COMPARISON OF ORDER PROJECTIONS IN ABSOLUTE MATRIX ORDER UNIT SPACES

By

AMIT KUMAR

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National Institute of Science Education and Research, Bhubaneswar

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DECLARATION

I hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

A. Kumar.

(Amit Kumar)

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Dedicated to My Family

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Contents

Summary

In the thesis, we have introduced absolute value preserving maps between two absolutely ordered spaces. We have discussed some of the elementary properties of these maps. We have proved that a linear map between two absolutely ordered spaces is absolute value preserving if and only if it is orthogonality preserving. We have also studied absolute value preserving maps between two absolute order unit spaces and proved the following result: Let $\phi : V \to W$ be a unital bijective linear map between two absolute order unit spaces V and W. Then ϕ is absolute value preserving if and only if it is an isometry. Since the self-adjoint parts of unital C*-algebras are absolute order unit spaces, this result can be considered an extension of the studies on surjective linear isometries between C*-algebras due to Kadison in [27, 29] and the characterization of surjective linear isometries between unital JB-algebras obtained by Wright and Youngson in [48] (cf. Corollary 4.2.2).

We have introduced matricial versions of absolutely ordered spaces and absolute order unit spaces namely absolutely matrix ordered spaces and absolute matrix order unit spaces in the context of matrix ordered spaces. We have studied some of the properties of these spaces. We have extended the notion of orthogonality to general elements in absolute matrix order unit spaces and related it to the orthogonality among positive elements. We have also generalized the notion of absolute value preserving maps between absolutely ordered spaces to completely

absolute value preserving maps between absolutely matrix ordered spaces.

We have defined the notion of partial isometry and some other related algebraic notions of C*-algebras in the order theoretic contexts in absolute matrix order unit spaces. Using them, we have introduced and studied comparison of order projections in absolute matrix order unit spaces. This idea is also an extension of comparison of projections in a C*-algebra. We have defined notions of infinite and properly infinite projections and studied characterizations of these notions.

Our proposed comparison theory culminates in formation of K_0-groups of absolute matrix order unit spaces. To introduce K_0-groups, we have described the matricial inductive limit of absolute matrix order unit spaces. We have studied order structure on K_0-groups. We have proved that K_0 is a functor from category of absolute matrix order unit spaces with morphisms as unital completely absolute value preserving maps to category of abelian groups. We have also defined orthogonality of completely positive maps and proved that sum of two orthogonal completely absolute value preserving maps is again a completely absolute value preserving map. We have further proved that K_0 is additive on orthogonal unital completely absolute value preserving maps.

Chapter 1

Introduction

Order structure is one of the basic ingredients of the C*-algebra theory. L. Kantorovitch initiated the order theoretic work in functional analysis in 1937 [32]. In 1941, Kakutani proved that an abstract M-space is precisely a concrete $C(K, \mathbb{R})$ space for a suitable compact and Hausdorff space K [31]. In 1943, Gelfand and Neumark proved that an abstract (unital) commutative C*-algebra is precisely a concrete $C(K, \mathbb{C})$ for a suitable compact and Hausdorrf space K [21]. Thus Gelfand-Neumark theorem for commutative C*-algebra, in the light of Kakutani theorem, yields that the self-adjoint part of a commutative C*-algebra is, in particular, a vector lattice having some other properties. As a contrast, Kadison's antilattice theorem [28] informs us that the self-adjoint part of non-commutative C*-algebra can not be a vector lattice. Thus the study of the order structure of a general C*-algebra opens an interesting area. The corresponding theory evolves as a study of ordered vector spaces not having any vector lattice structure.

In 1951, Kadison proved that the self-adjoint part of a unital C*-algebra A is isometrically order isomorphic to the space of continuous affine functions on the state space $S(A)$ of A [26]. This is known as Kadison's functional representation theorem. Actually, he proved this result for any operator system (that is, a unital self-adjoint subspace) in A. If K be a compact and convex set in a locally convex space E, and if $A(K)$ denote the space of all real valued continuous affine functions on K, then $A(K)$ is an order unit space. In particular, the self-adjont part of an operator system is an order unit space.

This work led to the emergence of the order theoretic (non-commutative) functional analysis. The duality theory of ordered Banach spaces was studied during 1950's and 60's in the works of Asimov, Bonsall, Edwards, Ellis and Ng and many others [5, 6, 12, 13, 16, 19, 54]. In 1964, D. A. Edwards introduced the notion of base normed space [16] and A. J. Ellis studied duality between order unit spaces and base normed spaces [19]. Details can be found in [1] and [23] and the references therein.

In spite of the anti-lattice nature of non-commutative C*-algebras, the order structure of a C*-algebra is rich with many properties. The works of Kadison, Effros, Størmer and Pedersen many others, highlight various aspects of order structure of a C*-algebra and encourages us to expect a 'non-commutative vector lattice' or a 'near lattice' structure in it. The monograph [55] (and references therein), for example, is a good source of information for this purpose.

In 1977, Effros found a relation between norm and order structure of a C*-algebra in the following sense: given an element a in a C*-algebra A, we have

$$\|a\| \leq 1 \text{ if and only if } \begin{bmatrix} 1 & a \\ a^* & 1 \end{bmatrix} \geq 0.$$

(See [17]). With the help of this result, in 1977 Choi and Effros characterized operator systems as *matrix order unit spaces* [14]. This characterization set a benchmark for the study of the (quantized) functional analysis.

With the introduction of matrix ordered spaces due to Choi and Effros in 1977 and that of L^∞-matricially normed space due to Ruan [62], the non-unital matrix ordered spaces were studied independently by Schreiner [64] and Karn and Vasudevan [42, 43, 44, 45]. They initiated the study of non-unital matrix ordered spaces and their duality. While Schreiner adopted the operator space duality, Karn and Vasudevan considered matrix (Choi-Effros) duality. However, Blecher and Neal noted that the operator space dual of a C*-algebra can not be order embedded in C*-algebras [11], in general. To overcome this problem, Karn introduced and studied the notion of matrix order smooth ∞-normed space [35, 36].

In a subsequent work, Karn characterized algebraic orthogonality in the terms of order and norm [37, 38, 39]. This characterization paved a way to introduce the notions of absolutely ordered space and absolute order unit space. For an element a in a C*-algebra A, we define the 'absolute value' of a as $|a| := (a^*a)^{\frac{1}{2}}$ and for an element v in a vector lattice V, we define the 'absolute value' of v as $|v| := v \vee (-v)$. The absolute values in two different contexts have a connection. We note that for a pair of positive elements a and b in A, $ab = 0$ if and only if $|a - b| = a + b$ [39]. Similarly, for a pair of positive elements u and v in V, $u \wedge v = 0$ if and only if $|u - v| = u + v$ [39]. Thus in both the cases, we can say that $a \perp b$ if and only if $|a - b| = a + b$. In other words, the two kinds of orthogonality relate to the same kind of relation in terms of absolute value. The definition of an absolutely ordered space is influenced by some of basic properties of the orthogonality which hold in both kinds of the above mentioned (ordered) spaces. The self-adjoint parts of unital C*-algebras and (unital) M-spaces are examples of absolute order unit spaces. It was shown that under an additional condition, an absolutely ordered space turns out to be a vector lattice [38, Theorem 4.12].

One can easily show that under the same condition, an absolute order unit space becomes an M-space. Therefore an absolutely ordered space may be termed as a 'non-commutative vector lattice'.

Let A and B be unital C*-algebras and let $\phi : A \to B$ be a surjective linear map. In [27], Kadison proved the following two results:

(1) If ϕ is Jordan $*$-isomorphism, then it is isometry [27, Theorem 5].

(2) If ϕ is isometry, then $\phi(1)$ is unitary and $\phi = \phi(1)\psi$ for some Jordan $*$-isomorphism $\psi : B \to B$ [27, Theorem 7]. In particular, if ϕ is unital, then it is a Jordan $*$-isomorphism.

Further, in [29], Kadison proved the following result: Let A and B be unital C*-algebras and let $\phi : A \to B$ be a unital linear map. If $\phi(|a|) = |\phi(a)|$ for all $a \in A_{sa}$, then ϕ is a Jordan $*$-homomorphism [29, Theorem 6]. If, we combine these three results, we can deduce the following: Let A and B be unital C*-algebras and $\phi : A \to B$ be unital bijective $*$-linear map. Then the following statements are equivalent:

(1) $\phi : A_{sa} \to B_{sa}$ is an isometry.

(2) ϕ is a Jordan isomorphism.

(3) $\phi : A_{sa} \to B_{sa}$ is absolute value preserving.

In this thesis, we introduce absolute value preserving maps between two absolutely ordered spaces. We discuss some of the elementary properties of these maps. We prove that a linear map between two absolutely ordered spaces is absolute value preserving if and only if it is orthogonality preserving. We also study absolute value preserving maps between two absolute order unit spaces and prove the following result: Let $\phi : V \to W$ be a unital bijective linear map between two absolute order unit spaces V and W. Then ϕ is absolute value preserving if and only if it is an isometry. Since the self-adjoint parts of unital C*-algebras are absolute order unit spaces, our result can be considered an extenstion of the studies on surjective linear isometries between C*-algebras due to Kadison.

In [48], Maitland Wright and Youngson proved that any surjective linear unital isometry $\phi : A \to B$ between unital JB-algebras A and B is a Jordan isomorphism [48, Theorem 4]. Since unital JB-algebras are also examples of absolute order unit spaces, therefore by our result, we may deduce the following: Let A and B be unital JB-algebras and let $\phi : A \to B$ be a unital bijective linear map. Then the following statements are equivalent:

1. ϕ is an isometry;

2. ϕ is a Jordan isomorphism;

3. ϕ is absolute value preserving.

Thus our result can also be considered an extension of the characterization of surjective linear isometries between unital JB-algebras obtained by Wright and Youngson.

We introduce matricial versions of absolutely ordered spaces and absolute order unit spaces namely absolutely matrix ordered spaces and absolute matrix order unit spaces in the context of matrix ordered spaces. We have studied some of the properties of these spaces. We extend the notion of orthogonality to general elements in absolute matrix matrix order unit spaces and relate it to the orthogonality among positive elements. We also generalize the notion of absolute value preserving maps between absolutely ordered spaces to completely absolute value preserving maps between absolutely matrix ordered spaces. We prove the following result: Let $\phi : V \to W$ be a unital bijective $*$-linear map between two absolute matrix order unit spaces. Then ϕ is completely absolute value preserving map if and only if it is a complete isometry. In particular, when restricted to C*-algebras, we can deduce the following result: Let A and B be unital C*-algebras and let $\phi : A \to B$ be unital, bijective, $*$-linear map. Then the following statements are equivalent:

1. ϕ is a complete isometry;

2. ϕ is a C*-algebra isomorphism;

3. ϕ is completely absolute value preserving.

It is worth noting that Blecher et al. have studied this kind of problems for operator algebras (non-selfadjoint in general) in [9, 10]. (See, in particular, Theorem 3.1, and Corollaries 3.2 and 3.4 of [9].)

In the theory of self-adjoint operator algebras, (self-adjoint) projections acquire a central role. For example, to each self-adjoint element a of a von Neumann algebra $\mathcal{M}$, there exists a spectral family of projections $\{p_\lambda : \lambda \in \mathbb{R}\}$ such that $a = \displaystyle\int_{-\|a\|}^{\|a\|} \lambda \, dp_\lambda$ and the integral converges in norm to a in the sense of Riemann. Thus for any Borel function $f : [-\|a\|, \|a\|] \to \mathbb{C}$, we have $f(a) = \displaystyle\int_{-\|a\|}^{\|a\|} f(\lambda) \, dp_\lambda$.

The germs of comparison theory of projections in $B(H)$ (set of all bounded linear operators on Hilbert space H) may be found in series of the fundamental papers by F. J. Murray and J. von Neumann, "On rings of operators I–IV" [49, 50, 51, 52, 53]. However this theory was formally studied for Banach algebras by I. Kaplansky [33]. The comparison theory of projections in C*-algebras grew to K-theory in operator algebras which is the main tool for classification of von Neumann algebras and also that of C*-algebras. We refer to [7, 8, 61, 66] and the references therein for further details.

Recently, Karn introduced the notion of order projections in absolute order unit spaces and studied a suitable spectral decomposition theorem [39]. In this thesis, we have chosen order projections as the central scheme.

We have defined the notion of partial isometry and some other related algebraic notions of C*-algebras in the order theoretic contexts in absolute matrix order unit spaces. Using them, we have introduced and studied comparison of order projections in absolute matrix order unit spaces. This is again an extension of comparison of projections in a C*-algebra. We define notions of infinite and properly infinite projections and prove characterization theorems for them.

In [39], Karn also introduced the notions of order unit property and absolute order unit property in absolute order unit spaces. He characterized order projections in absolute order unit spaces in terms of these properties. We extend these notions to matrix order unit property and absolute matrix order unit property in absolute matrix order unit spaces. We also characterize order projections in absolute matrix order unit spaces in terms of matrix order unit property and absolute matrix order unit property.

Our proposed comparison theory culminates in the formation of K_0 of absolute matrix order unit spaces. To introduce K_0-groups, we have described the matricial inductive limit of absolute matrix order unit spaces. We study order structure on K_0-groups. We prove that K_0 is a functor from category of absolute matrix order unit spaces with morphisms as unital completely absolute value preserving maps to category of abelian groups. We also define orthogonality of completely absolute value preserving maps and prove that sum of two orthogonal completely absolute value preserving maps is again a completely absolute value preserving map. We also prove that K_0 is additive on orthogonal unital completely absolute value preserving maps.

1.1 Chapterwise details of the thesis

Now we present a brief description of the remaining chapters.

In second chapter, we recall basic definitions and results which is required to follow this thesis. We recall the theory of real ordered vector spaces, direct limit of matrix ordered spaces and theory of absolutely ordered spaces.

In third chapter, we introduce the notions of absolutely matrix ordered spaces and absolute matrix order unit spaces. These spaces are matricial versions of absolutely ordered spaces and absolute order unit spaces respectively in the context of matrix ordered spaces. We extend the notion of orthogonality to the general elements of an absolute matrix order unit space and relate it to the orthogonality among positive elements [Propositions 3.3.1 and 3.3.5].

In fourth chapter, we define absolute value preserving maps between two absolutely ordered spaces. We prove that a linear map between two absolutely ordered spaces is absolute value preserving if and only if it is orthogonality preserving [Proposition 4.1.3]. With the help of this result, we proved that a unital, bijective linear map between two absolute order unit spaces is an isometry if and only if it is absolute value preserving [Theorem 4.2.1] which leads to provide a simple proof of well known result that every unital Jordan isomorphism between two unital JB-algebras is an isometry and hence we deduce that unital, bijective absolute value preseving maps between two unital JB-algebras are precisely Jordan isomorphisms [Corollary 4.2.2]. The notion of absolute compatibility was introduced by Karn in [39]. In Theorems 4.3.2 and 4.3.3, we prove relation of absolute compatibility with absolute value preservers. We extend the notion of absolute value preserving maps between absolutely ordered spaces to completely absolute value preserving maps between absolutely matrix ordered spaces. We prove that a unital, bijective $*$-linear map between absolute matrix order unit spaces is a complete isometry if and only if it is completely absolute value preserving [Theorem 4.4.2]. From this, we deduce that on (unital) C*-algebras such maps are precisely C*-algebra isomorphisms [Corollary 4.4.3]. We use a matricial trick which is apparently new. Latter, we define orthogonality of positive maps and prove that sum of two orthogonal completely absolute value preserving maps is again completely absolute value preserving [Theorem 4.5.4].

In fifth chapter, we generalize and study some C*-algebraic notions in order theoretic context. Karn [39] has recently studied order projections in an absolute order unit space, we extend the notion of order projections in an absolute matrix order unit space and introduce the notion of a partial isometry to describe the comparison of order projections. In this chapter, we mainly focus on order projections and partial isometries in absolute matrix order unit spaces. We also study finiteness of order projections and prove related Theorems 5.4.5 and 5.5.4.

In the last chapter, we define matrix order unit property and absolute matrix order unit property in absolute matrix order unit spaces which are matricial versions of order unit property and absolute order unit property in absolute order unit spaces. We describe the direct limit of absolute matrix order unit spaces. Using this, we introduce K_0-group of an absolute matrix order unit space [Theorem 6.3.5]. We observe that K_0 is functorial in nature. We prove that K_0 is a functor from category of absolute matrix order unit spaces with morphisms as unital completely absolute value preserving maps to category of abelian groups [Theorem 6.4.2]. We show that under a certain condition K_0 of an absolute matrix order unit space is an ordered abelian group with some distinguished order unit [Theorem 6.5.1 and Corollary 6.5.2]. We also prove that K_0 is additive on orthogonal unital completely absolute value preserving maps [Theorem 6.5.3(3)].

Chapter 2

Preliminaries

In this chapter, we recall the basic definitions and results. In the first section, we briefly recall theory of ordered vector spaces including definition of order ideal, order unit space, matrix ordered space, matrix order unit space, L^∞-matricially normed space, L^∞-matricially $*$-normed space, positive map, completely positive map, completely bounded map, types of orthogonality in order unit spaces and some results charactering these notions. In the second section, we recall the direct limit of matrix ordered spaces and its characterization in terms of non-degenerate ordered $\mathfrak{F}$-bimodule. In the third section, we recall the theory of absolutely ordered vector spaces, absolute order unit spaces and notion of orthogonality in absolutely ordered spaces. Later, we recall the notion of order projections in absolute order unit spaces and some characterizations of order projections.

2.1 Ordered vector spaces

We begin by recalling some basic order theoretic notions. Throughout this thesis $\mathbb{R}$ denotes the field of all real numbers and $\mathbb{C}$ denotes the field of all complex numbers. Let V be a real vector space. A non-empty subset V^+ of V is called a *cone*, if V^+ is closed under vectors' addition as well as scalar multiplication with non-negative real numbers. In this case, (V, V^+) is called a *real ordered vector space*.

Let (V, V^+) be a real ordered vector space. For $u, v \in V$, define $u \leq v$ if $v - u \in V^+$. Then $(V, \leq)$ is a *partially ordered space*, in a unique way, in the following sense that (i) $u \leq u$ for all $u \in V$ (ii) $u \leq w$ whenever $u \leq v$ and $v \leq w$ for $u, v, w \in V$ and (iii) $u + w \leq v + w$ and $ku \leq kv$ whenever $u \leq v$ for $u, v, w \in V$ and k is a positive real number.

The cone V^+ is said to be *proper*, if $V^+ \cap -V^+ = \{0\}$. It is said to be *generating*, if $V = V^+ - V^+$. Recall that V^+ is proper if and only if $\leq$ is anti-symmetric.

A positive element $e \in V^+$ is said to be an *order unit* for V, if for each $v \in V$, there is a positive real number k such that $ke \pm v \in V^+$. The cone V^+ is said to be *Archimedean*, if for any $v \in V$ with $ku + v \in V^+$ for a fixed $u \in V^+$ and all positive real numbers k, we have $v \in V^+$.

Let (V, V^+) be a real ordered vector space with an order unit e. Then V^+ is Archimedean if and only if for any $v \in V$ with $ke + v \in V^+$ for all positive real numbers k, we have $v \in V^+$.

Let W be a vector subspace of V. Then W is said to be an *order ideal* of (V, V^+), if for $v \in V^+, w \in W$ with $v \leq w$, we have $v \in W$.

Let W be a vector subspace of an ordered vector space (V, V^+). Consider the *quotient vector space* V/W of V by W. Put $(V/W)^+ = \{v + W : v \in V^+\}$. Then $(V/W, (V/W)^+)$ is an ordered vector space.

The above details can be found in [1, 23, 63, 67].

Next result characterizes an order ideal in terms of properness of a cone.

Proposition 2.1.1. *[1, Proposition II.1.1] Let W be a vector subspace of an* *ordered vector space* (V, V^+). *Then* $(V/W)^+$ *is proper if and only if* W *is an order ideal.*

Let (V, V^+) be a real ordered vector space with an order unit e such that V^+ is proper and Archimedean. Then e determines a norm on V given by

$$\|v\| := \inf\{k > 0 : ke \pm v \in V^+\}$$

in such a way that V^+ is norm-closed and for each $v \in V$, we have $\|v\|e \pm v \in V^+$. In this case, we say that V is an *order unit space* and denote it by (V, e) [1, Proposition II.1.2].

Remark 2.1.2. *Let* (V, e) *be an absolute order unit space. Also let* $u, v, w \in V$ *be such that* $u \leq v \leq w$. *Then* $\|v\| \leq max\{\|u\|, \|w\|\}$. *In fact, in this case,* $-\|u\|e \leq v \leq \|w\|e$ *for* $-\|u\|e \leq u \leq \|u\|e$ *and* $-\|w\|e \leq w \leq \|w\|e$. *Thus* $-max\{\|u\|, \|w\|\}e \leq v \leq max\{\|u\|, \|w\|\}e$ *so that* $\|v\| \leq max\{\|u\|, \|w\|\}$.

We recall two types of orthogonality in an order unit space.

Definition 2.1.3. *[39, Definition 3.6] Let* (V, e) *be an order unit space. Then*

(a) *For* $u, v \in V^+$, *we say that* u *is* ∞-orthogonal *to* v *(we write it,* $u \perp_\infty v$*), if* $\|k_1 u + k_2 v\| = max\{\|k_1 u\|, \|k_2 v\|\}$ *for all* $k_1, k_2 \in \mathbb{R}$;

(b) *For* $u, v \in V^+$, *we say that* u *is* absolutely ∞-orthogonal *to* v *(we write it,* $u \perp_\infty^a v$*), if* $u_1 \perp_\infty v_1$ *whenever* $0 \leq u_1 \leq u$ *and* $0 \leq v_1 \leq v$.

The following result provides a characterization of ∞-orthogonality in an order unit space:

Theorem 2.1.4. *[37, Theorem 3.3] Let* V *be an order unit space. Suppose that* $u, v \in V^+ \setminus \{0\}$. *Then* $u \perp_\infty v$ *if and only if* $\|\|u\|^{-1}u + \|v\|^{-1}v\| = 1$.

Let (V_i, V_i^+) be the ordered vector spaces for $i = 1, 2$ and let $\phi : V_1 \to V_2$ be a linear map. We say that ϕ is positive (we write it, $\phi \geq 0$), if

$$\phi(V_1^+) \subset V_2^+$$

[23, 63, 67].

Let (V, e_V) and (W, e_W) be order unit spaces and let $\phi : V \to W$ be a linear map such that $\phi(e_V) = e_W$. Then ϕ is positive if and only if ϕ is bounded with $\|\phi\| = 1$ [1, Proposition II.1.3].

Let V be a complex vector space. We denote by $M_{m,n}(V)$ the vector space of all the $m \times n$ matrices $v = [v_{i,j}]$ with entries $v_{i,j} \in V$ and by $M_{m,n}$ the vector space of all the $m \times n$ matrices $a = [a_{i,j}]$ with entries $a_{i,j} \in \mathbb{C}$. We write $0_{m,n}$ for zero element in $M_{m,n}(V)$. For $m = n$, we write $0_{m,n} = 0_n$. We define $av = \left[\sum_{k=1}^{m} a_{i,k}v_{k,j}\right]$ and $vb = \left[\sum_{k=1}^{n} v_{i,k}b_{k,j}\right]$ for $a \in M_{r,m}, v \in M_{m,n}(V)$ and $b \in M_{n,s}$. We write

$$v \oplus w = \begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix} \text{ for } v \in M_{m,n}(V), w \in M_{r,s}(V).$$

Here 0 denotes suitable rectangular matrix of zero entries from V [62].

Let V be a complex vector space. Then a map $* : V \to V$ is said to be *involution* on V, if it satisfies the following conditions:

(1) $(v + w)^* = v^* + w^*$ for all $v, w \in V$.

(2) $(\alpha v)^* = \bar{\alpha} v^*$ for all $\alpha \in \mathbb{C}, v \in V$.

(3) $(v^*)^* = v$ for all $v \in V$ [14].

A complex vector space with an involution is called a $*$-*vector space*. We write $V_{sa} = \{v \in V : v = v^*\}$. Then V_{sa} is a real vector space [14].

Definition 2.1.5. *[14] A* matrix ordered space *is a $*$-vector space V together with a sequence $\{M_n(V)^+\}$ with $M_n(V)^+ \subset M_n(V)_{sa}$ for each $n \in \mathbb{N}$ satisfying the following conditions:*

(a) $(M_n(V)_{sa}, M_n(V)^+)$ is a real ordered vector space, for each $n \in \mathbb{N}$; and

(b) $a^*va \in M_m(V)^+$ for all $v \in M_n(V)^+$, $a \in M_{n,m}$ and $n, m \in \mathbb{N}$.

It is denoted by $(V, \{M_n(V)^+\})$. If, in addition, $e \in V^+$ is an order unit in V_{sa} such that V^+ is proper and $M_n(V)^+$ is Archimedean for all $n \in \mathbb{N}$, then V is called a matrix order unit space *and is denoted by $(V, \{M_n(V)^+\}, e)$.*

Proposition 2.1.6. *Let $(V, \{M_n(V)^+\})$ be a matrix ordered space.*

(1) [42, Proposition 1.8]

(a) If V^+ is proper, then $M_n(V)^+$ is proper for all $n \in \mathbb{N}$.

(b) If V^+ is generating, then $M_n(V)^+$ is generating for all $n \in \mathbb{N}$.

(2) [42, Lemma 2.6] If $e \in V^+$ is an order unit for V_{sa}. Then e^n is an order unit for $M_n(V)_{sa}$ for all $n \in \mathbb{N}$ (where $e^n := e \oplus \cdots \oplus e \in M_n(V)$).

Let V and W be two vector spaces. A linear map $\phi : V \to W$ induces a sequence of linear maps $\{\phi_n\}, \phi_n : M_n(V) \to M_n(W)$ such that $\phi_n([v_{ij}]) = [\phi(v_{ij})]$ for all $[v_{ij}] \in M_n(V)$. Let V and W be $*$-vector spaces. Then ϕ is said to be *$*$-linear*, if $\phi(v^*) = \phi(v)^*$ for all $v \in V$ or equivalently $\phi(V_{sa}) \subset W_{sa}$. If ϕ is $*$-linear, then each ϕ_n is $*$-linear [14].

Let $(V, \{M_n(V)^+\})$ and $(W, \{M_n(W)^+\})$ be two matrix ordered spaces and let $\phi : V \to W$ be a $*$-linear map. Then ϕ is said to be *completely positive*, if $\phi_n : M_n(V)_{sa} \to M_n(W)_{sa}$ is positive for each $n \in \mathbb{N}$ [14].

An *L^∞-matricially normed space* denoted by $(V, \{\|\cdot\|_n\})$ is a complex vector space V together with a sequence of norms $\|\cdot\|_n$ such that

(1) $(M_n(V), \|\cdot\|_n)$ is a normed linear space for each $n \in \mathbb{N}$:

(2) $\|v \oplus w\|_{n+m} = \max\{\|v\|_n, \|w\|_m\}$; and

(3) $\|avb\|_n \leq \|a\|\|v\|_n\|b\|$

for all $v \in M_n(V), w \in M_m(V), a, b \in M_n$ and $m, n \in \mathbb{N}$ [62].

An L^∞*-matricially $*$-normed space* is an L^∞-matricially normed space $(V, \{\|\cdot\|_n\})$ such that V is a $*$-vector space and $\|v^*\|_n = \|v\|_n$ for all $v \in M_n(V), n \in \mathbb{N}$ [36, Definition 1.3].

Let $(V, \{\|\cdot\|_n\})$ and $(W, \{\|\cdot\|_n\})$ be L^∞-matricially normed spaces and let $\phi : V \to W$ be a linear map. Then

(1) ϕ is called *completely bounded*, if $\sup\{\|\phi_n\| : n \in \mathbb{N}\} < \infty$. If ϕ is completely bounded, we write $\phi_{cb} := \sup\{\|\phi_n\| : n \in \mathbb{N}\}$.

(2) ϕ is called *completely contractive*, if ϕ is completely bounded and we have $\phi_{cb} \leq 1$.

(3) ϕ is called *complete isometry*, if ϕ_n is an isometry for each $n \in \mathbb{N}$ [62].

Let $(V, \{M_n(V)^+\}, e)$ be a matrix order unit space. For each $n \in \mathbb{N}$, put

$$\|v\|_n := \inf \left\{ k > 0 : \begin{bmatrix} ke^n & v \\ v^* & ke^n \end{bmatrix} \in M_{2n}(V)^+ \right\} \quad \text{for all } v \in M_n(V).$$

Then $(V, \{M_n(V)^+\}, e)$ with $\{\|\cdot\|_n\}$ is an L^∞*-matricially $*$-normed space* [14, 62].

Let $(V, \{M_n(V)^+\}, e)$ be a matrix order unit space. For $m, n \in \mathbb{N}$ with $m \neq n$, we extend a norm $\|\cdot\|_{m,n}$ on $M_{m,n}(V)$ in the following way:

$$\|v\|_{m,n} := \begin{cases} \left\| \begin{bmatrix} v & 0 \end{bmatrix} \right\|_m & \text{if } m > n \\ \left\| \begin{bmatrix} v \\ 0 \end{bmatrix} \right\|_n & \text{otherwise} \end{cases}$$

for all $v \in M_{m,n}(V)$. In this case, we have

$$\|v\|_{m,n} := \inf \left\{ k > 0 : \begin{bmatrix} ke^m & v \\ v^* & ke^n \end{bmatrix} \in M_{m+n}(V)^+ \right\} \quad \text{for all } v \in M_{m,n}(V).$$

2.2 Direct limit of matrix ordered spaces

In this section, we recall the direct limit of complex vector spaces and the direct limit of matrix ordered spaces. We also recall the definition of non-degenerate ordered $\mathfrak{F}$-bimodule and the characterization of the direct limit of a matrix ordered space as a non-degenerate ordered $\mathfrak{F}$-bimodule.

Consider the family $\{M_n\}$. For each $n, m \in \mathbb{N}$ define $\sigma_{n,n+m} : M_n \to M_{n+m}$ given by $\sigma_{n,n+m}(a) = a \oplus 0_m$. Then $\sigma_{n,n+m}$ is a *vector space isomorphism* with

$$\sigma_{n,n+m}(ab) = \sigma_{n,n+m}(a)\sigma_{n,n+m}(b).$$

We observe that $\{M_n, \sigma_{n,n+m}, \mathbb{N}\}$ is a direct system. Let $\mathfrak{F}$ denote the set of all the $\infty \times \infty$ complex matrices having atmost finitely many non-zero entries. For each $n \in \mathbb{N}$, define $\sigma_n : M_n \to \mathfrak{F}$ given by $\sigma_n(a) = a \oplus \mathfrak{o}$ for all $a \in M_n$, where $\mathfrak{o}$ denotes the zero element in $\mathfrak{F}$. Then $\{\mathfrak{F}, \sigma_n\}$ is the *inductive limit* of

$\{M_n, \sigma_{n,n+m}, \mathbb{N}\}$. In fact, we have

$$\mathfrak{F} = \bigcup_{n=1}^{\infty} \sigma_n(M_n).$$

Let $\mathfrak{e}_{ij}$ denote the $\infty \times \infty$ matrix with 1 at the (i,j)th entry and 0 elsewhere. Then the collection $\{\mathfrak{e}_{ij}\}$ is called the set of matrix units in $\mathfrak{F}$. We write $\mathfrak{I}_n$ for $\sum_{i=1}^{n} \mathfrak{e}_{ii}$.

For $i,j,k,l \in \mathbb{N}$, we have $\mathfrak{e}_{ij}\mathfrak{e}_{kl} = \delta_{jk}\mathfrak{e}_{il}$ where

$$\delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise.} \end{cases}$$

Note that for any $\mathfrak{a} \in \mathfrak{F}$, there exist complex numbers a_{ij} such that

$$\mathfrak{a} = \sum_{i,j} a_{ij}\mathfrak{e}_{ij} \text{ (a finite sum).}$$

Thus $\mathfrak{F}$ is an *algebra.*

For $\mathfrak{a} = \sum_{i,j} a_{ij}\mathfrak{e}_{ij} \in \mathfrak{F}$, we define $\mathfrak{a}^* = \sum_{i,j} \bar{a}_{ji}\mathfrak{e}_{ij} \in \mathfrak{F}$. Then $\mathfrak{a} \longmapsto \mathfrak{a}^*$ is an *involution.* In other words, $\mathfrak{F}$ is a $*$*-algebra.*

The above details are available in [18].

Definition 2.2.1. *[18] Let V be a complex vector space. Consider the family $\{M_n(V)\}$. For each $n,m \in \mathbb{N}$, define $T_{n,n+m} : M_n(V) \to M_{n+m}(V)$ by $T_{n,n+m}(v) = v \oplus 0_m, 0_m \in M_m(V)$. Then $T_{n,n+m}$ is an injective homomorphism. Let $\{\mathfrak{V}, T_n\}$ be the inductive limit of the directed family $\{M_n(V), T_{n,n+m}, \mathbb{N}\}$ so that $T_n = T_{n+m} \circ T_{n,n+m}$ for all $m,n \in \mathbb{N}$. Then $\mathfrak{V}$ is an $\mathfrak{F}$-bimodule. We shall call $\mathfrak{V}$ the* matricial inductive limit or direct limit *of V.*

Definition 2.2.2. *[18] An $\mathfrak{F}$-bimodule $\mathfrak{V}$ is said to be* non-degenerate, *if for every $\mathfrak{v} \in \mathfrak{V}$ there exists $n \in \mathbb{N}$ such that $\mathfrak{I}_n\mathfrak{v}\mathfrak{I}_n = \mathfrak{v}$.*

The matricial inductive limit of a complex vector space may be characterized in the following sense:

Theorem 2.2.3. *[18] The matricial inductive limit of a complex vector space is a non-degenerate $\mathfrak{F}$-bimodule. Conversely, let $\mathfrak{V}$ be a non-degenerate $\mathfrak{F}$-bimodule. Put $V = \mathfrak{I}_1 \mathfrak{V} \mathfrak{I}_1$. Then V is a complex vector space and $\mathfrak{V}$ is its matricial inductive limit in the sense of Definition 2.2.1. Moreover,*

(a) $T_n(M_n(V)) = \mathfrak{I}_n \mathfrak{V} \mathfrak{I}_n$.

(b) $\mathfrak{V} = \bigcup_{n=1}^{\infty} T_n(M_n(V))$.

Let $\mathfrak{V}$ be a non-degenerate $\mathfrak{F}$-bimodule. Also let $\mathfrak{v} \in \mathfrak{V}$ and $\alpha \in \mathbb{C}$. We write $\alpha \mathfrak{v} = (\alpha \mathfrak{I}_n)\mathfrak{v}$ for some $n \in \mathbb{N}$ with $\mathfrak{I}_n \mathfrak{v} \mathfrak{I}_n = \mathfrak{v}$. Then $\alpha \mathfrak{v}$ is well-defined. Thus $\mathfrak{V}$ is a complex vector space.

Now we recall the notion of $\mathfrak{F}$-bimodule norm on a non-degenerate $\mathfrak{F}$-bimodule in the following sense:

Definition 2.2.4. *[58, Definition 1.4] Let $\mathfrak{V}$ be a non-degenerate $\mathfrak{F}$-bimodule. Let $\|\cdot\|$ be a norm on $\mathfrak{V}$. Then we say that $\|\cdot\|$ is an $\mathfrak{F}$-bimodule norm on $\mathfrak{V}$, if $\|\mathfrak{a}\mathfrak{v}\mathfrak{b}\| \leq \|\mathfrak{a}\|\|\mathfrak{v}\|\|\mathfrak{b}\|$ for any $\mathfrak{a}, \mathfrak{b} \in \mathfrak{F}$ and $\mathfrak{v} \in \mathfrak{V}$.*

Next, we describe the order theoretic aspect.

Definition 2.2.5. *[57, Definition 2.6] Let $\mathfrak{V}$ be an $\mathfrak{F}$-bimodule and let $* : \mathfrak{V} \longrightarrow \mathfrak{V}$ be a map satisfying the following conditions:*

1. $(\mathfrak{u} + \mathfrak{v})^* = \mathfrak{u}^* + \mathfrak{v}^*$ *for all* $\mathfrak{u}, \mathfrak{v} \in \mathfrak{V}$;

2. $(\mathfrak{a}\mathfrak{v})^* = \mathfrak{v}^* \mathfrak{a}^*$, $(\mathfrak{v}\mathfrak{a})^* = \mathfrak{a}^* \mathfrak{v}^*$ *for all* $\mathfrak{v} \in \mathfrak{V}, \mathfrak{a} \in \mathfrak{F}$;

3. $(\mathfrak{v}^*)^* = \mathfrak{v}$ *for all* $\mathfrak{v} \in \mathfrak{V}$.

Then $*$ *is called an* involution *on* $\mathfrak{V}$ *and in this case* $\mathfrak{V}$ *is called a* $*$-$\mathfrak{F}$-bimodule*.* *We put* $\mathfrak{V}_{sa} = \{\mathfrak{v} \in \mathfrak{V} : \mathfrak{v}^* = \mathfrak{v}\}$.

Definition 2.2.6. *[57, Definition 3.2] Let* $\mathfrak{V}$ *be a* $*$-$\mathfrak{F}$*-bimodule and let* $\mathfrak{V}^+ \subset \mathfrak{V}_{sa}$ *satisfying the following conditions:*

1. $\mathfrak{u} + \mathfrak{v} \in \mathfrak{V}^+$ *for all* $\mathfrak{u}, \mathfrak{v} \in \mathfrak{V}^+$;

2. $\mathfrak{a}^*\mathfrak{v}\mathfrak{a} \in \mathfrak{V}^+$ *for all* $\mathfrak{v} \in \mathfrak{V}^+, \mathfrak{a} \in \mathfrak{F}$.

Then $\mathfrak{V}^+$ *is called a* bimodule cone *and* $(\mathfrak{V}, \mathfrak{V}^+)$ *is called an* ordered $\mathfrak{F}$-bimodule*.*

Definition 2.2.7. *[57] Let* $(\mathfrak{V}, \mathfrak{V}^+)$ *be an ordered* $\mathfrak{F}$*-bimodule. We say that* $\mathfrak{V}^+$ *is* proper*, if* $\mathfrak{V} \cap (-\mathfrak{V}^+) = \{0\}$ *and* generating*, if given* $\mathfrak{v} \in \mathfrak{V}$ *there exist* $\mathfrak{v}_0, \mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3 \in \mathfrak{V}^+$ *such that* $\mathfrak{v} = \sum_{k=0}^{3} i^k \mathfrak{v}_k$*, where* $i^2 = -1$*. We say that* $\mathfrak{V}^+$ *is* Archimedean*, if for any* $\mathfrak{v} \in \mathfrak{V}_{sa}$ *with* $k\mathfrak{u} + \mathfrak{v} \in \mathfrak{V}^+$ *for a fixed* $\mathfrak{u} \in \mathfrak{V}^+$ *and all positive real numbers* k*, we have* $\mathfrak{v} \in \mathfrak{V}^+$*.*

Theorem 2.2.8. *[57, Theorem 3.4] Let* $(V, \{M_n(V)^+\})$ *be a matrix ordered space and let* $\mathfrak{V}$ *be the matricial inductive limit of* V*. Then* $T_{n,n+m}$ *and* T_n *are positive maps. Put* $\mathfrak{V}^+ = \bigcup_{n=1}^{\infty} T_n(M_n(V)^+)$*. Then* $(\mathfrak{V}, \mathfrak{V}^+)$ *is a non-degenerate ordered* $\mathfrak{F}$*-bimodule. Conversely, let* $(\mathfrak{V}, \mathfrak{V}^+)$ *be a non-degenerate ordered* $\mathfrak{F}$*-bimodule and put* $V = \mathfrak{I}_1\mathfrak{V}\mathfrak{I}_1$*. Define* $T_n : M_n(V) \to \mathfrak{V}$ *given by*

$$T_n([v_{i,j}]) = \sum_{i,j=1}^{n} \mathfrak{e}_{i,1} v_{i,j} \mathfrak{e}_{1,j}$$

for all $[v_{i,j}] \in M_n(V)$*. Then* T_n *is an injective homomorphism such that* $T_n(M_n(V)) = \mathfrak{I}_n\mathfrak{V}\mathfrak{I}_n$ *for all* $n \in \mathbb{N}$*. Set* $M_n(V)^+ = T_n^{-1}(\mathfrak{I}_n\mathfrak{V}^+\mathfrak{I}_n)$ *for all* $n \in \mathbb{N}$*. Then* $(V, \{M_n(V)^+\})$ *is a matrix ordered space and* $\mathfrak{V}$ *is its matricial inductive limit with* $\mathfrak{V}^+ = \bigcup_{n=1}^{\infty} T_n(M_n(V)^+)$*.*

Remark 2.2.9. *Let $(\mathfrak{V}, \mathfrak{V}^+)$ be a non-degenerate ordered $\mathfrak{F}$-bimodule and let $(V, \{M_n(V)^+\})$ be the corresponding matrix ordered space as in Theorem 2.2.8. Then*

(1) $\mathfrak{V}^+$ is proper if and only if V^+ is proper [57, Theorem 3.9].

(2) $\mathfrak{V}^+$ is generating if and only if V^+ is generating [57, Theorem 3.12].

(3) $\mathfrak{V}^+$ is Archimedean if and only if $M_n(V)^+$ is Archimedean for each $n \in \mathbb{N}$.

Proof. (3) First assume that $M_n(V)^+$ is Archimedean for all $n \in \mathbb{N}$. Let $\mathfrak{v} \in \mathfrak{V}_{sa}$ with $k\mathfrak{u} + \mathfrak{v} \in \mathfrak{V}^+$ for a fixed $\mathfrak{u} \in \mathfrak{V}^+$ and all positive real numbers k. Choose $n \in \mathbb{N}$ such that $\mathfrak{I}_n \mathfrak{u} \mathfrak{I}_n = \mathfrak{u}$ and $\mathfrak{I}_n \mathfrak{v} \mathfrak{I}_n = \mathfrak{v}$. Then $T_n^{-1}(\mathfrak{u}) \in M_n(V)^+, T_n^{-1}(\mathfrak{v}) \in M_n(V)_{sa}$ with $kT_n^{-1}(\mathfrak{u}) + T_n^{-1}(\mathfrak{v}) \in M_n(V)^+$ for all positive real numbers k. Since $M_n(V)^+$ is Archimedean, we get that $T_n^{-1}(\mathfrak{v}) \in M_n(V)^+$ so that $\mathfrak{v} = T_n(T_n^{-1}(\mathfrak{v})) \in \mathfrak{V}^+$. Thus $\mathfrak{V}^+$ is Archimedean.

Conversely, assume that $\mathfrak{V}^+$ is Archimedean. Fix $n \in \mathbb{N}$ and let $v \in M_n(V)_{sa}$ with $ku + v \in V^+$ for a fixed $u \in M_n(V)^+$ and all positive real numbers k. Then $T_n(u) \in \mathfrak{V}^+, T_n(v) \in \mathfrak{V}_{sa}$ with $kT_n(u) + T_n(v) \in \mathfrak{V}^+$ for all positive real numbers k. Thus by assumption, we get that $T_n(v) \in \mathfrak{I}_n \mathfrak{V}^+ \mathfrak{I}_n$ so that $v = T_n^{-1}(T_n(v)) \in M_n(V)^+$. Hence $M_n(V)^+$ is Archimedean for each $n \in \mathbb{N}$.

□

2.3 Absolutely ordered vector spaces

In this section, we recall the notions of absolutely ordered spaces and absolute order unit spaces. Before it, we recall some definitions and facts which may be seen as a fresh start for the theory of absolutely ordered spaces and absolute order unit spaces.

Let (V, V^+) be a real ordered vector space. For $v, w \in V$, we write $v \leq w$ if $w - v \in V^+$. Then V is said to be *vector lattice*, if given any v and w in V, supremum of v and w exists in V with respect to $\leq$. We denote the supremum of v and w in V by $v \vee w$. We also write $|v| = v \vee (-v)$ for all $v \in V$.

Let (V, V^+) be a vector lattice. For each pair $v, w \in V$, infimum of v and w also exists in V with respect to $\leq$. We denote it by $v \wedge w$. It is worth to notice that $v \wedge w = -((-v) \vee (-w))$ for all $v, w \in V$.

Let (V, V^+) be a vector lattice with a norm $\| \cdot \|$ such that $(V, \| \cdot \|)$ is a Banach space. Then (V, V^+) is said to an *M-space*, if it satisfies the following two conditions:

(1) For each pair $v, w \in V$ with $|v| \leq |w|$, we have $\|v\| \leq \|w\|$.

(2) $\|v \vee w\| = \max\{\|v\|, \|w\|\}$ for all $x, y \in V^+$.

For more details see [23, 63, 67].

Remark 2.3.1. *[39, Remark 3.3]*

(1) Let (V, V^+) be a vector lattice. Then

(a) $|v| = v$ for all $v \in V^+$.

(b) $|v| \pm v \in V^+$ for all $v \in V$.

(c) $|kv| = |k||v|$ for all $v \in V$ and $k \in \mathbb{R}$.

(d) If $u, v, w \in V$ with $u \wedge v = 0$ and $0 \leq w \leq v$, then $u \wedge w = 0$.

(e) If $u, v, w \in V$ with $u \wedge v = 0$ and $u \wedge w = 0$, then $u \wedge |v \pm w| = 0$.

(2) *Let A be a C^*-algebra and let A_{sa} and A^+ be the collections of all the self-adjoint and all the positive elements in A respectively. Define $|\cdot| : A_{sa} \to A^+$ given by $x \longmapsto (x^2)^{\frac{1}{2}}$. Then*

(a) $|x| = x$ *if* $x \in A^+$;

(b) $|x| \pm x \in A^+$ *for all* $x \in A_{sa}$;

(c) $|k \cdot x| = |k| \cdot |x|$ *for all* $x \in A_{sa}$ *and* $k \in \mathbb{R}$;

(d) *If* x, y *and* $z \in A_{sa}$ *with* $|x - y| = x + y$ *and* $0 \leq z \leq y$, *then* $|x - z| = x + z$;

(e) *If* x, y *and* $z \in A_{sa}$ *with* $|x - y| = x + y$ *and* $|x - z| = x + z$, *then* $|x - |y \pm z|| = x + |y \pm z|$.

These properties lead to the following notion:

Definition 2.3.2. *[39, Definition 3.4] Let (V, V^+) be a real ordered vector space and let $|\cdot| : V \to V^+$ be a mapping satisfying the following conditions:*

(a) $|v| = v$ *if* $v \in V^+$;

(b) $|v| \pm v \in V^+$ *for all* $v \in V$;

(c) $|k \cdot v| = |k| \cdot |v|$ *for all* $v \in V$ *and* $k \in \mathbb{R}$;

(d) *If* u, v *and* $w \in V$ *with* $|u - v| = u + v$ *and* $0 \leq w \leq v$, *then* $|u - w| = u + w$;

(e) *If* u, v *and* $w \in V$ *with* $|u - v| = u + v$ *and* $|u - w| = u + w$, *then* $|u - |v \pm w|| = u + |v \pm w|$.

Then $(V, V^+, |\cdot|)$ is said to be an absolutely ordered space.

Let $(V, V^+, |\cdot|)$ be an absolutely ordered space. We call that $|\cdot|$ is an absolute value on V.

Remark 2.3.3. *[39] Let $(V, V^+, |\cdot|)$ be an absolutely ordered space. Then*

(1) *The cone V^+ is* proper and generating. *In fact, if $\pm v \in V^+$, then by Definition 2.3.2(a) and (c), we get*

$$v = |v| = |-v| = -v$$

so that $v = 0$. Next, by Definition 2.3.2(b), for any $v \in V$, we have

$$v = \frac{1}{2}\left((|v| + v) - (|v| - v)\right) \in V^+ - V^+.$$

(2) *Let $u, v \in V$ be such that $|u - v| = u + v$. Then $u, v \in V^+$. For such a pair $u, v \in V^+$, we shall say that u is* orthogonal *to v and denote it by $u \perp v$.*

(3) *We write, $v^+ := \frac{1}{2}(|v| + v)$ and $v^- := \frac{1}{2}(|v| - v)$. Then $v^+ \perp v^-, v = v^+ - v^-$ and $|v| = v^+ + v^-$. This decomposition is unique in the following sense: If $v = v_1 - v_2$ with $v_1 \perp v_2$, then $v_1 = v^+$ and $v_2 = v^-$. In other words, every element in V has a* unique orthogonal decomposition *in V^+.*

The following result relates absolutely ordered space structure to a vector lattice structure:

Theorem 2.3.4. *[38, Theorem 4.12] Let $(V, V^+, |\cdot|)$ be an absolutely ordered space. For $v, w \in V$, put*

$$v \dot{\vee} w := \frac{1}{2}(v + w + |v - w|).$$

Then the following statements are equivalent:

(i) *$v \dot{\vee} w = sup\{v, w\}$ for all $v, w \in V$.*

(ii) $\dot{\vee}$ *is associative in* V.

(iii) $u \pm v \in V^+$ *implies* $|v| \leq u$ *for all* $u, v \in V$.

(iv) $|v + w| \leq |v| + |w|$ *for all* $v, w \in V$.

We have a relation among all three types of orthogonality mentioned above given in the next result.

Proposition 2.3.5. *[39, Proposition 3.7] Let* $(V, V^+, |\cdot|)$ *be an absolutely ordered vector space and assume that* $\| \cdot \|$ *is a norm on* V *determined by the order unit* e *such that* V^+ *is* $\| \cdot \|$*-closed. Then the following statements are equivalent:*

(A) For each $v \in V$, *we have*

$$\||v|\| = \|v\| = max\{\|v^+\|, \|v^-\|\};$$

(B) For each $u, v \in V^+$, *we have* $u \perp_\infty^a v$ *whenever* $u \perp v$*;*

(C) For each $u, v \in V^+$, *we have* $u \perp_\infty v$ *whenever* $u \perp v$*;*

(D) For each $v \in V$ *with* $\pm v \leq e$, *we have* $|v| \leq e$.

Definition of absolute order unit space is motivated by Proposition 2.3.5 and is given in the following manner:

Definition 2.3.6. *[39, Definition 3.8] Let* $(V, V^+, | \cdot |)$ *be an absolutely ordered space and let* $\| \cdot \|$ *be an order unit norm on* V *determined by the order unit* e *such that* V^+ *is* $\| \cdot \|$*-closed. Then* $(V, V^+, | \cdot |, e)$ *is called an* absolute order unit space, *if* $\perp = \perp_\infty^a$ *on* V^+.

The self-adjoint part of a unital C*-algebra is an absolute order unit space [39, Remark 3.9(1)]. More generally, every unital JB-algebra is an absolute order unit space.

Under the conditions of Theorem 2.3.4, an absolute order unit space becomes an M-space.

Orthogonality in C*-algebras or more generally, absolute ∞-orthogonality in absolute order unit space has a curious by-product. We recall the following result which relates absolute ∞-orthogonality with order unit and absolute value.

Proposition 2.3.7. *[39, Proposition 4.1] Let (V, e) be an absolute order unit space and let $u, v \in V$ be such that $0 \leq u, v \leq e$. Then $u \perp_\infty^a v$ if and only if $u + v \leq e$ and $|u - v| + |e - u - v| = e$.*

The later part of the Proposition 2.3.7 is segregated as a property and the definition is given in the following manner:

Let (V, e) be an absolute order unit space and let $u, v \in V^+$. We say that u is *absolutely compatible* with v (we write, $u \bigtriangleup v$), if $|u - v| + |e - u - v| = e$ [39, Definition 4.3].

Next result provides a bound for absolutely compatible elements.

Proposition 2.3.8. *[39, Proposition 4.2] Let (V, e) be an absolute order unit space and let $u, v \in V^+$. If $u \bigtriangleup v$, then $u, v \leq e$.*

Let us recall order unit property and absolute order unit property defined in [39, Definition 5.1].

Let V be an ordered vector space. For $u \in V^+$, we set

$$V_u = \{v \in V : ku \pm v \in V^+ \text{ for some } k > 0\}.$$

If (V, e) is an order unit space, then $u \in V^+$ is said to have the *order unit property* in V, if for any $v \in V_u$ we have $\pm v \leq \|v\| u$. Moreover, if $\|u\| = 1$ then (V_u, u) is also an order unit space and $\|v\|_u = \|v\|_e$ for each $v \in V_u$. Here $\| \cdot \|_u$ is the order unit norm on V_u determined by u.

If (V, e) is an absolute order unit space, then $u \in V^+$ is said to have the *absolute order unit property* in V, if for any $v \in V_u$ we have $|v| \leq \|v\| u$. Moreover, if $\|u| = 1$ then (V_u, u) is also an absolute order unit space and $\|v\|_u = \|v\|_e$ for each $v \in V_u$.

Finally, we recall the notion of order projections in absolute order unit spaces. Before it, we recall the following characterization of projections in a unital C*-algebra:

Theorem 2.3.9. *[39, Theorem 5.3] Let A be a unital C^*-algebra and let $x \in A$ be such that $0 \leq x \leq 1$. Then the following facts are equivalent:*

(1) x is a projection in A;

(2) x has the order unit property in A;

(3) $x \perp (1 - x)$.

By equivalence of (1) and (3) in the Theorem 2.3.9, the notion of order projections in absolute order unit spaces is defined in the following way:

Definition 2.3.10. *[39, Definition 5.2] Let (V, e) be an absolute order unit space. Let $p \in V$ be such that $0 \leq p \leq e$. We say that p is an* order projection*, if $p \perp e - p$. We write $\mathcal{OP}(V)$ for the set of all the order projections in V.*

We can characterize orthogonality among order projections in the following sense:

Proposition 2.3.11. *Let (V, e) be an absolute order unit space.*

(1) [39, Proposition 5.5] Let $p, q \in \mathcal{OP}(V)$. Then the following statements are equivalent:

(a) $p + q \leq e$;

(b) $p \perp q$;

(c) $p + q \in \mathcal{OP}(V)$; and

(d) $p \perp_\infty q$.

(2) [39, Proposition 5.6] Let $u, v \in V$ be such that $0 \leq u, v \leq e$. If $u + v \in \mathcal{OP}(V)$ with $u \perp v$, then $u, v \in \mathcal{OP}(V)$.

(3) Let $p, q \in \mathcal{OP}(V)$ be such that $p \leq q$. Then

(i) $q - p \in \mathcal{OP}(V)$ [39, Corollary 5.7].

(ii) $p \perp (q - p)$ [39, Remark 5.8].

Next result characterizes order projections in terms of the order unit property and the absolute order unit property.

Proposition 2.3.12. *[39, Proposition 6.1] Let (V, e) be an absolute order unit space and let $u \in V$ be such that $0 \leq u \leq e$. Then the following facts are equivalent:*

(a) $u \in \mathcal{OP}(V)$;

(b) u and $e - u$ have the order unit property;

(c) u and $e - u$ have the absolute order unit property.

Chapter 3

Absolute matrix order unit spaces

In the first section of this chapter, we define absolutely matrix ordered spaces. In the second section, we define absolute matrix order unit spaces. These notions are matricial versions of absolutely ordered spaces and absolute order unit spaces respectively in the context of matrix ordered space. We also study some properties of these spaces. In the third section, we extend the notion of orthogonality in absolutely matrix ordered spaces.

3.1 Absolutely matrix ordered spaces

In this section, we introduce absolutely matrix ordered spaces and prove some properties of these spaces. We also show that under a certain condition absolute value function in an absolutely matrix ordered space is positive definite. We begin with the following notion:

Definition 3.1.1. *Let* $(V, \{ M_n(V)^+\})$ *be a matrix ordered space and assume* *that* $|\cdot|_{m,n} : M_{m,n}(V) \to M_n(V)^+$ *for* $m, n \in \mathbb{N}$. *Let us write* $|\cdot|_{n,n} = |\cdot|_n$ *for every* $n \in \mathbb{N}$. *Then* $(V, \{M_n(V)^+\}, \{|\cdot|_{m,n}\})$ *is called an* absolutely matrix ordered space, *if it satisfies the following conditions:*

1. *For all* $n \in \mathbb{N}$, $(M_n(V)_{sa}, M_n(V)^+, |\cdot|_n)$ *is an absolutely ordered space;*

2. *For* $v \in M_{m,n}(V)$ *and* $a \in M_{l,m}$, *we have*

$$|av|_{l,n} \leq \|a\| |v|_{m,n};$$

3. *For* $v \in M_{m,n}(V)$ *and* $b \in M_{n,t}$, *we have*

$$|vb|_{m,t} = ||v|_{m,n} b|_{n,t};$$

4. *For* $v \in M_{m,n}(V)$ *and* $w \in M_{l,t}(V)$, *we have*

$$|v \oplus w|_{m+l,n+t} = |v|_{m,n} \oplus |w|_{l,t}.$$

Let $(V, \{ M_n(V)^+\}, \{|\cdot|_{m,n}\})$ be an absolutely matrix ordered space. We call that $|\cdot|_{m,n}$ is an absolute value on $M_{m,n}(V)$ for all $m, n \in \mathbb{N}$.

Proposition 3.1.2. *Let* $(V, \{M_n(V)^+\}, \{|\cdot|_{m,n}\})$ *be an absolutely matrix ordered space.*

1. *If* $a \in M_{l,m}$ *is an isometry i.e.* $a^*a = I_m$, *then* $|av|_{l,n} = |v|_{m,n}$ *for any* $v \in M_{m,n}(V)$.

2. *If* $v \in M_{m,n}(V)$, *then* $\left| \begin{bmatrix} 0_m & v \\ v^* & 0_n \end{bmatrix} \right|_{m+n} = |v^*|_{n,m} \oplus |v|_{m,n}$.

3. $\begin{bmatrix} |v^*|_{n,m} & v \\ v^* & |v|_{m,n} \end{bmatrix} \in M_{m+n}(V)^+$ *for any* $v \in M_{m,n}(V)$.

4. $|v|_{m,n} = \left| \begin{bmatrix} v \\ 0 \end{bmatrix} \right|_{m+l,n}$ *for any* $v \in M_{m,n}(V)$ *and* $l \in \mathbb{N}$.

5. $|v|_{m,n} \oplus 0_t = \left| \begin{bmatrix} v & 0 \end{bmatrix} \right|_{m,n+t}$ *for any* $v \in M_{m,n}(V)$ *and* $t \in \mathbb{N}$.

Proof. (1) Let $a \in M_{l,m}$ be an isometry. Then, using 3.1.1(2), we get that

$$|av|_{l,n} \leq \|a\| |v|_{m,n} = |a^*av|_{m,n} \leq \|a^*\| |av|_{l,n} = |av|_{l,n}.$$

Thus $|av|_{l,n} = |v|_{m,n}$.

(2) Put $a = \begin{bmatrix} 0_{n,m} & I_n \\ I_m & 0_{m,n} \end{bmatrix} \in M_{n+m}$. Then a is an isometry with

$$a \begin{bmatrix} 0_m & v \\ v^* & 0_n \end{bmatrix} = \begin{bmatrix} v^* & 0_n \\ 0_m & v \end{bmatrix}.$$

Now, by (1) and 3.1.1(4), it follows that

$$\left| \begin{bmatrix} 0_m & v \\ v^* & 0_n \end{bmatrix} \right|_{m+n} = \left| \begin{bmatrix} v^* & 0_n \\ 0_m & v \end{bmatrix} \right|_{m+n} = |v^*|_{n,m} \oplus |v|_{m,n}.$$

(3) As $\begin{bmatrix} 0_m & v \\ v^* & 0_n \end{bmatrix} \in M_{m+n}(V)_{sa}$, therefore by 2.3.2(b)

$$\begin{bmatrix} |v^*|_{n,m} & v \\ v^* & |v|_{m,n} \end{bmatrix} = \left| \begin{bmatrix} 0_m & v \\ v^* & 0_n \end{bmatrix} \right|_{m+n} + \begin{bmatrix} 0_m & v \\ v^* & 0_n \end{bmatrix} \in M_{m+n}(V)^+.$$

(4) $av = \begin{bmatrix} v \\ 0 \end{bmatrix} \in M_{m+l,n}$ for $a = \begin{bmatrix} I_m \\ 0 \end{bmatrix} \in M_{m+l,m}$. Since $a^*a = I_m$, by (1), we conclude that $|v|_{m,n} = \left| \begin{bmatrix} v \\ 0 \end{bmatrix} \right|_{m+l,n}$, if $v \in M_{m,n}(V)$ and $l \in \mathbb{N}$.

(5) For $a = \begin{bmatrix} I_m \\ 0 \end{bmatrix} \in M_{m+l,m}$, we get that $a \begin{bmatrix} v & 0 \end{bmatrix} = \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} \in M_{m+l,n+t}(V)$. Since $a^*a = I_m$, again using (1)

$$\left| \begin{bmatrix} v & 0 \end{bmatrix} \right|_{m,n+t} = \left| \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} \right|_{m+l,n+t} = \begin{bmatrix} |v|_{m,n} & 0 \\ 0 & 0 \end{bmatrix},$$

if $v \in M_{m,n}(V)$ and $l, t \in \mathbb{N}$.

□

Proposition 3.1.3. *Let V be an absolutely matrix ordered space and let $v \in M_{m,n}(V)$ for some $m, n \in \mathbb{N}$. Then $v = 0$ if and only if $|v|_{m,n} = 0$ and $|v^*|_{n,m} = 0$. If $M_{m+n}(V)^+$ is Archimedean, then $v = 0$ if and only if $|v|_{m,n} = 0$.*

Proof. If $v = 0$, then by the definition, we have $|v|_{m,n} = 0$ and that $|v^*|_{n,m} = 0$. Conversely assume that $|v|_{m,n} = 0$ and $|v^*|_{n,m} = 0$. By Proposition 3.1.2(3), we deduce that $\pm \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \in M_{m+n}(V)^+$. Since $M_{m+n}(V)^+$ is proper, we have $\begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} = 0$ so that $v = 0$.

Now, we assume that $M_{m+n}(V)^+$ is Archimedean. If $|v|_{m,n} = 0$, then as above, we have $\begin{bmatrix} |v^*|_{n,m} & v \\ v^* & 0 \end{bmatrix} \in M_{m+n}(V)^+$. Let $k > 0$ be any real number and consider $a = \begin{bmatrix} kI_m & 0 \\ 0 & k^{-1}I_n \end{bmatrix}$. Then

$$\begin{bmatrix} k^2|v^*|_{n,m} & v \\ v^* & 0 \end{bmatrix} = a^* \begin{bmatrix} |v^*|_{n,m} & v \\ v^* & 0 \end{bmatrix} a \in M_{m+n}(V)^+.$$

It follows that $\begin{bmatrix} k|v^*|_{n,m} & \pm v \\ \pm v^* & 0 \end{bmatrix} \in M_{m+n}(V)^+$ for all $k \in \mathbb{R}$, $k > 0$. Since $M_{m+n}(V)^+$ is Archimedean, we have $\pm \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \in M_{m+n}(V)^+$. Now as $M_{m+n}(V)^+$ is proper, we conclude that $v = 0$. □

Lemma 3.1.4. *Let V be an absolutely matrix ordered space and let $n \in \mathbb{N}$. Then $|a^*va|_n = a^*|v|_n a$, if $v \in M_n(V)$ and $a \in M_n$ is a unitary.*

Proof. By 3.1.1(2), we have

$$|a^*va|_n \leq \|a^*\|||v|_n a|_n = ||v|_n a|_n = |a(a^*|v|_n a)|_n \leq a^*|v|_n a.$$

Thus $|a^*va|_n \leq a^*|v|_n a$. Now replacing v by ava^*, we get $|v|_n \leq a^*|ava^*|_n a$ so that $a|v|_n a^* \leq |ava^*|_n$. Finally, interchanging a and a^*, we get that $a^*|v|_n a \leq |a^*va|_n$.

Hence $|a^*va|_n = a^*|v|_n a$. $\square$

3.2 Absolute matrix order unit spaces

In this section, we define absolute matrix order unit spaces and prove some facts about it. We observe that absolute value function in an absolute matrix order unit space is positive definite. We also prove that absolute value function is an isometry.

Definition 3.2.1. *Let $(V, \{M_n(V)^+\}, e)$ be a matrix order unit space such that*

(a) $(V, \{M_n(V)^+\}, \{|\cdot|_{m,n}\})$ is an absolutely matrix ordered space; and

(b) $\perp = \perp_\infty^a$ on $M_n(V)^+$ for all $n \in \mathbb{N}$.

Then $(V, \{M_n(V)^+\}, \{|\cdot|_{m,n}\}, e)$ is called an absolute matrix order unit space.

Example 3.2.2. *A unital C^*-algebra is an absolute matrix order unit space. Let A be a unital C^*-algebra with unity 1_A. Then, for each $n \in \mathbb{N}$, $M_n(A)$ is a C^*-algebra with unity element I_A^n (where $I_A^n := 1_A \oplus \cdots \oplus 1_A \in M_n(A)$). If $M_n(A)^+$ denotes the set of all the positive elements in $M_n(A)$, then $(A, \{M_n(A)^+\}_{n\in\mathbb{N}}, 1_A)$ is a matrix order unit space.*

For $m, n \in \mathbb{N}$, define $|\cdot|_{m,n} : M_{m,n}(A) \longrightarrow M_n(A)^+$ given by $|x|_{m,n} = (x^*x)^{\frac{1}{2}}$ for all $x \in M_{m,n}(A)$. We show that $(A, \{M_n(A)^+\}, \{|\cdot|_{m,n}\}, 1_A)$ is an absolute matrix order unit space.

1. Let $x \in M_{m,n}(A)$ and $a \in M_{l,m}$. Then

$$\begin{aligned} |ax|_{l,n}^2 &= (ax)^*(ax) \\ &= x^*(a^*a)x \\ &\leq \|a\|^2(x^*x) \\ &= \|a\|^2|x|_{m,n}^2. \end{aligned}$$

Thus $|ax|_{l,n} \leq \|a\||x|_{m,n}$.

2. *Let* $x \in M_{m,n}(A)$ *and* $b \in M_{n,t}$. *Then*

$$\begin{aligned} |xb|_{m,t}^2 &= (xb)^*(xb) \\ &= b^*x^*xb \\ &= b^*|x|_{m,n}^2 b \\ &= ((|x|_{m,n}b)^*(|x|_{m,n}b)) \\ &= ||x|_{m,n}b|_{n,t}^2. \end{aligned}$$

Thus $|xb|_{m,t} = ||x|_{m,n}b|_{n,t}$.

3. *Next, let* $x \in M_{m,n}(A)$ *and* $y \in M_{l,t}(A)$. *Then*

$$|x \oplus y|_{m+l,n+t}^2 = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}^* \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$$

$$\begin{aligned} &= \begin{bmatrix} x^*x & 0 \\ 0 & y^*y \end{bmatrix} \\ &= \begin{bmatrix} |x|_{m,n}^2 & 0 \\ 0 & |y|_{l,t}^2 \end{bmatrix} \\ &= \begin{bmatrix} |x|_{m,n} & 0 \\ 0 & |y|_{l,t} \end{bmatrix}^2. \end{aligned}$$

Thus $|x \oplus y|_{m+l,n+t} = |x|_{m,n} \oplus |y|_{l,t}$.

Remark 3.2.3. *Let* V *be an absolute order unit space and let* $v \in M_{m,n}(V)$ *for some* $m, n \in \mathbb{N}$. *Then* $v = 0$ *if and only if* $|v|_{m,n} = 0$.

Proposition 3.2.4. *Let* V *be an absolute matrix order unit space and let* $v \in M_n(V)$ *for some* $n \in \mathbb{N}$. *Then* $\| |v|_n \|_n = \|v\|_n = \| |v^*|_n \|_n$. *In particular, if* $v \in M_{m,n}(V)$ *for some* $m, n \in \mathbb{N}$, *then*

$$\| |v|_{m,n} \|_n = \|v\|_{m,n} = \| |v^*|_{n,m} \|_m.$$

Proof. If $v = 0$, then $|v|_n = 0$ and $|v^*|_n = 0$ so that the result holds trivially. Thus we may assume that $v \neq 0$. Then $|v|_n \neq 0$ and $|v^*|_n \neq 0$. As

$$\begin{aligned} \|v\|_n &= \left\| \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \right\|_{2n} \\ &= \left\| \left| \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \right|_{2n} \right\|_{2n} \\ &= \left\| \begin{bmatrix} |v^*|_n & 0 \\ 0 & |v|_n \end{bmatrix} \right\|_{2n} \end{aligned}$$

$$= \quad max\{\||v|_n\|_n, \||v^*|_n\|_n\},$$

we have $\||v|_n\|_n, \||v^*|_n\|_n \leq \|v\|_n$. Now, as in the proof of Proposition 3.1.3, we may deduce that $\begin{bmatrix} k|v^*|_n & \pm v \\ \pm v^* & k^{-1}|v|_n \end{bmatrix} \in M_{2n}(V)^+$ for all $k \in \mathbb{R}, k > 0$. Put $k = \sqrt{\frac{\||v|_n\|_n}{\||v^*|_n\|_n}}$. Since $|v|_n \leq \||v|_n\|_n e^n$ and since $|v^*|_n \leq \||v^*|_n\|_n e^n$, we may further conclude that $\begin{bmatrix} k_0 e^n & \pm v \\ \pm v^* & k_0 e^n \end{bmatrix} \in M_{2n}(V)^+$ where $k_0 = \sqrt{\||v|_n\|_n \||v^*|_n\|_n}$. Thus $\|v\|_n \leq \sqrt{\||v|_n\|_n \||v^*|_n\|_n}$. Now

$$\sqrt{\||v|_n\|_n \||v^*|_n\|_n} \leq max\{\||v|_n\|_n, \||v^*|_n\|_n\} = \|v\|_n$$

so that $\|v\|_n = \sqrt{\||v|_n\|_n \||v^*|_n\|_n}$. Hence $\||v|_n\|_n = \|v\|_n = \||v^*|_n\|_n$.

Next, let $v \in M_{m,n}(V)$ for some $m, n \in \mathbb{N}$ with $m \neq n$. Without loss of generality, we assume that $m > n$. Then

$$\begin{aligned} \|v\|_{m,n} &= \left\| \begin{bmatrix} v & 0 \end{bmatrix} \right\|_m \\ &= \left\| \left| \begin{bmatrix} v & 0 \end{bmatrix} \right|_m \right\|_m \\ &= \||v|_{m,n} \oplus 0\|_m \\ &= \||v|_{m,n}\|_n \end{aligned}$$

and

$$\begin{aligned}
\|v\|_{m,n} &= \left\| \begin{bmatrix} v & 0 \end{bmatrix} \right\|_m \\
&= \left\| \left| \begin{bmatrix} v & 0 \end{bmatrix}^* \right|_m \right\|_m \\
&= \left\| \left| \begin{bmatrix} v^* \\ 0 \end{bmatrix} \right|_m \right\|_m \\
&= \| |v^*|_{n,m} \|_m .
\end{aligned}$$

□

3.3 Extending orthogonality

3.3.1 Extending orthogonality on absolutely ordered spaces

In this subsection, we extend the notion of orthogonality to general elements in an absolutely ordered space. Let V be an absolutely ordered space and let $u, v \in V$. We say that u is *orthogonal* to v, we still write $u \perp v$, if $|u| \perp |v|$ (or equivalently $||u| - |v|| = |u| + |v|$).

In the next result, we characterize orthogonality of general elements in an absolutely ordered space in terms of the orthogonality among positive elements.

Proposition 3.3.1. *Let $(V, V^+, |\cdot|)$ be an absolutely ordered space and let $u, v \in V$. Then the following statements are equivalent:*

1. *$u \perp v$;*

2. *u^+, u^-, v^+, v^- are mutually orthogonal;*

3. *$|u \pm v| = |u| + |v|$.*

Proof. (1) implies (2): Let $u \perp v$ or equivalently $|u| \perp |v|$. As $0 \leq u^+, u^- \leq |u|$ and $0 \leq v^+, v^- \leq |v|$, a repeated use of the definition yields that u^+, u^-, v^+, v^- are mutually orthogonal.

(2) implies (1): Let u^+, u^-, v^+, v^- be mutually orthogonal. Then by the definition, we get $(u^+ + u^-) \perp (v^+ + v^-)$ for $|u^+ + u^-| = u^+ + u^-$ and $|v^+ + v^-| = v^+ + v^-$. That is, $|u| \perp |v|$ so that $u \perp v$.

(2) implies (3): Again, let u^+, u^-, v^+, v^- be mutually orthogonal. Then by the definition, once again, we have $(u^+ + v^+) \perp (u^- + v^-)$ and $(u^+ + v^-) \perp (u^- + v^+)$. Thus

$$\begin{aligned} |u + v| &= |u^+ - u^- + v^+ - v^-| \\ &= |(u^+ + v^+) - (u^- + v^-)| \\ &= u^+ + v^+ + u^- + v^- = |u| + |v| \end{aligned}$$

and

$$\begin{aligned} |u - v| &= |u^+ - u^- - v^+ + v^-| \\ &= |(u^+ + v^-) - (u^- + v^+)| \\ &= u^+ + v^- + u^- + v^+ = |u| + |v|. \end{aligned}$$

(3) implies (2): Finally, assume that $|u \pm v| = |u| + |v|$. Then as before, we may get that $(u^+ + v^+) \perp (u^- + v^-)$ and $(u^+ + v^-) \perp (u^- + v^+)$. Thus u^+, u^-, v^+, v^- are mutually orthogonal. $\square$

Corollary 3.3.2. *[39, Remark 3.5] Let $(V, V^+, |\cdot|)$ be an absolutely ordered space and let $u, v, w \in V^+$ be such that $u \perp v$ and $u \perp w$. Then $|u - (v \pm w)| = u + |v \pm w|$.*

Proof. Since $u \perp v$ and $u \perp w$, by Definition 2.3.2(e), we get that $u \perp |v \pm w|$.

Thus $u \perp (v \pm w)$ so that $|u - (v \pm w)| = u + |v \pm w|$. □

3.3.2 Matricial version of orthogonality

Now, we extend the notion of orthogonality to the general elements in an absolutely matrix ordered space.

Definition 3.3.3. *Let $(V, \{M_n(V)^+\}, \{|\cdot|_{m,n}\})$ be an absolutely matrix ordered space and let $u, v \in M_{m,n}(V)$ for some $m, n \in \mathbb{N}$. We say that u is* orthogonal *to v (we continue to write, $u \perp v$), if $\begin{bmatrix} 0 & u \\ u^* & 0 \end{bmatrix} \perp \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix}$ in $M_{m+n}(V)_{sa}$.*

Remark 3.3.4. *Let V be an absolutely matrix ordered space.*

1. *Let $u_1, u_2 \in M_m(V)^+$ and $v_1, v_2 \in M_n(V)^+$ for some $m, n \in \mathbb{N}$. Then $u_1 \oplus v_1 \perp u_2 \oplus v_2$ in $M_{m+n}(V)^+$ if and only if $u_1 \perp u_2$ and $v_1 \perp v_2$ in $M_m(V)^+$ and $M_n(V)^+$ respectively. In fact, by 3.1.1(4), we have $u_1 \perp u_2$ and $v_1 \perp v_2$ if and only if*

$$\begin{aligned} \left| \begin{bmatrix} u_1 & 0 \\ 0 & v_1 \end{bmatrix} - \begin{bmatrix} u_2 & 0 \\ 0 & v_2 \end{bmatrix} \right| &= \begin{bmatrix} |u_1 - u_2| & 0 \\ 0 & |v_1 - v_2| \end{bmatrix} \\ &= \begin{bmatrix} u_1 + u_2 & 0 \\ 0 & v_1 + v_2 \end{bmatrix} \\ &= \begin{bmatrix} u_1 & 0 \\ 0 & v_1 \end{bmatrix} + \begin{bmatrix} u_2 & 0 \\ 0 & v_2 \end{bmatrix} \end{aligned}$$

if and only if

$$\begin{bmatrix} u_1 & 0 \\ 0 & v_1 \end{bmatrix} \perp \begin{bmatrix} u_2 & 0 \\ 0 & v_2 \end{bmatrix}.$$

2. *In particular, for* $u, v \in M_{m,n}(V)$, *we have* $u \perp v$ *if and only if* $|u|_{m,n} \perp |v|_{m,n}$ *and* $|u^*|_{n,m} \perp |v^*|_{n,m}$ *(using Proposition 3.1.2(2)).*

Proposition 3.3.5. *Let* V *be an absolutely matrix ordered space and let* $u, v \in M_{m,n}(V)$ *for some* $m, n \in \mathbb{N}$.

1. *Then* $u \perp v$ *if and only if* $|u \pm v|_{m,n} = |u|_{m,n} + |v|_{m,n}$ *and* $|u^* \pm v^*|_{n,m} = |u^*|_{n,m} + |v^*|_{n,m}$.

2. *If* $u \perp v$, *then* $\left| \begin{bmatrix} u \\ v \end{bmatrix} \right|_{2m,n} = |u|_{m,n} + |v|_{m,n}$ *and* $\left| \begin{bmatrix} u & v \end{bmatrix} \right|_{m,2n} = |u|_{m,n} \oplus |v|_{m,n}$.

Proof. (1). First, assume that $u \perp v$. Then by Proposition 3.3.1, we have

$$\begin{aligned}
\begin{bmatrix} |u^* \pm v^*|_{n,m} & 0 \\ 0 & |u \pm v|_{m,n} \end{bmatrix} &= \left| \begin{bmatrix} 0 & u \\ u^* & 0 \end{bmatrix} \pm \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \right|_{m+n} \\
&= \left| \begin{bmatrix} 0 & u \\ u^* & 0 \end{bmatrix} \right|_{m+n} + \left| \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \right|_{m+n} \\
&= \begin{bmatrix} |u^*|_{n,m} & 0 \\ 0 & |u|_{m,n} \end{bmatrix} + \begin{bmatrix} |v^*|_{n,m} & 0 \\ 0 & |v|_{m,n} \end{bmatrix} \\
&= \begin{bmatrix} |u^*|_{n,m} + |v^*|_{n,m} & 0 \\ 0 & |u|_{m,n} + |v|_{m,n} \end{bmatrix}.
\end{aligned}$$

Thus $|u \pm v|_{m,n} = |u|_{m,n} + |v|_{m,n}$ and $|u^* \pm v^*|_{n,m} = |u^*|_{n,m} + |v^*|_{n,m}$. Now tracing back the proof, we can also prove the converse.

(2). First, we observe that

$$\left|\begin{bmatrix} u \\ 0 \end{bmatrix}^*\right|_{n,2m} = \begin{bmatrix} |u^*|_{n,m} & 0 \\ 0 & 0 \end{bmatrix} \perp \begin{bmatrix} 0 & 0 \\ 0 & |v^*|_{n,m} \end{bmatrix} = \left|\begin{bmatrix} 0 \\ v \end{bmatrix}^*\right|_{n,2m}$$

and that

$$\left|\begin{bmatrix} u \\ 0 \end{bmatrix}\right|_{2m,n} = |u|_{m,n} \perp |v|_{m,n} = \left|\begin{bmatrix} 0 \\ v \end{bmatrix}\right|_{2m,n}.$$

Thus by Remark 3.3.4(2) and by (1), we get

$$\left|\begin{bmatrix} u \\ v \end{bmatrix}\right|_{2m,n} = \left|\begin{bmatrix} u \\ 0 \end{bmatrix}\right|_{2m,n} + \left|\begin{bmatrix} 0 \\ v \end{bmatrix}\right|_{2m,n} = |u|_{m,n} + |v|_{m,n}.$$

Similarly, we have

$$\begin{aligned} \left|\begin{bmatrix} u & v \end{bmatrix}\right|_{m,2n} &= \left|\begin{bmatrix} u & 0 \end{bmatrix}\right|_{m,2n} + \left|\begin{bmatrix} 0 & v \end{bmatrix}\right|_{m,2n} \\ &= \begin{bmatrix} |u|_{m,n} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & |v|_{m,n} \end{bmatrix} \\ &= |u|_{m,n} \oplus |v|_{m,n} \end{aligned}$$

for

$$\left|\begin{bmatrix} u & 0 \end{bmatrix}\right|_{m,2n} = \begin{bmatrix} |u|_{m,n} & 0 \\ 0 & 0 \end{bmatrix} \perp \begin{bmatrix} 0 & 0 \\ 0 & |v|_{m,n} \end{bmatrix} = \left|\begin{bmatrix} 0 & v \end{bmatrix}\right|_{m,2n}$$

and

$$\left|\begin{bmatrix} u & 0 \end{bmatrix}^*\right|_{2n,m} = |u^*|_{n,m} \perp |v^*|_{n,m} = \left|\begin{bmatrix} 0 & v \end{bmatrix}^*\right|_{2n,m}.$$

$\square$

Chapter 4

Absolute value preserving maps

In the first section of this chapter, we define absolute value preserving maps between absolutely ordered spaces and study some properties of such maps. In the second section, we prove that a unital bijective linear map between two absolute order unit spaces is absolute value preserving if and only if it is an isometry. From here, we deduce the following known result: Unital, bijective absolute value preserving maps between two unital JB-algebras are precisely Jordan isomorphisms. In the third section, we extend the notion of absolute value preserving maps to completely absolute value preserving maps between absolutely matrix ordered spaces. We prove that a unital bijective $*$-linear map between two absolute matrix order unit spaces is completely absolute value preserving if and only if it is a complete isometry. Again from this, we deduce a known result that on (unital) C*-algebras, unital bijective completely absolute value preserving maps are precisely C*-algebra isomorphisms. We give a simple, order-theoretic proof using a trick which is apparently new. In the fourth section, we define notion of orthogonality of positive maps between absolutely ordered spaces. We also extend this notion to orthogonality of completely positive maps between absolutely matrix ordered spaces and prove that sum of two orthogonal completely absolute value preserving maps is a completely absolute value preserving map.

4.1 Absolute value preservers on absolutely ordered spaces

In this section, we define absolute value preserving maps between two absolutely ordered spaces and characterize such maps in terms of orthogonality of positive elements. We introduce the notions of absolutely ordered subspaces and absolute order ideals in absolutely ordered spaces and prove that kernel and image of absolutely preserving maps are absolute order ideal and absolutely ordered subspaces respectively. We also show that the quotient of an absolutely ordered space by the kernel of an absolute value preserving map is an absolutely ordered space.

Definition 4.1.1. *Let V and W be absolutely ordered spaces. A linear map $\phi : V \to W$ is said to be an* absolute value preserving map *($|\cdot|$-preserving map, in short), if $\phi(|v|) = |\phi(v)|$ for all $v \in V$.*

Remark 4.1.2. *Let $\phi : V \to W$ be a bijective, linear and $|\cdot|$-preserving map. Then ϕ^{-1} is also a bijective, linear and $|\cdot|$-preserving map. To see this, let $w \in W$. Since ϕ is bijective, there exists a unique $v \in V$ such that $\phi(v) = w$. Now*

$$\phi(|\phi^{-1}(w)|) = \phi(|v|) = |\phi(v)| = |w|$$

so that $|\phi^{-1}(w)| = \phi^{-1}(|w|)$. Hence ϕ^{-1} is $|\cdot|$-preserving.

The next result is an elementary characterization of $|\cdot|$-preserving maps.

Proposition 4.1.3. *Let V and W be absolutely ordered spaces and let $\phi : V \to W$ be a linear map. Then the following statements are equivalent:*

(1) ϕ *is* $|\cdot|$*-preserving;*

(2) $\phi \geq 0$ *and* $\phi(v_1) \perp \phi(v_2)$ *for all* $v_1, v_2 \in V^+$ *with* $v_1 \perp v_2$*;*

(3) $\phi(v^+) = \phi(v)^+$ *for all* $v \in V$*;*

(4) $\phi(v^-) = \phi(v)^-$ *for all* $v \in V$*.*

Proof. (1) $\Longrightarrow$ (2): Let $v \in V$ with $v \geq 0$. Now, as ϕ is an $|\cdot|$-preserving and by 2.3.2(a), we have $\phi(v) = \phi(|v|) = |\phi(v)| \geq 0$, therefore $\phi \geq 0$. Let $v_1, v_2 \in V^+$ be such that $v_1 \perp v_2$. Put $v = v_1 - v_2$. Then $|v| = v_1 + v_2$. Since ϕ is an additive $|\cdot|$-map (by (1)), we get $\phi(v_1) + \phi(v_2) = \phi(|v|) = |\phi(v)| = |\phi(v_1) - \phi(v_2)|$. Thus $\phi(v_1), \phi(v_2) \in W^+$ with $\phi(v_1) \perp \phi(v_2)$.

(2) $\Longrightarrow$ (3): Let $v \in V$. Then $v^+ \perp v^-$ so that by (2), $\phi(v^+) \perp \phi(v^-)$. As $\phi(v) = \phi(v^+) - \phi(v^-)$, we get $\phi(v)^+ = \phi(v^+), \phi(v)^- = \phi(v^-)$.

(3) $\Longrightarrow$ (4): If we use the fact, $v^- = (-v)^+$.

(4) $\Longrightarrow$ (1): Let $v \in V$. Then $|v| = v^+ + v^- = (-v)^- + v^-$. Thus by (4), we get

$$\phi(|v|) = \phi(-v)^- + \phi(v)^- = \phi(v)^+ + \phi(v)^- = |\phi(v)|.$$

$\square$

Definition 4.1.4. *Let* $(V, V^+, |\cdot|)$ *be an absolutely ordered space. Let* W *be a vector subspace of* V *and put* $W^+ := W \cap V^+$*. Then* W *is said to be an* absolutely ordered subspace *of* $(V, V^+, |\cdot|)$*, if* $|w| \in W^+$ *for all* $w \in W$*. A vector subspace* W *of* V *which is an order ideal of* (V, V^+) *and an absolutely ordered subspace of* $(V, V^+, |\cdot|)$ *is called an* absolute order ideal *of* $(V, V^+, |\cdot|)$*.*

Theorem 4.1.5. *Let* V *and* W *be absolutely ordered spaces and let* $\phi : V \to W$ *be a linear* $|\cdot|$*-preserving map. Then*

(1) $\ker(\phi)$ *is an absolute order ideal of* V*.*

(2) $\phi(V)$ *is an absolutely ordered subspace of* W*. In particular,* $\phi(V)^+ = \phi(V^+)$*.*

(3) For each $v \in V$*, we put* $|v + \ker(\phi)| = |v| + \ker(\phi)$*. Then*

$$(V/\ker(\phi), \left(V/\ker(\phi)\right)^+, |\cdot|)$$

is also an absolutely ordered space, where

$$\left(V/\ker(\phi)\right)^+ := \{v + \ker(\phi) : v \in V^+\}.$$

Proof. (1) Let $v \in \ker(\phi)$. Then $\phi(v) = 0$ so that $0 = |\phi(v)| = \phi(|v|)$. Thus $|v| \in \ker(\phi)$ and consequently, $\ker(\phi)$ is an absolutely ordered subspace of V. Now, let $v \in V^+, u \in \ker(\phi)$ such that $v \leq u$. Since $\phi \geq 0$, thus $0 \leq \phi(v) \leq \phi(u)$. But $\phi(u) = 0$ so that $\pm\phi(v) \in W^+$. As W^+ is proper, we get that $\phi(v) = 0$ *i.e.* $v \in \ker(\phi)$, therefore $\ker(\phi)$ is an order ideal.

(2) Let $w \in \phi(V)$, say $w = \phi(v)$ for some $v \in V$. Then $\phi(v^+) = \phi(v)^+ = w^+ \in \phi(V)$ and $\phi(v^-) = \phi(v)^- = w^- \in \phi(V)$. Thus $\phi(V)$ is an absolutely ordered subspace of W. Next, if $w \in \phi(V)^+$, then $\phi(v)^- = w^- = 0$ so that $w = \phi(v) = \phi(v^+)$. Thus $\phi(V)^+ \subset \phi(V^+)$. Now, being $|\cdot|$-preserving, $\phi \geq 0$ so that $\phi(V^+) \subset \phi(V)^+$. Hence $\phi(V^+) = \phi(V)^+$.

(3) By Proposition 2.1.1, we note that $\left(V/\ker(\phi)\right)^+$ is a proper cone of $V/\ker(\phi)$. We show that the absolute value is well defined in $V/\ker(\phi)$. To see this, let $u, v \in V$ such that $u + \ker(\phi) = v + \ker(\phi)$. Then $\phi(u) = \phi(v)$. Now, ϕ is $|\cdot|$-preserving so that $\phi(|u|) = |\phi(u)| = |\phi(v)| = \phi(|v|)$ and hence $|u| + \ker(\phi) = |v| + \ker(\phi)$.

(a) Let $v \in V$ with $v + \ker(\phi) \in \big(V/\ker(\phi)\big)^+$. There exists $v_0 \in V^+$ such that $v + \ker(\phi) = v_0 + \ker(\phi)$. Thus

$$|v + \ker(\phi)| = |v_0 + \ker(\phi)| = |v_0| + \ker(\phi) = v_0 + \ker(\phi) = v + \ker(\phi).$$

(b) Let $v \in V$. Then

$$|v + \ker(\phi)| \pm (v + \ker(\phi)) = (|v| \pm v) + \ker(\phi) \in \big(V/\ker(\phi)\big)^+.$$

(c) Let $k \in \mathbb{R}$. Then

$$\begin{aligned} |k(v + \ker(\phi))| &= |kv + \ker(\phi)| = |kv| + \ker(\phi) \\ &= |k||v| + \ker(\phi) = |k|(|v| + \ker(\phi)) \\ &= |k||v + \ker(\phi)|. \end{aligned}$$

(d) Let $u, v, w \in V$ such that $u + \ker(\phi), v + \ker(\phi), w + \ker(\phi) \in \big(V/\ker(\phi)\big)^+$ with $|u - v| + \ker(\phi) = u + v + \ker(\phi)$ and $w + \ker(\phi) \leq v + \ker(\phi)$. Then $|\phi(u - v)| = \phi(|u - v|) = \phi(u + v)$ and $0 \leq \phi(w) \leq \phi(v)$. Since $\phi(V)$ is an absolutely ordered space, we may conclude that

$$\phi(|u - w|) = |\phi(u) - \phi(w)| = \phi(u) + \phi(w).$$

Thus $|u - w| + \ker(\phi) = u + w + \ker(\phi)$.

(e) Let $|u - v| + \ker(\phi) = u + v + \ker(\phi)$ and $|u - w| + \ker(\phi) = u + w + \ker(\phi)$. Then $|\phi(u) - \phi(v)| = \phi(u) + \phi(v)$ and $|\phi(u) - \phi(w)| = \phi(u) + \phi(w)$. Since $\phi(V)$ is an absolutely ordered space, we may conclude that $|\phi(u) - |\phi(v) \pm$

$\phi(w)|| = \phi(u) + |\phi(v) \pm \phi(w)|$. Thus, it follows that $|u - |v \pm w|| + \ker(\phi) = u + |v \pm w| + \ker(\phi)$.

Hence $(V/\ker(\phi), (V/\ker(\phi))^+, |\cdot|)$ is an absolutely ordered space.

□

Corollary 4.1.6. *Let V and W be absolutely ordered spaces and let $\phi : V \to W$ be a linear $|\cdot|$-preserving map. Put* $\ker^+(\phi) := \{v \in V^+ : \phi(v) = 0\}$, *then*

(1) ϕ is injective if and only if $\ker^+(\phi) = \{0\}$.

(2) ϕ is surjective if and only if $\phi(V^+) = W^+$.

(3) The quotient map $\tilde{\phi} : V/\ker(\phi) \to \phi(V)$ is a bijective $|\cdot|$-preserving map.

Proof. (1) By Theorem 4.1.5(1), $\ker(\phi)$ is an absolutely ordered space. Thus we have $\ker(\phi) = \ker^+(\phi) - \ker^+(\phi)$. Now, the proof of (1) is immediate.

(2) If ϕ is surjective, it follows from Theorem 4.1.5(2) that $\phi(V^+) = W^+$. Conversely, assume that $\phi(V^+) = W^+$. If $w \in W$, by assumption there exist $v_1, v_2 \in V^+$ such that $\phi(v_1) = w^+, \phi(v_2) = w^-$. Put $v = v_1 - v_2$ so that $\phi(v) = w$. Hence ϕ is surjective.

(3) It is an immediate consequence of Theorem 4.1.5(3).

□

4.2 Absolute value preservers on absolute order unit spaces

In this section, we study absolute value preserving maps between absolute order unit spaces.

We begin with the following characterization of unital, bijective $|\cdot|$-preserving maps between absolute order unit spaces:

Theorem 4.2.1. *Let (V, e_V) and (W, e_W) be absolute order unit spaces and let $\phi : V \to W$ be a unital, bijective linear map. Then ϕ is $|\cdot|$-preserving if and only if it is an isometry.*

Proof. First, assume that ϕ is an $|\cdot|$-preserving map. Since ϕ is surjective, by Corollary 4.1.6(2), $\phi(V^+) = W^+$. As it is also an injection, it follows that $re_V \pm v \in V^+$ if and only if $re_W \pm \phi(v) \in W^+$ whenever $v \in V$ and $r \in \mathbb{R}$. Thus $\|v\| = \|\phi(v)\|$ for all $v \in V$ so that ϕ is an isometry.

Conversely, let ϕ be an isometry. Next, we show that ϕ preserves $|\cdot|$. Let $v \in V^+$ with $\|v\| \leq 1$. Then $0 \leq v \leq e_V$ so that $0 \leq e_V - v \leq e_V$. Thus $\|e_V - v\| \leq 1$. Since ϕ is an isometry, we get that $\|e_W - \phi(v)\| \leq 1$ and this implies $e_W - \phi(v) \leq e_W$. Then $\phi(v) \in W^+$ and hence $\phi \geq 0$. Now, as ϕ^{-1} is also an isometry, therefore $\phi^{-1} \geq 0$.

Let $v_1, v_2 \in V^+$ with $v_1 \perp v_2$. If $v_1 = 0$ or $v_2 = 0$, then $\phi(v_1) \perp \phi(v_2)$. Now, assume that $v_1 \neq 0, v_2 \neq 0$. Then $w_i = \phi(v_i) \in W^+ \setminus \{0\}, i = 1, 2$. Let $0 \leq u_i \leq w_i, i = 1, 2$. Then $0 \leq \phi^{-1}(u_i) \leq v_i, i = 1, 2$. Since $v_1 \perp v_2$, we have $v_1 \perp_\infty^a v_2$ and consequently, $\phi^{-1}(u_1) \perp_\infty \phi^{-1}(u_2)$. Thus, by Theorem 2.1.4, we have

$$\begin{aligned} 1 &= \left\| \|\phi^{-1}(u_1)\|^{-1}\phi^{-1}(u_1) + \|\phi^{-1}(u_2)\|^{-1}\phi^{-1}(u_2) \right\| \\ &= \left\| \|u_1\|^{-1}u_1 + \|u_2\|^{-1}u_2 \right\|. \end{aligned}$$

as ϕ^{-1} is an isometry. Again applying Theorem 2.1.4, we get that $u_1 \perp_\infty u_2$ so that $w_1 \perp_\infty^a w_2$. Now, by the definition of an absolute order unit space, we get that $w_1 \perp w_2$. Hence by Proposition 4.1.3, ϕ is $|\cdot|$-preserving. $\square$

Maitland Wright and Youngson proved that any surjective linear unital isometry $\phi : A \to B$ between unital JB-algebras A and B is a Jordan isomorphism [48, Theorem 4]. If we combine this result with Theorem 4.2.1, we may deduce the following:

Corollary 4.2.2. *Let A and B be unital JB-algebras and let $\phi : A \to B$ be a bijective linear map. Then the following statements are equivalent:*

1. *ϕ is a unital isometry;*
2. *ϕ is a unital $|\cdot|$-preserving map;*
3. *ϕ is a Jordan isomorphism.*

Proof. Let ϕ be a Jordan isomorphism. Let $\phi(1_A) = x_0 \in B$ and let $\phi^{-1}(1_B) = y_0 \in A$. Then

$$1_B = \phi(y_0) = \phi(1_A o y_0) = \phi(1_A) o \phi(y_0) = x_0 o 1_B = x_0$$

so that ϕ is unital. Also, ϕ is positive. In fact, if $x \in A^+$, then $x = (x^{\frac{1}{2}})^2$ so that $\phi(x) = \phi(x^{\frac{1}{2}})^2 \in B^+$. Now, for any $y \in A$, we have

$$|\phi(y)|^2 = \phi(y)^2 = \phi(y^2) = \phi(|y|^2) = \phi(|y|)^2$$

so that $\phi(|y|) = |\phi(y)|$ for all $y \in A$. Thus (3) implies (2). Now, by Theorem 4.2.1, the proof is complete. □

Corollary 4.2.3. *Let (V, e_V) and (W, e_W) be absolute order unit spaces and let $\phi : V \to W$ be a bijective linear map. Consider the three statements:*

(1) ϕ is unital;

(2) ϕ is an isometry; and

(3) ϕ is $|\cdot|$-preserving.

Then any two of these statements imply the third.

Proof. By Theorem 4.2.1, we have that (1) and (2) imply (3) and that (1) and (3) imply (2). Now, assume that (2) and (3) hold. Then by Remark 4.1.2, we note that ϕ^{-1} is also $|\cdot|$-preserving. Thus ϕ and ϕ^{-1} are positive isometries. Put $\phi(e_V) = w_0$. Then $w_0 \in W^+$ with $\|w_0\| = 1$ so that $w_0 \leq e_W$. Thus $e_V = \phi^{-1}(w_0) \leq \phi^{-1}(e_W)$. Since ϕ^{-1} is an isometry, we get that $\phi^{-1}(e_W) \leq e_V$ so that $\phi^{-1}(e_W) = e_V$. Hence ϕ is unital. □

4.3 Absolute compatibility

Absolute compatibility is an useful notion. It was applied to derive spectral decomposition in an absolute order unit space [39]. This property has been separately studied in the context of operator algebras [24, 25]. Here, we study absolute compatibility preservers in absolute order unit spaces and their relation with absolute value preservers.

Remark 4.3.1. *Let (V, e_V) and (W, e_W) be absolute order unit spaces. Then a unital $|\cdot|$-preserving map $\phi : V \to W$ preserves order projections. To see this, let $p \in \mathcal{OP}(V)$. Then $p \perp e_V - p$. As $\phi(e_V) = e_W$ and $\phi(|v|) = |\phi(v)|$ for all $v \in V$, by Proposition 4.1.3, we get $\phi(p) \perp e_W - \phi(p)$. Thus $\phi(p) \in \mathcal{OP}(W)$.*

Now, in the next Theorem 4.3.2(1), we generalize Remark 4.3.1.

Theorem 4.3.2. *Let V and W be absolute order unit spaces and let $\phi : V \to W$ be an $|\cdot|$-preserving map such that $\phi(e_V) \in \mathcal{OP}(W)$. Then*

(1) $\phi(\mathcal{OP}(V)) \subset \mathcal{OP}(W)$.

(2) For $u, v \in V^+$ *with* $u \bigtriangleup v$, *we have* $\phi(u) \bigtriangleup \phi(v)$.

Proof. (1) Put $\phi(e_V) = q$ and let $p \in \mathcal{OP}(V)$. Then

$$|e_V - 2p| = |(e_V - p) - p| = e_V.$$

Since ϕ is $|\cdot|$-preserving, we have

$$q = \phi(e_V) = |\phi(e_V) - 2\phi(p)| = |q - 2\phi(p)|.$$

Also, as $p \leq e_V$, we get that $\phi(p) \leq q$.

Now $(e_W - q) \perp q$ and $0 \leq \phi(p), q - \phi(p) \leq q$ so that, by Definition 2.3.2(d), we get that $(e_W - q) \perp \phi(p)$ and $(e_W - q) \perp q - \phi(p)$. Using Corollary 3.3.2, we have

$$\begin{aligned}
|e_W - 2\phi(p)| &= |(e_W - q) + (q - \phi(p)) - \phi(p))| \\
&= |(e_W - q)| + |q - \phi(p) - \phi(p)| \\
&= (e_W - q) + |q - 2\phi(p)| \\
&= e_W.
\end{aligned}$$

Hence $\phi(p) \in \mathcal{OP}(W)$.

(2) Let $u, v \in V^+$ be such that $u \bigtriangleup v$. Then $|u - v| + |e_V - u - v| = e_V$. By Proposition 2.3.8, we conclude that $0 \leq u, v \leq e_V$. Since $\phi \geq 0$, we have

$0 \leq \phi(u), \phi(v) \leq \phi(e_V)$. As ϕ is $|\cdot|$-preserving, we further get that

$$|\phi(u) - \phi(v)| + |\phi(e_V) - \phi(u) - \phi(v)| = \phi(e_V).$$

Now $\phi(e_V) \in \mathcal{OP}(W)$ so that $\phi(e_V) \perp e_W - \phi(e_V)$. Thus as $0 \leq \phi(u), \phi(v) \leq \phi(e_V)$ (by technique in proof of 2.3.8) and $\phi(e_V) \perp e_W - \phi(e_V)$, by Definition 2.3.2(d), we have $\phi(u) \perp e_W - \phi(e_V)$ and $\phi(v) \perp e_W - \phi(e_V)$. Now, by Definition 2.3.2(e), we get that $\phi(u) + \phi(v) \perp e_W - \phi(e_V)$. Thus applying Corollary 3.3.2, we obtain that

$$\begin{aligned} |e_W - \phi(u) - \phi(v)| &= |(e_W - \phi(e_V)) + (\phi(e_V) - \phi(u) - \phi(v))| \\ &= (e_W - \phi(e_V)) + |\phi(e_V) - \phi(u) - \phi(v)|. \end{aligned}$$

Therefore, we get

$$\begin{aligned} |\phi(u) &- \phi(v)| + |e_W - \phi(u) - \phi(v)| \\ &= |\phi(u) - \phi(v)| + (e_W - \phi(e_V)) + |\phi(e_V) - \phi(u) - \phi(v)| \\ &= e_W. \end{aligned}$$

so that $\phi(u) \bigtriangleup \phi(v)$.

$\square$

Theorem 4.3.3. *Let V and W be absolute order unit spaces and let $\phi : V \to W$ be a linear map such that $\phi \geq 0$. If ϕ is $\bigtriangleup$-preserving, then*

(1) ϕ is a contraction.

(2) ϕ is $|\cdot|$-preserving.

Proof. Assume that ϕ is $\triangle$-preserving.

(1) First, we show that ϕ is contractive on V^+. To see this, let $v \in V^+$. Without loss of generality, we may assume that $\|v\| \leq 1$. Since $v \perp 0$, by Proposition 2.3.7, we have $v \triangle 0$ so that $\phi(v) \triangle 0$. Now $\phi(v) \triangle 0$, by Proposition 2.3.8, we get that $0 \leq \phi(v) \leq e_W$. Thus ϕ is contrative on V^+. Now let $v \in V$ be an arbitrary element with $\|v\| \leq 1$. Consider the orthogonal decomposition $v = v^+ - v^-$. Then $v^+ \perp v^-$ so that $\max\{\|v^+\|, \|v^-\|\} = \|v\| \leq 1$. Also then $-v^- \leq v \leq v^+$. Since ϕ is positive, we have $-\phi(v^-) \leq \phi(v) \leq \phi(v^+)$. Thus $\|\phi(v)\| \leq \max\{\|\phi(v^-)\|, \|\phi(v^+)\|\} \leq 1$ as ϕ is contractive on V^+. Hence ϕ is a contraction on V.

(2) Let $u, v \in V^+$ with $u \perp v$ and assume that $\|u\| \leq 1, \|v\| \leq 1$. Then by Proposition 2.3.7, we have $u + v \leq e_V$ and $u \triangle v$. Since $\phi \geq 0$ and $\triangle$-preserving, we get $\phi(u) + \phi(v) \leq \phi(e_V)$ and $\phi(u) \triangle \phi(v)$. Also, by (1), $\phi(e_V) \leq e_W$. Thus again applying Proposition 2.3.7, we may conclude that $\phi(u) \perp \phi(v)$. Hence by Proposition 4.1.3, ϕ is $|\cdot|$-preserving map.

□

Remark 4.3.4. *Let V and W be absolute order unit spaces and let $\phi : V \to W$ be a linear map.*

(1) If $\phi \geq 0$ and $\phi(e_V) \in \mathcal{OP}(W)$, then ϕ is $\triangle$-preserving if and only if it $|\cdot|$-preserving.

(2) Let ϕ be a unital surjective isometry. Then for all $u, v \in V^+$, we have $\phi(u) \triangle \phi(v)$ if and only if $u \triangle v$.

4.4 Matricial version of absolute value preserving maps

In this section, we define completely $|\cdot|$-preserving maps between absolutely matrix ordered spaces which is extension of the notion of $|\cdot|$-preserving maps between absolutely ordered spaces. We prove that unital, bijective completely absolute value preserving maps between absolutely matrix ordered spaces are precisely complete isometries. We also deduce known result that on (unital) C*-algebras, such maps are C*-algebra isomorphisms.

Definition 4.4.1. *Let V and W be absolutely matrix ordered spaces and let $\phi : V \to W$ be a $*$-linear map. We say that ϕ is a* complete $|\cdot|$-preserving, *if $\phi_n : M_n(V) \to M_n(W)$ is an $|\cdot|$-preserving map for each $n \in \mathbb{N}$.*

Now we present the matricial version of Theorem 4.2.1.

Theorem 4.4.2. *Let (V, e_V) and (W, e_W) be absolute matrix order unit spaces and let $\phi : V \to W$ be a unital $*$-linear surjective isomorphism. Then ϕ is a complete isometry if and only if it is completely $|\cdot|$-preserving.*

Proof. First, let ϕ be a complete isometry. Fix $n \in \mathbb{N}$. Then $(M_{2n}(V)_{sa}, M_{2n}(V)^+, |\cdot|_{2n}, e_V^{2n})$ is an absolute order unit space and $\phi_{2n} : M_{2n}(V)_{sa} \to M_{2n}(W)_{sa}$ is a unital, bijective linear isometry, thus by Theorem 4.2.1, we get that $\phi_{2n}(|v|_{2n}) = |\phi_{2n}(v)|_{2n}$ for all $v \in M_{2n}(V)_{sa}$. Let $v \in M_n(V)$. Then $\begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \in M_{2n}(V)_{sa}$ so that

$$\phi_{2n}\left(\left|\begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix}\right|_{2n}\right) = \left|\phi_{2n}\left(\begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix}\right)\right|_{2n}.$$

Thus, by Proposition 3.1.2(2), we get

$$\begin{bmatrix} \phi_n(|v^*|_n) & 0 \\ 0 & \phi_n(|v|_n) \end{bmatrix} = \begin{bmatrix} |\phi_n(v)^*|_n & 0 \\ 0 & |\phi_n(v)|_n \end{bmatrix}$$

so that $\phi_n(|v|_n) = |\phi_n(v)|_n$. Therefore ϕ is completely $|\cdot|$-preserving.

Conversely, assume that ϕ is a completely $|\cdot|$-preserving map. Fix $n \in \mathbb{N}$. Then $\phi_{2n} : M_{2n}(V)_{sa} \to M_{2n}(W)_{sa}$ is an $|\cdot|$-preserving map. Since $\phi_{2n} : M_{2n}(V)_{sa} \to M_{2n}(W)_{sa}$ is a unital bijective $|\cdot|$-preserving map, again applying Theorem 4.2.1, we have that $\phi_{2n} : M_{2n}(V)_{sa} \to M_{2n}(W)_{sa}$ is an isometry. Let $v \in M_n(V)$. Then $\begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \in M_{2n}(V)_{sa}$ so that $\|v\|_n = \left\| \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \right\|_{2n} = \left\| \phi_{2n}\left(\begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \right) \right\|_{2n} = \left\| \begin{bmatrix} 0 & \phi_n(v) \\ \phi_n(v)^* & 0 \end{bmatrix} \right\|_{2n} = \|\phi_n(v)\|_n$. Hence $\phi : V \to W$ is a complete isometry. □

Corollary 4.4.3. *Let A and B be any two unital C^*-algebras and let $\phi : A \to B$ be a $*$-linear bijective map. Then the following facts are equivalent:*

1. *ϕ is a unital complete isometry;*
2. *ϕ is a unital completely $|\cdot|$-preserving map;*
3. *ϕ is a C^*-algebra isomorphism.*

Proof. Following Theorem 4.4.2, it suffices to show that (2) (or equivalently (1)) implies (3). Let ϕ be a unital completely $|\cdot|$-preserving map. Then $\phi_n : M_n(A)_{sa} \to M_n(B)_{sa}$ is a unital $|\cdot|$-preserving map for each $n \in \mathbb{N}$. Thus, by Corollary 4.2.2, $\phi_n : M_n(A)_{sa} \to M_n(B)_{sa}$ is a Jordan isomorphism for each $n \in \mathbb{N}$. In particular, $\phi_3(z^2) = \phi_3(z)^2$ for any $z \in M_3(A)_{sa}$. Let $x, y \in A$

and consider $z = \begin{bmatrix} 0 & x & 0 \\ x^* & 0 & y \\ 0 & y^* & 0 \end{bmatrix} \in M_3(A)_{sa}$. Then $\phi_3(z^2) = \phi_3(z)^2$ yields that $\phi(xy) = \phi(x)\phi(y)$. Thus ϕ is a C*-algebra isomorphism. $\square$

Remark 4.4.4. *It follows, from Corollary* 4.4.3*, that a unital surjective* $*$*-linear map between unital C*-algebras is complete isometry, if it is a* 3*-isometry.*

4.5 Orthogonality of positive maps

In this section, we define orthogonality of completely positive linear maps between absolutely matrix ordered spaces. We show that sum of two orthogonal completely $|\cdot|$-preserving maps is again a completely $|\cdot|$-preserving.

Definition 4.5.1. *Let V and W be absolutely ordered spaces and let $\phi, \psi : V \to W$ be positive linear maps. We say that ϕ is* orthogonal *to ψ (we write, $\phi \perp \psi$), if $\phi(u) \perp \psi(v)$ for all $u, v \in V^+$.*

Remark 4.5.2. *Let V and W be absolutely ordered spaces and let $\phi, \psi : V \to W$ be $|\cdot|$-preserving maps. Then $\phi \perp \psi$ if and only if $\phi(u) \perp \psi(v)$ for all $u, v \in V$. Thus we get that $|\phi(u) \pm \psi(v)| = |\phi(u)| + |\psi(v)|$ for all $u, v \in V$. In fact, if ϕ and ψ are orthogonal $|\cdot|$-preserving maps, then by Proposition 4.1.3 and by Definition 4.5.1, we have that $\phi(u)^+, \phi(u)^-, \psi(v)^+$ and $\psi(v)^-$ are mutually orthogonal. Thus by Proposition 3.3.1, we conclude that $\phi(u) \perp \psi(v)$ and $|\phi(u) \pm \psi(v)| = |\phi(u)| + |\psi(v)|$.*

Definition 4.5.3. *Let V and W be absolutely matrix ordered spaces and let $\phi, \psi : V \to W$ be completely positive maps. We say that ϕ is* completely orthogonal *to ψ (we continue to write, $\phi \perp \psi$), if $\phi_n \perp \psi_n$ for all $n \in \mathbb{N}$.*

Theorem 4.5.4. *Let V and W be absolutely matrix ordered spaces and $\phi, \psi : V \to W$ be completely $|\cdot|$-preserving maps such that $\phi \perp \psi$. Then $\phi + \psi$ is also completely $|\cdot|$-preserving map.*

Proof. Let $v \in M_n(V)$. Then $\begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \in M_{2n}(V)_{sa}$. Since ϕ is completely orthogonal to ψ, by Remark 4.5.2, we have $\phi_{2n}\left(\begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix}\right) \perp \psi_{2n}\left(\begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix}\right)$. Thus we get the following:

$$\begin{aligned}
\begin{bmatrix} |(\phi+\psi)_n(v^*)|_n & 0 \\ 0 & |(\phi+\psi)_n(v)|_n \end{bmatrix} &= \left|\begin{bmatrix} 0 & (\phi+\psi)_n(v) \\ (\phi+\psi)_n(v^*) & 0 \end{bmatrix}\right|_{2n} \\
&= \left|\begin{bmatrix} 0 & \phi_n(v)+\psi_n(v) \\ \phi_n(v^*)+\psi_n(v^*) & 0 \end{bmatrix}\right|_{2n} \\
&= \left|\begin{bmatrix} 0 & \phi_n(v) \\ \phi_n(v^*) & 0 \end{bmatrix} + \begin{bmatrix} 0 & \psi_n(v) \\ \psi_n(v^*) & 0 \end{bmatrix}\right|_{2n} \\
&= \left|\phi_{2n}\left(\begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix}\right) + \psi_{2n}\left(\begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix}\right)\right|_{2n} \\
&= \left|\phi_{2n}\left(\begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix}\right)\right|_{2n} + \left|\psi_{2n}\left(\begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix}\right)\right|_{2n} \\
&= \left|\begin{bmatrix} 0 & \phi_n(v) \\ \phi_n(v^*) & 0 \end{bmatrix}\right|_{2n} + \left|\begin{bmatrix} 0 & \psi_n(v) \\ \psi_n(v^*) & 0 \end{bmatrix}\right|_{2n} \\
&= \begin{bmatrix} |\phi_n(v^*)|_n & 0 \\ 0 & |\phi_n(v)|_n \end{bmatrix} + \begin{bmatrix} |\psi_n(v^*)|_n & 0 \\ 0 & |\psi_n(v)|_n \end{bmatrix}
\end{aligned}$$

$$= \begin{bmatrix} \phi_n(|v^*|_n) & 0 \\ 0 & \phi_n(|v|_n) \end{bmatrix} + \begin{bmatrix} \psi_n(|v^*|_n) & 0 \\ 0 & \psi_n(|v|_n) \end{bmatrix}$$

$$= \begin{bmatrix} (\phi+\psi)_n(|v^*|_n) & 0 \\ 0 & (\phi+\psi)_n(|v|_n) \end{bmatrix}$$

so that $|(\phi+\psi)(v)|_n = (\phi+\psi)(|v|_n)$. Hence $\phi+\psi$ is completely $|\cdot|$-preserving.

□

Chapter 5

Partial isometries in an absolute matrix order unit space

In this chapter, we generalize and study certain C*-algebraic notions in the order theoretic context. We define notions of unitaries, order projections, partial isometries, isometries and some other terms of a C*-algebra in absolute matrix order unit spaces. We mainly focus on order projections and partial isometries in this chapter. In the first section, we discuss some properties of partial isometries in absolute matrix order unit space. In the second section, we define partial isometric equivalence relation on the collection of all the order projections in absolute matrix order unit spaces. This equivalence relation generalizes the Murray-von Neumann equivalence relation among the projections in a C*-algebra. We also discuss some properties related to partial isometric equivalence relation. In the third section, we study unitary equivalence of two projections in absolute matrix order unit spaces. We relate it to partial isometric equivalence. In the fourth and fifth sections, we define the notions of infinite and properly infinite order projections respectively in absolute matrix order unit spaces which are generalizations of infinite and properly infinite projections respectively in a C*-algebra. We characterize these notions.

5.1 Partial isometry

We begin with the following definitions.

Definition 5.1.1. *Let (V, e) be an absolute matrix order unit space and let $v \in M_n(V)$ for some $n \in \mathbb{N}$.*

1. *v is said to be a* normal, *if $|v|_n = |v^*|_n$;*
2. *v is said to be a* unitary, *if $|v|_n = |v^*|_n = e^n$;*
3. *v is said to be a* symmetry, *if $v^* = v$ and $|v|_n = e^n$;*
4. *v is said to be an* order projection, *if $v^* = v$ and $|2v - e^n|_n = e^n$;*
5. *v is said to be a* partial unitary, *if $|v|_n = |v^*|_n$ is an order projection;*
6. *v is said to be a* partial symmetry, *if $v^* = v$ and $|v|_n$ is an order projection;*

   Now let $v \in M_{m,n}(V)$ for some $m, n \in \mathbb{N}$.

7. *v is said to be a* partial isometry, *if $|v|_{m,n}$ and $|v^*|_{n,m}$ are order projections;*
8. *v is said to be an* isometry, *if v is a partial isometry and $|v|_{m,n} = e^n$;*
9. *v is said to be a* co-isometry, *if v is a partial isometry and $|v^*|_{n,m} = e^m$.*

The set of all partial isometries in $M_{m,n}(V)$ will be denoted by $\mathcal{PI}_{m,n}(V)$ and the set of all order projections in $M_n(V)$ will be denoted by $\mathcal{OP}_n(V)$. For $m = n$, we write $\mathcal{PI}_{m,n}(V) = \mathcal{PI}_n(V)$. For $n = 1$, we shall write $\mathcal{PI}(V)$ for $\mathcal{PI}_1(V)$ and $\mathcal{OP}(V)$ for $\mathcal{OP}_1(V)$.

It may be noted that the statements in Propositions 2.3.11 and 2.3.12 remain valid in an absolute matrix order unit space V, if we replace $\mathcal{OP}(V)$ by $\mathcal{OP}_n(V)$ for any $n \in \mathbb{N}$.

Now, we prove some matricial properties of order projections.

Proposition 5.1.2. *Let V be an absolute matrix order unit space and let $m, n \in \mathbb{N}$. Then $p \in \mathcal{OP}_m(V), q \in \mathcal{OP}_n(V)$ if and only if $p \oplus q \in \mathcal{OP}_{m+n}(V)$.*

Proof. Let $p \oplus q \in \mathcal{OP}_{m+n}(V)$. Then

$$e^{m+n} = |2(p \oplus q) - e^{m+n}|_{m+n} = |2p - e^m|_m \oplus |2q - e^n|_n.$$

Thus $|2p - e^m|_m = e^m$ and $|2q - e^n|_n = e^n$ so that $p \in \mathcal{OP}_m(V)$ and $q \in \mathcal{OP}_n(V)$. Now tracing back the proof, we can prove the converse part. $\square$

Proposition 5.1.3. *Let (V, e) be an absolute matrix order unit space and let $a \in M_{m,n}$ such that $a^*a = I_n$. Then $aua^* \perp ava^*$ for $u, v \in M_n(V)^+$ with $u \perp v$. In particular, if $p \in \mathcal{OP}_n(V)$, then $apa^* \in \mathcal{OP}_m(V)$.*

Proof. Let $u, v \in M_n(V)^+$ such that $u \perp v$. Since $a^*a = I_n$, we have that $\text{rank}(a) = n$ so that $n \leq m$. Thus we can find a unitary $d \in M_m$ such that $a = d \begin{bmatrix} I_n \\ 0 \end{bmatrix}$. Then

$$aua^* = d \begin{bmatrix} I_n \\ 0 \end{bmatrix} u \begin{bmatrix} I_n & 0 \end{bmatrix} d^* = d(u \oplus 0)d^*.$$

Similarly $ava^* = d(v \oplus 0)d^*$. Now by Lemma 3.1.4, we get

$$\begin{aligned}
|aua^* - ava^*|_m &= |d(u \oplus 0)d^* - d(v \oplus 0)d^*|_m \\
&= d|(u - v) \oplus 0|_m d^* \\
&= d(|u - v|_n \oplus 0)d^* \\
&= d((u + v) \oplus 0)d^* \\
&= aua^* + ava^*
\end{aligned}$$

so that $aua^* \perp ava^*$.

Next let $p \in \mathcal{OP}_n(V)$ and put $r = p \oplus 0_{m-n}$. Then $r \in \mathcal{OP}_m(V)$ and $apa^* = drd^*$. Thus $r \perp (e^m - r)$ so that, by the first part, we have $drd^* \perp d(e^m - r)d^* = e^m - drd^*$. Hence $apa^* \in \mathcal{OP}_m(V)$. $\square$

Remark 5.1.4. *Let V be an absolute matrix order unit space and let $m, n \in \mathbb{N}$.*

1. *Let $u, v \in \mathcal{PI}_{m,n}(V)$ with $u \perp v$. Then $\begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{PI}_{2m,n}(V)$ and $\begin{bmatrix} u & v \end{bmatrix} \in \mathcal{PI}_{m,2n}(V)$. In fact, as $u \perp v$ by Proposition 3.3.5(2), we have $\left| \begin{bmatrix} u \\ v \end{bmatrix} \right|_{2m,n} = |u|_{m,n} + |v|_{m,n}$ and $\left| \begin{bmatrix} u & v \end{bmatrix} \right|_{m,2n} = |u|_{m,n} \oplus |v|_{m,n}$. Similarly $\left| \begin{bmatrix} u \\ v \end{bmatrix}^* \right|_{n,2m} = |u^*|_{n,m} \oplus |v^*|_{n,m}$ and $\left| \begin{bmatrix} u & v \end{bmatrix}^* \right|_{2n,m} = |u^*|_{n,m} + |v^*|_{n,m}$ for $u^* \perp v^*$. Thus by Propositions 2.3.11(1) and 5.1.2, we get that $\begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{PI}_{2m,n}(V)$ and $\begin{bmatrix} u & v \end{bmatrix} \in \mathcal{PI}_{m,2n}(V)$.*

2. *Let $v \in \mathcal{PI}_{m,n}(V)$. Then as in (1), we get $\begin{bmatrix} v & 0 \end{bmatrix} \in \mathcal{PI}_m(V)$, if $m > n$*

and $\begin{bmatrix} v \\ 0 \end{bmatrix} \in \mathcal{PI}_n(V)$, *if* $m < n$.

3. *Let* $v \in M_{m,n}(V)$. *Then the following facts are equivalent:*

   (a) $v \in \mathcal{PI}_{m,n}(V)$;

   (b) $\begin{bmatrix} v^* & 0 \\ 0 & v \end{bmatrix} \in \mathcal{PI}_{m+n}(V)$; *and*

   (c) $\begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \in \mathcal{PI}_{m+n}(V)$. *(That is,* $\begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix}$ *is a partial symmetry.)*

   In fact, we have

$$\left| \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \right|_{m+n} = \begin{bmatrix} |v^*|_{n,m} & 0 \\ 0 & |v|_{m,n} \end{bmatrix} = \left| \begin{bmatrix} v^* & 0 \\ 0 & v \end{bmatrix} \right|_{m+n}.$$

Proposition 5.1.5. *Let* V *be an absolute matrix order unit space and let* $v \in M_n(V)_{sa}$ *for some* $n \in \mathbb{N}$. *Then* $v \in \mathcal{PI}_n(V)$ *(that is,* v *is a partial symmetry) if and only if* $v^{\pm} \in \mathcal{OP}_n(V)$. *In particular, if* $v \in M_{m,n}(V)$ *for some* $m, n \in \mathbb{N}$, *then* $v \in \mathcal{PI}_{m,n}(V)$ *if and only if* $\begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix}^{\pm} \in \mathcal{OP}_{m+n}(V)$.

Proof. Since $v \in M_n(V)_{sa}$, let $v = v^+ - v^-$ be the orthogonal decomposition of v in $M_n(V)^+$. Thus $|v| = v^+ + v^-$. Now first assume that $v \in \mathcal{PI}_n(V)$. Then $|v| \in \mathcal{OP}_n(V)$, therefore by Proposition 2.3.11(2), we get that $v^{\pm} \in \mathcal{OP}_n(V)$ as $v^+ \perp v^-$.

Conversely assume that $v^{\pm} \in \mathcal{OP}_n(V)$. Again by Proposition 2.3.11(1), we get that $|v| = v^+ + v^- \in \mathcal{OP}_n(V)$ so that $v \in \mathcal{PI}_n(V)$.

Next, let $v \in M_{m,n}(V)$ for some $m, n \in \mathbb{N}$. Then $\begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \in M_{m+n}(V)_{sa}$. By Remark 5.1.4(3), we have that $v \in \mathcal{PI}_{m,n}(V)$ if and only if $\begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \in \mathcal{PI}_{m+n}(V)$. Hence $v \in \mathcal{PI}_{m,n}(V)$ if and only if $\begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix}^{\pm} \in \mathcal{OP}_{m+n}(V)$. $\square$

Corollary 5.1.6. *Let V be an absolute matrix order unit space and let $v_1, \ldots, v_k \in M_{m,n}(V)$ be mutually orthogonal vectors for some $k, m, n \in \mathbb{N}$. Then $v_1, \ldots, v_k \in \mathcal{PI}_{m,n}(V)$ if and only if $\sum_{i=1}^{k} v_i \in \mathcal{PI}_{m,n}(V)$.*

Proof. $\begin{bmatrix} 0 & v_i \\ v_i^* & 0 \end{bmatrix} \in M_{m+n}(V)_{sa}$ for all $i = 1, 2, \cdots, k$. Since $v_i \perp v_j$ for all $i \neq j$, by Proposition 3.3.1, we get that $\left\{ \begin{bmatrix} 0 & v_i \\ v_i^* & 0 \end{bmatrix}^{\pm} : i = 1, 2, \cdots, k \right\}$ is an orthogonal set so that

$$\begin{bmatrix} 0 & \sum_{i=1}^{k} v_i \\ \left(\sum_{i=1}^{k} v_i\right)^* & 0 \end{bmatrix}^{+} = \left(\sum_{i=1}^{k} \begin{bmatrix} 0 & v_i \\ v_i^* & 0 \end{bmatrix} \right)^{+} = \sum_{i=1}^{k} \begin{bmatrix} 0 & v_i \\ v_i^* & 0 \end{bmatrix}^{+}$$

and

$$\begin{bmatrix} 0 & \sum_{i=1}^{k} v_i \\ \left(\sum_{i=1}^{k} v_i\right)^* & 0 \end{bmatrix}^{-} = \left(\sum_{i=1}^{k} \begin{bmatrix} 0 & v_i \\ v_i^* & 0 \end{bmatrix} \right)^{-} = \sum_{i=1}^{k} \begin{bmatrix} 0 & v_i \\ v_i^* & 0 \end{bmatrix}^{-}.$$

Then by Remark 5.1.4(3) and by Propositions 2.3.11 and 5.1.5, it follows that $v_1, \ldots, v_k \in \mathcal{PI}_{m,n}(V)$ if and only if $\sum_{i=1}^{k} v_i \in \mathcal{PI}_{m,n}(V)$. $\square$

5.2 Comparison of order projections

In this section, we study comparison of order projections in absolute matrix order unit spaces. Recall that a pair of projections p and q in a C*-algebra A is said to (Murray-von Neumann) equivalent, if there exists a partial isometry $u \in A$ such that $p = u^*u$ and $q = uu^*$ or equivalently $p = |u|$ and $q = |u^*|$. The latter form provides room to extend the notion to absolute matrix order unit spaces.

Before we formally introduce the notion, let us look at the other aspects. Let us note that being multiplicative in nature, partial isometries in C*-algebras enjoy certain properties which we have not been able to prove order theoretically.

1. Let u be a partial isometry in a C*-algebra A so that $|u|$ and $|u^*|$ are projections and let p be projection in A. Then $p \leq |u|$ if and only if there exists a partial isometry $u_1 \in A$ with $u_1 \perp (u - u_1)$ such that $p = |u_1|$. In fact, if $p \leq |u|$, we put $u_1 = up$. Then u_1 is also a partial isometry in A with $u_1 \perp (u - u_1)$ such that $p = |u_1|$. Conversely, if u_1 is a partial isometry in A with $u_1 \perp (u - u_1)$ such that $p = |u_1|$, then $p \leq |u|$.

2. Let u and v be any two partial isometries in A such that $|u| = |v|$. Put $w = vu^*$. Then $w^*w = uu^*$ and $ww^* = vv^*$, that is, $|w| = |u^*|$ and $|w^*| = |v^*|$.

These properties can be extended to $M_{m,n}(A)$ for any $m, n \in \mathbb{N}$. We propose them in terms of the following conditions on an absolute matrix order unit space V:

(H) If $u \in \mathcal{PI}_{m,n}(V)$ and if $p \in \mathcal{OP}_n(V)$ with $p \leq |u|_{m,n}$, then there exists a $u_1 \in \mathcal{PI}_{m,n}(V)$ with $u_1 \perp (u - u_1)$ such that $p = |u_1|_{m,n}$.

(T) If $u \in \mathcal{PI}_{m,n}(V)$ and $v \in \mathcal{PI}_{l,n}(V)$ with $|u|_{m,n} = |v|_{l,n}$, then there exists a $w \in \mathcal{PI}_{m,l}(V)$ such that $|w^*|_{l,m} = |u^*|_{n,m}$ and $|w|_{m,l} = |v^*|_{n,l}$.

Definition 5.2.1. *Let V be an absolute matrix order unit space. We write*

$$\mathcal{OP}_\infty(V) = \{p : p \in \mathcal{OP}_n(V) \text{ for some } n \in \mathbb{N}\}.$$

Given $p \in \mathcal{OP}_m(V)$ and $q \in \mathcal{OP}_n(V)$ for some $m, n \in \mathbb{N}$, we say that p is partial isometric equivalent *to q (we write, $p \sim q$), if there exists $v \in \mathcal{PI}_{m,n}(V)$ such that $p = |v^*|_{n,m}$ and $q = |v|_{m,n}$.*

Lemma 5.2.2. *Let V be an absolute matrix order unit space satisfying condition (H) and let $p \in \mathcal{OP}_m(V)$ and $q \in \mathcal{OP}_n(V)$ for some $m, n \in \mathbb{N}$ with $p \sim q$. If $p_1 \leq p$ for some $p_1 \in \mathcal{OP}_m(V)$, then there exists $q_1 \in \mathcal{OP}_n(V)$ with $q_1 \leq q$ such that $p_1 \sim q_1$ and $(p - p_1) \sim (q - q_1)$.*

Proof. Since $p \sim q$, there exists $u \in \mathcal{PI}_{m,n}(V)$ such that $p = |u^*|_{n,m}$ and $q = |u|_{m,n}$. Next assume that $p_1 \leq p$ for some $p_1 \in \mathcal{OP}_m(V)$. As $p_1 \leq p = |u^*|_{n,m}$, by condition (H), there exists $v \in \mathcal{PI}_{m,n}(V)$ with $v \perp (u - v)$ such that $|v^*|_{n,m} = p_1$. Put $q_1 = |v|_{m,n}$. Then $p_1 \sim q_1$ and $q = |u|_{m,n} = |v|_{m,n} + |(u - v)|_{m,n} \geq q_1$. Further, $|(u - v)^*|_{n,m} = p - p_1$ and $|(u - v)|_{m,n} = q - q_1$ so that $(p - p_1) \sim (q - q_1)$. □

Remark 5.2.3. *Let V be an absolute matrix order unit space satisfying condition (H) and assume that $p_i, p \in \mathcal{OP}_m(V)$, $i = 1, \dots k$ and $q \in \mathcal{OP}_n(V)$ for some $k, m, n \in \mathbb{N}$. If $p_i \perp p_j$ for $i \neq j$ and if $p_i \leq p \sim q$ for each $i = 1, \dots k$, then there exists $q_i \in \mathcal{OP}_n(V)$, $i = 1, \dots k$ such that $q_i \leq q$, $p_i \sim q_i$ for all $i = 1, \dots k$ and $q_i \perp q_j$ whenever $i \neq j$.*

Proposition 5.2.4. *Let V be an absolute matrix order unit space satisfying the condition (T). Then $\sim$ is an equivalence relation on $\mathcal{OP}_\infty(V)$.*

Proof. (a) Let $p \in \mathcal{OP}_\infty(V)$. Considering $p \in \mathcal{PI}_\infty(V)$, we get $p \sim p$.

(b) Let $p \in \mathcal{OP}_m(V)$ and $q \in \mathcal{OP}_n(V)$ be such that $p \sim q$. Then there exists a $v \in \mathcal{PI}_{m,n}(V)$ such that $p = |v^*|_{n,m}$ and $q = |v|_{m,n}$. Now, for $w = v^* \in \mathcal{PI}_{n,m}(V)$, we have $q = |w^*|_{m,n}$ and $p = |w|_{n,m}$. Thus $q \sim p$.

(c) Let $p \in \mathcal{OP}_k(V)$, $q \in \mathcal{OP}_l(V)$ and $r \in \mathcal{OP}_m(V)$ be such that $p \sim q$ and $q \sim r$. Then there exists $u \in \mathcal{PI}_{k,l}(V)$ and $v \in \mathcal{PI}_{l,m}(V)$ such that $p = |u^*|_{l,k}$, $q = |u|_{k,l}$, $q = |v^*|_{m,l}$ and $r = |v|_{l,m}$. As $|u|_{k,l} = |v^*|_{m,l}$, by condition (T), there exists $w \in \mathcal{PI}_{k,m}(V)$ such that $|w|_{k,m} = |v|_{l,m} = r$ and $|w^*|_{m,k} = |u^*|_{l,k} = p$. Thus $p \sim r$. $\square$

Proposition 5.2.5. *Let V be an absolute matrix order unit space and let $p, q, r, p', q' \in \mathcal{OP}_\infty(V)$.*

1. *If $m, n \in \mathbb{N}$ and let $p \in \mathcal{OP}_m(V)$, then $p \sim p \oplus 0_n$ and $p \sim 0_n \oplus p$;*

2. *If $p \sim q$ and $p' \sim q'$ with $p \perp p'$ and $q \perp q'$, then $p + p' \sim q + q'$;*

3. *If $p \sim p'$ and $q \sim q'$, then $p \oplus q \sim p' \oplus q'$;*

4. *$p \oplus q \sim q \oplus p$;*

5. *If $p \perp q$, then $p + q \sim p \oplus q$;*

6. *$(p \oplus q) \oplus r = p \oplus (q \oplus r)$.*

Proof. 1. Let $m, n \in \mathbb{N}$ and let $p \in \mathcal{OP}_m(V)$. Put $v = \begin{bmatrix} p \\ 0_{n,m} \end{bmatrix}$. Then by Proposition 3.1.2(4) and (5), we have $|v| = p$ and $|v^*| = p \oplus 0_n$. Thus

$p \sim p \oplus 0_n$. Similarly, $w = \begin{bmatrix} 0_{n,m} \\ p \end{bmatrix}$ yields $|w| = p$ and $|w^*| = 0_n \oplus p$ so that $p \sim 0_n \oplus p$.

2. Let $p, p' \in \mathcal{OP}_m(V)$ and $q, q' \in \mathcal{OP}_n(V)$ be such that $p \perp p', q \perp q', p \sim q$ and $p' \sim q'$. Since $p \perp p'$ and $q \perp q'$, by Proposition 2.3.11(1), we have $p + p' \in \mathcal{OP}_m(V)$ and $q + q' \in \mathcal{OP}_n(V)$. Since $p \sim q$ and $p' \sim q'$, there exist $v, v' \in \mathcal{PI}_{m,n}(V)$ such that $p = |v^*|_{n,m}$, $q = |v|_{m,n}$, $p' = |v'^*|_{n,m}$ and $q' = |v'|_{m,n}$. As $p \perp p'$ and $q \perp q'$, by Remark 3.3.4(2), we get that $v \perp v'$. Now by Corollary 5.1.6, we may conclude that $v + v' \in \mathcal{PI}_{m,n}$. Also then

$$|v + v'|_{m,n} = |v|_{m,n} + |v'|_{m,n} = q + q'$$

and

$$|v^* + v'^*|_{n,m} = |v^*|_{n,m} + |v'^*|_{n,m} = p + p'$$

so that $p + p' \sim q + q'$.

3. Since $p \sim q$ and $p' \sim q'$, there exist $v \in \mathcal{PI}_{m,n}(V)$ and $v' \in \mathcal{PI}_{r,s}(V)$ such that $p = |v^*|_{n,m}$, $q = |v|_{m,n}$, $p' = |v'^*|_{s,r}$ and $q' = |v'|_{r,s}$. Put $w = \begin{bmatrix} v & 0 \\ 0 & v' \end{bmatrix}$. Then $|w^*|_{n+s,m+r} = p \oplus q$ and $|w|_{m+r,n+s} = p' \oplus q'$. Hence $p \oplus q \sim p' \oplus q'$.

4. Note that $\begin{bmatrix} 0 & I_m \\ I_n & 0 \end{bmatrix} \begin{bmatrix} 0 & q \\ p & 0 \end{bmatrix} = p \oplus q$ and $\begin{bmatrix} 0 & I_n \\ I_m & 0 \end{bmatrix} \begin{bmatrix} 0 & q \\ p & 0 \end{bmatrix}^* = q \oplus p$. Since $\begin{bmatrix} 0 & I_m \\ I_n & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & I_n \\ I_m & 0 \end{bmatrix}$ are isometries in M_{m+n}, by Proposition 3.1.2(1), we get $p \oplus q = \left| \begin{bmatrix} 0 & q \\ p & 0 \end{bmatrix} \right|_{m+n}$ and $q \oplus p = \left| \begin{bmatrix} 0 & q \\ p & 0 \end{bmatrix}^* \right|_{m+n}$. Hence $p \oplus q \sim q \oplus p$.

5. Let $p, q \in \mathcal{OP}_n(V)$ such that $p \perp q$. By (1), we get that $p \sim p \oplus 0_n$ and $q \sim 0_n \oplus q$. Also $p \oplus 0_n \perp 0_n \oplus q$. Thus by (2), we may conclude that $p + q \sim p \oplus q$.

Now, (6) can be proved in the same way. □

Definition 5.2.6. *Let V be an absolute matrix order unit space and let $p, q \in \mathcal{OP}_\infty(V)$. We say that $p \preceq q$, if there exists $r \in \mathcal{OP}_\infty(V)$ such that $p \sim r \leq q$.*

Proposition 5.2.7. *Let V be an absolute matrix order unit space satisfying conditions (T) and (H).*

1. *Then $\preceq$ is a reflexive and transitive relation on $\mathcal{OP}_\infty(V)$.*

2. *Let $p, q, r, s \in \mathcal{OP}_\infty(V)$. If $p \preceq r$ and $q \preceq s$, then $p \oplus q \preceq r \oplus s$.*

Proof. (1). Let $p, q, r \in \mathcal{OP}_\infty(V)$. Then $p \preceq p$ holds trivially. Next let $p \preceq q$ and $q \preceq r$. Then there exist $p_1, q_1 \in \mathcal{OP}_\infty(V)$ such that $p \sim p_1 \leq q$ and $q \sim q_1 \leq r$. Since $p_1 \leq q \sim q_1$, by Lemma 5.2.2, there exists $q_2 \leq q_1$ such that $p_1 \sim q_2$. Since $p \sim p_1$ and since $p_1 \sim q_2$, by Proposition 5.2.4, we get $p \sim q_2 \leq q_1 \leq r$. Thus $p \preceq r$.

(2). Let $p \preceq r$ and $q \preceq s$. Then there are $p_1, q_1 \in \mathcal{OP}_\infty(V)$ such that $p \sim p_1 \leq r$ and $q \sim q_1 \leq s$. Thus $p \oplus q \sim p_1 \oplus q_1 \leq r \oplus s$ so that $p \oplus q \preceq r \oplus s$. □

Proposition 5.2.8. *Let V be absolute matrix order unit space satisfying conditions (T) and (H) and let $p, q \in \mathcal{OP}_\infty(V)$. Then $p \preceq q$ if and only if $q \sim p \oplus p_0$ for some $p_0 \in \mathcal{OP}_\infty(V)$.*

Proof. First assume that $p \preceq q$. Then $p \sim r \leq q$ for some $r \in \mathcal{OP}_\infty(V)$. Put $p_0 = (q - r)$. Then by Proposition 2.3.11(3), we have that $p_0 \in \mathcal{OP}_\infty(V)$ with $p_0 \perp r$. Since $r \sim p$, by Proposition 5.2.5, we get that $q = r + p_0 \sim p \oplus p_0$.

Conversely assume that $q \sim p \oplus p_0$ for some $p_0 \in \mathcal{OP}_\infty(V)$. Then $p \oplus 0 \leq p \oplus p_0 \sim q$. Thus by Lemma 5.2.2, there exist $r \in \mathcal{OP}_\infty(V)$ such that $p \oplus 0 \sim r \leq q$. Now, by Propositions 5.2.4 and 5.2.5(1), we have $p \sim r$ so that $p \preceq q$. $\square$

Corollary 5.2.9. *Let V be an absolute matrix order unit space satisfying (T),(H) and let $p_1, p_2, q_1, q_2 \in \mathcal{OP}_\infty(V)$ such that $p_1 \perp p_2$ and $q_1 \perp q_2$. If $p_1 \preceq q_1$ and $p_2 \preceq q_2$, then $p_1 + p_2 \preceq q_1 + q_2$.*

Proof. Assume that $p_1 \preceq q_1$ and $p_2 \preceq q_2$. Thus by Proposition 5.2.7(2), we get $p_1 \oplus p_2 \preceq q_1 \oplus q_2$. As $p_1 \perp p_2$ and $q_1 \perp q_2$, by Proposition 5.2.5, we have $p_1 + p_2 \sim p_1 \oplus p_2$ and $q_1 + q_2 \sim q_1 \oplus q_2$ so that $p_1 + p_2 \preceq p_1 \oplus p_2$ and $q_1 \oplus q_2 \preceq q_1 + q_2$. Then again by Proposition 5.2.7(1), we may conclude that $p_1 + p_2 \preceq q_1 + q_2$. $\square$

5.3 Unitary equivalence

In this section, we define unitary equivalence of order projections in absolute matrix order unit spaces and relate it to partial isometric equivalence.

Definition 5.3.1. *Let V be an absolutely matrix ordered space and let $u, v, w \in M_n(V)$ for some $n \in \mathbb{N}$ such that $u = v + w$. We say that v and w are ortho-components of u, if $v \perp w$.*

Remark 5.3.2.

1. *Let V be an absolute matrix order unit space. Then a partial isometry in V may be ortho-component of more than one unitary in V. To see this, let $V = M_2(\mathbb{C})$ and put $v = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Also set $w_\alpha = \begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix}$ for $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. Note that v is a partial isometry with $v \perp w_\alpha$ and $u_\alpha := v + w_\alpha$ is a unitary.*

2. *Let V be an absolute matrix order unit space. Then every partial unitary*

is an ortho-component of some unitary. In fact, if v is partial unitary in $M_n(V)$ for some $n \in \mathbb{N}$, then $v \perp e^n - |v|_n$ so that $v \pm (e^n - |v|_n)$ is unitary.

Definition 5.3.3. *Let V be an absolute matrix order unit space and let $p, q \in \mathcal{OP}_n(V)$ for some $n \in \mathbb{N}$. We say that p is* unitary equivalent *to q (we write, $p \sim_u q$), if there exists a unitary $u \in M_n(V)$ and an ortho-component v of u such that $p = |v^*|_n$ and $q = |v|_n$.*

Note that if v is an ortho-complement of a unitary u, then v and $u - v$ are partial isometries. In the next result, we characterize unitary equivalence of projections in terms of partial isometric equivalence.

Proposition 5.3.4. *Let V be an absolute matrix order unit space and let $p, q \in \mathcal{OP}_n(V)$ for some $n \in \mathbb{N}$. Then $p \sim_u q$ if and only if $p \sim q$ and $e^n - p \sim e^n - q$.*

Proof. First assume that $p \sim_u q$. Then there exists an ortho-complement v of a unitary u in $M_n(V)$ such that $p = |v^*|_n$ and $q = |v|_n$. Put $w = u - v$. Then

$$|v|_n + |w|_n = |v + w|_n = |u|_n = e^n$$

and

$$|v^*|_n + |w^*|_n = |v^* + w^*|_n = |u^*|_n = e^n.$$

Thus $|w^*|_n = e^n - p$ and $|w|_n = e^n - q$ so that $p \sim q$ and $e^n - p \sim e^n - q$.

Conversely assume that $p \sim q$ and $e^n - p \sim e^n - q$. Then there exist partial isometries $v, w \in M_n(V)$ such that $p = |v^*|_n$, $q = |v|_n$, $e^n - p = |w^*|_n$ and $e^n - q = |w|_n$. Thus $|v|_n \perp |w|_n$ and $|v^*|_n \perp |w^*|_n$. Now, by Remark 3.3.4(2) and Proposition 3.3.5(1), we get that $v \perp w$ with $|v + w|_n = e^n = |v^* + w^*|_n$. Thus $u := v + w$ is a unitary so that $p \sim_u q$. □

Corollary 5.3.5. *Let V be an absolute matrix order unit space satisfying the condition (T).*

1. *Then $\sim_u$ is an equivalence relation on $\mathcal{OP}_n(V)$ for all $n \in \mathbb{N}$.*

2. *If $p \in \mathcal{OP}_m(V)$ and $q \in \mathcal{OP}_n(V)$, then $p \oplus q \sim_u q \oplus p$.*

3. *If $p, p' \in \mathcal{OP}_m(V)$ and $q, q' \in \mathcal{OP}_n(V)$ with $p \sim_u p'$ and $q \sim_u q'$, then $p \oplus q \sim_u p' \oplus q'$.*

Proof. (1) follows from Propositions 5.2.4 and 5.3.4.

(2). Let $p \in \mathcal{OP}_m(V)$ and $q \in \mathcal{OP}_n(V)$. It follows from Proposition 5.2.5(4) that $p \oplus q \sim q \oplus p$ and $(e^m - p) \oplus (e^n - q) \sim (e^n - q) \oplus (e^m - p)$. Thus $p \oplus q \sim_u q \oplus p$.

(3) follows from Propositions 5.2.5(3) and 5.3.4. □

Corollary 5.3.6. *Let V be an absolute matrix order unit space satisfying condition (T) and let $p, q \in \mathcal{OP}_n(V)$ for some $n \in \mathbb{N}$. Then the following statements are equivalent:*

1. *$p \sim q$;*

2. *$p \oplus 0_n \sim_u q \oplus 0_n$;*

3. *$0_n \oplus p \sim_u 0_n \oplus q$.*

Proof. (1) implies (2) and (3): Assume that $p \sim q$. Then $p = |v^*|_n$ and $q = |v|_n$ for some partial isometry $v \in M_n(V)$. Put $u = \begin{bmatrix} v^* & e^n - q \\ e^n - p & v \end{bmatrix}$ and consider $u_{11} = \begin{bmatrix} v^* & 0 \\ 0 & 0 \end{bmatrix}$, $u_{12} = \begin{bmatrix} 0 & e^n - q \\ 0 & 0 \end{bmatrix}$, $u_{21} = \begin{bmatrix} 0 & 0 \\ e^n - p & 0 \end{bmatrix}$ and $u_{22} = \begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}$. Then $u = u_{11} + u_{12} + u_{21} + u_{22}$. Also, we have $|u_{11}|_{2n} = p \oplus 0$, $|u_{12}|_{2n} = 0 \oplus (e^n - q)$, $|u_{21}|_{2n} = (e^n - p) \oplus 0$ and $|u_{22}|_{2n} = 0 \oplus q$. Similarly, $|u_{11}^*|_{2n} = q \oplus 0$, $|u_{12}^*|_{2n} =$

$(e^n - q) \oplus 0$, $|u_{21}^*|_{2n} = 0 \oplus (e^n - p)$ and $|u_{22}^*|_{2n} = 0 \oplus p$. Thus u_{11}, u_{12}, u_{21} and u_{22} are mutually orthogonal partial isometries in $M_{2n}(V)$ and subsequently, u is a unitary in $M_{2n}(V)$. In particular, $p \oplus 0_n \sim_u q \oplus 0_n$ and $0_n \oplus p \sim_u 0_n \oplus q$. Therefore, (1) implies (2) and (3).

(2) implies (1): Next, assume that $p \oplus 0 \sim_u q \oplus 0$. Then by Proposition 5.3.4, $p \oplus 0 \sim q \oplus 0$. Thus by Propositions 5.2.5(1) and 5.2.4, we may conclude that $p \sim q$. Thus (2) implies (1).

Similarly, as (2) implies (1), we can show that (3) implies (1). □

Proposition 5.3.7. *Let (V, e) be an absolute matrix order unit space and let $p \in \mathcal{OP}_n(V)$ for some $n \in \mathbb{N}$. If d is a unitary in M_n, then $p \sim_u d^*pd$.*

Proof. Let $d \in M_n$ be a unitary. Put $d^*pd = q$. Consider $v = d^*p$. Then $|v|_n = |d^*p|_n = p$ and $|v^*|_n = |pd|_n = |dq|_n = q$ for $pd = dq$. Thus $p \sim q$. Since $e^n - q = e^n - d^*pd = d^*(e^n - p)d$, by a similar argument, we can also show that $(e^n - p) \sim (e^n - q)$. Thus $p \sim_u q$. □

Corollary 5.3.8. *Let (V, e) be an absolute matrix order unit space satisfying condition (T) and let $p \in \mathcal{OP}_n(V)$ for some $n \in \mathbb{N}$. Then $apa^* \sim p$ for any $a \in M_{m,n}$ with $a^*a = I_n$.*

Proof. Let $a \in M_{m,n}$ with $a^*a = I_n$. Find a unitary $d \in M_m$ such that $a = d \begin{bmatrix} I_n \\ 0 \end{bmatrix}$. Then $apa^* = d(p \oplus 0)d^* \in \mathcal{OP}_m(V)$. Now, by Propositions 5.2.5(1) and 5.3.7, we have $d(p \oplus 0)d^* \sim_u (p \oplus 0) \sim p$. Thus by Propositions 5.2.4 and 5.3.4, we get that $apa^* \sim p$. □

5.4 Infinite projections

In this section, we define and study infinite and finite order projections in absolute matrix order unit spaces.

Definition 5.4.1. *Let V be an absolute matrix order unit space and let $p \in \mathcal{OP}_n(V)$ for some $n \in \mathbb{N}$. Then p is said to be* infinite, *if there exists $q \in \mathcal{OP}_n(V)$ with $q \leq p$ and $q \neq p$ such that $p \sim q$. We say that p is* finite, *if it is not infinite.*

Proposition 5.4.2. *Let V be an absolute matrix order unit space.*

1. *Let $p, q \in \mathcal{OP}_n(V)$ for some $n \in \mathbb{N}$ such that $p \leq q$. If q is finite, then so is p.*

2. *Let $p_1, p_2, q \in \mathcal{OP}_n(V)$ with $p_1 \leq q$, $p_2 \leq q$ and $p_1 \perp p_2$. If q is finite, then so is $p_1 + p_2$.*

Proof. (1) Let $r \in \mathcal{OP}_n(V)$ with $p \sim r \leq p$. Since $p \perp (q-p)$ and $r \leq p$, we get that $r \perp (q-p)$. By Proposition 5.2.5(2), we may conclude that $q = (p + (q-p)) \sim (r + (q-p))$. Now assume that q is finite. Then $q = r + (q-p)$ *i.e.* $p = r$ for $(r + (q-p)) \leq q$ and q is finite. Hence p is also finite.

(2) As $p_1, p_2 \leq q$ and $p_1 \perp p_2$, we get that $p_1 + p_2 \leq 2q$ and $\|p_1 + p_2\|_n = \max\{\|p_1\|_n, \|p_2\|_n\} \leq 1$. Then $p_1 + p_2 \in V_q$. By Proposition 2.3.12, we have that q has order unit property so that $p_1 + p_2 \leq \|p_1 + p_2\| q \leq q$. Now assume that q is finite. Hence by (1), we conclude that $p_1 + p_2$ is also finite.

□

The following result characterizes finite order projections:

Corollary 5.4.3. *Let V be an absolute matrix order unit space and let $n \in \mathbb{N}$. Then the following statements are equivalent:*

(1) e^n is finite;

(2) Every isometry in $M_n(V)$ is unitary;

(3) p is finite for all $p \in \mathcal{OP}_n(V)$.

Proof. The equivalence of (1) and (3) follows from Proposition 5.4.2(1).

(1) implies (2): Assume that e^n is finite. Let $v \in M_n(V)$ is an isometry. Then $|v^*|_n \sim |v|_n = e^n$. Since $|v^*|_n \in \mathcal{OP}(V)$, we have that $|v^*|_n \leq e^n$. By assumption, we get that $|v^*|_n = e^n$ so that v is unitary.

(2) implies (3): Finally assume that every isometry in $M_n(V)$ is unitary. Let $p, q \in \mathcal{OP}_n(V)$ such that $p \sim q \leq p$. Then there exists $v \in M_n(V)$ with $|v|_n = p$ and $|v^*|_n = q$. Put $w = v + (e^n - p)$. Since $q \leq p$, we get $v \perp (e^n - p)$ so that

$$|w|_n = |v|_n + (e^n - p) = e^n$$

and

$$|w^*|_n = |v^*|_n + (e^n - p) = q + (e^n - p).$$

By Proposition 2.3.11(1), we have $|w^*| \in \mathcal{OP}_n(V)$. Thus w is an isometry in $M_n(V)$. By assumption, w is a unitary in $M_n(V)$. Then $|w^*|_n = e^n$ so that $q = p$. Hence every projection in $M_n(V)$ is finite. □

In the next result, we characterize infinite projections. For this, we need the following result:

Lemma 5.4.4. *Let V be an absolute matrix order unit space satisfying conditions (T) and (H) and let $p \sim q$ for some $p \in \mathcal{OP}_m(V)$ and $q \in \mathcal{OP}_n(V)$, $m, n \in \mathbb{N}$. If p is infinite, then q is also infinite.*

Proof. Assume that p is infinite. Then $p \sim r \leq p$ with $r \neq p$ for some $r \in \mathcal{OP}_m(V)$. By Lemma 5.2.2, there exists $s \in \mathcal{OP}_n(V)$ such that $s \leq q$, $r \sim s$ and $(p-r) \sim (q-s)$. As $p \sim q$, $p \sim r$ and $r \sim s$, by Proposition 5.2.4, we have $q \sim s$. Since $(p-r) \sim (q-s)$, by Proposition 3.2.4, we have that $0 \neq \|p-r\|_m = \|q-s\|_n$. Thus $q \neq s$ so that q is also infinite. $\square$

Theorem 5.4.5. *Let V be an absolute matrix order unit space satisfying condition (T) and let $p \in \mathcal{OP}_n(V) \setminus \{0\}$ for some $n \in \mathbb{N}$. Then the following statements are equivalent:*

(1) p is an infinite projection;

(2) there exist $q, r \in \mathcal{OP}_n(V) \setminus \{0\}$ with $q \perp r$ such that $p = q + r$ and $p \sim q$;

(3) $p \oplus 0$ is an infinite projection;

(4) $0 \oplus p$ is an infinite projection;

(5) apa^* is an infinite projection whenever $a \in M_{m,n}$ with $a^*a = I_n$;

Moreover, if V also satisfies (H), then the above statements are also equivalent to:

(6) $p \oplus q \preceq p$ for some $q \in \mathcal{OP}_\infty(V) \setminus \{0\}$.

Proof. (1) is equivalent to (2) by the definition.

(1) is equivalent to (3): Let $p \in \mathcal{OP}_m(V)$ for some $m \in \mathbb{N}$. Also suppose that $r \leq p \oplus 0_n$ for some $r \in \mathcal{OP}_{m+n}(V), n \in \mathbb{N}$. Thus there exist $s \in M_m(V), u \in M_{m,n}(V)$ and $s_0 \in M_n(V)$ such that $r = \begin{bmatrix} s & u \\ u^* & s_0 \end{bmatrix}$. Put $a_0 = \begin{bmatrix} I_m \\ 0_n \end{bmatrix}$ and $b_0 = \begin{bmatrix} 0_m \\ I_n \end{bmatrix}$. Then

$$0_m \leq a_0^* r a_0 \leq a_0^*(p \oplus 0_n) a_0$$

and

$$0_n \leq b_0^* r b_0 \leq b_0^*(p \oplus 0_n) b_0$$

so that $0_m \leq s \leq p$ and $0_n \leq s_0 \leq 0_n$. Since $\pm s_0 \in M_n(V)^+$ and $M_n(V)^+$ is proper, we get that $s_0 = 0_n$. Let $s = [s_{i,j}]$ and $u = [u_{i,j}]$. For each $u_{i,j}$, we set $c_{ij} \in M_{m+n,2}$ such that

$$c_{ij} = \begin{cases} 1 & \text{at (i,1) and (m+j,2)} \\ 0 & \text{elsewhere.} \end{cases}$$

so that $\begin{bmatrix} s_{i,i} & u_{i,j} \\ u_{i,j}^* & 0 \end{bmatrix} = c_{ij}^* r c_{ij} \in M_2(V)^+$. Then $\begin{bmatrix} ks_{ii} & \pm u_{i,j} \\ \pm u_{i,j}^* & 0 \end{bmatrix} \in M_2(V)^+$ for all $k \in \mathbb{R}, k > 0$. As $M_2(V)^+$ is Archimedean, we get that $\pm \begin{bmatrix} 0 & u_{i,j} \\ u_{i,j}^* & 0 \end{bmatrix} \in M_2(V)^+$. By properness of $M_2(V)^+$, we have $\pm \begin{bmatrix} 0 & u_{i,j} \\ u_{i,j}^* & 0 \end{bmatrix} = 0$ so that $u_{i,j} = 0$. Thus $r = s \oplus 0_n$ and $s \leq p$. Now equivalence of (1) and (3) follows from Propositions 5.2.4, 5.2.5(1) and 5.1.2.

Similarly, we can show that (1) is equivalent to (4) as well.

(3) is equivalent to (5): Let $a \in M_{m,n}$ such that $a^* a = I_n$. Then $apa^* = d(p \oplus 0_{m-n}) d^*$ for some d unitary in M_m. Since d is unitary, we get that apa^* is finite if and only if $p \oplus 0_m$ is finite. Hence equivalence of (3) and (5) follows.

Now assume that V also satisfies (H).

(1) implies (6): Assume that p is infinite. Then there exists $r \in \mathcal{OP}_\infty(V)$ such that $p \sim r \leq p$ and $r \neq p$. Put $q = p - r$. Then $q \in \mathcal{OP}_\infty(V) \setminus \{0\}$. Now, by Proposition 5.2.5, we get that $p = r + q \sim p \oplus q$. Thus $p \oplus q \preceq p$.

(6) implies (1): Finally, assume that there exist $q \in \mathcal{OP}_m(V) \setminus \{0\}$ such that

$p \oplus q \preceq p$ for some $m \in \mathbb{N}$. Then by Proposition 5.2.8, we get that $p \sim p \oplus q \oplus q_0$ for some $q_0 \in \mathcal{OP}_l(V)$, $l \in \mathbb{N}$. Thus

$$p \oplus q \oplus q_0 \sim p \sim p \oplus 0_{m+l} \leq p \oplus q \oplus q_0.$$

Since $p \oplus 0_{m+l} \neq p \oplus q \oplus q_0$, we get that $p \oplus q \oplus q_0$ is an infinite projection. Hence by Lemma 5.4.4, p is also an infinite projection. ☐

Remark 5.4.6. *Let V be an absolute matrix order unit space satisfying (T) and let $p \in \mathcal{OP}_n(V), q \in \mathcal{OP}_m(V)$ for some $m, n \in \mathbb{N}$. If p or q is infinite, then so is $p \oplus q$. In fact, in this case, either $p \oplus 0_m$ or $0_n \oplus q$ is infinite. Thus by Proposition 5.4.2(1), we get that $p \oplus q$ is also infinite.*

Proposition 5.4.7. *Let V be an absolute matrix order unit space satisfying conditions (T) and (H) and assume that $p \in \mathcal{OP}_m(V)$ such that p is infinite. Then there exists a strictly decreasing sequence of infinite projections $\{r_n\}_{n=1}^{\infty}$ with $r_1 = p$ such that $r_n \sim p$ as well as $(r_n - r_{n+1}) \sim (r_{n+1} - r_{n+2})$ for all $n \in \mathbb{N}$.*

Proof. Let $p = r_1$. As r_1 is infinite, there exists $r_2 \in \mathcal{OP}_m(V)$ such that $r_2 \neq r_1$ and $r_2 \leq r_1 \sim r_2$. By Lemma 5.2.2, we can find a projection $r_3 \in \mathcal{OP}_m(V)$ with $r_3 \leq r_2$ such that $r_2 \sim r_3$ and $(r_1 - r_2) \sim (r_2 - r_3)$. Then by Proposition 5.2.4, we have that $r_3 \sim p$. Since $r_1 \neq r_2$ and $(r_1 - r_2) \sim (r_2 - r_3)$, we also have that $r_2 \neq r_3$. Now the result follows from induction on n. ☐

Remark 5.4.8. *Put $r_n - r_{n+1} = s_n$. As $r_{n+1} \leq r_n$ with $r_n \neq r_{n+1}$, by Proposition 2.3.11(3)(i), we have $s_n \in \mathcal{OP}_m(V) \setminus \{0\}$ for all $n \in \mathbb{N}$. Further, as $\sum_{n=1}^{k} s_n = r_1 - r_{k+1} \leq r_1 \leq e^m$ for all $k \in \mathbb{N}$, again by Proposition 2.3.11(1), we get that $s_l \perp s_n$ with $p = r_1 = \sum_{n=1}^{k} s_n + r_{k+1}$ for all $l, n \in \mathbb{N}, l \neq n$.*

Corollary 5.4.9. *Let V be an absolute matrix order unit space satisfying conditions (T) and (H) and let $q \preceq p$ for some $p, q \in \mathcal{OP}_\infty(V)$. If p is finite, then q is also finite.*

Proof. Since $q \preceq p$, by Proposition 5.2.8, we get that $p \sim q \oplus q_0$ for some $q_0 \in \mathcal{OP}_n(V), n \in \mathbb{N}$. Now, assume that p is finite. Then by Lemma 5.4.4, $q \oplus q_0$ is also finite. As $q \oplus 0_n \leq q \oplus q_0$, by Propositions 5.4.2(1) and 5.2.5(1) and by Lemma 5.4.4, we may conclude that q is also finite. □

Proposition 5.4.10. *Let V be an absolute matrix order unit space satisfying conditions (T) and (H) and let $p \preceq q$ and $q \preceq p$ for some $p, q \in \mathcal{OP}_\infty(V)$. If one of them is finite, then $p \sim q$.*

Proof. Without any loss of generality, we may assume that p is a finite order projection. As $p \preceq q$ and $q \preceq p$, by Proposition 5.2.8, we have $q \sim p \oplus p_0$ and $p \sim q \oplus q_0$ for some $p_0, q_0 \in \mathcal{OP}_\infty(V)$. Thus by Propositions 5.2.4 and 5.2.5, we get that $p \sim p \oplus (p_0 \oplus q_0)$. Since p is finite, by Lemma 5.4.4, we conclude that $p \oplus (p_0 \oplus q_0)$ is also finite. Also note that $p \oplus 0 \leq p \oplus (p_0 \oplus q_0)$ and $p \oplus 0 \sim p \oplus (p_0 \oplus q_0)$. Now $p \oplus (p_0 \oplus q_0)$ is finite so that $p_0 = 0$ and $q_0 = 0$. Thus $p \sim q \oplus 0 \sim q$ so that $p \sim q$. □

5.5 Properly infinite projections

In this section, we define notion of properly infinite projections in absolute matrix order unit spaces. The class of properly infinite projections is subclass of infinite projections. We also prove a result characterizing properly infinite order projections.

Definition 5.5.1. *Let V be an absolute matrix order unit space and let $p \in$*

$\mathcal{OP}_n(V) \setminus \{0\}$ *for some* $n \in \mathbb{N}$. *We say that* p *is* properly infinte, *if there exist* $r, s \in \mathcal{OP}_n(V)$ *such that* $r + s \leq p$ *and* $r \sim p \sim s$.

Remark 5.5.2. *Every properly infinite projection in an absolute matrix order unit space is an infinite projection. In fact, as* $r \sim p \sim s$ *and* $p \neq 0$, *by Proposition 3.2.4, we get that* $\|r\| = \|s\| = \|p\| \neq 0$. *Then* $s \neq 0$ *so that* $r \neq p$ *for* $r + s \leq p$. *Thus* $p \sim r \leq p$ *with* $r \neq p$ *so that* p *is infinite.*

Proposition 5.5.3. *Let* V *be an absolute matrix order unit space and let* $p, q \in \mathcal{OP}_n(V) \setminus \{0\}$ *for some* $n \in \mathbb{N}$. *If* p *and* q *both are properly infinite and if* $p \perp q$, *then* $p + q$ *is also properly infinite.*

Proof. Assume that p and q both are properly infinite. Then there exist $r_1, r_2, s_1, s_2 \in \mathcal{OP}_n(V)$ such that $(r_1 + s_1) \leq p$, $(r_2 + s_2) \leq q$, $r_1 \sim p \sim s_1$ and $r_2 \sim q \sim s_2$. Since $p \perp q$, we have $r_1 \perp r_2$ and $s_1 \perp s_2$. Now by Proposition 5.2.5(2), $(r_1 + r_2) \sim (p + q) \sim (s_1 + s_2)$. Also note that $(r_1 + r_2) + (s_1 + s_2) \leq p + q$ so that $p + q$ is properly infinite. □

Theorem 5.5.4. *Let* V *be an absolute matrix order unit space satisfying condition (T) and let* $p \in \mathcal{OP}_\infty(V) \setminus \{0\}$. *Then the following statements are equivalent:*

(1) p *is properly infinite;*

(2) $p \oplus 0$ *is properly infinite;*

(3) $0 \oplus p$ *is properly infinite;*

Moreover, if V *also satisfies (H), then the above statements are also equivalent to:*

(4) $p \oplus p \preceq p$.

Proof. It follows from the definition that (1) implies (2) and (3).

(2) implies (1): Let $p \oplus 0$ be properly infinite. Then there exist $s_1, s_2 \in \mathcal{OP}_\infty(V)$ with $s_1, s_2 \leq p \oplus 0$ such that $s_1 \sim p \oplus 0 \sim s_2$. Since $s_1, s_2 \leq p \oplus 0$, there exist $r_1, r_2 \in \mathcal{OP}_\infty(V)$ with $r_1, r_2 \leq p$ such that $s_1 = r_1 \oplus 0$ and $s_2 = r_2 \oplus 0$. Thus $r_1 \oplus 0 \sim p \oplus 0 \sim r_2 \oplus 0$. Now by Propositions 5.2.4 and 5.2.5(1), we get that $r_1 \sim p \sim r_2$. Thus p is also properly infinite.

Similarly, we can show that (3) implies (1) as well.

Now assume that V also satisfies (H).

(1) implies (4): Assume that p is properly infinite. Then there exist $r, s \in \mathcal{OP}_\infty(V)$ such that $r + s \leq p$ and $r \sim p \sim s$. Since $r \perp s$, by Propositions 5.2.4 and 5.2.5, we get that $(r + s) \sim p \oplus p$. Hence $p \oplus p \precsim p$.

(4) implies (1): Finally assume that $p \oplus p \precsim p$. Then by Proposition 5.2.8, we have $p \sim (p \oplus p) \oplus p_0$ for some $p_0 \in \mathcal{OP}_\infty(V)$. As

$$(p \oplus 0) \oplus 0 \leq (p \oplus p) \oplus p_0,$$

$$(0 \oplus p) \oplus 0 \leq (p \oplus p) \oplus p_0$$

and

$$(p \oplus 0) \oplus 0 \perp (0 \oplus p) \oplus 0,$$

by Lemma 5.2.2, there exist $p_1, p_2 \in \mathcal{OP}_\infty(V)$ with $p_1, p_2 \leq p$, and $p_1 \perp p_2$ such that $p_1 \sim (p \oplus 0) \oplus 0$ and $p_2 \sim (0 \oplus p) \oplus 0$. Thus $p_1 + p_2 \leq p$ and $p_1 \sim p \sim p_2$ for $(p \oplus 0) \oplus 0 \sim p \sim (0 \oplus p) \oplus 0$ so that p is properly infinite. $\square$

Remark 5.5.5. *Let V be an absolute matrix order unit space satisfying (T) and let $p \in \mathcal{OP}_n(V) \setminus \{0\}, q \in \mathcal{OP}_m(V) \setminus \{0\}$ for some $m, n \in \mathbb{N}$. If both p and q are properly infinite, then so is $p \oplus q$. In fact, in this case, $p \oplus 0_m$ and $0_n \oplus q$ are* *properly infinite order projections such that* $p \oplus 0_m \perp 0_n \oplus q$*. Thus by Proposition 5.5.3, we get that* $p \oplus q = (p \oplus 0_m) + (0_n \oplus q)$ *is properly infinite.*

Proposition 5.5.6. *Let* V *be an absolute matrix order unit space satisfying conditions (T) and (H) and let* $p, q \in \mathcal{OP}_\infty(V) \setminus \{0\}$ *be such that* $p \preceq q \preceq p$*. If* p *is properly infinite, then so is* q*.*

Proof. As $p \preceq q \preceq p$, by Proposition 5.2.8, we have $q \sim p \oplus p_0$ and $p \sim q \oplus q_0$ for some $p_0, q_0 \in \mathcal{OP}_\infty(V)$. Now assume that p is properly infinite. Then by Theorem 5.5.4, $p \oplus p \preceq p$. Thus by Proposition 5.2.8, there exists $r \in \mathcal{OP}_\infty(V)$ such that $p \sim (p \oplus p) \oplus r$. A repeated use of Proposition 5.2.5 yields that

$$\begin{aligned} q &\sim p \oplus p_0 \sim ((p \oplus p) \oplus r) \oplus p_0 \\ &\sim (((q \oplus q_0) \oplus (q \oplus q_0)) \oplus r) \oplus p_0 \\ &\sim (q \oplus q) \oplus (q_0 \oplus q_0 \oplus r \oplus p_0). \end{aligned}$$

Again applying Proposition 5.2.8 and Theorem 5.5.4, we may conclude that q is properly infinite. $\square$

Remark 5.5.7. *It follows from the proof of Proposition 5.5.6 that if* $p \sim q$ *and if* p *is properly infinite, then so is* q*.*

Chapter 6

K_0 of an absolute matrix order unit space

In this chapter, we describe K_0 of an absolute matrix order unit space. For this purpose, we need to discuss the direct limit of absolute matrix order unit spaces. In the first section, we describe the direct limit of absolute matrix order unit spaces. In the second section, we define and discuss matrix order unit property and absolute matrix order unit property. We study some properties of matrix order unit property and absolute matrix order unit property. We characterize order projections in absolute matrix order spaces in terms of matrix order unit property and absolute matrix order unit property. In the third section, we define K_0 explicitly. In the fourth section, we show that K_0 is a functor from category of absolute matrix order unit spaces with morphisms as unital completely $|\cdot|$-preserving maps to category of abelian groups. In the fifth section, we study order structure on K_0 and prove that under certain condition K_0 is an ordered abelian group. We also show that K_0 is additive on orthogonal unital completely $|\cdot|$-preserving maps.

6.1 Direct limit of absolute matrix order unit spaces

We start this section with the following definition:

Definition 6.1.1. *Let $\mathfrak{V}$ be a non-degenerate $\mathfrak{F}$-bimodule and let $\mathfrak{v} \in \mathfrak{V}$. Then the smallest $n \in \mathbb{N}$ such that $\mathfrak{I}_n \mathfrak{v} \mathfrak{I}_n = \mathfrak{v}$ is called the* order *of $\mathfrak{v}$ in $\mathfrak{V}$. We denote the order of $\mathfrak{v}$ in $\mathfrak{V}$ by $o(\mathfrak{v})$.*

Proposition 6.1.2. *Let $(\mathfrak{V}, \mathfrak{V}^+)$ be a non-degenerate ordered $\mathfrak{F}$-bimodule and assume that $\mathfrak{V}^+$ is proper and Archimedean. Then for $\mathfrak{u}, \mathfrak{v} \in \mathfrak{V}^+$ with $\mathfrak{u} \leq \mathfrak{v}$, we have $o(\mathfrak{u}) \leq o(\mathfrak{v})$.*

Proof. If possible, assume that $n = o(\mathfrak{u}) > o(\mathfrak{v})$. Then $\mathfrak{e}_{1,n} \mathfrak{v} \mathfrak{e}_{n,1} = 0$ so that $0 \leq \mathfrak{e}_{1,n} \mathfrak{u} \mathfrak{e}_{n,1} \leq \mathfrak{e}_{1,n} \mathfrak{v} \mathfrak{e}_{n,1} = 0$. As $\mathfrak{V}^+$ is proper, we get $\mathfrak{e}_{1,n} \mathfrak{u} \mathfrak{e}_{n,1} = 0$. Since $o(\mathfrak{u}) = n$, there exists $1 \leq j < n$ such that $\mathfrak{e}_{1,j} \mathfrak{u} \mathfrak{e}_{n,1} \neq 0$. Let $\lambda \in \mathbb{C}$ and put $\mathfrak{a} = (\lambda \mathfrak{e}_{j,1} + \mathfrak{e}_{n,1})$. Then $\mathfrak{a}^* \mathfrak{u} \mathfrak{a} \in \mathfrak{V}^+$ so that

$$|\lambda|^2 \mathfrak{e}_{1,j} \mathfrak{u} \mathfrak{e}_{j,1} + \bar{\lambda} \mathfrak{e}_{1,j} \mathfrak{u} \mathfrak{e}_{n,1} + \lambda \mathfrak{e}_{1,n} \mathfrak{u} \mathfrak{e}_{j,1} \in \mathfrak{V}^+$$

for all $\lambda \in \mathbb{C}$. Thus

$$k \mathfrak{e}_{1,j} \mathfrak{u} \mathfrak{e}_{j,1} + \bar{\alpha} \mathfrak{e}_{1,j} \mathfrak{u} \mathfrak{e}_{n,1} + \alpha \mathfrak{e}_{1,n} \mathfrak{u} \mathfrak{e}_{j,1} \in \mathfrak{V}^+$$

for all $k > 0$ and $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. As $\mathfrak{e}_{1,j} \mathfrak{u} \mathfrak{e}_{j,1} \in \mathfrak{V}^+$ and $\mathfrak{V}^+$ is Archimedean, we conclude that $\bar{\alpha} \mathfrak{e}_{1,j} \mathfrak{u} \mathfrak{e}_{n,1} + \alpha \mathfrak{e}_{1,n} \mathfrak{u} \mathfrak{e}_{j,1} \in \mathfrak{V}^+$ for all $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. Put $\alpha = \pm 1$ and $\pm i$. Then

$$\pm(\mathfrak{e}_{1,j} \mathfrak{u} \mathfrak{e}_{n,1} + \mathfrak{e}_{1,n} \mathfrak{u} \mathfrak{e}_{j,1}) \in \mathfrak{V}^+$$

and

$$\pm i(\mathfrak{e}_{1,j}\mathfrak{u}\mathfrak{e}_{n,1} - \mathfrak{e}_{1,n}\mathfrak{u}\mathfrak{e}_{j,1}) \in \mathfrak{V}^+.$$

Again using the fact that $\mathfrak{V}^+$ is proper, we have

$$\mathfrak{e}_{1,j}\mathfrak{u}\mathfrak{e}_{n,1} + \mathfrak{e}_{1,n}\mathfrak{u}\mathfrak{e}_{j,1} = 0$$

and

$$i(\mathfrak{e}_{1,j}\mathfrak{u}\mathfrak{e}_{n,1} - \mathfrak{e}_{1,n}\mathfrak{u}\mathfrak{e}_{j,1}) = 0$$

or

$$\mathfrak{e}_{1,j}\mathfrak{u}\mathfrak{e}_{n,1} - \mathfrak{e}_{1,n}\mathfrak{u}\mathfrak{e}_{j,1} = 0.$$

Thus $\mathfrak{e}_{1,j}\mathfrak{u}\mathfrak{e}_{n,1} = 0$ which is a contradiction. Hence $o(\mathfrak{u}) \leq o(\mathfrak{v})$. $\square$

Definition 6.1.3. *Let $\mathfrak{V}$ be a non-degenerate $\mathfrak{F}$-bimodule and let $\mathfrak{u}, \mathfrak{v} \in \mathfrak{V}$. Then $\mathfrak{u}$ and $\mathfrak{v}$ are said to be* $\mathfrak{F}$-independent *in $\mathfrak{V}$, if there exist $I, J \subset \mathbb{N}$ with $I \cap J = \phi$ such that $\mathfrak{r}\mathfrak{u}\mathfrak{r} = \mathfrak{u}$ and $\mathfrak{s}\mathfrak{v}\mathfrak{s} = \mathfrak{v}$ where $\mathfrak{r} = \sum_{i \in I} \mathfrak{e}_{i,i}$ and $\mathfrak{s} = \sum_{j \in J} \mathfrak{e}_{j,j}$.*

Remark 6.1.4. *Let $\mathfrak{V}$ be a non-degenerate $\mathfrak{F}$-bimodule and let $\mathfrak{u}, \mathfrak{v} \in \mathfrak{V}$. Then*

(a) $\mathfrak{o}$ is $\mathfrak{F}$-independent to every element.

(b) $\mathfrak{v}$ is $\mathfrak{F}$-independent to itself if and only if $\mathfrak{v} = 0$.

(c) $\mathfrak{u}$ is $\mathfrak{F}$-independent to $\mathfrak{v}$ if and only if $\mathfrak{u}^*$ is $\mathfrak{F}$-independent to $\mathfrak{v}$ if and only if $\mathfrak{u}$ is $\mathfrak{F}$-independent to $-\mathfrak{v}$.

A notation: Let $(\mathfrak{V}, \mathfrak{V}^+)$ be a non-degenerate ordered $\mathfrak{F}$-bimodule and let $\mathfrak{v}_1, \mathfrak{v}_2 \in \mathfrak{V}^+$ with $\mathfrak{I}_n\mathfrak{v}_1\mathfrak{I}_n = \mathfrak{v}_1$ and $\mathfrak{I}_n\mathfrak{v}_2\mathfrak{I}_n = \mathfrak{v}_2$ for some $n \in \mathbb{N}$. Put $\mathfrak{J}_n = \sum_{i=1}^{n} \mathfrak{e}_{i,n+i}$. Then we write $(\mathfrak{v}_1, \mathfrak{v}_2)_n^+ = \mathfrak{v}_1 + \mathfrak{J}_n^*\mathfrak{v}_2\mathfrak{J}_n$ and $sa_n(\mathfrak{v}) = \mathfrak{I}_n\mathfrak{v}\mathfrak{J}_n + \mathfrak{J}_n^*\mathfrak{v}^*\mathfrak{I}_n$ for some $\mathfrak{v} \in \mathfrak{V}$ with $\mathfrak{I}_n\mathfrak{v}\mathfrak{I}_n = \mathfrak{v}$.

Let $(\mathfrak{V}, \mathfrak{V}^+)$ be the matricial inductive limit of matrix ordered space $(V, \{M_n(V)^+\})$. Put $v_1 = T_n^{-1}(\mathfrak{v}_1), v_2 = T_n^{-1}(\mathfrak{v}_2)$ and $v = T_n^{-1}(\mathfrak{v})$. Then $\begin{bmatrix} v_1 & 0 \\ 0 & v_2 \end{bmatrix} = T_{2n}^{-1}((\mathfrak{v}_1, \mathfrak{v}_2)_n^+)$ and $\begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} = T_{2n}^{-1}(sa_n(\mathfrak{v}))$. Hence $\begin{bmatrix} v_1 & \pm v \\ \pm v^* & v_2 \end{bmatrix} = T_{2n}^{-1}((\mathfrak{v}_1, \mathfrak{v}_2)_n^+ \pm sa_n(\mathfrak{v}))$.

Definition 6.1.5. *Let $(\mathfrak{V}, \mathfrak{V}^+)$ be a non-degenerate ordered $\mathfrak{F}$-bimodule. Let $|\cdot| : \mathfrak{V} \longrightarrow \mathfrak{V}^+$ be a map satisfying the following conditions:*

1. $o(|\mathfrak{v}|) \leq o(\mathfrak{v})$;

2. $|\mathfrak{v}| = \mathfrak{v}$ *if* $\mathfrak{v} \in \mathfrak{V}^+$;

3. $(|\mathfrak{v}^*|, |\mathfrak{v}|)_{o(\mathfrak{v})}{}^+ + sa_{o(\mathfrak{v})}(\mathfrak{v}) \in \mathfrak{V}^+$ *for all* $\mathfrak{v} \in \mathfrak{V}$;

4. *For* $\mathfrak{v} \in \mathfrak{V}$ *and* $\mathfrak{a} \in \mathfrak{F}$
$$|\mathfrak{a}\mathfrak{v}| \leq \|\mathfrak{a}\||\mathfrak{v}|;$$

5. *For* $\mathfrak{v} \in \mathfrak{V}$ *and* $\mathfrak{b} \in \mathfrak{F}$
$$|\mathfrak{v}\mathfrak{b}| = ||\mathfrak{v}|\mathfrak{b}|;$$

6. *For* $\mathfrak{u}$ *and* $\mathfrak{v}$ $\mathfrak{F}$*-independent elements in* $(\mathfrak{V}, \mathfrak{V}^+)$*, we have*
$$|\mathfrak{u} + \mathfrak{v}| = |\mathfrak{u}| + |\mathfrak{v}|;$$

7. *For* $\mathfrak{u}, \mathfrak{v}$ *and* $\mathfrak{w} \in \mathfrak{V}^+$ *with* $|\mathfrak{u} - \mathfrak{v}| = \mathfrak{u} + \mathfrak{v}$ *and* $\mathfrak{o} \leq \mathfrak{w} \leq \mathfrak{v}$*, we have* $|\mathfrak{u} - \mathfrak{w}| = \mathfrak{u} + \mathfrak{w}$;

8. *For* $\mathfrak{u}, \mathfrak{v}$ *and* $\mathfrak{w} \in \mathfrak{V}^+$ *with* $|\mathfrak{u} - \mathfrak{v}| = \mathfrak{u} + \mathfrak{v}$ *and* $|\mathfrak{u} - \mathfrak{w}| = \mathfrak{u} + \mathfrak{w}$*, we have* $|\mathfrak{u} - |\mathfrak{v} \pm \mathfrak{w}|| = \mathfrak{u} + |\mathfrak{v} \pm \mathfrak{w}|$.

Then $(\mathfrak{V}, \mathfrak{V}^+, |\cdot|)$ *is said to be a* non-degenerate absolutely ordered $\mathfrak{F}$-bimodule*.*

Proposition 6.1.6. *Let* $(\mathfrak{V}, \mathfrak{V}^+, |\cdot|)$ *be a non-degenerate absolutely ordered* $\mathfrak{F}$*-bimodule. Then* $|\mathfrak{a}\mathfrak{v}| = |\mathfrak{v}|$ *for any* $\mathfrak{v} \in \mathfrak{V}$ *and* $\mathfrak{a} \in \mathfrak{F}$ *with* $\mathfrak{a}^*\mathfrak{a} = \mathfrak{I}_{o(\mathfrak{v})}$*.*

Proof. Let $\mathfrak{a} \in \mathfrak{F}$ with $\mathfrak{a}^*\mathfrak{a} = \mathfrak{I}_{o(\mathfrak{v})}$. Then, by Definition 6.1.5(4), we get that

$$|\mathfrak{a}\mathfrak{v}| \leq \|\mathfrak{a}\||\mathfrak{v}| = |\mathfrak{a}^*\mathfrak{a}\mathfrak{v}| \leq \|\mathfrak{a}^*\||\mathfrak{a}\mathfrak{v}| = |\mathfrak{a}\mathfrak{v}|.$$

so that $|\mathfrak{a}\mathfrak{v}| = |\mathfrak{v}|$. □

Remark 6.1.7. *Let* $(\mathfrak{V}, \mathfrak{V}^+, |\cdot|)$ *be a non-degenerate absolutely ordered* $\mathfrak{F}$*-bimodule. Then*

(a) $|\alpha\mathfrak{v}| = |\alpha||\mathfrak{v}|$ *for all* $\mathfrak{v} \in \mathfrak{V}$ *and* $a \in \mathbb{C}$*. To see this, let* $\mathfrak{v} \in \mathfrak{V}$ *and* $\alpha \in \mathbb{C}$*. Then* $\alpha\mathfrak{v} = (\alpha\mathfrak{I}_{o(\mathfrak{v})})\mathfrak{v}$*. Thus by 6.1.5(4), we have* $|\alpha\mathfrak{v}| = |\alpha||\mathfrak{v}|$*.*

(b) $\mathfrak{V}^+$ *is proper. To verify this, let* $\pm\mathfrak{v} \in \mathfrak{V}^+$*. Then by 6.1.5(2) and by (a), we have* $\mathfrak{v} = |\mathfrak{v}| = |-\mathfrak{v}| = -\mathfrak{v}$ *so that* $\mathfrak{v} = 0$*.*

(c) $\mathfrak{V}^+$ *is also generating. To see this, let* $\mathfrak{v} \in \mathfrak{V}_{sa}$*. Then by 6.1.5(3), we get that* $|\mathfrak{v}| \pm \mathfrak{v} \in \mathfrak{V}^+$ *and hence*

$$\mathfrak{v} = \frac{1}{2}\left((|\mathfrak{v}| + \mathfrak{v}) - (|\mathfrak{v}| - \mathfrak{v})\right) \in \mathfrak{V}^+ - \mathfrak{V}^+.$$

(d) Let $\mathfrak{u}, \mathfrak{v} \in \mathfrak{V}_{sa}$ *be such that* $|\mathfrak{u} - \mathfrak{v}| = \mathfrak{u} + \mathfrak{v}$*. Then* $\mathfrak{u}, \mathfrak{v} \in \mathfrak{V}^+$*. For such a pair* $\mathfrak{u}, \mathfrak{v} \in \mathfrak{V}^+$*, we shall say that* $\mathfrak{u}$ *is* orthogonal *to* $\mathfrak{v}$ *and denote it by* $\mathfrak{u} \perp \mathfrak{v}$*.*

(e) For $\mathfrak{v} \in \mathfrak{V}_{sa}$*, we write* $\mathfrak{v}^+ := \frac{1}{2}(|\mathfrak{v}| + \mathfrak{v})$ *and* $\mathfrak{v}^- := \frac{1}{2}(|\mathfrak{v}| - \mathfrak{v})$*. Then* $\mathfrak{v}^+ \perp \mathfrak{v}^-, \mathfrak{v} = \mathfrak{v}^+ - \mathfrak{v}^-$ *and* $|\mathfrak{v}| = \mathfrak{v}^+ + \mathfrak{v}^-$*. This decomposition is unique in the following sense: If* $\mathfrak{v} = \mathfrak{v}_1 - \mathfrak{v}_2$ *with* $\mathfrak{v}_1 \perp \mathfrak{v}_2$*, then* $\mathfrak{v}_1 = \mathfrak{v}^+$ *and*

$\mathfrak{v}_2 = \mathfrak{v}^-$. *In other words, every element in* $\mathfrak{V}_{sa}$ *has a* unique orthogonal decomposition *in* $\mathfrak{V}^+$.

(f) Condition (6) in Definition 6.1.5 can also be replaced by anyone of the following statements:

(i) For $\mathfrak{u}$ *and* $\mathfrak{v}$ $\mathfrak{F}$*-independent elements in* $(\mathfrak{V}, \mathfrak{V}^+)$*, we have*

$$|\mathfrak{u}^* + \mathfrak{v}| = |\mathfrak{u}^*| + |\mathfrak{v}|;$$

(ii) For $\mathfrak{u}$ *and* $\mathfrak{v}$ $\mathfrak{F}$*-independent elements in* $(\mathfrak{V}, \mathfrak{V}^+)$*, we have*

$$|\mathfrak{u} - \mathfrak{v}| = |\mathfrak{u}| + |\mathfrak{v}|;$$

Definition 6.1.8. *An element* $\mathfrak{a} \in \mathfrak{F}$ *is said to be a* local unitary*, if* $\mathfrak{a}^*\mathfrak{a} = \mathfrak{I}_{o(\mathfrak{a})} = \mathfrak{a}\mathfrak{a}^*$.

Proposition 6.1.9. *Let* $(\mathfrak{V}, \mathfrak{V}^+, |\cdot|)$ *be a non-degenerate absolutely ordered* $\mathfrak{F}$*-bimodule and let* $\mathfrak{v} \in \mathfrak{V}$*. Then* $|\mathfrak{a}^*\mathfrak{v}\mathfrak{a}| = \mathfrak{a}^*|\mathfrak{v}|\mathfrak{a}$ *for every local unitary element* $\mathfrak{a} \in \mathfrak{F}$ *with* $o(\mathfrak{v}) \leq o(\mathfrak{a})$.

Proof. Let $\mathfrak{a} \in \mathfrak{F}$ be a local unitary with $o(\mathfrak{v}) \leq o(\mathfrak{a})$. Since $o(|\mathfrak{v}|) \leq o(\mathfrak{v})$, by Definition 6.1.5(4), we have

$$\begin{aligned} |\mathfrak{a}^*\mathfrak{v}\mathfrak{a}| &\leq \|\mathfrak{a}^*\||\,|\mathfrak{v}|\mathfrak{a}| \\ &= |\,|\mathfrak{v}|\mathfrak{a}| \\ &= |\mathfrak{a}(\mathfrak{a}^*|\mathfrak{v}|\mathfrak{a})| \\ &\leq \|\mathfrak{a}\||\mathfrak{a}^*|\mathfrak{v}|\mathfrak{a}| \\ &= \mathfrak{a}^*|\mathfrak{v}|\mathfrak{a}. \end{aligned}$$

Similarly $|\mathfrak{a}\mathfrak{v}\mathfrak{a}^*| \leq \mathfrak{a}|\mathfrak{v}|\mathfrak{a}^*$. Now $o(\mathfrak{v}) \leq o(\mathfrak{a})$, thus replacing $\mathfrak{v}$ by $\mathfrak{a}^*\mathfrak{v}\mathfrak{a}$, we get $|\mathfrak{v}| \leq \mathfrak{a}|\mathfrak{a}^*\mathfrak{v}\mathfrak{a}|\mathfrak{a}^*$ so that $\mathfrak{a}^*|\mathfrak{v}|\mathfrak{a} \leq |\mathfrak{a}^*\mathfrak{v}\mathfrak{a}|$. As $\mathfrak{V}^+$ is proper, we conclude that $|\mathfrak{a}^*\mathfrak{v}\mathfrak{a}| = \mathfrak{a}^*|\mathfrak{v}|\mathfrak{a}$. □

Now, in next two Theorems 6.1.10 and 6.1.11, we generalize Theorem 2.2.8 in the context of absolutely matrix ordered space.

Theorem 6.1.10. *Let $(V, \{M_n(V)^+\}, \{|\cdot|_n\})$ be an absolutely matrix ordered space and let $(\mathfrak{V}, \mathfrak{V}^+)$ be the matricial inductive limit of $(V, \{M_n(V)^+\})$. For $\mathfrak{v} \in \mathfrak{V}$, we define $|\mathfrak{v}| := T_{o(\mathfrak{v})}(|T^{-1}_{o(\mathfrak{v})}(\mathfrak{v})|)$ and consider the map $|\cdot| : \mathfrak{V} \to \mathfrak{V}^+$ given by $\mathfrak{v} \longmapsto |\mathfrak{v}|$ for all $\mathfrak{v} \in \mathfrak{V}$. Then $(\mathfrak{V}, \mathfrak{V}^+, |\cdot|)$ is a non-degenerate absolutely ordered $\mathfrak{F}$-bimodule.*

Proof. Let $\mathfrak{v} \in \mathfrak{V}$. Then

$$I_{o(\mathfrak{v})} T^{-1}_{o(\mathfrak{v})}(\mathfrak{v}) I_{o(\mathfrak{v})} = T^{-1}_{o(\mathfrak{v})}(\mathfrak{v})$$

so that

$$I_{o(\mathfrak{v})} |T^{-1}_{o(\mathfrak{v})}(\mathfrak{v})|_{o(\mathfrak{v})} I_{o(\mathfrak{v})} = |T^{-1}_{o(\mathfrak{v})}(\mathfrak{v})|_{o(\mathfrak{v})}.$$

Now, it follows that $\mathfrak{I}_{o(\mathfrak{v})}|\mathfrak{v}|\mathfrak{I}_{o(\mathfrak{v})} = |\mathfrak{v}|$ so that $o(|\mathfrak{v}|) \leq o(\mathfrak{v})$ for all $\mathfrak{v} \in \mathfrak{V}$.

If $\mathfrak{v} \in \mathfrak{V}^+$, then $T^{-1}_{o(\mathfrak{v})}(\mathfrak{v}) \in M_{o(\mathfrak{v})}(V)^+$ so that $|T^{-1}_{o(\mathfrak{v})}(\mathfrak{v})|_{o(\mathfrak{v})} = T^{-1}_{o(\mathfrak{v})}(\mathfrak{v})$. Thus $|\mathfrak{v}| = \mathfrak{v}$ if $\mathfrak{v} \in \mathfrak{V}^+$.

Note that $\begin{bmatrix} |T^{-1}_{o(\mathfrak{v})}(\mathfrak{v}^*)|_{o(\mathfrak{v})} & T^{-1}_{o(\mathfrak{v})}(\mathfrak{v}) \\ T^{-1}_{o(\mathfrak{v})}(\mathfrak{v}^*) & |T^{-1}_{o(\mathfrak{v})}(\mathfrak{v})|_{o(\mathfrak{v})} \end{bmatrix} \in M_{2o(\mathfrak{v})}(V)^+$ for any $\mathfrak{v} \in \mathfrak{V}$ so that $(|\mathfrak{v}^*|, |\mathfrak{v}|)_{o(\mathfrak{v})}{}^+ + sa_{o(\mathfrak{v})}(\mathfrak{v}) \in \mathfrak{V}^+$.

Now, let $\mathfrak{v} \in \mathfrak{V}$ and $\mathfrak{a}, \mathfrak{b} \in \mathfrak{F}$. Put $n = \max\{o(\mathfrak{v}), o(\mathfrak{a})\}$ and $m = \max\{o(\mathfrak{v}), o(\mathfrak{b})\}$. Then

$$\begin{aligned} |T_n^{-1}(\mathfrak{a}\mathfrak{v})|_n &= |\sigma_n^{-1}(\mathfrak{a})T_n^{-1}(\mathfrak{v})|_n \\ &\leq \|\sigma_n^{-1}(\mathfrak{a})\||T_n^{-1}(\mathfrak{v})|_n \\ &= \|\mathfrak{a}\||T_n^{-1}(\mathfrak{v})|_n \end{aligned}$$

and

$$\begin{aligned} |T_m^{-1}(\mathfrak{v}\mathfrak{b})|_m &= |T_m^{-1}(\mathfrak{v})\sigma_m^{-1}(\mathfrak{b})|_m \\ &= ||T_m^{-1}(\mathfrak{v})|_m\sigma_m^{-1}(\mathfrak{b})|_m \\ &= |T_m^{-1}(T_m(|T_m^{-1}(\mathfrak{v})|_m)\mathfrak{b})|_m \\ &= |T_m^{-1}(|\mathfrak{v}|\mathfrak{b})|_m \end{aligned}$$

so that $|\mathfrak{a}\mathfrak{v}| \leq \|\mathfrak{a}\||\mathfrak{v}|$ and $|\mathfrak{v}\mathfrak{b}| = ||\mathfrak{v}|\mathfrak{b}|$ if $\mathfrak{v} \in \mathfrak{V}$ and $\mathfrak{a}, \mathfrak{b} \in \mathfrak{F}$.

Next, let $\mathfrak{u}$ and $\mathfrak{v}$ be $\mathfrak{F}$-independent elements in $(\mathfrak{V}, \mathfrak{V}^+)$. There exist $I, J \subset \mathbb{N}$ with $I \cap J = \phi$ such that $\mathfrak{r}\mathfrak{u}\mathfrak{r} = \mathfrak{u}$ and $\mathfrak{s}\mathfrak{v}\mathfrak{s} = \mathfrak{v}$ where $\mathfrak{r} = \sum_{i \in I} \mathfrak{e}_{i,i}$ and $\mathfrak{s} = \sum_{j \in J} \mathfrak{e}_{j,j}$. Let

$$I = \{i_1 < i_2 < \cdots < i_{k-1} < i_k\},$$

$$J = \{j_1 < j_2 < \cdots < j_{l-1} < j_l\}$$

and put $n = \max\{i_k, j_l\}$. Then $I \cup J \subset \{1, 2, 3, \cdots, n\}$. Let us write

$$G := \{1, 2, 3, \cdots, n\} \backslash (I \cup J) = \{r_1, r_2, r_3, \cdots, r_{m-1}, r_m\}$$

so that $k+l+m=n$. Put

$$\mathfrak{a}=\sum_{t=1}^{k}\mathfrak{e}_{i_t,t},\ \ \mathfrak{b}=\sum_{t=1}^{l}\mathfrak{e}_{j_t,t+k},\mathfrak{c}=\sum_{t=1}^{m}\mathfrak{e}_{r_t,t+k+l}$$

and $\mathfrak{d}=\mathfrak{a}+\mathfrak{b}+\mathfrak{c}$. It is routine to verify that $\mathfrak{a}^*\mathfrak{a}=\mathfrak{I}_k$, $\mathfrak{a}\mathfrak{a}^*=\mathfrak{r}$, $\mathfrak{b}^*\mathfrak{b}=0_k\oplus\mathfrak{I}_l$, $\mathfrak{b}\mathfrak{b}^*=\mathfrak{s}$, $\mathfrak{c}^*\mathfrak{c}=0_{k+l}\oplus\mathfrak{I}_m$, $\mathfrak{a}^*\mathfrak{b}=\mathfrak{a}\mathfrak{b}^*=\mathfrak{a}^*\mathfrak{c}=\mathfrak{a}\mathfrak{c}^*=\mathfrak{b}^*\mathfrak{c}=\mathfrak{b}\mathfrak{c}^*=\mathfrak{o}$ and $\mathfrak{d}^*\mathfrak{d}=\mathfrak{I}_n=\mathfrak{d}\mathfrak{d}^*$. Put $\mathfrak{u}_1=\mathfrak{a}^*\mathfrak{u}\mathfrak{a}$ and $\mathfrak{v}_1=\mathfrak{b}^*\mathfrak{v}\mathfrak{b}$. Then $\mathfrak{I}_k\mathfrak{u}_1\mathfrak{I}_k=\mathfrak{u}_1$ and $(0_k\oplus\mathfrak{I}_l)\mathfrak{v}_1(0_k\oplus\mathfrak{I}_l)=\mathfrak{v}_1$.

Let $T_k^{-1}(\mathfrak{u}_1)=u$ and $T_{k+l}^{-1}(\mathfrak{v}_1)=v_1$. Then $u\in M_k(V)$ and $v_1\in M_{k+l}(V)$. Also then

$$v_1=T_{k+l}^{-1}((0_k\oplus\mathfrak{I}_l)\mathfrak{v}_1(0_k\oplus\mathfrak{I}_l))=(0_k\oplus I_l)v_1(0_k\oplus I_l)$$

so that $v_1=0_k\oplus v$ for some $v\in M_l(V)$. Now $T_{k+l}^{-1}(\mathfrak{u}_1)=u\oplus 0_l$ and $T_{k+l}^{-1}(\mathfrak{v}_1)=0_k\oplus v$ so that $T_{k+l}^{-1}(\mathfrak{u}_1+\mathfrak{v}_1)=u\oplus v$. Thus

$$\begin{aligned}
|\mathfrak{u}_1+\mathfrak{v}_1| &= T_{k+l}(|u\oplus v|_{k+l})=T_{k+l}(|u|_k\oplus|v|_l)\\
&= T_{k+l}(|u|_k\oplus 0_k)+T_{k+l}(0_k\oplus|v|_l)\\
&= T_k(|u|_k)+T_{k+l}(|v_1|_{k+l})\\
&= |\mathfrak{u}_1|+|\mathfrak{v}_1|.
\end{aligned}$$

Since $\mathfrak{r}\mathfrak{d}=\mathfrak{a}\mathfrak{a}^*\mathfrak{a}$, we get

$$\begin{aligned}
\mathfrak{d}^*\mathfrak{u}\mathfrak{d} &= \mathfrak{d}^*(\mathfrak{r}\mathfrak{u}\mathfrak{r})\mathfrak{d} \\
&= (\mathfrak{d}^*\mathfrak{r})\mathfrak{u}(\mathfrak{r}\mathfrak{d}) \\
&= (\mathfrak{a}^*\mathfrak{a}\mathfrak{a}^*)\mathfrak{u}(\mathfrak{a}\mathfrak{a}^*\mathfrak{a}) \\
&= \mathfrak{a}^*(\mathfrak{a}\mathfrak{a}^*)\mathfrak{u}(\mathfrak{a}\mathfrak{a}^*)\mathfrak{a} \\
&= \mathfrak{a}^*(\mathfrak{r}\mathfrak{u}\mathfrak{r})\mathfrak{a} \\
&= \mathfrak{a}^*\mathfrak{u}\mathfrak{a} \\
&= \mathfrak{u}_1.
\end{aligned}$$

Similarly $\mathfrak{d}^*\mathfrak{v}\mathfrak{d} = \mathfrak{v}_1$. Thus by Proposition 6.1.9, we get

$$\begin{aligned}
\mathfrak{d}^*|\mathfrak{u}+\mathfrak{v}|\mathfrak{d} &= |\mathfrak{d}^*(\mathfrak{u}+\mathfrak{v})\mathfrak{d}| = |\mathfrak{u}_1+\mathfrak{v}_1| \\
&= |\mathfrak{u}_1| + |\mathfrak{v}_1| = \mathfrak{d}^*|\mathfrak{u}|\mathfrak{d} + \mathfrak{d}^*|\mathfrak{v}|\mathfrak{d} \\
&= \mathfrak{d}^*(|\mathfrak{u}|+|\mathfrak{v}|)\mathfrak{d}.
\end{aligned}$$

Hence $|\mathfrak{u}+\mathfrak{v}| = |\mathfrak{u}| + |\mathfrak{v}|$.

Now, let $|\mathfrak{u}-\mathfrak{v}| = \mathfrak{u}+\mathfrak{v}$ and $\mathfrak{o} \leq \mathfrak{w} \leq \mathfrak{v}$ for some $\mathfrak{u},\mathfrak{v},\mathfrak{w} \in \mathfrak{V}^+$. Put $n = \max\{o(\mathfrak{u}), o(\mathfrak{v}), o(\mathfrak{w})\}$. Let $T_n^{-1}(\mathfrak{u}) = u, T_n^{-1}(\mathfrak{v}) = v, T_n^{-1}(\mathfrak{w}) = w$. Then $u,v,w \in M_n(V)^+$ such that $|u-v|_n = u+v$ and $0 \leq w \leq v$. Thus $|u-w|_n = u+w$ so that $|\mathfrak{u}-\mathfrak{w}| = \mathfrak{u}+\mathfrak{w}$.

Finally, let $\mathfrak{u},\mathfrak{v},\mathfrak{w} \in \mathfrak{V}^+$ such that $|\mathfrak{u}-\mathfrak{v}| = \mathfrak{u}+\mathfrak{v}$ and $|\mathfrak{u}-\mathfrak{w}| = \mathfrak{u}+\mathfrak{w}$. Let $n = \max\{o(\mathfrak{u}), o(\mathfrak{v}), o(\mathfrak{w})\}$ and put $u = T_n^{-1}(\mathfrak{u}), v = T_n^{-1}(\mathfrak{v})$ and $w = T_n^{-1}(\mathfrak{w})$. Then $u,v,w \in M_n(V)^+$ and $|u-v|_n = u+v, |u-w|_n = u+w$. Thus $|u-|v\pm w|_n|_n = u+|v\pm w|_n$ so that $|\mathfrak{u}-|\mathfrak{v}\pm\mathfrak{w}|| = \mathfrak{u}+|\mathfrak{v}\pm\mathfrak{w}|$.

□

Next, we prove the converse of Theorem 6.1.10.

Theorem 6.1.11. *Let $(\mathfrak{V}, \mathfrak{V}^+, |\cdot|)$ be a non-degenerate absolutely ordered $\mathfrak{F}$-bimodule. Let $(V, \{M_n(V)^+\})$ be the matrix ordered space whose matricial inductive limit is $(\mathfrak{V}, \mathfrak{V}^+)$ as in Theorem 2.2.8. For each $v \in M_n(V), n \in \mathbb{N}$ we define $|v|_n := T_n^{-1}(|T_n(v)|)$ and consider the map $|\cdot|_n : M_n(V) \to M_n(V)^+$ given by $v \longmapsto |v|_n$. Then $(V, \{M_n(V)^+\}, \{|\cdot|_n\})$ is an absolutely matrix ordered space with $(\mathfrak{V}, \mathfrak{V}^+, |\cdot|)$ as its matricial inductive limit.*

Proof. Let $v \in M_n(V)^+$. Since $T_n(v) \in \mathfrak{V}^+$, we get that $|T_n(v)| = T_n(v)$. Thus $T_n^{-1}(|T_n(v)|) = v$ so that $|v|_n = v$.

For any $v \in M_n(V)_{sa}$ and $k \in \mathbb{R}$, we have that $|T_n(v)| \pm T_n(v) \in \mathfrak{V}^+$ and $|kT_n(v)| = |k||T_n(v)|$. Thus $|v|_n \pm v = T_n^{-1}(|T_n(v)| \pm T_n(v)) \in M_n(V)^+$ and $|kv|_n = T_n^{-1}(|kT_n(v)|) = T_n^{-1}(|k||T_n(v)|) = |k||v|_n$.

Now, let $|u - v|_n = u + v$ and $0 \leq w \leq v$ for some $u, v, w \in M_n(V)^+$. Then $T_n^{-1}(|T_n(u) - T_n(v)|) = u + v$ (or equivalently $|T_n(u) - T_n(v)| = T_n(u) + T_n(v)$) and $0 \leq T_n(w) \leq T_n(v)$. Thus $|T_n(u) - T_n(w)| = T_n(u) + T_n(w)$ so that $|u - w|_n = T_n^{-1}(|T_n(u) - T_n(w)|) = u + w$.

Next, let $u, v, w \in M_n(V)^+$ such that $|u - v|_n = u + v$ and $|u - w|_n = u + w$. Then

$$|T_n(u) - T_n(v)| = T_n(u) + T_n(v)$$

and

$$|T_n(u) - T_n(w)|_n = T_n(u) + T_n(w)$$

so that

$$|T_n(u) - |T_n(v) \pm T_n(w)|| = T_n(u) + |T_n(v) \pm T_n(w)|.$$

Thus

$$\begin{aligned}
|u - |v \pm w|_n|_n &= T_n^{-1}(|T_n(u) - |T_n(v) \pm T_n(w)||) \\
&= T_n^{-1}(T_n(u) + |T_n(v) \pm T_n(w)|) \\
&= u + |v \pm w|_n.
\end{aligned}$$

Let $v \in M_n(V)$ and $a, b \in M_n$. Then

$$\begin{aligned}
|T_n(av)| &= |\sigma_n(a)T_n(v)| \\
&\leq \|\sigma_n(a)\||T_n(v)| \\
&= \|a\||T_n(v)|
\end{aligned}$$

and

$$\begin{aligned}
|T_n(vb)| &= |T_n(v)\sigma_n(b)| \\
&= ||T_n(v)|\sigma_n(b)| \\
&= |T_n(T_n^{-1}(|T_n(v)|)b)| \\
&= |T_n(|v|_n b)|
\end{aligned}$$

so that $T_n^{-1}(|T_n(av)|) \leq \|a\|T_n^{-1}(|T_n(v)|)$ and $T_n^{-1}(|T_n(vb)|) = T_n^{-1}(|T_n(|v|_n b)|)$. Thus $|av|_n \leq \|a\||v|_n$ and $|vb|_n = ||v|_n b|_n$.

Finally, let $u \in M_m(V)$ and $v \in M_n(V)$. Put $v_1 = 0_m \oplus v$ and $a = \begin{bmatrix} 0 & I_m \\ I_n & 0 \end{bmatrix}$. Then a is unitary in M_{m+n} with $a^* v_1 a = v \oplus 0_m$ so that $\sigma_{m+n}(a)$ is local unitary in $\mathfrak{F}$ with $\sigma_{m+n}(a)^* T_{m+n}(v_1)\sigma_{m+n}(a) = T_n(v)$. By Proposition 6.1.9, we get that

$\sigma_{m+n}(a)^*|T_{m+n}(v_1)|\sigma_{m+n}(a) = |T_n(v)|$. Thus

$$\begin{aligned} a^*|v_1|_{m+n}a &= a^*T_{m+n}^{-1}(|T_{m+n}(v_1)|)a \\ &= T_{m+n}^{-1}(\sigma_{m+n}(a)^*|T_{m+n}(v_1)|\sigma_{m+n}(a)) \\ &= T_{m+n}^{-1}(|T_n(v)|) \\ &= |v|_n \oplus 0_m \end{aligned}$$

so that $|v_1|_{m+n} = 0_m \oplus |v|_n$. Now note that $T_m(u)$ and $T_{m+n}(v_1)$ are $\mathfrak{F}$-independent in $\mathfrak{V}$. Thus $|T_m(u) + T_{m+n}(v_1)| = |T_m(u)| + |T_{m+n}(v_1)|$ so that

$$\begin{aligned} |u \oplus v|_{m+n} &= |(u \oplus 0_n) + (0_m \oplus v)|_{m+n} \\ &= T_{m+n}^{-1}(|T_m(u) + T_{m+n}(v_1)|) \\ &= T_{m+n}^{-1}(|T_m(u)| + |T_{m+n}(v_1)|) \\ &= (|u|_m \oplus 0_n) + |v_1|_{m+n} \\ &= (|u|_m \oplus 0_n) + (0_m \oplus |v|_n) \\ &= |u|_m \oplus |v|_n. \end{aligned}$$

□

Definition 6.1.12. *Let $(\mathfrak{V}, \mathfrak{V}^+)$ be a non-degenerate $\mathfrak{F}$-bimodule. An element $\mathfrak{e} \in \mathfrak{V}^+$ is said to be a local order unit for $(\mathfrak{V}, \mathfrak{V}^+)$, if it satisfies the following two conditions:*

(i) $\mathfrak{I}_1 \mathfrak{e} \mathfrak{I}_1 = \mathfrak{e}$.

(ii) For each $\mathfrak{v} \in \mathfrak{I}_1\mathfrak{V}\mathfrak{I}_1$, there exists $k > 0$ such that

$$(k\mathfrak{e}, k\mathfrak{e})_1{}^+ + sa_1(\mathfrak{v}) \in \mathfrak{V}^+.$$

Note:- Let $\mathfrak{V}$ be a non-degenerate $\mathfrak{F}$-bimodule and let $\mathfrak{v} \in \mathfrak{V}$ with $\mathfrak{I}_1\mathfrak{v}\mathfrak{I}_1 = \mathfrak{v}$. For each $n \in \mathbb{N}$, we write $\mathfrak{v}^n = \sum_{i=1}^{n} \mathfrak{e}_{i,1}\mathfrak{v}\mathfrak{e}_{1,i}$.

Proposition 6.1.13. *Let $(\mathfrak{V}, \mathfrak{V}^+)$ be a non-degenerate $\mathfrak{F}$-bimodule with a local order unit $\mathfrak{e}$. Also let $(V, \{M_n(V)^+\})$ be a matrix ordered space whose matricial inductive limit is $(\mathfrak{V}, \mathfrak{V}^+)$. Then*

(1) For $\mathfrak{v} \in \mathfrak{I}_n\mathfrak{V}\mathfrak{I}_n$ and $k > 0$, we have $(k\mathfrak{e}^n, k\mathfrak{e}^n)_n{}^+ + sa_n(\mathfrak{v}) \in \mathfrak{V}^+$ if and only if $(k\mathfrak{e}^n, k\mathfrak{e}^n)_n{}^+ - sa_n(\mathfrak{v}) \in \mathfrak{V}^+$.

(2) $T^{-1}(\mathfrak{e})$ is an order unit for V.

(3) For each $\mathfrak{v} \in \mathfrak{I}_n\mathfrak{V}\mathfrak{I}_n$, there exists $k > 0$ such that $(k\mathfrak{e}^n, k\mathfrak{e}^n)_n{}^+ + sa_n(\mathfrak{v}) \in \mathfrak{V}^+$.

Proof. (1) Put $\mathfrak{a} = \mathfrak{I}_n - \mathfrak{J}_n^*\mathfrak{I}_n\mathfrak{J}_n$. Then $\mathfrak{a}$ is a local unitary with $(k\mathfrak{e}^n, k\mathfrak{e}^n)_n{}^+ - sa_n(\mathfrak{v}) = \mathfrak{a}^*((k\mathfrak{e}^n, k\mathfrak{e}^n)_n{}^+ + sa_n(\mathfrak{v}))\mathfrak{a}$. Thus by Definition 2.2.6(2), we have $(k\mathfrak{e}^n, k\mathfrak{e}^n)_n{}^+ + sa_n(\mathfrak{v}) \in \mathfrak{V}^+$ if and only if $(k\mathfrak{e}^n, k\mathfrak{e}^n)_n{}^+ - sa_n(\mathfrak{v}) \in \mathfrak{V}^+$.

(2) Since $\mathfrak{I}_1\mathfrak{e}\mathfrak{I}_1 = \mathfrak{e}$, we get that $T^{-1}(\mathfrak{e}) \in V^+$. Next, let $v \in V$. Then $T(v) \in \mathfrak{I}_1\mathfrak{V}\mathfrak{I}_1$. Now by definition and by (1), we have $(k\mathfrak{e}, k\mathfrak{e})_1{}^+ \pm sa_1(T(v)) \in \mathfrak{V}^+$. Note that $o((k\mathfrak{e}, k\mathfrak{e})_1{}^+ \pm sa_1(T(v))) \leq 2$. Thus $\begin{bmatrix} kT^{-1}(\mathfrak{e}) & \pm v \\ \pm v^* & kT^{-1}(\mathfrak{e}) \end{bmatrix} = T_2^{-1}((k\mathfrak{e}, k\mathfrak{e})_1{}^+ \pm sa_1(T(v)) \in M_2(V)^+$ so that $T^{-1}(\mathfrak{e})$ is an order unit for V.

(3) Let $\mathfrak{v} \in \mathfrak{I}_n\mathfrak{V}\mathfrak{I}_n$ and put $e = T^{-1}(\mathfrak{e})$. Since e is order unit for V and $T_n^{-1}(\mathfrak{v}) \in$

$M_n(V)$, we get that $\begin{bmatrix} ke^n & T_n^{-1}(\mathfrak{v}) \\ T_n^{-1}(\mathfrak{v})^* & ke^n \end{bmatrix} \in M_{2n}(V)^+$. Then $(k\mathfrak{e}^n, k\mathfrak{e}^n)_n{}^+ +$
$sa_n(\mathfrak{v}) = T_{2n}\left(\begin{bmatrix} ke^n & T_n^{-1}(\mathfrak{v}) \\ T_n^{-1}(\mathfrak{v})^* & ke^n \end{bmatrix}\right) \in \mathfrak{V}^+$. □

Definition 6.1.14. *A non-degenerate ordered $\mathfrak{F}$-bimodule $(\mathfrak{V}, \mathfrak{V}^+)$ with a local order unit $\mathfrak{e}$ is said to be a local $\mathfrak{F}$-order unit bimodule, if $\mathfrak{V}^+$ is proper and Archimedean. We denote it by $(\mathfrak{V}, \mathfrak{V}^+, \mathfrak{e})$.*

Remark 6.1.15. *Let $(V, \{M_n(V)^+\}, e)$ be a matrix order unit space and let $(\mathfrak{V}, \mathfrak{V}^+)$ be the matricial inductive limit of $(V, \{M_n(V)^+\})$ as in Theorem 2.2.8. Then $(\mathfrak{V}, \mathfrak{V}^+, T(e))$ is a Local $\mathfrak{F}$-order unit bimodule.*

Proposition 6.1.16. *Let $(\mathfrak{V}, \mathfrak{V}^+, \mathfrak{e})$ be a local $\mathfrak{F}$-order unit bimodule. For each $\mathfrak{v} \in \mathfrak{V}$, put*

$$\|\mathfrak{v}\| = \inf\{k > 0 : (k\mathfrak{e}^{o(\mathfrak{v})}, k\mathfrak{e}^{o(\mathfrak{v})})^+_{o(\mathfrak{v})} + sa_{o(\mathfrak{v})}(\mathfrak{v}) \in \mathfrak{V}^+\}.$$

Then $(\mathfrak{V}, \mathfrak{V}^+, \mathfrak{e})$ with $\|\cdot\|$ is a $\mathfrak{F}$-bimodule normed space. Moreover, for each $\mathfrak{v} \in \mathfrak{V}$, we have

$$\|\mathfrak{v}\| = \inf\{k > 0 : (k\mathfrak{e}^n, k\mathfrak{e}^n)^+_n + sa_n(\mathfrak{v}) \in \mathfrak{V}^+ \text{ for some } n \in \mathbb{N} \text{ with } \mathfrak{I}_n\mathfrak{v}\mathfrak{I}_n = \mathfrak{v}\}.$$

Proof. Let $(V, \{M_n(V)^+\})$ be the matrix ordered space corresponding to $(\mathfrak{V}, \mathfrak{V}^+)$ as in Theorem 2.2.8. Put $e = T^{-1}(\mathfrak{e})$. By Remark 2.2.9(1) and (3) and by Proposition 6.1.13(2), we get that $(V, \{M_n(V)^+\}, e)$ is a matrix order unit space. Also recall that $(V, \{M_n(V)^+\}, e)$ with $\{\|\cdot\|_n\}$ is an L^∞-matricially $*$-normed space, where each $\|\cdot\|_n$ is given in the following way:

$$\|v\|_n := \inf\left\{k > 0 : \begin{bmatrix} ke^n & v \\ v^* & ke^n \end{bmatrix} \in M_{2n}(V)^+\right\}$$

for all $v \in M_n(V)$.

Note that for any $\mathfrak{v} \in \mathfrak{V}$ and $k \in \mathbb{R}$, we have $(k\mathfrak{e}^n, k\mathfrak{e}^n)_n^+ + sa_n(\mathfrak{v}) \in \mathfrak{V}^+$ if and only if $\begin{bmatrix} ke^n & T_n^{-1}(\mathfrak{v}) \\ T_n^{-1}(\mathfrak{v})^* & ke^n \end{bmatrix} \in M_{2n}(V)^+$ for all $n \geq o(\mathfrak{v})$. Also note that for $n > o(\mathfrak{v})$, we have that

$$\begin{bmatrix} ke^{o(\mathfrak{v})} & T_{o(\mathfrak{v})}^{-1}(\mathfrak{v}) \\ T_{o(\mathfrak{v})}^{-1}(\mathfrak{v})^* & ke^{o(\mathfrak{v})} \end{bmatrix} \in M_{2o(\mathfrak{v})}(V)^+$$

if and only if

$$\begin{bmatrix} ke^n & T_n^{-1}(\mathfrak{v}) \\ T_n^{-1}(\mathfrak{v})^* & ke^n \end{bmatrix} \in M_{2n}(V)^+.$$

Thus $\|\mathfrak{v}\| = \|T_{o(\mathfrak{v})}^{-1}(\mathfrak{v})\|_{o(\mathfrak{v})}$ and $(k\mathfrak{e}^{o(\mathfrak{v})}, k\mathfrak{e}^{o(\mathfrak{v})})_{o(\mathfrak{v})}^+ + sa_{o(\mathfrak{v})}(\mathfrak{v}) \in \mathfrak{V}^+$ if and only if $(k\mathfrak{e}^n, k\mathfrak{e}^n)_n^+ + sa_n(\mathfrak{v}) \in \mathfrak{V}^+$. Since $\|v \oplus 0_m\|_{n+m} = \|v\|_n$ for all $v \in M_n(V)$ and $n, m \in \mathbb{N}$, we get that $\|\cdot\|$ determines a norm on $\mathfrak{V}$ and $\|\mathfrak{v}\| = \inf\{k > 0 : (k\mathfrak{e}^n, k\mathfrak{e}^n)_n^+ + sa_n(\mathfrak{v}) \in \mathfrak{V}^+$ for some $n \in \mathbb{N}$ with $\mathfrak{I}_n\mathfrak{v}\mathfrak{I}_n = \mathfrak{v}\}$.

Finally, let $\mathfrak{a}, \mathfrak{b} \in \mathfrak{F}$ and $\mathfrak{v} \in \mathfrak{V}$. Put $n = \max\{o(\mathfrak{a}), o(\mathfrak{b}), o(\mathfrak{v})\}$. Then

$$\begin{aligned} \|\mathfrak{a}\mathfrak{v}\mathfrak{b}\| &= \|T_{o(\mathfrak{a}\mathfrak{v}\mathfrak{b})}^{-1}(\mathfrak{a}\mathfrak{v}\mathfrak{b})\|_{o(\mathfrak{a}\mathfrak{v}\mathfrak{b})} = \|T_n^{-1}(\mathfrak{a}\mathfrak{v}\mathfrak{b})\|_n \\ &= \|\sigma_n^{-1}(\mathfrak{a})T_n^{-1}(\mathfrak{v})\sigma_n^{-1}(\mathfrak{b})\|_n \\ &\leq \|\sigma_n^{-1}(\mathfrak{a})\|\|T_n^{-1}(\mathfrak{v})\|_n\|\sigma_n^{-1}(\mathfrak{b})\| \\ &= \|\mathfrak{a}\|\|\mathfrak{v}\|\|\mathfrak{b}\|. \end{aligned}$$

so that $\|\cdot\|$ is a $\mathfrak{F}$-bimodule norm on $\mathfrak{V}$. Hence $(\mathfrak{V}, \mathfrak{V}^+, \mathfrak{e})$ with $\|\cdot\|$ is a $\mathfrak{F}$-bimodule normed space. ☐

Definition 6.1.17. *Let $(\mathfrak{V}, \mathfrak{V}^+)$ be a non-degenerate ordered $\mathfrak{F}$-bimodule and let $\|\cdot\|$ be a $\mathfrak{F}$-bimodule norm on $\mathfrak{V}$. Also let $\mathfrak{u}, \mathfrak{v} \in \mathfrak{V}^+$. We write $\mathfrak{u} \perp_\infty \mathfrak{v}$, if*

$$\|k_1\mathfrak{u} + k_2\mathfrak{v}\| = \max\{\|k_1\mathfrak{u}\|, \|k_2\mathfrak{v}\|\}$$

for all $k_1, k_2 \in \mathbb{R}$ and $\mathfrak{u} \perp_\infty^a \mathfrak{v}$, if $\mathfrak{u}_1 \perp_\infty \mathfrak{v}_1$ for all $\mathfrak{u}_1, \mathfrak{v}_1 \in \mathfrak{V}^+$ with $\mathfrak{u}_1 \leq \mathfrak{u}$ and $\mathfrak{v}_1 \leq \mathfrak{v}$.

Proposition 6.1.18. *Let $(\mathfrak{V}, \mathfrak{V}^+, \mathfrak{e})$ be a local $\mathfrak{F}$-order unit bimodule and let $\mathfrak{u}, \mathfrak{v} \in \mathfrak{V}^+$ such that $\|\mathfrak{u}\| = 1 = \|\mathfrak{v}\|$. Then $\mathfrak{u} \perp_\infty \mathfrak{v}$ if and only if $\|\mathfrak{u} + \mathfrak{v}\| = 1$.*

Proof. Let $(V, \{M_n(V)^+\})$ be the matrix ordered space corresponding to $(\mathfrak{V}, \mathfrak{V}^+)$ as in Theorem 2.2.8. Put $e = T^{-1}(\mathfrak{e})$. Then $(V, \{M_n(V)^+\}, e)$ is a matrix order unit space. By proof of Proposition 6.1.16, we get that $\|\mathfrak{v}\| = \|T_n^{-1}(\mathfrak{v})\|_n$ for any $\mathfrak{v} \in \mathfrak{V}$ and $n \geq o(\mathfrak{v})$. Also note that $\mathfrak{v} \in \mathfrak{V}^+$ if and only if $T_n^{-1}(\mathfrak{v}) \in M_n(V)^+$ for all $n \geq o(\mathfrak{v})$.

Next let $\mathfrak{u}, \mathfrak{v} \in \mathfrak{V}^+$ with $\|\mathfrak{u}\| = 1 = \|\mathfrak{v}\|$. Without loss of generality, we may assume that $o(\mathfrak{u}) \geq o(\mathfrak{v})$. Then $\mathfrak{u} \perp_\infty \mathfrak{v}$ in $\mathfrak{V}^+$ if and only if $T_{o(\mathfrak{u})}^{-1}(\mathfrak{u}) \perp_\infty T_{o(\mathfrak{u})}^{-1}(\mathfrak{v})$ in $M_{2o(\mathfrak{u})}(V)$. By Theorem 2.1.4, we have $T_{o(\mathfrak{u})}^{-1}(\mathfrak{u}) \perp_\infty T_{o(\mathfrak{u})}^{-1}(\mathfrak{v})$ if and only if $\|T_{o(\mathfrak{u})}^{-1}(\mathfrak{u}) + T_{o(\mathfrak{u})}^{-1}(\mathfrak{v})\|_{o(\mathfrak{u})} = 1$. Thus $\mathfrak{u} \perp_\infty \mathfrak{v}$ if and only if $\|\mathfrak{u} + \mathfrak{v}\| = 1$. ☐

Definition 6.1.19. *Let $(\mathfrak{V}, \mathfrak{V}^+, \mathfrak{e})$ is a Local $\mathfrak{F}$-order unit bimodule such that*

(a) $(\mathfrak{V}, \mathfrak{V}^+, |\cdot|)$ is a non-degenerate absolutely ordered $\mathfrak{F}$- bimodule. and

(b) $\perp = \perp_\infty^a$ on $\mathfrak{V}^+$.

Then $(\mathfrak{V}, \mathfrak{V}^+, \mathfrak{e}, |\cdot|)$ is said to be non-degenerate absolute order unital $\mathfrak{F}$-bimodule.

Next two results follow from Theorems 6.1.10 and 6.1.11 respectively.

Corollary 6.1.20. *Let $(V, \{M_n(V)^+\}, e, \{|\cdot|_n\})$ be an absolute matrix order unit space. Let $(\mathfrak{V}, \mathfrak{V}^+, |\cdot|)$ be non-degenerate absolutely ordered $\mathfrak{F}$-bimodule corresponding to $(V, \{M_n(V)^+\}, \{|\cdot|_n\})$ as in Theorem 6.1.10. Then $(\mathfrak{V}, \mathfrak{V}^+, T(e), |\cdot|)$ is a non-degenerate absolute order unital $\mathfrak{F}$-bimodule.*

Corollary 6.1.21. *Let $(\mathfrak{V}, \mathfrak{V}^+, \mathfrak{e}, |\cdot|)$ be a non-degenerate absolute order unital $\mathfrak{F}$-bimodule. Let $(V, \{M_n(V)^+\}, \{|\cdot|_n\})$ be absolute matrix ordered space corresponding to $(\mathfrak{V}, \mathfrak{V}^+, |\cdot|)$ as in Theorem 6.1.11. Then $(V, \{M_n(V)^+\}, T^{-1}(\mathfrak{e}), \{|\cdot|_n\})$ is an absolute matrix order unit space.*

6.2 Matrix order unit property

In this section, we define matrix order unit property and absolute matrix order unit property. These are matricial versions of order unit property and absolute order unit property which were introduced in [39].

Proposition 6.2.1. *Let (V, e) be an absolute matrix order unit space and let $p \in \mathcal{OP}_n(V) \setminus \{0\}$ for some $n \in \mathbb{N}$. Then $\|p\|_n = 1$.*

Proof. By Proposition 2.3.12, we get that $p \leq \|p\|_n p$. Since $p \neq 0_n$, we conclude that $\|p\|_n \geq 1$. By Remark 2.1.2, we have that $\|p\|_n \leq 1$ for $0_n \leq p \leq e^n$. Hence $\|p\|_n = 1$. □

Definition 6.2.2. *Let V be a matrix order unit space, then $u \in V^+$ is said to have the matrix order unit property, if u^n has the order unit property in $M_n(V)_{sa}$ for all $n \in \mathbb{N}$. If V is an absolute matrix order unit space, then $u \in V^+$ is said to have the absolute matrix order unit property, if u^n has the absolute order unit property in $M_n(V)_{sa}$ for all $n \in \mathbb{N}$.*

Proposition 6.2.3. *Let (V, e) be an absolute matrix order unit space and let $u \in V^+$.*

(1) u has the matrix order unit property if, and only if, for each $n \in \mathbb{N}$ and $\begin{bmatrix} ku^n & v \\ v^* & ku^n \end{bmatrix} \in M_{2n}(V)^+$ for some $v \in M_n(V)$ and $k \in \mathbb{R}$ with $k > 0$, we have that $\begin{bmatrix} \|v\|_n u^n & v \\ v^* & \|v\|_n u^n \end{bmatrix} \in M_{2n}(V)^+$.

(2) If $u \leq e$, then

(i) u and $e - u$ have the order unit property if and only if u and $e - u$ have the matix order unit property.

(ii) u and $e - u$ have the absolute order unit property if and only if u and $e - u$ have the absolute matix order unit property.

Proof. (1) First assume that u has matrix order unit property. Let $n \in \mathbb{N}$ and $v \in M_n(V)$ be such that $\begin{bmatrix} ku^n & v \\ v^* & ku^n \end{bmatrix} \in M_{2n}(V)^+$ for some $k \in \mathbb{R}$ with $k > 0$. Then we also get that $\begin{bmatrix} ku^n & -v \\ -v^* & ku^n \end{bmatrix} \in M_{2n}(V)^+$. By assumption, we have that u^{2n} has order unit property in $M_{2n}(V)_{sa}$ so that $\pm \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \leq \left\| \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \right\|_{2n} u^{2n} = \|v\|_n u^{2n}$. Thus $\begin{bmatrix} \|v\|_n u^n & v \\ v^* & \|v\|_n u^n \end{bmatrix} \in M_{2n}(V)^+$.

Now assume that converse statement is true. Let $n \in \mathbb{N}$ and $v \in M_n(V)_{sa}$ be such that $\pm v \leq ku^n$ for some $k > 0$. Then

$$\begin{bmatrix} ku^n & v \\ v & ku^n \end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix} I_n \\ I_n \end{bmatrix} (ku^n + v) \begin{bmatrix} I_n & I_n \end{bmatrix}\right) + \frac{1}{2}\left(\begin{bmatrix} I_n \\ -I_n \end{bmatrix} (ku^n - v) \begin{bmatrix} I_n & -I_n \end{bmatrix}\right)$$

so that

$$\begin{bmatrix} ku^n & v \\ v & ku^n \end{bmatrix} \in M_{2n}(V)^+.$$

By assumption, we have that $\begin{bmatrix} \|v\|_n u^n & v \\ v & \|v\|_n u^n \end{bmatrix} \in M_{2n}(V)^+$ so that

$$\|v\|u^n \pm v = \frac{1}{2}\left(\begin{bmatrix} I_n & \pm I_n \end{bmatrix} \begin{bmatrix} \|v\|_n u^n & v \\ v & \|v\|_n u^n \end{bmatrix} \begin{bmatrix} I_n \\ \pm I_n \end{bmatrix}\right) \in V^+.$$

Thus u^n has the order unit property in $M_n(V)_{sa}$ for all $n \in \mathbb{N}$. Hence u has the matrix order unit property.

(2) (i) Assume that $u \leq e$ and let $n \in \mathbb{N}$. By Propositions 2.3.12 and 5.1.2, we get that u and $e - u$ have the order unit property if and only if $u \in \mathcal{OP}(V)$ if and only if $u^n \in \mathcal{OP}(V)$ if and only if u^n and $e^n - u^n$ have the order unit property. Thus u and $e - u$ have the order unit property if and only if u and $e - u$ have the matrix order unit property.

Now, we can prove (ii) in a similar way. □

Theorem 6.2.4. *Let $(V, \{M_n(V)^+\}, e, \{|\cdot|_{m,n}\})$ be an absolute matrix order unit space and let $p \in \mathcal{OP}(V)$. Put*

$$V_p = \left\{ v \in V : \begin{bmatrix} kp & v \\ v^* & kp \end{bmatrix} \in M_2(V)^+ \text{ for some } k > 0 \right\}$$

and

$$M_{m,n}(V)_{p^m,p^n} = \left\{ v \in M_{m,n}(V) : \begin{bmatrix} kp^m & v \\ v^* & kp^n \end{bmatrix} \in M_{m+n}(V)^+ \text{ for some } k > 0 \right\}$$

for all $m, n \in \mathbb{N}$. *Let us write* $M_{n,n}(V)_{p^n,p^n} = M_n(V)_{p^n}$ *for every* $n \in \mathbb{N}$. *Then*

(1) $M_{m,n}(V_p) = M_{m,n}(V)_{p^m,p^n}$.

(2) $M_n(V_p)^+ = M_n(V_p) \bigcap M_n(V)^+$ *is a proper cone in* $M_n(V_p)$.

(3) p *is an order unit for* V_p.

(4) *Let* $n \in \mathbb{N}$ *and* $v \in M_n(V_p)$. *Put*

$$\|v\|_n^p = inf \left\{ k > 0 : \begin{bmatrix} kp^n & v \\ v^* & kp^n \end{bmatrix} \in M_{2n}(V)^+ \right\}.$$

Then $\|\cdot\|_n^p$ *is a norm on* $M_n(V_p)$.

(5) $M_n(V_p)^+$ *is* $\|\cdot\|_n^p$*-closed in* $M_n(V_p)$.

(6) *If* $m, n \in \mathbb{N}$ *and* $v \in M_{m,n}(V_p)$, *then we have* $|v|_{m,n} \in M_n(V_p)^+$.

Thus $|\cdot|_{m,n}^p := |\cdot|_{m,n}$ *determines a matrix absolute value* $\{|\cdot|_{m,n}^p\}$ *in* V_p *so that* $(V_p, \{M_n(V_p)^+\}, p, \{|\cdot|_{m,n}^p\})$ *is an absolute matrix order unit space.*

By matrix absolute value $\{|\cdot|_{m,n}^p\}$ in V_p, we mean that $|\cdot|_{m,n}^p : M_{m,n}(V_p) \to M_n(V_p)^+$ is an absolute value for all $m, n \in \mathbb{N}$.

Proof. The statement (2) is routine to verify. Next, we prove the other statements.

(1) Let $m, n \in \mathbb{N}$ and $v = [v_{i,j}] \in M_{m,n}(V_p)$. For each $v_{i,j} \in V_p$, there exists $k_{i,j} > 0$ such that $\begin{bmatrix} k_{i,j}p & v_{i,j} \\ v_{i,j}^* & k_{i,j}p \end{bmatrix} \in M_2(V)^+$. For $k_0 = \max\{k_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\}$, we have

$$\begin{bmatrix} k_0p & v_{i,j} \\ v_{i,j}^* & k_0p \end{bmatrix} \in M_2(V)^+$$

for all i, j.

For each $v_{i,j}$, let $c_{ij} \in M_{m+n,2}$ be as constructed in the proof of Theorem 5.4.5. Thus we get that

$$\begin{bmatrix} nk_0p^m & v \\ v^* & mk_0p^n \end{bmatrix} = \sum_{i,j} c_{ij} \begin{bmatrix} k_0p & v_{i,j} \\ v_{i,j}^* & k_0p \end{bmatrix} c_{ij}^* \in M_{m+n}(V)^+.$$

Put $k = \max\{nk_0, mk_0\}$. Then

$$\begin{bmatrix} kp^m & v \\ v^* & kp^n \end{bmatrix} \in M_{m+n}(V)^+$$

so that $v \in M_{m,n}(V)_{p^m,p^n}$.

Conversely, let $v \in M_{m,n}(V)_{p^m,p^n}$. Thus $\begin{bmatrix} kp^m & v \\ v^* & kp^n \end{bmatrix} \in M_{m+n}(V)^+$ for some $k > 0$. Then

$$\begin{bmatrix} kp & v_{i,j} \\ v_{i,j}^* & kp \end{bmatrix} = c_{ij}^* \begin{bmatrix} kp^m & v \\ v^* & kp^n \end{bmatrix} c_{ij} \in M_2(V)^+$$

so that

$$v_{i,j} \in V_p$$

for each i, j. Hence

$$M_{m,n}(V_p) = M_{m,n}(V)_{p^m,p^n}.$$

(3) Let $v \in M_n(V_p)$ for some $n \in \mathbb{N}$. Then by (1), we get that $\begin{bmatrix} kp^n & v \\ v^* & kp^n \end{bmatrix} \in M_{2n}(V)^+$. Thus p is an order unit for V_p.

(4) Let $v \in M_n(V)$ be such that $\|v\|_n^p = 0$. Then

$$k \begin{bmatrix} p^n & 0 \\ 0 & p^n \end{bmatrix} \pm \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \in M_{2n}(V)^+$$

for all $k > 0$. Since $M_{2n}(V)^+$ is Archimedean, we get that $\pm \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \in M_{2n}(V)^+$. By properness of $M_{2n}(V)^+$, we have that $\begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} = 0$ so that $v = 0$. Now let $v = 0$, then by definition $\|v\|_n^p = 0$. Thus $\|v\|_n^p = 0$ if and only if $v = 0$.

Next let $\alpha \in \mathbb{C} \setminus \{0\}$. Then $\alpha = |\alpha| e^{i\theta}$ for some $\theta \in \mathbb{R}$. Now observe that

$$\begin{bmatrix} k|\alpha|^{-1}p^n & v \\ v^* & k|\alpha|^{-1}p^n \end{bmatrix} = \begin{bmatrix} e^{-i\theta}|\alpha|^{-\frac{1}{2}} I_n & 0 \\ 0 & |\alpha|^{-\frac{1}{2}} I_n \end{bmatrix} \begin{bmatrix} kp^n & \alpha v \\ \bar{\alpha} v^* & kp^n \end{bmatrix} \begin{bmatrix} e^{i\theta}|\alpha|^{-\frac{1}{2}} I_n & 0 \\ 0 & |\alpha|^{-\frac{1}{2}} I_n \end{bmatrix}$$

for all $k \in \mathbb{R}$. Thus we conclude that $\|v\|_n^p = \frac{\|\alpha v\|_n^p}{|\alpha|}$ *i.e.* $\|\alpha v\|_n^p = |\alpha| \|v\|_n^p$.

Finally let $v_1, v_2 \in M_n(V)$ and also let $\epsilon > 0$. Then

$$\begin{bmatrix} (\|v_i\|_n^p + \frac{\epsilon}{2})p^n & v_i \\ v_i^* & (\|v_i\|_n^p + \frac{\epsilon}{2})p^n \end{bmatrix} \in M_{2n}(V)^+ \text{ for } i = 1, 2$$

so that $\begin{bmatrix} (\|v_1\|_n^p + \|v_2\|_n^p + \epsilon)p^n & v_1 + v_2 \\ v_1^* + v_2^* & (\|v_1\|_n^p + \|v_2\|_n^p + \epsilon)p^n \end{bmatrix} \in M_{2n}(V)^+$. By definition, we have that $\|v_1 + v_2\|_n^p \leq \|v_1\|_n^p + \|v_2\|_n^p + \epsilon$. Since $\epsilon > 0$ is arbitrary, we get that $\|v_1 + v_2\|_n^p \leq \|v_1\|_n^p + \|v_2\|_n^p$.

Hence $\| \cdot \|_n^p$ is a norm on $M_n(V)$.

(5) Let $\{v_m\}$ be a sequence in $M_n(V_p)^+$ converging to $v \in M_n(V_p)$ in $\| \cdot \|_n^p$. Since p has matrix order unit property in V, we have $\| \cdot \|_n^p = \| \cdot \|_n$ for all $n \in \mathbb{N}$. Then $\|v_m - v\|_n^p = \|v_m - v\|_n$. As $M_n(V)^+$ is $\| \cdot \|_n$-closed, we have $v \in M_n(V)^+$. Thus $M_n(V_p)^+$ is $\| \cdot \|_n^p$-closed.

(6) Let $v \in M_{m,n}(V_p)$. Then $\begin{bmatrix} kp^m & \pm v \\ \pm v^* & kp^n \end{bmatrix}$ for some $k > 0$. Since $\begin{bmatrix} p^m & 0 \\ 0 & p^n \end{bmatrix}$ is a projection in $M_{m+n}(V)$, by Proposition 2.3.12, we get that $\begin{bmatrix} |v^*|_{n,m} & 0 \\ 0 & |v|_{m,n} \end{bmatrix} = \left\| \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \right\|_{m+n} \leq k_0 \begin{bmatrix} p^m & 0 \\ 0 & p^n \end{bmatrix}$ for $k_0 = \left\| \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \right\|_{m+n}$. Thus $|v|_{m,n} \leq k_0 p^n$ so that $|v|_{m,n} \in M_n(V_p)^+$. Hence $| \cdot |_{m,n} : M_{m,n}(V_p) \to M_n(V_p)^+$ is well defined for all $m, n \in \mathbb{N}$.

Finally, we show that $\perp = \perp_\infty^a$ on $M_n(V_p)^+$ for all $n \in \mathbb{N}$. Let $u, v \in M_n(V_p)^+$ be such that $u \perp v$. By (2), we get that $u \perp v$ in $M_n(V)^+$. Let $0 \leq u_1 \leq u$ and $0 \leq v_1 \leq v$ in $M_n(V_p)^+$. As $\perp = \perp_\infty^a$ on $M_n(V)^+$, we have

$$\begin{aligned} \|k_1 u_1 + k_2 v_1\|_n^p &= \|k_1 u_1 + k_2 v_1\|_n \\ &= \max\{\|k_1 u_1\|_n, \|k_2 v_1\|_n\} \\ &= \max\{\|k_1 u_1\|_n^p, \|k_2 v_1\|_n^p\}. \end{aligned}$$

Thus $u \perp_\infty^a v$ in $M_n(V_p)^+$. Now tracing back the proof, we can show that $\perp = \perp_\infty^a$ on $M_n(V_p)^+$. Hence $(V_p, \{M_n(V_p)^+\}, p, \{|\cdot|_{m,n}^p\})$ is an absolute matrix order unit space. $\square$

Definition 6.2.5. *Let $(\mathfrak{V}, \mathfrak{V}^+, \mathfrak{e}, |\cdot|)$ be a non-degenerate absolute order unital $\mathfrak{F}$-bimodule. An element $\mathfrak{v} \in \mathfrak{V}^+$ is said to be an order projection, if $|\mathfrak{e}^n - 2\mathfrak{v}| = \mathfrak{e}^n$ whenever $\mathfrak{I}_n \mathfrak{v} \mathfrak{I}_n = \mathfrak{v}$ for some $n \in \mathbb{N}$. We denote the collection of all order projections in $\mathfrak{V}^+$ by $\mathcal{OP}(\mathfrak{V})$. Further, let $n \in \mathbb{N}$ and $\mathfrak{p} \in \mathcal{OP}(\mathfrak{V})$, then $\mathfrak{p}$ is said to be an n-projection, if $\mathfrak{I}_n \mathfrak{p} \mathfrak{I}_n = \mathfrak{p}$ and $\mathfrak{I}_m \mathfrak{p} \mathfrak{I}_m \neq \mathfrak{p}$ for all $m \in \mathbb{N}$ with $m < n$.*

Corollary 6.2.6. *Let $(V, \{M_n(V)^+\}, e, \{|\cdot|_n\})$ be an absolute matrix order unit space and let $p \in \mathcal{OP}(V)$. Assume that $(\mathfrak{V}, \mathfrak{V}^+, T(e), |\cdot|)$ is the matricial inductive limit of $(V, \{M_n(V)^+\}, e, \{|\cdot|_n\})$. Put $\mathfrak{p} = T(p)$,*

$$\mathfrak{V}_{\mathfrak{p}} = \{\mathfrak{v} \in \mathfrak{V} : (k\mathfrak{p}^n, k\mathfrak{p}^n)_n^+ + sa_n(\mathfrak{v}) \in \mathfrak{V}^+ \ for \ some \ k > 0 \ and \ n \geq o(\mathfrak{v})\}$$

and let $\mathfrak{V}_{\mathfrak{p}}^+ = \mathfrak{V}_{\mathfrak{p}} \cap \mathfrak{V}^+$. Then $(\mathfrak{V}_{\mathfrak{p}}, \mathfrak{V}_{\mathfrak{p}}^+, \mathfrak{p}, |\cdot|)$ is a non-degenerate absolute order unital $\mathfrak{F}$-bimodule and is the matricial inductive limit of $(V_p, \{M_n(V_p)^+\}, p, \{|\cdot|_n\})$.

Proof. Let $v \in M_n(V_p)$. By Theorem 6.2.4(1), we get that $\begin{bmatrix} kp^n & v \\ v^* & kp^n \end{bmatrix} \in M_{2n}(V)^+$ for some $k > 0$. Thus $(k\mathfrak{p}^n, k\mathfrak{p}^n)_n^+ + sa_n(T_n(v)) \in \mathfrak{V}^+$ so that $v \in \mathfrak{V}_{\mathfrak{p}}$.

Conversely let $\mathfrak{v} \in \mathfrak{V}_{\mathfrak{p}}$. Then $(k\mathfrak{p}^n, k\mathfrak{p}^n)_n^+ + sa_n(\mathfrak{v}) \in \mathfrak{V}^+$ for some $k > 0$. Therefore, we get $\begin{bmatrix} kp^n & T_n^{-1}(\mathfrak{v}) \\ T_n^{-1}(\mathfrak{v})^* & kp^n \end{bmatrix} \in M_{2n}(V)^+$ so that $T_n^{-1}(\mathfrak{v}) \in M_n(V)_{p^n}$. Hence $(\mathfrak{V}_{\mathfrak{p}}, \mathfrak{V}_{\mathfrak{p}}^+, \mathfrak{p}, |\cdot|)$ corresponds to $(V_p, \{M_n(V_p)^+\}, p, \{|\cdot|_n\}$. $\square$

The converse of Corollary 6.2.6 can be proved in a similar fashion.

Corollary 6.2.7. *Let $(\mathfrak{V}, \mathfrak{V}^+, \mathfrak{e}, |\cdot|)$ be a non-degenerate absolute order unital $\mathfrak{F}$-bimodule and let $\mathfrak{p} \in \mathcal{OP}(\mathfrak{V})$ with $\mathfrak{I}_1 \mathfrak{p} \mathfrak{I}_1 = \mathfrak{p}$. Put $p = T^{-1}(\mathfrak{p})$. If $(\mathfrak{V}, \mathfrak{V}^+, \mathfrak{e}, |\cdot|)$*

is the matricial inductive limit of $(V, \{M_n(V)^+\}, e, \{|\cdot|_n\})$*, then* $(\mathfrak{V}_\mathfrak{p}, \mathfrak{V}_\mathfrak{p}^+, \mathfrak{p}, |\cdot|)$ *is the matricial inductive limit of* $(V_p, \{M_n(V_p)^+\}, p, \{|\cdot|_n\})$.

6.3 K_0 of an absolute matrix order unit space

In this section, we show the existence of K_0 of an absolute matrix order unit space with its explicit form. We identify the direct limit of an absolute matrix order unit space V with $M_\infty(V) = \bigcup_{n=1}^{\infty} M_n(V)$. Under this identification, the corresponding set of projections may be identified with $\mathcal{OP}_\infty(V) = \bigcup_{n=1}^{\infty} \mathcal{OP}_n(V)$. We define the following relation on $\mathcal{OP}_\infty(V)$:

Definition 6.3.1. *Let V be an absolute matrix order unit space. For $p, q \in \mathcal{OP}_\infty(V)$, we say that $p \approx q$, if there exists $r \in \mathcal{OP}_\infty(V)$ such that $p \oplus r \sim q \oplus r$.*

Proposition 6.3.2. *Let V be an absolute matrix order unit space and let $p, q \in \mathcal{OP}_\infty(V)$.*

(1) $p \sim q$ implies $p \approx q$.

(2) If V satisfies (T), then

(i) $\approx$ is an equivalence relation.

(ii) $p \approx q$ if and only if $p \oplus e^m \sim q \oplus e^m$ for some $m \in \mathbb{N}$.

Proof. (1) Let $p \sim q$ and also let $r \in \mathcal{OP}_\infty(V)$. Since $r \sim r$, by Proposition 5.2.5(3), we get that $p \oplus r \sim q \oplus r$. Thus $p \approx q$.

(2) Assume that V satisfies (T).

(i) By Proposition 5.2.4 and by (1), it follows that $\approx$ is reflexive and symmetric. Next, we check transitivity of $\approx$. Let $p \approx q$ and also let

$r \in \mathcal{OP}_\infty(V)$ such that $q \approx r$. Then $p \oplus s \sim q \oplus s$ and $q \oplus t \sim r \oplus t$ for some $s, t \in \mathcal{OP}_\infty(V)$. Thus by Proposition 5.2.5(3) and (4), we get that $(p \oplus s) \oplus t \sim (q \oplus s) \oplus t$, $q \oplus (s \oplus t) \sim q \oplus (t \oplus s)$, $(q \oplus t) \oplus s \sim (r \oplus t) \oplus s$ and $r \oplus (t \oplus s) \sim r \oplus (s \oplus t)$. Since $\oplus$ is associative, by Proposition 5.2.4, we have that $p \oplus (s \oplus t) \sim r \oplus (s \oplus t)$ so that $p \approx r$.

(ii) Let $p \approx q$. Then there exists $r \in \mathcal{OP}_m(V), m \in \mathbb{N}$ such that $p \oplus r \sim q \oplus r$. Since $r \perp (e^m - r)$, by Proposition 5.2.5(5), we get that $e^m \sim r \oplus (e^m - r)$. Then again applying Proposition 5.2.5(3), we conclude that $p \oplus e^m \sim p \oplus (r \oplus (e^m - r))$, $(p \oplus r) \oplus (e^m - r) \sim (q \oplus r) \oplus (e^m - r)$, $q \oplus (r \oplus (e^m - r)) \sim q \oplus e^m$. Now $\oplus$ is associative, thus by Proposition 5.2.4, we have that $p \oplus e^m \sim q \oplus e^m$.

□

Proposition 6.3.3. *Let V be an absolute matrix order unit space and let $p, q, r, p', q' \in \mathcal{OP}_\infty(V)$.*

1. *If $m, n \in \mathbb{N}$ and let $p \in \mathcal{OP}_m(V)$, then $p \approx p \oplus 0_n$ and $p \approx 0_n \oplus p$;*
2. *If V satisfies (T), then*
   - *(i) If $p \approx q$ and $p' \approx q'$ with $p \perp p'$ and $q \perp q'$, then $p + p' \approx q + q'$;*
   - *(ii) If $p \approx p'$ and $q \approx q'$, then $p \oplus q \approx p' \oplus q'$;*
3. *$p \oplus q \approx q \oplus p$;*
4. *If $p \perp q$, then $p + q \approx p \oplus q$;*
5. *$(p \oplus q) \oplus r = p \oplus (q \oplus r)$.*

Proof. (1), (3), (4) and (5) are immediate by Proposition 5.2.5(1), (4), (5) and (6) respectively with Proposition 6.3.2(1).

(2) Assume that V satisfies (T).

(i) Let $p \approx q$, $p' \approx q'$, $p \perp p'$ and $q \perp q'$. Then $p \oplus r \sim q \oplus r$ and $p' \oplus s \sim q' \oplus s$ for some $r, s \in \mathcal{OP}_\infty(V)$. Without loss of generality, we can assume that $r \perp s$ so that $p \oplus r \perp p' \oplus s$ and $q \oplus r \perp q' \oplus s$. By Proposition 5.2.5(2), we get that $(p \oplus r) + (p' \oplus s) \sim (q \oplus r) + (q' \oplus s)$. Thus $(p + p') \oplus (r + s) \sim (q + q') \oplus (r + s)$ so that $p + p' \approx q + q'$.

(ii) Assume that $p \approx p'$ and $q \approx q'$. Then there exist $r, s \in \mathcal{OP}_\infty(V)$ such that $p \oplus r \sim p' \oplus r$ and $q \oplus s \sim q' \oplus s$. Thus by Proposition 5.2.5(3),(4) and (6) with Proposition 5.2.4, we conclude that $(p \oplus q) \oplus (r \oplus s) \sim (p' \oplus q') \oplus (r \oplus s)$. Hence $p \oplus q \approx p' \oplus q'$.

□

Proposition 6.3.4. *Let V be an absolute matrix order unit space satisfying (T). For each $p, q \in \mathcal{OP}_\infty(V)$, let $[p] = \{r \in \mathcal{OP}_\infty(V) : r \approx p\}$ and put $[p] + [q] = [p \oplus q]$. Then*

(1) $[p] + [0] = [p]$ for all $p \in \mathcal{OP}_\infty(V)$;

(2) $[p] + [q] = [q] + [p]$ for all $p,\ q \in \mathcal{OP}_\infty(V)$;

(3) $[p] + [r] = [q] + [r]$ for $p,\ q,\ r \in \mathcal{OP}_\infty(V)$, then $[p] = [q]$.

Thus $(\mathcal{OP}_\infty(V)/\approx, +)$ is a monoid in which cancellation law holds.

Proof. By Proposition 6.3.3(2)(ii), it follows that $+$ is well-defined in $\mathcal{OP}_\infty(V)/\approx$. Note that (1) and (2) immediately follow from 6.3.3(1) and (3) respectively.

(3) Let $p, q, r \in \mathcal{OP}_\infty(V)$ such that $[p] + [r] = [q] + [r]$. Then $p \oplus r \approx q \oplus r$ so that $p \oplus (r \oplus s) \sim q \oplus (r \oplus s)$ for some $s \in \mathcal{OP}_\infty(V)$. Thus $p \approx q$ so that $[p] = [q]$.

$\square$

Theorem 6.3.5. *Let (V, e) be an absolute matrix order unit space satisfying (T). Consider $\mathcal{OP}_\infty(V) \times \mathcal{OP}_\infty(V)$ and for all $p_1, p_2, q_1, q_2 \in \mathcal{OP}_\infty(V)$, define $(p_1, q_1) \equiv (p_2, q_2)$ if and only if $p_1 \oplus q_2 \approx p_2 \oplus q_1$. Then $\equiv$ is an equivalence relation on $\mathcal{OP}_\infty(V) \times \mathcal{OP}_\infty(V)$. Further, put $K_0(V) = \{[(p,q)] : p, q \in \mathcal{OP}_\infty\}$ where $[(p, q)]$ stands for the equivalence class of (p, q) in $(\mathcal{OP}_\infty(V) \times \mathcal{OP}_\infty(V), \equiv)$. For all $p_1, p_2, q_1, q_2 \in \mathcal{OP}_\infty(V)$, we write $[(p_1, q_1)] + [(p_2, q_2)] = [(p_1 \oplus p_2, q_1 \oplus q_2)]$, then $(K_0(V), +)$ is an abelian group.*

Proof. The relation $\equiv$ on $\mathcal{OP}_\infty(V) \times \mathcal{OP}_\infty(V)$ is reflexive and symmetric follows from Proposition 6.3.2(2)(i). Let $[(p_1, q_1)] \equiv [(p_2, q_2)]$ and $[(p_2, q_2)] \equiv [(p_3, q_3)]$ for some $p_1, p_2, p_3, q_1, q_2, q_3 \in \mathcal{OP}_\infty(V)$. Then $p_1 \oplus q_2 \approx p_2 \oplus q_1$ and $p_2 \oplus q_3 \approx p_3 \oplus q_2$. By Proposition 6.3.3(2)(ii),(3) and (5), we get that $(p_1 \oplus q_3) \oplus (p_2 \oplus q_2) \approx (p_3 \oplus q_1) \oplus (p_2 \oplus q_2)$ so that $[p_1 \oplus q_3] + [p_2 \oplus q_2] = [p_3 \oplus q_1] + [p_2 \oplus q_2]$. By Proposition 6.3.4(3), we conclude that $[p_1 \oplus q_3] = [p_3 \oplus q_1]$ so that $p_1 \oplus q_3 \approx p_3 \oplus q_1$. Thus $[(p_1, q_1)] \equiv [(p_3, q_3)]$ so that $\equiv$ satisfies transitivity. Hence $\equiv$ is an equivalence relation on $\mathcal{OP}_\infty(V) \times \mathcal{OP}_\infty(V)$.

Next, we show that $+$ is well-defined on $K_0(V)$. Let $(p_1, q_1) \equiv (p_1', q_1')$ and $(p_2, q_2) \equiv (p_2', q_2')$ in $\mathcal{OP}_\infty(V) \times \mathcal{OP}_\infty(V)$. Then $p_1 \oplus q_1' \approx p_1' \oplus q_1$ and $p_2 \oplus q_2' \approx p_2' \oplus q_2$. By Proposition 6.3.3(2)(ii),(3) and (5), we get that $(p_1 \oplus p_2) \oplus (q_1' \oplus q_2') \approx (p_1' \oplus p_2') \oplus (q_1 \oplus q_2)$. Thus $[(p_1, q_1)] + [(p_2, q_2)] = [(p_1', q_1')] + [(p_2', q_2')]$ so that $+$ is well-defined. Next we verify that $K_0(V)$ is an abelian group.

(1) **Commutativity:-** Let $p_1, q_1, p_2, q_2 \in \mathcal{OP}_\infty(V)$. By Proposition 6.3.3(2)(ii) and (3), we have that $(p_1 \oplus p_2) \oplus (q_2 \oplus q_1) \approx (p_2 \oplus p_1) \oplus (q_1 \oplus q_2)$. Thus $[(p_1, q_1)] + [(p_2, q_2)] = [(p_2, q_2)] + [(p_1, q_1)]$.

(2) **Existence of Identity:-** Let $p, q \in \mathcal{OP}_\infty(V)$. We claim that $[(0, 0)]$ is an

identity element in $K_0(V)$. By Proposition 6.3.3(1) and (2)(ii), it follows that $(p \oplus 0) \oplus q \approx p \oplus (q \oplus 0)$. Hence $[(p,q)] + [(0,0)] = [(p,q)]$.

(3) **Associativity:-** It is immediate by 6.3.3(5).

(4) **Existence of Inverse:-** Let $p, q \in \mathcal{OP}_\infty(V)$. We claim that $[(q,p)]$ is inverse element of $[(p,q)]$ in $K_0(V)$. Since $(p \oplus q) \oplus 0 \approx 0 \oplus (p \oplus q) \approx 0 \oplus (q \oplus p)$, by Proposition 6.3.2(2)(i), we get that $(p \oplus q) \oplus 0 \approx 0 \oplus (q \oplus p)$. Then $(p \oplus q, q \oplus p) \equiv (0,0)$ so that $[(p \oplus q, q \oplus p)] \equiv [(0,0)]$. Thus $[(p,q)] + [(q,p)] = [(0,0)]$.

$\square$

6.4 Functoriality of K_0

Let V be an absolute matrix order unit space satisfying (T). Then $p \longmapsto [(p,0)]$ defines a map $\chi_V : \mathcal{OP}_\infty(V) \to K_0(V)$. If V and W are complex vector spaces and if $\phi : V \to W$ be a linear map, we denote the corresponding $\mathcal{F}$-bimodule map from $M_\infty(V)$ to $M_\infty(W)$ again by ϕ. In this sense, $\phi_{|_{M_n(V)}} = \phi_n$ for all $n \in \mathbb{N}$. In the next result, we describe the functorial nature of K_0.

Theorem 6.4.1. *Let V and W be absolute matrix order unit spaces satisfying (T) and let $\phi : V \to W$ be a completely $|\cdot|$-preserving map such that $\phi(e_V) \in \mathcal{OP}(W)$. Then there exists a unique group homomorphism $K_0(\phi) : K_0(V) \to K_0(W)$ such that the following diagram commutes:*

$$\begin{array}{ccc}
\mathcal{OP}_\infty(V) & \xrightarrow{\phi} & \mathcal{OP}_\infty(W) \\
{\scriptstyle\chi_V}\big\downarrow & & \big\downarrow{\scriptstyle\chi_W} \\
K_0(V) & \xrightarrow[K_0(\phi)]{} & K_0(W)
\end{array}$$

Proof. Since $\phi(e) \in \mathcal{OP}(W)$, by Theorem 4.3.2(1) and by Proposition 5.1.2, we have that $\phi(\mathcal{OP}_\infty(V)) \subset \mathcal{OP}_\infty(W)$. Let $p, q \in \mathcal{OP}_\infty(V)$ such that $p \sim q$. Without loss of generality, we may assume that $p \in \mathcal{OP}_n(V)$ and $q \in \mathcal{OP}_m(V)$ with $n \geq m$. There exists $v \in M_{m,n}(V)$ such that $p = |v|_{m,n}$ and $q = |v^*|_{n,m}$. Put $r = n - m$ and $w = \begin{bmatrix} v \\ 0 \end{bmatrix} \in M_n(V)$. By Proposition 3.1.2(4) and (5), it follows that $p = |w|_n$ and $q \oplus 0_r = |w^*|_n$ so that $p \sim q \oplus 0_r$ in $\mathcal{OP}_n(V)$. As $\phi_n : M_n(V) \to M_n(W)$ is an $|\cdot|$-preserving map, we get that $\phi_n(p) = |\phi_n(w)|_n$ and $\phi_m(q) \oplus 0_r = |\phi_n(w)^*|_n$. Thus $\phi_n(p) \sim \phi_m(q) \oplus 0_r$. By Propositions 5.2.4 and 5.2.5(1), we have that $\phi_n(p) \sim \phi_m(q)$ so that $\phi(p) \sim \phi(q)$.

Next, put $K_0(\phi)([(p,q)]) = [(\phi(p), \phi(q))]$ for each $[(p,q)] \in K_0(V)$. We show that $K_0(\phi)$ is well-defined. Let $[(p_1, q_1)] = [(p_2, q_2)]$ for some $p_1, p_2, q_1, q_2 \in \mathcal{OP}_\infty(V)$. Then there exists $r \in \mathcal{OP}_\infty(V)$ such that $p_1 \oplus q_2 \oplus r \sim p_2 \oplus q_1 \oplus r$. Thus $\phi(p_1) \oplus \phi(q_2) \oplus \phi(r) \sim \phi(p_2) \oplus \phi(q_1) \oplus \phi(r)$ so that $[(\phi(p_1), \phi(q_1))] = [(\phi(p_2), \phi(q_2))]$. Hence $K_0(\phi)$ is well-defined. For all $[(p_1, q_1)], [(p_2, q_2)] \in K_0(V)$, we have that

$$\begin{aligned}
K_0(\phi)([(p_1, q_1)] + [(p_2, q_2)]) &= K_0(\phi)([(p_1 \oplus p_2, q_1 \oplus q_2)]) \\
&= [(\phi(p_1 \oplus p_2), \phi(q_1 \oplus q_2))] \\
&= [(\phi(p_1) \oplus \phi(p_2), \phi(q_1) \oplus \phi(q_2))] \\
&= [(\phi(p_1), \phi(q_1))] + [(\phi(p_2), \phi(q_2))] \\
&= K_0(\phi)([(p_1, q_1)]) + K_0(\phi)([(p_2, q_2)])
\end{aligned}$$

so that $K_0(\phi)$ is a group homomorphism. By construction K_0 satisfies the diagram.

Uniqueness of $K_0(\phi)$:- Let $\mathcal{H} : K_0(V) \to K_0(W)$ be a group homomorphism satisfying the same diagram. Then $K_0(\phi)(\chi_V(p)) = \chi_W(\phi(p)) = \mathcal{H}(\chi_V(p))$ for all $p \in \mathcal{OP}_\infty(V)$. Thus we get that

$$\begin{aligned} K_0(\phi)([(p,q)]) &= K_0(\phi)([(p,0)] - [(q,0)]) \\ &= K_0(\phi)(\chi_V(p) - \chi_V(q)) \\ &= K_0(\phi)(\chi_V(p)) - K_0(\phi)(\chi_V(q)) \\ &= \mathcal{H}(\chi_V(p)) - \mathcal{H}(\chi_V(q)) \\ &= \mathcal{H}(\chi_V(p) - \chi_V(q)) \\ &= \mathcal{H}([(p,q)]) \end{aligned}$$

for all $[(p,q)] \in K_0(V)$. Hence $K_0(\phi) = \mathcal{H}$. □

Let V and W be absolute matrix order unit spaces. We denote the zero map between V and W by $0_{W,V}$. Similarly, the identity map on V is denoted by I_V. Further, if V and W satisfy (T), then we denote the zero group homomorphism between $K_0(V)$ and $K_0(W)$ by $0_{K_0(W),K_0(V)}$ and the identity map on $K_0(V)$ is denoted by $I_{K_0(V)}$.

Corollary 6.4.2. *Let U, V and W be absolute matrix order unit spaces satisfying (T). Then*

(a) $K_0(I_V) = I_{K_0(V)}$;

(b) If $\phi : U \to V$ and $\psi : V \to W$ be unital completely $|\cdot|$-preserving maps, then $K_0(\psi \circ \phi) = K_0(\psi) \circ K_0(\phi)$;

(c) $K_0(0_{W,V}) = 0_{K_0(W),K_0(V)}$.

Proof. (a) Let $p, q \in \mathcal{OP}_\infty(V)$. Then

$$\begin{aligned} K_0(I_V)([(p,q)]) &= [(I_V(p), I_V(q))] \\ &= [(p,q)] \end{aligned}$$

so that by Theorem 6.4.1, $K_0(I_V) = I_{K_0(V)}$.

(b) For any $[(p,q)] \in K_0(U)$, we get that

$$\begin{aligned} K_0(\psi \circ \phi)([(p,q)]) &= [(\psi \circ \phi(p), \psi \circ \phi(q))] \\ &= [(\psi(\phi(p)), \psi(\phi(q)))] \\ &= K_0(\psi)[(\phi(p), \phi(q))] \\ &= K_0(\psi)(K_0(\phi)([(p,q)])) \\ &= K_0(\psi) \circ K_0(\phi)([(p,q)]). \end{aligned}$$

Thus by Theorem 6.4.1, we conclude that $K_0(\psi \circ \phi) = K_0(\psi) \circ K_0(\phi)$.

(c) $K_0(0_{W,V})([(p,q)]) = [(0_{W,V}(p), 0_{W,V}(q))] = [(0,0)]$ for all $[(p,q)] \in K_0(V)$. Thus again using 6.4.1, we get that $K_{0_{W,V}} = 0_{K_0(W),K_0(V)}$.

□

Remark 6.4.3. *It follows from Corollary 6.4.2 that K_0 is a functor from category of absolute matrix order unit spaces with morphisms as unital completely $|\cdot|$-preserving maps to category of abelian groups.*

Corollary 6.4.4. *Let V and W be isomorphic absolute matrix order unit spaces (isomorphic in the sense that there exists a unital, bijective completely $|\cdot|$-preserving map between V and W). Then $K_0(V)$ and $K_0(W)$ are group isomorphic.*

Proof. Let $\phi : V \to W$ be unital completely $|\cdot|$-preserving map. Then ϕ^{-1} is also unital completely $|\cdot|$-preserving map. Since $\phi^{-1} \circ \phi = I_V$ and $\phi \circ \phi^{-1} = I_W$, by Corollary 6.4.2(a) and (b), we get that $K_0(\phi^{-1}) \circ K_0(\phi) = I_{K_0(V)}$ and $K_0(\phi) \circ K_0(\phi^{-1}) = I_{K_0(W)}$. Thus $K_0(\phi) : K_0(V) \to K_0(W)$ is a surjective group isomorphism and $K_0(\phi)^{-1} = K_0(\phi^{-1})$. Hence $K_0(V)$ and $K_0(W)$ are group isomorphic. $\square$

6.4.1 Relation between $K_0(V_p)$ and $K_0(V)$

Let V be an absolute matrix order unit space and let $p \in \mathcal{OP}(V)$. By [39, Remark 6.2(3)], we get that

$$\begin{aligned} \mathcal{OP}_n(V_p) &= \{q \in \mathcal{OP}_n(V) : q \leq p^n\} \\ &= \mathcal{OP}_n(V) \cap M_n(V_p). \end{aligned}$$

for all $n \in \mathbb{N}$.

Proposition 6.4.5. *Let V be an absolute matrix order unit space and let $q, r \in \mathcal{OP}_\infty(V_p)$ for some $p \in \mathcal{OP}(V)$. Then $q \sim r$ in V_p if and only if $q \sim r$ in V.*

Proof. $q \sim r$ in V_p implies $q \sim r$ in V is obvious. Now assume that $q \sim r$ in V. Let $q \in \mathcal{OP}_m(V)$ and $r \in \mathcal{OP}_n(V)$ for some $m, n \in \mathbb{N}$. Then there exists $v \in M_{m,n}(V)$ such that $q = |v^*|_{n,m}$ and $r = |v|_{m,n}$. Since $p^m \geq q$ and $p^n \geq r$, we get that

$$\begin{aligned}\begin{bmatrix} p^m & 0 \\ 0 & p^n \end{bmatrix} &\geq \begin{bmatrix} |v^*|_{n,m} & 0 \\ 0 & |v|_{m,n} \end{bmatrix} \\ &= \left| \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \right|_{m+n} \\ &\geq \pm \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix}.\end{aligned}$$

Thus $\begin{bmatrix} p^m & v \\ v^* & p^n \end{bmatrix} \in M_{m+n}(V)^+$ so that $v \in M_{m,n}(V_p)$. Hence $q \sim r$ in V_p. $\square$

The following Corollary is immediate from Proposition 6.4.5.

Corollary 6.4.6. *Let V be an absolute matrix order unit space and let $p \in \mathcal{OP}(V)$. If V satisfies (T), then V_p also satisfies (T).*

Corollary 6.4.7. *Let V be an absolute matrix order unit space satisfying (T). Then $[(q,r)] \longmapsto [(q,r)]$ is a group homomorphism from $K_0(V_p)$ to $K_0(V)$.*

Proof. Let $i : V_p \hookrightarrow V$ be inclusion map. By Theorem 6.4.1, there exists a unique group homomorphism $K_0(i) : K_0(V_p) \to K_0(V)$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{OP}_\infty(V_p) & \xhookrightarrow{i} & \mathcal{OP}_\infty(V) \\ {\scriptstyle \chi_{V_p}}\big\downarrow & & \big\downarrow{\scriptstyle \chi_V} \\ K_0(V_p) & \xrightarrow[K_0(i)]{} & K_0(V) \end{array}$$

. Thus $K_0(i)(\chi_{V_p}(q)) = \chi_V(q)$ for all $q \in \mathcal{OP}_\infty(V_p)$.

Let $[(q,r)] \in K_0(V_p)$. Then

$$
\begin{aligned}
K_0(i)([(q,r)]) &= K_0(i)(\chi_{V_p}(q) - \chi_{V_p}(r)) \\
&= K_0(i)(\chi_{V_p}(q)) - K_0(i)(\chi_{V_p}(r)) \\
&= \chi_V(q) - \chi_V(r) \\
&= [(q,r)]
\end{aligned}
$$

so that by Theorem 6.4.1, $[(q,r)] \longmapsto [(q,r)]$ is a group homomorphism from $K_0(V_p)$ to $K_0(V)$.

□

6.5 Order structure on K_0

Let $(G,+)$ be a group and let G^+ be a non-empty subset of G such that $G^+ + G^+ \subset G^+$. Then G^+ is said to be a group cone in G. In this case, (G, G^+) is said to be an ordered group.

Let (G, G^+) be an ordered group. Given any g_1, g_2, we write $g_1 \leq g_2$ if and only if $g_2 - g_1 \in G^+$. Then $G^+ = \{g \in G : g \geq 0\}$.

Let (G, G^+) be an ordered group. Then

(1) G^+ is said to be proper, if $G^+ \cap -G^+ = \{0\}$.

(2) G^+ is said to be generating, if $G = G^+ - G^+$.

Let (G, G^+) be an ordered group. Then (G, G^+) is said to be an ordered abelian group, if G is abelian, proper and generating.

Let (G, G^+) be an ordered group. Then $g \in G^+$ is said to be an order unit for G, if given any $h \in G$ there exists $n \in \mathbb{N}$ such that $-ng \leq h \leq ng$.

An ordered abelian group (G, G^+) with an order unit g, is called an ordered abelian group with a distinguished order unit. For details please refer to [22, 61].

In this section, we show that given an absolute matrix order unit space V, $K_0(V)$ is an ordered abelian group with a distinguished order unit under the assumption that all order projections in V are finite.

Theorem 6.5.1. *Let V be an absolute matrix order unit space satisfying (T). Put $K_0(V)^+ = \{[(p, 0)] : p \in \mathcal{OP}_\infty(V)\}$. Then*

(a) $K_0(V)^+$ is a group cone in $K_0(V)$.

(b) If e^n is finite for all $n \in \mathbb{N}$, then $K_0(V)^+$ is proper.

(c) $K_0(V)^+$ is generating.

In other words, if e^n is finite for all $n \in \mathbb{N}$, then $(K_0(V), K_0(V)^+)$ is an ordered abelian group.

Proof. (a) Let $p, q \in \mathcal{OP}_\infty(V)$. Then $[(p, 0)] + [(q, 0)] = [(p \oplus q, 0)]$. Thus $K_0(V)^+ + K_0(V)^+ \subset K_0(V)^+$.

(b) Assume that e^n is finite for all $n \in \mathbb{N}$. Let $g \in K_0(V)^+ \cap -K_0(V)^+$. There exist $p \in \mathcal{OP}_m(V)$ and $q \in \mathcal{OP}_n(V)$ such that $g = [(p, 0)] = [(0, q)]$. Then $(p, 0) \equiv (0, q)$ so that $p \oplus q \approx 0_m \oplus 0_n$. Thus $p \oplus q \oplus r \sim 0_m \oplus 0_n \oplus r$ for some $r \in \mathcal{OP}_l(V)$. Since $p \oplus q \oplus r \sim 0_m \oplus 0_n \oplus r$ and $p \oplus q \oplus r \geq 0_m \oplus 0_n \oplus r$, by Corollary 5.4.3, we get that $p \oplus q \oplus r = 0_m \oplus 0_n \oplus r$. Then $p \oplus q \oplus 0_l = 0_{m+n+l}$ so that $p_m = 0_m$ and $q = 0_n$. Thus $g = 0$.

(c) For each $p, q \in \mathcal{OP}_\infty(V)$, we get that

$$\begin{aligned}[(p,q)] &= [(p,0)]+[(0,q)] \\ &= [(p,0)]-[(q,0)]\end{aligned}$$

so that $K_0(V)=K_0(V)^+-K_0(V)^+$.

□

Corollary 6.5.2. *Let (V,e) be an absolute matrix order unit space satisfying (T) and let e^n be finite for all $n\in\mathbb{N}$. Then $(K_0(V),K_0(V)^+)$ is an ordered abelian group with distinguished order unit $[(e,0)]$. In other words, for each $g\in K_0(V)$, there exists $n\in\mathbb{N}$ such that $-n[(e,0)]\leq g\leq n[(e,0)]$.*

Proof. By Theorem 6.5.1, $(K_0(V),K_0(V)^+)$ is an ordered abelian group. We show that $[(e,0)]$ is an order unit. First, let $r\in\mathcal{OP}_m(V)$ for some $m\in\mathbb{N}$. By Proposition 5.2.5(5), we have

$$\begin{aligned}[(r,0)] &\leq [(r,0)]+[(e^m-r,0)] \\ &= [(r\oplus(e^m-r),0)] \\ &= [(r+(e^m-r),0)] \\ &= [(e^m,0)] \\ &= m[(e,0)].\end{aligned}$$

Let $g\in K_0(V)$. Then by Theorem 6.5.1(c), we have $g=[(p,0)]-[(q,0)]$ for some $p,q\in\mathcal{OP}_\infty(V)$. Without any loss of generality, we may assume that $p,q\in\mathcal{OP}_n(V)$ for some $n\in\mathbb{N}$. Since $-[(q,0)]\leq g\leq[(p,0)]$, we get that $-n[(e,0)]\leq g\leq n[(e,0)]$. Hence $[(e,0)]$ is an order unit for $K_0(V)$. □

Theorem 6.5.3. *Let V and W be absolute matrix order unit spaces and $\phi, \psi : V \to W$ be completely $|\cdot|$-preserving maps such that $\phi \perp \psi$. If ϕ and ψ map $\mathcal{OP}_\infty(V)$ into $\mathcal{OP}_\infty(W)$, then*

(1) $\phi + \psi$ also maps $\mathcal{OP}_\infty(V)$ into $\mathcal{OP}_\infty(W)$.

(2) ϕ, ψ and $\phi+\psi$ are completely contractive maps with $\|\phi+\psi\|_{cb} = max\{\|\phi\|_{cb}, \|\psi\|_{cb}\}$.

(3) If V and W satisfy (T), then $K_0(\phi + \psi) = K_0(\phi) + K_0(\psi)$.

Proof. Assume that ϕ and ψ map $\mathcal{OP}_\infty(V)$ into $\mathcal{OP}_\infty(W)$.

(1) Let $p \in \mathcal{OP}_n(V)$. Then $\phi_n(p), \psi_n(p) \in \mathcal{OP}_n(W)$ with $\phi_n(p) \perp \psi_n(p)$. By Proposition 2.3.11(1), we get that $\phi_n(p) + \psi_n(p) \in \mathcal{OP}_n(V)$. Thus $\phi + \psi$ maps $\mathcal{OP}_\infty(V)$ into $\mathcal{OP}_\infty(W)$.

(2) Let $v \in M_n(V), n \in \mathbb{N}$. By Theorems 4.3.2(2) and 4.3.3(1), we get that

$$\begin{aligned}
\|\phi_n(v)\|_n &= \left\| \begin{bmatrix} 0 & \phi_n(v) \\ \phi_n(v)^* & 0 \end{bmatrix} \right\|_{2n} \\
&= \left\| \phi_{2n}\left(\begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \right) \right\|_{2n} \\
&\leq \left\| \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \right\|_{2n} \\
&= \|v\|_n.
\end{aligned}$$

Thus ϕ is completely contractive. Similarly, we can show that ψ is also completely contractive.

Further, by Remark 4.5.2, we have

$$\begin{bmatrix} 0 & \phi_n(v) \\ \phi_n(v)^* & 0 \end{bmatrix} = \phi_{2n}\left(\begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix}\right) \perp \psi_{2n}\left(\begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & \psi_n(v) \\ \psi_n(v)^* & 0 \end{bmatrix}$$

so that

$$\begin{aligned}
\|(\phi_n+\psi_n)(v)\|_n &= \left\|\begin{bmatrix} 0 & (\phi_n+\psi_n)(v) \\ (\phi_n+\psi_n)(v)^* & 0 \end{bmatrix}\right\|_{2n} \\
&= \left\|\left|\begin{bmatrix} 0 & (\phi_n+\psi_n)(v) \\ (\phi_n+\psi_n)(v)^* & 0 \end{bmatrix}\right|_{2n}\right\|_{2n} \\
&= \left\|\left|\begin{bmatrix} 0 & \phi_n(v) \\ \phi_n(v)^* & 0 \end{bmatrix} + \begin{bmatrix} 0 & \psi_n(v) \\ \psi_n(v)^* & 0 \end{bmatrix}\right|_{2n}\right\|_{2n} \\
&= \left\|\left|\begin{bmatrix} 0 & \phi_n(v) \\ \phi_n(v)^* & 0 \end{bmatrix}\right|_{2n} + \left|\begin{bmatrix} 0 & \psi_n(v) \\ \psi_n(v)^* & 0 \end{bmatrix}\right|_{2n}\right\|_{2n} \\
&= \max\left\{\left\|\left|\begin{bmatrix} 0 & \phi_n(v) \\ \phi_n(v)^* & 0 \end{bmatrix}\right|_{2n}\right\|_{2n}, \left\|\left|\begin{bmatrix} 0 & \psi_n(v) \\ \psi_n(v)^* & 0 \end{bmatrix}\right|_{2n}\right\|_{2n}\right\} \\
&= \max\left\{\left\|\begin{bmatrix} 0 & \phi_n(v) \\ \phi_n(v)^* & 0 \end{bmatrix}\right\|_{2n}, \left\|\begin{bmatrix} 0 & \psi_n(v) \\ \psi_n(v)^* & 0 \end{bmatrix}\right\|_{2n}\right\} \\
&= \max\left\{\|\phi_n(v)\|_n, \|\psi_n(v)\|_n\right\}.
\end{aligned}$$

Thus $\|\phi+\psi\|_{cb} = \max\{\|\phi\|_{cb}, \|\psi\|_{cb}\}$ and consequently $\phi+\psi$ is also completely contractive.

(3) Assume that V and W satisfy (T). Let $p \in \mathcal{OP}_n(V)$ for some $n \in \mathbb{N}$.

Since $\phi_n(p) \perp \psi_n(p)$, by Proposition 5.2.5(5), we get that $\phi_n(p) + \psi_n(p) \sim \phi_n(p) \oplus \psi_n(p)$. Then

$$\begin{aligned}
K_0(\phi + \psi)([(p, 0)]) &= [((\phi + \psi)_n(p), 0)] \\
&= [(\phi_n(p) + \psi_n(p), 0)] \\
&= [(\phi_n(p) \oplus \psi_n(p), 0)] \\
&= [(\phi_n(p), 0)] + [(\psi_n(p), 0)] \\
&= K_0(\phi)([(p, 0)]) + K_0(\psi)([(p, 0)]).
\end{aligned}$$

Thus by Theorem 6.5.1(c), we conclude that $K_0(\phi + \psi) = K_0(\phi) + K_0(\psi)$.

$\square$

Remark 6.5.4. *Let V and W be absolute matrix order unit spaces and let $\phi_i : V \to W$ be completely $|\cdot|$-preserving maps for $i = 1, 2, \cdots, n$ such that $\phi_i \perp \phi_j$ for all $i \neq j$. Then $\sum_{i=1}^{n} \phi_i$ is also a completely $|\cdot|$-preserving map. Moreover, if V and W satisfy (T) and if ϕ maps $\mathcal{OP}_\infty(V)$ into $\mathcal{OP}_\infty(W)$ for each i, then $K_0\left(\sum_{i=1}^{n} \phi_i\right) = \sum_{i=1}^{n} K_0(\phi_i)$.*

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Thesis Highlights

Name of the Student: **Amit Kumar**
Name of the CI/OCC: **NISER Bhubaneswar**
Enrolment No.: **MATH11201504001**
Thesis Title: **Comparison of order projections in absolute matrix order unit spaces**
Discipline: **Mathematical Sciences**
Sub-Area of Discipline: **Functional Analysis**
Date of Viva Voce: **22nd June 2020**

The main objective of this thesis is to study comparison theory of order projections in absolute matrix order unit spaces. In this thesis, we have introduced absolute value preserving maps between two absolute order unit spaces. We have proved that a unital bijective linear map between two absolute order unit spaces is absolute value preserving if and only if it is an isometry. Since the self-adjoint parts of unital C*-algebras are absolute order unit spaces, our result can be considered an extension of the studies on surjective linear isometries between C*-algebras due to Kadison and the characterization of surjective linear isometries between unital JB-algebras obtained by Wright and Youngson..

We have introduced matricial version of absolute order unit spaces namely absolute matrix order unit spaces. We have also generalized the notion of absolute value preserving maps to completely absolute value preserving maps between two absolute matrix order unit spaces.

We have define the notion of partial isometry and some other related algebraic notions of C*-algebras in order theoretic contexts in absolute matrix order unit spaces. Using them, we have introduced and studied comparison of order projections in absolute matrix order unit spaces. This idea is an extension of comparison of projections in a C*-algebra. We have defined notions of infinite and properly infinite projections and studied characterizations of these notions.

Our proposed comparison theory culminates in formation of K_0-groups. We have proved that K_0 is a functor from category of absolute matrix order unit spaces with morphisms as unital completely absolute value preserving maps to category of abelian groups. We have also defined orthogonality of completely absolute value preserving maps and proved that K_0 is additive on orthogonal unital completely absolute value preserving maps.