Combinatorial characterizations of point and line sets in $\mathrm{PG}(3, q)$ with respect to a quadric

By

PUSPENDU PRADHAN

Enrolment No. MATH11201704002

National Institute of Science Education and Research, Bhubaneswar

A thesis submitted to the

Board of Studies in Mathematical Sciences

In partial fulfillment of requirements

for the Degree of

DOCTOR OF PHILOSOPHY

of

HOMI BHABHA NATIONAL INSTITUTE

February, 2022

Homi Bhaba National Institute

Recommendations of the Viva Voce Committee

As members of the Viva Voce Committee, we certify that we have read the dissertation prepared by **Puspendu Pradhan** entitled **"Combinatorial characterizations of point and line sets in $\mathrm{PG}(3, q)$ with respect to a quadric"** and recommend that it may be accepted as fulfilling the thesis requirement for the award of Degree of Doctor of Philosophy.

Chairman - Dr. Brundaban Sahu	
Guide/Convener - Dr. Binod Kumar Sahoo	04/02/2022
Co-guide - Prof. Bart De Bruyn	
Examiner - Prof. N. S. Narasimha Sastry	
Member 1 - Dr. Kamal Lochan Patra	04.02.2022
Member 2 - Dr. Deepak Kumar Dalai	04/02/2022
Member 3 - Dr. Jaban Meher	Jaban Meher 04.02.22

Final approval and acceptance of this thesis is contingent upon the candidate's submission of the final copies of the thesis to HBNI.

We hereby certify that we have read this thesis prepared under our direction and recommend that it may be accepted as fulfilling the thesis requirement.

Date : February 04, 2022

Place : NISER

Signature	**Signature**
Co-guide	**Guide**

STATEMENT BY AUTHOR

This dissertation has been submitted in partial fulfilment of requirements for an advanced degree at Homi Bhabha National Institute (HBNI) and is deposited in the Library to be made available to borrowers under rules of the HBNI.

Brief quotations from this dissertation are allowable without special permission, provided that accurate acknowledgement of source is made. Requests for permission for extended quotation from or reproduction of this manuscript in whole or in part may be granted by the Competent Authority of HBNI when in his or her judgement the proposed use of the material is in the interests of scholarship. In all other instances, however, permission must be obtained from the author.

Puspendu Pradhan

(Puspendu Pradhan)

DECLARATION

I hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Puspendu Pradhan

(Puspendu Pradhan)

List of Publications arising from the thesis

Journals

1. P. Pradhan and B. Sahu, A characterization of the family of secant lines to a hyperbolic quadric in $\mathrm{PG}(3, q)$, q odd, *Discrete Math.* **343** (2020), no. 11, 112044, 8pp.

2. B. De Bruyn, P. Pradhan and B. K. Sahoo, Minimum size blocking sets of certain line sets with respect to an elliptic quadric in $\mathrm{PG}(3, q)$, *Contrib. Discrete Math.* **15** (2020), no. 2, 132–147.

3. B. De Bruyn, P. Pradhan and B. Sahu, Blocking sets of external, tangent and secant lines to a quadratic cone in $\mathrm{PG}(3, q)$, *Discrete Math.* **344** (2021), no. 6, 112352, 4pp.

4. B. De Bruyn, P. Pradhan and B. K. Sahoo, Blocking sets of tangent and external lines to an elliptic quadric in $\mathrm{PG}(3, q)$, *Proc. Indian Acad. Sci. Math. Sci.* **131** (2021), no. 2, article no. 39, 16 pp.

5. B. De Bruyn, P. Pradhan and B. K. Sahoo, Next-to-minimum size blocking sets of external lines to a nondegenerate quadric in $\mathrm{PG}(3, q)$, communicated.

Puspendu Pradhan

(Puspendu Pradhan)

Dedicated

to

My Parents and Brother

ACKNOWLEDGEMENTS

First of all, I wish to express my sincere gratitude to my thesis advisers Dr. Binod Kumar Sahoo and Prof. Bart De Bruyn for their inspiring guidance and valuable advice from the very early stage of my research. I have gained extraordinary experiences while working with them throughout my tenure as a doctoral student.

I would like to thank my doctoral committee members: Dr. Brundaban Sahu, Dr. Kamal Lochan Patra, Dr. Deepak Kumar Dalai and Dr. Jaban Meher for their encouragement and insightful comments/suggestions during my research work. I am extremely grateful to my senior, Dr. Bikramaditya Sahu, for the fruitful discussions we had and also for providing motivation and encouragement throughout my Ph.D period.

I am indebted to my friends: Sutapa Patra, Abhrojyoti Sen, Atibur Rahman, Anantadulal Paul, Rajeeb Ranjan Mohanta, Diptesh Kumar Saha, Nilkantha Das, Dinesh Pandey, Mrityunjoy Charan, for their friendship and constant support. They provided a fun environment to grow and learn during the tenure of my Ph.D programme.

I acknowledge the financial support provided by the Council of Scientific and Industrial Research (CSIR), Government of India, through a research fellowship to pursue my doctoral studies. I would also like to thank the National Institute of Science Education and Research (NISER), Bhubaneswar for providing the necessary facilities to carry out my research work.

Last but not the least, my deepest gratitude is to my beloved parents and my elder brother for their warm support which brings belief and hope into my life. They have been so caring to me all the time and a source of inspiration towards achieving the goals in life.

Contents

SUMMARY

Let $\mathrm{PG}(d,q)$ be the d-dimensional projective space defined over a finite field of order q. For a nonempty set $\mathcal{L}$ of lines of $\mathrm{PG}(d,q)$, a set B of points of $\mathrm{PG}(d,q)$ is called an *$\mathcal{L}$-blocking set* if each line of $\mathcal{L}$ meets B in at least one point. An $\mathcal{L}$-blocking set B in $\mathrm{PG}(d,q)$ is said to be *minimal* if B has no proper subset which is also an $\mathcal{L}$-blocking set in $\mathrm{PG}(d,q)$.

Blocking sets are combinatorial objects in finite geometry with several applications and have been the subject of investigation by many researchers with respect to varying sets of lines. The first step in this regard has been to determine the minimum size of a blocking set and then to characterize, if possible, all blocking sets of that cardinality. When $\mathcal{L}$ is the set of all lines of $\mathrm{PG}(d,q)$, a classical result by Bose and Burton [13, Theorem 1] says that if B is a blocking set in $\mathrm{PG}(d,q)$ with respect to all its lines, then $|B| \geqslant (q^d-1)/(q-1)$ and equality holds if and only if B is the point set of a hyperplane of $\mathrm{PG}(d,q)$.

In $\mathrm{PG}(3,q)$, consider an elliptic quadric $Q^-(3,q)$, a hyperbolic quadric $Q^+(3,q)$ and a quadratic cone $\mathcal{K}$ with base an irreducible conic in some plane of $\mathrm{PG}(3,q)$. Let $\mathcal{Q} \in \{Q^-(3,q), Q^+(3,q), \mathcal{K}\}$. Every line of $\mathrm{PG}(3,q)$ meets $\mathcal{Q}$ in either $0,1,2$ or $q+1$ points. A line L of $\mathrm{PG}(3,q)$ is called *external* if $|L\cap\mathcal{Q}|=0$, *secant* if $|L\cap\mathcal{Q}|=2$, and *tangent* if $|L\cap\mathcal{Q}|=1$ or $q+1$. We denote by $\mathcal{E}$, $\mathcal{S}$ and $\mathcal{T}$ the set of all lines of $\mathrm{PG}(3,q)$ that are external, secant and tangent, respectively, with respect to $\mathcal{Q}$.

In this thesis, for $\mathcal{Q} \in \{Q^-(3,q), \mathcal{K}\}$, we study the minimum size $\mathcal{L}$-blocking sets in $\mathrm{PG}(3,q)$, where the line set $\mathcal{L}$ is one of $\mathcal{E}$, $\mathcal{T}$, $\mathcal{S}$, $\mathcal{E} \cup \mathcal{T}$, $\mathcal{E} \cup \mathcal{S}$ and $\mathcal{T} \cup \mathcal{S}$. We note that, when $\mathcal{Q} = Q^+(3,q)$, the minimum size blocking sets in $\mathrm{PG}(3,q)$ with respect to such line sets have already been characterized in the papers [8, 9, 22, 23, 54, 55, 56], also see [57]. When $q = 2$ and $\mathcal{Q} \in \{Q^-(3,2), Q^+(3,2)\}$, we classify all minimal $\mathcal{E}$-blocking sets in $\mathrm{PG}(3,2)$, up to isomorphisms. When $q = 3$ and $\mathcal{Q} \in \{Q^-(3,3), Q^+(3,3)\}$, we classify all next-to-minimum size $\mathcal{E}$-blocking sets in $\mathrm{PG}(3,3)$, up to isomorphisms. We also give a characterization of the secant lines with respect to a hyperbolic quadric in $\mathrm{PG}(3,q)$ for odd $q \geqslant 7$ based on certain combinatorial properties.

Chapter 1

Preliminaries

In this chapter, we recall the basic definitions and properties of point-line geometries that are needed in the subsequent chapters. We also discuss substructures like ovals in a finite projective plane, irreducible conics in $\mathrm{PG}(2, q)$, and ovoids and quadrics in $\mathrm{PG}(3, q)$.

1.1 Point-line geometries

A *point-line geometry* is a triple $\mathcal{X} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, where $\mathcal{P}$ and $\mathcal{L}$ are disjoint sets with $\mathcal{P}$ nonempty and $\mathcal{I}$ is a subset of $\mathcal{P} \times \mathcal{L}$ such that for every $L \in \mathcal{L}$ there are at least two $x \in \mathcal{P}$ with $(x, L) \in \mathcal{I}$. The elements of $\mathcal{P}$ and $\mathcal{L}$ are called *points* and *lines* of $\mathcal{X}$ respectively and the set $\mathcal{I}$ is called the *incidence relation*. If $(x, L) \in \mathcal{I}$, then we say that x is incident with L, or that L is incident with x. Two distinct points of $\mathcal{X}$ are said to be *collinear* if they are incident with a common line. We say that two distinct lines of $\mathcal{X}$ *meet* or *intersect* if they are incident with a common point. If $\mathcal{P}$ is a finite set, then $\mathcal{X}$ is called a finite point-line geometry. If any two distinct points of $\mathcal{X}$ are incident with at most one line, then $\mathcal{X}$ is called a *partial linear space*. If any two distinct points of $\mathcal{X}$ are incident with exactly one line, then $\mathcal{X}$ is called a *linear space*. Clearly, every linear space is also a partial linear space. All the point-line geometries considered in this thesis are partial linear spaces.

Let $\mathcal{X} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a partial linear space. Every line $L \in \mathcal{L}$ can be identified with a unique subset $\{x \in \mathcal{P} : (x, L) \in \mathcal{I}\}$ of $\mathcal{P}$. Then for $(x, L) \in \mathcal{I}$, we also say that x is contained in L, or that L contains x, or that x lies on L, or that L passes through x. For two distinct collinear points x and y of $\mathcal{X}$, we denote by xy the unique line of $\mathcal{X}$ containing both x and y. A subset S of $\mathcal{P}$ is called a *subspace* of $\mathcal{X}$ if every line that contains at least two points of S has all its point in S. The intersection of any collection of subspaces of $\mathcal{X}$ is again a subspace. For a subset U of $\mathcal{P}$, define $\langle U \rangle$ to be the intersection of all subspaces of $\mathcal{X}$ containing U. Then $\langle U \rangle$ is called the *subspace of* $\mathcal{X}$ *generated by* U. Thus $\langle U \rangle$ is the smallest subspace of $\mathcal{X}$ containing U. If $\langle U \rangle = \mathcal{P}$, then U is called a *generating set* of $\mathcal{X}$. The *generating index* of $\mathcal{X}$ is the minimal size of a generating set of $\mathcal{X}$.

Let $\mathcal{X} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ and $\mathcal{X}' = (\mathcal{P}', \mathcal{L}', \mathcal{I}')$ be two partial linear spaces. An *isomorphism* from $\mathcal{X}$ to $\mathcal{X}'$ is a bijective map ϕ from $\mathcal{P} \cup \mathcal{L}$ to $\mathcal{P}' \cup \mathcal{L}'$ such that $\phi(\mathcal{P}) = \mathcal{P}'$, $\phi(\mathcal{L}) = \mathcal{L}'$, and $(x, L) \in \mathcal{I}$ if and only if $(\phi(x), \phi(L)) \in \mathcal{I}'$. We say that $\mathcal{X}$ and $\mathcal{X}'$ are *isomorphic*, denoted by $\mathcal{X} \cong \mathcal{X}'$, if there is an isomorphism between $\mathcal{X}$ and $\mathcal{X}'$. Any isomorphism from $\mathcal{X}$ to itself is called an *automorphism* of $\mathcal{X}$.

Let $\mathcal{X} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a partial linear space such that every point of $\mathcal{X}$ is contained in at least two lines. Then the *(point-line) dual of* $\mathcal{X}$ is the point-line geometry $\mathcal{X}^D = (\mathcal{P}^D, \mathcal{L}^D, \mathcal{I}^D)$, where $\mathcal{P}^D = \mathcal{L}$, $\mathcal{L}^D = \mathcal{P}$, and $\mathcal{I}^D \subseteq \mathcal{P}^D \times \mathcal{L}^D = \mathcal{L} \times \mathcal{P}$ such that $(L, x) \in \mathcal{I}^D$ if and only if $(x, L) \in \mathcal{I}$ for $x \in \mathcal{P}$ and $L \in \mathcal{L}$. We say that $\mathcal{X}$ is *self-dual* if $\mathcal{X}^D \cong \mathcal{X}$.

1.2 Projective planes

A *projective plane* is a linear space in which any two distinct lines meet at exactly one point and there exist four points such that no three of them are collinear. The generating index of a projective plane is three.

Let $\mathcal{X} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a projective plane. The point-line dual $\mathcal{X}^D$ of $\mathcal{X}$ is again a projective plane. For subsets $\mathcal{P}_0$ of $\mathcal{P}$ and $\mathcal{L}_0$ of $\mathcal{L}$, the triple $\mathcal{X}_0 = (\mathcal{P}_0, \mathcal{L}_0, \mathcal{I}_0)$ with $\mathcal{I}_0$ the restriction of $\mathcal{I}$ to $\mathcal{P}_0 \times \mathcal{L}_0$ is called a *subplane* of $\mathcal{X}$ if $\mathcal{X}_0$ itself is a projective plane. For every point-line pair $(x, L) \in \mathcal{P} \times \mathcal{L}$, there is a bijective correspondence between the set of lines of $\mathcal{X}$ that are incident with x and the set of points of $\mathcal{X}$ that are incident with L.

Now suppose that $\mathcal{X}$ is a finite projective plane. Then there exists a positive integer $n \geqslant 2$ such that every point of $\mathcal{X}$ is contained in exactly $n+1$ lines and every line of $\mathcal{X}$ contains exactly $n+1$ points. The integer n is called the *order* of $\mathcal{X}$ and we have $|\mathcal{P}| = |\mathcal{L}| = n^2 + n + 1$.

If $\mathcal{X}$ contains a proper subplane $\mathcal{X}_0$ of order $n_0 < n$, then either $n = n_0^2$ or $n \geqslant n_0^2 + n_0$ (see [31, Theorem 3.7] or [33, Theorem 1.12]). If $n = n_0^2$, then $\mathcal{X}_0$ is called a *Baer subplane* of $\mathcal{X}$. In that case, the lines of $\mathcal{X}_0$ are called *Baer lines*. Thus Baer subplanes of $\mathcal{X}$ cannot exist unless n is a perfect square. We have that $\mathcal{X}_0$ is a Baer subplane of $\mathcal{X}$ if and only if each line of $\mathcal{X}$ is incident with at least one point of $\mathcal{X}_0$. If $\mathcal{X}_0$ is a Baer subplane of $\mathcal{X}$ and x is a point of $\mathcal{X}$ but not a point of $\mathcal{X}_0$, then there is exactly one line of $\mathcal{X}$ through x that is also a line of $\mathcal{X}_0$ and each of the remaining lines of $\mathcal{X}$ through x is incident with exactly one point of $\mathcal{X}_0$, see [14]. We note that every known finite projective plane is of prime power order.

Ovals

Let $\mathcal{X}$ be a finite projective plane of order n. A *k-arc* in $\mathcal{X}$ is a set of k points such that no three of them are collinear. For every k-arc in $\mathcal{X}$, we have $k \leqslant n+1$ if n is odd, and $k \leqslant n+2$ if n is even. Any $(n+1)$-arc in $\mathcal{X}$ is called an *oval*. If n is even, then any $(n+2)$-arc in $\mathcal{X}$ is called a *hyperoval*. An oval can be obtained from a hyperoval by removing a point.

Let O be an oval in $\mathcal{X}$. We refer to the book [43] for the basics properties of points and lines of $\mathcal{X}$ with respect to O. Every line of $\mathcal{X}$ meets O in at most two points. A line L of $\mathcal{X}$ is called *external* (respectively, *tangent*, *secant*) with respect to O if $|L \cap O| = 0$ (respectively, $|L \cap O| = 1$, $|L \cap O| = 2$). There are $n(n+1)/2$ lines of $\mathcal{X}$ secant to O. Every point of O is contained in a unique tangent line and n secant lines of $\mathcal{X}$. Thus there are exactly $n+1$ lines of $\mathcal{X}$ tangent to O and so the number of external lines to O is $n^2+n+1-(n+1)-n(n+1)/2 = n(n-1)/2$.

First consider the case that n is even. Then all the $n+1$ tangent lines of $\mathcal{X}$ with respect to O meet in a unique common point, which is called the *nucleus* of O. There is a unique hyperoval in $\mathcal{X}$ containing O, and this hyperoval consists of the points of O and the nucleus of O. Every point of $\mathcal{X}$ different from the points of O and its nucleus is contained in one tangent line, $n/2$ external lines and $n/2$ secant lines.

Now consider the case that n is odd. Then every point of $\mathcal{X}$ is contained in at most two tangent lines. We call a point x of $\mathcal{X}$ *interior* (respectively, *absolute*, *exterior*) with respect to O if x is contained in 0 (respectively, 1, 2) tangent lines. The points of O are precisely the absolute points of $\mathcal{X}$. So there are $n+1$ absolute points of $\mathcal{X}$. There are $n(n+1)/2$ exterior points and $n(n-1)/2$ interior points of $\mathcal{X}$. Every exterior point of $\mathcal{X}$ is contained in two tangent lines, $(n-1)/2$ secant lines and $(n-1)/2$ external lines. Every interior point of $\mathcal{X}$ is contained in $(n+1)/2$ secant lines and $(n+1)/2$ external lines. Every tangent line of $\mathcal{X}$ contains one absolute point and n exterior points. Every external line of $\mathcal{X}$ contains $(n+1)/2$ exterior points and $(n+1)/2$ interior points. Every secant line of $\mathcal{X}$ contains two absolute points, $(n-1)/2$ exterior points and $(n-1)/2$ interior points.

1.3 Projective spaces

A linear space $\mathcal{X} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ is called a *projective space* if it satisfies the following conditions:

P1. Every line is incident with at least three points and there are at least two distinct lines.

P2. *Veblen-Young Axiom*: If w, x, y, a, b are five mutually distinct points of $\mathcal{X}$ such that the lines ab and xy intersect in a single point w, then the lines xa and yb also meet in a point.

In a projective space, the geometry induced on a subspace containing at least two distinct lines is again a projective space. The *dimension* of a subspace S of a projective space is defined to be one less than the generating index of S. The dimension of a projective space is at least two, and equality holds if and only if it is a projective plane.

Classical projective spaces

Consider a right vector space V of dimension at least three over a division ring $\mathbb{K}$. Let $\mathrm{PG}(V)$ denote the point-line geometry whose points are the one dimensional subspaces of V, lines are the two dimensional subspaces of V and incidence is containment. Then $\mathrm{PG}(V)$ is a projective space of dimension $\dim(V) - 1$. If V is of finite dimension $d + 1$ for some positive integer $d \geqslant 2$, then $\mathrm{PG}(V)$ is of dimension d and it is denoted by $\mathrm{PG}(d, \mathbb{K})$. If $\mathbb{K}$ is a finite division ring, then $\mathbb{K}$ is a field by a theorem of Wedderburn and so $\mathbb{K}$ is the finite field $\mathbb{F}_q$ for some prime power q. In this case, $\mathrm{PG}(d, \mathbb{K})$ will be denoted by $\mathrm{PG}(d, q)$. For $d = 2$, $\mathrm{PG}(2, q)$ is an example of a projective plane, but not every projective plane is of this form.

Let $\mathcal{X} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a projective space. Consider a Desargues configuration (in the sense of [63, Page 73]) $x, a_1, a_2, a_3, b_1, b_2, b_3$ consisting of seven distinct points of $\mathcal{X}$, that is, these points satisfy the following two conditions:

(i) the points a_1, a_2, a_3 are not collinear and the points b_1, b_2, b_3 are not collinear in $\mathcal{X}$;

(ii) the lines a_1b_1, a_2b_2, a_3b_3 are mutually distinct and intersect at the point x.

If the intersection points z_{12}, z_{13} and z_{23}, where $z_{ij} := a_ia_j \cap b_ib_j$ for $1 \leqslant i < j \leqslant 3$, are collinear in $\mathcal{X}$ for all possible Desargues configurations in $\mathcal{X}$, then the projective space $\mathcal{X}$ is called *Desarguesian*. Every Desarguesian projective space is isomorphic to $\mathrm{PG}(V)$ for some right vector space V of dimension at least three over a division ring. Every projective space of dimension at least three is Desarguesian. In particular, every finite projective space of dimension $d \geqslant 3$ is isomorphic to $\mathrm{PG}(d,q)$ for some prime power q. However, there are projective planes which are not isomorphic to $\mathrm{PG}(2,q)$.

Consider the projective space $\mathrm{PG}(d,q)$ associated with a $(d+1)$-dimensional vector space V over the field $\mathbb{F}_q$. If U is a $(t+1)$-dimensional vector subspace of V, then the set of one-dimensional subspaces of U forms a t-dimensional subspace of $\mathrm{PG}(d,q)$. Conversely, if S is a t-dimensional subspace of $\mathrm{PG}(d,q)$, then there exists a $(t+1)$-dimensional vector subspace U of V such that the points of S are precisely the one-dimensional subspace of U. The points and lines of $\mathrm{PG}(d,q)$ are precisely the subspaces of dimensions 0 and 1, respectively. The 2-dimensional subspaces of $\mathrm{PG}(d,q)$ are called *planes* and the $(d-1)$-dimensional subspaces of $\mathrm{PG}(d,q)$ are called *hyperplanes*. Each line of $\mathrm{PG}(d,q)$ meets every hyperplane. The number of points in $\mathrm{PG}(d,q)$ is equal to $(q^{d+1}-1)/(q-1)$. More generally, for $0 \leqslant t \leqslant d$, the number of t-dimensional subspaces of $\mathrm{PG}(d,q)$ is

$$\frac{(q^{d+1}-1)(q^{d+1}-q)\cdots(q^{d+1}-q^t)}{(q^{t+1}-1)(q^{t+1}-q)\cdots(q^{t+1}-q^t)}.$$

If $T : V \longrightarrow V$ is an invertible linear transformation, then T induces an automorphism of $\mathrm{PG}(d,q)$ which maps a point $\langle v\rangle$ to the point $\langle T(v)\rangle$ for $v \in V\backslash\{0\}$. Such automorphisms of $\mathrm{PG}(d,q)$ are called *projective transformations*. Two invertible linear transformations T_1 and T_2 of V induce the same projective transformation of $\mathrm{PG}(d,q)$ if and only if $T_2 = \alpha T_1$ for some nonzero element $\alpha \in \mathbb{F}_q$. The group of all projective transformations of $\mathrm{PG}(d,q)$ is denoted by $PGL(d+1,q)$. We say that two point sets in $\mathrm{PG}(d,q)$ are *projectively equivalent* if one of them can be mapped into the other by an element of $PGL(d+1,q)$.

Let $\mathrm{PG}(d,q)^*$ denote the point-line geometry whose points are the $(d-1)$-dimensional subspaces of $\mathrm{PG}(d,q)$ (that is, hyperplanes of $\mathrm{PG}(d,q)$), lines are the $(d-2)$-dimensional subspaces of $\mathrm{PG}(d,q)$ and a point U is incident with a line W in $\mathrm{PG}(d,q)^*$ if and only if the subspace U contains W in $\mathrm{PG}(d,q)$. Then $\mathrm{PG}(d,q)^*$ is a projective space of dimension d. Note that $\mathrm{PG}(d,q)^*$ is also often called the dual space of $\mathrm{PG}(d,q)$. A *duality* of $\mathrm{PG}(d,q)$ is an isomorphism from $\mathrm{PG}(d,q)$ to $\mathrm{PG}(d,q)^*$. A duality ϕ of $\mathrm{PG}(d,q)$ is called a *polarity* if the permutation of the set of subspaces of $\mathrm{PG}(d,q)$ induced by ϕ has order 2.

1.4 Bilinear and quadratic forms

Let V be a vector space of finite dimension $d+1$ over a field $\mathbb{F}$. A map $B : V \times V \longrightarrow \mathbb{F}$ is said to be a *bilinear form* on V if it is linear in both components, that is,

(i) $B(\bar{u}+\lambda\bar{v},\bar{w}) = B(\bar{u},\bar{w}) + \lambda B(\bar{v},\bar{w})$;

(ii) $B(\bar{u},\bar{v}+\lambda\bar{w}) = B(\bar{u},\bar{v}) + \lambda B(\bar{u},\bar{w})$

for all $\bar{u},\bar{v},\bar{w} \in V$ and $\lambda \in \mathbb{F}$.

Let B be a bilinear form on V. The form B is called *nondegenerate* if $\bar{u} = \bar{0}$ is the only vector in V for which $B(\bar{u},\bar{v}) = 0$ for all $\bar{v} \in V$, equivalently, if $\bar{u} = \bar{0}$ is the only vector in V for which $B(\bar{v},\bar{u}) = 0$ for all $\bar{v} \in V$. If B is not nondegenerate, then it is called *degenerate*. For a subspace W of V, the set $W^\perp := \{\bar{v} \in V \mid B(\bar{v},\bar{w}) = 0 \text{ for all } \bar{w} \in W\}$ is a subspace of V. We have $V^\perp = \{\bar{0}\}$ if and only if B is nondegenerate. If B is nondegenerate, then $\dim W + \dim W^\perp = d+1 = \dim V$ for every subspace W of V. A subspace W of V is called *totally isotropic* if $B(\bar{x},\bar{y}) = 0$ for all $\bar{x},\bar{y} \in W$, that is, if W is contained in $W^\perp$.

If $B(\bar{u},\bar{v}) = B(\bar{v},\bar{u})$ for all $\bar{u},\bar{v} \in V$, then B is called a *symmetric* form on V. If $B(\bar{v},\bar{v}) = 0$ for all $\bar{v} \in V$, then B is called an *alternating* form on V. If B is an alternating form, then $B(\bar{v},\bar{u}) = -B(\bar{u},\bar{v})$ for all $\bar{u},\bar{v} \in V$. If B is a nondegenerate alternating form on V, then $\dim V$ must be even and such a form is also called a *symplectic form* on V.

A *quadratic form* on V is a function $Q : V \longrightarrow \mathbb{F}$ such that the following two conditions are satisfied:

(i) $Q(\lambda\bar{u}) = \lambda^2 Q(\bar{u})$ for all $\bar{u} \in V$ and $\lambda \in \mathbb{F}$.

(ii) The map $B : V \times V \longrightarrow \mathbb{F}$ defined by $B(\bar{u},\bar{v}) := Q(\bar{u}+\bar{v}) - Q(\bar{u}) - Q(\bar{v})$ for $\bar{u},\bar{v} \in V$ is a symmetric bilinear form on V.

Let Q be a quadratic form on V and B be the associated symmetric bilinear form on V. If $V^{\perp} \cap Q^{-1}(0) = \{\bar{0}\}$, then Q is called *nondegenerate*. Thus Q is nondegenerate if for every nonzero $\bar{v} \in V$ with $Q(\bar{v}) = 0$, there exists $\bar{u} \in V$ with $B(\bar{v},\bar{u}) \neq 0$. Clearly, Q is nondegenerate if B is nondegenerate. The converse is true if the characteristic of $\mathbb{F}$ is odd. The converse statement need not be true if the characteristic of $\mathbb{F}$ is two. For example: Let $V = \mathbb{F}^3$, where $\mathbb{F}$ is of characteristic 2. The quadratic form Q on V defined by $Q(\bar{u}) := u_0^2 + u_1u_2$ for $\bar{u} = (u_0,u_1,u_2) \in V$ is nondegenerate. If B is the symmetric bilinear form on V associated with Q, then $B(\bar{u},\bar{v}) = u_1v_2 + u_2v_1$ for $\bar{u} = (u_0,u_1,u_2)$ and $\bar{v} = (v_0,v_1,v_2)$ in V. The form B is degenerate as $(1,0,0) \in V^{\perp}$.

A one-dimensional subspace $\langle\bar{v}\rangle$ of V is called *singular* or *nonsingular* according as $Q(\bar{v}) = 0$ or not. A subspace W of V is called *totally singular* if $Q(\bar{w}) = 0$ for all $\bar{w} \in W$. Every totally singular subspace of V is totally isotropic with respect to B. When $\mathbb{F}$ has odd characteristic, the converse is also true. If Q is nondegenerate and W is a totally singular subspace of V, then $\dim W \leqslant (d+1)/2 = \dim V/2$.

Let $\mathcal{Q}$ be the set of points $\langle\bar{v}\rangle$, $\bar{v} \in V \setminus \{\bar{0}\}$, of $\mathrm{PG}(d,\mathbb{F})$ such that $Q(\bar{v}) = 0$. The set $\mathcal{Q}$ is called a *quadric* in $\mathrm{PG}(d,\mathbb{F})$ with respect to Q. We say that $\mathcal{Q}$ is a *nondegenerate quadric* in $\mathrm{PG}(d, \mathbb{F})$ if Q is a nondegenerate quadratic form on V. If $d = 2$ (so that $\dim V = 3$), then the quadric $\mathcal{Q}$ in $\mathrm{PG}(2, \mathbb{F})$ is called a *conic*. Nondegenerate quadrics in $\mathrm{PG}(2, \mathbb{F})$ are called *irreducible conics*. If $\mathcal{Q}$ is a nondegenerate quadric, then the *Witt index* of $\mathcal{Q}$ is the maximum vector space dimension of a totally singular subspace contained in $\mathcal{Q}$.

If the symmetric bilinear map B on V associated with Q is nondegenerate, then the map $\tau : W \to W^{\perp}$ for subspaces W of V is a polarity of $\mathrm{PG}(d, \mathbb{F})$ as $\dim V = d + 1$ is finite. This polarity τ is called a *symplectic polarity* or an *orthogonal polarity* according as the characteristic of $\mathbb{F}$ is even or odd.

Let $\mathcal{Q}$ be a quadric in $\mathrm{PG}(d, q)$. Then every line of $\mathrm{PG}(d, q)$ meets $\mathcal{Q}$ in $0, 1, 2$ or $q + 1$ points. A line L of $\mathrm{PG}(d, q)$ is called *external* if $|L \cap \mathcal{Q}| = 0$, *secant* if $|L \cap \mathcal{Q}| = 2$, and *tangent* if $|L \cap \mathcal{Q}| = 1$ or $q + 1$. A tangent line of $\mathrm{PG}(d, q)$ is called an *outer tangent* or an *inner tangent* with respect to $\mathcal{Q}$ according as it meets $\mathcal{Q}$ in 1 or $q + 1$ points.

1.4.1 Irreducible conics in $\mathrm{PG}(2, q)$

Let $\mathcal{C}$ be an irreducible conic in $\mathrm{PG}(2, q)$. Then $\mathcal{C}$ is the set of points $\langle (u_0, u_1, u_2) \rangle$ of $\mathrm{PG}(2, q)$ satisfying a nonzero homogeneous quadratic polynomial equation $Q(X, Y, Z) = 0$ in three variables, where $Q(X, Y, Z) \in \mathbb{F}_q[X, Y, Z]$ is irreducible. By a suitable linear change of coordinates, $\mathcal{C}$ is equivalent to the conic defined by $XZ - Y^2 = 0$. Thus we can write

$$\mathcal{C} = \{\langle (1, t, t^2) \rangle : t \in \mathbb{F}_q\} \cup \{\langle (0, 0, 1) \rangle\}.$$

The conic $\mathcal{C}$ has $q + 1$ points and no three of them are collinear. Thus $\mathcal{C}$ is an oval in $\mathrm{PG}(2, q)$ and hence all the properties of points and lines with respect to an oval hold true with respect to $\mathcal{C}$ as well. If q is odd, then by a result of Segre [58] the irreducible conics are precisely the ovals in $\mathrm{PG}(2, q)$. If $q \in \{2, 4\}$, then every oval in $\mathrm{PG}(2, q)$ is also an irreducible conic [17, Theorem 4.9]. However, for $q \geqslant 8$

even, there are ovals in $\mathrm{PG}(2,q)$ which are not irreducible conics [17, Theorem 4.11]. The classification of these ovals is a challenging and ongoing problem.

1.4.2 Quadrics in $\mathrm{PG}(3,q)$

There are two types of nondegenerate quadrics in $\mathrm{PG}(3,q)$: (1) Elliptic quadrics which are of Witt index one, and (2) Hyperbolic quadrics which are of Witt index two. These are of special interest to us along with degenerate *quadratic cones*. We refer to [30] for the basic properties of points, lines and planes of $\mathrm{PG}(3,q)$ with respect to these quadrics.

Elliptic and hyperbolic quadrics

Let $V=\mathbb{F}_q^4$ be the four-dimensional vector space over $\mathbb{F}_q$ and let Q be a nondegenerate quadratic form on V. By a linear change of coordinates, Q is given by either $Q(x_0,x_1,x_2,x_3)=x_0x_1+ax_2^2+bx_2x_3+cx_3^2$; or $Q(x_0,x_1,x_2,x_3)=x_0x_1+x_2x_3$ for $(x_0,x_1,x_2,x_3)\in V$, where the quadratic polynomial $aX^2+bX+c\in\mathbb{F}_q[X]$ is irreducible over $\mathbb{F}_q$. The associated quadratic $Q^-(3,q)$ in $\mathrm{PG}(3,q)$ is of Witt index 1 in the first case and is called an *elliptic quadric*. In the latter case, the associated quadric $Q^+(3,q)$ is of Witt index 2 and is called a *hyperbolic quadric*.

Let $Q^-(3,q)$ be an elliptic quadric in $\mathrm{PG}(3,q)$. There are no inner tangent lines. Then $Q^-(3,q)$ contains q^2+1 points of $\mathrm{PG}(3,q)$ and it meets every line of $\mathrm{PG}(3,q)$ in at most two points. Every point of $Q^-(3,q)$ is contained in $q+1$ tangent lines, this gives $(q+1)(q^2+1)$ tangent lines to $Q^-(3,q)$. We also have $q^2(q^2+1)/2$ secant lines and then $(q^2+1)(q^2+q+1)-q^2(q^2+1)/2-(q+1)(q^2+1)=q^2(q^2+1)/2$ external lines to $Q^-(3,q)$. Every point of $Q^-(3,q)$ is contained in q^2 secant lines. Every point of $\mathrm{PG}(3,q)\setminus Q^-(3,q)$ is contained in $q+1$ tangent lines, $q(q-1)/2$ secant lines and $q(q+1)/2$ external lines.

Let $Q^+(3,q)$ be a hyperbolic quadric in $\mathrm{PG}(3,q)$. Then $Q^+(3,q)$ contains $(q+1)^2$ points of $\mathrm{PG}(3,q)$ and every line of $\mathrm{PG}(3,q)$ meets $Q^+(3,q)$ in $0,1,2$

or $q+1$ points. Every point of $Q^+(3,q)$ is contained in $q+1$ tangent lines (out of which 2 are inner tangents and $q-1$ are outer tangents) and q^2 secant lines. Every point of $\mathrm{PG}(3,q) \setminus Q^+(3,q)$ is contained in $q+1$ outer tangents, $q(q+1)/2$ secant lines and $q(q-1)/2$ external lines. There are $2(q+1)$ inner tangents, $(q-1)(q+1)^2$ outer tangents, $q^2(q+1)^2/2$ secant lines and $q^2(q-1)^2/2$ external lines to $Q^+(3,q)$.

With the quadric $Q^\epsilon(3,q)$, $\epsilon \in \{-,+\}$, there is naturally associated a polarity τ of $\mathrm{PG}(3,q)$ which is symplectic if q is even, and orthogonal if q is odd. For a point x of $\mathrm{PG}(3,q)$, the plane x^τ is called a *tangent* or *secant* plane according as x is a point of $Q^\epsilon(3,q)$ or not. For every point x of $\mathrm{PG}(3,q) \setminus Q^\epsilon(3,q)$, the secant plane x^τ intersects $Q^\epsilon(3,q)$ in an irreducible conic. The map $x \mapsto x^\tau \cap Q^\epsilon(3,q)$ defines a bijection between $\mathrm{PG}(3,q) \setminus Q^\epsilon(3,q)$ and the set of irreducible conics contained in $Q^\epsilon(3,q)$. For every point x of $Q^\epsilon(3,q)$, the tangent plane x^τ intersects $Q^\epsilon(3,q)$ at the point x if $\epsilon = -$, and in the union of the two inner tangents through x if $\epsilon = +$. In both cases, the $q+1$ tangent lines through x are precisely the lines through x contained in x^τ.

Suppose that q is odd. Then, for every point x of $\mathrm{PG}(3,q) \setminus Q^\epsilon(3,q)$, the secant plane x^τ does not contain the point x and the tangent lines through x are precisely the $q+1$ lines through x meeting the conic $x^\tau \cap Q^\epsilon(3,q)$.

Suppose that q is even. Then, for every point x of $\mathrm{PG}(3,q) \setminus Q^\epsilon(3,q)$, the secant plane x^τ contains the point x and the tangent lines contained in x^τ are precisely the $q+1$ lines of x^τ through x. Thus the point x is the nucleus of the conic $x^\tau \cap Q^\epsilon(3,q)$ in x^τ.

Consider the points, lines and planes of $\mathrm{PG}(3,q)$ with respect to $Q^-(3,q)$. There are q^2+1 tangent planes and q^3+q secant planes of $\mathrm{PG}(3,q)$ with respect to $Q^-(3,q)$. Every point of $Q^-(3,q)$ is contained in one tangent plane and q^2+q secant planes. Every point of $\mathrm{PG}(3,q) \setminus Q^-(3,q)$ is contained in $q+1$ tangent planes and q^2 secant planes. Every tangent line is contained in one tangent plane and q secant planes. Every secant line is contained in $q+1$ secant planes. Every external line is contained in two tangent planes and $q-1$ secant planes.

Now, consider the points, lines and planes of $\mathrm{PG}(3,q)$ with respect to $Q^+(3,q)$. There are $(q+1)^2$ tangent planes and q^3-q secant planes of $\mathrm{PG}(3,q)$ with respect to $Q^+(3,q)$. Every point of $Q^+(3,q)$ is contained in $2q+1$ tangent planes and $q(q-1)$ secant planes. Every point of $\mathrm{PG}(3,q) \setminus Q^+(3,q)$ is contained in $q+1$ tangent planes and q^2 secant planes. Every inner tangent is contained in $q+1$ tangent planes, every outer tangent is contained in one tangent plane and q secant planes, every external line is contained in $q+1$ secant planes, and every secant line is contained in two tangent planes and $q-1$ secant planes. Each tangent plane contains $q+1$ tangent lines and q^2 secant lines. Every secant plane contains $q+1$ tangent lines, $\frac{q(q+1)}{2}$ secant lines and $\frac{q(q-1)}{2}$ external lines. Every pencil of lines in a tangent plane contains 0 or q secant lines. If q is odd, then every pencil of lines in a secant plane contains $\frac{q-1}{2}$, $\frac{q+1}{2}$ or q secant lines.

1.4.3 Quadratic cones

Let π^* be a plane in $\mathrm{PG}(3,q)$, $\mathcal{C}$ be an irreducible conic in π^* and p^* be a point of $\mathrm{PG}(3,q) \setminus \pi^*$. A quadratic cone $\mathcal{K}$ in $\mathrm{PG}(3,q)$ with base $\mathcal{C}$ and kernel p^* is the set of points on the lines joining p^* with the points of $\mathcal{C}$. Thus $\mathcal{K}$ consists of the q^2+q+1 points of $\mathrm{PG}(3,q)$ that are contained in the $q+1$ lines through p^* meeting $\mathcal{C}$. Every line of $\mathrm{PG}(3,q)$ intersects $\mathcal{K}$ in either $0, 1, 2$ or $q+1$ points. There are $q^3(q+1)/2$ secant lines, q^3+2q^2+q+1 tangent lines and $q^3(q-1)/2$ external lines of $\mathrm{PG}(3,q)$ with respect to $\mathcal{K}$.

A plane π of $\mathrm{PG}(3,q)$ intersects $\mathcal{K}$ either in the point p^*, in a line (through p^*), in two lines (intersecting at p^*), or in an irreducible conic. If $\pi \cap \mathcal{K}$ is an irreducible conic, then we call π a *secant plane*. The planes of $\mathrm{PG}(3,q)$ not passing through p^* are precisely the secant planes, giving in total q^3 secant planes of $\mathrm{PG}(3,q)$. There are q^2 secant planes through a point of $\mathrm{PG}(3,q) \setminus \{p^*\}$, and q secant planes through a line not containing p^*.

1.5 Ovoids in PG$(3, q)$

An *ovoid* in PG$(3, q)$ is a set of $q^2 + 1$ points intersecting each plane in either a singleton or an oval. For $q > 2$, the ovoids in PG$(3, q)$ are precisely the subsets of the point set of PG$(3, q)$ of largest possible size with the property that no three points of the set are collinear. In PG$(3, 2)$, the complement of a plane is a subset of size $8 > 2^2 + 1$ in which no three points are collinear, but it is not an ovoid in PG$(3, 2)$.

Every elliptic quadric in PG$(3, q)$ is an example of an ovoid. In fact, if q is odd, then every ovoid in PG$(3, q)$ is also an elliptic quadric [5, 47]. If $q > 2$ is even and a nonsquare (that is, $q = 2^t$ for some odd integer $t \geqslant 3$), then another type of ovoids in PG$(3, q)$ is known which are called 'Tits ovoids', see [30, Section 16.4] for more on these ovoids. All ovoids in PG$(3, q)$ are classified for $q \in \{2, 4, 8, 16, 32\}$. By [27, 44, 45, 46], every ovoid in PG$(3, q)$ is an elliptic quadric if $q \in \{2, 4, 16\}$, and either an elliptic quadric or a Tits ovoid if $q \in \{8, 32\}$. However, classifying all ovoids in PG$(3, q)$ for other even q is still an open problem.

Let $\mathcal{O}$ be an ovoid in PG$(3, q)$. If L is a line of PG$(3, q)$, then L meets $\mathcal{O}$ in at most two points. We say that L is an *external*, *tangent* or *secant line* depending on whether $|L \cap \mathcal{O}|$ is equal to 0, 1 or 2.

Suppose that q is even. Then there is a symplectic polarity τ of PG$(3, q)$ associated with $\mathcal{O}$ [59]. Planes intersecting $\mathcal{O}$ in a singleton are called *tangent planes* while planes intersecting $\mathcal{O}$ in ovals are called *secant planes*. If $x \in \mathcal{O}$, then x^τ is a tangent plane through x and every other plane through x is a secant plane. Through x, there are $q + 1$ tangent lines (namely the $q + 1$ lines of x^τ through x) and q^2 secant lines. If $x \in \mathrm{PG}(3, q) \setminus \mathcal{O}$, then x^τ is a secant plane intersecting $\mathcal{O}$ in $q+1$ points, say $y_1, y_2, \ldots, y_{q+1}$. The planes $y_1^\tau, y_2^\tau, \ldots, y_{q+1}^\tau$ are the $q + 1$ tangent planes through x, while the remaining q^2 planes through x are secant planes. The point x is contained in precisely $q + 1$ tangent lines (namely $xy_1, xy_2, \ldots, xy_{q+1}$), $\frac{q^2-q}{2}$ secant lines and $\frac{q^2+q}{2}$ external lines. Also, every external line of $\mathrm{PG}(3,q)$ with respect to $\mathcal{O}$ is contained in two tangent planes and $q-1$ secant planes.

1.6 Generalized quadrangles

Let s and t be positive integers. A *generalized quadrangle of order* (s,t) is a partial linear space $\mathcal{X} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ satisfying the following axioms:

(Q1) Every line of $\mathcal{X}$ is incident with $s+1$ points and every point of $\mathcal{X}$ is incident with $t+1$ lines.

(Q2) For every point-line pair (x, L) with x not incident with L, there exists a unique line M of $\mathcal{X}$ incident with x and intersecting L.

If $s = t$, then the generalized quadrangle $\mathcal{X}$ is said to have order s. See [49] for background material on generalized quadrangles.

Let $\mathcal{X} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a generalized quadrangle of order (s,t). Then $|\mathcal{P}| = (s+1)(st+1)$ and $|\mathcal{L}| = (t+1)(st+1)$, see [49, 1.2.1]. A subset $\mathcal{O}$ of $\mathcal{P}$ such that every line of $\mathcal{X}$ contains exactly one point of $\mathcal{O}$ is called an *ovoid* of $\mathcal{X}$. If $\mathcal{X}$ has an ovoid $\mathcal{O}$, then $|\mathcal{O}| = st+1$ (this follows counting in two ways the number of point-line pairs (x, L), where $x \in \mathcal{O}$ and L is a line of $\mathcal{X}$ containing x).

Two points of $\mathcal{X}$ are called *collinear* if there is a line of $\mathcal{X}$ containing them. For a point set A of $\mathcal{X}$, the set $A^{\perp}$ consists of all points of $\mathcal{X}$ which are collinear with every point of A. Note that, for a point x, $x^{\perp} := \{x\}^{\perp}$ contains x. For two distinct points x, y, we have $|\{x,y\}^{\perp}| = s+1$ or $t+1$ according as x is collinear with y or not. For two noncollinear points x, y, the set $\{x,y\}^{\perp\perp}$ is called the *hyperbolic line* defined by x and y.

If $\mathcal{P}$ is a subset of the point set of some projective space $\mathrm{PG}(d,q)$, $\mathcal{L}$ is a set of lines of $\mathrm{PG}(d,q)$, $\mathcal{P}$ is the union of all lines in $\mathcal{L}$ and $\mathcal{I}$ is the incidence relation induced from $\mathrm{PG}(d,q)$, then $\mathcal{X}$ is called a *projective generalized quadrangle*. The points and the lines contained in a hyperbolic quadric in $\mathrm{PG}(3,q)$ form a (projective) generalized quadrangle of order $(q, 1)$. Conversely, any projective generalized quadrangle of order $(q, 1)$ with ambient space $\mathrm{PG}(3, q)$ is a hyperbolic quadric in $\mathrm{PG}(3, q)$, see [49, 4.4.8].

Let V be a 4-dimensional vector space over $\mathbb{F}_q$ and $B : V \times V \to \mathbb{F}_q$ be a symplectic form on V. The map $U \mapsto U^\perp = \{\bar{v} \in V : B(\bar{u}, \bar{v}) = 0 \text{ for all } \bar{u} \in U\}$ for subspaces U of V defines a symplectic polarity of $\mathrm{PG}(3, q)$. The point-line geometry whose point set is that of $\mathrm{PG}(3, q)$ and line set consisting of all the totally isotropic lines of $\mathrm{PG}(3, q)$ with respect to B is a generalized quadrangle of order q, denoted by $W(q)$. It has ovoids if and only if q is even, see [49, 3.2.1, 3.4.1]. By Segre [59]: Every ovoid of $\mathrm{PG}(3, q)$, q even, is an ovoid of some $W(q)$. The converse statement is due to Thas [62]: Every ovoid of $W(q)$, q even, is an ovoid of the ambient space $\mathrm{PG}(3, q)$.

1.7 Blocking sets

Let $\mathcal{X} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a partial linear space and $\mathbb{L}$ be a nonempty subset of $\mathcal{L}$. An $\mathbb{L}$-*blocking set* in $\mathcal{X}$ is a subset B of $\mathcal{P}$ such that every line in $\mathbb{L}$ is incident with at least one point of B. An $\mathbb{L}$-blocking set B is said to be *minimal* if B has no proper subset which is also an $\mathbb{L}$-blocking set in $\mathcal{X}$. Every $\mathbb{L}$-blocking set in $\mathcal{X}$ of minimum size is a minimal $\mathbb{L}$-blocking set, but the converse need not be true.

When $\mathcal{X} = \mathrm{PG}(d, q)$, blocking sets in $\mathrm{PG}(d, q)$ are combinatorial objects in finite geometry and have been the subject of investigation by many researchers with respect to varying sets of lines. The first step in this regard has been to determine the minimum size of a blocking set and then to characterize, if possible, all blocking sets of that cardinality. The following classical result was proved by Bose and Burton in [13, Theorem 1].

Proposition 1.7.1 ([13]). *Let $\mathcal{L}$ be the set of all lines of $\mathrm{PG}(d, q)$. If B is an $\mathcal{L}$-blocking set in $\mathrm{PG}(d, q)$, then $|B| \geqslant (q^d - 1)/(q - 1)$, and equality holds if and only if B is the point set of a hyperplane of $\mathrm{PG}(d, q)$.*

The above proposition is used frequently in the subsequent chapters while studying blocking sets in $\mathrm{PG}(d,q)$, $d \in \{2,3\}$, with respect to certain subsets of the line set of $\mathrm{PG}(d,q)$.

A blocking set in $\mathrm{PG}(d,q)$ with respect to all its lines is called *nontrivial* if it does not contain any hyperplane of $\mathrm{PG}(d,q)$, equivalently, if every hyperplane of $\mathrm{PG}(d,q)$ contains a point outside the blocking set. In view of Proposition 1.7.1, one aspect to the study of blocking sets in $\mathrm{PG}(d,q)$ is to characterize the minimal nontrivial blocking sets in $\mathrm{PG}(d,q)$. When $d = 2$, minimal nontrivial blocking sets in $\mathrm{PG}(2,q)$ have been extensively studied by several authors and many results are available in the literature, see [10, 37, 51, 52, 61] for example and the references therein.

The other aspect to the study of blocking sets in $\mathrm{PG}(d,q)$ is to characterize the minimal $\mathbb{L}$-blocking sets in $\mathrm{PG}(d,q)$ for proper subsets $\mathbb{L}$ of the line set of $\mathrm{PG}(d,q)$. If $\mathbb{L}$ is the set of all lines of $\mathrm{PG}(d,q)$ which are contained in a given non-degenerate quadric $\mathcal{Q}$ in $\mathrm{PG}(d,q)$, then K. Metsch [38, 39, 40, 41, 42] has studied the minimum size $\mathbb{L}$-blocking sets in $\mathrm{PG}(d,q)$. He proved that such blocking sets can be obtained as sets consisting of the nonsingular points of quadrics $H \cap \mathcal{Q}$ for suitable hyperplanes H of $\mathrm{PG}(d,q)$. In this thesis, we investigate the minimum size blocking sets in $\mathrm{PG}(3,q)$ of certain line sets with respect to a quadric in it.

There are several applications of blocking sets in finding solutions of geometrical problems, and problems in other related research areas. The theory of blocking sets are very useful in the study of the weights of the codewords of a q-ary linear code $C_k(d,q)$. The authors in [34] used properties of blocking sets to find upper bounds for the minimum weight of the dual code $C_k(d,q)^\perp$. Using the link between blocking sets and the codewords of $C_k(d,q)$, the authors in [35] computed a gap for the weights of the codewords. In [11], the authors provided a family of minimal linear codes arising from certain blocking sets. See [50] for more on applications of blocking sets in finite geometry and other related areas.

Chapter 2

Blocking sets in $\mathrm{PG}(2,q)$

In this chapter, we recall the known results available in the literature and prove some new results on blocking sets in $\mathrm{PG}(2,q)$, which are needed in the subsequent chapters. The following result was proved in [55, Lemma 2.4], also see [9, Proposition 3.1].

Proposition 2.0.1 ([9, 55])**.** *Let x be a point of $\mathrm{PG}(2,q)$ and $\mathbb{L}$ be the set of all lines of $\mathrm{PG}(2,q)$ not containing x. If A is an $\mathbb{L}$-blocking set in $\mathrm{PG}(2,q)$, then $|A| \geqslant q$ and equality holds if and only if $A = L \setminus \{x\}$ for some line L of $\mathrm{PG}(2,q)$ through x.*

Let $\mathcal{C}$ be an irreducible conic in $\mathrm{PG}(2,q)$. If q is even, then n denotes the nucleus of $\mathcal{C}$. In this chapter, we denote by $\mathcal{E}$, $\mathcal{S}$ and $\mathcal{T}$ the set of all lines of $\mathrm{PG}(2,q)$ that are external, secant and tangent, respectively, with respect to $\mathcal{C}$. The minimum size $\mathbb{L}$-blocking sets in $\mathrm{PG}(2,q)$, where the line set $\mathbb{L}$ is one of $\mathcal{E}$, $\mathcal{T}$, $\mathcal{S}$, $\mathcal{E}\cup\mathcal{T}$, $\mathcal{E}\cup\mathcal{S}$ and $\mathcal{T}\cup\mathcal{S}$, have been determined by the contributions of several authors. One can refer to [48] for a brief survey of the results obtained in this regard.

2.1 $\mathcal{E}$-blocking sets

The minimum size $\mathcal{E}$-blocking sets in $\mathrm{PG}(2,q)$ were characterized by Giulietti in [29, Theorems 1.1, 1.2] for q even and by Aguglia and Korchmáros in [3, Theorem 1.1] for q odd. When $q=3$, it was observed in [22, Theorem 2.1] that one more possibility occurs.

Proposition 2.1.1 ([3, 22])**.** *If A is an $\mathcal{E}$-blocking set in $\mathrm{PG}(2,q)$ with q odd, then $|A| \geqslant q-1$ and the following hold for equality case:*

(i) For $q \geqslant 9$, $|A| = q-1$ if and only if $A = L \setminus \mathcal{C}$ for some secant line L of $\mathrm{PG}(2,q)$.

(ii) For $q \in \{5,7\}$, $|A| = q-1$ if and only if one of the following two cases occurs:

(a) $A = L \setminus \mathcal{C}$ for some secant line L of $\mathrm{PG}(2,q)$.

(b) A is a suitable set of $q-1$ points interior to $\mathcal{C}$.

(iii) For $q=3$, $|A|=2$ if and only if one of the following two cases occurs:

(a) $A = L \setminus \mathcal{C}$ for some secant line L of $\mathrm{PG}(2,3)$.

(b) A consists of any two interior points to $\mathcal{C}$.

Proposition 2.1.2 ([29])**.** *If A is an $\mathcal{E}$-blocking set in $\mathrm{PG}(2,q)$ with q even, then $|A| \geqslant q-1$, and equality holds if and only if one of following three cases occurs:*

(i) $A = L \setminus \mathcal{C}$ for some secant line L of $\mathrm{PG}(2,q)$.

(ii) $A = L \setminus (\{n\} \cup \mathcal{C})$ for some tangent line of $\mathrm{PG}(2,q)$.

(iii) q is a square and $A = \Pi \setminus (\{n\} \cup \mathcal{C})$, where Π is a Baer subplane of $\mathrm{PG}(2,q)$ containing n such that $|\Pi \cap \mathcal{C}| = \sqrt{q}+1$.

The following is an immediate consequence of Propositions 2.1.1 and 2.1.2.

Corollary 2.1.3. *Let A be an $\mathcal{E}$-blocking set in* $\mathrm{PG}(2,q)$*. If $|A| = q-1$, then $A \cap \mathcal{C} = \emptyset$.*

2.2 $\mathcal{T}$-blocking sets

If q is even, then the proof of the following proposition is straightforward.

Proposition 2.2.1. *Let A be a $\mathcal{T}$-blocking set in* $\mathrm{PG}(2,q)$*, where q is even. Then $|A| \geqslant 1$, and equality holds if and only if $A = \{n\}$.*

For q odd, the minimum size $\mathcal{T}$-blocking sets in $\mathrm{PG}(2,q)$ were characterized by Patra et al. in [48, Section 3.1]. They proved the following.

Proposition 2.2.2 ([48])**.** *Let A be a $\mathcal{T}$-blocking set in* $\mathrm{PG}(2,q)$*, where q is odd. Then $|A| \geqslant (q+1)/2$, and equality holds if and only if A consists of $(q+1)/2$ exterior points to $\mathcal{C}$ such that the line through any two distinct points of A is not tangent to $\mathcal{C}$.*

2.3 $\mathcal{S}$-blocking sets

A set of four points in $\mathrm{PG}(2,q)$ with the property that no three of them are collinear is called a *quadrangle*. Let $\{x,y,z,w\}$ be a quadrangle in $\mathrm{PG}(2,q)$. Then the three points a,b,c defined by $a := xy \cap zw$, $b := xz \cap yw$ and $c := xw \cap yz$ are called the *diagonal points* of the quadrangle $\{x,y,z,w\}$. These three diagonal points are collinear in $\mathrm{PG}(2,q)$ if and only if q is even [36, 9.63, p.501].

The minimum size $\mathcal{S}$-blocking sets in $\mathrm{PG}(2,q)$ were studied by Aguglia et al. in [4, Theorem 1.1] for q even and in [1, Theorem 1.1] for q odd.

Proposition 2.3.1 ([1, 4])**.** *If A is an $\mathcal{S}$-blocking set in* $\mathrm{PG}(2,q)$*, then $|A| \geqslant q$. Moreover, the following hold:*

(1) If q is odd, then $|A| = q$ if and only if $|A \setminus \mathcal{C}| \in \{0,1,3\}$ and one of the following three cases occurs:

(i) $A = \mathcal{C} \setminus \{x\}$ *for some point* $x \in \mathcal{C}$.

(ii) $A = (\mathcal{C} \setminus \{x, y\}) \cup \{a\}$ *for two points* $x, y \in \mathcal{C}$, *and for some point* a *(different from* x *and* y*) on the secant line* xy *to* $\mathcal{C}$.

(iii) $A = (\mathcal{C} \setminus \{w, x, y, z\}) \cup \{a, b, c\}$ *for some quadrangle* $\{w, x, y, z\} \subseteq \mathcal{C}$ *with diagonal points* a, b, c.

(2) If q *is even and* $|A| = q$, *then the points of* $A \setminus \mathcal{C}$ *are contained in a line of* $\mathrm{PG}(2, q)$ *which is tangent to* $\mathcal{C}$. *The set* $L \setminus \{n\}$ *is an* $\mathcal{S}$*-blocking set of size* q *for every tangent line* L *of* $\mathrm{PG}(2, q)$.

We note that, for q even, the description of the $\mathcal{S}$-blocking sets in $\mathrm{PG}(2, q)$ of minimum size q is quite different. The statement in Proposition 2.3.1(2) above was obtained while proving the main result of [4] in Section 2 (see after case (3) on page 654 of that paper). The following is an immediate consequence of Proposition 2.3.1.

Corollary 2.3.2. *Let* A *be an* $\mathcal{S}$*-blocking set in* $\mathrm{PG}(2, q)$. *If* $|A| = q$, *then* $A \cap \mathcal{C} \neq \emptyset$, *except when* $q = 3$ *and* A *consists of all the three interior points to* $\mathcal{C}$.

2.4 $(\mathcal{E} \cup \mathcal{T})$-blocking sets

Let L be a secant line of $\mathrm{PG}(2, q)$ with respect to $\mathcal{C}$ and let $L \cap \mathcal{C} = \{a, b\}$. The *pole* of L is the intersection point of the two tangent lines through a and b.

The minimum size $(\mathcal{E} \cup \mathcal{T})$-blocking sets in $\mathrm{PG}(2, q)$ were studied by Aguglia and Giulietti in [1, Theorem 1.2] for q even and by Aguglia and Korchmáros in [2, theorem 1.1] for q odd.

Proposition 2.4.1 ([1, 2])**.** *Let* A *be an* $(\mathcal{E} \cup \mathcal{T})$*-blocking set in* $\mathrm{PG}(2, q)$. *Then* $|A| \geqslant q$, *and equality holds if and only if one of the following three cases occurs:*

(i) $A = L \setminus \mathcal{C}$ *for some tangent line* L *of* $\mathrm{PG}(2, q)$.

(ii) $A = (L \setminus \mathcal{C}) \cup \{x\}$ *for some secant line* L *of* $\mathrm{PG}(2,q)$*, where* x *is the pole of* L *for* q *odd, and* $x = n$ *is the nucleus of* $\mathcal{C}$ *for* q *even.*

(iii) q *is a square and* $A = \Pi \setminus (\Pi \cap \mathcal{C})$*, where* Π *is a Baer subplane of* $\mathrm{PG}(2,q)$ *such that* $\Pi \cap \mathcal{C}$ *is an irreducible conic in* Π*.*

The examples mentioned in (i) and (ii) are always $(\mathcal{E} \cup \mathcal{T})$-blocking sets in $\mathrm{PG}(2,q)$, while this is not always the case for the examples mentioned in (iii). In (iii), it is therefore implicitly assumed that the Baer subplane Π needs to be chosen in such a way that $A = \Pi \setminus (\Pi \cap \mathcal{C})$ is a blocking set, and this is always possible. We refer to Section 2.7, in particular to Remark 2.7.2, for a further discussion of this issue. The following is an immediate consequence of Proposition 2.4.1.

Corollary 2.4.2. *Let* A *be an* $(\mathcal{E} \cup \mathcal{T})$*-blocking set in* $\mathrm{PG}(2,q)$*. If* $|A| = q$*, then* $A \cap \mathcal{C} = \emptyset$*.*

2.5 $(\mathcal{E} \cup \mathcal{S})$-blocking sets

The minimum size $(\mathcal{E} \cup \mathcal{S})$-blocking sets in $\mathrm{PG}(2,q)$ were studied by Patra et al. in [48, Section 3.2]. The following two results were proved in [48, Theorems 3.2 and 3.3].

Proposition 2.5.1 ([48])**.** *Let* A *be an* $(\mathcal{E} \cup \mathcal{S})$*-blocking set in* $\mathrm{PG}(2,q)$*, where* q *is even. Then* $|A| \geqslant q$*, and equality holds if and only if* $A = L \setminus \{n\}$ *for some tangent line* L *of* $\mathrm{PG}(2,q)$*.*

Proposition 2.5.2 ([48])**.** *Let* A *be an* $(\mathcal{E} \cup \mathcal{S})$*-blocking set in* $\mathrm{PG}(2,q)$*, where* q *is odd. Then* $|A| \geqslant 3$ *for* $q = 3$ *and* $|A| \geqslant q+1$ *for* $q \geqslant 5$*. Moreover, the following hold:*

(i) For $q = 3$*,* $|A| = 3$ *if and only if* A *consists of all the three interior points to* $\mathcal{C}$*.*

(ii) For $q = 5$, $|A| = 6$ if and only if one of the following occurs:

(a) A is a line of $\mathrm{PG}(2,5)$.

(b) $A = I \setminus \{a_1, a_2, a_3, a_4\}$, where I is the set of interior points to $\mathcal{C}$ and $\{a_1, a_2, a_3, a_4\} \subseteq I$ is a quadrangle such that the line $a_i a_j$ is external to $\mathcal{C}$ for $1 \leqslant i \neq j \leqslant 4$.

(iii) For $q \geqslant 7$, $|A| = q + 1$ if and only if A is a line of $\mathrm{PG}(2,q)$.

2.6 $(\mathcal{T} \cup \mathcal{S})$-blocking sets

Every $(\mathcal{T} \cup \mathcal{S})$-blocking set in $\mathrm{PG}(2,q)$ is of size at least $q+1$. The conic $\mathcal{C}$ itself and lines of $\mathrm{PG}(2,q)$ are $(\mathcal{T} \cup \mathcal{S})$-blocking sets of size $q+1$. Every $(\mathcal{T} \cup \mathcal{S})$-blocking set in $\mathrm{PG}(2,q)$ of size $q+1$ which is disjoint from $\mathcal{C}$ must be an external line to $\mathcal{C}$. This was proved by Bruen and Thas in [15] for q even and by Segre and Korchmáros in [60] for all q. We refer to [12] for the description of all $(\mathcal{T} \cup \mathcal{S})$-blocking sets in $\mathrm{PG}(2,q)$ of size $q+1$ that are different from $\mathcal{C}$ and the lines of $\mathrm{PG}(2,q)$, also see [48, Section 2.3].

2.7 A new result

We shall need the following result in $\mathrm{PG}(2,q)$ while studying the blocking sets of external and tangent lines with respect to an elliptic quadric in $\mathrm{PG}(3,q)$.

Lemma 2.7.1. *Let Π be a Baer subplane of $\mathrm{PG}(2,q)$, q a square, such that $\Pi \cap \mathcal{C}$ is an irreducible conic in Π. Suppose that $A := \Pi \setminus (\Pi \cap \mathcal{C})$ is an $(\mathcal{E} \cup \mathcal{T})$-blocking set in $\mathrm{PG}(2,q)$ of size q. Let L be a line of $\mathrm{PG}(2,q)$ that is tangent to $\mathcal{C}$ in the point x. Then the following hold for L:*

(i) If $x \in \Pi \cap \mathcal{C}$, then L intersects Π in a Baer subline that is tangent to $\Pi \cap \mathcal{C}$ in the point x.

(*ii*) *If* $x \in \mathcal{C} \setminus \Pi$, *then* L *cannot intersect* Π *in a Baer subline.*

Proof. (i) Suppose $x \in \Pi \cap \mathcal{C}$. Then as L meets $A = \Pi \setminus (\Pi \cap \mathcal{C})$, it must intersect Π in a Baer subline and this Baer subline necessarily coincides with the line of Π that is tangent to $\Pi \cap \mathcal{C}$ in the point x.

(ii) Suppose $x \in \mathcal{C} \setminus \Pi$ and q is even. Let y_1 and y_2 be two distinct points of $\Pi \cap \mathcal{C}$. Let L_i with $i \in \{1, 2\}$ denote the unique line of $\mathrm{PG}(2, q)$ that is tangent to $\mathcal{C}$ in the point y_i. By (i), we know that $L_1 \cap \Pi$ and $L_2 \cap \Pi$ are two distinct lines of Π that are tangent to $\Pi \cap \mathcal{C}$. So, $L_1 \cap \Pi$ and $L_2 \cap \Pi$ meet in the nucleus m of the conic $\mathcal{C} \cap \Pi$ in Π. As $m \in L_1 \cap L_2$, the point m is also the nucleus of $\mathcal{C}$ and so $m = n$. Now, the line L must contain the nucleus n. If $L \cap \Pi$ were a Baer subline, then $L \cap \Pi$ would be a line of Π that is tangent to $\mathcal{C} \cap \Pi$ in a point y. As $y \in L \cap \mathcal{C}$, we have $x = y$, in contradiction with the fact that $x \notin \Pi$.

Suppose $x \in \mathcal{C} \setminus \Pi$ and q is odd. We choose coordinates such that Π consists of all points (X_1, X_2, X_3) of $\mathrm{PG}(2, q)$ with $X_1, X_2, X_3 \in \mathbb{F}_{\sqrt{q}}$ and $\Pi \cap \mathcal{C}$ consists of all points (X_1, X_2, X_3) of $\mathrm{PG}(2, q)$ with $X_1, X_2, X_3 \in \mathbb{F}_{\sqrt{q}}$ satisfying $X_1^2 + X_2X_3 = 0$. Suppose $\mathcal{C}$ has equation $\sum\limits_{1 \leqslant i \leqslant j \leqslant 3} a_{ij}X_iX_j = 0$ with respect to the same reference system, where $a_{ij} \in \mathbb{F}_q$ for all $i, j \in \{1, 2, 3\}$ with $i \leqslant j$. Since $(1, 1, -1)$, $(1, -1, 1)$, $(0, 1, 0)$ and $(0, 0, 1)$ belong to $\mathcal{C} \cap \Pi \subseteq \mathcal{C}$, the equation of $\mathcal{C}$ is of the form $a_{11}(X_1^2 + X_2X_3) + a_{12}(X_1X_2 + X_1X_3) = 0$, that is, of the form $X_1^2 + X_2X_3 + b(X_1X_2 + X_1X_3) = 0$ for some $b \in \mathbb{F}_q \setminus \{1, -1\}$ as $\mathcal{C}$ is irreducible. The line $\overline{K}$ of $\mathrm{PG}(2, q)$ that is tangent to $\mathcal{C}$ in the point $(0, 1, 0)$ has equation $X_3 + bX_1 = 0$, while the line K of Π that is tangent to $\Pi \cap \mathcal{C}$ in the point $(0, 1, 0)$ has equation $X_3 = 0$. By (i), we know that $\overline{K} \cap \Pi = K$. This implies that $b = 0$, that is, $\mathcal{C}$ has equation $X_1^2 + X_2X_3 = 0$ with respect to the same reference system (but coordinates are in $\mathbb{F}_q$).

Now, suppose that x has coordinates (u_1, u_2, u_3) and that L intersects Π in a

Baer subline. The line L has equation $a_1X_1 + a_2X_2 + a_3X_3 = 0$, where

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} := \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}. \tag{2.7.1}$$

Since $L \cap \Pi$ is a Baer subline, (a_1, a_2, a_3) is proportional to a nonzero element of $\mathbb{F}^3_{\sqrt{q}}$. This implies by (2.7.1) that (u_1, u_2, u_3) is also proportional to a nonzero element of $\mathbb{F}^3_{\sqrt{q}}$. As this is in contradiction with the fact that $x \in \mathcal{C} \setminus \Pi$, our assumption that L intersects Π in a Baer subline was wrong. □

Remark 2.7.2. *Let Π be a Baer subplane of $\mathrm{PG}(2, q)$, q a square, such that $\Pi \cap \mathcal{C}$ is an irreducible conic in Π. Then for $q \geqslant 16$, $A := \Pi \setminus (\Pi \cap \mathcal{C})$ is always an $(\mathcal{E} \cup \mathcal{T})$-blocking set in $\mathrm{PG}(2, q)$, while counter examples to that claim exist for $q = 4$ and $q = 9$.*

We justify Remark 2.7.2 in the following. As in the proof of Lemma 2.7.1, we can take a reference system in $\mathrm{PG}(2, q)$ with respect to which Π consists of all points (X_1, X_2, X_3) of $\mathrm{PG}(2, q)$ with $X_1, X_2, X_3 \in \mathbb{F}_{\sqrt{q}}$ and $\Pi \cap \mathcal{C}$ consists of all points (X_1, X_2, X_3) of $\mathrm{PG}(2, q)$ with $X_1, X_2, X_3 \in \mathbb{F}_{\sqrt{q}}$ satisfying $X_1^2 + X_2X_3 = 0$. Suppose $\mathcal{C}$ has equation $\sum_{1 \leqslant i \leqslant j \leqslant 3} a_{ij}X_iX_j = 0$ with respect to the same reference system. The fact that the points $(0, 1, 0)$, $(0, 0, 1)$ and $(1, \omega, -\omega^{-1})$ belong to $\mathcal{C}$ for every $\omega \in \mathbb{F}_{\sqrt{q}} \setminus \{0\}$ then implies that $a_{22} = a_{33} = 0$ and $a_{12}\omega^2 + (a_{11} - a_{23})\omega - a_{13} = 0$ for all $\omega \in \mathbb{F}_{\sqrt{q}} \setminus \{0\}$. For $q \geqslant 16$, this implies that $a_{12} = a_{13} = 0$, $a_{11} = a_{23}$ and so $\mathcal{C}$ has equation $X_1^2 + X_2X_3 = 0$ with respect to the same reference system[1]. It is now straightforward to verify that for every point $x \in \Pi \cap \mathcal{C}$, we have $K_x = \overline{K_x} \cap \Pi$, where $\overline{K_x}$ and K_x are the tangent lines of $\mathrm{PG}(2, q)$ and Π through x with respect to $\mathcal{C}$ and $\Pi \cap \mathcal{C}$, respectively. This property implies that $A = \Pi \setminus (\Pi \cap \mathcal{C})$ is an

[1] Another way to see this is as follows. For $q \geqslant 16$, we have $|\Pi \cap \mathcal{C}| = \sqrt{q} + 1 \geqslant 5$. As there is a unique irreducible conic through any given collection of five points of which no three are on the same line, we thus see that $\mathcal{C}$ coincides with the irreducible conic with equation $X_1^2 + X_2X_3 = 0$.

$(\mathcal{E} \cup \mathcal{T})$-blocking set in $\mathrm{PG}(2,q)$.

For $q \in \{4,9\}$, it is possible that $\mathcal{C}$ has equation $X_1^2 + X_2X_3 + \omega(X_1X_2 + X_1X_3) = 0$ with $\omega \in \mathbb{F}_q \setminus \mathbb{F}_{\sqrt{q}}$. We can then verify that $K \neq \overline{K} \cap \Pi$, where $\overline{K}$ and K are the tangent lines of $\mathrm{PG}(2,q)$ and Π through $(0,1,0)$ with respect to $\mathcal{C}$ and $\Pi \cap \mathcal{C}$, respectively. The line $\overline{K}$ does not contain any points of $A = \Pi \setminus (\Pi \cap \mathcal{C})$ and so A cannot be an $(\mathcal{E} \cup \mathcal{T})$-blocking set in $\mathrm{PG}(2,q)$.

2.8 Extending some results to ovals

Recall that ovals in $\mathrm{PG}(2,q)$ with q odd are precisely the irreducible conics. If q is even and at least 8, then there are ovals in $\mathrm{PG}(2,q)$ which are not irreducible conics, however, they have similar combinatorial properties. In the following proposition, we extend some of results of the previous sections to arbitrary ovals in $\mathrm{PG}(2,q)$ with q even.

Proposition 2.8.1. *Let O be an oval in $\mathrm{PG}(2,q)$, q even, and let n be the nucleus of O. Then the following hold:*

(i) If A is a blocking set in $\mathrm{PG}(2,q)$ with respect to the external lines to O, then $|A| \geqslant q-1$. If $|A| = q-1$, then $A \cap (O \cup \{n\}) = \emptyset$.

(ii) If A is a blocking set in $\mathrm{PG}(2,q)$ with respect to the external and tangent lines to O, then $|A| \geqslant q$.

(iii) The point sets of size q in $\mathrm{PG}(2,q)$ that block all tangent and external lines to O are precisely the sets of the form $A \cup \{n\}$, where A is a blocking set in $\mathrm{PG}(2,q)$ of size $q-1$ with respect to the external lines to O. In particular, such sets of points are always disjoint from O.

Proof. (i) It suffices to prove this claim for minimal blocking sets A with respect to the external lines to O. Since $A\setminus(O\cup\{n\})$ is also a blocking set in $\mathrm{PG}(2,q)$ with respect to the external lines to O, minimality of A implies that $A\cap(O\cup\{n\}) = \emptyset$.

So, every point of A is contained in $\frac{q}{2}$ external lines. As there are $\frac{q(q-1)}{2}$ external lines each containing at least one point of A, the number of points in A is at least $\frac{q(q-1)}{2}\left(\frac{q}{2}\right)^{-1} = q-1$. Hence, $|A| \geqslant q-1$.

If $|A| = q-1$, then A necessarily is a minimal blocking set and by the above, we then know that $A \cap (O \cup \{n\}) = \emptyset$.

(ii) Suppose that $|A| \leqslant q-1$. As A is also a blocking set with respect to the external lines to O, we know from (i) that $|A| = q-1$ and $A \cap (O \cup \{n\}) = \emptyset$. Each point of A is therefore contained in $\frac{q}{2}+1 = \frac{q+2}{2}$ lines that are external or tangent to O. In total, there are $\frac{q(q-1)}{2}+q+1 = \frac{q^2+q+2}{2}$ lines in $\mathrm{PG}(2,q)$ that are external or tangent to O. As each such line contains at least one point of A, we have $|A| \geqslant \frac{q^2+q+2}{2}\cdot\left(\frac{q+2}{2}\right)^{-1} > q-1$, a contradiction.

(iii) If A is a blocking set of size $q-1$ with respect to the external lines to O, then $A \cup \{n\}$ is a blocking set of size q with respect to the external and tangent lines to O (as each tangent line contains n). Conversely, suppose A' is a blocking set of size q with respect to the external and tangent lines to O. If $n \notin A'$, then each of the $q+1$ lines through n is a tangent line containing a point of A', proving that $|A'| \geqslant q+1$, a contradiction. Hence, $n \in A'$. Obviously, $A := A' \setminus \{n\}$ is a blocking set of size $q-1$ with respect to the external lines to O. By (i), we know that A is disjoint from O. Hence, also $A' = A \cup \{n\}$ is then disjoint from O. $\square$

Chapter 3

Blocking sets in $\mathrm{PG}(3, q)$: Elliptic quadrics

Let $Q^-(3, q)$ be an elliptic quadric in $\mathrm{PG}(3, q)$ and τ be the associated polarity of $\mathrm{PG}(3, q)$. So τ is symplectic if q is even, and orthogonal if q is odd. In this chapter, we denote by $\mathcal{E}$, $\mathcal{S}$ and $\mathcal{T}$ the set of all lines of $\mathrm{PG}(3, q)$ that are external, secant and tangent, respectively, with respect to $Q^-(3, q)$. We shall study the minimum size $\mathbb{L}$-blocking sets in $\mathrm{PG}(3, q)$, where the line set $\mathbb{L}$ is one of $\mathcal{E}$, $\mathcal{T}$, $\mathcal{S}$, $\mathcal{E} \cup \mathcal{T}$, $\mathcal{E} \cup \mathcal{S}$ and $\mathcal{T} \cup \mathcal{S}$. The contents of this chapter appear in [18] and [19].

All the tangent and secant planes of $\mathrm{PG}(3, q)$ considered in this chapter are with respect to the quadric $Q^-(3, q)$. Let $\mathbb{L} \in \{\mathcal{E}, \mathcal{T}, \mathcal{S}, \mathcal{E} \cup \mathcal{T}, \mathcal{E} \cup \mathcal{S}, \mathcal{T} \cup \mathcal{S}\}$. For a plane π of $\mathrm{PG}(3, q)$, we denote by $\mathbb{L}_\pi$ the set consisting of those lines of $\mathbb{L}$ that are contained in π. For a secant plane π of $\mathrm{PG}(3, q)$, we shall denote by $\mathcal{C}_\pi$ the irreducible conic $\pi \cap Q^-(3, q)$ in π. In that case, the lines of $\mathbb{L}_\pi$ are of the same type with respect to $\mathcal{C}_\pi$ as that of $\mathbb{L}$ with respect to $Q^-(3, q)$. Note that if B is an $\mathbb{L}$-blocking set in $\mathrm{PG}(3, q)$ and π is a secant plane of $\mathrm{PG}(3, q)$, then the set $B_\pi := \pi \cap B$ is an $\mathbb{L}_\pi$-blocking set in π.

3.1 $\mathcal{S}$-blocking sets

Let π be a secant plane of $\mathrm{PG}(3,q)$. For a given $\mathcal{S}_\pi$-blocking set A in π, we define the set $A(\pi)$ by

$$A(\pi) := A \cup (Q^-(3,q) \setminus \mathcal{C}_\pi),$$

which is a disjoint union. The proof of the following lemma is straightforward.

Lemma 3.1.1. *Let π be a secant plane of $\mathrm{PG}(3,q)$. If A is an $\mathcal{S}_\pi$-blocking set in π, then $A(\pi)$ is an $\mathcal{S}$-blocking set in $\mathrm{PG}(3,q)$ of size $q^2 - q + |A|$.*

In this section, we shall prove the following theorem which characterizes the minimum size $\mathcal{S}$-blocking sets in $\mathrm{PG}(3,q)$.

Theorem 3.1.2. *Let B be an $\mathcal{S}$-blocking set in $\mathrm{PG}(3,q)$. Then $|B| \geqslant q^2$, and equality holds if and only if $B = A(\pi)$ for some secant plane π of $\mathrm{PG}(3,q)$ and for some $\mathcal{S}_\pi$-blocking set A in π of size q.*

As a consequence of Theorem 3.1.2 and Proposition 2.3.1(1), we have the following.

Corollary 3.1.3. *Let B be an $\mathcal{S}$-blocking set in $\mathrm{PG}(3,q)$ of minimum size q^2. If q is odd, then $|B \setminus Q^-(3,q)| \in \{0,1,3\}$ and one of the following three cases occurs:*

(i) $B = Q^-(3,q) \setminus \{x\}$ *for some point* $x \in Q^-(3,q)$.

(ii) $B = (Q^-(3,q) \setminus \{x,y\}) \cup \{a\}$, *where x, y are two distinct points of $Q^-(3,q)$ and a is a point (different from x, y) on the secant line xy.*

(iii) $A = (Q^-(3,q) \setminus \{w,x,y,z\}) \cup \{a,b,c\}$, *where $\{w,x,y,z\}$ is a quadrangle contained in $\mathcal{C}_\pi$ for some secant plane π of $\mathrm{PG}(3,q)$ and a, b, c are the three diagonal points of this quadrangle.*

If π is a secant plane of $\mathrm{PG}(3,q)$ and A is an $\mathcal{S}_\pi$-blocking set in π of size q, then Lemma 3.1.1 implies that $A(\pi)$ is an $\mathcal{S}$-blocking set in $\mathrm{PG}(3,q)$ of size q^2. We prove the other parts of Theorem 3.1.2 in the rest of this section. Suppose that B is an $\mathcal{S}$-blocking set in $\mathrm{PG}(3,q)$ of minimum possible size.

3.1.1 General properties

For every point x of $Q^-(3,q)$, observe that $Q^-(3,q)\setminus\{x\}$ is an $\mathcal{S}$-blocking set in $\mathrm{PG}(3,q)$ of size q^2. Then the minimality of $|B|$ implies that $|B|\leqslant q^2$ and hence $Q^-(3,q)\setminus B$ is nonempty.

Lemma 3.1.4. *The following hold:*

(i) Every secant line through a point of $Q^-(3,q)\setminus B$ meets B in a unique point.

(ii) $|B|=q^2$.

Proof. Let w be a point of $Q^-(3,q)\setminus B$. Each of the q^2 secant lines through w meets B and two distinct such lines meet B at different points. This gives $|B|\geqslant q^2$. Since $|B|\leqslant q^2$, it follows that both (i) and (ii) hold. □

Suppose that $B\subseteq Q^-(3,q)$, then $|B|=q^2$ implies that $B=Q^-(3,q)\setminus\{x\}$ for some point x of $Q^-(3,q)$. Consider a secant plane π through the point x. Then $x\in\mathcal{C}_\pi$ and $A:=\mathcal{C}_\pi\setminus\{x\}$ is an $\mathcal{S}_\pi$-blocking set in π of size q. We also have $B=A\cup(Q^-(3,q)\setminus\mathcal{C}_\pi)=A(\pi)$. This proves Theorem 3.1.2 in this case.

From now on assume that $B\nsubseteq Q^-(3,q)$. Then both the sets $B\setminus Q^-(3,q)$ and $Q^-(3,q)\setminus B$ are nonempty and we have

$$|Q^-(3,q)\setminus B|=|B\setminus Q^-(3,q)|+1. \tag{3.1.1}$$

Lemma 3.1.5. *$B\cap Q^-(3,q)$ is nonempty.*

Proof. Suppose that $B\cap Q^-(3,q)$ is empty. There are $q(q-1)/2$ secant lines through every point of B. We also have $|B|=q^2$ and $|\mathcal{S}|=q^2(q^2+1)/2$. Since

B is an $\mathcal{S}$-blocking set, counting the cardinality of the set

$$R = \{(x, L) : x \in B, L \in \mathcal{S}, x \in L\}$$

in two ways, we get

$$q^2 \cdot q(q-1)/2 = |R| \geqslant |\mathcal{S}| \cdot 1 = q^2(q^2+1)/2,$$

which is not possible. Hence $B \cap Q^-(3,q)$ is nonempty. □

Corollary 3.1.6. *Every secant line through a point of $B \setminus Q^-(3,q)$ contains two points of either $B \cap Q^-(3,q)$ or $Q^-(3,q) \setminus B$.*

Proof. This follows from Lemma 3.1.4(i). □

Lemma 3.1.7. *The tangency point of every tangent line through a point of $B \setminus Q^-(3,q)$ is contained in $B \cap Q^-(3,q)$.*

Proof. Let x be a point of $B \setminus Q^-(3,q)$ and $y \in Q^-(3,q)$ be the tangency point of some tangent line through x. We show that y is a point of $B \cap Q^-(3,q)$.

Suppose to the contrary that y is a point of $Q^-(3,q) \setminus B$. Then there would go at least q^2+1 lines through y each containing a point of B, namely the q^2 secant lines (see Lemma 3.1.4) and the line yx, in contradiction with $|B| = q^2$. □

Corollary 3.1.8. *Every line joining a point of $B \setminus Q^-(3,q)$ and a point of $Q^-(3,q) \setminus B$ is a secant line which meets $Q^-(3,q) \setminus B$ in a second point.*

Proof. This follows from Lemma 3.1.7 and Corollary 3.1.6. □

Corollary 3.1.9. *$|Q^-(3,q) \setminus B|$ is even and hence $|B \setminus Q^-(3,q)|$ is odd.*

Proof. Since $B \setminus Q^-(3,q)$ is nonempty by our assumption, the first part follows from Corollary 3.1.8. The second part follows from (3.1.1) using the first part. □

Lemma 3.1.10. *Let L be a secant line containing two points of $Q^-(3,q) \setminus B$. If π is a secant plane containing L, then B_π is an $\mathcal{S}_\pi$-blocking set in π of size q.*

Proof. Let $\pi = \pi_0, \pi_1, \ldots, \pi_q$ be the $q+1$ secant planes through L. By Proposition 2.3.1, $|B_{\pi_i}| \geqslant q$ for every $i \in \{0, 1, \ldots, q\}$. We show that $|B_{\pi_i}| = q$ for each i and then the lemma will follow from this.

Let $|B_{\pi_i}| = q + s_i$, $0 \leqslant i \leqslant q$, for some nonnegative integer s_i. We have $|L \cap B| = 1$ by Lemma 3.1.4(i) and $\pi_i \cap \pi_j = L$ for distinct $i, j \in \{0, 1, \ldots, q\}$. Since $B = \bigcup_{i=0}^{q} B_{\pi_i}$, we get

$$q^2 = |B| = 1 + \sum_{i=0}^{q} (q + s_i - 1).$$

This gives

$$\sum_{i=0}^{q} s_i = 0.$$

Since each $s_i \geqslant 0$, we must have $s_i = 0$ for all i. □

3.1.2 The case q even

Lemma 3.1.11. *If q is even, then the line through two distinct points of $B \setminus Q^-(3, q)$ is tangent to $Q^-(3, q)$.*

Proof. Let x, y be two distinct points of $B \setminus Q^-(3, q)$. We show that xy is a tangent line. Consider a point $a \in Q^-(3, q) \setminus B$. Lemma 3.1.7 implies that the lines xa and ya are secant to $Q^-(3, q)$. Since $x, y \in B$ with $x \neq y$, Lemma 3.1.4(i) implies that the secant lines xa and ya are distinct. Then, by Corollary 3.1.6, there exist distinct points b and c of $Q^-(3, q) \setminus B$ such that $xa \cap Q^-(3, q) = \{a, b\}$ and $ya \cap Q^-(3, q) = \{a, c\}$.

Consider the plane $\pi := \langle a, b, c \rangle$ generated by the points a, b, c. Then π is a secant plane and x, y are points of π. Since $a, b \in Q^-(3, q) \setminus B$, applying Lemma 3.1.10 to the secant line $xa = ab$, we get that B_π is an $\mathcal{S}_\pi$-blocking set in π of minimum size q. So, by Proposition 2.3.1(2), all the points of $B_\pi \setminus \mathcal{C}_\pi$ are contained in a common line L which is tangent to $\mathcal{C}_\pi$ and hence tangent to $Q^-(3, q)$. Since

$x, y \in B_\pi \setminus \mathcal{C}_\pi$, it follows that the tangent line L contains both x and y. □

As a consequence of Lemma 3.1.11, we have the following.

Corollary 3.1.12. *If q is even, then any secant line contains at most one point of $B \setminus Q^-(3, q)$.*

Lemma 3.1.13. *If q is even, then all the points of $B \setminus Q^-(3, q)$ are contained in a common tangent line.*

Proof. The statement is clear if $|B \setminus Q^-(3, q)| = 1$. Since $|B \setminus Q^-(3, q)|$ is odd by Corollary 3.1.9, we assume that $|B \setminus Q^-(3, q)| \geqslant 3$. By Lemma 3.1.11, it is enough to show that any three distinct points x, y, z of $B \setminus Q^-(3, q)$ are contained in a line.

By Lemma 3.1.11, xy, xz and yz are tangent lines. Suppose that the line xy does not contain the point z. Then the plane π generated by the two tangent lines xy and xz is a secant plane. Since q is even, x must be the nucleus of the conic $\mathcal{C}_\pi$ in π and so all tangent lines contained in π meet at x. But the tangent line yz contained in π does not contain x, a contradiction. □

The following proposition proves Theorem 3.1.2 when q is even.

Proposition 3.1.14. *If q is even, then $B = A(\pi)$ for some secant plane π of* $\mathrm{PG}(3, q)$ *and for some $\mathcal{S}_\pi$-blocking set A in π of size q.*

Proof. By Lemma 3.1.13, there exists a tangent line T containing all the points of $B \setminus Q^-(3, q)$. Consider a point x of T which is in $B \setminus Q^-(3, q)$. Let L be a secant line through x meeting $Q^-(3, q) \setminus B$ at two points. Such a line L exists by Corollary 3.1.8 as $Q^-(3, q) \setminus B$ is nonempty. The plane π generated by the two intersecting lines T and L is a secant plane of $\mathrm{PG}(3, q)$. Applying Lemma 3.1.10 to the secant line L, the set B_π is an $\mathcal{S}_\pi$-blocking set in π of size q.

As every point of $B \setminus Q^-(3, q)$ is contained in $A := B_\pi$, we have $B \subseteq A \cup (Q^-(3, q) \setminus \mathcal{C}_\pi) = A(\pi)$. Since $|A(\pi)| = |A| + q^2 - q = q^2 = |B|$, we see that $B = A(\pi)$. □

3.1.3 The case q odd

Lemma 3.1.15. *If q is odd, then there is no line of* $\mathrm{PG}(3,q)$ *containing more than two points of $B \setminus Q^-(3,q)$.*

Proof. Suppose that L_1 is a line of $\mathrm{PG}(3,q)$ containing at least three points, say a, b, c, of $B \setminus Q^-(3,q)$. Let L_2 be a secant line through a which contains two points of $Q^-(3,q) \setminus B$. Such a line L_2 exists by Corollary 3.1.8 as $Q^-(3,q) \setminus B$ is nonempty. By Lemma 3.1.4(i), a is the only point of $B \setminus Q^-(3,q)$ contained in L_2. So $L_1 \neq L_2$. The plane π generated by the two intersecting lines L_1 and L_2 is a secant plane of $\mathrm{PG}(3,q)$. By Lemma 3.1.10, the set B_π is an $\mathcal{S}_\pi$-blocking set in π of size q. Then $|B_\pi \setminus \mathcal{C}_\pi| \leqslant 3$ by Proposition 2.3.1(1). Since the points a, b, c of L_1 are contained in $B_\pi \setminus \mathcal{C}_\pi$, we must have $B_\pi \setminus \mathcal{C}_\pi = \{a, b, c\}$.

By Proposition 2.3.1(1)(iii), a, b, c must be the three diagonal points of some quadrangle contained in $\mathcal{C}_\pi$. Then a, b, c can not be contained in any line of π as q is odd, contradicting that the line L_1 of π contains a, b, c. □

Lemma 3.1.16. *If q is odd, then $|Q^-(3,q) \setminus B| \leqslant 4$ and hence $|B \setminus Q^-(3,q)| \leqslant 3$.*

Proof. Suppose that $|Q^-(3,q) \setminus B| > 4$. Then $|Q^-(3,q) \setminus B| \geqslant 6$ as $|Q^-(3,q) \setminus B|$ is even by Corollary 3.1.9. Fix a point x of $B \setminus Q^-(3,q)$. Let L_1, L_2, L_3 be three secant lines through x each of which meets $Q^-(3,q) \setminus B$ at two points (use Corollary 3.1.8). Set $L_i \cap (Q^-(3,q) \setminus B) = \{a_i, b_i\}$ for $i \in \{1,2,3\}$. Since a_1, a_2, a_3 (respectively, b_1, b_2, b_3) are points of $Q^-(3,q)$, they are not contained in any line of $\mathrm{PG}(3,q)$. Thus the seven points $x, a_1, a_2, a_3, b_1, b_2, b_3$ form a Desarguesian configuration. So the three intersection points z_{12}, z_{13}, z_{23}, where $a_1a_2 \cap b_1b_2 = \{z_{12}\}$, $a_1a_3 \cap b_1b_3 = \{z_{13}\}$ and $a_2a_3 \cap b_2b_3 = \{z_{23}\}$, are contained in a line of $\mathrm{PG}(3,q)$.

Let π_{ij} be the plane generated by the two intersecting lines L_i and L_j, where $1 \leqslant i < j \leqslant 3$. Then π_{ij} is a secant plane and by Lemma 3.1.10, the set $B_{\pi_{ij}}$ is an $\mathcal{S}_{\pi_{ij}}$-blocking set in π_{ij} of size q. Since a_i, b_i, a_j, b_j are points of $\mathcal{C}_{\pi_{ij}} \setminus B_{\pi_{ij}}$, Proposition 2.3.1(1) implies that $\mathcal{C}_{\pi_{ij}} \setminus B_{\pi_{ij}} = \{a_i, b_i, a_j, b_j\}$ and the set $B_{\pi_{ij}} \setminus \mathcal{C}_{\pi_{ij}}$

consists of the diagonal points of the quadrangle $\{a_i, b_i, a_j, b_j\}$ contained in $\mathcal{C}_{\pi_{ij}}$. In particular, the point z_{ij} is contained in $B_{\pi_{ij}} \setminus \mathcal{C}_{\pi_{ij}}$ and hence in $B \setminus Q^-(3, q)$. It follows that the three points z_{12}, z_{13}, z_{23} of $B \setminus Q^-(3, q)$ are contained in a line of $\mathrm{PG}(3, q)$, contradicting Lemma 3.1.15.

Hence $|Q^-(3, q) \setminus B| \leqslant 4$ and then (3.1.1) implies that $|B \setminus Q^-(3, q)| \leqslant 3$. $\square$

The following proposition proves Theorem 3.1.2 when q is odd.

Proposition 3.1.17. *If q is odd, then $B = A(\pi)$ for some secant plane π of* $\mathrm{PG}(3, q)$ *and for some $\mathcal{S}_\pi$-blocking set A in π of size q.*

Proof. We have $|B \setminus Q^-(3, q)| \in \{1, 3\}$ by Lemma 3.1.16 and Corollary 3.1.9. First assume that $|B \setminus Q^-(3, q)| = 1$. Then $|Q^-(3, q) \setminus B| = 2$ by (3.1.1). Let $B \setminus Q^-(3, q) = \{a\}$ and $Q^-(3, q) \setminus B = \{x, y\}$. The secant line xy meets B at the point a. Consider any secant plane π containing the line xy. Then $x, y \in \mathcal{C}_\pi$ and $A := (\mathcal{C}_\pi \setminus \{x, y\}) \cup \{a\}$ is an $\mathcal{S}_\pi$-blocking set in π of size q. It can be seen that $B = A \cup (Q^-(3, q) \setminus \mathcal{C}_\pi) = A(\pi)$.

Now assume that $|B \setminus Q^-(3, q)| = 3$. Then $|Q^-(3, q) \setminus B| = 4$ by (3.1.1). Let $B \setminus Q^-(3, q) = \{a, b, c\}$ and $Q^-(3, q) \setminus B = \{w, x, y, z\}$. Using Corollary 3.1.8, there are exactly two secant lines through a point of $B \setminus Q^-(3, q)$ each of which meets $Q^-(3, q) \setminus B$ at two points. Conversely, any secant line through two points of $Q^-(3, q) \setminus B$ contains a unique point of $B \setminus Q^-(3, q)$ by Lemma 3.1.4(i). Thus the four points w, x, y, z generate a plane π of $\mathrm{PG}(3, q)$ which contains the points a, b, c as well. In fact, a, b, c are the three diagonal points of the quadrangle $\{w, x, y, z\}$ contained in the conic $\mathcal{C}_\pi$. Then $A := (\mathcal{C}_\pi \setminus \{w, x, y, z\}) \cup \{a, b, c\}$ is an $\mathcal{S}_\pi$-blocking set in π of size q. We also have $B = A \cup (Q^-(3, q) \setminus \mathcal{C}_\pi) = A(\pi)$ in this case. $\square$

3.2 $\mathcal{E}$-blocking sets

The following theorem characterizes the minimum size $\mathcal{E}$-blocking sets in $\mathrm{PG}(3, q)$.

Theorem 3.2.1. *Let B be an $\mathcal{E}$-blocking set in* $\mathrm{PG}(3,q)$*. Then $|B| \geqslant q^2$, and equality holds if and only if $B = \pi \setminus Q^-(3,q)$ for some secant plane π of* $\mathrm{PG}(3,q)$*.*

We note that Biondi et al. studied in [9, Section 3] the $\mathcal{E}$-blocking sets in $\mathrm{PG}(3,q)$ and proved Theorem 3.2.1 in [9, Theorem 3.5], with exception of the equality case for some small values of q, namely $q \in \{2,3,4,5,7,8\}$. We observed that their proof also works for all even q, but not for $q = 3,5,7$. By Propositions 2.1.1 and 2.1.2, the minimum size of a blocking set of the external lines with respect to an irreducible conic in $\mathrm{PG}(2,q)$ is $q-1$. For $q = 3,5,7$, there are examples of such blocking sets in $\mathrm{PG}(2,q)$ of size $q-1$ consisting of interior points (see Proposition 2.1.1) and the arguments used to prove [9, Theorem 3.5] does not cover such sporadic examples in the plane case.

In this section, our aim is to give an alternate proof of the equality case in Theorem 3.2.1 which works for all q, in particular, for $q = 3,5,7$. We implicitly borrow some of the arguments used in [9, Section 3] for our proof.

Let B be an $\mathcal{E}$-blocking set in $\mathrm{PG}(3,q)$ of minimum size. If π is a secant plane, then $\pi \setminus Q^-(3,q)$ is an $\mathcal{E}$-blocking set of size q^2. From the minimality of $|B|$ and the fact that $B \setminus Q^-(3,q)$ is an $\mathcal{E}$-blocking set, we have that $|B| \leqslant q^2$ and that $B \cap Q^-(3,q)$ is empty. Note that if π is a tangent plane, then $|B_\pi| \geqslant q$ using Proposition 2.0.1.

A proof of the following lemma can easily be extracted from [9, Proposition 3.4]. A proof is added here for reasons of completeness.

Lemma 3.2.2. *Let N_1 denote the number of tangent planes intersecting B in exactly q points. Then $N_1 \geqslant q+1$. If $N_1 = q+1$, then $|B| = q^2$ and each of the remaining $q^2 - q$ tangent planes meets B in exactly $q+1$ points.*

Proof. There are q^2+1-N_1 tangent planes intersecting B in at least $q+1$ points by Proposition 2.0.1. As each point of $\mathrm{PG}(3,q) \setminus Q^-(3,q)$ is contained in exactly $q+1$ tangent planes, a double counting of the incident point-tangent plane pairs

(x, π) with $x \in B$ yields

$$N_1 \cdot q + (q^2 + 1 - N_1) \cdot (q+1) \leqslant |B| \cdot (q+1) \leqslant q^2(q+1),$$

that is, $q+1 \leqslant N_1$. If $N_1 = q+1$, then the above implies that $|B| = q^2$ and that each of the $q^2 + 1 - N_1 = q^2 - q$ tangent planes that meets B in at least $q+1$ points meet it in precisely $q+1$ points. $\square$

In the sequel, let $\pi_1, \pi_2, \ldots, \pi_k$ with $k \geqslant q+1$ be all the tangent planes intersecting B in precisely q points. Let α_i with $i \in \{1, 2, \ldots, k\}$ be the tangency point of π_i. By Proposition 2.0.1, there exists a line U_i of π_i through α_i such that $B \cap \pi_i = U_i \setminus \{\alpha_i\}$.

Lemma 3.2.3. *Any two distinct lines in the collection $\{U_1, U_2, \ldots, U_k\}$ intersect in a singleton (outside $Q^-(3,q)$).*

Proof. Let $i_1, i_2 \in \{1, 2, \ldots, k\}$ with $i_1 \neq i_2$. The external line $L = \pi_{i_1} \cap \pi_{i_2}$ intersects B in a singleton $\{\beta\}$ which belongs to both U_{i_1} and U_{i_2}. Hence $U_{i_1} = \alpha_{i_1}\beta$ and $U_{i_2} = \alpha_{i_2}\beta$ meet in the singleton $\{\beta\}$. $\square$

Lemma 3.2.4. *We have $k = q+1$ and $U_1, U_2, \ldots, U_{q+1}$ are the $q+1$ tangent lines contained in a secant plane π^*.*

Proof. As $\{U_1, U_2, \ldots, U_k\}$ is a collection of mutually intersecting lines by Lemma 3.2.3, at least one of the following cases occurs:

(1) the lines $U_1, U_2, \ldots, U_k$ are contained in the same plane π^*;

(2) the lines $U_1, U_2, \ldots, U_k$ go through the same point x^*.

We show that case (1) occurs. Note that if case (2) occurs, then $x^* \notin Q^-(3,q)$ by Lemma 3.2.3.

If case (2) occurs with q even, then the tangent lines $U_1, U_2, \ldots, U_k$ are all contained in the secant plane $(x^*)^\tau$ through x^*, indeed showing case (1) occurs.

Suppose that case (2) occurs with q odd. Then the tangent lines through x^* are the $q+1$ lines through x^* containing a point of $(x^*)^\tau \cap Q^-(3,q)$. As $k \geqslant q+1$, we then have $k = q+1$. As $|B \cap (U_1 \cup U_2 \cup \ldots \cup U_{q+1})| = 1 + (q+1)(q-1) = q^2$ and $|B| \leqslant q^2$, we must have $B = (U_1 \cup U_2 \cup \ldots \cup U_{q+1}) \setminus Q^-(3,q)$. But this is impossible, as any external line contained in the secant plane $(x^*)^\tau$ would then be disjoint from B.

Thus, case (1) occurs. As the $k \geqslant q+1$ tangent lines $U_1, U_2, \ldots, U_k$ are all contained in the same plane and their tangency points are mutually distinct, we can conclude that $k = q+1$, π^* is a secant plane and that $U_1, U_2, \ldots, U_{q+1}$ are all the $q+1$ tangent lines contained in π^*. $\square$

The following is a consequence of Lemmas 3.2.2 and 3.2.4.

Corollary 3.2.5. *We have $|B| = q^2$. There are $q+1$ tangent planes meeting B in precisely q points and $q^2 - q$ tangent planes meeting B in precisely $q+1$ points.*

The following lemma completes the proof of Theorem 3.2.1.

Lemma 3.2.6. *Let π^* be the secant plane as in Lemma 3.2.4. Then $B = \pi^* \setminus Q^-(3,q)$.*

Proof. For q odd, let E_{π^*} and I_{π^*} denote the set of points of π^* that are exterior and interior, respectively, with respect to the conic $\mathcal{C}_{\pi^*}$. Lemma 3.2.4 implies that the set of points of B covered by the tangent lines contained in π^* is precisely $\pi^* \setminus Q^-(3,q)$ if q is even, and E_{π^*} if q is odd. Since $|B| = q^2 = |\pi^* \setminus Q^-(3,q)|$, the lemma follows for even q and it suffices to prove that $I_{\pi^*} \subseteq B$ for odd q.

Suppose to the contrary that there exists a point $x \in I_{\pi^*}$ which is not contained in B. There are $\frac{q^2+q}{2} - \frac{q+1}{2} = \frac{q^2-1}{2}$ external lines through x which are not contained in π^*. Each of these $\frac{q^2-1}{2}$ external lines contains at least one point of $B \setminus \pi^*$, implying that

$$|B| = |B \cap \pi^*| + |B \setminus \pi^*| \geqslant |E_{\pi^*}| + \frac{q^2-1}{2} = \frac{q^2+q}{2} + \frac{q^2-1}{2} = q^2 + \frac{q-1}{2},$$

in contradiction with $|B| = q^2$. $\square$

3.3 $(\mathcal{T} \cup \mathcal{S})$-blocking sets

We prove the following theorem which characterizes the minimum size $(\mathcal{T} \cup \mathcal{S})$-blocking sets in $\mathrm{PG}(3, q)$.

Theorem 3.3.1. *Let B be a $(\mathcal{T} \cup \mathcal{S})$-blocking set in $\mathrm{PG}(3, q)$. Then $|B| \geqslant q^2 + 1$ and equality holds if and only if $B = Q^-(3, q)$.*

Proof. Consider B to be a $(\mathcal{T} \cup \mathcal{S})$-blocking set in $\mathrm{PG}(3, q)$ of minimum possible size. Since the quadric $Q^-(3, q)$ contains $q^2 + 1$ points and it blocks the tangent and secant lines, the minimality of $|B|$ implies that $|B| \leqslant |Q^-(3, q)| = q^2 + 1$. We assert that $B = Q^-(3, q)$. It is enough to show that $Q^-(3, q) \subseteq B$.

Suppose that there exists a point x of $Q^-(3, q)$ which is not in B. Each line through x in $\mathrm{PG}(3, q)$ is either tangent or secant to $Q^-(3, q)$. Since $x \notin B$ and B is a $(\mathcal{T} \cup \mathcal{S})$-blocking set, the $q^2 + q + 1$ lines of $\mathrm{PG}(3, q)$ through x would meet B at different points. This gives $|B| \geqslant q^2 + q + 1$, a contradiction to that $|B| \leqslant q^2 + 1$. Thus $Q^-(3, q) \subseteq B$. $\square$

3.4 $(\mathcal{E} \cup \mathcal{S})$-blocking sets

In this section, we shall prove the following theorem which characterizes the minimum size $(\mathcal{E} \cup \mathcal{S})$-blocking sets in $\mathrm{PG}(3, q)$.

Theorem 3.4.1. *Let B be an $(\mathcal{E} \cup \mathcal{S})$-blocking set in $\mathrm{PG}(3, q)$. Then the following hold:*

(i) If $q = 2$, then $|B| \geqslant 6$ and equality holds if and only if $B = L \cup L^\tau$ for some line L secant to $Q^-(3, q)$.

(ii) If $q \geqslant 3$, then $|B| \geqslant q^2 + q + 1$ and equality holds if and only if B is a plane of $\mathrm{PG}(3, q)$.

When $q \geqslant 4$ is even, Theorem 3.4.1(ii) can also be seen from [56, Theorem 1.3] which was proved using properties of the symplectic generalized quadrangle $W(q)$ of order q. Here we give a different proof which works for all q.

The rest of this section is devoted to prove Theorem 3.4.1. Note that if π is a tangent plane, then there is no secant line in $(\mathcal{E} \cup \mathcal{S})_\pi$. If π is a secant plane, then $(\mathcal{E} \cup \mathcal{S})_\pi$ is precisely the set of external and secant lines with respect to the conic $\mathcal{C}_\pi = \pi \cap Q^-(3, q)$ in π.

Suppose that B is a minimum size $(\mathcal{E} \cup \mathcal{S})$-blocking set in $\mathrm{PG}(3, q)$. Then $|B| \leqslant q^2 + q + 1$, as each plane blocks every line of $\mathrm{PG}(3, q)$. For every plane π, the set $B_\pi = \pi \cap B$ is an $(\mathcal{E} \cup \mathcal{S})_\pi$-blocking set in π.

We first prove two results for q even, which are needed to show that any secant plane contains at least $q+1$ points of B for $q \geqslant 4$. Recall that, if q is even and x is a point of $\mathrm{PG}(3, q) \setminus Q^-(3, q)$, then x is a point of the secant plane x^τ. In fact, x is the nucleus of the conic $\mathcal{C}_{x^\tau}$ in the plane x^τ, in particular, x^τ is precisely the union of the $q+1$ tangent lines through x.

Lemma 3.4.2. *Suppose that q is even. Let L be a tangent line through a point $y \notin B$. Then $|B_{x^\tau}| \geqslant q$ for every point x of $L \setminus \{y\}$ and the points of B_{x^τ} are different from those of $L \cap B$.*

Proof. Let M be a line through y in x^τ which is different from L. If x is a point of $Q^-(3, q)$, then x^τ is a tangent plane and so the line M is external to $Q^-(3, q)$. If $x \notin Q^-(3, q)$, then x is the nucleus of the conic $\mathcal{C}_{x^\tau}$ in x^τ, implying that $L = xy$ is the unique tangent line of x^τ through y and that M is either a secant or an external line. In all cases, each such line M must meet B. Since $y \notin B$ and there are q possible choices for M, it follows that $|B_{x^\tau}| \geqslant q$ and the points of B_{x^τ} are different from those of $L \cap B$. □

Lemma 3.4.3. *Suppose that q is even and let x be a point of $B \cap Q^-(3, q)$. If there exists a tangent line L through x with $|L \cap B| = q$, then every tangent line through x contains at least q points of B. In particular, $|B_{x^\tau}| \geqslant 1 + (q+1)(q-1) = q^2$.*

Proof. Let M be a tangent line through x different from L. Since L has a point not in B, Lemma 3.4.2 implies that the tangent plane x^τ through L contains at least $2q$ points of B. Note that x^τ is also the tangent plane through M.

Suppose that $|M \cap B| \leqslant q-1$. Then M has at least two points which are not in B. Applying Lemma 3.4.2 carefully to the tangent line M, it follows that each of the q secant planes z^τ, $z \in M \setminus \{x\}$, through M contains at least q points of B which are different from those of $M \cap B$. Counting the points of B contained in the $q+1$ planes through M, we get

$$|B| \geqslant 2q + q^2 > q^2 + q + 1,$$

which is a contradiction to the fact that $|B| \leqslant q^2+q+1$. So $|M \cap B| \geqslant q$. $\square$

Lemma 3.4.4. *Let π be a plane of* $\mathrm{PG}(3,q)$*. Then the following hold:*

(i) Suppose that π is a tangent plane. Then $|B_\pi| \geqslant q$ and equality holds if and only if $B_\pi = L \setminus \{x\}$ for some tangent line L through x, where $\{x\} = \pi \cap Q^-(3,q)$.

(ii) Suppose that π is a secant plane. Then $|B_\pi| \geqslant q$. Further, if $q \geqslant 4$, then $|B_\pi| \geqslant q+1$.

Proof. (i) Since π is a tangent plane, $(\mathcal{E} \cup \mathcal{S})_\pi$ is precisely the set of all lines of π not containing the point x. Then (i) follows from Proposition 2.0.1.

(ii) Here B_π is an $(\mathcal{E} \cup \mathcal{S})_\pi$-blocking set in π. The first part for all q and the second part for odd $q \geqslant 5$ follow from Propositions 2.5.1 and 2.5.2.

Assume that $q \geqslant 4$ is even. Let $\pi = x^\tau$ for some point x of $\mathrm{PG}(3,q) \setminus Q^-(3,q)$. We have $|B_{x^\tau}| \geqslant q$ by Proposition 2.5.1. Suppose that $|B_{x^\tau}| = q$. Note that $(\mathcal{E} \cup \mathcal{S})_{x^\tau}$ is precisely the set of lines in x^τ not containing x. Then, by Proposition 2.5.1 again, $B_{x^\tau} = L \setminus \{x\}$ for some tangent line L through x. Let $L = \{x_0, x_1, \cdots, x_{q-1}, x_q = x\}$ with tangency point $x_0 \in Q^-(3,q)$. By Lemma 3.4.2, each of the secant planes x_i^τ, $1 \leqslant i \leqslant q-1$, through L contains at least

q points of B which are different from those of $L \cap B$. Also we have $|B_{x_0^\tau}| \geqslant q^2$ for the tangent plane x_0^τ by Lemma 3.4.3. Counting the points of B contained in the q planes x_i^τ through L for $i \in \{0, 1, \ldots, q-1\}$ and using our assumption that $q \geqslant 4$, we get

$$|B| \geqslant q^2 + (q-1)q > q^2 + q + 1,$$

which is a contradiction to the fact that $|B| \leqslant q^2+q+1$. Hence $|B_{x^\tau}| \geqslant q+1$. □

Corollary 3.4.5. $|B| \geqslant q^2 + q$.

Proof. If every tangent line meets B, then B would be a blocking set with respect to all lines of $\mathrm{PG}(3, q)$ and hence we must have $|B| \geqslant q^2 + q + 1$ by Proposition 1.7.1. Suppose that there is a tangent line L which is disjoint from B. Count the points of B contained in the $q + 1$ planes through L. Since $L \cap B = \emptyset$, we get $|B| \geqslant (q + 1)q = q^2 + q$ using Lemma 3.4.4. □

The following proposition proves Theorem 3.4.1 when $q \geqslant 4$.

Proposition 3.4.6. *If $q \geqslant 4$, then $|B| = q^2+q+1$ and B is a plane of* $\mathrm{PG}(3, q)$.

Proof. By Proposition 1.7.1, it is enough to show that every tangent line meets B. Suppose that there exists a tangent line L which is disjoint from B. Count the points of B contained in the $q + 1$ planes through L. There is one tangent plane and q secant planes containing L. Using the assumption that $q \geqslant 4$, Lemma 3.4.4 implies that

$$|B| \geqslant q + q(q + 1) = q^2 + 2q > q^2 + q + 1,$$

which is a contradiction to the fact that $|B| \leqslant q^2 + q + 1$. Hence every tangent line meets B. □

The following proposition proves Theorem 3.4.1 for $q = 2$.

Proposition 3.4.7. *If $q = 2$, then $|B| = 6$ and $B = L \cup L^\tau$ for some line L secant to $Q^-(3, q)$.*

Proof. By Corollary 3.4.5, we have $|B| \geqslant 6$. Let L be a secant line of $\mathrm{PG}(3,2)$ and $L \cap Q^-(3,2) = \{u, v\}$. Then L^τ is an external line which is common to the two tangent planes u^τ and v^τ. If w is the third point of L, then L^τ is the unique external line contained in the secant plane w^τ. If M is a secant line not containing u and v, then M contains two points of $\mathcal{C}_{w^\tau} = Q^-(3,2) \setminus \{u, v\}$ and so is a line of w^τ. In the plane w^τ, the lines M and L^τ meet. If M is an external line, then M meets the plane w^τ in at least one point of $\{w\} \cup L^\tau$. It follows that $L \cup L^\tau$ is an $(\mathcal{E} \cup \mathcal{S})$-blocking set in $\mathrm{PG}(3,2)$ of size 6. Thus $|B| = 6$ by the minimality of $|B|$.

Conversely, let B be an $(\mathcal{E} \cup \mathcal{S})$-blocking set in $\mathrm{PG}(3,2)$ of size 6. Since $B \setminus Q^-(3,2)$ is an $\mathcal{E}$-blocking set in $\mathrm{PG}(3,2)$, we have $|B \setminus Q^-(3,2)| \geqslant 4$ by Theorem 3.2.1. There are 10 secant lines to $Q^-(3,2)$, and every point of $\mathrm{PG}(3,2) \setminus Q^-(3,2)$ is contained in a unique secant line. If $|B \setminus Q^-(3,2)| = 6$ and $B \cap Q^-(3,2) = \emptyset$, then B blocks precisely 6 secant lines. If $|B \setminus Q^-(3,2)| = 5$ and $|B \cap Q^-(3,2)| = 1$, then B blocks at most $5 + 4 = 9$ secant lines. So we must have $|B \setminus Q^-(3,2)| = 4$ and $|B \cap Q^-(3,2)| = 2$. Hence $B \setminus Q^-(3,2) = \pi \setminus Q^-(3,2)$ for some secant plane π of $\mathrm{PG}(3,2)$ by Theorem 3.2.1. Let $B \cap Q^-(3,2) = \{x, y\}$. Since $|B| = 6$ and B blocks all secant lines, it can be seen that $Q^-(3,2) \setminus \mathcal{C}_\pi = \{x, y\}$. The secant line $L = xy$ meets the plane π in the nucleus of $\mathcal{C}_\pi$ and L^τ is precisely the unique external line contained in π. It follows that $B = L \cup L^\tau$. $\square$

In the rest of this section, we prove Theorem 3.4.1 for $q = 3$.

Lemma 3.4.8. *If $q = 3$, then $|B| = 13$.*

Proof. We have $|B| \leqslant 13$ and by Corollary 3.4.5, $|B| \geqslant 12$. Suppose that $|B| = 12$. Then Proposition 1.7.1 implies that there exists a tangent line L of $\mathrm{PG}(3,3)$ which is disjoint from B. By Lemma 3.4.4, each of the four planes through L contains at least three points of B. Since $L \cap B = \emptyset$ and $|B| = 12$, it follows that each plane through L contains exactly three points of B. Clearly, the points of B contained in the tangent plane through L are outside $Q^-(3,3)$. Proposition

2.5.2(i) implies that the points of B contained in a secant plane through L are also outside $Q^-(3,3)$. Thus B is disjoint from $Q^-(3,3)$.

There are three secant lines through each point of B, giving that B blocks at most 36 secant lines. But there are 45 lines which are secant to $Q^-(3,3)$. It follows that B does not block all the secant lines, a contradiction. Hence $|B| = 13$. $\square$

Lemma 3.4.9. *Suppose that $q = 3$. If $|B \cap Q^-(3,3)| = 1$, then B is a tangent plane.*

Proof. Since $|B| = 13$ by Lemma 3.4.8 and $|B \cap Q^-(3,3)| = 1$, we have $|B \setminus Q^-(3,3)| = 12$. There are three secant lines through each point of $B \setminus Q^-(3,3)$, implying that the points of $B \setminus Q^-(3,3)$ block at most 36 secant lines. There are nine secant lines through the point of $B \cap Q^-(3,3)$. Since B blocks each of the 45 secant lines, it follows that each secant line contains exactly one point of B. Thus, if $B \cap Q^-(3,3) = \{x\}$, then none of the secant lines through x contains a point of $B \setminus Q^-(3,3)$. This is equivalent to saying that the 12 points of $B \setminus Q^-(3,3)$ are contained in the tangent lines through x. It follows that B coincides with the tangent plane x^τ. $\square$

Lemma 3.4.10. *Suppose that $q = 3$. If $B \cap Q^-(3,3)$ contains exactly two points, say x_1, x_2, then every tangent line that is disjoint from B meets $x_1x_2 \setminus \{x_1, x_2\}$.*

Proof. Let K be a tangent line which is disjoint from B. Suppose that K does not meet $x_1x_2 \setminus \{x_1, x_2\}$. Then the planes $\pi_1 = \langle K, x_1 \rangle$ and $\pi_2 = \langle K, x_2 \rangle$ are distinct secant planes through K. Now, for every $i \in \{1, 2\}$, B_{π_i} is an $(\mathcal{E} \cup \mathcal{S})_{\pi_i}$-blocking set in π_i containing the point $x_i \in \mathcal{C}_{\pi_i}$. By Proposition 2.5.2, we then know that $|B_{\pi_i}| \geqslant 4$. Each of the two remaining planes π_3, π_4 through K distinct from π_1, π_2 contains at least three points of B by Lemma 3.4.4. As $B \cap K = \emptyset$, we have $|B| = \sum_{i=1}^{4} |B_{\pi_i}| \geqslant 4 + 4 + 3 + 3 = 14$, in contradiction with $|B| = 13$. $\square$

Proposition 3.4.11. *If $q = 3$, then B is a plane of $\mathrm{PG}(3, q)$.*

Proof. We have $|B| = 13$ by Lemma 3.4.8. There are three secant lines through a point of $\mathrm{PG}(3,3) \setminus Q^-(3,3)$. If B is disjoint from $Q^-(3,3)$, then $|B| = 13$ implies that B blocks at most 39 secant lines. Since there are 45 lines which are secant to $Q^-(3,3)$, it would follow that B does not block all the secant lines. So, $|B \setminus Q^-(3,3)| \leqslant 12$.

We show that every tangent line meets B. Then $|B| = 13$ and Proposition 1.7.1 would imply that B is a plane.

On the contrary suppose that there exists a tangent line L of $\mathrm{PG}(3,3)$ which is disjoint from B. Let $\pi_0, \pi_1, \pi_2, \pi_3$ be the four planes of $\mathrm{PG}(3,3)$ through L, where π_0 is the tangent plane and the other three are secant planes. By Lemma 3.4.4, $|B_{\pi_i}| \geqslant 3$ for each $i \in \{0,1,2,3\}$. Since $L \cap B = \emptyset$ and $|B| = 13$, it follows that exactly one of planes π_i contains 4 points of B and each of the remaining three planes contains 3 points of B. By Proposition 2.5.2(i), if π_i contains exactly three points of B for some $i \in \{1,2,3\}$, then B_{π_i} is disjoint from $Q^-(3,3)$. Since $|B \setminus Q^-(3,3)| \leqslant 12$ and the points of B_{π_0} are outside $Q^-(3,3)$, it follows that we must have $|B_{\pi_0}| = 3$.

Without loss of generality, we may assume that $|B_{\pi_1}| = 4$, $|B_{\pi_2}| = 3$ and $|B_{\pi_3}| = 3$. Since $B_{\pi_1} \setminus \mathcal{C}_{\pi_1}$ is an $\mathcal{E}_{\pi_1}$-blocking set in π_1, we have $|B_{\pi_1} \setminus \mathcal{C}_{\pi_1}| \geqslant 2$ by Proposition 2.1.1. Since $|B \setminus Q^-(3,3)| \leqslant 12$, the set $B \cap Q^-(3,3) = B \cap \mathcal{C}_{\pi_1} = B_{\pi_1} \cap \mathcal{C}_{\pi_1}$ is nonempty. So $|B_{\pi_1} \setminus \mathcal{C}_{\pi_1}| \leqslant 3$. Thus $|B_{\pi_1} \setminus \mathcal{C}_{\pi_1}| \in \{2,3\}$.

If $|B_{\pi_1} \setminus \mathcal{C}_{\pi_1}| = 3$, then $|B \setminus Q^-(3,3)| = 12$ and $|B \cap Q^-(3,3)| = 1$. In this case, Lemma 3.4.9 implies that B is a tangent plane which is not possible as $L \cap B = \emptyset$.

Thus $|B_{\pi_1} \setminus \mathcal{C}_{\pi_1}| = 2$. Then $|B \setminus Q^-(3,3)| = 11$ and $|B \cap Q^-(3,3)| = 2$. Put $B \cap Q^-(3,3) = \{x_1, x_2\}$.

Since L is disjoint from B, Lemma 3.4.10 implies that L meets $x_1x_2 \setminus \{x_1, x_2\}$ in a singleton. Denote by α the tangency point of L in $Q^-(3,3)$. Note that B_{π_0} is an $\mathcal{E}_{\pi_0}$-blocking set of size 3 in π_0. By Proposition 2.0.1, we then know that $B_{\pi_0} = U \setminus \{\alpha\}$ for some line U of π_0 through α distinct from L. If we denote by K a line of π_0 through α distinct from L and U, then K is another tangent line disjoint from B. By Lemma 3.4.10 again, we then know that K meets $x_1x_2 \setminus \{x_1, x_2\}$. But that is impossible: as $\pi_0 \cap x_1x_2 = L \cap x_1x_2$, L is the unique line through α in π_0 that meets x_1x_2. $\square$

3.5 $(\mathcal{T} \cup \mathcal{E})$-blocking sets

In this section, we shall prove the following theorem which characterizes the minimum size $(\mathcal{T} \cup \mathcal{E})$-blocking sets in $\mathrm{PG}(3, q)$.

Theorem 3.5.1. *Let B be a $(\mathcal{T} \cup \mathcal{E})$-blocking set in* $\mathrm{PG}(3, q)$*. Then $|B| \geqslant q^2 + q$, and equality holds if and only if $B = x^\tau \setminus \{x\}$ for some point x of $Q^-(3, q)$.*

3.5.1 Basic properties

Let B be a $(\mathcal{T} \cup \mathcal{E})$-blocking set in $\mathrm{PG}(3, q)$ and π be a plane of $\mathrm{PG}(3, q)$. If π is a tangent plane, then it does not contain any line of $\mathrm{PG}(3, q)$ that is secant to $Q^-(3, q)$ and so $(\mathcal{T} \cup \mathcal{E})_\pi$ is the set of all lines of π. In that case, Proposition 1.7.1 (with $d = 2$) implies that $|B_\pi| \geqslant q + 1$, and equality holds if and only if B_π is a line of π. If π is a secant plane, then $|B_\pi| \geqslant q$ by Proposition 2.4.1. We thus have the following.

Lemma 3.5.2. *Let B be a $(\mathcal{T} \cup \mathcal{E})$-blocking set in* $\mathrm{PG}(3, q)$ *and π be a plane of* $\mathrm{PG}(3, q)$*. Then the following hold:*

(i) If π is a secant plane, then $|B_\pi| \geqslant q$.

(ii) If π is a tangent plane, then $|B_\pi| \geqslant q + 1$ and equality holds if and only if B_π is a line of π.

Now let B be a $(\mathcal{T} \cup \mathcal{E})$-blocking set in $\mathrm{PG}(3, q)$ of minimum cardinality. For any point x of $Q^-(3, q)$, it is clear that the point set $x^\tau \setminus \{x\}$ blocks every line of $\mathcal{T} \cup \mathcal{E}$ and so is a $(\mathcal{T} \cup \mathcal{E})$-blocking set in $\mathrm{PG}(3, q)$ of size $q^2 + q$. Then minimality of $|B|$ implies that $|B| \leqslant q^2 + q$.

Lemma 3.5.3. *The following hold:*

(*i*) *There exists a secant line of* $\mathrm{PG}(3,q)$ *that is disjoint from* B. *If* L *is such a secant line, then each of the* $q+1$ *(secant) planes through* L *contains exactly* q *points of* B.

(*ii*) $|B| = q^2 + q$ *and* $B \cap Q^-(3,q) = \emptyset$.

Proof. Since $|B| \leqslant q^2 + q$, Proposition 1.7.1 implies that there exists a secant line L of $\mathrm{PG}(3,q)$ that is disjoint from B. Each plane through L, being a secant plane, contains at least q points of B by Lemma 3.5.2(i). Since L is disjoint from B, the $q+1$ secant planes through L together contain at least q^2+q points of B. It follows that $|B| = q^2 + q$ and that each secant plane through L contains exactly q points of B.

For a given secant plane π through L, $|B_\pi| = q$ implies that B_π is disjoint from the conic $\mathcal{C}_\pi$ in π by Corollary 2.4.2. Since B is a disjoint union of the sets B_π, and $Q^-(3,q)$ is the union of the conics $\mathcal{C}_\pi$ as π runs over all the secant planes through L, it follows that B is disjoint from $Q^-(3,q)$. □

Lemma 3.5.4. *If* π *is a tangent plane with* $|B_\pi| = q+1$, *then* B_π *is an external line.*

Proof. By Lemma 3.5.2(ii), B_π is a line of π. Since B is disjoint from $Q^-(3,q)$ by Lemma 3.5.3(ii), it follows that B_π is an external line. □

Lemma 3.5.5. *There are at least two tangent planes each containing* $q+1$ *points of* B.

Proof. By Lemma 3.5.2(ii), every tangent plane contains at least $q+1$ points of B. Let a be the number of tangent planes containing exactly $q+1$ points of B. Consider the set

$$Y = \{(x,\pi) : x \in B, \pi \text{ is a tangent plane containing } x\}.$$

Every point of $\mathrm{PG}(3,q) \setminus Q^-(3,q)$ is contained in $q+1$ tangent planes. Using Lemma 3.5.3(ii), we count $|Y|$ in two ways to get

$$(q^2+q)(q+1) = |B|(q+1) = |Y| \geqslant a(q+1) + (q^2+1-a)(q+2).$$

This gives $a \geqslant 2$. $\square$

Lemma 3.5.6. *If π_1 and π_2 are two distinct tangent planes with $|B_{\pi_1}| = q+1 = |B_{\pi_2}|$, then the external lines B_{π_1} and B_{π_2} are either equal or meet in a point.*

Proof. The external line $\pi_1 \cap \pi_2$ of π_2 contains a point of the line B_{π_2}. This point of B lies in π_1 and hence is contained in B_{π_1}. So, the lines B_{π_1} and B_{π_2} meet. $\square$

Lemma 3.5.7. *For every external line L, there are at most two tangent planes x^τ, $x \in Q^-(3,q)$, such that $B_{x^\tau} = L$.*

Proof. This follows from the fact that there are precisely two tangent planes through L. $\square$

Lemma 3.5.8. *Suppose π is a tangent plane with tangency point x. Then the number of secant lines through x disjoint from B is at least $|B_\pi| - q \geqslant 1$.*

Proof. By Lemma 3.5.2(ii), $|B_\pi| \geqslant q+1$ and so $|B_\pi| - q \geqslant 1$. We have $x \notin B$ and $|B| = q^2+q$ by Lemma 3.5.3(ii). There are $|B| - |B_\pi| = q^2 - (|B_\pi| - q)$ points of $B \setminus \pi$ and these are distributed over the q^2 (secant) lines through x not contained in π. So, at least $|B_\pi| - q$ of these q^2 lines are disjoint from B. $\square$

3.5.2 The case q odd

Lemma 3.5.9. *Suppose that q is odd. Then every tangent line meets B in either 1, $\sqrt{q}$ or q points.*

Proof. Consider a tangent line L with $L \cap Q^-(3,q) = \{x\}$. By Lemma 3.5.8, there exists a secant line K through x disjoint from B. Put $\pi := \langle K, L \rangle$, the secant plane generated by K and L. By Lemma 3.5.3(i), π intersects B in precisely q points. Since B_π is a $(\mathcal{T} \cup \mathcal{E})_\pi$-blocking set in π of size q, one of the following three possibilities occurs for B_π by Proposition 2.4.1.

(1) Suppose that $B_\pi = M \setminus \mathcal{C}_\pi$ for some line M of π tangent to $\mathcal{C}_\pi$. In π, if $L = M$, then $|L \cap B| = q$, and if $L \neq M$, then $|L \cap B| = 1$ as L and M intersect in a point not in $\mathcal{C}_\pi$.

(2) Suppose that $B_\pi = (M \setminus \mathcal{C}_\pi) \cup \{n\}$ for some line M of π secant to $\mathcal{C}_\pi$, where n is the pole of M with respect to $\mathcal{C}_\pi$. If $x \in M \cap \mathcal{C}_\pi$, then L passes through the pole n of M and so $|L \cap B| = 1$. If $x \notin M \cap \mathcal{C}_\pi$, then L does not pass through n and meets $M \setminus \mathcal{C}_\pi$ in a point, implying $|L \cap B| = 1$.

(3) Suppose that q is square and $B_\pi = \mu \setminus (\mu \cap \mathcal{C}_\pi)$, where μ is a Baer subplane of π such that $\mu \cap \mathcal{C}_\pi$ is an irreducible conic in μ. Put $K \cap Q^-(3, q) = \{x, y\}$. Note that every line of π meets μ in a singleton or a set of $\sqrt{q}+1$ points. Since K is disjoint from B, it intersects μ in a unique point, and this point must belong to $\mathcal{C}_\pi$. So, one of x, y belongs to $\mu \cap \mathcal{C}_\pi$ while the other belongs to $\mathcal{C}_\pi \setminus \mu$. If $x \in \mu \cap \mathcal{C}_\pi$ and $y \in \mathcal{C}_\pi \setminus \mu$, then L intersects B in $\sqrt{q}$ points by Lemma 2.7.1(i).

Assume therefore that $y \in \mu \cap \mathcal{C}_\pi$ and $x \in \mathcal{C}_\pi \setminus \mu$. Then Lemma 2.7.1(ii) implies that the tangent line L of π through x meets μ in a unique point not belonging to $\mathcal{C}_\pi$, that is, $|L \cap B| = 1$. $\square$

Remark 3.5.10. *The proof of Lemma 3.5.9 does not work for q even. Otherwise, in Step (2), we then have that $B_\pi = (M \setminus \mathcal{C}_\pi) \cup \{n\}$, where n is the nucleus of $\mathcal{C}_\pi$. In the case that $y = u$, we then have that L contains two points of B, namely the nucleus n and a point of $M \setminus \mathcal{C}_\pi$.*

For every $i \in \{1, \sqrt{q}, q\}$, we denote by N_i the number of tangent lines meeting B in exactly i points.

Lemma 3.5.11. *Suppose that q is odd. Then the following hold:*

(i) $N_1 \geqslant (q+1)(q^2 - \sqrt{q})$ *and* $N_{\sqrt{q}} + N_q \leqslant (\sqrt{q}+1)(q+1)$.

(ii) If $N_{\sqrt{q}} = 0$, then $N_1 = q^2(q+1)$ and $N_q = q+1$.

Proof. Since there are $q+1$ tangent lines through each point of $Q^-(3,q)$, we see that

$$N_1 + N_{\sqrt{q}} + N_q = |Q^-(3,q)| \cdot (q+1) = (q^2+1)(q+1). \tag{3.5.1}$$

Consider the set $Z = \{(b, L) : b \in B,\ L \text{ is a tangent line containing } b\}$. By Lemma 3.5.3(ii), we have $|B| = q^2+q$ and $B \subseteq \mathrm{PG}(3,q) \setminus Q^-(3,q)$. Recall also that each point of $\mathrm{PG}(3,q) \setminus Q^-(3,q)$ is contained in exactly $q+1$ tangent lines. Counting $|Z|$ in two ways, we have

$$N_1 + \sqrt{q} \cdot N_{\sqrt{q}} + q \cdot N_q = |B| \cdot (q+1) = (q^2+q)(q+1). \tag{3.5.2}$$

If $N_{\sqrt{q}} = 0$, then (3.5.1) and (3.5.2) imply that $N_1 = q^2(q+1)$ and $N_q = q+1$. This proves (ii). In the general case, we deduce from (3.5.1) and (3.5.2) that

$$(\sqrt{q}-1)N_1 - (q-\sqrt{q})N_q = (q+1) \cdot (\sqrt{q}(q^2+1) - (q^2+q)).$$

So,

$$(\sqrt{q}-1)N_1 \geqslant (q+1) \cdot ((\sqrt{q}-1)q^2 - \sqrt{q}(\sqrt{q}-1)),$$

that is, $N_1 \geqslant (q+1)(q^2-\sqrt{q})$. Then (3.5.1) implies that $N_{\sqrt{q}}+N_q \leqslant (\sqrt{q}+1)(q+1)$. This proves (i). $\square$

Let Γ denote the set of all points x of $Q^-(3,q)$ such that the tangent plane x^τ meets B in exactly $q+1$ points. By Lemma 3.5.5, $|\Gamma| \geqslant 2$. This bound is improved in the following lemma for odd q.

Lemma 3.5.12. *If q is odd, then $|\Gamma| \geqslant 2q > q+1$.*

Proof. For $x \in \Gamma$, B_{x^τ} is an external line by Lemma 3.5.4 and so every tangent line through x meets B in a singleton. If q is not a square, then $N_{\sqrt{q}} = 0$ and so $N_q = q+1$ by Lemma 3.5.11(ii). This implies that there are at most $q+1$ points of $Q^-(3,q)$ which are not contained in Γ. So,

$$|\Gamma| \geqslant (q^2+1) - (q+1) = q^2 - q \geqslant 2q,$$

where the last inequality holds as $q \geqslant 3$ for q being odd.

If q is a square, then $N_{\sqrt{q}} + N_q \leqslant (q+1)(\sqrt{q}+1)$ by Lemma 3.5.11(i). This implies that at most $(q+1)(\sqrt{q}+1)$ points of $Q^-(3,q)$ are not contained in Γ. So,

$$|\Gamma| \geqslant (q^2+1) - (q+1)(\sqrt{q}+1) \geqslant 2q,$$

where the second inequality holds as $q \geqslant 9$ for q being an odd square. $\square$

The following proposition proves Theorem 3.5.1 when q is odd.

Proposition 3.5.13. *If q is odd, then B is equal to the point set of a tangent plane minus its tangency point in $Q^-(3,q)$.*

Proof. Recall that, for $x \in \Gamma$, the tangent plane x^τ meets B in the external line $L_x := B_{x^\tau}$. The collection L_x, $x \in \Gamma$, of external lines mutually intersect by Lemma 3.5.6. There are two possibilities: all these lines contain a given point or are contained in a given plane.

Suppose first that all the lines of the collection L_x, $x \in \Gamma$, share a common point, say u. Then $u \in L_x \subseteq x^\tau$ for every $x \in \Gamma$. So, $x \in u^\tau$ for every $x \in \Gamma$, implying that the plane u^τ intersects $Q^-(3,q)$ in at least $|\Gamma| > q+1$ points (Lemma 3.5.12), a contradiction.

Therefore, the external lines L_x, $x \in \Gamma$, are all contained in the same plane, say μ^*. For every such external line L, there are at most two points $x \in \Gamma$ such that $L_x = L$ by Lemma 3.5.7. This fact and Lemma 3.5.12 imply that the number of mutually distinct lines in the collection L_x, $x \in \Gamma$, is at least $\frac{|\Gamma|}{2} \geqslant q$. Let

$L_1, L_2, \ldots, L_q$ be q mutually distinct lines of this collection. We then have

$$\begin{aligned} |B \cap \mu^*| &\geqslant |L_1| + |L_2 \setminus L_1| + |L_3 \setminus (L_1 \cup L_2)| \\ &\qquad + \cdots + |L_q \setminus (L_1 \cup L_2 \cup \cdots \cup L_{q-1})| \\ &\geqslant (q+1) + q + \cdots + 2 = \frac{q(q+3)}{2}. \end{aligned}$$

We claim that $\mu^* \setminus Q^-(3,q)$ is contained in B. Suppose there exists a point u of $\mu^* \setminus (Q^-(3,q) \cup B)$. There are $q+1$ tangent lines and $\dfrac{q^2+q}{2}$ external lines of $\mathrm{PG}(3,q)$ through u. So, there are at least $\dfrac{q^2+q}{2}$ tangent or external lines through u which are not contained in μ^*. Each of these lines contains a point of $B \setminus \mu^*$. So,

$$|B| = |B \cap \mu^*| + |B \setminus \mu^*| \geqslant \frac{q(q+3)}{2} + \frac{q^2+q}{2} = q^2 + 2q,$$

in contradiction with the fact that $|B| = q^2 + q$.

Thus $\mu^* \setminus Q^-(3,q) \subseteq B$. If μ^* is a tangent plane, then $|B| = q^2 + q$ implies that B is equal to the point set of the tangent plane μ^* minus its tangency point. In order to complete the proof of the proposition, we now show that μ^* cannot be a secant plane. If μ^* is a secant plane, then $B_{\mu^*} = \mu^* \setminus \mathcal{C}_{\mu*} = \mu^* \setminus Q^-(3,q)$ contains q^2 points. Since $|B| = q^2 + q$, we get

$$|B \setminus \mu^*| = q \geqslant 2.$$

In fact, for every $x \in \mathcal{C}_{\mu*}$, each of the q tangent lines through x not in μ^* contains a unique point of $B \setminus \mu^*$. This implies that $y \in x^\tau$ for every $x \in \mathcal{C}_{\mu*}$ and every $y \in B \setminus \mu^*$. This is impossible as the intersection $\bigcap_{x \in \mathcal{C}_{\mu*}} x^\tau$ is a singleton, namely $(\mu^*)^\tau$. $\square$

3.5.3 The case q even

Lemma 3.5.14. *Suppose that q is even. Let x be a point of $\mathrm{PG}(3, q) \setminus Q^-(3, q)$. If x is not a point of B, then $|B_{x^\tau}| \geqslant q + 1$.*

Proof. Since q is even, every line of x^τ through x is a tangent line. Each such line must contain a point of B, that is, a point of B_{x^τ}. Since $x \notin B$, it follows that $|B_{x^\tau}| \geqslant q + 1$. □

As a consequence of Lemma 3.5.14, we have

Corollary 3.5.15. *Suppose that q is even. Let x be a point of $\mathrm{PG}(3, q) \setminus Q^-(3, q)$. If the secant plane x^τ contains exactly q points of B, then $x \in B$.*

Lemma 3.5.16. *Suppose that q is even. Then the following hold for points x of $\mathrm{PG}(3, q) \setminus Q^-(3, q)$:*

(i) $x \in B$ if and only if $|B_{x^\tau}| = q$.

(ii) $x \notin B$ if and only if $|B_{x^\tau}| = q + 1$.

Proof. By Lemma 3.5.2(i), every secant plane of $\mathrm{PG}(3, q)$ contains at least q points of B. Let R be the collection of all points x of $\mathrm{PG}(3, q) \setminus Q^-(3, q)$ for which $|B_{x^\tau}| = q$.

(i) It is enough to show that $R = B$. We have $R \subseteq B$ by Corollary 3.5.15, and so $|R| \leqslant |B| = q^2 + q$. Consider the following set

$$X = \{(b, \pi) : b \in B,\ \pi \text{ is a secant plane containing } b\}.$$

The number of secant planes of $\mathrm{PG}(3, q)$ is $q^3 + q$ and each point of $\mathrm{PG}(3, q) \setminus Q^-(3, q)$ is contained in exactly q^2 secant planes. Counting $|X|$ in two ways using Lemma 3.5.3(ii), we get

$$(q^2 + q)q^2 = |B| \cdot q^2 = |X| \geqslant |R| \cdot q + (q^3 + q - |R|)(q + 1).$$

This gives $|R| \geqslant q^2 + q$. Thus $|R| = q^2 + q = |B|$ and hence $R = B$.

(ii) Let s denote the number of secant planes of $\mathrm{PG}(3, q)$ containing exactly $q+1$ points of B. Using (i), we have $s \leqslant q^3+q-|B| = q^3+q-(q^2+q) = q^3-q^2$. Consider again the set X defined in the proof of (i) above. We count $|X|$ in two ways to get

$$(q^2 + q)q^2 = |X| \geqslant (q^2 + q)q + s(q + 1) + (q^3 + q - (q^2 + q) - s)(q + 2).$$

This gives $s \geqslant q^3 - q^2$. Thus $s = q^3 - q^2$ and it follows that each of the $q^3 - q^2$ secant planes x^τ with $x \notin B$ meets B in exactly $q + 1$ points. $\square$

The following proposition proves Theorem 3.5.1 when q is even.

Proposition 3.5.17. *If q is even, then B is equal to the point set of a tangent plane minus its tangency point in $Q^-(3, q)$.*

Proof. Let π_0 be a tangent plane containing exactly $q + 1$ points of B. Such a plane π_0 exists by Lemma 3.5.5. Then B_{π_0} is an external line by Lemma 3.5.4 and so it is contained in two tangent planes and $q - 1$ secant planes. Let π_1 be the other tangent plane and $\pi_2, \ldots, \pi_q$ be the secant planes through B_{π_0}. Then B_{π_i} contains B_{π_0} for every $i \in \{0, 1, \ldots, q\}$. By Lemma 3.5.16, every secant plane contains q or $q+1$ points of B. This implies that $B_{\pi_j} = B_{\pi_0}$ for $2 \leqslant j \leqslant q$. Since $B = \bigcup_{i=0}^{q} B_{\pi_i}$, it now follows that $B = B_{\pi_1}$. Since B is disjoint from $Q^-(3, q)$ and $|B| = q^2 + q$ by Lemma 3.5.3(ii), we must have that $B = B_{\pi_1} = \pi_1 \setminus \{y\}$, where y is the tangency point of π_1 in $Q^-(3, q)$. $\square$

3.6 $\mathcal{T}$-blocking sets

We first prove the following lemma which gives a lower bound for the size of a $\mathcal{T}$-blocking sets in $\mathrm{PG}(3, q)$.

Lemma 3.6.1. *Let B be a $\mathcal{T}$-blocking set in* $\mathrm{PG}(3,q)$. *Then* $|B| \geqslant q^2+1$, *and equality holds if and only if every tangent line meets B in a unique point.*

Proof. Consider the set $X = \{(x, L) \mid x \in B, L \in \mathcal{T}, x \in L\}$. We count the cardinality of X in two ways. Every point of B is contained in $q+1$ tangent lines. This implies $|X| = |B|(q+1)$. Since each tangent line meets B and $|\mathcal{T}| = (q+1)(q^2+1)$, we get $|X| \geqslant (q+1)(q^2+1)$. Now it follows that $|B| \geqslant \frac{(q+1)(q^2+1)}{q+1} = q^2+1$. It is also clear that $|B| = q^2+1$ if and only if every tangent line contains a unique point of B. □

If q is even, then it is known that the point-line geometry $\mathcal{X} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, where $\mathcal{P}$ is the point set of $\mathrm{PG}(3,q)$, $\mathcal{L} = \mathcal{T}$ and $\mathcal{I}$ is the incidence relation induced from $\mathrm{PG}(3,q)$ is a generalized quadrangle of order q and it is isomorphic to $W(q)$. The following theorem characterizes the minimum size $\mathcal{T}$-blocking sets in $\mathrm{PG}(3,q)$ for q even.

Theorem 3.6.2. *Let B be a $\mathcal{T}$-blocking set in* $\mathrm{PG}(3,q)$, *q even. Then $|B| = q^2+1$ if and only if B is an ovoid of $\mathcal{X} \cong W(q)$.*

Proof. Since q is even, we know that $W(q)$ has ovoids. Every ovoid of $W(q)$ is of size q^2+1. By Lemma 3.6.1, $|B| = q^2+1$ if and only if every tangent line meets B in a unique point. The latter statement is equivalent to saying that B is an ovoid of $\mathcal{X} \cong W(q)$. □

Recall that ovoids in $W(q)$ (and hence in $\mathrm{PG}(3,q)$) with q even are not classified except for $q \in \{2, 4, 8, 16, 32\}$.

If q is odd, then the quadric $Q^-(3,q)$ itself is an example of a $\mathcal{T}$-blocking set in $\mathrm{PG}(3,q)$ of size q^2+1. The problem of classifying $\mathcal{T}$-blocking sets in $\mathrm{PG}(3,q)$, q odd, of size q^2+1 is under our investigation.

3.7 Generalizations to ovoids in PG$(3, q)$

Recall that if q is odd, then every ovoid in PG$(3, q)$ is an elliptic quadric, but more examples exist for q even. One can now wonder whether the main results on minimum size blocking sets obtained in the previous sections extend to ovoids in PG$(3, q)$, where q is even. We have observed that a general ovoid in PG$(3, q)$, q even, satisfies similar properties as an elliptic quadric. This raises the hope that some of the main results may be generalized to arbitrary ovoids.

First observe that the arguments used in the proof of Theorem 3.3.1 holds good with respect to any ovoid. We thus have the following.

Theorem 3.7.1. *Let $\mathcal{O}$ be an ovoid in* PG$(3, q)$*, q even, and B be a blocking set of the tangent and secant lines of* PG$(3, q)$ *with respect to $\mathcal{O}$. Then $|B| \geqslant q^2 + 1$, and equality holds if and only if $B = \mathcal{O}$.*

The proofs of Theorems 3.2.1 and 3.5.1 rely on the inequality $|A| \geqslant q - 1$ from Proposition 2.1.1, the inequality $|A| \geqslant q$ from Proposition 2.4.1 and Corollary 2.4.2. These results were proved in the literature for irreducible conics. In Proposition 2.8.1, we have extended these results to arbitrary ovals. Taking into account Proposition 2.8.1 and the properties of ovoids in PG$(3, q)$, q even, we can easily verify that the arguments used in Sections 3.2, 3.5.1 and 3.5.3 remain valid for general ovoids. So the conclusions of Theorems 3.2.1 and 3.5.1 remain valid for general ovoids in PG$(3, q)$, q even. We thus have the following theorems:

Theorem 3.7.2. *Let $\mathcal{O}$ be an ovoid in* PG$(3, q)$*, q even, and B be a blocking set of the external lines of* PG$(3, q)$ *with respect to $\mathcal{O}$. Then $|B| \geqslant q^2$, and equality holds if and only if $B = \pi \setminus \mathcal{O}$ for some secant plane π of* PG$(3, q)$ *with respect to $\mathcal{O}$.*

Theorem 3.7.3. *Let $\mathcal{O}$ be an ovoid in* PG$(3, q)$*, q even, and B be a blocking set of the external and tangent lines of* PG$(3, q)$ *with respect to $\mathcal{O}$. Then $|B| \geqslant q^2 + q$,* *and equality holds if and only if* $B = \pi \setminus \mathcal{O}$ *for some tangent plane* π *of* $\mathrm{PG}(3,q)$ *with respect to* $\mathcal{O}$.

For every ovoid in $\mathrm{PG}(3,q)$, q even, recall that there is associated a symplectic polarity of $\mathrm{PG}(3,q)$. Taking into account the properties of ovoids in $\mathrm{PG}(3,q)$, q even, we can easily verify that the arguments used in Sections 3.4 and 3.6 remain valid for general ovoids. So the conclusions of Theorem 3.4.1, Lemma 3.6.1 and Theorem 3.6.2 remain valid for ovoids in $\mathrm{PG}(3,q)$, q even. We thus have the following theorems:

Theorem 3.7.4. *Let* $\mathcal{O}$ *be an ovoid in* $\mathrm{PG}(3,q)$, *q even, and* τ *be the symplectic polarity associated with* $\mathcal{O}$. *If B is a blocking set of the external and secant lines of* $\mathrm{PG}(3,q)$ *with respect to* $\mathcal{O}$, *then the following hold:*

(i) If $q = 2$, *then* $|B| \geqslant 6$ *and equality holds if and only if* $B = L \cup L^\tau$ *for some line L secant to* $\mathcal{O}$.

(ii) If $q \geqslant 4$, *then* $|B| \geqslant q^2+q+1$ *and equality holds if and only if B is a plane of* $\mathrm{PG}(3,q)$.

Theorem 3.7.5. *Let* $\mathcal{O}$ *be an ovoid in* $\mathrm{PG}(3,q)$, *q even, and* $W(q)$ *be the generalized quadrangle of order q corresponding to the symplectic polarity associated with* $\mathcal{O}$. *If B is a blocking set of the tangent lines of* $\mathrm{PG}(3,q)$ *with respect to* $\mathcal{O}$, *then* $|B| \geqslant q^2+1$, *and equality holds if and only if B is an ovoid of* $W(q)$.

It is not clear whether the proof of Theorem 3.1.2 can be extended to ovoids in $\mathrm{PG}(3,q)$, q even, as it makes use of Proposition 2.3.1(2) for which no generalization to ovals is known.

Chapter 4

Blocking sets in $\mathrm{PG}(3,q)$: Quadratic cones

Let $\mathcal{K}$ be a quadratic cone in $\mathrm{PG}(3,q)$ with kernel the point p^* whose base is an irreducible conic $\mathcal{C}^*$ in a plane π^* not containing p^*. In this chapter, we denote by $\mathcal{E}$, $\mathcal{S}$ and $\mathcal{T}$ the set of all lines of $\mathrm{PG}(3,q)$ that are external, secant and tangent, respectively, with respect to $\mathcal{K}$. We shall study the minimum size $\mathcal{A}$-blocking sets in $\mathrm{PG}(3,q)$, where the line set $\mathcal{A}$ is one of $\mathcal{E}$, $\mathcal{T}$, $\mathcal{S}$, $\mathcal{E}\cup\mathcal{T}$, $\mathcal{E}\cup\mathcal{S}$ and $\mathcal{T}\cup\mathcal{S}$. All secant planes of $\mathrm{PG}(3,q)$ considered in this chapter are with respect to the quadratic cone $\mathcal{K}$.

4.1 Main result

Let $\mathcal{A}\in\{\mathcal{E},\mathcal{T},\mathcal{S},\mathcal{E}\cup\mathcal{T},\mathcal{E}\cup\mathcal{S},\mathcal{T}\cup\mathcal{S}\}$. With $\mathcal{A}$, we associate two parameters $N_{\mathcal{A}}$ and $\epsilon_{\mathcal{A}}$ as given in the following table.

$\mathcal{A}$	$N_{\mathcal{A}}$	$\epsilon_{\mathcal{A}}$
$\mathcal{E}$	$q-1$	0
$\mathcal{T}$	1 if q is even, and $\frac{q+1}{2}$ if q is odd	1
$\mathcal{S}$	q	0
$\mathcal{E}\cup\mathcal{T}$	q	1
$\mathcal{E}\cup\mathcal{S}$	q if q is even, 3 if $q=3$ and $q+1$ if $q>3$ is odd	0
$\mathcal{T}\cup\mathcal{S}$	$q+1$	1

Note that $N_{\mathcal{A}} \leqslant q+1$ and $\epsilon_{\mathcal{A}} = 1$ if and only if $\mathcal{T} \subseteq \mathcal{A}$. Denote by $\mathcal{A}'$ the set of lines of $\mathcal{A}$ that are contained in π^*. As we will see in Lemma 4.2.1, the number $N_{\mathcal{A}}$ equals the smallest size of an $\mathcal{A}'$-blocking set in π^*. The following theorem is the main result of this chapter which appears in [21]. It gives a uniform characterization of the minimum size $\mathcal{A}$-blocking sets in $\mathrm{PG}(3,q)$.

Theorem 4.1.1. *Let B be a minimum size $\mathcal{A}$-blocking set in* $\mathrm{PG}(3,q)$*, where $\mathcal{A}$ is one of the line sets $\mathcal{E}$, $\mathcal{T}$, $\mathcal{S}$, $\mathcal{E}\cup\mathcal{T}$, $\mathcal{E}\cup\mathcal{S}$ and $\mathcal{T}\cup\mathcal{S}$. Then $|B| = qN_{\mathcal{A}} + \epsilon_{\mathcal{A}}$ and the following hold:*

(i) If $\mathcal{A} \in \{\mathcal{E},\mathcal{S},\mathcal{E}\cup\mathcal{S}\}$, then $B = \mathcal{K}' \setminus \{p^*\}$, where $\mathcal{K}'$ is a cone with kernel p^* and base an $\mathcal{A}'$-blocking set of size $N_{\mathcal{A}}$ in π^*.

(ii) If $\mathcal{A} \in \{\mathcal{T},\mathcal{E}\cup\mathcal{T}\}$, then $B = \mathcal{K}'$, where $\mathcal{K}'$ is a cone with kernel p^* and base an $\mathcal{A}'$-blocking set of size $N_{\mathcal{A}}$ in π^*.

(iii) Suppose that $\mathcal{A} = \mathcal{T}\cup\mathcal{S}$.

(a) If $p^* \notin B$, then B is a secant plane.

(b) If $p^* \in B$, then $B = \mathcal{K}'$, where $\mathcal{K}'$ is a cone with kernel p^* and base an $\mathcal{A}'$-blocking set of size $N_{\mathcal{A}}$ in π^*.

4.2 Preliminaries

We note that, for a secant plane π of $\mathrm{PG}(3,q)$ with respect to the cone $\mathcal{K}$, the set of external (respectively, tangent, secant) lines of $\mathrm{PG}(3,q)$ with respect to $\mathcal{K}$ contained in π are precisely the set of external (respectively, tangent, secant) lines with respect to the irreducible conic $\pi \cap \mathcal{K}$ in π.

Let $\mathcal{A}$ be one of the line sets $\mathcal{E}$, $\mathcal{T}$, $\mathcal{S}$, $\mathcal{E} \cup \mathcal{T}$, $\mathcal{E} \cup \mathcal{S}$, $\mathcal{T} \cup \mathcal{S}$ in $\mathrm{PG}(3,q)$. Recall that $\mathcal{A}'$ denotes the set of lines of $\mathcal{A}$ that are contained in π^*. Note that $\mathcal{A}'$ consists of all the lines of π^* of the same type (as that of $\mathcal{A}$) with respect to the irreducible conic $\mathcal{C}^*$.

Lemma 4.2.1. *Every $\mathcal{A}'$-blocking set in π^* has size at least $N_{\mathcal{A}}$, and there exists an $\mathcal{A}'$-blocking set in π^* of size $N_{\mathcal{A}}$.*

Proof. In $\mathrm{PG}(2,q)$, blocking sets of minimum size with respect to various line sets (consisting of external/tangent/secant lines) determined by a given irreducible conic have been studied in the papers [1, 2, 3, 4, 12, 15, 22, 29, 48, 60]. Brief survey of the results obtained in this regard is given in Chapter 2 of this thesis. The present lemma is a consequence of these results. In particular, for $\mathcal{A} = \mathcal{E}$, the lemma is implied by Proposition 2.1.1 for q odd and by Proposition 2.1.2 for q even. For $\mathcal{A} = \mathcal{T}$, the lemma is implied by Proposition 2.2.1 for q even and by Proposition 2.2.2 for q odd. For $\mathcal{A} = \mathcal{S}$, the lemma is implied by Proposition 2.3.1. For $\mathcal{A} = \mathcal{E} \cup \mathcal{T}$, the lemma is implied by Proposition 2.4.1. For $\mathcal{A} = \mathcal{E} \cup \mathcal{S}$, the lemma is implied by Proposition 2.5.1 for q even and by Proposition 2.5.2 for q odd. Finally, for $\mathcal{A} = \mathcal{T} \cup \mathcal{S}$, the lemma is implied by Section 2.6. $\square$

Lemma 4.2.2. *Let X be an $\mathcal{A}'$-blocking set of size $N_{\mathcal{A}}$ in π^*. Let $\mathcal{K}'$ be the cone in* $\mathrm{PG}(3,q)$ *with kernel p^* and base $X \subseteq \pi^*$. We put $B' = \mathcal{K}' \setminus \{p^*\}$ if $\mathcal{A} \in \{\mathcal{E}, \mathcal{S}, \mathcal{E} \cup \mathcal{S}\}$ and $B' = \mathcal{K}'$ if $\mathcal{A} \in \{\mathcal{T}, \mathcal{E} \cup \mathcal{T}, \mathcal{T} \cup \mathcal{S}\}$. Then B' is an $\mathcal{A}$-blocking set in* $\mathrm{PG}(3,q)$ *of size $qN_{\mathcal{A}} + \epsilon_{\mathcal{A}}$.*

Proof. By construction, B' has size $qN_{\mathcal{A}} + \epsilon_{\mathcal{A}}$. We need to prove that B' is an

$\mathcal{A}$-blocking set in $\mathrm{PG}(3,q)$.

Let L be a line of $\mathcal{A}$. If $p^* \in L$, then we necessarily have $\mathcal{A} \in \{\mathcal{T}, \mathcal{E}\cup\mathcal{T}, \mathcal{T}\cup\mathcal{S}\}$, and in this case L contains the point p^* of B'. So, we may suppose that L does not contain p^*. The plane $\sigma := \langle p^*, L\rangle$ can have three possibilities depending upon L. If L is a tangent line meeting $\mathcal{K}$ in one point, then σ contains a line of $\mathcal{K}$. If L is a secant line, then σ contains two lines of $\mathcal{K}$ through p^*. If L is an external line, then σ contains only the point p^* of $\mathcal{K}$. In each of these three cases, it follows that σ intersects π^* in a line L' that belongs to $\mathcal{A}'$. Since X is an $\mathcal{A}'$-blocking set in π^*, L' contains a point x belonging to X. Then the unique point of L on the line p^*x belongs to B'. We conclude that B' is an $\mathcal{A}$-blocking set in $\mathrm{PG}(3,q)$. $\square$

4.3 Proof of the main result

Suppose B is an $\mathcal{A}$-blocking set in $\mathrm{PG}(3,q)$ of minimum possible size. By Lemmas 4.2.1 and 4.2.2, we then know that

$$|B| \leqslant qN_{\mathcal{A}} + \epsilon_{\mathcal{A}}. \tag{4.3.1}$$

Lemma 4.3.1. *The following hold:*

(a) *If $\mathcal{A} \in \{\mathcal{E}, \mathcal{S}, \mathcal{E}\cup\mathcal{S}\}$, then $p^* \notin B$.*

(b) *If $\mathcal{A} \in \{\mathcal{T}, \mathcal{E}\cup\mathcal{T}, \mathcal{T}\cup\mathcal{S}\}$ and $p^* \notin B$, then $N_{\mathcal{A}} = q+1$, $\epsilon_{\mathcal{A}} = 1$ and $|B| = q^2+q+1$.*

Proof. (a) If $\mathcal{A} \in \{\mathcal{E}, \mathcal{S}, \mathcal{E}\cup\mathcal{S}\}$, then $B \setminus \{p^*\}$ is also an $\mathcal{A}$-blocking set in $\mathrm{PG}(3,q)$. By the minimality of $|B|$, we then have $B \setminus \{p^*\} = B$, i.e. $p^* \notin B$.

(b) If $\mathcal{A} \in \{\mathcal{T}, \mathcal{E}\cup\mathcal{T}, \mathcal{T}\cup\mathcal{S}\}$ and $p^* \notin B$, then each of the q^2+q+1 (tangent) lines through p^* contains a point of B, implying that $|B| \geqslant q^2+q+1$. On the other hand, (4.3.1) implies that $|B| \leqslant qN_{\mathcal{A}} + \epsilon_{\mathcal{A}} \leqslant q(q+1)+1 = q^2+q+1$. So, $|B| = q^2+q+1$, $N_{\mathcal{A}} = q+1$ and $\epsilon_{\mathcal{A}} = 1$. □

Lemma 4.3.2. *Suppose $p^* \in B$ if $\mathcal{A} \in \{\mathcal{T}, \mathcal{E}\cup\mathcal{T}, \mathcal{T}\cup\mathcal{S}\}$. Then $|B| = qN_{\mathcal{A}} + \epsilon_{\mathcal{A}}$ and every secant plane meets B in exactly $N_{\mathcal{A}}$ points.*

Proof. By Lemma 4.3.1(a), we have $p^* \notin B$ if $\mathcal{A} \in \{\mathcal{E}, \mathcal{S}, \mathcal{E}\cup\mathcal{S}\}$. So,

$$|B \setminus \{p^*\}| = |B| - \epsilon_{\mathcal{A}} \leqslant qN_{\mathcal{A}} \tag{4.3.2}$$

by (4.3.1) and the definition of $\epsilon_{\mathcal{A}}$. We count the number N of pairs (x, π), where x is a point of $B \setminus \{p^*\}$ and π is a secant plane through x. As each point of $B \setminus \{p^*\}$ is contained in precisely q^2 secant planes, we have

$$N = |B \setminus \{p^*\}| \cdot q^2 \leqslant q^3 N_{\mathcal{A}} \tag{4.3.3}$$

using (4.3.2). On the other hand, if π is one of the q^3 secant planes, then π contains at least $N_{\mathcal{A}}$ points of $B \setminus \{p^*\}$ by Lemma 4.2.1, implying that

$$q^3 N_{\mathcal{A}} \leqslant N. \tag{4.3.4}$$

By (4.3.2), (4.3.3) and (4.3.4), we know that $N = q^3 N_{\mathcal{A}}$ and $|B| = qN_{\mathcal{A}} + \epsilon_{\mathcal{A}}$. Moreover, the reasoning leading to (4.3.4) then reveals that every secant plane contains exactly $N_{\mathcal{A}}$ points of B. □

Lemma 4.3.3. *We have $|B| = qN_{\mathcal{A}} + \epsilon_{\mathcal{A}}$.*

Proof. By Lemma 4.3.2, we can assume that $\mathcal{A} \in \{\mathcal{T}, \mathcal{E}\cup\mathcal{T}, \mathcal{T}\cup\mathcal{S}\}$ and $p^* \notin B$. By Lemma 4.3.1(b), we then know that $N_{\mathcal{A}} = q+1$, $\epsilon_{\mathcal{A}} = 1$ and $|B| = q^2+q+1$. Hence, $|B| = qN_{\mathcal{A}} + \epsilon_{\mathcal{A}}$ holds. □

Lemma 4.3.3 proves the first part of Theorem 4.1.1 regarding the minimum size of an $\mathcal{A}$-blocking set in $\mathrm{PG}(3, q)$. We next characterize the $\mathcal{A}$-blocking sets of minimum size $qN_{\mathcal{A}} + \epsilon_{\mathcal{A}}$ for which following result is needed.

Lemma 4.3.4. *Suppose $p^* \in B$ if $\mathcal{A} \in \{\mathcal{T}, \mathcal{E} \cup \mathcal{T}, \mathcal{T} \cup \mathcal{S}\}$. If π is a plane through p^* and L is a line of π not containing p^*, then*

$$|L \cap B| = \frac{qN_{\mathcal{A}} + |\pi \cap B| - |B|}{q} = \frac{|\pi \cap B| - \epsilon_{\mathcal{A}}}{q}.$$

Proof. By Lemma 4.3.2, the q secant planes through L contain in total $|L \cap B| + q\Big(N_{\mathcal{A}} - |L \cap B|\Big) = qN_{\mathcal{A}} - (q-1) \cdot |L \cap B|$ points of B. It follows that

$$qN_{\mathcal{A}} - (q-1) \cdot |L \cap B| = |B| - |\pi \cap B| + |L \cap B|.$$

Hence, $|L \cap B| = \frac{qN_{\mathcal{A}} + |\pi \cap B| - |B|}{q}$. By Lemma 4.3.2, we also know that this number is equal to $\frac{|\pi \cap B| - \epsilon_{\mathcal{A}}}{q}$. $\square$

Lemma 4.3.5. *Suppose $p^* \in B$ if $\mathcal{A} \in \{\mathcal{T}, \mathcal{E} \cup \mathcal{T}, \mathcal{T} \cup \mathcal{S}\}$. Then B can be obtained as in Lemma 4.2.2.*

Proof. By Lemma 4.3.1(a), it suffices to prove that if x is a point not belonging to $B \cup \{p^*\}$, then $p^*x \setminus \{p^*\}$ has no points in common with B. Consider a secant plane π through x. As $|\pi \cap B| = N_{\mathcal{A}} \leqslant q+1$ by Lemma 4.3.2, there exists a line L in π through x meeting B in $\eta \in \{0, 1\}$ points. Let π' be the plane $\langle p^*, L\rangle$. By Lemma 4.3.4, $B \cap \pi'$ is a set of $\eta q + \epsilon_{\mathcal{A}}$ points intersecting each line of π' not containing p^* in exactly η points. If $\eta = 0$, then necessarily $B \cap \pi' \subseteq \{p^*\}$ and the claim is valid. If $\eta = 1$, then $(B \cap \pi') \cup \{p^*\}$ is a set of $q+1$ points meeting each line and so $(B \cap \pi') \cup \{p^*\}$ is a line through p^* by Proposition 1.7.1. Also the claim is valid in this case. $\square$

Lemma 4.3.6. *Suppose $\mathcal{A} \in \{\mathcal{T}, \mathcal{E} \cup \mathcal{T}, \mathcal{T} \cup \mathcal{S}\}$ and $p^* \notin B$. Then $\mathcal{A} = \mathcal{T} \cup \mathcal{S}$ and B is a plane not containing p^*.*

Proof. By Lemma 4.3.1(b), we know that $N_{\mathcal{A}} = q+1$ and $|B| = q^2+q+1$. As $N_{\mathcal{A}} = q+1$, we necessarily have $\mathcal{A} = \mathcal{T} \cup \mathcal{S}$. If each external line meets B, we necessarily have that B is a plane by Proposition 1.7.1. So, we may assume that there exists an external line L that is disjoint from B. Each of the q secant planes through L contains at least $N_{\mathcal{A}} = q+1$ points of B. Also, the plane $\sigma := \langle p^*, L\rangle$ (where $\sigma \cap \mathcal{K} = \{p^*\}$) through L must contain at least $q+1$ points of B, namely at least one on each of the $q+1$ tangent lines of σ through p^*. So, $|B| \geqslant (q+1)^2 = q^2+2q+1$, a contradiction. $\square$

Remark. A classification of all $\mathcal{E}$-blocking sets (with respect to $\mathcal{K}$) in $\mathrm{PG}(3,q)$ of minimum size was already obtained in Theorem 4.3 of [9] in the case that $q \geqslant 9$. However, that theorem has counter examples in the case that q is even. Indeed, by Proposition 2.1.2, we know that there are three types of minimum size blocking sets of the external lines with respect to an irreducible conic in $\mathrm{PG}(2,q)$, q even. Via the construction mentioned in Lemma 4.2.2, we then obtain three types of $\mathcal{E}$-blocking sets of minimum size q^2-q. Theorem 4.3 of [9] only mentions one of these possibilities. The problem seems to originate from errors in the proof of [9, Proposition 4.2].

Chapter 5

Some minimal blocking sets in $\mathrm{PG}(3, q)$, $q \in \{2, 3\}$

Let $Q^-(3, q)$ be an elliptic quadric and $Q^+(3, q)$ be a hyperbolic quadric in $\mathrm{PG}(3, q)$. In this chapter, we shall denote by $\mathcal{E}^-$ and $\mathcal{E}^+$ the set of all external lines of $\mathrm{PG}(3, q)$ with respect to $Q^-(3, q)$ and $Q^+(3, q)$, respectively.

For $\epsilon \in \{-, +\}$, two $\mathcal{E}^\epsilon$-blocking sets X_1 and X_2 in $\mathrm{PG}(3, q)$ are said to be *isomorphic* if there is an automorphism of $\mathrm{PG}(3, q)$ stabilizing $Q^\epsilon(3, q)$ and mapping X_1 to X_2.

Recall from Theorem 3.2.1 that the following result holds for $\mathcal{E}^-$-blocking sets in $\mathrm{PG}(3, q)$:

Proposition 5.0.1. *Let X be an $\mathcal{E}^-$-blocking set in* $\mathrm{PG}(3, q)$. *Then $|X| \geqslant q^2$, and equality holds if and only if $X = \pi \setminus Q^-(3, q)$ for some secant plane π of* $\mathrm{PG}(3, q)$ *with respect to $Q^-(3, q)$.*

If π is a tangent plane of $\mathrm{PG}(3, q)$ with respect to $Q^-(3, q)$, then $\pi \setminus Q^-(3, q)$ is obviously also an $\mathcal{E}^-$-blocking set in $\mathrm{PG}(3, q)$. A straightforward counting shows that each point of $\pi \setminus Q^-(3, q)$ is contained in an external line that does not lie in π, implying that also this $\mathcal{E}^-$-blocking set in $\mathrm{PG}(3, q)$ is minimal. Its size is equal to $q^2 + q$. It can be of interest to search for new (families of) minimal

$\mathcal{E}^-$-blocking sets in $\mathrm{PG}(3,q)$ whose sizes are relatively small, say in the open interval $]q^2, q^2+q[$. In this chapter, we classify, up to isomorphisms, all minimal $\mathcal{E}^-$-blocking sets in $\mathrm{PG}(3,q)$ if $q=2$ and all minimal $\mathcal{E}^-$-blocking sets in $\mathrm{PG}(3,q)$ of size $q^2+1=10$ if $q=3$. We have not been able so far to describe an infinite family that contains these examples.

The minimum size $\mathcal{E}^+$-blocking sets in $\mathrm{PG}(3,q)$ were characterized in [8, Theorem 1.1] for even $q \geqslant 8$ and in [9, Theorem 2.4] for odd $q \geqslant 9$. Alternate proofs characterizing such blocking sets are given in [22, Section 2] for all odd q and in [54, Section 3] for all even q. The following result holds for $\mathcal{E}^+$-blocking sets in $\mathrm{PG}(3,q)$:

Proposition 5.0.2 ([8, 9, 22, 54])**.** *Let X be an $\mathcal{E}^+$-blocking set in* $\mathrm{PG}(3,q)$*. Then $|X| \geqslant q^2-q$, and equality holds if and only if $X = \pi \setminus Q^+(3,q)$ for some tangent plane π of* $\mathrm{PG}(3,q)$ *with respect to $Q^+(3,q)$.*

If π is a secant plane of $\mathrm{PG}(3,q)$ with respect to $Q^+(3,q)$, then $\pi \setminus Q^+(3,q)$ is obviously also an $\mathcal{E}^+$-blocking set in $\mathrm{PG}(3,q)$. It is straightforward to verify that each point of $\pi \setminus Q^+(3,q)$ is contained in an external line that is not in π if and only if $q \neq 2$. So, for $q>2$, this $\mathcal{E}^+$-blocking set in $\mathrm{PG}(3,q)$ is minimal and its size is equal to q^2. It can be of interest to search for new (families of) minimal $\mathcal{E}^+$-blocking sets in $\mathrm{PG}(3,q)$ whose sizes are relatively small, say in the open interval $]q^2-q, q^2[$. In this chapter, we also classify, up to isomorphisms, all minimal $\mathcal{E}^+$-blocking sets in $\mathrm{PG}(3,q)$ if $q=2$ and all minimal $\mathcal{E}^+$-blocking sets in $\mathrm{PG}(3,q)$ of size $q^2-q+1=7$ if $q=3$. We have not been able so far to describe an infinite family that contains these examples.

5.1 Description of the main results

The results of this chapter appear in [20]. Suppose first that $q=2$ and consider a hyperbolic quadric $Q^+(3,2)$ in $\mathrm{PG}(3,2)$. If π is a tangent plane of $\mathrm{PG}(3,2)$ with respect to $Q^+(3,2)$, then $\pi \setminus Q^+(3,2)$ is a minimal $\mathcal{E}^+$-blocking set in $\mathrm{PG}(3,2)$ of size 2 by Proposition 5.0.2. Such an $\mathcal{E}^+$-blocking set is also of the form $L \setminus Q^+(3,2)$ for some outer tangent L. We will prove the following.

Theorem 5.1.1. *Every minimal $\mathcal{E}^+$-blocking set in $\mathrm{PG}(3,2)$ is of the form $L \setminus Q^+(3,2)$ for some outer tangent L.*

Consider now an elliptic quadric $Q^-(3,2)$ in $\mathrm{PG}(3,2)$. Let V be the 4-dimensional vector space over $\mathbb{F}_2$ for which $\mathrm{PG}(3,2)$ is the associated projective space. The elliptic quadrics in $\mathrm{PG}(3,2)$ are precisely the frames of $\mathrm{PG}(3,2)$ implying that we can take an ordered basis $(\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4)$ in V with respect to which $Q^-(3,2)$ consists of the points with coordinates $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$ and $(1,1,1,1)$. Let Ω denote the set of all ω's of the form $(\{i,j\},k,l)$ with $\{i,j,k,l\} = \{1,2,3,4\}$ and $i < j$. For every such $\omega \in \Omega$, we define

$$B_\omega := \{\langle \bar{e}_i + \bar{e}_j \rangle, \langle \bar{e}_i + \bar{e}_j + \bar{e}_k \rangle, \langle \bar{e}_i + \bar{e}_j + \bar{e}_l \rangle, \langle \bar{e}_i + \bar{e}_k \rangle, \langle \bar{e}_j + \bar{e}_l \rangle\}.$$

The set $\mathcal{B} := \{B_\omega \,|\, \omega \in \Omega\}$ contains 12 elements which are all projectively equivalent under the stabilizer of $Q^-(3,2)$ inside $PGL(4,2)$, which is a group isomorphic to S_5. We will prove the following.

Theorem 5.1.2. *Each element of $\mathcal{B}$ is a minimal $\mathcal{E}^-$-blocking set in $\mathrm{PG}(3,2)$ of size 5.*

Theorem 5.1.3. *Up to isomorphisms, there are three minimal $\mathcal{E}^-$-blocking sets in $\mathrm{PG}(3,2)$. Each such blocking set is either of the form $\pi \setminus Q^-(3,2)$ for some secant plane π of $\mathrm{PG}(3,2)$ with respect to $Q^-(3,2)$, of the form $\pi \setminus Q^-(3,2)$ for some tangent plane π of $\mathrm{PG}(3,2)$ with respect to $Q^-(3,2)$ or belongs to $\mathcal{B}$.*

Next, we describe our obtained results for $q = 3$. For a given secant plane π of $\mathrm{PG}(3,3)$ with respect to the quadric $Q^\epsilon(3,3)$ with $\epsilon \in \{-,+\}$, we denote by E_π the set of six points of π that are exterior with respect to the irreducible conic $\pi \cap Q^\epsilon(3,3)$ in π. We will prove the following results.

Theorem 5.1.4. *Let $Q^-(3,3)$ be the elliptic quadric in $\mathrm{PG}(3,3)$ that has equation $X_1X_2 + X_3^2 + X_4^2 = 0$ with respect to a certain reference system in $\mathrm{PG}(3,3)$ and let π be the secant plane of $\mathrm{PG}(3,3)$ with equation $X_4 = 0$. Then*

$$Y := E_\pi \cup \{\langle(1,0,0,1)\rangle, \langle(0,-1,0,1)\rangle, \langle(1,-1,1,1)\rangle, \langle(1,-1,-1,1)\rangle\}$$

is a minimal $\mathcal{E}^-$-blocking set in $\mathrm{PG}(3,3)$ of size 10.

Theorem 5.1.5. *Let B be an $\mathcal{E}^-$-blocking set in $\mathrm{PG}(3,3)$ of size 10. Then B is one of the following:*

(1) *$B = (\pi \setminus Q^-(3,3)) \cup \{x\}$, where π is a secant plane of $\mathrm{PG}(3,3)$ with respect to $Q^-(3,3)$ and x is a point of $\mathrm{PG}(3,3)$ not belonging to $\pi \setminus Q^-(3,3)$.*

(2) *B is isomorphic to the $\mathcal{E}^-$-blocking set Y described in Theorem 5.1.4.*

Theorem 5.1.6. *Let w be a point of $\mathrm{PG}(3,3) \setminus Q^+(3,3)$ and π be the secant plane w^τ, where τ is the orthogonal polarity associated with $Q^+(3,3)$. Then $B := E_\pi \cup \{w\}$ is a minimal $\mathcal{E}^+$-blocking set in $\mathrm{PG}(3,3)$ of size 7.*

Theorem 5.1.7. *Let B be an $\mathcal{E}^+$-blocking set in $\mathrm{PG}(3,3)$ of size 7. Then B is one of the following:*

(1) *$B = (\pi \setminus Q^+(3,3)) \cup \{x\}$, where π is a tangent plane of $\mathrm{PG}(3,3)$ with respect to $Q^+(3,3)$ and x is a point of $\mathrm{PG}(3,3)$ not belonging to $\pi \setminus Q^+(3,3)$.*

(2) *B is as described in Theorem 5.1.6.*

5.2 Proofs for $q = 2$

5.2.1 Proof of Theorem 5.1.1

Suppose $q = 2$ and X is a minimal $\mathcal{E}^+$-blocking set in $\mathrm{PG}(3,2)$. Then $|X| \geqslant 2$ with equality if and only if $X = L \setminus Q^+(3,2)$ for some outer tangent L (see Proposition

5.0.2). Suppose therefore that $|X| \geqslant 3$. As X is minimal and $X \setminus Q^+(3,2)$ is an $\mathcal{E}^+$-blocking set, we know that $X \cap Q^+(3,2) = \emptyset$ and no two distinct points of X are on the same outer tangent.

Let x_1, x_2 and x_3 be three mutually distinct points of X. Suppose they are on an (external) line L of PG$(3,2)$. Let π be a (secant) plane through L and let y be the nucleus of the conic $\pi \cap Q^+(3,2)$. The unique external line through y is not contained in π and contains a point $x_3' \in X$. Upon replacing x_3 by x_3', we may thus assume that x_1, x_2 and x_3 are three points of X not on the same line of PG$(3,2)$. But then x_1x_2, x_1x_3 and x_2x_3 are three distinct external lines of the plane $\langle x_1, x_2, x_3 \rangle$. This is impossible as a plane only can contain 0 or 1 external line depending on whether it is a tangent or secant plane with respect to $Q^+(3,2)$.

5.2.2 Proof of Theorem 5.1.2

Suppose $q = 2$. We first give here a construction for minimal $\mathcal{E}^-$-blocking sets in PG$(3,2)$ of size 5 and prove subsequently in Lemma 5.2.1 that each element of $\mathcal{B}$ can be obtained via this construction.

Let π be a secant plane of PG$(3,2)$ with respect to $Q^-(3,2)$ and denote by k the nucleus of the irreducible conic $\mathcal{C}_\pi = \pi \cap Q^-(3,2)$. Let x be an arbitrary point of $\pi \setminus (\mathcal{C}_\pi \cup \{k\})$ and denote the two other points of $\pi \setminus (\mathcal{C}_\pi \cup \{k\})$ by y_1 and y_2. Through x, there are two external lines L_1 and L_2 not contained in π. The plane $\langle L_1, L_2 \rangle$ meets π in a third line L_3 through x. Since L_1 and L_2 are external lines, $\langle L_1, L_2 \rangle$ must be a tangent plane and L_3 a tangent line necessarily equal to kx. Let $x_1 \in L_1 \setminus \{x\}$ and $x_2 \in L_2 \setminus \{x\}$ such that the line x_1x_2 contains the tangency point x_3 of L_3 (so $L_3 = \{k, x, x_3\}$). By construction, $X := \{k, y_1, y_2, x_1, x_2\}$ is an $\mathcal{E}^-$-blocking set in PG$(3,2)$ of size 5. We now show that it is minimal. As L_i is an external line meeting X precisely in x_i, $X \setminus \{x_i\}$ is not an $\mathcal{E}^-$-blocking set for every $i \in \{1,2\}$. As $\langle L_1, L_2 \rangle$ is a tangent plane with tangency point x_3, the lines

x_1k and x_2k are external to $Q^-(3,2)$. As the third external line through k is not contained in π, we see that $X \setminus \{k\}$ is not an $\mathcal{E}^-$-blocking set. Suppose now that $X \setminus \{y_i\}$ is an $\mathcal{E}^-$-blocking set for some $i \in \{1,2\}$. Then y_ix_1 and y_ix_2 must be the two external lines through y_i not contained in π. As above, we then know that the plane $\langle y_ix_1, y_ix_2\rangle$ meets π in the line ky_i, in particular the line x_1x_2 must meet ky_i. This is obviously not the case here as the point $x_3 \in kx$ does not lie on ky_i. So, X must be a minimal $\mathcal{E}^-$-blocking set in $\mathrm{PG}(3,2)$.

Lemma 5.2.1. *Each $B \in \mathcal{B}$ is an $\mathcal{E}^-$-blocking set in* $\mathrm{PG}(3,2)$ *that can be obtained via the above construction.*

Proof. Let π be the secant plane through the points $(1,0,0,0)$, $(0,1,0,0)$ and $(0,0,1,0)$ of $Q^-(3,2)$. The nucleus k of $\mathcal{C}_\pi = \pi \cap Q^-(3,2)$ is equal to $(1,1,1,0)$ and we denote the three points of $\pi \setminus (\mathcal{C}_\pi \cup \{k\})$ by $x = (0,1,1,0)$, $y_1 = (1,1,0,0)$ and $y_2 = (1,0,1,0)$. The tangency point x_3 on the line $L_3 = kx$ is then equal to $(1,0,0,0)$. If $x_1 = (0,1,0,1)$ and $x_2 = (1,1,0,1)$, then $x_3 \in x_1x_2$ and $L_1 = xx_1$, $L_2 = xx_2$ are the two external lines through x not contained in π. We thus see that B_ω with $\omega = (\{1,2\},3,4)$, which is equal to the $\mathcal{E}^-$-blocking set $\{k, y_1, y_2, x_1, x_2\}$, can be obtained as in the above construction. The claim then follows from the fact that any two elements of $\mathcal{B}$ are isomorphic. □

5.2.3 Proof of Theorem 5.1.3

Suppose again that $q = 2$. We already know that $\pi \setminus Q^-(3,2)$ is a minimal $\mathcal{E}^-$-blocking set in $\mathrm{PG}(3,2)$ for every plane π. Let B be a minimal $\mathcal{E}^-$-blocking set in $\mathrm{PG}(3,2)$ that is not of the form $\pi \setminus Q^-(3,2)$ for some plane π. As B is minimal and $B \setminus Q^-(3,2)$ is a minimal $\mathcal{E}^-$-blocking set, we know that $B \cap Q^-(3,2) = \emptyset$ and B does not contain (nor is contained in) a set of the form $\pi \setminus Q^-(3,2)$ with π a plane.

Applying a permutation to the coordinates of the points (X_1, X_2, X_3, X_4) of $\mathrm{PG}(3,2)$ is an automorphism of $\mathrm{PG}(3,2)$ stabilizing $Q^-(3,2)$. The stabilizer of

$Q^-(3,2)$ inside $PGL(4,2)$ therefore contains a subgroup $S \cong S_4$. We will show that $B \in \mathcal{B}$. As $\mathcal{B}$ is stabilized by S, we are allowed to classify the sets B, up to isomorphisms in S.

The tangent plane π through the point $(1,1,1,1) \in Q^-(3,2)$ has equation $X_1 + X_2 + X_3 + X_4 = 0$. As B contains points outside π, i.e. points with weight 3, we may without loss of generality assume that $(1,1,1,0) \in B$. The external line $\{(1,1,0,0),(1,0,1,0),(0,1,1,0)\}$ must contain a point of B. Without loss of generality, we may assume that $(1,1,0,0) \in B$. We distinguish two cases.

Case (1). Suppose first that $(0,0,1,1) \notin B$. As each of the external lines $\{(0,0,1,1),(0,1,1,0),(0,1,0,1)\}$ and $\{(0,0,1,1),(1,0,1,0),(1,0,0,1)\}$ contains a point of B, at least one of $(0,1,1,0)$, $(0,1,0,1)$ belongs to B, as well as at least one of $(1,0,1,0)$, $(1,0,0,1)$. In the secant plane α with equation $X_4 = 0$, the points $(1,1,0,0)$ and $(1,1,1,0)$ belong to B, implying that at most one of the two remaining points $(0,1,1,0)$, $(1,0,1,0)$ in $\alpha \setminus Q^-(3,2)$ belongs to B. We will prove that precisely one of them belongs to B.

If this is not true, then none of $(0,1,1,0)$, $(1,0,1,0)$ belongs to B and we must have $(1,1,1,0),(1,1,0,0),(0,1,0,1),(1,0,0,1) \in B$. As the secant plane with equation $X_3 = 0$ contains at most three points of B (outside $Q^-(3,2)$), we have $(1,1,0,1) \notin B$. As each of the two external lines $\{(1,0,1,0),(0,1,1,1),(1,1,0,1)\}$ and $\{(0,1,1,0),(1,0,1,1),(1,1,0,1)\}$ contains a point of B, we then have that $(0,1,1,1)$ and $(1,0,1,1)$ belong to B. But that is impossible as it would imply that the tangent plane in the point $(0,0,1,0)$ with equation $X_1 + X_2 + X_4 = 0$ has all its points in B, with exception of $(0,0,1,0) \in Q^-(3,2)$.

So, precisely one of the points $(0,1,1,0)$, $(1,0,1,0)$ belongs to B. Up to isomorphisms in S, we may assume that $(1,1,1,0),(1,1,0,0),(1,0,1,0) \in B$ and $(0,1,1,0) \notin B$. The external line $\{(0,1,1,0),(0,1,0,1),(0,0,1,1)\}$ meets B. As $(0,0,1,1)$ and $(0,1,1,0)$ are not in B, we have that $(0,1,0,1) \in B$. The four points outside $Q^-(3,2)$ in the secant plane $X_1 = X_3$ are $(1,1,1,0)$, $(1,0,1,0)$, $(0,1,0,1)$, $(1,0,1,1)$. As not all these points can be contained in B, we have

$(1,0,1,1) \notin B$. As the external line $\{(0,1,1,0),(1,1,0,1),(1,0,1,1)\}$ contains points of B, we must have $(1,1,0,1) \in B$. So, B_ω with $\omega = (\{1,2\},3,4)$ is contained in B. The minimality of B then implies that $B = B_\omega$.

Case (2). Suppose next that $(0,0,1,1) \in B$. So, $(1,1,1,0)$, $(1,1,0,0)$ and $(0,0,1,1)$ are in B. The fourth point $(1,1,0,1)$ of $PG(3,2)\setminus Q^-(3,2)$ in the secant plane through $(1,1,1,0)$, $(1,1,0,0)$, $(0,0,1,1)$ cannot belong to B. Considering the external lines

$$\{(1,1,0,1),(0,1,1,1),(1,0,1,0)\} \text{ and } \{(1,1,0,1),(1,0,1,1),(0,1,1,0)\}$$

through $(1,1,0,1)$, we see that at least one of $(0,1,1,1)$, $(1,0,1,0)$ belongs to B, as well as at least one of $(1,0,1,1)$, $(0,1,1,0)$.

By considering the secant plane with equation $X_3 = X_4$ and taking into account that the points $(1,1,0,0)$, $(0,0,1,1)$ of B belong to that plane, we see that not both of $(0,1,1,1)$, $(1,0,1,1)$ can belong to B.

By considering the secant plane with equation $X_4 = 0$ and taking into account that the points $(1,1,1,0)$, $(1,1,0,0)$ of B belong to that plane, we see that not both of $(1,0,1,0)$, $(0,1,1,0)$ belong to B.

So, we either have $(0,1,1,1),(0,1,1,0) \in B$ or $(1,0,1,1),(1,0,1,0) \in B$. So, B_ω with ω equal to either $(\{2,3\},1,4)$ or $(\{1,3\},2,4)$ is contained in B. The minimality of B then implies that $B = B_\omega$.

5.3 Proofs for $q = 3$

5.3.1 Proofs of Theorems 5.1.4 and 5.1.5

Let B be an $\mathcal{E}^-$-blocking set in $PG(3,3)$ of size 10. If B is not minimal, then $B = B' \cup \{x\}$, where B' is an $\mathcal{E}^-$-blocking set in $PG(3,3)$ of size 9 and x is a point of $PG(3,3)$ not belonging to B'. By Proposition 5.0.1, $B' = \pi \setminus Q^-(3,3)$

for some secant plane π of PG$(3, 3)$ with respect to $Q^-(3, 3)$. We then have case (1) of Theorem 5.1.5. Therefore, we may assume that B is minimal. We divide the treatment into two cases:

- **Case I**: There exists a secant plane π of PG$(3, 3)$ with respect to $Q^-(3, 3)$ such that all the points of π exterior to the conic $\pi \cap Q^-(3, 3)$ are contained in B.

- **Case II**: There does not exist any secant plane π of PG$(3, 3)$ with respect to $Q^-(3, 3)$ such that all the points of π exterior to $\pi \cap Q^-(3, 3)$ are contained in B.

We will prove that the set Y defined in Theorem 5.1.4 is a minimal $\mathcal{E}^-$-blocking set in PG$(3, 3)$ of size 10 (Proposition 5.3.5) and that if Case I occurs, then B is isomorphic to Y (Proposition 5.3.6). We shall then prove that there are no examples of blocking sets corresponding to Case II. We repeatedly use the following lemma, mostly without mention.

Lemma 5.3.1. *We have* $B \cap Q^-(3, 3) = \emptyset$.

Proof. This follows from the minimality of B and the fact that $B \setminus Q^-(3, 3)$ is also an $\mathcal{E}^-$-blocking set in PG$(3, 3)$. □

Treatment of Case I

Here, we suppose that B is a minimal $\mathcal{E}^-$-blocking set in PG$(3, 3)$ of size 10 containing the set E_π of all points of a secant plane π that are exterior with respect to the conic $\mathcal{C}_\pi := \pi \cap Q^-(3, 3)$ in π. Let I_π denote the set of three points of π that are interior with respect to $\mathcal{C}_\pi$. Through each point of I_π, there are four lines belonging to $\mathcal{E}^-$ that are not lines of π.

Lemma 5.3.2. *The following hold:*

(1) $B \cap \pi = E_\pi$ *and* $|B \setminus \pi| = 4$.

(2) *If $x \in I_\pi$, then each of the four external lines through x not contained in π meets B in precisely one point.*

Proof. Since $\pi \setminus \mathcal{C}_\pi$ is an $\mathcal{E}^-$-blocking set in PG$(3,3)$ of size 9 containing E_π, the minimality of B with $|B| = 10$ implies that there must exist a point in I_π not belonging to B. Now, let x be an arbitrary point of $I_\pi \setminus B$. Since each of the four external lines through x not contained in π meets B, we must have $|B \setminus \pi| \geqslant 4$. As $|B| = 10$ and E_π is contained in B with $|E_\pi| = 6$, we thus see that the conclusions of the lemma must hold. □

Let Q denote the quadratic form defining the quadric $Q^-(3,3)$. We can choose a reference system in PG$(3,3)$ such that $Q^-(3,3)$ has equation

$$Q(X_1, X_2, X_3, X_4) = X_1X_2 + X_3^2 + X_4^2 = 0$$

and π has equation $X_4 = 0$. We then have:

$$\begin{aligned}
\mathcal{C}_\pi &= \{\langle(1,0,0,0)\rangle, \langle(0,1,0,0)\rangle, \langle(1,-1,1,0)\rangle, \langle(1,-1,-1,0)\rangle\},\\
E_\pi &= \{\langle(0,0,1,0)\rangle, \langle(1,1,0,0)\rangle, \langle(1,0,1,0)\rangle, \langle(1,0,-1,0)\rangle,\\
&\qquad\qquad \langle(0,1,1,0)\rangle, \langle(0,1,-1,0)\rangle\},\\
I_\pi &= \{\langle(1,-1,0,0)\rangle, \langle(1,1,1,0)\rangle, \langle(1,1,-1,0)\rangle\}.
\end{aligned}$$

In fact, $\mathcal{C}_\pi$ (respectively, E_π, I_π) consists of all points $\langle(X_1, X_2, X_3, 0)\rangle$ of π for which $Q(X_1, X_2, X_3, 0)$ is equal to 0 (respectively, 1, -1). If $f : \mathbb{F}_3^4 \times \mathbb{F}_3^4 \to \mathbb{F}_3$ is the symmetric bilinear form associated with Q, then

$$f((X_1, X_2, X_3, X_4), (Y_1, Y_2, Y_3, Y_4)) = X_1Y_2 + X_2Y_1 - X_3Y_3 - X_4Y_4$$

for all $(X_1, X_2, X_3, X_4), (Y_1, Y_2, Y_3, Y_4) \in \mathbb{F}_3^4$.

Lemma 5.3.3. *If $\langle(a,b,c,1)\rangle \in B \setminus \pi$, then $(a,b,c) \neq (0,0,0)$ and $\langle(a,b,c,0)\rangle \in \mathcal{C}_\pi$.*

Proof. Suppose that $(a, b, c) = (0, 0, 0)$ and let $\langle(u, v, w, 0)\rangle$ be an arbitrary point of I_π. As $Q(u, v, w, 1) = Q(u, v, w, 0)+Q(0, 0, 0, 1) = -1+1 = 0$, the line through $\langle(a, b, c, 1)\rangle = \langle(0, 0, 0, 1)\rangle$ and $\langle(u, v, w, 0)\rangle$ would not be external as required by Lemma 5.3.2. Hence, $(a, b, c) \neq (0, 0, 0)$.

We also have $\langle(a, b, c, 0)\rangle \notin I_\pi$ as otherwise $Q(a, b, c, 1) = Q(a, b, c, 0) + Q(0, 0, 0, 1) = -1 + 1 = 0$, in contradiction with $B \cap Q^-(3, 3) = \emptyset$ (Lemma 5.3.1).

Suppose that $\langle(a, b, c, 0)\rangle \in E_\pi$. Consider then the unique external line through $\langle(a, b, c, 0)\rangle$ contained in π. This external line contains precisely two points of I_π, say $\langle(u_1, v_1, w_1, 0)\rangle$ and $\langle(u_2, v_2, w_2, 0)\rangle$. Without loss of generality, we may suppose that we have chosen (u_1, v_1, w_1) and (u_2, v_2, w_2) in such a way that $(a, b, c) = (u_2, v_2, w_2) + \lambda(u_1, v_1, w_1)$ for some $\lambda \in \mathbb{F}_3$. The line through the points $\langle(a, b, c, 1)\rangle \in B \setminus \pi$ and $\langle(u_1, v_1, w_1, 0)\rangle \in I_\pi$ then contains the point $\langle(u_2, v_2, w_2, 1)\rangle$ belonging to $Q^-(3, 3)$ as $Q(u_2, v_2, w_2, 1) = Q(u_2, v_2, w_2, 0) + Q(0, 0, 0, 1) = -1 + 1 = 0$. But that is impossible as such a line must belong to $\mathcal{E}^-$ by Lemma 5.3.2. $\square$

Lemma 5.3.4. *Let* $(a_1, b_1, c_1, 0)$, $(a_2, b_2, c_2, 0)$, $(a_3, b_3, c_3, 0)$ *and* $(a_4, b_4, c_4, 0)$ *be mutually distinct elements of* $\mathbb{F}_3^4$ *such that each* $\langle(a_i, b_i, c_i, 0)\rangle$, $i \in \{1, 2, 3, 4\}$, *is a point of* $\mathcal{C}_\pi$. *Then*

$$X := \{\langle(a_i, b_i, c_i, 1)\rangle \,|\, i \in \{1, 2, 3, 4\}\} \cup E_\pi$$

is an $\mathcal{E}^-$*-blocking set in* $\mathrm{PG}(3, 3)$ *if and only if* $\langle(a_i - a_j, b_i - b_j, c_i - c_j, 0)\rangle \notin I_\pi$ *for all* $i, j \in \{1, 2, 3, 4\}$ *with* $i \neq j$.

Proof. We first prove that if $\langle(a, b, c, 0)\rangle \in \mathcal{C}_\pi$, then $\langle(a, b, c, 1), (a', b', c', 0)\rangle$ is an external line for every $\langle(a', b', c', 0)\rangle \in I_\pi$. Indeed, we have $Q(a, b, c, 1) = Q(a, b, c, 0) + Q(0, 0, 0, 1) = 0 + 1 \neq 0$ and $Q(a', b', c', 0) = -1$. The secant line $\langle(a, b, c, 0), (a', b', c', 0)\rangle$ in π contains only one point of I_π, namely $\langle(a', b', c', 0)\rangle$. If $(x, y, z) \in \{(a+a', b+b', c+c'), (a-a', b-b', c-c')\}$, then $\langle(x, y, z, 0)\rangle$ is a point of $\langle(a,b,c,0),(a',b',c',0)\rangle$ not belonging to I_π and hence the point $\langle(x,y,z,1)\rangle \in \langle(a,b,c,1),(a',b',c',0)\rangle$ does not belong to $Q^-(3,3)$ as $Q(x,y,z,1) = Q(x,y,z,0) + Q(0,0,0,1) = Q(x,y,z,0) + 1 \neq 0$.

Now, X is an $\mathcal{E}^-$-blocking set in PG$(3,3)$ if and only if for every $\langle(a',b',c',0)\rangle \in I_\pi$, the four lines $\langle(a',b',c',0),(a_i,b_i,c_i,1)\rangle$, $i \in \{1,2,3,4\}$, are distinct (external) lines. The latter statement precisely holds when the condition in the lemma is satisfied. $\square$

As an application of Lemma 5.3.4, we have the following proposition that also proves Theorem 5.1.4.

Proposition 5.3.5. *The sets*

$$Y := \{\langle(1,0,0,1)\rangle, \langle(0,-1,0,1)\rangle, \langle(1,-1,1,1)\rangle, \langle(1,-1,-1,1)\rangle\} \cup E_\pi$$

and

$$Z := \{\langle(-1,0,0,1)\rangle, \langle(0,1,0,1)\rangle, \langle(-1,1,-1,1)\rangle, \langle(-1,1,1,1)\rangle\} \cup E_\pi$$

are isomorphic minimal $\mathcal{E}^-$-blocking sets in PG$(3,3)$ *of size* 10.

Proof. Clearly, $|Y| = 10 = |Z|$. The facts that Y and Z are $\mathcal{E}^-$-blocking sets in PG$(3,3)$ follows from Lemma 5.3.4. Since the map

$$(X_1, X_2, X_3, X_4) \mapsto (X_1, X_2, X_3, -X_4)$$

defines an automorphism of PG$(3,3)$ stabilizing $Q^-(3,3)$ and fixing π pointwise, it follows that Y and Z are isomorphic.

If Y and Z were not minimal as $\mathcal{E}^-$-blocking sets, then they would be of the form $(\pi' \setminus Q^-(3,3)) \cup \{x\}$, where π' is a secant plane of PG$(3,3)$ with respect to $Q^-(3,3)$ and x is a point not belonging to $\pi' \setminus Q^-(3,3)$. This is obviously not the case here. $\square$

The following proposition proves Theorem 5.1.5 if Case I occurs.

Proposition 5.3.6. *B is equal to either Y or Z.*

Proof. Let $\mathcal{G}$ be the graph with vertex set

$$\begin{aligned} V(\mathcal{G}) &= \{(1,0,0,1),(0,-1,0,1),(1,-1,1,1),(1,-1,-1,1), \\ &\qquad (-1,0,0,1),(0,1,0,1),(-1,1,-1,1),(-1,1,1,1)\} \subseteq \mathbb{F}_3^4, \end{aligned}$$

where two distinct vertices $(a,b,c,1)$ and $(a',b',c',1)$ are adjacent whenever $\langle(a-a',b-b',c-c',0)\rangle \in I_\pi \cup \mathcal{C}_\pi$. Then $\mathcal{G}$ is isomorphic to the complete bipartite graph $K_{4,4}$ with the two parts

$$\{(1,0,0,1),(0,-1,0,1),(1,-1,1,1),(1,-1,-1,1)\},$$

and

$$\{(-1,0,0,1),(0,1,0,1),(-1,1,-1,1),(-1,1,1,1)\}$$

for which the set of all edges $\{(a,b,c,1),(a',b',c',1)\}$ satisfying $\langle(a-a',b-b',c-c',0)\rangle \in \mathcal{C}_\pi$ is a complete (that is, perfect) matching.

We know that $B \backslash E_\pi$ is a subset of $\{\langle(a,b,c,1)\rangle : (a,b,c,1) \in V(\mathcal{G})\}$ by Lemma 5.3.3 and that B must be obtained as in Lemma 5.3.4. The first paragraph of this proof in combination with Lemma 5.3.4 then immediately implies that B is either Y or Z. □

Treatment of Case II

Here, we suppose that B is a minimal $\mathcal{E}^-$-blocking set in $\mathrm{PG}(3,3)$ of size 10 satisfying the following:

> $(*)$ There is no secant plane π of $\mathrm{PG}(3,3)$ with respect to $Q^-(3,3)$ such that all the points of π exterior to the conic $\pi \cap Q^-(3,3)$ are contained in B.

We shall derive a contradiction at the end of this subsection.

Lemma 5.3.7. *Let π be a tangent plane of* $\mathrm{PG}(3,3)$ *with respect to* $Q^-(3,3)$. *Then $|\pi \cap B| \geqslant 3$, with equality if and only if $\pi \cap B = L \setminus Q^-(3,3)$ for some tangent line L contained in π.*

Proof. This follows from Proposition 2.0.1. □

Lemma 5.3.8. *A secant plane of* $\mathrm{PG}(3,3)$ *with respect to* $Q^-(3,3)$ *cannot contain more than five points of B.*

Proof. Suppose that there is a secant plane π containing at least six points of B. By our assumption $(*)$, there exists a point x in $\pi \setminus B$ exterior with respect to the conic $\pi \cap Q^-(3,3)$. Through x, there are five external lines not contained in π. Each of these five external lines contains an extra point of B, implying that $|B| \geqslant |\pi \cap B| + 5 \geqslant 6 + 5 = 11$, a contradiction. □

Lemma 5.3.9. *There does not exist any tangent plane of* $\mathrm{PG}(3,3)$ *with respect to* $Q^-(3,3)$ *meeting B in an external line.*

Proof. Suppose π_1 is a tangent plane meeting B in an external line L. Let π_2 be the other tangent plane through L, and π_3, π_4 be the two secant planes through L. Each of $\pi_3 \setminus L$ and $\pi_4 \setminus L$ contains at most one point of B by Lemma 5.3.8, implying that $\pi_2 \setminus L$ contains at least four points of B. Thus, the tangent plane π_2 contains at least eight points of B.

Now, let x be a point of $\pi_2 \setminus Q^-(3,3)$ not contained in B. There are three external lines through x not contained in π. Each of these three external lines contains a point of B, implying that $|B| \geqslant |\pi_2 \cap B| + 3 \geqslant 8 + 3 = 11$, a contradiction. □

Lemma 5.3.10. *There does not exist any tangent plane of* $\mathrm{PG}(3,3)$ *with respect to* $Q^-(3,3)$ *meeting B in three points.*

Proof. Suppose π is a tangent plane meeting B in three points. Then there exists a tangent line L in π such that $\pi \cap B = L \setminus Q^-(3,3)$ by Lemma 5.3.7.

By Lemma 5.3.8, each of the three secant planes through L contains besides the points of $L \setminus Q^-(3,3)$ at most two other points of B, giving that $|B| \leqslant |L \setminus Q^-(3,3)| + 3 \cdot 2 = 9$, a contradiction. $\square$

Lemma 5.3.11. *Each tangent plane of* $\mathrm{PG}(3,3)$ *with respect to* $Q^-(3,3)$ *intersects* B *in precisely four points.*

Proof. We count in two ways the number N of pairs (x, π), where $x \in B$ and π a tangent plane through x. As there are 10 tangent planes, we know by Lemmas 5.3.7 and 5.3.10 that $N \geqslant 40$, with equality if and only if every tangent plane contains precisely four points. On the other hand, we have $|B| = 10$ possibilities for x and we know by Lemma 5.3.1 that there are four possibilities for π for a given x. So, $N = 40$ and every tangent plane contains precisely four points of π. $\square$

Lemma 5.3.12. *Let* x *be a point of* $Q^-(3,3)$ *and* π *be the tangent plane of* $\mathrm{PG}(3,3)$ *with respect to* $Q^-(3,3)$ *through* x*. Then*

$$\pi \cap B = (L_x \setminus Q^-(3,3)) \cup \{z_x\}$$

for some (tangent) line L_x *of* π *through* x *and some point* $z_x \in \pi \setminus L_x$*.*

Proof. By Lemma 5.3.11, we have $|\pi \cap B| = 4$. First suppose that every tangent line contained in π meets B. Then $\pi \cap B$ is a blocking set of size 4 with respect to all lines of π and so $\pi \cap B$ is a line of π by Proposition 1.7.1. As $B \cap Q^-(3,3) = \emptyset$, $\pi \cap B$ must be an external line, in contradiction with Lemma 5.3.9.

Therefore, there is some tangent line in π disjoint from B. As $|B \cap \pi| = 4$, this implies that there is some tangent line L_x in π meeting B in at least two points, say x_1 and x_2. Let x_3 denote the unique point in $L_x \setminus Q^-(3,3)$ distinct from x_1 and x_2. If $x_3 \notin B$, then each of the three external lines through x_3

contained in π would contain another point of B, implying $|\pi \cap B| \geqslant 5$ which is not possible. So, $x_3 \in B$ and $L_x \setminus Q^-(3,3)$ is contained in B. Since $|\pi \cap B| = 4$ and $B \cap Q^-(3,3) = \emptyset$, there exists a unique point $z_x \in \pi \cap B$ not belonging to L_x. Thus, $\pi \cap B = (L_x \setminus Q^-(3,3)) \cup \{z_x\}$. □

Lemma 5.3.13. *There exist no four mutually distinct points x_1, x_2, x_3, x_4 in $Q^-(3,3)$ such that the tangent lines L_{x_1}, L_{x_2}, L_{x_3}, L_{x_4} obtained as in Lemma 5.3.12 meet in a common point.*

Proof. Suppose L_{x_1}, L_{x_2}, L_{x_3}, L_{x_4} meet in a common point y. Then the secant plane y^τ contains the points x_1, x_2, x_3 and x_4. The number of points of B contained in $L_{x_1} \cup L_{x_2} \cup L_{x_3} \cup L_{x_4}$ equals 9. There are also at least two points of B in y^τ which follows from Proposition 2.1.1. This would imply that $|B| \geqslant 11$, a contradiction. □

Lemma 5.3.14. *Each point of B is contained in precisely three tangent lines L_x, $x \in Q^-(3,3)$.*

Proof. We count in two ways the number N of pairs (x, y) with $x \in Q^-(3,3)$ and $y \in L_x \setminus Q^-(3,3) \subseteq B$. As $|Q^-(3,3)| = 10$, there are $N = 30$ such pairs. On the other hand, there are at most $|B| = 10$ possibilities for y. For a given y, there are at most three possibilities for x with $y \in L_x$ by Lemma 5.3.13. As $N = 30$, we thus see that every point of B is contained in precisely three lines L_x, $x \in Q^-(3,3)$. □

<u>Derivation of a contradiction</u>:

Fix a point $x \in B$ and let $L_{u_1}, L_{u_2}, L_{u_3}$ be the three mutually distinct tangent lines through x obtained in Lemma 5.3.14, where $u_1, u_2, u_3 \in Q^-(3,3)$. Put

$$L_{u_1} := \{x, y_1, z_1, u_1\},\ L_{u_2} := \{x, y_2, z_2, u_2\} \text{ and } L_{u_3} := \{x, y_3, z_3, u_3\}.$$

Then $\{x, y_1, z_1, y_2, z_2, y_3, z_3\} \subseteq B$. We thus have already found seven points of B. Each of the planes $\langle L_{u_1}, L_{u_2}\rangle$, $\langle L_{u_1}, L_{u_3}\rangle$ and $\langle L_{u_2}, L_{u_3}\rangle$ is a secant plane containing, by Lemma 5.3.8, no further points of B than those in $\{x, y_1, z_1, y_2, z_2, y_3, z_3\}$. So, for the point $y_1 \in B$, if

$$L_{u_1} = \{y_1, x, z_1, u_1\},\ L_{u_2'} := \{y_1, y_2', z_2', u_2'\} \text{ and } L_{u_3'} := \{y_1, y_3', z_3', u_3'\}$$

are the three mutually distinct tangent lines through y_1 obtained in Lemma 5.3.14 with $u_2', u_3' \in Q^-(3,3)$ and $y_2', z_2', y_3', z_3' \in B$, then none of the points y_2', z_2', y_3', z_3' can be contained in $\{x, y_1, z_1, y_2, z_2, y_3, z_3\}$. This would imply that $\{x, y_1, z_1, y_2, z_2, y_3, z_3, y_2', z_2', y_3', z_3'\}$ is a set of 11 points of B, a contradiction.

5.3.2 Proof of Theorem 5.1.6

In this subsection, w is a point of $\mathrm{PG}(3,3) \setminus Q^+(3,3)$, π is the secant plane w^τ of $\mathrm{PG}(3,3)$, where τ is the orthogonal polarity associated with $Q^+(3,3)$, and $B = E_\pi \cup \{w\}$.

Proof of Theorem 5.1.6. Clearly, we have $|B| = 7$. We need to show that B is a minimal $\mathcal{E}^+$-blocking set in $\mathrm{PG}(3,3)$. Through w, there are:

- four outer tangents, namely the lines wx with x a point in the conic $\pi \cap Q^+(3,3)$;

- three external lines with respect to $Q^+(3,3)$, namely the lines $wz \in \mathcal{E}^+$ with $z \in I_\pi$ (see [23, Corollary 2.4]), where I_π is the set of three points of π that are interior with respect to $\pi \cap Q^+(3,3)$;

- six secant lines with respect to $Q^+(3,3)$, namely the lines wy with $y \in E_\pi$.

Through a point $u \in I_\pi$, there are three lines belonging to $\mathcal{E}^+$. Two of them are contained in π and so each of them meets E_π. The remaining one external line necessarily coincides with uw.

Now, any external line $L \in \mathcal{E}^+$ is either contained in π, meets π in a point of E_π or meets π in a point of I_π. In the first case, L contains two points of E_π. In the second case, L contains one point of E_π but not the point w. In the third case, L contains the point w but not any of the points of E_π.

It follows from the above that B is an $\mathcal{E}^+$-blocking set in $\mathrm{PG}(3,3)$, but also that it is minimal as removing one point x of B would imply the existence of lines belonging to $\mathcal{E}^+$ that are disjoint from $B \setminus \{x\}$. Indeed, if $x = w$, then this would be the case for the external lines $wz \in \mathcal{E}^+$ with $z \in I_\pi$. If $x \in E_\pi$, then this would be the case for the two external lines through x not contained in π. This completes the proof. □

The rest of this subsection is devoted to prove a result (Corollary 5.3.16) which is needed while proving Theorem 5.1.7.

Let L^* be an outer tangent contained in π with tangency point x^* in $Q^+(3,3)$. As wx^* is also an outer tangent, the plane $\pi^* := \langle w, L^* \rangle$ is a tangent plane with tangency point x^*. Further, π^* meets B in four points among which three are on the same outer tangent L^* and the remaining one point, namely w is on the other outer tangent through x^*, that is, $\pi^* \cap B = (L^* \setminus \{x^*\}) \cup \{w\}$.

Lemma 5.3.15. *Any $\mathcal{E}^+$-blocking set in* $\mathrm{PG}(3,3)$ *of size* 7 *intersecting* π^* *in* $B \cap \pi^*$ *coincides with* B.

Proof. Let X be an $\mathcal{E}^+$-blocking set in $\mathrm{PG}(3,3)$ of size 7 such that $X \cap \pi^* = B \cap \pi^*$. We claim that $X = B$. It suffices to prove that there is at most one such $\mathcal{E}^+$-blocking set X.

Put $wx^* := \{w, x^*, u_1, u_2\}$. Then $w \in B$ and x^*, u_1, u_2 are not contained in B. Through the outer tangent wx^*, there are three secant planes, say π_1, π_2, π_3, of $\mathrm{PG}(3,3)$ with respect to $Q^+(3,3)$. For $i \in \{1,2,3\}$, the set $X \cap \pi_i$ is a blocking set in π_i of the external lines with respect to the conic $\mathcal{C}_i = \pi_i \cap Q^+(3,3)$. Then $|X \cap \pi_i| \geqslant 2$ by Proposition 2.1.1, and so each π_i contains at least one extra point of X besides w. But as $|X| = 7$ and $|X \cap \pi^*| = |B \cap \pi^*| = 4$, there are precisely three points in $X \setminus \pi^*$, showing that each π_i contains precisely one extra point of

X different from w. It suffices to show that this extra point in each π_i is uniquely determined.

As wx^* is tangent to the conic $\mathcal{C}_i$, the points u_1 and u_2 of π_i are exterior with respect to $\mathcal{C}_i$. So, each u_j with $j \in \{1,2\}$ is contained in a unique line M_{ij} of π_i external to $\mathcal{C}_i$. Each of the lines M_{i1} and M_{i2} contains a point of X. As π_i contains only one extra point of X besides w, this extra point of X in π_i must coincide with $M_{i1} \cap M_{i2}$. $\square$

We thus have also shown the following.

Corollary 5.3.16. *The $\mathcal{E}^+$-blocking sets in* PG$(3,3)$ *of size* 7 *disjoint from* $Q^+(3,3)$ *and intersecting a tangent plane in a set of four points three of which are on the same outer tangent and the remaining one is on the other outer tangent are precisely the $\mathcal{E}^+$-blocking sets described in Theorem 5.1.6.*

5.3.3 Proof of Theorem 5.1.7

Let B be an $\mathcal{E}^+$-blocking set in PG$(3,3)$ of size 7. If B is not minimal, then $B = B' \cup \{x\}$, where B' is an $\mathcal{E}^+$-blocking set in PG$(3,3)$ of size 6 and x is a point not belonging to B'. By Proposition 5.0.2, $B' = \pi \setminus Q^+(3,3)$ for some tangent plane π of PG$(3,3)$ with respect to $Q^+(3,3)$. We then have case (1) of Theorem 5.1.7. So, we may assume the following:

Assumption 1: B is minimal.

As $B \setminus Q^+(3,3)$ is also an $\mathcal{E}^+$-blocking set in PG$(3,3)$, we have the following by the minimality of B.

Lemma 5.3.17. $B \cap Q^+(3,3) = \emptyset$.

Each tangent plane of PG$(3,3)$ with respect to $Q^+(3,3)$ meets B thus in at most six points.

Lemma 5.3.18. *There is no tangent plane of* $\mathrm{PG}(3,3)$ *with respect to* $Q^+(3,3)$ *meeting* B *in precisely six points.*

Proof. If this were the case, then B would be as in (1) of Theorem 5.1.7, contrary to the assumption that B is minimal. $\square$

Lemma 5.3.19. *There exists no tangent plane of* $\mathrm{PG}(3,3)$ *with respect to* $Q^+(3,3)$ *intersecting* B *in precisely five points.*

Proof. Suppose π is a tangent plane intersecting B in precisely five points and let x be the unique point in $\pi \setminus (B \cup Q^+(3,3))$. Through x, there are three lines belonging to $\mathcal{E}^+$ not contained in π. Each of these lines contains at least one point of B, showing that $|B| = |B \cap \pi| + |B \setminus \pi| \geqslant 5 + 3 = 8$, a contradiction. $\square$

We may also make the following assumption.

> **Assumption 2.** There is no tangent plane of $\mathrm{PG}(3,3)$ with respect to $Q^+(3,3)$ intersecting B in a set of four points three of which are on the same outer tangent and the remaining one is on the other outer tangent.

Indeed, if this were not the case, then Corollary 5.3.16 implies that we would have case (2) of Theorem 5.1.7. We shall derive a contradiction at the end of this section under the Assumptions 1 and 2.

Lemma 5.3.20. *There exists no tangent plane of* $\mathrm{PG}(3,3)$ *with respect to* $Q^+(3,3)$ *intersecting* B *in precisely four points.*

Proof. Suppose π is a tangent plane intersecting B in precisely four points. By Lemma 5.3.17 and Assumption 2, we then know that each of the two outer tangents in π contains precisely two points of B. So, there exists a unique line L in π that is secant to $Q^+(3,3)$ and disjoint from B. There are two secant planes, say π_1, π_2, of $\mathrm{PG}(3,3)$ with respect to $Q^+(3,3)$ through L. Each of them contains at least two points of B by Proposition 2.1.1, showing that $|B| \geqslant |B \cap \pi| + |B \cap \pi_1| + |B \cap \pi_2| \geqslant 4 + 2 + 2 = 8$, a contradiction. $\square$

Lemma 5.3.21. *There exists no outer tangent intersecting B in precisely three points.*

Proof. Suppose L is an outer tangent meeting B in precisely three points. There is a unique tangent plane π through L and π contains besides L one other outer tangent L'. By Lemmas 5.3.17, 5.3.18, 5.3.19 and 5.3.20, we know that $L' \cap B = \emptyset$. There are three secant planes, say π_1, π_2, π_3, of $\mathrm{PG}(3,3)$ with respect to $Q^+(3,3)$ through L'. Each of them contains at least two points of B by Proposition 2.1.1, giving that $|B| \geqslant |B \cap \pi| + |B \cap \pi_1| + |B \cap \pi_2| + |B \cap \pi_3| \geqslant 3 + 2 + 2 + 2 = 9$, a contradiction. □

Lemma 5.3.22. *There exists no tangent plane of* $\mathrm{PG}(3,3)$ *with respect to* $Q^+(3,3)$ *meeting B in precisely three points.*

Proof. Suppose π is a tangent plane with tangency point x meeting B in precisely three points. By Lemma 5.3.21, there exists a unique outer tangent L_1 in π containing precisely two points of B and the other outer tangent L_2 in π contains a unique point of B. By Lemma 5.3.15, we know that there exists a unique $\mathcal{E}^+$-blocking set B^* in $\mathrm{PG}(3,3)$ of size 7 that meets π in $(L_1 \setminus \{x\}) \cup (L_2 \cap B)$.

Put $L_2 \cap B = \{y\}$ and $L_2 = \{x, u_1, u_2, y\}$. There are three secant planes, say π_1, π_2, π_3, of $\mathrm{PG}(3,3)$ with respect to $Q^+(3,3)$ through L_2. Each of them contains besides the point y at least one extra point of B by Proposition 2.1.1. As $((\pi_1 \cup \pi_2 \cup \pi_3) \setminus L_2) \cap B$ has size $|B| - |B \cap \pi| = 4$, two of these planes, say π_1 and π_2, contain precisely one extra point of B, while the remaining plane π_3 contains exactly two extra points of B (besides y). By the proof of Lemma 5.3.15, we know that each plane π_i, $i \in \{1,2,3\}$, contains a unique point $z_i \in B^*$ not belonging to L_2 (that is, $z_i \neq y$). If M_{ij} denotes the unique external line in π_i through the point u_j with $j \in \{1,2\}$, then z_i is obtained as the intersection point of M_{i1} and M_{i2}. Each of the external lines M_{i1} and M_{i2} must also contain a point of B, showing that z_1 and z_2 are the unique points of B in respectively $\pi_1 \setminus L_2$ and $\pi_2 \setminus L_2$. We claim that z_3 is not a point of B.

By the original construction of the $\mathcal{E}^+$-blocking set B^* (see Theorem 5.1.6, Lemma 5.3.15 and its proof), we know that $B^* = \{y\} \cup (L_1 \setminus \{x\}) \cup \{z_1, z_2, z_3\}$, where $(L_1 \setminus \{x\}) \cup \{z_1, z_2, z_3\}$ consists of the six points of $\pi^* := y^\tau$ that are exterior with respect to the conic $\pi^* \cap Q^+(3, 3)$.

Let z' denote the unique point of $L_1 \setminus (\{x\} \cup B)$. We know that there are at least four points of B in π^*, namely z_1, z_2 and the two points in $L_1 \setminus \{x, z'\}$. We also know that there are at least three points of B outside π^*, namely y and two points on the two external lines through z' not contained in π^* (note that yz' is a secant line). These must constitute all the seven points of B and it follows that z_3 is not a point of B.

Thus, z' and z_3 are the only points of B^* not contained in B. Consider the line $z'z_3$ in π^*. It cannot be a secant line as both z' and z_3 are exterior points with respect to $\pi^* \cap Q^+(3, 3)$. If $z'z_3$ is an external line containing besides z' and z_3 two interior points with respect to $\pi^* \cap Q^+(3, 3)$, then this external line would be disjoint from B, which is impossible. So, $z'z_3$ is an outer tangent in π^* and it must contain one of z_1 and z_2, say z_1, as the third exterior point with respect to $\pi^* \cap Q^+(3, 3)$. The exterior point z_1 is contained in two outer tangents of π^*. One of them is $z'z_3$. The other outer tangent through z_1 contains precisely three points of B, namely z_1, z_2 and one point of $L_1 \cap B$. But that is in contradiction with Lemma 5.3.21. This completes the proof. $\square$

Lemma 5.3.23. *Every outer tangent intersects B in either 0 or 1 point.*

Proof. Suppose that this is not the case. Then by Lemmas 5.3.17 and 5.3.21, there exists an outer tangent L meeting B in exactly two points. Let π be the unique tangent plane containing L. By Lemmas 5.3.17, 5.3.18, 5.3.19, 5.3.20 and 5.3.22, the other outer tangent L' in π is disjoint from B. Then each of the three secant planes, say π_1, π_2, π_3, through L' contains at least two points of B by Proposition 2.1.1, showing that $|B| \geqslant |B \cap \pi| + |B \cap \pi_1| + |B \cap \pi_2| + |B \cap \pi_3| \geqslant 2 + 2 + 2 + 2 = 8$, a contradiction. $\square$

Lemma 5.3.24. *The number of outer tangents disjoint from* B *is equal to* 4. *The number of outer tangents intersecting* B *in a singleton is equal to* 28.

Proof. Let λ_i with $i \in \{0, 1\}$ denote the number of outer tangents meeting B in exactly i points. Each point of $Q^+(3,3)$ is contained in precisely two outer tangents and so the total number of outer tangents is equal to $|Q^+(3,3)| \cdot 2 = 16 \cdot 2 = 32$. By Lemma 5.3.23, we thus have $\lambda_0 + \lambda_1 = 32$. As each point of B is outside $Q^+(3,3)$ by Lemma 5.3.17 and is contained in four outer tangents, counting the pairs (x, L) with L an outer tangent and $x \in L \cap B$ yields $\lambda_1 = 7 \cdot 4 = 28$. We then find $\lambda_0 = 32 - \lambda_1 = 4$. $\square$

Lemma 5.3.25. *If* π *is a secant plane of* $\mathrm{PG}(3,3)$ *with respect to* $Q^+(3,3)$, *then among the six points of* π *that are exterior with respect to the conic* $\pi \cap Q^+(3,3)$ *at most two can belong to* B.

Proof. If B contains at least three of the six exterior points of π, then there would exist an outer tangent in π containing at least two of these points of B, contradicting Lemma 5.3.23. $\square$

The following is a consequence of Lemmas 5.3.17 and 5.3.25.

Corollary 5.3.26. *If* π *is a secant plane of* $\mathrm{PG}(3,3)$ *with respect to* $Q^+(3,3)$, *then* $|\pi \cap B| \leqslant 5$. *If* $|\pi \cap B| = 5$, *then all three points of* π *interior to the conic* $\pi \cap Q^+(3,3)$ *are contained in* B *and precisely two of the six points of* π *exterior to* $\pi \cap Q^+(3,3)$ *are contained in* B.

Lemma 5.3.27. *There cannot exist secant planes of* $\mathrm{PG}(3,3)$ *with respect to* $Q^+(3,3)$ *meeting* B *in precisely five points.*

Proof. Suppose π is a secant plane meeting B in precisely five points. Let y_1 and y_2 denote the two points of B not contained in π. By Corollary 5.3.26, there exist four points x_1, x_2, x_3, x_4 in $\pi \setminus B$ that are exterior with respect to the conic $\pi \cap Q^+(3,3)$. Through each x_i, $i \in \{1, 2, 3, 4\}$, there are two lines belonging to

$\mathcal{E}^+$ not contained in π. As each of these lines contains a point of B, they have to coincide with x_iy_1 and x_iy_2. It follows that y_1x_1, y_1x_2, y_1x_3 and y_1x_4 are four external lines through y_1, in contradiction with the fact that there are only three external lines through y_1. □

Lemma 5.3.28. *There exists no tangent plane of* $\mathrm{PG}(3,3)$ *with respect to* $Q^+(3,3)$ *meeting* B *in precisely two points.*

Proof. Suppose π is a tangent plane intersecting B in precisely two points. By Lemma 5.3.23, each of the two outer tangents in π contains one point of B. Let L denote the secant line with respect to $Q^+(3,3)$ through the two points of $\pi \cap B$. We count the points of B contained in the four planes through L. By Lemmas 5.3.17, 5.3.18, 5.3.19, 5.3.20 and 5.3.22, each tangent plane contains at most two points of B. This implies that each of the two tangent planes through L contains precisely two points of B, namely the two points of $L \cap B = \pi \cap B$. By Corollary 5.3.26 and Lemma 5.3.27, each of the two secant planes through L contains at most four points of B (including the two points of $L \cap B$). It follows that $|B| \leqslant 6$, a contradiction. □

By Lemmas 5.3.17, 5.3.18, 5.3.19, 5.3.20, 5.3.22 and 5.3.28, we thus have the following:

Corollary 5.3.29. *Every tangent plane of* $\mathrm{PG}(3,3)$ *with respect to* $Q^+(3,3)$ *contains either* 0 *or* 1 *point of* B.

We are now ready to derive a contradiction (under the Assumptions 1 and 2). Let n_i with $i \in \{0,1\}$ denote the number of tangent planes meeting B in precisely i points. By Corollary 5.3.29, we have $n_0 + n_1 = 16$ and so $n_1 \leqslant 16$. The total number of outer tangents meeting B in a singleton is equal to n_1. Then $n_1 \leqslant 16 < 28$ contradicts Lemma 5.3.24.

Chapter 6

A characterization in $\mathrm{PG}(3, q)$

Characterizations of the family of external or secant lines with respect to an ovoid/quadric in $\mathrm{PG}(3, q)$ based on certain combinatorial properties have been given by several authors. A characterization of the family of secant lines to an ovoid in $\mathrm{PG}(3, q)$ was obtained in [28] for q odd and in [24] for $q > 2$ even, which was further improved in [26] for all $q > 2$. A characterization of the family of external lines to a hyperbolic quadric in $\mathrm{PG}(3, q)$ was given in [25] for all q (also see [32] for a different characterization in terms of a point-subset of the Klein quadric in $\mathrm{PG}(5, q)$) and to an ovoid in $\mathrm{PG}(3, q)$ was obtained in [26] for all $q > 2$. One can refer to [6, 7, 64, 65] for characterizations of external lines in $\mathrm{PG}(3, q)$ with respect to quadric cone, oval cone and hyperoval cone.

6.1 Main result

In this chapter, we give a characterization of the secant lines with respect to a hyperbolic quadric in $\mathrm{PG}(3, q)$ for odd $q \geqslant 7$. We prove the following result which appears in [53]:

Theorem 6.1.1. *Let $\mathcal{S}$ be a family of lines of $\mathrm{PG}(3, q)$, $q \geqslant 7$ odd, for which the following properties are satisfied:*

(P1) There are $\frac{q(q+1)}{2}$ *or* q^2 *lines of* $\mathcal{S}$ *through a given point of* $\mathrm{PG}(3,q)$*. Further, there exists a point which is contained in* $\frac{q(q+1)}{2}$ *lines of* $\mathcal{S}$ *and a point which is contained in* q^2 *lines of* $\mathcal{S}$*.*

(P2) Every plane π *of* $\mathrm{PG}(3,q)$ *contains at least one line of* $\mathcal{S}$ *and one of the following two cases occurs:*

(P2a) every pencil of lines in π *contains* 0 *or* q *lines of* $\mathcal{S}$*.*

(P2b) every pencil of lines in π *contains* $\frac{q-1}{2}$, $\frac{q+1}{2}$ *or* q *lines of* $\mathcal{S}$*.*

Then either $\mathcal{S}$ *is the set of all secant lines with respect to a hyperbolic quadric in* $\mathrm{PG}(3,q)$*, or the set of points each of which is contained in* q^2 *lines of* $\mathcal{S}$ *form a line* L *of* $\mathrm{PG}(3,q)$ *and* $\mathcal{S}$ *is a hypothetical family of* $\frac{q^4+q^3+2q^2}{2}$ *lines of* $\mathrm{PG}(3,q)$ *not containing* L*.*

Let m_i, $1 \leqslant i \leqslant k$, be k integers with $0 \leqslant m_1 < m_2 < \cdots < m_k \leqslant q+1$. A set K of points of $\mathrm{PG}(2,q)$ is said to be *of class* $[m_1, m_2, \ldots, m_k]$ if every line of $\mathrm{PG}(2,q)$ meets K in m_i points for some $i \in \{1, 2, \ldots, k\}$. We need the following result which was proved in [16, Theorem 4.6] for odd $q \geqslant 7$.

Proposition 6.1.2. *[16] Let* K *be a set of class* $\left[1, \frac{q+1}{2}, \frac{q+3}{2}\right]$ *in* $\mathrm{PG}(2,q)$*, where* $q \geqslant 7$ *is odd. Then* K *consists of an irreducible conic in* $\mathrm{PG}(2,q)$ *and its interior points.*

6.2 Combinatorial results

Let $\mathcal{S}$ be a family of lines of $\mathrm{PG}(3,q)$ for which the properties stated in Theorem 6.1.1 hold. A plane of $\mathrm{PG}(3,q)$ is said to be *tangent* or *secant* according as it satisfies the property (P2a) or (P2b). For a given plane π of $\mathrm{PG}(3,q)$, we denote by $\mathcal{S}_\pi$ the set of lines of $\mathcal{S}$ which are contained in π.

6.2.1 Tangent planes

Lemma 6.2.1. *If π is a tangent plane of* $\mathrm{PG}(3,q)$*, then* $|\mathcal{S}_\pi| = q^2$*.*

Proof. Fix a line L of $\mathcal{S}_\pi$. Every line of $\mathcal{S}_\pi$ meets L. By property (P2a), every point of L is contained in q lines of $\mathcal{S}_\pi$. Then the numbers of lines of $\mathcal{S}_\pi$ is $1 + (q+1)(q-1) = q^2$. □

Lemma 6.2.2. *Let π be a tangent plane. Then there are q^2+q points of π, each of which is contained in q lines of $\mathcal{S}_\pi$. Equivalently, there is only one point of π which is contained in no lines of $\mathcal{S}_\pi$.*

Proof. Let A_π (respectively, B_π) be the set of points of π each of which is contained in no lines (respectively, q lines) of $\mathcal{S}_\pi$. Then $|A_\pi| + |B_\pi| = q^2 + q + 1$. Consider the following set of point-line pairs:

$$X = \{(x, L) : x \in B_\pi, L \in \mathcal{S}_\pi, x \in L\}.$$

By Lemma 6.2.1, $|\mathcal{S}_\pi| = q^2$. Counting $|X|$ in two ways, we get $|B_\pi| \times q = |X| = q^2 \times (q+1)$. This gives $|B_\pi| = q^2 + q$ and hence $|A_\pi| = 1$. □

For a tangent plane π, we denote by p_π the unique point of π which is contained in no lines of $\mathcal{S}_\pi$ (Lemma 6.2.2) and call it the *pole* of π.

Corollary 6.2.3. *Let π be a tangent plane. Then the $q+1$ lines of π not contained in $\mathcal{S}_\pi$ are precisely the lines of π through the pole p_π.*

6.2.2 Secant planes

In the rest of this chapter, *we assume that q is odd and that $q \geqslant 7$.* As an application of Proposition 6.1.2, we have the following.

Lemma 6.2.4. *Let K be a set of class $\left[\frac{q-1}{2}, \frac{q+1}{2}, q\right]$ in* $\mathrm{PG}(2,q)$*. Then K consists of the exterior points of an irreducible conic in* $\mathrm{PG}(2,q)$*.*

Proof. The complement $\overline{K}$ of K in $\mathrm{PG}(2,q)$ is a set of class $\left[1, \frac{q+1}{2}, \frac{q+3}{2}\right]$. Since $q \geqslant 7$, $\overline{K}$ consists of an irreducible conic C and its interior points by Proposition 6.1.2. Then K consists of the exterior points of C. □

Lemma 6.2.5. *Let π be a secant plane. Then $\mathcal{S}_\pi$ consists of the secant lines of an irreducible conic in π and so $|\mathcal{S}_\pi| = \frac{q(q+1)}{2}$.*

Proof. Each point of π is contained in $(q-1)/2$, $(q+1)/2$ or q lines of $\mathcal{S}_\pi$. Therefore, in the dual plane π^D of π, $\mathcal{S}_\pi$ is a set of points of class $\left[\frac{q-1}{2}, \frac{q+1}{2}, q\right]$. Since $q \geqslant 7$, by Lemma 6.2.4, $\mathcal{S}_\pi$ consists of the exterior points of an irreducible conic C^D in π^D. The dual of C^D is an irreducible conic C in π and the lines of π contained in C^D are precisely the tangent lines of C. Since exterior points of C^D in π^D correspond to the secant lines of C in π, it follows that $\mathcal{S}_\pi$ is precisely the set of secant lines of C in π and hence $|\mathcal{S}_\pi| = \frac{q(q+1)}{2}$. □

For a secant plane π of $\mathrm{PG}(3,q)$, we denote by $\gamma(\pi)$ the irreducible conic obtained in Lemma 6.2.5 with respect to which $\mathcal{S}_\pi$ is the set of secant lines in π. The sets of exterior and interior points of $\gamma(\pi)$ in π are denoted by $\alpha(\pi)$ and $\beta(\pi)$, respectively. Then we have $|\gamma(\pi)| = q+1$, $|\alpha(\pi)| = \frac{q^2+q}{2}$ and $|\beta(\pi)| = \frac{q^2-q}{2}$. Note that $\alpha(\pi)$, $\beta(\pi)$ and $\gamma(\pi)$ are precisely the sets of points of π which are contained in $\frac{q-1}{2}$, $\frac{q+1}{2}$ and q lines of $\mathcal{S}_\pi$, respectively.

6.3 Black points

Recall that every point of $\mathrm{PG}(3,q)$ is contained in q^2 or $\frac{q(q+1)}{2}$ lines of $\mathcal{S}$ by property (P1). We call a point of $\mathrm{PG}(3,q)$ *black* if it is contained in q^2 lines of $\mathcal{S}$.

6.3.1 Existence of tangent and secant planes

We show that both tangent and secant planes exist for which the following result is needed.

Lemma 6.3.1. *The number of tangent planes through a line L of* $\mathrm{PG}(3,q)$ *is equal to the number of black points contained in L.*

Proof. Let t and b, respectively, denote the number of tangent planes through L and the number of black points contained in L. We count in two different ways the total number of lines of $\mathcal{S} \setminus \{L\}$ meeting L. We have $|\mathcal{S}_\pi| = q^2$ or $\frac{q(q+1)}{2}$ by Lemmas 6.2.1 and 6.2.5 for a plane π. Any line of $\mathcal{S}$ meeting L is contained in some plane through L. If $L \in \mathcal{S}$, then we get

$$t(q^2-1)+(q+1-t)\left(\frac{q(q+1)}{2}-1\right)=b(q^2-1)+(q+1-b)\left(\frac{q(q+1)}{2}-1\right).$$

If $L \notin \mathcal{S}$, then we get

$$tq^2+(q+1-t)\frac{q(q+1)}{2}=bq^2+(q+1-b)\frac{q(q+1)}{2}.$$

In both cases, it follows that $(t-b)\frac{q^2-q}{2}=0$ and hence $t=b$. □

Corollary 6.3.2. *Both tangent and secant planes exist.*

Proof. By property (P1), let x (respectively, y) be a point of $\mathrm{PG}(3,q)$ which is contained in q^2 (respectively, $\frac{q(q+1)}{2}$) lines of $\mathcal{S}$. Taking L to be the line through x and y, the corollary follows from Lemma 6.3.1 using the facts that x is a black point but y is not a black point. □

Corollary 6.3.3. *Every line of a tangent plane contains at least one black point.*

Corollary 6.3.4. *Every black point is contained in some tangent plane.*

6.3.2 Black points in secant planes

Lemma 6.3.5. *If π is a secant plane, then the set of black points in π is contained in the conic $\gamma(\pi)$. In particular, there are exactly q lines of $\mathcal{S}_\pi$ through a black point in π.*

Proof. Let x be a black point in π. Suppose that x is not contained in $\gamma(\pi)$. Fix a line L of $\mathcal{S}_\pi$ through x and consider the $q+1$ planes of $\mathrm{PG}(3,q)$ through L. There are q^2 lines of $\mathcal{S}$ through x and each of them is contained in some plane through L. Since $x \notin \gamma(\pi)$, the plane π contains at most $\frac{q+1}{2}$ lines of $\mathcal{S}$ through x. Each of the remaining q planes through L contains at most q lines of $\mathcal{S}$ through x. This implies that there are at most $\frac{q+1}{2} + q(q-1)$ lines of $\mathcal{S}$ through x. This is not possible, as $\frac{q+1}{2} + q(q-1) < q^2$. So $x \in \gamma(\pi)$. □

Lemma 6.3.6. *The number of black points in a given secant plane is independent of that plane.*

Proof. Let π be a secant plane and λ_π denote the number of black points in π. We count the total number of lines of $\mathcal{S}$. The lines of $\mathcal{S}$ are divided into two types:

(I) the $\frac{q(q+1)}{2}$ lines of $\mathcal{S}$ which are contained in π,

(II) those lines of $\mathcal{S}$ which meet π in a singleton.

Let θ be the number of type (II) lines of $\mathcal{S}$. In order to calculate θ, we divide the points of π into four groups:

(a) The λ_π black points contained in π: These points are contained in $\gamma(\pi)$ by Lemma 6.3.5. Out of the q^2 lines of $\mathcal{S}$ through such a point, q of them are contained in π.

(b) The $|\gamma(\pi)| - \lambda_\pi$ points of $\gamma(\pi)$ which are not black: Out of the $\frac{q(q+1)}{2}$ lines of $\mathcal{S}$ through such a point, q of them are contained in π.

(c) The points of $\alpha(\pi)$: Out of the $\frac{q(q+1)}{2}$ lines of $\mathcal{S}$ through such a point, $\frac{q-1}{2}$ of them are contained in π.

(d) The points of $\beta(\pi)$: Out of the $\frac{q(q+1)}{2}$ lines of $\mathcal{S}$ through such a point, $\frac{q+1}{2}$ of them are contained in π.

Since $|\gamma(\pi)| = q+1$, $|\alpha(\pi)| = \frac{q^2+q}{2}$ and $|\beta(\pi)| = \frac{q^2-q}{2}$, we get

$$\begin{aligned}\theta &= \lambda_\pi(q^2-q) + (q+1-\lambda_\pi)\left(\frac{q(q+1)}{2} - q\right) + |\alpha(\pi)|\left(\frac{q(q+1)}{2} - \frac{q-1}{2}\right) \\ &\qquad + |\beta(\pi)|\left(\frac{q(q+1)}{2} - \frac{q+1}{2}\right) \\ &= \lambda_\pi\left(\frac{q^2-q}{2}\right) + \frac{q^3(q+1)}{2}.\end{aligned}$$

Then $|\mathcal{S}| = \theta + \dfrac{q(q+1)}{2} = \lambda_\pi\left(\dfrac{q^2-q}{2}\right) + \dfrac{q^4+q^3+q^2+q}{2}$. Since $|\mathcal{S}|$ is a fixed number, it follows that λ_π is independent of the secant plane π. □

By Lemma 6.3.6, we denote by λ the number of black points in a secant plane. From the proof of Lemma 6.3.6, we thus have the following equation involving λ and $|\mathcal{S}|$:

$$\lambda\left(\frac{q^2-q}{2}\right) + \frac{q^4+q^3+q^2+q}{2} = |\mathcal{S}|. \tag{6.3.1}$$

As a consequence of Lemma 6.3.5, we have

Corollary 6.3.7. *$\lambda \leqslant q+1$.*

6.3.3 Black points in tangent planes

Lemma 6.3.8. *The number of black points in a given tangent plane is independent of that plane.*

Proof. Let π be a tangent plane with pole p_π and μ_π be the number of black points in π. We shall apply a similar argument as in the proof of Lemma 6.3.6 by calculating $|\mathcal{S}|$. The lines of $\mathcal{S}$ are divided into two types: (I) the q^2 lines of $\mathcal{S}$ which are contained in π, and (II) those lines of $\mathcal{S}$ which meet π in a singleton. Let θ be the number of type (II) lines of $\mathcal{S}$. In order to calculate θ, we divide the points of π into two groups:

(a) The μ_π black points contained in π,

(b) The $q^2+q+1-\mu_\pi$ points of π which are not black.

If x is a point of π which is different from p_π, then Lemma 6.2.2 implies that the number of lines of $\mathcal{S}$ through x which are not contained in π is q^2-q or $\frac{q(q+1)}{2}-q$ according as x is a black point or not. We consider two cases depending on p_π is a black point or not.

Case-1: p_π is a black point. In this case, Lemma 6.2.2 implies that none of the q^2 lines of $\mathcal{S}$ through p_π is contained in π. Then

$$\begin{aligned}\theta &= q^2 + (\mu_\pi - 1)(q^2-q) + (q^2+q+1-\mu_\pi)\left(\frac{q(q+1)}{2} - q\right)\\ &= \mu_\pi\left(\frac{q^2-q}{2}\right) + \frac{q^4+q}{2}.\end{aligned}$$

Case-2: p_π is not a black point. In this case, none of the $\frac{q(q+1)}{2}$ lines of $\mathcal{S}$ through p_π is contained in π by Lemma 6.2.2. Then

$$\begin{aligned}\theta &= \mu_\pi(q^2-q) + \frac{q(q+1)}{2} + (q^2+q-\mu_\pi)\left(\frac{q(q+1)}{2} - q\right)\\ &= \mu_\pi\left(\frac{q^2-q}{2}\right) + \frac{q^4+q}{2}.\end{aligned}$$

In both cases, $|\mathcal{S}| = \theta + q^2 = \mu_\pi\left(\frac{q^2-q}{2}\right) + \frac{q^4+2q^2+q}{2}$. Since $|\mathcal{S}|$ is a fixed number, it follows that μ_π is independent of the tangent plane π. $\square$

By Lemma 6.3.8, we denote by μ the number of black points in a tangent plane. From the proof of Lemma 6.3.8, we thus have the following equation involving μ and $|\mathcal{S}|$:

$$\mu\left(\frac{q^2-q}{2}\right) + \frac{q^4+2q^2+q}{2} = |\mathcal{S}|. \tag{6.3.2}$$

From equations (6.3.1) and (6.3.2), we have

$$\mu = \lambda + q. \tag{6.3.3}$$

6.3.4 Black points on a line

Lemma 6.3.9. *The following hold:*

(i) Every line of $\mathrm{PG}(3,q)$ *contains* $0, 1, 2$ *or* $q+1$ *black points.*

(ii) If a line of $\mathrm{PG}(3,q)$ *contains exactly two black points, then it is a line of* $\mathcal{S}$*.*

Proof. Let L be a line of $\mathrm{PG}(3,q)$ and b be the number of black points contained in L. Assume that $b > 2$. If there exists a secant plane π through L, then Lemma 6.3.5 implies that the line L contains $b \geqslant 3$ number of points of the conic $\gamma(\pi)$ in π, which is not possible. So all planes through L are tangent planes. Then all the $q+1$ points of L are black by Lemma 6.3.1. This proves (i).

If $b = 2$, then Lemma 6.3.1 implies that there exists a secant plane π through L. By Lemma 6.3.5, L is a secant line of the conic $\gamma(\pi)$ in π and hence a line of $\mathcal{S}_\pi$. This proves (ii). □

6.4 Proof of the main result

We shall continue with the notation used in the previous sections and the assumption that $q \geqslant 7$. We denote by $\mathcal{H}$ the set of all black points of $\mathrm{PG}(3,q)$, and by $\mathcal{H}_\pi$ the set of black points of $\mathrm{PG}(3,q)$ which are contained in a given plane π.

Lemma 6.4.1. $|\mathcal{H}| = \lambda(q+1)$. *In particular,* $|\mathcal{H}| \leqslant (q+1)^2$.

Proof. Fix a secant plane π. Let L be a line of π which is external to the conic $\gamma(\pi)$. By Lemma 6.3.5, none of the points of L is black. Then, by Lemma 6.3.1, each plane through L is a secant plane. The number of black points contained in a secant plane is λ. Counting all the black points contained in the $q+1$ planes through L, we get $|\mathcal{H}| = \lambda(q+1)$. Since $\lambda \leqslant q+1$ by Corollary 6.3.7, we have $|\mathcal{H}| \leqslant (q+1)^2$. □

Lemma 6.4.2. *If* π *is a tangent plane, then* $\mathcal{H}_\pi$ *contains a line.*

Proof. By Corollary 6.3.3, every line of π meets $\mathcal{H}_\pi$. By Proposition 1.7.1 (taking $d = 2$), we then have $|\mathcal{H}_\pi| \geqslant q + 1$, and equality holds if and only if $\mathcal{H}_\pi$ itself is a line of π.

Therefore, assume that $|\mathcal{H}_\pi| > q + 1$. Since q is odd, the maximum size of an arc in π is $q + 1$. So $\mathcal{H}_\pi$ cannot be an arc and hence there exists a line L of π which contains at least three points of $\mathcal{H}_\pi$. Then all points of L are black by Lemma 6.3.9(i) and so L is contained in $\mathcal{H}_\pi$. □

Lemma 6.4.3. *Let π be a tangent plane. Then $\mathcal{H}_\pi$ is either a line or union of two (intersecting) lines.*

Proof. Using Lemmas 6.3.9(i) and 6.4.2, observe that there are only four possibilities for the set $\mathcal{H}_\pi$:

(1) $\mathcal{H}_\pi$ is a line.

(2) $\mathcal{H}_\pi$ is the union of a line L and a point of π not contained in L.

(3) $\mathcal{H}_\pi$ is the union of two (intersecting) lines.

(4) $\mathcal{H}_\pi$ is the whole plane π.

We show that the possibilities (2) and (4) do not occur. If $\mathcal{H}_\pi$ is the whole plane π, then $\mu = q^2 + q + 1$ and so $\lambda = q^2 + 1$ by equation (6.3.3), which is not possible by Corollary 6.3.7.

Now suppose that $\mathcal{H}_\pi$ is the union of a line L and a point x not on L. If the pole p_π of π is different from x, then take T to be the line through p_π and x (note that p_π may or may not be on L). If $p_\pi = x$, then take T to be any line through $p_\pi = x$. Since π is a tangent plane, T is not a line of $\mathcal{S}_\pi$ by Corollary 6.2.3 and hence is not a line of $\mathcal{S}$. On the other hand, since T contains only two black points (namely, the point x and the intersection point of L and T), T is a line of $\mathcal{S}$ by Lemma 6.3.9(ii). This leads to a contradiction. □

Lemma 6.4.4. *Let π be a tangent plane. If $\mathcal{H}_\pi$ is a line of* $\mathrm{PG}(3,q)$*, then the following hold:*

(i) $\mathcal{H}_\pi$ is not a line of $\mathcal{S}$.

(ii) $\mathcal{H}_\pi = \mathcal{H}$.

(iii) $\mathcal{S}$ is a set of $\dfrac{q^4+q^3+2q^2}{2}$ lines of $\mathrm{PG}(3,q)$ *not containing the line $\mathcal{H}$.*

Proof. (i) Suppose that $\mathcal{H}_\pi$ is a line of $\mathcal{S}$. Then, by Corollary 6.2.3, the pole p_π of π must be a point of $\pi \setminus \mathcal{H}_\pi$. Fix a line M of π through p_π. Note that $M \notin \mathcal{S}$ again by Corollary 6.2.3. Let x be the point of intersection of M and $\mathcal{H}_\pi$. Since M contains only one black point (which is x), Lemma 6.3.1 implies that M is contained in one tangent plane (namely, π) and q secant planes. Since $x \neq p_\pi$, by Lemma 6.2.2, there are q lines of π through x which are contained in $\mathcal{S}$. In each of the q secant planes through M, x being a black point, there are q lines through x which are contained in $\mathcal{S}$ by Lemma 6.3.5. Since $M \notin \mathcal{S}$, we get $q(q+1) = q^2+q$ lines of $\mathcal{S}$ through x, which is not possible by property (P1).

(ii) Suppose that x is a black point which is not contained in $\mathcal{H}_\pi$. Let π' be the plane generated by the line $\mathcal{H}_\pi$ and the point x. We have $\pi \neq \pi'$ as x is not a black point of π. Each of the planes through the line $\mathcal{H}_\pi$ is a tangent plane by Lemma 6.3.1. In particular, π' is a tangent plane. Note that π contains $q+1$ black points, whereas π' contains at least $q+2$ black points. This contradicts Lemma 6.3.8.

(iii) The line $\mathcal{H}$ is not contained in $\mathcal{S}$ by (i) and (ii). Since the tangent plane π contains $q+1$ black points, we have $\mu = q+1$. Then equation (6.3.2) gives that $|\mathcal{S}| = \dfrac{q^4+q^3+2q^2}{2}$. □

Lemma 6.4.4 proves the second possibility as mentioned in Theorem 6.1.1 for the family $\mathcal{S}$. Theorem 6.1.1 for $q \in \{3,5\}$ and the family $\mathcal{S}$ of lines with $|\mathcal{S}| = \frac{q^4+q^3+2q^2}{2}$ and satisfying the conditions of Theorem 6.1.1 are under our investigation.

In the rest of this section, we assume that $\mathcal{H}_\pi$ is the union of two (intersecting) lines for every tangent plane π. So $\mu = 2q+1$ and then equation (6.3.3) gives that $\lambda = q+1$. From equation (6.3.2) and Lemma 6.4.1, we get

$$|\mathcal{S}| = \frac{q^2(q+1)^2}{2} \quad \text{and} \quad |\mathcal{H}| = (q+1)^2. \tag{6.4.1}$$

Lemma 6.4.5. *Let π be a tangent plane. If $\mathcal{H}_\pi$ is the union of the lines L and L' of π, then the pole p_π of π is the intersection point of L and L'.*

Proof. Let x be the intersection point of L and L'. Suppose that $p_\pi \neq x$. Let T be a line of π through p_π which does not contain x (note that p_π may or may not be contained in $L \cup L'$). Since π is a tangent plane, T is not a line of $\mathcal{S}_\pi$ by Corollary 6.2.3 and hence is not a line of $\mathcal{S}$. On the other hand, since T contains two black points (namely, the two intersection points of T with L and L'), it is a line of $\mathcal{S}$ by Lemma 6.3.9(ii). This leads to a contradiction. □

We call a line of $\mathrm{PG}(3,q)$ *black* if it is contained in $\mathcal{H}$.

Lemma 6.4.6. *Every black point is contained in at most two black lines.*

Proof. Let x be a black point. If possible, suppose that there are three distinct black lines L, L_1, L_2 each of which contains x. Let π (respectively, π') be the plane generated by L, L_1 (respectively, L, L_2). Each plane through L is a tangent plane by Lemma 6.3.1. So π and π' are tangent planes. Since $\mathcal{H}_\pi = L \cup L_1$ and $\mathcal{H}_{\pi'} = L \cup L_2$, it follows that $\pi \neq \pi'$. By Lemma 6.4.5, x is the pole of both π and π'. So the lines through x which are contained in π or π' are not lines of $\mathcal{S}$ by Corollary 6.2.3. Thus each line of $\mathcal{S}$ through x is contained in some plane through L which is different from both π and π'. It follows that the number of lines of $\mathcal{S}$ through x is at most $q(q-1)$, which contradicts to the fact that there are q^2 lines of $\mathcal{S}$ through x (being a black point). □

Lemma 6.4.7. *Every black point is contained in precisely two black lines.*

Proof. Let x be a black point and L be a black line containing x. The existence of such a line L follows from the facts that x is contained in a tangent plane (Corollary 6.3.4) and that the set of all black points in that tangent plane is a union of two black lines. By Lemma 6.3.1, let $\pi_1, \pi_2, \dots, \pi_{q+1}$ be the $q+1$ tangent planes through L. For $1 \leqslant i \leqslant q+1$, we have $\mathcal{H}_{\pi_i} = L \cup L_i$ for some black line L_i of π_i different from L. Let $\{p_i\} = L \cap L_i$. Lemma 6.4.6 implies that $p_i \neq p_j$ for $1 \leqslant i \neq j \leqslant q+1$, and so $L = \{p_1, p_2, \dots, p_{q+1}\}$. Since $x \in L$, we have $x = p_j$ for some $1 \leqslant j \leqslant q+1$. Thus, applying Lemma 6.4.6 again, it follows that x is contained in precisely two black lines, namely, L and L_j. □

The following two lemmas complete the proof of Theorem 6.1.1.

Lemma 6.4.8. *The points of $\mathcal{H}$ together with the black lines form a hyperbolic quadric in* $\mathrm{PG}(3, q)$.

Proof. We have $|\mathcal{H}| = (q+1)^2$ by (6.4.1). It is enough to show that the points of $\mathcal{H}$ together with the black lines form a projective generalized quadrangle of order $(q, 1)$. Clearly, this point-line geometry is a partial linear space. We shall verify the axioms (Q1) and (Q2) of a generalized quadrangle defined in Section 1.6.

Each black line contains $q+1$ points of $\mathcal{H}$. By Lemma 6.4.7, each point of $\mathcal{H}$ is contained in exactly two black lines. Thus the axiom (Q1) is satisfied with $s = q$ and $t = 1$.

We verify the axiom (Q2). Let $L = \{x_1, x_2, \dots, x_{q+1}\}$ be a black line and x be a black point not contained in L. By Lemma 6.4.7, let L_i be the second black line through x_i (different from L) for $1 \leqslant i \leqslant q+1$. If L_i and L_j intersect for $i \neq j$, then the tangent plane π generated by L_i and L_j contains L as well. This implies that $\mathcal{H}_\pi$ contains the union of three distinct black lines (namely L, L_i, L_j), which is not possible. Thus the black lines $L_1, L_2, \dots, L_{q+1}$ are pairwise disjoint. These $q+1$ black lines contain $(q+1)^2$ black points and hence their union must be equal to $\mathcal{H}$. In particular, x is a point of L_j for unique $j \in \{1, 2, \dots, q+1\}$. Then L_j is the unique black line containing x_j and intersecting L.

It now follows from the above that the points of $\mathcal{H}$ together with the black lines form a projective generalized quadrangle of order $(q, 1)$. $\square$

Lemma 6.4.9. *The lines of $\mathcal{S}$ are precisely the secant lines to the hyperbolic quadric $\mathcal{H}$.*

Proof. By (6.4.1), we have $|\mathcal{S}| = \dfrac{q^2(q+1)^2}{2}$, which is equal to the number of secant lines to $\mathcal{H}$. It is enough to show that every secant line to $\mathcal{H}$ is a line of $\mathcal{S}$. This follows from Lemma 6.3.9(ii), as every secant line to $\mathcal{H}$ contains exactly two black points. $\square$

Bibliography

[1] A. Aguglia and M. Giulietti, Blocking sets of certain line sets related to a conic, *Des. Codes Cryptogr.* **39** (2006), no. 3, 397–405.

[2] A. Aguglia and G. Korchmáros, Blocking sets of nonsecant lines to a conic in $\mathrm{PG}(2,q)$, q odd, *J. Combin. Des.* **13** (2005), no. 4, 292–301.

[3] A. Aguglia and G. Korchmáros, Blocking sets of external lines to a conic in $\mathrm{PG}(2,q)$, q odd, *Combinatorica* **26** (2006), no. 4, 379–394.

[4] A. Aguglia, G. Korchmáros and A. Siciliano, Minimal covering of all chords of a conic in $\mathrm{PG}(2,q)$, q even, *Bull. Belg. Math. Soc. Simon Stevin* **12** (2005), no. 5, 651–655.

[5] A. Barlotti, Un'estensione del teorema di Segre-Kustaanheimo, *Boll. Un. Mat. Ital. (3)* **10** (1955), 498–506.

[6] S. G. Barwick and D. K. Butler, A characterisation of the lines external to a quadric cone in $\mathrm{PG}(3,q)$, q odd, *Innov. Incidence Geom.* **8** (2008), 39–48.

[7] S. G. Barwick and D. K. Butler, A characterisation of the lines external to an oval cone in $\mathrm{PG}(3,q)$, q even, *J. Geom.* **93** (2009), no. 1-2, 21–27.

[8] P. Biondi and P. M. Lo Re, On blocking sets of external lines to a hyperbolic quadric in $\mathrm{PG}(3,q)$, q even, *J. Geom.* **92** (2009), no. 1-2, 23–27.

[9] P. Biondi, P. M. Lo Re and L. Storme, On minimum size blocking sets of external lines to a quadric in $\mathrm{PG}(3, q)$, *Beiträge Algebra Geom.* **48** (2007), no. 1, 209–215.

[10] A. Blokhuis, On the size of a blocking set in $\mathrm{PG}(2, p)$, *Combinatorica* **14** (1994), no. 1, 111–114.

[11] M. Bonini and M. Borello, Minimal linear codes arising from blocking sets, *J. Algebraic Combin.* **53** (2021), no. 2, 327–341.

[12] E. Boros, Z. Füredi and J. Kahn, Maximal intersecting families and affine regular polygons in $\mathrm{PG}(2, q)$, *J. Combin. Theory Ser. A* **52** (1989), no. 1, 1–9.

[13] R. C. Bose and R. C. Burton, A characterization of flat spaces in a finite geometry and the uniqueness of the Hamming and the MacDonald codes, *J. Combinatorial Theory* **1** (1966), 96–104.

[14] A. A. Bruen, Baer subplanes and blocking sets, *Bull. Amer. Math Soc.* **76** (1970), 342–344.

[15] A. A. Bruen and J. A. Thas, Flocks, chains and configurations in finite geometries, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8)* **59** (1975), no. 6, 744–748 (1976).

[16] F. De Clerck and N. De Feyter, A characterization of the sets of internal and external points of a conic, *European J. Combin.* **28** (2007), no. 7, 1910–1921.

[17] B. De Bruyn, An Introduction to Incidence Geometry, Frontiers in Mathematics, Birkhäuser/Springer, Cham, 2016.

[18] B. De Bruyn, P. Pradhan and B. K. Sahoo, Minimum size blocking sets of certain line sets with respect to an elliptic quadric in $\mathrm{PG}(3, q)$, *Contrib. Discrete Math.* **15** (2020), no. 2, 132–147.

[19] B. De Bruyn, P. Pradhan and B. K. Sahoo, Blocking sets of tangent and external lines to an elliptic quadric in $\mathrm{PG}(3,q)$, *Proc. Indian Acad. Sci. Math. Sci.* **131** (2021), no. 2, article no. 39, 16 pp.

[20] B. De Bruyn, P. Pradhan and B. K. Sahoo, Next-to-minimum size blocking sets of external lines to a nondegenerate quadric in $\mathrm{PG}(3,q)$, Communicated.

[21] B. De Bruyn, P. Pradhan and B. Sahu, Blocking sets of external, tangent and secant lines to a quadratic cone in $\mathrm{PG}(3,q)$, *Discrete Math.* **344** (2021), no. 6, 112352, 4 pp.

[22] B. De Bruyn, B. K. Sahoo and B. Sahu, Blocking sets of tangent and external lines to a hyperbolic quadric in $\mathrm{PG}(3,q)$, *Discrete Math.* **341** (2018), no. 10, 2820–2826.

[23] B. De Bruyn, B. K. Sahoo and B. Sahu, Blocking sets of tangent lines to a hyperbolic quadric in $\mathrm{PG}(3,3)$, *Discrete Appl. Math.* **266** (2019), 121–129.

[24] M. J. de Resmini, A characterization of the secants of an ovaloid in $\mathrm{PG}(3,q)$, q even, $q > 2$, *Ars Combin.* **16** (1983), B, 33–49.

[25] R. Di Gennaro, N. Durante and D. Olanda, A characterization of the family of lines external to a hyperbolic quadric of $\mathrm{PG}(3,q)$, *J. Geom.* **80** (2004), no. 1-2, 65–74.

[26] N. Durante and D. Olanda, A characterization of the family of secant or external lines of an ovoid of $\mathrm{PG}(3,q)$, *Bull. Belg. Math. Soc. Simon Stevin* **12** (2005), no. 1, 1–4.

[27] G. Fellegara, Gli ovaloidi in uno spazio tridimensionale di Galois di ordine 8, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8)* **32** (1962), 170–176.

[28] O. Ferri and G. Tallini, A characterization of the family of secant lines of an elliptic quadric in $\mathrm{PG}(3,q)$, q odd, Proceedings of the conference on combi-

natorial and incidence geometry: principles and applications (La Mendola, 1982), 297–305, Rend. Sem. Mat. Brescia, 7, Sci. Mat., 7, Vita e Pensiero, Milan, 1984.

[29] M. Giulietti, Blocking sets of external lines to a conic in $\mathrm{PG}(2, q)$, q even, *European J. Combin.* **28** (2007), no. 1, 36–42.

[30] J. W. P. Hirschfeld, Finite Projective Spaces of Three Dimensions, Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1985.

[31] D. R. Hughes and F. C. Piper, Projective Planes, Graduate Texts in Mathematics, Vol. 6, Springer-Verlag, New York-Berlin, 1973.

[32] S. Innamorati, M. Zannetti and F. Zuanni, A characterization of the lines external to a hyperbolic quadric in $\mathrm{PG}(3, q)$, *Ars Combin.* **103** (2012), 3–11.

[33] G. Kiss and T. Szőnyi, Finite Geometries, CRC Press, Boca Raton, FL, 2020.

[34] M. Lavrauw, L. Storme and G. Van de Voorde, On the code generated by the incidence matrix of points and k-spaces in $\mathrm{PG}(n, q)$ and its dual, *Finite Fields Appl.* **14** (2008), no. 4, 1020–1038.

[35] M. Lavrauw, L. Storme, P. Sziklai and G. Van de Voorde, An empty interval in the spectrum of small weight codewords in the code from points and k-spaces of $\mathrm{PG}(n, q)$, *J. Combin. Theory Ser. A* **116** (2009), no. 4, 996–1001.

[36] R. Lidl and H. Niederreiter, Finite Fields, Second edition, Encyclopedia of Mathematics and its Applications, 20, Cambridge University Press, Cambridge, 1997.

[37] G. Lunardon and O. Polverino, Linear blocking sets: a survey, Finite Fields and Applications (Augsburg, 1999), 356–362, Springer, Berlin, 2001.

[38] K. Metsch, A Bose-Burton theorem for elliptic polar spaces, *Des. Codes Cryptogr.* **17** (1999), no. 1-3, 219–224.

[39] K. Metsch, Bose-Burton type theorems for finite projective, affine and polar spaces, Surveys in Combinatorics, 1999 (Canterbury), 137–166, London Math. Soc. Lecture Note Ser. 267, Cambridge University Press, Cambridge, 1999.

[40] K. Metsch, On blocking sets of quadrics, *J. Geom.* **67** (2000), no. 1-2, 188–207.

[41] K. Metsch, A Bose-Burton type theorem for quadrics, *J. Combin. Des.* **11** (2003), no. 5, 317–338.

[42] K. Metsch, Blocking sets in projective spaces and polar spaces, *J. Geom.* **76** (2003), no. 1-2, 216–232.

[43] G. E. Moorhouse, Incidence Geometry, 2017, available online at http://ericmoorhouse.org/handouts/Incidence_Geometry.pdf

[44] C. M. O'Keefe and T. Penttila, Ovoids of $\mathrm{PG}(3, 16)$ are elliptic quadrics, *J. Geom.* **38** (1990), no. 1-2, 95–106.

[45] C. M. O'Keefe and T. Penttila, Ovoids of $\mathrm{PG}(3, 16)$ are elliptic quadrics, II, *J. Geom.* **44** (1992), no. 1-2, 140–159.

[46] C. M. O'Keefe, T. Penttila and G. F. Royle, Classification of ovoids in $\mathrm{PG}(3, 32)$, *J. Geom.* **50** (1994), no. 1-2, 143–150.

[47] G. Panella, Caratterizzazione delle quadriche di uno spazio (tridimensionale) lineare sopra un corpo finito, *Boll. Un. Mat. Ital. (3)* **10** (1955), 507–513.

[48] K. L. Patra, B. K. Sahoo and B. Sahu, Minimum size blocking sets of certain line sets related to a conic in $\mathrm{PG}(2, q)$, *Discrete Math.* **339** (2016), no. 6, 1716–1721.

[49] S. E. Payne and J. A. Thas, Finite Generalized Quadrangles, Second edition, EMS Series of Lectures in Mathematics, European Mathematical Society (EMS), Zürich, 2009.

[50] V. Pepe and L. Storme, The use of blocking sets in Galois geometries and in related research areas, Buildings, finite geometries and groups, 305–327, Springer Proc. Math., 10, Springer, New York, 2012.

[51] P. Polito and O. Polverino, On small blocking sets, *Combinatorica* **18** (1998), no. 1, 133–137.

[52] O. Polverino, Small blocking sets in $\mathrm{PG}(2, p^3)$, *Des. Codes Cryptogr.* **20** (2000), no. 3, 319–324.

[53] P. Pradhan and B. Sahu, A characterization of the family of secant lines to a hyperbolic quadric in $\mathrm{PG}(3, q)$, q odd, *Discrete Math.* **343** (2020), no. 11, 112044, 8 pp.

[54] B. K. Sahoo and B. Sahu, Blocking sets of tangent and external lines to a hyperbolic quadric in $\mathrm{PG}(3, q)$, q even, *Proc. Indian Acad. Sci. Math. Sci.* **129** (2019), no. 1, 14 pp.

[55] B. K. Sahoo and B. Sahu, Blocking sets of certain line sets related to a hyperbolic quadric in $\mathrm{PG}(3, q)$, *Adv. Geom.* **19** (2019), no. 4, 477–486.

[56] B. K. Sahoo and N. S. N. Sastry, Binary codes of the symplectic generalized quadrangle of even order, *Des. Codes Cryptogr.* **79** (2016), no. 1, 163–170.

[57] B. Sahu, Blocking Sets of Certain Line Sets in $\mathrm{PG}(3, q)$, Ph.D. Thesis, Homi Bhabha National Institute, India, 2018.

[58] B. Segre, Ovals in a finite projective plane, *Canadian J. Math.* **7** (1955), 414–416.

[59] B. Segre, On complete caps and ovaloids in three-dimensional Galois spaces of characteristic two, *Acta Arith.* **5** (1959), 315–332.

[60] B. Segre and G. Korchmáros, Una proprietà degli insiemi di punti di un piano di Galois caratterizzante quelli formati dai punti delle singole rette esterne ad una conica, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8)* **62** (1977), no. 5, 613–619.

[61] T. Szőnyi, Blocking sets in Desarguesian affine and projective planes, *Finite Fields Appl.* **3** (1997), no. 3, 187–202.

[62] J. A. Thas, Ovoidal translation planes, *Arch. Math. (Basel)* **23** (1972), 110–112.

[63] J. Ueberberg, Foundations of Incidence Geometry, Projective and Polar Spaces, Springer Monographs in Mathematics, Springer, Heidelberg, 2011.

[64] M. Zannetti, A characterization of the external lines of a hyperoval cone in $\mathrm{PG}(3, q)$, q even, *Discrete Math.* **311** (2011), no. 4, 239–243.

[65] F. Zuanni, A characterization of the external lines of a hyperoval in $\mathrm{PG}(3, q)$, q even, *Discrete Math.* **312** (2012), no. 6, 1257–1259.