STUDY OF NONLINEAR WAVES IN OCEANIC AND OTHER PHYSICAL SYSTEMS

By

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Abhik Mukherjee

List of Publications arising from the thesis

Journal

1. **" Modelling rogue waves through exact dynamical lump soliton controlled by ocean currents"**,

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2. **"Phase-modulated solitary waves controlled by a boundary condition at the bottom"**,

   A. Mukherjee and M.S Janaki, *Phys.Rev.E*, **2014**, *89*, 062903-1 - 0627-8

3. **"Quantum corrections to nonlinear ion acoustic wave with Landau damping"**,

   A. Mukherjee, A. Bose and M.S Janaki, *Phys.Plasmas*, **2014**, *21*, 072303-1 - 072303-6

4. **"A new (2+1) dimensional integrable evolution equation for an ion acoustic wave in a magnetized plasma"**,

   A. Mukherjee, M. S. Janaki and A. Kundu , *Phys.Plasmas*, **2015**, *22*, 072302-1 - 072302-11

5. **"Bending of solitons in weak and slowly varying inhomogeneous plasma "**,

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Synopsis

Near shore surging waves like tsunami, bore waves etc have hazardous nature when it propagates towards the shore. In some cases it affects coastal habitation or costly installations. Such events which can often trigger extremely hazardous effects have attracted intense attention over centuries and have been studied extensively from both theoretical and practical points of view. The main emphasis of the investigations was to work towards the development of early warning systems for minimizing the the loss of human lives. However, there are few situations where the installation of a passive warning system is not enough, while the demand is for more active intervention. This is particularly true for example, in protecting nuclear reactors and related installations which are located usually at the vicinity of the sea shore due to logistic reasons, against the tsunami threat. Therefore along with the traditional warning systems, it is desirable to find ways and means geared towards possible invasive procedures for taming of such hazardous wave phenomena. There are few suggestions for effective interventions, like plantation of Mangrove treas along the coastal lines , installation of breakwaters at strategic positions , stoppage of erosion by concrete bolders . However, these are mostly indirect ways to counter the surging waves. Ways to directly attack the problem has received much less attention. We have proposed in this thesis work, a direct method to control the near shore surging waves, which could reduce the hazardous effects. A leakage based method is proposed here, which would suck water at the bottom causing the properties of solitary waves to change. It has been shown analytically that if the vertical fluid velocity at the bottom is independent of the free surface wave profile then the phase of the solitary wave gets changed controlled by the bottom boundary condition, whereas the amplitude remaining constant. But if the leakage function depends on the surface wave profile then a variable KdV equation can be derived for slowly changing bathymetry function. An exact solitary wave solution is obtained for a finer balance between the variable depth function and leakage function .The amplitude of the solution decreases as it propagates towards the shore in spite of the surging nature due to decreasing bathymetry. This constitutes the first part of my thesis work.

In the second part of the thesis the deep water rogue wave phenomena has been modelled. The most popular nonlinear model for rogue waves is the Peregrine breather (PB) , which is an exact rational solution of Nonlinear Schrodinger equation(NLSE). But the most problematic point is that NLSE together with its different generalizations are equations in (1+1) dimensions and therefore all or their solutions, including the PB and its higher order generalizations can describe the time evolution of a wave only along a one-dimensional line. It can also be seen that the maximum amplitude of the PB solution describing a one-dimensional rogue wave is fixed and just three times that of the background waves. The steepness of the wave as well as the fastness of its appearance is also fixed, as this solution admits no free parameter. Existence of rogue waves have been analyzed in the Davey-Stewartson equation, which is an (2+1) dimensional nonlocal generalization of NLSE. However the single peak solution is the line rogue wave, hence it can be reduced to the PB solution by a simple rotation in the two dimensional plane. In such a situation, an analytical (2+1) dimensional rogue wave model is desperately needed where all of its dynamics i.e, higher amplitude, fastness of appearance, duration of stay, steepness etc can be successfully modelled. In this context, a new completely integrable, (2+1) dimensional, local, modulated, nonlinear, evolution, equation has been proposed which is derivable from the basic hydrodynamic equations. The most important part is that this equation possesses an exact two dimensional rational solution which is analogous with the Peregrine breather in terms of mathematical form. Also it contains two free parameters which can be chosen arbitrarily to model the full grown rogue wave. As as extension to the problem, we have also derived this new, (2+1)D, evolution equation in the propagation of nonlinear ion acoustic wave of lossless, magnetized plasma containing cold ions and hot isothermal electrons.

As an applications to these integrable models, which have been discussed, we explored different physical systems like inhomogeneous plasma, quantum plasma and derived various interesting results.

On a calm day, it is an enchanting sight to see the ocean waves continuously falling on the beach. Occasionally, the ocean reveals its fury, when, where and how, nobody knows, leading to devastation, loss of lives and destruction of property; the beauty of the ocean transforming into a tragedy. Scientists are always trying to fathom the reasons for such phenomena, the mysteries of nature are profound, and the scope to understand them is limitless. The motivation of this thesis is rooted in the author's passionate desire to contribute in this direction, a modicum of his own concepts perhaps, a drop in the ocean of knowledge, with the hope that such drops make up the ocean.

Chapter 1

Introduction

1.1 General Introduction

The propagation of water waves has fascinated laymen and scientists for centuries. Water waves come in a seemingly endless array of forms, shaped by ever-changing influences (e.g. the topography of the sea floor, the speed and direction of the wind, and the presence of an underlying current) and yet they are to some extent mathematically predictable.

The first successful water wave theories, discovered at the beginning of the nineteenth century by Cauchy and Laplace , were linear and dispersive and solvable by the ordinary, linear Fourier transform . Thereafter linear theories have dominated in explaining hydrodynamic and other fluid phenomena in different physical circumstances. However the difficulties faced by the linear models in explaining phenomena related to large amplitude waves, wave- particle, wave-wave interactions etc also concerned the scientific community and the importance of nonlinear theories has emerged. But it is evident that, as the nonlinearity of the partial differential equations becomes higher and complex, its exact solution becomes difficult giving only approximate analytical or numerical solutions. There lies the importance of nonlinear integrable systems as they have the generalized mathematical method to solve themselves. The origin of the nonlinear integrable equation came into picture long ago when John Scott Russell, a Scottish engineer and naval architect, observed a solitary wave in the Union Canal connecting two cities, Edinburgh and Glasgow . He identified a large solitary heap of water traveling in the shallow water canal with undiminished speed or shape [1].

In 1895, the Dutch professor Diederik Korteweg and his doctoral student Gustav de Vries derived a partial differential equation (PDE) which models the solitary wave that Russell had observed [2]. But, the solitary wave was considered a relatively unimportant curiosity in the field of nonlinear waves. That all changed in 1965, when Zabusky and Kruskal realized that the KdV equation arises as the continuum limit of a one dimensional anharmonic lattice used by Fermi, Pasta, and Ulam (1955) to investigate thermalization or how energy is distributed among the many possible oscillations in the lattice [3]. Zabusky and Kruskal (1965) simulated the collision of solitary waves in a nonlinear crystal lattice and observed that they retain their shapes and speed after collision. In analogy with colliding particles, they coined the word solitons to describe these elastically colliding waves. A new method called, inverse scattering method, was discovered by Gardner, Greene, Kruskal and Miura (1967) [4] as a means for solving the initial value problem for the KdV equation on the infinite line, for initial values that decay sufficiently rapidly at infinity. Subsequently this method has been significantly enhanced and extended to other integrable systems, which have been discovered in both (1+1) and (2+1) dimensions. The applicability of these models are expanded from hydrodynamics to other subjects like plasma physics, condensed matter physics, atomic and molecular physics, astrophysics etc. Though the study of nonlinear waves in oceanic systems has been done extensively both in shallow as well as deep water regime, there are certain problems which need to be addressed. We shall discuss these problems of oceanic wave phenomena in the following sections.

1.2 Shallow water wave propagation

A surface wave is said to be in shallow water if its wavelength is much larger than the local water depth. The KdV equation, which is a nonlinear dispersive wave equation, was originally derived to describe shallow water waves of long wavelength and small amplitude [2]. The nonlinear steepening of the water wave can be balanced by dispersion. If so, the result of these counteracting effects is a stable solitary wave with particle-like properties. A solitary wave has a finite amplitude and propagates at constant speed and without change in shape over a fairly long distance. But when the bottom topography varies, the amplitude ,speed, width of the solitary wave also changes. Specially when the wave moves towards the near shore coastal region, the water depth decreases slowly reaching zero magnitude and the amplitude of the solitary wave rises to give surging effects. In certain conditions such surging waves show destructing nature at the shore like tsunami waves, bore waves [5] But one of the important facts is that KdV dynamics is relevant in modelling tsunami only when the propagation distance is large enough, otherwise higher order effects would not be relevant. There are several equations which are also relevant in shallow water limit, analogous to KdV equation like modified KdV equation, Kadomtsev Petviashvili equation which is the two dimensional extension of KdV equation with weak transverse propagation, Boussinesq equation in both (1+ 1) and (2+1) dimension , Camassa Holm equation, Gardener equation, (2+1) Gardener equation, which is a relatively new shallow water wave equation characterized by enhanced nonlinearity and improved directional spreading , KP-Gardener equation etc. Each equations can be derived as an approximate model of the evolution of long waves of moderate amplitude propagating in shallow water of uniform depth and each of of them possesses a rich mathematical structure called complete integrability.

But in case of propagation in variable environment, certain changes to be made in the structure of the solution as well as in the evolution equation in the shallow water limit.

For example in case of propagation in shear flow or in slowly variable bathymetry, KdV equation with some modification terms can be derived.

If the variation of bathymetry occurs very slowly compared to the evolution scale of the solitary wave a modified KdV equation with variable coefficient was derived by Johnson[4]. Its approximate solitary wave solution shows that its amplitude varies inversely with the water depth. Hence as the wave propagates towards the shallower region, its depth decreases causing its amplitude to increase giving hazardous effects. Such surging wave events have attracted intense attention over centuries and have been studied extensively from both theoretical and practical points of view. The main emphasis of the investigations was to work towards the development of early warning systems for minimizing the the loss of human lives. The present development of the tsunami warning system has definitely been reached to a satisfactory level [6]-[11].

Therefore along with the traditional warning systems, it is desirable to find ways and means geared towards possible invasive procedures for taming of such hazardous wave phenomena. There are few suggestions for effective interventions, like plantation of Mangrove treas along the coastal lines [12], installation of breakwaters at strategic positions [13]-[16], stoppage of erosion by concrete bolders . However, these are mostly indirect ways to counter the surging waves, while we lack the proposals for directly attacking the problem, perhaps with the exception of the proposed bubble method, aiming to stop the incoming waves by a stream of fast and strong counter-waves, mixed with air bubbles [17]. Though the last method was proposed more than fifty years back, its feasibility and effectiveness has not been established yet. The attenuation of incident water waves by a curved vane like structure positioned beneath or at the surface of a body of water is described in a Patent [18] where the detailed design of the structure is given. An attempt was made to reduce the devastating effects of a tsunami waves by single and double submerged barrier was done in Tel Aviv University[19]. They performed their experiments in a basin 5 m in length and 10.5 cm in depth. The wavelength of the generated wave was about 3 m, which allows referring to it as a tsunami.

Thus the effective control of near shore hazardous surging wave phenomena is still an open problem in terms of theoretical and practical points of view. In recent time, its importance as well as necessity cannot be denied. For example, in protecting nuclear reactors and related installations which are located usually at the vicinity of the sea shore due to logistic reasons, against the tsunami threat.

In this thesis work, we have tried to work on this problem which will be discussed later.

1.3 Deep water wave propagation

In deep water, the nonlinear waves behave quite differently. For example, tsunami wave in deep water is quite small wave of amplitude ≈ 1 meter but as it propagates towards the shore it evolves completely to give hazardous surging effects. In deep water, where the ratio of water depth to the wavelength can no longer be taken small, Nonlinear Schrodinger equation (NLSE) describes the propagation of modulated wave packet in (1+1) dimension. Although, NLSE can be extrapolated to be valid at an arbitrary depth. It is a completely integrable system having rich mathematical structure. A two dimensional generalization of NLSE is Davey Stewartson (DS) equation [20] which in the infinite depth limit becomes NLSE in (2+1) dimensions and non-integrable. In shallow-water limit, DS equation is a coupled nonlocal equation having a localized solution called Dromion.

One of the most popular deep wave phenomena in the recent times is rogue wave. These mysterious ocean waves are reported to be observed in a calm sea, where they, as a localized and isolated surface waves, apparently appear from nowhere, make a sudden hole in the sea just before attaining surprisingly high amplitude and disappear again without a trace [21]. The concept of rogue waves is not confined only in the oceanography alone but also extended in the other branches of physics like condensed matter physics, plasma physics, astrophysics etc. Though it is mainly a deep wave phenomena but it can occur also in the shallow water [22].

As linear models of rogue waves the following mechanisms are mainly considered: dispersion enhancement of transient wave groups, geometrical focusing in basins of variable depth, and wave-current interaction[21]. Taking into account nonlinearity of the water waves, these mechanisms remain valid but should be modified.

The most popular nonlinear model for rogue waves is the Peregrine breather (PB) [23], which is an exact rational solution of Nonlinear Schrodinger equation. The solution is localized in both time and space, hence have the relevance with the oceanic rogue wave phenomena. But the most problematic point is that NLSE together with its different generalizations are equations in (1+1) dimensions and therefore all or their solutions, including the PB and its higher order generalizations can describe the time evolution of a wave only along a one-dimensional line. It can also be seen that the maximum amplitude of the PB solution describing a one-dimensional rogue wave is fixed and just three times that of the background waves. The steepness of the wave as well as the fastness of its appearance is also fixed, as this solution admits no free parameter. This situation can be improved to obtain higher amplitude and steepness of the PB model by using higher order rational solutions [24]. But the maximum amplitude and steepness reachable by this class of solutions are fixed owing to the absence of relevant free tunable parameter, making it difficult to adjust to the continuously varied range of shape and sizes of the observed oceanic rogue waves. However, recently higher order rational solutions to the NLSE allowing free parameters have been discovered[25] though they seem to represent multi-peak waves in the x-t plane for the nontrivial choice of parameters [25]. The single peak solution, which is suitable for describing rogue waves having a single appearance in time, is obtained unfortunately for a trivial choice of free parameters. The trigonometric breathers also contain free parameters, though such periodic solutions, as mentioned already are different in nature from the single crest RW event. The crucial fact however is that the one dimensional spatial nature remains the same for the whole class of PB solutions, including its higher order rational and trigonometric generalizations. Therefore, modelling of RW, which is a two dimensional surface wave, by these class of one dimensional PB solutions remains problematic.

Existence of rogue waves have been analyzed in the Davey-Stewarson equation, which is an (2+1) dimensional nonlocal generalization of NLSE. However the single peak solution is the line rogue wave, hence it can be reduced to the PB solution by a simple rotation in the two dimensional plane. The Boiti-Leon-Pempinelli equation is another (2+1) dimensional integrable equation, defined through two real coupled equations. Recently a RW type solution has been found in this equation allowing free parameters [26]. However the BPL equation describes wave propagation along a channel, its applicability in modelling the ocean RW is questionable.

In such a not so clear situation, an analytical (2+1) dimensional rogue wave model is desperately needed where all of its dynamics i.e, higher amplitude, fastness of appearance, duration of stay, steepness etc can be successfully modelled. This is an open problem in recent times which has been dealt in this thesis work.

1.4 Motivation and Methods used

Motivated by the nature of the problems, which are discussed in the previous sections, we have oriented the thesis work in two different directions.

In the first part of the thesis , we have dealt with shallow water wave phenomena. The main aim was to control the near shore surging waves by an active method, which could reduce the hazardous effects. A leakage based method is proposed here, which would suck water at the bottom causing the properties of solitary waves to change. It has been shown analytically that if the vertical fluid velocity at the bottom is independent of the free surface wave profile then the phase of the solitary wave gets changed controlled by the bottom boundary condition, whereas the amplitude remaining constant. But if the leakage function depends on the surface wave profile then a variable KdV equation can be derived for slowly changing bathymetry function. An exact solitary wave solution is obtained for a finer balance between the variable depth function and leakage function .The amplitude of the solution decreases as it propagates towards the shore in spite of the surging nature due to decreasing bathymetry.

In the second part of the thesis the deep water rogue wave phenomena has been modelled. In this context, a new completely integrable, (2+1) dimensional, nonlinear, local, modulated equation has been proposed which is derivable from the basic hydrodynamic equations. The most important part is that this equation possesses an exact two dimensional rational solution which is analogous with the Peregrine breather in terms of mathematical form. Also it contains two free parameters which can be chosen arbitrarily to model the full grown rogue wave. As as extension to the problem, we have also derived this new, (2+1)D, evolution equation in the propagation of nonlinear ion acoustic wave of lossless, magnetized plasma containing cold ions and hot isothermal electrons. We have applied the wave models used here to the other physical systems like inhomogeneous plasma, quantum plasma to retrieve interesting features.

In the whole thesis work some analytical methods have been used and the other numerical calculations have been done using the software Mathematica.

In deriving nonlinear evolution equations in various situations we have used multi -scale reductive perturbation technique [4].

In order to solve such equations and find exact solitary wave solutions we have used Hirota Bilinearization method [27].

For finding approximate solitary wave solution, we have used Bogoliubov Mitropolsky approximation method which is a multi-scale method to solve perturbed equations.

All of the variables in each chapter has its own meaning and do not coincide with the variables of the other chapters.

1.5 Summary

We can summarize our thesis in the following way.

In chapter 2, we have studied the propagation of shallow water, unidirectional nonlinear wave in constant depth with nontrivial bottom boundary conditions. We have shown analytically that for the choice of leakage velocity functions which are independent of the free surface wave profile, the solitary wave solution gets modified in phase where as the amplitude remains constant. Analytic solutions have been found out for different functional forms of the leakage velocities where we get bending of solitons in x-t plane.

In order to explore the effect of bottom boundary condition on the solitary wave amplitude which was absent in the chapter 2, the leakage functions are assumed to depend on the free surface wave profile in chapter 3. First the constant depth problem was investigated to identify the profile of leakage function which would induce maximum damping effects on the solitary wave amplitude. Taking this profile, the variable depth problem was studied where a variable KdV equation was derived where the bathymetry function varies slowly. For a finer balance between their Depth function and the leakage velocity function, an exact solitary wave solutions have been found out which decays as it propagates towards the shore in spite of the surging effects due to decreasing bathymetry. An application to this theoretical findings have been done to a real near shore bathymetry in Chennai, South India and the decay of amplitude due to leakage have been shown.

In chapter 4, the deep water rogue wave has been modelled using a new completely integrable,nonlinear, (2+1) dimensional equation, proposed by us and derivable from the basic hydrodynamic equations. An exact rational solution have been found out having two free parameters to model the full grown rogue wave. By the tunable free parameters the maximum amplitude of the rogue wave, steepness, position of holes can be determined. In order to explain its dynamical behavior an ocean current term has been introduced which will control the duration of staying of the rogue wave. Modulation instability associated with the new evolution equation has been found out showing asymmetric nature and directional preference.

In chapter 5, the new (2+1) dimensional integrable equation, which was introduced in the previous chapter, has been derived in the propagation of nonlinear ion acoustic waves in magnetized lossless plasma containing cold ions and hot isothermal electrons. A relation of the equation with the integrable KP equation have been established. Higher soliton solutions have been found out using Hirota method.

These above chapters contain the main part of the thesis. As an application of the integrable models used here we have explored other fluid systems like inhomogeneous plasma and quantum plasma.

In chapter 6 ,the propagation of ion acoustic soliton in weak and slowly varying inhomogeneous plasma has been studied. It has been shown that the dynamics of the nonlinear ion acoustic wave is controlled by KP equation. The two dimensional soliton of the evolution equation gets bend in the two dimensional plane controlled by the unperturbed ion number density, whereas the amplitude remains constant.

In chapter 7, a new field called quantum plasma has been explored using KdV model. A quantum corrections has been done in the semi-classical limit to the nonlinear ion acoustic wave with electron Landau damping. A new higher order KdV equation has been derived containing nonlinear quantum correction terms and the quantum correction to the Landau damping. Using Bogoliubov Mitropolsky approximation method the decay of amplitude due to the Landau damping term has been calculated.

Conclusions and outlook are given in chapter 8.

Chapter 2

Phase modulation of solitary waves controlled by bottom boundary conditions

2.1 Introduction

Near shore coastal regions often witness surging of the approaching waves including extreme events like tsunamis [5]. Such a natural phenomenon has also been observed, though in a miniature scale in few rivers around the world as bore waves [28, 29, 30]. Famous examples are the river Seine [28, 5] in France and the river Hoogli in India [28]. Such surging waves are suspected to be caused by nonlinear gravity waves, propagating over a decreasing depth bathymetry towards the shore or along upstream river. Such events which can often trigger extremely hazardous effects have attracted intense attention over centuries and have been studied extensively from both theoretical and practical points of view. The main emphasis of the investigations was to work towards the development of early warning systems for minimizing the loss of human lives. The present development of the tsunami warning system has definitely reached to a satisfactory level[6]-[11].

However, there are few situations where the installation of a passive warning system is not enough, and the demand is for more active intervention. This is particularly true for example, in protecting nuclear reactors and related installations which are located usually at the vicinity of the sea shore due to logistic reasons, against the tsunami threat. As we know the tsunami of 2004 which played the devastating effects spreading over many countries was a potential threat to the nuclear reactor at Kalpakkam of India. The tsunami of 2010 inflected real calamities in Fukushima nuclear reactors in Japan [31, 32].

Our main aim in this chapter is to put forward an innovative proposal based on theoretical study on the effect of sudden feedback boundary control at the bottom on the nonlinear surface waves, governed by nonlinear equations describing unidirectional gravitational waves, derived from basic hydrodynamic equations at the shallow water regime. For the sake of simplicity, we shall consider here the propagation of nonlinear waves over shallow water of constant depth.

The dynamics of the shallow water nonlinear, unidirectional, dispersive, gravity induced surface waves is described by the celebrated KdV equation[2] that admits solitary wave solutions. The derivation of such an equation assumes that the fluid is incompressible and inviscid, bounded below by a rigid, impermeable bottom and above by a free surface. The generalization of the KdV equation to higher order nonlinearities[33] and multi-dimensions[34] lead to a multitude of nonlinear equations that found potential applications[35] in various physical situations.

In this work, we encounter a series of forced KdV equations as the evolution equations of shallow water, nonlinear dispersive waves over non-trivial bottom boundary conditions. The different functional nature of this fixed bottom condition self-consistently generates different types of forced KdV equations. These types of nontrivial bottom boundary conditions can be considered to be generated artificially by a controlled feedback water leakage at the bottom.

In most of the realistic situations, water waves propagate over a porous bed so that one needs to consider the transformation of the waves brought about by the permeability of the bottom bed.

Mei[41] has developed several theoretical concepts needed to pursue the problems of wave induced stresses in a porous media using the boundary layer approximation that facilitates even the nonlinear modelling of seabeds. The dynamics of the linear water waves in a channel of permeable bottom has been one of the interesting research problems in water wave theory undertaken from the early times[42]-[43]. Rigorous development of mathematical models[44]-[45] for nonlinear, diffusive, weakly dispersive water waves interacting with a permeable bottom has begun only in the last decade, with the description based on the Boussinesq approximation. In shallow water, Boussinesq equation gives wave solutions propagating in both positive and negative directions. However, for unidirectional wave propagation in shallow water, the KdV equation appears as a reasonable dynamical equation when the vertical fluid velocity at the bottom is assumed to be zero.

In this work, we consider the non zero vertical fluid velocity at the bottom that leads to a series of forced KdV equations self-consistently where the functional forms of the leakage velocity appears as forcing function. But the leakage velocity at the bottom is not due to porosity, which occurs naturally. It is a controlled feedback leakage at the bottom which affects the solitary wave solution at the surface. The basic features of analysis of this work run parallel to the derivation of KdV equation[46] in a hard bottom channel. The novelty of the present work is that exact solitary wave solutions of the forced KdV equations have been obtained analytically for different leakage conditions. For example, constant, time dependent, space dependent or both space-time dependent forms of leakage velocity have been considered that control the phase modulation of the obtained solitary wave solutions leading to different types of dynamical behavior of such waves.

2.2 Derivation of the free surface evolution equation in presence of water leakage at the bottom

A one dimensional, unidirectional, surface wave motion propagating through a shallow water channel with bottom leakage is considered. The channel is of uniform cross section and of constant depth h. The fluid is assumed to be incompressible with the wavelength, amplitude and velocity of the wave represented by l, a and v respectively (as shown in Figure 2.1). The surface tension and viscosity have been neglected throughout this calculation. At an arbitrary (x, t) the free surface displacement is denoted by $\eta(x, t)$. Two natural small parameters $\epsilon = a/h$ and $\delta = h/l$ are introduced, both of which are very very less than 1 , and further $\epsilon \approx \delta^2$.

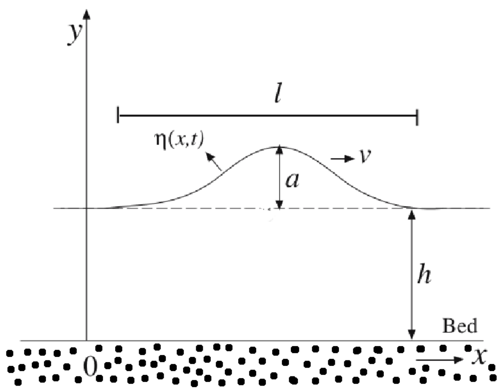


Figure 2.1: Shallow water solitary wave in a water channel with controlled leakage at the bottom

The fluid motion can be described by the velocity vector $\vec{V} = V_h\vec{i} + V_v\vec{j}$ where the subscripts h, v denote horizontal and vertical components of the fluid velocity respectively. From the condition of irrotational flow of the fluid we can introduce a velocity potential $\phi(x, y, t)$ such that $\vec{V} = \vec{\nabla}\phi$.

Since the fluid is incompressible, the mass conservation equation leads to

$$\nabla^2\phi = 0 \tag{2.1}$$

Again from the momentum conservation equation, i.e, Euler equation, we get,

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{p}{\rho} + gy = 0, \tag{2.2}$$

where ρ, p, g are density, pressure of the fluid and acceleration due to gravity respectively. Eq. (2.2) is well known as the Lagrange equation. When ϕ is independent of time, then the above equation is called Bernoulli's condition. Equations (2.1), (2.2) are the two main equations of the problem which must be supplemented by appropriate boundary conditions.

The fluid is bounded by two surfaces, one is the fixed bottom and other is the free boundary. Since at the upper free surface, $p = 0$, hence taking derivative of eq. (2.2) along the direction of propagation, we obtain

$$\frac{\partial V_h}{\partial t} + V_h \frac{\partial V_h}{\partial x} + V_v \frac{\partial V_v}{\partial x} + g\frac{\partial \eta}{\partial x} = 0 \tag{2.3}$$

Again, at the free surface

$$y(x,t) = h + \eta(x,t) \tag{2.4}$$

Taking time derivative of eq.(2.4), we get

$$V_v = \frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x} V_h \tag{2.5}$$

. These two equations (2.3) and (2.5) are defined at the free surface of the wave. Since the upper free surface is movable, hence these equations are called variable boundary conditions.

Since some amount of water is considered to be leaking through the fixed bottom of the channel, hence the downward vertical fluid velocity at the bottom is nonzero. This constitutes the fixed boundary condition defined by the equation

$$V_v(x,0,t) = \frac{\partial \phi(x,0,t)}{\partial y} = C(x,t) \tag{2.6}$$

where $C(x,t)$ is the vertical fluid velocity at the bottom of the channel. This equation is called the penetration condition. Here we consider the leakage velocity $C(x,t)$ at the bottom to be independent of the surface wave profile $\eta(x,t)$.

Thus ultimately we get two equations (2.1), (2.2) that are valid in the bulk of the fluid. Taking the derivative of eq. (2.2) and eq. (2.4) at the free boundary, we get two nonlinear boundary conditions (2.3), (2.5) respectively and the penetration condition given by (2.6).

The velocity potential ϕ is expanded in Taylor series as follows

$$\phi(x,y,t) = \sum_{n=0}^{\infty} y^n \phi_n(x,t) \tag{2.7}$$

Where $\phi_n(x,t) = \frac{\partial^n \phi}{\partial y^n}$ at y= 0. Substituting this in the Laplace's equation (2.1) the following recurrence relation is obtained

$$\frac{\partial^2 \phi_n}{\partial x^2} = -(n+2)(n+1)\phi_{n+2} \tag{2.8}$$

Using penetration condition (2.6) we can arrive at $\phi_1(x,t) = C(x,t)$ and using this in the recurrence relation (2.8) the expression for $\phi(x,y,t)$ is obtained as -

$$\phi(x,y,t) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} F_{2m} y^{2m} + \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} C_{2m} y^{2m+1} \tag{2.9}$$

where $F = \phi_0(x,t)$ and the subscript -(2m) in C and F denotes 2m-th order derivative w.r.to x. The horizontal and vertical components of the fluid velocity at the free surface are determined as,

$$V_h = yC_x - \frac{1}{3!}C_{xxx}y^3 + f - \frac{1}{2!}y^2 f_{xx} + h.o.t \tag{2.10}$$

$$V_v = C - \frac{1}{2!}C_{xx}y^2 - yf_x + \frac{1}{3!}y^3 f_{xxx} + h.o.t \tag{2.11}$$

where x in the subscript denotes partial derivative with respect to x, $f = \frac{\partial F}{\partial x}$ and h.o.t denotes higher order terms in y. The different dimensional variables that have made their appearance in the problem will be made dimensionless by incorporating the following scaling of variables so that small parameters ϵ, δ creep into the equations and smaller terms can be neglected in comparison to them.

2.1 Scaling of variables

All the dependent and independent variables occurring in the above equations are scaled in the following way by taking account of the smallness parameters ϵ, δ

$$x \Rightarrow lx', \eta \Rightarrow a\eta', t \Rightarrow \frac{l}{\sqrt{gh}}t', V_h \Rightarrow \epsilon\sqrt{gh}V_h', V_v \Rightarrow \epsilon\delta\sqrt{gh}V_v',$$

$$f \Rightarrow \epsilon\sqrt{gh}f', y \Rightarrow h(1+\epsilon\eta'), C \Rightarrow \epsilon\delta\sqrt{gh}C',$$

where the variables in prime are dimensionless and henceforth all terms $\backsimeq \epsilon\delta^2$ will be neglected by considering them to be small compared to terms of the order of ϵ or δ^2. Using this scaling in equations (2.10), (2.11) dimensionless velocity components are obtained as-

$$V_h' = \delta^2 C_{x'}' + f' - \frac{1}{2}\delta^2 f_{x'x'}' \tag{2.12}$$

$$V_v' = C' - \frac{1}{2}\delta^2 C_{x'x'}' - (1+\epsilon\eta')f_{x'}' + \frac{1}{6}\delta^2 f_{x'x'x'}' \tag{2.13}$$

Hence from the two nonlinear boundary conditions (2.5), (2.3) we get

$$\eta'_{t'} + f'_{x'} + \epsilon f'\eta_{x'} + \epsilon\eta' f'_{x'} - \frac{1}{6}\delta^2 f'_{x'x'x'} = C' - \frac{1}{2}\delta^2 C'_{x'x'} \tag{2.14}$$

$$\eta'_{x'} + f'_{t'} + \epsilon f' f'_{x'} - \frac{1}{2}\delta^2 f'_{x'x't'} = -\delta^2 C'_{x't'} \tag{2.15}$$

For notational convenience the prime symbol will be omitted in all the variables in the subsequent analysis, remembering however that all variables correspond to rescaled quantities. These are the equations related to the displacement of the free surface wave η, function related to velocity potential f and the leakage velocity C.

In order to formulate the problem in a more general way $C(x,t)$ is considered to have different forms. In the next section, C is considered to be a constant i.e., leakage velocity of water at the bottom is constant throughout its motion.

2.2.1 C is pure constant

Considering C to be constant, equations (2.14), (2.15) can be written as

$$\eta_t + f_x + \epsilon f\eta_x + \epsilon\eta f_x - \frac{1}{6}\delta^2 f_{xxx} = C \tag{2.16}$$

$$\eta_x + f_t + \epsilon f f_x - \frac{1}{2}\delta^2 f_{xxt} = 0 \tag{2.17}$$

Expanding f in a series of small parameters as

$$f = f^{(0)} + \epsilon f^{(1)} + \delta^2 f^{(2)} + h.o.t \tag{2.18}$$

and neglecting higher order terms in ϵ or δ^2, equations (2.16), (2.17) converge to -

$$\eta_t + f_x^{(0)} + \epsilon(f_x^{(1)} + \eta f_x^{(0)} + \eta_x f^{(0)}) + \delta^2(f_x^{(2)} - \frac{1}{6}f_{xxx}^{(0)}) = C \tag{2.19}$$

$$\eta_x + f_t^{(0)} + \epsilon(f_t^{(1)} + f^{(0)} f_x^{(0)}) + \delta^2(f_t^{(2)} - \frac{1}{2} f_{xxt}^{(0)}) = 0 \qquad (2.20)$$

In order that equations (2.19), (2.20) are self-consistent as evolution equations for a one-dimensional wave propagating along the positive x-axis, the following choice is made:

$$f^{(0)} = \eta - Ct + O(\epsilon\delta^2) \qquad (2.21)$$

where $O(\epsilon\delta^2)$ denote terms $\sim \epsilon\delta^2$. Thus from equations (2.19), (2.20), we get,

$$\tilde{\eta}_t + \tilde{\eta}_x + \epsilon(f_x^{(1)} + 2\tilde{\eta}\tilde{\eta}_x + \tilde{\eta}_x Ct) + \delta^2(f_x^{(2)} - \frac{1}{6}\tilde{\eta}_{xxx}) = 0 \qquad (2.22)$$

$$\tilde{\eta}_t + \tilde{\eta}_x + \epsilon(f_t^{(1)} + \tilde{\eta}\tilde{\eta}_x) + \delta^2(f_t^{(2)} - \frac{1}{2}\tilde{\eta}_{xxt}) = 0 \qquad (2.23)$$

where $\tilde{\eta} = \eta - Ct$.

Let $f^{(1)}, f^{(2)}$ be functions of $\tilde{\eta}$ and its spatial derivatives. This leads to $f_t^{(1)} = -\tilde{\eta}_x f_{\tilde{\eta}}^{(1)} + f_{\tilde{\eta}}^{(1)} O(\epsilon, \delta^2)$. where $\mathrm{O}(\epsilon, \delta^2)$ is the term proportional to ϵ or δ^2. Since terms of the order of $\epsilon\delta^2$ are being neglected in the present work, the following relations are obtained

$$f_t^{(1)} \approx -\tilde{\eta}_x f_{\tilde{\eta}}^{(1)} = -f_x^{(1)}$$

$$f_t^{(2)} \approx -\tilde{\eta}_x f_{\tilde{\eta}}^{(2)} = -f_x^{(2)}$$

$$\tilde{\eta}_{xxt} \approx -\tilde{\eta}_{xxx}$$

Using these results in equations (2.22) and (2.23), the condition for compatibility of these two equations leads to

$$f_x^{(1)} = -\frac{1}{2}\tilde{\eta}_x(\tilde{\eta} + Ct) \qquad (2.24)$$

$$f_x^{(2)} = \frac{1}{3}\tilde{\eta}_{xxx} \qquad (2.25)$$

These results when substituted into any of the equations (2.22), (2.23), the following single evolution equation is obtained as

$$\tilde{\eta}_t + \tilde{\eta}_x + \epsilon(\frac{3}{2}\tilde{\eta}\tilde{\eta}_x + \frac{1}{2}\tilde{\eta}_x Ct) + \delta^2(\frac{1}{6}\tilde{\eta}_{xxx}) = 0 \qquad (2.26)$$

Equation (2.26) can be converted to a forced KdV equation with a constant forcing term by redefining the dependent variable as $\bar{\eta} = 3\eta + Ct$.

Equation (2.26) is similar to KdV equation except for the 4-th term which comes from the leakage. A suitable transformation into a moving frame can remove this term so that the standard form of KdV equation is recovered. We use,

$$X = x - t - bt^2, T = t$$

where b is a constant denoting the acceleration of the frame. In the new frame (X, T), equation (2.26) will look like

$$\tilde{\eta}_T + \epsilon(\frac{3}{2}\tilde{\eta}\tilde{\eta}_X + \frac{1}{2}\tilde{\eta}_X CT) + \delta^2(\frac{1}{6}\tilde{\eta}_{XXX}) - 2bT\tilde{\eta}_X = 0 \qquad (2.27)$$

Choosing $b = \frac{\epsilon C}{4}$ and defining new variables $u = (3\epsilon/2\delta^2)\tilde{\eta}$, $T' = (\delta^2/6)T$ the following standard form of KdV equation is obtained which is in the accelerated frame.

$$u_{T'} + 6uu_X + u_{XXX} = 0 \qquad (2.28)$$

Nature of the solution

The well known one-Soliton solution of equation (2.28) is given by

$$u(X, T') = \frac{\beta}{2}Sech^2[\frac{\sqrt{\beta}}{2}(X - \beta T')] \qquad (2.29)$$

where β is a constant. Back boosting the solution to the rest frame we get

$$u(x,t) = \frac{\beta}{2} Sech^2 [\frac{\sqrt{\beta}}{2}(x - t\{(1 + \frac{\beta\delta^2}{6}) - \frac{\epsilon \mid C \mid}{4} t\})] \tag{2.30}$$

where $\mid C \mid$ is the absolute value of C. The argument of $u(x,t)$ contains linear and quadratic terms in t and the term inside the second bracket behaves like the velocity of the wave.

When t starts increasing from a very small value, the term inside the second parentheses decreases i.e. the wave retards.

At a critical time given by

$$t = t_c = \frac{4(1 + \frac{\beta\delta^2}{6})}{\epsilon \mid C \mid},$$

the second bracketed term vanishes and the wave stops. After t_c, the wave propagates in the negative x-axis with increasing speed. Since the wave moving in positive x axis is only of concern to us, this oppositely moving wave can be neglected in a practical situation. Since ϵ is a small parameter, hence t_c is large for small values of fluid leakage. For large leakage velocity, t_c become small causing reflection of wave at earlier time. A x-t plot of the solution (2.30) is shown in Figure 1.2 indicating that the wave gets reflected at time at t_c and then moves in opposite direction.

Expressing the surface wave in old profile η as $\tilde{\eta} = \eta - Ct$ we see that a background part $-|C|t$ develops behind the wave. This will cause the while wave profile, η, to decrease with time. But since we are concerned on the effect of controlled leakage on the solitary wave, this background part doesn't produce interesting result. The solitary wave would be affected only in phase, not in amplitude.

2.2.2 C is function of t only

In the previous section solitary wave solution has been obtained by considering a constant leakage velocity. The problem is now generalized by assuming C to be function of t only.

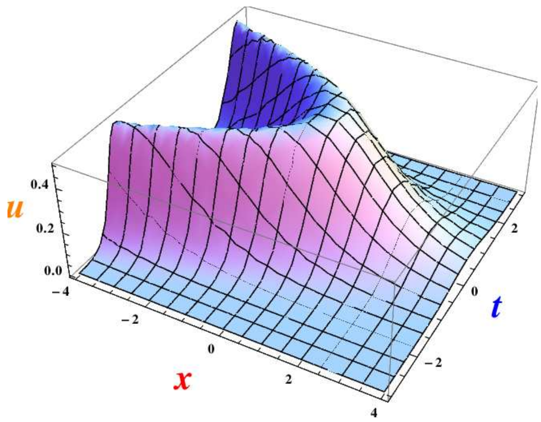


Figure 2.2: $x-t$ plot of the solution (2.30) with $\beta = 1$, $\mid C \mid= 40000$, $\delta = 0.01$, $\epsilon = 0.0001$. x and t are plotted in the 2 horizontal axes and $u(x,t)$ is plotted in the vertical axis.

From equations (2.14), (2.15)

$$\eta_t + f_x + \epsilon f \eta_x + \epsilon \eta f_x - \frac{1}{6}\delta^2 f_{xxx} = C(t) \tag{2.31}$$

$$\eta_x + f_t + \epsilon f f_x - \frac{1}{2}\delta^2 f_{xxt} = 0 \tag{2.32}$$

Carrying out series expansion for f as in equation (2.18) and neglecting higher order terms in ϵ or δ^2 the following equations are obtained from equations (2.31) and (2.32)

$$\eta_t + f_x^{(0)} + \epsilon(f_x^{(1)} + \eta f_x^{(0)} + \eta_x f^{(0)}) + \delta^2(f_x^{(2)} - \frac{1}{6}f_{xxx}^{(0)}) = C(t) \tag{2.33}$$

$$\eta_x + f_t^{(0)} + \epsilon(f_t^{(1)} + f^{(0)} f_x^{(0)}) + \delta^2(f_t^{(2)} - \frac{1}{2}f_{xxt}^{(0)}) = 0 \tag{2.34}$$

In order to make equations (2.33) and (2.34) self-consistent, the following choice is made

$$f^{(0)} = \eta - \int C(t)dt + O(\epsilon\delta^2) \tag{2.35}$$

Thus from equation (2.33), (2.34) the following equations are obtained -

$$\tilde{\eta}_t + \tilde{\eta}_x + \epsilon(f_x^{(1)} + 2\tilde{\eta}\tilde{\eta}_x + \tilde{\eta}_x B(t)) + \delta^2(f_x^{(2)} - \frac{1}{6}\tilde{\eta}_{xxx}) = 0 \tag{2.36}$$

$$\tilde{\eta}_t + \tilde{\eta}_x + \epsilon(f_t^{(1)} + \tilde{\eta}\tilde{\eta}_x) + \delta^2(f_t^{(2)} - \frac{1}{2}\tilde{\eta}_{xxt}) = 0 \tag{2.37}$$

where $\tilde{\eta} = \eta - B(t)$ where $B(t) = \int C(t)dt$.

Considering $f^{(1)}, f^{(2)}$ to be functions of $\tilde{\eta}$ and its spatial derivatives, $f_t^{(1)} = -\tilde{\eta}_x f_{\tilde{\eta}}^{(1)} + f_{\tilde{\eta}}^{(1)} O(\epsilon, \delta^2)$. Since terms of the order of $\epsilon\delta^2$ are neglected, the following relations are obtained

$$f_t^{(1)} \approx -\tilde{\eta}_x f_{\tilde{\eta}}^{(1)} = -f_x^{(1)}$$

$$f_t^{(2)} \approx -\tilde{\eta}_x f_{\tilde{\eta}}^{(2)} = -f_x^{(2)}$$

$$\tilde{\eta}_{xxt} \approx -\tilde{\eta}_{xxx}$$

Using these results in any of the equations (2.36) and (2.37) together with the compatibility condition leads to the following single equation

$$\tilde{\eta}_t + \tilde{\eta}_x + \epsilon(\frac{3}{2}\tilde{\eta}\tilde{\eta}_x + \frac{1}{2}\tilde{\eta}_x B(t)) + \delta^2(\frac{1}{6}\tilde{\eta}_{xxx}) = 0 \tag{2.38}$$

Equation (2.38) can be cast in the form of a forced KdV equation by redefining a new variable as given in the previous subsection with a time dependent forcing term. An analytical treatment of the influence of the time dependent random external noise on the propagation of nonlinear waves has been carried out by Orlowski[38] by considering a forced KdV equation. Considering the Gaussian character of the noise, the nature of deformation of the stationary solution of KdV-Burgers equation was studied[39]-[40]. during its propagation in randomly excited media.

In order to arrive at a standard form of the KdV equation from equation(2.38) a transformation to a moving frame given by

$$X = x - t - a(t), \; T = t \tag{2.39}$$

is carried out, where a(t) is a function dependent on t. This leads to

$$\tilde{\eta}_T + \frac{3\epsilon}{2}\tilde{\eta}\tilde{\eta}_X + \frac{\delta^2}{6}\tilde{\eta}_{XXX} + (\frac{\epsilon B(t)}{2} - \frac{\partial a}{\partial t})\tilde{\eta}_X = 0 \tag{2.40}$$

With the choice $\partial a/\partial t = \epsilon B(t)/2$, and defining $u = (3\epsilon/2\delta^2)\tilde{\eta}$, $T' = (\delta^2/6)T$, we get the standard form of KdV equation in the moving frame

$$u_{T'} + 6uu_X + u_{XXX} = 0 \tag{2.41}$$

Nature of the Solution

One-Soliton solution of equation (2.41) is of the standard form

$$u(X, T') = \frac{\alpha}{2}Sech^2[\frac{\sqrt{\alpha}}{2}(X - \alpha T')] \tag{2.42}$$

where α is a constant. Transforming this solution to the rest frame

$$u(x, t) = \frac{\alpha}{2}Sech^2[\frac{\sqrt{\alpha}}{2}(x - (1 + \frac{\alpha\delta^2}{6})t - \frac{\epsilon}{2}\int B(t)dt)] \tag{2.43}$$

It should be noted that when C is constant then the integral term inside the argument of $u(x, t)$ is $\frac{\epsilon C_{abs} t^2}{4}$ which is consistent with the solution of the previous case.

The functional form of $B(t)$ controls the motion of the solution (2.43). For different choices of the function $B(t)$ the 3D x-t plot of the solution will have different shapes. If the leakage velocity is dependent on the fluid velocity such that when a large upsurge of water arrives in a region, large leakage occurs and when t $\longrightarrow \pm\infty$ the leakage goes to zero i.e. leakage is localized in time. As an example we can choose the functional form of C as, $C(t) = b_0/(b_1 + b_2 t^2)$, where b_0, b_1,b_2 are constants , which is localized in time. The corresponding 3D plot of the solution is given as Figure (1.3). The solution gets curved due to the presence of nonlinear function of t.

Here also a background part will be developed behind the soliton which doesn't produce interesting results as described earlier.

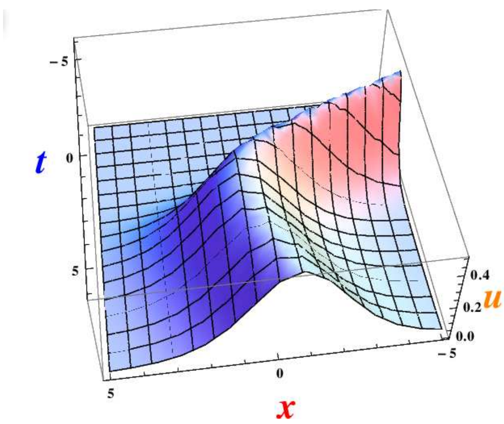


Figure 2.3: $x-t$ plot of the solution (2.43) with $\alpha = 1$, $\delta = 0.01$, $\epsilon = 0.0001$, $b_0 = -20000$, $b_1 = 1$,$b_2 = 1$. x and t are plotted in the 2 horizontal axes and $u(x,t)$ is plotted in the vertical axis.

2.3 C is a function of x only

The next case of interest deals with the situation where the leakage velocity is dependent on spatial coordinate only.

From equations (2.14), (2.15) we obtain

$$\eta_t + f_x + \epsilon f \eta_x + \epsilon \eta f_x - \frac{1}{6}\delta^2 f_{xxx} = C - \frac{1}{2}\delta^2 C_{xx} \tag{2.44}$$

$$\eta_x + f_t + \epsilon f f_x - \frac{1}{2}\delta^2 f_{xxt} = 0 \tag{2.45}$$

The function f is expanded in a perturbation series as in earlier sections and the following choice is made for $f^{(0)}$,

$$f^{(0)}(x,t) = \eta(x,t) + B(x) + O(\epsilon\delta^2) \tag{2.46}$$

where $B(x) = \int C(x)dx$.

The compatibility of the 2 equations (2.44) and (2.45) leads to,

$$f_x^{(1)} = -\frac{1}{2}(\eta_x \eta - BC) \tag{2.47}$$

$$f_x^{(2)} = \frac{1}{3}(\eta_{xxx}) - \frac{1}{6}C_{xx} \tag{2.48}$$

Substituting these values of $f_x^{(1)}, f_x^{(2)}$ in any of the equations (2.44) and (2.45) leads to a single evolution equation

$$\bar{\eta}_t + \bar{\eta}_x + \epsilon\frac{3}{2}\bar{\eta}\bar{\eta}_x + \frac{\delta^2}{6}\bar{\eta}_{xxx} = \frac{2}{3}B_x + \frac{\epsilon}{6}BB_x - \frac{\delta^2}{18}B_{xxx} \tag{2.49}$$

where $\bar{\eta} = \eta + \frac{2}{3}B$ and the equation(2.49) has the form of a forced KdV equation.

Nature of the Solution

For the case when the vertical fluid velocity at the bottom is a function of spatial coordinate only, the nonlinear equation for water wave propagation given by equation (2.49) is obtained as an inhomogeneous KdV equation. The right hand side of eq. (2.49) has a space KdV like form with the time coordinate replaced by the spatial coordinate (here x). The mathematical elegance of this equation enables one to obtain its solitary wave solutions in a simple manner considering the following two different cases:

Case (a)

A very simple situation occurs if the right hand side of the equation (2.49) is taken to be zero. Using the scaling ($X = x/\delta$, $B' = \epsilon B$), the inhomogeneous part of equation (2.49) reduces to the following:

$$12B'_X + 3B'B'_X - B'_{XXX} = 0 \tag{2.50}$$

A one-soliton like solution of this space KdV equation is obtained as

$$B'(X) = (-12)sech^2[\sqrt{3}(X - X_0] \tag{2.51}$$

and the corresponding leakage velocity is given by

$$C'(X) = \frac{\partial B'}{\partial X} = (24\sqrt{3})sech^2[\sqrt{3}(X - X_0)]tanh[\sqrt{3}(X - X_0)] \tag{2.52}$$

This leads to the conclusion that if B' has the functional form that satisfies (2.50), then the evolution equation for the waves would be the standard KdV equation. Hence its solitary wave solution will also be given by the standard KdV solitary wave solution moving with constant velocity. Thus for those functional forms of B' and the corresponding leakage velocity, the solitary wave solution will be practically unaffected by the leakage. The background wave generated behind the soliton will be unimportant since the controlled leakage doesn't affect the solitary wave. Since the solution of this case is the solution of standard KdV, it is not being shown explicitly. The leakage velocity given in equation (2.52) is plotted as Figure 1.4.

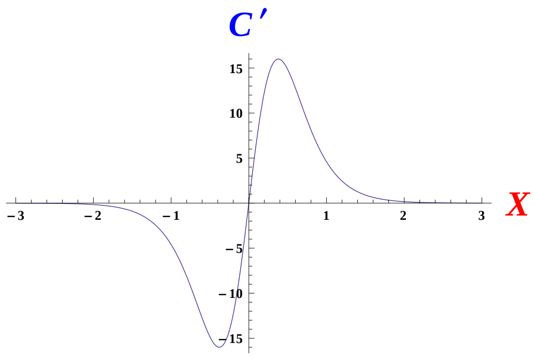


Figure 2.4: Leakage velocity profile for $X_0 = 0$ when B' satisfies (2.50)

Case (b):

An interesting analytic solution of the equation (2.49) arises if the leakage velocity function has a preassigned form. Considering the leakage velocity to be localized in space, (that is often consistent with certain physical condition), the following relevant form of $B(x)$ is chosen

$$B(x) = m \tanh(nx), \tag{2.53}$$

where m and n are two external small parameters dependent on the leakage profile.

Since m, n are small parameters, in the following calculation terms upto $\sim mn$ will be retained and terms beyond this order are neglected. Hence second and third terms in r.h.s of the equation (2.49) will produce higher order terms in m, n and hence these are neglected.

Transforming to a new variable $u = \bar{\eta} - 2B_x t/3$, the following equation is obtained from equation (2.49)

$$u_t + u_x + \epsilon\frac{3}{2}uu_x + \frac{\delta^2}{6}u_{xxx} + \epsilon B_x u_x t = 0 \tag{2.54}$$

Further, a transformation to a moving frame given by

$$X = x - \frac{\epsilon B_x}{2}t^2 - t, T = t$$

is carried out to obtain

$$u_T + (\epsilon\frac{3}{2})uu_X + (\frac{\delta^2}{6})u_{XXX} = 0 \tag{2.55}$$

The above equation is finally cast in the standard form of a KdV equation by defining the following variables $U = (3\epsilon/2\delta^2)u, T' = (\frac{\delta^2}{6})T$

$$U_{T'} + 6UU_X + U_{XXX} = 0 \tag{2.56}$$

The corresponding solitary wave solution in the rest frame is given by

$$\begin{aligned} U(x,t) &= \frac{\gamma}{2} Sech^2[\frac{\sqrt{\gamma}}{2}(x - \frac{\epsilon B_x}{2}t^2 - t(1+\frac{\gamma\delta^2}{6})] \\ &= \frac{\gamma}{2} Sech^2[\frac{\sqrt{\gamma}}{2}(x - \frac{\epsilon mnSech^2(nx)}{2}t^2 - t(1+\frac{\gamma\delta^2}{6})], \end{aligned} \tag{2.57}$$

where γ is a constant. The critical time t_c at which the velocity of the wave becomes zero is also a function of x. The 3D plot of the solution is shown in Figure (5). Since our main interest lies on the effect of the controlled leakage on solitary wave, the background part which develops behind the solitary wave doesn't produce interest. The solitary wave gets affected only in phase by the leakage, whereas its amplitude remains unaffected.

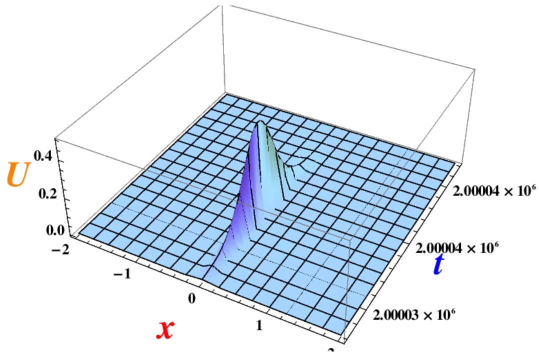


Figure 2.5: $x - t$ plot of the solution (2.57) with $m = -0.1$, $n = 0.1$, $\gamma = 1$

2.4 When C is function of both x and t

Finally the problem will be treated in the most general manner by treating the leakage velocity C as a function of both x and t. The mathematical analysis is carried out in the following way. As in all the earlier cases f is expanded as

$$f = f^{(0)} + \epsilon f^{(1)} + \delta^2 f^{(2)} + h.o.t, \tag{2.58}$$

where $f^{(i)}$'s are functions of η and its spatial derivatives. In the present case, the leakage velocity $C(x,t)$ is also expanded in a series of small parameters in the following way

$$C(x,t) = \epsilon C_1(x,t) + \delta^2 C_2(x,t) + h.o.t \tag{2.59}$$

Carrying out the same kind of mathematical analysis as in the earlier cases the following two equations are obtained from equations (2.14) and (2.15)

$$\eta_t + f_x^{(0)} + \epsilon(f_x^{(1)} + \eta f_x^{(0)} + \eta_x f^{(0)} - C_1) + \delta^2(f_x^{(2)} - \frac{1}{6}f_{xxx}^{(0)} - C_2) = 0 \tag{2.60}$$

$$\eta_x + f_t^{(0)} + \epsilon(f_t^{(1)} + f^{(0)} f_x^{(0)}) + \delta^2(f_t^{(2)} - \frac{1}{2}f_{xxt}^{(0)}) = 0 \tag{2.61}$$

In order to make equations (2.60), (2.61) self-consistent as evolution equation for a 1d wave propagating to the right, the following transformation is carried out

$$f^{(0)}(x,t) = \eta(x,t) + h.o.t \tag{2.62}$$

Since terms of the order of $\epsilon\delta^2$ are neglected, the following relations are obtained

$$f_t^{(1)} \approx -\eta_x f_\eta^{(1)} = -f_x^{(1)}$$

,

$$f_t^{(2)} \approx -\eta_x f_\eta^{(2)} = -f_x^{(2)}$$

$$\eta_{xxt} \approx -\eta_{xxx}$$

From the compatibility of the two equations (2.60) and (2.61) we get,

$$\begin{aligned} f_x^{(1)} &= \frac{1}{2}(C_1 - \eta_x \eta) \\ f_x^{(2)} &= \frac{1}{3}(\eta_{xxx}) + \frac{1}{2}C_2 \end{aligned} \tag{2.63}$$

Finally, a single evolution equation is obtained utilizing these functional forms of $f_x^{(1)}, f_x^{(2)}$ in any of the equations (2.60), (2.61) as:

$$\eta_t + \eta_x + (\epsilon\frac{3}{2})\eta\eta_x + (\frac{\delta^2}{6})\eta_{xxx} = \frac{1}{2}(\epsilon C_1 + \delta^2 C_2) \tag{2.64}$$

For the sake of mathematical simplicity the condition $C_2(x,t) = 0$ is assumed. Since $\epsilon \approx \delta^2$, it does not break the generality of the problem. Using Galilean transformation $\xi = x - t$ and $\tau = (\delta^2/6)t$ and defining the variable $u(\xi,\tau) = (\frac{3\epsilon}{2\delta^2})\eta(\xi,\tau)$, the final form of the evolution equation is obtained as,

$$u_\tau + 6uu_\xi + u_{\xi\xi\xi} = \frac{9}{2}C_1(\xi,\tau) \tag{2.65}$$

The above equation has the form of a forced kdV equation.

2.5.1 Nature of Solutions

For the sake of notational simplicity equation (3.17) is expressed in variables (x,t) as

$$u_t + 6uu_x + u_{xxx} = \frac{9}{2}\ C_1(x,t) \tag{2.66}$$

In order to obtain a solitary wave solution of the forced KdV equation, the bilinearization technique[47] is used.

Assuming

$$\frac{9}{2}C_1(x,t) = \frac{\partial D(x,t)}{\partial x} \tag{2.67}$$

and using the bilinear transformation

$$u = 2\frac{\partial^2}{\partial x^2}[\ln(F)], \quad D = \frac{G}{F}$$

equation(2.66) transforms to the following bilinear equation

$$FF_{xt} - F_xF_t + FF_{xxxx} + 3F_{xx}^2 - 4F_xF_{xxx} = GF \tag{2.68}$$

In order to obtain similar solitary wave solutions in this case also the following choice for the functions G and F have to be made,

$$G = h(t)Sech[x - vt - p(t)] \tag{2.69}$$

$$F = h_1(t)(\exp[x - vt - q(t)] + \exp[-x + vt + q(t)]), \tag{2.70}$$

where $h(t), h_1(t), p(t)$ and $q(t)$ are all arbitrary functions of time. Substituting the expressions for F and G in equation (2.68) leads to

$$p(t) = q(t), \quad h_1(t) = -\frac{h(t)}{2(-4 + v + \frac{\partial p}{\partial t})} \tag{2.71}$$

Thus, the analytic solution of the forced KdV equation (2.66) is obtained as

$$u = 2(Sech[x - vt - p(t)])^2 \tag{2.72}$$

with the forcing term D given by

$$D(x,t) = -(-4 + v + \frac{\partial q}{\partial t})(Sech[x - vt - p(t)])^2 \tag{2.73}$$

Hence the leakage velocity $C_1(x,t)$ is obtained in the form

$$C_1(x,t) = -2(-4 + v + \frac{\partial p}{\partial t})(Sech[x - vt - p(t)])^2 \tanh[x - vt - p(t)] \tag{2.74}$$

Since there is an arbitrary function $p(t)$ in the leakage velocity, as well as in the solution, one can observe different types of waves excited by different forcing sources, i.e. different functional forms of $p(t)$. Here also the amplitude of the wave solution remains constant.

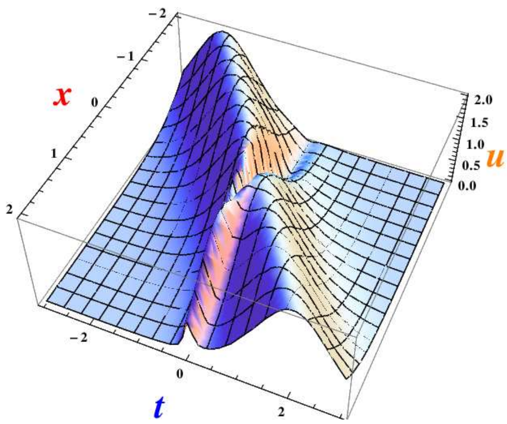


Figure 2.6: $x - t$ plot of the solution (2.72) for $q(t) = Sech(10t), v = 1$

2.5 Summary

The problem of shallow water, unidirectional, weakly nonlinear, surface wave propagation in a water channel is considered (in a way distinct from existing literature) by incorporating water leakage at the bottom in the form of a non-trivial penetration condition. The controlled feedback leakage is also considered to be a function independent of the surface wave profile η. When the vertical fluid velocity at the bottom is constant then it is shown that the evolution equation is given by a forced KdV equation with a constant forcing term. The solitary wave solutions of such equation will contain a constant retardation term in the argument of the function, while the amplitude will remain constant. When the leakage is a function of time, a time dependent retardation appears in the argument of the solution while the amplitude still remains constant. When the leakage velocity is only space dependent, two different kinds of solitary wave solutions are obtained analytically. For those functional forms of B(x), satisfying the stated space KdV like equation, the solution will remain unaffected by the leakage and becomes identical to the standard KdV soliton with constant velocity. When the leakage velocity is a slowly varying function localized in space a soliton solution with constant amplitude is obtained under certain approximation related to the slowness of variation of the leakage velocity profile. When the vertical fluid velocity at the bed is assumed to be function of both space and time, the bilinearization technique has been employed to obtain the solutions to the evolution equation. The technique yields a solitary wave solution with a constant amplitude and an arbitrary function of time appearing in the argument of the solution as well as in the argument of the leakage velocity profile. The nature of the solution can be modulated by choosing different forms of this arbitrary function.

But the main problem in the results we obtained is that the amplitude of the solitary wave solution remains constant whereas its phase gets modified by the feedback leakage at the bottom which is independent of the surface wave profile. Hence the effect of the leakage on the surface solitary wave amplitude is not observed. Again we have considered here the nonlinear wave propagation at the constant water depth which is also an idealistic assumption. In real situation, the bottom topography varies with distance causing the amplitude of the solitary wave to change with propagation. These conditions will be considered in the next chapter and the control of solitary wave amplitude with the bottom leakage in a shallow water channel of variable bottom will also be discussed in detail.

Chapter 3

Control of nonlinear surging waves through bottom boundary conditions

3.1 Introduction:

In the previous chapter, we have studied the effect of controlled leakage at the bottom which is independent of the surface wave profile on the surface solitary wave. Analytical solitary wave solutions showed that the phase of the solitary wave gets modified controlled by the feedback bottom leakage whereas its amplitude remains constant. Also we have considered the propagation of solitary wave in shallow water of constant depth which is also an idealistic approximation. In real situation the bottom topography varies and the near shore waves show surging nature due to the decreasing bathymetry. In certain conditions such surging waves show destructive nature at the shore like tsunami waves, bore waves [5].

In a relatively smaller scale, the near shore waves and bore waves caused many devastating effects to the coastal habitats and in-land rivers throughout the century . Therefore along with the traditional warning systems, it is desirable to find ways and means geared towards possible invasive procedures for taming of such hazardous wave phenomena. There are few suggestions for effective interventions, like plantation of Mangrove treas along the coastal lines [12], installation of breakwaters at strategic positions [13]-[16], stoppage of erosion by concrete bolders . However, these are mostly indirect ways to counter the surging waves, while we lack the proposals for directly attacking the problem, perhaps with the exception of the proposed bubble method, aiming to stop the incoming waves by a stream of fast and strong counter-waves, mixed with air bubbles [17]. Though the last method was proposed more than fifty years back, its feasibility and effectiveness has not been established yet. The attenuation of incident water waves by a curved vane like structure positioned beneath or at the surface of a body of water is described in a Patent [18] where the detailed design of the structure is given. An attempt was made to reduce the devastating effects of a tsunami waves by single and double submerged barrier was done in Tel Aviv University[19]. They performed their experiments in a basin 5 m in length and 10.5 cm in depth. The wavelength of the generated wave was about 3 m, which allows referring to it as a tsunami.

Our aim here is to put forward an innovative proposal based on a theoretical study on the effect of sudden feedback boundary control at the bottom on the surging surface wave amplitude, governed by nonlinear equations describing unidirectional gravitational waves, derived from the basic hydrodynamic equations at the shallow water regime. The key factor responsible for surging of the approaching nonlinear waves to the shore (or in upstream rivers) is the decreasing depth bathymetry which triggers the amplitude surge of the surface waves.

Our strategy is similar as that of the previous chapter. The main aim is to study first the effect of the bottom boundary condition on the nonlinear solitary surface waves of the well-known perturbed KdV equation propagating in shallow water of constant depth. The vertical fluid velocity at the bottom is taken as a function of surface wave profile to identify subsequently through theoretical analysis, the linear dependence as the optimal case inducing maximum amplitude damping to the surface waves. This knowledge is applied through slowly varying bathymetry which without the leakage condition, as we know would result to solitary wave solution with increasing amplitude with the water depth decreasing along its propagation. However when the controlled bottom leakage with optimal feedback wave profile is imposed, the surging amplitude of the wave meets the counter damping effect, resulting to a managed propagating waves towards the shore with reduced hazardous effect due to the effective damping of the wave. We would like to emphasize that there could be various natural bottom boundary effects inducing damping of the surface wave amplitudes, like porosity [42]- [45], irregularities, uneven heights, periodic topography [53], friction [52] apart from the fluid viscosity[51] etc. while the long obstacle can induce fission of the solitary waves[52]. However our aim here is to induce damping effect artificially through controlled mechanism.

The privilege of our theoretical result is the exact nature of the solutions we obtain, in spite of the variable depth bathymetry, which is rather a rare achievement. Our theoretical results with exact solutions allows to extract finer details and precise predictions. Our findings are extended to cover different cases of the controlled bottom leakage conditions, ranging from space dependent to time dependent, from vanishing of effective leakage velocity to a desirable leakage conditions etc. Our theoretical findings for the possible control of the surging waves like tsunamis and bore waves based on our exact results are applied next to real sea shore bathymety. We have focused in particular two high risk coastal zones of bay of Bengal near the city of Chennai in south India as presented in the recent in depth study of the subject [58].

Our analysis shows that a significant upsurge could have been experienced by a future tsunami wave approaching towards these coastal points. For example at the identified northern coastal point (N 13° $10.5'$ - E 80° $18.75'$) a wave of nearby 1 meters built at a distance of 10.5 km from the shore would have been developed to a killing height of 30 meter at the shore without any control. Similarly at a southern point (N 13° $0'$ - E 80° $16.2'$) the bathymetry would induces equally devastating upsurge for a wave of 1 m created at a distance of 11.1 Km away, to develop into a 30 meter killer wave at the shore. Applications of feedback controlled method through bottom boundary condition, that we propose here is found to be able to regulate such upsurging waves to a considerable extent, minimizing its hazardous effects. In particular, the surging waves at the northern point could be regulated at the height of 1.23 meters if the leakage installation could be made starting from a distance of 900 meters from the shore. A smaller distance could result to a higher amplitude though significantly lesser than that without control. Similarly the surging waves at the southern coast could be controlled to a wave amplitude of 0.4 meter, if the installation starts at a distance of 900 meters. Thus the hazardous effects of tsunami like surging waves could possibly be neutralized to some extent through a controlled bottom leakage condition tuned by a linearly dependent wave profile, created through a feedback mechanism.

3.2 Effect of feedback bottom boundary condition on nonlinear surface wave in constant depth

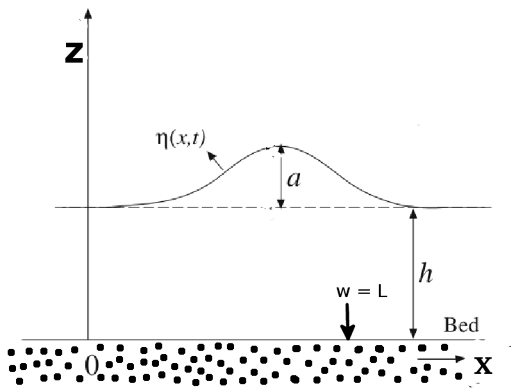


Figure 3.1: Solitary wave in shallow water of constant depth with leakage at the bottom

The purpose of this section is to study the effect of bottom BC with controlled leakage, designed with a feedback from the surface wave,where the leakage function would depend on the wave profile and its spatial derivatives. As is well known that, the nonlinear free surface gravity waves propagating in a shallow water in constant depth with the traditional hard bed boundary condition in the form of solitary waves retain their constant amplitude profile with a high degree of stability [4, 41]. However, when the boundary condition is changed to a leakage function dependent on the wave profile itself, as we find here, the solitary wave propagating on the surface suffer an amplitude damping along its propagation. Different forms of the leakage velocity function at the bottom induce different types of damping. Such a controlled leakage at the bottom may be arranged using a functional feedback from the profile of the wave appearing on the surface over that location and at that instant of time. Our motivation for this study is to analyze different damping effects corresponding to different leakage functions and identify the case when the damping would be maximum, which is the most desirable feature in the present context.

In the following subsections we derive the corresponding free surface wave equation and investigate the nature of the solitary wave solution with damping caused by different cases of the bottom leakage condition.

3.2.1 Surface wave evolution equation with leakage boundary condition

We consider here the shallow water nonlinear surface-gravity wave , propagating along the positive x-direction in a constant water depth with the viscosity and the surface tension of the fluid, which is assumed to be incompressible, are neglected in what follows, We start from the dimensionless basic hydrodynamic equations [4]:

$$u_t + \epsilon(uu_x + wu_z) = -p_x, \quad \delta^2[w_t + \epsilon(uw_x + ww_z)] = -p_z, \tag{3.1}$$

along the x and the z axis, respectively, which are reducible from the Euler equation in the present case.

Here u, w, p, η are horizontal and vertical fluid velocity components, pressure and the surface wave profile, respectively, with the subscripts denoting partial derivatives. ϵ is the amplitude parameter defined by $\epsilon = \frac{a}{h}$ and $\delta = \frac{h}{l}$ is the shallowness parameter, expressed through the maximum amplitude a, the water depth h and the wavelength l (see FIG 3.1). ϵ and δ are natural parameters supposed to be small, which is consistent with the long wave and the shallow water limit. The continuity equation of the fluid yields

$$u_x + w_z = 0. \tag{3.2}$$

Nonlinear variable boundary conditions, valid at the free boundary $z = 1 + \epsilon\eta$, on the other hand, gives

$$p = \eta, \quad w = \eta_t + \epsilon u \eta_x, \tag{3.3}$$

while we take the boundary condition for the vertical component of the water velocity at the bottom: $z = 0$ as

$$w = -\epsilon\tilde{\alpha}G(\eta, \eta_x, ...), \tag{3.4}$$

where $G(\eta, \eta_x, ...)$ is assumed, in general, to be an arbitrary function of η and its spatial derivatives and α' is a positive constant with ϵ being a small parameter as defined above. It is important to note here, that usual hard bed scenario with no leakage one would have $w = 0$ at the bottom whereas in our choice the nontrivial leakage function G may depend functionally on the surface wave profile which could be designed through a feedback route, sensing the surface movement. The leakage is considered here to be in the ϵ order. Note that the negative sign in equation (3.4) appears because the leakage velocity occurs along the negative z-direction, i.e, vertically downward. In order to model shallow water solitary waves, there must be an appropriate balance between nonlinearity and dispersion, i.e, $\delta^2 = O(\epsilon)$ as ϵ tends to zero. Thus for any δ, there exists a region in $(x,t)-$ plane with ϵ tending to zero, where this balance remains valid. This region of our interest may be defined by a scaling of independent variables as $x \to \frac{\delta}{\sqrt{\epsilon}}x$, $t \to \frac{\delta}{\sqrt{\epsilon}}t$ and $w \to \frac{\sqrt{\epsilon}}{\delta}w$ for any values of ϵ and δ. The set of equations (3.1-3.4) thus becomes,

$$u_t + \epsilon(uu_x + wu_z) = -p_x, \quad \epsilon[w_t + \epsilon(uw_x + ww_z)] = -p_z, \quad u_x + w_z = 0, \tag{3.5}$$

together with the boundary conditions

$$p = \eta, \quad w = \eta_t + \epsilon u\eta_x, \tag{3.6}$$

$$w = -\epsilon\alpha G(\eta, \eta_x, ...). \tag{3.7}$$

valid at the free surface and at the bottom, respectively, where $\alpha = \tilde{\alpha}\frac{\delta}{\sqrt{\epsilon}}$, with a net outcome of the transformation is to replace δ^2 by ϵ in equations (3.1-3.4). Introducing a new frame of reference with stretched time $\xi = x - t$, $\tau = \epsilon t$, we seek an asymptotic solution of the system of equations and boundary conditions in the form

$$q(\xi,\tau,z;\epsilon) \sim \sum_{n=0}^{\infty} \epsilon^n q_n(\xi,\tau,z), \quad \eta(\xi,\tau;\epsilon) \sim \sum_{n=0}^{\infty} \epsilon^n \eta_n(\xi,\tau), \tag{3.8}$$

where q (and related q_n) represents each of the functions u, w and p for the corresponding expansion.

Now to deduce the final evolution equation from the set of complicated nonlinear equations (3.5-3.7) involving several variables, we have to make the asymptotic multi-scale expansions as explained above. We carry out an explicit order by order calculation to demonstrate the process.

Result at ϵ^0 order

At ϵ^0 order the above set of equations (3.5)-(3.7) is reduced respectively to the following set

$$u_{0\xi} = p_{0\xi}, \quad p_{0z} = 0, \quad u_{0\xi} + w_{0z} = 0 \tag{3.9}$$

$$p_0 = \eta_0, \quad w_0 = -\eta_{0\xi}, \tag{3.10}$$

$$w_0 = 0, \tag{3.11}$$

with equation (3.10) valid at $z = 1$ and (3.11) at $z = 0$. These equations lead to the solutions expressed through η_0 as $p_0 = \eta_0, \;\; u_0 = \eta_0, \;\; w_0 = -z\eta_{0\xi},$ with the appearance of η caused only by the passage of the wave has been imposed, i.e, $u_0 = 0$, whenever $\eta_0 = 0$.

Result at ϵ order

In this order of approximation, two free boundary conditions at $z = 1 + \epsilon\eta$ are evaluated by performing Taylor expansions of the functions u, w, p around the point $z = 1$. Consequently the following set of equations are obtained from (3.5)-(3.7):

$$-u_{1\xi} + u_{0\tau} + u_0 u_{0\xi} + w_0 u_{0z} = -p_{1\xi}, \quad p_{1z} = w_{0\xi}, \quad u_{1\xi} + w_{1z} = 0 \tag{3.12}$$

with the boundary conditions:

$$p_1 + \eta_0 p_{0z} = \eta_1, \quad w_1 + \eta_0 w_{0z} = -\eta_{1\xi} + \eta_{0\tau} + u_0 \eta_{0\xi}, \tag{3.13}$$

valid at $z = 1$. We also get from the BC at the bottom: $z = 0$, the relation

$$w_1 = -\alpha G_0(\eta_0, \eta_{0\xi}, ...) \tag{3.14}$$

where G_0 is the contribution of the leakage function at ϵ^0 order. Using the above result, w_1 can be expressed now as

$$w_1 = -(\eta_{1\xi} + \eta_{0\tau} + \eta_0\eta_{0\xi} + \frac{1}{2}\eta_{0\xi\xi\xi})z + \frac{1}{6}z^3\eta_{0\xi\xi\xi} - \alpha G_0, \tag{3.15}$$

giving thus all other functions expressed through the fields η_0 and η_1 only, in this order of approximation. Finally eliminating η_1 we obtain the free surface wave equation as

$$2\eta_{0\tau} + \frac{1}{3}\eta_{0\xi\xi\xi} + 3\eta_0\eta_{0\xi} + \alpha G_0 = 0, \tag{3.16}$$

with an additional term due to the wave profile dependent bottom leakage function appearing in the well known integrable KdV equation, which however spoils the integrability of the system, in general. With a scaling of the variables as $U = 9\eta_0, T = \tau/6$ equation (7.35) takes a normalized form

$$U_T + UU_\xi + U_{\xi\xi\xi} + \beta G_0 = 0, \tag{3.17}$$

where α is scaled to β and $G_0(U, U_\xi, ..)$ is an arbitrary smooth function, originating from the wave profile dependent leakage velocity. It is fascinating to note, that the condition, we impose for the fluid velocity at the bottom through a boundary condition with wave profile dependence makes it way to the nonlinear evolution equation at the surface.

Notice that equation (3.17) is an extension of the KdV equations with arbitrary higher nonlinearity, which in general represents a non integrable system. However an approximate method due to Bogoliubov and Mitropolsky [55, 50, 51] could be applied here for extracting analytic solutions for the wave equation (3.17) ,in general, in an implicit form. For explicit analytic solution, one needs to make suitable choices for function G_0. We focus below on some of such choices with lower order nonlinearities, e.g. $G_0 = U,\ U^2,\ U^3,\ U_\xi^2$ though this set, in principle, can be extended further. We do not put emphasis on the physical meaning for the individual forms of the leakage function, since our main motivation is to compare theoretically the result of the corresponding wave solutions, to identify the case that would induce maximum damping of the wave amplitude. It is intriguing to note, that similar equations for some of the cases considered by us were obtained earlier [51, 52], though in completely different physical set-ups.

In order that this approximation scheme to be consistent with the condition for the validity of (3.17), it is required that the leakage coefficient β should be a small parameter of order higher than ϵ as $1 \gg \beta \gg \epsilon$.

Introducing a phase coordinate $\phi(\xi, T, \beta) = \sqrt{\frac{N(T,\beta)}{12}}(\xi - \frac{1}{3}\int_0^T N(T, \beta)dT)$, through a time-dependent function $N(T, \beta)$, assumed to vary slowly with time, with two different time scales $t_0 = T$, $t_1 = \beta T$, we seek a solution of the wave equation following [55]. By expanding $U(\phi, \beta, T)$ in small parameter β as

$$U(\phi, \beta, T) = U_0(\phi, t_0, t_1) + \beta U_1(\phi, t_0) + O(\beta^2), \tag{3.18}$$

valid for long times(as large as $T \sim O(1/\beta)$), we obtain using (3.17) an equation containing different powers of β. Equating coefficients of the same powers of β, equations at different orders are derived, which need to be solved at each order.

3.2.2 Case $G_0 = U$

We explore this case with some details for demonstrating the applicability of the Bogoliubov method for solving perturbed KdV equation and for identifying the quantitative trend in the influence of the bottom leakage G_0 on the amplitude of the surface waves.

Note that the equation obtained in this case mathematically coincides with the dissipation induced evolution considered in the context of ion-sound waves damped by ion-neutral collisions [51].

Integrating equation (3.17) for $G_0 = U$, over the whole range of ξ we can solve for the total wave amplitude $I(T) = \int_{-\infty}^{\infty} U d\xi$ and the total intensity of the wave $P(T) = \int_{-\infty}^{\infty} U^2 d\xi$ to get the explicit expressions as $I(T) = I(0)\, \exp(-\beta T)$ and $P(T) = P(0) \exp(-2\beta T)$, respectively, where $U(\xi, T)$ and its higher order ξ derivatives are assumed to vanish at infinity. It is also evident from the exponentially decaying nature, that the wave intensity is not conserved in time, confirming that the integrability of the perturbed KdV equation (3.17) in this case is lost due to the leakage we have considered here.

Since estimating the damping of the solitary water waves is the main concern of our problem, we take the following relations as the required initial and boundary conditions: $U(\phi, 0, \beta) = N_0 sech^2(\phi)$, $U(\pm\infty, T, \beta) = 0$. The lowest order equation takes the form

$$\rho \frac{\partial U_0}{\partial t_0} + \frac{\partial^3 U_0}{\partial \phi^3} - 4\frac{\partial U_0}{\partial \phi} + \frac{12}{N} U_0 \frac{\partial U_0}{\partial \phi} = 0, \tag{3.19}$$

where $\rho = \frac{12\sqrt{12}}{N\sqrt{N}}$ with $N(t_1)$ as an arbitrary function of t_1, except for the initial condition $N(0) = N_0$. Solving this equation we obtain

$$U_0(\phi, t_0, t_1) = N(t_1) sech^2(\phi), \tag{3.20}$$

while the β order equation takes the form

$$\frac{\partial U_1}{\partial t_0} + L[U_1] = M[U_0], \tag{3.21}$$

where,

$$M[U_0] = -\frac{\partial U_0}{\partial t_1} - \frac{\phi}{2N}\frac{\partial U_0}{\partial \phi}\frac{dN}{dt_1} - U_0, \quad L[U_1] = \frac{1}{\rho}\frac{\partial^3 U_1}{\partial \phi^3} - \frac{4}{\rho}\frac{\partial U_1}{\partial \phi} + \frac{12}{N\rho}\frac{\partial (U_0 U_1)}{\partial \phi}. \quad (3.22)$$

The boundary and initial conditions for U_1 are $U_1(\pm\infty, t_0) = 0, U_1(\phi, 0) = 0$ and it is required that $U_1(\phi, t_0)$ should not behave secularly with t_0. To eliminate secular behavior of U_1 it is necessary that $M[U_0]$ be orthogonal to all solutions $g(\phi)$ of $L^+[g] = 0$, where the function $g(\phi)$ should satisfy $g(\pm\infty) = 0$. Here L^+ is the operator adjoint to L given by,

$$L^+ = -\frac{1}{\rho}\frac{\partial^3}{\partial \phi^3} + \frac{4}{\rho}\frac{\partial}{\partial \phi} - \frac{12}{\rho} sech^2(\phi)\frac{\partial}{\partial \phi}. \quad (3.23)$$

One can show, that the only possible solution of $L^+[g] = 0$, with $g(\pm\infty) = 0$, is in the solitonic form $g(\phi) = sech^2(\phi)$.

Thus from the orthogonality requirement we get

$$\int_{-\infty}^{\infty} sech^2(\phi) M[U_0] d\phi = 0, \quad (3.24)$$

which yields a simple first order differential equation for $N(t_1)$, the solution of which is

$$N(t_1) = N(0) exp(-\frac{4t_1}{3}), \quad t_1 = \beta T \quad (3.25)$$

for positive small leakage parameter β at large time T. Therefore we obtain the final result as

$$U = N(t_1) sech^2 \phi(\xi, t_1) + O(\beta), \quad \phi(\xi, t_1) = \sqrt{\frac{N(t_1)}{12}}(\xi + \frac{1}{4\beta}N(t_1)). \quad (3.26)$$

The wave solution of equation (3.17) thus obtained for $G_0 = U$, shows that the amplitude of the solitary wave would decrease with time following (3.25).

Recall that similar dissipative soliton solution was derived earlier in many different physical situations [51, 52].

3.2.3 Case $G_0 = U^2$

We take up this case for comparison and find that the same Bogoliubov- Mitropolsky method discussed above is applicable also in this case with the wave equation taking the form of a perturbed KdV equation

$$U_T + UU_\xi + U_{\xi\xi\xi} + \beta U^2 = 0. \tag{3.27}$$

Notice, that equation (3.17) with the choice for our leakage velocity function, coincides formally with the dissipation due to friction at the bottom (Chezy law) [52], though for completely different origin.

Using the same approximation technique, details of which we omit, the decay law of the solitary wave amplitude for equation (3.27) can be derived as

$$N(T) = \frac{N(0)}{[1 + \frac{16N(0)\beta}{15}T]}. \tag{3.28}$$

Observe, that in comparison with the linear choice of the leakage velocity the amplitude decay with time becomes weaker in this nonlinear case. To confirm this trend,which is rather anti-intuitive we take up new cases with enhanced nonlinearity and derivatives.

Interestingly, the choice of leakage function as $G_0 = -U_{\xi\xi}$ would lead to very similar decay law (3.28) and would also coincide formally with the effect of magnetosonic waves damped by electron collisions [51].

3.2.4 Case $G = U^3$

Such a choice of leakage velocity condition with cubic dependence on wave profile would give rise to the equation

$$U_T + UU_\xi + U_{\xi\xi\xi} + \beta U^3 = 0, \tag{3.29}$$

representing a new perturbed KdV equation, apparently ignored earlier. The same approximate treatment leads to the decay law of the solitary wave amplitude of (3.29) as

$$N = \frac{N(0)}{\sqrt{[1 + \frac{32N(0)^2\beta}{35}T]}}, \tag{3.30}$$

decreasing with time as shown in FIG 3.2. Since here for cubic nonlinearity we get the

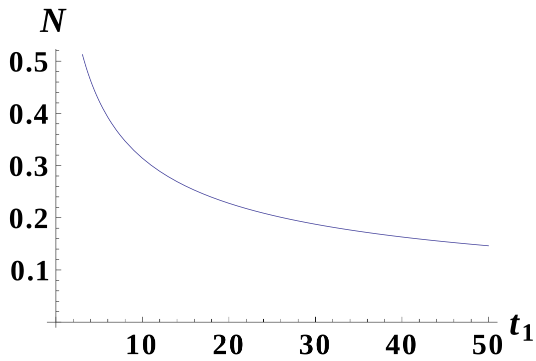


Figure 3.2: Plot showing the dependence of the soliton amplitude N on t_1, for the solution (3.30) with $N(0) = 1$. The decaying nature of $N(t_1)$ is explicit.

decay rate in inverse square root power as seen from (3.30), we notice again that the same trend of the weaker decay of the soliton amplitude with higher nonlinear dependence of the wave profile on the leakage velocity function, continues confirming the anti intuitive trend noticed above.

3.2.5 Case $G = U_\xi^2$

For this choice of the leakage velocity function the perturbed KdV equation reduces to

$$U_T + UU_\xi + U_{\xi\xi\xi} + \beta U_\xi^2 = 0, \tag{3.31}$$

apparently not investigated earlier. Through similar procedure we can derive the damped solitary wave amplitude of (3.31) as

$$N = \frac{N(0)}{\sqrt{[1 + \frac{8N(0)^2\beta}{45}T]}}, \tag{3.32}$$

which is graphically represented in FIG 3.3.

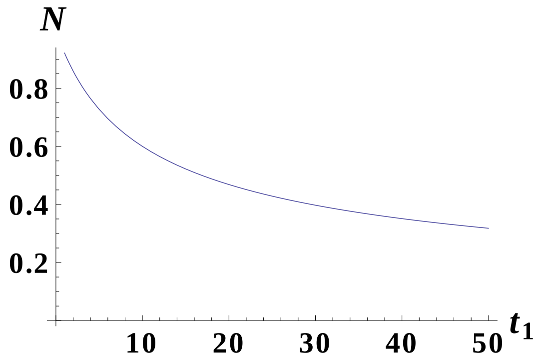


Figure 3.3: The solitonic wave amplitude $N(t_1)$ (3.32) as decays with time t_1 for $N(0) = 1$.

Comparing (3.28) with (3.32) we may conclude, that the increase of nonlinearity as well as derivatives, of the wave profile in the leakage velocity function weakens the decay rate of the solitonic amplitude. Analyzing the above results for linear and nonlinear choices of G_0, we may conclude that the leakage with the linear dependence on the profile $G_0 = U$ is the optimal one capable of inducing maximum decay rate on the soliton amplitude as exponential functions, compared to all other cases considered here. Therefore in the next sections we take up this particular case, being the most desirable one, for controlling the surging waves in a decreasing depth scenario.

3.3 Effect of leakage BC on nonlinear shallow water surface wave in variable depth bathymetry

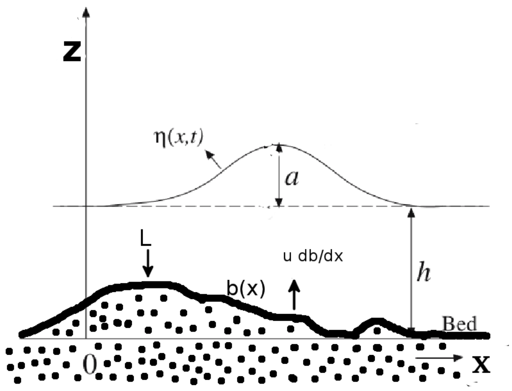


Figure 3.4: Solitary wave in a shallow water of slowly varying depth with leakage at the bottom

Propagation of nonlinear shallow water unidirectional waves over variable depth topography has been studied intensively with rich results [4, 52, 5, 53].

It is known that the slowly variable depth in comparison to the evolution scale of the wave, can lead to the upsurging wave amplitude, for decreasing depth, which occurs when the wave approaches to the shore. In this section we intend to focus on such a situation due to its potentially hazardous consequences and look for its possible regulation through bottom leakage. Since in the previous section we have identified the maximum damping effect of surface waves for leakage velocity function depending linearly on the wave profile, we will apply this particular leakage condition to achieve maximal damping effect. Therefore we take up the problem of nonlinear wave propagation over shallow water of slowly varying depth, in the framework of KdV equation, together with a nontrivial leakage condition at the bottom with a leakage function proportional to the surface wave profile, sensed through a feedback mechanism.

This problem targeted towards controlling the surging waves due to decreasing depth bathymetry has not received the needed attention.

3.3.1 Derivation of nonlinear surface wave evolution equation with slowly variable depth under bottom boundary leakage condition

Under this physical situation one has to start with the same basic dimensionless hydrodynamic equations considered in the previous section as:

$$u_t + \epsilon(uu_x + wu_z) = -p_x, \quad \epsilon[w_t + \epsilon(uw_x + ww_z)] = -p_z, \quad u_x + w_z = 0, \tag{3.33}$$

together with the surface boundary conditions $p = \eta, \quad w = \eta_t + \epsilon u \eta_x$ valid at $z = 1 + \epsilon\eta$. However, the effect of variable depth and the leakage condition enter through a more general boundary condition at the bottom, varying as $z = b(x)$:

$$w = u\frac{db}{dx} - \epsilon g(\epsilon x)G(\eta, \eta_x, ...). \tag{3.34}$$

Note that in comparison with the previous case (3.4) together with the variable depth function an additional leakage function $g(\epsilon x)$ independent of the wave profile η appears with G similar to the feedback leakage function as considered in the previous section. The bathymetry function b is assumed to depend on the small parameter ϵ, such that $b(x) = B(\epsilon x)$. As we have identified in previous section, we assume $G = \eta$ to get the maximum benefit of damping due to leakage. For detailed investigation we introduce a new set of variables

$$\xi = \frac{1}{\epsilon}\chi(X) - t, \quad X = \epsilon x, \tag{3.35}$$

where $\chi(X)$ will be determined later in equation (3.38). For solving the above set of equations we would represent the asymptotic solutions as we have used earlier.

We stress again that the hydrodynamic equations involved here are the same as those used in the previous section in dealing with the constant depth problem, except the crucial BC at the bottom.

Result at ϵ^0 order

At ϵ^0 order, the above equations are reduced to

$$u_{0\xi} = \chi' p_{0\xi}, \quad p_{0z} = 0, \quad \chi' u_{0\xi} + w_{0z} = 0, \tag{3.36}$$

together with the boundary conditions $p_0 = \eta_0, \quad w_0 = -\eta_{0\xi},$ valid at the surface and $w_0 = 0,$ at the variable bottom $z = B(X)$.

Using the above bulk equations and the boundary conditions we obtain

$$p_0 = \eta_0, \quad u_0 = \chi' \eta_0, \quad w_0 = \chi'^2 \eta_{0\xi}(B - z), \quad \chi'^2 = \frac{1}{D(X)}, \tag{3.37}$$

where $D(X) = 1 - B(X)$ and χ' is the derivative of χ with respect to X. χ can be solved explicitly through the bathymetry function for the right moving wave as

$$\chi(X) = \int_0^X \frac{dX_1}{\sqrt{D(X_1)}}. \tag{3.38}$$

ϵ order approximation

In next order approximation we obtain the set of equations

$$-u_{1\xi} + \chi' u_0 u_{0\xi} + w_0 u_{0z} = -\chi' p_{1\xi} - p_{0X}, \quad p_{1z} = w_{0\xi}, \quad \chi' u_{1\xi} + u_{0X} + w_{1z} = 0 \tag{3.39}$$

together with the surface boundary conditions

$$p_1 = \eta_1, \quad w_1 + \eta_0 w_{0z} = -\eta_{1\xi} + u_0 \chi' \eta_{0\xi}, \tag{3.40}$$

and the condition

$$w_1 = u_0 B'(X) - g(X)\eta_0, \tag{3.41}$$

valid at the variable bottom with $B'(X)$ denoting derivative in X. Our aim is to express other field variables only through the wave functions η_0 and η_1 as

$$p_1 = \eta_1 + \frac{1}{D}\eta_{0\xi\xi}[\frac{1}{2}(1-z^2) + B(z-1)] \tag{3.42}$$

and

$$w_1 = (\frac{B'}{\sqrt{D}} - g)\eta_0 + \frac{(B-z)}{\sqrt{D}}\eta_{0X} + (B-z)(\frac{\eta_0}{\sqrt{D}})_X + \frac{(B-z)}{D}\eta_{1\xi} + \frac{(B-z)}{D^2}\eta_0\eta_{0\xi}$$
$$-\frac{\eta_{0\xi\xi\xi}}{D^2}[B(\frac{z^2}{2} - z) + \frac{(z - \frac{z^3}{3})}{2} - \frac{B^3}{3} + B^2 - \frac{B}{2}]. \tag{3.43}$$

Using the above expressions we can finally derive the surface wave evolution equation

$$2\sqrt{D}\eta_{0X} + \frac{3}{D}\eta_0\eta_{0\xi} + (\frac{D'}{2\sqrt{D}} + g)\eta_0 + \frac{D}{3}\eta_{0\xi\xi\xi} = 0. \tag{3.44}$$

Note that this variable coefficient KdV equation contains explicitly the bathymetry function $D(X)$ linked to the variable depth as well as the function $g(X)$ related to the leakage at the bottom. This variable coefficient KdV equation containing the combined effect of variable depth and the leakage is an important result we have derived here. Different types of variable coefficient KdV like equations were studied earlier for analyzing the possible solutions [56, 57].

3.3.2 Nature of the solitary wave solution

It is evident that in the absence of the leakage ($g = 0$), our equation (3.44) would reduce to the KdV equation with variable depth [4, 52, 5]:

$$2\sqrt{D}\eta_{0X} + \frac{3}{D}\eta_0\eta_{0\xi} + (\frac{D'}{2\sqrt{D}})\eta_0 + \frac{D}{3}\eta_{0\xi\xi\xi} = 0. \tag{3.45}$$

When the depth variation occurs in a scale slower than the evolution scale of the wave, the solitary wave solution of equation (3.45),as is wellknown, can be expressed as an approximate solution

$$\eta_0 = \frac{A_0}{D}sech^2[\sqrt{\frac{3A_0}{4D^3}}(\xi - \frac{D^{-(\frac{5}{2})}A_0X}{2})], \tag{3.46}$$

as given in [4]. Here A_0 is the amplitude of the wave for constant depth ($D = 1$). It is clearly seen that the amplitude of the solitary wave increases as D decreases i.e. the channel becomes shallower, showing that such waves would approach the shore with surging amplitude. Note that for exponentially decreasing depth D the growing of wave amplitude will also be exponential. This particular case will be considered in more details in the Appendix.

It is intriguing to note that for variable bathymetry with uneven depth, irregular depth or periodic topography in place of growing amplitude one gets a damping wave amplitude as explained in [53]. We will be concerned however with the surging waves caused by a smoothly decreasing depth due to their hazardous effects.

Now we will analyze the solution of equation (3.44) with nontrivial boundary leakage, rewriting it in a more general form

$$a(X)\eta_{0X} + b(X)\eta_0\eta_{0\xi} + c(X)\eta_0 + d(X)\eta_{0\xi\xi\xi} = 0, \tag{3.47}$$

where we have denoted $a(X) = 2\sqrt{D}, b(X) = \frac{3}{D}, c(X) = (\frac{D'}{2\sqrt{D}} + g)$ and $d(X) = \frac{D}{3}$. Dividing (3.47) by $d(X)$ and defining $\eta_0 = \frac{U}{b_1}$ where $a_1 = \frac{a}{d}, b_1 = \frac{b}{d}$ and $c_1 = \frac{c}{d}$, respectively, the equation (3.47) can be transformed to

$$a_1 U_X + UU_\xi + U_{\xi\xi\xi} + (c_1 - a_1 \frac{b_{1X}}{b_1})U = 0, \tag{3.48}$$

which in general cannot be solved exactly. However, we may notice, that for a finer balance tuned between the variable depth bathymetry and the controlled leakage velocity function giving the condition

$$g = -\frac{9D'}{2\sqrt{D}}, \tag{3.49}$$

the last term of (3.48) vanishes reducing the equation to a more simple form of variable coefficient KdV equation

$$a_1 U_X + UU_\xi + U_{\xi\xi\xi} = 0, \tag{3.50}$$

where $a_1 = \frac{a(X)}{d(X)}$. It is interesting to note, that the tuning condition (3.49) relating the leakage function with the bathymetry function is exactly same as the solvability condition used in [57] for obtaining analytic solutions of a general variable coefficient KdV equation, considered in a formal mathematical setting.

Defining a new coordinate $T = \int \frac{\sqrt{D(X)}}{6} dX$ equation (3.50) can be transformed into the standard constant coefficient KdV equation

$$U_T + UU_\xi + U_{\xi\xi\xi} = 0, \tag{3.51}$$

admitting the well known solitary wave solution $U = N_0 sech^2[\sqrt{\frac{N_0}{12}}(\xi - \frac{N_0}{3}\int \frac{\sqrt{D}}{6} dX)]$. Expressing in terms of the original field variable we get finally the wave solution

$$\eta_0 = \frac{D^2}{9} N_0 sech^2[\sqrt{\frac{N_0}{12}}(\xi - V(X)], \quad V(X) = \frac{N_0}{3}\int \frac{\sqrt{D}}{6} dX \tag{3.52}$$

with the depth function $D(X)$ and leakage velocity function $g(X)$ are tuned as (3.49).

Note that for decreasing depth D, which without leakage would make the wave amplitude to surge as in (3.46), due to the controlled tuning of the leakage the resultant solitonic wave function would suffer a damping of its amplitude as evident from (3.52). Moreover the solitonic wave flattens down with a change in its velocity along its propagation (see FIG 5). Thus we have achieved control over a surging wave approaching to the shore by inducing combination of feedback and a controlled tuning of the the leakage at the bottom.

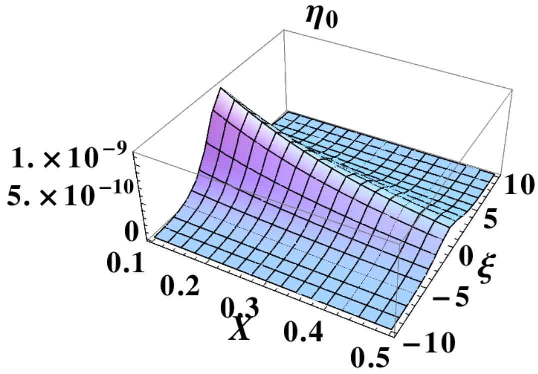


Figure 3.5: 3D plot of the solitary wave solution (3.52) in the (ξ, X) plane. For demonstrating the nature of the solution, we have assumed $N(0) = 1$ $\alpha = 0.1$, $g = \exp[-X]$, showing exponential damping of the wave amplitude with a change in its width and velocity along its propagation.

3.4 Application of the exact result to real near shore bathymetry

In the previous sections, we have first discussed the effect of wave profile dependent leakage to the solitary wave amplitude at constant water depth. Applying similar mathematical procedure to a slowly changing bathymetry, we have derived next a variable coefficient KdV equation containing terms due to both leakage and variable depth. Though in general such equations are non integrable, a finer balance between the leakage and variable depth function miraculously solves the equation exactly, giving a solitary wave like solution. Its amplitude, which without leakage would increase giving surging effects, decreases as the wave moves towards the shallower region. These theoretical findings of exact nature with an intension to control near shore surging waves, by creating artificial leakage, would gain ground when it is implemented to a real sea shore bathymetry. Therefore, in this section we apply previously obtained exact results to a near shore bathymetry in order to see the effectiveness of our findings.

One should remember the fact that according to the estimates of the United Nations in 1992, more than half of the population lives within 60 km of the shoreline. Urbanization and rapid growth of coastal cities have also been dominant population trends over the last few decades, leading to the development of numerous mega cities in all coastal regions around the world.

Our study region is the coastal zone of Chennai district of the Tamil Nadu state, in southeast coast of India which was one of the worst affected areas during 2004 Indian Ocean tsunami. A Coastal Vulnerability Index was developed for this region in [58] using eight relative risk variables including near shore bathymetry to know the high and low vulnerable areas. According to one of those risk variables, bathymetry at about 29.11 km of coastline in that area has a high risk rating having high vulnerability ,while about 18.55 km of coastline has medium risk rating and about 10.54 km shows low risk rating, which are displayed in FIG 3.6.

The depth contour of Chennai coastline, which is constructed from the Naval Hydrographic Charts for 2002, is also given in [58] and is displayed in FIG 3.7. Now to implement our exact results on this coastline, we chose one of the high risk points (N 13° 10.5′ - E 80° 18.75′) , which is denoted by the red line in FIG 3.6.

We have drawn the near shore bathymetry following the depth contour (FIG.3.7) of this shoreline point along the latitude which is given as FIG 3.8.This diagram shows that at the near shore region, the depth function flattens down denoting a slow variation along X. Hence the soliton gets enough time to evolve and give the surging effects. Note also

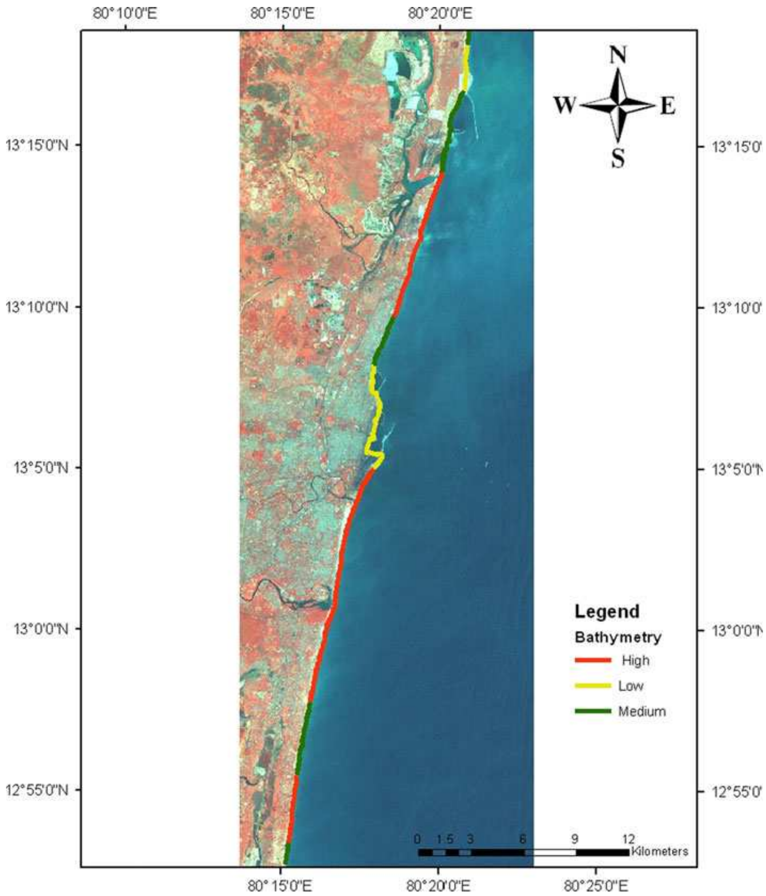


Figure 3.6: Risk zones of Chennai coastline bathymetry ,taken from [58] with permission

that, the variation along X is in Km whereas variation along $D(X)$ is in meter. Hence the depth function is very slowly varying which is consistent with our theoretical assumptions.

Note that in the absence of the leakage at the bottom, the solitary wave amplitude would increase following (3.46) with the amplitude as

$$A_1 = \frac{A_0}{D}. \tag{3.53}$$

We see from FIG.3.8 that as the wave approaches the near shore region, the depth function flattens out and therefore the soliton amplitude A_1 develops rapidly to give surging effects. Now if at a certain position in the near shore bathymetry, an artificial leakage following our theoretical findings (3.52), is turned on then the amplitude would decrease as $A_2 = \frac{N_0 D^2}{9}$,

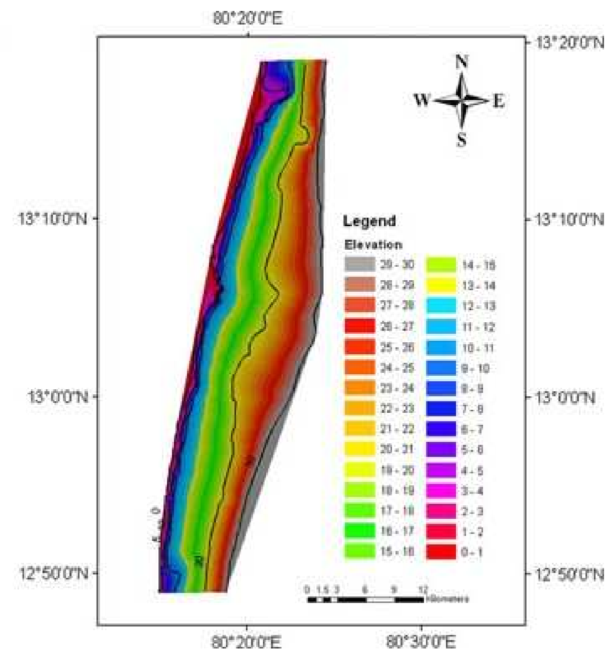


Figure 3.7: Depth contour of the Chennai coastline,taken from [58] with permission

where N_0 is a free constant as described earlier. The effectiveness of the of the amplitude decay of the solitary waves by the leakage would be stronger, if the leakage starts at a longer distance away from the shore. Note that the amplitude starts growing rapidly at 1.2 Km away from the shore, from where the depth function starts flattening.

Therefore, if a solitary wave of amplitude of nearly 1 meter starts approaching towards the shore from around 10.5 Km, then it would ultimately grow to a surging wave of amplitude $\sim$ 30 meter at the coast. It is obvious that such a huge wave will produce devastating effects on coastal habitation and costly installations.

However if we implement now an artificial leakage based feedback method linked to the surface wave profile as discussed in the previous section with exact result (3.52) the surging amplitude would decrease when propagating towards the shore. with damping amplitude given as (3.52)

$$A_2 = \frac{N_0 D^2}{9}, \tag{3.54}$$

where N_0 is a free constant. One checks that if the leakage is implemented in a region of

0.9 Km from the shore, the wave amplitude of 1 meter which would otherwise increase to 30 meter without any leakage, would decrease to an amplitude of ~ 1.23 meter as shown in FIG 3.9, where we have chosen $N_0 = 11.07$.

If the leakage installation is implemented from a nearer point from the shore, the wave amplitude decrease would also be less which is also displayed in FIG 3.9. For optimal estimation however, the cost effectiveness and the concrete requirements should be taken into account in deciding the range of such proposed installations. The main emphasis should possibly be on the protection of sensitive installations like nuclear reactors at the sea coast against the danger of tsunami like waves. The options known for the protection of the Chennai coast area are dune afforestation, mangrove restoration and management, periodic beach nourishment and building seawalls and groins etc. Our control mechanism for the possible management of the potentially hazardous near shore waves, proposed here, could be a new option, which may be implemented only in limited strategic areas surrounding costly installations, for reducing the intensity of the approaching wave to a safer limit.

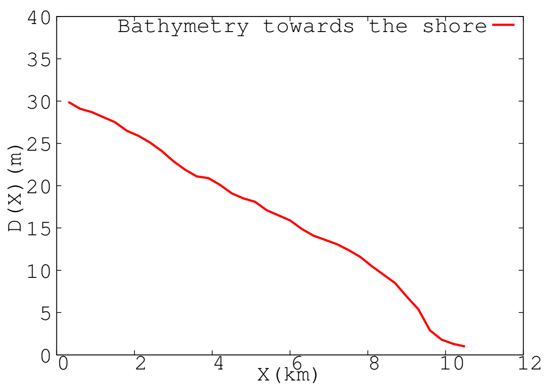


Figure 3.8: Bathymetry towards the shore of the shoreline point (N 13° 10.5′ - E 80° 18.75′)

The same methodology can be applied to another high risk point (N 13° 10.5′ - E 80° 18.75′) at the shoreline, the near shore bathymetry of which is shown in FIG 10. The increase of amplitude without leakage, and its damping due to leakage is explicit in FIG.11.

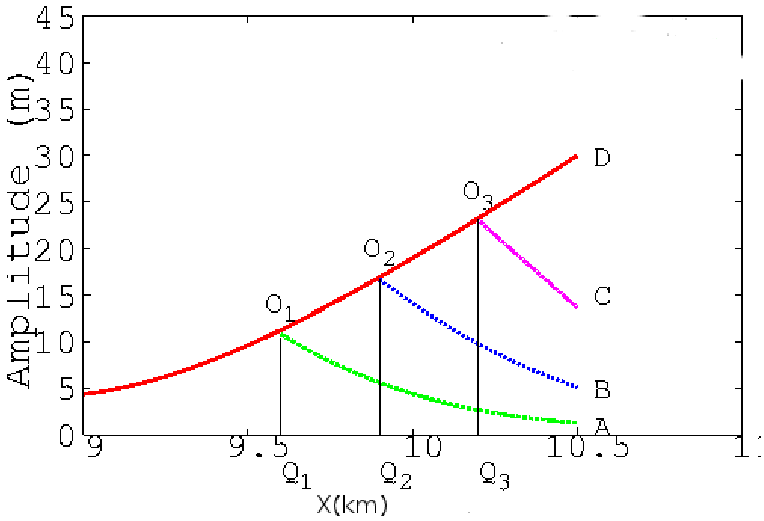


Figure 3.9: Surging amplitude A_1 without leakage moving towards the shoreline point (N 13° 10.5′ - E 80° 18.75′) and growing upto the point D (30 m) following eq.(3.53). Figure also demonstrates the damping of the amplitude A_2 due to leakage following eq. (3.54) . Installations of the leakage starting from different points to the shore Q_1 (9.6 km), Q_2 (9.9 km), and Q_3 (10.2 km), would damp the amplitude A_2 to different values (A (1.23 m), B (5.15m) and C (13.5m) respectively). N_0, a free constant appearing in eq. (3.54) is chosen as 11.60, 46.29 and 122.89 respectively at these points. It is evident that, the further the leakage is from the shore, the more the decay of the amplitude.

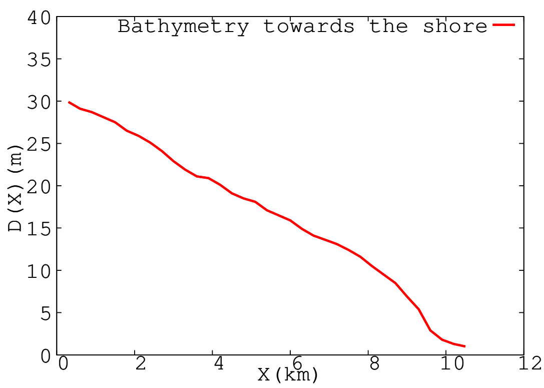


Figure 3.10: Bathymetry towards the shore of the shoreline point (N 13° 0′ - E 80° 16.2′)

3.5 Summary

The focus of our investigation is in an innovative possibility of controlling the intensity of near shore surging waves including tsunamis and bore waves by inducing damping effect through a specially designed leakage mechanism at the water bed. In the previous chapter

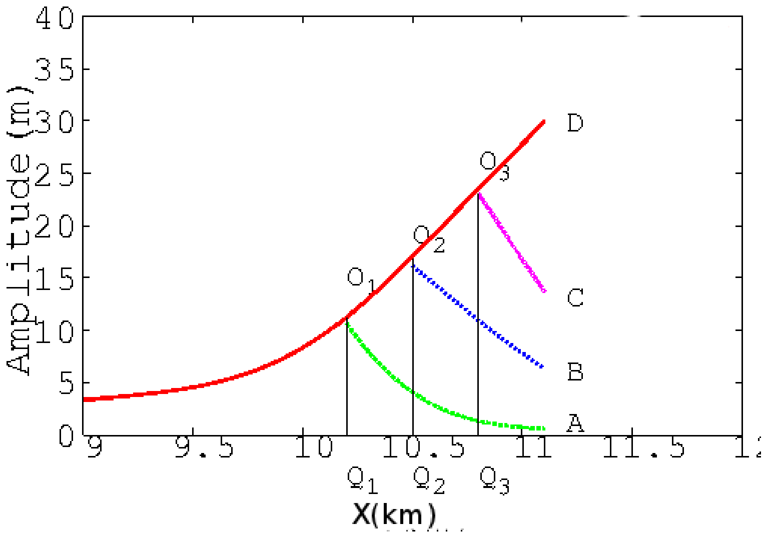


Figure 3.11: Surging amplitude A_1 without leakage moving towards the shoreline point (N 13° 10.5′ - E 80° 18.75′) and growing upto the point D (30 m) following eq.(3.53). Figure also demonstrates the damping of the amplitude A_2 due to leakage following eq. (3.54) . Installations of the leakage starting from different points Q_1(10.2 km), Q_2 (10.5 km) and Q_3 (10.8 km), would damp the amplitude A_2 to different values (A (0.53m), B(6.24m) and C(13.57m) respectively). N_0, a free constant appearing in eq. (3.54) is chosen as 5.44, 56.91 and 122.89 respectively at these points. It is evident that, the further the leakage is from the shore, the more the decay of the amplitude.

[54] we have seen that the leakage function which is independent to the free surface wave profile, affects the phase of the solitary wave solution whereas the amplitude remains constant. Dependence of leakage on the amplitude of the surface solitary wave, which would give surging effects near the shore, has not been found out.

The majority of the earlier studies, concentrated on the damping of the waves occurring due to natural effect like viscosity, bottom roughness, sand porosity etc. In contrast, our main motivation here is to analyze the impact of artificially created bottom boundary condition on the swelling wave approaching the shore with an aim to reduce the hazardous effect of such near shore wave phenomenon.

Our crucial observation is that the surging of approaching waves caused by decreasing water depth bathymetry may be thought of to be triggered by effective vertical fluid flow proportional to the gradient of the depth profile acting as a virtual *source* emerging from the bottom. Our key idea for controlling the growing amplitude of the surface wave is to counter this source by an effective *sink* through such leakage mechanism creating a downward fluid velocity.

We have considered the propagation of an unidirectional, shallow water, nonlinear free surface gravity wave based on the basic hydrodynamic equations at the shallow water regime and identified first that a feedback leakage function at the bottom, dependent linearly on the surface wave profile could induce maximum desirable damping effect on the amplitude of the surface wave. This knowledge is then applied to the problem of regulating the surging solitary waves propagating towards the shore due to the slowly decreasing depth. The corresponding evolution equation for the combined effect of leakage and the variable bathymetry turns out to be in the form of a variable depth KdV equation different from the variable coefficient KdV equation obtained earlier. Though in general this is a nonintegrable system, we have found that for a controlled tuning between the topography and leakage velocity function, the equation becomes exactly solvable allowing solitary wave solutions with damping amplitude.

A strong point of our result is its exact nature which allows one to access precise and finer effects and make more accurate predictions. We have applied the result obtained to real data from the bathymetry map of the high risk near shore regions on the Bay of Bengal in India and tested the implications, range and predictions of our theoretical result. As shown by the real bathymetry assessment, the more extensive installations starting from a further distance into the sea would result to a more effective control of the incoming surging waves, however, the cost effectiveness and the concrete requirements should be taken into account in deciding the range of such proposed installations. The main emphasis should possibly be on the protection of sensitive installations like nuclear reactors at the sea cost against the danger of tsunami like waves. Therefore the control mechanism for the possible management of the potentially hazardous near shore waves, proposed here, may be implemented only in limited strategic areas surrounding costly installations, for reducing the intensity of the approaching wave to a safer limit.

We have studied also various possible extensions of the leakage boundary conditions and their corresponding effects in modifying the nature of the surging solitary waves which might be of practical importance of different other situations.(This material is included as appendix.)

3.6 Appendix: Extension of boundary leakage condition with variable bathymetry

Though we have achieved our major goals in taming the surging waves as reported in the main text, we consider below few extensions of this result for understanding the effect of bottom boundary leakage condition on the surface wave solution, which might be of applicable interest in other physical situation. In particular ,we have investigated

A) Leakage function at the bottom with a combination of both wave profile dependent and independent functions,

B) Leakage condition linked to effective zero fluid velocity at the bottom with the specific bathymetry profile.

C)Leakage function related to time.

All the studies yielding analytic result of different nature though all of them having the effect of amplitude damping of the waves, surging otherwise due to decreasing depth bathymetry.

3.6.1 Leakage function at the bottom with a combination of both wave profile dependent and independent functions

In our previous paper [54], we considered the leakage function to be independent of the wave profile that yielded a forced KdV like equation as the surface wave equation. Its solitary wave solution exhibits phase modification leading its velocity to change whereas the amplitude remaining constant. In order to explore the effect of the bottom leakage on the solitary wave amplitude we have considered in the main text, the leakage function to be dependent on the free surface wave profile which exhibited damping of amplitude.

Now in this section, we have extended the problem such that the leakage velocity at the bottom depends both on the wave profile dependent and independent functions as

$$w = u\frac{db}{dx} - \epsilon g(\epsilon x)G(\eta, \eta_x, ...) + \epsilon C(X). \tag{3.55}$$

on $z = B$. Here the second term in (3.55) is the wave profile dependent term whereas the third one is the wave profile independent term.

As we have mentioned we assume $G = \eta$ to get the maximum benefit of damping due to leakage. After a bit of mathematical calculations we can finally derive the surface wave evolution equation

$$2\sqrt{D}\eta_{0X} + \frac{3}{D}\eta_0\eta_{0\xi} + (\frac{D'}{2\sqrt{D}} + g)\eta_0 + \frac{D}{3}\eta_{0\xi\xi\xi} = -C(X). \tag{3.56}$$

Note that this variable coefficient KdV equation contains explicitly the bathymetry function $D(X)$ linked to the variable depth as well as the function $g(X)$ and $C(X)$ related to the leakage at the bottom.

Now after applying the same balancing condition (3.49) the equation can be transformed into

$$a_1 U_X + UU_\xi + U_{\xi\xi\xi} = -E_1 \tag{3.57}$$

where we have denoted $a_1(X) = \frac{6}{\sqrt{D}}$, $E_1 = \frac{27C(X)}{D^3}$ and $\eta_0 = \frac{U}{b_1}$.

Defining a new coordinate $T = \int \frac{\sqrt{D(X)}}{6} dX$ equation (3.57) can be transformed into the standard constant coefficient KdV equation with a forcing term

$$U_T + UU_\xi + U_{\xi\xi\xi} = E_1(T), \tag{3.58}$$

admitting the well known solitary wave solution $U = N_0 sech^2[\sqrt{\frac{N_0}{12}}(\xi - \frac{N_0}{3}\int \frac{\sqrt{D}}{6}dX) - f(T)] - \int E_1 dT$.

Expressing in terms of the original field variable we get finally the wave solution

$$\eta_0 = (D^2/9)[N_0 sech^2\{\sqrt{\frac{N_0}{12}}(\xi - \frac{N_0}{3}\int \frac{\sqrt{D}}{6}dX - f(T))\} - \int E_1 dT] \tag{3.59}$$

where $\frac{\partial^2 f(T)}{\partial T^2} = -E_1$.

Note that if we neglect the wave profile independent part $C(X)$, then automatically we get $C_1 = E_1 = F = 0$ and $f =$ constant. Thus the solution (3.59) converges to the solution of earlier case (3.52).

3.6.2 Balancing through effective hard bottom condition with leakage giving exact result

Here we stick to a particular choice of decreasing bathymetry $D = \exp(-\sigma X)$, for the wave approaching to the shore. Such solitary waves without any leakage condition would result to an exponentially surging waves carrying potential hazards. Our aim here would be to control such wave through bottom leakage condition inducing necessary damping. For this purpose we consider a different balancing effect of the leakage condition, obtained from an effective hard bottom condition amounting to the vertical fluid velocity at the water bed w to be zero. This leads at the leading order to $w_0 = 0$, $w_1 = u_0 B'(X) - g(X)\eta_0 = 0$, at $z = B$, which gives a new balance between the leakage and the variable depth function as $g = -\frac{D'}{\sqrt{D}}$ at $z = B$. For this effective hard bottom condition, we follow again similar mathematical procedure as presented in the previous section, which leads to the surface wave evolution equation

$$2\sqrt{D}\eta_{0X} + \frac{3}{D}\eta_0\eta_{0\xi} - \frac{D'}{2\sqrt{D}}\eta_0 + \frac{D}{3}\eta_{0\xi\xi\xi} = 0, \quad D = D(X). \tag{3.60}$$

Note that this variable coefficient KdV equation is different from the variable bathymetry equation (3.45) obtained earlier [4]. As such this equation is also difficult to solve analytically. However interestingly for a special choice of bathymetry function $D = \exp(-\sigma X)$, with D decreasing with the increase of X, which is consistent with the wave propagating towards shallower region, we can find an exact wave solution for equation (3.60).

Dividing equation (3.60) by $\frac{D}{3}$ and redefining the field as $\eta_0 = \frac{D^2}{9}H$, the equation with our specific choice of D can be converted to

$$6\exp(\sigma X/2)H_X + HH_\xi + H_{\xi\xi\xi} = (\frac{21\sigma}{2})\exp(\sigma X/2)H. \tag{3.61}$$

Defining a new coordinate variable as $T = -\frac{\exp(-\sigma X/2)}{3\sigma}$, equation (3.61) can be transformed now to a convenient form of the so called concentric KdV equation

$$H_T + HH_\xi + H_{\xi\xi\xi} + \frac{7}{2T}H = 0, \tag{3.62}$$

which is a known integrable equation derivable from the hydrodynamic equations with cylindrical symmetry [4]. An exact solution of the variable coefficient KdV equation (3.62) is presented in [59] in the rational form as $H = \frac{(c-\frac{5}{2}\xi)}{T}$. Using the relation with our original field: $\eta_0 = \frac{D^2}{9}H$ and reverting to our old coordinates ξ, X we can transform back the solution to obtain the required exact solution for the surface wave

$$\eta_0 = -\frac{\sigma}{3}(c - \frac{5\xi}{2})\exp(-3\sigma X/2), \tag{3.63}$$

with an arbitrary constant c. Note that this is a rational solution, not of solitonic type and it behaves differently for different values of ξ. For $\xi < \frac{2c}{5}, \quad \eta_0 < 0$, for $\xi > \frac{2c}{5}$, $\eta_0 > 0$, while at $\xi = \frac{2c}{5}, \quad \eta_0 = 0$ (see Figure 3.12). Solution (3.63) shows that, the amplitude decays down due to the exponential damping factor, as the wave propagates along the positive X direction. Thus the surging waves are controlled to damping wave through balancing with the leakage at the bottom as we have aimed at. At $\xi \to \pm\infty$ the wave profile shows divergent nature. However since our intention is to consider the wave propagation towards the shore the damping effect obtained along X is the relevant factor.

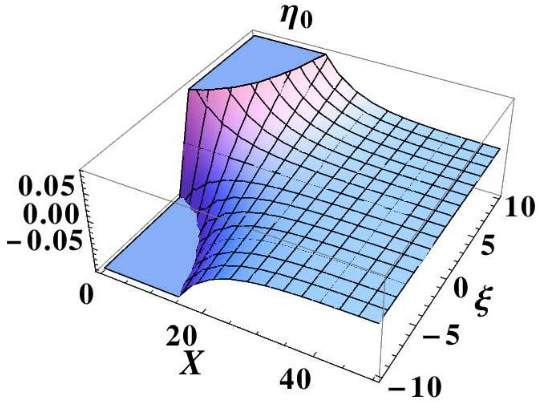


Figure 3.12: 3D plot of the exact wave solution (3.63) in the ξ, X plane with exponentially decreasing depth with X and a bottom leakage with $\sigma = 0.1$. The amplitude decay with the distance traveled along X, is evident. The divergent nature of the solution in ξ can be detected from the figure.

3.6.3 Time dependent leakage:

In all the previously discussed cases, the leakage function g is assumed to depend slowly on the space variable x as $g(\epsilon x)$. As an extension to the problem, we consider here a special kind of leakage function, which is depends slowly on time, as $g(\epsilon t)$. Hence, the bottom boundary condition at the variable bathymetry $z = B(X)$ becomes

$$w = u\frac{db}{dx} - \epsilon g(\epsilon t)G(\eta). \tag{3.64}$$

where the leakage function g depends slowly on time t. As we have mentioned we assume $G = \eta$ to get the maximum benefit of damping due to leakage. For detailed investigation we introduce a new set of variables

$$\xi = \frac{1}{\epsilon}\chi(X) - t, \quad X = \epsilon x, \Theta = \epsilon t \tag{3.65}$$

Note that, here we have introduced a new slow time variable Θ which also depends slowly on time. After a bit of calculations we can finally derive the surface wave evolution equation

$$2\sqrt{D}\eta_{0X} + \frac{3}{D}\eta_0\eta_{0\xi} + 2\eta_{0\Theta} + (\frac{D'}{2\sqrt{D}} + g(\Theta))\eta_0 + D\eta_{0\xi\xi\xi} = 0. \tag{3.66}$$

Note that two extra terms arise due to the slow time Θ which can be canceled in the following way.

Let us consider a new transformation $\eta_0 = f(\Theta)\phi(\xi, X)$. We consider $g(\Theta)$ to be such that the extra two terms which arose due to the slow time Θ cancels each other such that

$$2\eta_{0\Theta} + g(\Theta)\eta_0 = 0 \tag{3.67}$$

which finally gives $f = A\exp^{-\frac{1}{2}\int gd\Theta}$, where A is a constant. The equation satisfied by the function ϕ is nothing but that obtained by Johnson (3.45). Hence using their solution (3.46) as given in [4] the final solution can be written as

$$\eta_0 = \frac{A}{D}\exp^{-\frac{1}{2}\int gd\Theta} sech^2[\sqrt{\frac{3A_0}{4D^3}}(\xi - \frac{D^{-(\frac{5}{2})}A_0X}{2})], \tag{3.68}$$

The dynamics of the solution (3.68) can be explained like follows. As the wave propagates towards the shallower region, due to the factor $1/D$ the wave amplitude increases, whereas due to the exponentially decaying factor, which depends on time the amplitude increase is compensated to some extent. But the leakage function $g(\epsilon t)$ should be synchronized in such a way that as the wave starts increasing it starts working. Such physical mechanism and installations can be used in the other physical situations as required.

Chapter 4

Modelling rogue waves through exact dynamical lump soliton controlled by ocean currents

4.1 Introduction

In the previous two chapters, we have dealt shallow water wave phenomena. But as we move into deep water, waves behave quite differently. For example a wave of amplitude ≈ 1 meter in deep sea would evolve to give surging tsunami wave at the shore. In this chapter, we will a discuss deep water wave phenomena called rogue waves which can also be observed at shallow water. The mysterious ocean rogue waves (RWs) are reported to being observed in a relatively calm sea, where they, as a localized and isolated surface waves, apparently appear from nowhere, make a sudden hole in the sea just before attaining surprisingly high amplitude and disappear again without a trace [60, 21, 61, 62, 63]. This elusive freak wave caught the imagination of the broad scientific community quite recently [64, 65, 66, 67, 68, 69, 72, 73], triggering off an upsurge in theoretical [76, 75, 74, 63] and experimental [61, 64, 65, 66, 67, 68, 69, 72, 73] studies of this unique phenomenon. For identifying such extreme waves the suggested signature of these rare events is a deviation of the probability distribution function (PDF) of the wave amplitude from its usual random Gaussian distribution (GD), by having a long-tail, indicating that the appearance of high intensity pulses more often, has much higher probability than that predicted by the GD [77, 63, 67]. In conformity with this definition RWs were detected in a photonic crystal fiber [68], in a multi-stable state of an erbium doped fiber laser [73], in chaotic but deterministic regime of optical injected semiconductor lasers [64, 61], in nonlinear optical cavity [67], in acoustic turbulence in He II [65] and other set ups [69, 70, 71] including other physical systems like plasma, Bose-Einstein condensates etc. On the formation of the ocean RWs a number of supporting linear as well as nonlinear theories have been developed [21]. Among various possible factors contributing to the creation of the RW, the modulation instability (MI) supported by the nonlinear effect is believed to play a crucial role, by inducing preliminary amplification of water wave height , which may trigger self attractive nonlinear interaction, initiating the RW formation [78]. The MI can also cause wave-wave interaction leading to the four-wave mixing at matching frequencies and wave numbers, inducing resonance effect which might also develop into a RW [63, 69, 84]. Though RWs have been found both in shallow [21, 22] and deep water, in this work we shall concentrate on the oceanic rogue waves which preferably occurs at the deep sea. Like the four-wave nonlinear interaction, a leading order nonlinear effect in deep-sea waves, is found also to be a dominant interaction in the nonlinear Schrödinger (NLS) equation

$$iq_t = q_{xx} + 2|q|^2 q, \tag{4.1}$$

with the subscripts denoting partial differentiation. One of the previous works of explanation of RW by NLS equation was done by Smith [85] in the presence of Agulhas Current where it was stated that the giant waves occur where the wave groups are reflected by the current. The NLS based nonlinear models are the most accepted ones for the RW, though often with certain modifications to include higher oder dispersion or ocean currents, which are suspected to have a deciding role in the formation of the RW [62]. In extended space dimensional systems the nonlinear effect due to the MI in combination with a space-asymmetry, directional spectra and broken symmetry due to nonlocal coupling is suspected to be the major causes of such extreme waves [66, 67, 74]. The NLS equation (4.1) is a well known evolution equation with integrability properties like having a Lax pair and exact soliton solutions [88]. Some models of RW generalize the NLS equation with the addition of extra terms on physical grounds, like ocean current [62], nonlinear dispersion [76, 89] etc. However such modifications of the NLS equation (4.1) make the system nonintegrable, allowing only numerical solutions. The most popular 1D RW model is a unique analytic rational solution of the original NLS equation (4.1) [79], given by the Peregrine breather (PB) [68, 75, 72] or its higher order versions [24]-[25] and the trigonometric variants [90, 23]. However, since the RW is an aperiodic event with a single appearance, the trigonometric breather solutions, due to their periodic nature, are not much suitable for a direct description of the RW. Nevertheless, interestingly these breather solutions, periodic in time [90] or in space [23], degenerate to the rational PB solution (4.2) at their periods going to infinity[94, 102].

Note that the conventional soliton solution of the NLS equation (4.1), representing a localized translational wave behaves like a stable particle and unlike a RW propagates with unchanged shape and amplitude. Tsunami waves, though highly devastating, also exhibits different nature than the ocean RW. The ocean RW are mainly deep sea waves with 2D character, localized in both space dimensions and appears as a single-peak event for a short interval of time. Tsunami waves on the other hand are manifested only in shallow water near the sea shore, though generated in the deep sea and propagate across a long distance. In the deep sea tsunami waves behave like 1D translatory wave, moving very fast with insignificant amplitude[80]. Therefore tsunamis and the RWs exhibit different features and dynamics and need different types of modelling which for the RW is still an open problem. More details on the progress in the study of the ocean RW can be found in some excellent reviews on the subject [21].

4.1.1 RW model on a 1D line

In contrast to the soliton or the trigonometric breather solutions of the NLS equation (4.1), its exact rational PB solution

$$\begin{aligned} q_P(x,t) &= e^{-2it}(u+iv),\ u = G-1,\ v = -4tG, \\ \text{where } G &= 1/F(x,t),\ F(x,t) = x^2 + 4t^2 + \frac{1}{4}, \end{aligned} \tag{4.2}$$

represents a breather mode with unit amplitude at both distant past and future. The amplitude of the wave rises suddenly at $t = 0$, attaining its maximum at $x = 0$, though subsiding with time again to the same breathing state. This intriguing behavior makes the PB a popular candidate for the RW [68, 75, 72].

Since the characteristics of the envelop wave is the most significant in the description for the RW, the modulus of the PB solution (4.2)

$$|q_P(x,t)| = (u^2 + v^2)^{\frac{1}{2}} = [(G-1)^2 + (4tG)^2]^{\frac{1}{2}}, \tag{4.3}$$

with G as in (4.2), is used in describing the RW profile. The full grown 1D RW at $t = 0$ therefore may be represented by

$$|q_P(x,0)| = (G-1)|_{t=0} = [\frac{1}{x^2 + \frac{1}{4}} - 1], \tag{4.4}$$

as shown in Fig. 4.1. The maximum amplitude as seen from (4.4) is attained at $x = 0$ as

$|q_P(0,0)| = 3$. The modular inclination defined as

$$S_P^x(x) = \frac{\partial}{\partial x}|q_P(x,0)| = -\frac{2x}{(x^2+\frac{1}{4})^2} \tag{4.5}$$

attains its maximum $S_{Pmax}^x(x_m) = 3\sqrt{3}$ at $x_m = \pm\frac{1}{2\sqrt{3}}$.

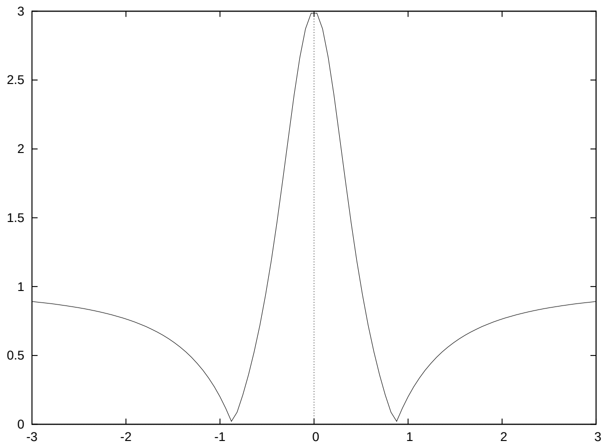


Figure 4.1: Amplitude variation of the full grown 1D rogue wave, modelled by the modulus of the static Peregrine breather $|q_P(x,0)|$. The maximum amplitude 3 is attained at $x = 0$, while it goes to its asymptotic value 1 at $x \to \pm\infty$. The maximum inclination attainable is $3\sqrt{3}$ at $x = \frac{\sqrt{3}}{6}$, and becomes 0 both at $x = 0$ and $x \to \pm\infty$.

Notice however that, the NLS equation (4.1) together with its different generalizations are equations in $(1+1)$-dimensions and therefore all of their solutions, including the PB and its higher order generalizations, can describe the time evolution of a wave only along an one dimensional line (as in Fig. 4.1). Looking more closely into the PB we also realize that the maximum amplitude of this solution describing an 1D RW is fixed, and just three times that of the background waves (see Fig.4.1). The modular inclination of this wave as well as the fastness of its appearance are also fixed, since solution (4.2) admits no free parameters. This situation can be improved to obtain higher amplitude and modular inclination of the PB model by using higher order rational solutions [24]. For example, the next higher order PB known also as Akhmediev-Peregrine breather can enhance the maximum wave elevation by a factor of five, while the next one by a factor of seven and so on, with an intriguing enhancement of factors by increasing odd numbers. Thus, the maximum amplitude occurs at the origin and have the magnitude $2j+1$ where j is the order of rational solution. Such increments in amplitude however are discrete and could be achieved at the cost of going to new solutions with increasingly complicated structures involving higher and higher order polynomials [24]. The maximum amplitude and modular inclination reachable by this class of solutions are fixed due to the absence of relevant free tunable parameters, making it difficult to adjust to the continuously varied range of shape and sizes of the observed oceanic RWs. However recently higher order rational solutions to the NLS equation allowing free parameters have been discovered [82, 25], though they seem to represent multi-peak wave in the $x-t$ plane for the nontrivial choice of parameters [25]. It is shown that general $N-$th order rogue wave contains $N-1$ free irreducible complex parameters. For different values of these free parameters, these rogue waves can exhibit other solution dynamics such as arrays of fundamental RW arising at different time and spatial positions and forming interesting patterns. The single-peak solution which is suitable for describing oceanic RW having a single appearance in time, is obtained unfortunately for a trivial choice of the free parameters. The trigonometric breathers [90, 23] also contain free parameters [94], though such periodic solutions, as mentioned already, are different in nature than the single crest RW event. Along with NLSE, RW solution was also discovered in other (1+1)D integrable equations like Hirota equation[86] and DNLS equation [87]. The crucial fact however is that, the 1D spatial nature remains the same for the whole class of the solutions, including its higher order rational and trigonometric generalizations. Therefore modelling an ocean RW, which is a 2D surface wave, by this class of 1D solutions remains problematic.

4.1.2 Need for a RW model on a 2D plane

Therefore, though the well accepted class of PB or other solutions of the generalized NLS equation could fit into the working definition of the ocean RW, saying any wave with height more than twice the nearby *significant height* (average height among one-third of the highest waves) could be treated as the RW [77], they perhaps, with their restricted characteristics, can explain successfully only fixed and moderately intense RW-like events on a 1D line, as observed in water channels [72], optical fibers [68, 73] or optical lasers [61, 64], but seem to be not satisfactory for modelling the ocean surface RWs.

Oceanic RWs are said to be consist of an almost vertical wall of water preceded by a trough so deep, that it was referred to as a hole in the sea [83]. In march 2001, two reputed ships named as Bremen and Calendonian Star, carrying hundreds of tourists across the South Atlantic, had a devastating encounter with RW like events. It is reported by the witnesses that a giant isolated wave of around 30 meter high, fell upon the ship like a wall of water, out of no where and disappeared again without a trace [91]. At the initiatives of 11 organizations involving several countries in EU tasked the Earth - scanning satellites, named ERS-1 and ERS-2, to send images from a localized area of $10 \times 5\ km^2$ on the sea surface at certain locations to spot the possible occurrence of rogue waves [91].

All these available facts and information suggest that unlike the tsunami and internal waves, pictures of which can be seen through satellite images [92], ocean RWs with the hole states must have a 2D character, localized in both the space dimensions. In 2D water basin experiments as well as in the related simulations the amplitude and the modular inclination of the RWs were found to be higher [66, 69, 74, 75] than those predicted and observed in 1D [72, 75].

The above arguments should be convincing enough to go beyond the 1D equations and search for a suitable (2+1) dimensional equations, to find a 2D alternative to the PB and other solutions of the 1D NLS equation, for constructing a more realistic model for the ocean RWs.

There are many nonlinear equations known in (2+1) dimensions having fruitful applications in various fields. Some of them allow exact analytic solutions, while others permit only approximate numerical simulation.

The well known KP equation is an integrable extension of the KdV equation to 2D space [118, 119] describing the dynamics of a real field. However the KP equation like the KdV is a shallow water model, whereas the oceanic RW is naturally a deep water phenomenon.

There are also several equations extending the 1D NLS equation to (2+1) dimensions. From the basic hydrodynamic equations, by taking the perturbation analysis to a higher order, Dysthe has derived for the deep water waves a 2D evolution equation [94, 120] which is applicable for a more rough sea. The Dysthe equation in general is non integrable .

The Davey-Stewartson equation [20] is a 2D generalization of NLS equation where the existence of rogue wave has been analyzed [116, 96]. However such fundamental rogue wave solutions of Davey-Stewartson system are line rogue waves arising from constant background in a line profile and retreat back to constant background again. Hence, it is reducible to the PB solution by a simple rotation in the plane. BLP equation [117] is another $(2 + 1)$ dimensional integrable equation, defined through two real coupled equations. Recently a RW type solution has been found in this equation allowing a free parameter [26]. However since the BPL equation describes wave propagation along an infinite narrow channel of constant depth , its applicability in modelling the ocean RW is questionable.

Zakharov have proposed several 2D equations[63], some of them are integrable [122, 121] while others are not [84, 97]. Though these equations are applicable in other fields [97], the model proposed in [84] seem to be a successful model for the RW.

A straightforward 2D extension of the NLS equation :

$$iq_t = d_1 q_{xx} - d_2 q_{yy} + 2|q|^2 q, \tag{4.6}$$

where $q(x, y, t)$ is a slowly varying envelop and d_1, d_2 represents linear dispersion coefficients [74] of the deep water gravity wave, was proposed in connection with RW [74, 75].

Note however that the 2D NLS equation (4.6) is not an integrable system and gives only approximate numerical solutions with no stable soliton. Nevertheless, this unlikely candidate is found to exhibit RW like structures numerically, with higher amplitude and modular inclination and with an intriguing directional preference [66, 74] with broken spatial symmetry [67, 75]. However though the experimental and theoretical studies on nonlinear systems in 2D space have shown promises in describing more realistic situations in the formation of 2D ocean RWs, unfortunately, all of them can give only approximate numerical results and most of this models could not consider the effect of ocean current which is supposed to play a crucial role in the formation of ocean RWs [75, 74].

4.2 Proposed integrable 2D NLS equation

In the light of not so satisfactory present state in modelling the deep sea RWs, we propose an *integrable* extension of the 2D NLS equation:

$$iq_t \;=\; d_1 q_{xx} - d_2 q_{yy} + 2iq(\sqrt{d_1}j^x - \sqrt{d_2}j^y), \quad j^a \equiv q q_a^* - q^* q_a, \tag{4.7}$$

allowing an exact lump-soliton as a suitable RW model. In (4.7) the linear dispersion relation is exactly same as the conventional water wave dispersion as described in (4.6), with the only difference from this well known 2D NLS equation being in the nonlinear term. Notice that, when the conventional *amplitude*-like nonlinear term in the non-integrable equation (4.6) is replaced by a nonlinear *current*-like term (expressed through j^x, j^y), the resulting equation (4.7) miraculously becomes a completely integrable system with all its characteristic properties, which is much rarer in 2D than in 1D. Before proceeding further observe, that through scaling and a $\frac{\pi}{4}$ rotation on the plane : $(x, y) \to (\bar{x},\ \bar{y})$ with $\bar{x} = \frac{1}{2}(-\frac{x}{\sqrt{d_1}} + \frac{1}{\sqrt{d_2}}y)$, $\bar{y} = \frac{1}{2}(\frac{x}{\sqrt{d_1}} + \frac{1}{\sqrt{d_2}}y)$ and $\bar{t} = 2t$, our 2D NLS equation (4.7) can be simplified to

$$iq_t + q_{xy} + 2iq(qq_x^* - q^*q_x) = 0, \tag{4.8}$$

where the *bar* over the coordinates is omitted. Encouragingly, our 2D NLS equation (4.8), at par with the well known 1D NLS equation is derivable from the more fundamental hydrodynamic equations and exhibits MI together with a nonlinear frequency correction, as we show below. Equation (4.8) admits also exact soliton and breather solutions through the standard formalism of Hirota's bilinearization and an associated Lax pair as well as an infinite set of conserved charges [104], proving thus the integrability of this nonlinear equation . More satisfactorily, equation (4.8), as we see below, admits an exact 2D generalization of the PB with the desirable properties of a realistic surface RW. It is promising that many characteristic properties like directional preference, MI, appearance of higher amplitude etc observed theoretically and experimentally in connection with the formation of RW in 2D models [66, 67, 75, 98, 99, 100, 74], which remained as numerical approximations, get confirmed through analytic result in our model based on the integrable equation (4.8).

4.2.1 Nonlinear frequency correction and modulation instability

Instability of a planer wave, appearing due to the interplay between dispersion and nonlinear effect called Benjamin Feir or MI [109], which has been in the continuous focus for many years [107, 108], has gained more importance recently in the context of the RW. MI was first discovered by Lighthill[110], developed independently by Benjamin- Feir[109] and Zakharov [111] and first observed experimentally by Feir[112]. The nonlinearity and the MI are supposed to be the basic reason behind the formation of RWs. Therefore, before progressing further with our 2D NLS equation (4.8), we focus on the correction of its linear frequency induced by the nonlinear effect and the appearance of the MI mediated by such nonlinearity in the system. For investigating the contributions to the frequency due to the linear dispersive and the nonlinear term in (4.8), we insert the plane wave solution $q_0 = A_0\ e^{i(\omega t + k^x x + k^y y)}$, with A_0 as the real constant amplitude, ω as frequency and (k^x, k^y) as the wave vector . For the plane wave to be an exact solution of (4.8), the frequency should be $\omega = \omega_L + \omega_{NL},\ \omega_L = -k^x k^y,\ \omega_{NL} = 4A_0^2 k^x$, where ω_L is the frequency due to linear dispersion and ω_{NL} is its nonlinear correction, which depends on the amplitude of the wave as well as on the x component of the wave vector.

Now to explore the onset of MI in the system affecting this plane wave solution, we perturb it by a small parameter function $\epsilon(x, y, t)$. Note that the perturbation is considered in both the space directions since its importance in the instability in 2D is emphasized in the context of RW formation [75]. The solution

$$q_\epsilon = (A_0 + \epsilon)\ e^{i(\omega t + k^x x + k^y y)}, \tag{4.9}$$

neglecting the higher order terms in ϵ yields from (4.8) a linear equation for ϵ as

$$i\epsilon_t + \epsilon_{xy} + i(k^y \epsilon_x + k^x \epsilon_y) + 2iA_0^2(\epsilon_x^* - \epsilon_x) + 4A_0^2 k^x(\epsilon^* + \epsilon) = 0. \tag{4.10}$$

The appearance of the last two terms in equation (4.10) is due to the nonlinearity. For detecting the instability of the perturbation we represent $\epsilon = c_1 e^{i(\omega_m t + k_m^x x + k_m^y y)} + c_2 e^{-i(\omega_m t + k_m^x x + k_m^y y)}$ Inserting this form of perturbation in equation (4.10) and arranging the independent terms we get a set of two homogeneous equations for the arbitrary coefficients c_1, c_2, nontrivial solutions of which can exist only when the determinant of the matrix vanishes leading to the necessary relation $\bar{\omega}_m^2 = K^2 - \Omega_c$, where $\bar{\omega}_m = \omega_m - \omega_0$, and $\omega_0 = 2A_0^2 k_m^x - k^x k_m^y - k_m^x k^y,\ K = k_m^x k_m^y - 4A_0^2 k^x,\ \Omega_c = 4A_0^4(4k^{x2} - k_m^{x\,2})$, which gives finally

$$\omega_m = \omega_0 \pm i\omega_I,\ \omega_I = (\Omega_c - K^2)^{\frac{1}{2}}. \tag{4.11}$$

Therefore, under the condition $K^2 < \Omega_c$ with $\Omega_c > 0$, i.e when $|k_m^x| < 2|k^x|$ the modulation frequency ω_m can acquire an imaginary part ω_I, initiating an exponential growth of perturbation with time t and hence onsetting the MI. ω_I is the growth rate of the instability given by (4.11), a graphical form of which is presented in Fig. 4.2, showing its dependence on the longitudinal and transverse directions through k_x^m, and k_y^m respectively.

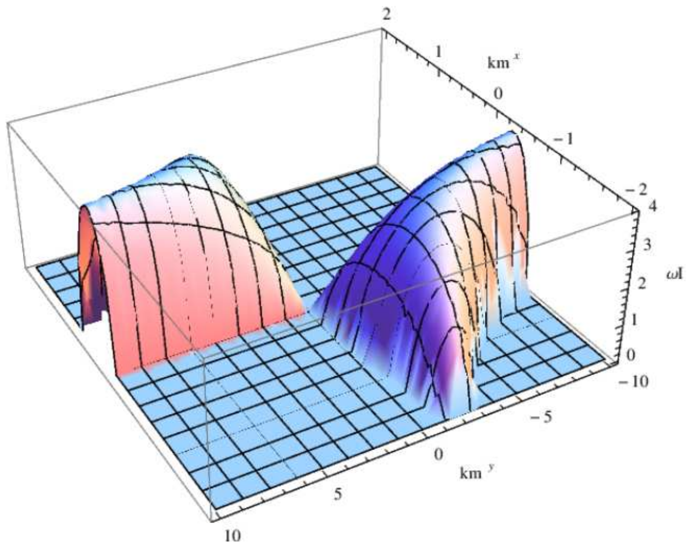


Figure 4.2: The growth rate ω_I of the MI given by (4.11), arising in our 2DNLS equation, exhibiting how it changes (for $A_0 = 1.0, k^x = 1.0$) along the longitudinal (k_m^x) and transverse (k_m^y) directions, showing a strong directional preference.

Both these figures show clearly, that the behavior of MI as well as the growth rate has a strong directional preference and range as observed also earlier in 2D models [66, 75, 113, 99, 100, 74]. We have confirmed such properties through exact analytic result showing explicitly that in the MI as well as in the growth rate the components (k_m^x, k_m^y) of the wave vector do not enter symmetrically, in addition with a directional range $|k_m^x| < 2|k^x|$.

A comparison here with the analysis of MI in case of the known 1D NLS equation [81] may be illuminating. The condition for the onset of instability in the 1D case involves only the nonlinear amplitude A_0 expressed in the form $|k_m^x| < 2A_0$, while in the present situation the condition is more complicated involving all components k_m^x, k_m^y, k^x apart from A_0, together with an allowed range on the wave vector component, as found above analytically and shown graphically in Fig. 4.3. Similar situation is also true for the growth

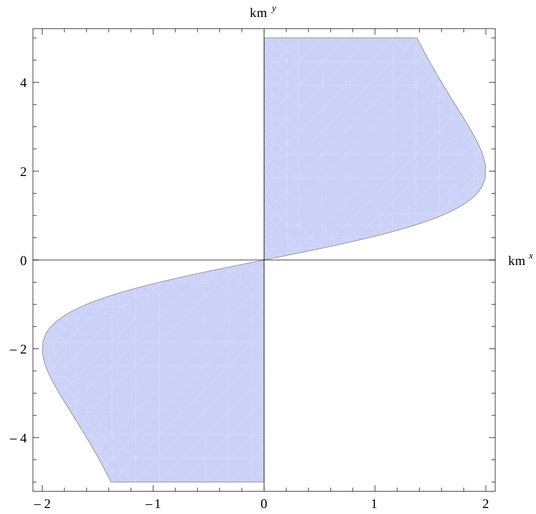


Figure 4.3: Graphical representation of the MI region, where the instability can occur only within the shaded area (for fixed values of $A_0 = 1.0, K^x = 1.0$). The instability region, showing dependence on the wave vector (k_x^m, k_y^m), varies asymmetrically along the longitudinal and transverse direction, as seen clearly from the figure.

rates, where in the 1D case it is given by $\omega_I = \ |k_m^x|[(2A_0)^2 - (k_m^x)^2]^{\frac{1}{2}}$ [81], while in the present case the form of ω_I is more complicated and depends on both longitudinal and transverse directions, as shown above.

Thus the overall picture for the onset of the MI is similar to that occurring in the 1D NLS equation [81], though in the case of the 2D NLS equation (4.8) the details are different and more intricate with a directional preference and range, as seen also for the MI, initiating RW formation in some other systems in higher space-dimensions [66, 74, 67, 75, 101]. We emphasize however, that in place of approximate numerical result obtained earlier [114], we found here similar properties in exact analytic form in our model. This is a strong point of our exact model. As in the case of the well studied 1D NLS model, we may expect the MI to play a key role in the creation of RWs based on our 2D NLS model (4.8).

4.3 Modelling of 2D rogue waves

Apart from finding a novel 2D integrable equation (4.8), our aim, relevant to the present problem, is to construct a 2D RW model as an exact solution of this equation.

4.3.1 Static lump soliton

Before presenting the dynamical lump solution related to (4.8) we consider first its static 2D lump-like structure:

$$q_{P(2d)}(x,y) = e^{4iy}(u+iv),\ u = G-1,\ v = -4yG,$$
$$\text{where } G \equiv \frac{1}{F(x,y)},\ F(x,y) = \alpha x^2 + 4y^2 + c, \tag{4.12}$$

localized in both space directions and describing a fully developed RW. One can check by direct insertion that (4.12), having two arbitrary parameters α and c, is an exact static solution of the 2D nonlinear equation (4.8). Solution (4.12), in spite of its close resemblance with the well known PB solution (4.2), marks some important differences. The static wave profile $|q_P(x,0)|$ (4.4), obtained from PB solution (4.2) at time $t=0$ is a curve, representing full blown 1D RWs admitting no free parameters of relevance. On the other hand

$$|q_{P(2D)}(x,y)| = (u^2+v^2)^{\frac{1}{2}} = [(G-1)^2 + (4yG)^2]^{\frac{1}{2}}, \tag{4.13}$$

obtained from the static solution (4.12) represents a 2D lump with two independent free parameters, significance of which will be is explained below and shown in Fig. 2(a-d).

4.3.2 Rogue wave with adjustable amplitude, inclination and hole waves

Note, that the static lump soliton (4.12), can be obtained from the dynamical RW solution (4.19) at the static point $t = 0$ (Fig. 3c) similar to static PB profile obtained from (4.2), and hence it physically represents a full grown RW solution as shown in Fig. 2 (d). Looking more closely into solution (4.12), for understanding the physical relevance of its free parameters c and α, we notice that the wave attains its maximum amplitude: $|q_{P(2D)}(0,0)| \equiv A_{rog}(c) = (\frac{1}{c} - 1)$, at the center $(x = 0,\ y = 0)$, while at large distances $(|x| \to \infty,\ |y| \to \infty)$ the wave goes to the background plane wave, with its amplitude decreasing to $A_\infty = 1$. Therefore the maximum amplitude reachable by our RW solution, relative to that of the background wave is $\frac{A_{rog}(c)}{A_\infty} = (\frac{1}{c} - 1)$. Consequently, the amplitude of the full grown RW described by the lump soliton can be changed continuously by changing parameter c (with $A_{rogue}(c)$ increasing with decreasing c) and could therefore be adjusted to fit the heights of any observed RW. Consequently, the maximum RW amplitude in our model can be made as high as desired, by decreasing the value of an arbitrary smooth parameter c (see Fig. 2 (a-d), for particular examples). Comparing this situation with the conventional 1D RW model given by the PB (4.2) and its generalizations [24], as mentioned above, we conclude that, in the well known class of PB solutions, the maximum amplitudes reachable by the 1D RW are given by the fixed discrete odd numbers $2j + 1$, with $j = 1, 2, 3, \ldots$ and can be obtained by going only to different higher solutions involving more and more complicated higher order polynomials. The higher rational solutions having free parameters [82, 25] become parameterless for a single-peak RW solution [25]. On the other hand, in our 2D RW model the maximum amplitude $A_{rog}(c)$ can be varied continuously and increased as required, by tuning an arbitrary parameter c in the same single-peak, first order solution (4.12) or its dynamical extension (4.19) (as shown in Fig. 2 (a-d)), making the model suitable for RWs with a diverse range of heights, anywhere in the range 17-30 meters in calm sea [21]-[62], as observed in deep sea 2D RWs.

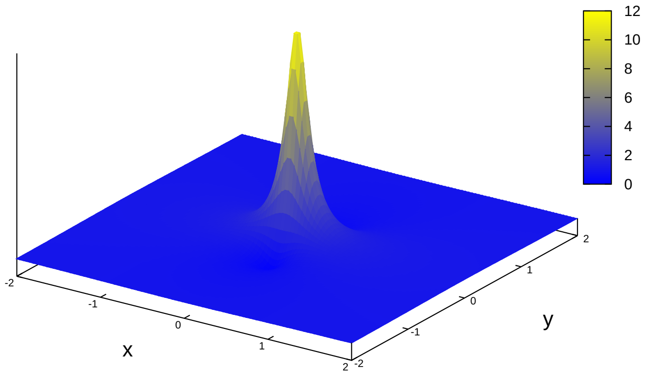
Extending the modular inclination in case of 1D: $S^x_P(x)$, as defined in (4.5) we get for the full grown 2D RW solution (4.12) the modular inclination as

$$S^x_{P(2D)}(x,y) = \frac{\partial}{\partial x}|q_{P_{(2D)}}(x,y)|, \quad S^y_{P(2D)}(x,y) = \frac{\partial}{\partial y}|q_{P_{(2D)}}(x,y)| \tag{4.14}$$

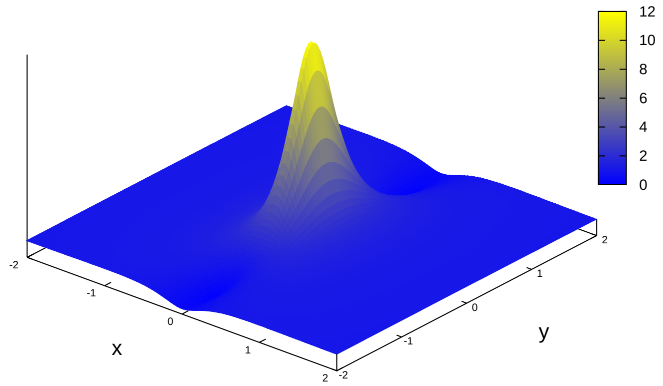
Focusing on the inclination $S^x_{P(2D)}(x,0)$ as observed at the middle of the wave front, we notice, that it is linked also to another free parameter α and attains its maximum

$$S^x_{P(2D)max}(x_m,0) = -2\alpha x_m G^2(x_m,0) \tag{4.15}$$

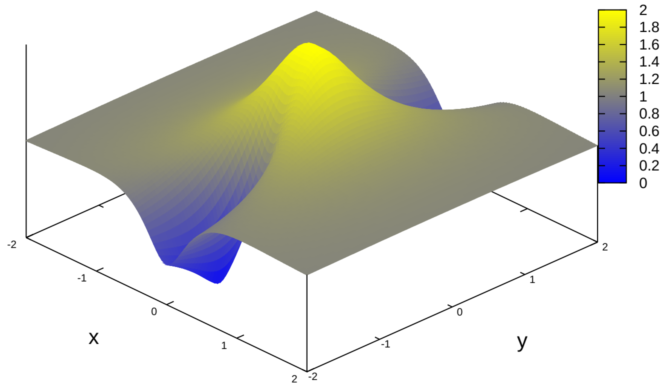
at $x_m = \frac{\sqrt{c}}{\sqrt{3\alpha}}$ with function $G(x,y)$ as defined in (4.12).In [63], it is mentioned that the real RWs are more steep than that predicted from NLS equation. We see that the maximum modular inclination of a full grown 2D RW in our model depends on both the parameters c and α in an intricate way and can be changed continuously by varying two arbitrary parameters to fit varied situations (Fig. 4(a-d)). Note that this inclination will be influenced by the physical steepness of the wave, contributing from the wave vector of the career wave. We can identify another intriguing feature of our solution, by noting that the amplitude of the wave (4.13) falls to its minimum: $A_0 = 0$, at $y = 0$, $x = \pm x_0$ where $x_0 = \sqrt{\frac{1}{\alpha}(1-c)}$, which depends again on two free parameters. This significant feature emerging from our RW model, as will be demonstrated below in Fig. 3(a,b), is related to the *hole-wave* formation observed during ocean RWs [84, 83],[75].



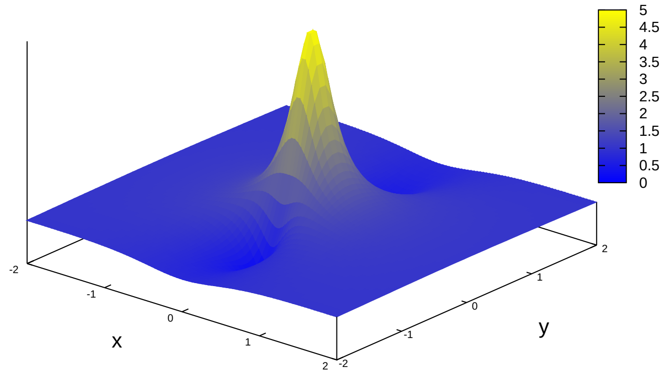
(a) High amplitude (12) and high modular inclination, for $c = 1/13,\ \alpha = 4.0$.



(b) High amplitude (12) but low modular inclination, for $c = 1/13,\ \alpha = 0.4$



(c) Low amplitude (2) but low modular inclination, for $c = 1/3,\ \alpha = 0.4$



(d) Moderate amplitude (5) and moderate modular inclination, for $c = 1/6,\ \alpha = 1.2$. The last situation is the same as figure 5c, obtained at $t = 0$.

FIG.2: **Full grown two-dimensional rogue wave modelled by the modulus ($|q_{p(2D)}|$ of the static lump soliton (4.12) with different shapes and sizes, generated from the same single- peak solution. The maximum amplitude and modular inclination are tunable through two parameters c and α.**

4.3.3 Topological consideration

Though the static lump-solution (4.12) can describe the profile of a full grown 2D RW, for modelling an evolving realistic RW, we need to find a time-dependent solution, which would smoothly go to its static form (4.12) at the moment $t = 0$. Our next aim therefore is to construct a dynamical lump soliton out of the static lump-solution, to create a true picture of a RW which can appear and disappear fast with time. However, for constructing such a solution we have to clarify first, whether it is possible in principle for our lump soliton to disappear without a trace, i.e. whether the soliton is free from all topological restrictions, which otherwise would prevent such a vanishing. The reason for such suspicion is due an interesting lesson from topology stating that, when a complex field $q(x, y)$ is defined on a 2D space with non-vanishing boundary condition $|q| \to 1$ at large distances, but having vanishing values $q \to 0$ close to the center, we can define a unit vector $\hat{\phi} = \frac{q}{|q|}$ on an 1-sphere S^1. However, this vector $\hat{\phi} = (\phi^1, \phi^2)$ is well defined only at the space boundaries: $\partial \mathrm{R}^2 \sim S^1$ (since $q = 0$ at inner points), realizing a smooth map: $S^1 \to S^1$ with possible nontrivial topological charge $Q = n$. This charge with integer values $n = 0, 1, 2, \ldots$, labels the distinct homotopy classes and is defined as the degree of the map, which unlike a Nöther charge is conserved irrespective of the dynamics of the system. Such a situation occurs for example in type II superconductors with the charge linked to the quantized flux of vortices for the magnetic field $\mathbf{B}(x, y)$ [106] :

$$2\pi Q = \int d\mathbf{S} \cdot \mathbf{B} = \int_C \mathbf{dl} \cdot \mathbf{A}, \tag{4.16}$$

where $\mathbf{B} = \text{curl}\, \mathbf{A} = \hat{\mathbf{z}}(\partial_x \phi^1 \partial_y \phi^2 - \partial_x \phi^2 \partial_y \phi^1)$. Notice that, our complex field solution $q_{P(2d)}(x, y)$ possesses clearly the features of $\hat{\phi}$ discussed above, since (4.12) goes to a constant modulation $-e^{4iy}$ at large distances and vanishes at points $(0, \pm x_0)$. Note that, such a solution related to a sphere to sphere map can not go to a trivial configuration , if it belongs to a homotopy class with nontrivial topological charge: $Q = n, n = 1, 2, 3, ...,$

due to conservation of the charge, with the only exception for the class with zero charge $Q = 0$. Therefore, for confirming the possible appearance/disappearance property of a RW for solution (4.12), we have to establish first that in spite of defining a nontrivial topological map, it belongs nevertheless to the sector with topological charge: $Q = 0$, i.e. our lump soliton is indeed shrinkable to the *vacuum* solution. For this we calculate explicitly the topological charge (4.16) associated with (4.12) as

$$2\pi Q = \int_C \mathbf{dl} \cdot \mathbf{A} = \int (dxA_x + dyA_x), , \tag{4.17}$$

where $A_a = \phi^1 \partial_a \phi^2$, $\phi_1 = \text{Re } q/|q|$, $\phi_2 = \text{Im } q/|q|$, where the contour integral along x and y are taken along a closed square at the boundaries of the plane. Substituting explicit form of solution $q(x, y)$ from (4.12) and arguing about the oddness and evenness of the integrand with respect to x, y or checking directly by any analytic computational package one can show that the related charge is indeed $Q = 0$ and therefore the solution belongs to the trivial topological sector as we wanted. The intriguing reason behind this fact is that, the two holes appearing here have opposite charges resulting to their combined charge being zero.

4.3.4 Construction of dynamical lump soliton

For constructing a dynamical extension of the 2D static lump soliton (4.12) we realize that, a sudden change of amplitude with time, as necessary to mimic the 2D RW behavior, might result to a non-conservation of energy. This however can not be described by an integrable equation alone, since the integrability demands a strict conservation of all charges and therefore our integrable equation (4.8) needs certain modification for allowing the appearing/disappearing nature of its lump-solution. On the other hand, the importance of ocean currents in the formation of RWs is documented and repeatedly emphasized [62, 84, 78, 103], which however is absent in equation (4.8). This motivates us to solve both these problems in one go, by modifying equation (4.8) with the inclusion of the effect of an *ocean current*, as in [62], by adding a term in the form $I = -iU_c q_x$. For obtaining an exact dynamical RW solution to the modified 2D NLS equation, we choose the current flowing along longitudinal directions and changing with time and location as $U_c(x,t) = \frac{\mu t}{\alpha x}$. Looking closely into the structure of this current term for the RW solution (4.19):

$$I(x,y,t) = i(\frac{\mu t}{ax})\frac{\partial}{\partial x}[q_{P(2D)}(x,y,t)] = -2\mu t(4y-i)G^2\ e^{4iy}, \tag{4.18}$$

with G as defined in (4.19), it becomes apparent, that the currents would flow to the center of formation of the RW ($x = y = 0$) from both of the longitudinal and the transverse sides, though with a directional preference, with their magnitude $|I(x,y,t)|$ increasing as they approach to the center, however stopping completely at the moment of the full surge at $t = 0$. The picture gets reversed after the RW event with currents flowing back quickly, away from the center with the intensity of the current $|I|$ diminishing as the distance increases. Such an inflow and outflow of energy seems to be physically consistent with the formation of a 2D ocean RW. Note that, though the current factor U_c looks ill-defined, the multiplicative factor q_x makes the term $I(x,y,t)$ well-behaved on the RW solution (4.19), with the ocean current term becoming a smooth and bounded function in all space and time variables, as evident from (4.18). It has been suspected in earlier studies, that spatially nonuniform current should be responsible in the development of ocean rogue waves [78]. Such a nonuniform dependence on space variables can be seen in our current term $I(x,y,t)$. Interestingly, the modified 2D NLS equation ((4.8) with the inclusion of the current term I) admits now an exact dynamical 2D extension of the Peregrine soliton in the analytic form, though the modified equation loses its integrability in the sense, discussed earlier. The dynamical RW solution has a similar form as (4.12), only with the function G becoming dynamical by the inclusion of time variable :

$$q_{P(2d)}(x,y,t) = e^{4iy}[-1 + (1 - i4y)G],$$
$$G \equiv \frac{1}{F(x,y,t)}, F(x,y,t) = \alpha x^2 + 4y^2 + \mu t^2 + c. \tag{4.19}$$

The arbitrary parameter μ appearing in the solution (4.19) is related to the ocean current and can control how fast the RW would appear and how long it would stay. Note again that (4.19) at $t = 0$, representing a full grown RW (see Fig 3c)) coinciding with the exact static lump-solution (4.12) of the 2D NLS equation (4.8) (as in Fig. 2d)), justifying the physical relevance of the static lump solution. At this stage, a comparison between 1D PB soliton

$$q_P(x,t) = [-1 + \frac{(1 - 4it)}{x^2 + 4t^2 + \frac{1}{4}}]e^{-2it} \tag{4.20}$$

and our 2D lump soliton

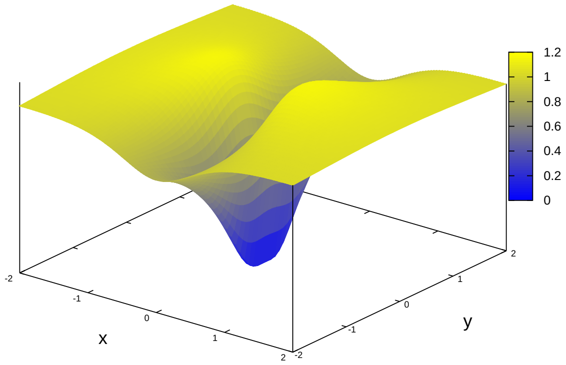
$$q_{P(2D)}(x,0,t) = [-1 + \frac{(1)}{\alpha x^2 + \mu t^2 + c}], \text{ at } y = 0, \tag{4.21}$$

might be interesting. This shows that though there is some similarity between these two solutions, there are many differences as well at $y = 0$. In the absence of the transverse coordinate, the 2D solution (4.21) of our modified equation, becomes real, though still having 3 independent free parameters. The 1D PB soliton on the other hand is complex with a breathing mode, but without any free parameter.

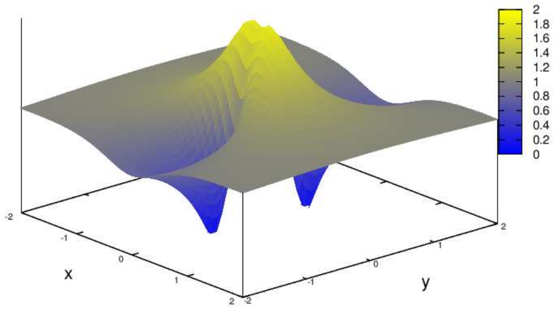
We should mention here, that the 2D extension of PB solution (4.19), unlike the standard 1D PB, unfortunately could not be derived as a limiting case from the breather solution of 2D NLS equation (4.8), due to the two-dimensional nature of the solution and has to be constructed by direct insertion through an ansatz.

4.3.5 Proposed 2D rogue wave model and its dynamics

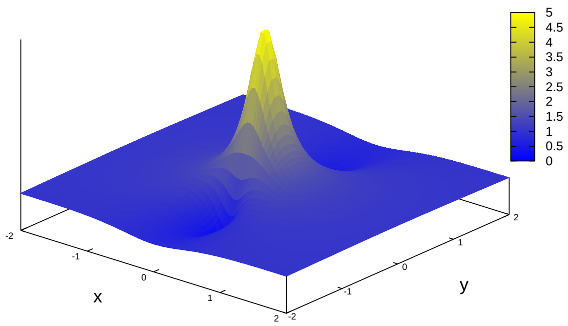
It is convincingly demonstrated in Fig. 3 (fixing the free parameters to certain values), how the envelop wave $|q_{P(2D)}(x, y, t)|$ corresponding to the exact dynamical 2D lump-soliton (4.19), dependent on time t and two space variables x and y on a plane, evolves from a background plane wave existing in the distance past and how it could acquire a sudden 2D hole at the centre ($x = 0, y = 0$) at the moment $t_h = -\sqrt{\frac{1}{\mu}(1-c)}$, ($t_h = -0.83$ for $c = 1/6, \mu = 1.2$ in Fig. 3a), as told in marine-lore [63, 75, 83]. The hole subsequently splits into two and shift apart from the centre (Fig. 3b)), to make space for a high steep upsurge of the lump forming the full grown RW (Fig. 3c) at time $t = 0$. Note that we have derived analytically the exact positions of these holes in the previous subsection. With the passage of time the picture gets reversed and the 2D RW disappears fast into the background waves with the 2D holes merging at the centre and vanishing again. Thus our model describes vividly well the reported picture of the ocean surface RWs [62, 73, 84] as well as those found in large scale 2D experiments [66]. Since our model is an exact one , we could work out these details analytically. The surface RWs modelled by our solution (4.19) and as visible from Fig. 3 (similarly from solution (4.12) and Fig. 2), shows a distinct directional preference and an asymmetry between the two space variables x, y, (similar to the report of [62, 73]). The maximum amplitude attained by the full grown RW (as shown in Fig 3c) is five-times that of the background waves, due to our choice $c = 1/6$. Examples of other amplitudes and modular inclination of full grown RWs for some other choices of the free parameters c and α, as modelled by the static solution (4.12), are presented already in Fig. 2 (a-d).



(a) At $t = t_h = -0.83$: creation of two dimensional hole at the centre.



(b) At t=-0.40: the hole splits into two, which are drifting away from the centre.



(c) At $t = 0.0$: The full grown RW corresponds also to the static lump soliton (4.12), as shown in fig 4d.

FIG.3: Snap shots of a two dimensional RW with two dimensional holes during its formation in different times, described by the modulus $|q_{P(2D)}(x, y, t)|$ of the dynamical lump soliton 4.19 with parameter values $c = 1/6, \alpha = 1.2, \mu = 1.2$ at three crucial moments of time.

4.4 Physical origin of the proposed 2D NLS equation and its integrability

We have shown that the 2D NLS equation (4.8) which is equivalent to nonlinear equation (4.7) can give an exact model, considerably successful in describing realistic 2D rogue waves. In this section we show the direct link of the 2D NLS equation with basic hydrodynamic equations. Moreover we show the underlying integrable structures of the proposed equation.

4.4.1 Derivation of the integrable 2D NLS equation from basic hydrodynamic equations

For emphasizing the physical significance of our main nonlinear integrable equation (4.8), on which the ocean rogue wave model is based, we show its direct link with basic hydrodynamic equations. The procedure is based on the asymptotic multi-scale expansion, at par with the celebrated equations like KdV, NLS etc [105, 123], though one should include here an extra space dimension with an asymmetric scaling in space variables, considering the perturbative expansion to the next higher order. This is consistent however with the modelling of an ocean rogue wave, which is a surface phenomena with a likely broken space symmetry and directional preference [67]. Before entering into the detailed calculation, three dimensionless entities: $\epsilon = \frac{a}{h_0}$, $\delta = \frac{h_0}{\lambda_x}$ and $\mu = \frac{\lambda_x}{\lambda_y}$ are defined where a is the maximum amplitude, h_0 is the constant water depth, λ_x and λ_y are the wavelengths of the surface wave along longitudinal and transverse directions. The nonlinear parameter ϵ is responsible for the slow evolution of a harmonic wave of wavenumber k_x, k_y. The wave is thus slowly modulated as ϵ tends to 0 and therefore this small parameter can be used for perturbative expansion. Smallness of ϵ is consistent with the deep water limit with $a << h_0$ and hence with the formation of oceanic rogue waves. Note that parameters ϵ and δ are similar to those appearing in the derivation of the well known 1D NLS equation, with ϵ small and δ without any restriction since h_0 and λ_x both are large quantities for deep water and long wavelength limit, as also true in our case. However, an additional parameter μ, also without any restriction on its value appears in our 2D case, due to the presence of an additional transverse direction.

The first step in the derivation is to write the basic hydrodynamic equations for inviscid, irrotational and incompressible fluid in dimensionless variables,

for the velocity potential field $\phi(t, x, y)$ and the gravity wave $\eta(t, x, y)$ as the free surface displacement above the mean water depth h_0 in the form

$$\phi_{zz} + \delta^2(\phi_{xx} + \mu^2\phi_{yy}) = 0, \tag{4.22}$$

at $0 < z < 1 + \epsilon\eta$, which comes from continuity equation. The equation

$$\phi_z = \delta^2[\eta_t + \epsilon(\phi_x\eta_x + \mu^2\phi_y\eta_y)], \tag{4.23}$$

called kinematic condition, is valid on z = 1 + $\epsilon\eta$. The equation

$$\phi_t + \eta + \frac{1}{2}\epsilon[\frac{\phi_z^2}{\delta^2} + \phi_x^2 + \mu^2\phi_y^2] = 0 \tag{4.24}$$

is also another free surface boundary condition valid at z = 1 + $\epsilon\eta$, while

$$\phi_z = 0 \tag{4.25}$$

is the fixed boundary condition valid at z = 0, i.e. at the bottom.

We introduce new variables with different scaling through ϵ as

$$\xi = k_x x + k_y y - \omega t,\ \zeta = \epsilon(x - M_x t),\ Y = \epsilon^2 y,\ \tau = \epsilon^3 t, \tag{4.26}$$

where ω, M_x are frequency and velocity parameters to be determined later. Note, that the two space variables are treated with a non-symmetric scaling and using these set of variables, equations (4.22 -4.25) become

$$\phi_{zz} + \delta^2(k_x^2\phi_{\xi\xi} + \epsilon^2\phi_{\zeta\zeta} + 2\epsilon k_x\phi_{\zeta\xi}) + \mu^2\delta^2(k_y^2\phi_{\xi\xi} + \epsilon^4\phi_{YY} + 2\epsilon^2 k_y\phi_{Y\xi}) = 0 \tag{4.27}$$

$$\begin{aligned}\phi_z = \delta^2[-\omega\eta_\xi - \epsilon M_x\eta_\zeta + \epsilon^3\eta_\tau] + \epsilon\delta^2(k_x\phi_\xi + \epsilon\phi_\zeta)(k_x\eta_\xi + \epsilon\eta_\zeta) +& \\ \mu^2\epsilon\delta^2(k_y\phi_\xi + \epsilon^2\phi_Y)(k_y\eta_\xi + \epsilon^2\eta_Y)&\end{aligned} \tag{4.28}$$

and

$$[-\omega\phi_\xi - \epsilon M_x\phi_\zeta + \epsilon^3\phi_\tau] + \eta + (\frac{\epsilon}{2\delta^2})\phi_z^2 + \frac{\epsilon}{2}(k_x\phi_\xi + \epsilon\phi_\zeta)^2 + \frac{\mu^2\epsilon}{2}(k_y\phi_\xi + \epsilon^2\phi_Y)^2 = 0 \tag{4.29}$$

both valid at $z = 1 + \epsilon\eta$, while

$$\phi_z = 0,\ \text{at } z = 0. \tag{4.30}$$

Seeking asymptotic solution of these equations in the series form

$$\phi = \sum_{n=0}^{\infty}\epsilon^n\phi_n(\xi,\zeta,Y,\tau,z),\quad \eta = \sum_{n=0}^{\infty}\epsilon^n\eta_n(\xi,\zeta,Y,\tau) \tag{4.31}$$

and using this expansion of dependent variables along with the scaled independent variables, different sets of equations are obtained from the basic set (4.27)-(4.30) at different powers of ϵ. In each ϵ order different equations are obtained for various powers of E and E^*. We would consider these equations sequentially at each order of parameter ϵ.

1) ϵ^0 order : The solution of interest in this case takes the form

$$\phi_0 = f_0 + F_0 E + F_0^* E^*, \eta = A_0 E + A_0^* E^*, \tag{4.32}$$

where $F_0(\zeta, Y, \tau, z)$, $A_0(\zeta, Y, \tau)$ are complex functions with F_0^*, A_0^* as complex conjugates, while $f_0(\zeta, Y, \tau)$ is a real function and $E = exp(i\xi)$.

Using (4.27) and (4.30), F_0, can be determined as

$$F_0 = G_0 \cosh{(\delta K_1 z)}, \text{ where } G_0 = \frac{-iA_0\omega\delta}{K_1 \sinh{(\delta K_1)}}, K_1 = \sqrt{k_x^2 + \mu^2 k_y^2}. \tag{4.33}$$

Using other two nonlinear boundary conditions (4.28), (4.29) we obtain the dispersion relation $\omega^2 = \frac{K_1}{\delta} \tanh{(\delta K_1)}$

2) ϵ order : Expanding ϕ_n, η_n as

$$\phi_n = \sum_{m=0}^{n+1} F_{nm} E^m + c.c, \quad \eta_n = \sum_{m=0}^{n+1} A_{nm} E^m + c.c, \tag{4.34}$$

where $F_{nm}(\zeta, Y, \tau, z)$ and $A_{nm}(\zeta, Y, \tau)$ are to be determined for various powers of E, at each powers of ϵ .

At ϵ order ,the components $F_{10}, F_{11}, F_{12}, A_{10}, A_{11}, A_{12}$ and the velocity parameter M_x are determined from the equations corresponding to E, E^2 and E^0, explicit forms of which are appended in $A1$.

3) ϵ^2 order: At this order a NLS type equation (Space coordinate Y replacing the time coordinate) is obtained, collecting the coefficients of E from (4.28), (4.29) and by using the quantities, already determined. Before calculating the final form of this equation some other components namely $F_{21}, F_{20}, f_{0\zeta}$ at this order need to be evaluated, which are given in the appendix $A2$.

The final form of the NLS like equation is obtained eliminating the unknown terms and expressing other terms through the single function A_0 as

$$i\alpha_1 A_{0Y} + \alpha_2 A_{0\zeta\zeta} + \beta_2 |A_0|^2 A_0 = 0, \tag{4.35}$$

where the constant coefficients α_1, α_2 and β_2 are also given in appendix $A22$

Following the same procedure the components F_{22},A_{22}, A_{20},f_{0Y} are determined ,which we are not furnishing here due to their cumbersome expressions.

4) ϵ^3 order: In this order an evolution equation is obtained, for which some relevant components i.e. F_{31},F_{30} etc, are also determined by continuing with the same procedure. The explicit forms of these coefficients presented in $A3$.

The evolution equation obtained by using equation (4.28), (4.29) and collecting coefficients of E takes the form

$$\begin{aligned} iaA_{0\tau} + \alpha_{31} A_{0\zeta Y} + i\bar{\beta}_{32} A_0^2 A_{0\zeta}^* + i\bar{\beta}_{31} |A_0|^2 A_{0\zeta} + ieG_{11}^* A_0^2 + ifG_{11} |A_0|^2 + \\ i\alpha_{32} A_{0\zeta\zeta\zeta} = 0, \end{aligned} \tag{4.36}$$

where $a, \alpha_{31}, \bar{\beta}_{32}, \bar{\beta}_{31}, e, f, \alpha_{32}$ are real constants dependent on parameters k_x, k_y, μ, δ.

If it is assumed, that the term G_{11} depends also on A_0 like the other terms as $F_0 \sim A_0$ and $G_{12}, A_{12} \sim A_0^2$ etc. (see Appendix) , then the only consistent relation would be $G_{11} = P_1 A_{0\zeta}$, where P_1 is a real constant, dependent only on k_x, k_y, μ, δ. Using this relation in (4.36) one simplifies it in the form

$$iaA_{0\tau} + \alpha_{31} A_{0\zeta Y} + i\alpha_{32} A_{0\zeta\zeta\zeta} + i(\beta_{31} |A_0|^2 A_{0\zeta} + \beta_{32} A_0^2 A_{0\zeta}^*) = 0, \tag{4.37}$$

where β_{31}, β_{32}, are another set of constant coefficients expressed through earlier coefficients. Notice that the above equation (4.37) is similar to but not the same as our integrable 2D NLS equation due to the appearance of the term $i\alpha_{32}A_{0\zeta\zeta\zeta}$. However fortunately we have another equation (4.35) at our disposal, obtained at a lower order. Taking derivative of (4.35) with respect to ζ we derive the relation

$$i\alpha_2 A_{0\zeta\zeta\zeta} = \alpha_1 A_{0\zeta Y} - i\beta_2(|A_0|^2 A_0)_\zeta \tag{4.38}$$

using which we can eliminate this unwanted term from (4.37) to obtain an equation in the form

$$iC_0 A_{0\tau} + C_1 A_{0\zeta Y} + iC_2 A_0(A_0 A^*_{0\zeta} - A^*_0 A_{0\zeta}) = 0, \tag{4.39}$$

under the condition on the coefficients of the original equation as

$$\frac{\beta_2}{\alpha_2} = \frac{(\beta_{32} + \beta_{31})}{3\alpha_{32}} \tag{4.40}$$

Rescaling ζ, Y and τ and renaming A_0 equation (4.39) goes directly to the 2D NLS equation (4.8), which is equivalent to (4.7) proposed by us. Note that constraint (4.40), we have to impose for deriving our integrable 2D NLS equation from the basic hydrodynamic equations, though does not hold for general water wave problems, this loss of generality is compensated for by the gain of our important exact results. This in general is true for all integrable models.

4.4.2 Integrable structures of the proposed equation:

We present here the associated integrability properties of equation (4.8). The one-soliton solution of this equation is given in the form $q_{s(2d)}(x, y, t) = \text{sech}\kappa(y + \rho x - vt)e^{i(k_1 x + k_2 y + \omega t)}$, while allowing also higher soliton solutions(given in the next chapter in details) and infinite set of conserved quantities . One can also find the associated linear system

$$\Phi_y = U(\lambda)\Phi, \Phi_t = V(\lambda)\Phi,$$

with a Lax pair given by

$$U(\lambda) \equiv V_2(\lambda) = 2\lambda V_1(\lambda) + V_2^{(0)},\ V(\lambda) \equiv V_3(\lambda) = 2\lambda V_2(\lambda) + V_3^{(0)} \tag{4.41}$$

where

$$\begin{aligned} V_1(\lambda) &= i(\lambda\sigma^3 + U^{(0)}),\ V_2^{(0)} = \sigma^3(U_x^{(0)} - iU^{(0)2}) \\ V_3^{(0)} &= -\sigma^3 U_y^{(0)} - [U^{(0)}, U_x^{(0)}],\ U^{(0)} = q\sigma^+ + q^*\sigma^-, \end{aligned} \tag{4.42}$$

with $\sigma^a, a = \pm, 3$, Pauli matrices, the flatness condition: $U_t - V_y + [U, V] = 0$, of which generates our 2D NLS equation (4.8). Note that unlike the known Lax pair of the 1D NLS, the pair $U(\lambda), V(\lambda)$ associated to our system have higher order dependence on the spectral parameter λ. It is not difficult to show, that the flatness condition yields from (4.42) different relations at different powers of λ. The equation linked to the λ corresponds to our $(2 + 1)$-dimensional NLS equation (4.8), while the relation with λ^0 gives another intriguing nonlinear equation

$$iq_{xt} + q_{yy} + 2i|q|^2 q_y + 2q_x(qq_x^* - q^* q_x) = 0. \tag{4.43}$$

Our main concern here however is the 2D NLS equation (4.8), which we intend to use for constructing a 2D rogue wave model. Note however, the modification of (4.8) by the addition of the current term as considered in the Sect. 3 though yields exact analytic RW solution no longer remains integrable in the sense described here.

Systems with infinite degrees of freedom like the 2D-NLS equation (4.8), when integrable, should have infinite set of independent conserved quantities. We generate here the related infinite set of conserved charges $C_n,\ n = 1, 2, \ldots$ in the explicit form, demonstrating again an important feature of the 2D NLS equation linked to its integrability. In analogy with the 1D NLS equation we start from the linear system , but use now the Lax equation along the y-direction: $\Phi_y = U(\lambda)\Phi$. Note, that for the wave function $\Phi(\lambda, y) = (\phi, \tilde{\phi})$, the component

$$\phi(y, \lambda) = e^{\int_{-\infty}^{y} \rho(\lambda, y')dy'},$$

with $\int_{-\infty}^{+\infty} dy'\rho(\lambda, y') = \sum_{n=1}^{\infty} C_n\lambda^{-n}$ acts as a generator of the conserved quantities, yielding

$$\ln\phi(y = \infty, \lambda) = \sum_{n=1}^{\infty} C_n\lambda^{-n}.$$

Therefore using $U(\lambda)$ as in (4.41) or in more explicit form in the Lax equation $\Phi_y = U(\lambda)\Phi$, we can build systematically the infinite set of conserved charges: $C_n,\ n = 1, 2, \ldots$ through a recurrence relation giving

$$\begin{aligned} C_1 &= i\int dy(q^* q_x - q_x^* q),\ C_2 = \int dy(i\frac{1}{2}(q_y^* q - q^* q_y) + q_x^* q_x + |q|^4),\ C_3 = \int dy(q_y^* q_x + q_x^* q_y), \\ C_4 &= \int dy\left(iq_{xy}^* q_x + q_y^* q_y - i|q|^2(q^* q_y - q_y^* q) - 2|q|^2\ q_x^* q_x + (q^{*2} q_x^2 + q_x^{*2} q^2)\right), \end{aligned} \tag{4.44}$$

and so on. We note the involvement of both the space-variables x, y in this series of independent conserved quantities, which also gives another strong argument in favor of the integrability of the 2D nonlinear equation (4.8). Taking these conserved quantities as Hamiltonians $H \equiv C_n$ we can generate the integrable hierarchy for this 2D NLS equation.

4.5 Summary

We conclude by listing a few distinguishing features of our proposed dynamical lump soliton (4.19), which are important for a realistic ocean RW model.

1) This is the first 2D dynamical deep water RW model given in an analytic form.

2) It is a 2D extension of Peregrine like soliton, representing an exact lump solution linked to a novel $(2+1)$-dimensional integrable NLS equation, derivable from the basic hydrodynamic equations.

3) The dynamics of the RW solution is induced by a ocean current term and controlled by it. Importance of the current in the formation of RW is strongly emphasized [66, 62], though perhaps for the first time this effect in 2D is attempted to be analyzed analytically in our model.

4) Both the height and the inclination of the single peak RW are adjustable by two independent free parameters present of our model.

5) The fastness of appearance of the RW and the duration of its stay can be regulated by yet another parameter linked to the ocean current.

6) The proposed solution and MI exhibit broken spatial symmetry as well as a directional preference, which are suspected to be the crucial features in the formation of a 2D RW [66, 67, 74, 75]. Note again that these features obtained earlier through observation or numerical simulation, found and confirmed in our model through exact analytic result.

7) Strange appearance (and disappearance) of a 2D hole just before (and after) the formation of the rogue wave [75, 84, 83] is also confirmed in our model, graphically as well as by analytic findings.

In comparison the original Peregrine soliton (4.2) (together with its higher order solutions), by far the most popular model of the rogue wave, does not exhibit most of these essential properties, due to its inherently one-dimensional nature and absence of free parameters. Therefore, while the class of Peregrine solitons are successful in modelling 1D rogue wave like structures observed in many experiments, the two-dimensional rogue wave model reported here should complement it, to stand close to a realistic model for ocean surface rogue waves. We hope that, this breakthrough in describing large ocean RWs by an analytic dynamical lump-soliton with adjustable height, inclination and duration would also be valuable for experimental findings of two-dimensional RWs in other systems like capillary fluid waves [69] optical cavity waves [67] and basin water waves [66]. Derivation of our exact lump soliton from the breather solution of the integrable 2D NLS equation presented here, in a systematic way as well as to find higher order rational lump solutions would be challenging theoretical problems.

We can conclude by stating the fact that, the new two dimensional equation which we have introduced (4.8) demands its applications like other integrable equations as KdV, NLS, KP, DS to other physical systems like plasma, Bose-Einstein condensates etc. Since the equation (4.8) is an integrable two dimensional extension of NLS equation its various integrable properties might be interesting in explaining various physical properties of those systems. Motivated by this fact we will derive and establish this 2D-NLS equation in the propagation of an ion acoustic wave in lossless magnetized plasma of cold ions and hot electrons in the next chapter and its various physical properties will be explored.

4.6 Appendix:

A1: Coefficients appearing in order ϵ :

$$F_{10} = G_{10}(\zeta, Y, \tau), \quad A_{10} = M_x f_{0\zeta} - \tfrac{2\delta K_1}{\sinh(2\delta K_1)}|A_0|^2$$

$$F_{12} = G_{12}\cosh(2\delta K_1 z), \text{ where } G_{12} = \tfrac{-3i\omega\delta^2}{4\sinh^4(\delta K_1)}A_0^2,$$

$$A_{12} = \tfrac{\delta K_1\cosh(\delta K_1)}{2[\sinh(\delta K_1)]^3}[1 + 2\cosh^2(\delta K_1)]A_0^2$$

$$F_{11} = G_{11}\cosh(\delta K_1 z) - \tfrac{i\delta k_x}{K_1}G_{0\zeta} z\sinh(\delta K_1 z),$$

$A_{11} = i\omega[G_{11}\cosh(\delta K_1) - \frac{i\delta k_x}{K_1}G_{0\zeta}\sinh(\delta K_1)] + M_x[G_{0\zeta}\cosh(\delta K_1)]$

The velocity parameter: $M_x = \frac{\omega k_x}{2K_1^2}[1 + \frac{2\delta K_1}{\sinh(2\delta K_1)}]$

A2: Coefficients appearing in order ϵ^2 :

$F_{21} = G_{21}\cosh(\delta K_1 z) - \frac{i\delta k_x}{K_1}G_{11\zeta}z\sinh(\delta K_1 z) - \frac{i\delta k_y}{K_1}\mu^2 G_{0Y}z\sinh(\delta K_1 z) + G_{0\zeta\zeta}[(-\frac{\delta}{2K_1})z\sinh(\delta K_1 z) + (\frac{\delta k_x^2}{2K_1^3})z\sinh(\delta K_1 z) - (\frac{\delta^2 k_x^2}{2K_1^2})z^2\cosh(\delta K_1 z)]$,

$f_{0\zeta} = \frac{1}{(1-M_x^2)}[-\frac{2M_x\delta K_1}{\sinh(2\delta K_1)} - \frac{2\omega\delta k_x\coth(\delta K_1)}{K_1}]|A_0|^2$,

$F_{20} = -\delta^2 f_{0\zeta\zeta}\frac{z^2}{2} + G_{10}(\zeta, Y, \tau)$,

$\alpha_1 = -k_y\mu^2\tanh(K_1\delta)\frac{[2K_1\delta + \sinh(2K_1\delta)]}{2\omega^3\cosh^2(K_1\delta)}$,

$\alpha_2 = \frac{\delta}{2\omega K_1^3}[K_1^3\delta\{2M_x^2 - \frac{1}{\cosh^2(K_1\delta)}\} - K_1^2\tanh(K_1\delta) + k_x^2\tanh(K_1\delta) + 4K_1k_xM_x\delta^3\omega^3 - K_1k_x^2\delta\{1 + \tanh^2(K_1\delta)\}]$,

$\beta_2 = -\frac{\delta^2}{\omega(1-M_x^2)}[4k_x^2 + K_1\delta\frac{1}{\sinh(2K_1\delta)}\{K_1^2((-1+M_x^2)(8+\cosh(4K_1\delta)\frac{1}{\sinh^2(K_1\delta)} + 2\frac{1}{\cosh^2(K_1\delta)}) + 8k_xM_x\omega\}]$

A3: Coefficients appearing in order ϵ^3 :

$$F_{30} = -\delta^2 F_{10\zeta\zeta}\frac{z^2}{2} + G_{30}(\zeta, Y, \tau), \tag{4.45}$$

$$F_{31} = G_{31}\cosh(\delta K_1 z) + G_{21\zeta}\{(\frac{-i\delta k_x}{K_1})z\sinh(\delta K_1 z)\} + G_{11\zeta\zeta}\{(\frac{i\delta k_x}{K_1})^2\frac{z^2}{2}\cosh(\delta K_1 z) - (\frac{i\delta k_x}{K_1})^2(\frac{z}{2\delta K_1})\sinh(\delta K_1 z) - (\frac{\delta}{2K_1})z\sinh(\delta K_1 z)\} + (\frac{i\delta k_x}{K_1})G_{0\zeta\zeta\zeta}\{(\frac{\delta}{2K_1})(z^2/2)\cosh(\delta K_1 z) - (\frac{\delta}{2K_1})(\frac{z}{2\delta K_1})\sinh(\delta K_1 z) - (\frac{\delta k_x^2}{2K_1^3})\frac{z^2}{2}\cosh(\delta K_1 z) + \frac{\delta k_x^2}{2K_1^3}(z/2\delta K_1)\sinh(\delta K_1 z) + (\frac{\delta}{2K_1})\frac{z^2}{2}\cosh(\delta K_1 z) - (\frac{\delta}{2K_1})(\frac{z}{2\delta K_1})\sinh(\delta K_1 z) + \frac{\delta^2 k_x^2}{2K_1^2}\frac{z^3}{3}\sinh(\delta K_1 z) - \frac{\delta^2 k_x^2}{2K_1^2}(z^2/2\delta K_1)\cosh(\delta K_1 z) + \frac{\delta^2 k_x^2}{2K_1^2}(z/2\delta^2 K_1^2)\sinh(\delta K_1 z)\} + G_{0\zeta Y}(\frac{i\delta k_x}{K_1})(\frac{i\delta k_y}{K_1})[\mu^2 z^2\cosh(\delta K_1 z) - \mu^2\sinh(\delta K_1 z)(\frac{z}{\delta K_1}) + G_{11Y}[-(\frac{i\delta k_y}{K_1})\mu^2 z\sinh(\delta K_1 z)]$$

Chapter 5

A new (2+1) dimensional integrable evolution equation for an ion acoustic wave in a magnetized plasma

5.1 Introduction

In the previous chapter, we have introduced a new (2+1) dimensional completely integrable nonlinear evolution equation for modelling oceanic rogue wave phenomena. Since this equation can be regarded as the integrable generalization of the (2+1) dimensional Nonlinear Schrodinger Equation (NLSE), it demands applications like the other integrable models, to the various physical systems. Motivated by this fact we shall try to develop this new (2+1) dimensional integrable NLSE in the propagation of modulated ion acoustic wave packet in the lossless magnetized plasma system consisting of cold ions and hot isothermal electrons. Now before entering into the main subject it is necessary to mention the importance of the integrable models in explaining the nonlinear plasma systems.

Active research on nonlinear phenomena in plasma physics has grown extensively and gained much importance over the past few decades due to failure of linear theory in explaining phenomena related to large amplitude waves, wave- particle, wave-wave interactions etc [124]. However, the complexity of the associated nonlinear partial differential equations makes the system less well understood in most of the cases. There lies the importance of the integrable nonlinear equations because of their rich analytical beauty and availability of generalized mathematical techniques for solving them [125]. Washimi and Taniuti [126] were first to derive the completely integrable Korteweg de Vries (KdV) equation for small but finite amplitude ion acoustic solitary waves for collision less plasma composed of cold ions and hot electrons. Since then the plasma physics community has been actively involved in nonlinear phenomena related structures such as solitons, shocks, instabilities, wave-wave and wave-particle interactions etc. The first experimental observation of ion acoustic soliton has been made by Ikezi et.al [128, 127]. Since then the KdV model has been used extensively in various branches like dusty plasma[129, 130, 131], Bose Einstein gravitationally condensed gas[132], weakly relativistic magnetized plasma[133], non-thermal plasma[134], dense plasma with degenerate electron fluids[135] in planar as well as nonplanar geometry[136, 137] and also in other branches. Other nonlinear equations like Boussinesq equation[127], Benjamin-Bona-Mahony (BBM) equation[138], which are not integrable also find their applications in plasma physics. It is well known that nonlinear wave propagation is generally subject to an amplitude modulation due to carrier wave self interaction resulting in a slow modulation of monochromatic plane wave leading to the formation of an envelope soliton, which may be described by the Nonlinear Schrödinger(NLS) equation, also a completely integrable system. This equation is also investigated extensively in various areas of plasma systems like dusty plasma[139, 140], multicomponent plasma[141], in explaining rogue waves[140, 142, 143], relativistic laser plasma interactions[144] and various other fields. Peregrine soliton of NLS equation which is used to describe rogue waves is experimentally observed in a multicomponent plasma with negative ions[145]. A complex Ginzberg-Landau equation is derived in compressional dispersive Alfvenic waves in a collisional magnetoplasma[143] which reduces to standard NLS equation in a collisionless plasma. The discussed equations are all (1+1) dimensional, but in practical circumstances the waves observed in laboratory and space are certainly not bounded in one dimension. Franz[146] et.al have shown that a purely 1D model cannot account for the observed features in the auroral region, especially at higher polar altitudes. The best known 2D generalization of KdV equation are Kadomtsev-Petviashvili (KP) equation and Zakharov- Kuznetsov(ZK) equation. A completely integrable generalization of the KdV equation is the KP equation [119] which has also been used in various branches of plasma such as inhomogeneous plasma with finite temperature drifting ions[147], ultracold quantum magnetospheric plasma[148], electron positron ion plasma [149] and also in other areas. The stability of their solutions under transverse perturbations was also studied [150, 148]. The ZK equation [97] which is more isotropic in transverse direction was first derived for describing weakly nonlinear ion acoustic waves in strongly magnetized lossless plasma in 2D[152]. It was also reported that this equation is not integrable under inverse scattering method[153, 154] and till date only three polynomial conservation laws have been given [155, 156]. This equation was also explored vastly in the last few decades [160, 159, 158, 157] and higher dimensional solitons were derived [162, 161]. A 2D generalization of NLS equation is DS equation which was also derived for electrostatic ion waves [163], electron acoustic wave [164], space and laboratory dusty plasma [165], and in cylindrical geometry [166]. For special choice of coefficients DS equation converges to DS1 equation which is analytically integrable and admits dromion solutions with localized structure in higher dimensions [164, 163, 165, 167, 168]. But in case of DS1 equation additional fields are coupled in the interacting term which could be related to the basic fields only through nonlocal transformations. Hence due to the presence of a few integrable equations (both in 1D and 2D) in plasma systems, there is always a requirement for the discovery of new integrable equations (specially in 2D).

In this work we have derived a completely integrable (2+1) dimensional nonlinear evolution equation in lossless magnetized plasma with asymmetric scaling on transverse variable. This equation involving only local interactions of dependent variables was derived earlier in hydrodynamic system [169] in explaining oceanic RW phenomena . The 2D generalizations of NLS equation available in the literature involve either nonlocal interactions, or are non-integrable, whereas the equation presented here is the completely integrable, local, (2+1) dimensional generalization of NLS equation which however possesses many properties similar to (1+1) dimensional NLS equation.

5.2 Derivation of (2+1) dimensional integrable equation for electrostatic waves propagating in a magnetized plasma

A new (2+1) dimensional integrable evolution equation for the propagation of nonlinear ion acoustic waves in magnetized plasma is derived in this section. We consider the propagation of electrostatic waves in a magnetized plasma with the magnetic field $B = B_0\hat{e_z}$, where B_0 is a constant. For situations where plasma pressure is much smaller than the magnetic pressure, plasma wave excitation is in general electrostatic. The focus is on a plasma composed of two components, ions and electrons that are described by fluid equations under collision-free conditions in the cartesian coordinates, which include conservation of mass and momentum together with Poisson's equation given by

$$\frac{\partial n}{\partial t}+\overrightarrow{\bigtriangledown}\cdot(n\overrightarrow{v})=0,\ \ \frac{\partial \overrightarrow{v}}{\partial t}+(\overrightarrow{v}\cdot\overrightarrow{\bigtriangledown})\overrightarrow{v}+\overrightarrow{\bigtriangledown}\phi-\alpha(\overrightarrow{v}\times\overrightarrow{b})=0,\ \ \bigtriangledown^2\phi=n_e-n, n_e=\exp(\phi) \tag{5.1}$$

where n_e, n, v, B, ϕ are electron, ion number densities and ion fluid velocity, magnetic field and electrostatic potential respectively and α is a dimensionless parameter given by ω_{ci}/ω_{pi}. For convenience, we have used the following normalization resulting in dimensionless parameters: electron and ion densities normalized by n_0, coordinates by electron Debye length $\lambda_{De} = v_{te}/\omega_{pe}$, fluid velocity by the acoustic speed $c_s = \sqrt{k_B T_e/m_i}$; time by ion plasma period ω_{pi}^{-1}, and magnetic field B by $B_0 = m_e\omega_{pi}/e$, where $v_{te}, \omega_{pe}, \omega_{pi}, \omega_{ci}$ are the electron thermal velocity, electron and ion plasma frequencies and ion cyclotron frequency respectively.

In the above equations, the ions are assumed to be cold and on the slow ion time scale, the electrons are assumed to be in local thermodynamic equilibrium. When the electron inertia is neglected, the electrons can be considered to follow a Boltzmann distribution if the propagation vector has a small component along the magnetic field, such that the angle χ between the wave vector normal to the magnetic field and the wave vector is larger than $\sqrt{m_e T_i/m_i T_e}$, so that as a special case we can take $k_z \to 0$. This enables us to consider propagation perpendicular to the magnetic field with the wave vector $k = (k_x, k_y, 0)$. The linear propagation of electrostatic ion cyclotron waves propagating perpendicular to the magnetic field is governed by the dispersion relation

$$\omega^2 = k^2 c_s^2/(1 + k^2\lambda_{De}^2) + \omega_{ci}^2$$

where $k^2 = k_x^2 + k_y^2$.

In order to derive the nonlinear evolution equation governing the propagation of the electrostatic ion cyclotron waves, we assume perturbation of the form $\sim$ exp[i($\overrightarrow{k} \cdot \overrightarrow{r}$ - ω t)], and adopt the reductive perturbation expansion technique. All the physical quantities are expanded about their equilibrium values as-

$$n = 1 + \sum_{m=1}^{\infty} \epsilon^m \sum_{l=-m}^{m} n_l^{(m)} exp[il(\overrightarrow{k} \cdot \overrightarrow{r} - \omega t)] \tag{5.2}$$

$$\phi = \sum_{m=1}^{\infty} \epsilon^m \sum_{l=-m}^{m} \phi_l^{(m)} exp[il(\overrightarrow{k} \cdot \overrightarrow{r} - \omega t)] \tag{5.3}$$

$$v_j = \sum_{m=1}^{\infty} \epsilon^m \sum_{l=-m}^{m} v_{jl}^{(m)} exp[il(\overrightarrow{k} \cdot \overrightarrow{r} - \omega t)] \tag{5.4}$$

where j denotes the x and y components of ion velocities. We have introduced the following stretched variables with asymmetric scaling on transverse direction as

$$\xi = \epsilon(x - M_x t), \quad \eta = \epsilon^2 y, \quad \tau = \epsilon^3 t \tag{5.5}$$

where M_x is the group velocity in the x axis. The scaling used here is different from the scaling involved in the derivation of Davey-Stewartson equation that has a symmetric dependence on all the space variables. The stretching in this case is asymmetric with respect to one of the space variables. Such a situation may arise in some experimental scenario where there is a possibility of weak dependence in one of the directions.

Transforming all independent variables by equation (5.5), we expand equations (5.1) and carry out a systematic balancing of terms at each order of ϵ. The coefficients appearing at different orders are all given at the appendix.

At $\epsilon : \mathbf{l = 1}$ order we get

$$\phi_1 = K_1 n_1^{(1)}, \quad v_{x1}^{(1)} = A_1^{(1)} n_1^{(1)}, \quad v_{y1}^{(1)} = B_1^{(1)} n_1^{(1)} \tag{5.6}$$

Combining the above expressions leads to the linear dispersion relation for the ion acoustic wave-

$$\omega^2 = |k|^2 K_1 + \alpha^2, K_1 = 1/(1 + \mid k \mid^2) \tag{5.7}$$

Similiarly at $\epsilon^{\mathbf{2}} : \mathbf{l = 0}$; we get,

$$v_{x0}^{(2)} = A_0^{(2)} |n_1|^2, \quad v_{y0}^{(2)} = B_0^{(2)} |n_1|^2, \quad \phi_0^{(2)} = -K_1^2 |n_1^{(1)}|^2 + n_0^{(2)} \tag{5.8}$$

At $\epsilon^{\mathbf{2}} : \mathbf{l} = \mathbf{1}$; we obtain,

$$\phi_1^{(2)} = K_1 n_1^{(2)} + 2ik_x K_1^2 \frac{\partial n_1^{(1)}}{\partial \xi}, \quad v_{x1}^{(2)} = A_1^{(2)} n_1^{(2)} + B_1^{(2)} \frac{\partial n_1^{(1)}}{\partial \xi}, \quad v_{y1}^{(2)} = C_1^{(2)} n_1^{(2)} + D_1^{(2)} \frac{\partial n_1^{(1)}}{\partial \xi} \tag{5.9}$$

The group velocity along x axis, M_x, can be found out from this order of calculation as

$$M_x = \frac{(A_1^{(1)}(\omega^2 - \alpha^2) + k_x K_1 \omega - iK_1 k_y \alpha - 2K_1^2 k_x \omega |k|^2}{(-\alpha^2 + \omega^2 + iB_1^{(1)} k_x \alpha + A_1^{(1)} k_x \omega - iA_1^{(1)} k_y \alpha + B_1^{(1)} k_y \omega)} \tag{5.10}$$

At $\epsilon^{\mathbf{2}} : \mathbf{l} = \mathbf{2}$;

$$\phi_2^{(2)} = D_2^{(2)} (n_1^{(1)})^2, \quad v_{x2}^{(2)} = A_2^{(2)} (n_1^{(1)})^2, \quad v_{y2}^{(2)} = B_2^{(2)} (n_1^{(1)})^2, \quad n_2^{(2)} = C_2^{(2)} (n_1^{(1)})^2 \tag{5.11}$$

At $\epsilon^{\mathbf{3}} : \mathbf{l} = \mathbf{1}$; order, an NLS-type equation (space co-ordinate η replacing the time co-ordinate) is obtained as-

$$iA_1^{(3)} \frac{\partial n_1^{(1)}}{\partial \eta} + B_1^{(3)} \frac{\partial^2 n_1^{(1)}}{\partial \xi^2} + C_1^{(3)} |n_1^{(1)}|^2 n_1^{(1)} = 0 \tag{5.12}$$

The above space-type NLS equation has resulted because in the present work, we have scaled the transverse variable y in the same way as time is scaled in the derivation of NLS equation.

At $\epsilon^{\mathbf{3}} : \mathbf{l} = \mathbf{0}$; order we find $n_0^{(2)} = 0$ since $n \to 0$ as $\xi \to \infty$. The other quantities determined are-

$$v_{x0}^{(3)} = A_0^{(3)} n_1^{(1)*} \frac{\partial n_1^{(1)}}{\partial \xi} + B_0^{(3)} n_1^{(1)} \frac{\partial n_1^{(1)*}}{\partial \xi} + C_0^{(3)} n_1^{(1)*} n_1^{(2)} + C_0^{(3)} n_1^{(1)} n_1^{(2)*} \tag{5.13}$$

$$v_{y0}^{(3)} = E_0^{(3)} n_1^{(1)*} \frac{\partial n_1^{(1)}}{\partial \xi} + F_0^{(3)} n_1^{(1)} \frac{\partial n_1^{(1)*}}{\partial \xi} + G_0^{(3)} n_1^{(1)*} n_1^{(2)} + H_0^{(3)} n_1^{(1)} n_1^{(2)*} \tag{5.14}$$

$$n_0^{(3)} = I_0^{(3)} n_1^{(1)*} \frac{\partial n_1^{(1)}}{\partial \xi} + J_0^{(3)} n_1^{(1)} \frac{\partial n_1^{(1)*}}{\partial \xi} + K_0^{(3)} n_1^{(1)*} n_1^{(2)} + L_0^{(3)} n_1^{(1)} n_1^{(2)*} \tag{5.15}$$

$$\phi_0^{(3)} = M_0^{(3)} n_1^{(1)*} \frac{\partial n_1^{(1)}}{\partial \xi} + P_0^{(3)} n_1^{(1)} \frac{\partial n_1^{(1)*}}{\partial \xi} + N_0^{(3)} n_1^{(1)*} n_1^{(2)} + Q_0^{(3)} n_1^{(1)} n_1^{(2)*} \tag{5.16}$$

Detailed mathematical forms of all the coefficients occurring in the above equations are given in the Appendix. Similarly the $\epsilon^{\mathbf{3}} : \mathbf{l} = \mathbf{2}$; order quantities like $v_{x2}^{(3)}, v_{y2}^{(3)}$ etc can be determined by the same procedure but the exact expressions cannot be given in view of their extreme cumbersome nature.

Finally at $\epsilon^{\mathbf{4}} : \mathbf{l} = \mathbf{1}$; order a two dimensional evolution equation is obtained in the form

$$\begin{aligned} iA_1^{(4)} \frac{\partial n_1^{(1)}}{\partial \tau} + B_1^{(4)} \frac{\partial^2 n_1^{(1)}}{\partial \xi \partial \eta} + iC_1^{(4)} \frac{\partial^3 n_1^{(1)}}{\partial \xi^3} + iD_1^{(4)} (n_1^{(1)})^2 \frac{\partial n_1^{(1)*}}{\partial \xi} + iE_1^{(4)} |n_1^{(1)}|^2 \frac{\partial n_1^{(1)}}{\partial \xi} \\ + F_1^{(4)} |n_1^{(1)}|^2 n_1^{(2)} + G_1^{(4)} (n_1^{(1)})^2 n_1^{(2)*} + iH_1^{(4)} \frac{\partial n_1^{(2)}}{\partial \eta} + I_1^{(4)} \frac{\partial^2 n_1^{(2)}}{\partial \xi^2} = 0 \end{aligned} \tag{5.17}$$

where the coefficients $A_1^{(4)}$ - $I_1^{(4)}$, which are real constants dependent on parameters k_x, k_y and α, are too cumbersome to be expressed in an explicit form. This is a general two dimensional non-integrable equation of two dependent variables $n_1^{(1)}$ and $n_1^{(2)}$. If it is assumed that the term $n_1^{(2)}$ depends on $n_1^{(1)}$ like the other terms as $v_{x1}^{(1)}, v_{y1}^{(1)} \sim n_1^{(1)}$, $v_{x2}^{(2)}, v_{y2}^{(2)}, n_2^{(2)} \sim (n_1^{(1)})^2$ etc, then the only possible consistent relation between $n_1^{(1)}$ and $n_1^{(2)}$ would be $n_1^{(2)} \sim \frac{\partial n_1^{(1)}}{\partial \xi}$. Hence we consider $n_1^{(2)} = iP_1 \frac{\partial n_1^{(1)}}{\partial \xi}$ where P_1 is a constant dependent on k_x, k_y, α. Now using (5.12) in (5.17) we see that the general nonintegrable equation (5.17) turns into the form

$$iC_0 \frac{\partial n_1^{(1)}}{\partial \tau} + C_1 \frac{\partial^2 n_1^{(1)}}{\partial \xi \partial \eta} + 2iC_2 n_1^{(1)} (n_1^{(1)} \frac{\partial n_1^{(1)*}}{\partial \xi} - n_1^{(1)*} \frac{\partial n_1^{(1)}}{\partial \xi}) = 0 \tag{5.18}$$

for the choice of the constant

$$P_1 = \frac{[\frac{3C_1^{(3)}C_1^{(4)}}{B_1^{(3)}} - D_1^{(4)} - E_1^{(4)}]}{[F_1^{(4)} - G_1^{(4)} - \frac{3I_1^{(4)}C_1^{(3)}}{B_1^{(3)}}]} \tag{5.19}$$

where C_0, C_1, C_2 depends on the parameters k_x, k_y, α. In case of the multidimensional extension of modulated ion acoustic wave by Nishinari[163], general multidimensional coupled equations were obtained which were converted to the integrable DS1 equation for the specific choice ($k_x \to 0$, $\alpha_1\alpha_2 > 0$ and all the variables independent of ζ [163]). Similarly, in our case, for the specific choice of P_1 as given in (5.19), the general nonlinear nonintegrable equation becomes completely integrable (5.18). As the explicit representation of the coefficients C_0, C_1 and C_2 are too cumbersome, we will show their behavior graphically. In Figure (5.1), we plot the variation of the coefficients C_0, C_1 and C_2 with α for $k_x = 1, k_y = 1$.

Now rescaling the variables τ, ξ and η in (5.18) we get

$$i\frac{\partial n_1^{(1)}}{\partial_t} + \frac{\partial^2 n_1^{(1)}}{\partial_x \partial_y} + 2in_1^{(1)}(n_1^{(1)}\frac{\partial n_1^{(1)*}}{\partial_x} - n_1^{(1)*}\frac{\partial n_1^{(1)}}{\partial_x}) = 0 \tag{5.20}$$

and renaming $n_1^{(1)}$ as u it gives

$$iu_t + u_{xy} + 2iu(uu_x^* - u^*u_x) = 0 \tag{5.21}$$

which is our new (2+1) dimensional completely integrable evolution equation that has been obtained at a higher perturbation order compared to the NLS equation, hence expected to address weaker effects. Similar equation was derived in the context of water waves[169] in order to model oceanic rogue waves. It has structural similarity with the NLS equation, where the nonlinearity comes from the ponderomotive force that depends on the square of modulus and not on the phase of u. The present equation(5.21) has

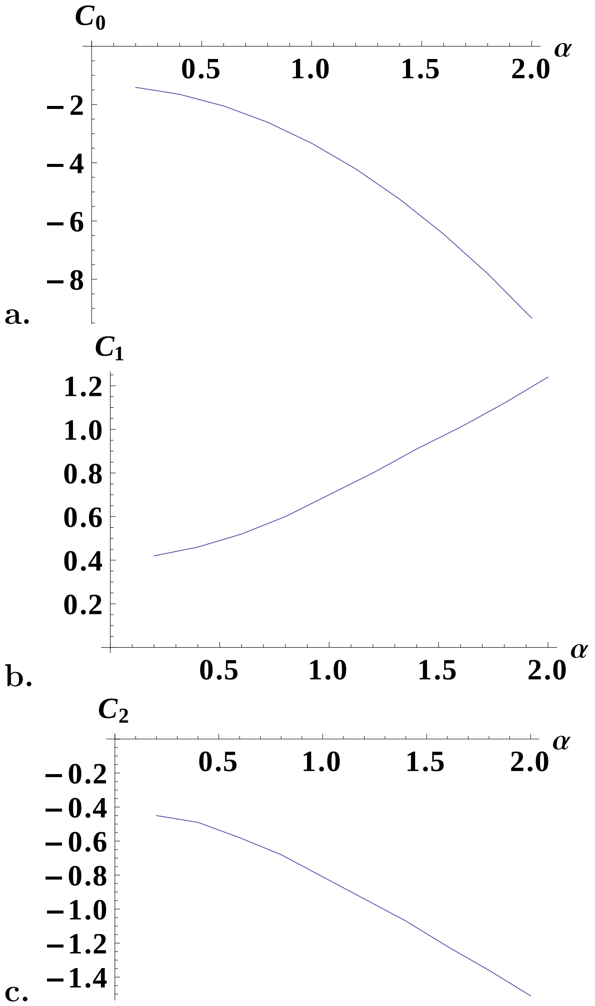


Figure 5.1: Variation of coefficients C_0, C_1, C_2 with α

slightly different characteristics, with the nonlinear potential dependent on the square of modulus of the wave profile as well as on the x-derivative of phase. The dispersive term, is a cross derivative term dependent on both longitudinal and transverse directions.

The study of propagation of modulated ion acoustic waves in the presence of a magnetic field has been extensively done using the NLS equation that restricts the study to one dimension. A multidimensional generalization of the NLS equation for a modulated ion acoustic wave packet propagating in a magnetized plasma leads to the Davey Stewartson equation [163, 165]. However, since all the spatial directions are scaled symmetrically, this equation certainly does not describe weak transverse propagation. In the long wave length regime, the KP equation is well known to describe the propagation of such waves when weak transverse perturbation is considered [147, 150, 148]. In an effort to obtain a 2D extension with weak transverse dependence of modulated ion wave packets propagating in a magnetized plasma, we obtain an asymmetric (2+1) dimensional novel equation(5.21) along with a space-like NLS equation (5.12). Hence, we observe that our equation has resemblance to the KP equation from the point of weak transverse propagation, and to the DS equation from its modulated structure. In case of DS or KP, the system reduces to NLS or KdV equation respectively when the transverse coordinate is neglected, but the present equation given in (5.21) does not reduce to the standard NLS equation in such limit, indicating its distinctive asymmetric nature.

Being a completely integrable system, equation (5.21) possesses Lax pair, infinite number of conserved quantities, higher soliton solutions etc some of which are discussed in the previous chapter. In the following section we shall derive its higher soliton solutions using Hirota bilinearisation procedure and explore its various features.

5.3 Soliton solutions

In this section we will elaborately discuss its multisoliton solutions, which can be derived by many methods e.g, inverse scattering transform, Hirota method and various dressing methods. The IST method is more powerful (it can handle general, initial conditions) and at the same time more complicated. But, if one just wants to find soliton solutions, Hirota's method is fastest in producing results [27, 125]. Hence, following the method, we use the bilinearizing transformation given by

$$u(x, y, t) = \frac{G(x, y, t)}{F(x, y, t)}, \tag{5.22}$$

where $G(x,y,t)$ and $F(x,y,t)$ are complex and real functions respectively. Using (5.21) one derives a pair of bilinear equations:

$$i(FG_t - GF_t) + (FG_{xy} + GF_{xy} - G_xF_y - G_yF_x) = 0, 2i(GG_x^* - G^*G_x) + 2(F_xF_y - FF_{xy}) = 0. \tag{5.23}$$

Multisoliton solutions are obtained by finite perturbation expansions as

$$F = 1 + \epsilon_0^2 F_2 + \epsilon_0^4 F_4 + \cdots, G = \epsilon_0 G_1 + \epsilon_0^3 G_3 + \cdots, \tag{5.24}$$

where ϵ_0 is formal expansion parameter need not to be small. Collecting like powers of ϵ_0, we obtain the following series of equations:

$$(\epsilon_0): \quad iG_{1t} + G_{1xy} = 0 \tag{5.25}$$

$$\mathrm{O}(\epsilon_0^2): \quad 2F_{2xy} = 2i[G_1G_{1x}^* - G_1^*G_{1x}] \tag{5.26}$$

$$\mathrm{O}(\epsilon_0^3): \quad iG_{3t} + G_{3xy} = i[G_1F_{2t} - F_2G_{1t}] - [F_2G_{1xy} + G_1F_{2xy} - G_{1x}F_{2y} - G_{1y}F_{2x}] = 0 \tag{5.27}$$

$$\mathrm{O}(\epsilon_0^4): \quad 2F_{4xy} = 2i[G_3G_{1x}^* + G_1G_{3x}^* - G_3^*G_{1x} - G_1^*G_{3x}] + 2[F_{2x}F_{2y} - F_2F_{2xy}] \tag{5.28}$$

and similarly higher order equations.

5.3.1 1-soliton

To construct 1-soliton solution for (5.21) we assume the ansatz

$$G_1 = e^{\eta_1}, \ \eta_1 = k_1x + p_1y - w_1t + \eta_1^0 \tag{5.29}$$

where k_1, p_1, w_1, η_1^0 are complex constants. From equation (5.25) therefore one obtains the associated dispersion relation $w_1 = -ik_1p_1$, using which the equation (5.26) is solved easily to yield

$$F_2 = i(k_1^* - k_1)\frac{e^{(\eta_1+\eta_1^*)}}{(p_1+p_1^*)(k_1+k_1^*)}. \tag{5.30}$$

We can verify using (5.29) and (5.30), that all higher order terms in ϵ beyond G_1 and F_2 trivially vanish. Absorbing ϵ in arbitrary constant η_1^0, we construct from (5.22) using (5.29) and (5.30) the 1 soliton solution in the form

$$u(x,y,t) = \frac{G_1}{1+F_2} = \frac{e^{\eta_1}}{1+\alpha e^{(\eta_1+\eta_1^*)}} \tag{5.31}$$

where α depends on the parameter k_1, p_1. One can identify the interesting 2d nature of our equation (5.21) by making $p_1 = 0$, then from eq. (5.30) we can see that F_2 will diverge and no soliton solution can be found. If additionally we use the dispersion relation of the constraint equation

$$iu_y + u_{xx} + 2|u|^2u = 0 \tag{5.32}$$

which comes from (5.12) after rescaling, as $p_1 = -ik_1^2$, the soliton solution (5.31) simplifies to yield the conventional form

$$q(x,y,t) = \text{sech}\xi\ e^{i\theta},\ \text{with}\ \xi = \eta(x+v_yy+vt),\ \theta = (k_xx+k_yy+\omega t). \tag{5.33}$$

A frozen picture of the modulus of our traveling soliton solution (5.33) at time $t = 0$ is shown in Fig.5.2.

5.3.2 2-Soliton

For obtaining 2-soliton solution we start with the standard procedure assuming

$$G_1 = e^{\eta_1} + e^{\eta_2}, \eta_1 = k_1x + p_1y - w_1t + \eta_1^0,\ \eta_2 = k_2x + p_2y - w_2t + \eta_2^0, \tag{5.34}$$

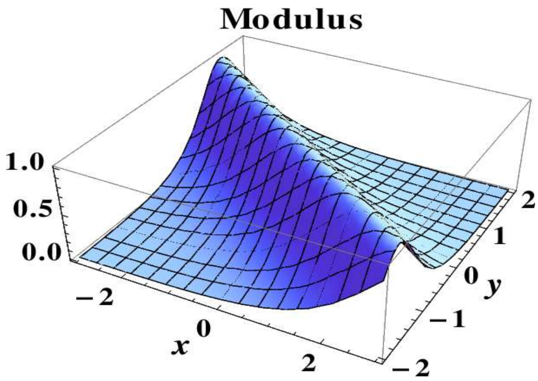


Figure 5.2: Modulus of 1 soliton with $k_{1r} = 1, k_{1i} = 1, \eta_{1r}^0 = 1, \eta_{1i}^0 = -1$ and at t=0

where the parameters involved are complex numbers. Applying similar dispersion relations as earlier we get $w_1 = -ik_1p_1,\ w_2 = -ik_2p_2$ and obtain from (5.26)

$$F_2 = [e^{(\eta_1+\eta_1^*+R_1)} + e^{(\eta_2+\eta_2^*+R_2)} + e^{(\eta_2+\eta_1^*+\delta_0)} + e^{(\eta_1+\eta_2^*+\delta_0^*)}], \tag{5.35}$$

where all the constant parameters can be worked out explicitly (see Appendix). Similarly equation (5.27) at higher order expansion gives

$$G_3 = e^{(\eta_1+\eta_1^*+\eta_2+\delta_1)} + e^{(\eta_1+\eta_2^*+\eta_2+\delta_2)}, \tag{5.36}$$

where the relevant parameter details are given in Appendix . Using further equation (5.28) one obtains

$$F_4 = e^{(\eta_1+\eta_1^*+\eta_2+\eta_2^*+R_3)}, \tag{5.37}$$

with the relevant parameters presented in Appendix. For simplifying the expressions,as mentioned earlier, we can use the constraint equation (5.32), imposing the relations between k_1 , p_1 and k_2 , p_2 as $p_1 = -ik_1^2$, $p_2 = -ik_2^2$ (see Appendix). Here we find again, that the higher order terms in ϵ beyond G_3 and F_4 trivially vanish, leaving the exact 2-soliton solution in the form

$$u(x, y, t) = \frac{G_1 + G_3}{1 + F_2 + F_4} \tag{5.38}$$

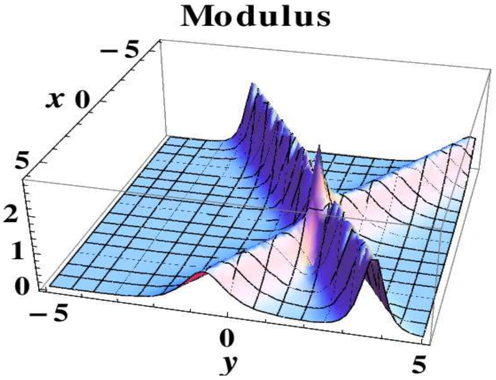


Figure 5.3: Modulus of 2 soliton with $k_{1r} = 1, k_{1i} = 1, k_{2r} = 2, k_{2i} = -1, \eta^0_{1r} = 1, \eta^0_{1i} = 1, \eta^0_{2r} = 1, \eta^0_{2i} = 1$ and at t=2

Here also we can see that for no transverse dependence i.e. $p_1 = p_2 = 0$, all the quantities determined F_2, G_3, F_4 diverges and two soliton solution cannot be found which indicates the strict 2d nature of the equation. A graphical plot of the modulus of this solution in $(2+1)$-dimensions, frozen at time $t = 2$, is shown in Fig. 5.3, where the 2-soliton as two interacting 1-solitons is clearly seen on a 2D (x, y)-plane. Following the same procedure the higher soliton solutions of our equation can be evaluated.

These one and two soliton solutions of (5.21) have similarities with the soliton solutions of NLS equation with an additional transverse dependence. Since a purely 1D model cannot account for the observed features of many physical situations, specially in auroral region with higher polar altitudes[146] corresponding 2D model is necessary. There lies the importance of the soliton solutions of our equation (5.21), which could be applied to many research areas of this field.

5.4 Exact static 2D lump solution:

Localized wave structure in (2+1) dimensional systems are very important in terms theoretical and experimental aspects of plasma. Localized rational structure following KP-I equation have been found by Janaki et.al in the propagation of oblique magnetosonic wave in warm collisional plasma system[170], where the wave profile decays algebraically in both directions. Whereas, in case of modulated wave equation like Davey Stewartson system, an exponentially localized solution called Dromion, which moves with time has been found in magnetized electron acoustic wave system [164]. Our equation (5.21) which has some similarity with both DS and KP equations, also possesses an exact localized wave solution which has been discussed in the previous chapter in details, decaying rationally in both spatial directions.

The static 2D rational lump solution is given by

$$u_{static}(x,y) = exp(4iy)(\frac{1-4iy}{c+\alpha_1 x^2+4y^2} - 1) \tag{5.39}$$

where c , α_1 are 2 free parameters. From this we can see that the wave attains the maximum amplitude

$$A_{max} = \frac{1-c}{c} \tag{5.40}$$

at the centre x=0, y=0 which can be controlled by c. At large distances($|x| \to \infty$,$|y| \to \infty$) the amplitude goes to unity. The steepness of this static wave solution as observed from the front is $\frac{\partial}{\partial y}u_{static}$ is related to another free parameter α_1.The amplitude of the wave falls to its minimum at x=0, $y = \pm y_0$, with $y_0 = \sqrt{\frac{1-c}{\alpha_1}}$. Hence the density gets localized at the centre x=0,y=0 and the concentration can be controlled by the free parameter c. This is an interesting feature because in actual physical situation the ion density can change which need to be controlled by the free parameters. This is absent in the (1+1) dimensional NLS system, where Peregrine breather which is used to describe density localization in the x-t plane, have no free parameters and hence can achieve a fixed maximum value (3 times than the background). Whereas our solution, having two free parameters c and α_1 can achieve any amplitude and steepness, relevant to the actual physical condition.

Thus unlike the exponentially decaying dromion solution of DS1 equation, our system (5.21) provides a rational solution in both space directions having similar structure to

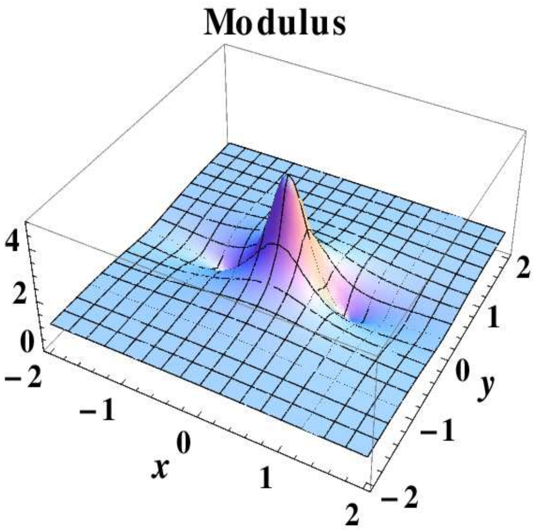


Figure 5.4: Absolute value of the solution (5.39) with c=1/6, $\alpha = 6/5$

the rational solution of KP-I equation. There lies another connection of the equation (5.21) with KP which has similar scaling and weaker dependence on transverse directions as (5.21). Unfortunately, the time evolution of the rational solution (5.39) could not be found which can be explored in future.

Time evolution of the rational solution of our equation (5.21) is presented in [172] by the one-fold Darboux transformation from a non-zero seed solution.

5.5 Connection with Kadomtsev- Petviashvili (KP) equation

It is interesting to note that the new integrable equation (5.21) together with the space NLS equation derived in (5.12), has a deep connection with another well known (2+1) dimensional evolution equation. The equation (5.21) along with (5.12) is a complex equation denoting the propagation of ion acoustic wave with modulation. But if we are concerned with the modulus of the wave, then our system also provides another well known, integrable equation.

Equation (5.21) is our new evolution equation after scaling, while the space NLS equation (5.12) at the same scaling becomes

$$iu_y = Bu_{xx} + A|u^2|u, \tag{5.41}$$

where A, B are two real constants dependent on C_1, C_2 as $B = -\frac{B_1^{(3)}C_2}{A_1^{(3)}C_1}$, $A = -\frac{C_1^{(3)}C_1}{A_1^{(3)}C_2}$. Now, if (5.21) is multiplied by u and its complex conjugate equation by u^* and subtracted from one another, then taking derivative w.r.t x we get another equation containing quadratic power in u. Now using (5.41) and its complex conjugate equation, that equation can be simplified to yield

$$[4B\phi_t - 6AB\phi\phi_x - B^2\phi_{xxx}]_x + 3\phi_{yy} = 0, \tag{5.42}$$

where $\phi = uu^*$.

Equation (5.42) is nothing but the well known KP equation where ϕ being the square of modulus of the wave u, the dependent variable of equations (5.21) and (5.41). It means that the square of the absolute value of the wave without modulation satisfies another real equation which is also integrable and weak in the transverse perturbation. Note that, in the derivation of (5.21) we have implied weak scaling on the transverse coordinate, similar as the scaling involved in the derivation of KP equation. Hence our equation (5.21) though being a modulated equation in two dimensions, has a different structure than the DS equation which is symmetric in both the spatial variables and has a connection with another two dimensional long wave equation which is asymmetric in the transverse direction.

For ion acoustic wave, KP equation has been derived and its stability properties under transverse perturbations have been discussed in many plasma systems like quantum electron ion plasma[148] or in electron-positron-ion plasma with high energy tail electron and positron distribution[149] and in other fields and using Sagdeev potential approach conditions of existence of stable solitary waves have been obtained. Unlike KdV equation where the form of soliton solution does not depend on the sign of the dispersion term, the form of soliton solution of KP equation is directly determined by the dispersion sign. Again the stability of the soliton depends on the coefficients of the KP equation which in this case also depends on the coefficients of (5.18) and (5.12). Depending on the signs of A, B equation (5.42) can be transformed into KP-I or KP-II equations admitting different types of solutions . The square root of various solutions of (5.42) can be used for constructing the solution of (5.21) together with a phase factor connected with (5.41), which we plan to explore in the future. The connection of our equation (5.21) with KP equation stresses the two dimensional and asymmetric nature of the equation.

5.6 Summary

In this chapter a completely integrable, (2+1) dimensional, modulated , nonlinear evolution equation has been derived in the propagation of an ion acoustic wave of magnetized collisionless plasma system. It has been obtained at a higher perturbation order compared to the NLS equation, hence expected to address weaker effects. The two spatial directions are scaled asymmetrically allowing weak transverse perturbation. Thus the equation derived (5.21), has connection with the DS equation in terms of its modulated structure, with KP equation in terms of its weak transverse dependence. It is shown that the square of modulus of the dependent variable of the equation satisfies KP equation along with the space NLS like constraint(5.12) equation. Using Hirota bilinearization scheme its higher soliton solutions are calculated which has similar structure with the 1D soliton with an additional dependence on transverse direction. The exact algebraic lump solution of (5.21), carrying 2 free parameters is also given and density localization of the system is discussed. Applications of this important and novel equation to other physical systems and identification of its different features may pave a new direction of research in this field.

5.7 Appendix:

i) Coefficients appearing in the derivation of the two dimensional integrable evolution equation (5.21):

$K_1 = \frac{1}{1+|k|^2}$, $|k|^2 = (k_x^2+k_y^2)$,$\kappa_1 = \frac{1}{1+4|k|^2}$, $A_1^{(1)} = -\frac{(iK_1k_y\alpha+K_1k_x\omega)}{(\alpha^2-\omega^2)}$, $B_1^{(1)} = \frac{(iK_1k_x\alpha-K_1k_y\omega)}{(\alpha^2-\omega^2)}$

$A_0^{(2)} = \frac{-2\omega k_x}{|k|^2}$, $B_0^{(2)} = \frac{-2\omega k_y}{|k|^2}$,

$A_1^{(2)} = -\frac{(iK_1k_y\alpha+K_1k_x\omega)}{(\alpha^2-\omega^2)}$, $C_1^{(2)} = -\frac{(-iK_1k_x\alpha^3+K_1k_y\alpha^2\omega+iK_1k_x\alpha\omega^2-K_1k_y\omega^3)}{(-\alpha^2+\omega^2)^2}$,

$D_1^{(2)} = -\frac{(-iK_1k_yM_x\alpha^2-K_1\alpha^3+2K_1^2k_x^2\alpha^3-2K_1k_xM_x\alpha\omega+2iK_1^2k_xk_y\alpha^2\omega-iK_1k_yM_x\omega^2+K_1\alpha\omega^2-2K_1^2k_x^2\alpha\omega^2-2iK_1^2k_xk_y\omega^3)}{(-\alpha^2+\omega^2)^2}$,

$B_1^{(2)} = -\frac{\{-2K_1^2k_xk_y\alpha-iK_1\omega+2iK_1^2k_x^2\omega-(iK_1k_xM_x\alpha^2)/(\alpha^2-\omega^2)+(2K_1k_yM_x\alpha\omega)/(\alpha^2-\omega^2)-(iK_1k_xM_x\omega^2)/(\alpha^2-\omega^2)\}}{(\alpha^2-\omega^2)}$,

$A_2^{(2)} = \{-2K_1^3k_x|k|^4\kappa_1 + K_1|k|^2(5k_x\alpha^2 + 3ik_y\alpha\omega + 4k_x^3\kappa_1 + 4k_xk_y^2\kappa_1)$
$+ \alpha^2(3k_x\alpha^2 + 3ik_y\alpha\omega + 4k_x|k|^2\kappa_1) + K_1^2|k|^2(2k_x|k|^2 - 2k_x\alpha^2\kappa_1 - ik_y\alpha\omega\kappa_1)\}/$
$\{3\alpha^2\omega|k|^2 + 4|k|^4\omega(K_1 - \kappa_1)\}$,

$B_2^{(2)} = \{-2K_1^3k_y|k|^2\kappa_1 + K_1|k|^2(5k_y\alpha^2 - 3ik_x\alpha\omega + 4k_x^2k_y\kappa_1 + 4k_y^3\kappa_1)$
$+ \alpha^2(3k_y\alpha^2 - 3ik_x\alpha\omega + 4k_y|k|^2\kappa_1) + K_1^2|k|^2(2k_y|k|^2 - 2k_y\alpha^2\kappa_1 + ik_x\alpha\omega)\}/$
$\{3\alpha^2\omega|k|^2 + 4|k|^4\omega(K_1 - \kappa_1)\}$,

$C_2^{(2)} = 2\frac{\{3\alpha^2+K_1|k|^2(3-K_1\kappa_1)\}}{\{3\alpha^2+4|k|^2(K_1-\kappa_1)\}}$,

$D_2^{(2)} = \frac{\{-4K_1(-3+K_1^2)|k|^2-3(-4+K_1^2)\alpha^2\}\kappa_1}{\{6\alpha^2+8|k|^2(K_1-\kappa_1)\}}$,

$A_1^{(3)} = 2K_1k_y\omega(|k|^2K_1 - 1),$

$B_1^{(3)} = \frac{-K_1(K_1|k|^2-1)\{K_1(-k_y^2+3K_1k_x^2|k|^2)+(4K_1k_x^2-1)\alpha^2\}}{\omega},$

$C_1^{(3)} = |k|^2\omega[K_1\{4K_1(K_1^3-3)|k|^2+3(K_1^3-4)\alpha^2\}+\{4K_1(12+K_1^2(K_1-3)(K_1+2)\}|k|^2+ 3(k_1^2-4)^2\alpha^2\}\kappa_1]/\{6\alpha^2+8|k|^2(K_1-\kappa_1)\},$

$A_0^{(3)} = -\frac{1}{\alpha}[A_1^{(1)*}B_1^{(1)} - iB_1^{(2)}B_1^{(1)*}k_x + iA_1^{(1)*}D_1^{(2)}k_x - B_0^{(2)}M_x]$

$B_0^{(3)} = -\frac{1}{\alpha}[A_1^{(1)}B_1^{(1)*} - iB_1^{(1)}B_1^{(2)*}k_x - iA_1^{(1)}D_1^{(2)*}k_x - B_0^{(2)}M_x]$

$C_0^{(3)} = \frac{i}{\alpha}[A_1^{(2)}B_1^{(1)*} - A_1^{(1)*}C_1^{(2)}]k_x$

$D_0^{(3)} = -\frac{i}{\alpha}[A_1^{(2)*}B_1^{(1)} - A_1^{(1)}C_1^{(2)*}]k_x$

$E_0^{(3)} = \frac{1}{\alpha}[A_1^{(1)}A_1^{(1)*} - K_1^2 + iB_1^{(2)}B_1^{(1)*}k_y - iA_1^{(1)*}D_1^{(2)}k_y - A_0^{(2)}M_x]$

$F_0^{(3)} = \frac{1}{\alpha}[A_1^{(1)}A_1^{(1)*} - K_1^2 - iB_1^{(1)}B_1^{(2)*}k_y + iA_1^{(1)}D_1^{(2)*}k_y - A_0^{(2)}M_x]$

$G_0^{(3)} = \frac{i}{\alpha}[A_1^{(2)}B_1^{(1)*} - A_1^{(1)*}C_1^{(2)}]k_y$

$H_0^{(3)} = -\frac{i}{\alpha}[A_1^{(2)*}B_1^{(1)} - A_1^{(1)}C_1^{(2)*}]k_y$

$L_0^{(3)} = \frac{1}{M_x\alpha}[A_1^{(1)}\alpha + A_1^{(2)*}\alpha + iA_1^{(1)}C_1^{(2)*}k_x - iA_1^{(2)*}B_1^{(1)}k_x]$

$K_0^{(3)} = \frac{1}{M_x\alpha}[A_1^{(2)}\alpha + A_1^{(1)*}\alpha + iA_1^{(2)}B_1^{(1)*}k_x - iA_1^{(1)*}C_1^{(2)}k_x]$

$J_0^{(3)} = \frac{1}{M_x\alpha}[-A_1^{(1)}B_1^{(1)*} - iB_1^{(1)}B_1^{(2)*}k_x + iA_1^{(1)}D_1^{(2)*}k_x + B_0^{(2)}M_x + B_1^{(2)*}\alpha]$

$I_0^{(3)} = \frac{1}{M_x\alpha}[-B_1^{(1)}A_1^{(1)*} + iB_1^{(1)*}B_1^{(2)}k_x - iA_1^{(1)*}D_1^{(2)}k_x + B_0^{(2)}M_x + B_1^{(2)}\alpha]$

$Q_0^{(3)} = -\frac{1}{M_x\alpha}[iA_1^{(2)*}B_1^{(1)}k_x - iA_1^{(1)}C_1^{(2)*}k_x - A_1^{(1)}\alpha - A_1^{(2)*}\alpha + K_1^2M_x\alpha]$

$P_0^{(3)} = \frac{1}{M_x\alpha}[-A_1^{(1)}B_1^{(1)*} - iB_1^{(1)}B_1^{(2)*}k_x + iA_1^{(1)}D_1^{(2)*}k_x + B_0^{(2)}M_x + B_1^{(2)*}\alpha + 2iK_1^3k_xM_x\alpha]$

$M_0^{(3)} = \frac{1}{M_x\alpha}[-A_1^{(1)*}B_1^{(1)} - iB_1^{(2)}B_1^{(1)*}k_x - iA_1^{(1)*}D_1^{(2)}k_x + B_0^{(2)}M_x + B_1^{(2)}\alpha - 2iK_1^3k_xM_x\alpha]$

$N_0^{(3)} = -\frac{1}{M_x\alpha}[-iA_1^{(2)}B_1^{(1)*}k_x + iA_1^{(1)*}C_1^{(2)}k_x - A_1^{(2)}\alpha - A_1^{(1)*}\alpha + K_1^2M_x\alpha]$

ii) Coefficients appearing in the Hirota bilinearization procedure in solving equation (5.21):

$$e^{R_1} = i\frac{(k_1^* - k_1)}{(p_1 + p_1^*)(k_1 + k_1^*)}, e^{R_2} = i\frac{(k_2^* - k_2)}{(p_2 + p_2^*)(k_2 + k_2^*)},$$
$$e^{\delta_0} = i\frac{(k_1^* - k_2)}{(p_2 + p_1^*)(k_2 + k_1^*)}, e^{\delta_0^*} = i\frac{(k_2^* - k_1)}{(p_1 + p_2^*)(k_1 + k_2^*)}$$

$$\begin{aligned} e^{\delta_1} &= \frac{i}{[(k_2 + k_1^*)(p_1 + p_1^*) + (k_1 + k_1^*)(p_2 + p_1^*)]}[\frac{(k_1^* - k_1)(p_2 - p_1)}{(p_1 + p_1^*)} + \\ &+ \frac{(k_1^* - k_1)(k_2 - k_1)}{(k_1 + k_1^*)} + \frac{(k_1^* - k_2)(p_1 - p_2)}{(p_2 + p_1^*)} + \frac{(k_1^* - k_2)(k_1 - k_2)}{(k_2 + k_1^*)}] \end{aligned} \quad (5.43)$$

$$\begin{aligned} e^{\delta_2} &= \frac{i}{[(k_1 + k_2^*)(p_2 + p_2^*) + (k_2 + k_2^*)(p_1 + p_2^*)]}[\frac{(k_2^* - k_2)(p_1 - p_2)}{(p_2 + p_2^*)} + \\ &+ \frac{(k_2^* - k_2)(k_1 - k_2)}{(k_2 + k_2^*)} + \frac{(k_2^* - k_1)(p_2 - p_1)}{(p_1 + p_2^*)} + \frac{(k_2^* - k_1)(k_2 - k_1)}{(k_1 + k_2^*)}] \end{aligned} \quad (5.44)$$

$$\begin{aligned} e^{R_3} &= \frac{1}{(k_1 + k_1^* + k_2 + k_2^*)(p_1 + p_1^* + p_2 + p_2^*)}[\{ie^{\delta_2}(k_1^* - k_1 - k_2 - k_2^*) \\ &+ ie^{\delta_1}(k_2^* - k_1 - k_2 - k_1^*) + ie^{\delta_1^*}(k_1^* + k_2^* + k_1 - k_2) + ie^{\delta_2^*}(k_1^* + k_2^* + k_2 - k_1)\} \\ &+ \{(k_2 + k_2^* - k_1 - k_1^*)(p_1 + p_1^*) + (k_1 + k_1^* - k_2 - k_2^*)(p_2 + p_2^*)\}e^{R_1 + R_2} \\ &+ \{(k_2 + k_1^* - k_1 - k_2^*)(p_1 + p_2^*) + (k_1 + k_2^* - k_2 - k_1^*)(p_2 + p_1^*)\}e^{\delta_0 + \delta_0^*}] \end{aligned} \quad (5.45)$$

For simplifying the expressions we can impose the relations between k_1 , p_1 and k_2 , p_2 as $p_1 = -ik_1^2$, $p_2 = -ik_2^2$, which would yield

$$e^{R_1} = \frac{1}{(k_1 + k_1^*)^2}, e^{R_2} = \frac{1}{(k_2 + k_2^*)^2}, e^{\delta_0} = \frac{1}{(k_1 + k_2^*)^2}, e^{\delta_0^*} = \frac{1}{(k_2 + k_1^*)^2},$$

$$e^{\delta_1} = \frac{(k_1 - k_2)^2}{(k_1 + k_1^*)^2(k_2 + k_1^*)^2}, e^{\delta_2} = \frac{(k_2 - k_1)^2}{(k_2 + k_2^*)^2(k_1 + k_2^*)^2},$$

$$e^{R_3} = \frac{|(k_1 - k_2)|^4}{(k_1 + k_1^*)^2(k_2 + k_2^*)^2(|(k_1 + k_2^*|)^4}$$

Chapter 6

Bending of solitons in weak and slowly varying inhomogeneous plasma

6.1 Introduction

In chapters 1 and 2 we have discussed the modified form of completely integrable KdV equation in modelling shallow water wave phenomena . In chapter 3 we have introduced a new (2+1) dimensional completely integrable NLS equation in describing deep water oceanic RW phenomena and in chapter 4 we derived this equation in the propagation of modulated ion acoustic wave in collision-less magnetized plasma. These constitute the main portion of the thesis. In the forgoing two chapters we shall investigate different plasma systems namely inhomogeneous and quantum plasmas respectively and apply these nonlinear integrable equations to explore various features. In this chapter we shall investigate a weak and slowly varying inhomogeneous plasma and explore the bending of ion acoustic soliton by this weak and slowly varying inhomogeneity. Before entering into the main work, it is necessary to briefly outline the important works carried out in the field of inhomogeneous plasma.

In a homogeneous plasma, an ion acoustic soliton travels without change in shape, amplitude and speed [124, 173]. But in actual experimental conditions, we encounter inhomogeneities in plasma at the edges or boundaries of the system or in the presence of density gradient. The propagation of ion acoustic KdV solitons in an inhomogeneous plasma was first considered by Nishikawa and Kaw [174] who presented a WKB solution when its spatial width is very small as compared to density gradient scale length. Gell and Gomberoff [175] reconsidered the situation and showed that amplitude, velocity and width of the soliton are proportional to the fractional powers of ion density which was verified experimentally by John and Saxena[176] and modified by Rao and Verma [177] by taking into account ion drift velocity, but allowing terms proportional to the stretched variable ξ in their first order equations. These inconsistencies were later removed by Kuehl and Imen [178] and their results are found to be in good agreement with those of Chang et.al[179]. One of the most important features of ion acoustic soliton is its reflection by plasma inhomogeneity. This phenomenon was first observed experimentally by Dahiya et.al[180] from the sheath around a negatively biased grid, where the density gradient is high. Popa and Oertl found reflection of ion acoustic soliton from a bipolar potential wall structure[181], Nishida [182] and Imen - Kuehl [183] found from a finite plane boundary, Nagasawa [184] found from a metallic mesh electrode showing nonlinear Snell's law and Yi and Cooney et.al found from a sheath in a negative ion plasma [185, 186]. Kuehl investigated theoretically the reflection of ion acoustic soliton, and showed that a shelf develops behind the soliton and the reflected wave is small compared with both trailing shelf and soliton amplitude decrease due to energy transfer to the shelf[187, 188, 189]. Then after, many authors took the problem of soliton propagation in inhomogeneous plasma in different physical situations like plasma with finite ion temperature[190, 191], with negative ions [192, 193, 194, 195], with dust[196] and trapped electrons [197], in magnetic field [198, 199], with non isothermal electrons [200], with ionization [201, 202], with electron inertia contribution [203] and also in other contexts [204, 205, 186].

These discussed cases are all (1+1) dimensional, but in practical circumstances the waves observed in laboratory and space are certainly not bounded in one dimension. Nevertheless, the two dimensional propagation of ion acoustic waves in inhomogeneous plasma has received much less attention. Zakharov- Kuznetsov (ZK) equation, which is the more isotropic 2 dimensional generalization of KdV equation, was obtained in modified form in magnetized dusty inhomogeneous plasma with non-extensive electrons [206], with dust charge fluctuation[207], with quantum effects [208], with non thermal ions and dust charge variation[209] and in other situations. But if weak transverse propagation is considered then the possible 2 dimensional generalization of KdV model is Kadomtsev- Petviashvili (KP) equation which was first derived in the context of plasma[118]. Malik et.al derived KP equation in modified form in inhomogeneous plasma with finite temperature drifting ions [147] and solved it for constant density gradient. Later in quantum inhomogeneous plasma a modified KP equation was also obtained [211, 212, 213] and line soliton solutions were presented. Along with reflection and transmission of line solitons in inhomogeneous plasma, its bending in two dimensional plane is also a possible relevant phenomenon which was not explored in literature considered earlier as well as in [214, 215] in two dimensions as far as our knowledge goes.

In this chapter, we have taken up this problem by considering ion acoustic soliton propagation in unmagnetized plasma containing cold plasma and hot isothermal electrons. Using reductive perturbation technique, a modified form of KP equation is obtained for weak transverse propagation and weak and slowly varying inhomogeneous ion number density . Exact solitary wave solutions were presented showing the bending of ion acoustic solitons in two dimensional plane. The soliton is modified in phase which is controlled by a function related to equilibrium ion number density, causing soliton bending in two dimensional plane, whereas the amplitude remains constant.

6.2 Derivation of two dimensional evolution equation for an ion acoustic wave propagating in a weak and slowly varying inhomogeneous plasma

We consider a two dimensional, collisionless, unmagnetized, weak and slowly varying spatially inhomogeneous plasma consisting of hot isothermal electrons and cold ions ($T_i = 0$). The plasma is weakly inhomogeneous with a slow variation of the equilibrium ion density along one spatial direction. The ion continuity and momentum equations together with Poisson's equation and the electron Boltzmann distribution can be written in the dimensionless form as

$$\frac{\partial n}{\partial t} + \overrightarrow{\bigtriangledown} \cdot (n\overrightarrow{u}) = 0, \quad \frac{\partial \overrightarrow{u}}{\partial t} + (\overrightarrow{u} \cdot \overrightarrow{\bigtriangledown})\overrightarrow{u} + \overrightarrow{\bigtriangledown}\phi = 0, \quad \bigtriangledown^2\phi = n_e - n, n_e = \exp(\phi) \quad (6.1)$$

In equation (6.1), $u \equiv (u_x, u_y)$ is the ion fluid velocity normalized by the ion acoustic speed $c_s = \sqrt{\frac{T_e}{m_i}}$, n and n_e are ion and electron number densities respectively normalized by unperturbed ion number density $\tilde{n_0}$ at an arbitrary reference point in plasma which we chose to be $x = 0$, ϕ is the electrostatic potential normalized by $\frac{T_e}{e}$ where T_e, m_i, e are electron temperature, ion mass and electronic charge respectively. All the spatial co-ordinates x, y are normalized by the Debye length $\lambda_D = \sqrt{\frac{\epsilon_0 T_e}{\tilde{n_0} e^2}}$ at $x = 0$ and time by inverse of the ion plasma frequency $\omega_{pi} = \sqrt{\frac{\tilde{n_0} e^2}{\epsilon_0 m_i}}$ at $x = 0$, where ϵ_0 is the permettivity of free space. We have assumed that the equilibrium electron and ion number densities are equal at $x = 0$ (quasi-neutrality) and that the zero reference of the equilibrium potential is at $x = 0$. In the above equations, the ions are assumed to be cold and on the slow ion time scale, the electrons are assumed to be in local thermodynamic equilibrium. When the electron inertia is neglected, the electrons can be considered to follow a Boltzmann distribution. Under these assumptions, in the absence of any equilibrium drift, the ion acoustic waves follow the dispersion relation given by

$$\omega = kc_s, \tag{6.2}$$

where ω, k, c_s are angular frequency, wave vector and ion acoustic speed respectively. In order to study the ion-acoustic wave propagation and its two dimensional evolution as a solitary wave in weak and slowly varying inhomogeneous plasma, we consider the following appropriate stretched co-ordinates

$$\xi = \epsilon^{\frac{1}{2}}(x - Mt), \quad \lambda = \epsilon y, \quad \eta = \epsilon^{\frac{3}{2}}x, \tag{6.3}$$

where M is a constant and ϵ is a small expansion parameter. Generally, phase velocity is taken to be a function of x in the literatures of inhomogeneous plasma, but here we have taken M to be a constant which is similar as the scaling used by Gell in [175]. This assumption will be shown to be consistent with the calculations for the chosen unperturbed ion number density profile.

Chang et.al, in their experimental studies of propagation of ion acoustic solitons in an inhomogeneous plasma [179], created a definite ion number density profile as shown in FIG 6.1, in a large multi dipole plasma device. To create local inhomogeneity in a previously homogeneous quiscent plasma, a perturbing object was inserted far from the excitation region. The left portion of FIG 6.1, where the density variation is slow was shown to be the host environment for studying soliton characteristics. It was also reported that the experiment had a pronounced two dimensional character.

We have followed the experimental results done by Chang et.al and numerical solutions obtained by Kuehl [8],[13] in the context of the propagation of ion acoustic wave in inhomogeneous plasma. The ion density plot, which was reported in their paper is reproduced here as FIG 6.1.

It is evident from the figure that, before insertion of the perturbing structure the

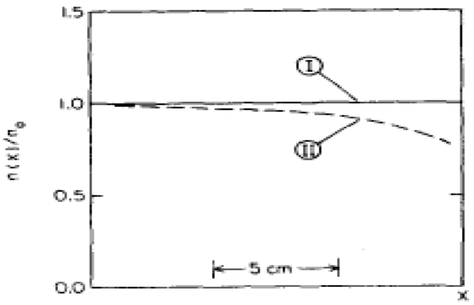


Figure 6.1: Measured ion number density profile in the target chamber for the experiment done by Chang et.al [179]. Continuous line denotes the profile if the perturbing structure is absent and the broken line denotes if it is present at the right edge. The figure is taken from the paper [179]
.

density was homogeneous (continuous curve) and the presence of the structure caused a density inhomogeneity as shown by the broken curve. This plot is consistent with the numerical solutions of the equilibrium density done in [8] and [13].

Following the above stated environment for soliton propagation, we have taken the unperturbed ion number density profile to be of the form $\tilde{n_0}(\eta) = 1 + \delta f_0(\eta)$, where δ is a small parameter, having the same features of the left portion of FIG 6.1 . The inhomogeneity is weak as well as slowly varying along η as shown in FIG 6.1, so that the plasma is nearly homogeneous. Our entire work is based on this region of weak and slowly varying inhomogeneity, showing more finer effects on the propagation of soliton. Experimental methods of producing such density gradients have been discussed in earlier works[176, 180].

From the steady state condition of ion continuity equation we get

$$\frac{\partial}{\partial \eta}[\tilde{n_0}\tilde{u_0}] = 0, \tag{6.4}$$

where $\tilde{u_0}$ is the equilibrium ion velocity. Hence after integration $\tilde{u_0}$ can be determined as

$$\tilde{u_0} = \frac{c_1}{\tilde{n_0}} = c_1[1 - \delta f_0], \tag{6.5}$$

where c_1 is an integration constant and higher order terms are neglected due to smallness.

Now from the steady state condition of the x component of momentum equation we get

$$\frac{\partial}{\partial \eta}[\tilde{u_0}^2 + \tilde{\phi_0}] = 0, \tag{6.6}$$

where $\tilde{\phi_0}$ is the equilibrium potential. After integration $\tilde{\phi_0}$ can be determined as

$$\tilde{\phi_0} = C_2 - \tilde{u_0}^2, \tag{6.7}$$

where C_2 is another integration constant. Choosing $C_2 = c_1^2$, we get

$$\tilde{\phi_0} = 2c_1^2 \delta f_0(\eta), \tag{6.8}$$

where also higher order terms are neglected due to smallness. Choosing this, we can also see that the steady state condition of Poisson's equation is also satisfied for these functions of $\tilde{\phi_0}, \tilde{n_0}$ if the higher order terms are neglected due to smallness.

These equilibrium quantities are obtained self consistently from the fluid equations (6.1). To create equilibrium in a real experimental situation, external electric fields are imposed by using appropriate biasing arrangements inside the plasma. Details of the setup are found in [176, 180]. This also gives rise to steady drift that is space dependent in presence of density gradients.

We give stress upon the point that equilibrium electron number density is also inhomogeneous. The inhomogeneity of the of the equilibrium electron density can be expressed clearly from equation (6.1) , from where we get

$$\tilde{n_{e0}} = e^{\tilde{\phi_0}(\eta)}, \tag{6.9}$$

hence it is also inhomogeneous.

A reductive perturbation method is carried out with ϵ as the expansion parameter to obtain the two dimensional nonlinear evolution equation with weak transverse propagation. δ is a small parameter which is controlled externally to form the equilibrium density profile. For the sake of this work we take here δ to be $\approx \epsilon$.

All the variables are expanded as

$$n = 1 + \epsilon f_0(\eta) + \epsilon n_1(\xi, \eta, \lambda) + \epsilon^2 n_2(\xi, \eta, \lambda) + ... \tag{6.10}$$

$$\phi = 2c_1^2 \epsilon f_0(\eta) + \epsilon \phi_1(\xi, \eta, \lambda) + \epsilon^2 \phi_2(\xi, \eta, \lambda) + ... \tag{6.11}$$

$$u_x = c_1[1 - \epsilon f_0(\eta)] + \epsilon u_1(\xi, \eta, \lambda) + \epsilon^2 u_2(\xi, \eta, \lambda) + ... \tag{6.12}$$

$$u_y = \epsilon^{\frac{3}{2}} v_1(\xi, \eta, \lambda) + \epsilon^{\frac{5}{2}} v_2(\xi, \eta, \lambda) + ... \tag{6.13}$$

The set of stretched quantities and the expansion of the physical quantities given by (6.3)and (6.10)-(6.13) are used in the fluid equations (6.1) and the coefficients of different powers of ϵ are collected and set to zero.

At the lowest order ϵ, we get

$$\phi_1 + 2c_1^2 f_0 = n_1 + f_0 \tag{6.14}$$

At $\epsilon^{\frac{3}{2}}$ we get,

$$(M - c_1)\frac{\partial n_1}{\partial \xi} = \frac{\partial u_1}{\partial \xi}, \quad \frac{\partial \phi_1}{\partial \xi} = (M - c_1)\frac{\partial u_1}{\partial \xi}, \tag{6.15}$$

from where we obtain $(M-c_1)n_1 = u_1$ and $(M-c_1)u_1 = \phi_1$, where it is assumed that as $\xi \to \pm\infty$, $n_1, u_1 \to 0$. Using (6.14) and (6.15) we get $(M-c_1)^2 = 1$ and $2c_1^2 = 1$ which gives $c_1 = \frac{1}{\sqrt{2}}, M = 1 + \frac{1}{\sqrt{2}}$ and $\phi_1 = n_1 = u_1$.

Because of the presence of drift, the equilibrium dispersion relation in normalized variable is given by $M = 1 + \frac{1}{\sqrt{2}}$ At $\epsilon^{\mathbf{2}}$; we obtain,

$$\frac{\partial v_1}{\partial \xi} = \frac{\partial \phi_1}{\partial \lambda}, \quad \frac{\partial^2 \phi_1}{\partial \xi^2} + n_2 = \phi_2 + \frac{1}{2}(f_0 + \phi_1)^2 \tag{6.16}$$

Finally at $\epsilon^{\frac{5}{2}}$ order we get

$$-\frac{\partial n_2}{\partial \xi} + \frac{\partial}{\partial \xi}(-c_1 f_0 n_1 + u_1 n_0) + \frac{\partial u_2}{\partial \xi} + \frac{\partial}{\partial \xi}(u_1 n_1) - c_1 \frac{\partial f_0}{\partial \eta} + \frac{\partial u_1}{\partial \eta} + \frac{\partial v_1}{\partial \lambda} + c_1 \frac{\partial f_0}{\partial \eta} + c_1 \frac{\partial n_1}{\partial \eta} = 0, \tag{6.17}$$

$$-\frac{\partial u_2}{\partial \xi} + \frac{\partial}{\partial \xi}(-c_1 f_0 n_1) + \frac{\partial \phi_2}{\partial \xi} + n_1 \frac{\partial}{\partial \xi}(n_1) - c_1 \frac{\partial f_0}{\partial \eta} + c_1 \frac{\partial u_1}{\partial \eta} + 2c_1^2 \frac{\partial f_0}{\partial \eta} + \frac{\partial n_1}{\partial \eta} = 0, \tag{6.18}$$

combination of which using (6.16), we get the final evolution equation

$$\frac{\partial}{\partial \xi}[(2 + 2c_1)\frac{\partial n_1}{\partial \eta} + 2n_1 \frac{\partial n_1}{\partial \xi} + \frac{\partial^3 n_1}{\partial \xi^3}] + \frac{\partial^2}{\partial \lambda^2}(n_1) - 2c_1 f_0 \frac{\partial^2 n_1}{\partial \xi^2} = 0, \tag{6.19}$$

with $c_1 = \frac{1}{\sqrt{2}}$, which is nothing but Kadomtsev-Petviashvili (KP) equation with an extra term appearing due to inhomogeneity . Here we have considered the simplest configuration of unmagnetized plasma with cold ions and isothermal electrons, but the similar equation with different coefficients can be derived for more complexities like ion temperature, presence of magnetic field etc for the chosen equilibrium ion number density profile. Note that, in [210] the modified KP equation was derived, considering the fact that the scale length of the plasma inhomogeneity is much larger than the width of the soliton, and solitary wave solution is given for constant density gradient. Here,the equation (6.19) is the evolution equation for the nonlinear ion acoustic wave in two dimension where the unperturbed ion number density profile is taken to be slowly varying and weak.

Moving into the new frame

$$X = \xi + a(\eta), \quad Y = \lambda, \quad T = \eta, \tag{6.20}$$

with

$$a(\eta) = (\frac{c_1}{1+c_1}) \int f_0(\eta) d\eta, \tag{6.21}$$

with $c_1 = \frac{1}{\sqrt{2}}$. Equation (6.19) can be transformed to the standard constant coefficient KP equation

$$\frac{\partial}{\partial X}[\frac{\partial U}{\partial \tau} + 6U\frac{\partial U}{\partial X} + \frac{\partial^3 U}{\partial X^3}] + \frac{\partial^2 U}{\partial Y^2} = 0, \tag{6.22}$$

where $U = \frac{n_1}{3}$ and $\tau = T/(2+2c_1)$. This is a standard completely integrable KP equation which can be solved exactly giving soliton solutions. But due to the presence of the term $a(\eta)$ which is related to $f_0(\eta)$ via (6.21), in the new co-ordinate X, bending of solitons in the two dimensional plane occurs which will be shown in the next section.

6.3 Bending of solitons

One soliton solution of (6.22) is given by [217, 218],

$$U = \frac{k_1^2}{2} Sech^2[\frac{1}{2}(k_1 X + m_1 Y - \frac{k_1^4 + m_1^2}{k_1}\tau)]. \tag{6.23}$$

Expressing the solution in old variables we get,

$$n_1 = (\frac{3k_1^2}{2}) Sech^2[\frac{1}{2}\{k_1\xi + k_1 a(\eta) + m_1\lambda - \frac{k_1^4 + m_1^2}{2k_1(1+c_1)}\eta\}], \tag{6.24}$$

with $c_1 = \frac{1}{\sqrt{2}}$,where $a(\eta)$ is given by (6.21) and k_1, m_1 are arbitrary constants.

Due to the presence of the quantity $a(\eta)$ in the phase, the bending of soliton occurs controlled by $f_0(\eta)$, related to the inhomogeneous ion number density. Here the plasma is inhomogeneous due to the presence of the function $f_0(\eta)$. For different choices of f_0, the inhomogeneities are different. For the choice of $f_0 = 0$ the plasma becomes homogeneous which have the usual line solitons which is represented in FIG 6.2. Thus this trivial choice of f_0 in the equilibrium ion number density profile reproduces homogeneous plasma from the chosen inhomogeneity profile. For different functional forms of f_0 dependent on the slowly varying co-ordinate η, different types of bending occurs, which are shown in FIG. 6.2,6.3.

Since for the sake of this problem, we have chosen $\delta = \epsilon$, which is the small perturbation parameter of our calculation. The solitary wave solution is independent of ϵ,which is here $= \delta$. The solution depends on the function $a(\eta)$ which is related to the function $f_0(\eta)$ through equation (6.21) , causing the soliton to bend in the two dimensional plane. But the amplitude of the solitary wave solution remains constant.

Similarly, the two soliton solution is given by [217, 218],

$$n_1 = 6\frac{\partial^2}{\partial \xi^2}(\ln F_2), \tag{6.25}$$

with,

$$\begin{aligned} F_2 &= 1 + e^{\eta_1} + e^{\eta_2} + A_{12}e^{\eta_1+\eta_2}, \\ A_{12} &= \frac{(K_1 - K_2)^2 - (M_1 - M_2)^2}{(K_1 + K_2)^2 - (M_1 - M_2)^2}, \\ \eta_1 &= K_1[\xi + a(\eta) + \sqrt{3}M_1\lambda - \frac{(K_1^2 + 3M_1^2)}{(2 + 2c_1)}\eta], \\ \eta_2 &= K_2[\xi + a(\eta) + \sqrt{3}M_2\lambda - \frac{(K_2^2 + 3M_2^2)}{(2 + 2c_1)}\eta], \end{aligned} \tag{6.26}$$

where K_1, K_2, M_1, M_2 are arbitrary constants. Bending of two soliton solution for different functional forms of $f_0(\eta)$ are also shown in FIG. 6.2,6.3. Since f_0 should reach zero value at $\eta = 0$ following FIG 6.1, it has been chosen accordingly. Now it is required to determine how much bending is taking place by varying f_0 i.e, what the condition is for larger bending.

Let us start from the one soliton solution (6.24). The amplitude of the 'Sech' function is maximum when its argument goes to zero. For static case ($\xi = 0$), the locus of the highest amplitude of the solution is of the form

$$\frac{1}{2}\{k_1 a(\eta) + m_1 \lambda - \frac{k_1^4 + m_1^2}{2k_1(1+c_1)}\eta\} = 0, \tag{6.27}$$

where $c_1 = \frac{1}{\sqrt{2}}$.

Note that, for homogeneous plasma f_0 is zero making $a(\eta)$ to be also zero determined from equation (6.21). Hence the locus is straight line giving line solitons for homogeneous plasma.

Now taking derivative w.r.to η twice in the above equation (6.27) we get

$$\frac{ds}{d\eta} = -\frac{k_1 c_1}{m_1(1+c_1)}\frac{df_0}{d\eta} \tag{6.28}$$

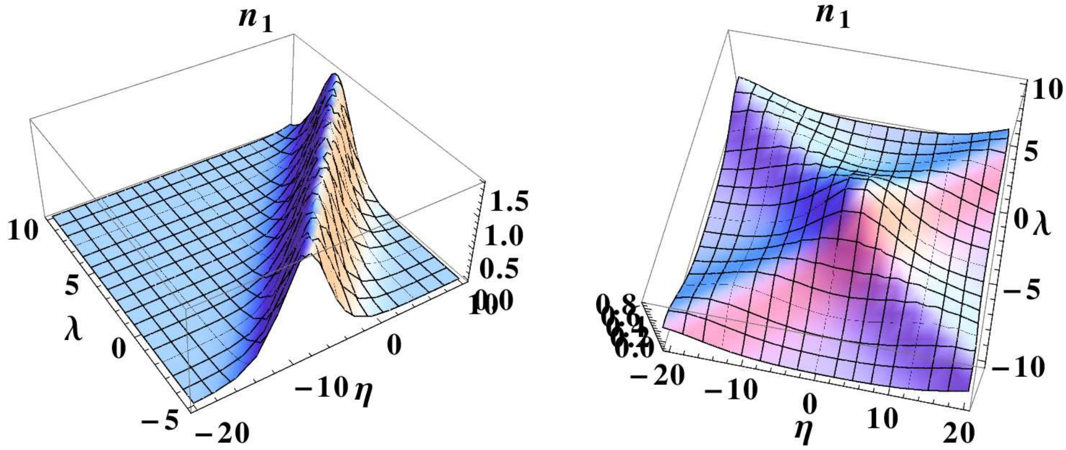
where $s = \frac{d\lambda}{\partial\eta}$ is the slope of the locus of the maximum amplitude. We choose k_1, m_1 such that

$$\frac{ds}{d\eta} = \frac{df_0}{d\eta} \tag{6.29}$$

We see from the above equation that for higher value of RHS, rate of variation of slope will also be higher. Hence the slope of the maximum amplitude curve will vary large for traversing unit distance in η. Larger rate of variation of slope describes larger bending.

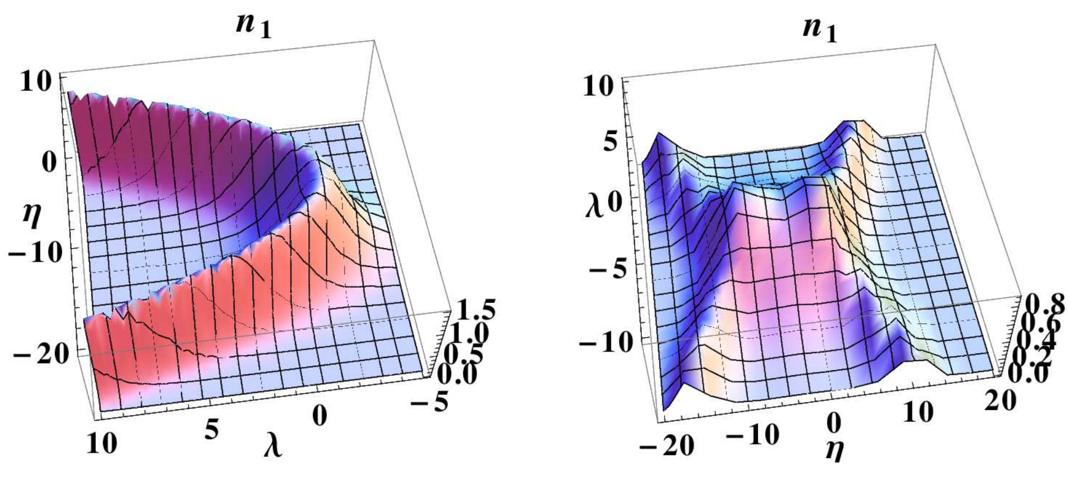
Hence for large bending of solitons to take place, the first derivative of f_0 w.r.to η must also be high. This is incorporated in FIG 6.3 where bending of solitons occur for different choices of f_0.

For FIG 6.3(a), if we increase the amplitude of f_0 then the parabola will steepen causing larger bending. Similar thing can be observed for FIG 6.3(b) where the sine function becomes more rapid . Now if we increase the wave vector of f_0 in 6.3(b) then also the bending will become larger. Increase/decrease of both amplitude as well as wave vector of f_0 will increase/decrease the first derivative of f_0 causing more/less bending. The same analysis can be extended to the other figures 6.3(c), 6.3(d) too.

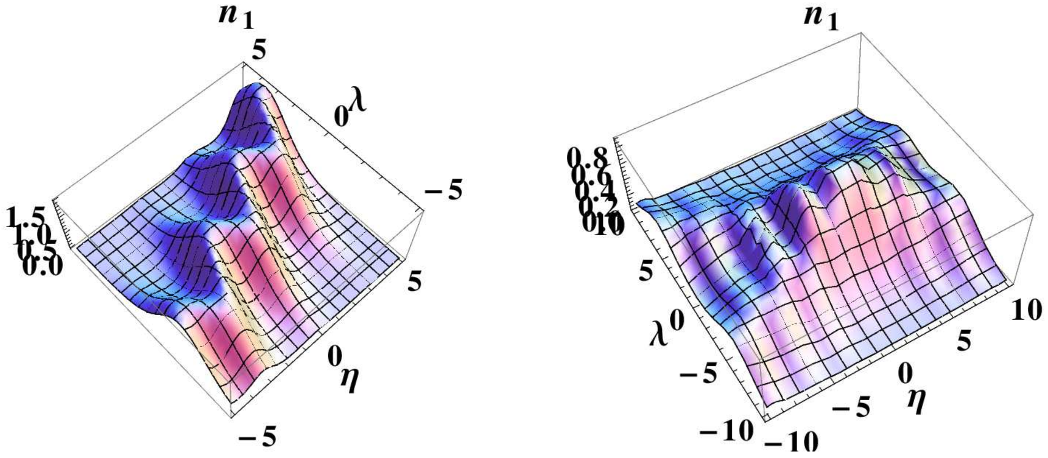


(a) One soliton (b) Two soliton

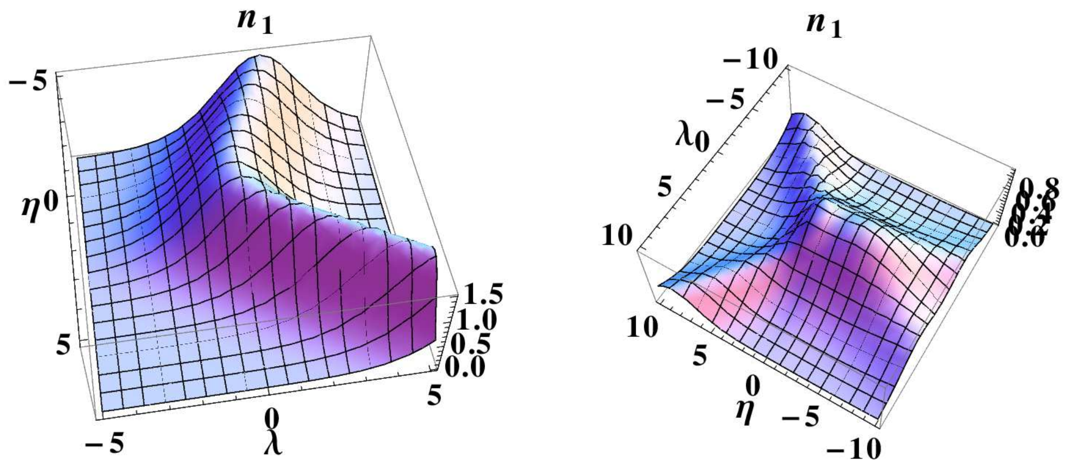
FIG 6.2: **Static picture of one and two soliton solutions given by (6.24) and (6.25) of the two dimensional ion acoustic wave at** $\xi = 0$ **for** $k_1 = 1, m_1 = 1, K_1 = K_2 = \frac{1}{2}, P_1 = -P_2 = \frac{2}{3}$ **and** $f_0 = 0$**. For this choice of** f_0**, the solution converges to the usual line solitons as observed in the homogeneous plasma.**



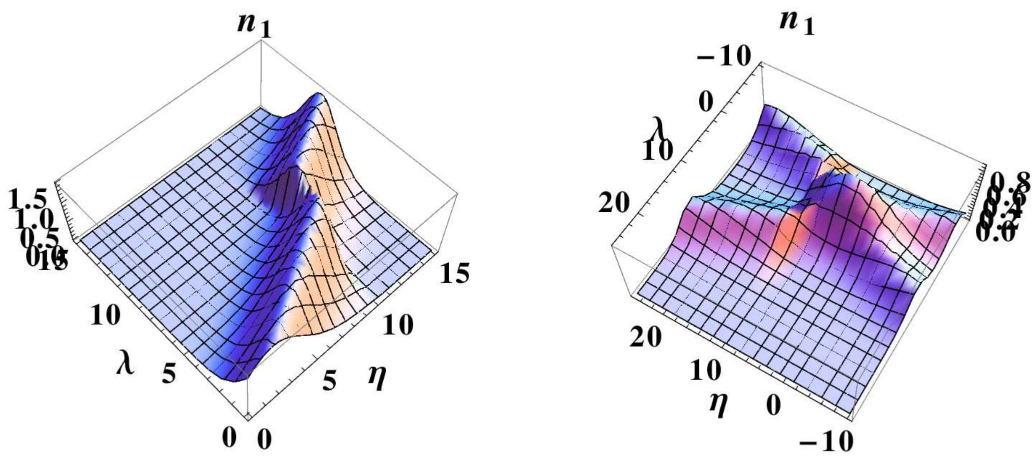
(a) Bending of one and two soliton for $f_0 = -\eta/3$



(b) Bending of one and two soliton for $f_0 = -4\sin(2\eta)$



(c) Bending of one and two soliton for $f_0 = -(1+\sqrt{2})\tanh(\eta)$



(d) Bending of one and two soliton for $f_0 = -5(\sqrt{2}+1)(Sech(5(\eta - 10)))^2$

FIG 6.3: **Static picture of one and two soliton solutions given by (6.24) and (6.25) of the two dimensional ion acoustic wave at** $\xi = 0$ **for** $k_1 = 1, m_1 = 1, K_1 = K_2 = \frac{1}{2}, P_1 = -P_2 = \frac{2}{3}$ **and for the specified functions of** f_0 **which is related to unperturbed ion number density. The different functional forms of** f_0 **causes the phase of the solitary wave to change which causes bending in the two dimensional plane, whereas the amplitude remains constant.**

Frycz and Infeld obtained the bending of soliton [216] by studying numerically the nonlinear stability analysis of KP equation. The characteristics of KP equation state that the initial condition must fulfill an infinite set of constraints if the solution is to remain localized. Just adding a perturbation to one soliton solution would violate this constraint. Thus bending is a natural perturbation which is a choice for initial condition of this numerical simulation. But in our work, the bending of solitons were obtained analytically showing dependence on f_0 which is related to inhomogeneous ion number density. We have exactly solved the KP equation (6.19) obtained for the two dimensional propagation of ion acoustic wave for weak and slowly varying inhomogeneity, related to the arbitrary function $f_0(\eta)$. Since we have transformed the evolution equation into a standard constant coefficient KP equation, its each and every solution faces the same phase modification controlled by f_0, causing the shape of the solution to change in the two dimensional plane. The amplitude of the soliton solutions is found to remain constant. This is in view of the weak and slowly varying inhomogeneous ion number density, so that all variations appear only in the phase of the soliton.

We see that the weak and slowly varying equilibrium potential, which exists in the plasma, is a function of f_0, directed in the x axis (i.e, η axis). Hence due to this time independent potential an electric field develops which exert force on the ions, constituting the soliton. But due to the inhomogeneity of the equilibrium potential function, different ions situated at different positions are attracted (or repelled) differently. Again an equilibrium ion drift velocity also exists, which is also directed in the x axis and inhomogeneous. Due to the superposed effects of the inhomogeneous equilibrium and also the time dependent quantities, the ions change their positions. This causes the ion acoustic soliton to bend in the two dimensional plane. Since the potential drop is weak as well as slow, the number of ions forming soliton do not change drastically. Hence the amplitude of the soliton remains constant causing its phase to vary with $f_0(\eta)$.

We see that, the one soliton solution of our evolution equation (6.19) contains the inhomogeneous function $a(\eta)$ in the phase. We see from the solution that the function reaches its maximum value when the phase factor turns to be zero. Hence as the soliton propagates in the two dimensional plane, η changes causing $a(\eta)$ to change nonlinearly depending on $f_0(\eta)$. Now if we fix the time variable ξ, then the transverse variable λ has to adjust itself in order to make the phase factor of the "Sech" function zero causing soliton bending.

These bending features of the solitons is very relevant and important in the context of inhomogeneous plasmas along with the other features like reflection, transmission etc. But such a feature has not been explored till now. We see in this work that if the equilibrium density variation is slow and weak, which is very close to the homogeneous value then these bending features can be seen. Hence a more accurate experiment could reveal such finer effects.

6.4 Summary

In this work, we have obtained the bending of ion acoustic solitary wave in the two dimensional plane for the propagation in unmagnetized plasma with cold ions and isothermal electrons with weak and slowly varying density inhomogeneity. We have obtained a modified KP equation with an extra term arising due to inhomogeneous equilibrium ion number density . We have exactly solved the KP equation giving a solitary wave solution in which the phase of the soliton gets modified by a function f_0, which is related to unperturbed ion number density, causing soliton bending, where as the amplitude remains constant. The bending features of the solitons are very relevant and important in inhomogeneous plasma along with the other features like reflection, transmission etc. More accurate and precession experiments could reveal such finer and interesting features.

Chapter 7

Quantum corrections to nonlinear ion acoustic wave with Landau damping

7.1 Introduction

As an application to the nonlinear integrable models, which constitute the main part of the thesis, we explore in this chapter, a new branch of plasma system, called quantum plasma. Before entering into the main work, we briefly outline some basic facts of quantum plasma.

The study of plasmas, is in general limited to the domain of classical physics where temperature is high and particle density is low. In recent years, the study of plasmas such as dense astrophysical plasmas [219], laser plasmas [220] as well as miniature electronic devices that are under extreme physical conditions [221],[222] requires quantum mechanical effects to be taken into account. In such systems, the scale length becomes comparable to the particle de Broglie wavelength rendering classical transport models unsuitable and quantum mechanical effects to be relevant.

In broad aspect there are mainly two approaches to model quantum plasmas which are quantum hydrodynamic approach[223],[224] and quantum kinetic approach[225] i.e, Wigner equation approach. The plasma fluid equations with the inclusion of quantum diffraction and statistical pressure effects give rise to new physical phenomena in the context of linear and nonlinear waves and instabilities [227]. Haas[226] et al. have examined quantum quasilinear plasma turbulence using quasilinear equation derived from Wigner-Poisson system.

The quantum fluid equations being macroscopic in nature are relatively simple and are easily accessible for nonlinear calculations. However, working with such macroscopic models leads to loss of understanding in the situations where effects like Landau damping are important which take into account the resonant interaction of many particles with the electrostatic wave. Under certain conditions, the average effect of this interaction appears as a damping of the wave that can be explored by moving into a kinetic picture. The collision less damping phenomenon of electron plasma waves was first predicted by Landau[229] in 1946 for Langmuir oscillation. At the beginning, people once thought that Landau damping is just a mathematical result, and it did not exist in physics. Fortunately, Dawson derived it from the perspective of energy exchange between particles and waves, and meanwhile Malmberg and Wharton[230] confirmed it experimentally in 1965. Since derived in many ways and confirmed experimentally , Landau damping has become perhaps one of the most important phenomena in plasma physics. Landau damping of ion-acoustic wave [234] is investigated thoroughly in different set ups like in highly ionized [237], multicomponent [238], dusty [240] plasmas with shock solutions[235], with ion Landau damping [233] etc. The investigation in Alfven waves [231], magnetosonic wave [236], electromagnetic waves and electron acoustic waves [241] also received wide attention and interest.

The kinetic description of plasma possessing quantum mechanical features is provided by the Wigner equation that can be considered as the quantum analogue of the Vlasov equation. It describes the evolution of the quantum mechanical phase space distribution function given by the Wigner-Moyal distribution and can be a useful tool to look into the microscopic nature of the system. The Wigner function is called quasi-distribution as it can have negative values although its velocity moments give rise to various physical variables such as density, current etc. Gardner[228] derived the full three-dimensional quantum hydrodynamic (QHD) model for the first time by a moment expansion of the Wigner-Boltzmann equation. So far nonlinear problems like KdV equation and BGK modes have been tackled successfully in classical plasma. Recently, Lange et al.[232] have provided a quantum generalization of the classical BGK modes by obtaining a solution of the stationary Wigner-Poisson equation.

In this work we have attempted to look into the quantum KdV problem in the semi-classical limit. For a classical plasma [50] Ott and Sudan have modeled nonlinear ion acoustic wave in a kinetic picture taking the mass of electron into account. They obtained a KdV equation together with a Landau damping term as an evolution equation for the ion acoustic wave. In order to explore the quantum corrections to the nonlinear evolution of an ion acoustic wave in presence of Landau damping terms we have to replace the Vlasov equation by the Wigner equation.

In this chapter we have tried to investigate, in the semi-classical limit, the quantum corrections to nonlinear ion acoustic wave with Landau damping. We have derived a higher order KdV equation which has higher order nonlinear quantum corrections with the usual classical Landau damping term and a term containing the quantum corrections due to Landau damping as the dynamical evolution equation. The equation converges to the same equation as derived by Ott and Sudan in the classical limit i.e, when $\hbar$ tends to zero. The equation shows some features like conservation of total ion number , decay of initial waveform due to Landau damping etc. In the next stage we have carried out the perturbative approach of Bogoliubov and Mitropolsky to get the decay nature of KdV solitary wave amplitude. For this purpose we have assumed the Landau damping parameter α_1 to be of the order of the quantum factor Q. The procedure reveals that the amplitude decays inversely with the square of time depending on the factor Q.

7.2 Derivation of the dynamical equation

The Wigner distribution function is a function of the phase-space variables (x, v) and time, which, is given by N single particle wave function $\psi_\alpha(x,t)$ each characterized by a probability P_α satisfying $\sum_{\alpha=1}^{N} P_\alpha = 1$.

It is given as,

$$f(x,v,t) = \sum_{\alpha=1}^{N} \frac{m}{2\pi\hbar} P_\alpha \int_{-\infty}^{\infty} \psi_\alpha^*(x+\lambda/2,t)\psi_\alpha(x-\lambda/2,t)e^{\frac{imv\lambda}{\hbar}}d\lambda, \tag{7.1}$$

where m is the mass of the particle. The Wigner function follows the following evolution equation called the Wigner equation

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{em}{2i\pi\hbar^2}\iint [\phi(x+\lambda/2) - \phi(x-\lambda/2)]f(x,v',t)e^{\frac{im\lambda(v-v')}{\hbar}}d\lambda dv' = 0, \tag{7.2}$$

where $\hbar, \phi$ are the reduced Planck's constant and self- consistent electrostatic potential. Considering semi-classical limit, we develop the integral up to $O(\hbar^2)$ and neglect all higher order terms containing $\hbar$ to obtain

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{e}{m}\frac{\partial \phi}{\partial x}\frac{\partial f}{\partial v} - (\frac{e\hbar^2}{24m^3})\frac{\partial^3 \phi}{\partial x^3}\frac{\partial^3 f}{\partial v^3} = 0 \tag{7.3}$$

We can see from (7.3) that the Vlasov equation is recovered in the limit $\hbar \to 0$.

In our work, we consider a situation where ions are cold ($T_i = 0$) and electrons have finite temperature and the quantum effects are relevant for electrons only. Therefore, we consider the usual fluid equations for describing the dynamics of ions and the Wigner equation for describing the electrons.

Hence in this case the relevant normalized system of one-dimensional equations are -

$$\frac{\partial n}{\partial t} + \frac{\partial (nu)}{\partial x} = 0, \tag{7.4}$$

which is the continuity equation for ions. The momentum conservation equation for the ions is given by

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{\partial \phi}{\partial x}, \tag{7.5}$$

$$(\frac{\lambda_D}{L})^2 \frac{\partial^2 \phi}{\partial x^2} = n_e - n, \tag{7.6}$$

which is the Poisson's equation appropriate for the description of dispersive ion acoustic waves. The electron number density if obtained as the velocity space average of the single particle distribution function f

$$n_e = \int_{-\infty}^{\infty} f dv, \tag{7.7}$$

that is described by the Wigner equation in the semi-classical limit

$$(\frac{m_e}{m_i})^{\frac{1}{2}} \frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial x}\frac{\partial f}{\partial v} - Q\frac{\partial^3 \phi}{\partial x^3}\frac{\partial^3 f}{\partial v^3} = 0, , \tag{7.8}$$

where n_e, n, u are the electron number density, ion number density and ion velocity respectively, $\lambda_D = \sqrt{KT_e/4\pi n_0 e^2}$ is the Debye Length, L is the characteristic length for variations of n, u, ϕ, n_e, f and Q is the quantum parameter $= \hbar^2/24m^2c_s^2L^2$.

The variables in the above equations (7.4) - (7.8) are normalized dimensionless variables. Here the following normalization scheme has been used:

$$x = \frac{\tilde{x}}{L}, t = \frac{c_0\tilde{t}}{L}, v = \frac{\tilde{v}}{c_s}, \phi = \frac{e\tilde{\phi}}{K_B T_e}, n = \frac{\tilde{n}}{n_0}, f = \frac{\tilde{f}}{n_0}, u = \frac{\tilde{u}}{c_0}, \tag{7.9}$$

where c_0 is the ion acoustic sound speed $= \sqrt{KT_e/m_i}$, c_s is the electron thermal velocity $= \sqrt{KT_e/m_e}$, n_0 is the ambient number density of electrons (ions) and T_e is the electron temperature. The tilde variables appearing in (7.9) are original dimensional variables which are made dimensionless by multiplying with appropriate scale factors.

As in case of [50], here also three basic parameters enter into the problem which are parameters due to Landau damping by electrons, measure of nonlinearity and measure of dispersive effects. In this calculation we do not neglect the electron to ion mass ratio and since $T_i = 0$, the Landau damping is provided solely by electrons. We consider all these three effects i.e., Landau damping, nonlinearity and dispersion to be small but of the same order of magnitude.

1) $\sqrt{(m_e/m_i)} = \alpha_1\epsilon$, effect due to Landau damping by electrons.

2) $\triangle\, n/n_0 = \alpha_2\epsilon$, measure of the strength of nonlinearity.

3) $(\lambda_D/L)^2 = 2\alpha_3\epsilon$, measure of strength of dispersive effects.

Here ϵ is smallness parameter. As is the usual mathematical procedure we transform our co-ordinates to a moving frame with a stretched time as

$$\xi = x - t,\ \ \tau = \epsilon t, \tag{7.10}$$

and expand the dependent variables for small nonlinearity as

$$\begin{aligned}
n &= 1 + \alpha_2\epsilon n^{(1)} + \alpha_2^2\epsilon^2 n^{(2)} + ...,\\
u &= \alpha_2\epsilon u^{(1)} + \alpha_2^2\epsilon^2 u^{(2)} + ...,\\
\phi &= \alpha_2\epsilon \phi^{(1)} + \alpha_2^2\epsilon^2 \phi^{(2)} + ...,\\
n_e &= 1 + \alpha_2\epsilon n_e^{(1)} + \alpha_2^2\epsilon^2 n_e^{(2)} + ...,\\
f &= f^{(0)} + \alpha_2\epsilon f^{(1)} + \alpha_2^2\epsilon^2 f^{(2)} + ...
\end{aligned} \tag{7.11}$$

Considering semi-classical limit, the form of $f^{(0)}$ is chosen as

$$f^{(0)}(v) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-v^2}{2}\right) \tag{7.12}$$

Substituting Eqns. (7.10), (7.11), (7.12) in (7.4)-(7.8) and equating coefficients of ϵ, ϵ^2 to zero we get first and second order equations which need to be solved.

7.2.1 ϵ order calculation:

From Eqns (7.4)-(7.6) we get

$$\frac{\partial n^{(1)}}{\partial \xi} = \frac{\partial u^{(1)}}{\partial \xi} = \frac{\partial \phi^{(1)}}{\partial \xi}, n^{(1)} = n_e^{(1)} \tag{7.13}$$

From equation (7.8) we get

$$v\frac{\partial f^{(1)}}{\partial \xi} = v\frac{\partial \phi^{(1)}}{\partial \xi} f^{(0)} + Q\frac{\partial^3 \phi^{(1)}}{\partial \xi^3}(3v - v^3) f^{(0)}, \tag{7.14}$$

which yields

$$\frac{\partial f^{(1)}}{\partial \xi} = \frac{\partial \phi^{(1)}}{\partial \xi} f^{(0)} + Q\frac{\partial^3 \phi^{(1)}}{\partial \xi^3}(3 - v^2) f^{(0)} + \lambda(\xi, \tau)\delta(v), \tag{7.15}$$

where $\delta(v)$ is the Dirac delta function and $\lambda(\xi, \tau)$ is an arbitrary function of ξ, τ. Here also the problem of non-uniqueness arises as in case of [50],[242] which can be removed by taking a τ derivative term from higher ϵ order. Thus, we write

$$(\alpha_1 \epsilon^2)\frac{\partial f_\epsilon^{(1)}}{\partial \tau} + v\frac{\partial f_\epsilon^{(1)}}{\partial \xi} = v\frac{\partial \phi^{(1)}}{\partial \xi} f^{(0)} + Q\frac{\partial^3 \phi^{(1)}}{\partial \xi^3}(3v - v^3) f^{(0)}, \tag{7.16}$$

where the first term of (7.16) has been taken from order ϵ^3 equation. Once $f_\epsilon^{(1)}$ is known, $f^{(1)}$ can be determined uniquely by :

$$f^{(1)} = \lim_{\epsilon \to 0} f_\epsilon^{(1)} \tag{7.17}$$

We introduce Fourier transform in ξ and τ as

$$\widehat{f_\epsilon^{(1)}}(\omega, k) = \frac{1}{2\pi}\int_{\xi=-\infty}^{\infty}\int_{\tau=0}^{\infty} f_\epsilon^{(1)}(\xi, \tau)\exp[i(\omega\tau - k\xi)]d\xi d\tau \tag{7.18}$$

Now ,

$$(\widehat{\frac{\partial f_\epsilon^{(1)}}{\partial \xi}})(\omega, k) = (ik)\widehat{f_\epsilon^{(1)}}(\omega, k), \tag{7.19}$$

and

$$(\widehat{\frac{\partial f_\epsilon^{(1)}}{\partial \tau}})(\omega, k) = -(i\omega)\widehat{f_\epsilon^{(1)}}(\omega, k) - \frac{1}{2\pi}\int_{\xi=-\infty}^{\infty} \exp[-ik\xi] f_\epsilon^{(1)}|_{\tau=0} d\xi, \tag{7.20}$$

and

$$\widehat{\frac{\partial^3 \phi^{(1)}}{\partial \xi^3}}(\omega, k) = (-ik^3)\widehat{\phi^{(1)}}(\omega, k) \tag{7.21}$$

Now applying these Fourier transforms on (7.16), letting $\epsilon \to 0$ and using

$$\lim_{\epsilon \to 0} \frac{1}{(kv - \omega\alpha_1\epsilon^2)} = P(\frac{1}{kv}) + i\pi\delta(kv) \tag{7.22}$$

we get,

$$\widehat{f^{(1)}}(\omega, k) = \widehat{\phi^{(1)}}(\omega, k) f^{(0)} - Qk^2(3 - v^2)\widehat{\phi^{(1)}}(\omega, k) f^{(0)}, \tag{7.23}$$

where P is the principal part of the integral. Taking inverse Fourier transform we get the form of $f^{(1)}$ as,

$$f^{(1)} = \phi^{(1)} f^{(0)} + Q(3 - v^2)\frac{\partial^2 \phi^{(1)}}{\partial \xi^2} f^{(0)} \tag{7.24}$$

The first term of (7.24) is same with the classical case whereas the second term is the quantum correction term. Thus the procedure yields that $\lambda(\xi, \tau)$ appearing in (7.15) is zero.

7.2.2 ϵ^2 order calculation:

From equations (7.4)- (7.6), we can obtain in a straightforward way,

$$2\frac{\partial n^{(1)}}{\partial \tau} + 3\alpha_2 n^{(1)}\frac{\partial n^{(1)}}{\partial \xi} + 2\alpha_3\frac{\partial^3 n^{(1)}}{\partial \xi^3} = \alpha_2\frac{\partial}{\partial \xi}(n_e^{(2)} - \phi^{(2)}) \tag{7.25}$$

From equation (7.8) we get,

$$(\alpha_1\epsilon^2)\frac{\partial f_\epsilon^{(2)}}{\partial\tau}+v\frac{\partial f_\epsilon^{(2)}}{\partial\xi}-vf^{(0)}\frac{\partial\phi^{(2)}}{\partial\xi}-Q\frac{\partial^3\phi^{(2)}}{\partial\xi^3}\frac{\partial^3 f^{(0)}}{\partial v^3}=C(\xi,\tau,v), \tag{7.26}$$

where the τ derivative term is taken from ϵ^4 order and terms which are product of quantum term and second order perturbation term are neglected as small compared to other terms. Here $C(\xi,\tau,v)$ is defined as

$$C(\xi,\tau,v)=[C_a(\xi,\tau)+C_b(\xi,\tau)v+C_c(\xi,\tau)v^2+C_d(\xi,\tau)v^3]f^{(0)}, \tag{7.27}$$

where

$$C_a(\xi,\tau)=(\frac{\alpha_1}{\alpha_2})[\frac{\partial\phi^{(1)}}{\partial\xi}+3Q\frac{\partial^3\phi^{(1)}}{\partial\xi^3}] \tag{7.28}$$

$$C_b(\xi,\tau)=[\phi^{(1)}\frac{\partial\phi^{(1)}}{\partial\xi}+5Q\frac{\partial^2\phi^{(1)}}{\partial\xi^2}\frac{\partial\phi_1}{\partial\xi}+3Q\phi^{(1)}\frac{\partial^3\phi^{(1)}}{\partial\xi^3}] \tag{7.29}$$

$$C_c(\xi,\tau)=-Q(\frac{\alpha_1}{\alpha_2})[\frac{\partial^3\phi^{(1)}}{\partial\xi^3}] \tag{7.30}$$

$$C_d(\xi,\tau)=[-Q\frac{\partial^2\phi^{(1)}}{\partial\xi^2}\frac{\partial\phi_1}{\partial\xi}-Q\phi^{(1)}\frac{\partial^3\phi^{(1)}}{\partial\xi^3}] \tag{7.31}$$

Introducing Fourier transform in (7.26) and letting ϵ tends to zero we get,

$$\begin{aligned}\widehat{f^{(2)}}(\omega,k)-f^{(0)}\widehat{\phi^{(2)}}(\omega,k)=&-i\widehat{C_a}[P(\frac{1}{kv})+i\pi\delta(kv)]f^{(0)}-iv\widehat{C_b}P(\frac{1}{kv})f^{(0)}\\&-iv^2\widehat{C_c}P(\frac{1}{kv})f^{(0)}-iv^3\widehat{C_d}P(\frac{1}{kv})f^{(0)}\end{aligned} \tag{7.32}$$

Multiplying by (ik) and integrating over v yields

$$ik\widehat{n^{(2)}}-ik\widehat{\phi^{(2)}}=i\sqrt{\frac{\pi}{2}}\widehat{C_a}sgn(k)+\widehat{C_b}+\widehat{C_d}, \tag{7.33}$$

where we have used $k\delta(kv) = sgn(k)\delta(v)$.

Now taking inverse Fourier transform of equation (7.33) we obtain,

$$\frac{\partial}{\partial \xi}(n^{(2)} - \phi^{(2)}) = C_b + C_d - \frac{1}{\sqrt{2\pi}}[P\int_{-\infty}^{\infty}(\frac{\alpha_1}{\alpha_2})\frac{\partial n^{(1)}}{\partial \xi'}\frac{d\xi'}{\xi - \xi'} + P\int_{-\infty}^{\infty}(\frac{3Q\alpha_1}{\alpha_2})\frac{\partial^3 n^{(1)}}{\partial \xi'^3}\frac{d\xi'}{\xi - \xi'}], \tag{7.34}$$

Now using (7.25) and (7.34) we get finally,

$$\frac{\partial n^{(1)}}{\partial \tau} + \alpha_2 n^{(1)}\frac{\partial n^{(1)}}{\partial \xi} + \alpha_3\frac{\partial^3 n^{(1)}}{\partial \xi^3} - Q\alpha_2\frac{\partial}{\partial \xi}[\frac{\partial n^{(1)}}{\partial \xi}]^2 - Q\alpha_2 n^{(1)}\frac{\partial^3 n^{(1)}}{\partial \xi^3} + \frac{\alpha_1}{\sqrt{8\pi}}[P\int_{-\infty}^{\infty}\frac{\partial n^{(1)}}{\partial \xi'}\frac{d\xi'}{(\xi - \xi')}] + \frac{3\alpha_1 Q}{\sqrt{8\pi}}[P\int_{-\infty}^{\infty}\frac{\partial^3 n^{(1)}}{\partial \xi'^3}\frac{d\xi'}{(\xi - \xi')}], = 0 \tag{7.35}$$

which is the main equation of interest of this work. This equation implies the evolution equation of motion of nonlinear ion acoustic wave taking into account the Landau damping effect with quantum corrections arising from semi-classical kinetic approach i.e, the Wigner equation approach. Quantum correction to linear Landau damping of electron plasma waves have been investigated earlier in [239]. The fourth and fifth terms of (7.35) are nonlinear quantum corrections and the last term of the LHS is the quantum correction on the Landau damping. We can see that the equation converges exactly to the equation derived by Ott and Sudan [50] in the limit $\hbar \to 0$. The equation is like a higher order KdV equation which have higher order nonlinear quantum correction terms and Landau damping term with its quantum correction. Due to the nature of the equation we can show that it conserves total number of particles. The presence of Landau damping terms also assure that the amplitude of soliton must decay with time. These relevant facts are derived in the next section.

7.3 Some relevant properties

7.3.1 Conservation of ion number

The equation (7.35) is the higher order KdV equation with Landau damping terms. Integrating (7.35)w.r.to ξ and assuming $n^{(1)}, \partial n^{(1)}/\partial\xi, \partial^2 n^{(1)}/\partial\xi^2 = 0$ at $\xi = \pm\infty$ and renaming $n^{(1)} = U$, we can show that

$$\frac{\partial}{\partial\tau}\int_{-\infty}^{\infty} U d\xi = 0 \tag{7.36}$$

Here we have used the fact that

$$P\int_{-\infty}^{\infty} \frac{d\xi}{\xi - \xi'} = 0 \tag{7.37}$$

Hence ion number is conserved.

7.3.2 Decay of solitary wave

Ott and Sudan in their paper [50] considered α_1 to be a small perturbation parameter and used the fact that due to Landau damping the amplitude of KdV solitary wave will decrease with time. Then using Bogoliubov- Mitropolsky approximation method, which has been discussed in chapter 3, they found the decay rate of amplitude, which depends on the small parameter α_1 . In (7.35),we see that there are higher order KdV terms with Landau damping term and its quantum correction. But since exact Sech- solitary wave solution of a general higher order KdV equation of above form is possible only when (coefficient of the term $\frac{\partial U}{\partial\xi}[\frac{\partial U}{\partial\xi}]^2$) = -2 (coefficient of the term $U\frac{\partial^3 U}{\partial\xi^3}$), which is not present in (7.35), hence the exact solitary wave solution of the higher order KdV equation and its decay due to Landau damping terms cannot be worked out here. Also it can be seen that (7.35) contains 2 small parameters α_1 and Q where α_2, α_3 are assumed to be ≈ 1. Hence in the subsequent part of the work, the quantum correction terms and the Landau damping term are treated as perturbation term to the KdV equation. But since perturbation with multiple small parameters will include multiple time scales in the calculation, hence it will be too complicated to be computed analytically. In order to simplify the case and find out the nature of decay of the KdV solitary wave amplitude we will assume that $\alpha_1 \approx \alpha_2 Q$. For example, in the case of hydrogen plasma α_1 is approximately 0.025 and in [243, 244], the factor Q is taken to be equal to be order of 0.01. Assuming this relation between small parameters we can consider that the quantum correction to the Landau damping term which appears as the last term of (7.35) is $\alpha_1 \approx Q^2$, and hence it can be neglected as small compared to the other terms.

Now we have to apply the well known method of Bogoliubov and Mitropolsky with $\alpha_2 Q = C$ as small perturbation parameter. Hence α_1 can be taken as $\alpha_1 = \beta C$ where β is any number $\approx$ unity. In order for the perturbation analysis to be consistent with the condition of validity of (7.35) it is also required that $1 \gg C \gg \epsilon$. Assuming a new phase co-ordinate to have the form

$$\phi(\xi, \tau) = \sqrt{\frac{N(\tau)\alpha_2}{12\alpha_3}}(\xi - \frac{\alpha_2}{3}\int_0^\tau N(\tau)d\tau), \tag{7.38}$$

where $N(\tau)$ is assumed to vary slowly with time.

We introduce two time scales following [50] as

$$t_0 = \tau, t_1 = C\tau, \tag{7.39}$$

and $N = N(C, \tau)$ and shall seek a solution of the form

$$U(\phi, C, \tau) = U_0(\phi, t_0, t_1) + O(C), \tag{7.40}$$

where (7.40) is to be valid for long times,i.e., times as large as $\tau \sim O(1/C)$. In order to find such a solution, valid for long times, we first expand $u(\phi,\tau,C)$ to $O(C)$:

$$U(\phi,\tau,C)=U_0(\phi,t_0,t_1)+CU_1(\phi,t_0)+O(C^2) \tag{7.41}$$

Using (7.38), (7.39), (7.41) in (7.35) we get an equation containing different powers of C and equating coefficients of each power of C we get different order equations which need to be solved.

Since we are interested in the damping of solitary waves, we have the following initial and boundary conditions:

$$\begin{aligned} U(\phi,0,C)&=N_0sech^2(\phi),\\ U(\pm\infty,\tau,C)&=0 \end{aligned} \tag{7.42}$$

Solving the order unity equation which is

$$\rho\frac{\partial U_0}{\partial t_0}+\frac{\partial^3U_0}{\partial\phi^3}-4\frac{\partial U_0}{\partial\phi}+\frac{12}{N}U_0\frac{\partial U_0}{\partial\phi}=0, \tag{7.43}$$

we get

$$U_0(\phi,t_0,t_1)=N(t_1)sech^2(\phi), \tag{7.44}$$

where $\rho=24\sqrt{3\alpha_3}/(N\alpha_2)\sqrt{N\alpha_2}$ and $N(t_1)$ is an arbitrary function of t_1 except for the initial condition $N(0)=N_0$. Hence U_0 doesn't depend on t_0.

The order C equation is

$$\frac{\partial U_1}{\partial t_0}+L[U_1]=M[U_0], \tag{7.45}$$

where

$$
\begin{aligned}
M[U_0] &= -\frac{\partial U_0}{\partial t_1} - \frac{\phi}{2N}\frac{\partial U_0}{\partial \phi}\frac{dN}{dt_1} && (7.46)\\
&+ \frac{1}{(\rho\alpha_3)}[\frac{\partial^3 U_0}{\partial \phi^3}U_0 + 2\frac{\partial U_0}{\partial \phi}\frac{\partial^2 U_0}{\partial \phi^2}] \\
&- \frac{\beta}{\sqrt{8\pi}}[P\int_{-\infty}^{\infty}\sqrt{\frac{N(\tau)\alpha_2}{12\alpha_3}}\frac{\partial U_0}{\partial \phi'}\frac{d\xi'}{\xi-\xi'}],
\end{aligned}
$$

(7.47)

$$L[U_1] = \frac{1}{\rho}\frac{\partial^3 U_1}{\partial \phi^3} - \frac{4}{\rho}\frac{\partial U_1}{\partial \phi} + \frac{12}{(N\rho)}\frac{\partial (U_0 U_1)}{\partial \phi} \tag{7.48}$$

Again the boundary and initial conditions are

$$U_1(\pm\infty, t_0) = 0, U_1(\phi, 0) = 0 \tag{7.49}$$

In order that (7.41) to be valid for times as large as $\tau \sim O(1/C)$ it is required that $U_1(\phi, t_0)$ does not behave secularly with t_0. To eliminate secular behavior of U_1 it is necessary that $M[U_0]$ be orthogonal to all solutions, $g(\phi)$, of $L^+[g] = 0$ which satisfy (7.49)[i.e, $g(\pm\infty) = 0$], where L^+ is the operator adjoint to L given by,

$$L^+ = -\frac{1}{\rho}\frac{\partial^3}{\partial \phi^3} + \frac{4}{\rho}\frac{\partial}{\partial \phi} - \frac{12}{\rho}sech^2(\phi)\frac{\partial}{\partial \phi}. \tag{7.50}$$

The only solution of $L^+[g] = 0$, $g(\pm\infty) = 0$, is $g(\phi) = sech^2(\phi)$.

Thus,

$$\int_{-\infty}^{\infty} sech^2(\phi) M[U_0] d\phi = 0 \tag{7.51}$$

In order to evaluate this integral we have to consider term by term of (7.47). The first 2 terms of $M[U_0]$ together give $-dN/dt_1$ after integration. The third and fourth terms which come from the nonlinear quantum correction terms give zero after integration due to the odd nature of the integrand. Finally using

$$P\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} sech^2(\phi)\frac{\partial(sech^2(\phi'))}{\partial\phi'}d\phi\frac{d\phi'}{(\phi-\phi')} = (24/\pi^2)\zeta(3) = 2.92 \tag{7.52}$$

we get a first order differential equation in N, solving which we get

$$N = \frac{N(0)}{[1+(\frac{1}{2}\beta_1\alpha_2 QN(0)^{(\frac{1}{2})})\tau]^2}, \tag{7.53}$$

where $\beta_1 = (2.92)\beta\sqrt{\alpha_2/96\pi\alpha_3}$.

From eqn (7.53) we see that the decay law of amplitude depends on the quantum factor Q. A full numerical computation of (7.35) could reveal the total dynamical nature of the solution.

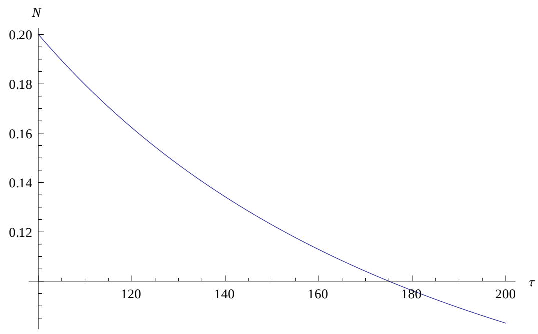


Figure 7.1: Decay of soliton amplitude with time when when $Q = 0.01, N(0) = 1, \alpha_3 = 1, \alpha_2 = 6$, and $\beta = 1$

7.4 Summary

In this work we have extended the methodology of the work of Ott and Sudan to include the semi-classical quantum effects to obtain a new evolution equation in the context of a nonlinear ion acoustic wave. This equation is of the form of a higher order KdV equation having higher order nonlinear terms as quantum corrections, together with a classical Landau damping term as well as quantum contribution coming from resonant particle effects.

Using the fluid equations for ions and the classical kinetic Vlasov equation for electrons, Ott and Sudan obtained a KdV equation with a Landau damping term as the evolution equation for the nonlinear ion acoustic wave. In order to introduce the quantum corrections, the classical Vlasov equation is replaced by an appropriate quantum analog i.e, the Wigner equation. In a similar approach using the Wigner equation in place of the Vlasov equation gives rise to our higher order KdV equation with Landau damping terms. The equation exactly converges to the equation done in [50] when $\hbar$ tends to zero i.e, in the classical limit. The mathematical nature of the equation shows that it conserves the total number of ions. The importance of the higher order KdV equation derived here, lies in the fact that its solution would give the quantum modification of the KdV solitary wave. But unfortunately, exact solitary wave solutions of this equation cannot be obtained. Since there are two small parameters in the equation, α_1 and Q, we treat the quantum corrections as well as the Landau damping terms as perturbation to the KdV equation. In order to carry out the Bogoliubov and Mitropolsky approximation technique, multiple time scales stretched by these small parameters have to be introduced. Such a technique is too complicated to comprehend analytically. Hence in order to get a useful analytical result , we have assumed $\alpha_1 \approx Q$. Hence, the quantum correction to Landau damping term turns out to be of the order of Q^2 and therefore neglected.

In the perturbative approach, the contribution to the decay rate coming from the nonlinear quantum correction terms turns out to be zero because of the odd nature of the integrand. The final contribution to the decay of solitary wave amplitude comes from the classical Landau terms, whose coefficient, due to the perturbation scheme, turns out to be of the order of Q. The amplitude is shown to decay inversely with the square of time depending on the quantum factor Q. In our final equation of decay rate no terms come from the quantum correction, i.e quantum nonlinear part goes to zero when the integration over ϕ is performed and the quantum Landau damping terms being of order Q^2

are neglected. This is due to our chosen scheme, and application of perturbation scheme with multiple time scales could give rise to solutions with more appropriate dependance on quantum effects. But the importance of the equation cannot be turned down and could be the initiator of numerical computation that would reveal the entire dynamical nature of the solution with the inclusion of quantum mechanical effect.

Chapter 8

Conclusion & Outlook

The thesis contains modelling of both deep and shallow water wave phenomena and applications of those models in other fluid systems.

In the first part of my thesis, we have studied the propagation of shallow water, unidirectional nonlinear wave with nontrivial bottom boundary conditions. Our aim was to study the effect of the controlled leakage at the bottom to the surface solitary wave. We have shown analytically that for the choice of leakage velocity functions which are independent of the free surface wave profile, the solitary wave solution gets modified in phase where as the amplitude remains constant. These studies are done in the shallow water of constant depth whereas the actual bathymetry varies with position. In order to explore the effect of bottom boundary condition on the surface solitary wave amplitude which was absent in the chapter 1, the leakage functions are assumed to depend on the free surface wave profile in chapter 2. First the constant depth problem was investigated to identify the profile of leakage function which would induce maximum damping effects on the solitary wave amplitude. Taking this profile, the variable depth problem was studied where a variable KdV equation was derived where the bathymetry function varies slowly. For a finer balance between the depth function and the leakage velocity function, exact solitary wave solutions have been found out which decay as it propagates towards the shore in spite of the surging effects due to decreasing bathymetry.

We emphasize that obtaining exact solution in variable bathymetry in presence of controlled leakage is a rare achievement. In place of approximate or numerical result obtained earlier, we found here the decay of solitary wave amplitude in exact analytic form. This is the strong point of our findings. Since our model gives a possible control mechanism of near shore surging waves along with the other methods like plantation of Mangrove treas along the coastal lines, installation of breakwaters at strategic positions , stoppage of erosion by concrete bolders etc, it demands application to a near sea shore bathymetry which has been done in Chennai, South India and the decay of amplitude due to leakage have been shown. These are the basic things we have done in the first part, which can be extended in future to involve realistic chaotic leakage and transverse perturbation. Such investigations would require numerical modelling of the phenomena.

In the second part of my thesis, the deep water oceanic rogue waves have been modelled. Rogue waves are reported to being observed in a relatively calm sea, where they, as a localized and isolated surface waves, apparently appear from nowhere, make a sudden hole in the sea just before attaining surprisingly high amplitude and disappear again without a trace. In this regard, we have introduced a new completely integrable, (2+1) dimensional, Nonlinear Schrodinger equation which is derivable from the basic hydrodynamic equations, to model oceanic rogue waves. Showing the discrepancies of the existing models, an exact rational two dimensional solution containing two free parameters of the new equation describes the full grown rogue wave. Its maximum amplitude, steepness, position of holes can be determined by those free tunable parameters. In order to explain its dynamical behavior an ocean current term has been introduced which will control the duration of the rogue wave. Modulation instability associated with the new evolution equation has been found out showing asymmetric nature and directional preference.

Since a completely integrable equation is very rich in terms of its mathematical behavior, its discovery in any physical system demands exploration of its various properties. We have derived its Lax pair structure, infinite conserved quantities, soliton solutions etc. Its dynamic rational solution, as the superposition of breather solutions could not be derived which could be explored in future. He et.al [172] presented the first order rational solution of our new equation by the one-fold Darboux transformation from a nonzero "seed" solution. They also have discussed localization of rogue waves in their paper which is related to the amplitude of the seed solution. We have also shown that our new 2 dimensional NLS equation is related to another completely integrable KP equation which shows the strong 2d nature and directional preference of our equation.

The newly discovered equation demands applications in various physical systems to explore various features. As an application, it has been derived in the context of propagation of nonlinear ion acoustic waves in magnetized lossless plasma containing cold ions and hot isothermal electrons. The discovery of this novel, (2+1) dimensional integrable NLS type equation should pave a new direction of research in the field.

As an application of the integrable models derived here we have explored other fluid systems like inhomogeneous plasma. The propagation of ion acoustic soliton in weak and slowly varying inhomogeneous plasma has been studied. It has been shown that the dynamics of the nonlinear ion acoustic wave is controlled by KP equation. The two dimensional soliton of the evolution equation gets bend in the two dimensional plane controlled by the unperturbed ion number density, whereas the amplitude remains constant.

A new field called quantum plasma has been explored using KdV model. A quantum corrections has been done in the semi-classical limit to the nonlinear ion acoustic wave with electron Landau damping. A new higher order KdV equation has been derived containing nonlinear quantum correction terms and the quantum correction to the Landau damping. Using Bogoliubov Mitropolsky approximation method the decay of amplitude due to the Landau damping term has been calculated.

Bibliography

[1] G.S Emmerson *John Scott Russell: A great Victorian engineer and naval archi- tect.* John Murray Publishers, London (1977) 19

[2] D.J. Korteweg and G. de Vries, *Philos. Mag.* **39**(1895) 422. 19, 20, 29

[3] F Fermi et al *Studies of nonlinear problems I. Los Alamos Sci Lab Rep LA- 1940 Reproduced in: AC Newell (ed) (1974) Nonlinear wave motion.* AMS, Providence, Rhode Island(1955) 19

[4] R. S Johnson, *A Modern Introduction to the Mathematical Theory of Water Waves*, Cambridge University Press, Cambridge, (1997) and *references therein.* 19, 21, 25, 56, 67, 71, 84, 86

[5] A Kundu *Tsunami and Nonlinear Waves* Springer-Verlag Berlin Heidelberg, (2007) 20, 28, 52, 67, 71

[6] A Rudloff et. al *Nat.Haz.Earth.Sys, Sci* **9**(2009) 1381 21, 28

[7] S. Nayak and T.S Kumar *Int. Archives.Photogrammetry* **XXXVII**(2008)B4

[8] J Lauterjung, U Munch & A. Rudloff *Nat.Haz.Earth.Sys. Sci* **10**(2010)641

[9] D.E Barrick *Remote Sensing of Environment* **8**(1979) 353

[10] G Blewitt et.al *J. Geod* **83** (2009) 335

[11] E. Marris *Nature* **433** (2005) 3 21, 28

[12] K. Katheresan & N. Rajendran *Estuarine, Coastal and Shelf Sciences***65** (2005) 601 21, 53

[13] N.K Liang, J.S Huang & C.F Li *Ocean Engineering***31**(2004) 43 21, 53

[14] W.Koo *Ocean Engineering* **36** (2009) 723

[15] L. Martinelli, P. Ruol & B. Zanuttigh *Applied Ocean Research* **30** (2008)199

[16] C. Michailides & D. C Angelides *Applied Ocean Research* **35** (2012) 77 21, 53

[17] G. Taylor *Proc. R. Soc. A* **231** (1955) 466; G.I. Taylor, *Admiralty Scientific research Dept. ATR/Misc/1259* 1943. 21, 53

[18] D. L. Hammond *United States Patent* (1974) 3,785,159 21, 53

[19] A. M Fridman et.al *Physics-Uspekhi* **53(8)** (2010) 809 21, 53

[20] A.Davey & K. Stewartson *Proc.R.Soc.Lond.A* **338** (1974) 101. 22, 95

[21] C. Kharif & E. Pelinovsky *Eur. J. Mech.*B -Fluid **22** (2003) 603; K. Dysthe, H. E. Krofstat & P. Muller *Annu. Rev. Fluid Mech.* **40** (2008) 287; C. Kharif C, E. Pelinovsky & A. Slunyaev *Rogue waves in Ocean* (Springer-Verlag, Berlin Heidelberg, 2009) (with refrences therein); A. R. Osborne *Nonlinear Ocean waves and the inverse scattering transform*, Int. Geophys. Ser. **97** (Acad. Press,2010). 22, 23, 88, 89, 91, 103

[22] E. Pelinovsky, T. Talipova & C. Kharif *Physica D* **147** (2000) 83. 23, 89

[23] N. N. Akhmediev, V. M. Eleonskii & N. E. Kulagin first-order solutions of the non-linear Schrodinger equation. *Theor. Math. Phys.* **72** (1987) 809. 23, 90, 93

[24] N. Akhmediev, A. Ankiewicz & J. M. Soto-Crespo *Phys. Rev.* **E 80**(2009) 026601. 23, 90, 92, 93, 102

[25] A. Ankiewicz, D.J. Kedziora & N. Akhmediev *Phys. Lett. A* **375** (2011) 2782. 23, 90, 93, 102

[26] G. Mu, Z.Dai & Z. Zhao *Pramana J. Phys.* **81** (2013) 367 . 24, 95

[27] R Hirota *Phys.Rev. Lett* **27** (1971) 1192 25, 134

[28] E. N Dolgopolova *Water Resources***40** (2013)16 28

[29] H. C. Miahr *J.Hydraulic. Res***43**(2005) 234 28

[30] S. Furuyama & H Chanson *Coastal Eng. J***52** 215 28

[31] L.T Dauer et.al *J. Nuclear Medicine***52**(2011)9 29

[32] T Takemura wt.al *SOLA* **7**(2011)101 29

[33] G.I. Burde *Phys. Rev. E* **84** (2011) 026615 . 29

[34] E. Infeld, A. Senatorski, & A. A. Skorupski *Phys. Rev. E* **51** (1995) 3183 . 29

[35] M.J. Ablowitz & D.E. Baldwin *Phys. Rev. E* **86** (2012) 036305 . 29

[36] T.Y. Wu *J. Fluid Mech.* **184** (1987) 75 .

[37] W.K. Melville & K.R. Helfrich *J. Fluid Mech.* **178**(1987) 31.

[38] A. Orlowski *Phys. Rev. E* **49** (1994) 2465 . 40

[39] M. Wadati, *Phys. Soc. Jpn.* **52** (1983) 2642. 40

[40] N. Zahibo, E. Pelinovsky & A. Sergeeva, *Chaos, Solitons and Fractals* **39** (2009) 1645 . 40

[41] C.C. Mei *The applied dynamics of ocean surface waves* (World Scientific, Singapore, 1989). 30, 56

[42] R. O Reid & K. Kajiura *Trans. Am. Geophys. Union* **38** (1957) 662 30, 54

[43] J. M Hunt *J. Geophys. Res.* **64** (1959) 437 30

[44] P.L. Liu & I.C. Chan *J. Fluid. Mech.* **579** (2007) 467. 30

[45] P.L. Liu & J. Wen *J. Fluid. Mech.* **347** (1997) 119. 30, 54

[46] M. Lakshmanan & S. Rajasekar, *Nonlinear dynamics: Integrability, Chaos and Patterns*, (Springer, Berlin). 2003. 30

[47] R. Hirota, *Direct Method in Soliton Theory* Springer, Berlin (1980). 48

[48] Z. Jun-Xiao & G. Bo-Ling *Commun. Theor. Phys. (Beijing, China)* **52** (2009) 279.

[49] M. J. Ablowitz & D. E. Baldwin, *Phys. Rev.E* **86** (2012) 036305 .

[50] E. Ott & R.N. Sudan *Phys. Fluids* **12** (1969) 11 . 60, 166, 169, 170, 173, 174, 175, 179

[51] E. Ott & R.N. Sudan, *Phys. Fluids* **13** (1970) 6. 54, 60, 61, 62, 64

[52] L.A. Ostrovsky, *Int. J. Nonlin. Mechanics* **11** (1976) 401. 54, 61, 64, 67, 71

[53] O. Nakoulima, N. Zahibo, E. Pelinovsky, T. Talipova & A. Kurkin, *Chaos* **15** (2005) 037107. 54, 67, 71

[54] A. Mukherjee & M.S. Janaki, *Phys. Rev. E* **89** (2014) 062903 . 79, 81

[55] N.N. Bogoliubov and Y.A. Mitropolsky, *Asymptotic Methods in the Theory of Nonlinear Oscillations* (Gordon and Breach Science Publishers, Inc, New York, 1961). 60, 61

[56] Zhi-Yuan Sun, Yi-Tian Gao, Ying Liu & Xin Yu, *Phys. Rev. E* **84** (2011) 026606 . 70

[57] Xin Yu, Yi-Tian Gao, Zhi-Yuan Sun & Ying Liu, *Phys. Rev. E* **83** (2011) 056601 . 70, 72

[58] A.A Kumar & P.D Kunte *Nat Hazards* **64** (2012) 853 . 11, 54, 74, 75, 76

[59] A.G. Johnpillai, C.M. Khalique & A. Biswas, *Appl. Math. Comp.* **216** (2010) 3114 . 84

[60] http://news.bbc.co.uk/2/hi/8548547.stm 88

[61] C. Bonatto et al. *Phys. Rev. Lett.* **107** (2011) 053901. 88, 89, 94

[62] M. Onorato, D. Proment & A. Toffoli *Phys. Rev. Lett.* **107** (2011) 184502 (and references therein). 88, 90, 103, 107, 108, 110, 120

[63] V. E. Zakharov, A. I. Dyachenkov & R. V. Shamin *Eur.Phys. J. Special Topics* **185** (2010) 113124 88, 89, 95, 103, 110

[64] D. R. Solli, C. Ropers, P. Koonath & B. Jalali *Nature* **450** (2007) 06402 88, 89, 94

[65] A. N. Ganshin et al *Phys. Rev. Lett.* **101** (2008) 065303 88, 89

[66] M. Onorato et al *Phys. Rev. Lett.* **102** (2009) 114502 88, 90, 94, 96, 97, 99, 100, 110, 120, 121

[67] A. Montina, et al *Phys. Rev. Lett.* **103** (2009) 173901. URL http://link.aps.org/doi/10.1103/PhysRevLett.103.173901. 88, 89, 90, 96, 97, 100, 112, 120, 121

[68] Kibler B, et al. *Nature Physics* **6** (2010) 790. 88, 89, 90, 91, 94

[69] M. Shats, H. Punzmann, & H. Xia *Phys. Rev. Lett.* **104** (2010) 104503. http://link.aps.org/doi/10.1103/PhysRevLett.104.104503. 88, 89, 94, 121

[70] F.T Arecchi, U. Bortolozzo, A. Montina & S. Residori *Phys. Rev. Lett.* **106** (2011) 153901. 89

[71] K. Hammani, B. Kibler, C. Finot & A. Picozzi *Phys. Lett. A* **374** (2010) 3585. 89

[72] A. Chabchoub, N. P. Hoffmann & N. Akhmediev *Phys. Rev. Lett.* **106** (2011) 204502. URL http://link.aps.org/doi/10.1103/PhysRevLett.106.204502. 88, 90, 91, 94

[73] A. N. Pisarchik, et al R. Jaimes-Reategui R, Sevilla-Escoboza R, Huerta-Cuellar R & Taki M *Phys. Rev. Lett.* **107** (2011) 274101. http://link.aps.org/doi/10.1103/PhysRevLett.107.274101. 88, 89, 94, 110

[74] B. Eliasson, & P. K. Shukla *Phys. Rev. Lett.* **105** (2010) 014501. URL http://link.aps.org/doi/10.1103/PhysRevLett.105.014501. 88, 90, 94, 95, 96, 97, 99, 100, 120

[75] A. R. Osborne, M. Onorato & M. Serio *Phys. Lett.* A **275** (2000) 386. URL-http://www.sciencedirect.com/science/article/pii/S0375960100057514. 88, 90, 91, 94, 95, 96, 97, 98, 99, 100, 103, 110, 120

[76] M. J. Ablowitz, J. Hammack, D. Henderson & C. M. Schober *Phys. Rev. Lett.* **84** (2000) 887. URLhttp://link.aps.org/doi/10.1103/PhysRevLett.84.887. 88, 90

[77] N. Akhmediev & E. Pelinovsky *Eur.Phys.J.Special Topics* **185** (2010) 1. 89, 94

[78] V. P. Ruban *Phys. Rev. Lett.* **99** (2007) 044502. 89, 107, 108

[79] D. H. Peregrine *Austral. Math. Soc.* B 25 (1983) 16. 90

[80] M. Lakshmanan in *Tsunami and Nonlinear Waves* (Ed. A. Kundu, Springer Verlag, NY, 2006) p. 31 90

[81] G. P. Agarwal *Nonlinear fiber optics* (Acad. Press, London, 2007), ch. 5. 99, 100

[82] P. Dubard & V. B. Matveev *Nat. Hazards Earth Syst. Sci.* **11**(2011) 667. 93, 102

[83] http://wikipedia.org/wiki/$Rogue_Wave$ 94, 103, 110, 120

[84] A. I. Dyachenko & V. E. Zakharov *JETP Lett.* **81** (2005) 6. 89, 95, 103, 107, 110, 120

[85] R. Smith *J. Fluid Mech.* **77** (1976) 417. 89

[86] A. Ankiewicz, J. M. Soto-Crespo & N. Akhmediev *Phys. Rev. E* **81** (2010) 046602. 93

[87] S. Xu, J. He & L. Wang *J. Phys. A: Math.Teor* **44** (2011) 305203. 93

[88] M. Ablowitz & H. Segur *Solitons and the Inverse Scattering Transform*, (SIAM, Philadelphia, 1981). 90

[89] A. Calini & C. M. Schober *Phys. Lett. A* **298** (2002) 335349. 90

[90] Y. C. Ma *Stud. Appl. Math.* **60** (1979) 43. 90, 93

[91] http://news.bbc.co.uk/2/hi/3917539.stm 94

[92] http://earthobservatory.nasa.gov/IOTD/view.php?id=44567 94

[93] Akhmediev N, Korneev V I 1986 *Theor. Math. Phys.* **69**, 1089

[94] K. B. Dysthe & K. Trulsen *Phys. Scr.* **82** (1999) 48. 90, 93, 95

[95] Y. Ohta & J. Yang *J.Phys. A:Math. Theor* **46** (2013) 105202.

[96] Y. Ohta & J. Yang *Phys. Rev. E* **86** (2012) 036604. 95

[97] V. E. Zakharov & E.A Kuznetsov *Sov.Phys* **39**(1974) 285. 95, 126

[98] M. Onorato, A.R Osborne & M. Serio *Phys. Fluids* **14**(2002) L25. 97

[99] M. Onorato et al. *J. Fluid. Mech* **627**(2009) 235. 97, 99

[100] O. Gramstad & K. Trulsen *J. Fluid. Mech* **582**(2007) 463. 97, 99

[101] K. Trulsen & I. Kliakhandler *Phys. Fluids***12**(2000)2432 100

[102] N. Akhmediev, J. M Soto-Crespo & A. Ankiewicz *Phys. Rev. A* **80**(2009)043818. 90

[103] S. White & B. Fornberg *J. Fluid. Mech* **355** (1998) 113. 107

[104] A. Kundu, arXiv: 1201.0627 [nlin.SI, (2012) A. Kundu & A. Mukherjee, arXiv:1305.4023 [nlin.SI] (2013). 97

[105] R. S. Johnson *A Modern Introduction to the Mathematical Theory of Water Waves*, (Cambridge Texts in Appl. Math., 2004). 112

[106] Z. Parsa *Am. J. Phys.* **47** (1979) 56. 106

[107] J.W McLean *J.Fluid Mech* **114** (1982) 331. 97

[108] C. Kharif & A. Ramamonjiarisoa *Phys.Fluids* **31** (1988) 1286. 97

[109] T.B Benjamin & J.E. Feir *J.Fluid Mech* **27** (1967) 417. 97

[110] M.J Lighthill *J. Inst. Math. Appl***1**(1965) 269. 97

[111] V.E Zakharov *Sov. Phys. JETP***24**(1967) 455. 97

[112] J.E Feir *Proc. Royal. Soc. A***54**(1967) 299. 97

[113] P.K Shukla et al *Phys. Rev. Lett* **97**(2006)094501 99

[114] K.L Henderson, D.H Peregrine & J.W Dold *Wave Motion***29**(1999) 341 100

[115] K.B.Dysthe *Proc.R.Soc.Lond.A* **369** (1979) 105.

[116] Y. Ohta & J. Yang *J.Phys A:Math. Theor.* **46** (2013) 105202. 95

[117] M. Boiti, J. Leon & F. Pempinelli *Inv. Prob.* **3** (1987) 37. 95

[118] B. B Kadomtsev & V.I Petviashvili *Sov.Phys.* **15** (1970) 539. 95, 149

[119] E. Infeld, A.Senatorski & A.A. Skorupski *Phys. Rev. E.* **51** (1995) 4. 95, 126

[120] K Trulsen & K.B.Dysthe *Wave Motion* **24** (1996) 381. 95

[121] V. E. Zakharov, in *Solitons* (Ed. R. K. Bullough and P. J. Caudrey, Springer, Berlin, 1980) 95

[122] V. E. Zakharov *JETP Lett.* **35** (1972) 908. 95

[123] M. J. Ablowitz *Nonlinear Dispersive waves,Asymptotic Analysis and Solitons*, (Cambridge University Press. 2011). 112

[124] F.F Chen *Introduction to Plasma physics and controlled fusion* , (Plenum Press, New York and London, 1984). 125, 148

[125] M. Lakshmanan & S. Rajasekar *Nonlinear dynamics: Integrability, Chaos and Patterns*, (Springer, Berlin, 2003). 125, 134

[126] H. Washimi & T. Taniuti *Phys. Rev. Lett* **17** (1966) 19. 125

[127] M. Q. Tran *Phys. Scr.* **20** (1979) 317 125

[128] H. Ikezi, R Taylor & D. Bekar *Phys. Rev. Lett* **25** (1970) 11. 125

[129] S.E Cousens et. al *Phys. Rev. E* **86** (2012) 066404 . 125

[130] A.A Mamun, *Phys. Rev. E* **77** (2008) 026406. 125

[131] M. Bacha, M.Tribeche & P.K.Shukla *Phys. Rev. E* **85** (2012) 056413 . 125

[132] S. Ghosh & N. Chakrabarti *Phys. Rev. E* **84** (2011) 046601 . 125

[133] H. K. Malik, *Phys. Rev. E* **54** (1996) 5. 125

[134] F. Verheest, M.A Hellberg & W.A Hereman, *Phys. Rev. E* **86** (2012) 036402 . 125

[135] P.K Shukla, A.A Mamun & D.A Mendis, *Phys. Rev. E* **84** (2011) 026605 . 125

[136] G. Huang & M.G. Velarde *Phys. Rev. E* **53** (1996) 3. 125

[137] S. Maxon & J. Vicelli, *Phys. Fluids* **17** (1974) 1614. 125

[138] L.F.J Broer & F.W. Sluijter *Phys. Fluids* **20** (1977) 1458 . 125

[139] M.R Amin, G.E Morfill & P.K Shukla, *Phys. Rev. E* **58** (1998) 5. 125

[140] W.M Moslem, R. Sabry, S.K El-Labany & P.K Shukla *Phys. Rev. E* bf 84 (2011) 066402 . 125

[141] R Sabry, W.M Moslem, P.K Shukla & H. Saleem , *Phys. Rev. E* **79** (2009) 056402 . 125

[142] W.M Moslem, *Phys. Plasmas* **18** (2011) 032301. 125

[143] R. Sabry, W.M Moslem & P.K Shukla *Phys. Rev. E* **86** (2012) 036408 . 125, 126

[144] N.S Javan & F. Adili, *Phys. Rev. E* **88** (2013) 043102 . 125

[145] H. Bailung, S.K Sharma & Y. Nakamura *Phys. Rev. Lett* **107** (2011) 255005 . 126

[146] J.R Franz, P.M Kintner & J.S Pickett *Geophys. Res. Lett* **25** (1998) 2041. 126, 138

[147] H.K. Malik & R.P. Dahiya, *Phys. Lett. A* **195** (1994) 369 . 126, 134, 149

[148] A. Mushtaq & S.A. Khan *Phys. Plasma* **14** (2007) 052307 . 126, 134, 141

[149] M. Shahmansouri & E. Astaraki *J Theor. Appl. Phys* **8** (2014) 189 . 126, 141

[150] M. Kako & G. Rowlands, *Plasma. Phys.* **18** (1976) 165 . 126, 134

[151] V.E Zakharov & E.A Kuznetsov *Zh. Eksp. Teor. Fiz* **66** (1974) 594 .

[152] V.E Zakharov & E.A Kuznetsov *Sov. Phys.* **39** (1974) 285 . 126

[153] B.K Shivamoggi, *Phys. Scripta* **42** (1990) 641 . 126

[154] E Infeld & P Fryczs , *J. Plasma. Phys.* **37** (1987) 97 . 126

[155] E Infeld *J. Plasma Phys.* **33** (1985) 171 . 126

[156] B.K Shivamoggi , *J. Plasma Phys.* **41** (1989) 83 . 126

[157] A.S Bains, M Tribeche, N.S Saini & T.S Gill *Phys. Plasmas* **18** (2011) 104503. 126

[158] S.K El-Labany, W.F.El- Tribany & O.M El-Abbasy , *Phys. Plasmas* **12** (2005) 092304 . 126

[159] N.S Saini, B.S Chahal, A.S Bains & C.Bedi *Phys. Plasmas* **21** (2014) 022114 . 126

[160] H.L Zhen, B Tian, Y.F.Wang, H.Zhong & W.R Sun *Phys. Plasmas* **21** (2014) 012304 . 126

[161] A.M Wazwaz *Comm. Non. Sc. Num. Sim* **10** (2005) 597. 126

[162] R.L Mace & M. A Hellberg *Phys. Plasmas* **8** (2001) 2649 . 126

[163] K. Nishinari, K. Abe & J. Satsuma *Phys. Plasmas* **1** (1994) 2559 . 126, 132, 133

[164] S.S Ghosh, A Sen & G.S Lakhina *Non. Processes. Geophys* **9** (2002) 463 . 126, 139

[165] K. Annou & R. Annou *Phys. Plasmas* **19** (2012) 043705 . 126, 133

[166] J.K Xue *Phys. Plasmas* **12** (2005) 092107 . 126

[167] K Nishinari & T. Yajima, *Phys. Rev. E* **51** (1995) 5 . 126

[168] M Boiti, J.J.P Leon, L Martina & F. Pempinelli *Phys. Lett. A* **132** (1988). 126

[169] A Kundu, A Mukherjee & T Naskar *Proc. R. Soc. A* **470** (2014) 20130576 . 127, 132

[170] M.S Janaki, B.K Som, B. Dasgupta & M.R. Gupta *J.Phys.Soc.Jpn* **60** (1991) 2977 138

[171] A Kundu & A Mukherjee arxive:1305.4023

[172] D. Qiu, Y. Zhang & J. He *Commun Nonlinear Sci Numer Simulat* **30** (2016) 307 140, 183

[173] F. Aziz *Thesis: Ion-acoustic solitons: Analytical, experimental and numerical studies*, (2011) and references therein 148

[174] N. Nishikawa & and P. K.Kaw *Phys. Lett A* **50** (1975) 455. 148

[175] Y. Gell & L. Gomberoff *Phys. Lett. A* **60** (1977) 125. 148, 151

[176] P.I John & Y.C Saxena *Phys. Lett. A* **56** (1976) 385. 148, 152, 153

[177] N. N Rao & R. K Verma *Phys. Lett. A* **70** (1979) 9. 148

[178] H.H Kuehl & K Imen, *Phys. Fluids.* **28** (1985) 2375. 148

[179] H.Y Chang, S. Raychaudhuri, J.Hill, E.K Tsikis & K.E Lonngren *Phys. Fluids.* **29** (1986) 294. 13, 148, 151, 152

[180] R. P Dahiya, P.I John & Y. C Saxena *Phys. Lett. A* **65** (1978) 323. 148, 152, 153

[181] G. Popa & M.Oertl *Phys. Lett. A* **98** (1983) 110. 148

[182] Y. Nishida *Phys. Fluids* **27** (1984) 2176. 148

[183] K.Imen & H.H Kuehl *Phys. Fluids* **30** (1987) 73. 148

[184] T. Nagasawa & Y. Nishida *Phys. Rev. Lett* **56** (1986) 2688. 148

[185] S. Yi, J.L Cooney, H.Kim, A. Amin, Y. El-Zein & K.E Lonngren *Phys. Plasmas* **3** (1996) 529. 148

[186] J.L Cooney, M.T Gavin, J.E Williams, D. W Aossey & K.E Lonngren *Phys. Fluids B* **3** (1991) 3277. 148, 149

[187] H.H Kuehl *Phys. Fluids* **26** (1983) 1577. 148

[188] K.Ko & H.H Kuehl *Phys. Rev. Lett* **40** (1978) 233. 148

[189] K. Ko & H.H Kuehl *Phys. Fluids* **23** (1980) 834. 148

[190] S.Singh & R.P Dahiya *Phys. Fluids B* **3** (1991) 255. 148

[191] S.Singh & R.P Dahiya *J.Plasma Phys* **41** (1989) 185. 148

[192] H.K Malik & R.P Dahiya *Phys.Plasma* **1** (1994) 2872. 148

[193] D.K Singh & H.K Malik *Phys.Plasma* **13** (2006) 082104. 148

[194] S.S Chauhan,H.K. Malik & R.P. Dahiya, *Phys. Plasma* **3** (1996) 3932. 148

[195] D.K Singh & H.K Malik *Phys. Plasma* **14** (2007) 062113. 148

[196] D.Xiao, J.X Ma, Y.Li, Y.Xia & M.Y Yu *Phys. Plasma 13* (2006) 052308 148

[197] R.Kumar, H.K Malik & S.Kawata *Physica. D* **240** (2011) 310 148

[198] H.K Malik *Phys. Lett. A* **365** (2007) 224. 148

[199] E Infeld Q.Haque & S.Mahmood *Phys. Plasma* **15** (2008) 034501 148

[200] H.K Malik *Phys. Plasma* **15** (2008) 072105 148

[201] Jyoti & H.K Malik *Phys. Plasmas* **18** (2011) 102116. 148

[202] H.K Malik, Jyoti & R.Kumar *J.Theor. Appl Phys* **8** (2014) 123. 148

[203] K.Singh, V.Kumar & H.K Malik *Phys. Plasmas* **12** (2005) 072302. 149

[204] W.Duan & J.Zhao *Phys. Plasmas* **6** (1999) 3484. 149

[205] I. Ibrahim & H.H Kuehl *Phys.Fluids* **27** (1984) 962 149

[206] W.F El-Taibany,M.M Selim, N.A El-Bedwehy & O.M Al-Abbasy *Phys. Plasmas* **21** (2014) 073710. 149

[207] A.P Misra & A.R Chowdhury *Phys. Plasmas* **13** (2006) 062307. 149

[208] W.Masood *Phys. Plasmas* **17** (2010) 052312. 149

[209] W.F.El-Taibany, M.Wadati & R. Sabry *Phys. Plasmas* **14** (2007) 032304. 149

[210] H.K Malik, S.Singh & R.P dahiya *Phys. Lett. A* **195** (1994) 369. 155

[211] W. Masood *Phys. Lett.A* **373** (2009) 1455. 149

[212] W.Masood, S.Karim, H.A Shah & M.Siddiq *Non. Phys.plasma* **16** (2009) 042108. 149

[213] W.Masood, S.Karim & H.A Shah *Phys. Scr.* **82** (2010) 045503. 149

[214] J.R Yang, X.Y Tang, X.N Gao, X.P Cheng & S.Y. Lou *EPL* **12** (2011) 45001. 149

[215] L.P. Zhang & J.K. Xue *Commun Nonlinear Sci Numer Simulat* **15** (2010) 3379. 149

[216] P. Frycz & E.Infeld *Phys. Rev. A* **41** (1990) 3375. 161

[217] A.M Wazwaz *Appl.Math.Comput* **190** (2007) 633. 156, 157

[218] M.J ablowitz & D.E Baldwin *Phys.Rev.E* **86** (2012) 036305. 156, 157

[219] Y. D. Jung *Phys. Plasma* **8** (2001) 3842 164

[220] D. Kremp, Th. Bornath, M. Bonitz, & M. Schlanges *Phys. Rev. E* **60** (1999) 4725 164

[221] N. C. Kluksdahl, A. M. Kriman, D. K. Ferry, & C. Ringhofer *Phys. Rev. B* **39** (1989) 7720 164

[222] A. A. G. Driskill- Smith, D.G.Hasko & H.Ahmed *Appl. Phys. Lett* **75** (1999) 2845 164

[223] F. Haas, Quantum Plasmas- An hydrodynamic approach, Springer, New York (2011) 165

[224] G. Manfredi & F. Haas *Phys. Rev. B* **64** (2001) 075316. 165

[225] M. Bonitz *AIP Conf. Proc.* **1421** (2012) 135 165

[226] F. Haas, B. Eliasson, P. K. Shukla & G. Manfredi, *Phys. Rev. E* **78** (2008) 056407 . 165

[227] F. Haas, L.G Garcia, J. Goedert & G. Manfredi *Phys.Plasmas***10** (2003) 3858 165

[228] C.L Gardner *SIAM. J.Appl. Math* **54(2)** (1994) 409. 166

[229] L.D Landau *J.Phys (Moscow)* **10** (1946) 25 165

[230] J.H Malmberg & C.B Wharton *Phys.Rev.Lett* **13** (1964) 184502 165

[231] J.V Hollweg *Phys.Rev.Lett* **27** (1971) 1349 165

[232] H. Lange, B. Toomire & P.F. Zweifel *Trans. theor. Stat .Phys.* **25(6)** (1996) 713 166

[233] T. Taniuti *J. Phys. Soc. Japan* **33** (1972) 277 165

[234] J.W Vandam & T. Taniuti *J. Phys. Soc. Japan* **35** (1973) 897 165

[235] Y. Saitou & Y. Nakamura *Phys. Plasmas* **10** (2003) 4265 165

[236] M.S Janaki, B. Dasgupta, M.R Gupta & B.K Som *Physica Scripta* **45** (1992) 368 165

[237] A.Y Wong, R.W Motley & N. D'Angelo *Phys.Rev* **133(2A)** (1964) A436 165

[238] S.N Paul, B.Paul, C.Das, S.K. Bhattacharya & B.Ghosh *Bulg.J.Phys* **37** (2010) 16 165

[239] J. Zhu, P.Ji & N.Lu *Phys.Plasma* **16** (2009) 032105 173

[240] A.Barman & A.P Misra *Phys. Plasma* (2014) 073708 165

[241] A.R. Chowdhuri, G.Pakira, S.N Paul *Physica C* **151** (1988) 518 165

[242] A. Bandyopadhyay & K.P. Das *Phys. Plasma* **9** (2002) 2. 170

[243] A. Luque, H. Schamel & R. Fedele *Phys. Lett. A* **324** (2004) 185 175

[244] D. Jovanovic & R. Fedele *Phys. Lett. A* **364** (2007) 304 175

[245] M. Salimullah, M. Jamil, I. Zeba, Ch. Uzma, & H. A. Shah, *Phys. Plasma.* **16**(2009) 034593

[246] H. Ren, Z. Wu, J. Cao & P. K. Chu *J. Phys. A* **41** (2008) 115501.