NONLINEAR PLASMA WAVE EXCITATION AND ITS BREAKING PHENOMENA

by

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Mithun Karmakar

List of Publications arising from the thesis

Peer reviewed journals

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2. *Wave-breaking amplitudes of relativistic upper-hybrid oscillations in a cold magnetized plasma.*

   M. Karmakar, C. Maity, and N. Chakrabarti, *Physics of Plasmas*, **2016**, 23, 064503.

3. *Plasma wakefield excitation in a cold magnetized plasma for particle acceleration*

   M. Karmakar, N. Chakrabarti, and S. Sengupta, *Physics of Plasmas*, **2017**, 24, 052111.

4. *Phase-mixing of large amplitude electron oscillations in a cold inhomogeneous plasma.*

   M. Karmakar, C. Maity, N. Chakrabarti, and S. Sengupta, *Physics of Plasmas*, **2018**, 25, 022102.

5. *Relativistic electron plasma oscillations in an inhomogeneous ion background.*

   M. Karmakar, C. Maity, and N. Chakrabarti, *Phys. Scr.*, **2018**, 93, 065601.

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Dedicated to my respected teacher

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Contents

Synopsis

The primary concern of this thesis is to unravel the physics of charged particle beam driven plasma wake field excitation and to determine the breaking amplitude of such plasma waves. A detailed theoretical investigation has been performed to study the wake field structures of electron or proton driven plasma wake field accelerator incorporating both effects of external magnetic field and non-relativistic dynamics of ion fluid. Since the maximum amplitude of the electric field (wave breaking amplitude) that can be supported by plasma is an important entity in this acceleration process, a proper understanding of the wave breaking phenomena of different plasma modes is an important issue in this context. We have performed a thorough investigations on breaking phenomena of different plasma modes (relativistic electron plasma wave, upper hybrid wave etc.) in two or three component plasma systems in different physical situations. Admittedly, this studies have certain relevant implications in advancing the field of nonlinear relativistic wave dynamics as well as plasma based particle acceleration process.

There exist different kinds of mechanisms of plasma wake field excitation by laser or charged particle beams, such as, the plasma wake field accelerator (PWFA), the plasma beat-wave accelerator (PBWA), the laser wake field accelerator (LWFA) etc.[1] However, our main focus in this thesis has been centered around PWFA technology. The concept of PWFA was first proposed by Chen, Huff and Dawson in 1984.[2] In this acceleration scheme a charged particle is accelerated by the strong plasma wave excited behind a relativistically propagating electron beam through plasma. The driving beam electrons in the course of its propagation repel the plasma electrons to create an electron free positive space charge at the wake of the beam. The expelled electrons then snap back to the original positions and then overshoot to create a longitudinal electrostatic strong plasma wave. A late coming trailing beam of electrons launched at proper phase will then be accelerated to very high energy.

In an earlier theoretical investigation on PWFA, Rosenzweig discusses the generation of nonlinear wake waves behind a rectangular electron pulse.[3] He has obtained an analytical expression of transformer ratio (R), i.e., the ratio of the maximum accelerating electric field amplitude outside the beam to the maximum decelerating field inside the beam for the unmagnetized beam driven plasma. Motivated by this work, we have extended his analysis to magnetized plasma systems. By constructing a travelling wave solution, we get some significant results which can influence in improving the efficiency of the particle acceleration. We have reported here the excitation mechanism of wake field behind a relativistic, specially shaped electron beam (rectangular or Gaussian) passing through the plasma.[4] Also, the effects of magnetic field and variation of peak beam density on the transformer ratio (R) are discussed elaborately. The results corresponding to the limit of zero magnetic field have been readily reproduced from our solution. Earlier, it has been speculated by Rosenzweig that inclusion of ion motion could have some effects on the particle acceleration process.[3] In order to show its effect, a numerical investigation has been performed and it has been established that consideration of non-relativistic ion motion does not affect much on the stationary structures of wake-wave electric field. However, we conjecture that the inclusion of relativistic motion of ion may have some effects in the wake field profiles which we hope to report in the near future.

In single stage acceleration process, in the laser pulse or electron bunch driver schemes, it is not possible to reach TeV order of energy. Moreover, the energy gain in this case is limited by the energy carried by the electron driver which is very small ($\sim$ 100 J). To overcome these shortcomings one may use a proton beam to excite the strong plasma wake wave. A proton bunch carrying energy of the order of kJ is capable of producing such energy in a single plasma stage. Because of their higher energy and mass, proton can drive wake fields over a very longer plasma lengths. Since protons are positively charged and much heavier than electrons, the physics of proton driven wake field accelerator is different from electron beam driven plasma. In case of negatively charged driver, background plasma electrons are repelled to cerate a blow out regime where the wake field is produced. Proton beam, on the other hand, sucks in the plasma electrons towards the propagation axis and creates the wake wave electric field. A trailing witness bunch of electrons then extracts energy from the drive beam and thereby get accelerated to relativistic energies. Although a large number of experimental as well as numerical simulations have been reported in the recent past to explore PDPWFA scheme,[6, 7] there exists a limited number of theoretical investigations carried out to unfold the physics of proton driven plasma wake field acceleration mechanism. Such an attempt has been made in one of our works where similar to the electron beam driven plasma a travelling wave solution for the PDPWFA has been obtained. We consequently consider the effect of an external focusing magnetic field on the wake field structures.

As stated earlier, one of the significant entities which determine the maximum energy gain in this acceleration process is the breaking amplitude of the excited plasma wave. For a plasma wave there is always a maximum limiting amplitude of the electric field beyond which the wave loses its coherent nature. At this critical amplitude limit, the plasma fluids' velocity at the crest of wave exceeds the phase velocity of the wave and, consequently, a wave breaks. In such a situation, gradient of wave electric field becomes infinite and ordering of oscillators constituting the wave gets destroyed leading to multi-stream flow. Breaking of plasma wave can also be possible even long before attaining this limiting amplitude by a process called 'phase mixing'. Phase mixing is physically attributed to the space dependency in the characteristic frequency of the plasma wave which happens due to different types of nonlinearities coming from inhomogeneity in the background density or in the magnetic field, relativistic electron mass variation effect, ion motion etc.

The studies on wave breaking began with the theoretical investigation of relativistic electron plasma wave by Akheizher and Polovin in 1956 who have obtained an expression of maximum electric field amplitude sustained by such plasma wave.[8] Thereafter, a large number of theoretical and experimental works are performed to demonstrate the physics of wave breaking. Evidently, the wave which is excited behind a relativistic electron beam or an intense laser pulse is nothing but the high frequency relativistic electron plasma wave (Akheizher-Polovin waves) or upper hybrid plasma wave (magnetized plasma). So it is fundamentally a pertinent issue to investigate elaborately the breaking phenomena of such plasma waves/oscillations in various physical situations.

Our next work is dedicated to study the breaking phenomena of one-dimensional relativistically strong electrostatic electron plasma wave in cold unmagnetized electron-positron-ion (EPI) plasmas which is still an unexplored area in this field till date. EPI plasmas are encountered in various astrophysical situations, including early universe[9], pulsar magnetosphere[10], also in laser matter interaction in Laboratory experiment.[11] We have adopted the well known Pseudo Potential technique[5] and transformed the problem of nonlinear plasma wave phenomena into a simplified classical mechanical problem of a fictitious single particle motion in a potential well. It is indeed a generalized model in which the dynamics of all the three species are taken to be relativistic. Consideration of ion motion can be justified due to the fact that the use of highly intense laser beam or ultra-relativistic charged particle beam can induce even the ion to follow relativistic dynamics along with the other two lighter species. The maximum permissible electric field amplitude before wave-breaking (wave-breaking amplitude) has been derived. We found that increasing the ratio of ion to electron density (α) has an effect to reduce the maximum supported amplitude of the plasma wave in the three component plasma system. It has been established that the breaking amplitude is maximum for the electron-positron plasma and minimum for the electron-ion plasma while for electron-positron-ion plasma the limiting amplitude is intermediate between these two. We have also reported in this work the effect of α on the wavelength of the plasma wave.

Next, we have extended our analysis of wave breaking incorporating the effect of magnetic field on the nonlinear plasma wave phenomena in electron-ion plasma system. There are different kinds of modes which are developed in presence of an external magnetic field in plasma. The high frequency relativistic upper hybrid oscillation (RUHO) is one of such modes, the space time evolution of which has been discussed in one of the recent works of Maity *et al.* [12] However, this analysis does not provide us any explicit information about the breaking electric field amplitude for such high frequency electrostatic wave. In order to give an analytical expression of maximum field amplitude sustainable by RUHO, we have obtained a travelling wave solution for such high frequency mode. It is found that the wave-breaking amplitude of RUHOs for a fixed phase velocity, decrease with the increase of the strength of the ambient magnetic field. We have also constructed this travelling wave solution from the exact space-time dependent solution of RUHOs by appropriately choosing initial conditions. It is established that these stationary waves are very sensitive to a small deviation of the initial conditions. A slight longitudinal perturbation causes such waves to break at arbitrary amplitudes via the phase-mixing process.

Most of the earlier investigation on wave breaking have been performed with the consideration of homogeneous ion background. However, presence of inhomogeneity in the ion density can cause the wave to break at arbitrarily small ampltudes. In our next work we have discussed the wave breaking phenomena in such an inhomogeneous plasma system. A non-relativistic analysis of electron plasma oscillations in presence of a time stationary but space periodic ion density profile has been performed. Here, the main difference from most of the earlier analyses is that, in presence of ion inhomogeneity, instead of treating a uniform initial electron density we have considered a finite amplitude electron perturbation. In order to investigate the mode-coupling effect of electron plasma wave, Kaw et al. have chosen this type of initial condition.[13] The presence of inhomogeneity in the ion density can make the characteristic frequency of the plasma wave to acquire space dependency. Thereby, different parts of the fluid elements start to oscillate with difference frequencies destroying the phase coherency which ultimately leads to the breaking of the wave. We have adopted Lagrangian fluid technique to obtain an exact non-stationary parametric solution to describe wave breaking via phase mixing. Also we have performed a homotopy perturbation analysis which explicitly gives us the dependence of phase-mixing time on the initial conditions. The phase-mixing time is found to depend on the amplitudes of both ion density fluctuation and electron density perturbation as well as on the scale length ratio of their variations in space.

Our investigation of phase mixing process is further extended to include the relativistic mass variation effect of electron. The initial condition has been chosen similar to the non-relativistic case described above. Inhomogeneous ion along with the relativistic variation of electron mass make the characteristic frequency of the wave to acquire a space dependency and thereby it breaks at arbitrarily small amplitude due to phase mixing. A fully exact analytical solution for the nonlinear relativistic electron plasma wave in inhomogeneous plasma system encounters with significant mathematical complexities. So we have obtained an approximate space time dependent solution in the weakly relativistic limit by Bogoliuboff and Kryloff method[14] of averaging by making an acceptable simplifying assumption. We find that the change in the ion density perturbation and also the relativistic electron mass variation have significant effect in modifying the time at which phase mixing occurs.

The theoretical studies made in this thesis on plasma wave excitation by relativistic charged particle beam and its breaking phenomena have wide range of applicability. For example, the plasma wake field excitation by the injection of driving electron in solar coronal and chromospheric plasmas is the basic process by which solar flare electrons are accelerated to extreme high energies.[15] Also, in the experimental context, our theoretical investigation has relevant importance in the ongoing Advanced Proton Driven Plasma Wakefield Acceleration Experiment (AWAKE) project at CERN.[16] On the other hand, in plasma heating process, in the generation of fast electron and also in self ignition process, breaking of plasma wave plays a very significant role.

The submission of this synopsis is recommended and approved by the Doctoral committee.

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Chapter 1

Introduction

This thesis is primarily aimed to provide a theoretical description of strong plasma wave excitation process and plasma wave breaking phenomena. The generation mechanism of such wave at the wake of relativistically propagating electron or proton beam along with the investigation of the effect of external magnetic field on wake field structures have been studied extensively. Also, since the breaking amplitude of plasma wave plays an important role in the plasma based wake field acceleration process, breaking phenomena have been thoroughly analyzed for two different high frequency nonlinear plasma modes (Langmuir wave and upper hybrid wave). Breaking amplitudes have been analytically derived for the electron plasma wave in three component electron-position-ion plasma as well as for the upper hybrid wave in two component electron ion plasma. Moreover, the phase mixing which acts as the potential mechanism responsible for high frequency plasma wave breaking has been discussed incorporating the effect of both relativistic mass variation and inhomogeneity in the background ion density.

1.1 General Overview

Plasma, an ensemble of ionized particles interacting collectively via electric and magnetic field, holds a rich varieties of characteristics and supports various fundamental modes.[1–3] The excitation mechanism and the dynamical evolution of such modes have long been studied both from the theoretical and experimental perspective because of its wide range of practical applicabilities.

One of the key applications of relativistically strong plasma wave is to produce ultra high energy charged particles for the purpose of high energy physics research. The superiority of using plasma as accelerating medium instead of vacuum emerges due to its capability to produce high electric field over a very short distance (several hundreds of Giga-Volts/meter). This large electric field is excited due to the collective response of plasma particles to the strong electric field of laser pulse or charged particle beam. The use of laser pulse to excite plasma wake field was first proposed by Tajima and Dawson in 1979.[4] The process of plasma beat wave (PBW) excitation, proposed earlier by Rosenbluth and Liu,[5] is relevant to the idea of Tajima and Dawson and is a mechanism worth to be mentioned here. PBW excitation requires two laser pulses and a careful tuning of the laser frequencies such that the difference in frequencies equals the plasma frequency. The plasma wake field is resonantly excited by the beat wave produced by the two laser pulses. An obvious advantage in advancing from PBW to laser wake field (LWF) accelerator is that, in LWF excitation instead of two, the ponderomotive force of a single, short, ultra-intense laser pulse is used to drive a plasma wave.[6–8] This considerably simplifies the experimental requirements. Alternatively, instead of using a highly intense laser pulse, excitation of a relativistically strong plasma wake fields can also be done by driving an ultra-relativistic bunch of charged particles (plasma wake field accelerator [PWFA]) through plasma.[9–15] Exploration of the physics of such PWFA mechanism is one of the key issues in this thesis.

The highest energy that can be gained in this wake field acceleration process is determined by the maximum permissible electric field amplitude that can be supported by the plasma system. This limiting amplitude is dictated by a process called plasma *wave breaking* which occurs due to presence of different types of non-linearities associated with relativistic electron mass variation,[16–20] background density inhomogeneity,[21–23] ion motion[24] etc. Usually, due to such nonlinear effects a plasma wave starts to lose its periodic sinusoidal nature and transforms into triangular wave shape followed by occurrence of an infinite gradient in the electric field profile and eventually the breaking happens. In the hydrodynamic definition, it is an indication of attaining a critical amplitude of a plasma wave beyond which wave coherence is destroyed and the particles become completely random. Physically speaking, at the onset of breaking, the plasma fluids' velocity at the crest of the wave exceeds its phase velocity. In such a situation, ordering of oscillators constituting the wave gets destroyed and the multi-stream flow is generated.

In this thesis, our theoretical studies are aimed to investigate nonlinear plasma wave excitation and breaking phenomena of such waves in varied physical contexts with the purpose to get a more clear vision of the physics involved in such processes. Before presenting the detailed illustration of our investigation, here we will provide the general background and the primary motivation of performing such analysis.

1.2 Plasma Wake Field Acceleration

There are basically two different charged particle beam driven wake field excitation schemes. One is to use relativistically propagating electron beam to excite the wake wave and the other is proton beam driven plasma wake field accelerator (PDPWFA). Because of the differences in their masses and charges of electron and proton, the physical processes involved in the wake field excitation and the characteristic features of the wake field structures differ appreciably in these two different charged particle acceleration methods.

1.2.1 Electron Beam Driver Scheme

In this acceleration scheme, the energy of a relativistically strong electron beam (driving beam) is first transferred to excite plasma wave and then extracting this energy a trailing beam of electrons (driven witness beam) is accelerated. As discussed, unlike the laser driven wake field accelerator, where a ponderomotive force is generated, here a space charge is developed behind the electron beam to excite the wake wave.[8, 25] The Coulomb force of the driving beam's space charge repels the plasma electrons while the beam passes through the plasma medium. These expelled electrons snap back to their original positions to regain the charge neutrality and consequently they overshoot again to set up a longitudinal plasma oscillation that trails behind the driving beam. Finally, a late-coming beam of electrons launched at a proper phase can be accelerated to very high energy by the excited wake wave.

The successful implementation of the idea of plasma wave excitation by launching relativistic charged particle beam through plasma medium was first made by

Chen, Huff and Dawson in 1985.[26] They have employed a series of short electron bunches as the driving beam in order to excite the wake wave. It was found that in this acceleration process energy gain is only limited to $2\gamma_b mc^2$, where $\gamma_b mc^2$ is the drive beam energy with $\gamma_b = [1 - (v_b/c)^2)]^{-1/2}$ being the relativistic Lorentz factor associated with beam velocity v_b. This limitation arises due to various effects associated with non ideal bunch shapes, transverse plasma dynamics, dephasing of accelerated particles etc.[14, 25, 27] Bane *et al.* have proposed that this limit can be overcome by using properly shaped driving bunch of finite longitudinal extent.[14] It was confirmed later by the theoretical investigation of T. Katsouleas who has examined the physical processes involved in PWFA with realistic and experimentally realizable choice of beam shapes.[25] Thereafter, a large number of experiments have been performed during the last few decades, to reach the energy range of several GeV by this PWFA scheme.[8, 10, 12, 25, 28] In a very recent experiment, Blumenfeld *et al.* successfully accelerated electrons from the tail of the driving beam of energy 42 GeV to maximum energy of 85 GeV at SLAC (Stanford Linear Accelerator Center).[29]

In our research work we have performed a theoretical investigation of such electron beam driven wake field acceleration process in presence of an external magnetic field and also discussed the effect of drive beam amplitude and shape on the wake field structures.

1.2.2 Proton Beam Driver Scheme

Over the last few decades, researches on the plasma based acceleration process were mainly focused in creating the wake field by launching a highly relativistic electron beam or an intense laser pulse into the plasma.[4, 5, 11, 30–32] In electron beam driven PWFA scheme, an energy range of the order of several GeV was reported.[29] However, it is not obvious how to reach the present day energy frontier of the particle physics i.e. teraelectronvolt regime by these schemes. First of all, it requires multiple stage acceleration which encounters a great deal of technological difficulties.[33, 34] On the other hand, the energy gain is limited by the energy carried by the electron driver which is very small ($\sim$ 100 J). So, in such cases an alternative approach is to use proton beam as the driving beam instead of electron. The availability of proton beam with the energy of several TeV makes it possible to excite plasma wake wave which can accelerate electrons in this high energy range. Moreover, a proton bunch carrying energy of the order of kJ is capable of producing such energy in a single plasma stage.[35, 36] Because of their higher energy and mass, proton can drive wake fields over a very longer plasma lengths. This proton-driver scheme is therefore much more superior compared to the other accelerators.

Since protons are positively charged and much heavier than electrons, the physics of proton driven wake field accelerator is different from electron beam driven plasma. In case of a negatively charged driver, background plasma electrons are repelled to create a blow out regime where the wake field is produced. Proton beams, on the other hand, suck in the plasma electrons towards the propagation axis and create the wake wave electric field. The linear analysis of proton beam driven scheme reveals that except for a difference in phase factor the excited wake electric field distribution is same as that of electron beam driven case.[37] However, significant differences are observed in the nonlinear structures for these two different acceleration schemes. During the recent past a great deal of attention has been given to investigate the proton driven charged particle acceleration method both experimentally and also in computer simulation.[36, 38–43] In 2009, Caldwell *et al.* have first proposed the scheme of proton-bunch driven plasma wake field acceleration (PDPWFA) and discussed its potentiality of producing TeV range energy in single plasma stage.[35, 36, 38] Later, a detailed numerical investigation has been performed by K. V. Lotov to identify the main effects limiting the energy efficiency in this scheme. In 2011, A. Caldwell and K. V. Lotov discussed the wake field excitation process by the modulated proton bunches.[39] A recent experiment at CERN-the AWAKE has been performed to understand the detailed physical processes involved in PDPWFA.[35] However, a well defined theoretical model to describe such phenomena has not still been witnessed till date. We have attempted to provide an analytical investigations of nonlinear wave dynamics and developed systematic studies on the dynamical evolution of such proton beam driven wake wave with the inclusion of the effect of plasma ion dynamics and magnetic field on the wake field structures.

In the next few sections we will discuss some of the notable characteristic features of the charged particle beam as well as several nonlinear effects encountered in these wake field acceleration processes.

1.2.3 Some Attributes of Wake Field Accelerator

Phase Slippage

The efficient excitation of the wake field and successful electron acceleration to high energy depend on some key effects. One such effect is the occurrence of ‘phase slippage’.[26, 39] In the wake field acceleration process, once the accelerating electrons catch up with the plasma wave and start to accelerate, they can outrun it after traversing a certain distance, called the dephasing length.[34] In the proton driver scheme, this phase slippage can occur between the proton driving bunch and the electron witness bunch. Proton bunch traveling through the plasma may slow down and the phase relationship with the light electron bunch will begin to change.[39, 40] Unless special care is taken to minimize this phase slippage, there may be a possibility of the degradation in energy gain.

Beam Characteristics

In unmagnetized plasma it was experimentally observed that the final amount of accelerated charge is appreciably lower than the injected charge.[44] This is due to the fact that it is not possible for all of the charged particles in the beam to couple with the plasma wake field. Also, it may happen that due to some instability caused during the propagation of the beam, the beam head get expanded laterally which can cause the large spread in final accelerated particle energy.[38, 45] The transverse electric field at the wake of the beam as well as the presence of an external magnetic field can minimize this lateral expansion by focusing the propagating beam. Recently, Litos *et al.* performed an experiment and have shown that this spread in the beam energy distribution can be minimized with certain extent with a suitable arrangement.[10]

One of the significant disadvantages in the proton beam driven acceleration scheme is that the proton bunches available today are much more longer in size compared to the plasma wavelength. So they are not resonant and excitation of strong wake field is not so efficient. However, a process called self modulational instability can cause such long proton bunch to split over a large number of microbunches which then efficiently excite the plasma wake wave.[39–43] In the recent past, a numerical analysis has been performed to analyze the excitation mechanism of wake field by such trains of equidistant particle bunches.[40] Caldwell and Lotov have discussed the splitting mechanism of long proton bunches by the modulational instability.[39] In our theoretical investigation we have also discussed the excitation of wake wave by using such proton microbunches.

Transformer Ratio

In the wake field acceleration process a parameter called 'transformer ratio' (R) has been introduced to quantify the energy efficiency.[25, 27, 30] It is defined as the ratio of the maximum energy gained by the accelerated beam to the initial drive beam energy. Alternatively, it can be defined as the ratio of the maximum accelerating electric field (E_+) behind the driving bunch, to the maximum retarding electric field (E_-) within the bunch. In the wake field acceleration process, the drive beam loses all its energy by this maximum decelerating electric field and the driven witness beam gains the energy by the maximum accelerating field. So, the energy is actually transferred from the drive to driven accelerated beam with the transformer ratio $R = (E_+/E_-)$. T. Katsouleas has presented a physical derivation of this transformer ratio in the context of wake field acceleration.[25] Rosenzweig has provided an analytical expression of the transformer ratio (R) in investigating the generation of wake wave behind a rectangular electron pulse.[30]

1.2.4 Magnetic Field Effect on the particle acceleration

One way to control the phase slippage between the particles and the accelerating electric field is to apply a perpendicular magnetic field which can make the particles to deflect across the wave front; prevents them to outrun the wave. The effect of such an external magnetic field has rarely been reported in the context of particle acceleration by plasma wake wave. In presence of an external magnetic field, an electrostatic relativistic upper hybrid wave is excited in the wake of a relativistic electron beam passing through plasma.[46–48] In contrast to the unmagnetized case, this fast electrostatic wave can accelerate the electrons to arbitrarily high energy.

The usefulness of an external magnetic field has been discussed by Katsouleas and Dawson in the context of charged particle acceleration.[13] It was shown that unlimited electron acceleration is possible, at least from the theoretical point of view, by the relativistic upper-hybrid (UH) wave electric fields due to the surfatron process. Presence of an external magnetic field also helps to confine the trailing beam charge at the wavefront of the wake wave and thereby to enhance its coupling with the wake field over a very long distance. Furthermore, magnetic field also controls over the focusing characteristics of the witness beam which accounts for the very low energy spread of the final accelerated beam.

Keeping in mind the importance of an external magnetic field, a complete understanding of the effect of such magnetic field on the wake field structures is therefore pre-requisite.

1.3 Breaking of Nonlinear Plasma Wave

1.3.1 Historical Development

Small amplitude electron plasma oscillation is a well-understood process and has been studied quite extensively. However, if the amplitude is high, some of the nonlinear effects come into play, which have a very significant role in the generation and breaking of electrostatic electron plasma wave. Nonlinear plasma wave theory began with the investigation to discuss the effect of the nonlinearity associated with the relativistic electron mass variation by Akheizher and Polovin in 1956.[49] They have provided a general theoretical study for longitudinal relativistic electron plasma waves and determined that the breaking amplitude of such waves was $\sqrt{2(\gamma-1)}$, where, $\gamma = [1-(v_{ph}/c)^2)]^{-1/2}$ and v_{ph} is the phase velocity of the wave. Later, Dawson has introduced the notion of non-relativistic wave breaking limit and demonstrated that the non-relativistic limit of Akheizher Polovin description directly follows from the space time dependent Lagrangian solution.[21] Couple of years later, he has described the phenomena of wave breaking in terms of sheet model.[50] In this novel theoretical model, the dynamics are described by following electron trajectories instead of the fluid elements. The point at which two different electron trajectories start to cross each other, wave breaking occurs. Davidson and Schram have then obtained an exact space time dependent Lagrangian solution of the nonlinear non-relativistic Langmuir wave.[51] They have found that there is a restriction in the choice of initial electron density perturbation amplitude for which the coherent plasma oscillation is maintained indefinitely over the region of initial excitation. Particularly, if the perturbation amplitude exceeds half of the equilibrium density, then multi-stream flow is developed in the system which leads to the destruction of the wave. However, this limitation emerges solely from the mathematical limitation on the uniqueness of the transformation from the Eulerian to Lagrangian co-ordinates but do not represent any physical limitation. Thereafter, thermal corrections are incorpoarted in the expression for non-relativistic wave breaking amplitude by Coffey.[52] The corresponding analysis is then extended in the relativistic situation by Katsoleas and Mori.[53] Both these investigations reveal that the thermal effects reduce the maximum electric field amplitude of the electrostatic waves.

1.3.2 Homogeneous Plasma: Consideration of Ion Motion

The nonlinearity in the homogeneous plasma system either comes from the ion motion or from the consideration of relativistic electron mass variation effect.[19, 24, 49] In majority of the previous studies the ions were considered as stationary background of positive charge.[2, 21, 49, 52, 54–56] However, for large amplitude electric field encountered in relativistic plasma wave excited behind high intense laser pulse or ultra-relativistic electron/proton, even the heavy mass plasma ion follow relativistic dynamics. So, in such situations the assumption of immobile ion background may be violated.[8, 12, 25, 28] In order to understand the effect of ion motion on the breaking amplitude of electron plasma wave in two component plasma with arbitrary mass ratio, Khachatrayan has considered the relativistic ion dynamics.[57] It has been shown that even though the ion motion affects weakly on the wave breaking field amplitude, it drastically changes the wavelength of the nonlinear plasma wave. Later, Gorbunov *et al.* have performed similar analysis with the consideration of the non-relativistic ion motion and investigated the general properties of nonlinear plasma wave in electron ion plasma.[58] In 2005, G. N. Kichigin has discussed the steady state nonlinear properties of electron plasma wave with the allowance of the ion motion.[59]

1.3.3 Breaking in Presence of an External Magnetic Field

The incorporation of an external magnetic field is an important issue in the studies of nonlinear plasma wave phenomena. Strong magnetic field can be generated in the plasma medium due to various reasons. In the laser matter interaction process magnetic field is produced by the hot electron current.[60] Also, circularly polarized laser radiation can induce axial magnetic field in the plasma. Magnetized plasma is encountered in various astrophysical situations, in laboratory experiments and also in the laser produced plasma systems.[61–65] Simple linear analysis shows that in presence of an external magnetic field, various modes like Alfven waves, lower and upper hybrid wave, magnetosonic wave can be excited in the plasma medium. The presence of such external magnetic field drastically modifies the nature of the nonlinear plasma wave. In the recent past, Maity *et al.* have studied the wave breaking and phase mixing process specifically for the above mentioned two hybrid modes.[46, 66, 67] Extending the analysis of wave breaking of upper hybrid mode, we have provided an analytical estimation of the breaking field amplitude of such high frequency mode in our investigation.

1.3.4 Wave Breaking via Phase Mixing

The solution obtained by Akheizher and Polovin represents the traveling wave for relativistically propagating electron plasma wave.[49] The wave which is excited behind a strong laser pulse or ultra-relativistic particle beam can be well-described by this type of stationary wave solution. The maximum supported electric field amplitude of such wave is limited by the wave breaking field amplitude given by $E_{WB} = \sqrt{2(\gamma - 1)}$ [AP limit], where $\gamma = [1-(v_{ph}/c)^2]^{-1/2}$. It is important to note from the expression of AP limit that as $v_{ph} \rightarrow c$, this implies $E_{WB} \rightarrow \infty$.

Almost four decades later, Infeld and Rowland have provided an exact analytical solution for relativistic longitudinal electron plasma wave in terms of Lagrangian co-ordinates.[18] They have provided an expression for the frequency of the nonlinear plasma wave which not only depends on the amplitude but also acquires a space dependency. The observed relativistic bursts is a result of such position dependent frequency. It is found that except for a choice of particular initial condition to excite the wave, the relativistic electron plasma waves always break at arbitrarily small amplitude long before it reaches to the limit imposed by Akheizher and Polovin. This has been confirmed later by the analysis of Verma et al. who have constructed traveling wave type Akeizher Polovin solution by freezing the Infeld-Rowland's exact space time dependent solution.[68] Corresponding non-relativistic situations has also been discussed earlier by Albriton and Rowland.[69] They have commented that in the non-relativistic situation, it is possible to construct traveling wave solution from the space time dependent solution for the non-linear cold plasma wave by choice of an appropriate initial condition. Hence, a general conclusion that comes out from these studies is that, only with the choice of such unique initial condition one obtains a stationary wave solution. A slight deviation originating from the noise present in the system can make the wave to break at an arbitrarily small amplitude. This small amplitude breaking or nonlinear damping is nothing but a result of phase mixing.

Phase mixing is physically associated with the space dependent frequency which occurs due to presence of different types of nonlinearities originating from the inhomogeneity in the background ion density, relativistic electron mass variation, ion motion etc. When the frequency becomes position dependent, different fluid elements oscillate with different frequencies and the crossing of electron trajectories occurs leading to phase mixing. In the non-relativistic situation phase mixing of nonlinear electron plasma wave has been studied in presence of ion density cavities.[22, 70] Such cavities are frequently encountered in the auroral region excited due to ion cyclotron wave motion. Nappi *et al.* has also obtained an approximate analytical solution of the nonlinear electron plasma wave in presence of sinusoidal time stationary inhomogeneous ion density background.[71] They have estimated the phase mixing time and discussed its dependency on the amplitude of the ion density inhomogeneity. Later, Sengupta *et al.* have investigated the phase mixing of relativistically strong plasma wave in homogeneous plasma system. [19]

Incorporation of the relativistic electron mass variation in the process of this phase mixing in inhomogeneous plasma system is an important issue. A general theoretical model which allows us to investigate the effect of the nonlinearity associated with the inhomogeneity and relativity in the phase mixing process will definitely contribute to the knowledge of plasma wave breaking phenomena.

1.4 Stationary Wave Ansatz and Lagrangian Fluid Technique

The basic studies on nonlinear plasma dynamics deal with solving some nonlinear partial differential equations. There exists no precise general technique of solving these differential equations. One way to handle with such equations is to adopt plane wave ansatz which is used to describe stationary properties of the nonlinear plasma system. On the other hand, one can use Lagrangian fluid technique to obtain exact space time dependent solution of the problem. Our investigation of wave breaking and wave excitation have been performed by using these useful fluid techniques. Here we provide a brief introduction of these two mathematical formulations.

Stationary wave: The nonlinear stationary wave solution can be obtained by assuming every dynamical dependent variables of the problem to depend on $\xi = x - vt$; a special combination of space and time. Here v is the constant velocity of a moving frame. Even though this kind of analysis does not provide any information about the exact space time evolution of the system, this method is used most often and give various insightful results for systems which are stable for long time. One of the significant advantage in this moving frame analysis is that with this co-ordinate transformation one can reduce the partial differential equations into a set of ordinary differential equations which are comparatively easier to handle with. *'Pseudo potential'* approach is one of such methods where solution for the basic equations describing nonlinear plasma wave phenomena can be easily obtained. In this novel technique all dynamical variables viz. electron density, fluid velocity are expressed in terms of electrostatic potential and then the whole nonlinear wave dynamics is being converted to a classical mechanical problem of a fictitious single particle motion in a pseudo potential. By the comparison of the bounded and unbounded motion of that particle in the potential well, the periodic wave motion and the breaking phenomena of plasma wave can be described. In investigating wave breaking of electrostatic electron plasma wave in three component electron-positron-ion plasma we have adopted this *'Pseudo potential'* method. Also, in the study of wave breaking in magnetized plasma system as well as in the charged particle beam driven wake field excitation process, this kind of stationarity is assumed.

Lagrangian method: A generic structure of studying the dynamics of any physical system is to specify its initial state and observing how the state evolves with time. In this context the Lagrangian fluid technique is a very useful method. This powerful mathematical tool has previously been extensively employed in the description of nonlinear electron plasma waves.[3, 51, 72, 73] In contrast to the Eulerian method, here the dynamics of nonlinear plasma wave is studied by tracking the individual fluid elements.

The Lagrangian variables $\{\xi, \tau\}$ is related to the Eulerian variables $\{x, t\}$ by the following transformation equations:

$$\xi = x - \int_0^{\tau} v(\xi, \tau')d\tau' \quad , \quad \tau = t. \tag{1.1}$$

With this transformation one can reduce the convective derivative term $D \equiv \frac{\partial}{\partial t} + v\frac{\partial}{\partial x}$ into local time derivative $\frac{\partial}{\partial \tau}$. This considerably simplifies the problem and makes it analytically tractable. This transformation will allow us to have a glimpse over the exact space time evolution of the nonlinear plasma dynamical situation encountered in various problems of plasma physics. Specifically, to study wave breaking by phase mixing this transformation is very useful. In this physical process the information about the initial condition from which the system evolves is an important issue. This formulation not only allows us to investigate the spatio-temporal evolution of the nonlinear plasma wave dynamics but also provides the knowledge of its initial state. A more detailed discussion on Lagrangian method can be found in the textbook of Davidson.[3]

1.5 Motivation

In the electron or proton beam driven wake field excitation process, the necessity of an external magnetic field is manifold. The magnetic field not only helps to reduce the beam charge loss but also helps the beam charge to couple efficiently with the excited plasma wave. Also, in the production of quasi mono-energetic high energy charged particle, it plays a significant role. It is quite obvious that in presence of such magnetic field, the electric field structure of the wake wave will modify from its unmagnetized profile. This will greatly influence on the overall energy efficiency in the acceleration process. The effect of such magnetic field has still not been discussed elaborately in the context of particle acceleration by the strong plasma wave. Therefore, a detailed study of wake wave excitation in presence of an external magnetic field will definitely make a significant contribution in the development of the production of high energy charged particle by plasma wave.

Furthermore, wave breaking of different nonlinear plasma modes has been studied quite extensively over the decades mostly in two component electron-ion plasma systems. However, it is not very easy and straightforward to extend the analysis of wave breaking in three component plasma systems like electron-positron-ion plasmas. During the last few years, there has been considerable interest on the studies of nonlinear wave phenomena in EPI plasmas [74–79]. EPI plasmas are found in various astrophysical environments, including early Universe [80], pulsar magnetosphere [81] etc. Such plasmas have also been produced in high intensity laser-matter interaction in laboratory [2]. So it is of particular interest to investigate wave breaking in such plasma systems. But consideration of the complex collective dynamics of three species in the wave propagation makes the problem difficult to investigate analytically. Also, as obvious from the previous discussion, it is needed to incorporate the relativistic dynamics of all the three species. Nevertheless, one way to study wave breaking in such plasma systems is to adopt stationary wave ansatz. Even if this method do not provide any information of the exact space time evolution, it enables us to estimate the breaking field amplitude of electron plasma waves in such plasma systems.

As mentioned earlier phase mixing is one of the strong potential mechanism by which the plasma wave breaks. This phase mixing occurs mainly due to nonlinearities associated with the relativistic mass variation of electron or because of the existing inhomogeneity in the background ion density. In most of the earlier investigations, the studies on wave breaking in inhomogeneous plasma system have been performed in the non-relativistic situations only. It is interesting to see how the physics of phase mixing modifies when these two nonlinear effects act together on the wave dynamics.

1.6 Outline of the thesis

A brief outline of the thesis is described below:

In chapter-II, a theoretical investigation has been performed to discuss the effect of an external magnetic field on the stationary wake field structures of non-linear relativistic electron beam driven plasma system. Experimentally realizable different shapes of the driving electron beam (Rectangular and Gaussian) have been considered for the excitation of plasma wake wave. The significance of this studies are discussed in the laboratory context of particle acceleration or in the study of generation of ultrahigh accelerating charged particle by strong plasma wave in astrophysical situations.

We have also extended our analysis to describe the physics of proton driven plasma wake field accelerator (PDPWFA) by constructing a travelling wave solution of the problem. The wake field excitation by single long proton beam as well as a train of equidistant proton micro-bunches produced due to self modulational instability has been discussed. Also, considering the necessity of the external magnetic field to control over the focusing characteristics of the beam and also to reduce the diffraction of beam head, studies on the effect of magnetic field on the wake field structures have been performed.

In **chapter-III**, the maximum permissible amplitude of the electric field has been derived for the relativistic electrostatic electron plasma wave in three component electron-positron-ion (EPI) plasmas incorporating the relativistic dynamics of all the three plasma species. We have adopted Pseudo potential technique to discuss the breaking phenomena in such plasma system. The dependence of the wave breaking amplitude on the relativistic Lorentz factor associated with the phase velocity of the plasma wave, on the electron/positron to ion mass ratio, and on the ratio of equilibrium ion density to equilibrium electron/positron density has been discussed elaborately.

In **chapter-IV**, we have provided a theoretical investigation on the stationary wave solution for the nonlinear relativistic upper hybrid wave. The wave breaking field amplitude of such waves has been derived. Furthermore, we seek the initial condition that should be chosen properly to freeze the exact space time dependent solution into stable stationary type solution.

In chapter-V, the wave breaking via phase mixing has been studied for the non-relativistic electron plasma wave in presence of ion density inhomogeneity. An exact spatio-temporal solution in terms of Lagrangian variables has been derived. Also, an approximate homotopy perturbative solution has been obtained in order to give an analytical expression for the phase mixing time.

We have extended our analysis on phase mixing **in chapter-VI** to incorporate the relativistic electron mass variation effect along with the background ion density inhomogeneity. These two types of nonlinearities acting together speed up the process of phase mixing which is reflected in the expression of phase mixing time as obtained in our theoretical study.

The theoretical investigation made in the thesis is not meant to understand very complicated studies of wave breaking research. Rather, it deals with very simple problems to explore the essential features of wave breaking and phase mixing process in magnetized and unmagnetized plasma systems. For the researchers embarking in the field of plasma wave breaking, phase mixing and also in the branch of plasma based particle accelerator physics, this thesis can be a starting point.

Chapter 2

Charged Particle Beam driven Plasma Wakefield Excitation

A theoretical study has been performed to find the stationary wave solution for the relativistically propagating electron as well as proton beam driven cold magnetized plasma system. The effect of the magnetic field on the transformer ratio (the ratio of energy gain to the drive beam energy) has been discussed independently for both these schemes. Also the effect of ion motion and different beam shapes on the wake field structures are analyzed in the electron beam driver case. Consideration of both single long proton beam and micron sized train of small particle bunches to excite the plasma wake wave have been incorporated in our investigation.

2.1 Introduction

As discussed in the introductory section there are basically two different charged particle beam driven wake field excitation schemes. One is to use relativistically propagating electron beam and the other is proton beam driven plasma wake field accelerator (PDPWFA). Here we report the excitation mechanism of wake field behind a relativistic, specially shaped electron beam (rectangular or Gaussian) [25] passing through a magnetized plasma system. Also, the effects of magnetic field and variation of peak beam density on the transformer ratio are discussed elaborately. Analytical work of Rosenzweig discusses the generation of wake wave behind a rectangular electron pulse.[30] He obtained an analytical expression of transformer ratio (R) i.e. the ratio of the maximum energy gain of accelerated particles to the initial energy of driving particles for the unmagnetized beam driven plasma. Extending his analysis to magnetized plasma system, we get some significant results which can influence on improving the efficiency of the particle acceleration.

Electron beam driven wake field accelerator is not very suitable for accelerating charged particle in the TeV energy range. A newly introduced scheme which uses proton beam to excite plasma wake field enables us to reach this energy range in a single plasma stage with high efficiency. There exists a lot of numerical and experimental works performed recently to understand the basic physics of PDPWFA.[38–40, 43, 45] However, a well defined theoretical model to describe such phenomena has still not been witnessed till date. In this chapter, such an attempt has been made where incorporating the proton beam density in the Poisson equation, the basic fluid Maxwell's equation have been solved and the results are discussed.

2.2 Nonlinear solution for charged particle beam driven wake in magnetized plasma

The basic equations describing the wake wave generation in the electron/proton beam driven cold magnetized plasma are the following fluid Maxwell equations:

$$\frac{\partial \mathbf{p}}{\partial t} + \mathbf{v}.\nabla\mathbf{p} = -eE - e\beta_{\mathbf{e}} \times \mathbf{B}. \tag{2.1}$$

$$\nabla.\mathbf{E} = 4\pi e(n_0 - n + qn_b). \tag{2.2}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}. \tag{2.3}$$

$$\nabla.\mathbf{B} = 0. \tag{2.4}$$

$$\nabla \times \mathbf{B} = 4\pi e(-n\beta_{\mathbf{e}} + qn_b\beta_{\mathbf{b}}) + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \tag{2.5}$$

where, $\beta_{\mathbf{e}}$ and $\beta_{\mathbf{b}}$ are respectively the electron fluid velocity (v_e) and beam velocity (v_b), both normalized by free space light speed c. The symbol 'q' takes negative sign for electron beam and positive sign for proton beam; e being the value of electronic charge. The electric field is $\mathbf{E} = E\hat{e}_x$, where $\hat{e}_x$ is the unit vector along the x axis. The external magnetic field is $\mathbf{B} = B_0\hat{e_z}$, where $\hat{e_z}$ is the unit vector along the z axis. Other variables have their usual meanings. Here, for the purpose of simplicity in analysis, the heavier mass ions are assumed to be static in the time scale of the electron dynamics.

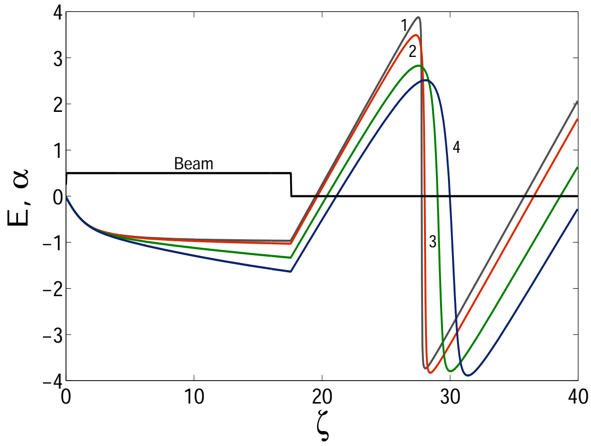

Figure 2.1: Variation of normalized electric field with and without magnetic field in electron beam driven plasma[$\beta_{ph} = 1$, $\beta_b = 1$], with beam density [$\alpha = 0.5$ for $0 \leqslant \zeta \leqslant 5.6\pi$ and zero otherwise], $1 \to \Omega = 0$, $2 \to \Omega = 0.2$, $3 \to \Omega = 0.5$, $4 \to \Omega = 0.7$.

In search for a travelling wave solution of Eqs. (2.1)-(2.5), it is convenient to introduce a variable transformation $\zeta = \frac{\omega_{pe}}{v_{ph}}(x - v_{ph}t)$, where $\omega_{pe} = \sqrt{4\pi n_0 e^2/m}$ with n_0 being the equilibrium electron density and v_{ph} is the phase velocity of the plane wave. In the newly introduced coordinate system two momentum equations take the form:

$$\frac{dp_x}{d\zeta} = \beta_{ph}\frac{E\sqrt{1+p^2} + \Omega p_y}{\beta_{ph}\sqrt{1+p^2} - p_x}, \tag{2.6}$$

$$\frac{dp_y}{d\zeta} = -\Omega\beta_{ph}\frac{p_x}{\beta_{ph}\sqrt{1+p^2} - p_x}. \tag{2.7}$$

Combination of Eq.(2.2) and Eq.(2.5), gives us the electric field evolution equation:

$$\frac{dE}{d\zeta} = -\beta_{ph}\frac{(1-q\alpha)p_x - q\alpha\beta_b\sqrt{1+p^2}}{\beta_{ph}\sqrt{1+p^2} - p_x}. \tag{2.8}$$

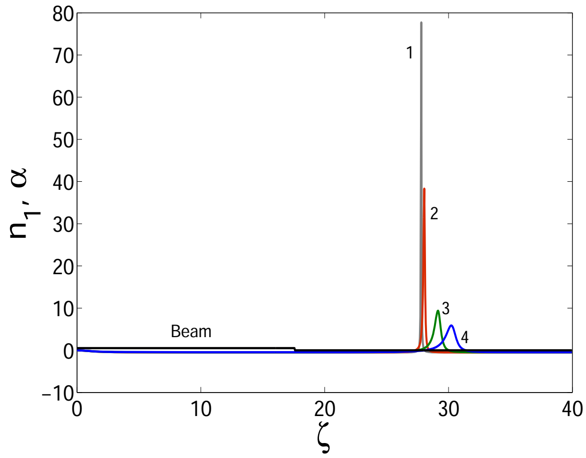

Figure 2.2: Variation of normalized perturbed electron density with and without magnetic field in electron beam driven plasma [$\beta_{ph} = 1$, $\beta_b = 1$], with beam density [$\alpha = 0.5$ for $0 \leqslant \zeta \leqslant 5.6\pi$ and zero otherwise], $1 \to \Omega = 0$, $2 \to \Omega = 0.2$, $3 \to \Omega = 0.5$, $4 \to \Omega = 0.7$.

Here, $\alpha = n_b/n_0$ is the normalized electron/proton beam density and $\Omega = \omega_c/\omega_{pe}$ with $\omega_c = eB_0/mc$ being the electron cyclotron frequency. The normalized variables we have used are $E \to eE/(m\omega_{pe}c)$, $p_x \to p_x/mc$, $p_y \to p_y/mc$, $n \to n/n_0$ and $\beta_{ph} = v_{ph}/c$. Also, we defined $p_x^2 + p_y^2 = p^2$. From the electron continuity equation it is easy to show that

$$n = \beta_{ph}\frac{\sqrt{1+p^2}}{\beta_{ph}\sqrt{1+p^2} - p_x}. \tag{2.9}$$

It is not an easy task to find an exact solution for the above coupled nonlinear Eqs.(2.6)-(2.8) analytically. Therefore, we have solved these differential equations by 4^{th} order Runge-Kutta method and obtained the solutions for the wake wave electric field, electron density etc. separately for the electron driver and proton driver scheme. Since these waves are encountered in the high energy physics accelerator, we have assumed the electron beam velocity as well as the phase velocity of the wake wave to take free space light velocity.

Electron beam driver scheme: In the electron beam driver case, first we

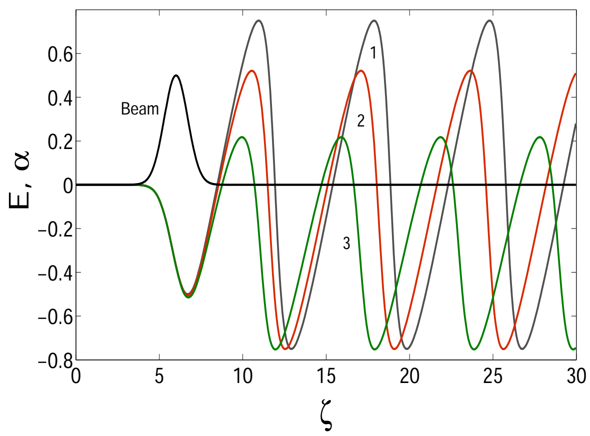

Figure 2.3: Variation of normalized electric field with and without magnetic field in a Gaussian shaped electron beam driven plasma $[A_0 = 0.5, \zeta_c = 6, \sigma_x = 1.0]$ and $\beta_{ph} = 1$, $\beta_b = 1$; $1 \rightarrow \Omega = 0$, $2 \rightarrow \Omega = 0.4$, $3 \rightarrow \Omega = 0.7$.

have considered rectangular profile of the drive beam with the following longitudinal variation of its density:

$$\begin{aligned} \alpha &= \alpha_0 \text{ for } 0 \leqslant \zeta \leqslant l_b, \\ &= 0, \text{otherwise}; \end{aligned} \tag{2.10}$$

i.e. the bunch is flat over the full beam length l_b with α_0 being a constant quantity.

Fig.(2.1) and Fig.(2.2) show the stationary electric field and perturbed density profiles for different strengths of the applied magnetic fields as obtained from our numerical investigation. In obtaining these results the electron beam density are taken to be half of the equilibrium plasma density ($\alpha_0 = 0.5$). The sawtooth like structures of electric field and electron density spikes are observed in case of zero magnetic field limit. These results have indeed a clear resemblance to the analytical solution for the wake wave profiles excited by rectangular electron pulse in the absence of any external magnetic field.[30] In this paper, Rosenzweig has provided an exact analytical expression in 1D for the wake wave electric field excited by an

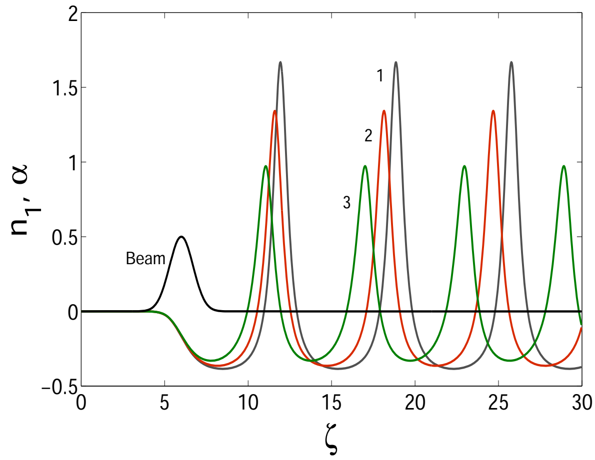

Figure 2.4: Variation of normalized perturbed electron density with and without magnetic field in a Gaussian shaped electron beam driven plasma $[A_0 = 0.5, \zeta_c = 6, \sigma_x = 1.0]$ and $\beta_{ph} = 1$, $\beta_b = 1$; $1 \rightarrow \Omega = 0$, $2 \rightarrow \Omega = 0.4$, $3 \rightarrow \Omega = 0.7$.

ultra-relativistic rectangular shaped electron beam in the unmagnetized plasma systems. Later, similar kind of analysis has been performed by Bera *et al.* who have obtained space time dependent solution of relativistic electron beam driven wake field in a cold, homogeneous plasma using 1D-fluid simulation techniques.[12] Presence of an external magnetic field has an effect to change the nature of these stationary structures. It is evident from these figures that in presence of the magnetic field the electric field behind the pulse gradually loses its sawtooth like shape and becomes sinusoidal with the increasing strength of magnetic field. Consequently, the peak amplitude of the perturbed density is also observed to decrease. It is to be noted here that the external magnetic field also affects the transformer ratio (R) which is defined to be the ratio of maximum accelerating electric field behind the beam and the maximum decelerating field inside the beam. With the increasing strength of magnetic field the transformer ratio gradually decreases. So, despite the fact that externally applied magnetic field can help to avoid phase slippage between the driven electrons and the plasma wave, it reduces the transformer ratio of charged particle acceleration.

Our numerical simulation process is then extended to show the stationary structures of the relativistic magnetized wake wave with a Gaussian electron beam density profile described by,

$$\alpha = A_0 \exp\{-(\zeta - \zeta_c)^2/\sigma_x^2\}. \tag{2.11}$$

where A_0, ζ_c and σ_x^2 are the amplitude, expectation value and variance of the distribution function respectively.

It is observed in Fig.(2.3) and Fig.(2.4) that with this Gaussian beam profile the electric field behind the beam are sinusoidal and perturbed density spikes are not very high for the slightly nonlinear case ($A_0 = 0.5$). In contrast to these results, the steepening of the electric field and occurrence of high density spikes can be seen in Fig.(2.5) and Fig.(2.6) for the highly nonlinear situation ($A_0 = 2.0$). However, for both the rectangular and Gaussian beam density profile the maximum amplitude of the electric field behind the pulse get reduced with the increase of the strength of the external magnetic field.

Proton beam driver scheme: We have also discussed the effect of an external magnetic field on the wake field structures of proton beam driven acceleration process. The drive beam profile has similar rectangular profile as considered in the electron driver case [Eq. (2.10)]. The stationary electric field and perturbed density profiles for different strengths of the applied magnetic fields are shown respectively in Fig.(2.7) and Fig.(2.8). In contrast to the electron beam driver case, here we see that with the increase in the magnetic field strength the maximum electric field behind the pulse gradually increases.

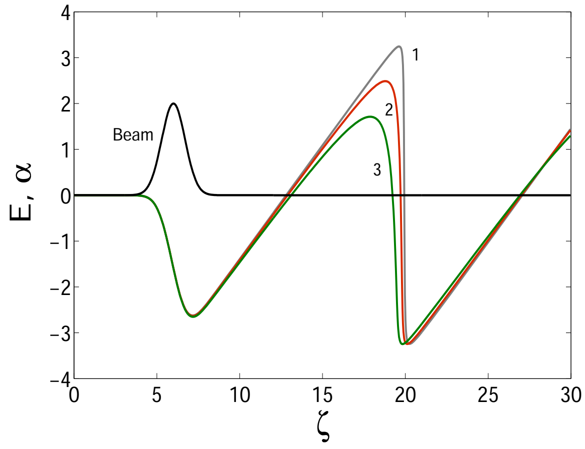

Figure 2.5: Variation of normalized electric field with and without magnetic field in a Gaussian shaped electron beam driven plasma $[A_0 = 2.0, \zeta_c = 6, \sigma_z = 1.0]$ and $\beta_{ph} = 1$, $\beta_b = 1$; $1 \rightarrow \Omega = 0$, $2 \rightarrow \Omega = 0.5$, $3 \rightarrow \Omega = 0.9$.

2.3 Effect of ion motion on electron/proton beam driven wake-field acceleration

Investigations of ultra-relativistic charged particle beam driven wake wave excitation process are normally performed with the assumption of stationary ion background. This assumption might be violated. In the generation process of plasma waves, due to their heavier mass the ions carry the main part of the momentum of the source (electron/proton beam). In the strong field excited behind electron or proton beam, the plasma ions can reach a velocity which is sufficient to make a contribution in the process of charge separation and thereby this can significantly influence the wake field structures.[57, 58] Thus, in our investigations we have included the plasma ion motion as well.

It has been speculated earlier that inclusion of ion motion could have some effects on the particle acceleration process in unmagnetized plasma.[11] In order to show its effect, it is convenient to adopt pseudo-potential approach to solve

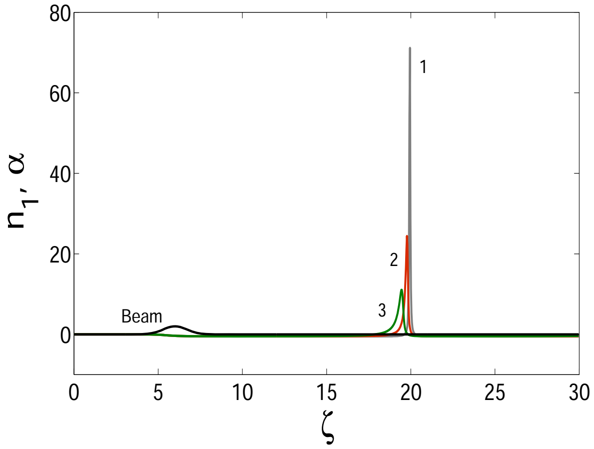

Figure 2.6: Variation of normalized perturbed electron density with and without magnetic field in a Gaussian shaped electron beam driven plasma $[A_0 = 2.0, \zeta_c = 6, \sigma_z = 1.0]$ and $\beta_{ph} = 1$, $\beta_b = 1$; $1 \rightarrow \Omega = 0$, $2 \rightarrow \Omega = 0.5$, $3 \rightarrow \Omega = 0.9$.

the problem.[82] In this method we express each of the dynamical variables of the system viz. species velocities, densities as function of electrostatic potential (φ) and then obtain a second order differential equation for φ.

When we include ion motion with an electron/proton beam passing through the unmagnetized plasma, the basic equations describing the system are:

The relativistic electron or ion momentum equations,

$$\left(\frac{\partial}{\partial t} + v_j \frac{\partial}{\partial x}\right)(\gamma_j v_j) = \frac{q_j E}{m_j}, \tag{2.12}$$

the continuity equations for electron, and ion fluids

$$\frac{\partial n_j}{\partial t} + \frac{\partial}{\partial x}(v_j n_j) = 0, \tag{2.13}$$

and the Poissons equation

$$\frac{\partial E}{\partial x} = 4\pi \left[\sum_j q_j n_j - q e n_b\right], \tag{2.14}$$

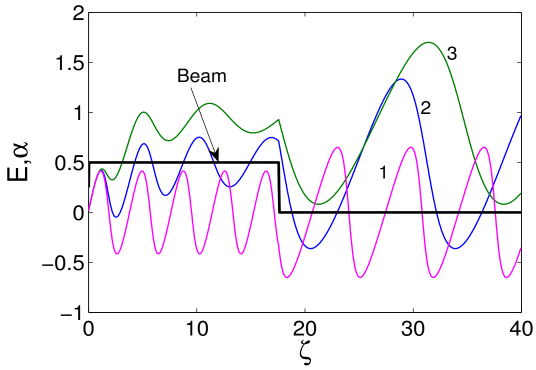

Figure 2.7: Variation of normalized electric field with and without magnetic field in proton beam driven plasma [$\beta_{ph} = 0.995$, $\beta_b = 0.995$], with beam density [$\alpha = 0.5$ for $0 \leq \zeta \leq 5.6\pi$ and zero otherwise], $1 \to \Omega = 0$, $2 \to \Omega = 0.5$, $3 \to \Omega = 0.9$.

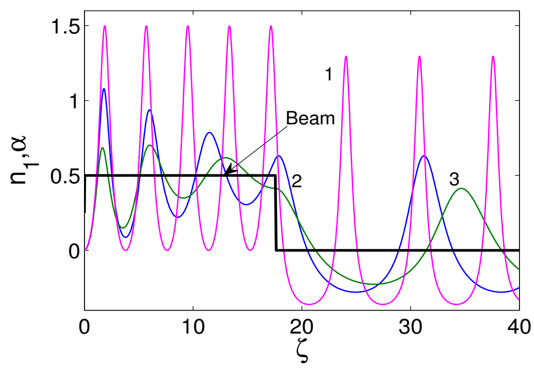

Figure 2.8: Variation of normalized perturbed electron density with and without magnetic field in proton beam driven plasma [$\beta_{ph} = 0.995$, $\beta_b = 0.995$], with beam density [$\alpha = 0.5$ for $0 \leq \zeta \leq 5.6\pi$ and zero otherwise], $1 \to \Omega = 0$, $2 \to \Omega = 0.5$, $3 \to \Omega = 0.9$.

where n_j, v_j, q_j, and m_j are the densities, velocities, charges, and masses of either electrons or ions, respectively, with $q_j = -e$ for electrons, $q_j = e$ for ions. Indeed, 'j' indicates the species index, with the signs $-$, and i refer to electrons, and ions, respectively. Here $\gamma_j = [1 - (v_j/c)^2)]^{-1/2}$ are the relativistic Lorentz factors associated with different species motions. For the purpose of simplicity in analysis, we have considered the motion of the plasma ion to follow non-relativistic dynamics so that γ_i is taken to be unity. Here n_b is the electron beam density.

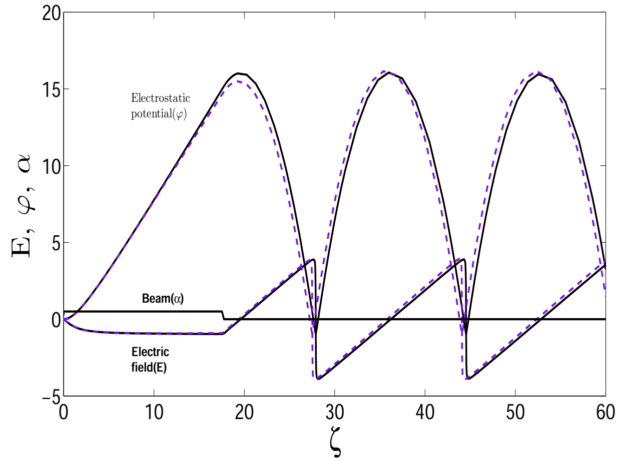

Figure 2.9: Stationary solution for electric field and potential in electron beam driven plasma with non-relativistic ion motion (dashed lines) and for static ions(continuous lines).

The wave electric field is along the x direction, i.e., $\mathbf{E} = E\hat{e}_x$; $\hat{e}_x$ is the unit vector along the x axis.

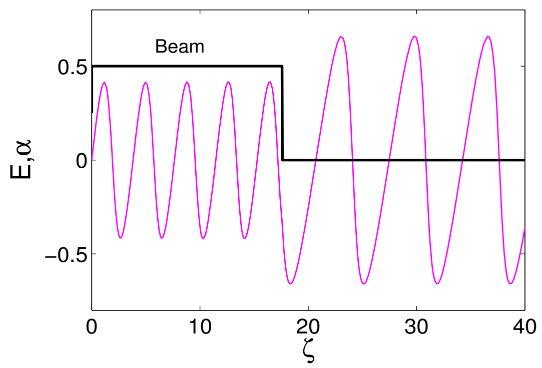

Figure 2.10: Variation of normalized electric field in a proton beam driven plasma [$\beta_{ph} = 0.995$, $\beta_b = 0.995$], with beam density [$\alpha = 0.5$ for $0 \leq \zeta \leq 5.6\pi$ and zero otherwise].

A travelling wave solution of Eqs.(2.12)-(2.14) can be obtained by introducing a variable defined as $\zeta = k_p(x - v_{ph}t)$, where $k_p = \omega_p/v_{ph}$ with $\omega_p = \sqrt{4\pi n_{0-}e^2/m_-}$; n_{0-} is the equilibrium density of electrons and v_{ph} is the phase velocity of the plane wave. We have also defined $\beta_- = v_-/c$ and $\beta_i = v_i/c$ as dimensionless velocities.

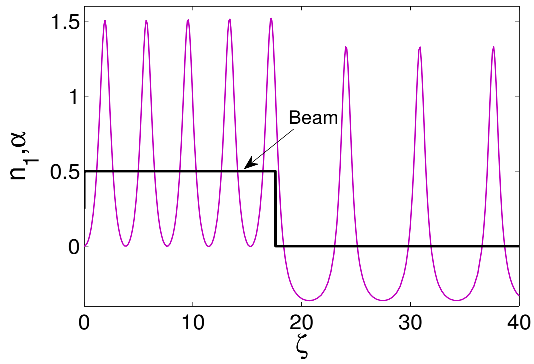

Figure 2.11: Variation of normalized perturbed electron density in a proton beam driven plasma [$\beta_{ph} = 0.995$, $\beta_b = 0.995$], with beam density [$\alpha = 0.5$ for $0 \leq \zeta \leq 5.6\pi$ and zero otherwise].

Therefore, $\gamma_- = (1 - \beta_-^2)^{-1/2}$ and $\gamma_i = (1 - \beta_i^2)^{-1/2}$. And, $m_-/m_i = \mu$ is the electron to ion mass ratio.

Following the methods as described by Karmakar et al.[82], the electron and ion densities normalized by their respective equilibrium values, can be expressed as,

$$N_e = \beta_{ph}\gamma^2 \left[\frac{\varphi_e}{(\varphi_e^2 - \gamma^{-2})^{1/2}} - \beta_{ph}\right], \tag{2.15}$$

$$N_i = \frac{\beta_{ph}}{\sqrt{\beta_{ph}^2 + 2\varphi_i}}, \tag{2.16}$$

where $\varphi_i = -\mu\varphi$ and $\varphi_e = 1 + \varphi$ and $\gamma = [1 - (v_{ph}/c)^2)]^{-1/2}$ is the relativistic Lorentz factor associated with phase velocity of the plasma wave. Using these expression for species densities, from the Poisson's equation, we obtain a second order differential equation for φ as,

$$\frac{d^2\varphi}{d\zeta^2} = -\frac{\beta_{ph}^3}{\sqrt{\beta_{ph}^2 + 2\varphi_i}} + \frac{\beta_{ph}^3\gamma^2\varphi_e}{\sqrt{\varphi_e^2 - \gamma^{-2}}} - \beta_{ph}^4\gamma^2 + q\alpha\beta_{ph}^2,$$

where, $\alpha = n_b/n_0$ is the normalized electron/proton beam density. We solve this second order differential equation numerically assuming a rectangular beam profile whose longitudinal extension is same as in the earlier simulation process for the magnetized case [Eq. (2.10)].

In case of electron driver, the solution for the electrostatic potential as well as the electric field is shown in the Fig.(2.9). There is no significant change that can be observed from this figure with the inclusion of the non-relativistic ion motion. So the assumption of static ion background in obtaining the relevant results for both the magnetized and unmagnetized beam plasma cases is quite justified.

Next, we have discussed the wake field excitation by using proton beam incorporating plasma ion dynamics. The solution for the wake field excited inside and behind the single proton bunch as well as corresponding perturbed electron density have been shown in the Fig. (2.10) and Fig. (2.11). The transformer ratio (R) which determines the overall energy efficiency of the accelerated particles can be calculated from this field profile.

2.4 Wake field excitation by trains of proton bunches

Due to self modulational instability, the long proton bunch can be split into long chain of equi-spaced micro-bunches. It is of fundamental interest to see how strong wake field can be excited behind this multi-beams.

The electric field structure and corresponding perturbed electron density profile for the excited wake field by the train of equidistant rectangular particle bunches with peak beam density $0.5n_0$ have been shown in Fig. (2.12) and Fig. (2.13)

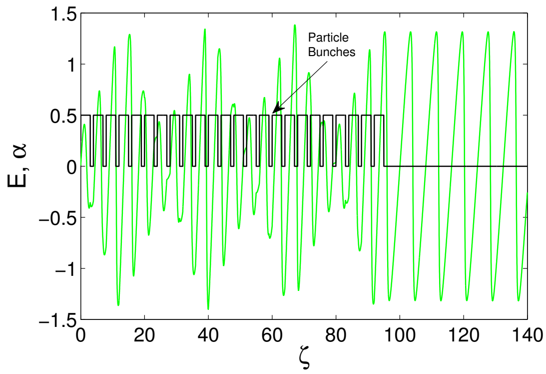

Figure 2.12: Variation of normalized electric field in absence of magnetic field driven by a train of equidistant proton micro-bunches [$\beta_{ph} = 0.995$, $\beta_b = 0.995$], with peak beam density $\alpha_0 = 0.5$.

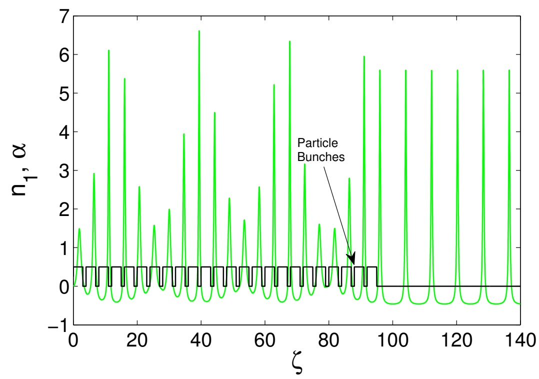

Figure 2.13: Variation of normalized perturbed electron density in absence of magnetic field driven by a train of equidistant proton micro-bunches [$\beta_{ph} = 0.995$, $\beta_b = 0.995$], with peak beam density $\alpha = 0.5$.

respectively as obtained by solving Eqs.(2.12)-(2.14). From the Fig. (2.12), it is observed that electric field amplitude can not grow indefinitely with the increase of the number of beams. Rather, the field saturates due to the amplitude dependent frequency as given in the one dimensional analytical solution reported by Akheizher and Polovin viz.[49]

$$\tau \simeq \tau_0 \left[1 + \frac{3}{16}(E_m/E_0)^2\right], \tag{2.17}$$

where, E_m and E_0 are respectively the maximum field amplitude and non-relativistic wave breaking limit. τ_0 is the wave period in absence of nonlinearity. The wave period (τ) is seen to increase with the maximum field amplitude. This change in wavelength can cause the bunches to fall under the decelerating field and thereby the field stops growing further.

2.5 Summary

We have shown that an external magnetic field plays a significant role in accelerating charged particle by plasma wake wave excited by electron as well as proton beam. The numerical simulation for the upper hybrid plasma wake wave has been performed to show the stationary electric field profiles which enables us to find the transformer ratio in presence of an external magnetic field. It has been established that, even though the magnetic field plays a crucial role in controlling the dephasing length, the transformer ratio which determines the overall energy gain of the accelerated particle is being reduced with the increase of the strength of magnetic field in the electron driver scheme. The external magnetic field confines trailing witness beam charge to the wavefront of the wake wave and controls its focusing nature. Thereby it helps to produce a highly collimated and quasi mono-energetic final accelerated beam. Effects of electron beam shape and density on the stationary structures of plasma variables are also shown to be of particular relevance in the particle acceleration process. It has been shown that consideration of non-relativistic ion motion does not affect much on the stationary structures of wake-wave electric field.

Consideration of both single long proton beam or micron sized train of small particle bunches to excite the plasma wake wave have been incorporated in our studies. Recently, K. V. Lotov has performed a numerical investigation on the two dimensional plasma wake field excitation process by trains of equidistant particle bunches which has been the motivation behind the use of such beam train in our theoretical analysis.[40] This analytical solution can facilitate the understanding of the underlying physical mechanisms in the production of high energy charged particles in laboratory experiment on PDPWFA. Particularly, it will certainly contribute to the theoretical knowledge of the ongoing "AWAKE [Advanced Proton Driven Plasma Wakefield Acceleration Experiment] project.[38, 83]

Admittedly, we have not considered the evolution of the beam (rigid beam case) throughout our calculations. Our whole analysis of wake field excitation has been carried out in the limit of $\beta_b \to 1$. In this limit, the beam behaves like a rigid charged rod, as has been shown by Bera et al.[12] by studying the space time evolution of a beam plasma system (relevant to PWFA) using fluid simulation techniques. Specifically, they have shown that as long as $\beta_b > 0.99$, the beam does not evolve within the time frame of interest. Our analysis of plasma wake field excitation in the limit of $\beta_b \to 1$ is further justified by the fact that the present day PWFA experiments use beam energies in the range of $20 \sim 40$ Gev,[29, 44] which amounts to a value $\beta_b \to 1$. Thus our results are relevant to present day PWFA experiments.

Chapter 3

Relativistic wave-breaking limit of electrostatic waves in cold electron-positron-ion plasmas

A one-dimensional nonlinear propagation of relativistically strong electrostatic waves in cold electron-positron-ion (EPI) plasmas has been analyzed. The motion of all the three species, namely, electron, positron, and ion has been treated to be relativistic. The maximum permissible electric field amplitude - so called "wave-breaking limit" of such an electrostatic wave before wave-breaking has been derived, showing its dependence on the relativistic Lorentz factor associated with the phase velocity of the plasma wave, on the electron/positron to ion mass ratio, and on the ratio of equilibrium ion density to equilibrium electron/positron density.

3.1 Introduction

Studies on "wave-breaking"[3, 21, 51] of nonlinear oscillations and/or waves in plasmas have been an active area of research in nonlinear plasma theory over the past years owing to its number of potential applications like plasma heating[84], particle acceleration by wake-fields[85, 86], etc.

As mentioned earlier, it is not possible to have a *coherent* wave motion in plasmas with an amplitude greater than its critical amplitude. The value of the critical amplitude may be physically different in the cold-nonrelativistic[21] or cold-relativistic[49] or warm-nonrelativistic[52] or warm-relativistic [53] plasma situations. Relativistic effects have been found to increase the critical amplitudes, whereas, thermal effects decrease the critical amplitudes. The critical amplitude beyond which a plasma wave breaks is well-known as the "wave-breaking amplitude". Over the past decades, much theoretical progress has been made on the understandings of the wave-breaking amplitudes for electron plasma waves in electron-ion plasmas.[2, 21, 49, 52–56, 87] Understanding the wave-breaking amplitude and corresponding electric field threshold (the maximum coherent electric field which a plasma can consistently sustain) is important for the plasma-based particle acceleration schemes.[4, 11] Because the wave-breaking amplitude is as one of the critical parameters that determines the maximum energy gain of the accelerated particles.[88]

In the recent years, there has been much discussion of wave-breaking of relativistically strong electron plasma waves regarding particle acceleration to high energies.[8, 9, 89, 90] A wave of sufficiently large amplitude is generally said to be relativistically strong if it can induce, at least, relativistic motions of lighter plasma species. In this regard, the ratio of quiver velocity of electrons (say) to the velocity of light in vacuum can become comparable to unity, i.e., $v \sim c$. Thus the longitudinal electric fields associated with such waves can be extremely large, i.e., $eE/(m_e \omega c) \sim 1$, where e is the charge of an electron, E is the electric field of the wave, m_e is the mass of an electron, and ω is the frequency of wave. The energy gain by an electron over a wave period could be made comparable to its rest mass energy while it is trapped within and in phase with such a plasma wave.

The investigation on maximum electric field of a relativistically strong electron plasma wave in cold electron-ion (EI) plasmas was first done by Akhiezer and Polovin (AP), where massive ions were assumed to form a fixed charge neutralizing background.[49] Surprisingly, in their classic paper, AP derived the corresponding wave-breaking limit as $eE_{\max}/m_e \omega_p v_{ph} = \sqrt{2(\gamma - 1)}/\beta$, even without mentioning "wave-breaking", where ω_p is the electron plasma frequency, v_{ph} is the phase velocity of the plasma wave, and γ is the relativistic Lorentz factor associated with the phase velocity of the plasma wave, i.e., $\gamma = (1 - \beta^2)^{-1/2}$ with $\beta = v_{ph}/c$. A few researchers then extended the analysis of AP by taking into account the ion motion, and thus they reported the wave-breaking amplitudes of relativistic oscillations in arbitrary mass ratio two component cold plasmas.[57, 58] In the context of plasma wave generation in the wakes of laser pulses or an electron beam, it was subsequently shown that ion being the more massive candidate compared to the other species they can carry the main part of the momentum transferred by the laser pulse.[58] Besides, we find from their work that the wave-breaking limit increases with the increase of the electron to ion mass ratio, and thus an equal mass electron-positron (EP) plasma can support higher amplitude relativistically strong coherent electrostatic waves compared to an electron-ion plasma.

In this chapter, we investigate the wave-breaking amplitude of one-dimensional relativistically strong electrostatic waves in cold unmagnetized electron-positron-ion (EPI) plasmas. In addition to oppositely charged and same mass species electrons and positrons, EPI plasmas contain a fraction of massive ions with an overall charge neutrality in equilibrium states. The present work is a generalized model where the system is a three component electron-positron-ion (EPI) plasma in which the dynamics of all the three species are taken to be relativistic.

3.2 Determination of the wave-breaking amplitude

We consider a cold unmagnetized electron-positron-ion (EPI) plasma having an overall charge neutrality in its equilibrium state, i.e., $n_{0-} = n_{0+} + n_{0i}$, where n_{0-}, n_{0+}, and n_{0i} are the equilibrium densities of electrons, positrons, and ions, respectively. Massive ions will be allowed to take part in the relativistically strong high frequency wave dynamics in such a multi-species plasma. And, the motion of all the three species, namely, electron, positron, and ion will also be taken to be relativistic. In one space-dimension, the basic equations that describe the nonlinear propagation of relativistically strong electrostatic waves in cold EPI plasmas are the continuity equations for electron, positron, and ion fluids

$$\frac{\partial n_j}{\partial t} + \frac{\partial}{\partial x}(v_j n_j) = 0, \tag{3.1}$$

the momentum equations for electron, positron, and ion fluids

$$\left(\frac{\partial}{\partial t} + v_j \frac{\partial}{\partial x}\right)(\gamma_j v_j) = \frac{q_j E}{m_j}, \tag{3.2}$$

and the Poissons equation

$$\frac{\partial E}{\partial x} = 4\pi \sum_j q_j n_j, \tag{3.3}$$

where n_j, v_j, γ_j, q_j, and m_j are the densities, velocities, relativistic Lorentz factors, charges, and masses of either electrons or positrons or ions, respectively, with $q_j = -e$ for electrons, $q_j = e$ for both positrons and ions. Indeed, 'j' indicates the species index, with the signs $-$, $+$, and i refer to electrons, positrons, and ions, respectively. The wave electric field is along the x direction, i.e., $E = E\hat{e}_x$; $\hat{e}_x$ is the unit vector along the x axis.

Looking for a travelling wave solution of Eqs. (3.1)-(3.3), it is convenient to introduce a variable transformation $\xi = k_p(x - v_{ph}t)$, where $k_p = \omega_p/v_{ph}$ with $\omega_p = \sqrt{4\pi n_{0-}e^2/m_-}$; n_{0-} is the equilibrium density of electrons and v_{ph} is the phase velocity of the plane wave. We also define $\beta_\pm = v_\pm/c$ and $\beta_i = v_i/c$ as dimensionless velocities. Therefore, $\gamma_\pm = (1 - \beta_\pm^2)^{-1/2}$, $\gamma_i = (1 - \beta_i^2)^{-1/2}$ and, $m_-/m_i = \mu$ is the electron to ion mass ratio.

In the transformed co-ordinate system the continuity and momentum equations, respectively, take the form as

$$\frac{d}{d\xi}[n_j(\beta - \beta_j)] = 0, \tag{3.4}$$

and

$$(\beta - \beta_j)\,\frac{d}{d\xi}(\beta_j\gamma_j) = \varepsilon_j \beta^2 E, \tag{3.5}$$

where $\beta = v_{ph}/c$, and ε_j is a constant; $\varepsilon_- = 1$ for electron, $\varepsilon_+ = -1$ for positron, and $\varepsilon_i = -\mu$ for ion. Here $\gamma_j = [1-(v_j/c)^2)]^{-1/2}$ are the relativistic Lorentz factors associated with different species motions.

Now using the quasi-neutrality condition at the equilibrium, viz., $n_{0-} = n_{0+} + n_{0i}$ and normalizing the species densities by their respective equilibrium values, i.e., $N_j \to n_j/n_{0j}$, the Poisson's equation in the new variable ξ becomes

$$\frac{dE}{d\xi} = \alpha N_i - N_- + (1-\alpha)N_+, \tag{3.6}$$

where $\alpha = n_{0i}/n_{0-}$, and the electric field is normalized on the non-relativistic wave-breaking field $m_-\omega_p v_{ph}/e$ and obeys $E(\xi) = -(1/\beta^2)(d\varphi/d\xi)$, where the rescaled potential, $\varphi = \varphi_0 + (e\Phi/m_-c^2)$ with Φ denoting the unnormalized electric potential. Here φ_0 is a constant potential in the unperturbed plasma state where the whole plasma is neutral and all the three species are at rest. Physically, electrons, positrons, and ions stop at the same point where the potential of the plasma wave is equal to this constant potential φ_0. Without much loss of generality, one may set $\varphi_0 = 1$, and henceforth, $\varphi = 1 + (e\Phi/m_-c^2)$. It should further be mentioned here that the unnormalized electric potential Φ has been assumed to be equal to zero when the plasma density is equal to the equilibrium density.

Now carrying out some simple algebra, we obtain from Eqs. (3.4)-(3.6) the normalized velocities and densities of the three species as

$$\beta_j = \frac{\beta - \varphi_j(\varphi_j^2 - \gamma^{-2})^{1/2}}{\beta^2 + \varphi_j^2}, \tag{3.7}$$

and

$$N_j = \beta\gamma^2 \left[\frac{\varphi_j}{(\varphi_j^2 - \gamma^{-2})^{1/2}} - \beta\right], \tag{3.8}$$

respectively, where $\varphi_- \equiv \varphi$, $\varphi_+ = 2 - \varphi$, and $\varphi_i = 1 + \mu(1-\varphi)$. Here $\gamma = [1-(v_{ph}/c)^2)]^{-1/2}$ is the relativistic Lorentz factor associated with the phase velocity of the plasma wave. It is to be noted here that φ is the potential of the plasma waves in the perturbed state, and φ_+ and φ_i are the new variables introduced which are related to φ. Furthermore, we note that, at the equilibrium state, each of φ_j takes the same value φ_0.

Substituting the values of N_j in Eq. (3.6) followed by using $E(\xi) = -(1/\beta^2)(d\varphi/d\xi)$, we obtain the following second order differential equation for φ as

$$\frac{d^2\varphi}{d\xi^2} + \alpha\beta^3\gamma^2 \left[\frac{\varphi_i}{(\varphi_i^2 - \gamma^{-2})^{1/2}} - \frac{\varphi_+}{(\varphi_+^2 - \gamma^{-2})^{1/2}}\right] + \beta^3\gamma^2 \left[\frac{\varphi_+}{(\varphi_+^2 - \gamma^{-2})^{1/2}} - \frac{\varphi}{(\varphi^2 - \gamma^{-2})^{1/2}}\right] = 0. \tag{3.9}$$

In the general regime $\varphi_j^2 > \gamma^{-2}$, Eq. (3.9) is solved numerically, and a typical wave form solution is depicted in Fig. (3.1) for different values of α with $\mu = 1/1836$ and $\gamma = 10$. From the figure it is evident that the wavelength of the plasma wave increases with the the ratio of equilibrium ion density to equilibrium electron/positron density α.

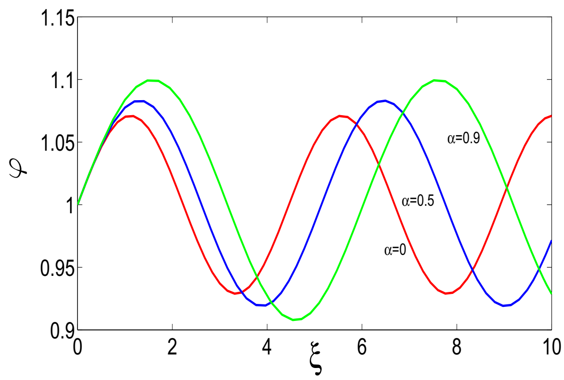

Figure 3.1: Normalized electrostatic potential of the plasma wave φ vs. ξ obtained by numerical solving of the potential equation for different values of α with $\mu = 1/1836$ and $\gamma = 10$.

Now we re-write Eq. (3.9) in the following form:

$$\frac{d^2\varphi}{d\xi^2} + \frac{dU}{d\varphi} = 0, \tag{3.10}$$

where

$$U = \beta^3\gamma^2 \left[-\frac{\alpha}{\mu}(\varphi_i^2 - \gamma^{-2})^{1/2} + (\alpha - 1)(\varphi_+^2 - \gamma^{-2})^{1/2} \right]$$
$$+\beta^3\gamma^2 \left[\left(\frac{\alpha}{\mu} - \alpha + 2 \right) (1 - \gamma^{-2})^{1/2} - (\varphi^2 - \gamma^{-2})^{1/2} \right]. \tag{3.11}$$

Here $U(\varphi)$ is chosen to be equal to zero at a point $\varphi = 1$ where it reaches a minimum. The Eq. (3.10) describes the one dimensional motion of a particle in a field with potential $U(\varphi)$; the values φ and E correspond to the coordinate and velocity of this fictitious particle of unit mass, respectively.

Next, from Eq. (3.10) we obtain the first integral of motion as

$$\frac{1}{2}\left(\frac{d\varphi}{d\xi}\right)^2 + U(\varphi) = I, \tag{3.12}$$

which can be re-written as

$$\frac{d\varphi}{d\xi} = \pm\sqrt{2(I - U)}, \tag{3.13}$$

where I is an integration constant and can be identified as the total energy of the fictitious particle having unit mass. The first term on the left hand side of Eq. (3.12) signifies the kinetic energy and the second term is the potential energy.

In order to have a periodic solution of the nonlinear plasma waves, the potential $U(\varphi)$ should have to be real restricting the allowed value of φ. In case of EPI and/or EP plasmas, the nonlinear periodic solution exists in a very small range $(1/\gamma) \leq \varphi \leq 2 - (1/\gamma)$. It is clearly seen in Fig. (3.2) that the potential U(φ) gradually loses its symmetry with the increase of the value of α. One may reasonably expect that such dependence of potential on α have direct effects on the wave-breaking amplitudes of relativistically strong waves in two or three species plasmas.

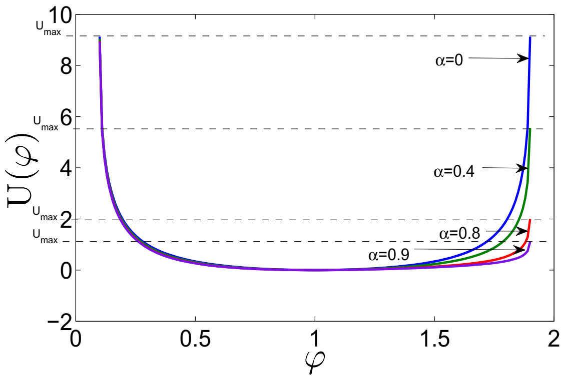

Figure 3.2: Variation of normalized potential for different values of α in EPI ($0 < \alpha < 1$) and EP ($\alpha = 0$) plasmas with $\mu = 1/1836$ and $\gamma = 10$. The potential $U(\varphi)$ is real in the range $1/\gamma \leq \varphi \leq 2 - (1/\gamma)$. All the three species are relativistic.

The restriction on the allowed values of $U(\varphi)$ inhibits to achieve arbitrarily large amplitude of the electric field. In order to obtain the maximum achievable electric field supported by the plasma viz. wave-breaking field, we need to find out the maximum "permissible" value of U(φ) ($U_{\max}$). We find from Ref. [57] that, in case of EP plasma ($\alpha = 0$), $U_{\max}$ is calculated at $\varphi = 1/\gamma$. Due to the symmetry of $U(\varphi)$ about $\varphi = 1$ for $\alpha = 0$, $U_{\max}$ could have been calculated at $\varphi = 2 - (1/\gamma)$. In case of EPI plasma ($0 < \alpha < 1$), $U(\varphi)$ loses its symmetry and thereby $U_{\max}$ should be calculated at $\varphi = 2-(1/\gamma)$. As we can see from Fig. (3.2) that, $U_{\max}$ gradually decreases with the increase of α. The fictitious particle of our problem can oscillate within the potential well with the maximum amplitude determined by the maximum permissible value of $U(\varphi)$ at $\varphi = 2-(1/\gamma)$. For any other values of $U(\varphi)$ (calculated at points beyond the restricted range of φ) greater than $U_{\max}$ calculated at $\varphi = 2 - (1/\gamma)$, the motion of the fictitious particle would be unbounded.

In the restricted range of φ, when $U(\varphi)$ reaches its minimum value at $\varphi = 1$, i.e., $U(\varphi) \mid_{\varphi=1}= 0$, then the kinetic energy takes its maximum value. Therefore, at $\varphi = 1$, the normalized electric field E of the plasma wave which equals to $-(1/\beta^2)(d\varphi/d\xi)$ also reaches its maximum value $E_{\max}$. Therefore, from Eq. (3.12) we obtain

$$E_{\max} = \frac{1}{\beta^2}\sqrt{2I}. \tag{3.14}$$

But I can not be arbitrarily large. A nonlinear periodic plasma wave exists if the right hand side of Eq. (3.13) is real, i.e., $I \geq U(\varphi)$. In case of EPI and/or EP plasmas, I, at most, can be equal to $U(\varphi = 2 - (1/\gamma)) \equiv U_{\max}$. With this value of I, we get the wave-breaking amplitude as

$$E_{\text{wb}} = \frac{1}{\beta^2}\sqrt{2U_{\max}}. \tag{3.15}$$

Now the maximum permissible value of $U(\varphi)$, reaching the point $\varphi = 2-(1/\gamma)$, is obtained from Eq. (3.11) as

$$U_{\max} = \beta^4\gamma^2\left[\left(\frac{\alpha}{\mu} - \alpha + 2\right) - \frac{\alpha}{\sqrt{\gamma+1}}\sqrt{\zeta_1\zeta_2} - 2\sqrt{\frac{\gamma}{\gamma+1}}\right], \tag{3.16}$$

where $\zeta_1 = (1-\mu^2)/\mu^2$ and $\zeta_2 = 1 + \gamma(1-\mu)/(1+\mu)$. Therefore, by using the above expression for $U_{\max}$, we finally obtain the wave-breaking electric field for relativistically strong electrostatic waves in EPI plasma from Eq. (3.15) as

$$E_{\text{wb}} = \sqrt{2}\gamma\left[\left(\frac{\alpha}{\mu} - \alpha + 2\right) - \frac{\alpha}{\sqrt{\gamma+1}}\sqrt{\zeta_1\zeta_2} - 2\sqrt{\frac{\gamma}{\gamma+1}}\right]^{1/2}. \tag{3.17}$$

The variation of wave-breaking field with α for different values of γ is shown in Fig. (3.3). It is seen that the wave-breaking field amplitudes are gradually being decreased with the increase of the value of α.

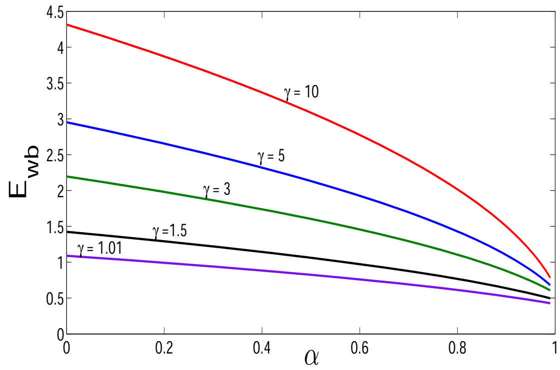

Figure 3.3: Variation of normalized wave-breaking electric field amplitude with α for $\mu = 1/1836$ for different γ. All the three species are relativistic.

The wave-breaking limit for the relativistic plasma waves in cold EP plasmas can be obtained by putting $\alpha = 0$ in Eq. (3.17) as

$$E_{\rm wb} = 2\gamma \left(1 - \frac{\sqrt{\gamma}}{\sqrt{\gamma + 1}}\right)^{1/2}. \tag{3.18}$$

Nevertheless, the wave-breaking limit for the relativistic plasma waves in cold electron-ion (EI) plasmas can not be recovered from Eq. (3.17) by merely putting $\alpha = 1$. Because in case of electron-ion plasmas the range of φ for which $U(\varphi)$ takes its real value is widened compared to the range in EPI and/or EP plasmas, i.e., $1/\gamma \le \varphi \le 1 + \frac{1}{\mu}\{1 - (1/\gamma)\}$. In this case, $U(\varphi)$ reaches its maximum allowed value at $\varphi = 1/\gamma$. Thus, by putting $\alpha = 1$ and $\varphi = 1/\gamma$ in Eq. (3.11), we obtain the maximum permissible value of $U(\varphi)$ in case of EI plasmas as

$$U_{\rm max} = \beta^4\gamma^2 \left[1 + \frac{1 - \sqrt{\eta_1\eta_2}}{\mu}\right], \tag{3.19}$$

where $\eta_1 = 1 + \mu$ and $\eta_2 = 1 + \mu(\gamma - 1)/(\gamma + 1)$. And, consequently, the wave-breaking amplitude for an EI plasma where both the electron and ion follow the relativistic dynamics simply reads as [57]

$$E_{\rm wb} = \sqrt{2}\gamma \left(1 + \frac{1 - \sqrt{\eta_1\eta_2}}{\mu}\right)^{1/2}. \tag{3.20}$$

Interestingly, in the equal mass limit ($\mu = 1$), from Eq. (3.20) one can recover the expression for $E_{\rm wb}$ given in Eq. (3.18). Moreover, the relativistic wave-breaking amplitude in an EI plasma with stationary ion background can be recovered from Eq. (3.20) by taking the appropriate limit $\mu \to 0$ as

$$E_{\rm wb} = \frac{\sqrt{2(\gamma - 1)}}{\beta}, \tag{3.21}$$

which is the well-known AP limit.[49]

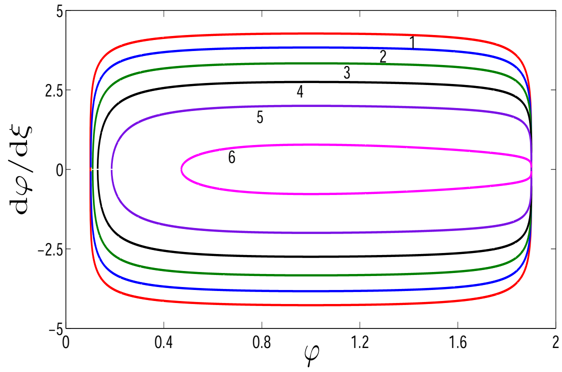

Figure 3.4: Phase portrait in case of EPI and/or EP plasmas: $1 \to \alpha = 0$, $2 \to \alpha = 0.2$, $3 \to \alpha = 0.4$, $4 \to \alpha = 0.6$, $5 \to \alpha = 0.8$, $6 \to \alpha = 0.99$ with $\mu = 1/1836$ and $\gamma = 10$.

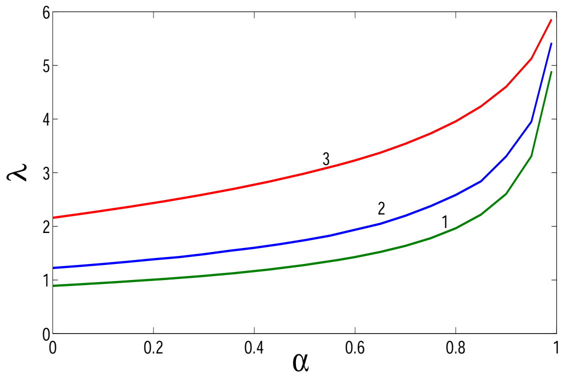

Figure 3.5: Variation of normalized plasma wavelength with the equilibrium ion to electron concentration ratio α: ($1 \to \gamma = 1.01$, $2 \to \gamma = 5$, $3 \to \gamma = 10$). The electron to ion mass ratio $\mu = 1/1836$ in all cases.

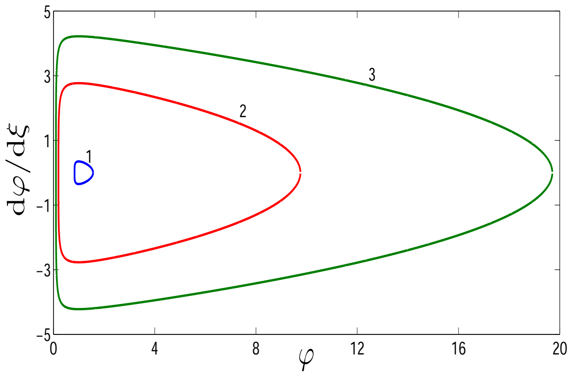

Figure 3.6: Phase portrait in case of electron-ion plasmas for ($1 \rightarrow \gamma = 1.2$, $2 \rightarrow \gamma = 5$, $3 \rightarrow \gamma = 10$). The electron to ion mass ratio $\mu = 1/1836$ in all cases.

Now we proceed to find the wavelength λ of relativistic plasma waves. The wavelength can be defined as twice the distance between minimum and maximum points of the electrostatic potential φ. Thus, in units of k_p^{-1}, wavelength is given by

$$\lambda = 2\int_{\varphi_-}^{\varphi_+} \frac{d\varphi}{d\varphi/d\xi}, \tag{3.22}$$

where $\varphi_\pm$ are the roots of the right hand side of Eq. (3.13). The roots can be found out from the phase portrait of Eq. (3.13). The phase portrait is shown in Fig. (3.4) with different values of equilibrium ion to electron concentration ratio α, and it bounds the possible real solution of Eq. (3.13). Indeed, $d\varphi/d\xi = 0$ determines the roots $\varphi_\pm$. Then by using the formula given in Eq. (3.22), we have calculated the wavelengths of relativistic plasma waves. The variation of wavelength with α in EPI and/or EP plasmas is shown in Fig. (3.5). We find that the wavelength gradually increases with the increase of ion concentration in EPI plasmas.

In case of EI plasmas ($\alpha = 1$), the phase portrait is drawn separately for three different values of γ in Fig. (3.6). Then by using (3.22) we have calculated the wavelengths, which are 1.6051, 12.6074 and, 17.5930 in units of k_p^{-1} for $\gamma = 1.2$,

$\gamma = 5$, and $\gamma = 10$, respectively.

3.3 summary

In summary, we have analyzed a one-dimensional nonlinear propagation of relativistically strong plasma waves in cold three component electron-positron-ion (EPI) plasmas. We have allowed the relativistic motion of massive ions along with the other two species of the system. Consideration of ion motion is quite reasonable in the context of wake-field generation by intense laser pulses or an electron beam, etc., where the ion fluid also follows relativistic dynamics. The maximum permissible electric field amplitude before wave-breaking ("wave-breaking amplitude") has been derived. It has been shown that the wave-breaking electric field amplitude in EPI plasmas depends on the ratio of equilibrium ion density to equilibrium electron/positron density along with on the relativistic Lorentz factor associated with the phase velocity of the plasma wave and on the ratio of the electron/positron mass to ion mass. Subsequently, it is found that the wave-breaking amplitude decreases with the increase of the ratio of equilibrium ion density to equilibrium electron density, and thus, the value of maximum amplitude possible before wave-breaking gets lowered due to the presence of a fraction of massive ions in a pure electron-positron plasma.

Chapter 4

Wave-breaking amplitudes of relativistic upper-hybrid oscillations

A travelling wave solution is presented for relativistic upper-hybrid oscillations (RUHOs) in a cold magnetized plasma. An expression for the wave-breaking amplitudes of RUHOs is derived. The wave-breaking amplitudes of RUHOs are found to decrease with the increase of the strength of an ambient magnetic field. These results will be of relevance to the laboratory context of particle acceleration by wake-fields in which magnetic field plays a central role.

4.1 Introduction

There has been an extensive theoretical progress on the understandings of the 'wave-breaking amplitudes' for electron plasma waves in the last few decades.[2, 21, 49, 51, 52, 54–56, 82, 87, 89] Much analytical information on the electrostatic wave-breaking has significantly improved our understanding of plasma-based particle acceleration schemes.[4, 8, 28, 30, 85, 86, 89, 90] Nevertheless, there has not been much investigation regarding wave-breaking of relativistic electrostatic plasma oscillations in an ambient magnetic field. About a few decades ago, Katsouleas and Dawson discussed the 'surfatron' mechanism for the energization of electrons to arbitrarily high energies by the relativistic upper-hybrid (UH) wave electric fields.[13] In a recent past, a comprehensive study of relativistic UH waves has been made showing an exact space-time dependent solution in Lagrangian co-ordinates.[46] Furthermore, the importance of RUHOs has also been discussed in the context of electron surfing acceleration in a self-consistent simulation for astrophysical applications.[91]

In this chapter, we obtain an expression for the wave-breaking amplitudes of relativistic upper-hybrid oscillations (RUHOs) in a cold magnetized plasma. A travelling wave solution for the wave electric field, electron density, and relativistic momenta of electrons associated with RUHOs is also presented. Moreover, we construct travelling wave solution from the exact space-time dependent solution of RUHOs by appropriately choosing initial conditions. It is well known that, in plasma-based particle acceleration schemes, wave-breaking amplitude serves as one of the critical parameters that determines the maximum energy gain of the accelerated particles.[4, 8] In such experiments, the waves which get excited in the wake of powerful sources are nothing but electrostatic travelling waves.[92] These waves are very sensitive to a small deviation of the initial conditions; a slight longitudinal perturbation causes such waves to break at arbitrary amplitudes via the phase-mixing process.[93, 94]

4.2 Travelling wave solution for upper hybrid wave

The basic 1-D equations that govern the relativistic electrostatic plasma oscillations in an ambient magnetic field are the electron continuity equation, the x and y components of relativistic electron momentum equations, and the electric field evolution equation, respectively,

$$\left(\frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x}\right) n = -n \frac{\partial v_x}{\partial x}, \tag{4.1}$$

$$\left(\frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x}\right) p_x = -eE - \frac{eB_0}{c} v_y, \tag{4.2}$$

$$\left(\frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x}\right) p_y = \frac{eB_0}{c} v_x, \tag{4.3}$$

and

$$\left(\frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x}\right) E = 4\pi e n_0 v_x, \tag{4.4}$$

where n is the electron density, $p_x = \gamma m v_x$ and $p_y = \gamma m v_y$ denote components of relativistic electron momentum along the x and y directions, respectively, and $\gamma = \{1 - (v_x^2 + v_y^2)/c^2\}^{-1/2}$ represents the relativistic Lorentz factor. The electric field is $\mathbf{E} = E\hat{e}_x$, where $\hat{e}_x$ is the unit vector along the x axis. The ambient magnetic field is $\mathbf{B} = B_0\hat{e_z}$, where $\hat{e_z}$ is the unit vector along the z axis. The

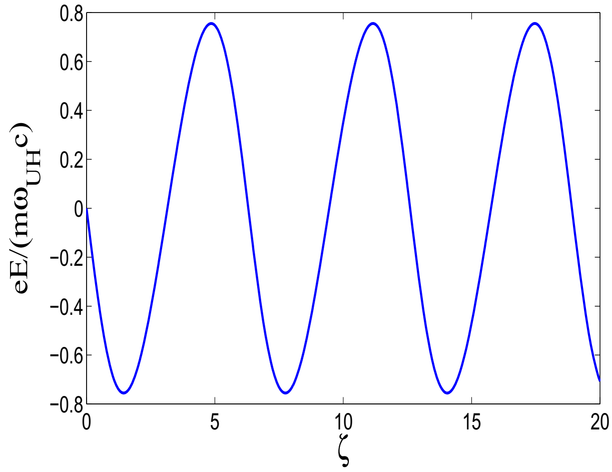

Figure 4.1: Variation of normalized electric field of RUHOs, with $\beta = 0.5$, $\gamma_{ph} = 10$, and $\alpha = 1.5$.

rest of the symbols have their usual meanings. The massive ions are assumed to be static and they constitute a charge neutralizing background with a constant density n_0.

In search for a travelling wave solution of Eqs. (4.1)-(4.4), it is convenient to introduce a variable transformation $\zeta = k_p(x - v_{ph}t)$, where $k_p = \omega_{uh}/v_{ph}$ with $\omega_{uh} = \sqrt{\omega_{pe}^2 + \Omega_e^2}$, $\omega_{pe} = \sqrt{4\pi n_0 e^2/m}$ and $\Omega_e = eB_0/mc$; v_{ph} is the phase velocity of the plane wave. In this coordinate system the transformed equations become

$$\left(1 - \frac{\beta_x}{\bar{v}_{ph}}\right)\frac{dp_x}{d\zeta} = E + \beta\beta_y, \tag{4.5}$$

$$\left(1 - \frac{\beta_x}{\bar{v}_{ph}}\right)\frac{dp_y}{d\zeta} = -\beta\beta_x, \tag{4.6}$$

$$(\bar{v}_{ph} - \beta_x)\frac{dn}{d\zeta} = n\frac{d\beta_x}{d\zeta}, \tag{4.7}$$

and

$$(\bar{v}_{ph} - \beta_x)\frac{dE}{d\zeta} = -\bar{v}_{ph}\bar{\omega}_p^2\beta_x, \tag{4.8}$$

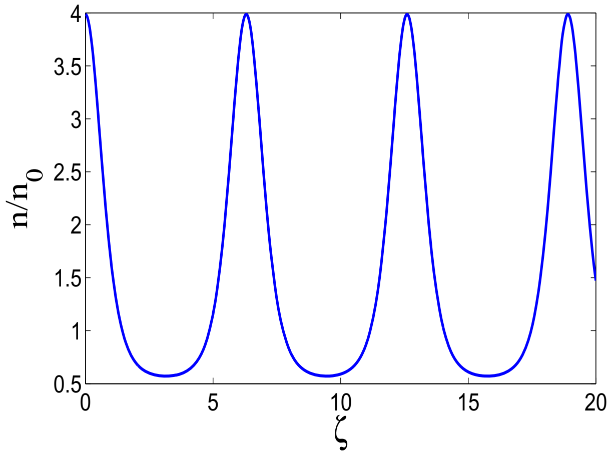

Figure 4.2: Variation of normalized electron density associated with RUHOs, with $\beta = 0.5$, $\gamma_{ph} = 10$, and $\alpha = 1.5$.

respectively, where $\beta_x = v_x/c$, $\beta_y = v_y/c$, $\bar{v}_{ph} = v_{ph}/c$, $\beta = \Omega_e/\omega_{uh}$, $\bar{\omega}_p = \omega_{pe}/\omega_{uh}$ with $\bar{\omega}_p^2 + \beta^2 = 1$. And, the normalized variables we have used are $E \to eE/(m\omega_{uh}c)$, $p_x \to p_x/mc$, $p_y \to p_y/mc$, and $n \to n/n_0$. Now the solutions for the x and y components of the normalized momenta are obtained as

$$p_x = \pm \left[\left(\alpha - \frac{E^2}{2\bar{\omega}_p^2} \right)^2 - 1 - \frac{\beta^2}{\bar{\omega}_p^4} E^2 \right]^{1/2}, \tag{4.9}$$

and

$$p_y = \frac{\beta}{\bar{\omega}_p^2} E, \tag{4.10}$$

respectively. Here $\alpha = \sqrt{1 + p^2} + E^2/(2\bar{\omega}_p^2)$ signifies the total energy of the system, where $p^2 = p_x^2 + p_y^2$. Now combining Eqs. (4.8) and (4.9) we obtain

$$\frac{dE}{d\zeta} = -\bar{\omega}_p^2 \frac{\sqrt{(E^2 - a)^2 - b^2}}{(2\alpha\bar{\omega}_p{}^2 - E^2)}, \tag{4.11}$$

where $a = 2(\alpha\bar{\omega}_p{}^2 + \beta^2)$ and $b = 2\sqrt{1 - 2\bar{\omega}_p{}^2\beta^2(1 - \alpha)}$.

Next, we introduce two new variables θ and r through the following definitions: $E^2 = (a - b)\sin^2\theta$ and $r^2 = (a - b)/(a + b)$. After some simple algebra, we obtain

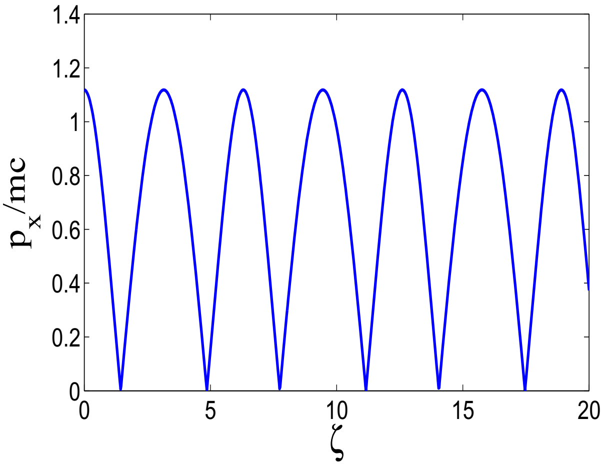

Figure 4.3: Variation of normalized x-component of relativistic electron momentum, with $\beta = 0.5$, $\gamma_{ph} = 10$, and $\alpha = 1.5$.

the following relation for the travelling wave solution:

$$-\bar{\omega}_p^2 \zeta = A(r)F(r,\theta) + B(r)G(r,\theta) + C(r)\sin\theta, \tag{4.12}$$

where

$$A(r) = \sqrt{\frac{2}{b(1-r^2)}} \left\{ \sqrt{\bar{\omega}_p^4 (1-r^2)^2 + r^2 b^2} - b \right\},$$
$$B(r) = \sqrt{\frac{2b}{1-r^2}}, \quad C(r) = -\left(\frac{r}{\bar{v}_{ph}}\right) \sqrt{\frac{2b}{1-r^2}}, \tag{4.13}$$

and $F(r,\theta)$ and $G(r,\theta)$ are the incomplete elliptical integrals of the first and second kind, respectively.

For typical parameter values $\beta = 0.5$, $\gamma_{ph} = 10$ [$\gamma_{ph} = (1 - \bar{v}_{ph}^2)^{-1/2}$ signifying the Lorentz factor associated with the phase velocity of RUHOs], and $\alpha = 1.5$, the variations of E, n, p_x, and p_y as a function of ζ are shown in Fig. (4.1), Fig. (4.2), Fig. (4.3), and Fig. (4.4), respectively.

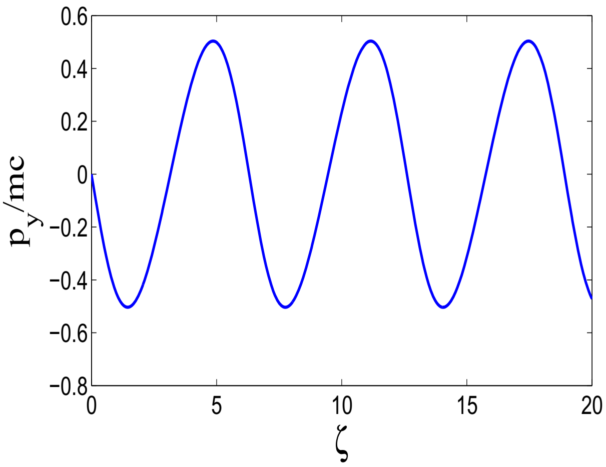

Figure 4.4: Variation of normalized y-component of relativistic electron momentum, with $\beta = 0.5$, $\gamma_{ph} = 10$, and $\alpha = 1.5$.

4.3 Determination of breaking field amplitude

We now proceed to find the wave-breaking amplitudes of RUHOs. It is well known that, at the wave-breaking, both the electron density and gradient of electric field become infinite.[18, 21] Thus the maximum allowable electric field amplitude - so called - the 'wave-breaking amplitude' of RUHOs can be found out if one sets the denominator of the electron density expression to zero. Now the expression of electron density in terms of the x-component of electron velocity can be obtained from Eq. (4.7) as

$$n = \frac{\bar{v}_{ph}}{\bar{v}_{ph} - \beta_x}, \tag{4.14}$$

from which we obtain the wave-breaking amplitudes of RUHOs, E_{wb}, as

$$\frac{eE_{wb}}{m\omega_{pe}c} = \frac{1}{\sqrt{1-\beta^2}} \Big[2\{\alpha(1-\beta^2) + \beta^2\gamma_{ph}^2\} \pm 2\left\{\beta^4\gamma_{ph}^4 + 2\alpha\beta^2\gamma_{ph}^2(1-\beta^2) + \gamma_{ph}^2(1-\beta^2)^2\right\}^{1/2}\Big]^{1/2}. \tag{4.15}$$

Now we observe that in absence of magnetic field ($\beta = 0$) the above expression of

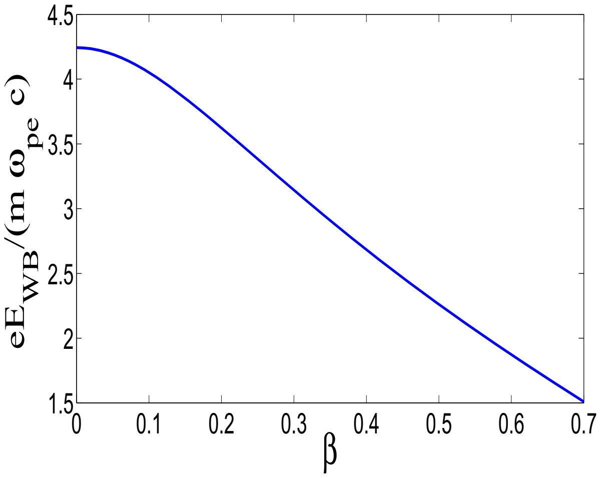

Figure 4.5: Variation of normalized wave-breaking field with magnetic field ($\beta = \Omega_e/\omega_{uh}$) of RUHOs, with $\gamma_{ph} = 10$.

E_{wb} becomes

$$\frac{eE_{wb}}{m\omega_{pe}c} = \sqrt{2(\alpha \pm \gamma_{ph})}, \tag{4.16}$$

signifying the wave-breaking limit of relativistic electron plasma oscillations (REPOs) - so called - the 'Akhiezer-Polovin (AP) limit'. The well-known 'AP limit'[49] $(eE_{wb}^{app}/m\omega_{pe}c) = \sqrt{2(\gamma_{ph} - 1)}$ can be recovered by taking the negative sign as well as by setting $\alpha = 2\gamma_{ph} - 1$ in the expression of Eq. (4.16). If we would've taken positive sign then α would take unphysical value($\alpha = -1$ is absurd). Taking negative sign in the expression of wave breaking field [Eq. (4.15)] and $\alpha = 2\gamma_{ph} - 1$ the variation of the wave-breaking electric field with the ambient magnetic field is shown in Fig. (4.5). It is evident that, for a fixed value of γ_{ph}, the wave-breaking electric field gradually decreases with the increase of the strength of an ambient magnetic field. This fact could've been anticipated if we would've looked at the expression of α, where applying an external magnetic field in the plasma system introduces an extra degree of freedom. At the point of wave breaking the x-component of the fluid velocity matches with the fixed phase velocity of the plasma wave in case of magnetized as well as for the unmagnetized plasma system. But due to the extra degree of freedom in the magnetized case (y-component of fluid velocity) the wave breaking field amplitude decreases from its unmagnetized value in order to conserve the total energy of the system. That's the reason why the wave-breaking amplitudes of REPOs always remain higher in comparison with those of RUHOs.[46]

4.4 Relation between Lagrangian and travelling wave solution

We now construct travelling wave solution from the exact non-stationary solution of RUHOs presented in Ref. [21]. The dynamics (an explicit time dependence) of RUHOs can be readily obtained as[46]

$$\bar{\omega}_p^2 \omega_{uh} \tau = \{A(r)F(r,\theta) + B(r)G(r,\theta)\} + \Phi(\xi), \tag{4.17}$$

where $\Phi(\xi)$ is an integration constant, and (ξ, τ) are Lagrangian variables:[18, 21, 51, 72, 95] $\xi = x - \int_0^\tau v_x(\xi, \tau')d\tau'$, $\tau = t$. Now we subtract $(\bar{\omega}_p^2 \omega_{uh}/\bar{v}_{ph} c)x$ from both sides of Eq. (4.17) and replace τ by t to get

$$\begin{aligned} & -\bar{\omega}_p^2 \left\{ -\omega_{uh} t + \left(\frac{\omega_{uh}}{\bar{v}_{ph} c} \right) x \right\} \\ &= \{A(r)F(r,\theta) + B(r)G(r,\theta)\} + \Phi(\xi) - \left(\frac{\bar{\omega}_p^2 \omega_{uh}}{\bar{v}_{ph} c} \right) x. \end{aligned} \tag{4.18}$$

Noting that $\zeta = -\omega_{uh} t + (\omega_{uh}/\bar{v}_{ph} c)x$ and using the following coordinate transformation relation[46]

$$x = \xi + \frac{cr}{\omega_{uh} \bar{\omega}_p^2} \sqrt{\frac{2b}{1 - r^2}} \sin\theta, \tag{4.19}$$

we can re-write Eq. (4.18) as

$$-\bar{\omega}_p^2\zeta = A(r)F(r,\theta) + B(r)G(r,\theta) + C(r)\sin\theta + \left\{\Phi(\xi) - \left(\frac{\bar{\omega}_p^2\omega_{uh}}{\bar{v}_{ph}c}\right)\xi\right\}. \tag{4.20}$$

Comparing Eq. (4.20) with the travelling wave solution of RUHOs [Eq. (4.12)], we find the following condition to freeze the exact space-time dependent solution into the travelling wave solution:

$$\Phi(\xi) = \left(\frac{\bar{\omega}_p^2\omega_{uh}}{\bar{v}_{ph}c}\right)\xi. \tag{4.21}$$

We notice that in absence of magnetic field ($\beta = 0$ and thus $\bar{\omega}_p = 1$) the above initial condition (4.21) becomes

$$\Phi(\xi) = \frac{\omega_{pe}\xi}{\bar{v}_{ph}c}, \tag{4.22}$$

This is the same initial condition that has been obtained by Verma *et al.* in the analytical investigation of relativistic electron plasma oscillations.[68] The knowledge of such initial condition is very important to study the phase mixing process discussed in the up-coming chapters. P.S. Verma *et al.* have clearly demonstrated this using a one-dimensional simulation based on the Dawson sheet model.[93] It has been shown that a slight deviation from this initial condition can cause AP longitudinal waves to break through the process of phase mixing at an amplitude well below the breaking amplitude for AP waves. We conjecture that similar kind of numerical solution can be easily performed for the magnetized case also.

4.5 Summary

In summary, a travelling wave solution is presented for the relativistic cold plasma waves in an ambient magnetic field. The stationary solutions for the different fluid-field quantities associated with relativistic upper-hybrid oscillations (RUHOs) viz., the electron density, relativistic electron momenta, and wave electric field are obtained. The travelling wave analysis provides an expression for the wave-breaking amplitude of RUHOs which, for a fixed phase velocity, is found to decrease with the increase of the strength of an ambient magnetic field. Consequently, one may conclude that, for a fixed phase velocity, wave-breaking amplitudes of RUHOs always remain smaller in comparison with those of relativistic electron plasma oscillations (REPOs). Another interesting aspect is that, from the exact space-time dependent solution of RUHOs, we have constructed the travelling wave solution by making a special choice of initial conditions. These results are of particular interest in order to study phase-mixing effects of RHUOs by perturbing the system on its initial stationary structures. Furthermore, the results of our investigation could contribute to the understanding of the plasma-based particle energization schemes in which magnetic fields play a central role.

Chapter 5

Phase-mixing of large amplitude electron plasma oscillations in presence of ion inhomogeneity

Phase-mixing of large amplitude non-relativistic electron oscillations around an inhomogeneous background of massive ions has been studied in a cold plasma. For our purposes, a space periodic but time independent ion density profile along with a perturbation in the electron density are considered. An exact space-time dependent solution is presented in parametric form by using Lagrangian coordinates. An inhomogeneity in the ion density causes the characteristic plasma frequency to acquire spatial dependency, leading to phase-mixing and thus breaking of excited oscillations at arbitrary amplitudes. The effects of finite amplitude electron density perturbation on the process of phase-mixing have also been discussed.

5.1 Introduction

During the last few decades, large number of theoretical and experimental investigation have been performed to explore the physics of wave breaking in various physical situations.[3, 4, 8, 16, 18, 21, 30, 46, 48, 49, 51–53, 55, 67, 82, 85, 86, 90, 93, 96, 97] But those analysis were mostly done with the assumption of homogeneous stationary ion background. However, if inhomogeneity in the ion density is considered, the characteristic frequency of the plasma wave becomes space dependent. And when it happens, the phase mixing occurs due to the self-intersection of electron trajectories, leading to breaking of the wave.

The phase-mixing/wave-breaking phenomena of nonlinear electron oscillations around a time stationary but inhomogeneous background of massive ions have been studied by several authors.[22, 70, 71] In 1989, Infeld *et al.* have investigated the phase-mixing process in presence of sinusoidal time stationary ion density inhomogeneity in the nonrelativistic plasma system.[22] Later, similar analysis has been done for the nonlinear Langmuir oscillation against single-ion pulse or cavity background.[70] In the present paper, a non-relativistic analysis of electron plasma oscillations in presence of a time stationary but space periodic ion density profile has been performed. Here, the main difference from most of the earlier analyses is that, in presence of ion inhomogeneity, instead of treating a uniform initial electron density we have considered a finite amplitude electron perturbation. Such type of initial condition has been adopted by Kaw *et al.* to investigate the mode-coupling effect of electron plasma waves.[98] Also, such initial conditions that we are considering here may occur in experiments involving beam or laser induced plasma oscillations in presence of pre-existing ion waves in the plasma background. In recent years, studies on phase-mixing and nonlinear dynamics of plasma oscillations/waves are being extensively reported by several authors.[94, 97, 99–105]

By transforming into Lagrangian coordinates, we have obtained an exact non-stationary solution of the problem. Nevertheless, an exact calculation of spatio-temporal evolution of the nonrelativistic electron plasma waves in the inhomogeneous plasma background does not result in the physical understanding of the novel phase-mixing phenomenon. In order to expose the underlying physics, one may need to do a perturbative calculation or to adopt a reasonable approximation, which gives an explicit expression for the time scale in which the wave phase-mixes and eventually breaks (phase-mixing time). Keeping this in mind, we have performed a homotopy perturbation analysis which explicitly gives us the dependence of phase-mixing time on the initial conditions; a result which is simply difficult to obtain from the exact result.

5.2 Basic Equations and Linear Analysis

The space-time evolution of large amplitude electron oscillations around an inhomogeneous background of massive ions in a cold collisionless plasma can be described fairly by the following 1-D electron fluid-Maxwell's equations:

$$\frac{\partial n_e}{\partial t} + \frac{\partial n_e v_e}{\partial x} = 0, \tag{5.1}$$

$$\frac{\partial v_e}{\partial t} + v_e \frac{\partial v_e}{\partial x} = -eE/m_e, \tag{5.2}$$

$$\frac{\partial E}{\partial x} = 4\pi e(n_i - n_e), \tag{5.3}$$

where all the symbols have their usual meanings. In writing the above equations, we have treated the electron fluid non-relativistically. We further assume that, in the equilibrium plasma state, a pre-existing ion wave induces an inhomogeneity in the background ion density. For the sake of simplicity, we consider the ion density profile to be time independent but spatially periodic

$$n_i(x,t) = n_0(1+\delta_i \cos k_i x), \tag{5.4}$$

where δ_i and k_i, respectively, denote the amplitude and wavenumber of the ion inhomogeneity. Moreover, we take the perturbation in the electron density as

$$n_e(x,0) = n_0(1+\delta_e \cos k_e x), \tag{5.5}$$

where δ_e and k_e, respectively, signify the strength and inverse of scale length of the perturbation. Thus, an initial charge imbalance between electron and ion creates an electric field which can be obtained from Eq. (5.3) as

$$E(x,0) = 4\pi e n_0 \left(\delta_i {k_i}^{-1} \sin k_i x - \delta_e {k_e}^{-1} \sin k_e x\right). \tag{5.6}$$

This initial electric field drives the plasma system in a nonlinear electron plasma mode. Before performing a nonlinear analysis, we first linearize Eqs. (5.1)-(5.3) in order to extract the essential features of our problem. Following a standard procedure for the linearization, we find the perturbed electric field $\tilde{E}$ to obey the equation

$$\frac{\partial^2 \tilde{E}}{\partial t^2} + \omega_p^2 \left(1+\delta_i \cos k_i x\right) \tilde{E} = 0, \tag{5.7}$$

where $\omega_p = \sqrt{4\pi n_0 e^2/m_e}$ is the electron plasma frequency. With the initial condition given in Eq. (5.6) and keeping in mind that $\dot{\tilde{E}}(x,0) = 0$, a solution of the above equation is given by

$$\tilde{E}(x,t) = 4\pi e n_0 \left(\delta_i {k_i}^{-1} \sin k_i x - \delta_e {k_e}^{-1} \sin k_e x\right) \cos\omega t, \tag{5.8}$$

where $\omega = \omega_p\sqrt{1+\delta_i \cos k_i x}$, signifying that the characteristic frequency of the oscillation acquires a spatial dependency due to the ion density inhomogeneity. This in turn indicates an onset of phase-mixing in the excited oscillations, which leads to the breaking of the oscillations at arbitrary amplitudes. Moreover, the mode-coupling effect can also be explicitly seen if we express the solution given in Eq. (5.8) in terms of Bessel function of first kind J_l as

$$\begin{aligned}\tilde{E}(x,t) = \frac{4\pi e n_0 \delta_i}{2k_i} \sum_{-\infty}^{\infty} J_l\left(\frac{\delta_i \omega_p t}{2}\right) &[\cos\varphi\{\sin(l+1)k_i x - \sin(l-1)k_i x\} \\ - \sin\varphi\{\cos(l+1)k_i x - \cos(l-1)k_i x\} - \mu\cos\varphi\{&\sin(k_e + lk_i)x + \sin(k_e - lk_i)x\} \\ &+\mu\sin\varphi\{\cos(k_e+lk_i)x - \cos(k_e - lk_i)x\}], \end{aligned} \tag{5.9}$$

where $\varphi = \omega_p t + (l\pi/2)$ and $\mu = (\delta_e/\delta_i)(k_i/k_e)$. Thus we see that an ion inhomogeneity essentially causes the mode-coupling where initial energy given to primary "k_e" flows irreversibly towards higher coupled modes "$k_e \pm lk_i$" as time goes on.[98] And, the time when the amplitude of the primary mode drops appreciably from its initial value is approximately given by $\omega_p t \sim 2/\delta_i$, which commonly refers to the phase-mixing time. This suggests that an increase in the strength of ion inhomogeneity results in a decrease in the phase-mixing time. Moreover, in the limit $\delta_i \to 0$, the phase-mixing time becomes infinite. Physically, in such a situation, wave coherency can be maintained indefinitely, provided the value of electron perturbation amplitude is kept below its critical value.[3]

5.3 Nonlinear analysis: exact space-time dependent Lagrangian solution

Next, to find an exact space-time dependent solution of the problem, we introduce Lagrangian coordinates (ξ, τ) through an auxiliary variable ψ:[51, 72, 73, 94–96, 106, 107] $\xi = x - \psi(\xi, \tau)$, $\psi = \int_0^\tau v_e(\xi, \tau) d\tau$, $\tau = t$. Therefore, the electron continuity equation (5.1) simplifies to give $n_e(\xi, \tau) = n_e(\xi, 0)\left(1 + \partial\psi/\partial\xi\right)^{-1}$. Moreover, expressing Eqs. (5.2) and (5.3) in terms of Lagrangian variables followed by combining them we find a single evolution equation for ψ as

$$\frac{\partial^3 \psi}{\partial \tau^3} + \omega_p^2 \left[1 + \delta_i \cos k_i(\xi + \psi)\right] \partial_\tau \psi = 0. \tag{5.10}$$

With the prescribed initial conditions, two successive integrations of Eq. (5.10) yield

$$\frac{\partial^2 \phi}{\partial \bar{\tau}^2} + \left[\phi + \delta_i \kappa^{-1} \sin \kappa(\bar{x} + \phi) - \delta_e \sin \bar{x}\right] = 0, \tag{5.11}$$

and

$$\left(\frac{\partial \phi}{\partial \bar{\tau}}\right)^2 = -\phi^2 + 2\delta_i \kappa^{-2}[\cos \kappa(\bar{x} + \phi) - \cos \kappa \bar{x}] + 2\delta_e \phi \sin \bar{x}, \tag{5.12}$$

respectively. In the above two equations, we have introduced the variables: $\phi = k_e \psi$, $\bar{x} = k_e \xi$, $\bar{\tau} = \omega_p \tau$. And the parameter $\kappa = k_i/k_e$ denotes the wave number ratio of ion inhomogeneity to electron perturbation. Thus an exact analytical solution in simple parametric form $t = t(\bar{x}, \phi)$, $x = x(\bar{x}, \phi)$, and $n_e = n_e(\bar{x}, \phi)$ can immediately be given as

$$\omega_p t = \int_0^{\phi} \frac{d\phi}{\left[2\delta_e\phi\sin\bar{x} - \phi^2 + \frac{2\delta_i}{\kappa^2}\{\cos\kappa(\bar{x}+\phi) - \cos\kappa\bar{x}\}\right]^{1/2}},$$
$$k_e x = \bar{x} - \phi, \text{ and } n_e(\bar{x},\phi) = n_0(1+\delta_e\cos\bar{x})\left(1+\frac{\partial\phi}{\partial\bar{x}}\right)^{-1}, \tag{5.13}$$

where

$$\frac{\partial\phi}{\partial\bar{x}} = \left[2\delta_e\phi\sin\bar{x} - \phi^2 + \frac{2\delta_i}{\kappa^2}\{\cos\kappa(\bar{x}+\phi) - \cos\kappa\bar{x}\}\right]^{1/2} \times$$
$$\int_0^{\phi} d\phi\ \left[-\delta_e\phi\cos\bar{x} - \frac{\delta_i}{k}\{\sin\kappa\bar{x} - \sin\kappa(\bar{x}+\phi)\}\right]$$
$$\left[2\delta_e\sin\bar{x} - \phi^2 + \frac{2\delta_i}{\kappa^2}\{\cos\kappa(\bar{x}+\phi) - \cos\kappa\bar{x}\}\right]^{-3/2}.$$

5.4 Approximate nonlinear solution: Homotopy Perturbation Method

Admittedly, it is difficult to analyse the exact results. In particular, we note that an estimation of the phase-mixing time is turned out to be analytically difficult from the exact parametric solutions. To make the nonlinear results more transparent, we now adopt a reasonable approximation that the scale length of the ion inhomogeneity is much larger than the electron perturbation scale length, i.e., $\kappa \equiv k_i/k_e \ll 1$. For our purposes, we start analyzing the Eq. (5.11). In this situation, $\sin\kappa(\bar{x}+\phi) \approx \sin\kappa\bar{x}\,[1-(\kappa^2\phi^2)/2] + \kappa\phi\cos\kappa\bar{x}$, and so the equation (5.11) becomes

$$\frac{\partial^2\phi}{\partial\bar{\tau}^2} + f_1\phi - f_2\phi^2 + f_3 \approx 0, \tag{5.14}$$

where $f_1 = 1+\delta_i\cos\kappa\bar{x},\ f_2 = (\delta_i\kappa/2)\sin\kappa\bar{x},\ f_3 = (\delta_i/\kappa)\sin\kappa\bar{x} - \delta_e\sin\bar{x}$. We now proceed to solve Eq. (5.14) by employing the homotopy perturbation method.[66]

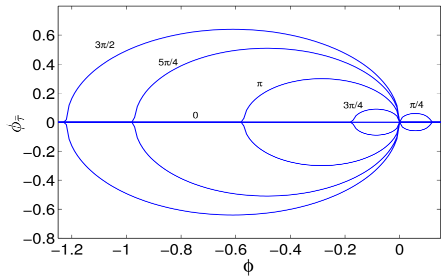

Figure 5.1: Phase-space curves from exact solution for different values of $\bar{x}$. Here $\delta_e = 0.2$, $\delta_i = 0.1$, and $\kappa = 0.1$.

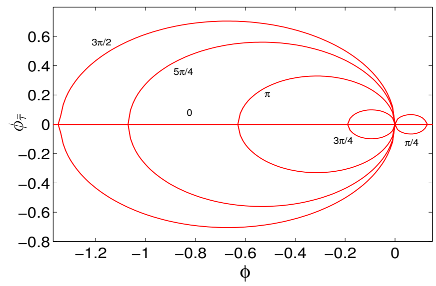

Figure 5.2: Phase-space curves from homotopy perturbation solution for different values of $\bar{x}$. Here $\delta_e = 0.2$, $\delta_i = 0.1$, and $\kappa = 0.1$.

The homotopy of Eq. (5.14) can be constructed by introducing a small parameter $p \in [0, 1]$ as

$$\frac{\partial^2 \phi}{\partial \bar{\tau}^2} + \alpha^2 f_1 \phi + p[(1-\alpha^2) f_1 \phi - f_2 \phi^2 + f_3] = 0. \tag{5.15}$$

Notice that, when $p = 0$, it takes the linearized form, although the value of a parameter α introduced here is to be found out. And if $p = 1$, it transformed into the original equation (5.14). Looking for the periodic solution, we expand ϕ in powers of small parameter p, i.e., $\phi = \sum_{i=0}^{\infty} p^i \phi_i$. Substituting ϕ into Eq. (5.15), we collect various powers of p:

$$\begin{aligned}
p^0 &: \frac{\partial^2\phi_0}{\partial\bar{\tau}^2} + \alpha^2 f_1\phi_0 = 0, \\
p^1 &: \frac{\partial^2\phi_1}{\partial\bar{\tau}^2} + \alpha^2 f_1\phi_1 + (1-\alpha^2)f_1\phi_0 - f_2\phi_0^2 + f_3 = 0, \\
p^2 &: \frac{\partial^2\phi_2}{\partial\bar{\tau}^2} + \alpha^2 f_1\phi_2 - 2f_2\phi_0\phi_1 + (1-\alpha^2)f_1\phi_1 = 0, \\
\ldots &
\end{aligned} \tag{5.16}$$

With the initial conditions $\phi(\bar{x},0) = \dot{\phi}(\bar{x},0) = 0$, we obtain the following solutions for ϕ_0 and ϕ_1

$$\phi_0 = 0, \text{ and } \phi_1 = \frac{f_3}{f_1\alpha^2}\left[\cos(\alpha\sqrt{f_1}\,\bar{\tau}) - 1\right], \tag{5.17}$$

respectively.

Inserting the solutions for ϕ_0 and ϕ_1 in the evolution equation of ϕ_2 we obtain

$$\frac{\partial^2\phi_2}{\partial\bar{\tau}^2} + \alpha^2 f_1\phi_2 + (\alpha^{-2} - 1)f_3\left[\cos(\alpha\sqrt{f_1}\,\bar{\tau}) - 1\right] = 0. \tag{5.18}$$

To remove secular term in ϕ_2, we require vanishing co-efficient of $\cos(\alpha\sqrt{f_1}\,\bar{\tau})$ which makes $\alpha = 1$. Thus the normalized characteristic frequency of oscillation, $\bar{\omega}$ (normalized by ω_p), turns out to be

$$\bar{\omega} \equiv \alpha\sqrt{f_1} = \sqrt{1 + \delta_i\cos\kappa\bar{x}}, \tag{5.19}$$

which clearly shows a space dependency of the frequency, indicating phase-mixing of the oscillations. This is certainly caused due to an inhomogeneity in the background ion concentration. Physically, the position dependent frequency causes the different electrons situated at different locations in space to oscillate at different frequencies which leads to (complete) destruction of the wave coherency in a finite time.

Now the solution can be obtained up to the first order (sufficient in the present context) as

$$\phi(\bar{x},\bar{\tau}) = \lim_{p\to 1}[\phi_0 + p\,\phi_1] = \frac{f_3}{\bar{\omega}^2}\,(\cos\bar{\omega}\bar{\tau} - 1)\,. \tag{5.20}$$

For small δ_i, we can safely set $\bar{\omega} \approx 1 + (\delta_i/2)\cos\kappa\bar{x}$. Then an expression for the electron density becomes

$$n_e(\bar{x},\bar{\tau}) = n_0(1 + \delta_e\cos\bar{x})/D, \tag{5.21}$$

where $D = 1 + f_4(\cos\bar{\omega}\bar{\tau} - 1) + \delta_i\kappa f_3\tau\sin\kappa\bar{x}\sin\bar{\omega}\bar{\tau}$. Here $f_4 = (\delta_i\cos\kappa\bar{x} - \delta_e\cos\bar{x})$. And, the expression for the electric field becomes

$$\bar{E} = f_3(\cos\bar{\omega}\bar{\tau} - 1) + \frac{\delta_i}{\kappa}\sin\kappa[\bar{x} + f_3(\cos\bar{\omega}\bar{\tau} - 1)], \tag{5.22}$$

where $\bar{E} \equiv (k_e E)/(4\pi e n_0)$. Clearly, in the limit of $\delta_e = 0$, these expressions for density and electric field exactly match with the solution obtained by Nappi *et al.*[71] Finally, the co-ordinate transformation relation can be expressed as

$$k_e x = \bar{x} + f_3(\cos\bar{\omega}\bar{\tau} - 1). \tag{5.23}$$

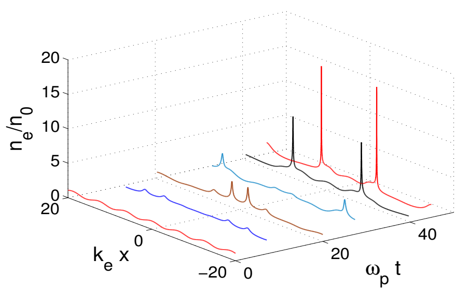

Figure 5.3: Space-time evolution of normalized electron density n_e/n_0 with $\delta_e = 0.2$, $\delta_i = 0.2$, and $\kappa = 0.1$. The phase-mixing/wave-breaking time $\omega_p t_{\rm mix} \sim 45$.

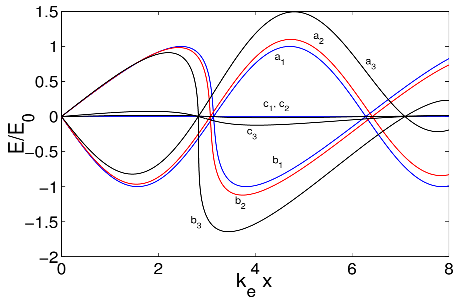

Figure 5.4: Normalized electric field profiles as a function of $k_e x$ ($E_0 = 0.45 m_e \omega_p^2/k_e e$) for $\delta_e = 0.45$ and $\kappa = 0.1$. Here a_1, a_2, and a_3 correspond to $\delta_i = 0, 0.01$, and 0.05, respectively, at $\omega_p t = \pi$. And, b_1, b_2, and b_3 correspond to $\delta_i = 0, 0.01$, and 0.05, respectively, at $t = 0, 2\pi$ along with c_1, c_2, and c_3 is at $t = \pi/2, 3\pi/2$ for $\delta_i = 0, 0.01$, and 0.05, respectively.

We now show that the approximate solutions given in Eqs. (5.20)-(5.23) can be used with confidence for small values of κ. We first construct an equation for ϕ that would yield phase space curves from the approximate solution for ϕ given in Eq. (5.20). The approximate equation for phase space curves is obtained as

$$\left(\frac{\partial \phi}{\partial \bar{\tau}}\right)^2 + 2\bar{\omega}^2 \left(\delta_i \kappa^{-1} \sin \kappa \bar{x} - \delta_e \sin \bar{x}\right) \phi + \bar{\omega}^2 \phi^2 = 0. \tag{5.24}$$

The phase space curves corresponding to Eq. (5.12) and Eq. (5.24) have been drawn in Figs. (5.1) and (5.2) , respectively, by using the same parameter values $\delta_e = 0.2$, $\delta_i = 0.1$, and $\kappa = 0.1$. From these figures, it is evident that Eq. (5.24) is a quite good approximation to the exact Eq. (5.12). Physically, these phase portraits exhibit the periodic motion of individual fluid elements. As different fluid elements are characterized by different values of $\bar{x}$, in the present context, the time periods of their oscillations depend on $\bar{x}$. This is in contrast to the case of homogeneous plasma system where the period of oscillation is independent of $\bar{x}$. Due to this space dependency, as time goes on, neighboring fluid elements start to cross their trajectories which indicates an onset of fine scale mixing of oscillations, leading to the breaking of excited oscillations at a finite time. We further stress here that, in absence of ion inhomogeneity ($\delta_i = 0$), the above approximate solutions presented in Eqs. (5.20)-(5.23) exactly match with the results of nonlinear *coherent* electron oscillations in a cold homogeneous plasma as outlined in the Davidson's book.[3]

Next, using the approximate solution, the space-time evolution of the electron density is shown in Fig. (5.3). This clearly shows an appearance of sharp peaks in the electron density profile at a finite time, indicating breaking of the oscillations via phase-mixing phenomena. Moreover, by setting typical parameter values, the electric field profiles are shown in Fig. (5.4). The steepening of the electric field is evident from this figure.

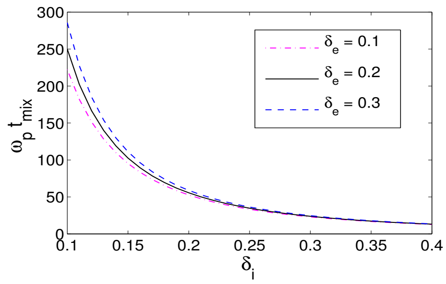

Figure 5.5: Variation of approximate phase-mixing time $\omega_p t_{\rm mix}$ with δ_i for different values of δ_e, with $\kappa = 0.1$.

Now the phase-mixing time can be estimated considering the point $\partial\phi/\partial\bar{x} = -1$ at which singularity in the electron density is observed. The expression for the approximate phase-mixing time is obtained as

$$\omega_p t_{\rm mix} \simeq \frac{2}{\delta_i(\delta_i - \kappa\delta_e)}. \tag{5.25}$$

Fig. (5.5) shows the variation of phase-mixing time $\omega_p t_{\rm mix}$ with the amplitude of ion inhomogeneity δ_i for different values of perturbation amplitude δ_e. Since the background ion density fluctuation makes the frequency space dependent, so increasing its amplitude accelerates the process of phase-mixing, and thereby phase-mixing time decreases with the increase of plasma inhomogeneity. Only for the particular situation when $\delta_i = 0$, the phase-mixing time becomes infinity and the wave coherency is preserved indefinitely as expected.

5.5 Conclusion

In conclusion, the results of our investigation show some interesting features which immensely replenish the earlier studies on nonlinear electron plasma oscillations. The perturbative solution of the problem has been obtained with the consideration of a physically realistic condition $k_i \ll k_e$. The assumption of large scale space variation in the background ion density is quite justified due to the fact that ions are less mobile than the electrons. Instead of treating an initial electron density profile to be uniform, we have given a perturbation to the electron density in the inhomogeneous ion background. Thus, we were able to see the modifications in the electron density and electric field profiles arising due to variations of electron perturbation amplitudes, ion inhomogeneity strengths, etc. Here, we have also estimated the phase-mixing time which is found to depend on the amplitudes of both ion density fluctuation and electron density perturbation as well as on the scale length ratio of their variations in space.

Chapter 6

Phase-mixing of relativistic electron plasma oscillations with background ion inhomogeneity

Combined effects of relativistic electron mass variation and background ion inhomogeneity on the phase-mixing process of large amplitude electron oscillations in cold plasmas have been analyzed by using Lagrangian coordinates. For the purposes, an inhomogeneity in the ion density is assumed to be time-independent but spatially periodic, and a periodic perturbation in the electron density is considered as well. An approximate space-time dependent solution is obtained in the weakly-relativistic limit by employing the Bogoliubov and Krylov method of averaging. It is shown that phase-mixing process of relativistically corrected electron oscillations is strongly influenced by the presence of pre-existing ion density ripple in the plasma background.

6.1 Introduction

As discussed in the previuos chapter, the wave breaking of electron plasma wave in inhomogeneous plasma is generally caused due to *phase mixing* process. This phase mixing is associated with the space dependence of the characteristic plasma frequency which can arise because of the existing inhomogeneity in the background ion density. Relativity also plays a significant role in the phase mixing process. In this context, the relativistic bursts solution of Infeld and Rowlands[18] is worth to be mentioned. They have provided an exact analytical solution for relativistic longitudinal electron plasma wave in terms of Lagrangian co-ordinates. It has been shown that because of the relativistic electron mass variation the frequency of the nonlinear plasma oscillation becomes both amplitude and space dependent. As a result, an explosive behavior in the excited plasma wave is generally observed and the wave breaks at arbitrarily small amplitude long before it reaches to the limit imposed by Akhiezher and Polovin.[49] In this chapter, we incorporates the effects of nonlinearities associated with both the relativistic mass variation of electron and an inhomogeneity in the background ion density. These two nonlinear effects acting together greatly modifies the physics of phase mixing.

6.2 The basic equations and nonlinear analysis

We start with the following equations describing the dynamics of large amplitude relativistic electron plasma oscillations:

$$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x}(n_e v_e) = 0, \tag{6.1}$$

$$\frac{\partial p_e}{\partial t} + v_e \frac{\partial p_e}{\partial x} = -\frac{eE}{m_e}, \tag{6.2}$$

$$\frac{\partial E}{\partial x} = 4\pi e(n_i - n_e), \tag{6.3}$$

where $p_e = \gamma m_e v_e$ denotes the relativistic electron momentum along the x direction, with $\gamma = (1 - v_e^2/c^2)^{-1/2}$ representing the relativistic Lorentz factor associated with nonlinear plasma wave. Here, the spatial variations are assumed to be one dimensional. The wave electric field is $\mathbf{E} = E\ \hat{e}_x$, where $\hat{e}_x$ is the unit vector along the x axis. And, the rest of the symbols have their usual meanings.

Instead of considering a homogeneous positive ion background, in our investigation, we assume that the plasma system initially has a time independent but space periodic ion density profile

$$n_i(x,t) = n_0(1 + \delta_i \cos k_i x); \tag{6.4}$$

where δ_i and k_i, respectively, denote the amplitude and inverse of scale length of the ion inhomogeneity. Along with this we consider an initial sinusoidal perturbation in the electron density profile

$$n_e(x,0) = n_0(1 + \delta_e \cos k_e x), \tag{6.5}$$

δ_e and k_e, respectively, signify the strength and inverse of scale length of the perturbation. This will give rise to an initial electric field profile obtained from Eq.(6.3) as

$$E(x,0) = 4\pi e n_0 \left(\delta_i {k_i}^{-1} \sin k_i x - \delta_e {k_e}^{-1} \sin k_e x\right). \tag{6.6}$$

We now switch from Eulerian to Lagrangian description in order to elucidate the dynamical evolution of the system starting from the physically realizable initial state as mentioned above. The Eulerian variables (x,t) are related to Lagrangian variables (ξ,τ) as:[3, 51, 72, 73, 95]

$\xi = x - \psi(\xi, \tau),\ \tau = t$, where $\psi = \int_0^\tau v_e(\xi, \tau)\, d\tau$. Thus, in the newly introduced co-ordinate system the transformed equations take the form as

$$\frac{\partial}{\partial \tau}\left[n_e\left(1 + \partial_\xi \psi\right)\right] = 0 \implies n_e = n_e(\xi, 0)(1 + \partial_\xi \psi)^{-1}, \tag{6.7}$$

$$\frac{\partial p_e}{\partial \tau} = -\frac{eE}{m_e}, \tag{6.8}$$

and

$$\frac{\partial E}{\partial \tau} = 4\pi e n_i v_e, \tag{6.9}$$

respectively.

With the specified initial electric field profile, we can reduce the above set of equations into a single nonlinear differential equation of the following form:

$$\frac{\ddot{\psi}}{\left(1 - \frac{\dot{\psi}^2}{c^2}\right)^{3/2}} + \omega_p^2 \left[\psi + \frac{\delta_i}{k_i} \sin k_i(\xi + \psi) - \frac{\delta_e}{k_e} \sin k_e \xi\right] = 0, \tag{6.10}$$

where 'dot' over ψ denotes partial differentiation w.r.t. τ and $\omega_p = \sqrt{4\pi n_0 e^2 / m_e}$ denotes the electron plasma frequency.

Since the massive ions are treated here as motionless, we assume $k_i \ll k_e$. In order to gain some insight into the relativistic effects on the excited longitudinal plasma wave in an inhomogeneous ion background, it is sufficient to retain terms up to first order in $k_i \psi$ in the following expansion: $\sin k_i(\xi + \psi) \approx \sin k_i \xi + \kappa(k_e \psi) \cos k_i \xi$ as $(k_i / k_e) \equiv \kappa \ll 1$. This further simplifies Eq. (6.10) to obtain

$$\begin{aligned} \ddot{\psi} + \omega_p^2 \Bigg[\psi &+ \frac{\delta_i}{k_i}\left\{\sin k_i \xi + \kappa(k_e \psi) \cos k_i \xi\right\} \\ &- \frac{\delta_e}{k_e} \sin k_e \xi\Bigg]\left(1 - \frac{3}{2}\frac{\dot{\psi}^2}{c^2}\right) \approx 0. \end{aligned} \tag{6.11}$$

Next, introducing normalized variables $k_e \psi = \phi$, $k_e \xi = \bar{x}$, $\omega_p \tau = \bar{\tau}$, we get

$$\frac{d^2 \phi}{d\bar{\tau}^2} + f_1 \phi - f_2 \phi \dot{\phi}^2 - f_3 \dot{\phi}^2 + f_4 \approx 0, \tag{6.12}$$

where $f_1 = 1 + \delta_i \cos\kappa\bar{x}$, $f_2 = 3\beta^2(1 + \delta_i \cos\kappa\bar{x})/2$, $f_3 = 3\beta^2(\delta_i\kappa^{-1}\sin\kappa\bar{x} - \delta_e \sin\bar{x})/2$, $f_4 = \delta_i\kappa^{-1}\sin\kappa\bar{x} - \delta_e\sin\bar{x}$, with $\beta = \omega_p/(k_e c)$. In the next section, we present an approximate solution of the Eq. (6.12) by employing the Bogoliubov-Krylov method of averaging.[108]

6.3 Nonlinear solution in the weakly relativistic limit: Bogoliubov-Krylov method

Before employing the perturbation method, we further simplify Eq. (6.12) by introducing a new variable $\chi = \phi + (f_4/f_1)$ to obtain

$$\frac{d^2\chi}{d\bar{\tau}^2} + f_1\chi - f_2\chi\left(\frac{\partial\chi}{\partial\bar{\tau}}\right)^2 = 0. \tag{6.13}$$

Now treating f_2 as a small parameter, a solution of the above equation by employing the method of Bogoliubov and Krylov[108] is obtained as

$$\chi(\bar{x},\bar{\tau}) = \chi_0(\bar{x})\sin[\bar{\omega}\bar{\tau} + \theta(\bar{x})], \tag{6.14}$$

where

$$\bar{\omega} = (1 + \delta_i\cos\kappa\bar{x})^{1/2}\left[1 - \frac{3\beta^2}{16}\frac{[(\delta_i/\kappa)\sin\kappa\bar{x} - \delta_e\sin\bar{x}]^2}{1 + \delta_i\cos\kappa\bar{x}}\right].$$

It is clearly seen from the above expression that the characteristic frequency of the nonlinear plasma mode acquires spatial dependency due to the relativistic variation of electron mass and the ion density inhomogeneity. As a result, different electrons situated at different locations in space oscillate with their local frequencies and they gradually go out of phase with each other in course of nonlinear oscillations. At a finite time, their trajectories cross, leading to loss of coherency of the excited oscillations (wave-breaking) at arbitrarily low amplitudes.

The full solution of the problem depends upon the two unknown functions $\chi_0(\bar{x})$ and $\theta(\bar{x})$ which can be determined by our prescribed initial condition, viz., $\phi(\bar{x},0) = \dot{\chi}(\bar{x},0) = 0$. This gives us $\theta(\bar{x}) = \pi/2$, and $\chi_0(\bar{x}) = f_4/f_1$. Thus we obtain the approximate solution of ϕ truncated to the second order in δ_i as

$$\phi(\bar{x},\bar{\tau}) = f_4(1-\delta_i\cos\kappa\bar{x})[\cos(\bar{\omega}\bar{\tau})-1]. \tag{6.15}$$

From the solution we calculate $\partial\phi/\partial\bar{x}$ and obtain an expression for electron density as

$$n_e(\bar{x},\bar{\tau}) = \frac{n_0[1+\delta_e\cos\kappa\bar{x}]}{1+A(\bar{x})(\cos\bar{\omega}\bar{\tau}-1)-B(\bar{x})\bar{\tau}\sin\bar{\omega}\bar{\tau}},$$

where,

$$A(\bar{x}) = (\delta_i\cos\kappa\bar{x}-\delta_e\cos\bar{x})(1-\delta_i\cos\kappa\bar{x})+\delta_i\kappa f_4\sin\kappa\bar{x}$$

and

$$B(\bar{x}) = -\frac{\delta_i\kappa f_4\sin\kappa\bar{x}}{2}(1-\delta_i\cos\kappa\bar{x})+\frac{\delta_i^2\kappa f_4}{8}\sin(2\kappa\bar{x}) \\ -\frac{3}{8}\beta^2 f_4^2(\delta_i\cos\kappa\bar{x}-\delta_e\cos\bar{x}).$$

In Fig. (6.1) we have shown time evolution of electron density for different values of relativistic parameter β, with $\delta_e = 0.3$, $\delta_i = 0.2$, $\kappa = 0.1$ and $\bar{x} = \pi$. An appearance of sharp peaks is observed, indicating breaking of the oscillations through the process of phase-mixing.

Next, from the solution of ϕ, we construct an equation that would yield the phase-space trajectories of different fluid elements as

$$\dot{\phi}^2+\bar{\omega}^2\phi^2+2\bar{\omega}^2 f_4(1-\delta_i\cos\kappa\bar{x})\phi \approx 0. \tag{6.16}$$

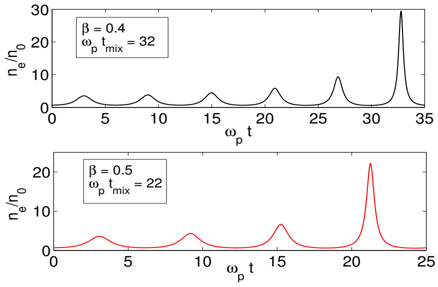

Figure 6.1: Electron density profiles. Here $\delta_e = 0.3$, $\delta_i = 0.2$, $\kappa = 0.1$ and $\bar{x} = \pi$.

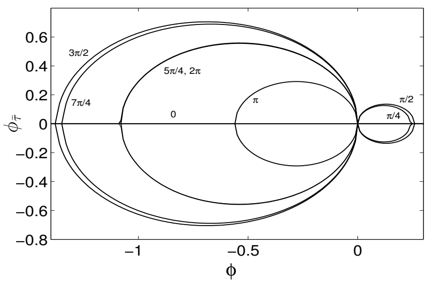

Figure 6.2: Phase-space curves for different values of $\kappa\bar{x}$. Here $\beta = 0.4$, $\delta_e = 0.3$, $\delta_i = 0.1$ and $\kappa = 0.1$.

Moreover, we simplify $\bar{\omega}$ and it is expressed as

$$\bar{\omega} \approx 1 + \frac{\delta_i}{2}\cos(\kappa\bar{x}) - \frac{\delta_i^2}{8}\cos^2(\kappa\bar{x}) - \frac{3}{8}\beta^2 f_4^2. \tag{6.17}$$

It is evident from the phase portrait Fig. (6.2) that individual electrons execute periodic motion, but the time period of oscillation depends on $\bar{x}$. This results in the crossing of their trajectories at a finite time.

The occurrence of secular term in the denominator of the electron density expression due to the space dependency of the frequency causes the density to blow up in a time scale roughly determined by the following expression of the phase mixing time

$$\omega_p t_{\rm mix} \simeq (\delta_i - \kappa\delta_e)^{-1}\left[\frac{3\delta_i^2}{8} - \frac{\delta_i}{2} - \frac{3\beta^2}{16\kappa^2}(\delta_i^2 - 2\delta_i\delta_e + \kappa\delta_e^2)\right]^{-1}. \tag{6.18}$$

The variation of phase mixing time with δ_i is shown in Fig. (6.3) for different values of β, with fixed $\delta_e = 0.3$ and $\kappa = 0.1$. From this figure it is evident that relativistically corrected electron oscillations phase mix away quickly in presence of background ion density inhomogeneity.

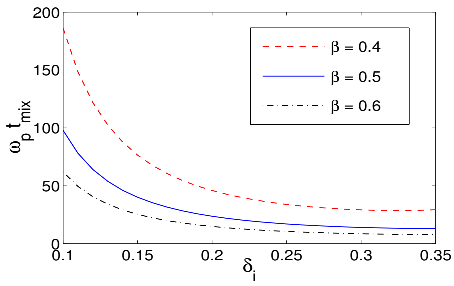

Figure 6.3: Phase-mixing time vs. δ_i for different values of β. Here $\delta_e = 0.3$ and $\kappa = 0.1$.

We now analyze our results in different limiting cases.

(i) Weakly-relativistic electron plasma oscillations in homogeneous plasmas with finite amplitude electron perturbations ($\beta \neq 0$, $\delta_i = 0$, $\delta_e \neq 0$):

In this case, the expressions for nonlinear frequency shifting and the phase-mixing time turn out to be

$$\omega \simeq \left(1 - \frac{3}{16}\beta^2\delta_e^2 \sin^2\bar{x}\right)\omega_p,$$

and

$$\omega_p t_{\rm mix} \simeq [(3/16)\beta^2\delta_e^3]^{-1},$$

respectively. These are the same expressions as obtained by Infeld *et. al.*[18] and Sengupta *et. al.*[19] Furthermore, in the non-relativistic limit, i.e., $\beta \to 0$, we obtain $\omega_p t_{\rm mix} \to \infty$, which physically represents that the phase coherence of neighboring electrons can be maintained indefinitely.[3, 51]

(ii) Non-relativistic electron plasma oscillations in presence of inhomogeneous ion background with finite amplitude electron perturbations ($\beta = 0$, $\delta_i \neq 0$, $\delta_e \neq 0$): In such a situation, keeping terms up to the second order in δ_i^2 in Eq. (6.18), we get $\omega_p t_{\rm mix} \simeq 2/[\delta_i(\delta_i - \kappa\delta_e)]$.[109] In addition, when we set $\delta_e = 0$, we recover the expression of the phase-mixing time as obtained by Nappi *et. al.*[71] $\omega_p t_{\rm mix} \simeq 2/\delta_i^2$.

6.4 Conclusion

In conclusion, within a fluid description, an analysis of relativistic electron plasma wave breaking has been carried out in presence of ion density inhomogeneity. Admittedly, the development of the fully relativistic theory of nonlinear electron plasma wave in presence of ion density inhomogeneity encounters with significant mathematical difficulties. For the sake of simplicity, we have treated background ion density profile to be spatially periodic but time-independent. In addition, we have considered a finite amplitude electron perturbation. In order to follow an explicit time evolution of the excited electron plasma modes, the Lagrangian coordinates are introduced, and analytic solutions are presented in the weakly-relativistic limit by employing the Bogoliubov and Krylov method of averaging. An expression for the characteristic frequency of the excited oscillations is derived in the weakly-relativistic limit. A rough estimate of the phase-mixing time is also provided. It is demonstrated that the nonlinear effects associated with relativistic electron mass variation and ion inhomogeneity acting together substantially modifies the process of phase mixing.

Chapter 7

Conclusion

In this chapter, we have summarized the results obtained in this thesis. Some future prospects of our investigation have also been stated in addition.

7.1 Summary

To summarize, throughout this thesis we have developed a detailed theoretical investigation to describe the charged particle beam driven wake field excitation process and wave breaking of electron plasma wave in magnetized as well as in unmagnetized plasma systems.

- In chapter II, we have discussed the effect of magnetic field on relativistic electron beam as well as proton beam driven plasma wake wave dynamics. Our theoretical solution reveals several interesting features which have significant usefulness in the determination of the energy gain in the charged particle beam driven acceleration process. The effect of magnetic field on the wake field structures with the consideration of its importance in minimization of phase slippage of accelerated electrons has been discussed. In addition, the wake field excitation mechanism by trains of proton micro-bunches has been demonstrated in our investigation.

- In chapter III, stationary wave solution is obtained to provide an analytical estimation of the wave-breaking amplitude of electrostatic waves in three component electron-positron-ion plasma. Consideration of relativistic dynamics of all the three species of the plasma system has been made. The effect of ion motion on the limiting electric field amplitude and also on the wavelength of the plasma wave have been discussed elaborately.

- In chapter IV, a travelling wave solution has been obtained in order to estimate the breaking field amplitude of relativistic upper hybrid wave (UHW). The wave breaking amplitude for such mode is observed to decrease with the increase in the strength of the external magnetic field. Also, we have constructed travelling wave solution from the exact space time dependent solution for UHW by choosing a particular initial condition.

- In Chapter V, the phase mixing process arising due to ion density inhomogeneity in the nonrelativistic cold plasma system has been studied. An exact solution along with an approximate perturbative solution have been presented. In contrast to the earlier studies, instead of treating a uniform initial electron density we have considered a finite amplitude electron perturbation in presence of ion density inhomogeneity. Thereby, it has been possible to see the modifications in the electron density and electric field profiles arising due to variations of electron perturbation amplitudes, ion inhomogeneity strengths, etc. These results can be of certain relevance in the basic studies of wave breaking phenomena and also in laboratory context.

- In chapter VI, within a fluid description, an investigation of relativistic electron plasma wave breaking has been carried out in presence of ion density inhomogeneity. An expression for the phase mixing time has been derived in the weakly-relativistic limit. It is observed that the nonlinear effects associated with relativistic electron mass variation and ion inhomogeneity acting together substantially modifies the process of phase mixing.

7.2 Future scope of the work

We would like to provide here a brief sketch of the future prospects of our investigation on wave breaking and plasma wake wave excitation process.

- In this thesis, the studies on wave breaking so far have been done in the cold plasma system. However, we can extend our analysis of wave breaking by considering thermal effects also.

- In the recent past several works have been performed to obtain the stationary wave solution for electron beam driven plasma wake field accelerator. However, there is no evidence of any theoretical works performed to investigate the exact space time evolution of electron beam driven non-linear plasma oscillations. In the near future we propose to give an exact spatio-temporal solution for the electron beam driven plasma by introducing Lagrangian variables.

  In the preliminary analysis, we observe that the exact parametric solution of the problem can be expressed in terms of incomplete Elliptic integrals. We will extend our analysis further to obtain the full nonlinear Lagrangian solutions of the electron beam driven wake wave generation.

- We are also planning to provide the exact space time dependent Lagrangian solution for the proton driven plasma wake field accelerator (PDPWFA). This investigation will certainly contribute to the theoretical knowledge of the ongoing experimental AWAKE [Advanced Proton Driven Plasma Wakefield Acceleration Experiment] project at CERN dealing with the description of the plasma properties of wake field accelerator.

- Lagrangian variables are very beneficial in order to obtain simple analytical solutions that lead to basic understanding of the wake wave dynamics and help with designing large scale simulations of the problem. However, in reality, a proper theoretical model of three dimensional systems encounters significant mathematical difficulties. Geometry also adds additional complexities. In order to investigate three dimensional problems we propose to apply plasma simulations based on the PIC (Particle in Cell) method.[110, 111] Such simulations may also provide additional support for the analytical results in simpler geometry. In this approach, the charge density is deposited in the mesh of a spatial grid. Then the self-consistent electric fields are calculated on the grid points. This simulation will provide us the scope of observing the time variation of the accelerating beam and the witness beam as well as the modifications of the wave profiles.

- In solar coronal and chromospheric plasmas injection of driving electrons produces wake fields. This is considered to be an important mechanism by which solar flare electrons are accelerated to extreme high energies.[112, 113] Thus our investigations can also be extended to such astrophysical situations.

Finally, we believe that our studies on plasma wave breaking and wake field excitation process, have certain relevance in advancing the research field of nonlinear relativistic wave dynamics and also hold promises to generate significant social and scientific impact in the quest for viable schemes for the production of high energy charged particle accelerators in the near future.

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