EVOLUTION OF COSMOLOGICAL PERTURBATIONS THROUGH BOUNCE IN COVARIANT PERTURBATION THEORY AND TESTS OF LINEARITY

By

ATANU KUMAR

Enrollment No. PHYS05200904001

Saha Institute of Nuclear Physics, Kolkata

A thesis submitted to the

Board of Studies in Physical Sciences

In partial fulfillment of requirements

for the Degree of

DOCTOR OF PHILOSOPHY

Of

HOMI BHABA NATIONAL INSTITUTE

February, 2016

Homi Bhabha National Institute[1]

Recommendations of the Viva Voce Committee

As members of the Viva Voce Committee, we certify that we have read the dissertation prepared by Atanu Kumar, entitled ``Evolution of cosmological perturbations through bounce in covariant perturbation theory and tests of linearity" and recommend that it may be accepted as fulfilling the thesis requirement for the award of Degree of Doctor of Philosophy.

Name	Designation	Signature	Date
Prof. Parthasarathi Mitra	Chairman	P. Mitra	25. 7. 16
Prof. Amit Ghosh	Guide & Convener		25/07/2016
Prof. L. Sriramkumar	Examiner	L. Sriramkumar	25/7/16
Prof. Debades Bandyopadhyay	Member 1	Debades Bandyopadhyay	25.7.2016
Prof. Asit K. De	Member 2	Asit K De	25/7/16

Final approval and acceptance of this thesis is contingent upon the candidate's submission of the final copies of the thesis to HBNI.

I hereby certify that I have read this thesis prepared under my direction and recommend that it may be accepted as fulfilling the thesis requirement.

Date: 25/07/2016

Place: SINP, Kolkata.

Guide

1. This page is to be included only for final submission after successful completion of viva voce

Version approved during the meeting of Standing Committee of Deans held during 29-30 Nov 2013

STATEMENT BY AUTHOR

This dissertation has been submitted in partial fulfillment of requirements for an advanced degree at Homi Bhabha National Institute (HBNI) and is deposited in the Library to be made available to borrowers under rules of the HBNI.

Brief quotations from this dissertation are allowable without special permission, provided that accurate acknowledgement of source is made. Requests for permission for extended quotation from or reproduction of this manuscript in whole or in part may be granted by the Competent Authority of HBNI when in his or her judgment the proposed use of the material is in the interests of scholarship. In all other instances, however permission must be obtained from the author.

Atanu Kumar

Atanu Kumar

DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Atanu Kumar

Atanu Kumar

List of Publications arising from the thesis

Journal

1. "Growth of covariant perturbations in the contracting phase of a bouncing universe", Atanu Kumar, *Physical Review D*, **2012**, *86*, 123522 (1-9).

2. "Covariant perturbations through a simple non-singular bounce", Atanu Kumar, *Physical Review D*, **2014**, *89*, 084059 (1-17)

Atanu Kumar

Atanu Kumar

ACKNOWLEDGMENTS

It is my pleasure to thank my supervisor Amit Ghosh for his guidance, encouragement and discussions. It was impossible to complete the research without his persistent support throughout. Discussions with Parthasarathi Mitra and Palash Baran Pal enriched me. Long discussions with my friend Avirup Ghosh helped me in understanding some key issues. I am indebted to the teachers of Hooghly Mohsin College specially Narayan Banerjee and Ajay Chowdhury who had built my base in Physics. I have been enriched from the classes of Binayak Dutta Roy, Samir Kr. Paul and Amitabha Lahiri in SNBNCBS.

I also like to thank all staffs of SINP and in particular Theory Division for their help and cooperation. I express my gratitude to the co-researchers in the Room No. 363 for providing a cheerful environment and helping me in every possible way. I also like to thank my colleagues in Chandernagore College for their cooperation in continuing my research.

I like to thank my entire family and the friends for encouraging me. I would be in nowhere without the sacrifice and support of my parents Amitadyuti Kumar and Nibedita Kumar. And it is my wife Srirupa, without whose constant support the thesis would not be completed.

Contents

Synopsis

The standard cosmological model (SCM) provides an almost accurate description of the evolution of the universe through a span of approximately 13.7 billion years. The main hypotheses on which the model is based are the following:

1. Gravity is described by General Relativity.

2. The universe obeys the Cosmological Principle, i.e., the universe is homogeneous and isotropic above a certain length scale as viewed by the comoving observers. As a consequence, all the relevant quantities depend only on global comoving time.

3. Above this scale, the matter content of the universe is described by a continuous distribution of matter and/or energy which to a first approximation, is described by a perfect fluid.

In spite of its success, the SCM suffers from several difficulties, such as the horizon problem, flatness problem, monopole problem, possible existence of dark energy and dark matter and the initial singularity [1],[2]. Some of these difficulties have been resolved totally or partially by assuming an inflationary phase in the early period of expansion of the universe [4]. Inflation also explained the origin of inhomogeneities that seed the structure formation and scale invariance of the spectrum of density perturbations in Cosmic microwave background radiation (CMBR). Another drawback of SCM is the initial singularity. Friedmann's equations tell that an ever expanding universe starts from an initial singularity when curvature and energy density blow up and therefore, the description of the space-time in terms of classical physics breaks down. Initial singularity is not removed by inflation, it is only pushed to a further past [5].

A possible way to avoid the initial singularity is to consider both the contracting and expanding branch of Friedmann's equations. Models of nonsingular universes that have an initial contracting phase followed by a phase of re-expansion after attaining a minimal size, called bounce, are being studied for a long time as alternatives to the standard big-bang inflationary models [6]-[9]. These bouncing models could not attract the attention of the community because bounce does not fit into classical general relativity if matter content of the universe obeys certain energy conditions. According to the singularity theorems of Hawking And Penrose, *a contracting universe (Hubble constant $H < 0$) with a matter content that obeys $\mu + 3p > 0$ (μ and p being energy density and pressure) and vanishing acceleration and vorticity will collapse to a singularity within a time less than H^{-1}* [10]. But recently, observations of accelerating universe show that nature is not so strict about the energy conditions. Also it is understood that when the size of a contracting universe shrinks to the Plank length, the classical general relativity should be replaced by some quantum gravity.

Besides solving horizon problem and removing the initial singularity, bouncing cosmology with a matter dominated contracting phase also provides a mechanism of generating scale invariant spectrum of perturbations. For those perturbations that exit the Hubble radius with a bluish spectrum, contractions boost longer wavelengths more than the shorter wavelengths and thus producing a scale invariant spectrum [13].

For being observed in the expanding phase of the Universe, perturbations must evolve through the bounce. But the perturbations have growing modes in the contracting universe. For example, in a dust dominated flat Friedmann-Lemaitre-Robertson-Walker (FLRW) background, the gauge invariant Bardeen potential Φ has a constant mode as well as a growing mode:

$$\Phi_k(\eta) = C_1(k) + C_2(k)\eta^{-5}. \tag{0.0.1}$$

In singular models, the perturbations diverge as $a \to 0$. In nonsingular bounce, the perturbations are not necessarily diverging but still strongly growing in the contracting phase. These growing modes of perturbations raise doubts on the validity of linear regime of perturbation theory near bounce [15, 16] and preservation of scale invariant spectrum of the perturbations. In the new ekpyrotic bouncing model [17], the adiabatic modes of perturbations are observed to be amplified exponentially at the turning points, i.e., the boundary of contracting phase and bouncing phase, resulting in the breakdown of perturbation theory and spoiling scale invariance of the spectrum [19],[20]. In conventional coordinate based perturbation theory, the perturbations are observed to grow in some gauges while they remain small in some other gauges [23]. It is also found that the commonly defined gauges are not well defined near and at the bounce. Hence linear perturbation theory is valid if one uses only the well defined gauge.

Naturally, a question arises whether these results are real or gauge artefacts. In this regards we think that the alternative approach, considered by Hawking [24] and developed by Ellis, Bruni and others [25], is an efficient tool to investigate the validity of linear perturbation theory near bounce. In this alternative approach, known as covariant perturbation theory, we start from a physical manifold with arbitrary metric and matter configurations which are close to a background spacetime (generally, an FLRW one). We choose a suitable family of timelike observers with four velocity u^a, provided the worldlines of these observers coincide with that of the comoving observers in the background FLRW spacetime. The first covariant derivative of the 4-velocity can be written in terms as

$$\nabla_b u_a = \frac{1}{3}\theta h_{ab} + \sigma_{ab} + \omega_{ab} - u_b \nu_a, \tag{0.0.2}$$

where the trace part $\theta = \nabla_a u^a$ is the expansion scalar, the traceless symmetric part σ_{ab} is called the shear tensor and the antisymmetric part ω_{ab} is the vorticity. Accelerations of the worldlines are represented by ν_a. The matter content of the universe is represented by an energy momentum tensor T_{ab} for an imperfect fluid moving with velocity u^a:

$$T_{ab} = (\mu + p)u_a u_b + p g_{ab} + 2u_{(a}q_{b)} + \pi_{ab}, \tag{0.0.3}$$

where $\mu = T_{ab}u^a u^b$ and $p = \frac{1}{3}h^{ab}T_{ab}$ are energy density and pressure respectively. The anisotropic part involves the energy flux $q_a = -h_a^{\ b}T_{bc}u^c$ and the anisotropic stress $\pi_{ab} = h_a^{\ c}h_b^{\ d}T_{cd} - p h_{ab}$. The projection tensor h_{ab}, defined by

$$h_{ab} = g_{ab} + u_a u_b, \tag{0.0.4}$$

is the metric of the constant time hypersurfaces perpendicular to the u^a if the worldlines are hypersurface orthogonal. An FLRW spacetime is a conformal spacetime (i.e. Weyl tensor C_{abcd} is zero), characterized by vanishing shear, vorticity and acceleration. In such situation one can define a cosmic time t as $u_a = -\nabla_a t$. Then μ, p and θ are function of only t.

Now according to Stewart Walker Lemma [61] a variable is gauge invariant if vanishes on the background (in our case FLRW) spacetime. So under the above characterization of FLRW spacetime one can define certain gauge invariant quantities:

1. Shear (σ_{ab}), vorticity (ω_{ab}) and acceleration(ν_a),

2. "Electric" and "magnetic" parts of the Weyl tensor:

$$E_{ab} = C_{acbd}u^c u^d, \quad H_{ab} = \frac{1}{2}C_{acpq}\eta^{pq}{}_{bd}u^c u^d. \tag{0.0.5}$$

3. Spatial gradients of the energy density (μ), pressure (p) and expansion (θ):

$$X_a = \kappa h_a^{\ b}\nabla_b\mu, \quad Y_a = \kappa h_a^{\ b}\nabla_b p, \quad Z_a = h_a^{\ b}\nabla_b\theta. \tag{0.0.6}$$

A closed set of nonlinear evolution equations are obtained for these variables. Then we linearize the equations for "almost FLRW" spacetimes by just giving up the terms appeared as products of first order variables.

In coordinate based perturbation theory, one can also construct linear combination of perturbation variables, that are invariant under any infinitesimal coordinate transformation and hence are gauge invariant [26]. However, physical interpretation of such variables is clear only if one choose some specific gauge. On the other hand the covariant perturbations are defined through variables that vanish at the background spacetime. Hence they carry physical meaning without choice of any coordinate system. At the linear order, however, one can establish relations between covariant variables used in covariant perturbation theory and ordinary gauge invariant variables used in conventional perturbation theory and the physical interpretation of the latter is possible via such correspondence without choosing any coordinate system [27]. Although the perturbations and their evolution equations are completely gauge invariant and covariant, apparently a new type of ambiguity may arise due to choice of observers. If the background spacetime is spatially homogeneous and isotropic, there exists a preferred family of observers, the comoving observers, which are also observers of the spatial homogeneity and isotropy. In the physical inhomogeneous manifold no such preferred observer exist. However, under the change of observers, the variables and their evolution equations do not change, only their interpretation is altered.

So far, it has been checked whether perturbations are sufficiently small compared to their background quantities. If that criterion holds, second order perturbations are assumed to be even smaller and linear perturbation theory is considered to be a good approximation. For example we can write the perturbed metric $g_{\mu\nu}$ as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}. \tag{0.0.7}$$

The $\bar{g}_{\mu\nu}$ is the metric of background FLRW spacetime,

$$\bar{ds}^2 = a^2(\eta)\left(-d\eta^2 + \gamma_{ij}dx^i dx^j\right), \tag{0.0.8}$$

where γ_{ij} being the metric of maximally symmetric 3 dimensional spatial hypersurface with normalized curvature $K = 0, \pm 1$ and $\delta g_{\mu\nu}$ represents the scalar metric perturbation [28],

$$\delta g_{00} = -2a^2\phi, \quad \delta g_{0i} = a^2 D_i\mathcal{B}, \quad \delta g_{ij} = 2a^2\left(-\psi\gamma_{ij} + D_i D_j\mathcal{E}\right), \tag{0.0.9}$$

where D_i is the covariant derivative operator, compatible with γ_{ij}. The linear perturbations must satisfy

$$\phi \ll 1, \quad \mathcal{B} \ll 1, \quad \psi \ll 1, \quad \mathcal{E} \ll 1. \tag{0.0.10}$$

Such conditions depend on the chosen gauge just because the variables ϕ, $\mathcal{B}$, ψ, $\mathcal{E}$ are not gauge invariant. But this test will not work in covariant formalism as the background values of the gauge invariant perturbations are zero. We suggest an alternative treatment to check the validity of linear approximation [29].

We focus on the evolution equations for energy density perturbation X_a which is coupled with Z_a at linear order:

$$\begin{aligned} a^{-4}h_a^{\ b}(a^4X_b)\dot{} &= -\kappa(\mu+p)Z_a - (\sigma^b_{\ a}+\omega^b_{\ a})X_b, & (0.0.11) \\ a^{-3}h_a^{\ b}(a^3Z_b)\dot{} &= \mathcal{R}\nu_a - \frac{1}{2}X_a + A_a + 2^{(3)}\nabla_a(\omega^2-\sigma^2) \\ & \qquad -(\sigma^b_{\ a}+\omega^b_{\ a})Z_b, & (0.0.12) \end{aligned}$$

where $A = \nabla_a\nu^a$, $A_a = {}^{(3)}\nabla_a A$ and $\mathcal{R} = \kappa\mu - \frac{1}{3}\theta^2 + A + 2(\omega^2-\sigma^2)$. The time derivative, represented by overdot, is defined as the covariant derivative along u^a. We consider only the perfect fluid perturbations, i.e. $q_a = 0$ and $\pi_{ab} = 0$. In the right hand side of the Eq. (0.0.11), the term $\kappa(\mu+p)Z_a$ is of first order (linear) whereas $\sigma^b_{\ a}X_b$ and $\omega^b_{\ a}X_b$ are higher order terms. According to the Friedmann equations $\kappa\mu = \frac{1}{3}\theta^2$, thus $\mathcal{R}$ is a first order variable. So in the right hand side of the Eq. (0.0.12), there are two first order terms $\frac{1}{2}X_a$ and A_a and five second order terms $\mathcal{R}\nu_a$, $2^{(3)}\nabla_a(\sigma^2)$, $2^{(3)}\nabla_a(\omega^2)$, $\sigma^b_{\ a}Z_b$ and $\omega^b_{\ a}Z_b$.

To compare the nonlinear terms of these equations with the linear terms of the same equations, we define following parameters:

$$\begin{aligned} \varepsilon_1 &= \frac{|\omega^b_{\ a}X_b|}{|\kappa(\mu+p)Z_a|}, \quad \varepsilon_2 = \frac{|\sigma^b_{\ a}X_b|}{|\kappa(\mu+p)Z_a|}, \\ \varepsilon_3 &= \frac{|\mathcal{R}\nu_a|}{|\frac{1}{2}X_a|}, \quad \varepsilon_4 = \frac{|2h_a^{\ b}\nabla_b\omega^2|}{|\frac{1}{2}X_a|}, \quad \varepsilon_5 = \frac{|2h_a^{\ b}\nabla_b\sigma^2|}{|\frac{1}{2}X_a|}, \\ \varepsilon_6 &= \frac{|\omega^b_{\ a}Z_b|}{|\frac{1}{2}X_a|}, \quad \varepsilon_7 = \frac{|\sigma^b_{\ a}Z_b|}{|\frac{1}{2}X_a|}, \\ \tilde{\varepsilon}_3 &= \frac{|\mathcal{R}\delta u_a|}{|A_a|}, \quad \tilde{\varepsilon}_4 = \frac{|2h_a^{\ b}\nabla_b\omega^2|}{|A_a|}, \quad \tilde{\varepsilon}_5 = \frac{|2h_a^{\ b}\nabla_b\sigma^2|}{|A_a|}, \\ \tilde{\varepsilon}_6 &= \frac{|\omega^b_{\ a}Z_b|}{|A_a|}, \quad \tilde{\varepsilon}_7 = \frac{|\sigma^b_{\ a}Z_b|}{|A_a|}. & (0.0.13) \end{aligned}$$

The linear perturbation theory is valid for the solutions of (0.0.11) and (0.0.12) provided the following conditions are satisfied throughout the regime under consideration:

$$(\mathbf{1})\ \varepsilon_1, \varepsilon_2 \ll 1,$$

$$(\mathbf{2})\ \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7 \ll 1,\ \text{and/or}\ \tilde{\varepsilon}_3, \tilde{\varepsilon}_4, \tilde{\varepsilon}_5, \tilde{\varepsilon}_6, \tilde{\varepsilon}_7 \ll 1.$$

Then the linear perturbation equations are solved for a collapsing FLRW background near the bounce. The solutions are used to compute the linearity parameters. It is found that some of those parameters grow beyond order unity near the bounce in both radiation and dust dominated collapsing universe. That means the nonlinear terms are comparable to the linear terms. So unless some special initial conditions are imposed on the variables such as shear and vorticity, perturbations may not be linear near the bounce.

Thus we conclude that perturbations may not be linear near the bounce and linear perturbation theory may not be adequate to give proper evolution of perturbations through the bounce. Our results are independent of the choice of gauge. We used gauge invariant variables that were not assumed to be small with respect to background. So one can evolve them through the bounce and match with corresponding quantities in the expanding phase—but this would require the full nonlinear analysis.

This formalism is applied to a simple nonsingular bouncing model in [31]. We take a toy model for the flat FLRW bouncing universe filled with a two-component perfect fluid, one component is a normal fluid with a dustlike equation of state, henceforth referred to as fluid-1, and the other component has a negative energy density and pressure, henceforth referred to as fluid-2 [30]. Away from the bounce, the contribution of fluid-2 in the total energy budget is negligible and hence, the contraction of the universe is essentially guided by fluid-1. However, close to the bounce, fluid-2 becomes dominant and as a result the collapse slows down by minimizing the Hubble parameter H. At turning point $\dot{H}$ becomes zero. Eventually the bouncing point $H = 0$ is reached and the universe starts to re-expand. Again at another turning point $\dot{H}$ vanishes and subsequently fluid-1 starts to dominate. Between the two turning points the null energy condition,

$$T_{\mu\nu}k^{\mu}k^{\nu} \geq 0 \quad \text{for any null vector } k^{\mu}, \tag{0.0.14}$$

is violated by the composite fluid.

Evolution of vector perturbations ω_a and r_a is rather simple. But the analytic solutions for scalar and tensor perturbations in the entire range of time are obtained only for zero wave number mode. For $q \neq 0$ the equations are simplified to get analytic solutions in three different regions, namely long before bounce, at the turning point and at the bounce. The scalar perturbations are smooth across the bounce but diverge at the turning point. The shear σ_{ab} is decomposed into scalar, vector and pure tensor parts. The gravitational wave, i.e. pure tensor part of shear shows oscillating behavior both at the bounce and at the turning point. At the turning point, scalar and vector parts dominate over the gravitational wave. The comoving curvature perturbation ζ^S has a nonadiabatic growing mode at the turning point, besides its adiabatic constant mode. The growth rates of the linearity parameters are computed at the turning point. It is observed that many of these parameters diverge. So the perturbations cease to be linear at the turning point in this simple nonsingular bouncing model [31].

We also consider a specific initial condition for scalars in which the entropic perturbation is absent and the adiabatic perturbations are originated from quantum fluctuations of the Bunch-Davis vacuum state in the matter dominated era. Using a numerical analysis we evolve the perturbations through the bounce. Divergence of the linearity parameters remains unaltered even in the presence of these special initial conditions.

As a byproduct of our analysis we study the matching conditions and spectra of the perturbations. We have studied the matching conditions for scalar variables. It has been shown that the spectrum of perturbations after the bounce can be obtained by employing the sound matching conditions. Despite the divergence at the turning points and the growth of amplitudes, the scale invariant spectrum of the perturbations is preserved after the bounce. Our numerical analysis shows that the variable $\mathcal{V}$ should be matched across the transition surface to get the correct spectra, while matching $\mathcal{X}$ leads to a wrong spectra. Since $\mathcal{V}$ and $\mathcal{X}$ are related to spatial curvature perturbation ($\delta\mathcal{R}$) and the Bardeen potential (Φ), these results coincide with the ones obtained in [53].

List of Figures

*

Chapter 1

Introduction

The standard cosmological model (SCM) provides a more or less accurate description of the evolution of the universe over a span of approximately 13.7 billion years. The main hypotheses on which the model is based are the following:

1. Gravity is described by Einstein's theory.

2. The universe obeys the Cosmological Principle, namely the Universe is homogeneous and isotropic above a certain distance scale as viewed by comoving observers. As a consequence, all relevant quantities depend only on a global time coordinate called comoving time.

3. Above this distance scale the matter content of the universe is described by a continuous, homogeneous and isotropic distribution of matter/energy which is modeled by a perfect fluid.

In spite of its great success, SCM suffers from several inherent difficulties, such as the horizon problem, flatness problem, monopole problem, etc. [1],[2]. Some of these difficulties are totally or partially resolved by assuming an early inflationary phase, a phase of rapid and accelerated expansion of the universe [3],[4]. Inflation also explains the origin of inhomogeneities that seed the structure formation and scale invariance of the spectrum of density perturbations in the Cosmic Microwave Background Radiation (CMBR). Another drawback of SCM is the initial singularity. Friedmann equations tell us that any expanding universe when traced back in time shows an initial singularity—a three dimensional surface where spacetime curvature and matter energy density blow up everywhere on the surface and therefore, the evolution of the universe in terms of Friedmann equations breaks down. Initial singularity can not be removed by inflation, only it is pushed back in the past [5].

A possible way to get rid of the initial singularity is to consider both the contracting and expanding solutions of the Friedmann equations and try to join them together by a nonsingular bounce. Models of nonsingular universes with an initial contracting phase followed by a phase of re-expansion after attaining a minimum size (bounce) have been studied extensively [6]-[9]. However, the bounce models could not attract the attention of a larger community because they do not fit in the realm of *standard* physics as the matter required to produce bounce in such models violates certain energy conditions. As per the singularity theorems of Hawking and Penrose, *a contracting universe (Hubble constant $H < 0$) with a matter content obeying $\mu + 3p > 0$ (μ and p being energy density and pressure of the fluid) and vanishing acceleration and vorticity will collapse to a singularity within a time less than H^{-1}* [10]. But recent observations related to acceleration of the universe hint that one should relax some conditions of the singularity theorems. Also when the size of a contracting universe shrinks to the order of a Planck length, classical physics is to be replaced by quantum physics and in particular quantum gravity should not be ignored.

Besides solving the horizon problem and removing the initial singularity, bouncing cosmological models with a matter dominated contracting phase also provide a mechanism of generating scale invariant perturbations. As perturbations exit the Hubble radius with a bluish spectrum, contractions boost longer wavelengths more than the shorter wavelengths, thus producing a scale invariant spectrum [11]-[13].

For being observed in the expanding phase of the universe, perturbations must evolve through bounce [14]. But the perturbations have growing modes in the contracting universe. For example, in a dust dominated flat Friedmann-Lemaitre-Robertson-Walker (FLRW) background, the gauge invariant Bardeen potential Φ has a constant mode as well as a growing mode:

$$\Phi_k(\eta) = C_1(k) + C_2(k)\eta^{-5}, \tag{1.0.1}$$

where η is the conformal time. In singular models the perturbations diverge as $a \to 0$. In nonsingular bounce the perturbations do not necessarily diverge but grows rapidly in the contracting phase. These growing modes of perturbations raise some doubts about the validity of linear perturbation theory close to bounce and preservation of scale invariant spectrum of perturbations [15, 16]. In ekpyrotic bouncing model [17], the adiabatic modes of perturbations are observed to be amplified exponentially at the turning points, namely the boundary between the contracting phase and bouncing phase, resulting in a breakdown of perturbation theory and scale invariant spectrum [18]-[20]. In the conventional coordinate based perturbation theory, the perturbations are observed to grow in some gauges while they remain small in some other gauges [21]-[23]. It is also found that the commonly defined gauges are not well defined close to and at the bounce. Hence, linear perturbation theory is valid only if one uses some well-defined gauges.

Naturally, a question arises whether these results are physical or gauge artifacts. In this connection we find that the alternative approach, proposed by Hawking [24] and later developed by Ellis, Bruni and other collaborators [25], is a more appropriate tool in investigating the validity of linear perturbation theory close to bounce. In this approach, commonly called as the covariant perturbation theory, one deals with completely gauge invariant quantities that are defined in a coordinate independent manner. In particular, the quantities that vanish or remain constant (in the case of a scalar matter) in the background spacetime are used as gauge invariant variables. In the coordinate based perturbation theory, one can also construct linear combination of perturbation variables, that are invariant under any infinitesimal coordinate transformation and hence are gauge invariant [26]. However, physical interpretation of such variables is clear only if one chooses a specific gauge. On the other hand the covariant perturbations are defined through variables that vanish at the background spacetime. Hence, they are directly physical. Moreover, this approach does not involve perturbation expansion of variables and thus the dynamical equations are nonlinear.

However, at linear order one can establish a map between the covariant variables used in covariant perturbation theory and the usual gauge invariant variables used in the conventional perturbation theory. A physical interpretation of the latter variables is provided via this map without the recourse of choosing a coordinate system [27].

So far whether perturbations are sufficiently small compared with their background quantities has been checked. If this scenario holds, second order perturbations may be assumed to be even smaller and linear perturbation theory can be considered to be a good approximation. For example, we can write the perturbed metric $g_{\mu\nu}$ as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}. \tag{1.0.2}$$

The $\bar{g}_{\mu\nu}$ is the metric of the background FLRW spacetime,

$$\bar{ds}^2 = a^2(\eta)\left(-d\eta^2 + \gamma_{ij}dx^i dx^j\right), \tag{1.0.3}$$

where γ_{ij} is the metric of a maximally symmetric 3-dimensional spatial hypersurface with normalized curvature parameter $K = 0, \pm 1$ and $\delta g_{\mu\nu}$ represents the scalar metric perturbation [28],

$$\delta g_{00} = -2a^2\phi, \quad \delta g_{0i} = a^2 D_i\mathcal{B}, \quad \delta g_{ij} = 2a^2\left(-\psi\gamma_{ij} + D_iD_j\mathcal{E}\right), \tag{1.0.4}$$

where D_i is the covariant derivative compatible with γ_{ij}. The linear perturbation must satisfy

$$\phi \ll 1, \quad \mathcal{B} \ll 1, \quad \psi \ll 1, \quad \mathcal{E} \ll 1. \tag{1.0.5}$$

Such conditions depend on the choice of gauge because the variables involved, ϕ, $\mathcal{B}$, ψ and $\mathcal{E}$, are not gauge invariant. Moreover, these conditions are not usable in the covariant framework as the background values of the gauge invariant perturbations are zero. We suggest an alternative treatment to verify the validity of linear approximation [29]. We consider the evolution equations of the perturbations and examine whether the nonlinear terms are truly small compared to the linear terms. For this purpose, some linearity parameters are defined as the ratio of nonlinear and linear terms. For dust and radiation dominated collapsing backgrounds, these parameters are found to diverge as $a \to 0$.

This formalism is applied to a simple and nonsingular bounce. The bounce is observed in a model having two components of perfect fluids—one is a normal fluid component with a dust-like equation of state and the other has a negative energy density and radiation-like equation of state. The fluid violates the null energy condition close to the bounce. Gauge invariant perturbations are defined in terms of the comoving observers. The tensor perturbations remain finite and well behaved at and close to the bounce but scalar and vector perturbations become singular at the turning point which is the boundary of the spacetime region where the null energy condition is violated. Besides the constant adiabatic mode, the comoving curvature perturbation has a non-adiabatic mode that diverges at the turning point. By computing the growth of linearity parameters it has been shown that the perturbations do not remain linear at the turning point. We also study the matching conditions of scalar perturbations and $\mathcal{V}$ ($\mathcal{V}$ is related to the spatial curvature perturbation) is found to be the appropriate variable to match across the transition surface.

In the thesis we first discuss some characteristics of bouncing cosmology and emergence of scale invariant spectrum from a bounce. Then followed by a brief introduction to the covariant perturbation theory we set up some conditions for linearity of perturbations and study the evolution of cosmological perturbations through a simple toy model for bounce.

Chapter 2

Bouncing scenario as an alternative to inflation

2.1 Introduction

In this chapter we discuss some important issues regarding bouncing cosmology. In order to provide a viable alternative to inflation, a model should, at least, do as well as inflation in many respects. Inflation has successfully answered the following puzzles of standard hot big bang model: "why is the universe isotropic on the largest accessible scales?", "Why does the content of the universe sum up in the exactly required fashion so as to make its spatial curvature negligible?" and "Why do we not observe an absurdly large number of primordial monopoles that should have been copiously produced during the grand unification transition?". However, inflation cannot remove the initial singularity. In Sec. (2.2) we discuss how the the bouncing models address these puzzles.

The attractive nature of gravity bars a collapsing universe to stop its contraction and bounce to an expanding phase as long as the Einsteins general relativity dictates the dynamics of universe. The issue of singularity and the possibilities of a bounce are discussed in the Sec. (2.3) with a description of various energy conditions. With a brief introduction to the theory of cosmological perturbations the origine of scale invariant spectrum of density fluctuation and matching the perturbations in contracting phase with those in expanding phase are discussed in the Sec. (2.4). Finally a discussion on the validity of linear perturbation theory near bounce is given in the Sec. (2.5).

2.2 Cosmological puzzles

2.2.1 Horizon problem

The temperature of the cosmic microwave radiation backgraound in two different patches of the sky is same in at least one part in 10^{-5}. This observation is consistent with the assumption of isotropy in the FLRW cosmology. However, such high degree of anisotropy posed a problem with the causality. The portions of the universe that we are observing today should not be in causal contact with each other at early universe if the scale factor grows as power law of cosmic time, i.e. the univrese is matter or radiation dominated. At the time of last scattering the horizon size was

$$d_H(t_L) \approx \frac{1}{H_0(1+z_L)^{3/2}}, \tag{2.2.1}$$

where z_L is the redshift at the time of last scattering and H_0 is the present value of Hubble parameter. The angular diameter distance from the surface of last scattering is

$$d_A(t_L) \approx \frac{1}{H_0(1+z_L)}. \tag{2.2.2}$$

So a patch of horizon size in the surface of last scattering subtends an angle

$$\frac{d_H}{d_A} \approx \frac{1}{(1+z_L)^{1/2}} \tag{2.2.3}$$

in radians. From the redshift at last scattering $z_L \approx 1101$, this angle becomes $1.6°$. So the patches of the universe seperated by angle more than this were not in causal contact and hence should not be in same tempetrature.

The problem will be solved if we have a mechanism that amplify the horizon size such that $d_H(t_L) > d_A(t_L)$. Inflation have done this job successfully. In bouncing cosmology, as the initial time surface goes to $-\infty$, the integral

$$d_H(t_L) = a(t_L) \int_{-\infty}^{t_L} \frac{dt}{a(t)} \tag{2.2.4}$$

diverges. Hence, the entire spatial hypersurface at the time of recombination which we observe today in the CMB is within the causal horizon and there is no Horizon problem.

2.2.2 Flatness problem

Observation tells that the present energy density of the universe is very close to the critical energy density

$$\mu_0 \approx \frac{3H_0^2}{\kappa},$$

so that the absolute value of curvature parameter $\Omega_K = -K/a_0^2 H_0^2$ is less than unity and the constant time spatial hypersurfaces are almost flat. From the time the temperature dropped to about 10^4 K until near the present, $a(t)$ has been increasing as $t^{2/3}$, so $|K|/a^2H^2$ has also been increasing as $t^{2/3} \propto T^{-1}$. Thus, if $|\Omega_K| < 1$ at present, then at 10^4K the curvature parameter $|K|/a^2H^2$ could not have been greater than about 10^{-4}. Earlier, $a(t)$ was increasing as $t^{1/2}$, so $|\frac{K}{a^2H^2}|$

was increasing as $t \propto T^{-2}$. In order for $|K|/a^2H^2$ at 10^4K to be no greater than about 10^{-4} , it is necessary that $|K|/a^2H^2$ was at most about 10^{-16} at the temperature $T \approx 10^{10}$K of electron-positron annihilation (roughly, the beginning of the period of neutron–proton conversion that results in the observed helium abundance), and even smaller at earlier times.

This is not a paradox. There is no reason why universe can not start with $K = 0$. However, presence of an inflationary phase precceding the radiation dominated phase of evolution makes room for a universe starting with non zero K. During inflation H remains roughly constant and the curvature parameter falls exponentially. So at the beginning of radiation dominated expansion, the curvature parameter $|K|/a^2H^2$ is negligiblly small.

A bouncing cosmology is "neutral" with respect to the Flatness problem: if we postulate a similar degree of spatial flatness in the contracting phase at an equal amount of time prior to the bounce point as is observed today a certain time interval after the bounce point, then the observations can be explained.

2.2.3 Absence of magnetic monopoles

All grand unified theories predict that there should be, in the spectrum of possible particles, extremely massive particles carrying a net magnetic charge. By combining grand unified theories with classical cosmology without inflation, Preskill [32] found that magnetic monopoles would be produced so copiously that they would outweigh everything else in the universe by a factor of about 10^{12} . A mass density this large would cause the inferred age of the universe to drop to about 30 000 years! Inflation is certainly the simplest known mechanism to eliminate monopoles from the visible universe, even though they are still in the spectrum of possible particles. The monopoles are eliminated simply by arranging the parameters so that inflation takes place after (or during) monopole production, so the monopole density is diluted to a completely negligible level.

In a bouncing scenario however this problem resurfaces in an acute manner, in particular for a matter bounce, if a subdominant thermal component is present. Since (p)reheating has not been studied in bouncing cosmologies beyond simple estimates, it is unknown whether or not thermal relics are formed. Nevertheless, the defect question becomes a crucial one, not on considerations of energy density and relative contribution, but more fundamentally, because of the initial conditions they demand: if many Higgs fields are originally present in the large and cold universe, some of them must have vacuum expectation values, which in turns means, for most of those, arbitrary phases. As far as we know, there seems to be no natural (and accepted) way to set up these phases: it is an open question whether or not it can actually be achieved at all.

2.2.4 Initial singularity problem

Ever expanding cosmologies have been shown to be past incomplete, so that, as far as classical gravity is concerned, the Universe began with an initial singularity, featured by the divergence of Ricci curvature scalar and the energy density

$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right), \quad \mu = \frac{M}{a^{3(1+w)}} \tag{2.2.5}$$

as $a \to 0$, w being the eqution of state parameter. At the singularity, description of spacetime in terms of classical physics breaks down. This problem is not directly addressed in the inflationary framework, as it is then postulated that the physics describing cosmological evolution begins at a stage for which the relevant theories are well understood, assuming previous phases to be based on quantum gravity and to have a limited influence on the relevant scales.

In many models, the inflation is considered to last forever, i.e. the universe is eternal to the future [33]. If the model is extended to the past, then universe would be in a state of eternal inflation without any beginning. However, it is proved under reasonable assumptions that the inflating region must be incomplete in past directions, so some physics other than inflation is needed to describe the past boundary of the inflating region [5].

It is perfectly reasonable to assume the primordial singularity to have been somehow resolved; one way to do so actually would be to connect the currently expanding Universe to a previous contracting phase through a bounce. In this sense, studying bouncing solutions addresses an extremely relevant question ignored (or perhaps overlooked) by the inflationary paradigm.

2.3 Energy conditions and singularities

2.3.1 Singularity theorem

The issue of the initial singularity of the FRLW solution was debated for a long time, since it was not clear if this singular state was an inherent trace of the universe or just a consequence of the high degree of symmetry of the model. works of Lifshitz and others wrongly suggested that the singularity was not unavoidable, but a consequence of the special symmetries of the FLRW solution [34]. From a completely different point of view, Hawking, Penrose, Geroch and others developed theorems that give global conditions under which timelike and null geodesics cannot be extended beyond a certain (singular) point [35]. In their analysis, the singularity is viewed as abrupt termination of geodesics, not as divergence of some functional of metric. The singularity theorem can be stated as follows:

Theorem 1. *The following requirements cannot all be true for a given space–time* M*:*

1. *There exists a compact spacelike hypersurface (without boundary)* H*;*

2. *The expansion* θ *of the unit normals to* H *is positive at every point of* H*;*

3. $R_{ab}\xi^a\xi^b \geq 0$ *for every non-spacelike vector* u^a *;*

4. M *is geodesically complete in past timelike directions.*

Link of this theorem with physics comes through condition (3) via Einstein equation, yielding a statement about the energy–momentum tensor T_{ab}

$$T_{ab}\xi^a\xi^b + \frac{1}{2}T \geq 0, \tag{2.3.1}$$

called the strong energy condition (SEC). the strength of this theorem is the generality of their assumptions, while their weakness is that they give little information about how the approach to the singularity is described in terms of the dynamics of the theory or about the nature of the singularity.

A simpler version of the the therorem is given as follows. Let us consider a congruence of smooth timelike curves, generated by a smooth vector field u^a in a open subset O of the spacetime manifold S, such that through every point $p \in O$ there passes one and only one curve of the family. As will be discussed in the Sec. (3.3.1), u^a is the velocity of preferred observers in S. The key equation to describe the singularity is the Raychaudhuri equation (3.4.6). If the curves are hypersurface orthogonal geodesics then the vorticity ω_{ab} and the acceleration ν_a

vanish. Further if the cosmological constant Λ is negative and $\mu + 3p > 0$, we can write

$$\frac{d\theta}{d\tau} + \frac{1}{3}\theta^2 \leq 0 \Rightarrow \frac{d}{d\tau}\left(\theta^{-1}\right) \geq \frac{1}{3}, \tag{2.3.2}$$

where τ is the proper time of the preferred observers. Hence

$$\theta^{-1} \geq \theta_0^{-1} + \frac{1}{3}\tau, \tag{2.3.3}$$

where θ_0 is the initial value of the expansion. If the congruence is converging, i.e. $\theta_0 < 0$, then θ^{-1} must pass through zero, i.e. $\theta \to \infty$ within a proper time $\tau \leq \frac{3}{|\theta_0|}$. So the singularity theorem can be stated as

Theorem 2. *In a universe where $\mu + 3p \geq 0$ is valid, $\Lambda \leq 0$, and $\nu_a = \omega_{ab} = 0$ at all times, if at any instant Hubble parameter ($H = \frac{1}{3}\theta$) takes a negative value, there must have been a time less than $\tau_0 \leq 1/H$ such that $a \to 0$ as $\tau \to \tau_0$. A spacetime singularity occurs at $\tau = \tau_0$, in such a way that energy density and the temperature diverge.*

2.3.2 Energy conditions

Before searching for possibilities of avoiding the singularity let us now discuss different energy conditions that are assumed to be satisfied by physically reasonable matters. In particular we consider energy conditions on matter described by a perfect fluid energy monemtum tensor:

$$T_{ab} = (\mu + p)u_a u_b + p g_{ab}.$$

Weak energy condition

$T_{ab}\xi^a\xi^b$ is the energy density meausured by an observer with 4-velocity ξ^a. This energy density is non negative for any timelike observer obeying the weak energy condition (WEC):

$$T_{ab}\xi^a\xi^b \geq 0, \quad \text{for all timelike } \xi^a. \tag{2.3.4}$$

For perfect fluid case this reduce to

$$\mu \geq 0, \quad \mu + p \geq 0. \tag{2.3.5}$$

Strong energy condition

In Raychoudhury equation the matter is coupled via the term

$$R_{ab}\xi^a\xi^b = \kappa\left(T_{ab}\xi^a\xi^b + \frac{1}{2}T\right). \tag{2.3.6}$$

The trace of T_{ab} is not so large and negative such that the right hand side of (2.3.6) becomes negative, according to strong energy condition (2.3.1). In the perfect fluid case:

$$\mu + p \geq 0, \quad \mu + 3p \geq 0. \tag{2.3.7}$$

Null energy condition

The null energy condition (NEC) states that, for any null vector k_a, $T_{ab}k^ak^b$ is nonnegative. In terms of perfect fluid energy momentum tensor this reduces to

$$\mu + p \geq 0. \tag{2.3.8}$$

Dominant energy conditions

For any timelike observer with 4-velocity ξ_a, the vector $-T^a_b\xi^b$ represents the energy-momentum 4-current density. The dominant energy condition states that this vector is a future directed timelike or null vector. Physically this signifies that the speed of energy flow is always less than the speed of light.

2.3.3 Possibilities of bounce

We have seen that singularity is unavoidable for an FLRW universe that obeys the energy conditions and the dynamically dictated by the general relativity. The term $\mu + 3p$ acts as active gravitational mass in the Raychoudhury equation and the energy conditions are consequence of the attractive nature of gravity. The vorticity, acceleration and the cosmological constant are repulsive and thus act against the collapse. The attractive shear terms aids the collapse. To obtain a bounce we should search for ways to avoid the singularity. Let us consider a homogeneous and isotropic FLRW spacetime that obeys the Einstein equation.

Bounce is defined locally in terms of Tolman wormhole as a collapsing universe somehow stops its contraction, attain a minimum size and then reexpands, thus avoiding the big crunch singularity [36]. A bouncing FLRW universe implies that the scale factor a is minimum at the bounce $t = t_b$, i.e.,

$$\dot{a}_b = \dot{a}(t_b) = 0 \Rightarrow H_b = H(t_b) = 0$$

and

$$\ddot{a}_b = \ddot{a}(t_b) > 0 \Rightarrow \dot{H}_b = \dot{H}(t_b) > 0.$$

For $t = t_b$ to be a true minimum there must exist a Δt such that $\ddot{a}_b > 0$ for $t \in (t_b - \Delta t) \bigcup (t_b + \Delta t)$.

Now from the Friedmann equations we have

$$\begin{aligned} \kappa\mu &= 3\left(H^2 + \frac{K}{a^2}\right) && (2.3.9) \\ \kappa(\mu + p) &= 2\left(-\dot{H} + \frac{K}{a^2}\right) && (2.3.10) \\ \kappa(\mu + 3p) &= -6\frac{\ddot{a}}{a}. && (2.3.11) \end{aligned}$$

From the Eq. (2.3.11) it is clear that violation of SEC is necessary for a bounce.

On the other hand violation of NEC depends on the value of spatial curvature. NEC must be violated for a bounce in hyperbolic or flat universes ($K = 0, -1$). For spherical universe ($K = 1$), bounce with $\ddot{a}_b > a_b^{-1}$ requires violation of the NEC. Note that the violation of NEC implies violation of all energy conditions. It follows that

$$\exists \text{ Bounce } + [K \neq +1] \Rightarrow \text{ NEC violated },$$
$$\exists \text{ Bounce } \Rightarrow \text{ SEC violated }.$$

The case that minimizes the violations of the energy conditions can be stated as

$$\exists \text{ Bounce } + \left[K = +1, \ddot{a}_b \leq a_b^{-1}\right] \Rightarrow \text{ NEC, WEC, DEC satisfied, SEC violated.}$$

This result is expected since the curvature term with $K = +1$ acts like a negative energy density in Friedmann's equation. The restriction to a FLRW model was lifted and the analysis in a general case was done following standard techniques taken from the ordinary wormhole case [37], [38]. It was found that even in the case of a geometry with no particular symmetries, the SEC must be violated if there is to be a bounce in GR.

To summarize what was discussed up to now, we can say that there is a "window of opportunity" to avoid the initial singularity in FLRW models at a classical level by one or a combination of the following assumptions:

1. Violating SEC in the realm of GR.

2. Working with a new gravitational theory, as for instance those that add scalar degrees of freedom to gravity or by adopting an action built with higher-order invariants.

3. Changing the way gravity couples to matter.
4. Using a non-perfect fluid as a source.
5. Considering quantum gravitational effects.

2.4 Scale invariant spectrum from bounce

2.4.1 Background

In the bouncing universe, the cosmic time t runs from $-\infty$ to ∞. The bounce point can be taken to be $t = 0$. For negative times the universe is contracting. In the absence of entropy production at the bounce point it is logical to assume that the contracting phase is the mirror inverse of the expanding phase of the standard big bang cosmology, i.e. at very early times the universe is dominated by pressure-less matter, and at a time $t = -t_{eq}$ (where t_{eq} is the time of equal matter and radiation in the expanding phase) there is a transition to a radiation-dominated phase. If there is entropy production at the bounce, then the transition from matter to radiation domination will occur closer to the bounce point than $-t_{eq}$.

In Figure 1 we show a space-time sketch of the bouncing background. The horizontal axis is the comoving spatial coordinate and the vertical is conformal time. Wavelength of a fluctuation mode, indicated by the vertical line, is compared with the comoving Hubble radius $\mathcal{H}^{-1} = a'/a$ (prime denotes derivative with respect to conformal time). The Hubble radius is the maximal scale on which causal and local microphysics can generate fluctuations. In order to have a causal generation mechanism, for fluctuations, it is thus important that wavelengths which are observed today originate on sub-Hubble scales. Fixed comoving scales start out

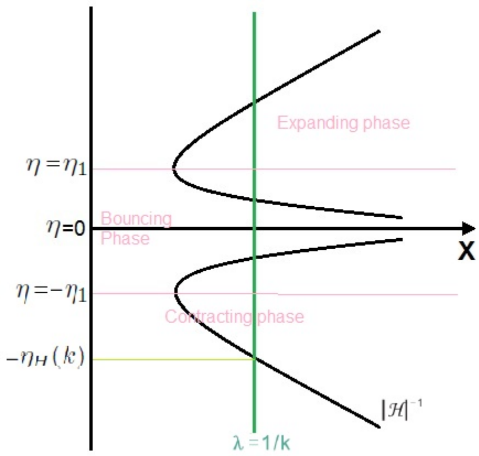

Figure 2.1: Space-time sketch of a non-singular Matter Bounce

The vertical axis is conformal time, the horizontal axis corresponds to comoving spatial coordinates. The vertical line indicates the wavelength for some fixed perturbation mode. The comoving Hubble radius is shown.

with a wavelength smaller than the Hubble radius. In the contracting phase the comoving Hubble radius decreases as the universe collapse and the fluctuations come out of the Hubble radius. Hence a causal generation mechanism for fluctuations is possible in a bouncing cosmology, like it is in inflationary cosmology. In inflationary cosmology it is the exponential decrease of the comoving Hubble radius during the inflationary phase which allows for a causal generation mechanism. We see that in a contracting universe dominated by regular matter a similar decrease in the comoving Hubble radius occurs.

In this section we discuss how a scale-invariant spectrum of adiabatic curvature fluctuations emerges from initially vacuum perturbations. Whereas in inflationary cosmology there is a symmetry argument (time translation invariance during the inflationary phase) which underlies the scale-invariance of the cosmological fluctuations, in a contracting universe there is no such symmetry. As we will see below, it is only vacuum fluctuations which exit the Hubble radius in a phase of matter domination which end up with a scale-invariant spectrum.

2.4.2 Theory of cosmological perturbations

The fluctuations which we are interested in are inhomogeneities in the curvature of space-time which are induced by perturbations in matter. Relative amplitudes of the fluctuations, observed today on large cosmological scales, are very small (One part in 10^{-5}). Hence, evolution of these fluctuations are well described by linearizing the Einstein equation about background FLRW solutions. There are two approaches of studying these fluctuation. The standard coordinate based approach, based on the perturbative expansion of the spacetime metric and the energy momentum tensor, are described in [28]. In the following we summarize the essentials. The alternative approach, known as the covariant perturbation theory will be described in the next chapter.

Fluctuations in the space-time metric in cosmology are classified into scalar, vector and tensor perturbations according to their transformation under spatial rotations. In total there are ten modes. However, four of the modes are nonphysical since those correspond to coordinate transformations. There are only two scalar, two vector and two tensor modes which are physical. The tensor and vector modes do not couple to curvature perturbations at linear order. In an expanding universe, vector modes decay, and this yields an additional reason to neglect them. However, in a contracting universe they grow, and thus neglecting them is only justifiable at linear order. Only scalar modes at linear order couple with matter inhomogeneities.

The background FLRW metric and the scalar metric perturbations are given by the Eqs. (1.0.3) and (1.0.4) respectively. The physical scalar modes can be formed by taking linear combinations of these perturbations as follows:

$$\Phi = \phi + \frac{1}{a}\left(a(\mathcal{B} - \mathcal{E}')\right)', \quad \Psi = \psi - \frac{a'}{a}(\mathcal{B} - \mathcal{E}').$$

For simple forms of matter such as scalar fields and perfect fluids there is no anisotropic stress at linear order and this leads to $\Psi = \Phi$, leaving one degree of freedom which describes curvature fluctuations. In Newtonian gauge (defined by $\mathcal{B} = \mathcal{E} = 0$), the metric including the scalar fluctuation mode $\Phi(t, \vec{x})$ can be written as

$$ds^2 = a^2(\eta)\left[-(1 + 2\Phi(\eta, \vec{x}))d\eta^2 + (1 - 2\Phi(\eta, \vec{x}))\gamma_{ij}dx^i dx^j\right], \tag{2.4.1}$$

where $a(\eta)$ is the cosmological scale factor and we have made use of conformal time η related to physical time t via $a(\eta)d\eta = dt$.

The fluctuations of spatial curvature are described in terms of the variables ζ and $\delta\mathcal{R}$ defined by

$$\zeta = \frac{2}{3}\frac{\Phi + \mathcal{H}^{-1}\Phi'}{1 + w} + \Phi, \quad \delta\mathcal{R} = \zeta + \frac{2}{3}\frac{\vec{\nabla}^2\Phi}{\kappa(\bar{\mu} + \bar{p})a^2}, \tag{2.4.2}$$

where $\bar{\mu}$ and $\bar{p}$ are background values of energy density and pressure. For adiabatic perturbations, ζ and $\delta\mathcal{R}$ are conserved outside the horizon.

By expanding the full action (Einstein-Hilbert action plus action for matter) to second order about the classical background cosmology, one can obtain the action for cosmological perturbations which yields the linearized equations of motion. This action can be canonically quantized. By the logic of the previous paragraph, the resulting action can be written in terms of a single dynamical variable which, in addition, can be canonically normalized. The formalism is described in terms of the canonical fluctuation variable v, derived by Mukhanov [39] and Sasaki [40]. For a scalar field matter v is simply related to the variable $\delta\mathcal{R}$:

$$v = z\delta\mathcal{R}, \tag{2.4.3}$$

where $z(\eta) = a\varphi_0^{'}/\mathcal{H} = a^2\mathcal{H}^{-1}\sqrt{\bar{\mu}+\bar{p}}$ is a function of the cosmological background. If the equation of state of matter is time-independent, then $z(\eta)$ is proportional to $a(\eta)$. For a scalar field matter v takes the following form:

$$v \, = \, a\big[\delta\varphi + \frac{\varphi_0^{'}}{\mathcal{H}}\Phi\big] \,, \tag{2.4.4}$$

where $\delta\varphi$ is the fluctuation of the matter field. The action for v takes the following form

$$S^{(2)} \, = \, \frac{1}{2}\int d^4x\big[v'^2 - (\vec{\nabla}v)^2 + \frac{z''}{z}v^2\big] \,. \tag{2.4.5}$$

The equation of motion which follows from this action (2.4.5) is (in momentum space)

$$v_k^{''} + k^2 v_k - \frac{z^{''}}{z}v_k \, = \, 0 \,, \tag{2.4.6}$$

where v_k is the k'th Fourier mode of v. We see that each Fourier mode satisfies a harmonic oscillator equation of motion with a time-dependent mass, the time dependence being given by the background cosmology. The mass term in the above equation is in general given by the Hubble expansion rate. Thus, we see that the Hubble radius plays a key role in the evolution of fluctuations. The mode k whose wavelength at time t is equal to the Hubble radius will be denoted by $k_H(t)$. it follows that on small length scales, i.e. for $k > k_H$, the solutions for v_k are constant amplitude oscillations . These oscillations freeze out at Hubble radius crossing, i.e. when $k = k_H$. On longer scales ($k \ll k_H$), there is a mode of v_k which scales as z. This mode is the dominant one in an expanding universe, but not in a contracting one.

Canonical quantization of the action for cosmological perturbations corresponds to imposing canonical commutation relations for each Fourier mode v_k. If we impose vacuum initial conditions at some time η_i, this implies

$$v_k(\eta_i) = \frac{1}{\sqrt{2k}}, \quad v_k^{'}(\eta_i) = \frac{\sqrt{k}}{\sqrt{2}} \tag{2.4.7}$$

Before applying the above formalism to initial vacuum perturbations in the matter bounce scenario, we will review how a scale-invariant spectrum emerges in inflationary cosmology. The definition of scale-invariance of the curvature power spectrum is

$$P_{\delta\mathcal{R}} \equiv \frac{1}{2\pi^2} z^{-2} k^3 |v_k|^2 \sim k^{n_s - 1} \sim \text{const}\,, \tag{2.4.8}$$

i.e. $n_s = 1$, where n_s is called the spectral index of scalar metric fluctuations. Note that the vacuum spectrum, i.e. the spectrum obtained with the values (2.4.7) is not scale invariant. Rather, it is blue with $n_s = 3$ (more power on short wavelengths).

In inflationary cosmology the Hubble radius is constant while the wavelength of comoving modes expands exponentially. Thus, provided the period of inflation is sufficiently long, all modes which are currently observed originated on sub-Hubble scales during inflation. A mode with wavenumber k exits the Hubble radius at a time $\eta_H(k)$ given by

$$a^{-1}(\eta_H(k))k \,=\, H\,. \tag{2.4.9}$$

In inflationary cosmology, any classical fluctuations existing at the beginning of the period of inflation are red-shifted and leave behind a vacuum state. Thus, it makes sense to start fluctuations on sub-Hubble scales in their vacuum. The fluctuations will oscillate with constant amplitude while on sub-Hubble scales.

However, on super-Hubble scales v_k will increase as $a(\eta)$. Since long wavelengths spend more time outside the Hubble radius they experience a bigger growth. Thus, the final spectrum will be less blue. When measured at a fixed final time η, the increase in the amplitude of v_k will be

$$v_k(\eta) \approx \frac{v_k(\eta_H(k))a(\eta)}{a(\eta_H(k))}$$

,which from (2.4.9) is proportional to $k^{-3/2}$. Hence, the slope of the power spectrum changes by $\delta n_s = -2$, converting the vacuum spectrum into a scale-invariant one.

We will now see that a similar mechanism converts a vacuum spectrum into a scale-invariant one in the Matter Bounce scenario.

2.4.3 Vacuum fluctuations in the contracting phase

In this section we consider the modes which originate as quantum vacuum fluctuations on sub-Hubble scales at early times and which cross the Hubble radius during the phase of matter dominated contraction. The vacuum spectrum has index $n_s = 3$. To convert it into a scale-invariant spectrum we require a mechanism which boosts long wavelengths compared to short wavelengths. Since v_k grows on super-Hubble scales, such a mechanism naturally arises in a contracting universe. As we see below, in a matter-dominated phase of contraction the boost factor is exactly the right one to turn the vacuum spectrum into a scale-invariant one [11]-[13].

From the equation of motion (2.4.6) it follows that v_k will oscillate with constant amplitude proportional to $1/\sqrt{k}$ until the scale exits the Hubble radius at conformal time $\eta_H(k)$ given by

$$\eta_H(k) \sim k^{-1}. \tag{2.4.10}$$

In a matter-dominated phase we have $a(\eta) \sim \eta^2$. Hence, it follows by solving (2.4.6) outside the horizon, where one can neglect the term k^2 compared with z''/z, that the dominant mode of v_k scales as η^{-1}. So the amplitude of v_k at a later time η when the modes of interest are outside the Hubble radius is given by

$$|v_k(\eta)|^2 \approx |v_k(\eta_H(k))|^2 \left(\frac{\eta_H(k)}{\eta}\right)^2 \approx k^{-3}\eta^{-2}. \tag{2.4.11}$$

Thus, the power spectrum of curvature fluctuations

$$P_{\delta\mathcal{R}}(k,\eta) \sim k^3 |v_k(\eta)|^2 a^{-2}(\eta)$$

at time η acquire a spectral index $n_s = 1$.

Thus, we have shown that vacuum perturbations which exit the Hubble radius during the matter-dominated phase of contraction acquire a scale-invariant spectrum. After the transition to a radiation-dominated phase of contraction (if such a phase exists) all modes are boosted by the same factor, and hence the scale-invariance of the spectrum persists at least until the bounce phase begins.

2.4.4 Matching conditions

We have shown that before the bounce the spectrum of the curvature perturbation, that crossed the horizon at matter dominated contracting phase is scale-invariant. However, we need to determine the curvature fluctuations in the expanding phase, i.e. after the bounce. Since the curvature fluctuations (in the case of adiabatic perturbations) are constant in time on super-Hubble scales in an expanding universe, it is sufficient to compute the spectrum immediately after the bounce at the onset of the period of *standard cosmological* evolution.

The transfer of curvature fluctuations through a non-singular bouncing phase is a non-trivial topic due to the appearance of new physics. In most of the models with only adiabatic fluctuations it is found that the power specrtum of the v do not change on length scales longer than the time duration of bounce [41]-[44]. In the presence of entropy fluctuations changes are to be expected [45].

There are two ways of following fluctuations through the bouncing phase. The first is the explicit numerical integration. In any given realization of the matter bounce we know what the equations for the fluctuations are, and we can integrate them. However, to obtain a good understanding of the results, it is important to have an analytical method. This method uses matching conditions at the transition surface from one phase to the next. Matching conditions were introduced by Israel [46] in the context of matching two solutions of Einstein's equations across a time-like surface. Those conditions were then generalized to the problem of matching across a space-like boundary hypersurface in cosmology [47].

We can devide the total spacetime into three regions, namely the contracting phase, bouncing phase and expanding phase and $\eta = -\eta_1$ and $\eta = \eta_1$ are the transition surfaces that seperate these regions. $\eta = 0$ is taken as the bounce point. The bounce phase is directed by the new physics or dominated by the unusual matter that causes the bounce. While matching the solutions in contracting phase and expanding phase we neglect the details of this transition and match our solution at $\eta = -\eta_1$ in the contracting universe to the solution at $\eta = +\eta_1$ in the expanding univerese. In other words we suppose that the slice of space-time "squeezed" between $\eta = -\eta_1$ and $\eta = \eta_1$ is so thin compared to the scales we are interested in, that it can be replaced by a spacelike hypersurface. So the thin shell formalism and the Israel junction conditions [46] for the surface layers on the $\eta = \pm\eta_1$ can be applied to match the spacetime before and after the bounce.

The matching conditions state that at the boundary between one phase and another the induced metric on the boundary hypersurface is continuous and the extrinsic curvature is either continuous or (in presence of surface layers) jumps according to the surface energy of the hypersurface. Applied to the case of cosmological perturbations, one must first make sure that the background satisfies the matching conditions [45]. In an FLRW universe, the continuity of induced metric h_{ab} implies scal factor takes same value on hypersurfaces, i.e. $a(-\eta_1) = a(\eta_1)$. On the other hand the extrinsic curvature $K_{ab} = \frac{1}{2}\left(h_a^{\ c}\nabla_c n_b + h_b^{\ c}\nabla_c n_a\right)$, where n_a is the normal to the hypersurface, is proportional to $\mathcal{H}$. Since $\mathcal{H}$ changes sign in transition from contracting to expanding phase, K_{ab} is discontinuous if we simply glue the contracting phase at $\eta = -\eta_1$ with the expanding phase at $\eta = \eta_1$. This discontinuity is interepreted as the surface stress tensor of the layer.

Once the matching of the background is successfully achieved, the matching conditions are imposed on the metric fluctuation variable Φ and the extrinsic curvature fluctuation on the transition surfaces. If the matching surface is a constant energy density hypersurface, then the continuity of the extrinsic curvature implies the continuity of v [45].

2.5 Validity of linear perturbation theory in a bouncing universe

Study of evolution of cosmological fluctuations in expanding universe is generally based on the perturbative techniques. This approach is supported by the observed small inhomogeneities in the microwave sky. However, in a bouncing spacetime the smallness of the fluctuations should not be taken for granted. For singular models, such as the original ekpyrotic scenario, cosmological scalar perturbations diverge as $a \to 0$ [15], casting doubt on the perturbative treatment. In such scenarios, it is understood that at some point the classical 4D theory breaks down, and one has to resort to the full description in string theory. One might hope that the singularity and thus the divergence of perturbations is absent. However, merely going to the higher dimensional setting of the ekpyrotic universe [48], and properly incorporating perturbations of the 5D metric, does not necessarily resolve this problem, as shown in [49]. A similar problem is present for vector perturbations in a contracting universe [16].

For a nonsingular bounce, perturbations are not necessarily divergent, but still strongly growing in the contracting phase. It is found that linear perturbation theory fails in some gauges, such as the longitudinal [50], [15] and comoving ones [51], while it remains valid in others, for example, in the uniform χ field gauge [22]. As shown in [18], [19], the curvature perturbation and anisotropy grow rapidly during contraction, questioning again the viability of perturbation theory.

One of the first occurrences where this problem surfaced was the pre-big bang scenario [52]. In [53] it was argued that one could go to a gauge where scalar perturbations are at most logarithmically growing, but it should be noted that the usual gauge invariant variables still obey a power law growth.

The question regarding the validity of the perturbative linear regime during the bounce is particularly evident in the Newtonian gauge due to the rapidly growing mode of Bardeen potential Φ [26] which, in this gauge, is identiacal to the metric perturbation function ϕ defined in(1.0.4). Such a growing contribution is absent in other gauges [22].

A generalization of these ideas to a large class of models was attempted in

[23], where a set of conditions for linearity is obtained that allows the perturbative expansion to be valid. The spectrum of modes considered in [23] became frozen during a matter dominated contracting phase, but the actual bounce was kept general. The conditions arise by demanding that the metric perturbation components are small than the corresponding components of background metric and the terms in perturbed Einstein equation are small compared with the corresponding terms of background Einstein equations. The first condition is mathematically stated in the Eq. (1.0.5) and the second condition leads to:

$$\left|\frac{a\delta\theta}{\mathcal{H}}\right| \ll 1, \left|\frac{\left(D^2+3K\right)\psi}{K}\right| \ll 1, \left|\frac{D^2\delta\sigma}{a\mathcal{H}}\right| \ll 1, \left|\frac{D^2\phi}{\mathcal{H}'-\mathcal{H}^2}\right| \ll 1, \tag{2.5.1}$$

where $\delta\theta$ and $\delta\sigma$ are given by

$$\begin{aligned} a\delta\theta &= -D^2\left(\mathcal{E}'-\mathcal{B}\right)+3\left(\psi'+\mathcal{H}\phi\right) \\ \delta\sigma &= -a\left(\mathcal{E}'-\mathcal{B}\right) \end{aligned} \tag{2.5.2}$$

and D_i is covariant derivative operator compatible with spatial metric γ_{ij}. If the background spacetime is spatially flat, i.e. $K=0$ then the second condition of (2.5.1) will be replaced by:

$$\left|\frac{D^2\psi}{\mathcal{H}^2}\right| \ll 1, \quad \left|\frac{D^2\psi}{2\mathcal{H}'+\mathcal{H}^2}\right| \ll 1. \tag{2.5.3}$$

Above conditions involve non-physical modes and hence these are not gauge invariant. Since Weyl tensor vanishes on an FLRW spacetime, its perturbations are gauge invariant (see Sec. (3.2)). By comparing the Weyl tensor components $C_{i0}^{\ \ j0}$ with the components of Ricci tensor $R_i^{\ j}$ one obtains a gauge invariant condition:

$$\left|\frac{D^2(\Phi+\Psi)}{2K+\mathcal{H}'+2\mathcal{H}^2}\right| \ll 1, \tag{2.5.4}$$

that is also derivable from (2.5.1). However, this condition is a necessary but not sufficient condition for the linearity of the perturbations [23]. Hence one must consider particular gauges in order to check whether the linearity conditions are satisfied or not. In the Newtonian gauge $\phi = \Phi$, therefore condition (1.0.5) is not satisfied and linear perturbation theory breaks down in this gauge because Φ grows larger than 1 in collapsing phase near bounce. On the other hand in uniform curvature gauge one has

$$\phi = \frac{3(1+w)}{2}\zeta$$

for a single fluid dominated universe. Therefore, in this gauge ϕ follows the evolution of ζ instead of Φ. ζ grows in the contracting phase until it attains, near the bounce, an amplitude approximately equal to the constant mode of Φ. In the expanding phase, ζ also has a decaying mode, but in this case this mode is always smaller than the constant one. Hence ϕ remains small in this gauge. The other conditions are also satisfied in this gauge. Similarly it can be shown that exactly at the point when bounce occurs it is neither Newtonian,nor uniform curvature gauge but the synchronous gauge in which the conditions (1.0.5) and (2.5.1) are satisfied.

2.6 Conclusion

Bounce with a matter dominated contracting phase is an alternative to cosmological inflation as a mechanism for generating an almost scale invariant spectrum of cosmological fluctuations. As in the case of inflationary cosmology, the spectrum has a slight red tilt since smaller scales exit the Hubble radius when the radiation component of matter is more important and the vacuum slope of the spectrum (which corresponds to $n_s = 3$) is rearing its head.

However, implementing a bounce give rise to new challenges that need to be addressed. As explained earlier the matter content of the universe should be such that violates certain energy conditions if the bounce is to be happened in the realm of GR. Otherwise bounce can occur in the framework of modified gravity or quantum gravity. Another serious problem is the growth of anisotropies in the contracting phase. The shear term in the Raychoudhury equation is attractive and grows as a^{-6}, which can takeover any repulsive term and thus spoil the bounce; however, this problem is easily solved by assuming an ekpyrotic contraction phase before the bounce with an equation of state parameter $w \gg 1$.

The evolution of cosmological fluctuations are analyzed using linear pertutbation theory. The growing mode of perturbations raise doubts whether the perturbations remain linear near bounce. Results from conventional coordinate based perturbation theory shows that the condition for linearity of perturbations depends crucially on the choice of gauge. Hence it is necessary to formulate the linearity conditions in gauge invariant way.

Chapter 3

Covariant Perturbation Theory

3.1 Introduction

The standard model of cosmology with a handful of parameters, such as the expansion rate, the temperature of the present microwave background radiation, the density of visible matter, dark matter and dark vacuum energy successfully describes the average expansion of the universe at large scales according to Einstein's general theory of relativity. It also explains the evolution of the universe from a hot and dense initial state dominated by radiation to a cool and low density state dominated by non-relativistic matter and apparently also by vacuum energy at present.

But a homogeneous model is incapable of explaining the complexity of the actual distribution of matter and energy in our observed universe where stars and galaxies form clusters and superclusters of galaxies across a wide range of scales. For this, we need spatial inhomogeneity and anisotropy. But there are only some exact solutions of Einstein's equations known that incorporate spatially inhomogeneous and anisotropic matter and hence, geometry. Also the extreme degree of isotropy observed in the CMBR (temperature fluctuations are less than 1 part in 10^{-5} on all angular scales) indicates that the large scale structures evolve somehow from an initially small density perturbations in FLRW spacetime. This perturbations may be primordial (fed in the initial conditions of the classical universe at plank time) or spontaneous (created by some physical process such as quantum fluctuations of Bunch-Davis vacuum). In any case, they evolve linearly for a long while. Therefore, a major tusk to understand the structure formation is to develop a relativistic theory of linear perturbations in expanding FLRW universe.

Classical relativistic theory of cosmological perturbations, developed by Lifshitz and Khalatnikov [54], suffered from gauge ambiguities which result in the presence of nonphysical modes in the evolution of perturbations [55]. In an attempt to resolve this problem, Hawking used curvature variables rather than perturbations of metric components [24]. But the analysis of density perturbation was based on the gauge dependent $\delta\mu/\mu$. Later, a completely gauge invariant formulation of perturbations was presented by Bardeen [26]. Unfortunately, most of Bardeen's variables do not have simple geometrical meaning. In standard approaches of perturbation theory, perturbation is defined by usual splitting of any tensorial quantity T: $T = \bar{T} + \epsilon\delta T$ where $\epsilon \ll 1$ is a small number and δT is considered to be a field in a background spacetime (the backgraound metric $\bar{g}$ is defined in a similar way, namely $g = \bar{g} + \delta g$). However, such an approximation to linear order in ϵ reduces the equivalence class of δT to $\delta T \sim \delta T + L_\xi \bar{T}$, so to linear order, δT is not a generally covariant object [56]. Bardeen's gauge invariant variables are thus constructed as gauge invariant linear combinations of gauge dependent variables, knowing their transformation rules under gauge transformations. Geometrical and physical meaning of such variables are often obscure unless a particular hypersurface codition (time gauge) is chosen.

A completely gauge independent and covariant formulation of cosmological perturbation was developed by Ellis, Bruni and others [25],[57]-[60],[27]. According to the Stewart and Walker lemma, the simplest example of gauge invariant variables are scalars that are constant in the background universe or tensors that vanish there. In both cases gauge changes are irrelevant because they all define the same perturbation. The only other possibility is a tensor that can be written as linear combination of products of the Kronecker deltas with constant coefficients. The same general criteria also apply to second order perturbations, but this time the Stewart and Walker requirements must be satisfied by the first-order variables [61].

Most cosmological applications deal with FLRW models. One would therefore like to know which quantities satisfy this criterion on Friedmannian backgrounds. Since the only invariantly defined constant is the cosmological constant and because constant products of the Kronecker deltas do not occur naturally, the only remaining option is to look for quantities that vanish in FLRW environments. Given the symmetries of the Friedmann models, any variable that describes spatial inhomogeneity or anisotropy must vanish there and therefore its linear perturbation should remain invariant under gauge transformations.

3.2 Perturbations and gauge dependence

3.2.1 Cosmological perturbations

Physics in general relativity is covariant under general coordinate transformation. So a generally covariant theory is defined on a differential manifold with no preferred coordinate chart. But in a particular class of problems we use some suitable coordinates. For example, metric $\bar{g}_{\mu\nu}$ of the FLRW manifold $\bar{S}$ takes the simple homogeneous and isotropic form

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-Kr^2} + r^2 d\Omega^2\right) \tag{3.2.1}$$

in the rest frame of comoving observers. The respective values of K for closed, flat and open universes are 1, 0 and -1. Matter content of the FLRW universe is described by an homogeneous and isotropic perfect fluid energy-momentum tensor,

$$\bar{T}_{\mu\nu} = (\bar{\mu} + \bar{p})\bar{u}_\mu \bar{u}_\nu + \bar{p}\bar{g}_{\mu\nu}, \tag{3.2.2}$$

where $\bar{\mu}(t)$ and $\bar{p}(t)$ are energy density and pressure as observed by the comoving observer with four velocity components $\bar{u}^\mu = (1, 0, 0, 0)$.

To define the perturbations let us consider our physical manifold S with metric $g_{\mu\nu}$ and energy momentum tensor $T_{\mu\nu}$ and let the background is the FLRW manifold $\bar{S}$. The perturbations are defined as the difference between the corresponding quantities and their background values:

$$\delta g_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}, \quad \delta T_{\mu\nu} = T_{\mu\nu} - \bar{T}_{\mu\nu}. \tag{3.2.3}$$

The perturbations $\delta g_{\mu\nu}$ and $\delta T_{\mu\nu}$ are "small" in some suitable sense. The background metric must be a solution of the Einstein equation (zeroth order equation of motion) with the background energy momentum equation as the source term. Then the $g_{\mu\nu}$ and $T_{\mu\nu}$ are substituted in the Einstein equation,

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \quad \kappa = 8\pi G. \tag{3.2.4}$$

Subtracting the zeroth order terms and neglecting higher order terms one obtains the linear equations of motion for the perturbations.

However, (3.2.3) is meaningful only if we specify a correspondence between S and $\bar{S}$. Since S and $\bar{S}$ are two different manifolds, we can not subtract a tensor defined on $\bar{S}$ from a tensor defined on S. That requires a point-wise identification map which tells which point in S is obtained for a given point in $\bar{S}$. If we have such a map, subtracting tensors of two manifolds is possible. The choice of a particular map from the background spacetime $\bar{S}$ to the physical spacetime S can be referred to as gauge ambiguities. In general, the choice of such a map is completely arbitrary, although some choices may be suitable for particular purposes.

To express the gauge transformation in terms of coordinates, consider an infinitesimal spacetime coordinate transformation

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x), \tag{3.2.5}$$

with ϵ^μ small in the same sense that $\delta g_{\mu\nu}$, $\delta T_{\mu\nu}$ are small. Under this transformation, the metric tensor will be transformed to

$$g'_{\mu\nu}(x') = g_{\lambda\rho}(x)\frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial x^\rho}{\partial x'^\nu} \tag{3.2.6}$$

In contrast with such transformations, which affect the coordinates and unperturbed fields as well as the perturbations to the fields, gauge transformations affect only the field perturbations. After making the coordinate transformation (3.2.5), we relabel coordinates by dropping the prime on the coordinate argument, and we attribute the whole change in $g_{\mu\nu}(x)$ to a change in the perturbation $\delta g_{\mu\nu}(x)$. The field equations should thus be invariant under the gauge transformation $\delta g_{\mu\nu} \to \delta g_{\mu\nu} + \delta_\epsilon g_{\mu\nu}$,

$$\delta_\epsilon g_{\mu\nu}(x) = g'_{\mu\nu}(x) - g_{\mu\nu}(x) \tag{3.2.7}$$

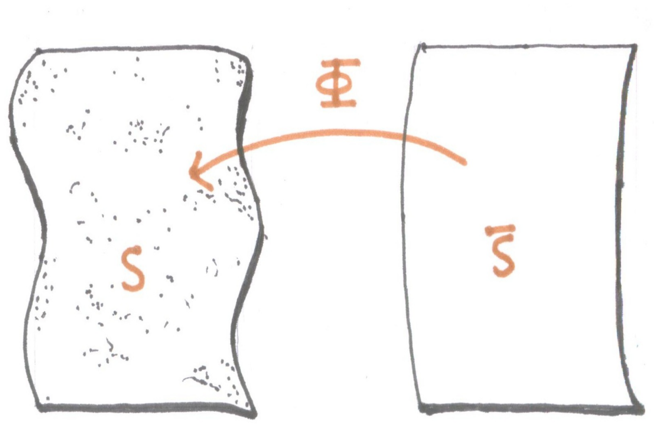

Figure 3.1: Mapping between background and physical spacetimes

with the original FLRW metric $\bar{g}_{\mu\nu}$ left unchanged. Up to first order,

$$\delta_\epsilon g_{\mu\nu} = -\mathcal{L}_\epsilon \bar{g}_{\mu\nu}. \tag{3.2.8}$$

In the above analysis we have not use the fact that $g_{\mu\nu}$ is the metric tensor. Hence the equation (3.2.8) can be applied to the perturbation of any tensor $\mathcal{T}_{\mu\nu\ldots}$:

$$\delta_\epsilon \mathcal{T}_{\mu\nu\ldots} = -\mathcal{L}_\epsilon \bar{\mathcal{T}}_{\mu\nu\ldots}. \tag{3.2.9}$$

3.2.2 Gauge specification

The obvious way of investigating the perturbations in general relativity is often misleading. Following the procedure sketched in the previous section, the energy density perturbation is

$$\delta\mu = \mu - \bar{\mu} \tag{3.2.10}$$

This approach suggests that the background model $\bar{S}$ is related to the physical model S in some special way, which is not quite true. Although we can always perturb away from a given background spacetime, recovering the smooth metric from a given perturbed one is not a uniquely defined process. This is a problem because it is always possible to choose an alternative background and therefore arrive at different perturbation values.

Definition of $\bar{S}$ in S is equivalent to define a map Φ from $\bar{S}$ to S that can pull back any tensor quantity defined in S to its image in $\bar{S}$. For example the energy density μ is a scalar field on the physical manifold S. At any point $x \in S$ its value is $\mu(x)$. Let $x = \Phi(\bar{x})$ is the image of $\bar{x} \in \bar{S}$ under Φ. Then $\Phi^*\mu$, the pull back of μ, is defined by,

$$\Phi^*\mu(\bar{x}) = \mu(\Phi(\bar{x})). \tag{3.2.11}$$

The $\Phi^*\mu$ is the image of μ under Φ. Since $\Phi^*\mu$ and $\bar{\mu}$ both are scalar fields defined on the background manifold, we can define the energy density perturbation as the difference between these quantities:

$$\delta\mu = \Phi^*\mu - \bar{\mu}. \tag{3.2.12}$$

This proccess can be generalized to any tensor field $\mathcal{T}_{ab...}$ on S. Perturbation of $\mathcal{T}_{ab...}$ is then

$$\delta\mathcal{T}_{ab...} = \Phi^*\mathcal{T}_{ab...} - \bar{\mathcal{T}}_{ab...}. \tag{3.2.13}$$

Perturbation defined in this way is entirely dependent on the map Φ. So far the map is completely arbitrary; however, in order that the image of smooth tensor fields be also smooth in $\bar{S}$ the map Φ should be a local diffeomorphism. Thus the choice of a map is equivalent to choosing a coordinate chart in S. To see this consider two such maps Φ_1 and Φ_2 which map two different points $\bar{x}_1 \in \bar{S}$ and $\bar{x}_2 \in \bar{S}$ respectively to $x \in S$:

$$x = \Phi_1(\bar{x}_1) = \Phi_2(\bar{x}_2). \tag{3.2.14}$$

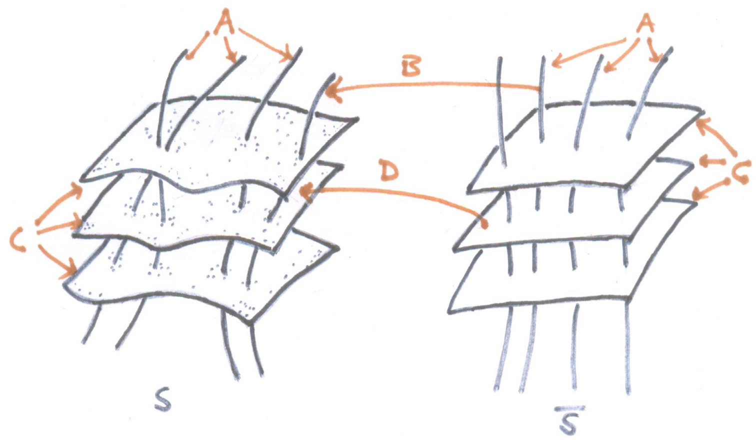

Figure 3.2: Four aspects of the map Φ

The map Φ has four aspects: (A) choice of families of world lins in $\bar{S}$ and S, (B) correspondence between world lines between two spacetimes, (C) choice of spacelike surfaces in $\bar{S}$ and S, (D) correspondence between particular surfaces two spacetimes.

The $\bar{x}_1$ lies in an open set $\bar{\mathcal{O}}_1 \subset \bar{S}$ and has a coordinate chart $\bar{\psi}_1 : \bar{\mathcal{O}}_1 \to \bar{\mathcal{U}}_1$, where $\bar{\mathcal{U}}_1$ is an open subset of $\mathbf{R}^4$. Now $x \in \mathcal{O}_1 \subset S$ where $\mathcal{O}_1 = \Phi_1 \left[\bar{\mathcal{O}}_1\right]$. So there is a natural coordinate chart $\psi_1 : \mathcal{O}_1 \to \bar{\mathcal{U}}_1$ defined by $\psi_1 = \bar{\psi}_1 \cdot \Phi_1^{-1}$. Similarly $\bar{x}_2$ has a coordinate chart $\bar{\psi}_2 : \bar{\mathcal{O}}_2 \to \bar{\mathcal{U}}_2$, that induces another coordinate chart $\psi_2 : \mathcal{O}_2 \to \bar{\mathcal{U}}_2$ in S, defined by $\psi_2 = \bar{\psi}_2 \cdot \Phi_2^{-1}$.

Thus the actual situation is that what we are given to study is real, physical Universe s, and we define the perturbed quantities and their evolution by the way we specify a mapping Φ of the background spacetime $\bar{S}$ into S. The determination of the best way to make this correspondence can be called "Fitting Problem" in cosmology [62]; there are various way to do this, so the answer is not unique. Once we completely specify the map Φ, there is no arbitrariness in $\delta\mathcal{T}_{ab\ldots}$; insofar as Φ is unspecified, that quantity is arbitrary. It is convenient to think of this map as having four aspects:

(A) We define a family of world lines $\bar{\gamma}$ in $\bar{S}$ and a corresponding family of world lines γ in S. This determines the world lines in each spacetime along which we will compare the evolution of density perturbations. If the background spacetime is FLRW one, there is an obvious choice for $\bar{\gamma}$, namely the world lines of comoving observers. This can also be a choice in S, but other choices such as flow lines normal to the constant energy density surface or constant expansion surface may be convenient.

(B) We define a specific correspondence between individual world lines $\bar{\gamma}_i$ in $\bar{S}$ and individual world lines γ_i in S. This specifies which specific observers observations we shall compare with which. In the case where $\bar{S}$ is an FLRW spacetime, this choice does not matter because of the spatial homogeneity of $\bar{S}$.

(C) We define a family of spacelike surfaces $\bar{\Sigma}$ in $\bar{S}$ and a corresponding family Σ in S; this are the "time surfaces" in each spacetime. There is an obvious choice in FLRW spacetime, namely the surfaces of homogeneity ($\bar{t}$ = constant). Since in a homogeneous spacetime the energy density $\bar{\mu}$ is function of only $\bar{t}$, the image of this surfaces in S are the idealized surfaces of constant energy density ($\bar{\mu}$ = constant) we use to define the density perturbations

(D) We define a correspondence between the particular spacelike surfaces $\bar{\Sigma}_i$ in the family $\bar{\Sigma}$ in $\bar{S}$ and the particular spacelike surfaces Σ_i in the family Σ in S. This specifies which point $\bar{x}$ in $\bar{S}$ corresponds to which point x in S and complete the specification of the map.

We understand that by (C) we choose constant time surfaces in the physical manifold S and by (D) we assign values of time to each surfaces via $t = \bar{t}$.

3.2.3 Arbitrariness of $\delta\mu$

It is clear from the previous section that the definition of $\delta\mu$ depends crucially on the choice of the map between the background manifold and the physical manifold. We can always "remove" a real density perturbation or "produce" a fictitous density perturbation merely by fixing the map. For example we can set the $\bar{t} = t$ and make the $\delta\mu$ spatially uniform by choosing the Σ to be family of surfaces of constant μ. Since Σ_i are also $\bar{\mu} =$ constant, we get $\delta\mu = \delta\mu(t)$. Now, given a choice of the family of surfaces Σ in S, we can assign any value to $\delta\mu$ through the gauge freedom (D), by changing the assignation of values $\bar{\mu}$ to the surfaces Σ. In particular choosing $\bar{\mu} = \mu$ on any surface Σ_i in the family Σ, we can set $\delta\mu = 0$ on Σ_i.

How this propagates along the chosen timeline then depends on the gauge choice and the fluid equation of state. We can choose a gauge where $\delta\mu$ vanishes at every point of γ by assigning the mapping of densities to satisfy the condition $\bar{\mu} = \mu$ on γ. This is seen by Bardee's formalism [26].

Once the mapping Φ is fixed, we can say, in general, that the background energy density $\bar{\mu} = \bar{\mu}(t)$ is constant on any Σ_i, but the $\mu = \mu(t, \vec{x})$ vary on Σ_i, $\vec{x}$ being the coordinates on Σ_i, fixed by (B):

$$\mu(t, \vec{x}) = \bar{\mu}(t) + \delta\mu(t, \vec{x}) = \bar{\mu}(t) \left(1 + \delta(t, \vec{x})\right), \quad \delta = \frac{\delta\mu}{\bar{\mu}}. \tag{3.2.15}$$

Using the freedom (C) we choose a different family of time surfaces Σ, which is equivalent to choose a new time variable t'. Assuming the difference between t and t' is of the same order as δ, the coordinate transformation can be written, following (3.2.5)

$$t' = t + \epsilon^0(t, \vec{x}), \quad \vec{x}' = \vec{x}. \tag{3.2.16}$$

Then since μ is a scalar,

$$\mu'(t', \vec{x}') = \mu(t, \vec{x}) = \mu(t' - \epsilon^0, \vec{x}') = \mu(t'\vec{x}') - \epsilon^0 \partial_t \mu. \tag{3.2.17}$$

Removing primes on coordinates, using (3.2.15), and keeping upto first order,

$$\mu'(t, \vec{x}) = \bar{\mu}(t) \left(1 + \delta + 3(1+w)H\epsilon^0\right) = \bar{\mu}(t)(1 + \delta'), \tag{3.2.18}$$

where we have used the energy conservation equation

$$\dot{\mu} + 3H(\mu + p) = 0 \tag{3.2.19}$$

and the fluid equation of state $p = w\mu$. If we choose the arbitrary function ϵ^0 as

$$\epsilon^0 = -\frac{\delta}{3H(1+w)} \tag{3.2.20}$$

then $\delta' = 0$, i.e. the energy density perturbation vanishes along γ in the new gauge. This gauge is called zero density perturbation gauge. This choice of cource does not mean that the spatial variation of energy density is absent. The spatial variation of μ is coded in the fact that the proper time seperation between two surfaces, measured along the normal to the surfaces varies spatially.

3.2.4 Geometric description of perturbation

In previous sections we have outlined the concept of the choice of gauge as a mapping between the background spacetime and the physical spacetime. The choice of the mapping is equivalent to the choice of coordinate chart. It is shown in terms of the coordinates that the effect of gauge transformation on any tensor can be represented as lie derivative of the background. Here we present a brief coordinate independent description of perturbation following [63], [61].

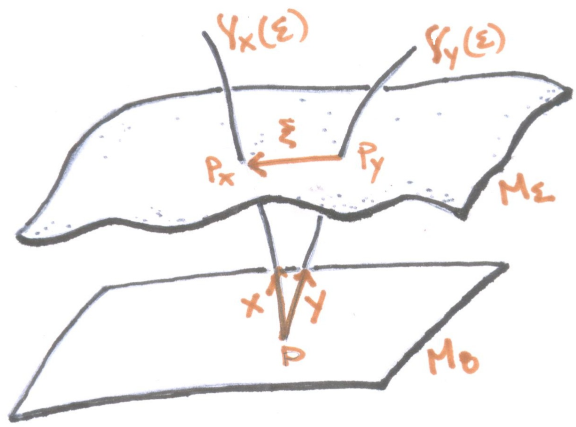

Figure 3.3: Description of Φ as vector fields in a 5D embedding manifold

The point identification map Φ can be described by vector fields X or Y in the 5-dimensional manifold N, in which background spacetime M_0 and the physical spacetime M_ϵ are embedded. The point $p \in M_0$ is identified with $p_X \in M_\epsilon$ and $p_Y \in M_\epsilon$ by two different choices X and Y respectively.

Let us consider a one parameter family of 4-manifolds M_ϵ embedded in a 5-dimensional manifold N. Each M_ϵ represents corresponding spacetime. We consider M_0 to be the background spacetime $\bar{S}$ and M_ϵ, for small value of ϵ, to be the perturbed spacetime S. Then the point identification map

$$\Phi_\epsilon : M_0 \to M_\epsilon \tag{3.2.21}$$

specifies which point in S is same to a point in $\bar{S}$.

Now consider a smooth, nowhere vanishing vector field X in N which is everywhere transverse (nowhere tangent) to the family $\{M_\epsilon\}$. Then the map Φ_ϵ, associated with X can be defined in the following way. The point $p_X \in M_\epsilon$ is the image of $p \in M_0$ under Φ_ϵ if p_X and p lie on the same integral curve γ of X. Introducing the coordinates x^A in N ($A = 0, ..., 4$) and parametrizing the curves by ϵ, we can set

$$X^A = \frac{dx^A}{d\epsilon} \tag{3.2.22}$$

Consider a tensor field $\bar{\mathcal{T}}$ in M_0. The corresponding field in M_ϵ is represented by $\mathcal{T}$. The pull back of $\bar{\mathcal{T}}$ by Φ_ϵ is expanded in order of ϵ as

$$\Phi_\epsilon^* \mathcal{T} = \bar{\mathcal{T}} + \epsilon \mathcal{L}_X \bar{\mathcal{T}} + \mathcal{O}(\epsilon^2). \tag{3.2.23}$$

The perturbation of $\mathcal{T}$, defined by (3.2.13) up to linear order is

$$\delta \mathcal{T} = \epsilon \mathcal{L}_X \bar{\mathcal{T}}. \tag{3.2.24}$$

The point identification map is now represented by X and dependence of $\delta\mathcal{T}$ on X in now explicit. Since X is an arbitrary vector field, we can choose any other vector field Y in N which is also nowhere vanishing on N and everywhere transverse on the family $\{M_\epsilon\}$. The perturbation, defined in terms of Y reads $\delta\mathcal{T} = \epsilon \mathcal{L}_Y \bar{\mathcal{T}}$. The change of perturbations, arising due to change of vector field is then the gauge transformation:

$$\delta_\xi \mathcal{T} = \epsilon \mathcal{L}_Y \bar{\mathcal{T}} - \epsilon \mathcal{L}_X \bar{\mathcal{T}} = -\mathcal{L}_\xi \bar{\mathcal{T}},$$

where $\xi = \epsilon(X - Y)$. Adopting the local coordinates $x^A = (x^\alpha, \epsilon)$, where x^α are coordinates on M_ϵ, we get $X^4 = Y^4 = 1$. Hence ξ is a vector field on each M_ϵ. Finally taking the limit $\epsilon \to 0$, we obtain on M_0:

$$\delta_{\bar{\xi}} \mathcal{T} = -\mathcal{L}_{\bar{\xi}} \bar{\mathcal{T}}, \tag{3.2.25}$$

which is equivalent to (3.2.9). Thus the effect of gauge transformation on tensor fields due to change in vector fields is same as that described by infinitesimal coordinate transformation. Given (3.2.25) (or (3.2.9)) we can now state the following lemma due to Stewart and Walker:

The perturbation $\delta\mathcal{T}$ of the background quantity $\bar{\mathcal{T}}$ is gauge invariant if one of the following conditions hold:

1. $\bar{\mathcal{T}}$ vanishes.
2. $\bar{\mathcal{T}}$ is a constant scalar.
3. $\bar{\mathcal{T}}$ is a constant linear combination of Kronecker Deltas.

3.3 Gauge invariant variables

3.3.1 Kinematics

In the context of cosmolgy the matter content is described by a fluid, whose non equilibrium state is described by the energy momentum tensor T_{ab}, particle flux N_a and the entropy flux S_a. The T_{ab} and N_a satisfy the conservation equations:

$$\nabla^b T_{ab} = 0, \quad \nabla^a N_a = 0 \tag{3.3.1}$$

and S_a obeys the second law of thermodynamics:

$$\nabla^a S_a \geq 0. \tag{3.3.2}$$

If the energy density is nonnegative, i.e. $T_{ab}n^a n^b \geq 0$ for any timelike observer with velocity n^a, then T_{ab} has a unique timelike eigenvector, u^E_a:

$$h^E_{ab} T^{bc} u^E_c = 0, \quad u^E_a u^{Ea} = -1 \tag{3.3.3}$$

where $h^E_{ab} = g_{ab} + u^E_a u^E_b$ is the projector tensor with respect to u^E_a. The u^E_a defines the world lines representing the observers, at rest with respect to the fluid volume element.

In equilibrium u_a^E, N_a and S_a are all parallel. Then there exists a unique family of comoving observers with 4-velocity u_a:

$$u_a = u_a^E = \frac{N_a}{\sqrt{-N_b N^b}} = \frac{S_a}{\sqrt{-S_b S^b}} \tag{3.3.4}$$

and the fluid can described as perfect one:

$$T_{ab} = (\mu + p)u_a u_b + p g_{ab}. \tag{3.3.5}$$

For imperfect fluid, the choice of "hydrodynamical velocity" is ambiguous. We assume that a preferred choice of u^a is made for the fluid describing the matter content of the physical universe. Note that the background spacetime is FLRW, so the 4-velocity of the comoving observers is unique.

It τ be the proper time measured along the world lines of the preferred (fundamental) observers then $u^a = (\partial/\partial\tau)^a$ and its components are

$$u^\alpha = \frac{dx^\alpha}{d\tau}. \tag{3.3.6}$$

At each point p of the spacetime we have a subspace H_p of the tangent space T_p. The projection tensor

$$h_{ab} = g_{ab} + u_a u_b \tag{3.3.7}$$

projects any vector to the H_p. If u_a is hypersurface orthogonal then h_{ab} is the metric of the 3 dimensional surface orthogonal to u_a.

Time derivative of any tensor $S^{a...}_{b...}$ along fluid flow lines is the covariant derivative along u^a:

$$\dot{S}^{a...}_{b...} = u^c \nabla_c S^{a...}_{b...}. \tag{3.3.8}$$

This represetnts the rate of change of the quantity with respect to the proper time of the fundamental observer. Similarly we can define the spatial derivative using h_{ab} as

$${}^{(3)}\nabla_a S^{b...}_{c...} = h_a^{\ d} h^b_{\ e} ... h_c^{\ f} ... \nabla_d S^{e...}_{f...}. \tag{3.3.9}$$

In the physical spacetime the world lines of fundamental observers are not geodesics. So there is a nonzero acceleration defined by

$$\nu^a = \dot{u}^a = u^b \nabla_b u^a. \tag{3.3.10}$$

From (3.3.6), it follows that u^a is a unit timelike vector,

$$u^a u_a = -1 \tag{3.3.11}$$

and hence

$$u^a \nu_a = 0. \tag{3.3.12}$$

Let us denote the family of fundamental world lines by $\gamma_s(\tau)$, such that for each $s \in \mathbb{R}$, γ_s is a worldline, parametrized by the affine parameter τ. Then $\eta^a = (\partial/\partial s)^a$ represents the deviation between nearby world lines. It can be chosen to be prependicular to u^a. Now the map $(s, \tau) \to \gamma_s(t)$ is smooth, is one to one and has smooth inverse. So the curves $\gamma_s(\tau)$ span a two dimensional manifold, with s and τ as possible choice of coordinates. Since u^a and η^a both are coordinate vector fields, they commute [10]. So the time rate of change of η_a can be written as

$$\dot{\eta}_a = u^b \nabla_b \eta_a = \eta^b \nabla_b u_a. \tag{3.3.13}$$

So the first covariant derivative of four velocity is a significant quantity and it is conveniently decomposed in the following way:

$$\nabla_b u_a = {}^{(3)}\nabla_b u_a - u_b \nu_a, \quad {}^{(3)}\nabla_b u_a = \frac{1}{3}\theta h_{ab} + \sigma_{ab} + \omega_{ab}. \tag{3.3.14}$$

The trace $\nabla^a u_a = \theta$ of $\nabla_b u_a$ is called the expansion scalar and it represents the isotropic part of the expansion. For instance, action of θ on a sphere changes it to a sphere larger (if $\theta > 0$) or smaller (if $\theta < 0$) radius. We can define a scale $a(\tau)$, determined up to a multiplicative constant, along each world line as

$$\theta = 3\frac{\dot{a}}{a} = \frac{1}{a^3}\frac{d}{d\tau}a^3. \tag{3.3.15}$$

a represents average distance behavior of the fluid and the volume varies as a^3. This quantity is the generalization of Robertson-Walker scale factor in FLRW spacetime.

The spatial traceless symmetric part

$$\sigma_{ab} = \left(h_{(a}^{\ c} h_{b)}^{\ d} - \frac{1}{3} h_{ab} h^{cd} \right) \nabla_d u_c \tag{3.3.16}$$

is the shear tensor. By the action of σ_{ab} alone, a sphere, which is Lie transported along u_a is distorted to an ellipsoid, keeping the volume unchanged. Principal axes of the ellipsoid are given by the eigenvectors of $\sigma^a_{\ b}$. The antisymmetric part

$$\omega_{ab} = h_{[a}^{\ c} h_{b]}^{\ d} \nabla_d u_c \tag{3.3.17}$$

is called the vorticity and it measures the rotation. Both σ_{ab} and ω_{ab} are orthogonal to u^a:

$$\sigma_{ab} u^b = 0 = \omega_{ab} u^b. \tag{3.3.18}$$

If ω_{ab} vanishes then u^a becomes hypersurface orthogonal and $-{}^{(3)}\nabla_b u_a$ represents the exritrinsic curvature.

3.3.2 Geometry and matter

Curvature

The curvature of spacetime is measured by the Riemann tensor R_{abcd}. It is defined by the commutation relation of covariant derivatives on any vector v^a:

$$\nabla_a \nabla_b v^c - \nabla_b \nabla_a v^c = R_{ab\ d}^{\ \ c} v^d \tag{3.3.19}$$

Due to the symmetry properties

$$R_{[ab][cd]} = R_{abcd} = R_{cdab}, \quad R_{a[bcd]} = 0, \tag{3.3.20}$$

Riemann tensor has only 20 independent components. It can be decomposed into its trace Ricci tensor (10 independent components),

$$R_{ab} = R_{acb}{}^{c} \tag{3.3.21}$$

and the traceless part, the Weyl tensor C_{abcd} (10 independent components) as

$$R_{abcd} = C_{abcd} + g_{a[c}R_{d]b} - g_{b[c}R_{d]a} - \frac{1}{3}g_{a[c}g_{d]b}R, \tag{3.3.22}$$

where $R = g^{ab}R_{ab}$ is the Ricci scalar.

The Weyl tensor can be further splitted into the so called "electric" and "magnetic" parts, defined by

$$E_{ab} = C_{acbd}u^c u^d, \quad H_{ab} = \frac{1}{2}C_{acpq}\eta^{pq}{}_{bd}u^c u^d. \tag{3.3.23}$$

E_{ab} and H_{ab} both are spatial, traceless, symmetric tensor:

$$E_{ab} = E_{(ab)}, \quad H_{ab} = H_{(ab)}, \quad E^a_{\ a} = H^a_{\ a} = 0, \quad E_{ab}u^b = H_{ab}u^b = 0. \tag{3.3.24}$$

The Bianchi identity

$$\nabla_{[e}R_{ab]cd} = 0 \tag{3.3.25}$$

yields

$$\nabla_d C^{abcd} = \nabla^{[a}R^{b]c} - \frac{1}{6}g^{c[a}\nabla^{b]}R. \tag{3.3.26}$$

Constant time hypersurfaces

If the vorticuty ω_{ab} is zero we have a family of hypersurfaces Σ_T, everywhere orthogonal to the fluid veclocity u^a. Then we can express u^a as a 4-gradient

($u_a = -\nabla_a t$) and the surfaces become t=constant surfaces. The metric h_{ab} is compatible with the spatial derivative operator ${}^{(3)}\nabla_a$ defined in (3.3.9):

$$ {}^{(3)}\nabla_a h_{bc} = 0. \qquad (3.3.27) $$

Consequently we can raise and lower indices in the equations involving ${}^{(3)}\nabla_a$ by h_{ab}, h^{ab}. So the operator ${}^{(3)}\nabla_a$ acts as covariant derivative operator on the constant time hypersurfaces. Then all the usual commutation relation holds and we can define the curvature tensrors and scalar. For example in a general fluid flow we can define the quantity

$$ \mathcal{K} = 2\left(\kappa\mu - \frac{1}{3}\theta^2 + \sigma^2 - \omega^2 + \Lambda\right), \qquad (3.3.28) $$

which becomes the Ricci scalar on Σ_T for $\omega_{ab} = 0$.

In presence of nonzero vorticity no such orthogonal hypersurface exists. However, we can construct normalized comoving coordinates (t, x^i), characterized by the condition

$$ \dot{t} = u^a\nabla_a t = 1, \quad \dot{x}^i = u^a\nabla_a x^i = 0. \qquad (3.3.29) $$

Using such coordinates, the t=constant surfaces (Σ_T) can be set orthogonal to a particular chosen world line γ and almost orthogonal to neighboring world lines by the remaining gauge freedom:

$$ t \to t' = t + f(x^i), \qquad (3.3.30) $$

where $f(x^i)$ is an arbitrary function of the spatial coordinates. for example we can choose an initial surface ($t = t_0$) to be generated by orthogonal geodesics emanating from γ. Then $\mathcal{K}$ will be nearly the Ricci-scalar of these three-spaces on and near γ. More generally, if u^a is not too different from the normal n^a to a family of surfaces, then $\mathcal{K}$ will not be too different from the Ricci scalar of those three-spaces.

Although h_{ab} is still preserved by ${}^{(3)}\nabla_a$, we can not treat this operator as the covariant derivative on the three hypersurfaces when $\omega_{ab} \neq 0$ because the usual commutation relations are not valid in this case. For example the commutator of two vector fields A_a and B_a on Σ_T do not lie on Σ_T:

$$[A, B]^a - h^a_{\ b}[A, B]^b = u^a \omega_{ab} A^a B^b. \tag{3.3.31}$$

In these case we can define the "generalized curvature tensor" from the following commutation relation:

$$ {}^{(3)}\nabla_a {}^{(3)}\nabla_b A^c - {}^{(3)}\nabla_b {}^{(3)}\nabla_a A^c + \omega_{ab} h^c_{\ d} \dot{A}^d = {}^{(3)}R_{ab\ d}^{\ \ c} A^d. \tag{3.3.32}$$

As a consequence the spatial derivative operators do not commute on a scalar function:

$$ {}^{(3)}\nabla_{[a} {}^{(3)}\nabla_{b]} f = -\omega_{ab} \dot{f}. \tag{3.3.33}$$

Matter

The matter content of the universe is described by the energy momentum tensor of an imperfect fluid with 4-velocity u^a

$$T_{ab} = (\mu + p) u_a u_b + p g_{ab} + 2u_{(a} q_{b)} + \pi_{ab}, \tag{3.3.34}$$

where $\mu = T_{ab} u^a u^b$ and $p = \frac{1}{3} h^{ab} T_{ab}$ are energy density and pressure respectively. The anisotropic part involves the energy flux $q_a = -h_a^{\ b} T_{bc} u^c$ and the anisotropic stress $\pi_{ab} = h_a^{\ c} h_b^{\ d} T_{cd} - p h_{ab}$. The energy-momentum tensor obeys the conservation equation

$$\nabla^b T_{ab} = 0. \tag{3.3.35}$$

Einstein's equation

The connection between geometry and matter is established via Einstein's equations:

$$G_{ab} + \Lambda g_{ab} = R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} = \kappa T_{ab}, \tag{3.3.36}$$

where $\kappa = 8\pi G$, G is Newton's constant and Λ is the cosmological constant.

3.3.3 Gauge invariant variables

The model consists of a perfect fluid in a FLRW spacetime characterized by the following conditions [64]:

- Local isotropy holds everywhere. So

  $$\sigma_{ab} = \omega_{ab} = \dot{u}^a = 0, \tag{3.3.37}$$

  which implies that there exists a global time coordinate t defined by $u_a = -\nabla_a t$ and the kinematical quantities are function of only t:

  $$\mu = \mu(t), \quad p = p(t), \quad \theta = \theta(t). \tag{3.3.38}$$

- The Weyl tensor vanishes:

  $$C_{abcd} = 0, \tag{3.3.39}$$

  i.e. this space-time is conformally flat.

From Stewart and Walker lemma, a tensor is gauge invariant if it vanishes in the background spacetime. Thus, the above characterizations of FLRW spacetime lead us to some simple gauge invariant variables:

1. Shear, vorticity and acceleration:

$$\sigma_{ab}, \quad \omega_{ab}, \quad \nu_a. \tag{3.3.40}$$

2. "Electric" and "magnetic" parts of the Weyl tensor:

$$E_{ab}, \quad H_{ab}. \tag{3.3.41}$$

3. Spatial gradients of the energy density, pressure density and expansion:

$$X_a = \kappa h_a^{\ b}\nabla_b\mu, \quad Y_a = \kappa h_a^{\ b}\nabla_b p, \quad Z_a = h_a^{\ b}\nabla_b\theta. \tag{3.3.42}$$

4. Anisotropic components of the energy momentum tensor:

$$q_a, \quad \pi_{ab}. \tag{3.3.43}$$

As all of these first order variables vanish in the exact FLRW model, they all are uniquely defined GI variables provided u^a is uniquely defined in the realistic almost-FLRW Universe model. Thus, we obtain a set of 1 + 3 covariant and gauge invariant variables characterizing departures from a FLRW geometry. Because these are tensors defined in the real spacetime, we can evaluate them in any local coordinate system we like in that spacetime.

Two simple gauge-invariant quantities give us the information of time evolution of energy density fluctuations. The basic quantities we start with are the orthogonal projections of the energy density gradient, i.e., the vector X_a, and of the expansion gradient, i.e., the vector Z_a. The first can be determined (a) from virial theorem estimates and large scale structure observations, (b) by observing gradients in the numbers of observed sources and estimating the mass-to-light ratio and (c) by gravitational lensing observations. However, these do not directly correspond to the quantities usually calculated, but two closely related quantities do. The first is the matter-comoving fractional energy density gradient

$$\mathcal{D}_a = a\frac{X_a}{\kappa\mu}, \tag{3.3.44}$$

which is gauge-invariant and dimensionless, and represents the spatial energy density variation over a fixed comoving scale. Note that a, and so D_a , is defined only up to a constant; this allows it to represent the energy density variation between any neighbouring worldlines. The vector D_a can be separated into a magnitude D and direction e_a

$$\mathcal{D}_a = \mathcal{D}e_a, e_a e^a = 1, e_a u^a = 0 \Rightarrow \mathcal{D} = \sqrt{\mathcal{D}_a\mathcal{D}^a}. \tag{3.3.45}$$

The magnitude $\mathcal{D}$ is the gauge-invariant variable that most closely corresponds to the intention of the usual $\delta = \delta\mu/\mu$. The crucial difference from the usual definition is that $\mathcal{D}$ represents a (real) spatial fluctuation, rather than a (fictitious) time fluctuation, and does so in a GI manner. An important auxiliary variable in what follows is the matter-comoving spatial expansion gradient:

$$\mathcal{Z}_a = aZ_a. \tag{3.3.46}$$

The Ricci scalar of the homogeneous an isotropic spacelike hypersurfaces of the FLRW spacetime is

$$^{(3)}\bar{R} = \frac{6K}{a^2}. \tag{3.3.47}$$

So in the case of vanishing vorticity, the Ricci scalar $^{(3)}\mathcal{R}$ of the orthogonal three-spaces is gauge invariant if and only if the homogeneous space sections in S are flat ($K = 0$), i.e. , if that idealized universe is at the critical density. However, its spatial gradient is always gauge invariant. Thus for a general fluid flow we define the gauge-invariant quantity:

$$\mathcal{K}_a = {}^{(3)}\nabla_a \mathcal{K} = 2\left(X_a - \frac{2}{3}\theta Z_a + {}^{(3)}\nabla_a(\sigma^2 - \omega^2)\right). \tag{3.3.48}$$

If the pressure is given in terms of the energy density alone then Y_a is not an independent variable. It can be expressed in terms of X_a:

$$Y_a = \frac{dp}{d\mu} X_a. \tag{3.3.49}$$

However, if the equation of state is found to be

$$p = p(\mu, S), \tag{3.3.50}$$

i.e. the pressure is function of both energy density and entropy density S, then small change in p is given by

$$\delta p = c_s^2 \delta\mu + \tau\delta S, \tag{3.3.51}$$

where $c_s^2 = \left(\frac{\partial p}{\partial \mu}\right)_S$ is adiabatic speed of sound and $\tau = \left(\frac{\partial p}{\partial S}\right)_\mu$. Since in absence of dissipation, entropy is conserved along fluid flow lines, i.e. $\dot{S} = 0$,

$$c_s^2 = \frac{\dot{p}}{\dot{\mu}}. \tag{3.3.52}$$

However, in presence of dissipation (such as viscocity), the above equality does not hold. The spatial gradient of p is then written as a combination of spatial gradient of energy density plus a non-adiabatic term:

$$Y_a = c_s^2 X_a + \Gamma_a, \tag{3.3.53}$$

where

$$\Gamma_a = \kappa\tau^{(3)}\nabla_a S. \tag{3.3.54}$$

There are many other gauge invariant quantities, which vanish in the background model. We are not going to list all such variables. Our goal is to set up a closed set of (nonlinear) equations that contains the whole information of the Einstein equations and study the non-linear evolution of density perturbations and the gravitational waves. For that purpose we mention some more variables that will appear in the equations. One scalar and one vector are constructed from the acceleration,

$$A = \nabla^a \nu_a, \quad A_a = {}^{(3)}\nabla_a A, \tag{3.3.55}$$

and two vectors are constructed from the vorticity,

$$\omega_a = \frac{1}{2}\epsilon_{abc}\omega^{bc}, \quad r_a = \nabla^b \omega_{ab}, \tag{3.3.56}$$

where ϵ_{abc} is Levi-Civita tensor in 3-hypersurface defined by $\epsilon_{abc} = \eta_{abcd}u^d$.

Note that the gauge invariant variables defined in this section are not restricted to any order of perturbations and in the next section we will show that it is possible to set nonlinear evolution equations for these variables. Hence covariant formalism can be used to study the nonlinear evolution of the perturbations. Covariantly defined comoving curvature perturbation ζ_a, deined as

$$\zeta_a = W_a + \frac{X_a}{3\kappa(\mu + p)}, \quad W_a = {}^{(3)}\nabla_a \ln a, \tag{3.3.57}$$

is suitable for that job [65]. This variable is closely related to the gauge invariant perturbation ζ, used in the standard perturbation theory.

The non-vanishing quantities μ, p and θ are zeroth order variables, whereas those defined in (3.3.40)-(3.3.55) are called first order variables. We can construct higher order variables from first order ones as

$$\omega^2 = \frac{1}{2}\omega_{ab}\omega^{ab}, \;\; \sigma^2 = \frac{1}{2}\sigma_{ab}\sigma^{ab}, \;\; \sigma_{ab}E^{ab}, \;\; \sigma_{ab}H^{ab} \;\; . \tag{3.3.58}$$

3.4 Dynamical equations and constraints

3.4.1 Exact eqations

Conservation equation

The conservation equation (3.3.35) can be seperated into time (energy conservation) and space (momentum conservation) components. Inserting the energy momentum tensor (3.3.34) in (3.3.35), we get

$$\begin{aligned} u_a\left[\dot{(\mu+p)}+\theta(\mu+p)+\nabla^b q_b\right]+\nabla_a p+\dot{q}_a+\nabla^b\pi_{ab} \\ +(\mu+p)\nu_a+\left[\frac{4}{3}\theta h_{ab}+\sigma_{ab}+\omega_{ab}\right]q^b=0. \end{aligned} \tag{3.4.1}$$

Projecting (3.4.1) along u^a we obtain the energy (non)conservation equation:

$$\dot{\mu}+\theta(\mu+p)+\nabla^a q_a+\nu^a q_a+\sigma^{ab}\pi_{ab}=0.$$

Similarly projecting the (3.4.1) by h_{ab} we get the equation for momentum (non) conservation:

$$(\mu+p)\nu_a+{}^{(3)}\nabla p+h_a^{\ b}\left(\dot{q}_b+\nabla^c\pi_{bc}\right)+\left(\frac{4}{3}\theta h_{ab}+\sigma_{ab}+\omega_{ab}\right)q^b=0.$$

The (non) conservation equations can be written as

$$\dot{\mu}+\theta(\mu+p)=-\varrho, \tag{3.4.2}$$

$$\kappa(\mu+p)\nu_a+Y_a=-\kappa\alpha_a, \tag{3.4.3}$$

where $\varrho=\nabla^a q_a+\nu^a q_a+\sigma^{ab}\pi_{ab}$ and $\alpha_a=h_a^{\ b}\left(\dot{q}_b+\nabla^c\pi_{bc}\right)+\left(\frac{4}{3}\theta h_{ab}+\sigma_{ab}+\omega_{ab}\right)q^b$ are dissipating terms that vansih in the case of perfect fluid perturbations. Once the equation of state $p=p(\mu)$ is given, evolution equation for p is found from (3.4.2).

Hydrodynamic equations

Taking the time derivative of $\nabla_b u_a$ and using the commutation realation (3.3.19) we get

$$u^c \nabla_c \nabla_b u_a = \nabla_b \nu_a - (\nabla_b u^c)(\nabla_c u_a) - R_{acbd} u^c u^d. \tag{3.4.4}$$

Taking trace of (3.4.4) we get

$$\dot{\theta} = A - \left(\frac{1}{3}\theta^2 + 2\sigma^2 - 2\omega^2\right) - R_{ab} u^a u^b. \tag{3.4.5}$$

Using the Einstein equation (3.3.36) and the form of energy momentum tensor we obtain the Raychoudhury equation:

$$\dot{\theta} + \frac{1}{3}\theta^2 - A + \frac{1}{2}\kappa(\mu + 3p) - \Lambda + 2(\sigma^2 - \omega^2) = 0, \tag{3.4.6}$$

where $\sigma^2 = \frac{1}{2}\sigma_{ab}\sigma^{ab}$ and $\omega^2 = \frac{1}{2}\omega_{ab}\omega^{ab}$ .The Raychoudhury equation is a fundamental equation to study singularity. It reveals that $\mu + 3p$ plays the roll of active gravitational mass of fluid. The cosmological constant acts as constant repulsive force. The vorticity and acceleration divergence act also give repulsive forces. On the other hand shear tend to shrink the volume.

The equations for σ_{ab} and ω_{ab} are also obtained from (3.4.4). Let us define the following notations [66]:

$$\begin{aligned}
&\lambda_{(ab)} = \frac{1}{2}(\lambda_{ab} + \lambda_{ba}), \quad \lambda_{[ab]} = \frac{1}{2}(\lambda_{ab} - \lambda_{ba}), \\
&\lambda_{\langle ab \rangle} = h_a^{\ c} h_b^{\ d} \lambda_{(cd)} - \frac{1}{3} h_{ab} h^{cd} \lambda_{cd}, \\
&\mathrm{Curl}\lambda_{ab} = h^e_{\ (a}\epsilon_{b)cd} \nabla^d \lambda_e^{\ c},
\end{aligned}$$

where λ_{ab} is any $(0, 2)$ tensor.

Consider the traceless symmetric part of the equation (3.4.4):

$$\begin{aligned}\dot{\sigma}_{ab} &= \nabla_{\langle a}\nu_{b\rangle} - \frac{2}{3}\theta\sigma_{ab} + \sigma_{c\langle a}\sigma_{b\rangle}^{\ c} - \omega_{c\langle a}\omega_{b\rangle}^{\ c} \\ &\qquad -C_{acbd}u^c u^d + \frac{1}{2}R_{\langle ab\rangle}. \end{aligned} \tag{3.4.7}$$

From (3.3.36) we get

$$R_{ab} = \kappa\left(\frac{1}{2}(\mu+3p)u_a u_b + \frac{1}{2}(\mu-p)h_{ab} + q_a u_b + q_b u_a + \pi_{ab}\right) + \Lambda g_{ab}, \tag{3.4.8}$$

which yields

$$R_{\langle ab\rangle} = h_a^{\ c}h_b^{\ d}R_{(cd)} - \frac{1}{3}h_{ab}h^{cd}R_{cd} = \kappa\pi_{ab}. \tag{3.4.9}$$

So the equation for σ_{ab} becomes

$$a^{-2}h_a^{\ c}h_b^{\ d}(a^2\sigma_{cd})\dot{} = -E_{ab} + \nabla_{\langle b}\nu_{a\rangle} + \sigma_{c\langle a}\sigma_{b\rangle}^{\ c} - \omega_{c\langle a}\omega_{b\rangle}^{\ c} + \nu_a\nu_b + \frac{1}{2}\kappa\pi_{ab}. \tag{3.4.10}$$

The antisymmetric part of (3.4.4) yields the equation for vorticity:

$$a^{-2}h_a^{\ c}h_b^{\ d}(a^2\omega_{cd})\dot{} = h_a^{\ c}h_b^{\ d}\nabla_{[d}\nu_{c]} + 2\sigma_{c[a}\omega_{b]}^{\ c} \tag{3.4.11}$$

The equations (3.4.6), (3.4.10) and (3.4.11) also follow from (3.3.19). There are three further equations obtained from (3.3.19):

$$h_a^{\ c}\nabla_b(\omega^b_{\ c} + \sigma^b_{\ c}) - \nu^b(\omega_{ab} + \sigma_{ab}) + q_a = \frac{2}{3}Z_a, \tag{3.4.12}$$

$$\nabla_a\omega^a = 2\nu_a\omega^a, \tag{3.4.13}$$

$$\mathrm{Curl}\,\omega_{ab} + \mathrm{Curl}\,\sigma_{ab} = -H_{ab}, \tag{3.4.14}$$

These are the constraint realation, not the dynamic equations because these relations do not involve time derivatives.

Maxwell like equations for Weyl tensor components

By the Bianchi identity (3.3.26) the Weyl tensor components are related to the Ricci tensor and Ricci Scalar. The decomposition of Weyl tensor (3.3.23) into "electric" and "magnetic" parts allows us to extract four equations from (3.3.26) that are analogus to Maxwell equations in electrodynamics. In the perfect fluid case these equations are:

$$
\begin{aligned}
a^{-3}h_a^{\ c}h_b^{\ d}\dot{(a^3E_{cd})} = -\mathrm{Curl}H_{ab} - \frac{1}{2}\kappa(\mu+p)\sigma_{ab} + E^c_{\ (a}\omega_{b)c} \\
+E^c_{\ (a}\sigma_{b)c} + \epsilon_{acd}\epsilon_{bpq}\sigma^{cp}E^{dq} - 2H^c_{\ (a}\epsilon_{b)cd}\nu^d, \qquad (3.4.15) \\
a^{-3}h_a^{\ c}h_b^{\ d}\dot{(a^3H_{cd})} = \mathrm{Curl}E_{ab} + H^c_{\ (a}\omega_{b)c} \\
+H^c_{\ (a}\sigma_{b)c} + \epsilon_{acd}\epsilon_{bpq}\sigma^{cp}H^{dq} - 2H^c_{\ (a}\epsilon_{b)cd}\nu^d, \qquad (3.4.16) \\
h_a^{\ c}\nabla^bE_{bc} + 3H_{ab}\omega^b - \epsilon_{abc}\sigma^b_{\ d}H^{cd} = \frac{1}{3}X_a, \qquad (3.4.17) \\
h_a^{\ c}\nabla^bH_{bc} - 3E_{ab}\omega^b - \epsilon_{abc}\sigma^b_{\ d}H^{cd} = \kappa(\mu+p)\omega_a. \qquad (3.4.18)
\end{aligned}
$$

Equations for acceleration, density gradient and expansion gradient

We can obtain the evolution equations for ν_a, X_a and Z_a form (3.4.2), (3.4.3) and (3.4.6). Taking the derivative of (3.4.3) with respect to proper time,

$$
\dot{[\kappa(\mu+p)\nu_a]} + \dot{Y}_a = -\kappa\dot{\alpha}_a. \qquad (3.4.19)
$$

The differnt terms are calculated below:

$$
\begin{aligned}
\dot{[(\mu+p)\nu_a]} &= \kappa(\mu+p)\left[\dot{\nu}_a - (c_s^2+1)\theta\nu_a\right] \\
&= \kappa(\mu+p)\left[\dot{\nu}_a - c_s^2\theta\nu_a\right] + \theta Y_a,
\end{aligned}
$$

$$
\begin{aligned}
\dot{Y}_a &= \kappa u^c \nabla_c h_a^{\ b} \nabla_b p \\
&= \kappa h_a^{\ b} \nabla_b (u^c \nabla_c p) - \kappa h_a^{\ b} \left(\nabla_b u^c\right) \nabla_c p + \kappa \dot{p} \nu_a + u_a \nu_b Y^b \\
&= -\kappa(\mu + p) \left[h_a^{\ b} \nabla_b (c_s^2 \theta) - \frac{1}{3} \theta \nu_a - (\sigma^b\ a + \omega^b\ a) \nu_a \right] \\
&\qquad\qquad\qquad\qquad\qquad - c_s^2 \theta X_a + u_a \nu_b Y^b.
\end{aligned}
$$

Putting these in (3.4.19) and taking projection orthogonal to u^a we obtain

$$
\begin{aligned}
h_a^{\ b} \dot{\nu}_b - \left(c_s^2 - \frac{1}{3} \right) \theta \nu_a - h_a^{\ b} \nabla_b (c_s^2 \theta) + (\sigma^b_{\ a} + \omega^b_{\ a}) \nu_a & \\
= -\frac{\theta \Gamma_a}{\kappa(\mu + p)} - \frac{\kappa h_a^{\ b} \dot{\alpha}_b}{\kappa(\mu + p)}&, \qquad (3.4.20)
\end{aligned}
$$

where Γ_a is the non-adiabatic perturbation defined by (3.3.53) which vanishes in the case of adiabatic perturbations, when the pressure can be expressed as function of only the energy density.

The equation for the density gradient is found from the spatial gradient of (3.4.2).

$$
\kappa h_a^{\ b} \nabla_b \dot{\mu} + \kappa h_a^{\ b} \nabla_b \theta (\mu + p) + \kappa h_a^{\ b} \nabla_b \varrho = 0.
$$

Now

$$
\begin{aligned}
\kappa h_a^{\ b} \nabla_b \dot{\mu} &= \kappa \left[u^c \nabla_c (h_a^{\ b} \nabla_b \mu) - \nabla_b \mu \left(\nu_a u^b + \nu_b u^a \right) + h_a^{\ b} \left(\nabla_b u^c \right) \nabla_c \mu \right] \\
&= h_a^{\ b} \dot{X}_a + \left(\frac{1}{3} \theta h^b_{\ a} + \sigma^b_{\ a} + \omega^b_{\ a} \right) X_b + \kappa(\mu + p) \theta \nu_a,
\end{aligned}
$$

and

$$
\kappa h_a^{\ b} \nabla_b \theta (\mu + p) = \theta (X_a + Y_a) + \kappa(\mu + p) Z_a.
$$

Then using (3.4.3) we obtain

$$
h_a^{\ b} \dot{X}_a + \left(\frac{4}{3} \theta h^b_{\ a} + \sigma^b_{\ a} + \omega^b_{\ a} \right) X_b + \kappa(\mu + p) Z_a - \kappa \theta \alpha_a + \kappa^{(3)} \nabla_a \varrho = 0.
$$

Using (3.3.15) we can cast the equation in following form:

$$a^{-4}h_a^{\ b}\dot{(a^4X_b)} = -\kappa(\mu+p)Z_a - (\sigma^b_{\ a} + \omega^b_{\ a})X_b + \kappa\left(\theta\alpha_a - {}^{(3)}\nabla_a\varrho\right). \quad (3.4.21)$$

The equation for expansion gradient is obtained from the spatial gradient of (3.4.6) in similar fashion:

$$\begin{aligned} a^{-3}h_a^{\ b}\dot{(a^3Z_b)} &= \mathcal{R}\nu_a - \frac{1}{2}X_a + A_a + 2^{(3)}\nabla_a(\omega^2-\sigma^2) \\ &\quad -(\sigma^b_{\ a}+\omega^b_{\ a})Z_b + \frac{3}{2}\kappa\alpha_a, \end{aligned} \quad (3.4.22)$$

where

$$\mathcal{R} = \kappa\mu - \frac{1}{3}\theta^2 + A + 2(\omega^2-\sigma^2) + \Lambda = \frac{1}{2}\mathcal{K} + A + 3(\omega^2-\sigma^2), \quad (3.4.23)$$

with $\mathcal{K}$, defined in Eq. (3.3.28), is the Ricci curvature scalar of the spacelike hypersurfaces orthogonal to the fluid flow when the vorticity vanishes.

Given the perfect fluid approximation ($q_a = 0, \pi_{ab} = 0$) and the adiabatic approximation ($\Gamma_a = 0$), the equations (3.4.10), (3.4.11), (3.4.15), (3.4.16), (3.4.20), (3.4.21) and (3.4.22) provides a closed set of equations for the propation of the gauge invariant variables X_a, Z_a, ν_a, ω_{ab}, σ_{ab}, E_{ab} and H_{ab}. The density gradient X_a, expansion gradient Z_a and the acceleration ν_a are coupled at the linear order via the equations (3.4.20-3.4.21). These equations have nonlinear coupling terms with the shear σ_{ab} and the vorticity ω_{ab}, whoose evolution are ditermined by (3.4.10)-(3.4.11). Since (3.4.10) contains the "Electric field" E_{ab}, we have to take account of the Maxwell like equations (3.4.15)-(3.4.16). Finally the constraint relations must be satisfied on each constant time hypersurface.

Equation for comoving curvature perturbations

The evolution equation for ζ_a is easily obtained in terms of Lie derivative. Lie derivative of any vector $\mathcal{A}_a$ along u^a is

$$\mathcal{L}_u \mathcal{A}_a = u^b \nabla_b \mathcal{A}_a + \mathcal{A}_b \nabla_a u^b = h_a{}^b \dot{\mathcal{A}}_b + \frac{1}{3}\theta \mathcal{A}_a + \left(\sigma^a{}_a + \omega^a{}_a\right) \mathcal{A}_b. \qquad (3.4.24)$$

The action of $\mathcal{L}_u$ on a scalar ϕ is simply the time derivative along fluid flow lines.

$$\mathcal{L}_u \phi = u^c \nabla_c \phi = \dot{\phi}. \qquad (3.4.25)$$

The spatial derivative ${}^{(3)}\nabla_a$ and Lie derivative $\mathcal{L}_u$ on a scalar do not commute but their difference is proportional to ν_a:

$${}^{(3)}\nabla_a \dot{\phi} = \mathcal{L}_u {}^{(3)}\nabla_a \phi - \nu_a \dot{\phi}. \qquad (3.4.26)$$

From (3.4.26) we establish the relation between Z_a and W_a:

$$Z_a + \theta \nu_a = 3\mathcal{L}_u W_a. \qquad (3.4.27)$$

Using (3.4.24), the equation (3.4.21) can be written as

$$\mathcal{L}_u X_a + \theta X_a + \kappa(\mu + p) Z_a = \kappa \left(\theta \alpha_a - {}^{(3)}\nabla_a \varrho\right). \qquad (3.4.28)$$

Taking Lie derivative of (3.3.57),

$$\begin{aligned}
\mathcal{L}_u \zeta_a &= \mathcal{L}_u W_a + \frac{\mathcal{L}_u X_a}{3\kappa(\mu+p)} - \frac{(\dot{\mu}+\dot{p})}{\kappa(\mu+p)^2} X_a \\
&= \frac{1}{3}\left(Z_a + \theta \nu_a\right) + \frac{\mathcal{L}_u X_a}{3\kappa(\mu+p)} + \frac{(1+c_s^2)\theta X_a}{3\kappa(\mu+p)} \\
&= \frac{\mathcal{L}_u X_a + \theta X_a + \kappa(\mu+p) Z_a + \theta\kappa(\mu+p)\nu_a + c_s^2 \theta X_a}{3\kappa(\mu+p)} \\
&= \frac{\kappa\left(\theta\alpha_a - {}^{(3)}\nabla_a \varrho\right) - \theta(Y_a + \kappa\alpha_a) + c_s^2 \theta X_a}{3\kappa(\mu+p)},
\end{aligned}$$

where we have used (3.4.27), (3.4.28) and (3.4.3). Then we obtain [67]

$$\mathcal{L}_u \zeta_a = -\frac{\theta \left(\Gamma_a + \kappa^{(3)} \nabla_a \varrho \right)}{3\kappa(\mu + p)}. \tag{3.4.29}$$

So for the perfect fluid and adiabatic perturbations, the ζ_a is Lie dragged along the fluid flow lines. The equation (3.4.29) can be written as

$$a^{-1} h_a^{\ b} \dot{(a\zeta_b)} = -\frac{\theta \left(\Gamma_a + \kappa^{(3)} \nabla_a \varrho \right)}{3\kappa(\mu + p)} - (\sigma^b_{\ a} + \omega^b_{\ a})\zeta_b. \tag{3.4.30}$$

The set of full non linear equations contains the complete information of the Einstein equation. We only choose new variables, suitable to study the density inhomogeneities. No background spacetime is introduced while developing the equations. In order to solve the equations we need a more restrictive approach. One step for this job is to linearize the equations about a background FLRW spacetime.

3.4.2 Linearization for almost FLRW universe

The variables defined in the Sec. 3.3.3 are covariantly defined exact quantities. Thus they bear clear physical and geometrical meaning. Exact evolution equations for these variables (which are infact Einsteins equation, written in different form) are derived in the Sec. 3.4.1. We now want to put our attention on a real physiacal spacetime that is close to the FLRW one. such spacetimes are referred to “almost FLRW”. Instead of starting from an exact FLRW model and perturbing it in standard way we consider the physical universe as a subset in the whole space of solutions of Einstein’s equation, surrounding the exact FLRW solutions. By considering the almost FLRW model we can speak of zeroth order and first order variables.

Zeroth order variables: The background FLRW spacetime can be taken as zeroth order approximation of the physiacal almost FLRW universe. The variables that survives in the background spacetimes such as the energy density μ, pressure p and expansion $\theta = 3H$ are called zeroth order variables. If the background is a closed or open universe then the spatial curvature $\mathcal{R}$ or $\mathcal{K}$ is also a zeroth order variable.

First order variables: First order variables are those vanish on the background FLRW spacetime, i.e. the gauge invariant variables defined in Sec. 3.3.3. Any product of first order variables give rise to higher order or nonlinear term such as

$$\sigma_{ab}X^b, \quad \omega_{ab}X^b, \quad \sigma_{ab}Z^b, \quad \omega_{ab}Z^b, \quad {}^{(3)}\nabla_a\omega^2, \quad {}^{(3)}\nabla_a\sigma^2.$$

For a flat background $\mathcal{R}$ is first order, hence $\mathcal{R}\nu_a$ is a higher order term.

The linearized form of the exact equations are obtained by dropping the first and higher order terms from zeroth order equation and dropping higher order terms from the first order equations. By zeroth order equations we mean those equations which are not identically zero on the background spacetime. These are the equations for zeroth order variables. the first order equations are the dynamic equations of the first order variables. We also assume perfect fluid perturbations, i.e. $\pi_{ab} = 0$ and $q_a = 0$.

So the zeroth order equations, obtained from (3.4.2) and (3.4.6), are

$$\dot{\mu} + \theta(\mu + p) = 0, \tag{3.4.31}$$

$$\dot{\theta} + \frac{1}{3}\theta^2 + \frac{1}{2}\kappa(\mu + 3p) - \Lambda = 0. \tag{3.4.32}$$

These equations are simply the Friedmann equations. Given the equation of state, we can solve the above equations to find the energy density μ and the Robertson-Walker scale factor a as a function of cosmic time t.

The linearized form of the first order equations are

$$
\begin{aligned}
a^{-4}(a^4 X_a)^{\cdot} &= -\kappa(\mu+p)Z_a, && (3.4.33)\\
a^{-3}(a^3 Z_a)^{\cdot} &= \frac{1}{2}\mathcal{K}\nu_a - \frac{1}{2}X_a + A_a, && (3.4.34)\\
a^{-2}(a^2 \omega_{ab})^{\cdot} &= {}^{(3)}\nabla_{[b}\nu_{a]}, && (3.4.35)\\
a^{-2}(a^2 \sigma_{ab})^{\cdot} &= -E_{ab} + \nabla_{\langle b}\nu_{a\rangle}, && (3.4.36)\\
a^{-3}(a^3 E_{ab})^{\cdot} &= -\mathrm{Curl}H_{ab} - \frac{1}{2}\kappa(\mu+p)\sigma_{ab}, && (3.4.37)\\
a^{-3}(a^3 H_{ab})^{\cdot} &= \mathrm{Curl}E_{ab}. && (3.4.38)
\end{aligned}
$$

The acceleration ν_a is found from the momentum conservation equation:

$$\kappa(\mu+p)\nu_a + Y_a = 0. \qquad (3.4.39)$$

With the (3.4.39) and the equation of state, (3.4.33) and (3.4.34) formed two coupled 1st order differential equations for X_a and Z_a. The evolution of vorticity is obtained from (3.4.35). The equtions for shear, “electric field” and “magnetic field” are given in (3.4.36)-(3.4.38).

Similarly we obtain the linearized constraint equations:

$$
\begin{aligned}
&{}^{(3)}\nabla_b({\omega^b}_a + {\sigma^b}_a) = \frac{2}{3}Z_a, \qquad {}^{(3)}\nabla_a\omega^a = 0, && (3.4.40)\\
&H_{ab} = -\mathrm{Curl}\omega_{ab} - \mathrm{Curl}\sigma_{ab}, && (3.4.41)\\
&{}^{(3)}\nabla^b E_{ab} = \frac{1}{3}X_a, \qquad {}^{(3)}\nabla^b H_{ab} = \kappa(\mu+p)\omega_a. && (3.4.42)
\end{aligned}
$$

While the constraint equations are not needed to determine the propagation of perturbations along the flow lines, they must of course be satisfied at some initial time on each world line. This gives interesting information about what is and is not possible. For example consider the first constraint of (3.4.40). This shows that if θ varies spatially, i.e., $Z_a \neq 0$, then either the shear or the vorticity must also be nonzero. Conversely only restricted shear and vorticity perturbations will be compatible with Z_a remaining zero. Similarly the first of the (3.4.42) shows that the electric part E_{ab} of the Weyl tensor must be nonzero if there is a nonzero density gradient (i.e. if $X_a \neq 0$).

3.5 Towards the solution of linearized equations

3.5.1 ADM decomposition

Any vector V_a on H_p can be decomposed as a sum of a transverse vector V_a^T and spatial gradient of a scalar ϕ:

$$V_a = V_a^T + {}^{(3)}\nabla_a \phi, \tag{3.5.1}$$

where the first term has vanishing spatial divergenec (${}^{(3)}\nabla^a V_a^T = 0$) and following (3.3.33) the curl of the second term

$$\text{Curl}^{(3)}\nabla_a \phi = \epsilon_a{}^{bc(3)}\nabla_b{}^{(3)}\nabla_c \phi = -\frac{1}{2}\epsilon_{abc}\omega^{ab}\phi \tag{3.5.2}$$

vanishes in linear approximation. Taking 3-divergence of (3.5.1), we obtain the Poisson's equation for ϕ,

$${}^{(3)}\nabla^2 \phi = {}^{(3)}\nabla^a V_a \tag{3.5.3}$$

which can be solved with given boundary conditions. Then V_a^T is obtained from (3.5.1). So the decomposition is not unique but depends on the boundary conditions chosen. Similarly any $(0,2)$ tensor P_{ab} on H_p can be decomposed as follows:

$$P_{ab} = P_{ab}^T + {}^{(3)}\nabla_a V_b^T + {}^{(3)}\nabla_b V_a^T + {}^{(3)}\nabla_{\langle a}{}^{(3)}\nabla_{b\rangle}\phi + \frac{1}{3}h_{ab}\psi + W_{ab}, \tag{3.5.4}$$

where $W_{ab} = P_{[ab]}$ is the antisymmetric part and $\psi = P_a{}^a$ is the trace and

$$P_{\langle ab\rangle} = P_{ab}^T + {}^{(3)}\nabla_a V_b^T + {}^{(3)}\nabla_b V_a^T + {}^{(3)}\nabla_{\langle a}{}^{(3)}\nabla_{b\rangle}\phi \tag{3.5.5}$$

is the trace-less symmetric part of P_{ab}. P^T_{ab} and V^T_a obey

$$ {}^{(3)}\nabla^b P^T_{ab} = 0, \quad {}^{(3)}\nabla^a V^T_a = 0, \quad h^{ab} P^T_{ab} = 0. \tag{3.5.6} $$

By virtue of (3.5.5) we can decompose any traceless symmetric tensor into a scalar part ${}^{(3)}\nabla_{\langle a}{}^{(3)}\nabla_{b\rangle}\phi$, a vector part $2{}^{(3)}\nabla_{(a}V^T_{b)}$ and a tensor part P^T_{ab}. Such decomposition allow us to classify the gauge invariant variables defined in the Sec. (3.3.3) into three groups. We will refer all scalars, vectors, constructed as gradients of scalars and traceless symmetric tensors, constructed from gradients of scalars, as "scalar perturbations". All divergence-less vectors and traceless symmetric tensors, constructed from divergence-less vectors, will be referred as "vector perturbations". Finally all divergenceless, traceless, symmetric tensors will be called "tensor perturbations". The equations for scalar, vector and tensor perturbations decouple from each other at linear order.

Under such characterization, the spatial gradients X_a, Y_a, Z_a, Γ_a, A_a, $\mathcal{K}_a$, W_a, ζ_a are simply scalar perturbations. So is the acceleration ν_a, from (3.4.39). Before searching for vector perturbations let us mention that the total divergence and spatial divergence of a vector v_a on H_p differ by a nonlinear term:

$$ \nabla_a v^a = {}^{(3)}\nabla_a v^a + \nu_a v^a. \tag{3.5.7} $$

Hence vanishing of total divergence is sufficient to tagging a variable as vector perturbation. According to the 2nd of constraints (3.4.40) the ω_a is a divergenceless vector. The divergence of r_a is

$$ \nabla^a r_a = \nabla^a \nabla^b \omega_{ab} = \nabla^{[a}\nabla^{b]}\omega_{ab} = -R^{ab}\omega_{ab} = 0. \tag{3.5.8} $$

So r_a is also a vector perturbations. So the vector perturbations are directly linked with the rotation of the fluid. The traceless symmetric tensors σ_{ab}, E_{ab} and H_{ab}

can be decomposed into scalar, vector and tensor parts:

$$\sigma_{ab} = \sigma^S_{ab} + \sigma^V_{ab} + \sigma^T_{ab},$$
$$E_{ab} = E^S_{ab} + E^V_{ab} + E^T_{ab}, \quad H_{ab} = H^S_{ab} + H^V_{ab} + H^T_{ab}.$$

3.5.2 Harmonics

The standard harmonic decomposition of first order perturbations is usually carried out explicitly using harmonic functions defined as eigenfunctions of certain differential operators on well established spaces. Usually in cosmology we deal with harmonics which are eigenfunctions of a Laplace-Beltrami operator on 3-hypersurface of constant curvature, i.e. on the homogeneous spatial sections of FLRW univereses. In the standard approach to linear perturbation theory one split each quantity into the zeroth order part, which is the value of the quantity in background FLRW universe, and the first order part, omitting all higher order terms and uses covariant derivative with respect to the background FLRW metric. In the covariant approach instead we consider only quantities defined in the real almost FLRW universes. In doing this we emphasize the fluid velocity u^a rather than an arbitrarily chosen spatial slicing and we define spatial quantities on projecting orthogonal to u^a with h_{ab}. In a general spacetime we have a spatial deivative ${}^{(3)}\nabla_a$ with ${}^{(3)}\nabla_a h_{bc} = 0$, which however is not the covariant derivative in a hypersurface , unless $\omega_{ab} \neq 0$. Accordingly we want to use harmonics $Q(k)$ of order k defined through the operator ${}^{(3)}\nabla_a$ derivatives, and constany along flow lines(i.e. independent of proper time) [27]. The tensor eigenfunctions (harmonics) of the spatial Laplacian ${}^{(3)}\nabla^2 = {}^{(3)}\nabla_a {}^{(3)}\nabla^a$ are solutions of the tensor Helmholtz equation:

$$ {}^{(3)}\nabla^2 Q_{ab...c}(k) + \frac{k^2}{a^2} Q_{ab...c}(k) = 0. \quad (3.5.9) $$

In the almost FLRW universe we can expand the first order quantities in terms of Qs. For example a first orde scalar variable S can be expanded as

$$ S = \sum_k S_k Q(k), \quad (3.5.10) $$

where S_k being the component of S with vanishing spatial derivative. Then action of ${}^{(3)}\nabla_a$ results in

$$ {}^{(3)}\nabla_a S = \sum_k S_k {}^{(3)}\nabla_a Q(k). \quad (3.5.11) $$

Scalar harmonics

The scalar harmonics are solutions of scalar Helmholtz equation:

$$ {}^{(3)}\nabla^2 Q^{(0)} + \frac{k^2}{a^2} Q^{(0)} = 0. \quad (3.5.12) $$

$Q^{(0)}$ are constructed as time independent ($\dot{Q}^{(0)} = 0$) and dimensionless. From now on we drop the argument (k) from Qs. From the scalar eigenfunction $Q^{(0)}$ we can construct a vector:

$$ Q_a^{(0)} = -\frac{a}{k} {}^{(3)}\nabla_a Q^{(0)} \quad (3.5.13) $$

and a traceless symmetric tensor:

$$ \begin{aligned} Q_{ab}^{(0)} &= -\left(\frac{a}{k}\right) \nabla_{\langle a} Q_{b\rangle}^{(0)} \\ &= \left(\frac{a}{k}\right)^2 \nabla_{\langle a} \nabla_{b\rangle} Q^{(0)} \\ &= \left(\frac{a}{k}\right)^2 {}^{(3)}\nabla_{(a} {}^{(3)}\nabla_{b)} Q^{(0)} + \frac{1}{3} h_{ab} Q^{(0)}. \end{aligned} \quad (3.5.14) $$

The $\frac{a}{k}$ factors ensure that the $Q^{(0)}_a$ and $Q^{(0)}_{ab}$ are also time independent and dimensionless. Then applying the commutation relations (3.3.32)-(3.3.33) we get

$$\begin{aligned} {}^{(3)}\nabla_{[a}{}^{(3)}\nabla_{b]}Q^{(0)} &= -\omega_{ab}\dot{Q}^{(0)} = 0, && (3.5.15)\\ {}^{(3)}\nabla_{[a}{}^{(3)}\nabla_{b]}Q^{(0)}_c &= \frac{K}{2a^2}\left(h_{ac}Q^{(0)}_b - h_{bc}Q^{(0)}_a\right), && (3.5.16)\\ {}^{(3)}\nabla_{[a}{}^{(3)}\nabla_{b]}Q^{(0)}_{cd} &= \frac{K}{2a^2}\left(h_{ac}Q^{(0)}_{bd} - h_{bc}Q^{(0)}_{ad}\right)\\ &\quad + \frac{K}{2a^2}\left(h_{ad}Q^{(0)}_{bc} - h_{bd}Q^{(0)}_{ac}\right). && (3.5.17) \end{aligned}$$

Using the derivative ${}^{(3)}\nabla_a$ we derive various properties satisfied by the above-defined scalar harmonics:

$$a^{(3)}\nabla^a Q^{(0)}_a = kQ^{(0)}, \quad a^{2(3)}\nabla^2 Q^{(0)}_a = -(k^2-2K)Q^{(0)}_a, \tag{3.5.18}$$

$$a^{(3)}\nabla_b Q^{(0)}_a = -k\left(Q^{(0)}_{ab} - \tfrac{1}{3}h_{ab}Q^{(0)}\right), \tag{3.5.19}$$

$$h^{ab}Q^{(0)}_{ab} = 0, \quad a^{(3)}\nabla^b Q^{(0)}_{ab} = -\tfrac{2}{3}\left(\tfrac{3K}{k} - k\right)Q^{(0)}_a, \tag{3.5.20}$$

$$a^{2(3)}\nabla^{a(3)}\nabla^b Q^{(0)}_{ab} = \tfrac{2}{3}\left(k^2-3K\right)Q^{(0)}, \tag{3.5.21}$$

$$a^{2(3)}\nabla_b{}^{(3)}\nabla^c Q^{(0)}_{ac} = -\tfrac{2}{3}\left(k^2-3K\right)\left(Q^{(0)}_{ab} - \tfrac{1}{3}h_{ab}Q^{(0)}\right), \tag{3.5.22}$$

$$a^{2(3)}\nabla^2 Q^{(0)}_{ab} = -(k^2-6K)Q^{(0)}_{ab}. \tag{3.5.23}$$

Vector harmonics

Vector harmonics are vector eigen functions of Laplace-Beltrami operator:

$${}^{(3)}\nabla^2 Q^{(1)}_a = -\frac{k^2}{a^2}Q^{(1)}_a, \tag{3.5.24}$$

such that $Q^{(1)}_a$ are solenoidal vector,

$${}^{(3)}\nabla^a Q^{(1)}_a = 0. \tag{3.5.25}$$

With this we can construct a traceless symmetric (0,2) tensor:

$$Q^{(1)}_{ab} = -\frac{a}{k}{}^{(3)}\nabla_{(a}Q^{(1)}_{b)}. \tag{3.5.26}$$

Both $Q^{(1)}_a$ and $Q^{(1)}_{ab}$ are constant along fluid flow lines and satisfies the relations (3.5.16) and (3.5.17) respectively. Actions of spatial derivatives on $Q^{(1)}_{ab}$ are found from following properties:

$$h^{ab}Q^{(1)}_{ab} = 0, \quad a^{(3)}\nabla^b Q^{(0)}_{ab} = -\frac{1}{2}\left(\frac{2K}{k} - k\right) Q^{(1)}_a, \tag{3.5.27}$$

$$a^2 \left({}^{(3)}\nabla_b{}^{(3)}\nabla^c Q^{(1)}_{ac} + {}^{(3)}\nabla_a{}^{(3)}\nabla^c Q^{(1)}_{bc}\right) = -\left(k^2 - 2K\right) Q^{(1)}_{ab}, \tag{3.5.28}$$

$$a^{2(3)}\nabla^2 Q^{(1)}_{ab} = -(k^2 - 4K)Q^{(1)}_{ab}. \tag{3.5.29}$$

Tensor harmonics

Again tensor harmonics are symmetric second rank tensor eigen functions of Laplace-Beltrami operator:

$${}^{(3)}\nabla^2 Q^{(2)}_{ab} = -\frac{k^2}{a^2}Q^{(2)}_{ab}, \tag{3.5.30}$$

constant along fluid flow lines, traceless and divergenceless,

$$h^{ab}Q^{(2)}_{ab} = 0, \quad {}^{(3)}\nabla^b Q^{(2)}_{ab} = 0, \tag{3.5.31}$$

and satisfies the equation (3.5.17).

3.5.3 Scalar perturbations

For a barotropic fluid $p = p(\mu)$, all scalar perturbations can be expressed in terms of only two variables, X_a and Z_a. In other words the density fluctuations are represented by scalar perturbations and the evolution of linear density perturbations is completely decoupled from vector and tensor perturbations. So the dynamics of scalar perturbations are described completely by Eqs. (3.4.33) and (3.4.34). With the help of the Eq. (3.4.39) the first and third terms of the right hand side of

Eq. (3.4.34) is modified as follows:

$$\frac{1}{2}\mathcal{K}\nu_a = -\frac{\mathcal{K}}{2\kappa(\mu+p)}Y_a = -\frac{\mathcal{K}c_s^2}{2\kappa(\mu+p)}X_a, \tag{3.5.32}$$

$$A_a = {}^{(3)}\nabla_a{}^{(3)}\nabla^b\nu_b = -\frac{c_s^2}{\kappa(\mu+p)}{}^{(3)}\nabla_a{}^{(3)}\nabla^b X_b. \tag{3.5.33}$$

Multiplying both sides of the Eq. (3.4.33) by a^3 and taking derivative with respect to time we obtain

$$\begin{aligned}[a^2(a^4X_a)\dot{}]\dot{} + c_s^2\theta a^2(a^4X_a)\dot{} &= \frac{1}{2}\left(\kappa(\mu+p)+\mathcal{K}c_s^2\right)a^6X_a \\ &\quad + c_s^{2(3)}\nabla_a{}^{(3)}\nabla^b(a^6X_b).\end{aligned} \tag{3.5.34}$$

Then we can extract a second order differential equation for X_a [57]:

$$\begin{aligned}\ddot{X}_a + \left(\frac{10}{3}+c_s^2\right)\theta\dot{X}_a + \left[\frac{4}{3}\left(\frac{5}{3}+c_s^2\right)\theta^2 - \frac{7}{6}\kappa\mu - \frac{5}{2}\kappa p + \frac{4}{3}\Lambda - \frac{1}{2}\mathcal{K}c_s^2\right]X_a \\ - c_s^{2(3)}\nabla_a{}^{(3)}\nabla^b X_b = 0.\end{aligned} \tag{3.5.35}$$

This equation can be solved if we specify the vector fields X_a and $\dot{X}_a$ on an initial spacelike hypersurface. Then Z_a is obtained from the Eq. (3.4.33). Before looking at the solutions for different choices of background FLRW models we comment on some general properties of these equations.

1. Inhomogeneity on a world line γ is indicated by at least one of X_a, Z_a being nonzero. Because the equations governing its evolution are homogeneous, inhomogeneity cannot arise spontaneously: if both X_a and Z_a are zero at any point p on γ, then they both are zero at all points on γ; if either is nonzero at any event on γ, they are both nonzero at almost all points on γ (one or the other maybe zero at exceptional points).

2. In general, X and Z are not parallel. However, if they are parallel at one point p on γ, they are parallel at all points on γ; and if either vanishes at any event q on γ, they are parallel at all points on γ where they are nonzero.

3. The sign of the gravitational term is positive in the equations, (for example the term $\frac{1}{2}\kappa(\mu+p)a^6X_a$ in the right hand side of (3.5.34)) expressing the feature of gravitational instability of inhomogeneities. However, in the expanding universe the expansion, expressed in the factor a^6 works against this instability.

Since X_a is a scalar perturbation, we can expand it in terms of the harmonics $Q^{(0)}_a$ as

$$X_a = \sum_k X(k,t)Q^{(0)}_a. \tag{3.5.36}$$

Then the last term of (3.5.35) reduces to

$$\begin{aligned}
-c_s^{2(3)}\nabla_a{}^{(3)}\nabla^b X_b &= -c_s^{2(3)}\nabla_a{}^{(3)}\nabla^b \sum_k X(k,t)Q^{(0)}_b \\
&= -c_s^{2(3)}\nabla_a{}^{(3)}\nabla^b \sum_k X(k,t)\left(-\frac{a}{k}{}^{(3)}\nabla_b Q^{(0)}\right) \\
&= c_s^{2(3)}\nabla_a \sum_k \frac{a}{k} X(k,t)^{(3)}\nabla^2 Q^{(0)} \\
&= c_s^{2(3)}\nabla_a \sum_k \frac{a}{k} X(k,t)\left(-\frac{k^2}{a^2}\right)Q^{(0)} \\
&= c_s^2 \sum_k \frac{k^2}{a^2} X(k,t)Q^{(0)}_a
\end{aligned}$$

Now the Eq. (3.5.35) can be solved for the fourier modes $X(k,t)$ for each wave number k. Corresponding modes of Z_a are the given by

$$Z(k,t) = -\frac{[a^4X(k,t)]^{\dot{}}}{\kappa(\mu+p)a^4}. \tag{3.5.37}$$

As mentioned in Sec. (3.5.1), the shear has a scalar part σ^S_{ab}, which is obtained from the first of the constraint (3.4.40). Left hand side of the constraint is

$$^{(3)}\nabla_b(\omega^b{}_a + \sigma^b{}_a) = -r_a + {}^{(3)}\nabla^b\sigma^S_{ab} + {}^{(3)}\nabla^b\sigma^V_{ab} \tag{3.5.38}$$

Expanding the scalar σ^S_{ab} and the vectors r_a and σ^V_{ab} in harmonics as

$$r_a = \sum_k r(k,t)Q^{(1)}_a, \quad \sigma^S_{ab} = \sum_k \sigma^S(k,t)Q^{(0)}_{ab}, \quad \sigma^V_{ab} = \sum_k \sigma^V(k,t)Q^{(1)}_{ab} \tag{3.5.39}$$

and using the properties of harmonics, shown in the second of (3.5.20) and the second of (3.5.27) we obtain

$$\begin{aligned} {}^{(3)}\nabla^b\sigma^S_{ab} &= \sum_k \sigma^S(k,t)^{(3)}\nabla^b Q^{(0)}_{ab} = \sum_k \frac{2}{3}\left(k - \frac{3K}{k}\right)\sigma^S(k,t)Q^{(0)}_{ab}, \\ {}^{(3)}\nabla^b\sigma^V_{ab} &= \sum_k \sigma^V(k,t)^{(3)}\nabla^b Q^{(1)}_{ab} = \sum_k \frac{1}{2}\left(k - \frac{2K}{k}\right)\sigma^V(k,t)Q^{(1)}_{ab}. \end{aligned}$$

Putting this in (3.4.40) and equating the coefficients of each $Q^{(0)}_{ab}$ and $Q^{(1)}_{ab}$ we get

$$\sigma^S(k,t) = \frac{a}{k}\frac{Z(k,t)}{1-\frac{3K}{k^2}}, \quad \sigma^V(k,t) = \frac{a}{k}\frac{2r(k,t)}{1-\frac{2K}{k^2}}. \tag{3.5.40}$$

Let us demonstrate the solution for a flat FLRW background with zero cosmological constant and matter sector holding simple equation of state

$$p = w\mu, \tag{3.5.41}$$

with w is a constant. The time dependence of the zeroth order quantities are obtained from the Eqs. (3.4.31) and (3.4.32):

$$\mu = \frac{M}{a^{3(1+w)}}, \quad a = \left(\frac{t}{t_*}\right)^{\frac{2}{3(1+w)}}, \quad \theta = \frac{2}{1+w}\frac{1}{t}, \tag{3.5.42}$$

where M and t_* are constants. Then the Eq. (3.5.35) reduces to:

$$\ddot{X}_a + \frac{10+3w}{3}\theta\dot{X}_a + \frac{11+3w}{6}\theta^2 X_a - w^{(3)}\nabla_a{}^{(3)}\nabla^b X_b = 0 \tag{3.5.43}$$

or in terms of Fourier modes

$$\ddot{X} + \frac{10+3w}{3}\theta\dot{X} + \left(\frac{11+3w}{6}\theta^2 + w\frac{k^2}{a^2}\right)X = 0. \tag{3.5.44}$$

In terms of the conformal time η, defined as

$$\eta = \int_0^t \frac{dt}{a(t)}, \tag{3.5.45}$$

the Eq. (3.5.44) becomes

$$X'' + 3(3+w)\mathcal{H}X' + \left(\frac{3(11+3w)}{2}\mathcal{H}^2 + wk^2\right)X = 0, \tag{3.5.46}$$

where prime denotes differentiation with respect to η and $\mathcal{H} = a'/a = \frac{1}{3}a\theta$. Using (3.5.42) we can perform the integration (3.5.45) to obtain a and $\mathcal{H}$ as function of η:

$$a = \left(\frac{1+3w}{3(1+w)}\frac{\eta}{t_*}\right)^{\frac{2}{1+3w}}, \quad \mathcal{H} = \frac{2}{1+3w}\frac{1}{\eta}. \tag{3.5.47}$$

Then the Eq. (3.5.46) reduces to:

$$X'' + 6\frac{3+w}{1+3w}\frac{1}{\eta}X' + \left(6\frac{11+3w}{(1+3w)^2}\frac{1}{\eta^2} + wk^2\right)X = 0. \tag{3.5.48}$$

The Eq. (3.5.48) is readily solved in terms of the Bessel functions

$$X(k,\eta) = (k\eta)^{-\frac{17+3w}{2(1+3w)}}\left(C_1(k)Y_\alpha\left(\sqrt{w}k\eta\right) + C_2(k)J_\alpha\left(\sqrt{w}k\eta\right)\right), \tag{3.5.49}$$

where

$$\alpha = \frac{5+3w}{2(1+3w)}. \tag{3.5.50}$$

On large scale ($k \ll \mathcal{H}$) X shows a power law solution:

$$X(k,\eta) = \mathbb{X}_1(k)\eta^{-(11+3w)/(1+3w)} + \mathbb{X}_2(k)\eta^{-6/(1+3w)}. \tag{3.5.51}$$

From (3.5.37) we obtain for large scale:

$$Z(k,\eta) = \mathbb{Z}_1(k)\eta^{-8/(1+3w)} + \mathbb{Z}_2(k)\eta^{-3(1-w)/(1+3w)}, \tag{3.5.52}$$

where

$$\begin{aligned}
\mathbb{Z}_1(k) &= \frac{9}{1+3w}\theta_*^{-2}\mathbb{X}_1(k), \\
\mathbb{Z}_2(k) &= -\frac{6}{(1+w)(1+3w)}\theta_*^{-2}\mathbb{X}_2(k),
\end{aligned} \tag{3.5.53}$$

with θ_* is the value of θ at $t = t_*$. Note that if the equation of state is dust like, i.e. $w = 0$, the solutions (3.5.51) and (3.5.52) are valid for all scales.

3.5.4 Vector perturbations

Let us consider the evolution of vorticity. Using the energy conservation equation (3.4.31), momentum conservation equation (3.4.39) and the commutation relation (3.3.33), the right hand side of the Eq. (3.4.35) becomes

$$^{(3)}\nabla_{[b}\nu_{a]} = -\frac{^{(3)}\nabla_{[b}Y_{a]}}{\kappa(\mu+p)} = -\frac{^{(3)}\nabla_{[b}{}^{(3)}\nabla_{a]}p}{\mu+p} = \omega_{ba}\frac{\dot{p}}{\mu+p} = c_s^2\theta\omega_{ab}.$$

Then the Eq. (3.4.35) is solved to obtain a simple evolution of ω_{ab}:

$$\omega_{ab} = \frac{\Omega_{ab}}{a^2}\exp\left(\int c_s^2\theta dt\right), \tag{3.5.54}$$

where Ω_{ab} is an antisymmetric tensor, constant along the fluid flow lines ($\dot{\Omega}_{ab} = 0$). Taking the divergence of (3.5.59) we get

$$r_a = \frac{^{(3)}\nabla^b\Omega_{ab}}{a^2}\exp\left(\int c_s^2\theta dt\right).$$

But $^{(3)}\nabla^b\Omega_{ab}$ is not constant along the flow lines. To derive correct growth rate of r_a, let us consider the linearized commutation relation between the spatial divergence and the time derivative:

$$(^{(3)}\nabla^b\lambda_{ab})\dot{} = {}^{(3)}\nabla^b a(a^{-1}\lambda_{ab})\dot{}, \tag{3.5.55}$$

where λ_{ab} is a first order variable orthogonal to u^a. The Eq. (3.5.55) suggests

$$\dot{\Omega}_{ab} = 0 \Rightarrow \left[{}^{(3)}\nabla^b (a\Omega_{ab})\right]^{\dot{}} = 0. \tag{3.5.56}$$

So we can write

$$r_a = \frac{R_a}{a^3} \exp\left(\int c_s^2 \theta dt\right), \quad R_a = {}^{(3)}\nabla^b (a\Omega_{ab}). \tag{3.5.57}$$

The integration can be performed once the equation of state is given. For example we consider a simple equation of state

$$p = w\mu, \tag{3.5.58}$$

where w is a constant. Then inserting $c_s^2 = w$ and (3.3.15) in (3.5.54) we get

$$\omega_{ab} = \Omega_{ab} a^{-2+3w}, \quad r_a = R_a a^{-3+3w}. \tag{3.5.59}$$

In expanding universe the vector perturbations decay very quickly and thus they are not very interesting.

3.5.5 Tensor perturbations

The tensor perturbations describe gravitational waves, which are the degrees of freedom of the gravitational field itself. In the linear approximation the gravitational waves do not induce any perturbations in the perfect fluid.

In the covariant approach, the study of tensor perturbations was first considered by Hawking [24]. In his paper, the electric part of the Weyl tensor was used as the variable to characterize them. Later the magnetic part of the Weyl tensor was found to be a better choice, because it has no analogue in Newtonian theory where gravity is propagated instantaneously. Hence the magnetic part obviously plays an important role in describing gravitational waves, but given the correspondence with electromagnetism, where neither the electric nor magnetic fields provide a complete description of EM waves, it was suggested that both electric and magnetic parts of the Weyl tensor are required for a full understanding of tensor perturbations [69]. Indeed it is their curls that characterize gravitational waves, as we will see below.

The E_{ab} and H_{ab} are coupled with shear σ_{ab} at linear order as observed from (3.4.37). Thus we have to consider the tensor part of shear σ^T_{ab} for complete description of gravitational waves. To obtain the equations for tensor perturbations we set all the scalar and vector perturbation to zero,

$$X_a = Y_a = Z_a = 0, \quad \omega_{ab} = 0.$$

Then the constraints (3.4.40)-(3.4.42) implies

$$^{(3)}\nabla^b \sigma^T_{ab} = {}^{(3)}\nabla^b E^T_{ab} = {}^{(3)}\nabla^b H^T_{ab} = 0, \quad H^T_{ab} = -\mathrm{Curl}\sigma^T_{ab}. \tag{3.5.60}$$

The Eqs (3.4.36)-(3.4.38) are recast in the form:

$$\triangle\sigma^T_{ab} + \frac{5}{3}\theta\dot{\sigma}^T_{ab} + \left(\frac{4}{9}\theta^2 - \frac{\kappa(5\mu+9p)}{6} + \frac{2}{3}\Lambda\right)\sigma^T_{ab} = 0, \tag{3.5.61}$$

$$\triangle E^T_{ab} + \frac{7}{3}\theta\dot{E}^T_{ab} + \left(\theta^2 - \kappa(\mu+2p) + \Lambda\right)E^T_{ab} - \frac{1+3c_s^2}{6}\theta\kappa(\mu+p)\sigma^T_{ab} = 0, \tag{3.5.62}$$

$$\triangle H^T_{ab} + \frac{7}{3}\theta\dot{H}^T_{ab} + \left(\theta^2 - \kappa(\mu+2p) + \Lambda\right)H^T_{ab} = 0, \tag{3.5.63}$$

where $\triangle P^T_{ab} = \ddot{P}^T{}_{ab} - {}^{(3)}\nabla^2 P^T_{ab}$ for any traceless divergenceless symmetric tensor P^T_{ab}, orthogonal to u^a. We have used the identities:

$$\mathrm{Curl}^2 P^T_{ab} = -{}^{(3)}\nabla^2 P^T_{ab}, \quad (\mathrm{Curl}P^T_{ab})^{\dot{}} = \mathrm{Curl}a(a^{-1}P^T_{ab})^{\dot{}}. \tag{3.5.64}$$

So we get a second order differential equation for σ^T_{ab} and a second order differential equation for H^T_{ab}. However, the Eq. (3.5.62) contains a term having σ^T_{ab}. Elimination of σ^T_{ab} results in a third order differntial equation for the fourier modes of E_{ab}. To see this let us decompose E^T_{ab}, H^T_{ab} and σ^T_{ab} in terms of the tensor harmonics:

$$
\begin{aligned}
E^T_{ab} &= \sum_k E^T(k,t) Q^{(2)}_{ab}, \\
H^T_{ab} &= \sum_k H^T(k,t) Q^{(2)}_{ab}, \\
\sigma^T_{ab} &= \sum_k \sigma^T(k,t) Q^{(2)}_{ab},
\end{aligned}
\tag{3.5.65}
$$

such that all ${}^{(3)}\nabla^2$ in (3.5.61)-(3.5.63) are replaced by $-\frac{k^2}{a^2}$. Then differentiating (3.5.62) w.r.t. time and using the linearized shear evolution equation (3.4.36), specialized for tensor perturbation, we obtain

$$
\begin{aligned}
\dddot{E}^T + \left[3\theta - \frac{\dot{\mathcal{B}}}{\mathcal{B}}\right]\ddot{E}^T + \frac{7}{3}\left[\frac{1}{3}\theta^2 - \frac{1}{2}\kappa(\mu+3p) + \Lambda + \frac{3}{7}\mathcal{A} - \theta\frac{\dot{\mathcal{B}}}{\mathcal{B}}\right]\dot{E}^T \\
+ \left[\dot{\mathcal{A}} + \left(\frac{2}{3}\theta - \frac{\dot{\mathcal{B}}}{\mathcal{B}}\right)\mathcal{A} - \mathcal{B}\right] E^T = 0,
\end{aligned}
\tag{3.5.66}
$$

where

$$
\mathcal{A} = \theta^2 - \kappa(\mu+3p) + \Lambda + \frac{k^2}{a^2}, \quad \mathcal{B} = -\frac{1+3c_s^2}{6}\theta\kappa(\mu+p). \tag{3.5.67}
$$

In contrast, the shear and “magnetic” part of the Weyl tensor have evolution equation of second order. That is evident from (3.5.61) and (3.5.63).

The appearance of a third-order equation for E_{ab} is very surprising. Force laws are expected to be formulated as second-order evolution equations. In the standard coordinate based perturbation theory, the evolution equations of gravitational waves are obtained by linearizing the Einsteins equations and since Einstein equations are of second order for metric, the equations of gravitational waves are found to be second order. That means that all solutions of the Eq. (3.5.66) do not satisfy the original system of first order equations (3.4.36)-(3.4.38).

That is seen directly as follows. From the Eq (3.5.61) we obtain a general solution of $\sigma^T(k,t)$ with two arbitrary constants. Then substituting that solution in Eq. (3.4.36) we get $E^T = -a^{-2}(a^2\sigma^T\dot{)}$ with two arbitrary constants. Hence the system of equations (3.4.36)-(3.4.38) permits only two linearly independent solution for E^T and third solution of (3.5.66) is unphysical.

Now let us consider a flat FLRW background with zero cosmological constant and simple equation of state (3.5.41). Then the second order equations for σ^T and H^T are

$$\ddot{\sigma}^T + \frac{5}{3}\theta\dot{\sigma}^T + \left(\frac{1-3w}{6}\theta^2 + \frac{k^2}{a^2}\right)\sigma^T = 0, \tag{3.5.68}$$

$$\ddot{H}^T + \frac{7}{3}\theta\dot{H}^T + \left(\frac{2}{3}(1-w)\theta^2 + \frac{k^2}{a^2}\right)H^T = 0. \tag{3.5.69}$$

Using the conformal time η defined in (3.5.45), this equations are reduced to the following form:

$$\sigma^{T\prime\prime} + \frac{8}{1+3w}\frac{1}{\eta}\sigma^{T\prime} + \left(6\frac{1-3w}{(1+3w)^2}\frac{1}{\eta^2} + k^2\right)\sigma^T = 0, \tag{3.5.70}$$

$$H^{T\prime\prime} + \frac{12}{1+3w}\frac{1}{\eta}H^{T\prime} + \left(24\frac{1-w}{(1+3w)^2}\frac{1}{\eta^2} + k^2\right)H^T = 0. \tag{3.5.71}$$

The solutions of (3.5.70) and (3.5.71) are

$$\sigma^T(k,\eta) = (k\eta)^{-\frac{7-3w}{2(1+3w)}}\left(D_1(k)Y_\alpha(k\eta) + D_2(k)J_\alpha(k\eta)\right), \tag{3.5.72}$$

$$H^T(k,\eta) = (k\eta)^{-\frac{11-3w}{2(1+3w)}}\left(D_1^H(k)Y_\alpha(k\eta) + D_2^H(k)J_\alpha(k\eta)\right), \tag{3.5.73}$$

where α is given by (3.5.50). In the large scale limit ($k \ll \mathcal{H}$) we obtain power law solutions:

$$\sigma^T(k,\eta) = \Sigma_1(k)\eta^{-6/(1+3w)} + \Sigma_2(k)\eta^{-(1-3w)/(1+3w)}, \tag{3.5.74}$$

$$H^T(k,\eta) = \mathbb{H}_1(k)\eta^{-8/(1+3w)} + \mathbb{H}_2(k)\eta^{-3(1-w)/(1+3w)}. \tag{3.5.75}$$

The third order equation for E in terms of η is

$$\begin{aligned} E''' + \frac{1}{2}(7+3w)\mathcal{H}E'' + \left[6(5+2w)\mathcal{H}^2 + k^2\right]E' \\ +\frac{3}{2}\left[\left(16-(1+w)(1+3w)\right)\mathcal{H}^2 + 3(1+w)k^2\right]\mathcal{H}E = 0. \end{aligned} \tag{3.5.76}$$

Solution of the equation for large scale is obtained as

$$\begin{aligned} E^T(k,\eta) &= \mathbb{E}_1(k)\eta^{-4/(1+3w)} + \mathbb{E}_2(k)\eta^{-3(3+w)/(1+3w)} \\ &\quad +\mathbb{E}_3(k)\eta^{-(5-3w)/(1+3w)}. \end{aligned} \tag{3.5.77}$$

The first two of these modes are obtained from the (3.5.74) via (3.4.36). Hence the third mode in unphysical because it is not a solution of original system of first order equation.

3.6 Choice of observers

The gauge invariant variables, used in the covariant formulation of cosmological perturbation theory, are completely independent of the choice of coordinate system; however, as we have seen in the Sec. (3.3), the variables are defined with respect to some observers, known as “fundamental observers”. If the matter content of the univerese is described by a perfect fluid, there exists a preferred choice of observers, i.e. comoving observers. But there exists many other possible choices of timelike observers.

However, that choice is not completely arbitrary if we consider the physical manifold is an "almost FLRW" one. All the variables defined in the Sec. (3.3) are gauge invariant because they vanish on the background spacetime, which is possible only if the world lines of fundamental observers coincide with those of the comoving observers in the background FLRW spacetime. In other words, for any two possible choice of observers with 4-velocities u^a and $\tilde{u}^a$, $u^a - \tilde{u}^a$ is a first order variable that vanishes on the background. Let us decompose $\tilde{u}_a$ in components parallel and perpendicular to u^a as:

$$\tilde{u}_a = \sqrt{1+\beta^2}u_a + \beta_a, \tag{3.6.1}$$

where $\beta_a = h^b_a \tilde{u}_b$ and $\beta^2 = (u^a\tilde{u}_a)^2 - 1 = \beta_a\beta^a$. The vector β_a is a first order variable. The projection tensor $\tilde{h}_{ab} = g_{ab} + \tilde{u}_a\tilde{u}_b$ is related to h_{ab} by

$$\tilde{h}_{ab} = h_{ab} + \beta^2 u_a u_b + \sqrt{1+\beta^2}\,(u_a\beta_b + \beta_a u_b) + \beta_a\beta_b. \tag{3.6.2}$$

Then the energy momentum tensor in the frame of $\tilde{u}^a$ is written as

$$T_{ab} = (\tilde{\mu}+\tilde{p})\tilde{u}_a\tilde{u}_b + \tilde{p}g_{ab} + 2\tilde{u}_{(a}\tilde{q}_{b)} + \tilde{\pi}_{ab}, \tag{3.6.3}$$

where $\tilde{\mu} = T_{ab}\tilde{u}^a\tilde{u}^b$, $\tilde{p} = \frac{1}{3}\tilde{h}^{ab}T_{ab}$, $\tilde{q}_a = -\tilde{h}_a{}^b T_{bc}\tilde{u}^c$ and $\tilde{\pi} = \tilde{h}_a{}^c\tilde{h}_b{}^d T_{cd} - \tilde{p}\tilde{h}_{ab}$ are combinations of the quantities defined in the frame of u^a and the vector β_a. For example:

$$\tilde{\mu} = \mu + \beta^2(\mu+p) - 2\sqrt{1+\beta^2}q_a\beta^a + \pi_{ab}\beta^a\beta^b, \tag{3.6.4}$$

$$\tilde{p} = p + \frac{1}{3}\left[\beta^2(\mu+p) - 2\sqrt{1+\beta^2}q_a\beta^a + \pi_{ab}\beta^a\beta^b\right]. \tag{3.6.5}$$

$$\tag{3.6.6}$$

So the differences in energy densities or pressure measured in two frames are of higher order. We can construct gauge invariant variables in the frame of $\tilde{u}^a$

just as in the Sec. (3.3) and again those variables will be some combination of the gauge invariant variables defined in the frame of u^a. Let us consider two special observers: One has the four velocity u^E_a, defined by the Eq. (3.3.3) with zero energy flux ($q^E_a = 0$) and other one has the four velocity u_a, normal to constant energy density hypersurface:

$$u_a = -\frac{\nabla_a \mu}{\sqrt{-\nabla_b \mu \nabla^b \mu}}. \tag{3.6.7}$$

In the frame of u^a, $X_a = 0$. We can write relations between the two veloicities in a way similar to (3.6.1):

$$u_a = \sqrt{1+\beta^2} u^E_a + \beta^E_a, \quad u^E_a = \sqrt{1+\beta^2} u_a + \beta_a,$$

where β_a and β^E_a have following relations:

$$\beta_a = -\beta^2 u^E_a - \sqrt{1+\beta^2} \beta^E_a \approx -\beta^E_a, \quad \beta_a \beta^a = \beta^{Ea} \beta^E_a = \beta^2, \tag{3.6.8}$$

the simbol $\approx$ stands for relations keeping up to first order terms. Various zeroth and first order quantities in the two frames are related as:

$$\mu \approx \mu^E, \quad p \approx p^E, \quad \theta \approx \theta^E + \nabla^a \beta^E_a, \tag{3.6.9}$$

$$\begin{aligned} q_a \approx q^E_a - (\mu + p)\beta^E_a, \quad \pi_{ab} \approx \pi^E_{ab} \\ X_a \approx X^E_a + \kappa \dot{\mu} \beta^E_a, \\ Z_a \approx Z^E_a + \dot{\theta} \beta^E_a + {}^{(3)}\nabla_a \nabla^c \beta^E_c, \\ \nu_a \approx \nu^E_a + \dot{\beta}^E_a + \frac{1}{3}\theta \beta^E_a. \end{aligned} \tag{3.6.10}$$

Using $q_a^E = 0$ and $X_a = 0$ we have

$$
\begin{aligned}
q_a &\approx -\frac{X_a^E}{\kappa\theta} \approx (\mu + p)\beta_a, \\
Z_a &\approx Z_a^E + \frac{\dot{\theta}X_a^E + {}^{(3)}\nabla_a \nabla^b X_b^E}{\theta\kappa(\mu + p)}, \\
\nu_a &\approx \nu_a^E + \frac{\dot{X}_a^E + \left[\left(\frac{4}{3} + c_s^2\right)\theta - \frac{\dot{\theta}}{\theta}\right] X_a^E}{\kappa(\mu + p)\theta}.
\end{aligned} \tag{3.6.11}
$$

Now let us consider the linearized form of evolution equations for density gradient (3.4.21) and expansion gradient (3.4.22) in the frame of u^a:

$$
\kappa(\mu + p)Z_a - \kappa\theta\left(\dot{q}_a + \frac{4}{3}\theta q_a + {}^{(3)}\nabla^c \pi_{ac}\right) + \kappa^{(3)}\nabla_a{}^{(3)}\nabla^b q_b = 0, \tag{3.6.12}
$$

$$
\dot{Z}_a + \theta Z_a - \mathcal{R}\nu_a - A_a - \frac{3}{2}\kappa\left(\dot{q}_a + \frac{4}{3}\theta q_a + {}^{(3)}\nabla^c \pi_{ac}\right) = 0. \tag{3.6.13}
$$

Using the relations (3.6.10) and (3.6.11) to replace q_a, Z_a, π_a and ν_a by X_a^E, Z_a^E, π_a^E and ν_a^E, the above equations are reduced to:

$$
\dot{X}_a^E + \frac{4}{3}\theta X_a^E + \kappa(\mu + p)Z_a^E - \kappa\theta^{(3)}\nabla^c \pi_{ac}^E = 0, \tag{3.6.14}
$$

$$
\dot{Z}_a^E + \theta Z_a^E - \mathcal{R}\nu_a^E + \frac{1}{2}X_a^E - A_a^E - \frac{3}{2}\kappa^{(3)}\nabla^c \pi_{ac}^E = 0, \tag{3.6.15}
$$

which are linearized form of (3.4.21) and (3.4.22) in the frame of u_a^E.

Hence the gauge invariant variables defined in the frame of u^a are combinations of the variables defined in the frame of u_a^E and these "new" variables satisfy the equations, same as the evolution equations of the "old" variables.

3.7 Relation with the ordinary gauge invariant variables

In the coordinate based perturbation theory, we consider small fluctuations of spacetime metric about the background, which in our case is a flat FLRW metric:

$$\bar{ds}^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = a^2(\eta)\left(-d\eta^2 + dx^i dx^i\right) \tag{3.7.1}$$

and similar fluctuations of energy-momentum tensor about an homogeneous and isotropic perfect fluid energy-momentum tensor:

$$\bar{T}_{\mu\nu} = (\bar{\mu} + \bar{p})\bar{u}_\mu \bar{u}_\nu + \bar{p}\bar{g}_{\mu\nu}, \tag{3.7.2}$$

where $\bar{\mu}(\eta)$ and $\bar{p}(\eta)$ are energy density and pressure as observed by a comoving observer with velocity $\bar{u}^\mu$:

$$\bar{u}_\mu \bar{u}^\mu = -1, \quad \bar{u}_\mu = \left(\frac{1}{a}, \vec{0}\right), \quad \bar{u}^\mu = \left(-a, \vec{0}\right). \tag{3.7.3}$$

Metric perturbations are defined as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}.$$

The spatial isotropy and homogeneity of the background metric and energy-momentum tensor allow us to simplify these results by decomposing the perturbations into scalars, divergenceless vectors, and divergenceless traceless symmetric tensors with respect to coordinate transformation on the homogeneous and isotropic spacelike (constant time) hypersurfaces in the background spacetime. These scalar, vector and tensor modes are not coupled to each other by the linearized field equations or conservation equations. Perturbations of the metric can always be put in the form:

$$\delta g_{\mu\nu} = a^2 \begin{pmatrix} -2\phi & \partial_i \mathcal{B} - \mathcal{B}_i \\ \partial_i \mathcal{B} - \mathcal{B}_i & -2\psi\delta_{ij} + 2\partial_i\partial_j\mathcal{E} + \partial_i\mathcal{E}_j + \partial_j\mathcal{E}_i + \mathcal{E}_{ij} \end{pmatrix}, \tag{3.7.4}$$

where $\phi, \mathcal{B}, \psi$ and $\mathcal{E}$ are scalar, $\mathcal{B}_i$ and $\mathcal{E}_i$ are divergenceless vectors and $\mathcal{E}_{ij}$ is a divergenceless, traceless, symmetric tensor. The metric perturbations produce perturbations in the Christofell symbols:

$$\delta g^{\mu}_{\nu\lambda} = \frac{1}{2}\bar{g}^{\mu\rho}\left(-2\delta g_{\rho\sigma}\bar{\Gamma}^{\sigma}_{\nu\lambda} + \partial_\lambda \delta g_{\rho\nu} + \partial_\nu \delta g_{\rho\lambda} - \partial_\rho \delta g_{\lambda\nu}\right), \tag{3.7.5}$$

where $\bar{\Gamma}^{\mu}_{\nu\lambda}$ is the Christofell symbol of backgraound spacetime. Its nonzero components are

$$\bar{\Gamma}^0_{00} = \mathcal{H}, \quad \bar{\Gamma}^i_{0j} = \bar{\Gamma}^0_{ij} = \mathcal{H}\delta_{ij}. \tag{3.7.6}$$

The nonzero components of $\delta\Gamma^{\mu}_{\nu\lambda}$ are

$$\begin{aligned}
&\delta\Gamma^0_{00} = \phi', \quad \delta\Gamma^0_{0i} = \partial_i(\phi + \mathcal{H}\mathcal{B}), \\
&\delta\Gamma^0_{ij} = -\left(\psi' + 2\mathcal{H}(\psi + \phi)\right)\delta_{ij} - \partial_i\partial_j(\mathcal{B} - \mathcal{E}' - 2\mathcal{H}\mathcal{E}), \\
&\delta\Gamma^i_{00} = \partial_i(\phi + \mathcal{B}' + \mathcal{H}\mathcal{B}), \quad \delta\Gamma^i_{0j} = -\psi'\delta_{ij} + \partial_i\partial_j\mathcal{E}', \\
&\delta\Gamma^i_{jk} = \delta_{jk}\partial_i(\psi - \mathcal{H}\mathcal{B}) - \delta_{ik}\partial_j\psi - \delta_{ij}\partial_k\psi - \partial_i\partial_j\partial_k\mathcal{E}.
\end{aligned}$$

We use the notations, $()' = \frac{d}{d\eta}$, $\dot{()} = \frac{d}{dt} = \bar{u}^\mu\bar{\nabla}_\mu$, $\mathcal{H} = \frac{a'}{a}$. The perfect fluid perturbations are described in terms of perturbations in energy density, pressure and the velocity:

$$\begin{aligned}
p(\eta, \vec{x}) = \bar{p}(\eta) + \delta p(\eta, \vec{x}), \quad \mu(\eta, \vec{x}) = \bar{\mu}(\eta) + \delta\mu(\eta, \vec{x}), \\
\bar{u}^\mu = u^\mu + \delta u^\mu.
\end{aligned} \tag{3.7.7}$$

From the condition $u_\mu u^\mu = -1$ we have

$$\delta u^0 = -\frac{\phi}{a}, \quad \delta u_0 = -a\phi \tag{3.7.8}$$

and

$$\begin{aligned} \delta u_i &= \partial_i \mathcal{U} + \mathcal{U}_i, \\ \delta u^i &= \frac{1}{a^2}\left[\partial_i\left(\mathcal{U} - a\mathcal{B}\right) + \left(\mathcal{U}_i + a\mathcal{B}_i\right)\right], \end{aligned} \tag{3.7.9}$$

where again we have used standard decomposition of the perturbations of the energy momentum tensor in terms of the scalars $\delta\mu, \delta p, \mathcal{U}$ and the divergenceless vector $\mathcal{U}_i$. We consider only perfect fluid perturbations. Hence the anisotropic stresses are zero. All of the above variables are not invariant under infinitesimal coordinate (gauge) transformation. But we can construct some gauge invariant variables as follows:

$$\begin{aligned} \Phi = \phi + \tfrac{1}{a}\left(a(\mathcal{B} - \mathcal{E}')\right)', &\quad \Psi = \psi - \tfrac{a'}{a}(\mathcal{B} - \mathcal{E}'), \\ \mathcal{U}^{\mathrm{GI}} = \mathcal{U} - a(\mathcal{B} - \mathcal{E}'), &\quad \delta\mu^{\mathrm{GI}} = \delta\mu + \bar{\mu}'(\mathcal{B} - \mathcal{E}'), \\ \delta p^{\mathrm{GI}} &= \delta p + \bar{p}'(\mathcal{B} - \mathcal{E}'), \end{aligned} \tag{3.7.10}$$

$$\mathcal{B}_i^{\mathrm{GI}} = \mathcal{B}_i + \mathcal{E}_i. \tag{3.7.11}$$

$\mathcal{U}_i$ and $\mathcal{E}_{ij}$ are gauge invariant.

The expansion θ can be written as

$$\theta = \nabla_\mu u^\mu = \bar{\theta} + \delta\theta, \quad \bar{\theta} = \frac{3\mathcal{H}}{a}, \tag{3.7.12}$$

$$\delta\theta = -\frac{3}{a}\left(\psi' + \mathcal{H}\phi\right) + \frac{1}{a^2}\vec{\nabla}^2\left(\mathcal{U} + a(\mathcal{E}' - \mathcal{B})\right). \tag{3.7.13}$$

Covariant perturbations are defined in terms of the projection tensor h_{ab} in the physical manifold. Components of this tensor are

$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu = \bar{h}_{\mu\nu} + \delta g_{\mu\nu} + \bar{u}_\mu \delta u_\nu + \bar{u}_\nu \delta u_\mu,$$

where $\bar{h}_{ab}$ is the metric of constant time 3-hypersurfaces in the background space-time:

$$\bar{h}_{\mu\nu} = \bar{g}_{\mu\nu} + \bar{u}_\mu \bar{u}_\nu = \begin{pmatrix} 0 & 0 \\ 0 & a^2\delta_{ij} \end{pmatrix}.$$

The components of $h_a^{\ b}$ are

$$\begin{aligned} h_\mu^{\ \nu} &= \bar{h}_\mu^{\ \nu} + \bar{u}_\mu \delta u^\nu + \bar{u}^\nu \delta u_\mu \\ &= \begin{pmatrix} 0 & -\frac{1}{a}\partial_i(\mathcal{U} - a\mathcal{B}) - \frac{1}{a}(\mathcal{U}_i + a\mathcal{B}_i) \\ \frac{1}{a}(\partial_i \mathcal{U} + \mathcal{U}_i) & \delta_{ij} \end{pmatrix}. \end{aligned}$$

The components of shear σ_{ab} are

$$\sigma_{\mu\nu} = h^\alpha_{(\mu} h^\beta_{\nu)} \nabla_\beta u_\alpha - \frac{1}{3} h_{\mu\nu}\theta, \tag{3.7.14}$$

$$\sigma_{00} = 0, \quad \sigma_{0i} = 0, \quad \sigma_{ij} = \sigma^S_{ij} + \sigma^V_{ij} + \sigma^T_{ij}, \tag{3.7.15}$$

where

$$\sigma^S_{ij} = \partial_i\partial_j \mathcal{U}^{\mathrm{GI}} - \frac{1}{3}\vec{\nabla}^2 \mathcal{U}^{\mathrm{GI}} \delta_{ij}, \tag{3.7.16}$$

$$\sigma^V_{ij} = \frac{1}{2}\left[\partial_i\left(\mathcal{U}_j + a\mathcal{B}^{GI}_j\right) + \partial_j\left(\mathcal{U}_i + a\mathcal{B}^{GI}_i\right)\right],$$

$$\sigma^T_{ij} = \frac{1}{2}\mathcal{E}_{ij}. \tag{3.7.17}$$

In the same way we calculate the components of the vorticity ω_{ab}:

$$\omega_{\mu\nu} = h^\alpha_{[\mu} h^\beta_{\nu]} \nabla_\beta u_\alpha, \tag{3.7.18}$$

$$\omega_{00} = 0, \quad \omega_{0i} = 0, \quad \omega_{ij} = \partial_j \mathcal{U}_i - \partial_i \mathcal{U}_j \tag{3.7.19}$$

and the divergenceless vector r_a:

$$r_i = \frac{1}{a^2}\vec{\nabla}^2\mathcal{U}_i. \tag{3.7.20}$$

Components of the density gradient X_a and the expansion gradient Z_a are

$$\begin{aligned} X_i &= \kappa\left(\partial_i\delta\mu + \frac{1}{a}\bar{\mu}'\delta u_i\right), \\ &= \kappa\partial_i\left(\delta\mu + \bar{\mu}'\frac{\mathcal{U}}{a}\right) + \kappa\bar{\mu}'\frac{\mathcal{U}_i}{a} \\ &= \kappa\partial_i\left(\delta\mu^{\rm GI} + \frac{\bar{\mu}'}{a}\mathcal{U}^{\rm GI}\right) + \frac{\kappa\bar{\mu}'}{a}\mathcal{U}_i, \end{aligned} \tag{3.7.21}$$

$$\begin{aligned} Z_i &= \partial_i(\delta\theta + \dot{\theta}\mathcal{U}) + \dot{\theta}\mathcal{U}_i \\ &= \partial_i\left[-\frac{3}{a}\left(\Psi' + \mathcal{H}\Phi\right) + \frac{1}{a^2}\vec{\nabla}^2\mathcal{U}^{\rm GI} - \frac{3}{2}\kappa(\bar{\mu}+\bar{p})\mathcal{U}^{\rm GI}\right] \\ &\qquad -\frac{3}{2}\kappa(\bar{\mu}+\bar{p})\mathcal{U}_i^{\rm GI}. \end{aligned} \tag{3.7.22}$$

To obtain the above relations we have used the background Friedmann equations:

$$\begin{aligned} \dot{\bar{\theta}} = \frac{3}{a^2}(\mathcal{H}' - \mathcal{H}^2) &= -\frac{3}{2}\kappa(\bar{\mu}+\bar{p}), \\ \frac{1}{3}\bar{\theta}^2 = \frac{3\mathcal{H}^2}{a^2} &= \kappa\bar{\mu} \end{aligned} \tag{3.7.23}$$

and the conservation equation:

$$\bar{\mu}' + 3\mathcal{H}(\bar{\mu}+\bar{p}) = 0. \tag{3.7.24}$$

The expressions (3.7.21) and (3.7.22) can be further simplified using the perturbation equations used in coordinate based perturbation theory. For perfect fluid perturbations (anisotropic stresses are absent) $\Phi = \Psi$,

$$\vec{\nabla}^2\Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Phi) = \frac{1}{2}\kappa a^2\delta\mu^{\rm GI}, \tag{3.7.25}$$

$$(a\Phi)' = -\frac{1}{2}\kappa a^2(\bar{\mu}+\bar{p})\mathcal{U}^{\rm GI}, \tag{3.7.26}$$

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = \frac{1}{2}\kappa a^2\delta p^{\rm GI}. \tag{3.7.27}$$

Using (3.7.26) we obtain

$$\mathcal{U}^{\mathrm{GI}} = \frac{a}{\mathcal{H}} (\Phi - \zeta), \tag{3.7.28}$$

where ζ is the comoving curvature perturbation:

$$\zeta = \frac{2}{3} \frac{\Phi + \mathcal{H}^{-1}\Phi'}{1+w} + \Phi. \tag{3.7.29}$$

Note that $w = p/\mu$ is not a constant in Eq. (3.7.29). Then,

$$\begin{aligned} X_i &= \frac{2}{a^2}\partial_i \vec{\nabla}^2 \Phi - \frac{3\mathcal{H}}{a}\kappa(\bar{\mu}+\bar{p})\mathcal{U}_i, & (3.7.30) \\ Z_i &= \frac{1}{a\mathcal{H}}\partial_i \vec{\nabla}^2 (\Phi - \zeta) - \frac{3}{2}\kappa(\bar{\mu}+\bar{p})\mathcal{U}_i. & (3.7.31) \end{aligned}$$

The scalar covariant perturbations are related not only to the scalar perturbations but also to the vector perturbations of coordinate based perturbation theory. The reason is that in the coordinate based perturbation theory, the 3+1 decomposition is done with respect to the world lines of the background comoving observers, whereas in the covariant perturbation theory, we use the world lines of the comoving observers of the physical spacetime.

It is not very surprising that the spatial gradient of $\mathcal{K}$ which is the curvature of 3-hypersurfaces orthogonal to fluid flow lines in vanishing vorticity case is directly related to the comoving curvature perturbation ζ:

$$\mathcal{K}_i = \frac{2}{a^2}\partial_i \vec{\nabla}^2 \zeta. \tag{3.7.32}$$

However, ζ is also related to the comoving curvature perturbation ζ_a, defind in (3.3.57).

$$\begin{aligned} W_i &= \partial_i \left(\delta \ln a + \dot{(\ln a)} \mathcal{U} \right) + \dot{(\ln a)} \mathcal{U}_i \\ &= \partial_i \left(\frac{\delta a}{a} + \frac{1}{3}\bar{\theta}\mathcal{U} \right) + \frac{1}{3}\bar{\theta}\mathcal{U}_i. & (3.7.33) \end{aligned}$$

Now,

$$\theta = 3\frac{d}{dt}\ln a, \quad \bar{\theta} = 3\frac{d}{d\bar{t}}\ln \bar{a}, \tag{3.7.34}$$

where t is the time of the comoving observers in physical spacetime and $\bar{t}$ is that of the comoving observers in background FLRW spacetime. derivatives with respect to t and $\bar{t}$ are defined as:

$$\frac{d}{dt} = u^a \nabla_a, \quad \frac{d}{d\bar{t}} = \bar{u}^a \bar{\nabla}_a. \tag{3.7.35}$$

$$\theta = 3u^a \nabla_a \ln a = 3\left(\frac{d}{d\bar{t}} + \delta u^\mu \partial_\mu\right)\left(\ln \bar{a} + \frac{\delta a}{a}\right) = \bar{\theta}(1-\phi) + 3\frac{d}{d\bar{t}}\frac{\delta a}{a}.$$

Then using (3.7.13),

$$3\frac{d}{d\bar{t}}\frac{\delta a}{a} = \delta\theta + \bar{\theta}\phi = -\frac{3}{a}\psi' + \frac{1}{a^2}\vec{\nabla}^2\mathcal{U}^{\mathrm{GI}}. \tag{3.7.36}$$

Since $d\bar{t} = a d\eta$,

$$\frac{\delta a}{a} = -\psi + \frac{1}{3}\int \frac{1}{a}\vec{\nabla}^2\mathcal{U}^{\mathrm{GI}} d\eta. \tag{3.7.37}$$

From (3.7.33) we get

$$\begin{aligned} W_i &= \partial_i\left(-\Psi + \frac{1}{3}\theta\mathcal{U}^{\mathrm{GI}} + \frac{1}{3}\int \frac{1}{a}\vec{\nabla}^2\mathcal{U}^{\mathrm{GI}} d\eta\right) + \frac{1}{3}\theta\mathcal{U}_i \\ &= -\partial_i\left(\zeta + \frac{1}{3}\int d\eta\mathcal{H}^{-1}\vec{\nabla}^2\left(\zeta - \Phi\right)\right) + \frac{1}{3}\theta\mathcal{U}_i. \end{aligned} \tag{3.7.38}$$

Then using (3.7.30) and (3.7.38), we obtain the relation between components of ζ_a and ζ:

$$\zeta_i = \partial_i\left(-\zeta + \frac{2\vec{\nabla}^2\Phi}{\kappa(\bar{\mu} + \bar{p})a^2} - \frac{1}{3}\int d\eta\mathcal{H}^{-1}\vec{\nabla}^2\left(\zeta - \Phi\right)\right). \tag{3.7.39}$$

When spatial derivatives are small, $\zeta_i \approx -\partial_i \zeta$ and $\mathcal{K}_i \approx -\frac{4}{a^2}\vec{\nabla}^2\zeta_i$.

The spatial curvature perturbation $\delta\mathcal{R}$ is defined as

$$\delta\mathcal{R} = \zeta + \frac{2}{3}\frac{\vec{\nabla}^2\Phi}{\kappa(\bar{\mu}+\bar{p})a^2}. \tag{3.7.40}$$

It can be shown readily that $\delta\mathcal{R}$ is related to the covariant variable

$$V_a = \frac{1}{2}\mathcal{K}_a + \frac{2}{3}\frac{{}^{(3)}\nabla^2 X_a}{\kappa(\bar{\mu}+\bar{p})} + \frac{2}{3}\theta r_a \tag{3.7.41}$$

as

$$V_i = \frac{2}{a^2}\partial_i \vec{\nabla}^2 \delta\mathcal{R}. \tag{3.7.42}$$

3.8 Conclusion

In this chapter we have presented a description of the covariant perturbation theory, as an alternative to the standard coordinate based approach to study cosmological inhomogeneities. In the standard approach one starts from a ideal background spacetime with a background metric and matter configuration and then perturb them to obtain a more realistic physical universe. Perturbations defined in this way bear many nonphysical degrees of freedom due to general coordinate transformation in the background spacetime. Such non uniqueness in the definition of perturbations is known as gauge ambiguity. The gauge transformation on any tensor quantity can be represented in terms of lie derivative of that variable along any arbitrary vector field. Thus vanishing of lie derivative along arbitrary vector field implies gauge invariance. This is the precisely the Stewart-Walker lemma that a quantity is gauge invariant if in the background spacetime it is zero or a constant scalar or a constant linear combination of Kronecker deltas.

In covariant perturbation theory we starts from a physical manifold with arbitrary metric and matter configuration, which is close to a background spacetime (generally an FLRW one). We choose a suitable family of timelike observers with four velocity u^a, provided the worldlines of these observers coincide with that of comoving observers in the background FLRW spacetime. Then we define certain tensor quantities that vanish on the background spacetime, such that these can be regarded as gauge invariant variables according to Stewart Walker lemma. A closed set of non-linear evolution equations are obtained for these variables. Then we linearize the equations for an "almost FLRW" spacetime, by just giving up the terms appeared as products of first order variables. The linearized equations are solved following decomposition of first order variables into scalar, divergenceless vector and traceless, divergenceless tensor.

Although the perturbations and their evolution equations are completely gauge invariant and covariant, apparently a new type of ambiguity may arise due to choice of observers. If the background spacetime is spatially homogeneous and isotropic, there exists a preferred family of observers, i.e. the comoving observers, which are also observer of spatial homogeneity and isotropy. In the physical inhomogeneous manifold no such prefered observers exist. However, under the change of observers, the variables and their evolution equations do not change, only their interpretation is altered.

Finally we have presented the relations between the covariant perturbations and the ordinary gauge invariant variables, used in the coordinate based perturbation theory.

Chapter 4

Perturbations in the contracting phase of a bouncing universe

4.1 Introduction

We have seen in Sec. (2.5) that the perturbations have growing modes in a collapsing universe, which can invalidate the linear perturbation theory at or near the bounce and give rise to nonlinear effects, which in turn may affect the power spectrum of the perturbations. Hence one should be cautious in using the linear perturbation theory to evolve the scale invariant perturbations originated in the matter dominated phase of the contracting branch through the bounce. The calculations in standard coordinate based perturbation theory have shown that the perturbations grow in some gauges, while they remain small in some other gauges. For example the perturbation remains linear close to the bounce if one uses the uniform curvature gauge instead of the Newtonian gauge. Similarly, *at the bounce*, where Hubble's constant goes to zero, neither the Newtonian nor the uniform curvature gauge but the synchronous gauge preserves the linearity of the perturbation.

Similar phenomena is observed in the expanding universe. Scalar perturbations grow in the longitudinal gauge in inflationary phase. But that growing mode can be “gauged down” by choosing an appropriate gauge and the spectrum of perturbations are computed in that gauge or using the covariant and gauge invariant variables [70].

The purpose of this chapter, based on [29], is to analyze the problem in the covariant approach to the perturbation theory. In this approach, the dynamical variables are fully gauge invariant and hence, the analysis is completely independent of the choice of gauge conditions. Moreover, in contrast with the perturbative expansion of Einstein equations used in standard perturbation theory, the dynamical equations are exact and nonlinear. One can relate the variables used in the covariant analysis with the gauge invariant variables used in the standard perturbation theory. For example, the vector ζ_a, defined in Eq. (3.3.57) of Sec. (3.3.3), can be seen as a generalization of the curvature perturbation ζ, used in linear theory. In particular ζ_a coincides with the usual ζ for long wavelengths, but the two quantities differ on small scales, where spatial derivatives cannot be neglected.

We consider adiabatic and perfect fluid perturbations in a collapsing radiation and dust dominated flat FLRW background. For the equation of state $p = w\mu$, the zeroth order quantities in flat FLRW background are defined by the Eq (3.5.42) with t_* is replaced by $-t_*$:

$$\mu = \frac{M}{a^{3(1+w)}}, \quad a = \left(-\frac{t}{t_*}\right)^{\frac{2}{3(1+w)}}, \quad \theta = \frac{2}{1+w}\frac{1}{t}.$$

We assume the bounce occurs at $t = -t_b$ such that t_b is of the order of Plank scale. The initial time $t = -t_*$ is well inside the regime dominated by the single fluid considered here such that, $a_b = a(-t_b) << 1$. The constants M and t_* have the following relation:

$$\kappa M t_*^2 = \frac{4}{3(1+w)^2}. \tag{4.1.1}$$

First, the linear perturbation equations are solved in these backgrounds near the bounce. Then these solutions are used to compare the linear and nonlinear terms in the full nonlinear perturbation equations to investigate the validity of the linearized approximations. Our discussion is general and not specific to a particular model of bounce.

So far it has been checked whether perturbations are sufficiently small compared with their background quantities. If that criterion holds, second order perturbations are assumed to be even smaller and linear perturbation theory is considered to be a good approximation. But this test will not work in covariant formalism as the background values of the gauge invariant perturbations are zero. In the present work, we have checked whether higher order perturbations are truly smaller compared with 1st order ones. We will show that the answer is not positive for all perturbation modes near bounce.

The outline of this chapter is as follows. In Sec. 4.2, the conditions for validity of the linear approximation for adiabatic perturbations in a flat FLRW backgraound are set up. In Sec. 4.3 we present the solutions of the linearized equations for a background with matter equation of state $p = w\mu$. In Sec. 4.4 we compare the first and higher order terms in the full nonlinear equations.

4.2 Conditions for linearity

In the standard perturbation theory, linearization is justified if the perturbations are small with respect to corresponding background quantities. The gauge invariant variables, defined in Sec. (3.3.3), are nonlinear because we have not assumed any "smallness" to define these quantities. However, when we linearize the system of equations to study the linear evolution, we assume that the perturbations are "small" in some sense. It is not possible to describe that "smallness" in the same mathematical language as used in coordinate based perturbation theory because the variables, corresponding to the gauge invariant variables in physical spacetime, are zero by definition on the background spacetime. The use of dimensionless variable [71] is not unique, because one can multiply any power of scale factor a and that variable remains dimensionless. The natural way is to demand that the higher order terms of perturbation equations remain small with respect to the first order terms.

Let us concentrate on the evolution of density perturbations X_a. The X_a is coupled with Z_a up to linear order by the Eqs. (3.4.21) and (3.4.22). We consider only perfect fluid perturbations, i. e., $q_a = 0$ and $\pi_{ab} = 0$. Hence these equations become:

$$a^{-4} h_a^{\ b} \dot{(a^4 X_b)} = -\kappa(\mu + p) Z_a - (\sigma^b_{\ a} + \omega^b_{\ a}) X_b, \tag{4.2.1}$$

$$a^{-3} h_a^{\ b} \dot{(a^3 Z_b)} = \mathcal{R}\nu_a - \frac{1}{2} X_a + A_a + 2^{(3)}\nabla_a(\omega^2 - \sigma^2) - (\sigma^b_{\ a} + \omega^b_{\ a}) Z_b. \tag{4.2.2}$$

In the right hand side of the first equation, the term $\kappa(\mu + p)Z_a$ is of first order (linear) whereas $\sigma^b_{\ a} X_b$ and $\omega^b_{\ a} X_b$ are higher order terms. According to the Friedmann equations $\kappa\mu = \frac{1}{3}\theta^2$ for a flat FLRW spacetime; thus $\mathcal{R}$ is a first order variable. So in the right hand side of the second equation, there are two first order terms X_a and Z_a and five second order terms $\mathcal{R}\nu_a$, $2^{(3)}\nabla_a(\sigma^2)$, $2^{(3)}\nabla_a(\omega^2)$, $\sigma^b_{\ a} Z_b$ and $\omega^b_{\ a} Z_b$.

To compare the nonlinear terms of these equations with the linear terms of the same equations, we define following parameters:

$$\begin{aligned}
\varepsilon_1 &= \frac{\left|\omega^b{}_a X_b\right|}{|\kappa(\mu+p)Z_a|}, \quad \varepsilon_2 = \frac{\left|\sigma^b{}_a X_b\right|}{|\kappa(\mu+p)Z_a|}, \\
\varepsilon_3 &= \frac{|\mathcal{R}\nu_a|}{\left|\frac{1}{2}X_a\right|}, \quad \varepsilon_4 = \frac{\left|2h_a{}^b\nabla_b\omega^2\right|}{\left|\frac{1}{2}X_a\right|}, \quad \varepsilon_5 = \frac{\left|2h_a{}^b\nabla_b\sigma^2\right|}{\left|\frac{1}{2}X_a\right|}, \\
\varepsilon_6 &= \frac{\left|\omega^b{}_a Z_b\right|}{\left|\frac{1}{2}X_a\right|}, \quad \varepsilon_7 = \frac{\left|\sigma^b{}_a Z_b\right|}{\left|\frac{1}{2}X_a\right|}, \\
\tilde{\varepsilon}_3 &= \frac{|\mathcal{R}\delta u_a|}{|A_a|}, \quad \tilde{\varepsilon}_4 = \frac{\left|2h_a{}^b\nabla_b\omega^2\right|}{|A_a|}, \quad \tilde{\varepsilon}_5 = \frac{\left|2h_a{}^b\nabla_b\sigma^2\right|}{|A_a|}, \\
\tilde{\varepsilon}_6 &= \frac{\left|\omega^b{}_a Z_b\right|}{|A_a|}, \quad \tilde{\varepsilon}_7 = \frac{\left|\sigma^b{}_a Z_b\right|}{|A_a|}.
\end{aligned} \tag{4.2.3}$$

The linear perturbation theory is valid for the solutions of (4.2.1) and (4.2.2), if the following conditions are satisfied throughout the regime under consideration:

$(\mathbf{1})$ $\varepsilon_1, \varepsilon_2 \ll 1$,

$(\mathbf{2})$ $\varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7 \ll 1$, and/or $\tilde{\varepsilon}_3, \tilde{\varepsilon}_4, \tilde{\varepsilon}_5, \tilde{\varepsilon}_6, \tilde{\varepsilon}_7 \ll 1$.

4.3 Linear perturbations in a contracting universe

In the Sec. (3.5), solutions of the linear perturbation equations are demonstrated for simple cases. We are concerned with those perturbations which exit the Hubble radius in matter dominated contracting phase, far away from the bounce. At the end of the contracting phase almost all modes move outside the horizon. So we consider the superhorizon modes of perturbations such that

$$\frac{k}{a} \ll |H|\,. \tag{4.3.1}$$

The long wavelength solutions for the scalar perturbations are given in the Eqs. (3.5.51) and (3.5.52). Using (3.5.47) we can write X and Z as functions of the scale factor a:

$$X(k,t) = X^{(1)}(k)a^{-(11+3w)/2} + X^{(2)}(k)a^{-3}, \tag{4.3.2}$$

$$Z(k,t) = Z^{(1)}(k)a^{-4} + Z^{(2)}(k)a^{-3(1-w)/2}, \tag{4.3.3}$$

where

$$Z^{(1)}(k) = \frac{3}{2}\theta_*^{-1}X^{(1)}(k), \quad Z^{(2)}(k) = -\frac{1}{1+w}\theta_*^{-1}X^{(2)}(k). \tag{4.3.4}$$

The acceleration is expressed in terms of X_a as

$$\nu = -\frac{w}{1+w}\frac{X}{\kappa\mu} = -\frac{3w}{1+w}\theta_*^{-2}\left(X^{(1)}(k)a^{-(5-3w)/2} + X^{(2)}(k)a^{3w}\right). \tag{4.3.5}$$

The variables A and A_a are found using the properties of scalar harmonics (3.5.13) and (3.5.18):

$$A = {}^{(3)}\nabla^a\nu_a = \sum_k \nu^{(3)}\nabla^a Q_a^{(0)} = \sum_k \frac{k}{a}\nu Q^{(0)}, \tag{4.3.6}$$

$$A_a = {}^{(3)}\nabla_a A = \sum_k \frac{k}{a}\nu^{(3)}\nabla_a Q^{(0)} = -\sum_k \frac{k^2}{a^2}\nu Q_a^{(0)}. \tag{4.3.7}$$

The scalar perturbations X_a and Z_a are coupled with the vector and tensor perturbations in nonlinear order. So in order to calculate the growth rate of linearity parameters we need the ω_{ab} and σ_{ab}. The vecor perturbations ω_{ab} and r_a are given by the Eq. (3.5.59) and the tensor part of σ_{ab} is given by Eq. (3.5.72). Writing in terms of a:

$$\sigma^T(k,t) = \Sigma_T^{(1)}(k)a^{-3} + \Sigma_T^{(2)}(k)a^{-(1-3w)/2}. \tag{4.3.8}$$

The fourier components of the scalar and vector parts of σ_{ab} are obtained from the relations (3.5.40):

$$\sigma^V(k,t) = 2\frac{a}{k}r(k,t) = 2\frac{R(k)}{k}a^{-2+3w}, \tag{4.3.9}$$

$$\sigma^S(k,t) = \frac{a}{k}Z(k,t) = \frac{Z^{(1)}(k)}{k}a^{-3} + \frac{Z^{(2)}(k)}{k}a^{-(1-3w)/2}. \tag{4.3.10}$$

4.4 Comparison of linear and nonlinear terms

We have seen that each variable has growing mode(s) near the bounce. But this growing mode(s) can be absorbed by a mere redefinition of that variable. For example, if a variable f behaves as $f \sim a^{-n}$ near the bounce, then we can construct a new variable $\tilde{f} = a^m f$, such that $m \geq n$, which remains finite. So the growing modes of perturbations do not rule out their linear evolution. In order to investigate whether the perturbations remain linear near the bounce one needs to compare linear and nonlinear terms of the perturbation equations using the behavior of linearity parameters defined in Sec. 4.2.

4.4.1 Radiation dominated case

Let us consider the background to be a radiation dominated flat FLRW universe. The equation of state is $p = \frac{1}{3}\mu$. So putting $w = \frac{1}{3}$ in the equations (4.3.2)-(4.3.7) and keeping only the dominant modes near the bounce, we obtain:

$$\begin{aligned} &X_a = \bar{X}_a a^{-6}, \quad Z_a = \bar{Z}_a a^{-4}, \\ &\nu_a = U_a a^{-2}, \quad A = \bar{A} a^{-3}, \quad A_a = \bar{A}_a a^{-4}. \end{aligned} \tag{4.4.1}$$

The evolution of the vorticity is obtained by putting $w = \frac{1}{3}$ in Eq. (3.5.59). The shear σ_{ab} is composed of tansor, vector and scalar parts, given respectively by (4.3.8), (4.3.9) and (4.3.10). Taking only the dominant modes we can write

$$\omega_{ab} = \Omega_{ab} a^{-1}, \quad \sigma_{ab} = \Sigma_{ab} a^{-3}. \tag{4.4.2}$$

The higher order value of curvature perturbation (3.4.23) is

$$\begin{aligned}\mathcal{R} &= A + 2(\omega^2 - \sigma^2) \\ &= \bar{A}a^{-3} + 2\Omega^2 a^{-2} - 2\Sigma^2 a^{-6} \\ &= -\bar{\mathcal{R}}a^{-6}, \end{aligned} \tag{4.4.3}$$

where, $\bar{X}_a$, $\bar{Z}_a$, U_a, $\bar{A}$, $\bar{A}_a$, Ω_{ab}, Σ_{ab} and $\bar{\mathcal{R}}$ are time independent.

Using the commutation of time and spatial derivatives,

$$h_a^{\ b}\nabla_b\omega^2 = \Omega_a a^{-3}, \quad h_a^{\ b}\nabla_b\sigma^2 = \Sigma_a a^{-7}, \tag{4.4.4}$$

with $\dot{\Omega}_a = \dot{\Sigma}_a = 0$. Now substituting the perturbations from (4.4.1)-(4.4.2) in (4.2.3), we obtain the growth rates of the linearity parameters with the scale factor a:

$$\varepsilon_1 = \frac{\left|\Omega^b_{\ a}\bar{X}_b\right|}{\left|\frac{4}{3}\kappa M\bar{Z}_a\right|}a, \quad \varepsilon_2 = \frac{\left|\Sigma^b_{\ a}\bar{X}_b\right|}{\left|\frac{4}{3}\kappa M\bar{Z}_a\right|}a^{-1}, \tag{4.4.5}$$

$$\varepsilon_3 = \frac{\left|\bar{\mathcal{R}}U_a\right|}{\left|\frac{1}{2}\bar{X}_a\right|}a^{-2},\ \varepsilon_4 = \frac{|2\Omega_a|}{\left|\frac{1}{2}\bar{X}_a\right|}a^3,\ \varepsilon_5 = \frac{|2\Sigma_a|}{\left|\frac{1}{2}\bar{X}_a\right|}a^{-1}, \tag{4.4.6}$$

$$\varepsilon_6 = \frac{\left|\Omega^b_{\ a}\bar{Z}_b\right|}{\left|\frac{1}{2}\bar{X}_a\right|}a,\ \varepsilon_7 = \frac{\left|\Omega^b_{\ a}\bar{Z}_b\right|}{\left|\frac{1}{2}\bar{X}_a\right|}a^{-1}, \tag{4.4.7}$$

$$\tilde{\varepsilon}_3 = \frac{\left|\bar{\mathcal{R}}U_a\right|}{\left|\bar{A}_a\right|}a^{-4},\ \tilde{\varepsilon}_4 = \frac{|2\Omega_a|}{\left|\bar{A}_a\right|}a,\ \ \tilde{\varepsilon}_5 = \frac{|2\Sigma_a|}{\left|\bar{A}_a\right|}a^{-3}, \tag{4.4.8}$$

$$\tilde{\varepsilon}_6 = \frac{\left|\Omega^b_{\ a}\bar{Z}_b\right|}{\left|\bar{A}_a\right|}a^{-1},\ \tilde{\varepsilon}_7 = \frac{\left|\Omega^b_{\ a}\bar{Z}_b\right|}{\left|\bar{A}_a\right|}a^{-3}. \tag{4.4.9}$$

Let at some time slice $t = -t_1$, such that $t_* \gg t_1 \gg t_b$, linearity conditions are satisfied. So $\varepsilon_2(-t_1) = \frac{\left|\Sigma^b_{\ a}\bar{X}_b\right|}{\left|\frac{4}{3}\kappa M\bar{Z}_a\right|}a_1^{-1} \ll 1$. If we consider another time slice $t = -t_2$, which is close to t_b, i. e., $t_1 \gg t_2 \gtrsim t_b$, then

$$\varepsilon_2(-t_2) = \varepsilon_2(-t_1)\frac{a_1}{a_2}. \tag{4.4.10}$$

Since $a_2 \ll a_1$, the parameter ε_2 may become order 1 at $t = -t_2$ and the condition (1) no longer holds. Similar arguments can be given for ε_3, ε_5, ε_7, $\tilde{\varepsilon_3}$, $\tilde{\varepsilon_5}$, $\tilde{\varepsilon_6}$ and $\tilde{\varepsilon_7}$.

4.4.2 Dust dominated case

For a dust dominated flat FLRW background, the equation of state is $p = 0$. As a consequence Y_a, ν_a, A and A_a vanishes identiacally. Evolution of other gauge invariant variables are found by putting $w = 0$:

$$
\begin{aligned}
&X_a = \bar{X}_a a^{-11/2}, \quad Z_a = \bar{Z}_a a^{-4,}. \\
&\omega_{ab} = \Omega_{ab} a^{-2}, \quad \sigma_{ab} = \Sigma_{ab} a^{-3}, \\
&h_a^{\ b} \nabla_b \omega^2 = \Omega_a a^{-5}, \quad h_a^{\ b} \nabla_b \sigma^2 = \Sigma_a a^{-7}.
\end{aligned} \tag{4.4.11}
$$

Since $A_a = 0$, $\tilde{\varepsilon_3}$-$\tilde{\varepsilon_7}$ are undefined. To preserve linearity, all ε_1-ε_7 must be much less than 1. The growth rates of the linearity parameters are

$$
\varepsilon_1 = \frac{\left|\Omega^b{}_a \bar{X}_b\right|}{\left|\kappa M \bar{Z}_a\right|} a^{-\frac{1}{2}}, \quad \varepsilon_2 = \frac{\left|\Sigma^b{}_a \bar{X}_b\right|}{\left|\kappa M \bar{Z}_a\right|} a^{-\frac{3}{2}}, \tag{4.4.12}
$$

$$
\varepsilon_3 = 0, \ \varepsilon_4 = \frac{\left|2\Omega_a\right|}{\left|\frac{1}{2}\bar{X}_a\right|} a^{\frac{1}{2}}, \ \varepsilon_5 = \frac{\left|2\Sigma_a\right|}{\left|\frac{1}{2}\bar{X}_a\right|} a^{-\frac{3}{2}}, \tag{4.4.13}
$$

$$
\varepsilon_6 = \frac{\left|\Omega^b{}_a \bar{Z}_b\right|}{\left|\frac{1}{2}\bar{X}_a\right|} a^{-\frac{1}{2}}, \ \varepsilon_7 = \frac{\left|\Sigma^b{}_a \bar{Z}_b\right|}{\left|\frac{1}{2}\bar{X}_a\right|} a^{-\frac{3}{2}}. \tag{4.4.14}
$$

So, in this case also some of the parameters ε_1, ε_2, ε_5, ε_6, ε_7 may become order 1, near the bounce.

4.5 Conclusion

In order to investigate the issue of validity of linear perturbation theory near bounce, we used the covariant approach. We focus on the evolution equations for density perturbation X_a. The validity conditions of linear approximation of the (nonlinear) density perturbation equations are set in terms of some linearity parameters. Then the linear perturbation equations are solved for a collapsing FLRW background near the bounce. The solutions are used to compute the linearity parameters. It is found that some of those parameters grow beyond order unity near the bounce in both radiation and dust dominated cases. That means the nonlinear terms are comparable to the linear terms. So unless some special initial conditions are imposed on the variables such as shear and vorticity, perturbations may not be linear near the bounce.

Thus we conclude that perturbations may not be linear near the bounce and linear perturbation theory may not be adequate to give proper evolution of perturbations through the bounce. Our results are independent of choice of gauge. We used gauge invariant variables that were not assumed to be small with respect to background. So one can evolve them through the bounce and match with corresponding quantities in the expanding phase—but this would require the full nonlinear analysis.

In this work, we consider only the contracting branch and used general relativity with usual matter distribution as a correct theory to describe the dynamics of the universe. To investigate the nonlinearity of perturbations in a concrete manner, we have to take specific models of bounce.

Chapter 5

Covariant perturbations through a simple nonsingular bounce

5.1 Introduction

In the previous chapter, it is shown that in a single fluid dominated contracting branch of a bouncing universe the higher order perturbations grow more rapidly in comparison to the linear order perturbations. However, in order to investigate the behavior of perturbations at the bounce, we need to study a specific model of the nonsingular and bouncing universe. as discussed in Sec. (2.3), existance of a bouncing solution demands either a theory beyond Einsteins general relativity or presence of some unusual matter that violates certain energy conditions. In this chapter, based on [31], we choose a model that makes use the latter option.

We take a toy model for the flat Friedmann-Lemaitre-Robertson-Walker (FLRW) bouncing universe filled with a two-component perfect fluid, one component is a normal fluid with a dustlike equation of state, henceforth referred to as fluid-1,

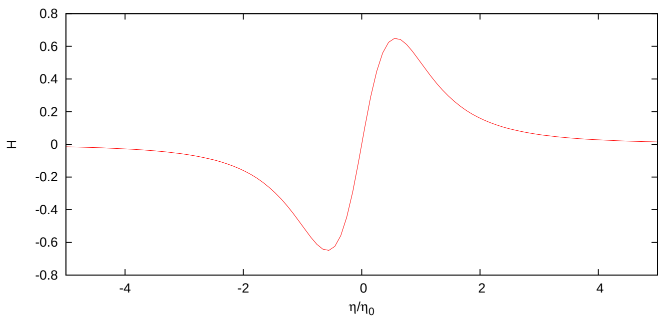

Figure 5.1: Plot of Hubble parameter as a function of conformal time η

and the other component has a negative energy density and pressure, henceforth referred to as fluid-2 [30]. Away from the bounce, the contribution of fluid-2 in the total energy budget is negligible and hence, the contraction of the universe is essentially guided by fluid-1. But close to the bounce, fluid-2 becomes dominant and as a result the collapse slows down by minimizing the Hubble parameter H. At turning point $\dot{H}$ becomes zero. Eventually the bouncing point $H = 0$ is reached and the universe starts to re-expand. Again at another turning point $\dot{H}$ vanishes and subsequently fluid-1 starts to dominate. Between the two turning points the null energy condition (NEC) is violated by the composite fluid. Variation of Hubble parameter as function of conformal time η is shown in Figure(5.1).

In this paper we study the evolution of perturbations through the bounce in the covariant approach. It turns out that the scalar and vector perturbations diverge not at the bouncing point but at the turning point; whereas the tensor perturbations oscillate at the bounce as well as at the turning point. At the turning point, we investigate the validity of linear perturbation theory. The linearity parameters (the ratio of the nonlinear and linear terms in perturbation equations) diverge at the turning point, confirming the appearance of nonlinearity in perturbations. The comoving curvature perturbation is conserved for adiabatic perturbations; however, in our model a nonadiabatic mode of perturbation exists. We have computed the nonadiabatic mode of covariantly defined comoving curvature perturbation and have shown that that mode is singular at the turning point.

We also consider a specific initial condition for scalars in which the entropic perturbation is absent and the adiabatic perturbations are originated from quantum fluctuations of the Bunch-Davis vacuum state in the matter dominated era. Using a numerical analysis we evolve the perturbations through the bounce. Divergence of the linearity parameters remains unaltered even in the presence of these special initial conditions. The scale invariance of the spectra are preserved well after the bounce. The correct spectra are obtained from the matching of $\mathcal{V}$ and not the $\mathcal{X}$ across the transition surface.

The paper is organized as follows. In Sec. (5.2), we describe the background bouncing model. In Sec. (5.3) the gauge invariant perturbations are defined covariantly. The equations are set up in Sec. (5.4). In Sec. (5.5), we demonstrate the solutions of linear perturbation equations. In Sec. (5.6), the behavior of comoving curvature perturbations are discussed. In Sec. (5.7), we compute the linearity parameters at the turning point. The matching conditions are discussed in Sec. (5.8) and the numerical analysis is demonstrated in Sec. (5.9).

5.2 Background

We consider a flat FLRW universe with a two component perfect fluid [30]. The two components have the same 4-velocity u^a which is taken to be the velocity of the comoving observers,

$$u^a = \frac{dx^a}{d\tau}, \quad u^a u_a = -1, \tag{5.2.1}$$

where τ is the proper time along the world lines of comoving observers. The two components of the fluid must have identical velocity at least in the background spacetime, because otherwise the background ceases to be an isotropic one. We assume here that the fluids' velocity is the same in the physical spacetime also. Although this assumption may lead to some loss of accuracy, our aim in this paper is not to calculate the cosmological parameters accurately but to understand the physical consequences of bounce on the evolution of perturbations. We hope such an assumption does not significantly alter the qualitative results. Note that such assumptions are often taken into consideration for matter-radiation transition in expanding universes [1].

The dynamical evolution is determined by the Einstein equation

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R = \kappa T_{ab}, \tag{5.2.2}$$

where $\kappa = 8\pi G$. T_{ab} is the total energy-momentum tensor:

$$T_{ab} = \mu u_a u_b + p h_{ab}, \quad h_{ab} = g_{ab} + u_a u_b, \tag{5.2.3}$$

$$\mu = \mu_1 - \mu_2, \quad p = p_1 - p_2. \tag{5.2.4}$$

Fluid-1 is a normal fluid, whereas fluid-2 violates the strong and weak energy condition. Each component satisfies the energy conservation condition separately:

$$\dot{\mu}_1 + \theta(\mu_1 + p_1) = 0, \quad \dot{\mu}_2 + \theta(\mu_2 + p_2) = 0. \tag{5.2.5}$$

The overdot is representing the covariant derivative along world lines of comoving observers and $\theta = \nabla_a u^a$ is the expansion of neighboring world lines of comoving observers. The scale factor $a(\tau)$ along each world line is defined by the Eq. (3.3.15).

Equation (5.2.5) together with the equations of state, $p_1 = w_1\mu_1$, $p_2 = w_2\mu_2$, give the evolution of energy densities,

$$\mu_1 = \frac{M_1}{a^{n_1}}, \quad \mu_2 = \frac{M_2}{a^{n_2}}, \tag{5.2.6}$$

where $n_1 = 3(1+w_1)$ and $n_2 = 3(1+w_2)$.

If w_1 and w_2 bears the following relation,

$$w_2 = 2w_1 + \frac{1}{3} \Leftrightarrow n_2 = 2(n_1 - 1), \tag{5.2.7}$$

then the Friedmann's equations yield a simple bouncing solution,

$$a(\eta) = \epsilon \left(1 + \frac{\eta^2}{\eta_0^2}\right)^{\alpha}, \tag{5.2.8}$$

where $\eta = \int a^{-1} dt$ is conformal time and

$$\epsilon = \left(\frac{M_2}{M_1}\right)^{\alpha}, \quad \alpha = \frac{1}{n_2 - n_1} = \frac{1}{n_1 - 2}. \tag{5.2.9}$$

At any point on the manifold, a perfect fluid is characterized completely by energy density μ, entropy density S and the velocity 4-vector u_a. The pressure can be expressed as a function of μ and S via equation of state

$$p = p(\mu, S). \tag{5.2.10}$$

So the small change in pressure is given by

$$\delta p = c_s^2 \delta\mu + \tau\delta S, \tag{5.2.11}$$

where $c_s^2 = \left(\frac{\partial p}{\partial \mu}\right)_S$ is adiabatic speed of sound and $\tau = \left(\frac{\partial p}{\partial S}\right)_\mu$. Since in absence of dissipation, entropy is conserved along fluid flow lines, i.e., $\dot{S} = 0$,

$$c_s^2 = \frac{\dot{p}}{\dot{\mu}} = -\frac{\dot{p}}{\theta(\mu + p)}. \tag{5.2.12}$$

This shows that if $\mu + p$ vanishes, but $\dot{p}$ remains nonzero, then the speed of sound blows up.

Let us consider the normal fluid is dustlike, i. e., $w_1 = 0$. Then the relation (5.2.7) constrains the fluid-2 to be radiationlike ($w_2 = \frac{1}{3}$):

$$\mu_1 = \frac{M_1}{a^3}, \quad \mu_2 = \frac{M_2}{a^4}. \tag{5.2.13}$$

In terms of the dimensionless quantity $x = \eta/\eta_0$, we have

$$\begin{aligned} a(x) &= \epsilon(1+x^2), \quad \epsilon = \frac{M_2}{M_1}, \quad \kappa M_1 \eta_0^2 = 12\epsilon, & (5.2.14)\\ \mathcal{H} &= \frac{a'}{a} = \frac{2x}{1+x^2}, \quad \mathcal{H}' = 2\frac{1-x^2}{1+x^2}, & (5.2.15)\\ \frac{a''}{a} &= \mathcal{H}' + \mathcal{H}^2 = \frac{2}{1+x^2}. & (5.2.16) \end{aligned}$$

Primes are representing derivatives with respect to x.

The scalar curvature $R = \frac{6}{\eta_0^2 a^2}\frac{a''}{a}$ remains finite for the entire range of x:

$$\mu + p = \frac{M_1}{a^4}(a - \beta) = \frac{M_1}{\epsilon^3}\frac{3x^2-1}{3(x^2+1)^4}, \quad \beta = \frac{4}{3}\epsilon. \tag{5.2.17}$$

So, at $x = \pm\frac{1}{\sqrt{3}}$, i.e. $a = \beta$, $\mu + p$ vanishes. The null energy condition, which in the case of perfect fluid means $\mu + p \geq 0$, is satisfied for $|x| \geq \frac{1}{\sqrt{3}}$, but it is violated for $|x| < \frac{1}{\sqrt{3}}$. The spacelike hypersurfaces at $x = \pm\frac{1}{\sqrt{3}}$, which form the boundary between the two regions, are called turning points.

Speed of sound in this model diverges at the turning points,

$$c_s^2 = -\frac{1}{3}\frac{\beta}{a-\beta} = -\frac{4}{3}\frac{1}{3x^2-1}. \tag{5.2.18}$$

5.3 Perturbations

In covariant perturbation theory, as gauge invariant perturbations, we consider the variables, which vanish in the background FLRW manifold. Some of those variables which form a closed set of equations are listed below [59]:

(1) Shear (σ_{ab}), vorticity (ω_{ab}) and acceleration (ν_a),

(2) "Electric" (E_{ab}) and "magnetic" (H_{ab}) parts of the Weyl tensor,

(3) Spatial gradients of the energy densities, pressure densities and expansion,

$$\begin{aligned} X_a &= \kappa h_a^{\ b}\nabla_b\mu, \quad Y_a = \kappa h_a^{\ b}\nabla_b p, \quad Z_a = h_a^{\ b}\nabla_b\theta, \\ X_{1a} &= \kappa h_a^{\ b}\nabla_b\mu_1, \quad X_{2a} = \kappa h_a^{\ b}\nabla_b\mu_2, \\ Y_{1a} &= \kappa h_a^{\ b}\nabla_b p_1, \quad Y_{2a} = \kappa h_a^{\ b}\nabla_b p_2, \\ X_a &= X_{1a} - X_{2a}, \quad Y_a = Y_{1a} - Y_{2a}. \end{aligned} \tag{5.3.1}$$

Using equations of state, $Y_{1a} = 0, Y_{2a} = \frac{1}{3}X_{2a}$.

The nonadiabatic mode of perturbation is

$$\Gamma_a = \kappa\tau h_a^{\ b}\nabla_b S = Y_a - c_s^2 X_a = \frac{\beta X_{1a} - aX_{2a}}{3(a-\beta)}. \tag{5.3.2}$$

All these variables defined in and their derivatives are considered to be linear or first order variables. These first order variables can be translated to the ordinary gauge invariant perturbations used in coordinate based perturbation theory. Some of those relations are shown in Sec. 3.7. Any quantity which is quadratic in first order variables is higher order.

5.4 Dynamic equations and constraints

We assume the two components do not exchange energy but exchange momentum among themselves. So the momentum conservation equation must be satisfied for the two fluids together:

$$\kappa(\mu + p)\nu_a + Y_a = 0. \tag{5.4.1}$$

Taking time derivative of X_{1a} and using the Eqs. (3.3.14), (5.4.1) and first of the Eq. (5.2.5):

$$\begin{aligned}
\dot{X}_{1a} &= \kappa u^b \nabla_b (h_a^{\ c} \nabla_c \mu_1) \\
&= \kappa h_a^{\ c} u^b \nabla_c \nabla_b \mu_1 + \kappa \nabla_c \mu_1 u^b \nabla_b (u_a u^c) \\
&= \kappa h_a^{\ c} [\nabla_c (u^b \nabla_b \mu_1) - (\nabla_b \mu_1)(\nabla_c u^b)] + \kappa \nabla_c \mu_1 (\nu_a u^c + u_a \nu^c) \\
&= \kappa h_a^{\ c} \left[\nabla_c \left(-\theta\left(\mu_1 + p_1\right)\right) - \left(\nabla_b \mu_1\right) \left({}^{(3)}\nabla_c u^b - u_c \nu^b\right)\right] \\
&\qquad\qquad +\kappa \left[\nu_a \dot{\mu}_1 + u_a \nu^c \nabla_c \mu_1\right] \\
&= -\kappa(\mu_1 + p_1) Z_a - \theta(X_{1a} + Y_{1a}) \\
&\qquad - \left(\frac{1}{3} h^b_{\ a} \theta + \sigma^b_{\ a} + \omega^b_{\ a} - u_a \nu^b\right) X_{1b} \\
&\qquad + \frac{Y_a}{\kappa(\mu + p)} \kappa \theta (\mu_1 + p_1) + u_a \nu^b X_{1b} \\
&= -\kappa(\mu_1 + p_1) Z_a - \frac{4}{3} \theta X_{1a} - \theta \left(Y_{1a} - \frac{\mu_1 + p_1}{\mu + p} Y_a\right) \\
&\qquad -(\sigma^b_{\ a} + \omega^b_{\ a} - u_a \nu^b) X_{1b}.
\end{aligned}$$

Projecting on the 3-hypersurface:

$$\begin{aligned}
a^{-4} h_a^{\ b} (a^4 X_{1a})^{\dot{}} &= -\kappa(\mu_1 + p_1) Z_a - \theta \left(Y_{1a} - \frac{\mu_1 + p_1}{\mu + p} Y_a\right) \\
&\qquad -(\sigma^b_{\ a} + \omega^b_{\ a}) X_{1b}.
\end{aligned} \tag{5.4.2}$$

Similarly using the second equation of (5.2.5) we obtain the evolution equation of X_{2a}:

$$\begin{aligned}
a^{-4} h_a^{\ b} (a^4 X_{2a})^{\dot{}} &= -\kappa(\mu_2 + p_2) Z_a - \theta \left(Y_{2a} - \frac{\mu_2 + p_2}{\mu + p} Y_a\right) \\
&\qquad -(\sigma^b_{\ a} + \omega^b_{\ a}) X_{2b}.
\end{aligned} \tag{5.4.3}$$

The equations (5.4.2) and (5.4.3) can be put in more convenient form by replacing Y_a by $Y_{1a} - Y_{2a}$

$$\begin{aligned} Y_{1a} - \frac{\mu_1 + p_1}{\mu + p} Y_a &= Y_{2a} - \frac{\mu_2 + p_2}{\mu + p} Y_a \\ &= -\frac{\mu_2 + p_2}{\mu + p} Y_{1a} + \frac{\mu_1 + p_1}{\mu + p} Y_{2a}, \end{aligned} \tag{5.4.4}$$

which leads to the following evolution equations for X_{1a} and X_{2a}:

$$\begin{aligned} a^{-4} h_a^{\ b} (a^4 X_{1a}\dot{)} = \theta \left(\frac{\mu_2 + p_2}{\mu + p} Y_{1a} - \frac{\mu_1 + p_1}{\mu + p} Y_{2a} \right) - \kappa(\mu_1 + p_1) Z_a \\ -(\sigma^b_{\ a} + \omega^b_{\ a}) X_{1b}, \end{aligned} \tag{5.4.5}$$

$$\begin{aligned} a^{-4} h_a^{\ b} (a^4 X_{2a}\dot{)} = \theta \left(\frac{\mu_2 + p_2}{\mu + p} Y_{1a} - \frac{\mu_1 + p_1}{\mu + p} Y_{2a} \right) - \kappa(\mu_2 + p_2) Z_a \\ -(\sigma^b_{\ a} + \omega^b_{\ a}) X_{2b}. \end{aligned} \tag{5.4.6}$$

Subtracting (5.4.6) from (5.4.5) we obtain,

$$a^{-4} h_a^{\ b} (a^4 X_a\dot{)} = -\kappa(\mu + p) Z_a - (\sigma^b_{\ a} + \omega^b_{\ a}) X_b. \tag{5.4.7}$$

which is same as the Eq. (3.4.21) for perfect fluid perturbations, i.e. with $q_a = 0$ and $\pi_{ab} = 0$.

Evolution equations for other variables Z_a, σ_{ab}, ω_{ab}, E_{ab} and H_{ab} are given by the equations (3.4.22), (3.4.10), (3.4.11), (3.4.15) and (3.4.16) respectively with $q_a = 0$ and $\pi_{ab} = 0$. The constraint relations that must be satisfied at some initial time on each world line are given by (3.4.12)-(3.4.14), (3.4.17) and (3.4.18).

5.5 Solutions of linearized equations

To study the linear evolution of perturbations we will use the usual classification of perturbations in terms of scalar, vector and tensor modes, described in the Sec. (3.5.1). Under such characterization X_as, Y_as, Z_a, Γ_a, and ν_a are scalar perturbations and ω_{ab} and r_a are vector perturbations. The σ_{ab}, E_{ab} and H_{ab} are traceless symmetric tensors, which can be decomposed into scalar, vector and pure tensor perturbation.

5.5.1 Scalar perturbations

Linearized forms of (5.4.5), (5.4.6) and (3.4.22), in our background model, are

$$
\begin{aligned}
a^{-4}(a^4 X_{1a}\dot{)} &= -\frac{\kappa M_1}{a^3} Z_a - \frac{1}{3}\theta \frac{a}{a-\beta} X_{2a}, && (5.5.1)\\
a^{-4}(a^4 X_{2a}\dot{)} &= -\frac{\beta \kappa M_1}{a^4} Z_a - \frac{1}{3}\theta \frac{a}{a-\beta} X_{2a}, && (5.5.2)\\
a^{-3}(a^3 Z_a\dot{)} &= -\frac{1}{2}(X_{1a} - X_{2a}) + A_a. && (5.5.3)
\end{aligned}
$$

To solve the equations, let us expand the variables in Fourier modes on the 3-hypersurface,

$$
S_a = \sum_k S(k,t) Q_a^{(0)}, \tag{5.5.4}
$$

where S stands for any scalar perturbations. $Q_a^{(0)}$ are the eigenfunctions of spatial Laplacian, explained in the Sec. 3.5.2.

Using (5.4.1),

$$
\begin{aligned}
A_a &= h_a^{\ b} \nabla_b \nabla^c \nu_c \\
&= \frac{1}{\kappa M_1} \frac{a^4}{a-\beta} \sum_k \frac{k^2}{a^2} Y(k,t) Q_a^{(0)} \\
&= -\frac{1}{3\kappa M_1} \frac{a^4}{a-\beta} \sum_k \frac{k^2}{a^2} X_2(k,t) Q_a^{(0)}. && (5.5.5)
\end{aligned}
$$

Using dimensionless quantities

$$
\mathcal{X}_1 = \eta_0^3 a^4 X_1, \quad \mathcal{X}_2 = \eta_0^3 a^4 X_2, \quad \mathcal{Z} = \eta_0^2 a^3 Z, \tag{5.5.6}
$$

Eqs. (5.5.1)-(5.5.3) in Fourier modes become

$$\begin{aligned}
\mathcal{X}_1' &= -\frac{9\beta}{a}\mathcal{Z} - \frac{a'}{a-\beta}\mathcal{X}_2, && (5.5.7)\\
\mathcal{X}_2' &= -\frac{9\beta^2}{a^2}\mathcal{Z} - \frac{a'}{a-\beta}\mathcal{X}_2, && (5.5.8)\\
\mathcal{Z}' &= -\frac{1}{2}(\mathcal{X}_1 - \mathcal{X}_2) - \frac{q^2a^2}{27\beta(a-\beta)}\mathcal{X}_2, && (5.5.9)
\end{aligned}$$

where prime denotes the derivative with respect to dimensionless conformal time $x = \eta/\eta_0$ and $q = k\eta_0$ is the dimensionless wave number.

Eliminating $\mathcal{Z}$ from (5.5.7) and (5.5.8),

$$\beta\mathcal{X}_1' = (a\mathcal{X}_2)' \quad \Longrightarrow \mathcal{X}_2(q,x) = \frac{\beta}{a}(\mathcal{X}_1(q,x) - C_1(q)). \tag{5.5.10}$$

Using new variable $\mathcal{W} = \mathcal{X}_1 - C_1$, Eqs. (5.5.7)-(5.5.9) are reduced to:

$$\begin{aligned}
\mathcal{W}' &= -\frac{a'}{a}\frac{\beta}{a-\beta}\mathcal{W} - \frac{9\beta}{a}\mathcal{Z}, && (5.5.11)\\
\mathcal{Z}' &= -\left(\frac{1}{2}\frac{a-\beta}{a} + \frac{q^2}{27}\frac{a}{a-\beta}\right)\mathcal{W} - \frac{1}{2}C_1. && (5.5.12)
\end{aligned}$$

The arbitrary constant $C_1(q)$ is related to the initial spectrum of nonadiabatic mode of perturbation, defined in (5.3.2):

$$\Gamma(q,x) = \frac{\beta C_1(q)}{3\eta_0^3 a^4(a-\beta)}. \tag{5.5.13}$$

This shows that entropy perturbation decays far away from bounce ($a \gg \beta$) as a^{-5} and diverges at $a \sim \beta$.

From (5.5.11) and (5.5.12) we extract a second order inhomogeneous differential equation for $\mathcal{W}$:

$$\begin{aligned}
\mathcal{W}'' + \frac{a'}{a-\beta}\mathcal{W}' + \frac{\beta}{a}\left(\frac{a''}{a} - \frac{a'^2}{(a-\beta)^2} - \frac{9}{2}\frac{a-\beta}{a} - \frac{q^2}{3}\frac{a}{a-\beta}\right)\mathcal{W}& \\
= \frac{9\beta C_1(q)}{2a}& \qquad (5.5.14)
\end{aligned}$$

or

$$\mathcal{W}'' + P_1(q,x)\mathcal{W}' + P_0(q,x)\mathcal{W} = C_1(q)P(q,x), \tag{5.5.15}$$

where

$$\begin{aligned} P_1(q,x) &= \frac{6x}{3x^2-1}, \\ P_0(q,x) &= -\left(\frac{2(9x^2+1)(3x^4-2x^2+3)}{(x^2+1)^2(3x^2-1)^2} + \frac{4}{3}\frac{q^2}{3x^2-1}\right), \\ P(q,x) &= \frac{6}{x^2+1}. \end{aligned} \tag{5.5.16}$$

$\mathcal{X}_1$, $\mathcal{X}_2$ and $\mathcal{Z}$ can be expressed in terms of $\mathcal{W}$:

$$\begin{aligned} &\mathcal{X}_1 = \mathcal{W} + C_1, \quad \mathcal{X}_2 = \frac{\beta}{a}\mathcal{W} = \frac{4}{3(x^2+1)}\mathcal{W}, \\ &\mathcal{Z} = -\frac{a}{9\beta}\left(\mathcal{W}'' + \frac{a'}{a}\frac{a}{a-\beta}\mathcal{W}\right) = -\frac{1}{9}\frac{a^2}{\beta(a-\beta)}\left(\frac{a-\beta}{a}\mathcal{W}\right)'. \end{aligned}$$

All scalar perturbations are given by

$$\begin{aligned} X_1(q,x) = \eta_0^{-3}a^{-4}\mathcal{X}_1(q,x), \quad X_2(q,x) = \eta_0^{-3}a^{-4}\mathcal{X}_2(q,x), \\ Z(q,x) = \eta_0^{-2}a^{-3}\mathcal{Z}(q,x), \quad Y(q,x) = -\tfrac{1}{3}X_2(q,x), \\ \nu(q,x) = -\tfrac{Y(q,x)}{\kappa(\mu+p)}, \quad A(q,x) = -\left(\tfrac{k}{a}\right)^2\nu(q,x). \end{aligned} \tag{5.5.17}$$

Scalar part of shear is given by the Eq. (3.5.40),

$$\sigma^S(q,x) = Z(q,x)\frac{a}{k} = \eta_0^{-1}a^{-2}\frac{\mathcal{Z}(q,x)}{q}. \tag{5.5.18}$$

For $q=0$, Eq. (5.5.15) has a general solution:

$$\begin{aligned} \mathcal{W}(0,x) &= -C_1(0)\frac{3(x^2+1)}{3x^2-1} + C_2(0)\frac{3x}{(x^2+1)(3x^2-1)} \\ &\quad + C_3(0)\frac{9x^6+25x^4+15x^2+15}{3(x^2+1)(3x^2-1)}. \end{aligned} \tag{5.5.19}$$

To solve for modes with nonzero momentum, we will concentrate on different regions of interest. We are working in a collapsing FLRW universe undergoing a nonsingular bounce. Long before the bounce ($a \gg \epsilon$), the energy density of fluid-1 dominates over the energy density of fluid-2 and we have a dust dominated collapsing FLRW background. Let us call this region as region A. The neighborhood of the turning point $x \sim -\frac{1}{\sqrt{3}}$ is region B. Another region of interest is the point of bounce, characterized by vanishing of the Hubble parameter and corresponds to the time $x = 0$. This is the region C.

Region A

In this region, $|x| \gg 1$ and $a(x) \simeq \frac{3}{4}\beta x^2$. Changing the variable x to $z = \frac{1}{x}$, Eq. (5.5.15) takes form

$$\frac{d^2\mathcal{W}_A}{dz^2} + P_{A1}(q,z)\frac{d\mathcal{W}_A}{dz} + P_{A0}(q,z)\mathcal{W}_A = C_1(q)P_A(q,z), \qquad (5.5.20)$$

where

$$\begin{aligned} P_{A1}(q,z) &= \frac{2}{z} - \frac{1}{z^2}P_1(q,\frac{1}{z}), \\ P_{A0}(q,z) &= \frac{1}{z^4}P_0(q,\frac{1}{z}), \\ P_A(q,z) &= \frac{1}{z^4}P(q,\frac{1}{z}). \end{aligned}$$

Expanding the coefficients P_{A1}, P_{A0}, P_A in Taylor series around $z = 0$:

$$\begin{aligned} P_{A1}(q,z) &= -\frac{2}{3}\left(z + \frac{z^3}{3} + \frac{z^5}{9} + \cdots\right), \\ P_{A0}(q,z) &= -\left(6 + \frac{4q^2}{9}\right)\frac{1}{z^2} + \left(\frac{34}{3} - \frac{4q^2}{27}\right) - \left(22 + \frac{4q^2}{81}\right)z^2 + \cdots, \\ P_A(q,z) &= 6\left(\frac{1}{z^2} - 1 + z^2 - \cdots\right). \end{aligned} \qquad (5.5.21)$$

We obtain the power series solution of (5.5.20) as

$$
\begin{aligned}
\mathcal{W}_A(q,\frac{1}{z}) &= -\frac{C_1(q)}{1+\frac{2q^2}{27}}\left[1+\frac{4}{3}\frac{9-q^2}{9+q^2}z^2+\cdots\right] \\
&\quad +C_2^A(q)z^{3+\delta}\left[1-\frac{28-7\delta-\delta^2}{6(7+2\delta)}z^2+\cdots\right] \\
&\quad +\frac{C_3^A(q)}{z^{2+\delta}}\left[1+\frac{38-3\delta-\delta^2}{6(3+2\delta)}z^2+\cdots\right], \qquad (5.5.22)
\end{aligned}
$$

where

$$
\delta = \frac{5}{2}\left(\sqrt{1+\left(\frac{4q}{15}\right)^2}-1\right). \qquad (5.5.23)
$$

In the limit $z \to 0$, behaviors of the variables $\mathcal{Z}$, $\mathcal{X}_1$ and $\mathcal{X}_2$ are found to be

$$
\begin{aligned}
\mathcal{Z}_A(q,x) &= \frac{12q^2}{(27+2q^2)(9+q^2)}\frac{C_1(q)}{x}+\frac{3+\delta}{12}\frac{C_2^A(q)}{x^{2+\delta}}-\frac{2+\delta}{12}C_3^A(q)x^{3+\delta}, \\
\mathcal{X}_{1A}(q,x) &= \frac{2q^2}{27+2q^2}C_1(q)+C_2^A(q)x^{-3-\delta}+C_3^A(q)x^{2+\delta}, \\
\mathcal{X}_{2A}(q,x) &= -\frac{36}{27+2q^2}C_1(q)x^{-2}+\frac{4}{3}C_2^A(q)x^{-5-\delta}+\frac{4}{3}C_3^A(q)x^{\delta}. \qquad (5.5.24)
\end{aligned}
$$

Region B

In this region, $x \sim -\frac{1}{\sqrt{3}}$. In terms of a new variable, $y=\sqrt{3}x+1$, (5.5.15) takes the following form:

$$
\frac{d^2\mathcal{W}_B}{dy^2}+P_{B1}(q,y)\frac{d\mathcal{W}_B}{dy}+P_{B0}(q,y)\mathcal{W}_B = C_1(q)P_B(q,y). \qquad (5.5.25)
$$

Again the coefficients obtained as a Taylor series around $y=0$,

$$
\begin{aligned}
P_{B1}(q,y) &= \frac{1}{y}\left[1-\frac{1}{2}y-\frac{1}{4}y^2+\cdots\right], \\
P_{B0}(q,y) &= -\frac{1}{y^2}\left[1+\left(\frac{1}{2}-\frac{2q^2}{9}\right)y+\left(\frac{1}{4}-\frac{q^2}{9}\right)y^2+\cdots\right], \\
P_B(q,y) &= \frac{3}{2}\left[1+\frac{1}{2}y+\cdots\right]. \qquad (5.5.26)
\end{aligned}
$$

The general solution of (5.5.25) in the limit $y \to 0$ is

$$\mathcal{W}_B(q,y) = \frac{1}{2}C_1(q)y^2\left[1+\frac{1}{8}\left(3-\frac{2q^2}{9}\right)y+\cdots\right] + C_2^B(q)y\left[1+\frac{1}{3}\left(1-\frac{2q^2}{9}\right)y+\cdots\right] + \frac{C_3^B(q)}{y}\left[1+\frac{2q^2}{9}y+\cdots\right]. \tag{5.5.27}$$

Keeping only the leading terms,

$$\mathcal{X}_{1B}(q,y) = C_1(q) + C_2^B(q)y + \frac{C_3^B(q)}{y}, \tag{5.5.28}$$

$$\mathcal{X}_{2B}(q,y) = \frac{1}{2}C_1(q)y^2 + C_2^B(q)y + \frac{C_3^B(q)}{y}, \tag{5.5.29}$$

$$\mathcal{Z}_B(q,y) = -\frac{1}{\sqrt{3}}\left(\frac{1}{2}C_1(q)y + \frac{2}{3}C_2^B(q) + \frac{2q^2}{27}\frac{C_3^B(q)}{y}\right). \tag{5.5.30}$$

So the scalar perturbations diverge as y^{-1} at the turning point. Though both $\mathcal{X}_1$ and $\mathcal{X}_2$ diverge as y^{-1} near the turning point, the combination $\mathcal{X} = \mathcal{X}_1 - \mathcal{X}_2$ remains finite. So $X(q,x) = \eta_0^{-3}a^{-4}\mathcal{X}$ is also finite and well behaved at the turning point.

Region C

This is the region near bounce, i.e., $x = 0$. The Eq. (5.5.15) becomes

$$\frac{d^2\mathcal{W}_C}{dx^2} + P_{C1}(q,x)\frac{d\mathcal{W}_C}{dx} + P_{C0}(q,x)\mathcal{W}_C = C_1(q)P_C(q,x), \tag{5.5.31}$$

where

$$\begin{aligned} P_{C1}(q,x) &= -6x(1+3x^2+9x^4+\cdots), \\ P_{C0}(q,x) &= -\left(6-\frac{4q^2}{3}\right) - \left(74-4q^2\right)x^2 \\ &\quad - \left(278-12q^2\right)x^4 - \cdots, \\ P_C(q,x) &= 6(1-x^2+x^4-\cdots). \end{aligned} \tag{5.5.32}$$

Solutions in the limit $x \sim 0$,

$$
\begin{aligned}
\mathcal{W}_C(q,x) &= C_1(q)\left[3x^2 + \left(4 - \frac{q^2}{3}\right)x^4 + \cdots\right] \\
&\quad + C_2^C(q)\left[x + 2\left(1 - \frac{q^2}{9}\right)x^3 + \cdots\right] \\
&\quad + C_3^C(q)\left[1 + \left(3 - \frac{2q^2}{3}\right)x^2 + \left(32 - 5q^2 + \frac{2q^4}{9}\right)\frac{x^4}{3} + \cdots\right], \\
\mathcal{X}_{1C}(q,x) &= C_1(q)\left[1 + 3x^2 + \cdots\right] + C_2^C(q)\left[x + 2\left(1 - \frac{q^2}{9}\right)x^3 + \cdots\right] \\
&\quad + C_3^C(q)\left[1 + \left(3 - \frac{2q^2}{3}\right)x^2 + \cdots\right], \\
\mathcal{X}_{2C}(q,x) &= \frac{4}{3}C_1(q)\left[3x^2 + \left(1 - \frac{q^2}{3}\right)x^2 + \cdots\right] \\
&\quad + \frac{4}{3}C_2^C(q)\left[x + \left(1 - \frac{2q^2}{9}\right)x^3 + \cdots\right] \\
&\quad + \frac{4}{3}C_3^C(q)\left[1 + 2\left(1 - \frac{q^2}{3}\right)x^2 + \cdots\right], \\
\mathcal{Z}_C(q,x) &= -\frac{C_1(q)}{6}\left[3x - \left(1 + \frac{2q^2}{3}\right)x^3\right] \\
&\quad - \frac{C_2^C(q)}{12}\left[1 - \left(1 + \frac{2q^2}{3}\right)x^2 + \cdots\right] \\
&\quad + \frac{C_3^C(q)}{6}\left[\left(1 + \frac{2q^2}{3}\right)x + \left(47 + 4q^2 - \frac{4q^4}{9}\right)\frac{x^3}{3} + \cdots\right],
\end{aligned} \tag{5.5.33}
$$

show that the scalar perturbations remain finite and well behaved near bounce.

5.5.2 Vector perturbations

The evolution of the vector perturbation ω_{ab} is given by the Eq. (3.5.54):

$$
\omega_{ab} = \Omega_{ab}\frac{1}{\eta_0 a^2}e^{\int c_s^2\theta dt} = \frac{\Omega_{ab}}{\eta_0 a(a-\beta)}, \quad \dot{\Omega}_{ab} = 0, \tag{5.5.34}
$$

where the factor of $1/\eta_0$ is added to keep Ω_{ab} dimensionless. The vector part of shear (σ_{ab}^V) is obtained from (3.5.40). Let us define a dimensionless and spatial derivative operator D_a as

$$D_a = a{\eta_0}^{(3)}\nabla_a, \quad D^2 = a^2\eta_0^{2(3)}\nabla^2.$$

D_a commutes with the derivative along fluid flow lines $u^a\nabla_a$.

Then

$$r_a = \nabla^b\omega_{ab} = \eta_0^{-1}a^{-1}D^b\omega_{ab} = \frac{R_a}{\eta_0^2 a^2(a-\beta)}, \tag{5.5.35}$$

where $R_a = D^b\Omega_{ab}$, $\dot{R}_a = 0$.

So

$$\sigma^V = 2\frac{{\eta_0}^{-2}R}{a^2(a-\beta)}\frac{a}{k} = \frac{2R}{\eta_0 qa(a-\beta)}, \tag{5.5.36}$$

where $R(q)$ is the Fourier mode of R_a, defined by

$$R_a = \sum_q R(q)Q_a^{(1)}. \tag{5.5.37}$$

5.5.3 Gravitational waves

The pure tensor parts of σ_{ab}^T, E_{ab}^T and H_{ab}^T are the gravitational waves. The linearized equation for σ_{ab}^T is obtained from (3.4.10), (3.4.15) and (3.4.16) by setting $X_{ia} = Z_a = 0, \omega_{ab} = 0$,

$$\triangle\sigma_{ab}^T + \frac{5}{3}\theta\dot{\sigma^T}_{ab} + \frac{1}{6}(\theta^2 - 9\kappa p)\sigma_{ab}^T = 0. \tag{5.5.38}$$

E_{ab}^T and H_{ab}^T are given by,

$$E_{ab}^T = -a^{-2}\dot{(a^2\sigma^T)}_{ab}, \quad H_{ab}^T = -\mathrm{Curl}\sigma_{ab}^T. \tag{5.5.39}$$

Using dimensionless variables, (5.5.38) takes the following form:

$$\sigma^{T\prime\prime}(q,x) + \frac{8x}{x^2+1}\sigma^{T\prime}(q,x) + \left(\frac{6}{x^2+1} + q^2\right)\sigma^T(q,x) = 0. \tag{5.5.40}$$

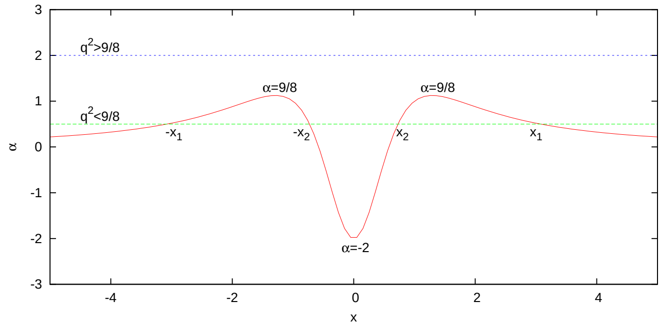

Figure 5.2: Plot of α as a function of x

α has a minimum value $\alpha_{\text{min}} = -2$ at the bounce and two maxima $\alpha_{\text{max}} = \frac{9}{8}$ at $x = \pm\sqrt{\frac{5}{3}}$. For $q^2 < \frac{9}{8}$, there are two regions where $q^2 - \alpha < 0$. But for $q^2 > \frac{9}{8}$, $q^2 - \alpha$ is always positive.

The general solution for the $q = 0$ mode is

$$\sigma^T(0, x) = D_1(0)\frac{x(3x^4 + 10x^2 + 15)}{3(x^2+1)^3} + D_2(0)\frac{1}{(x^2+1)^3}. \tag{5.5.41}$$

Using the variable $f = (1 + x^2)\sigma^T$, Eq. (5.5.40) becomes

$$f'' + \left[q^2 - \alpha(x)\right] f = 0, \quad \alpha(x) = 2\frac{3x^2 - 1}{(x^2+1)^2}. \tag{5.5.42}$$

For $x^2 \gg \frac{6}{q^2}$, f oscillates with frequency q. If $q^2 < \frac{9}{8}$, the equation

$$q^2 - \alpha(x) = 0 \tag{5.5.43}$$

has four roots, $\pm x_1(q), \pm x_2(q)$. For $x_2 < |x| < x_1$, $q^2 - \alpha(x)$ is negative, but f oscillates again for $-x_2 < x < x_2$. If, however, $q^2 > \frac{9}{8}$, $q^2 - \alpha$ is positive always and f shows oscillatory behavior over the whole range of x. The frequency of oscillation is maximum at the point of bounce $x = 0$. Graphical representation of $\alpha(x)$ is shown in Figure(5.2). In any case f and hence σ^T never blow up at the bounce or at the turning points.

Region A: In this region, (5.5.40) becomes

$$x^2\sigma_A^{T\prime\prime} + 8x\sigma_A^{T\prime} + \left(q^2x^2 + 6\right)\sigma_A^T = 0. \tag{5.5.44}$$

The general solution of σ^T in this region is

$$\sigma_A^T(q,x) = (qx)^{-7/2}\left[D_1^A(q)J_{5/2}(qx) + D_2^A(q)Y_{5/2}(qx)\right], \tag{5.5.45}$$

where J and Y are the Bessel function and the Neumann function respectively.

Region B: Using the variable $y = \sqrt{3}x + 1$ in region B, Eq. (5.5.40) is simplified to

$$\frac{d^2\sigma_B^T}{dy^2} - 2\frac{d\sigma_B^T}{dy} + \left(\frac{3}{2} + \frac{q^2}{3}\right)\sigma_B^T = 0 \tag{5.5.46}$$

and its general solution is

$$\sigma_B^T(q,y) = e^y\left[D_1^B(q)\cos\left(m_q y\right) + D_2^B(q)\sin\left(m_q y\right)\right], \tag{5.5.47}$$

where,

$$m_q = \sqrt{\frac{1}{2} + \frac{q^2}{3}}. \tag{5.5.48}$$

Region C: At the bounce $(x \to 0)$, as explained earlier, σ^T oscillates with frequency $\sqrt{2+q^2}$:

$$\sigma_C^T(q,x) = D_1^C(q)\cos\left(\sqrt{2+q^2}x\right) + D_2^C(q)\sin\left(\sqrt{2+q^2}x\right). \tag{5.5.49}$$

5.6 Comoving curvature perturbation

The comoving curvature perturbation is defined by the Eq. (3.3.57) [65, 68]. This variable is related to the comoving curvature perturbation ζ, used in the coordinate based perturbation theory [28]. In particular, since ζ_a is a spatial gradient of scalar up to first order, we can write

$$\zeta_a = h_a^{\ b} \nabla_b \zeta^S. \tag{5.6.1}$$

In the Sec. 3.7, it has been shown that ζ^S is equal to $-\zeta$ on the large scale. ζ^S is conserved on all scales for adiabatic perturbation, whereas ζ is conserved on the large scale only. However, in our model adiabatic modes are present. So the evolution of ζ_a is determined by the following equation:

$$\mathcal{L}_u \zeta_a = -\frac{\theta}{3\kappa(\mu+p)} \Gamma_a, \tag{5.6.2}$$

$$\Rightarrow a^{-1} h_a^{\ b} \dot{(a\zeta_b)} = -\frac{\theta}{3\kappa(\mu+p)} \Gamma_a - (\sigma^b_{\ a} + \omega^b_{\ a}) \zeta_b, \tag{5.6.3}$$

$\mathcal{L}_u$ being the Lie derivative with respect to u^a. Up to first order, using (5.5.13),

$$a^{-1} \dot{(a\zeta_q)} = -\frac{\theta}{3\kappa(\mu+p)} \Gamma = -\frac{\dot{a}}{a} \frac{C_1}{27\eta_0 (a-\beta)^2}. \tag{5.6.4}$$

Integrating,

$$\zeta_q = \frac{1}{27\eta_0} \frac{1}{a} \left(\frac{C_1}{a-\beta} + \tilde{C}_2 \right). \tag{5.6.5}$$

So, besides the nonadiabatic constant mode of $\zeta^S \sim -\frac{a}{k}\zeta_q$ there is an adiabatic mode which diverges at the turning point.

5.7 Validity of linear treatment at the turning point

The speed of sound and different perturbation variables become infinite at the turning point, not at the bounce. Existence of these growing modes raised doubts on the validity of linear perturbation theory. In the coordinate based perturbation theory, linear perturbation treatment is justified if the perturbations remain small compared with background quantities. However, in covariant perturbation theory, background values of all gauge invariant variables are zero. So in this case we demand that the higher order terms in a perturbation equation must be small compared with the first order terms. Let us consider the equations for scalar perturbations (3.4.21) and (3.4.22). We have defined some linearity parameters $\varepsilon_1 - \varepsilon_7$ and $\tilde{\varepsilon}_3 - \tilde{\varepsilon}_7$ as the ratio of nonlinear to the linear terms in these equations in the Sec. 4.2. The linear perturbation theory for the scalar perturbations is valid, if the following conditions are satisfied throughout the regime under consideration:

$$
\begin{aligned}
&(\mathbf{1})\ \varepsilon_1, \varepsilon_2 \ll 1,\\
&(\mathbf{2})\ \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7 \ll 1 \text{ and/or } \tilde{\varepsilon}_3, \tilde{\varepsilon}_4, \tilde{\varepsilon}_5, \tilde{\varepsilon}_6, \tilde{\varepsilon}_7 \ll 1.
\end{aligned}
$$

Using Eq. (5.5.17) and the solutions (5.5.27), (5.5.30), (5.5.34), (5.5.18), (5.5.36) and (5.5.47) of the perturbation equations at region B, the dominant mode of different variables that appear in (3.4.21) and (3.4.22) can be written as

$$
\begin{aligned}
&X_a = \frac{1}{\eta_0^3\beta^4}\left(\Upsilon_a + \Xi_a\right), \quad Z_a = \frac{4}{27\eta_0^2\beta^3}\frac{D^2\Xi_a}{y}, \quad \nu_a = \frac{4}{27\eta_0\beta^2}\frac{\Xi_a}{y^2},\\
&A_a = \frac{4}{27\eta_0^3\beta^4}\frac{D^2\Xi_a}{y^2}, \quad \omega_{ab} = -\frac{2}{\eta_0\beta^2}\frac{\Omega_{ab}}{y},\\
&\sigma_{ab} = \frac{4}{27\eta_0\beta^2}\frac{D_{<a}\Xi_{b>} + D_{<a}\Lambda_{b>}}{y},\\
&\mathcal{R} = \frac{4}{27\eta_0^2\beta^3}\left(D^a\Xi_a + \frac{4}{27\beta}|D_{<a}\Xi_{b>} + D_{<a}\Lambda_{b>}|^2 - \frac{4}{\beta}|\Omega_{ab}|^2\right)\frac{1}{y^2},\\
&{}^{(3)}\nabla_a\sigma^2 = \frac{2}{27\eta_0^3\beta^5}\frac{D_a\left(|D_{<c}\Xi_{d>} + D_{<c}\Lambda_{d>}|^2\right)}{y^2},\\
&{}^{(3)}\nabla_a\omega^2 = \frac{2}{\eta_0^3\beta^5}\frac{D_a\left(|\Omega_{cd}|^2\right)}{y^2},
\end{aligned}
\tag{5.7.1}
$$

where

$$\Upsilon_a = \sum_k C_1(q) Q_a^{(0)}, \quad \Xi_a = -\tfrac{1}{2}\sum_k C_3^B(q) Q_a^{(0)},$$
$$\Lambda_a = 27\sum_k \tfrac{R(q)}{q^2} Q_a^{(1)}. \tag{5.7.2}$$

Near the turning point ($y \sim 0$), (5.2.17) reduces to

$$\kappa(\mu + p) = -\frac{\kappa M_1}{2\beta^3} y = -\frac{9}{2}\eta_0^{-2}\beta^{-2} y. \tag{5.7.3}$$

Then the linearity parameters for (3.4.21) and (3.4.22) are found to be

$$\begin{aligned}
\varepsilon_1 &= \frac{|\omega^b{}_a X_b|}{|\kappa(\mu+p)Z_a|} = \frac{3}{\beta}\frac{|\Omega^b{}_a(\Upsilon_b + \Xi_b)|}{|D^2\Xi_a|} y^{-1}, \\
\varepsilon_2 &= \frac{|\sigma^b{}_a X_b|}{|\kappa(\mu+p)Z_a|} = \frac{2}{9\beta}\frac{|h^{bc}(D_{<a}\Xi_{b>} + D_{<a}\Lambda_{b>})(\Upsilon_c + \Xi_c)|}{|D^2\Xi_a|} y^{-1}, \\
\varepsilon_3 &= \frac{|\mathcal{R}\nu_a|}{|\frac{1}{2}X_a|} \\
&= \left(\frac{2}{3}\right)^5 \frac{\left|D^c\Xi_c + \frac{4}{27\beta}|D_{<c}\Xi_{d>} + D_{<c}\Lambda_{d>}|^2 - \frac{4}{\beta}|\Omega_{cd}|^2\right| |\Xi_a|}{3\beta|\Upsilon_a + \Xi_a|} y^{-4}, \\
, \varepsilon_4 &= \frac{|2h_a{}^b\nabla_b\omega^2|}{|\frac{1}{2}X_a|} = \frac{8}{\beta}\frac{|D_a(|\Omega_{cd}|^2)|}{|\Upsilon_a + \Xi_a|} y^{-2}, \\
\varepsilon_5 &= \frac{|2h_a{}^b\nabla_b\sigma^2|}{|\frac{1}{2}X_a|} = \left(\frac{2}{3}\right)^3 \frac{1}{\beta}\frac{|D_a(|D_{<c}\Xi_{d>} + D_{<c}\Lambda_{d>}|^2)|}{|\Upsilon_a + \Xi_a|} y^{-2}, \\
\varepsilon_6 &= \frac{|\omega^b{}_a Z_b|}{|\frac{1}{2}X_a|} = \left(\frac{2}{3}\right)^3 \frac{2}{\beta}\frac{|\Omega^b{}_a D^2\Xi_b|}{|\Upsilon_a + \Xi_a|} y^{-2}, \\
\varepsilon_7 &= \frac{|\sigma^b{}_a Z_b|}{|\frac{1}{2}X_a|} = \left(\frac{2}{3}\right)^3 \frac{1}{3\beta}\frac{|h^{bc}(D_{<a}\Xi_{b>} + D_{<a}\Lambda_{b>}) D^2\Xi_b|}{|\Upsilon_a + \Xi_a|} y^{-2}.
\end{aligned} \tag{5.7.4}$$

Other sets of parameters $\tilde{\varepsilon}_3$-$\tilde{\varepsilon}_7$ are related to the ε_3-ε_7 via

$$\tilde{\varepsilon}_I = \frac{1}{2}\frac{|X_a|}{|A_a|}\varepsilon_I = \left(\frac{3}{2}\right)^3 \frac{|\Upsilon_a + \Xi_a|}{|D^2\Xi_a|} y^2 \varepsilon_I, \quad \text{for I=3 to 7.} \tag{5.7.5}$$

$$\tilde{\varepsilon}_3 = \frac{4\left|D^c\Xi_c + \frac{4}{27\beta}|D_{<c}\Xi_{d>} + D_{<c}\Lambda_{d>}|^2 - \frac{4}{\beta}|\Omega_{cd}|^2\right||\Xi_a|}{27\beta\,|D^2\Xi_a|}y^{-2},$$
$$\tilde{\varepsilon}_4 = \frac{27\,|D_a\,(|\Omega_{cd}|^2)|}{\beta\,|D^2\Xi_a|},\ \tilde{\varepsilon}_5 = \frac{|D_a\,(|D_{<c}\Xi_{d>} + D_{<c}\Lambda_{d>}|^2)|}{\beta\,|D^2\Xi_a|},$$
$$\tilde{\varepsilon}_6 = \frac{2\,|\Omega^b{}_a D^2\Xi_b|}{\beta\,|D^2\Xi_a|},\ \tilde{\varepsilon}_7 = \frac{\left|h^{bc}\,(D_{<a}\Xi_{b>} + D_{<a}\Lambda_{b>})\,D^2\Xi_b\right|}{3\beta\,|D^2\Xi_a|}. \qquad (5.7.6)$$

The ε_1 and ε_2 diverge at the turning point as $y \to 0$. So the condition (1) is not satisfied at the turning point. Although $\tilde{\varepsilon}_4 - \tilde{\varepsilon}_7$ remain finite at the turning point, $\tilde{\varepsilon}_3$ diverges. So the condition (2) is also not satisfied.

5.8 Matching conditions

We have seen that even for this simple model analytical solutions for the perturbations throughout the bounce are not available. One can obtain the solutions by numerical integration, but to have a good understanding on the result one needs some analytical methods as discussed in the Sec. (2.4.4). Such methods involve matching of the variables across the transition surfaces. In the nonbouncing cases it is well known that the spatial metric on the hypersurface and the extrinsic curvature must be continuous across the boundary separating the two regions [47]. However for the bouncing models one should find the appropriate variables, which should be matched to get a correct spectrum. In a nonsingular bouncing background the spatial curvature perturbation $\delta\mathcal{R}$ is found to be the appropriate variable (rather than the Bardeen potential Φ) which is to be matched in order to get good agreement with the numerical results [53]. In the Sec. 3.7 we have shown that $\delta\mathcal{R}$ and Φ are related to V_a and X_a respectively. We now investigate whether matching of these variables will lead to the correct spectrum after the bounce.

Considering only scalar variables, we have

$$X_i = \frac{2}{a^2}\partial_i\vec{\nabla}^2\Phi, \quad V_i = \frac{2}{a^2}\partial_i\vec{\nabla}^2\delta\mathcal{R}. \tag{5.8.1}$$

We consider the perturbation modes that exit the horizon in deep matter dominated era ($|x| \gg 1$). If $x = -x_{\text{exit}}$ is the value of x at the horizon exit, then

$$q = |\mathcal{H}_{\text{exit}}| \quad \Rightarrow \quad x_{\text{exit}} = \frac{2}{q}. \tag{5.8.2}$$

Since $x_{\text{exit}} \gg 1$, q must be much less than order unity. Expanding V_a and $\delta\mathcal{R}$ in Fourier modes and considering only scalar modes,

$$V_a = \sum_k \eta_0^{-3} a^{-a}\mathcal{V}Q_a^{(0)}, \quad \delta\mathcal{R} = \sum_k \delta\mathcal{R}_q Q^{(0)}. \tag{5.8.3}$$

Then (5.8.1) leads to

$$\mathcal{V} \approx 2q^3\delta\mathcal{R}_q a. \tag{5.8.4}$$

$\mathcal{V}$ can be written in terms of $\mathcal{X}$ and $\mathcal{Z}$ as

$$\mathcal{V} = \left(1 + \frac{2q^2a^2}{27\beta(a-\beta)}\right)\mathcal{X} - 2\mathcal{H}\mathcal{Z}. \tag{5.8.5}$$

The Mukhanov-Sasaki variable is defined as $v = \delta\mathcal{R}_q z$, where

$$z = 3a\theta^{-1}\sqrt{\kappa(\mu + p)}.$$

In our model,

$$z = \sqrt{3}a\sqrt{\frac{a-\beta}{a-\epsilon}}, \tag{5.8.6}$$

$$\mathcal{V} \approx \frac{2}{\sqrt{3}}q^3\sqrt{\frac{a-\epsilon}{a-\beta}}v. \tag{5.8.7}$$

The initial values of v and its derivative are given by the quantum vacuum initial condition at the time of horizon exit:

$$v \sim \sqrt{\frac{1}{2q}}, \quad v' \sim i\sqrt{\frac{q}{2}}. \tag{5.8.8}$$

In this region, $a \gg \beta, \epsilon$. So, $\mathcal{V} \approx \frac{2}{\sqrt{3}} q^3 v$ and the initial conditions on $\mathcal{V}$ are obtained as

$$\mathcal{V} \sim \sqrt{\frac{2}{3}q^5}, \quad \mathcal{V}' \sim i\sqrt{\frac{2}{3}q^7}. \tag{5.8.9}$$

Now in region A of contracting phase,

$$\mathcal{X}^{(-)} = \frac{2q^2}{27}C_1^{(-)} + \frac{C_2^{A(-)}}{x^3} + C_3^{A(-)}x^2, \tag{5.8.10}$$

$$\begin{aligned}\mathcal{V}^{(-)} = {} & \frac{4}{3}\frac{C_1^{(-)}}{x^2}\left(1+\frac{q^2x^2}{36}\right)\left(1+\frac{q^2x^2}{18}\right) + \frac{q^2}{36}\frac{C_2^{A(-)}}{x} \\ & + \frac{5}{3}C_3^{A(-)}x^2\left(1+\frac{q^2x^2}{60}\right).\end{aligned} \tag{5.8.11}$$

In the expanding phase perturbations have similar evolution but with different constants,

$$\mathcal{X}^{(+)} = \frac{2q^2}{27}C_1^{(+)} + \frac{C_2^{A(+)}}{x^3} + C_3^{A(+)}x^2, \tag{5.8.12}$$

$$\begin{aligned}\mathcal{V}^{(+)} = {} & \frac{4}{3}\frac{C_1^{(+)}}{x^2}\left(1+\frac{q^2x^2}{36}\right)\left(1+\frac{q^2x^2}{18}\right) + \frac{q^2}{36}\frac{C_2^{A(+)}}{x} \\ & + \frac{5}{3}C_3^{A(+)}x^2\left(1+\frac{q^2x^2}{60}\right).\end{aligned} \tag{5.8.13}$$

The relation between the constants are obtained by proper matching of the variables in the boundary of the bouncing phase. We want to study such matching conditions on the surfaces $x = \pm 1$, which are the boundary of week energy condition ($\mu + 3p \geq 0$) violated region. First we deduce the spectrum of perturbations using two matching conditions, namely the continuity of $\mathcal{V}$ and $\mathcal{X}$ across the transitions surface and then calculate the same spectrum from numerical computation.

Since the entropy perturbation is obtained for all values of a, we get a matching condition:

$$C_1^{(+)} = C_1^{(-)}. \tag{5.8.14}$$

Matching $\mathcal{V}$ and $\mathcal{V}'$ on these surfaces, we get

$$\begin{aligned} \frac{q^2}{12} C_2^{A(+)} &= -\frac{16}{3} C_1^{(-)} - \frac{q^2}{36} C_2^{A(-)} + \frac{20}{3} C_3^{A(-)} \\ 5 C_3^{A(+)} &= \frac{16}{3} C_1^{(-)} - \frac{q^2}{18} C_2^{A(-)} - \frac{5}{3} C_3^{A(-)}. \end{aligned} \tag{5.8.15}$$

To know the correct spectrum of perturbations, we need the initial conditions on non adiabatic perturbations. For simplicity, let us assume $\Gamma_a = 0$, which implies, by (5.5.13), $C_1^{(-)} = 0$. Then the initial conditions (5.8.9) give

$$C_2^{A(-)} \approx (i-1) 8 \sqrt{\frac{2}{3}} q^{-1/2}, \quad C_3^{A(-)} \approx \frac{2i-1}{8} \sqrt{\frac{2}{3}} q^{9/2}. \tag{5.8.16}$$

Then (5.8.15) leads to

$$C_1^{(+)} = 0, \quad C_2^{A(+)} \approx (1-i) \frac{8}{3} \sqrt{\frac{2}{3}} q^{-1/2}, \tag{5.8.17}$$

$$C_3^{A(+)} \approx (1-i) \frac{4}{45} \sqrt{\frac{2}{3}} q^{3/2}. \tag{5.8.18}$$

Using this constants in (5.8.13), we get

$$\mathcal{V}^{(+)} \approx (1-i) \frac{2}{27} \sqrt{\frac{2}{3}} q^{3/2} \left(\frac{1}{x} + 2x^2 \left(1 + \frac{q^2 x^2}{60} \right) \right). \tag{5.8.19}$$

In the deep matter dominated phase,

$$\left| \mathcal{V}^{(+)} \right|^2 \approx q^3 \left| 1 + \frac{q^2 x^2}{60} \right|^2. \tag{5.8.20}$$

Using (5.8.4) the spectrum of $\delta \mathcal{R}_q$ is found to be

$$P_\zeta \approx q^3 \left| \delta \mathcal{R}_q \right|^2 \approx \left| 1 + \frac{q^2 x^2}{60} \right|^2. \tag{5.8.21}$$

So the power spectrum of $\delta \mathcal{R}$, obtained from this matching condition is nearly scale invariant, provided $q^2 x^2 < 60$, which is satisfied even after the horizon reentry ($qx = 2$). Using this matching we can also calculate the spectrum of $\mathcal{X}$. From (5.8.12),

$$\mathcal{X}^{(+)} = -\frac{4}{3}\sqrt{\frac{2}{3}}(i-1)\left(2q^{-1/2}x^{-3} + \frac{q^{3/2}x^2}{15}\right). \tag{5.8.22}$$

So in the deep matter dominated era of the expanding phase,

$$\left|\mathcal{X}^{(+)}\right|^2 \approx q^3. \tag{5.8.23}$$

Now we use a different matching condition, i.e., matching of $\mathcal{X}$. Matching of $\mathcal{X}$ and its derivative on the matching surfaces yields:

$$5C_2^{A(+)} = C_2^{A(-)} + 4C_3^{A(-)}, \quad 5C_3^{A(+)} = -6C_2^{A(-)} + C_3^{A(-)}. \tag{5.8.24}$$

The initial conditions (5.8.16) lead to:

$$C_1^{(+)} = 0, \quad C_2^{A(+)} \approx (1-i)\frac{8}{5}\sqrt{\frac{2}{3}}q^{-1/2}, \tag{5.8.25}$$

$$C_3^{A(+)} \approx -(1-i)\frac{16}{15}\sqrt{\frac{2}{3}}q^{-1/2}. \tag{5.8.26}$$

Then the $\mathcal{V}^{(+)}$ and $\mathcal{X}$ after the bounce behave as

$$\mathcal{V}^{(+)} \approx (i-1)\left(\frac{2}{3}\right)^{3/2} q^{3/2}\left[\frac{1}{15x} - \frac{8}{3}\frac{x^2}{q^2}\left(1 + \frac{q^2x^2}{60}\right)\right],$$

$$\mathcal{X}^{(+)} \approx \frac{8}{5}\sqrt{\frac{2}{3}}q^{-1/2}\left(\frac{1}{x^3} + \frac{2}{3}x^2\right).$$

The spectra of $\mathcal{V}$ and $\mathcal{X}$ are

$$\left|\mathcal{V}^{(+)}\right|^2 \approx q^{-1}\left|1 + \frac{q^2x^2}{60}\right|^2, \quad \left|\mathcal{X}^{(+)}\right|^2 \approx q^{-1}. \tag{5.8.27}$$

When $q^2x^2 < 60$, $\left|\mathcal{V}^{(+)}\right|^2 \approx q^{-1}$, the $\delta\mathcal{R}_q$ does not have a scale invariant spectrum; however, at much later time, $q^2x^2 > 60$, $\left|\mathcal{V}^{(+)}\right|^2$ behaves as q^3 and the power spectrum of $\delta\mathcal{R}$ is scale invariant again. We will find that the numerical results agree with (5.8.20) and (5.8.23), not with (5.8.27).

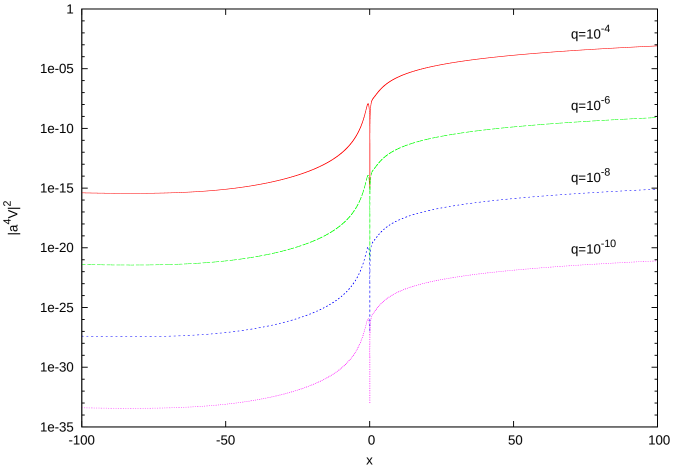

Figure 5.3: Time evolution of $\mathcal{V}$ for different values of the wave number

5.9 Numerical Analysis

We solve the coupled set of differential equations (5.5.7)-(5.5.9) by the Runge-Kutta method. The initial conditions are chosen as follows. The perturbations exit the horizon at $x = -x_{\text{exit}}$ in the matter dominated era. At a later time $x = -x_0$, but still within the matter dominated era, $\mathcal{V}$ and $\mathcal{V}'$ are given by

$$\begin{aligned} \mathcal{V}(-x_0) &\approx \frac{x_{\text{exit}}}{x_0}\mathcal{V}(-x_{\text{exit}}) = \frac{2}{x_0}\sqrt{\frac{2}{3}q^3} \\ \mathcal{V}'(-x_0) &\approx \left(\frac{x_{\text{exit}}}{x_0}\right)^2 \mathcal{V}'(-x_{\text{exit}}) = i\frac{2}{x_0}\sqrt{\frac{2}{3}q^3}, \end{aligned} \tag{5.9.1}$$

where we have used the initial conditions (5.8.9).

Now since $C_1 = 0$,

$$\mathcal{X}_2 = \frac{\beta}{a}\mathcal{X}_1, \quad \mathcal{X} = \frac{a-\beta}{a}\mathcal{X}_1. \tag{5.9.2}$$

From (5.8.5) and using (5.5.7)-(5.5.9) we get

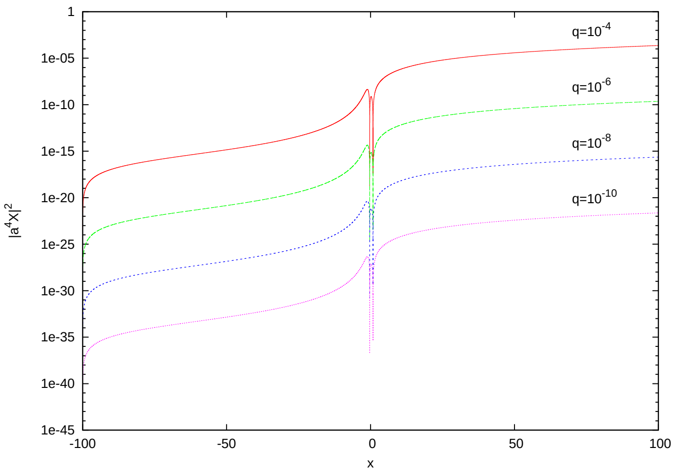

Figure 5.4: Time evolution of $\mathcal{X}$ for different values of the wave number

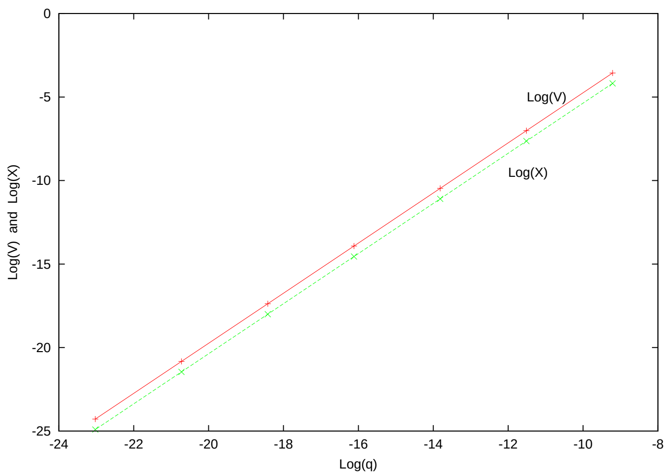

Figure 5.5: Spectral distribution of $\mathcal{X}$ and $\mathcal{V}$ at a fixed time $x = 100$.

$$\mathcal{V} = \mathbf{A}\mathcal{X}_1 + \mathbf{B}\mathcal{Z},$$

$$\mathcal{V}' = \mathbf{C}\mathcal{X}_1 + \mathbf{D}\mathcal{Z},$$

where

$$\mathbf{A} = \frac{a-\beta}{a} + \frac{2q^2}{27}\frac{a}{\beta}, \quad \mathbf{B} = -2\mathcal{H},$$
$$\mathbf{C} = \frac{2q^2}{27}\frac{a}{\beta}, \quad \mathbf{D} = -6\frac{\beta(a-\epsilon)}{a^2} - \frac{2}{3}q^2.$$

So,

$$\mathcal{X}_1 = \frac{\mathbf{D}\mathcal{V} - \mathbf{B}\mathcal{V}'}{\mathbf{AD} - \mathbf{BC}}, \quad \mathcal{Z} = -\frac{\mathbf{C}\mathcal{V} - \mathbf{A}\mathcal{V}'}{\mathbf{AD} - \mathbf{BC}}. \tag{5.9.3}$$

Substituting (5.9.1) in (5.9.3) we get the values of $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{Z}$ at $x = -x_0$. We take $x_0 = 100$. The results of numerical computation are shown in Figure(5.3)-(5.5). In Figure(5.3) and Figure(5.4) the time evolution of $\mathcal{V}$ and $\mathcal{X}$ is shown for wave numbers $q = 10^{-4}, 10^{-6}, 10^{-8}, 10^{-10}$. It is seen that the spectrum of both variables behave as $q^{3/2}$ in agreement with (5.8.20) and (5.8.23). We have also plotted $\log|\mathcal{X}|$ and $\log|\mathcal{V}|$ as a function of $\log|q|$ in Figure(5.5) at a time $x = 100$ in the expanding phase when all modes are outside the horizon. This gives

$$\frac{\delta \log|\mathcal{V}|}{\delta \log|q|} = \frac{\delta \log|\mathcal{X}|}{\delta \log|q|} = 1.5. \tag{5.9.4}$$

We have also plotted the behavior of perturbations in region B in Figure(5.6). It is observed that X_1 and X_2 grow as y^{-1} near turning point, but X and Z remain constant. However, according to (5.5.30), the growing mode of Z starts to dominate at nearer to the turning point for smaller frequencies. It is evident from Figure(5.7) that Z also grows as y^{-1} very close to the turning point.

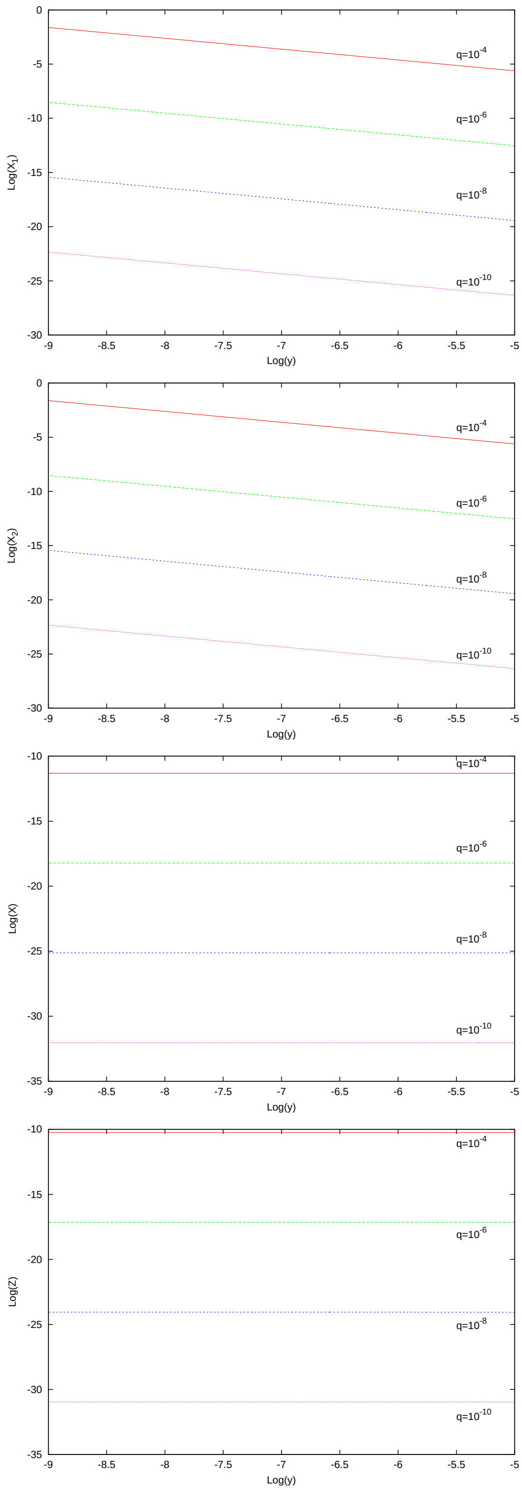

Figure 5.6: Evolution of perturbations near the turning point

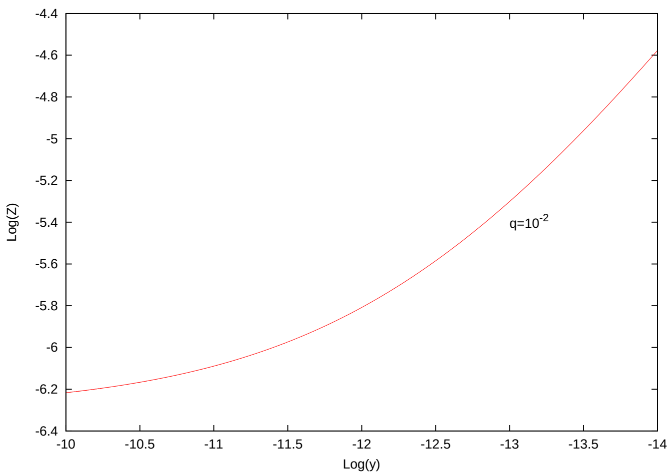

Figure 5.7: Evolution of Z very close to the turning point for $q = 0.01$.

Hence, the numerical analysis with our special initial conditions support our analytical results (5.5.28)-(5.5.30). Since the growth rates of scalar variables in (5.7.1) are derived from (5.5.28)-(5.5.30), the results in Sec. (5.7) involving scalar variables are still valid.

5.10 Conclusion

We have studied evolution of cosmological perturbations in a toy model of nonsingular and bouncing universe using the techniques of covariant perturbation theory. The matter sector is a two-component perfect fluid. The dust-like normal fluid drives the contraction and expansion and the radiation-like fluid with a negative energy density generates the bounce.

Evolution of vector perturbations ω_a and r_a are simple. But the analytic solutions for scalar and tensor perturbations in the entire range of time are obtained only for zero wave number mode. For $q \neq 0$, the equations are simplified to get analytic solutions in three different regions, namely long before bounce, at the turning point and at bounce. The scalar perturbations are smooth across the bounce but diverge at the turning point. The shear σ_{ab} is decomposed into scalar, vector and pure tensor parts. The gravitational wave, i.e., the pure tensor part of shear shows oscillations both at bounce and at the turning point. At the turning point, scalar and vector parts dominate over the gravitational wave. At the turning point, the comoving curvature perturbation ζ^S has a non-adiabatic growing mode besides its constant adiabatic mode.

The growth rates of linearity parameters are computed at the turning point. It is observed that many of these parameters diverge. So the perturbations fail to be linear at the turning point even in this simple nonsingular bouncing model.

The perturbation variables are defined here in terms of velocity u^a of the comoving observers in physical spacetime. This choice is not unique as discussed in Sec. (3.6). In order that the perturbations are gauge independent, the variables must vanish in the background spacetime, which means that the world lines of observers in the physical spacetime must not differ too much from that of the comoving observers in the background spacetime in the following precise sense: One can choose any arbitrary family of observers having velocity $\tilde{u}^a$ (for example, observers whose velocity is normal to the constant energy density hypersurface) such that $\tilde{u}^a - u^a$ vanish in the background spacetime. Let $\tilde{X}_a$, $\tilde{Y}_a$, $\tilde{Z}_a$ etc. be the perturbations, covariantly defined in terms of $\tilde{u}^a$. Then these new variables can be written in terms of the old ones X_a, Y_a, Z_a etc. So although our results are completely independent of the choice of gauge, they are tied to some choice of observers.

We have studied the matching condition for scalar variables. It has been shown that the spectrum of perturbations after bounce can be obtained by employing sound matching conditions. Despite the divergence at the turning point and the growth of amplitude, the scale invariant spectrum of perturbations is preserved after bounce. Our numerical analysis shows that the variable $\mathcal{V}$ should be matched across the transition surface to get the correct spectra, while matching $\mathcal{X}$ will lead to a wrong spectra. Since $\mathcal{V}$ and $\mathcal{X}$ are related to spatial curvature perturbation ($\delta\mathcal{R}$) and the Bardeen potential (Φ), these results coincide with the ones obtained in [53].

However, one may ask whether this spectrum is disrupted by the appearance of nonlinearity at the turning point. The y^{-n} dependence of linearity parameters implies that the nonlinearity effect may last only for a very short interval of time. Moreover, the interval may be shorter for larger wavelengths, as indicated by Eq. (5.5.30) and the numerical analysis. To address the question of whether the temporary nonlinearity can alter the future evolution of perturbations substantially, one requires to perform a full nonlinear analysis as has been performed in [72] for adiabatic perturbations.

Chapter 6

Summary and discussion

Inflation and bouncing cosmology are two important tools to describe our universe in terms of FLRW model and fluctuations over it. The merits of bouncing cosmology over inflation depend on many factors. A crucial one is whether the scale invariant spectrum of perturbations that is generated in the contracting phase can be continued to the expanding phase through bounce. Growing modes of perturbations may invalidate the linear perturbation theory. Also the emergence of new physics beyond classical general relativity or energy condition violating matter (which is necessary for bounce) can alter the spectrum and linearity of perturbations. Hence, one needs to verify the linearity of perturbations in a bouncing model in order to present it as an alternative scenario to inflation.

In standard coordinate based perturbation theory such tests depend on gauge choices. In some gauges perturbations are observed to grow while in some other gauges they remain small. In this report we have presented an alternative test of linearity using covariant perturbation theory. In this approach, perturbations are nonlinear and gauge invariant. Unlike gauge invariant perturbations in coordinate based approach, the covariant perturbations carry clear physical meaning without reference to any coordinate system. We have constructed some linearity parameters based on the fact that the nonlinear terms in an evolution equation are smaller compared to the linear terms.

Some of the linearity parameters grow in dust and radiation dominated contracting phase of a bouncing universe. In a simple bouncing universe, modeled by a two-component perfect fluid, the linearity parameters diverge at the turning point, i.e., the spacelike surface separating the region of spacetime obeying null energy condition with the region violating it. The results indicate that the linear perturbation theory is not adequate for the description of perturbations through a cosmological bounce.

Although the covariant perturbation variables are independent of the gauge choice, they are defined with reference to a family of observers. Choice of a different family of observers will leads to a different set of covariant variables. These new variables are expressible in terms of old ones. It will be an interesting and important study to investigate whether and how much the linearity conditions depend on the choice of observers.

We hope that the test of linearity of perturbations using covariant perturbation theory, presented in this report, will help in the search of proper model of bounce that serves as a viable alternative to inflation. Also the techniques can be applied in the expanding phase also. The linear perturbation theory must break down at some stage of the evolution of the Universe, otherwise the nonlinear structure formation did not take place. But question is when and how? The linearity conditions can be a useful tool to address that question.

Bibliography

[1] S. Weinberg, "Cosmology", Oxford, UK: Oxford Univ. Pr. (2008) 593 p

[2] V. Mukhanov, "Physical foundations of cosmology," Cambridge, UK: Univ. Pr. (2005) 421 p

[3] N. Turok, "A critical review of inflation," Class. Quant. Grav. **19**, 3449 (2002).

[4] A. R. Liddle and D. H. Lyth, "Cosmological inflation and large scale structure,"

[5] A. Borde, A. H. Guth and A. Vilenkin, "Inflationary space-times are incomplete in past directions," Phys. Rev. Lett. **90**, 151301 (2003) [gr-qc/0110012].

[6] R. C. Tolman, "On the theoretical requirements for the periodic behavior of the universe", Phys. Rev. **38**, 1758 (1931).

[7] M. Novello and J. M. Salim, "Nonlinear Photons In The Universe," Phys. Rev. D **20**, 377 (1979).

[8] V. N. Melnikov and S. V. Orlov, "Nonsingular Cosmology As A Quantum Vacuum Effect," Phys. Lett. A **70**, 263 (1979).

[9] V. F. Mukhanov and R. H. Brandenberger, "A Nonsingular universe", Phys. Rev. Lett. **68**, 1969 (1992).

[10] R. M. Wald, "General Relativity," Chicago, USA: Univ. Pr. (1984) 491p

[11] D. Wands, "Duality invariance of cosmological perturbation spectra," Phys. Rev. D **60**, 023507 (1999) [gr-qc/9809062].

[12] J. Khoury, B. A. Ovrut, P. J. Steinhardt and N. Turok, "Density perturbations in the ekpyrotic scenario," Phys. Rev. D **66**, 046005 (2002) [hep-th/0109050].

[13] F. Finelli and R. Brandenberger, "On the generation of a scale invariant spectrum of adiabatic fluctuations in cosmological models with a contracting phase", Phys. Rev. D **65**, 103522 (2002) [hep-th/0112249].

[14] V. Bozza and G. Veneziano, "Regular two-component bouncing cosmologies and perturbations therein," JCAP **0509**, 007 (2005) [gr-qc/0506040].

[15] D. H. Lyth, "The Failure of cosmological perturbation theory in the new ekpyrotic scenario" Phys. Lett. B **526**, 173 (2002) [hep-ph/0110007]

[16] T. J. Battefeld and R. Brandenberger, "Vector perturbations in a contracting universe", Phys. Rev. D **70**, 121302 (2004)

[17] J. Khoury, B. A. Ovrut, P. J. Steinhardt and N. Turok, "The Ekpyrotic universe: Colliding branes and the origin of the hot big bang", Phys. Rev. D **64**, 123522 (2001) [hep-th/0103239].

[18] B. Xue and P. J. Steinhardt, "Unstable growth of curvature perturbation in nonsingular bouncing cosmologies", Phys. Rev. Lett. **105**, 261301 (2010) [arXiv:1007.2875 [hep-th]].

[19] B. Xue and P. J. Steinhardt, "Evolution of curvature and anisotropy near a nonsingular bounce", Phys. Rev. D **84**, 083520 (2011) [arXiv:1106.1416 [hep-th]].

[20] Y. -F. Cai, E. McDonough, F. Duplessis and R. Brandenberger, "Two Field Matter Bounce Cosmology," JCAP **1310**, 024 (2013) [arXiv:1305.5259 [hep-th]].

[21] M. Gasperini, M. Giovannini and G. Veneziano, "Perturbations in a nonsingular bouncing universe", Phys. Lett. B **569**, 113 (2003) [hep-th/0306113].

[22] L. E. Allen and D. Wands, "Cosmological perturbations through a simple bounce", Phys. Rev. D **70**, 063515 (2004) [astro-ph/0404441].

[23] Sandro Dias Pinto Vitenti and N. Pinto-Neto, "Large Adiabatic Scalar Perturbations in a Regular Bouncing Universe", Phys. Rev. D **85**, 023524 (2012) [arXiv:1111.0888 [astro-ph.CO]].

[24] S. W. Hawking, "Perturbations of an expanding universe", Astrophys. J. **145**, 544 (1966).

[25] G. F. R. Ellis and M. Bruni, "Covariant And Gauge Invariant Approach To Cosmological Density Fluctuations", Phys. Rev. D **40**, 1804 (1989).

[26] J. M. Bardeen, "Gauge Invariant Cosmological Perturbations", Phys. Rev. D **22**, 1882 (1980).

[27] M. Bruni, P. K. S. Dunsby and G. F. R. Ellis, "Cosmological perturbations and the physical meaning of gauge invariant variables," Astrophys. J. **395**, 34 (1992).

[28] V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, "Theory of cosmological perturbations", Phys. Rept. **215**, 203 (1992).

[29] A. Kumar, "Growth of covariant perturbations in the contracting phase of a bouncing universe", Phys. Rev. D **86**, 123522 (2012) [arXiv:1205.6637 [gr-qc]].

[30] F. Finelli, P. Peter and N. Pinto-Neto, "Spectra of primordial fluctuations in two-perfect-fluid regular bounces" Phys. Rev. D **77**, 103508 (2008) [arXiv:0709.3074 [gr-qc]].

[31] A. Kumar, "Covariant perturbations through a simple non-singular bounce," Phys. Rev. D **89**, 084059 (2014). [arXiv:1310.2374 [gr-qc]].

[32] J. Preskill, "Cosmological Production of Superheavy Magnetic Monopoles," Phys. Rev. Lett. **43**, 1365 (1979).

[33] A. H. Guth, "Eternal inflation and its implications," J. Phys. A **40**, 6811 (2007) [hep-th/0702178 [HEP-TH]].

[34] V. A. Belinsky, I. M. Khalatnikov and E. M. Lifshitz, "Oscillatory approach to a singular point in the relativistic cosmology," Adv. Phys. **19**, 525 (1970).

[35] S. W. Hawking and G. F. R. Ellis, "The Large scale structure of space-time," Cambridge University Press, Cambridge, 1973

[36] C. Molina-Paris and M. Visser, "Minimal conditions for the creation of a Friedman-Robertson-Walker universe from a 'bounce'," Phys. Lett. B **455**, 90 (1999) [gr-qc/9810023].

[37] D. Hochberg and M. Visser, "Geometric structure of the generic static traversable wormhole throat," Phys. Rev. D **56**, 4745 (1997) [gr-qc/9704082].

[38] D. Hochberg, C. Molina-Paris and M. Visser, "Tolman wormholes violate the strong energy condition," Phys. Rev. D **59**, 044011 (1999) [gr-qc/9810029].

[39] V. F. Mukhanov, "Quantum Theory of Gauge Invariant Cosmological Perturbations," Sov. Phys. JETP **67**, 1297 (1988) [Zh. Eksp. Teor. Fiz. **94N7**, 1 (1988)].

[40] M. Sasaki, "Large Scale Quantum Fluctuations in the Inflationary Universe," Prog. Theor. Phys. **76**, 1036 (1986).

[41] R. Brandenberger, H. Firouzjahi and O. Saremi, "Cosmological Perturbations on a Bouncing Brane," JCAP **0711**, 028 (2007) [arXiv:0707.4181 [hep-th]].

[42] Y. F. Cai, T. Qiu, R. Brandenberger, Y. S. Piao and X. Zhang, "On Perturbations of Quintom Bounce," JCAP **0803**, 013 (2008) [arXiv:0711.2187 [hep-th]]

[43] Y. F. Cai, T. t. Qiu, R. Brandenberger and X. m. Zhang, "A Nonsingular Cosmology with a Scale-Invariant Spectrum of Cosmological Perturbations from Lee-Wick Theory," Phys. Rev. D **80**, 023511 (2009) [arXiv:0810.4677 [hep-th]]

[44] X. Gao, Y. Wang, R. Brandenberger and A. Riotto, "Cosmological Perturbations in Horava-Lifshitz Gravity," Phys. Rev. D **81**, 083508 (2010) [arXiv:0905.3821 [hep-th]].

[45] R. Durrer and F. Vernizzi, "Adiabatic perturbations in pre - big bang models: Matching conditions and scale invariance," Phys. Rev. D **66**, 083503 (2002) [hep-ph/0203275].

[46] W. Israel, "Singular hypersurfaces and thin shells in general relativity," Nuovo Cim. B **44S10**, 1 (1966).

[47] N. Deruelle and V. F. Mukhanov, "On matching conditions for cosmological perturbations," Phys. Rev. D **52**, 5549 (1995) [gr-qc/9503050].

[48] K. Enqvist, E. Keski-Vakkuri and S. Rasanen, "Hubble law and brane matter after ekpyrosis," Nucl. Phys. B **614**, 388 (2001) [hep-th/0106282]

[49] T. J. Battefeld, S. P. Patil and R. Brandenberger, "Non-singular perturbations in a bouncing brane model," Phys. Rev. D **70**, 066006 (2004) [hep-th/0401010]

[50] J. Martin, P. Peter, N. Pinto Neto and D. J. Schwarz, "Passing through the bounce in the ekpyrotic models," Phys. Rev. D **65**, 123513 (2002) [hep-th/0112128]

[51] H. Kodama and T. Hamazaki, "Evolution of cosmological perturbations in a stage dominated by an oscillatory scalar field," Prog. Theor. Phys. **96**, 949 (1996) [gr-qc/9608022].

[52] M. Gasperini and G. Veneziano, “The Pre - big bang scenario in string cosmology,” Phys. Rept. **373**, 1 (2003) [hep-th/0207130].

[53] M. Gasperini, M. Giovannini and G. Veneziano, “Cosmological perturbations across a curvature bounce,” Nucl. Phys. B **694**, 206 (2004) [hep-th/0401112].

[54] E. Lifshitz, “On the Gravitational stability of the expanding universe,” J. Phys. (USSR) **10**, 116 (1946).

[55] E. M. Lifshitz and I. M. Khalatnikov, “Investigations in relativistic cosmology,” Adv. Phys. **12**, 185 (1963).

[56] V. Faraoni, “On the gauge dependence of some results in linearized general relativity,” Phys. Lett. A **153**, 67 (1991).

[57] G. F. R. Ellis, J. Hwang and M. Bruni, “Covariant and Gauge Independent Perfect Fluid Robertson-Walker Perturbations,” Phys. Rev. D **40**, 1819 (1989).

[58] M. Bruni, G. F. R. Ellis and P. K. S. Dunsby, “Gauge invariant perturbations in a scalar field dominated universe,” Class. Quant. Grav. **9**, 921 (1992).

[59] P. K. S. Dunsby, M. Bruni and G. F. R. Ellis, “Covariant Perturbations in a multifluid cosmological medium,” Astrophys. J. **395** (1992) 54.

[60] P. K. S. Dunsby and M. Bruni, “Conserved quantities in perturbed inflationary universes,” Int. J. Mod. Phys. D **3**, 443 (1994). [gr-qc/9405008].

[61] J. M. Stewart, “Perturbations of Friedmann-Robertson-Walker cosmological models,” Class. Quant. Grav. **7**, 1169 (1990).

[62] G. F. R. Ellis and W. Stoeger, “The ’fitting problem’ in cosmology,” Class. Quant. Grav. **4**, 1697 (1987).

[63] J. M. Stewart and M. Walker, “Perturbations of spacetimes in general relativity,” Proc. Roy. Soc. Lond. A **341**, 49 (1974).

[64] G. F. R. Ellis and H. van Elst, “Cosmological models: Cargese lectures 1998,” NATO Sci. Ser. C **541**, 1 (1999) [gr-qc/9812046].

[65] D. Langlois and F. Vernizzi, “Evolution of nonlinear cosmological perturbations”, Phys. Rev. Lett. **95**, 091303 (2005) [astro-ph/0503416].

[66] R. Maartens and B. A. Bassett, “Gravitoelectromagnetism”, Class. Quant. Grav. **15**, 705 (1998) [gr-qc/9704059].

[67] D. Langlois and F. Vernizzi, “Conserved non-linear quantities in cosmology,” Phys. Rev. D **72**, 103501 (2005) [astro-ph/0509078].

[68] D. Langlois and F. Vernizzi, “A geometrical approach to nonlinear perturbations in relativistic cosmology”, Class. Quant. Grav. **27**, 124007 (2010) [arXiv:1003.3270 [astro-ph.CO]].

[69] P. K. S. Dunsby, B. A. C. C. Bassett and G. F. R. Ellis, “Covariant analysis of gravitational waves in a cosmological context”, Class. Quant. Grav. **14**, 1215 (1997) [gr-qc/9811092].

[70] R. Brustein, M. Gasperini, M. Giovannini, V. F. Mukhanov and G. Veneziano, “Metric perturbations in dilaton driven inflation,” Phys. Rev. D **51**, 6744 (1995) doi:10.1103/PhysRevD.51.6744 [hep-th/9501066].

[71] N. Goheer, P. K. S. Dunsby, A. Coley and M. Bruni, “Large-scale perturbations on the brane and the isotropy of the cosmological singularity,” Phys. Rev. D **70**, 123517 (2004) [hep-th/0408092].

[72] B. Xue, D. Garfinkle, F. Pretorius and P. J. Steinhardt, “Nonperturbative analysis of the evolution of cosmological perturbations through a nonsingular bounce,” Phys. Rev. D **88**, 083509 (2013) [arXiv:1308.3044 [gr-qc]].