Transport Coefficients of Hot Hadronic Matter

By

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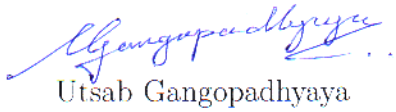

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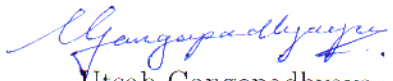

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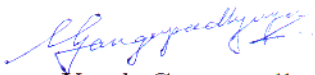
2. "*The Transport Co-efficients of Two Component Hot Hadronic Matter*", Utsab Gangopadhyaya, Snigdha Ghosh, Sukanya Mitra, Sourav Sarkar; Proceedings of DAE Symp. On Nucl. Phys. 60 E9(2015), 722

3. "*Medium Effects on the Thermal and Diffusion Coefficients of a Hot Hadronic Two Component Mixture*", Utsab Gangopadhyaya, Snigdha Ghosh, Sourav Sarkar, Sukanya Mitra.; Proceedings of DAE Symp. On Nucl. Phys. 62 D9(2017), 752



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Dedicated to

the important ones,

they know who they are.

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Contents

List of Publications **i**

Synopsis **ix**

List of Figures **3**

1 Introduction **5**

1.1 Nuclear interaction 6

1.2 Motivation to study high energy heavy ion collision 7

1.3 Signatures of QGP 8

1.3.1 Electromagnetic probes 8

1.3.2 Quarkonia suppression 9

1.3.3 Jet Quenching 10

1.3.4 Elliptic flow 11

1.4 Collective flow 12

1.4.1 Flow coefficient 12

1.4.2 Determining flow coefficient 13

1.5 Theoretical methods to treat heavy ion collision 16

1.5.1 Hydrodynamics 17

1.5.2 Kinetic Theory 19

1.6 Scope and Organisation of present work 20

2 Kinetic Theory **27**

2.1 Introduction 27

2.1.1 Single particle distribution function 28

2.1.2 Particle four-flow 30

2.1.3 Energy-momentum tensor 31

2.1.4 Entropy four-flow 32

2.1.5 Hydrodynamic four-velocity 32

2.1.6 Physical quantities in local rest frame 35

2.1.7 Mixture 36

2.2 Relativistic Boltzmann Transport Equation 38

2.2.1 Assumptions of Boltzmann's equation 38

2.2.2 Boltzmann's equation 39

2.3 Conservation Equations 43

2.3.1 Conservation of number 44

2.3.2 Conservation of energy-momentum 45

2.4 Equations of fluid dynamics 45

2.4.1 Time derivative and gradient 46

2.4.2 Continuty equation 47

2.4.3 Equation of motion 48

2.4.4 Equation of energy 48

2.5 Equilibrium distribution function 49

3 Transport Coefficients of a hot Pion Gas **53**

3.1 Linear Theory: Enskog Expansion 56

3.2 Chapman-Enskog Method 61

3.2.1 Transport Coefficients of Pion Gas 61

3.3 Grad's 14 Moment Method - Evaluation of relaxation time of flows 71
3.3.1 Determining $\boldsymbol{A}$, $\boldsymbol{B_\mu}$ and $\boldsymbol{C_{\mu\nu}}$ for the Pion Gas 75
3.3.2 Equation of motion of dissipative fluxes 79
3.4 Appendix-A 81
3.5 Appendix-B 82
3.6 Appendix C 83
3.7 Appendix D 85

4 Transport Coefficients of a Pion-Nucleon Gas 89
4.1 Linearisation for a Non-reactive Mixture 90
4.2 Relaxation Time Approximation 91
4.2.1 Transport Coefficients of a two component system 92
4.3 Appendix-A 99
4.4 Appendix-B 100

5 Dynamic Inputs for determining Transport coefficients 103
5.1 $\pi\pi$ scattering 104
5.1.1 Interaction Lagrangian 104
5.1.2 Amplitude of scattering due to the exchange of ρ and σ meson 105
5.1.3 $\pi\pi$ cross-section at finite temperature 107
5.1.4 Temperature dependent pion chemical potential 111
5.2 πN scattering 112
5.2.1 The Δ self-energy in the medium 112
5.2.2 The π-N Cross Section 118
5.2.3 Numerical results 120

6 Numerical Results 126
6.1 Viscous coefficients and Thermal conductivity of Pion Gas 128

6.2 Relaxation time of flows 130
6.3 Transport coefficient of a mixture consisting of Pions and Nucleons 133
6.3.1 Viscous Coefficients 135
6.3.2 Coefficients of Thermal Conduction and Diffusion 139

7 Summary and Outlook **147**

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SYNOPSIS OF Ph.D. THESIS

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SYNOPSIS

Introduction and Motivation

The possibility that the exotic matter that might have existed in the microsecond old early universe, or exists inside the core of a neutron star could be recreated terrestrially has led us to ultra relativistic heavy ion collision. The exotic matter is known as quark gluon plasma (QGP), a de-confined state with coloured degrees of freedom. Experiments at Super Proton Synchrotron (SPS) at CERN, strongly suggested the formation of QGP [1]; confirmation of QGP phase was obtained at RHIC [2–10] and LHC energies [11–14]. The hot dense nuclear matter created in these heavy ion collisions may be close to local thermodynamic equilibrium. If the interaction between the quarks and gluons is strong enough to maintain local thermodynamic equilibrium,

then the subsequent hadronic phase will also have well-defined transport properties. From the experimentally measured energy and momentum of the particle that reaches the detector after kinetic freeze out (the state when the particles are so far apart that they no longer interact with each other), it is not easy to extract the transport properties. Thus to study the transport phenomena of QGP the properties of all the stages in the heavy ion collisions is to be studied, starting from the collision of two Lorentz contracted heavy ions, followed by QGP formation and its thermalization, evolution; phase transition/crossover to hadronic phase , the evolution of the hadronic phase and eventually freeze-out.

Transport coefficients go into the hydrodynamic equation used to describe the system. These coefficients can be estimated microscopically using either linear response theory where they appear in the form of two-point functions or in the kinetic theory approach which is more popular owing to better computational efficiency. Shear viscosity has been derived using a hadron resonance gas model in Ref. [16–19]. In the kinetic theory approach viscosities have been estimated using parametrized cross section extracted from empirical data in Refs. [20–23], or using lowest order chiral perturbation theory in Refs. [24, 25].

The scattering cross-section that appears in the collision integral is the dynamical input in the kinetic theory approach and it is highly suggestive that it contains the effect of the hot and/or dense medium. For the case of a pion gas the consequences of an in-medium cross-section on the temperature dependence of the transport coefficients were extensively discussed in [26–28]. Using effective interactions and the techniques of thermal field theory the $\pi\pi$ scattering amplitudes evaluated with self-energy corrected ρ and σ meson propagators in the internal lines caused a significant modification in the cross-section and consequently the viscosities [26], thermal conductivity [27] and relaxation times of flows [28].

We consider a two component system constituting of pions and nucleons. Analogous to the ρ and σ mesons mediating the $\pi\pi$ interaction we consider πN scattering to proceed by exchange of the lightest baryon resonance, the Δ, which is close to an ideally elastic πN resonance, decaying almost entirely into pions and nucleons. We obtain the Δ self-energy at finite temperature and baryon density evaluating several one-loop diagrams with π, ρ, N and Δ in the internal lines

using standard thermal field theoretic methods. The in-medium propagator of the Δ baryon is then used in the scattering amplitudes to obtain the πN cross-section. The Novelty in our approach is the use of in-medium cross-section, while almost all the earlier works in this field dealing with transport coefficients of hadronic matter have been done using vacuum scattering amplitudes. This feature attributes a realistic nature to the evaluation of these quantities and makes the formalism more complete.

The transport equations for the pion and nucleon are solved to obtain the temperature and density dependence of the shear viscosity, bulk viscosity, the thermal conductivity and diffusion coefficients. Compared to the viscosities, the thermal conductivity and diffusion coefficients have received much less attention. This may be due to the absence of a conserved quantum number, the baryon number being insignificantly small for systems produced at RHIC and LHC. However at FAIR energies [29] or in the Beam Energy Scan (BES) program at RHIC, the baryon chemical potential is expected to be significant and baryon number will play a significant role in determining the thermal and diffusion coefficients. A careful analysis also reveals that a system in which the total number of particle is conserved can also sustain thermal conduction or diffusion if the number of particles of individual species is also conserved. Such a scenario is reached as the system expands and cools and reaches chemical freeze out as the collisions become mostly elastic. Diffusion coefficients arise in the treatment of a multicomponent gas in addition to thermal conductivity [30]. For the pion-nucleon system under study there arise two thermal coefficients (thermal conductivity and Dufour coefficient) and two diffusion coefficients (diffusion and thermal diffusion coefficient). We have not encountered an elaborate study of the temperature and density dependence of these coefficients in the literature.

Formalism

We work within the framework of relativistic kinetic theory, and the process used to derive the transport coefficients of the two component system constituting of nucleons and pions is

relaxation time approximation method. The salient features of this method will be discussed in this section.

Relaxation Time Approximation

The system is described by two Boltzmann transport equation one for each component, the covarient form of the equation have been used.

$$p_k^\mu \partial_\mu f_k(x,p_k) = \sum_{l=1}^{2} \frac{g_l}{1+\delta_{kl}}\, C_{kl}(x,p_k) \tag{1}$$

$$\begin{aligned} C_{kl}(x,p) = \int \frac{d^3p_l}{(2\pi)^3 p_l^0}\frac{d^3p_k'}{(2\pi)^3 p_k'^0}\frac{d^3p_l'}{(2\pi)^3 p_l'^0}[f_k' f_l'(1\pm f_k)(1\pm f_l) \\ - f_k f_l(1\pm f_k')(1\pm f_l)]W(p_k,p_k|p_k',p_l'). \qquad k,l=1,2 \end{aligned} \tag{2}$$

Here subscript $l = 1, 2$ corresponds to nucleon and pion respectively. For the relaxation time approximation the right hand side of the Boltzmann equation becomes $-(p_k \cdot U)\delta f_k/\tau_k$, U^μ being the hydrodynamic four velocity.

$$\big[\tau_k(p_k)\big]^{-1} = \sum_{l=1}^{N} \big[\tau_{kl}(p_k)\big]^{-1} \tag{3}$$

$$\big[\tau_{kl}(p_k)\big]_{-1} = \frac{g_l}{1+\delta_{kl}}\frac{csh(\epsilon_k/2)}{E_k}\int d\omega_l d\omega_k' d\omega_l' W_{kl} \tag{4}$$

where $d\omega_k = d\Gamma_{p_k}/[2csh(\epsilon_k/2)]$, $d\Gamma_{p_k} = \frac{dp_k}{(2\pi)^3 p_k^0}$, $\epsilon_k = \big(E_k - \mu_k\big)$ and the function $csh(\alpha_k) = \cosh(\alpha_k)$ if k^{th} specie is a fermion, and if boson $csh(\alpha_k) = \sinh(\alpha_k)$.

We introduce the non-uniformity parameter ε in the boltzmann transport equation and expand the distribution function and its derivative in a series in terms of the non-uniformity parameter.

$$p_k^\mu U_\mu D f_k = -\varepsilon\; p_k^\mu \nabla_\mu f_k + \sum_{l=1}^{2} \frac{g_l}{1+\delta_{kl}}\; C_{kl}(x,p_k). \tag{5}$$

$$f = f^0 + \varepsilon f^1 + \varepsilon^2 f^2 + ... + \varepsilon^r f^r + ..., \tag{6}$$

$$Df = \varepsilon\big(Df\big)^1 + \varepsilon^2\big(Df\big)^2 + ... + \varepsilon^r\big(Df\big)^r + ... \tag{7}$$

Where $D = U^\mu \partial_\mu$, $\nabla^\mu = \Delta^{\mu\nu}\partial_\mu$ and f^0 is the equilibrium distribution function, function of only the macroscopic parameters like temperature, chemical potential and the hydrodynamic four velocity, while the higher times like $f^1, f^2, ...$ are function of the space derivative of the macroscopic parameters. The non-uniformity parameter keeps track of the order of the space derivative of the macroscopic parameters used to define the system. We substitute the expanded form of the distribution function and its derivative in eqn.(5) and restrict ourselves to $r = 1$, thus obtaining a reduced form of the Bolzmann transport equation. We use this transport equation to derive the different hydrodynamic equations like the energy equation, equation of motion and continuity equation. These equation turns out to be same as that used to describe ideal fluid. These conservation equations along with the Gibbs Duhem relation is used to represent the reduced form of the Boltzmann equation in terms of the thermodynamic forces.

$$\frac{f_k^{(0)}(1+f_k^{(0)})}{T}\Bigg\{\Big[p_k\cdot U - h + (\delta_{kN}-x_N)T^2\Big(\frac{\partial}{\partial T}\Big(\frac{\mu_N}{T}\Big)_{px_N} - \frac{\partial}{\partial T}\Big(\frac{\mu_\pi}{T}\Big)_{px_N}\Big)\Big]p_k^\mu\frac{\nabla_\mu T}{T} + \frac{(\delta_{kN}-x_N)}{x_\pi}\Big(\frac{\partial\mu_N}{\partial x_N}\Big)_{PT}p_k^\mu\nabla_\mu x_N + Q_\pi\partial_\nu U^\nu - \langle p_k^\mu p_k^\nu\rangle\langle\partial_\mu U_\nu\rangle\Bigg\} = -\frac{\delta f_k}{\tau_k}E_k \tag{8}$$

Here the subscript N represents nucleon, while k can be either nucleon or pion, h represents the enthalpy per particle and x_k is the concentration of the k^{th} specie. From the above equation the expression of $\delta f_k = f_k - f_k^0 = f_k^1$ can be determined, and using the expression we can determine the irreversible flows. The hydrodynamic form of the irreversible flow are expressed as a combination of the thermodynamic forces and flow coefficients, and they are expressed as follows,

$$\bar{I}^\mu = I^\mu - \sum_{k=1}^{2} h_k N_k^\mu; \quad \bar{I}^\mu = \lambda\nabla^\mu T + D_T' n x_1 T\Big(\frac{\partial\mu_1}{\partial x_1}\Big)_{PT}\nabla^\mu x_1 \tag{9}$$

$$I_1^\mu = N_1^\mu - x_1 N^\mu; \quad I_1^\mu = Dn\nabla^\mu x_1 + D_T n x_1 x_2 \nabla^\mu T \tag{10}$$

$$T^{\mu\nu} = enU^\mu U^\nu - p\,\triangle^{\mu\nu} + \Pi^{\mu\nu}; \quad \Pi^{\mu\nu} = 2\eta\langle\partial^\mu U^\nu\rangle + \zeta\big(\partial\cdot U\big)\,\triangle^{\mu\nu} \tag{11}$$

The irreversible flow occurs when the system is not in equilibrium and these flows helps to bring the system back to equilibrium hence these flows should be solely depended on the part

of distribution function that represents the deviation from equilibrium.

$$\bar{I}_q^{\mu} = \sum_{k=1}^{2} \int d\Gamma_k (p_k^{\nu} U_{\nu} - h_k) p_k^{\mu} \delta f_k. \tag{12}$$

$$I_1^{\mu} = \sum_{k=1}^{2} \int d\Gamma_k (\delta_{1k} - x_k) p_k^{\mu} \delta f_k \tag{13}$$

$$\Pi^{\mu\nu} = \mathring{\Pi}^{\mu\nu} + \Pi \triangle^{\mu\nu} = \sum_{k=1}^{2} \int d\Gamma_k \{ \triangle_{\sigma}^{\mu} \triangle_{\tau}^{\nu} - \frac{1}{3} \triangle_{\sigma\tau} \triangle^{\mu\nu} \} p_k^{\sigma} p_k^{\tau} \ \delta f_k + \sum_{k=1}^{2} \frac{1}{3} \int d\Gamma_k \triangle_{\sigma\tau} \triangle^{\mu\nu} p_k^{\sigma} p_k^{\tau} \ \delta f_k \tag{14}$$

From the above two set of equations (one set representing the hydrodynamic theory form and the other representing kinetic theory form of the irreversible flow), it is quite easy to derive the expression for the flow coefficients, The thermal conductivity λ, Dufour coefficient D_T', diffusion coefficient D and thermal diffusibility D_T.

$$\lambda = \frac{L_{qq}}{T} \tag{15}$$

$$D_T^{'} = \frac{L_{qN}}{n x_{\pi} x_N T} \tag{16}$$

$$D = \frac{L_{NN}}{n x_{\pi}} \Big(\frac{\partial \mu_N}{\partial x_N} \Big)_{pT} \tag{17}$$

$$D_T = \frac{L_{Nq}}{n x_{\pi} x_N T} \qquad , \tag{18}$$

$$\eta = \frac{1}{15T} \sum_{k=1}^{2} \int \frac{d^3 p_k}{(2\pi)^3} \frac{\tau_k}{E_k^2} \mid \vec{p_k} \mid^4 f_k^{(0)} (1 \pm f_k^{(0)}) \tag{19}$$

$$\zeta = \frac{1}{T} \sum_{k=1}^{2} \int \frac{d^3 p_k}{(2\pi)^3} \frac{\tau_k}{E_k^2} \{Q_k\}^2 f_k^{(0)} (1 \pm f_k^{(0)}) \tag{20}$$

Where,

$$L_{qq} = \frac{1}{6\pi^2 T}\sum_{k=1}^{2} g_k \int \frac{p_k^4 dp_k}{E_k^2}(p_k \cdot U - h_k)\Big[p_k.U - h + (\delta_{k1} - x_1)T^2\beta\Big]\tau_k f_k^{(0)}(1 \pm f_k^{(0)}), \tag{21}$$

$$L_{q1} = \frac{1}{6\pi^2 T}\sum_{k=1}^{2} g_k \int \frac{p_k^4 dp_k}{E_k^2}(p_k \cdot U - h_k)(\delta_{k1} - x_1)\tau_k f_k^{(0)}(1 \pm f_k^{(0)}), \tag{22}$$

$$L_{11} = \frac{1}{6\pi^2 T}\sum_{k=1}^{2} g_k \int \frac{p_k^4 dp_k}{E_k^2}(\delta_{k1} - x_1)^2\tau_k f_k^{(0)}(1 \pm f_k^{(0)}), \tag{23}$$

$$L_{1q} = \frac{1}{6\pi^2 T}\sum_{k=1}^{2} g_k \int \frac{p_k^4 dp_k}{E_k^2}\Big[p_k \cdot U - h + (\delta_{k1} - x_1)T^2\beta\Big](\delta_{k1} - x_1)\tau_k f_k^{(0)}(1 \pm f_k^{(0)}). \tag{24}$$

Here $\beta = \frac{\partial}{\partial T}\left(\frac{\mu_1}{T}\right)_{Px_1} - \frac{\partial}{\partial T}\left(\frac{\mu_2}{T}\right)_{Px_1}$.

Dynamic Input

The kinetic theory approach is quite general, the expression of the transport coefficients are same for any hydrodynamic system. What differentiates one system from the another is the interaction between the constituent particles, which enter the expression of the coefficients through the scattering amplitude in the relaxation time.

For the $\pi\pi$ scattering, we assume it to proceed via ρ and σ meson exchange. Using the interaction $\mathcal{L} = g_\rho \vec{\rho}^\mu \cdot \vec{\pi} \times \partial_\mu \vec{\pi} + \frac{1}{2} g_\sigma m_\sigma \vec{\pi} \cdot \vec{\pi}\sigma$ with $g_\rho = 6.05$ and $g_\sigma = 2.5$, the matrix elements in the isoscalar and isovector channels are given by

$$\begin{aligned}\mathcal{M}_{I=0} &= 2g_\rho^2\left[\frac{s-u}{t-m_\rho^2} + \frac{s-t}{u-m_\rho^2}\right] \\ &+ g_\sigma^2 m_\sigma^2\left[\frac{3}{s-m_\sigma^2+\sum_\sigma} + \frac{1}{t-m_\sigma^2} + \frac{1}{u-m_\sigma^2}\right]\end{aligned} \tag{25}$$

$$\begin{aligned}\mathcal{M}_{I=1} &= g_\rho^2\left[\frac{2(t-u)}{s-m_\rho^2+\sum_\rho} + \frac{t-s}{u-m_\rho^2} - \frac{u-s}{t-m_\rho^2}\right] \\ &+ g_\sigma^2 m_\sigma^2\left[\frac{1}{t-m_\sigma^2} - \frac{1}{u-m_\sigma^2}\right] .\end{aligned} \tag{26}$$

where effective propagators obtained by a Dyson-Schwinger sum of one-loop self-energy diagrams in vacuum has replaced the corresponding s-channel propagators. The cross-section and the isospin averaged amplitude is defined as, $\sigma = \frac{1}{64\pi^2 s}\int \overline{|\mathcal{M}|^2} d\Omega$ and $\overline{|\mathcal{M}|^2} = \frac{1}{9}\sum(2I+1)\overline{|\mathcal{M}_I|^2}$. The vacuum self energies $\sum_\rho$ and $\sum_\sigma$ are replaced with in-medium ones evaluated using thermal field theory [31, 32]. For the σ meson only the $\pi\pi$ loop graph is evaluated in the medium whereas in case of the ρ meson in addition to the $\pi\pi$ loop diagram, $\pi\omega$, πh_1, πa_1 self-energy diagrams are included [33]. The imaginary part of the self-energy is given by

$$\begin{aligned}\mathrm{Im}\sum(q_0,\vec{q}) = -\pi\int\frac{d^3k}{(2\pi)^3 4\omega_\pi\omega_h}\times \\ [L_1(1+n_+(\omega_\pi)+n_+(\omega_h))\delta(q_0-\omega_\pi-\omega_h) \\ + L_2(n_-(\omega_\pi)-n_+(\omega_h))\delta(q_0+\omega_\pi-\omega_h)]\end{aligned} \tag{27}$$

where $n_\pm(\omega) = \frac{1}{e^{(\omega\mp\mu)/T}-1}$ is the distribution function for boson, $\omega_\pi = \sqrt{\vec{k}^2+m_\pi^2}$ and $\omega_h = \sqrt{(\vec{q}-\vec{k})^2+m_h^2}$. The angular integration is done using the δ-functions which define the kinematic domains for occurrence of scattering and decay processes leading to loss or gain of ρ (or σ) mesons in the medium. The term with L_1 arises from the unitary cut and corresponds to formation and decay in the medium weighted by Bose enhancement factors, while the second term corresponds to the so-called Landau cut contribution arising from resonant scattering in the medium. The self-energy function is convoluted with their spectral functions in order to account for the substantial 3π and $\rho\pi$ branching ratios of some of the unstable particles in the loop.

The πN scattering is treated analogously. It is taken to proceed via the exchange of the Δ-baryon which is the lightest baryon resonance. We use the well-known interaction $\mathcal{L}_{\pi N\Delta} = \frac{f_{\pi N\Delta}}{m_\pi}\bar{\Delta}^\mu\vec{T}^\dagger\partial_\mu\vec{\pi}\psi + H.c.$ with $f_{\pi N\Delta} = 2.8$ to evaluate the scattering matrix elements. Averaging over isospin, the squared invariant amplitude for the process $\pi(k)\ N(p) \to\ \pi(k')\ N(p')$ turns out to be,

$$\begin{aligned}\overline{|\mathcal{M}|^2} \ = \ \frac{1}{3}\left(\frac{f_{\pi N\Delta}}{m_\pi}\right)^4\Bigg[\frac{F^4(k,p)T_s}{\left|s-m_\Delta^2-\Pi\right|^2}+\frac{F^4(k,p')T_u}{\left(u-m_\Delta^2\right)^2} \\ +\frac{2F^2(k,p)F^2(k,p')T_m(s-m_\Delta^2-\mathrm{Re}\Pi)}{3(u-m_\Delta^2)\left|s-m_\Delta^2-\Pi\right|^2}\Bigg].\end{aligned} \tag{28}$$

where T_s, T_u and T_m can be read off from Ref. [34]. At each vertex we consider the form factor [34]

$$F(p,k) = \frac{\Lambda^2}{\Lambda^2 + (\frac{p\cdot k}{m_p})^2 - k^2}$$

where p and k denote the momenta of the fermion and boson respectively. The cut-off is taken as $\Lambda = 600$ MeV [34]. At finite temperature additional contributions coming from on-shell particles in the medium is considered, by evaluating πN, ρN, $\pi\Delta$ and $\rho\Delta$ self-energies using the real time method. The expression for the spin averaged imaginary part of self-energy is given by

$$\begin{aligned} \mathrm{Im}\Pi = \quad & -\pi \int \frac{d^3k}{(2\pi)^3}\frac{1}{4\omega_k\omega_p} \\ & [N_1(1+n_+(\omega_k)-\tilde{n}_+(\omega_p))\delta(q_0-\omega_k-\omega_p) \\ & + N_2(n_-(\omega_k)+\tilde{n}_+(\omega_p))\delta(q_0+\omega_k-\omega_p)] \end{aligned} \tag{29}$$

where the distribution function for the fermions is given by $\tilde{n}_\pm(\omega) = \frac{1}{e^{\beta(\omega\mp\mu)}+1}$. As before the first term is the contribution from decay and formation of the Δ baryon weighted by thermal factors. The second term is a result of scattering processes in the medium leading to the absorption of the Δ. These processes contribute significantly to the imaginary part which leads to the suppression of the πN cross-section.

Results

Here we present a few interesting recent results of work done during my PhD tenure. On comparing the relaxation time calculated using in-medium cross section with those calculated using the vacuum cross section we find a significant change. The value of the relaxation time for both pion and nucleon as seen in Fig. 1, are found to be enhanced after the introduction of the medium effect, this is due to the suppression of the cross section in the medium due to the enhancement of the imaginary part of the ρ and Δ self energies. This surge in the value of the relaxation time is reflected in the transport coefficients as well, since they are directly proportional to the relaxation time of pion as well as nucleon. The introduction of

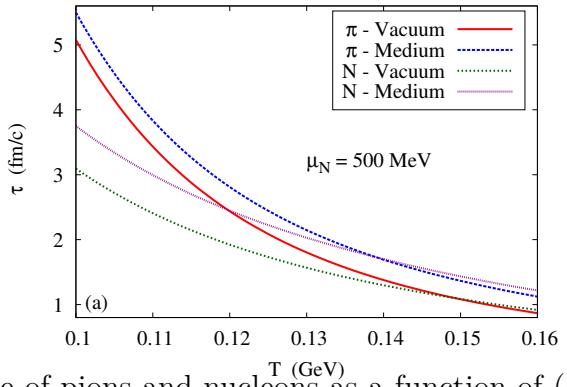

Figure 1: The relaxation time of pions and nucleons as a function of (a) T for $\mu_N = 500$ MeV and $\mu_\pi = 80$ MeV

nucleons to the system also substantially changes the transport coefficients of the system. The effect of the introduction of the medium effect on the shear viscosity, bulk viscosity, thermal conductivity and diffusion coefficient is quite significant, but the effect on the Dufour coefficient and thermal diffusibility is not that pronounced. The variation of these transport coefficients with temperature have been studied for different value of the nucleon chemical potential. The value of the bulk and the shear viscosity goes up with increase in temperature and nucleon chemical potential. The value of the the thermal conductivity, Dufour coefficient, Thermal diifusibility goes down with increase in temperature and nucleon chemical potential. In Fig. 2(a) we can see that the diffusion coefficient decreases with increase in temperature, and in Fig. 2(b) we see it remains almost unchanged with increase in nucleon chemical potential up till $\mu_N =$ 0.3GeV then it goes up for temperature $T = 100$ MeV, or goes down for $T = 160$ MeV or higher value.

Summary

With an aim to elucidate the effect of a hot and dense medium produced during the later stages of relativistic heavy ion collision on the temperature and density dependence of transport coefficients, we have discussed in detail the framework of relativistic kinetic theory in particular the collision integral where in medium $\pi\pi$ and πN cross-section calculated using thermal field

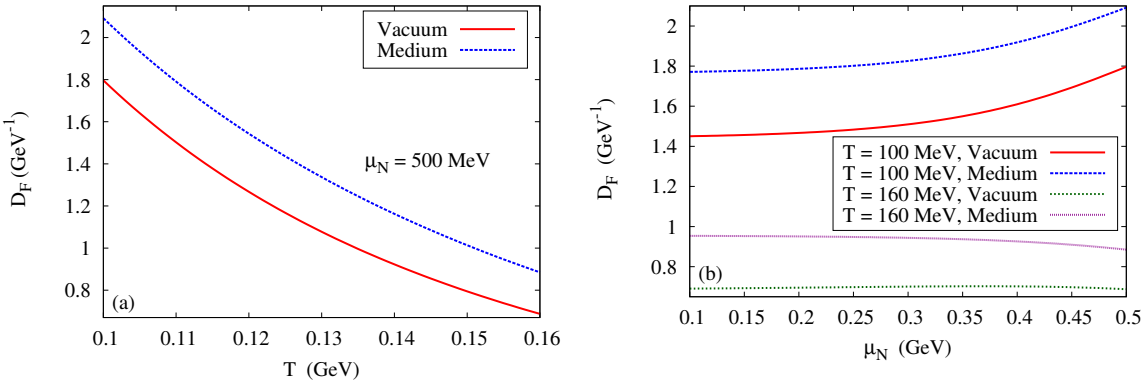

Figure 2: The diffusion coefficient as a function of (a) temperature and (b) nucleon density

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theoretic techniques have been incorporated. The temperature and density dependence of shear and bulk viscosity and thermal conductivity for the case of a πN gas have been discussed in details. These are expected to have a non-trivial effect on the hydrodynamic evolution of the later stages of heavy ion collisions.

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List of Figures

1 The relaxation time of pions and nucleons as a function of (a) T for $\mu_N = 500$ MeV and $\mu_\pi = 80$ MeV xviii

2 The diffusion coefficient as a function of (a) temperature and (b) nucleon density xix

1.1 Energy loss of a away side jet [22]. 11

1.2 Differential elliptic flow v_2 vs $p_\perp$, for ALICE collaboration. Top panel: Results for mid-peripheral events, for two-particle correlations (blue asterisks) and for four-particle correlations (red triangles); where the 'non-flow' effects are suppressed. The result for the STAR collaboration is represented by the grey band. Bottom panel: elliptic flow for different centralities calculated with four-particle correlations. The elliptic flow increases with the centrality, having larger values for peripheral events. Figures taken from [26]. Copyright 2010 by The American Physical Society. 13

1.3 Left panel: Integrated elliptic flow v_2 vs centrality. The blue dots and the red dots represents the results for two-particle and four-particle correlation respectively. The STAR results are represented by the blue and red lines. Right panel: Integrated elliptic flow for the centrality class $20\% - 30\%$ as a function of the beam CM energy. Figures taken from [26]. Copyright 2010 by The American Physical Society. 16

1.4 Comparison of hydrodynamic models to experimental data on charged hadron integrated (left) and minimum bias (right) elliptic flow by PHOBOS and STAR, respectively. STAR event plane data has been reduced by 20 per cent to estimate the removal of non-flow contributions [30]. 17

5.1 $\pi\pi$ scattering with self-energy corrections 107

5.2 The $\pi\pi$ cross-section as a function of $E_{c.m.}$. The dashed line corresponds to scattering in the vacuum. The Dot-dashed line refers to the in-medium cross-section involving only the pion loop for the σ and ρ mesons. The solid line corresponds to the additional loops in the ρ meson self-energy. 111

5.3 The temperature dependent pion chemical potential [9]. 112

5.4 Feynman Diagrams for Δ-Self Energy. 114

5.5 The Landau-cut contribution to the imaginary part of the self-energy function for different T and μ_N. [24] 119

5.6 The unitary cut contribution to the imaginary part of the Δ self-energy at $T =$ 100 MeV and $\mu_N = 200$ MeV. [24] 120

5.7 The spectral function of the Δ for q_0 from 1 to 1.8 GeV. [24] 121

5.8 The Δ spectral function vs q_0: (a) in comparison with [16] and (b) for different form factors. The results of Ref. [16] are represented by symbols. [24] 122

5.9 The π-N elastic scattering cross-section with medium-effects [24]. 123

6.1 The bulk viscosity as a function of T. The upper set of lines (with circles), middle set of lines (with triangles) and lower sets of lines correspond to $\mu_\pi = 85$ MeV, $\mu_\pi = \mu_\pi(T)$ and $\mu_\pi = 0$ MeV respectively. In each set the solid line represents the use of vacuum cross-section, the dotted line for in-medium modification due to pion loop and the dashed line for loops with heavy particles in addition. . . . 127

6.2 η as a function of T for a temperature dependent pion chemical potential. The lower one represents the one with vacuum cross-section, while the other two represents the one with in-medium cross-section. The middle line for in-medium modification due to pion loop and the topmost line for loops with heavy particles in addition. 129

6.3 λT as a function of T evaluated using Chapman-Enskog approximation. 129

6.4 Relaxation time of bulk viscous flow as a function of T. 130

6.5 Relaxation time of heat flow as a function of T. 131

6.6 Relaxation time of shear viscous flow as a function of temperature. 132

6.7 The relaxation time of pions in a hadron gas mixture of pions and nucleons, for nucleon chemical potential $\mu_N = 200$ and 500 MeV; with and without medium-effects. 133

6.8 The relaxation time of nucleons in a hadron gas mixture of pions and nucleons, for nucleon chemical potential $\mu_N = 200$ and 500 MeV; with and without medium-effects. 134

6.9 The relaxation time of pions and nucleons as a function of μ for $T = 160$ MeV for $\mu_\pi = 80$ MeV. 135

6.10 Shear viscosity as a function of T for various μ_N with and without medium-effects. Legends 1,2,3 and 4 indicate μ_N =200, 300, 400 and 500 MeV respectively. The pion chemical potential $\mu_\pi = 80$ MeV. 136

6.11 Shear viscosity over entropy density as a function of T for diffrent nucleon chemical potential μ_N expressed in MeV. 137

6.12 The bulk viscosity as a function of T with and without medium-effects. Legends 1,2,3 and 4 indicate $\mu N = 200, 300, 400$ and 500 MeV respectively. 138

6.13 Ratio of bulk viscosity and entropy density vs T for different μ_N expressed in MeV. 139

6.14 The thermal conductivity for a mixture constituting of nucleons and pions as a function of (a) temperature and (b) nucleon density. 140

6.15 The Dufour coefficient as a function of (a) temperature and (b) nucleon density. 141

6.16 The thermal diffusion coefficient for a mixture constituting of nucleons and pions as a function of (a) temperature and (b) nucleon density. 142

6.17 The diffusion coefficient as a function of (a) temperature and (b) nucleon density. 144

Chapter 1

Introduction

Humans have always been intrigued by the nature that surrounds it, that affects its daily life, and one of the key things that constitute nature is matter. Humans from the very beginning have tried to fathom what are the basic building blocks that make all the matters that are surrounding him; it is as if they knew somehow from the very beginning that all matter no matter how complex have a somewhat similar substructure. Nuclear physics is a modern approach to this very ancient obsession of man.

Late nineteenth and early twentieth-century discoveries by Thompson, Rutherford, and Chadwick confirmed the electron, the proton and the neutron to be the building blocks of matter - the fundamental particles. Post-1947 however, a huge number of fundamental particles began to get discovered. So much so, William Lamb observed in his Nobel Acceptance Speech - "*finders of new elementary particles used to be rewarded with a Nobel prize; but such a discovery now ought to be punished by a* $\$10,000$ *fine*". The Deep Inelastic Scattering [DIS] experiments at Stanford Linear Accelerator Center [SLAC] in 1968 revealed protons to have point-like internal structure, these point-like structures were named partons by Feynman. However, these point-like structures had been proposed earlier by Murray Gell-Mann [1], who had named them Quarks. The pursuit to probe deeper and deeper into matter requires energy. The more energy one has, the deeper into the sub-nuclear structure one can probe. Thus was born a new

branch of study - High Energy Nuclear Physics. As per our present understanding, the basic constituents of matter are quarks, leptons, gauge bosons, Higgs Bosons, and their anti-particles.

1.1 Nuclear interaction

The interaction of the elementary particles in the Standard Model (SM), is governed by the Electroweak theory and Quantum Chromodynamics (QCD). QCD the theory of strong interaction, governs the interaction between the quarks, anti-quarks, and gluons. These quarks carry color, also known as color charge, which plays a role analogous to electric charge in QED (Quantum Electrodynamics). There are three different colors that a quark can carry. The gluons are the gauge bosons of QCD and they are bicolored (carrying one positive unit of color and one negative unit). They mediate the interaction between the quarks and anti-quarks. The gauge bosons of QCD interact within themselves unlike the gauge bosons of QED (photons), since they themselves have charge (color charge). QCD is thus a non-Abelian gauge theory. The coupling depends on the scale of momentum transfer q by the relation,

$$\alpha\left(q^2\right) = \frac{\alpha_0}{1 + \alpha_0 \frac{\left(33-2n_f\right)}{12\pi} ln\left(\frac{-q^2}{\mu^2}\right)}, \tag{1.1}$$

where α_0 and n_f are the coupling constant of momentum transfer μ and number of flavours respectively. The running coupling of QCD shows two unique features.

For a small momentum transfer (or large distance) we have high coupling; hence the interaction is strong. This is the reason why quarks are always confined within a hadrons (protons, pions, kaons etc.). This phenomenon is known as *Colour Confinement*.

On the opposite end of the spectrum, i.e., for large momentum transfer (small distance) the coupling asymptotically decreases, leading to weak interaction between the partons. This is known as *Asymptotic freedom*. Thus if we are able to produce a nuclear matter which is at high temperature and hence high momentum transfer and/or high density, the long-range interaction

can be dynamically screened thus enabling the quarks and gluons to freely move in the nuclear mass.

1.2 Motivation to study high energy heavy ion collision

Due to the properties of confinement and asymptotic freedom, QCD matter will behave like a gas of interacting hadrons at low energies, but at high energies and or high density in the domain of weak coupling the nuclear matter behaves as a weakly coupled gas of gluons and quarks. This particular state in which the quarks and gluons are no longer bound inside a hadron and they propagate through a large volume is known as Quark-Gluon Plasma (QGP). There is expected to be a phase transition between these two states. It is accepted that just after the Big Bang the entire universe has undergone such a phase transition at around some critical temperature (170 MeV). In fact, the entire phase diagram is of immense physical interest, the phase boundary the critical endpoint, etc. In particular, the zone at zero baryonic chemical potential is of great relevance, as the early structure of the Universe has cooled down through this zone.

QGP can also be found at the core of a neutron star. The Neutron star is the remnant of a regular star after the supernova explosion. The gravitational field of the star collapses it into a body of extremely high density, the density is so high at the centre of the neutron star, that the quarks and gluons may not remain confined inside a particular hadron thus creating a QGP phase with high baryonic density.

To produce QGP, nuclear matter at high temperature and density needs to be produced. In order to do so heavy nuclei are collided after accelerating them to ultra-relativistic velocities. This is achieved with the help of particle accelerators. Experiments at Super Proton Synchrotron (SPS) at CERN, strongly suggested the formation of QGP [2]; confirmation of QGP phase was obtained at RHIC energies [3–11] and LHC energies [12–15]. The hot and dense nuclear matter created in these heavy-ion collisions may be close to thermodynamic equilibrium (local), with thermodynamic and transport properties. If the interaction between the quarks

and gluons is strong enough to maintain local thermodynamic equilibrium in the subsequent phases, then those phases will also have definite transport properties. Since experimentally it is only possible to measure the energy and momentum of the particle that reaches the detector after kinetic freeze out (the state when the particles are so far apart that they no longer interact with each other), it is not easy to extract the transport properties. Thus to study the transport phenomenon of QGP the properties of all the stages in the heavy ion collisions are to be studied and modelled, starting from the two Lorentz contracted heavy ion nuclei, followed by QGP formation and its thermalisation, evolution; phase transition to hadronic phase, and the evolution of the hadronic phase.

Thus to study the transport properties of QGP the transport properties of the latter stages should also be studied in order to verify our findings through experiments. Here we have concentrated on the transport phenomena of the hadronic state.

1.3 Signatures of QGP

In order to study the properties of QGP and the interactions governing it, we need to identify whether this exotic matter is produced in the heavy ion collision experiment. And since we are only able to measure the energy and momentum of the particles that reach the detectors long after hadronisation, various signatures or probes have been identified based on the emmision spectra of these particles. These signatures or probes would indicate whether QGP was really created during the early stages of the heavy ion collision, and will provide us with relevant information. Here we will discuss some of these signatures.

1.3.1 Electromagnetic probes

In the hot QGP, real and virtual photons are created due to the interaction between the charged quarks and anti-quarks. These virtual photons, in turn, give rise to the lepton pairs. These photons and dileptons are known as the electromagnetic probe. These probes are also produced

due to electromagnetic interaction among the hadrons. Electromagnetic probes are one of the cleanest probes of the QGP as they do not interact with the strongly interacting medium and have a mean free path which is quite large compared to the size of the system, and so will be emitted mostly unaffected carrying the information of the whole evolution process. These probe helps us to see the earliest and the hottest part of the evolution [16, 17]. One of the major problems with this probe is that they can be emitted from any stages of the evolution process so there will be a lot of background emission from hadronic decay, if we can differentiate the early stage radiation from the background it will be a good signal of QGP [18, 19].

1.3.2 Quarkonia suppression

The bound states of heavy quark anti-quark pairs ($c\bar{c}$, $b\bar{b}$) are known as Quarkonia. They can only be produced during the early stages of the heavy ion collision before the formation of QGP. The resonance interaction of the $c\bar{c}$ system will then lead to J/ψ production. After the formation of the QGP medium, the J/ψ will find itself in a de-confined medium with high temperature. If the Debye screening radius for strong interaction is smaller than the size of J/ψ, the resonance interaction will not be operative and the produced $c\bar{c}$ will not result into J/ψ, rather they will propagate separately inside the medium. For a better understanding let us consider the vacuum $c\bar{c}$ potential

$$V(r) = \sigma r - \alpha_{eff}/r,$$

where σ and α_{eff} are the string tension and the Coulomb interaction coupling respectively. For an isolated $c\bar{c}$ system at $T = 0$ the magnitude of the J/ψ radius comes out to be $r_{J/\psi} = 0.2\ fm$, where mass of charm has been taken to be $1.5\ GeV$. On varying the various parameters like the mass, string tension and α_{eff}, the radius turns out to be in the range: $0.2 \leq r_{J/\psi} \leq 0.5 fm$. The Screening potential at $T \geq T_C$, when there is no string constant (as $\sigma(T_c) = 0$) is given by

$$V(r) = -\frac{\alpha_{eff}}{r} exp(-r/r_D), \tag{1.2}$$

where r_D is the Debye screening radius. Using this formula, we can show that the smallest radius which permits bound state is,

$$r_D^{min} = [0.84 m \alpha_{eff}(T)]^{-1}. \tag{1.3}$$

If the Debye screening radius goes below this value then the J/ψ will dissociate, and it has been observed from different studies that this will happen when $T/T_c = 1.2$ or even less. There are other quarkonia states having different sizes and binding energies. Therefore, at first, the larger and more loosely bound excited states are dissolved and finally the smallest and the most tightly bound states. This sequential suppression of quarkonia is good probe of the formation of QGP hence a good probe [20, 21].

1.3.3 Jet Quenching

Hard scattering during heavy ion collision gives rise to fast partons, which in turn develops a shower of particles around it due to strong interaction. This fast-moving parton along with the shower constitutes a collimated beam of particles known as jet. These partons become observable as jets of hadrons after they hadronize. When these jets of partons propagates through the fireball it loses energy; the energy loss depends on the density of the medium the type of interaction and the distance travelled in the medium. If the medium created is strongly interacting (QGP) the quarks and gluons are able to interact strongly with it. When a quark or gluon jet propagates through a QGP it loses energy by i) radiating gluons after the collision in its course of propagation and, ii) by elastic colliding with medium particles elastically. Gluon bremsstrahlung is a major mechanism of losing energy in the QCD medium.

As shown in Fig.1.1 the two back to back jets come out of the fireball with very different energy, simply because one of the Jet had to cover a longer distance through the medium and so experience greater energy loss. If the medium created is truly QGP then the energy difference will be pronounced. Due to various low p_T collective effects such as colour screening or Cronin enhancement this effect should be most pronounced at high p_T. This suppression of high p_T particles is known as Jet Quenching. To quantify this effect the heavy ion yields are compared

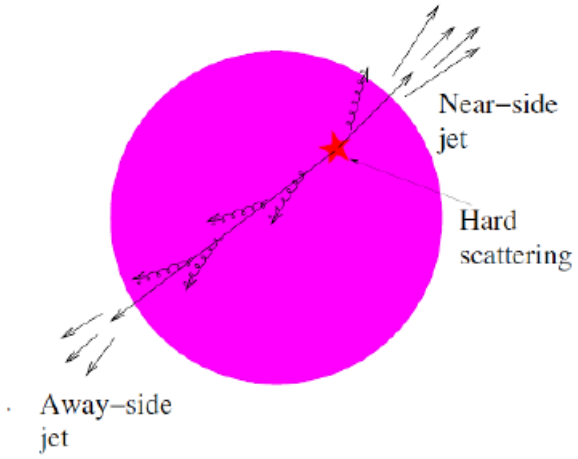

Figure 1.1: Energy loss of a away side jet [22].

to the scaled yields of $p+p$ collisions at the same energy. This is expressed through the nuclear modification factor R_{AA},

$$R_{AA} = \frac{d^2N_{AA}/dp_T d\eta}{\langle T_{AA}\rangle d^2\sigma_{pp}/dp_T d\eta}, \tag{1.4}$$

where $d^2N_{AA}/dp_T d\eta$ and $d^2\sigma_{pp}/dp_T d\eta$ are the differential particle yield in nucleus-nucleus collisions and the cross-section in proton-proton collisions respectively. The nucleus overlap function $\langle T_{AA}\rangle$ is obtained from the Glauber model and is proportional to the number of binary collisions. At high p_T, and in the absence of medium effects, R_{AA} is expected to be unity. In the region of low transverse momentum, the soft scatterings are the dominant processes, and so R_{AA} deviates from unity.

1.3.4 Elliptic flow

The elliptic flow coefficient v_2 is a measure of anisotropy of the momentum distribution of the produced particles for non-central nucleus-nucleus collision [23]. The exact form and the way to calculate this coefficient will be discussed in the upcoming section. The overlap region in a non-central heavy ion collision is deformed. This initial spatial anisotropy of the almond shape overlap region of the colliding nuclei is transformed into an anisotropy in momentum

space through interactions between the particles. Therefore the elliptic flow is sensitive to the collision dynamics in the early stages.

Comparing experimental values of elliptic flow with hydrodynamically calculated results, we find that they match well if we consider thermalisation at an early stage of heavy ion collision. This early thermalisation suggests that at the early stage, collisions are dominated by strongly interacting matter, with short constituent mean free paths essentially a perfect liquid with small viscosity.

1.4 Collective flow

For a non-central collision (i.e. finite value of b), the particle multiplicity has an azimuthal ϕ dependence. The overlap region at the time of collision is characterized by the spatial eccentric parameter ε_x.

$$\varepsilon_x(b) = \frac{\langle y^2 - x^2 \rangle}{\langle y^2 + x^2 \rangle}, \tag{1.5}$$

where the average is over the average energy ne (discussed in the next chapter). Fig. **??** depicts a non-central collision, the overlapping region is composed of participating nucleons, and the non-overlapping region are composed of spectator nucleons. After the spectator nucleons have left, the geometric anisotropy of the overlapping region, causes greater pressure gradient to develop along the along y-axis. This, in turn, gives rise to momentum anisotropy [23].

1.4.1 Flow coefficient

The azimuthal momentum distribution can be expanded in Fourier series,

$$E_p \frac{dN}{d^3p} = \frac{1}{2\pi} \frac{dN}{p_\perp dp_\perp dy_p} \Big[1 + \sum_{n=1}^{\infty} v_n(p_\perp, y_p) \cos(n\phi - n\Psi_R) \Big]. \tag{1.6}$$

Where Ψ_R is the reaction plane, and v_n is the n^{th} flow coefficient. The flow coefficient for $n = 1$, $n = 2$ and $n = 3$ are known as direct flow, elliptic flow and the triangular flow respectively.

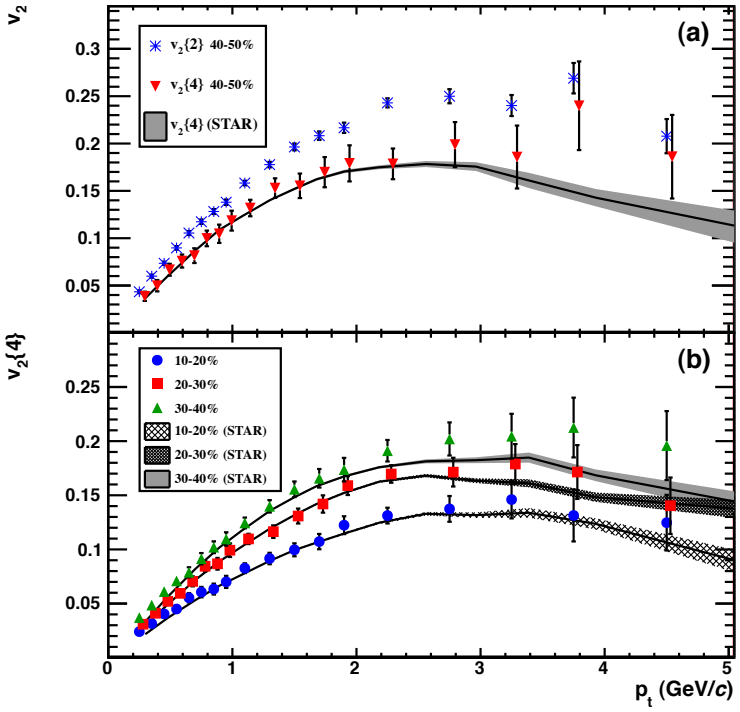

Figure 1.2: Differential elliptic flow v_2 vs $p_\perp$, for ALICE collaboration. Top panel: Results for mid-peripheral events, for two-particle correlations (blue asterisks) and for four-particle correlations (red triangles); where the 'non-flow' effects are suppressed. The result for the STAR collaboration is represented by the grey band. Bottom panel: elliptic flow for different centralities calculated with four-particle correlations. The elliptic flow increases with the centrality, having larger values for peripheral events. Figures taken from [26]. Copyright 2010 by The American Physical Society.

The dominant flow coefficient is the elliptic flow. The odd harmonics are forbidden as has been expressed in optical Glauber mode; however fluctuations in the initial state [24] gives rise to nontrivial value of odd harmonics, this can be computed using Monte Carlo Glauber Model, with the help of which random initial random initial positions of the nucleons is generated following the Woods-Saxon distribution.

1.4.2 Determining flow coefficient

The flow coefficients are determined from experimental data employing various techniques:

1. Event plane method: In this method n^{th} flow coefficient is given by expression

$$v_n\{EP\} = \frac{\int f_1(\boldsymbol{p}) \cos[n(\phi - \Psi_R)] d^3p}{\int f_1(\boldsymbol{p}) d^3p}. \tag{1.7}$$

Where $f_1(\boldsymbol{p}) = dN/d^3p$ is determined experimentally. To find the flow the reaction plane should also be known, here we replace the observable reaction plane Ψ_R with the reconstructed event plane Ψ_n, as its is expected that $\Psi_R \simeq \Psi_n$. The reconstructed event plane is found by plotting the angular distribution of the final particles, and choosing the angle with the maximum number of particles.

$$(\cos 2\Psi_n, \sin 2\Psi_n) = \frac{\boldsymbol{Q}}{|\boldsymbol{Q}|}, \qquad \boldsymbol{Q} = \Big(\sum_i \cos 2\phi_i, \sum_i \sin 2\phi_i \Big). \tag{1.8}$$

Where the summation over i represents the summation over all the final particles.

2. Two particle correlation: The flow coefficients can also be determined using multi particle correlation. In this method we don't need to determine the reaction plane. The simplest method is to use two particle correlation. In order to cancel the dependence of the reaction plane we combine the azimuthal distribution of the two particles

$$\langle \cos[n(\phi_1 - \phi_2)] \rangle = \frac{\int f_2(\boldsymbol{p}_1, \boldsymbol{p}_2) \cos[n(\phi_1 - \phi_2)] d^3p_1 d^3p_2}{\int f_2(\boldsymbol{p}_1, \boldsymbol{p}_2) d^3p_1 d^3p_2}, \tag{1.9}$$

where the function $f_2(\boldsymbol{p}_1, \boldsymbol{p}_2)$ is the probability of finding two particles one with momentum $\boldsymbol{p}_1$ and the other with $\boldsymbol{p}_2$, the two belonging to the same event

$$f_2(\boldsymbol{p}_1, \boldsymbol{p}_2) = f(\boldsymbol{p}_1) f(\boldsymbol{p}_2) + f_c(\boldsymbol{p}_1, \boldsymbol{p}_2). \tag{1.10}$$

The first term to the left represents the uncorrelated part while the second term represents the correlated part. The correlation part would be present even in the absence of the reaction plane, and mainly comes from resonance decay, jets, etc. The correlated part of the two-particle distribution function is suppressed by $1/N_{ev}$ [25] , where N_{ev} is the event multiplicity. Thus the Eqn.(1.9) assumes the form,

$$\langle \cos[n(\phi_1 - \phi_2)] \rangle = v_n^2 + \mathcal{O}\Big(\frac{1}{N_{ev}}\Big). \tag{1.11}$$

The second term is known as the non-flow contribution. Thus the method is a good method to calculate the flow if the condition $v_n \gg 1/\sqrt{N_{ev}}$ is satisfied. For RHIC this is not a good method to calculate the elliptic flow v_2, since v_2 is something around 0.2 while $N_{ev} \sim 100$ [25].

3. Many particle correlation, $v_2\{4\}$, $v_2\{6\}$, etc: By taking correlation on a larger number of particles we can disentangle the non-flow effects in harmonic coefficients. For four particle correlation we have,

$$\langle\langle\cos[n(\phi_1+\phi_2-\phi_3-\phi_4)]\rangle\rangle = \langle\cos[n(\phi_1+\phi_2-\phi_3-\phi_4)]\rangle$$
$$-\langle\cos[n(\phi_1-\phi_3)]\rangle\langle\cos[n(\phi_2-\phi_4)]\rangle - \langle\cos[n(\phi_1-\phi_4)]\rangle\langle\cos[n(\phi_2-\phi_3)]\rangle. \quad (1.12)$$

This expression reduces to,

$$\langle\langle\cos[n(\phi_1+\phi_2-\phi_3-\phi_4)]\rangle\rangle = -v_n^4 + \mathcal{O}\Big(\frac{1}{N_{ev}^3}\Big) + \mathcal{O}\Big(\frac{v_{2n}^2}{N_{ev}^4}\Big). \quad (1.13)$$

Since higher order flow coefficient are much smaller ($v_n \gg v_{2n}$), the condition to suppress non-flow effects is

$$v_n \gg \frac{1}{N_{ev}^{3/4}}. \quad (1.14)$$

This condition is satisfied at RHIC. Using a higher number of particle correlation, the non-flow effects are suppressed more. Since at LHC the energy is greater than RHIC, a greater number of particles are produced, and hence non-flow effects are suppressed more.

Fig. 1.2 shows the differential elliptic flow with four-particle correlation as function of $p_\perp$ for events with $40-50\%$ centrality, calculated using two particle correlation (top plot), and four particle correlation (bottom plot) for LHC ($\sqrt{s_{NN}} = 2.76$TeV) [26] and STAR ($\sqrt{s_{NN}} = 200$GeV). The lower panel is for higher centrality; it is evident that for peripheral collisions the elliptic flow becomes larger. The Fig. 1.3 shows the integrated elliptic flow between $p_\perp \epsilon (0.2, 5.9)$GeV as a function of centrality, for different particle correlation. The elliptic flow at STAR is found to be less than that at LHC; the integrated elliptic flow is larger for collisions with higher collision energy. The full and open markers show respectively the

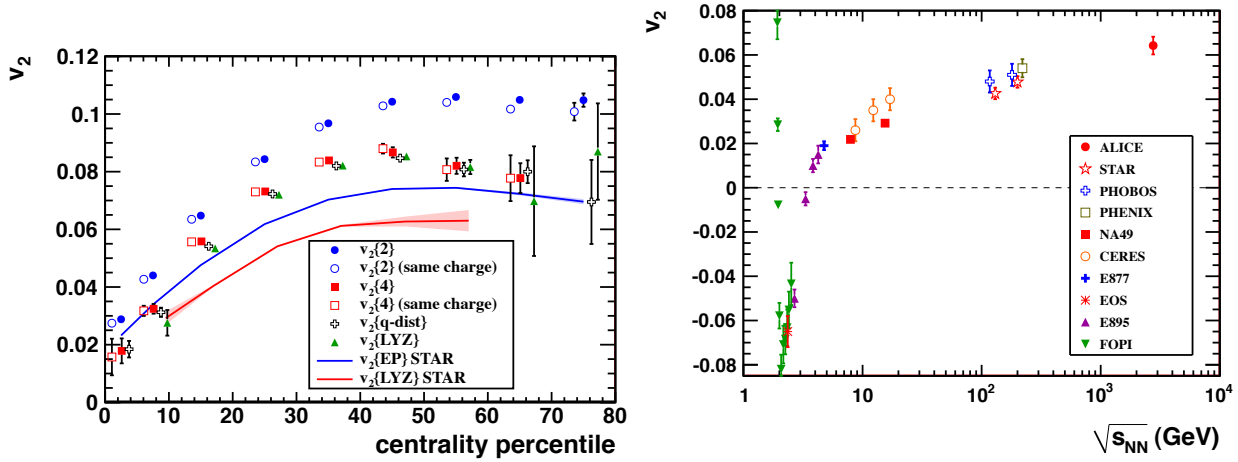

Figure 1.3: Left panel: Integrated elliptic flow v_2 vs centrality. The blue dots and the red dots represents the results for two-particle and four-particle correlation respectively. The STAR results are represented by the blue and red lines. Right panel: Integrated elliptic flow for the centrality class $20\% - 30\%$ as a function of the beam CM energy. Figures taken from [26]. Copyright 2010 by The American Physical Society.

differences when doing the multi-particle correlations among all particles and among particles with the same charge.

1.5 Theoretical methods to treat heavy ion collision

The two main ways to treat the system created in a heavy ion collision are i) covariant transport theory and ii) relativistic hydrodynamics. These processes deal with different episodes in the evolution of the matter created in a heavy ion collision. The transport theory is used to study the early pre-equilibrium and the late hadronic freeze-out state in a heavy ion collision. A Multi-Phase Transport (AMPT) model [27] and Ultra-Relativistic Quantum Molecular Dynamic (URQMD) model [28] are some of the transport based models used to study heavy ion collision.

Between the early pre-equilibrium and the final decoupling stage, exists an extended period over which hydrodynamics is applicable. Here in this thesis, we will be dealing with a part of this period of applicability of hydrodynamics, the hadronic phase.

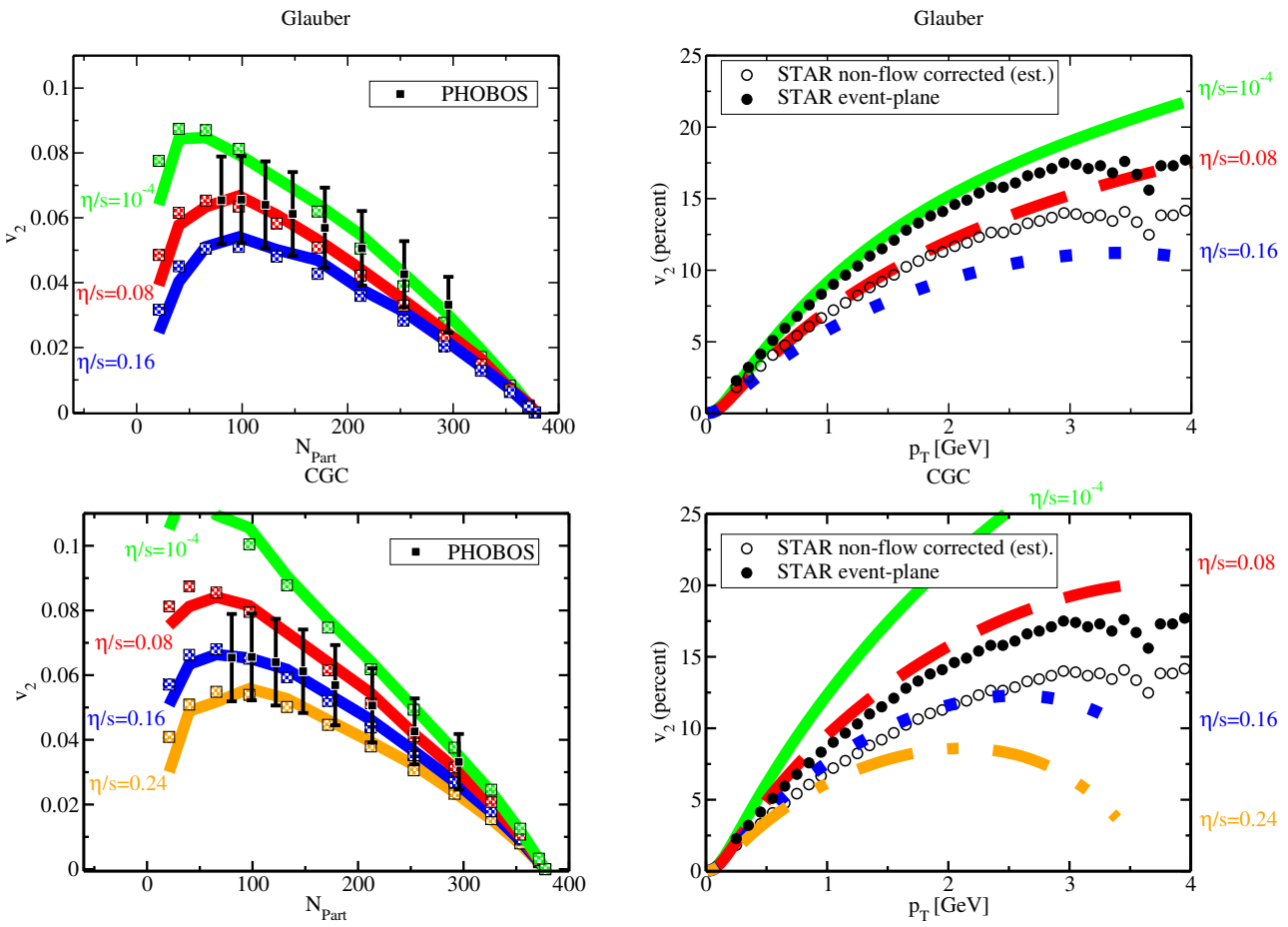

Figure 1.4: Comparison of hydrodynamic models to experimental data on charged hadron integrated (left) and minimum bias (right) elliptic flow by PHOBOS and STAR, respectively. STAR event plane data has been reduced by 20 per cent to estimate the removal of non-flow contributions [30].

1.5.1 Hydrodynamics

The many-body systems that are produced, in a heavy ion collision, is extremely difficult to deal with if we try to analyse the evolution of each particle constituting the system. A very efficient way to bypass this difficulty is to use hydrodynamics to study such a system, where we need a macroscopic description of the system. The macroscopic quantities are an average of some microscopic quantity (energy, momentum, baryon number etc.), overall the constituent particles. Hydrodynamics is only applicable if the magnitudes of these macroscopic quantities

are far greater than their respective fluctuations. For our case, we will need a covariant form of hydrodynamics.

Relativistic hydrodynamics have been quite successful in explaining many collective phenomena in high energy physics. Quite an extensive study has been done considering ideal hydrodynamics, which neglects viscous effects. But the system created in a heavy ion collision undergoes irreversible processes, due to quantum mechanical uncertainty [29] dissipation is always present, the ideal fluid results serve only as a benchmark. For this, we will need relativistic dissipative fluid dynamics. In Fig.1.4 we can see clearly that ideal hydrodynamics ($\eta/s = 10^{-4}$) overpredicts the data, both for STAR and Phobos. Introduction of a significant amount of viscosity brings down the value of elliptic flow v_2 and helps to match it more accurately to experimental data.

The first-order dissipative hydrodynamics known as Navier Stokes theory, formulated mainly by Eckart [31] and Landau-Lifshitz [32], suffers from acausality and numerical instability. The reason for the acausality is the parabolic form of the equations. This difficulty has been dealt with in second-order hydrodynamics Israel Stewart theory; in this theory, the dissipative quantities become independent dynamic variables and obeying equations that describe their relaxation. These equations are hyperbolic in nature and are able to preserve causality.

Hydrodynamic simulation of a relativistic heavy ion collision is done using computational methods. The hydrodynamic equations are solved numerically and are matched to the momentum distribution as observed in the experiments. If ideal hydrodynamics is employed then the input parameters are adjusted in order to match the experimental data for radial flow. The initial energy density of the system is fixed so that the final multiplicity matches with the experimental value. Glauber Model and Colour-Glass Condensate model are the two models mainly used to describe the initial energy density. In the Glauber model, the energy density profile follows the nucleon distribution. However, the Colour Glass Condensate model uses the number density of gluons in binary collision [33, 34]. Colour Glass Condensate model has recently attracted more interest recently because it includes information about QCD.

For dissipative hydrodynamics; viscous coefficients, thermal conductivity go in as input. The shear viscous coefficient normalized by entropy density (η/s) is one of the most important coefficients and is responsible for the property of the fluid. These collective effects generate flow-coefficients. For second order hydrodynamics, which is causal unlike the first-order hydrodynamics (Navier-Stokes) relaxation times of different viscous and thermal flows also go in as input.

1.5.2 Kinetic Theory

Kinetic theory is another way of simplifying the problem of multi-body dynamics. Unlike the previous process here we look from the microscopic point of view. Here instead of studying the evolution of each particle, we take a statistical approach, and the interaction between the particles determine how these statistical distributions evolve with time. Hydrodynamic equations can be derived from kinetic theory, and hence the viscous and thermal coefficients. Kinetic theory approach has been used to estimate the transport coefficients and their respective relaxation time for QGP [35, 36], and hadrons [7].

Here we start from the *Lioüville's* equation for N particles constituting the system. Since the number N is extremely large this equation in this form is completely useless for all practical purposes. So we instead try to find the evolution of the probability distribution function of a single particle, which we find is governed by *Lioüville's* equation with an extra correction term that depends on two-particle distribution function. And similarly, the evolution of the two-particle distribution function will be governed by *Lioüville's* equation with a correction term that depends on three particle distribution function. In this way we get N coupled equations, this is known as BBGKY hierarchy prescribed by Bogoliubov, Born, Green, Kirkwood and Yvon to treat the time evolution of the system. Though this doesn't simplify the problem, it provides a scheme for approximation. The simplest equation obtained from this is the Boltzmann transport equation which governs the evolution of the single particle distribution function. Being an integrodifferential equation this equation is also very difficult to solve. Some of the approximation methods employed to solve this equation will be discussed in this thesis.

1.6 Scope and Organisation of present work

The scattering cross-section that appears in the collision integral is the dynamical input in the kinetic theory approach, and it is highly suggestive that it contains the effect of the hot and dense medium. But in earlier literature viscous coefficient of a hadronic gas mixture has been evaluated using parametrized cross-section extracted from empirical data [7,9,14,38]. In [41,42] quasi-particle model has also been used. In Kinetic theory approach NJL model [43] has also been used to predict the viscous coefficients. Scattering amplitudes evaluated using lowest order chiral perturbation theory have been used in [44, 45], and a unitarized cross-section has been used in [46] using the inverse amplitude method to obtain an estimate of η. In [47], the bulk viscosity of a pion gas, has been computed including number-changing inelastic processes using chiral perturbation theory. In [48] the behaviour of ζ has been demonstrated around the point of phase transition using the linear sigma model.

In the present work, The effect of the medium has been taken into consideration, for a gas mixture constituting of pions and nucleons. Using effective interactions and the techniques of thermal field theory the $\pi\pi$ scattering amplitudes evaluated with self-energy corrected ρ and σ meson propagators in the internal lines cause a significant modification in the cross-section. Similarly, for πN scattering the self-energy of the mediator Δ, at finite temperature and baryon density is obtained, evaluating several one-loop diagrams with π, ρ, N and Δ in the internal lines using standard thermal field theoretic methods. Due to the upcoming CBM experiment at FAIR, nucleons and finite baryon potential have been included. The viscous coefficients, thermal conductivity and diffusion coefficients have been calculated for this system constituting of pions and nucleons, using relaxation time approximation. Compared to the viscosities, the thermal conductivity and diffusion coefficients have received much less attention. This may be due to the absence of a conserved quantum number, the baryon number being insignificantly small for systems produced at RHIC and LHC, however, at FAIR energies or in the Beam Energy Scan (BES) program, the baryon chemical potential is expected to be significant. The relaxation times of flows appearing in second-order hydrodynamics have also been calculated

with in-medium cross-section, for a system constituted of only pions, using Grads 14-moment method.

In Chapter 2, we discuss systems that can be described under a kinetic theory and the hydrodynamic model. Different parameters describing a hydrodynamic system have been defined in terms of the single particle distribution function. Relativistic Boltzmann equation has been derived and using it conservation equations and thus the hydrodynamic equations have been derived. The expression of local equilibrium distribution has been derived.

In Chapter 3, transport coefficients of a single component system for the first-order as well as second-order hydrodynamics have been introduced. Here we have discussed how to obtain the shear viscosity, bulk viscosity and thermal conductivity of a pion gas using Chapman Enskog approximation. The relaxation times of flows for a pion gas have been derived using Grad's 14 Moment method.

In Chapter 4, transport coefficients of a two-component system for first-order hydrodynamics have been introduced. Here we derive the shear viscosity, bulk viscosity, thermal conductivity, thermal diffusibility, Diffusion coefficient and Dufour coefficient of a hadronic gas composed of pions and nucleons. Relaxation time approximation has been employed to solve the transport equation to derive these coefficients.

In Chapter 5, the in-medium cross-section of $\pi\pi$ scattering and πN scattering have been derived using finite temperature field theory. The results have been compared to the analytically derived vacuum cross-section, and experimental data. Finally, in Chapter 6, we summarize our results. Here the different transport coefficients have been studied for different temperature and nucleon chemical potential, for a mixture of pions and nucleons. While relaxation time of flows as a function of temperature and pion chemical potential, have been studied for a system consisting of only pions. Here how the introduction of the medium effects in the cross-section affects these transport coefficients have been studied in full.

In Chapter 7, we conclude the thesis, by summarising the main features and discussions in each chapter. In this chapter, we also discuss some of the ways by which the work presented in this thesis can be improved.

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Chapter 2

Kinetic Theory

2.1 Introduction

Kinetic theory is a way of looking at the macroscopic properties and its evolution with time, of a system of particles from a microscopic view point. It helps us to understand the thermodynamic and or hydrodynamic properties of a system from the basic interaction between the particles.

A system constituting of N (a very large number) of particle contains $3N$ spatial co-ordinates and $3N$ momentum components. Calculating the evolution of these coordinates with time is extremely difficult as the equations defining their evolution are coupled. In order to get over these problem we take a statistical approach. This approach causes a great deal of loss of information, but since the number of macroscopic parameters is far less than the microscopic co-ordinates, this loss of information doesn't pose much of a trouble.

The basic formulation of a system of many particles in kinetic theory is essentially the same, the difference in their macroscopic properties arises from the difference in the basic interaction between the particle and the conserved quantum numbers.

2.1.1 Single particle distribution function

The macroscopic description of a fluid in equilibrium or close to equilibrium is given by the specification of only a few macroscopic parameters namely the energy, pressure, temperature and the number of particles. While on the other hand the microscopic description needs the specification of large number of co-ordinates (i.e. $6N$ coordinates if there are N particles that make up the system), namely the position $\boldsymbol{x}_i(t)$ and the momentum $\boldsymbol{p}_i(t)$ of all the particles. The micro-state thus corresponds to a point $\mu(t)$ in the $6N$- dimensional phase space.

This many-to-one correspondence suggests the introduction of statistical ensemble of micro-state. Consider an ensemble of $\mathcal{N}$ system representing the same macro-state but different micro-state each described by a different representative point $\mu(t)$ in the phase space Γ. Let $d\mathcal{N}(\boldsymbol{x}, \boldsymbol{p}, t)$ be the number of representative points in an infinitesimal phase space volume $d\Gamma = \prod_{i=1}^{N} d^3\boldsymbol{x}_i d^3\boldsymbol{p}_i$ around the point $(\boldsymbol{p}, \boldsymbol{x})$, where $\boldsymbol{x} = \{\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_N\}$ and $\boldsymbol{p} = \{\boldsymbol{p}_1, \boldsymbol{p}_2, ..., \boldsymbol{p}_N\}$ respectively, are the sets comprised of all position and momentum co-ordinates of all the particles. The phase space density is then defined by.

$$\rho(\boldsymbol{p}, \boldsymbol{x}, t) = lim_{\mathcal{N}\to\infty} \frac{d\mathcal{N}(\boldsymbol{p}, \boldsymbol{x}, t)}{\mathcal{N}} \tag{2.1}$$

From the above equation it is evident $\int d\Gamma \rho = 1$, making ρ is a properly normalized probability density. When μ is specified then the system is said to be in the pure state, on the other hand when our knowledge of the system is probabilistic it is said to be in the mixed state. Equilibrium is defined in terms of the mixed state by examining the evolution of phase space density $\rho(t)$.

The full phase space density contains more information than generally needed to determine the macroscopic parameters like pressure, number density, etc. from microscopic consideration. Single particle distribution is sufficient to calculate these macroscopic quantities. Single particle distribution function $f((\boldsymbol{x}, \boldsymbol{p}, t))$ is the probability of finding any of the N particle constituting

the system around the location $\boldsymbol{x}$ with momentum around $\boldsymbol{p}$ at time t, which is given by.

$$\begin{aligned} f((\boldsymbol{x},\boldsymbol{p},t)) &= \Big\langle \sum_{i=1}^{N} \delta^3(\boldsymbol{p}-\boldsymbol{p}_i(t))\delta^3(\boldsymbol{x}-\boldsymbol{x}_i(t)) \Big\rangle \\ &= N \int \prod_{i=2}^{N} d^3\boldsymbol{p}_i d^3\boldsymbol{x}_i \; \rho(\boldsymbol{p}_1=\boldsymbol{p},\boldsymbol{x}_1=\boldsymbol{x},\boldsymbol{p}_2,\boldsymbol{x}_2,.....,\boldsymbol{p}_N,\boldsymbol{x}_N,t) \end{aligned} \tag{2.2}$$

The second identity is obtained by assuming that the density is symmetric with respect to permuting particle (i.e. same for all particles).

It can be shown that the single particle distribution function is a Lorentz scalar. Let us consider a function

$$\mathfrak{N}(x,p) = \frac{1}{p_0}\delta\big(p^0 - \sqrt{\boldsymbol{p}^2+m^2}\big) f(x,p) = 2\theta(p^0)\delta(p^2-m^2)f(x,p) \tag{2.3}$$

Here x and p are the set of all space-time co-ordinates and the four momentum respectively, and $\theta(p^0)$ is a step function. By using Eqn. (2.2) the above function can be written as.

$$\mathfrak{N}(x,p) = \Big\langle \sum_{i=1}^{N} \frac{1}{p_i^0(t)} \; \delta^4(p-p_i(t)) \; \delta^3(\boldsymbol{x}-\boldsymbol{x}_i(t)) \Big\rangle \tag{2.4}$$

Introducing an additional integration we get.

$$\mathfrak{N}(x,p) = \Big\langle \sum_{i=1}^{N} \int dt_i \frac{1}{p_i^0(t_i)} \; \delta(t-t_i) \; \delta^4(p-p_i(t)) \; \delta^3(\boldsymbol{x}-\boldsymbol{x}_i(t)) \Big\rangle \tag{2.5}$$

We then change to the proper time τ_i from time t_i, using relation

$$d\tau_i = \frac{m}{p_i^0(t)} dt_i. \tag{2.6}$$

and $x_i^0 = t_i$ the relation for $\mathfrak{N}(x,p)$ becomes.

$$\mathfrak{N}(x,p) = \frac{1}{m}\Big\langle \sum_{i=1}^{N} \int d\tau_i \; \delta^4(p-\hat{p}_i(\tau)) \; \delta^4(x-\hat{x}_i(\tau)) \Big\rangle \tag{2.7}$$

where $x_i(t) = \hat{x}_i(\tau)$ and $p_i(t) = \hat{p}_i(\tau)$. From the above formula it is evident that $\mathfrak{N}(x,p)$ is a Lorentz scalar as all the terms on the right are scalar quantities. And on account of the Eqn. (2.3), we can see that $f(x,p)$ is also a Lorentz scalar.

2.1.2 Particle four-flow

To describe a non-uniform system we need a local density, a function of space time co-ordinates $n'(\boldsymbol{x}, t)$. Multiplying this quantity with infinitesimal volume $\triangle^3 x$ (i.e. $n'(\boldsymbol{x}, t)\triangle^3 x$) gives the average number of particles in the infinitesimal volume at point $\boldsymbol{x}$ at time t. To describe a non-uniform system we also need a space and time dependent particle flow current, $\boldsymbol{j}(\boldsymbol{x}, t)$. The quantity $\boldsymbol{j}(\boldsymbol{x}, t) \cdot \triangle^2 \boldsymbol{\sigma}$ gives the average number of particles passing through the infinitesimal surface $\triangle^2 \boldsymbol{\sigma}$ at point $\boldsymbol{x}$ per unit time at time t. In a covariant theory these two quantities combines to form a four vector, namely the particle four-flow.

$$N^\mu(x) = \big(n'(\boldsymbol{x}, t), \boldsymbol{j}(\boldsymbol{x}, t)\big) \tag{2.8}$$

The index μ runs from 0 to 3; while $x = x^\mu = (t, \boldsymbol{x})$ denotes time-space point, and $p = p^\mu = (p^0, \boldsymbol{p})$ is the four momenta of a particle. If the particle has mass m and momentum p then the relativistic energy p^0 is given by,

$$p^0 = \sqrt{\boldsymbol{p}^2 + m^2} \tag{2.9}$$

Since the distribution function $f(x, p)$ gives the probability of finding any particle constituting a system around the point $(t, \boldsymbol{x})$ with momentum around $(p^0, \boldsymbol{p})$, the number density and the particle current are given by,

$$n'(x) = \int \frac{d^3p}{(2\pi)^3} f(x, p) \tag{2.10}$$

$$\boldsymbol{j}(x) = \int \frac{d^3p}{(2\pi)^3}\, \boldsymbol{v} f(x, p) \tag{2.11}$$

where $v = \boldsymbol{p}/p^0$ is the relativistic velocity of particle. As a consequence of the above formulas and the fact that, the distribution function and the quantity d^3p/p^0 are Lorentz scalar, while p^μ is a Lorentz vector we have the particle four-flow; a Lorentz vector.

$$N^\mu = \int \frac{d^3p}{(2\pi)^3 p^0}\, p^\mu f(x, p) \tag{2.12}$$

2.1.3 Energy-momentum tensor

The energy momentum density and their currents will also be a function of space and time just like other thermodynamic quantities of a non-uniform system. The energy and momentum density are given by;

$$n'e' = T^{00}(x) = \int \frac{d^3p}{(2\pi)^3}\ p^0 f(x,p) \tag{2.13}$$

$$T^{i0}(x) = \int \frac{d^3p}{(2\pi)^3}\ p^i f(x,p), \qquad i = 1,2,3. \tag{2.14}$$

where the quantity e' is the average energy per particle, and T^{i0} is the i^{th} component of momentum density. Just like particle flow, the energy and momentum flow are given by.

$$T^{0i}(x) = \int \frac{d^3p}{(2\pi)^3}\ p^0\ v^i f(x,p) = \int \frac{d^3p}{(2\pi)^3 p^0}\ p^0 p^i f(x,p), \quad i = 1,2,3 \tag{2.15}$$

$$T^{ij}(x) = \int \frac{d^3p}{(2\pi)^3}\ p^i\ v^j f(x,p) = \int \frac{d^3p}{(2\pi)^3 p^0}\ p^i p^j f(x,p), \quad i,j = 1,2,3 \tag{2.16}$$

Where $v^i = p^i/p^0$. Here T^{0i} is the average energy flow in the i^{th} direction: and T^{ij} is the flow of the i^{th}component of average momentum of a fluid cell in the j^{th} direction. The formulas (2.13 - 2.16) may be written in a compact and covariant form

$$T^{\mu\nu} = \int \frac{d^3p}{(2\pi)^3 p^0}\ p^\mu p^\nu f(x,p) \tag{2.17}$$

where $\mu, \nu = 0,1,2,3$. Thus the energy-momentum density and their flows, form a symmetric second rank tensor, known as energy-momentum tensor. It is to be noted, that the energy-momentum tensor takes only the rest energy and the particle energy into account, we have assumed that the system is dilute and the interaction energy of the particles is small compared to their kinetic energy.

2.1.4 Entropy four-flow

The local entropy density is given by

$$S^0(x) = -\int \frac{d^3p}{(2\pi)^3} f(x,p)\big[\log f(x,p) - 1\big], \tag{2.18}$$

and the entropy flow by,

$$\boldsymbol{S}(x) = -\int \frac{d^3p}{(2\pi)^3} \boldsymbol{v} f(x,p)\big[\log f(x,p) - 1\big]. \tag{2.19}$$

Thus the entropy four flow can be defined as.

$$S^\mu(x) = -\int \frac{d^3p}{(2\pi)^3 p^0} p^\mu f(x,p)\big[\log f(x,p) - 1\big]. \tag{2.20}$$

2.1.5 Hydrodynamic four-velocity

A fluid in motion is characterized by the vector field $U^\mu(x)$, the hydrodynamic four-velocity. It is the velocity of an infinitesimal fluid cell at space time point $(t, \boldsymbol{x})$. The vector just like any other velocity vector in relativistic mechanics is time like, with unit length

$$U^\mu(x) U_\mu(x) = 1 \tag{2.21}$$

hence the derivative of its length with respect to any of the space time co-ordinate, $\partial_\nu = \partial/\partial x^\nu$ yields zero.

$$U^\mu(x) \partial_\nu U_\mu(x) = 0 \tag{2.22}$$

The hydrodynamic velocity is a bit arbitrary, as will be discussed at the end of this section. So, choice of a particular hydrodynamic velocity sets the rest frame of a fluid cell, and so the hydrodynamic velocity is used to evaluate the particular value of any tensor or vector quantity in the local rest frame (i.e the rest frame of the fluid cell at the space-time point under consideration). To do this two projection operator has been defined, U^μ and $\triangle^{\mu\nu}$.

$$\triangle^{\mu\nu} = g^{\mu\nu} - U^\mu U^\nu \tag{2.23}$$

where $g^{\mu\nu} = diag(1, -1 - 1 - 1)$ is the metric tensor. Since in the local rest frame the hydrodynamic velocity is parallel to the time co-ordinate, $U^{\mu} = (1, 0, 0, 0)$; contracting it with any vector will give the component of the time component of the vector in the local rest frame (indicated by index **LR**). The operator $\triangle^{\mu\nu}$ annihilates the part parallel to the hydrodynamic velocity, hence contracted with any vector will give the spatial component of the vector in the local rest frame.

$$\triangle_{\boldsymbol{LR}\mu\nu} = \triangle^{\boldsymbol{LR}}_{\mu\nu} = diag(0, -1, -1, -1), \qquad \triangle^{\mu}_{\boldsymbol{LR}\ \nu} = diag(0, 1, 1, 1) \tag{2.24}$$

The velocity of a fluid cell, is essentially the average velocity of the $\mathcal{N}$ particles constituting the fluid cell. Depending on how the velocity of these particles are weighted, gives rise to different definitions of velocity. In literature two choice are quite popular.

Eckart's definition

In this definition of hydrodynamic velocity, the velocity of each particle is given a weightage of one.

$$U^{\mu} = \frac{\sum_{i=1}^{\mathcal{N}} v_i^{\mu}}{\sum_{i=1}^{\mathcal{N}} i} \tag{2.25}$$

where $v_i^{\mu} = p_i^{\mu}/p_i^0$ is the four velocity of the i^{th} particle in infinitesimal fluid cell at x, and $\mathcal{N}$ is the total number of particles in the fluid cell. This definition makes the hydrodynamic velocity parallel to the particle four-flow, and since U^{μ} has unit length, we can express it as

$$U^{\mu} = \frac{N^{\mu}}{\sqrt{N^{\nu}N_{\nu}}} = \frac{N^{\mu}}{N^{\nu}U_{\nu}} \tag{2.26}$$

Now since we have chosen the hydrodynamic velocity to be parallel to N^{μ} projecting N^{μ} perpendicular to U^{μ} will give us zero.

$$\triangle^{\mu\nu}N_{\nu} = 0 \tag{2.27}$$

Thus by adopting Eckart's definition of hydrodynamic four-velocity, in the local rest frame the spatial component of N^μ vanishes.

$$N^i_{\boldsymbol{LR}} = 0, \qquad i = 1, 2, 3 \tag{2.28}$$

Landau and Lifshitz's definition

In this case the velocity of individual particles are given a weightage equal to the relativistic mass of the particles.

$$U^\mu = \frac{\sum_{i=1}^{\mathcal{N}} m'_i \; v_i^\mu}{\sum_{i=1}^{\mathcal{N}} m'_i} \tag{2.29}$$

the relativistic mass, $m'_i = m_i/\sqrt{1 - \boldsymbol{v}_i^2}$; where m_i is the rest mass of the particle. This makes the hydrodynamic four velocity parallel to the momentum density or equivalently the energy flow at any space time point inside the fluid. Thus, the hydrodynamic velocity can be expressed as

$$U^\mu = \frac{T^{\mu\nu}U_\nu}{\sqrt{U_\rho T^{\rho\sigma} T_{\sigma\alpha} U^\alpha}} = \frac{T^{\mu\nu}U_\nu}{U_\rho T^{\rho\sigma} U_\sigma} \tag{2.30}$$

Under this definition, applying the projector operator $\triangle^{\mu\nu}$ on the above equation we get,

$$\triangle^{\mu\nu} T_{\nu\sigma} U^\sigma = 0 \tag{2.31}$$

Thus in the local rest frame under this definition of hydrodynamic velocity, the momentum density and the energy current vanishes.

$$T^{0i}_{\boldsymbol{LR}} = T^{i0}_{\boldsymbol{LR}} = 0, \qquad i = 1, 2, 3 \tag{2.32}$$

2.1.6 Physical quantities in local rest frame

Once a definition of the hydrodynamic velocity have been chosen, one may define the relevant thermodynamic quantities relative to the local rest frame,like particle density, energy density, heat flow , pressure tensor and the entropy density. The definitions as mentioned here are quite general, the exact form of these quantities in Eckart's definition and Landau's definition can be obtained by using Eqn.(2.27) and Eqn.(2.31) respectively.

The particle density $n(x)$ is defined as the total number of particles in the local rest frame.

$$n(x) = n'_{\boldsymbol{LR}}(x) = N^{\mu}U_{\mu} = N^{0}_{\boldsymbol{LR}} \tag{2.33}$$

where $n'(x)$ and N^{μ} have been defined in Eqn. (2.10) and Eqn. (2.12). similarly the scaler energy density is the energy density in the local rest frame.

$$e(x)n(x) = e'_{\boldsymbol{LR}}(x)n'_{\boldsymbol{LR}}(x) = U_{\mu}T^{\mu\nu}U_{\nu} \tag{2.34}$$

where e is the average energy per particle in the fluid cell at x, in the local rest frame. The heat flow has been defined as the difference of energy flow and the enthalpy flow in the local rest frame.

$$I^{\mu}_{q} = (U_{\nu}T^{\nu\sigma} - hN^{\sigma})\triangle^{\mu}_{\sigma} \tag{2.35}$$

where $h = e + pn^{-1}$ is the enthalpy per particle, and p the hydrostatic pressure in the local rest frame. The pressure tensor is given by.

$$P^{\mu\nu} = \triangle^{\mu}_{\sigma}\, T^{\sigma\tau}\triangle^{\nu}_{\tau} \tag{2.36}$$

the pressure tensor has a reversible part and an irreversible part.

$$P^{\mu\nu} = -p\triangle^{\mu\nu} + \Pi^{\mu\nu} \tag{2.37}$$

The p is the hydrostatic pressure (reversible part) and $\Pi^{\mu\nu}$ is the viscous-pressure tensor (irreversible part). Using the above quantities in the local rest frame the entire energy-momentum

can be decomposed into three important parts.

$$U_\mu T^{\mu\nu} U_\nu = en \tag{2.38}$$

$$U_\nu T^{\nu\sigma} \triangle^\mu_\sigma = I^\mu_q + h \triangle^{\mu\nu} N_\nu \tag{2.39}$$

$$\triangle^\mu_\sigma T^{\sigma\tau} \triangle^\nu_\sigma = -p \triangle^{\mu\nu} + \Pi^{\mu\nu} \tag{2.40}$$

The energy-momentum tensor can also be decomposed into the reversible and irreversible part.

$$T^{\mu\nu} = T^{(0)\mu\nu} + T^{(1)\mu\nu} \tag{2.41}$$

with the reversible $T^{(0)\mu\nu}$, and irreversible $T^{(1)\mu\nu}$ contribution given as

$$T^{(0)\mu\nu} = enU^\mu U^\nu - p \triangle^{\mu\nu} \tag{2.42}$$

$$T^{(1)\mu\nu} = \left[(I^\mu_q + h \triangle^{\mu\sigma} N_\sigma) U^\nu + (I^\nu_q + h \triangle^{\nu\sigma} N_\sigma) U^\mu \right] + \Pi^{\mu\nu} \tag{2.43}$$

The particle four flow can also be split up into two parts.

$$N^\mu = nU^\mu + \triangle^{\mu\nu} N_\nu = nU^\mu + V^\mu \tag{2.44}$$

These forms will play an important role in the derivation of macroscopic laws.

2.1.7 Mixture

The formulas mentioned above are those for a system consisting of particles of a single type, they can be quite easily generalized to fit situations where there are more than one type of particles (i.e. mixture). Let the fluid mixture be composed of $\mathcal{N}$ types of particles. The four-flow for each particle species can be represented as,

$$N^\mu_k = \int \frac{d^3 p_k}{(2\pi)^3 p^0_k} \, p^\mu_k f_k(x, p_k), \qquad k = 1, 2, 3..., \mathcal{N} \tag{2.45}$$

where k specifies the species, of the particle. And $f_k(x, p_k)$ is the distribution function of the k^{th} specie, and the four-momentum p_k has length m_k; where m_k is the particle rest mass of the

particular specie. The total particle flow is given by.

$$N^{\mu} = \sum_{k=1}^{\mathcal{N}} N_k^{\mu}(x) \tag{2.46}$$

The total mass flow and mass flow of the k species are

$$M^{\mu}(x) = \sum_{k=1}^{\mathcal{N}} M_k^{\mu}(x) \tag{2.47}$$

and

$$M_k^{\mu}(x) = m_k N_k^{\mu} \tag{2.48}$$

respectively. Similarly the total energy-momentum tensor will be the sum of energy momentum tensor for each specie.

$$T^{\mu\nu}(x) = \sum_{k=1}^{\mathcal{N}} T_k^{\mu\nu}(x) = \sum_{k=1}^{\mathcal{N}} \int \frac{d^3p_k}{(2\pi)^3 p_k^0}\, p_k^{\mu} p_k^{\nu} f_k(x, p_k) \tag{2.49}$$

The particle density n_k of species k in the local rest frame, and its relation to the total particle density are give by,

$$n_k = N_k^{\mu} U_{\mu}, \qquad k = 1, 2, ..., \mathcal{N} \tag{2.50}$$

and

$$n = N^{\mu} U_{\mu} = \sum_{k=1}^{\mathcal{N}} N_k^{\mu} U_k = \sum_{k=1}^{\mathcal{N}} n_k \tag{2.51}$$

respectively. the concentration x_k of the k component, is defined as,

$$x_k = n_k/n, \qquad k = 1, 2, ..., \mathcal{N} \tag{2.52}$$

and the diffusion flow of k component is the difference of the four-flow of the k component and concentration of the k component times the total four-flow.

$$I_k^{\mu} = N_k^{\mu} - x_k N^{\mu}, \qquad k = 1, 2, ..., \mathcal{N} \tag{2.53}$$

From the above definition it is evident that the sum of concentration is unity, and the sum of diffusion flow vanishes.

$$\sum_{k=1}^{\mathcal{N}} x_k = 1 \tag{2.54}$$

$$\sum_{k=1}^{\mathcal{N}} I_k^{\mu} = 0 \tag{2.55}$$

2.2 Relativistic Boltzmann Transport Equation

The aim of Boltzmann equation is to study the evolution of the single particle distribution with time. Since the single particle distribution function is enough to determine the necessary macroscopic quantity, needed for the complete description of a fluid, we can follow the evolution of a fluid in its path to global equilibrium once the initial condition are known.

The Boltzmann transport equation is a non-linear integro-differential equation. To solve this equation, various approximation methods have been developed, a few of them will be discussed in the next chapter. The application of Boltzmann transport equation is limited to only those systems which satisfy certain assumptions that are used during its derivation. Here we will derive the Boltzmann's equation for a simple system consisting of only one type of particle, then we will generalize it to describe mixtures.

2.2.1 Assumptions of Boltzmann's equation

The assumptions necessary for the derivation are same as that required for non-relativistic case. These are:

1. The medium is rarefied hence chances of more than two particle collision (interaction) is extremely low, and so not taken into consideration.

2. Hypothesis of molecular chaos: the velocities or the momentum of the colliding particle are uncorrelated. The number of binary collision will be proportional to the distribution

function of the colliding particles. For quantum statistics the collision rate will also depend on the states of the particles after collision (Bose enhancement and Pauli blocking).

3. The single particle distribution function varies gradually over space and time. The change of distribution function will be significant over a length of space and time which is far greater than the characteristic interaction length and time.

2.2.2 Boltzmann's equation

Using the particle four-vector we can construct the scalar quantity $\triangle N$

$$\triangle N(x) = \int_{\triangle^3\sigma} d^3\sigma_\mu N^\mu(x) = \iint_{\triangle^3\sigma} d^3\sigma_\mu \frac{d^3p}{(2\pi)^3 p^0} p^\mu f(x,p) \tag{2.56}$$

where $\triangle^3\sigma_\mu$ is the time-like area four-vector of an small segment of a space-like three dimensional hyperplane σ, situated at x. In the frame where the $d^3\sigma$ is completely time-like, it has components $(d^3x, 0, 0, 0)$. In this frame the expression gets the form.

$$\triangle N(x) = \iint_{\triangle^3 x} d^3x \frac{d^3p}{(2\pi)^3} f(x,p) \tag{2.57}$$

The quantity mentioned in the above equation is just the number of particles in the volume $\triangle^3 x$, or the number of world lines crossing the segment $\triangle^3\sigma$. Thus the number of world-lines crossing the segment $\triangle^3\sigma$ and having momentum around p in the range $\triangle^3 p$, is given by,

$$\triangle N(x) = \int_{\triangle^3 x} \int_{\triangle^3 p} d^3x \frac{d^3p}{(2\pi)^3} f(x,p). \tag{2.58}$$

After some time the world line that crosses segment $\triangle^3\sigma$ will cross another segment $\triangle^3\hat{\sigma}$. If the particles do not collide or interact with each other then the momentum of each particle will remain unchanged (i.e. around p in the range $\triangle^3 p$). Thus

$$\int_{\triangle^3\hat{\sigma}} \int_{\triangle^3 p} d^3\sigma_\mu \frac{d^3p}{(2\pi)^3 p^0} p^\mu f(x,p) - \int_{\triangle^3\sigma} \int_{\triangle^3 p} d^3\sigma_\mu \frac{d^3p}{(2\pi)^3 p^0} p^\mu f(x,p) = 0 \tag{2.59}$$

Consider the tube like surface that connects the two segments $\triangle^3\hat{\sigma}$ and$\triangle^3\sigma$, no world line crosses this surface since we are consider a case without collision. These three surface forms a

closed surface $\triangle^3\sigma'$; thus the above equation can be expressed as,

$$\oint_{\triangle^3\sigma'} \int_{\triangle^3 p} d^3\sigma_\mu \frac{d^3p}{(2\pi)^3 p^0} p^\mu f(x,p) = 0 \tag{2.60}$$

Using Gauss's theorem we get;

$$\int_{\triangle^4 x} \int_{\triangle^3 p} d^3\sigma_\mu \frac{d^3p}{(2\pi)^3 p^0} p^\mu \ \partial_\mu f(x,p) = 0 \tag{2.61}$$

where $\partial_\mu = \partial/\partial x^\mu = (\partial_t, \nabla)$. Since the volume $\triangle^4 x$ and the momentum range $\triangle^3 p$ are arbitrary, so it follows that

$$p^\mu \partial_\mu f(x,p) = 0, \tag{2.62}$$

this is the Boltzmann equation for collision less case.

Now if there are collisions between the particles, then the number of particle with momentum around p that crosses the segment $\triangle^3\sigma$ is not necessarily same as the number of particles that crosses the segment $\triangle^3\hat{\sigma}$ with the same momentum. The change in number is given by,

$$\triangle^4 x \frac{\triangle^3 p}{(2\pi)^3 p^0} C(x,p). \tag{2.63}$$

Let us consider a collision of the type $p + p_1 \to p' + p'_1$, the number of collision in Minkowski-space element $\triangle^4 x$ around x as stated in assumption (2) of the previous section is directly proportional to:

1. the average density of the colliding particles (i.e. $f(x,p)\triangle^3 p/(2\pi)^3$ and $f(x,p_1)\triangle^3 p_1/(2\pi)^3$),

2. the interval $\triangle^3 p'/(2\pi)^3$, $\triangle^3 p'_1(2\pi)^3$ and $\triangle^4 x$. For quantum statistics $\triangle^3 p'/(2\pi)^3$ and $\triangle^3 p'_1/(2\pi)^3$ will be replaced with $(1 \pm f(x,p'))\triangle^3 p'/(2\pi)^3$ and $(1 \pm f(x,p'_1))\triangle^3 p'_1/(2\pi)^3$ respectively, because of Bose enhancement and Pauli blocking.

The proportionality factor will be denoted by $W(p,p_1|p',p'_1)/p^0 p_1^0 p'^0 p_1'^0$. The quantity $W(p,p_1|p',p'_1)$ known as transition rate is a Lorentz scalar depending only on the four momentum.

The particle with momentum p changes to a state with a different momentum because of collision of the form $p + p_1 \rightarrow p' + p'_1$; where p_1, p' and p'_1 can acquire any value that is permissible. Hence the number of particles with momentum around p in the range $\triangle^3 p$ lost due to collision is given by,

$$\frac{1}{2}\triangle^4 x \frac{\triangle^3 p}{(2\pi)^3 p^0} \int \frac{d^3 p_1}{(2\pi)^3 p_1^0} \frac{d^3 p'}{(2\pi)^3 p'^0} \frac{d^3 p'_1}{(2\pi)^3 p_1'^0} \; f(x,p) f(x,p_1) W(p, p_1 | p', p'_1) \tag{2.64}$$

and for quantum statistics,

$$\frac{1}{2}\triangle^4 x \frac{\triangle^3 p}{(2\pi)^3 p^0} \int \frac{d^3 p_1}{(2\pi)^3 p_1^0} \frac{d^3 p'}{(2\pi)^3 p'^0} \frac{d^3 p'_1}{(2\pi)^3 p_1'^0} \; f(x,p) f(x,p_1)[1 \pm f(x,p')] \tag{2.65}$$
$$[1 \pm f(x,p'_1)] W(p, p_1 | p', p'_1).$$

Similarly the gain of particle will be due to collision of the type $p' + p'_1 \rightarrow p + p_1$. And the number of collision that will cause the gain of particle with momentum p for classical and quantum statistics are given by,

$$\frac{1}{2}\triangle^4 x \frac{\triangle^3 p}{(2\pi)^3 p^0} \int \frac{d^3 p_1}{(2\pi)^3 p_1^0} \frac{d^3 p'}{(2\pi)^3 p'^0} \frac{d^3 p'_1}{(2\pi)^3 p_1'^0} \; f(x,p') f(x,p'_1) W(p', p'_1 | p, p_1) \tag{2.66}$$

and

$$\frac{1}{2}\triangle^4 x \frac{\triangle^3 p}{(2\pi)^3 p^0} \int \frac{d^3 p_1}{(2\pi)^3 p_1^0} \frac{d^3 p'}{(2\pi)^3 p'^0} \frac{d^3 p'_1}{(2\pi)^3 p_1'^0} \; f(x,p') f(x,p'_1)[1 \pm f(x,p)] \tag{2.67}$$
$$[1 \pm f(x,p_1)] W(p', p'_1 | p, p_1).$$

respectively. Thus the net change of particles in the interval $\triangle^4 x$ and $\triangle^3 p$ is the amount (2.66) minus amount (2.64), or amount (2.68) minus amount (2.66) depending on the statistics. Using the theory of detailed balance the collision function $C(x,p)$ thus has the form;

$$C(x,p) = \frac{1}{2} \int \frac{d^3 p_1}{(2\pi)^3 p_1^0} \frac{d^3 p'}{(2\pi)^3 p'^0} \frac{d^3 p'_1}{(2\pi)^3 p_1'^0} [f(x,p') f(x,p'_1) \tag{2.68}$$
$$- f(x,p) f(x,p_1)] W(p, p_1 | p', p'_1)$$

or,

$$C(x,p) = \frac{1}{2}\int \frac{d^3p_1}{(2\pi)^3 p_1^0}\frac{d^3p'}{(2\pi)^3 p'^0}\frac{d^3p_1'}{(2\pi)^3 p_1'^0}\{f(x,p')f(x,p_1')[1\pm f(x,p)][1\pm f(x,p_1)] \\ -f(x,p)f(x,p_1)[1\pm f(x,p')][1\pm f(x,p_1')]\}W(p,p_1|p',p_1') \tag{2.69}$$

depending on the type of statistics. The distribution function and the transition rate will be determined by the underlying dynamics of the system. Thus taking into account the collision between the constituent particles the Eqn. (2.61) becomes,

$$\int_{\triangle^4 x}\int_{\triangle^3 p} d^3\sigma_\mu \frac{d^3p}{(2\pi)^3 p^0} p^\mu \ \partial_\mu f(x,p) = \triangle^4 x \frac{\triangle^3 p}{(2\pi)^3 p^0} C(x,p). \tag{2.70}$$

And since we know that the intervals $\triangle^4 x$ and $\triangle^3 p$ are arbitrary, the above equation can be written as,

$$p^\mu \partial_\mu f(x,p) = C(x,p). \tag{2.71}$$

This is the single particle Boltzmann transport equation. The Boltzmann equation for single particle can be readily generalized to a mixture containing $\mathcal{N}$ species of particles. The Boltzmann equation for the k^{th} specie is given by,

$$p_k^\mu \partial_\mu f_k(x,p_k) = \sum_{l=1}^{\mathcal{N}} \frac{g_l}{1+\delta_{kl}}\ C_{kl}(x,p_k). \tag{2.72}$$

where g_l is the l^{th} species and $C_{kl}(x,p)$ is the collision term for elastic collision (i.e. collision of the form $k+l \to k+l$) is given by,

$$C_{kl}(x,p) = \int \frac{d^3p_l}{(2\pi)^3 p_l^0}\frac{d^3p_k'}{(2\pi)^3 p_k'^0}\frac{d^3p_l'}{(2\pi)^3 p_l'^0}[f_k' f_l' - f_k f_l] \\ W(p_k,p_k|p_k',p_l'). \quad k,l = 1,2,...,\mathcal{N} \tag{2.73}$$

and

$$C_{kl}(x,p) = \int \frac{d^3p_l}{(2\pi)^3 p_l^0}\frac{d^3p_k'}{(2\pi)^3 p_k'^0}\frac{d^3p_l'}{(2\pi)^3 p_l'^0}[f_k' f_l'(1\pm f_k)(1\pm f_l) \\ -f_k f_l(1\pm f_k')(1\pm f_l)]W(p_k,p_k|p_k',p_l'). \qquad k,l = 1,2,...,\mathcal{N} \tag{2.74}$$

depending on whether statistics is classical (Maxwell Boltzmann) or quantum (Fermi-Dirac or Bose-Einstein). The quantities f_k, f_l, f'_k and f'_l are $f_k(x, p_k)$, $f_k(x, p_l)$, $f_k(x, p'_k)$, and $f_k(x, p'_l)$ respectively. The quantity $\gamma_{kl} = 1 - \frac{1}{2}\delta_{kl}$ ensures that the above expression are true for both identical particle ($k = l$) as well as different species. Similarly for a reactive mixture the collision term is given by,

$$C_{kl} = \frac{1}{2}\sum_{i,j=1}^{\mathcal{N}} \int \frac{d^3p_l}{(2\pi)^3 p_l^0} \frac{d^3p'_i}{(2\pi)^3 p_i'^0} \frac{d^3p'_j}{(2\pi)^3 p_j'^0} (f_i f_j W_{ij|kl} - f_k f_l W_{kl|ij}), \qquad k, l = 1, 2, ..\mathcal{N}. \quad (2.75)$$

We have restricted ourselves to classical (Maxwell Boltzmann) statistics and reactions in which particle number is conserved (i.e. reactions of the type $k + l \rightarrow i + j$). The collision term for quantum statistics can be derived easily by incorporating the Bose enhancement or Pauli blocking

2.3 Conservation Equations

The macroscopic quantities that are used to give a macroscopic description of a system follow several conservation laws. These conservation laws have to be postulated in a macroscopic theory, but from the stand point of microscopic theory such as Kinetic theory these laws can de derived using more basic laws.

Here for the sake of convenience we will use classical statistics to derive the conservation laws. For cases where quantum statistics is applicable the same laws can be derived following similar steps that are followed here, all we have to do is choose the appropriate collision term in the Boltzmann equation.

Let us consider a reactive mixture made up of $\mathcal{N}$ species of particles. A reactive mixture is one were inelastic collisions take place along with elastic collisions, between the constituent particles. An inelastic collision

$$k + l \rightarrow i + j + ... \quad (2.76)$$

is a reaction in which two or more particles of which, at least one is of a different species than the colliding particle are produced. In the above reaction particle of species k and l transforms into particles of species i,j,... . Here we restrict ourselves to only those reactions, in which the total number of particle remains the same. The distribution function $f_k = f_k(x, p_k),\ \ k = 1, 2, .., \mathcal{N}$ of each particle evolve according to the Eqn. (2.72) and the collision term is given by Eqn.(2.75). Then the following equation holds true (The proof is in Appendix-A).

$$\sum_{k,l=1}^{\mathcal{N}} \int \frac{d^3 p_k}{p_k^0} \psi_k(x, p_k) C_{kl}(x, p_k) = 0 \tag{2.77}$$

Where the variable ψ_k is given by,

$$\psi_k(x, p_k) = a_k(x) + b_\mu(x) p_k^\mu \tag{2.78}$$

where the functions $a_k(x)$ and $b_\mu(x)$ are arbitrary except for the constrain that during the reaction $(k + l \rightarrow i + j)$ the following relation should hold true.

$$a_k(x) + a_l(x) = a_i(x) + a_j(x) \tag{2.79}$$

For a non-reactive mixture it can be shown (Appendix-A),

$$\int \frac{d^3 p_k}{p_k^0} \psi_k(x, p_k) C_{kl}(x, p_k) = 0 \tag{2.80}$$

Using Eqn.(2.77) and Eqn.(2.80) the conservation equations can be derived.

2.3.1 Conservation of number

Now let us take $b^\mu(x)$ equal to zero and $a_k(x)$ equal to one for all particles. Then Eqn.(2.77) becomes.

$$\sum_{k,l=1}^{\mathcal{N}} \int \frac{d^3 p_k}{p_k^0} C_{kl}(x, p_k) = 0 \tag{2.81}$$

Now if we replace $\sum_{l=1}^{\mathcal{N}} C_{kl}(x, p_k)$ using Eqn.(2.72) we get,

$$\sum_{k=1}^{\mathcal{N}} \int \frac{d^3 p_k}{p_k^0} p_k^\mu \partial_\mu f_k(x, p_k) = \partial_\mu \sum_{k=1}^{\mathcal{N}} \int \frac{d^3 p_k}{p_k^0} p_k^\mu f_k(x, p_k) = 0. \tag{2.82}$$

Using Eqn.(2.45) and Eqn.(2.46) the above expressions can be reduced to,

$$\partial_\mu \sum_{k=1}^{\mathcal{N}} N_k^\mu(x) = \partial_\mu N^\mu(x) = 0 \tag{2.83}$$

This equation expresses that the total number of particle is conserved. For non-reactive mixture using Eqn.(2.80) it can be shown than number of particle of each species is conserved.

$$\partial_\mu N_k^\mu(x) = 0, \qquad k = 1, 2, ..., \mathcal{N} \tag{2.84}$$

2.3.2 Conservation of energy-momentum

Now taking $a_k(x)$ equal to zero and we find from Eqn.(2.77) ,

$$\sum_{k,l=1}^{\mathcal{N}} \int \frac{d^3 p_k}{p_k^0} p_k^\mu C_{kl}(x, p_k) = 0. \tag{2.85}$$

Again replacing $\sum_{l=1}^{\mathcal{N}} C_{kl}(x, p_k)$ with the left side of the transport equation we get,

$$\partial_\mu \sum_{k=1}^{\mathcal{N}} \int \frac{d^3 p_k}{p_k^0} p_k^\mu p_k^\nu f_k(x, p_k) = \partial_\mu T^{\mu\nu} = 0. \tag{2.86}$$

For $\mu = 0$ the above equation expresses the law of conservation of energy, and for $\mu = 1, 2, 3$ conservation of momentum.

2.4 Equations of fluid dynamics

Using the conservation equations derived in the previous section we can derive the fluid dynamic equations, the time and spatial evolution of the macroscopic quantities, namely the particle

density, the hydrodynamic four velocity and the energy density. Since the macroscopic quantities are defined in the local rest frame the resulting equations will depend on the choice of the rest frame, hence the definition of the hydrodynamic velocity used.

2.4.1 Time derivative and gradient

The projection operators helps to decompose the time-space derivative into a time-like and a space like part.

$$\partial^\mu = U^\mu U^\nu \partial_\nu + \big(g^{\mu\nu} - U^\mu U^\nu\big)\partial_\nu = U^\mu D + \nabla^\mu. \tag{2.87}$$

where,

$$D = U^\nu \partial_\nu, \qquad\qquad \nabla^\mu = \triangle^{\mu\nu}\partial_\nu. \tag{2.88}$$

The convective time derivative D in the local rest frame represents a pure time derivative.

$$D_{\boldsymbol{LR}} = \frac{\partial}{\partial t} \tag{2.89}$$

Similarly the ∇^μ is purely spatial in the local rest frame,

$$\nabla^0_{\boldsymbol{LR}} = 0, \qquad\qquad \nabla^i_{\boldsymbol{LR}} = -\frac{\partial}{\partial x^i}, \quad i = 1, 2, 3 \tag{2.90}$$

On contracting the time-space derivative with the particle four-flow we get what is known as the substantial time derivative.

$$\mathscr{D} = N^\mu \partial_\mu = nD + N^\mu \nabla_\mu \tag{2.91}$$

It is n times the time derivative in the frame in which the spatial part of the hydrodynamic four-flow vanishes (rest frame as according to Eckart's definition).

2.4.2 Continuty equation

The total particle four-flow using the projection operators can be split into two parts, one parallel to the hydrodynamic velocity and the other orthogonal to it.

$$N^{\mu} = nU^{\mu} + V^{\mu}, \qquad V^{\mu} = \triangle^{\mu\nu} N_{\nu} \tag{2.92}$$

Where n is the particle density as defined earlier. Since in Eckart's definition the hydrodynamic velocity is parallel to the particle four-flow the quantity V^{μ} is zero; in Landau's definition $V^{\mu} = -I_k^{\mu}$. Replacing the above expression in Eqn.(2.83) we get the equation of continuity.

$$Dn = -n\nabla_{\mu} U^{\mu} - \nabla_{\mu} V^{\mu} + v_{\mu} DU^{\mu} \tag{2.93}$$

Similarly for a mixture the particle four-flow for the k^{th} specie can be expressed as,

$$N_k^{\mu} = n_k U^{\mu} + V_k^{\mu}, \qquad k = 1, 2, ..., \mathcal{N} \tag{2.94}$$

Using Eqn.(2.84), Eqn.(2.53) and the above equation we get the continuity equation for a non-reactive mixture.

$$nDx_k = -\nabla_{\mu} I_k^{\mu} - V^{\mu} \nabla_{\mu} x_k + I_k^{\mu} DU_{\mu} \tag{2.95}$$

Depending on the definition of Hydrodynamic four-flow we choose the value of V^{μ} and replace them in the continuity equations. On neglecting the transport quantities V^{μ} and I_k^{μ} we get the zeroth order equation so called Euler type.

$$Dn = -n\nabla_{\mu} U^{\mu} \tag{2.96}$$

and

$$Dx_k = 0 \tag{2.97}$$

The above equation is valid for perfect fluids.

2.4.3 Equation of motion

Contracting the energy-momentum conservation equation (2.86) with the projection operator $\triangle^{\mu\nu}$ gives us the equation of motion.

$$\triangle^{\mu}_{\nu}\partial_{\sigma}T^{\nu\sigma} = 0 \tag{2.98}$$

Using the decomposition of the energy-momentum tensor as expressed in Eqn.(2.42) and Eqn.(2.43) we get,

$$hnDU^{\mu} = \nabla^{\mu}p - \triangle^{\mu}_{\nu}\nabla_{\sigma}\Pi^{\nu\sigma} + \Big(\Pi^{\mu\nu}DU_{\nu} - \triangle^{\mu}_{\nu}DW^{\nu} - W^{\mu}\nabla_{\nu}U^{\nu} - W^{\nu}\nabla_{\nu}U^{\mu}\Big). \tag{2.99}$$

Where $W^{\mu} = U^{\nu}T_{\nu\sigma}\triangle^{\sigma\mu} = I^{\mu}_{q} + hV^{\mu}$, and h the average enthalpy per particle. Depending on whether we have chosen Eckart's definition or Landau's definition, we take $V^{\mu} = 0$ or $W^{\mu} = 0$ respectively. For perfect fluids we neglect the transport terms $\Pi^{\mu\nu}$ and W^{μ}.

$$DU^{\mu} = \frac{1}{hn}\nabla^{\mu}p \tag{2.100}$$

2.4.4 Equation of energy

The equation of energy is obtained by contracting the energy-momentum conservation equation (2.86) with U^{μ}.

$$U_{\mu}\partial_{\nu}T^{\mu\nu} = 0 \tag{2.101}$$

Now just like in the previous section using Eqn.(2.42)and Eqn.(2.43) along with the expression for W^{μ} we get the energy equation.

$$nDe = -p\nabla_{\mu}U^{\mu} + \Pi^{\mu\nu}\nabla_{\nu}U_{\mu} - \nabla_{\mu}W^{\mu} + e\nabla_{\mu}V^{\mu} + (2W^{\mu} - eV^{\mu})DU_{\mu} \tag{2.102}$$

Similarly just like the previous cases, we neglect the transport terms to get the equation valid for perfect fluids.

$$nDe = -p\nabla_\mu U^\mu \tag{2.103}$$

2.5 Equilibrium distribution function

Any system left to itself approaches a definite limit, a steady state known as equilibrium. The distribution function describing such state is the equilibrium distribution function. The equilibrium state is the state of maximum entropy, hence at equilibrium the entropy production stops. This condition along with the fact that the distribution function satisfies the transport equation uniquely determines the form of the equilibrium distribution function.

The entropy density per particle s is defined in terms of the entropy four-flow S^μ as,

$$sn = S^\mu U_\mu. \tag{2.104}$$

The substantial time derivative of the average entropy per particle is expressed as,

$$\mathscr{D}s = \partial_\mu s N^\mu = -\partial_\mu (S^\mu - sN^\mu) + \partial_\mu S^\mu. \tag{2.105}$$

In the above equation we have used the conservation of particle equation (2.83). The above equation can be rewritten as,

$$\mathscr{D}s = -\partial_\mu I_s^\mu + \sigma. \tag{2.106}$$

where $I_s^\mu = S^\mu - sN^\mu$ and $\sigma = \partial_\mu S^\mu$. The quantity I_s^μ is the change of entropy per particle due to entropy-flow, while σ is the entropy production. Using the definition of entropy four-flow Eqn.(2.20) and the Boltzmann transport equation (2.71)the entropy production can be

expressed as,

$$\sigma(x) = -\int \frac{d^3p}{(2\pi)^3 p^0}[\log f(x,p)]p^\mu \partial_\mu f(x,p) = \int \frac{d^3p}{(2\pi)^3 p^0}[\log f(x,p)]C(x,p). \tag{2.107}$$

Clearly the entropy production stops when the collision term $C(x,p)$ vanishes.

Classical statistics: The collision term for classical statistics is given by Eqn.(2.69),

$$C(x,p) = \frac{1}{2}\int \frac{d^3p_1}{(2\pi)^3 p_1^0}\frac{d^3p'}{(2\pi)^3 p'^0}\frac{d^3p_1'}{(2\pi)^3 p_1'^0}[f(x,p')f(x,p_1') - f(x,p)f(x,p_1)]W(p,p_1|p',p_1'). \tag{2.108}$$

From the above equation it is clear that the collision term for a classical statistics is zero when,

$$f(x,p)f(x,p_1) = f(x,p')f(x,p_1') \tag{2.109}$$

where the momentum are connected by relation,

$$p + p_1 = p' + p_1' \tag{2.110}$$

A distribution function that satisfies the Eqn.(2.109) will be the equilibrium distribution function as discussed above, it will be indicated by $f^0(x,p)$. The Eqn.(2.109) can be rewritten as,

$$\log f^0(x,p) + \log f^0(x,p_1) = \log f^0(x,p') + \log f^0(x,p_1') \tag{2.111}$$

To ensure that the above relation is true for all momentum it is clear that the term $\log f(x,p)$ should be such that it is same before and after the collision. Since we know what quantities are conserved during collision, we construct the most general form out of those quantities.

$$\log f^0(x,p) = \beta(x)\big[\mu(x) - p^\mu U_\mu(x)\big] \tag{2.112}$$

The quantity $\mu(x)$ is so chosen, that integrating $f(x,p)$ over entire momentum range will give the total number of particle per unit volume; and $\beta(x) = 1/T(x)$. Thus the distribution

function can be written as,

$$f^0(x,p) = \exp\Big(\frac{\mu(x) - p^\mu U_\mu(x)}{T(x)}\Big). \tag{2.113}$$

The above distribution function describes the local equilibrium, though the change in distribution function due to collision vanishes, the change due to flow of particles still continues.

Quantum statistics: For quantum statistics the collision term is given by Eqn.(2.70).

$$\begin{aligned} C(x,p) = \frac{1}{2}\int \frac{d^3p_1}{(2\pi)^3 p_1^0}\frac{d^3p'}{(2\pi)^3 p'^0}\frac{d^3p_1'}{(2\pi)^3 p_1'^0}\{f(x,p')f(x,p_1')[1\pm f(x,p)][1\pm f(x,p_1)] \\ -f(x,p)f(x,p_1)[1\pm f(x,p')][1\pm f(x,p_1')]\}W(p,p_1|p',p_1') \end{aligned} \tag{2.114}$$

Here the collision term vanishes when,

$$f(x,p)f(x,p_1)[1\pm f(x,p')][1\pm f(x,p_1')] = f(x,p')f(x,p_1')[1\pm f(x,p)][1\pm f(x,p_1)] \tag{2.115}$$

Just like in the previous paragraph the above equation can be written as,

$$\log\Big(\frac{f^0(x,p)}{1\pm f^0(x,p)}\Big) + \log\Big(\frac{f^0(x,p_1)}{1\pm f^0(x,p_1)}\Big) = \log\Big(\frac{f^0(x,p')}{1\pm f^0(x,p')}\Big) + \log\Big(\frac{f^0(x,p_1'}{1\pm f^0(x,p_1')}\Big) \tag{2.116}$$

For the above condition to hold for any momentum, each log term should satisfy,

$$\log\Big(\frac{f(x,p)}{1\pm f(x,p)}\Big) = \beta(x)\big[\mu(x) - p^\mu U_\mu(x)\big] \tag{2.117}$$

Thus the local equilibrium distribution function for quantum statistics is given by,

$$f^0(x,p) = \frac{1}{\exp\Big(\frac{p^\mu U_\mu(x)-\mu(x)}{T(x)}\Big)\mp 1}. \tag{2.118}$$

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Chapter 3

Transport Coefficients of a hot Pion Gas

A hydrodynamic system in local equilibrium evolves from one macro state to another and finally achieves global equilibrium, due to presence of "transport phenomena ". Transport phenomena involve the flow of any physical quantity, viz. energy, momentum or simply any conserved quantum number. Transport phenomena is caused by the interaction of the constituent particles that make up the system, and since the matter created during a heavy ion collision is believed to behave like a hydrodynamic system (except in early pre-equilibrium state characterized by high expansion rate, and the later de-coupled state where the mean free path becomes greater than the Hubble radius of the expanding fireball), the study of transport phenomena is of great interest and it provides us with a glimpse of the interactions that exist between the particles.

In the regime where both kinetic theory and hydrodynamics are applicable, the regime in which we are interested here, the hydrodynamic equations can be derived using the Boltzmann transport equation. For a relativistic fluid the hydrodynamic equations describe the conservation of particle number (or for reactive mixture any other conserved quantum number) and energy momentum. They are,

$$\partial_\mu N^\mu = 0, \qquad \partial_\mu T^{\mu\nu} = 0. \tag{3.1}$$

The energy momentum tensor and the particle four-flow is given by,

$$T^{\mu\nu} = T^{(0)\mu\nu} + T^{(1)\mu\nu},$$
$$T^{(0)\mu\nu} = enU^{\mu}U^{\nu} - p\triangle^{\mu\nu},$$
$$T^{(1)\mu\nu} = \left[(I_q^{\mu} + h\triangle^{\mu\sigma}N_{\sigma})U^{\nu} + (I_q^{\nu} + h\triangle^{\nu\sigma}N_{\sigma})U^{\mu}\right] + \Pi^{\mu\nu},$$

and,

$$N^{\mu} = nU^{\mu} + \triangle^{\mu\nu}N_{\nu} = nU^{\mu} + V^{\mu}. \tag{3.2}$$

In the above equations the viscous pressure tensor $\Pi^{\mu\nu}$ and the heat flow tensor I_q^{μ} constitute the irreversible flows. These irreversible flows along with the number density and energy density have 14 unknown fields, but the conservation equations Eqn.(3.1) provides us with only 5 equations. Thus the set of equations are not closed and we need 9 additional equations of motion. These equations are obtained from the Boltzmann transport equation employing various approximation schemes. One of the methods for obtain such additional equations is Chapman-Enskog expansion [1]. Here the single particle distribution function in local equilibrium is assumed to be a function of temperature, chemical potential and the three components of hydrodynamic velocity. The correction to the equilibrium distribution function is arranged symmetrically in terms of expanding power of Knudsen number. The first truncation leads to the Navier-Stokes theory, where the dissipative flows are directly proportional to the thermodynamic forces (spatial gradient of macroscopic quantities like temperature T, Pressure P etc.),

$$Thermodynamic \;\; flow = C \;\times\; Thermodynamic \;\; forces. \tag{3.3}$$

Here the term C represents the proportionality constant known as transport coefficient. These thermodynamic forces or the spatial non-uniformity give rise to the thermodynamic forces which drive the system away from equilibrium. The second law of thermodynamics $\partial_{\mu}S^{\mu} \geq 0$ requires viscous pressure and the heat flow to be,

$$\Pi^{\mu\nu} = 2\eta\langle\partial^{\mu}U^{\nu}\rangle + \zeta\,\triangle^{\mu\nu}\,\partial\cdot U,$$

and,

$$I_q^\mu = \lambda(\partial_\sigma T - TDU_\sigma)\triangle^{\mu\sigma},$$

respectively, where $\langle A^{\mu\nu}\rangle = \frac{1}{2}\left(\triangle^{\mu\sigma}\triangle^{\nu\rho} + \triangle^{\mu\rho}\triangle^{\nu\sigma} - \frac{2}{3}\triangle^{\mu\nu}\triangle^{\sigma\rho}\right)A_{\sigma\rho}$, is the trace-less part of tensor $A^{\mu\nu}$. The Navier stokes equation though stable in the non relativistic case, is unstable for relativistic generalization. Keeping the second and the higher order terms we get the Burnett and the super-Burnett equations respectively [2], these equations are unstable for non-relativistic cases. For relativistic Navier-Stokes theory the equation of motion and the energy equation turn out to be,

$$(e+P)DU^\alpha - \nabla^\alpha P + \triangle^\alpha_\nu \partial_\mu \Pi^{\mu\nu} = 0$$

and,

$$De + (e+P)\partial_\mu U^\mu - \Pi^{\mu\nu}\nabla_\mu U_\nu = o$$

respectively. Now introducing a perturbation of the form $e = e_0 + \delta e(x,t)$ and $U^\mu = (1,0) + \delta U^\mu(x,t)$ in the above equations [6] we get diffusion type evolution for perturbation $\delta U^\mu(x,t)$,

$$\partial_t \delta U^\mu - \frac{\eta_0}{e_0+P_0}\partial_x^2 U^\mu = \mathcal{O}(\delta^2). \tag{3.4}$$

Individual modes of this diffusion processes we insert a Laplace-Fourier wave ansatz in the equation above,

$$\delta U^\mu(t,x) = \exp(-\omega t + ikx) f_{\omega,k}. \tag{3.5}$$

Thus the "dispersion-relation" of the diffusion equation turns out to be,

$$\omega = \frac{\eta_0}{e_0+P_0}k^2. \tag{3.6}$$

Thus, the speed of diffusion of mode is given by

$$v_T(k) = \frac{d\omega}{dk} = 2\frac{\eta_0}{e_0+P_0}k. \tag{3.7}$$

Hence the velocity increases without bound with increase in mode, and at some value of k exceeds the speed of light which violates causality. Thus, even though the Navier-Stokes equation can describe the transport phenomenon, it is not causal.

The problem of un-stability in second-order hydrodynamics as found in Burnett and super-Burnett equations , vanish in moments method developed by Grad [3, 4]. The relativistic generalization is given by Muller-Israel-Stewart's theory. In these theories, along with the bulk viscous coefficient ζ, shear viscous coefficient η and thermal conductivity λ, we have relaxation time for bulk viscous pressure τ_ζ, shear viscous pressure τ_η, and heat flow τ_λ.

In the sections that follow we will try to derive these transport coefficients from a microscopic approach or kinetic theory, since these quantities go into the hydrodynamic equations as input and they cannot be estimated from macroscopic methods. In this chapter we will be deriving the expression for the transport coefficients for hot pion gas. We will derive the first-order transport coefficients i.e. shear viscosity, bulk viscosity and the thermal conductivity' using the Chapman-Enskog process, and the second-order coefficients (the relaxation time of flows) using the moments method.

3.1 Linear Theory: Enskog Expansion

When the system is not far away from equilibrium it is possible to obtain a linearised form of the transport equation. The linearisation can be performed around the global equilibrium where the parameters governing the distribution function are constant, or the local equilibrium where the parameters governing the distribution function are a function of space and time. In both cases the collision term becomes a more tractable integral operator with a symmetric kernel. In order to solve the transport equation, a special form of distribution function is chosen.

On dividing the Boltzmann transport equation

$$p^\mu U_\mu Df = -p^\mu \nabla_\mu f + C[f,f], \tag{3.8}$$
$$C = \frac{1}{2}\int \frac{d^3p_1}{(2\pi)^3 p_1^0}\frac{d^3p'}{(2\pi)^3 p'^0}\frac{d^3p_1'}{(2\pi)^3 p_1'^0}\Big(f'f_1' - ff_1\Big)W(p',p_1'|p,p_1),$$

by $mf(x,p)$ we get the dimension of reciprocal length. The length L_{hydro}, that is associated with the left term and the first term on the right characterizes the spatial non-uniformity of the system, i.e. the length over which the macroscopic fluid variables that parametrize the local equilibrium distribution function vary. This length can be defined in many ways that give quantitatively different but similar order of magnitude results: $L^{-1}_{hydro} \sim \partial_\mu U^\mu \sim |\partial_\mu e|/e$ etc. So the derivative of any function that is dependent on the parameters such as U^μ, e or T, will have a factor with an order of magnitude equal to that of L^{-1}_{hydro}, while the second term to the right is associated with the mean free path of the particles. The later length is much smaller than the former in the hydrodynamic regime, the ratio of these two lengths is used to expand the Boltzmann equation. The distribution function is taken to be a function of the macroscopic parameters like the temperature, density and their spatial gradient.

The Boltzmann equation is expressed as follows

$$p^\mu U_\mu Df = -\varepsilon\ p^\mu \nabla_\mu f + C[f,f]. \tag{3.9}$$

The parameter ε is known as the non uniformity parameter (or Knudsen number, which acts as a book-keeping factor). It keeps track of the order of derivative of the macroscopic parameters like U^μ, T and μ, and hence the order of L^{-1}_{hydro}. To find the solution of the above equation the distribution function is expanded as follows

$$f = f^0 + \varepsilon f^1 + \varepsilon^2 f^2 + \dots\ , \tag{3.10}$$

and the convective time derivative is expanded as,

$$Df = \varepsilon\big(Df\big)^1 + \varepsilon^2\big(Df\big)^2 + \dots\ .. \tag{3.11}$$

Substituting the above two expansions in Eqn.(3.9) and equating for equal powers of ε we obtain,

$$C[f^0, f^0] = 0, \tag{3.12}$$

$$p^\mu U_\mu \big(Df\big)^r = -p^\mu \nabla_\mu f^{r-1} + \sum_{s=0}^{r} C[f^s, f^{r-s}], \qquad r \geq 1. \tag{3.13}$$

The term $(Df)^0$ turns out to be zero when equated for the coefficient of ε^0, and so has been left out in Eqn.(3.11). The function f^0 has the form of local equilibrium distribution function, depending on parameters μ,U^μ and T that are functions of space and time. Thus we have the relation

$$f^0(x,p)f^0(x,p_1)\left[1\pm f^0(x,p')\right]\left[1\pm f^0(x,p_1')\right] = f^0(x,p')f^0(x,p_1') \left[1\pm f^0(x,p)\right]\left[1\pm f^0(x,p_1)\right]. \tag{3.14}$$

Now choosing the $\mathcal{L}[\phi^r]$ as

$$-\mathcal{L}[\phi^r] = C[f^0,f^r]+C[f^r,f^0] \tag{3.15}$$

$$\mathcal{L}[\phi^r] = \frac{1}{2}\int \frac{d^3p_1}{(2\pi)^3p_1^0}\frac{d^3p_1'}{(2\pi)^3p'^0}\frac{d^3p_1'}{(2\pi)^3p_1'^0} f^0f_1^0[1\pm f'^0][1\pm f_1'^0]\big(\phi^r+\phi_1^r-\phi'^r-\phi_1'^r\big)W(p',p_1'|p,p_1) \tag{3.16}$$

where $\phi^r = f^r/f^0$, the Boltzmann equation can be rewritten as

$$p^\mu U_\mu\big(Df\big)^r + p^\mu\nabla_\mu f^{r-1} - \sum_{s=0}^{r-1} C[f^s,f^{r-s}] = -\mathcal{L}[\phi^r]\,. \tag{3.17}$$

This is also known as the Chapman-Enskog hierarchy. If we are dealing with classical statistics we will not have the terms $[1\pm f'^0]$ and $[1\pm f_1'^0]$ in the integrand. The macroscopic parameters that describe the system are completely determined by the zeroth order distribution function f^0. They are,

$$n = \int \frac{d^3p}{(2\pi)^3p^0}p^\mu U_\mu f^0, \tag{3.18}$$

$$ne = \int \frac{d^3p}{(2\pi)^3p^0}(p^\mu U_\mu)^2 f^0. \tag{3.19}$$

And due to the form of the zeroth order distribution function we will have,

$$\triangle^{\mu\nu}N_\nu = \int \frac{d^3p}{(2\pi)^3p^0}\triangle^{\mu\nu}p_\nu f^0 = 0 \tag{3.20}$$

and

$$\triangle^{\mu\nu}T_{\nu\sigma}U^\sigma = \int \frac{d^3p}{(2\pi)^3p^0}\triangle^{\mu\nu}p_\nu p_\sigma U^\sigma f^0 = 0. \tag{3.21}$$

The above four relations are satisfied if we impose on function f^r for all $r \geq 0$ the conditions

$$\int \frac{d^3p}{(2\pi)^3 p^0} p^\mu U_\mu f^r = 0, \tag{3.22}$$

$$\int \frac{d^3p}{(2\pi)^3 p^0} (p^\mu U_\mu)^2 f^r = 0, \tag{3.23}$$

and,

$$\triangle^{\mu\nu} N_\nu = \int \frac{d^3p}{(2\pi)^3 p^0} \triangle^{\mu\nu} p_\nu f^r = 0 \tag{3.24}$$

or,

$$\triangle^{\mu\nu} T_{\nu\sigma} U^\sigma = \int \frac{d^3p}{(2\pi)^3 p^0} \triangle^{\mu\nu} p_\nu p_\sigma U^\sigma f^r = 0, \tag{3.25}$$

depending on whether Eckart's or Landau's definition of hydrodynamic velocity was considered. The above four equations are known as the condition of fit.

First Approximation Obtaining the conservation equation following the same steps as mentioned in previous chapter from Eqn.(3.17) and equating for same power of ε we get,

$$\int \frac{d^3p}{(2\pi)^3 p^0} p^\mu U_\mu \big(Df\big)^r = -\int \frac{d^3p}{(2\pi)^3 p^0} p^\mu \nabla_\mu f^{r-1}, \tag{3.26}$$

$$\int \frac{d^3p}{(2\pi)^3 p^0} p^\mu p^\nu U_\nu \big(Df\big)^r = -\int \frac{d^3p}{(2\pi)^3 p^0} p^\mu p^\nu \nabla_\nu f^{r-1}. \tag{3.27}$$

Since the distribution function is dependent on the independent variable $n(x)$, $T(x)$ and $U^\mu(x)$ and its spatial gradients, we can write

$$\begin{aligned} \big(Df\big)^r = \sum_{s=1}^{r} \left[\frac{\partial f^{r-s}}{\partial n} \big(Dn\big)^s + \frac{\partial f^{r-s}}{\partial T} \big(DT\big)^s + \frac{\partial f^{r-s}}{\partial U^\mu} \big(DU^\mu\big)^s \right] \\ + \sum_{s=2}^{r} \left[\frac{\partial f^{r-s}}{\partial (\nabla^\mu n)} \big(D\nabla^\mu n\big)^s + ... + ... \right] + ... \ , \end{aligned} \tag{3.28}$$

replacing this expansion in Eqn.(3.26), Eqn.(3.27) and restricting ourselves to $r = 1$ we get the Euler type equations, as discussed in the previous chapter'

$$\left(Dn\right)^1 = -n\nabla_\mu U^\mu, \tag{3.29}$$

$$\left(DU^\mu\right)^1 = \frac{1}{hn}\nabla^\mu p, \tag{3.30}$$

$$nDe = -p\nabla_\mu U^\mu. \tag{3.31}$$

Using the relations

$$N^{(0)}_\mu = nU_\mu \tag{3.32}$$

and,

$$T^{(0)\mu\nu} = enU^\mu U^\nu - p\triangle^{\mu\nu}. \tag{3.33}$$

Where,

$$N^{(r)}_\mu = \int \frac{d^3p}{(2\pi)^3 p^0} p_\mu f^r, \tag{3.34}$$

$$T^{(r)\mu\nu} = \int \frac{d^3p}{(2\pi)^3 p^0} p^\mu p^\nu f^r. \tag{3.35}$$

The condition of fit for the first approximation is given by,

$$\int \frac{d^3p}{(2\pi)^3 p^0} p^\mu U_\mu f^0 \phi^1 = 0, \tag{3.36}$$

$$\int \frac{d^3p}{(2\pi)^3 p^0} (p^\mu U_\mu)^2 f^0 \phi^1 = 0, \tag{3.37}$$

and,

$$\int \frac{d^3p}{(2\pi)^3 p^0} \triangle^{\mu\nu} p_\nu f^0 \phi^1 = 0 \tag{3.38}$$

or,

$$\int \frac{d^3p}{(2\pi)^3 p^0} \triangle^{\mu\nu} p_\nu p_\sigma U^\sigma f^0 \phi^1 = 0 \tag{3.39}$$

where $\phi^1 = f^1/f^0$.

3.2 Chapman-Enskog Method

This method is applicable only for systems slightly away from local equilibrium. Here in order to extract the hydrodynamic coefficients from the microscopic interaction of the particles, we truncate the Chapman expansion after the first-order (i.e. terms higher than f^1 are neglected). Employing the expansion in Eqn.(3.28) the Boltzmann equation takes the form:

$$(p.U)\Big[\frac{p.U}{T^2}DT + D\Big(\frac{\mu}{T}\Big) - \frac{p^\mu}{T}DU_\mu\Big] + p^\mu\Big[\frac{p.U}{T^2}\nabla_\mu T + \nabla_\mu\Big(\frac{\mu}{T}\Big) - \frac{p^\nu}{T}\nabla_\mu U_\nu\Big] = -\frac{\mathcal{L}[\phi^1]}{f^0(1+f^0)}\ . \tag{3.40}$$

The term ϕ parametrizes the deviation of the distribution function from the local equilibrium, and is expressed in terms of the thermodynamic forces multiplied with coefficients with appropriate tensorial rank so that it stays a Lorentz scalar.

$$\phi = \sum A^{\alpha_1\ldots\alpha_k} X_{\alpha_1\ldots\alpha_k}. \tag{3.41}$$

Here $X_{\alpha_1\ldots\alpha_k}$ is the thermodynamic force and $A^{\alpha_1\ldots\alpha_k}$ is the associated coefficient . The coefficient $A^{\alpha_1\ldots\alpha_k}$ is composed of a scalar $A(\tau)$ that is a function of $\tau = \frac{p.U}{T}$and a general tensor that can be constructed using p^μ. The term $A(\tau)$ is expanded in a series before it is substituted back into Eqn.(3.40).

3.2.1 Transport Coefficients of Pion Gas

Here Chapman Enskog approximation will be employed to derive the transport coefficient of a hadronic gas composed only of pions. For our convenience we will consider all pions to be identical and their collision cross-section will be averaged over isospin.

The local equilibrium distribution function f^0 determines the macroscopic parameters as discussed in the previous section. Using the form of the local equilibrium distribution function discussed in the previous chapter i.e. $f^0 = \frac{1}{\exp\frac{p\cdot U-\mu}{T}}$, the relevant macroscopic parameters turn

out to be,

$$n = g \int d\Gamma_p p^\mu U_\mu f^0 = \frac{g}{2\pi^2} z^2 T^3 S_2^1(z) \ , \tag{3.42}$$

$$e = \frac{g}{n} \int d\Gamma_p \big(p^\mu U_\mu\big)^2 f^0 = \frac{T}{S_2^1(z)} \big[z_\pi S_3^1(z) - S_2^2(z)\big] \ , \tag{3.43}$$

$$P = g \int d\Gamma_p \frac{\boldsymbol{p}^2}{3} f^0 = \frac{g}{2\pi^2} z^2 T^3 S_2^2(z) \ \ , \tag{3.44}$$

$$h = e + \frac{P}{n} = zT \frac{S_3^1(z)}{S_2^1(z)} \ , \tag{3.45}$$

where g is the degeneracy of pion, $z = m/T$, $d\Gamma_p = \frac{dp}{(2\pi)^3 p^0}$ and $S_n^\alpha(z) = \sum_{k=1}^{\infty} e^{k\mu/T} k^{-\alpha} K_n(kz)$, $K_n(kz)$ denoting the modified Bessel function of order n. The detailed method of calculation and expression for $K_n(kz)$ has been discussed in Appendix-A.

As we have seen in the previous section, truncation of the Enskog expansion after the first term gives Euler type equations so the continuity equation, equation of motion, and energy equation for pion gas turn out to be.

$$Dn = -n\nabla_\mu U^\mu, \tag{3.46}$$

$$DU^\mu = \frac{1}{nh} \nabla^\mu P, \tag{3.47}$$

$$De = -\frac{P}{n} \partial_\mu U^\mu, \tag{3.48}$$

respectively. The Gibbs-Duhem relation can be expressed as,

$$\partial_\nu P = nT\partial_\nu(\frac{\mu}{T}) + nhT^{-1}\partial_\nu T. \tag{3.49}$$

Using the energy equation, the Gibbs- Duhem relation along with the expression for the macroscopic parameters e and h from Eqn.(3.43) and Eqn.(3.45) respectively, to remove the time derivatives of the macroscopic parameters in Eqn.(3.40)we get,

$$f^0(1+f^0)\big[Q\partial_\nu U^\nu + p_\mu \triangle^{\mu\nu} (p\cdot U - h)(T^{-1}\partial_\nu T - DU_\nu) - \langle p_\mu p_\nu\rangle\langle\partial^\mu U^\nu\rangle\big] = T\mathcal{L}[\phi], \tag{3.50}$$

where, $Q = -\frac{1}{3}m_\pi^2 + (p\cdot U)^2\{\frac{4}{3} - \gamma'\} + (p\cdot U)\{(\gamma'' - 1)h - \gamma''' T\}$. The detailed calculation needed to get Eqn. (3.40), and the expression for γ', γ'' and γ''' has been discussed in Appendix-B. This form of the Bolzmann equation that contains only space derivative of the macroscopic parameters is desirable because, the transport equations that are to be derived using it contain only space derivative of these parameters, and no time derivatives. Now taking into account the thermodynamic forces that appear in the Boltzmann equation Eqn.(3.50) we construct accordingly the term ϕ. Since the integration on the right of Eqn.(3.50) acts on the momentum only, and for the right hand side to reproduce the left hand side the parameter ϕ must be a linear combination of the thermodynamic forces with coefficients dependent on τ,

$$\phi = A\partial\cdot U + B_\mu \triangle^{\mu\nu}\,(T^{-1}\partial_\nu T - DU_\nu) - C_{\mu\nu}\langle\partial^\mu U^\nu\rangle. \tag{3.51}$$

Substituting this expression in Eqn.(3.50) and equating the coefficients of the thermodynamic forces we get,

$$\mathcal{L}[A] = -Qf^0(p)\{1 + f^0(p)\}/T, \tag{3.52}$$

$$\mathcal{L}[B_\mu] = -\triangle_{\mu\sigma}\,p^\sigma(p\cdot U - h)f^0(p)\{1 + f^0(p)\}/T, \tag{3.53}$$

$$\mathcal{L}[C_{\mu\nu}] = -\langle p_\mu p_\nu\rangle f^0(p)\{1 + f^0(p)\}/T. \tag{3.54}$$

Where $C_{\mu\nu} = C(\tau)\langle p_\mu p_\nu\rangle$, $B_\mu = B(\tau)\,\triangle_{\mu\nu}\,p^\mu$ and $A = A(\tau)$. The hydrodynamic representation of the viscous tensor and the heat conduction is given by.

$$\Pi^{\mu\nu} = \mathring{\Pi}^{\mu\nu} + \Pi\triangle^{\mu\nu} = 2\eta\langle\partial^\mu U^\nu\rangle + \zeta(\partial\cdot U)\triangle^{\mu\nu}, \tag{3.55}$$

$$\Delta I^\mu = \lambda\,\triangle^{\mu\alpha}\,(T^{-1}\partial_\alpha T - DU_\alpha), \tag{3.56}$$

and the kinetic theory definition of the viscous tensor and the heat conduction is given by,

$$\Pi^{\mu\nu} = \int \frac{d^3p}{(2\pi)^3 p^0}\,\triangle^\mu_\sigma\,\triangle^\nu_\tau p^\sigma p^\tau f^0(1 + f^0)\phi, \tag{3.57}$$

$$I^\mu = \int \frac{d^3p}{(2\pi)^3 p^0}(p\cdot U - h)p^\sigma\,\triangle^\mu_\sigma\,f^0(1 + f^0)\phi. \tag{3.58}$$

Substituting the expression of ϕ from Eqn.(3.51) comparing the two set of equations Eqn.(3.55), Eqn.(3.56) and Eqn.(3.57), Eqn.(3.58) and equating the coefficients with appropriate tensorial

ranks (thermodynamic forces) we get the expression of the hydrodynamic coefficients. The thermodynamic flow and forces of different rank do not couple, the inner product of two irreducible tensor gives zero; this is known as the Curie law in the framework of relativistic kinetic theory. We finally get

$$\eta = -\frac{1}{10}\int \frac{d^3p}{(2\pi)^3 p^0} f^0(1+f^0) C\langle p^\alpha p^\beta\rangle\langle p_\alpha p_\beta\rangle, \tag{3.59}$$

$$\zeta = -\int \frac{d^3p}{(2\pi)^3 p^0} QAf^0(1+f^0), \tag{3.60}$$

$$\lambda = \frac{1}{3T}\int \frac{d^3p}{(2\pi)^3 p^0} B_\mu \triangle^\mu_\nu\, p^\nu (p\cdot U - h) f^0(1+f^0). \tag{3.61}$$

The conditions of fit have been employed in deriving the Eqn.(3.60); this take care of the arbitrariness of the coefficient A.

Coefficient of Bulk viscosity

To get the bulk viscosity we expand the coefficient $A(\tau)$ in terms of the Laguerre polynomial of order $1/2$ and degree $m = 0, 1, 2, ...$

$$A(\tau) = \sum_{m=0}^{\infty} a_m L_m^{1/2}(\tau). \tag{3.62}$$

Now multiplying Eqn.(3.52) with $L_n^{1/2}(\tau)$ and integrating over entire momentum space of p, we get.

$$[A(\tau), L_n^{1/2}(\tau)] = \frac{\alpha_n}{n}, \qquad n = 0, 1, 2, ... \tag{3.63}$$

$$\alpha_n = -\frac{1}{nT}\int d\Gamma_p f^0(p)[1+f^0(p)] Q L_n^{1/2}(\tau). \tag{3.64}$$

Where,

$$[F, G] = \frac{1}{4n^2}\int d\Gamma_p d\Gamma_{p_1} d\Gamma_{p'} d\Gamma_{p'_1} f_p^0(1+f_p^0)\ \delta(F)\ \delta(G)\ W. \tag{3.65}$$

is known as the bracket quantity in transport theory containing the dynamic cross-section, and

$$\delta(Z) = Z(p) + Z(p_1) - Z(p') - Z(p'_1). \tag{3.66}$$

Substituting the expression of A from Eqn.(3.62) into Eqn.(3.63) we obtain the following summation,

$$\sum_{m=0}^{\infty} a_m a_{mn} = \frac{\alpha_n}{n}, \tag{3.67}$$

where $a_{mn} = [L_m^{1/2}(\tau), L_n^{1/2}(\tau)]$ is known as the collision bracket. The above expression represents infinite set of equations, an approximate solution can be obtained by limiting the infinite number of equation to finite number r. The truncated sum is given by,

$$\sum_{m=2}^{r+1} a_m^{(r)} a_{mn} = \frac{\alpha_n}{n}. \qquad r = 2, 3, ..., r \tag{3.68}$$

The sum begins from $m = 2$ because the solubility condition states $\int d\Gamma f^0(1+f^0)Q\phi = 0$ when ϕ is some constant or p^μ. So,

$$\begin{aligned} &a_{0n} = a_{n0} = 0, \\ &a_{1n} = a_{n1} = 0, \qquad (n = 0, 1, 2, ...) \\ &a_0 = a_1 = 0. \end{aligned} \tag{3.69}$$

Now substituting the expression for A from Eqn.(3.62) into Eqn.(3.60)we get.

$$\begin{aligned} \zeta &= -\sum_{m=0}^{\infty} a_m \int \frac{d^3p}{(2\pi)^3 p^0} f^0(1+f^0) L_m^{1/2}(\tau) Q \\ &= nT \sum_{m=0}^{\infty} a_m \alpha_m. \end{aligned} \tag{3.70}$$

So the lowest order approximation of bulk viscosity is,

$$[\zeta]_1 = nT a_2^{(1)} \alpha_2 = T \frac{\alpha^2}{a_{22}}, \tag{3.71}$$

where,

$$\alpha_2 = -\frac{1}{nT}\int d\Gamma_p f^0(1+f^0)QL_2^{1/2}(\tau), \tag{3.72}$$

and

$$a_{22} = [L_2^{1/2}(\tau), L_2^{1/2}(\tau)]. \tag{3.73}$$

Evaluation of α_2: Using the properties of Laguerre polynomial $L_n^{\alpha}(x+y) = \sum_{k=0}^{n} L_{n-k}^{\alpha+k}(x)\frac{(-y^k)}{k!}$ we get,

$$\alpha_n = -\frac{1}{nT}\sum_{k=2}^{n}(-1)^k\frac{z^k}{k!}L_{n-k}^{1/2+k}(-z)U_{\nu_1\ldots\nu_k}Q^{\nu_1\ldots\nu_k}, \tag{3.74}$$

where $U_{\nu_1\ldots\nu_k} = U_{\nu_1}\ldots U_{\nu_k}$, and $Q^{\nu_1\ldots\nu_k} = \frac{1}{m_k}\int d\Gamma_p f^0(1+f^0)p^{\nu_1\ldots\nu_k}Q$. Thus α_2 becomes.

$$\alpha_2 = -\frac{1}{nT}L_0^{5/2}(-z)\frac{1}{2!}z^2U_{\mu\nu}Q^{\mu\nu}. \tag{3.75}$$

Now defining the moment of distribution function as,

$$F_{\nu_1\ldots\nu_k} = \int d\Gamma_p f^0(1+f^0)p_{\nu_1}p_{\nu_k}, \tag{3.76}$$

where,

$$F_{\nu_1\ldots\nu_k} = \frac{\rho(m)^n}{mTS_2^1}\sum_{l=0}^{[\frac{n}{2}]}a_{nl}(\triangle U)_{nl}, \tag{3.77}$$

$$a_{nl} = \sum_{k=1}^{\infty}k\sum_{s=0}^{[\frac{n}{2}-l]}(-1)^s(2l+2s-1)!![^{l+s}C_s][^nC_{2l+2s}]\frac{k_{n-l-s+l(kz)}}{(kz)^{l+s+1}}\exp\{\frac{k\mu}{T}\}, \tag{3.78}$$

$$(\triangle U)_{nl} = \triangle_{(\alpha_1\alpha_2)}\ldots\triangle_{(\alpha_{2l-1}\alpha_{2l})}U_{\alpha_{2l+1}}U_{\alpha_{2l+2}}\ldots U_{\alpha_n}. \tag{3.79}$$

Now we can write $t_{(\alpha_1 \ldots \alpha_n)} = \frac{1}{n!}\sum_P t_{P(\alpha_1 \ldots \alpha_n)}$, where the summation is extended over all permutation P of the indices. Thus the expression of α_2 comes to be.

$$\alpha_2 = \frac{z^3}{2}\Big\{\frac{1}{3}\Big(\frac{S_3^0}{S_2^1} - z^{-1}\Big) + \Big(\frac{S_2^0}{S_2^1} + \frac{3}{z}\frac{S_3^1}{S_2^1}\Big)\Big[(1-\gamma^{''})\frac{S_2^0}{S_2^1} + \gamma^{'''}z^{-1}\Big] - \Big(\frac{4}{3} - \gamma^{'}\Big)\Big(\frac{S_3^0}{S_2^1} + \frac{15}{z^2}\frac{S_3^2}{S_2^1} + \frac{2}{z}\Big)\Big\} \tag{3.80}$$

Evaluation of a_{22}: According to Eqn.(3.73) the quantity a_{22} is expressed as;

$$a_{22} = \frac{1}{4n^2}\int d\Gamma_p d\Gamma_{p_1} d\Gamma_{p'} d\Gamma_{p_1'} f_p^0(1+f_p^0)\ \delta\{L_2^{1/2}(\tau)\}\ \delta\{L_2^{1/2}(\tau)\}\ W. \tag{3.81}$$

To solve this integral we define the quantities;

$$g_\alpha = \frac{1}{2}(p_{1\alpha} - p_\alpha), \qquad g'_\alpha = \frac{1}{2}(p'_{1\alpha} - p'_\alpha), \tag{3.82}$$

and,

$$P_\alpha = (p_{1\alpha} + p_\alpha) = (p'_{1\alpha} - p'_\alpha) = P'_[\alpha]. \tag{3.83}$$

Using these the value of a_{22} comes to be,

$$a_{22} = \frac{z^2}{4} I_3. \tag{3.84}$$

Where the quantity $I_\alpha(z)$ is,

$$I_\alpha(z) = \frac{8z^4}{[s_2^1(z)]^2} e^{(-2\mu_{\mu/T})} \int_0^\infty d\psi\ \cosh^3\phi\ \sinh^7\psi \int_0^\pi d\Theta\ \sin\Theta\Big[\frac{1}{2}\frac{d\sigma}{d\Omega}(\phi,\Theta)\Big]\int_0^{2\pi} d\phi \int_0^\infty d\chi \sinh^{(2\alpha)}\chi \int_0^\pi d\theta \sin\theta \frac{e^{2z\cosh\psi\cosh\chi}}{(e^E-1)(e^F-1)(e^G-1)(e^H-1)} M_\alpha(\theta,\Theta). \tag{3.85}$$

The quantity $M_\alpha(\theta,\Theta)$ is,

$$M_1(\theta,\Theta) = 1 - \cos^2\Theta, \tag{3.86}$$

$$M_2(\theta,\Theta) = \cos^2\theta + \cos^2\theta' - 2\cos\theta\cos\theta'\cos\Theta, \tag{3.87}$$

$$M_3(\theta,\Theta) = [\cos^2\theta - \cos^2\theta']^2. \qquad . \tag{3.88}$$

The details of the variable used in the $I_\alpha(z)$ will be discussed in Appendix-C and Appendix-D.

Coefficient of Thermal Conductivity

The procedure for deriving the thermal conductivity is similar to the one used for deriving bulk viscosity. Here also we expand the coefficient $C(\tau)$ in terms of Laguerre polynomial of order 3/2.

$$B(\tau) = \sum_{m=0}^{\infty} b_m L_m^{3/2}(\tau). \tag{3.89}$$

Now multiplying Eqn.(3.53) with $L_n^{3/2}(\tau) \triangle_{\alpha\mu} p^\mu$ we get,

$$[Bp^\alpha, L_n^{3/2}(\tau) \triangle_{\alpha\mu} p^\mu] = \frac{T}{n}\beta_n, \qquad n = 0, 1, 2, ... \tag{3.90}$$

$$\beta_n = -\frac{1}{nT^2} \int \Gamma_p f^0(1 + f^0) L_n^{3/2}(\tau)(p \cdot U - h) \triangle_{\mu\nu} p^\mu p^\nu. \tag{3.91}$$

Substituting the expanded form of $B(\tau)$ in the above equation we get,

$$\sum_{m=0}^{\infty} b_m b_{mn} = \frac{\beta_n}{\rho}, \tag{3.92}$$

where $b_{mn} = \frac{1}{mT}[L_m^{3/2}(\tau)p^\mu, L_n^{3/2}(\tau) \triangle_{\mu\nu} p^\nu]$. As in the case of bulk viscosity the solubility condition dictates, $b_{0n} = b_{n0} = 0$ for $n = 0, 1, 2, ..$ and $\beta_0 = 0$. Approximate solution is obtained by restricting the number of equations in the above expression to r,

$$\sum_{m=1}^{r} b_m^{(r)} b_{mn} = \frac{\beta_n}{\rho}. \tag{3.93}$$

Substituting the value of B in Eqn.(3.61) we get,

$$\begin{aligned} \lambda &= -\frac{1}{3} \sum_{m=1}^{\infty} b_m \int d\Gamma_p f^0(1 + f^0) L_m^{3/2}(\tau) \triangle_{\mu\nu} p^\mu p^\nu T^{-1}(p \cdot U - h) \\ &= \frac{1}{3} nT \sum_{m=1}^{\infty} b_m \beta_m. \end{aligned} \tag{3.94}$$

So under first approximation (i.e. restricting ourselves to $r = 1$) we get,

$$[\lambda]_1 = -\frac{1}{3} nT b_1^{(1)} \beta_m 1 = -\frac{T}{3m} \frac{\beta_1^2}{b_{11}}, \tag{3.95}$$

where,

$$\beta_1 = -\frac{1}{nT^2}\int \Gamma_p f^0(p)(1+f^0(p))L_1^{3/2}(\tau)(p\cdot U-h)\triangle_{\mu\nu}\, p^\mu p^\nu, \tag{3.96}$$

and

$$b_{11} = \frac{1}{mT}\triangle_{\mu\nu}\,[L_1^{3/2}(\tau)p^\mu, L_1^{3/2}(\tau)p^\nu]. \tag{3.97}$$

Evaluation of $\boldsymbol{\beta_1}$ Using the properties of Laguerre polynomial in Eqn.(3.91) we get,

$$\beta_n = -\frac{1}{nT^2}\sum_{k=1}^{n}\frac{(-1)^k}{k!}z^k L_{n-k}^{3/2+k}(-z)U_{\nu_1\ldots\nu_k}G^{\nu_1\ldots\nu_k}, \tag{3.98}$$

where, $G^{\nu_1\ldots\nu_k} = m^{-k}\int d\Gamma_p f^0(1+f^0)p^{\nu_1}\ldots p^{\nu_k}G$, and $G = (p\cdot U-h)\triangle_{\mu\nu}p^\mu p^\nu$. Thus β_1 becomes.

$$\beta_1 = \left(\frac{1}{nT^2}\right)zU_\mu G^\mu, \qquad G^\mu = m^{-1}\triangle_{\alpha\beta}\{U_\nu F^{\mu\nu\alpha\beta} - hF^{\mu\alpha\beta}\}. \tag{3.99}$$

The moment calculation yields,

$$\beta_1 = 3z^2\left[1+5z^{-1}\frac{s_3^2}{s_2^1} - \left(\frac{s_3^2}{s_2^1}\right)^2\right]. \tag{3.100}$$

Evaluation of $\boldsymbol{b_{11}}$ The expression of b_{11} is given as,

$$\begin{aligned} b_{11} &= \frac{1}{mT}\triangle_{\alpha\beta}\,[L_1^{3/2}(\tau)p^\alpha, L_1^{3/2}(\tau)p^\beta] \\ &= \frac{1}{4n^2mT}\int \Gamma_p\Gamma_{p_1}\Gamma_{p'}\Gamma_{p_1'}f^0 f_1^0(1+f'^0(1+f_1'^0))W(pp_1|p'p_1') \\ &\qquad \triangle_{\alpha\beta}\delta\{L_1^{3/2}(\tau)p^\alpha\}\delta\{L_1^{3/2}(\tau)p^\beta\}. \end{aligned} \tag{3.101}$$

Using the same geometric choices and process employed for a_{22} we get,

$$b_{11} = -z\{I_2(z)+I_2(z)\}, \tag{3.102}$$

where $I_n(z)$ has the same meaning as discussed earlier.

Coefficient of Shear Viscosity

Just as in the previous cases, here too we expand C in terms of Laguerre polynomials,

$$C(\tau) = \sum_{m=0}^{\infty} c_m L_m^{5/2}(\tau). \tag{3.103}$$

Following the same procedure as employed earlier we multiply $L_n^{5/2}(\tau)\langle p^\mu p^\nu\rangle$ and integrate

$$[C\langle p_\mu p_\nu\rangle, L_n^{5/2}(\tau)\langle p^\mu p^\nu\rangle] = \frac{mT}{n}\gamma_n, \qquad n = 0, 1, 2, \dots , \tag{3.104}$$

$$\gamma_n = -\frac{1}{\rho T^2}\int d\Gamma_p f^0(p)[1 + f^0(p)]L_n^{5/2}(\tau)\langle p^\mu p^\nu\rangle. \tag{3.105}$$

Now substituting the expanded version of C in above equation we get,

$$\sum_{m=0}^{\infty} c_m c_{mn} = \frac{1}{\rho T}\gamma_n, \tag{3.106}$$

where $c_{mn} = \frac{1}{(mT)^2}[L_n^{5/2}(\tau)\langle p_\mu p_\nu\rangle, L_n^{5/2}(\tau)\langle p^\mu p^\nu\rangle]$ as discussed in the previous cases is the bracket quantity. An approximate solution is obtained by limiting the infinite number of equations to finite number r,

$$\sum_{m=0}^{(r-1)} c_m^{(r)} c_{mn} = \frac{1}{\rho T}\gamma_n, \qquad (n = 0, 1, 2, ..., (r-1)). \tag{3.107}$$

Substituting the expansion of C in Eqn.(3.59) we get,

$$\eta = -\frac{1}{10}\sum_{m=0}^{\infty} c_m \int d\Gamma_p f^0(1 + f^0)L_n^{5/2}(\tau)\langle p_\mu p_\nu\rangle\langle p^\mu p^\nu\rangle \tag{3.108}$$

$$= \frac{\rho T^2}{10}\sum_{m=0}^{\infty} c_m\gamma_n. \tag{3.109}$$

Hence the approximate value of shear viscosity we get after restricting ourselves to $r = 1$ is,

$$[\eta]_1 = \frac{\rho T^2}{10}c_0^{(1)}\gamma_0 = \frac{T}{10}\frac{\gamma_0^2}{c_{00}}, \tag{3.110}$$

where,

$$\gamma_0 = -\frac{1}{\rho T^2}\int d\Gamma_p f^{(0)}(p)[1+f^{(0)}(p)]L_n^{5/2}(\tau)\langle p_\mu p_\nu\rangle\langle p^\mu p^\nu\rangle, \tag{3.111}$$

$$c_{00} = \frac{1}{(mT)^2}[L_n^{5/2}(\tau)\langle p_\mu p_\nu\rangle, L_n^{5/2}(\tau)\langle p^\mu p^\nu\rangle]. \tag{3.112}$$

Evaluation of γ_0 Using the properties of Laguerre polynomial in the definition of γ_n we get,

$$\gamma_n = -\frac{1}{\rho T^2}\sum_{k=0}^{n}\frac{(-1)^k}{k!}z^k L_{n-k}^{5/2+k}(-z)\ \triangle_{\alpha\beta\gamma\delta}\ U_{\nu_1\ldots\nu_k}F^{\alpha\beta\gamma\delta\nu_1\ldots\nu_k}. \tag{3.113}$$

So the expression for γ_0 turns out to be,

$$\gamma_0 = -\frac{1}{\rho T^2}L_0^{5/2}(-z)\ \triangle_{\alpha\beta\gamma\delta}\ F^{\alpha\beta\gamma\delta} = -10\frac{S_3^2}{S_2^1}. \tag{3.114}$$

Evaluation of c_{00} The expanded form of the quantity c_{00} is given by,

$$\begin{aligned} c_{00} &= \frac{1}{(mT)^2}[L_0^{5/2}(\tau)\langle p_\mu p_\nu\rangle, L_0^{5/2}(\tau)\langle p^\mu p^\nu\rangle] \\ &= \frac{1}{4\rho^2T^2}\int \Gamma_p\Gamma_{p_1}\Gamma_{p'}\Gamma_{p_1'}f^0 f_1^0(1+f'^0)(1+f_1'^0)W(pp_1|p'p_1') \\ &\qquad \delta\{L_0^{5/2}(\tau)\langle p_\mu p_\nu\rangle\}\delta\{L_0^{5/2}(\tau)\langle p^\mu p^\nu\rangle\}. \end{aligned} \tag{3.115}$$

The quantities $\delta\{L_0^{5/2}(\tau)\langle p_\mu p_\nu\rangle\}$ and $\delta\{L_0^{5/2}(\tau)\langle p^\mu p^\nu\rangle\}$ are same as defined earlier. Using the geometric choises as discussed in Appendix-C and Appendix-D we get.

$$c_{00} = 2I_1(z) + 2I_2(z) + \frac{2}{3}I_3(z). \tag{3.116}$$

3.3 Grad's 14 Moment Method - Evaluation of relaxation time of flows

The parabolic hydrodynamic equations that we got from the Chapman-Enskog ran into the problem of causality violation, while Grad's 14-moment method provides hyperbolic equations resulting in finite time scale for the thermodynamic flows. The Chapman-Enskog method of

linearising the Boltzmann transport equations is applicable only when the mean free path of the constituent particles are negligibly small, compared to the characteristic macroscopic length. But when this condition is strictly not satisfied we have to use the Grads-14 method. Here the single particle distribution function was expanded around the local equilibrium distribution function in terms of complete set of Hermite polynomials, and then truncated so that the single particle distribution function is dependent on variables like temperature, chemical potential and the hydrodynamic velocity, along with the irreversible flows like heat current and the shear stress tensor, a total of fourteen moments of the distribution function, and hence the name 14-moment method. A major flaw of this process is the lack of a small parameter, like the Knudsen number in Chapan-Enskog expansion, which prevents us from power counting and hence there is no way to improve the approximation. The relativistic generalization of this process was attempted by Israel and Stewart [5, 6]. They expanded the distribution function around the local equilibrium in terms of a series of (reducible) Lorentz tensors formed with particle four momentum p^μ, i.e., 1, p^μ, $p^\mu p^\nu$, ... The series was truncated after the second-order in momentum, i.e., keeping only tensors 1, p^μ and $p^\mu p^\nu$ with 14 unknown coefficients to describe the distribution function. There are only 14 coefficients because the trace of $p^\mu p^\nu$ is equal to m^2, where m is the rest mass of the particles constituting the system. The problem was revisited and Denicol-Neime-Molnar-Rischke theory [7] was formed, it was found that the 9 extra equations (known as the relaxation equation of flows) take a more general form than previously derived by Israel-Stewart,

$$\begin{aligned}
\tau_\Pi D\Pi + \Pi &= -\zeta\partial_\mu U^\mu + \mathcal{J} + \mathcal{K} + \mathcal{R}, \\
\tau_\lambda D I_q^\mu + I_q^\mu &= \lambda\left(\nabla^\mu - \frac{T}{nh}\nabla^\mu P\right) + \mathcal{J}^\mu + \mathcal{K}^\mu + \mathcal{R}^\mu, \\
\tau_\eta D\langle\Pi^{\mu\nu}\rangle + \langle\Pi^{\mu\nu}\rangle &= 2\eta\langle\nabla^\mu U^\nu\rangle + \mathcal{J}^{\mu\nu} + \mathcal{K}^{\mu\nu} + \mathcal{R}^{\mu\nu}.
\end{aligned} \tag{3.117}$$

All the cursive term on the right are of second-order in combined power of Knudsen number and Reynold number associated each dissipative flow.

$$R_\Pi^{-1} = \frac{|\Pi|}{P}, \quad R_\Pi^{-1} = \frac{\sqrt{I^\mu I_\mu}}{P}, \quad R_\pi^{-1} = \frac{\sqrt{\Pi_{\mu\nu}\Pi^{\mu\nu}}}{P}. \tag{3.118}$$

These Reynold's numbers characterize the response of the system, to the thermodynamic forces that drives the system away from equilibrium. The Israel-Stewarts equation can be obtained from the more general set of equations mentioned above if we take all the $\mathcal{K}$, $\mathcal{R}$ terms to be zero and $\mathcal{J} = -\frac{4}{3}\tau_{\Pi}(\partial_{\mu}U^{\mu})\Pi$, $\mathcal{J}^{\mu} = \tau_{\lambda}(\partial_{\nu}U^{\nu})I^{\mu}$ and $\mathcal{J}^{\mu\nu} = -\frac{4}{3}\tau_{\eta}(\partial_{\sigma}U^{\sigma})\Pi^{\mu\nu}$.

The method we employ here to derive the expression for the relaxation time of flows is very similar to the one employed by Denicol et al. Here we expand the distribution function using irreducible tensors, 1, $\langle\Pi^{\mu}\rangle$, $\langle\Pi^{\mu}\Pi^{\nu}\rangle$, ... ; unlike Israel-Stewart. The evolution of the distribution function will be described by the the transport equation,

$$p^{\mu}\partial_{\mu}f(x,p) = C[f]. \tag{3.119}$$

Just as with the previous methods, this method is also applicable when the system is slightly away from equilibrium. Thus the distribution function of the pions $f(x,p)$ is expanded around the local equilibrium,

$$f(x,p) = f^{0}(x,p)[1+\phi(x,p)]. \tag{3.120}$$

The term $f^{0}(x,p)$ is the zeroth order distribution function describing the local equilibrium condition, hence a Juttner's form is chosen,

$$f^{0}(x,p) = \Big[\exp\big(\frac{p^{\mu}U_{\mu}(x) - \mu_{\pi}(x)}{T(x)}\big) - 1\Big]^{-1}. \tag{3.121}$$

Where $T(x)$, $U_{\mu}(x)$ and $\mu_{\pi}(x)$ are the local temperature, hydrodynamic four velocity and the pion chemical potential, respectively. Replacing the expanded form of the distribution function in the transport equation we get,

$$p^{\mu}\partial_{\mu}f^{0} + f^{0}(1+f^{0})p^{\mu}\partial_{\mu}\phi = -\mathcal{L}[\phi] \tag{3.122}$$

Where,

$$\mathcal{L}[\phi] = f^{0}(x,p)\int d\Gamma_{k}d\Gamma_{p'}d\Gamma_{k'}f^{0}(x,p)[1+f^{0}(x,p')][1+f^{0}(x,k')] \\ \big[\phi(x,p)+\phi(x,k)-\phi(x,p')-\phi(x,k')\big]W. \tag{3.123}$$

Unlike the Chapman-Enskog method of approximation we do not neglect the the term containing the derivative of ϕ, since this quantity may be small but its derivative over time and space may not be small and hence cannot be neglected and should be incorporated in this second-order theory. The deviation function that parametrizes the deviation of the distribution function from local equilibrium, depends on space-time coordinates x^μ and momentum p^μ, and is a scalar. Thus ϕ can be written as a sum of scalar products of tensors formed with p^μ and tensors depending on x^μ. Thus ϕ constructed out of irreducible tensors assumes the form,

$$\phi(x,p) = A(x,\tau) - B_\mu(x,\tau)\langle\Pi^\mu\rangle + C_{\mu\nu}(x,\tau)\langle\Pi^\mu\Pi^\nu\rangle + ... \ , \tag{3.124}$$

where $\tau = (p^\mu U_\mu)/T$, $\Pi^\mu = p^\mu/T$, $\langle\Pi^\mu\rangle = \triangle^{\mu\nu}\Pi_{nu}$ and $\langle\Pi^\mu\Pi^\nu\rangle = \Big[\frac{1}{2}(\triangle^{\mu\alpha}\triangle^{\nu\beta} + \triangle^{\nu\alpha}\triangle^{\mu\beta}) - \frac{1}{3}\triangle^{\mu\nu}\triangle^{\alpha\beta}\Big]\Pi_{\alpha\beta}$. The zeroth order distribution defines the number density, energy density and hydrodynamic four velocity, just like in Chapman-Enskog method,

$$n = g\int\frac{d^3p}{(2\pi)^3p^0}p^\mu U_\mu f = g\int\frac{d^3p}{(2\pi)^3p^0}p^\mu U_\mu f^0, \tag{3.125}$$

$$en = g\int\frac{d^3p}{(2\pi)^3p^0}(p^\mu U_\mu)^2 f = g\int\frac{d^3p}{(2\pi)^3p^0}(p^\mu U_\mu)^2 f^0. \tag{3.126}$$

Thus just as in the Chapman-Enskog method we have the conditions of fit.

$$\int\frac{d^3p}{(2\pi)^3p^0}p^\mu U_\mu f^0(1+f^0)\phi = 0, \tag{3.127}$$

$$\int\frac{d^3p}{(2\pi)^3p^0}(p^\mu U_\mu)^2 f^0(1+f^0)\phi = 0, \tag{3.128}$$

$$\int\frac{d^3p}{(2\pi)^3p^0}\langle p^\mu\rangle f^0(1+f^0)\phi = 0 \quad . \tag{3.129}$$

Here we have chosen Eckart's definition of hydrodynamic velocity, (i.e. $\triangle^{\mu\nu}N_\nu = \int\frac{d^3p}{(2\pi)^3p^0}\triangle^{\mu\nu}p_\nu f^0 = 0$). On substituting the expression of Φ from Eqn.(3.124) we get,

$$\frac{d^3p}{(2\pi)^3p^0}\tau A(x,\tau)f^0(1+f^0) = 0, \tag{3.130}$$

$$\frac{d^3p}{(2\pi)^3p^0}\tau^2 A(x,\tau)f^0(1+f^0) = 0, \tag{3.131}$$

$$\frac{d^3p}{(2\pi)^3p^0}\langle\Pi^\mu\rangle B_\nu(x,\tau)\langle\Pi^\nu\rangle f^0(1+f^0) = 0. \tag{3.132}$$

Here we have used the fact that the inner product of irreducible tensors of different ranks vanishes. The continuity equation, equation of motion and energy equation can be derived from the transport equation as expressed by, Eqn.(3.122) using the same procedure as employed in the case of Chapman Enskog process. Using Eckarts definition as mentioned earlier the equation turns out to be,

$$Dn = -n\partial \cdot U, \tag{3.133}$$

$$hnDU^{\mu} = \nabla^{\mu} P - \triangle^{\mu}_{\nu} \nabla_{\sigma} \Pi^{\nu\sigma} - \triangle^{\mu}_{\nu} DI^{\nu}_{q}, \tag{3.134}$$

$$nDe = -P\partial.U - \nabla_{\nu} I^{\nu}_{q}. \tag{3.135}$$

Where,

$$\Pi_{\mu\nu} = \int \frac{d^3p}{(2\pi)^3 p^0} p^{\mu} p^{\nu} f^0 (1 + f^0) \phi, \tag{3.136}$$

and,

$$I^{\mu}_{q} = \int \frac{d^3p}{(2\pi)^3 p^0} p^{\sigma} \triangle^{\mu}_{\sigma} (p \cdot U - h) f^0 (1 + f^0) \phi; \tag{3.137}$$

just as in the previous approximation method. Similarly, we also get the Gibs-Duhem relationship,

$$Dh = TD\Big(\frac{\mu}{T}\Big) + hT^{-1} DT - \frac{1}{n} \nabla_{\nu} I^{\nu}_{q}. \tag{3.138}$$

3.3.1 Determining A, B_{μ} and $C_{\mu\nu}$ for the Pion Gas

The time derivatives of the macroscopic parameters like the temperature T, hydrodynamic four velocity U^{μ} and the chemical potential μ_{π} that appear in the transport equation Eqn.(3.122) after the introduction of the chosen expression for $f(x, p)$, are expressed in terms of space derivatives, using the equation of motion, energy equation and the Gibbs-Duhem relation. The equation of motion Eqn.(3.134) gives the expression for the time derivative of the hydrodynamic velocity. To obtain the expression for the time derivative of T and μ/T we solve Eqn.(3.135)

and Eqn.(3.138),

$$T^{-1}DT = (1-\gamma^{'})\Big[\partial\cdot U + \frac{\delta}{P}\nabla_{\nu}I_{q}^{\nu}\Big], \tag{3.139}$$

$$TD\Big(\frac{\mu}{T}\Big) = [(\gamma^{''}-1)h - \gamma^{''}T]\Big[\partial\cdot U + \frac{\delta}{P}\nabla_{\nu}I_{q}^{\nu}\Big] - \frac{\delta^{'}}{n}\nabla_{\nu}I_{q}^{\nu}. \tag{3.140}$$

Where,

$$\gamma^{'} = \frac{(S_2^0/S_2^1)^2 - (S_3^0/S_2^1)^2 + 4z^{-1}S_2^0S_3^1/(S_2^1)^2 + z^{-1}(S_3^0/S_2^1)}{(S_2^0/S_2^1)^2 - (S_3^0/S_2^1)^2 + 3z^{-1}S_2^0S_3^1/(S_2^1)^2 + 2z^{-1}(S_3^0/S_2^1) - z^{-2}}, \tag{3.141}$$

$$\gamma^{''} = 1 + \frac{z^{-2}}{(S_2^0/S_2^1)^2 - (S_3^0/S_2^1)^2 + 3z^{-1}S_2^0S_3^1/(S_2^1)^2 + 2z^{-1}(S_3^0/S_2^1) - z^{-2}}, \tag{3.142}$$

$$\gamma^{'''} = \frac{(S_2^0/S_2^1) + 5z^{-1}(S_3^1/S_2^1) - S_3^0S_3^1/(S_2^1)^2}{(S_2^0/S_2^1)^2 - (S_3^0/S_2^1)^2 + 3z^{-1}S_2^0S_3^1/(S_2^1)^2 + 2z^{-1}(S_3^0/S_2^1) - z^{-2}}, \tag{3.143}$$

and,

$$\delta = \frac{S_2^2S_2^0/(S_2^1)^2}{1 - z[S_3^0S_2^1 - S_3^1S_2^0]/(S_2^1)^2}, \tag{3.144}$$

$$\delta^{'} = -\frac{1}{1 - z[S_3^0S_2^1 - S_3^1S_2^0]/(S_2^1)^2}. \tag{3.145}$$

The terms z and S_n^{α} has the same meaning as in the previous section (Chapman-Enskog method). Using these equations the first term on the left hand side of the transport equation becomes,

$$\Pi^{\mu}\partial_{\mu}f^0 = f^0(1+f^0)\Big[(\tau-\hat{h})\Pi_{\alpha}\frac{\nabla^{\alpha}T}{T} + \frac{1}{Tn}\Pi_{\alpha}\alpha\nabla^{\alpha}P - \langle\Pi_{\mu}\Pi_{\nu}\rangle\langle\nabla^{\mu}U^{\nu}\rangle - \tau\Pi_{\mu}DU^{\mu} + \hat{Q}\partial\cdot U$$
$$+\tau\Big[\{\tau(1-\gamma^{'}) + (\gamma^{''}-1)\hat{h} - \gamma^{'''}\}\frac{\delta}{P}\nabla_{\alpha}I_q^{\alpha} - \frac{\delta^{'}}{nT}\nabla_{\alpha}I_q^{\alpha}\Big]\Big]. \tag{3.146}$$

Where, $\Pi^{\mu} = \frac{p^{\mu}}{T}$, $\hat{h} = h/T$, $\hat{Q} = \frac{Q}{T^2}$ and $Q = -\frac{1}{3}m_{\pi}^2 + (p\cdot U)^2[\frac{4}{3} - \gamma^{'}] + (p\cdot U)[(\gamma^{''}-1)h - \gamma^{'''}T]$. Thus we see that, the viscous pressure and the heat flow are also macroscopic parameters that describe the system. So in Grads-14 moment method we truncate the series for ϕ in such a way that the viscous pressure and heat flow are retained as state variables in addition to the usual five hydrodynamic variables. Thus we are left with,

$$\phi(x,p) = A(x,\tau) - B_{\mu}(x,\tau)\langle\Pi^{\mu}\rangle + C_{\mu\nu}(x,\tau)\langle\Pi^{\mu}\Pi^{\nu}\rangle. \tag{3.147}$$

The coefficients of the term that remain are expanded in terms of τ, up to the first term that has non-vanishing contribution to the irreversible flows,

$$A(x,\tau) = A_0(x) + A_1(x)\tau + A_2(x)\tau^2 = \sum_{s=0}^{2} A_s(x)\tau^s, \tag{3.148}$$

$$B_\mu(x,\tau) = (B_0)_\mu(x) + (B_1)_\mu(x)\tau = \sum_{s=0}^{1} (B_s)_\mu(x)\tau^s, \tag{3.149}$$

$$C_{\mu\nu}(x,\tau) = (C_0)_{\mu\nu}(x). \tag{3.150}$$

As before, we find the viscous flows and the heat flow in terms of the coefficients of ϕ. The bulk viscous pressure as defined before, is

$$\Pi = \frac{1}{3}\int \frac{d^3p}{(2\pi)^3 p^0} \triangle_{\mu\nu}\, p^\mu p^\nu f^0(1+f^0)\phi. \tag{3.151}$$

Substituting the value of ϕ along with the expanded form of the coefficients, and using the condition of fit just like it has been used to derive Eqn.(3.60), the above can be expressed so,

$$\Pi = -A_2 \int \frac{d^3p}{(2\pi)^3 p^0} Q\tau^2 f^0(1+f^0). \tag{3.152}$$

The terms with the coefficients A_0 and A_1 vanishes due the properties of summation invariant. Thus we have,

$$\Pi = nT\alpha_2 A_2, \tag{3.153}$$

where, $\alpha_n = -\frac{1}{nT}\int \frac{d^3p}{(2\pi)^3 p^0} f^0(1+f^0) Q\tau^n$. Similarly for heat flow we have

$$I_q^\mu = \int \frac{d^3p}{(2\pi)^3 p^0} p^\sigma \triangle_\sigma^\mu\, (p\cdot U - h) f^0(1+f^0)\phi. \tag{3.154}$$

Substituting the expanded form of ϕ and noting the fact that the term containing $(B_1)_\nu$ vanishes we get,

$$I_q^\mu = -T^2(B_{1\nu}) \int \frac{d^3p}{(2\pi)^3 p^0} p^\sigma \triangle_\sigma^\mu\, (p\cdot U - h)\tau\langle\Pi^\nu\rangle f^0(1+f^0) = \frac{1}{3} nT(B_1)_\nu \triangle^{\mu\nu}\, \beta_1, \tag{3.155}$$

where $\beta_n = -\frac{1}{Tn}\int \frac{d^3p}{(2\pi)^3 p^0} f^0(1+f^0)\tau^n(\tau-h)\triangle^{\mu\nu}\, p_\mu p_\nu$. Finally, the viscous pressure tensor,

$$\langle \Pi^{\mu\nu}\rangle = \int \frac{d^3p}{(2\pi)^3 p^0}\langle p^\mu p^\nu\rangle f^0(1+f^0)\phi, \tag{3.156}$$

after substitution of the expression of ϕ turns out to be,

$$\langle \Pi^{\mu\nu}\rangle = -\frac{\rho\gamma_0}{5}\langle C^{\mu\nu}\rangle. \tag{3.157}$$

Here we use the definition, $\gamma_n = -\frac{1}{\rho T^2}\int \frac{d^3p}{(2\pi)^3 p^0} f^0(1+f^0)\tau^n \langle p_\mu p_\nu\rangle\langle p^\mu p^\nu\rangle$.

Now, though the coefficients A_0, A_1 and $(B_0)_\mu$ do not contribute to the irreversible flows, they cannot be put equal to zero. To find the value of those terms we introduce Eqn.(3.148) and Eqn.(3.149) in the condition of fit. Doing so, we get

$$a_1A_0 + a_2A_1 + a_3A_2 = 0, \tag{3.158}$$

$$a_2A_0 + a_3A_1 + a_4A_2 = 0, \tag{3.159}$$

$$B^0_\nu \triangle^{\mu\nu}\, b_0 + B^1_\nu \triangle^{\mu\nu}\, b_1 = 0, \tag{3.160}$$

where $a_n = \int \frac{d^3p}{(2\pi)^3 p^0} f^0(1+f^0)\tau^n$ and $\triangle^{\mu\nu} b_n = \int \frac{d^3p}{(2\pi)^3 p^0} f^0(1+f^0)\tau^n \langle \Pi^\mu\rangle\langle \Pi^\nu\rangle$. Solving the above equations we get the complete set of coefficients in terms of the thermodynamic flows. They are given by,

$$A_0 = \frac{(a_2a_4 - a_3^2)}{(a_1a_3 - a_2^2)}\frac{\Pi}{nT\alpha_2}, \tag{3.161}$$

$$A_1 = \frac{(a_1a_4 - a_2a_3)}{(a_2^2 - a_1a_3)}\frac{\Pi}{nT\alpha_2}, \tag{3.162}$$

$$A_2 = \frac{\Pi}{nT\alpha_2}, \tag{3.163}$$

$$(B_0)_\nu = \frac{I^\mu_q \triangle^{\mu\nu}}{nT\beta_1}\Big(-\frac{b_1}{b_0}\Big), \tag{3.164}$$

$$(B_1)_\nu = \frac{I^\mu_q \triangle^{\mu\nu}}{nT\beta_1}, \tag{3.165}$$

$$\langle (C_0)^{\mu\nu}\rangle = -\frac{5}{\rho\gamma_0}\langle \Pi^{\mu\nu}\rangle. \tag{3.166}$$

Thus, the above expressions completely define the term ϕ, and hence the distribution function in terms of the macroscopic parameters which include the viscous pressure and the heat flow.

3.3.2 Equation of motion of dissipative fluxes

The equation of continuity, motion and energy are a set of five equations, whereas we have fourteen unknowns, including the hydrodynamic variable, heat flow and the viscous pressure. We need nine additional equations to completely determine the evolution of the system.

Bulk viscous pressure equation

To construct the bulk viscous pressure equation we take inner product on both sides of Eqn.(3.122) with τ^2. Applying the variation approach proposed by Galerkin [10] and using the inner product properties, we get

$$\begin{aligned}\Pi = \zeta\Bigg[\nabla_\mu U^\mu - \frac{1}{n^2\alpha_2^2}\Big\{\frac{a_3^3 - 2a_2a_3a_4 + a_1a_4^2}{a_2^2 - a_1a_3} + a_5\Big\}D\Pi \\ - \frac{1}{n^2\alpha^2}\Bigg\{\frac{3}{\beta_1}\Big(\frac{b_1b_2}{b_0} - b_3\Big) + (1-\gamma^{'})\delta\Big(\frac{S_2^1}{S_2^2}\Big)a_4 \\ + \Big\{\big(\hat{h}(\gamma^{''}-1) - \gamma^{'''}\big)\delta\Big(\frac{S_2^1}{S_2^2}\Big) - \delta^{'}\Big\}a_3\Bigg\}\nabla_\mu I_q^\mu\Bigg].\end{aligned} \tag{3.167}$$

Thus the equation indeed is hyperbolic, containing time derivative of the bulk viscous pressure. From the above equation the relaxation time of the bulk viscous pressure turns out to be,

$$\tau_\zeta = \zeta\frac{1}{n^2\alpha_2^2}\Big[\frac{a_3^3 - 2a_2a_3a_4 + a_1a_4^2}{a_2^2 - a_1a_3} + a_5\Big], \tag{3.168}$$

where,

$$\begin{aligned}a_1 &= \frac{n}{T}\Big(\frac{S_2^0}{S_2^1}\Big),\\ a_2 &= \frac{n}{T}\Big(z\frac{S_3^0}{S_2^1} - 1\Big),\\ a_3 &= \frac{n}{T}z^2\Big(\frac{S_2^0}{S_2^1} + 3z^{-1}\frac{S_3^1}{S_2^1}\Big),\\ a_4 &= \frac{n}{T}z^3\Big(15z^{-2}\frac{S_3^2}{S_2^1} + 2z^{-1} + \frac{S_3^0}{S_2^1}\Big),\\ a_5 &= \frac{n}{T}z^4\Big[6z^{-1}\Big(\frac{S_3^1}{S_2^1} + 15z^{-2}\frac{S_3^3}{S_2^1}\Big) + \Big(\frac{S_2^0}{S_2^1} + 15z^{-2}\frac{S_2^2}{S_2^1}\Big)\Big];\end{aligned} \tag{3.169}$$

and

$$\alpha_2 = z^3\Big[\frac{1}{3}\Big(\frac{S_3^0}{S_2^1} - z^{-1}\Big) + \Big(\frac{S_2^0}{S_2^1} + \frac{3}{z}\frac{S_3^1}{S_2^1}\Big)\Big\{(1-\gamma^{''})\frac{S_3^1}{S_2^1} + \gamma^{'''}z^{-1}\Big\} - \Big(\frac{4}{3} - \gamma^{'}\Big)\Big\{\frac{S_3^0}{S_2^1} + 15z^{-2}\frac{S_3^2}{S_2^1} + 2z^{-1}\Big\}\Big]. \tag{3.170}$$

Heat flow equation

Just as in the previous case the variation approach leads us to the heat flow equation. Taking inner product on both sides of Eqn.(3.122) with $\langle \Pi^{\mu} \rangle$ we get,

$$I_q^{\mu} = T\lambda\Big[\Big(\frac{\nabla^{\mu}T}{T} - \frac{\nabla^{\mu}P}{nh}\Big) - \frac{1}{nT}\big(\beta^{''}DI_q^{\mu} + \gamma^{''}\nabla_{\nu}\langle\Pi^{\mu\nu}\rangle + \alpha^{''}\nabla^{''}\Pi\big)\Big], \tag{3.171}$$

$$\beta^{''} = -\frac{1}{\beta_1}\Big[\frac{9T}{n\beta_1}\Big(b_3 - \frac{b_1 b_2}{b_0}\Big) - \frac{3T}{n}\frac{b_2}{\hat{h}}\Big], \tag{3.172}$$

$$\gamma^{''} = \frac{1}{\beta_1}\Big[\frac{\gamma_1}{\gamma_0} + \frac{3T}{n}\frac{b_2}{\hat{h}}\Big], \tag{3.173}$$

$$\alpha^{''} = \frac{3T}{n}\frac{1}{\beta_1}\Big[\frac{1}{\alpha_2}\Big(b_1\frac{a_2a_4 - a_3^2}{a_1a_3 - a_2^2} + b_2\frac{a_1a_4 - a_2a_3}{a_2^2 - a_1a_3} + b_3\Big)\frac{b_2}{\hat{h}}\Big]. \tag{3.174}$$

The first linear term on the right hand side of Eqn.(3.171), and the coefficient of the time derivative of the heat flow gives the relaxation time,

$$\tau_{\lambda} = \lambda T\frac{1}{nT}\beta^{''}, \tag{3.175}$$

where,

$$b_0 = -\frac{n}{T}, \tag{3.176}$$

$$b_1 = -\frac{n}{T}z\frac{S_3^1}{S_2^1}, \tag{3.177}$$

$$b_2 = -\frac{n}{T}\Big[5z\frac{S_3^2}{S_2^1} + z^2\Big], \tag{3.178}$$

$$b_3 = -\frac{n}{T}\Big[30z\frac{S_3^3}{S_2^1} + 5z^2\frac{S_2^2}{S_2^1} + z^3\frac{S_3^1}{S_2^1}\Big]; \tag{3.179}$$

and

$$\beta_1 = 3z^2\Big[1 + 5z^{-1}\frac{S_3^2}{S_2^1} - \Big(\frac{S_3^1}{S_2^1}\Big)^2\Big]. \tag{3.180}$$

Shear viscous pressure equation

For the shear viscous pressure equation we take an inner product of Eqn.(3.122) with $\langle \Pi^{\mu}\Pi^{\nu} \rangle$. The rest of the procedure is similar to the above. We get,

$$\langle \Pi^{\mu\nu} \rangle = \eta \Big[2\langle \nabla^{\mu} U^{\nu} \rangle - \frac{1}{nT} \Big(\gamma^{'''} D \langle \Pi^{\mu\nu} \rangle \beta^{'''} \nabla^{\mu} I_q^{\nu} \Big) \Big], \tag{3.181}$$

with,

$$\gamma^{'''} = \frac{z^2 \Big[\frac{S_2^2}{S_2^1} + 6z^{-1} \frac{S_3^3}{S_2^1} \Big]}{\Big[z \frac{S_3^2}{S_2^1} \Big]^2}, \tag{3.182}$$

$$\beta^{''} = \frac{6}{\beta_1} \Big[\hat{h} - \Big(6 \frac{S_3^3}{S_3^2} + z \frac{S_2^2}{S_3^2} \Big) \Big]. \tag{3.183}$$

The coefficient of the time derivative gives the relaxation time for shear viscosity,

$$\tau_{\eta} = \eta \frac{1}{nT} \gamma^{'''}. \tag{3.184}$$

3.4 Appendix-A

Here the derivation for the expression of n_{π} has been discussed in detail. The expression for the other quantities can be derived using the same method.

$$n_{\pi} = g_{\pi} \int \frac{d^3 p_{\pi}}{(2\pi)^2 p_{\pi}^0} p_{\pi}^{\mu} U_{\mu} f_{\pi}^0 \tag{3.185}$$

For our convenience we introduce two dimensionless quantities,

$$z_{\pi} = \frac{m_{\pi}}{T}, \qquad \zeta_{\pi} = \frac{p_{\pi}^{\mu} U_{\mu}}{T} = \frac{1}{T} \big[\boldsymbol{p}_{\pi}^2 + m_{\pi}^2 \big], \tag{3.186}$$

Here we have used the fact, that in the local rest frame $U^{\mu} = (1, 0, 0, 0)$. Using the above two quantities, we can write.

$$\frac{d^3 p_{\pi}}{p_{\pi}^0} = T^2 \big(\zeta_{\pi}^2 - z_{\pi}^2 \big)^{1/2} d\zeta_{\pi} d\Omega, \tag{3.187}$$

where $\Omega = d(\cos\theta)d\phi$ is the differential solid angle. Using the expansion identity $\frac{1}{z^{-1}e^{x}-1} = \sum_{k=1}^{\infty}(ze^{-x})^{k}$ we express the above integral as a sum

$$n_{\pi} = \frac{g_{\pi}}{2\pi^{2}} z_{\pi}^{2} T^{3} \sum_{k=1}^{\infty} e^{k\mu_{\pi}/T} k^{-1} K_{2}(kz_{\pi}), \tag{3.188}$$

where $K_n(z)$ is the modified Bessel function of second kind.

$$K_{n}(z) = \frac{2^{n}n!}{(2n)!z^{n}} \int d\zeta \left(\zeta^{2} - z^{2}\right)^{n-1/2} e^{-\zeta} = \frac{2^{n}n!(2n-1)}{(2n)!z^{n}} \int \zeta d\zeta \left(\zeta^{2} - z^{2}\right)^{n-3/2} e^{-\zeta} \tag{3.189}$$

3.5 Appendix-B

The energy equation, and the Gibb's-Duhem equations for a pion gas in hydrodynamic regime slightly away from equilibrium are:

$$De = -\frac{P}{n}\partial_{\mu}U^{\mu} \ , \tag{3.190}$$

$$\partial_{\nu}P = nT\partial_{\nu}(\frac{\mu}{T}) + nhT^{-1}\partial_{\nu}T, \tag{3.191}$$

respectively. The Gibb's-Duhem relation on contracting with the hydrodynamic four velocity U^{ν} can be rewritten as,

$$Dh = TD(\frac{\mu}{T}) + hT^{-1}DT \ . \tag{3.192}$$

Expanding Eqn.(3.190) and Eqn.(3.192) in terms of derivative of temperature and chemical potential over temperature, we get.

$$\left(\frac{\partial e}{\partial T}\right)_{\mu/T} DT + \left(\frac{\partial e}{\partial(\mu/T)}\right)_{T} D\left(\frac{\mu}{T}\right) = -\frac{P}{n}\partial_{\mu}U^{\mu} \tag{3.193}$$

$$\left[\left(\frac{\partial h}{\partial T}\right)_{\mu/T} - hT^{-1}\right] DT + \left[\left(\frac{\partial h}{\partial(\mu/T)}\right)_{T} - T\right] D\left(\frac{\mu}{T}\right) = 0 \tag{3.194}$$

Using Eqn.(3.43) and Eqn.(3.45) we can derive the following quantities,

$$\Big(\frac{\partial h}{\partial T}\Big)_{\mu/T} = z\left[5\frac{S_3^1}{S_2^1} + z\frac{S_2^0}{S_2^1} - z\frac{S_3^1 S_3^0}{(S_2^1)^2}\right] \tag{3.195}$$

$$\Big(\frac{\partial e}{\partial T}\Big)_{\mu/T} = 4z\frac{S_3^1}{S_2^1} + z\frac{S_2^2 S_3^0}{(S_2^1)^2} - \frac{S_2^2}{S_2^1} + z^2\left[\frac{S_2^0}{S_2^1} - \frac{S_3^1 S_3^0}{(S_2^1)^2}\right] \tag{3.196}$$

$$\Big(\frac{\partial h}{\partial(\mu/T)}\Big)_{T} = Tz\left[\frac{S_3^0}{S_2^1} - \frac{S_3^1 S_2^0}{(S_2^1)^2}\right] \tag{3.197}$$

$$\Big(\frac{\partial e}{\partial(\mu/T)}\Big)_{T} = -T\left[1 - \frac{S_2^2 S_2^0}{(S_2^1)^2}\right] + Tz\left[\frac{S_3^0}{S_2^1} - \frac{S_3^1 S_2^0}{(S_2^1)^2}\right] . \tag{3.198}$$

Replacing these four quantities in Eqn.(3.193) and Eqn.(3.194) and solving for DT and $D\Big(\frac{\mu}{T}\Big)$ we get,

$$T^{-1}DT = (1-\gamma')\partial_\mu U^\mu \tag{3.199}$$

$$TD\Big(\frac{\mu}{T}\Big) = [(\gamma'' - 1)h - \gamma'' T]\partial_\mu U^\mu \tag{3.200}$$

where,

$$\gamma' = \frac{(S_2^0/S_2^1)^2 - (S_3^0/S_2^1)^2 + 4z^{-1}S_2^0 S_3^1/(S_2^1)^2 + z^{-1}(S_3^0/S_2^1)}{(S_2^0/S_2^1)^2 - (S_3^0/S_2^1)^2 + 3z^{-1}S_2^0 S_3^1/(S_2^1)^2 + 2z^{-1}(S_3^0/S_2^1) - z^{-2}}, \tag{3.201}$$

$$\gamma'' = 1 + \frac{z^{-2}}{(S_2^0/S_2^1)^2 - (S_3^0/S_2^1)^2 + 3z^{-1}S_2^0 S_3^1/(S_2^1)^2 + 2z^{-1}(S_3^0/S_2^1) - z^{-2}}, \tag{3.202}$$

$$\gamma''' = \frac{(S_2^0/S_2^1) + 5z^{-1}(S_3^1/S_2^1) - S_3^0 S_3^1/(S_2^1)^2}{(S_2^0/S_2^1)^2 - (S_3^0/S_2^1)^2 + 3z^{-1}S_2^0 S_3^1/(S_2^1)^2 + 2z^{-1}(S_3^0/S_2^1) - z^{-2}}. \tag{3.203}$$

3.6 Appendix C

In a collision of the type $p + p_1 \to p' + p_1'$ the total momentum of the colliding particles is defined as,

$$P_\mu = p_\mu + p_{1\mu} = p'_\mu + p'_{1\mu} = P'_\mu. \tag{3.204}$$

The relative momenta are defined as,

$$g_\mu = \frac{1}{2}(p_{1\mu} - p_\mu), \qquad g'_\mu = \frac{1}{2}(p'_{1\mu} - p'_\mu). \tag{3.205}$$

As a result of the above definitions we have $g_\mu P^\mu = g'_\mu P^\mu = 0$. Now after going to the local rest frame (depending on the definition of the hydrodynamic four velocity) we express the above vectors in the following form.

$$p^\mu = P(\cosh\chi, \sinh\chi\bar{e}), \qquad \bar{e} = (\sin\bar{\theta}\cos\bar{\phi}, \sin\bar{\theta}\sin\bar{\phi}, \cos\bar{\theta}). \tag{3.206}$$

Any vector can be expressed in Minkowski space in this way; the different values of χ, $\bar{\theta}$ and $\bar{\phi}$ will give rise to different vectors. The angles $\bar{\theta}$ and $\bar{\phi}$ give the spatial orientation of the vector in three dimensional space while the parameter χ defines the type of four vector it is (i.e. space-like or time-like). In order to define g^μ using the same parameters $\bar{\theta}$ and $\bar{\phi}$ we define a set of three orthogonal vectors,

$$\begin{aligned} \vec{e}_1 &= (\cos\bar{\theta}\cos\bar{\phi}, \cos\bar{\theta}\cos\bar{\phi}, -sin\bar{\theta}) \\ \vec{e}_2 &= (-\sin\bar{\phi}, \cos\bar{\phi}, 0) \\ \vec{e}_3 &= (\cosh\chi\bar{e}). \end{aligned} \tag{3.207}$$

The vector $\vec{g}$ is represented as

$$\vec{g} = g\{\vec{e}_1\sin\theta\cos\phi + \vec{e}_2\sin\theta\sin\phi + \vec{e}_2\cos\theta\}, \tag{3.208}$$

while g is expressed as,

$$g = m\sinh\psi. \tag{3.209}$$

Using this definition, the quantity P can be shown to be,

$$P = 2m\cosh\psi. \tag{3.210}$$

Now in order to express $\vec{g'}$ we need another set of orthogonal vectors defined using the vector set $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$,

$$\hat{E}_1 = \hat{e}_1\cos\theta\cos\phi + \hat{e}_2\cos\theta\sin\psi - \hat{e}_3\sin\theta$$

$$\hat{E}_2 = -\hat{e}_1\sin\phi + \hat{e}_2\cos\phi \tag{3.211}$$

$$\hat{E}_3 = \frac{\vec{g}}{g}, \tag{3.212}$$

Thus the expression of $\vec{g'}$ turns out to be

$$\frac{\vec{g}}{g} = \hat{E}_1 \sin\Theta\cos\Phi + \hat{E}_2 \sin\Theta\sin\Phi + \hat{E}_3\cos\Theta, \tag{3.213}$$

where angles Θ and Φ parametrise the orientation of the vector. Using these expressions we can write,

$$\frac{\vec{g}\cdot\vec{g}}{g} = \cos\Theta. \tag{3.214}$$

Thus the angle Θ is the scattering angle in the centre of mass frame. With the help of the parameters used to express the quantities P^μ, P'^μ, g^μ and g'^μ we express the quantities p, p_1, p' and p'_1. Doing so we have

$$\frac{d^3p}{p^0}\frac{d^3p_1}{p_1^0} = 8m^4(\sinh\chi\cosh\psi\sinh\psi)^2\sin\theta\sin\bar{\theta}\ d\theta\ d\bar{\theta}\ d\phi\ d\bar{\phi}\ d\psi\ d\chi. \tag{3.215}$$

Similarly it can be proved,

$$\frac{d^3p}{p^0} = \frac{1}{2}\tanh\phi\ d^4P'\ \sin\Theta\ d\Theta\ d\Psi. \tag{3.216}$$

Thus, using all these expression the collision bracket can be expressed as,

$$\begin{aligned}&\int FW(pp_1|p'p'_1)\frac{d^3p}{(2\pi)^3p^0}\frac{d^3p_1}{(2\pi)^3p_1^0}\frac{d^3p'}{(2\pi)^3p'^0}\frac{d^3p'_1}{(2\pi)^3p_1'^0} = \\ &2\frac{m^6}{\pi^4}\int F\frac{d\sigma}{d\Omega}(\phi,\Theta)\sinh^2\chi(\sinh\phi\cosh\phi)^3\sin\theta\sin\Theta\ d\chi\ d\phi\ d\theta\ d\Theta\ d\Phi.\end{aligned} \tag{3.217}$$

Where, $F = f^0f_1^0(1+f'^0)(1+f_1'^0)\{G(p)+G(p_1)-G(p')-G(p'_1)\}\{H(p)+H(p_1)-H(p')-H(p'_1)\}$.

3.7 Appendix D

Using the co-ordinates used to express the quantities P^μ, P'^μ, g^μ and g'^μ as discussed in Appendix-C, the product of the distribution function can be expressed as,

$$f^0f_1^0(1+f'^0)(1+f_1'^0) = \frac{e^{\frac{2\mu}{T}}e^{2z\cosh\psi\cosh\xi}}{(e^E-1)(e^F-1)(e^G-1)(e^H-1)}. \tag{3.218}$$

Where,

$$
\begin{aligned}
E &= z(\cosh\psi\cosh\chi - \sinh\psi\sinh\chi\cos\theta) - \frac{\mu}{T} \\
F &= z(\cosh\psi\cosh\chi - \sinh\psi\sinh\chi\cos\theta^{'}) - \frac{\mu}{T} \\
G &= E + 2z\sinh\psi\sinh\chi\cos\theta \\
H &= F + 2z\sinh\psi\sinh\chi\cos\theta^{'},
\end{aligned}
\tag{3.219}
$$

The quantity $\theta^{'}$is defined as follows,

$$
\theta^{'} = \cos^{-1}(\cos\theta\cos\Theta - \sin\theta\sin\Theta\cos\Phi). \tag{3.220}
$$

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Chapter 4

Transport Coefficients of a Pion-Nucleon Gas

In this chapter we shall investigate, the transport coefficients of a hot hadronic gas constituting of pions and nucleons. In view of the upcoming CBM experiment at FAIR it is natural to ask how the presence of a finite baryon density is likely to affect the transport property of the system created in later stage of a heavy ion collision, that constitute mainly of pions. To study such effects one has to include nucleons (anti nucleons have not been considered here). We shall be deriving only the transport coefficients that appear in first-order hydrodynamics, and due to the added complexity of an extra species of particle (i.e. nucleons) we will be using the relaxation time approximation approach.

As seen earlier for a multi component system, each component will be described by its own single particle distribution function f_k $(k = 1, 2, 3, ...N)$ giving the probability of finding any particle of that particular species in the accessible phase space. The evolution of this distribution function will be governed by N coupled Boltzmann transport equations.

$$p^\mu U_\mu D f_k(x, p_k) = -p^\mu \nabla_\mu f_k(x, p_k) + \sum_{l=1}^{\mathcal{N}} \frac{g_l}{1 + \delta_{kl}} \, C_{kl}(x, p_k). \tag{4.1}$$

$$\begin{aligned} C_{kl}(x, p) = \int \frac{d^3p_l}{(2\pi)^3 p_l^0} \frac{d^3p'_k}{(2\pi)^3 p'^0_k} \frac{d^3p'_l}{(2\pi)^3 p'^0_l} [f'_k f'_l (1 \pm f_k)(1 \pm f_l) \\ - f_k f_l (1 \pm f'_k)(1 \pm f_l)] W(p_k, p_k | p'_k, p'_l). \qquad k, l = 1, 2, ..., \mathcal{N} \end{aligned} \tag{4.2}$$

In order to solve these equations we need to linearise the equation as we did earlier.

4.1 Linearisation for a Non-reactive Mixture

The way to generalize the linearisation process for a multi component system starting from a single component system is as follows. Here we use the Enskog expansion of the Boltzmann equation and the distribution function to linearise,

$$p^{\mu}U_{\mu}Df_k(x,p_k) = -p^{\mu}\varepsilon\nabla_{\mu}f_k(x,p_k) + \sum_{l=1}^{\mathcal{N}} \frac{g_l}{1+\delta_{kl}}\, C_{kl}(x,p_k), \tag{4.3}$$

$$f_k = f_k^0 + \varepsilon f_k^1 + \varepsilon^2 f_k^2 + ..., \tag{4.4}$$

$$Df_k = \varepsilon\big(Df_k\big)^1 + \varepsilon^2\big(Df_k\big)^2 + ... \ \ , \tag{4.5}$$

where f_k^0 has the form of local equilibrium distribution function as in the case for single species, which satisfies,

$$f_k^0(x,p_k)f_l^0(x,p_l)\left[1 \pm f_k^0(x,p_k')\right]\left[1 \pm f_k^0(x,p_l')\right] = f_k^0(x,p_k')f_k^0(x,p_l') \\ \left[1 \pm f_k^0(x,p_k)\right]\left[1 \pm f_l^0(x,p_l)\right]. \tag{4.6}$$

The relevant macroscopic parameters are determined completely by f_k^0,

$$n_k = \int \frac{d^3p_k}{(2\pi)^3 p_k^0} p_k^{\mu}U_{\mu}f_k^0, \tag{4.7}$$

$$ne = \sum_k n_k e_k^0 = \sum_k \int \frac{d^3p_k}{(2\pi)^3 p_k^0}\big(p_k^{\mu}U_{\mu}\big)^2 f_k^0. \tag{4.8}$$

We restrict ourself to first approximation, that is we truncate the expansion of the distribution function after the second term in Eqn.(4.4),

$$f_k(x,p_k) = f_k^0(x,p_k) + \varepsilon\ f_k^0(x,p_k)\phi_k^1(x,p_k), \qquad k = 1,2,..,N. \tag{4.9}$$

Substituting Eqn.(4.9) and Eqn.(4.5) in Eqn.(4.3) and equating for the coefficients of ε we get,

$$p_k^{\mu}U_{\mu}\big(Df_k\big)^1 + p_k^{\mu}\nabla_{\mu}f_k^0 = -\sum_{l=1}^{N}\mathcal{L}_{kl}[\phi_k^1], \qquad k = 1,2,...,N \tag{4.10}$$

$$\mathcal{L}_{kl}[\phi_k] = \frac{g_l}{1+\delta_{kl}}\int \frac{d^3p_l}{(2\pi)^3 p_l^0}\frac{d^3p_k'}{(2\pi)^3 p_k'^0}\frac{d^3p_l'}{(2\pi)^3 p_l'^0} f_k^0 f_l^0[1 \pm f_k'^0][1 \pm f_l'^0] \\ \big(\phi_k + \phi_l - \phi_k' - \phi_l'\big)W_{kl}(p_k',p_l'|p_k,p_l). \tag{4.11}$$

The solutions to this equation are subjected to the conditions of fit.

$$\int \frac{d^3p_k}{(2\pi)^3 p_k^0} p_k^\mu U_\mu f_k^0 \phi_k^1 = 0, \tag{4.12}$$

$$\sum_k \int \frac{d^3p_k}{(2\pi)^3 p_k^0} (p_k^\mu U_\mu)^2 f_k^0 \phi_k^1 = 0 \tag{4.13}$$

and,

$$\sum_k \int \frac{d^3p_k}{(2\pi)^3 p_k^0} \triangle_\nu^\mu p_k^\nu f_k^0 \phi_k^1 = 0 \tag{4.14}$$

or,

$$\sum_k \int \frac{d^3p_k}{(2\pi)^3 p_k^0} \triangle_\nu^\mu p_k^\nu p_k^\sigma U_\sigma f_k^0 \phi_k^1 = 0. \tag{4.15}$$

The conservation equations from Chapter 2 help us to derive the Euler type equations for a non-reactive mixture. They are given by,

$$\big(Dn_k\big)^1 = -n_k \nabla_\mu U^\mu, \tag{4.16}$$

$$\big(DU^\mu\big)^1 = \frac{1}{hn} \nabla^\mu p, \tag{4.17}$$

$$\big(De\big)^1 = \sum_k x_k \big(De_k\big)^1 = -p \nabla_\mu U^\mu = -\bigg(\sum_k P_k \bigg) \nabla_\mu U^\mu, , \tag{4.18}$$

where x_k and P_k are the concentration and the partial pressure of the k^{th} species of particle.

4.2 Relaxation Time Approximation

Relaxation time approximation is the most simple method of linearising the Boltzmann equation. In this method the collision part $C[f, f]$ on the right hand side of the Boltzmann equation reduces to $E_p\ \delta f/\tau$. Where τ is the relaxation time of collision and $E_p = p^0$, and $\delta f = f - f^0$. For a mixture $\sum_{l=1}^{N} \frac{g_l}{1+\delta_{kl}} C_{kl}$ is replaced with $E_k\ \delta f_k/\tau_k$ where $E_k = p_k^0$, $\delta f_k = f_k - f_k^0$ and τ_k is the relaxation time of the k^{th} specie of particle. Expanding the distribution function and its derivative using Chapman expansion, and equating it for the same power of ε and restricting ourselves to $r = 1$ we get,

$$p^\mu U_\mu \big(Df\big)^1 + p^\mu \nabla_\mu f^0 = -E_p \frac{\delta f}{\tau} = -E_p \frac{f^1}{\tau}, \tag{4.19}$$

$$p^\mu U_\mu \big(Df_k\big)^1 + p^\mu \nabla_\mu f_k^0 = -E_k \frac{\delta f_k}{\tau_k} = -E_k \frac{f_k^1}{\tau_k}. \tag{4.20}$$

This approximated form of the collision term will keep the conservation equation, and thus the hydrodynamic equations as discussed in the previous chapter intact.

The above equation helps us to find the expression of the term f^1 in terms of the spatial and temporal derivative of f^0, and hence time and spatial derivative of the macroscopic parameters such as T, U^μ and, n_k or $\frac{\mu_k}{T}$. The time derivative of the parameters will be replaced by space derivatives using the hydrodynamic equations. The expression $E_k\ \delta f_k/\tau$ can be obtained from the collision integral [8],

$$\sum_{l=1}^{N}\mathcal{L}_{kl}[\phi_k^1] = \sum_{l=1}^{N}\frac{g_l}{1+\delta_{kl}}\int\frac{d^3p_l}{(2\pi)^3p_l^0}\frac{d^3p_k'}{(2\pi)^3p_k'^0}\frac{d^3p_l'}{(2\pi)^3p_l'^0}f_k^0f_l^0[1\pm f_k'^0][1\pm f_l'^0] \\ \big(\phi_k^1+\phi_l^1-\phi_k'^1-\phi_l'^1\big)W_{kl}(p_k',p_l'|p_k,p_l), \tag{4.21}$$

by using $\phi_l^1=\phi_k'^1=\phi_l'^1=0$ and $\phi_k^1=f_k^1/f_k^0$. The relaxation time of the k^{th} species of particle is thus given by,

$$\big[\tau_k(p_k)\big]^{-1} = \sum_{l=1}^{N}\big[\tau_{kl}(p_k)\big]^{-1}, \tag{4.22}$$

$$\big[\tau_{kl}(p_k)\big]^{-1} = \frac{g_l}{1+\delta_{kl}}\frac{csh(\epsilon_k/2)}{E_k}\int d\omega_l d\omega_k' d\omega_l' W_{kl}, \tag{4.23}$$

where $d\omega_k = d\Gamma_{p_k}/[2csh(\epsilon_k/2)]$, $d\Gamma_{p_k}=\frac{dp_k}{(2\pi)^3p_k^0}$, $\epsilon_k=\big(E_k-\mu_k\big)$ and the function $csh(\alpha_k)=\cosh(\alpha_k)$ if k^{th} specie is a fermion, and if boson $csh(\alpha_k)=\sinh(\alpha_k)$.

4.2.1 Transport Coefficients of a two component system

In this subsection we derive the expression for the transport coefficients of a two component system using relaxation time approximation. The system we consider is constituted of pions and nucleons. For our convenience we will consider all pions to be identical and their collision cross section will be averaged over isospin. The same thing will be done in the case of nucleons.

The functions f_π and f_N signify the distribution function of the pions and the nucleons respectively. The distribution functions are expanded using the Chapman expansion, and we will restrict ourselves to the first-order approximation. Thus,

$$f_\pi(x,p_\pi) = f_\pi^0(x,p_\pi)+\varepsilon\ f_\pi^1(x,p_\pi), \tag{4.24}$$

$$f_N(x,p_N) = f_N^0(x,p_N)+\varepsilon\ f_N^1(x,p_N), \tag{4.25}$$

where $f_\pi^0(x,p_\pi) = 1/\Big(exp\big(p_\pi^\mu U_\mu-\mu_\pi\big)/T-1\Big)$ and $f_N^0(x,p_N) = 1/\Big(exp\big(p_N^\mu U_\mu-\mu_N\big)/T+1\Big)$, and the macroscopic parameters U^μ, μ_π, μ_N and T are functions of position and time.

The distribution functions f_π^0 and f_N^0 defines completely the macroscopic parameters the pion density n_π, nucleon density n_N, energy per pion e_π, energy per nucleon e_N . Using the formula $\left[a-1\right]^{-1} = \sum_{n=1}^{\infty}\left(a^{-1}\right)^n$ to expand f_π^0 and f_N^0 in Eqn.(4.7) and eqn(4.8) we get,

$$n_\pi = g_\pi \int d\Gamma_{p_\pi} p_\pi^\mu U_\mu f_\pi^0 = \frac{g_\pi}{2\pi^2} z_\pi^2 T^3 S_2^1(z_\pi), \tag{4.26}$$

$$n_N = g_N \int d\Gamma_{p_N} p_N^\mu U_\mu f_N^0 = \frac{g_N}{2\pi^2} z_N^2 T^3 T_2^1(z_N), \tag{4.27}$$

$$e_\pi = \frac{g_\pi}{n_\pi} \int d\Gamma_{p_\pi} \left(p_\pi^\mu U_\mu\right)^2 f_\pi^0 = \frac{T}{S_2^1(z_\pi)}\left[z_\pi S_3^1(z_\pi) - S_2^2(z_\pi)\right], \tag{4.28}$$

$$e_N = \frac{g_N}{n_N} \int d\Gamma_{p_N} \left(p_N^\mu U_\mu\right)^2 f_N^0 = \frac{T}{T_2^1(z_N)}\left[z_N T_3^1(z_N) - T_2^2(z_N)\right], \tag{4.29}$$

where $z_\pi = m_\pi/T$, $z_N = m_N/T$, $S_n^\alpha(z_\pi) = \sum_{k=1}^{\infty} e^{k\mu_\pi/T} k^{-\alpha} K_n(kz_\pi)$ and $T_n^\alpha(z_N) = \sum_{k=1}^{\infty}(-1)^{k-1} e^{k\mu_N/T} k^{-\alpha} K_n(kz_N)$, $K_n(x)$ denoting the modified Bessel function of order n. The methods employed to calculate this quantities have been discussed earlier in Appendix-A of Chapter 2. Similarly, the partial pressure of each component is given by.

$$P_\pi = g_\pi \int d\Gamma_{p_\pi} \frac{\boldsymbol{p}_\pi^2}{3} f_\pi^0 = \frac{g_\pi}{2\pi^2} z_\pi^2 T^3 S_2^2(z_\pi)\ , \tag{4.30}$$

$$P_N = g_N \int d\Gamma_{p_N} \frac{\boldsymbol{p}_N^2}{3} f_N^0 = \frac{g_N}{2\pi^2} z_N^2 T^3 T_2^2(z_N), \tag{4.31}$$

hence the enthalpy per pion and nucleon is given by,

$$h_\pi = e_\pi + \frac{P_\pi}{n_\pi} = z_\pi T \frac{S_3^1(z_\pi)}{S_2^1(z_\pi)}, \tag{4.32}$$

$$h_N = e_N + \frac{P_N}{n_N} = z_N T \frac{T_3^1(z_N)}{T_2^1(z_N)}. \tag{4.33}$$

The Boltzmann equation for the pion and nucleon under the relaxation time approximation after employing Chapman's expansion turns out to be,

$$\begin{aligned} f_\pi^0(1+f_\pi^0)\Bigg\{ (p_\pi.U)\Big[\frac{p_\pi.U}{T^2} DT + D\left(\frac{\mu_\pi}{T}\right) - \frac{p_\pi^\mu}{T} DU_\mu\Big] \\ + p_\pi^\mu\Big[\frac{p_\pi.U}{T^2}\nabla_\mu T + \nabla_\mu\left(\frac{\mu_\pi}{T}\right) - \frac{p_\pi^\nu}{T}\nabla_\mu U_\nu\Big]\Bigg\} = -\frac{\delta f_\pi}{\tau_\pi} E_\pi, \end{aligned} \tag{4.34}$$

$$\begin{aligned} f_N^0(1-f_N^0)\Bigg\{ (p_N.U)\Big[\frac{p_N.U}{T^2} DT + D\left(\frac{\mu_N}{T}\right) - \frac{p_N^\mu}{T} DU_\mu\Big] \\ + p_N^\mu\Big[\frac{p_N.U}{T^2}\nabla_\mu T + \nabla_\mu\left(\frac{\mu_N}{T}\right) - \frac{p_N^\nu}{T}\nabla_\mu U_\nu\Big]\Bigg\} = -\frac{\delta f_N}{\tau_N} E_N. \end{aligned} \tag{4.35}$$

Here we have employed the expansion mentioned in Eqn.(3.28). The term $E_k = p_k.U = p^0_{k\boldsymbol{LR}}$ is the energy of the particle in the local rest frame.

The above equations are supposed to give rise to the transport equations, but the transport equations do not contain time derivative of the macroscopic parameters; they involve only space derivatives. To replace the time derivatives with space derivatives we employ the continuity equation, the energy equation and the equation of motion. From the previous section we have seen that restricting ourselves to $r = 1$ (i.e. the case we have at hand) we get Euler type equations,

$$Dn_\pi = -n_\pi \nabla_\mu U^\mu, \tag{4.36}$$

$$Dn_N = -n_N \nabla_\mu U^\mu, \tag{4.37}$$

$$De = n_\pi De_\pi + n_N De_N = -\Big(P_\pi + P_N\Big)\nabla_\mu U^\mu, \tag{4.38}$$

$$DU^\mu = \frac{1}{hn}\nabla^\mu p = \frac{1}{n_\pi h_\pi + n_N h_N}\nabla^\mu\big(P_\pi + P_N\big), \tag{4.39}$$

Using the expression of n_π, n_N, e_π, e_N, P_π and P_N from Eqn.(4.26), Eqn.(4.27), Eqn.(4.28), Eqn.(4.29), Eqn.(4.30) and Eqn.(4.31) respectively and replacing them in Eqn.(4.36), Eqn.(4.37) and Eqn.(4.38) and taking T, μ_π/T, μ_N/T, U^μ to be the independent macroscopic parameters we get the expression of DT, $D(\mu_\pi/T)$ and $D(\mu_N/T)$ in terms of the space derivative of U^μ as,

$$T^{-1}DT = \Big(1 - \gamma^{'}\Big)\,\partial_\mu U^\mu, \tag{4.40}$$

$$TD\left(\frac{\mu_\pi}{T}\right) = T\left[\Big(\gamma_\pi^{''} - 1\Big)\,\hat{h_\pi} - \gamma_\pi^{'''}\right], \tag{4.41}$$

$$TD\left(\frac{\mu_N}{T}\right) = T\left[\Big(\gamma_N^{''} - 1\Big)\,\hat{h_N} - \gamma_N^{'''}\right]. \tag{4.42}$$

The detailed calculation and the expressions for DT, $D(\mu_\pi/T)$ and $D(\mu_N/T)$ are given in Appendix-B. Replacing this in Eqn.(4.34) and Eqn.(4.35) we get,

$$\frac{f_\pi^0(1+f_\pi^0)}{T}\Bigg[Q_\pi\partial_\nu U^\nu - \langle p_\pi^\mu p_\pi^\nu\rangle\langle\partial_\mu U_\nu\rangle + (p_\pi^\sigma U_\sigma)p_\pi^\mu\Big(\frac{\bigtriangledown_\mu T}{T} - \frac{\bigtriangledown_\mu p}{nh}\Big) + Tp_\pi^\mu\,\bigtriangledown_\mu\left(\frac{\mu_\pi}{T}\right)\Bigg] = -\frac{\delta f_\pi}{\tau_\pi}E_\pi, \tag{4.43}$$

$$\frac{f_N^0(1+f_N^0)}{T}\Bigg[Q_N\partial_\nu U^\nu - \langle p_N^\mu p_N^\nu\rangle\langle\partial_\mu U_\nu\rangle + (p_N^\sigma U_\sigma)p_N^\mu\Big(\frac{\bigtriangledown_\mu T}{T} - \frac{\bigtriangledown_\mu p}{nh}\Big) + Tp_N^\mu\,\bigtriangledown_\mu\left(\frac{\mu_N}{T}\right)\Bigg] = -\frac{\delta f_N}{\tau_N}E_N. \tag{4.44}$$

Here $Q_k = T^2[-\frac{1}{3}z_k^2 + \zeta_k^2(\frac{4}{3} - \gamma) + \zeta_k\{(\gamma_k'' - 1)\hat{h}_k - \gamma_k'''\}]$, where $z_k = \frac{m_k}{T}$, $\zeta_k = \frac{(p_k\cdot U)}{T}$ and $\hat{h}_k = \frac{h_k}{T}$.

The Gibbs Duhem relation for a two component mixture is given by,

$$n^{-1}\partial_\nu p = s\partial_\nu T + \sum_{k=1}^{2} x_k \partial_\nu \mu_k, \tag{4.45}$$

where s is the entropy per particle. The above equation can be rewritten as,

$$h\Big(\frac{\partial_\nu T}{T} - \frac{\partial_\nu p}{nh}\Big) + \sum_{k=1}^{2} x_k T \partial_\nu \Big(\frac{\mu_k}{T}\Big) = 0. \tag{4.46}$$

The Gibbs Duhem relation represents the fact that for a two-component system which is at rest there are only three independent intrinsic thermodynamic parameters. The Eqn.(4.45) represents the derivative of pressure in terms of the independent thermodynamic parameters T, μ_1/T and μ_2/T. If the independent parameters are changed to temperature T, pressure p and the concentration of first species x_1, Eqn.(4.46) can be rewritten as,

$$h\Big(\frac{\nabla_\nu T}{T} - \frac{\nabla_\nu p}{nh}\Big) + \sum_{k=1}^{2} x_k \Big\{ T\Big(\frac{\partial}{\partial T}\frac{\mu_k}{T}\Big)_{px_1} \nabla_\nu T + \Big(\frac{\partial \mu_k}{\partial P}\Big)_{Tx_1} \nabla_\nu p + \Big(\frac{\partial \mu_k}{\partial x_1}\Big)_{Tp} \nabla_\nu x_1 \Big\} = 0. \tag{4.47}$$

Since p, T and x_1 are independent parameters, the coefficients of $\nabla_\nu T$, $\nabla_\nu p$ and $\nabla_\nu x_1$ must be individually zero. Hence from the coefficient of $\nabla_\nu x_1$ we get

$$x_1\Big(\frac{\partial \mu_1}{\partial x_1}\Big)_{Tp} + x_2\Big(\frac{\partial \mu_2}{\partial x_1}\Big)_{Tp} = 0. \tag{4.48}$$

Taking the first species of particles to be nucleons, and the second species to be pion and using Eqn.(4.47), Eqn.(4.43), Eqn.(4.44) and Eqn.(4.48) we get,

$$\begin{aligned}
\frac{f_\pi^{(0)}(1+f_\pi^{(0)})}{T}\Big\{ \Big[p_\pi \cdot U - h + (0-x_N)T^2\Big(\frac{\partial}{\partial T}\Big(\frac{\mu_N}{T}\Big)_{px_N} - \frac{\partial}{\partial T}\Big(\frac{\mu_\pi}{T}\Big)_{px_N}\Big)\Big] p_\pi^\mu \frac{\nabla_\mu T}{T} \\
+ \frac{(0-x_N)}{x_\pi}\Big(\frac{\partial \mu_N}{\partial x_N}\Big)_{PT} p_\pi^\mu \nabla_\mu x_N + Q_\pi \partial_\nu U^\nu - \langle p_\pi^\mu p_\pi^\nu \rangle \langle \partial_\mu U_\nu \rangle \Big\} = -\frac{\delta f_\pi}{\tau_\pi} E_\pi, \qquad (4.49) \\
\frac{f_N^{(0)}(1+f_N^{(0)})}{T}\Big\{ \Big[p_N \cdot U - h + (1-x_N)T^2\Big(\frac{\partial}{\partial T}\Big(\frac{\mu_N}{T}\Big)_{px_N} - \frac{\partial}{\partial T}\Big(\frac{\mu_\pi}{T}\Big)_{px_N}\Big)\Big] p_N^\mu \frac{\nabla_\mu T}{T} \\
+ \frac{(1-x_N)}{x_\pi}\Big(\frac{\partial \mu_N}{\partial x_N}\Big)_{pT} p_N^\mu \nabla_\mu x_N + Q_N \partial_\nu U^\nu - \langle p_N^\mu p_N^\nu \rangle \langle \partial_\mu U_\nu \rangle \Big\} = -\frac{\delta f_N}{\tau_N} E_N. \qquad (4.50)
\end{aligned}$$

Here we have restricted ourselves to a situation where the system is in mechanical equilibrium (i.e. $\nabla^\mu p = 0$).

As discussed earlier, we know that the relation between the different flows and the thermodynamic forces for a linear hydrodynamic theory of a two-component mixture that satisfies the

second law of thermodynamics is given by,

$$\bar{I}^{\mu} = I^{\mu} - \sum_{k=1}^{2} h_k N_k^{\mu}; \qquad \bar{I}^{\mu} = \lambda \nabla^{\mu} T + D_T^{'} n x_1 T \Big(\frac{\partial \mu_1}{\partial x_1}\Big)_{PT} \nabla^{\mu} x_1 \tag{4.51}$$

$$I_1^{\mu} = N_1^{\mu} - x_1 N^{\mu}; \qquad I_1^{\mu} = D' n \nabla^{\mu} x_1 + D_T n x_1 x_2 \nabla^{\mu} T \tag{4.52}$$

$$T^{\mu\nu} = enU^{\mu}U^{\nu} - p \bigtriangleup^{\mu\nu} + \Pi^{\mu\nu}; \qquad \Pi^{\mu\nu} = 2\eta \langle \partial^{\mu} U^{\nu} \rangle + \zeta \big(\partial \cdot U\big) \bigtriangleup^{\mu\nu} \tag{4.53}$$

where λ, $D_T^{'}$, D', D_T, η and ζ are the thermal conductivity, Dufour coefficient, diffusion coefficient, thermal diffusion coefficient, shear viscosity and bulk viscosity respectively. The irreversible flow occurs when the system is not in equilibrium and these flows help to bring the system back to equilibrium. Hence these flows should be solely dependent on the part of the distribution function that represents the deviation from equilibrium. The flows are

$$\bar{I}_q^{\mu} = \sum_{k=1}^{2} \int d\Gamma_k (p_k^{\nu} U_{\nu} - h_k) p_k^{\mu} \delta f_k, \tag{4.54}$$

$$I_1^{\mu} = \sum_{k=1}^{2} \int d\Gamma_k (\delta_{1k} - x_k) p_k^{\mu} \delta f_k \tag{4.55}$$

and,

$$\Pi^{\mu\nu} = \mathring{\Pi}^{\mu\nu} + \Pi \bigtriangleup^{\mu\nu} = \sum_{k=1}^{2} \int d\Gamma_k \{ \bigtriangleup_{\sigma}^{\mu} \bigtriangleup_{\tau}^{\nu} - \frac{1}{3} \bigtriangleup_{\sigma\tau} \bigtriangleup^{\mu\nu} \} p_k^{\sigma} p_k^{\tau} \ \delta f_k + \sum_{k=1}^{2} \frac{1}{3} \int d\Gamma_k \bigtriangleup_{\sigma\tau} \bigtriangleup^{\mu\nu} p_k^{\sigma} p_k^{\tau} \ \delta f_k. \tag{4.56}$$

Taking the 1^{st} and the 2^{nd} species to be nucleons and pions respectively the value of δf_{π} and δf_N are,

$$\delta f_{\pi} = f_{\pi}^1 = f_{\pi}^0 \big(1 + f_{\pi}^0\big) \phi_{\pi}, \tag{4.57}$$

$$\delta f_N = f_N^1 = f_N^0 \big(1 + f_N^0\big) \phi_N. \tag{4.58}$$

The quantities ϕ_{π} and ϕ_N parametrise the deviation of the pion and nucleon distribution functions from equilibrium. The right hand sides of Eqn.(4.54), Eqn.(4.55) and Eqn.(4.56) involve integration over the particle three momenta and in order that they conform to the form as expressed on the left hand side of Eqn.(4.51), Eqn.(4.52) and Eqn.(4.53) respectively; ϕ_{π} and ϕ_N must be linear combinations of the thermodynamic forces with proper coefficients having

appropriate tensorial ranks. Thus,

$$\phi_\pi = A_\pi \partial \cdot U - C_\pi^{\mu\nu}\langle \partial_\mu U_\nu \rangle - B_\mu^{(\pi)q}\frac{\nabla^\mu T}{T} - B_\mu^{(\pi N)}\frac{1}{x_\pi}\Big(\frac{\partial \mu_N}{\partial x_N}\Big)_{pT}\nabla^\mu x_N, \tag{4.59}$$

$$\phi_N = A_N \partial \cdot U - C_N^{\mu\nu}\langle \partial_\mu U_\nu \rangle - B_\mu^{(N)q}\frac{\nabla^\mu T}{T} - B_\mu^{(NN)}\frac{1}{x_\pi}\Big(\frac{\partial \mu_N}{\partial x_N}\Big)_{pT}\nabla^\mu x_N. \tag{4.60}$$

Substituting the value of ϕ_π and ϕ_N in Eqn.(4.54), Eqn.(4.55) and Eqn.(4.56) with the help of Eqn.(4.57) and Eqn.(4.57) and then comparing the coefficients of space derivatives of temperature, concentration and velocity having the same tensorial ranks, we get

$$\lambda = \frac{L_{qq}}{T}, \tag{4.61}$$

$$D_T^{'} = \frac{L_{qN}}{n x_\pi x_N T}, \tag{4.62}$$

$$D' = \frac{L_{NN}}{n x_\pi}\Big(\frac{\partial \mu_N}{\partial x_N}\Big)_{pT}, \tag{4.63}$$

$$D_T = \frac{L_{Nq}}{n x_\pi x_N T}, \qquad , \tag{4.64}$$

$$\eta = -\frac{1}{10}\sum_{k=1}^{2}\int d\Gamma_k \langle p_{k\mu} p_{k\nu}\rangle f_k^{(0)}[1 \pm f_k^{(0)}]C_k^{\mu\nu}, \tag{4.65}$$

$$\zeta = \frac{1}{3}\sum_{k=1}^{2}\int d\Gamma_k \bigtriangleup_{\mu\nu} p_k^\mu p_k^\nu f_k^{(0)}[1 \pm f_k^{(0)}]A_k., \tag{4.66}$$

where,

$$L_{qq} = -\frac{1}{3}\sum_{k=1}^{2}\int d\Gamma_k (p_k \cdot U - h_k) p_k^\sigma \Delta_\sigma^\alpha B_\alpha^{(k)q} f_k^{(0)}[1 \pm f_k^{(0)}], \tag{4.67}$$

$$L_{qN} = -\frac{1}{3}\sum_{k=1}^{2}\int d\Gamma_k (p_k \cdot U - h_k) p_k^\sigma \Delta_\sigma^\alpha B_\alpha^{(k1)} f_k^{(0)}[1 \pm f_k^{(0)}], \tag{4.68}$$

$$L_{NN} = -\frac{1}{3}\sum_{k=1}^{2}\int d\Gamma_k (\delta_{k1} - x_k) p_k^\sigma \Delta_\sigma^\alpha B_\alpha^{(k1)} f_k^{(0)}[1 \pm f_k^{(0)}], \tag{4.69}$$

$$L_{Nq} = -\frac{1}{3}\sum_{k=1}^{2}\int d\Gamma_k (\delta_{k1} - x_k) p_k^\sigma \Delta_\sigma^\alpha B_\alpha^{(k)q} f_k^{(0)}[1 \pm f_k^{(0)}]. \tag{4.70}$$

The coefficients $B_\alpha^{(N)q}$, $B_\alpha^{(\pi)q}$, $B_\alpha^{(NN)}$, $B_\alpha^{(\pi N)}$, $C_\pi^{\mu\nu}$, $C_N^{\mu\nu}$, A_π and A_N can be obtained by substituting the expression of δf_π and δf_N from Eqn.(4.57) and Eqn.(4.58) into Eqn.(4.49) and Eqn.(4.50) using expression for ϕ_π and ϕ_N and then comparing the coefficients of the thermodynamic forces (i.e. space derivative of temperature T, concentration of nucleon x_N and

hydrodynamic four velocity U^μ). The coefficients turn out to be,

$$B_\nu^{(\pi)q}\triangle^{\mu\nu} = \frac{\tau_\pi}{E_\pi T}\Big[p_\pi\cdot U - h + (0-x_N)T^2\Big(\frac{\partial}{\partial T}\Big(\frac{\mu_N}{T}\Big)_{px_N} - \frac{\partial}{\partial T}\Big(\frac{\mu_\pi}{T}\Big)_{px_N}\Big)\Big]\triangle^{\mu\nu}\,p_{\pi\nu}, \tag{4.71}$$

$$B_\nu^{(N)q}\triangle^{\mu\nu} = \frac{\tau_N}{E_N T}\Big[p_N\cdot U - h + (1-x_N)T^2\Big(\frac{\partial}{\partial T}\Big(\frac{\mu_N}{T}\Big)_{px_N} - \frac{\partial}{\partial T}\Big(\frac{\mu_\pi}{T}\Big)_{px_N}\Big)\Big]\triangle^{\mu\nu}\,p_{N\nu}, \tag{4.72}$$

$$B_\nu^{(\pi N)}\triangle^{\mu\nu} = \frac{\tau_\pi}{E_\pi T}(0-x_N)\triangle^{\mu\nu}\,p_{\pi\nu}, \tag{4.73}$$

$$B_\nu^{(NN)}\triangle^{\mu\nu} = \frac{\tau_N}{E_N T}(0-x_N)\triangle^{\mu\nu}\,p_{N\nu}, \tag{4.74}$$

$$A_\pi = -\frac{\tau_\pi}{TE_\pi}Q_\pi, \tag{4.75}$$

$$A_N = -\frac{\tau_N}{TE_N}Q_N, \tag{4.76}$$

$$C_\pi^{\mu\nu} = -\frac{\tau_\pi}{TE_\pi}\langle p_\pi^\mu p_\pi^\nu\rangle, \tag{4.77}$$

$$C_N^{\mu\nu} = -\frac{\tau_N}{TE_N}\langle p_N^\mu p_N^\nu\rangle\ . \tag{4.78}$$

Substituting the expressions of these coefficients in Eqn.(4.65) to Eqn.(4.70) we get,

$$\eta = \frac{1}{15T}\sum_{k=1}^{2}\int\frac{d^3p_k}{(2\pi)^3}\frac{\tau_k}{E_k^2}\mid\vec{p_k}\mid^4 f_k^{(0)}(1\pm f_k^{(0)}), \tag{4.79}$$

$$\zeta = \frac{1}{T}\sum_{k=1}^{2}\int\frac{d^3p_k}{(2\pi)^3}\frac{\tau_k}{E_k^2}\{Q_k\}^2 f_k^{(0)}(1\pm f_k^{(0)}), \tag{4.80}$$

$$L_{qq} = \frac{1}{6\pi^2 T}\sum_{k=1}^{2}g_k\int\frac{p_k^4 dp_k}{E_k^2}(p_k\cdot U - h_k)\Big[p_k.U - h + (\delta_{k1}-x_1)T^2\beta\Big]\tau_k f_k^{(0)}(1\pm f_k^{(0)}), \tag{4.81}$$

$$L_{q1} = \frac{1}{6\pi^2 T}\sum_{k=1}^{2}g_k\int\frac{p_k^4 dp_k}{E_k^2}(p_k\cdot U - h_k)(\delta_{k1}-x_1)\tau_k f_k^{(0)}(1\pm f_k^{(0)}), \tag{4.82}$$

$$L_{11} = \frac{1}{6\pi^2 T}\sum_{k=1}^{2}g_k\int\frac{p_k^4 dp_k}{E_k^2}(\delta_{k1}-x_1)^2\tau_k f_k^{(0)}(1\pm f_k^{(0)}) \tag{4.83}$$

and,

$$L_{1q} = \frac{1}{6\pi^2 T}\sum_{k=1}^{2}g_k\int\frac{p_k^4 dp_k}{E_k^2}\Big[p_k\cdot U - h + (\delta_{k1}-x_1)T^2\beta\Big](\delta_{k1}-x_1)\tau_k f_k^{(0)}(1\pm f_k^{(0)}), \tag{4.84}$$

where $\beta = \frac{\partial}{\partial T}\Big(\frac{\mu_1}{T}\Big)_{Px_1} - \frac{\partial}{\partial T}\Big(\frac{\mu_2}{T}\Big)_{Px_1}$.The expressions for $\frac{\partial}{\partial T}\Big(\frac{\mu_1}{T}\Big)_{Px_1}$, $\frac{\partial}{\partial T}\Big(\frac{\mu_2}{T}\Big)_{Px_1}$ and $\Big(\frac{\partial\mu_1}{\partial x_1}\Big)_{PT}$ needed to calculate the transport coefficients has been derived in Appendix-B.

4.3 Appendix-A

The continuity equations and the energy equations of a mixture of pions and nucleons are,

$$\begin{aligned} Dn_\pi &= -n_\pi \nabla_\mu U^\mu, \\ Dn_N &= -n_N \nabla_\mu U^\mu, \\ De = n_\pi De_\pi + n_N De_N &= -\Big(P_\pi + P_N\Big)\nabla_\mu U^\mu. \end{aligned}$$

Expanding the equations in terms of derivative of temperature and chemical potential over temperature we get,

$$\begin{aligned} \frac{\partial n_\pi}{\partial T} DT + \frac{\partial n_\pi}{\partial\left(\mu_\pi/T\right)} D\left(\frac{\mu_\pi}{T}\right) + 0 \cdot D\left(\frac{\mu_N}{T}\right) &= -n_\pi \partial_\mu U^\mu, \\ \frac{\partial n_\pi}{\partial T} DT + 0 \cdot D\left(\frac{\mu_\pi}{T}\right) + \frac{\partial n_N}{\partial\left(\mu_N/T\right)} D\left(\frac{\mu_N}{T}\right) &= -n_N \partial_\mu U^\mu, \\ \left[n_\pi \frac{\partial e_\pi}{\partial T} + n_N \frac{\partial e_N}{\partial T}\right] DT + n_\pi \frac{\partial e_\pi}{\partial\left(\mu_\pi/T\right)} D\left(\frac{\mu_\pi}{T}\right) + n_N \frac{\partial e_N}{\partial\left(\mu_N/T\right)} D\left(\frac{\mu_N}{T}\right) &= -P \partial_\mu U^\mu. \end{aligned} \tag{4.85}$$

Using the expression of n_π, n_N, $e_\pi$and e_N from Eqn.(4.26), Eqn.(4.27), Eqn.(4.28) and Eqn.(4.29) we find the expression of its derivative with respect to T, μ_π/T and μ_N/T. They are,

$$\frac{\partial e_\pi}{\partial T} = 4 z_\pi \frac{S_3^1}{S_2^1} + z_\pi \frac{S_2^2 S_3^0}{(S_2^1)^2} - \frac{S_2^2}{S_2^1} + z_\pi^2 \left[\frac{S_2^0}{S_2^1} - \frac{S_3^1 S_3^0}{(S_2^1)^2}\right]; \frac{\partial e_N}{\partial T} = 4 z_N \frac{T_3^1}{T_2^1} + z_N \frac{T_2^2 T_3^0}{(T_2^1)^2} - \frac{T_2^2}{T_2^1} + z_N^2 \left[\frac{T_2^0}{T_2^1} - \frac{T_3^1 T_3^0}{(T_2^1)^2}\right]$$

$$\frac{\partial e_\pi}{\partial(\mu_\pi/T)} = -T\left[1 - \frac{S_2^2 S_2^0}{(S_2^1)^2}\right] + T z_\pi \left[\frac{S_3^0}{S_2^1} - \frac{S_3^1 S_2^0}{(S_2^1)^2}\right]; \frac{\partial e_N}{\partial(\mu_N/T)} = -T\left[1 - \frac{T_2^2 T_2^0}{(T_2^1)^2}\right] + T z_N \left[\frac{T_3^0}{T_2^1} - \frac{T_3^1 T_2^0}{(T_2^1)^2}\right]$$

$$\frac{\partial n_\pi}{\partial T} = \frac{4\pi}{(2\pi)^2} T^2 \left[-z_\pi^2 S_2^1 + z_\pi^3 S_3^0\right]; \frac{\partial n_N}{\partial T} = \frac{4\pi}{(2\pi)^2} T^2 \left[-z_N^2 T_2^1 + z_N^3 T_3^0\right]$$

$$\frac{\partial n_\pi}{\partial\left(\mu_\pi/T\right)} = \frac{4\pi}{(2\pi)^3} z_\pi^2 T^2 S_2^0; \frac{\partial n_N}{\partial\left(\mu_N/T\right)} = \frac{4\pi}{(2\pi)^3} z_N^2 T^2 T_2^0.$$

Substituting the derivatives in the continuity equations and the energy equation, and solving for DT, $D(\mu_\pi/T)$ and $D(\mu_N/T)$ making use of the expression for h_π and h_N from Eqn.(4.32)

and Eqn.(4.33) we get,

$$T^{-1}DT = \left(1-\gamma^{'}\right)\partial_{\mu}U^{\mu}, \tag{4.86}$$

$$TD\left(\frac{\mu_{\pi}}{T}\right) = T\left[\left(\gamma_{\pi}^{''}-1\right)\hat{h_{\pi}}-\gamma_{\pi}^{'''}\right], \tag{4.87}$$

$$TD\left(\frac{\mu_{N}}{T}\right) = T\left[\left(\gamma_{N}^{''}-1\right)\hat{h_{N}}-\gamma_{N}^{'''}\right], \tag{4.88}$$

where,

$$\gamma^{'} = \frac{1}{|A|}\{g_{\pi}\left[z_{\pi}^{3}\left(4S_{2}^{0}S_{3}^{1}T_{2}^{0}+S_{2}^{1}S_{3}^{0}T_{2}^{0}\right)+z_{\pi}^{4}\left(\left(S_{2}^{0}\right)^{2}T_{2}^{0}-\left(S_{3}^{0}\right)^{2}T_{2}^{0}\right)\right] + g_{N}\left[z_{N}^{3}\left(4S_{2}^{0}T_{2}^{0}T_{3}^{1}+S_{2}^{0}T_{2}^{1}T_{3}^{0}\right)+z_{N}^{4}\left(S_{2}^{0}\left(T_{2}^{0}\right)^{2}-S_{2}^{0}\left(T_{3}^{0}\right)^{2}\right)\right]\}, \tag{4.89}$$

$$\gamma_{\pi}^{''} = \frac{1}{|A|}\{g_{\pi}\left[-5z_{\pi}^{2}\left(S_{2}^{1}\right)^{2}T_{2}^{0}+z_{\pi}^{3}\left(3S_{2}^{0}S_{3}^{1}T_{2}^{0}+3S_{2}^{1}S_{3}^{0}T_{2}^{0}\right)+z_{\pi}^{4}\left(\left(S_{2}^{0}\right)^{2}T_{2}^{0}-\left(S_{3}^{0}\right)^{2}T_{2}^{0}\right)\right] + g_{N}\left[-z_{N}^{2}S_{2}^{0}\left(T_{2}^{1}\right)^{2}+z_{N}^{3}\left(3S_{2}^{0}T_{2}^{0}T_{3}^{1}+2S_{2}^{0}T_{2}^{1}T_{3}^{0}\right)+z_{N}^{4}\left(S_{2}^{0}\left(T_{2}^{0}\right)^{2}-S_{2}^{0}\left(T_{3}^{0}\right)^{2}\right)\right]\}, \tag{4.90}$$

$$\gamma_{\pi}^{'''} = \frac{1}{|A|}\{g_{\pi}\left[z_{\pi}^{4}S_{2}^{1}S_{2}^{0}T_{2}^{0}\right]+g_{N}[z_{N}^{3}\left(4S_{2}^{1}T_{2}^{0}T_{3}^{1}+S_{2}^{1}T_{2}^{1}T_{3}^{0}\right)-z_{\pi}z_{N}^{2}S_{3}^{0}\left(T_{2}^{1}\right)^{2} + z_{N}^{4}\left(S_{2}^{1}\left(T_{2}^{0}\right)^{2}-S_{2}^{1}\left(T_{3}^{0}\right)^{2}\right)+z_{\pi}z_{N}^{3}\left(S_{3}^{0}T_{2}^{1}T_{3}^{0}-S_{3}^{0}T_{2}^{0}T_{3}^{1}\right)]\}, \tag{4.91}$$

and,

$$|A| = g_{\pi}\left[-z_{\pi}^{2}\left(S_{2}^{1}\right)^{2}T_{2}^{0}+z_{\pi}^{3}\left(3S_{2}^{0}S_{3}^{1}T_{2}^{0}+2S_{2}^{1}S_{3}^{0}T_{2}^{0}\right)+z_{\pi}^{4}\left(\left(S_{2}^{0}\right)^{2}T_{2}^{0}-\left(S_{3}^{0}\right)^{2}T_{2}^{0}\right)\right] + g_{N}\left[-z_{N}^{2}S_{2}^{0}\left(T_{2}^{1}\right)^{2}+z_{N}^{3}\left(3S_{2}^{0}T_{2}^{0}T_{3}^{1}+2S_{2}^{0}T_{2}^{1}T_{3}^{0}\right)+z_{N}^{4}\left(S_{2}^{0}\left(T_{2}^{0}\right)^{2}-S_{2}^{0}\left(T_{3}^{0}\right)^{2}\right)\right]. \tag{4.92}$$

The corresponding expressions of $\gamma_{N}^{''}$ and $\gamma_{N}^{'''}$ are obtained by replacing S_{β}^{α} with T_{β}^{α} and vice versa in $\gamma_{\pi}^{''}$ and $\gamma_{\pi}^{'''}$ respectively.

4.4 Appendix-B

Using Eqn.(4.26),Eqn.(4.27), Eqn.(4.30) and Eqn.(4.31) we can calculate the partial derivative of x_{π}, x_{N}, P_{π} and P_{N}, with respect to the independent thermodynamic parameters; μ_{π}/T, μ_{N}/T and T directly. Now if we change the independent thermodynamic parameters to x_{N}, T,

P then

$$\Big(\frac{\partial P}{\partial T}\Big)_{Px_N} = \Big(\frac{\partial P}{\partial T}\Big)_{\frac{\mu_\pi}{T}\frac{\mu_N}{T}} + \Big(\frac{\partial P}{\partial(\mu_\pi/T)}\Big)_{\frac{\mu_N}{T}T}\Big(\frac{\partial}{\partial T}\frac{\mu_\pi}{T}\Big)_{Px_N} + \Big(\frac{\partial P}{\partial(\mu_N/T)}\Big)_{\frac{\mu_\pi}{T}T}\Big(\frac{\partial}{\partial T}\frac{\mu_N}{T}\Big)_{Px_N} = 0, \quad (4.93)$$

$$\Big(\frac{\partial x_N}{\partial T}\Big)_{Px_N} = \Big(\frac{\partial x_N}{\partial T}\Big)_{\frac{\mu_\pi}{T}\frac{\mu_N}{T}} + \Big(\frac{\partial x_N}{\partial(\mu_\pi/T)}\Big)_{\frac{\mu_N}{T}T}\Big(\frac{\partial}{\partial T}\frac{\mu_\pi}{T}\Big)_{Px_N} + \Big(\frac{\partial x_N}{\partial(\mu_N/T)}\Big)_{\frac{\mu_\pi}{T}T}\Big(\frac{\partial}{\partial T}\frac{\mu_N}{T}\Big)_{Px_N} = 0. \quad (4.94)$$

Using the above two equations we can calculate the value of $\Big(\frac{\partial}{\partial T}\frac{\mu_\pi}{T}\Big)_{Px_N}$ and $\Big(\frac{\partial}{\partial T}\frac{\mu_N}{T}\Big)_{Px_N}$. They turn out to be

$$\Big(\frac{\partial}{\partial T}\frac{\mu_\pi}{T}\Big)_{Px_N} = \frac{[g_N x_\pi m_N^3 T_3^0 - g_\pi x_N m_\pi^3 S_3^0]x_N - g_N h x_\pi m_N^2 T_2^0}{T^2[g_\pi x_N^2 m_\pi^2 S_2^0 + g_N x_\pi^2 m_N^2 T_2^0]}, \quad (4.95)$$

$$\Big(\frac{\partial}{\partial T}\frac{\mu_N}{T}\Big)_{Px_N} = \frac{[g_\pi x_N m_\pi^3 S_3^0 - g_N x_\pi m_N^3 T_3^0]x_\pi - g_\pi h x_N m_\pi^2 S_2^0}{T^2[g_\pi x_N^2 m_\pi^2 S_2^0 + g_N x_\pi^2 m_N^2 T_2^0]}. \quad (4.96)$$

Using the above equations we can calculate β. Now:

$$\Big(\frac{\partial x_\pi}{\partial \mu_\pi}\Big)_{T\mu_N}\Big(\frac{\partial \mu_\pi}{\partial x_N}\Big)_{PT} + \Big(\frac{\partial x_\pi}{\partial \mu_N}\Big)_{T\mu_N}\Big(\frac{\partial \mu_N}{\partial x_N}\Big)_{PT} = 1, \quad (4.97)$$

$$\Big(\frac{\partial P}{\partial \mu_\pi}\Big)_{T\mu_N}\Big(\frac{\partial \mu_\pi}{\partial x_N}\Big)_{PT} + \Big(\frac{\partial P}{\partial \mu_N}\Big)_{T\mu_N}\Big(\frac{\partial \mu_N}{\partial x_N}\Big)_{PT} = 0. \quad (4.98)$$

Using the above two equations we get

$$\Big(\frac{\partial \mu_N}{\partial x_N}\Big)_{PT} = \frac{2\pi^2 n_\pi}{[g_\pi x_N^2 m_\pi^2 S_2^0 + g_N x_\pi^2 m_N^2 T_2^0]}. \quad (4.99)$$

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Chapter 5

Dynamic Inputs for determining Transport coefficients

The formulas derived in Chapter 3, can be used for any single component fluid, where particle number is conserved. Similarly, expressions for transport coefficients in Chapter 4 is applicable for any two-component fluid. But for these expressions to represent transport coefficients of pion gas and pion-nucleon gas, we must incorporate in them the interaction between the constituent particles. These interactions differentiate one fluid from the other and are introduced through the scattering cross-section of the constituent particles. Thus these cross-sections play a significant role in determining the transport coefficients. Most of the earlier works in this field have used vacuum cross-section, but for a more realistic picture, we have incorporated medium-effect in the cross-sections. To do this, we have employed *Real Time Formalism* of *Thermal Field Theory*.

5.1 $\pi\pi$ scattering

5.1.1 Interaction Lagrangian

To evaluate the $\pi\pi$ cross-section we make use of an effective Lagrangian in which the coupling of the ρ meson to the pions have been introduced through gauge covariant derivative of the pion field operator [1].

$$\mathcal{L}_{\rho\pi\pi} = \frac{ig_{\rho\pi\pi}}{4}\mathrm{Tr}\big[V^{\mu}, [\partial_{\mu}\Phi, \Phi]\big], \tag{5.1}$$

where Φ collects the pion field in the form $\begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^+ & -\pi^0 \end{pmatrix}$ while V^{μ} collects the ρ fields . Thus the interaction Lagrangian reduces to,

$$\mathcal{L}_{\rho\pi\pi} = \frac{ig_{\rho\pi\pi}}{4}\mathrm{Tr}\big[\rho_{\pi}, \{\vec{\pi} \times \vec{\pi}\}\big]. \tag{5.2}$$

Now using $\pi_1 = \frac{1}{\sqrt{2}}(\pi_+ + \pi_-)$ and $\pi_2 = \frac{1}{\sqrt{2}}(\pi_+ - \pi_-)$ the above expression becomes,

$$\begin{aligned}\mathcal{L}_{\rho\pi\pi} = {} & ig_{\rho\pi\pi}\{\rho_{+\mu}\pi_0\partial^{\mu}\pi_- + \rho_{-\mu}\pi_+\partial^{\mu}\pi_0 + \rho_{0\mu}\pi_-\partial^{\mu}\pi_+\} \\ & -ig_{\rho\pi\pi}\{\rho_{+\mu}\pi_-\partial^{\mu}\pi_0 + \rho_{-\mu}\pi_0\partial^{\mu}\pi_+ + \rho_{0\mu}\pi_+\partial^{\mu}\pi_-\},\end{aligned} \tag{5.3}$$

where $g_{\rho\pi\pi}$ is the coupling constant for $\rho \to \pi\pi$ decay, and it is determined using the decay width of ρ meson. The decay width of the decay of ρ meson into two pions is given by,

$$\Gamma = \frac{1}{2E_p}\int \frac{d^3p_1}{(2\pi)^3 2E_1}\frac{d^3p_2}{(2\pi)^3 2E_2}(2\pi)^4\delta^4(p - p_1 - p_2)|\overline{M_{\rho\to\pi\pi}}|^2. \tag{5.4}$$

p_1 and p_2 being the momentum of the two pions produced due to the decay of ρ with momentum p. The decay amplitude of the different modes of the decay are,

$$M_{\rho^0\to\pi^+\pi^-} = g^2_{\rho\pi\pi}(m^2_{\rho} - 4m^2_{\pi}), \tag{5.5}$$

$$M_{\rho^+\to\pi^+\pi^0} = g^2_{\rho\pi\pi}(m^2_{\rho} - 4m^2_{\pi}), \tag{5.6}$$

$$M_{\rho^-\to\pi^-\pi^0} = g^2_{\rho\pi\pi}(m^2_{\rho} - 4m^2_{\pi}). \tag{5.7}$$

Using these amplitudes the expression for decay width Γ is given by,

$$\begin{aligned}\Gamma &= \frac{g^2_{\rho\pi\pi}}{8\pi m_\rho}(m^2_\rho - 4m^2_\pi)\frac{\lambda^{1/2}(s, m^2_\pi, m^2_\pi)}{2s} \\ &= \frac{g^2_{\rho\pi\pi} m_\rho}{48\pi}\Big[1 - \frac{4m^2_\pi}{m^2_\rho}\Big]^{3/2}. \end{aligned} \tag{5.8}$$

The value of Γ in vacuum turns out to be 150 MeV [2], from which we get $g_{\rho\pi\pi} = 6.05$.

5.1.2 Amplitude of scattering due to the exchange of ρ and σ meson

We will be using an isospin averaged amplitudes to derive the transport coefficients. These are given by

$$|\overline{M_{\pi\pi}}|^2 = \frac{1}{\sum_I (2I+1)}\Big[\sum_2^{I=0}(2I+1)(M^I_{\pi\pi})^2\Big], \tag{5.9}$$

where I represents the isospin of different channels. The total isospin of two colliding pions can be $\vec{0}$, $\vec{1}$ and $\vec{2}$, since isospin of each pion is $\vec{1}$. Thus the above expression for the isospin average becomes,

$$|\overline{M_{\pi\pi}}|^2 = \frac{1}{9}[|M^0_{\pi\pi}|^2 + 3|M^1_{\pi\pi}|^2 + 5|M^2_{\pi\pi}|^2]. \tag{5.10}$$

The amplitude for total isospin $\vec{I} = \vec{0}$ is,

$$\begin{aligned} M^{I=0}_{\pi\pi} = \frac{1}{3}\big\{ &\langle\pi^+\pi^-|\mathcal{L}|\pi^+\pi^-\rangle + \langle\pi^+\pi^-|\mathcal{L}|\pi^-\pi^+\rangle + \langle\pi^+\pi^-|\mathcal{L}|\pi^0\pi^0\rangle \\ &+\langle\pi^-\pi^+|\mathcal{L}|\pi^+\pi^-\rangle + \langle\pi^-\pi^+|\mathcal{L}|\pi^-\pi^+\rangle + \langle\pi^-\pi^+|\mathcal{L}|\pi^0\pi^0\rangle \\ &+\langle\pi^0\pi^0|\mathcal{L}|\pi^+\pi^-\rangle + \langle\pi^0\pi^0|\mathcal{L}|\pi^-\pi^+\rangle + \langle\pi^0\pi^0|\mathcal{L}|\pi^0\pi^0\rangle\big\}. \end{aligned} \tag{5.11}$$

Similarly the amplitude corresponding to total isospin $\vec{I} = \vec{1}$ and $I_z = +1$ is,

$$M^{I=1}_{\pi\pi} = \frac{1}{2}\big\{\langle\pi^+\pi^0|\mathcal{L}|\pi^+\pi^0\rangle - \langle\pi^+\pi^0|\mathcal{L}|\pi^0\pi^+\rangle - \langle\pi^0\pi^+|\mathcal{L}|\pi^+\pi^0\rangle + \langle\pi^0\pi^+|\mathcal{L}|\pi^0\pi^+\rangle\big\}, \tag{5.12}$$

and for total isospin $\vec{I} = \vec{2}$ and $I_z = +2$,

$$M^{I=2}_{\pi\pi} = \langle\pi^+\pi^+|\mathcal{L}|\pi^+\pi^+\rangle. \tag{5.13}$$

Now to obtain the amplitude for each value of I we need to evaluate a set of Feynman's diagrams corresponding to each term in Eqn.(5.11), Eqn.(5.12) and Eqn.(5.13) using Lagrangian in Eqn.(5.3) and doing so we get,

$$M_{\pi\pi}^{I=0} = 2g_{\rho\pi\pi}^2\Big[\frac{s-u}{t-m_\rho^2} + \frac{s-t}{u-m_\rho^2}\Big], \tag{5.14}$$

$$M_{\pi\pi}^{I=1} = g_{\rho\pi\pi}^2\Big[2\frac{t-u}{s-m_\rho^2} + \frac{t-s}{u-m_\rho^2} - \frac{u-s}{t-m_\rho^2}\Big], \tag{5.15}$$

$$M_{\pi\pi}^{I=2} = g_{\rho\pi\pi}^2\Big[\frac{u-s}{t-m_\rho^2} + \frac{t-s}{u-m_\rho^2}\Big]. \tag{5.16}$$

Substituting the above expressions in Eqn.(5.10) we get the isospin averaged $\pi\pi$ scattering amplitude due to ρ meson exchange.

To describe $\pi\pi$ scattering at low energies we need to include σ-exchange as well. For σ exchange we use a Lagrangian,

$$\mathcal{L}_{\sigma\pi\pi} = \frac{1}{2}g_{\sigma\pi\pi}m_\sigma\vec{\pi}\cdot\vec{\pi}\sigma, \tag{5.17}$$

where $g_{\sigma\pi\pi} = 2.5$. Following the same steps as we did for ρ we obtain the amplitude for respective isospin channels.

$$M_{\pi\pi}^{I=0} = g_{\sigma\pi\pi}^2 m_\sigma^2\Big[\frac{3}{s-m_\sigma^2} + \frac{1}{t-m_\sigma^2} + \frac{1}{u-m_\sigma^2}\Big], \tag{5.18}$$

$$M_{\pi\pi}^{I=1} = g_{\sigma\pi\pi}^2 m_\sigma^2\Big[\frac{1}{t-m_\sigma^2} - \frac{1}{u-m_\sigma^2}\Big], \tag{5.19}$$

$$M_{\pi\pi}^{I=2} = g_{\sigma\pi\pi}^2 m_\sigma^2\Big[\frac{1}{t-m_\sigma^2} + \frac{1}{u-m_\sigma^2}\Big]. \tag{5.20}$$

We then introduce the vacuum width of the ρ and σ mesons in the corresponding s-channel diagram. The total amplitude for binary elastic scattering for pions is obtained as,

$$M_{\pi\pi}^{I=0} = 2g_{\rho\pi\pi}^2\Big[\frac{s-u}{t-m_\rho^2} + \frac{s-t}{u-m_\rho^2}\Big] + g_{\sigma\pi\pi}^2 m_\sigma^2\Big[\frac{3}{s-m_\sigma^2+im_\sigma\Gamma_\sigma} + \frac{1}{t-m_\sigma^2} + \frac{1}{u-m_\sigma^2}\Big], \tag{5.21}$$

$$M_{\pi\pi}^{I=1} = g_{\rho\pi\pi}^2\Big[\frac{2(t-u)}{s-m_\rho^2+im_\rho\Gamma_\rho(s)} + \frac{t-s}{u-m_\rho^2} - \frac{u-s}{t-m_\rho^2}\Big] + g_{\sigma\pi\pi}^2 m_\sigma^2\Big[\frac{1}{t-m_\sigma^2} - \frac{1}{u-m_\sigma^2}\Big], \tag{5.22}$$

$$M_{\pi\pi}^{I=2} = g_{\rho\pi\pi}^2\Big[\frac{u-s}{t-m_\rho^2} + \frac{t-s}{u-m_\rho^2}\Big] + g_{\sigma\pi\pi}^2 m_\sigma^2\Big[\frac{1}{t-m_\sigma^2} + \frac{1}{u-m_\sigma^2}\Big], \tag{5.23}$$

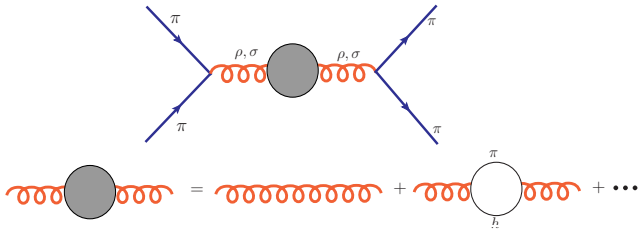

Figure 5.1: $\pi\pi$ scattering with self-energy corrections

where $m_\sigma = 450$ MeV and $\Gamma_\sigma = 550$ MeV, these are in conformity with [3]. The isospin averaged amplitude for $\pi\pi$ scattering via ρ and σ can be obtained using Eqn.(5.9) and thus we can calculate the cross-section using the following equation,

$$\sigma = \frac{1}{16\pi}\int_0^{-s} \frac{|M|^2}{(s^2 - 4m_\pi^2 s)} dt. \tag{5.24}$$

In this calculation the ρ propagator $D^0_{\mu\nu} = (-g_{\mu\nu} + q_\mu q_\nu/m_\rho^2)/(q^2 - m_\rho^2 + i\epsilon)$ has been modified, $i\epsilon$ has been replaced with $im_\rho\Gamma(s)$. Where $\Gamma_\rho(s)$ has been obtained from Eqn.(5.8). This modification has been done only for the s-channel for total isospin $I = 1$, since it's the only channel that contributes to the resonance structure.

5.1.3 $\pi\pi$ cross-section at finite temperature

The self-energy is used to introduce medium-effects. It is evaluated by perturbative methods, using effective interaction. Then the exact propagators are obtained using the Dyson equation Fig. 5.1. In real time thermal field theory, the two-point functions assume a 2×2 matrix form [6] which can be diagonalized. The diagonal component of such matrix also obey the Dyson equation [7], and they are used to obtain the full propagator $D_{\mu\nu}$,

$$D_{\mu\nu} = D^{(0)}_{\mu\nu} + D^{(0)}_{\mu\sigma}\Pi^{\sigma\lambda}D_{\lambda\nu}. \tag{5.25}$$

Here $D_{\mu\nu}$ is vacuum propagator for ρ meson and $\Pi^{\sigma\lambda}$ is the self-energy function obtained from one-loop diagram as represented in fig 5.1. We can write the in-medium self-energy in terms of

longitudinal and transverse part, just like [4, 6]

$$\Pi_{\mu\nu} = P_{\mu\nu}\Pi^T + Q_{\mu\nu}\Pi^L. \tag{5.26}$$

Here,

$$P_{\mu\nu} = -g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} - \frac{q^2}{\bar{q}^2}\tilde{u}_\mu\tilde{u}_\nu, \tag{5.27}$$

is the longitudinal projection tensor, where $\tilde{u}_\mu = u_\mu - (u\cdot q)q_\mu/q^2$; while

$$Q_{\mu\nu} = \frac{q^4}{\bar{q}}\tilde{u}_\mu u_\nu, \tag{5.28}$$

is the transverse projection tensor, where $\bar{q}^2 = (u.q)^2 - q^2$, where u_μ represents the velocity of the thermal bath. Thus we can see that,

$$P_{\mu\nu} + Q_{\mu\nu}/q^2 = -g_{\mu\nu} + q_\mu q_\nu/q^2. \tag{5.29}$$

Note that while P and Q are four dimensional transverse, Q is longitudinal while P is three-dimensional transverse. Solving Eqn.(5.25) the ρ propagator turns out to be,

$$D_{\mu\nu}(q_0,\vec{q}) = -\frac{P_{\mu\nu}}{q^2 - m_\rho^2 - \Pi^T} - \frac{Q_{\mu\nu}/q^2}{q^2 - m_\rho^2 - q^2\Pi^L} + \frac{q_\mu q_\nu}{m_\rho^2 q^2}, \tag{5.30}$$

where,

$$\Pi^T = -\frac{1}{2}(\Pi^\mu_\mu + \frac{q^2}{\bar{q}^2}\Pi_{00}), \qquad \Pi^L = \frac{1}{\bar{q}^2}\Pi_{00}, \qquad \Pi_{00} = U^\mu U^\nu \Pi_{\mu\nu}, \tag{5.31}$$

with

$$\Pi = \frac{1}{3}(2\Pi^T + q^2\Pi^L) \tag{5.32}$$

being the polarisation averaged self-energy function. Using this expressions in Eqn.(5.30), and neglecting the non-pole piece we get,

$$D_{\mu\nu}(q_0,\vec{q}) = \frac{-g_{\mu\nu} + q_\mu q_\nu/q^2}{q^2 - m_\rho^2 - \text{Re }\Pi(q_0,\vec{q}) + i\text{Im }\Pi(q_0,\vec{q})}. \tag{5.33}$$

The self-energy term in Eqn.(5.33) assumes a 2×2 matrix structure, in the real time formalism of thermal field theory, its 11-component is given by,

$$\Pi^{11}_{\mu\nu}(q) = i\int \frac{d^4k}{(2\pi)^4} N_{\mu\nu}(q,k) D^{11}_{\pi}(k) D^{11}_{h}(q-k), \tag{5.34}$$

where $D^{11}(k) = \triangle(k) + 2\pi i f^{(0)}(k)\delta(k^2 - m^2)$ is the 11-component of the scalar propagator which constitutes the internal line of the loop, and $\triangle(k)$ is the vacuum part. The expression for $N_{\mu\nu}$ is available in [4], where the interaction has been taken from chiral perturbation theory, and includes the tensor structure of the vector propagator and the two vertices . The real and imaginary part of the self-energy function in Eqn.(5.33) can can be expressed as [6, 7],

$$\begin{aligned} \mathrm{Re}\Pi_{\mu\nu} &= \mathrm{Re}\ \Pi^{11}_{\mu\nu} \\ \mathrm{Im}\Pi_{\mu\nu} &= \epsilon(q_0)\tanh(\beta q_0/2)\mathrm{Im}\ \Pi^{11}_{\mu\nu}. \end{aligned} \tag{5.35}$$

Integrating over suitable contour we get,

$$\begin{aligned} \Pi^{\mu\nu}(q_0,\vec{q}) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{4\omega_\pi\omega_h}\Big[&\frac{(1+f^0(\omega_\pi))N_1^{\mu\nu} + f^0(\omega_h)N_3^{\mu\nu}}{q_0-\omega_\pi-\omega_h+i\eta\epsilon(q_0)} + \frac{-f^0(\omega_\pi)N_1^{\mu\nu} + f^0(\omega_h)N_4^{\mu\nu}}{q_0-\omega_\pi+\omega_h+i\eta\epsilon(q_0)} \\ &+\frac{f^0(\omega_\pi)N_2^{\mu\nu} - f^0(\omega_h)N_3^{\mu\nu}}{q_0+\omega_\pi-\omega_h+i\eta\epsilon(q_0)} + \frac{-f^0(\omega_\pi)N_2^{\mu\nu} - (1+f^0(\omega_h))N_4^{\mu\nu}}{q_0+\omega_\pi+\omega_h+i\eta\epsilon(q_0)}\Big]. \end{aligned} \tag{5.36}$$

Where, $f^0(\omega) = \frac{1}{e^{\omega-\mu_\pi/T}-1}$, $\omega_\pi = \sqrt{\vec{k}^2 + m_\pi^2}$ and $\omega_h = \sqrt{(\vec{q}-\vec{k})^2 + m_h^2}$. The subscript (1 to 4) in the above equation corresponds to $k_0 = \omega_\pi, -\omega_\pi, q_0-\omega_h$ and $q_0+\omega_h$ respectively. The processes like decay, scattering and regeneration of ρ which results in its gain or loss are embodied in the imaginary part of the self-energy. The δ-functions in the imaginary part defines the region in q_0 and $\vec{q}$ where these process can occur [4], and using them the angular integration is carried out.

The real part of the self-energy modifies the position of the pole of the spectral function. The discontinuities of the self-energies in the complex energy plane provide us with the imaginary parts for $\pi\pi$, $\pi\omega$, πh_1 and πa_1 loops. Since the mesons ω, h_1 and a_1 have negative G-parity and substantial 3π and $\rho\pi$ decay widths [2], the polarization averaged self-energy of the loop

containing the particles have been folded with their spectral functions [5]

$$\Pi(q, m_h) = \frac{1}{N_h} \int_{(m_h-2\Gamma_h)^2}^{(m_h+2\Gamma_h)^2} dM^2 \frac{1}{\pi} \mathrm{Im}\Big[\frac{1}{M^2 - m_h^2 + iM\Gamma_h(M)}\Big] \Pi(q, M). \tag{5.37}$$

Where,

$$N_h = \int_{(m_h-2\Gamma_h)^2}^{(m_h+2\Gamma_h)^2} dM^2 \frac{1}{\pi} \mathrm{Im}\Big[\frac{1}{M^2 - m_h^2 + iM\Gamma_h(M)}\Big]. \tag{5.38}$$

The contribution from loops containing the heavy mesons (the πh loops) are considered as multi pion contribution to the ρ self-energy.

Similarly we take the effective propagator of σ as,

$$D(q_0, \vec{q}) = \frac{1}{q^2 - m_\sigma^2 - \mathrm{Re}\ \Pi(q_0, \vec{q}) + i\mathrm{Im}\ \Pi(q_0, \vec{q})}. \tag{5.39}$$

Following similar steps, as taken in in the previous case, the self-energy of σ turns out to be,

$$\begin{aligned}\Pi(q_0, \vec{q}) = N \int \frac{d^3k}{(2\pi)^3} \frac{1}{4\omega_\pi \omega'_\pi} \Big[&\frac{1 + f^0(\omega_\pi) + f^0(\omega'_\pi)}{q_0 - \omega_\pi - \omega'_\pi + i\eta\epsilon(q_0)} + \frac{f^0(\omega'_\pi) - f^0(\omega_\pi)}{q_0 - \omega_\pi + \omega'_\pi + i\eta\epsilon(q_0)} \\ &+ \frac{f^0(\omega_\pi) - f^0(\omega'_\pi)}{q_0 + \omega_\pi - \omega'_\pi + i\eta\epsilon(q_0)} - \frac{1 + f^0(\omega_\pi) + f^0(\omega'_\pi)}{q_0 + \omega_\pi + \omega'_\pi + i\eta\epsilon(q_0)} \Big],\end{aligned} \tag{5.40}$$

where, $\omega'_\pi = \sqrt{(\vec{q} - \vec{k})^2 + m_\pi^2}$.

In Fig. 5.2 we plot the total $\pi\pi$ cross-section defined by $\sigma(s) = \frac{1}{2} \int d\Omega \frac{|M|}{64\pi^2 s}$. Here we see the effect of the introduction of the in-medium ρ propagator in place of the vacuum propagator $D^0_{\mu\nu}$ along with in-medium σ propagator, on the cross-section of the $\pi\pi$ collision. We see the suppression of the cross-section at the resonance and a shift of the resonance peak. The suppression of the peak becomes greater with the introduction of multi pions. This change is mainly due to the temperature dependence of the real and imaginary parts of the self-energy. The width occurring in the denominator of the propagator increases with the temperature, thus bringing down the resonance peak.

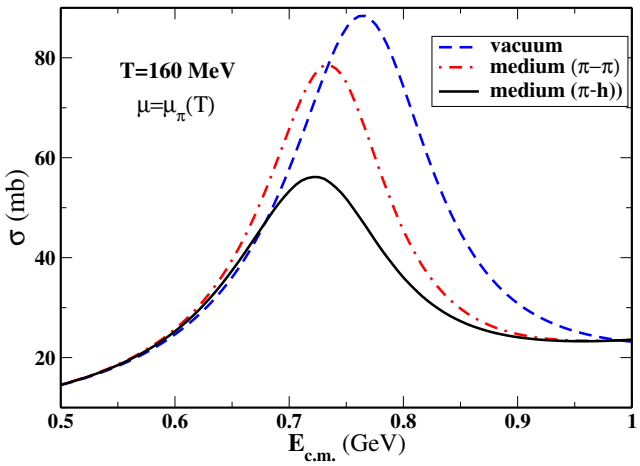

Figure 5.2: The $\pi\pi$ cross-section as a function of $E_{c.m.}$. The dashed line corresponds to scattering in the vacuum. The Dot-dashed line refers to the in-medium cross-section involving only the pion loop for the σ and ρ mesons. The solid line corresponds to the additional loops in the ρ meson self-energy.

5.1.4 Temperature dependent pion chemical potential

In relativistic heavy ion collisions, below the crossover temperature, inelastic reactions cease and, this leads to chemical freeze-out of hadrons. Since only elastic collisions occur, the number-density gets fixed at this temperature and to conserve it a phenomenological chemical potential is introduced which increases with decreasing temperature until kinetic freezeout is reached [8]. In this work, we use the numerical results of the temperature-dependent pion chemical potential from the work of [9] where the above scenario is implemented. It is depicted by the parametric form

$$\mu_T = a + bT + cT^2 + dT^3, \tag{5.41}$$

where $a = 0.824$, $b = 3.04$, $c = 0.028$, $d = 6.0510^5$ and T, μ in MeV. By fixing the ratio s/n, s being the entropy density and n being the number density, to the value at chemical freeze-out where $\mu = 0$, one can go down in temperature up to 100 MeV the kinetic freeze-out by increasing the pion chemical potential, thus leading to the temperature dependence leading to $\mu(T)$ as depicted in Fig. 5.3

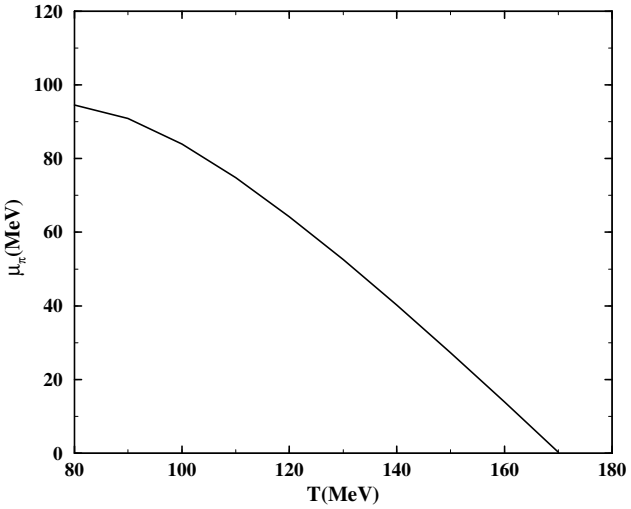

Figure 5.3: The temperature dependent pion chemical potential [9].

5.2 πN scattering

5.2.1 The Δ self-energy in the medium

The in-medium propagators in the real time formalism

All two-point functions in real time formalism of thermal field can essentially be described by a single analytic function, even though the two-point function assumes the form of 2×2 matrix [6, 10]. This is achieved by diagonalising the matrix. Since the function can be given by any one of the components of the matrix, we obtain only that component of the self-energy matrix. Here we will specify only the 11-component of the thermal propagator for the particle participating in the one loop graph.

The 11-component of a free thermal propagator matrix for a particle is composed of two parts. The first part is the vacuum propagator and the second part is a term which is determined by the on-shell distribution function of particles of the same type as the propagator in the medium through which it propagates. The 11-component of the thermal pion propagator is expressed as,

$$D_{11}(k, m_\pi) = \Delta(k, m_\pi) + 2\pi i N_1^2(k, m_\pi)\delta(k^2 - m_\pi^2) \tag{5.42}$$

where $\Delta(k,m) = \frac{-1}{k^2-m^2+i\eta}$ and, $N_1(k,m) = \theta(k^0)\sqrt{n_+^k} + \theta(-k^0)\sqrt{n_-^k}$ with $n_\pm^k = \frac{1}{e^{\beta(\omega_k \mp \mu_k)}-1}$, $\omega_k = \sqrt{\vec{k}^2+m^2}$, and $\theta(k^0) = 1$ for $k^0 > 0$ and 0 for $k^0 < 0$. The 11-component of the thermal ρ propagator is expressed as,

$$D_{11}^{\mu\nu}(k,m_\rho) = \left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{m_\rho^2}\right) D_{11}(k,m_\rho) \ . \tag{5.43}$$

Nucleon and Δ being fermions their thermal matrix part is different than those for bosons, and the 11-component of their propagator is given by,

$$S^{11}(p) = (\not{p} + m_N) E^{11}(p,m_N) \tag{5.44}$$

and

$$\begin{aligned} S_{\mu\nu}^{11}(p) = (\not{p} + m_\Delta) \quad & \left\{ -g_{\mu\nu} + \frac{2}{3m_\Delta^2} p_\mu p_\nu + \frac{1}{3}\gamma_\mu\gamma_\nu \right. \\ & \left. + \frac{1}{3m_\Delta}(\gamma_\mu p_\nu - \gamma_\nu p_\mu) \right\} E^{11}(p,m_\Delta) \end{aligned} \tag{5.45}$$

respectively, where $E_{11}(p,m) = \Delta(p,m) - 2\pi i \tilde{N}_1^2 \delta(p^2-m^2)$, with $\tilde{N}_1(p_0) = \theta(p_0)\sqrt{\tilde{n}_+^p} + \theta(-p_0)\sqrt{\tilde{n}_-^p}$. The complete propagator $\boldsymbol{S'}$ in terms of the free fermion propagator $\boldsymbol{S}$ and self-energy $\boldsymbol{\Pi}$ is given by the Dyson equation,

$$\boldsymbol{S'} = \boldsymbol{S} - \boldsymbol{S}\,\boldsymbol{\Pi}\,\boldsymbol{S'}. \tag{5.46}$$

each bold terms represents a 2×2 matrix in the thermal indices. These matrices can be diagonalised to obtained the respective analytic function denoted by a bar, so

$$\overline{S'} = \overline{S} - \overline{S}\,\overline{\Pi}\,\overline{S'} \ . \tag{5.47}$$

Any single component of the self-energy matrix can provide the self-energy function $\overline{\Pi}$. It is related to the 11-component by

$$\begin{aligned} \mathrm{Im}\bar{\Pi}(p) &= \epsilon(p_0)\coth[\beta(p_0-\mu_p)/2]\mathrm{Im}\Pi_{11}(p), \\ \mathrm{Re}\bar{\Pi}(p) &= \mathrm{Re}\Pi_{11}(p), \end{aligned} \tag{5.48}$$

where $\epsilon(p_0) = +1$ for $p_0 > 0$ and -1 for $p_0 < 0$.

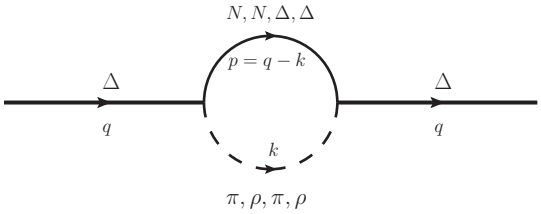

Figure 5.4: Feynman Diagrams for Δ-Self Energy.

The Δ self-energy

The effective Lagrangian for $\pi N\Delta$, $\rho N\Delta$, $\pi\Delta\Delta$ and $\rho\Delta\Delta$ are given by the well known interactions [11]

$$\mathcal{L}_{\pi N\Delta} = \frac{f_{\pi N\Delta}}{m_\pi}\bar{\Delta}_\alpha \mathcal{O}^{\alpha\mu}\vec{T}^\dagger \partial_\mu \vec{\pi}\psi + H.c. \tag{5.49}$$

$$\mathcal{L}_{\rho N\Delta} = -i\frac{f_{\rho N\Delta}}{m_\rho}\bar{\Delta}_\alpha \mathcal{O}^{\alpha\mu}\gamma^5\gamma^\nu \vec{T}^\dagger \vec{\rho}_{\mu\nu}\psi + H.c. \tag{5.50}$$

$$\mathcal{L}_{\pi\Delta\Delta} = \frac{f_{\pi\Delta\Delta}}{m_\pi}\bar{\Delta}^\alpha \mathcal{O}_{\alpha\mu}\gamma^5\gamma^\nu \vec{T}\Delta^\mu \partial_\nu \vec{\pi} \tag{5.51}$$

$$\mathcal{L}_{\rho\Delta\Delta} = -f_{\rho\Delta\Delta}\bar{\Delta}^\beta \mathcal{O}_{\alpha\beta}\left[\gamma^\mu - \frac{\kappa_{\rho\Delta\Delta}}{2m_\Delta}\sigma^{\mu\nu}\partial_\nu\right]\vec{\rho}_\mu \vec{T}\Delta^\alpha \tag{5.52}$$

where $f_{\pi N\Delta} = 2.8$, $f_{\rho N\Delta} = 16.03$, $f_{\pi\Delta\Delta} = 1.78$, $f_{\rho\Delta\Delta} = 7.67$, $\kappa_{\rho\Delta\Delta} = 6.1$ and $\mathcal{O}_{\alpha\beta} = g_{\alpha\beta} - a\gamma_\alpha\gamma_\beta$. The second term of $\mathcal{O}$ contributes only when the spin-3/2 is off the mass shell, thus keeping the coupling constants unchanged. The form factor at the vertex has been taken as

$$F(p,k) = \frac{\Lambda^2}{\Lambda^2 + (\frac{p\cdot k}{m_p})^2 - k^2}, \tag{5.53}$$

where p and k denote the momenta of the fermion and boson respectively. By fitting the phase shift and vacuum cross-section for πN scattering we get $a = 0.002$ and $\Lambda = 600$ MeV. We consider an exponential form factor (Form Factor-II),

$$F(p) = \exp[-(p^2 - (m_N + m_\pi)^2)/\Lambda^2] \tag{5.54}$$

where $a = 0.002$, $\Lambda = 1.25\ GeV$ and p represents the momentum of Δ. Using the interaction expressed by the above set of Lagrangian the vacuum self-energy for the one loop diagrams

shown in Fig. 5.4 turns out to be

$$\Pi^{\mu\nu}_{vac} = \sum_{i\epsilon\{\pi,\rho\}} \sum_{j\epsilon\{N,\Delta\}} \Pi^{\mu\nu}_{ij\Delta} = \Pi^{\mu\nu}_{\pi N\Delta} + \Pi^{\mu\nu}_{\rho N\Delta} + \Pi^{\mu\nu}_{\pi\Delta\Delta} + \Pi^{\mu\nu}_{\rho\Delta\Delta} \tag{5.55}$$

where,

$$\Pi^{\mu\nu}_{\pi N\Delta} = i\frac{f^2_{\pi N\Delta}}{m^2_\pi}\int \frac{d^4k}{(2\pi)^4} F^2(p,k)\mathcal{O}^{\nu\beta}k_\beta S^0(p)\mathcal{O}^{\alpha\mu}k_\alpha D^0(k) \tag{5.56}$$

$$\begin{aligned}\Pi^{\mu\nu}_{\rho N\Delta} &= i\frac{f^2_{\rho N\Delta}}{m^2_\rho}\int \frac{d^4k}{(2\pi)^4} F^2(p,k)\mathcal{O}^{\nu\eta}\gamma^5\gamma^\phi(g_{\beta\phi}k_\eta - g_{\beta\eta}k_\phi) \\ &\qquad S^0(p)\gamma^5\gamma^\lambda(g_{\alpha\lambda}k_\sigma - g_{\alpha\sigma}k_\lambda)\mathcal{O}^{\mu\sigma}D^{\alpha\beta}_0(k)\end{aligned} \tag{5.57}$$

$$\Pi^{\mu\nu}_{\pi\Delta\Delta} = i\frac{f^2_{\pi\Delta\Delta}}{m^2_\pi}\int \frac{d^4k}{(2\pi)^4} F^2(p,k)\mathcal{O}^{\nu\chi}\gamma^5\gamma^\beta k_\beta \mathcal{O}^{\psi\sigma}g_{\chi\psi}S^0_{\lambda\sigma}(p)\mathcal{O}^{\lambda\eta}\gamma^5\gamma^\alpha k_\alpha \mathcal{O}^{\phi\mu}g_{\eta\phi}D^0(k) \tag{5.58}$$

$$\begin{aligned}\Pi^{\mu\nu}_{\rho\Delta\Delta} &= if^2_{\rho\Delta\Delta}\int \frac{d^4k}{(2\pi)^4} F^2(p,k)\mathcal{O}^{\nu\chi}(\gamma^\beta + i\frac{\kappa_{\Delta\Delta\rho}}{2m_\Delta}\sigma^{\beta\epsilon}k_\epsilon)\mathcal{O}^{\psi\sigma}g_{\chi\psi} \\ &\qquad S^0_{\lambda\sigma}(p)\mathcal{O}^{\lambda\eta}(\gamma^\alpha - i\frac{\kappa_{\Delta\Delta\rho}}{2m_\Delta}\sigma^{\alpha\delta}k_\delta)\mathcal{O}^{\phi\mu}g_{\eta\phi}D^0_{\alpha\beta}(k)\end{aligned} \tag{5.59}$$

with scalar propagator $D^0(k) = \Delta(k, m_\pi)$, vector propagator $D^0_{\mu\nu}(k) = A_{\mu\nu}(k)\Delta(k, m_\rho)$, Dirac field propagator $S^0(p) = (\not{p} + m)\Delta(p, m_N)$ and the Rarita-Schwinger field propagator $S^0_{\mu\nu}(p) = \Sigma_{\mu\nu}(p)\Delta(p, m_\Delta)$. The vacuum self-energy of Δ can be expressed as,

$$\Pi^{\mu\nu}_{vac} = \sum_{i\epsilon\{\pi,\rho\}} \sum_{j\epsilon\{N,\Delta\}} i\int \frac{d^4k}{(2\pi)^4} N^{\mu\nu}_{ij\Delta}(k_i, p_j)D_F(k_i, m_i)D_F(p_j, m_j) \tag{5.60}$$

where,

$$N^{\mu\nu}_{\pi N\Delta} = \frac{f^2_{\pi N\Delta}}{m^2_\pi}F^2(p,k)\left[k_\alpha k_\beta \mathcal{O}^{\nu\beta}(\not{p} + m_p)\mathcal{O}^{\alpha\mu}\right] \tag{5.61}$$

$$N^{\mu\nu}_{\rho N\Delta} = \frac{f^2_{\rho N\Delta}}{m^2_\rho}F^2(p,k)\left[\mathcal{O}^{\nu\eta}\gamma^5\left(\gamma_\beta k_\eta - g_{\beta\eta}\not{k}\right)(\not{p} + m_p)\gamma^5\left(\gamma_\alpha k_\sigma - g_{\alpha\sigma}\not{k}\right)\mathcal{O}^{\mu\sigma}A^{\alpha\beta}\right] \tag{5.62}$$

$$N^{\mu\nu}_{\pi\Delta\Delta} = \frac{f^2_{\pi\Delta\Delta}}{m^2_\pi}F^2(p,k)\left[g_{\chi\psi}g_{\eta\phi}\mathcal{O}^{\nu\chi}\gamma^5\not{k}\mathcal{O}^{\psi\sigma}\Sigma_{\lambda\sigma}(p)\mathcal{O}^{\lambda\eta}\gamma^5\not{k}\mathcal{O}^{\phi\mu}\right] \tag{5.63}$$

$$\begin{aligned}N^{\mu\nu}_{\rho\Delta\Delta} &= f^2_{\rho\Delta\Delta}F^2(p,k)\left[g_{\chi\psi}g_{\eta\phi}\mathcal{O}^{\nu\chi}\left(\gamma^\beta + i\frac{\kappa_{\rho\Delta\Delta}}{2m_\Delta}\sigma^{\beta\epsilon}k_\epsilon\right)\mathcal{O}^{\psi\theta}\Sigma_{\lambda\theta}(p)\right. \\ &\qquad \left.\mathcal{O}^{\lambda\eta}\left(\gamma^\alpha - i\frac{\kappa_{\rho\Delta\Delta}}{2m_\Delta}\sigma^{\alpha\delta}k_\delta\right)\mathcal{O}^{\phi\mu}A_{\alpha\beta}\right]\end{aligned} \tag{5.64}$$

with

$$A_{\alpha\beta}(k) \;=\; -g_{\alpha\beta} + \frac{k_\alpha k_\beta}{m_k^2} \tag{5.65}$$

and

$$\Sigma_{\alpha\beta}(q) \;=\; (\not{q} + m_q)\left[-g_{\alpha\beta} + \frac{1}{3m_q^2}q_\alpha q_\beta + \frac{1}{3}\gamma_\alpha\gamma_\beta + \frac{1}{3m_q}\left(\gamma_\alpha q_\beta - \gamma_\beta q_\alpha\right)\right] \;. \tag{5.66}$$

To write down the corresponding expressions in the medium, only the 11-component $\Pi_{11}^{\mu\nu}$ needs to be calculated, as we have discussed in previous section. Thus the self-energy in the medium is expressed as

$$\Pi_{11}^{\mu\nu}(q) = i\int \frac{d^4k}{(2\pi)^4} N^{\mu\nu} E_{11}(p) D_{11}(k) \;. \tag{5.67}$$

On expanding D_{11} and E_{11} in the above equation we get four terms, the first term is $\Pi_{vac}^{\mu\nu}$ the second and the third terms are linear in thermal distribution function, while the fourth term is non-linear in the distribution function which is purely imaginary. We obtain the imaginary and real parts of the self-energy function after performing the k^0 integral and using Eqn.(5.48),

$$\begin{aligned}
\mathrm{Im}\bar{\Pi}^{\mu\nu}(q) \;=\; & -\pi\epsilon(q_0)\int \frac{d^3k}{(2\pi)^3}\frac{1}{4\omega_k\omega_p} \times \\
& [N^{\mu\nu}(k^0=\omega_k)\{(1+n_+^k-\tilde{n}_+^p)\delta(q_0-\omega_k-\omega_p) + (-n_+^k-\tilde{n}_-^p)\delta(q_0-\omega_k+\omega_p)\} + \\
& N^{\mu\nu}(k^0=-\omega_k)\{(-1-n_-^k+\tilde{n}_-^p)\delta(q_0+\omega_k+\omega_p) + (n_-^k+\tilde{n}_+^p)\delta(q_0+\omega_k-\omega_p)\}]
\end{aligned} \tag{5.68}$$

and

$$\begin{aligned}
\mathrm{Re}\bar{\Pi}^{\mu\nu}(q) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k\omega_p} \quad & \mathcal{P}\left[\left(\frac{n_+^k\omega_p N^{\mu\nu}(k^0=\omega_k)}{(q_0-\omega_k)^2-\omega_p^2}\right) + \left(\frac{n_-^k\omega_p N^{\mu\nu}(k^0=-\omega_k)}{(q_0+\omega_k)^2-\omega_p^2}\right) - \right. \\
& \left.\left(\frac{\tilde{n}_+^p\omega_k N^{\mu\nu}(k^0=q_0-\omega_p)}{(q_0-\omega_p)^2-\omega_k^2}\right) - \left(\frac{\tilde{n}_-^p\omega_k N^{\mu\nu}(k^0=q_0+\omega_p)}{(q_0+\omega_p)^2-\omega_k^2}\right)\right]
\end{aligned} \tag{5.69}$$

where $\omega_k = \sqrt{m_k^2+\vec{k}^2}$ and $\omega_p = \sqrt{m_p^2+(\vec{q}-\vec{k})^2}$. Each of the four terms in the imaginary part corresponds to a scattering and decay of the Δ-baryon, and the delta functions defines

the kinematic domain in which the processes occur. The regions in the complex q_0 plane where these terms are non-vanishing are known as branch cuts. The region $q^2 > (m_k + m_p)^2$ known as the unitary cut, already present in vacuum, is the range where the first and the third terms are non-vanishing. While the region $q^2 < (m_p - m_k)^2$ where the second and the fourth terms are non-vanishing is known as the Landau cut, and is purely a medium-effect. We confine to the region $q_0 > 0$ and $q^2 > 0$, here only the first and the fourth terms contribute. The first term represents the decay of Δ into baryon-pair $N\pi$, $N\rho$ etc, and the Bose enhancement of the meson and Pauli blocking of the baryon in these processes are indicated by the thermal factor $1 + n_+^k - \tilde{n}_+^p = (1 + n_+^k)(1 - \tilde{n}_+^p) + n_+^k \tilde{n}_+^p$. The fourth term corresponds to the absorption of Δ due to scattering by a meson, leading to the production baryon and vice versa, which is evident from the thermal weight factor $n_-^k + \tilde{n}_+^p = n_-^k(1 - \tilde{n}_+^p) + \tilde{n}_+^p(1 + n_-^k)$. The self-energy functions are folded with their spectral function in order to account for the finite width of unstable particles, and so the sharp thresholds of the branch cuts get smeared. For unstable mesons h we use [12],

$$\Pi(q, m_h) = \frac{1}{N_h} \int_{(m_h - 2\Gamma_h)^2}^{(m_h + 2\Gamma_h)^2} dM^2 \frac{1}{\pi} \mathrm{Im} \left[\frac{1}{M^2 - m_h^2 + iM\Gamma_h(M)} \right] \Pi(q, M), \tag{5.70}$$

where $N_h = \displaystyle\int_{(m_h - 2\Gamma_h)^2}^{(m_h + 2\Gamma_h)^2} dM^2 \frac{1}{\pi} \mathrm{Im} \left[\frac{1}{M^2 - m_h^2 + iM\Gamma_h(M)} \right]$ and $\Gamma_h = \Gamma_h(m_h)$. Here $h \equiv \rho$ so,

$$\Gamma_\rho(M) = \Gamma_{\rho \to \pi\pi}(M) = \left[\frac{g_{\rho\pi\pi}^2}{48\pi M^3} \right] \left[M^2 - 4m_\pi^2 \right] \lambda^{\frac{1}{2}} \left(M^2, m_\pi^2, m_\pi^2 \right) \tag{5.71}$$

where $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2(xy + yz + zx)$. In order to obtain this expression we have used $\mathcal{L}_{\rho\pi\pi} = g_{\rho\pi\pi} \vec{\rho}_\mu \cdot (\vec{\pi} \times \partial^\mu \vec{\pi})$ with $g_{\rho\pi\pi} = 6.05$. For baryons R with non-trivial decay width we use

$$\Pi(q, m_R) = \frac{1}{N_R} \int_{m_R - 2\Gamma_R}^{m_R + 2\Gamma_R} dM \frac{1}{\pi} \mathrm{Im} \left[\frac{1}{M - m_R + \frac{i}{2}\Gamma_R(M)} \right], \Pi(q, M) \tag{5.72}$$

in the loops, where $N_R = \displaystyle\int_{m_R - 2\Gamma_R}^{m_R + 2\Gamma_R} dM \frac{1}{\pi} \mathrm{Im} \left[\frac{1}{M - m_R + \frac{i}{2}\Gamma_R(M)} \right]$ and $\Gamma_R = \Gamma_R(m_R)$. In this case $R \equiv \Delta$ for which the decay formula is given by Eqn.(5.73).

5.2.2 The π-N Cross Section

Our aim is to construct a dynamical framework by which medium-effects can be incorporated using thermal field theoretic methods into the πN cross-section, which at the same time is normalized to the experimental data in vacuum. considering the $\pi N\Delta$ interaction (5.49) we compare it with the phase shift date, defining $\tan(\delta_{33}) = \frac{\mathrm{Im}f}{\mathrm{Re}f}$ with the partial wave amplitude given by $f(E) \sim 1/[E^2 - m_\Delta^2 + im_\Delta\Gamma_\Delta(E)]$, where the $\Delta \to \pi N$ decay width which follows from the imaginary part of Eqn.(5.56) is given by,

$$\Gamma_\Delta(E) = \frac{1}{24\pi}\left(\frac{f_{\pi N\Delta}}{m_\pi}\right)^2 F^2(E)\frac{\vec{p}^{\,3}}{E^2}[(E+m_N)^2 - m_\pi^2] \tag{5.73}$$

with the c.m. momentum $\vec{p}^{\,2} = [E^2 - (m_N + m_\pi)^2][E^2 - (m_N - m_\pi)^2]/4E^2$. The phase shift thus calculated is in good agreement with the data using $\Lambda = 600$ MeV and $m_\Delta = 1234$ MeV. We calculate the invariant amplitudes for elastic πN scattering in the isospin basis, here we replace the vacuum Δ propagator with an effective propagator containing the vacuum self-energy due to the loop diagrams mentioned above. The squared invariant amplitude for the process $\pi(k)\ N(p) \to\ \pi(k')\ N(p')$ averaged over isospin is expressed as,

$$\begin{aligned}
|\bar{\mathcal{M}}|^2 &= \frac{\sum(2I+1)|\mathcal{M}_I|^2}{\sum(2I+1)} \\
&= \frac{1}{3}\left(\frac{f_{\pi N\Delta}}{m_\pi}\right)^4\left[\frac{F^4(k,p)T_s}{\left|s - m_\Delta^2 - \Pi\right|^2} + \frac{F^4(k,p')T_u}{\left(u - m_\Delta^2\right)^2}\right. \\
&\qquad \left. + \frac{2F^2(k,p)F^2(k,p')T_m(s - m_\Delta^2 - \mathrm{Re}\Pi)}{3(u - m_\Delta^2)\left|s - m_\Delta^2 - \Pi\right|^2}\right]
\end{aligned} \tag{5.74}$$

where,

$$T_s = \mathrm{Tr}\left[(\not{p}' + m_N)D_s(\not{p} + m_N)\gamma^0 D_s^\dagger \gamma^0\right] \tag{5.75}$$

$$T_u = \mathrm{Tr}\left[(\not{p}' + m_N)D_u(\not{p} + m_N)\gamma^0 D_u^\dagger \gamma^0\right] \tag{5.76}$$

$$T_m = \mathrm{Tr}\left[(\not{p}' + m_N)D_s(\not{p} + m_N)\gamma^0 D_u^\dagger \gamma^0\right] \tag{5.77}$$

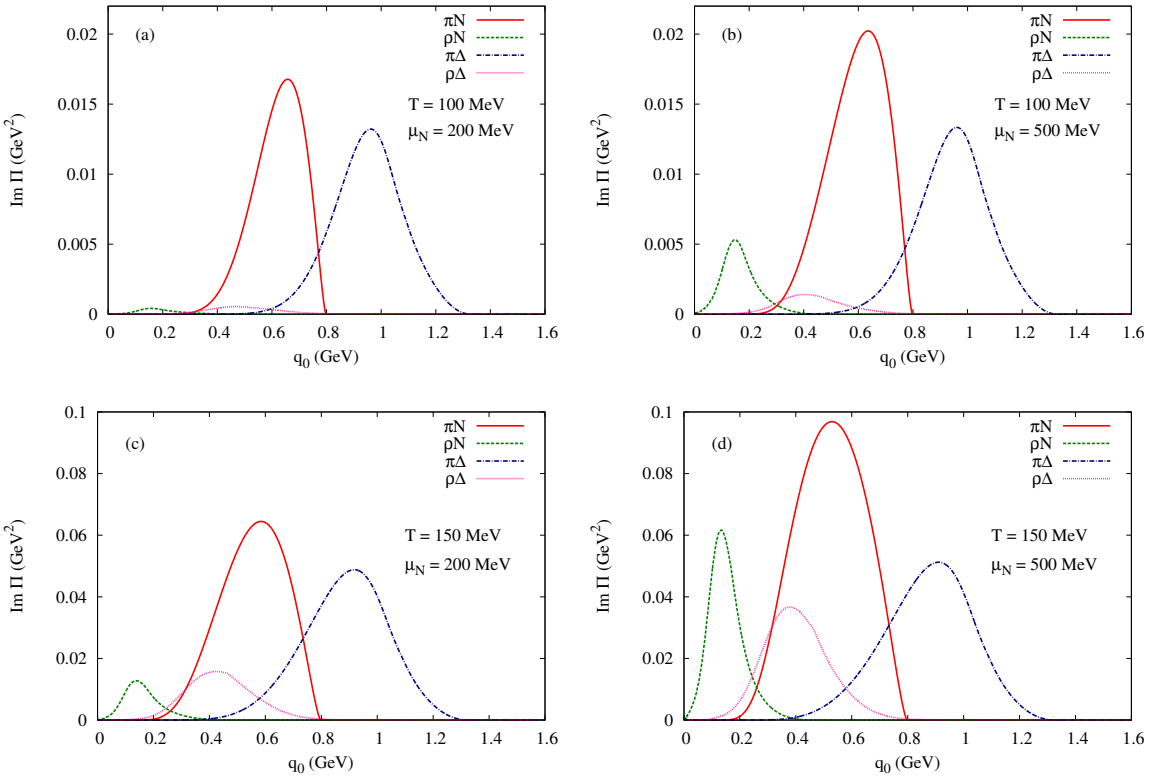

Figure 5.5: The Landau-cut contribution to the imaginary part of the self-energy function for different T and μ_N. [24]

with,

$$D_s = k_\alpha k'_\beta \mathcal{O}^{\beta\nu} \Sigma_{\mu\nu}(q_s) \mathcal{O}^{\mu\alpha} \tag{5.78}$$

$$D_u = k'_\alpha k_\beta \mathcal{O}^{\beta\nu} \Sigma_{\mu\nu}(q_u) \mathcal{O}^{\mu\alpha}. \tag{5.79}$$

The term $\Sigma_{\mu\nu}$ has been defined in Eqn.(5.66). With these we can calculate the cross-section using the expression,

$$\sigma(s) = \frac{1}{64\pi^2 s} \int |\bar{\mathcal{M}}|^2 d\Omega. \tag{5.80}$$

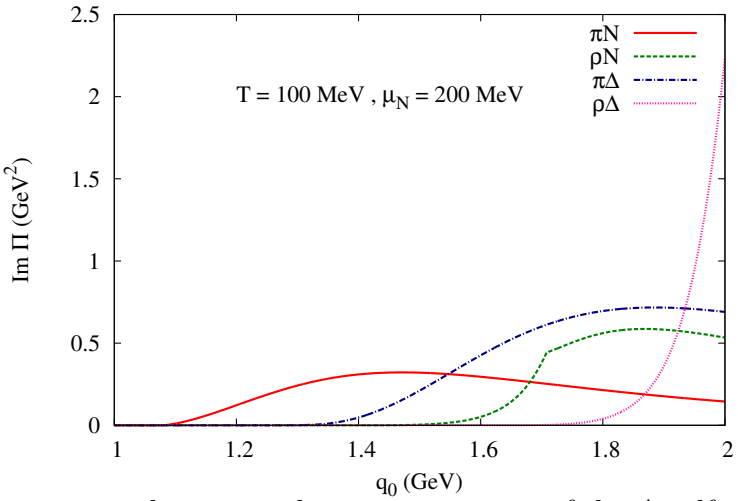

Figure 5.6: The unitary cut contribution to the imaginary part of the Δ self-energy at $T = 100$ MeV and $\mu_N = 200$ MeV. [24]

5.2.3 Numerical results

First we show the numerical result for the spin averaged imaginary part of the function Π [14,15],

$$\Pi = \frac{1}{4} \sum_{s_\Delta} \bar{\Psi}_\mu \bar{\Pi}^{\mu\nu} \Psi_\nu \tag{5.81}$$

where $\bar{\Psi}_\mu$ denote Rarita-Schwinger spinors. The factors $N^{\mu\nu}$ of Eqn.(5.64) then go over to $\frac{1}{4}\mathrm{Tr}[N^{\mu\nu}\Sigma_{\mu\nu}]$.

The Landau cut contribution to the imaginary part coming from the different loop graphs have been represented in Fig. 5.5. The upper and the lower set of panels have been calculated for $T = 100$ MeV and $T = 150$ MeV respectively, for the panel (a) and (c) we have taken $\mu_N = 200$ MeV, while for the panel (b) and (d) we have taken $\mu_N = 500$ MeV. From these, it is evident that the ρN and $\rho\Delta$ loops start contributing to the imaginary part only at larger baryon densities.

The unitary cut contributions from the four loops at $T = 100$ MeV for different values of μ_N have been represented in Fig. 5.6. These are the same as in vacuum apart from the Pauli blocking and Bose enhancement factors in the final state. The monotonous rise of these contributions at higher q_0 has been suppressed by the form factor. The contributions to the imaginary part by

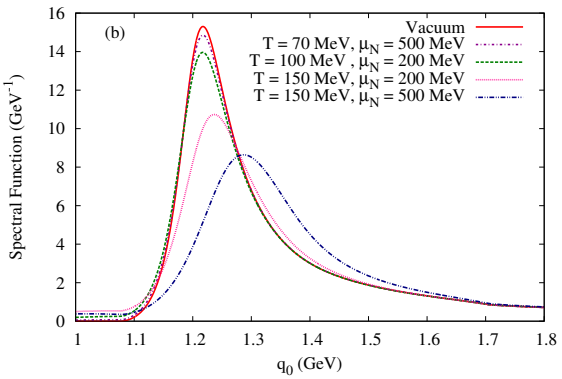

Figure 5.7: The spectral function of the Δ for q_0 from 1 to 1.8 GeV. [24]

the unitary cut is found to be much larger in magnitude than the contribution by the Landau cut.

The unitary cut contribution in high q_0 region in the vicinity of the bare Δ mass consisting of the contributions from the unitary cuts is shown in Fig. 5.7. The temperature and chemical potential of the medium are found to have a significant influence over the spectral density. Due to the larger imaginary parts in the denominator of the in-medium propagator a gradual suppression of the peak with increasing temperature is observed.

A reasonable agreement is observed between the present work and that of [16], when the Δ spectral function obtained by both the process are plotted side by side in Fig. 5.8 panel (a), where continuous lines and symbols represent the one calculated by our process and by [16] respectively . The slight disparity that does arise is due to the differences in the Lagrangian and associated parameters, as well as due to higher order effects introduced through dressed nucleon and pion propagators in the Δ self-energy considered in [16] wherein vertex corrections were included through Migdal parameters in the pion propagator.

In Fig. 5.8 panel (b) we plot the Δ spectral function after the introduction of phenomenological hadronic form factors, to take into account the finite size of the vertices. The introduction of the Form Factor I with $\Lambda = 600$ MeV produces a good fit to the phase shift data and πN cross-section. On changing to $\Lambda = 700$ MeV, we find a small reduction at the peak, though the πN cross-section remains largely unchanged. We have also plotted the spectral function

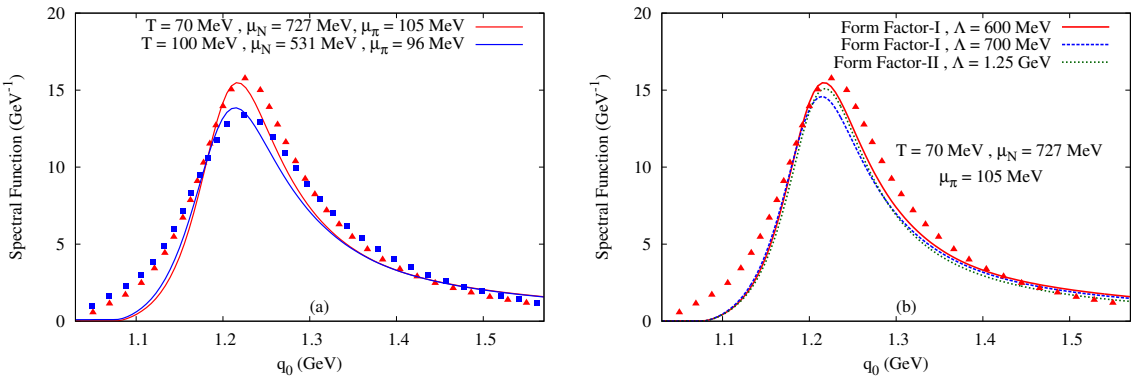

Figure 5.8: The Δ spectral function vs q_0: (a) in comparison with [16] and (b) for different form factors. The results of Ref. [16] are represented by symbols. [24]

calculated using Form Factor II with $\Lambda = 1.25$ GeV (as used in [17] with $\Lambda = 0.97$ GeV), but no appreciable difference is found with the one with Form Factor I and $\Lambda = 600$ MeV. The symbols in the plot denote the results of [16].

In Fig. 5.9 we plot the cross-section calculated using our process and those by [5] (obtained using phase shift and inelasticity data from [23] and [22]). The results are in good agreement with each other. Here in the evaluation of the scattering amplitudes, we have considered only $\Delta(1232)$ exchange so that we can fix the parameters, and thus obtaining a baseline for estimating the effect of the modified Δ propagator on the cross-section. After normalising the framework with the experimental data, we introduce medium-effect. On replacing the vacuum self-energy by the in-medium ones evaluated in the real-time formalism a significant suppression of the peak with increasing temperature is obtained, this is due to the increase in the imaginary part. A small upward shift is also seen at higher baryon densities, which is due to the small positive contribution of the real part of the self-energy.

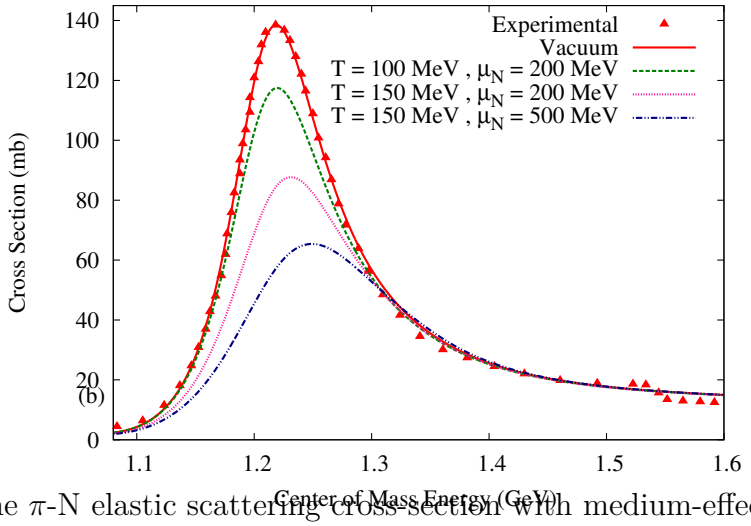

Figure 5.9: The π-N elastic scattering cross-section with medium-effects [24].

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Chapter 6

Numerical Results

In this chapter, we present the results of numerical calculations. We emphasise that most of the literature discussing transport coefficients from kinetic theory perspective has mainly used vacuum interaction cross-sections. The temperature dependence there comes only from the phase space factors, the finite temperature contribution from the dynamics is generally neglected. In the previous chapter, we have discussed the medium-effect on the $\pi\pi$ and πN cross-section. Here in this chapter, we will discuss how the suppression of the cross-section due to the introduction of medium-effect will affect the transport coefficients. The medium-effects on the NN cross-section have not been considered, since the cross-section doesn't have a resonance peak.

In the first section of this chapter, we review the effect of the medium-effect on the viscous coefficients and the thermal conductivity of a pion gas as done in [1, 15, 16]. Chapman-Enskog approximation along with the cross-section derived in the previous chapter has been employed to calculate these transport coefficients. The behaviour of these transport coefficients of the pion gas with varying temperature for various values of pion chemical potential is discussed. These coefficients are used as inputs for the calculation of the relaxation time of flows.

The second section of this chapter deals with the relaxation time of flows that appear in second-order hydrodynamics for a pionic gas [1]. The process employed is the relativistic version of the

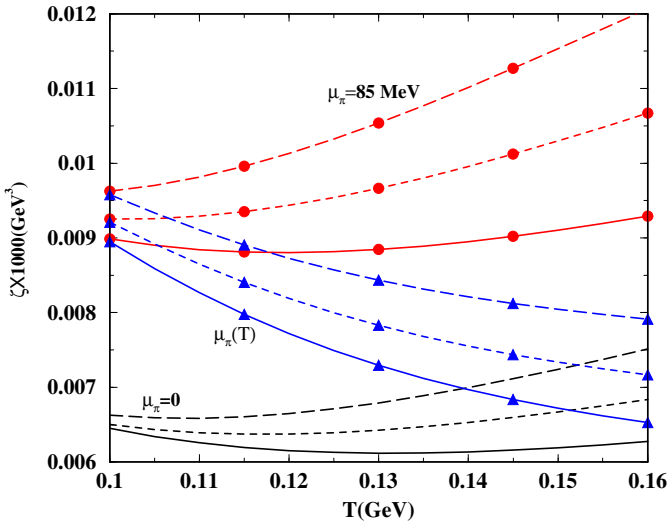

Figure 6.1: The bulk viscosity as a function of T. The upper set of lines (with circles), middle set of lines (with triangles) and lower sets of lines correspond to $\mu_\pi = 85$ MeV, $\mu_\pi = \mu_\pi(T)$ and $\mu_\pi = 0$ MeV respectively. In each set the solid line represents the use of vacuum cross-section, the dotted line for in-medium modification due to pion loop and the dashed line for loops with heavy particles in addition.

Grads-14 moment method, first proposed by Israel-Stewart [2, 3], and then refined by Denicol etal [4]. The property of the relaxation time has been studied for different temperature T and pion chemical potential μ_π. Further, the effect of temperature-dependent pion chemical potential on the relaxation times of flow has been studied.

In the third section, we deal with a two-component hadronic gas composed of pions and nucleons. The transport coefficients that appear in first-order hydrodynamics (Navier-Stoke's theory), namely shear viscosity, bulk viscosity, thermal conductivity, diffusion coefficient, thermal diffusion coefficient and Dufour coefficient have been studied. These coefficients have been derived using the relaxation time approximation as discussed in the previous chapter. The behaviour of these coefficients at different temperature T and nucleon chemical potential μ_N have been studied.

6.1 Viscous coefficients and Thermal conductivity of Pion Gas

First, we consider the bulk viscosity of pion gas taken from [1, 15]. Fig. 6.1 represents the variation of the bulk viscous coefficient with temperature. The figure has three sets of curves, each set corresponds to a different value of pion chemical potential μ_π. The lowest set of curves corresponds to $\mu_\pi = 0$ while the uppermost with circles corresponds to $\mu_\pi = 85$ MeV, these values are representative of the kinetic and chemical freeze-out in heavy ion collisions respectively. The middle set of lines with triangles represent the case where μ_π is a function of temperature, as discussed in the previous chapter. Each set contains three curves, the lowest bold line has been calculated using the vacuum cross-section, for the middle dotted lines medium-effect was incorporated by introducing pion loops in the ρ propagator, and the long dashed line in each set corresponds to the situation when the heavy mesons are included i.e. for πh loops where $h = \pi, \omega, h_1, a_1$. The clear separation between the curves in each set displays a significant effect brought about by the medium dependence of the cross-section. A large dependence on the pion chemical potential is also inferred since the three sets of curves appear nicely separated.

We now turn to the shear viscosity [1,15]. Shown in Fig. 6.2 is the shear viscosity as a function of T where the results with $\pi\pi$ and πh loops are contrasted with the case where the vacuum cross-section is used. The result with the vacuum cross-section agrees with [48] and [7] for $\mu_\pi = 0$. A noticeable medium-effect is observed as indicated by the short and long-dashed lines.

We next turn to the results of thermal conductivity [1,16]. In Fig. 6.3 we plot λT as a function of T evaluated in the Chapman-Enskog approach. The dashed line shows results where the vacuum cross-section is used. For $\mu_\pi = 0$ this result agrees with those of [5,7]. The long dashed line represents the case where medium-effects have been introduced. A substantial medium-effect is seen even for $\mu_\pi = 0$ and this is seen to increase with the increase of temperature. The bold line represents λT calculated using temperature dependent chemical potential $\mu_\pi(T)$,

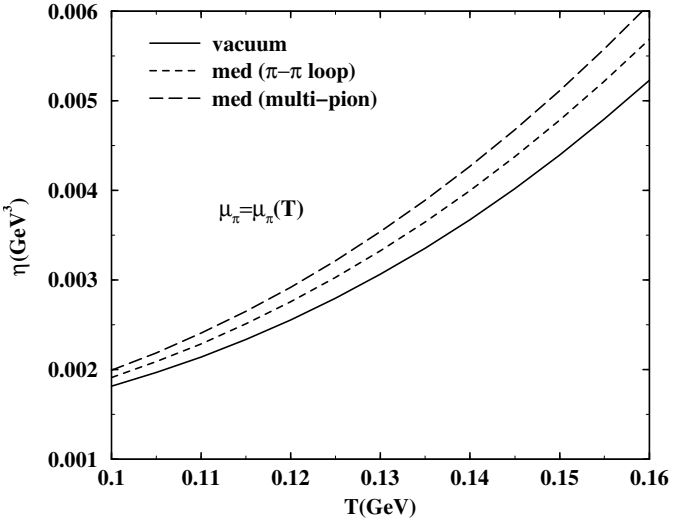

Figure 6.2: η as a function of T for a temperature dependent pion chemical potential. The lower one represents the one with vacuum cross-section, while the other two represents the one with in-medium cross-section. The middle line for in-medium modification due to pion loop and the topmost line for loops with heavy particles in addition.

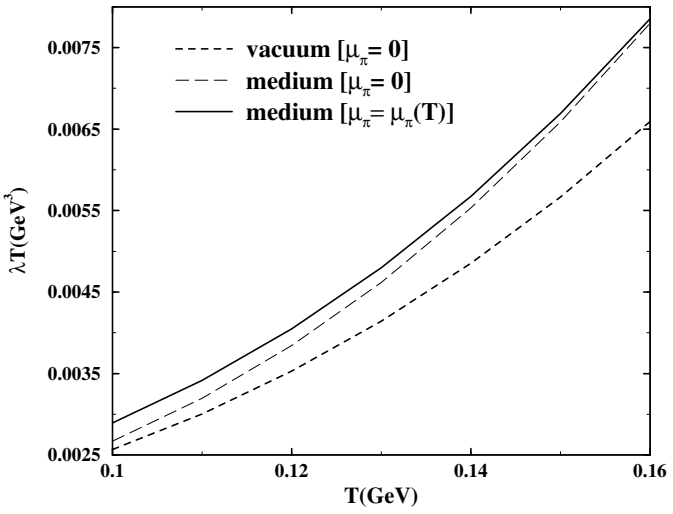

Figure 6.3: λT as a function of T evaluated using Chapman-Enskog approximation.

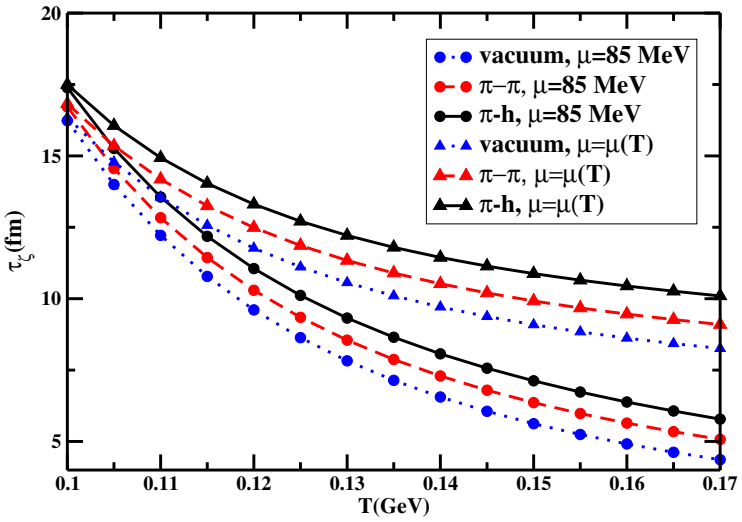

Figure 6.4: Relaxation time of bulk viscous flow as a function of T.

with medium-effects taken into consideration. Comparing with the long-dashed line the effect of chemical freeze-out is seen to be more at lower temperatures since the value of $\mu_\pi(T)$ increases as one approaches kinetic freeze-out.

6.2 Relaxation time of flows

Here in this section, we discuss how the relaxation time of flows is affected by the introduction of in-medium cross-section [17]. Fig. 6.4 represents the variation of the relaxation time of bulk viscosity with temperature. The two set of curves in the figure correspond to two different values of the pion chemical potential. The lower set of curves with circles are evaluated with a constant value of pion chemical potential $\mu = 85$ MeV, this value of the chemical potential is a representative of the kinetic freeze-out in heavy ion collisions. The upper set of curves with triangles represent the case for a temperature T dependent chemical potential $\mu = \mu(T)$. There are three different curves in each set, the lowest one represents the one calculated using vacuum cross-section, while the upper two represents the behaviour of the relaxation time after the introduction of the medium-effect. We have checked for the vacuum case with a constant value of chemical potential with [5, 7], and find that it agrees. The middle curves in each set

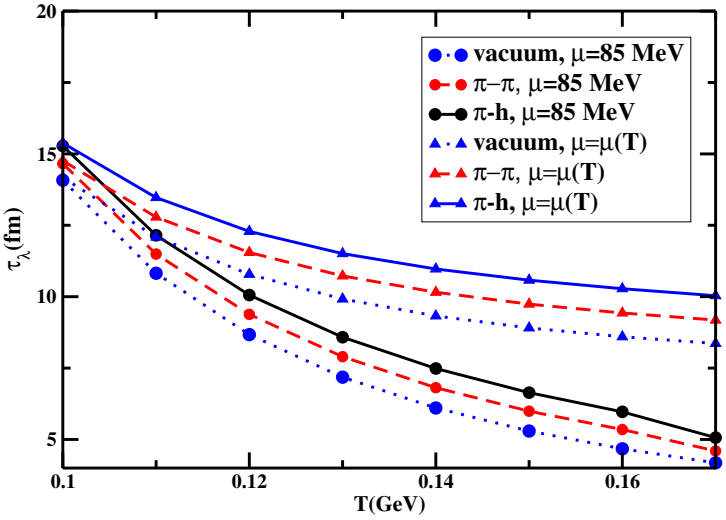

Figure 6.5: Relaxation time of heat flow as a function of T.

depict the effect of the medium corresponding to the pion loop in the σ and ρ propagators. The effect of the thermal medium on τ_ζ, is inflation in its value. Finally, the uppermost solid curves correspond to the situation when the heavy mesons are included in the propagator, i.e. for πh loops where $h = \pi$, ω, h_1, a_1. The increase in the value of the relaxation time τ_ζ is brought about by the suppression of the cross-section which appears in the denominator. As is apparent from the separation of the curves the introduction of medium-effect has caused a significant change. Since the value of the time-varying chemical potential at kinetic freeze-out ($T = 100$ MeV) is $\mu = 85$ MeV the two set of curve merges at $T = 100$ MeV.

Next, we turn our attention to the relaxation time of the irreversible heat flow. In Fig. 6.5 the relaxation time for the irreversible heat flow, τ_λ is plotted against temperature for the same pair of values of pion chemical potentials mentioned above. Just like in the previous case the two choice of pion chemical potential produces two distinct set of curves which merges at temperature $T = 100$ MeV. In each set, the curves are plotted for different $\pi\pi$ cross-sections, with the lowest indicating the one with the vacuum cross-section and the highest with a cross-section that includes the heavy mesons in the ρ propagator. Similar to the earlier case here also we notice that the introduction of the medium-effect in the cross-sections, evaluated at finite

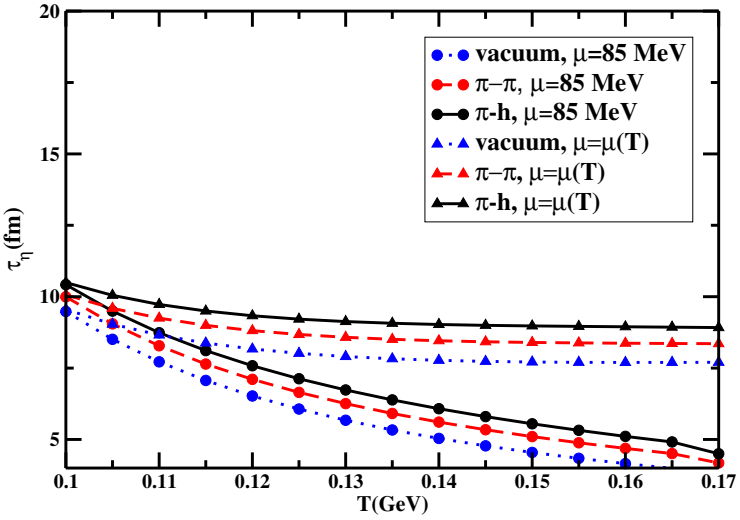

Figure 6.6: Relaxation time of shear viscous flow as a function of temperature.

temperature causes an appreciable increase in the value of τ_λ. The multi-pion loop contribution due to heavier mesons in the ρ propagator turns out to be more significant than the $\pi\pi$ loop.

Finally, we discuss the relaxation time of shear viscous flow τ_η, the variation with temperature is plotted in Fig. 6.6. The curves are seen to follow the same trend corresponding to the cases described above though the magnitudes are a little lower. The lower set of curves are those for the constant value of μ, this has already been seen in [5, 7]. The variation of τ_η is small as seen from the figure, this is expected since $\tau_\eta = \eta\beta_\eta/nT$; the increase of η with T is largely compensated by the decrease in $1/nT$, $\beta\eta$ thus remaining approximately constant in the temperature range shown. When $\mu = \mu(T)$ is used, an almost insignificant variation of τ_η with temperature is noted. This is because the increase in n with the increase in temperature is compensated with the decrease in μ at higher temperature.

Note though in the first-order theory we found that the magnitude of the coefficients ζ, λ and η differs quite significantly, the respective relaxation time as plotted on the same scale in Fig. 6.4, Fig. 6.5 and Fig. 6.6 come out to be of similar magnitude. The bulk viscosity ζ is generally much smaller than η as seen in e.g. [15,16]. Consequently, bulk viscosity and thermal conductivity are usually ignored in the set of hydrodynamic equations.

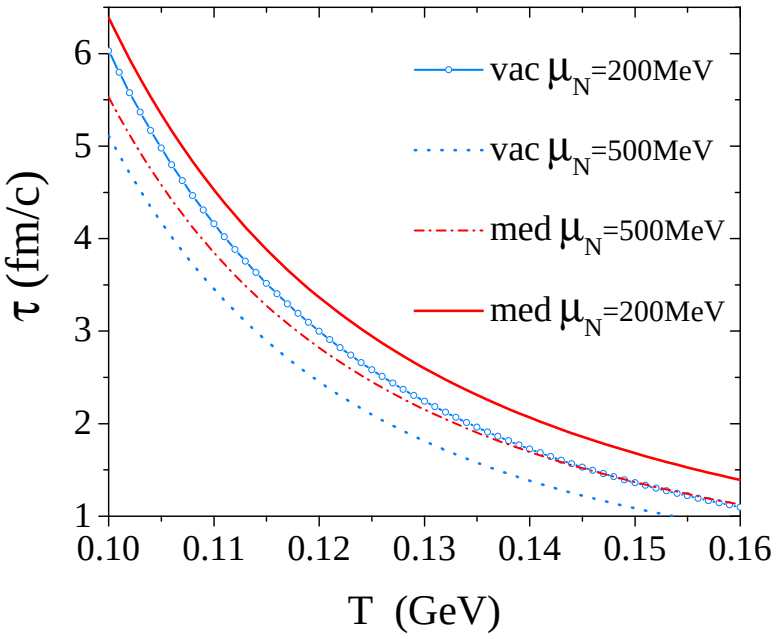

Figure 6.7: The relaxation time of pions in a hadron gas mixture of pions and nucleons, for nucleon chemical potential $\mu_N = 200$ and 500 MeV; with and without medium-effects.

6.3 Transport coefficient of a mixture consisting of Pions and Nucleons

To understand the behaviour of the transport coefficients derived using relaxation time approximation method we must first study the behaviour of the average relaxation time at different temperature and chemical potential. The average has been taken over the entire momentum space of the projectile particle. The relaxation time contains the dynamic information of the medium, that is embedded in the binary collision leading to the transport phenomena. The $\pi\pi$ and the πN cross-section determines the relaxation time of pion τ_π and; NN and πN cross-section determines the relaxation time of nucleon τ_N. In Fig. 6.7 and Fig. 6.8 the temperature dependence of the relaxation time of pions and nucleons in the mixture have been plotted. The curves with vacuum cross-section are of similar order of magnitude as given in [5]. The cross-section decreases with the introduction of medium-effect as discussed in the previous chapter,

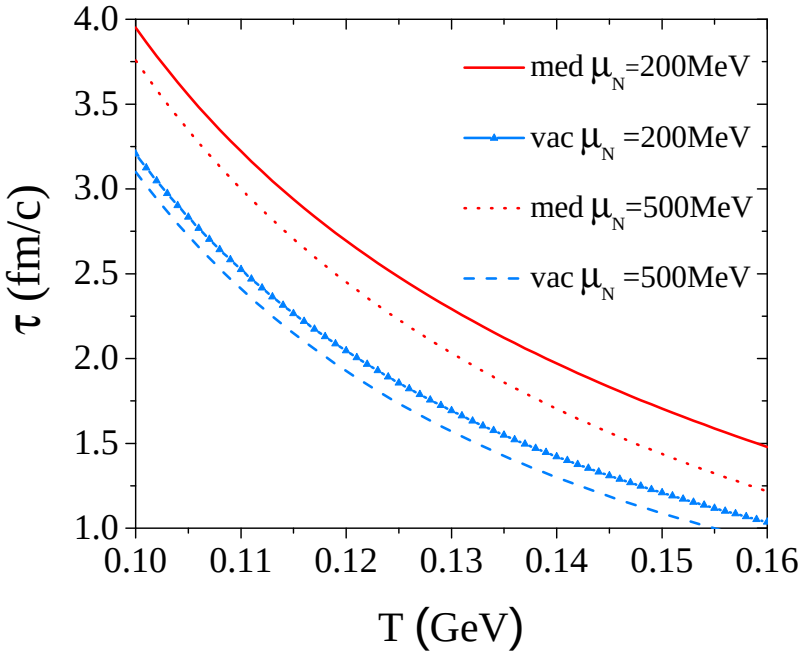

Figure 6.8: The relaxation time of nucleons in a hadron gas mixture of pions and nucleons, for nucleon chemical potential $\mu_N = 200$ and 500 MeV; with and without medium-effects.

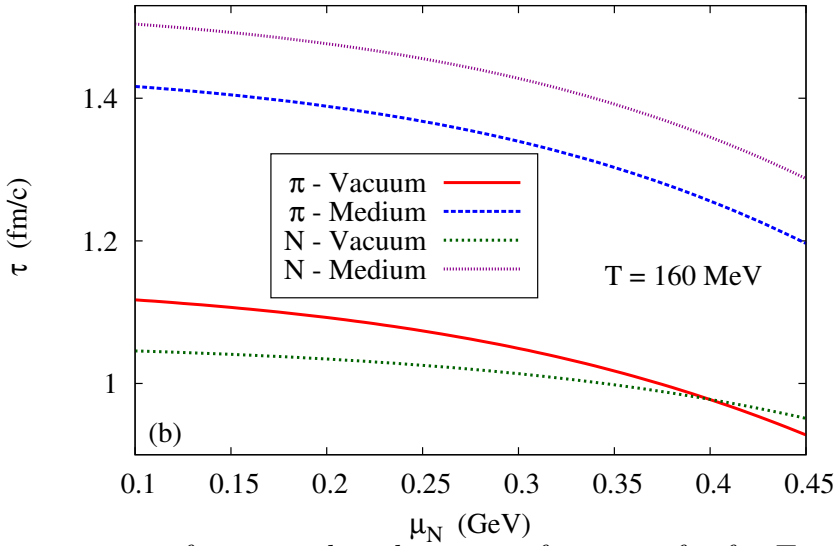

Figure 6.9: The relaxation time of pions and nucleons as a function of μ for $T = 160$ MeV for $\mu_\pi = 80$ MeV.

leading to an increase in the relaxation time. Though at lower temperatures near kinetic freeze-out the magnitude of the nucleon relaxation time is about a factor of two lower than that of the pions, they are found to even out at higher temperatures. In Fig. 6.9 the dependence of the relaxation time of pions and nucleons on the nucleon chemical potential has been shown. Here too the relaxation times decrease with the increase in chemical potential, but the rate is much slower. The larger gap between the results with vacuum and medium cross-sections shows the dominant role played by temperature especially in the later stage of the evolution.

6.3.1 Viscous Coefficients

Here we show how the medium modified $\pi\pi$ and πN cross-sections discussed above are reflected in the viscosities of the system, evaluated numerically. We will show results for the temperature range 100 to 160 MeV which is typical of a hadron gas produced in the later stages of heavy ion collisions, between kinetic and chemical freeze-out. Accordingly, we consider a non-zero value of the pion chemical potential [8] in addition to the chemical potential for nucleons.

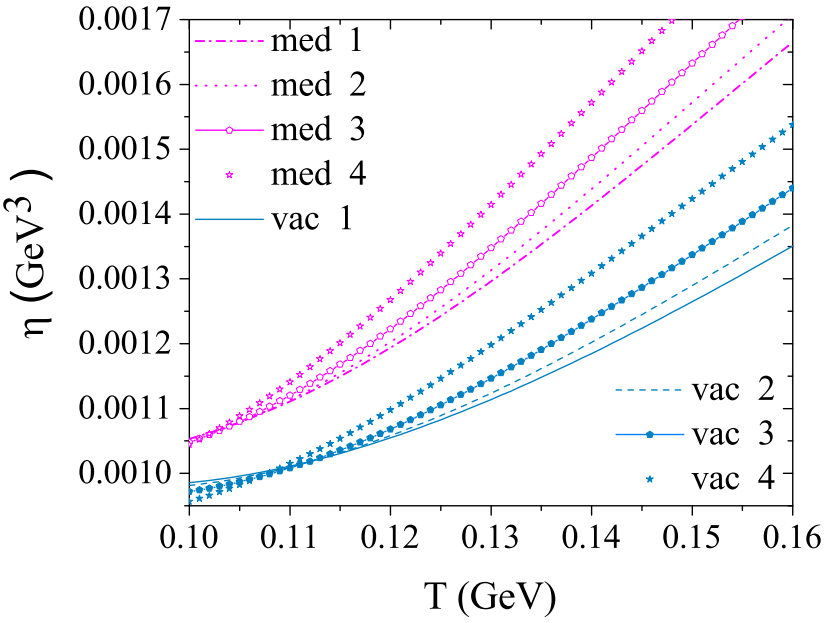

Figure 6.10: Shear viscosity as a function of T for various μ_N with and without medium-effects. Legends 1,2,3 and 4 indicate μ_N =200, 300, 400 and 500 MeV respectively. The pion chemical potential $\mu_\pi = 80$ MeV.

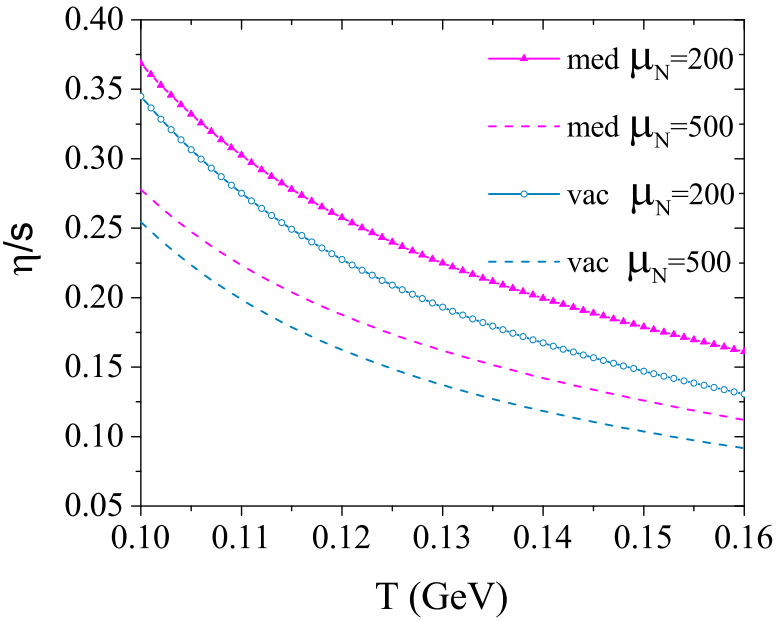

Figure 6.11: Shear viscosity over entropy density as a function of T for diffrent nucleon chemical potential μ_N expressed in MeV.

The shear viscous coefficient as a function of temperature is depicted in Fig. 6.10. In the lower set of curves are the ones evaluated using the vacuum cross-sections, while the upper set is evaluated taking into consideration the medium-effect on the $\pi\pi$ and πN cross-sections. The different curves in each set correspond to different values of the nucleon chemical potential μ_N while the pion chemical potential μ_π is taken to be 80 MeV [9]. It is apparent from the graph that both in vacuum and medium the shear viscosity appears to increase with increasing πN and is the result of interplay of various factors. This was already noted in [10] where by means of a simplified estimate of the viscosity of the mixture this feature could be understood as resulting from an enhancement of the nucleon component with increasing μ_N . Moreover, due to the introduction of the medium-effect a considerable change in the value of the shear viscous coefficient is seen. The increase in the relaxation time and hence the viscosity as can be seen in the upper set of curves, is due to the decrease of the in-medium cross-section with increasing T and μ_N.

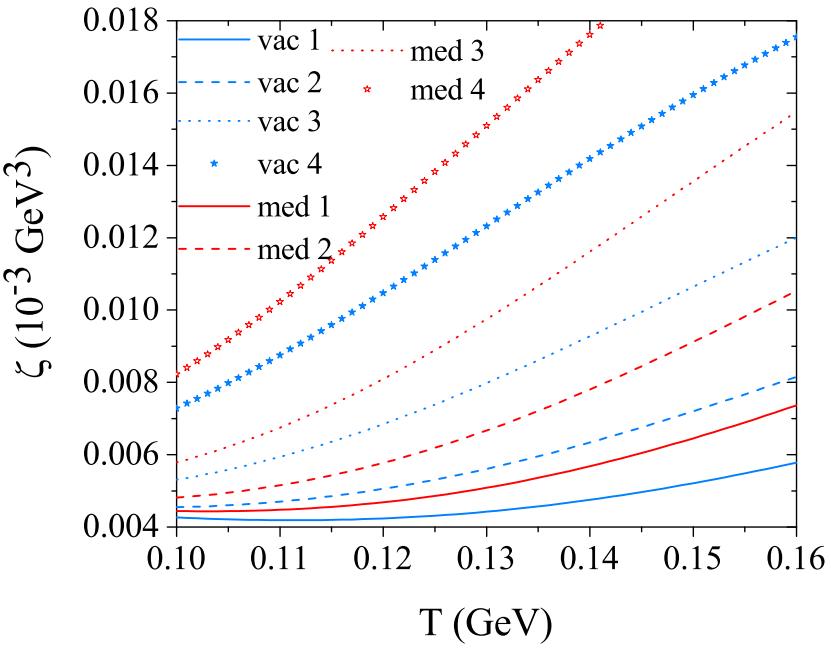

Figure 6.12: The bulk viscosity as a function of T with and without medium-effects. Legends 1,2,3 and 4 indicate $\mu N = 200, 300, 400$ and 500 MeV respectively.

A significant influence of the medium on the kinematic shear viscosity (i.e. η/s; s is the entropy density of the system) is also found. Fig. 6.11 depicts the variation of kinematic viscosity with temperature for nucleon chemical potential 200 MeV and 500 MeV represented by lines with and without symbols respectively. The lower curve in each set corresponds to vacuum cross-section while the higher one corresponds to the one calculated taking into account the medium-effect for the same nucleon chemical potential. The monotonous decrease in η/s with T may be attributed to the increase in the entropy density with T [10]. The decrease however, respects the lower bound [11] around the transition temperature.

The temperature dependence of the bulk viscous coefficient is shown in Fig. 6.12. Proceeding from the bottom of the diagram each pair of graphs corresponds to a different value of nucleon chemical potential. The lower curve for each pair corresponds to the one where vacuum cross-section has been used to calculate the relaxation time while the upper curve corresponds to the one where the medium-effects have been taken into consideration. Although the magnitude of bulk viscosity appears to be much smaller than that of shear viscosity the effects of the

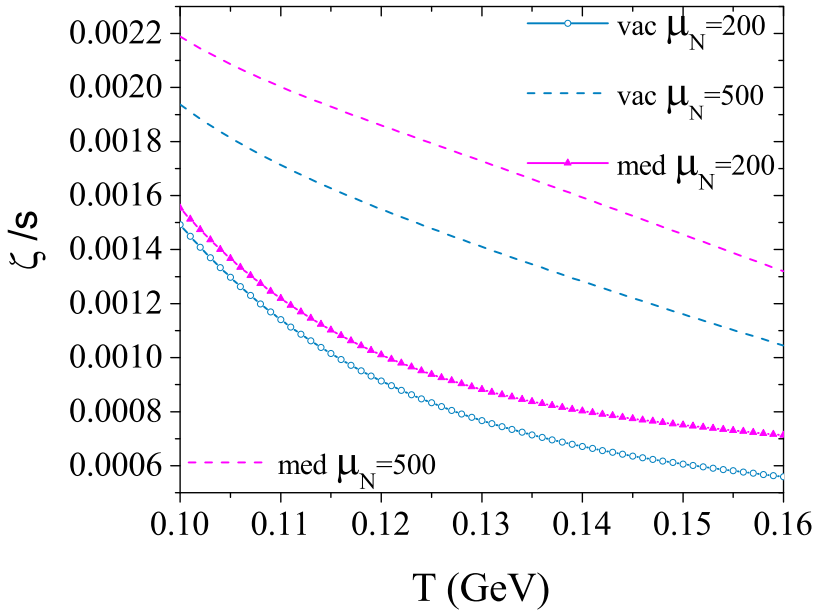

Figure 6.13: Ratio of bulk viscosity and entropy density vs T for different μ_N expressed in MeV.

thermal medium are quite significant in this case too. Fig. 6.13 depicts the variation of the kinematic bulk viscosity (i.e, ζ/s), where the lines correspond to the same combination of parameters as in Fig. 6.11. The dependence with T and μ_N in the vacuum and in the medium show similar features as η/s. A significant difference is observed between the magnitude of viscosities calculated with vacuum and medium cross-sections.

6.3.2 Coefficients of Thermal Conduction and Diffusion

The variation of thermal conductivity with temperature is depicted in Fig. 6.14(a), where the pion chemical potential is taken as 80 MeV which is its value at kinetic freeze-out. The nucleon chemical potential μ_N is taken as 500 MeV. The coefficient of thermal conductivity is found to decrease with the increase in temperature. The introduction of medium-effect not only causes a significant change in the value of the thermal conductivity but also changes its behaviour with temperature though it changes only slightly with the rise in temperature after 130 MeV.

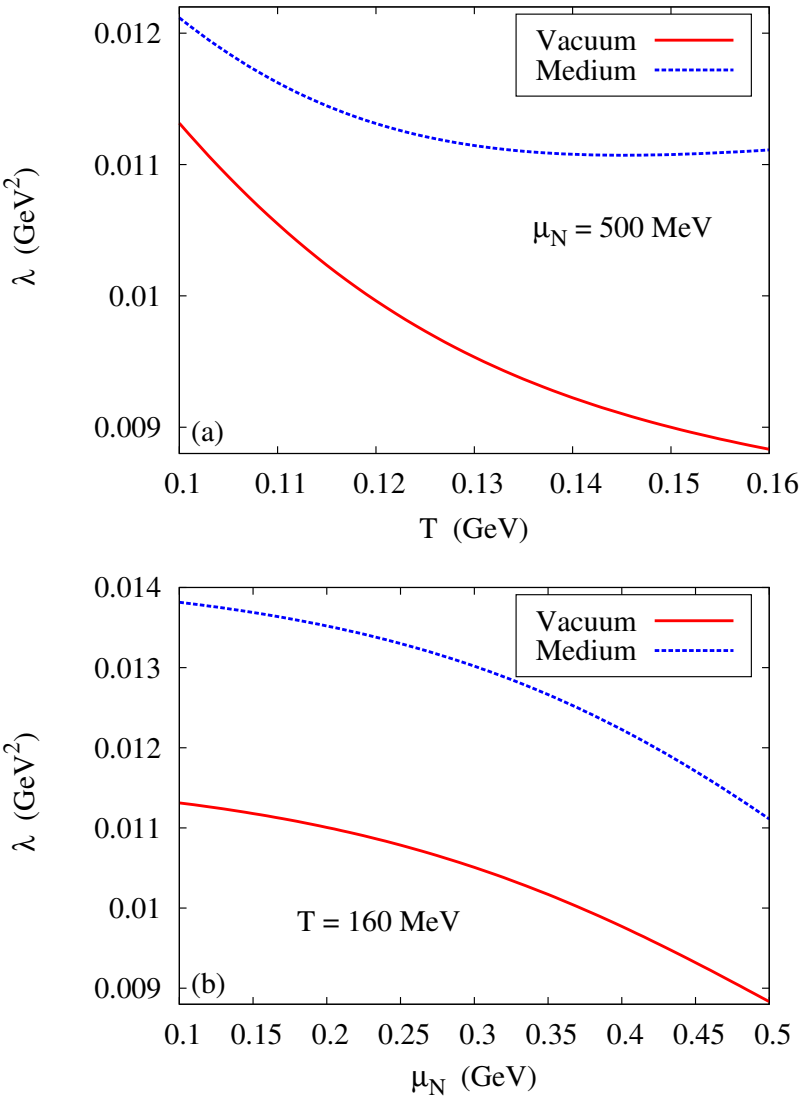

Figure 6.14: The thermal conductivity for a mixture constituting of nucleons and pions as a function of (a) temperature and (b) nucleon density.

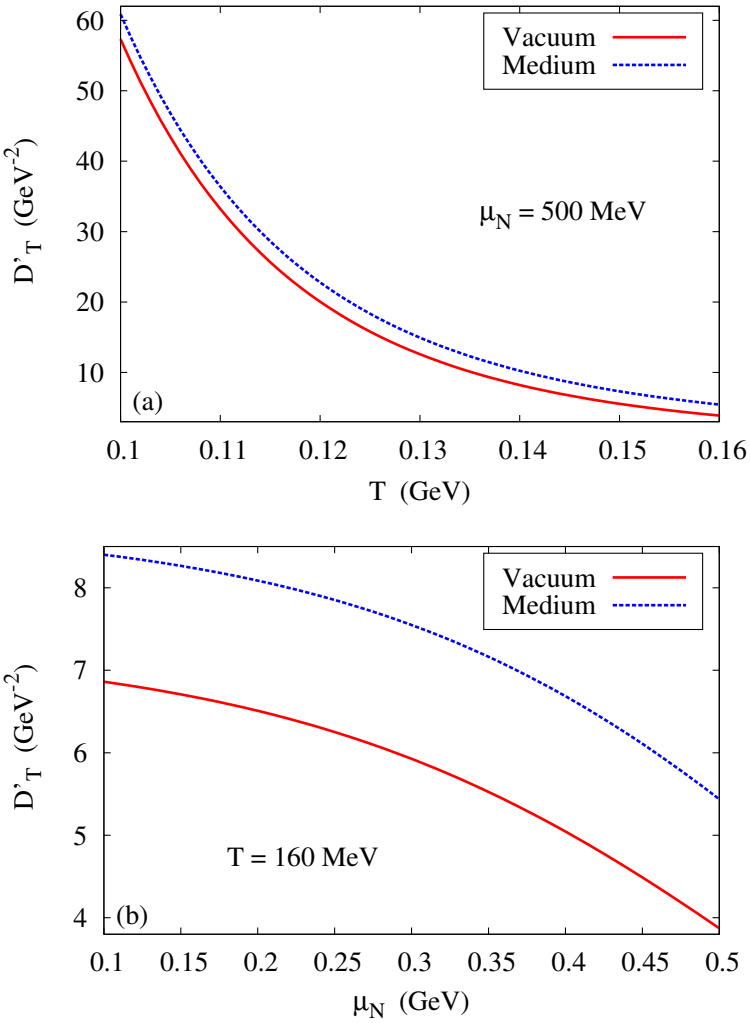

Figure 6.15: The Dufour coefficient as a function of (a) temperature and (b) nucleon density.

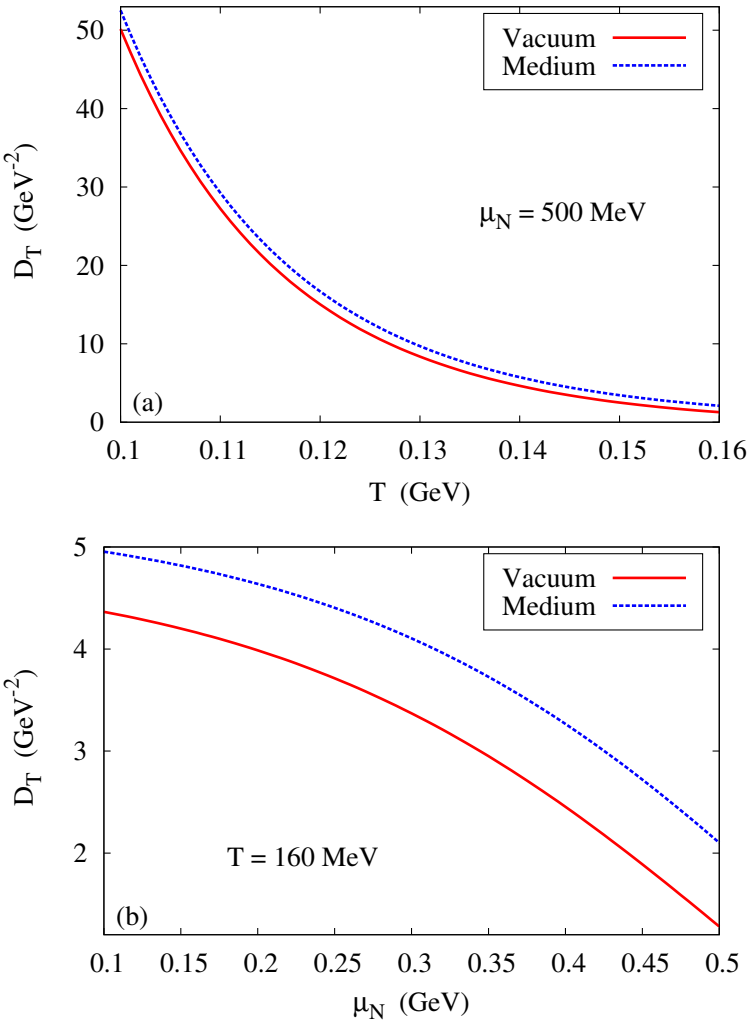

Figure 6.16: The thermal diffusion coefficient for a mixture constituting of nucleons and pions as a function of (a) temperature and (b) nucleon density.

The variation of the thermal conductivity with the nucleonic chemical potential at temperature 160 MeV is shown in Fig. 6.14(b). Here also we find a significant change in its value with the introduction of medium-effect. Note that the increase in average energy with temperature is in fact cancelled by the $1/T$ factor in the expression of λ and hence the coefficient of thermal conductivity follows the trait of relaxation time. Increase in density (chemical potential) is found to have very little effect on its behaviour.

The temperature dependence of the Dufour and the Thermal diffusion coefficients is presented in Fig. 6.15(a) and Fig. 6.16(a) respectively. Here we see that the two coefficients are approximately of similar magnitude (in fact, the values would have been identical if the Chapman-Enskog formalism was used to derive the values of these coefficients [12, 13]). We find that the introduction of medium-effect has very little effect on the value of these coefficients though there is a slight increase in their magnitudes. With the increase in temperature, the coefficients are found to fall sharply; the presence of $1/T$ in their expression causes the rate of decrease to be sharper than that of the relaxation time. Fig. 6.15(b) and Fig. 6.16(b) shows the behaviour of the Dufour coefficient and the thermal diffusion as a function of the nucleon chemical potential at $T = 160$ MeV. The trend followed by the coefficients is similar to that of the relaxation time.

From Fig. 6.17(a) we can see that the diffusion coefficient decreases with temperature. The introduction of medium-effects in the relaxation time causes a significant surge in its value. The dependence of the diffusion coefficient on the nucleon chemical potential is depicted in Fig. 6.17(b). It is found to behave quite differently from the relaxation time of collision and its value depend strongly on the density of nucleons (which goes up with the increase of nucleon chemical potential). We plot for two values of T, in this case, to show its different behaviour at lower and higher values resulting from the interplay of the two factors. At 160 MeV temperature the value of the diffusion coefficient is found to be quite steady up to $\mu_N = 450$ MeV after which it decreases slowly. In contrast, at lower temperatures (100 MeV) the value of the diffusion coefficient goes up with an increase in chemical potential.

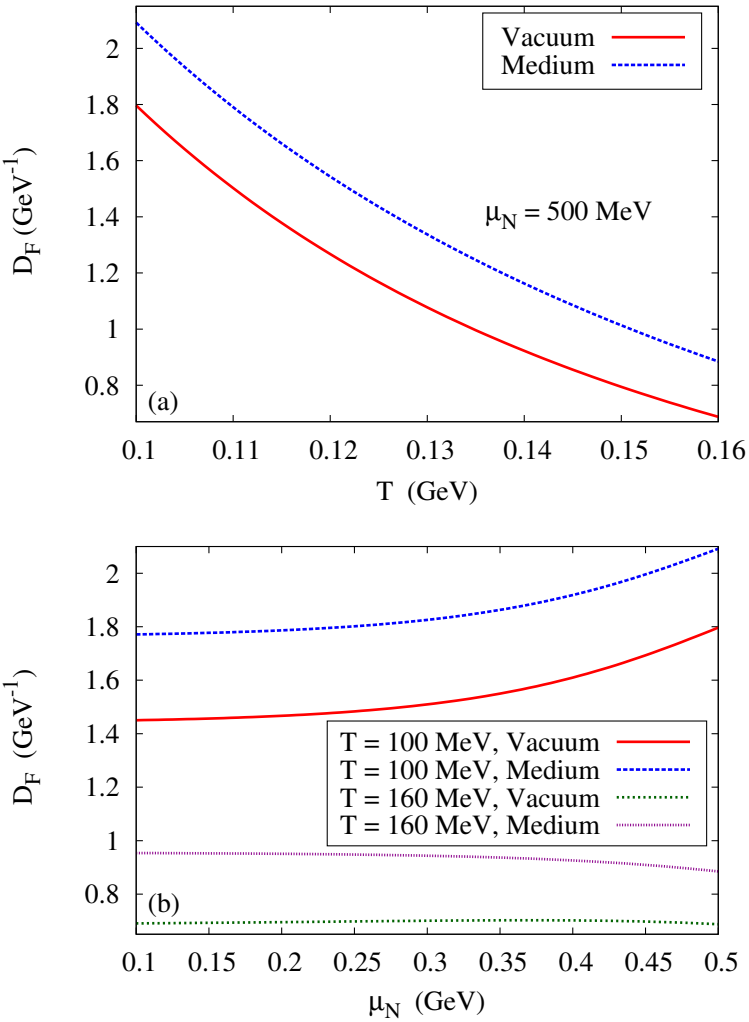

Figure 6.17: The diffusion coefficient as a function of (a) temperature and (b) nucleon density.

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Chapter 7

Summary and Outlook

The main aim of the thesis is the study of transport properties of the later stage of a heavy ion collision, mainly after chemical freeze-out, from a kinetic theory approach. Transport coefficients in first-order hydrodynamics for a mixture of pions and nucleons are investigated, for a finite baryonic chemical potential. This study is motivated by the BES (Beam Energy Scan) program at RHIC, where the baryon chemical potential is expected to be significant and thus is expected to play an important role in determining the transport coefficients. The method employed to calculate these transport coefficients of first-order hydrodynamics is the *Relaxation Time Approximation*. Here we have also evaluated the transport coefficients of second-order hydrodynamics for a system of pions. These coefficients have been calculated using the *Grad's 14-Moment Method*.

As mentioned before in most of the earlier studies of transport coefficients of hadronic gas within the kinetic theory framework, vacuum cross-sections were employed. These cross-sections which appear in the collision integral is the dynamic input; through these cross-sections, the interaction between the constituent particles are introduced. But for a more realistic picture we have incorporated medium-effect in our cross-section, as the system we are studying presumably, is at a high temperature and density. This is one of the novelties of the work reported in this thesis. We summarise below the main findings and discussions from each chapter.

Chapter 2 : In this chapter, we have discussed the basics of *Relativistic Kinetic Theory*. Here we have described how the statistical description of the microscopic constituents of a system is given by the single particle distribution function $f(x, p)$. And how this single particle distribution can be used to obtain parameters like hydrodynamic velocity U^μ, energy-momentum tensor $T^{\mu\nu}$, concentration etc., which are used to provide a macroscopic description of the system, thus acting as a bridge between microscopic and macroscopic viewpoint. We have discussed here the *Boltzmann Transport Equation* and how it describes the evolution of the single particle distribution function. We have also discussed the limitation of the transport equation, thus describing the domain where this equation is applicable. Using the Boltzmann equation we have obtained the first-order *Relativistic Hydrodynamic Equations*, thus completing the full macroscopic description of the system starting from the microscopic viewpoint. With the help of the conservation equation, we also obtain a form of the equilibrium distribution function.

Chapter 3 : Here we evaluate the transport coefficients in second-order hydrodynamics for a single component gas of pions using *Grads 14-Moment Method*. We have reviewed the *Enskog* expansion, where the single particle distribution function is expanded around its equilibrium value, and how it is used to linearise the Boltzmann transport equation. Here we have concisely discussed the *Chapman-Enskog* method, and have obtained expressions for shear viscosity η, bulk viscosity ζ, and thermal conductivity λ. We then discuss the second-order hydrodynamic equations and how they eliminate the acausality of the first-order hydrodynamic equations. The term ϕ used to parametrise the deviation of the single particle distribution function has been expanded using irreducible tensors, 1, $\langle \Pi^\mu \rangle$, $\langle \Pi^\mu \Pi^\nu \rangle$, ...; constructed out of momentum p^μ. Using the expressions for the transport coefficients obtained using Chapman-Enskog method we derive the coefficients of the above mentioned irreducible tensors in ϕ, which in turn has been expanded in terms of $\tau = (p^\mu U_\mu)/T$. Applying variation approach proposed by Gelarken we obtain the general form of the second-order relativistic hydrodynamic equation. From these we identify the relaxation time of flows τ_η, τ_ζ and τ_λ which appear in Israel-Stewart equation. Using the value of the coefficients of the irreducible tensors we obtain the expressions for these relaxation times of flows.

Chapter 4 : We discuss here the formalism to evaluate the transport coefficients in first-order hydrodynamics for a gas constituting of nucleons and pions using *Relaxation Time Approximation*. Here we review the coupled Boltzmann equations for a mixture. The Boltzmann equations are linearised using the same Enskog expansion. The collision integral is simplified by considering only the projectile particles to be away from equilibrium and the rest in equilibrium. Doing so the collision integral is expressed in terms of the relaxation times τ_π and τ_N, and parameters ϕ_π and ϕ_N. The terms ϕ_π parametrises the deviation of pion single particle distribution function from equilibrium, and ϕ_N does the same for nucleons. ϕ_π and ϕ_N are expressed as linear combination of the thermodynamic forces with proper coefficients having appropriate tensorial rank such that, ϕ_π and ϕ_N turn out to be Lorentz scalars. Expanding the left-hand side of the Boltzmann equation, and equating the coefficients of the thermodynamic forces, we get the coefficients in the expressions of ϕ_π and ϕ_N. Using these coefficients we derive the expression for the shear viscosity η, bulk viscosity ζ, thermal conductivity λ, Diffusion coefficient D', thermal diffusion coefficient D_T and Dufour coefficient D'_T.

Chapter 5 : In this chapter, we have calculated the in-medium cross-section of $\pi\pi$ and πN scattering. To construct $\pi\pi$ cross-section in the medium we employ an effective Lagrangian approach which incorporates ρ and σ exchange and agrees well with experimental data. Then we introduce medium-effect by modifying the propagator in the medium. We have quantified the effect of a thermal medium on the ρ and σ propagator through the self-energy evaluated at finite temperature using *real time formalism* of *Thermal Field Theory*. In the self-energy loop of σ we consider two pions, while for ρ we consider one-loop self energy involving two pions or one pion and another heavy meson, like ω, h_1 and a_1. The cross-section calculated with medium-effect incorporated is found to be suppressed compared to the vacuum cross-section. We have also discussed a time-varying pion chemical potential which takes care of pion number conservation. For in-medium πN cross-section we follow the same procedure. Here we have incorporated Δ-baryon exchange. The self-energy loop of Δ comprises of π, ρ, N, and Δ. Here also we find that the introduction of the medium-effect causes an appreciable amount of suppression.

Chapter 6 : In the first half of this chapter, we study the transport coefficients of a pion gas at different temperatures. We find that on changing the vacuum cross-section with an in-medium cross-section, the temperature dependence of all the transport coefficients are significantly modified. In the second half of this chapter, we study the relaxation time of pion and nucleon in the pion-nucleon gas at different temperatures and different nucleon chemical potentials. Here we see that the suppression in the cross-section causes an increase in the value of the relaxation time. This increase in the value of relaxation time is also reflected in the transport coefficients calculated using them. We also find that on changing nucleon chemical potential the transport coefficients change significantly.

We conclude by tabulating a few aspects of the studies reported here where possible improvements could be made.

1. In Chapter 4 we have discussed the limitation of the first order hydrodynamics. It is known to be acausal. So to overcome this problem second-order hydrodynamics was introduced, which restored causality. But though the second-order equations are causal, they are not guaranteed to be stable. Thus higher order hydrodynamics is needed. And the medium-effects are expected to significantly modify the magnitudes of the transport coefficients of higher-order hydrodynamic theory.

2. In Chapter 5 we have derived the Dufour coefficient D'_T and the thermal diffusion coefficient D_T. According to Onsager's reciprocal relation, these two coefficients are supposed to be symmetric, but as far as our calculations suggest, they are not so. This is a limitation of the relaxation time approximation, and this arises because there is no way to implement the "condition of fit". So for a more accurate result for these cross terms, we need to employ Chapman-Enskog approximation since in this process we have a provision to implement the "condition of fit".

3. We have considered a mixture of pion and nucleon, but for a more realistic picture, we will have to incorporate other significant degrees of freedom like Kaons.