# Exact solutions of some quantum integrable systems associated with polarized spin reversal operators

*By*

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# DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Chitralekha Datta

## List of Publications arising from the thesis

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2. "Supersymmetric analogue of $BC_N$ type rational integrable models with polarized spin reversal operators, "

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3. " Super Rogers-Szegö polynomials associated with $BC_N$ type of Polychronakos spin chains"

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# Chapters in books and lectures notes

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## Conferences

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## Others

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Chitralekha Datta

# DEDICATIONS

To , ma and baba
who value honesty above all.

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# Contents

# Synopsis

Remarkale progress has been made in recent years in the computation of exact spectra, partition functions and correlation functions of one-dimentional quantum integrable spin chains and dynamical models as well as their supersymmetric generalizations. These quantum integrable systems are closely related to different areas of physics and mathematics such as quantum optics, condensed matter systems exhibiting generalized exclusion statistics, random matrix theory, quantum Hall effect, Dunkl operators, $N = 4$ super Yang-Mills theory and Yangian quantum groups. Since these models are exactly solvable, many statistical properties related to the spectra of these models can be studied from their exact solutions.

Broadly speaking, there exist two different types of quantum integrable models—quantum integrable models with short-range interaction where the particles or spins interact locally within a short distance and quantum integrable models with long-range interaction where all particles or spins interact with each other. One dimentional systems like $\delta$-function bose gas, Hubbard model, isotropic and anisotropic versions of Heisenberg spin-$\frac{1}{2}$ chain are some examples of quantum integrable models with short-range interactions [1]. On the other hand, one dimentional systems like Calogero model, Sutherland model, Haldane-Shastry (HS), Polychronakos spin chain are the examples of quantum integrable models with long-ranged interaction.

The study of quantum integrable and exactly solvable models with long-range interactions was pioneered by Calogero [2]. He introduced the exactly solvable Calogero model in which particles move on a line and interact with each other through a two-body potential proportional to the inverse square of their relative distances. Sutherland investigated a similar system in which particles move on a circle, instead of moving on a line and interact with each other through a two-body potential proportional to the inverse square of their chord distances [3]. Apart from these dynamical models, Haldane and Shastry have derived the exact spectrum of

a spin-$\frac{1}{2}$ chain with lattice sites equally spaced on a circle and spins interacting through pairwise interactions inversely proportional to the square of their chord distances [4]. A close relation between the $su(m)$ generalization of this HS spin chain and the $su(m)$ spin Sutherland model has been established by using the 'freezing trick', which we briefly describe in the following. In contrast to the case of HS spin chain where lattice sites are fixed at equidistant positions on a circle, the particles of the spin Sutherland model can move on a circle and they contain both coordinate as well as spin degrees of freedom. However, in the strong coupling limit, the dynamics of the decoupled spin degrees of freedom of the $su(m)$ spin Sutherland model is governed by the Hamiltonian of the $su(m)$ HS model. Application of the freezing trick to the spin (rational) Calogero model leads to the quantum integrable Polychronakos spin chain, which is also known in the literature as the Polychronakos-Frahm (PF) spin chain. The site of such rational PF spin chain are inhomogeneously spaced on a line and, in fact, they coincide with the zeros of the Hermite polynomial [5].

My research topic is mainly focussed on study of $BC_N$ and $D_N$ types of rational quantum integrable models associated with polarized spin reversal operators (PSRO). In this context one may note that, it is often possible to construct many variants of a quantum integrable system by using different classical root systems such as $A_{N-1}$, $B_N$, $C_N$, $BC_N$ and $D_N$. These different root systems are related to the generators of the corresponding simple Lie algebras. One of the fundamental features of the $BC_N$ root system and its $B_N$ and $C_N$ degenerations is the fact that the corresponding Weyl algebras contain a family of reflection operators $S_i (i = 1, \cdots, N)$ satisfying $S_i^2 = 1$. (In the case of $D_N$ root system, the Weyl group only contains products $S_i S_j$ with $i \neq j$.) In the spin chains studied earlier by some authors [6, 7, 8, 9, 10], the operators $S_i$ are represented by spin reversal operators $P_i$ (acting on the Hilbert space of the $i$-th spin), However, such a representation of $S_i$ is by no

means the only possible choice. In the spin Calogero model of $BC_N$ type and its corresponding PF chain studied in Ref.[11], the operators $S_i$ are represented instead by arbitrarily polarized spin reversal operators (PSRO) $P_i^{(m_1,m_2)}$, which act as the identity on the first $m_1$ elements of the spin basis and act as minus the identity on the rest. These operators are equivalent under a similarity transformation to the usual spin reversal $P_i$ only when $m_1 = m_2$ or $m_1 = m_2 \pm 1$, i.e., when there is minimal polarization.

In our first paper, we have studied the spin Calogero model of $D_N$ type with polarized spin reversal operators and its associated spin chain namely PF spin chain of $D_N$ type with PSRO. We have solved the former model and from the spectrum of this model we are able to compute its partition function in closed form, which yields the partition function of the related spin chain via Polychronakos's freezing trick. More preciously, we have shown that the latter partition function can be expressed in terms of the partition function of the PF chain of the type-$A$. Since the type-$A$ partition function can be efficiently evaluated by using a simple recursion formula that we have also derived in this paper, we are able to explicitly compute the exact spectrum of the $D_N$ -type chain for relatively high number of lattice sites of the system. We have studied several global properties of the spectrum of this chain. We have provided strong numerical evidence showing that its energy levels are a sequence of consecutive integers and its level density becomes normally distributed when the number of spins tends to infinity. We have also determined the number of distinct energy levels of the spin chain, showing that it is a second-degree polynomial in $N$, as is the case with PF chain of $A_{N-1}$ type. For the spin chains of HS type related to the $A_{N-1}$ root system, it is known that the polynomial growth of the number of distinct levels is a consequence of the fact that these models are equivalent to a Yangian-invariant vertex model with linear energy function and

polynomial dispersion relation. Our results strongly suggest that this is also the case for the present model, a conjecture which certainly deserves further study.

It is well known that the $SU(m|n)$ supersymmetric extension of the $A_{N-1}$ type of HS and PF models are also possible where each site is occupied by either one of the $m$ type of bosonic states or one of the $n$ type of fermionic states [12, 13, 14]. In our second paper, we have derived the exact spectra as well as partition functions for a class of $BC_N$ type of spin Calogero models, whose Hamiltonians are constructed by using supersymmetric analogues of polarized spin reversal operators (SAPSRO). The strong coupling limit of these spin Calogero models yields $BC_N$ type of PF spin chains with SAPSRO. We have obtained an exact expression for the partition functions of such PF spin chains by applying the freezing trick. We have also derived a formula, which expresses such a partition function in terms of known partition functions of several $A_K$ type of supersymmetric PF spin chains, where $K \leq N-1$. By using this formula, we have analyzed statistical properties like level density distribution in the spectra of spin chains with sufficiently large number of lattice sites. It turns out that, in analogy with the case of many other integrable systems with long-range interactions, the level density of PF spin chains with SAPSRO follows the gaussian distribution and the cumulative nearest neighbour spacings distribution obeys the so called 'square root of a logarithm' law. In this paper, we have also shown that the partition functions of PF spin chains with SAPSRO obey an interesting type of duality relation. To this end, we have considered a new quantum number which measures the parity of the spin states under the action of SAPSRO. It has been found that the partition functions of these spin chains satisfy an 'extended' boson-fermion duality relation, which involvs not only the exchange of bosonic and fermionic degrees of freedom, but also the exchange of positive and negative parity degrees of freedom associated with SAPSRO. As an application of

this duality relation, we have computed the highest energy levels of these spin chains from their ground state energies. We have found that partition functions of a large class of integrable and nonintegrable spin chains with Hamiltonians of a similar form satisfy this type of extended boson-fermion duality relation.

We have also computed the partition functions of the $BC_N$ type of spin Calogero model with SAPSRO through the grand-cannonical approach. Finally, by using the freezing trick, we have obtained an alternative formula for the cannonical partition function of the PF chain with SAPSRO and have shown that such partition functions obey a set of recursion relations. These recursion relations can be used to compute the exact spectra of PF spin chains with SAPSRO in an efficient way.

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# CHAPTER 1

# Introduction

In the last few decades, significant progress has been made in understanding the behaviour of quantum integrable many-body systems and their supersymmetric generalisations. Initial research interest in this field started with the discovery of exact solution of the one-dimensional isotropic Heisenberg spin chain by Bethe [1]. The two-dimensional Ising model and related vertex models [2,3], the Lieb-Liniger model with zero-range contact interactions in free space [4], the Calogero model with inverse square interactions [5] are some other prominent examples of exactly solvable classical statistical models and quantum integrable systems. Usually the integrability of a quantum system is found in one spatial dimension. However, such one dimensional systems might be useful to approximately describe various features of some quantum systems in higher dimension that are relevant for physical applications. Exact solutions of those quantum systems in higher dimension can not be computed analytically in most cases. Hence, numerical calculations and perturbative methodologies are the only ways to handle such problems. In those scenarios, rigorous analysis of one-dimensional integrable models can provide a testing ground for numerical analysis to understand the properties of real physical systems. Thus the significance of exact solutions of quantum integrable models can be justified in a larger context to test the relevance of various approximate formalisms. More-

over, recent experiments involving optical lattice of ultra-cold Rydberg atoms and trapped ions, neutral atoms in optical cavities, etc., have opened up the possibility of realizing various theoretical models of lower-dimensional spin systems in a very precise way [6–10]. In such experiments it has been observed that, in contrast to the case of spin chains with nearest or next-to-nearest neighbour interactions, spin chains with long-range interactions often exhibit interesting physical phenomena such as the realisation of quantum spin glasses, quantum crystals, and high-speed propagation of correlations exceeding the Lieb-Robinson bound [7–12].

Quantum integrable models are classified into two types depending on the nature of their interaction — quantum integrable models with short-range interaction and quantum integrable models with long-range interaction. For the case of quantum integrable models with short-range interaction, particles or spins interact locally like within nearest neighbour or next-to-nearest neighbour sites in a lattice system. One dimensional systems like $\delta$-function Bose gas [4, 13, 14], Hubbard model [13, 15–17], isotropic and anisotropic variants of Heisenberg spin-$\frac{1}{2}$ chain [1, 13, 14, 17–19] are some examples of quantum integrable models with short-range interactions, exact solutions of which can be obtained by using the Bethe ansatz. This type of integrable systems have found applications in physical phenomena like electromagnetic field propagation in quantum optics [20–22], strongly interacting ultracold atomic gases [23–30] etc.

For the case of quantum integrable models with long-range interactions, all particles or spins interact with each other. Calogero has pioneered the study of quantum integrable and exactly solvable models with long-range interactions [5,31]. In particular, he found the exact spectrum of the well known Calogero model with harmonic confining interaction, in which particles move on a line and interact with each other through a two-body potential proportional to the inverse square of their relative distances. Sutherland investigated a similar exactly solvable system in which particles

move on a circle and interact with each other through a two-body potential proportional to the inverse square of their chord distances [32, 33]. Subsequently Khare found that a variant of the Calogero model, where the harmonic confining potential is replaced by a Coulomb-like interaction, also represents an exactly solvable system [34, 35]. Apart from these dynamical models, Haldane and Shastry derived the exact spectrum of a spin-$\frac{1}{2}$ chain with long-range interaction, where the lattice sites are equally spaced on a circle and the spins interact with each other through pairwise interactions inversely proportional to the square of their chord distances [36, 37]. A close relation between the $su(m)$ generalization of this Haldane-Shastry (HS) spin chain and the $su(m)$ spin Sutherland model has been established by using the 'freezing trick' [38, 39]. Application of the freezing trick to the $su(m)$ spin Calogero model with confining harmonic potential leads to the quantum integrable $su(m)$ Polychronakos spin chain, which is also known in the literature as the Polychronakos-Frahm (PF) spin chain [38, 40, 41]. The sites of such $su(m)$ PF spin chain are inhomogeneously spaced on a line and, in fact, they coincide with the zeros of the Hermite polynomial. Quantum integrable spin chains and dynamical models with long-range interaction have found applications in diverse areas of physics and mathematics like condensed matter systems exhibiting generalized exclusion statistics [42–45], quantum Hall effect [46], Dunkl operators related to various root systems [47–51], random matrix theory [52], conformal field theory [53–56], super Yang-Mills theory [57–59], quantum electric transport phenomena [60, 61] and Yangian quantum groups [42, 62–67]. In this context it may be noted that, although Dunkl operators were introduced in Ref. [47] in a purely mathematical setting, the first explicit use of such operators for Calogero-like systems and their integrability was presented in Ref. [48]. Subsequently, such Dunkl operators were used in Ref. [49] to derive spin Calogero systems.

Let us first briefly review the concept of classical and quantum integrability. Liouville introduced the concept of integrability in the context of classical mechan-

ics [68, 69]. A system characterized by a $2N$- dimensional phase space $\Gamma$ with coordinates $\zeta \equiv (q_i, p_i), \quad i = 1, \cdots, N$ and Hamiltonian $H(q, p)$ is called completely integrable if their exist $N$ number of independent functions $I_i(q, p)$ such that they have vanishing Poisson bracket among themselves:

$$\{I_i, I_j\} = 0, \tag{1.1.1}$$

and $H$ can be written as a function of $I_i$'s, i.e.

$$H(q, p) = \tilde{H}(I). \tag{1.1.2}$$

Therefore, it follows that $\{H(q, p), I_i\} = 0$. Hence, $I_i$'s are called the commuting integrals of motion or conserved quantities of the integrable system. For an integrable system, one can in principle find a canonical transform of the form

$$(q, p) \to (\phi, I) \tag{1.1.3}$$

such that the following relations are satisfied

$$\{\phi_i, I_j\} = \delta_{ij}. \tag{1.1.4}$$

Therefore it follows that

$$\{\phi_i, \tilde{H}\} = \frac{\partial \tilde{H}}{\partial I_i} \equiv \omega_i(I). \tag{1.1.5}$$

In terms of new canonical variables, the equations of motion are given by

$$\dot{I}_i = \{I_i, \tilde{H}\} = 0, \qquad \dot{\phi}_i = \{\phi_i, \tilde{H}\} = \omega_i(I). \tag{1.1.6}$$

Since $\omega_i(I)$ does not depend on $t$, these equations of motion can be integrated to find explicit solutions as

$$I_i = c_i; \qquad \phi_i(t) = \phi_i(t=0) + \omega_i t. \tag{1.1.7}$$

where $c_i$'s are some constants. These $(\phi, I)$ are known as the 'action-angle' variables. For the case of classical integrable field models, one obtains infinite number of conserved quantities in involution.

In quantum mechanical treatment, one replaces the Poisson brackets by the corresponding commutator brackets. However, the definition of a quantum integrable system is not so straightforward as in the classical case [15]. In particular, some definition of integrability similar to the classical case can not be used for quantum systems with finite dimensional Hilbert space (which do not have any classical limit), such as spin chains. There will be two major problems regarding such definition of integrability. First, what will be the number $N$? Whether it will be the number of spins or it will be the dimension of the direct product space constituted from the Hilbert spaces of individual spin degrees of freedom. And secondly, what will be the nature of $N$ independent conserved commuting operators? It is not clear that independence of such operators means linear independence or algebraic independence. Indeed for a quantum mechanical system associated with a finite dimensional Hilbert space, the root idea behind the existence of $N$ number of algebraically independent commuting operators does not seem to be very well-defined for $N > 1$ [15].

To avoid the above mentioned problems, a different approach may be taken to extend the idea of classical integrability to the quantum case. Such an approach is

known as scattering without diffraction [15]. However, this approach is only applicable to quantum systems that support scattering, which means that the particles of the system will fly apart into pieces and move as free particles unless bound in a finite region of space. The pieces are the particles or bound states of cluster of particles. In the process of scattering without diffraction, the number of pieces $N$ will be conserved for an integrable system. Consequently, for a non-diffractive scattering process involving $N$ number of pieces, local conserved charges like $I_k$, $k = 1, 2, 3, ...., N$ can be constructed and each of those charges would be a symmetric polynomial function of the asymptotic momenta $p_k, k = 1, 2, 3, ....., N$. The information about the total momentum and total energy of the system is included in this set of conserved charges. The higher order charges can be related to the hidden symmetries of the Hamiltonian. Such new definition of quantum integrability seems to be more appropriate as the corresponding classical definition can be directly related to it.

The conserved charges of an integrable system can be determined through an analytical technique which requires the knowledge of the Lax pair corresponding to the model. This technique was first used by Calogero [68, 69] and Moser [70] to prove the integrability of some classical systems with pairwise long-range interaction. Remarkably, this Lax pair technique of proving integrability is also applicable for Calogero like quantum systems with long-range interaction [15, 71–73]. It can be shown that there exists a deep connection between the Hamiltonian of the original classical or quantum Calogero model and the root system of the $A_{N-1}$ Lie algebra. Such a connection leads to the construction of different variants of original Calogero model related to the other root systems of Lie algebra [74]. For the purpose of explaining this point, let us briefly review the concept of root system.

Even though the root systems are closely connected to the Lie algebras, they can be defined independently in the following way. For any two vectors $\alpha$, $\beta \in \mathbb{R}^N$, the

Weyl reflection of $\beta$ through the hyperplane perpendicular to $\alpha$ is defined as

$$\sigma_\alpha(\beta) = \beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha, \tag{1.1.8}$$

where $(\beta,\alpha)$ is the scalar product of $\beta$ and $\alpha$ in $\mathbb{R}^N$. Note that $\sigma_\alpha^2(\beta) = \beta$. A finite set $\Sigma$ of non-zero vectors in $\mathbb{R}^N$ gives a non-reduced root system, if it satisfies the following three conditions:

1) $\Sigma$ spans $\mathbb{R}^N$ or a subspace of $\mathbb{R}^N$.

2) $2\frac{(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$, for any $\alpha, \beta \in \Sigma$.

3) $\sigma_\alpha\Sigma = \Sigma$, for any $\alpha \in \Sigma$.

The elements of $\Sigma$ are called as roots. The dimension of the vector space spanned by $\Sigma$ is called the rank of this root system. The group generated by the reflections $\sigma_\alpha$ (i.e, by combining these reflections in all distinct ways) is called the Weyl group [75] $W$ associated with the root system $\Sigma$. It may be noted that:

1) Since $\sigma_\alpha(\alpha) = -\alpha$, $-\alpha \in \Sigma$ if $\alpha \in \Sigma$.

2) Since each element of $W$ permutes the set $\Sigma$, $W$ is a subgroup of the permutation group defined on the elements of $\Sigma$.

A subset $\Gamma \in \Sigma$ is called a simple root system, if $\Gamma$ forms a basis of $\Sigma$, and any $\rho \in \Sigma$ can be expressed as $\rho = \sum_{\alpha\in\Gamma} \rho_\alpha\alpha$, where all $\rho_\alpha$'s are either non-negative ($\geqslant 0$) integers or non-positive ($\leqslant 0$) integers. The elements of $\Gamma$ are called as simple roots. If all $\rho_\alpha$'s are non-negative integers in the expansion $\rho = \sum_\alpha \rho_\alpha\alpha$, $\rho$ is called a positive root. The collection of all such positive roots is denoted by $\Sigma_+$. $\Gamma$ is a subset of $\Sigma_+$ and for $\alpha, \beta \in \Gamma$, $2\frac{(\beta,\alpha)}{(\alpha,\alpha)} \leqslant 0$. Starting from $\Gamma$, one can construct the whole root system $\Sigma$ by giving all possible Weyl reflections till their closure condition is satisfied.

Let us denote by $\{e_1, e_2, \cdots, e_N\}$ the usual orthonormal basis in $\mathbb{R}^N$. The simple root systems $\Gamma = \{\alpha_1, \alpha_2, \cdots, \alpha_{N-1} \text{ or } \alpha_N\}$ for four classical Lie algebras are given by:

1) For $A_{N-1}$, i.e. $su(N)$: $\alpha_i = e_i - e_{i+1}$ with $1 \leqslant i \leqslant N-1$.

2) For $B_N$, i.e. $so(2N+1)$: $\alpha_i = e_i - e_{i+1}$ with $1 \leqslant i \leqslant N-1$ and $\alpha_N = e_N$.

3) For $C_N$, i.e. $sp(2N)$: $\alpha_i = e_i - e_{i+1}$ with $1 \leqslant i \leqslant N-1$ and $\alpha_N = 2e_N$.

4) For $D_N$, i.e. $so(2N)$: $\alpha_i = e_i - e_{i+1}$ with $1 \leqslant i \leqslant N-1$ and $\alpha_N = e_{N-1} + e_N$.

However, all root systems are not directly related to the Lie algebras. In addition to the above mentioned four root systems, one can construct another (non-reduced) $BC_N$ type of root system, whose simple root system is obtained through the union of simple roots corresponding to the $B_N$ and $C_N$ root systems. In the following, we give a list of positive roots $\Sigma_+$ associated with various root systems:

1) For $A_{N-1}$, i.e. $su(N)$: $e_i - e_j$ $(1 \leqslant i < j \leqslant N)$.

2) For $B_N$, i.e. $so(2N+1)$: $e_i$ $(i = 1, \cdots, N)$, $e_i \pm e_j$ $(1 \leqslant i < j \leqslant N)$.

3) For $C_N$, i.e. $sp(2N)$: $2e_i$ $(i = 1, \cdots, N)$, $e_i \pm e_j$ $(1 \leqslant i < j \leqslant N)$.

4) For $D_N$, i.e. $so(2N)$: $e_i \pm e_j$ $(1 \leqslant i < j \leqslant N)$.

5) For $BC_N$: $e_i$, $2e_i$ $(i = 1, \cdots, N)$, $e_i \pm e_j$ $(1 \leqslant i < j \leqslant N)$.

The general form of quantum integrable Calogero model (with harmonic confinement), where particles on a line interact with each other through a rational Weyl-invariant potential, can be expressed for any root system as [76]

$$H = \sum_{i=1}^{N} \left( -\frac{\partial^2}{\partial x_i^2} + \omega^2 x_i^2 \right) + \frac{1}{2} \sum_{\rho \in \Sigma_+} \frac{g_\rho (g_\rho - 1) \rho^2}{(\rho \cdot \mathbf{r})^2}, \tag{1.1.9}$$

where $\mathbf{r} = \sum_i x_i e_i \in \mathbb{R}^N$ and $g_\rho$'s are real positive coupling constants which are

defined on the orbits of the Weyl group. For the case of $A_{N-1}$ root system, $H$ in (1.1.9) reduces to the original Calogero Hamiltonian given by [5]

$$H^{(A)} = \sum_{i=1}^{N}\left(-\frac{\partial^2}{\partial x_i^2}+\omega^2 x_i^2\right)+\nu(\nu-1)\sum_{1\leqslant i\neq j\leqslant N}\frac{1}{(x_i-x_j)^2}. \tag{1.1.10}$$

For the case of $B_N$, $C_N$ or $BC_N$ root system, $H$ in (1.1.9) yields a Hamiltonian of the form

$$\begin{aligned} H^{(B)} = & \sum_{i=1}^{N}\left(-\frac{\partial^2}{\partial x_i^2}+\omega^2 x_i^2\right)+\nu(\nu-1)\sum_{1\leqslant i\neq j\leqslant N}\left\{\frac{1}{(x_i-x_j)^2}+\frac{1}{(x_i+x_j)^2}\right\} \\ & +\rho(\rho-1)\sum_{j=1}^{N}\frac{1}{x_j^2}. \end{aligned} \tag{1.1.11}$$

For the case of $D_N$ root system, $H$ in (1.1.9) yields

$$H^{(D)} = \sum_{i=1}^{N}\left(-\frac{\partial^2}{\partial x_i^2}+\omega^2 x_i^2\right)+\nu(\nu-1)\sum_{1\leqslant i\neq j\leqslant N}\left\{\frac{1}{(x_i-x_j)^2}+\frac{1}{(x_i+x_j)^2}\right\}. \tag{1.1.12}$$

Comparing Eq. (1.1.10) with Eqs. (1.1.11) and (1.1.12), we observe that only $A_{N-1}$ type of Calogero Hamiltonian possesses the translational symmetry and this symmetry is broken for the corresponding $BC_N$ and $D_N$ type of models. Furthermore, it may be noted that, the Hamiltonian (1.1.12) of the $D_N$ type of Calogero model can be obtained from its $B_N$ counterpart (1.1.11) by taking $\rho \to 0$ limit. Nevertheless, the spectrum of the $D_N$ type of Calogero model can not be reproduced from that of its $B_N$ counterpart as a special case [73, 77].

The general form of quantum integrable Sutherland model, where particles moving on a circle interact with each other through a trigonometric Weyl-invariant potential, is given by [76]

$$H = -\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2}+\frac{1}{2}\sum_{\rho\in\Sigma_+}\frac{g_\rho(g_\rho-1)\rho^2}{\sin^2(\rho\cdot\mathbf{r})}. \tag{1.1.13}$$

where $\mathbf{r} = \sum_i x_i e_i \in \mathbb{R}^N$ and $g_\rho$'s are real positive coupling constants which are

defined on the orbits of the Weyl group. The Hamiltonian of the original Sutherland model associated with the $A_{N-1}$ root system can be obtained from the general Weyl-invariant Hamiltonian (1.1.13) as

$$H^{(A)} = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + a(a-1) \sum_{i \neq j} \frac{1}{\sin^2(x_i - x_j)}. \tag{1.1.14}$$

Different variants of this Sutherland model associated with other root systems can also be derived from the general Hamiltonian (1.1.13) in a similar way.

The study of quantum integrable spin chains with long-range interactions was pioneered by Haldane and Shastry, who derived the exact spectrum of a spin-$\frac{1}{2}$ HS spin chain with lattice sites equally spaced on a circle and spins interacting through pairwise exchange interactions inversely proportional to the square of their chord distances [36, 37]. It has been found that, the exact ground state wave function of this su(2) symmetric HS spin chain coincides with the $U \to \infty$ limit of Gutzwiller's variational wave function describing the ground state of the one-dimensional Hubbard model [78–80]. The Hamiltonian of the su($m$) HS spin chain is given by

$$\mathcal{H}_{HS} = \frac{1}{2} \sum_{1 \leqslant i < j \leqslant N} \frac{1 - \epsilon P_{ij}}{\sin^2 \frac{\pi}{N}(i-j)}, \tag{1.1.15}$$

where $\epsilon = +1$ $(-1)$ corresponds to the ferromagnetic (anti-ferromagnetic) case, the spin exchange operator $P_{ij}$ is defined as

$$P_{ij}|s_1, \cdots, s_i, \cdots, s_j, \cdots, s_N\rangle = |s_1, \cdots, s_j, \cdots, s_i, \cdots, s_N\rangle, \tag{1.1.16}$$

with $s_i \in \{M, M-1, \cdots, -M\}$ and $M = \frac{m-1}{2}$. One of the key properties of the HS chain—already noted by Haldane and Shastry in their original papers—is its intimate connection with the scalar (trigonometric) Sutherland model [32, 33]. This connection was subsequently elucidated by Polychronakos in Ref. [38], who showed

how to derive the su($m$) HS chain from the su($m$) spin Sutherland model [49,72,81] by a technique that he called the 'freezing trick'. The particles of the su($m$) spin Sutherland model possess dynamical as well as spin degrees of freedom, and the corresponding Hamiltonian is given by

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i \neq j} \frac{a(a - \epsilon P_{ij})}{\sin^2(x_i - x_j)}. \tag{1.1.17}$$

The main idea behind this freezing trick is to note that when the coupling constant $a$ in the spin Sutherland model goes to infinity the particles tend to concentrate on the coordinates of the (essentially unique) minimum of the scalar part of the interaction potential, which are precisely the sites of the HS chain. Thus, in this limit the dynamical and the spin degrees of freedom decouple, and the latter are governed by the chain's Hamiltonian. Using this idea it is straightforward, for instance, to obtain the first integrals of the HS chain from their well-known counterparts for the spin Sutherland model.

It may noted that, the Hamiltonian of the su($m$) spin Calogero model of $A_{N-1}$ type is given by [49]

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + a^2 \sum_i x_i^2 + a \sum_{i \neq j} \frac{a - \epsilon P_{ij}}{(x_i - x_j)^2}\,, \tag{1.1.18}$$

where $\epsilon = +1$ $(-1)$ corresponds to the ferromagnetic (anti-ferromagnetic) case. Application of the freezing trick to this su($m$) spin Calogero model leads to the quantum integrable su($m$) PF spin chain with Hamiltonian given by [38]

$$\mathcal{H}_{PF} = \sum_{i<j} \frac{1 - \epsilon P_{ij}}{(\rho_i - \rho_j)^2}\,, \tag{1.1.19}$$

where the sites of this chain, i.e., $\rho_i$'s are the coordinates of the unique minimum point of the scalar part of the potential of the spin Calogero Hamiltonian (1.1.18).

Such minimum of the scalar part of the potential is determined by the algebraic system of equations like

$$\rho_i = 2 \sum_{j=1\,(j\neq i)}^{N} \frac{1}{(\rho_i - \rho_j)^3}, \qquad i = 1, ...., N, \tag{1.1.20}$$

which are satisfied by the roots of the Hermite polynomial of degree $N$ [82]. Hence, the lattice sites of the PF spin chain (1.1.19) are given by the roots of the Hermite polynomial of degree $N$. The spectrum of this PF spin chain was first studied numerically by Frahm [41] and then exactly computed by Polychronakos [40], who also derived an exact formula for the partition function by means of the freezing trick. On the other hand, the partition function of the su($m$) HS chain was only evaluated more than a decade later by Finkel and González-López [83].

Both of the above mentioned spin Sutherland and spin Calogero models (and their corresponding HS and PF spin chains) are associated with the $A_{N-1}$ root system, where $N$ is the number of particles. Indeed, in these models the interactions only depend on the difference between the coordinates, and the spin operators appearing in the Hamiltonian are permutation operators, and thus generate a realization of the Weyl group of $A_{N-1}$ type. In fact, there are versions of the Sutherland and Calogero models associated to any (extended) classical root system [74]. Among these systems, those of $BC_N$, $B_N$, $C_N$ and $D_N$ type are by far the most studied in the literature, since they make it possible to construct integrable models with an arbitrary number of particles. By applying the freezing trick to the spin version of these models one obtains the corresponding generalizations of the HS and PF chains, that we shall collectively refer to as spin chains of HS type [84–90]. One of the fundamental features of the $BC_N$ root system and its $B_N$ and $C_N$ degenerations is the fact that its Weyl algebra contains a family of reflection operators $S_i$ ($i = 1, \ldots, N$) satisfying $S_i^2 = 1$. (In the case of the $D_N$ root system, the Weyl group only contains products $S_iS_j$ with $i \neq j$.) In the spin chains studied in Refs. [84–88],

the operators $S_i$ are represented by spin reversal operators $P_i$ (acting on the Hilbert space of the $i$-th particle), which acts on the states as

$$P_i|s_1, \cdots, s_i, \cdots, s_N\rangle = |s_1, \cdots, -s_i, \cdots, s_N\rangle. \tag{1.1.21}$$

By using this $P_i$, one can define the Hamiltonian of the $BC_N$ type of spin Calogero model [84] as

$$H^{(B)} = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + a\sum_{i\neq j}\left[\frac{a+P_{ij}}{(x_{ij}^-)^2} + \frac{a+\tilde{P}_{ij}}{(x_{ij}^+)^2}\right] + b\sum_i \frac{b-\epsilon P_i}{x_i^2} + \frac{a^2}{4}r^2\,, \tag{1.1.22}$$

where $\epsilon = \pm 1$, $x_{ij}^- \equiv x_i - x_j$, $x_{ij}^+ \equiv x_i + x_j$, $r^2 = \sum_{i=1}^N x_i^2$, and $\tilde{P}_{ij} = P_iP_jP_{ij}$. By setting $b = \beta a$ (here $\beta$ is a positive parameter) in the Hamiltonian (1.1.22) and using the freezing trick at $a \to \infty$ limit, one obtains the PF spin chain of $BC_N$ type, with Hamiltonian given by

$$\mathcal{H}_{PF}^{(B)} = \sum_{i\neq j}\left[\frac{1+P_{ij}}{(\xi_i-\xi_j)^2} + \frac{1+\tilde{P}_{ij}}{(\xi_i+\xi_j)^2}\right] + \beta\sum_i \frac{1-\epsilon P_i}{\xi_i^2}\,, \tag{1.1.23}$$

where $\xi_i = \sqrt{2y_i}$ and $y_i$ represents the $i$-th zero point of the generalized Laguerre polynomial $L_N^{\beta-1}$.

The $D_N$ type of Calogero models associated with the spin reversal operator $P_i$ have also been studied in the literature [86]. The Hamiltonian of the such spin Calogero model of $D_N$ type is obtained by setting $b = 0$ in its $BC_N$ counterpart (1.1.22) as

$$H^{(D)} = -\sum_i \frac{\partial^2}{\partial x_i^2} + a\sum_{i\neq j}\left(\frac{a+P_{ij}}{(x_{ij}^-)^2} + \frac{a+\widetilde{P}_{ij}}{(x_{ij}^+)^2}\right) + \frac{a^2}{4}r^2. \tag{1.1.24}$$

Taking the $a \to \infty$ limit of this spin Calogero Hamiltonian (1.1.24) and using the

freezing trick, the Hamiltonian of the PF chain of $D_N$ type can be derived as

$$\mathcal{H}_{PF}^{(D)} = \sum_{i \neq j} \left[ \frac{1 + P_{ij}}{(\xi_i - \xi_j)^2} + \frac{1 + \widetilde{P}_{ij}}{(\xi_i + \xi_j)^2} \right], \tag{1.1.25}$$

where the lattice sites of this chain are obtained from those of its $BC_N$ counterpart (1.1.23) in the limit $\beta \to 0$. However, it should be noted that, the Hamiltonian (1.1.23) of the PF chain of $BC_N$ type does not reduce to its $D_N$ variant (1.1.25) in the above mentioned limit. To explain this fact it may be noted that, all roots of the equation $L_N^{\beta-1}(y) = 0$, except the smallest one, tend to some finite nonzero values in the limit $\beta \to 0$. Consequently, terms like $\beta(1 - \epsilon P_i)/\xi_i^2$, which appear in the r.h.s. of Eq. (1.1.23), vanish for $i = 2, \cdots, N$. However it can be shown that, at $\beta \to 0$ limit, the smallest root $\xi_1$ of the equation $L_N^{\beta-1}(y) = 0$ satisfies the relation $\lim_{\beta\to 0}(\frac{2\beta}{\xi_1^2}) = N$ [86]. Substituting this limiting value in (1.1.23), it is easy to see that the Hamiltonians of the $BC_N$ and $D_N$ types of PF chains are related as

$$\lim_{\beta \to 0} \mathcal{H}_{PF}^{(B)} = \mathcal{H}_{PF}^{(D)} + \frac{N}{2}(1 - \epsilon P_1). \tag{1.1.26}$$

It may be noted that, although the Hamiltonian (1.1.24) of the $D_N$ type of su($m$) spin Calogero model can be obtained by taking $b \to 0$ limit of its $BC_N$ counterpart (1.1.22), the Hilbert space of the former Hamiltonian is drastically different from its $BC_N$ counterpart [86]. Indeed, it can be shown that the Hilbert space of the $D_N$ model gets "doubled" in comparison with the $BC_N$ one. More precisely, the Hilbert space of the $D_N$ spin Calogero model can be written as a direct sum of the Hilbert spaces associated to two different $BC_N$ models with opposite "chiralities" corresponding to $\epsilon = \pm 1$. Due to this remarkable property of the $D_N$ model, the spectra of this model and its associated spin chain cannot be obtained as a limiting case of their $BC_N$ counterparts.

However it should be noted that, for the purpose of constructing $BC_N$ and $D_N$

types of integrable spin models, the above mentioned representation of the operators $S_i$ as spin reversal operators $P_i$ is by no means the only possible choice. As a matter of fact, in the novel version of the spin Calogero model of $BC_N$ type and its corresponding (PF) chain introduced in Ref. [91], the operators $S_i$ are represented instead by arbitrarily polarized spin reversal operators (PSRO) $P_i^{(m_1,m_2)}$, which act as the identity on the first $m_1$ elements of the spin basis and as minus the identity on the remaining $m_2$ elements of the spin basis. For the purpose of more precisely describing such action of $P_i^{(m_1,m_2)}$, let us define the total spin space $\mathcal{S}$ through a set of orthonormal basis vectors as

$$\mathcal{S} = \Big\langle |s_1, \cdots, s_N\rangle | s_i \in \{1, 2, \cdots, m\} \Big\rangle, \tag{1.1.27}$$

where $m = m_1 + m_2$. The action of $P_i^{(m_1,m_2)}$ on these basis vectors can be written as

$$P_i^{(m_1,m_2)} |s_1, \cdots, s_i, \cdots, s_N\rangle = (-1)^{f(s_i)} |s_1, \cdots, s_i, \cdots, s_N\rangle, \tag{1.1.28}$$

where

$$f(s_i) = \begin{cases} 0, & 1 \leqslant s_i \leqslant m_1 \\ 1, & m_1 + 1 \leqslant s_i \leqslant m_1 + m_2. \end{cases} \tag{1.1.29}$$

The Hamiltonian of the $BC_N$-type Calogero model with PSRO is defined as [91]

$$\begin{aligned} H_{B,\epsilon}^{(m_1,m_2)} &= -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + a \sum_{i \neq j} \left( \frac{a + P_{ij}}{(x_{ij}^-)^2} + \frac{a + \widetilde{P}_{ij}^{(m_1,m_2)}}{(x_{ij}^+)^2} \right) \\ &\quad + \beta a \sum_{i=1}^{N} \frac{\beta a - \epsilon P_i^{(m_1,m_2)}}{x_i^2} + \frac{a^2}{4} r^2 , \end{aligned} \tag{1.1.30}$$

where $a > \frac{1}{2}$, $\beta > 0$, $\epsilon = \pm 1$, $x_{ij}^{\pm} = x_i \pm x_j$, $r^2 = \sum_i x_i^2$, and $\widetilde{P}_{ij}^{(m_1,m_2)} = P_i^{(m_1,m_2)} P_j^{(m_1,m_2)} P_{ij}$. It can be shown that when $m$ is even (resp. odd) and $m_1 = m_2$ (resp. $m_1 = m_2 \pm 1$), the PSRO in (1.1.28) is equivalent via a similarity transformation to the usual spin reversal operator $P_i$. As a result, for these special choices of

$m_1$ and $m_2$, the Hamiltonian (1.1.30) reduces to that of the standard spin Calogero model of $BC_N$ type studied in Ref. [84]. However, for the remaining values of the discrete parameters $m_1$ and $m_2$, $H_{B,\epsilon}^{(m_1,m_2)}$ in (1.1.30) differs from the standard spin Calogero model of $BC_N$-type. From the spin dynamical model (1.1.30) one can construct a PF chain of $BC_N$ type with PSRO by applying the freezing trick. The Hamiltonian of this chain is given by

$$\mathcal{H}_{B,\epsilon}^{(m_1,m_2)} = \sum_{i\neq j}\left[\frac{1+P_{ij}}{(\xi_i-\xi_j)^2}+\frac{1+\widetilde{P}_{ij}^{(m_1,m_2)}}{(\xi_i+\xi_j)^2}\right]+\beta\sum_i\frac{1-\epsilon P_i^{(m_1,m_2)}}{\xi_i^2}\,, \tag{1.1.31}$$

where the lattice sites $\xi_i$ are related to the zeros $y_i$ of the Laguerre polynomial $L_N^{\beta-1}$ by $y_i=\xi_i^2/2$. The exact partition function of the chain (1.1.31) has also been computed in Ref. [91] by exploiting its connection with the spin dynamical model (1.1.30).

It is often possible to construct su($m|n$) supersymmetric generalisation of the integrable spin chains with long-range interaction, in which each lattice site is occupied by either one of the $m$ number of 'bosonic' spins or one of the $n$ number of 'fermionic' spins [92–94]. Such supersymmetric spin chains with different types of interactions play an important role in describing some quantum impurity problems and disordered systems in condensed matter physics, where holes moving in the dynamical background of spins behave as bosons, and spin-1/2 electrons behave as fermions [95–99]. It is worth noting that the Hamiltonians (1.1.15) and (1.1.19) of the $A_{N-1}$ type of HS and PF spin chains admit natural su($m|n$) supersymmetric generalizations. For example, the Hamiltonian of the su($m|n$) supersymmetric HS spin chain is given by [92, 100]:

$$\mathcal{H}_{HS}^{(m|n)} = \frac{1}{2}\sum_{1\leqslant j<k\leqslant N}\frac{1-\hat{P}_{ij}^{(m|n)}}{\sin^2\frac{\pi}{N}(i-j)}, \tag{1.1.32}$$

where the supersymmetric exchange operator $\hat{P}_{ij}^{(m|n)}$ is defined as

$$\hat{P}_{ij}^{(m|n)} = \sum_{\alpha,\beta=1}^{m+n} C_{i,\alpha}^{\dagger} C_{j,\beta}^{\dagger} C_{i,\beta} C_{j,\alpha}. \tag{1.1.33}$$

Here the creation and annihilation operators, $C_{i,\alpha}^{\dagger}$ and $C_{i,\alpha}$ are assumed to be bosonic for $\alpha = 1, \cdots, m$ and fermionic for $\alpha = m+1, \cdots, m+n$. These creation and annihilation operators act on a restricted Hilbert space where each lattice site is occupied by only one spin, i.e., $\sum_{\alpha=1}^{m+n} C_{i,\alpha} C_{i,\alpha} = 1$ for all $i$. It may be observed that for the special case $m \neq 0$, $n = 0$, the supersymmetric Hamiltonian (1.1.32) reduces to the standard su($m$) HS spin chain (1.1.15) with $\epsilon = 1$. Similar to the case of supersymmetric HS spin chain, the Hamiltonian of the supersymmetric PF chain is defined as [93]

$$\mathcal{H}_{PF}^{(m|n)} = \sum_{1 \leqslant i < j \leqslant N} \frac{1 - \hat{P}_{ij}^{(m|n)}}{(\rho_i - \rho_j)^2}, \tag{1.1.34}$$

where $\rho_i$'s are the zeros of the $N$-th order Hermite polynomial. The partition functions of these supersymmetric PF and HS spin chains have been calculated in closed form by using the freezing trick [93, 94, 100].

We have mentioned earlier that, the Weyl algebra associated with the $BC_N$ root system contains a family of reflection operators $S_i$ $(i = 1, \ldots, N)$ satisfying $S_i^2 = 1$. By taking $S_i$ as the spin reversal operator on a superspace which contains both 'bosonic' and 'fermionic' spin states, the Hamiltonian of a supersymmetric analogue of the PF chain associated with the $BC_N$ root system has been constructed in Ref. [101]. The Hamiltonian of the spin dynamical model of Calogero type, which yields the above mentioned PF spin chain in the strong coupling limit, has also been introduced in the later reference. Furthermore, the partition function of such $BC_N$ type of PF chain containing supersymmetric analogue of spin reversal operator (SASRO) has been computed by using the freezing trick.

It may be noted that multivariate Rogers-Szegö (RS) polynomials play an important

role in analyzing the spectra and partition functions of the $A_{N-1}$ type of PF spin chains [62,102]. The classical Rogers-Szegö (RS) polynomial in a single variable (say, $x$) is defined as $\mathbb{H}_N(x,q) = \sum_{k=0}^{N} \left[\begin{smallmatrix} N \\ k, N-k \end{smallmatrix}\right]_q x^k$ [103]. This RS polynomial has been studied in connection with the well known Rogers-Ramanujan identities in number theory. Moreover, this RS polynomial can be viewed as a $q$-deformed version of the Hermite polynomial, which provides a basis for the coordinate representation of the $q$-oscillator algebra [104, 105]. Different types of homogeneous and inhomogeneous multivariate generalizations of the classical RS polynomial have also been studied in the literature. In particular, Hikami has observed that the homogeneous multivariate RS polynomials (depending on only one type of variables) of the form

$$\mathbb{H}^{(m)}_{A,N}(x_1, x_2, \cdots, x_m; q) = \sum_{\substack{\sum_{i=1}^{m} a_i = N \\ a_i \geqslant 0}} \left[\begin{smallmatrix} N \\ a_1, a_2, \cdots, a_m \end{smallmatrix}\right]_q x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m} , \qquad (1.1.35)$$

reproduce the partition function of the $A_{N-1}$ type of su($m$) PF spin chain in the limit $x_1 = x_2 = \cdots = x_m = 1$ [62, 102, 106]. Consequently, a representation of the motifs associated with the Yangian quantum group symmetry of the $A_{N-1}$ type of su($m$) PF spin chain can be constructed by using a recursion relation satisfied by the RS polynomials (1.1.35). In a similar way, super RS (SRS) polynomials containing two different types of variables have been studied for the purpose of analyzing the spectra and partition functions of the supersymmetric PF spin chains (1.1.34) on the basis of their super Yangian symmetry [94].

In this thesis our aim is to study some rational quantum integrable spin dynamical models and spin chains associated with PSRO and supersymmetric analogues of PSRO (SAPSRO). In particular, in Chapter 2 of this thesis, we study the spin Calogero model of $D_N$ type with PSRO, as well as its associated PF spin chain, both in the antiferromagnetic and ferromagnetic cases [107]. We compute the spectrum and the partition function of the former model in closed form, from which we derive an exact formula for the chain's partition function in terms of products of partition

functions of PF spin chains of type $A$. Using a recursion relation for the latter partition functions, we are able to numerically evaluate the partition function, and thus the spectrum, of the $D_N$-type spin chain for relatively high values of the number of spins $N$. We analyze several global properties of the chain's spectrum, such as the asymptotic level density, the distribution of consecutive spacings of the unfolded spectrum, and the average degeneracy.

In Chapter 3 of this thesis, we derive the exact spectra as well as partition functions for a class of $BC_N$ type of spin Calogero models, whose Hamiltonians are constructed by using SAPSRO [108]. The strong coupling limit of these spin Calogero models yields $BC_N$ type of PF spin chains with SAPSRO. By applying the freezing trick, we obtain an exact expression for the partition functions of such PF spin chains. We also derive a formula which expresses the partition function of any $BC_N$ type of PF spin chain with SAPSRO in terms of partition functions of several $A_K$ type of supersymmetric PF spin chains, where $K \leqslant N-1$. Subsequently we show that an extended boson-fermion duality relation is obeyed by the partition functions of the $BC_N$ type of PF chains with SAPSRO. Some spectral properties of these spin chains, like level density distribution and nearest neighbour spacing distribution, are also studied.

In Chapter 4, we construct SRS polynomials associated with the $BC_N$ type of PF spin chains with SAPSRO [109]. To this end, we compute the grand canonical partition functions of the $BC_N$ type of ferromagnetic as well as anti-ferromagnetic spin Calogero models with SAPSRO, and expand those grand canonical partition functions as some power series of the fugacity parameter to obtain the corresponding canonical partition functions. Applying the freezing trick, subsequently we derive new expressions for the canonical partition functions of the related $BC_N$ type of PF spin chains with SAPSRO. Inspired by the form of such partition functions, we introduce novel multivariate SRS polynomials depending on four types of variables.

We construct the generating functions for such SRS polynomials and show that these polynomials can be written as some bilinear combinations of the $A_{N-1}$ type of SRS polynomials. We also use the above mentioned generating functions to derive a set of recursion relations for the partition functions of the $BC_N$ type of PF spin chains involving different numbers of lattice sites and internal degrees of freedom.

In Chapter 5 of this thesis, we make some concluding remarks.

# CHAPTER 2

# Rational quantum integrable systems of $D_N$ type with polarized spin reversal operators

## 2.1 Introduction

As mentioned in Chapter 1, a class of $BC_N$ type of spin Calogero models and their associated spin chains were introduced in Ref. [91]. The distinguishing feature of these models is that they are constructed using a new representation of the Weyl group of the $BC_N$ root system, obtained by replacing the standard spin reversal operators by PSRO. As shown in the latter reference, these models are exactly solvable for all such representations and, in particular, the partition function of the spin chains can be exactly computed by using Polychronakos's freezing trick [38,40].

In this Chapter, we shall introduce the spin Calogero model of $D_N$-type with PSRO and its corresponding spin chain of HS type, i.e., the PF chain of $D_N$ type with PSRO. As explained in Ref. [86], $D_N$ type of models with spin reversal operators are singular limits of their corresponding $BC_N$ counterparts, so that their spectrum

cannot be obtained by setting to zero the parameter $\beta$ in the latter models. This is also apparent at the level of the Hilbert space, which is the direct sum of the Hilbert spaces of two $BC_N$ models with opposite chiralities. Similarly, the $D_N$ type of models which will be studied in this Chapter are not limiting cases of their $BC_N$ versions in Ref. [91].

The main result of this Chapter is the derivation of a closed-form expression for the partition function of the PF chain of $D_N$ type with PSRO in terms of products of partition functions of type-$A$ PF chains. Our approach is based on the computation of the spectrum and partition function of the corresponding spin Calogero model, from which the chain's partition function follows by a standard freezing trick argument. The structure of this partition function turns out to be more involved than that of its $BC_N$ counterpart. In particular, it is not manifest that it is a polynomial in $q \equiv \mathrm{e}^{-1/(k_{\mathrm{B}}T)}$, as follows from the freezing trick. Using the explicit expression for the partition function, we shall study several global properties of the chain's spectrum, such as the behaviour of the level density and the average degeneracy when the number of spins tends to infinity.

This Chapter is organized as follows. In Section 2.2 we recall the definition and main properties of the polarized spin reversal operators $P_i^{(m_1,m_2)}$, and construct the Hamiltonians of the $D_N$-type spin Calogero model with PSRO and its associated spin chain. Section 2.3 is devoted to the derivation of the closed-form expression of the chain's partition function, as explained in the previous paragraph. Using this expression, in Section 2.4 we analyze several global properties of the spectrum, providing strong numerical evidence of the Gaussian character of the level density when the number of spins is large enough. In Section 2.5 we extend the above results to the ferromagnetic version of the models under consideration. This Chapter ends with a short technical Appendix establishing a useful recursion relation for the partition function of the PF chain of type $A_{N-1}$.

## 2.2 Construction of the models

For the purpose of describing the $D_N$-type Calogero model with polarized spin reversal operators, let us briefly recapitulate the construction of its $BC_N$ counterpart [91]. To this end, let

$$\mathcal{S} = \left\langle |s_1, \ldots, s_N\rangle \;\middle|\; s_i \in \{1, 2, \ldots, m\} \right\rangle . \tag{2.2.1}$$

denote the internal spin space for $N$ particles. As usual, the action of the spin exchange operator $P_{ij}$ on $\mathcal{S}$ is defined as

$$P_{ij}|s_1, \ldots, s_i, \ldots, s_j, \ldots, s_N\rangle = |s_1, \ldots, s_j, \ldots, s_i, \ldots, s_N\rangle . \tag{2.2.2}$$

Let us denote the PSRO associated with the $i$-th particle as $P_i^{(m_1,m_2)}$, where $m_1$ and $m_2$ are two nonnegative integers satisfying the relation $m_1 + m_2 = m$. The action of $P_i^{(m_1,m_2)}$ on $\mathcal{S}$ is given by [91]

$$P_i^{(m_1,m_2)}|s_1, \ldots, s_i, \ldots, s_N\rangle = (-1)^{f(s_i)}|s_1, \ldots, s_i, \ldots, s_N\rangle , \tag{2.2.3}$$

where

$$f(s_i) = \begin{cases} 0, & 1 \leqslant s_i \leqslant m_1 \\ 1, & m_1 + 1 \leqslant s_i \leqslant m_1 + m_2. \end{cases} \tag{2.2.4}$$

In terms of these operators, the Hamiltonian of the $BC_N$-type Calogero model with PSRO is defined as [91]

$$\begin{aligned} H_{B,\epsilon}^{(m_1,m_2)} \;=\; & -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + a \sum_{i \neq j} \left( \frac{a + P_{ij}}{(x_{ij}^-)^2} + \frac{a + \widetilde{P}_{ij}^{(m_1,m_2)}}{(x_{ij}^+)^2} \right) \\ & + \beta a \sum_{i=1}^{N} \frac{\beta a - \epsilon P_i^{(m_1,m_2)}}{x_i^2} + \frac{a^2}{4} r^2 , \end{aligned} \tag{2.2.5}$$

where the sums run from 1 to $N$, $a > \frac{1}{2}$, $\beta > 0$, $\epsilon = \pm 1$, $x_{ij}^{\pm} = x_i \pm x_j$, $r^2 = \sum_i x_i^2$, and

$$\widetilde{P}_{ij}^{(m_1,m_2)} = P_i^{(m_1,m_2)} P_j^{(m_1,m_2)} P_{ij}.$$

It can be shown that when $m$ is even (resp. odd) and $m_1 = m_2$ (resp. $m_1 = m_2 \pm 1$), the PSRO in (2.2.3) is equivalent via a similarity transformation to the usual spin reversal operator $P_i$, which changes $s_i$ into $m - s_i + 1$. As a result, for these special choices of $m_1$ and $m_2$, the Hamiltonian (1.1.30) reduces to that of the standard $su(m)$ spin Calogero model of $BC_N$ type studied in Ref. [84].

Since $H_{\mathrm{B},\epsilon}^{(m_1,m_2)}$ contains the discrete parameters $m_1$, $m_2$ and $\epsilon$, it is natural to inquire whether there exists any relation between models (2.2.5) with different sets of parameters. In fact, we shall now show that $H_{\mathrm{B},\epsilon}^{(m_1,m_2)}$ is equivalent to $H_{\mathrm{B},-\epsilon}^{(m_2,m_1)}$ through a unitary transformation. To this end, consider the unitary operator $T$ whose action on the spin space $\mathcal{S}$ is given by

$$T|s_1, \ldots, s_i, \ldots, s_N\rangle = |s_1', \ldots, s_i', \ldots, s_N'\rangle, \tag{2.2.6}$$

where

$$s_i' = \begin{cases} s_i + m_1, & 1 \leqslant s_i \leqslant m_2 \\ s_i - m_2, & m_2 + 1 \leqslant s_i \leqslant m_1 + m_2. \end{cases} \tag{2.2.7}$$

Using Eqs. (2.2.3) and (2.2.6) we easily obtain

$$T^\dagger P_i^{(m_1,m_2)} T|s_1, \ldots, s_i, \ldots, s_N\rangle = (-1)^{f(s_i')}|s_1, \ldots, s_i, \ldots, s_N\rangle\,. \tag{2.2.8}$$

From Eq. (2.2.7) it follows that $s_i' \in \{m_1 + 1, \ldots, m_1 + m_2\}$ for $1 \leqslant s_i \leqslant m_2$ and $s_i' \in \{1, \ldots, m_1\}$ for $m_2 + 1 \leqslant s_i \leqslant m_2 + m_1$, so that

$$f(s_i') = \begin{cases} 1, & 1 \leqslant s_i \leqslant m_2 \\ 0, & m_2 + 1 \leqslant s_i \leqslant m_2 + m_1. \end{cases} \tag{2.2.9}$$

Equations (2.2.8) and (2.2.9) clearly imply that

$$T^\dagger P_i^{(m_1,m_2)} T = -P_i^{(m_2,m_1)} \,. \tag{2.2.10}$$

It is also obvious from Eqs. (2.2.2) and (2.2.6) that

$$T^\dagger P_{ij} T = P_{ij} \,. \tag{2.2.11}$$

From Eqs. (2.2.5), (2.2.10) and (2.2.11) we readily obtain

$$T^\dagger H_{\mathrm{B},\epsilon}^{(m_1,m_2)} T = H_{\mathrm{B},-\epsilon}^{(m_2,m_1)}, \tag{2.2.12}$$

as claimed. In view of the above relation, it suffices to study the Hamiltonian (2.2.5) in the case $\epsilon = 1$. However, in the following we shall intentionally keep the parameter $\epsilon$ in $H_{\mathrm{B},\epsilon}^{(m_1,m_2)}$ in order to facilitate the comparison with its $D_N$ counterpart that we shall introduce below.

Due to the nature of the singularities of the Hamiltonian $H_{\mathrm{B},\epsilon}^{(m_1,m_2)}$, its configuration space can be taken as one of the Weyl chambers of the $BC_N$ root system, i.e., one of the maximal open subsets of $\mathbb{R}^N$ on which the functions $x_i \pm x_j$ and $x_i$ have constants signs. We shall choose this configuration space as the principal Weyl chamber

$$C^{(\mathrm{B})} = \left\{ \mathbf{x} \in \mathbb{R}^N : 0 < x_1 < x_2 < \cdots < x_N \right\}, \tag{2.2.13}$$

where $\mathbf{x} \equiv (x_1, \ldots, x_N)$. The Hamiltonian $H_{\mathrm{B},\epsilon}^{(m_1,m_2)}$ is thus defined on an appropriate dense subset of the Hilbert space $L^2(C^{(B)}) \otimes \mathcal{S}$. When $\epsilon = 1$, the spectrum of $H_{\mathrm{B},\epsilon}^{(m_1,m_2)}$ was computed in Ref. [91] by constructing a suitable (non-orthogonal) basis of this Hilbert space in which this Hamiltonian acts triangularly.

As explained in the latter reference, from the spin dynamical model (2.2.5) one can construct a PF chain of $BC_N$ type with PSRO by applying the freezing trick. The

Hamiltonian of this chain is given by

$$\mathcal{H}_{\mathrm{B},\epsilon}^{(m_1,m_2)} = \sum_{i\neq j}\left[\frac{1+P_{ij}}{(\zeta_i-\zeta_j)^2}+\frac{1+\widetilde{P}_{ij}^{(m_1,m_2)}}{(\zeta_i+\zeta_j)^2}\right]+\beta\sum_i\frac{1-\epsilon P_i^{(m_1,m_2)}}{\zeta_i^2}\,, \tag{2.2.14}$$

where the lattice sites $\zeta_i$ are related to the zeros $y_i$ of the Laguerre polynomial $L_N^{\beta-1}$ by $y_i = \zeta_i^2/2$. The exact partition function of the chain (2.2.14) has also been computed in Ref. [91] by exploiting its connection with the spin dynamical model (2.2.5).

The Hamiltonian $H^{(m_1,m_2)}$ of the $D_N$-type spin Calogero model with PSRO is naturally defined by dropping the term related to the roots $x_i$ in $H_{\mathrm{B},\epsilon}^{(m_1,m_2)}$, i.e., by setting $\beta = 0$ in Eq. (2.2.5). We thus obtain

$$H^{(m_1,m_2)} = -\sum_i\frac{\partial^2}{\partial x_i^2}+a\sum_{i\neq j}\left[\frac{a+P_{ij}}{(x_{ij}^-)^2}+\frac{a+\widetilde{P}_{ij}^{(m_1,m_2)}}{(x_{ij}^+)^2}\right]+\frac{a^2}{4}r^2\,. \tag{2.2.15}$$

It should be noted that, unlike its $BC_N$ counterpart, the latter Hamiltonian does not depend on $\epsilon$. Just as before, from Eqs. (2.2.10) and (2.2.11) it follows that $H^{(m_2,m_1)}$ is unitarily equivalent to $H^{(m_1,m_2)}$ under $T$:

$$T^\dagger H^{(m_1,m_2)}T = H^{(m_2,m_1)}\,. \tag{2.2.16}$$

Thus we can impose without loss of generality the restriction $m_1 \geqslant m_2$. Consequently, for any given $m$ one can construct $\lfloor m/2+1\rfloor$ inequivalent spin Calogero models of $D_N$ type with PSRO, where $\lfloor\cdot\rfloor$ denotes the integer part. Among these models, only those with $m_1 = m_2$ (for even $m$) or $m_1 = m_2+1$ (for odd $m$) are unitarily equivalent to the su($m$) spin Calogero model of $D_N$ type with standard spin reversal operators introduced in Ref. [86].

As is the case with the latter model, the configuration space $C$ of the Hamiltonian (2.2.15) can be taken as one of the maximal open subsets of $\mathbb{R}^N$ on which the

linear functionals $x_i \pm x_j$ have constant signs. We shall again take $C$ as the principal Weyl chamber of the $D_N$ root system, namely

$$C = \left\{ \mathbf{x} \in \mathbb{R}^N : |x_1| < x_2 < \cdots < x_N \right\}. \tag{2.2.17}$$

Note that this configuration space contains its $BC_N$ counterpart (2.2.13) as a subset. As before, the Hamiltonian (2.2.15) is defined on a suitable dense subspace of the Hilbert space $L^2(C) \otimes \mathcal{S}$.

We shall next explain in detail how to construct the $D_N$-type PF chain with PSRO associated to the spin dynamical model (2.2.15) by means of Polychronakos's freezing trick. To begin with, note that the Hamiltonian $H^{(m_1,m_2)}$ can be decomposed as

$$H^{(m_1,m_2)} = H^{\mathrm{sc}} + a\hat{\mathcal{H}}^{(m_1,m_2)}(\mathbf{x}), \tag{2.2.18}$$

where

$$H^{\mathrm{sc}} = -\sum_i \frac{\partial^2}{\partial x_i^2} + a(a-1)\sum_{i\neq j}\left[\frac{1}{(x_{ij}^-)^2} + \frac{1}{(x_{ij}^+)^2}\right] + \frac{a^2}{4}r^2 \tag{2.2.19}$$

is the Hamiltonian of the scalar $D_N$ Calogero model and

$$\hat{\mathcal{H}}^{(m_1,m_2)}(\mathbf{x}) = \sum_{i\neq j}\left[\frac{1+P_{ij}}{(x_{ij}^-)^2} + \frac{1+\widetilde{P}_{ij}^{(m_1,m_2)}}{(x_{ij}^+)^2}\right] \tag{2.2.20}$$

is a spin-dependent multiplication operator. On the other hand, in the strong coupling limit $a \to \infty$ the coefficient of the dominant term (of order $a^2$) in the Hamiltonian (2.2.15) is given by

$$U(\mathbf{x}) = \sum_{i\neq j}\left[\frac{1}{(x_{ij}^-)^2} + \frac{1}{(x_{ij}^+)^2}\right] + \frac{r^2}{4}. \tag{2.2.21}$$

Hence as $a \to \infty$ the particles concentrate at the coordinates $\xi_i$ of the unique mini-

mum $\boldsymbol{\xi}$ of the potential $U(\mathbf{x})$ in the configuration space $C$ [76], and the coordinate degrees of freedom of $H^{(m_1,m_2)}$ decouple from the internal ones. By Eq. (2.2.18), in this limit the eigenvalues of $H^{(m_1,m_2)}$ are approximately given by

$$E_{ij} \simeq E_i^{\mathrm{sc}} + a\mathcal{E}_j, \tag{2.2.22}$$

where $E_i^{\mathrm{sc}}$ and $\mathcal{E}_j$ are two arbitrary eigenvalues of $H^{\mathrm{sc}}$ and

$$\mathcal{H}^{(m_1,m_2)} \equiv \hat{\mathcal{H}}^{(m_1,m_2)}(\boldsymbol{\xi}) = \sum_{i\neq j}\left[\frac{1+P_{ij}}{(\xi_i-\xi_j)^2} + \frac{1+\widetilde{P}_{ij}^{(m_1,m_2)}}{(\xi_i+\xi_j)^2}\right]. \tag{2.2.23}$$

We shall take Eq. (2.2.23) as the precise definition of the Hamiltonian of the $D_N$-type PF chain with PSRO. In fact, using Eqs. (2.2.10) and (2.2.11), it is easy to show that the Hamiltonians $\mathcal{H}^{(m_1,m_2)}$ and $\mathcal{H}^{(m_2,m_1)}$ are related by

$$T^{\dagger}\mathcal{H}^{(m_1,m_2)}T = \mathcal{H}^{(m_2,m_1)}. \tag{2.2.24}$$

Thus, we may assume without loss of generality that $m_1 \geqslant m_2$, so that there are again $\lfloor m/2+1 \rfloor$ inequivalent PF chains of $D_N$ type with PSRO. Since the sites of these chains depend only on the scalar potential (2.2.21), the above models reduce to the su($m$) PF chain of $D_N$ type with standard spin reversal operators [86] when $m_1 = m_2$ (for even $m$) or $m_1 = m_2+1$ (for odd $m$). See, e.g., Fig. 2.1 for a comparison of the spectra of the $D_N$ chain with PSRO (2.2.23) with $m_1 = 3, m_2 = 1$ and the su(4) $D_N$-type PF chain with standard time-reversal operators (corresponding to $m_1 = m_2 = 2$) for $N = 10$ spins.

A brief remark on the relation between the $D_N$ and $BC_N$ spin chains with PSRO in Eqs. (2.2.23) and (2.2.14) is now in order. As shown in [86], the lattice sites of the former chain are given by $\xi_1 = 0$ and $\xi_i = \sqrt{2y_{i-1}}$ $(2 \leqslant i \leqslant N)$, where $y_k > 0$ denotes the $k$-th root of the generalized Laguerre polynomial $L_{N-1}^1$. From the well-known identity $NL_N^{-1}(y) = -yL_{N-1}^1(y)$ and the previous characterization of the sites $\zeta_i$ of

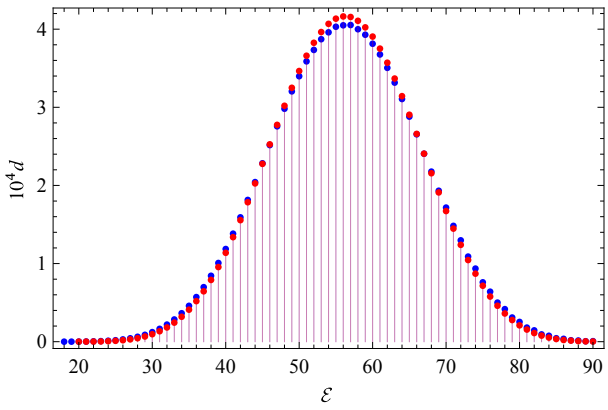


**Figure 2.1:** Degeneracy $d$ (in units of $10^4$) versus energy $\mathcal{E}$ of the $D_N$ chain (2.2.23) with $m_1 = 3$, $m_2 = 1$ (blue), compared to the su(4) $D_N$-type PF chain with standard time-reversal operators (red), for $N = 10$ spins.

the $BC_N$ chain (2.2.14), it immediately follows that $\boldsymbol{\xi} = \lim_{\beta\to 0}\boldsymbol{\zeta}$. Although one may naively think that the Hamiltonian $\mathcal{H}^{(m_1,m_2)}$ is simply the $\beta \to 0$ limit of its $BC_N$ counterpart $\mathcal{H}_{\mathrm{B},\epsilon}^{(m_1,m_2)}$, this is certainly not the case. The point is that, although the roots $\zeta_i$ with $2 \leqslant i \leqslant N$ tend to finite nonzero limits when $\beta \to 0$, the first root $\zeta_1$ tends to 0 in this limit. As a consequence, the $i = 1$ term of the last sum in Eq. (2.2.14) need not vanish as $\beta \to 0$, and in fact it can be shown [86] that

$$\lim_{\beta\to 0} \frac{\beta}{\zeta_1^2} = \frac{N}{2}. \tag{2.2.25}$$

Letting $\beta \to 0$ in Eq. (2.2.14) and using the latter identity we immediately obtain

$$\lim_{\beta\to 0} \mathcal{H}_{\mathrm{B},\epsilon}^{(m_1,m_2)} = \mathcal{H}^{(m_1,m_2)} + \frac{N}{2}\big(1 - \epsilon P_1^{(m_1,m_2)}\big)\,. \tag{2.2.26}$$

Thus, the $\beta \to 0$ limit of the Hamiltonian $\mathcal{H}_{\mathrm{B},\epsilon}^{(m_1,m_2)}$ differs from its $D_N$ counterpart $\mathcal{H}^{(m_1,m_2)}$ by the surface term or impurity interaction $N\big(1 - \epsilon P_1^{(m_1,m_2)}\big)/2$. It is easy to see that this term vanishes only for $\epsilon = 1$, $m_1 = m$, $m_2 = 0$ (or, equivalently, $\epsilon = -1$, $m_1 = 0$, $m_2 = m$). For any other choice of $m_1$ and $m_2$, this surface term is nonzero and does not commute with the Hamiltonian $\mathcal{H}^{(m_1,m_2)}$. Thus, except in the previously noted special cases, the spectrum of $\mathcal{H}^{(m_1,m_2)}$ cannot be obtained from that of $\mathcal{H}_{\mathrm{B},\epsilon}^{(m_1,m_2)}$ by taking the $\beta \to 0$ limit. This fact is illustrated in Fig. 2.2, which

shows that the spectra of these chains with $m_1 = 3$, $m_2 = 1$ and $N = 10$ spins are clearly different.

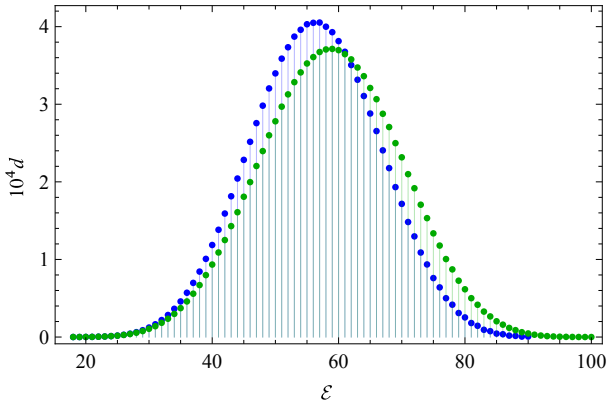


**Figure 2.2:** Degeneracy $d$ (in units of $10^4$) versus energy $\mathcal{E}$ of the $D_N$ chain (2.2.23) with $m_1 = 3$, $m_2 = 1$ and $N = 10$ spins (blue), compared to its $BC_N$ counterpart in Eq. (2.2.14) (green; recall that the spectrum of the latter chain does not depend on $\beta$)

## 2.3 Spectrum and partition function

In this section, we shall compute in closed form the spectrum and partition function of the spin Calogero model of $D_N$ type with PSRO in Eq. (2.2.15). This will enable us to compute the partition function $\mathcal{Z}^{(m_1,m_2)}$ of the $D_N$-type PF chain with PSRO (2.2.23) by a standard freezing trick argument. Indeed, from Eq. (2.2.22) it is straightforward to derive the following exact formula for $\mathcal{Z}^{(m_1,m_2)}$ in terms of the partition functions $Z^{(m_1,m_2)}$ and $Z$ of the spin dynamical model (2.2.15) and of its scalar counterpart (2.2.19):

$$\mathcal{Z}^{(m_1,m_2)}(T) = \lim_{a\to\infty} \frac{Z^{(m_1,m_2)}(aT)}{Z(aT)}. \tag{2.3.1}$$

Since $Z$ has already been computed in Ref. [86], Eq. (2.3.1) provides an effective way of evaluating $\mathcal{Z}^{(m_1,m_2)}$ once $Z^{(m_1,m_2)}$ is known.

The key idea for deriving the spectrum of the spin Hamiltonian (2.2.15) is to observe

that it can be obtained by applying a suitable projection to a simpler differential-difference operator $H'$ acting on scalar functions. The spectrum of $H'$ can be readily computed by constructing a (non-orthogonal) basis of its Hilbert space on which this operator acts triangularly. The spectrum of $H^{(m_1,m_2)}$ is then easily determined by projecting onto the Hilbert space of the latter operator.

More precisely, the auxiliary operator $H'$ is given by [86]

$$H' = -\sum_i \frac{\partial^2}{\partial x_i^2} + a\sum_{i\neq j}\left[\frac{a}{(x_{ij}^-)^2}(a - K_{ij}) + \frac{a}{(x_{ij}^+)^2}(a - \tilde{K}_{ij})\right] + \frac{a^2}{4}r^2\,, \tag{2.3.2}$$

where $K_{ij}$ and $K_i$ are coordinate exchange and sign reversing operators, defined by

$$\begin{aligned} K_{ij}f(x_1,\dots,x_i,\dots,x_j,\dots,x_N) &= f(x_1,\dots,x_j,\dots,x_i,\dots,x_N)\,,\\ K_i f(x_1,\dots,x_i,\dots,x_N) &= f(x_1,\dots,-x_i,\dots,x_N)\,, \end{aligned}$$

and $\widetilde{K}_{ij} \equiv K_iK_jK_{ij}$. The domain of the operator $H'$ is of course a suitable dense subset of the Hilbert space $L^2(\mathbb{R}^N)$. The operator $H'$ can be expressed in terms of the $D_N$-type rational Dunkl operators [47]

$$J_i^- = \frac{\partial}{\partial x_i} + a\sum_{j;j\neq i}\left[\frac{1}{x_{ij}^-}(1 - K_{ij}) + \frac{1}{x_{ij}^+}(1 - \tilde{K}_{ij})\right] \tag{2.3.3}$$

as [110]

$$H' = \rho(\mathbf{x})\left[-\sum_i (J_i^-)^2 + a\sum_i x_i\frac{\partial}{\partial x_i} + E_0\right]\rho(\mathbf{x})^{-1}\,, \tag{2.3.4}$$

where

$$\rho(\mathbf{x}) = \mathrm{e}^{-\frac{a}{4}r^2}\prod_{i<j}|x_i^2 - x_j^2|^a$$

is the ground state of the scalar Calogero model of $D_N$-type and

$$E_0 = Na\big(a(N-1) + \tfrac{1}{2}\big) \tag{2.3.5}$$

is its ground-state energy. A basis of this Hilbert space on which $H'$ acts triangularly is provided by the functions

$$\phi_{\mathbf{n}}(\mathbf{x}) = \rho(\mathbf{x}) \prod_i x_i^{n_i}\,, \qquad \mathbf{n} \equiv (n_1, \ldots, n_N), \tag{2.3.6}$$

where the $n_i$'s are arbitrary non-negative integers. Indeed, since $J_i^-$ lowers the degree $|\mathbf{n}| \equiv n_1 + \cdots + n_N$ of any monomial $\prod_i x_i^{n_i}$, from Eqs. (2.3.3) and (2.3.4) it immediately follows that

$$H'\,\phi_{\mathbf{n}}(\mathbf{x}) = E'_{\mathbf{n}}\,\phi_{\mathbf{n}}(\mathbf{x}) + \sum_{|\mathbf{m}|<|\mathbf{n}|} c_{\mathbf{mn}}\,\phi_{\mathbf{m}}(\mathbf{x})\,, \tag{2.3.7}$$

where the coefficients $c_{\mathbf{mn}}$ are real constants and

$$E'_{\mathbf{n}} = a|\mathbf{n}| + E_0\,. \tag{2.3.8}$$

As the diagonal elements of any upper triangular operator coincide with its eigenvalues, the spectrum of $H'$ is given by Eq. (2.3.8).

The spectrum of the spin Hamiltonian $H^{(m_1,m_2)}$ can be derived from that of $H'$ by noting that these Hamiltonians are formally related by

$$H^{(m_1,m_2)} = H'|_{K_{ij}\to -P_{ij},\, K_iK_j\to P_i^{(m_1,m_2)}P_j^{(m_1,m_2)}}. \tag{2.3.9}$$

In order to take advantage of this observation, we introduce the operator $\Lambda^{(m_1,m_2)}$ projecting the Hilbert space $L^2(\mathbb{R}^N)\otimes\mathcal{S}$ onto states that are antisymmetric under particle permutations and symmetric under the action of $K_iK_jP_i^{(m_1,m_2)}P_j^{(m_1,m_2)}$ for any $i \neq j$. In other words, the projector $\Lambda^{(m_1,m_2)}$ is determined by

$$\pi_{ij}\Lambda^{(m_1,m_2)} = -\Lambda^{(m_1,m_2)}, \qquad \pi_i^{(m_1,m_2)}\pi_j^{(m_1,m_2)}\Lambda^{(m_1,m_2)} = \Lambda^{(m_1,m_2)}, \tag{2.3.10}$$

where

$$\pi_{ij} \equiv K_{ij}P_{ij}\,, \qquad \pi_i^{(m_1,m_2)} \equiv K_i P_i^{(m_1,m_2)}\,, \tag{2.3.11}$$

so that

$$K_{ij}\Lambda^{(m_1,m_2)} = -P_{ij}\Lambda^{(m_1,m_2)}, \qquad K_iK_j\Lambda^{(m_1,m_2)} = P_i^{(m_1,m_2)}P_j^{(m_1,m_2)}\Lambda^{(m_1,m_2)}. \tag{2.3.12}$$

We shall now outline the construction of the projector $\Lambda^{(m_1,m_2)}$ in terms of the analogous projectors $\Lambda_{\mathrm{B},\pm}^{(m_1,m_2)}$ for the $BC_N$-type spin Calogero model with PSRO (2.2.14) with chirality $\varepsilon = \pm 1$ (cf. [91]). To this end, recall that $\Lambda_{\mathrm{B},\pm}^{(m_1,m_2)}$ projects from the Hilbert space $L^2(\mathbb{R}^N)\otimes\mathcal{S}$ onto spin wavefunctions antisymmetric under particle permutations and with parity $\pm 1$ under $\pi_i^{(m_1,m_2)}$, i.e.,

$$\pi_{ij}\Lambda_{\mathrm{B},\pm}^{(m_1,m_2)} = -\Lambda_{\mathrm{B},\pm}^{(m_1,m_2)}, \qquad \pi_i^{(m_1,m_2)}\Lambda_{\mathrm{B},\pm}^{(m_1,m_2)} = \pm\Lambda_{\mathrm{B},\pm}^{(m_1,m_2)}, \tag{2.3.13}$$

The projector $\Lambda_{\mathrm{B},\pm}^{(m_1,m_2)}$ can then be expressed as

$$\Lambda_{\mathrm{B},\pm}^{(m_1,m_2)} = \frac{1}{2^N N!}\prod_{j=1}^{N}\left(1 \pm \pi_j^{(m_1,m_2)}\right).\sum_{l=1}^{N!}\varepsilon_l\mathcal{P}_l\,, \tag{2.3.14}$$

where $\mathcal{P}_l$ denotes an element of the realization of the permutation group generated by the operators $\pi_{ij}$ and $\varepsilon_l$ is the signature of $\mathcal{P}_l$. From Eqs. (2.3.10) and (2.3.13) we conclude that

$$\Lambda^{(m_1,m_2)} = \Lambda_{\mathrm{B},+}^{(m_1,m_2)} + \Lambda_{\mathrm{B},-}^{(m_1,m_2)}\,. \tag{2.3.15}$$

Indeed, the right-hand side of the latter equation is clearly a projector, since

$$\Lambda_{\mathrm{B},+}^{(m_1,m_2)}\Lambda_{\mathrm{B},-}^{(m_1,m_2)} = \Lambda_{\mathrm{B},-}^{(m_1,m_2)}\Lambda_{\mathrm{B},+}^{(m_1,m_2)} = 0\,,$$

and it satisfies (2.3.10) on account of (2.3.13). Thus the space

$$V \equiv \Lambda^{(m_1,m_2)}(L^2(\mathbb{R}^N) \otimes \mathcal{S})$$

decomposes as the direct sum

$$V = V_{\mathrm{B},+} \oplus V_{\mathrm{B},-}, \qquad V_{\mathrm{B},\pm} \equiv \Lambda^{(m_1,m_2)}_{\mathrm{B},\pm}(L^2(\mathbb{R}^N) \otimes \mathcal{S}). \tag{2.3.16}$$

We have already mentioned that, due to the impenetrable nature of the singularities of the Hamiltonian $H^{(m_1,m_2)}$, its Hilbert space can be taken as the space $L^2(C)\otimes\mathcal{S}$ of spin wavefunctions square integrable on the open set $C$ in Eq. (2.2.17). On the other hand, any point in $\mathbb{R}^N$ not lying on the singular subset $x_i \pm x_j = 0$, $1 \leqslant i < j \leqslant N$, can be mapped in a unique way to a point in $C$ by a suitable element of the $D_N$ Weyl group, which is generated by coordinate permutations and sign reversals of an *even* number of coordinates [111]. Using this fact, it can be shown that $L^2(C) \otimes \mathcal{S}$ is actually isomorphic to the space $V$, and $H^{(m_1,m_2)}$ is equivalent to its natural extension to the latter space which (with a slight abuse of notation) we shall also denote by $H^{(m_1,m_2)}$. With this identification, in view of Eq. (2.3.12) we can write

$$H^{(m_1,m_2)} = H^{(m_1,m_2)}\Lambda^{(m_1,m_2)} = H'\Lambda^{(m_1,m_2)}\,, \tag{2.3.17}$$

where $H'$ acts trivially (as the identity) on $\mathcal{S}$.

We shall now explain how the spectrum of $H^{(m_1,m_2)}$ can be derived from that of $H'$ using the previous equation. To this end, note that by Eq. (2.3.16) the Hilbert space $V$ is the closure of the linear subspace spanned by the spin wavefunctions

$$\psi^{\epsilon}_{\mathbf{n},\mathbf{s}}(\mathbf{x}) = \Lambda^{(m_1,m_2)}_{\mathrm{B},\epsilon}(\phi_{\mathbf{n}}(\mathbf{x})|\mathbf{s}\rangle)\,, \qquad \epsilon = \pm\,, \tag{2.3.18}$$

where $|\mathbf{s}\rangle \equiv |s_1, \ldots, s_N\rangle$ is an arbitrary element of the canonical spin basis. In fact,

the wavefunctions (2.3.18) with *fixed* $\epsilon$ span a subspace whose closure is the Hilbert space $V_{\mathrm{B},\epsilon}$. Clearly, the functions (2.3.18) are not linearly independent. Indeed, using Eq. (2.3.13) it is easy to show that these functions satisfy the relations

$$\psi^{\epsilon}_{\mathbf{n},\mathbf{s}}(\mathbf{x}) = -\psi^{\epsilon}_{\mathbf{n}',\mathbf{s}'}(\mathbf{x})\,, \qquad \psi^{\epsilon}_{\mathbf{n},\mathbf{s}}(\mathbf{x}) = \epsilon(-1)^{n_i+f(s_i)}\psi^{\epsilon}_{\mathbf{n},\mathbf{s}}(\mathbf{x})\,, \tag{2.3.19}$$

where $\mathbf{n}'$ and $\mathbf{s}'$ are respectively obtained from $\mathbf{n}$ and $\mathbf{s}$ by permuting any two of their components (the same for both). Due to these identities, the sets $\{\psi^{+}_{\mathbf{n},\mathbf{s}}(\mathbf{x})\}$ and $\{\psi^{-}_{\mathbf{n},\mathbf{s}}(\mathbf{x})\}$ are both linearly independent provided that the following three conditions are imposed on the quantum numbers $\mathbf{n}$ and $\mathbf{s}$:

1. To avoid overcounting, and for later convenience, we shall order the components of $\mathbf{n}$ as follows:

$$\mathbf{n} \equiv (\mathbf{n}_{\mathrm{e}}, \mathbf{n}_{\mathrm{o}}) = \big(\overbrace{2p_1, \ldots, 2p_1}^{k_1}, \ldots, \overbrace{2p_s, \ldots, 2p_s}^{k_s}, \overbrace{2q_1+1, \ldots, 2q_1+1}^{l_1}, \ldots, \overbrace{2q_t+1, \ldots, 2q_t+1}^{l_t}\big)\,,$$

   where $0 \leqslant s, t \leqslant N$, $p_1 > p_2 > \cdots > p_s \geqslant 0$ and $q_1 > q_2 > \cdots > q_t \geqslant 0$.

2. By the second equation in (2.3.19), the allowed values of $s_i$ corresponding to each $n_i$ are given by

$$s_i \in \begin{cases} \{1, 2, \ldots, m_1\}\,, & \text{for even } n_i\,, \\ \{m_1+1, m_1+2, \ldots, m_1+m_2\}\,, & \text{for odd } n_i\,, \end{cases}$$

   for the set $\{\psi^{+}_{\mathbf{n},\mathbf{s}}(\mathbf{x})\}$, and by

$$s_i \in \begin{cases} \{1, 2, \ldots, m_1\}\,, & \text{for odd } n_i\,, \\ \{m_1+1, m_1+2, \ldots, m_1+m_2\}\,, & \text{for even } n_i\,. \end{cases}$$

for the set $\{\psi^{-}_{\mathbf{n},\mathbf{s}}(\mathbf{x})\}$.

3. If $n_i = n_j$ and $i < j$ we shall take $s_i > s_j$, again to avoid overcounting.

If the above conditions are satisfied, each of the sets $\{\psi^{\epsilon}_{\mathbf{n},\mathbf{s}}(\mathbf{x})\}$ ($\epsilon = \pm$) is a non-orthogonal basis of the corresponding subspace $V_{\mathrm{B},\epsilon}$, and the union of these sets provides a non-orthogonal basis of the whole Hilbert space $V$ by Eq. (2.3.16). We shall next show that $H^{(m_1,m_2)}$ leaves invariant each of the subspaces $V_{\mathrm{B},\epsilon}$, and that it acts triangularly on the corresponding basis $\{\psi^{\epsilon}_{\mathbf{n},\mathbf{s}}(\mathbf{x})\}$ provided that we (partially) order it by the total degree $|\mathbf{n}|$. Indeed, using Eqs. (2.3.15) and (2.3.17), and taking into account that $[H', \Lambda^{(m_1,m_2)}_{\mathrm{B},\epsilon}] = 0$ we obtain

$$H^{(m_1,m_2)}\psi^{\epsilon}_{\mathbf{n},\mathbf{s}}(\mathbf{x}) = \Lambda^{(m_1,m_2)}_{\mathrm{B},\epsilon}\big((H'\phi_{\mathbf{n}}(\mathbf{x}))|\mathbf{s}\rangle\big). \tag{2.3.20}$$

From this equation and Eqs. (2.3.7) and (2.3.18) it readily follows that

$$H^{(m_1,m_2)}\psi^{\epsilon}_{\mathbf{n},\mathbf{s}}(\mathbf{x}) = E'_{\mathbf{n}}\psi^{\epsilon}_{\mathbf{n},\mathbf{s}}(\mathbf{x}) + \sum_{|\mathbf{m}|<|\mathbf{n}|} C_{\mathbf{mn}}\psi^{\epsilon}_{\mathbf{m},\mathbf{s}'}(\mathbf{x})\,, \tag{2.3.21}$$

where the $C_{\mathbf{mn}}$'s are real constants and $\mathbf{s}'$ is a permutation of $\mathbf{s}$ such that $(\mathbf{m},\mathbf{s}')$ satisfies conditions i)–iii) above; see [86] for more details. By Eq. (2.3.21), the action of $H^{(m_1,m_2)}$ on the whole Hilbert space $V = V_{\mathrm{B},+}\oplus V_{\mathrm{B},-}$ is the direct sum of two upper triangular actions on each of the subspaces $V_{\mathrm{B},\pm}$. Consequently, the eigenvalues of this operator are given by

$$E^{\epsilon}_{\mathbf{n},\mathbf{s}} = E'_{\mathbf{n}} = a|\mathbf{n}| + E_0\,, \tag{2.3.22}$$

where $\epsilon = \pm$ and $(\mathbf{n},\mathbf{s})$ satisfies conditions i)–iii) above. Since the RHS of Eq. (2.3.22) does not depend on $\epsilon$ and $\mathbf{s}$, the eigenvalue associated with the quantum number $\mathbf{n}$ has an *intrinsic degeneracy* $d^{(m_1,m_2)}_{\mathbf{n}}$ coming from the two possible chiralities and

the spin degrees of freedom. This intrinsic degeneracy is in fact the sum

$$d_{\mathbf{n}}^{(m_1,m_2)} = d_{\mathbf{n},+}^{(m_1,m_2)} + d_{\mathbf{n},-}^{(m_1,m_2)}, \tag{2.3.23}$$

where $d_{\mathbf{n},\epsilon}^{(m_1,m_2)}$ is the number of spin states satisfying conditions i)–iii) for the given $\mathbf{n}$ and $\epsilon$. Using these conditions we readily obtain [91]

$$d_{\mathbf{n},+}^{(m_1,m_2)} = \prod_{i=1}^{s} \binom{m_1}{k_i} \prod_{j=1}^{t} \binom{m_2}{l_j}, \qquad d_{\mathbf{n},-}^{(m_1,m_2)} = d_{\mathbf{n},+}^{(m_2,m_1)}, \tag{2.3.24}$$

and therefore

$$d_{\mathbf{n}}^{(m_1,m_2)} = \prod_{i=1}^{s} \binom{m_1}{k_i} \prod_{j=1}^{t} \binom{m_2}{l_j} + \prod_{i=1}^{s} \binom{m_2}{k_i} \prod_{j=1}^{t} \binom{m_1}{l_j}. \tag{2.3.25}$$

Thus the spectrum of the $D_N$-type spin Calogero model with PSRO (2.2.15) is given by the RHS of Eq. (2.3.22), where each level possesses an intrinsic degeneracy given by Eq. (2.3.25). Of course, the actual degeneracy of an energy $a|\mathbf{n}| + E_0$ is the sum $\sum_{|\mathbf{n}'|=|\mathbf{n}|} d_{\mathbf{n}'}^{(m_1,m_2)}$, where the sum is over all multiindices $\mathbf{n}'$ satisfying condition i) above.

It is worth mentioning at this point that the spectrum of the $BC_N$-type spin Calogero model with PSRO and chirality $\epsilon$ in Eq. (2.2.5) is also given by the RHS of Eq. (2.3.22), with $E_0$ replaced by [91]

$$E_{0,\mathrm{B}} = E_0 + N\beta a^2 .$$

Moreover, the intrinsic degeneracy of the energy $a|\mathbf{n}| + E_{0,\mathrm{B}}$ is given by $d_{\mathbf{n},\epsilon}^{(m_1,m_2)}$. It follows from Eq. (2.3.23) that the $D_N$ spin Hamiltonian $H^{(m_1,m_2)}$ is (up to a constant) the direct sum of two $BC_N$-type spin Calogero models of opposite chiralities with PSRO. Using Eqs. (2.3.22) and (2.3.25), the canonical partition function of the

$D_N$-type spin Calogero model with PSRO can be written as

$$Z^{(m_1,m_2)}(aT) = q^{E_0/a} \sum_{\mathbf{n}} d_{\mathbf{n}}^{(m_1,m_2)} q^{|\mathbf{n}|} , \qquad q \equiv \mathrm{e}^{-1/(k_{\mathrm{B}}T)} , \tag{2.3.26}$$

where the sum ranges over all multi-indices $\mathbf{n}$ satisfying condition i) above. Similarly, the partition functions of the corresponding $BC_N$-type models (2.2.5) are given by

$$Z_{B,\pm}^{(m_1,m_2)}(aT) = q^{(E_{0,\mathrm{B}})/a} \sum_{\mathbf{n}} d_{\mathbf{n},\pm}^{(m_1,m_2)} q^{|\mathbf{n}|} \equiv Z_{B,\mp}^{(m_2,m_1)} . \tag{2.3.27}$$

From Eq. (2.3.23) it then follows that

$$\begin{aligned} q^{-E_0/a} Z^{(m_1,m_2)}(aT) &= q^{-(E_{0,\mathrm{B}})/a} \big[ Z_{B,+}^{(m_1,m_2)}(aT) + Z_{B,-}^{(m_1,m_2)}(aT) \big] \\ &= q^{-(E_{0,\mathrm{B}})/a} \big[ Z_{B,+}^{(m_1,m_2)}(aT) + Z_{B,+}^{(m_2,m_1)}(aT) \big] . \end{aligned} \tag{2.3.28}$$

In order to apply the freezing trick formula (2.3.1), we need only recall the expression for the partition function $Z$ of the scalar Calogero model of $D_N$-type derived in Ref. [86], namely

$$q^{-E_0/a} Z(aT) = \big(1+q^N\big) \prod_i (1-q^{2i})^{-1} = q^{-(E_{0,\mathrm{B}})/a} \big(1+q^N\big) Z_{\mathrm{B}}(aT) , \tag{2.3.29}$$

where $Z_{\mathrm{B}}$ denotes the partition of the scalar Calogero model of $BC_N$ type. Dividing Eq. (2.3.28) by Eq. (2.3.29) and applying the analog of the freezing trick formula (2.3.1) for the partition function $\mathcal{Z}_{\mathrm{B},+}^{(m_1,m_2)}$ of the PF spin chain of $BC_N$ type (2.2.14) we finally obtain

$$\mathcal{Z}^{(m_1,m_2)}(q) = \big(1+q^N\big)^{-1} \Big[ \mathcal{Z}_{\mathrm{B},+}^{(m_1,m_2)}(q) + \mathcal{Z}_{\mathrm{B},+}^{(m_2,m_1)}(q) \Big] , \tag{2.3.30}$$

where from now on we shall use the variable $q = \mathrm{e}^{-1/(k_{\mathrm{B}}T)}$ in place of $T$. The partition function $\mathcal{Z}_{\mathrm{B},+}^{(m_1,m_2)}$ can in turn be expressed in terms of the partition function $\mathcal{Z}_{\mathrm{A},k}^{(m)}(q)$

of the su($m$) PF chain of type $A$ with $k$ spins with Hamiltonian

$$\mathcal{H}_A^{(m)} = \sum_{1\leqslant i<j\leqslant k} \frac{1+P_{ij}}{(\rho_i-\rho_j)^2}\,, \tag{2.3.31}$$

where $\rho_i$ is the $i$-th zero of the Hermite polynomial of degree $k$. Indeed, it is shown in Ref. [91] that for $m_2 > 0$ we have

$$\mathcal{Z}_{\mathrm{B},+}^{(m_1,m_2)}(q) = \sum_{k=0}^{N} q^{N-k} \begin{bmatrix} N \\ k \end{bmatrix}_{q^2} \mathcal{Z}_{\mathrm{A},k}^{(m_1)}(q^2)\, \mathcal{Z}_{\mathrm{A},N-k}^{(m_2)}(q^2) \qquad (m_2>0), \tag{2.3.32}$$

where the $q$-binomial coefficient $\left[\begin{smallmatrix} N \\ k \end{smallmatrix}\right]_{q^2}$ is defined as

$$\begin{bmatrix} N \\ k \end{bmatrix}_{q^2} = \frac{(q^2)_N}{(q^2)_k (q^2)_{N-k}}\,, \qquad (q^2)_j \equiv \prod_{i=1}^{j}(1-q^{2i})\,. \tag{2.3.33}$$

Combining Eqs. (2.3.30) and (2.3.32) we finally arrive at the following expression for the partition function of the $D_N$-type PF chain with PSRO (2.2.23) in terms of its type $A$ counterpart as

$$\mathcal{Z}^{(m_1,m_2)}(q) = \sum_{k=0}^{N} f_{N,k}(q)\, \mathcal{Z}_{\mathrm{A},k}^{(m_1)}(q^2)\, \mathcal{Z}_{\mathrm{A},N-k}^{(m_2)}(q^2) \qquad (m_2>0), \tag{2.3.34}$$

where $f_{N,k}(q)$ is given by

$$f_{N,k}(q) = \frac{q^{N-k}+q^k}{1+q^N} \begin{bmatrix} N \\ k \end{bmatrix}_{q^2}. \tag{2.3.35}$$

The case $m_2 = 0$, for which $P_i^{(m_1,0)} = 1$ and the Hamiltonian (2.2.23) reduces to the rational version of the (trigonometric) Simons–Altshuler chain [112], deserves special attention. Indeed, in this case by Eq. (2.3.19) the components of the multiindex $\mathbf{n}$ are all even (resp. odd) for the eigenfunctions $\psi^{+}_{\mathbf{n},\mathbf{s}}$ (resp. $\psi^{-}_{\mathbf{n},\mathbf{s}}$). As shown in Ref. [91],

this entails that for $m_2 = 0$ Eq. (2.3.32) should be replaced by

$$\mathcal{Z}_{\mathrm{B},+}^{(m_1,0)}(q) = \mathcal{Z}_{\mathrm{A},N}^{(m_1)}(q^2)\,. \tag{2.3.36}$$

On the other hand, since $P_i^{(0,m_2)} = -1$ we have

$$\mathcal{H}_{\mathrm{B},+}^{(0,m_2)} = \mathcal{H}_{\mathrm{B},+}^{(m_1,0)} + \sum_i \frac{2\beta}{\xi_i^2} = \mathcal{H}_{\mathrm{B},+}^{(m_1,0)} + N$$

by Eqs. (A2)-(A5) of Ref. [84]. From (2.3.36) it then follows that

$$\mathcal{Z}_{\mathrm{B},+}^{(0,m_2)}(q) = q^N \mathcal{Z}_{\mathrm{A},N}^{(m_1)}(q^2)\,, \tag{2.3.37}$$

and substituting into Eq. (2.3.30) we finally obtain

$$\mathcal{Z}^{(m_1,0)}(q) = \mathcal{Z}_{\mathrm{A},N}^{(m_1)}(q^2)\,. \tag{2.3.38}$$

Note that, as shown in Ref. [91], the RHS of the latter equation also coincides with the partition function of the $BC_N$-type chain (2.2.14) with $\varepsilon = 1$ and $m_2 = 0$. This was to be expected, as the latter model reduces to its $D_N$ counterpart (2.2.23) when $m_2 = \beta = 0$ and its spectrum does not depend on $\beta$.

As is well known, several equivalent closed-form expressions for the partition function of the $A_{k-1}$-type PF chain (2.3.31) exist in the in the literature [40, 64, 66, 113]. For instance, Polychronakos [40] showed that this function is given by

$$\mathcal{Z}_{\mathrm{A},k}^{(m)}(q) = \sum_{k_1+\cdots+k_m=k} q^{\frac{1}{2}\sum_{i=1}^{m} k_i(k_i-1)} [k_1,\ldots,k_m]_q\,, \tag{2.3.39}$$

where the $q$-multinomial coefficient $[k_1, \ldots, k_m]_q$ is defined by

$$[k_1, \ldots, k_m]_q = \frac{(q)_{k_1+\cdots+k_m}}{\prod_{i=1}^{m} (q)_{k_i}} \,.$$

Another well-known expression for the partition function $\mathcal{Z}_{\mathrm{A},k}^{(m)}$ was derived in Ref. [113], namely

$$\mathcal{Z}_{\mathrm{A},k}^{(m)}(q) = \sum_{\mathbf{f}\in\mathcal{P}_k} d_m(\mathbf{f})\, q^{\sum_{j=1}^{r-1} \mathcal{F}_j} \prod_{j=1}^{k-r} (1 - q^{\mathcal{F}'_j}) \,. \tag{2.3.40}$$

Here $\mathcal{P}_k$ represents the set of all ordered partitions $\mathbf{f} \equiv \{f_1, f_2, \ldots, f_r\}$ of the integer $k$, $d_m(\mathbf{f}) = \prod_{i=1}^{r} \binom{m}{f_i}$, $\mathcal{F}_j = \sum_{i=1}^{j} f_i$ are the partial sums of $\mathbf{f}$, and the complementary partial sums are defined as $\{\mathcal{F}'_1, \mathcal{F}'_1, \ldots, \mathcal{F}'_{k-r}\} \equiv \{1, 2, \ldots, k\} \backslash \{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_r\}$. A related expression for the partition function of the chain (2.3.31) can be obtained by exploiting its connection with a one-dimensional classical vertex model consisting of $k+1$ vertices connected by $k$ intermediate bonds [64]. Any possible state for this vertex model can be represented by a path configuration given by

$$\vec{s} \equiv \{s_1, s_2, \ldots, s_k\} \,, \tag{2.3.41}$$

where $s_i \in \{1, 2, \cdots, m\}$ denotes the spin state of the $i$-th bond. The energy function associated with this spin path configuration $\vec{s}$ is defined as

$$E^{(m)}(\vec{s}) = \sum_{j=1}^{k-1} j\, \theta(s_j - s_{j+1}) \,, \tag{2.3.42}$$

where $\theta$ is Heaviside's step function, defined as

$$\theta(x) = \begin{cases} 0 \,, & \text{if } x < 0 \,, \\ 1 \,, & \text{if } x \geqslant 0 \,. \end{cases} \tag{2.3.43}$$

Using the Yangian quantum group symmetry of the model (2.3.31), it can be shown

that its partition function coincides with that of the one-dimensional vertex model with energy function (2.3.42) (cf. [64]). Thus $\mathcal{Z}^{(m)}_{\mathrm{A},k}(q)$ can be expressed as

$$\mathcal{Z}^{(m)}_{\mathrm{A},k}(q) = \sum_{\vec{s}} q^{E^{(m)}(\vec{s})}\,, \tag{2.3.44}$$

where the sum has runs over all possible $m^k$ spin path configurations. In particular, from Eq. (2.3.44) it follows that

$$\mathcal{Z}^{(1)}_{\mathrm{A},k}(q) = q^{\frac{1}{2}k(k-1)}\,.$$

Thus the partition function (2.3.34) with $m_2 = 1$ reduces to

$$\mathcal{Z}^{(m_1,1)}(q) = \sum_{k=0}^{N} q^{(N-k)(N-k-1)} f_{N,k}(q)\, \mathcal{Z}^{(m_1)}_{\mathrm{A},k}(q^2)\,. \tag{2.3.45}$$

It is obvious from any of the expressions (2.3.39), (2.3.40) or (2.3.44) that the partition function $\mathcal{Z}^{(m)}_{\mathrm{A},k}(q)$ is a polynomial in $q$. In particular, from Eq. (2.3.38) it follows that $\mathcal{Z}^{(m_1,0)}$ is an even polynomial in $q$, and its energies are therefore even nonnegative integers. By Eq. (2.3.34), to show that the partition function of the $D_N$-type PF chain with PSRO is a polynomial in $q$ when $m_2 > 0$ it suffices to prove that the coefficients $f_{N,k}(q)$ in Eq. (2.3.35) depend polynomially on $q$. Although it is well known that the $q$-binomial coefficient $\left[\begin{smallmatrix} N \\ k \end{smallmatrix}\right]_{q^2}$ in (2.3.33) is indeed an even polynomial in $q$ of degree $2k(N-k)$ [124], it is not clear whether $f_{N,k}(q)$ is also a polynomial. In fact, we have verified that this is the case for a wide range of values of $N$ and all $k \leqslant N$. We conjecture that this is true in general, so that when $m_2 > 0$ the energies of the spin chain (2.2.23) are also nonnegative integers. Note that the latter fact also follows from the freezing trick formula (2.2.22), Eq. (2.3.22) for the spectrum of the spin dynamical model (2.2.15) and the analogous formula for the scalar $D_N$-type Calogero model in Ref. [86].

## 2.4 Statistical properties of the spectrum

A characteristic property of *all* spin chains of Haldane–Shastry type is the fact that their level density approaches a Gaussian distribution as the number of spins tends to infinity. This property has been rigorously proved for the chains of $A_{N-1}$ type and their related one-dimensional vertex models [77, 114], and has been numerically checked for the $B_N$, $BC_N$ and $D_N$ type chains with standard spin reversal operators [84–86, 88]. More recently, it has been established that the level density of the $BC_N$-type PF chain with PSRO shows a similar behaviour [91]. It is therefore of interest to ascertain whether the level density of the $D_N$-type spin chain with PSRO in Eq. (2.2.23) becomes normally distributed as the number of spins tends to infinity. In fact, Figs. 2.1 and 2.2 clearly suggest that this is actually the case. We shall restrict ourselves in the rest of this section to the case $m_2 > 0$, since for $m_2 = 0$ the spectrum of the chain (2.2.23) is twice that of an su($m_1$) PF chain of $A_{N-1}$ type (with the same degeneracies) on account of Eq. (2.3.38).

The spectrum of the spin chain (2.2.23) can be determined for any fixed $N$ by evaluating its partition function (2.3.34) with the help of, e.g., MATHEMATICA. It turns out that the most efficient way to compute the partition function $\mathcal{Z}^{(m)}_{\mathrm{A},k}$ appearing in the latter equation is using the recursion relation

$$\mathcal{Z}^{(m)}_{\mathrm{A},k}(q) = \sum_{l=1}^{\min(m,k)} \binom{m}{l} q^{k-l} \prod_{i=1}^{l-1} (1-q^{k-i}) \cdot \mathcal{Z}^{(m)}_{\mathrm{A},k-l}(q) \tag{2.4.1}$$

with the initial condition $\mathcal{Z}^{(m)}_{\mathrm{A},0}(q) = 1$ (see 2.5). In this way it is possible to evaluate the partition function $\mathcal{Z}^{(m_1,m_2)}(q)$ on a standard desktop computer for relatively high values of $N$ (of the order of 50) and, say, $m_1 + m_2 \leqslant 4$. Our computations show that the energy levels of the $D_N$-type spin chain with PSRO are always a set of consecutive integers. This result is consistent with the fact that the spectrum of

all previously studied rational spin chains of HS type is a set of consecutive integers [40, 84, 86], including the rational spin chain of $BC_N$ type with PSRO introduced in Ref. [91]. For this reason, in order to test the Gaussian character of the level density of the chain (2.2.23) as $N \to \infty$ one can compare directly its normalized level density

$$f(\mathcal{E}) = m^{-N} \sum_{i=1}^{L} d_i \, \delta(\mathcal{E} - \mathcal{E}_i) \,, \qquad m \equiv m_1 + m_2 \,, \tag{2.4.2}$$

where $\mathcal{E}_1 < \cdots < \mathcal{E}_L$ are the distinct energy levels and $d_i$ is the degeneracy of $\mathcal{E}_i$, with the Gaussian distribution

$$g(\mathcal{E}) = \frac{1}{\sqrt{2\pi}\sigma} \, \mathrm{e}^{-\frac{(\mathcal{E}-\mu)^2}{2\sigma^2}} \tag{2.4.3}$$

with parameters $\mu$ and $\sigma$ given by the mean and standard deviation of the spectrum, respectively. More precisely, the level density of the chain (2.2.23) is asymptotically Gaussian provided that

$$\frac{d_i}{m^N} \simeq g(\mathcal{E}_i) \,, \qquad N \gg 1.$$

In order to check the validity of the latter equation for any given $m_1$, $m_2$ and $N$ we need to compute the corresponding values of $\mu$ and $\sigma$. We shall next show that, as is the case with other spin chains of HS type, these parameters can be easily evaluated in closed form from their definition

$$\mu = m^{-N} \, \mathrm{tr} \, \mathcal{H}^{(m_1,m_2)}, \qquad \sigma^2 = m^{-N} \, \mathrm{tr} \left[ \left( \mathcal{H}^{(m_1,m_2)} \right)^2 \right] - \mu^2 \,. \tag{2.4.4}$$

The traces appearing in (2.4.4) can be computed in essentially the same way as for the $BC_N$-type PF chain with PSRO (2.2.14), using the traces of the spin operators

$P_{ij}$, $P_i^{(m_1,m_2)}$ and $\widetilde{P}_{ij}^{(m_1,m_2)}$ given in Ref. [91]. Proceeding in this way we obtain

$$\mu = \left(1+\frac{1}{m}\right)\sum_{i\neq j}(h_{ij}+\widetilde{h}_{ij}), \tag{2.4.5}$$

$$\sigma^2 = 2\left(1-\frac{1}{m^2}\right)\sum_{i\neq j}(h_{ij}^2+\widetilde{h}_{ij}^2)+\frac{4}{m^2}(t^2-1)\sum_{i\neq j}h_{ij}\widetilde{h}_{ij}\,, \tag{2.4.6}$$

where $t \equiv m_1 - m_2$ and

$$h_{ij} = (\xi_i - \xi_j)^{-2}\,, \qquad \widetilde{h}_{ij} = (\xi_i + \xi_j)^{-2}\,.$$

The sums in Eqs. (2.4.5)-(2.4.6) can be evaluated by taking the $\beta \to 0$ limit of the corresponding formulas in Appendix A of Ref. [84]. We thus obtain

$$\mu = \frac{1}{2}\left(1+\frac{1}{m}\right)N(N-1), \tag{2.4.7}$$

$$\sigma^2 = \frac{1}{36}\left(1-\frac{1}{m^2}\right)N(N-1)(4N+1)+\frac{1}{4m^2}N(N-1)(t^2-1)\,. \tag{2.4.8}$$

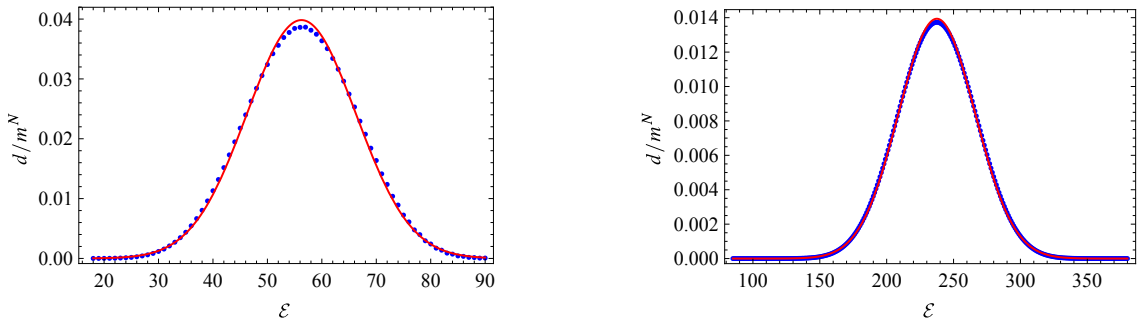


**Figure 2.3:** Left: level density of the chain (2.2.23) with $m_1 = 3$, $m_2 = 1$ and $N = 10$ (blue dots) compared to the Gaussian distribution (2.4.3) (continuous red line). Right: analogous plot for $N = 20$ spins.

We have checked that the normalized level density of the spin chain (2.2.23) is indeed in excellent agreement with the Gaussian distribution (2.4.3) for different values of $m_1$, $m_2$, and even moderately large values of $N \gtrsim 15$. As an example, in Fig. 2.3 we compare the normalized level density of the chain (2.2.23) with $m_1 = 3$,

$m_2 = 1$, for $N = 10$ and $N = 20$ spins, respectively, with the corresponding Gaussian distribution (2.4.3). It is apparent from these plots that the fit, already quite good for $N = 10$, improves significantly for $N = 20$. This is confirmed by computing the RMSE errors for both fits, which are respectively equal to $3.66\times10^{-2}$ and $2.18\times10^{-2}$. For comparison purposes, we note that this error decreases to $1.11\times10^{-2}$ for $N = 50$ spins.

Another interesting property of the spectrum of the chain (2.2.23) is connected to the distribution of the spacings between consecutive levels of the unfolded spectrum [115], which in this case is given by

$$s_i = (\eta_{i+1} - \eta_i)/\Delta, \qquad i = 1, \ldots, L-1,$$

where $\Delta = (\eta_L - \eta_1)/(L-1)$, $\eta_i = \eta(\mathcal{E}_i)$, and

$$\eta(\mathcal{E}) = \int_{-\infty}^{\mathcal{E}} g(\mathcal{E}')\mathrm{d}\mathcal{E}' = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{\mathcal{E}-\mu}{\sqrt{2}\sigma}\right)\right].$$

According to a celebrated conjecture due to Berry and Tabor [116], the distribution of these spacings for a "generic" quantum integrable system should be Poissonian, i.e., $p(s) = \mathrm{e}^{-s}$. On the other hand, a fundamental conjecture in quantum chaos due to Bohigas, Giannoni and Schmit [117] posits that the spacings distribution for a fully chaotic quantum system invariant under time reversal should follow Wigner's law

$$p(s) = (\pi s/2)\exp(-\pi s^2/4)\,,$$

characteristic of the Gaussian orthogonal ensemble in random matrix theory [118]. In fact, it has been shown that the spacings distribution of a large class of integrable spin chains of Haldane–Shastry type follows neither Poisson's nor Wigner's law [83, 84, 91, 113, 119]. More precisely, it is shown in Refs. [84, 101, 113] that the cumulative spacings density $P(s) \equiv \int_0^s p(s')\mathrm{d}s'$ of a quantum system with equispaced

energy levels and asymptotically Gaussian level density follows the "square root of a logarithm law"

$$P(s) \simeq 1 - \frac{2}{\sqrt{\pi} s_{\max}} \sqrt{\log\left(\frac{s_{\max}}{s}\right)}\,, \qquad s_{\max} \equiv \frac{\mathcal{E}_L - \mathcal{E}_1}{\sqrt{2\pi}\,\sigma}\,, \tag{2.4.9}$$

provided that a few mild technical conditions are satisfied. We have just shown that the energy levels of the rational $D_N$ chain with PSRO (2.2.23) are equispaced and its level density is asymptotically Gaussian, and it can be easily checked using the formulas for $\mathcal{E}_1$ and $\mathcal{E}_L$ below that the technical assumptions in Ref. [101] are satisfied. Thus the spacings distribution of this chain is again approximately given by Eq. (2.4.9). It should be noted that for a more precise test of the validity of the Berry–Tabor conjecture one should restrict oneself to eigenspaces with well-defined quantum numbers corresponding to the main symmetries of the model. On the other hand, the fact that the spacings distribution of the whole spectrum is not Poissonian suggests that the Berry–Tabor conjecture does not hold in these eigenspaces, since the superposition of even a small number of Poissonian distributions is also Poissonian [120].

One of the characteristic properties of both the original Haldane–Shastry and the Polychronakos–Frahm spin chains of $A_{N-1}$ type is their invariance under the quantum group $Y(\mathrm{sl}(m))$. From the existence of such a large symmetry group one should expect that the spectrum of these chain exhibits a high degree of degeneracy. In fact, it is shown in Ref. [121] that the spectrum of these models is far more degenerate than that of a generic Yangian-invariant system, due to their equivalence to a vertex model of the form (2.3.42) with a very simple dispersion relation. Indeed, as shown in the latter reference, the number $\nu^{(m)}$ of distinct levels of a generic $Y(\mathrm{sl}(m))$-invariant spin system with a large number of sites $N$ behaves as $\lambda_m^N$, where $1 < \lambda_m < 2$ is the highest real root of the polynomial $\lambda^m - \lambda^{m-1} - \cdots - 1$. In contrast, $\nu^{(m)}$ grows as a polynomial in $N$ for all spin chains of HS type associated with

the $A_{N-1}$ root system. For instance, in the case of the type $A_{N-1}$ PF chain this polynomial is simply given by $\mathcal{E}_L - \mathcal{E}_1 + 1$, since its spectrum is a set of consecutive integers. From the explicit expressions for the maximum and minimum energies of this model in Ref. [113] we easily obtain

$$\nu^{(m)} = \frac{1}{2}\left(1 - \frac{1}{m}\right)N^2 + \frac{l(m-l)}{2m} + 1 \qquad \text{(PF chain)}\,, \tag{2.4.10}$$

where $l = N \bmod m$. The situation is far less clear for spin chains of HS type associated to other root systems, with either standard or polarized spin reversal operators. On the one hand, the presence of these spin reversal operators breaks su($m$) invariance, so that it is not obvious whether these models are invariant under a suitable quantum group, let alone $Y(\mathrm{sl}(m))$. On the other hand, it has been observed that the spectrum of some of these chains is also highly degenerate, which seems to indicate the presence of a large symmetry group.

In the particular case of the $D_N$-type chain with PSRO in Eq. (2.2.23), the number of distinct energy levels can again be exactly computed under the assumption (which we have numerically checked) that the spectrum consists of consecutive integers. Indeed, it suffices to evaluate the maximum and minimum energies $\mathcal{E}_{\mathrm{max}}^{(m_1,m_2)}$ and $\mathcal{E}_{\mathrm{min}}^{(m_1,m_2)}$, in terms of which the number $\nu^{(m_1,m_2)}$ of distinct energy levels is given by

$$\nu^{(m_1,m_2)} = \mathcal{E}_{\mathrm{max}}^{(m_1,m_2)} - \mathcal{E}_{\mathrm{min}}^{(m_1,m_2)} + 1.$$

In the first place, the maximum energy can be easily computed by taking into account that $P_{ij}$ and $\tilde{P}_{ij}^{(m1,m_2)}$ are self-adjoint operators whose square is the identity, so that their eigenvalues are $\pm 1$. Moreover, it is clear that a state of the form $|s, s, \ldots, s\rangle$ is a simultaneous eigenvector of all the operators $P_{ij}$ and $\tilde{P}_{ij}^{(m1,m_2)}$ with

eigenvalue 1. Hence the maximum energy of the chain (2.2.23) is given by

$$\mathcal{E}_{\rm max}^{(m_1,m_2)} = 2\sum_{i\neq j}\left[(\xi_i-\xi_j)^{-2}+(\xi_i+\xi_j)^{-2}\right] = N(N-1), \tag{2.4.11}$$

where the sum was evaluated in [86]. On the other hand, by Eq. (2.3.30) the minimum energy is given by

$$\mathcal{E}_{\rm min}^{(m_1,m_2)} = \min\left(\mathcal{E}_{\rm B,+}^{(m_1,m_2)}, \mathcal{E}_{\rm B,+}^{(m_2,m_1)}\right), \tag{2.4.12}$$

where $\mathcal{E}_{\rm B,+}^{(m_1,m_2)}$ is the minimum energy of the $BC_N$-type chain (2.2.14) with $\epsilon=+1$. The latter energy was computed in Ref. [91], with the result

$$\mathcal{E}_{\rm B,+}^{(m_1,m_2)} = (N-l)(N+l-m_1)/m + (l-m_1)\theta(l-m_1) \tag{2.4.13}$$

where $l\equiv N \bmod m$ and $\theta$ is Heaviside's function (cf. Eq. (2.3.43)). Using the above relation it is straightforward to check that if $m_1\geqslant m_2$ we have $\mathcal{E}_{\rm B,+}^{(m_1,m_2)}\leqslant\mathcal{E}_{\rm B,+}^{(m_2,m_1)}$, and therefore

$$\mathcal{E}_{\rm min}^{(m_1,m_2)} = \mathcal{E}_{\rm B,+}^{(m_1,m_2)}, \qquad m_1\geqslant m_2. \tag{2.4.14}$$

From Eqs. (2.4.11) and (2.4.14), and the assumption that the energy levels are equispaced, we finally obtain the following closed formula for the number of distinct energy levels of the $D_N$ chain (2.2.23):

$$\nu^{(m_1,m_2)} = \left(1-\frac{1}{m}\right)N^2 - \frac{m_2}{m}\,N + \frac{l(l-m_1)}{m} - (l-m_1)\theta(l-m_1) + 1\,. \tag{2.4.15}$$

Thus, it is apparent that $\nu^{(m_1,m_2)}$ is a quadratic polynomial in $N$, as is the case with the PF chain of $A_{N-1}$ type (cf. Eq. (2.4.10)). In particular, the spectrum of the chain (2.2.23) exhibits a very high degeneracy, much larger than that of a generic Yangian-invariant su($m$) spin model; see, e.g., Fig. 2.4.

The low number $\nu^{(m_1,m_2)}$ of distinct energy levels of the model (2.2.23) entails an

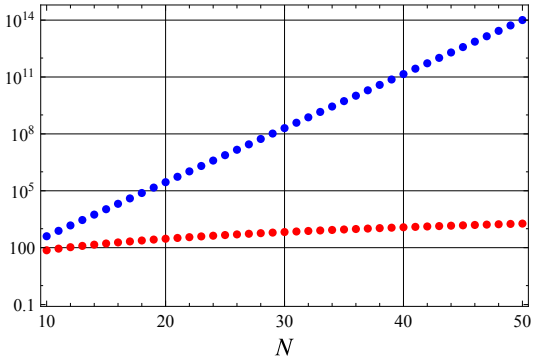


**Figure 2.4:** Logarithmic plot of the number of distinct energy levels of the rational $D_N$-type chain with PSRO (2.2.23) with $m_1 = 3$, $m_2 = 1$ (blue dots) and of a generic su(4) Yangian spin model (red dots) for $10 \leqslant N \leqslant 50$.

extremely high average degeneracy $d^{(m_1,m_2)} \equiv m^N/\nu^{(m_1,m_2)}$, which in turn suggests the existence of a large symmetry group. More precisely, it was shown in Ref [121] that the polynomial growth of the number of distinct energy levels of the spin chains of HS type associated to the $A_{N-1}$ root system is ultimately due to the equivalence of these chains to a Yangian-invariant vertex model of the form (2.3.42) with a suitable dispersion relation. This observation makes it reasonable to conjecture that the $D_N$-type spin chain with PSRO (2.2.23) is also invariant under a suitable Yangian group, and that its spectrum coincides with that of a vertex model analogous to (2.3.42) with an appropriate energy function.

## 2.5 The ferromagnetic models

We shall consider in this section the ferromagnetic counterparts of the $D_N$-type spin Calogero model with PSRO (2.2.15) and its corresponding spin chain (2.2.23), with Hamiltonians respectively given by

$$H_{\rm F}^{(m_1,m_2)} = -\sum_i \frac{\partial^2}{\partial x_i^2} + a\sum_{i\neq j}\left[\frac{a-P_{ij}}{(x_{ij}^-)^2} + \frac{a-\widetilde{P}_{ij}^{(m_1,m_2)}}{(x_{ij}^+)^2}\right] + \frac{a^2}{4}r^2 \qquad (2.5.1)$$

and

$$\mathcal{H}_{\rm F}^{(m_1,m_2)} = \sum_{i\neq j}\left[\frac{1-P_{ij}}{(\xi_i-\xi_j)^2} + \frac{1-\widetilde{P}_{ij}^{(m_1,m_2)}}{(\xi_i+\xi_j)^2}\right]. \tag{2.5.2}$$

The spectrum of the ferromagnetic spin Calogero model (2.5.1) can be studied in a similar way as its antiferromagnetic counterpart, following the procedure described in Section 2.3. To begin with, we note that the Hamiltonian (2.5.1) and the auxiliary operator (2.3.2) are related by

$$H_{\rm F}^{(m_1,m_2)} = H'\big|_{K_{ij}\to P_{ij},\, K_iK_j\to P_i^{(m_1,m_2)}P_j^{(m_1,m_2)}}. \tag{2.5.3}$$

Hence, the operator $\Lambda^{(m_1,m_2)}$ in Section 2.3 should be replaced by the projector $\Lambda_{\rm s}^{(m_1,m_2)}$ onto states *symmetric* under simultaneous exchange of the particles' spatial and spin coordinates, and with parity $+1$ under the product of an *even* number of operators $\pi_i^{(m_1,m_2)}$ (cf. (2.3.11)). The new projection operator is the sum

$$\Lambda_{\rm s}^{(m_1,m_2)} = \Lambda_{\rm B,s,+}^{(m_1,m_2)} + \Lambda_{\rm B,s,-}^{(m_1,m_2)}$$

of the symmetric analogs of the $BC_N$-type projectors in Section 2.3, determined by

$$\pi_{ij}\Lambda_{\rm B,s,\pm}^{(m_1,m_2)} = \Lambda_{\rm B,s,\pm}^{(m_1,m_2)}, \qquad \pi_i^{(m_1,m_2)}\Lambda_{\rm B,s,\pm}^{(m_1,m_2)} = \pm\Lambda_{\rm B,s,\pm}^{(m_1,m_2)}. \tag{2.5.4}$$

As explained in Section 2.3 for the antiferromagnetic case, the operator $H_{\rm F}^{(m_1,m_2)}$ is equivalent to its natural extension to the Hilbert space

$$V = V_{\rm B,s,+}\oplus V_{\rm B,s,-}, \qquad V_{\rm B,s,\pm} \equiv \Lambda_{\rm B,s,\pm}^{(m_1,m_2)}(L^2(\mathbb{R}^N)\otimes\mathcal{S}). \tag{2.5.5}$$

A set of (non-orthogonal) vectors whose linear span is dense in each of the Hilbert spaces $V_{\rm B,s,\pm}$ can be constructed in much the same way as in the antiferromagnetic case, replacing $\Lambda_{\rm B,\pm}^{(m_1,m_2)}$ by $\Lambda_{\rm B,s,\pm}^{(m_1,m_2)}$ in (2.3.18). Due to the symmetry of $\Lambda_{\rm B,s,\pm}^{(m_1,m_2)}$ under permutations, in order to obtain a basis of these Hilbert spaces we must

replace condition iii) in Section 2.3 by

iii′) $s_i \geqslant s_j$ if $n_i = n_j$ and $i < j$ .

As a result, the spectrum of the ferromagnetic model (2.5.1) is still given by Eq. (2.3.22), but the corresponding degeneracy factor $d_{\mathbf{n}}^{(m_1,m_2)}$ in (2.3.25) should be replaced by

$$d_{\mathrm{F},\mathbf{n}}^{(m_1,m_2)} = \prod_{i=1}^{s} \binom{m_1 + k_i - 1}{k_i} \prod_{j=1}^{t} \binom{m_2 + l_j - 1}{l_j} + \prod_{i=1}^{s} \binom{m_2 + k_i - 1}{k_i} \prod_{j=1}^{t} \binom{m_1 + l_j - 1}{l_j} . \quad (2.5.6)$$

Using this formula for the degeneracy factor and proceeding as in Section 2.3, we find that the partition function of the $D_N$-type ferromagnetic spin chain (2.5.2) is given by the following analog of Eq. (2.3.30):

$$\mathcal{Z}_{\mathrm{F}}^{(m_1,m_2)}(q) = \left(1 + q^N\right)^{-1} \left[\mathcal{Z}_{\mathrm{B,F,+}}^{(m_1,m_2)}(q) + \mathcal{Z}_{\mathrm{B,F,+}}^{(m_2,m_1)}(q)\right] , \quad (2.5.7)$$

where $\mathcal{Z}_{\mathrm{B,F,+}}^{(m_1,m_2)}$ denotes the partition function of the ferromagnetic counterpart of the rational $BC_N$-type chain (2.2.14). Proceeding as in Ref. [91] one can readily prove the ferromagnetic version of Eq. (2.3.32), namely

$$\mathcal{Z}_{\mathrm{B,F,+}}^{(m_1,m_2)}(q) = \sum_{k=0}^{N} q^{N-k} \begin{bmatrix} N \\ k \end{bmatrix}_{q^2} \mathcal{Z}_{\mathrm{A,F},k}^{(m_1)}(q^2)\, \mathcal{Z}_{\mathrm{A,F},N-k}^{(m_2)}(q^2) \qquad (m_2 > 0) . \quad (2.5.8)$$

Here $\mathcal{Z}_{\mathrm{A,F},k}^{(m)}$ denotes the partition function of the ferromagnetic version of the su($m$) PF chain of type $A_{N-1}$ (2.3.31) with $k$ spins, obtained replacing $P_{ij}$ by $-P_{ij}$ in the latter equation. Finally, from Eqs. (2.5.7) and (2.5.8) we immediately obtain the following explicit formula for the partition function of the ferromagnetic chain (2.5.2):

$$\mathcal{Z}_{\mathrm{F}}^{(m_1,m_2)}(q) = \sum_{k=0}^{N} f_{N,k}(q)\, \mathcal{Z}_{\mathrm{A,F},k}^{(m_1)}(q^2)\, \mathcal{Z}_{\mathrm{A,F},N-k}^{(m_2)}(q^2) \qquad (m_2 > 0) , \quad (2.5.9)$$

where $f_{N,k}(q)$ is again given by (2.3.35). For $m_2 = 0$, proceeding exactly as in Section 2.3 we obtain

$$\mathcal{Z}_{\mathrm{F}}^{(m_1,0)}(q) = \mathcal{Z}_{\mathrm{A,F},N}^{(m_1)}(q^2)\,. \tag{2.5.10}$$

Several explicit expressions for the partition function $\mathcal{Z}_{\mathrm{A,F},k}^{(m)}$ of the ferromagnetic PF chain of $A_{k-1}$ type appearing in the previous formulas are known in the literature. The first of these expressions is the analog of Eq. (2.3.39), namely

$$\mathcal{Z}_{\mathrm{A,F},k}^{(m)}(q) = \sum_{k_1+\cdots+k_m=k} [k_1,\ldots,k_m]_q\,.$$

Alternatively, $\mathcal{Z}_{\mathrm{A,F},k}^{(m)}$ may be obtained from Eq. (2.3.40) replacing $d_m(\mathbf{f})$ by its ferromagnetic version $d_{\mathrm{F},m}(\mathbf{f}) \equiv \prod_{i=1}^{r} \binom{m+f_i-1}{f_i}$. Finally, $\mathcal{Z}_{\mathrm{A,F},k}^{(m)}$ is also given by the RHS of Eq. (2.3.44) with $\theta(x)$ replaced by $1-\theta(x)$ in the definition (2.3.42) of $E^{(m)}(\vec{s})$. From any of these explicit formulas for $\mathcal{Z}_{\mathrm{A,F},k}(q)^{(m)}$, it follows that this function is a polynomial in $q$. By Eqs. (2.5.9)-(2.5.10) the same is true for the partition function of the chain (2.2.23), provided that the coefficient $f_{N,k}(q)$ is a polynomial in $q$.

As is well known, the partition functions of the $A_{N-1}$-type ferromagnetic and antiferromagnetic PF spin chains satisfy a certain duality relation [40,93,94]. In fact, a similar relation also holds for PF chains associated with other root systems [84, 86, 91]. In order to establish a duality relation between the partition functions of the ferromagnetic and antiferromagnetic spin chains of $D_N$ type with PSRO, it suffices to observe that their Hamiltonians (2.5.2) and (2.2.23) are related by

$$\mathcal{H}_{\mathrm{F}}^{(m_1,m_2)} + \mathcal{H}^{(m_1,m_2)} = 2\sum_{i\neq j}\left[(\xi_i-\xi_j)^{-2} + (\xi_i+\xi_j)^{-2}\right] = N(N-1) \tag{2.5.11}$$

(cf. Eq. (2.4.11)). This obviously implies that the eigenvalues of $\mathcal{H}_{\mathrm{F}}^{(m_1,m_2)}$ and $\mathcal{H}^{(m_1,m_2)}$ are also related by (2.5.11), so that their partition functions satisfy the

duality relation

$$\mathcal{Z}_{\rm F}^{(m_1,m_2)}(q) = q^{N(N-1)}\mathcal{Z}^{(m_1,m_2)}(q^{-1}). \tag{2.5.12}$$

## Appendix A : Recursion relation for the partition function of the PF chain of $A_{k-1}$ type

We shall provide in this Appendix a short derivation of the recursion relation (2.4.1) satisfied by the partition function $\mathcal{Z}_{{\rm A},k}^{(m)}$ of the su($m$) PF chain of $A_{k-1}$ type. The main idea behind the proof is to decompose the multiindex $\mathbf{f} \in \mathcal{P}_k$ in Eq. (2.3.40) as

$$\mathbf{f} = (f_1, \ldots, f_{r-1}, l) \equiv (\tilde{\mathbf{f}}, l)\,,$$

with $1 \leqslant l \leqslant \min(m,k)$ and $\tilde{\mathbf{f}} \in \mathcal{P}_{k-l}$. Setting $s = r-1$ we have

$$\mathcal{F}_1 + \cdots + \mathcal{F}_{r-1} = \tilde{\mathcal{F}}_1 + \cdots + \tilde{\mathcal{F}}_{s-1} + \mathcal{F}_{r-1} = \tilde{\mathcal{F}}_1 + \cdots + \tilde{\mathcal{F}}_{s-1} + k - l\,, \tag{A.1}$$

and therefore

$$\left\{\mathcal{F}'_1, \ldots, \mathcal{F}'_{k-r}\right\} = \left\{\tilde{\mathcal{F}}'_1, \ldots \tilde{\mathcal{F}}'_{k-l-s}\right\} \cup \left\{k-l+1, \ldots, k-1\right\}. \tag{A.2}$$

Substituting (A.1) and (A.2) into Eq. (2.3.40) we obtain

$$\begin{aligned}\mathcal{Z}_{{\rm A},k}^{(m)}(q) &= \sum_{l=1}^{\min(m,k)} \binom{m}{l} q^{k-l} \prod_{i=1}^{l-1}(1-q^{k-i}) \cdot \sum_{\tilde{\mathbf{f}}\in\mathcal{P}_{k-l}} \prod_{i=1}^{s} \binom{m}{\tilde{f}_i} q^{\tilde{\mathcal{F}}_1+\cdots+\tilde{\mathcal{F}}_{s-1}} \prod_{i=1}^{k-l-s}\left(1-q^{\tilde{\mathcal{F}}'_i}\right) \\ &\equiv \sum_{l=1}^{\min(m,k)} \binom{m}{l} q^{k-l} \prod_{i=1}^{l-1}(1-q^{k-i}) \cdot \mathcal{Z}_{{\rm A},k-l}^{(m)}(q)\,,\end{aligned}$$

as claimed. As to the initial condition, from Eq. (2.3.40) with $k=1$ it easily follows that $\mathcal{Z}_{{\rm A},1}^{(m)}(q) = m$. From the recursion relation (2.4.1) with $k=1$ we easily obtain $\mathcal{Z}_{{\rm A},0}^{(m)}(q) = 1$.

# CHAPTER 3

# Supersymmetric analogue of $BC_N$ type rational integrable models with polarized spin reversal operators

## 3.1 Introduction

In this Chapter, our aim is to study a class of $BC_N$ type of spin Calogero models and related PF chains, whose Hamiltonians are constructed by using SAPSRO. We have already mentioned that a class of exactly solvable spin Calogero models of $BC_N$ type and the corresponding PF chains have been introduced in Ref. [91], where the reflection operators are represented by PSRO (which is denoted by $P_i^{(m_1,m_2)}$). For the particular case $m_2 = 0$ and $m_1 = m$, $P_i^{(m_1,m_2)}$ reduces to the identity operator and leads to a novel su($m$) invariant spin chain, which is described by the Hamiltonian

$$\mathcal{H}^{(m,0)} = \sum_{1\leqslant i\neq j\leqslant N} \frac{y_i + y_j}{(y_i - y_j)^2} \left(1 - \epsilon P_{ij}^{(m)}\right), \tag{3.1.1}$$

where $\epsilon = \pm 1$, $y_i$ denotes the $i$-th zero of the generalized Laguerre polynomial $L_N^{\beta-1}$. Thus, the lattice sites of $\mathcal{H}^{(m,0)}$ implicitly depend on the real positive parameter

$\beta$. Computing the partition function of the spin chain (3.1.1) by using the freezing trick and analyzing such partition function, it has been found that the spectrum of this spin chain coincides (up to a scale factor) with that of the original PF model (1.1.19) [91].

Even though the spectrum and partition function of the supersymmetric generalization of the $A_{N-1}$ type of PF spin chain (1.1.19) have been computed earlier [93, 94], no such result is available till now for the supersymmetric generalization of the spin chain (3.1.1). In this context it is interesting to ask whether it is possible to compute the partition function for the supersymmetric version of the spin chain (3.1.1) by using the freezing trick, and whether the corresponding spectrum can be related in a simple way with that of the supersymmetric PF spin chain. In the present Chapter, we try to answer these questions by constructing SAPSRO, which would satisfy the $BC_N$ type of Weyl algebra. By using such SAPSRO, we obtain a rather large class of exactly solvable spin Calogero models and PF chains of $BC_N$ type. In a particular case where polarization is minimal, SAPSRO reduce to SASRO and lead to the spin Calogero models as well as PF chains of $BC_N$ type which have been studied earlier [101]. However, in all other cases, these SAPSRO can be used to generate novel exactly solvable spin Calogero models and PF chains of $BC_N$ type. In particular, for the case where polarization is maximal, we find that SAPSRO reduces to the trivial identity operator and lead to a supersymmetric extension of the spin chain (3.1.1), whose partition function and spectrum can be computed by using the freezing trick.

Another interesting topic which we shall address in this Chapter is a modification of the usual boson-fermion duality relation which is satisfied by the partition functions of $A_{N-1}$ type of spin chains. This type of modified duality relation has been studied earlier for the special case of $BC_N$ type of PF chains associated with SASRO [101]. It has been observed in the later reference that this duality relation not only involves

the exchange of bosonic and fermionic degrees freedom, but also certain changes of the two discrete parameters which appear in the corresponding Hamiltonian. However, the full significance for such change of the two discrete parameters has not been explored till now. We find that the underlying reason for such change of the discrete parameters can be understood in a natural way if one studies the duality relation for $BC_N$ type of PF chains in the broader context of SAPSRO. Indeed, in this Chapter we consider a new quantum number which measures the parity of the spin states under the action of SAPSRO. Curiously, it turns out that the partition functions of the spin chains now satisfy an 'extended' boson-fermion duality relation, which involves not only the exchange of bosonic and fermionic degrees of freedom, but also the exchange of positive and negative parity degrees of freedom associated with the SAPSRO.

The arrangement of this Chapter is as follows. In Section 2, we construct SAPSRO which, along with the supersymmetric spin exchange operators, lead to new representations of the $BC_N$ type of Weyl algebra and related PF spin chains with open boundary conditions. Next, in Section 3, we consider $BC_N$ type of spin Calogero models associated with SAPSRO, which in the strong coupling limit yield the above mentioned class of PF spin chains. We derive the exact spectra as well as partition functions of these $BC_N$ type of spin Calogero models with SAPSRO. By applying the freezing trick, subsequently we obtain an exact expression for the partition functions of the related PF spin chains. In Section 4, we derive a formula which expresses the partition function of any $BC_N$ type of PF spin chain with SAPSRO in terms of partition functions of several $A_K$ type of supersymmetric PF spin chains, where $K \leqslant N-1$. By taking a particular limit of the above mentioned formula, we find that the partition function of the supersymmetric extension of the spin chain (3.1.1) coincides with that of a $A_{N-1}$ type of supersymmetric PF spin chain. In Section 5, we derive an extended boson-fermion duality relation for the $BC_N$ type of PF chains with SAPSRO. In Section 6, we compute the ground state and the

highest state energies of these spin chains. Some spectral properties of these spin chains, like level density distribution and nearest neighbour spacing distribution, are studied in Section 7.

## 3.2 $BC_N$ type of Weyl algebra and related PF chains

As is well known, different representations of the $BC_N$ type of Weyl algebra play a key role in constructing exactly solvable variants of HS and PF spin chains with open boundary conditions. This $BC_N$ type of Weyl algebra is generated by the elements $\mathcal{W}_{ij}$ and $\mathcal{W}_i$ satisfying the relations

$$\mathcal{W}_{ij}^2 = \mathbb{1}\,, \quad \mathcal{W}_{ij}\mathcal{W}_{jk} = \mathcal{W}_{ik}\mathcal{W}_{ij} = \mathcal{W}_{jk}\mathcal{W}_{ik}\,, \quad \mathcal{W}_{ij}\mathcal{W}_{kl} = \mathcal{W}_{kl}\mathcal{W}_{ij}\,, \tag{3.2.1a}$$

$$\mathcal{W}_{i}^2 = \mathbb{1}\,, \quad \mathcal{W}_{i}\mathcal{W}_{j} = \mathcal{W}_{j}\mathcal{W}_{i}\,, \quad \mathcal{W}_{ij}\mathcal{W}_{k} = \mathcal{W}_{k}\mathcal{W}_{ij}\,, \quad \mathcal{W}_{ij}\mathcal{W}_{j} = \mathcal{W}_{i}\mathcal{W}_{ij}\,, \tag{3.2.1b}$$

where $i,\ j,\ k,\ l$ are all different indices. Let us assume that the Hermitian operators $\mathcal{P}_{ij}$ and $\mathcal{P}_i$ yield a realization of the elements $\mathcal{W}_{ij}$ and $\mathcal{W}_i$ respectively on an appropriate spin space. Motivated by the earlier works [84, 90, 91, 101], we define a general form of Hamiltonian for the $BC_N$ type of PF spin chain as

$$\mathcal{H} = \sum_{i\neq j}\left[\frac{1-\mathcal{P}_{ij}}{(\xi_i-\xi_j)^2} + \frac{1-\widetilde{\mathcal{P}}_{ij}}{(\xi_i+\xi_j)^2}\right] + \beta\sum_{i=1}^{N}\frac{1-\mathcal{P}_i}{\xi_i^2}\,, \tag{3.2.2}$$

where $\beta$ is a positive parameter, $\widetilde{\mathcal{P}}_{ij} = \mathcal{P}_i\mathcal{P}_j\mathcal{P}_{ij}$, $\xi_i = \sqrt{2y_i}$ and $y_i$ represents the $i$-th zero point of the generalized Laguerre polynomial $L_N^{\beta-1}$. In the following, at first we shall briefly discuss how this general form of Hamiltonian yields already known PF spin chains associated with the $BC_N$ root system for different choices of the operators $\mathcal{P}_{ij}$ and $\mathcal{P}_i$. Subsequently, we shall construct SAPSRO which, along

with the supersymmetric spin exchange operators, would lead to a new class of representations for the $BC_N$ type of Weyl algebra and the related PF chains.

In the case of a non-supersymmetric spin chain with $N$ number of lattice sites, the total internal space $\mathbf{\Sigma}^{(m)}$ is expressed as

$$\mathbf{\Sigma}^{(m)} \equiv \underbrace{\mathcal{C}_m \otimes \mathcal{C}_m \otimes \cdots \otimes \mathcal{C}_m}_{N}\,, \tag{3.2.3}$$

where $\mathcal{C}_m$ denotes a $m$-dimensional complex vector space. In terms of orthonormal basis vectors, $\mathbf{\Sigma}^{(m)}$ may be written as

$$\mathbf{\Sigma}^{(m)} = \left\langle |s_1, \cdots, s_N\rangle \;\middle|\; s_i \in \{-M, -M+1, \cdots, M\};\; M = \frac{m-1}{2} \right\rangle . \tag{3.2.4}$$

The spin exchange operator $P_{ij}^{(m)}$ and the spin reversal operator $P_i$ act on these orthonormal basis vectors as

$$P_{ij}^{(m)} |s_1\,, \cdots, s_i\,, \cdots, s_j\,, \cdots, s_N\rangle = |s_1\,, \cdots, s_j\,, \cdots, s_i\,, \cdots, s_N\rangle\,, \tag{3.2.5a}$$

$$P_i |s_1\,, \cdots, s_i\,, \cdots, s_N\rangle = |s_1\,, \cdots, -s_i\,, \cdots, s_N\rangle\,. \tag{3.2.5b}$$

It is easy to check that $\epsilon P_{ij}^{(m)}$ and $\epsilon' P_i$ (where $\epsilon, \epsilon' = \pm 1$ are two independent signs) yield a realization of the $BC_N$ type of Weyl algebra (3.2.1). Substituting $\epsilon P_{ij}^{(m)}$ and $\epsilon' P_i$ in the places of $\mathcal{P}_{ij}$ and $\mathcal{P}_i$ respectively in the general form of Hamiltonian (3.2.2), one obtains an exactly solvable $BC_N$ type of non-supersymmetric PF spin chain whose partition function has been computed by using the freezing trick [84].

For the purpose of generalizing the above mentioned spin chain through PSRO, it is convenient to define the space $\mathbf{\Sigma}^{(m)}$ through a different set of orthonormal basis vectors as

$$\mathbf{\Sigma}^{(m)} = \left\langle |s_1, \cdots, s_N\rangle \;\middle|\; s_i \in \{1, 2, \cdots, m\} \right\rangle . \tag{3.2.6}$$

The action of spin exchange operator $P_{ij}^{(m)}$ on these orthonormal basis vectors is

again given by an equation of the form (3.2.5a). However, the spin reversal operator is replaced by PSRO (denoted by $P_i^{(m_1,m_2)}$ for the $i$-th lattice site) which acts on these orthonormal basis vectors as [91]

$$P_i^{(m_1,m_2)}|s_1,\cdots,s_i,\cdots,s_N\rangle = (-1)^{f(s_i)}|s_1,\cdots,s_i,\cdots,s_N\rangle, \tag{3.2.7}$$

where

$$f(s_i) = \begin{cases} 0, & \text{if } s_i \in \{1,2,\cdots,m_1\}, \\ 1, & \text{if } s_i \in \{m_1+1,\cdots,m_1+m_2\}, \end{cases}$$

and $m_1$ and $m_2$ are two arbitrary non-negative integers satisfying the relation $m_1 + m_2 = m$. Using Eqs. (3.2.5a) and (3.2.7), it is easy to check that $\epsilon P_{ij}^{(m)}$ and $P_i^{(m_1,m_2)}$ yield a realization of $BC_N$ type of Weyl algebra (3.2.1). Substituting $\epsilon P_{ij}^{(m)}$ and $P_i^{(m_1,m_2)}$ (in places of $\mathcal{P}_{ij}$ and $\mathcal{P}_i$, respectively) in the general form of Hamiltonian (3.2.2) and taking different possible values of $m_1$ and $m_2$, one obtains a class of exactly solvable $BC_N$ type of PF spin chains with PSRO [91]. Using a similarity transform it has been shown in the latter reference that, in the special case given by $m_1 = m_2$ ($m_1 = m_2 + \epsilon'$) for even (odd) values of $m$, the operator $P_i^{(m_1,m_2)}$ becomes equivalent to $\epsilon' P_i$. Consequently, PF spin chain associated with PSRO reduces to PF spin chain associated with spin reversal operators in this special case. It may also be observed that, in another special case given by $m_1 = m,\ m_2 = 0$, $P_i^{(m_1,m_2)}$ in (3.2.7) reduces to the trivial identity operator and the corresponding Hamiltonian (3.2.2) yields the exactly solvable $su(m)$ invariant spin chain (3.1.1) which has been discussed earlier.

Next, for the purpose of discussing representations of the $BC_N$ type of Weyl algebra (3.2.1) on a superspace, we consider a set of operators like $C_{j\alpha}^\dagger(C_{j\alpha})$ which creates (annihilates) a particle of species $\alpha$ on the $j$-th lattice site. The parity of these

operators are defined as

$$\pi(C_{j\alpha}) = \pi(C^{\dagger}_{j\alpha}) = 0 \text{ for } \alpha \in [1, 2, ...., m]\,,$$
$$\pi(C_{j\alpha}) = \pi(C^{\dagger}_{j\alpha}) = 1 \text{ for } \alpha \in [m+1, m+2, ...., m+n]\,,$$

i.e, they are assumed to be bosonic when $\alpha \in [1, 2, ...., m]$ and fermionic when $\alpha \in [m+1, m+2, ...., m+n]$. These operators satisfy commutation (anti-commutation) relations given by

$$[C_{j\alpha}, C_{k\beta}]_{\pm} = 0\,,\ [C^{\dagger}_{j\alpha}, C^{\dagger}_{k\beta}]_{\pm} = 0\,,\ [C_{j\alpha}, C^{\dagger}_{k\beta}]_{\pm} = \delta_{jk}\delta_{\alpha\beta}\,, \tag{3.2.8}$$

where $[C, D]_{\pm} \equiv CD - (-1)^{\pi(C)\pi(D)}DC$. On a subspace of the corresponding Fock space, where each lattice site is occupied by only one particle (i.e., $\sum_{\alpha=1}^{m+n} C^{\dagger}_{j\alpha}C_{j\alpha} = 1$ for all $j$), the supersymmetric exchange operator is defined as

$$\hat{P}^{(m|n)}_{ij} \equiv \sum_{\alpha,\beta=1}^{m+n} C^{\dagger}_{i\alpha}C^{\dagger}_{j\beta}C_{i\beta}C_{j\alpha}\,. \tag{3.2.9}$$

This supersymmetric exchange operator can equivalently be described as an operator on a spin space in the following way. Let us assume that each lattice site of a spin chain is occupied by either one of the $m$ number of 'bosonic' spins or one of the $n$ number of 'fermionic' spins. Hence, the total internal space associated with such spin chain can be expressed as

$$\mathbf{\Sigma}^{(m|n)} \equiv \underbrace{\mathcal{C}_{m+n} \otimes \mathcal{C}_{m+n} \otimes \cdots \otimes \mathcal{C}_{m+n}}_{N}\,. \tag{3.2.10}$$

Using the notation of Ref. [101], the orthonormal basis vectors of $\mathbf{\Sigma}^{(m|n)}$ may be denoted as $|s_1, \cdots, s_N\rangle$, where $s_i \equiv (s^1_i, s^2_i)$ is a vector with two components taking

values within the range

$$s_i^1 \equiv \pi(s_i) = \begin{cases} 0, & \text{for bosons,} \\ 1, & \text{for fermions,} \end{cases} \tag{3.2.11a}$$

$$s_i^2 \in \begin{cases} \{-\frac{m-1}{2}, -\frac{m-1}{2}+1, \cdots, \frac{m-1}{2}\}, & \text{if } \pi(s_i)=0, \\ \{-\frac{n-1}{2}, -\frac{n-1}{2}+1, \cdots, \frac{n-1}{2}\}, & \text{if } \pi(s_i)=1. \end{cases} \tag{3.2.11b}$$

Thus the component $s_i^1 \equiv \pi(s_i)$ denotes the type of spin (bosonic or fermionic) and the component $s_i^2$ denotes the numerical value of the spin. A supersymmetric spin exchange operator $P_{ij}^{(m|n)}$ has been defined earlier on the space $\mathbf{\Sigma}^{(m|n)}$ as [65, 100]

$$P_{ij}^{(m|n)}|s_1, \cdots, s_i, \cdots, s_j, \cdots, s_N\rangle = (-1)^{\alpha_{ij}(\mathbf{s})}|s_1, \cdots, s_j, \cdots, s_i, \cdots, s_N\rangle, \tag{3.2.12}$$

where $\alpha_{ij}(\mathbf{s}) = \pi(s_i)\pi(s_j) + (\pi(s_i)+\pi(s_j))\ h_{ij}(\mathbf{s})$ and $h_{ij}(\mathbf{s}) = \sum_{k=i+1}^{j-1} \pi(s_k)$ denotes the number of fermions in between the $i$-th and $j$-th spins. From Eq. (3.2.12) it follows that, the exchange of two bosonic (fermionic) spins produces a phase factor of $1(-1)$. However, the exchange one bosonic spin with one fermionic spin (or, vice versa) produces a phase factor of $(-1)^{h_{ij}(\mathbf{s})}$. Using the commutation (anti-commutation) relations in (3.2.8), it can be shown that $\hat{P}_{ij}^{(m|n)}$ in (3.2.9) is completely equivalent to $P_{ij}^{(m|n)}$ in (3.2.12) [65, 100].

It may be noted that, SASRO can also be defined on the space $\mathbf{\Sigma}^{(m|n)}$ [101]. While acting on the basis vectors of $\mathbf{\Sigma}^{(m|n)}$, this SASRO (denoted by $P_i^{\epsilon\epsilon'}$) reverses the value of the $i$-th spin without affecting its type and multiplies the state by a sign factor. More precisely, the action of $P_i^{\epsilon\epsilon'}$ is given by

$$P_i^{\epsilon\epsilon'}|s_1, \cdots, s_i, \cdots, s_N\rangle = \rho(s_i)|s_1, \cdots, s_i^-, \cdots, s_N\rangle, \tag{3.2.13}$$

where $s_i^- = (s_i^1, -s_i^2)$, $\rho(s_i) = \epsilon\ (\epsilon')$ for $\pi(s_i) = 0\ (1)$, and $\epsilon, \epsilon' = \pm 1$ are two independent signs. With the help of (3.2.12) and (3.2.13), one can easily check

that $P_{ij}^{(m|n)}$ and $P_i^{\epsilon\epsilon'}$ yield a realization of the $BC_N$ type of Weyl algebra (3.2.1). Substitution of $P_{ij}^{(m|n)}$ and $P_i^{\epsilon\epsilon'}$ in Eq. (3.2.2) yields an exactly solvable Hamiltonian given by [101]

$$\mathcal{H}_{\epsilon\epsilon'}^{(m|n)} = \sum_{i\neq j}\left[\frac{1-P_{ij}^{(m|n)}}{(\xi_i-\xi_j)^2}+\frac{1-\widetilde{P}_{ij}^{(m|n)}}{(\xi_i+\xi_j)^2}\right]+\beta\sum_{i=1}^{N}\frac{1-P_i^{\epsilon\epsilon'}}{\xi_i^2}\,, \tag{3.2.14}$$

where $\widetilde{P}_{ij}^{(m|n)} = P_i^{\epsilon\epsilon'}P_j^{\epsilon\epsilon'}P_{ij}^{(m|n)}$. However, since $\mathcal{H}_{\epsilon\epsilon'}^{(m|n)}$ in the above equation does not reduce to $\mathcal{H}^{(m,0)}$ in (3.1.1) for the special case $n=0$ (and for any possible choice of $\epsilon$ and $\epsilon'$), the former Hamiltonian can not be considered as a supersymmetric extension of the later one.

At present our aim is to construct SAPSRO which would satisfy the $BC_N$ type of Weyl algebra (3.2.1). To this end, we denote the total internal space of the related spin system as $\mathbf{\Sigma}^{(m_1,m_2|n_1,n_2)}$, where $m_1,\ m_2,\ n_1,\ n_2$ are some arbitrary non-negative integers satisfying the relations $m_1+m_2=m$ and $n_1+n_2=n$. This $\mathbf{\Sigma}^{(m_1,m_2|n_1,n_2)}$ can be expressed in a direct product form exactly like (3.2.10), but each $s_i$ within the corresponding basis vectors now possess an extra quantum number associated with the action of SAPSRO. More precisely, $\mathbf{\Sigma}^{(m_1,m_2|n_1,n_2)}$ is spanned by orthonormal state vectors like $|s_1,\cdots,s_N\rangle$, where $s_i\equiv(s_i^1,s_i^2,s_i^3)$ is a vector with three components taking values within the range

$$s_i^1\equiv\pi(s_i)=\begin{cases}0, & \text{for bosons,}\\ 1, & \text{for fermions,}\end{cases} \tag{3.2.15a}$$

$$s_i^2\equiv f(s_i)=\begin{cases}0, & \text{for positive parity under SAPSRO}\\ 1, & \text{for negative parity under SAPSRO,}\end{cases} \tag{3.2.15b}$$

$$s_i^3\in\begin{cases}\{1,2,\cdots,m_1\}, & \text{if } \pi(s_i)=0 \text{ and } f(s_i)=0,\\ \{1,2,\cdots,m_2\}, & \text{if } \pi(s_i)=0 \text{ and } f(s_i)=1,\\ \{1,2,\cdots,n_1\}, & \text{if } \pi(s_i)=1 \text{ and } f(s_i)=0,\\ \{1,2,\cdots,n_2\}, & \text{if } \pi(s_i)=1 \text{ and } f(s_i)=1.\end{cases} \tag{3.2.15c}$$

Indeed, we define the action of SAPSRO (denoted by $P_i^{(m_1,m_2|n_1,n_2)}$) on these state vectors as

$$P_i^{(m_1,m_2|n_1,n_2)}|s_1,\cdots,s_i,\cdots,s_N\rangle = (-1)^{f(s_i)}|s_1,\cdots,s_i,\cdots,s_N\rangle, \tag{3.2.16}$$

which shows that $s_i^2 \equiv f(s_i)$ is determined through the parity of the spin $s_i$ under the action of SAPSRO. As before, the action of supersymmetric spin exchange operator $P_{ij}^{(m|n)}$ on the space $\mathbf{\Sigma}^{(m_1,m_2|n_1,n_2)}$ is given by an equation of the form (3.2.12), where the phase factor $\alpha_{ij}(\mathbf{s})$ depends on the first components of the spins like $s_k^1 \equiv \pi(s_k)$. Using Eqs. (3.2.12) and (3.2.16), we find that $P_{ij}^{(m|n)}$ and $P_i^{(m_1,m_2|n_1,n_2)}$ yield a realization of the $BC_N$ type of Weyl algebra (3.2.1). Substituting these operators in the general form of Hamiltonian (3.2.2), we obtain the Hamiltonian for a large class of $BC_N$ type of PF spin chains as

$$\mathcal{H}^{(m_1,m_2|n_1,n_2)} = \sum_{i\neq j}\left[\frac{1-P_{ij}^{(m|n)}}{(\xi_i-\xi_j)^2}+\frac{1-\widetilde{P}_{ij}^{(m_1,m_2|n_1,n_2)}}{(\xi_i+\xi_j)^2}\right]+\beta\sum_{i=1}^{N}\frac{1-P_i^{(m_1,m_2|n_1,n_2)}}{\xi_i^2}\,, \tag{3.2.17}$$

where $\widetilde{P}_{ij}^{(m_1,m_2|n_1,n_2)} \equiv P_i^{(m_1,m_2|n_1,n_2)}P_j^{(m_1,m_2|n_1,n_2)}P_{ij}^{(m|n)}$.

It is worth noting that the Hamiltonian (3.2.17) can reproduce all of the previously studied $BC_N$ type of PF spin chains at certain limits. For example, in the presence of only bosonic or fermionic spins, i.e., when either $n_1 = n_2 = 0$ or $m_1 = m_2 = 0$, $\mathcal{H}^{(m_1,m_2|n_1,n_2)}$ reduces to the non-supersymmetric PF spin chain associated with PSRO [91]. Next, let us assume that the discrete parameters $m_1,\ m_2,\ n_1,\ n_2$ in the Hamiltonian (3.2.17) satisfy the relations

$$m_1 = \frac{1}{2}\left(m+\epsilon\,\tilde{m}\right),\ m_2 = \frac{1}{2}\left(m-\epsilon\,\tilde{m}\right),\ n_1 = \frac{1}{2}\left(n+\epsilon'\,\tilde{n}\right),\ n_2 = \frac{1}{2}\left(n-\epsilon'\,\tilde{n}\right), \tag{3.2.18}$$

where $\epsilon,\epsilon' = \pm 1$, $\tilde{m} \equiv m \mod 2$ and $\tilde{n} \equiv n \mod 2$. One can easily check that, for these particular values of the discrete parameters, the trace of $P_i^{(m_1,m_2|n_1,n_2)}$ in

(3.2.16) would coincide with that of $P_i^{\epsilon\epsilon'}$ in (3.2.13). Furthermore, it would be possible to construct an unitary transformation which maps $P_i^{(m_1,m_2|n_1,n_2)}$ to $P_i^{\epsilon\epsilon'}$ and keeps $P_{ij}^{(m|n)}$ invariant. Consequently, for the special case given in (3.2.18), $\mathcal{H}^{(m_1,m_2|n_1,n_2)}$ in (3.2.17) becomes equivalent to the exactly solvable Hamiltonian $\mathcal{H}_{\epsilon\epsilon'}^{(m|n)}$ in (3.2.14).

Except for the two particular cases which are discussed above, the Hamiltonian in (3.2.17) represents novel class of $BC_N$ type of PF spin chains associated with SAPSRO. For example, if we choose the discrete parameters as $m_1 = m,\ m_2 = 0,\ n_1 = n,\ n_2 = 0$, then Eqs. (3.2.15c) and (3.2.16) imply that $P_i^{(m,0|n,0)} = \mathbb{1}$ and $\widetilde{P}_{ij}^{(m,0|n,0)} = P_{ij}^{(m|n)}$. Consequently, for this particular case, $\mathcal{H}^{(m_1,m_2|n_1,n_2)}$ in (3.2.17) yields a supersymmetric spin chain of the form

$$\mathcal{H}^{(m,0|n,0)} = \sum_{i\neq j} \frac{y_i + y_j}{(y_i - y_j)^2} \left(1 - P_{ij}^{(m|n)}\right) , \tag{3.2.19}$$

which has not been studied previously in the literature. It is interesting to observe that, for the special case $n = 0$, the above Hamiltonian reduces to $\mathcal{H}^{(m,0)}$ in (3.1.1) with $\epsilon = 1$. On the other hand, by putting $n = 0$ after interchanging $m$ and $n$ in (3.2.19), one easily gets $\mathcal{H}^{(m,0)}$ with $\epsilon = -1$. Therefore, the Hamiltonian $\mathcal{H}^{(m,0|n,0)}$ in (3.2.19) can be considered as a supersymmetric extension of $\mathcal{H}^{(m,0)}$ in (3.1.1).

We would like to make a comment at this point. The integrability of the Hamiltonian $\mathcal{H}^{(m_1,m_2|n_1,n_2)}$ in (3.2.17) can be established by using a procedure similar to that of Ref. [90] in the non-supersymmetric case. However, there exists an important difference between the symmetry algebra of spin chains associated with the $BC_N$ root system and that of spin chains associated with the $A_{N-1}$ root system. As is well known, the Hamiltonian (1.1.19) of the $A_{N-1}$ type of PF spin chain exhibit global su($m$) symmetry along with more general $Y(gl(m))$ Yangian quantum group symmetry [62]. Moreover, the supersymmetric extension of this $A_{N-1}$ type of PF spin exhibit global su($m|n$) supersymmetry as well as $Y(gl(m|n))$ Yangian symmetry

[94]. On the other hand, PF spin chains associated with the $BC_N$ root system do not, in general, exhibit global su($m$) symmetry or su($m|n$) supersymmetry. For example, the presently considered Hamiltonian $\mathcal{H}^{(m_1,m_2|n_1,n_2)}$ in (3.2.17), which depends on operators like $P_{ij}^{(m|n)}$ and $P_i^{(m_1,m_2|n_1,n_2)}$, does not commute with all generators of the su($m|n$) super Lie algebra for arbitrary values of the discrete parameters $m_1$, $m_2$, $n_1$ and $n_2$. This happens because, while $P_{ij}^{(m|n)}$ commutes with all generators of the su($m|n$) super Lie algebra, $P_i^{(m_1,m_2|n_1,n_2)}$ defined in (3.2.16) does not commute with those generators for arbitrary values of the discrete parameters. However, we have already mentioned that in the particular case given by $m_1 = m$, $m_2 = 0$, $n_1 = n$, $n_2 = 0$, $P_i^{(m_1,m_2|n_1,n_2)}$ reduces to the trivial identity operator. Consequently, the corresponding Hamiltonian $\mathcal{H}^{(m,0|n,0)}$ in (3.2.19) commutes with all generators of the su($m|n$) super Lie algebra.

## 3.3 Spectra and partition functions of $BC_N$ type models with SAPSRO

In the following, our aim is to compute the partition functions of the $BC_N$ type of PF spin chains (3.2.17) for all possible choice of the corresponding discrete parameters. To this end, we shall consider a class of $BC_N$ type of spin Calogero models with SAPSRO and, by using the freezing trick, show that the strong coupling limit of such spin Calogero models leads to the Hamiltonian $\mathcal{H}^{(m_1,m_2|n_1,n_2)}$ in (3.2.17). Next, we shall find out the exact spectra for the above mentioned $BC_N$ type of spin Calogero models with SAPSRO and also compute the corresponding partition functions in the strong coupling limit. Finally, by 'modding out' the contribution of the coordinate degrees of freedom from the above mentioned partition functions, we shall obtain an exact expression for the partition functions of the $BC_N$ type of PF spin chains (3.2.17).

By using SAPSRO in (3.2.16), let us define the Hamiltonian for a class of $BC_N$ type of spin Calogero models as

$$H^{(m_1,m_2|n_1,n_2)} = -\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + \frac{a^2}{4}r^2 + a\sum_{i\neq j}\left[\frac{a - P_{ij}^{(m|n)}}{(x_{ij}^-)^2} + \frac{a - \widetilde{P}_{ij}^{(m_1,m_2|n_1,n_2)}}{(x_{ij}^+)^2}\right] + \beta a\sum_{i=1}^{N}\frac{\beta a - P_i^{(m_1,m_2|n_1,n_2)}}{x_i^2} \quad (3.3.1)$$

where $a > \frac{1}{2}$, $\beta > 0$ are real coupling constants and the notations $x_{ij}^- \equiv x_i - x_j$, $x_{ij}^+ \equiv x_i + x_j$, $r^2 \equiv \sum_{i=1}^N x_i^2$ are used. It should be noted that this Hamiltonian contains both coordinate and spin degrees of freedom. Similar to the case of $BC_N$ type of spin Calogero models considered earlier [84, 90, 91, 101], the potentials of $H^{(m_1,m_2|n_1,n_2)}$ in (3.3.1) become singular in the limits $x_i \pm x_j \to 0$ and $x_i \to 0$. Therefore, the configuration space of this Hamiltonian can be taken as one of the maximal open subsets of $\mathbb{R}^N$ on which linear functionals $x_i \pm x_j$ and $x_i$ have constant signs. Let us choose this configuration space as the principal Weyl chamber of the $BC_N$ root system given by

$$C = \{\mathbf{x} \equiv (x_1, x_2, \cdots, x_N) : \ 0 < x_1 < x_2 < \ldots < x_N\}\,. \quad (3.3.2)$$

Next, we express $H^{(m_1,m_2|n_1,n_2)}$ (3.3.1) in powers of the coupling constant $a$ as

$$H^{(m_1,m_2|n_1,n_2)} = -\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + a^2\,U(\mathbf{x}) + O(a)\,, \quad (3.3.3)$$

with

$$U(\mathbf{x}) = \sum_{i\neq j}\left[\frac{1}{(x_{ij}^-)^2} + \frac{1}{(x_{ij}^+)^2}\right] + \beta^2\sum_{i=1}^{N}\frac{1}{x_i^2} + \frac{r^2}{4}. \quad (3.3.4)$$

Since the $a^2$ order term in (3.3.3) dominates in the strong coupling limit $a \to \infty$, the particles of $H^{(m_1,m_2|n_1,n_2)}$ concentrate at the coordinates $\xi_i$ of the minimum $\boldsymbol{\xi}$ of the potential $U(\mathbf{x})$ in $C$. As a result, the coordinate and spin degrees of freedom of these particles decouple from each other and the Hamiltonian $H^{(m_1,m_2|n_1,n_2)}$ in

(3.3.1) can be written in $a \to \infty$ limit as

$$H^{(m_1,m_2|n_1,n_2)} \approx H_{sc} + a\,\mathfrak{H}^{(m_1,m_2|n_1,n_2)}|_{\mathbf{x}\to\boldsymbol{\xi}}\,, \tag{3.3.5}$$

where $H_{sc}$ is the scalar (spinless) Calogero model of $BC_N$ type given by

$$H_{sc} = -\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + \frac{a^2}{4}\,r^2 + a(a-1)\sum_{i\neq j}\left[\frac{1}{(x_{ij}^-)^2} + \frac{1}{(x_{ij}^+)^2}\right] + \sum_{i=1}^{N}\frac{a\beta(a\beta-1)}{x_i^2}\,, \tag{3.3.6}$$

and

$$\mathfrak{H}^{(m_1,m_2|n_1,n_2)} = \sum_{i\neq j}\left[\frac{1-P_{ij}^{(m|n)}}{(x_i-x_j)^2} + \frac{1-\widetilde{P}_{ij}^{(m_1,m_2|n_1,n_2)}}{(x_i+x_j)^2}\right] + \beta\sum_{i=1}^{N}\frac{1-P_i^{(m_1,m_2|n_1,n_2)}}{x_i^2}\,. \tag{3.3.7}$$

The uniqueness of the unique minimum $\boldsymbol{\xi}$ of the potential $U$ (3.3.4) within the configuration space $C$ (3.3.2) has been established in Ref. [76] by expressing this potential in terms of the logarithm of the ground state wave function of the scalar Calogero model (3.3.6). The ground state wave function of this scalar Calogero model, with ground state energy

$$E_0 = Na\Big(\beta a + a(N-1) + \frac{1}{2}\Big)\,, \tag{3.3.8}$$

is given by

$$\mu(\mathbf{x}) = e^{-\frac{a}{4}r^2}\prod_i |x_i|^{\beta a}\prod_{i<j}|x_i^2 - x_j^2|^a\,. \tag{3.3.9}$$

Using the fact that the sites $\xi_i$ coincide with the coordinates of the (unique) critical point of $\log\mu(\mathbf{x})$ in $C$, one obtains a set of relations among these sites as [76, 84]

$$\sum_{\substack{j=1\\(j\neq i)}}^{N}\frac{2y_i}{y_i-y_j} = y_i - \beta\,, \tag{3.3.10}$$

where $\xi_i = \sqrt{2y_i}$ and $y_i$'s denote the zeros of the generalized Laguerre polynomial

$L_N^{\beta-1}$. Consequently, the operator $\mathfrak{H}^{(m_1,m_2|n_1,n_2)}|_{\mathbf{x}\to\boldsymbol{\xi}}$ in (3.3.5) coincides with the Hamiltonian $\mathcal{H}^{(m_1,m_2|n_1,n_2)}$ (3.2.17) of PF spin chains with SAPSRO. Furthermore, due to Eq. (3.3.5), eigenvalues of $H^{(m_1,m_2|n_1,n_2)}$ are approximately given by

$$E_{ij}^{(m_1,m_2|n_1,n_2)} \simeq E_i^{sc} + a\,\mathcal{E}_j^{(m_1,m_2|n_1,n_2)}\,, \tag{3.3.11}$$

where $E_i^{sc}$ and $\mathcal{E}_j^{(m_1,m_2|n_1,n_2)}$ are two arbitrary eigenvalues of $H_{sc}$ and $\mathcal{H}^{(m_1,m_2|n_1,n_2)}$ respectively. With the help of Eq. (3.3.11), we obtain an exact formula for the partition function $\mathcal{Z}_N^{(m_1,m_2|n_1,n_2)}(T)$ of the spin chain (3.2.17) at a given temperature $T$ as

$$\mathcal{Z}_N^{(m_1,m_2|n_1,n_2)}(T) = \lim_{a\to\infty} \frac{Z_N^{(m_1,m_2|n_1,n_2)}(aT)}{Z_N(aT)}\,, \tag{3.3.12}$$

where $Z_N^{(m_1,m_2|n_1,n_2)}(T)$ represents the partition function of the $BC_N$ type of spin Calogero Hamiltonian (3.3.1) and $Z_N(T)$ represents that of the scalar model (3.3.6).

An exact expression for the partition function of the scalar model (3.3.6) has been obtained earlier as [84]

$$Z_N(aT) = \frac{q^{\frac{E_0}{a}}}{\prod_{j=1}^{N}(1-q^{2j})}\,, \tag{3.3.13}$$

where $q = e^{-1/(k_B T)}$. Therefore, for the purpose of evaluating the partition function $\mathcal{Z}_N^{(m_1,m_2|n_1,n_2)}(T)$ of the spin chain (3.2.17) by using Eq. (3.3.12), it is required to compute the spectrum and partition function of spin Calogero Hamiltonian $H^{(m_1,m_2|n_1,n_2)}$ in (3.3.1). To this end, we start with the $BC_N$ type of auxiliary operator given by [84]

$$\mathbb{H} = -\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + a\sum_{i\neq j}\left[\frac{a-K_{ij}}{(x_{ij}^-)^2} + \frac{a-\widetilde{K}_{ij}}{(x_{ij}^+)^2}\right] + \beta a\sum_{i=1}^{N}\frac{\beta a - K_i}{x_i^2} + \frac{a^2}{4}\,r^2\,, \tag{3.3.14}$$

where $K_{ij}$ and $K_i$ are coordinate permutation and sign reversing operators, defined

by

$$(K_{ij}f)(x_1,\ldots,x_i,\ldots,x_j,\ldots,x_N)=f(x_1,\ldots,x_j,\ldots,x_i,\ldots,x_N), \tag{3.3.15a}$$

$$(K_i f)(x_1,\ldots,x_i,\ldots,x_N) = f(x_1,\ldots,-x_i,\ldots,x_N)\,, \tag{3.3.15b}$$

and $\widetilde{K}_{ij} = K_i K_j K_{ij}$. As shown in the latter reference, the auxiliary operator (3.3.14) can be written as

$$\mathbb{H} = \mu(\mathbf{x})\left[-\sum_i \left(J_i\right)^2 + a\sum_i x_i\frac{\partial}{\partial x_i} + E_0\right]\mu^{-1}(\mathbf{x})\,, \tag{3.3.16}$$

where $J_i$'s are $BC_N$ type of Dunkl operators given by

$$J_i = \frac{\partial}{\partial x_i} + a\sum_{j\neq i}\left[\frac{1}{x_{ij}^-}(1-K_{ij}) + \frac{1}{x_{ij}^+}(1-\widetilde{K}_{ij})\right] + \beta a\,\frac{1}{x_i}(1-K_i)\,, \tag{3.3.17}$$

with $i \in \{1,2,\ldots,N\}$. Let us now consider a Hilbert space spanned by a set of basis vectors like

$$\phi_{\mathbf{r}}(\mathbf{x}) = \mu(\mathbf{x})\prod_i x_i^{r_i}, \tag{3.3.18}$$

with $r_i$'s being arbitrary non-negative integers, and (partially) order these basis vectors according to their total degree $|\mathbf{r}| \equiv r_1 + r_2 + \cdots + r_N$. Since the Dunkl operators (3.3.17) clearly map any monomial $\prod_i x_i^{r_i}$ into a polynomial of total degree $r_1+r_2+\cdots+r_N-1$, it follows from Eq. (3.3.16) that $\mathbb{H}$ acts as an upper triangular matrix in the aforementioned non-orthonormal basis:

$$\mathbb{H}\phi_{\mathbf{r}}(\mathbf{x}) = E_{\mathbf{r}}\phi_{\mathbf{r}}(\mathbf{x}) + \sum_{|\mathbf{r}'|<|\mathbf{r}|} c_{\mathbf{r}'\mathbf{r}}\,\phi_{\mathbf{r}'}(\mathbf{x})\,, \tag{3.3.19}$$

where

$$E_{\mathbf{r}} = a|\mathbf{r}| + E_0\,, \tag{3.3.20}$$

and the coefficients $c_{\mathbf{r}'\mathbf{r}}$ are some real constants. Hence the spectrum of $\mathbb{H}$ is given

by the diagonal entries of this upper triangular matrix, i.e., $E_{\mathbf{r}}$'s in Eq. (3.3.20), where $r_i$'s can be taken as arbitrary non-negative integers.

In the following, we shall compute the spectrum of the spin Calogero Hamiltonian $H^{(m_1,m_2|n_1,n_2)}$ from that of $\mathbb{H}$ by taking advantage of the fact that these two operators are related through formal substitutions like

$$H^{(m_1,m_2|n_1,n_2)} = \mathbb{H}|_{K_{ij}\to P_{ij},\, K_i\to P_i^{(m_1,m_2|n_1,n_2)}}\,. \tag{3.3.21}$$

Due to the impenetrable nature of the singularities of the spin Calogero Hamiltonian $H^{(m_1,m_2|n_1,n_2)}$, its Hilbert space can be taken as the space $L^2(C)\otimes\mathbf{\Sigma}^{(m_1,m_2|n_1,n_2)}$ of wave functions square integrable on the set $C$ in Eq. (3.3.2). However, any point in $\mathbb{R}^N$ not lying within the singular subset $x_i \pm x_j = 0,\ x_i = 0,\ 1 \leqslant i < j \leqslant N$, can be mapped in a unique way to a point in $C$ by an element of the $BC_N$ Weyl group [111]. Using this fact, it can be shown that $L^2(C)\otimes\mathbf{\Sigma}^{(m_1,m_2|n_1,n_2)}$ is isomorphic to the Hilbert space $\mathbb{V}$ defined as

$$\mathbb{V} \equiv \Lambda^{(m_1,m_2|n_1,n_2)}(L^2(\mathbb{R}^N)\otimes\mathbf{\Sigma}^{(m_1,m_2|n_1,n_2)})\,, \tag{3.3.22}$$

with $\Lambda^{(m_1,m_2|n_1,n_2)}$ being a projector which satisfies the relations

$$\Pi_{ij}^{(m|n)}\,\Lambda^{(m_1,m_2|n_1,n_2)} = \Lambda^{(m_1,m_2|n_1,n_2)}\,\Pi_{ij}^{(m|n)} = \Lambda^{(m_1,m_2|n_1,n_2)}, \tag{3.3.23a}$$

$$\Pi_{i}^{(m_1,m_2|n_1,n_2)}\,\Lambda^{(m_1,m_2|n_1,n_2)} = \Lambda^{(m_1,m_2|n_1,n_2)}\,\Pi_{i}^{(m_1,m_2|n_1,n_2)} = \Lambda^{(m_1,m_2|n_1,n_2)}, \tag{3.3.23b}$$

where $\Pi_{ij}^{(m|n)} \equiv K_{ij}P_{ij}^{(m|n)}$ and $\Pi_i^{(m_1,m_2|n_1,n_2)} \equiv K_iP_i^{(m_1,m_2|n_1,n_2)}$. Following the usual procedure of constructing projectors associated with the $BC_N$ type of Weyl algebra [122, 123], we obtain an expression for $\Lambda^{(m_1,m_2|n_1,n_2)}$ satisfying (3.3.23) as

$$\Lambda^{(m_1,m_2|n_1,n_2)} = \frac{1}{2^N\cdot N!}\left\{\prod_{j=1}^{N}\left(1+\Pi_j^{(m_1,m_2|n_1,n_2)}\right)\right\}\sum_{l=1}^{N!}\mathcal{P}_l\,, \tag{3.3.24}$$

where $\mathcal{P}_l$ denotes the realization of an element of the permutation group (for $N$ number of particles) through the operators $\Pi_{ij}^{(m|n)}$. For example, in the simplest $N = 2$ case, Eq. (3.3.24) yields

$$\Lambda^{(m_1,m_2|n_1,n_2)} = \frac{1}{8}\Big(1 + \Pi_1^{(m_1,m_2|n_1,n_2)}\Big)\Big(1 + \Pi_2^{(m_1,m_2|n_1,n_2)}\Big)(1 + \Pi_{12}^{(m|n)}).$$

It may be noted that $\Lambda^{(m_1,m_2|n_1,n_2)}$ in (3.3.24) commutes with the auxiliary operator in (3.3.14):

$$\big[\Lambda^{(m_1,m_2|n_1,n_2)}, \mathbb{H}\big] = 0\,. \tag{3.3.25}$$

Since $H^{(m_1,m_2|n_1,n_2)}$ is equivalent to its natural extension to the space $\mathbb{V}$ (3.3.22), with a slight abuse of notation we also denote the latter operator as $H^{(m_1,m_2|n_1,n_2)}$. Thus, by using the relations (3.3.23), we can transform Eq. (3.3.21) into an operator relation given by

$$H^{(m_1,m_2|n_1,n_2)}\Lambda^{(m_1,m_2|n_1,n_2)} = \mathbb{H}\Lambda^{(m_1,m_2|n_1,n_2)}\,. \tag{3.3.26}$$

We shall now explain how the operator relation (3.3.26) plays an important role in finding the spectrum of $H^{(m_1,m_2|n_1,n_2)}$ from that of $\mathbb{H}$. To this end, it may be noted that the Hilbert space $\mathbb{V}$ in (3.3.22) is the closure of the linear subspace spanned by the wave functions of the form

$$\psi_{\mathbf{r}}^{\mathbf{s}} \equiv \psi_{r_1,\ldots,r_i,\ldots,r_j,\ldots,r_N}^{s_1,\ldots,s_i,\ldots,s_j,\ldots,s_N} = \Lambda^{(m_1,m_2|n_1,n_2)}\,(\phi_{\mathbf{r}}(\mathbf{x})|\mathbf{s}\rangle)\,, \tag{3.3.27}$$

where $\phi_{\mathbf{r}}$ is given in (3.3.18) and $|\mathbf{s}\rangle \equiv |s_1, \cdots, s_N\rangle$ is an arbitrary basis element of the spin space $\mathbf{\Sigma}^{(m_1,m_2|n_1,n_2)}$. However, $\psi_{\mathbf{r}}^{\mathbf{s}}$'s defined in Eq. (3.3.27) do not form a set of linearly independent state vectors. Indeed, by using (3.3.23a), (3.3.15a) and an equation of the form (3.2.12) for the basis elements of $\mathbf{\Sigma}^{(m_1,m_2|n_1,n_2)}$, we find that

$\psi^{\mathbf{s}}_{\mathbf{r}}$'s satisfy the condition

$$\psi^{s_1,\ldots,s_i,\ldots,s_j,\ldots,s_N}_{r_1,\ldots,r_i,\ldots,r_j,\ldots,r_N} = (-1)^{\alpha_{ij}(\mathbf{s})}\, \psi^{s_1,\ldots,s_j,\ldots,s_i,\ldots,s_N}_{r_1,\ldots,r_j,\ldots,r_i,\ldots,r_N} \,. \tag{3.3.28}$$

Moreover, by using (3.3.23b), (3.3.15b) and (3.2.16), we obtain

$$\psi^{s_1,\ldots,s_N}_{r_1,\ldots,r_N} = (-1)^{r_i+f(s_i)}\, \psi^{s_1,\ldots,s_N}_{r_1,\ldots,r_N} \,. \tag{3.3.29}$$

Due to Eqs. (3.3.28) and (3.3.29) it follows that, $\psi^{\mathbf{s}}_{\mathbf{r}}$'s defined through Eq. (3.3.27) would be nontrivial and linearly independent if the following three conditions are imposed on the corresponding $r_i$'s and $s_i$'s.

1) An ordered form of $\mathbf{r}$, which separately arranges its even and odd components into two non-increasing sequences, i.e.,

$$\begin{aligned} \mathbf{r} \equiv (\mathbf{r_e}, \mathbf{r_o}) = (&\overbrace{2l_1, \ldots, 2l_1}^{k_1}, \ldots, \overbrace{2l_s, \ldots, 2l_s}^{k_s}, \\ &\overbrace{2p_1+1, \ldots, 2p_1+1}^{g_1}, \ldots, \overbrace{2p_t+1, \ldots, 2p_t+1}^{g_t}) \,, \end{aligned} \tag{3.3.30}$$

where $0 \leqslant s, t \leqslant N$, $l_1 > l_2 > \ldots > l_s \geqslant 0$ and $p_1 > p_2 > \ldots > p_t \geqslant 0$, is chosen as the lower index of $\psi^{\mathbf{s}}_{\mathbf{r}}$. It may be noted that, any given $\mathbf{r}$ can be brought in the ordered form (3.3.30) through an appropriate permutation of its components. Therefore, as a consequence of Eq. (3.3.28), we can choose the ordered form (3.3.30) in the lower index of independent state vectors.

2) Using Eq. (3.3.29), we find that the second component of $s_i$ corresponding to each $r_i$ is given by

$$s_i^2 \equiv f(s_i) = \begin{cases} 0, & \text{for } r_i \in \mathbf{r_e} \,, \\ 1 \,, & \text{for } r_i \in \mathbf{r_o} \,. \end{cases} \tag{3.3.31}$$

3) Let us consider the special case where $r_i = r_j$ for $i < j$. Then, due to the con-

dition 2), the second components of the corresponding spins $s_i$ and $s_j$ must have the same value. In this special case, we can further use Eq. (3.3.28) along with the definition of $\alpha_{ij}(\mathbf{s})$ which appears just after Eq. (3.2.12), and arrange the first components of $s_i$ and $s_j$ (and also their third components in some cases) associated with independent state vectors such that

$$\text{i)} \;\; \pi(s_i) \leqslant \pi(s_j) \;,$$
$$\text{ii)} \;\; s_i^3 \geqslant s_j^3 + \pi(s_j) \;, \text{ if } \pi(s_i) = \pi(s_j).$$

All linearly independent $\psi^{\mathbf{s}}_{\mathbf{r}}$'s (3.3.27), satisfying the above mentioned three conditions, may now be taken as a set of (non-orthonormal) basis vectors for the Hilbert space $\mathbb{V}$ in (3.3.22). Let us define a partial ordering among these basis vectors as: $\psi^{\mathbf{s}}_{\mathbf{r}} > \psi^{\mathbf{s}'}_{\mathbf{r}'}$, if $|\mathbf{r}| > |\mathbf{r}'|$. Applying the key relation (3.3.26) along with (3.3.27), we obtain

$$H^{(m_1,m_2|n_1,n_2)}\psi^{\mathbf{s}}_{\mathbf{r}} = \Lambda^{(m_1,m_2|n_1,n_2)}\left((\mathbb{H}\phi_{\mathbf{r}}(\mathbf{x}))\,|\mathbf{s}\rangle\right).$$

Using this equation as well as (3.3.25) and (3.3.19), we find that $H^{(m_1,m_2|n_1,n_2)}$ in (3.3.1) acts on the above mentioned partially ordered basis vectors of $\mathbb{V}$ as

$$H^{(m_1,m_2|n_1,n_2)}\,\psi^{\mathbf{s}}_{\mathbf{r}} = E^{\mathbf{s}}_{\mathbf{r}}\,\psi^{\mathbf{s}}_{\mathbf{r}} + \sum_{|\mathbf{r}'|<|\mathbf{r}|} C_{\mathbf{r}'\mathbf{r}}\,\psi^{\mathbf{s}'}_{\mathbf{r}'}\,, \tag{3.3.32}$$

where $C_{\mathbf{r}'\mathbf{r}}$'s are real constants, $\mathbf{s}'$ is a suitable permutation of $\mathbf{s}$ and

$$E^{\mathbf{s}}_{\mathbf{r}} = a|\mathbf{r}| + E_0\,. \tag{3.3.33}$$

Due to such upper triangular matrix form of $H^{(m_1,m_2|n_1,n_2)}$, all eigenvalues of this Hamiltonian are given by Eq. (3.3.33), where the quantum number $\mathbf{r}$ satisfies the condition 1) and the quantum number $\mathbf{s}$ satisfies the conditions 2) and 3). Since the RHS of Eq. (3.3.33) does not depend on the spin quantum number $\mathbf{s}$, the eigenvalue

associated with the quantum number $\mathbf{r}$ in Eq. (3.3.30) has an *intrinsic degeneracy* $d_{\mathbf{k},\mathbf{g}}^{(m_1,m_2|n_1,n_2)}$ which counts the number of all possible choice of corresponding spin degrees of freedom. Using the conditions 2) and 3), we compute this intrinsic spin degeneracy associated with the quantum number $\mathbf{r}$ as

$$d_{\mathbf{k},\mathbf{g}}^{(m_1,m_2|n_1,n_2)} = \prod_{i=1}^{s} d_{m_1,n_1}(k_i) \prod_{j=1}^{t} d_{m_2,n_2}(g_j), \tag{3.3.34}$$

where the function $d_{x,y}(\nu)$ is given by

$$d_{x,y}(\nu) = \sum_{i=0}^{\min(\nu,y)} \binom{y}{i} \binom{x+\nu-i-1}{\nu-i}. \tag{3.3.35}$$

Due to Eq. (3.3.33), the actual degeneracy of an energy $aE_1+E_0$ is evidently obtained by summing over the intrinsic degeneracy (3.3.34) for all multi-indices $\mathbf{r}$ in (3.3.30) with fixed order $E_1$. Consequently, the actual degeneracy factors for the energy levels of spin Calogero Hamiltonian $H^{(m_1,m_2|n_1,n_2)}$ in (3.3.1) would depend on the discrete parameters $m_1$, $m_2$, $n_1$ and $n_2$.

Let us now calculate the partition function for the Hamiltonian $H^{(m_1,m_2|n_1,n_2)}$. Since $|\mathbf{r}|$ corresponding to the multi-index $\mathbf{r}$ in (3.3.30) is given by $2\sum_{i=1}^{s} l_i k_i + 2\sum_{j=1}^{t} p_j g_j + \sum_{j=1}^{t} g_j$, we can express the energy eigenvalues (3.3.33) of $H^{(m_1,m_2|n_1,n_2)}$ as

$$E_{\mathbf{r}}^{\mathbf{s}} = 2a \sum_{i=1}^{s} l_i k_i + 2a \sum_{j=1}^{t} p_j g_j + a \sum_{j=1}^{t} g_j + E_0 \,. \tag{3.3.36}$$

By using Eq. (3.3.30), we obtain the numbers of the even and the odd components of $\mathbf{r}$ (denoted by $N_1$ and $N_2$ respectively) as

$$N_1 = \sum_{i=1}^{s} k_i, \qquad N_2 = \sum_{j=1}^{t} g_j \,,$$

which satisfy the condition $N_1+N_2 = N$. Hence, we can write $\mathbf{k} \equiv \{k_1, k_2, \ldots, k_s\} \in$

$\mathcal{P}_{N_1}$ and $\mathbf{g} \equiv \{g_1, g_2, \ldots, g_t\} \in \mathcal{P}_{N_2}$, where $\mathcal{P}_{N_1}$ and $\mathcal{P}_{N_2}$ denote the sets of all ordered partitions of $N_1$ and $N_2$ respectively. Next, we compute the sum over the Boltzmann weights corresponding to all $\mathbf{r}$'s of the form (3.3.30) with energy eigenvalues (3.3.36) and intrinsic degeneracy factors (3.3.34). Thus, we obtain the canonical partition function for the $BC_N$ type of spin Calogero model (3.3.1) with SAPSRO as

$$Z_N^{(m_1,m_2|n_1,n_2)}(aT) = q^{\frac{E_0}{a}} \sum_{\substack{N_1,N_2 \\ (N_1+N_2=N)}} \sum_{\mathbf{k}\in\mathcal{P}_{N_1},\mathbf{g}\in\mathcal{P}_{N_2}} d_{\mathbf{k},\mathbf{g}}^{(m_1,m_2|n_1,n_2)} \times \sum_{l_1>\cdots>l_s\geqslant 0} \sum_{p_1>\cdots>p_t\geqslant 0} q^{2\sum_{i=1}^{s} l_i k_i + 2\sum_{j=1}^{t} p_j g_j + N_2}. \quad (3.3.37)$$

It may be noted that, the summations over $l_i$'s and $p_j$'s appearing in the above equation can be performed through appropriate change of variables [84]. As a result, we get a simpler expression for $Z_N^{(m_1,m_2|n_1,n_2)}(aT)$ in (3.3.37) as

$$Z_N^{(m_1,m_2|n_1,n_2)}(aT) = q^{\frac{E_0}{a}} \sum_{\substack{N_1,N_2 \\ (N_1+N_2=N)}} \sum_{\mathbf{k}\in\mathcal{P}_{N_1},\mathbf{g}\in\mathcal{P}_{N_2}} d_{\mathbf{k},\mathbf{g}}^{(m_1,m_2|n_1,n_2)} q^{-(N+\kappa_s)} \times \prod_{i=1}^{s} \frac{q^{2\kappa_i}}{1-q^{2\kappa_i}} \prod_{j=1}^{t} \frac{q^{2\zeta_j}}{1-q^{2\zeta_j}}, \quad (3.3.38)$$

with $\kappa_i \equiv \sum_{l=1}^{i} k_l$ and $\zeta_j \equiv \sum_{l=1}^{j} g_l$ representing the partial sums associated with the sets $\mathbf{k}$ and $\mathbf{g}$ respectively. Inserting the expressions for $Z_N^{(m_1,m_2|n_1,n_2)}(aT)$ in (3.3.38) and $Z_N(aT)$ in (3.3.13) to the relation (3.3.12), we derive the partition functions for the $BC_N$ type of PF spin chains with SAPSRO (3.2.17) as

$$\mathcal{Z}_N^{(m_1,m_2|n_1,n_2)}(q) = \prod_{l=1}^{N}(1-q^{2l}) \sum_{\substack{N_1,N_2 \\ (N_1+N_2=N)}} \sum_{\mathbf{k}\in\mathcal{P}_{N_1},\mathbf{g}\in\mathcal{P}_{N_2}} d_{\mathbf{k},\mathbf{g}}^{(m_1,m_2|n_1,n_2)}\, q^{-(N+\kappa_s)} \times \prod_{i=1}^{s} \frac{q^{2\kappa_i}}{1-q^{2\kappa_i}} \prod_{j=1}^{t} \frac{q^{2\zeta_j}}{1-q^{2\zeta_j}}, \quad (3.3.39)$$

where from now on we shall use the variable $q = e^{-1/kT}$ instead of $T$. Let us now try

to write the above partition function as a polynomial function of $q$, which is expected for the case of any spin system with finite number of lattice sites. To this end, we define complementary sets of the two sets $\{\kappa_1, \kappa_2, \ldots, \kappa_s\}$ and $\{\zeta_1, \zeta_2, \ldots, \zeta_t\}$ as $\{\kappa'_1, \kappa'_2, \ldots, \kappa'_{N_1-s}\} \equiv \{1, 2, \ldots, N_1-1, N_1\} \backslash \{\kappa_1, \kappa_2, \ldots, \kappa_s\}$ and $\{\zeta'_1, \zeta'_2, \ldots, \zeta'_{N_2-t}\} \equiv \{1, 2, \ldots, N_2-1, N_2\} \setminus \{\zeta_1, \zeta_2, \ldots, \zeta_t\}$, respectively. Using the elements of the sets $\{\kappa_1, \kappa_2, \ldots, \kappa_s\}$ and $\{\zeta_1, \zeta_2, \ldots, \zeta_t\}$, along with the elements of their complementary sets, the partition function in (3.3.39) can be explicitly written as a polynomial in $q$ as

$$\mathcal{Z}_N^{(m_1,m_2|n_1,n_2)}(T) = \sum_{\substack{N_1,N_2 \\ (N_1+N_2=N)}} \sum_{\mathbf{k}\in\mathcal{P}_{N_1},\,\mathbf{g}\in\mathcal{P}_{N_2}} d_{\mathbf{k},\mathbf{g}}^{(m_1,m_2|n_1,n_2)} \begin{bmatrix} N \\ N_1 \end{bmatrix}_{q^2} q^{N_2+2\sum_{i=1}^{s-1}\kappa_i+2\sum_{j=1}^{t-1}\zeta_j} \times \prod_{i=1}^{N_1-s}(1-q^{2\kappa'_i}) \prod_{j=1}^{N_2-t}(1-q^{2\zeta'_j})\,. \quad (3.3.40)$$

In the above expression, $\begin{bmatrix} N \\ N_1 \end{bmatrix}_{q^2}$ denotes a $q$-binomial coefficient given by

$$\begin{bmatrix} N \\ N_1 \end{bmatrix}_{q^2} = \frac{\prod_{l=1}^{N}(1-q^{2l})}{\prod_{i=1}^{N_1}(1-q^{2i}) \prod_{j=1}^{N-N_1}(1-q^{2j})}\,,$$

which can be expressed as an even polynomial of degree $2N_1(N-N_1)$ in $q$ [124].

## 3.4 Connection with $A_K$ type of supersymmetric PF chains

In the following, our aim is to establish a connection between the partition function (3.3.40) and the partition functions of some supersymmetric PF spin chains of type $A$. To this end, we note that the Hamiltonian of the $A_{N-1}$ type of $su(m|n)$

supersymmetric PF spin chain is given by [93, 94]

$$\mathcal{H}_{\rm PF}^{(m|n)} = \sum_{1\leqslant i<j\leqslant N} \frac{1-P_{ij}^{(m|n)}}{(\rho_i-\rho_j)^2}\,. \tag{3.4.1}$$

It is evident that, for the special case $n = 0$, the above Hamiltonian reduces to $\mathcal{H}_{\rm PF}^{(m)}$ in (1.1.19) with $\epsilon = 1$. Moreover, by putting $n = 0$ after interchanging $m$ and $n$ in (3.4.1), one gets $\mathcal{H}_{\rm PF}^{(m)}$ with $\epsilon = -1$. There exists a few different but equivalent expressions for the partition function of the $su(m|n)$ supersymmetric spin chain (3.4.1) in the literature [64, 66, 93, 94]. One such expression for the partition function of the spin chain (3.4.1) is given by [66]

$$\mathcal{Z}_{(A)\,N}^{(m|n)}(q) = \sum_{\mathbf{f}\in\mathcal{P}_N} d^{(m|n)}(\mathbf{f})\, q^{\sum_{j=1}^{r-1}\mathcal{F}_j} \prod_{j=1}^{N-r}(1-q^{\mathcal{F}'_j})\,. \tag{3.4.2}$$

where $\mathbf{f} \equiv \{f_1, f_2 \cdots f_r\}$, the partial sums are given by $\mathcal{F}_j = \sum_{i=1}^{j} f_i$, and the complementary partial sums are defined as $\{\mathcal{F}'_1, \mathcal{F}'_1, \cdots, \mathcal{F}'_{N-r}\} \equiv \{1, 2, \cdots, N\} - \{\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_r\}$. Moreover, $d^{(m|n)}(\mathbf{f})$ in the above expression is defined through $d_{x,y}(\nu)$ in (**??**) as

$$d^{(m|n)}(\mathbf{f}) = \prod_{i=1}^{r} d_{m,n}(f_i)\,. \tag{3.4.3}$$

Using Eq. (3.4.3), one can express the spin degeneracy factor $d_{\mathbf{k},\mathbf{g}}^{m_1,m_2}$ in (3.3.34) as

$$d_{\mathbf{k},\mathbf{g}}^{(m_1,m_2|n_1,n_2)} = d^{(m_1|n_1)}(\mathbf{k})\, d^{(m_2|n_2)}(\mathbf{g})\,.$$

Substituting this factorised form of $d_{\mathbf{k},\mathbf{g}}^{(m_1,m_2|n_1,n_2)}$ to Eq. (3.3.40), we obtain

$$\begin{aligned}\mathcal{Z}_N^{(m_1,m_2|n_1,n_2)}(q) = \sum_{\substack{N_1,N_2\\(N_1+N_2=N)}} q^{N_2} \begin{bmatrix} N \\ N_1 \end{bmatrix}_{q^2} \left( \sum_{\mathbf{k}\in\mathcal{P}_{N_1}} d^{(m_1|n_1)}(\mathbf{k})\, q^{2\sum_{j=1}^{s-1}\kappa_j} \prod_{j=1}^{N_1-s}(1-q^{2\kappa'_j}) \right) \\ \times \left( \sum_{\mathbf{g}\in\mathcal{P}_{N_2}} d^{(m_2|n_2)}(\mathbf{g})\, q^{2\sum_{j=1}^{t-1}\zeta_j} \prod_{j=1}^{N_2-t}(1-q^{2\zeta'_j}) \right).\end{aligned} \tag{3.4.4}$$

Using the expression of $\mathcal{Z}^{(m|n)}_{(A)\,N}(q)$ in (3.4.2) for all nontrivial cases where $N \geqslant 1$ and $m+n \geqslant 1$, and also assuming that $\mathcal{Z}^{(m|n)}_{(A)\,0}(q) = 1$ and $\mathcal{Z}^{(0|0)}_{(A)\,N}(q) = \delta_{N,0}$, we finally rewrite $\mathcal{Z}^{(m_1,m_2|n_1,n_2)}_{N}(q)$ in (3.4.4) as

$$\mathcal{Z}^{(m_1,m_2|n_1,n_2)}_{N}(q) = \sum_{N_1=0}^{N} q^{N-N_1} \begin{bmatrix} N \\ N_1 \end{bmatrix}_{q^2} \mathcal{Z}^{(m_1|n_1)}_{(A)\,N_1}(q^2)\, \mathcal{Z}^{(m_2|n_2)}_{(A)\,N-N_1}(q^2)\,. \tag{3.4.5}$$

Thus we find that the partition function of the $BC_N$ type of PF spin chain with SAPSRO (3.2.17) can be expressed in an elegant way through the partition functions of several $A_K$ type of supersymmetric PF spin chains, where $K \leqslant N-1$.

We have previously mentioned that, for a particular choice of the discrete parameters given by $m_1 = m,\ m_2 = 0,\ n_1 = n,\ n_2 = 0$, $\mathcal{H}^{(m_1,m_2|n_1,n_2)}$ in (3.2.17) reduces to $\mathcal{H}^{(m,0|n,0)}$ in (3.2.19). Applying Eq. (3.4.5) for this particular choice of the discrete parameters and also using $\mathcal{Z}^{(0|0)}_{(A)\,N-N_1}(q^2) = \delta_{N_1,N}$, we obtain

$$\begin{aligned} \mathcal{Z}^{(m,0|n,0)}_{N}(q) &= \sum_{N_1=0}^{N} q^{N-N_1} \begin{bmatrix} N \\ N_1 \end{bmatrix}_{q^2} \mathcal{Z}^{(m|n)}_{(A)\,N_1}(q^2)\, \mathcal{Z}^{(0|0)}_{(A)\,N-N_1}(q^2) \\ &= \mathcal{Z}^{(m|n)}_{(A)\,N}(q^2)\,. \end{aligned} \tag{3.4.6}$$

Hence, replacing $q$ by $q^2$ in the RHS of (3.4.2), it is possible to get an explicit expression for the partition function of $\mathcal{H}^{(m,0|n,0)}$ in (3.2.19). Since $\mathcal{Z}^{(m|n)}_{(A)\,N}(q)$ in (3.4.2) can be expressed as a polynomial function of $q$, Eq. (3.4.6) also implies that the spectrum of $\mathcal{H}^{(m,0|n,0)}$ would coincide with that of the following Hamiltonian $\widetilde{\mathcal{H}}^{(m|n)}_{\rm PF}$, which is obtained by multiplying $\mathcal{H}^{(m|n)}_{\rm PF}$ in (3.4.1) by a factor of two:

$$\widetilde{\mathcal{H}}^{(m|n)}_{\rm PF} = \sum_{1\leqslant i\neq j\leqslant N} \frac{1-P^{(m|n)}_{ij}}{(\rho_i-\rho_j)^2}\,. \tag{3.4.7}$$

As shown in Ref. [94], the spectrum of such $su(m|n)$ supersymmetric PF spin chain can be expressed through Haldane's motifs which characterize the irreducible representations of the $Y(gl(m|n)$ Yangian quantum group. The motif $\boldsymbol{\delta}$ for the spin chain

(3.4.7) is given by a $(N-1)$ sequence of 0's and 1's, i.e. $\boldsymbol{\delta} = (\delta_1, \delta_2, \cdots, \delta_{N-1})$, with $\delta_i \in \{0, 1\}$. In the non-supersymmetric case where the value of $n$ is taken as zero, the motifs of the spin chain (3.4.7) obey a 'selection rule' which forbids the appearance of $m$ number of consecutive 1's. On the other hand, $\delta_i$'s can freely take the values 0 or 1 for supersymmetric spin chains with $m \geqslant 1$ and $n \geqslant 1$. Consequently, it is possible to construct $2^{N-1}$ number of distinct motifs in the case of supersymmetric spin chains. All energy levels of the spin chain (3.4.7), in the supersymmetric as well as non-supersymmetric cases, can be expressed through the corresponding motifs as [94]

$$E_{\boldsymbol{\delta}} = 2 \sum_{i=1}^{N-1} j\delta_j \,. \tag{3.4.8}$$

Hence, due to Eq. (3.4.6), it follows that the spectrum of $\mathcal{H}^{(m,0|n,0)}$ in (3.2.19) is also be given by $E_{\boldsymbol{\delta}}$ in the above equation. In particular, for the supersymmetric case, the motif $\boldsymbol{\delta} = (0, 0, \cdots, 0)$ gives the ground state energy of this Hamiltonian as $\mathcal{E}_{min}^{(m,0|n,0)} = 0$ and the motif $\boldsymbol{\delta} = (1, 1, \cdots, 1)$ gives the corresponding highest state energy as $\mathcal{E}_{max}^{(m,0|n,0)} = N^2 - N$. The degeneracy of each energy level in (3.4.8) can also be computed for all possible values of $m$ and $n$, by taking appropriate limits of the supersymmetric Schur polynomials [94]. Thus it is possible to find out the full spectrum of the supersymmetric spin chain (3.2.19), by using our key result that this spectrum coincides with that of the $A_{N-1}$ type of $su(m|n)$ supersymmetric PF spin chain (3.4.7).

We have already mentioned that, the lattice sites of $\mathcal{H}^{(m,0|n,0)}$ in (3.2.19) and $\widetilde{\mathcal{H}}_{\rm PF}^{(m|n)}$ in (3.4.7) are determined through the zero points of the generalized Laguerre polynomial $L_N^{\beta-1}$ and the zero points of the Hermite polynomial $H_N$ respectively. Thus the lattice sites of these two Hamiltonians are quite different in nature. However, since $\mathcal{H}^{(m,0|n,0)}$ and $\widetilde{\mathcal{H}}_{\rm PF}^{(m|n)}$ share exactly same spectrum, these two Hamiltonians

must be related through a unitary transformation like

$$\mathcal{H}^{(m,0|n,0)} = \mathcal{S}_{\beta}^{(m|n)}\, \widetilde{\mathcal{H}}_{\rm PF}^{(m|n)} \left(\mathcal{S}_{\beta}^{(m|n)}\right)^{\dagger} . \tag{3.4.9}$$

Even though we do not know the explicit form of $\mathcal{S}_{\beta}^{(m|n)}$, it is possible to find out the asymptotic form of this operator at $\beta \to \infty$ limit by using the following conjecture. For any $N \geqslant 2$, let us order the zero points of the of the Hermite polynomial $H_N$ and the generalized Laguerre polynomial $L_N^{\beta-1}$ on the real line as $\rho_1 > \rho_2 > \cdots > \rho_N$ and $y_1 > y_2 > \cdots > y_N$ respectively. Then, based on numerical results, it has been conjectured that these zero points would satisfy the asymptotic relations given by [91]

$$\lim_{\beta\to\infty} \frac{y_i + y_j}{(y_i - y_j)^2} = \frac{1}{(\rho_i - \rho_j)^2} , \tag{3.4.10}$$

where $1 \leqslant i < j \leqslant N$. Using this conjecture, it is easy to see that the $\beta \to \infty$ limit of $\mathcal{H}^{(m,0|n,0)}$ in (3.2.19) yields $\widetilde{\mathcal{H}}_{\rm PF}^{(m|n)}$ in (3.4.7). Hence Eq. (3.4.9) would be satisfied in this limit if we take the asymptotic form of $\mathcal{S}_{\beta}^{(m|n)}$ as $\lim_{\beta\to\infty} \mathcal{S}_{\beta}^{(m|n)} = \mathbb{1}$.

## 3.5 Extended boson-fermion duality for $BC_N$ type of PF chains with SAPSRO

Boson-fermion duality relations involving the partition functions of various $A_{N-1}$ type of supersymmetric spin chains with long-range interaction have been established in the literature [66, 93, 94, 100]. Subsequently, a similar type of duality relation has been studied for the case of $BC_N$ type of PF spin chains associated with SASRO [101]. More precisely, it has been found in the latter reference that

$$\mathcal{Z}_{\epsilon,\epsilon'}^{(m|n)}(q) = q^{N^2}\, \mathcal{Z}_{-\epsilon',-\epsilon}^{(n|m)}(q^{-1}) , \tag{3.5.1}$$

where $\mathcal{Z}^{(m|n)}_{\epsilon,\epsilon'}(q)$ represents the partition function for the Hamiltonian $\mathcal{H}^{(m|n)}_{\epsilon,\epsilon'}$ in (3.2.14). It is evident that the duality relation (3.5.1) not only involves the exchange of bosonic and fermionic degrees freedom, but also the exchange of the two discrete parameters $\epsilon$ and $\epsilon'$ along with their sign change. For the purpose of gaining some deeper understanding for such change of the two discrete parameters, in the following we aim to study the duality relation for the case of $BC_N$ type of PF chains (3.2.17) associated with SAPSRO.

To begin with, we define the star operator $\mathcal{S}$: $\mathbf{\Sigma}^{(m_1,m_2|n_1,n_2)} \to \mathbf{\Sigma}^{(m_1,m_2|n_1,n_2)}$ as

$$\mathcal{S}|s_1, s_2, \cdots, s_N\rangle = (-1)^{\sum_{j=1}^{N} j\pi(s_j)}|s_1, s_2, \cdots, s_N\rangle\,. \tag{3.5.2}$$

It is easy to verify that $\mathcal{S}$ operator is self-adjoint and $\mathcal{S}\circ\mathcal{S}$ is the identity in $\mathbf{\Sigma}^{(m_1,m_2|n_1,n_2)}$. Next, we consider the Hilbert space $\mathbf{\Sigma}^{(n_2,n_1|m_2,m_1)}$, and denote the corresponding supersymmetric spin exchange operator and the SAPSRO as $P^{(n|m)}_{ij}$ and $P^{(n_2,n_1|m_2,m_1)}_{i}$ respectively. The Hamiltonian $\mathcal{H}^{(n_2,n_1|m_2,m_1)}$ associated with this Hilbert space is evidently obtained from $\mathcal{H}^{(m_1,m_2|n_1,n_2)}$ in (3.2.17) through the replacements: $m_1 \to n_2$, $m_2 \to n_1$, $n_1 \to m_2$ and $n_2 \to m_1$. In analogy with the basis vectors of $\mathbf{\Sigma}^{(m_1,m_2|n_1,n_2)}$ and the ranges of the corresponding spin components in (3.2.15), we assume that $\mathbf{\Sigma}^{(n_2,n_1|m_2,m_1)}$ is spanned by orthonormal state vectors like $|\bar{s}_1, \cdots, \bar{s}_N\rangle$, where the components of $\bar{s}_i \equiv (\bar{s}^1_i, \bar{s}^2_i, \bar{s}^3_i)$ are taking values within

the ranges

$$\bar{s}_i^1 \equiv \pi(\bar{s}_i) = \begin{cases} 0, & \text{for bosons,} \\ 1, & \text{for fermions,} \end{cases} \tag{3.5.3a}$$

$$\bar{s}_i^2 \equiv f(\bar{s}_i) = \begin{cases} 0, & \text{for positive parity under SAPSRO,} \\ 1, & \text{for negative parity under SAPSRO,} \end{cases} \tag{3.5.3b}$$

$$\bar{s}_i^3 \in \begin{cases} \{1, 2, \cdots, n_2\}, \text{ if } \pi(s_i) = 0 \text{ and } f(s_i) = 0, \\ \{1, 2, \cdots, n_1\}, \text{ if } \pi(s_i) = 0 \text{ and } f(s_i) = 1, \\ \{1, 2, \cdots, m_2\}, \text{ if } \pi(s_i) = 1 \text{ and } f(s_i) = 0, \\ \{1, 2, \cdots, m_1\}, \text{ if } \pi(s_i) = 1 \text{ and } f(s_i) = 1. \end{cases} \tag{3.5.3c}$$

It is evident that the spaces $\mathbf{\Sigma}^{(m_1,m_2|n_1,n_2)}$ and $\mathbf{\Sigma}^{(n_2,n_1|m_2,m_1)}$ have the same dimension given by $(m+n)^N$. Let us now define an invertible operator $\chi^{(m_1,m_2|n_1,n_2)}$: $\mathbf{\Sigma}^{(m_1,m_2|n_1,n_2)} \to \mathbf{\Sigma}^{(n_2,n_1|m_2,m_1)}$ by

$$\chi^{(m_1,m_2|n_1,n_2)} |s_1, s_2, \cdots, s_N\rangle = |\bar{s}_1, \bar{s}_2, \cdots, \bar{s}_N\rangle\,, \tag{3.5.4}$$

where

$$\bar{s}_i^1 = 1 - s_i^1,\ \bar{s}_i^2 = 1 - s_i^2,\ \bar{s}_i^3 = s_i^3.$$

From the above relation it is clear that, if $s_i$ represents a bosonic (fermionic) spin with parity $\pm 1$ under SAPSRO, then $\bar{s}_i$ would represent a fermionic (bosonic) spin with parity $\mp 1$ under SAPSRO. Using Eq. (3.5.4), it is easy to check that $\chi^{(m_1,m_2|n_1,n_2)^\dagger} = \chi^{(n_2,n_1|m_2,m_1)}$ and $\chi^{(n_2,n_1|m_2,m_1)} \circ \chi^{(m_1,m_2|n_1,n_2)}$ is the identity in $\mathbf{\Sigma}^{(m_1,m_2|n_1,n_2)}$. Subsequently, we define the operator $\mathcal{U}^{(m_1,m_2|n_1,n_2)}$: $\mathbf{\Sigma}^{(m_1,m_2|n_1,n_2)} \to \mathbf{\Sigma}^{(n_2,n_1|m_2,m_1)}$ as the composition

$$\mathcal{U}^{(m_1,m_2|n_1,n_2)} = \chi^{(m_1,m_2|n_1,n_2)} \circ \mathcal{S}. \tag{3.5.5}$$

By using the above mentioned properties of $\mathcal{S}$ and $\chi^{(m_1,m_2|n_1,n_2)}$, it is easy to show

that $\mathcal{U}^{(m_1,m_2|n_1,n_2)}$ in (3.5.5) is an unitary operator satisfying the relation

$$\mathcal{U}^{(m_1,m_2|n_1,n_2)^\dagger} = \mathcal{U}^{(m_1,m_2|n_1,n_2)^{-1}} = \mathcal{S} \circ \chi^{(n_2,n_1|m_2,m_1)}. \tag{3.5.6}$$

Using Eqs. (3.5.2) and (3.5.4), and closely following the procedure of Ref. [66] for establishing boson-fermion duality relation in the case of $A_{N-1}$ type of supersymmetric HS spin chain, it is straightforward to show that $\mathcal{U}^{(m_1,m_2|n_1,n_2)} P_{ij}^{(m|n)} = -P_{ij}^{(n|m)} \mathcal{U}^{(m_1,m_2|n_1,n_2)}$, or equivalently

$$\mathcal{U}^{(m_1,m_2|n_1,n_2)^{-1}} P_{ij}^{(n|m)} \mathcal{U}^{(m_1,m_2|n_1,n_2)} = -P_{ij}^{(m|n)}. \tag{3.5.7}$$

Next, by using Eqs. (3.2.16), (3.5.2), (3.5.4) and (3.5.5), we find that

$$\mathcal{U}^{(m_1,m_2|n_1,n_2)} P_i^{(m_1,m_2|n_1,n_2)} |s_1, \cdots, s_N\rangle = (-1)^{f(s_i)} (-1)^{\sum_{j=1}^{N} j\pi(s_j)} |\bar{s_1}, \cdots, \bar{s_N}\rangle\,, \tag{3.5.8}$$

and

$$P_i^{(n_2,n_1|m_2,m_1)} \mathcal{U}^{(m_1,m_2|n_1,n_2)} |s_1, \cdots, s_N\rangle = (-1)^{f(\bar{s}_i)} (-1)^{\sum_{j=1}^{N} j\pi(s_j)} |\bar{s_1}, \cdots, \bar{s_N}\rangle\,. \tag{3.5.9}$$

Since, due to Eqs. (3.5.4), it follows that $(-1)^{f(s_i)} = -(-1)^{f(\bar{s}_i)}$, comparing Eq. (3.5.8) with Eq. (3.5.9) we find that

$$\mathcal{U}^{(m_1,m_2|n_1,n_2)} P_i^{(m_1,m_2|n_1,n_2)} = -P_i^{(n_2,n_1|m_2,m_1)} \mathcal{U}^{(m_1,m_2|n_1,n_2)}\,,$$

or, equivalently

$$\mathcal{U}^{(m_1,m_2|n_1,n_2)^{-1}} P_i^{(n_2,n_1|m_2,m_1)} \mathcal{U}^{(m_1,m_2|n_1,n_2)} = -P_i^{(m_1,m_2|n_1,n_2)}. \tag{3.5.10}$$

With the help of Eqs. (3.2.17), (3.5.7) and (3.5.10), we obtain

$$\begin{aligned}&\mathcal{H}^{(m_1,m_2|n_1,n_2)}+\mathcal{U}^{(m_1,m_2|n_1,n_2)^{-1}}\mathcal{H}^{(n_2,n_1|m_2,m_1)}\mathcal{U}^{(m_1,m_2|n_1,n_2)}\\&=2\sum_{i\neq j}\left[(\xi_i-\xi_j)^{-2}+(\xi_i+\xi_j)^{-2}\right]+2\beta\sum_i\xi_i^{-2}=N^2,\end{aligned}\tag{3.5.11}$$

where the last sum has been derived in Ref. [84]. Since the Hamiltonians $\mathcal{H}^{(n_2,n_1|m_2,m_1)}$ and $\mathcal{U}^{(m_1,m_2|n_1,n_2)^{-1}}\mathcal{H}^{(n_2,n_1|m_2,m_1)}\mathcal{U}^{(m_1,m_2|n_1,n_2)}$ are isospectral, Eq. (3.5.11) implies that the spectra of $\mathcal{H}^{(m_1,m_2|n_1,n_2)}$ and $\mathcal{H}^{(n_2,n_1|m_2,m_1)}$ are 'dual' to each other. More precisely, the eigenvalues of $\mathcal{H}^{(m_1,m_2|n_1,n_2)}$ and $\mathcal{H}^{(n_2,n_1|m_2,m_1)}$ are related as

$$\mathcal{E}_i^{(m_1,m_2|n_1,n_2)}=N^2-\mathcal{E}_i^{(n_2,n_1|m_2,m_1)}\,.\tag{3.5.12}$$

Using the above equation, we obtain a novel type of duality relation between the partition functions of $\mathcal{H}^{(m_1,m_2|n_1,n_2)}$ and $\mathcal{H}^{(n_2,n_1|m_2,m_1)}$ as

$$\mathcal{Z}^{(m_1,m_2|n_1,n_2)}(q)=q^{N^2}\,\mathcal{Z}^{(n_2,n_1|m_2,m_1)}(q^{-1}).\tag{3.5.13}$$

It is interesting to observe that this duality relation not only involves the exchange of bosonic and fermionic degrees of freedom, but also involves the exchange of positive and negative parity degrees of freedom associated with SAPSRO. Therefore, the duality relation (3.5.13) can be interpreted as a nontrivial extension of the usual boson-fermion duality relation which holds for the case of $A_{N-1}$ type of supersymmetric spin chains. It is also interesting to note that, applying the relation (3.5.12) in the special case where $n_1=m_2$ and $n_2=m_1$, the spectrum of the Hamiltonian $\mathcal{H}^{(m_1,m_2|m_2,m_1)}$ can be shown to be invariant under $\mathcal{E}\mapsto N^2-\mathcal{E}$, i.e., to be symmetric about the mean energy $N^2/2$.

We have mentioned in Sec. 2 that, for the special values of discrete parameters appearing in (3.2.18), it is possible to construct an unitary transformation which maps

$P_i^{(m_1,m_2|n_1,n_2)}$ to $P_i^{\epsilon,\epsilon'}$ and keeps $P_{ij}^{(m|n)}$ invariant. It is interesting to observe that Eq. (3.2.18) remains invariant under the simultaneous transformations given by: $m_1 \to n_2$, $m_2 \to n_1$, $n_1 \to m_2$, $n_2 \to m_1$ and $\epsilon \to -\epsilon'$, $\epsilon' \to -\epsilon$. Hence, it is also possible to construct an unitary transformation which would map $P_i^{(n_2,n_1|m_2,m_1)}$ to $P_i^{-\epsilon',-\epsilon}$ and keep $P_{ij}^{(n|m)}$ invariant. Due to the existence of such unitary transformations in the special case (3.2.18), $\mathcal{H}^{(m_1,m_2|n_1,n_2)}$ in (3.2.17) and related $\mathcal{H}^{(n_2,n_1|m_2,m_1)}$ become equivalent to the Hamiltonians $\mathcal{H}_{\epsilon,\epsilon'}^{(m|n)}$ in (3.2.14) and related $\mathcal{H}_{-\epsilon',-\epsilon}^{(n|m)}$ respectively. Consequently, for the special values of discrete parameters given in (3.2.18), our duality relation (3.5.13) would naturally reproduce the previously obtained duality transformation (3.5.1).

Next, let us now investigate whether extended boson-fermion duality relation like (3.5.13) holds for some other quantum spin chains associated with SAPSRO. To this end, we consider a class of one dimensional spin chains with Hamiltonian of the form

$$\begin{aligned}\hat{\mathcal{H}}^{(m_1,m_2|n_1,n_2)} = \quad & \sum_{i\neq j}\Big[w_{ij}(1-P_{ij}^{(m|n)})+\tilde{w}_{ij}(1-\widetilde{P}_{ij}^{(m_1,m_2|n_1,n_2)})\Big] \\ & +\sum_i w_i\left(1-P_i^{(m_1,m_2|n_1,n_2)}\right), \end{aligned} \tag{3.5.14}$$

where $w_{ij}$, $\tilde{w}_{ij}$, $w_i$ are arbitrary real parameters. Clearly, the above Hamiltonian would represent a non-integrable system for almost all values of these parameters. Using again Eqs. (3.5.7) and (3.5.10), we find that

$$\hat{\mathcal{H}}^{(m_1,m_2|n_1,n_2)}+\mathcal{U}^{(m_1,m_2|n_1,n_2)^{-1}}\hat{\mathcal{H}}^{(n_2,n_1|m_2,m_1)}\mathcal{U}^{(m_1,m_2|n_1,n_2)} = W, \tag{3.5.15}$$

where $W = 2(\sum_{i\neq j}(w_{ij}+\tilde{w}_{ij})+\sum_i w_i)$. Using this relation and proceeding as before, we obtain a duality relation given by

$$\hat{\mathcal{Z}}^{(m_1,m_2|n_1,n_2)}(q) = q^W\,\hat{\mathcal{Z}}^{(n_2,n_1|m_2,m_1)}(q^{-1}), \tag{3.5.16}$$

where $\hat{\mathcal{Z}}^{(m_1,m_2|n_1,n_2)}(q)$ denotes the partition function of $\hat{\mathcal{H}}^{(m_1,m_2|n_1,n_2)}$. Hence, the extended boson-fermion duality relation can be applied to a wide range of spin chains of the form (3.5.14). In the following, however, we shall restrict its application only for the case of $BC_N$ type of PF chains (3.2.17) associated with SAPSRO. Indeed, in the next section, at first we shall compute the ground state energies for the spin chains (3.2.17) with the help of the freezing trick and subsequently derive the corresponding highest state energies by using this duality relation.

## 3.6 Ground state and highest state energies for PF chains with SAPSRO

It is well known that the spectra of the $A_{N-1}$ type of PF spin chain (1.1.19) and its supersymmetric generalization (3.4.1) are equispaced within the corresponding lowest and highest energy levels. This result follows from the fact that corresponding partition functions can be expressed as some polynomials in $q$, where all consecutive powers of $q$ (within the allowed range) appear with positive integer coefficients. It has been shown in Ref. [91] that spectrum for the $BC_N$ type of PF chains (3.2.17) are also equispaced in the special case where either bosonic or fermionic spins are present. Using the expression of the partition function (3.4.5) and following the arguments of the later reference, it can be shown that the spectra for the $BC_N$ type of PF chains (3.2.17) are also equispaced when both of the bosonic and fermionic spins are present, i.e., when $m, n \geqslant 1$. At present, our aim is to compute the lower and the upper limits of such equispaced spaced spectra, i.e., the ground state and the highest state energies of the Hamiltonian $\mathcal{H}^{(m_1,m_2|n_1,n_2)}$ in (3.2.17) for the cases where $m, n \geqslant 1$.

In Sec. 4 it has been shown that, for the particular choice of the discrete parameters

given by $m_1 = m,\ m_2 = 0,\ n_1 = n,\ n_2 = 0$, the spectrum of the Hamiltonian $\mathcal{H}^{(m_1,m_2|n_1,n_2)}$ coincides with that of $\widetilde{\mathcal{H}}^{(m|n)}_{\rm PF}$ in (3.4.7). By using such coincidence, we have found the ground state and the highest state energies of the Hamiltonian $\mathcal{H}^{(m,0|n,0)}$ as $\mathcal{E}^{(m,0|n,0)}_{min} = 0$ and $\mathcal{E}^{(m,0|n,0)}_{max} = N^2 - N$, respectively. The above mentioned method of calculating the ground state and the highest state energies is clearly not applicable for more general cases where $m_2$ or $n_2$ takes nontrivial value. However, by using the freezing trick, it is possible to compute the ground state energy of $\mathcal{H}^{(m_1,m_2|n_1,n_2)}$ in (3.2.17) for all cases where $m, n \geqslant 1$. To this end, we consider Eq. (3.3.11) which implies that

$$\mathcal{E}^{(m_1,m_2|n_1,n_2)}_{min} = \lim_{a\to\infty} \frac{1}{a}(E^{(m_1,m_2|n_1,n_2)}_{min} - E_0), \tag{3.6.1}$$

where $E_0$ is the known ground state energy (3.3.8) of the $BC_N$ type of scalar Calogero model and $E^{(m_1,m_2|n_1,n_2)}_{min}$ represents the ground state energy of the $BC_N$ type of spin Calogero model (3.3.1). Using Eq. (3.3.33), we can express the latter ground state energy as $E^{(m_1,m_2|n_1,n_2)}_{min} = a|\mathbf{r}|_{min} + E_0$, where $|\mathbf{r}|_{min}$ denotes the minimum value of $|\mathbf{r}|$ for all possible choice of the multi-index $\mathbf{r}$ compatible with the conditions $1) - 3)$ of Sec. 3. Substituting this expression of $E^{(m_1,m_2|n_1,n_2)}_{min}$ in Eq. (3.6.1), we find that the ground state energy of the spin chain (3.2.17) is given by

$$\mathcal{E}^{(m_1,m_2|n_1,n_2)}_{min} = |\mathbf{r}|_{min}\,. \tag{3.6.2}$$

For the purpose of finding out the explicit value of $\mathcal{E}^{(m_1,m_2|n_1,n_2)}_{min}$, in the following we divide the spin chains (3.2.17) with $m, n \geqslant 1$ into two distinct classes.

**Case I**: Here, we consider all spin chains (3.2.17) with $m_1 \geqslant 1$ and $n \geqslant 1$. In this case, there exists at least one type of bosonic spin with positive parity (under SAPSRO). From the conditions 2) and 3) of Sec. 3 it follows that, all $s_i$'s can be

filled up by this type of spin if we choose the corresponding $\mathbf{r}$ as $(0, 0, \cdots, 0)$. So, using (3.6.2) we obtain

$$\mathcal{E}_{min}^{(m_1,m_2|n_1,n_2)} = 0. \tag{3.6.3}$$

**Case II**: Let us consider all spin chains (3.2.17) with $m_1 = 0$, $m_2 \geqslant 1$ and $n \geqslant 1$. In this case, there exist $m_2$ types of bosonic spins with negative parity. Furthermore, if $n_1 > 0$, there exist $n_1$ types of fermionic spins with positive parity. Due to the condition 2) of Sec. 3, $s_i$'s can be filled up by only these $n_1$ types of spin states corresponding to $r_i = 0$. Since these are fermionic spin states, due to the condition 3) of Sec. 3, at most $n_1$ number of consecutive $r_i$'s are allowed to take the zero value. Now if $N \leqslant n_1$, then it is evident that $\mathcal{E}_{min} = 0$. For $N > n_1$, we can take $r_i = 1$ for the remaining $N - n_1$ number of positions, and fill up the corresponding $s_i$'s by any of the $m_2$ types of bosonic spins with negative parity. Consequently, we find that the configuration

$$\mathbf{r} = (\overbrace{0, \ldots, 0}^{n_1}, \overbrace{1, \ldots, 1}^{N-n_1})$$

yields $|\mathbf{r}|_{min}$ in Eq. (3.6.2). Thus for all possible spin chains with $m_1 = 0$ and $n \geqslant 1$, we obtain

$$\mathcal{E}_{min}^{(m_1,m_2|n_1,n_2)} = max\,\{N - n_1, 0\}. \tag{3.6.4}$$

It is interesting to observe that the highest eigenvalue of $\mathcal{H}^{(m_1,m_2|n_1,n_2)}$ can be determined in terms of the lowest eigenvalue of $\mathcal{H}^{(n_2,n_1|m_2,m_1)}$ by using the duality relation (3.5.12). Hence, for the purpose of computing the highest energy eigenvalues of the spin chains (3.2.17) for $m, n \geqslant 1$, it is convenient to divide these spin chains into following two distinct classes. At first, we consider all spin chains (3.2.17) with $n_1 \geqslant 1$, $n_2 = 0$ and $m \geqslant 1$. With the help of Eqs. (3.5.12) and (3.6.4), we find that the highest energy eigenvalues for this class of spin chains are given by

$$\mathcal{E}_{max}^{(m_1,m_2|n_1,n_2)} = N^2 - max\,\{N - m_2, 0\}. \tag{3.6.5}$$

Finally, we consider all spin chains (3.2.17) with $n_2 \geqslant 1$ and $m \geqslant 1$. Using Eqs. (3.5.12) and (3.6.3), we obtain the highest energy eigenvalues for this class of spin chains as

$$\mathcal{E}_{max}^{(m_1,m_2|n_1,n_2)} = N^2. \tag{3.6.6}$$

## 3.7 Some spectral properties of PF spin chains with SAPSRO

It may be noted that, with the help of symbolic software package like Mathematica, the partition function $\mathcal{Z}_N^{(m_1,m_2|n_1,n_2)}(q)$ in (3.4.5) can be explicitly written as a polynomial of $q$ for a wide range of values of the parameters $m_1$, $m_2$, $n_1$, $n_2$, and $N$. If the term $q^{\mathcal{E}_i}$ appears in such a polynomial with (positive) integer valued coefficient $c(\mathcal{E}_i)$, then $\mathcal{E}_i$ would represent an energy level with degeneracy factor or 'level density' $c(\mathcal{E}_i)$ in the corresponding spectrum. Since the sum of these degeneracy factors for the full spectrum is given by the dimension of the corresponding Hilbert space, the normalized level density $d(\mathcal{E}_i)$ is obtained through the relation $d(\mathcal{E}_i) = c(\mathcal{E}_i)/(m+n)^N$. In this way, it is possible to compute the level density distribution for the $BC_N$ type of PF chains with SAPSRO. By using such procedure, it has been found earlier that the level densities of both $A_{N-1}$ type of PF spin chain (1.1.19) and its supersymmetric extension (3.4.1) follow the Gaussian distribution with high degree of accuracy for sufficiently large number of lattice sites [113, 119]. Furthermore, the level densities of the $BC_N$ type of PF chain with usual spin reversal operator and its extension on a superspace (3.2.14) have been found to satisfy the Gaussian distribution for sufficiently large values of $N$ [84, 101]. The Gaussian behaviour of the level density distributions at $N \to \infty$ limit has also been established analytically for the case of several $A_{N-1}$ type of spin chains and related vertex models [77, 114].

In this section, at first we shall study the level density distributions of the $BC_N$ type of PF spin chains with SAPSRO (3.2.17) for the case of finite but sufficiently large number of lattice sites. However it has been mentioned earlier that, for the special case (3.2.18), $\mathcal{H}^{(m_1,m_2|n_1,n_2)}$ in (3.2.17) becomes equivalent to the previously studied Hamiltonian $\mathcal{H}^{(m|n)}_{\epsilon\epsilon'}$ in (3.2.14). We have also shown that, in another special case given by $m_1 = m,\ m_2 = 0,\ n_1 = n,\ n_2 = 0$, the spectrum of the Hamiltonian $\mathcal{H}^{(m_1,m_2|n_1,n_2)}$ coincides with that of the $A_{N-1}$ type of supersymmetric PF spin chain (3.4.7). For the purpose of excluding these two special cases for which spectral properties are already known, in the following we shall restrict our attention to the spin chains (3.2.17) where $m_1$, $m_2$, $n_1$ and $n_2$ are taken as positive integers satisfying the conditions $|m_1 - m_2| > 1$ and $|n_1 - n_2| > 1$. To begin with, let us compute the mean ($\mu$) and the variance ($\sigma$) for the spectrum of the Hamiltonian $\mathcal{H}^{(m_1,m_2|n_1,n_2)}$, which are given by the relations

$$\mu = \frac{\mathrm{tr}\left[\mathcal{H}^{(m_1,m_2|n_1,n_2)}\right]}{(m+n)^N}, \qquad \sigma^2 = \frac{\mathrm{tr}\left[(\mathcal{H}^{(m_1,m_2|n_1,n_2)})^2\right]}{(m+n)^N} - \mu^2\,. \tag{3.7.1}$$

Defining four parameters such as $\tau_1 \equiv m_1 + m_2 + n_1 + n_2$, $\tau_2 \equiv m_1 - m_2 + n_1 - n_2$, $\tau_3 \equiv m_1 + m_2 - n_1 - n_2$, and $\tau_4 \equiv m_1 - m_2 - n_1 + n_2$, and applying Eqs. (3.2.12) as well as (3.2.16), we obtain a set of trace relations given by

$$\begin{aligned}
&\mathrm{tr}\,[\mathbb{1}] = \tau_1^N,\ \ \mathrm{tr}\left[P_i^{(m_1,m_2|n_1,n_2)}\right] = \tau_2\,\tau_1^{N-1},\ \ \mathrm{tr}\,[P_{ij}] = \mathrm{tr}\left[\widetilde{P}_{ij}^{(m_1,m_2|n_1,n_2)}\right] = \tau_3\,\tau_1^{N-2},\\
&\mathrm{tr}\left[P_{ij}P_i^{(m_1,m_2|n_1,n_2)}\right] = \mathrm{tr}\left[\widetilde{P}_{ij}^{(m_1,m_2|n_1,n_2)}P_i^{(m_1,m_2|n_1,n_2)}\right] = \tau_4\,\tau_1^{N-2},\\
&\mathrm{tr}\left[P_{ij}P_k^{(m_1,m_2|n_1,n_2)}\right] = \mathrm{tr}\left[\widetilde{P}_{ij}^{(m_1,m_2|n_1,n_2)}P_k^{(m_1,m_2|n_1,n_2)}\right] = \tau_2\,\tau_3\,\tau_1^{N-3},\\
&\mathrm{tr}\,[P_{ij}P_{jl}] = \mathrm{tr}\left[P_{ij}\widetilde{P}_{jl}^{(m_1,m_2|n_1,n_2)}\right] = \mathrm{tr}\left[\widetilde{P}_{ij}^{(m_1,m_2|n_1,n_2)}\widetilde{P}_{jl}^{(m_1,m_2|n_1,n_2)}\right] = \tau_1^{N-2},\\
&\mathrm{tr}\,[P_{ij}P_{kl}] = \mathrm{tr}\left[P_{ij}\widetilde{P}_{kl}^{(m_1,m_2|n_1,n_2)}\right] = \mathrm{tr}\left[\widetilde{P}_{ij}^{(m_1,m_2|n_1,n_2)}\widetilde{P}_{kl}^{(m_1,m_2|n_1,n_2)}\right] = \tau_3^2\,\tau_1^{N-4},\\
&\mathrm{tr}\left[P_{ij}\widetilde{P}_{ij}^{(m_1,m_2|n_1,n_2)}\right] = \mathrm{tr}\left[P_i^{(m_1,m_2|n_1,n_2)}P_j^{(m_1,m_2|n_1,n_2)}\right] = \tau_2^2\,\tau_1^{N-2},
\end{aligned}$$

where it is assumed that $i, j, k, l$ are all different indices. Substituting the explicit

form of $\mathcal{H}^{(m_1,m_2|n_1,n_2)}$ in (3.2.17) to Eq. (3.7.1) and using the aforementioned trace formulae, we get

$$\mu = \left(1 - \frac{\tau_3}{\tau_1^2}\right) \sum_{i\neq j} (h_{ij} + \widetilde{h}_{ij}) + \left(1 - \frac{\tau_2}{\tau_1}\right) \sum_{i=1}^{N} h_i \,, \tag{3.7.2}$$

and

$$\begin{aligned}\sigma^2 = {} & 2\left(1 - \frac{\tau_3^2}{\tau_1^4}\right) \sum_{i\neq j} (h_{ij}^2 + \widetilde{h}_{ij}^2) + 4\left(\frac{\tau_1^2\tau_2^2 - \tau_3^2}{\tau_1^4}\right) \sum_{i\neq j} h_{ij}\widetilde{h}_{ij} + \left(1 - \frac{\tau_2^2}{\tau_1^2}\right) \sum_{i=1}^{N} h_i^2 \\ & + \frac{4(\tau_1\tau_4 - \tau_2\tau_3)}{\tau_1^3} \sum_{i\neq j} (h_{ij} + \widetilde{h}_{ij}) h_i + \frac{16\,mn}{\tau_1^4} \sum_{i,j,k}{}' (h_{ij} + \widetilde{h}_{ij})(h_{jk} + \widetilde{h}_{jk}),\end{aligned} \tag{3.7.3}$$

where $h_{ij} \equiv 1/(\xi_i - \xi_j)^2$, $\widetilde{h}_{ij} \equiv 1/(\xi_i + \xi_j)^2$, $h_i \equiv \beta/\xi_i^2$, and the symbol $\sum_{i,j,k}'$ denotes summation over $i \neq j \neq k \neq i$. Using equations (3.7.2) and (3.7.3) along with the identities given by [84, 125, 126]

$$\begin{aligned}&\sum_{i\neq j} (h_{ij} + \widetilde{h}_{ij}) = \frac{N}{2}(N-1), \quad \sum_{i=1}^{N} h_i = \frac{N}{2}, \\ &\sum_{i\neq j} (h_{ij}^2 + \widetilde{h}_{ij}^2) = \frac{N(N-1)}{72(1+\beta)} \left[2\beta(2N+5) + 4N + 1\right], \\ &\sum_{i=1}^{N} h_i^2 = \frac{N(N+\beta)}{4(1+\beta)}, \quad \sum_{i\neq j} h_{ij}\widetilde{h}_{ij} = \frac{N(N-1)}{16(1+\beta)}, \quad \sum_{i\neq j} (h_{ij} + \widetilde{h}_{ij}) h_i = \frac{N}{4}(N-1), \\ &\sum_{i,j,k}{}' (h_{ij} + \widetilde{h}_{ij})(h_{jk} + \widetilde{h}_{jk}) = \frac{2}{9} N(N-1)(N-2)\,,\end{aligned} \tag{3.7.4}$$

we finally express $\mu$ and $\sigma^2$ as some functions of the discrete parameters $m_1$, $m_2$, $n_1$, $n_2$, and $N$:

$$\mu = \left(1 - \frac{\tau_3}{\tau_1^2}\right) \frac{N}{2}(N-1) + \left(1 - \frac{\tau_2}{\tau_1}\right) \frac{N}{2}, \tag{3.7.5}$$

$$\begin{aligned}\sigma^2 = {} & \frac{1}{36}\left(1 - \frac{\tau_3^2}{\tau_1^4}\right) N(4N^2 + 6N - 1) + \frac{32mn}{9\tau_1^4} N(N-1)(N-2) \\ & + \frac{(\tau_1\tau_4 - \tau_2\tau_3)}{\tau_1^3} N(N-1) + \frac{1}{4\tau_1^2}\left(\frac{\tau_3^2}{\tau_1^2} - \tau_2^2\right) N.\end{aligned} \tag{3.7.6}$$

Since the Gaussian distribution (normalized to unity) corresponding these $\mu$ and $\sigma$ is given by

$$G(\mathcal{E}) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\mathcal{E}-\mu)^2}{2\sigma^2}}, \tag{3.7.7}$$

now it is possible to easily check whether the normalized level density of the spin chain (3.2.17) satisfies the condition $d_i \simeq G(\mathcal{E}_i)$ for sufficiently large numbers of lattice sites. Indeed, by taking different sets of positive integer values of $m_1$, $m_2$, $n_1$ and $n_2$ satisfying the conditions $|m_1 - m_2| > 1$ and $|n_1 - n_2| > 1$, we find that the normalized level density of the spin chain (3.2.17) is in excellent agreement with the Gaussian distribution (3.7.7) for moderately large values of $N$ ($N \geqslant 15$). As an example, in Fig. 1 we compare the normalized level density with the Gaussian distribution for the case $m_1 = 3$, $m_2 = 1$, $n_1 = 4$, $n_2 = 1$ and $N = 20$. We also calculate the mean square error (MSE) for the above mentioned case and find it to be as low as $1.34 \times 10^{-8}$. Furthermore, this MSE reduces to $1.86 \times 10^{-10}$ when we take $N = 40$ and keep all other parameters unchanged. Thus the agreement between normalized level density of the spin chain (3.2.17) and the Gaussian distribution (3.7.7) improves with the increasing value of $N$.

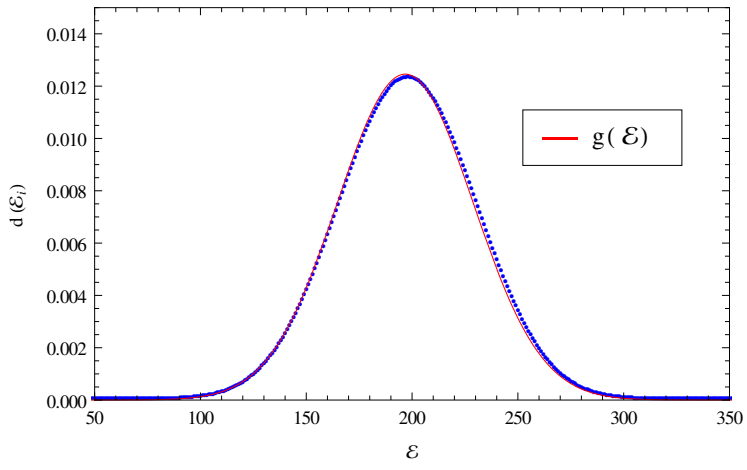


**Figure 3.1:** Continuous red curve represents the Gaussian distribution and blue dots represent the level density distribution of the spin chain (3.2.17) with $m_1 = 3$, $m_2 = 1$, $n_1 = 4$, $n_2 = 1$ and $N = 20$.

Next, we shall study the distribution of spacing between consecutive energy levels

for the spin chain (3.2.17). For the purpose of eliminating the effect of local level density variation in the distribution of spacing between energy levels, an unfolding mapping is usually employed to the 'raw' spectrum [115]. Since the level density of the spin chain (3.2.17) obeys Gaussian distribution for large number of lattice sites, one can express the corresponding cumulative level density $\eta(\mathcal{E})$ through the error function as

$$\eta(\mathcal{E}) = \int_{-\infty}^{\mathcal{E}} G(x)dx = \frac{1}{2}\left[1 + \text{erf}\left(\frac{\mathcal{E}-\mu}{\sqrt{2}\sigma}\right)\right]. \tag{3.7.8}$$

For the case of spin chain (3.2.17), this cumulative level density function is applied to map the energy levels $\mathcal{E}_i, \quad i = 1, ..., l$, into unfolded energy levels of the form $\eta_i \equiv \eta(\mathcal{E}_i)$. The cumulative level spacing distribution for such unfolded energy levels is obtained through the relation

$$P(s) = \int_0^s p(x)dx\,, \tag{3.7.9}$$

where $p(s_i)$ denotes the probability density of normalized spacing $s_i$ given by $s_i = (\eta_{i+1} - \eta_i)/\Delta$ and $\Delta = (\eta_l - \eta_1)/(l-1)$ is the mean spacing between unfolded energy levels. According to a well-known conjecture by Berry and Tabor, the density of normalized spacing for a 'generic' quantum integrable system should obey the Poisson's law given by $p(s) = e^{-s}$ [116]. However, it has been observed earlier that $p(s)$ does not exhibit this Poissonian behaviour for a large class of quantum integrable spin chains with long-range interactions [83–85, 91, 113, 119].

To explain the above mentioned anomalous behaviour in the spectra of quantum integrable spin chains with long range interactions, it has been analytically shown in Ref. [84] that if the discrete spectrum of a quantum system satisfies the following four conditions:

i) the energy levels are equispaced, i.e., $\mathcal{E}_{i+1} - \mathcal{E}_i = \delta$, for $i = 1, 2, \ldots, l-1$,

ii) the level density is approximately Gaussian,

iii) $\mathcal{E}_{max} - \mu,\ \mu - \mathcal{E}_{min} \gg \sigma$,

iv) $|\mathcal{E}_{max} + \mathcal{E}_{min} - 2\mu| \ll \mathcal{E}_{max} - \mathcal{E}_{min}$ ,

then the corresponding cumulative level spacing distribution is approximately given by

$$\tilde{P}(s) \simeq 1 - \frac{2}{\sqrt{\pi} s_{max}} \sqrt{\ln\left(\frac{s_{max}}{s}\right)}\,, \tag{3.7.10}$$

where

$$s_{max} = \frac{\mathcal{E}_{max} - \mathcal{E}_{min}}{\sqrt{2\pi}\ \sigma}\,. \tag{3.7.11}$$

Since, the spectra of many quantum integrable spin chains with long-range interactions satisfy the above mentioned four conditions with reasonable accuracy, the cumulative level density of such spin chains obey the 'square root of a logarithm' law (3.7.10). In the case of presently considered spin chain (3.2.17), it has been already found that the conditions i) and ii) are satisfied. For the purpose of analyzing the remaining conditions, we use Eqs. (3.6.3), (3.6.4), (3.6.5) and (3.6.6) to obtain $\mathcal{E}_{min} = O(N)$ and $\mathcal{E}_{max} = N^2 + O(N)$. Moreover, with the help of Eqs. (3.7.2) and (3.7.3), we find that

$$\mu = \frac{1}{2}\left(1 - \frac{\tau_3}{\tau_1^2}\right) N^2 + O(N), \qquad \sigma^2 = \frac{1}{9}\left[1 - \frac{\tau_3^2 - 32mn}{\tau_1^4}\right] N^3 + O(N^2).$$

Since $\tau_1 = m + n$ and $\tau_3 = m - n$, the leading order contributions to mean and variance in the above equation interestingly depend only on the values of $m$ and $n$. Using the leading order contributions to $\mathcal{E}_{min}$, $\mathcal{E}_{max}$, $\mu$ and $\sigma^2$, it is easy to check that the conditions iii) is also obeyed for the spectrum of the spin chain (3.2.17) with $N \gg 1$, whereas condition iv) holds only in the case when $m = n$. However, it can be shown that even if condition iv) is dropped, Eq. (3.7.10) is still obeyed within a slightly smaller range of $s$ [101]. Hence, it is expected that $P(s)$ in (3.7.9) would follow the analytical expression $\tilde{P}(s)$ in (3.7.10) for the case of spin chain (3.2.17). With the help of Mathematica, we compute $P(s)$ by taking different sets of positive integer values of $m_1$, $m_2$, $n_1$ and $n_2$ satisfying the conditions $|m_1 - m_2| > 1$ and $|n_1 - n_2| > 1$, and for moderately large values of $N$.

It turns out that $P(s)$ obeys the analytical expression (3.7.10) with remarkable accuracy in all of these cases. As an example, in Fig. 2 we compare $P(s)$ with $\tilde{P}(s)$ for the particular case $m_1 = 3$, $m_2 = 1$, $n_1 = 4$, $n_2 = 1$ and $N = 20$.

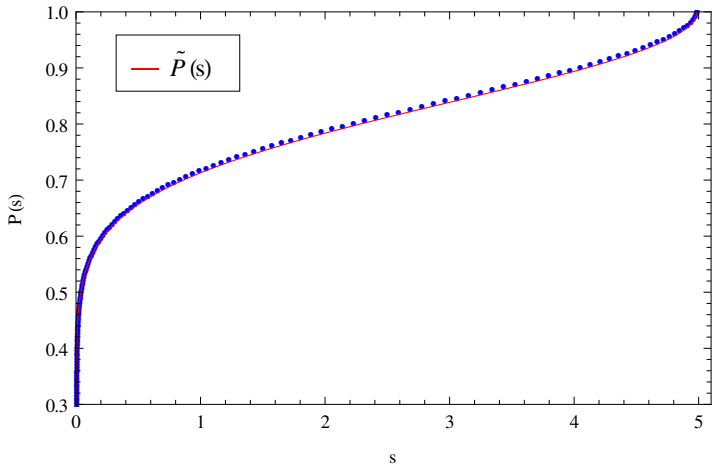


**Figure 3.2:** Blue dots represent cumulative level spacing distribution $P(s)$ for the spin chain with $m_1 = 3$, $m_2 = 1$, $n_1 = 4$, $n_2 = 1$ and $N = 20$, while continuous red line is the corresponding analytic approximation $\tilde{P}(s)$.

# CHAPTER 4

# Super Rogers-Szegö polynomials associated with $BC_N$ type of Polychronakos spin chains

## 4.1 Introduction

In this Chapter our aim is to construct multivariate SRS polynomials associated with the $BC_N$ type of PF spin chains with SAPSRO and study some properties of such polynomials. As we have mentioned earlier, by applying the freezing trick to the $A_{N-1}$ type of su($m$) spin Calogero model with confining harmonic potential, one can obtain the corresponding PF spin chain. The sites of this $A_{N-1}$ type of su($m$) PF spin chain, which are inhomogeneously spaced on a line, coincide with the zeros of the Hermite polynomial. The Hamiltonian of such ferromagnetic su($m$) PF spin chain is given by

$$\mathcal{H}_{\rm PF}^{(m)} = \sum_{1\leqslant i<j\leqslant N} \frac{1-P_{ij}^{(m)}}{(\rho_i-\rho_j)^2}\,, \tag{4.1.1}$$

where $P_{ij}^{(m)}$ denotes the exchange operator which interchanges the spins of the $i$-th and $j$-th lattice sites, and $\rho_i$ is the $i$-th zero point of the Hermite polynomial of degree $N$. A remarkable feature of $A_{N-1}$ type of su($m$) PF and HS spin chains is that they exhibit Yangian quantum group symmetry even for finite number of lattice sites. As a result, the energy eigenvalues of these spin chains can be expressed in an elegant way by using certain sequences of the binary digits '0' and '1', which are called as 'motifs' in the literature [62, 63, 127]. Furthermore, the complete spectra of these PF and HS spin chains, including the degeneracy factors of all energy levels, can be reproduced from the energy functions of some one-dimensional classical vertex models [64].

Due to the decoupling of the spin and coordinate degrees of freedom in the case of su($m$) spin Calogero model for large values of its coupling constant, the canonical partition function of the su($m$) PF spin chain (4.1.1) can be derived by using the freezing trick. More precisely, this partition function can be obtained by dividing the canonical partition function of the su($m$) spin Calogero model through that of the spinless Calogero model. Thus, for the purpose of deriving the partition function of the su($m$) PF spin chain by using the freezing trick, it is necessary to calculate at first the canonical partition function of the su($m$) spin Calogero model. This partition function has been computed in the literature by using two different approaches — a direct one and an indirect one. Polychronakos has originally computed this partition function in an indirect way by expanding the corresponding grand canonical partition function (which can be obtained easily from the grand canonical partition function of the spinless Calogero model) as a power series of the fugacity parameter [40]. Finally, by applying the freezing trick, the partition function of the ferromagnetic su($m$) PF spin chain (4.1.1) has been derived in the form

$$\mathcal{Z}_{A,N}^{(m)}(q) = \sum_{\substack{\sum_{i=1}^{m} a_i = N \\ a_i \geqslant 0}} \begin{bmatrix} N \\ a_1, a_2, \cdots, a_m \end{bmatrix}_q , \tag{4.1.2}$$

where $q \equiv e^{-1/(k_B T)}$, summation is taken over all $a_i$ (which are non-negative integers) satisfying the condition $\sum_{i=1}^{m} a_i = N$, and the $q$-multinomial coefficients are defined as

$$\begin{bmatrix} N \\ a_1, a_2, \cdots, a_m \end{bmatrix}_q = \frac{(q)_N}{(q)_{a_1}(q)_{a_2}\cdots(q)_{a_m}},$$

with $(q)_n \equiv (1-q)(1-q^2)\cdots(1-q^n)$. Since each $q$-multinomial coefficient is a polynomial of $q$, the partition function $\mathcal{Z}_{A,N}^{(m)}(q)$ in (4.1.2) can also be expressed as a polynomial of $q$. A supersymmetric generalization of the PF spin chain (4.1.1), containing both bosonic and fermionic spin degrees of freedom, has also been studied and the corresponding partition function has been computed with the help of the freezing trick via the indirect approach as described above [93]. However, it is also possible to directly compute the canonical partition function of the su($m$) spin Calogero model from the knowledge of its spectrum. Proceeding in this way and subsequently applying the freezing trick, Barba et al. have derived [113] the canonical partition function of the su($m$) PF spin chain (4.1.1) in a form which apparently looks quite different from $\mathcal{Z}_{A,N}^{(m)}(q)$ in (4.1.2).

In this context it may be noted that, the classical RS polynomial in a single variable (say, $x$) is defined as $\mathbb{H}_N(x,q) = \sum_{k=0}^{N} \begin{bmatrix} N \\ k, N-k \end{bmatrix}_q x^k$ [103]. This RS polynomial has been studied in connection with the well known Rogers-Ramanujan identities in number theory. Moreover, this RS polynomial can be viewed as a $q$-deformed version of the Hermite polynomial, which provides a basis for the coordinate representation of the $q$-oscillator algebra [104, 105]. Different types of homogeneous and inhomogeneous multivariate generalizations of the classical RS polynomial have also been studied in the literature. In particular, Hikami has observed that the homogeneous multivariate RS polynomials (depending on only one type of variables) of the form

$$\mathbb{H}_{A,N}^{(m)}(x_1, x_2, \cdots, x_m; q) = \sum_{\substack{\sum_{i=1}^{m} a_i = N \\ a_i \geqslant 0}} \begin{bmatrix} N \\ a_1, a_2, \cdots, a_m \end{bmatrix}_q x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m}, \tag{4.1.3}$$

reproduce the partition function (4.1.2) of the PF spin chain in the limit $x_1 = x_2 = \cdots = x_N = 1$ [62, 102, 106]. Consequently, a representation of the $Y(gl_m)$ Yangian invariant motifs associated with the PF spin chain (4.1.1) can be constructed by using a recursion relation satisfied by the RS polynomials (4.1.3). It may also be noted that, SRS polynomials containing two different types of variables have been proposed in Ref. [94] for the purpose of analyzing the spectra and partition functions of the supersymmetric PF spin chains on the basis of their $Y(gl_{(m|n)})$ super Yangian symmetry. From the above discussion it is clear that RS and SRS polynomials play an important role in the study of PF spin chains and their supersymmetric generalizations. However, since the partition function of the su($m$) PF spin chain obtained by using the freezing trick via the direct approach appears to be quite different from (4.1.2), such a form of the partition function cannot be connected with the RS polynomials (4.1.3) in a straightforward way.

As is well known, root systems associated with Lie algebras are widely used in the classification of quantum integrable systems with long-range interaction. In particular, the above discussed su($m$) PF spin chains with $N$ number of lattice sites and their supersymmetric generalization are related to the $A_{N-1}$ root system. However, it is possible to construct exactly solvable variants of the PF spin chain (4.1.1) associated with the $BC_N$ and $D_N$ root systems [84, 86, 90]. One remarkable feature of the Hamiltonians of the PF spin chains associated with the latter root systems is that they contain reflection operators like $S_i$ $(i = 1, \ldots, N)$, which satisfy the relation $S_i^2 = \mathbb{1}$ and yield a representation of some elements appearing in the $BC_N$ or $D_N$ type of Weyl algebra. In the special case when $S_i$ is taken as the spin reversal operator $P_i$, which changes the sign of the spin component on the $i$-th lattice site, the partition functions of PF spin chains associated with the $BC_N$ and $D_N$ root systems have been computed by using the freezing trick via the direct approach [91, 107]. Furthermore, by taking reflection operators as supersymmetric analogue of spin reversal operators (SASRO), partition functions of PF spin chains

associated with the $BC_N$ root system have been computed by using the freezing trick via the direct as well as indirect approaches [108].

However, it has been found recently that reflection operators can be chosen in more general way than the above mentioned spin reversal operators and their supersymmetric analogues. For example, choosing the reflection operators as such PSRO, new exactly solvable spin Calogero models of $BC_N$ and $D_N$ type have been constructed and the partition functions of the related PF chains have also been computed by using the freezing trick via the direct approach [91,107]. Furthermore, exactly solvable spin Calogero models of $BC_N$ type have been constructed by taking reflection operators as SAPSRO [108]. The strong coupling limit of such spin Calogero models yields a large class of $BC_N$ type of PF spin chains with SAPSRO, which can reproduce all of the previously studied $BC_N$ type of PF spin chains at certain limits.

In spite of the above mentioned developments on different variants of the $BC_N$ type of PF spin chains, it is not clear till now whether the spectra of these spin chains can be described by some motif like objects related to the symmetry of these spin chains. Furthermore, one may interesting ask whether there exist some one-dimensional classical vertex models whose energy functions would generate the complete spectra of these $BC_N$ type of PF spin chains. However it is known that, in the cases of $A_{N-1}$ type of PF spin chains and their supersymmetric generalizations, multivariate RS and SRS polynomials play a key role in solving such problems. Hence, as a first step towards solving these problems for the case of $BC_N$ type of PF spin chains, at present our aim is to construct the corresponding multivariate SRS polynomials and explore some of their properties. Since all of the previously studied PF spin chains of $BC_N$ type can be obtained by taking certain limits of the PF spin chains with SAPSRO, it is expected that canonical partition functions of the later spin chains would help us in finding out the general form of the $BC_N$ type of multivariate SRS polynomials. In this context it may be noted that, the canonical partition functions

of the $BC_N$ type of PF chains with SAPSRO have been computed earlier by using the freezing trick via the direct approach [108]. However it may be recalled that, for the case of $A_{N-1}$ type of PF spin chains, partition functions obtained by using the above mentioned procedure cannot be connected with the multivariate RS polynomials in a straightforward way. Therefore, in this Chapter we shall derive a new expression for the canonical partition functions of the $BC_N$ type of PF chains with SAPSRO by using the freezing trick via the indirect approach, and subsequently use those partition functions to construct the corresponding multivariate SRS polynomials.

The arrangement of this Chapter is as follows. In Sec. 2 we shall briefly review some results of Ref. [108] which are relevant for our purpose, like the construction of SAPSRO by using the $BC_N$ type of Weyl algebra and the method of generating PF spin chains with SAPSRO from the related spin Calogero models by applying the freezing trick. Furthermore, we shall describe the Hilbert space associated with the $BC_N$ type of spin Calogero models with SAPSRO and the procedure of deriving the spectra of these models by choosing a partially ordered set of basis vectors where the corresponding Hamiltonians can be expressed in a triangular form. In Sec. 3, we shall compute the grand canonical partition functions of the $BC_N$ type of ferromagnetic as well as anti-ferromagnetic spin Calogero models with SAPSRO, and expand those grand canonical partition functions as some power series of the fugacity parameter to obtain the corresponding canonical partition functions. Applying the freezing trick, subsequently we shall derive novel expressions for the canonical partition functions of the $BC_N$ type of ferromagnetic and anti-ferromagnetic PF chains with SAPSRO. Inspired by the form of such partition functions, in Sec. 4 we shall define $BC_N$ type of homogeneous multivariate SRS polynomials and also find out the corresponding generating functions. Using these generating functions, we shall show that the $BC_N$ type of SRS polynomials can be expressed as some bilinear combinations of the $A_{N-1}$ type of SRS polynomials. Furthermore, we shall derive a set of recursion relations for the partition functions of the $BC_N$ type of PF spin chains involving different

numbers of lattice sites and internal degrees of freedom.

## 4.2 $BC_N$ type of spin models with SAPSRO

It is well known that the $BC_N$ type of Weyl algebra is generated by the elements like $\mathcal{W}_{ij}$ and $\mathcal{W}_i$ , which satisfy the relations

$$\begin{aligned} &\mathcal{W}_{ij}^2 = \mathbb{1}\,, \quad \mathcal{W}_{ij}\mathcal{W}_{jk} = \mathcal{W}_{ik}\mathcal{W}_{ij} = \mathcal{W}_{jk}\mathcal{W}_{ik}\,, \quad \mathcal{W}_{ij}\mathcal{W}_{kl} = \mathcal{W}_{kl}\mathcal{W}_{ij}\,, \\ &\mathcal{W}_i^2 = \mathbb{1}\,, \quad \mathcal{W}_i\mathcal{W}_j = \mathcal{W}_j\mathcal{W}_i\,, \quad \mathcal{W}_{ij}\mathcal{W}_k = \mathcal{W}_k\mathcal{W}_{ij}\,, \quad \mathcal{W}_{ij}\mathcal{W}_j = \mathcal{W}_i\mathcal{W}_{ij}\,, \end{aligned} \tag{4.2.1}$$

where $i,\ j,\ k,\ l \in \{1, 2, \cdots, N\}$ are all different indices. Representations of this Weyl algebra play an important role in constructing $BC_N$ type of quantum integrable spin models with long-range interaction. For the purpose of describing a class of representations of the $BC_N$ type of Weyl algebra (4.2.1) on a superspace, let us consider a set of operators like $C_{j\alpha}^\dagger$ $(C_{j\alpha})$ which creates (annihilates) a particle of species $\alpha$ on the $j$-th lattice site. These creation (annihilation) operators are assumed to be bosonic when $\alpha \in [1, 2, ...., m]$ and fermionic when $\alpha \in [m+1, m+2, ...., m+n]$. Hence, the parity of these operators are defined as

$$\begin{aligned} \pi(C_{j\alpha}) = \pi(C_{j\alpha}^\dagger) = 0 \text{ for } \alpha \in [1, 2, ...., m]\,, \\ \pi(C_{j\alpha}) = \pi(C_{j\alpha}^\dagger) = 1 \text{ for } \alpha \in [m+1, m+2, ...., m+n]\,, \end{aligned}$$

and they satisfy commutation (anti-commutation) relations given by

$$[C_{j\alpha}, C_{k\beta}]_\pm = 0\,,\ [C_{j\alpha}^\dagger, C_{k\beta}^\dagger]_\pm = 0\,,\ [C_{j\alpha}, C_{k\beta}^\dagger]_\pm = \delta_{jk}\delta_{\alpha\beta}\,, \tag{4.2.2}$$

where $[C, D]_{\pm} \equiv CD - (-1)^{\pi(C)\pi(D)}DC$. Let us now consider a finite dimensional subspace of the related Fock space, where each lattice site is occupied by only one particle, i.e., $\sum_{\alpha=1}^{m+n} C_{j\alpha}^{\dagger} C_{j\alpha} = 1$ for all $j \in \{1, 2, \cdots, N\}$. The supersymmetric exchange operator $\hat{P}_{ij}^{(m|n)}$ is defined on such subspace of the Fock space as [42]

$$\hat{P}_{ij}^{(m|n)} \equiv \sum_{\alpha,\beta=1}^{m+n} C_{i\alpha}^{\dagger} C_{j\beta}^{\dagger} C_{i\beta} C_{j\alpha} \,. \tag{4.2.3}$$

The supersymmetric exchange operator (4.2.3) can equivalently be expressed as an operator on the total internal space of $N$ number of spins, which is defined in the following way [65, 108]. Let us denote such total internal space as $\mathbf{\Sigma}^{(m_1,m_2|n_1,n_2)}$, where $m_1$, $m_2$, $n_1$, $n_2$ are some arbitrary non-negative integers satisfying the relations $m_1 + m_2 = m$ and $n_1 + n_2 = n$ . The space $\mathbf{\Sigma}^{(m_1,m_2|n_1,n_2)}$ is spanned by orthonormal state vectors of the form $|s_1, \cdots, s_i, \cdots, s_N\rangle$, where $s_i \equiv (s_i^1, s_i^2, s_i^3)$ has three components which take discrete values like $s_i^1 \equiv \pi(s_i) \in \{0, 1\}$, $s_i^2 \equiv f(s_i) \in \{0, 1\}$, and

$$s_i^3 \in \begin{cases} \{1, 2, \cdots, m_1\}, & \text{if } \pi(s_i) = 0 \text{ and } f(s_i) = 0, \\ \{1, 2, \cdots, m_2\}, & \text{if } \pi(s_i) = 0 \text{ and } f(s_i) = 1, \\ \{1, 2, \cdots, n_1\}, & \text{if } \pi(s_i) = 1 \text{ and } f(s_i) = 0, \\ \{1, 2, \cdots, n_2\}, & \text{if } \pi(s_i) = 1 \text{ and } f(s_i) = 1. \end{cases} \tag{4.2.4}$$

Hence, each local spin vector $s_i$ may be chosen in $(m+n)$ number of different ways and $\mathbf{\Sigma}^{(m_1,m_2|n_1,n_2)}$ can be expressed in a direct product form given by

$$\mathbf{\Sigma}^{(m_1,m_2|n_1,n_2)} \equiv \underbrace{\mathcal{C}_{m+n} \otimes \mathcal{C}_{m+n} \otimes \cdots \otimes \mathcal{C}_{m+n}}_{N} \,, \tag{4.2.5}$$

where $\mathcal{C}_{m+n}$ denotes an $(m+n)$-dimensional complex vector space. It is evident that this $\mathbf{\Sigma}^{(m_1,m_2|n_1,n_2)}$ is isomorphic to the subspace of the Fock space, on which $\hat{P}_{ij}^{(m|n)}$ in (4.2.3) is defined. A supersymmetric spin exchange operator $P_{ij}^{(m|n)}$ is defined on

the space $\mathbf{\Sigma}^{(m_1,m_2|n_1,n_2)}$ as

$$P_{ij}^{(m|n)}|s_1,\cdots,s_i,\cdots,s_j,\cdots,s_N\rangle = (-1)^{\alpha_{ij}(\mathbf{s})}|s_1,\cdots,s_j,\cdots,s_i,\cdots,s_N\rangle, \quad (4.2.6)$$

where $\alpha_{ij}(\mathbf{s}) = \pi(s_i)\pi(s_j) + (\pi(s_i)+\pi(s_j))\ h_{ij}(\mathbf{s})$ and $h_{ij}(\mathbf{s}) = \sum_{k=i+1}^{j-1}\pi(s_k)$. From Eq. (4.2.6) it follows that, the exchange of two spins with $\pi(s_i) = \pi(s_j) = 0$ or $\pi(s_i) = \pi(s_j) = 1$ produces a phase factor of 1 or $-1$ respectively. So we may call $s_i$ as a 'bosonic' spin if $s_i^1 \equiv \pi(s_i) = 0$ and a 'fermionic' spin if $s_i^1 \equiv \pi(s_i) = 1$. However, it should be noted the exchange one bosonic spin with one fermionic spin (or, vice versa) produces a nontrivial phase factor of $(-1)^{h_{ij}(\mathbf{s})}$, where $h_{ij}(\mathbf{s})$ represents the number of fermionic spins within the $i$-th and $j$-th lattice sites. Using the commutation (anti-commutation) relations in (4.2.2), it can be shown that $\hat{P}_{ij}^{(m|n)}$ in (4.2.3) is completely equivalent to $P_{ij}^{(m|n)}$ in (4.2.6) [65]. The action of SAPSRO (denoted by $P_i^{(m_1,m_2|n_1,n_2)}$) is defined on the space $\mathbf{\Sigma}^{(m_1,m_2|n_1,n_2)}$ as [108]

$$P_i^{(m_1,m_2|n_1,n_2)}|s_1,\cdots,s_i,\cdots,s_N\rangle = (-1)^{f(s_i)}|s_1,\cdots,s_i,\cdots,s_N\rangle. \quad (4.2.7)$$

Hence, the second component of the spin $s_i$ determines its parity under the action of SAPSRO. It is easy to verify that $P_{ij}^{(m|n)}$ in (4.2.6) and $P_i^{(m_1,m_2|n_1,n_2)}$ in (4.2.7) respectively yield representations of the elements $\mathcal{W}_{ij}$ and $\mathcal{W}_i$ appearing in the $BC_N$ type of Weyl algebra (4.2.1). These representations of the Weyl algebra can be used to construct a large class of exactly solvable $BC_N$ type of ferromagnetic PF spin chains with Hamiltonians given by

$$\mathcal{H}^{(m_1,m_2|n_1,n_2)} = \sum_{i\neq j}\left[\frac{1-P_{ij}^{(m|n)}}{(\xi_i-\xi_j)^2} + \frac{1-\widetilde{P}_{ij}^{(m_1,m_2|n_1,n_2)}}{(\xi_i+\xi_j)^2}\right] + \beta\sum_{i=1}^{N}\frac{1-P_i^{(m_1,m_2|n_1,n_2)}}{\xi_i^2}, \quad (4.2.8)$$

where $\beta > 0$ is a real parameter, $\xi_i = \sqrt{2y_i}$ with $y_i$ being the $i$-th zero point of the generalized Laguerre polynomial $L_N^{\beta-1}$, and $\widetilde{P}_{ij}^{(m_1,m_2|n_1,n_2)} \equiv P_i^{(m_1,m_2|n_1,n_2)}P_j^{(m_1,m_2|n_1,n_2)}P_{ij}^{(m|n)}$. It may be noted that, the Hamiltonian (4.2.8) can reproduce all of the previously

studied $BC_N$ type of PF spin chains for some specific values of the discrete parameters $m_1,\ m_2,\ n_1$ and $n_2$. For example, in the presence of only bosonic or fermionic spins, i.e., when either $n_1 = n_2 = 0$ or $m_1 = m_2 = 0$, $\mathcal{H}^{(m_1,m_2|n_1,n_2)}$ reduces to the non-supersymmetric PF spin chain associated with PSRO [91]. In another special case, where the discrete parameters in (4.2.8) satisfy the relations

$$m_1 = \frac{1}{2}\left(m + \epsilon\,\tilde{m}\right),\ m_2 = \frac{1}{2}\left(m - \epsilon\,\tilde{m}\right),\ n_1 = \frac{1}{2}\left(n + \epsilon'\,\tilde{n}\right),\ n_2 = \frac{1}{2}\left(n - \epsilon'\,\tilde{n}\right), \tag{4.2.9}$$

with $\epsilon, \epsilon' = \pm 1$, $\tilde{m} \equiv m \mod 2$ and $\tilde{n} \equiv n \mod 2$, the exactly solvable Hamiltonian (which depends on the parameters $m, n, \epsilon, \epsilon'$) of the $BC_N$ type of PF spin chains with SASRO [101] can be obtained from $\mathcal{H}^{(m_1,m_2|n_1,n_2)}$ through a unitary transformation [108].

Applying the freezing trick, the Hamiltonians (4.2.8) of the $BC_N$ type of PF spin chains with SAPSRO can be derived from those of $BC_N$ type of spin Calogero models containing both coordinate and spin degrees of freedom. The Hamiltonians of such spin Calogero models are given by

$$\begin{aligned} H^{(m_1,m_2|n_1,n_2)} = -\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + \frac{a^2}{4}r^2 \quad & +a\sum_{i\neq j}\left[\frac{a - P_{ij}^{(m|n)}}{(x_{ij}^-)^2} + \frac{a - \widetilde{P}_{ij}^{(m_1,m_2|n_1,n_2)}}{(x_{ij}^+)^2}\right] \\ & +\beta a\sum_{i=1}^{N}\frac{\beta a - P_i^{(m_1,m_2|n_1,n_2)}}{x_i^2}, \end{aligned} \tag{4.2.10}$$

where $a > \frac{1}{2}$ is a real coupling constant, $x_{ij}^- \equiv x_i - x_j$, $x_{ij}^+ \equiv x_i + x_j$ and $r^2 \equiv \sum_{i=1}^N x_i^2$. The coefficient of the $a^2$ order term in the r.h.s. of (4.2.10) may be written as

$$U(\mathbf{x}) = \sum_{i\neq j}\left[\frac{1}{(x_{ij}^-)^2} + \frac{1}{(x_{ij}^+)^2}\right] + \beta^2\sum_{i=1}^{N}\frac{1}{x_i^2} + \frac{r^2}{4}. \tag{4.2.11}$$

Since this $a^2$ order term in $H^{(m_1,m_2|n_1,n_2)}$ dominates in the strong coupling limit $a \to \infty$, the particles of this spin Calogero model concentrate at the minimum $\boldsymbol{\xi}$

of the potential $U(\mathbf{x})$. Consequently, the coordinate and spin degrees of freedom of these particles decouple from each other. Furthermore one can show that, within the configuration space of the Hamiltonian (4.2.10), the coordinates $\xi_i$ of the unique minimum $\boldsymbol{\xi}$ of the potential $U(\mathbf{x})$ are given by $\xi_i = \sqrt{2y_i}$, where $y_i$'s denote the zeros of the generalized Laguerre polynomial $L_N^{\beta-1}$ [76]. Consequently, $H^{(m_1,m_2|n_1,n_2)}$ in (4.2.10) can be written in $a \to \infty$ limit as

$$H^{(m_1,m_2|n_1,n_2)} \approx H_{sc} + a\,\mathcal{H}^{(m_1,m_2|n_1,n_2)}\,, \tag{4.2.12}$$

where $H_{sc}$ is the scalar (spinless) Calogero model of $BC_N$ type given by

$$H_{sc} = -\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + \frac{a^2}{4}\,r^2 + a(a-1)\sum_{i\neq j}\left[\frac{1}{(x_{ij}^-)^2} + \frac{1}{(x_{ij}^+)^2}\right] + \sum_{i=1}^{N}\frac{\beta a(\beta a-1)}{x_i^2}\,. \tag{4.2.13}$$

Thus the Hamiltonians (4.2.8) of the $BC_N$ type of PF spin chains with SAPSRO emerge naturally from the strong coupling limit of the corresponding spin Calogero models (4.2.10). Due to Eq. (4.2.12), the eigenvalues of $H^{(m_1,m_2|n_1,n_2)}$ satisfy the relation

$$E_{ij}^{(m_1,m_2|n_1,n_2)} \simeq E_i^{sc} + a\,\mathcal{E}_j^{(m_1,m_2|n_1,n_2)}\,, \tag{4.2.14}$$

where $E_i^{sc}$ and $\mathcal{E}_j^{(m_1,m_2|n_1,n_2)}$ are two eigenvalues of $H_{sc}$ and $\mathcal{H}^{(m_1,m_2|n_1,n_2)}$ respectively. With the help of Eq. (4.2.14), one can derive an exact formula for the canonical partition function $\mathcal{Z}_{B,N}^{(m_1,m_2|n_1,n_2)}(T)$ of the $BC_N$ type of PF spin chain (4.2.8) at a given temperature $T$ as

$$\mathcal{Z}_{B,N}^{(m_1,m_2|n_1,n_2)}(T) = \lim_{a\to\infty}\frac{Z_{B,N}^{(m_1,m_2|n_1,n_2)}(aT)}{Z_{B,N}(aT)}\,, \tag{4.2.15}$$

where $Z_{B,N}(aT)$ and $Z_{B,N}^{(m_1,m_2|n_1,n_2)}(aT)$ represent canonical partition functions (at the temperature $aT$) of the $BC_N$ type of spinless Calogero model (4.2.13) and spin Calogero model (4.2.10) respectively.

The $BC_N$ type of spinless Calogero model (4.2.13) is a well known exactly solvable system with ground state wave function of the form

$$\mu(\mathbf{x}) = e^{-\frac{a}{4}r^2} \prod_i |x_i|^{\beta a} \prod_{i<j} |x_i^2 - x_j^2|^a \,, \tag{4.2.16}$$

and ground state energy given by

$$E_0 = Na\Big(\beta a + a(N-1) + \frac{1}{2}\Big) \,. \tag{4.2.17}$$

An exact expression for the canonical partition function of the $BC_N$ type of spinless Calogero model (4.2.13) has been derived earlier as [84]

$$Z_{B,N}(aT) = \frac{1}{\prod_{j=1}^{N}(1-q^{2j})} \,, \tag{4.2.18}$$

where $q \equiv e^{-1/(k_B T)}$ and the contribution from the ground state energy has been ignored without any loss of generality.

The exact spectrum of the $BC_N$ type of spin Calogero Hamiltonian (4.2.10) can be computed by expressing it in a triangular form while acting on a partially ordered set of basis vectors of the corresponding Hilbert space [108]. As found in the later reference, the Hilbert space associated with this spin Calogero Hamiltonian is the closure of the linear subspace spanned by the wave functions of the form

$$\psi^{\mathbf{s}}_{\mathbf{r}} \equiv \psi^{s_1,\ldots,s_i,\ldots,s_j,\ldots,s_N}_{r_1,\ldots,r_i,\ldots,r_j,\ldots,r_N} = \Lambda^{(m_1,m_2|n_1,n_2)} \left(\phi_{\mathbf{r}}(\mathbf{x})|\mathbf{s}\rangle\right) \,, \tag{4.2.19}$$

where $r_i$'s are arbitrary non-negative integers, $\phi_{\mathbf{r}}(\mathbf{x}) \equiv \mu(\mathbf{x}) \prod_{i=1}^{N} x_i^{r_i}$, $|\mathbf{s}\rangle \equiv |s_1, \cdots, s_N\rangle$ represents an arbitrary basis element of the spin space $\mathbf{\Sigma}^{(m_1,m_2|n_1,n_2)}$, and $\Lambda^{(m_1,m_2|n_1,n_2)}$ is a completely symmetric projector related to the $BC_N$ type of Weyl algebra. It

can be shown that $\psi_{\mathbf{r}}^{\mathbf{s}}$'s in (4.2.19) satisfy the symmetry conditions

$$\psi_{r_1,\ldots,r_i,\ldots,r_j,\ldots,r_N}^{s_1,\ldots,s_i,\ldots,s_j,\ldots,s_N} = (-1)^{\alpha_{ij}(\mathbf{s})}\, \psi_{r_1,\ldots,r_j,\ldots,r_i,\ldots,r_N}^{s_1,\ldots,s_j,\ldots,s_i,\ldots,s_N}\,, \tag{4.2.20}$$

and

$$\psi_{r_1,\ldots,r_i,\ldots,r_N}^{s_1,\ldots,s_i,\ldots,s_N} = (-1)^{r_i+f(s_i)}\; \psi_{r_1,\ldots,r_i,\ldots,r_N}^{s_1,\ldots,s_i,\ldots,s_N}\,, \tag{4.2.21}$$

where $\alpha_{ij}(\mathbf{s})$ is defined after (4.2.6) and $1 \leqslant i < j \leqslant N$. Due to these symmetry conditions, $\psi_{\mathbf{r}}^{\mathbf{s}}$'s corresponding to all possible values of $\mathbf{r}$ and $\mathbf{s}$ do not form a set of linearly independent basis vectors for the Hilbert space associated with the spin Calogero Hamiltonian $H^{(m_1,m_2|n_1,n_2)}$. However, $\psi_{\mathbf{r}}^{\mathbf{s}}$'s in (4.2.19) would lead to a complete set of basis vectors if the following three conditions are imposed on the possible values of $\mathbf{r}$ and $\mathbf{s}$.

i) The lower index $\mathbf{r}$ in $\psi_{\mathbf{r}}^{\mathbf{s}}$ is chosen in an ordered form which separately arranges its even and odd components into two non-increasing sequences:

$$\begin{aligned}\mathbf{r} \;\equiv\; (\mathbf{r_e}, \mathbf{r_o}) \;=\; (&\overbrace{2l_1, \ldots, 2l_1}^{k_1}, \ldots, \overbrace{2l_s, \ldots, 2l_s}^{k_s},\\ &\overbrace{2p_1+1, \ldots, 2p_1+1}^{g_1}, \ldots, \overbrace{2p_t+1, \ldots, 2p_t+1}^{g_t})\,,\end{aligned} \tag{4.2.22}$$

where $0 \leqslant s, t \leqslant N$, $l_1 > l_2 > \ldots > l_s \geqslant 0$ and $p_1 > p_2 > \ldots > p_t \geqslant 0$. Since any given $\mathbf{r}$ can be brought in the ordered form (4.2.22) through an appropriate permutation of its components, one can choose this ordered form as a consequence of the symmetry condition (4.2.20).

ii) The second component of $s_i$ corresponding to each $r_i$ is given by

$$s_i^2 \equiv f(s_i) = \begin{cases} 0, & \text{for } r_i \in \mathbf{r_e}\,, \\ 1\,, & \text{for } r_i \in \mathbf{r_o}\,. \end{cases} \tag{4.2.23}$$

This is a direct consequence of the symmetry condition (4.2.21).

iii) If $r_i = r_j$ for $i < j$, then from Eq. (4.2.23) it follows that the second components of the corresponding spins $s_i$ and $s_j$ must have the same value. In that case, one can further apply Eq. (4.2.20) to obtain an ordering among $s_i$ and $s_j$, by using the rule $\pi(s_i) \leqslant \pi(s_j)$, and subsequently, for the case $\pi(s_i) = \pi(s_j)$, by using the rule $s_i^3 \geqslant s_j^3 + \pi(s_j)$.

All $\psi_{\mathbf{r}}^{\mathbf{s}}$'s in (4.2.19), satisfying the above mentioned three conditions, represent a set of (non-orthonormal) basis vectors of the Hilbert space associated with the spin Calogero Hamiltonian in (4.2.10). If a partial ordering is defined among these basis vectors like $\psi_{\mathbf{r}}^{\mathbf{s}} > \psi_{\mathbf{r}'}^{\mathbf{s}'}$, for $|\mathbf{r}| > |\mathbf{r}'|$, where $|\mathbf{r}| \equiv \sum_{i=1}^{N} r_i$, it can be shown that $H^{(m_1,m_2|n_1,n_2)}$ in (4.2.10) acts as an upper triangular matrix on such partially ordered basis vectors:

$$H^{(m_1,m_2|n_1,n_2)}\,\psi_{\mathbf{r}}^{\mathbf{s}} = E_{\mathbf{r}}^{\mathbf{s}}\,\psi_{\mathbf{r}}^{\mathbf{s}} + \sum_{|\mathbf{r}'|<|\mathbf{r}|} C_{\mathbf{r}'\mathbf{r}}\,\psi_{\mathbf{r}'}^{\mathbf{s}'}\,, \tag{4.2.24}$$

where $C_{\mathbf{r}'\mathbf{r}}$'s are real constants, $\mathbf{s}'$ is a suitable permutation of $\mathbf{s}$ and

$$E_{\mathbf{r}}^{\mathbf{s}} = a|\mathbf{r}| + E_0\,. \tag{4.2.25}$$

Consequently, all eigenvalues in the spectrum of $H^{(m_1,m_2|n_1,n_2)}$ are given by $E_{\mathbf{r}}^{\mathbf{s}}$ in (4.2.25), where the quantum numbers $\mathbf{r}$ and $\mathbf{s}$ satisfy the conditions i)-iii). Since the r.h.s. of (4.2.25) does not depend on the quantum number $\mathbf{s}$, $E_{\mathbf{r}}^{\mathbf{s}}$ has an 'intrinsic degeneracy' which is obtained by counting the number of all possible choice of spin degrees of freedom corresponding to a given $\mathbf{r}$. By using the energy levels (4.2.25) and corresponding intrinsic degeneracy factors, it is possible to directly compute the canonical partition function $Z_{B,N}^{(m_1,m_2|n_1,n_2)}(aT)$ of the spin Calogero model (4.2.10). Furthermore, by inserting such expression of $Z_{B,N}^{(m_1,m_2|n_1,n_2)}(aT)$ and $Z_{B,N}(aT)$ given in (4.2.18) to the relation (4.2.15), one can evaluate the canonical partition function $\mathcal{Z}_{B,N}^{(m_1,m_2|n_1,n_2)}(T)$ of the spin chains (4.2.8) [108].

However the partition functions of the $BC_N$ type of PF spin chains with SAPSRO,

obtained in the above mentioned way, have a rather complicated form which can not be expressed through the $q$-multinomial coefficients in a straightforward way. Hence, for the purpose of constructing $BC_N$ type of multivariate SRS polynomials, in the next section we shall derive a new expression for the canonical partition functions of the $BC_N$ type of PF spin chains (4.2.8) through the indirect approach. More precisely, we shall first compute the grand canonical partition functions of the $BC_N$ type of spin Calogero models with SAPSRO (4.2.10) and expand such grand canonical partition functions as a power series of the fugacity parameter to obtain the corresponding the canonical partition functions. Substitution of those canonical partition functions to the relation (4.2.15) would lead to the desired expressions for the canonical partition functions of the spin chains (4.2.8).

## 4.3 Partition functions of $BC_N$ type of spin models with SAPSRO

A remarkable feature of the grand canonical partition functions associated with the $A_{N-1}$ type of spin Calogero models (with harmonic confining potentials) and their supersymmetric generalizations is that such partition functions can be expressed as some simple products of the corresponding 'basic modes'. For example, the grand canonical partition function $\mathbb{Z}_A^{(m|0)}$ of the $m$-flavor bosonic spin Calogero model can be written through the corresponding basic mode, i.e., the grand canonical partition function of the one-flavor (spinless) bosonic Calogero model as [40]

$$\mathbb{Z}_A^{(m|0)} = \left(\mathbb{Z}_A^{(1|0)}\right)^m . \tag{4.3.1}$$

It is well known that, up to a constant shift of all energy levels, the spectrum of one-flavor bosonic Calogero model of $A_{N-1}$ type coincides with that of the $N$ number

of free bosonic oscillators. Dropping the zero-point energy of these oscillators and using an identity given by

$$\sum_{k_1 \geqslant k_2 \geqslant \cdots \geqslant k_N \geqslant 0} q^{\sum_{i=1}^{N} k_i} = \frac{1}{(q)_N}\,, \tag{4.3.2}$$

the canonical partition function $Z_{A,N}^{(1|0)}(q)$ of such one-flavor bosonic Calogero model with $N$ number of particles can be obtained as

$$Z_{A,N}^{(1|0)}(q) = \frac{1}{(q)_N}\,. \tag{4.3.3}$$

As a result, the corresponding grand canonical partition function may be expressed as

$$\mathbb{Z}_A^{(1|0)} = \sum_{N=0}^{\infty} y^N \cdot Z_{A,N}^{(1|0)}(q) = \sum_{N=0}^{\infty} \frac{y^N}{(q)_N}\,, \tag{4.3.4}$$

where $y \equiv q^{-\mu}$ (with $\mu$ being the chemical potential) denotes the fugacity parameter. Inserting this expression of $\mathbb{Z}_A^{(1|0)}$ into Eq. (4.3.1), one can derive the grand canonical partition function of the $m$-flavor bosonic spin Calogero model [40].

The grand canonical partition function of the su$(m|n)$ supersymmetric spin Calogero model of $A_{N-1}$ type can also be written as the product of two types of basic modes as [93]

$$\mathbb{Z}_A^{(m|n)} = \left(\mathbb{Z}_A^{(1|0)}\right)^m \left(\mathbb{Z}_A^{(0|1)}\right)^n\,, \tag{4.3.5}$$

where $\mathbb{Z}_A^{(0|1)}$ represents the grand canonical partition function of the one-flavor (spinless) fermionic Calogero model. By using the identity

$$\sum_{k_1 > k_2 > \cdots > k_N \geqslant 0} q^{\sum_{i=1}^{N} k_i} = q^{\frac{N(N-1)}{2}} \cdot \frac{1}{(q)_N}\,, \tag{4.3.6}$$

the canonical partition function of such Calogero model with $N$ number of particles

can be derived as

$$Z_{A,N}^{(0|1)}(q) = q^{\frac{N(N-1)}{2}} \cdot \frac{1}{(q)_N}\,, \tag{4.3.7}$$

and the corresponding grand canonical partition function may be obtained as

$$\mathbb{Z}_A^{(0|1)} = \sum_{N=0}^{\infty} y^N Z_{A,N}^{(0|1)}(q) = \sum_{N=0}^{\infty} y^N\, q^{\frac{N(N-1)}{2}} \cdot \frac{1}{(q)_N}. \tag{4.3.8}$$

Inserting $\mathbb{Z}_A^{(1|0)}$ in (4.3.4) and $\mathbb{Z}_A^{(0|1)}$ in (4.3.8) into Eq. (4.3.5), one can derive the grand canonical partition function of the su($m|n$) supersymmetric spin Calogero model of $A_{N-1}$ type [93].

It may be noted that, grand canonical partition functions of the $BC_N$ type of spin Calogero models with SASRO have been computed earlier in Ref. [101]. Those spin Calogero models with SASRO may be considered as some special cases of the $BC_N$ type of spin Calogero models with SAPSRO (4.2.10), since the former models can be obtained from the latter ones by imposing the condition (4.2.9) and also using a unitary transformation. However it has been found in the later reference that, instead of only $BC_N$ type of basic modes, both $BC_N$ and $A_{N-1}$ types of basic modes appear in the expressions of grand canonical partition functions of the spin Calogero models with SASRO. Such a mixture of two different types of basic modes in the expression of the grand canonical partition function is clearly not suitable for our present purpose of constructing $BC_N$ type of multivariate RS polynomials. In the following, our aim is to derive the grand canonical partition functions of the $BC_N$ type of spin Calogero models with SAPSRO (4.2.10) as simple products of only $BC_N$ types of basic modes.

In the previous section it has been mentioned that $\psi_{\mathbf{r}}^{\mathbf{s}}$'s in (4.2.19), with indices $\mathbf{r}$ and $\mathbf{s}$ satisfying the rules i)-iii), represent a set of (non-orthonormal) basis vectors of the Hilbert space associated with the spin Calogero Hamiltonian with SAPSRO (4.2.10). While these rules for ordering $\mathbf{r}$ and $\mathbf{s}$ are very convenient for computing

the canonical partition function of the Hamiltonian (4.2.10), they are not suitable for computing the corresponding grand canonical partition function and they do not uniquely follow from the symmetry conditions (4.2.20) and (4.2.21). Indeed, for the purpose of computing the grand canonical partition function of the Hamiltonian (4.2.10) from its spectrum, it is necessary to order at first the upper index $\mathbf{s}$ of $\psi^{\mathbf{s}}_{\mathbf{r}}$ in an appropriate way and subsequently find out the rules which the lower index $\mathbf{r}$ should obey. Hence, instead of using the rules i)-iii), we order the indices $\mathbf{s}$ and $\mathbf{r}$ of the state vectors (4.2.19) by using the following equivalent set of rules to obtain essentially the same set of complete basis vectors:

1) Let us define the difference between two local spin vectors $s \equiv (s^1, s^2, s^3)$ and $\bar{s} \equiv (\bar{s}^1, \bar{s}^2, \bar{s}^3)$ as $s - \bar{s} = (s^1 - \bar{s}^1, s^2 - \bar{s}^2, s^3 - \bar{s}^3)$, and assume that $s < \bar{s}$ if the first non-vanishing component of $s - \bar{s}$ is negative. Using the symmetry condition (4.2.20), we order the index $\mathbf{s} \equiv (s_1, s_2, \cdots, s_N)$ such that $s_i \leqslant s_j$ for $i < j$.

2) If $s_i = s_j$ for $i < j$, then by using (4.2.20) we order the components $r_i$ and $r_j$ within $\mathbf{r} \equiv (r_1, r_2, \cdots, r_N)$ such that $r_i \geqslant r_j + \pi(s_i)$.

3) Due to the condition (4.2.21), $r_i$ is taken as an even non-negative integer if $s_i^2 \equiv f(s_i) = 0$, and $r_i$ is taken as an odd positive integer if $s_i^2 \equiv f(s_i) = 1$.

As before, a partial ordering may be defined among these relabeled basis vectors as: $\psi^{\mathbf{s}}_{\mathbf{r}} > \psi^{\mathbf{s}'}_{\mathbf{r}'}$, for $|\mathbf{r}| > |\mathbf{r}'|$. It is evident that, in analogy with (4.2.24), the spin Calogero Hamiltonian (4.2.10) would act as an upper triangular matrix on such partially ordered basis vectors. As a result, all eigenvalues in the spectrum of $H^{(m_1,m_2|n_1,n_2)}$ can equivalently be given by $E^{\mathbf{s}}_{\mathbf{r}}$ in (4.2.25), where the indices $\mathbf{r}$ and $\mathbf{s}$ are ordered by using the new set of rules 1)-3).

Next, we assume that the local spin $s \equiv (s^1, s^2, s^3)$ occurs $\gamma^{s^1,s^2,s^3}$ times in the

configuration $\mathbf{s} \equiv (s_1, s_2, \cdots, s_N)$. It is evident that $N$ can be written as

$$N = \sum_{s^1,s^2,s^3} \gamma^{s^1,s^2,s^3} \,. \tag{4.3.9}$$

Using the condition 1), we explicitly order the configuration $\mathbf{s}$ as

$$\mathbf{s} = (S_1, S_2, S_3, S_4) \,, \tag{4.3.10}$$

where

$$\begin{aligned}
S_1 &= \underbrace{(001), \cdots, (001)}_{\gamma^{001}}, \cdots\cdots, \underbrace{(00m_1), \cdots, (00m_1)}_{\gamma^{00m_1}}, \\
S_2 &= \underbrace{(011), \cdots, (011)}_{\gamma^{011}}, \cdots\cdots, \underbrace{(01m_2), \cdots, (01m_2)}_{\gamma^{01m_2}}, \\
S_3 &= \underbrace{(101), \cdots, (101)}_{\gamma^{101}}, \cdots\cdots, \underbrace{(10n_1), \cdots, (10n_1)}_{\gamma^{10n_1}}, \\
S_4 &= \underbrace{(111), \cdots, (111)}_{\gamma^{111}}, \cdots\cdots, \underbrace{(11n_2), \cdots, (11n_2)}_{\gamma^{11n_2}} \,.
\end{aligned} \tag{4.3.11}$$

Let $r_i^{s^1,s^2,s^3}$ be the local lower index corresponding to the $i$-th copy of the local upper index $s \equiv (s^1, s^2, s^3)$, where $i \in \{1, 2, \cdots, \gamma^{s^1,s^2,s^3}\}$. Due to the condition 2), we obtain a restriction on $r_i^{s^1,s^2,s^3}$ as

$$r_i^{s^1,s^2,s^3} \geqslant r_{i+1}^{s^1,s^2,s^3} + s^1 \,, \tag{4.3.12}$$

where $i \in \{1, 2, \cdots, \gamma^{s^1,s^2,s^3} - 1\}$. Furthermore, by using the condition 3), we can express any $r_i^{s^1,s^2,s^3}$ in the form

$$r_i^{s^1,s^2,s^3} = 2k_i^{s^1,s^2,s^3} + \delta_{s^2,1} \,, \tag{4.3.13}$$

where $k_i^{s^1,s^2,s^3}$ is a non-negative integer. Hence, the lower index $\mathbf{r}$ corresponding to

the upper index **s** in (4.3.10) and (4.3.11) may be written as

$$\mathbf{r} = (R_1, R_2, R_3, R_4)\,, \tag{4.3.14}$$

where

$$\begin{aligned}
R_1 &= \underbrace{2k_1^{001}, \cdots, 2k_{\gamma^{001}}^{001}}_{\gamma^{001}}, \cdots\cdots, \underbrace{2k_1^{00m_1}, \cdots, 2k_{\gamma^{00m_1}}^{00m_1}}_{\gamma^{00m_1}}\,,\\
R_2 &= \underbrace{2k_1^{011}+1, \cdots, 2k_{\gamma^{011}}^{011}+1}_{\gamma^{011}}, \cdots\cdots, \underbrace{2k_1^{01m_2}+1, \cdots, 2k_{\gamma^{01m_2}}^{01m_2}+1}_{\gamma^{01m_2}}\,,\\
R_3 &= \underbrace{2k_1^{101}, \cdots, 2k_{\gamma^{101}}^{101}}_{\gamma^{101}}, \cdots\cdots, \underbrace{2k_1^{10n_1}, \cdots, 2k_{\gamma^{10n_1}}^{10n_1}}_{\gamma^{10n_1}}\,,\\
R_4 &= \underbrace{2k_1^{111}+1, \cdots, 2k_{\gamma^{111}}^{111}+1}_{\gamma^{111}}, \cdots\cdots, \underbrace{2k_1^{11n_2}+1, \cdots, 2k_{\gamma^{11n_2}}^{11n_2}+1}_{\gamma^{11n_2}}
\end{aligned} \tag{4.3.15}$$

Due to Eqs. (4.3.12) and (4.3.13), we obtain a restriction on $k_i^{s^1,s^2,s^3}$ appearing in Eq. (4.3.15) as

$$k_i^{s^1,s^2,s^3} \geqslant k_{i+1}^{s^1,s^2,s^3} + s^1\,, \tag{4.3.16}$$

where $i \in \{1, 2, \cdots, \gamma^{s^1,s^2,s^3} - 1\}$.

Let us now try to evaluate the grand canonical partition function of the $BC_N$ type of spin CS model with SAPSRO (4.2.10). To this end, we note that $|\mathbf{r}|$ can be written as

$$|\mathbf{r}| = \sum_{s^1,s^2,s^3} \sum_{i=1}^{\gamma^{s^1,s^2,s^3}} r_i^{s^1,s^2,s^3}\,. \tag{4.3.17}$$

By using the above form of $|\mathbf{r}|$, the expression of $N$ given in (4.3.9) and the energy eigenvalue relation (4.2.25), we define the grand canonical partition function associated with the Hamiltonian (4.2.10) as

$$\mathbb{Z}_B^{(m_1,m_2|n_1,n_2)}(aT,\mu) = \sum_{\{\gamma^{t^1,t^2,t^3}\}\geqslant 0} \acute{\sum_{\{r_j^{t^1,t^2,t^3}\}\geqslant 0}} q^{|\mathbf{r}|-\mu N}\,, \tag{4.3.18}$$

where the symbol $\sum_{\{\gamma^{t^1,t^2,t^3}\}\geqslant 0}$ implies multiple sums over all $\gamma^{t^1,t^2,t^3}$ ranging from 0 to $\infty$, the symbol $\acute{\sum}_{\{r_j^{t^1,t^2,t^3}\}\geqslant 0}$ implies restricted multiple sums over all $r_j^{t^1,t^2,t^3}$ (ranging from 0 to $\infty$) which satisfy the conditions (4.3.12) and (4.3.13), and the contribution from the ground state energy has been ignored as before. Using (4.3.13), one can rewrite $|\mathbf{r}|$ in (4.3.17) as

$$|\mathbf{r}| = \sum_{s^1,s^2,s^3} \left\{ \sum_{i=1}^{\gamma^{s^1,s^2,s^3}} 2k_i^{s^1,s^2,s^3} + \delta_{s_2,1}\ \gamma^{s^1,s^2,s^3} \right\}.$$

Inserting the above expression of $|\mathbf{r}|$ along with $N$ given in (4.3.9) into Eq. (4.3.18), we obtain

$$\begin{aligned}
&\mathbb{Z}_B^{(m_1,m_2|n_1,n_2)} \\
&= \sum_{\{\gamma^{t^1,t^2,t^3}\}\geqslant 0} \acute{\sum_{\{k_j^{t^1,t^2,t^3}\}\geqslant 0}} q^{\sum\limits_{s^1,s^2,s^3} \left\{ \sum\limits_{i=1}^{\gamma^{s^1,s^2,s^3}} 2k_i^{s^1,s^2,s^3} + (\delta_{s_2,1}-\mu)\ \gamma^{s^1,s^2,s^3} \right\}} \\
&= \sum_{\{\gamma^{t^1,t^2,t^3}\}\geqslant 0} \acute{\sum_{\{k_j^{t^1,t^2,t^3}\}\geqslant 0}} \prod_{s^1,s^2,s^3} q^{\sum\limits_{i=1}^{\gamma^{s^1,s^2,s^3}} 2k_i^{s^1,s^2,s^3} + (\delta_{s_2,1}-\mu)\ \gamma^{s^1,s^2,s^3}},
\end{aligned} \tag{4.3.19}$$

where $\mathbb{Z}_B^{(m_1,m_2|n_1,n_2)} \equiv \mathbb{Z}_B^{(m_1,m_2|n_1,n_2)}(aT,\mu)$, and the symbol $\acute{\sum}_{\{k_j^{t^1,t^2,t^3}\}\geqslant 0}$ implies restricted multiple sums over all $k_j^{t^1,t^2,t^3}$ (ranging from 0 to $\infty$) which satisfy the condition (4.3.16). It is possible to express $\mathbb{Z}_B^{(m_1,m_2|n_1,n_2)}$ in (4.3.19) in a factorized form like

$$\begin{aligned}
&\mathbb{Z}_B^{(m_1,m_2|n_1,n_2)} \\
&= \prod_{s^1,s^2,s^3} \sum_{\gamma^{s^1,s^2,s^3}=0}^{\infty} q^{(\delta_{s_2,1}-\mu)\gamma^{s^1,s^2,s^3}} \left\{ \acute{\sum_{k_1^{s^1,s^2,s^3},\cdots,k_{\gamma^{s^1,s^2,s^3}}^{s^1,s^2,s^3}\geqslant 0}} q^{\sum\limits_{i=1}^{\gamma^{s^1,s^2,s^3}} 2k_i^{s^1,s^2,s^3}} \right\},
\end{aligned} \tag{4.3.20}$$

where $\acute{\sum}_{k_1^{s^1,s^2,s^3},\cdots,k_{\gamma^{s^1,s^2,s^3}}^{s^1,s^2,s^3}\geqslant 0}$ implies restricted multiple sums over the variables $k_1^{s^1,s^2,s^3}$, $k_2^{s^1,s^2,s^3}$, $\cdots$, $k_{\gamma^{s^1,s^2,s^3}}^{s^1,s^2,s^3}$ (ranging from 0 to $\infty$) which satisfy the condition (4.3.16).

Let us now rewrite Eq. (4.3.20) in the form

$$\mathbb{Z}_B^{(m_1,m_2|n_1,n_2)} = \prod_{s^1,s^2} \left( \prod_{s_3} \mathbb{Z}_B^{s^1,s^2,s^3} \right) , \tag{4.3.21}$$

where $\mathbb{Z}_B^{s^1,s^2,s^3}$ is given by

$$\mathbb{Z}_B^{s^1,s^2,s^3} = \sum_{\gamma^{s^1,s^2,s^3}=0}^{\infty} q^{(\delta_{s_2,1}-\mu)\,\gamma^{s^1,s^2,s^3}} \left\{ \acute{\sum_{k_1^{s^1,s^2,s^3},\cdots,k_{\gamma^{s^1,s^2,s^3}}^{s^1,s^2,s^3}\geqslant 0}} q^{\sum_{i=1}^{\gamma^{s^1,s^2,s^3}} 2k_i^{s^1,s^2,s^3}} \right\} . \tag{4.3.22}$$

Even though the above expression of $\mathbb{Z}_B^{s^1,s^2,s^3}$ implicitly depends on $s_1$ through the condition (4.3.16) and explicitly depends on $s_2$, it does not depend at all on the value of $s_3$. Consequently, by replacing each value of $s_3$ with 1, we can express Eq. (4.3.21) in a factorized form like

$$\mathbb{Z}_B^{(m_1,m_2|n_1,n_2)} = \left\{\mathbb{Z}_B^{0,0,1}\right\}^{m_1} \left\{\mathbb{Z}_B^{0,1,1}\right\}^{m_2} \left\{\mathbb{Z}_B^{1,0,1}\right\}^{n_1} \left\{\mathbb{Z}_B^{1,1,1}\right\}^{n_2} . \tag{4.3.23}$$

For small values of the discrete parameters satisfying the condition $m_1+m_2+n_1+n_2=1$ (like $m_1=1, m_2=n_1=n_2=0$), Eq. (4.3.23) leads to the relations

$$\mathbb{Z}_B^{(1,0|0,0)} = \mathbb{Z}_B^{0,0,1},\ \mathbb{Z}_B^{(0,1|0,0)} = \mathbb{Z}_B^{0,1,1},\ \mathbb{Z}_B^{(0,0|1,0)} = \mathbb{Z}_B^{1,0,1},\ \mathbb{Z}_B^{(0,0|0,1)} = \mathbb{Z}_B^{1,1,1}. \tag{4.3.24a,b,c,d}$$

Hence, we can write the grand canonical partition function (4.3.23) of the $BC_N$ type of spin CS model (4.2.10) completely in terms of the corresponding basic modes as

$$\mathbb{Z}_B^{(m_1,m_2|n_1,n_2)} = \left\{\mathbb{Z}_B^{(1,0|0,0)}\right\}^{m_1} \left\{\mathbb{Z}_B^{(0,1|0,0)}\right\}^{m_2} \left\{\mathbb{Z}_B^{(0,0|1,0)}\right\}^{n_1} \left\{\mathbb{Z}_B^{(0,0|0,1)}\right\}^{n_2} . \tag{4.3.25}$$

Let us now try to evaluate the four $BC_N$ type of basic modes appearing in the above relation. Using Eqs. (4.3.24a) and (4.3.22), along with the condition (4.3.16) (for

the case $s^1 = 0, s^2 = 0, s^3 = 1$), we obtain

$$\mathbb{Z}_B^{(1,0|0,0)} = \sum_{N=0}^{\infty} y^N \sum_{k_1^{001} \geqslant k_2^{001} \geqslant \cdots \geqslant k_N^{001} \geqslant 0} q^{\sum_{i=1}^{N} 2k_i^{001}} .$$

Using the identity (4.3.2), the above equation can be written as

$$\mathbb{Z}_B^{(1,0|0,0)} = \sum_{N=0}^{\infty} y^N \frac{1}{(q^2)_N} . \tag{4.3.26}$$

Next, by using Eqs. (4.3.24b) and (4.3.22), along with the condition (4.3.16) (for the case $s^1 = 0, s^2 = 1, s^3 = 1$), we obtain

$$\mathbb{Z}_B^{(0,1|0,0)} = \sum_{N=0}^{\infty} y^N q^N \sum_{k_1^{011} \geqslant k_2^{011} \geqslant \cdots \geqslant k_N^{011} \geqslant 0} q^{\sum_{i=1}^{N} 2k_i^{011}} .$$

Again, using the identity (4.3.2), the above equation can be written as

$$\mathbb{Z}_B^{(0,1|0,0)} = \sum_{N=0}^{\infty} y^N \frac{q^N}{(q^2)_N} . \tag{4.3.27}$$

Next, by using Eqs. (4.3.24c) and (4.3.22), along with the condition (4.3.16) (for the case $s^1 = 1, s^2 = 0, s^3 = 1$), we obtain

$$\mathbb{Z}_B^{(0,0|1,0)} = \sum_{N=0}^{\infty} y^N \sum_{k_1^{101} > k_2^{101} > \cdots > k_N^{101} \geqslant 0} q^{\sum_{i=1}^{N} 2k_i^{101}} ,$$

which, due to the identity (4.3.6), leads to

$$\mathbb{Z}_B^{(0,0|1,0)} = \sum_{N=0}^{\infty} y^N \frac{q^{N(N-1)}}{(q^2)_N} . \tag{4.3.28}$$

Finally, by using Eqs. (4.3.24d) and (4.3.22), along with the condition (4.3.16) (for

the case $s^1 = 1, s^2 = 1, s^3 = 1$), we obtain

$$\mathbb{Z}_B^{(0,0|0,1)} = \sum_{N=0}^{\infty} y^N q^N \sum_{k_1^{111} > k_2^{111} > \cdots > k_N^{111} \geqslant 0} q^{\sum_{i=1}^{N} 2k_i^{111}} ,$$

which, due to the identity (4.3.6), leads to

$$\mathbb{Z}_B^{(0,0|0,1)} = \sum_{N=0}^{\infty} y^N \frac{q^{N^2}}{(q^2)_N} . \tag{4.3.29}$$

The grand canonical partition function $\mathbb{Z}_B^{(m_1,m_2|n_1,n_2)}$ can be formally expanded as a power series of the fugacity parameter $y$ as

$$\mathbb{Z}_B^{(m_1,m_2|n_1,n_2)} = \sum_{N=0}^{\infty} y^N \, Z_{B,N}^{(m_1,m_2|n_1,n_2)}(aT) . \tag{4.3.30}$$

Inserting the expressions of the basic modes given in Eqs. (4.3.26), (4.3.27), (4.3.28) and (4.3.29) into the r.h.s. of Eq. (4.3.25), and comparing this r.h.s. of the latter equation with that of Eq. (4.3.30), we obtain a new expression for the canonical partition function of the $BC_N$ type of spin CS model (4.2.10) as

$$Z_{B,N}^{(m_1,m_2|n_1,n_2)}(aT) = \sum_{\substack{\sum_{i=1}^{m_1} a_i + \sum_{j=1}^{m_2} b_j + \sum_{k=1}^{n_1} c_k + \sum_{l=1}^{n_2} d_l = N \\ a_i \geqslant 0,\ b_j \geqslant 0,\ c_k \geqslant 0,\ d_l \geqslant 0}} \frac{q^{\sum_{j=1}^{m_2} b_j + \sum_{k=1}^{n_1} c_k(c_k-1) + \sum_{l=1}^{n_2} d_l^2}}{\prod_{i=1}^{m_1} (q^2)_{a_i} \prod_{j=1}^{m_2} (q^2)_{b_j} \prod_{k=1}^{n_1} (q^2)_{c_k} \prod_{l=1}^{n_2} (q^2)_{d_l}} . \tag{4.3.31}$$

Inserting the above expression of $Z_{B,N}^{(m_1,m_2|n_1,n_2)}(aT)$, along with $Z_{B,N}(aT)$ given in (4.2.18), into the relation (4.2.15), we also get a new expression for the canonical partition function of the $BC_N$ type of ferromagnetic PF chain (4.2.8) as

$$\mathcal{Z}_{B,N}^{(m_1,m_2|n_1,n_2)}(q) = \sum_{\substack{\sum_{i=1}^{m_1} a_i + \sum_{j=1}^{m_2} b_j + \sum_{k=1}^{n_1} c_k + \sum_{l=1}^{n_2} d_l = N \\ a_i \geqslant 0,\ b_j \geqslant 0,\ c_k \geqslant 0,\ d_l \geqslant 0}} \frac{(q^2)_N \cdot q^{\sum_{j=1}^{m_2} b_j + \sum_{k=1}^{n_1} c_k(c_k-1) + \sum_{l=1}^{n_2} d_l^2}}{\prod_{i=1}^{m_1} (q^2)_{a_i} \prod_{j=1}^{m_2} (q^2)_{b_j} \prod_{k=1}^{n_1} (q^2)_{c_k} \prod_{l=1}^{n_2} (q^2)_{d_l}} , \tag{4.3.32}$$

where, for the sake of convenience, the variable $q$ is used (instead of $T$) as the argument of $\mathcal{Z}_{B,N}^{(m_1,m_2|n_1,n_2)}$. It may be noted that, $\mathcal{Z}_{B,N}^{(m_1,m_2|n_1,n_2)}(q)$ in (4.3.32) can be rewritten by using the $q^2$-multinomial coefficients as

$$\begin{aligned}
&\mathcal{Z}_{B,N}^{(m_1,m_2|n_1,n_2)}(q) \\
&= \sum_{\substack{\sum_{i=1}^{m_1} a_i + \sum_{j=1}^{m_2} b_j + \sum_{k=1}^{n_1} c_k + \sum_{l=1}^{n_2} d_l = N \\ a_i \geqslant 0,\ b_j \geqslant 0,\ c_k \geqslant 0,\ d_l \geqslant 0}} q^{\sum_{j=1}^{m_2} b_j + \sum_{k=1}^{n_1} c_k(c_k-1) + \sum_{l=1}^{n_2} d_l^2} \left[ \begin{matrix} N \\ \{a\}_{m_1}\{b\}_{m_2}\{c\}_{n_1}\{d\}_{n_2} \end{matrix} \right]_{q^2},
\end{aligned} \tag{4.3.33}$$

where the notations $\{a\}_{m_1} \equiv a_1, \cdots, a_{m_1}$, $\{b\}_{m_2} \equiv b_1, \cdots, b_{m_2}$, $\{c\}_{n_1} \equiv c_1, \cdots, c_{n_1}$ and $\{d\}_{n_2} \equiv d_1, \cdots, d_{n_2}$ are used.

We like to make a comment here on the rather surprising appearance of both $BC_N$ and $A_{N-1}$ types of basic modes in the grand canonical partition functions of the $BC_N$ type of spin Calogero models with SASRO, as found in Ref. [101]. Choosing $m$ as any odd number, $n$ as any even number and taking two discrete parameters as $\epsilon = \epsilon' = 1$, the grand canonical partition function $\mathbb{Z}_{11}^{(m|n)}$ of the $BC_N$ type of spin Calogero model with SASRO has been computed in the latter reference as

$$\mathbb{Z}_{11}^{(m|n)} = \mathbb{Z}_B^{(1,0|0,0)} \left\{ \mathbb{Z}_A^{(1|0)} \right\}^{\frac{m-1}{2}} \left\{ \mathbb{Z}_A^{(0|1)} \right\}^{\frac{n}{2}}, \tag{4.3.34}$$

where we have used the notations of the present Chapter for all basic modes appearing in the r.h.s. of the above equation. Since spin Calogero models with SASRO can be reproduced from the more general class of spin Calogero models with SAPSRO (4.2.10) by imposing the condition (4.2.9), it should be possible to obtain the grand canonical partition functions of the former models from those of the later models by imposing the same condition. If $m$ is an odd number, $n$ is an even number and $\epsilon = \epsilon' = 1$, the condition (4.2.9) yields $m_1 = (m+1)/2$, $m_2 = (m-1)/2$ and $n_1 = n_2 = n/2$. Substituting these values of $m_1$, $m_2$, $n_1$ and $n_2$ in Eq. (4.3.25), and

also replacing the corresponding $\mathbb{Z}_B^{(m_1,m_2|n_1,n_2)}$ with the notation $\mathbb{Z}_{11}^{(m|n)}$, we find that

$$\mathbb{Z}_{11}^{(m|n)} = \mathbb{Z}_B^{(1,0|0,0)} \left\{ \mathbb{Z}_B^{(1,0|0,0)} \mathbb{Z}_B^{(0,1|0,0)} \right\}^{\frac{m-1}{2}} \left\{ \mathbb{Z}_B^{(0,0|1,0)} \mathbb{Z}_B^{(0,0|0,1)} \right\}^{\frac{n}{2}} . \tag{4.3.35}$$

Equating the r.h.s. of Eq. (4.3.34) with that of Eq. (4.3.35), we obtain novel relations like

$$\mathbb{Z}_A^{(1|0)} = \mathbb{Z}_B^{(1,0|0,0)} \mathbb{Z}_B^{(0,1|0,0)} , \qquad \mathbb{Z}_A^{(0|1)} = \mathbb{Z}_B^{(0,0|1,0)} \mathbb{Z}_B^{(0,0|0,1)} , \tag{4.3.36a,b}$$

which connects the $A_{N-1}$ and $BC_N$ types of basic modes associated with the grand canonical partition functions of the corresponding one-flavor Calogero models. Inserting the expressions of $\mathbb{Z}_A^{(1|0)}$ in (4.3.4), $\mathbb{Z}_B^{(1,0|0,0)}$ in (4.3.26), and $\mathbb{Z}_B^{(0,1|0,0)}$ in (4.3.27) into Eq. (4.3.36a), and comparing the coefficients of $q^N$ from both sides of the later equation, we obtain a $q$-identity of the form

$$\frac{1}{(q)_N} = \sum_{r=0}^{N} \frac{q^{N-r}}{(q^2)_r \cdot (q^2)_{N-r}} . \tag{4.3.37}$$

Similarly, inserting the expressions of $\mathbb{Z}_A^{(0|1)}$ in (4.3.8), $\mathbb{Z}_B^{(0,0|1,0)}$ in (4.3.28), and $\mathbb{Z}_B^{(0,0|0,1)}$ in (4.3.29) into Eq. (4.3.36b), and comparing the coefficients of $q^N$ from both sides of the later equation, we obtain another $q$-identity of the form

$$\frac{1}{(q)_N} = \sum_{r=0}^{N} \frac{q^{\frac{1}{2}(N-2r)(N-2r+1)}}{(q^2)_r \cdot (q^2)_{N-r}} . \tag{4.3.38}$$

One can easily verify that, for all possible choice of the parameters $m, n, \epsilon$ and $\epsilon'$, the grand canonical partition functions of the spin Calogero models with SASRO [101] can be reproduced in a similar way from Eq. (4.3.25) by using the condition (4.2.9) and the relations (4.3.36a,b). Hence, the appearance of both $BC_N$ and $A_{N-1}$ types of basic modes in the grand canonical partition functions of the $BC_N$ type of spin Calogero models with SASRO can be explained by employing the relations (4.3.36a,b).

It may be noted that, following a procedure similar to the case of $BC_N$ type of ferromagnetic PF chain (4.2.8), one can also calculate the partition function of the $BC_N$ type of anti-ferromagnetic PF chain with Hamiltonian given by

$$\widetilde{\mathcal{H}}^{(m_1,m_2|n_1,n_2)} = \sum_{i\neq j}\left[\frac{1+P_{ij}^{(m|n)}}{(\xi_i-\xi_j)^2}+\frac{1+\widetilde{P}_{ij}^{(m_1,m_2|n_1,n_2)}}{(\xi_i+\xi_j)^2}\right]+\beta\sum_{i=1}^{N}\frac{1+P_i^{(m_1,m_2|n_1,n_2)}}{\xi_i^2}\,. \tag{4.3.39}$$

By using the freezing trick, the above Hamiltonian can be obtained from the $BC_N$ type of spin Calogero Hamiltonian like

$$\begin{aligned}\widetilde{H}^{(m_1,m_2|n_1,n_2)} = -\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2}+\frac{a^2}{4}r^2 \quad &+a\sum_{i\neq j}\left[\frac{a+P_{ij}^{(m|n)}}{(x_{ij}^-)^2}+\frac{a+\widetilde{P}_{ij}^{(m_1,m_2|n_1,n_2)}}{(x_{ij}^+)^2}\right]\\ &+\beta a\sum_{i=1}^{N}\frac{\beta a+P_i^{(m_1,m_2|n_1,n_2)}}{x_i^2}\,.\end{aligned} \tag{4.3.40}$$

Hence, we can derive an analogue of Eq. (4.2.15) as

$$\widetilde{\mathcal{Z}}_{B,N}^{(m_1,m_2|n_1,n_2)}(T) = \lim_{a\to\infty}\frac{\widetilde{Z}_{B,N}^{(m_1,m_2|n_1,n_2)}(aT)}{Z_{B,N}(aT)}\,, \tag{4.3.41}$$

where $\widetilde{\mathcal{Z}}_{B,N}^{(m_1,m_2|n_1,n_2)}(T)$ and $\widetilde{Z}_{B,N}^{(m_1,m_2|n_1,n_2)}(aT)$ represent canonical partition functions of the $BC_N$ type of anti-ferromagnetic PF spin chain (4.3.39) and spin Calogero model (4.3.40) respectively. The Hilbert space of $\widetilde{H}^{(m_1,m_2|n_1,n_2)}$ in (4.3.40) can be obtained as the closure of the linear subspace spanned by the wave functions which are quite similar to their ferromagnetic counterparts (4.2.19). More precisely, the state vectors associated with the Hilbert space of $\widetilde{H}^{(m_1,m_2|n_1,n_2)}$ are given by

$$\tilde{\psi}_{\mathbf{r}}^{\mathbf{s}} \equiv \tilde{\psi}_{r_1,\ldots,r_i,\ldots,r_j,\ldots,r_N}^{s_1,\ldots,s_i,\ldots,s_j,\ldots,s_N} = \tilde{\Lambda}^{(m_1,m_2|n_1,n_2)}\left(\phi_{\mathbf{r}}(\mathbf{x})|\mathbf{s}\rangle\right)\,, \tag{4.3.42}$$

where the completely symmetric projector $\Lambda^{(m_1,m_2|n_1,n_2)}$ in (4.2.19) is replaced by the completely antisymmetric projector $\tilde{\Lambda}^{(m_1,m_2|n_1,n_2)}$ related to the $BC_N$ type of Weyl algebra. Due to this change of the projector, the symmetry conditions (4.2.20) and

(4.2.21) of the wave functions are now modified as

$$\tilde{\psi}^{s_1,\ldots,s_i,\ldots,s_j,\ldots,s_N}_{r_1,\ldots,r_i,\ldots,r_j,\ldots,r_N} = (-1)^{\alpha_{ij}(\mathbf{s})+1}\, \tilde{\psi}^{s_1,\ldots,s_j,\ldots,s_i,\ldots,s_N}_{r_1,\ldots,r_j,\ldots,r_i,\ldots,r_N}\,, \tag{4.3.43}$$

and

$$\tilde{\psi}^{s_1,\ldots,s_i,\ldots,s_N}_{r_1,\ldots,r_i,\ldots,r_N} = (-1)^{r_i+f(s_i)+1}\, \tilde{\psi}^{s_1,\ldots,s_i,\ldots,s_N}_{r_1,\ldots,r_i,\ldots,r_N}\,, \tag{4.3.44}$$

where $1 \leqslant i < j \leqslant N$. Due to these modified symmetry conditions, it is possible to obtain a set of (non-orthonormal) basis vectors for the Hilbert space of $\widetilde{H}^{(m_1,m_2|n_1,n_2)}$ by ordering the indices $\mathbf{s}$ and $\mathbf{r}$ of the state vectors $\tilde{\psi}^{\mathbf{s}}_{\mathbf{r}}$ in a suitable way. More precisely, the rule 1) described in this section for ordering $\mathbf{s}$ in the ferromagnetic case remains unchanged in the present case, while the rules 2) and 3) for ordering $\mathbf{r}$ in the ferromagnetic case are modified in the following way:

2′) If $s_i = s_j$ for $i < j$, then by using (4.3.43) the components $r_i$ and $r_j$ within $\mathbf{r} \equiv (r_1, r_2, \cdots, r_N)$ are ordered such that $r_i \geqslant r_j + 1 - \pi(s_j)$.

3′) Due to the condition (4.3.44), $r_i$ is taken as an odd positive integer if $s_i^2 \equiv f(s_i) = 0$ and $r_i$ is taken as an even nonnegative integer if $s_i^2 \equiv f(s_i) = 1$.

If one defines a partial ordering among the above mentioned basis vectors as: $\tilde{\psi}^{\mathbf{s}}_{\mathbf{r}} > \tilde{\psi}^{\mathbf{s}'}_{\mathbf{r}'}$, for $|\mathbf{r}| > |\mathbf{r}'|$, then $\widetilde{H}^{(m_1,m_2|n_1,n_2)}$ in (4.3.40) would act as an upper triangular matrix on such partially ordered basis vectors. Consequently, all eigenvalues in the spectrum of $\widetilde{H}^{(m_1,m_2|n_1,n_2)}$ are given by $E^{\mathbf{s}}_{\mathbf{r}}$ in (4.2.25), where the indices $\mathbf{r}$ and $\mathbf{s}$ are now ordered by using the set of rules 1), 2′), and 3′). Appropriately modifying the procedure described in this section for the ferromagnetic case, in accordance with this new set of rules, we find that the grand canonical partition function of the $BC_N$ type of spin CS model (4.3.40) can be expressed as

$$\widetilde{\mathbb{Z}}_B^{(m_1,m_2|n_1,n_2)} = \left\{\widetilde{\mathbb{Z}}_B^{(1,0|0,0)}\right\}^{m_1} \left\{\widetilde{\mathbb{Z}}_B^{(0,1|0,0)}\right\}^{m_2} \left\{\widetilde{\mathbb{Z}}_B^{(0,0|1,0)}\right\}^{n_1} \left\{\widetilde{\mathbb{Z}}_B^{(0,0|0,1)}\right\}^{n_2}, \tag{4.3.45}$$

where the anti-ferromagnetic basic modes appearing in the r.h.s. of the above equation are related to their ferromagnetic counterparts as

$$\widetilde{\mathbb{Z}}_B^{(1,0|0,0)} = \mathbb{Z}_B^{(0,0|0,1)},\ \widetilde{\mathbb{Z}}_B^{(0,1|0,0)} = \mathbb{Z}_B^{(0,0|1,0)},\ \widetilde{\mathbb{Z}}_B^{(0,0|1,0)} = \mathbb{Z}_B^{(0,1|0,0)},\ \widetilde{\mathbb{Z}}_B^{(0,0|0,1)} = \mathbb{Z}_B^{(1,0|0,0)}. \tag{4.3.46}$$

Expanding the grand canonical partition function (4.3.45) as a power series of the fugacity parameter, we derive the corresponding canonical partition function as

$$\widetilde{Z}_{B,N}^{(m_1,m_2|n_1,n_2)}(aT) = \sum_{\substack{\sum_{i=1}^{m_1} a_i + \sum_{j=1}^{m_2} b_j + \sum_{k=1}^{n_1} c_k + \sum_{l=1}^{n_2} d_l = N \\ a_i \geqslant 0,\ b_j \geqslant 0,\ c_k \geqslant 0,\ d_l \geqslant 0}} \frac{q^{\sum_{i=1}^{m_1} a_i^2 + \sum_{j=1}^{m_2} b_j(b_j-1) + \sum_{k=1}^{n_1} c_k}}{\prod_{i=1}^{m_1}(q^2)_{a_i} \prod_{j=1}^{m_2}(q^2)_{b_j} \prod_{k=1}^{n_1}(q^2)_{c_k} \prod_{l=1}^{n_2}(q^2)_{d_l}}. \tag{4.3.47}$$

Substituting this expression of $\widetilde{Z}_{B,N}^{(m_1,m_2|n_1,n_2)}(aT)$, along with $Z_{B,N}(aT)$ given in (4.2.18), to the relation (4.3.41), we finally obtain the canonical partition function of the $BC_N$ type of anti-ferromagnetic PF chain (4.3.39) as

$$\widetilde{\mathcal{Z}}_{B,N}^{(m_1,m_2|n_1,n_2)}(q) = \sum_{\substack{\sum_{i=1}^{m_1} a_i + \sum_{j=1}^{m_2} b_j + \sum_{k=1}^{n_1} c_k + \sum_{l=1}^{n_2} d_l = N \\ a_i \geqslant 0,\ b_j \geqslant 0,\ c_k \geqslant 0,\ d_l \geqslant 0}} \frac{(q^2)_N \cdot q^{\sum_{i=1}^{m_1} a_i^2 + \sum_{j=1}^{m_2} b_j(b_j-1) + \sum_{k=1}^{n_1} c_k}}{\prod_{i=1}^{m_1}(q^2)_{a_i} \prod_{j=1}^{m_2}(q^2)_{b_j} \prod_{k=1}^{n_1}(q^2)_{c_k} \prod_{l=1}^{n_2}(q^2)_{d_l}}. \tag{4.3.48}$$

In the next section, we shall use the expressions (4.3.32) and (4.3.48) for the partition functions of the $BC_N$ type of PF spin chains for the purpose of constructing corresponding multivariate SRS polynomials.

## 4.4 SRS polynomials associated with $BC_N$ type of PF spin chains

We have seen earlier that the homogeneous multivariate RS polynomial (4.1.3) is closely related to the partition function (4.1.2) of the $A_{N-1}$ type of su($m$) PF spin chain. Before starting the discussion on the $BC_N$ type of homogeneous multivariate SRS polynomial, let us briefly review the connection between the partition function of the $A_{N-1}$ type of su($m|n$) supersymmetric PF spin chain and the corresponding SRS polynomial [94]. The partition functions of the $A_{N-1}$ type of su($m|n$) supersymmetric PF spins have been computed by using the freezing trick as [93]

$$\mathcal{Z}_{A,N}^{(m|n)}(q) = \sum_{\substack{\sum_{i=1}^{m} a_i + \sum_{j=1}^{n} b_j = N \\ a_i \geqslant 0,\ b_j \geqslant 0}} \frac{(q)_N \cdot q^{\sum_{j=1}^{n} \frac{b_j(b_j-1)}{2}}}{\prod_{i=1}^{m}(q)_{a_i} \prod_{j=1}^{n}(q)_{b_j}} \,. \tag{4.4.1}$$

It may be noted that, in the absence of the fermionic spin degrees of freedom (i.e., for the case $n = 0$), $\mathcal{Z}_{A,N}^{(m|n)}(q)$ reduces to $\mathcal{Z}_{A,N}^{(m)}(q)$ in (4.1.2). Motivated by the form of the partition functions (4.4.1), the $A_{N-1}$ type of SRS polynomials have been defined as [94]

$$\mathbb{H}_{A,N}^{(m|n)}(x,y;q) = \sum_{\substack{\sum_{i=1}^{m} a_i + \sum_{j=1}^{n} b_j = N \\ a_i \geqslant 0,\ b_j \geqslant 0}} (q)_N \cdot q^{\sum_{j=1}^{n} \frac{b_j(b_j-1)}{2}} \prod_{i=1}^{m} \frac{x_i^{a_i}}{(q)_{a_i}} \prod_{j=1}^{n} \frac{y_j^{b_j}}{(q)_{b_j}} \,, \tag{4.4.2}$$

(along with $\mathbb{H}_{A,0}^{(m|n)}(x,y;q) = 1$), where $x \equiv x_1, x_2, \cdots, x_m$ and $y \equiv y_1, y_2, \cdots, y_n$ represent two different types of variables. It is evident that the partition functions (4.4.1) can be obtained from the SRS polynomials (4.4.2) as $\mathcal{Z}_{A,N}^{(m|n)}(q) = \mathbb{H}_{A,N}^{(m|n)}(x = 1, y = 1; q)$. Moreover, for the special case $n = 0$, the SRS polynomial (4.4.2) reduces

to its bosonic counterpart (4.1.3). By using the relation

$$(q^{-1})_l = (-1)^l \, q^{-\frac{l(l+1)}{2}} \, (q)_l \,, \tag{4.4.3}$$

the SRS polynomials (4.4.2) may be rewritten as

$$\mathbb{H}^{(m|n)}_{A,N}(x,y;q) = \sum_{\substack{\sum_{i=1}^{m} a_i + \sum_{j=1}^{n} b_j = N \\ a_i \geqslant 0,\ b_j \geqslant 0}} (q)_N \cdot \prod_{i=1}^{m} \frac{x_i^{a_i}}{(q)_{a_i}} \prod_{j=1}^{n} \frac{(-q^{-1}y_j)^{b_j}}{(q^{-1})_{b_j}} \,. \tag{4.4.4}$$

The above form of the SRS polynomials can be obtained from a power series expansion of the generating function given by [94]

$$\mathcal{G}^{(m|n)}_{A}(x,y;q,t) = \frac{1}{\prod_{i=1}^{m}(tx_i;q)_\infty \cdot \prod_{j=1}^{n}(-tq^{-1}y_j;q^{-1})_\infty} \,, \tag{4.4.5}$$

where $(t;q)_0 \equiv 1$ and $(t;q)_l \equiv (1-t)(1-qt)\cdots(1-q^{l-1}t)$ for $l > 0$. Indeed, by using the identity [103]

$$\frac{1}{(t:q)_\infty} = \sum_{N=0}^{\infty} \frac{t^N}{(q)_N} \,, \tag{4.4.6}$$

it is easy to check that the generating function in (4.4.5) can be expanded as a power series of the parameter $t$ as

$$\mathcal{G}^{(m|n)}_{A}(x,y;q,t) = \sum_{N=0}^{\infty} \frac{\mathbb{H}^{(m|n)}_{A,N}(x,y;q)}{(q)_N} \, t^N \,. \tag{4.4.7}$$

Inspired by the form of partition functions (4.3.32) of the $BC_N$ type of ferromagnetic PF chains with SAPSRO, we define $BC_N$ type of homogeneous multivariate SRS

polynomials of the first kind as

$$\mathbb{H}_{B,N}^{(m_1,m_2|n_1,n_2)}(x,\bar{x},y,\bar{y};q) = \sum_{\substack{\sum_{i=1}^{m_1} a_i+\sum_{j=1}^{m_2} b_j+\sum_{k=1}^{n_1} c_k+\sum_{l=1}^{n_2} d_l=N \\ a_i\geqslant 0,b_j\geqslant 0,c_k\geqslant 0,d_l\geqslant 0}} (q^2)_N \cdot q^{\sum_{j=1}^{m_2} b_j+\sum_{k=1}^{n_1} c_k(c_k-1)+\sum_{l=1}^{n_2} d_l^2} \cdot \prod_{i=1}^{m_1} \frac{x_i^{a_i}}{(q^2)_{a_i}} \prod_{j=1}^{m_2} \frac{(\bar{x}_j)^{b_j}}{(q^2)_{b_j}} \prod_{k=1}^{n_1} \frac{y_k^{c_k}}{(q^2)_{c_k}} \prod_{l=1}^{n_2} \frac{(\bar{y}_l)^{d_l}}{(q^2)_{d_l}}, \tag{4.4.8}$$

and set $\mathbb{H}_{B,0}^{(m_1,m_2|n_1,n_2)}(x,\bar{x},y,\bar{y};q)=1$, where $x\equiv x_1,x_2,\cdots,x_{m_1}$, $\bar{x}\equiv\bar{x}_1,\bar{x}_2,\cdots,\bar{x}_{m_2}$, $y\equiv y_1,y_2,\cdots,y_{n_1}$ and $\bar{y}\equiv\bar{y}_1,\bar{y}_2,\cdots,\bar{y}_{n_2}$ represent four different types of variables. The partition functions in (4.3.32) can be obtained from these $BC_N$ type of SRS polynomials as

$$\mathcal{Z}_{B,N}^{(m_1,m_2|n_1,n_2)}(q) = \mathbb{H}_{B,N}^{(m_1,m_2|n_1,n_2)}(x=1,\bar{x}=1,y=1,\bar{y}=1;q)\,. \tag{4.4.9}$$

In the absence of the fermionic spin degrees of freedom, i.e., for the case $n_1=n_2=0$, the $BC_N$ type of SRS polynomials (4.4.8) reduce to that type of RS polynomials of the form

$$\mathbb{H}_{B,N}^{(m_1,m_2)}(x,\bar{x};q) = \sum_{\substack{\sum_{i=1}^{m_1} a_i+\sum_{j=1}^{m_2} b_j=N \\ a_i\geqslant 0,\ b_j\geqslant 0}} (q^2)_N \cdot q^{\sum_{j=1}^{m_2} b_j} \cdot \prod_{i=1}^{m_1} \frac{x_i^{a_i}}{(q^2)_{a_i}} \prod_{j=1}^{m_2} \frac{(\bar{x}_j)^{b_j}}{(q^2)_{b_j}}\,. \tag{4.4.10}$$

Interestingly, in another special case like $m_1=m$, $m_2=0$, $n_1=n$, $n_2=0$, the $BC_N$ type of SRS polynomials (4.4.8) can be connected with the $A_{N-1}$ type of SRS polynomials (4.4.2) as

$$\mathbb{H}_{B,N}^{(m,0|n,0)}(x,y;q) = \mathbb{H}_{A,N}^{(m|n)}(x,y;q^2)\,. \tag{4.4.11}$$

By using the relation (4.4.3), we express the $BC_N$ type of SRS polynomials (4.4.8)

in a more compact form as

$$\mathbb{H}_{B,N}^{(m_1,m_2|n_1,n_2)}(x,\bar{x},y,\bar{y};q)$$
$$= \sum_{\substack{\sum_{i=1}^{m_1} a_i+\sum_{j=1}^{m_2} b_j+\sum_{k=1}^{n_1} c_k+\sum_{l=1}^{n_2} d_l=N \\ a_i\geqslant 0,b_j\geqslant 0,c_k\geqslant 0,d_l\geqslant 0}} (q^2)_N \cdot \prod_{i=1}^{m_1} \frac{x_i^{a_i}}{(q^2)_{a_i}} \prod_{j=1}^{m_2} \frac{(q\bar{x}_j)^{b_j}}{(q^2)_{b_j}} \prod_{k=1}^{n_1} \frac{(-q^{-2}y_k)^{c_k}}{(q^{-2})_{c_k}} \prod_{l=1}^{n_2} \frac{(-q^{-1}\bar{y}_l)^{d_l}}{(q^{-2})_{d_l}} \,. \tag{4.4.12}$$

Let us now define a generating function of the form

$$\mathcal{G}_B^{(m_1,m_2|n_1,n_2)}(x,\bar{x},y,\bar{y};q,t) = \mathcal{G}_1^{(m_1)}(x;q,t)\cdot\mathcal{G}_2^{(m_2)}(\bar{x};q,t)\cdot\mathcal{G}_3^{(n_1)}(y;q,t)\cdot\mathcal{G}_4^{(n_2)}(\bar{y};q,t), \tag{4.4.13}$$

where

$$\mathcal{G}_1^{(m_1)}(x;q,t) = \frac{1}{\prod_{i=1}^{m_1}(tx_i;q^2)_\infty}\,, \tag{4.4.14a}$$
$$\mathcal{G}_2^{(m_2)}(\bar{x};q,t) = \frac{1}{\prod_{j=1}^{m_2}(tq\bar{x}_j;q^2)_\infty}\,, \tag{4.4.14b}$$
$$\mathcal{G}_3^{(n_1)}(y;q,t) = \frac{1}{\prod_{k=1}^{n_1}(-tq^{-2}y_k;q^{-2})_\infty}\,, \tag{4.4.14c}$$
$$\mathcal{G}_4^{(n_2)}(\bar{y};q,t) = \frac{1}{\prod_{l=1}^{n_2}(-tq^{-1}\bar{y}_l;q^{-2})_\infty}\,. \tag{4.4.14d}$$

Expanding all terms appearing in the r.h.s. of Eq. (4.4.13) by using the identity (4.4.6) and subsequently using the expression of the $BC_N$ type of SRS polynomials given in (4.4.12), we obtain

$$\mathcal{G}_B^{(m_1,m_2|n_1,n_2)}(x,\bar{x},y,\bar{y};q,t) = \sum_{N=0}^{\infty} \frac{\mathbb{H}_{B,N}^{(m_1,m_2|n_1,n_2)}(x,\bar{x},y,\bar{y};q)}{(q^2)_N}\, t^N\,. \tag{4.4.15}$$

Thus $\mathcal{G}_B^{(m_1,m_2|n_1,n_2)}(x,\bar{x},y,\bar{y};q,t)$ in (4.4.13) represents the generating function of the

$BC_N$ type of SRS polynomials.

We have already seen in Eq. (4.4.11) that, in a particular case, the $BC_N$ type of SRS polynomial can be expressed through the $A_{N-1}$ type of SRS polynomial. For the purpose of exploring such connection between the $BC_N$ and $A_{N-1}$ types of SRS polynomials in a general case, we use Eqs. (4.4.14a), (4.4.14c) and (4.4.5) to find that

$$\mathcal{G}_1^{(m_1)}(x;q,t)\cdot\mathcal{G}_3^{(n_1)}(y;q,t)=\mathcal{G}_A^{(m_1|n_1)}(x,y;q^2,t). \tag{4.4.16}$$

Hence, by using the power series expansion (4.4.7), we obtain

$$\mathcal{G}_1^{(m_1)}(x;q,t)\cdot\mathcal{G}_3^{(n_1)}(y;q,t)=\sum_{N_1=0}^{\infty}\frac{\mathbb{H}_{A,N_1}^{(m_1|n_1)}(x,y;q^2)}{(q^2)_{N_1}}\,t^{N_1}. \tag{4.4.17}$$

Next, by using Eqs. (4.4.14b), (4.4.14d) and (4.4.5), we find that

$$\mathcal{G}_2^{(m_2)}(\bar{x};q,t)\cdot\mathcal{G}_4^{(n_2)}(\bar{y};q,t)=\mathcal{G}_A^{(m_2|n_2)}(\tilde{x},\tilde{y};q^2,t), \tag{4.4.18}$$

where $\tilde{x}\equiv q\cdot\bar{x}$ and $\tilde{y}\equiv q\cdot\bar{y}$. Hence, by using (4.4.7), we obtain

$$\mathcal{G}_2^{(m_2)}(\bar{x};q,t)\cdot\mathcal{G}_4^{(n_2)}(\bar{y};q,t)=\sum_{N_2=0}^{\infty}\frac{\mathbb{H}_{A,N_2}^{(m_2|n_2)}(\tilde{x},\tilde{y};q^2)}{(q^2)_{N_2}}\,t^{N_2}. \tag{4.4.19}$$

Since $\mathbb{H}_{A,N_2}^{(m_2|n_2)}(\tilde{x},\tilde{y};q^2)$ is a homogeneous polynomial of the variables $\tilde{x}$, $\tilde{y}$ of order $N_2$, the above equation can be rewritten as

$$\mathcal{G}_2^{(m_2)}(\bar{x};q,t)\cdot\mathcal{G}_4^{(n_2)}(\bar{y};q,t)=\sum_{N_2=0}^{\infty}\frac{q^{N_2}}{(q^2)_{N_2}}\,\mathbb{H}_{A,N_2}^{(m_2|n_2)}(\bar{x},\bar{y};q^2)\,t^{N_2}. \tag{4.4.20}$$

Inserting the series expansions (4.4.17) and (4.4.20) in Eq. (4.4.13), it is easy to find

that

$$\mathcal{G}_B^{(m_1,m_2|n_1,n_2)}(x,\bar{x},y,\bar{y};q,t)$$
$$=\sum_{N=0}^{\infty}t^N\sum_{N_1=0}^{N}\frac{q^{N-N_1}}{(q^2)_{N_1}\cdot(q^2)_{N-N_1}}\mathbb{H}_{A,N_1}^{(m_1|n_1)}(x,y;q^2)\cdot\mathbb{H}_{A,N-N_1}^{(m_2|n_2)}(\bar{x},\bar{y};q^2). \quad (4.4.21)$$

Comparing the coefficients of $t^N$ in the r.h.s. of (4.4.15) and (4.4.21), we finally obtain a relation between the $BC_N$ and $A_{N-1}$ types of SRS polynomials as

$$\mathbb{H}_{B,N}^{(m_1,m_2|n_1,n_2)}(x,\bar{x},y,\bar{y};q,t)$$
$$=\sum_{N_1=0}^{N}q^{N-N_1}\begin{pmatrix}N\\N_1\end{pmatrix}_{q^2}\mathbb{H}_{A,N_1}^{(m_1|n_1)}(x,y;q^2)\cdot\mathbb{H}_{A,N-N_1}^{(m_2|n_2)}(\bar{x},\bar{y};q^2)\,, \quad (4.4.22)$$

where the notation $\binom{N}{N_1}_{q^2}\equiv\begin{bmatrix}N\\N_1,N-N_1\end{bmatrix}_{q^2}$ has been used.

It may be noted that, even though the $BC_N$ type of partition function given in (4.3.32) is rather complicated in form, it can be easily computed for arbitrary values of $N$ and for some small values of the discrete parameters $m_1,\ m_2,\ n_1$ and $n_2$. In particular, by using (4.3.32), it is easy to find that

$$\mathcal{Z}_{B,N}^{(1,0|0,0)}(q)=1,\ \mathcal{Z}_{B,N}^{(0,1|0,0)}(q)=q^N,\ \mathcal{Z}_{B,N}^{(0,0|1,0)}(q)=q^{N(N-1)},\ \mathcal{Z}_{B,N}^{(0,0|0,1)}(q)=q^{N^2}. \quad (4.4.23)$$

In this context it is interesting to ask whether there exists some recursion relations such that, by taking the partition functions given in (4.4.23) as the initial conditions, it is possible to compute $\mathcal{Z}_{B,N}^{(m_1,m_2|n_1,n_2)}(q)$ for arbitrarily values of the discrete parameters $m_1,\ m_2,\ n_1,\ n_2$ and $N$. To answer this question, we define a generating function as

$$\mathcal{G}_B^{(m_1,m_2|n_1,n_2)}(q,t)\equiv\mathcal{G}_B^{(m_1,m_2|n_1,n_2)}(x=1,\bar{x}=1,y=1,\bar{y}=1;q,t)\,. \quad (4.4.24)$$

Inserting $x=1,\ \bar{x}=1,\ y=1,\ \bar{y}=1$ in (4.4.15) and also using (4.4.9), one can

expand $\mathcal{G}_B^{(m_1,m_2|n_1,n_2)}(q,t)$ in a power series of $t$ as

$$\mathcal{G}_B^{(m_1,m_2|n_1,n_2)}(q,t) = \sum_{N=0}^{\infty} \frac{\mathcal{Z}_{B,N}^{(m_1,m_2|n_1,n_2)}(q)}{(q^2)_N}\, t^N\,, \tag{4.4.25}$$

where it is assumed that $\mathcal{Z}_{B,0}^{(m_1,m_2|n_1,n_2)}(q) = 1$. Therefore, $\mathcal{G}_B^{(m_1,m_2|n_1,n_2)}(q,t)$ may be considered as the generating function for the partition function $\mathcal{Z}_{B,N}^{(m_1,m_2|n_1,n_2)}(q)$. Next, by using Eqs. (4.4.13), (4.4.14) and (4.4.24), we find that

$$\begin{aligned}&\mathcal{G}_B^{(m_1,m_2|n_1,n_2)}(q,t)\\ &\quad= \frac{1}{\{(t;q^2)_\infty\}^{m_1}\cdot\{(tq;q^2)_\infty\}^{m_2}\cdot\{(-tq^2;q^{-2})_\infty\}^{n_1}\cdot\{(-tq^{-1};q^{-2})_\infty\}^{n_2}}\,.\end{aligned}$$

Consequently, this generating function satisfies a factorization relation given by

$$\mathcal{G}_B^{(m_1+m_1',m_2+m_2'|n_1+n_1',n_2+n_2')}(q,t) = \mathcal{G}_B^{(m_1,m_2|n_1,n_2)}(q,t)\cdot\mathcal{G}_B^{(m_1',m_2'|n_1',n_2')}(q,t). \tag{4.4.26}$$

Expanding both sides of the above equation by using (4.4.25) and comparing the coefficients of $t^N$, we find that

$$\mathcal{Z}_{B,N}^{(m_1+m_1',m_2+m_2'|n_1+n_1',n_2+n_2')}(q) = \sum_{N_1=0}^{N}\begin{pmatrix}N\\N_1\end{pmatrix}_{q^2}\mathcal{Z}_{B,N-N_1}^{(m_1,m_2|n_1,n_2)}(q)\cdot\mathcal{Z}_{B,N_1}^{(m_1',m_2'|n_1',n_2')}\,. \tag{4.4.27}$$

Appropriately choosing the values of discrete variables $m_1'$, $m_2'$, $n_1'$, $n_2'$ in Eq. (4.4.27)

and also using Eq. (4.4.23), we derive a set of recursion relations like

$$\begin{aligned}
\mathcal{Z}_{B,N}^{(m_1+1,m_2|n_1,n_2)}(q) &= \sum_{N_1=0}^{N} \binom{N}{N_1}_{q^2} \cdot \mathcal{Z}_{B,N-N_1}^{(m_1,m_2|n_1,n_2)}(q)\,, \\
\mathcal{Z}_{B,N}^{(m_1,m_2+1|n_1,n_2)}(q) &= \sum_{N_1=0}^{N} \binom{N}{N_1}_{q^2} \cdot q^{N_1} \cdot \mathcal{Z}_{B,N-N_1}^{(m_1,m_2|n_1,n_2)}(q)\,, \\
\mathcal{Z}_{B,N}^{(m_1,m_2|n_1+1,n_2)}(q) &= \sum_{N_1=0}^{N} \binom{N}{N_1}_{q^2} \cdot q^{N_1(N_1-1)} \cdot \mathcal{Z}_{B,N-N_1}^{(m_1,m_2|n_1,n_2)}(q)\,, \\
\mathcal{Z}_{B,N}^{(m_1,m_2|n_1,n_2+1)}(q) &= \sum_{N_1=0}^{N} \binom{N}{N_1}_{q^2} \cdot q^{N_1^2} \cdot \mathcal{Z}_{B,N-N_1}^{(m_1,m_2|n_1,n_2)}(q)\,.
\end{aligned} \tag{4.4.28}$$

By using this set of recursion relations and also using the initial conditions in (4.4.23), it is possible to compute $\mathcal{Z}_{B,N}^{(m_1,m_2|n_1,n_2)}(q)$ for arbitrarily values of the discrete parameters $m_1,\ m_2,\ n_1,\ n_2$ and $N$. Furthermore, it is easy to check that, all initial conditions appearing in (4.4.23) can also be derived from the recursion relations (4.4.28) by using only one initial condition given by

$$\mathcal{Z}_{B,N}^{(0,0|0,0)}(q) = \delta_{N,0}\,. \tag{4.4.29}$$

Next, we define $BC_N$ type of homogeneous multivariate SRS polynomials of the second kind as

$$\begin{aligned}
&\widetilde{\mathbb{H}}_{B,N}^{(m_1,m_2|n_1,n_2)}(x,\bar{x},y,\bar{y};q) \\
&= \sum_{\substack{\sum_{i=1}^{m_1} a_i+\sum_{j=1}^{m_2} b_j+\sum_{k=1}^{n_1} c_k+\sum_{l=1}^{n_2} d_l=N \\ a_i\geqslant 0,b_j\geqslant 0,c_k\geqslant 0,d_l\geqslant 0}} (q^2)_N \cdot q^{\sum_{i=1}^{m_1} a_i^2+\sum_{j=1}^{m_2} b_j(b_j-1)+\sum_{k=1}^{n_1} c_k} \cdot \prod_{i=1}^{m_1} \frac{x_i^{a_i}}{(q^2)_{a_i}} \prod_{j=1}^{m_2} \frac{(\bar{x}_j)^{b_j}}{(q^2)_{b_j}} \prod_{k=1}^{n_1} \frac{y_k^{c_k}}{(q^2)_{c_k}} \prod_{l=1}^{n_2} \frac{(\bar{y}_l)^{d_l}}{(q^2)_{d_l}},
\end{aligned} \tag{4.4.30}$$

which is related to the partition function (4.3.48) associated with the $BC_N$ type of

anti-ferromagnetic PF spin chain as

$$\widetilde{\mathcal{Z}}_{B,N}^{(m_1,m_2|n_1,n_2)}(q) = \widetilde{\mathbb{H}}_{B,N}^{(m_1,m_2|n_1,n_2)}(x=1,\bar{x}=1,y=1,\bar{y}=1;q)\,. \tag{4.4.31}$$

Comparing (4.4.30) with (4.4.8), we find that $BC_N$ type of SRS polynomials of the first kind and the second kind are related as

$$\widetilde{\mathbb{H}}_{B,N}^{(m_1,m_2|n_1,n_2)}(x,\bar{x},y,\bar{y};q) = \mathbb{H}_{B,N}^{(n_2,n_1|m_2,m_1)}(\bar{y},y,\bar{x},x;q)\ . \tag{4.4.32}$$

It may be noted that, by using Eqs. (4.4.32) and (4.4.22), one can easily derive a relation between $BC_N$ type of SRS polynomials of the second kind and $A_{N-1}$ type of SRS polynomials. Let us now try to find out the generating function for $\widetilde{\mathbb{H}}_{B,N}^{(m_1,m_2|n_1,n_2)}(x,\bar{x},y,\bar{y};q)$, which would satisfy the relation

$$\widetilde{\mathcal{G}}_{B}^{(m_1,m_2|n_1,n_2)}(x,\bar{x},y,\bar{y};q,t) = \sum_{N=0}^{\infty} \frac{\widetilde{\mathbb{H}}_{B,N}^{(m_1,m_2|n_1,n_2)}(x,\bar{x},y,\bar{y};q)}{(q^2)_N}\, t^N\,. \tag{4.4.33}$$

Using Eqs. (4.4.15), (4.4.32) and (4.4.33), it is easy to find that

$$\widetilde{\mathcal{G}}_{B}^{(m_1,m_2|n_1,n_2)}(x,\bar{x},y,\bar{y};q,t) = \mathcal{G}_{B}^{(n_2,n_1|m_2,m_1)}(\bar{y},y,\bar{x},x;q,t)\,. \tag{4.4.34}$$

By using the above relation along with (4.4.13), we get an expression for this generating function as

$$\widetilde{\mathcal{G}}_{B}^{(m_1,m_2|n_1,n_2)}(x,\bar{x},y,\bar{y};q,t) = \mathcal{G}_{1}^{(n_2)}(\bar{y};q,t)\cdot\mathcal{G}_{2}^{(n_1)}(y;q,t)\cdot\mathcal{G}_{3}^{(m_2)}(\bar{x};q,t)\cdot\mathcal{G}_{4}^{(m_1)}(x;q,t)\,, \tag{4.4.35}$$

where the factors appearing in the r.h.s. can be obtained from Eq. (4.4.14). Using the $x=\bar{x}=y=\bar{y}=1$ limit of this generating function and following a procedure similar to the ferromagnetic case, it can be shown that $\widetilde{\mathcal{Z}}_{B,N}^{(m_1,m_2|n_1,n_2)}(q)$ satisfies a relation exactly of the form (4.4.27). For some small values of the discrete parameters

$m_1,\ m_2,\ n_1,\ n_2$, and for arbitrary values of $N$, Eq. (4.3.48) yields

$$\widetilde{\mathcal{Z}}_{B,N}^{(1,0|0,0)}(q) = q^{N^2},\ \mathcal{Z}_{B,N}^{(0,1|0,0)}(q) = q^{N(N-1)},\ \mathcal{Z}_{B,N}^{(0,0|1,0)}(q) = q^{N},\ \mathcal{Z}_{B,N}^{(0,0|0,1)}(q) = 1. \tag{4.4.36}$$

Using these partition functions and an equation of the form (4.4.27) corresponding to the anti-ferromagnetic case, we derive a set of recursion relations like

$$\begin{aligned}
\widetilde{\mathcal{Z}}_{B,N}^{(m_1+1,m_2|n_1,n_2)}(q) &= \sum_{N_1=0}^{N} \binom{N}{N_1}_{q^2} \cdot q^{N_1^2} \cdot \widetilde{\mathcal{Z}}_{B,N-N_1}^{(m_1,m_2|n_1,n_2)}(q)\,, \\
\widetilde{\mathcal{Z}}_{B,N}^{(m_1,m_2+1|n_1,n_2)}(q) &= \sum_{N_1=0}^{N} \binom{N}{N_1}_{q^2} \cdot q^{N_1(N_1-1)} \cdot \widetilde{\mathcal{Z}}_{B,N-N_1}^{(m_1,m_2|n_1,n_2)}(q)\,, \\
\widetilde{\mathcal{Z}}_{B,N}^{(m_1,m_2|n_1+1,n_2)}(q) &= \sum_{N_1=0}^{N} \binom{N}{N_1}_{q^2} \cdot q^{N_1} \cdot \widetilde{\mathcal{Z}}_{B,N-N_1}^{(m_1,m_2|n_1,n_2)}(q)\,, \\
\widetilde{\mathcal{Z}}_{B,N}^{(m_1,m_2|n_1,n_2+1)}(q) &= \sum_{N_1=0}^{N} \binom{N}{N_1}_{q^2} \cdot \widetilde{\mathcal{Z}}_{B,N-N_1}^{(m_1,m_2|n_1,n_2)}(q)\,.
\end{aligned} \tag{4.4.37}$$

By using this set of recursion relations and the initial conditions given in (4.4.36) (or, alternatively, a single initial condition of the form (4.4.29)), in principle it is possible to compute $\widetilde{\mathcal{Z}}_{B,N}^{(m_1,m_2|n_1,n_2)}(q)$ for arbitrarily values of the discrete parameters $m_1,\ m_2,\ n_1,\ n_2$ and $N$. Indeed by using the symbolic software package Mathematica we have seen that, in comparison to the direct use of the expressions (4.3.32) and (4.3.48), it is much more efficient to use the set of corresponding recursion relations (4.4.28) and (4.4.37) to obtain explicit forms of the ferromagnetic and anti-ferromagnetic partition functions as some polynomials of the variable $q$. Hence, the set of recursion relations (4.4.28) and (4.4.37) might be useful in studying various spectral properties like level density distribution and nearest neighbour spacing distribution for the $BC_N$ type of ferromagnetic and anti-ferromagnetic PF chains.

# CHAPTER 5

# Conclusion

In this thesis we compute the exact spectra and partition functions of some rational quantum integrable systems like $D_N$ type of spin Calogero models as well as PF spin chains associated with PSRO, and $BC_N$ type of spin Calogero models as well as PF spin chains associated with SAPSRO. Furthermore, we construct some novel $BC_N$ type of multivariate homogeneous SRS polynomials which are closely connected with the partition functions of the $BC_N$ type of PF spin chains with SAPSRO and investigate various properties of such SRS polynomials.

In the Chapter 2 of this thesis, we introduce the $D_N$ type of spin Calogero model with PSRO and its associated spin chain of HS type, namely the $D_N$ type of PF chain with PSRO. We solve the former model by finding a suitable (non-orthonormal) basis of its Hilbert space on which its Hamiltonian acts triangularly. From the spectrum of this model we are able to compute its partition function in closed form, which yields the partition function of the spin chain via Polychronakos's freezing trick. More precisely, we show that the latter partition function can be expressed in terms of the partition function of the type-$A$ PF chain. Since the type-$A$ partition function can be efficiently evaluated by using a simple recursion formula, we are able to exactly compute the spectrum of the $D_N$-type chain for relatively high values of

$N$. In this way, we are able to study several global properties of the spectrum of the latter chain. In particular, we provide strong numerical evidence showing that its energy levels are a sequence of consecutive integers, and that its level density becomes normally distributed when the number of spins tends to infinity. From these facts we conclude that the spacings between consecutive levels of the unfolded spectrum follows a "square-root-of-a-logarithm" distribution, characteristic of most spin chains of HS type.

In the Chapter 3 of this thesis, we construct SAPSRO which satisfy the $BC_N$ type of Weyl algebra and lead to a novel class of spin Calogero models as well as related PF chains with reflecting ends. We compute the exact spectra of these $BC_N$ type of spin Calogero models, by using the fact that their Hamiltonians can be represented in triangular forms while acting on some partially ordered sets of basis vectors of the corresponding Hilbert spaces. Since the strong coupling limit of these spin Calogero models yields $BC_N$ type of PF chains with SAPSRO, we apply the freezing trick to obtain the partition functions of this type of PF spin chains in a closed form. We also derive a formula (3.4.5) which expresses such a partition function in terms of known partition functions of several $A_K$ type of supersymmetric PF spin chains, where $K \leqslant N-1$. By using this formula, we analyze statistical properties like level density distribution and nearest neighbour spacings distribution in the spectra of spin chains with sufficiently large number of lattice sites. It turns out that, in analogy with the case of many other integrable systems with long-range interactions, the level density of PF spin chains with SAPSRO follows the Gaussian distribution and the cumulative nearest neighbour spacings distribution obeys the 'square root of a logarithm' law.

In Chapter 3, we also show that the partition functions of the $BC_N$ type of PF spin chains with SAPSRO obey an interesting type of duality relation. To this end, we consider a new quantum number which measures the parity of the spin states under

the action of SAPSRO. It is found that the partition functions of these spin chains satisfy an 'extended' boson-fermion duality relation (3.5.13), which involves not only the exchange of bosonic and fermionic degrees of freedom, but also the exchange of positive and negative parity degrees of freedom associated with SAPSRO. As an application of this duality relation, we compute the highest energy levels of these spin chains from their ground state energies. Moreover, we find that partition functions of a large class of integrable and nonintegrable spin chains with Hamiltonians of the form (3.5.14) satisfy this type of duality relation.

In Chapter 4 of this thesis, we derive the canonical partition functions of the $BC_N$ type of PF spin chains with SAPSRO by employing the freezing trick via the indirect approach, and subsequently construct the related $BC_N$ type of homogeneous multivariate SRS polynomials. More precisely, we compute the grand canonical partition functions of the $BC_N$ type of ferromagnetic as well as anti-ferromagnetic spin Calogero models with SAPSRO, and expand those grand canonical partition functions as some power series of the fugacity parameter to obtain the corresponding canonical partition functions. Applying the freezing trick, subsequently we derive novel expressions for the canonical partition functions of the related $BC_N$ type of PF spin chains. Inspired by the form of such partition functions, we define $BC_N$ type of homogeneous multivariate SRS polynomials and also find out the corresponding generating functions. Using these generating functions, we show that the $BC_N$ type of SRS polynomials can be expressed as some bilinear combinations of the $A_{N-1}$ type of SRS polynomials. We also use the above mentioned generating functions to derive a set of recursion relations (4.4.28) and (4.4.37) for the partition functions of the $BC_N$ type of PF spin chains involving different numbers of lattice sites and internal degrees of freedom.

It is worth noting that, the grand canonical partition functions of the $BC_N$ type of spin Calogero models with SAPSRO are expressed as some simple products of

only $BC_N$ types of basic modes in Eqs. (4.3.25) and (4.3.45). Such expressions of the grand canonical partition functions play an important role in our construction of the $BC_N$ type of SRS polynomials. Even though the grand canonical partition functions of the $BC_N$ type of spin Calogero models with SASRO have been computed earlier [101], it was found that both $BC_N$ and $A_{N-1}$ types of basic modes appear in the expressions of such grand canonical partition functions. Comparing our results with this earlier work, we find novel relations like (4.3.36a,b), which connect the basic modes of the $A_{N-1}$ and $BC_N$ types of grand canonical partition functions and also lead to interesting $q$-identities of the form (4.3.37) and (4.3.38).

The works done in this thesis suggest some possible future developments. Among them, the most natural one would be to study the spin Sutherland (both trigonometric and hyperbolic) models of $BC_N$, $B_N$ and $D_N$ type and their related spin chains with PSRO and SAPSRO. Another interesting point is that, in Chapter 2 of this thesis we determine the number of distinct energy levels of the $D_N$ type of PF spin chain with PSRO, showing that it is a second-degree polynomial in $N$, as is the case with the PF chain of $A_{N-1}$ type. For spin chains of HS type related to the $A_{N-1}$ root system, it is known [121] that the polynomial growth of the number of distinct levels is a consequence of the fact that these models are equivalent to a Yangian-invariant vertex model with linear energy function and polynomial dispersion relation. Our results strongly suggest that this is also the case for the present model, a conjecture which certainly deserves further study. In particular, the validity of this conjecture would also point out at the existence of a suitable Yangian symmetry for both the $D_N$-type spin chain and the spin Calogero model with PSRO, as is the case with the rational and trigonometric Calogero–Sutherland models of $A_{N-1}$-type and their associated spin chains.

As we have discussed in Chapter 3 of this thesis, $BC_N$ type of PF spin chains with SAPSRO do not exhibit global su($m|n$) supersymmetry for arbitrary values

of the related discrete parameters. However, for a particular choice of these discrete parameters, SAPSRO reduce to the trivial identity operator and lead to the su($m|n$) supersymmetric Hamiltonian $\mathcal{H}^{(m,0|n,0)}$ in (3.2.19). Curiously, we find that the partition function of this $\mathcal{H}^{(m,0|n,0)}$ coincide with those of $A_{N-1}$ type of su($m|n$) supersymmetric PF chain with Hamiltonian (3.4.7). Consequently, the spectrum of $\mathcal{H}^{(m,0|n,0)}$ can be expressed through Haldane's motifs as given in (3.4.8). It would be interesting to find out whether some modification of these motifs can be used to describe the spectra of $BC_N$ type of PF spin chains with SAPSRO for other possible choice of the related discrete parameters.

As has been mentioned earlier, in Chapter 4 of this thesis we derive a set of recursion relations (4.4.28) and (4.4.37) for the partition functions of the $BC_N$ type of PF spin chains involving different numbers of lattice sites and internal degrees of freedom. In this context it may noted that, another type of recursion relations, involving different numbers of lattice sites and fixed values of the internal degrees of freedom, have been computed earlier for the partition functions of the $A_{N-1}$ type of PF (resp. supersymmetric PF) spin chains and the corresponding RS (resp. SRS) polynomials [62,94,102,128]. The later type of recursion relations play a key role in expressing the spectra of the $A_{N-1}$ type of PF spin chains and their supersymmetric generalizations through the motifs and in constructing the related one-dimensional vertex models. Therefore, it would be interesting to investigate whether the $BC_N$ type of SRS polynomials studied by us also satisfy the later type of recursion relations, involving different values of $N$ and fixed values of the internal parameters $m_1$, $m_2$, $n_1$ and $n_2$. By using such recursion relations, it might be possible to describe the spectra of $BC_N$ type of PF spin chains with SAPSRO through some motif like objects similar to the case of $A_{N-1}$ type of spin chains. Furthermore, it might be possible to construct some one-dimensional classical vertex models whose energy functions would generate the complete spectra of these $BC_N$ type of PF spin chains.

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