SOME LOCAL CALCULATIONS OF TEMPERATURES OF BLACK HOLE HORIZONS

By

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Bhramar Chatterjee

To

my parents

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Contents

SYNOPSIS

A classical black hole absorbs everything and nothing comes out of its horizon to the outside world. But the relation between the mass, area of the horizon and the surface gravity of a black hole resembles the first law of thermodynamics suggesting the black hole as a thermal body with the area and the surface gravity proportional to the entropy and the temperature respectively. The scale was fixed after the theoretical discovery of thermal radiation from a black hole horizon by Hawking.

The original derivation of Hawking radiation and the other methods like Hartle - Hawking formalism or the Euclidean method which subsequently followed all depend on the entire spacetime geometry of a black hole. But a black hole horizon can be defined locally, depending only on the local geometrical properties of the spacetime. Moreover the laws of black hole mechanics is also applicable for these types of local horizons. So some derivation of Hawking effect using only the local geometry was in order.

The first attempt in this direction was the tunnelling method of calculating Hawking temperature associated with a black hole horizon. In this formalism one calculates the probability of classically forbidden process of s-wave emission across a black hole horizon which has the form of a Boltzmann distribution and thus gives the Hawking temperature. The emission probability is obtained by using WKB approximation near the horizon where the classical action S for the trajectory satisfies the Hamilton-Jacobi equation. This method works well for different types of horizons including the conventional black hole spacetimes and more exotic types of horizons. As it describes an essentially across the horizon phenomenon, a good set of coordinates is required for the calculations which is regular at the horizon. Though singular coordinates can be used, we have shown that in that case the choice of boundary conditions is crucial in obtaining the correct temperature. We have also found that as long as the metric remains stationary no higher order corrections to the Hawking temperature can be obtained in this method which is the effect of backreaction.

Though very successful, the tunnelling method has one drawback which is the use of WKB approximation and hence it can not be applied to situations where this approximation does not hold for example the extremal spacetimes. Because of this we have proposed a more general derivation of Hawking effect which is free of such kind of assumptions and depends on the behaviour of the field modes near the horizon. Our idea is to construct single particle states only outside the horizon using the field equations and continue them inside the horizon. While crossing the horizon the outgoing modes acquire a logarithmic singularity which can be avoided by considering the distributional properties of the modes. Then the conditional probability that a particle emits when incident on the other side of the horizon gives the expected Hawking temperature associated with the horizon. This method works for the conventional black hole horizons and as well as for the extremal spacetimes in each case producing the corresponding temperature. We have also applied this method to a fully dynamical spacetime and found the temperature of this dynamical horizon.

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Chapter 1

Introduction

A black hole is a region in spacetime from which no classical particles can escape and which is separated from the rest of the spacetime by a boundary called an event horizon. An observer outside the black hole is unable to observe events on the other side of the event horizon because light rays from inside cannot propagate outside, hence the name black hole. These objects appear as solutions of Einstein's equations in general relativity. Schwarzschild wrote down the black hole metric as the vacuum spacetime outside a spherically symmetric body immediately after the discovery of the field equations,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{1.1}$$

But the significance of the solution was not understood then. At first it was thought there is a singularity at $r = 2M$ and hence the solution is unphysical for $r \leq 2M$. Much later it was realized that this singularity is merely a coordinate artefact and the physical quantities are quite well behaved there. With this came the understanding that there is indeed a true curvature singularity at $r = 0$ which can not be removed by any change of coordinates. Furthermore, the region $r < 2M$ describes a black hole. Subsequently other types of black hole solutions were found. The Reissner-Nordström metric describes a static, spherically symmetric black hole of mass M possessing an electric charge Q. A rotating black hole solution that is stationary and axially-symmetric was found by Kerr.

A black hole is a macroscopic object which can be described exactly by a few parameters; its mass, angular momentum and charge. A set of three laws governing the behaviour of black holes were formulated during the early seventies [1, 2, 3]. They give a relation between the parameters of two black hole solutions having masses M and $M + \delta M$, angular momenta J and $J + \delta J$ and charges Q and $Q + \delta Q$. These laws are,

- Zeroth law : the surface gravity κ of a stationary black hole is uniform over the entire event horizon. The surface gravity is defined as the force that is needed at infinity in order to keep an object suspended at the horizon.

- First law : the changes in mass (M), angular momentum (J) and surface area (A) of a stationary black hole, when it is going through a quasi-static process, are related by

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J \tag{1.2}$$

  where Ω_H is the angular velocity of the horizon.

- Second law : the second law of black hole mechanics states that if the null energy condition is satisfied then the surface area of a black hole never decrease, i.e. $\delta A \geq 0$. This is also known as the area theorem for black hole.

These laws have striking resemblance with the laws of thermodynamics with the mass (M), area (A) and κ playing the roles of internal energy, entropy and temperature respectively. This analogy was first perceived to be purely formal and coincidental because from the point of view of classical gravity, a black hole is a perfect absorber (nothing comes out of the horizon) and hence should be assigned a zero temperature.

At this point Hawking made a crucial discovery[4]. He showed that quantum effects near the horizon causes a black hole to emit thermal radiation at a temperature $T_H = \kappa/2\pi$. This result showed that black holes do indeed behave as thermodynamic systems in thermal equilibrium with other systems. Also the entropy (S) of a black hole is $S = A/4$.

Without the rigorous mathematical detail, Hawking's effect can be understood by an intuitive picture of creation of particle-antiparticle pairs from vacuum fluctuations close to the horizon. According to quantum field theory, the physical vacuum is in general a complex entity populated by virtual particles which are constantly being created, interacted among themselves and then annihilated. In absence of external fields, the vacuum is usually stable in the sense that virtual particles are not able to survive long enough to become real. However, in presence of external fields these virtual particles can be converted into real ones with the fields supplying the energy required to propagate further.

In a black hole spacetime if a pair is created just outside the event horizon due to vacuum fluctuations, the negative energy particle is in a classically forbidden region. But it can tunnel through the event horizon to the region inside the black hole where the Killing vector, which is timelike outside the horizon, is spacelike and hence the negative energy particle can exist as a real particle with a timelike momentum vector even though its energy relative to an asymptotic observer as measured by the timelike Killing vector is negative. The other particle of the pair, having a positive energy, can escape to infinity where it constitutes a part of the thermal emission described in Hawking effect. In this way there is a net negative flux across the horizon to the inside of the black hole which balances the positive energy flux emitted to infinity and the mass of the black hole is decreased.

The original derivation of Hawking effect did not involve a full quantum theory of gravity [4]. It considered quantum fields in the background spacetime geometry describing the formation of a stationary black hole due to gravitational collapse and comparing the initial and final states of incoming and outgoing radiation it was shown that an asymptotic observer i.e an observer at infinity will perceive a thermal radiation coming out of the black hole with temperature $T_H = \kappa/2\pi$.

Over the years, several other techniques have been developed to study spontaneous particle emission and the Hawking temperature for more general spacetimes. In the Hartle-Hawking proposal [5] a path integral study was made of a scalar particle moving in the background geometry of a stationary black hole and the amplitude for the black hole to emit a particle in a particular mode was found by analytic continuation in the complexified spacetime. It was shown that $P_{emission} = \exp(-\beta E)P_{absorption}$ where $P_{emission}$ and $P_{absorption}$ are the emission and absorption probabilities respectively and $\beta = 2\pi/\kappa$ is the inverse temperature of the horizon.

In the Euclidean approach [7, 8] depending on the methods of finite temperature quantum field theory an analytic continuation to the time coordinate of the black hole metric was performed and the periodicity β was chosen to remove a conical singularity that would otherwise be present. The black hole was then considered to be in equilibrium with the quantum field that has inverse temperature β at infinity.

Although these standard formulations of Hawking effect are quite elegant they do not indicate how such a thermal state may arise as a result of some physical process. Also these derivations all require knowledge about the global properties of the spacetime. As a result they are quite inapplicable to study the thermal properties of any horizon defined locally. On the other hand the laws of black hole mechanics apply equally well to local horizons, black hole horizons which can be described using only local geometrical properties without any assumptions on the global development of the spacetime in which the horizon is embedded [59, 60, 64, 65]. It has also been established that such horizons can be assigned an entropy proportional to the area of the local horizon [68, 69]. Thus, it seems to be a reasonable physical expectation that even with a local definition of black hole horizon one should be able to establish the analogy to thermodynamics. More precisely, such horizons should have a temperature of $T = \kappa/2\pi$ and it seems some derivation of Hawking effect was required involving only the local properties of the spacetime.

The first successful attempt in this area was the work by Parikh, Wilczek and Kraus

[9, 10, 11] who described Hawking radiation as a tunnelling phenomena across the event horizon of a black hole. This method was subsequently developed with contributions from others with a variation in the formalism [13, 14, 15, 17, 18, 19, 20] and eventually the tunnelling formalism as an alternative way of deriving Hawking effect began to take shape. As this method is more intuitive and the derivation much simpler than Hawking's original approach, it became very popular over the years

Hawking radiation as tunnelling across the event horizon of a black hole is very easy to visualize as we have mentioned earlier. Nontheless the derivation is not so straightforward because of some conceptual problems. In the standard picture of quantum mechanical tunnelling there exists a potential barrier and two separated classical turning points on both sides of the barrier which are joined by a trajectory in complex time. In the WKB or geometrical optics limit, the probability of tunnelling is related to the imaginary part of the action for the classically forbidden trajectory by,

$$\Gamma \sim \exp(-2\text{Im}S), \tag{1.3}$$

where S is the action for the trajectory. In the case of a black hole horizon there seems to be no such barrier! If a particle is even infinitesimally outside the horizon, it can escape classically. The turning points thus seem to have no separation at all.

Parikh and Wilczek countered this problem by considering energy conservation during the process of radiation. They argued that if a particle with energy E tunnells out of event horizon then the turning points are $r_{in} = r_h$, the initial radius of the horizon and $r_{out} = r_h - E$, the final radius of the horizon after the emission of the particle. Thus, in their own words, "it is the tunnelling particle itself that secretly defines the barrier" [12]. Since a black hole loses mass during Hawking radiation and its horizon shrinks, this argument seems justified. Moreover by enforcing the energy conservation during the process they were able to calculate non-thermal corrections to the emitted spectrum. Still there are some points of doubt as this is not the usual picture of tunnelling in the quantum mechanical sense. Also the results were obtained by using a particular set of coordinates (the Painleve-Gullstrand coordinates) and it is not clear why this coordinate is preferred as the results of Hawking effect is independent of coordinate choice.

Another approach was taken by the authors of [13, 14, 15] to describe particle production in Schwarzschild-like spacetimes by using the method of complex path analysis which is used to describe tunnelling processes in semiclassical quantum mechanics [16]. In this case using the standard Schwarzschild metric it was shown that the coordinate singularity present at the horizon manifests itself as a singularity in the expression for the semiclassical propagator and regularizing this singularity Hawking's results were recovered. As our work is mainly related to this method let us discuss this in a little detail.

This method is loosely based on Hartle-Hawking formalism of Hawking effect but instead of the path integral mechanism a more simpler semiclassical approach is taken. A relativistic scalar particle Φ in a stationary black hole background, which satisfies the Klein-Gordon equation, can be written in the semiclassical approximation as,

$$\Phi = \exp(-iS), \tag{1.4}$$

where S is the classical action for the scalar particle. Under the WKB approximation near the horizon it is found that S satisfies the Hamilton-Jacobi equation and using the symmetry of the spacetime a solution for S is obtained. A singular integral appears in the expression for S as one considers emission and absorption of particles at the horizon as no classical path exists from inside the horizon to outside it. This singular integral is treated by using the method of complex paths and contributes an imaginary term in the classical action S. Finally using the Hartle-Hawking formula for emission and absorption probabilities it is found that,

$$\frac{P_{emission}}{P_{absorption}} = e^{-2\text{Im}S} = e^{-\beta E}, \tag{1.5}$$

which gives the inverse temperature β of the horizon.

This formalism works in different coordinate settings including singular coordinates like Schwarzschild and non-singular coordinates like Painleve-Gullstrand or Lemaitre coordinates and gives the same result emphasizing the fact that Hawking effect is indeed independent of coordinate choice. But a problem remains. The singular integral appears in both the incoming and outgoing modes and while the outgoing particle experiences difficulty in escaping from the black hole, no such problem exists for the incoming ones as all the classical paths lead into the horizon. So the incoming mode of S should not contain any imaginary part. To solve this problem the authors in [17, 18] took another approach which is in fact a modified version of the previous method.

The idea is that of particles (massive or massless bosons or fermions) tunnelling out of the black hole horizon and giving rise to a thermal radiation. Considering again a relativistic scalar particle in a stationary black hole background, it is assumed that the probability for the classically forbidden process of s-wave emission across the horizon is given by the standard semiclassical tunnelling probability, which depends on the classical action S of the particle,

$$\Gamma = e^{-2\text{Im}S}. \tag{1.6}$$

This is equated to the Boltzmann factor for emission $e^{-\beta E}$ where β is the inverse Hawking temperature associated with the horizon. This implies,

$$T_H = \frac{E}{2\text{Im}S}. \tag{1.7}$$

So, all is required is the imaginary part of the action S. As we have mentioned earlier, the action S satisfies the relativistic Hamilton-Jacobi(HJ) equation. Now from the symmetries of the metric an appropriate form of the action is picked and solving the HJ equation $\mathrm{Im}S$ is obtained. Here the boundary conditions are set in such a way that the incoming modes are well behaved at the horizon and the absorption probability is unity by default. Since this method involves mainly solving the Hamilton-Jacobi equation under various circumstances, it became known as the "Hamilton-Jacobi variant" of the tunnelling formalism. As the calculations are much simpler, now a days this has become the standard method employed for doing tunnelling calculations across a black hole horizon.

The main difference between the two variants of tunnelling formalism i.e the Parikh-Wilczek method and the Hamilton-Jacobi method, is that in the former case the back reaction of the emitted particle is taken into account resulting a correction in the Hawking temperature while no such correction appears in the Hamilton-Jacobi method because the effect of self gravitation is not considered here. Throughout the calculations the spacetime metric is held fixed as in the case of the other formulations of Hawking effect. Some authors claimed that despite the absence of back reaction, corrections to the Hawking temperature can still be obtained considering higher order terms in the semiclassical action S but this was proved to be an error in the calculations and the fact was established that without the effect of self gravitation, no corrections can appear in the Hawking temperature [25, 54].

Before going into the mathematical detail let us first discuss some features of the tunnelling formulation. First of all we see that the tunnelling method works with the particle interpretation of the emission process. This is the most natural way of explaining the loss of energy of the radiating black hole. But there is also a problem. The notion of particles in general relativity is not in general unique, there are being many inequivalent choice of time in a curved spacetime and a different vacuum can be defined with respect to each and every one of them. In the case of tunnelling across the black hole horizon, it will always be possible to choose any coordinate system, as long as it is regular across the horizon, and use it to define an observer-dependent vacuum relative to which a particle definition is feasible. And we are concerned only with the probability that such an observer dependent notion of particle be emitted from the horizon of the black hole. If this probability is a coordinate scalar it will not depend on what particle concept one employs. This is indeed the case as this method gives the same result in different coordinate settings.

Though the final result does not depend on the coordinates used, the choice of coordinates plays an important role in tunnelling formulation. As it describes an across horizon phenomenon, a good set of coordinates are required for the calculations which are regular at the horizon. Fortunately this is always available. For example for the Schwarzschild black hole there are several choice of coordinates like Painleve-Gullstrand coordinates, Eddington-Finkelstein coordinates, Lemaitre coordinates, Kruskal coordinates etc all of which are well behaved at the horizon. But if one uses coordinates which are singular at the horizon, the boundary conditions has to be set carefully to distinguish the black hole horizon from the white hole horizon. Otherwise there will be a problem in obtaining the correct temperature as we shall see later.

The main advantage of the tunnelling picture is that the calculation is completely local. The method uses only the classical action of a single particle and relates the particle emission probability to an imaginary contribution to the classical action localized at the horizon, which only comes from the local geometry. The global structure of the spacetime has nothing to do with it. With this advantage, the tunnelling method can be applied to any local horizon to calculate the temperature associated with it.

Now the tunnelling formalism can not give us much more information about the radiation process than was already known besides understanding Hawking effect from a different viewpoint as it is mainly involved with the semiclassical emission rate. The advantage of applying this method over the other techniques is that the calculations are much simpler and easy to keep track with. With this idea, this method has been extensively used in recent years for more intricate spacetime geometries where the original derivations of Hawking effect can not be applied easily as well as for the more conventional types of black holes. A few examples include anti-de Sitter(AdS) [41], de Sitter(dS) [35, 36, 37, 38, 39, 40], BTZ [42, 43], higher dimensional black holes and some exotic spacetimes [44, 46, 47] where the corresponding Hawking temperature has been produced correctly by applying tunnelling method. It has also been applied to past horizons and white holes, in which case a clear notion of temperature emerges in complete analogy to black holes [24]. Also as the calculations involve only local geometry it has been used to obtain temperature for some locally defined horizons, for example the cosmological and weakly isolated horizons [36, 45] and dynamical horizons [20, 29]. Usually the method is applied to a scalar field in the black hole background, but one can also consider fermionic field [31, 32, 33, 34]. Though the calculations are a bit complicated it is essentially the same.

On the other hand, the tunnelling formulation depends heavily on the semiclassical approximation. Although it is argued that the WKB approximation holds near the horizon where tunnelling takes place there are some arguments regarding the validity of this approximation in the strong field limit [48, 49]. So it would be much better to devise a more general formalism without requiring this kind of approximations. Also there is one example where the tunnelling method fails due to the breakdown of semiclassical approximation i.e the case of extremal black holes. An extremal black hole should have zero temperature as its surface gravity is zero. But employing the tunnelling method one obtains two contradictory results. The Hamilton-Jacobi variant gives an infinite term in the real part of the action S and a zero imaginary part which naively implies a zero β and hence an infinite temperature for the extremal spacetimes while the Parikh-Wilczek method gives a non-zero finite temperature which is quantized in units of the temperature of a Schwarzschild black hole [21]! Neither of this corresponds to the real situation which led people to believe that the tunnelling approach does not work for extremal black holes because the WKB approximation does not hold near an extremal horizon.

To avoid this, we have proposed another formulation of the Hawking effect which has all the advantage of the tunnelling picture yet does not require the WKB approximation to compute the emission rate. This is based on quantum field theory in curved spacetime.

Our idea is to construct single particle states only outside the horizon using the field equations and somehow continue these states to 'inside' keeping in mind that the horizon or the surface from where particle escapes to infinity is not a null hypersurface, but something like Hayward's timelike trapping horizon [60, 67]. A black hole spacetime contains a Killing vector which is timelike outside the event horizon, null on the horizon and spacelike inside it. One can define a vacuum with respect to this Killing vector outside the horizon and single particle states can be constructed. Considering a scalar field in a black hole background we have calculated the positive frequency modes of this Killing vector directly from the field equations using the metric near the horizon. As these modes cross the horizon the outgoing mode acquires a logarithmic singularity at the horizon. This can be treated by keeping in mind that the field modes are essentially distribution valued and these distributions are quite well behaved at the horizon. This behaviour of the field modes was first noticed by Damour and Ruffini [55]. Then calculating the probability current coming out of the horizon we have found that the conditional probability that a particle emits when it is incident on the surface from the other side gives the required Boltzmann factor,

$$P_{(emission|incident)} = \frac{P_{(emission \cap incident)}}{P_{(incident)}} = e^{-\beta E}, \tag{1.8}$$

and the Hawking temperature associated with the horizon [56].

Since this is essentially a horizon crossing phenomenon, a good set of coordinates is required which is regular across the horizon, and we have used Kruskal coordinates. The advantage of using Kruskal coordinates is that near the horizon the metric is flat which simplifies the calculations considerably. In these calculations we have never been concerned about the asymptotic properties of the spacetime. To construct the field modes only the near horizon metric is required. So this method can be also applied to any horizon defined locally like the tunnelling approach and on the positive sides we do not require any WKB like approximations either.

We have applied this formalism to the Reissner-Nordström (RN) and Kerr black holes to calculate the Hawking temperature. The main reason for choosing Kerr and RN spacetime is that both exhibit extremal limits. Both spacetimes have two horizons : outer (r_+) and inner (r_-) and the extremal limit is achieved as $r_+ \rightarrow r_-$. We have calculated the temperature of the two horizons separately and also for the extremal black holes. No difficulty arises for the extremal case contrary to the tunnelling formalism and so we hope this gives a more sound formulation of Hawking effect.

The main problem with this type of calculations is the nature of the event horizon. An event horizon is a null hypersurface which does not evolve during any physical process and to define the event horizon of a black hole spacetime one requires the knowledge about the global structure of the spacetime. To consider the event horizon as the radiating surface thus poses a difficulty. In real situation a black hole must grow while it is absorbing matter and shrink while radiating so that the horizon must evolve. Also being a null hypersurface, an event horizon can not allow any particle to come out of it without violating causality because the Killing vector, which is timelike outside, is zero on the horizon and spacelike inside it. So particle states can not be defined using this Killing vector.

To solve these problems people have tried the idea of dynamical horizons associated with a black hole spacetime as the real surface where all the physical processes, like absorption and emission of matter, take place [60, 71, 65]. A dynamical horizon can be defined locally, the global properties of the spacetime is not required. Also they are quite different in nature from the event horizon as they evolve during any processes. It was shown that while a black hole is absorbing matter, its dynamical horizon is a spacelike hypersurface [71] and for a radiating black hole the dynamical horizon is timelike in nature. For a timelike hypersurface, the Kodama vector (it is this vector which plays the role of Killing vector in a dynamical spacetime [70]) is timelike both inside and outside of it and there is no difficulty in defining particle states.

We have applied our technique to the dynamical situation where the horizon is a future outer trapping horizon (FOTH) [60]. Constructing the positive frequency modes of the Kodama vector as before and calculating the probability current we have found that the temperature associated with this types of horizons is $T = \kappa/2\pi$, as expected, where κ is the dynamical surface gravity of the spacetime.

In fact the tunnelling method have been used to calculate the temperature of a FOTH [20, 33]. But there are some technical difficulties with the calculations. Also the issue of evolution of the horizon was not addressed. In our calculations we have shown explicitly that a radiating horizon shrinks in accordance with the outgoing flux as expected.

This thesis is organized as follows. In chapter 2 we shall discuss the tunnelling method in its Hamilton-Jacobi variant. We shall develop the mathematical techniques and then apply them to Schwarzschild and Kerr black holes in different coordinate settings including both singular and non-singular ones. The coordinate invariant nature of Hawking effect and the issue of choice of boundary conditions will be clear in this way. Also we shall discuss quantum absorption into white holes which is found by applying tunnelling method to the past horizons. In the later part of this chapter we shall discuss the issue of higher order corrections to the Hawking temperature in the tunnelling picture. In chapter 3 we shall develop our new formulation of Hawking effect considering first Reissner-Nordström and then Kerr black holes in each case treating the two horizons of these spacetimes separately. It will be shown that the inner horizon is in equilibrium at a higher temperature than the outer one and one can obtain an effective temperature for these two spacetimes which corresponds to the temperature of the outer horizon if the black hole is large and is zero in the extremal limit as expected. Later considering an extremal metric we shall show that the temperature is indeed zero but the findings have to be interpreted carefully. Radiation from dynamical horizons will be treated in chapter 4. First the definition and geometrical structure of a future outer trapping horizon will be discussed. Then applying our technique we shall calculate the temperature of this horizon. Finally we shall show that the horizon loses area while radiating and the amount of area loss corresponds exactly to the outgoing flux of radiation. In the last chapter we shall summarize our results and discuss different aspects of these two local formulations of Hawking effect.

We shall use the following conventions throughout. The metric signature is the standard $(-, +, +, +)$. The constants c, G, k_B and $\hbar$ are set equal to unity unless explicitly stated.

Chapter 2

Tunnelling from black holes in the Hamilton-Jacobi approach

In this chapter we shall discuss the tunnelling of scalar particles from a black hole horizon using the Hamilton-Jacobi method. As we have mentioned in the previous chapter, the Hamilton-Jacobi method originated from the work of Padmanabhan and his collaborators [13, 14, 15] and developed further by Vanzo et al [17, 18, 50] and is one of the two methods employed to calculate the classical action required to find the tunnelling probability for emitted particles in Hawking effect. This method works with the assumption that the classical action S of the radiated particles satisfies the relativistic Hamilton-Jacobi equation.

We start with a relativistic scalar particle Φ (taken massless for simplicity) in a curved background, described by the Klein-Gordon equation

$$\frac{1}{\sqrt{-g}}\partial_\mu \left(g^{\mu\nu}\sqrt{-g}\partial_\nu \Phi\right) = 0. \tag{2.1}$$

Before deriving the Hamilton-Jacobi equation from equation (2.1) using the WKB approximation, let us see to what extent this approximation is justified near the horizon of a black hole $(r = r_h)$. If a distant observer detects a wave with frequency ω_{ob}, this has been emitted with frequency

$$\omega_{em} \propto (1 - r_h/r)^{-1/2} \omega_{ob}. \tag{2.2}$$

In the limit $r \rightarrow r_h$ where the emission occurs, the wavelength of the radiation is infinitely blue shifted and the WKB approximation can be used without any problem.

Expanding $\Phi = exp(-iS + \cdots)$, we obtain from equation (2.1) to leading order in Planck's constant the Hamilton-Jacobi equation

$$g^{\mu\nu} \partial_\mu S \partial_\nu S = 0. \tag{2.3}$$

S is the classical action of the particle. We have neglected the higher order terms in equation (2.3), but we shall return to them later to calculate the corrections arising from these terms.

Finally to find S we shall use the symmetries of the system under consideration to pick an appropriate form of S and solve equation (2.3). Usually there are two mode solutions - incoming and outgoing. The emission and absorption probability at the horizon, the modulus square of amplitude Φ, are found from the outgoing and incoming modes respectively and the Hawking-Hartle relation

$$P_{emission} = e^{-\beta E} P_{absorption}, \tag{2.4}$$

gives the Hawking temperature β^{-1}.

The main advantage of the Hamilton-Jacobi formulation is that this method applies to any well behaved coordinate system across the horizon, unlike the Parikh-Wilczek method (known as the "null-geodesic variant" [21]) which strongly relies on a very specific choice of coordinates (regular across the horizon, particularly the Painleve-Gullstrand coordinate) which is not good from the point of view of general relativity. Physical observables like the flux of radiation in Hawking effect should be independent of any choice of coordinates.

The tunnelling formulation of Hawking effect essentially describes an across horizon phenomenon. So it is expected to perform the calculation in a coordinate system which is well-behaved across the horizon. But this is not necessary in the case of Hamilton-Jacobi version of the tunnelling formulation. We shall show that this method applies to regular coordinates across the horizon and also to singular coordinates as well. But special care has to be taken to retrieve the standard result in singular coordinates. This is because singular coordinate systems unlike the regular ones can not distinguish between the future and the past horizons, and to obtain the result for a specific case the boundary conditions has to be set precisely.

In this chapter we shall emphasize mainly on two issues, the first one regarding the choice of appropriate boundary conditions for both singular and non-singular coordinates in the tunnelling formulation. We shall consider several coordinate systems belonging to both classes and in each case calculate the Hawking temperature using the Hamilton-Jacobi technique outlined above. The coordinate independent nature of the Hawking effect will be clear in this way. Also in the course of calculation we shall show that not only the black hole horizon is capable of quantum radiation, but the white hole horizon which is the time reversed version of black hole can also absorb quantum particles [24].

The second issue that we shall address in this chapter is the calculation of higher order corrections to Hawking temperature in the Hamilton-Jacobi method. Recently some authors claimed that higher-order quantum corrections can be found using the Hamilton-Jacobi version of the tunnelling method [51]. The idea is quite simple: in the Hamilton-Jacobi method, using the semi-classical limit $\hbar \to 0$ only the leading-order terms are retained in the Hamilton-Jacobi equation to calculate the standard Hawking temperature. The claim is that if one keeps the higher-order terms, then non-zero higher-order corrections to the Hawking temperature are found. These corrections are later associated with back-reaction effects and a conformal trace anomaly. This result was first shown for scalars [51] before being expanded to fermions [52] and photons [53], finally dealing with the thermodynamics of various spacetimes

These corrections are odd for many reasons. First, the authors claimed these "quantum corrections" are related to the back-reaction effect though what they study is actually a free field on a fixed, stationary background metric. So, what is calculated cannot be a true back-reaction. Moreover, the Hawking temperature can be calculated exactly through various other methods, and no higher-order quantum corrections are found as long as the metric remains fixed.

In this chapter we shall consider the full equation and taking the action as a power series in Planck's constant, calculate the terms upto third order. In this way we have found that no such "quantum corrections" appear in the Hawking temperature and what the others have found is in fact an error in the calculations which was misinterpreted [25]. This is true for fermions also.

We shall first discuss tunnelling of scalar particles from a Schwarzschild black hole using both singular and non-singular coordinates. Then we shall consider a rotating Kerr black hole which has an axisymmetric geometry. Finally we shall discuss the higher order corrections in the action S and conclude.

2.1 Schwarzschild black hole

Let us take the spacetime metric $g_{\mu\nu}$ to be the Schwarzschild solution which gives a spherically symmetric static black hole

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.5}$$

There are several different coordinates which describe a Schwarzschild black hole, some singular and some others regular at the horizon. We shall consider them one by one to calculate the tunnelling probability of scalar particles from the horizon in Hamilton-Jacobi formalism with the target of eventually finding the Hawking temperature associated with a Schwarzschild black hole. We begin with a generic coordinate representation for Schwarzschild spacetime which is singular at the horizon.

2.1.1 Singular coordinates

Schwarzschild like coordinates

If the black hole horizon is a Killing horizon such that there is a Killing vector $\chi \equiv \partial_\tau$ which is timelike in the vicinity of the horizon and becomes null on the horizon, a surface gravity κ can be defined as follows:

$$\chi \cdot \nabla \chi^\mu = \kappa \chi^\mu, \tag{2.6}$$

on the horizon. With this definition we can conveniently introduce a spatial coordinate λ by

$$\chi^2 = -2\kappa\lambda. \tag{2.7}$$

Then near the horizon the metric takes the form [22]

$$ds^2 = -2\kappa\lambda d\tau^2 + \frac{d\lambda^2}{2\kappa\lambda} + \text{other terms}. \tag{2.8}$$

For the Schwarzschild metric $\lambda = r - 2M$ and $\kappa = 1/4M$. Since the metric is spherically symmetric, we assume an appropriate form for S independent of the other (angular) spatial coordinates:

$$S = E\tau + C + S_0(\lambda), \tag{2.9}$$

where E is to be interpreted as the energy and C is a constant. Substituting S in equation (2.3) the equation for S_0 then reads,

$$-\frac{E^2}{2\kappa\lambda} + (2\kappa\lambda)S_0'(\lambda)^2 = 0. \tag{2.10}$$

The formal solution of this equation is

$$S_0(\lambda) = \pm E \int^{\lambda} \frac{d\lambda}{2\kappa\lambda}. \tag{2.11}$$

The sign ambiguity comes from the square root and corresponds to the fact that there can be incoming/outgoing solutions. This integral has a pole at the horizon $\lambda = 0$ and to obtain a solution across the horizon we need to remove this singularity. Using the standard

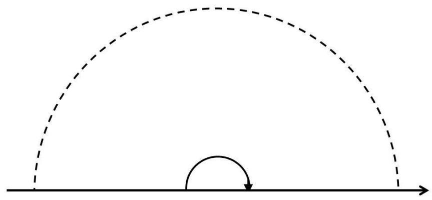

Figure 2.1: *The semi-circular contour for Schwarzschild like coordinates*

procedure of skirting the pole by changing λ to $\lambda - i\epsilon$, finally we get,

$$S_{out} = E\tau + C - \frac{E}{2\kappa}\left[i\pi + \int^{\lambda} d\lambda\, P\left(\frac{1}{\lambda}\right)\right], \tag{2.12}$$

$$S_{in} = E\tau + C + \frac{E}{2\kappa}\left[i\pi + \int^{\lambda} d\lambda\, P\left(\frac{1}{\lambda}\right)\right], \tag{2.13}$$

where $P()$ denotes the principal value. Now the metric in equation (2.8) does not distinguish between the future and the past horizon. So we need to choose the boundary conditions carefully. Radiation occurs at the future horizon which is the black hole horizon. For that the incoming probability has to be unity, since classically a black hole absorbs everything. So we fix the constant C such that the imaginary part of S_{in} is zero,

$$C = -i\pi \frac{E}{2\kappa} + \mathrm{Re}\, C \tag{2.14}$$

This implies,

$$S_{out} = E\tau - \frac{E}{2\kappa}\left[2i\pi + \int^{\lambda} d\lambda\ P\left(\frac{1}{\lambda}\right)\right] + \mathrm{Re}\, C. \tag{2.15}$$

So the imaginary part of the action is $\pi E/\kappa$ which implies a decay factor of $\exp(-\pi E/\kappa)$ in the amplitude and a factor of $\exp(-2\pi E/\kappa)$ in the probability and we get the Hawking temperature $T_H = \kappa/2\pi$.

Isotropic coordinates

The isotropic coordinates for the Schwarzschild black hole is obtained by changing the radial coordinate r to another radial coordinate ρ where

$$r = 2M + \frac{(\rho - M/2)^2}{\rho}, \tag{2.16}$$

and the metric is given by

$$ds^2 = -\frac{\left(1 - \frac{M}{2\rho}\right)^2}{\left(1 + \frac{M}{2\rho}\right)^2} dt^2 + \left(1 + \frac{M}{2\rho}\right)^4 \left(d\rho^2 + \rho^2 d\Omega^2\right). \tag{2.17}$$

The time t and angular coordinates remain unchanged. Clearly the horizon is situated at $\rho = M/2$. Moreover the coordinate ρ is complex inside the horizon $r = 2M$ [26, 50]. Since this metric is also spherically symmetric, we assume a form for S independent of the other (angular) spatial coordinates:

$$S = Et + C + S_0(\rho), \tag{2.18}$$

where E is to be interpreted as the energy and C is a constant. Substituting S in equation (2.3) we get S_0 as,

$$S_0(\rho) = \pm E \int^{\rho} \frac{\left(1 + \frac{M}{2\rho}\right)^3}{\left(1 - \frac{M}{2\rho}\right)} d\rho, \tag{2.19}$$

which near the horizon $\rho = M/2$ takes the form

$$S_0(\rho) \approx \pm 4ME \int^{\rho} \frac{d\rho}{\rho - \frac{M}{2}}. \tag{2.20}$$

Now there is a singularity at $\rho = M/2$ as before, but we have to remember the coordinate

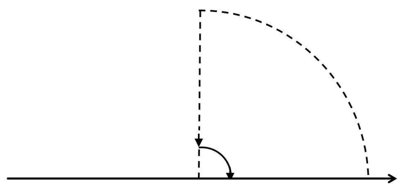

Figure 2.2: *The quarter contour for isotropic coordinates*

ρ itself is complex for $\rho < M/2$. So to perform the integration, instead of taking the contour extending from $-\infty$ to ∞ with a semicircle of infinitesimal radius ϵ at $\rho = M/2$ to avoid the pole we have to take a quarter circle of infinitesimal radius ϵ extending from real axis to the imaginary axis. This gives,

$$S_0(\rho) = \pm 4ME \left[i\frac{\pi}{2} + \int^{\rho} d\rho \, P \left(\frac{1}{\rho - M/2} \right) \right]. \tag{2.21}$$

Imposing the boundary condition that the absorption probability is unity we require S_{in} to be real, which implies, $C = -2i\pi ME$ and the outgoing mode turns out to be,

$$S_{out} = Et - 4i\pi ME - 4ME \int^{\rho} d\rho \, P \left(\frac{1}{\rho - M/2} \right). \tag{2.22}$$

This gives a decay factor of $\exp(-8\pi ME)$ in the emission probability which produces the correct Hawking temperature for the Schwarzschild black hole.

However the use of singular coordinates to describe any across the horizon phenomenon is not reliable and the procedure of choosing the proper boundary conditions can best be regarded as an ad hoc one. No such problems arise if one takes the coordinate system to be regular at the horizon as we shall show now. For regular coordinates the appropriate boundary conditions are set by default and we do not require to put them by hand in the solutions.

2.1.2 Regular coordinates

Eddington-Finkelstein coordinates

The Eddington-Finkelstein metric for a Schwarzschild spacetime is obtained by replacing t with the radial null coordinate v in the Schwarzschild metric. On radial null geodesics in Schwarzschild spacetime

$$dt^2 = \frac{dr^2}{\left(1 - \frac{2M}{r}\right)^2} \equiv dr_*^2, \tag{2.23}$$

where $r_* = r + 2M \ln |\frac{r-2M}{2M}|$ is the Regge-Wheeler tortoise coordinate. Defining the radial null coordinate v by

$$dv = dt + \frac{dr}{1 - \frac{r_h}{r}}, \tag{2.24}$$

and rewriting the Schwarzschild metric in advanced Eddington-Finkelstein coordinate (v, r, θ, ϕ) we get

$$ds^2 = -(1 - \frac{2M}{r})dv^2 + 2dvdr + r^2 d\Omega^2. \tag{2.25}$$

This metric is non-singular at $r = 2M$ and is defined for all $r > 0$.

The metric is spherically symmetric and the Hamilton-Jacobi equation for the action S becomes,

$$2\frac{\partial S}{\partial v}\frac{\partial S}{\partial r} + (1 - \frac{2M}{r})\left(\frac{\partial S}{\partial r}\right)^2 = 0 \tag{2.26}$$

Since the Killing vector is ∂_v in this coordinate, we can put $\frac{\partial S}{\partial v} = E$ and the outgoing / incoming solutions are,

$$
\begin{aligned}
S_{out} &= Ev - 2E\int \frac{dr}{1-\frac{2M}{r}} && (2.27)\\
S_{in} &= Ev && (2.28)
\end{aligned}
$$

The singular integral in the outgoing mode gives an imaginary part $-4\pi ME$ and the Hawking temperature $T_H = 1/8\pi M$. The incoming solution does not contain any imaginary part and the incoming probability is unity by default. This is because the metric here only describes a black hole horizon which is different from the Schwarzschild case.

There exists also the retarded Eddington-Finkelstein type of coordinate which describes the past horizon, the time reversed version of the black hole horizon. In this case we need the outgoing radial null coordinate u defined by

$$du = dt - \frac{dr}{1-\frac{2M}{r}}. \tag{2.29}$$

The metric in the retarded Eddington-Finkelstein coordinates (u, r, θ, ϕ) becomes

$$ds^2 = -(1-\frac{2M}{r})du^2 - 2dudr + r^2 d\Omega^2. \tag{2.30}$$

With this metric the Hamilton-Jacobi equation for spherically symmetric S is

$$-2\frac{\partial S}{\partial u}\frac{\partial S}{\partial r} + (1-\frac{2M}{r})\left(\frac{\partial S}{\partial r}\right)^2 = 0, \tag{2.31}$$

so that,

$$
\begin{aligned}
S_{out} &= Eu && (2.32)\\
S_{in} &= Eu + 2E\int \frac{dr}{1-\frac{2M}{r}} && (2.33)
\end{aligned}
$$

Here, S_{out} has no imaginary part but S_{in} has an imaginary part $4\pi ME$, which means that outward emission is complete, but absorption is inhibited by a factor $\exp(-4\pi ME)$ in the amplitude, and a factor $\exp(-8\pi ME)$ in the probability. The situation is just the reverse from that of the black hole event horizon. This is because the retarded Eddington-Finkelstein metric actually describes the exterior region with a white hole interior.

We have found that just as black holes may absorb matter classically but can emit matter only quantum mechanically and with a specific temperature, white holes may emit matter classically but can absorb matter only quantum mechanically and again with a specific temperature.

This will become more evident by considering other examples of regular coordinates for a Schwarzschild black hole. For example, we shall take the Painleve-Gullstrand coordinate and the Lemaitre coordinate alongwith Kruskal coordinate and in each case we shall explicitly calculate the tunnelling probability for both the future (black hole) and the past (white hole) horizons.

Painleve-Gullstrand coordinates

In this section, we consider Painleve coordinates, another coordinate representation of the Schwarzschild spacetime which is of our interest. The metric in these coordinates is non-singular at the horizon just as in the previous case and is stationary [15, 24]. The coordinate transformation from the Schwarzschild coordinates (t, r, θ, ϕ) to the Painleve coordinates (t_p, r, θ, ϕ) is given by

$$t_p = t \pm 2\sqrt{2Mr} \pm 2M \ln\left(\frac{\sqrt{r} - \sqrt{2M}}{\sqrt{r} + \sqrt{2M}}\right). \tag{2.34}$$

The line element in these new coordinates is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt_p^2 \pm 2\sqrt{\frac{2M}{r}} dt_p dr + dr^2 + r^2 d\Omega^2. \tag{2.35}$$

The metric is stationary and the constant time slice is just flat Euclidean space. The upper (lower) sign corresponds to the black hole (white hole) horizon. We shall first see what happens in the case of black hole and return to the white hole horizon later.

The Hamilton-Jacobi equation for spherically symmetric S is

$$-\left(\frac{\partial S}{\partial t_p}\right)^2 + 2\sqrt{\frac{2M}{r}}\frac{\partial S}{\partial t_p}\frac{\partial S}{\partial r} + \left(1-\frac{2M}{r}\right)\left(\frac{\partial S}{\partial r}\right)^2 = 0. \tag{2.36}$$

Since the Killing vector is ∂_{t_p} we use the ansatz $S = Et_p + S_0(r)$ to separate the variables and obtain the solution for $S_0(r)$,

$$S_0(r) = E\int^r dr\frac{\left(-\sqrt{\frac{2M}{r}} \pm 1\right)}{1-\frac{2M}{r}}. \tag{2.37}$$

The ±1 comes from solving the quadratic equation. Again there is a singularity at the horizon and with the same treatment as before we get the outgoing and incoming solutions respectively,

$$S_{out} = Et_p - E\left[i4\pi M + \int^r drP\left(\frac{1}{r-2M}\right)r\left(1 \pm \sqrt{\frac{2M}{r}}\right)\right], \tag{2.38}$$

$$S_{in} = Et_p + E\left[\int^r drP\left(\frac{1}{r-2M}\right)r\left(1 \mp \sqrt{\frac{2M}{r}}\right)\right]. \tag{2.39}$$

The outgoing mode contains an imaginary part $(-4\pi ME)$ which reproduces the Hawking temperature while no imaginary term appears in the incoming mode implying complete absorption.

Now let us consider the other line element corresponding to the white hole horizon. In this case the Hamilton-Jacobi equation for spherically symmetric S is given by

$$-\left(\frac{\partial S}{\partial t_p}\right)^2 - 2\sqrt{\frac{2M}{r}}\frac{\partial S}{\partial t_p}\frac{\partial S}{\partial r} + \left(1-\frac{2M}{r}\right)\left(\frac{\partial S}{\partial r}\right)^2 = 0. \tag{2.40}$$

Again separating the variables as before we get the r-dependent part of the action,

$$S_0(r) = E \int^r dr \frac{\left(\sqrt{\frac{2M}{r}} \pm 1\right)}{1 - \frac{2M}{r}}. \tag{2.41}$$

Here also the ± 1 comes from the quadratic equation. Solving the singular integral we find the mode solutions,

$$\begin{aligned} S_{out} &= Et_p - E\left[\int^r drP\left(\frac{1}{r-2M}\right) r\left(1 \pm \sqrt{\frac{2M}{r}}\right)\right], && (2.42)\\ S_{in} &= Et_p + E\left[i4\pi M + \int^r drP\left(\frac{1}{r-2M}\right) r\left(1 \mp \sqrt{\frac{2M}{r}}\right)\right]. && (2.43)\end{aligned}$$

We find that emission is complete as there is no imaginary part in the outgoing mode, but absorption is inhibited by the factor of $\exp(-4\pi ME)$ in the amplitude and the factor of $\exp(-8\pi ME)$ in the probability.

Lemaitre coordinates

We now consider the Lemaitre coordinates which is another set of non-static coordinates for Schwarzschild spacetime [15, 24]. The coordinate transformations from the Schwarzschild coordinates (t, r) to Lemaitre coordinates (τ, R) are

$$\begin{aligned} \tau &= \pm t \pm \sqrt{2M} \int dr \frac{\sqrt{r}}{r-2M}, && (2.44)\\ R &= t + \frac{1}{\sqrt{2M}} \int dr \frac{r\sqrt{r}}{r-2M}. && (2.45)\end{aligned}$$

Using these two coordinates the Schwarzschild line element becomes,

$$ds^2 = -d\tau^2 + \frac{dR^2}{\left[\frac{3}{4M}(R \mp \tau)\right]^{2/3}} + (2M)^2 \left[\frac{3}{4M}(R \mp \tau)\right]^{4/3} d\Omega^2. \tag{2.46}$$

There are two metrics and like the Painleve coordinates they correspond to two different regions of Schwarzschild spacetime : the upper sign gives the future horizon and the lower sign gives the past horizon respectively. It is easy to see from the transformations in equation (2.44) that the Schwarzschild r coordinate is given by the relation

$$\frac{r}{2M} = \left[\frac{3}{4M}(R \mp \tau)\right]^{2/3}. \tag{2.47}$$

So the horizons are located at $\frac{3}{4M}(R \mp \tau) = 1$.

Let us first consider the radiation from the black hole horizon. The Hamilton-Jacobi equation for spherically symmetric S corresponding to the black hole metric is given by

$$-\left(\frac{\partial S}{\partial \tau}\right)^2 + \left[\frac{3}{4M}(R - \tau)\right]^{2/3}\left(\frac{\partial S}{\partial R}\right)^2 = 0. \tag{2.48}$$

To solve this equation let us separate the variables as

$$S = S_-(R - \tau) + S_+(R + \tau). \tag{2.49}$$

Now we use the ansatz $S'_+ = E/2$, where "$\prime$" denotes derivative w.r.t $(R + \tau)$. The normalization is fixed by noting that, $\tau = 1/2(R+\tau) - 1/2(R-\tau)$ and in the asymptotic region the dR^2 piece drops out. With this the Hamilton-Jacobi equation becomes,

$$\left(\frac{E}{2} + S'_-\right)^2 \left[\frac{3}{4M}(R - \tau)\right]^{2/3} = \left(\frac{E}{2} - S'_-\right)^2. \tag{2.50}$$

Solving for S_- we get,

$$S_- = \frac{E}{2}\left(\frac{4M}{3}\right)\int d\zeta_-^3 \frac{-(\zeta_-^2 + 1) \pm 2\zeta_-}{\zeta_-^2 - 1}, \tag{2.51}$$

where $\zeta_- = \left[\frac{3}{4M}(R - \tau)\right]^{1/3}$. The horizon is located at $\zeta_- = 1$. So the incoming and outgoing modes become

$$S_{in} = \frac{E}{2}(R+\tau) + \frac{E}{2}\left(\frac{4M}{3}\right)\int d\zeta_-^3 \frac{1-\zeta_-}{1+\zeta_-}, \tag{2.52}$$

$$S_{out} = \frac{E}{2}(R+\tau) + \frac{E}{2}\left(\frac{4M}{3}\right)\int d\zeta_-^3 \frac{1+\zeta_-}{1-\zeta_-}. \tag{2.53}$$

As before there is no singularity in the incoming solution implying complete absorption and a singularity appears at the horizon in the outgoing solution. This singular integral gives an imaginary part $(-4\pi ME)$ which produces the Hawking temperature in the emission probability.

For the white hole horizon the metric is given by the lower sign in the line element. In this case the Hamilton-Jacobi equation for spherically symmetric S is given by

$$-\left(\frac{\partial S}{\partial \tau}\right)^2 + \left[\frac{3}{4M}(R+\tau)\right]^{2/3}\left(\frac{\partial S}{\partial R}\right)^2 = 0. \tag{2.54}$$

To find the solution we again separate the variables as

$$S = S_-(R-\tau) + S_+(R+\tau), \tag{2.55}$$

And use the ansatz $S'_- = -E/2$. The normalization is fixed in a similar manner as we have used in the case of black hole horizon. The Hamilton-Jacobi equation becomes,

$$\left(-\frac{E}{2} + S'_+\right)^2 \left[\frac{3}{4M}(R+\tau)\right]^{2/3} = \left(\frac{E}{2} + S'_+\right)^2 \tag{2.56}$$

Solving for S_+ we finally get the incoming and outgoing mode solutions

$$S_{in} = -\frac{E}{2}(R-\tau) + \frac{E}{2}\left(\frac{4M}{3}\right)\int d\zeta_+^3 \frac{1+\zeta_+}{\zeta_+-1}, \tag{2.57}$$

$$S_{out} = -\frac{E}{2}(R-\tau) + \frac{E}{2}\left(\frac{4M}{3}\right)\int d\zeta_+^3 \frac{\zeta_+-1}{1+\zeta_+}. \tag{2.58}$$

Here $\zeta_+ = \left[\frac{3}{4M}(R+\tau)\right]^{1/3}$. The horizon is situated at $\zeta_+ = 1$.

Again we have found that the outward emission is complete for the white hole horizon as there is no imaginary term in the outgoing mode but absorption is inhibited by a factor of $\exp(-8\pi ME)$ in the absorption probability which comes from the singular integral in the incoming mode.

Kruskal coordinates

The Kruskal coordinates (U, V) represents the maximal extension of Schwarzschild spacetime. The metric has no singularity at the horizon. The emission and absorption properties of black and white holes can be more evident in these coordinates [24, 31]. The metric is written as,

$$ds^2 = -\frac{32M^3}{r}e^{-\frac{r}{2M}}dUdV + r^2 d\Omega^2, \tag{2.59}$$

where the Schwarzschild coordinate r is related to U, V by

$$UV = \left(1 - \frac{r}{2M}\right)e^{\frac{r}{2M}} \tag{2.60}$$

The metric is flat near the horizon. It describes both the past and the future horizon. The future (i.e the black hole event horizon) horizon is given by $V = \text{constant}$ while $U = \text{constant}$ gives the past (white hole) horizon. The Hamilton-Jacobi equation for a spherically symmetric S in these coordinates is,

$$\frac{\partial S}{\partial V}\frac{\partial S}{\partial U} = 0. \tag{2.61}$$

S can depend either on U (outgoing mode) or on V(incoming mode). For the black hole horizon since V is constant, we have,

$$S_{out} = -4ME\int\frac{dU}{U}. \tag{2.62}$$

An imaginary factor $i\pi$ arises in this integral and produces the Hawking temperature $1/8\pi M$ for emission. Similarly for the past horizon $(U = \text{constant})$ the action is given by

$$S_{in} = 4ME \int \frac{dV}{V}, \tag{2.63}$$

so that absorption is inhibited by the factor $\exp(-4\pi ME)$ in the amplitude and $\exp(-8\pi ME)$ in the probability.

2.2 Kerr black hole

Till now we have been discussing Schwarzschild spacetime in different coordinate settings. Now we shall consider rotating Kerr spacetime which has a slightly more complicated geometrical structure than the Schwarzschild case [21, 24]. The Kerr metric can also be written in a number of different ways. Here we shall write it in the standard Boyer-Lindquist form as,

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right) dt^2 - \frac{4Mar\sin^2\theta}{\rho^2} dt d\phi + \frac{\Sigma}{\rho^2}\sin^2\theta d\phi^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2. \tag{2.64}$$

where,

$$\begin{aligned} \rho^2 &= r^2 + a^2\cos^2\theta. \\ \Delta &= r^2 + a^2 - 2Mr. \\ \Sigma &= \left(r^2 + a^2\right)^2 - a^2\Delta\sin^2\theta. \end{aligned}$$

The metric is stationary and axially symmetric, it therefore admits two Killing vectors ∂_t and ∂_ϕ. It is also asymptotically flat like the Schwarzschild metric. M is the spacetime's mass and $J \equiv aM$ is the angular momentum. The metric is singular at $\Delta(r) = 0$ and $\rho^2(r, \theta) = 0$. But the singularities at $\Delta(r) = 0$ are just coordinate singularities while

$\rho^2(r,\theta) = 0$ admits a true curvature singularity.

The Kerr metric has two horizons : outer (r_+) and inner (r_-), situated at $\Delta(r) = 0$ and are given by $r_\pm = M \pm \sqrt{M^2 - a^2}$. The outer horizon (r_+) is the event horizon of the spacetime and we shall consider emission and absorption phenomena near this horizon only.

In applying the Hamilton-Jacobi formalism for calculating the tunnelling probability of scalar particles from the outer horizon of Kerr black hole we have to first note that we can not consider emission of generic spherical waves in this case because the metric components depend on the angle θ. Instead we will examine rings of scalar waves at fixed $\theta = \theta_0$. In the end we shall discover that the temperature does not depend on the choice of θ as it should be.

For fixed $\theta = \theta_0$ the metric becomes

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 - \frac{4Mar\sin^2\theta}{\rho^2}dt d\phi + \frac{\Sigma}{\rho^2}\sin^2\theta d\phi^2 + \frac{\rho^2}{\Delta}dr^2, \tag{2.65}$$

where all the metric components are calculated at $\theta = \theta_0$. The Hamilton-Jacobi equation $g^{\mu\nu}\partial_\mu S \partial_\nu S = 0$ takes the form

$$g^{rr}(\partial_r S)^2 + g^{tt}\left[(\partial_t S)^2 + 2\frac{g^{t\phi}}{g^{tt}}\partial_t S \partial_\phi S + \frac{g^{\phi\phi}}{g^{tt}}(\partial_\phi S)^2\right] = 0. \tag{2.66}$$

Evaluating the metric components near the horizon $r = r_+$ we get

$$\frac{g^{t\phi}}{g^{tt}} = \frac{2Mar}{\Sigma} \equiv \Omega, \tag{2.67}$$

$$\frac{g^{\phi\phi}}{g^{tt}} = \frac{a^2}{\Sigma} \equiv \Omega^2, \tag{2.68}$$

$$-\frac{g^{rr}}{g^{tt}} = \frac{\Delta^2}{\Sigma} \equiv \frac{4\left[(r_+ - M)(r - r_+)\right]^2}{(r_+^2 + a^2)^2}. \tag{2.69}$$

So the equation (2.66) becomes,

$$\frac{4\left[(r_+ - M)(r - r_+)\right]^2}{(r_+^2 + a^2)^2}\left(\partial_r S\right)^2 - \left[\partial_t S + \Omega\partial_\phi S\right]^2 = 0. \tag{2.70}$$

Since the Killing vector which is timelike outside and null on the horizon for the Kerr spacetime is $\partial_\chi = \partial_t + \Omega\partial_\phi$, we use the ansatz $S(t, r, \phi, \theta_0) = \omega\chi + S(r, \theta_0)$ where $\omega = E - J\Omega$ and $E = \omega_\infty$ is the frequency at infinity. Finally we get

$$S(t, r, \phi, \theta_0) = \omega\chi \pm \omega\frac{r_+^2 + a^2}{2(r_+ - M)}\int\frac{dr}{r - r_+}, \tag{2.71}$$

where the $\pm$ sign corresponds to the incoming/outgoing modes. The singular integral gives a factor of πi in addition to the principal part. Again we see to get complete absorption for the black hole horizon we need to fix the boundary conditions by adding an imaginary term $-i\omega(r_+^2 + a^2)\pi/2(r_+ - M)$ in the incoming mode so that the emission probability becomes

$$P_{emission} = \exp\left(-\frac{2\pi(r_+^2 + a^2)\omega}{(r_+ - M)}\right), \tag{2.72}$$

and we get the Hawking temperature for the Kerr black hole as,

$$T_H = \frac{r_+ - M}{2\pi(r_+^2 + a^2)}. \tag{2.73}$$

2.2.1 Eddington-Finkelstein coordinates for Kerr black hole

Let us now consider Eddington-Finkelstein coordinates for Kerr spacetime. This metric is regular across the horizon. Like the Schwarzschild case, there are two types of coordinates : advanced and retarded which describes the future (black hole) and the past (white hole) horizon respectively. We shall first discuss the advanced metric. The transformation from Boyer-Lindquist coordinates (t, r, θ, ϕ) to advanced Eddington-Finkelstein coordinates

(v, r, θ, ψ) is given by

$$dv = dt + \frac{r^2 + a^2}{\Delta} dr \tag{2.74}$$

$$d\psi = d\phi + \frac{a}{\Delta} dr \tag{2.75}$$

The coordinates (r, θ) remain unchanged. In these new coordinates the line element can be written as

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right) dv^2 + 2dvdr - 2a\sin^2\theta d\psi dr - \frac{4Mar\sin^2\theta}{\rho^2} dvd\psi + \frac{\Sigma}{\rho^2}\sin^2\theta d\psi^2 + \rho^2 d\theta^2.$$

Now we can write the Hamilton-Jacobi equation in these coordinates for a fixed $\theta = \theta_0$ and solve for S. Near the horizon $S(v, r, \psi)$ satisfies,

$$\left(\frac{\partial S}{\partial r} + \frac{r_+^2 + a^2}{\Delta}\frac{\partial S}{\partial v} + \frac{a}{\Delta}\frac{\partial S}{\partial \psi}\right)^2 = \left(\frac{r_+^2 + a^2}{\Delta}\right)^2 \left(\frac{\partial S}{\partial v} + \Omega\frac{\partial S}{\partial \psi}\right)^2. \tag{2.76}$$

The solutions are of the form,

$$S_{out} = Ev + J\psi - 2(E + \Omega J)\int dr \frac{r_+^2 + a^2}{\Delta}, \tag{2.77}$$

$$S_{in} = Ev + J\psi. \tag{2.78}$$

There is no imaginary part in the incoming mode and the absorption probability is unity as expected. The singular integral in the outgoing mode produces a decay factor of $\exp\left(-(E + \Omega J)\pi(r_+^2 + a^2)/(r_+ - M)\right)$ and the Hawking temperature is $T_H = \frac{r_+ - M}{2\pi(r_+^2 + a^2)}$.

One can describe emission and absorption phenomena at the past horizon by considering retarded Eddington-Finkelstein coordinates in a similar manner. The coordinate transformations in this case is given by

$$
\begin{aligned}
du &= dt - \frac{r^2 + a^2}{\Delta} dr, && (2.79) \\
d\chi &= d\phi - \frac{a}{\Delta} dr. && (2.80)
\end{aligned}
$$

After writing the Hamilton-Jacobi equation in these coordinates the solutions near the horizon for a fixed θ are found to be

$$
\begin{aligned}
S_{out} &= Eu + J\chi && (2.81) \\
S_{in} &= Eu + J\chi + 2(E + \Omega J) \int dr \frac{r_+^2 + a^2}{\Delta} && (2.82)
\end{aligned}
$$

Again as expected, the outgoing mode shows complete emission while the incoming mode produces a decay factor of $\exp\left(-(E + \Omega J)\pi(r_+^2 + a^2)/(r_+ - M)\right)$ indicating inhibited absorption at the white hole horizon of the Kerr spacetime.

2.3 Higher order calculations

In the previous section the action S has been treated only to the lowest order in Plancks constant. In this section we shall consider the higher order terms in S [25]. We consider a scalar field in a spherically symmetric stationary black hole background as before. In the Klein-Gordon equation putting $\Phi = \exp(-iS)$, the full equation for the action S is

$$
g^{\mu\nu}\partial_\mu S \partial_\nu S = -i \left[g^{\mu\nu}\partial_\mu \partial_\nu S + \frac{1}{\sqrt{-g}} \partial_\mu \left(g^{\mu\nu}\sqrt{-g} \right) \partial_\nu S \right]. \quad (2.83)
$$

The Hamilton-Jacobi equation is obtained by setting the r.h.s equal to 0. In the Schwarzschild metric, this equation reads

$$r^2\left[\frac{\dot{S}^2}{\left(1-\frac{2M}{r}\right)}-\left(1-\frac{2M}{r}\right)S'^2\right]=-i\left[r^2\frac{\ddot{S}}{\left(1-\frac{2M}{r}\right)}-\left(r^2\left(1-\frac{2M}{r}\right)S'\right)'\right]. \tag{2.84}$$

The r^2 factors in the l.h.s of this equation comes from $(g)^{1/2}$ in three space dimensions which is not included in [51] and thus differs from us. Since we are trying to calculate higher order corrections let us assume,

$$S = Et + C + S_0(r) + S_1(r) + S_2(r) + \cdots . \tag{2.85}$$

where, $S_n(r)$ is of n^{th} degree in $\hbar$. $S_0(r)$ gives the standard Hawking temperature as was discussed in the previous section. The terms of n^{th} degree in $\hbar$ for $n > 0$ yield

$$r^2\left(1-\frac{2M}{r}\right)\sum_{r=0}^{n}S'_rS'_{n-r} = -i\left[r^2\left(1-\frac{2M}{r}\right)S'_{n-1}\right]' \tag{2.86}$$

This equation gives $S_n(r)'$ in terms of lower order S'_n-s and can be solved recursively. In particular we calculate the first three terms,

$$S'_1 = -\frac{i}{r}. \tag{2.87}$$

$$S'_2 = \mp\frac{2M}{2Er^3}. \tag{2.88}$$

$$S'_3 = \frac{i2M}{4E^2r^4}\left[\frac{2M}{r}-3\left(1-\frac{2M}{r}\right)\right]. \tag{2.89}$$

We find odd order $S_n(r)$ are purely imaginary and of the same sign for outgoing and incoming solutions and even order $S_n(r)$ are real and differ in sign for the two solutions.

Now let us see what happens when the background metric is regular at the horizon for e.g the advanced Eddington-Finkelstein metric. In this case the full equation for the action S is

$$r^2\left(1-\frac{2M}{r}\right)S'^2+2r^2ES'+i\left[2rE+\left(r^2\left(1-\frac{2M}{r}\right)S'\right)'\right]=0. \quad (2.90)$$

Again as before assuming the action has the form,

$$S=Ev+S_0(r)+S_1(r)+S_2(r)+\cdots, \quad (2.91)$$

we find the terms in n^{th} degree for $n>0$ yield the equation,

$$r^2\left(1-\frac{2M}{r}\right)\sum_{r=0}^{n}S'_rS'_{n-r}+2r^2ES'_n=-i\left[2rE+\left(r^2\left(1-\frac{2M}{r}\right)S'_{n-1}\right)'\right]. \quad (2.92)$$

This equation also can be solved recursively and the first three terms turns out to be the same as was obtained for a Schwarzschild metric,

$$S'_1 = -\frac{i}{r}. \quad (2.93)$$

$$S'_2 = \mp\frac{2M}{2Er^3}. \quad (2.94)$$

$$S'_3 = \frac{i2M}{4E^2r^4}\left[\frac{2M}{r}-3\left(1-\frac{2M}{r}\right)\right]. \quad (2.95)$$

As regards the significance of these corrections to S we first note that the even order terms are all real and hence do not contribute to any correction to the temperature since Hawking temperature comes from the imaginary part of the action S. The odd order terms are imaginary but they have same sign for both the incoming and the outgoing modes. Now let us consider these terms one by one. We see that the 1st order term is not really a correction since,

$$S_1=-i\ln r, \quad (2.96)$$

which means Φ gets a multiplicative factor of $1/r$ which is to be expected for a spherical wave in three space dimensions. The first correction comes from the 3rd order term S_3 as,

$$S_3 = i\frac{r_h}{4E^2r^3}\left(1 - \frac{r_h}{r}\right). \tag{2.97}$$

This term changes both the incoming and outgoing modes. However S_3 not only depends on the energy E as $1/E^2$ but also on r and vanishes at the horizon and at infinity. So, this does not really bring any corrections to the Hawking temperature. In general

$$S_n \propto E^{1-n}, \tag{2.98}$$

apart from the functions of r and this changes ϕ in a complicated way though no corrections to the Hawking temperature can be seen.

It is also interesting to see what happens in case of Fermions. In the case of Fermions the Klein-Gordon equation has to be replaced by Dirac equation and a spin connection has to be introduced. In the massless case, two component wave functions are appropriate. There are no solutions independent of the angular variables [27] , but one can take the angular part to be $\frac{1}{\sqrt{\sin\theta}}$. In this simple case, the wave functions can be written as

$$\Psi = \frac{(1-\frac{r_h}{r})^{-1/4}}{r\sqrt{\sin\theta}}e^{-iEt}\begin{pmatrix} F(r) \\ G(r) \end{pmatrix}. \tag{2.99}$$

The functions $F(r)$ and $G(r)$ satisfy coupled first order equations,

$$EF = \left(1 - \frac{r_h}{r}\right)G', \tag{2.100}$$

$$EG = -\left(1 - \frac{r_h}{r}\right)F', \tag{2.101}$$

which can be solved exactly. The incoming solution has the form,

$$F = -iG \propto -ie^{-iE\int\frac{dr}{1-\frac{r_h}{r}}}, \tag{2.102}$$

while the outgoing solution is of the form,

$$F = iG \propto ie^{iE\int \frac{dr}{1-\frac{r_h}{r}}}, \tag{2.103}$$

Both solutions contain damping factors $\exp(-\pi E r_h)$ which lead to factors $\exp(-2\pi E r_h)$ in the probability. But there are no corrections to the dependence on E or to the Hawking temperature. For more involved angular dependence, the equations for $F(r)$,$G(r)$ have extra terms [27] and corrections to the above solution start from S_2, with $S_n \propto E^{1-n}$ once again, changing the distribution in a complicated r-dependent way. But this cannot be interpreted as a correction to the Hawking temperature.

A few words are in order here regarding the corrections to Hawking temperature obtained by the authors of [51]. While deriving the full equation for the semiclassical action S we have mentioned that a factor of r^2 was missing in the derivation of [51] which comes from considering Hawking radiation in 4-dimensional spacetimes. The authors of [51] argued that since the temporal and the radial part of the action contain all the important features of Hawking radiation, one can essentially take the metric to be $(1 + 1)$ dimensional which simplifies the calculations. Though the argument is correct, one has to take care for a 4-dimensional black hole because in that case the angular part of the metric also contain the radial function r. If one takes the metric naively to be $(1 + 1)$ dimensional for a 4-dimensional black hole, one gets an incorrect equation for the action S as was obtained by the authors of [51]. As a result they get the same equation for all higher order terms in S in the form,

$$\partial_t \, S_i = \pm C \, \partial_r \, S_i, \tag{2.104}$$

where C is a constant. This implies all the higher order terms in S are proportional to the zeroth order term S_0, but the proportionality constant can not be fixed from these calculations. Later it was also shown [54] that these “corrections“ are indeed vacuous because of a wrong definition of energy in the calculations and finally people were convinced that no quantum corrections to Hawking temperature can be obtained by considering Hawking radiation in a fixed background metric.

2.4 Discussions

In this work we have analyzed tunnelling across black hole horizons in the Hamilton-Jacobi approach. Considering both singular and non-singular coordinates at the horizon for the Schwarzschild and Kerr spacetimes we have shown that not only there is a possibility of quantum radiation from the future horizon, but the past horizon is also capable of quantum absorption at the same temperature.

In this regard let us make the issue of choice of boundary conditions a bit more clear. Previously people have used singular coordinates to calculate the Hawking temperature of a black hole in the tunnelling framework. And noted the difficulties arising in obtaining the correct result. For example in [13, 14, 15] the authors have found an imaginary term both in the incoming and outgoing modes of the action S which gives the absorption probability as $P_{absorption} = \exp(\beta E/2)$ and the emission probability as $P_{emission} = \exp(-\beta E/2)$. The correct Hawking temperature is obtained by taking ratio of these two probabilities. But conceptually this is quite unacceptable. First of all an imaginary term in the incoming mode of S means there is no classical path which is not at all true for incoming particles as all the classical paths lead inside the horizon. Secondly a probability can never be greater than unity while here the absorption probability is an exponentially large term. It appears that though the correct result is obtained, the techniques are not sound. In [17] the authors have tried a different approach. Instead of the radial coordinate r they introduced a "proper radial coordinate" σ and using the properties of distributions it was shown that σ is quite well behaved at the horizon and poses no problem in getting the correct temperature.

In fact the use of singular coordinates in tunnelling picture led to the so called "factor of two" problem in Hawking temperature [50] which is not a real problem, but actually a confusion arising due to improper boundary conditions.

Here we have taken another way out of this problem. Introducing a constant of integration C, which can arise in general from the reconstruction of S, in both the incoming and outgoing modes of S we have enforced the boundary condition that absorption probability is unity. This fixes C and also the emission probability gives the correct Hawking temperature for the corresponding spacetime. Some people have criticized this treatment [50] by stating that this method does not work for some types of singular coordinates for example the isotropic coordinates. But in that case the catch is in choosing the proper contour of integration as we have mentioned in the discussion for isotropic coordinates. A naive calculation using the wrong contour (in this case the semicircle in the upper half plane) will give half of the standard value of Hawking temperature. But once the proper contour is identified, there is no problem with our method.

In fact the invariance of the action integral in the tunnelling method can be established for a general class of coordinate transformations which is singular at the horizon [26]. To illustrate this let us consider any coordinate transformation of the form $r = r(R)$ where r is the radial coordinate in the Schwarzschild metric. The equation satisfied by the radial part of the action $S_0(R)$ becomes

$$-\frac{E^2}{1-\frac{2M}{r(R)}} + \frac{\left(1-\frac{2M}{r(R)}\right)}{r'(R)^2} S_0'(R)^2 = 0. \tag{2.105}$$

This yields,

$$\begin{aligned} S_0(R) &= \pm E \int^R \frac{r'(R)\, dR}{1-\frac{2M}{r(R)}} \\ &= \pm E \int^{r(R)} \frac{dr(R)}{1-\frac{2M}{r(R)}} = S_0(r(R)). \end{aligned} \tag{2.106}$$

This shows the formal invariance of the integral. We can also express this integral in terms of R. Let R_0 be the value of R at the horizon i.e $2M = r(R_0)$. Continuity requires that, $r \to 2M$ as $R \to R_0$. A large class of transformations which satisfy such a condition have

$$r - 2M \approx C\,(R - R_0)^{\alpha} \tag{2.107}$$

near the horizon where $C,\ \alpha$ are non-vanishing constants. For example for isotropic coordinates $\alpha = 2$. So near the horizon

$$r'(R) \approx \alpha\, C\,(R - R_0)^{\alpha - 1}\,, \tag{2.108}$$

and the integral for S_0 becomes,

$$S_0(R) = \pm 2ME\,\alpha \int^{R} \frac{dR}{R - R_0}. \tag{2.109}$$

Apparently it seems that the factor α in front of the integral creates a problem in obtaining the correct temperature. But this is not really the case as can be seen from equation (2.107),

$$\frac{dr}{r - 2M} = \alpha\, \frac{dR}{R - R_0}. \tag{2.110}$$

This establishes the invariance of the integral $S_0(R) = S_0(r(R))$ for this type of coordinate transformations.

For regular coordinates there is no need for such a constant because the absorption probability turns out to be unity by default. What can we make of this? One has to note that in case of singular coordinates, like the Schwarzschild one, the spacetime metric does not distinguish between the future and the past horizons. Let us consider the extended Schwarzschild spacetime in Kruskal coordinates. As can be seen from figure 2.3, the Schwarzschild metric only describes the region I and both the future and the past horizon are included in the line element. So if one wants to consider a particular region of the

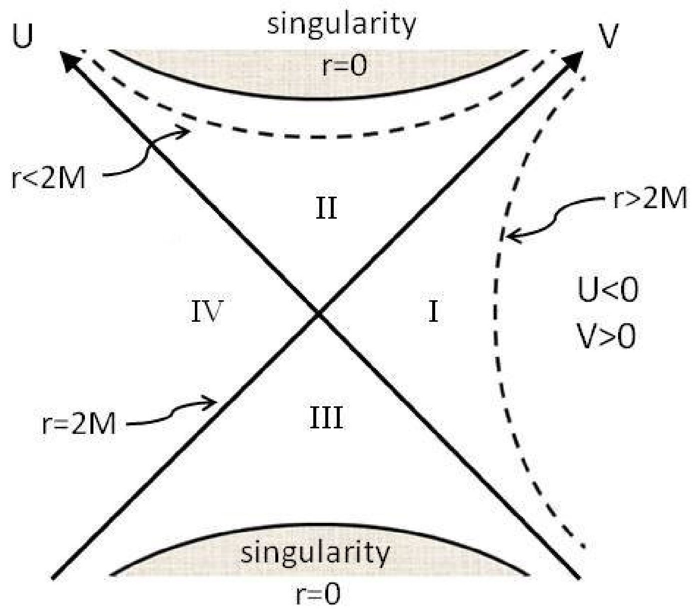

Figure 2.3: *The Schwarzschild spacetime in Kruskal coordinates*

spacetime then the appropriate boundary conditions are to be put by hand. The case is different for regular coordinates. Here the line element is different for future or the past horizon and choosing a particular line element means the boundary conditions are set from the beginning, so there is no need to put them by hand at the end of calculations. In the case of the three regular coordinates that we have discussed, the advanced Eddington-Finkelstein and the upper metric in Painleve-Gullstrand and Lemaitre coordinates describe the regions I and II in the Kruskal diagram (figure 2.3) excluding the past horizon and the retarded Eddington-Finkelstein and the lower metric in Painleve-Gullstrand and Lemaitre coordinates describe the regions I and III in the Kruskal diagram excluding the future horizon. In the case of Kruskal coordinates (U, V) the future horizon is chosen by setting $0 \leq V \leq \infty$, and for the past horizon one chooses $-\infty \leq U \leq 0$. It shows that the calculation involving coordinates which are regular at the horizon is less complicated since we do not need to impose the boundary conditions by hand in these cases which can best be regarded as an ad hoc one.

Considering the line element for the past horizon which describes the exterior and interior of a white hole, we have found the phenomena of quantum absorption into white holes at the same temperature. Hawking has argued [6] that while black holes may be created by classical collapse, their disintegration is a quantum phenomenon. By time-reversal, the creation of white holes is a quantum phenomenon, but their disintegration is classical. This is consistent with our results, though of course we have considered only eternal black or white holes, which are used only as providing gravitational backgrounds.

In the last section we have calculated higher order terms in the action S taking the full equation into consideration and we have shown that no quantum corrections appear in the Hawking temperature if one considers higher order terms in the semiclassical action while the background metric is stationary. The order by order "corrections" others obtained [51, 53] is simply an error in calculations. Later it was also shown that these "corrections" arise due to a misinterpretation of energy [54] and finally people were convinced that no such "corrections" exist for a stationary black hole.

Chapter 3

A new formulation of Hawking radiation

In this chapter we propose another formulation of the Hawking effect based on quantum field theory in curved spacetime. Our aim is to calculate the temperature associated with a horizon considering only the local geometrical properties of the horizon so that we do not require any knowledge about the global properties of the spacetime in which the horizon is embedded. If this can be done then this method can be applied to find temperature associated with any local horizon without bothering about their asymptotic properties.

The tunnelling formalism can deal with local horizons since only the near horizon geometry is required to calculate the action S from the Hamilton-Jacobi equation and the emission probability. But this method is quite restrictive as it depends entirely on the WKB approximation to compute the emission rate and hence is not applicable to systems where this approximation breaks down. One such example is the extremal black hole. In this case the Hamilton-Jacobi method gives contradictory results because the WKB approximation does not hold near an extremal horizon. Because of these inherent difficulties in the tunnelling formalism we have tried to develope another method of calculating temperature of a horizon which will dispense the WKB approximation completely.

Our idea is to construct the modes directly from the field equation near the horizon without using any WKB like approximation and calculate the emission rate from these solutions. Since this is essentially a horizon crossing phenomenon, a good set of coordinates is required which is regular across the horizon. In this treatment we have used Kruskal coordinates. One advantage of using Kruskal coordinates is that this type of double null coordinates can be easily constructed for any local horizon. Also the metric is flat near the horizon which simplifies calculations considerably. We shall construct the field modes using the near horizon metric only and the asymptotic properties of the metric is not required. So this is also a local calculation of Hawking effect since we need not be bothered about the global structure of the spacetime.

A black hole horizon admits a Killing vector which is timelike outside the horizon and null on the horizon. One can define a vacuum w.r.t this Killing vector and construct single particle states outside the horizon. In this formalism we shall consider a scalar field in a black hole background geometry and construct the positive frequency modes of the Killing vector near the horizon. After constructing the modes it is found that the incoming modes are quite well behaved but the outgoing modes have a logarithmic singularity at the horizon. But we have to keep in mind that the field modes are essentially distribution valued and it is shown that the distributions are quite well behaved at the horizon. This nature of the field modes has previously been noted by Damour and Ruffini [55].

After constructing the modes, we shall calculate the probability current coming out of the horizon using the standard field theoretic formula. It is found that the conditional probability that a particle emits from the horizon when it is incident on the surface from the other side has the form of a Boltzmann distribution

$$P_{(emission|incident)} = \frac{P_{(emission \cap incident)}}{P_{(incident)}} = e^{-\beta E}, \tag{3.1}$$

which gives the inverse temperature β associated with the horizon.

There is one problem in this formalism which is the nature of the horizon from where the emission occurs. A black hole event horizon is a null surface and the Killing vector behaves differently in different regions of the spacetime; it is timelike outside the horizon, null on the surface and spacelike inside the horizon. So one can not define a global vacuum and single particle states with respect to this Killing vector. To solve this problem we have assumed that the surface from where particle escapes to infinity is not the null event horizon, but something like Hayward's timelike trapping horizon [60, 67]. If the horizon is a timelike surface then the Killing vector is timelike both inside and outside and it is possible to define a vacuum and construct particle states outside the horizon and continue these states to inside. In real situation the process of radiation is a dynamical one and the horizon evolves during the process. It is shown that [71] if a horizon absorbs particle it evolves to a spacelike one and if it radiates then it becomes a timelike surface. So our assumption is quite justified considering the radiation as a real dynamical process. This will be more clear later when we shall discuss Hawking radiation in the context of a dynamical horizon.

In this chapter we shall apply this new method to the Reissner-Nordström(RN) and the Kerr black holes. The main reason for choosing Kerr and/or RN spacetime is that both exhibit extremal limits. Both metrics have two horizons: the outer $(r = r_+)$ and the inner $(r = r_-)$, having different properties. They do not even have the same temperature, the temperature of the inner horizon is higher than the outer horizon. The extremal limit is achieved as $r_+ \rightarrow r_-$. In this chapter we shall calculate the probability flux coming out of both the horizon separately and also see what happens in the extremal case. It is found that if one considers the extremal solution as a limiting case of the non-extremal one, then the results agree with the standard picture, i.e, the temperature of an extremal black hole is zero. On the other hand if one takes an extremal metric from the beginning, then the findings are to be interpreted carefully.

This chapter is organized as follows: in the next section we shall discuss Reissner-

Nordström solution, constructing the Kruskal coordinates for both the horizons and then computing the scalar modes and the probability current for both the cases. Kerr black hole will be considered next, emphasizing again on the different sets of Kruskal coordinates for the outer and the inner horizons, the field modes using distributions and finally the probability flux coming out of both the horizons. We will show how the extremal solution as a limiting case of the non-extremal one produces the standard results at the end of the next section. In the last section we shall consider an extremal metric and using the aforementioned formulation calculate the probability flux across the horizon and discuss the results.

3.1 Reissner-Nordström black hole

3.1.1 The metric and Kruskal coordinates

The Reissner-Nordström (RN) metric describes a static, spherically symmetric black hole of mass M possessing an electric charge Q and is a solution of Einstein-Maxwell equation. The line element is given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2. \tag{3.2}$$

$$f(r) = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right). \tag{3.3}$$

Here, M is the total mass of the spacetime and Q is the electric charge of the black hole. The horizons are situated at, $f(r) = 0$, i.e,

$$r_\pm = M \pm \sqrt{M^2 - Q^2}. \tag{3.4}$$

The outer horizon r_+ is the event horizon and the inner horizon r_- is the apparent horizon for the Reissner-Nordström spacetime. The extremal limit is obtained from $Q \to M$, or

$r_+ \rightarrow r_-$.

The coordinates t, r are singular at the outer horizon $(r = r_+)$ and one can introduce Kruskal-like coordinates to extend the metric across this surface. However, these coordinates fail to be regular at the inner horizon and so another coordinate system is needed to extend the metric beyond the inner horizon. So the coordinates are specific to a given horizon and even two coordinate patches fail to cover the entire Reissner-Nordström manifold. Let us consider the two horizons separately.

The outer horizon

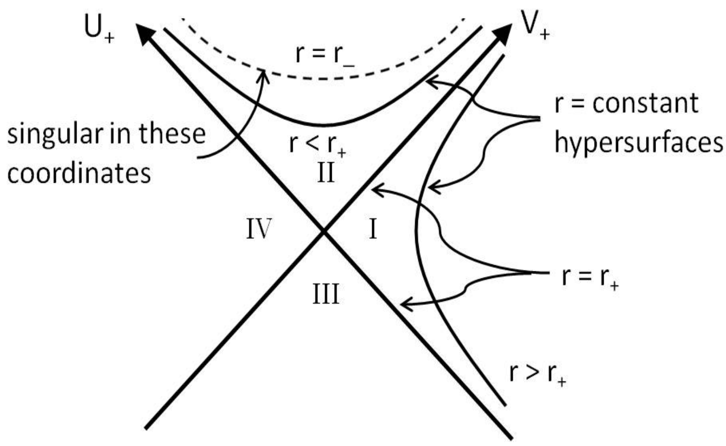

Figure 3.1: *The Kruskal coordinates for the outer horizon of Reissner-Nordström black hole*

As, $r \rightarrow r_+$,

$$f(r) \approx 2\kappa_+ (r - r_+) \,. \tag{3.5}$$

where,

$$\kappa_+ \equiv \frac{1}{2} f'(r_+) = \frac{r_+ - r_-}{2r_+^2}. \tag{3.6}$$

is the surface gravity at the outer horizon. It follows that near $r = r_+$,

$$r_* \equiv \int \frac{dr}{f(r)} \cong \frac{1}{2\kappa_+} \ln |\kappa_+ (r - r_+)|. \tag{3.7}$$

Introducing the null coordinates $u = t - r_*$ and $v = t + r_*$, the surface $r = r_+$ appears at $v - u = -\infty$ and we define the Kruskal-like coordinates U_+ and V_+ by,

$$U_+ = \mp e^{-\kappa_+ u}, \quad V_+ = e^{\kappa_+ v}. \tag{3.8}$$

Here the upper sign refers to $r > r_+$ and the lower sign refers to $r < r_+$. The future outer horizon is defined as $U_+ = 0, V_+ > 0$. The metric is regular at the outer horizon as seen from the near horizon form,

$$ds^2 \simeq -\frac{2}{\kappa_+^2} dU_+ dV_+ + r_+^2 d\Omega^2. \tag{3.9}$$

But $r_* \to \infty$ at the inner horizon which is located at $v - u = \infty$ or $U_+ V_+ = \infty$ and the Kruskal coordinates are singular there. This coordinate patch can be used for $r_1 < r < \infty$, where, $r_1 > r_-$. Thus, we need another set of Kruskal coordinates to extend the spacetime beyond $r = r_-$.

The inner horizon

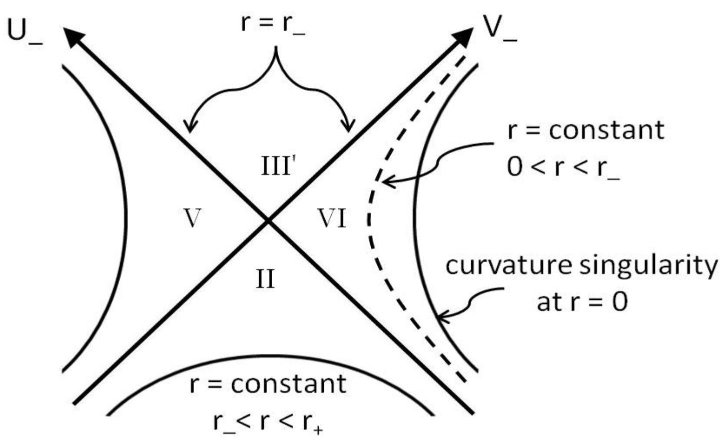

Figure 3.2: *The Kruskal coordinates for the inner horizon of Reissner-Nordström black hole*

The new set of Kruskal coordinates for the inner horizon can be constructed in a similar manner. As $r \to r_-$, the function $f(r)$ becomes,

$$f(r) \approx -2\kappa_- (r - r_-) . \tag{3.10}$$

where,

$$\kappa_- \equiv \frac{1}{2}|f'(r_-)| = \frac{r_+ - r_-}{2r_-^2}. \tag{3.11}$$

Near $r = r_-$,

$$r_* \equiv \int \frac{dr}{f(r)} \cong -\frac{1}{2\kappa_-} \ln |\kappa_- (r - r_-) |. \tag{3.12}$$

With $u = t - r_*$ and $v = t + r_*$, the surface $r = r_-$ appears at $v - u = +\infty$ and we define the new Kruskal coordinates by,

$$U_- = \mp e^{\kappa_- u}, \quad V_- = -e^{-\kappa_- v}. \tag{3.13}$$

Here, the upper sign refers to $r > r_-$ and the lower sign refers to $r < r_-$. The future inner horizon is defined as $U_- = 0, V_- < 0$. Then $f \simeq -2U_-V_-$ and the metric becomes

$$ds^2 \simeq -\frac{2}{\kappa_-^2} dU_- dV_- + r_-^2 d\Omega^2. \tag{3.14}$$

which is regular at $r = r_-$.

3.1.2 Scalar modes and the probability current

A scalar field Φ satisfies the covariant Klein-Gordon equation and in this background the modes can be separated as,

$$\Phi_{\omega lm} = \frac{1}{\sqrt{4\pi\omega}} \frac{\Phi_\omega(r_*, t)}{r} Y_{lm}(\theta, \phi) . \tag{3.15}$$

We shall consider only the positive frequency $(\omega > 0)$ modes which satisfies

$$i\partial_t \Phi_\omega = \omega \Phi_\omega. \tag{3.16}$$

In Kruskal coordinates (for the outer horizon), the Killing vector ∂_t becomes,

$$\partial_t = -\kappa_+ U_+ \partial_{U_+} + \kappa_+ V_+ \partial_{V_+}. \tag{3.17}$$

So, the U_+ and V_+ modes are decoupled,

$$\Phi_\omega = \left[f_\omega \left(U_+ \right) + g_\omega \left(V_+ \right) \right]. \tag{3.18}$$

And the solutions are,

$$f_\omega \left(U_+ \right) = N_\omega |U_+|^{\frac{i\omega}{\kappa_+}}. \tag{3.19}$$

$$g_\omega \left(V_+ \right) = N_\omega (V_+)^{-\frac{i\omega}{\kappa_+}}. \tag{3.20}$$

where N_ω is the normalization constant. The V_+ modes are ingoing in the outer horizon and is well behaved across the horizon.

The U_+ modes are outgoing and are not well behaved close to the horizon because they oscillate infinitely rapidly. However, to calculate the emission probability we shall need these modes only.

The probability current is positive definite for positive frequency modes and associated with the U_+ modes it is

$$j^{out} = -i \left[-\kappa_+ U_+ \left(\partial_{U_+} \overline{\Phi_\omega} \right) \Phi_\omega + \kappa_+ U_+ \overline{\Phi_\omega} \left(\partial_{U_+} \Phi_\omega \right) \right]. \tag{3.21}$$

The U_+ modes are defined inside and outside the horizon, but as it approaches the horizon at $U_+ = 0$ the modes pick up a logarithmic singularity and is not differentiable. So

j^{out} cannot be calculated naively. But actually, these modes are distribution valued as mentioned earlier and not to be interpreted as ordinary functions. As distributions, they are well defined and infinitely differentiable at the horizon. The distributions are of the form [57],

$$f_\omega = \lim_{\epsilon\to 0} N_\omega |U_+ + i\epsilon|^{\frac{i\omega}{\kappa_+}} = \begin{cases} N_\omega \left(U_+\right)^{\frac{i\omega}{\kappa_+}} & \text{for } U_+ > 0, \\ N_\omega |U_+|^{\frac{i\omega}{\kappa_+}} e^{-\frac{\pi\omega}{\kappa_+}} & \text{for } U_+ < 0. \end{cases} \tag{3.22}$$

and the complex conjugate distribution is,

$$\overline{f_\omega} = \lim_{\epsilon\to 0} N^*_\omega \left|U_+ - i\epsilon\right|^{-\frac{i\omega}{\kappa_+}} = \begin{cases} N^*_\omega \left(U_+\right)^{-\frac{i\omega}{\kappa_+}} & \text{for } U_+ > 0, \\ N^*_\omega \left|U_+\right|^{-\frac{i\omega}{\kappa_+}} e^{-\frac{\pi\omega}{\kappa_+}} & \text{for } U_+ < 0. \end{cases} \tag{3.23}$$

These distributions are uniquely associated with the U_+-modes if we impose the additional condition that these are well behaved for large frequencies, $\omega \to \infty$. The derivatives of the distributions are also uniquely determined,

$$\partial_{U_+} f_\omega = N_\omega \left(\frac{i\omega}{\kappa_+}\right) \lim_{\epsilon\to 0} \left|U_+ + i\epsilon\right|^{\frac{i\omega}{\kappa_+}-1}. \tag{3.24}$$

$$\partial_{U_+} \overline{f_\omega} = N^*_\omega \left(-\frac{i\omega}{\kappa_+}\right) \lim_{\epsilon\to 0} \left|U_+ - i\epsilon\right|^{-\frac{i\omega}{\kappa_+}-1}. \tag{3.25}$$

So, the probability current associated with the outgoing modes is,

$$j^{out} = \omega U_+ |N_\omega|^2 \lim_{\epsilon\to 0} \left[\frac{1}{U_+ - i\epsilon} + \frac{1}{U_+ + i\epsilon}\right] \left(U_+ - i\epsilon\right)^{-\frac{i\omega}{\kappa_+}} \left(U_+ + i\epsilon\right)^{\frac{i\omega}{\kappa_+}}. \tag{3.26}$$

Now, $U_+ \left(U_+ \pm i\epsilon\right)^{-1}$ gives the identity distribution, because $\left(U_+ \pm i\epsilon\right)^{-1} = PV(1/U_+) \mp i\pi\delta(U_+)$ and $U_+\delta(U_+) = 0$. Finally,

$$\lim_{\epsilon\to 0} \left(U_+ \mp i\epsilon\right)^{\mp\frac{i\omega}{\kappa_+}} = \lim_{\epsilon\to 0} e^{\mp\frac{i\omega}{\kappa_+}\ln(U_+ \mp i\epsilon)} = e^{\mp\frac{i\omega}{\kappa_+}\left(\ln|U_+| \mp i\pi\theta(-U_+)\right)}. \tag{3.27}$$

As a result, we get the outgoing probability current,

$$j^{out} = \begin{cases} |N_\omega|^2 & \text{for } U_+ > 0, \\ |N_\omega|^2 e^{-\frac{2\pi\omega}{\kappa_+}} & \text{for } U_+ < 0. \end{cases} \tag{3.28}$$

In a similar manner, using the Kruskal coordinates on the inner horizon one can calculate the outgoing probability current inside and outside of the inner horizon. In this case,

$$j^{out} = \begin{cases} |\tilde{N}_\omega|^2 & \text{for } U_- > 0, \\ |\tilde{N}_\omega|^2 e^{-\frac{2\pi\omega}{\kappa_-}} & \text{for } U_- < 0. \end{cases} \tag{3.29}$$

3.2 Kerr black hole

3.2.1 The metric and Kruskal coordinates

The Kerr metric gives a rotating black hole solution to the Einstein field equations. There are a number of different ways of writing this metric. In the standard Boyer-Lindquist coordinates the line element is given by,

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right) dt^2 - \frac{4Mar\sin^2\theta}{\rho^2} dt d\phi + \frac{\Sigma}{\rho^2}\sin^2\theta d\phi^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2. \tag{3.30}$$

where,

$$\rho^2 = r^2 + a^2\cos^2\theta. \tag{3.31}$$

$$\Delta = r^2 + a^2 - 2Mr. \tag{3.32}$$

$$\Sigma = \left(r^2 + a^2\right)^2 - a^2\Delta\sin^2\theta. \tag{3.33}$$

The horizons are situated at $\Delta = 0$, i.e at,

$$r_\pm = M \pm \sqrt{M^2 - a^2}. \tag{3.34}$$

Construction of Kruskal coordinates for the Kerr spacetime is very similar to the Reissner-Nordström spacetime though a bit more complicated. Let us define the u, v coordinates for the Kerr spacetime by using the following transformations,

$$r_* = \int \frac{r^2 + a^2}{\Delta} dr. \tag{3.35}$$

$$\tilde{\phi} = \phi - \frac{at}{2Mr_+}. \tag{3.36}$$

And as usual,

$$u = t - r_*. \tag{3.37}$$

$$v = t + r_*. \tag{3.38}$$

Let us now define the Kruskal like coordinates for the Kerr spacetime,

$$U_+ = \mp e^{-\kappa_+ u}, \quad V_+ = e^{\kappa_+ v}. \tag{3.39}$$

where

$$U_+ V_+ = \mp e^{2\kappa_+ r} (r - r_+) (r - r_-)^{-\frac{\kappa_+}{\kappa_-}}. \tag{3.40}$$

$$\kappa_+ = \frac{1}{2} \left(\frac{r_+ - r_-}{r_+^2 + a^2} \right). \tag{3.41}$$

$$\kappa_- = \frac{1}{2} \left(\frac{r_+ - r_-}{r_-^2 + a^2} \right). \tag{3.42}$$

κ_+ is the surface gravity at the outer horizon. Here again, the upper sign is for $r > r_+$, and the lower sign is for $r < r_+$. The future outer horizon is defined as $U_+ = 0, V_+ > 0$.

The metric near the horizon$(r \rightarrow r_+, U_+ \rightarrow 0)$ in the Kruskal coordinates takes the form

$$\begin{aligned} ds^2 = & \mp \; 4\rho_+^2 \left(r_+ - r_-\right)^{\frac{\kappa_+}{\kappa_-}-1} e^{-2\kappa_+ r_+} dU_+ dV_+ + \rho_+^2 d\theta^2 \\ & + \; 4a^2 \sin^2\theta \left(r_+ - r_-\right)^{2\frac{\kappa_+}{\kappa_-}-2} \left(\frac{r_+^2}{\rho_+^2} + \frac{r_+^2 - a^2}{r_+^2 + a^2}\right) e^{-4\kappa_+ r_+} V_+^2 dU_+^2 \\ & \pm \; 2a \sin^2\theta \left(r_+ - r_-\right)^{\frac{\kappa_+}{\kappa_-}} \left(1 + \frac{r_+}{\kappa_+ \rho_+^2}\right) e^{-2\kappa_+ r_+} V_+ dU_+ d\tilde{\phi} \\ & + \; \frac{\left(r_+^2 + a^2\right)^2}{\rho_+^2} \sin^2\theta d\tilde{\phi}^2. \end{aligned} \tag{3.43}$$

So, the coordinate singularity is removed and the metric is regular at the horizon in the U_+, V_+ coordinates.

Just as for the RN spacetime, the coordinates (U_+, V_+) are singular at the inner horizon $(r = r_-)$, and another coordinate patch is required to extend the Kerr metric beyond this horizon.

For the inner horizon we define

$$r_* = \int \frac{r^2 + a^2}{\Delta} dr. \tag{3.44}$$

$$\tilde{\psi} = \phi - \frac{at}{2Mr_-}. \tag{3.45}$$

With $u = t - r_*$ and $v = t + r_*$, the appropriate choice for Kruskal coordinates are,

$$U_- = \mp e^{\kappa_- u}, V_- = -e^{-\kappa_- v}. \tag{3.46}$$

The upper sign refers to $r > r_-$ and the lower sign refers to $r < r_-$. Near the horizon

$(r \to r_-, U_- \to 0)$ the metric becomes

$$\begin{aligned} ds^2 = & \mp 4\rho_-^2 \left(r_+ - r_-\right)^{\frac{\kappa_-}{\kappa_+}-1} e^{2\kappa_- r_-} dU_- dV_- + \rho_-^2 d\theta^2 \\ & + 4a^2 \sin^2\theta \left(r_+ - r_-\right)^{2\frac{\kappa_-}{\kappa_+}-2} \left(\frac{r_-^2}{\rho_-^2} + \frac{r_-^2 - a^2}{r_-^2 + a^2}\right) e^{4\kappa_- r_-} V_-^2 dU_-^2 \\ & \pm 2a \sin^2\theta \left(r_+ - r_-\right)^{\frac{\kappa_-}{\kappa_+}} \left(\frac{r_+}{\kappa_+ \rho_+^2} - 1\right) e^{2\kappa_- r_-} V_- dU_- d\tilde{\psi} \\ & + \frac{\left(r_-^2 + a^2\right)^2}{\rho_-^2} \sin^2\theta d\tilde{\psi}^2. \end{aligned} \tag{3.47}$$

The metric is regular at $r = r_-$, and the spacetime can be extended beyond this horizon.

3.2.2 Scalar modes and the probability current

For the Kerr metric the scalar modes can be separated as [58]

$$\Psi_{\omega m} = \frac{1}{\sqrt{4\pi\omega}} \frac{\Psi_\omega\left(r_*, t\right)}{r} e^{im\phi} \Theta\left(\theta\right). \tag{3.48}$$

The Killing vector in this case is $(\partial_t + \Omega_H \partial_\phi)$ and considering only the positive frequency solutions as before we get

$$i\left(\partial_t + \Omega_H \partial_\phi\right) \Psi_{\omega m} = \omega \Psi_{\omega m}. \tag{3.49}$$

Here, $\omega = E - m\Omega_H$ where $E = \omega_\infty$ is the frequency at infinity.

In Kruskal coordinates the Killing vector becomes

$$\begin{aligned} \partial_t + \Omega_H \partial_\phi &= \partial_u + \partial_v + \frac{\partial\tilde{\phi}}{\partial t}\partial_{\tilde{\phi}} + \Omega_H \frac{\partial\tilde{\phi}}{\partial\phi}\partial_{\tilde{\phi}} \\ &= -\kappa_+ U_+ \partial_{U_+} + \kappa_+ V_+ \partial_{V_+}. \end{aligned} \tag{3.50}$$

just the same as the RN spacetime, and hence the U_+ and the V_+ modes are decoupled,

$$\Psi_\omega = \left[f_\omega(U_+) + g_\omega(V_+)\right]. \tag{3.51}$$

Concerning ourselves with only the outgoing U_+ modes, we get the solutions

$$f_\omega\left(U_+\right) = N_\omega |U_+|^{\frac{i\omega}{\kappa_+}}. \tag{3.52}$$

Again as before, these modes are well behaved both inside and outside the horizon and is not differentiable at the horizon because of the logarithmic singularity. And we have to resort back to the distributions to calculate the probability current for emission. The distributions have the same form as in the case of RN spacetime, and the probability current through the outer horizon is

$$j^{out} = \begin{cases} |N_\omega|^2 & \text{for } U_+ > 0, \\ |N_\omega|^2 e^{-\frac{2\pi\omega}{\kappa_+}} & \text{for } U_+ < 0. \end{cases} \tag{3.53}$$

For the inner horizon, using the specific Kruskal coordinates, the probability current is the same as the RN inner horizon

$$j^{out} = \begin{cases} |\tilde{N}_\omega|^2 & \text{for } U_- > 0, \\ |\tilde{N}_\omega|^2 e^{-\frac{2\pi\omega}{\kappa_-}} & \text{for } U_- < 0. \end{cases} \tag{3.54}$$

3.2.3 Effective temperature and extreme limit of non extremal solutions

A non-extremal spacetime with an outer (r_+) and an inner (r_-) horizon becomes extremal as $r_+ \to r_-$. The two horizons are in equilibrium at two different temperatures and as a result, the outgoing fluxes are also different. As was shown earlier, both for the Reissner-

Nordström and Kerr metrics, the temperature of the outer horizon is, $T_{out} = \frac{\hbar\kappa_+}{2\pi}$ and that of the inner horizon is $T_{in} = \frac{\hbar\kappa_-}{2\pi}$. Since $k_- > k_+$ the inner horizon is in equilibrium at a higher temperature than the outer one. So the outgoing flux from the outer horizon, given by (3.28) and (3.53) for the R-N and Kerr spacetimes respectively, are less than the incoming flux through the inner horizon given by (3.29) and (3.54) respectively. The results are consistent with expectations. In the extremal limit as $r_+ \to r_-$, $T_{out} \to 0$ as $\kappa_+ = 0$. So no thermality is observed at the outer horizon as expected. This can be shown more clearly by considering an effective temperature for the outer horizon.

For both RN and Kerr black holes, since the spacetime between the two horizons for $(r_- < r < r_+)$ is vacuum, the fluxes have to match. This implies,

$$|\tilde{N_\omega}|^2 e^{-\frac{2\pi\omega}{\kappa_-}} = |N_\omega|^2. \tag{3.55}$$

So, the effective flux coming out of the outer horizon is given by,

$$j^{out} = |N_\omega|^2 e^{-\frac{2\pi\omega}{\kappa_+}} = |\tilde{N_\omega}|^2 e^{-2\pi\omega\left(\frac{1}{\kappa_+}+\frac{1}{\kappa_-}\right)}. \tag{3.56}$$

This gives an effective temperature of the outer horizon as the harmonic mean of κ_+ and κ_-,

$$\beta_{eff} = \frac{2\pi}{\hbar}\left(\frac{1}{\kappa_+} + \frac{1}{\kappa_-}\right). \tag{3.57}$$

Far from extremality, $M \gg Q$, as a result $r_+ \gg r_-$ and $\kappa_- \gg \kappa_+$. Thus, $T_{eff} \approx T_{out}$.

But in the extremal limit, as $r_- \to r_+$, $T_{eff} = 0$. This shows that the outgoing flux approaches zero not only on the outer horizon but on the inner horizon as well. This limit is consistent with what we expect from an extremal solution.

3.3 Extremality

The nature of the extremal metric is quite different from other stationary solutions. Still we can calculate the scalar modes and the outgoing probability current through the horizon following the same procedure as that of a stationary metric. Naturally, the mode solutions near the horizon is different and this leads to a different type of distributions for the extremal case. We found that though the flux vanishes precisely at the horizon, leading to a zero temperature for the horizon, it is non-zero both outside and inside of the horizon, which is not physically acceptable. So to retrieve the standard result we have to look more carefully into the spacetime structure of the extremal black hole. However, no such problems arise if we consider the extremal spacetime as a limiting case of a non-extremal one.

3.3.1 Extremal solution

The extreme Reissner-Nordström metric is

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \frac{dr^2}{\left(1 - \frac{M}{r}\right)^2} + r^2 d\Omega^2. \tag{3.58}$$

where M is the mass of the black hole. The horizon is at $r = M$. Introducing the null coordinates $u = t - r_*$, $v = t + r_*$,with

$$r_* \equiv \int \frac{dr}{\left(1 - \frac{M}{r}\right)^2} = r + 2M \ln|r - M| - \frac{M^2}{r - M}. \tag{3.59}$$

the surface $r = M$ appears at $v - u = -\infty$. The Kruskal coordinates U, V are given by the implicit relations,

$$u = -M \cot U. \tag{3.60}$$

$$v = -M \tan V. \tag{3.61}$$

The future horizon is located at $U = 0, V < \frac{\pi}{2}$.

The Killing vector ∂_t is

$$\partial_t = \frac{1}{M}\left[\sin^2 U \partial_U - \cos^2 V \partial_V\right]. \quad (3.62)$$

Again the U and the V modes are decoupled and the positive frequency solutions for a scalar field are found to be

$$\Psi_\omega(U) = N_\omega e^{i\omega M \cot U}. \quad (3.63)$$

$$\Psi_\omega(V) = N_\omega e^{i\omega M \tan V}. \quad (3.64)$$

The ingoing V-modes are regular across the horizon. Near the horizon the outgoing U-modes takes the form

$$\Psi_\omega(U) = N_\omega e^{i\omega M \cot U} \simeq N_\omega e^{\frac{i\omega M}{U}}. \quad (3.65)$$

The singularity at the horizon can be removed by using distributions. In this case the appropriate distributions are found by taking logarithm of the modes,

$$\ln \Psi_\omega(U) = \ln N_\omega + \lim_{\epsilon\to 0} \frac{i\omega M}{U - i\epsilon}. \quad (3.66)$$

$$\ln \overline{\Psi}_\omega(U) = \ln N^*_\omega - \lim_{\epsilon\to 0} \frac{i\omega M}{U + i\epsilon}. \quad (3.67)$$

The extremal solution for the Kerr spacetime is obtained by setting $a = M$ in the Kerr metric. The line element becomes,

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right) dt^2 - \frac{4M^2 r \sin^2\theta}{\rho^2} dt d\phi + \frac{\Sigma}{\rho^2} \sin^2\theta d\phi^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2. \quad (3.68)$$

with,

$$\rho^2 = r^2 + M^2 \cos^2\theta. \tag{3.69}$$

$$\Delta = (r - M)^2 . \tag{3.70}$$

$$\Sigma = \left(r^2 + M^2\right)^2 - M^2 \Delta \sin^2\theta. \tag{3.71}$$

The horizon is situated at $r = M$. Introducing the null coordinates $u = t - r_*, v = t + r_*$ as before, with

$$r_* \equiv \int \frac{r^2 + M^2}{(r - M)^2} dr = r + 2M \ln |r - M| - \frac{2M^2}{r - M}. \tag{3.72}$$

$$\tilde{\phi} = \phi - \frac{1}{2M} t = \phi - \Omega t \tag{3.73}$$

the surface $r = M$ appears at $v - u = -\infty$. The Kruskal coordinates U, V are the same as in the case of extremal Reissner-Nordström metric,

$$u = -M \cot U. \tag{3.74}$$

$$v = -M \tan V. \tag{3.75}$$

The future horizon is located at $U = 0, V < \frac{\pi}{2}$.

The Killing vector $\partial_t + \Omega \partial_\phi$ is

$$\begin{aligned} \partial_t + \Omega \partial_\phi &= \partial_u + \partial_v + \frac{\partial \tilde{\phi}}{\partial t} \partial_{\tilde{\phi}} + \Omega \partial_{\tilde{\phi}} \\ &= \frac{1}{M} \left[\sin^2 U \partial_U - \cos^2 V \partial_V \right] . \end{aligned} \tag{3.76}$$

So, the mode solutions for the extremal Kerr metric are similar to the Reissner-Nordström case.

To find the outgoing probability flux across the horizon, we have calculated

$$\begin{aligned} \frac{j^{out}}{\overline{\Psi}_\omega(U)\Psi_\omega(U)} &= -\frac{iU^2}{M}\partial_U\left[\ln\overline{\Psi}_\omega(U) - \ln\Psi_\omega(U)\right] \\ &= \omega U^2\left[\frac{1}{(U+i\epsilon)^2} + \frac{1}{(U-i\epsilon)^2}\right]. \end{aligned} \tag{3.77}$$

Now, $(U \pm i\epsilon)^{-2} = PV(1/U^2) \pm i\pi\delta'(U)$ and $U^2\delta'(U) = -2U\delta(U) = 0$, so the above quantity is equal to 2ω. Finally taking logarithm of both sides again we get the flux

$$\ln j^{out} = \ln|N_\omega|^2 - 2\pi\omega M\delta(U). \tag{3.78}$$

Clearly, for all $N \neq 0$, $j^{out} = |N_\omega|^2$, namely no thermality is observed. It is as if the horizon is a transparent membrane. To calculate its 'value' at $U = 0$, we regularize the delta-function; in the limit $\epsilon \to 0$

$$j^{out} = |N_\omega|^2 \exp\left(-2M\omega\frac{\epsilon}{U^2+\epsilon^2}\right). \tag{3.79}$$

Then $j^{out} = |N_\omega|^2 \exp(-2M\omega/\epsilon)$ which approaches zero in the limit $\epsilon \to 0$. So there is a finite discontinuity of flux at the horizon and no thermal effect is noted. However, one can argue that this discontinuity is a coordinate artefact. Recall that for an extremal black hole, be it an extremal Reissner-Nordström or extremal Kerr, the proper radial distance from the horizon to any point however close to the horizon outside or inside, is infinite. Thus, it is impossible for any particle state to cross the horizon when incident from inside or outside. This is the basic reason why an extremal black hole cannot absorb or emit particle states. The coordinates U_+, V_+ remove the coordinate singularity at the horizon. But unlike the non-extremal cases, U_+ fails to be a proper distance. So a discontinuity in U_+ is not a physical discontinuity. The flux gets infinite proper time to become zero at the horizon when it is incident either from inside or outside. When equated to the Boltzmann factor, it implies an infinite β. This is equivalent to setting a zero temperature for the black hole.

3.4 Discussions

We have investigated Hawking radiation for near extremal Reissner-Nordström and Kerr black holes from a different viewpoint. First, we have developed a method of calculating temperature associated with a horizon based on quantum field theory in a curved background. We have constructed the field modes (in this case a scalar field) near the horizon and computed the probability current coming out of the horizon using the modes. The problem of horizon crossing is taken care of by the distributional nature of these modes. This can be regarded as an improvement over the conventional tunnelling calculations since we do not need to calculate any semiclassical action using WKB approximation to find the emission probability. These approximations are never precisely established in the conventional analysis apart from the fact that they give the 'correct' Hawking temperatures. Our analysis shows that the results are more robust and independent of such approximations. The only approximation we have used is the flatness of the metric near the horizon which is justifiably so in well-defined coordinates.

Second, we have considered radiation from both the outer and the inner horizon for these two spacetimes. It was found that the two horizons are in thermal equilibrium at different temperatures, the temperature of the inner horizon is higher than that of the outer horizon, as a result the incoming flux through the inner horizon is less than the outgoing flux through the outer horizon. Matching the current in between the two horizons we have found an effective temperature for the outer horizon which is such that when the spacetime is far from extremality $(r_+ \gg r_-)$ the effective temperature is the temperature of the outer horizon for massive black holes. On the other hand, in the extremal limit, when $r_+ \to r_-$, the effective temperature is zero (actually, both temperatures vanish in this limit) and the emission probability is also zero. This is consistent with the expectations.

In the case of an extremal metric it was found that the flux is the same both outside and inside the horizon and zero at the horizon. This indicates a zero temperature for the pure extremal black hole, though there is a finite discontinuity of flux which is not observed in the extremal limit of the non-extremal spacetimes. In this sense the extremal spacetime as the limiting case of the non-extremal one is more physically acceptable.

In the past, extremal black holes have been considered by several authors in the context of tunnelling and contradictory results were obtained. Both in the "Hamilton-Jacobi" variant of the tunnelling approach and the "null geodesic" approach the real part of the action S is divergent. In the Hamilton-Jacobi approach one gets a vanishing imaginary part [17] (which naively implies a zero β, hence a divergent Hawking temperature) and in the "null geodesic" approach the imaginary part of the action is non-vanishing and the temperature of an extremal black hole is found to be "quantized in units of the temperature of a Schwarzschild black hole" [21]. These lead to the conclusions that as if the tunnelling formulation breaks down for extremal cases because it fails to reproduce the expected results. However, in our formulation such complications do not arise because our method does not involve any semiclassical approximation or action. In the distributional sense our result is consistent with the standard viewpoint.

Chapter 4

Hawking radiation from dynamical horizons

Dynamical horizons

In this chapter we shall discuss Hawking radiation from a dynamical horizon using the formalism that we have developed in the previous chapter. Till now we have considered radiation from static spacetimes only. But even for a static spacetime the idea of Hawking radiation is that the black hole loses mass and its horizon shrinks during the process of radiation eventually completely evaporating to a fate till debated. It is then expected that any realistic model of Hawking effect should be able to incorporate these dynamical features of the process. To do this we need to consider a completely dynamical spacetime instead of a static one.

But in case of a dynamical spacetime the issue of defining the surface of a black hole is a tricky one. We can not naively take the event horizon as the radiating surface because the concept of event horizon is absolutely global and teleological. An event horizon is defined as the future boundary of the causal past of future null infinity and is inherently related to asymptotically flat spacetime. Also the entire future evolution of the spacetime must be known to determine the position of the event horizon. This is not desirable and we wish to define the surface of a black hole from only local geometrical properties of the spacetime whose location at any given time depends only on the properties of the spacetime at that time and which is not a null hypersurface at all like the event horizon but evolves accordingly during any dynamical process.

Fortunately several definitions of dynamical horizons are available in literature. For example dynamical horizons (Ashtekar and Krishnan), slowly evolving horizons (Booth and Fairhurst), trapping horizons (Hayward) etc. All these horizons have some nice properties : they are defined locally and are not endowed with teleological features. For our purpose we shall consider the concept of trapping horizons by Hayward. Trapping horizons are locally defined and have physical properties such as mass, angular momentum and surface gravity, satisfying conservation laws [61, 62, 63, 72]. They are a geometrically natural generalization of Killing horizons, which are stationary trapping horizons. A non-stationary trapping horizon is not null, but still has infinite red-shift.

In fact the tunnelling method in its Hamilton-Jacobi variant has been applied to dynamical black hole horizons and the temperature is found to be $\kappa/2\pi$ where κ is the dynamical surface gravity[20, 50]. Still there are some problems with the method itself apart from the use of WKB like approximation. While calculating the imaginary part of the semiclassical action S from the Hamilton-Jacobi equation, a singular integral appears with a pole at the horizon as we have seen earlier. In the case of a dynamical horizon it is not clear how the integration is to be performed since the position of the horizon is not fixed as in the case of a static horizon. Also the issue of the evolution of the horizon remained unanswered in this method.

In this work we have used trapping horizon which is of *future outer* type (FOTH) : the term "*future*" denotes the horizon associated with a black hole spacetime rather than a white hole one and "*outer*" distinguishes this black hole horizon from a cosmological one. Once this radiating surface is defined and its properties are discussed, we shall then proceed to find the positive frequency modes of the Kodama vector which plays the analogous role of Killing vector in a dynamical spacetime. It is found that for a radiating spacetime the FOTH is timelike in nature. Thus no problem arises in continuing the modes inside the horizon and in constructing single particle states. The appearance of a logarithmic singularity in the outgoing mode is dealt with by taking into account the distributional nature of the field modes. The Hawking temperature is determined if one equates the conditional probability, that modes incident on one side is emitted to the other side, to the Boltzmann factor as we have done before. Since this method does not depend on the entire evolution of the field modes in the spacetime, it is ideally suited for defining temperature locally. After finding the temperature associated with the dynamical FOTH we shall address the issue of evolution of the horizon during radiation. Various discussions and speculations are found in literature on how shall a dynamical black hole will evolve during Hawking radiation but no precise mathematical formulation exists. Here we shall calculate the outgoing flux of radiation through the timelike FOTH along the direction of the Kodama vector and show that the horizon shrinks in accordance with this outgoing flux.

4.1 Future outer trapping horizon

4.1.1 Definition of FOTH

We begin with definitions. We shall follow the conventions of [60]. Consider a four dimensional spacetime $\mathscr{M}$ with signature $(-,+,+,+)$. A three-dimensional submanifold Δ in $\mathscr{M}$ is said to be a *future outer trapping horizon* (FOTH) if

- It is foliated by a preferred family of topological two-spheres such that, on each leaf S, the expansion θ_+ of a null normal l_+^a vanishes and the expansion θ_- of the other null normal l_-^a is negative definite,

- The directional derivative of θ_+ along the null normal l_-^a (i.e., $\mathscr{L}_{l_-}\theta_+$) is negative definite.

Thus, Δ is foliated by marginally trapped two-spheres. According to a theorem due to Hawking, the topology of S is necessarily spherical in order that matter or gravitational flux across Δ is non-zero. If these fluxes are identically zero then Δ becomes a Killing or isolated horizon.

Even though our arguments will remain local, for definiteness, we choose a spherically symmetric background metric

$$ds^2 = -2e^{-f}dx^+dx^- + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{4.1}$$

where both f and r are smooth functions of $x^\pm$. The expansions $\theta_\pm$ of the two null normals are found to be

$$\theta_\pm = \frac{2}{r}\partial_\pm r \tag{4.2}$$

respectively where $\partial_\pm = \partial/\partial x^\pm$. In this coordinate system, the second requirement for FOTH translates to $\partial_-\theta_+ < 0$ on Δ.

4.1.2 Condition for a timelike FOTH

Let the vector field $t^a = l_+^a + h\, l_-^a$ be tangential to the FOTH for some smooth function h. Then the Raychaudhuri equation for l_+^a and the Einstein equation implies

$$\partial_+\theta_+ = -h\partial_-\theta_+ = -8\pi T_{++}. \tag{4.3}$$

where $T_{++} = T_{ab}\, l_+^a l_+^b$ and T_{ab} is the energy momentum tensor. Several consequences follow from this equation.

- First, the FOTH is degenerate (or null) if and only if $T_{++} = 0$ on Δ. In that case, the FOTH is generated by l_+^a.

Degenerate FOTH is not interesting for Hawking radiation because this implies $\partial_+ r = 0$. As a consequence, the area, $A = 4\pi r^2$ of S, and the Misner-Sharp energy for this spacetime, given by $E = \frac{1}{2}r$, also remains unchanged.

- Secondly, since $t^2 = -2h\, e^{-f}$, a FOTH becomes spacelike if and only if $T_{++} > 0$ and is timelike if and only if $T_{++} < 0$.

For a timelike FOTH, $\theta_+ = 0$ implies $\partial_+ r = 0$ and $\theta_- < 0$ implies $\partial_- r < 0$. As a result, $\mathscr{L}_t r < 0$, and hence, Δ is timelike if and only if the area A and the Misner-Sharp energy E decreases along the horizon. This is also expected on general grounds since the horizon receives an incoming flux of negative energy, $T_{++} < 0$. This means the dominant energy condition (DEC) is violated during Hawking radiation.

4.1.3 The positive frequency modes of the Kodama vector and probability current

In dynamical spherically symmetric spacetimes the Kodama vector field (K) plays the analog role of the Killing vector field in stationary spacetimes. For such spacetimes the metric can be written as,

$$ds^2 = g_{ij} dx^i dx^j + r^2 (d\theta^2 + \sin^2\theta d\phi^2). \tag{4.4}$$

The Kodama vector field (K) associated with this metric is defined as,

$$K^i = \epsilon^{ij} \partial_j r, \tag{4.5}$$

where ϵ_{ij} denotes the volume form associated with the two metric g_{ij}. For the spacetime (4.1), the Kodama vector is given by

$$K = e^f\,(\partial_- r)\,\partial_+ - e^f\,(\partial_+ r)\,\partial_-. \tag{4.6}$$

Using the Einstein's equations it can be shown that the vector field $J^a = G^{ab} K_b$ is divergence free where G^{ab} is the Einstein tensor. This implies that even though the spacetime is completely dynamical, J^a is the locally conserved energy flux. So the Kodama vector field generates a preferred flow of time in spherically symmetric dynamical spacetimes, suggesting the name dynamic time. Using the Kodama vector (K) the dynamical surface gravity (κ) is defined as

$$K^a \nabla_{[b} K_{a]} = \kappa \, K_b, \tag{4.7}$$

and in the metric (4.1) it is $\kappa = -e^f \, \partial_- \partial_+ r$. The FOTH condition $\partial_- \theta_+ < 0$ implies $\kappa > 0$.

Let us now determine the positive frequency modes of the Kodama vector. It is easy to see that any smooth function of r is a zero-mode of the Kodama vector. Once, a zero-mode is obtained, we can evaluate other positive frequency eigenmodes using

$$iK \, Z_\omega = \omega \, Z_\omega \tag{4.8}$$

Here, Z_ω are the eigenfunctions corresponding to the positive frequency ω. To simplify the calculations let us introduce new coordinates, $y = x^-$ and r. Then

$$\begin{aligned} \partial_+ &= (\partial_+ r) \, \partial_r \\ \partial_- &= \partial_y + (\partial_- r) \, \partial_r . \end{aligned} \tag{4.9}$$

With this coordinate transformation, we introduce two new functions, $\bar{Z}_\omega(y, r) = Z_\omega(x^+, x^-)$ and $e^f \, (\partial_+ r) = G(y, r)$. As a result, the eigenvalue equation (4.8) reduces to

$$G \, \partial_y \bar{Z}_\omega = i\omega \, \bar{Z}_\omega . \tag{4.10}$$

Integrating and transforming back to old coordinates, the above equation gives

$$Z_\omega = F(r)\exp\left(i\omega\int_r \frac{dx^-}{e^f\partial_+ r}\right), \tag{4.11}$$

where $F(r)$ is an arbitrary smooth function of r and the subscript r under the integral sign denotes that while doing the integration r is kept fixed. To evaluate the integral in (4.11), we multiply the numerator and the denominator by $(\partial_-\theta_+)$ and use the fact that for any fixed r surface, $e^f\,(\partial_-\theta_+) = -2\kappa/r$, (although the strict interpretation of κ as the surface gravity holds only for surfaces with $\theta_+ = 0$, it exists as a function in any neighbourhood of the horizon). Thus, in some neighbourhood of the horizon we get

$$\int_r \frac{dx^-\,\partial_-\theta_+}{e^f\,\partial_+ r\partial_-\theta_+} = -\int_r \frac{d\theta_+}{\kappa\theta_+}, \tag{4.12}$$

where the subscript r in the integral indicates that the integral is to be evaluated on a constant $r =$ surface. We now assume (this is the only assumption we make in this calculation) that during the dynamical evolution κ is a slowly varying function in some small neighbourhood of the horizon (the zeroth law takes care of it on the horizon, but we also assume it to hold in a small neighbourhood of the horizon). This gives

$$Z_\omega = F(r)\begin{cases}\theta_+^{-\frac{i\omega}{\kappa}} & \text{for } \theta_+ > 0\\ (|\theta_+|)^{-\frac{i\omega}{\kappa}} & \text{for } \theta_+ < 0.\end{cases} \tag{4.13}$$

where the spheres are not trapped 'outside the trapping horizon' ($\theta_+ > 0$) and fully trapped 'inside' ($\theta_+ < 0$). These are precisely the modes which are defined outside and inside the dynamical horizon respectively but not on the horizon. Now we have to keep in mind that the modes (4.13) are not ordinary functions, but are distribution-valued.

Using the standard results [57], we find for

$$(\theta_+ + i\epsilon)^\lambda = \begin{cases} \theta_+^\lambda & \text{for } \theta_+ > 0 \\ |\theta_+|^\lambda e^{i\lambda\pi} & \text{for } \theta_+ < 0 \end{cases} \tag{4.14}$$

for the choice $\lambda = -i\omega/\kappa$. We have discussed the same distributions for spherically symmetric static case in the previous chapter. The distribution (4.14) is well-defined for all values of θ_+ and λ, and it is differentiable to all orders. The modes Z^*_ω are given by the complex conjugate distribution

$$(\theta_+ - i\epsilon)^{\lambda^*} = \begin{cases} \theta_+^{\lambda^*} & \text{for } \theta_+ > 0 \\ |\theta_+|^{\lambda^*} e^{-i\lambda^*\pi} & \text{for } \theta_+ < 0 \end{cases} \tag{4.15}$$

Let us now calculate the probability density in a single particle Hilbert space for positive frequency solutions across the dynamical horizon

$$\varrho(\omega) = -\frac{i}{2}\Big[Z^*_\omega K Z_\omega - K Z^*_\omega Z_\omega\Big] = \omega Z^*_\omega Z_\omega. \tag{4.16}$$

A straightforward calculation gives, apart from a positive function of r,

$$\begin{aligned} \varrho(\omega) &= \omega(\theta_+ + i\epsilon)^{-\frac{i\omega}{\kappa}}(\theta_+ - i\epsilon)^{\frac{i\omega}{\kappa}}. \\ &= \begin{cases} \omega & \text{for } \theta_+ > 0 \\ \omega e^{\frac{2\pi\omega}{\kappa}} & \text{for } \theta_+ < 0. \end{cases} \end{aligned} \tag{4.17}$$

The conditional probability that a particle emits when it is incident on the horizon from inside is,

$$P_{(emission|incident)} = e^{-\frac{2\pi\omega}{\kappa}} \tag{4.18}$$

This gives the correct Boltzmann weight with the temperature $\kappa/2\pi$, which is the desired value for this dynamical horizon.

4.1.4 Evolution of the horizon

We shall now show that as the horizon evolves, the radius of the 2-sphere foliating the horizon shrinks in precise accordance with the amount of flux radiated by the horizon. To study the flux equation, let us consider new coordinates, $(x^+, x^-) \mapsto (\theta_+, \tilde{x}^-)$ where $\tilde{x}^- = x^-$. So we get,

$$dx^- = d\tilde{x}^- \tag{4.19}$$

$$dx^+ = \left(\frac{1}{\partial_+\theta_+}\right) d\theta_+ - \left(\frac{\partial_-\theta_+}{\partial_+\theta_+}\right) d\tilde{x}^-. \tag{4.20}$$

On FOTH, $(\partial_-\theta_+)/(\partial_+\theta_+) = -(\partial_-\partial_+ r)/(4\pi r\, T_{++})$ and is negative definite. As a result, the derivatives are related to each other by

$$\tilde{\partial}_- = \partial_- + \left(\frac{\partial_-\partial_+ r}{4\pi r\, T_{++}}\right) \partial_+. \tag{4.21}$$

It is not difficult to show that $\tilde{\partial}_-$ is proportional to the tangent vector t^a to the FOTH. To do this we first observe that the normal one-form to Δ must be proportional to $(dr - 2\, dE)$. When evaluated on the horizon this normal one-form becomes

$$dr - 2dE = 8\pi e^f r^2\, T_{++}\partial_- r dx^+ - 2re^f \partial_-\partial_+ r \partial_- r dx^-. \tag{4.22}$$

In arriving at the above identity we have made use of two Einstein's equations [60]

$$r\,\partial_-\partial_+ r + \partial_+ r\,\partial_- r + \frac{1}{2}e^{-f} = 4\pi r^2\, T_{-+}, \tag{4.23}$$

$$\partial_+^2 r + \partial_+ f\,\partial_+ r = -4\pi r\, T_{++},$$

and energy equations

$$\partial_\pm E = 2\pi e^f r^3 (T_{-+}\,\theta_\pm - T_{\pm\pm}\,\theta_\mp). \tag{4.24}$$

As a result, the normal vector n^a is proportional to

$$\partial_+ - \left(\frac{4\pi r\, T_{++}}{\partial_-\partial_+ r}\right)\partial_- = \partial_+ - h\partial_-, \tag{4.25}$$

so that the tangent vector $t^a = \partial_+^a + h\partial_-^a$, which is clearly proportional to (4.21).

So $\tilde{x}^-, \theta, \phi$ are natural coordinates on FOTH. The line-element (4.1) induces a line-element on Δ

$$ds^2 = -2e^{-f}h^{-1}(d\tilde{x}^-)^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{4.26}$$

Consequently, the volume element on the FOTH is given by

$$d\mu = \sqrt{2e^{-f}h^{-1}}r^2 \sin\theta\, d\tilde{x}^- d\theta d\phi. \tag{4.27}$$

We can now calculate the flux of matter energy that crosses the dynamical horizon along the direction of the Kodama vector—it is an integral on a slice of horizon bounded by two spherical sections S_1 and S_2

$$\mathscr{F} = \int d\mu\; T_{ab}\hat{n}^a K^b \tag{4.28}$$

where $\hat{n}^a$ is the unit normal vector

$$\hat{n}^a = \frac{1}{\sqrt{2he^{-f}}}(\partial_+^a - h\partial_-^a) \tag{4.29}$$

and K^a is the Kodama vector. Using spherical symmetry, eqn. (4.21) and eqn. (4.23), we get

$$\begin{aligned}\mathscr{F} &= \int d\tilde{x}^-\ 4\pi r^2(\frac{1}{h}T_{++} - T_{+-})e^f\partial_- r \\ &= \int d\tilde{x}^-\ 4\pi r^2(\frac{1}{4\pi r}\partial_+\partial_- r - T_{+-})e^f\partial_- r.\end{aligned} \tag{4.30}$$

Making use of the Einstein equation (4.23) on the horizon and (4.21), we get

$$\begin{aligned}\mathscr{F} &= -\int d\tilde{x}^-\ \frac{1}{2}\partial_- r = -\int d\tilde{x}^-\ \frac{1}{2}\tilde{\partial}_- r \\ &= -\frac{1}{2}(r_2 - r_1)\end{aligned} \tag{4.31}$$

where r_1, r_2 are respectively the two radii of S_1, S_2. Since the area is decreasing along the horizon, $r_2 < r_1$ where S_2 lies in the future of S_1. As a result, the outgoing flux of matter energy radiated by the dynamical horizon is positive definite (and the ingoing flux of matter energy is negative definite). The flux formula (4.31) differs from the one given in [71] in an important way: since the Kodama vector field provides a timelike direction and is null on the horizon, it is appropriate to use K^a in the flux formula for the dynamical horizon.

4.1.5 Discussions

Our derivation of Hawking temperature and the flux law depends on two assumptions. The first is the existence of the Kodama vector field and the Misner-Sharp energy. For spherically symmetric spacetimes, the Kodama vector field exists unambiguously and the Misner-Sharp energy is well defined. For more general spacetimes, a Kodama-like vector field is not known. However, one can still define some mass for such cases that reduces to the Misner-Sharp energy in the spherical limit [73]. The second assumption, is the slow variation of the dynamical surface gravity κ during evolution. For large black holes, the horizon evolves slowly enough so that the surface gravity function should vary slowly in some small neighbourhood of the horizon. In other words, this derivation implies that the Hawking temperature for a dynamically evolving large black hole is $\kappa/2\pi$ if the dynamical surface gravity is slowly varying in the vicinity of the horizon.

The set-up described in this chapter can be further developed to model dynamically evaporating black hole horizons through Hawking radiation, analytically as well as numerically. Over the years, several models have been constructed which study radiating black holes, formed in a gravitational collapse, based on the imploding Vaidya metric with a negative energy-momentum tensor, show that a timelike apparent horizon forms due to violation of energy conditions [74]. However, such models are based on global considerations of event horizons, while local structures like that used in this work might be useful for a better understanding of Hawking radiation and computations of quantum field theoretic effects (see also [75, 76]).

Chapter 5

Conclusions

In this thesis we have worked with two different local formulations of calculating temperature associated with a black hole horizon, the tunnelling method in its Hamilton-Jacobi variant and the field theoretic method that we have developed as an improvement over the tunnelling formulation.

First we have discussed the tunnelling method which have been profoundly used in recent years to calculate temperature of different types of black hole horizons. We have tried to clarify the issue of choosing appropriate boundary conditions in this method while working with singular and non-singular coordinates. In the course of the work we have shown that not only the black hole horizon is capable of radiation but its time-reversed version, the white hole horizon can also absorb quantum particles at the same temperature as the black hole. We have also addressed the issue of obtaining higher order corrections in Hawking temperature from higher order terms in the semiclassical action S and shown that no such "corrections" appear as long as the spacetime metric remains stationary.

We have also proposed another local formulation of Hawking effect which can dispense the WKB like approximations used in the tunnelling approach. This method is based on quantum field theoretic ideas in a curved background. We have applied this new method to Reissner-Nordström (RN) and Kerr black holes which are stationary spacetimes with multiple horizons and to extremal spacetimes and calculated the temperature associated with these horizons.

In this section let us make a few observations regarding the advantages and disadvantages of these two formulations of Hawking effect and their differences. The tunnelling method, though very popular, has some limitations because it is inherently an approximate calculation. The main advantage of tunnelling is that the calculations are much simpler than any other methods of deriving Hawking temperature of a black hole and so it can be applied more easily to intricate black hole solutions. It fails in situations where WKB approximation breaks down, for example the extremal black hole solutions as we have mentioned before, but otherwise this method produces correct Hawking temperature for almost all known black hole horizons.

On the other hand the new formulation that we have developed can be applied to the standard stationary black hole horizons with the added advantage that no WKB like approximations are required. This method even works for extremal spacetimes, the only example where the application of tunnelling formulation is debatable.

Regarding the differences of these two methods first of all we notice that the tunnelling formulation deals with the single particle picture of emission process. The treatment is essentially based on relativistic quantum mechanics. In the new formulation we considered a field system, not only the emission of a single particle from the horizon. One constructs the field modes and using them calculates the field current out of the horizon. So this is essentially quantum field theory in a curved spacetime.

A surprising aspect of these two formulations that we have noticed is that the solutions of the field equations and the solutions of the single particle wave equation in tunnelling considering only the zeroth order term in the action S are similar. To solve the field equations we have only used the near horizon form of the metric and the solutions are exact without any approximations whereas the solutions of the wave equation in tunnelling is obtained by using WKB approximation near the horizon. At this stage we can not offer any other explanation for this similarity between the solutions except this is merely a mathematical coincidence. Some further work is needed on this issue to get a more satisfactory answer.

So far we have considered only stationary black hole solutions. Both the tunnelling and the field theoretic method work equally well for stationary black holes, but their application can be extended to local dynamical horizons also. People have worked with dynamical horizons in the tunnelling framework but there are some technical problems as we have discussed in the last chapter. There are several existing definitions of dynamical horizon associated with a black hole spacetime. We have adopted Hayward's notion of future outer trapping horizon (FOTH) as the radiating surface of the black hole and applied the field theoretic method of calculating temperature of these type of horizons. It is found that the temperature is given by $T = \kappa/2\pi$, where κ is the dynamical surface gravity of the horizon. We have also been able to show that the horizon loses area in accordance with the flux radiated by it. This behaviour of the evolving horizon during Hawking radiation were expected but as far as we know no mathematical proof existed in literature on this issue. The flux formula we have derived is different from that of [71]. For a timelike dynamical horizon the emitted flux have no definite sign in [71] though the horizon area decreases monotonically. We have considered flow of energy along the direction of the Kodama vector for the timelike FOTH. Since this vector field provides a timelike direction it seems appropriate to consider flux along this direction. And the choice is justified as the emitted flux has definite sign and the area of the horizon decreases in accordance with the outgoing flux.

While more interesting and deeper issues can only be understood in a full quantum theory of gravity, the present framework can elucidate the suggestions of [78] and provide a better understanding of the Hawking radiation process.

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1. **“Tunnelling from black holes in the Hamilton Jacobi approach”**

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