Certain congruences among Hermitian Jacobi forms and Hermitian modular forms

By

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MATH11201504002

National Institute of Science Education and Research, Bhubaneswar

A thesis submitted to the

Board of Studies in Mathematical Sciences

In partial fulfillment of requirements

for the Degree of

DOCTOR OF PHILOSOPHY

of

HOMI BHABHA NATIONAL INSTITUTE

March, 2020

Homi Bhabha National Institute

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DECLARATION

I hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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List of Publications arising from the thesis

Journal

- *Congruences in Hermitian Jacobi and Hermitian modular forms*, J. Meher and S. K. Singh, Forum Math. **32** (2020), 501–523.

Other Publications

- *On the parity of the Fourier coefficients of the Hauptmoduln $j_N(z)$ and $j_N^+(z)$*, M. Kumari and S. K. Singh, Acta Arith. **188** (2019), 171–182.

Sujeet Kumar Singh

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Dedicated to My Father

ACKNOWLEDGEMENTS

First and Foremost, I would like to express my sincere gratitude to my supervisor Prof. Jaban Meher for introducing me to the wonderful area of Automorphic forms. I would like to thank him for his continuous support, patience guidance, motivation, enthusiasm and knowledge during my Ph. D study and research. His guidance helped me in all the time of research and writing of this thesis.

I would like to thank Prof. B. Ramakrishnan for teaching me various courses during my Ph.D. I am also grateful to Prof. Soumya Das for mathematical discussions and suggestions. I am also grateful to many other mathematicians, Prof. Brundaban Sahu, Prof. M. Manickam, Prof. Karam Deo Shankadhar who have helped me and taught me. I wish to thank the other members of my doctoral committee, Prof. A. Karn, Prof. Senthil Kumar and Prof. M. R. Sahoo. I also sincerely thank the faculty members of NISER for their patience and encouragement during my course work.

I take this opportunity to thank my seniors Abhash, Moni, Bikram, Anindya and my batch mate Amit for their academic and non-academic helps, during my stay at NISER. I would like to thank my friends Priyaranjan, Anup, Lalit, Mohit, Abhishek and Prabhat for their support.

I would like to thank my father, late Shri Syurya Bhan Singh and my mother Usha Singh for supporting me throughout my studies and allowing me the freedom to pursue my own interests. I would like to thank my brothers Amit bhaiya and Abhishek, my sister Chitralekha didi for their love, support and constant encouragement.

Above all I would like to thank my wife Rimpa for her love and constant support. Thank you for taking care of me in every aspect of life. But most of all, thank you for being my best friend. I owe you everything.

Contents

Chapter 1

Preliminaries

1.1 Introduction

In this chapter we recall the definitions of a modular form, a Jacobi form of matrix index, a Hermitian Jacobi form over $\mathbb{Q}(i)$, a Siegel modular form and a Hermitian modular form of degree 2 over $\mathbb{Q}(i)$. We also recall different Heat operators on these automorphic functions.

1.2 Modular forms

Let $\mathcal{H} = \{\tau \in \mathbb{C} : \Im(\tau) > 0\}$ be the complex upper-half plane. The full modular group $SL_2(\mathbb{Z})$ is defined by

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z} \text{ with } ad - bc = 1 \right\}.$$

The group $SL_2(\mathbb{Z})$ acts on $\mathcal{H}$ by

$$g\tau = \frac{a\tau + b}{c\tau + d}, \qquad \text{where } \ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \text{ and } \tau \in \mathcal{H}.$$

Using the above action of $SL_2(\mathbb{Z})$ on $\mathcal{H}$, we define a family of actions of $SL_2(\mathbb{Z})$ on the set of functions from $\mathcal{H}$ to $\mathbb{C}$. Let $f : \mathcal{H} \to \mathbb{C}$ be a function. For each positive integer k, the action of $SL_2(\mathbb{Z})$ on the set of functions from $\mathcal{H}$ to $\mathbb{C}$ is defined by

$$(f \mid_k g)(\tau) = (c\tau + d)^{-k} f(g\tau),$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $\tau \in \mathcal{H}$.

Definition 1.2.1. *A holomorphic function $f : \mathcal{H} \to \mathbb{C}$ is called a modular form of weight k on $SL_2(\mathbb{Z})$ if*

- *$f \mid_k g = f$ for all $g \in SL_2(\mathbb{Z})$,*
- *f has a Fourier expansion of the form*

$$f(\tau) = \sum_{n=0}^{\infty} a(f;n) q^n, \qquad q = e^{2\pi i \tau}.$$

The function f is called a cusp form if in addition to the above two conditions, we also have $a(f;0) = 0$ in the above Fourier expansion of f.

For any even integer $k \geq 4$, the Eisenstein series E_k is defined by

$$E_k(\tau) = \frac{1}{2} \sum_{\substack{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\} \\ (m,n)=1}} \frac{1}{(m\tau + n)^k}, \qquad \tau \in \mathcal{H}.$$

The Fourier expansion of E_k is given by

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \tag{1.2.1}$$

where B_k is the kth Bernoulli number, $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$. It is well known that for $k \geq 4$, E_k is a modular form of weight k on $SL_2(\mathbb{Z})$. One also can define $E_2(\tau)$ by the Fourier expansion given in $(1.2.1)$. Although E_2 is not a modular form, it plays a vital role in number theory. The Ramanujan delta function is defined by

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n.$$

It is well known that $\Delta(\tau)$ is a cusp form of weight 12 on $SL_2(\mathbb{Z})$. We denote by $M_k(SL_2(\mathbb{Z}))$ the complex vector space of all modular forms of weight k on $SL_2(\mathbb{Z})$. For a ring $R \subset \mathbb{C}$, we denote by $M_k(SL_2(\mathbb{Z}), R)$ the set of all modular forms of weight k on $SL_2(\mathbb{Z})$ with all the Fourier coefficients in R. Let $M_*(SL_2(\mathbb{Z})) = \bigoplus_k M_k(SL_2(\mathbb{Z}))$ be the graded ring of all modular forms on $SL_2(\mathbb{Z})$. Let $M_*(SL_2(\mathbb{Z}), R) = \bigoplus_k M_k(SL_2(\mathbb{Z}), R)$ be the graded ring of all modular forms on $SL_2(\mathbb{Z})$ with all the Fourier coefficients in R.

1.2.1 Heat Operator

Ramanujan theta operator or heat operator in modular forms is defined by

$$\theta = q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau}.$$

If $f \in M_k(SL_2(\mathbb{Z}))$ has Fourier expansion $f(\tau) = \sum_{n=0}^{\infty} a(f;n)q^n$, then the Fourier expansion of $\theta(f)$ is given by

$$\theta(f)(\tau) = \sum_{n=0}^{\infty} na(f;n)q^n.$$

We have the following well known result [26, Theorem 5.3] .

Lemma 1.2.2. *If $f \in M_k(SL_2(\mathbb{Z}))$ then*

$$\hat{f} = 12\theta(f) - kE_2 f \in M_{k+2}(SL_2(\mathbb{Z})).$$

1.3 Jacobi forms

The theory of Jacobi forms was introduced by Eichler and Zagier [16]. They systematically studied Jacobi forms of integer index. Later, Ziegler [42] introduced Jacobi forms of matrix index. The Jacobi group $\Gamma^l = SL_2(\mathbb{Z}) \ltimes (\mathbb{Z}^l \times \mathbb{Z}^l)$ acts on $\mathcal{H} \times \mathbb{C}^l$ as follows

$$(g, (\lambda, \mu)) \cdot (\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d} \right),$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, $\tau \in \mathcal{H}$, $\lambda = (\lambda_1, \cdots, \lambda_l)^t, \mu = (\mu_1, \cdots, \mu_l)^t \in \mathbb{Z}^l$ and $z = (z_1, \cdots, z_l)^t \in \mathbb{C}^l$. This action extends to a family of actions on the set of functions from $\mathcal{H} \times \mathbb{C}^l$ to $\mathbb{C}$. Let $\psi : \mathcal{H} \times \mathbb{C}^l \to \mathbb{C}$ be a function and let k be a positive integer. Suppose that M is a symmetric, positive definite, half-integral $l \times l$ matrix with integral diagonal entries and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. The action of

$SL_2(\mathbb{Z})$ on ψ is defined by

$$(\psi \mid_{k,M} g)(\tau, z_1, \cdots, z_l) := (c\tau + d)^{-k} e^{-2\pi i \frac{cM[z]}{c\tau+d}} \psi\left(\frac{a\tau + b}{c\tau + d}, \frac{z_1}{c\tau + d}, \cdots, \frac{z_l}{c\tau + d}\right).$$

Let $\lambda = (\lambda_1, \cdots, \lambda_l) \in \mathbb{Z}^l, \mu = (\mu_1, \cdots, \mu_l) \in \mathbb{Z}^l$. The action of $\mathbb{Z}^l \times \mathbb{Z}^l$ on ψ is defined by

$$(\psi \mid_M [\lambda, \mu])(\tau, z_1, \cdots, z_l) := e^{2\pi i(\tau M[\lambda] + 2\lambda^t M z)} \psi(\tau, z_1 + \lambda_1\tau + \mu_1, \cdots, z_l + \lambda_l\tau + \mu_l).$$

Definition 1.3.1. *A holomorphic function $\psi : \mathcal{H} \times \mathbb{C}^l \longrightarrow \mathbb{C}$ is a Jacobi form of weight k and index M on $SL_2(\mathbb{Z})$ if for each $g \in SL_2(\mathbb{Z})$ and $\lambda, \mu \in \mathbb{Z}^l$, we have*

- $$\psi \mid_{k,M} g = \psi, \tag{1.3.1}$$

- $$\psi \mid_M [\lambda, \mu] = \psi, \tag{1.3.2}$$

- *ψ has a Fourier expansion of the form*

$$\psi(\tau, z_1, \cdots, z_l) = \sum_{\substack{0 \le n \in \mathbb{Z}, r \in \mathbb{Z}^l \\ 4\det(M)n - M^{\#}[r] \ge 0}} c(\psi; n, r) q^n \zeta^r, \tag{1.3.3}$$

where $q = e(\tau)$, $\zeta^r = e(r^t z)$ and $M^{\#}$ is the adjugate of M.

We denote by $J_{k,M}(\Gamma^l)$ the complex vector space of all Jacobi forms of weight k and matrix index M on Γ^l. For a ring $R \subset \mathbb{C}$, we denote by $J_{k,M}(\Gamma^l, R)$ the set of all Jacobi forms of weight k and matrix index M on Γ^l with all the Fourier coefficients in R. Let $J_{*,M}(\Gamma^l) = \bigoplus_k J_{k,M}(\Gamma^l)$ be the graded ring of all Jacobi forms of matrix index M on Γ^l. Let $J_{*,M}(\Gamma^l, R) = \bigoplus_k J_{k,M}(\Gamma^l, R)$ be the graded ring of

all Jacobi forms of matrix index M on Γ^l with all the Fourier coefficients in R. The rings $J_{*,M}(\Gamma^l)$ and $J_{*,M}(\Gamma^l, R)$ are modules over $M_*(SL_2(\mathbb{Z}))$ and $M_*(SL_2(\mathbb{Z}), R)$ respectively.

1.3.1 Heat Operator

When $l = 1$, the above Definition 1.3.1 gives the definition of a Jacobi form of integer index which was introduced by Eichler and Zagier [16]. For $\phi \in J_{k,m}(\Gamma^1)$, the heat operator on ϕ is defined by

$$K_m := \frac{1}{(2\pi i)^2}\left(8\pi i m \frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial z^2}\right).$$

If ϕ has Fourier expansion of the form

$$\phi(\tau, z) = \sum_{\substack{n,r\in\mathbb{Z} \\ 4nm-r^2\geq 0}} c(\phi; n, r) q^n \zeta^r,$$

then the Fourier expansion of $K_m(\phi)$ is

$$K_m(\phi)(\tau, z) = \sum_{\substack{n,r\in\mathbb{Z} \\ 4nm-r^2\geq 0}} (4nm - r^2) c(\phi; n, r) q^n \zeta^r.$$

We refer to Richter [34, p. 869] for the following lemma.

Lemma 1.3.2. *If $\phi \in J_{k,m}(\Gamma^1)$ then*

$$\hat{\phi} = K_m(\phi) - \frac{(2k-1)m}{6}\phi E_2 \in J_{k+2,m}(\Gamma^1)$$

1.4 Hermitian Jacobi forms

Haverkemp [19, 20] and Das [10, 11] have studied Hermitian Jacobi forms over $\mathbb{Q}(i)$. Later, Richter and Senadheera [36] introduced the corrected definition of a Hermitian Jacobi forms over $\mathbb{Q}(i)$. We recall the new definition of Hermitian Jacobi forms over $\mathbb{Q}(i)$ introduced by Richter and Senadheera.

Let $\mathcal{O} = \mathbb{Z}[i]$ be the ring of integers of Gaussian field of rational number $\mathbb{Q}(i)$. Let $\mathcal{O}^{\#} = \frac{i}{2}\mathcal{O}$ be the inverse different of $\mathbb{Q}(i)$ over $\mathbb{Q}$. Let $\mathcal{O}^{\times} = \{1, -1, i, -i\}$ be the set of units in $\mathcal{O}$. The Hermitian Jacobi group over $\mathcal{O}$ is defined by

$$\Gamma^J(\mathcal{O}) = \Gamma(\mathcal{O}) \ltimes \mathcal{O}^2,$$

where $\Gamma(\mathcal{O}) = \{\epsilon M \mid M \in SL_2(\mathbb{Z}), \epsilon \in \mathcal{O}^{\times}\}$. The Hermitian Jacobi group $\Gamma^J(\mathcal{O})$ acts on $\mathcal{H} \times \mathbb{C}^2$ as follow

$$(\epsilon g, (\lambda, \mu)) \cdot (\tau, z_1, z_2) = \left(\frac{a\tau + b}{c\tau + d}, \epsilon\frac{z_1 + \lambda\tau + \mu}{c\tau + d}, \overline{\epsilon}\frac{z_2 + \overline{\lambda}\tau + \overline{\mu}}{c\tau + d}\right),$$

where $\epsilon \in \mathcal{O}^{\times}$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, $\tau \in \mathcal{H}$, $z_1, z_2 \in \mathbb{C}$. This action extends to a family of actions on the set of functions from $\mathcal{H} \times \mathbb{C}^2$ to $\mathbb{C}$. Let $\phi : \mathcal{H} \times \mathbb{C}^2 \to \mathbb{C}$ be any function and let k be a non-negative integer. Let $\delta \in \{+, -\}$ and let m be an integer. Suppose $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $\epsilon \in \mathcal{O}^{\times}$. We define the action of $\Gamma(\mathcal{O})$ on ϕ by

$$(\phi \mid_{k,m,\delta} \epsilon g)(\tau, z_1, z_2) := \sigma(\epsilon)\epsilon^{-k}(c\tau + d)^{-k} e^{\frac{-2\pi i m c z_1 z_2}{c\tau + d}} \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{\epsilon z_1}{c\tau + d}, \frac{\overline{\epsilon} z_2}{c\tau + d}\right),$$

where

$$\sigma(\epsilon) = \begin{cases} 1 & \text{if} \quad \delta = +, \\ \epsilon^2 & \text{if} \quad \delta = -. \end{cases}$$

Let $\lambda, \mu \in \mathcal{O}$. We define the action of $\mathcal{O}^2$ on ϕ by

$$(\phi \mid_m [\lambda, \mu])(\tau, z_1, z_2) := e^{2\pi i m(\lambda\overline{\lambda}\tau + \overline{\lambda}z_1 + \lambda z_2)} \phi\left(\tau, z_1 + \lambda\tau + \mu, z_2 + \overline{\lambda}\tau + \overline{\mu}\right).$$

Definition 1.4.1. *A holomorphic function $\phi : \mathcal{H} \times \mathbb{C}^2 \longrightarrow \mathbb{C}$ is a Hermitian Jacobi form of weight k, index m and parity $\delta \in \{+, -\}$ on $\Gamma^J(\mathcal{O})$ if for each $g \in SL_2(\mathbb{Z})$, $\epsilon \in \mathcal{O}^\times$ and $\lambda, \mu \in \mathcal{O}$, we have*

- $$\phi \mid_{k,m,\delta} \epsilon g = \phi, \tag{1.4.1}$$

- $$\phi \mid_m [\lambda, \mu] = \phi, \tag{1.4.2}$$

- *ϕ has a Fourier expansion of the form*

$$\phi(\tau, z_1, z_2) = \sum_{\substack{n \in \mathbb{Z}, r \in \mathcal{O}^\# \\ nm - N(r) \geq 0}} c(\phi; n, r) q^n \zeta_1^r \zeta_2^{\overline{r}}, \tag{1.4.3}$$

where $q = e(\tau)$, $\zeta_1 = e(z_1)$, $\zeta_2 = e(z_2)$.

We denote by $HJ^\delta_{k,m}(\Gamma^J(\mathcal{O}))$ the complex vector space of all Hermitian Jacobi forms of weight k, index m and parity δ on $\Gamma^J(\mathcal{O})$. For a ring $R \subset \mathbb{C}$, we denote by $HJ^\delta_{k,m}(\Gamma^J(\mathcal{O}), R)$ the set of all Hermitian Jacobi forms of weight k, index m and parity δ on $\Gamma^J(\mathcal{O})$ with all the Fourier coefficients in R.

1.5 Jacobi forms on $\Gamma^1(\mathcal{O})$

Consider the Jacobi group $\Gamma^1(\mathcal{O}) = SL_2(\mathbb{Z}) \ltimes \mathcal{O}^2$. A Jacobi form of weight k and index m on the group $\Gamma^1(\mathcal{O})$ is a holomorphic function from $\mathcal{H} \times \mathbb{C}^2$ to $\mathbb{C}$ which satisfies the transformation property $(1.4.1)$ with $\epsilon = 1$, transformation property $(1.4.2)$ and it has a Fourier expansion of the form given in $(1.4.3)$. We refer to [11, 31] for more details on Jacobi forms on $\Gamma^1(\mathcal{O})$. We denote by $J^1_{k,m}(\Gamma^1(\mathcal{O}))$ the complex vector space of all Jacobi forms of weight k and index m on $\Gamma^1(\mathcal{O})$. For a ring $R \subset \mathbb{C}$, we denote by $J^1_{k,m}(\Gamma^1(\mathcal{O}), R)$, the set of all Jacobi forms of weight k and index m on $\Gamma^1(\mathcal{O})$ with all the Fourier coefficients in R. Let $J^1_{*,m}(\Gamma^1(\mathcal{O})) = \bigoplus_k J^1_{k,m}(\Gamma^1(\mathcal{O}))$ be the graded ring of all Jacobi forms of index m on $\Gamma^1(\mathcal{O})$. Let $J^1_{*,m}(\Gamma^1(\mathcal{O}), R) = \bigoplus_k J^1_{k,m}(\Gamma^1(\mathcal{O}), R)$ be the graded ring of all Jacobi forms of index m with all the Fourier coefficients in R. The graded rings $J^1_{*,m}(\Gamma^1(\mathcal{O}))$ and $J^1_{*,m}(\Gamma^1(\mathcal{O}), R)$ are modules over $M_*(SL_2(\mathbb{Z}))$ and $M_*(SL_2(\mathbb{Z}), R)$ respectively. We observe that

$$HJ^{\delta}_{k,m}(\Gamma^J(\mathcal{O})) \subset J^1_{k,m}(\Gamma^1(\mathcal{O})) \quad \text{for each} \quad \delta \in \{+, -\}. \tag{1.5.1}$$

For a given $\phi \in J^1_{k,m}(\Gamma^1(\mathcal{O}))$, we can construct a Hermitian Jacobi form of weight k, index m and parity δ using the averaging operator

$$A : J^1_{k,m}(\Gamma^1(\mathcal{O})) \to HJ^{\delta}_{k,m}(\Gamma^J(\mathcal{O}))$$

is defined by

$$\phi \mapsto \sum_{\epsilon \in \mathcal{O}^\times} \phi \mid_{k,m,\delta} \epsilon I, \tag{1.5.2}$$

where I is the identity matrix.

1.5.1 Heat Operator

Let $\phi : \mathcal{H} \times \mathbb{C}^2 \longrightarrow \mathbb{C}$ be a function. Suppose ϕ has a Fourier expansion of the form

$$\phi(\tau, z_1, z_2) = \sum_{\substack{n\in\mathbb{Z}, r\in\mathcal{O}^\# \\ nm-N(r)\geq 0}} c(\phi; n, r) q^n \zeta_1^r \zeta_2^{\overline{r}}. \tag{1.5.3}$$

The heat operator on ϕ is defined by

$$L_m := -\frac{1}{\pi^2}\left(2\pi i m \frac{\partial}{\partial\tau} - \frac{\partial^2}{\partial z_1 \partial z_2}\right).$$

The Fourier expansion of $L_m(\phi)$ is given by

$$L_m(\phi)(\tau, z_1, z_2) = \sum_{\substack{n\in\mathbb{Z}, r\in\mathcal{O}^\# \\ nm-N(r)\geq 0}} 4(nm - N(r)) c(\phi; n, r) q^n \zeta_1^r \zeta_2^{\overline{r}}.$$

The following lemma gives the actions of L_m on the spaces $J_{k,m}(\Gamma^1(\mathcal{O}))$ and $HJ^\delta_{k,m}(\Gamma^J(\mathcal{O}))$. We refer to Senadheera [37, Lemma 5.1] for a proof.

Lemma 1.5.1. *For a holomorphic function $\phi : \mathcal{H} \times \mathbb{C}^2 \to \mathbb{C}$ which has a Fourier expansion of the form* (1.5.3)*, let*

$$\hat{\phi} = L_m(\phi) - \frac{(k-1)m}{3} E_2 \phi. \tag{1.5.4}$$

Then

- *if $\phi \in J^1_{k,m}(\Gamma^1(\mathcal{O}))$ then $\hat{\phi} \in J^1_{k+2,m}(\Gamma^1(\mathcal{O}))$;*
- *if $\phi \in HJ^\delta_{k,m}(\Gamma^J(\mathcal{O}))$ then $\hat{\phi} \in HJ^{-\delta}_{k+2,m}(\Gamma^J(\mathcal{O}))$.*

1.6 Siegel modular forms

For an integer $n \geq 1$, we define the Siegel upper half-space of degree n by

$$\mathbb{H}_n = \{Z \in M_n(\mathbb{C}) \mid Z = Z^t, \frac{Z - \overline{Z}}{2i} > 0\}.$$

Let $J_{2n} = \begin{pmatrix} \mathbf{0}_n & -I_n \\ I_n & \mathbf{0}_n \end{pmatrix}$, where I_n and $\mathbf{0}_n$ denotes the $n \times n$ identity matrix and zero matrix respectively. The full Siegel modular group of degree n is defined by

$$\Gamma_{2n} = Sp_{2n}(\mathbb{Z}) = \{M \in M_{2n}(\mathbb{Z}) \mid MJ_{2n}M^t = J_{2n}\}.$$

The group Γ_{2n} acts on $\mathbb{H}_n$ by the fractional transformation

$$Z \mapsto M \cdot Z = (AZ + B)(CZ + D)^{-1},$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_{2n}$ and $Z \in \mathbb{H}_n$.

Definition 1.6.1. *A Siegel modular form of degree $n > 1$ and weight k on Γ_{2n} is a holomorphic function $F : \mathbb{H}_n \to \mathbb{C}$ satisfying*

$$F(M \cdot Z) = (\det(CZ + D))^k F(Z),$$

for all $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_{2n}$.

A Siegel modular form possesses a Fourier expansion of the form

$$F(Z) = \sum_{T = T^t \geq 0} A_F(T) e(tr(TZ)),$$

where the summation is over all symmetric, semi-positive definite, half-integral $n \times n$ matrices T with integral diagonal entries. We denote by $M_k(\Gamma_{2n})$ the complex vector space of all Siegel modular forms of degree n and weight k on Γ_{2n}. For a ring $R \subset \mathbb{C}$ we denote by $M_k(\Gamma_{2n}, R)$ the set of Siegel modular forms of degree n and weight k on Γ_{2n} with all the Fourier coefficients in R.

If $n = 2$, we write $Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$ with $\tau, \tau' \in \mathcal{H}$, $z \in \mathbb{C}$ and $\Im(z)^2 < \Im(\tau)\Im(\tau')$. We have $T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}$ with $n, m, r \in \mathbb{Z}$, $n, m \geq 0$, $4nm - r^2 \geq 0$ and we write the Fourier expansion of any $F \in M_k(\Gamma_4)$ as

$$F(\tau, z, \tau') = \sum_{\substack{n,m,r \in \mathbb{Z} \\ 4nm - r^2 \geq 0}} A_F(n, r, m) q^n \zeta^r (q')^m,$$

where $q = e(\tau), \zeta = e(z), q' = e(\tau')$. The Siegel modular form F has the Fourier-Jacobi expansion of the form

$$F(\tau, z, \tau') = \sum_{m=0}^{\infty} \phi_{k,m}(\tau, z)(q')^m,$$

where

$$\phi_{k,m}(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z} \\ 4nm - r^2 \geq 0}} A_F(n, r, m) q^n \zeta^r \in J_{k,m}(\Gamma^1).$$

1.6.1 Heat Operator

The heat operator on a Siegel modular forms of degree 2 is defined by

$$\mathbb{P} = \frac{1}{(2\pi i)^2} \left(4 \frac{\partial}{\partial \tau} \frac{\partial}{\partial \tau'} - \frac{\partial^2}{\partial z^2} \right).$$

If $F \in M_k(\Gamma_4)$ has the Fourier expansion

$$F(\tau, z, \tau') = \sum_{\substack{n,r,m\in\mathbb{Z} \\ 4nm-r^2\geq 0}} A_F(n,r,m) q^n \zeta^r (q')^m,$$

then the Fourier expansion of $\mathbb{P}(F)$ is given by

$$\mathbb{P}(F)(\tau, z, \tau') = \sum_{\substack{n,r,m\in\mathbb{Z} \\ 4nm-r^2\geq 0}} (4nm - r^2) A_F(n,r,m) q^n \zeta^r (q')^m.$$

1.7 Hermitian modular forms of degree 2

The Hermitian upper half-space of degree 2 is defined by

$$\mathcal{H}_2 = \left\{ Z = \begin{pmatrix} \tau & z_1 \\ z_2 & \tau' \end{pmatrix} \in M_2(\mathbb{C}) \mid \frac{1}{2i}(Z - \overline{Z}^t) > 0 \right\}.$$

The Hermitian modular group $\Gamma^2(\mathcal{O})$ of degree 2 over $\mathbb{Q}(i)$ is defined by

$$\Gamma^2(\mathcal{O}) = \{ M \in M_4(\mathcal{O}) \mid \overline{M}^t J_4 M = J_4 \}.$$

The group $\Gamma^2(\mathcal{O})$ acts on $\mathcal{H}_2$ by the fractional transformation

$$Z \longmapsto M \cdot Z = (AZ + B)(CZ + D)^{-1},$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^2(\mathcal{O})$ and $Z \in \mathcal{H}_2$. Let F be a complex valued function on $\mathcal{H}_2$. For a positive integer k, we define

$$(F \mid_k M)(Z) = (\det(CZ + D))^{-k} F(M \cdot Z),$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL_4(\mathbb{Q}(i))$. For $k \in \mathbb{Z}$, let ν_k denote the abelian characters on $\Gamma^2(\mathcal{O})$ satisfying $\nu_k \cdot \nu_{k'} = \nu_{k+k'}$.

Definition 1.7.1. *A holomorphic function $F : \mathcal{H}_2 \to \mathbb{C}$ is called a Hermitian modular form of weight k and character ν_k on $\Gamma^2(\mathcal{O})$ if*

$$F \mid_k M = \nu_k(M)F \qquad \textit{for all} \;\; M \in \Gamma^2(\mathcal{O}).$$

Writing $Z = \begin{pmatrix} \tau & z_1 \\ z_2 & \tau' \end{pmatrix}$, a Hermitian modular form F has a Fourier expansion of the form

$$F(Z) = \sum_{T \in \Delta_2} A_F(T) e(tr(TZ)) = \sum_{\substack{n,m \in \mathbb{Z}, r \in \mathcal{O}^{\#} \\ nm - N(r) \geq 0}} A_F(n,r,m) q^n \zeta_1^r \zeta_2^{\overline{r}} (q')^m, \qquad (1.7.1)$$

where

$$\Delta_2 = \left\{ T = \begin{pmatrix} n & r \\ \overline{r} & m \end{pmatrix} \geq 0 \mid n, m \in \mathbb{Z}, n \geq 0, m \geq 0, r \in \mathcal{O}^{\#} \right\},$$

$q = e(\tau)$, $\zeta_1 = e(z_1)$, $\zeta_2 = e(z_2)$, $q' = e(\tau')$. A Hermitian modular form F is called a Hermitian cusp form if the sum in $(1.7.1)$ runs over all positive-definite matrices $T \in \Delta_2$. We denote by $M_k(\Gamma^2(\mathcal{O}), \nu_k)$ the complex vector space of all Hermitian modular forms of weight k and character ν_k on $\Gamma^2(\mathcal{O})$. A Hermitian modular form $F \in M_k(\Gamma^2(\mathcal{O}), \nu_k)$ is called symmetric if

$$F(Z^t) = F(Z)$$

for all $Z \in \mathcal{H}_2$. We denote by $M_k(\Gamma^2(\mathcal{O}), \nu_k)^{sym}$ the subspace of $M_k(\Gamma^2(\mathcal{O}), \nu_k)$

consisting of all symmetric Hermitian modular forms of weight k and character ν_k on $\Gamma^2(\mathcal{O})$. Any $F \in M_k(\Gamma^2(\mathcal{O}), \nu_k)$ has the Fourier-Jacobi expansion of the form:

$$F(\tau, z_1, z_2, \tau') = \sum_{\substack{n\in\mathbb{Z}, r\in\mathcal{O}^{\#} \\ nm-N(r)\geq 0}} A_F(n,r,m) q^n \zeta_1^r \zeta_2^{\overline{r}} (q')^m = \sum_{m\geq 0} \phi_{k,m}(\tau, z_1, z_2)(q')^m, \tag{1.7.2}$$

where

$$\phi_{k,m}(\tau, z_1, z_2) = \sum_{\substack{n\in\mathbb{Z}, r\in\mathcal{O}^{\#} \\ nm-N(r)\geq 0}} A_F(n,r,m) q^n \zeta_1^r \zeta_2^{\overline{r}} \in HJ_{k,m}^{\delta}(\Gamma^J(\mathcal{O}))$$

for some $\delta \in \{+,-\}$. We are interested in the case when $\nu_k = \det^{k/2}$ (k even), where the character $\det^{k/2}$ on $\Gamma^2(\mathcal{O})$ is defined by $M \mapsto \det(M)^{k/2}$. Following a similar proof given by Haverkemp [19, Theorem 7.1], we have the following result.

Theorem 1.7.2. *Let the Fourier-Jacobi expansion of $F \in M_k(\Gamma^2(\mathcal{O}), \det^{k/2})$ be given by*

$$F(\tau, z_1, z_2, \tau') = \sum_{m\geq 0} \phi_{k,m}(\tau, z_1, z_2)(q')^m.$$

Then $\phi_{k,m}$ is a Hermitian Jacobi form of weight k, index m and parity δ on $\Gamma^J(\mathcal{O})$, where

$$\delta = \begin{cases} + & \text{if } k \equiv 0 \pmod 4, \\ - & \text{if } k \equiv 2 \pmod 4. \end{cases}$$

Let $M(\Gamma^2(\mathcal{O}), \det)^{sym} = \bigoplus_{k\in 2\mathbb{Z}} M_k(\Gamma^2(\mathcal{O}), \det^{k/2})^{sym}$ be the graded ring of all symmetric Hermitian modular forms of even weights on $\Gamma^2(\mathcal{O})$. For any ring $R \subseteq \mathbb{C}$, we denote by $M_k(\Gamma^2(\mathcal{O}), \det^{k/2}, R)$ the set of all Hermitian modular forms of weight k and character $\det^{k/2}$ on $\Gamma^2(\mathcal{O})$ with all the Fourier coefficients in R. We denote by $M_k(\Gamma^2(\mathcal{O}), \det^{k/2}, R)^{sym}$ the subset of $M_k(\Gamma^2(\mathcal{O}), \det^{k/2}, R)$ consisting of all sym-

metric Hermitian modular forms of weight k and character $\det^{k/2}$ on $\Gamma^2(\mathcal{O})$ with all the Fourier coefficients in R. Let $M(\Gamma^2(\mathcal{O}), \det, R)^{sym} = \bigoplus_{k\in 2\mathbb{Z}} M_k(\Gamma^2(\mathcal{O}), \det^{k/2}, R)^{sym}$ be the graded ring of all symmetric Hermitian modular forms of even weights on $\Gamma^2(\mathcal{O})$ with all the Fourier coefficients in R.

The Hermitian Eisenstein series of degree 2 and even weight $k \geq 6$ on $\Gamma^2(\mathcal{O})$ is defined by

$$H_k(Z) = \sum_M (\det M)^{k/2} \mathsf{det}(CZ + D)^{-k},$$

where $M = \begin{pmatrix} * & * \\ C & D \end{pmatrix}$ runs over a set of representatives of $\left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \backslash \Gamma^2(\mathcal{O})$. The Hermitian Eisenstein series H_4 of degree 2 and weight 4 on $\Gamma^2(\mathcal{O})$ has been constructed by the Maass lift by Krieg [25]. It is well-known that for even $k \geq 4$,

$$H_k \in M_k(\Gamma^2(\mathcal{O}), \det^{k/2})^{sym}.$$

Using the Hermitian Eisenstein series, we obtain the symmetric Hermitian cusp forms

$$\chi_8 = -\frac{61}{230400}(H_8 - H_4^2),$$
$$F_{10} = -\frac{277}{2419200}(H_{10} - H_4 H_6),$$

and

$$F_{12} = -\frac{34910011}{2002662144000}H_{12} - \frac{34801}{1009152000}H_4^3 + \frac{414251}{9082368000}H_4 H_8 + \frac{50521}{8010648576}H_6^2$$

of weights 8, 10 and 12 respectively.

We state the following result by Kikuta and Nagaoka [23, Theorem 4.3, Theorem 5.1] which gives the structure of symmetric Hermitian modular forms of degree 2 over

$\mathbb{Q}(i)$. Let $\mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ at the prime ideal $p\mathbb{Z}$.

Theorem 1.7.3. *The symmetric Hermitian modular forms H_4, H_6, χ_8, F_{10}, F_{12} are algebraically independent over $\mathbb{C}$. If $F \in M_k(\Gamma^2(\mathcal{O}), \det^{k/2})^{sym}$, then there exists a polynomial $M_F(X_1, X_2, X_3, X_4, X_5) \in \mathbb{C}[X_1, X_2, X_3, X_4, X_5]$ such that*

$$F = M_F(H_4, H_6, \chi_8, F_{10}, F_{12}).$$

Therefore we have,

$$\bigoplus_{k \in 2\mathbb{Z}} M_k(\Gamma^2(\mathcal{O}), \det^{k/2})^{sym} = \mathbb{C}[H_4, H_6, \chi_8, F_{10}, F_{12}].$$

Moreover, the symmetric Hermitian modular forms H_4, H_6, χ_8, F_{10}, F_{12} have integral Fourier coefficients. If $F \in M_k(\Gamma^2(\mathcal{O}), \det^{k/2}, \mathbb{Z}_{(p)})^{sym}$, then there exists a polynomial $P_F(X_1, X_2, X_3, X_4, X_5) \in \mathbb{Z}_{(p)}[X_1, X_2, X_3, X_4, X_5]$ such that

$$F = P_F(H_4, H_6, \chi_8, F_{10}, F_{12}).$$

Therefore we have,

$$\bigoplus_{k \in 2\mathbb{Z}} M_k(\Gamma^2(\mathcal{O}), \det^{k/2}, \mathbb{Z}_{(p)})^{sym} = \mathbb{Z}_{(p)}[H_4, H_6, \chi_8, F_{10}, F_{12}].$$

1.7.1 Heat Operator

The heat operator on Hermitian modular forms is defined by

$$\mathbb{D} = -\frac{1}{\pi^2}\left(\frac{\partial^2}{\partial\tau\partial\tau'} - \frac{\partial^2}{\partial z_1 \partial z_2}\right).$$

If $F \in M_k(\Gamma^2(\mathcal{O}), \det^{k/2})$ has Fourier expansion of the form

$$F(Z) = \sum_{\substack{n,m\in\mathbb{Z}, r\in\mathcal{O}^{\#} \\ nm-N(r)\geq 0}} A_F(n,r,m) q^n \zeta_1^r \zeta_2^{\overline{r}} (q')^m,$$

then the Fourier expansion of $\mathbb{D}(F)$ is given by

$$\mathbb{D}(F)(\tau, z_1, z_2, \tau') = \sum_{\substack{n,m\in\mathbb{Z}, r\in\mathcal{O}^{\#} \\ nm-N(r)\geq 0}} 4(nm - N(r)) A_F(n,r,m) q^n \zeta_1^r \zeta_2^{\overline{r}} (q')^m$$

1.7.2 Rankin-Cohen brackets

Martin and Sendheera [27] have defined Rankin-Cohen brackets for Hermitian modular forms. Rankin-Cohen brackets have been constructed for others automorphic forms also (see for example [2, 4, 5, 6, 7, 8, 15]). In this subsection, we recall the definition of Rankin-Cohen brackets for Hermitian modular forms of degree 2 over $\mathbb{Q}(i)$. We need the definition of first Rankin-Cohen bracket. Suppose that $F_1 \in M_{k_1}(\Gamma^2(\mathcal{O}), \det^{k_1/2})$ and $F_2 \in M_{k_2}(\Gamma^2(\mathcal{O}), \det^{k_2/2})$. We define the first Rankin-Cohen bracket of F_1, F_2 by

$$\begin{aligned}[F_1, F_2]_1 = (k_1-1)(k_2-1)\mathbb{D}(F_1F_2) - (k_2-1)(k_1+k_2-1)\mathbb{D}(F_1)F_2 \\ -(k_1-1)(k_1+k_2-1)F_1\mathbb{D}(F_2).\end{aligned}$$

Then we have $[F_1, F_2]_1 \in M_{k_1+k_2+2}(\Gamma^2(\mathcal{O}), \det^{(k_1+k_2+2)/2})$. We note down here that our definition of the first Rankin-Cohen bracket is slightly different from the definition of Martin and Senadheera. But up to some constant multiple both the definitions are same. We have the following lemma. The proof follows by a simple computation.

Lemma 1.7.4. *If* $F_1 \in M_{k_1}(\Gamma^2(\mathcal{O}), \det^{k_1/2}, \mathbb{Z}_{(p)})^{sym}$ *and* $F_2 \in M_{k_2}(\Gamma^2(\mathcal{O}), \det^{k_2/2}, \mathbb{Z}_{(p)})^{sym}$,

then

$$[F_1, F_2]_1 \in M_{k_1+k_2+2}(\Gamma^2(\mathcal{O}), \det^{(k_1+k_2+2)/2}, \mathbb{Z}_{(p)})^{sym}.$$

Chapter 2

Certain Congruences among automorphic functions

2.1 Introduction

There has been a great amount of research on the congruences of the Fourier coefficients of modular forms and in general of different automorphic functions. There are two kinds of congruences namely, $U(p)$ congruences and Ramanujan-type congruences which have attracted many mathematicians due to their various applications in number theory. Both $U(p)$ congruences and Ramanujan-type congruences are nice applications of the theory developed by Serre [38] and Swinnerton-Dyer [41] on modular forms modulo a prime p. $U(p)$ congruences involve Atkin's U-operator. On the other hand, Ramanujan-type congruences are certain kinds of congruences which were first studied by Ramanujan for the partition function $p(n)$. $U(p)$ congruences in elliptic modular forms were studied by Ahlgren and Ono [1], Elkies, Ono and Yang [17] and Guerzhoy [18]. Ramanujan-type congruences in elliptic modular forms were studied by Cooper, Wage and Wang [9], Dewar [12, 13] and Sinick [39]. $U(p)$ congru-

ences and Ramanujan-type congruences in Jacobi forms have been studied by Richter [33, 34] and Dewar and Richter [14] respectively by using the theory of Jacobi forms modulo p. $U(p)$ congruences and Ramanujan-type congruences in Siegel modular forms of degree 2 have been studied by Choi, Choie and Richter [3], and Dewar and Richter [14] respectively. Raum and Richter [32] have also studied $U(p)$ congruences in Siegel modular forms of arbitrary degree.

In this chapter we review certain results on $U(p)$ congruences and Ramanujan-type congruences in modular forms. We also review the generalization of these results to Jacobi forms and Siegel modular forms.

2.2 Modular forms modulo p

Throughout the thesis we assume that $p \geq 5$ is a prime. Let $\mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ at the prime ideal $p\mathbb{Z}$. Let $\mathbb{F}_p$ be the finite field $\mathbb{Z}/p\mathbb{Z}$ with p elements.

Let $f = \sum_{n\geq 0} a(f;n)q^n \in M_k(SL_2(\mathbb{Z}), \mathbb{Z}_{(p)})$. We define

$$\overline{f} = \sum_{n\geq 0} \overline{a}(f;n)q^n,$$

where $\overline{a}(f;n)$ is the reduction of $a(f;n)$ modulo $p\mathbb{Z}_{(p)}$ (also written as $a(f;n)$ modulo p). We call $\overline{f}$ as modular form reduced modulo p. Let $g = \sum_{n\geq 0} a(g;n)q^n \in M_{k'}(SL_2(\mathbb{Z}), \mathbb{Z}_{(p)})$. We say that $f \equiv g \pmod{p}$ (or $\overline{f} = \overline{g}$) if $a(f;n) \equiv a(g;n) \pmod{p}$ (or $\overline{a}(f;n) = \overline{a}(g;n)$) for all $n \geq 0$. We define the collection of modular forms of weight k reduced modulo p by

$$M_k(SL_2(\mathbb{Z}), \mathbb{F}_p) = \{\overline{f} \mid f \in M_k(SL_2(\mathbb{Z}), \mathbb{Z}_{(p)})\}.$$

We define the algebra of modular forms reduced modulo p over $\mathbb{F}_p$ by

$$M_*(SL_2(\mathbb{Z}), \mathbb{F}_p) = \sum_k M_k(SL_2(\mathbb{Z}), \mathbb{F}_p).$$

Swinnerton-Dyer [26, Theorem 7.4] have determined the structure of the algebra of modular forms modulo p. Using this structure, Swinnerton-Dyer [26, Theorem 7.5, Corollary] proved the following result.

Theorem 2.2.1. *Suppose that $f \in M_k(SL_2(\mathbb{Z}), \mathbb{Z}_{(p)})$ and $g \in M_{k'}(SL_2(\mathbb{Z}), \mathbb{Z}_{(p)})$ such that $f \equiv g \not\equiv 0 \pmod{p}$. Then $k \equiv k' \pmod{p-1}$. Moreover if $(g_k)_k$ is a finite family of modular forms with $g_k \in M_k(SL_2(\mathbb{Z}), \mathbb{Z}_{(p)})$ such that $\sum_k g_k \equiv 0 \pmod{p}$, then for each $\alpha \in \mathbb{Z}/(p-1)\mathbb{Z}$ we have*

$$\sum_{k \in \alpha + (p-1)\mathbb{Z}} g_k \equiv 0 \pmod{p}.$$

Definition 2.2.2. *Let $f : \mathcal{H} \to \mathbb{C}$ be a function which has Fourier expansion of the form*

$$f = \sum_{n \geq 0} a(f; n) q^n.$$

If $f \equiv g \pmod{p}$ for some $g \in M_k(SL_2(\mathbb{Z}), \mathbb{Z}_{(p)})$, we define the filtration of f modulo p by

$$u(f) = \inf\{l \mid g \in M_l(SL_2(\mathbb{Z}), \mathbb{Z}_{(p)}) \text{ and } \overline{f} = \overline{g}\}.$$

Using the structure of modular forms modulo p, Swinnerton-Dyer [26, Theorem 7.5] also proved the following result which is very useful to calculate the filtration of a modular form. Let $\mathbb{Z}_{(p)}[X, Y]$ be the polynomial ring in two variables X, Y with coefficients in $\mathbb{Z}_{(p)}$. Suppose that $F(X, Y) = \sum c_{a,b} X^a Y^b \in \mathbb{Z}_{(p)}[X, Y]$. We define

$\overline{F}(X,Y) = \sum \overline{c}_{a,b} X^a Y^b$, where $\overline{c}_{a,b}$ is the reduction of $c_{a,b}$ modulo p.

Lemma 2.2.3. *If $f \in M_k(SL_2(\mathbb{Z}), \mathbb{Z}_{(p)})$ then there exists a polynomial $F(X,Y) \in \mathbb{Z}_{(p)}[X,Y]$ such that*

$$f = F(E_4, E_6).$$

Let $A(X,Y) \in \mathbb{Z}_{(p)}[X,Y]$ be such that $E_{p-1} = A(E_4, E_6)$. Then $u(f) < k$ if and only if $\overline{A}(X,Y) \nmid \overline{F}(X,Y)$ in the polynomial ring $\mathbb{F}_p[X,Y]$.

We state the following result [26, Theorem 7.1, Theorem 7.3] which will be used in Chapter 3.

Lemma 2.2.4. *We have following congruence relations among Eisenstein series on $SL_2(\mathbb{Z})$.*

- $E_{p-1} \equiv 1 \pmod{p}$.
- $E_{p+1} \equiv E_2 \pmod{p}$.
- *Let $A, B \in \mathbb{Z}_{(p)}[X,Y]$ be such that $E_{p-1} = A(E_4, E_6)$ and $E_{p+1} = B(E_4, E_6)$ respectively. Then the polynomials $\overline{A}(X,Y)$ and $\overline{B}(X,Y)$ are relatively prime.*

It is easy to check that if $f \in M_k(SL_2(\mathbb{Z}), \mathbb{Z}_{(p)})$ then $\theta(f)$ is not a modular form. Swinnerton-Dyer proved that $\theta(f)$ is congruent to a modular form modulo p and hence we can calculate its filtration modulo p. Using the structure of the algebra of modular forms [26, Theorem 7.4], Swinnerton-Dyer [26, Theorem 8.1] proved the following result on the filtration of $\theta(f)$.

Theorem 2.2.5. *If $f \in M_k(SL_2(\mathbb{Z}), \mathbb{Z}_{(p)})$ then there exists $g \in M_{k'}(SL_2(\mathbb{Z}), \mathbb{Z}_{(p)})$ for some integer k' such that $\overline{\theta(f)} = \overline{g}$. If $f \not\equiv 0 \pmod{p}$ then*

$$u(\theta(f)) \leq u(f) + p + 1,$$

with equality if and only if $p \nmid u(f)$.

2.2.1 $U(p)$ congruences and Ramanujan-type congruences in modular forms

Let $f \in M_k(SL_2(\mathbb{Z}), \mathbb{Z}_{(p)})$ with Fourier expansion

$$f(\tau) = \sum_{n \geq 0} a(f; n) q^n.$$

Using Fermat-Little theorem we see that $\theta^p(f) \equiv \theta(f) \pmod{p}$. We call the finite sequence $\overline{\theta(f)}, \cdots, \overline{\theta^{p-1}(f)}$ the theta cycle or the heat cycle of f. Tate initiated the study of such theta cycle, which was based on the study of the filtrations of the elements of the theta cycle. The book of Ono [30] is a good reference for this section.

Definition 2.2.6. *Suppose* $f \in M_k(SL_2(\mathbb{Z}), \mathbb{Z}_{(p)})$ *has Fourier expansion*

$$f = \sum_{n \geq 0} a(f; n) q^n.$$

The Atkin's $U(p)$ *operator on* f *is defined by*

$$f \mid U(p) = \sum_{\substack{n \geq 0 \\ p \mid n}} a(f; n) q^n.$$

We say that f *has* $U(p)$ *congruence if* $f \mid U(p) \equiv 0 \pmod{p}$.

Tate's theory of theta cycle (see [21, Section 7]) yields the following equivalent conditions for the existence of $U(p)$ congruence in terms of filtration.

Theorem 2.2.7. *Let* $f \in M_k(SL_2(\mathbb{Z}), \mathbb{Z}_{(p)})$ *with* $f \not\equiv 0 \pmod{p}$.

- *Then f has $U(p)$ congruence if and only if*

$$\theta^{p-1}(f) \equiv f \pmod{p}.$$

- *If $p > k$, then*

$$u(\theta^{p-k+1}(f)) = \begin{cases} 2p-k+2 & \text{if } f \mid U(p) \not\equiv 0 \pmod{p}, \\ p-k+3 & \text{if } f \mid U(p) \equiv 0 \pmod{p}. \end{cases}$$

Definition 2.2.8. *Suppose $f \in M_k(SL_2(\mathbb{Z}), \mathbb{Z}_{(p)})$ has Fourier expansion*

$$f = \sum_{n \geq 0} a(f;n) q^n.$$

We say that f has Ramanujan-type congruence at $b \not\equiv 0 \pmod{p}$ if $a(f;n) \equiv 0 \pmod{p}$ whenever $n \equiv b \pmod{p}$.

Dewar [12, Lemma 2.4, Theorem 1.1] has proved the following result on the existence of Ramanujan-type congruences.

Theorem 2.2.9. *Suppose $f \in M_k(SL_2(\mathbb{Z}), \mathbb{Z}_{(p)})$ is such that $f \not\equiv 0 \pmod{p}$.*

- *Then f has Ramanujan-type congruence at $b \pmod{p}$ if and only if*

$$\theta^{\frac{p+1}{2}}(f) \equiv -\left(\frac{b}{p}\right)\theta(f) \pmod{p},$$

  where $\left(\frac{\bullet}{p}\right)$ is the Legendre symbol.

- *Let $r \geq 0$ and $s, t \in \mathbb{Z}$. If $E_2^r E_4^s E_6^t$ has Ramanujan-type congruence at $b \pmod{p}$, then $p \leq 2r + 8|s| + 12|t| + 21$.*

In Chapter 4 we shall generalize Theorem 2.2.7 and Theorem 2.2.9 to the case of Hermitian Jacobi forms and Hermitian modular forms. The main ingredients required to prove Theorem 2.2.7 and Theorem 2.2.9 are Theorem 2.2.1 and Theorem 2.2.5.

2.3 Jacobi forms modulo p

Sofer has generalized Theorem 2.2.1 to the case of Jacobi forms of integer index. The result of Sofer [40, Lemma 2.1] is the following.

Theorem 2.3.1. *Suppose $\phi \in J_{k,m}(\Gamma^1, \mathbb{Z}_{(p)})$ and $\psi \in J_{k',m'}(\Gamma^1, \mathbb{Z}_{(p)})$ are such that $\phi \equiv \psi \not\equiv 0 \pmod{p}$. Then $m = m'$ and $k \equiv k' \pmod{p-1}$. Moreover, if m is fixed and $(\phi_k)_k$ is a finite family of Jacobi forms with $\phi_k \in J_{k,m}(\Gamma^1, \mathbb{Z}_{(p)})$ such that $\sum_k \phi_k \equiv 0 \pmod{p}$, then for all $\alpha \in \mathbb{Z}/(p-1)\mathbb{Z}$ we have*

$$\sum_{k \in \alpha + (p-1)\mathbb{Z}} \phi_k \equiv 0 \pmod{p}.$$

Definition 2.3.2. *Let $\phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ be a function which has Fourier expansion of the form*

$$\phi(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z} \\ 4nm - r^2 \geq 0}} c(\phi; n, r) q^n \zeta^r.$$

If $\phi \equiv \psi \pmod{p}$ for some $\psi \in J_{k,m}(\Gamma^1, \mathbb{Z}_{(p)})$, we define the filtration of ϕ modulo p by

$$v(\phi) = \inf\{l \mid \psi \in J_{l,m}(\Gamma^1, \mathbb{Z}_{(p)}) \text{ and } \overline{\phi} = \overline{\psi}\}.$$

The following result is by Richter [34, Proposition 2]. This result generalizes Theorem 2.2.5 to the case of Jacobi forms of integer index.

Theorem 2.3.3. *Let $\phi \in J_{k,m}(\Gamma^1, \mathbb{Z}_{(p)})$. There exists $\psi \in J_{k+p+1,m}(\Gamma^1, \mathbb{Z}_{(p)})$ such*

that $K_m(\phi) \equiv \psi \pmod{p}$. *If* $\phi \not\equiv 0 \pmod{p}$, *then*

$$v(K_m(\phi)) \leq v(\phi) + p + 1,$$

with equality if and only if $p \nmid (2v(\phi) - 1)m$.

Raum and Richter have further generalized Theorem 2.3.1 and Theorem 2.3.3 to the case of Jacobi forms of matrix index [32, Proposition 2.6, Proposition 2.15].

2.3.1 $U(p)$ Congruences and Ramanujan-type congruences in Jacobi forms

Definition 2.3.4. *Suppose* $\phi \in J_{k,m}(\Gamma^1, \mathbb{Z}_{(p)})$ *has Fourier expansion*

$$\phi(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z} \\ 4nm - r^2 \geq 0}} c(\phi; n, r) q^n \zeta^r.$$

The Atkin's $U(p)$ *operator on* ϕ *is defined by*

$$\phi \mid U(p) = \sum_{\substack{n,r \in \mathbb{Z} \\ 4nm - r^2 \geq 0 \\ p \mid 4nm - r^2}} c(\phi; n, r) q^n \zeta^r.$$

We say that ϕ *has* $U(p)$ *congruence if* $\phi \mid U(p) \equiv 0 \pmod{p}$.

The following result which was proved by Richter [33, Proposition 3] characterizes $U(p)$ congruences in terms of filtrations of Jacobi forms of integer index.

Theorem 2.3.5. *Let* $\phi \in J_{k,m}(\Gamma^1, \mathbb{Z}_{(p)})$ *be such that* $\phi \not\equiv 0 \pmod{p}$. *Assume*

that $p \nmid m$. *If* $p > 2k-5$, *then* $\phi \mid U(p) \not\equiv 0 \pmod{p}$. *If* $k < p < 2k-5$, *then*

$$v(K_m^{\frac{3p+3}{2}-k}(\phi)) = \begin{cases} 3p-k+3 & \text{if } \phi \mid U(p) \not\equiv 0 \pmod{p}, \\ 2p-k+4 & \text{if } \phi \mid U(p) \equiv 0 \pmod{p}. \end{cases}$$

Raum and Richter have generalized Theorem 2.3.5 to the case of Jacobi forms of matrix index [32, Theorem 2.17].

Definition 2.3.6. *Suppose* $\phi \in J_{k,m}(\Gamma^1, \mathbb{Z}_{(p)})$ *has Fourier expansion*

$$\phi(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z} \\ 4nm - r^2 \geq 0}} c(\phi; n, r) q^n \zeta^r.$$

We say that ϕ *has Ramanujan-type at* $b \not\equiv 0 \pmod{p}$ *if* $c(\phi; n, r) \equiv 0 \pmod{p}$ *whenever* $4nm - r^2 \equiv b \pmod{p}$.

We have the following result of Dewar and Richter [14, Proposition 2.7] on the non-existence of Ramanujan-type congruences in Jacobi forms of integer index.

Theorem 2.3.7. *Let* $\phi \in J_{k,m}(\Gamma^1, \mathbb{Z}_{(p)})$ *be such that* $K_m(\phi) \not\equiv 0 \pmod{p}$. *If* $k \geq 4$, $p > k$ *and* $p \nmid m$, *then* ϕ *does not have Ramanujan-type congruence at* $b \pmod{p}$.

2.4 Siegel modular forms modulo p

Nagaoka has generalized Theorem 2.2.1 to the case of Siegel modular forms of degree 2. The result of Nagaoka [29, Corollary 4.6] is the following.

Theorem 2.4.1. *Suppose that* $F \in M_k(\Gamma_4, \mathbb{Z}_{(p)})$ *and* $G \in M_{k'}(\Gamma_4, \mathbb{Z}_{(p)})$ *are such that* $F \equiv G \not\equiv 0 \pmod{p}$. *Then* $k \equiv k' \pmod{p-1}$. *Moreover if* $(G_k)_k$ *is a finite*

family of modular forms with $G_k \in M_k(\Gamma_4, \mathbb{Z}_{(p)})$ *such that* $\sum_k G_k \equiv 0 \pmod{p}$, *then for each* $\alpha \in \mathbb{Z}/(p-1)\mathbb{Z}$ *we have*

$$\sum_{k \in \alpha + (p-1)\mathbb{Z}} G_k \equiv 0 \pmod{p}.$$

Definition 2.4.2. *Let* $F : \mathbb{H}_2 \to \mathbb{C}$ *be a function which has Fourier expansion of the form*

$$F = \sum_{\substack{n,m,r \in \mathbb{Z} \\ 4nm - r^2 \geq 0}} A_F(n,r,m) q^n \zeta^r (q')^m.$$

If $F \equiv G \pmod{p}$ *for some* $G \in M_k(\Gamma_4, \mathbb{Z}_{(p)})$, *we define the filtration of* F *modulo* p *by*

$$w(F) = \inf\{l \mid G \in M_l(\Gamma_4, \mathbb{Z}_{(p)}) \text{ and } \overline{F} = \overline{G}\}.$$

The following result is by Choi, Choie and Richter [3, Proposition 4]. This result generalizes Theorem 2.2.5 to the case of Siegel modular forms of degree 2.

Theorem 2.4.3. *Let* $F \in M_k(\Gamma_4, \mathbb{Z}_{(p)})$. *There exists* $G \in M_{k+p+1}(\Gamma_4, \mathbb{Z}_{(p)})$ *such that* $\mathbb{P}(F) \equiv G \pmod{p}$. *Suppose that there exists a Fourier-Jacobi coefficient* $\phi_{k,m}$ *of* F *with* $p \nmid m$ *such that* $w(F) = v(\phi_{k,m})$. *Then*

$$w(\mathbb{P}(F)) \leq w(F) + p + 1,$$

with equality if and only if $p \nmid (2w(F) - 1)$.

Raum and Richter have further generalized Theorem 2.4.1 and Theorem 2.4.3 to Siegel modular forms of degree > 2 [32, Proposition 3.1, Proposition 3.7].

2.4.1 $U(p)$ congruences and Ramanujan-type congruences in Siegel modular forms

Definition 2.4.4. *Suppose $F \in M_k(\Gamma_4, \mathbb{Z}_{(p)})$ has Fourier expansion*

$$F = \sum_{\substack{n,r,m\in\mathbb{Z} \\ 4nm-r^2\geq 0}} A_F(n,r,m)q^n\zeta^r(q')^m.$$

The Atkin's $U(p)$ operator on F is defined by

$$F \mid U(p) = \sum_{\substack{n,r,m\in\mathbb{Z} \\ 4nm-r^2\geq 0 \\ p\mid 4nm-r^2}} A_F(n,r,m)q^n\zeta^r(q')^m.$$

We say that F has $U(p)$ congruence if $F \mid U(p) \equiv 0 \pmod{p}$.

We have the following result of Choi, Choie and Richter [3, Theorem 1] which gives a characterization of $U(p)$ congruences in terms of filtrations of Siegel modular forms of degree 2.

Theorem 2.4.5. *Let*

$$F = \sum_{\substack{n,r,m\in\mathbb{Z} \\ 4nm-r^2\geq 0}} A_F(n,r,m)q^n\zeta^r(q')^m \in M_k(\Gamma_4, \mathbb{Z}_{(p)}).$$

Assume that there exists integers n, r, m with $p \nmid nm$ and $A_F(n,r,m) \not\equiv 0 \pmod{p}$. If $p > 2k-5$, then $F \mid U(p) \not\equiv 0 \pmod{p}$. If $k < p < 2k-5$, then

$$w(\mathbb{P}^{\frac{3p+3}{2}-k}(\phi)) = \begin{cases} 3p-k+3 & \text{if } F \mid U(p) \not\equiv 0 \pmod{p}, \\ 2p-k+4 & \text{if } F \mid U(p) \equiv 0 \pmod{p}. \end{cases}$$

The above theorem has been generalized by Raum and Richter to the case of

Siegel modular forms of degree > 2 [32, Theorem 1.4].

Definition 2.4.6. *Suppose $F \in M_k(\Gamma_4, \mathbb{Z}_{(p)})$ has Fourier expansion*

$$F = \sum_{\substack{n,r,m \in \mathbb{Z} \\ 4nm - r^2 \geq 0}} A_F(n,r,m) q^n \zeta^r (q')^m.$$

We say that F has Ramanujan-type congruence at $b \not\equiv 0 \pmod{p}$ if $A_F(n,r,m) \equiv 0 \pmod{p}$ whenever $4nm - r^2 \equiv b \pmod{p}$.

We have the following result by Dewar and Richter [14, Theorem 1.2] on the non-existence of Ramanuujan-type congruences in Siegel modular forms of degree 2.

Theorem 2.4.7. *Let*

$$F = \sum_{\substack{n,r,m \in \mathbb{Z} \\ 4nm - r^2 \geq 0}} A_F(n,r,m) q^n \zeta^r (q')^m \in M_k(\Gamma_4, \mathbb{Z}_{(p)}).$$

If $p > k$, $p \neq 2k-1$ and there exist integers n, r, m with $p \nmid gcd(n,m)$ and $A_F(n,r,m) \not\equiv 0 \pmod{p}$, then F does not have Ramanujan-type congruence at $b \pmod{p}$.

Chapter 3

Hermitian Jacobi and Hermitian modular forms modulo p

3.1 Introduction

Serre [38] and Swinnerton-Dyer [41] introduced the theory of modular forms modulo a prime p. This theory has a great impact in studying the congruences of Fourier coefficients of modular forms. The theory of Jacobi forms of integer index modulo p has been studied by Richter [33, 34] and the theory of Jacobi forms of matrix index modulo p has been studied by Raum and Richter [32]. The theory of Siegel modular forms of degree 2 modulo p has been studied by Nagaoka [29] and the theory of Siegel modular forms of general degree modulo p has been studied by Raum and Richter [32]. The theory of Hermitian Jacobi forms over $\mathbb{Q}(i)$ modulo p was initiated by Richter and Senadheera [36]. But because of the lack of some structural property of Hermitian Jacobi forms of arbitrary index, they have studied only the mod p theory of Hermitian Jacobi forms of index 1. In this chapter we study the mod p theory of Hermitian Jacobi forms of arbitrary index over $\mathbb{Q}(i)$. Using the results of Kikuta and

Nagaoka [23, 24] on the structure of symmetric Hermitian modular forms of degree 2 modulo p and our results on Hermitian Jacobi forms modulo p, we also study the mod p theory of Hermitian modular forms of degree 2 over $\mathbb{Q}(i)$.

3.2 Hermitian Jacobi forms modulo p

Let $\phi \in HJ^{\delta}_{k,m}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$ with Fourier expansion

$$\phi(\tau, z_1, z_2) = \sum_{\substack{n\in\mathbb{Z}, r\in\mathcal{O}^{\#} \\ nm-N(r)\geq 0}} c(\phi; n, r) q^n \zeta_1^r \zeta_2^{\overline{r}}.$$

We define the reduction $\overline{\phi}$ of ϕ modulo p by

$$\overline{\phi}(\tau, z_1, z_2) = \sum_{\substack{n\in\mathbb{Z}, r\in\mathcal{O}^{\#} \\ nm-N(r)\geq 0}} \overline{c}(\phi; n, r) q^n \zeta_1^r \zeta_2^{\overline{r}},$$

where $\overline{c}(\phi; n, r)$ is the reduction of $c(\phi; n, r)$ modulo $p\mathbb{Z}_{(p)}$ (also written as $c(\phi; n, r)$ modulo p). Let $\psi \in HJ^{\delta'}_{k',m'}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$ with Fourier expansion

$$\psi(\tau, z_1, z_2) = \sum_{\substack{n\in\mathbb{Z}, r\in\mathcal{O}^{\#} \\ nm'-N(r)\geq 0}} c(\psi; n, r) q^n \zeta_1^r \zeta_2^{\overline{r}}.$$

We say that $\phi \equiv \psi \pmod{p}$ (or $\overline{\phi} = \overline{\psi}$) if $c(\phi; n, r) \equiv c(\psi; n, r) \pmod{p}$ (or $\overline{c}(\phi; n, r) = \overline{c}(\psi; n, r)$) for all n, r. Let

$$HJ^{\delta}_{k,m}(\Gamma^J(\mathcal{O}), \mathbb{F}_p) = \{\overline{\phi} : \phi \in HJ^{\delta}_{k,m}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})\},$$

be the collection of all Hermitian Jacobi forms of weight k, index m and parity δ reduced modulo p. It can be seen that the space $HJ^{\delta}_{k,m}(\Gamma^J(\mathcal{O}), \mathbb{F}_p)$ forms a vector

space over $\mathbb{F}_p$. We have the following result [28, Theorem 3.1] which is an extension of Theorem 2.2.1 to the case of Hermitian Jacobi forms of integer index over $\mathbb{Q}(i)$.

Theorem 3.2.1. *If $\phi \in HJ^{\delta}_{k,m}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$ and $\psi \in HJ^{\delta'}_{k',m'}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$ are such that $\phi \equiv \psi \not\equiv 0 \pmod{p}$, then $m = m'$ and $k \equiv k' \pmod{p-1}$. Moreover, if m is fixed and $(\psi_k)_k$ is a finite family of Hermitian Jacobi forms with $\psi_k \in HJ^{\delta_k}_{k,m}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$ such that $\sum_k \psi_k \equiv 0 \pmod{p}$, then for each $\alpha \in \mathbb{Z}/(p-1)\mathbb{Z}$ we have*

$$\sum_{k \in \alpha + (p-1)\mathbb{Z}} \psi_k \equiv 0 \pmod{p}.$$

The aim of the remaining part of this section is to prove Theorem 3.2.1. We follow the idea of Raum and Richter [32, Proposition 2.6] to prove Theorem 3.2.1.

Let $\psi \in HJ_{k,m}(\Gamma^J(\mathcal{O}))$. For $s \in \mathcal{O}$ and $z \in \mathbb{C}$, we define

$$\psi[s](\tau, z) = \psi(\tau, sz, \overline{s}z). \tag{3.2.1}$$

Using the transformation property and Fourier expansion of ψ, we observe that $\psi[s] \in J_{k,N(s)m}(\Gamma^1)$. If the Fourier expansion of ψ is

$$\psi(\tau, z_1, z_2) = \sum_{\substack{n \in \mathbb{Z}, r \in \mathcal{O}^{\#} \\ nm - N(r) \geq 0}} c(\psi; n, r) q^n \zeta_1^r \zeta_2^{\overline{r}},$$

then the Fourier expansion of $\psi[s]$ is given by

$$\psi[s](\tau, z) = \sum_{\substack{n \in \mathbb{Z}, r \in \mathcal{O}^{\#} \\ nm - N(r) \geq 0}} c(\psi; n, r) q^n \zeta^{2\Re(sr)} = \sum_{\substack{n \in \mathbb{Z}, a \in \mathbb{Z} \\ a^2 \leq 4N(s)mn}} c(\psi[s]; n, a) q^n \zeta^a,$$

where $\zeta = e(z)$ and

$$c(\psi[s]; n, a) = \sum_{\substack{r \in \mathcal{O}^{\#}, nm - N(r) \geq 0 \\ 2\Re(sr) = a}} c(\psi; n, r). \tag{3.2.2}$$

Therefore, we observe that if $\psi \in HJ^{\delta}_{k,m}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$, then $\psi[s] \in J_{k,N(s)m}(\Gamma^1, \mathbb{Z}_{(p)})$.

Proposition 3.2.2. *Let $(\psi_k)_k$ be a finite family of Hermitian Jacobi forms with $\psi_k \in HJ^{\delta_k}_{k,m}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$. If $0 \leq n_0$ is a fixed integer, then there exists an element $s \in \mathcal{O}$ such that for all $n \leq n_0$ and $r \in \mathcal{O}^{\#}$ satisfying $nm - N(r) \geq 0$, we have*

$$c(\psi_k[s]; n, 2\Re(sr)) = c(\psi_k; n, r). \tag{3.2.3}$$

Moreover, if $\psi_k \not\equiv 0 \pmod{p}$ for all k, then there exists an element $s \in \mathcal{O}$ such that $\psi_k[s] \not\equiv 0 \pmod{p}$ for all k.

Proof. We follow the idea of Raum and Richter [32, Proposition 2.5]. We choose an integer b such that

$$b > \max\left\{ |a_j| : r = \frac{a_1}{2} + \frac{a_2}{2} i \in \mathcal{O}^{\#}, n_0 m - N(r) \geq 0 \right\}. \tag{3.2.4}$$

Let $s = 1 + 4bi$. Assume r_1, $r_2 \in \mathcal{O}^{\#}$ and $n \geq 0$ is an integer such that $n \leq n_0$ and $nm - N(r_j) \geq 0$ for $j = 1, 2$. We shall prove that $2\Re(sr_1) = 2\Re(sr_2)$ if and only if $r_1 = r_2$. Then by (3.2.2), (3.2.3) follows. Let

$$r_1 = \frac{a_1}{2} + \frac{a_2}{2} i \quad \text{and} \quad r_2 = \frac{b_1}{2} + \frac{b_2}{2} i,$$

where a_1, a_2, b_1, b_2 are integers. If $r_1 = r_2$ then $2\Re(sr_1) = 2\Re(sr_2)$ follows trivially. Conversely, if $2\Re(sr_1) = 2\Re(sr_2)$, then $a_1 - b_1 = 4b(a_2 - b_2)$. Since $n_0 m - N(r_j) \geq 0$

for $j = 1, 2$, by (3.2.4) we have

$$|a_2 - b_2| = \frac{1}{4b}|a_1 - b_1| < \frac{1}{2}.$$

Hence we deduce that $r_1 = r_2$. In the second part of the theorem we assume that $\psi_k \not\equiv 0 \pmod{p}$ for all k. For each k, let n_k be the smallest integer such that there exists $r_k \in \mathcal{O}^{\#}$ with $c(\psi_k; n_k, r_k) \not\equiv 0 \pmod{p}$. Let n_0 be an integer such that $n_0 > \max\{n_k\}$. Then by the first assertion of the proposition, there exists $s \in \mathcal{O}$ such that for all $n \leq n_0$ and $r \in \mathcal{O}^{\#}$ satisfying $nm - N(r) \geq 0$, we have

$$c(\psi_k[s]; n, 2\Re(sr)) = c(\psi_k; n, r)$$

for all k. In particular, we have $c(\psi_k[s], n_k, 2\Re(sr_k)) \not\equiv 0 \pmod{p}$ for all k. Hence $\psi_k[s] \not\equiv 0 \pmod{p}$ for all k. □

Proof of Theorem 3.2.1

To prove $m = m'$, we follow the idea of Sofer [40, Lemma 2.1]. Let $\beta, \gamma \in \mathcal{O}^{\#}$ with $\beta \neq 0$. Replacing z_1 by $z_1 + \beta\tau + \gamma$ and z_2 by $z_2 + \overline{\beta}\tau + \overline{\gamma}$ in $\phi \equiv \psi \pmod{p}$ and using the transformation property (1.4.2), we have

$$(q^{|\beta|^2}\zeta_1^{\overline{\beta}}\zeta_2^{\beta})^{-m}\phi \equiv (q^{|\beta|^2}\zeta_1^{\overline{\beta}}\zeta_2^{\beta})^{-m'}\psi \pmod{p}. \tag{3.2.5}$$

Therefore we have

$$(q^{|\beta|^2}\zeta_1^{\overline{\beta}}\zeta_2^{\beta})^{-m}\phi \equiv (q^{|\beta|^2}\zeta_1^{\overline{\beta}}\zeta_2^{\beta})^{-m'}\phi \pmod{p}$$

for every $\beta \in \mathcal{O}^{\#}$ and hence $m = m'$. The congruence $k = k' \pmod{p-1}$ follows from the second assertion of the theorem. Therefore we shall prove the second

assertion of the theorem. For a fixed m, let $(\psi_k)_k$ be a family of Hermitian Jacobi forms with $\psi_k \in HJ^{\delta_k}_{k,m}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$. For any fixed $\rho \in \mathcal{O}$, we have $\psi_k[\rho] \in J_{k,N(\rho)m}(\Gamma^1, \mathbb{Z}_{(p)})$. If $\sum_k \psi_k \equiv 0 \pmod{p}$, then

$$\sum_k \psi_k[\rho] \equiv 0 \pmod{p}.$$

Then by Theorem 2.3.1 we have

$$\sum_{k \in \alpha + (p-1)\mathbb{Z}} \psi_k[\rho] \equiv 0 \pmod{p}. \tag{3.2.6}$$

By Proposition 3.2.2, if $0 \leq n_0$ is a fixed integer, then there exists an element $s \in \mathcal{O}$ such that for all $n \leq n_0$ and $r \in \mathcal{O}^{\#}$ satisfying $nm - N(r) \geq 0$, we have $c(\psi_k[s]; n, 2\Re(sr)) = c(\psi_k; n, r)$ for all k. Therefore by (3.2.6), for arbitrary n and r with $r \in \mathcal{O}^{\#}$ and $nm - N(r) \geq 0$, we have

$$\sum_{k \in \alpha + (p-1)\mathbb{Z}} c(\psi_k; n, r) \equiv 0 \pmod{p},$$

and hence we have

$$\sum_{k \in \alpha + (p-1)\mathbb{Z}} \psi_k \equiv 0 \pmod{p}.$$

3.3 Isomorphism between Jacobi forms

We prove the following isomorphism [28, Theorem 2.3] which is one of the main ingredients in studying the mod p theory of Hermitian Jacobi forms of integer index over $\mathbb{Q}(i)$.

Theorem 3.3.1. *For an integer $m \geq 1$, let B denote the matrix $\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$. Then*

the space $J^1_{k,m}(\Gamma^1(\mathcal{O}))$ is isomorphic to the space $J_{k,B}(\Gamma^2)$ as a vector space over $\mathbb{C}$.

Proof. For $\phi(\tau, z_1, z_2) \in J^1_{k,m}(\Gamma^1(\mathcal{O}))$, define

$$\hat{\phi}(\tau, z_1, z_2) = \phi(\tau, z_1 + iz_2, z_1 - iz_2).$$

Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and let $(z_1, z_2)^t \in \mathbb{C}^2$.

$$\begin{aligned}\left(\hat{\phi} \mid_{k,B} g\right)(\tau, z_1, z_2) &= (c\tau + d)^{-k} e\left(-\frac{mc(z_1^2 + z_2^2)}{c\tau + d}\right) \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z_1 + iz_2}{c\tau + d}, \frac{z_1 - iz_2}{c\tau + d}\right) \\ &= (\phi \mid_{k,m} g)(\tau, z_1 + iz_2, z_1 - iz_2) = \phi(\tau, z_1 + iz_2, z_1 - iz_2) = \hat{\phi}(\tau, z_1, z_2).\end{aligned}$$

Therefore $\hat{\phi}$ satisfy (1.3.1). Let $\lambda = (\lambda_1, \lambda_2)^t$, $\mu = (\mu_1, \mu_2)^t \in \mathbb{Z}^2$.

$$\begin{aligned}\left(\hat{\phi} \mid_B [\lambda, \mu]\right)(\tau, z_1, z_2) &= e\left(m(\lambda_1^2 + \lambda_2^2)\tau + 2m(\lambda_1 z_1 + \lambda_2 z_2)\right) \hat{\phi}(\tau, z_1 + \lambda_1\tau + \mu_1, z_2 + \lambda_2\tau + \mu_2) \\ &= (\phi \mid_m [\lambda, \mu])(\tau, z_1 + iz_2, z_1 - iz_2) = \phi(\tau, z_1 + iz_2, z_1 - iz_2) = \hat{\phi}(\tau, z_1, z_2).\end{aligned}$$

Therefore $\hat{\phi}$ satisfy (1.3.2). Suppose that the Fourier expansion of ϕ is given by

$$\phi(\tau, z_1, z_2) = \sum_{\substack{n \in \mathbb{Z}, r \in \mathcal{O}^\# \\ nm - N(r) \geq 0}} c(n, r) e(n\tau + rz_1 + \overline{r} z_2).$$

Then

$$\hat{\phi}(\tau, z_1, z_2) = \phi(\tau, z_1 + iz_2, z_1 - iz_2) = \sum_{\substack{n \in \mathbb{Z}, r \in \mathcal{O}^\# \\ nm - N(r) \geq 0}} c(n, r) e(n\tau + (z_1 + iz_2)r + (z_1 - iz_2)\overline{r}).$$

Let $r = \frac{\alpha}{2} + i\frac{\beta}{2}$, where $\alpha, \beta \in \mathbb{Z}$. Then define $s = (\alpha, -\beta)^t \in \mathbb{Z}^2$. The corre-

spondence $r = \frac{\alpha}{2} + i\frac{\beta}{2} \mapsto s = (\alpha, -\beta)^t$ from $\mathcal{O}^\#$ to $\mathbb{Z}^2$ is bijective. Therefore we have

$$\hat{\phi}(\tau, z_1, z_2) = \sum_{\substack{n\in\mathbb{Z}, r\in\mathcal{O}^\# \\ mn-N(r)\geq 0}} c(n,r)e(n\tau+\alpha z_1-\beta z_2) = \sum_{\substack{n\in\mathbb{Z}, s\in\mathbb{Z}^2 \\ 4\det(B)n-B[s]\geq 0}} c(n,r)e(n\tau+\alpha z_1-\beta z_2).$$

Thus $\hat{\phi}$ has a Fourier expansion of the form given in (1.3.3). Therefore the map

$$\Theta_1 : J^1_{k,m}(\Gamma^1(\mathcal{O})) \to J_{k,B}(\Gamma^2)$$

defined by

$$\phi(\tau, z_1, z_2) \mapsto \phi(\tau, z_1 + iz_2, z_1 - iz_2)$$

is a well-defined linear map. For $\psi \in J_{k,B}(\Gamma^2)$, define

$$\hat{\psi}(\tau, z_1, z_2) = \psi\left(\tau, \frac{z_1+z_2}{2}, \frac{z_1-z_2}{2}\right)$$

By doing a similar calculation one proves that $\hat{\psi}$ satisfies the transformation property (1.4.1) with $\epsilon = 1$ and the transformation property (1.4.2). It also has a Fourier expansion of the form (1.4.3). Therefore the map

$$\Theta_2 : J_{k,B}(\Gamma^2) \to J^1_{k,m}(\Gamma^1(\mathcal{O}))$$

is a well defined linear map. Now it can be easily checked that $\Theta_2 \circ \Theta_1 = I_1$ and $\Theta_1 \circ \Theta_2 = I_2$, where I_1 and I_2 are the identity maps on the vector spaces $J^1_{k,m}(\Gamma^1(\mathcal{O}))$ and $J_{k,B}(\Gamma^2)$ respectively. □

Next we note down some immediate consequences of the above theorem.

Corollary 3.3.2. *The space $J^1_{*,m}(\Gamma^1(\mathcal{O}))$ is isomorphic to $J_{*,B}(\Gamma^2)$ as a module*

over $M_*(SL_2(\mathbb{Z}))$.

Corollary 3.3.3. *$J^1_{k,m}(\Gamma^1(\mathcal{O}), \mathbb{Z}_{(p)})$ is isomorphic to $J_{k,B}(\Gamma^2, \mathbb{Z}_{(p)})$ as modules over $\mathbb{Z}_{(p)}$. Further, the space $J^1_{*,m}(\Gamma^1(\mathcal{O}), \mathbb{Z}_{(p)})$ is isomorphic to $J_{*,B}(\Gamma^2, \mathbb{Z}_{(p)})$ as a module over $M_*(SL_2(\mathbb{Z}), \mathbb{Z}_{(p)})$.*

The structure of Jacobi forms with matrix index has been studied by Raum and Richter [32]. We now state the following two results on the structure of Jacobi forms of matrix index $B = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$. These results are particular cases of three results of Raum and Richter [32, Theorem 2.8, Proposition 2.11, Theorem 2.14].

Lemma 3.3.4. *The space $J_{*,B}(\Gamma^2, \mathbb{Z}_{(p)})$ is a free module over $M_*(SL_2(\mathbb{Z}), \mathbb{Z}_{(p)})$of rank $4m^2$. Moreover, it has a basis $\{\varphi_1 \cdots, \varphi_{4m^2}\}$ such that $\varphi_i \in J_{k_i,B}(\Gamma^2, \mathbb{Z}_{(p)})$ for some integer k_i for $1 \leq i \leq 4m^2$.*

Lemma 3.3.5. *Let $\{\varphi_1, \cdots, \varphi_{4m^2}\}$ be the basis of $J_{*,B}(\Gamma^2, \mathbb{Z}_{(p)})$ given in the previous lemma. If $\phi = \sum_{i=1}^{4m^2} f_i\varphi_i \in J_{k,B}(\Gamma^2, \mathbb{Z}_{(p)})$ with $f_i \in M_{k-k_i}(SL_2(\mathbb{Z}), \mathbb{Z}_{(p)})$ and $\psi = \sum_{i=1}^{4m^2} g_i\varphi_i \in J_{k',B}(\Gamma^2, \mathbb{Z}_{(p)})$ with $g_i \in M_{k'-k_i}(SL_2(\mathbb{Z}), \mathbb{Z}_{(p)})$ are such that $0 \not\equiv \phi \equiv \psi \pmod{p}$, then $f_i \equiv g_i \pmod{p}$ for all $1 \leq i \leq 4m^2$.*

Now applying the isomorphism stated in Corollary 3.3.3, Lemma 3.3.4 and Lemma 3.3.5, we get the following immediate result.

Corollary 3.3.6. *The space $J^1_{*,m}(\Gamma^1(\mathcal{O}), \mathbb{Z}_{(p)})$ is a free module over $M_*(SL_2(\mathbb{Z}), \mathbb{Z}_{(p)})$ of rank $4m^2$. This space has a basis $\{\Upsilon_1, \cdots, \Upsilon_{4m^2}\}$ such that $\Upsilon_i \in J^1_{k_i,m}(\Gamma^1(\mathcal{O}), \mathbb{Z})$ for some integer k_i with $1 \leq i \leq 4m^2$. Moreover, if $\phi = \sum_{i=1}^{4m^2} f_i\Upsilon_i \in J^1_{k,m}(\Gamma^1(\mathcal{O}), \mathbb{Z}_{(p)})$ with $f_i \in M_{k-k_i}(SL_2(\mathbb{Z}), \mathbb{Z}_{(p)})$ and $\psi = \sum_{i=1}^{4m^2} g_i\Upsilon_i \in J^1_{k',m}(\Gamma^1(\mathcal{O}), \mathbb{Z}_{(p)})$ with $g_i \in M_{k'-k_i}(SL_2(\mathbb{Z}), \mathbb{Z}_{(p)})$ are such that $0 \not\equiv \phi \equiv \psi \pmod{p}$, then $f_i \equiv g_i \pmod{p}$ for all $1 \leq i \leq 4m^2$.*

In the following result we extend Theorem 3.2.1 to the case of Jacobi forms on $\Gamma^1(\mathcal{O})$.

Corollary 3.3.7. *If $\phi \in J^1_{k,m}(\Gamma^1(\mathcal{O}), \mathbb{Z}_{(p)})$ and $\psi \in J^1_{k',m'}(\Gamma^1(\mathcal{O}), \mathbb{Z}_{(p)})$ are such that $\phi \equiv \psi \not\equiv 0 \pmod{p}$, then $m = m'$ and $k \equiv k' \pmod{p-1}$. Moreover, if m is fixed and $(\psi_k)_k$ is a finite family of Jacobi forms with $\psi_k \in J^1_{k,m}(\Gamma^1(\mathcal{O}), \mathbb{Z}_{(p)})$ such that $\sum_k \psi_k \equiv 0 \pmod{p}$, then for each $\alpha \in \mathbb{Z}/(p-1)\mathbb{Z}$ we have*

$$\sum_{k \in \alpha + (p-1)\mathbb{Z}} \psi_k \equiv 0 \pmod{p}.$$

Proof. Let $\hat{\phi}(\tau, z_1, z_2) = \phi(\tau, z_1 + iz_2, z_1 - iz_2)$ and let $\hat{\psi} = \psi(\tau, z_1 + iz_2, z_1 - iz_2)$. By Theorem 3.3.1, we have $\hat{\phi} \in J_{k,B}(\Gamma^2, \mathbb{Z}_{(p)})$ and $\hat{\psi} \in J_{k',B}(\Gamma^2, \mathbb{Z}_{(p)})$, where $B = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$. If $\phi \equiv \psi \not\equiv 0 \pmod{p}$ then $\hat{\phi} \equiv \hat{\psi} \not\equiv 0 \pmod{p}$. Therefore by [32, Proposition 2.6] we have $m = m'$ and $k \equiv k' \pmod{p-1}$. This proves the first part of the corollary. Now suppose $(\psi_k)_k$ is a finite family of Jacobi forms with $\psi_k \in J^1_{k,m}(\Gamma^1(\mathcal{O}), \mathbb{Z}_{(p)})$. Then for each k, $\hat{\psi_k} = \psi_k(\tau, z_1 + iz_2, z_1 - iz_2) \in J_{k,B}(\Gamma^2, \mathbb{Z}_{(p)})$. If $\sum_k \psi_k \equiv 0 \pmod{p}$ then $\sum_k \hat{\psi_k} \equiv 0 \pmod{p}$. Therefore by [32, Proposition 2.6], for each $\alpha \in \mathbb{Z}/(p-1)\mathbb{Z}$, we have

$$\sum_{k \in \alpha + (p-1)\mathbb{Z}} \hat{\psi_k} \equiv 0 \pmod{p}.$$

This implies that for each $\alpha \in \mathbb{Z}/(p-1)\mathbb{Z}$, we have

$$\sum_{k \in \alpha + (p-1)\mathbb{Z}} \psi_k \equiv 0 \pmod{p}.$$

□

Remark 3.3.8. *The above Corollary 3.3.7 can also be proved by using a similar*

argument as in the proof of Theorem 3.2.1.

3.4 Filtrations and Heat operators in Jacobi forms on $\Gamma^1(\mathcal{O})$

Definition 3.4.1. *Let $\phi : \mathcal{H} \times \mathbb{C}^2 \to \mathbb{C}$ be a function which has Fourier expansion of the form*

$$\phi(\tau, z_1, z_2) = \sum_{\substack{n \in \mathbb{Z}, r \in \mathcal{O}^{\#} \\ nm - N(r) \geq 0}} c(\phi; n, r) q^n \zeta_1^r \zeta_2^{\overline{r}}.$$

If $\phi \equiv \psi \pmod{p}$ for some $\psi \in J^1_{k,m}(\Gamma^1(\mathcal{O}), \mathbb{Z}_{(p)})$, we define the filtration of ϕ modulo p by

$$\omega(\phi) = \inf\{l \mid \psi \in J_{l,m}(\Gamma^1(\mathcal{O}), \mathbb{Z}_{(p)}) \text{ and } \overline{\phi} = \overline{\psi}\}.$$

Lemma 3.4.2. *Let $\phi = \sum_{i=1}^{4m^2} f_i \Upsilon_i \in J^1_{k,m}(\Gamma^1(\mathcal{O}), \mathbb{Z}_{(p)})$, where $\Upsilon_i \in J^1_{k_i,m}(\Gamma^1(\mathcal{O}), \mathbb{Z})$ and $f_i \in M_{k-k_i}(SL_2(\mathbb{Z}), \mathbb{Z}_{(p)})$. For $1 \leq i \leq 4m^2$, let $F_i \in \mathbb{Z}_{(p)}[X, Y]$ be such that $f_i = F_i(E_4, E_6)$. Then $\omega(\phi) < k$ if and only if*

$$\overline{A} \mid \overline{F_i} \quad \text{in} \quad \mathbb{F}_p[X, Y] \quad \text{for all} \quad 1 \leq i \leq 4m^2.$$

Proof. We have

$$\phi = \sum_{i=1}^{4m^2} f_i \Upsilon_i \in J^1_{k,m}(\Gamma^1(\mathcal{O}), \mathbb{Z}_{(p)}),$$

where $f_i \in M_{k-k_i}(SL_2(\mathbb{Z}), \mathbb{Z}_{(p)})$. We shall first prove that $\omega(\phi) < k$ if and only if $u(f_i) < k - k_i$ for all $1 \leq i \leq 4m^2$. Suppose that $\omega(\phi) < k$. This implies that

there exists an integer $k' < k$ and

$$\psi = \sum_{i=1}^{4m^2} g_i \Upsilon_i \in J^1_{k',m}(\Gamma^1(\mathcal{O}), \mathbb{Z}_{(p)})$$

with $g_i \in M_{k'-k_i}(SL_2(\mathbb{Z}), \mathbb{Z}_{(p)})$ such that $\phi \equiv \psi \pmod{p}$. Therefore by Corollary 3.3.6, we have

$$f_i \equiv g_i \pmod{p} \tag{3.4.1}$$

for all $1 \leq i \leq 4m^2$. Since $k' - k_i < k - k_i$, we have $u(f_i) < k - k_i$ for all $1 \leq i \leq 4m^2$. Conversely, suppose that $u(f_i) < k - k_i$ for all $1 \leq i \leq 4m^2$. Then for each $1 \leq i \leq 4m^2$, let us assume that $u(f_i) = t_i - k_i$. This implies that for each $1 \leq i \leq 4m^2$, there exists $g_i \in M_{t_i - k_i}(SL_2(\mathbb{Z}), \mathbb{Z}_{(p)})$ such that $f_i \equiv g_i \pmod{p}$. Then by Theorem 2.2.1, we have $k - t_i \equiv 0 \pmod{p-1}$ for all $1 \leq i \leq 4m^2$. Let $t = \max\{t_i \mid 1 \leq i \leq 4m^2\}$. Since $t_i < k$ for all $1 \leq i \leq 4m^2$, we have $t < k$. Since $k - t_i \equiv 0 \pmod{p-1}$ for all $1 \leq i \leq 4m^2$, we have $t - t_i \equiv 0 \pmod{p-1}$ for all $1 \leq i \leq 4m^2$. For all $1 \leq i \leq 4m^2$, let $t - t_i = \alpha_i(p-1)$, where $\alpha_i \geq 0$. Then consider

$$\psi = \sum_{i=1}^{4m^2} E_{p-1}^{\alpha_i} g_i \Upsilon_i \in J^1_{t,m}(\Gamma^1(\mathcal{O}), \mathbb{Z}_{(p)}).$$

By the first part of Lemma 2.2.4 and (3.4.1), we have $\phi \equiv \psi \pmod{p}$. This implies that $\omega(\phi) < k$. This proves that $\omega(\phi) < k$ if and only if $u(f_i) < k - k_i$ for all $1 \leq i \leq 4m^2$. Now by Lemma 2.2.3, we conclude that $\omega(\phi) < k$ if and only if

$$\overline{A} \mid \overline{F_i} \ \text{ in } \ \mathbb{F}_p[X, Y] \ \text{ for all } \ 1 \leq i \leq 4m^2.$$

□

Next we extend Theorem 2.3.3 to the case of Jacobi forms on $\Gamma^1(\mathcal{O})$ [28, Propo-

sition 3.7].

Proposition 3.4.3. *If $\phi \in J^1_{k,m}(\Gamma^1(\mathcal{O}), \mathbb{Z}_{(p)})$, then there exists $\psi \in J^1_{k',m}(\Gamma^1(\mathcal{O}), \mathbb{Z}_{(p)})$ for some integer k' such that $\overline{L_m(\phi)} = \overline{\psi}$. Moreover, if $\phi \not\equiv 0 \pmod{p}$, then*

$$\omega(L_m(\phi)) \leq \omega(\phi) + p + 1,$$

with equality if and only if $p \nmid (\omega(\phi) - 1)m$.

Proof. We shall follow the idea of Richter [34, Proposition 2] to prove this proposition. By Lemma 1.5.1 we have

$$\hat{\phi} = L_m(\phi) - \frac{(k-1)m}{3} E_2\phi \in J^1_{k+2,m}(\Gamma^1(\mathcal{O}), \mathbb{Z}_{(p)}). \tag{3.4.2}$$

Therefore by Lemma 2.2.4 we have

$$L_m(\phi) \equiv E_{p-1}\hat{\phi} + \frac{(k-1)m}{3} E_{p+1}\phi \pmod{p} \tag{3.4.3}$$

and $E_{p-1}\hat{\phi} + \frac{(k-1)m}{3} E_{p+1}\phi \in J^1_{k+p+1,m}(\Gamma^1(\mathcal{O}), \mathbb{Z}_{(p)})$. This proves the first assertion of the proposition. Now assume that $\phi \not\equiv 0 \pmod{p}$. If $\omega(\phi) < k$, let $\psi \in J_{k',m}(\Gamma^1(\mathcal{O}), \mathbb{Z}_{(p)})$ be such that $\phi \equiv \psi \pmod{p}$. This implies that $L_m(\phi) \equiv L_m(\psi) \pmod{p}$. Hence $\omega(L_m(\phi)) = \omega(L_m(\psi))$. Therefore without loss of generality we assume that $\omega(\phi) = k$. Then from (3.4.3) we have

$$\omega(L_m(\phi)) \leq k + p + 1.$$

If $p \mid (k-1)m$, then from (3.4.3) we have $\omega(L_m(\phi)) \leq k+2 < k+p+1$. Conversely assume that $\omega(L_m(\phi)) < k+p+1$. We need to show that $p \mid (k-1)m$. Assume on the contrary that $p \nmid (k-1)m$. Then by (3.4.2) we have $\omega\left(\frac{(k-1)m}{3} E_2\phi\right) < k+p+1$.

We shall prove that $\omega(E_2\phi) = k + p + 1$. This will lead to a contradiction. By Corollary 3.3.6 we have

$$\phi = \sum_{i=1}^{4m^2} f_i \Upsilon_i,$$

where $f_i \in M_{k-k_i}(SL_2(\mathbb{Z}), \mathbb{Z}_{(p)})$. Let $F_i \in \mathbb{Z}_{(p)}[X, Y]$ be such that $f_i = F_i(E_4, E_6)$ for $1 \leq i \leq 4m^2$. Since $E_2\phi \equiv E_{p+1}\phi \pmod{p}$ and $E_{p+1}\phi \in J^1_{k+p+1,m}(\Gamma^1(\mathcal{O}), \mathbb{Z}_{(p)})$, we can calculate the filtration of $E_2\phi$. We have

$$E_{p+1}\phi = \sum_{i=1}^{4m^2} E_{p+1} f_i \Upsilon_i.$$

Since $\omega(\phi) = k$, by Lemma 3.4.2, there exists j such that $\overline{A} \nmid \overline{F}_j$. By the third part of Lemma 2.2.4 we have $\overline{A} \nmid \overline{BF}_j$. Therefore by Lemma 3.4.2 we have $\omega(E_2\phi) = \omega(E_{p+1}\phi) = k + p + 1$. □

3.5 Filtrations and Heat operators in Hermitian Jacobi forms

Definition 3.5.1. *Let $\phi : \mathcal{H} \times \mathbb{C}^2 \to \mathbb{C}$ be a function which has Fourier expansion of the form*

$$\phi(\tau, z_1, z_2) = \sum_{\substack{n \in \mathbb{Z}, r \in \mathcal{O}^\# \\ nm - N(r) \geq 0}} c(\phi; n, r) q^n \zeta_1^r \zeta_2^{\overline{r}}.$$

If $\phi \equiv \psi \pmod{p}$ for some $\psi \in HJ^\delta_{k,m}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$, we define the filtration of ϕ modulo p by

$$\Omega(\phi) = \inf\{l \mid \ \psi \in HJ^{\delta'}_{l,m}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)}) \text{ for some } \delta' \text{ and } \overline{\phi} = \overline{\psi}\}.$$

The heat operator L_m acts on a Hermitian Jacobi form $\phi \in HJ^\delta_{k,m}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$.

It is easy to check that $L_m(\phi)$ is not a Hermitian Jacobi form. We shall prove that $L_m(\phi)$ is congruent to a Hermitian Jacobi form of index m modulo p and hence we can calculate its filtration modulo p. Our next theorem which will be proved later, is an extension of Theorem 2.3.3 to the case of Hermitian Jacobi forms of integer index over $\mathbb{Q}(i)$.

Theorem 3.5.2. *If $\phi \in HJ^{\delta}_{k,m}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$, then there exists $\psi \in HJ^{\delta'}_{k',m}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$ for some integer k' and $\delta' \in \{+,-\}$ such that $\overline{L_m(\phi)} = \overline{\psi}$. If $\phi \not\equiv 0 \pmod{p}$, then*

$$\Omega(L_m(\phi)) \leq \Omega(\phi) + p + 1,$$

with equality if and only if $p \nmid (\Omega(\phi) - 1)m$.

If $\phi \in HJ^{\delta}_{k,m}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$, then since $HJ^{\delta}_{k,m}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)}) \subset J^1_{k,m}(\Gamma^1(\mathcal{O}), \mathbb{Z}_{(p)})$, both $\Omega(\phi)$ and $\omega(\phi)$ are defined. We prove in the following proposition [28, Proposition 3.8] that in fact, both are same.

Proposition 3.5.3. *If $\phi \in HJ^{\delta}_{k,m}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$, then $\Omega(\phi) = \omega(\phi)$.*

Proof. Since $HJ^{\delta}_{k,m}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)}) \subset J^1_{k,m}(\Gamma^1(\mathcal{O}), \mathbb{Z}_{(p)})$, we always have

$$\omega(\phi) \leq \Omega(\phi).$$

Suppose that $\omega(\phi) = l$. To prove $\omega(\phi) = \Omega(\phi)$, it is sufficient to prove that there exists a Hermitian Jacobi form $\psi \in HJ_{l,m}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$ for some $\delta \in \{+,-\}$ such that

$$\phi \equiv \psi \pmod{p}.$$

Since $\omega(\phi) = l$, there exists $\varphi \in J^1_{l,m}(\Gamma^1(\mathcal{O}), \mathbb{Z}_{(p)})$ such that

$$\phi(\tau, z_1, z_2) \equiv \varphi(\tau, z_1, z_2) \pmod{p}. \tag{3.5.1}$$

By Corollary 3.3.7, we have $k - l = a(p-1)$ for some integer a. Let $k - l \equiv 0 \pmod 4$. Let $\epsilon \in \mathcal{O}^\times$, replacing z_1 by ϵz_1 and z_2 by $\overline{\epsilon} z_2$ in (3.5.1) we have

$$\phi(\tau, \epsilon z_1, \overline{\epsilon} z_2) \equiv \varphi(\tau, \epsilon z_1, \overline{\epsilon} z_2) \pmod p.$$

Using the transformation property (1.4.1) of ϕ in the above congruence, we obtain

$$\phi(\tau, z_1, z_2) \equiv \sigma(\epsilon)\epsilon^{-k}\varphi(\tau, \epsilon z_1, \overline{\epsilon} z_2) \equiv \epsilon^{l-k}\sigma(\epsilon)\epsilon^{-l}\varphi(\tau, \epsilon z_1, \overline{\epsilon} z_2) \pmod p,$$

which implies

$$\phi(\tau, z_1, z_2) \equiv \varphi \mid_{l,m,\delta} \epsilon I \pmod p.$$

Let us define

$$\psi(\tau, z_1, z_2) = \frac{1}{4} \sum_{\epsilon \in \mathcal{O}^\times} \varphi \mid_{l,m,\delta} \epsilon I.$$

Then from (1.5.2) we have $\psi(\tau, z_1, z_2) \in HJ^{\delta}_{l,m}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$. Also it is clear that $\phi(\tau, z_1, z_2) \equiv \psi(\tau, z_1, z_2) \pmod p$. This proves that $\Omega(\phi) = \omega(\phi)$ if $k - l \equiv 0 \pmod 4$. Let $k - l \equiv 2 \pmod 4$. Replacing z_1 by ϵz_1 and z_2 by $\overline{\epsilon} z_2$ in (3.5.1) and using the transformation property (1.4.1) of ϕ we have

$$\phi(\tau, z_1, z_2) \equiv \sigma(\epsilon)\epsilon^{-k}\varphi(\tau, \epsilon z_1, \overline{\epsilon} z_2) \equiv \epsilon^{l-k+2}\sigma(\epsilon)\epsilon^{-2}\epsilon^{-l}\varphi(\tau, \epsilon z_1, \overline{\epsilon} z_2) \pmod p,$$

which implies

$$\phi(\tau, z_1, z_2) \equiv \varphi \mid_{l,m,-\delta} \epsilon I \pmod p.$$

Let us define

$$\psi(\tau, z_1, z_2) = \frac{1}{4} \sum_{\epsilon \in \mathcal{O}^\times} \varphi \mid_{l,m,-\delta} \epsilon I.$$

Then from (1.5.2) we have $\psi(\tau, z_1, z_2) \in HJ^{-\delta}_{l,m}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$. Also it is clear that

$\phi(\tau, z_1, z_2) \equiv \psi(\tau, z_1, z_2) \pmod{p}$. This proves that $\Omega(\phi) = \omega(\phi)$ if $k - l \equiv 2 \pmod{4}$. $\square$

We are now well equipped to prove Theorem 3.5.2.

Proof of Theorem 3.5.2

Let $\phi \in HJ^{\delta}_{k,m}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$. We shall prove that there exists

$$\psi \in \begin{cases} HJ^{\delta}_{k+p+1}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)}) & \text{if} \quad p \equiv 3 \pmod{4}, \\ HJ^{-\delta}_{k+p+1}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)}) & \text{if} \quad p \equiv 1 \pmod{4} \end{cases} \tag{3.5.2}$$

such that $L_m(\phi) \equiv \psi \pmod{p}$. From Lemma 1.5.1, we have

$$L_m(\phi) = \hat{\phi} + \frac{(k-1)m}{3} E_2 \phi,$$

where $\hat{\phi} \in HJ^{-\delta}_{k+2}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$. Then by Lemma 2.2.4 we have

$$L_m(\phi) \equiv E_{p-1}\hat{\phi} + \frac{(k-1)m}{3} E_{p+1}\phi \pmod{p}.$$

Let $\psi = E_{p-1}\hat{\phi} + \frac{(k-1)m}{3} E_{p+1}\phi$. We observe that ψ satisfies $(1.4.2)$ and $(1.4.3)$ with weight $k+p+1$ and index m. We need to prove that ψ satisfies $(1.4.1)$ for some δ'. Let $p \equiv 3 \pmod{4}$. We shall prove that $\psi \in HJ^{\delta}_{k+p+1,m}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$. We observe that it is sufficient to prove that

$$\psi \mid_{k+p+1,m,\delta} \epsilon I = \psi, \tag{3.5.3}$$

for every $\epsilon \in \mathcal{O}^{\times}$. To prove $(3.5.3)$ we shall prove that

$$E_{p-1}\hat{\phi} \mid_{k+p+1,m,\delta} \epsilon I = E_{p-1}\hat{\phi} \quad \text{and} \quad E_{p+1}\phi \mid_{k+p+1,m,\delta} \epsilon I = E_{p+1}\phi.$$

We shall prove the above transformation for $E_{p-1}\hat{\phi}$. One can prove similarly for $E_{p+1}\phi$ also. We have

$$\left(E_{p-1}\hat{\phi} \mid_{k+p+1,m,\delta} \epsilon I\right)(\tau, z_1, z_2) = \sigma(\epsilon)\epsilon^{-(k+p+1)} E_{p-1}(\tau)\hat{\phi}\left(\tau, \epsilon z_1, \overline{\epsilon} z_2\right),$$

Applying the transformation property $(1.4.1)$ of $\hat{\phi}$ in the above identity we have

$$\left(E_{p-1}\hat{\phi} \mid_{k+p+1,m,\delta} \epsilon I\right)(\tau, z_1, z_2) = \epsilon^{-(p-3)} E_{p-1}(\tau)\hat{\phi}(\tau, z_1, z_2) = E_{p-1}(\tau)\hat{\phi}(\tau, z_1, z_2).$$

Therefore we have proved $(3.5.3)$ for ψ. Let $p \equiv 1 \pmod 4$. By doing a similar calculation one proves that

$$\psi \mid_{k+p+1,m,-\delta} \epsilon I = \psi.$$

This proves the first assertion of the theorem. Now by Proposition 3.5.3, we have

$$\Omega(\phi) = \omega(\phi) \quad \text{and} \quad \Omega(L_m(\phi)) = \omega(L_m(\phi)),$$

Therefore by Proposition 3.4.3, Theorem 3.5.2 follows.

3.6 Hermitian modular forms modulo p

The theory of Hermitian modular forms modulo p has been studied by Kikuta and Nagaoka [23]. They have studied the structure of symmetric Hermitian modular forms modulo p. In this section our main aim is to prove a theorem like Theorem 3.2.1 in the case of Hermitian modular forms of degree 2 over $\mathbb{Q}(i)$.

Let $F \in M_k(\Gamma^2(\mathcal{O}), \det^{k/2}, \mathbb{Z}_{(p)})$ with Fourier expansion

$$F = \sum_{T \in \Delta_2} A_F(T) e(tr(TZ)).$$

We define

$$\overline{F} = \sum_{T \in \Delta_2} \overline{A_F(T)} e(tr(TZ)),$$

where $\overline{A_F(T)}$ is the reduction of $A_F(T)$ modulo p. Let $G = \sum_{T \in \Delta_2} A_G(T) e(tr(TZ))$. We say that $F \equiv G \pmod{p}$ (or $\overline{F} = \overline{G}$) if $A_F(T) \equiv A_G(T) \pmod{p}$ (or $\overline{A_F(T)} = \overline{A_G(T)}$) for all $T \in \Delta_2$. Let

$$M_k(\Gamma^2(\mathcal{O}), \det^{k/2}, \mathbb{F}_p) = \{\overline{F} \mid F \in M_k(\Gamma^2(\mathcal{O}), \det^{k/2}, \mathbb{Z}_{(p)})\},$$

$$M_k(\Gamma^2(\mathcal{O}), \det^{k/2}, \mathbb{F}_p)^{sym} = \{\overline{F} \mid F \in M_k(\Gamma^2(\mathcal{O}), \det^{k/2}, \mathbb{Z}_{(p)})^{sym}\},$$

$$M(\Gamma^2(\mathcal{O}), \det, \mathbb{F}_p)^{sym} = \sum_k M_k(\Gamma^2(\mathcal{O}), \det^{k/2}, \mathbb{F}_p)^{sym}.$$

We have the following theorem [28, Theorem 7.1]. A more general result for symmetric Hermitian modular forms has been proved by Kikuta [22, Theorem 1.4]. But our method of proof is different and we prove it for Hermitian modular forms.

Theorem 3.6.1. *Suppose $F \in M_k(\Gamma^2(\mathcal{O}), \det^{k/2}, \mathbb{Z}_{(p)})$ and $G \in M_{k'}(\Gamma^2(\mathcal{O}), \det^{k'/2}, \mathbb{Z}_{(p)})$ are such that $F \equiv G \not\equiv 0 \pmod{p}$. Then $k \equiv k' \pmod{p-1}$. Moreover, if $(F_k)_k$ is a finite family of Hermitian modular forms with $F_k \in M_k(\Gamma^2(\mathcal{O}), \det^{k/2}, \mathbb{Z}_{(p)})$ such that $\sum_k F_k \equiv 0 \pmod{p}$, then for each $\alpha \in \mathbb{Z}/(p-1)\mathbb{Z}$ we have*

$$\sum_{k \in \alpha + (p-1)\mathbb{Z}} F_k \equiv 0 \pmod{p}.$$

Proof. The first assertion of the theorem follows from second. Therefore we shall

prove the second assertion of the theorem. Let us denote by $\phi_{k,m}$ the mth Hermitian Jacobi form in the Fourier-Jacobi expansion of F_k (see (1.7.2)). Then by the Fourier Jacobi expansion of F_k we observe that

$$\sum_k F_k \equiv 0 \pmod{p}$$

if and only if

$$\sum_k \phi_{k,m} \equiv 0 \pmod{p},$$

for all $m \geq 0$. Then by Theorem 3.2.1 we have

$$\sum_{k \in \alpha + (p-1)\mathbb{Z}} \phi_{k,m} \equiv 0 \pmod{p}$$

for all $m \geq 0$. This implies that

$$\sum_{k \in \alpha + (p-1)\mathbb{Z}} F_k \equiv 0 \pmod{p}.$$

□

3.7 Filtrations and Heat operator in Hermitian modular forms

In this section our main aim is to prove a theorem like Theorem 3.5.2 in the case of symmetric Hermitian modular forms of degree 2 over $\mathbb{Q}(i)$.

Definition 3.7.1. *Let $F : \mathcal{H}_2 \to \mathbb{C}$ be a function which has Fourier expansion of*

the form

$$F(Z) = \sum_{\substack{n\in\mathbb{Z}, r\in\mathcal{O}^{\#} \\ nm-N(r)\geq 0}} A_F(n,r,m) q^n \zeta_1^r \zeta_2^{\overline{r}} (q')^m.$$

If $F \equiv G \pmod{p}$ *for some* $G \in M_k(\Gamma^2(\mathcal{O}), \det^{k/2}, \mathbb{Z}_{(p)})^{sym}$*, we define the filtration of* F *modulo* p *by*

$$\mho(F) = \inf\{l : G \in M_l(\Gamma^2(\mathcal{O}), \det^{l/2}, \mathbb{Z}_{(p)})^{sym} \quad and \quad \overline{F} = \overline{G}\}$$

We have the following result by Kikuta and Nagaoka [24, Theorem 3].

Theorem 3.7.2. *Let* $F \in M_k(\Gamma^2(\mathcal{O}), \det^{k/2}, \mathbb{Z}_{(p)})^{sym}$*. There exists a cusp form* $G \in M_{k+p+1}(\Gamma^2(\mathcal{O}), \det^{(k+p+1)/2}, \mathbb{Z}_{(p)})^{sym}$ *such that*

$$\mathbb{D}(F) \equiv G \pmod{p}.$$

By the above result, for $F \in M_k(\Gamma^2(\mathcal{O}), \det^{k/2}, \mathbb{Z}_{(p)})^{sym}$, we can calculate the filtration of $\mathbb{D}(F)$. Our next theorem is a result analogous to Theorem 3.5.2 for symmetric Hermitian modular forms of degree 2 over $\mathbb{Q}(i)$ [28, Proposition 7.7].

Theorem 3.7.3. *Let* $F \in M_k(\Gamma^2(\mathcal{O}), \det^{k/2}, \mathbb{Z}_{(p)})^{sym}$*. Suppose that there exists an integer* m *such that* $p \nmid m$ *and the* m^{th} *Fourier Jacobi coefficient* $\phi_{k,m}$ *of* F *satisfies* $\Omega(\phi_{k,m}) = \mho(F)$*. Then*

$$\mho(\mathbb{D}(F)) \leq \mho(F) + p + 1,$$

with equality if and only if $p \nmid (\mho(F) - 1)$*.*

The remaining part of this section is devoted to the proof of Theorem 3.7.3. We shall first recall a result on the structure of symmetric Hermitian modular forms of degree 2 over $\mathbb{Q}(i)$.

Let $T(X_1, X_2, X_3, X_4, X_5) = \sum c_{a,b,c,d,e} X_1^a X_2^b X_3^c X_4^d X_5^e \in \mathbb{Z}_{(p)}[X_1, X_2, X_3, X_4, X_5]$ be a polynomial in the variables X_1, X_2, X_3, X_4, X_5. The reduction of T modulo a prime p is defined by

$$\overline{T} = \sum \overline{c}_{a,b,c,d,e} X_1^a X_2^b X_3^c X_4^d X_5^e \in \mathbb{F}_p[X_1, X_2, X_3, X_4, X_5],$$

where $\overline{c}_{a,b,c,d,e}$ is the reduction of $c_{a,b,c,d,e}$ modulo p. With this definition, we recall the following result [23, Proposition 5.1, Theorem 5.2].

Theorem 3.7.4. *Let $F \in M_k(\Gamma^2(\mathcal{O}), \det^{k/2}, \mathbb{Z}_{(p)})^{sym}$. There exists a symmetric Hermitian modular form $G \in M_{p-1}(\Gamma^2(\mathcal{O}), \det^{(p-1)/2}, \mathbb{Z}_{(p)})^{sym}$ such that*

$$G \equiv 1 \pmod{p}.$$

Furthermore, if $P_G(X_1, X_2, X_3, X_4, X_5) \in \mathbb{Z}_{(p)}[X_1, X_2, X_3, X_4, X_5]$ is such that $G = P_G(H_4, H_6, \chi_8, F_{10}, F_{12})$, then $(\overline{P}_G - 1)$ is irreducible in the polynomial ring $\mathbb{F}_p[X_1, X_2, X_3, X_4, X_5]$ and

$$M(\Gamma^2(\mathcal{O}), \det, \mathbb{F}_p)^{sym} \cong \mathbb{F}_p[X_1, X_2, X_3, X_4, X_5]/(\overline{P}_G - 1). \tag{3.7.1}$$

As a consequence of the above theorem we obtain the following important corollary. The proof of the corollary is similar to the proof of Lemma 2.2.3. Therefore we omit the proof here.

Corollary 3.7.5. *Let $F \in M_k(\Gamma^2(\mathcal{O}), \det^{k/2}, \mathbb{Z}_{(p)})^{sym}$ be such that $F \not\equiv 0 \pmod{p}$. If $P_F \in \mathbb{Z}_{(p)}[X_1, X_2, X_3, X_4, X_5]$ is such that $F = P_F[H_4, H_6, \chi_8, F_{10}, F_{12}]$, then $\mho(F) < k$ if and only if $\overline{P}_G \mid \overline{P}_F$ in the polynomial ring $\mathbb{F}_p[X_1, X_2, X_3, X_4, X_5]$, where P_G is as in Theorem 3.7.4.*

Now using the above corollary we obtain the following result [28, Lemma 7.4]

which will be used in the proof of Theorem 3.7.3.

Lemma 3.7.6. *Let $F \in M_k(\Gamma^2(\mathcal{O}), \det^{k/2}, \mathbb{Z}_{(p)})^{sym}$ be such that $F \not\equiv 0 \pmod{p}$. Let $\mho(F) = k$. Suppose that $P_F \in \mathbb{Z}_{(p)}[X_1, X_2, X_3, X_4, X_5]$ is such that $F = P_F(H_4, H_6, \chi_8, F_{10}, F_{12})$. Then there exists a positive integer k' and a Hermitian modular form $R \in M_{k'}(\Gamma^2(\mathcal{O}), \det^{k'/2}, \mathbb{Z}_{(p)})^{sym}$ such that $p \nmid k'(k'-1)$ and $\mho(R) = k'$. Also if $P_R \in \mathbb{Z}_{(p)}[X_1, X_2, X_3, X_4, X_5]$ is such that $R = P_R(H_4, H_6, \chi_8, F_{10}, F_{12})$ then $\overline{P}_G$ does not divide the product $\overline{P}_F\overline{P}_R$, where P_G is as in Theorem 3.7.4.*

Proof. We first consider the case when $\gcd(\overline{P}_F, \overline{P}_G) = \overline{P}_R \neq 1$. Let

$$P_R(X_1, X_2, X_3, X_4, X_5) \in \mathbb{Z}_{(p)}[X_1, X_2, X_3, X_4, X_5]$$

be such that $\overline{P}_R$ is the reduction of the polynomial $P_R(X_1, X_2, X_3, X_4, X_5)$ modulo p. Then it can be easily checked that $P_R(X_1, X_2, X_3, X_4, X_5)$ is a graded polynomial, i.e., $R = P_R(H_4, H_6, \chi_8, F_{10}, F_{12}) \in M_{k'}(\Gamma^2(\mathcal{O}), \det^{k'/2}, \mathbb{Z}_{(p)})^{sym}$ for some integer $k' > 0$. Since $\mho(F) = k$, by Corollary 3.7.5 we have $\overline{P}_R \neq \overline{P}_G$. Since $\overline{P}_R$ is a non-trivial factor of $\overline{P}_G$, we have $k' < p-1$ and $\mho(R) = k'$ by Theorem 3.6.1. Therefore $p \nmid k'(k'-1)$. Also since $\overline{P}_R$ is a non trivial factor of $\overline{P}_G$ and $\overline{P}_R \neq \overline{P}_G$, we observe that $\overline{P}_G \nmid \overline{P}_R\overline{P}_F$. Now consider the case $\gcd(\overline{P}_F, \overline{P}_G) = 1$. Let us first assume that $p > 5$. Since the Fourier coefficient of H_4 corresponding to the zero matrix is 1, $H_4 \not\equiv 0 \pmod{p}$ for any prime p. Also since $p > 5$, by Theorem 3.6.1 we have $\mho(H_4) = 4$. Thus if we consider $R = H_4$, then by Corollary 3.7.5, $\overline{P}_G$ does not divide $\overline{P}_R$. Since $\gcd(\overline{P}_F, \overline{P}_G) = 1$, we have $\overline{P}_G$ does not divide $\overline{P}_F\overline{P}_R$. Now suppose that $p = 5$. From [23, Lemma 4.3], we have $A_{\chi_8}(1, (1+i)/2, 1) = 1$, hence $\chi_8 \not\equiv 0 \pmod{5}$. From Theorem 3.6.1, the possible values of $\mho(\chi_8)$ are 4 and 8. If $\mho(\chi_8) = 4$, then

$$\chi_8 \equiv aH_4 \pmod{5}$$

for some $a \in \mathbb{Z}_{(5)}$. The above congruence relation is not possible since the Fourier coefficient corresponding to the zero matrix of H_4 is 1 where as that of χ_8 is 0. Therefore $\mho(\chi_8) = 8$. Then from Corollary 3.4.2, we deduce that $\overline{P}_G$ does not divide $\overline{P}_R$. Since $\gcd(\overline{P}_R, \overline{P}_G) = 1$, $\overline{P}_G$ does not divide $\overline{P}_F\overline{P}_R$. □

We are now well equipped to prove Theorem 3.7.3.

Proof of Theorem 3.7.3

If $\mho(F) < k$, then there exists a positive integer $k' < k$ and $R \in M_{k'}(\Gamma^2(\mathcal{O}), \det^{k'/2}, \mathbb{Z}_{(p)})^{sym}$ such that $F \equiv R \pmod{p}$. Then $\mathbb{D}(F) \equiv \mathbb{D}(R) \pmod{p}$ and $\mho(\mathbb{D}(F)) = \mho(\mathbb{D}(R))$. Therefore without loss of generality we assume that $\mho(F) = k$. Let

$$F(\tau, z_1, z_2, \tau') = \sum_{m \geq 0} \phi_{k,m}(\tau, z_1, z_2, \tau')(q')^m,$$

be the Fourier Jacobi expansion of F with $\phi_{k,m} \in J^{\delta}_{k,m}(\Gamma^2(\mathcal{O}), \det^{k/2}, \mathbb{Z}_{(p)})$, where

$$\delta = \begin{cases} + & \text{if } \ k \equiv 0 \pmod{4}, \\ - & \text{if } \ k \equiv 2 \pmod{4}. \end{cases}$$

Then we have

$$\mathbb{D}(F) = \sum_{m \geq 0} L_m(\phi_{k,m})(q')^m.$$

By the hypothesis of the theorem, there is an integer m such that $p \nmid m$ and $\Omega(\phi_{k,m}) = k$. If $p \nmid (k-1)$, then by Theorem 3.5.2, we have

$$\Omega(L_m(\phi_{k,m})) = k + p + 1.$$

It can be trivially observed that for all m, we have $\Omega(L_m(\phi_{k,m})) \leq \mho(\mathbb{D}(F))$. There-

fore

$$\mho(\mathbb{D}(F)) \geq k+p+1.$$

Also from Theorem 3.7.2 we have

$$\mho(\mathbb{D}(F)) \leq k+p+1.$$

Hence $\mho(\mathbb{D}(F)) = k+p+1$. Conversely, assume that $p \mid (k-1)$. We need to show that $\mho(\mathbb{D}(F)) < k+p+1$. Suppose on the contrary that $\mho(\mathbb{D}(F)) = k+p+1$. Let $S \in M_{k+p+1}(\Gamma^2(\mathcal{O}), \det^{(k+p+1)/2}, \mathbb{Z}_{(p)})^{sym}$ be such that $\mathbb{D}(F) \equiv S \pmod{p}$. Let $P_S \in \mathbb{Z}_{(p)}[X_1, X_2, X_3, X_4, X_5]$ be such that $S = P_S(H_4, H_6, \chi_8, F_{10}, F_{12})$. Since $\mho(S) = k+p+1$, $S \not\equiv 0 \pmod{p}$. Then by Lemma 3.7.6 there exists a positive integer k' and a Hermitian modular form $R \in M_{k'}(\Gamma^2(\mathcal{O}), \det^{k'/2}, \mathbb{Z}_{(p)})^{sym}$ such that $p \nmid k'(k'-1)$ and $\mho(R) = k'$. If $P_R \in \mathbb{Z}_{(p)}[X_1, X_2, X_3, X_4, X_5]$ is such that $R = P_R(H_4, H_6, \chi_8, F_{10}, F_{12})$ then $\overline{P}_G \nmid \overline{P}_S\overline{P}_R$, where P_G is as in Theorem 3.7.4. By Lemma 1.7.4 we have $[F, R]_1 \in M_{k+k'+2}(\Gamma^2(\mathcal{O}), \det^{(k+k'+2)/2}, \mathbb{Z}_{(p)})^{sym}$ and

$$[F, R]_1 \equiv -(k'-1)k'\mathbb{D}(F)R \equiv -(k'-1)k'SR \pmod{p}.$$

Since $\overline{P}_G \nmid \overline{P}_S\overline{P}_R$, by Corollary 3.7.5 and the above identity, we obtain

$$\mho([F, R]_1) = k+k'+p+1. \tag{3.7.2}$$

But since $[F, R]_1 \in M_{k+k'+2}(\Gamma^2(\mathcal{O}), \det^{(k+k'+2)/2}, \mathbb{Z}_{(p)})^{sym}$, we have $\mho([F, R]_1) \leq k+k'+2$, a contradiction to (3.7.2).

Chapter 4

Congruences among Hermitian Jacobi forms and Hermitian modular forms

4.1 Introduction

In this chapter we characterize $U(p)$ cogruences in Hermitian Jacobi forms and Hermitian modular forms of degree 2 over $\mathbb{Q}(i)$ in terms of filtrations. We also study the existence and non-existence of Ramanujan-type congruences in Hermitian Jacobi forms and Hermitian modular forms of degree 2 over $\mathbb{Q}(i)$.

4.2 $U(p)$ Congruences and Ramanujan-type congruences in Hermitian Jacobi forms

In this section we shall characterize $U(p)$ congruences and study Ramanujan-type congruences in Hermitian Jacobi forms over $\mathbb{Q}(i)$.

4.2.1 Heat cycles

Let $\phi \in HJ^{\delta}_{k,m}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$ with Fourier expansion

$$\phi(\tau, z_1, z_2) = \sum_{\substack{n\in\mathbb{Z}, r\in\mathcal{O}^{\#} \\ nm-N(r)\geq 0}} c(\phi; n, r) q^n \zeta_1^r \zeta_2^{\overline{r}}.$$

The heat operator L_m on ϕ is given by

$$L_m(\phi)(\tau, z_1, z_2) = \sum_{\substack{n\in\mathbb{Z}, r\in\mathcal{O}^{\#} \\ nm-N(r)\geq 0}} (4(nm - N(r)))c(\phi; n, r) q^n \zeta_1^r \zeta_2^{\overline{r}}.$$

Therefore we have

$$L_m^p(\phi) \equiv L_m(\phi) \pmod{p}.$$

Similarly, we have

$$L_m^{i+p-1}(\phi) \equiv L_m^i(\phi) \pmod{p}$$

for any integer $i \geq 1$. We call the finite sequence $\overline{L_m^1(\phi)} = \overline{L_m(\phi)}, \cdots, \overline{L_m^{p-1}(\phi)}$, the heat cycle of ϕ. We say that ϕ is in its own heat cycle if $\overline{L_m^{p-1}(\phi)} = \overline{\phi}$.

Suppose that $L_m(\phi) \not\equiv 0 \pmod{p}$. For an integer $i \geq 1$, we call $\overline{L_m^i(\phi)}$ a high point of the heat cycle if $\Omega(L_m^i(\phi)) \equiv 1 \pmod{p}$ and $\overline{L_m^{i+1}(\phi)}$ a low point of the heat cycle. If $\overline{L_m^i(\phi)}$ is a high point of a heat cycle, then by Theorem 3.5.2, we have

$$\Omega(L_m^{i+1}(\phi)) < \Omega(L_m^i(\phi)) + p + 1.$$

Therefore by Theorem 3.2.1 we have

$$\Omega(L_m^{i+1}(\phi)) = \Omega(L_m^i(\phi)) + p + 1 - s(p-1), \tag{4.2.1}$$

for some integer $s \geq 1$. We now prove the following result [28, Lemma 4.1] which will be applied to prove results on $U(p)$ congruences and Ramanujan-type congruences.

Lemma 4.2.1. *Let $\phi \in HJ_{k,m}^{\delta}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$. Suppose that $p \nmid m$ and $L_m(\phi) \not\equiv 0 \pmod{p}$. Then we have*

- *For any integer $i \geq 1$, we have $\Omega(L_m^i(\phi)) \not\equiv 2 \pmod{p}$.*
- *The heat cycle of ϕ has only one low point if and only if there is some $i \geq 1$ such that $\Omega(L_m^i(\phi)) \equiv 3 \pmod{p}$. In this case the low point is $\overline{L_m^i(\phi)}$.*
- *For any integer $i \geq 1$, we have $\Omega(L_m^{i+1}(\phi)) \neq \Omega(L_m^i(\phi)) + 2$.*
- *The number of low points in the heat cycle of ϕ is either one or two.*

Proof. Suppose that for some integer $i \geq 1$, we have $\Omega(L_m^i(\phi)) \equiv 2 \pmod{p}$.

Then $p \nmid (\Omega(L_m^i(\phi)) - 1)m$. Applying Theorem 3.5.2 inductively we obtain

$$\Omega(L_m^{i+n}(\phi)) = \Omega(L_m^i(\phi)) + n(p+1)$$

for $1 \leq n \leq p-1$. For $n = p-1$ we have $L_m^{i+p-1}(\phi) \equiv L_m^i(\phi) \pmod{p}$. Therefore

$$\Omega(L_m^{i+p-1}(\phi)) = \Omega(L_m^i(\phi)) + (p-1)(p+1) = \Omega(L_m^i(\phi)),$$

a contradiction. This proves the first assertion of the lemma.

Suppose that $\Omega(L_m^i(\phi)) \equiv 3 \pmod{p}$ for some integer $i \geq 1$. Then the sequence $\overline{L_m^i(\phi)}$, $\overline{L_m^{i+1}(\phi)}$, $\cdots$, $\overline{L_m^{i+p-2}(\phi)}$ is a cyclic permutation of the heat cycle of ϕ. Since $\Omega(L_m^i(\phi)) \not\equiv 1 \pmod{p}$, applying Theorem 3.5.2 inductively we have

$$\Omega(L_m^{i+n}(\phi)) = \Omega(L_m^i(\phi)) + n(p+1), \tag{4.2.2}$$

for $1 \leq n \leq p-2$. Also for $n = p-2$, we observe that

$$\Omega(L_m^{i+p-2}(\phi)) \equiv 1 \pmod{p}. \tag{4.2.3}$$

Therefore from (4.2.2) and (4.2.3), we observe that $\overline{L_m^{i+p-2}(\phi)}$ is the only high point and $\overline{L_m^{i+p-1}(\phi)}$ is the only low point of the heat cycle. Since $\overline{L_m^{i+p-1}(\phi)} = \overline{L_m^i(\phi)}$, we get the required result. Conversely, assume that the heat cycle of ϕ has only one low point. Let it be $\overline{L_m^i(\phi)}$. Then the sequence $\overline{L_m^i(\phi)}, \cdots, \overline{L_m^{i+p-2}(\phi)}$ is a cyclic permutation of the heat cycle of ϕ. Since $\overline{L_m^i(\phi)}$ is the only low point, $\overline{L_m^{i+p-2}(\phi)}$ must be the only high point of the heat cycle. Therefore applying Theorem 3.5.2 inductively, we have

$$\Omega(L_m^{i+p-2}(\phi)) = \Omega(L_m^i(\phi)) + (p-2)(p+1). \tag{4.2.4}$$

Since $\overline{L_m^{i+p-2}(\phi)}$ is the high point, we have $\Omega(L_m^{i+p-2}(\phi)) \equiv 1 \pmod{p}$. Therefore from (4.2.4), we have $\Omega(L_m^i(\phi)) \equiv 3 \pmod{p}$. This proves the second assertion of the lemma.

Suppose that for some integer $i \geq 1$ we have $\Omega(L_m^{i+1}(\phi)) = \Omega(L_m^i(\phi)) + 2$. By Theorem 3.5.2 we must have

$$\Omega(L_m^i(\phi)) \equiv 1 \pmod{p}.$$

Therefore $\Omega(L_m^{i+1}(\phi)) \equiv 3 \pmod{p}$. Applying Theorem 3.5.2 inductively we have

$$\Omega(L_m^{i+1+n}(\phi)) = \Omega(L_m^{i+1}(\phi)) + n(p+1),$$

for any integer n with $1 \leq n \leq p-2$. In particular for $n = p-2$, we have

$$\Omega(L_m^i(\phi)) = \Omega(L_m^{i+(p-1)}(\phi)) = \Omega(L_m^i(\phi)) + 2 + (p-2)(p+1).$$

The above identity gives a contradiction. This proves the third assertion of the lemma.

Suppose that the heat cycle of ϕ has more than one low point. Let $t \geq 2$ be such that $\overline{L_m^{i_1}(\phi)}$, $\overline{L_m^{i_2}(\phi)}$, $\cdots$, $\overline{L_m^{i_t}(\phi)}$ are all the high points of the heat cycle. We assume $i_{t+1} = i_1 + (p-1)$ for our convenience. By the third assertion of the lemma and (4.2.1), for each integer j with $1 \leq j \leq t$, we have

$$\Omega(L_m^{i_j+1}(\phi)) = \Omega(L_m^{i_j}(\phi)) + (p+1) - s_j(p-1) \equiv 2 + s_j \pmod{p}, \qquad (4.2.5)$$

where $s_j \geq 2$. Let $i_{j+1} - i_j = l_j$. Then by (4.2.5) and applying Theorem 3.5.2

repeatedly, we obtain

$$\Omega(L_m^{i_{j+1}}(\phi)) = \Omega(L_m^{i_j+l_j}(\phi)) = \Omega(L_m^{i_j}(\phi)) + l_j(p+1) - s_j(p-1). \tag{4.2.6}$$

Therefore we have

$$\Omega(L_m(\phi)) = \Omega(L_m^{1+p-1}(\phi)) = \Omega(L_m(\phi)) + (p-1)(p+1) - \sum_{j=1}^{t} s_j(p-1).$$

From the above identity we deduce that $\sum_{j=1}^{t} s_j = p+1$. Since $\overline{L_m^{i_j}(\phi)}$ is a high point for each integer j with $1 \leq j \leq t$, we observe from (4.2.6) that

$$l_j + s_j \equiv 0 \pmod{p}.$$

Since $s_j \geq 2$, $1 \leq l_j \leq p-1$ and $\sum_{j=1}^{t} s_j = p+1$, we deduce that $l_j = p - s_j$. Now

$$p - 1 = i_{t+1} - i_1 = \sum_{j=1}^{t}(i_{j+1} - i_j) = \sum_{j=1}^{t}(p - s_j) = tp - (p+1),$$

which implies $t = 2$. □

4.2.2 $U(p)$ congruences

Definition 4.2.2. *Suppose $\phi \in HJ_{k,m}^{\delta}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$ has Fourier expansion*

$$\phi(\tau, z_1, z_2) = \sum_{\substack{n\in\mathbb{Z}, r\in\mathcal{O}^{\#} \\ nm - N(r)\geq 0}} c(\phi; n, r) q^n \zeta_1^r \zeta_2^{\overline{r}}.$$

The Atkin's $U(p)$ operator on ϕ is defined by

$$\phi \mid U(p) = \sum_{\substack{n\in\mathbb{Z}, r\in\mathcal{O}^{\#} \\ nm-N(r)\geq 0 \\ p\mid 4(nm-N(r))}} c(\phi; n, r) q^n \zeta_1^r \zeta_2^{\overline{r}}.$$

We say that ϕ has $U(p)$ congruence if $\phi \mid U(p) \equiv 0 \pmod{p}$.

In the following theorem we give a characterization of $U(p)$ congruences in terms of the filtrations of Hermitian Jacobi forms [28, Theorem 4.3]. This generalizes the result of Richter and Senadheera [36, Theorem 1.2] on Hermitian Jacobi forms of index 1 to Hermitian Jacobi forms of any integer index.

Theorem 4.2.3. *Let $\phi \in HJ^{\delta}_{k,m}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$ be such that $\phi \not\equiv 0 \pmod{p}$.*

- *Then ϕ has $U(p)$ congruence if and only if*

$$L_m^{p-1}(\phi) \equiv \phi \pmod{p}.$$

- *If $k \geq 4$, $p > k$ and $p \nmid m$, then*

$$\Omega(L_m^{p+2-k}(\phi)) = \begin{cases} 2p+4-k & \text{if } \phi \mid U(p) \not\equiv 0 \pmod{p}, \\ p+5-k & \text{if } \phi \mid U(p) \equiv 0 \pmod{p}. \end{cases}$$

Proof. Let $\phi \in HJ^{\delta}_{k,m}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$ with Fourier expansion

$$\phi(\tau, z_1, z_2) = \sum_{\substack{n\in\mathbb{Z}, r\in\mathcal{O}^{\#} \\ nm-N(r)\geq 0}} c(\phi; n, r) q^n \zeta_1^r \zeta_2^{\overline{r}}.$$

We have

$$L_m^{p-1}(\phi)(\tau, z_1, z_2) = \sum_{\substack{n\in\mathbb{Z}, r\in\mathcal{O}^{\#} \\ nm-N(r)\geq 0}} (4(nm - N(r)))^{p-1} c(\phi; n, r) q^n \zeta_1^r \zeta_2^{\overline{r}}.$$

Thus

$$\phi - L_m^{p-1}(\phi) \equiv \phi \mid U(p) \pmod{p}.$$

Therefore $\phi \mid U(p) \equiv 0 \pmod{p}$ if and only if $L_m^{p-1}(\phi) \equiv \phi \pmod{p}$. This proves the first assertion of the theorem.

Suppose that $k \geq 4$, $p > k$ and $p \nmid m$. Since $p > k$, $\Omega(\phi) \not\equiv 1 \pmod{p}$. Therefore by Theorem 3.5.2,

$$\Omega(L_m(\phi)) = \Omega(\phi) + p + 1 > 0. \tag{4.2.7}$$

Hence $L_m(\phi) \not\equiv 0 \pmod{p}$. The possible values of $\Omega(\phi)$ are 0 and k. If $\Omega(\phi) = 0$ then $\phi \equiv a \pmod{p}$ for some $a \in \mathbb{Z}_{(p)}$ with $a \not\equiv 0 \pmod{p}$. This implies $L_m(\phi) \equiv 0$, which is not true. Therefore $\Omega(\phi) = k$.

Let us assume that $\phi \mid U(p) \equiv 0 \pmod{p}$. Since $L_m^{p-1}(\phi) \equiv \phi \pmod{p}$, ϕ is in its own heat cycle. Now we claim that $\overline{\phi}$ is a low point of the heat cycle. Assume on the contrary that $\overline{\phi}$ is not a low point of the heat cycle. Then $\overline{L_m^{p-2}(\phi)}$ is not a high point of the heat cycle. Therefore $\Omega(L_m^{p-2}(\phi)) \not\equiv 1 \pmod{p}$. Suppose that $\Omega(L_m^{p-2}(\phi)) = t > 0$. Then by Theorem 3.5.2 we have

$$\Omega(\phi) = \Omega(L_m^{p-1}(\phi)) = t + p + 1 > k,$$

a contradiction. Therefore $\overline{\phi}$ is a low point of the heat cycle. By Lemma 4.2.1, the heat cycle of ϕ has either one or two low points. Suppose that the heat cycle

of ϕ has one low point. Since $\overline{\phi}$ is the low point, by the second assertion of Lemma 4.2.1, we have $\Omega(\phi) \equiv 3 \pmod{p}$. Since $\Omega(\phi) = k$, $k \geq 4$ and $p > k$ this is not true. Therefore the heat cycle of ϕ has two low points. Since $\overline{\phi}$ is one of the low points, $\overline{L_m^{p-2}(\phi)}$ is a high point. Let i_1 be an integer with $1 \leq i_1 < p-2$ such that $\overline{L_m^{i_1}(\phi)}$ is the other high point of the heat cycle. Applying Theorem 3.5.2 inductively we have

$$\Omega(L_m^{i_1}(\phi)) = k + i_1(p+1) \equiv k + i_1 \equiv 1 \pmod{p}.$$

Since $1 \leq i_1 < p-2$ and $4 \leq k < p$, by the above congruence identity, the only possible value of i_1 is $p+1-k$. Let $s_1, s_2 \geq 1$ be integers such that

$$\Omega(L_m^{i_1+1}(\phi)) = \Omega(L_m^{i_1}(\phi)) + p + 1 - s_1(p-1)$$

and

$$\Omega(L_m^{p-2+1}(\phi)) = \Omega(L_m^{p-2}(\phi)) + p + 1 - s_2(p-1).$$

In the proof of the fourth assertion of Lemma 4.2.1, we have proved that $s_1 + s_2 = p+1$ and $p-2-i_1 = p-s_1$. Thus we have $s_1 = p-k+3$, $s_2 = k-2$ and

$$\Omega(L_m^{i_1+1}(\phi)) = \Omega(L_m^{p+2-k}(\phi)) = k+(p+2-k)(p+1)-(p-k+3)(p-1) = p+5-k.$$

Now assume that $\phi \mid U(p) \not\equiv 0 \pmod{p}$. Then we claim that $\overline{L_m(\phi)}$ is a low point of the heat cycle of ϕ. Suppose on the contrary that it is not true. Then consider the function

$$\psi(\tau, z_1, z_2) = \sum_{\substack{n \in \mathbb{Z}, r \in \mathcal{O}^{\#} \\ nm - N(r) \geq 0 \\ p \nmid 4(nm - N(r))}} c(\phi; n, r) q^n \zeta_1^r \zeta_2^{\overline{r}}.$$

We observe that $L_m^{p-1}(\phi) \equiv \psi \pmod{p}$. Therefore the filtration of ψ is defined. Also we observe that $L_m(\psi) \equiv L_m(\phi) \pmod{p}$. This implies

$$\Omega(L_m(\psi)) = \Omega(L_m(\phi)). \tag{4.2.8}$$

Since $\overline{L_m(\phi)}$ is not a low point, $\overline{L_m^{p-1}(\phi)}$ is not a high point. Therefore $\Omega(L_m^{p-1}(\phi)) = \Omega(\psi) \not\equiv 1 \pmod{p}$. Suppose that $\Omega(\psi) = l$. Then $\Omega(L_m(\psi)) = l + p + 1$. By (4.2.7) and (4.2.8), we have

$$\Omega(\phi) = \Omega(\psi) = k. \tag{4.2.9}$$

We observe that

$$\phi - \psi \equiv E_{p-1}^{p+1}\phi - L_m^{p-1}(\phi) \pmod{p}.$$

We shall prove that there exists $\varphi \in HJ^{\delta}_{k+(p-1)(p+1),m}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$ such that

$$E_{p-1}^{p+1}\phi - L_m^{p-1}(\phi) \equiv \varphi \pmod{p}. \tag{4.2.10}$$

Let $p \equiv 1 \pmod{4}$. In the proof of Theorem 3.5.2, we have seen that if $\phi \in HJ^{\delta}_{k,m}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$ then $L_m(\phi) \equiv \varphi_1 \pmod{p}$ for some $\varphi_1 \in HJ^{-\delta}_{k+p+1,m}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$. Therefore inductively we conclude that $L_m^{p-1}(\phi) \equiv \varphi_2 \pmod{p}$ for some $\varphi_2 \in HJ^{\delta}_{k+(p-1)(p+1),m}(\Gamma^J(\mathcal{O}, \mathbb{Z}_{(p)}))$. Also $E_{p-1}^{p+1}\phi \in HJ^{\delta}_{k+(p-1)(p+1),m}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$. Therefore we have proved (4.2.10) if $p \equiv 1 \pmod{4}$. Similarly one proves (4.2.10) when $p \equiv 3 \pmod{4}$. Therefore $\Omega(\phi - \psi)$ is defined. By (4.2.9) we deduce that $\omega(\phi - \psi)$ is either 0 or k. Therefore by Proposition 3.5.3 we conclude that $\Omega(\phi - \psi)$ is either 0 or k. This implies that $\Omega(\phi - \psi) \not\equiv 1 \pmod{p}$. Since $\phi \mid U(p) \not\equiv 0 \pmod{p}$, we have $\phi - \psi \not\equiv 0 \pmod{p}$ by the first assertion of the theorem. Therefore by

Theorem 3.5.2, we have

$$\Omega(L_m(\phi-\psi)) = \Omega(\phi-\psi)+p+1>0,$$

a contradiction since $L_m(\phi-\psi)\equiv 0 \pmod p$. Therefore $\overline{L_m(\phi)}$ is a low point in the heat cycle of ϕ. By Lemma 4.2.1, the heat cycle of ϕ has one or two low points. Suppose that the heat cycle of ϕ has one low point. Then $\overline{L_m(\phi)}$ is that low point. By the second part of Lemma 4.2.1, $\Omega(L_m(\phi)) = k+p+1\equiv 3 \pmod p$. This implies that $k=2$ is the only possibility. Since $k\geq 4$, this implies that the heat cycle of ϕ has two low points. We know that $\overline{L_m^{p-1}(\phi)}$ is a high point. Let for some integer i_1 with $1\leq i_1<p-1$ be such that $\overline{L_m^{i_1}(\phi)}$ is the other high point. Let $s_1, s_2\geq 1$ be integers such that

$$\Omega(L_m^{i_1+1}(\phi)) = \Omega(L_m^{i_1}(\phi))+p+1-s_1(p-1),$$

and

$$\Omega(L_m(\phi)) = \Omega(L_m^{p-1+1}(\phi)) = \Omega(L_m^{p-1}(\phi))+p+1-s_2(p-1).$$

Applying Theorem 3.5.2 inductively we have

$$\Omega(L_m^{i_1}(\phi)) = \Omega(L_m(\phi))+(i_1+1)(p+1) = k+p+1+(i_1-1)(p+1)\equiv k+i_1\equiv 1 \pmod p.$$

Then as done previously, we deduce that $i_1 = p+1-k$ and $s_1 = p-k+2$. Therefore we have

$$\Omega(L_m^{i_1+1}(\phi)) = \Omega(L_m^{p+2-k}(\phi)) = p+k+1+(p+1-k)(p+1)-(p-k+2)(p-1) = 2p+4-k.$$

□

4.2.3 Examples

In this subsection we give some examples of Hermitian Jacobi forms over $\mathbb{Q}(i)$ which have $U(p)$ congruences. Let $\phi \in HJ^{\delta}_{k,m}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$. Suppose that for a given prime $p \geq 5$ we want to check if $\phi \mid U(p) \equiv 0 \pmod{p}$. If $k \geq 4$, $k < p$ and $p \nmid m$ we apply the second part of Theorem 4.2.3. Otherwise we need to check if $L_m^{p-1}(\phi) \equiv \phi \pmod{p}$. We also explain how one gets more examples of Hermitian Jacobi forms of index > 1 having $U(p)$ congruences from Hermitian Jacobi forms of index 1 having $U(p)$ congruences.

Let

$$x = 1 + 2\sum_{n=1}^{\infty} q^{\frac{n^2}{2}}, \qquad y = 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{\frac{n^2}{2}}, \qquad z = 2q^{1/8}\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}.$$

Let $S = \left\{0, \frac{1}{2}, \frac{i}{2}, \frac{(1+i)}{2}\right\}$ be the set of coset representatives of $\mathcal{O}^{\#}/\mathcal{O}$. For $s \in S$, we define

$$\theta_{1,s}(\tau, z_1, z_2) = \sum_{r \in \mathcal{O}^{\#},\ r \equiv s \pmod{\mathcal{O}}} q^{N(r)} \zeta_1^r \zeta_2^{\overline{r}}.$$

We define the Hermitian Jacobi forms $\phi^+_{k,1} \in HJ_{k,1}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$ for $k = 4, 8, 10$ by

$$\phi^+_{4,1} = \frac{1}{2}(x^6 + y^6)\theta_{1,0} + \frac{1}{2}z^6(\theta_{1,1/2} + \theta_{1,i/2}) + \frac{1}{2}(x^6 - y^6)\theta_{1,(1+i)/2},$$

$$\phi^+_{8,1} = \frac{1}{2}(x^{14} + y^{14})\theta_{1,0} + \frac{1}{2}z^{14}(\theta_{1,1/2} + \theta_{1,i/2}) + \frac{1}{2}(x^{14} - y^{14})\theta_{1,(1+i)/2},$$

$$\phi^+_{10,1} = \frac{1}{64}x^6y^6z^6(\theta_{1,1/2} - \theta_{1,i/2}).$$

Sasaki [35] provides the theta decomposition of these Hermitian Jacobi forms. Richter and Senadheera [36, Lemma 2.4] have constructed the following Hermitian Jacobi

form of negative parity:

$$\phi_{6,1}^{-} = h_0\theta_{1,0} + h_{1/2}\theta_{1,1/2} + h_{i/2}\theta_{1,i/2} + h_{(1+i)/2}\theta_{1,(1+i)/2} \in HJ_{6,1}^{-}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)}),$$

where

$$h_0 = -\frac{1}{2}(x^2+y^2)(x^8 - x^6y^2 - x^4y^4 - x^2y^6 + y^8),$$

$$h_{1/2} = \frac{1}{2}z^6(z^4 - 2x^4),$$

$$h_{i/2} = \frac{1}{2}z^6(z^4 - 2x^4),$$

$$h_{(1+i)/2} = -\frac{1}{2}(x^2-y^2)(x^8 + x^6y^2 - x^4y^4 + x^2y^6 + y^8).$$

Consider the Hermitian Jacobi form $\phi_{10,1}^{+} \in HJ_{10,1}^{+}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$. We check if $\phi_{10,1}^{+} \mid U(p) \equiv 0 \pmod{p}$ for $p = 5, 7$. Since $E_4 \equiv 1 \pmod{5}$ and $E_2 \equiv E_6 \pmod{5}$, a direct calculation shows that

$$L_m^4(\phi_{10,1}) \equiv \phi_{10,1}^{+} \pmod{5}.$$

Therefore $\phi_{10,1}^{+} \mid U(5) \equiv 0 \pmod{5}$. Since $E_6 \equiv 1 \pmod{7}$ and $E_2 \equiv E_4^2 \pmod{7}$, we have

$$L_1^6(\phi_{10,1}^{+}) \equiv (4E_2^5 - E_2^3E_4 + 3E_2E_4^2 + 4E_2^2 - 2E_4)\phi_{10,1}^{+} \not\equiv \phi_{10,1}^{+} \pmod{7}.$$

Therefore $\phi_{10,1}^{+} \mid U(7) \not\equiv 0 \pmod{7}$.

For $\rho \in \mathcal{O}$, the index raising operator

$$\pi_\rho : HJ_{k,m}^{\delta}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)}) \longrightarrow HJ_{k,N(\rho)m}^{\delta}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$$

is defined by

$$\phi(\tau, z_1, z_2) \longmapsto \phi(\tau, \rho z_1, \overline{\rho} z_2).$$

Therefore if $\rho \in \mathcal{O}$ is such that $p \nmid N(\rho)$, $\phi \in HJ^{\delta}_{k,m}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$ and $\phi \mid U(p) \equiv 0 \pmod{p}$, then $\pi_\rho(\phi) \mid U(p) \equiv 0 \pmod{p}$. We know that $\phi^+_{10,1} \mid U(5) \equiv 0 \pmod{5}$. Therefore $\pi_{(1+i)}(\phi^+_{10,1}) = \phi^+_{10,1}(\tau, (1+i)z_1, (1-i)z_2) \in HJ^+_{10,2}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(5)})$ and $\pi_{(1+i)}(\phi^+_{10,1}) \mid U(5) \equiv 0 \pmod{5}$.

4.2.4 Ramanujan-type congruences

Definition 4.2.4. *Suppose $\phi \in HJ^{\delta}_{k,m}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$ has Fourier expansion*

$$\phi(\tau, z_1, z_2) = \sum_{\substack{n \in \mathbb{Z}, r \in \mathcal{O}^{\#} \\ nm - N(r) \geq 0}} c(\phi; n, r) q^n \zeta_1^r \zeta_2^{\overline{r}}.$$

We say that ϕ has Ramanujan-type congruence at $b \not\equiv 0 \pmod{p}$ if $c(\phi; n, r) \equiv 0 \pmod{p}$ whenever $4(nm - N(r)) \equiv b \pmod{p}$.

It is easy to observe that ϕ has Ramanujan-type congruence at $b \pmod{p}$ if and only if $q^{\frac{-b}{4m}}\phi$ has $U(p)$ congruence. Therefore ϕ has Ramanujan-type congruence at $b \pmod{p}$ if and only if $L_m^{p-1}(q^{\frac{-b}{4m}}\phi) \equiv q^{\frac{-b}{4m}}\phi \pmod{p}$. We now prove the following result which gives an equivalent condition on the existence of Ramanujan-type congruences in Hermitian Jacobi forms [28, Proposition 4.5]. A similar result in the case of Jacobi forms with integer index has been proved by Dewar and Richter [14, Proposition 2.4].

Proposition 4.2.5. *If $\phi \in HJ_{k,m}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$ then ϕ has Ramanujan-type congruence at $b \pmod{p}$ if and only if*

$$L_m^{\frac{p+1}{2}}(\phi) \equiv -\left(\frac{b}{p}\right) L_m(\phi) \pmod{p}.$$

Proof. We know that ϕ has Ramanujan-type congruence at $b \pmod p$ if and only if $L_m^{p-1}(q^{\frac{-b}{4m}}\phi) \equiv q^{\frac{-b}{4m}}\phi$. Since $b \not\equiv 0 \pmod p$, we have

$$L_m^{p-1}(q^{-\frac{b}{4m}}\phi) \equiv q^{-\frac{b}{4m}} \sum_{i=0}^{p-1} b^{p-1-i} L_m^i(\phi) \pmod p.$$

Therefore ϕ has Ramanujan-type congruence at $b \pmod p$ if and only if

$$\sum_{i=1}^{p-1} b^{p-1-i} L_m^i(\phi) \equiv 0 \pmod p. \tag{4.2.11}$$

Since $\phi \in HJ_{k,m}^{\delta}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)}) \subset J_{k,m}^1(\Gamma^1(\mathcal{O}), \mathbb{Z}_{(p)})$, by Corollary 3.3.6 we have

$$\phi = \sum_{j=1}^{4m^2} f_j \Upsilon_j$$

where $\Upsilon_j \in J_{k_j,m}^1(\Gamma^1(\mathcal{O}), \mathbb{Z})$ and $f_j \in M_{k-k_j}(SL_2(\mathbb{Z}), \mathbb{Z}_{(p)})$. From the proof of Theorem 3.5.2, we see that for each integer $i \geq 1$, there exists $\phi_i \in HJ_{k+i(p+1),m}^{\delta_i}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$ for some $\delta_i \in \{+, -\}$ such that $L_m^i(\phi) \equiv \phi_i \pmod p$. Let $F_{i,j} \in M_{k+i(p+1)-k_j}(SL_2(\mathbb{Z}), \mathbb{Z}_{(p)})$ be such that

$$\phi_i = \sum_{j=1}^{4m^2} F_{i,j} \Upsilon_j.$$

Then

$$L_m^i(\phi) \equiv \sum_{j=1}^{4m^2} F_{i,j} \Upsilon_j \pmod p. \tag{4.2.12}$$

Substituting (4.2.12) in (4.2.11) we deduce that ϕ has Ramanujan-type congruence at $b \pmod p$ if and only if

$$\sum_{j=1}^{4m^2} \left(\sum_{i=1}^{p-1} b^{p-1-i} F_{i,j} \right) \Upsilon_j \equiv 0 \pmod p.$$

Therefore by Corollary 3.3.6, ϕ has Ramanujan-type congruence at $b \pmod p$ if

and only if

$$\sum_{i=1}^{p-1} b^{p-1-i} F_{i,j} \equiv 0 \pmod{p}$$

By Theorem 2.2.1, the above equation is equivalent to

$$b^{(p-1)/2-i} F_{i+\frac{p-1}{2},j} + b^{p-1-i} F_{i,j} \equiv 0 \pmod{p}$$

for all $1 \leq j \leq 4m^2$ and $1 \leq i \leq \frac{p-1}{2}$, which is equivalent to the statement

$$F_{i+\frac{p-1}{2},j} \equiv -\left(\frac{b}{p}\right) F_{i,j} \pmod{p} \tag{4.2.13}$$

for all $1 \leq j \leq 4m^2$ and $1 \leq i \leq \frac{p-1}{2}$. Therefore by (4.2.12), (4.2.13) is equivalent to

$$L_m^{i+\frac{p-1}{2}}(\phi) = \sum_{j=1}^{4m^2} F_{i+\frac{p-1}{2},j} \Upsilon_j \equiv \sum_{j=1}^{4m^2} -\left(\frac{b}{p}\right) F_{i,j} \Upsilon_j \equiv -\left(\frac{b}{p}\right) L_m^i(\phi) \pmod{p} \tag{4.2.14}$$

for all $1 \leq i \leq \frac{p-1}{2}$. In particular for $i = 1$ we obtain

$$L_m^{\frac{p+1}{2}}(\phi) \equiv -\left(\frac{b}{p}\right) L_m(\phi) \pmod{p}. \tag{4.2.15}$$

Conversely if (4.2.15) holds, then by applying L_m repeatedly on both sides of (4.2.15), we obtain (4.2.14) for each integer i with $1 \leq i \leq \frac{p-1}{2}$. This proves the proposition. □

As a consequence of the above proposition, we note down the follwing corollary [28, Corollary 4.6].

Corollary 4.2.6. *Suppose that $\phi \in HJ_{k,m}^{\delta}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$ has Ramanujan-type congruence at $b \pmod{p}$ and $L_m(\phi) \not\equiv 0 \pmod{p}$. Then the heat cycle of ϕ has two*

low points. Moreover, if $\Omega(\phi) = Ap + B$ *with* $1 < B \leq p-1$*, then*

$$\frac{p+3}{2} \leq B \leq A + \frac{p+6}{2}.$$

Proof. Suppose that ϕ has a Ramanujan-type congruence at $b \pmod p$. Then by Proposition 4.2.5, we have

$$\Omega(L_m(\phi)) = \Omega(L_m^{\frac{p+1}{2}}(\phi)) = \Omega(L_m^p(\phi)).$$

Therefore there must be one low point in the first half of the heat cycle of ϕ and another low point in the second half of the heat cycle of ϕ. Let $\overline{L_m^{i_1}(\phi)}$ and $\overline{L_m^{i_2}(\phi)}$ be the high points of the heat cycle of ϕ, where $1 \leq i_1 \leq \frac{p-1}{2}$ and $\frac{p+1}{2} \leq i_2 \leq p-1$. By (4.2.1) we have

$$\Omega(L_m^{i_1+1}(\phi)) = \Omega(L_m^{i_1}(\phi)) + p + 1 - s_1(p-1)$$

and

$$\Omega(L_m^{i_2+1}(\phi)) = \Omega(L_m^{i_2}(\phi)) + p + 1 - s_2(p-1)$$

for some integer $s_1, s_2 \geq 1$. Now by Proposition 4.2.5 and (4.2.1) we have

$$\Omega(L_m(\phi)) = \Omega(L_m^{\frac{p+1}{2}}(\phi)) = \Omega(L_m(\phi)) + \frac{p-1}{2}(p+1) - s_1(p-1).$$

This implies $s_1 = \frac{p+1}{2}$. Similarly one proves that $s_2 = \frac{p+1}{2}$. Now suppose $\Omega(\phi) = Ap + B$ with $1 < B \leq p-1$. Since $\overline{L_m^{i_1}(\phi)}$ is a high point, we have

$$\Omega(L_m^{i_1}(\phi)) = Ap + B + i_1(p+1) \equiv B + i_1 \equiv 1 \pmod p.$$

This implies that $B+i_1 = p+1$ and $B \geq \frac{p+3}{2}$. Also the filtration of the low point $\overline{L_m^{p-B+2}(\phi)}$ is given by

$$\Omega(L_m^{p-B+2}(\phi)) = Ap + B + (p-B+2)(p+1) - \frac{p+1}{2}(p-1).$$

Since $\Omega(L_m^{p-B+2}(\phi)) > 0$, from the above identity we have

$$B \leq A + \frac{p+6}{2}.$$

□

Our next result is on the non-existence of Ramanujan-type congruences in Hermitian Jacobi forms [28, Theorem 4.7].

Theorem 4.2.7. *Suppose $\phi \in HJ_{k,m}^{\delta}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$ with $L_m(\phi) \not\equiv 0 \pmod{p}$. If $p > k$, $p \neq 2k-3, 2k-5$ and $p \nmid m$, then ϕ does not have Ramanujan-type congruence at $b \pmod{p}$.*

Proof. Assume on the contrary that ϕ has a Ramanujan-type congruence at $b \pmod{p}$. Then by Theorem 3.2.1, $\Omega(\phi) = 0$ or k. Since $L_m(\phi) \not\equiv 0 \pmod{p}$, $\Omega(\phi) \neq 0$. Therefore $\Omega(\phi) = k$. Suppose $\Omega(\phi) = k = 1$ then by Theorem 3.5.2 and Theorem 3.2.1, $\Omega(L_m(\phi)) = 1 + p + 1 - s(p-1)$ for some $s \geq 1$. Since $\Omega(L_m(\phi)) \geq 0$, we have $s = 1$ and hence $\Omega(L_m(\phi)) = 3$. Therefore by the third part of Lemma 4.2.1, we deduce that the heat cycle of ϕ has only one low point. This gives a contradiction to Corollary 4.2.6. Therefore $\Omega(\phi) = k \neq 1$. Writing $\Omega(\phi) = Ap+B$ as in Corollary 4.2.6, we have $A = 0$ and $B = k$ with $1 < k \leq p-1$. Therefore by Corollary 4.2.6 we have

$$\frac{p+3}{2} \leq k \leq \frac{p+6}{2}.$$

Hence the possible values of p are $2k-3$ and $2k-5$. This gives a contradiction to the hypothesis of the theorem. $\square$

4.2.5 Examples

We first state two results which will be used to obtain examples of Hermitian Jacobi forms which have Ramanujan-type congruences. By (3.5.2) and Proposition 4.2.5 we have the following result.

Theorem 4.2.8. *Let $\phi \in HJ^{\delta}_{k,m}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$. If*

$$\psi := L_m^{\frac{p+1}{2}}(\phi) + \left(\frac{b}{p}\right) L_m(\phi),$$

then there exists $\varphi \in HJ^{\delta'}_{k+\frac{(p+1)^2}{2},m}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$ for some $\delta' \in \{+,-\}$ such that $\psi \equiv \varphi \pmod{p}$. Moreover, ϕ has Ramanujan-type congruence at $b \not\equiv 0 \pmod{p}$ if and only if $\psi \equiv 0 \pmod{p}$.

Our next result is on the Sturm bound of Hermitian Jacobi forms in characteristic p. Sturm bound for Hermitian Jacobi forms in characteristic 0 has been proved by Das [11, Proposition 6.2]. The proof of Das is applicable in the case of characteristic p too. Therefore we omit the proof here. To state the result, define

$$\eta(k,m) = \left[\frac{4m^2(k-1)}{3} \prod_{q|4m} \left(1 - \frac{1}{q^2}\right) + \frac{m}{2}\right],$$

where q runs over all the prime divisors of $4m$.

Proposition 4.2.9. *Let $\phi \in HJ^{\delta}_{k,m}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$ with Fourier expansion*

$$\phi(\tau, z_1, z_2) = \sum_{\substack{n\in\mathbb{Z}, r\in\mathcal{O}^{\#} \\ nm-N(r)\geq 0}} c(\phi; n, r) q^n \zeta_1^r \zeta_2^{\overline{r}}.$$

If $c(\phi; n, r) \equiv 0 \pmod{p}$ *for* $0 \leq n \leq \eta(k, m)$, *then* $\phi \equiv 0 \pmod{p}$.

Let $\phi \in HJ^{\delta}_{k,m}(\Gamma^J(\mathcal{O}), \mathbb{Z}_{(p)})$. Suppose ϕ has Ramanujan-type congruence at $b \pmod{p}$. Using Theorem 4.2.8 we get ψ. Now ϕ has Ramanujan-type congruence at $b \pmod{p}$ if and only if $\psi \equiv 0 \pmod{p}$. To check $\psi \equiv 0 \pmod{p}$, we apply Proposition 4.2.9. Also by Theorem 4.2.7, the values of p for which Ramamnujan-type congruences may exist are $p \leq k$, $p = 2k-3$ and $p = 2k-5$. Following table provides some examples of Hermitian Jacobi forms having Ramanujan-type congruences.

Hermitian Jacobi forms	$b \pmod{p}$
$\phi^{+}_{8,1}$	$b \equiv 1, 2, 4 \pmod{7}$
$\phi^{+}_{8,1}$	$b \equiv 1, 3, 4, 9, 10, 12 \pmod{13}$
$(E_6\phi^{+}_{4,1} - E_4\phi^{-}_{6,1})/24$	$b \equiv 1, 2, 4 \pmod{7}$

4.3 $U(p)$ congruences and Ramanujan-type congruences in Hermitian modular forms

In this section we shall characterize $U(p)$ congruences and study Ramanujan-type congruences in Hermitian modular forms of degree 2 over $\mathbb{Q}(i)$.

4.3.1 Heat cycles

Let $F \in M_k(\Gamma^2(\mathcal{O}), \mathbb{Z}_{(p)})^{sym}$ with Fourier expansion

$$F(\tau, z_1, z_2, \tau') = \sum_{\substack{n,m \in \mathbb{Z}, r \in \mathcal{O}^{\#} \\ nm - N(r) \geq 0}} A_F(n, r, m) q^n \zeta_1^r \zeta_2^{\overline{r}} (q')^m.$$

The heat operator $\mathbb{D}$ on F is given by

$$\mathbb{D}(F)(\tau, z_1, z_2, \tau') = \sum_{\substack{n\in\mathbb{Z}, r\in\mathcal{O}^{\#} \\ nm-N(r)\geq 0}} (4(nm - N(r)))A_F(n, r, m)q^n\zeta_1^r\zeta_2^{\overline{r}}(q')^m.$$

Therefore we have

$$\mathbb{D}^p(F) \equiv \mathbb{D}(F) \pmod{p}.$$

Similarly, we have

$$\mathbb{D}^{i+p-1}(F) \equiv \mathbb{D}^i(F) \pmod{p}$$

for any integer $i \geq 1$. We call the finite sequence $\overline{\mathbb{D}^1(F)} = \overline{\mathbb{D}(F)}, \cdots, \overline{\mathbb{D}^{p-1}(F)}$, the heat cycle of F. We say that F is in its own heat cycle if $\overline{\mathbb{D}^{p-1}(F)} = \overline{F}$.

Suppose that $\mathbb{D}(F) \not\equiv 0 \pmod{p}$. For an integer $i \geq 1$, we call $\overline{\mathbb{D}^i(F)}$ a high point of a heat cycle if $\Omega(\mathbb{D}^i(F)) \equiv 1 \pmod{p}$ and $\overline{\mathbb{D}^{i+1}(F)}$ a low point of the heat cycle. If $\overline{\mathbb{D}^i(F)}$ is a high point of a heat cycle, then by Theorem 3.7.3, we have

$$\mho(\mathbb{D}^{i+1}(F)) < \mho(\mathbb{D}^i(F)) + p + 1.$$

Therefore by Theorem 3.6.1 we have

$$\mho(\mathbb{D}^{i+1}(F)) = \mho(\mathbb{D}^i(F)) + p + 1 - s(p-1) \tag{4.3.1}$$

for some integer $s \geq 1$. We now state the following lemma which will be applied to prove results on $U(p)$ congruences and Ramanujan-type congruences in the following subsections. The proof of this lemma is similar to the proof of Lemma 4.2.1. Therefore we omit the proof here.

Lemma 4.3.1. *Let $F \in M_k(\Gamma^2(\mathcal{O}), \det^{k/2}, \mathbb{Z}_{(p)})^{sym}$ and let $\mathbb{D}(F) \not\equiv 0 \pmod{p}$. Suppose that there exists an integer m such that $p \nmid m$ and the mth Fourier Jacobi*

coefficient $\phi_{k,m}$ *of* F *satisfies* $\Omega(\phi_{k,m}) = \mho(F)$. *Then we have the following.*

- *For any integer* $i \geq 1$, *we have* $\mho(\mathbb{D}^i(F)) \not\equiv 2 \pmod{p}$.
- *The heat cycle of* F *has only one low point if and only if there is some* $i \geq 1$ *such that* $\mho(\mathbb{D}^i(\phi)) \equiv 3 \pmod{p}$. *In this case the low point is* $\overline{\mathbb{D}^i(F)}$.
- *For any integer* $i \geq 1$, *we have* $\mho(\mathbb{D}^{i+1}(F)) \neq \mho(\mathbb{D}^i(F)) + 2$.
- *The number of low points in the heat cycle of* F *is either one or two.*

4.3.2 $U(p)$ congruences

Definition 4.3.2. *Suppose* $F \in M_k(\Gamma^2(\mathcal{O}), \det^{k/2}, \mathbb{Z}_{(p)})$ *has Fourier expansion*

$$F(\tau, z_1, z_2, \tau') = \sum_{\substack{n,m\in\mathbb{Z}, r\in\mathcal{O}^\# \\ nm-N(r)\geq 0}} A_F(n,r,m) q^n \zeta_1^r \zeta_2^{\overline{r}} (q')^m.$$

The Atkin's $U(p)$ *operator on* F *is defined by*

$$F \mid U(p) = \sum_{\substack{n,m\in\mathbb{Z}, r\in\mathcal{O}^\# \\ nm-N(r)\geq 0 \\ p\mid 4(nm-N(r))}} A_F(n,r,m) q^n \zeta_1^r \zeta_2^{\overline{r}} (q')^m.$$

We say that F *has* $U(p)$ *congruence if* $F \mid U(p) \equiv 0 \pmod{p}$.

In the following theorem we give a characterization of $U(p)$ congruence in terms of filtrations of symmetric Hermitian forms [28, Theorem 8.2]. This theorem generalizes Theorem 2.4.5 to symmmetric Hermitian modular forms of degree 2 over $\mathbb{Q}(i)$.

Theorem 4.3.3. *Let*

$$F(\tau, z_1, z_2, \tau') = \sum_{\substack{n,m\in\mathbb{Z}, r\in\mathcal{O}^\# \\ nm-N(r)\geq 0}} A_F(n,r,m) q^n \zeta_1^r \zeta_2^{\overline{r}} (q')^m \in M_k(\Gamma^2(\mathcal{O}), \det^{k/2}, \mathbb{Z}_{(p)})^{sym}.$$

- *If* $F \not\equiv 0 \pmod{p}$ *then* F *has* $U(p)$ *congruence if and only if*

$$\mathbb{D}^{p-1}(F) \equiv F \pmod{p}.$$

- *If* $p > k$ *and there exist* $n, m \in \mathbb{Z}$ *and* $r \in \mathcal{O}^{\#}$ *such that* $p \nmid nm$ *and* $A_F(n, r, m) \not\equiv 0 \pmod{p}$. *Then*

$$\mathcal{U}(\mathbb{D}^{p+2-k}(F))) = \begin{cases} 2p + 4 - k & \text{if} \quad F \mid U(p) \not\equiv 0 \pmod{p}, \\ p + 5 - k & \text{if} \quad F \mid U(p) \equiv 0 \pmod{p}. \end{cases}$$

Proof. Let $F \in M_k(\Gamma^2(\mathcal{O}), \det^{k/2}, \mathbb{Z}_{(p)})^{sym}$ with Fourier expansion

$$F(\tau, z_1, z_2, \tau') = \sum_{\substack{n,m \in \mathbb{Z}, r \in \mathcal{O}^{\#} \\ nm - N(r) \geq 0}} A_F(n, r, m) q^n \zeta_1^r \zeta_2^{\overline{r}} (q')^m.$$

We have

$$\mathbb{D}^{p-1}(F)(\tau, z_1, z_2, \tau') = \sum_{\substack{n,m \in \mathbb{Z}, r \in \mathcal{O}^{\#} \\ nm - N(r) \geq 0}} (4(nm - N(r)))^{p-1} A_F(n, r, m) q^n \zeta_1^r \zeta_2^{\overline{r}} (q')^m.$$

Thus

$$F - \mathbb{D}^{p-1}(F) \equiv F \mid U(p) \pmod{p}.$$

Therefore $F \mid U(p) \equiv 0 \pmod{p}$ if and only if $\mathbb{D}^{p-1}(F) \equiv F \pmod{p}$. This proves the first assertion of the theorem.

To prove the second assertion of the theorem, we need to use Lemma 4.3.1 and follow similar arguments as in the proof of the second assertion of the Theorem

4.2.3. Let the Fourier-Jacobi expansion of F be given by

$$F(\tau, z_1, z_2, \tau') = \sum_{m=0}^{\infty} \phi_{k,m}(\tau, z_1, z_2)(q')^m.$$

To apply Lemma 4.3.1, it is required to show that there exists an integer m with $p \nmid m$ such that $\mho(F) = \Omega(\phi_{k,m})$. Suppose on the contrary that for every integer m with $p \nmid m$, we have $\Omega(\phi_{k,m}) < \mho(F)$. By the hypothesis of the theorem, we have $F \not\equiv A_F(0,0,0) \pmod{p}$. Therefore since $p > k$, by Theorem 3.6.1 we have $\mho(F) = k$. Thus $\Omega(\phi_{k,m}) < k$ for each integer m with $p \nmid m$. Thus by Theorem 3.2.1 we have $\phi_{k,m} \equiv 0 \pmod{p}$ for each m with $p \nmid m$. This implies that $A_F(n,r,m) \equiv 0 \pmod{p}$ for each m with $p \nmid m$. Also since $F(\tau, z_1, z_2, \tau') = F(\tau', z_1, z_2, \tau)$, we have $A_F(n,r,m) = A_F(m,r,n)$. Therefore we deduce that $A_F(n,r,m) \equiv 0 \pmod{p}$ for $p \nmid nm$. This gives a contradiction to the hypothesis of the theorem. Therefore there exists an integer m with $p \nmid m$ such that $\Omega(\phi_{k,m}) = \mho(F)$. □

4.3.3 Examples

Let $F \in M_k(\Gamma^2(\mathcal{O}), \det^{k/2}, \mathbb{Z}_{(p)})^{sym}$. Suppose one wants to check if $F \mid U(p) \equiv 0 \pmod{p}$. If $p > k$ and there exist $n, m \in \mathbb{Z}$ and $r \in \mathcal{O}^{\#}$ such that $p \nmid nm$ and $A_F(n,r,m) \not\equiv 0 \pmod{p}$, then we can apply the second part of Theorem 4.3.3. Otherewise, we need to check if $\mathbb{D}^{p-1}(F) \equiv F \pmod{p}$. We have the following proposition which follows from Theorem 3.7.2 and Theorem 3.7.4.

Proposition 4.3.4. *Let $F \in M_k(\Gamma^2(\mathcal{O}), \det^{k/2}, \mathbb{Z}_{(p)})^{sym}$. If*

$$G = \mathbb{D}^{p-1}(F) - F,$$

then there exists $H \in M_{k+(p+1)(p-1)}(\Gamma^2(\mathcal{O}), \det^{(k+(p+1)(p-1))/2}, \mathbb{Z}_{(p)})^{sym}$ *such that* $G \equiv H \pmod{p}$. *Moreover,* $F \mid U(p) \equiv 0 \pmod{p}$ *if and only if* $G \equiv 0 \pmod{p}$.

Let

$$\Delta_2(\mathbb{Q}(i)) = \left\{ T = \begin{pmatrix} n & r \\ \overline{r} & m \end{pmatrix} \mid n, m \in \mathbb{Z}, r \in \mathcal{O}^{\#} \right\}.$$

Note that $\Delta_2 \subset \Delta_2(\mathbb{Q}(i))$. Let

$$T = \begin{pmatrix} m & a+bi \\ a-bi & n \end{pmatrix} \quad \text{and} \quad T' = \begin{pmatrix} m' & a'+b'i \\ a'-b'i & n' \end{pmatrix}$$

be different elements of $\Delta_2(\mathbb{Q}(i))$. We say that $T \succ T$ if and only if any one of the following condition holds

1. $tr(T) > tr(T')$.
2. $tr(T) = tr(T')$, $m > m'$.
3. $tr(T) = tr(T')$, $m = m'$, $a > a'$.
4. $tr(T) = tr(T')$, $m = m'$, $a = a'$, $b > b'$.

Let $F \in M_k(\Gamma^2(\mathcal{O}), \det^{k/2}, \mathbb{Z}_{(p)})$. The order of F with respect to the prime p is defined by

$$\mathrm{ord}_p(F) = \inf\{T \in \Delta_2 \mid A_F(T) \not\equiv 0 \pmod{p}\}.$$

Kikuta and Nagaoka [24, Theorem 2] have proved the following result on Sturm bound for symmetric Hermitian modular forms of degree 2 over $\mathbb{Q}(i)$.

Theorem 4.3.5. *Let* $F \in M_k(\Gamma^2(\mathcal{O}), \det^{k/2}, \mathbb{Z}_{(p)})^{sym}$. *Let*

$$A = \begin{pmatrix} [\frac{k}{8}] & 2[\frac{k}{8}] \\ 2[\frac{k}{8}] & [\frac{k}{8}] \end{pmatrix} \in \Delta_2(\mathbb{Q}(i)).$$

If $\mathrm{ord}_p(F) \succ A$ *then* $F \equiv 0 \pmod{p}$.

We consider the Hermitian cusp form $\chi_8 \in M_8(\Gamma^2(\mathcal{O}), \det^4, \mathbb{Z})^{sym}$. We check if $\chi_8 \mid U(p) \equiv 0 \pmod{p}$ for $p = 5, 7, 11$. Let $p = 5$. Let $G = \mathbb{D}^4(\chi_8) - \chi_8$ Now using the Sturm bound given in Theorem 4.3.5, we deduce that $G \equiv 0 \pmod{5}$. Therefore by Proposition 4.3.4, we have $\chi_8 \mid U(5) \equiv 0 \pmod{5}$. If $p = 7$, then by Proposition 3.7.3, $\mho(\mathbb{D}(\chi_8)) < 16$. Thus the possible values of $\mho(\mathbb{D}(\chi_8))$ are 4 and 10. Since H_4 is a non-cusp form, $\mho(\mathbb{D}(\chi_8)) \neq 4$. Therefore $\mho(\mathbb{D}(\chi_8)) = 10$. Now by applying Theorem 3.7.3 repeatedly, we deduce that $\mho(\mathbb{D}^6(\chi_8)) = 50 \neq \mho(\chi_8) = 8$. Thus by the first part of Theorem 4.3.3, $\chi_8 \mid U(7) \not\equiv 0 \pmod{7}$. If $p = 11$, then by the second part of Theorem 4.3.3, we deduce that the possible values of $\mho(\mathbb{D}^5(\chi_8))$ are 8 and 18. If $\mho(\mathbb{D}^5(\chi_8)) = 8$, then $\mathbb{D}^5(\chi_8) \equiv \beta\chi_8 \pmod{11}$ for some $\beta \in \{0, 1, \cdots, 10\}$. We know from [23, Lemma 4.3] that $A_{\chi_8}(1, (1+i)/2, 1) = 1$ and $A_{\chi_8}(1, -1/2, 1) = -486$. Therefore $\mathbb{D}^5(\chi_8) \not\equiv \beta\chi_8 \pmod{11}$ for any $\beta \in \{0, 1, \cdots, 10\}$. Thus $\mho(\mathbb{D}^5(\chi_8)) \neq 8$. Hence $\mho(\mathbb{D}^5(\chi_8)) = 18$ and $\chi_8 \mid U(11) \not\equiv 0 \pmod{11}$ by Theorem 4.3.3.

4.3.4 Ramanujan-type congruences

Definition 4.3.6. *Suppose* $F \in M_k(\Gamma^2(\mathcal{O}), \mathbb{Z}_{(p)})$ *has Fourier expansion*

$$F(\tau, z_1, z_2, \tau') = \sum_{\substack{n,m \in \mathbb{Z}, r \in \mathcal{O}^{\#} \\ nm - N(r) \geq 0}} A_F(n, r, m) q^n \zeta_1^r \zeta_2^{\overline{r}} (q')^m.$$

We say that F *has Ramanujan-type congruence at* $b \not\equiv 0 \pmod{p}$ *if* $A_F(n, r, m) \equiv 0 \pmod{p}$ *whenever* $4(nm - N(r)) \equiv b \pmod{p}$.

In the next theorem, we prove results on the existence and non-existence of Ramanujan-type congruences in symmetric Hermitian modular forms [28, Theorem

8.4]. The following theorem generalizes Theorem 2.4.7 to symmmetric Hermitian modular forms of degree 2 over $\mathbb{Q}(i)$.

Theorem 4.3.7. *Let*

$$F(\tau, z_1, z_2, \tau') = \sum_{\substack{n,m\in\mathbb{Z}, r\in\mathcal{O}^{\#} \\ nm-N(r)\geq 0}} A_F(n,r,m) q^n \zeta_1^r \zeta_2^{\bar{r}} (q')^m \in M_k(\Gamma^2(\mathcal{O}), \det^{k/2}, \mathbb{Z}_{(p)})^{sym}.$$

- *Then F has a Ramanujan-type congruence at $b \pmod p$ if and only if*

$$\mathbb{D}^{\frac{p+1}{2}}(F) \equiv -\left(\frac{b}{p}\right) \mathbb{D}(F) \pmod p.$$

- *If $p > k$, $p \neq 2k-3, 2k-5$ and there exist integers n and m such that $p \nmid nm$ and $A_F(n,r,m) \not\equiv 0 \pmod p$, then F does not have a Ramanujan-type congruence at $b \pmod p$.*

Proof. Let the Fourier-Jacobi expansion of F be given by

$$F(\tau, z_1, z_2, \tau') = \sum_{m=0}^{\infty} \phi_{k,m}(\tau, z_1, z_2) e(\tau').$$

We observe that F has Ramanujan-type congruence at $b \pmod p$ if and only if $\phi_{k,m}$ has Ramanujan-type congruence at $b \pmod p$ for all m. By Proposition 4.2.5, it is equivalent to the statement that for each m, we have

$$L_m^{\frac{p+1}{2}}(\phi_{k,m}) \equiv -\left(\frac{b}{p}\right) L_m(\phi_{k,m}) \pmod p. \tag{4.3.2}$$

Since

$$\mathbb{D}(F) = \sum_{m=0}^{\infty} L_m(\phi_{k,m})(q')^m,$$

we deduce that F has Ramanujan-type congruence at $b \pmod p$ if and only if

$$\mathbb{D}^{\frac{p+1}{2}}(F) = \sum_{m=0}^{\infty} L_m^{\frac{p+1}{2}}(\phi_{k,m})(q')^m \equiv -\left(\frac{b}{p}\right) \sum_{m=0}^{\infty} L_m(\phi_{k,m})(q')^m \equiv -\left(\frac{b}{p}\right) \mathbb{D}(F) \pmod p.$$

This proves the first part of the theorem. Now we prove the second part of the theorem. By the hypothesis of the theorem we have $F \not\equiv A_F(0,0,0) \pmod p$ and hence $\mho(F) \neq 0$. Therefore $\mho(F) = k$ as $p > k$. Also by the same reason, there exists an integer $m > 0$ with $p \nmid m$ such that $\phi_{k,m} \not\equiv 0 \pmod p$ and $\Omega(\phi_{k,m}) = k$. Then by Theorem 3.5.2, $\Omega(L_m(\phi_{k,m})) = k + p + 1$. In particular, we have $L_m(\phi_{k,m}) \not\equiv 0 \pmod p$. Now applying Theorem 4.2.7, we deduce that $\phi_{k,m}$ does not have Ramanujan-type congruence at $b \pmod p$. This implies that F does not have Ramanujan-type congruence at $b \pmod p$. □

4.3.5 Examples

We use the following result to obtain some examples of Hermitian modular forms having Ramanujan-type congruences. By Theorem 3.7.2 and Theorem 4.3.7, we have the following result.

Theorem 4.3.8. *Let $F \in M_k(\Gamma^2(\mathcal{O}), \det^{k/2}, \mathbb{Z}_{(p)})^{sym}$. If*

$$G := \mathbb{D}^{\frac{p+1}{2}}(F) + \left(\frac{b}{p}\right) \mathbb{D}(F),$$

then there exists $H \in M_{k+\frac{(p+1)^2}{2}}(\Gamma^2(\mathcal{O}), \det^{\frac{k}{2}+\frac{(p+1)^2}{4}}, \mathbb{Z}_{(p)})^{sym}$ such that $G \equiv H \pmod p$. Moreover, F has Ramanujan-type congruence at $b \not\equiv 0 \pmod p$ if and only if $G \equiv 0 \pmod p$.

By Theorem 4.3.7, if $F \in M_k(\Gamma^2(\mathcal{O}), \det^{k/2}, \mathbb{Z}_{(p)})^{sym}$ has a Ramanujan-type congruence at $b \pmod p$, then $p \leq k$ or $p = 2k-3$ or $p = 2k-5$. Therefore we use

Theorem 4.3.8 and the Sturm bound given in Theorem 4.3.5 to get some examples of Hermitian modular forms which have Ramanujan-type congruences. The following table consists of some examples of Hermitian modular forms of weight ≤ 14 which have Ramanujan-type congruences.

Hermitian modular forms	$b \pmod p$
$F = \chi_8 - 6H_4^2$, $F \not\equiv 0 \pmod 7$, $\mathbb{D}(F) \equiv 0 \pmod 7$	$b \equiv 1, 2, 3, 4, 5, 6 \pmod 7$
F_{10}	$b \equiv 1, 4 \pmod 5$
$H_4 F_{10}$	$b \equiv 1, 4 \pmod 5$
$H_4^2 H_6 + H_6 \chi_8$	$b \equiv 1, 4 \pmod 5$

Bibliography

[1] S. Ahlgren and K. Ono, *Arithmetic of singular moduli and class polynomials*, Compos. Math. **141** (2005), 293–312.

[2] S. Bocherer and S. Das, *On holomorphic differential operators equivariant for the inclusion of* $Sp(n, \mathbf{R})$ *in* $U(n, n)$, Int. Math. Res. Not. IMRN (2013), 2534–2567.

[3] D. Choi, Y. Choie and O. K. Richter, *Congruences for Siegel modular forms*, Ann. Inst. Fourier (Grenoble) **61** (2011), 1455–1466.

[4] Y. Choie, Jacobi forms and the heat operator, Math. Z. **225** (1997), 95–101.

[5] Y. Choie, Jacobi forms and the heat operator II, Illinois J. Math. **42** (1998), 179–186.

[6] Y. Choie and W. Eholzer, *Rankin-Cohen operators for Jacobi and Siegel forms*, J. Number Theory **68** (1998), 160–177.

[7] Y. Choie and H. Kim, *Differential operators on Jacobi forms of several variables*, J. Number Theory **82** (2000), 140–163.

[8] Y. Choie, H. Kim and O. K. Richter, *Differential operators on Hilbert modular forms*, J. Number Theory **122** (2007), 25–36.

[9] Y. Cooper, N. Wage and I. Wang, *Congruences for modular forms of non-positive weight*, Int. J. Number Theory **4** (2008), 1–13.

[10] S. Das, *Note on Hermitian Jacobi forms*, Tsukuba Journal of Mathematics **34** (2010), 59–78.

[11] S. Das, *Some aspects of Hermitian Jacobi forms*, Arch. Math. (Basel) **95** (2010), 423–437.

[12] M. Dewar, *On the non-existence of simple congruences for quotients of Eisenstein series*, Acta Arith. **145** (2010), 33–41.

[13] M. Dewar, *Non-existence of Ramanujan congruences in modular forms of level four*, Canad. J. Math. **63** (2011), 1284–1306.

[14] M. Dewar and O. K. Richter, *Ramanujan congruences for Siegel modular forms*, Int. J. Number Theory **6** (2010), 1677–1687.

[15] W. Eholzer and T. Ibukiyama, *Rankin-Cohen type differential operators for Siegel modular forms*, Internat. J. Math. **9** (1998), 443–463.

[16] M. Eichler and D. Zagier, *The theory of Jacobi forms*, Progress in Mathematics **55**. Birkhäuser Boston, Inc., Boston, MA, 1985.

[17] N. Elkies, K. Ono and T. Yang, *Reduction of CM elliptic curves and modular function congruences*, Int. Math. Res. Not. (2005), 2695–2707.

[18] P. Guerzhoy, *On $U(p)$-congruences*, Proc. Amer. Math. Soc. **135** (2007), 2743–2746.

[19] K. Haverkamp, *Hermitesche Jacobiformen*, Schriftenreihe Math. Inst. Univ. Münster **3** (1995).

[20] K. Haverkamp, *Hermitian Jacobi forms*, Results Math. **29** (1996), 78–89.

[21] N. Jochnowitz, *A study of the local components of the Hecke algebra mod l*, Trans. Amer. Math. Soc. **270** (1982), 253–267.

[22] T. Kikuta, *Congruences for Hermitian modular forms of degree 2*, J. Number Theory **131** (2011), 1461–1469.

[23] T. Kikuta and S. Nagaoka, *On Hermitian modular forms mod p*, J. Math. Soc. Japan **63** (2011), 211–238.

[24] T. Kikuta and S. Nagaoka, *On the theta operator for Hermitian modular forms of degree 2*, Abh. Math. Semin. Univ. Hambg. **87** (2017), 145–163.

[25] A. Krieg, *The Maass spaces on the Hermitian half-space of degree 2*, Math. Ann., **289** (1991), 663–681.

[26] S. Lang, Introduction to modular forms, Springer-Verlag, Berlin-New York, 1976.

[27] J. Martin and J. Senadheera, Differential operators for Hermitian Jacobi forms and Hermitian modular forms, Ramanujan J. **42** (2017), 443–451.

[28] J. Meher and S. K. Singh, *Congruences in Hermitian Jacobi and Hermitian modular forms*, Forum Math. **32** (2020), 501–523.

[29] S. Nagaoka, *Note on mod p Siegel modular forms*, Math. Z. **235** (2000), 405–420.

[30] K. Ono, *The web of modularity: Arithmetic of the coefficients of modular forms and q-series*, CBMS Regional Conference Series in Mathematics **102**. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2004.

[31] S. Raghavan and J. Sengupta, *A Dirihlet series for Hermitian modular forms of degree* 2, Acta Arith. **58** (1991), 181–201.

[32] M. Raum and O. K. Richter, *The structure of Siegel modular forms modulo* p *and* $U(p)$ *congruences*, Math. Res. Lett. **22** (2015), 899–928.

[33] O. K. Richter, *On congruences of Jacobi forms*, Proc. Amer. Math. Soc. **136** (2008), 2729–2734.

[34] O. K. Richter, *The action of the heat operator on Jacobi forms*, Proc. Amer. Math. Soc. **137** (2009), 869–875.

[35] R. Sasaki, *Hermitian Jacobi forms of index one*, Tsukuba J. Math. **31** (2007), 301–325.

[36] O. K. Richter and J. Senadheera, *Hermitian Jacobi forms and* $U(p)$ *congruences*, Proc. Amer. Math. Soc. **143** (2015), 4199–4210.

[37] J. Senadheera, *Hermitian Jacobi forms and congruences*, Thesis (Ph.D.)–University of North Texas, 2014, 66 pp. ISBN: 978-1339-12165-9.

[38] J. -P. Serre, *Formes modulaires et fonctions zêta* p*-adiques*, Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, 1972), pp. 191–268. Lecture Notes in Math., Vol. 350, Springer, Berlin, 1973.

[39] J. Sinick, *Ramanujan congruences for a class of eta quotients*, Int. J. Number Theory **6** (2010), 835–847.

[40] A. Sofer, p*-adic aspects of Jacobi forms*, J. Number Theory **63** (1997), 191–202.

[41] H. P. F. Swinnerton-Dyer, *On* l*-adic representations and congruences for coefficients of modular forms*, Modular functions of one variable, III (Proc. Internat.

Summer School, Univ. Antwerp, 1972), pp. 1–55. Lecture Notes in Math., Vol. 350, Springer, Berlin, 1973.

[42] C. Ziegler, *Jacobi forms of higher degree*, Abh. Math. Sem. Univ. Hamburg **59** (1989), 191–224.

Notations

Let $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ be the set of positive integers, integers, rational numbers, real numbers and complex numbers respectively. For $z \in \mathbb{C}$, $\Re(z)$ denotes the real part of z and $\Im(z)$ denotes the imaginary part of z. For any $z \in \mathbb{C}$, we denote $e^{2\pi i z}$ by $e(z)$. For any $r \in \mathbb{Q}(i)$, the norm of r is defined by $N(r) := r\overline{r}$. For $R \subset \mathbb{C}$, $M_n(R)$ denotes the set of all $n \times n$ matrices with entries from R. We denote by $GL_n(R)$ the set of all $n \times n$ matrices with positive determinant and entries from R. We write M^t for the transpose of the matrix M, $\overline{M}$ for the complex conjugate of the matrix M, $\overline{M}^t$ for the transpose conjugate of the matrix M, $r(M)$ for the rank of the matrix M and $U[V]$ for $V^t U V$, where the matrices U and V are of appropriate sizes. We write $tr(M)$ for the trace of the matrix M. For a matrix M, the notation "$M \geq 0$" means M is a positive semi-definite matrix. Similarly the notion "$M > 0$" means M is a positive definite matrix. We write $\det(M)$ for the determinant of the matrix M. For $i = \sqrt{-1}$, let $\mathbb{Q}(i)$ be the field of Gaussian rational numbers. For $x \in \mathbb{R}$, let $[x]$ denote the greatest integer less than or equal to x.

Thesis Highlight

Name of the student: Sujeet Kumar Singh

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Thesis Title: Certain congruences among Hermitian Jacobi forms and Hermitian modular forms

Discipline: Mathematical Sciences Sub-Area of Discipline: Number Theory

Date of viva voice: June 11, 2020

There has been a great amount of research on the congruences of the Fourier coefficients modular forms and in general of different automorphism functions due to their various application in Mathematics and Physics. There are two kind of congruences namely, U(p) congruences and Ramanujan-type congruences which have attracted many mathematicians. These congruences have been studied in the case of modular forms, Jacobi forms and Seigel modular forms by several mathematicians. We study U(p) congruences and Ramanujan-type congruences in the case of Hermitian Jacobi forms and Hermitian modular forms of degree 2 over $\mathbb{Q}(i)$.

Hermitian Jacobi forms occur as the Fourier Jacobi coefficients in the Fourier-Jacobi expansion of Hermitian modular forms of degree 2. We develop the theory of Hermitian Jacobi forms over $\mathbb{Q}(i)$ modulo a prime p. The theory of Hermitian Jacobi forms over $\mathbb{Q}(i)$ modulo p was started by Richter and Senadheera but due to lack of some crucial structure property of Hermitian Jacobi forms of arbitrary index, they restricted their study to the index 1 case. We prove an isomorphism between certain spaces of Jacobi forms to get the required structure property of Hermitian Jacobi forms of arbitrary index. We then generalize the Tate's theory of heat cycle in the case of Hermitian Jacobi forms. We apply these results to characterize U(p) congruences for Hermitian Jacobi forms over $\mathbb{Q}(i)$ in terms of filtrations. We also study the existence and non-existence of Ramanujan-type congruences for Hermitian Jacobi forms over $\mathbb{Q}(i)$. We illustrate some examples to explain U(p) congruences and Ramanujan-type congruences in Hermitian Jacobi forms over $\mathbb{Q}(i)$.

We develop the theory of Hermitian modular forms of degree 2 over $\mathbb{Q}(i)$ modulo a prime p by using the fact that Hermitian Jacobi forms occur as the Fourier-Jacobi coefficients in the Fourier Jacobi expansion of Hermitian modular forms of degree 2. Using results on the structure of symmetric Hermitian modular forms modulo p, we prove various results on filtration in symmetric Hermitian modular forms over $\mathbb{Q}(i)$. We generalize the Tate's theory of heat cycle in the case of symmetric Hermitian modular forms of degree 2 over $\mathbb{Q}(i)$. We apply these results to characterize U(p) congruences for symmetric Hermitian modular forms of degree 2 over $\mathbb{Q}(i)$ in terms of filtrations. We also study the existence and non-existence of Ramanujan-type congruences for symmetric Hermitian modular forms of degree 2 over $\mathbb{Q}(i)$. We illustrate some examples to explain U(p) congruences and Ramanujan-type congruences in symmetric Hermitian modular forms of degree 2 over $\mathbb{Q}(i)$.