarise the above results in the theorem below.

Theorem 5.3.4. *Let $m \geq 1$ and $\nu = (n_1, \ldots, n_s)$ where $s \geq 2$. Let $\nu_o := |\{1 \leq j \leq s \mid n_j \equiv 1 \pmod 2\}|$. Then $H^{\mathrm{ev}}(P(m,\nu);\mathbb{Z}) \cong \mathbb{Z}^{b_e} \oplus \mathbb{Z}_2^{b'_e}$ and $H^{\mathrm{odd}}(P(m,\nu);\mathbb{Z}) \cong \mathbb{Z}^{b_o} \oplus \mathbb{Z}_2^{b'_o}$ where b_e, b_o, b'_e, b'_o are as follows:*

(i) When m is odd,

$$b_e = b_o = \ell_e = \begin{cases} \binom{n}{\nu}/2, & \text{if } \nu_o \geq 2 \\ (\binom{n}{\nu} + \binom{\lfloor n/2 \rfloor}{\lfloor \nu/2 \rfloor})/2, & \text{if } \nu_o \leq 1. \end{cases}$$

When m is even, $b_e = \binom{n}{\nu}$ and $b_o = 0$.

(ii) $b'_e = \beta_o = \lfloor m/2 \rfloor \cdot \ell_e$, $\quad b'_o = \beta_e = \lfloor (m+1)/2 \rfloor \cdot \ell_o$.

In particular, $H^{ev}(P(1,\nu);\mathbb{Z})$ has no torsion. □

The following remark gives some partial information about the ring structure of the integral cohomology of $P(m,\nu)$.

Remark 5.3.5. *(i) It is readily seen that $\mathbb{C}G(\nu)$ equals the Schubert variety $X(\mathbf{w}_0)$ that corresponds to the permutation $w_0 = (n, n-1, \ldots, 2, 1) \in S_n$ (in the one-line notation). So σ is orientation preserving if and only if $d = \dim_{\mathbb{C}} \mathbb{C}G(\nu) = \sum_{1 \leq i < j \leq s} n_i n_j$ is even, if and only if $\binom{\nu_o}{2}$ is even. It follows that $P(m,\nu)$ is orientable if and only if $m + \binom{\nu_o}{2}$ is odd. (See Remark 3.2.7.) When $P(m,\nu)$ is orientable, the natural orientation on it corresponds to the fundamental class $[X(m, \mathbf{w}_0)]$.*

(ii) Since the bundle $p : P(m,\nu) \to \mathbb{R}P^m$ admits a cross-section $s : \mathbb{R}P^m \to P(m,\nu)$, $p^* : H^*(\mathbb{R}P^m;\mathbb{Z}) \to H^*(P(m,\nu);\mathbb{Z})$ is a monomorphism of rings and its image is a direct summand.

(iii) Let $\iota : \mathbb{C}G(\nu) \hookrightarrow P(m,\nu)$ denote the fibre-inclusion of the bundle p :

$P(m,\nu) \to \mathbb{R}P^m$ over a point, say $[e_1] \in \mathbb{R}P^m$. Since $\{[X^+(0,\mathbf{i})] \mid \mathbf{i} \in I_e(\nu)\}$ is a $\mathbb{Z}$-basis for $\bigoplus_{q\geq 0} H_{4q}(\mathbb{C}G(\nu);\mathbb{Z}) \subset H_*(\mathbb{S}^m \times \mathbb{C}G(\nu);\mathbb{Z})$ and since $\iota_*([X(\mathbf{i})]) = \pi_*[X^+(0,\mathbf{i})] = [X(0,\mathbf{i})]$, it follows that $\iota_*(\bigoplus H_{4q}(\mathbb{C}G(\nu);\mathbb{Z})) \cong \mathbb{Z}^{\ell_e}$ is a direct summand of $H_{4q}(P(m,\nu);\mathbb{Z})$. Therefore

$$\iota^* : \bigoplus_{q\geq 0} H^{4q}(P(m,\nu);\mathbb{Z}) \to \bigoplus_{q\geq 0} H^{4q}(\mathbb{C}G(\nu);\mathbb{Z}) \tag{5.6}$$

is a surjective ring homomorphism. Since $\bigoplus_{q\geq 0} H^{4q}(\mathbb{C}G(\nu);\mathbb{Z})$ is torsion free, the kernal of ι^* contains the torsion subgroup of $\bigoplus_{q\geq 0} H^{4q}(P(m,\nu);\mathbb{Z})$.

(iv) Let $\{x(m,\mathbf{i})\}$ in $Hom(H_*(P(m,\nu);\mathbb{Z}),\mathbb{Z})$ be dual to $[X(m,\mathbf{i})]$ with respect to the basis $\mathcal{B} = \bigcup_{q\geq 0} \mathcal{B}_q$ of $H_*(P(m,\nu);\mathbb{Z})/\text{torsion}$. We have

$$\begin{aligned}
\langle \iota^*(x(m,\mathbf{i})), [X(\mathbf{j})]\rangle &= \langle x(m,\mathbf{i}), \iota_*([X(\mathbf{j})])\rangle \\
&= \langle x(m,\mathbf{i}), [X(0,\mathbf{j})]\rangle \\
&= 0,
\end{aligned}$$

for all $[X(\mathbf{j})] \in H_*(\mathbb{C}G(\nu);\mathbb{Z})$. Thus the kernel $\iota^* : H^*(P(m,\nu);\mathbb{Z}) \to H^*(\mathbb{C}G(\nu);\mathbb{Z})$ contains $x(m,\mathbf{i})$.

(iv) The cup-product $x(m,\mathbf{i}) \smile x(m,\mathbf{j})$ is a torsion element. This is obvious when any one of the factors is a torsion element. Suppose that both are of infinite order. Now $\pi^*(x(m,\mathbf{i})), \pi^*(x(m,\mathbf{j})) \in H^m(\mathbb{S}^m;\mathbb{Z}) \otimes H^*(\mathbb{C}G(\nu);\mathbb{Z})$. Hence $x(m,\mathbf{i}) \smile x(m,\mathbf{j}) \in \ker \pi^*$. By Proposition 3.2.6, $\ker(\pi^*)$ consists only of torsion elements, and so our assertion follows.

5.4 The ring structure of $H^*(P(\mathbb{S}^m, \mathbb{C}G(\nu)); \mathbb{Z}[1/2])$

Let R be a commutative ring in which 2 is a unit. From Theorem 5.3.1 we know that as an R-module, $H^q(P(m,\nu); R)$ is free of rank $|\mathcal{B}_q|$. We now turn to the ring structure of $H^*(P(m,\nu); R)$. We shall first describe it as a subring of $H^*(\mathbb{S}^m \times \mathbb{C}G(\nu); R)$ via the homomorphism π^* induced by the double covering projection $\pi : \mathbb{S}^m \times \mathbb{C}G(\nu) \to P(m,\nu)$. Note that since 2 is a unit in R, π^* is a monomorphism; see Proposition 3.2.6. Then, in Theorem 5.4.2 we shall describe $H^*(P(m,\nu); R)$ as a quotient of a polynomial algebra.

It will be convenient to use Chern classes of the canonical bundles over $\mathbb{C}G(\nu)$ as ring generators of $H^*(\mathbb{C}G(\nu); \mathbb{Z})$.

Let $1 \le j \le s$ and let γ_j denote the complex n_j-plane bundle over $\mathbb{C}G(\nu)$ whose fibre over $\mathbf{L} = (L_1, \ldots, L_s)$ is L_j. Then

$$\bigoplus_{1\le j\le s} \gamma_j = n\epsilon_{\mathbb{C}}.$$

Denote by $c(\omega, t)$ the total Chern polynomial $1 + c_1(\omega)t + c_2(\omega)t^2 + \cdots + c_r(\omega)t^r$ in the indeterminate t, where $c_j(\omega)$ is the jth Chern class of ω and r is the rank of ω. The following relation holds in $H^*(\mathbb{C}G(\nu); \mathbb{Z})$, in view of the above vector bundle isomorphism, where

$$\sum_{0\le r\le n} f_p t^p := \prod_{1\le j\le s} c(\gamma_j, t) = 1. \tag{5.7}$$

Recall that

$$H^*(\mathbb{C}G(\nu); \mathbb{Z}) = \mathbb{Z}[c_{r,j}; 1 \le r \le n_j, 1 \le j \le s]/\langle f_1, \cdots, f_n\rangle$$

where the f_j are defined by the equation

$$\sum_{0\le r\le n} f_r t^r = \prod_{1\le j\le s}(1+c_{1,j}t+\cdots+c_{n_j,j}t^{n_j}) \tag{5.8}$$

under an isomorphism that maps $c_{r,j}$ to $c_r(\gamma_j)$. See [Bor53b, Proposition 31.1].

Denote by $u_m \in H^m(\mathbb{S}^m;\mathbb{Z})$ the positive generator (with respect to the standard orientation on the sphere).

We now describe, using Proposition 5.3.1, the subring fixed by θ^* in $H^*(\mathbb{S}^m \times \mathbb{C}G(\nu);R) \cong R[u_m]\otimes R[c_{r,j}; 1\le r\le n_j, 1\le j\le s]/\langle u_m^2,\ f_p, 1\le p\le n\rangle$. Since π^* defines an isomorphism of $H^*(P(m,\nu);R)\cong \mathrm{Fix}(\theta^*)$, we have the following result. We omit the proof, which involves a straightforward verification.

Theorem 5.4.1. *Suppose that 2 is a unit in R. Then $H^*(P(m,\nu))$ is isomorphic to the subalgebra Fix(θ^*) $\subset H^*(\mathbb{S}^m\times\mathbb{C}G(\nu);R)$ which is generated by the following elements:*

Case (1): m is even.

$$\begin{aligned} u_m c_{2p-1}(\gamma_r),&\quad 1\le 2p-1\le n_r,\ 1\le r\le s;\\ c_{2j}(\gamma_r),&\quad 1\le j\le n_r,\ 1\le r\le s;\ \textit{and}\\ c_{2p-1}(\gamma_i)c_{2q-1}(\gamma_j),&\quad 1\le 2p-1\le n_i,\ 1\le 2q-1\le n_j,\ 1\le i\le j\le s.\end{aligned}$$

Case (2): m is odd.

$$\begin{aligned} u_m;\ \ c_{2j}(\gamma_r),&\quad 1\le j\le n_r,\ 1\le r\le s;\ \textit{and}\\ c_{2p-1}(\gamma_i)c_{2q-1}(\gamma_j),&\quad 1\le 2p-1\le n_i,\ 1\le 2q-1\le n_j,\ 1\le i\le j\le s.\end{aligned}$$

□

Next, we shall present $H^*(P(m,\nu);R)$ as a quotient of a polynomial algebra over R. We introduce commuting 'variables' $u_m, c_{2p,i}, c'_{2p-1,i,}, c''_{2p-1,2q-1,i,j}$, $1\le 2p-1\le n_i, 1\le 2q-1\le n_j, 1\le i\le j\le s$. Their degrees are assigned as follows:

$|u_m| = m, |c_{2p,i}| = 4p, |c'_{2p-1,i}| = m + 4p - 2, |c''_{2p-1,2q-1,i,j}| = 4(p + q - 1).$

It will be tacitly assumed that $c''_{2p-1,2q-1,i,j}$ is the same as the variable $c''_{2q-1,2p-1,j,i}$.

Recall the polynomials $f_r, 1 \leq r \leq n$, defined in Equation 5.8. When $r = 2r'$ is even, one may express f_r as a polynomial in $c_{2p,i}, c_{2p-1,2q-1,i,j}$, possibly in more than one way. Choose one such expression and define F_r to be the resulting polynomial.

When m is even, the same procedure applied to $c_{2t-1,l}f_r, 2p-1 \leq n_l$, for r odd yields the polynomials $F_{r,2t-1,l}$ in the variables $c_{2q,j}, c'_{2p-1,i}, c''_{2p'-1,2q'-1,i',j'}$, $1 \leq 2p' - 1, 2p - 1 \leq n_{i'}, 1 \leq 2q' - 1, 2q \leq n_j, 1 \leq i, j \leq s$. Similarly, assuming that r is odd, $u_m f_r$ can be expressed as a polynomial in the variables $c_{2q,j}, c'_{2p-1,i}, c''_{2p'-1,2q'-1,i',j'}$ in several ways. We choose one such expression and denote the resulting polynomial by $F'_{r,m}$.

Theorem 5.4.2. *We keep the above notations. Let R be a commutative ring in which 2 is a unit. Then $H^*(P(m, \nu); R)$ is isomorphic, as a graded R-algebra to $\mathcal{R}/\mathcal{J}$ where $\mathcal{R}$ is a polynomial algebra $\mathcal{R}$ whose description depends on the parity of m and is defined below.*

Case (1). *Suppose that $m \equiv 0 \pmod 2$. Then*

$$\mathcal{R} := R[c_{2t,i};\ c'_{2p-1,i};\ c''_{2p-1,2q-1,i,j};\ 1 \leq 2t, 2p-1 \leq n_i,\ 1 \leq 2q-1 \leq n_j,\ 1 \leq i, j \leq s],$$

with $|c_{2t,i}| = 4t, |c'_{2p-1,i}| = m + 4p - 2, |c''_{2p-1,2q-1,i,j}| = 4p + 4q - 4$. The ideal $\mathcal{J} \subset \mathcal{R}$ is the ideal generated by the following elements:

(i) $c'_{2p-1,i}c'_{2q-1,j};\quad c'_{2p-1,i}c''_{2q-1,2q'-1,j,j'} - c'_{2q'-1,j'}c''_{2p-1,2q-1,i,j};$

(ii) $c''_{2p-1,2q-1,i,j}c''_{2p'-1,2q'-1,i',j'} - c''_{2p-1,2p'-1,i,i'}c''_{2q-1,2q'-1,j,j'};$

(iii) $F_{2r},\ F'_{2k-1,m},\ F_{2k-1,2t-1,l},\ 1 \leq 2k-1, 2r \leq n,\ 1 \leq 2t-1 \leq n_l,\ 1 \leq l \leq s;$

where $2p - 1 \leq n_i,\ 2p' - 1 \leq n_{i'},\ 2q - 1 \leq n_j,\ 2q' - 1 \leq n_{j'},\ 1 \leq i, j, i', j' \leq s$.

The isomorphism $\eta : \mathcal{R}/\mathcal{J} \to H^*(P(m, \nu); R)$ is established by

$$c_{2p,i} \mapsto c_{2p}(\gamma_i),\ \ c'_{2p-1,i} \mapsto u_m c_{2p-1}(\gamma_i),\ \ c''_{2p-1,2q-1,i,j} \mapsto c_{2p-1}(\gamma_i)c_{2q-1}(\gamma_j).$$

Case (2). *Suppose that* $m \equiv 1 \pmod 2$. *Then*

$$\mathcal{R} := R[u_m;\ c_{2p,i}, 1 \le 2p \le n_i;\ \ c''_{2p-1,2q-1,i,j},\ 2p-1 \le n_i,\ 2q-1 \le n_j,\ 1 \le i,j \le s],$$

with $|u_m| = m$, $|c_{2p,i}| = 4p$, $|c''_{2p-1,2q-1,i.j}| = 4p+4q-4$. *The ideal* $\mathcal{J} \subset \mathcal{R}$ *is generated by the following elements:*

(i) u_m^2; $\ c''_{2p-1,2q-1,i,j}c''_{2p'-1,2q'-1,i',j'} - c''_{2p-1,2p'-1,i,i'}c''_{2q-1,2q'-1,j,j'}$;

(ii) $F_{2r},\ F_{2k-1,2t-1,l},\ 1 \le 2t-1 \le n_l,\ 1 \le l \le s$;

where $2p-1 \le n_i,\ 2p'-1 \le n_{i'},\ 2q-1 \le n_j,\ 2q'-1 \le n_{j'},\ 1 \le i,j,i',j' \le s$.

An isomorphism $\eta : \mathcal{R}/\mathcal{J} \to H^*(P(m,\nu);R)$ *is obtained as*

$$u_m \mapsto u_m,\ \ c_{2p,i} \mapsto c_{2p}(\gamma_i),\ \ c''_{2p-1,2q-1,i,j} \mapsto c_{2p-1}(\gamma_i)c_{2q-1}(\gamma_j).$$

Proof. It is clear η is a well-defined R-algebra homomorphism. Also image of η equals $\mathrm{Fix}(\theta^*)$. Since 2 is a unit in R, we have $H^*(P(m,\nu);R) \cong \mathrm{Fix}(\theta^*) \subset H^*(\mathbb{S}^m \times \mathbb{C}G(\nu);R)$ and so, to complete the proof, We only need to show that it is a monomorphism of R-modules.

Let $\tilde{\mathcal{R}}_0$ be the polynomial algebra $\widetilde{\mathcal{R}}_0 = R[c_{q,i} \mid 1 \le q \le n_i; 1 \le i \le s]$ and let $\mathcal{R}_0$ be the R-subalgebra of $\tilde{\mathcal{R}}_0$ generated by the $c_{2q,i},\ \ c_{2p-1,2q-1,i,j} := c_{2p-1,i}c_{2q-1,j}$. The algebra $\tilde{\mathcal{R}}_0$ is graded by $|c_{q,i}| = 2q\ \forall i$. Let $\tilde{\sigma} : \tilde{\mathcal{R}}_0 \to \tilde{\mathcal{R}}_0$ be the R-algebra involution defined by $c_{q,i} \mapsto (-1)^q c_{q,i}$. Then $\mathcal{R}_0 = \bigoplus_{t\ge 0} \widetilde{R}^{4t} = \mathrm{Fix}(\tilde{\sigma})$. The cohomology algebra $H^*(\mathbb{C}G(\nu);R)$ is isomorphic to $\widetilde{R}_0/\tilde{\mathcal{J}}_0$ where $\tilde{\mathcal{J}}_0$ is generated by $f_1, \cdots, f_n$ defined as in Equation 5.7. The isomorphism is obtained by sending $c_{q,i}$ to $c_q(\gamma_i) \in H^{2q}(\mathbb{C}G(\nu);R)$.

The ideal $\tilde{\mathcal{J}}_0$ is graded and we have that $\tilde{\mathcal{J}}_0 \cap \mathcal{R}_0 = \oplus_{t\ge 0}\tilde{\mathcal{J}}_0^{4t}$. So $\mathcal{J}_0 := \mathcal{R}_0 \cap \tilde{\mathcal{J}}_0 \subset \mathcal{R}_0$ is generated as an ideal of by the following elements:

$$f_{2r},\ c_{2p-1,i}f_{2r-1} \in \mathrm{Fix}(\tilde{\sigma}), 1 \le 2p-1 \le n_i, 1 \le i \le s.$$

We may express each of these elements as polynomials in the chosen generators of $\mathcal{R}_0$. The non-uniqueness of such expressions for a given ideal generator arises from the equality $c_{2p-1,2q-1,i,j}c_{2p'-1,2q'-1,i',j'} = c_{2p-1,2p'-1,i,i'}c_{2q-1,2q'-1,j,j'}$.

Consider the polynomial algebra over R generated by the indeterminates $c_{2p,i}, c''_{2p-1,2q-1,i,j}$. Then the quotient algebra

$$R[c_{2p,i}, c''_{2p-1,2q-1,i,j}]/\langle c''_{2p-1,2q-1,i,j}c''_{2p'-1,2q'-1,i',j'} - c''_{2p-1,2p'-1,i,i'}c''_{2q-1,2q'-1,j,j'}\rangle$$

is isomorphic to $\mathcal{R}_0 \subset \tilde{\mathcal{R}}_0$ via the R-algebra homomorphism

$$c_{2p,i} \mapsto c_{2p,i}, \;\; c''_{2p-1,2q-1,i,j} \mapsto c_{2p-1,2q-1,i,j}.$$

Set $R_0 := R[c_{2p,i}, c''_{2p-1,2q-1,i,j}]$ and let $J_0 \subset R_0$ be the ideal generated by

$$c''_{2p-1,2q-1,i,j}c''_{2p'-1,2q'-1,i',j'} - c''_{2p-1,2p'-1,i,i'}c''_{2q-1,2q'-1,j,j'}, \quad F_{2r}, \quad F_{2r-1,2t-1,l}.$$

Then

$$R_0/J_0 \cong \mathcal{R}_0/\mathcal{J}_0 \cong H^{4*}(\mathbb{C}G(\nu); R) \cong \mathrm{Fix}(\sigma^*). \tag{5.9}$$

The ring $\mathcal{R}/\mathcal{J}$ is an $\mathcal{R}_0/\mathcal{J}_0$-module whose (definition and) structure depends on the parity of m.

First, consider the case when m is odd. In this case, $\mathcal{R}/\mathcal{J}$ is a free $\mathcal{R}_0/\mathcal{J}_0$-module with basis $\{1, u_m\}$. Also, $H^*(\mathbb{S}^m; R) = R[u_m]/\langle u_m^2\rangle$ is fixed by α^*.

Since $u_m^2 \in \mathcal{J}$, we have the following isomorphisms of R-algebras $\mathcal{R}/\mathcal{J} \cong (\mathcal{R}_0/\mathcal{J}_0)[u_m]/\langle u_m^2\rangle \cong R[u_m]/\langle u_m^2\rangle \otimes_R (\mathcal{R}_0/\mathcal{J}_0) \cong H^*(\mathbb{S}^m; R) \otimes \mathrm{Fix}(\sigma^*) \cong \mathrm{Fix}(\theta^*) \cong H^*(P(m,\nu); R)$. This completes the proof when m is odd.

Now let m be even. We have $\alpha^*(u_m) = -u_m$ and

$$\text{Fix}(\theta^*) = \bigoplus_{t\geq 0} H^{4t}(\mathbb{C}G(\nu); R) \oplus \left(\bigoplus_{t\geq 0} Ru_m \otimes H^{4t+2}(\mathbb{C}G(\nu); R) \right). \tag{5.10}$$

Since each $H^{4t+2}(\mathbb{C}G(\nu); R)$ is a free R-module for any $t \geq 0$, so is $Ru_m \otimes H^{4t+2}(\mathbb{C}G(\nu); R) \cong H^{4t+2}(\mathbb{C}G(\nu); R)$.

On the other hand, $\mathcal{R}/\mathcal{J}$ is generated as a $\mathcal{R}_0/\mathcal{J}_0$-algebra by the $c'_{2p-1,i}$. Since $c'_{2p-1,i}c'_{2q-1,j} = 0$, we see that $\mathcal{R}/\mathcal{J}$ is generated as an $\mathcal{R}_0/\mathcal{J}_0$-module by $\{1, c'_{2p-1,i}\}$. Further, in view of the relation $c'_{2p-1,i}c''_{2q-1,2q'-1,j,j'} = c'_{2q'-1,j'}c''_{2p-1,2q-1,i,j}$ in $\mathcal{R}/\mathcal{J}$, if $x \in \mathcal{R}/\mathcal{J}$ is a monomial in the generators $c''_{2p-1,2q-1,i,j}$ and if r such that $1 \leq 2r-1 \leq n_l, 1 \leq l \leq s$, then $c'_{2r-1,l}x$ is *uniquely* expressible as $c'_{2r_0-1,l_0}y$ with y a monomial in the generators $c''_{2p-1,2q-1,i,j}$ and (r_0, l_0) is the *least* in lexicographical order among such expressions. This shows that, as R-submodules of $\mathcal{R}/\mathcal{J}$,

$$\sum (\mathcal{R}_0/\mathcal{J}_0)c'_{2p-1,i} \cong \bigoplus_{t\geq 0} Ru_m \otimes H^{4t+2}(\mathbb{C}G(\nu); R).$$

It follows that $\mathcal{R}/\mathcal{J} \cong \mathcal{R}_0/\mathcal{J}_0 \oplus (\bigoplus_{t\geq 0}(Ru_m \otimes H^{4t+2}(\mathbb{C}G(\nu); R)$ is an isomorphism. Thus, in view of Equation 5.10, we have the R-algebra isomorphism $\mathcal{R}/\mathcal{J} \cong \text{Fix}(\theta^*) \cong H^*(P(m,\nu); R)$ defined by η. $\square$

Chapter 6

K-theory of $P(\mathbb{S}^m, \mathbb{C}G(\nu))$

In this chapter, our goal is to determine the ring structure of the complex K-theoretic ring $K^*(P(m,\nu))$ for any $m \geq 1$ and $\nu = (n_1, \ldots, n_s), s \geq 2$. However, we have been greeted with limited success in our efforts. Our results are complete up to a 'finite ambiguity' in a sense that will be made precise later.

The complex K-ring of the classical Dold manifolds $P(m,n)$ was determined by Fujii in [Fuj66], [Fuj69]. If ω is a complex vector bundle over a finite CW complex B and $\mathbb{C}G(\nu)$ is a complex flag manifold, one has the associated $\mathbb{C}G(\nu)$-bundle $\mathbb{C}G(\omega;\nu) \to B$. In this case, the complex K-ring of $\mathbb{C}G(\omega;\nu)$ is well-known. See [Kar08, §3,Chapter IV] and also [Ati67]. However, it is easily seen that $P(m,\nu)$ is not homeomorphic to $\mathbb{C}G(\omega;\nu)$ for any complex vector bundle ω over $\mathbb{R}P^m$. Indeed, the flag manifold bundle $\mathbb{C}G(\nu) \hookrightarrow \mathbb{C}G(\omega;\nu) \to \mathbb{R}P^m$ admits a cohomology extension of the fibre over $\mathbb{Z}$ and so, by the Leray-Hirsch theorem [Spa82, §7, Chapter 5], $H^*(\mathbb{C}G(\omega;\nu);\mathbb{Z})$ is a free $H^*(\mathbb{R}P^m;\mathbb{Z})$-module of rank equal to $\mathrm{rank}_{\mathbb{Z}}(H^*(\mathbb{C}G(\nu);\mathbb{Z}))$. This implies, in particular, that the rank of $H^2(\mathbb{C}G(\omega;\nu);\mathbb{Z})$ is positive. However, $H^2(P(m,\nu);\mathbb{Z})$ has rank 0—see Theorem 5.3.1—and so $P(m,\nu)$ is not homeomorphic to $\mathbb{C}G(\omega;\nu)$ for any complex vector bundle ω over $\mathbb{R}P^m$. Also, it should be noted that when $m \geq 1$, $P(m,\nu)$ is not a homogeneous space of $Spin(m) \times U(n)$, although $\mathbb{S}^m \times \mathbb{C}G(\nu)$ is. As such the computation of the K-theory of $P(m,\nu)$ is a nontrivial and interesting problem.

We now proceed to describe our results on (a) the additive structure of $K^*(P(m,\nu))$, (b) the ring structure of $K^*(P(m,\nu)) \otimes \mathbb{Q}$, and, (c) a subring of $K^0(P(m,\nu))$ generated by the classes of certain canonical (complex) vector bundles on $P(m,\nu)$.

6.1 The additive structure of $K^*(P(\mathbb{S}^m, \mathbb{C}G(\nu)))$

Recall, from [AH61], the Atiyah-Hirzebruch spectral sequence $(E_r^{p,q}, d_r)$, which abuts to $K^*(B)$ for a finite CW complex B. The E_2-page is defined as $E_2^{p,q} = H^p(B; K^q(\{*\}))$ where $\{*\}$ denotes a one-point space. The differential is of bidegree $d_r : E_r^{p,q} \to E_r^{p+r,q+1-r}$ has bidegree $(r, 1-r)$. As such $d_r = 0$ if r is even, since $K^q(\{*\}) = 0$ for q odd.

Let ω be a complex vector bundle over B. Recall that the Chern character $\mathrm{ch}(\omega) \in H^*(B;\mathbb{Q})$ is defined as $\mathrm{ch}(\omega) = \sum_{k\geq 0}(\sum_{1\leq j\leq r} x_i^k/k!)$ where $x_j, 1 \leq j \leq r$ are the 'Chern roots' of ω. Thus, the x_j are defined by a formal factorization of the Chern polynomial:

$$c(\omega, t) = \sum_{0\leq p\leq r} c_p(\omega)t^p = \prod_{1\leq j\leq r}(1 + x_j t).$$

Atiyah and Hirzebruch [AH61] proved that the Chern character $\mathrm{ch} : K^0(B) \otimes \mathbb{Q} \to H^{\mathrm{ev}}(B;\mathbb{Q}) = \bigoplus_{q\geq 0} H^{2q}(B;\mathbb{Q})$ defined by $[\omega] \mapsto \mathrm{ch}(\omega)$ is an isomorphism of rings. In particular, the rank of $K^0(B)$ equals that of $H^{\mathrm{ev}}(B;\mathbb{Z})$. It follows that if $H^*(B;\mathbb{Z})$ has no p-torsion for a prime p, neither does $K^0(X)$. The same result applied to the suspension $S(B)$ of B yields an isomorphism $K^1(B) \otimes \mathbb{Q} \to H^{\mathrm{odd}}(B;\mathbb{Q}) = \bigoplus_{q\geq 1} H^{2q-1}(B;\mathbb{Q})$.

Taking B to be $P(m,\nu)$ we obtain the following. Denote by $\xi_{\mathbb{C}}$ the complexification of the Hopf line bundle ξ over $\mathbb{R}P^m$. Clearly $\xi_{\mathbb{C}} \otimes \xi_{\mathbb{C}} \cong \epsilon_{\mathbb{C}}$ and so $([\xi_{\mathbb{C}}]-1)^2 = -2([\xi_{\mathbb{C}}]-1)$. Adams [Ada62] showed that $K^0(\mathbb{R}P^m) \cong \mathbb{Z}[y]/\langle y^2+2y, y^{r+1}\rangle = \mathbb{Z}\oplus\mathbb{Z}_{2^r}y$ where $y = [\xi_{\mathbb{C}}]-1$ and $r = \lfloor m/2 \rfloor$. We shall denote by the same symbol $\xi_{\mathbb{C}}$ the pull-back $p^!(\xi_{\mathbb{C}})$ on $P(m,\nu)$.

Theorem 6.1.1. *Let $m \geq 1$, $\nu = (n_1,\ldots,n_s), s \geq 2$. Let b_e, b_o, b'_e, b'_o be as in Theorem 5.3.4. Then:*
(i) $K^0(P(m,\nu)) \cong \mathbb{Z}^{b_e} \oplus A_0$ where A_0 is a finite abelian group of order 2^k for some k, with $0 \leq k \leq b'_e$. The group A_0 contains a summand $\mathbb{Z}_{2^{\lfloor m/2\rfloor}}$ is generated by $y = [\xi_{\mathbb{C}}]-1$.
(ii) $K^1(m,\nu) \cong \mathbb{Z}^{b_o} \oplus A_1$, where A_1 is a finite abelian group of order 2^t for some $0 \leq t \leq b'_o$.
In particular, $K^1(P(m,\nu))$ is a torsion group if m is even, and, $K^0(P(1,\nu))$ is a torsion-free group.

Proof. Since $\mathrm{ch} : K^*(P(m,\nu)) \otimes \mathbb{Q} \to H^*(P(m,\nu);\mathbb{Q})$ is an isomorphism and since $K^*(P(m,\nu))$ is finitely generated group, it follows that the ranks of $K^0(P(m,\nu)), K^1(P(m,\nu))$ are the same as those of $H^{\mathrm{ev}}(P(m,\nu);\mathbb{Z}), H^{\mathrm{odd}}(P(m,\nu);\mathbb{Z})$ respectively. This shows that the ranks of $K^0(P(m,\nu)), K^1(P(m,\nu))$ as asserted.

Since the Atiyah-Hirzebruch spectral sequence converges to $K^*(P(m,\nu)), K^0(P(m,\nu))$ has a filtration so that the associated graded group is isomorphic to $\bigoplus_{p\geq 0} E_\infty^{p,-p}$. Since the torsion subgroup of $\bigoplus_{p\geq 0} E_2^{2p,-2p} = H^{\mathrm{ev}}(P(m,\nu);\mathbb{Z})$ is a 2-group of order $2^{b'_e}$, we obtain that the torsion subgroup of $K^0(P(m,\nu))$ is a 2-group whose orders are bounded above by $2^{b'_e}$. Similar argument yields that the torsion subgroup of $K^1(P(m,\nu))$ is a 2-group of order at most $2^{b'_o}$.

To see that $K^0(P(m,\nu))$ contains a summand isomorphic to $2^{\lfloor m/2\rfloor}$, we note that $p : P(m,\nu) \to \mathbb{R}P^m$ admits a cross-section, namely $s([v]) = [v,\mathbf{E}]$ where $\mathbf{E} = (\mathbb{C}^{n_1},\ldots,\mathbb{C}^{n_s})$ in which the standard basis vectors $e_{n_1+\cdots+n_{j-1}+1},\ldots,e_{n_1+\cdots+n_j}$ span the j-th coordinate $\mathbb{C}^{n_j}$. It follows that the composition $s^! \circ p^!$ is the identity map of $K^0(\mathbb{R}P^m)$. Hence $p^!$ is a monomorphism whose image is a summand of $K^0(P(m,\nu)$. By the work of Adams recalled above $\widetilde{K}^0(\mathbb{R}P^m) = \mathbb{Z}_{2^{\lfloor m/2 \rfloor}} y$. This completes the proof. □

We pause for an example.

Example 6.1.2. *(i) In the case of classical Dold manifolds, $\nu = (1, n-1)$ and $P(m,\nu) = P(m,n-1)$. The group $K^*(P(m,n-1))$ had been completely determined by Fujii [Fuj66, Theorem 3.14]. The table below summarises Fujii's results (in our notations) yielding the rank and the order of the torsion groups of $K^0(P(m,n-1)), K^1(P(m,n-1))$. Our results are consistent with Fujii's work but we have not been able to determine the groups A_0, A_1.*

$P(m,n-1)$		K^0		K^1	
m	$n-1$	b_e	$o(A_0)$	b_o	$o(A_1)$
$2r$	$2t$	$2t+1$	2^r	0	0
$2r+1$	$2t$	$t+1$	2^r	$t+1$	0
$2r$	$2t+1$	$2t+2$	2^r	0	2^r
$2r+1$	$2t+1$	$t+1$	2^r	$t+1$	2^{r+1}

Table 2: The values of $b_e, b_o, o(A_0), o(A_1)$ for $P(m,n-1)$.

(ii) When $\mathbb{C}G(\nu)$ is the complete flag manifold, that is, when $\nu = (1,\ldots,1)$, $\nu_o = n$. We leave out the trivial case when $n = 1$ and assume $n \geq 2$. Now the values of b_e, b'_e, b_o, b'_o for $P(m,(1,\ldots,1))$ in Theorem 6.1.1 are described in the table below.

$P(m,\nu)$	K^0		K^1	
m	b_e	b'_e	b_o	b'_o
$2r$	$n!/2$	$rn!/2$	$n!/2$	$rn!/2$
$2r+1$	$n!$	$rn!/2$	0	$(r+1)n!/2$

Table 3: The values of b_e, b'_e, b_o, b'_o for $P(m,(1,\ldots,1))$.

6.2 The ring structure of $K^0(P(\mathbb{S}^m, \mathbb{C}G(\nu)))$.

Recall that $\gamma_j, 1 \leq j \leq s$, denotes the canonical complex n_j-plane bundle over $\mathbb{C}G(\nu)$. It is a σ-conjugate bundle and so we obtain the real vector bundle $\hat{\gamma}_j$ over $P(m,\nu)$. Denote by $\hat{\gamma}_{j\mathbb{C}}$ the complexification $\hat{\gamma}_j \otimes_{\mathbb{R}} \mathbb{C}$.

Let $f : \mathbb{C}G(\nu) \to P(m,\nu)$ be the inclusion of the fibre over the base point $[e_1] \in \mathbb{R}P^m$. Thus $f(\mathbf{L}) = [e_1, \mathbf{L}]$. Since $f^!(\hat{\gamma}_j) = \gamma_j$ as *real* vector bundles, it follows that $f^!(\hat{\gamma}_{j\mathbb{C}}) = \gamma_j \otimes \mathbb{C} \cong \gamma_j \oplus \bar{\gamma}_j$ as complex vector bundles. Note that complexification commutes with Whitney sums, tensor products, and exterior products. So, we have $f^!(\Lambda^k(\hat{\gamma}_{j\mathbb{C}})) = \Lambda^k(\gamma_j \oplus \bar{\gamma}_j) \cong \bigoplus_{p+q=k} \Lambda^p(\gamma_j) \otimes \Lambda^q(\bar{\gamma}_j) \cong \bigoplus_{p+q=k} \Lambda^p(\gamma_j) \otimes \overline{\Lambda^q(\gamma_j)}$ for all $k \geq 0, 1 \leq j \leq s$.

We recall the following description of the complex K-ring of $\mathbb{C}G(\nu)$. See [Kar08, Theorem 3.6, Chapter-IV].

Theorem 6.2.1. *The ring $K^0(\mathbb{C}G(\nu))$ is isomorphic to the polynomial ring generated by $\lambda_{p,j}$, $1 \leq p \leq n_j$, $1 \leq j \leq s$, modulo the ideal generated by the set $\{h_p - \binom{n}{p} \mid 1 \leq p \leq n\}$, where $h_p = h_p(\lambda_{q,j})$ is the coefficient of t^p in the following polynomial in the variable t:*

$$\sum_{0 \leq p \leq n} h_p t^p = \prod_{1 \leq j \leq s} (\sum_{0 \leq r \leq n_j} \lambda_{r,j} t^r).$$

Here $h_0 = 1, \lambda_{0,j} = 1 \ \forall j$. The isomorphism is defined by sending $\lambda_{p,j}$ to $[\Lambda^p(\gamma_j)]$. Moreover, $K^1(\mathbb{C}G(\nu)) = 0$. □

We shall refer to the generators $\lambda_{p,j}$ as the *canonical generators* of $K^0(\mathbb{C}G(\nu))$.

Remark 6.2.2. *For any compact connected Lie group G and a closed subgroup*

$H \subset G$, the α-construction yields a λ-ring homomorphism $RH \to K^0(G/H)$ defined by that sends $[V] \in RH$ to the class of the associated vector bundle with projection $G \times_H V \to G/H$. (See [AH61], [Hus94].) The bundle γ_j is associated to the representation $\lambda_{1,j} : S(U(\nu)) := S(U(n_1) \times \cdots \times U(n_s)) \to U(n_j)$, the projection to the jth coordinate, via the so-called 'α-construction'. Taking the pth exterior power, we obtain that $\Lambda^p(\gamma_j)$ is associated to $\Lambda^p(\lambda_{1,j}) = \lambda_{p,j}$ in the representation ring $RS(U(\nu))$. Using the known description of the representation of the unitary group (see [Hus94, Chapter 14]), it is readily seen that $RS(U(\nu)) = \mathbb{Z}[\lambda_{p,j}, \lambda^{-1}_{n_j,j} \mid 1 \leq j \leq n_j]/\langle \prod_{1\leq j\leq s} \lambda_{n_j,j} - 1\rangle$. Note that $\lambda^{-1}_{n_j,j} = \prod_{i\neq j} \lambda_{n_i,i}$ in $RS(U(\nu))$. Denote by λ_1 the standard representation of $SU(n)$ on $\mathbb{C}^n$. We denote by $\rho([V]) \in RS(U(\nu))$ the restriction of the class $[V]$ of a $S(U(\nu))$ representation V. Thus $\rho(\lambda_1) = \sum_{1\leq j\leq s} \lambda_{1,j}$. Set $\Lambda_t([V]) := \sum_{1\leq j\leq r} [\Lambda^q(V)]t^q$. Then $\Lambda_t(\lambda_1) = \prod_{1\leq j\leq s} \Lambda_t(\lambda_{1,j})$.

If a complex representation of $S(U(\nu))$ arises as the extension of a representation of $SU(n)$, then the associated vector bundle is trivial. It follows that $\rho(h_p) = \binom{n}{p}$. So ρ defines a ring homomorphism $\bar{\rho} : RS(U(\nu))/\mathcal{I} \to K^0(\mathbb{C}G(\nu))$ where $\mathcal{I} \subset RS(U(\nu))$ is the ideal in $\langle h_p - \binom{n}{p} \mid 1 \leq p \leq n-1\rangle$. The above theorem is equivalent to the assertion that $\bar{\rho}$ is an isomorphism.

As a corollary, we obtain the following.

Lemma 6.2.3. *$K(\mathbb{C}G(\nu))$ is generated as an abelian group by the classes of σ-conjugate complex vector bundles.*

Proof. We need only observe that if ξ and η are σ-conjugate complex vector bundles so are $\xi \otimes \eta, \Lambda^q(\xi), q \geq 0$, and $\xi \otimes \eta$. (See [NS19, Example 2.2(iv), Lemma 2.3(ii)].) Since the canonical bundles $\gamma_j, 1 \leq j \leq s$, are all σ-conjugate bundles, the lemma follows from the above theorem. □

Since $\pi = \pi \circ \theta$, we have $\theta^! \circ \pi^! = (\pi \circ \theta)^! = \pi^!$. So, for any complex vector bundle ω over $P(m, \nu)$, $\pi^!(\omega)$ is a complex vector bundle over $\mathbb{S}^m \times \mathbb{C}G(\nu)$ such that $\theta^!(\pi^!(\omega)) =$

$\pi^!(\omega)$. This implies that the image of $\pi^! : K^0(P(m,\nu)) \to K^0(\mathbb{S}^m \times \mathbb{C}G(\nu))$ is contained in the fixed subring $\mathrm{Fix}(\theta^!)$.

Since $K^1(\mathbb{C}G(\nu)) = 0$, we have isomorphisms given by the exterior tensor products of vector bundles. See [Ati67, Corollary 2.7.15.], [Kar08, Prop. 3.24, Chapter IV].

$$K^0(\mathbb{S}^m \times \mathbb{C}G(\nu)) = K^0(\mathbb{S}^m) \otimes K^0(\mathbb{C}G(\nu)), \tag{6.1}$$

and,

$$K^1(\mathbb{S}^m \times \mathbb{C}G(\nu)) = K^1(\mathbb{S}^m) \otimes K^0(\mathbb{C}G(\nu)). \tag{6.2}$$

Also, if m is even $K^1(\mathbb{S}^m) = 0$ and if m is odd, $\widetilde{K}(\mathbb{S}^m) = 0$.

We shall identify $y \in K^0(\mathbb{C}G(\nu))$ with $pr_2^!(y) = 1 \otimes y \in K^0(\mathbb{S}^m \times \mathbb{C}G(\nu))$ and $x \in K^0(\mathbb{S}^m)$ with $x \otimes 1 = pr_1^!(x)$ where pr_i denotes the projection to the ith factor of $\mathbb{S}^m \times \mathbb{C}G(\nu)$. Thus $x \otimes y$ is identified with $xy \in K^0(\mathbb{S}^m \times \mathbb{C}G(\nu))$. A similar notation holds for the $K^0(\mathbb{S}^m \times \mathbb{C}G(\nu))$-module $K^1(\mathbb{S}^m \times \mathbb{C}G(\nu))$.

Let $m = 2r$ be even. By Theorem 2.5.1, the element $[\xi^+] - 2^{r-1} = 2^{r-1} - [\xi^-]$ generates $\ker(\mathrm{rank} : K^0(\mathbb{S}^m) \to \mathbb{Z}) \cong \mathbb{Z}$.

Definition 6.2.4. *Let $m \geq 0$. We set*

$$\delta_m := \begin{cases} [\xi^+] - 2^{r-1} = 2^{r-1} - [\xi^-] & \text{if } m \equiv 0 \pmod 2 \\ 0 & \text{if } m \equiv 1 \pmod 2. \end{cases}$$

We have $\theta^!(\delta_m) = \alpha^!(\delta_m) = -\delta_m$. (This follows from the naturality of the Chern character.)

For any $x \in K^0(\mathbb{C}G(\nu))$, $\theta^!(x) = \theta^!(1 \otimes x) = \sigma^!(x) = \bar{x}$.

Lemma 6.2.5. *Let $m \geq 1$ be arbitrary. Let $\mathcal{K}$ be the subring of $K^0(\mathbb{S}^m \times \mathbb{C}G(\nu))$ generated by the elements (i) $x + \bar{x}$ and, (ii) $\delta_m(x - \bar{x})$ when m is even, as x varies in $K^0(\mathbb{C}G(\nu))$. Then the subring $Fix(\theta^!) \subset K^0(\mathbb{S}^m \times \mathbb{C}G(\nu))$ contains $\mathcal{K}$. Moreover, if $z \in Fix(\theta^!)$, then $2z \in \mathcal{K}$. Thus*

$$2Fix(\theta^!) \subset \mathcal{K} \subset Fix(\theta^!).$$

Proof. It is evident that, for any $x \in K^0(\mathbb{C}G(\nu))$, $x + \bar{x} \in \mathrm{Fix}(\theta^!)$.

When m is even, $\theta^!(\delta_m(x - \bar{x})) = \alpha^!(\delta_m)(\bar{x} - x) = -\delta_m(\bar{x} - x) = \delta_m(x - \bar{x})$ and so $\delta_m(x - \bar{x}) \in \mathrm{Fix}(\theta^!)$.

This shows that $\mathcal{K} \subset \mathrm{Fix}(\theta^!)$ for any $m \geq 1$.

Let $z \in \mathrm{Fix}(\theta^!)$ where we assume that $\mathrm{rank}(z) = 0$. Write $z = z_0 + \delta_m z_1$ where $z_i \in \widetilde{K}^0(\mathbb{C}G(\nu))$ (recall that $\delta_m = 0$ if m is odd). Then $z = \theta^!(z) = \bar{z}_0 - \delta_m z_1$ and so $2z = z + \theta^!(z) = z_0 + \bar{z}_0 + \delta_m(z_1 - \bar{z}_1) \in \mathcal{K}$. □

Remark 6.2.6. *(i). If $y \in K^0(\mathbb{C}G(\nu))$, then, $y^2 + \bar{y}^2, (y + \bar{y})^2 \in \mathcal{K}$. Therefore $2y\bar{y} = (y + \bar{y})^2 - y^2 - \bar{y}^2 \in \mathcal{K}$.*
(ii) Since $\bar{\lambda}_{n_j,j} = \lambda_{n_j,j}^{-1} = \prod_{i \neq j} \lambda_{n_i,i}$, we may write any $x \in K^0(\mathbb{C}G(\nu))$ as $x = P(\lambda_{p,j})$ as a polynomial in the canonical generators $\lambda_{p,j}, 1 \leq p \leq n_j, 1 \leq j \leq s$, of $K^0(\mathbb{C}G(\nu))$. In particular, the exponents of $\lambda_{n_j,j}, 1 \leq j \leq s$, that occur in $P(\lambda_{p,j})$ is non-negative for each j. Furthermore, we may assume that, in each monomial, at least one of the $\lambda_{n_j,j}$ does not occur. This is possible since $\prod_{1 \leq j \leq s} \lambda_{n_j,j} = 1$.

Recall from §2.5 that, when $m \equiv 0 \pmod 2$, the complex vector bundles ξ^+, ξ^- over $\mathbb{S}^m$ are associated to the half-spin representations, and the bundle η^- is defined as the pull-back $\alpha^!(\xi^+)$. Also we obtained bundle maps $\tilde{\alpha}^- : \eta^- \to \xi^+$ and $\tilde{\alpha}^+ : \xi^+ : \xi^+ \to \eta^-$ both covering α such that $\tilde{\alpha}^- \circ \tilde{\alpha}^+$ and $\tilde{\alpha}^+ \circ \tilde{\alpha}^-$ are the identity isomorphisms id_{ξ^+} and id_{η^-} respectively.

Let $\tilde{\sigma} : E(\omega) \to E(\omega)$ be a σ-conjugate bundle involution that covers $\sigma : \mathbb{C}G(\nu) \to \mathbb{C}G(\nu)$. We regard it as a complex linear isomorphism $\omega \to \bar{\omega}$ covering σ. We have a complex vector bundle isomorphism $\tilde{\theta}^+ : E(\xi^+ \otimes \omega) \to E(\eta^- \otimes \bar{\omega})$ that covers θ. Explicitly, $\tilde{\theta}^+(e^+ \otimes u) = \tilde{\alpha}^+(e^+) \otimes \tilde{\sigma}(u)$ and $\tilde{\theta}^-(e^- \otimes v) = \tilde{\alpha}^-(e^-) \otimes \tilde{\sigma}(v)$. The bundle isomorphism $\tilde{\theta}^- : E(\eta^- \otimes \bar{\omega}) \to E(\xi^+ \otimes \omega)$ covering θ is defined analogously. We have

$$\tilde{\theta}^+ \circ \tilde{\theta}^- = id_{\eta^- \otimes \bar{\omega}}, \text{ and } \tilde{\theta}^- \circ \tilde{\theta}^+ = id_{\xi^+ \otimes \omega}. \tag{6.3}$$

Set $\tilde{\xi}(\omega) := \xi^+ \otimes \omega \oplus \eta^- \otimes \bar{\omega}$ for any σ-conjugate vector bundle ω.

Lemma 6.2.7. *With the above notations, we have:*
(i) For any $m \geq 1$, we have $\pi^!(\hat{\omega}_{\mathbb{C}}) \cong \omega \oplus \bar{\omega}$.
(ii) Let $m \equiv 0 \pmod 2$. Then there exists a complex vector bundle $\xi^0(\omega)$ over $P(m, \nu)$ such that $\pi^!(\xi^0(\omega)) \cong \tilde{\xi}(\omega)$.

Proof. We have identified ω with the bundle $pr_2^!(\omega)$ on $\mathbb{S}^m \times \mathbb{C}G(\nu)$ in the statement and will do so in the proof as well.

(i) We shall prove the stronger statement that $\pi^!(\hat{\omega}) \cong \omega_{\mathbb{R}}$, the real vector bundle underlying ω. Let $\hat{\sigma}$ be the σ-conjugate vector bundle morphism of ω that covers $\sigma : \mathbb{C}G(\nu) \to \mathbb{C}G(\nu)$. Recall that the total space $E(\hat{\omega}) = P(\mathbb{S}^m, E(\omega))$ of $\hat{\omega}$ is obtained as the quotient of $\mathbb{S}^m \times E(\omega)$ where (v, e) is identified to $(-v, \hat{\sigma}(e))$. The vector bundle projection $p_{\hat{\omega}} : E(\hat{\omega}) \to P(m, \nu)$ maps $[v, e]$ to $[v, p_\omega(e)]$. We have a commuting diagram where the horizontal maps are quotient maps and the vertical maps are bundle projections:

$$\begin{array}{ccc} \mathbb{S}^m \times E(\omega) & \xrightarrow{\hat{\pi}} & E(\hat{\omega}) \\ {\scriptstyle id \times p_\omega}\downarrow & & \downarrow{\scriptstyle p_{\hat{\omega}}} \\ \mathbb{S}^m \times \mathbb{C}G(\nu) & \xrightarrow{\pi} & P(m, \nu) \end{array} \tag{6.4}$$

Since $\hat{\pi}$ is an $\mathbb{R}$-linear isomorphism on each fibre $v \times E_x(\omega)$, it follows that $\pi^!(\hat{\omega}) \cong \omega_{\mathbb{R}}$.

(ii) We have a bundle involution $\tilde{\theta} : E(\tilde{\xi}(\omega)) \to E(\tilde{\xi}(\omega))$ that covers θ defined as:

$$\tilde{\theta}(e^+ \otimes u, e^- \otimes v) = (\tilde{\theta}^-(e^- \otimes v), \tilde{\theta}^+(e^+ \otimes u)),$$

for $e^+ \otimes u \in E_b(\xi^+ \otimes \omega), e^- \otimes v \in E_b(\eta^- \otimes \bar{\omega})$, $b \in \mathbb{S}^m \times \mathbb{C}G(\nu)$. Note that $\tilde{\theta}$ is fixed point free. So we obtain a vector bundle $\xi^0(\omega)$ over $P(m, \nu)$, whose total space is $E(\tilde{\xi}(\omega))/\langle\tilde{\theta}\rangle$ and we have the following commuting diagram in which the double covering projection $\tilde{\pi}$ is a bundle map:

$$\begin{array}{ccc} E(\tilde{\xi}(\omega)) & \xrightarrow{\tilde{\pi}} & E(\xi^0(\omega)) \\ {\scriptstyle \tilde{p}}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbb{S}^m \times \mathbb{C}G(\nu) & \xrightarrow{\pi} & P(m, \nu) \end{array} \qquad (6.5)$$

Therefore $\pi^!(\xi^0(\omega)) = \tilde{\xi}(\omega)$. □

It is not clear whether there is a *unique* bundle (up to isomorphism) $\xi^0(\omega)$ such that $\pi^!(\xi^0(\omega)) \cong \tilde{\xi}(\omega)$. (However, $\xi_{\mathbb{C}} \otimes \xi^0(\omega) \cong \xi^0(\omega)$; see Lemma 6.2.10(ii) below.) In the sequel, $\xi^0(\omega)$ will always denote the bundle as constructed in the above proof. The following lemma implies, that the class of any other such bundle differs from $[\xi^0(\omega)]$ by a torsion element in $K^0(P(m, \nu))$.

Lemma 6.2.8. *Let $m \geq 1$ be arbitrary. With notations as in Lemma 6.2.5, $\ker \pi^! : K^0(P(m, \nu)) \to K^0(\mathbb{S}^m \times \mathbb{C}G(\nu))$ is precisely the torsion subgroup of $K^0(P(m, \nu))$ and $Im(\pi^!)$ contains $\mathcal{K}$.*

Proof. By the naturality of the Chern character, we have the following commutative diagram:

$$\begin{array}{ccc} K^0(P(m,\nu)) & \xrightarrow{\pi^!} & K^0(\mathbb{S}^m \times \mathbb{C}G(\nu)) \\ {\scriptstyle \mathrm{ch}}\downarrow & & \downarrow{\scriptstyle \mathrm{ch}} \\ H^*(P(m,\nu);\mathbb{Q}) & \xrightarrow{\pi^*} & H^*(\mathbb{S}^m \times \mathbb{C}G(\nu);\mathbb{Q}) \end{array} \tag{6.6}$$

Since $\mathrm{ch} : K^0(\mathbb{S}^m \times \mathbb{C}G(\nu)) \to H^*(\mathbb{S}^m \times \mathbb{C}G(\nu);\mathbb{Q})$ is a monomorphism, and since $\pi^* : H^*(P(m,\nu);\mathbb{Q}) \to H^*(\mathbb{S}^m \times \mathbb{C}G(\nu);\mathbb{Q})$ is a monomorphism, $\ker \pi^!$ equals the kernel of $\mathrm{ch} : K^0(P(m,\nu)) \to H^*(P(m,\nu);\mathbb{Q})$, which is precisely the torsion subgroup of $K^0(P(m,\nu))$.

It remains to show that $\mathcal{K} \subset Im(\pi^!)$. Recall that, by definition, $\delta_m = 0$ when m is odd. Since $\mathcal{K}$ is generated as a subring by $x + \bar{x}, \delta_m(x - \bar{x})$ where x varies in $K^0(\mathbb{C}G(\nu))$, we need only to show that $x + \bar{x}$ and $\delta_m(x - \bar{x})$ are in $Im(\pi^!)$.

By Lemma 6.2.3, any $x \in \widetilde{K}(\mathbb{C}G(\nu))$ may be expressed $[\omega] - d$ where ω is a σ-conjugate complex vector bundle and $d = \mathrm{rank}(\omega)$. Thus $\theta^!(x) = \sigma^!(x) = [\bar{\omega}] - d = \bar{x}$. Consider the complex vector bundle $\hat{\omega}_{\mathbb{C}} = \hat{\omega} \otimes_{\mathbb{R}} \mathbb{C}$, the complexification of $\hat{\omega}$. We have $\pi^!([\hat{\omega}_{\mathbb{C}}] - 2d) = [\omega] + [\bar{\omega}] - 2d = x + \bar{x}$. Thus $x + \bar{x} \in Im(\pi^!)$. This proves that $\mathcal{K} \subset Im(\pi^!)$ when m is odd.

Let $m \equiv 0 \pmod 2$. Recall from Remark 2.5.2 that $[\xi^-] = [\eta^-]$ in $K(\mathbb{S}^m)$ and so $\delta_m = [\xi^+] - 2^{r-1} = -([\eta^-] - 2^{r-1})$ in $K^0(\mathbb{S}^m \times \mathbb{C}G(\nu))$. Therefore $[\tilde{\xi}(\omega)] = [\xi^+][\omega] + [\eta^-][\bar{\omega}] = \delta_m([\omega] - [\bar{\omega}]) + 2^{r-1}([\omega] + [\bar{\omega}])$. By Lemma 6.2.7, $[\tilde{\xi}(\omega)] \in Im(\pi^!)$. It follows that $\delta_m(x - \bar{x}) = \delta_m \cdot ([\omega] - [\bar{\omega}]) \in Im(\pi^!)$ since $[\omega] + [\bar{\omega}] \in Im(\pi^!)$. Hence $\mathcal{K} \subset Im(\pi^!)$. $\square$

Recall that $\xi_{\mathbb{C}}$ denotes the complexification of the Hopf line bundle ξ over $\mathbb{R}P^m$. We shall denote by the same symbol ξ (resp. $\xi_{\mathbb{C}}$) the bundle $p^!(\xi)$ (resp.$p^!(\xi_{\mathbb{C}})$) over $P(m,\nu)$.

Definition 6.2.9. *Let $m \geq 1$. With notations as above, define $\mathcal{K}^0 \subset K^0(P(m,\nu))$*

to be the subring generated by the following elements:

(i) $[\xi_{\mathbb{C}}]$, $[\hat{\omega}_{\mathbb{C}}]$, *and, (ii) when* m *is even,* $[\xi^0(\omega)]$,

where ω *varies over* σ*-conjugate complex vector bundles over* $\mathbb{C}G(\nu)$.

We have the following proposition which gives partial information on the multiplicative structure of $K^0(P(m,\nu))$.

Proposition 6.2.10. *Let* $m \geq 1$. *The following relations hold in* $K^0(P(m,\nu))$ *for any* σ*-conjugate vector bundles* ω_1, ω_2 *and* ω *over* $\mathbb{C}G(\nu)$.

(i) $([\xi_{\mathbb{C}}]-1)[\hat{\omega}_{\mathbb{C}}] = 0$, *and, the (additive) order of* $([\xi_{\mathbb{C}}]-1)$ *equals* $2^{\lfloor m/2 \rfloor}$.

Furthermore, when m *is even, we have*

(ii) $([\xi_{\mathbb{C}}]-1)[\xi^0(\omega)] = 0$, $[\xi^0(\omega_1)] + [\xi^0(\omega_2)] = [\xi^0(\omega_1 \oplus \omega_2)]$;

(iii) $[\xi^0(\omega)] + [\xi^0(\bar{\omega})] \equiv 2^r[\hat{\omega}_{\mathbb{C}}]$ *modulo torsion;*

(iv) $[\xi^0(\omega_1)][\xi^0(\omega_2)] \equiv 2^r[\xi^0(\omega_1 \otimes \omega_2)] - 2^{2r-2}([\omega_1]-[\bar{\omega}_1])([\omega_2]-[\bar{\omega}_2])$ *modulo torsion.*

Proof. (i) The assertion that $([\xi_{\mathbb{C}}]-1)[\hat{\omega}_{\mathbb{C}}] = 0$ holds since $\hat{\omega} \otimes \xi \cong \hat{\omega}$. (See [NS19, Lemma 2.7].) That $2^{\lfloor m/2 \rfloor}$ equals the additive order of $([\xi_{\mathbb{C}}]-1) = 0$ follows readily from the work of Adams [Ada62] in view of the fact that the projection $p : P(m,\nu) \to \mathbb{R}P^m$ induces a monomorphism $p^! : K^0(\mathbb{R}P^m) \to K^0(P(m,\nu))$ since p admits a cross-section.

(ii) The second assertion of (ii) is readily seen to be valid. We shall construct a bundle isomorphism $\tilde{\lambda} : \epsilon_{\mathbb{C}} \otimes \tilde{\xi}(\omega) \to \tilde{\xi}(\omega)$ that covers $\theta : \mathbb{S}^m \times \mathbb{C}G(\nu) \to \mathbb{S}^m \times \mathbb{C}G(\nu)$ and such that it descends to yield a bundle isomorphism $\xi_{\mathbb{C}} \otimes \xi^0(\omega) \cong \xi^0(\omega)$ on $P(m,\nu)$. This readily implies the first assertion of (ii).

We have $E(\xi_{\mathbb{C}}) = \mathbb{S}^m \times \mathbb{C}G(\nu) \times \mathbb{C}/\langle \mu \rangle$ where $\rho : \epsilon_{\mathbb{C}} \to \epsilon_{\mathbb{C}}$ is the involutive bundle isomorphism that covers θ defined as $\rho(b;t) = (\theta(b); -t)$ $\forall b \in \mathbb{S}^m \times \mathbb{C}G(\nu), t \in \mathbb{C}$. We shall denote by $[b;t] \in E_{[b]}(\xi_{\mathbb{C}})$ the image of $(b;t) \in E_b(\epsilon_{\mathbb{C}})$ under bundle map $\epsilon_{\mathbb{C}} \to \xi_{\mathbb{C}}$ covering the projection $\mathbb{S}^m \times \mathbb{C}G(\nu) \to P(m,\nu)$. Thus $[b;t] = [\theta(b); -t]$.

The total space $E(\xi^0(\omega))$ of $\xi^0(\omega)$ was described in the course of the proof of Lemma 6.2.7 as $E(\tilde{\xi})/\langle\tilde{\theta}\rangle$. We shall denote by $(b; x, y) \in E_b(\tilde{\xi}(\omega))$ the element $(x, y) \in E_b(\xi^+ \otimes \omega) \oplus E_b(\eta^- \otimes \bar{\omega})$ and by $[b; x, y]$ its image in $E_{[b]}(\xi^0(\omega))$ under the projection $E(\tilde{\xi}(\omega)) \to E(\xi^0(\omega))$. Thus $[b; x, y] = [\theta(b); \tilde{\theta}^-(y), \tilde{\theta}^+(x)]$.

The total space of $\xi_{\mathbb{C}} \otimes \xi^0(\omega)$ has the following description: The fibre over $[b] \in P(m, \nu)$ is the vector space $E_{[b]}(\xi_{\mathbb{C}}) \otimes E_{[b]}(\xi^0(\omega))$. Let $t \in \mathbb{C}, x \in E_b(\xi^+ \otimes \omega), y \in E_b(\eta^- \otimes \bar{\omega})$ so that $[b; t] \in E_{[b]}(\xi_{\mathbb{C}}), [b; x, y] \in E_{[b]}(\xi^0(\omega))$. Then the vector $[b; t] \otimes [b; x, y] \in E_{[b]}(\xi_{\mathbb{C}} \otimes \xi^0(\omega)) = E_{[b]}(\xi_{\mathbb{C}}) \otimes E_{[b]}(\xi^0(\omega))$ will be denoted $[(b; t) \otimes (x, y)]$. Note that

$$[(b; t) \otimes (x, y)] = [(\theta(b); -t) \otimes (\tilde{\theta}^-(y), \tilde{\theta}^+(x))]. \tag{6.7}$$

Consider the bundle map $\tilde{\lambda} : E(\epsilon_{\mathbb{C}} \otimes \tilde{\xi}(\omega)) \to E(\tilde{\xi}(\omega))$ defined as follows: For $b \in \mathbb{S}^m \times \mathbb{C}G(\nu), t \in \mathbb{C}, x \in E_b(\xi^+ \otimes \omega), y \in E_b(\eta^- \otimes \bar{\omega})$,

$$\tilde{\lambda}((b; t) \otimes (b; x, y)) = (b; -tx, ty). \tag{6.8}$$

Then $\tilde{\lambda}$ is a complex vector bundle isomorphism that covers θ. Moreover,

$$\begin{aligned} \tilde{\lambda} \circ (\rho \otimes \tilde{\theta})((b; t) \otimes (b; x, y)) &= \tilde{\lambda}((\theta(b); -t) \otimes (\theta(b); \tilde{\theta}^-(y), \tilde{\theta}^+(x))) \\ &= (\theta(b); t\tilde{\theta}^-(y), -t\tilde{\theta}^+(x)) \\ &= \tilde{\theta}(b; -tx, ty) \\ &= \tilde{\theta}(\tilde{\lambda}((b; t) \otimes (b; x, y))) \end{aligned}$$

Therefore $\tilde{\lambda}$ yields a well-defined bundle isomorphism $\lambda : \xi_{\mathbb{C}} \otimes \xi^0(\omega) \to \xi^0(\omega)$, covering the identity map of $P(m, \nu)$, defined as $[(b; t) \otimes (x, y)] \mapsto [b; -tx, ty]$. This shows that $\xi_{\mathbb{C}} \otimes \xi^0(\omega) \cong \xi^0(\omega)$.

(iii) In view of the fact that $\ker(\pi^!)$ equals the torsion subgroup of $K^0(P(m, \nu))$, it suffices to show that $\pi^!([\xi^0(\omega)] + [\xi^0(\bar{\omega})]) = 2^r \pi^!([\hat{\omega}_{\mathbb{C}}])$. Since $\pi^!(\hat{\omega}_{\mathbb{C}}) = ([\omega] + [\bar{\omega}])$ by

Lemma 6.2.7 and since $\pi^!(\xi^0(\omega)) = \tilde{\xi}(\omega)$, we need only to show that $[\tilde{\xi}(\omega)]+[\tilde{\xi}(\bar{\omega})] = 2^r([\omega] + [\bar{\omega}])$. It is clear that $\tilde{\xi}(\omega) \oplus \tilde{\xi}(\bar{\omega}) = (\xi^+ \oplus \eta^-) \otimes (\omega \oplus \bar{\omega}) \cong 2^r(\omega \oplus \bar{\omega})$ since we have a complex vector bundle isomorphism $\xi^+ \oplus \eta^- \cong \xi^+ \oplus \xi^- \cong 2^r\epsilon_{\mathbb{C}}$ on $\mathbb{S}^m$.

(iv) We have

$$\begin{aligned} \pi^!(\xi^0(\omega_1) \otimes \xi^0(\omega_2)) &= \tilde{\xi}(\omega_1) \otimes \tilde{\xi}(\omega_2) \\ &= (\tilde{\xi^+} \otimes \omega_1 + \eta^- \otimes \bar{\omega}_1) \otimes (\tilde{\xi^+} \otimes \omega_2 + \eta^- \otimes \bar{\omega}_2) \\ &= (\xi^+)^2 \otimes (\omega_1 \otimes \omega_2) + (\eta^-)^2 \otimes (\bar{\omega}_1 \otimes \bar{\omega}_2) + \xi^+\eta^- \cdot (\omega_1 \otimes \bar{\omega}_2 \oplus \bar{\omega}_1 \otimes \omega_2). \end{aligned}$$

Now by Theorem 2.5.1 and Remark 2.5.2, $[\xi^+]^2 = 2^r[\xi^+]-2^{2r-2}$, $[\eta^-]^2 = 2^r[\eta^-]-2^{2r-2}$ and $[\xi^+\eta^-] = 2^{2r-2}$. Substituting in the above equation and using $\tilde{\xi}(\omega) = \xi^+ \otimes \omega \oplus \eta^- \otimes \bar{\omega}$ (with $\omega = \omega_1 \otimes \omega_2$) we obtain:

$$\begin{aligned} \pi^!([\xi^0(\omega_1) \otimes \xi^0(\omega_2)]) &= 2^r([\tilde{\xi}(\omega_1 \otimes \omega_2)]) - 2^{2r-2}([\omega_1 \otimes \omega_2] + [\bar{\omega}_1 \otimes \bar{\omega}_2]) \\ &\quad +2^{2r-2}([\omega_1 \otimes \bar{\omega}_2] + [\bar{\omega}_1 \otimes \omega_2]) \\ &= 2^r\pi^!([\xi^0(\omega_1 \otimes \omega_2)] - 2^{2r-2}([\omega_1] - [\bar{\omega}_1])([\omega_2] - [\bar{\omega}_2]). \end{aligned}$$

Since $\ker(\pi^! : K(P(m,\nu)) \to K(\mathbb{S}^m \times \mathbb{C}G(\nu)))$ consists only of torsion elements, our assertion follows. □

Theorem 6.2.11. *Let $m \geq 1$ and let $r = \lfloor m/2 \rfloor$. The homomorphism $\pi^!$: $K^0(P(m,\nu)) \to K^0(\mathbb{S}^m \times \mathbb{C}G(\nu))$ defines a surjective ring homomorphism $\mathcal{K}^0 \to \mathcal{K}$, again denoted $\pi^!$, whose kernel equals the torsion ideal $\mathcal{T}^0 \subset \mathcal{K}^0$. The quotient group $K^0(P(m,\nu))/\mathcal{K}^0$ is a finite abelian 2-group.*

Proof. We have $\pi^!([\hat{\omega}_{\mathbb{C}}]) = [\omega]+[\bar{\omega}] \in \mathcal{K}$, and, when m is even, $\pi^!([\xi^0(\omega)]) = [\tilde{\xi}(\omega)] = [\xi^+][\omega] + [\eta^-][\omega] = [\xi^+]([\omega - \bar{\omega}]) + 2^r[\bar{\omega}] = \delta_m([\omega] - [\bar{\omega}]) + 2^{r-1}\pi^!([\hat{\omega}_{\mathbb{C}}])$. Therefore $\delta_m([\omega] - [\bar{\omega}])$ is in $\pi^!(\mathcal{K}^0)$. This shows that the homomorphism $\pi^! : \mathcal{K}^0 \to \mathcal{K}$ is surjective.

Since, by Lemma 6.2.8, the kernel of $\pi^! : K^0(P(m,\nu)) \to K^0(\mathbb{S}^m \times \mathbb{C}G(\nu))$

consists only of torsion elements, it follows that the same is true of $\mathcal{K}^0 \to \mathcal{K}$. By the same lemma, $\mathrm{rank}(\mathcal{K}) = \mathrm{rank}(K^0(P(m,\nu)))$, and since $\mathcal{K}^0 \subset K^0(P(m,\nu))$, it follows from the surjectivity of $\mathcal{K}^0 \to \mathcal{K}$ that $\mathrm{rank}(\mathcal{K}^0) = \mathrm{rank}(K^0(P(m,\nu))$. Therefore $K^0(P(m,\nu))/\mathcal{K}^0$ is a torsion group. Since $H^*(P(m,\nu);\mathbb{Z})$ has no odd torsion, it follows (for example by using the Atiyah-Hirzebruch spectral sequence) that $K^0(P(m,\nu))$ has no odd torsion. This proves the last assertion of the theorem.

□

Remark 6.2.12. *It is clear that $\mathbb{Z}_{2^r}y$ is contained in the torsion subgroup of $K^0(P(m,\nu))$. We have not been able to determine the torsion part of $K^0(P(m,\nu))$. This seems to us to be a rather difficult, albeit an interesting problem.*

Chapter 7

Applications

In this chapter, we apply our results on cohomology of generalized Dold spaces obtained in the earlier chapters, to mod-2 equivariant cohomology of certain (X, σ), fixed-point of self-maps of $P(\mathbb{S}^m, \mathbb{C}G_{n,k})$ and degrees of maps between $P(\mathbb{S}^m, \mathbb{C}G_{n,k})$ and $P(\mathbb{S}^r, \mathbb{C}G_{s,t})$ (under some restrictions).

7.1 Equivariant cohomology of (X, σ)

As an application of Theorems 4.2.4, 4.3.4 we obtain the $\mathbb{Z}_2$-equivariant cohomology $H^*_{\mathbb{Z}_2}(X; \mathbb{Z}_2)$ of (X, σ) when X is either a torus manifold whose torus quotient is a homology polytope, or, is a complex flag manifold F_ν. At least in the case of complex flag manifolds, this result is perhaps known to experts but we could not find an explicit reference.

When $S = \mathbb{S}^\infty$ with antipodal action, the space $P(\mathbb{S}^\infty, X)$ is identical to the Borel construction $\mathbb{S}^\infty \times_{\mathbb{Z}_2} X$ (since $\mathbb{S}^\infty$ is contractible). Therefore the equivariant cohomology algebra $H^*_{\mathbb{Z}_2}(X; \mathbb{Z}_2)$ equals $H^*(P(\mathbb{S}^\infty, X); \mathbb{Z}_2)$. When $H^*(X; \mathbb{Z}_2)$ is generated by mod 2 reduction of Chern classes of finitely many σ-conjugate vector bundles (ω_j, σ_j), Proposition 3.1.2 is applicable and we obtain that $H^*_{\mathbb{Z}_2}(X; \mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2[y] \otimes H^*(X;\mathbb{Z}_2)$ as a $H^*(\mathbb{R}P^\infty;\mathbb{Z}_2) = \mathbb{Z}_2[y]$-module. The inclusion $\mathbb{S}^m \hookrightarrow \mathbb{S}^\infty$ induces an inclusion $P(\mathbb{S}^m, X) \hookrightarrow P(\mathbb{S}^\infty, X)$ which is an $(m-1)$-equivalence. It follows that $H^i_{\mathbb{Z}_2}(X;\mathbb{Z}_2) \cong H^i(P(\mathbb{S}^\infty, X);\mathbb{Z}_2) \to H^i(P(\mathbb{S}^m, X);\mathbb{Z}_2)$ induced by inclusion is an isomorphism for all $i < m$. Therefore $H^*_{\mathbb{Z}_2}(X;\mathbb{Z}_2)$ is isomorphic to the inverse limit of graded $\mathbb{Z}_2$-algebras $\{H^*(P(\mathbb{S}^m, X);\mathbb{Z}_2)\}_{m\geq 2}$. As an illustration, we obtain the following result as an immediate consequence of Theorems 4.2.4 and 4.3.4.

Theorem 7.1.1. *We keep the above notations from §4.2.*
(i) Let $X = X(Q,\Lambda)$ be a T-torus manifold where $Q = X/T$ is a homology polytope. Then $H^*_{\mathbb{Z}_2}(X;\mathbb{Z}_2)$ is isomorphic to the A-algebra $R(Q,\Lambda)$ where $A = H^*(\mathbb{R}P^\infty;\mathbb{Z}_2) \cong \mathbb{Z}_2[y]$.
(ii) Let $\nu = n_1 \leq \cdots < n_s, n = \sum n_j$. Then $H^*_{\mathbb{Z}_2}(F_\nu;\mathbb{Z}_2)$ is isomorphic to $\mathcal{R}_\nu/\mathcal{I}_\nu$ where $\mathcal{R}_\nu = A[\hat{w}_{2i,j}; 1 \leq i \leq n_j, 1 \leq j < s]$ where $A = H^*(\mathbb{R}P^\infty;\mathbb{Z}_2) = \mathbb{Z}_2[y]$ and $\mathcal{I}_\nu \subset \mathcal{R}_\nu$ is the ideal generated by the $a_{2i} \in \mathcal{R}_\nu, n_s < i \leq n$. □

7.2 Fixed-points of self maps of $P(\mathbb{S}^m, \mathbb{C}G_{n,k})$

A topological space X is said to have fixed-point property if for any continuous map f on X, there exists $x \in X$ such that $f(x) = x$. Here we shall explore the fixed-point property of generalized Dold spaces. We shall make use of our earlier results on cohomology of generalized Dold spaces to deduce some conditions such that continuous maps satisfying them have fixed-points. Let us begin by the following result which gives a necessary criterion for a generalized Dold space to have fixed-point property.

Proposition 7.2.1. *A generalized Dold space $P(S,\alpha,X,\sigma)$ does not have fixed-point property if any of the following holds:*
(i) $Y = S/\!\sim_\alpha$ does not have the fixed-point property.

(ii) There exists a map $f : X \to X$ having no fixed-point and $f \circ \sigma = \sigma \circ f$.

Proof. (i) Let $g : Y \to Y$ be fixed-point free. Then $\phi := s \circ g \circ p$ on $P(S, X)$ also has no fixed-points, where p is the X-bundle projection and s is a section.

(ii). Suppose that $f : X \to X$ has no fixed-points and that $f \circ \sigma = \sigma \circ f$. Now define $\psi : P(S, X) \to P(S, X)$ as $[s, x] \mapsto [s, f(x)]$. The well-definedness follows because $\psi([\alpha(s), \sigma(x)]) = [\alpha(s), f \circ \sigma(x)] = [\alpha(s), \sigma \circ f(x)] = [s, f(x)] = \psi([s, x])$. Clearly, ψ has no fixed-points. This completes the proof. □

Remark 7.2.2. *(i) Let $P(m, n)$ be a classical Dold manifold such that at least one of m and n is odd. Then $P(m, n)$ does not have fixed-point property.*

Proof. When m is odd, the proof immedietly follows from the above proposition. When n is odd, we can construct a map $f : \mathbb{C}P^n \to \mathbb{C}P^n$ defined by

$$[x_1 : x_2 : \cdots : x_n, x_{n+1}] \mapsto [x_2 : -x_1 : \cdots : x_{n+1}, -x_n]$$

which does not have any fixed-point. Clearly, $f \circ \sigma = \sigma \circ f$, where σ is the involution on $\mathbb{C}P^n$ induced from complex conjugation on $\mathbb{C}^n$. Thus, again using Proposition 7.2.1, $P(m, n)$ does not have fixed-point property. □

Now we provide a criterion for the existence of a fixed-point of a self map on the generalized Dold manifolds $P(m, n, k) := \mathbb{S}^m \times \mathbb{C}G_{n,k}/\sim$, where $(x, \mathbf{L}) \sim (-x, \bar{\mathbf{L}})$. We have Proposition 7.2.4 as an application of Theorem 5.4.2.

In order to prove the Proposition 7.2.4, we need the following known result due to Glover and Homer (see [GH78]).

Theorem 7.2.3 (Glover-Homer). *Assume that either (i) $k \leq 3$ and $n > 2k$, or (ii) $k > 3$ and $n > 2k^2 - 1$ holds. Then every graded ring homomorphism φ of $H^*(\mathbb{C}G_{n,k}; \mathbb{Z})$ is an Adams map. i.e., there exists an integer λ such that $\varphi(c_i) = \lambda^i c_i$,* *where c_i denotes the i-th Chern class of the canonical complex k-plan bundle $\gamma_{n,k}$ over $\mathbb{C}G_{n,k}$.*

Proposition 7.2.4. *Let m be odd and either (i) $k \leq 3$ and $n > 2k$, or (ii) $k > 3$ and $n > 2k^2 - 1$ hold. Then for any continuous map $f : P(m, n, k) \to P(m, n, k)$ with the degree $\deg(p \circ f \circ s) \neq 1$, there exists a fixed-point.*

Note that the generalized Dold manifolds considered in the above statement does not have fixed-point property, using the fact that $\mathbb{R}P^m$ has fixed-point property if and only if m is even and the Proposition 7.2.1.

Proof. Observe that the map $f \circ \pi$ has a lift $\tilde{f}$ to $\mathbb{S}^m \times \mathbb{C}G_{n,k}$ for the double covering $\pi : \mathbb{S}^m \times \mathbb{C}G_{n,k} \to P(m, n, k)$, since $f_* \circ \pi_*(\pi_1(\mathbb{S}^m \times \mathbb{C}G_{n,k})) \subseteq \pi_*(\pi_1(\mathbb{S}^m \times \mathbb{C}G_{n,k}))$.

Recall from Theorem 5.4.2, setting $\nu = (k, n-k)$ and $\gamma_1 = \gamma_{n,k}, \gamma_2 = \beta_{n,k}$, that $\pi^* : H^*(P(m, n, k); \mathbb{Q}) \to \mathrm{Fix}H^*(\theta; \mathbb{Q}))$ is an isomorphism, where $\mathrm{Fix}H^*(\theta; \mathbb{Q})) \subset H^*(\mathbb{S}^m \times \mathbb{C}G_{n,k}; \mathbb{Q}) \cong H^*(\mathbb{S}^m; \mathbb{Q}) \otimes H^*(\mathbb{C}G_{n,k}; \mathbb{Q})$. Note that $\mathrm{Fix}H^*(\theta; \mathbb{Q})) \cong H^*(\mathbb{S}^m; \mathbb{Q}) \otimes \mathrm{Fix}H^*(\sigma; \mathbb{Q}))$. Then we have the following commutative diagram:

$$
\begin{array}{ccccc}
H^*(P(m,n,k);\mathbb{Q}) & \xrightarrow{\pi^*} & \mathrm{Fix}H^*(\theta;\mathbb{Q})) & \longrightarrow & H^*(\mathbb{S}^m \times \mathbb{C}G_{n,k};\mathbb{Q}) \\
\downarrow f^* & & \downarrow \tilde{f}^* & & \downarrow \tilde{f}^* \\
H^*(P(m,n,k);\mathbb{Q}) & \xrightarrow{\pi^*} & \mathrm{Fix}H^*(\theta;\mathbb{Q})) & \longrightarrow & H^*(\mathbb{S}^m \times \mathbb{C}G_{n,k};\mathbb{Q}).
\end{array}
\tag{7.1}
$$

Since m is odd and $H^{od}(\mathbb{C}G_{n,k}; \mathbb{Q}) = \bigoplus_{q \geq 0} H^{2q+1}(\mathbb{C}G_{n,k}; \mathbb{Q}) = 0$, because of degree reasons, there exist two graded endomorphisms $\varphi_1 : H^*(\mathbb{S}^m; \mathbb{Q}) \to H^*(\mathbb{S}^m; \mathbb{Q})$ and $\varphi_2 : H^*(\mathbb{C}G_{n,k}; \mathbb{Q}) \to H^*(\mathbb{C}G_{n,k}; \mathbb{Q})$ such that $\tilde{f}^* = \varphi_1 \otimes \varphi_2$. We regard f^* as $\varphi_1 \otimes \varphi_2$ restricted to $\mathrm{Fix}H^*(\theta; \mathbb{Q}))$.

Now Theorem 7.2.3 ensures the existence of an integer λ such that $\varphi_2(c_i) = \lambda^i c_i$, where c_i denotes the i-th Chern class of the canonical complex k-plane bundle $\gamma_{n,k}$ over $\mathbb{C}G_{n,k}$. Also there exist an integer μ such that $\varphi_1(u_m) = \mu u_m$, where u_m denotes a generator of $H^m(\mathbb{S}^m; \mathbb{Z})$. Note that $\deg(p \circ f \circ s) = \mu$ and so, $\mu \neq 1$.

The diagram 7.1 and the fact that $\pi_* : H^*(P(m,n,k);\mathbb{Q}) \to \mathrm{Fix}H^*(\theta;\mathbb{Q}))$ is an isomorphism, the Lefschetz numbers computed for f^* and the restriction of $\tilde{f}^*$ to $\mathrm{Fix}H^*(\theta;\mathbb{Q}))$ are equal. Let us denote $\mathrm{Fix}H^i(\theta,\mathbb{Q}))$ by T^i and $\mathrm{Fix}H^i(\sigma;\mathbb{Q}))$ by F^i. Thus the Lefschetz number of f:

$$\begin{aligned}
L(f) &= \sum(-1)^i tr(f^* : H^i(P(m,n,k);\mathbb{Q}) \to H^i(P(m,n,k);\mathbb{Q})) \\
&= \sum(-1)^i tr(\tilde{f}^* : \mathrm{Fix}H^i(\theta;\mathbb{Q})) \to \mathrm{Fix}H^i(\theta;\mathbb{Q}))) \\
&= \textstyle\sum_{i\geq 0} tr(\tilde{f}^*|_{T^{2i}}) - \sum_{i\geq 0} tr(\tilde{f}^*|_{T^{2i+1}}) \\
&= \textstyle\sum_{i\geq 0} tr(\varphi_2|_{F^{2i}}) - \mu \sum_{i\geq 0} tr(\varphi_2|_{F^{2i}}) \\
&\quad [\because \text{since } m \text{ is odd and } \varphi_1(u_m) = \mu u_m] \\
&= (1-\mu)\textstyle\sum_{i\geq 0} tr(\varphi_2|_{F^{2i}}) \\
&= (1-\mu)\textstyle\sum_{i\geq 0} d_{4i}\lambda^{2i} \text{ [where } d_{4i} \text{ is the dimension of } H^{4i}(\mathbb{C}G_{n,k};\mathbb{Q}) \text{]} \\
&\neq 0 \; [\because \mu = \deg(p\circ f\circ s) \neq 1].
\end{aligned}$$

This shows that the Lefschetz number of f is nonzero and consequently, f has a fixed-point. This completes the proof. □

7.3 Degrees of maps between certain generalized Dold manifolds

Here we see some applications of the results on cohomology of GDS on the degrees of maps between two oriented generalized Dold manifolds of the same dimension. Recall that $P(m,n,k) = P(\mathbb{S}^m, \mathbb{C}G_{n,k}) := \mathbb{S}^m \times \mathbb{C}G_{n,k}/\sim$, where $(x,\mathbf{L}) \sim (-x,\bar{\mathbf{L}})$. We have the following proposition as an application of Theorem 5.4.2.

Proposition 7.3.1. *Let $f : P(m,n,k) \to P(r,s,t)$ be a continuous map between two oriented and same-dimensional generalized Dold manifolds. Further, assume that (i) $\lfloor k/2\rfloor < \lfloor t/2\rfloor$ if $m = r$ is odd, and $k < t$ if $m = r$ is even,*
(ii) $\mathbb{C}G_{s,t} \neq \mathbb{C}G_{2,1} = \mathbb{C}P^1$ if $r \neq m = 2$.

Then the degree of f is zero.

The proof of Proposition 7.3.1 is divided into two following lemmas, namely Lemma 7.3.3 and Lemma 7.3.5, depending on the cases (i) $m \neq r$ and (ii) $m = r$.

Notations: Let us fix some notations before proceeding further. We denote the canonical complex k-plane bundle of a complex Grassmannian manifold $\mathbb{C}G_{n,k}$ by $\gamma_{n,k}$. Denote the i-th Chern classs of $\gamma_{n,k}$ and $\gamma_{s,t}$ by c_i and d_i respectively. Let u_m denote a generator of $H^m(\mathbb{S}^m;\mathbb{Q})$. Define the degrees of c_i, d_i to be $2i$ and the degree of u_m to be m. Denote both the coverings $\mathbb{S}^m \times \mathbb{C}G_{n,k} \to P(m,n,k)$ and $\mathbb{S}^r \times \mathbb{C}G_{s,t} \to P(r,s,t)$ by π. Also, we shall use the same symbol θ to denote the involutions on $\mathbb{S}^m \times \mathbb{C}G_{n,k}$ and $\mathbb{S}^r \times \mathbb{C}G_{s,t}$.

It is well-known that $H^*(\mathbb{C}G_{n,k};\mathbb{Z}) \cong \mathbb{Z}[c_1, c_2, \ldots, c_k]/\mathcal{I}$, where $\mathcal{I}$ is generated by the relations $f_l, n-k+1 \leq l \leq n$ for some homogeneous polynomials f_l in $c_1, c_2, ..., c_k$ of degree $2l$, considering the degrees of c_i to be $2i$. Now we make the following remark for future usage.

Remark 7.3.2. *There is no algebraic relation among $c_1, c_2, \ldots, c_k$ up to degree $2(n-k)$ in $\mathbb{Z}[c_1, c_2, \ldots, c_k]/\mathcal{I} \cong H^*(\mathbb{C}G_{n,k};\mathbb{Z})$, where degree of c_i is $2i$. Similar observation hold for $\mathbb{C}G_{s,t}$.*

In many places of the following proofs, we will use the fact that for a continuous map $f : M \to N$ between two closed, oriented manifolds of the same dimension, with $\deg(f) \neq 0$, the induced map on rational cohomology $f^* : H^*(N;\mathbb{Q}) \to H^*(M;\mathbb{Q})$ is a monomorphism.

It is known that $c_1^{k(n-k)} \neq 0$ in $H^*(\mathbb{C}G_{n,k};\mathbb{Q})$, where c_1 denotes the first Chern class of $\gamma_{n,k}$. See [§15, [Hir95]].

Lemma 7.3.3. *Consider a continuous map $f : P(m,n,k) \to P(r,s,t)$ where the generalized Dold manifolds are oriented and of the same dimension. Further assume that (i) $m \neq r$, and, (ii) $(m, 2t(s-t)) \neq (2,2)$. Then the degree of f is zero.*

Proof. Recall that the fundamental group of $P(m,n,k)$ isomorphic to $\mathbb{Z}$ if $m=1$ and $\mathbb{Z}_2$ if $m>1$. Using this and the fact that $\pi_*:\pi_1(\mathbb{S}^m\times\mathbb{C}G_{n,k})\to\pi_1(P(m,n,k))$ is a monomorphism, it is not hard to see that $(f\circ\pi)_*(\pi_1(\mathbb{S}^m\times\mathbb{C}G_{n,k}))\subset\pi_*(\pi_1(\mathbb{S}^r\times\mathbb{C}G_{s,t}))$. It follows that f can be lifted to $\tilde{f}:\mathbb{S}^m\times\mathbb{C}G_{n,k}\to\mathbb{S}^r\times\mathbb{C}G_{s,t}$ such that $\pi\circ f=\tilde{f}\circ\pi$. This implies $deg(f)=deg(\tilde{f})$, since $\deg(\pi)=2$. Thus it is enough to show that $\deg(\tilde{f})=0$.

Observe that $\tilde{f}^*(d_1)=au_m+bc_1$ for some $a,b\in\mathbb{Q}$ where $a=0$ if $m\neq 2$.

It is important to note that if $k(n-k),t(s-t)$ differ by an odd number, then one of the generalized Dold manifolds $P(m,n,k)$ and $P(r,s,t)$ is non-orientable. This follows from the fact that $\dim P(m,n,k)=m+2k(n-k)=r+2t(s-t)=\dim P(r,s,t)$ and $P(m,n,k)$ is orientable if and only if $m+k(n-k)$ is odd (see Remark 3.2.7 (ii)). Thus $m_1\neq m_2$ implies $|k(n-k)-t(s-t)|\geq 2$. Define $n_1:=k(n-k)$ and $n_2:=t(s-t)$. It is well-known that $0\neq c_1^{n_1}\in H^*(\mathbb{C}G_{n,k};\mathbb{Q})$ and $0\neq d_1^{n_2}\in H^*(\mathbb{C}G_{s,t};\mathbb{Q})$ (see [§15, [Hir95]]).

Case 1: $\underline{n_2-n_1\geq 2}$. Using the notations above, we have

$$0\neq d_1^{n_2}\mapsto\tilde{f}^*(d_1^{n_2})=n_2ab^{n_2-1}c_1^{n_2-1}u_m+b^{n_2}c_1^{n_2}=0,$$

since $n_2-n_1\geq 2$ and $u_m^2=0$. Consequently, $\tilde{f}^*$ is not a monomorphism.

Case 2: $\underline{n_1-n_2\geq 2}$. In this case, $\tilde{f}^*(d_1)=au_m+bc_1$ ($a=0$ unless $m=2$) implies

$$0=d_1^{n_1}\mapsto\tilde{f}^*(d_1^{n_1})=n_1ab^{n_1-1}c_1^{n_1-1}u_m+b^{n_1}c_1^{n_1},$$

using $n_1-n_2\geq 2$ and $u_m^2=0$. Since $c_1^{n_1},c_1^{n_1-1}u_m$ are $\mathbb{Q}$-linearly independent in $H^*(\mathbb{S}^m\times\mathbb{C}G_{n,k};\mathbb{Q})$, the right-hand side of above is zero implies $b=0,ab=0$.
If $a=0$, $\tilde{f}^*(d_1)=0$ and so $\tilde{f}^*$ is not a monomorphism.
If $a\neq 0$, $\tilde{f}^*(d_1)=au_2$. Now we have two cases: (a) $d_1^2\neq 0$ and (b) $d_1^2=0$.
In case (a), $0\neq d_1^2\mapsto\tilde{f}^*(d_1^2)=a^2u_2^2=0$ implies $\tilde{f}^*$ is not a monomorphism. In case

(b), $d_1^2 = 0$ implies $\mathbb{C}G_{s,t} = \mathbb{C}G_{2,1}$; and consequently, $(2, 2t(s-t)) = (2,2)$ which is not possible because of hypothesis (ii).

Therefore, $\tilde{f}^*$ is not a monomorphism and hence, $deg(\tilde{f}) = deg(f) = 0$. □

Remark 7.3.4. *The hypothesis (ii) in the above lemma may not be necessary. In fact, if $(m, t(s-t)) = (2,2)$, then we have $f : P(2,n,k) \to P(r,2,1)$. The equality of the dimensions of $P(2,n,k)$ and $P(r,2,1)$ implies $r = 2k(n-k)$. Additionally, $\mathbb{C}\mathbb{G}_{2,1} = \mathbb{C}\mathbb{P}^1$, which is homeomorphic to $\mathbb{S}^2$. One can see that there exist non-zero degree maps from $\mathbb{S}^2 \times \mathbb{C}\mathbb{G}_{n,k}$ to $\mathbb{S}^{2k(n-k)} \times \mathbb{S}^2$, using the degree 1 map from $\mathbb{C}G_{n,k} \to \mathbb{S}^{2k(n-k)}$ which collapses the $(2k(n-k)-1)$-th skeleton of $\mathbb{C}G_{n,k}$. However, it requires further investigation to determine whether $\tilde{f} : \mathbb{S}^2 \times \mathbb{C}\mathbb{G}_{n,k} \to \mathbb{S}^{2k(n-k)} \times \mathbb{S}^2$ can have non-zero degree.*

We have the following lemma where we consider $m = r$ for studying the degree of a map $f : P(m,n,k) \to P(r,s,t)$. See Theorem 2 in [RS97] for a similar result in the context of the complex Grassmannians.

Lemma 7.3.5. *Consider a continuous map $f : P(m,n,k) \to P(m,s,t)$ where the generalized Dold manifolds are oriented and of the same dimension. Further assume that $1 \le k \le n/2, 1 \le t \le s/2$ and $m \ge 1$. Then the degree of f is zero in each of the following cases:*
(i) m is odd and $\lfloor k/2 \rfloor < \lfloor t/2 \rfloor$,
(ii) m is even and $k < t$.

Proof. We shall use the notation $\#^l$ to mean "the number of *distinct* monomials of degree l" throughout the proof.

Using Theorem 5.4.2, we know that $\pi^* : H^*(P(m,n,k);\mathbb{Q}) \to \text{Fix}H^*(\theta;\mathbb{Q}))$ is an isomorphism, and, since $\text{Fix}H^*(\theta;\mathbb{Q})) \subset H^*(\mathbb{S}^m;\mathbb{Q}) \otimes H^*(\mathbb{C}G_{n,k};\mathbb{Q})$, the generators of $\text{Fix}H^*(\theta;\mathbb{Q}))$ involves the Chern classes of $\gamma_{n,k}$ and u_m (using Theorem 5.4.1 and Remark 7.3.2). Thus it makes sense to say that the generators of

$H^*(P(m,n,k);\mathbb{Q}) \cong \mathrm{Fix}H^*(\theta;\mathbb{Q}))$ involve certain Chern classes and u_m. Similar observation holds for $\pi^* : H^*(P(m,s,t);\mathbb{Q}) \to \mathrm{Fix}H^*(\theta;\mathbb{Q})))$.

(i) Assume (i) holds. We have $H^*(P(m,n,k)) \cong \mathrm{Fix}H^*(\theta;\mathbb{Q})) \cong H^*(\mathbb{S}^m;\mathbb{Q}) \otimes \mathrm{Fix}H^*(\sigma;\mathbb{Q}))$, since m is odd. The Remark 7.3.2 ensures that $\mathrm{Fix}H^*(\sigma;\mathbb{Q}))$ is generated by monomials in $c_1, c_2, \dots, c_k$, where each monomial's degree is a multiple of 4. Thus, we have

$$\begin{aligned}\dim(H^l(P(m,n,k);\mathbb{Q})) &= \#^l \text{ in } \{c_i\}_{i=1}^k \text{ if } l \leq 2(n-k)\\ &\leq \#^l \text{ in } \{c_i\}_{i=1}^k \text{ if } l > 2(n-k).\end{aligned}$$

A similar observation is true for $P(r,s,t)$. Combining all these, we get

$$\begin{aligned}\dim(H^{4\lfloor t/2\rfloor}(P(m,s,t);\mathbb{Q})) &= \#^{4\lfloor t/2\rfloor} \text{ in } \{d_i\}_{i=1}^t, \text{ since } t \leq s/2\\ &> \#^{4\lfloor t/2\rfloor} \text{ in } \{c_i\}_{i=1}^k, \text{ since } \lfloor k/2\rfloor < \lfloor t/2\rfloor\\ &\geq \dim(H^{4\lfloor t/2\rfloor}(P(m,n,k);\mathbb{Q})).\end{aligned}$$

This shows that $f^* : H^{4\lfloor t/2\rfloor}(P(m,s,t);\mathbb{Q}) \to H^{4\lfloor t/2\rfloor}(P(m,n,k);\mathbb{Q})$ is not a monomorphism and hence $deg(f) = 0$.

(ii) Assume (ii) holds. We proceed with two cases: (a) t is even and (b) t is odd.

(a) Suppose that t is even. Here $k < t$ implies $\lfloor k/2\rfloor < \lfloor t/2\rfloor$. Now using Theorem 5.4.1 and Remark 7.3.2, proceeding similarly as above, we have

$$\begin{aligned}\dim(H^{4\lfloor t/2\rfloor}(P(m,s,t);\mathbb{Q})) &= \#^{4\lfloor t/2\rfloor} \text{ in } \{d_i\}_{i=1}^t \cup \{u_m\}\ [\because t \leq s/2 \leq s-t]\\ &> \#^{4\lfloor t/2\rfloor} \text{ in } \{c_i\}_{i=1}^k \cup \{u_m\}, \text{ [since } \lfloor k/2\rfloor < \lfloor t/2\rfloor,\\ &\quad \text{the left hand side counts an extra monomial } d_{2\lfloor t/2\rfloor}.]\\ &\geq \dim(H^{4\lfloor t/2\rfloor}(P(m,n,k);\mathbb{Q})).\end{aligned}$$

This shows that f^* is not a monomorphism; consequently, the degree of f is zero.

(b) Suppose that t is odd. Now we consider $(m+2t)$-th cohomology and similarly observe that

$$\begin{aligned}
\dim(H^{m+2t}(P(m,s,t);\mathbb{Q})) &= \#^{m+2t} \text{ in } \{d_i\}_{i=1}^{t} \cup \{u_m\} \\
&> \#^{m+2t} \text{ in } \{c_i\}_{i=1}^{k} \cup \{u_m\}, \text{ [since left hand side counts} \\
&\quad \text{an extra monomial } u_m d_t \text{ as } k < t \text{ and } t \text{ is odd.]} \\
&\geq \dim(H^{m+2t}(P(m,n,k);\mathbb{Q})).
\end{aligned}$$

Thus, f^* is not a monomorphism and so, $deg(f) = 0$.

Now we are done with all the cases and this completes the proof. □

Bibliography

[Ada62] J. F. Adams. Vector fields on spheres. *Ann. of Math.*, 75:603–632, 1962.

[AH61] M. F. Atiyah and F. Hirzebruch. Vector bundles and homogeneous spaces. In *Proc. Sympos. Pure Math., Vol. III*, pages 7–38. Amer. Math. Soc., Providence, RI, 1961.

[Ati67] M. F. Atiyah. *K-theory.* W. A. Benjamin, Inc., New York-Amsterdam, 1967. Lecture notes by D. W. Anderson.

[Bor53a] A. Borel. La cohomologie mod 2 de certains espaces homogènes. *Comment. Math. Helv.*, 27:165–197, 1953.

[Bor53b] Armand Borel. Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts. *Ann. of Math.*, 57:115–207, 1953.

[Bot69] Raoul Bott. *Lectures on $K(X)$.* Mathematics Lecture Note Series. W. A. Benjamin, Inc., New York-Amsterdam, 1969.

[Bre72] Glen E. Bredon. *Introduction to compact transformation groups*, volume Vol. 46 of *Pure and Applied Mathematics.* Academic Press, New York-London, 1972.

[Bri05] Michel Brion. Lectures on the geometry of flag varieties. In *Topics in cohomological studies of algebraic varieties*, Trends Math., pages 33–85. Birkhäuser, Basel, 2005.

[BT82] Raoul Bott and Loring W. Tu. *Differential forms in algebraic topology*, volume 82 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982.

[CF64] P. E. Conner and E. E. Floyd. *Differentiable periodic maps*, volume Band 33 of *Ergebnisse der Mathematik und ihrer Grenzgebiete, (N.F.)*. Springer-Verlag, Berlin-Göttingen-Heidelberg; Academic Press, Inc., Publishers, New York, 1964.

[Dav10] Donald M. Davis. Projective product spaces. *J. Topol.*, 3(2):265–279, 2010.

[DJ91] Michael W. Davis and Tadeusz Januszkiewicz. Convex polytopes, Coxeter orbifolds and torus actions. *Duke Math. J.*, 62(2):417–451, 1991.

[Dol56] Albrecht Dold. Erzeugende der Thomschen Algebra $\mathfrak{N}$. *Math. Z.*, 65:25–35, 1956.

[Fuj66] Michikazu Fujii. K_U-groups of Dold manifolds. *Osaka Math. J.*, 3:49–64, 1966.

[Fuj69] Michikazu Fujii. Ring structures of K_U-cohomologies of Dold manifolds. *Osaka Math. J.*, 6:107–115, 1969.

[GH78] Henry Glover and Bill Homer. Endomorphisms of the cohomology ring of finite Grassmann manifolds. In *Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), I*, volume 657 of *Lecture Notes in Math.*, pages 170–193. Springer, Berlin-New York, 1978.

[Hir95] Friedrich Hirzebruch. *Topological methods in algebraic geometry*. Classics in Mathematics. Springer-Verlag, Berlin, english edition, 1995. Translated from the German and Appendix One by R. L. E. Schwarzenberger, Appendix Two by A. Borel, Reprint of the 1978 edition.

[HM03] Akio Hattori and Mikiya Masuda. Theory of multi-fans. *Osaka J. Math.*, 40(1):1–68, 2003.

[Hus94] Dale Husemoller. *Fibre bundles*, volume 20 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 1994.

[Kar08] Max Karoubi. *K-theory*. Classics in Mathematics. Springer-Verlag, Berlin, 2008. An introduction, Reprint of the 1978 edition, With a new postface by the author and a list of errata.

[LR08] Venkatramani Lakshmibai and Komaranapuram N. Raghavan. *Standard monomial theory*, volume 137 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2008.

[Mas99] Mikiya Masuda. Unitary toric manifolds, multi-fans and equivariant index. *Tohoku Math. J. (2)*, 51(2):237–265, 1999.

[MP06] Mikiya Masuda and Taras Panov. On the cohomology of torus manifolds. *Osaka J. Math.*, 43(3):711–746, 2006.

[MS74] John W. Milnor and James D. Stasheff. *Characteristic classes*, volume No. 76 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1974.

[MS22] Manas Mandal and Parameswaran Sankaran. Cohomology of generalized Dold spaces. *Topology Appl.*, 310:Paper No. 108040, 16, 2022.

[MT91] Mamoru Mimura and Hirosi Toda. *Topology of Lie groups. I, II*, volume 91 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, japanese edition, 1991.

[NS19] Avijit Nath and Parameswaran Sankaran. On generalized Dold manifolds. *Osaka J. Math.*, 56(1):75–90, 2019.

[RS97] Vimala Ramani and Parameswaran Sankaran. On degrees of maps between Grassmannians. *Proc. Indian Acad. Sci. Math. Sci.*, 107(1):13–19, 1997.

[San08] Parameswaran Sankaran. K-rings of smooth complete toric varieties and related spaces. *Tohoku Math. J. (2)*, 60(4):459–469, 2008.

[Spa82] Edwin H. Spanier. *Algebraic topology*. Springer-Verlag, New York, [1982].

[SZ22] Soumen Sarkar and Peter Zvengrowski. On generalized projective product spaces and Dold manifolds. *Homology Homotopy Appl.*, 24(2):265–289, 2022.

[Ucc65] J. J. Ucci. Immersions and embeddings of Dold manifolds. *Topology*, 4:283–293, 1965.