BLACK HOLES IN LOOP QUANTUM GRAVITY - ENTROPY, THERMAL STABILITY AND ENERGY SPECTRUM

By

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A thesis submitted to the

Board of Studies in Physical Sciences

In partial fulfillment of requirements

for the Degree of

DOCTOR OF PHILOSOPHY

of

HOMI BHABHA NATIONAL INSTITUTE

October, 2014

STATEMENT BY AUTHOR

Abhishek Majhi

List of Publications arising of the Thesis

1. A. Majhi and P. Majumdar, *Class. Quant. Grav.* **29** (2012) 135013; arXiv:1108.4670
2. A. Majhi, *Class. Quant. Grav.* **30** (2013) 055001, arXiv:1112.3457
3. A. Majhi and P. Majumdar, *Class. Quant. Grav.* **31** (2014) 195003, arXiv:1301.4553
4. A. Majhi, *Class. Quant. Grav.* **31** (2014) 095002, arXiv:1205.3487v2
5. A. Majhi, *Phys. Rev.* **D88** 024010 (2013), arXiv:1303.4829
6. A. Majhi, *Energy Spectrum of equilibrium Black Holes*, arXiv:1303.4832

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Abhishek Majhi

DEDICATION

This thesis is dedicated to my dear parents – Mr. Susanta Kumar Majhi and Mrs. Malina Majhi, to my wife – Mrs. Amrita Majumder, to the person who transfigured me as a human being – Prof. Parthasarathi Majumdar, to Late Mr. Mrinmoy Mitra who inspired me during my school days, to Mr. Arun Kumar Mukhopadhyay who motivated me during my college days.

ACKNOWLEDGEMENTS

First of all, I want to express my sincere gratitude to my thesis supervisor Prof. Parthasarathi Majumdar for giving me immense freedom in my research work, for offering me his vast wealth of knowledge and experience through numerous lengthy and fruitful discussions throughout my Ph.D. years and of course, most importantly for providing me moral and spiritual support whenever I needed it during the odds.

I want to thank Prof. Debades Bandyopadhyay for being my official thesis advisor in Saha Institute of Nuclear Physics. I want to thank all my colleagues in Saha Institute of Nuclear Physics for being so nice to me during my stay in the institute – especially, Prof. Pijushpani Bhattecherjee, Prof. Debasish Majumdar and Prof. Ambar Ghosal. I owe many thanks to Dr. Srijit Bhattecherjee, Chandrachur Chakraborty, Dr. Partha Pratim Pradhan, Atanu Kumar and Avirup Ghosh for spending with me some of their precious time for the academic interactions that I had with them at different times. I want to thank Prof. Romesh Kaul for the academic support that he provided me. I also want to thank all my friends in Saha Institute of Nuclear Physics of the Post M.Sc. batch 2009 for helping me in numerous issues during my stay at the institute.

It am grateful to Prof. Parthasarathi Mitra for being my mentor for a review project on Entropic Gravity, which ultimately enabled me to get the chance to opt for doing research on black hole physics.

Finally, I want to thank my parents for supporting me to pursue my research career. Last but not the least, I want to express my sincere gratitude to my wife, Mrs. Amrita Majumder, for standing by me during all odds.

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Chapter 1

Synopsis

In modern day literature an equilibrium black hole horizon is modeled as an **isolated horizon (IH)** [1, 2]. IHs generalize the notion of stationary event horizons by incorporation of radiation in their vicinity which makes the ambient spacetime non-stationary. They therefore provide a completely *local* description of black hole horizons, in contrast to the *globally defined* notion of event horizons. The zeroth and the first laws of black hole mechanics are completely realizable in this local framework. Consequently, the thermodynamics of such horizons depends crucially on the boundary conditions defining IH rather than the specific spacetime metric in the bulk. Although the thermodynamics of a system is described in classical terms, the foundations lie deep inside the underlying quantum structure and the corresponding statistical physics of the relevant system. Hence, the knowledge of the quantum theory of a system forms the basis of our understanding of the origin of the thermodynamic phenomena manifested by the system. This is not an exception as far as IH thermodynamics is concerned. In **loop quantum gravity (LQG)**, which is arguably a strong candidate of the nonperturbative quantum theory of gravity, the structure of a **quantum isolated horizon (QIH)** is depicted as an IH punctured by the edges of the spin network graphs which span the bulk quantum geometry. This quantum structure of the IH enables us to apply the statistical mechanical tools in the context of equilibrium black holes to explore the corresponding thermodynamic consequences. We emphasize that this QIH framework is background independent and generic, which does not refer to any particular known classical black hole spacetime metric.

Usually the microcanonical entropy of an IH used to be calculated for a fixed classical area and arbitrary number of punctures[13, 14]. The recent proposal in ref. [15] where an energy spectrum proportional to the area is used based on semiclassical gleanings, has an added feature : the notion of the number of punctures of a QIH represented as a 'quantum hair'. The issue is whether such a characterization of a QIH can be obtained directly from the quantum **Chern-Simons(CS)** theory governing the dynamics of the horizon. In the second chapter of the thesis, we deal with the analysis of the derivation of the microcanonical entropy directly from the quantum statistics of the QIH in the light of the recently proposed idea of 'quantum hair'[15] and its corresponding consequence on the fate of the **Barbero-Immirzi(BI)** parameter(γ). The classical phase space analysis of the IH reveals that there is a three (2+1) dimensional CS theory on it [1] and the quantum degrees of freedom of the QIH are that of the CS theory coupled to the punctures (which act as sources) made by the bulk spin network on the IH. Thus, the QIH framework provides us with the self contained quantum mechanical structure of an equilibrium black hole horizon, based on which we can implement the statistical mechanical tools to unravel the thermodynamical consequences. As far as the calculation of the microcanonical entropy of a QIH is concerned, the microstates have been usually counted for fixed classical area only (or equivalently the CS coupling constant), until in [15] where it was first proposed that the total number of punctures (N) can also be considered as a macroscopic parameter, termed as 'quantum hair'. We articulate [5] the fact that the macrostates of a QIH can be characterized in terms of two independent integer-valued parameters, viz,. the coupling constant(k) of the source coupled quantum $SU(2)$ CS theory describing QIH dynamics and the total number of punctures (N) on the QIH. Taking the expression for the number of microstates of a QIH for arbitrary spins [16], we demonstrate that the microcanonical entropy of macroscopic (both parameters k and N assuming very large values) QIHs can be directly obtained using standard statistical mechanical methods, without having to additionally postulate the horizon as an ideal gas of punctures, or to incorporate any additional classical or semi-classical input from general relativity vis-a-vis the functional dependence of the IH mass on its area, unlike what was done in [15] in course of the proposal of the idea of 'quantum hair'. The logarithmic correction to the **Bekenstein-Hawking Area Law (BHAL)**[58] obtained a decade ago by R. Kaul and P. Majumdar in [17] (considering only spin 1/2 punctures), ensues straightforwardly, with precisely the coefficient $-3/2$, making it a signature of the LQG approach to black hole entropy.

In this setup where we consider the total number of punctures (N) as a macroscopic parameter alongside the CS level (k) or equivalently classical area (A_{cl}), if we want that the final form of entropy be given by the BHAL (ignoring log correction), then the BI parameter can take any real positive value. This is actually revealed from the fact that the BI parameter should be equated to a function of the macroscopic parameters (k, N) in the process, which takes continuous values from zero to infinity. However, if we demand that the microcanonical entropy has the form

$$S_{MC} = A_{cl}/4\ell_p^2 + N\sigma(\gamma) \tag{1.1}$$

which has been reported in [15] to be observed by a local stationary observer, it is valid only for values of the BI parameter(γ) greater than a certain number [6]. It can be actually argued that the term $N\sigma(\gamma)$ must be negative definite, which leads to the bound on γ. Two qualitative arguments are presented in favor of this fact which can be described briefly as follows. Firstly, from the knowledge of the kinematical Hilbert space of an IH of fixed classical area, considering the total number of punctures (N, which has no classical analogue and hence named to be 'quantum' hair) as a macroscopic parameter and fixing it to define the microcanonical ensemble is tantamount to providing quantum information about the physical system. Hence, fixing N is like a constraint on the full Hilbert space. Since it is already known that the consideration of all the quantum microstates of the full Hilbert space yields the BHAL, it is quite obvious that the entropy corresponding to some fixed-N subspace of the full Hilbert space must be less than the BHAL. Consequently the term $N\sigma(\gamma)$ should be a negative definite addition to the area law.

Our second argument is based on the observer dependence of measurements. The idea of considering the total number of punctures N to be a macroscopic parameter was based on a local observer framework i.e. this quantum hair will be seen only by an observer very close to the horizon. The topological defects will act like a gas and N will behave as a thermodynamic parameter akin to the total number of particles of a gas. The entropy measured by the local observer will contain the $N\sigma(\gamma)$ term in addition to the area law. But the effect of this 'quantum hair' will not be observed at the asymptotic infinity. Thus we can argue that the topological defects on the IH, which appear to be only quantum fluctuations for an asymptotic observer, become accessible and apparently 'classical-like' macroscopic degrees of freedom. Since, entropy is the measure of uncertainty in a system, it is quite obvious that the asymptotic entropy must be greater than the locally measured 'entropy' of the same system.

Finally, it may be mentioned that considerations related to the Entropy Bound [18], as 'covariant'-ized in [19] and sharpened within LQG in [20], places our qualitative arguments in favor of $\sigma(\gamma) < 0$ on a stronger footing. As it has been pointed out in course of the presentation of our arguments, the idea of quantum hair (N) is an observer dependent notion[15] i.e. N can be considered as a macroscopic thermodynamic variable only by a local observer very close to the horizon and there is no such notion of quantum hair for an asymptotic observer. Now, it is already known in the literature that there is a covariant (observer independent) entropy bound associated with a closed two-surface[18, 19] which has been proved quantum geometrically in [20] leading to a tighter bound for a QIH. For a closed spatial two-surface of area A_{cl} the maximum associated entropy can only be $A_{cl}/4\ell_p^2$ (ignoring the logarithmic correction[20]). Since the horizon entropy is nothing but the entropy associated with the closed two-surface cross-sections, it is evident that whatever observer dependent entropy one can calculate, cannot be greater than $A_{cl}/4\ell_p^2$. Hence, the observer dependent notion of quantum hair N can only give rise to a negative contribution to the entropy and thus there is no other choice than to impose the condition $\sigma(\gamma) < 0$.

Furthermore, a quantitative estimate of the numerical value of the bound on the BI parameter is provided through explicit graph plots of relevant functions which can be debriefed as follows. In the process of the calculation of the microcanonical entropy, the Lagrange multipliers σ and λ are related by the following relationship

$$e^{\sigma} = \frac{2}{\lambda^2}\left(1 + \frac{\sqrt{3}}{2}\lambda\right) e^{-\frac{\sqrt{3}}{2}\lambda}$$

If one plots e^{σ} as a function of λ, it is seen that the value of e^{σ} falls below 1 i.e. σ becomes negative for $\lambda > 1.200$. Since one has to fix $\gamma = \lambda(k/N)/2\pi$ in the process to obtain the required form of entropy, one can obtain the bound on γ by dividing the allowed range of λ by 2π, which results in $\gamma > 0.191$.

However, all the above arguments and justifications are valid only if we demand that the final form of the entropy has to be given by eq.(1.1). But, given that we have the luxury to choose γ as par our convenience, it can as well be argued that the entropy is just given by the BHAL after making suitable choice of γ. As a matter of fact, using the relation between k and N at equilibrium, the entropy can also be expressed in the form $S_{MC} = \tilde{\lambda}(k/N)k/2$, where $\tilde{\lambda} = \lambda - \left(\sigma/\frac{d\sigma}{d\lambda}\right)$. Now, requiring that the BHAL must follow one has to fit $\gamma = \tilde{\lambda}(k/N)/2\pi$. As there is no additional term to the BHAL, therefore there is no question of any further arguments. Study of the function $\tilde{\lambda}$ reveals that it can take values from 0 to ∞ thus resulting in no bound on γ.

Given that IHs represent thermal equilibrium configurations, albeit in isolation, how does one address the issue of thermal stability of black holes in general ? In standard semiclassical approaches to black hole thermodynamics, the event horizon of an asymptopically flat Schwarzschild spacetime is patently unstable, since Hawking radiation from it is characterized by a Hawking temperature which decreases inversely as the mass of the black hole. As the mass decreases due to radiation, the temperature continues to rise, leading to a runaway situation[]. In contrast, an anti-de Sitter Schwarzschild black hole exhibits stable thermal behaviour within a certain range of parameters., beyond which a Hawking-Page phase transition may occur.

The first work reported in this thesis addresses the issue as to whether some general criteria for thermal stability exists, of thermal stability of radiant black holes, independent of classical metrical properties. Generalizing an earlier work [12] for spherical charge-less IH, we *derive* some criteria of thermal stability for electrically charged quantum black holes having a large horizon area (compared to the Planck area). We use key results of LQG and equilibrium statistical mechanics of a grand canonical ensemble, with Gaussian fluctuations around an equilibrium thermal configuration assumed here to be a QIH [3]. It is to be emphasized that the analysis is completely based on the generic QIH framework, without reference to any sort of classical background metric. Hence, the stability criteria can be treated as investigating tools for the thermal stability of charged, non-rotating black holes with an arbitrary semiclassical energy spectrum. Inputting a specific energy spectrum (as a function of horizon area) leads to a response specifying whether the horizon is thermally stable.

The Hilbert space of a quantum spacetime with boundary has the tensor product structure $\mathcal{H} = \mathcal{H}_v \otimes \mathcal{H}_b$, with the subscript v (b) denoting the bulk (boundary) component. The quantum CS theory on the inner boundary coupled to the spin network edges, which span the bulk quantum geometry, provide the boundary Hilbert. The states belonging to this Hilbert space provide the boundary degrees of freedom. Thus, any generic state in quantum geometry, $|\Psi\rangle$, admits the expansion $|\Psi\rangle = \sum_{v,b} C_{vb} \, |\psi_v\rangle \otimes |\chi_b\rangle$. In presence of electromagnetic fields, one can consider $|\psi_v\rangle$ (resp. $|\chi_b\rangle$) to be the composite quantum gravity + quantum electrodynamics bulk (resp. boundary) state. The bulk states are annihilated by the *full* bulk Hamiltonian : $\hat{H}_v|\psi_v\rangle \equiv [\hat{H}_{g,v} + \hat{H}_{e,v}]|\psi_v\rangle = 0$; this is the quantum version of the classical Hamiltonian constraint. The total Hamiltonian operator acting on the generic state $|\Psi\rangle$ has the form $\hat{H}_T|\Psi\rangle = (\hat{H}_v \otimes I_b + I_v \otimes \hat{H}_b)|\Psi\rangle$ where, $I_v(I_b)$ corresponds to the identity operator on $\mathcal{H}_v(\mathcal{H}_b)$. The primacy of the boundary partition function of a grand canonical ensemble, in situations where the bulk Hamiltonian is a constraint, is then quite trivial to establish considering the above information in the usual definition of the grand canonical partition function in quantum statistics. The boundary partition function is then evaluated using saddle point approximation, choosing an IH as the equilibrium configuration. Valid existence of the saddle point is shown to lead to the thermal stability criteria in terms of second-order partial differential inequalities involving the equilibrium mass and the microcanonical entropy. No aspect of classical black hole geometry is used to deduce the stability criteria. Since no particular form of the mass function is used *a priori*, our stability criteria provide a platform to test the thermal stability of a black hole with a given mass function. The mass functions of the two most familiar charged black hole solutions, namely Reissner-Nordstrom and AdS Reissner-Nordstrom, are tested as a fiducial check. The Reissner-Nordstrom black hole is seen to be locally unstable against Gaussian thermal fluctuations, whereas AdS Reissner-Nordstrom black hole is seen to be locally stable against Gaussian thermal fluctuations for certain range of parameters. Further, we also discuss the validity of the saddle-point approximation used to incorporate thermal fluctuations. Moreover, the equilibrium Hawking temperature is shown to have an additional quantum correction over the usual semiclassical value.

Next, we extend our analysis of thermodynamic stability of black holes in the context of a recent research proposal. Recently it has been argued in the literature [15], using a semiclassical approximation, that the area and energy of a black hole are proportional as observed by an observer very close to (in terms of proper distance) and stationary with respect to the horizon and hence the energy spectrum of the QIH is nothing but the area spectrum scaled by a proportionality constant. We show that a QIH, with such an energy spectrum, is locally unstable as a thermodynamic system [4]. The result is derived in two different ways. Firstly, the specific heat of the QIH is shown to be negative definite through a quantum statistical analysis. Then, it is shown, in the thermal holographic approach, that the canonical partition function of the QIH diverges under Gaussian thermal fluctuations of such energy spectrum, implying local instability of such a QIH as a thermodynamic system. The energy spectrum simply violates the thermal stability criterion.

In the final section of the thesis we investigate the possible structure of the Hamiltonian operator and the corresponding energy spectrum associated with the QIH. Even though there is a notion of classical energy associated with the IH satisfying the first law[9], there is not yet any well defined Hamiltonian operator or an energy spectrum for the QIH.

Instead of quantizing the classical energy associated with the IH, we propose the most general structure of the Hamiltonian operator for the QIH. The proposal is based on simple and strong physical motivations and supported by well justified arguments which can be briefly restated as follows :-

- The punctures are the most fundamental and elementary constituents of the QIH which collectively provide an effective description of the IH in the correspondence limit.

- The model Hamiltonian shares all the necessary and relevant properties of the area operator e.g. gauge-invariance, self-adjointness, etc.
- The model Hamiltonian and the area operator associated with the QIH have simultaneous eigenstates which are those of the CS theory coupled to punctures.
- The structure of the model Hamiltonian ensures that the constant area property of IH emerges in the correspondence limit.

The proposed structure of the Hamiltonian operator can be written as

$$\hat{H}_S \equiv \sum_{n=0}^{\Lambda} p_n \left(\hat{A}^n_{j_1} \otimes \hat{I}_{j_2} \otimes \cdots \otimes I_{j_N} + \hat{I}_{j_1} \otimes \hat{A}^n_{j_2} \otimes \cdots \otimes \hat{I}_{j_N} + \cdots + \hat{I}_{j_1} \otimes \hat{I}_{j_2} \otimes \cdots \otimes \hat{A}^n_{j_N} \right)$$

whose spectrum can be explicitly written as $\sum_{n=1}^{\Lambda} \sum_{l=1}^{N} p_n (8\pi\gamma\ell_p^2)^n [j_l(j_l+1)]^{n/2}$, where the coefficients (p-s) carry the burden of endowing the Hamiltonian operator with the correct dimensionality and Λ is a required cut-off. It is straightforward to see that the structure of the Hamiltonian operator ensures that all the requisite conditions discussed above are taken care of. Now, having a well defined Hamiltonian operator for the QIH we are able to write down the canonical partition function in the usual *energy ensemble* containing the actual Boltzmann factor $\exp[-\beta E]$, instead of using the area ensemble containing a 'Boltzmann-like' factor $\exp[-\alpha A]$ involving a fictitious parameter α conjugate to area A [36, 22, 23]. Ignoring the thermal fluctuations and the stability issues, we continue to work in the microcanonical ensemble defined by the relevant macroscopic parameters, namely k and N as discussed previously. Calculations of the statistical expectation values of the Hamiltonian and the area operators of the QIH and using it in the expression for the entropy yields the expression $S_{MC} = \frac{\lambda}{8\pi\gamma\xi\ell_p} E^\star + N\sigma$, where ξ is a complicated function containing the unknown coefficients of the Hamiltonian. Identification of the above expression with the usual form of the entropy $S = \beta E^\star + N\sigma$ reveals the form of the equilibrium temperature $T = 1/\beta$. It is to be mentioned that this form of the entropy is completely at par with the first law of thermodynamics associated with the QIH in the set up where N is considered to be a mcroscopic variable i.e. $T\ \delta S = \ \delta E + \mu\ \delta N$, where $\mu = -\sigma/T$ is the 'chemical potential' corresponding to the 'quantum hair' N[15]. Now, applying the γ-fit and making suitable choices of the unknown coefficients of the proposed Hamiltonian, the equilibrium temperature is now given by $T = \eta(1+\Lambda)\ell_p$ (Boltzmann constant is set to unity). In this process, the unknown coefficients of the proposed model Hamiltonian are fixed. This yields the relevant Hamiltonian operator and the corresponding energy spectrum of the QIH which can be written as[7, 8]

$$\hat{H}_S |\{s_j\}\rangle = \eta\Gamma(3, \sqrt{3}\pi\gamma)\ell_p \sum_j \sum_{n=0}^{\Lambda} \frac{(2\pi\gamma)^n}{\Gamma\left(n+2\ ,\ \sqrt{3}\pi\gamma\right)}\ s_j\{j(j+1)\}^{n/2}\ |\{s_j\}\rangle$$

where $|\{s_j\}\rangle$ is the spin configuration describing a quantum state of the QIH which is also a simultaneous eigenstate of the corresponding area operator; η and Λ are parameters which are required to be chosen to yield the correct temperature or surface gravity which is associated with the classical first law for the IH; γ is the Barbero-Immirzi parameter and ℓ_p is the Planck length.

Chapter 2

Introduction

Perhaps the clearest description of the quantum states of generic (extremal or non-extremal) four dimensional black hole horizons in the presence of matter and/or radiation (which do not cross the horizons) is in terms of the quantization of the classical Isolated Horizon (IH) phase space which yields the kinematical Hilbert space for the Quantum Isolated Horizon (QIH) [13, 14] derived from loop quantum gravity (LQG). In this description, at the classical level, the IH is considered as an inner boundary of spacetime with boundary conditions imposed upon it which (a) do not require that the ambient spacetime be stationary and (b) hold the area of spatial foliations (time-slice) of the IH fixed, precluding matter or radiation from crossing it. It has been shown in [1] that originating from these boundary conditions is a Hamiltonian structure of the degrees of freedom of a generic IH and their dynamics, in terms of an $SU(2)$ Chern-Simons (CS) theory, where the CS connection (on the chosen time-slice) belongs to a one-parameter family of linear combination of components of the usual Levi-Civita connection of general relativity, pulled back to the spatial slice. The CS connection couples to the bulk spacetime geometry, namely the bulk triads on the spatial slice of bulk spacetime, which act as sources for the CS connection, with the CS coupling being proportional to the classical area of the spatial slice of the IH. A somewhat more direct approach to the problem, albeit restricted to a static spacetime, namely, the Schwarzschild spacetime, has been given in [21] where the same geometric variables as used in [1] are evaluated explicitly, with the manifest emergence of the $SU(2)$ CS description of horizon degrees of freedom and their dynamics, namely their coupling to bulk geometry. Classically therefore, a black hole horizon is nothing but an $SU(2)$ CS theory of connection fields on an IH (a generalization, to reiterate, of an event horizon to non-stationary ambient spacetimes in the bulk) which are minimally coupled to sources consisting of bulk spacetime tetrads. Since tetrads are formally like the 'square root' of the metric, at the classical level, the relationship to the spacetime metric is clear.

In the quantum description, the physical states of a QIH are the gauge singlet states of an $SU(2)$ Chern Simons (CS) theory coupled to the punctures endowed with SU(2) spins deposited on the QIH by the intersecting edges of the bulk spin network describing the bulk quantum geometry. In other words, the QIH is taken to have the topology of a two-sphere with punctures carrying spins induced by the floating lattice known as the spin network whose edges carry spins. Such configurations are expected to arise as solutions of the full quantum dynamical equations describing bulk quantum geometry, in particular the quantum Hamiltonian constraint. Whether or not this actually happens remains a question for the future. In this situation, it is worthwhile to investigate properties of the kinematical Hilbert space to see how the issue of black hole entropy can be addressed.

There are a plethora of approaches to black hole entropy, many semiclassical ones with diverse claims for the black hole entropy having logarithmic corrections to the area law [24, 25, 26, 27, 28, 29, 30, 31], with various coefficients – some positive and some negative. In all these computations, one usually computes the entropy of quantum 'matter' fields (including gravitons) coupled to the classical background spacetime metric of a black hole. Even in approaches where the classical metric is 'integrated over', the functional integral over metrics is invariably saturated by the classical black hole metric. In other words, non-perturbatively large quantum fluctuations of the spacetime - which cannot meaningfully be separated into a classical background metric and its fluctuations - are usually ignored in these computations. Thus, these computations do *not* take into account the entirety of quantum states corresponding to quantum spacetime fluctuations of a black hole spacetime. Rather, they account for non-gravitational states *entangled* with the horizon, as discussed in ref. [32, 33]. In contrast, the LQG approach focuses on the quantum states describing the *quantum* geometry of the horizon, without assuming any entanglement with quantum matter states in its vicinity. The classical metric of the black hole plays no role in this approach. The LQG computation thus yields what one may call the *gravitational* (or 'spacetime') entropy of a black hole, as opposed to the non-gravitational entanglement entropy computed in other approaches, which depends rather directly on the classical black hole metric in all cases. Indeed, the two together would most likely constitute the total entropy of a quantum black hole spacetime, so that the results of both directions of computation are complimentary rather than competitively comparable. In other words, the entanglement entropy considered in the other approaches must be an additional contribution to the horizon entropy, over and above the spacetime entropy computed within the LQG approach.

The description of a QIH in terms of CS states coupled to bulk quantum geometry is itself a radical departure from the general relativistic description of a classical event horizon. Of course, a QIH does not radiate or accrete matter/radiation, and can only be a part of a more complete description of a radiant black hole comprising of Trapping [34] or Dynamical [10] horizons. Recently, such horizons have been argued to emit Hawking radiation [35] with the standard black body spectrum, raising hopes that a deeper understanding of black holes within the LQG approach might be around the corner. However, there are semi-classical assumptions that appear to be necessary to supplement the premises of LQG, which manifest a lingering incompleteness in basic understanding of the quantum geometry of IHs.

Chapter 3

Microcanonical Entropy of Black Hole Horizon in LQG : A Paradigm Shift and its Consequences

3.1 Introduction

The subject matter of this chapter is to re-articulate the fact that a complete description of the macrostates of a QIH is possible with two integer-valued quantum parameters ('quantum hairs'), namely the CS coupling constant k related to the classical horizon area A_{cl}, and the number N of punctures to which the CS fields couple. Thus, recent arguments regarding the number of punctures as a 'quantum hair' (see, e.g., [15]) based on semiclassical reasoning are subsumed within the Chern-Simons description of QIHs. Indeed, the microcanonical entropy can be obtained directly from the formula derived fourteen years ago [16] for the total number of $SU(2)$ singlet states coupled to a fixed set of spins $j_1, j_2, ..., j_N$, $j_i \in [1/2, k/2] \forall i = 1, ...N$. The independence of these two parameters k and N has been quite apparent at the formulation stage, while deriving this formula, even though their physical role and the precise path whereby a relation between the two emerges only becomes obvious through ref. [15], (albeit upon their invoking semiclassical arguments which we have made no use of). Summing this formula over all spins $\{j_i\}$ and using the multinomial expansion of elementary algebra, this degeneracy can be expressed in terms of sums over spin *configurations*, i.e., number of punctures $s_j \in [0, N]$ for spins $j = 1/2, ..., k/2$. This upper bound on allowed spin-values follows directly from the unitarity of the two dimensional conformal field theory (viz., the $SU(2)_k$ WZW model) living on the boundary S^2, whose conformal blocks capture the degeneracy information of the CS states on the QIH.

Using standard tenets of equilibrium statistical mechanics, one then looks for the 'Most Probable Spin Configuration' by maximizing the microcanonical entropy subject to the constraints [39] of a fixed N and fixed large QIH area differing from the classical area only to $O(\ell_p^2)$, where ℓ_p is the Planck length. This extremization procedure leads to an equilibrium 'equation of state' relating the two quantum parameters k and N describing the macrostates. Requiring that the resulting formula for the microcanonical entropy of a macroscopic ($k \ggg 1,\ N \ggg 1$) QIH reproduce the Bekenstein-Hawking area law in terms of classical IH area A_{cl} to leading order, imposes a restriction on the Barbero-Immirzi parameter γ in the definition of $k \equiv A_{cl}/4\pi\gamma\ell_p^2$, that this must lie within a specific interval on the real line [6]. Subleading corrections to the area law ensue naturally from the derived formula, especially the leading logarithmic (in *classical* area) correction with coefficient $-3/2$ found longer than a decade ago by Kaul and Majumdar [17] for a dominant class of spins, as we shall see in the sequel. These results have also been rederived recently in ref. [40, 41, 42, 43]

It may be asked how precisely this work builds upon from extant literature on microcanonical entropy by counting of CS states in LQG. We wish to note here that in those earlier papers, the formulation of QIH states in terms of two independent parameters (the so-called 'quantum hair' as espoused in [15]) was not used explicitly. In some papers, the entropy is computed simply by counting states of an ideal gas of punctures, so that the CS underpinning of the QIH states is not used explicitly. In some of the very early LQG literature, the computation does not quite use the connection with two dimensional $SU(2)_k$ WZW models which is crucial to our approach here. Still in some other papers, the counting is restricted to a 'dominant' spin configuration on the punctures, so that even though the approximation is not incorrect, the approach appears to lack generality.

The important value-addition in this work is to provide a direct link of the 'quantum hair' scenario [15] to the LQG formulation of QIH macrostates in terms of $SU(2)$ CS states coupled to punctures carrying spin, without any trappings of a semi-classical nature (like [15]) depending on classical metrical properties, nor any restriction to a specific class of spin values at the punctures. This connection of the CS formulation of the QIH to the conceptual understanding gained recently in ref. [15] of macrostates in terms of two integer-valued parameters k and N, leading eventually to the same microcanonical entropy as found earlier [17] thus ties up the older formulations with contemporary ideas in this field on a unified footing. While many of the extant papers in the recent literature on QIH entropy within the LQG approach simply count states of an ideal gas of spins, without much allusion to the underlying theory of QIHs in terms of quantum CS states, our approach here has been to underline the CS theory underpinning, and to make the argument as self-contained as possible within the original LQG approach pioneered by Ashtekar and coworkers [44, 45], while concomitantly relating to more recent aspects of the literature. This work has some overlap with the recent review of Kaul [46] as far as some parts of the calculations are concerned, but the emphasis on direct link-up with the CS theory perspective and some of the details are different and are, hopefully, of inherent merit.

3.2 The Hilbert Space and Physical Chern-Simons states of the QIH

The Hilbert space of a quantum spacetime admitting QIH as an inner boundary is given by $\mathcal{H} = \mathcal{H}_V \otimes \mathcal{H}_S$ modulo gauge transformations, where V denotes bulk and S denotes boundary(QIH) at a particular time slice[13, 14]. Mathematically, if the 4d spacetime $(\mathbf{R}\otimes\Sigma)$ admits a 3d IH (Δ) as null inner boundary, then $S \equiv \Delta \cap \Sigma$ denotes a cross-section of the IH [10]. Hence, a generic quantum state of the spatial geometry of such a spacetime can be written as $|\Psi\rangle = |\Psi_V\rangle \otimes |\Psi_S\rangle$, where $|\Psi_V\rangle$ is the wave function[1] corresponding to the volume(V) or bulk states represented

[1]Strictly speaking, these are actually *functionals* of the $SU(2)$ spatial connection variables and a smooth *function* of generalized gauge-invariant connections, the holonomies along the edges of the oriented graph [47], popularly known as the spin network.

by an oriented graph, say Γ, consisting of edges and vertices [47] and $|\Psi_S\rangle$ denotes a generic quantum state of the QIH. $|\Psi_S\rangle \in \mathcal{H}_S \equiv$ the Hilbert space of the CS theory coupled to the punctures $\{\mathcal{P}\}$ made by the bulk spin network Γ with the IH endowing them with the spin representations carried by the respective piercing edges which are solely responsible for all the relevant features of the QIH, the most important being the quantum area spectrum of the QIH. To be precise, for a given N number of punctures, with spins $(j_1, \cdots, j_N)$, the QIH Hilbert space is given by $\mathcal{H}_S \equiv \mathrm{Inv}(\otimes_{i=1}^{N} \mathcal{H}_{j_i})$ where 'Inv' denotes the invariance under the local $SU(2)$ gauge transformations on the QIH. Now, as it is seen that at the quantum level the full Hilbert space is the direct product space of the bulk and boundary Hilbert spaces, a generic quantum state of the QIH (boundary) can be written in terms of basis states on $\mathcal{H}_S$, independent of the bulk wave function. Hence, one should understand that a basis state of the QIH Hilbert space is actually a generic quantum state of the full Hilbert space, since the bulk part of the wave function is a linear combination of the basis states of the bulk geometry. In other words, a given *spin configuration* on the QIH admit all possible graphs (Γ-s) in the bulk consistent with the given configuration. This spin configurations provide the area eigenstate basis, which is the all important material in the context of QIH entropy. Such a basis state of the QIH Hilbert space is denoted by the ket $|\{s_j\}\rangle$. This is an eigenstate of the area operator associated with the QIH, having the area eigenvalue given by $\hat{A}_S|\{s_j\}\rangle = 8\pi\gamma\ell_p^2 \sum_{j=1/2}^{k/2} s_j \sqrt{j(j+1)}|\{s_j\}\rangle$. Such a spin configuration (eigenstate) has a $(N!/\prod_j s_j!)$-fold degeneracy due to the possible arrangement of the spins yielding the same area eigenvalue. Hence, a generic quantum state of the QIH can be written as

$$|\Psi_S\rangle = \sum_{\{s_j\}} c[\{s_j\}] |\, \{s_j\}\rangle$$

where $|c[\{s_j\}]|^2 = \omega[\{s_j\}]$(say) is the probability that the QIH is found in the state $|\, \{s_j\}\rangle$. Hence, a generic quantum state of the spacetime, admitting QIH as an inner boundary, may now be written as $|\Psi\rangle \equiv |\Psi_V\rangle \otimes \sum_{\{s_j\}} c[\{s_j\}] |\, \{s_j\}\rangle$.

The computation of the microcanonical entropy formula of Kaul and Majumdar [16] proceeds from the expression for the number of conformal blocks of $SU(2)_k$ WZW model on a

2-sphere with marked points (punctures) carrying spin which, in the seminal work of Witten [48], has been shown to give the dimensionality of the $SU(2)$ singlet part of the Hilbert space of CS states on $\mathbf{R} \otimes S^2$ coupled to punctures on the S^2. Using this remarkable connection, the fusion algebra and the Verlinde formula, the degeneracy of the microstates is expressed as [16]

$$\Omega(j_1, \cdots, j_N) = \frac{2}{k+2} \sum_{a=1}^{k+1} \frac{\sin \frac{a\pi(2j_1+1)}{k+2} \cdots \sin \frac{a\pi(2j_N+1)}{k+2}}{\left(\sin \frac{a\pi}{k+2}\right)^{N-2}} \tag{3.1}$$

which can be alternatively recast as a linear combination of Kronecker deltas [16], explicitly manifesting the singlet nature of the physical states :

$$\Omega(j_1, \cdots, j_N) \;=\; \sum_{m_1=-j_1}^{j_1} \cdots \sum_{m_N=-j_N}^{j_N} \left[\, \delta_{(\sum_{p=1}^{N} m_p),0} - \frac{1}{2} \;\; \delta_{(\sum_{p=1}^{N} m_p),1} - \frac{1}{2} \;\; \delta_{(\sum_{p=1}^{N} m_p),-1} \right]$$

For the calculations we shall use the expression (3.1). To obtain the total number of conformal blocks, this expression must be summed over all possible spin values at each puncture.

$$\Omega(N, k) = \sum_{j_1, \cdots, j_N} \Omega(j_1, \cdots, j_N) \tag{3.2}$$

Now, since we will apply the method of most probable distribution to find the microcanonical entropy of the QIH, it is convenient to recast eq.(3.2) as sum over spin-configurations, i.e., the number of punctures carrying a specific spin j, for all possible values of j, becomes the dynamical variable.

$$\Omega(N, k) = \sum_{\{s_j\}} \Omega[\{s_j\}] \tag{3.3}$$

where

$$\Omega[\{s_j\}] = \frac{N!}{\prod_j s_j!} g[\{s_j\}] \tag{3.4}$$

where j runs from $1/2$ to $k/2$ as usual and

$$g[\{s_j\}] = \frac{2}{k+2} \sum_{a=1}^{k+1} \sin^2 \frac{a\pi}{k+2} \prod_j \left\{ \frac{\sin \frac{a\pi(2j+1)}{k+2}}{\sin \frac{a\pi}{k+2}} \right\}^{s_j} \tag{3.5}$$

The combinatorial factor in eq.(3.4) reflects the *statistical* distinguishability of the punctures, a property inherited by the punctures in the quantization procedure of the classical IH due to the nontrivial holonomies of the Chern-Simons connection on the IH along disjoint closed loops about the punctures[14].

3.2.1 Multinomial Expansion : The Link

It is worth mentioning that, even though eq.(3.4) resembles the formula for the microstates of an ideal gas of spins obeying Maxwell-Boltzmann statistics, it has been derived directly from the formula (3.1) within the present scenario of QIH. One should keep in mind that eq.(3.4) is just another form of eq.(3.2) being written in a different basis for convenience of the statistical formulation. Hence, for the sake of clarity, let us derive eq.(3.4) directly from eq.(3.2). Using eq.(3.1) one can write eq.(3.2) in the following explicit form

$$\begin{aligned}
\Omega(N,k) &= \frac{2}{k+2} \sum_{a=1}^{k+1} \sin^2 \frac{a\pi}{k+2} \sum_{j_1,\cdots,j_N} \prod_{r=1}^{N} \left\{ \frac{\sin \frac{(2j_r+1)a\pi}{k+2}}{\sin \frac{a\pi}{k+2}} \right\} \\
&= \frac{2}{k+2} \sum_{a=1}^{k+1} \sin^2 \frac{a\pi}{k+2} \prod_{r=1}^{N} \sum_{j_1,\cdots,j_N} \left\{ \frac{\sin \frac{(2j_r+1)a\pi}{k+2}}{\sin \frac{a\pi}{k+2}} \right\} \\
&= \frac{2}{k+2} \sum_{a=1}^{k+1} \sin^2 \frac{a\pi}{k+2} \left[\sum_j \left\{ \frac{\sin \frac{(2j+1)a\pi}{k+2}}{\sin \frac{a\pi}{k+2}} \right\} \right]^N
\end{aligned}$$

Using Multinomial expansion, the above expression can be recast into the following form

$$\begin{aligned}
\Omega(N,k) &= \frac{2}{k+2} \sum_{a=1}^{k+1} \sin^2 \frac{a\pi}{k+2} \sum_{\{s_j\}} \frac{N!}{\prod_j s_j!} \prod_j \left\{ \frac{\sin \frac{(2j+1)a\pi}{k+2}}{\sin \frac{a\pi}{k+2}} \right\}^{s_j} \\
&= \sum_{\{s_j\}} \left[\frac{2}{k+2} \sum_{a=1}^{k+1} \sin^2 \frac{a\pi}{k+2} \frac{N!}{\prod_j s_j!} \prod_j \left\{ \frac{\sin \frac{(2j+1)a\pi}{k+2}}{\sin \frac{a\pi}{k+2}} \right\}^{s_j} \right]
\end{aligned} \tag{3.6}$$

This is exactly eq.(3.3), which is nothing but eq.(3.2) written as sum over spin-configurations.

3.3 Microcanonical Ensemble with fixed number of punctures

Here we wish to calculate the microcanonical entropy of the QIH for given k and N and study the consequences. To obtain the microcanonical entropy, one must maximize its expression with respect to the spin configurations, to determine the most probable spin configuration, subject to the restriction that the mean QIH area must equal the classical area up to $O(\ell_p^2)$ and the number N of punctures is fixed.

The area eigenvalue equation for a particular eigenstate of the QIH in the configuration basis can be written as

$$\hat{A}|\{s_j\}\rangle = 8\pi\gamma\ell_p^2 \sum_j s_j \sqrt{j(j+1)}\ |\{s_j\}\rangle \tag{3.7}$$

Hence, the expectation value of the area operator for the QIH is given by

$$\langle\hat{A}\rangle = \langle\Psi_S|\hat{A}|\Psi_S\rangle \ = \ 8\pi\gamma\ell_p^2 \sum_{\{s_j\}} \omega[\{s_j\}] \sum_j s_j \sqrt{j(j+1)} \ = \ A_{cl} \pm O(\ell_p^2) \tag{3.8}$$

where A_{cl} is the area of the classical IH closely represented by the QIH. Scaling the equation by $8\pi\gamma\ell_p^2$ and using $\langle\hat{A}\rangle/8\pi\gamma\ell_p^2 \approx A_{cl}/8\pi\gamma\ell_p^2 = k/2$ [14, 36], we obtain

$$\sum_{\{s_j\}} \omega[\{s_j\}] \sum_j s_j \sqrt{j(j+1)} = \frac{k}{2} \tag{3.9}$$

In this process we can also avoid the involvement of the ambiguous parameter γ which will be ultimately fixed at the end by the usual argument of the validity of the BHAL.

Apart from this, the expectation value of the number of punctures for the QIH is given by

$$\langle\hat{N}\rangle = \langle\Psi_S|\hat{N}|\Psi_S\rangle = \sum_{\{s_j\}} \omega[\{s_j\}] \sum_j s_j \ = \ N \tag{3.10}$$

where $\hat{N}$ can be considered as the operator for the number of punctures for a QIH. One should note that unlike the case of area the expectation value of $\hat{N}$ is *exactly* equal to the total number of punctures N. Now, since we are doing equilibrium statistical mechanics (thus stability is assumed) of QIH with large area ($A_{cl} \gg \ell_p^2$) and large number of punctures ($N \gg 1$), for all practical purposes, we can neglect the fluctuations and also consider that the dominant contribution to the entropy comes from a most probable configuration, $\{s_j^\star\}$ i.e. $\omega[\{s_j^\star\}] \simeq 1$ [39]. Thus, every spin configuration $\{s_j\}$ must obey the following constraints

$$\mathcal{C}_1 : \sum_j s_j = N \tag{3.11a}$$

$$\mathcal{C}_2 : \sum_j s_j \sqrt{j(j+1)} = \frac{k}{2} \tag{3.11b}$$

of which $\{s_j^\star\}$ will be the most probable one. Hence, we define a microcanonical ensemble of QIHs by assigning fixed values of k and N respectively. The obvious next step is the computation of the microcanonical entropy of a QIH whose *macrostates* are characterized by k and N.

3.4 Microcanonical Entropy

Having defined the microcanonical ensemble appropriately, we shall now derive the microcanonical entropy of a QIH. The microcanonical entropy of a QIH for given values of k and N is written as

$$S_{MC} = \log \Omega(N, k) \simeq \log \Omega[\{s_j^\star\}] \tag{3.12}$$

where we have set the Boltzmann constant to unity. Variation of $\log \Omega[\{s_j\}]$ with respect to s_j, subject to the constraints $\mathcal{C}_1$ and $\mathcal{C}_2$, yields the distribution function for the most probable configuration $\{s_j^\star\}$ which maximizes the entropy of the QIH. In other words, $s_j^\star$ satisfies the variational equation written as

$$\delta \log \Omega[\{s_j\}] - \sigma \sum_j \delta s_j - \lambda \sum_j \delta s_j \sqrt{j(j+1)} = 0 \tag{3.13}$$

where δ represents variation with respect to s_j, σ and λ are the Lagrange multipliers for $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively. This yields the most probable distribution given by

$$s_j^\star = N \exp\left[-\lambda\sqrt{j(j+1)} - \sigma + \frac{\delta}{\delta s_j} \log g[\{s_j\}]\right] \tag{3.14}$$

To proceed further we calculate $g[\{s_j\}]$ explicitly using saddle point approximation in the limit $k, N \to \infty$, which is appropriate for large black holes. First of all, we rewrite eq.(3.5) replacing the summation over a by integration as

$$\begin{aligned} g[\{s_j\}] &\simeq \frac{2}{k+2}\int_1^{k+1} \sin^2 \frac{a\pi}{k+2} \prod_j \left\{\frac{\sin\frac{a\pi(2j+1)}{k+2}}{\sin\frac{a\pi}{k+2}}\right\}^{s_j} da \\ &= \frac{2}{\pi}\int_\epsilon^{\pi-\epsilon} \sin^2\theta \prod_j \left\{\frac{\sin(2j+1)\theta}{\sin\theta}\right\}^{s_j} d\theta \end{aligned} \tag{3.15}$$

where we have applied a change in the integration variable as $a\pi/(k+2) = \theta$ and for which the limits follow with $\epsilon = \pi/(k+2)$. Now, for $k \to \infty$, $\epsilon \to 0$. Hence, we can safely write

$$\begin{aligned} g[\{s_j\}] &\simeq \frac{2}{\pi}\int_0^{\pi} \sin^2\theta \prod_j \left\{\frac{\sin(2j+1)\theta}{\sin\theta}\right\}^{s_j} d\theta \\ &= \frac{1}{\pi}\int_0^\pi \exp\left[G(\theta,k)\right] d\theta - \frac{1}{\pi}\int_0^\pi \exp\left[\ln(\cos 2\theta) + G(\theta,k)\right] d\theta \end{aligned} \tag{3.16}$$

where $G(\theta,k) = \sum_j s_j \log\left\{\frac{\sin(2j+1)\theta}{\sin\theta}\right\}$. The above two integrations can be performed by the saddle point method. To begin with, it is straightforward to show that

$$\lim_{\theta_0\to 0} G'(\theta,k)|_{\theta_0} = 0 \tag{3.17a}$$

$$\lim_{\theta_0\to 0} \{-2\tan 2\theta + G'(\theta,k)\}|_{\theta_0} = 0 \tag{3.17b}$$

which implies that the saddle point $\theta_0 \simeq 0$ ($'$ denotes partial derivative with respect to θ). Now, Taylor expanding $G(\theta,k)$ about the saddle point θ_0 up to second order and applying the saddle point conditions from eqs.(3.17), we have

$$g[\{s_j\}] \simeq \frac{1}{\pi}\prod_j (2j+1)^{s_j}\left[\int_0^{\pi} e^{-\frac{1}{2}\alpha\xi^2}d\xi - \int_0^{\pi} e^{-\frac{1}{2}(4+\alpha)\xi^2}d\xi\right] \tag{3.18}$$

where we have used $\xi = \theta - \theta_0$ and the following limits

$$\lim_{\theta_0\to 0} \exp[G(\theta_0,k)] = \prod_j (2j+1)^{s_j}$$
$$\lim_{\theta_0\to 0} G''(\theta,k)|_{\theta_0} = -4\sum_j s_j\, j(j+1) \equiv -\alpha$$
$$\lim_{\theta_0\to 0} \sec^2 2\theta_0 = 1$$

Evaluation of the integral yields

$$g[\{s_j\}] \simeq \frac{1}{\sqrt{2\pi}}\prod_j (2j+1)^{s_j}\left(\sqrt{\frac{1}{\alpha}}\ \mathrm{Erf}\left[\pi\sqrt{\alpha/2}\right] - \sqrt{\frac{1}{4+\alpha}}\ \mathrm{Erf}\left[\pi\sqrt{(4+\alpha)/2}\right]\right) \tag{3.19}$$

The quantity α results from the second order approximation. Hence, while calculating α we can only use the results up to first order i.e. we use $s_j^\star$ in place of s_j whose expression will be given by

$$s_j^\star \simeq N(2j+1)\exp[-\lambda\sqrt{j(j+1)} - \sigma] \tag{3.20}$$

which follows from the fact that $g[\{s_j\}] \simeq \frac{2C}{\pi}\prod_j (2j+1)^{s_j}$ neglecting the second order corrections, C being some constant. Now, using eq.(3.20) in eq.(3.11a) and eq.(3.11b), one obtains

$$\exp[\sigma] = \sum_{j=1/2}^{k/2} (2j+1)\exp[-\lambda\sqrt{j(j+1)}] \tag{3.21a}$$

$$k/2 = N\sum_{j=1/2}^{k/2} \sqrt{j(j+1)}(2j+1)\exp[-\lambda\sqrt{j(j+1)} - \sigma] \tag{3.21b}$$

Using eq.(3.20), eq.(3.21a) and eq.(3.21b), it is straightforward to show that $\alpha \simeq 8N(d^2\sigma/d\lambda^2)$. Thus, $\alpha \to \infty$ for $N \to \infty$. Hence, we can approximately write $\mathrm{Erf}\left[\pi\sqrt{(4+\alpha)/2}\right] \approx$

$\text{Erf}\left[\pi\sqrt{\alpha/2}\right]$. Plotting the function $\text{Erf}\left[\pi\sqrt{\alpha/2}\right]$ with α one can see that the function attains a constant value for large α. Hence, we can take $\lim_{\alpha\to\infty}\text{Erf}\left[\pi\sqrt{(4+\alpha)/2}\right] \simeq \lim_{\alpha\to\infty}\text{Erf}\left[\pi\sqrt{\alpha/2}\right] \simeq K$, some constant. Therefore, from eq.(3.19) it follows that

$$\begin{aligned} g[\{s_j\}] &\simeq \frac{K}{\sqrt{2\pi}}\prod_j (2j+1)^{s_j}\left(\sqrt{\frac{1}{\alpha}} - \sqrt{\frac{1}{4+\alpha}}\right) \\ &\simeq K\sqrt{\frac{2}{\pi}}\prod_j (2j+1)^{s_j}\alpha^{-\frac{3}{2}} \end{aligned}$$

Therefore, we have

$$\frac{\delta}{\delta s_j}\log g[\{s_j\}] = \log(2j+1) - \frac{6}{\alpha}j(j+1)$$

Hence, considering variation of α resulting from the inclusion of the quadratic fluctuations, the distribution for the most probable configuration given by eq.(3.20) gets modified into

$$s_j^\star \simeq N(2j+1)\exp[-\lambda\sqrt{j(j+1)} - \sigma - \frac{6}{\alpha}j(j+1)] \tag{3.22}$$

Now, using the most probable distribution given by eq.(3.20) or eq.(3.22) (result will differ by a constant only) we calculate the microcanonical entropy of a QIH for given values of k and N using Stirling approximation and the result comes out to be

$$S_{MC} = \frac{\lambda k}{2} + N\sigma - \frac{3}{2}\log N - \frac{3}{2}\log(d^2\sigma/d\lambda^2) + \cdots \tag{3.23}$$

where λ and σ satisfy the two equations (3.21a) and (3.21b).

Using eq.(3.21a) and eq.(3.21b) and also finding $d\sigma/d\lambda$ from eq.(3.21a) one can immediately show that

$$k = -2N(d\sigma/d\lambda) \tag{3.24a}$$

or, using $k = A_{cl}/4\pi\gamma\ell_p^2$,

$$A_{cl} = -8\pi\gamma\ell_p^2 N(d\sigma/d\lambda) \tag{3.24b}$$

Eq.(3.24a) can be regarded as the 'equation of state' for the QIH at equilibrium, whereas, eq.(3.24b) has the significance lying in the fact that there is indeed a deep underlying relationship between the so called 'quantum hair' defined only in the quantum domain and the physically measurable quantity A_{cl} which is purely classical. Hence it is justified why N should play a very fundamental role in the thermodynamics of black holes if one tries to investigate by beginning from the underlying quantum theory provided by the QIH framework in LQG.

It may be noted that the concept of 'quantum hair' N is really nothing new, because it has been quite implicit in the large body of work on QIHs [13, 14, 17, 46, 41, 42, 44, 45, 50, 51, 52, 53]. A careful analysis of the derivation of the *microcanonical* entropy of QIH for given k and N reveals that $\gamma > 0.191$ for the *microcanonical* entropy to be given by

$$S_{MC} = \frac{A_{cl}}{4\ell_p^2} + N\sigma(\gamma) \tag{3.25}$$

The bound on γ follows from the fact that the term $N\sigma(\gamma)$ must be a negative definite quantity. In this work, our emphasis is to discuss the implications in the context of *microcanonical* entropy calculation *if* one accepts the idea and considers N to be an additional macroscopic parameter for a QIH, alongside k or A_{cl}.

3.4.1 The Lagrange Multipliers

In the limit $k \to \infty$, eq.(3.21a) and eq.(3.21b) can be approximated to be

$$e^{\sigma} = \frac{2}{\lambda^2}\left(1 + \frac{\sqrt{3}}{2}\lambda\right) e^{-\frac{\sqrt{3}}{2}\lambda} \tag{3.26}$$

$$\frac{k}{N} = 1 + \frac{2}{\lambda} + \frac{4}{\lambda(\sqrt{3}\lambda + 2)} \tag{3.27}$$

Since we are dealing with the *microcanonical* ensemble, k and N are the given quantities and the Lagrange multipliers λ and σ can be obtained as the solutions of the equations (3.26) and (3.27). It can be checked explicitly in the following way. Eq.(3.27) is actually a cubic equation in λ written as

$$\lambda\left[\sqrt{3}\,(k/N-1)\,\lambda^2+\left(k/N-1-\sqrt{3}\right)\lambda-8\right]=0 \tag{3.28}$$

Excluding the trivial root $\lambda = 0$ of the above equation for obvious reasons[2], the other two nontrivial roots of the above equation are given by

$$\lambda=\frac{1}{\sqrt{3}(k/N-1)}\left[\left(\sqrt{3}+1-k/N\right)\pm\sqrt{k^2/N^2+\left(6\sqrt{3}-2\right)k/N+\left(4-6\sqrt{3}\right)}\right] \tag{3.29}$$

of which we shall again exclude the one with the '$-$' sign because it will yield negative values of λ for all $k/N > 0$ and hence leading to negative values of BI parameter (it will be clear shortly). Hence, we shall consider only the one with the '+' sign as this will only give the positive values of λ for $k/N > 1$. To see this, one can plot[3] λ as a function of k/N considering the expression with the '+' sign. The resulting graph is shown in in FIG.(3.1). Using the desired solution of λ as a function of k/N in eq.(3.26) it is trivial to obtain σ as a function of k/N. Hence, in the *microcanonical* ensemble, $\lambda \equiv \lambda(k/N)$ and $\sigma \equiv \sigma(k/N)$ are functions of k and N. Eq.(3.27) can be considered to be the equation of state relating λ, k and N only at the equilibrium and hence can be attained only after finding the most probable distribution giving the equilibrium configuration. It should be noted that *there is no freedom to choose λ in the microcanonical ensemble.* Now, the *microcanonical* entropy is given by $S_{MC} = \log \sum_{\{s_j\}} \Omega[\{s_j\}]$. Taking into account that the dominant contribution comes from the most probable configuration $\{s_j^\star\}$ which maximizes the entropy and taking the limit $N, s_j^\star \to \infty$ so as to apply the Stirling approximation, one can calculate the *microcanonical* entropy as

$$\begin{aligned} S_{MC} &\simeq \lim_{N,s_j^\star\to\infty} \log\Omega\left[\left\{s_j^\star\right\}\right] \\ &= \lambda(k/N)k/2+\sigma(k/N)N \end{aligned} \tag{3.30}$$

where one has to use also the eq.(3.11a) and eq.(3.11b). Thus, once we define the *microcanonical* ensemble of QIHs by giving k and N, the *microcanonical* entropy is completely known and given by eq.(3.30)

[2] $\lambda = 0$ leads to $\sigma \to \infty$ which will yield infinite entropy.
[3] All the graph plots shown in this work are performed with MATHEMATICA.

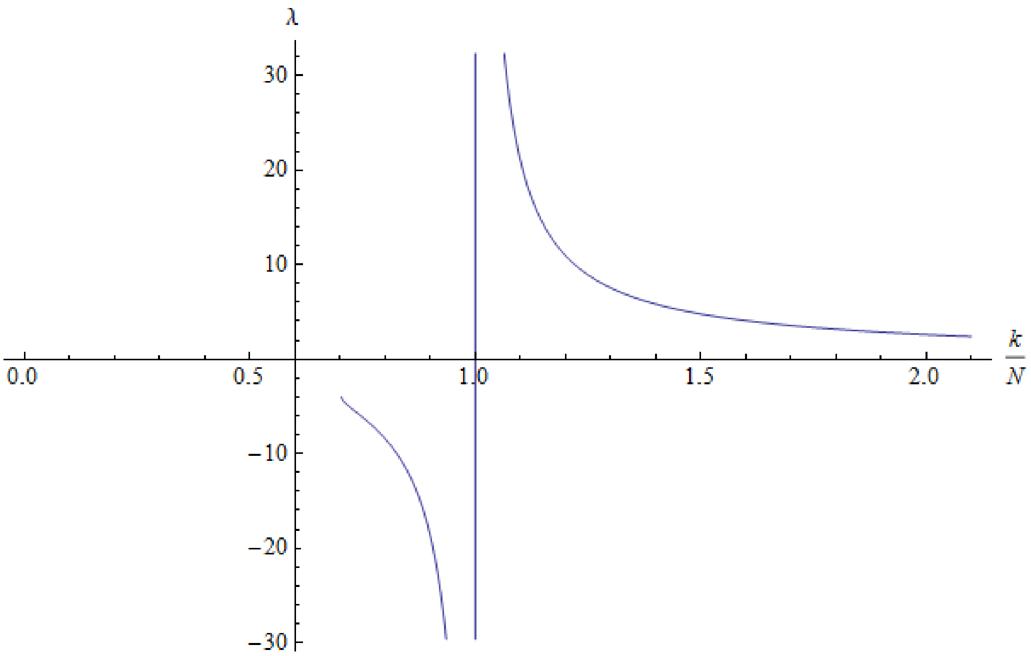

Figure 3.1: The plot shows the variation of λ with k/N for the solution of λ with the '+' sign in eq.(3.29). It is quite clear that the value of λ has a discontinuity at $k/N = 1$ and has positive values only for $k/N > 1$.

Digression : The above scenario is analogous to the case of an ideal gas whose equation of state is given by $E = \frac{3}{2}NT$(considering Boltzmann constant to be unity and the meaning of E, N and T are obvious). Since the *microcanonical* ensemble is defined by given values of E and N, T is a derived quantity and should be viewed as $T \equiv T(E, N)$. Thus the *microcanonical* entropy of an ideal gas for given E and N must be written as

$$S_{MC} = \beta(E, N)E + N\alpha(E, N) \tag{3.31}$$

where $\beta = 1/T$ and α are the Lagrange multipliers solved for given E and N [39]. It is only in the *canonical* ensemble one can say that T can be chosen because the ensemble is defined by specifying the equilibrium temperature (T) and the total number of particles(N). In this case T and N are the given quantities and $E \equiv E(T, N)$ becomes a derived quantity i.e. we *calculate* the mean energy of the system at a desired temperature and for a desired number of particles[39]. The *canonical* entropy of an ideal gas for given T(or equivalently β) and N must be written as

$$S_C = \beta E(\beta, N) + N\alpha(\beta, N) \tag{3.32}$$

It is a very crucial point to be noted that even though the structure of the thermodynamic equations, such as the form of entropy in eq.(3.31) and eq.(3.32), the equation of state, etc. are *independent* of the ensemble we use, the point which is often overlooked is that the roles of the parameters (E, N, T) indeed change with the ensemble as discussed above.

Now, we can write eq.(3.30), by replacing k with $A_{cl}/4\pi\gamma\ell_p^2$, in the following form

$$S_{MC} = \frac{\lambda(k/N)}{2\pi\gamma}\frac{A_{cl}}{4\ell_p^2} + \sigma(k/N)N \tag{3.33}$$

Our goal is to obtain the form of the *microcanonical* entropy given by the expression (3.25) from the expression (3.33). Usually, in calculation of entropy only for fixed k or A_{cl}(e.g. see [46]) (i.e. the scenario which appears by putting $\sigma = 0$ in the present case), a fixed numerical value of γ is determined by demanding the BHAL. In that case λ comes out to be a number and γ is chosen to get the desired result, which is consistent with the fact that γ has to have a specific numerical value so as to have an unambiguous LQG theory. But, in the present scenario with an additional macroscopic parameter N, λ is a function of k/N. Hence, there is no other way than to accept that $\gamma = \lambda(k/N)/2\pi$, a function of k and N, so as to obtain the *microcanonical* entropy of the form given by the expression (3.25).

This particular point may be further clarified as follows. Let us define the *microcanonical* ensemble by assigning values $k = k_1$ and $N = N_1$, for which we have $\lambda = \lambda(k_1/N_1) \equiv \lambda_1$. Now, we claim that $\gamma = \lambda_1/2\pi = \gamma_1$(say) so that we can obtain the first term of (3.33) to be given by $A_{cl1}/4\ell_p^2$, where $A_{cl1} = 4\pi\gamma_1 k_1 \ell_p^2$. Similarly, one can make another choice $k = k_2$ and $N = N_2$, such that $(k_1/N_1) \neq (k_2/N_2)$, for which there exists a corresponding γ_2 and A_{cl2} so as to obtain the first term of (3.25) to be $A_{cl2}/4\ell_p^2$. To be precise for every such choice of k/N there exists an unique value of γ, given by $\lambda(k/N)/2\pi$, which results in the *microcanonical* entropy given by (3.25).

Hence, for the *microcanonical* entropy to be given by (3.25), we must have $\gamma = \lambda(k/N)/2\pi$ and there is no way one can obtain a specific universal value of γ and it is indeed a *function* of

k/N in this scenario where N is considered to be a macroscopic parameter for a QIH alongside k [4]. By now we can conclude that for each value of k/N, there exists a unique value of γ for which the *microcanonical* entropy takes the form of expression (3.25) reported in [15]. The allowed values of γ is restricted by the bound : $\gamma > 0.191$. This bound obviously needs an explanation which is the subject matter of the next section.

3.5 Bound on γ

In this section we shall argue from two different viewpoints that the term $N\sigma(\gamma)$ should be a negative definite quantity from which the bound on γ will follow. First of all we shall explain this by looking at the kinematical Hilbert space structure of the QIH, which is usually studied for calculating black hole entropy in LQG framework. The second argument originates from the comparison of the entropy measured by a local stationary (with respect to the horizon) observer with the one measured by the observer at asymptotic infinity for the same QIH. The section ends with a quantitative estimate of the bound on γ.

3.5.1 Constrained kinematical Hilbert space

Imposing constraints on a system implies availability of more information about that system. Since, entropy is a measure of unavailability of information about a system[59], thus imposition of more constraints will result in decrement of the entropy. This is what happens also in the case of black hole entropy which, in the LQG framework, is calculated by taking the logarithm of the dimensionality of the associated kinematical Hilbert space. As far as the full kinematical Hilbert space of a QIH is concerned[13, 14, 40], it is interesting to note that there is actually a *sum over all possible sets of punctures* which encodes the information that the full Hilbert space of the QIH takes into account all possible values of N compatible with a given k :

[4]Here one may wonder if this problematic fixation of γ is a result of considering k to be a macroscopic parameter preferred to A_{cl}. But one can remain assured that this is not the actual reason and to get convinced (s)he may check by repeating this whole calculation by fixing A_{cl} instead of k, alongside N to define the *microcanonical* ensemble; the results and conclusions will still remain unaltered.

$$\mathcal{H}^k_{QIH} = \bigoplus_{\{\mathcal{P}\}} \text{Inv}\left(\bigotimes_{l=1}^{N} \mathcal{H}_{j_l}\right) \tag{3.34}$$

where $\{\mathcal{P}\} \equiv N; \frac{1}{2} \leq j_l \leq \frac{k}{2} \forall l \in [1, N] \ni \sum_{l=1}^{N} \sqrt{j_l(j_l+1)} = \frac{k}{2} \pm \mathscr{O}(\frac{1}{8\pi\gamma})$ and 'Inv' stands for the gauge invariance. Prior to the advent of the concept of quantum hair in [15], the dimensionality of this full Hilbert space was considered which gave the total number of horizon microstates for a given k and the entropy used to come out to be the BHAL for a unique value of γ[13, 14, 44, 45]. Now, *if* one considers N as an independent macroscopic parameter other than A_{cl} or k and N is specified to define the *microcanonical* ensemble, then the resulting *microcanonical* entropy will be that of a fixed-N subspace of the full kinematical Hilbert space. Since the dimensionality of this subspace is bound to be less than that of the full kinematical Hilbert space, the resulting entropy must be less than the BHAL i.e. the term $N\sigma(\gamma)$ should only appear as a negative term so as to lower the entropy below BHAL.

3.5.2 Local vs Asymptotic Views

As has been clearly explained in [15] that, the proposal of the quantum hair N has been given from the local stationary observer perspective i.e. an observer at a proper distance of few Planck lengths from the horizon and stationary with respect to the horizon, will realize the existence of the quantum hair N. It implies that *only* the local observer can treat the total number of punctures N as a macroscopic thermodynamic parameter, but the asymptotic observer does not realize the existence of this quantum hair N. The fluctuations of N appear to the asymptotic observer as small quantum fluctuations, which has no effect on the thermodynamics at asymptotic infinity, as opposed to the local observer who can treat N as a macroscopic thermodynamic parameter because the fluctuations of N indeed appear to the local observer as particle like excitations on the horizon. This is why the chemical potential conjugate to N which exists for the local observer, must vanish at asymptotic infinity [15]. For the *same* system i.e. the QIH, there are two observers and hence two different observations. The local observer gives us a fine grained view whereas the asymptotic observer gives us a coarse grained view of the *same* system. The local observer has an access to larger amount of information than the asymptotic observer has about the *same* system, the QIH and that is why N can be treated as a macroscopic parameter only by the local observer and not by the asymptotic observer. Thus the entropy of the QIH measured by the local observer must be less than the entropy of the same QIH measured by the asymptotic observer. Hence, the $N\sigma(\gamma)$ term, which is seen by the local observer only, must be negative definite i.e. $\sigma(\gamma) < 0$.

Remarks : One should note that this above argument stands only because we know or accept that the asymptotic observer must observe the BHAL. Due to the presence of this 'reference' measurement we could argue that the entropy of the QIH measured by the local observer must be less than this 'reference' BHAL. In general gas thermodynamics no such difference in observations is made and the $N\sigma$ like term that appears there can be anything : positive, negative or zero. This is a crucial point to be noted.

3.5.3 Covariant Entropy Bound

Finally, we may mention that considerations related to the Entropy Bound [18], as 'covariant'-ized in [19] and sharpened within LQG in [20], places our qualitative arguments in favor of $\sigma(\gamma) < 0$ on a stronger footing. As it has been pointed out in course of the presentation of our arguments, the idea of quantum hair (N) is an observer dependent notion[15] i.e. N can be considered as a macroscopic thermodynamic variable only by a local observer very close to the horizon and there is no such notion of quantum hair for an asymptotic observer. Now, it is already known in the literature that there is a covariant (observer independent) entropy bound associated with a closed two-surface[18, 19] which has been proved quantum geometrically in [20] leading to a tighter bound for a QIH. For a closed spatial two-surface of area A_{cl} the maximum associated entropy can only be $A_{cl}/4\ell_p^2$ (ignoring the logarithmic correction[20]). Since the horizon entropy is nothing but the entropy associated with the closed two-surface cross-sections, it is evident that whatever observer dependent entropy one can calculate, cannot be greater than $A_{cl}/4\ell_p^2$. Hence, the observer dependent notion of quantum hair N can only give rise to a negative contribution to the entropy and thus there is no other choice than to impose the condition $\sigma(\gamma) < 0$.

3.5.4 An estimate of the bound on γ

Following the above qualitative arguments in favour of the boundedness of γ resulting from the bound $\sigma(\gamma) < 0$, it is the turn to show off a quantitative analysis on behalf of the claim. From eq.(3.26) it is quite easy to get an estimate of the bound on γ. If one plots e^{σ} as a function of λ, it is seen that the value of e^{σ} falls below 1 i.e. σ becomes negative for $\lambda > 1.200$. Now, following the previous arguments regarding the fixation of γ, one can obtain the bound on γ by dividing the allowed range of λ by 2π, which results in $\gamma > 0.191$.

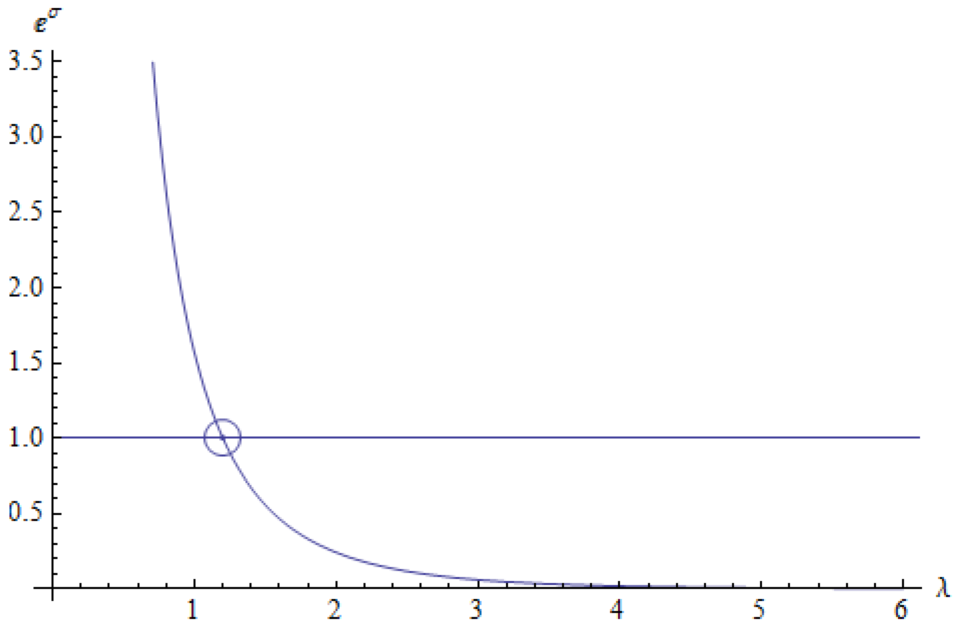

Figure 3.2: In the plot of e^{σ} as a function of λ, the coordinates of the marked point in the graph are $(1.200, 1.000)$. Therefore, one can conclude that $e^{\sigma} < 1 \Rightarrow \sigma < 0$ for $\lambda > 1.200$. Since $\gamma = \lambda(k/N)/2\pi$, we obtain the required bound on the BI parameter i.e. $\gamma > 0.191$.

3.5.5 Commentary on the boundedness of γ

The aim of all these analyses and arguments is to assert that *if* one accepts N as a 'quantum hair' of QIH and define the microcanonical ensemble for given k or A_{cl} and N, then to obtain the *microcanonical* entropy of a QIH given by eq.(3.25), we must have $\gamma > 0.191$. This is not quite in agreement with [15], according to which γ is a *free* parameter i.e. it can take any value. Moreover, there is no way one can obtain a unique value of γ in this particular scenario.

The BI parameter, γ, being of utmost importance in LQG, any further work following the idea of 'quantum hair' N must be performed with careful attention to the bound on γ which has been somehow overlooked in [15]. It is to be noted that the bound on γ may be calculated more precisely by numerical methods, but the motto of this work is to catch the essence of the boundedness of γ, which is the most essential physics content in the present context and it is not the mathematical accuracy of the numbers that we are after. It should be reminded that the pivotal point of this work consists of the arguments in favor of the condition $N\sigma(\gamma) < 0$ which results from viewing the problem from a very different perspective. In general, while studying the thermodynamics of a system, we do not talk precisely about the observer and all the measurements made are considered to be unique. But the topic of black hole thermodynamics which is related to general relativity, the observer must play a crucial role in the measurements. Since we have accepted by heart and soul that the observer at asymptotic infinity will measure the entropy to be nothing other than the BHAL, then, whatever observer and corresponding measurement we consider, there has to be a consistency with the known measured value at asymptotic infinity. Our arguments simply stand on this ground. If there *were* no BHAL, then we could not have presented any of our arguments in favor of $N\sigma(\gamma) < 0$. Then, it could have had arbitrary sign and γ would have been a free parameter.

In the present context the following few words are worth mentioning. The idea of a complex BI parameter arising from a formulation of general relativity based on the self-dual Sen-Ashtekar connection is an intriguing possibility. However, such a formulation necessarily deals with a complex configuration space which leads to mathematical difficulties when quantization is attempted [47]. As far as the black hole entropy computation is concerned, it may be noted that the comparison of the QIH entropy, derived from a purely quantum statistical calculation, with the semiclassical BHAL may be fraught with a slight danger since there is as yet no complete semiclassical formulation derived from the coherent states of LQG. There is indeed the need for an appropriate effective action of the theory which may result in a renormalized BI parameter. The situation is reminiscent of the θ-parameter in QCD, because of the topological character of the BI parameter [60]. In QCD, too, the comparison of phenomenological results based on θ-vacua with observations *assumes* an effective 'renormalized' θ parameter.

3.6 Freeing γ

However, it is indeed possible to render the BI parameter to be *free* even in this setup where N is considered as an independent macroscopic parameter alongside k, if one naively demands that the entropy be given by the BHAL only. It is only a question of fixation of γ. Using eq.(3.24b) and the relation $k = A_{cl}/4\pi\gamma\ell_p^2$ the microcanonical entropy can be expressed as

$$S_{MC} = \left[\frac{f(\lambda)}{2\pi\gamma}\right]\frac{A_{cl}}{4\ell_p^2} - \frac{3}{2}\log\frac{A_{cl}}{4\ell_p^2} + \cdots \tag{3.35}$$

where $f(\lambda) = \lambda - \left(\sigma/\frac{d\sigma}{d\lambda}\right)$. It is quite clear from the above expression for the *microcanonical* entropy that somehow we have to set the factor $f(\lambda)/2\pi\gamma$ to be unity to obtain the Bekenstein-Hawking area law(BHAL). Let us see how we can do that.

As mentioned earlier, since we are dealing with the *microcanonical* ensemble, k and N are the given quantities which determine the Lagrange multipliers λ and σ from the solutions of the above equations (3.26) and (3.27). Hence, in the *microcanonical* ensemble, $\lambda \equiv \lambda(k/N)$ and $\sigma \equiv \sigma(k/N)$ are functions of k and N. It should be noted that *there is no freedom to choose λ (hence $f(\lambda)$) or σ in the microcanonical ensemble.* Strictly speaking, we should write the function f as $f(k/N)$ and corresponding to every value of k/N there exists a unique value of f. To retrieve the BHAL from eq.(3.35) we can choose $\gamma = f(k/N)/2\pi$, which results in

$$S_{MC}(A_{cl}) = \frac{A_{cl}}{4\ell_p^2} - \frac{3}{2}\log\frac{A_{cl}}{4\ell_p^2} + \cdots \tag{3.36}$$

Hence, the range of allowed values of γ will be dictated by the range of f. Now, one can find $f(\lambda)$ to be given by the following expression

$$f(\lambda) = \lambda\left[1 + \frac{2(2+\sqrt{3}\lambda)\{\log 2 - 2\log\lambda + \log(1+\sqrt{3}\lambda/2) - \sqrt{3}\lambda/2\}}{\sqrt{3}\lambda^2 + 2(\sqrt{3}+1)\lambda + 8}\right] \tag{3.37}$$

From eq.(3.27) it is evident that for any value of λ between 0 and ∞, k/N remains positive and from the above expression it is evident that for $0 < \lambda < \infty$, we have $0 < f(\lambda) < \infty$. Hence, without finding the explicit form of $f(k/N)$ one can remain assured that f takes values from 0 to ∞ for positive values of k/N. It follows that γ can take value from 0 to ∞ for the leading term of the *microcanonical* entropy to follow the BHAL.

3.7 Discussion

As mentioned in the Introduction, the LQG formulation of a QIH in terms of a CS theory coupled to spins directly involves the use of the integer parameters ('quantum hairs') k and N. Strictly speaking, if the classical area A_{cl} is taken to be a hair, the BI parameter γ (the coefficient of a topological contribution to the classical action from the Nieh-Yan invariant [49]) is considered as an independent coupling parameter, then the only new parameter that has appeared in the quantum theory is the number of punctures N. Equivalently, one can, as we have in this work, take k and N to be the parameters characterizing the quantum theory. It is significant that the requirement that the microcanonical entropy of a QIH yields the area law for large k , N does not only allow the BI parameter γ to take any positive value on the real line,, it also yields the subleading logarithmic correction derived in earlier literature with the universal coefficient $-3/2$. Thus, the complete characterization of the *macrostate* of a QIH is given by two independent parameters, namely, k and N. The limit of large k and N can thus be taken to be the approximately the semiclassical domain, since it is in this limit that the computation reliably yields an answer for the microcanonical entropy which can be explicitly expressed entirely in terms of the BHAL. Although the 'classical limit' of bulk LQG has certain ambiguities in extraction of a classical metric from expectation values of geometrical observables within coherent states, as far as classical horizons are concerned, such an ambiguity can perhaps be avoided by working with the idea of an effective QIH in this limit and comparing with semiclassical behaviour. Of course, our approach does not take into account the entanglement of quantum matter states in the bulk and the boundary, not entanglement between quantum matter and quantum spacetime states, but accounts only for entanglement between bulk and boundary quantum spacetime states. The inclusion of quantum matter together with quantum geometry thus remains one of the key points of our research agenda for the immediate future.

The departure from recent literature is the direct link established in this work of these results and interpretation within the CS formulation of a QIH. The point here is that the computation of the microcanonical entropy of a QIH is *not* merely a combinatoric exercise in statistical mechanics of an ideal gas of punctures with some restrictions gleaned out of LQG as in some part of the recent literature [15, 50, 51, 52, 53, 44, 45], with at best a weak link to the complete formulation as a CS theory that has been available for years. Here we have shown that the original formulation yields an understanding that is complete as a quantum theory.

An extension of the foregoing analysis to Trapping horizons and Entanglement entropy of matter and radiation fields in their vicinity, would amount to truly new physics of radiant quantum horizons beyond general relativity. This has within it the potential to surpass, because of its firmer quantum geometric underpinning, recently proposed speculative ideas of a somewhat ad hoc nature based on semiclassical analysis [54, 68, 56], about how classical horizon geometry must change (become a 'fuzzball' or an 'energetic curtain' or a 'firewall') so as to allow the existence of well-defined scattering amplitudes for quantum matter fields. Whether or not such a structure emerges from LQG in an appropriate limit is not known at the moment, since there is still no complete quantum geometric analysis of Hawking radiation and its various conundra from an LQG standpoint. The problem at hand involves a quantization of the kinematical phase space of a Trapping/Dynamical horizon along the lines of [13, 14] which may not be technically so simple as a QIH because of the transient behaviour inherent in the geometry. However, perhaps a perturbation of the CS description by a set of appropriately chosen matter field operators might serve as a first approximation to the problem. The strength of the LQG approach lies in the transparent manner in which the CS symplectic structure emerges from the boundary conditions in the incipient formulation [14], without having to rely on conjectured results. The relation of the CS Hilbert space to the conformal blocks of the WZW model on the QIH is also not a matter of conjecture for large black holes. Thus, the LQG analysis of a QIH geometry throws up holographic structures as *emergent*, without any prior notion that they have to be there. One expects that generalizations to Trapping horizons to centre around this theme so as to reap the benefits of the 4 dimensional gravity - 2 dimensional conformal field theory relation found in ref. [16]. Note however that the thermal stability of radiant trapping horizons which approach an equilibrium QIH has been discussed within the LQG framework yielding a criterion of stability involving the equilibrium mass and the microcanonical entropy of the QIH in ref. [57, 12, 3].

Chapter 4

Thermal Stability Analysis of Charged, Non-rotating Black Holes

4.1 Introduction

The theory of QIH provides a self-contained platform for the application of the statistical mechanical techniques to understand the microscopic origin of the entropy of black hole horizons, as we have seen in the previous chapter. The physical essence of the microcanonical ensemble analysis is that the fluctuations of the macroscopic variables, which are regarded as thermal, are completely disallowed and what we deal with are purely quantum mechanical fluctuations of the system. Now, we shall pass on to the grand canonical ensemble scenario, to study the effects of thermal fluctuations on the thermal stability of black hole horizons in a situation where the black hole is allowed to interact with a thermal environment.

4.2 Thermal Holography

Quantum black holes not isolated from an ambient thermal reservoir have been considered in the past [62], [64, 65, 57], [12]. In this approach one uses certain key results of LQG like the discrete spectrum of the area operator [76, 77] and the central assumption that the thermal equilibrium configuration is indeed an IH whose microcanonical entropy, including quantum spacetime fluctuations have already been computed via LQG. The idea here has been the study of the interplay of thermal and quantum fluctuations, and a criterion for thermal stability of such horizons has been obtained [57, 12, 72], using a 'thermal holographic' description involving a canonical ensemble and incorporating Gaussian thermal fluctuations. The generalization to horizons carrying charge has also been attempted, using a grand canonical ensemble, even though a somewhat ad hoc mass spectrum has been assumed [65].

Here, we attempt to re-derive a thermal stability criterion for charged quantum horizons, *without* any ad hoc assumptions on the mass spectrum. With the benefit of hindsight, arguments which place the earlier formulation on a more solid footing are presented, together with novel aspects which enable us to sidestep earlier restrictions. A comparison with semiclassical thermal stability analyses of black holes [74] is made wherever possible. The range of validity of the saddle point approximation around the equilibrium configuration is examined to ensure the self-consistency of the Gaussian approximation.

4.2.1 Horizon Energy

For a consistent Hamiltonian evolution for spacetimes admitting internal boundaries (isolated horizons, representing black holes at equilibrium) there must be a first law associated with each internal boundary(b), assumed to be a null hypersurface with the properties of a 'one-way membrane' [9, 10] given by

$$\delta E_b^t = \frac{\kappa^t}{8\pi}\delta A_b + \Phi^t \delta Q_b \tag{4.1}$$

where E_b^t is the classical energy function associated with the horizon, κ^t and Φ^t are the surface gravity and the electric potential respectively of the horizon, Q_b is the horizon charge. All the quantities are defined for a particular choice of time evolution vector field t^μ. The family of time evolution vector fields $[t^\mu]$ satisfying such first laws on the horizon are the permissible time evolution vector fields. These evolution vector fields also need to satisfy other boundary conditions. Each of these time evolution vector fields associates a classical energy function with the horizon which is a function of area and charge for Einstein-Maxwell theory. In arbitrary non-stationary cases (radiation may be present arbitrarily close to the horizon) for a particular time evolution vector field (t), the Hamiltonian formulation yields

$$H^t = E^t_{ADM} - M^t_b \tag{4.2}$$

where H^t = Hamiltonian associated with the spacetime region in between the black hole boundary(b) and the boundaries at infinity, M^t_b = the mass associated with the horizon (b), E^t_{ADM} = the usual ADM energy associated with the spatial boundary at infinity for a permissible vector field t^μ. H^t is the Hamiltonian of the covariant phase space, which is the space of various class of solutions of the Einstein equations admitting internal boundaries. For stationary spacetimes the global timelike Killing field (ξ^μ) is the time evolution vector field. On physical ground one can say that there is nothing between the internal boundary and the boundary at infinity for stationary spacetimes, hence $H^\xi = 0$. On mathematical ground one can argue that in the Hamiltonian framework, for the stationary black hole solutions, the total Hamiltonian function H^ξ(which generates evolution along ξ^μ), must vanish as a first class constraint on the phase space [1, 10]. This gives $M^\xi_b = E^\xi_{ADM}$. This is exactly what has been proved in another manner in the literature : that in stationary black hole spacetimes the ADM mass equals the energy of the black hole. Hence it is logical to identify E^ξ_b with the horizon mass M_b in the stationary case. The difference for an arbitrary non-stationary case is that $H^t \neq 0$. Thus it can be called as the mass associated with the Isolated Horizon in an active sense, that can change from one dynamic equilibrium situation to another satisfying the first law. Here, one should be careful that this mass associated with the isolated horizon is completely physical and is not to be confused with the Hamiltonian of the $SU(2)$ Chern-Simons theory on the IH. The Hamiltonian for Chern-Simons theory vanishes identically, since the theory is topological and insensitive to arbitrary metric deformations.

Clearly, the horizon mass is *not* affected by boundary conditions at asymptopia. It is defined *locally* on the horizon without referring to the asymptotic structure at all. The asymptotic conditions only modify the energy associated with the boundary at infinity and the bulk equation of motion (Einstein's equations) [10, 71]. The Hamiltonian framework discussed above is equally applicable for asymptotically flat and AdS spacetimes.

4.2.2 Quantum Geometry

For a classical spacetime with boundary, boundary conditions determine the boundary degrees of freedom and their dynamics. For a quantum spacetime, on the other hand, fluctuations of the boundary degrees of freedom have a 'life' of their own (see for instance ref.[13, 14]). Consequently, the Hilbert space of a quantum spacetime with boundary has the tensor product structure $\mathcal{H} = \mathcal{H}_v \otimes \mathcal{H}_b$, with the subscript v (b) denoting the bulk (boundary) component. Thus, any generic state in quantum geometry, $|\Psi\rangle$, admits the expansion

$$|\Psi\rangle = \sum_{v,b} C_{vb}\,|\psi_v\rangle \otimes |\chi_b\rangle\ . \tag{4.3}$$

In presence of electromagnetic fields, one can consider $|\psi_v\rangle$ (resp. $|\chi_b\rangle$) to be the composite quantum gravity + quantum electrodynamics bulk (resp. boundary) state. The bulk states are annihilated by the *full* bulk Hamiltonian : $\hat{H}_v|\psi_v\rangle \equiv [\hat{H}_{g,v} + \hat{H}_{e,v}]|\psi_v\rangle = 0$; this is the quantum version of the classical Hamiltonian constraint [77]. The total Hamiltonian operator acting on the generic state $|\Psi\rangle$ has the form

$$\hat{H}_T|\Psi\rangle = (\hat{H}_v \otimes I_b + I_v \otimes \hat{H}_b)|\Psi\rangle \tag{4.4}$$

where, $I_v(I_b)$ corresponds to the identity operator on $\mathcal{H}_v(\mathcal{H}_b)$.

While defining the grand canonical partition function, the charge operator ($\hat{Q}$) for the black hole is also needed. It can be written in a similar fashion like the Hamiltonian as

$$\hat{Q}|\Psi\rangle = (\hat{Q}_v \otimes \hat{I}_b + \hat{I}_v \otimes \hat{Q}_b)|\Psi\rangle \tag{4.5}$$

where $\hat{Q}_v$ and $\hat{Q}_b$ are corresponding charge operators for the bulk states $|\psi_v\rangle$ and the boundary states $|\chi_b\rangle$, respectively. In the classical theory the charge of a black hole is defined on the horizon i.e the internal boundary of the four dimensional spacetime (e.g. one can see how charge can be properly defined for spacetimes admitting internal boundaries in Einstein-Maxwell or Einstein-Yang-Mills theories in [9]). There is *no* charge associated with the bulk black hole spacetime, i.e. $Q_v \approx 0$, which is basically the Gauss law constraint for electrodynamics. Hence, its quantum version is *assumed* to be

$$\hat{Q}_v|\psi_v\rangle = 0 \tag{4.6}$$

Combining the quantum constraints on the Hamiltonian and charge operators we can define a new quantum constraint as

$$\hat{H}'_v|\psi_v\rangle = 0 \tag{4.7}$$

where $\hat{H}'_v \equiv \hat{H}_{Tv} - \Phi\hat{Q}_v$ and Φ may be any function. But in our case it is a physically significant quantity which will be defined in the next paragraph. The implications of these quantum constraints will be seen during the construction of the grand canonical partition function.

4.2.3 The Partition Function

Let us consider a grand canonical ensemble of massive charged black holes immersed in a heat bath at some finite temperature with which it can exchange energy and charge as well. We construct a partition function for the thermodynamic system. Using the usual definition of the grand canonical partition function we write

$$Z_G \;=\; Tr\;\exp -\beta\hat{H}_T + \beta\Phi\hat{Q} \tag{4.8}$$

where the trace is taken over all states. Φ is the electrostatic potential and $\hat{Q}$ is the charge operator for the black hole. To write it in the explicit form first we write down a general quantum state of the black hole as follows

$$|\Psi\rangle \;=\; \sum_{v,b} c_{vb}\;|\psi_v\rangle \otimes |\chi_b\rangle \tag{4.9}$$

Now, we can write the partition function as

$$
\begin{aligned}
Z_G &= \sum_{v,b} |c_{vb}|^2 \langle\chi_b| \otimes \langle\psi_v| \exp -\beta\hat{H}_T + \beta\Phi\hat{Q}|\psi_v\rangle \otimes |\chi_b\rangle \\
&= \sum_{v,b} |c_{vb}|^2 \langle\chi_b| \otimes \langle\psi_v| \exp -\beta\hat{H}'|\psi_v\rangle \otimes |\chi_b\rangle
\end{aligned} \tag{4.10}
$$

where, $\hat{H}' = \hat{H}_T - \Phi\hat{Q}$. Writing the new operator H' as $(\hat{H}'_v \otimes \hat{I}_b + \hat{I}_v \otimes \hat{H}'_b)$ and using $\hat{H}'_v|\psi_v\rangle = 0$, the partition function comes out to be equal to the boundary partition function only i.e. $Z_G = Z_{Gb}$, where Z_{Gb} is the boundary partition function for the charged isolated horizon, given by

$$
Z_{Gb} = Tr_b \exp -\beta(\hat{H}_b - \Phi\hat{Q}_b)
$$

where it is assumed that the boundary states can be normalized through the squared norm $\sum_v |c_{vb}|^2 \langle\psi_v|\psi_v\rangle = |C_b|^2$. This is analogous to the canonical ensemble scenario described in [12].

Now, the spectrum of the boundary Hamiltonian operator is still unknown in LQG. So we *assume* that the spectrum of the boundary Hamiltonian operator is a function of the discrete area spectrum and the charge spectrum associated with the horizon, respectively[1]. The charge spectrum is equispaced on general physics grounds of charge quantization.

The area spectrum, in LQG, can be *approximately* taken to be equispaced for large area black holes due to the following reason. For large area black holes, we have already seen that the major contribution to the entropy comes from the lowermost spins. Hence, only spin 1/2 contribution for *all punctures* is taken into account which yields $A \sim N$ for a total of N , $N \gg 1$ punctures on the horizon. This leads to the equispaced area spectrum as an *approximation*. Of course the higher spins contribute, but their contribution is exponentially suppressed. A more rigorous calculation accounting for the contributions from higher spins may correct the final results, but only in a minor way. Hence, the inclusion of the higher spins would be more of

[1] Actually this second assumption follows from the discussion in Subsection(4.2.1) [9, 10] for spacetimes admitting weakly isolated horizons where there exists a mass function determined by the area and charge associated with the horizon. This is an extension of that assumption to the quantum domain.

technical importance rather than physical.

In a basis in which both area and charge operators are diagonal, the partition function can be written as

$$Z_G = \sum_{k,l} g(k,l) \exp -\beta \left[E(A_k, Q_l) - \Phi Q_l \right] \tag{4.11}$$

where $g(k,l)$ is the degeneracy corresponding to the area eigenvalue A_k and the charge eigenvalue Q_l. k, l are the area and charge quantum numbers respectively. As we are interested in regime of the large area and charge eigenvalues ($k \gg 1, l \gg 1$), Poisson resummation formula is applied [65] to approximate the summation to an integration given by

$$Z_G = \int dx\ dy\ \exp -\beta \left\{ E\left[A(x), Q(y) \right] - \Phi Q(y) \right\}\ g\left[A(x), Q(y) \right] \tag{4.12}$$

where x and y are area and charge quantum numbers in the continuum limit of k and l respectively. Since $A = A(x)$ and $Q = Q(y)$, we can write $dx = \frac{dA}{A_x}$ and $dy = \frac{dQ}{Q_y}$ to write the partition function in terms of area and charge as free variables as follows

$$Z_G \approx \int dA\ dQ\ e^{S(A) - \beta E(A,Q) + \beta \Phi Q} \tag{4.13}$$

where $S(A)$, being the microcanonical entropy, is a function of horizon area alone, as has been established within LQG [13, 14, 17]. Here, we have dropped out the irrelevant scaling constants and the suffix to the area variable.

4.3 Stability Against Gaussian Thermal Fluctuations

4.3.1 Saddle Point Approximation(S.P.A.)

Having a well defined partition function, we investigate its finiteness under Gaussian thermal fluctuations about stable equilibrium configurations of the black hole given by the saddle points $\{\bar{A}, \bar{Q}\}$. Taylor expanding $(S(A) - \beta E(A,Q) + \beta \Phi Q)$ about a saddle point $(\bar{A}, \bar{Q})$ and applying the saddle point conditions one can rewrite the partition function as

$$\begin{aligned} Z_G &= e^{\left[S(\bar{A})-\beta M(\bar{A},\bar{Q})+\beta\Phi\bar{Q}\right]} \\ &\times \int e^{-\frac{1}{2}\left[-\left\{S_{AA}(\bar{A})-\beta M_{AA}(\bar{A},\bar{Q})\right\}a^2+\beta M_{QQ}(\bar{A},\bar{Q})q^2+2\beta M_{AQ}(\bar{A},\bar{Q})aq\right]}\ da\ dq \end{aligned} \tag{4.14}$$

where, we have used $M(\bar{A},\bar{Q})$ to indicate the equilibrium isolated horizon mass as a function of the area and charge. The saddle point conditions imply that the coefficients of $a = (A - \bar{A})$ and $q = (Q - \bar{Q})$ must vanish, which yield

$$\beta(\bar{A},\bar{Q}) = \frac{S_A(\bar{A})}{M_A(\bar{A},\bar{Q})}\ ,\ \Phi(\bar{A},\bar{Q}) = M_Q(\bar{A},\bar{Q}) \tag{4.15}$$

4.3.2 Quantum Corrected Surface Gravity

An interesting result of this statistical mechanical approach is that, it gives rise to a quantum correction to the surface gravity which is a direct consequence of the logarithmic corrections of the microcanonical entropy $S = \frac{\bar{A}}{4} - \frac{3}{2}\log\frac{\bar{A}}{4}$ from loop quantum gravity. If one calculates β from the saddle point conditions and use it to find the quantum surface gravity ($\kappa_{quantum}$) in terms of classical surface gravity ($\kappa_{classical}$), there appear additional correction terms. One does this by calculating β from (4.15) and then using it in the expression $\kappa_{quantum} = \frac{2\pi}{\beta}$(more appropriately, this β can be replaced by $\beta_{quantum}$). The quantum corrections to the classical surface gravity is found to be

$$\kappa_{quantum} \approx \kappa_{classical}\left(1 + \frac{6}{\bar{A}}\right) \tag{4.16}$$

where higher order terms are neglected for large black holes ($\bar{A} >> 1$), $\bar{A}$ being the area of the weakly isolated horizon in Planck units. One can easily check the formula by applying it to the Reissner-Nordstrom and AdS Reissner-Nordstrom cases. Since the formulation is not dependent on any particular situation (symmetry, etc.), it will also be valid for other massive charged black holes also.

4.3.3 Stability Criteria

For the integral (4.14) to be convergent, the Hessian matrix H, given by (4.17) has to be *positive definite* where

$$H = \begin{pmatrix} \beta M_{AA}(\bar{A}, \bar{Q}) - S_{AA}(\bar{A}) & \beta M_{AQ}(\bar{A}, \bar{Q}) \\ \beta M_{AQ}(\bar{A}, \bar{Q}) & \beta M_{QQ}(\bar{A}, \bar{Q}) \end{pmatrix} \tag{4.17}$$

The necessary and sufficient condition for the *real symmetric square matrix* H to be positive definite can be stated as follows [78] - *'Determinants of all the principal submatrices, including the determinant, of H are positive.'* It is also a crucial point that the inverse temperature β must be positive. Hence, the necessary and sufficient conditions for the positive definiteness of the Hessian matrix lead to the following stability criteria

$$\beta \equiv \frac{S_A(\bar{A})}{M_A(\bar{A}, \bar{Q})} > 0 \tag{4.18}$$

$$\beta M_{AA}(\bar{A}, \bar{Q}) - S_{AA}(\bar{A}) > 0 \tag{4.19}$$

$$\det H \equiv \left\{\beta M_{AA}(\bar{A}, \bar{Q}) - S_{AA}(\bar{A})\right\} \beta M_{QQ}(\bar{A}, \bar{Q}) - \beta^2 M_{AQ}^2(\bar{A}, \bar{Q}) > 0 \tag{4.20}$$

It should be noted that, for $M_{QQ}(\bar{A}, \bar{Q}) > 0$, it will suffice to check only conditions (4.18) and (4.20) (which will be the case for RN and AdS RN black holes).

Here, one may wonder that how these stability criteria (4.18)–(4.20) are related to the convexity property of the entropy function, which is the usual notion for thermodynamic stability. It is true that the usual notion of thermodynamic stability is related to the convexity property of the entropy function. It is also true that this convexity property of the entropy function follows from the requirement of the convergence of the partition function under Gaussian thermal fluctuations[65, 74, 61]. Our stability criterion, likewise, follows from equations (4.18)–(4.20) which constitute the necessary and sufficient conditions for the grand canonical partition function to be well defined. Physically, the conditions to be satisfied by the entropy Hessian in [65],[74] and [61] (which imply convexity of entropy function) and the conditions (4.18)–(4.20) lead to the same conclusion i.e. the finiteness of the partition function under Gaussian thermal fluctuations. In fact, one can check that the apparent dissimilarity in the mathematical structure of the Hessian in [65] and eq. (4.17) is just a manifestation of the different variables which are summed over[2]. The equivalence between our conditions and the convexity of the entropy function is obvious for the charge-less case i.e. canonical ensemble, discussed in details in [12].

4.3.4 The Classical Metrics

Reissner-Nordstrom black hole : In this paragraph we investigate the stability of massive charged Reissner-Nordstrom black holes against Gaussian thermal fluctuations. The classical Reissner-Nordstrom metric which is given by

$$ds^2 = -(1 - \frac{2M}{r} + \frac{Q^2}{r^2})dt^2 + (1 - \frac{2M}{r} + \frac{Q^2}{r^2})^{-1}dr^2 + r^2 d\Omega^2 \tag{4.21}$$

The mass of the black hole in terms of area and charge as independent variables and it is given by

$$M = \frac{1}{2}\left(\frac{A}{4\pi}\right)^{\frac{1}{2}}(1+\rho) \tag{4.22}$$

where $\rho = \frac{4\pi Q^2}{A}$ is a dimensionless parameter in the chosen units. The microcanonical entropy including the logarithmic correction term from loop quantum gravity is given by

$$S = \frac{A}{4} - \frac{3}{2}\log\frac{A}{4} \tag{4.23}$$

Using (4.22) and (4.23) one finds that

$$\beta = 2\sqrt{\pi}A^{\frac{1}{2}}\frac{(1-\frac{6}{A})}{(1-\bar{\rho})} \tag{4.24}$$

$$detH = -\frac{\pi}{2\bar{A}}\frac{(1-\frac{6}{A})(1+\frac{6}{A})}{(1-\bar{\rho})} \tag{4.25}$$

[2]To understand the motivation behind this choice of variables one can see Subsection (4.2.1) where a detailed explanation is given.

where $\bar{\rho} = \frac{4\pi\bar{Q}^2}{\bar{A}}$. Conditions (4.18) and (4.24) together imply $\bar{\rho} < 1$, whereas, (4.19) and (4.25) imply $\bar{\rho} > 1$. From this contradiction one can conclude that the Reissner-Nordstrom black hole is *locally unstable* against Gaussian thermal fluctuations.[3]

AdS Reissner-Nordstrom black hole : In this paragraph we investigate the stability of massive charged AdS Reissner-Nordstrom black holes against thermal fluctuations. The classical AdS Reissner-Nordstrom metric which is given by

$$ds^2 = -(1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2)dt^2 + (1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2)^{-1}dr^2 + r^2 d\Omega^2 \tag{4.26}$$

where, $\frac{\Lambda}{3} = -\frac{1}{l^2}$. The mass of the AdS Reissner-Nordstrom black hole in terms of area and charge as the independent variables is given by

$$M = \frac{1}{2}\left(\frac{A}{4\pi}\right)^{\frac{1}{2}}(1 + \rho + \sigma) \tag{4.27}$$

where we have set $\frac{\Lambda}{3} = -\frac{1}{l^2}$ as Λ is negative for AdS spacetimes and introduced the parameter $\sigma = \frac{A}{4\pi l^2} = \frac{A}{A_\Lambda}$. Here also the entropy is given by (4.23). Calculating the required quantities for AdS Reissner-Nordstrom case from (4.23) and (4.27) one finds that

$$\beta = 2\sqrt{\pi}\bar{A}^{\frac{1}{2}}\frac{(1 - \frac{6}{\bar{A}})}{(1 - \bar{\rho} + 3\bar{\sigma})}$$
$$detH = \frac{\pi}{2\bar{A}}\frac{(1 - \frac{6}{\bar{A}})(1 + \frac{6}{\bar{A}})}{(1 - \bar{\rho} + 3\bar{\sigma})^2}\left[\bar{\rho} - 1 + 3\bar{\sigma}\left(1 - \frac{18}{\bar{A}}\right)\left(1 + \frac{6}{\bar{A}}\right)^{-1}\right]$$

where $\bar{\sigma} = \frac{\bar{A}}{4\pi l^2} = \frac{\bar{A}}{A_\Lambda}$. For the conditions (4.18) and (4.20) to be satisfied simultaneously the parameters have the following bound given by

$$1 - 3\bar{\sigma}\left(1 - \frac{18}{\bar{A}}\right)\left(1 + \frac{6}{\bar{A}}\right)^{-1} < \bar{\rho} < 1 + 3\bar{\sigma}$$

[3] This result is in agreement with what has been told in [63] i.e. *an asymptotically flat, non-extremal black hole can never achieve a state of thermal equilibrium.* This work also involves a statistical mechanical formulation without any classical metric but do not have a sound mathematical basis as a justification of the quantum statistical formulation which has been presented in our work.

which in the first order approximation reduces to the bound

$$1 - 3\bar{\sigma}\left(1 - \frac{24}{\bar{A}}\right) < \bar{\rho} < 1 + 3\bar{\sigma}$$

Hence, AdS Reissner-Nordstrom black holes are *locally stable* against thermal fluctuations for the above range of parameters.

4.4 Validity of S.P.A.

To ensure the validity of the saddle point approximation, the relative r.m.s. fluctuations about the saddle point are to be checked i.e. $\Delta A_{rms}/\bar{A} = (\Delta A^2)^{1/2}/\bar{A}, \Delta Q_{rms}/\bar{Q} = (\Delta Q^2)^{1/2}/\bar{Q}$ where

$$\Delta A^2 = (H^{-1})_{11} = \frac{\beta M_{QQ}}{detH} \quad , \qquad \Delta Q^2 = (H^{-1})_{22} = \frac{\beta M_{AA} - S_{AA}}{detH}$$

4.4.1 Reissner-Nordstrom black hole

The mean square fluctuations of area and charge in this case come out to be negative due to the presence of the determinant of the Hessian matrix. This emphasizes the instability of Reissner-Nordstrom black holes against Gaussian thermal fluctuations.

4.4.2 AdS Reissner-Nordstrom black hole

The relative r.m.s. fluctuations of area and charge in this case are found to be

$$\frac{\Delta A_{rms}}{\bar{A}} = \left[\frac{8(1-\bar{\rho}+3\bar{\sigma})}{\left(1+\frac{6}{\bar{A}}\right)\left\{\rho - 1 + 3\bar{\sigma}\left(1-\frac{18}{\bar{A}}\right)\left(1+\frac{6}{\bar{A}}\right)^{-1}\right\}}\right]^{\frac{1}{2}} \bar{A}^{-\frac{1}{2}} \tag{4.28}$$

$$\frac{\Delta Q_{rms}}{\bar{Q}} = \frac{\sqrt{3}}{\Phi^2}\left[\frac{\left(1-\bar{\rho}+3\bar{\sigma}\right)\left(1-\frac{2}{\bar{A}}\right)\left\{\bar{\rho}+\bar{\sigma}\left(1-\frac{18}{\bar{A}}\right)\left(1-\frac{2}{\bar{A}}\right)^{-1} - \frac{1}{3}\left(1+\frac{6}{\bar{A}}\right)\left(1-\frac{2}{\bar{A}}\right)^{-1}\right\}}{\left(1-\frac{6}{\bar{A}}\right)\left(1+\frac{6}{\bar{A}}\right)\left\{\rho - 1 + 3\sigma\left(1-\frac{18}{\bar{A}}\right)\left(1+\frac{6}{\bar{A}}\right)^{-1}\right\}}\right]^{\frac{1}{2}} \bar{A}^{-\frac{1}{2}} \tag{4.29}$$

where the equilibrium charge has been replaced in terms of the electric potential Φ which can be easily calculated from the saddle point conditions (4.15). It is clearly seen from the above expressions that the fluctuations fall off in the large area regime. As far as the positivity of the mean square fluctuations(r.m.s. fluctuation to be real) is concerned one needs to be more careful with the lower bound of $\bar{\rho}$. One can show that there arise the following two cases :

1)For $\bar{\sigma} < \frac{1}{3}\left(1+\frac{24}{A}\right)$, the correct lower bound for $\bar{\rho}$ is what has been obtained i.e. $1 - 3\bar{\sigma}\left(1-\frac{24}{A}\right)$.

2)For $\bar{\sigma} > \frac{1}{3}\left(1+\frac{24}{A}\right)$, the correct lower bound for $\bar{\rho}$ is given by $1-3\bar{\sigma}\left(1-\frac{16}{A}\right)+\frac{8}{A}$.

4.5 A General Discussion

The first thing to say about this approach is that its origin is purely based on quantum aspects of spacetime. During the build up of the formalism, *no classical metric* is used. The construction of the partition function is purely based on the ideas and results of LQG e.g. the use of Chern-Simons states, the splitting up of the total Hilbert space, etc. and also on the Hamiltonian formulation of spacetimes admitting weakly isolated horizons. The entropy correction also follows from the quantum theory. The classical metrics come to the picture only to be tested.

In course of this heuristic statistical mechanical approach of stability analysis of black holes, broadly two assumptions are made. In classical Hamiltonian GR it is known that the total Hamiltonian (gravity + matter) vanishes. So, it is very logical to consider that the quantum total Hamiltonian operator annihilates the bulk states of quantum matter coupled spacetime. A similar argument follows for the assumption of the quantum constraint on the volume charge operator. These two assumptions may be considered to be one due to their fundamental similarity and they ultimately give rise to a single quantum constraint.

In Section(4.2), a second assumption is made regarding the eigenvalue spectrum of the energy of the black hole. It is already mentioned (in a footnote) that this second assumption is not a strong one. If one studies [9] carefully, the classical mass function associated with the horizon is stipulated to be a function of horizon area and charge. Again, this horizon area and charge are the functions of the local fields on the horizon. Proper quantization of the classical horizon area and charge will obviously lead to a well defined boundary Hamiltonian operator. The fact that there exists a quantum boundary Hamiltonian operator which acts on the boundary Hilbert space of the black hole is an assumption, since the exact form of such a Hamiltonian operator is still unknown. But the fact that its eigenvalue spectrum is a function of eigenvalue spectra of the area and charge operators is most likely a valid assumption, as it is bound to happen if such a boundary Hamiltonian operator exists. It follows from the classical analog - the mass associated with the horizon must be a function of the horizon area and charge for a consistent Hamiltonian evolution[9].

In ref. [65] where a similar approach has been taken, a particular functional form of the mass in terms of the area and charge had been used on an ad hoc basis. Such an ad hoc assumption has been shown here to be quite redundant. This, therefore, is a significant strength of this work, relative to the earlier assay. Thus, the statistical mechanical approach adopted in this work, though similar, now stands on a far stronger ground than in the previous version.

This statistical mechanical approach gives us a new quantum correction to the surface gravity arising from the loop quantum gravity corrections to the microcanonical entropy. One can easily check its validity. Moreover, it predicts local thermodynamic instability of the Reissner-Nordstrom black hole and local thermodynamic stability of the AdS Reissner-Nordstrom black hole as was shown classically in [73]. As far as the relative charge fluctuations are concerned, in literature [75] there are problems for AdS RN black holes as it does not fall off for any condition. This problem has been solved in this work(Section4.4). Last but not the least, once more a word is worth mentioning that this formulation *does not* involve any classical metric. The whole thing depends on quantum aspects of spacetime. One can generalize this to study rotating and even charged-rotating black holes because there is no use of symmetry in the theory. We look forward to give those analyses in future.

There is a more crucial issue which can be of utmost importance to have a deeper understanding of black hole thermodynamics [79]. As far as the saddle-point approximation is concerned, the Euclidean path integral approach does look similar to our thermal holographic approach. But there is a crucial difference between the two approaches. The path integral approach, within the saddle point approximation, requires information of the full black hole spacetime (bulk and horizon) as given by the classical black hole metric solution chosen to be the saddle point. Thus one needs global information away from horizon. In the thermal holographic approach which we adopt, one needs only *local information* associated with the equilibrium isolated horizon geometry interpreted as an inner null boundary. Thus, no detailed knowledge of the full classical spacetime is needed. The mass of the isolated horizon is an *unspecified* function of the area and charge, and we never need to specify this function to derive our results, except as fiducial checks [Subsection (4.3.4)] appropriate to given classical metrics. This insensitivity of our approach to an explicit classical black hole metric is a key feature of our work and can be taken to mean that our results are in a sense more general than those computed from the Euclidean path integral.

A further distinction is that, in contrast to the Euclidean path integral approach, where quantum fluctuations around a *classical* metric are considered, our saddle point is a *quantum* isolated horizon whose quantum states and their (non-perturbative) dynamics are described by a quantum Chern Simons theory. Consequently, the *equilibrium* entropy is the microcanonical entropy computed in earlier work (ref. [13, 14, 16, 17]) based on LQG, and already has an infinite series of corrections (including those logarithmic in horizon area) beyond the Bekenstein-Hawing area law, incorporating quantum spacetime fluctuations. In this work, *additional thermal fluctuations* (and their physical effects) are considered, over and above the quantum spacetime fluctuations already incorporated for the equilibrium configuration. Quantum and thermal fluctuations are thus, treated somewhat distinctly in our approach, and the result is an interesting interplay between them. In the Euclidean path integral, such distinctions are not as clear. It might be of future interest to see better how these two somewhat disparate approaches may be related.

4.6 A local observer perspective

Recently, there have been proposal of a quantum energy spectrum of QIH [15] based on semi-classical approximations and arguments from classical Schwarzschild black hole, which are used as inputs from outside the *pure* quantum theory, so as to avoid a true quantization of the black hole energy. The proposed energy spectrum of the QIH in [15] is given by

$$\widehat{E}\,|j_1,\cdots,j_N\rangle = \frac{1}{8\pi\ell}\,\widehat{A}\,|j_1,\cdots,j_N\rangle \tag{4.30}$$

where $\widehat{E}$ and $\widehat{A}$ are the Hamiltonian and area operators for the QIH respectively. The proposal of the energy spectrum in eq.(5.21) follows form the semiclassical relation $E = A/8\pi\ell$, elaborated in [92], where $E =$ the classical energy of the black hole , $A =$ the classical area of the horizon and ℓ is a *constant* length scale characterizing the proper distance of a class of stationary local observers from the horizon, introduced in [92]. Similar expression for local energy ($E = A/8\pi\ell$) also appears in [93, 94] with explanations on different grounds [4]. Since only [15] discusses the thermodynamic stability of the QIH and thus, more closely related to the subject matter of this paper, we shall only refer to [15] at the relevant places.

Here, we present a detailed stability analysis of a QIH having the energy spectrum as in eq.(5.21), to argue that such a spectrum in fact leads to the local thermodynamic *instability* of the QIH. The stability analysis is carried out using two methods – one that is similar to the one followed in [15] and the other which is a completely independent approach, namely the thermal holographic method introduced in [12, 3]. Both methods lead to the same conclusion which is the key result of this paper: *An uncharged, non-rotating QIH, as observed by the local observers discussed in [15, 92], is locally unstable as a thermodynamic system.*

4.6.1 Energy Spectrum of QIH

In this section we shall discuss about the energy spectrum for the QIH observed by *a local observer*, which have been proposed in [15, 92] as a model of a black hole. Since this work is focused towards studying the thermodynamics of *quantum* Isolated Horizon having the energy spectrum proposed in [15], we shall be very brief in discussing the aspects of *classical* general relativity related to this work (e.g. definition of local observers, approximations of energy

[4] As far as this length scale ℓ is concerned there is a conflict between [15, 92] and [93]. According to [15, 92], $\ell \sim \ell_p$. On the other hand, according to [93], $\ell \gg \ell_p$, ℓ_p being the Planck length.

expression) to avoid unnecessary lengthening of the paper. Of course we shall point out the proper references where the ideas have been discussed in complete details.

Local observers : The definition of local observers used in [15, 92] originates from the ideas extensively discussed in the textbooks on General Relativity such as [96], [97] etc. These preferred class of observers are *stationary* with respect to the horizon of the black hole, which makes them ideal for the observation of the thermodynamics of horizons according to [92]. The four velocity of such an observer is given by $u = \frac{\xi}{\sqrt{|\xi.\xi|}}$, where ξ is the time-like Killing vector associated with the black hole spacetime and alternatively, the generator of the one-parameter group of isometry ($\mathscr{L}_\xi g_{ab} = 0$) for the associated black hole spacetime with the metric g_{ab}. While infinitesimally close to the black hole event horizon these observers represent the ZAMOs of [98]. For more details on this account one may look into [92] and the references therein.

Energy observed by a local observer : The energy spectrum used in [15] follows from the definition of energy of a classical Schwarzschild black hole from a local observer's perspective. The definition of local energy [15, 92] is given by

$$E_r = -\frac{1}{8\pi}\int_{S_r} \nabla^a u^b dS_{ab} \tag{4.31}$$

where $u = \frac{\xi}{\sqrt{|\xi.\xi|}}$ is the four velocity of the local observer and S_r is a 2-sphere of radius $r > 2M$. For a Schwarzschild black hole, it is straightforward to show that $E_r = \frac{M}{2}\left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}}$. The near horizon limit at $r = 2M + \epsilon$ is obtained to be

$$E \equiv E_{2M+\epsilon} \approx \frac{M}{2}\left(\frac{2M}{\epsilon}\right)^{\frac{1}{2}} = \frac{A}{8\pi\ell} \tag{4.32}$$

where one has to use $\ell = 2(2M\epsilon)^{\frac{1}{2}}$, $A = 16\pi M^2$ and consider $\epsilon \ll 2M$. The above result has been generalized for the case of Kerr-Newman black hole in [92].

Following the definition of energy given by eq.(4.32), the spectrum of the Hamiltonian operator is proposed [15] to be given by eq.(5.21), which in terms of the area spectrum of the QIH in LQG [76, 77] can be written as

$$\widehat{H}|j_1, \cdots, j_N\rangle = \frac{\gamma \ell_p^2}{\ell} \sum_l \sqrt{j_l(j_l+1)})\ |j_1, \cdots, j_N\rangle \tag{4.33}$$

where j_l taking values from the set $\{\frac{1}{2}, 1, \frac{3}{2}, \cdots, \frac{k}{2}\}$ is the spin associated with the l-th puncture. The length scale ℓ is the new object introduced, which does not belong to the quantum theory, namely LQG. It denotes the proper distance of a stationary observer from the event horizon of a Schwarzschild black hole at the radial coordinate $r = 2M + \epsilon$ and having an acceleration $1/\ell$. Detailed explanation of the derivation of the energy spectrum is available in [15, 92]. As far as our work is concerned the crucial point to be noted is that *the derivation of the energy spectrum in eq.(5.21) is dependent on the frame of a class of local observers introduced in [92], characterized by the length scale ℓ. Hence, the energy spectrum of the QIH given by eq.(5.21) is 'A Local Observer's view'.* It follows that the results of the thermodynamic analysis of a QIH with such an energy spectrum, which forms the core matter of this paper, will be the observation of 'a local observer'. Thus, the title of this paper is justified.

Motivation for the use of the spectrum : The Hamiltonian formulation of the classical phase space of spacetimes admitting internal boundaries (classical isolated horizons(CIH) at equilibrium) shows that there exists an energy associated with each CIH satisfying a first law[9]. A correct quantization of such a theory must lead to a horizon energy spectrum expressed in terms of the spectra of the operators corresponding to the other extensive variables of the first law (namely area, charge, angular momentum, etc.). Unfortunately, such things have not been done up till now. On the other hand, in quantum geometry[13, 14], the full Hilbert space of a quantum black hole can be written as $\mathcal{H} = \mathcal{H}_\mathcal{V} \otimes \mathcal{H}_\mathcal{S}$ modulo some constraints, where $\mathcal{V}(\mathcal{S})$ stands for volume (surface). Thus, any generic state $|\Psi\rangle$, of the quantum black hole can be written as $|\Psi\rangle = |\Psi_\mathcal{V}\rangle \otimes |\Psi_\mathcal{S}\rangle$. Hence, any operator which acts on the states of the Hilbert space $\mathcal{H}$, say the Hamiltonian $\widehat{H}$, must have a form [12] $\widehat{H} = (\widehat{H}_\mathcal{V} \otimes \widehat{I}_\mathcal{S} + \widehat{I}_\mathcal{V} \otimes \widehat{H}_\mathcal{S})$ where $\widehat{I}$ represents identity operator. But the spectrum of this Hamiltonian is unknown. So, clearly there is a missing link between the classical and quantum theories of IH as far as the energy spectrum is concerned. As already mentioned in [15], the effort made in [15, 92] is aimed to provide this missing link by an input from the classical theory.

The Canonical Partition function : Now, using quantum geometry one can show[12, 3] that the partition function of a quantum black hole is completely determined by surface states i.e. $Z = Z_{\mathcal{S}} = Tr_{\mathcal{S}} \exp -\beta \widehat{H}_{\mathcal{S}}$. This is nothing but the partition function for the QIH. Now, the generic wave function of the QIH can be written as $|\Psi_{\mathcal{S}}\rangle = \sum_{j_1,\cdots,j_N} C(j_1,\cdots,j_N)|j_1,\cdots,j_N\rangle$, where $|C(j_1,\cdots,j_N)|^2$ is the probability of finding the QIH in the eigenstate $|j_1,\cdots,j_N\rangle$ having the spin sequence $\{j_1,\cdots,j_N\}$. Hence, following eq.(4.32), the Hamiltonian acting on the generic wave function of the QIH can be written as [15]

$$\widehat{H}_{\mathcal{S}}|\Psi_{\mathcal{S}}\rangle = \frac{1}{8\pi\ell}\widehat{A}|\Psi_{\mathcal{S}}\rangle \tag{4.34}$$

where $\widehat{A}$ is the area operator in LQG [76, 77]. The above equation can be explicitly written in terms of the spectrum of $\widehat{H}_{\mathcal{S}}$ which can be written in terms of the spectrum of $\widehat{A}$ following eq.(5.21) as

$$\widehat{H}_{\mathcal{S}}|j_1,\cdots,j_N\rangle = \left(\frac{\gamma\ell_p^2}{\ell}\sum_{l=1}^{N}\sqrt{j_l(j_l+1)}\right)|j_1,\cdots,j_N\rangle \tag{4.35}$$

where γ is the Immirzi parameter and ℓ_p is the Planck length. j_l is the spin associated with the l th puncture, N being the total number of punctures. $|j_1,\cdots,j_N\rangle$ is a microstate of the SU(2) CS theory on the QIH, designated by the spin sequence $\{j_1,\cdots,j_N\}$ and also an eigenstate of the Hamiltonian $\widehat{H}_{\mathcal{S}}$. Hence, to be more appropriate $\widehat{H} \equiv \widehat{H}_{\mathcal{S}}$ in the eq.(5.10) which has been proposed in [15]. Also, the trace over the surface states in the partition function discussed above is the sum over all possible spin sequences.

Remarks : The energy spectrum used in [15] is based on an approximation of the energy, observed by a stationary observer near the horizon of a classical Schwarschild black hole, resulting from some *ad hoc* arguments put forward in [92]. Obviously this is not a true quantization of horizon energy. Nevertheless, the thermodynamic aspects of a QIH with the energy spectrum of [15], as observed by a local observer, is worth studying as far as its importance in the current literature is concerned, especially [93] and [94] besides several other works (not to be listed here). In [93], the notion of a quantum Rindler horizon is introduced whose classical version describes the near-horizon geometry of a non-extremal black hole as seen by a stationary local observer. The dynamics of the quantum surface, describing this system, is generated by the boost Hamiltonion of Lorentzian Spinfoams. The crucial point is that the expectation value of this boost Hamiltonian results in the local horizon energy introduced in [92]. On the other hand, in [94] it has been shown that the energy expression given by eq.(4.32) comes out to be equal to the canonical energy associated with the boundary term of the Holst action, alongside other relevant consequences as far as horizon thermodynamics is concerned. In a nutshell, even though the energy spectrum of a QIH given by eq.(5.21) is a very specific one, there is much reason to pay attention to it as far as its physical consequences are concerned.

4.6.2 Thermodynamics of QIH

This section is dedicated to an exhaustive thermodynamic stability analysis of a QIH having the energy spectrum given by eq.(5.21), as observed by a local observer.

Explicit Quantum Statistical Stability Analysis

To get an insight of the thermodynamic properties of the particular model of quantum black hole, we explore the canonical ensemble scenario where the total number of punctures (N) is kept *fixed* and the energy (E) is allowed to fluctuate. The canonical partition function can be written as a sum over spin configurations as

$$Z(\beta, N) = \sum_{\{s_j\}} d[\{s_j\}] e^{-\beta E_{\{s_j\}}} \approx d[\{\bar{s}_j\}] e^{-\beta E} \tag{4.36}$$

where E_j = energy associated with a spin j, $\sum_j \bar{s}_j E_j = E$ = energy of the QIH for $\{\bar{s}_j\}$(thermal equilibrium). The contributions from the sub-dominant configurations are neglected. β is the inverse temperature of the QIH given by $\beta = \partial S_{MC}/\partial E|_N$ which results in

$$\beta = 2\pi\ell\left(1 - \frac{6}{A}\right) \tag{4.37}$$

To calculate relevant thermodynamic quantities one needs to calculate the logarithm of the partition function in the appropriate limits ($\bar{s}_j, k \to \infty$). A straightforward calculation using equations (5.21), (4.36) and (4.37) yields

$$\log Z = N\sigma(\gamma) - \frac{3}{2}\log A + \frac{3}{2} \tag{4.38}$$

The *average energy* of the QIH in the canonical ensemble can be calculated from (4.38) using the usual thermodynamical relation $\langle E \rangle = -\frac{\partial}{\partial\beta}\log Z$. Using $d\beta/dA = 12\pi\ell/A^2$ and $E = A/8\pi\ell$ it is straightforward to show that the average energy of the QIH is equal to its equilibrium energy i.e. $\langle E \rangle = E$. Following this, the *specific heat* of the QIH can be calculated using the usual thermodynamic formula $C = -\beta^2\partial\langle E \rangle/\partial\beta$. A few steps of algebra lead to

$$C = -\frac{\beta^2 A^2}{96\pi^2\ell^2}$$

where one has to use $\langle E \rangle = E = A/8\pi\ell$ and $d\beta/dA = 12\pi\ell/A^2$. The specific heat being *negative definite* one can conclude that a QIH, having energy spectrum as in eq.(5.21), is locally *unstable* as a thermodynamic system.

In this context, the validity of *the first law* can be checked using (3.30) and (5.23) and one can easily show that $dE = TdS$ is indeed satisfied. Also, from (4.37) one can find the local *horizon temperature* to be

$$T = \frac{1}{2\pi\ell}\left(1 + \frac{6}{A} + ...\right) \qquad [k_B = 1] \tag{4.39}$$

This local horizon temperature contains a series which is identical to the correction terms obtained for the horizon temperature in [3] considering Gaussian thermal fluctuations about the equilibrium. The connection between these two may be a future issue of interest.

NOTE : Let us have a closer look at the canonical partition function. The *exact* canonical partition function, without any approximation, can be written as

$$Z(\beta, N) = \bar{Z}(\beta, N) + \delta(\beta, N) \tag{4.40}$$

where $\delta(\beta, N)$ is the contribution from *thermal fluctuations* (sub-dominant configurations $\{s_j\}$s other than $\{\bar{s}_j\}$) about equilibrium value ($\bar{Z}$) of the canonical partition function coming from the dominant configuration $\{\bar{s}_j\}$ whose spin distribution is given by eq.(3.14). Truly speaking, eq.(4.36) is only $\bar{Z}$ and *not* Z. The effect of the thermal fluctuations is completely neglected ($\delta = 0$) in eq.(4.36). This has a profound implication.

The canonical entropy is given by $S_C = \log Z + \beta \langle E \rangle$, which can be recast as

$$S_C = \bar{S}_C + \log(1 + \delta/\bar{Z})$$

where $\bar{S}_C = \log \bar{Z} + \beta E$ and $\langle E \rangle = E$ is used. If one calculates $\log \bar{Z} + \beta E$, a few steps of algebra leads to $\bar{S}_C = S_{MC}$. Therefore,

$$S_C = S_{MC} + \log(1 + \delta/\bar{Z}) \tag{4.41}$$

If we do not take the effects of thermal fluctuations in canonical ensemble i.e. $\delta = 0$, then it is obvious that $S_C = S_{MC}$ at *all* temperatures. This is a very general result concerning a thermodynamic system [61] which also applies for black holes as has been shown earlier in the literature [95]. In fact this extra contribution from thermal fluctuations plays a very important role in analyzing the thermodynamic stability of the QIH [12] which we will discuss briefly in the next subsection. Sitting at the equilibrium and ignoring the thermal fluctuations lead to a physically incomplete scenario which apparently looks to give us an *ensemble independent* result [15]. Hence, to get the complete picture, we must take into account quantum and thermal fluctuations both.

4.6.3 Thermal Fluctuations in Canonical Ensemble

If one calculates the partition function including the Gaussian thermal fluctuations and using $E = A/8\pi\ell$, it comes out to be

$$Z \approx \frac{1}{4\pi} e^{S_{MC}(\bar{A}) - \beta E(\bar{A})} \int_0^\infty e^{\frac{3}{4\bar{A}^2} a^2} da \tag{4.42}$$

where $\bar{A}$ is the horizon area at equilibrium(saddle point) and a is the fluctuation variable. The partition function is clearly undefined due to the infinite integral. Also, if one calculates the canonical entropy(S_C) taking the Gaussian thermal fluctuations into account[95], it comes out to be

$$S_C = S_{MC} - \frac{1}{2}\log\Delta$$

where

$$\Delta = \frac{K}{\partial E/\partial A}\left[\frac{\partial^2 E}{\partial A^2}\frac{\partial S_{MC}}{\partial A} - \frac{\partial^2 S_{MC}}{\partial A^2}\frac{\partial E}{\partial A}\right] \tag{4.43}$$

evaluated at the saddle point(equilibrium configuration), K being an irrelevant positive constant. For the canonical entropy to be well defined we must have $\Delta > 0$. But, using the energy spectrum given by (5.21) it is straightforward to show that $\Delta < 0$. Thus the canonical entropy can not be defined for a QIH having an energy spectrum as in eq.(5.21) which implies nothing but the instability of the QIH as a thermodynamic system[12] from the perspective of a local observer. (N does not play any role as it is kept fixed in canonical ensemble.)

In this Gaussian approximation method, the inverse temperature of the QIH at equilibrium is given by $\beta = \frac{\partial S_{MC}/\partial A}{\partial E/\partial A}$. Using eq.(5.21) and eq.(3.30) in the expression for β, it is easy to find that $\beta = 2\pi\ell\,(1 - 6/A)$. Comparing this result with eq.(4.37), one can see that the equilibrium temperature comes out to be the same in both the approaches. This is a consistency check.

4.6.4 Few remarks

Let us conclude with a few remarks on the thermodynamic stability analysis of the particular model of a QIH presented in this section. The final conclusion is the local thermodynamic *instability* of a QIH having energy spectrum given by the eq.(5.21). The result has been derived in two different approaches shown in the subsections (4.6.2) and (4.6.3). The crucial role of thermal fluctuations behind the thermodynamic instability of this particular model of the QIH is evident from the analyses. In fact, though the quantum statistical analysis gives a negative specific heat, the physical picture becomes much clearer in the *thermal holographic* analysis where the Gaussian thermal fluctuations manifestly control the convergence criterion of the partition function. In this approach we can actually see that any arbitrary QIH is *not* thermodynamically unstable, but only those which fail to satisfy the convergence condition for the partition function are unstable. The particular energy spectrum of the QIH given by eq.(5.21) considered in this paper is only one such example. This is *not* the stability analysis of a generic QIH, for which we need the information about the quantum energy spectrum of a QIH derived from a true quantization resulting from the fundamental quantum theory. It can be only considered to be a stability analysis of the QIH from the perspective of a local observer.

Now, an alert reader will surely wonder : *Why shall we make a thermodynamic analysis of such a very specific energy spectrum of a QIH ?* The answer is very simple : *The particular energy spectrum studied in this paper is of utmost importance as far as current literature is concerned.* Some of the important aspects of the particular definition of energy of a QIH studied in this work has been mentioned in the "Remarks" at end of section(4.6.1) in connection with current literature. But the most important consequence has been reported in [15]. Using the energy spectrum in eq.(5.21), it has been claimed in [15] that *" as a thermodynamic system the Isolated Horizon is locally stable"*. Hence, the purpose of an extensive thermodynamic analysis of a QIH having the particular energy spectrum given by eq.(5.21) is now clear enough and it does not need much of an effort to understand that our results are in complete contradiction with the above claim.

Chapter 5

Energy Spectrum of Equilibrium Black Holes in LQG : Model Hamiltonian

5.1 Introduction

Although it is a known fact that the IH is associated with a classical notion of energy which satisfies a first law [9], but there does not exist any quantized energy spectrum for a black hole. In this work, we take an upside down approach to find the energy spectrum of a QIH. Instead of quantizing the classical energy, taking the quantum theory as the starting point and using our knowledge of the area operator in LQG[13, 14, 88, 81], we propose the most generic structure of the Hamiltonian operator associated with the QIH. The motivations behind this approach can be stated as follows. We always pass to the quantum theory only through the quantization of the classical one. But we never originally formulate the quantum theory and pass on to the classical one by studying the correspondence limit of the quantum theory[1]. To study the correspondence limit of a quantum theory is only 'a consistency check' for us and *not* a 'method of derivation' of the laws of classical physics. The first law of thermodynamics for an IH, which results from the *classical* theory of IH, is nothing but the proportionality

[1]Nature is the way it is. We are the ones who can see it classically only, which is the sole reason for this order of passage from classical to quantum. If human mind were sharp enough to formulate the quantum theory on the first hand, then we would have done so. Then, many fundamental problems which arose in classical physics and later solved in quantum theory, would not have risen at all.

of the variation of classical energy to that of the classical area, the proportionality constant being related to the surface gravity of the IH i.e. $\delta E^t_{IH} = (\kappa^t_{IH}/8\pi)\,\delta A_{IH}$ where t denotes the choice of time evolution vector field[9]. Wearing the 'quantum spectacles' one can view that on the right hand side of the first law, the classical area A_{IH} is the expectation value of the QIH area operator. Then the classical energy E_{IH} on the left hand side must also result from the expectation value of some Hamiltonian operator for the QIH. This clearly indicates that there must be a gauge invariant true Hamiltonian operator in the quantum theory which will give rise to this notion of classical energy associated with the IH in the correspondence limit. Consideration of the quantum sources, the punctures, to play the fundamental role in the QIH theory[13, 14], along with a close observation of the structure of the area operator [88, 81], at once reveals the most general structure of such a Hamiltonian operator associated with the QIH which is gauge-invariant, self-adjoint and commutes with the area operator so as to yield the constant classical area property of IH. Finally, demanding that the first of IH thermodynamics should follow in the classical limit the unknown coefficients and parameters of the Hamiltonian operator can be fixed to get the final result.

5.2 Beginning from the quantum theory

5.2.1 The area operator revisited : Some crucial observations

The area operator is a gauge invariant, self-adjoint observable in *loop quantum gravity*(**LQG**) defined for any arbitrary two dimensional surface (S) embedded in the three dimensional spatial manifold (Σ) obtained from a specific foliation of the four dimensional spacetime manifold ($M \equiv R \times \Sigma$) by some preferred time evolution vector field (t)[88, 81]. The wave function of the spatial geometry is a functional of the spatial SU(2) connection A. These functionals are equipped with some specific properties and are called *cylindrical functions*[47]. The spatial quantum geometry is denoted by a graph consisting of edges (links) and vertices (nodes) known as spin network. Each edge is associated with a spin representation of the SU(2) group. The wave function of the quantum geometry is written as

$$\Psi_{\Gamma,\psi}[A] = \psi(h_{\rho_1}(A), h_{\rho_2}(A), \cdots, h_{\rho_n}(A)) \tag{5.1}$$

where $h_\rho(A)$-s are the holonomies along the edges of the spin network, Γ is the collection of the ordered oriented paths and ψ is a smooth function on $[\mathrm{SU}(2)]^n$[81, 47]. The holonomy of the connection along a curve ρ embedded in Σ is given by

$$h(A,\rho) \equiv \mathcal{P}\exp\int_\rho A \tag{5.2}$$

which is a gauge invariant quantity. The momentum (E) conjugate to the connection variable (A) [47] acts on the wave function as operators given by the following functional derivative :

$$\frac{1}{8\pi\gamma G}\hat{E}^a_i(\tau,x)\Psi_{\Gamma,\psi}[A] \quad = \quad -i\hbar\frac{\delta\Psi_{\Gamma,\psi}[A]}{\delta A^i_a(\tau,x)} \tag{5.3}$$

The momentum (E) being a two form can be naturally integrated over a two surface. The corresponding operator can also be smeared on a two surface which leads to the flux operator in LQG which plays the key role in endowing the surface (S) with a quantum area [88, 81, 47, 76]. Since, from eq.(5.1), it is evident that $\Psi_{\Gamma,\psi}[A]$ consists of the holonomies along the edges of the spin network, the action of the momentum operator on the holonomy is the most crucial step, which is given by the functional derivative of the holonomy with respect to the connection as follows

$$\frac{\delta h(A,\rho)}{\delta A^i_a(\tau,x)} \quad = \quad \int_0^1 ds\ \dot{\rho}^a(s)\ \ \delta^3(\rho(s),x)\times[h(A,\rho_1)\tau_i h(A,\rho_2)] \tag{5.4}$$

The quantity in the expression (5.4) is a two dimensional distribution and yields a well defined operator $\hat{E}_i(S)$ when smeared over a two dimensional surface (S), embedded in the three dimensional spatial manifold [88, 81, 76]. Geometrically, the operator $\hat{E}_i(S)$ signifies that when the path ρ intersects the surface S at a point Q(say), separating the path into two paths ρ_1 and ρ_2, Q gets associated with a matrix $\pm i8\pi\gamma G\hbar\tau_i$ in the particular spin representation carried by the holonomy of the path ρ. The signature depends on the relative orientation of the path (ρ) and the surface (S). Without any intersection the result is zero. The gauge invariant operator that can be constructed from $\hat{E}_i(S)$ and relevant for the construction of the area operator is simply $\sum_i \hat{E}_i^2(S)$. Its action on the holonomy of a path ρ carrying spin-j representation and intersecting S at a point is given by

$$\sum_i \hat{E}_i^2(S)\ ^{(j)}h(A,\rho) \quad = \quad ^{(j)}h(A,\rho_1)\left[-(8\pi\gamma G\hbar)^2 \sum_i\ ^{(j)}\tau_i^2\right]\ ^{(j)}h(A,\rho_2)$$

Using $-\sum_i\ ^{(j)}\tau_i^2 = j(j+1)\times \mathbf{I}$, the Casimir of the SU(2) group in the spin-j representation and the property $^{(j)}h(A,\rho) =\ ^{(j)}h(A,\rho_1)\ .\ ^{(j)}h(A,\rho_2)$ for $\rho = \rho_1 \cup \rho_2$ on the right hand side of the above equation, one obtains an eigenvalue equation with the eigenvalue $(8\pi\gamma G\hbar)^2 j(j+1)$. The gauge-invariance of the operator $\sum_i \hat{E}_i^2(S)$ is manifested by the insertion of the matrix Casimir of the group and the real eigenvalue manifests the self-adjoint property, *which are all associated only with the intersection point.* This is the most crucial fact which will play the pivotal role in the phenomenology presented in this paper. Now, comparing with the classical definition of area of the surface S given by $\int_S dS\sqrt{n_a E_i^a n_b E_i^b}$ (n being normal to the surface S), it can be said that the path ρ upon intersecting the surface S provides it with a quantum of area $8\pi\gamma G\hbar\sqrt{j(j+1)}$.

Now, let there be N such intersections of the surface with the spin network edges. The edges are $\rho_1, \rho_2, \cdots, \rho_N$ and the corresponding spin representations carried by the edges be $j_1, j_2, \cdots, j_N$. But for multiple intersections on a surface, matrices at different points get contracted spoiling the gauge invariance of the operator. To tackle this problem, the surface is partitioned into N pieces S_l, $l \in [1, N]$ such that $\cup_l S_l = S$. In the limit $N \to \infty$, the pieces get small enough to guarantee that there is a single intersection with each piece with a single edge. It should be remembered that the spin network is also dense enough so as to make large number of intersections with the surface S to make the above regularization[2] scheme valid. The area operator associated with the surface S is given by

$$\hat{A}_S \equiv \lim_{N\to\infty} \sum_l \left[\sum_i \hat{E}_i^2(S_l)\right]^{1/2} \tag{5.5}$$

[2] See [81, 47] for detailed account on the regularization issue.

Hence the eigenvalue spectrum of the area operator will be given by

$$\hat{A}_S|\cdots\rangle = 8\pi\gamma\ell_p^2 \sum_{l=1}^{N} \sqrt{j_l(j_l+1)}\;|\cdots\rangle \tag{5.6}$$

where $|\cdots\rangle$ denote a spin network state making intersections with the surface S and $\ell_p^2 = G\hbar$. j_l are half-integers[3] The real eigenvalue spectrum tells us that the area operator is self-adjoint and the gauge-invariance has been guaranteed in the construction itself, specifically by the Casimirs appearing at the intersections. The interesting property of the area operator that we are interested in is that *all the properties of the area operator are actually carried by the individual intersection points where the edges of the spin network pierce the surface S.*

5.2.2 Area spectrum and Hilbert space of QIH : Punctures are the building blocks

The LQG area spectrum, in general, is unbounded above along with a vanishing lower bound for any arbitrary surface. But, the spectrum of the area operator belonging to a QIH Hilbert space has a positive lower bound along with a finite upper bound owing to the physical properties of the QIH. The Hilbert space of a QIH is that of a three dimensional SU(2) *Chern-Simons*(**CS**) theory coupled to *punctures*(sources), carrying spin representations of the corresponding edges of the bulk spin network which pierce the IH (a null inner boundary of spacetime with the topology $R \times S^2$). The level of the source coupled CS theory is given by $k \equiv A_{cl}/4\pi\gamma\ell_p^2 \in I$, A_{cl} being the classical area of the IH. The physical states of the QIH belong to the singlet part of the Hilbert space which is given by $\mathcal{H}_S \equiv \mathrm{Inv}(\otimes_{l=1}^{N}\mathcal{H}_l)$, where 'Inv' stands for gauge invariance on the QIH. The area spectrum of a QIH with N punctures and CS level k can be written as $8\pi\gamma\ell_p^2 \sum_{l=1}^{N} \sqrt{j_l(j_l+1)}$ with $1/2 \le j_l \le k/2\ \forall l \in [1, N]$ [89]. Following the structure of $\mathcal{H}_S$, which is a gauge invariant direct product space of the Hilbert spaces associated with the *individual* punctures, we can further write $\hat{A}_S$ as

[3]The spectrum discussed here is only a part of the full area spectrum in LQG[81], which is generally considered to be relevant in the literature of quantum black holes and that is what we are interested in.

$$\hat{A}_S \equiv \hat{A}_{j_1} \otimes \hat{I}_{j_2} \otimes \cdots \otimes I_{j_N} + \hat{I}_{j_1} \otimes \hat{A}_{j_2} \otimes \cdots \otimes \hat{I}_{j_N} + \cdots + \hat{I}_{j_1} \otimes \hat{I}_{j_2} \otimes \cdots \otimes \hat{A}_{j_N} \quad (5.7)$$

Even though, this above expression for the area operator usually does not appear in the literature, but it is a trivial structure to write down to ensure that $\hat{A}_S$ acts on a quantum state of the QIH $|\phi_S\rangle \in \mathcal{H}_S \equiv \text{Inv}(\otimes_{l=1}^{N}\mathcal{H}_l)$, which can also be written as $|\phi_S\rangle \equiv \otimes_{l=1}^{N}|\phi_l\rangle$ following that the punctures are non-interacting and distinguishable[13, 14]. This particular way of writing the area operator is also motivated by the fact that the properties of the area operator of the QIH are actually carried by each *individual* punctures; recall that the Casimir of SU(2) is inserted only at the puncture where an edge of the bulk spin network intersects the surface. Hence, the punctures are the building blocks of the QIH in the LQG framework and must play the roles of fundamentally important individuals while constructing an operator associated with the QIH.

5.2.3 Proposal of the model Hamiltonian

Motivated by all the facts discussed up till now and with a view to approach the problem from a quantum viewpoint, it is prompting to write down the Hamiltonian operator for the QIH in the form of the area operator associated with the same. Hence, we write the QIH Hamiltonian as

$$\hat{H}_S \equiv \hat{H}_{j_1} \otimes \hat{I}_{j_2} \otimes \cdots \otimes I_{j_N} + \hat{I}_{j_1} \otimes \hat{H}_{j_2} \otimes \cdots \otimes \hat{I}_{j_N} + \cdots + \hat{I}_{j_1} \otimes \hat{I}_{j_2} \otimes \cdots \otimes \hat{H}_{j_N} \quad (5.8)$$

very similar to the area operator of the QIH given by eq.(5.7). Further, we propose that *any gauge invariant, self-adjoint operator associated with the QIH and which commutes with the area operator of the QIH, the contribution from a single puncture, carrying a spin-j representation, must be a polynomial of* $\hat{A}_j$. All these three properties are essential for a true Hamiltonian operator associated with the QIH, of which the reason for the requirement of commutativity is justified by two reasons. First of all, the area operator and the Hamiltonian associated with the QIH will have the simultaneous eigenstates, which are the states of the CS theory coupled to the punctures. Secondly, we must ensure that the expectation value of the area operator, which is equal to the classical area of the corresponding classical IH, must be a constant of motion, implying that the quantum theory properly leads to the most crucial property of the IH in the correspondence limit. So, we propose that the contribution to the Hamiltonian from a single puncture of a QIH carrying a spin-j representation must be of the form

$$\hat{H}_j \equiv \sum_{n=0}^{\Lambda} p_n \hat{A}_j^n \tag{5.9}$$

where the coefficients (p-s) carry the burden of endowing the Hamiltonian operator with the correct dimensionality and Λ is a cut-off. Hence, the Hamiltonian operator associated with the QIH with N punctures can be written as

$$\hat{H}_S \equiv \sum_{n=0}^{\Lambda} p_n \left(\hat{A}_{j_1}^n \otimes \hat{I}_{j_2} \otimes \cdots \otimes I_{j_N} + \hat{I}_{j_1} \otimes \hat{A}_{j_2}^n \otimes \cdots \otimes \hat{I}_{j_N} + \cdots + \hat{I}_{j_1} \otimes \hat{I}_{j_2} \otimes \cdots \otimes \hat{A}_{j_N}^n \right) \tag{5.10}$$

whose spectrum can be explicitly written as $\sum_{n=1}^{\Lambda} \sum_{l=1}^{N} p_n (8\pi\gamma\ell_p^2)^n [j_l(j_l+1)]^{n/2}$. Now, it is straightforward to see from expression (5.10) that $\left[\hat{H}_S, \hat{A}_S\right] \equiv \hat{0}$ which guarantees the fulfillment of the requirements that $\hat{H}_S$ and $\hat{A}_S$ have the simultaneous eigenstates which are that of the QIH Hilbert space and the expectation value of the area operator of a QIH is a constant of motion. If there is some evolution parameter ξ which parametrizes the QIH then it is evident that $\frac{d}{d\xi}\langle \hat{A}_S \rangle = \frac{i}{\hbar}\langle \left[\hat{H}_S, \hat{A}_S\right] \rangle = 0$ i.e. in the correspondence limit the classical IH has *constant area*[1, 2, 9, 10, 11].

5.2.4 Spectrum of the Hamiltonian operator

Now, from (5.9), the single puncture contribution to the energy spectrum can be written as

$$E_j = \ell_p \sum_{n=0}^{\Lambda} b_n {A_j}^n = \ell_p \sum_{n=0}^{\Lambda} a_n(\gamma) {C_j}^n \tag{5.11}$$

where $\mathrm{Dim}[b_n] = \ell_p^{-2n}$ and $a_n(\gamma) = b_n (8\pi\gamma\ell_p^2)^n$. It follows that the spectrum of a QIH with N punctures looks like

$$\hat{H}_S|j_1, \cdots, j_N\rangle = \ell_p \sum_{l=1}^{N} \sum_{n=0}^{\Lambda} a_n(\gamma) C_{j_l}{}^n |j_1, \cdots, j_N\rangle \tag{5.12}$$

The above form of the energy spectrum spectrum has been written just for the sake of clarity. We shall work in the spin configuration basis in what follows, as it will be convenient for our calculations. For simplicity, in our model, we consider n to take only integral values and $n \geq 0$ to ensure incremental monotonicity of the energy spectrum with the area contribution of a single puncture. Since this kind of model Hamiltonian or energy spectrum of the QIH has not been studied previously in literature and any property of such spectrum is hitherto unknown, to avoid any problem with the convergence of such spectrum we have used the cut-off parameter(Λ) *a priori*. We shall see that, at least from the thermodynamic viewpoint, we can assert that such a cut-off is indeed required and should emerge automatically from a true quantization of the horizon energy, if can be done anyhow. This is because, as we proceed, the parameter Λ will come out to be directly related to the equilibrium temperature of the QIH and one does not expect it to diverge for thermodynamically stable systems.

5.3 Compatibility with the Classical Results

5.3.1 Fixation of the model by matching with the classical results

It follows from eq.(5.11) that the energy eigenvalue of the QIH in a state designated by the spin configuration $\{s_j\}$ will be given by

$$\hat{H}_S|\{s_j\}\rangle = \ell_p \sum_{j} \sum_{n=0}^{\Lambda} a_n(\gamma) s_j C_j{}^n |\{s_j\}\rangle \tag{5.13}$$

It is further supported by the fact that the Hamiltonian and the area operators have the simultaneous eigenstates which follow from the fact that they commute. Since $|\{s_j\}\rangle$ is an eigenstate of the QIH, a generic quantum state of the QIH can be written as $|\Psi_S\rangle = \sum_{\{s_j\}} c_{\{s_j\}} |\{s_j\}\rangle$. Hence, the expectation value of the Hamiltonian operator or the mean energy for the QIH is given by

$$
\begin{aligned}
\langle \hat{H}_S \rangle &= \langle \Psi_S | \hat{H}_S | \Psi_S \rangle \\
&= \ell_p \sum_{\{s_j\}} \omega[\{s_j\}] \sum_j \sum_{n=0}^{\Lambda} a_n(\gamma) s_j {C_j}^n \\
&= \ell_p \sum_j \sum_{n=0}^{\Lambda} a_n(\gamma) s_j^\star {C_j}^n + \text{ sub-dominant contributions} \\
&= E^\star \pm \mathscr{O}(\ell_p) \qquad (5.14)
\end{aligned}
$$

where $\omega[\{s_j\}] = |c[\{s_j\}]|^2$ is the quantum mechanical probability of the QIH to be found in the state $|\{s_j\}\rangle$. Now, we can calculate $E^\star$ explicitly by using $s_j^\star$ given by eq.(3.20) in the following way

$$
\begin{aligned}
E^\star &\simeq \ell_p \sum_j \sum_{n=0}^{\Lambda} a_n(\gamma) s_j^\star {C_j}^n \\
&= \ell_p N \sum_j \sum_{n=0}^{\Lambda} a_n(\gamma)(2j+1){C_j}^n \exp(-\lambda C_j - \sigma) \\
&\simeq \ell_p N \exp(-\sigma) \sum_{n=0}^{\Lambda} a_n(\gamma) \int_{1/2}^{\infty} (2x+1)\,[x(x+1)]^{n/2} \exp(-\lambda\sqrt{x(x+1)})\; dx \\
&\qquad\qquad [\text{taking the limit } k \to \infty \text{ and replacing the sum over } j \text{ by integration over } x] \\
&= \ell_p N \exp(-\sigma) \sum_{n=0}^{\Lambda} a_n(\gamma) \frac{2}{\lambda^{n+2}} \int_{\frac{\lambda\sqrt{3}}{2}}^{\infty} y^{n+1} e^{-y} dy \\
&\qquad\qquad [\text{applying the change of variable } \lambda\sqrt{x(x+1)} = y\ ] \\
&= \ell_p N \exp(-\sigma) \sum_{n=0}^{\Lambda} a_n(\gamma) \frac{2}{\lambda^{n+2}}\, \Gamma\left(n+2\ ,\ \lambda\sqrt{3}/2\right) \\
&= \exp(-\sigma) F(\Lambda, \lambda, \gamma) \ell_p N \qquad (5.15)
\end{aligned}
$$

where $F(\Lambda, \lambda, \gamma) = \sum_{n=0}^{\Lambda} a_n(\gamma) \frac{2}{\lambda^{n+2}}\, \Gamma\left(n+2\ ,\ \lambda\sqrt{3}/2\right)$, which is obviously a positive definite function of λ. As we are interested in the classical limit and want to match the results with the first law derived from the classical theory in [9] devoid of any effect of quantum hair, we must evaluate the above result for $\sigma = 0, \lambda = 1.2 = \lambda_0$(say) and $\gamma = \lambda_0/2\pi = 0.191 = \gamma_0$(say) for the BHAL to follow. To check these values one has to go back to the second chapter on microcanonical entropy. Thus eq.(5.15), in the appropriate limit, reduces to

$$E_{IH} \simeq E^{\star} = F(\Lambda, \lambda_0, \gamma_0)\ell_p N \tag{5.16}$$

Hence an infinitesimal change in the energy of the IH can be written as

$$\delta E_{IH} \;\; = \;\; F(\Lambda, \lambda_0, \gamma_0)\ell_p \;\; \delta N \tag{5.17}$$

Similar to the above calculation one can also derive the expression for the mean area as follows

$$\begin{aligned}
A^{\star} &\simeq 8\pi\gamma\ell_p^2 \sum_j s_j^{\star} C_j \\
&= 8\pi\gamma\ell_p^2 N \sum_j (2j+1) C_j \exp(-\lambda C_j - \sigma) \\
&\simeq 8\pi\gamma\ell_p^2 N \exp(-\sigma) \int_{1/2}^{\infty} (2x+1)\,[x(x+1)]^{1/2} \exp(-\lambda\sqrt{x(x+1)})\; dx \\
&\qquad\qquad [\text{taking the limit } k \to \infty \text{ and replacing the sum over } j \text{ by integration over } x] \\
&= 8\pi\gamma\ell_p^2 N \exp(-\sigma) \frac{2}{\lambda^3} \int_{\frac{\lambda\sqrt{3}}{2}}^{\infty} y^2 e^{-y} dy \\
&\qquad\qquad [\text{applying the change of variable } \lambda\sqrt{x(x+1)} = y\ ] \\
&= \exp(-\sigma)\frac{16\pi\gamma}{\lambda^3}\,\Gamma\left(3, \lambda\sqrt{3}/2\right) \ell_p^2 N
\end{aligned} \tag{5.18}$$

Again taking the appropriate limit similar to the earlier calculation, eq.(5.18) reduces to

$$A_{IH} \simeq A^{\star} = \frac{16\pi\gamma_0}{\lambda_0^3}\,\Gamma\left(3, \lambda_0\sqrt{3}/2\right) \ell_p^2 N \tag{5.19}$$

Hence an infinitesimal change in the area of the IH can be written as

$$\delta A_{IH} \;\; = \;\; \frac{16\pi\gamma_0}{\lambda_0^3}\,\Gamma\left(3, \lambda_0\sqrt{3}/2\right) \ell_p^2 \;\; \delta N \tag{5.20}$$

Combining eq.(5.17) and eq.(5.20), and setting $\ell_p = 1$ henceforth, yields

$$\delta E_{IH} \;\; = \;\; \xi(\Lambda, \lambda_0, \gamma_0) \;\; \delta A_{IH} \tag{5.21}$$

where the quantity $\xi(\Lambda, \lambda_0, \gamma_0) = \frac{\lambda_0^3}{16\pi\gamma_0} \frac{F(\Lambda,\lambda_0,\gamma_0)}{\Gamma(3,\lambda\sqrt{3}/2)}$ can be explicitly written as

$$\xi(\Lambda, \lambda_0, \gamma_0) = \frac{\lambda_0^3}{16\pi\gamma_0 \Gamma(3, \lambda_0\sqrt{3}/2)} \sum_{n=0}^{\Lambda} a_n(\gamma_0) \frac{2}{\lambda_0^{n+2}} \Gamma\left(n+2\ ,\ \lambda_0\sqrt{3}/2\right) \tag{5.22}$$

Now, on identifying the eq.(5.21), with the first law of IH thermodynamics[9] and considering that the equilibrium temperature associated with the IH is given by $T_{IH} = \kappa_{IH}/2\pi$, we have

$$T_{IH} = \frac{1}{\Gamma(3, \sqrt{3}\pi\gamma_0)} \sum_{n=0}^{\Lambda} \frac{a_n(\gamma_0)}{(2\pi\gamma_0)^n} \Gamma\left(n+2\ ,\ \sqrt{3}\pi\gamma_0\right) \tag{5.23}$$

where the relation $\lambda_0 = 2\pi\gamma_0$ has been used.

5.3.2 Fixing the unknown coefficients

The expression for the temperature in eq.(5.23) is plagued with the unknown coefficients of the Hamiltonian operator which originates from the single puncture contributions. From a naive analogy with gas thermodynamics, where the results are independent of the single particle energy spectrum of the gas, we can expect, and argue as well, that the temperature of the IH must be independent of the single puncture energy spectrum. The form of the coefficients which will serve the purpose can be explicitly written as

$$a_n(\gamma_0) = \frac{\eta\Gamma(3, \sqrt{3}\pi\gamma_0)(2\pi\gamma_0)^n}{\Gamma\left(n+2\ ,\ \sqrt{3}\pi\gamma_0\right)} \tag{5.24}$$

where η is some unknown positive constant which needs to be fixed. Using these above coefficients the relevant quantities to calculate the temperature can be written down as

$$\begin{aligned} F(\Lambda, \gamma_0) &= \frac{\Gamma(3, \sqrt{3}\pi\gamma_0)}{2\pi^2\gamma_0^2}\eta(1+\Lambda) \\ \xi(\Lambda, \gamma_0) &= \frac{1}{4}\eta(1+\Lambda) \end{aligned}$$

and hence, the expression of the equilibrium temperature given by eq.(5.23) reduces to

$$T = \eta(1+\Lambda)\ell_p \tag{5.25}$$

The spectrum of the Hamiltonian operator takes the form as below :

$$\hat{H}_S|\{s_j\}\rangle = \eta\Gamma(3,\sqrt{3}\pi\gamma_0)\ell_p \sum_j \sum_{n=0}^{\Lambda} \frac{(2\pi\gamma_0)^n}{\Gamma\left(n+2\ ,\ \sqrt{3}\pi\gamma_0\right)}\ s_j C_j{}^n|\{s_j\}\rangle \tag{5.26}$$

It is evident from the expression for the temperature given by eq.(5.25) that the cut-off imposed on the sum over n is justified because the temperature diverges for $\Lambda \to \infty$. Hence, we can not model the single puncture energy spectrum with some series with infinite terms and the cut off (Λ) seems to be absolutely necessary.

5.3.3 Fixing the other parameters

After fixing the coefficients, we are left with two independent parameters η and Λ of the model. The fixation of these two is not unique because for every different choice of Λ, η can be suitably redefined. So, the case we investigate here is one specific choice.

The structure of the Hamiltonian operator implies Λ should be an integer. On the other hand, the theory of QIH requires k to be an integer and this is the only fundamental variable in the quantum CS theory. Hence we fix $(1+\Lambda) = k$. Therefore, the expression for the temperature reduces to $T_{IH} = \eta k = \eta A_{IH}/4\pi\gamma_0$.

Strictly speaking, so far we did not use any input from the classical theory and we have one more parameter left to be fixed i.e. η. This is the same sort of ambiguity that is also present in the derivation of the first law of IH thermodynamics in [9] which is related to the time evolution vector fields. For each and every choice of the time evolution vector fields there is a local first law of thermodynamics associated with the IH. This ambiguity is fixed by matching the definition of the classical energy of the IH with the one defined at asymptotic infinity for stationary bulk e.g. Schwarzschild or Reissner Nordstrom black hole, where the locally defined horizon mass is equal to the one defined at asymptotic infinity.

Here also we fix the remaining ambiguous parameter η, by matching the temperature with that of a known black hole, which can be considered as an input from the classical theory.

Here we consider the Schwarzschild black hole, for which the temperature associated with the horizon is $T_{Schw} = 1/8\pi M$, M being the mass associated with the horizon but defined at asymptotic infinity i.e. for a Schwarzschild black hole $E_{IH} = M$. Matching T_{Schw} with the T_{IH} and considering that $A_{IH} = 16\pi M^2$, it is obtained that $\eta = \gamma_0/32\pi M^3$. Expressing η in terms of k, which is the only macroscopic parameter in the QIH theory, the energy spectrum of a QIH is given by

$$\hat{H}_S|\{s_j\}\rangle = \frac{k^{-3/2}\gamma_0^{-1/2}}{4\pi}\sum_{j=1/2}^{k/2}\sum_{n=0}^{k-1}\frac{s_j\Gamma(3,\sqrt{3}\pi\gamma_0)(2\pi\gamma_0 C_j)^n}{\Gamma\left(n+2\ ,\ \sqrt{3}\pi\gamma_0\right)}|\{s_j\}\rangle$$

The spectrum of the Hamiltonian is bounded both below and above due to the bounds on spins and the bound on the series sum of the contribution from individual punctures, as can be seen trivially from the expression.

5.4 Discussion

The thermodynamics associated with the kind of energy spectrum of a QIH considered here, has never been studied earlier in literature. Generally, one considers the horizon energy as a function of the horizon area (e.g. power law). This intuition works in our mind due to our instinctive affinity to look at a quantum theory through the classical spectacles. To be more explicit, in numerous cases of the study of black holes the mass formula for known black hole solutions are expressed in terms of the area and addressed as the mass spectrum of the black hole [57, 65, 83, 66, 84, 12]. In fact, many a times, in such formulae, the area spectrum of LQG is used directly in the classical formula and the mass of the black hole is considered to be quantized [65, 66, 90, 91] which is of course not a true quantization of the horizon energy and also devoid of any physical justification, apart from being an ad hoc assumption. A genuine energy spectrum for a QIH should be derived by quantization of the classical notion of horizon energy similar to the quantization of area, volume and length resulting in the corresponding operators in quantum gravity [88, 81, 85, 86, 87]. The other alternative is to propose one, based on solid physical arguments, knowledge of the fundamental structures in the quantum theory and to show that the known results follow in the classical limit, which has been done here.

Now, from the perspective of thermodynamics, it is worth mentioning that unlike [82, 22, 23] we can now deal with usual canonical *energy* ensemble as we have an explicit structure of the Hamiltonian. We need not use a 'Boltzmann-like' factor $e^{-\alpha A}$ in the canonical partition function[82], accompanied by a fictitious conjugate parameter α, alongside the Boltzmann factor $e^{-\beta E}$. As far as [22, 23] are concerned, it is a pure area ensemble involving only $e^{-\alpha A}$ in the canonical partition function and devoid of the Boltzmann factor $e^{-\beta E}$. All of these approaches are significant and interesting by their own virtue. But none of them actually attacks the problem of black hole horizon thermodynamics following usual canonical energy ensemble due to the lack of knowledge of the Hamiltonian and the energy spectrum associated with the horizon. This is where the use of the proposed model Hamiltonian reap the benefits and allow us to follow usual canonical *energy* ensemble approach to thermodynamic analysis of QIHs .

We shall end with a few remarks on earlier works on energy spectrum of equilibrium black holes and how this present work improves upon the earlier ones in spite of being only a model. The approach taken in this paper to find the energy spectrum of black holes may look very awkward and it is an obvious question to ask that what is the problem if we take the energy-area relation of a horizon and quantize that. We shall discuss some issues regarding this straightforward approach and try to answer this question which will justify the reason to pursue this upside down approach in this work against the method of quantizing the classical expression for the horizon energy which had been attempted earlier in [90, 91] and most recently within the LQG framework in [99].

As far as the energy spectrum of black holes discussed in [90, 91] are concerned, these were only heuristic proposals made on the ground that a moderately satisfactory theory of quantum gravity or more appropriately a quantum theory for black horizon was absent at that time. They were based mainly on the ansatz that the horizon area is linearly quantized and the quantization of the mass follows trivially. Hence, those works should also be considered as some sort of model rather than a true quantization of the black hole mass from an underlying quantum theory.

However, the quantization of the black hole horizon energy carried out in [99] within the LQG framework needs to be discussed in the present context. For keeping things simple we shall not consider charge and angular momentum and carry on our analysis with Schwarzschild balck hole horizon whose mass-area relation goes as $M = \frac{1}{4\sqrt{\pi}}\sqrt{A}$. There is a very fundamental issue for which the quantization of black hole horizon energy shown in [99] is inconsistent with the quantum theory of IH [13, 14]. Actually, the operator $\hat{M}$ can not act on the quantum states of the horizon unless it has the form given by eq.(5.8). In fact this is true for any operator belonging to the Hilbert space of the IH and this is simply because of the many body, Fock-space-like structure of the Hilbert space of the IH. Due to the same reason, eventhough the classical energy is a function of area of the horizon, the commutativity of the operator $\hat{M}$ with the area operator is not manifest. As long as the operator corresponding to the energy spectrum of the black hole horizon is shown to be consistent with the available and well established quantum theory of geometry of IH, the candidature of the operator as a true Hamiltonian operator for the horizon, remains in jeopardy.

Note : It should be noted that in recent years there have been some investigations regarding certain kind of operators in loop quantum gravity framework which preserve the quantum area of a two surface[100]. However, those operators are of least relevance in the present context which is solely related to the case of black hole horizons. For an operator to be a candidate Hamiltonian for the isolated horizon Hilbert space, it is not sufficient to commute with the area operator. There are certain other necessary properties which that operator has to possess and these have been elaborately explained in this work. The most important thing is to check that whether the eigenvalue spectrum of the operator is compatible with the first law of of isolated horizon mechanics. In case of these recent works[100], these particular aspects of those area preserving operators have not been investigated and hence it remains a question whether those are at all relevant in the context of black hole physics within the loop quantum gravity framework. However, it would be an interesting problem for future to look upon, whether the operators investigated in [100] can be really considered as a Hamiltonian operator for black hole horizons.

Chapter 6

Outlook and Future Directions

Before going on to discuss some potential future problems, I would like to put forward my own view about the work presented in this thesis in brief. As far as the calculation of microcanonical entropy of black hole in LQG is concerned, it appears to me that the issue of fixation of N to define the microcanonical ensemble has been a retrogression instead of a progress. Although, in this thesis, it has been shown that the relevant theoretical calculations in the modified paradigm can be done completely within the Chern-Simons framework, but the physical motivation to fix N a priori to define the ensemble and to regard it as a macroscopic variable still remains unknown. Over and above there are strong reaons to demotivate the idea of fixing N, which can be explained as follows. As opposed to the classical macroscopic variables of the theory like area, charge, angular momentum, etc. which are calculable from the classical theory, the variable N is a quantization artifact. It originates from the method of point splitting regularization of the area operator defined over a two surface[47]. There is no way to calculate N. This is also understandable when we look at the Hilbert space of the IH which admits arbitrary N. Further it should be noted that the IH framework is locally defined and it is the strength of this framework that the local microscopic degrees of freedom are well defined covariantly (observer independent). So whatever observer measures the entropy, it should be the same. Hence, it defies the framework itself if someone proposes that the fixation of N is an observer dependent phenomenon (e.g. [15]). I shall prefer to work with arbitrary N, as was the case before, in whatever I shall do in future regarding QIHs. This concludes my viewpoint towards the issue of considering N as a macroscopic variable. Now, I shall proceed towards discussing some potential future research problems which are relevant in the present context.

The problems of quantum gravity are intimately related to the problems of black hole physics because black holes are the physical objects understanding which, a theory of quantum gravity becomes most demanding. This is one of the prime reasons why researchers pursue their queries in this direction. The work of this thesis is a little addition to the extant literature. There is so much to be done.

One of the major problems of the theory of loop quantum gravity is the semi-classical limit of the theory where the smooth background spacetime geometry will emerge and this is believed to be provided by the discovery of the proper coherent states of the theory. Although there have been some efforts in this direction, but there has not been much success. However, in the context of black holes, there has been hardly any attempt made till date to find the coherent states of isolated horizon. Considering the progress in this field of research, this is a problem which needs an immediate attention. As we know that there is an SU(2) Chern-Simons theory on the isolated horizon, a naive attempt will be to look at the coherent state quantization of SU(2) Chern-Simons theory. This will definitely be a step forward in understanding the issue of black hole entropy.

Another interesting and potential future research problem is to look for the generalization of the Chern-Simons field equations of motion on an isolated horizon in case of a dynamical horizon. Solving this problem, one will definitely be able to have a better understanding of a dynamical black hole which is either growing or shrinking due to intake of matter or due to Hawking radiation respectively. Quantization of the field equations will be the next step further.

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