EXACTLY SOLVABLE DRIVEN INTERACTING PARTICLE SYSTEMS

by

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STATEMENT BY AUTHOR

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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List of Publications arising from the thesis

Peer reviewed journals

1. *Cluster-factorized steady states in finite-range processes*, **Amit Chatterjee**, P. Pradhan and P. K. Mohanty, *Phys. Rev. E*, **2015**, *92*, 032103.

2. *Zero range and finite range processes with asymmetric rate functions*, **Amit Kumar Chatterjee** and P. K. Mohanty, *J. Stat. Mech.*, **2017**, *2017*, 093201.

3. *Matrix product states for interacting particles without hardcore constraints*, **Amit Kumar Chatterjee** and P. K. Mohanty, *J. Phys. A: Math and Theor.*, **2017**, *50*, 495001.

4. *Negative differential mobility in interacting particle systems*, **Amit Kumar Chatterjee**, U. Basu and P. K. Mohanty, *Phys. Rev. E*, **2018**, *97*, 052137.

5. *Assisted exchange models in one dimension.* **Amit Kumar Chatterjee** and P. K. Mohanty, *Phys. Rev. E*, **2018**, *98*, 062134.

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Other Publications

a. Publications from works outside the thesis

1. *Phase coexistence and spatial correlations in reconstituting k-mer models*, **Amit Chatterjee**, B. Daga and P. K. Mohanty, *Phys Rev. E*, **2016**, *94*, 012121.

b. Book/Book Chapters: NIL

c. Conference/Symposium:

1. **Poster Presentation** at STATPHYS-KOLKATA VIII in December 2015, entitled with *Cluster-factorized steady states in finite-range processes.*

2. **Poster Presentation** at Statphys Kolkata IX in December 2016, entitled with *Phase coexistence and spatial correlations in reconstituting k-mer models.*

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Dedicated to Ma, Baba, Nupur and my supervisor Prof. P. K. Mohanty

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Synopsis

Introduction:

The stationary measure of equilibrium states is uniquely characterized by Gibbs measure or Boltzmann distribution. Given a system with a Hamiltonian, it is straightforward to write down the stationary probabilities of all the configurations in equilibrium and from there, one is always able, in principle, to compute the partition function and consequently the macroscopic observables of interest. But, in nature, there are numerous processes where there is a net flow of some observable through the system in some resultant direction, this net flow or *current* indicates the onset of *non-equilibrium* phenomena. To start with, the problem that intrigues us in context of non-equilibrium processes [1], is that, in sharp contrast to the equilibrium states, there is no Gibbs-Boltzmann like general formula to determine the non-equilibrium steady states. For each given dynamics one must solve the Master Equation under stationary conditions to obtain the steady state weights.

We address the question of how to obtain the steady states of a class of interacting particle systems [2] driven out-of-equilibrium [3]. In this connection, we introduce several non-equilibrium stochastic processes particularly in one dimension and solve the corresponding steady states exactly using techniques like matrix product ansatz and others. We further calculate analytically the partition function and observables like spatial correlation functions, particle current etc. Interesting features like current reversal and negative differential mobility of particles has been observed in these exactly solvable non-equilibrium models. Also, we discuss novel phenomena like condensation of particles in some of the stochastic processes introduced here.

Models and their steady states:

Finite Range Processes(FRP)

In comparison to the study of systems in equilibrium where the starting point is some Hamiltonian containing the interaction between different microscopic components of the systems, the non-equilibrium processes are often defined in terms of the explicit dynamics executed by the components of the model under consideration. For example, in this section, we consider a one dimensional periodic lattice with L sites labeled by $i = 1, 2, \ldots L$ where each site i can either be vacant or one or more particles can reside there. Clearly, the particles are free of hard core repulsion so that the occupation number n_i for a site i can take any non-negative integer value $0, 1, 2 \ldots N$ where N is the total number of particles. A particle from a randomly chosen site i can hop to its right neighbor $(i+1)$ with a rate $u(n_{i-K}, n_{i-K+1}, \ldots, n_i, \ldots n_{i+K-1}, n_{i+K})$ that depends not only on the occupancy of the departure and arrival sites but also on that of K number of right neighbors and K number of left neighbors of the departure site i. Here the interaction between the particles is manifested through the hop rates which expresses the fact that the motion of any particle depends on the presence of other particles within a range K. We name this stochastic process as " finite range process"(FRP) [4]. The dynamics conserves the density $\rho = \frac{N}{L}$. One can express the FRP dynamics as follows

$$(\ldots, n_{i-1}, n_i, n_{i+1}, \ldots) \longrightarrow (\ldots, n_{i-1}, n_i - 1, n_{i+1} + 1, \ldots)$$
$$\text{with rate} \quad u(n_{i-K}, \ldots, n_i, \ldots, n_{i+K}).$$

The unidirectional motion of the particles ensures the steady state of the system to be an out of equilibrium one. So, one may now ask what does the probability $P(\{n_i\})$ of any possible configuration $\{n_i\} \equiv (n_1, \ldots, n_i, \ldots, n_L)$ looks like in the steady state. The master equation describing the time evolution of the probabilities can be expressed as " $\frac{\mathrm{d}P(\{n_i\})}{\mathrm{dt}}$ = total influx or gain to $\{n_i\}$ − total outflux or loss from $\{n_i\}$." Clearly, the steady state is defined by $\frac{\mathrm{d}P(\{n_i\})}{\mathrm{dt}} =$ 0- this in turn means, in order to achieve the steady state, we have to find cancellation techniques or schemes that balance the total outflux with the total influx for any configuration. Before discussing briefly the specific technique we used to solve exactly the steady state of FRP, first we look for steady state solution of the following form

$$P(\{n_i\}) \propto \prod_{i=1}^{L} g(n_i, n_{i+1}, \ldots n_{i+K}) \delta(\sum_{i=1}^{L} n_i - N)$$

where the cluster weight function $g(.)$, with $(K+1)$ variables, is connected to the rate $u(.)$(having $(2K+1)$ variables) through the relation
$u(n_{i-K}, \ldots, n_i, \ldots n_{i+K}) = \prod_{j=0}^{K} \frac{g(\bar{n}_{i-K+j}, \bar{n}_{i-K+1+j}, \ldots \bar{n}_{i+j})}{g(n_{i-K+j}, n_{i-K+1+j}, \ldots n_{i+j})}$ with $\bar{n}_j = n_j - \delta_{ji}$.

Technique used to obtain the steady state:

For FRP, in order to cancel the total influx with the total outflux for each configuration $\{n_i\}$, we use *pairwise balance condition.* For any configuration $(\dots n_{i-1}, n_i, n_{i+1} \dots)$, it can come from a configuration $(\dots n_{i-1}+1, n_i-1, n_{i+1} \dots)$ and it can go to a configuration $(\dots n_{i-1}, n_i-1, n_{i+1}+1 \dots)$. These local fluxes cancel as

$$\begin{aligned} & u(\dots n_{i-1}+1, n_i-1, n_{i+1} \dots)\ P(\dots n_{i-1}+1, n_i-1, n_{i+1} \dots) \\ & \qquad = \ u(\dots n_{i-1}, n_i, n_{i+1} \dots)\ P(\dots n_{i-1}, n_i, n_{i+1} \dots). \end{aligned}$$

The above condition is the pairwise balance condition that, through local balancing of fluxes between a triple of configuration, ensures that the total influx and outflux for each configuration cancels out resulting in the steady state.

Finite range process with asymmetric rate functions(AFRP)

In this section, we introduce a generalization of the finite range process in a sense that instead of unidirectional motion, now the particles on a one dimensional periodic lattice can hop to both right and left with different rate functions $u_R(.)$ and $u_L(.)$ respectively i.e. depending on which direction the particle would move, the rates are described by different rate functions in general. A particle from a randomly chosen site i hops either to its right neighbor with a rate $u_R(n_{i-K}, \dots n_i, \dots n_{i+K})$ or to its left neighbor with rate $u_L(n_{i-K}, \dots n_i, \dots n_{i+K})$. The conditions under which the rate functions $u_R(.)$ and $u_L(.)$ give rise to cluster factorized steady states of the form

$$P(\{n_i\}) \propto \prod_{i=1}^{L} g(n_i, n_{i+1}, \dots n_{i+K}) \delta(\sum_{i=1}^{L} n_i - N),$$

are quite complicated and have been derived in [5]. Here we briefly discuss only a special case where $K = 0$ resulting in factorized steady states.

K=0: Asymmetric zero range process

For $K = 0$, a particle from a randomly chosen site i can hop to its right with rate $u_R(n_i)$ or to its left at rate $u_L(n_i)$- the particles interacts only within the departure site- the spatial range of interaction is zero confirming that we have a zero range process(ZRP) with asymmetric rate functions. The simple ZRP [6] has a factorized steady state(FSS) $P(\{n_i\}) \propto \prod_{i=1}^{L} f(n_i)$ where $f(n) = \prod_{i=1}^{n} \frac{1}{u(i)}$- irrespective of the form of the rate $u(n)$. But, for ZRP with asymmetric rate functions(AZRP), we can have an FSS of the form $P(\{n_i\}) \propto \prod_{i=1}^{L} f(n_i)$ only when the rate functions $u_R(n)$ and $u_L(n)$ satisfy the constraint $\frac{u_L(n+1)u_R(1)-u_R(n+1)u_L(1)}{[u_R(n)+u_L(n)][u_R(n+1)+u_L(n+1)]} = C$ where C is a constant independent of n, here $f(n) = \prod_{i=1}^{n} \frac{1}{u_R(i)+u_L(i)}$ along with $f(0) = 1$.

Technique used to obtain the steady state:

To cancel the total influx with the total outflux for each configuration $\{n_i\}$ in order to reach the steady state, here we have used the following local cancellation scheme

$$u_R(n_{i+1}) + u_L(n_{i+1}) - u_R(n_i+1)\frac{f(n_i+1)f(n_{i+1}-1)}{f(n_i)f(n_{i+1})}$$
$$-u_L(n_{i+1}+1)\frac{f(n_i-1)f(n_{i+1}+1)}{f(n_i)f(n_{i+1})} = h(n_i) - h(n_{i+1}),$$

where $h(n)$ is some undetermined function. Fore an FSS with constrained rates described in the previous section, we obtain consistently $h(n) = h(0) - u_L(1)\frac{f(n-1)f(1)}{f(n)f(0)}$.

Finite range process with asymmetric rate functions along with different range of neighbors

Until now, both the hop rates $u_{R,L}(n_{i-K}, \ldots, n_i, \ldots, n_{i+K})$ were functions of the same number of variables $(2K+1)$ such that the number(K) of right and left neighbors of the departure site in both the rates is exactly same. In this section we further generalize the AFRP by introducing in the rates a further asymmetry which is between the right and left range of the departure site. More precisely, a particle from a randomly chosen site i can hop to its right neighbor with rate $u_R(n_{i-K_l}, \ldots n_i, \ldots n_{i+K_r})$ whereas it may hop to its left neighbor with rate $u_L(n_{i-K'_l}, \ldots n_i, \ldots n_{i+K'r})$. Clearly the previous model is a special case of this one with $K_l = K_r = K'_l = K'_r = K$.

For these general class of stochastic processes with rate functions $u_R(n_{i-K_l}, \ldots n_i, \ldots n_{i+K_r})$ and $u_L(n_{i-K'_l}, \ldots n_i, \ldots n_{i+K'r})$, we construct matrix product steady states [8] where each site containing some particles or vacancy is represented by a matrix and consequently the whole configuration is represented as a product of such L matrices each indicating the occupation of single sites. Clearly, since each site can contain any integer number of particles, it seems that we need an infinite number of such matrices which eventually gives rise to set of algebra of infinite number of matrix equations that has to be solved to obtain the matrix product state. To get the steady state solution, it has been shown in [9] that for a very large class of hop rates, the matrix algebra can be reduced to a single functional relation that we solve for the matrix $A(n)$ as a function of the variable n. So, we obtain a matrix product steady state where the probability of any configuration $\{n_i\}$ is expressed as $P(\{n_i\}) \propto \prod_{i=1}^{L} A(n_i)$ where $A(n_i)$ represents the state of the site i occupied by n_i particles.

Among all such models [9], we would like to quote an example where $K_l = 1 \neq K_l' = 2$ and $K_r = 2 \neq K_r' = 0$ such that $u_R(n_{i-1}, n_i, n_{i+1}, n_{i+2}) = u(n_{i-1}, n_i, n_{i+1}) + v(n_i, n_{i+1}, n_{i+2})$ and $u_L(n_{i-2}, n_{i-1}, n_i) = v(n_{i-2}, n_{i-1}, n_i)$. This model leads to a non-equilibrium matrix product steady state $P(\{n_i\}) \propto \prod_{i=1}^{L} A(n_i)$ with $A(n) = |\beta(n)\rangle\langle\alpha(n)|$ when $u_L(n_{i-2}, n_{i-1}, n_i) = \frac{\langle\alpha(n_{i-2})|\beta(n_{i-1})+1\rangle\langle\alpha(n_{i-1}+1)|\beta(n_i)\rangle}{\langle\alpha(n_{i-2})|\beta(n_{i-1})\rangle\langle\alpha(n_{i-1})|\beta(n_i)\rangle}$ and $u_R(n_{i-1}, n_i, n_{i+1}, n_{i+2}) = \frac{\langle\alpha(n_{i-1})|\beta(n_i)-1\rangle\langle\alpha(n_i-1)|\beta(n_{i+1})\rangle}{\langle\alpha(n_{i-1})|\beta(n_i)\rangle\langle\alpha(n_i)|\beta(n_{i+1})\rangle} + u_L(n_i, n_{i+1}, n_{i+2})$, where $|\beta(n)\rangle$ is any $d-$dimensional ($d > 0$ is a positive integer) column vector and $\langle\alpha(n)|$ is some $d-$dimensional row vector of the variable n.

Technique used to obtain the steady state:

The Master equation for the specific model described above, can be equivalently written as

$$\sum_{i=1}^{L} \mathrm{Tr}[\ldots A(n_{i-2})\mathbf{F}(n_{i-1}, n_i, n_{i+1})A(n_{i+2})\ldots] \;=\; 0,$$

where,

$$\begin{aligned}
\mathbf{F}(n_{i-1}, n_i, n_{i+1}) \;&=\; [u(n_{i-1}, n_i, n_{i+1})A(n_{i-1})A(n_i)A(n_{i+1}) \\
&- \; u(n_{i-1}, n_i+1, n_{i+1}-1)A(n_{i-1})A(n_i+1)A(n_{i+1}-1)] \\
&+ \; [v(n_{i-1}, n_i, n_{i+1})A(n_{i-1})A(n_i)A(n_{i+1}) \\
&- \; v(n_{i-1}+1, n_i-1, n_{i+1})A(n_{i-1}+1)A(n_i-1)A(n_{i+1})] \\
&+ \; [v(n_{i-1}, n_i, n_{i+1})A(n_{i-1})A(n_i)A(n_{i+1}) \\
&- \; v(n_{i-1}, n_i-1, n_{i+1}+1)A(n_{i-1})A(n_i-1)A(n_{i+1}+1)].
\end{aligned}$$

We follow the cancellation scheme

$$\begin{aligned}
F(n_{i-1}, n_i, n_{i+1}) = A(n_{i-1})\widetilde{A}(n_i)A(n_{i+1}) - A(n_{i-1})A(n_i)\widetilde{A}(n_{i+1}) + \widehat{A}(n_{i-1})\overline{A}(n_i)A(n_{i+1}) \\
-A(n_{i-1})\widehat{A}(n_i)\overline{A}(n_{i+1}) + A(n_{i-1})\overline{A}(n_i)\widehat{A}(n_{i+1}) - \overline{A}(n_{i-1})\widehat{A}(n_i)A(n_{i+1}),
\end{aligned}$$

where $\widetilde{A}(n), \widehat{A}(n), \overline{A}(n)$ are auxiliary matrices to be chosen suitably and $F(.)$ is a function of the rates and site occupation matrices given in [9]. In our case we find that $\widetilde{A}(n) = A(n-1), \widehat{A}(n) = \theta(n)A(n), \overline{A}(n) = A(n+1)$.

$\mathbf{K_r = 1, K_l = 0, K'_r = 0, K'_l = 1}$: Asymmetric misanthrope process (AMAP)

For $K_r = 1, K_l = 0, K'_r = 0, K'_l = 1$, the hop rates $u_R(n_i, n_{i+1})$ and $u_L(n_{i-1}, n_i)$ are functions of the occupation numbers at departure and arrival sites, making it a misanthrope process [7] *but* with asymmetric rate functions. As shown in [5], AMAP possesses FSS of the form $P(\{n_i\}) \propto \prod_{i=1}^{L} f(n_i)$ where $f(n) = f(0) \prod_{i=1}^{n} \frac{1}{w(i)}$ with $w(m) = \frac{u_R(m,0)+u_L(0,m)}{u_R(1,m-1)+u_L(m-1,1)}$ only if the right and left rate functions satisfy a condition described in details in [5].

Technique used to obtain the steady state:

The FSS of AMAP for some constrained rate functions $u_{R,L}(.)$ is obtained using the following local cancellation condition

$$u_R(n_{i-1}, n_i) + u_L(n_{i-1}, n_i) - u_R(n_{i-1}+1, n_i-1)\frac{f(n_{i-1}+1)f(n_i-1)}{f(n_{i-1})f(n_i)}$$
$$-u_L(n_{i-1}-1, n_i+1)\frac{f(n_{i-1}-1)f(n_i+1)}{f(n_{i-1})f(n_i)} = h(n_{i-1}) - h(n_i).$$

The desired FSS with the rate functions conditioned suitably ultimately gives $h(n) = u_R(1, n-1)\frac{u_R(n,0)+u_L(0,n)}{u_R(1,n-1)+u_L(n-1,1)}$. Note that, the same steady state of AMAP can also be obtained starting from the matrix product ansatz and using some cancellation scheme involving auxiliary matrices, only what happens is that we end up with both the site occupation matrix $A(n)$ and auxiliary matrices $\widetilde{A}(n)$ being scalars.

K-species assisted exchange models

All through the previous sections, we have dealt with particles that are free of hard core repulsion so that more than one particle can occupy the same lattice site. In this section, we are going to talk about K-species assisted exchange models on a one dimensional periodic lattice where the particles obey hard core exclusion which ensures the fact that any lattice site can either be vacant or it can be occupied by at most one single particle. Actually, in this model we have K (which can take any finite positive integer value) different species of hard core particles, denoted by $s = 0, 1, 2, \ldots K$, (s denotes the type of particle residing at individual lattice sites), executing exchange dynamics among themselves i.e. a particle with state s_i at a randomly chosen site i can exchange its position with its right nearest neighbor at state s_{i+1}(provided $s_i \neq s_{i+1}$) with an exchange rate $u(s_{i-1}, s_i, s_{i+1})$ or it may exchange its position with its left neighbor at state s_{i-1} with rate $u(s_{i-2}, s_{i-1}, s_i)$.

Both the dynamics can be clubbed into a single dynamics as $XIJ \xrightarrow{u(XIJ)} XJI$. Since, the exchange rates not only involve the two sites among which the exchange is taken place but also includes nearest neighbors(left nearest neighbor of the rightward moving particle), the exchange process is assisted by neighbors and we call this stochastic process as K-species *assisted* exchange process. We have a factorized steady state $P(\{s_i\}) \propto \prod_{i=1}^{L} f(s_i)$ for this model as a very special case when $u(XIJ) = u(IJ) = u(JI) = u(XJI)$. But the factorized being devoid of spatial correlations, we investigate the possibility of pair factorized steady states for general class of assisted exchange rates $u(s_{i-1}, s_i, s_{i+1})$. Indeed, for K-species assisted exchange model, we have a pair factorized steady state of the form $P(\{s_i\}) \propto \prod_{i=1}^{L} g(s_i, s_{i+1}) \delta(\sum_{i=1}^{L} \delta_{s_i,1} - N_1) \delta(\sum_{i=1}^{L} \delta_{s_i,2} - N_2) \ldots \delta(\sum_{i=1}^{L} \delta_{s_i,K} - N_K)$ where the rates are given as $u(s_{i-1}, s_i, s_{i+1}) = \frac{g(s_{i-1}, s_{i+1})}{g(s_{i-1}, s_i) g(s_i, s_{i+1})}$.

Technique used to obtain the steady state:

The desired pair factorized steady state with proper rates for the K-species assisted exchange model is reached through the following local cancellation scheme

$$u(s_{i-1}, s_{i+1}, s_i) \frac{g(s_{i-1}, s_{i+1}) g(s_{i+1}, s_i) g(s_i, s_{i+2})}{g(s_{i-1}, s_i) g(s_i, s_{i+1}) g(s_{i+1}, s_{i+2})} - u(s_{i-1}, s_i, s_{i+1})$$
$$= h(s_{i-1}, s_i, s_{i+1}) - h(s_i, s_{i+1}, s_{i+2}),$$

where the function $h(.)$ has to be determined consistently. For, $u(s_{i-1}, s_i, s_{i+1}) = \frac{g(s_{i-1}, s_{i+1})}{g(s_{i-1}, s_i) g(s_i, s_{i+1})}$, we obtain $h(s_{i-1}, s_i, s_{i+1}) = -u(s_{i-1}, s_i, s_{i+1})$ for $s_i \neq s_{i+1}$ and $h(s_{i-1}, s_i, s_{i+1}) = 0$ otherwise.

Observables and special features

Spatial correlation

Spatial correlation is often a quantity of interest as it manifests the interaction strength between the components of the system. For example, the hop rate or interaction between particles is bound to a single site for ZRP by virtue of which the corresponding steady state is factorized and the particles at different sites remain spatially uncorrelated. But, for finite range processes, the rates depend on several lattice sites extending the range of interaction between particles at different sites and for a large class of these hop rates, as discussed earlier, we have cluster factorized steady states which indicate existence of spatial correlation. The spatial correlation functions are calculated analytically [4] by using transfer matrix methods and we find damped oscillatory behavior or exponentially decaying correlations depending on the dynamics.

Particle current

Another very important observable that indicates the onset of non-equilibrium phenomena is the non-zero net flux of observables e.g. particle current [10] in contrast to the equilibrium systems which are zero-current states. The particle current can have several interesting features. Below we discuss very briefly a few of them.

Current reversal

Let a non-equilibrium system with a given dynamics has a non-zero current flowing through it along some fixed direction. Now, keeping the rates fixed, if we tune the particle density and after some particular value of the density, the current starts flowing in a different direction compared to the previous direction- then we have density dependent current reversal. A simple example is AZRP(AFRP with $K = 0$) with rates $u_R(n) = \delta$ for $n = 1$ and α for $n > 1$ whereas $u_L(n) = 1-\delta$ for $n = 1$ and $1 - \alpha$ for $n > 1$. The corresponding current [5] is $J = \frac{\rho}{(1+\rho)^2}\left[2\delta - 1 + \rho(2\alpha - 1)\right],$ for $\alpha > \frac{1}{2}$ and $\delta < \frac{1}{2}$, changes its direction of flow as the particle density(ρ) crosses the point of reversal $\rho^* = \frac{1-2\delta}{2\alpha-1}$.

Negative differential mobility

For systems fairly close to equilibrium, the response of an observable to a small enough perturbation is dictated by the equilibrium fluctuation of that observable meaning that the response acts monotonically with the perturbation. But, for systems driven out of equilibrium, absolute negative mobility [11](current flowing in the opposite direction of the bias) and negative differential mobility [12](current decreases with increasing bias) have been detected. In context of finite range processes, we have observed the phenomena of negative differential mobility [13] when some current carrying modes are slowed down by the applied bias. For example, in AMAP(AFRP with $K = 1$) with rates $u_R(m, n) = 1 + (\psi - 1)\delta_{n,0}$

and $u_L(m,n) = \psi\delta_{m,1} + (\frac{1}{2} + (e^{-\epsilon} - \frac{1}{2})\delta_{n,0})(1 - \delta_{m,1})$, the particle current shows a non-monotonic behavior and decreases with increasing bias(ϵ) for $\epsilon > 3$ when $\psi = \frac{1}{(1+\epsilon)}$.

Condensation

An intriguing phenomena occurring for particles without hardcore exclusion even in one dimension is the real space condensation [6, 14]. Among the models we studied, in case of AMAP(AFRP with $K = 1$)[5] with rates $u_L(m,n) = u_R(n+1, m-1)\frac{1+\frac{b}{m}}{1+\frac{b}{n+1}}$, the system can macroscopically distribute any number of particles if $b \leqslant 2$, but, for $b > 2$, the maximum allowed density is $\rho_c = \frac{1}{b-2}$ and if $\rho > \rho_c$, a macroscopic number, $(\rho - \rho_c)L$, of particles gather on some particular lattice site resulting in the formation of a single site condensate.

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Chapter 1

Introduction

Statistical mechanics deals with many particle systems with large number of degrees of freedom. The state of such a system at any instant of time can be represented by a set of generalized coordinates- commonly termed as a *configuration*. Then the time evolution of the system can be equivalently thought as the system changing its configuration from one to another in the configuration space. The description of a macroscopic system with a huge number of configurations demands a probabilistic approach where at each instant of time, the system has a probability of being in a particular configuration and these probabilities actually evolve with time describing the time evolution of the system. How fast a system changes its configuration to move to a different configuration- is quantified by *dynamical rates*. So the time evolution of the probabilities of all possible configurations of the system is described through an equation known as the *Master equation* that involves all the relevant dynamical rates and probabilities of the configurations as expected. In this connection, we should mention that the form of the Master equation we consider in the present thesis is Markovian in nature, meaning,*the probability of finding the system in a particular configuration at the next instant of time depends* *only on the configuration of the system at the present time but not on the system's configurations at any of the previous instants of time.*

We are often interested in a stationary state of the system where after these probabilities reach an asymptotic limit, the probability that the system, evolving according to some dynamics, visits any particular configuration attains a steady value, independent of time (though this constant probability value for any particular configuration is in general different from that of any other configuration)- we call this state of the system as the *steady state*. From the viewpoint of the Master equation, the steady state is a balance condition so that for every configuration, the total incoming probability to that configuration from all other configurations and the total outgoing probability flux from that configuration cancel each other.

The steady states in general can be categorized in two types- equilibrium and non-equilibrium states depending on the particulars of the dynamics or more precisely, dynamical transition rates. Particularly, if the system dynamics obeys detailed balance, then the corresponding steady state is an *equilibrium steady state*(ESS). The steady state measures or functional form of the probabilities in equilibrium are given by the famous Gibbs-Boltzmann distribution [30, 172]. This prior knowledge about the functional form of the probabilities takes us one step forward in calculating the physical observables of interest. More elaborately, with the information of the stationary probabilities, one can next calculate the partition function that in turn forms an expression for a free energy and the derivatives (of different orders) of the free energy give the thermodynamic state variables and response functions of interest. We should note that, for detailed balance, the microscopic probability flux between any two configurations in the configuration space happens to be exactly zero. This in turn implies that the net flux of any macroscopic observable, known as *current*, through the system in equilibrium is also zero, i.e., equilibrium state is essentially a *zero current state*.

On the contrary, all other steady states, which are reached through some dynamics that do not satisfy detailed balance, are called as *non-equilibrium steady state*(NESS). A careful look at the Master equation reveals that other than detailed balance, there are, in general, many possible ways to cancel the total incoming flux with the total outgoing flux. Each of these ways, which we justifiably call as *flux cancellation schemes*, leads to a non-equilibrium steady state. The steady state measures of non-equilibrium systems have no Gibbs-Boltzmann like general formula in sharp contrast to systems in equilibrium. This statement actually raises the *question: how to obtain the steady states of non-equilibrium systems ?* Obtaining non-equilibrium steady state measures has always been a subject of interest. Also another point to note, again in contrast to the equilibrium systems, the microscopic probability flux between configurations in non-equilibrium systems are nonzero in general and this results in the possibility of having a nonzero current of some macroscopic observables. Since current is absent in equilibrium, calculating current takes a step forward to use it to characterize non-equilibrium systems. There are many other observables of interest in context of non-equilibrium models, e.g., spatial correlations between several components or fluctuation of observables in systems out of equilibrium.

It has been realized that exact solution of steady state measures for certain non-equilibrium systems and analytical calculation of observables brings much insight to the understanding of the corresponding systems. In context of the exactly solvable interacting non-equilibrium systems, there exists many successful models. Below we briefly discuss in a qualitative and informative way some of the most powerful and handy schemes used in a profound manner to solve exactly the steady state and analyze characteristic observables of various class of non-equilibrium models.

1.1 Schemes for obtaining exact steady state measures

1.1.1 Matrix Product Ansatz

A prototypical example of non-equilibrium processes is the totally asymmetric simple exclusion process(TASEP) with open boundaries that has been very useful in cultivating several features of non-equilibrium phenomena. The steady state of the TASEP with open boundaries was obtained exactly by Derrida et. al. in Ref.[56] using *matrix product ansatz* (MPA). In case of matrix product ansatz, each microscopic constituent of the system is represented a-priori by a matrix; in steady state, the Master equation forces these matrices to satisfy certain matrix equations (formally known as the matrix algebra) involving the dynamical transition rates. Finally, the ansatz works if one succeeds in finding the representation of the matrices consistently, though there are specific examples where observables can be calculated using the matrix algebra without having explicit representations [56]. After successful implementation of MPA in TASEP with open boundaries [56], it has been used extensively to solve the steady states of different generalizations of TASEP, e.g., TASEP with multiple species of particles [69], TASEP with internal degrees of freedom [18]; non-conserved systems with deposition, evaporation, coagulation-decoagulation like dynamics [103] and also to subjects as diverse as quantum information [156]. A nice review on the role of matrix product ansatz in solving general non-equilibrium exclusion processes with numerous examples can be found in [27]. In context of multi-species TASEP with periodic boundaries, it has been shown in [76] that the steady state of N-species (N is any positive integer) TASEP can be written in matrix product form where for $N = 3$, the matrices representing different species of particles and vacancies can be represented as tensor product of elements of 2-species TASEP with periodic boundaries and for $N > 3$, the matrices can be similarly obtained in a recursive manner from matrices of multi-species TASEP with lower values of N. This results have been extended for particles moving in both directions in one dimension i.e., partially asymmetric multi-species exclusion process with periodic boundary conditions in [160] where also the steady states can be represented in matrix product forms with the matrices for N-species being obtained recursively from the matrices of $(N - 1)$-species. In [7], the authors have studied the matrix product ansatz for N-species TASEP on periodic lattices with inhomogeneous hopping rates. Also, for $N = 2$, the steady state of TASEP for some specific open boundary conditions has been determined in [49] and depending on the boundary rates, shock waves, maximal current and low/high density phases have been found.

1.1.2 Pairwise balance condition

Now if we shift from TASEP like models with hardcore exclusion to interacting particle systems where the constituents are free of hardcore repulsion, for example, the well studied model of zero range process (ZRP) introduced by Spitzer [171], the corresponding steady state can be achieved using *pairwise balance condition*

(PWB) where one uses the following principle: for every pair of configuration C and C', there exists another configuration C'' such that the flux coming to C from C' is exactly balanced by the flux going from C to C''. Note that a special case of the pairwise balance condition is the detailed balance when $C'' = C'$. For PWB to hold, a necessary condition is that the number of distinct incoming fluxes to any configuration must be equal to the number of distinct outgoing fluxes from that configuration. A different model of driven lattice gas with drop-push dynamics [166] gives rise to a product measure steady state (like ZRP) obtained explicitly by the pairwise balance condition. Even after introducing spatial disorder within this drop-push process [176], the corresponding steady state could still be obtained through the PWB. In Ref.[168], the authors have studied TASEP with extended objects as a model for protein synthesis and commented that the steady state measure could have been achieved from pairwise balance condition.

1.1.3 h-balance scheme

Continuing on the discussion on systems of particles, free from hardcore repulsion, there are many other stochastic process apart from ZRP, like misanthrope process, inclusion process etc. Misanthrope process is introduced by Cocozza-Thivent in [47] and later studied extensively in context of mass condensation in factorized steady states (FSS) in [77]. There, to obtain a factorized steady state measure, the dynamical rates must be constrained. More precisely, the corresponding FSS has been obtained by imposing a condition that the rates $u(.,.)$ (which are functions of two variables) satisfy a condition depending on an undetermined function $h(.)$ which is a function of only one variable. Subsequently the functional form of $h(.)$

is obtained suitably using boundary conditions. This particular scheme, we name conveniently as "h-balance scheme" which will be used extensively in this thesis for exact solvability in many different non-equilibrium models.

1.1.4 Bethe ansatz

Bethe ansatz (BA) was first introduced by Hans Bethe in [25] to study the energy eigenvalues and eigenfunctions of a linear chain of interacting atoms as a step forward to model the behavior of metals. A lucid introduction to the elements of algebraic Bethe ansatz for diagonalizing the Hamiltonian of the spin $\frac{1}{2}$ isotropic Heisenberg quantum closed chain can be found in [152] whereas Ref.[170] deals with BA for integrable open spin chains. A more elaborated discussion on the role of Bethe ansatz in the study of $1d$ quantum systems including Lieb-Liniger model, anisotropic XXZ chains etc. is the subject matter of [81]. Remarkably, as pointed out by Batchelor in [20], the Bethe ansatz "finds its way into everything from superconductors to string theory". For example, using BA for interacting fermions, one dimensional ($1d$) Hubbard model was solved by Lieb and Wu [136]. In the context of systems in classical statistical mechanics, the eigenvectors of the transfer matrix of the six-vertex model [137] (also known as the Ice model because of its resemblance to the crystalline structure of ice) can be achieved through the Bethe ansatz. Moreover, as shown in Ref.[21], the Heisenberg model Hamiltonian commutes with the transfer matrix of the ice model (both solved using BA) manifesting a general connection between $1d$ quantum and $2d$ classical statistical mechanical problems. Among various exclusion processes, Bethe ansatz has been incorporated to obtain dynamical quantities like velocity, diffusion constant etc.

of a defect particle in partially asymmetric simple exclusion process on a ring [59]. An extensive discussion on the use of Bethe ansatz for analytical derivation of the spectral gap, generating function of current etc. can be found in [145]. Also, for the totally asymmetric simple exclusion process in contact with two reservoirs, the eigenvalues and eigenvectors of the corresponding Markov matrix are obtained through the Bethe ansatz in [48].

1.1.5 Large deviation theory

As the name suggests, the scheme of large deviation theory (LDT) is used for the study of the rare events that, though take place with negligible frequency, may have huge impact on the system. The mathematical formulation of both equilibrium and non-equilibrium statistical mechanics in the language of LDT has been illustrated beautifully in [175]. For systems in equilibrium, the conditions of maximum entropy for an isolated system or that of minimum free energy for a closed or open system appear naturally in the framework of the large deviation theory with the entropy or free energy being the rate function in context of LDT. In relation to non-equilibrium systems where net flux of some observable- termed as current is often the observable of interest. The large deviation function (LDF) for the time integrated current for TASEP on a ring has been studied in Ref.[54]- where the authors determined the expression of LDF (similar looking as the pressure for an ideal Bose gas or Fermi gas in $3d$) that helps in calculating the exact diffusion constant of the particles and higher order cumulants of the current. Later, for open systems connected to reservoirs at unequal densities or temperature, the large deviation function for current has been calculated [29] from a simple additivity principle.

Along with this additivity principle, how the macroscopic fluctuation theory help us to calculate the LDF of current in exclusion processes, is discussed extensively in [60]- where more importantly, it has been argued how the large deviation of current may generalize the notion of free energy in non-equilibrium systems. In Ref.[6], the authors show that for symmetric simple exclusion process on a ring with a large system size, the LDFs of both current and activity take scaling forms and they further generalize the result by showing that this scaling form is actually universal for any arbitrary diffusive system with a single conserved field. Since the rare events are highly suppressed, an important step in this direction is the study of the problem of conditioning a Markov process on a rare event and representation of this conditioned process by an effective conditioning-free process [42]. The role of dissipation in putting bound on the current fluctuation has been studied in [88] and correspondingly two inequalities for the rate function of the LDF of current in Markov processes (used for modeling molecular motors, chemical reaction networks etc.) have been stated and proved. Apart from these classical non-equilibrium systems, in [83], large deviation theory has been applied to quantum non-equilibrium systems where LDT allows to treat ensemble of trajectories and one may observe there features of dynamical phase transitions, also a very important mapping is made here between two dynamical systems where the rare trajectories of one are the typical trajectories of the other that helped a lot in generating rare events of large fluctuations.

1.2 Schemes we have used

Among the schemes we have just discussed about, it may be noted that the introduction to the first three schemes namely matrix product ansatz, pairwise balance condition and h-balance scheme have been much shorter with fewer number of references and applications mentioned in comparison to the last two schemes Bethe ansatz and large deviation theory. This is done purposefully, because from now onwards, we will focus on the first three, i.e., matrix product ansatz, pairwise balance condition and h-balance scheme and discuss them in details in the next chapters to show how these schemes can be incorporated to analyze the non-equilibrium models we are going to introduce. Since all these schemes have been widely used, first let us make clear what are we going to discuss in particular.

(i) Firstly, in Chapter 3. we are going to introduce a class of driven diffusive systems with particles free of hardcore repulsion and the dynamical rate of hopping of particles to nearest neighbors depending on the occupation number of the departure site and some of its neighbors- this model is somewhat similar to the well-known model of zero range process (ZRP) *except* for the fact that in ZRP the hop rate depends only on the occupation number of the departure site where as in our case it depends on occupation of the departure site and that of a finite number of sites within a specified distance from the departure site. This model can be thought of as a generalization of ZRP where now particles at different sites interact with each other and we formally call this process as the *finite range process* (FRP). We have pointed out earlier that the steady state of ZRP can be obtained using the pairwise balance condition- in chapter 3. we are going to show that *pairwise balance also helps in obtaining the exact steady state measure of FRP, when the* *transition rates are constrained.* Unlike ZRP where the steady state is factorized for any choice of the jump rate, the steady state of FRP in product measure form can be obtained only when the rates satisfy certain conditions.

(*ii*) In Chapter 4., we are going to consider a more general scenario of interacting particles in FRP where the hopping of the particles along different spatial directions occur with rates whose functional form depend on the direction. Accordingly the corresponding stochastic processes are named as *finite range processes with asymmetric rate functions* or in short, *asymmetric finite range process* (AFRP). As we may remember of the h-balance scheme that has earlier been used in context of totally asymmetric misanthrope processes, in this chapter we explicitly show that use of *the h-balance scheme gives rise to the steady state measure of several models belonging to AFRP class of models.*

(*iii*) The matrix product ansatz has been widely used for exactly solving the steady states of several non-equilibrium models, but, only in the context of exclusion processes where the constituent particles obey hardcore repulsion. In Chapter 5., we introduce matrix product ansatz for interacting particles without hardcore exclusion and discuss in details how to overcome possible difficulties rising from the fact that the particles do not obey hardcore exclusion. In this connection, we reproduce some results using this method, which are obtained earlier in the previous chapters using other schemes. Also, *we discuss some non-equilibrium models that do not fall in the framework of FRP and AFRP, but the steady state measures for these very different class of models can be achieved from the matrix product ansatz introduced here.*

(*iv*) In chapter 6. we study a class of assisted exchange models (AEM) with multiple species of particles but each of them now obeying hardcore exclusion, the dynamics is assisted in the sense that the hop rate depends not only on the pair of particles which are exchanged but also on their neighbors. We will show that *the steady state of the AEM, for a broad class of hop rates, can be obtained by using the h-balance scheme.*

With this brief introduction to the schemes we are going to use to solve exactly the steady state measures of several class of non-equilibrium models. Let us now move on to a brief discussion on the physical observables of interest and possibility of phase transitions in non-equilibrium systems.

1.3 Observables and phase transitions in NESS

For systems in equilibrium with given Hamiltonian, since the steady state weights are given by Gibbs-Boltzmann distribution, in principle one can always calculate the partition function and subsequently define the free energy as the logarithm of the partition function. Then the observables of interest can be derived from the free energy. The formulation of thermodynamics in case of non-equilibrium systems is not so straightforward, as there is no general framework exists for this purpose. At the first place there is no generic measure of the non-equilibrium steady states and moreover, even if one can obtain the steady state measure for a particular non-equilibrium system, the existence of state variables like free energy is not guaranteed because of the presence of current in the configuration space and the spatial current. Though in special cases, for example with systems having certain kind of short range interactions, it is possible to define the notion of zeroth law of thermodynamics and free energy [39]. Study of non-equilibrium thermodynamics is a topic of immense interest in recent years. But, in the following discussions, we will not consider this topic much, rather we will concentrate on observables like particle current and spatial correlations.

Current, specially in the context of exclusion processes, has been studied extensively in the past few decades and has contributed much in understanding of non-equilibrium phenomena. Fluctuations of current through a bond in partially asymmetric simple exclusion process on a ring grows linearly with time [57] and the corresponding proportionality constant actually gives the diffusion constant, a quantity of interest for diffusive systems. Also exact large deviation functional of current in totally asymmetric simple exclusion process on a ring has been obtained that not only recovers the diffusion constant and higher order cumulants, but surprisingly, has an expression that imitates the pressure as a function of density of an ideal Bose or Fermi gas in three dimensions. Apart from these periodic lattices, large deviation function of current for weakly asymmetric exclusion process in contact with two reservoirs at the ends that mimics transport processes has been studied [28]. Importantly, the properties of these fluctuations and large deviations of current obtained exactly using matrix ansatz, Bethe ansatz, additivity principles etc. in non-equilibrium steady states can be compared with systems in equilibrium, e.g., the non-convexity of the large deviation function actually generalizes the notion of free energy so that the large deviation function of current (in non-equilibrium) may be viewed as the analog of equilibrium free energy. In context of particle current in our works, we can analytically calculate this observable for certain class of finite range processes, asymmetric finite range processes and assisted exchange models. As we will show later, the particle current in these models exhibits many interesting features like density dependent current reversal, negative differential mobility etc. Reversing the direction of current only by changing the particle density with the dynamical parameters kept constant, gives rise to the phenomena of current reversal. Negative differential mobility is observed when the particle current decreases with increasing bias. Apart from the current, we also calculate spatial correlations between different components of the system analytically using transfer matrix methods.

Irrespective of whether the system is in equilibrium or not, the study of phase transitions has always been a central focus of statistical mechanics. In this connection, one should note that the possibility of phase transitions in *one dimension* is ruled out for equilibrium systems with short range interactions; this is because, systems with state variables having finite number of choices and interacting through short range interaction can be described by finite dimensional symmetric transfer matrices for which the largest eigenvalue is non-degenerate. However, several different types of phase transitions have been observed in different models of non-equilibrium systems with short range interactions, *even in one dimension* [19, 46, 73]. A much celebrated example is the totally asymmetric simple exclusion process with open boundaries which exhibits novel phases namely, the low density phase, high density phase and maximal current phase. Phase transition between these phases can occur when boundary injection and extraction rates of particles are varied. Another example is the creation of phase separated states of the component particles in ABC model leading to phase separation transition [46, 73]. Absorbing state phase transitions from active state to absorbing configurations of the system where the system gets trapped, has also been studied extensively in context of self-organized criticality. In the thesis, we primarily concentrate on the condensation transition in systems of particles without hardcore exclusion, for example finite range processes and asymmetric finite range processes, where for specific choice of rates a macroscopic fraction of particles can gather at a single lattice site leading to the formation of a condensate.

1.4 Brief outline of the thesis

Here is a brief overview on the contents of the following chapters in the thesis. In chapter 2. we start our discussion by introducing the Master equation and thereby defining the steady state. Particularly, detailed balance leading to equilibrium has been summarized and then we briefly discuss the mathematical structure of the flux balance schemes like pairwise balance condition, h-balance scheme, matrix product ansatz and how to incorporate these tools in obtaining the exact steady states of some model systems like zero range process, two interacting random walkers model etc. This chapter serves as an introduction to the flux balance schemes that would be used extensively in the next chapters for more complicated systems.

In chapter 3. we will introduce the model and dynamics of the finite range processes. Starting from the Master equation, we will show how the pairwise balance condition leads to the steady state of FRP when the rates satisfy specific constraints. Then we calculate partition function and subsequently spatial correlations in these models which are not that straightforward, but can be done analytically for special form of the steady state weight functions. Also, formation of single site condensates and extended condensates for specific choice of the dynamics are studied extensively in this chapter.

The next chapter 4. focuses on generalizing the dynamics of finite range processes by invoking asymmetric rate functions. Here we discuss the h-balance scheme that has been used for obtaining the steady states. In particular, the asymmetric zero range process (AZRP) and asymmetric misanthrope process (AMAP) are studied extensively in this chapter. Moreover, using special choice of rates in AMAP and AZRP, we illustrate the phenomena of current reversal and negative differential mobility.

In chapter 5, the asymmetric finite range process (AFRP) is further generalized to include different interaction range in different directions with respect to the departure site from where the particle hops. We first formulate matrix product ansatz (MPA) for particles free of hardcore repulsions and show how MPA helps in obtaining the exact steady states of AFRP with different range of neighbors. Particularly, we study some examples that do not belong to the classes of FRP or AFRP.

We devote chapter 6. to study some models where there are multiple species of particles maintaining hardcore repulsion. Number of particles of each species is conserved in these models as the dynamics only exchange particles at consecutive sites with rates that depend on the type of pair of particles taking part in the exchange and the type of particle at the immediate left site of the pair. We show that the schemes like h-balance, matrix product ansatz etc. that have been useful to find the exact steady states of finite range processes without hardcore exclusion, also turn out to be very handy in obtaining the steady states of the models in the present chapter. Also, phenomena like density dependent current reversal, negative differential mobility, phase separation transitions can be observed in these models.

Finally, we summarize in chapter 7. and pose some open problems which can be of interest in context of transport processes and non-equilibrium thermodynamics.

Chapter 2

Prologue: obtaining exact steady states in simple examples

Before going to the detailed description of the exactly solvable steady state measures of the non-equilibrium models to be introduced in the subsequent chapters, for completeness we devote the present chapter to discuss in a nutshell the Master equation (a starting point for the systematic study of the systems having many configurations in the configuration space), quantitative definition of the steady state (equilibrium and non-equilibrium). We also discuss a few simple examples of non-equilibrium processes whose steady states can be achieved exactly using some suitable schemes.

Let us start with a many particle system that transits from one configuration to another in the configuration space. So what is the configuration of the system at a given instant is stochastic in nature. Probability of a system to be in a specific configuration at a given time, evolves with time. In long time limit, however, the fraction of time spent in a particular configuration with respect to the total time reaches a definite limit and can be interpreted as the stationary probability.

We start with denoting by $P(C,t)$ the probability that the system is in configuration C at time t- of course these probabilities change with time as the system at some configuration C can move to another configuration C' with a dynamical rate $w(C, C')$- and the corresponding time evolution is dictated by the Master equation. The Master equation is given as

$$\frac{d}{dt}P(C,t) = \left[\sum_{C'} w(C', C)P(C', t) - w(C, C')P(C, t)\right], \tag{2.1}$$

for each configuration C in the configuration space, where the first sum accounts for the total in-flux or gain to the configuration C from all other configurations C' and the second sum can be interpreted as the total out-flux or loss from the configuration C to all other configurations C'.

The *steady state* is defined as the state where the probability of all the configurations in the configuration space attain some limiting value. More specifically $P(C, t) \equiv P(C)$ for every configuration C. Now, as one can see from Eq. (2.1), there are in general plenty of positive terms in both the in-flux sum and the out-flux sum, so one has to balance the total in-flux with the total out-flux to reach the steady state, i.e., in steady state, total in-flux = total out-flux for each configuration in the configuration space. It is evident that there may exist numerous ways through which the flux can be canceled with each other giving rise to the steady state. Each of these possible ways, we call them as a *flux cancellation scheme*. In the following, we discuss a few of these schemes.

2.1 Detailed balance: road to equilibrium

A quick look at the right hand side of the Master equation (2.1) reveals that, if each term inside the sum equals to zero, then the whole sum automatically becomes zero. Mathematically,

$$w(C', C)P(C') = w(C, C')P(C), \tag{2.2}$$

that is, *for every pair of configurations C and C', the flux coming from C' to C is exactly balanced by the flux going from C to C'* - this is the *detailed balance* condition and it necessarily gives rise to *equilibrium* steady state. The name "detailed" is very much justified as the balance occurs in detail i.e. it takes place between every pair of configurations. The existence of detailed balance implies that the probability current $J_{C,C'} = w(C',C)P(C',t) - w(C,C')P(C,t)$ equals to zero between every pair of configurations C and C'- since these probability currents actually constitutes the particle current in real space, so the detailed balance in (2.2) suggests that equilibrium is essentially a *zero current state.*

Apart from the detailed balance which is a necessary and sufficient condition for an equilibrium states, any other flux cancellation scheme, if gives rise to a stationary state, the corresponding state is in fact a non-equilibrium steady state. Below we briefly study some of these schemes.

2.2 All configurations equally likely: single random walker on a periodic lattice

The simplest case is may be the one when all the configurations are equally probable, i.e., $P(C)$= constant=$\frac{1}{\Omega}$ for all configurations C, where Ω is the total number of configurations in the configuration space for the system under consideration, from equation (2.1) it is clear that the corresponding flux cancellation scheme is

$$\sum_{C'} w(C',C) = \sum_{C''} w(C,C''), \tag{2.3}$$

meaning that the *total inward rates from all other configurations C' to a particular configuration C = total outward rates from the configuration C to all other configurations C''* . Note that the flux balance scheme in Eq. (2.3) is very much different from the detailed balance in (2.2). Let us illustrate this with a small example. Consider a single random walker on a periodic lattice with L sites $i = 1, 2 \dots L$, the walker at site i can move to the right neighbor $(i + 1)$ with rate p or can move to the left neighbor $(i - 1)$ with rate q. If we denote the random walker by 1 and any vacant site by 0, then this dynamics can be represented by

$$10 \underset{q}{\overset{p}{\rightleftharpoons}} 01. \tag{2.4}$$

Clearly, here for any configuration $C \equiv \{\dots, 0, 1, 0, \dots\}$, we have $\sum_{C'} w(C', C) = (p+q) = \sum_{C''} w(C, C'')$. So, all the configurations of the single random walker are equally probable with probability $P(C) = \frac{1}{L}$ for every configuration C since the total number of configurations here is L. Also, note that the steady state becomes equilibrium only when $p = q$.

2.3 Two random walkers: matrix product ansatz (MPA)

Let us include another random walker denoted by 2, and introduce hardcore interaction between them, i.e., both the random walkers cannot stay at the same lattice site at a given instant of time. The random walker 1 (2) moves to right or left with rates p and q (α and β) respectively. So, now the dynamics of this two interacting random walkers model looks as

$$10 \underset{q}{\overset{p}{\rightleftharpoons}} 01, \quad 20 \underset{\beta}{\overset{\alpha}{\rightleftharpoons}} 02. \tag{2.5}$$

Let us denote the configuration of the system as $C \equiv \{s_i\}$ with $i = 1, 2 \dots, L$ and $s_i = 0, 1, 2$ represents whether the site i is vacant or occupied by walker 1 or occupied by walker 2 respectively. Firstly, consider a configuration like $C \equiv$

$\{\ldots,0,1,2,0,\ldots\}$- clearly the total inward rates to this configuration $\sum_{C'} w(C',C) = (p+\beta)$ is not equal to the total outward rates to this configuration which is $\sum_{C''} w(C,C'') = (q+\alpha)$. So, we cannot have an "equally likely" steady state as the flux cancellation scheme (2.3) fails here.

At this point, we take help of the *matrix product ansatz* (MPA). Here the ansatz is to express the probability of any configuration by the trace of product of matrices where each matrix represents the state(1 or 2 or 0) at corresponding lattice site. Mathematically,

$$P(C) \propto \mathrm{Tr}\left[\prod_{i=1}^{L} X_i\right], \tag{2.6}$$

where X_i is the matrix representing the state s_i at the i-th lattice site. More precisely, $X_i = D_1$ or D_2 or E if the site is occupied by random walker $s_i = 1$ or random walker $s_i = 2$ or the site is vacant ($s_i = 0$) respectively. The Master equation for this particular example takes the form

$$\begin{aligned}
&\frac{d}{dt}P(\{s_i\},t)\\
=&\sum_{i=1}^{L}\left[w(s_is_{i-1},s_{i-1}s_i)P(\ldots s_{i-2},s_i,s_{i-1},s_{i+1}\ldots) - w(s_{i-1}s_i,s_is_{i-1})P(\ldots s_{i-2},s_{i-1},s_i,s_{i+1}\ldots)\right].\\
=&\sum_{i=1}^{L}\left[X_1X_2\ldots X_{i-2}\left(w(s_is_{i-1},s_{i-1}s_i)X_iX_{i-1} - w(s_{i-1}s_i,s_is_{i-1})X_{i-1}X_i\right)X_{i+1}\ldots X_L\right]
\end{aligned} \tag{2.7}$$

The next step forward is to construct a suitable flux cancellation scheme that would help us to satisfy the steady state condition when probabilities are given in the form of (2.6). We device the following flux cancellation scheme

$$w(s_is_{i-1},s_{i-1}s_i)X_iX_{i-1} - w(s_{i-1}s_i,s_is_{i-1})X_{i-1}X_i = \widetilde{X}_{i-1}X_i - X_{i-1}\widetilde{X}_i\,, \tag{2.8}$$

where we have introduced *auxiliary* matrices $\widetilde{X}_i$ that help to ensure that the steady state condition is satisfied. It can be seen directly that if we sum over the index

i from 1 to L in Eq. (2.8), the sum vanishes. For the dynamics (2.5), the above cancellation scheme (2.8) generates the following set of equations

$$\begin{aligned} \widetilde{D}_1 D_1 - D_1 \widetilde{D}_1 = 0,\ \widetilde{D}_2 D_2 - D_2 \widetilde{D}_2 = 0,\ \widetilde{E} E - E \widetilde{E} = 0, \\ \widetilde{D}_1 D_2 - D_1 \widetilde{D}_2 = 0,\ \widetilde{D}_2 D_1 - D_2 \widetilde{D}_1 = 0, \\ p D_1 E - q E D_1 = \widetilde{E} D_1 - E \widetilde{D}_1 = D_1 \widetilde{E} - \widetilde{D}_1 E \\ \alpha D_2 E - \beta E D_2 = \widetilde{E} D_2 - E \widetilde{D}_2 = D_2 \widetilde{E} - \widetilde{D}_2 E\ , \end{aligned} \tag{2.9}$$

which we formally denote as the *matrix algebra*. What is needed further for the MPA to work, is the representation of the matrices X_i and $\widetilde{X}_i$ which satisfy the algebra (2.9). We make further simplifications by taking $\widetilde{X}_i$ as scalars $\widetilde{D}_1 = 0, \widetilde{D}_2 = 0, \widetilde{E} = 1$. The above matrix algebra then simplifies to

$$p D_1 E - q E D_1 = D_1,\ \alpha D_2 E - \beta E D_2 = D_2,\ . \tag{2.10}$$

We also have to put an explicit condition

$$D_1^2 = 0,\ D_2^2 = 0\ , \tag{2.11}$$

which ensures that there is exactly one walker of type 1 and only one walker of type 2. Thus any configuration having more than one 1 or 2 has weight equal to zero. Note that if instead of matrices, the walkers 1 and 2 and vacancy 0 were represented by scalars d_1, d_2, e respectively, then $d_1^2 = 0 = d_2^2$ would also imply $d_1 = 0 = d_2$ meaning that a scalar representation is useless in this particular example. Now, the last but non-trivial step is to find the representation of the matrices D_1, D_2, E satisfying the algebra in (2.10) and (2.11). A consistent set of matrices that satisfy the algebra are

$$D_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\ ,\ D_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\ ,\ E = \begin{pmatrix} \frac{p+\beta}{\alpha p - \beta q} & 0 \\ 0 & \frac{q+\alpha}{\alpha p - \beta q} \end{pmatrix}\ . \tag{2.12}$$

Of course, the matrix product has been used extensively in many particle systems in different contexts as discussed elaborately in [27]. In chapter 5. we will return to MPA to study its role in solving the steady state of systems without hard core repulsion.

2.4 Pairwise balance condition: zero range process (ZRP)

We may remember that in case of systems in equilibrium, detailed balance is satisfied where the flux cancellation scheme is such that the flux is canceled between *every pair of configurations.* Another flux cancellation scheme which involves three configurations C', C, C'', such that the *flux coming from C' to C is exactly balanced by the flux going from C to C''*; if this condition is satisfied for every triple of configurations in the configuration space, then we have a non-equilibrium steady state resulting from this *pairwise balance condition.* Mathematically,

$$w(C', C)P(C') = w(C, C'')P(C). \tag{2.13}$$

As an example let us consider the zero range process (ZRP) on a periodic lattice with L sites $i = 1, 2 \ldots L$ where each site i can be vacant or it can be occupied by one or more particles $n_i \leq N$, $N = \sum_{i=1}^{L} n_i$ being the total number of particles in the system. Any possible configuration of this system can be denoted as $C \equiv \{n_i\} = \{\ldots n_{i-1}, n_i, n_{i+1}, \ldots\}$. Clearly this multiple occupancy of a lattice site implies that these particles do not obey hardcore constraint. They actually interact through the particle hopping rate $w(\{n_i\}) = u(n_i)$ that depends on the number of particles at the departure site. In other words, the dynamics is such that a particle from a randomly chosen site i can hop to its right neighbor $(i + 1)$ with rate $u(n_i)$ that depends explicitly on the occupation number n_i of the departure site i. From the structure of the hopping dynamics, one expects no correlation between occupation on different lattice sites in the thermodynamic limit, it is reasonable to start with a hypothesis that this system has a factorized steady state (FSS) of the form

$$P(n_1, n_2 \ldots, n_L) \propto \prod_{i=1}^{L} f(n_i), \tag{2.14}$$

where the single site weight factor $f(n)$ depends on the single site occupation variable n. Now, the task is to verify if such an FSS is possible at all and whether it extends to all possible hop rate $u(n)$. Finally, we must know how $f(n)$ is related to $u(n)$. To answer these questions, we take help of the pairwise balance condition described in (2.13). Let us consider a representative triple of configurations in the form $C' = \{\ldots n_{i-1}+1, n_i - 1, n_{i+1} \ldots\}$, $C = \{\ldots n_{i-1}, n_i, n_{i+1} \ldots\}$ and $C'' = \{\ldots n_{i-1}, n_i - 1, n_{i+1}+1 \ldots\}$. Then, the flux balance scheme in (2.13) along with Eq. (2.14) together give the following equation

$$\begin{aligned} u(n_{i-1}+1) f(n_1) \ldots f(n_{i-2}) f(n_{i-1}+1) f(n_i - 1) f(n_{i+1}) \ldots f(n_L) \\ = u(n_i) f(n_1) \ldots f(n_{i-2}) f(n_{i-1}) f(n_i) f(n_{i+1}) \ldots f(n_L) \\ \Rightarrow u(n_{i-1}+1) f(n_{i-1}+1) f(n_i - 1) = u(n_i) f(n_{i-1}) f(n_i) \\ \Rightarrow u(n_{i-1}+1) \frac{f(n_{i-1}+1)}{f(n_{i-1})} = u(n_i) \frac{f(n_i)}{f(n_i - 1)}. \end{aligned} \tag{2.15}$$

Since the left hand side of the above equation is a function of the site $(i-1)$ whereas the right hand side is a function of the site i, this necessarily means

$$u(n) \frac{f(n)}{f(n-1)} = \text{constant} \Rightarrow u(n) = \frac{f(n-1)}{f(n)} \Rightarrow f(n) = \prod_{k=1}^{n} \frac{1}{u(k)}, \tag{2.16}$$

where $f(0) = 1$ without any loss of generality. So, in equation (2.16), we have expressed the single site weight factor $f(n)$ in terms of the hop rate $u(n)$. Also, this equation ensures that for any functional form of the rate $u(.)$, we can have a factorized steady state of the form (2.14). We will introduce interaction between particles at different lattice sites in chapter 3. and use the pairwise balance condition to solve the corresponding steady state.

2.5 h-balance scheme: zero range process (ZRP) revisited

In this section we discuss about another flux cancellation scheme which we call to be the h-balance scheme, the justification of the name comes from the fact that, just like the auxiliary matrices introduced in the flux balance scheme (2.8) in context of matrix product ansatz, here also we would introduce an undetermined function $h(.)$ in a suitable way so that the corresponding cancellation scheme satisfies the steady state condition. To illustrate this point, we reconsider the example of the zero range process. The Master equation for any configuration $C \equiv \{n_1, \dots n_{i-1}, n_i, n_{i+1} \dots, n_L\} \equiv \{n_i\}$ in this system can be written as

$$\begin{aligned} \frac{d}{dt}P(\{n_i\}) &= \sum_{i=1}^{L} u(n_{i-1}+1)\dots f(n_{i-1}+1)f(n_i-1)f(n_{i+1})\dots \\ &- \sum_{i=1}^{L} u(n_i)\dots f(n_{i-1})f(n_i)f(n_{i+1})\dots \,, \end{aligned} \tag{2.17}$$

where we have considered a priori the factorized form in (2.14). In steady state, the total in-flux and total out-flux has to be equal which means, for ZRP,

$$\begin{aligned} \sum_{i=1}^{L} \left[u(n_i+1)\frac{f(n_i+1)f(n_{i+1}-1)}{f(n_i)f(n_{i+1}-1)} - u(n_i) \right] = 0 \\ \Rightarrow \sum_{i=1}^{L} F(n_i, n_{i+1}) = 0\,, \end{aligned} \tag{2.18}$$

where we have arrived from Eq. (2.17) to Eq. (2.18) by shifting the index $(i-1) \to i$ in the first sum of (2.17) and then dividing both sides by the product $\prod_{i=1}^{L} f(n_i)$.

Also we have used a short hand notation $F(n_i, n_{i+1}) = (u(n_i+1)\frac{f(n_i+1)f(n_{i+1}-1)}{f(n_i)f(n_{i+1}-1)} - u(n_i))$. Now, we are going to introduce the following flux cancellation scheme

$$F(n_i, n_{i+1}) = u(n_i+1)\frac{f(n_i+1)f(n_{i+1}-1)}{f(n_i)f(n_{i+1})} - u(n_i) \ = \ h(n_{i+1}) - h(n_i) \ , \quad (2.19)$$

where the undetermined function $h(n)$ has been introduced in such a way that if we sum over the index i from 1 to L in equation (2.19), then we reach to the steady state condition in (2.18). The next task is to find $f(n)$ and $h(n)$ by using the boundary conditions in Eq. (2.19), e.g., if we put $n_{i+1} = 0$, we have

$$u(n_i) \ = \ h(n_i) - h(0), \quad (2.20)$$

whereas if we put $n_i = 0$ in (2.19), we get

$$\begin{aligned} u(1)\frac{f(1)}{f(0)}\frac{f(n_{i+1}-1)}{f(n_{i+1})} \ &= \ h(n_{i+1}) - h(0) \\ \Rightarrow u(1)\frac{f(1)}{f(0)}\frac{f(n_i-1)}{f(n_i)} \ &= \ h(n_i) - h(0) \ . \end{aligned} \quad (2.21)$$

Comparing the above two equations, we have

$$u(n) = \frac{f(n-1)}{f(n)} \Rightarrow f(n) = \prod_{k=1}^{n} \frac{1}{u(k)}, \quad (2.22)$$

along with $h(n) - h(0) = u(n)$, we have considered $f(0) = 1$ as earlier. So, using the h-balance scheme, we have reproduced the steady state of the zero range process (as can be seen both from (2.16) and (2.22)) using both the pairwise balance condition in the previous section and h-balance scheme in this section. Apart from this simple example of ZRP, there are other examples like the misanthrope process whose factorized steady state with specific condition on the rates cannot be obtained from pairwise balance condition, but can be obtained using the h-balance scheme [77]. The h-balance scheme in a generalized form will be proved to be handy in the subsequent chapters 4 and 6 of the thesis. It would be very interesting to find some connection between the pairwise balance condition and h-balance scheme. As already shown in case of ZRP, there exist more models (e.g. several finite range processes) where both these techniques work equally good to determine the steady state measures. So, further work in this direction to explore any deeper connection between h-balance scheme and pairwise balance condition would be a matter of interest.

Chapter 3

Finite Range Process(FRP)

We start our journey through the exactly solvable driven interacting particle systems with the finite range processes (FRP). In these models hop rate of particles depends on the occupation number of departure site and that of the sites within a "finite" distance measured from the departure site; this justifies the name finite range process. A special case of FRP, when the hop rate depends only on the departure site, is the well known zero range process (ZRP). Before going into the details of FRP, let us briefly introduce some intriguing features of ZRP which, as will be seen shortly, serves as a primary building block for the finite range processes. The zero range process (ZRP), a lattice gas model without any hardcore exclusions, exhibits nontrivial static and dynamic properties in the steady state. The ZRP was introduced [171] as a mathematical model for interacting diffusing particles and, since then, has found applications in different branches of science [70, 71], such as in describing phase separation criterion in driven lattice gases [112], network re-wiring [4, 149], statics and dynamics of extended objects [51, 97], etc. Interestingly, ZRP shows a condensation transition for some specific choices of particle hop rates when the density becomes larger than a critical density, a macroscopic number of particles accumulate on a single lattice site - representing condensation of macroscopic number of particles in real space.

In the ZRP, as we discussed in chapter 2., the particles hop stochastically to one of the nearest neighbors with a rate that depends only on the number of particles on the departure site. We have shown in the previous chapter; either by using pairwise balance condition or h-balance scheme, that ZRP has a factorized steady state (FSS) in the form of Eq. (2.14) along with (2.16), which is amenable to exact analytic studies. However, when the hop rate depends on the neighboring sites, the steady state does not factorize in general [72, 78]. In such situations, one may naturally ask for the possibility of a cluster-factorized steady state (CFSS) which is a straightforward generalization of the factorized steady state (FSS), where the steady state weight is a product of cluster-weight functions (see Eq. (3.3)) of several variables, i.e., the occupation numbers at *two* or more consecutive sites.

In this chapter, we study a class of non-equilibrium lattice models where particles hop in a particular direction, say from a site to its right nearest neighbor, where hop rates not only depend on the occupation of the departure site but also on the occupation of all of its neighbors within a range K; hereafter, K being a finite positive integer (for discrete lattice, would be real number for continuum), we refer to this process as *finite range process (FRP)*. We demonstrate that, in one spatial dimension, one can have a CFSS for various specific choices of hop rates; what we mean by the CFSS here is that the steady state probability weight can be written as a product of functions of $(K + 1)$ variables, each of them being an occupation number in the cluster of K consecutive sites. A special case of the CFSS with $K = 1$, called pair-factorized steady state (PFSS), was recently proposed and studied in [72] where it was shown that, for a particular class of PFSS, the system can also undergo a condensation transition. Later, the PFSS has been found in continuous mass-transfer models [24, 182], in systems with open boundaries [151] and in random graphs [180], etc. However, non-trivial spatial structure, which is not present in a FSS, has not been explored before.

We show that, for a broad class of systems having a CFSS with any K, there exists a finite dimensional transfer-matrix representation of the steady state. Being finite dimensional, these matrices are quite convenient to manipulate and help in exact calculations of spatial correlation functions of any order. Moreover, we propose a sufficient criterion for the hop rates that can give rise to condensation transition in FRP in general. Surprisingly, we find that a small perturbation to a FSS could destroy condensation transition, if any.

3.1 Model

The model is defined on a one dimensional periodic lattice with sites labeled by $i = 1, 2, \dots L$. Each site i has a non- negative integer variable n_i representing the number of particles at that site (for a vacant site $n_i = 0$). Particle from any randomly chosen site i can hop to one of its nearest neighbors, say the right neighbor, with a rate that depends on the number of particles at all the sites which are within a range K with respect to the departure site:

$$\begin{aligned} (\dots, n_{i-1}, n_i, n_{i+1}, \dots) \quad & \longrightarrow (\dots, n_{i-1}, n_i - 1, n_{i+1} + 1, \dots) \\ \text{with } \textbf{rate} \quad & u(n_{i-K}, \dots, n_i, \dots, n_{i+K}). \end{aligned} \tag{3.1}$$

Clearly the total number of particles $N = \sum_i n_i$, or the density $\rho = N/L$, is conserved by this dynamics.

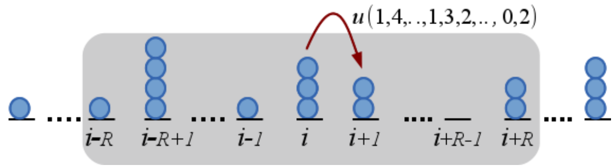

Figure 3.1: In one dimensional finite range process (FRP) a particle hop from a site i to its neighbor with a rate that depends on occupation of site i (here $n_i = 3$) all its neighbors within a range. The lattice model, for certain hop rate, can have a $(K + 1)$-cluster-factorized steady state.

For $K = 0$, this model is identical to the zero range process (ZRP) [71] with hop rate $u(n_i)$, and as shown already in (2.14) along with (2.16), it evolves to a factorized steady state (FSS)

$$P(\{n_i\}) \propto \prod_{i=1}^{L} f(n_i)\delta(\sum_i n_i - N), \tag{3.2}$$

with $f(n) = \prod_{m=1}^{n} u(m)^{-1}$. The Dirac delta function used here is to take care of the conservation of the total number of particles. The process we consider here is a generalized version of the ZRP and hereafter we refer to it as finite range process (FRP).

For $K > 0$, the steady state of FRP in general cannot have a FSS as there are nonzero spatial correlations; however, there can be exceptions in specific cases. We provide explicit proof, later in this chapter that, for $K = 1$, the factorized steady state can be achieved when $u(k, m, n) = w(m, n)$. This special case where the hop rate depends on the number of particles in both departure and arrival sites, is known as misanthrope process (MAP [77]). Since a FRP with $K \geq 1$ includes $K = 1$ as a special case, one expects that, except for the ZRP and the MAP, there cannot be a factorized steady state (FSS) for these class of systems. Actually, as K increases, more and more neighboring site particles are involved in the interaction meaning, the interaction range between the particles increases with increasing K. Consequently, nonzero spatial correlations between more number of neighboring lattice sites come into existence as K is increased. Since nonzero spatial correlations rule out the possibility of having factorized steady states, so, for $K \geq 1$ we do not expect to have factorized steady states.

3.2 Steady state: pairwise balance condition

For the FRP, we first try whether a $(K+1)$- cluster-factorized form,

$$P(\{n_i\}) \propto \prod_{i=1}^{L} g(n_i, n_{i+1}, \ldots n_{i+K}) \delta(\sum_i n_i - N) \tag{3.3}$$

with cluster-weight function $g(.)$ of $(K+1)$ occupation variables, can be a steady state weight for Master equation

$$\begin{aligned} \frac{d}{dt} P(\{n_i\}) &= \sum_{i=1}^{L} u(n_{i-K}, \ldots, n_i, \ldots, n_{i+K}) P(\{n_i\}) \\ &- \sum_{i=1}^{L} u(n_{i-K}, \ldots, n_i + 1, n_{i+1} - 1, .., n_{i+K}) \\ &\qquad \times P(\ldots, n_{i-1} + 1, n_i - 1, \ldots). \end{aligned} \tag{3.4}$$

Now, one can verify that a cluster-factorized form of steady state, as in Eq. (3.3), is indeed possible when the hop rate at site i satisfies the following condition

$$\begin{aligned} & u(n_{i-K}, \ldots, n_i, \ldots, n_{i+K}) \\ & \quad = \prod_{k=0}^{K} \frac{g(\bar{n}_{i-K+k}, \bar{n}_{i-K+1+k}, \ldots, \bar{n}_{i+k})}{g(n_{i-K+k}, n_{i-K+1+k}, \ldots, n_{i+K})}, \end{aligned} \tag{3.5}$$

where $\bar{n}_j = n_j - \delta_{ji}$. A simple way to prove this is to use the *pair-wise balance condition* defined in Eq. (2.13) which has already been proved handy to give the steady state of ZRP. For any configuration $C = \{\ldots, n_{i-1}, n_i, n_{i+1}, ..\}$, a particle hopping from site i can be balanced by taking $C' = \{\ldots, n_{i-1}+1, n_i - 1, n_{i+1}, ..\}$ with hopping from $i-1$. Equation (3.5) is important as it says that any desired cluster-factorized state can be obtained in FRP by a choosing a suitable K-range hop rate $u(n_{i-K}, \ldots, n_i, \ldots, n_{i+K})$. But, also note that, in contrast to ZRP where any given rate leads to an FSS shown in (2.16), the CFSS in FRP can not be achieved for any functional form of $u(.)$, rather only when the rate satisfies the condition (3.5).

In the rest of this chapter, we discuss various features of the cluster-factorized steady state and their applications.

3.3 2-clusters : Pair factorized steady state (PFSS)

Let us start with $K = 1$, for which the steady state is factorized as product of 2-site clusters, commonly known as the pair-factorized steady state (PFSS). In this case, particles hop from a site i to $i+1$ with rate $u(n_{i-1}, n_i, n_{i+1})$ that depends on the occupation of departure site and its neighbors. To have a pair-factorized steady state of the form

$$P(\{n_i\}) = \frac{1}{Z_{L,N}} \prod_{i=1}^{L} g(n_i, n_{i+1}) \delta(\sum_i n_i - N) \tag{3.6}$$

with a canonical partition function

$$Z_{L,N} = \sum_{\{n_i\}} \prod_{i=1}^{L} g(n_i, n_{i+1}) \delta\left(\sum_i n_i - N\right),$$

the hop rate must satisfy Eq. (3.5) with $K = 1$,

$$u(n_{i-1}, n_i, n_{i+1}) = \frac{g(n_{i-1}, n_i - 1)}{g(n_{i-1}, n_i)} \frac{g(n_i - 1, n_{i+1})}{g(n_i, n_{i+1})}. \tag{3.7}$$

Unlike the FSS, the PFSS inherently generates spatial correlations and, like the FSS, it can lead to real-space condensation for certain hop rate [72]. This study has been later generalized on arbitrary graphs [180], open boundaries [151] and for studying mass transport processes and condensation transition therein for discrete (particle) as well as continuous mass [182], etc. None of these studies, however, attempted to calculate the spatial correlations in these systems. In fact, the presence of spatial correlations can change the nature of transitions by creating spatially extended condensates with or without tunable shapes [64, 181].

3.3.1 Observables

To calculate spatial correlation functions and other observables of interest, we use the transfer matrix formulation which is possible for a large class of systems having a CFSS. For the purpose of illustration we mainly discuss this approach elaborately for the PFSS. Since the PFSS with any arbitrary cluster-weight function $g(n_i, n_{i+1})$ can be obtained from a suitable hop rate $u(n_{i-1}, n_i, n_{i+1})$ (as in Eq. (3.7)), we rather focus on the functional form of $g(n_i, n_{i+1})$, not on the hop rate. In fact, any arbitrary function $g(n_i, n_{i+1})$ is an element of the infinite dimensional matrix

$$G = \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} g(n, n') |n\rangle\langle n'| \tag{3.8}$$

where $\{|n\rangle\}$ are the standard infinite dimensional basis vectors which satisfy a completeness relation $\langle n|n'\rangle = \delta_{nn'}$. Then, in the grand canonical ensemble (GCE), where a fugacity z controls the density ρ, the partition sum can be written as

$$\mathcal{Z}_L(z) = \sum_{N=0}^{\infty} Z_{L,N} z^N = Tr[T^L] \tag{3.9}$$

where the transfer matrix T has element $\langle n|T|n'\rangle = z^{(n+n')/2} g(n, n')$. In the thermodynamic limit $Z_L(z) \simeq \lambda_{max}^L$ (when λ_{max}, the largest eigenvalue (in absolute value) of T is real and non-degenerate). Once we know the grand partition sum, we can calculate various observables; for example, all the moments for occupation number n at a site,

$$\langle n^k \rangle = \frac{1}{L} \frac{1}{\mathcal{Z}_L(z)} \left(z \frac{d}{dz} \right)^k \mathcal{Z}_L(z). \tag{3.10}$$

For $k = 1$, we get density of the system $\rho = \langle n \rangle = \frac{1}{L} \frac{d}{dz} \ln \mathcal{Z}_L(z)$; by inverting this density-fugacity relation, one can express other observables as a function of ρ.

This matrix formulation is quite general and works for any form of weight function $g(n_i, n_{i+1})$; however managing infinite dimensional matrices is a challenging task. In the following, we show that, for a large class of weight functions, one can have a finite dimensional representation of T which, in some cases, can even be extended to $K > 1$.

3.3.2 Finite dimensional transfer matrices: analytical calculation of n-point correlations

Let us consider a weight function which has the following form

$$g(n_i, n_i + 1) = \sum_{\kappa=0}^{K} a_\kappa(n_i) b_\kappa(n_{i+1}), \tag{3.11}$$

where $a_\kappa(n), b_\kappa(n)$ are arbitrary functions, not necessarily analytic. It is evident that $g(n_i, n_{i+1})$ can be written as an inner product of two $(K + 1)$-dimensional vectors,

$$g(n_i, n_{i+1}) = \langle \alpha(n_i) | \beta(n_{i+1}) \rangle, \tag{3.12}$$

where

$$\begin{aligned} \langle \alpha(n) | &= (a_0(n), a_1(n), \ldots, a_K(n)) \\ \langle \beta(n) | &= (b_0(n), b_1(n), \ldots, b_K(n)). \end{aligned} \tag{3.13}$$

Then the partition sum in grand canonical ensemble is $\mathcal{Z}_L(z) = Tr[T(z)^L]$ with

$$T(z) = \sum_{n=0}^{\infty} z^n |\beta(n)\rangle\langle\alpha(n)| \tag{3.14}$$

a $(K+1)$-dimensional matrix. Now the partition sum and the stationary correlation functions can be calculated easily.

To illustrate this, let us consider a simple example by setting $K = 1$, $b_0(.) = 1 = a_1(.)$, and renaming functions $a_0(.), b_1(.)$ as $f_0(.), f_1(.)$ respectively. The weight function is now,

$$g(n_i, n_i + 1) = f_0(n_i) + f_1(n_{i+1}). \tag{3.15}$$

which we refer to as *sum-form.* This particular choice, *i.e.,* a pair-factorized steady state with a weight function in sum-form, does not lead to condensation transition, which we discuss later in section 3.6.1.4. Also, in section 3.6.1, we consider a general case of Eq. (3.11), which gives condensation transition, and we develop a possible criterion for the transition.

For any functional form of $f_0(n)$ and $f_1(n)$ we always have an infinite dimensional representation given by Eq. (3.8). However, interestingly in this case, we can do away with the infinite dimensional representation and get a simple 2-dimensional representation by taking,

$$\langle\alpha(n)| = (f_0(n), 1) \;\text{ and }\; \langle\beta(n)| = (1, f_1(n)). \tag{3.16}$$

The partition sum in GCE is then $\mathcal{Z} = Tr[T(z)^L]$, where

$$T(z) = \sum_{n=0}^{\infty} z^n C(n);\; C(n) = \begin{pmatrix} f_0(n) & 1 \\ f_0(n)f_1(n) & f_1(n) \end{pmatrix}. \tag{3.17}$$

To see how the spatial correlation functions can be obtained, let us take a specific form of the functions $f_0(.)$ and $f_1(.)$,

$$g(n_i, n_{i+1}) = \frac{\bar{q}}{(n_i+1)^\nu} + \frac{q}{(n_{i+1}+1)^\nu}, \tag{3.18}$$

where parameters $\nu > 0$ and $0 \leq q \leq 1$ tune the hop rate of particles and $\bar{q} = 1-q$, corresponding to $f_0(n)/\bar{q} = f_1(n)/q = (n+1)^{-\nu}$. In this case, the desired hop rate, for which the PFSS with weight-function as in Eq. 3.18 is realized, is given by

$$u(n_{i-1}, n_i, n_{i+1}) = \left(1 + \frac{1}{n_i}\right)^{2\nu} \left[\frac{\bar{q}n_i^\nu + q(n_{i-1}+1)^\nu}{\bar{q}(n_i+1)^\nu + q(n_{i-1}+1)^\nu}\right] \times \left[\frac{\bar{q}(n_{i+1}-1)^\nu + q(n_i)^\nu}{\bar{q}(n_{i+1}+1)^\nu + q(n_i+1)^\nu}\right].$$

In the extreme limits $q = 0$ and $q = 1$, the model reduces to zero range process (details will be discussed in section 3.6).

The transfer matrix , following Eq. (3.17), becomes

$$T(z) = \frac{1}{z}\begin{pmatrix} \bar{q}Li_\nu(z) & q\bar{q}\frac{z}{1-z} \\ Li_{2\nu}(z) & qLi_\nu(z) \end{pmatrix} \tag{3.19}$$

where $Li_\nu(z)$ are the Polylog functions. The eigenvalues of T are

$$\lambda_\pm = \frac{Li_\nu(z)}{2z}\left(1 \pm \sqrt{(q-\bar{q})^2 - \frac{4q\bar{q}Li_{2\nu}(z)}{(1-z)Li_\nu(z)^2}}\right). \tag{3.20}$$

The partition function $\mathcal{Z}_L(z) = \lambda_+^L + \lambda_-^L$ in the thermodynamic limit $(L \to \infty)$ becomes $\mathcal{Z}_L(z) \simeq \lambda_+^L$ and thus the density

$$\rho(z) = z\frac{d}{dz}\ln\lambda_+. \tag{3.21}$$

Throughout the discussion, we calculate observables only in the thermodynamic limit. Let us consider

$$q = \frac{1}{2},\ \nu = -1 \tag{3.22}$$

(results for different q and ν are discussed in section 3.6); here $\lambda_\pm = \frac{1}{2}(1 \pm \sqrt{1+z})/(1-z)^2$ and the density is

$$\rho = \frac{2}{1-z} - \frac{1}{2\sqrt{1+z}} - \frac{3}{2}. \tag{3.23}$$

Now we proceed to calculate the correlation functions, first the two-point correlation function and later the higher order. The two point correlation function is defined by

$$C(r) = \langle n_i n_{i+r} \rangle - \langle n_i \rangle \langle n_{i+r} \rangle. \tag{3.24}$$

For $r > 0$ we have

$$C(r) = \frac{Tr[T' T^{r-1} T' T^{L-r-1}]}{Tr[T^L]} - \rho^2 \tag{3.25}$$

where $T' = dT/d(\ln z)$. For Eq. (3.22), we get

$$C(r) = \rho^2 \frac{z(3+z)^2}{4(1+z)(1-z)^2} e^{-r/\xi} \tag{3.26}$$

with $\xi^{-1} = |\ln \frac{\lambda_-}{\lambda_+}| = |\ln \frac{1-\sqrt{1+z}}{1+\sqrt{1-z}}|$ being the inverse correlation length. The correlation function for $r = 0$ is nothing but the variance $\sigma^2(\rho)$ of single-site occupation variable n_i, i.e.,

$$C(0) \equiv \sigma^2(\rho) = \langle n_i^2 \rangle - \langle n_i \rangle^2 = \frac{Tr[T'' T^{L-1}]}{Tr[T^L]} - \rho^2 \tag{3.27}$$

where $T'' = d^2T/d(\ln z)^2$ and, again for (3.22),

$$C(0) = \frac{z}{4(1-z)^2} \left[\frac{z^2 + 14z + 17}{(1+z)} - \frac{8}{\sqrt{1+z}} \right]. \tag{3.28}$$

Now, we turn our attention to higher order correlation functions. The 3-point correlation function, for example, is defined as

$$\mathcal{C}(r_1, r_2) = \langle n_i n_{i+r_1} n_{i+r_1+r_2} \rangle - \langle n_i \rangle \langle n_{i+r_1} \rangle \langle n_{i+r_1+r_2} \rangle$$

which, in terms of transfer matrix, can be evaluated from the expression

$$\mathcal{C}(r_1, r_2) = \frac{Tr[T' T^{r_1-1} T' T^{r_2-1} T' T^{L-r-1}]}{Tr[T^L]} - \rho^3. \tag{3.29}$$

We find that the three-point correlation function can be written in terms of the two-point correlation functions as

$$\mathcal{C}(r_1, r_2) = \rho\,[C(r_1) + C(r_2) - B(z)C(r_1)C(r_2)] \tag{3.30}$$

where $B(z)$ also depend on the parameters q and ν; for Eq. (3.22), we get $B(z) = 1 + 8z(1+z)/(3+z)^2$. In a similar way, one can calculate all the higher order correlation functions exactly.

To conclude, when the weight function $g(n_i, n_{i+1})$ is a sum of two functions as in Eq. (3.15), the correlation length $\xi = |\ln \frac{\lambda_-}{\lambda_+}|$ remains finite at any density as $\lambda_- < \lambda_+$ for any choice of q and ν.

3.4 Generalizations

3.4.1 3-clusters and general (K+1)-clusters

In this section we consider some specific models of FRP with $K > 1$ which give rise to $(K+1)$-cluster-factorized steady state. Corresponding partition function in the grand canonical ensemble would require contraction of a tensor product which is usually a hard task [155]. Our aim here would be to obtain, if possible, a matrix formulation that can accommodate some cluster- factorized steady states for any $K > 1$. For $K = 2$ we have a 3-site cluster factorized steady state,

$$P(\{n_i\}) = \prod_{i=1}^{L} g(n_i, n_{i+1}, n_{i+2})\delta\left(\sum_{i=1}^{L} n_i - N\right).$$

For illustration we consider a cluster-weight function,

$$g(n_i, n_{i+1}, \ldots n_{i+K}) = \sum_{\kappa=0}^{K} f_\kappa(n_{i+\kappa}) \tag{3.31}$$

which is a simple generalization of the sum-form given in Eq. (3.15). We will now show that a grand partition function of a finite range process which has a $(K+1)$-cluster-factorized steady state with a weight function given by Eq. (3.31) can be written as $\mathcal{Z}_L(z) = Tr[T^L]$ where z is the fugacity and T is a 2^K-dimensional transfer matrix. Since we intend to obtain the transfer matrix iteratively, let us rewrite the transfer matrix given by Eq. (3.17) for $K = 1$ in a convenient form,

$$T_1(z) = \sum_{n=0}^{\infty} z^n \mathbf{F}_1(n);\ \mathbf{F}_1(n) = \begin{pmatrix} f_0(n) & 1 \\ f_0(n)f_1(n) & f_1(n) \end{pmatrix}. \tag{3.32}$$

In a similar way, we extend to $K > 1$ and write $T_K = \sum_{n=0}^{\infty} z^n \mathbf{F}_K(n)$ where the 2^K- dimensional matrix can be written as

$$\mathbf{F}_K = \begin{pmatrix} \mathbf{F}_{K-1} & A_{K-1}\mathbf{F}_{K-1} \\ f_K\mathbf{F}_{K-1} & f_K A_{K-1}\mathbf{F}_{K-1} \end{pmatrix}, \tag{3.33}$$

using a constant matrix

$$A_K = \begin{pmatrix} 0 & 0 \\ I_{2^{K-1}} & 0 \end{pmatrix},$$

where $I_{2^{K-1}}$ is a 2^{K-1}-dimensional identity matrix. For $K = 0$, we take $A_0 = 1$. Since $K = 0$ corresponds to the ZRP which has a factorized steady state, we have $\mathbf{F}_0(n) = f_0(n)$, which is a scalar. Clearly $\mathbf{F}_1$ in Eq. (3.32) satisfy Eq. (3.33). A little more algebra would show that the transfer matrix for $K = 2$ is

$$T_2 = \sum_n z^n \begin{pmatrix} \mathbf{F}_1 & A_1\mathbf{F}_1 \\ f_2\mathbf{F}_1 & f_2A_1\mathbf{F}_1 \end{pmatrix} = \sum_n z^n \mathbf{F}_2(n). \tag{3.34}$$

From the transfer matrix, one can, in principle, calculate the expectation value of any desired observable. We will not discuss further the finite range process $K > 1$; the finite dimensional transfer matrix is expected to generate spatial correlations which was absent in the ZRP. We discuss some of the models in details which undergo condensation transitions (see section 3.6).

3.4.2 Continuous mass model

Until now, we have studied CFSS on a one dimensional lattice with each site having a discrete variable, called the occupation variables or number of particles. The model and the matrix formulation can be extended, without any particular difficulty, to systems with continuous mass m. As an example, let us consider

$$g(m_i, m_{i+1}, m_{i+2}) = m_i + m_{i+1} + m_{i+2}. \tag{3.35}$$

A 3-cluster-factorized steady state with above weight-function can be obtained when ϵ amount of mass is transferred from site i to $i+1$ with rate

$$\begin{aligned} &u(m_{i-2}, m_{i-1}, m_i, m_{i+1}, m_{i+2}) \\ &\quad = \prod_{k=0}^{2} \left[1 - g(m_{i-2+k}, m_{i-1+k}, m_{i+k})^{-1}\right]. \end{aligned} \tag{3.36}$$

For small ϵ, the model is equivalent to a discrete model where mass is measured in units of ϵ. In fact, the residual mass (actual mass modulo ϵ) at any site does not change during evolution. The residual masses, each being smaller than a pre-defined value ϵ which can be made arbitrary small, does not contribute to the asymptotic form of the hop rate. Thus we would obtain a transfer matrix T_2 discussed in the previous section, with $f_{0,1,2}(m) = m$, but the sum $\sum_m$ will now be replaced by an integral $\int dm$. Defining, a chemical potential μ (where $z = e^{\mu}$), we get the transfer matrix, as in Eq. (3.34),

$$T(\mu) = \frac{1}{\mu^2} \begin{pmatrix} 1 & \mu & 0 & 0 \\ \frac{2}{\mu} & 1 & 1 & \mu \\ \frac{2}{\mu} & 1 & 0 & 0 \\ \frac{6}{\mu^2} & \frac{2}{\mu} & \frac{2}{\mu} & 1 \end{pmatrix}. \tag{3.37}$$

This matrix has eigenvalues $\frac{1}{\mu^2}\{\lambda, \lambda_1 e^{\pm i\theta}, \lambda_2\}$, where λ (the largest eigenvalue), λ_1, θ and λ_2 are independent of μ, and their approximate numerical values are

$\lambda \approx 3.86841$, $\lambda_1 \approx 1.10465$, $\theta \approx 1.87254$ and $\lambda_2 \approx -0.21184$. In the thermodynamic limit, the partition function is $\mathcal{Z}_L = (\lambda/\mu^2)^L$, and density $\rho = -2/\mu$. The two-point correlation function for $r > 0$ is

$$\begin{aligned} C(r) &= \langle m_i m_{i+r} \rangle - \rho^2 \\ &= \rho^2 \left[c_2 \left(\frac{\lambda_2}{\lambda} \right)^r + 2c_1 \left(\frac{\lambda_1}{\lambda} \right)^r \cos(r\theta + \alpha) \right] \end{aligned} \tag{3.38}$$

where, $c_1 = 0.3380$, $c_2 = -0.0375$ and $\alpha = 0.1804$. And, for $r = 0$, the correlation (actually the variance of the mass distribution) is $C(0) = \langle m^2 \rangle - \rho^2 = 0.6704\rho^2$.

3.5 Impossibility of FSS in general for $K > 0$

In this section, we provide an argument that FRP can have a factorized steady state only for $K = 0$ (namely the ZRP) and for some specific misanthrope process (special cases of $K = 1$). For any $K > 1$, however, one cannot have a factorized steady state in general. First we consider $K = 1$ and show that, in this case, the hop rate reduces to those in the ZRP or in the misanthrope process, when one demands a factorized steady state. One can construct a general proof in a similar way, that condition of FSS would reduce the hop rate of FRP with $K > 1$ to the ZRP or the misanthrope process. Also, thereafter we provide a proof of the above for the hop rates which can be written in a product form.

3.5.1 $K = 1$: FSS not possible, only except misanthrope process

In this section, we show that, for $K = 1$, one cannot have a factorized steady state for the general hop rate $u(n_{i-1}, n_i, n_{i+1})$. The Master equation for FRP for general

$K > 0$ is

$$\frac{d}{dt}P(\{n_i\}) = \sum_{i=1}^{L} F(n_{i-K}, \ldots, n_i, \ldots, n_{i+K}), \tag{3.39}$$

where

$$\begin{aligned} F(n_{i-K}, .., n_i, .., n_{i+K}) &= u(n_{i-K}, .., n_i, .., n_{i+K})P(\{n_i\}) \\ &- u(n_{i-K}, .., n_i + 1, n_{i+1} - 1, .., n_{i+K}) \\ &\times P(.., n_i + 1, n_{i+1} - 1, ..). \end{aligned} \tag{3.40}$$

In the steady state, right hand side of Eq. (3.39) must vanish, which can happen if

$$\begin{aligned} F(n_{i-K}, \ldots, n_i, \ldots, n_{i+K}) &= h(n_{i-K}, \ldots, n_i, \ldots, n_{i+K-1}) \\ &- h(n_{i-K+1}, \ldots, n_i, \ldots, n_{i+K}) \end{aligned} \tag{3.41}$$

for some arbitrary function h of $2K$ arguments. Note, that the above cancellation scheme is only a sufficient condition.

Now let us consider $K = 1$, and demand that the steady state has a factorized form given by Eq. (3.2). Then

$$\begin{aligned} u(n_{i-1}, n_i + 1, n_{i+1} - 1)\frac{f(n_i + 1)}{f(n_i)}\frac{f(n_{i+1} - 1)}{f(n_{i+1})} \\ -u(n_{i-1}, n_i, n_{i+1}) = h(n_i, n_{i+1}) - h(n_{i-1}, n_i) \end{aligned} \tag{3.42}$$

where h is an arbitrary function, yet to be determined. Since the hop rate $u(n_{i-1}, n_i, n_{i+1}) = 0$ when $n_i = 0$ and we must have a boundary condition $f(m < 0) = 0$, we can use specific values of n_is in Eq. (3.42) to find recursion relation for h. For $n_i = 0 = n_{i+1}$ equation (3.42) in $h(n_{i-1}, 0) = h(0, 0)$. Again putting $n_{i+1} = 0 = n_{i-1}$ we get $h(0, n_i) - h(0, 0) = u(0, n_i, 0)$.

These two conditions leaves Eq. (3.42) for $n_i = 0$ as

$$u(n_{i-1}, 1, n_{i+1} - 1)\frac{f(1)}{f(0)}\frac{f(n_{i+1} - 1)}{f(n_{i+1})} = u(0, n_{i+1}, 0).$$

Clearly, in order to be consistent, $u(n_{i-1}, 1, n_{i+1})$ must be independent of n_{i-1}. For convenience, without any loss of generality, lets set $u(n_{i-1}, 1, n_{i+1}) = u(0, 1, n_{i+1})$. Thus, to have the factorized steady state for $K = 1$, the hop rate $u(n_{i-1}, n_i, n_{i+1})$ must satisfy

$$\begin{aligned} u(n_{i-1}, n_i + 1, n_{i+1} - 1)&\frac{u(0, 1, n_i)}{u(0, n_i + 1, 0)}\frac{u(0, n_{i+1}, 0)}{u(0, 1, n_{i+1} - 1)} \\ -u(n_{i-1}, n_i, n_{i+1}) &= u(n_i, n_{i+1}, 0) - u(n_{i-1}, n_i, 0). \end{aligned}$$

Now if we take $n_i = 1$ and use $u(n_{i-1}, 1, n_{i+1}) = u(0, 1, n_{i+1})$ in the above equation to rearrange the terms, we have

$$\begin{aligned} u(n_{i-1}, 2, n_{i+1} - 1)&\frac{u(0, 1, 1)}{u(0, 2, 0)}\frac{u(0, n_{i+1}, 0)}{u(0, 1, n_{i+1} - 1)} \\ -u(0, 1, n_{i+1}) &= u(1, n_{i+1}, 0) - u(0, 1, 0), \end{aligned}$$

which implies that $u(n_{i-1}, 2, n_{i+1})$ must be independent of n_{i-1}. A similar recursion would result that $u(n_{i-1}, n_i, n_{i+1})$ must be independent of n_{i-1}. This again reflects the fact that a factorized steady state is possible for $K = 1$ only when hop rate is $u = u(n_i, n_{i+1})$, i.e., the process is a *misanthrope* process.

3.5.2 $K \geq 1$: FSS not possible for product form hop rates, only except misanthrope process

In this section, we show that the FRP, for any $K > 0$, cannot have a FSS when the hop rate has the following product form,

$$u(n_{i-K}, \ldots, n_i, \ldots, n_{i+K}) = \prod_{j=-K}^{K} v_j(n_{i+j}). \tag{3.43}$$

The Master equation along with a demand of a factorized steady state of the form (3.2), and then Eqs. (3.40) and (3.41) together, implies

$$v_{-K} \dots v_{-1} v_2 \dots v_K G(n_i, n_{i+1}) = 0 \tag{3.44}$$

where $v_k \equiv v_k(n_{i+k})$ and

$$\begin{aligned} G(n_i, n_{i+1}) = & - \quad v_0(n_i+1) v_1(n_{i+1}) \frac{f(n_i+1) f(n_{i+1}-1)}{f(n_i) f(n_{i+1})} \\ & + \quad v_0(n_i) v_1(n_{i+1}). \end{aligned}$$

Now differentiating both sides of Eq. (3.44) with respect to n_{i-K} and n_{i+K}, we have

$$\frac{\partial v_{-K}}{\partial n_{i-K}} \frac{\partial v_K}{\partial n_{i+K}} v_{-K+1} \dots v_{-1} v_2 \dots v_{K-1} G(n_i, n_{i+1}) = 0.$$

This implies that, either $v_{-K}(n_{i-K})$ or $v_K(n_{i+K})$ must be a constant, because the other solution $f(n) = 1/v_0(n) = 1/v_1(n)$ cannot be accepted as it means $v_1(0) = v_0(0) = 0$, i.e., a particle cannot be transferred to a vacant neighboring site. So, let us proceed with v_{-K} =constant (say 1). Then Eq. (3.41) reads as

$$\begin{aligned} v_{1-K}..v_{-1} v_2..v_K G(n_i, n_{i+1}) = & \, h(n_{i-K}, .., n_i, .., n_{i+K-1}) \\ & - h(n_{i-K+1}, .., n_i, .., n_{i+K}). \end{aligned}$$

Clearly for this equation to be valid its right hand side must be independent of n_{1-K} and that in turn leads to $h(x_1, x_2, \dots, x_k) = h(x_2, \dots, x_k)$.

This way we can eliminate one variable at each step until finally we reach to

$$\begin{aligned} v(n_{i-K}, \dots, n_i, \dots, n_{i+K}) = v(n_i, n_{i+1}) = v_0(n_i) v_1(n_{i+1}), \\ \text{and } v_0(n_i+1) v_1(n_{i+1}-1) \frac{f(n_i+1)}{f(n_i)} \frac{f(n_{i+1}-1)}{f(n_{i+1})} \\ -v_0(n_i) v_1(n_{i+1}) = h(n_{i+1}) - h(n_i). \end{aligned}$$

This is exactly the criterion for having a factorized steady state in misanthrope process with a hop rate that has a product form $u(n_i, n_{i+1}) = v_0(n_i)v_1(n_{i+1})$ [77].

3.6 Applications

3.6.1 Condensation transition

One important feature in these simple one dimensional models is that they can exhibit condensation transition at a finite density when one or more parameters in the rate functions are tuned. To demonstrate the possibility of a condensation transition in the CFSS, for any K, we consider the weight of $(K+1)$-cluster to be,

$$g(n_i, n_{i+1}, ..., n_{i+K}) = \frac{\left[q + \sum_{j=0}^{K} n_{i+j}\right]^\gamma}{(n_i+1)^\nu}, \tag{3.45}$$

where γ, ν and q are positive and γ is an integer. This steady state weight can be generated from a hop rate given by Eq. (3.5),

$$u = \left(1 + \frac{1}{n_i}\right)^\nu \left[\prod_{k=0}^{K}\left(1 - \frac{1}{q + \sum_{j=0}^{K} n_{i-j+k}}\right)\right]^\gamma. \tag{3.46}$$

3.6.1.1 Case with $K = 1$ (PFSS)

We first consider $K = 1$. It is easy to see that for any γ, the weight function Eq. (3.45) can be expressed as Eq. (3.11) with suitable choice of $a_\kappa(n)$ and $b_\kappa(n)$ where κ varies from 0 to γ, leading to a $(\gamma+1)$ dimensional transition matrix. We further set the parameters $\gamma = 1 = q$; this gives rise to a PFSS, as in Eq. (3.6), with $g(n_i, , n_{i+1}) = (n_i + n_{i+1} + 1)/(n_i + 1)^\nu$, which can be realized when a particle hops out from a site i (to the right neighbor), having $n_i > 0$ particles, with the following rate

$$u = \left(1 + \frac{1}{n_i}\right)^\nu \frac{n_i + n_{i-1}}{1 + n_i + n_{i-1}} \frac{n_i + n_{i+1}}{1 + n_i + n_{i+1}}. \tag{3.47}$$

For this case, we can obtain exact results following the transfer matrix formulation developed here. First we write $g(m,n) = \langle \alpha(m)|\beta(n)\rangle$ where $\langle\alpha(m)| = ((m+1)^{-\nu}, (m+1)^{1-\nu})$, $\langle\beta(n)| = (n,1)$. Thus the grand partition function can be written as $Z(z) = Tr(T^L)$ with

$$T = \sum_{n=0}^{\infty} |\beta(n)\rangle\langle\alpha(n)| z^n = \frac{1}{z}\begin{pmatrix} Li_{\nu-1}(z) & Li_{\nu-2}(z) \\ Li_{\nu}(z) & Li_{\nu-1}(z) \end{pmatrix}$$

The eigenvalues of T are

$$\lambda_{\pm}(z) = \frac{1}{z}\left(Li_{\nu-1}(z) \pm \sqrt{Li_{\nu}(z)Li_{\nu-2}(z)}\right),$$

which leads to the density-fugacity relation $\rho(z) = z\lambda'_{+}(z)/\lambda_{+}(z)$ and the critical density $\rho_c = \lim_{z\to 1}\rho(z)$. It turns out that for $\nu \leq 4$, ρ_c diverges $-$ indicating a fluid phase for any density. For $\nu > 4$ we get,

$$\rho_c = \frac{\xi_1(\nu-1) - 2\xi_2(\nu) + \xi_3(\nu)}{2\xi_2(\nu) + 2\zeta(\nu-1)\sqrt{\xi_2(\nu)}} + \frac{\zeta(\nu-2) - \zeta(\nu-1)}{\sqrt{\xi_2(\nu)} + \zeta(\nu-1)}$$

where $\xi_k(\nu) = \zeta(\nu)\zeta(\nu-k)$ and $\zeta(\nu)$ are Riemann zeta functions. Thus, for $\nu > 4$ we have a condensate when density exceeds this critical value. Unlike the ZRP, where particles at different sites are not correlated, here we have non-vanishing correlation that extends up to a length scale $\xi(z) = |\ln\frac{\lambda_-(z)}{\lambda_+(z)}|^{-1}$ which is finite throughout.

3.6.1.2 Case with $K \geq 2$ (CFSS)

It is straightforward to extend the matrix formalism to $K > 1$ when $\gamma = 1$. First, let us take $\nu = 0$. In this case, the weight function g takes a sum-form given by Eq. (3.31), for which we have already constructed a general transfer-matrix. For $\nu > 0$, the dimension of the transfer matrix remains the same as in $\nu = 0$; it is only that each element of $\mathbf{F}_K$ in Eq. (3.34) will be multiplied by an extra factor

Table 3.1: Critical density ρ_c for $\gamma = 1$.

ν	$q = 1$ $K = 1$	$q = 1$ $K = 2$	$q = 2$ $K = 1$
5	0.3254	∞	0.1591
6	0.1054	0.2773	0.0544
7	0.0429	0.0981	0.0228

$(n_i + 1)^{-\nu}$. We omit the exact analytic expressions of the density-fugacity relation and the critical density - the calculations are straightforward but the expressions are very long. Only the numerical values of critical density are tabulated in Table 3.1 for different parameters.

3.6.1.3 Criterion for condensation transition

For the ZRP, it is well known that, provided the hop rate $u_0(n)$ has an asymptotic form

$$u_0(n) = 1 + \frac{b}{n^\sigma} + \dots \tag{3.48}$$

condensation occurs at a finite density, when $\sigma < 1$, or when $\sigma = 1$ but $b > 2$. This criterion can be extended to any other system (without any constraint on occupation number) when the steady state has a factorized form (3.2); one needs to consider an effective rate function $u_0(n) \equiv f(n-1)/f(n)$ and find its asymptotic form. This criterion determines whether a model can undergo a condensation transition and helps in understanding phase coexistence in hardcore lattice gas models [51, 112].

Such a criterion for cluster-factorized steady state would be very useful. At present, we do not have a general criterion, but the examples studied above suggest a sufficient condition for CFSS to have condensation. If the rate function can be expanded as

$$u(n_{i-K},..,n_{i+K}) = \sum_{\nu=0}^{\infty} \frac{B_\nu(n_{i-K},..,n_{i-1},n_{i+1}..,n_{i+K})}{n_i^\nu},$$

the condensation transition occurs for large particle density ρ (i.e when ρ becomes larger than some critical threshold density ρ_c) when both the conditions

$$\begin{aligned} &(i) \qquad \text{both } B_0 \text{ and } B_1 \text{ are constant} \\ &(ii) \qquad B_1/B_0 > 2 \end{aligned} \tag{3.49}$$

are satisfied. This is only a simple generalization of the criterion of condensation in the ZRP. Effectively, B_1/B_0 plays the role of b in Eq. (3.48). As the hop rate in Eq. (3.46) can be expanded as

$$u(\dots, n_{i-1}n_i, n_{i+1}\dots) = 1 + \frac{\nu - \gamma(K+1)}{n_i} + \mathcal{O}(\frac{1}{n_i^2}),$$

and thus $B_0 = 1$ and $B_2 = \nu - \gamma(K+1)$, the criterion correctly predicts the condensation which occurs only when $\nu > \gamma(K+1)+2$. This is same as the usual condensation criterion in the ZRP if we treat $b \equiv B_2/B_0$. In this particular case, we have also checked that moments $\langle n^k \rangle$ as a function of z, in leading order, are the same as that in the ZRP with corresponding b (see Eq. (3.48)). This criterion, however, cannot be applied to some of the following cases studied recently, such as, the misanthrope process [77] and the PFSS [72]. For the first case, B_0 and B_1 are not constants and, for the later case, hop rates are not analytic functions. A criterion of condensation, which can apply to a cluster-factorized steady state in general is desirable and remains a challenge.

We end this section with the following remark. The condensation transition here is different from that obtained for PFFS by Evans *et. al.* [72]. There, one

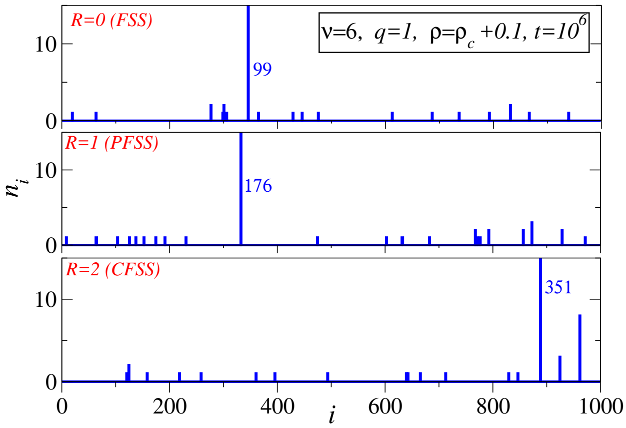

Figure 3.2: Particle distribution in FRP with weight function (3.45) after $t = 10^6$ MCS, starting from a random distribution of particles. Density is $\rho = \rho_c + 0.01$. The critical density for $K = 0$ is $\rho_c = 0.01925$; the same for $K > 0$ are taken from table 3.1. Clearly, for all cases, the condensate is localized to a single site. The condensate size is written beside the condensate site.

observes an extended condensate where both the size and the spatial extent of condensate scales with system size as $\sqrt{L}$. This indicates that the transition is associated with a diverging spatial correlation length. Whereas for the PFSS (and the CFSS) studied here, the correlation length remains finite throughout and the transition is characterized by a diverging mass fluctuation, as in the ZRP. The condensate is also localized to a single site (see Fig. 3.2).

3.6.1.4 No condensation for PFSS with weight function in sum-form

In this section, we first show that a pair factorized steady state with weight function $g(n_i, n_{i+1}) = f_0(n_i) + f_1(n_{i+1})$, which we refer to as *sum-form,* cannot give rise to condensation. Then, we demonstrate this considering a perturbation to a ZRP that converts the existing factorized steady state of the ZRP to a PFSS with weight function in the sum-form. For the PFSS with weight function in the sum-form, the transfer matrix $T(z)$ is given by Eq. (3.17).

The largest eigenvalue of the matrix $\lambda_+ = \frac{1}{2}(T_{11} + T_{22} + \sqrt{(T_{11} + T_{22})^2 - 4\mathcal{D}},$

where $\mathcal{D}$ is the determinant of T can be used in Eq. (3.21) to get the density $\rho(z)$. With some straightforward algebraic manipulations, one can show that the maximum density at $z = z_c = 1$ is,

$$\rho_c = \lim_{z\to 1} \rho(z) = \lim_{z\to 1} \frac{1}{2}\left[\frac{1}{T_{12}}\frac{dT_{12}}{dz} + \frac{1}{1-z}\right].$$

Clearly ρ_c diverges independent of the first term, leading to a conclusion that there can not be a condensation transition at any finite density. Thus, a PFSS *cannot* have condensation transition if the weight function has a sum-form. To illustrate this, we consider a simple zero range process with weight function $f(n) = 1/(n+1)^\nu$ or hop rate $u(n) = f(n-1)/f(n) = n^\nu/(n+1)^\nu$ and add a perturbative term get a new weight function

$$g(n_i, n_{i+1}) = (1-q)f(n_i) + qf(n_{i+1}) \tag{3.50}$$

which depends on occupation of two consecutive sites. Here $0 \le q \le 1, \bar{q} = 1-q$ and we choose $f(n) = 1/(n+1)^\nu$. A pair-factorized state, as in Eq. (3.6), with the above weight function occurs when particles hop rate is

$$u(n_{i-1}, n_i, n_{i+1}) = \frac{\bar{q}f(n_{i-1}) + qf(n_i - 1)}{\bar{q}f(n_{i-1}) + qf(n_i)} \times \frac{\bar{q}f(n_i - 1) + qf(n_{i+1})}{\bar{q}f(n_i) + qf(n_{i+1})}.$$

For both $q = 0$ and $q = 1$ we have a factorized steady state, as in Eq. (3.2), which corresponds to the ZRP with particle hop rate

$$u(n) = \frac{f(n-1)}{f(n)} = \left(1 + \frac{1}{n}\right)^\nu \simeq 1 + \frac{\nu}{n} + \mathcal{O}(\frac{1}{n^2}). \tag{3.51}$$

Thus we expect a condensation transition for $q = 0, 1$ when $\nu > 2$ and the density is larger than a critical value ρ_c. In this case the $\mathcal{Z}(z) = F(z)^L$ (the transfer matrix

$T(z)$ reduces to a scalar), where $F(z) = \sum_{n=0}^{\infty} h(n)z^n = Li_\nu(z)$. The density is $\rho = z\frac{d}{dz}F(z)$ and thus the critical density for $q = 0, 1$ is

$$\rho_c = \lim_{z\to 1} \rho(z) = \begin{cases} \infty & \text{for } \nu \leq 2, \\ \frac{\zeta(\nu-1)}{\zeta(\nu)} - 1 & \text{for } \nu > 2. \end{cases} \tag{3.52}$$

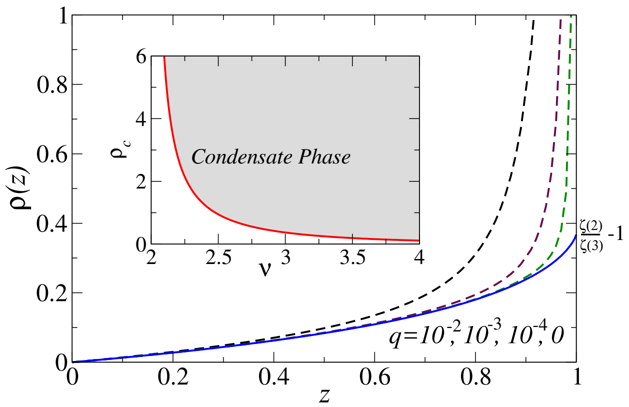

Figure 3.3: Small perturbation to the ZRP: For small $q = 10^{-2}, 10^{-3}$ or 10^{-4}, density $\rho(z)$ diverges when $z \to 1$. However for $q = 0$ or for $q = 1$, $\rho_c = \rho(1) = \frac{\zeta(\nu-1)}{\zeta(\nu)} - 1$ is finite, leading to a condensation transition when $\rho > \rho_c$. Inset shows the phase diagram for $\nu = 3$.

The phase-diagram of the condensation transition in the ρ-ν plane is shown in Fig. 3.3. The critical line ρ_c separates the condensate phase from the fluid phase. For a general $0 < q < 1$, we need to calculate the density using Eqs. (3.19), (3.20), and (3.21),

$$\rho(z) = \frac{1}{a(a-(1-z)Li_\nu(z))}[Li_{\nu-1}(z)(\bar{q}^2(1-z)^2 Li_\nu(z) - (1-z)a) + 2\bar{q}qz(Li_{2\nu}(z) + (1-z)Li_{2\nu-1}(z))] - 1$$

where

$$a(q,z) = \sqrt{(q-\bar{q})^2(1-z)^2 Li_\nu^2(z) + 4q\bar{q}z(1-z)Li_{2\nu}(z)}.$$

In the limit $z \to 1$, $\rho(z)$ diverges for all $\nu > 0$ and thus the condensation transition is destroyed. It is somewhat surprising why for any non-zero q however small, the condensation transition is destroyed. It seems that this perturbation, which takes the factorized steady state of the ZRP to a pair-factorized steady state, forces the condensation to disappear. One could understand this following the criterion (3.49). For $\nu = 3$, the rate for general q has an asymptotic form (i.e., when $n_i \to \infty$))

$$u(n_{i-1}, n_i, n_{i+1}) = 1 + 3\frac{q^2(1+n_{i-1})^3 + \bar{q}^2(1+n_{i+1})^3}{q\bar{q}n_i^4}.$$

Thus, here $B_1/B_0 = 0$ and therefore we should not expect condensation for this hop rate. It can be shown easily that for any $\nu \geq 1$ the asymptotic form of the hop rate does not satisfy condition (ii) of ansatz (3.49) ruling out the possibility of a condensation transition.

3.6.2 Subsystem mass distribution

It was argued in recent works [24, 39, 40] that, for systems with short-ranged interaction, irrespective of whether they are in equilibrium or not, one could obtain a state function which plays the role of a free energy function. It was shown in [39, 40] that the steady state distribution $P_v(m)$ of mass in a subsystems of volume $v \gg \xi$ can be determined from the functional dependence of the scaled variance $\sigma^2(\rho) = (\langle m^2\rangle - \langle m\rangle^2)/v$, in the limit of large v, on the mass density ρ. When $\sigma^2(\rho) \propto \rho^2$ is a quadratic function of density ρ, the subsystem mass distribution can be characterized through a gamma distribution, i.e., $P_v(m) \propto m^{\eta-1}\exp(\mu m)$, where $\mu = -\eta/\rho$ is an equilibrium-like chemical potential. The exponent η however depends on the details of the model and it can be calculated from the knowledge of two-point correlation function only. The matrix formulation developed here for the CFSS can thus help in determination of η.

To illustrate this, let us consider a continuous finite range process with $K = 1$, and calculate explicitly the variance of the subsystem mass. Consider the following homogeneous weight function for a pair-factorized steady state,

$$g(m_i, m_{i+1}) = m_i^{\delta} + c\ m_i^{\gamma} m_{i+1}^{\delta-\gamma} \tag{3.53}$$

The grand partition sum is $\mathcal{Z} = Tr[T^L]$ where the transfer matrix $T(\mu)$ ($\mu = \ln(z)$ is the corresponding chemical potential) is given below

$$T(\mu) = \frac{1}{\mu^{1+\delta}} \begin{pmatrix} \Gamma(\delta+1) & c\frac{\Gamma(\gamma+1)}{\mu^{\gamma-\delta}} \\ \frac{\Gamma(2\delta-\gamma+1)}{\mu^{\delta-\gamma}} & c\Gamma(\delta+1) \end{pmatrix}, \tag{3.54}$$

where $\Gamma(.)$ are Gamma functions. Eigenvalues of $T(\mu)$ are $\lambda_{\pm} = \Lambda_{\pm}(\delta,\gamma,c)/\mu^{1+\delta}$ where

$$\begin{aligned} 2\Lambda_{\pm}(\delta,\gamma,c) = &(1+c)\Gamma(\delta+1) \\ &\pm \sqrt{(\delta+1)^2(1-c)^2 + 4c\Gamma(2\delta-\gamma+1)\Gamma(\gamma+1)} \end{aligned}$$

and the particle density is

$$\rho = \frac{\partial}{\partial\mu}\ln\lambda_+ = -\frac{\delta+1}{\mu}, \tag{3.55}$$

implying a fluctuation-response (FR) relation

$$\frac{d\rho}{d\mu} = \sigma^2(\rho), \tag{3.56}$$

analogous to the fluctuation-dissipation theorem in equilibrium. Now, as shown below, one can check the above FR relation by explicitly calculating both sides of Eq. (3.56). The r.h.s of Eq. (3.56) can be calculated by integrating two-point correlation function $\sigma^2(\rho) = \sum_{r=-\infty}^{r=\infty} C(r)$, using Eq. (3.25),

$$C(r) = \langle n_i n_{i+r}\rangle - \rho^2 = \rho^2\left[A(r) - 1\right] \tag{3.57}$$

where, for $r > 0$,

$$A(r) = 1 + \left(\frac{\Lambda_-}{\Lambda_+}\right)^r \frac{(\delta-\gamma)^2/(\delta-1)^2}{1 - \frac{\Gamma(\delta+1)^2}{\Gamma(2\delta-\gamma+1)\Gamma(\gamma+1)}}$$

and

$$A(0) = \frac{\delta+2}{\delta+1} + \frac{2c}{\Lambda_+}\frac{(\delta-\gamma)^2}{(\delta+1)^2}\frac{\Gamma(2\delta-\gamma+1)\Gamma(\gamma+1)}{2\Lambda_+ - (1+c)\Gamma(\delta+1)}.$$

In this system, the *gap* $(\lambda_+ - \lambda_-)$ between the two eigenvalues is nonzero and the correlation length $\xi = |\ln \frac{\Lambda_-}{\Lambda_+}|^{-1}$ is finite. Therefore, following Ref. [40], the subsystem mass distribution $P_v(m)$, for $v \gg \xi$, is a gamma distribution where the exponent η can be written, using Eq. (3.21), as

$$\eta^{-1} = \sum_{r=-\infty}^{\infty} (A(r) - 1), = \frac{1}{\delta+1}. \tag{3.58}$$

Note that the exponent η depends only on the homogeneity exponent δ but neither on γ nor on c. The l.h.s, the compressibility $d\rho/d\mu$, of Eq. (3.56) gives the same $\eta = \rho^2(\frac{d\rho}{d\mu})^{-1} = \delta+1$, by differentiating the expression $\rho = -(\delta+1)/\mu$ in Eq. (3.55) w.r.t. μ; this is a proof that fluctuation-response relation indeed holds here and also consistent with additivity property proposed earlier for these systems [24, 40].

In principle, the single-site mass distribution (for $v = 1$) can be calculated straightforwardly from the moments, but the exact closed form expression is hard to obtain. In this regard, this formulation [40, 52] for obtaining subsystem mass distribution from two-point correlation function is quite useful in obtaining the macroscopic behavior of the system.

3.7 Summary

In the present chapter, we have introduced a class of non-equilibrium models, namely finite range processes (FRP), where particles on a one dimensional periodic lattice can hop in a particular direction, from one site to one of its nearest neighbors, with a rate that depends on the occupation of all the sites within a range K starting from the departure site. We show that, for certain specific functional forms of the hop rates, the FRP has a cluster-factorized steady state (CFSS), i.e., the steady state probability of a configuration can be written as a product of cluster-weight functions g having $(K+1)$ arguments - the occupation numbers of $(K+1)$ consecutive sites. The model with $K=0$ reduces to the familiar zero range process (ZRP), which has factorized steady state.

The CFSS with $K=1$ reduces to the pair-factorized steady state (PFSS) and its steady state can always be represented by an infinite-dimensional transfer matrix. However, for the CFSS with $K>1$, a transfer matrix formulation is *not* guaranteed. We show that, for a large class of systems having CFSS with $K>0$, there exists a *finite* dimensional transfer matrix representation. Being finite dimensional, these matrices are easy to deal with and thus help in exact calculation of the n-point correlation functions for any n. The two-point correlation function $(n=2)$ can be utilized to characterize the subsystem mass distribution in these non-equilibrium systems in terms of a non-equilibrium chemical potential and a free-energy function, which are obtained through a fluctuation-response relation [24, 40] - analogous to the equilibrium fluctuation-dissipation theorem.

Even though the finite range process is defined on a one dimensional lattice, it can exhibit condensation phase transition. We obtain a sufficient condition for the condensation transition for a particular class of hop rates in the FRP in general. The nature of the condensation transition studied here are however different from those studied in systems having a PFSS [72]; the condensate here remains localized, as in the ZRP, in contrast to the extended condensate observed in [72, 180, 182].

So, in finite range process, the interaction between particles extends over several lattice sites in contrast to the simple ZRP where there is no correlation between particles at different lattice sites. But, in this chapter we have considered a *uni-directional motion* of the particles so that the system is necessarily driven out of equilibrium by construction. So, one may ask, what happens when the particles in these one dimensional models hop in both the directions with different rates- how can one obtain the steady state measures of such systems- the *asymmetric motion of particles undergoing finite range process* is the subject matter of our next chapter.

Chapter 4

Finite range processes with asymmetric rate functions(AFRP)

The previous chapter dealt with a natural extension of the zero range process (ZRP), which is the finite range process (FRP), where the hop rate of the particles depends on the occupation number of not only the departure site but also that of all other sites within a specified distance; clearly in FRP, inter particle interaction extends to a finite number of neighboring lattice sites. For these systems, in fact, one can obtain a cluster factorized steady states (CFSS) described in Eq. (3.3), for certain rates that satisfy a specific condition mentioned in (3.5). An interesting consequence of the spatial correlations appearing between particles at different lattice sites in FRP in contrast to absence of site-site correlations in ZRP and misanthrope process, is that the condensates formed in the factorized steady states are single site condensates whereas it is possible in FRP, e.g., pair factorized states to observe condensates extending over a region in space [72, 181] due to spatial correlations. Now, as we have noticed that the FRP we have discussed in chapter 3. concerns only unidirectional motion of the particles, in this chapter we want to generalize the finite range process by introducing a general spatially asymmetric motion of the constituents of a given system. Our aim is to study how the steady state measure is modified in these systems, and to find possible characteristic features of observables.

In usual ZRP and related models, the hop rates do not depend on the direction along which the particles move. Although, recently some simple examples [41, 177] have been studied in two dimension ($2d$), where the rate functions are different in x- and y- directions, but it was observed that the two point correlations are finite indicating that the steady state is not factorized. Later, a generalized zero range processes was introduced [93] where more than one particle can hop from a site and the hop rates may depend on direction of hopping. A sufficient condition for having FSS in these models, which is also conjectured as the necessary condition, showed explicitly that indeed models described in [177] cannot have factorized steady states. Moreover, these models in $1d$ (with one hop at a time) reduce to an asymmetric ZRP where particles hop to right or left neighbor with rates $u_R(n) = pu(n), u_L(n) = qu(n)$ (R, L stands for right and left respectively) respectively; notably, the steady state weights of these models do not depend on p, q and the asymmetry parameter $\frac{p}{q}$ only redefines the fugacity of the system in grand canonical ensemble.

In this chapter we introduce a class of one dimensional interacting particle systems with *asymmetric rate functions*, i.e., the right hop rate $u_R(n)$ is an independent function, not just a constant multiple of the left hop rate $u_L(n)$. It is *a priori* not clear, whether a factorized steady state is at all possible for this asymmetric zero range process (AZRP). We derive a sufficient condition for AZRP to have a factorized steady state. Generalization of these asymmetric models to asymmetric misanthrope process (AMAP) and asymmetric finite range process (AFRP) are also investigated to find sufficient conditions on the rate functions that lead to factorized steady state in AMAP and cluster factorized form for AFRP. The h-balance scheme we discussed briefly in chapter 2. helps us here to obtain exactly these steady states as we will discuss in details in the subsequent sections. Interestingly, even though the steady state of both AZRP and AMAP are similar to that of ZRP, particle currents here show current-reversal as the density of the system is changed - a feature which cannot be observed in ZRP with rates $u_R(n) = pu(n), u_L(n) = qu(n)$. Moreover, the current, for specific form of hop rates in AMAP, exhibits negative differential mobility meaning decrease in current with increase in bias. We also address the possibility of condensation transitions in these systems and find that the onset of condensation can be tuned by a factor that merely controls how often the particle chooses to move right, compared to its left hops.

The asymmetric hopping models which we discuss here are interesting in their own right. In addition, there are physical situations which may correspond to the asymmetric diffusion proposed here. It is well known that geometry [169] or the mean forces [110, 122] acting on the particles may induce asymmetry across membrane channels and influence the particle fluxes across artificial or natural-biological pores. Such asymmetry is important for analyzing the dynamics of particle translocation [119, 120] in biological channels. Also, this asymmetric diffusion effect may be utilized [82] to regulate transport and distribution of motile microorganisms in irregular confined environments, such as wet soil or biological tissues.

4.1 Asymmetric zero range process (AZRP)

4.1.1 Model

Let us consider a system of N particles on a one dimensional periodic lattice with L sites labeled by $i = 1, 2, \ldots, L$. Each site i can accommodate $n_i \geq 0$ number of particles. The dynamics of the system is as follows. From each site i, having $n_i > 0$ particles, one particle is transferred either to the right neighbor $(i + 1)$ with a rate $u_R(n_i)$ or to its left neighbor $(i - 1)$ with a different rate function $u_L(n_i)$. Thus, the total number of particles $\sum_{i=1}^{L} n_i = N$ or the density $\rho = N/L$ is conserved. This stochastic process is a zero range process with asymmetric rate functions and hereafter we refer to it in short as asymmetric zero range process (AZRP). Clearly, in AZRP, particles at any given lattice site interact with other particles at the same site through the hop rates which explicitly depend on the occupation number; interaction between particles at different sites is invoked only via the global conservation of N. In the following we show that this interacting particle system can have a factorized steady state if the rate functions satisfy certain constraints.

4.1.2 Steady state: h-balance scheme and criterion on the rates

A special case of the model with $u_R(n) = pu(n), u_L(n) = qu(n)$ is the well known zero range process [71] which describe symmetric (when $p = q$) or asymmetric (when $p \neq q$) transfer of particles. In this case, the steady state has a factorized form for any choice of rate function $u(n)$, and for arbitrary values of p, q

$$P_N(\{n_i\}) \sim \prod_{i=1}^{L} f(n_i)\delta(\sum_{i=1}^{L} n_i - N), \tag{4.1}$$

where $f(n) = \prod_{m=1}^{n} u(m)^{-1}$. We now ask, if such a factorized form is possible when rate functions for right and left hops are different, i.e., $\frac{u_R(.)}{u_L(.)}$ is not independent of n. The Master equation for AZRP is

$$\begin{aligned}\frac{d}{dt}P(\{n_i\}) = \sum_{i=1}^{L} &[u_R(n_{i-1}+1)P(\dots n_{i-1}+1, n_i-1, n_{i+1}\dots) \\ &+u_L(n_{i+1}+1)P(\dots n_{i-1}, n_i-1, n_{i+1}+1\dots)] \\ -\sum_{i=1}^{L} &[u_R(n_i)+u_L(n_i)]\ \ P(n_1,\dots,n_{i-1},n_i,n_{i+1},\dots n_L)\end{aligned} \tag{4.2}$$

which governs how the probability $P(\{n_i\})$ of configuration $\{n_i\}$ evolves with time. Let us assume that the steady state of AZRP has a factorized form as in Eq. (4.1) and check whether the steady state condition $\frac{d}{dt}P(\{n_i\}) = 0$ is satisfied consistently. With a FSS, the steady state Master equation for any arbitrary configuration of AZRP reads as,

$$\begin{aligned}&\textstyle\sum_{i=1}^{L} [u_R(n_i)+u_L(n_i)]\ \ f(n_1)\dots f(n_{i-1})f(n_i)f(n_{i+1})\dots f(n_L) \\ &-[\textstyle\sum_{i=1}^{L} u_R(n_{i-1}+1)\dots f(n_{i-1}+1)f(n_i-1)\dots \\ &+\textstyle\sum_{i=1}^{L} u_L(n_{i+1}+1)\dots f(n_i-1)f(n_{i+1}+1)\dots] = 0.\end{aligned} \tag{4.3}$$

Now by shifting the index $i \to (i-1)$ in the last sum we get an equation $\sum_{i=1}^{L} F(n_{i-1}, n_i) = 0$, where

$$\begin{aligned}F(m,n) &= u_R(n)+u_L(n)-u_R(m+1)\frac{f(m+1)f(n-1)}{f(m)f(n)} \\ &\quad -u_L(n+1)\frac{f(m-1)f(n+1)}{f(m)f(n)}.\end{aligned} \tag{4.4}$$

Clearly we have a stationary measure if we can construct a single site function $h(n)$ that satisfies $F(m,n) = h(m) - h(n)$. Existence of such a function $h(n)$

ensures that $\sum_{i=1}^{L} F(n_{i-1}, n_i) = 0$ and thereby guarantees a factorized stationary measure. Since m, n are non-negative integers, let us first find what restrictions are imposed on $h(.)$ from the boundary values. When $m = 0 = n$, from Eq. (4.4) we have $F(0,0) = 0$, as $u_{R,L}(0) = 0$ (particle hopping is prohibited if the departure site is vacant) and $f(-1) = 0$ (a boundary condition that assigns zero weight for configurations having negative occupation numbers); thus $F(m,n) = h(m) - h(n)$ is automatically satisfied. For other cases,

$$\begin{aligned} n = 0, m > 0 \quad &: \quad u_L(1)\frac{f(m-1)f(1)}{f(m)f(0)} = h(0) - h(m) \\ n > 0, m = 0 \quad &: \quad u_R(n) + u_L(n) - u_R(1)\frac{f(n-1)f(1)}{f(n)f(0)} = h(0) - h(n). \end{aligned} \tag{4.5}$$

Since the right hand side of above equations are same, these equations are consistent if

$$f(n) = \frac{f(1)}{f(0)}\left[\frac{u_R(1) + u_L(1)}{u_R(n) + u_L(n)}\right]f(n-1), \text{ and } h(n) = h(0) - u_L(1)\frac{f(n-1)f(1)}{f(n)f(0)}. \tag{4.6}$$

Finally, a factorized steady state will be guaranteed if the above expressions of $h(n)$ and $f(n)$ consistently satisfy $F(m,n) = h(m) - h(n)$ for all $m > 0, n > 0$. This requirement actually constraints the right and left hop rates $u_{R,L}(n)$ to satisfy the following condition (from Eqs. (4.4) and (4.6)) ,

$$\frac{u_L(n+1)u_R(1) - u_R(n+1)u_L(1)}{[u_R(n) + u_L(n)]\,[u_R(n+1) + u_L(n+1)]} = C, \tag{4.7}$$

where C is a constant independent of n. This completes the proof: *AZRP has a factorized steady state if the hop rates $u_{R,L}(n)$ satisfy Eq. (4.7).* The weight factors $f(n)$ can be calculated from the recursion relation Eq. (4.6)

$$f(n) = [f(1)v(1)]^n \prod_{m=1}^{n} \frac{1}{v(m)} \text{ ; where } v(m) = u_R(m) + u_L(m), \tag{4.8}$$

where we set $f(0) = 1$, without loss of generality. Note a striking similarity of the weight factor $f(n)$ in AZRP with that of the ZRP. In Eq. (4.8) if one sets $f(1) = \frac{1}{v(1)}$, then the steady state of AZRP with specified hop rates $u_{R,L}(n)$ which satisfy Eq. (4.7) is exactly the same as that of the ordinary ZRP with hop rate $u_R(n) + u_L(n)$.

Note that, although validity of Eq. (4.7) is sufficient for AZRP to have a FSS, it is not *a priori* clear if there exists any rate functions which satisfy this condition. To obtain a desired FSS as in Eq. (4.1) where

$$f(n) = \prod_{m=1}^{n} \frac{1}{v(m)} \quad \text{along with } f(0) = 1, \tag{4.9}$$

one can show, following Eqs. (4.8) and (4.7), that the asymmetric rate functions have the following generic functional form for $n \geq 1$,

$$u_R(n) = v(n)\left[\delta - \gamma v(n-1)\right] \;\; ; \;\; u_L(n) = v(n)\left[1 - \delta + \gamma v(n-1)\right]. \tag{4.10}$$

Clearly for $n = 0$, $u_R(0) = 0 = u_L(0)$ meaning $v(0) = 0$. This functional form satisfies Eq. (4.7) with $C = v(1)\gamma$. Thus, now we have a family of asymmetric hop rates, characterized by two independent parameters $0 \leq \delta \leq 1$ and $0 \leq \gamma \leq \delta/v(n)|_{max}$ [1], which gives rise to a unique invariant measure described by Eqs. (4.1) and (4.9).

Some specific examples of AZRP will be discussed in the following sections. A simple situation is when $\gamma = 0$, where $u_R(n) = \delta v(n)$ and $u_L(n) = (1 - \delta)v(n)$. Since $\delta < 1$, the model is identical to an ordinary ZRP where a particle chooses the right (or the left) neighbor as a target site with probability δ (or $1 - \delta$) and then hops to that site with rate $v(n)$. Obviously, $\delta = 0, 1$ corresponds to the usual ZRP where particles hop along a unique direction.

[1] The range of δ and γ are fixed by the condition that the rates $u_{R,L}(n)$ must be positive.

For any conserved system (N particles in L sites) with a factorized steady state

$$P_N(\{n_i\}) = \frac{1}{Q_N^L}\prod_{i=1}^{L} f(n_i)\delta(\sum_{i=1}^{L} n_i - N), \text{ with } f(n) = \prod_{m=1}^{n}\frac{1}{v(m)}, \tag{4.11}$$

$$\text{where } Q_N^L = \sum_{\{n_i\}}\prod_{i=1}^{L} f(n_i)\delta(\sum_{i=1}^{L} n_i - N) \tag{4.12}$$

is the canonical partition function, one can calculate the steady state average of any local observable straightforwardly. For completeness let us describe the procedure briefly. The grand partition function of the system is

$$Z_L(z) = \sum_{N=0}^{\infty} Q_N^L z^N = F(z)^L; \quad F(z) = \sum_{n=0}^{\infty} f(n)z^n, \tag{4.13}$$

where the fugacity z controls the average density of the system $\rho(z) = zF'(z)/F(z)$. The steady state average value of any local observable $O(n_i)$ is then

$$\langle O \rangle = \frac{1}{F(z)}\sum_{n=0}^{\infty} O(n)f(n)z^n, \tag{4.14}$$

which is a function of z. One can get the corresponding value for the conserved system with a given density $\rho = \rho^*$ by setting z to a specific value z^* which satisfy $\rho(z^*) = \rho^*$.

4.1.3 Condensation transition

The most interesting thing that happens in ZRP with a hop rate $v(n)$, or for any other model which has a factorized steady state given by Eq. (4.11), is the condensation transition. If the asymptotic form of $v(n)$ is

$$v(n) = v(\infty)\left(1 + \frac{b}{n^\sigma} + \dots\right), \tag{4.15}$$

condensation occurs for large densities either when $\sigma < 1$, or when $\sigma = 1$ and $b > 2$ [71]. It turns out that higher order terms in the series expansion are irrelevant in deciding the possibility of a condensation transition; they only play a role in determining the exact critical density above which the system forms a condensate. Since there are many exclusion models that have exact or approximate ZRP correspondence, the above criteria is extensively used for determining the possibility of phase separation transition [112]. A particularly simple case of (4.15), where the critical density ρ_c can be calculated exactly [95], is

$$v(n) = 1 + \frac{b}{n} \tag{4.16}$$

that results in a condensation transition for $b > 2$, when density ρ of the system crosses a critical value $\rho_c = \frac{1}{b-2}$.

In AZRP, to have a FSS given by (4.11) with $v(n) = 1 + \frac{b}{n}$ for $n \geqslant 1$ ($v(0) = 0$ by definition as already mentioned) the rate functions must follow Eq. (4.10). For this choice of $v(n)$, the model has three parameters $b > 0$, $0 < \delta \leq 1$ and γ; here γ must be in the range $0 \leq \gamma \leq \frac{\delta}{v(n)|_{max}} = \frac{\delta}{1+b}$, so that the rates in Eq. (4.10) remain positive for all $n > 0$. Let us parametrize (b, δ, γ) in terms of three other parameters (b_R, b_L, α) as follows,

$$b = \alpha b_R + \bar{\alpha} b_L \ ; \quad \delta = \alpha(2 - \frac{b_R}{\alpha b_R + \bar{\alpha} b_L}) \ ; \quad \gamma = \alpha(1 - \frac{b_R}{\alpha b_R + \bar{\alpha} b_L}), \tag{4.17}$$

where we use $\bar{\alpha} \equiv 1-\alpha$ for notational convenience. The purpose of such parametrization will become clear in a moment. With these new parameters the hop rates of the model for the choice $v(n) = 1 + \frac{b}{n}$ can be written (using Eq. (4.10)) as

$$u_R(n) = \alpha \tilde{u}_R(n), u_L(n) = \bar{\alpha} \tilde{u}_L(n) \tag{4.18}$$

where for $n = 1$,

$$\tilde{u}_R(1) = (2 - \frac{b_R}{\alpha b_R + \bar{\alpha} b_L})\,[1 + \alpha b_R + \bar{\alpha} b_L]$$

$$\tilde{u}_L(1) = (1 - \frac{b_R}{\alpha b_R + \bar{\alpha} b_L})\left[1 + \alpha b_R + \bar{\alpha} b_L\right] \tag{4.19}$$

and for $n > 1$

$$\begin{aligned}
\tilde{u}_R(n) &= (1 + \frac{\alpha b_R + \bar{\alpha} b_L}{n})\left[1 - \bar{\alpha}\frac{b_L - b_R}{n-1}\right] \\
\tilde{u}_L(n) &= (1 + \frac{\alpha b_R + \bar{\alpha} b_L}{n})\left[1 + \alpha\frac{b_L - b_R}{n-1}\right].
\end{aligned} \tag{4.20}$$

It is easy to see that the asymptotic forms of $\tilde{u}_{R,L}(n)$ are

$$\tilde{u}_R(n) = 1 + \frac{b_R}{n} + \ldots \ ; \quad \tilde{u}_L(n) = 1 + \frac{b_L}{n} + \ldots . \tag{4.21}$$

The new parameters α, b_R, b_L are all familiar to us: $b_{R,L}$, are coefficients of $\frac{1}{n}$ in the asymptotic expansion of the rates $\tilde{u}_{R,L}(.)$ which normally take part in determining possibility of a condensation transition, and α may be considered as the probability that a particle chooses the right neighbor as the target site (note that $\alpha = \gamma - \delta$ varies in the range $(0, 1)$ for any $b > 0$). Thus, for the model in hand, particles choose to move right (or left) with probability α (or $1 - \alpha$) and hop there with rate $\tilde{u}_{R,L}(.)$ respectively.

For $\alpha = 0$, particles in this model move only to left with rate $\tilde{u}_L(n) = 1 + \frac{b_L}{n}$ leading to a factorized steady state and a condensation for large densities when $b_L > 2$. Similarly for $\alpha = 1$, condensation occurs for $b_R > 2$. It is interesting to ask, 'for a given fixed $b_{R,L}$, is it possible to observe a condensation transition by changing α ?' Note that α determines how often the system chooses to hop right and a condensation transition, if appears by tuning only α, is exciting as it has not been observed earlier in ZRP or related models.

The difficulty, however, lies with the fact that for any given $b_{R,L}$ we do not have exact steady state measure (within this formalism [79]) for all $\alpha \in (0, 1)$. The constraint comes from the requirement that the rate functions obtained in Eq. (4.18)-(4.20) must be positive valued for $n > 0$, which in turn restricts the value of α for which one can obtain the steady state weights exactly. In other words, for some $b_{R,L}$, it may not be possible to find $u_{R,L}(n)$ for which the steady state is factorized for any arbitrary $0 \leq \alpha \leq 1$. When both b_R and b_L are larger than 2, we have $b = \alpha b_R + (1-\alpha)b_L > 2$; this case is less interesting because, even if we find certain suitable hop rates that describe this situation, and result in a FSS as in Eq. (4.11) with $v(n) = 1 + \frac{b}{n}$, the system will remain in the condensate phase for all α. Similarly, for $b_R < 2, b_L < 2$, condensation transition is not possible as b is smaller than 2 for any $0 < \alpha < 1$. Thus, we focus on the case where $b_R < 2$ and $b_L > 2$ (the other alternative $b_R > 2$ and $b_L < 2$ can be described in the same manner). For any fixed value of b_R the minimum and the maximum accessible values of α, for which one can have exact FSS with rate functions $u_{R,L}(n)$ given by Eq. (4.18)-(4.20) are respectively

$$\alpha_{min} = \max\{0, \frac{b_L - b_R - 1}{b_L - b_R}\}; \quad \alpha_{max} = \min\{1, \frac{1}{2}\frac{b_L}{b_L - b_R}\}. \tag{4.22}$$

These conditions on α are calculated simply by demanding positivity of the hop rates in (4.18)-(4.20).

To demonstrate the possibility of a condensation transition tuned by α, we consider AZRP with hop rates $u_{R,L}(n)$ given by (4.18)-(4.20), in two separate cases $b_R = \frac{3}{2}$ and $\frac{1}{2}$. The maximum and minimum values of α now depends on b_L; in Fig. 4.1(a) and (b) we have plotted α_{min} and α_{max} in dashed lines for $b_R = \frac{3}{2}$ and $\frac{1}{2}$ respectively. The regions for $\alpha > \alpha_{max}$ and $\alpha < \alpha_{min}$ are shaded to indicate that within this formalism [79] the steady state does not have a factorized form in these regions. In the rest of regions, we have a factorized steady state given by

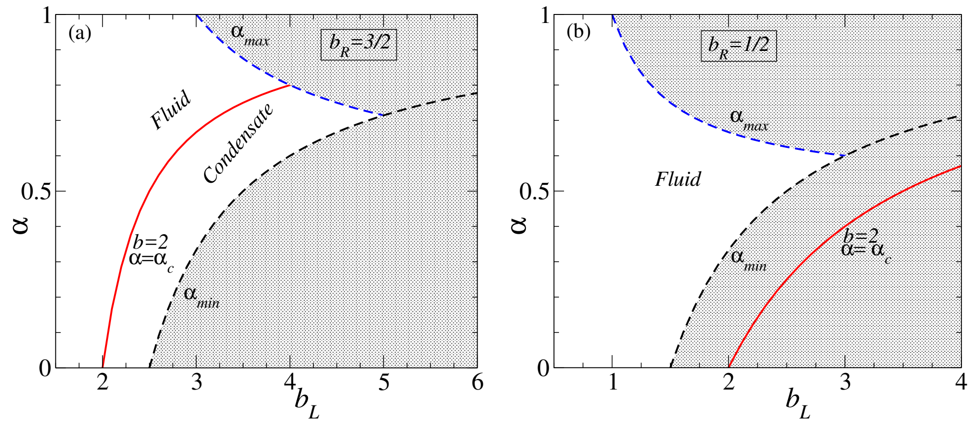

Figure 4.1: Condensation transition for AZRP dynamics given by Eqs. (4.18)-(4.20). For any given b_R, b_L the steady state has a factorized form when $\alpha \in (\alpha_{min}, \alpha_{max})$. Plots of α_{min} and α_{max} as function of b_L ($> b_R$) are shown here for (a) $b_R = \frac{3}{2}$ and (b) $b_R = 1/2$; we do not have exact steady state solution in the shaded regions where $\alpha > \alpha_{max}$ or $\alpha < \alpha_{min}$. The condensation transition occurs for large densities when $b = \alpha b_L + (1-\alpha) b_R$ is larger than 2. In (a), this transition line $b = 2$, which separates the fluid phase from the condensate one, lies in the region where we have the exact (factorized) steady state.

Eqs. (4.11) and (4.16) and a condensation transition occurs here for large densities ($\rho > \frac{1}{b-2}$) when b is greater than 2, which corresponds to $\alpha > \alpha_c$ where

$$\alpha_c = \frac{b_L - 2}{b_L - b_R}. \tag{4.23}$$

In Fig. 4.1 we have also shown $\alpha = \alpha_c$ as a solid line, marked as $b = 2$ and correspondingly $\alpha = \alpha_c$. In the left panel ($b_R = \frac{3}{2}$) this line lies in the exactly solvable regime separating the fluid phase from the condensate one. For $b_R = 1/2$, we could not conclude if there is a condensation transition as the exact steady state measure in the neighborhood of $\alpha = \alpha_c$ line is not known. In fact, with some simple algebra one can show that for any $1 < b_R < 2$ the transition line lies in the exactly solvable regime, which is not the case when $0 < b_R \leq 1$.

As an explicit example, let us consider $b_R = \frac{3}{2}, b_L = \frac{9}{4}$; in this case clearly α can vary freely in the range $(0, 1)$, which can be seen from Fig. 4.1 (a). The rate functions, from Eq. (4.18)-(4.20), are now $u_R(n) = \alpha\tilde{u}_R(n), u_L(n) = (1-\alpha)\tilde{u}_L(n)$ with

$$\tilde{u}_R(n) = \begin{cases} \frac{(13-3\alpha)(2-\alpha)}{2(3-\alpha)} & n=1 \\ \frac{(4n-3\alpha+9)(4n+3\alpha-7)}{16n(n-1)} & n>1 \end{cases} \qquad \tilde{u}_L(n) = \begin{cases} \frac{(13-3\alpha)(3-2\alpha)}{4(3-\alpha)} & n=1 \\ \frac{(4n-3\alpha+9)(4n+3\alpha-4)}{16n(n-1)} & n>1 \end{cases}$$

It is easy to check that these functions result in the FSS given by Eq. (4.11) along with (4.16) where $b = \alpha b_R + (1-\alpha)b_L$. For $\alpha = 1$, we have $b = b_R = \frac{3}{2}$ and the system remains in the fluid phase for all densities whereas for $\alpha = 0$, condensation occurs as $b = b_L = 9/4$. Interestingly for any arbitrary $0 < \alpha < 1$, $b = \frac{3}{4}(3-\alpha)$ and a condensation transition takes place when α is decreased below $\alpha_c = \frac{1}{3}$ (from Eq. (4.23)). For any $\alpha > \alpha_c$, the system sets in the condensate phase only when the density of the system is increased above $\rho_c = \frac{4}{1-3\alpha}$.

4.1.4 Current reversal

Another interesting thing that happens in AZRP is the current reversal, where the direction of current depends on the particle density of the system. When AZRP with hop rates $u_{R,L}(n)$ has a factorized steady state given by Eq. (4.11) with $v(n) = u_R(n) + u_L(n)$, the steady state current in the system can be written as

$$J = \frac{1}{F(z)} \sum_{n=1}^{\infty} [u_R(n) - u_L(n)] f(n) z^n = \langle u_R(n)\rangle - \langle u_L(n)\rangle \tag{4.24}$$

where $F(z) = \sum_{n=0}^{\infty} z^n f(n)$. As we have discussed, a sufficient condition required for having a factorized steady state in AZRP is that $u_{R,L}(n)$ must have a form given by Eq. (4.10), with some $0 \le \delta \le 1$ and $0 \le \gamma \le \delta/v(n)|_{max}$. Then $(u_R(n) - u_L(n)) = v(n)[2\delta - 1 - 2\gamma v(n-1)]$ and thus

$$\begin{aligned} J &= (2\delta-1)\langle v(n)\rangle - 2\gamma\langle v(n)v(n-1)\rangle \\ &= (2\delta-1)z - 2\gamma z^2. \end{aligned} \tag{4.25}$$

In the last step we used $v(n) = \frac{f(n-1)}{f(n)}$ to calculate $\langle v(n) \rangle = \frac{1}{F(z)} \sum_{n=1}^{\infty} v(n) f(n) z^n = z$ and similarly, $\langle v(n) v(n-1) \rangle = z^2$.

In a simple ZRP with hop rates $u_R(n) = \alpha v(n)$ and $u_L(n) = (1-\alpha) v(n)$, which corresponds to the choice $\delta = \alpha, \gamma = 0$, Eq. (4.25) leads to $J = (2\alpha - 1)z$. Thus, in ZRP, the direction of current J *cannot* be changed by changing the density ρ (or equivalently the fugacity z); the direction is fixed only by α, i.e., J is positive (or negative) when $\alpha > \frac{1}{2}$ ($\alpha < \frac{1}{2}$). The change of density can only increase or decrease the magnitude of current, it cannot change the direction of the flow. But surprisingly density dependent current reversal is possible in AZRP: for a fixed $u_{R,L}(n)$ the direction of the current may get reversed when the density of the system is changed. It is clear from Eq. (4.24) that such a reversal is not possible when $(u_R(n) - u_L(n))$ has the *same sign* for all $n > 0$. In the following, we illustrate with a simple example that direction of current can be tuned by the density, when $u_R(n) > u_L(n)$ for all n except $n = 1$ where $u_R(n) < u_L(n)$. To this end, we consider AZRP with rate functions

$$u_R(n) = \begin{cases} \delta & n = 1 \\ \alpha & n > 1 \end{cases} ; \quad u_L(n) = \begin{cases} 1 - \delta & n = 1 \\ 1 - \alpha & n > 1 \end{cases} , \tag{4.26}$$

which follow Eq. (4.10) with $\alpha = \delta - \gamma$ varying in the range $(0, 1)$ and $v(n) = 1 \;\; \forall\, n > 0$ (and $v(0) = 0$). In this model isolated particles hop with a different rate than the rest. We also consider $\alpha > \frac{1}{2}$ and $\delta < \frac{1}{2}$ so that isolated particles hop preferentially in a different direction (here towards left) compared to particles from sites having two or more particles which preferentially move towards right. In this case, the flow direction of current can depend on the density of the system. For very large density there are only a few sites which contain isolated particles and the current is expected to be positive (towards right) whereas for very low density most particles are isolated and one expects a negative current. Let us see if the direction of the current can be reversed when the density ρ of the system falls below a critical threshold ρ^*.

Since, $v(n) = 1 \ \forall \ n > 0$, this dynamics results in a FSS with $f(n) = 1 \ \forall \, n \geqslant 0$. Correspondingly $F(z) = \frac{1}{1-z}$ and $\rho = zF'(z)/F(z) = \frac{z}{1-z}$, which in turn implies $z = \frac{\rho}{1+\rho}$. Thus the current, from Eq. (4.25), is

$$J = \frac{\rho}{(1+\rho)^2} \left[2\delta - 1 + \rho(2\alpha - 1) \right]. \tag{4.27}$$

Since $\alpha > \frac{1}{2}$, and $\delta < \frac{1}{2}$, the current J flows in the negative direction if density ρ falls below $\rho^* = \frac{1-2\delta}{2\alpha-1}$.

For the class of AZRP with rate functions represented by (4.10), current reversal is expected at fugacity $z^* = \frac{2\delta-1}{2\gamma}$. But the crucial point that one must keep in mind is, z^* must lie in the range $0 < z^* < v(\infty)$ so that $z(\rho^*) = z^*$ would solve for a physically realizable density $\rho* > 0$.

It is worth mentioning that, at the point of reversal (z^* or eqivalently $\rho^\star$), the average current J is zero but the steady state of the system is far different from the equilibrium one which also is characterized by zero current. For the model we discussed here, all the configurations are equally likely i.e., occur with equal probability as $f(n) = 1$ for all $n \geq 0$. Now, as discussed earlier, the detailed balance is satisfied if the ratio of the probabilities ($P(.)$) of every pair of configuration is equal to the inverse ratio of the corresponding transition rates ($w(.)$) i.e., $\frac{P(C')}{P(C)} = \frac{w(C \to C')}{w(C' \to C)}$. So, in the present examples, probabilities of all configurations being equal this implies necessarily $w(C \to C') = w(C' \to C)$. Now, if we consider a right hop from a site occupied by one particle to its vacant right neighbor and the corresponding left jump from the site containing one particle to its vacant left neighbor, then $w(C \to C') = w(C' \to C)$ implies $\delta = 1 - \delta$ meaning $\delta = \frac{1}{2}$. Similarly, one can check, hops between other configurations will imply $\alpha = 1 - \alpha$ i.e. $\alpha = \frac{1}{2}$ if detailed balance has to hold. So, clearly, in the present model exhibiting current reversal, one obtains equilibrium *only* for $\delta = \alpha = \frac{1}{2}$ whereas the point of reversal $\rho^\star = \frac{1-2\delta}{2\alpha-1}$ has a finite value for any $(\alpha > 1/2, \delta < 1/2)$ which correspond to a non-equilibrium scenario as the detailed balance condition is violated.

4.2 Asymmetric misanthrope process (AMAP)

4.2.1 Model

The Misanthrope process (MAP) is an interacting particle system, where hop rate of particles depends on both, the occupation of departure site and the arrival site. In contrast to ZRP, here particles at the departure site not only interact among themselves, they also explicitly interact with particles at the arrival site. This model can have a factorized steady state in $1d$ if the hop-rate satisfies certain conditions; for a periodic lattice with L sites $i = 1, 2, \ldots, L$, each site i containing n_i particles, if particles move to their right neighbor with rate $u(n_i, n_{i+1})$, the condition for having a FSS reads as [77],

$$u(m,n) \; = \; u(m+1, n-1)\frac{u(1,m)u(n,0)}{u(m+1,0)u(1,n-1)} \; + \; u(m,0) \; - \; u(n,0). \qquad (4.28)$$

In this section we generalize the misanthrope process to include asymmetric rate functions $u_{R,L}(., *)$, where the subscripts R, L stands for right, left and the arguments "." and "$*$" correspond to occupation number of departure and arrival sites respectively. We ask if the steady state of this asymmetric misanthrope process (AMAP) can have a factorized form and if so, what would be the corresponding condition on the hop-rates ?

Like AZRP, the present section deals with a one dimensional periodic lattice with L sites labeled by $i = 1, 2, \ldots L$. Each site i contains $n_i (\geqslant 0)$ number of particles as earlier but the hop rates in AMAP depend not only on the occupancy of the departure site but also on the arrival site. More precisely, a particle from a randomly chosen site i, provided $n_i > 0$, can either hop to its right neighbor $(i+1)$ with a rate $u_R(n_i, n_{i+1})$ or it can move to its left neighbor $(i-1)$ with a rate $u_L(n_i, n_{i-1})$.

4.2.2 FSS: h-balance scheme and criterion on the rates

To study whether AMAP can have a FSS, as before, we start with the ansatz that the steady state has a factorized form $P(\{n_i\}) \sim \prod_{i=1}^{L} f(n_i) \delta(\sum_{i=1}^{L} n_i - N)$ and look for conditions on the rate functions that satisfy $\frac{d}{dt} P(\{n_i\}) = 0$ in steady state where $P(\{n_i\})$, the probability of each configuration $\{n_i\}$, follows the Master equation

$$\begin{aligned}
\tfrac{d}{dt} P(\{n_i\}) &= \textstyle\sum_{i=1}^{L} u_R(n_{i-1}+1, n_i-1) \ldots f(n_{i-1}+1) f(n_i-1) f(n_{i+1}) \ldots \\
&\quad + \textstyle\sum_{i=1}^{L} u_L(n_{i+1}+1, n_i-1) \ldots f(n_{i-1}) f(n_i-1) f(n_{i+1}+1) \ldots \\
&\quad - \textstyle\sum_{i=1}^{L} [u_R(n_i, n_{i+1}) + u_L(n_i, n_{i-1})] \; \ldots f(n_{i-1}) f(n_i) f(n_{i+1}) \ldots .
\end{aligned}$$

Let us collect all the terms from the right hand side of the above equation that contain both n_i and n_{i-1} as arguments of rate functions, and write them as $h(n_{i-1}) - h(n_i)$, where function $h(.)$ is yet to be determined,

$$\begin{aligned}
u_R(n_{i-1}, n_i) + u_L(n_{i-1}, n_i) - u_R(n_{i-1}+1, n_i-1) \frac{f(n_{i-1}+1) f(n_i-1)}{f(n_{i-1}) f(n_i)} \\
- u_L(n_i+1, n_{i-1}-1) \frac{f(n_{i-1}-1) f(n_i+1)}{f(n_{i-1}) f(n_i)} &= h(n_{i-1}) - h(n_i). \qquad (4.29)
\end{aligned}$$

Clearly, existence of a function $h(.)$ ensures that $\frac{d}{dt} P(\{n_i\}) = \sum_i h(n_{i-1}) - h(n_i) = 0$. Now let us check for the boundary conditions, i.e., when either of n_i, n_{i-1} or both are zero. Equation (4.29) is automatically satisfied when $n_i = n_{i-1} = 0$. When $n_i = 0, n_{i-1} = m > 0$, we have

$$h(m) = u_R(m,0) + u_L(m,0) - u_L(1,m-1)\frac{f(m-1)}{f(m)} \ . \tag{4.30}$$

Here we have used the facts that $u_{R,L}(0,*) = 0$ (particles cannot hop from vacant sites), $f(-1) = 0$ as $n_i > 0$, $f(1)/f(0) = 1$ (without loss of generality) and $h(0) = 0$ as the function $h(.)$ in Eq. (4.29) is defined up to an arbitrary additive constant. Similarly, $n_{i-1} = 0, n_i = m > 0$ results in

$$h(m) = u_R(1,m-1)\frac{f(m-1)}{f(m)}. \tag{4.31}$$

Solving the above two equations for $f(m)$ and $h(m)$, we obtain

$$h(m) = u_R(1,m-1)w(m) \ ; \ f(m) = \frac{f(m-1)}{w(m)} = f(0)\prod_{k=1}^{m}\frac{1}{w(k)} \tag{4.32}$$
$$\text{where } w(m) = \frac{u_R(m,0) + u_L(m,0)}{u_R(1,m-1) + u_L(1,m-1)}.$$

Clearly, for any given $u_{R,L}(n,m)$, the steady state of AMAP is the same as that of a simple ZRP with hop rate $w(m) = \frac{u_R(m,0)+u_L(m,0)}{u_R(1,m-1)+u_L(1,m-1)}$; the function $w(m)$, however satisfies $w(1) = 1$ (from above definition). The ZRP correspondence is not surprising, as we know that a factorized steady state (4.11) of any model can always be obtained from a simple ZRP with hop rate $\frac{f(m-1)}{f(m)}$. Finally using $f(m)$ and $h(m)$ in Eq. (4.29) we get the following condition on hop rates that ensures a FSS in AMAP,

$$\begin{aligned} u_R(m,n) \ &+ \ u_L(n,m) = \left[\frac{u_R(m+1,n-1)}{w(m+1)} - u_R(1,n-1)\right]w(n) + u_R(m,0) \\ &+ \ \left[\frac{u_L(n+1,m-1)}{w(n+1)} - u_L(1,m-1)\right]w(m) + u_L(n,0). \end{aligned} \tag{4.33}$$

When particles move only to the right, i.e., $u_L(.,*) = 0$ and $u_R(.,*) = u(.,*)$ this equation reduces to the condition Eq. (4.28) required for the usual totally asymmetric misanthrope process to have an FSS. In summary, a stochastic process on a $1d$ periodic lattice where particles (without obeying hardcore exclusion) hop to right or left with different rate functions $u_{R,L}(m,n)$ that depend on the occupation numbers m and n of departure and arrival site respectively, has a factorized steady state, as in Eq. (4.11) if the rate functions obey Eq. (4.33).

Equation (4.33) is more complicated than the corresponding condition (4.7) for AZRP. For AMAP with any given rate function $u_{R,L}(m,n)$ one can easily check if they obey Eq. (4.33), but obtaining a generic form of hop rates that satisfy this condition is rather difficult. In the following we consider three different class of models which obey Eq. (4.33).

A very special class, is the equilibrium AMAP. If rate functions are related as follows

$$u_L(m,n) \ = \ u_R(n+1,m-1)\,\frac{w(m)}{w(n+1)}, \tag{4.34}$$

they surely satisfy (4.33) required for having a FSS, at the same time they also obey the condition of detailed balance. Equation (4.34) clearly describes a class of generic equilibrium AMAP models in the sense that $u_R(n+1,m-1)$ can still be chosen freely.

Another class of AMAP models that has factorized steady state is

$$u_R(m,n) = \delta u(m,n) + \gamma u(m,0)u(1,n);\; u_L(m,n) = \gamma u(m,0)u(1,n). \tag{4.35}$$

These rates, when used in Eq. (4.33) result in Eq. (4.28), which is the condition required for an ordinary misanthrope process with hop rate $u(m,n)$ to have a FSS. Thus, Eq. (4.35) describes a family of models, parametrized by two positive constants δ,γ and a positive-valued function $u(m,n)$ with $u(0,n)=0$. In this case detailed balance is not satisfied and this class of models leads to a unique non equilibrium steady state having a factorized from as in Eq. (4.11) with weight function,

$$f(m) = \prod_{k=1}^{m} \frac{u(k,0)}{u(1,k-1)}. \tag{4.36}$$

Yet another interesting class of models are, when the rate functions are in product form,

$$u_{R,L}(m,n) = [c_{R,L}(m) - c_{R,L}(0)]\, c_{R,L}(n) \tag{4.37}$$

where the rate functions are in product form with $c_{R,L}(.)$ being positive functions for $m \geqslant 0$; presence of $c_{R,L}(0)$ ensures that $u_{R,L}(0,.) = 0$. A factorized steady state like in (4.32) can be obtained in this case when these rate-functions satisfy the constraint (4.33), which in turn impose the following recursion relations on $c_{R,L}(.)$,

$$c_{R,L}(m) = \; c_{R,L}(0) \; + \; w(m)\; c_{R,L}(m-1) \; \left(\frac{c_{R,L}(1)}{c_{R,L}(0)} - 1 \right). \tag{4.38}$$

In section 4.2.5 we discuss two specific models of AMAP where hop rates are given by Eq. (4.34) and Eq. (4.37) respectively. In the following section, we consider a model which does not belong to any of the three class of models (4.34), (4.35) or (4.37), but still leads to a factorized steady state and exhibits density dependent current reversal.

4.2.3 Current reversal in AMAP

Like AZRP, it is possible to reverse the direction of the average current J in AMAP, only by tuning the number density ρ. Let us consider the following rate functions,

$$u_R(m,n) = \begin{cases} p & n=0 \\ p_1 & n>0, \quad m=1 \\ p_2 & n>0, \quad m>1 \end{cases} \; ; \; u_L(m,n) = \begin{cases} q & n=0 \\ q_1 & n>0, \quad m=1 \\ q_2 & n>0, \quad m>1 \end{cases} \tag{4.39}$$

It is easy to check that the rates (4.39) satisfy the constraint (4.33) only if

$$q_2 = p_2 - q + q_1 + \frac{((p+q)q_1)}{(p_1+q_1)} - \frac{(p(p_1+q_1))}{(p+q)} \qquad (4.40)$$

With this choice of q_2 we have a factorized steady state given by Eq. (4.11) where

$$f(n) = \begin{cases} 1 & n = 0, 1 \\ \alpha^{n-1} & n \geqslant 2 \end{cases} \; ; \; \alpha = \frac{p_1+q_1}{p+q}. \qquad (4.41)$$

It is interesting to note that the steady state weight does not depend on p_2; any value of p_2 generates the same steady state as long as q_2 defined in Eq. (4.40) is positive. One must also note that though the rates in this model obey the generic constraint (4.33), they do not satisfy detailed balance and are not in the form of Eq. (4.34), also they do not belong to the special class of models defined in Eq. (4.35).

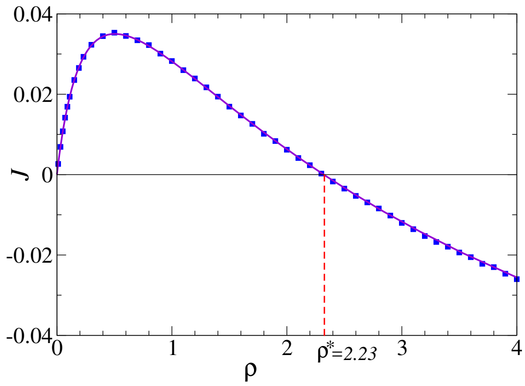

Figure 4.2: Current reversal in AMAP. Current J as a function of density ρ, measured from Monte Carlo simulation (symbols) of AMAP dynamics (4.39) with $(p = \frac{1}{2}, q = \frac{1}{4}, p_1 = \frac{1}{2}, q_1 = \frac{3}{4}, p_2 = 53/60, q_2 = 1)$ on a system of size L, is compared with exact results (lines) given by Eq. (4.44). As expected, current reversal occurs at density $\rho^* = 2.32$.

In the grand canonical ensemble, the partition function is $Z_L = F(z)^L$ with $F(z) = \sum_{n=0}^{\infty} f(n) z^n = \frac{1+(1-\alpha)z}{1-\alpha z}$, where the fugacity z lies in the range $(0, 1/\alpha)$, as the radius of convergence of $F(z)$ is $z_c = 1/\alpha$. The density of the system is now

$$\rho(z) = z\frac{F'(z)}{F(z)} = \frac{z}{(1-\alpha z)(1+(1-\alpha)z)} \tag{4.42}$$

$$\text{or } z = \frac{1+\rho(2\alpha-1)-\sqrt{(1-\rho)^2+4\alpha\rho}}{2\rho\alpha(\alpha-1)}. \tag{4.43}$$

The current in this system can be written as

$$\begin{aligned} J &= \frac{1}{F(z)^2}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left[u_R(m,n)-u_L(m,n)\right] z^{m+n} f(m) f(n) \\ &= \left[(p-q)\ +\ (p_1-q_1)z\ +\ (p_2-q_2)(F(z)-z-1)\right]\frac{F(z)-1}{F(z)^2}\ . \end{aligned} \tag{4.44}$$

If J needs to reverse its direction at some density ρ^*, the corresponding fugacity $z = z^\star$ must be such that $J|_{z=z^*} = 0$; using Eq. (4.44) this leads to

$$z^* = \frac{1}{\alpha-1}\left[1-\sqrt{\frac{p(p_1-p)+q(q_1-q)}{p_1q_1-pq}}\right] \tag{4.45}$$

The above value of z^* will correspond to a feasible density only if $0 < z^* < 1/\alpha$; and then, one can obtain the corresponding density $\rho^* = \rho(z^*)$ using Eq. (4.42).

Now let us consider some specific cases, say $\alpha = \frac{5}{3}$. This may be obtained from, say, $(p = \frac{1}{2}, q = \frac{1}{4}, p_1 = \frac{1}{2}, q_1 = \frac{3}{4})$ with $q_2 = p_2 + \frac{7}{60}$ (from Eq. (4.40)). In this case $z_c = \frac{1}{\alpha} = \frac{3}{5}$ and the fugacity at the reversal point $z^* = \frac{3}{4}(2-\sqrt{2}) < z_c$. So, for this choice of rates, the particle current changes its direction when density of the system crosses a threshold value $\rho^* = \rho(z^*) = \frac{3}{7}(4+\sqrt{2}) \approx 2.32$. In Fig. 6.2, we have shown a plot of the average current as a function of density; for very low density current flows towards right and increases as ρ is increased. Beyond a certain density where J reaches its maximum value, it decreases with ρ and finally starts flowing towards left as soon as the density becomes larger than $\rho^* \approx 2.32$.

Another interesting case is $\alpha = 1 = p + q$. In this case when $q_2 = p_2 + 1 - 2p_1$, we have a factorized steady state with a weight function $f(n) = 1\ \forall\ n > 0$. Thus,

$F(z) = \frac{1}{1-z}$, and $z = \frac{\rho}{1+\rho}$. Now the current in the system is, from Eq. (4.44),

$$J = \frac{\rho}{(1+\rho)^2} \left[2p - 1 + (2p_1 - 1)\rho\right] \tag{4.46}$$

which changes its direction at $\rho^* = -\frac{2p-1}{2p_1-1}$. Thus reversal is possible at density $\rho = \rho^*$, when $p > \frac{1}{2}, p_1 < \frac{1}{2}$ or when $p < \frac{1}{2}, p_1 > \frac{1}{2}$. The noticeable point here is that the current in (4.46) is exactly similar to that of the AZRP current in (4.27) with $p \to \delta$ and $p_1 \to \alpha$, so is the point of reversal ρ^*; but the dynamics of AMAP is very different from that of AZRP. The similarity originates from the fact that the stationary state of both models are factorized with identical weight function $f(n) = 1 \ \forall\ n \geq 0$.

4.2.4 Negative differential mobility in AMAP

Negative differential mobility (NDM) refers to the situation when the current decreases with increasing drive [67, 68, 185]. It has been observed in various electronic systems [50, 127, 129, 143], and in context of particle [15, 108, 185] and thermal transport [11, 105, 135]. In particular, the occurrence of NDM of driven tracer particles in presence of static obstacles [10, 14, 16, 61, 132] or in steady laminar flow [164] or crowded medium [22, 23] have been studied extensively in recent years. However, plausible mechanism by virtue of which a system may exhibit negative differential mobility in interacting many particle systems has still been very much an issue of interest. Recently it has been proposed in [35] that for an interacting many particle system with multiple current carrying modes (e.g. different particle types that hop with different hop rates), slowing down some non-driven modes (that are not biased along any direction) through biasing of other modes can give rise to NDM. Actually, in general, in interacting driven systems, each degree of freedom or mode could experience the bias differently; in particular, the bias might affect the time-scales of modes which are not directly biased, and in turn influence the behavior of the current. So, for specific choices of hop rates in several interacting particle systems, it may happen that the dynamical activity of certain non-driven modes can be decreased as the bias on driven modes increases and due to inter-particle interaction between different current carrying modes, ultimately the total particle current also decreases with increasing bias giving rise to the phenomena of negative differential mobility.

In this section, we would like to discuss an example of AMAP with specific choice of rates that give rise to negative differential response of the particles and try to validate the criterion for NDM proposed in [35] for this system. Following the local detailed balance condition, we can define the driving fields or bias in terms of the asymmetric rate functions as $E_{mn} = \ln \frac{u_R(m,n)}{u_L(n+1,m-1)}$ acting on bonds with local configurations (m,n). Clearly, if $E_{mn} = 0 \;\; \forall \; m,n$, we have $u_R(m,n) = u_L(n+1,m-1)$ and the system is in equilibrium satisfying detailed balance condition with all configurations being equally likely.

We now choose a set of specific rate functions,

$$u_R(m,n) = \left\{ \begin{array}{ll} \psi & n=0 \\ 1 & n>0 \end{array} \right. ; \; u_L(m,n) = \left\{ \begin{array}{ll} \psi & m=1 \\ e^{-\varepsilon} & m>1, n=0 \\ \frac{1}{2} & m>1, n>0 \end{array} \right. ,$$
$$\text{implying, } E_{mn} = \left[\ln 2 + (\varepsilon - \ln 2)\delta_{m,1}\right](1-\delta_{n,0}). \tag{4.47}$$

Here, hopping of isolated particles to *vacant* neighbors are not biased, i.e., $E_{10} = 0$, as both the rightward hop and corresponding reverse hop occur with same rate ψ; we consider them as non-driven modes. Jumps to occupied neighbors are however biased by an external field which depends on the occupation of the departure site: ε when the departure site has only one particle or otherwise a constant field $\ln 2$.

To explore the possibility of NDM in this system we did Monte Carlo simulation with $\psi(\varepsilon) = 1/(1+\varepsilon)$. Figure 4.3 shows the particle current j versus ε (symbols) which depicts a non-monotonic behavior, i.e., after a certain finite value of the bias, as the bias is further increased the current starts decreasing. Now, with this example, to verify the criterion for NDM described in [35], let us first identify the non-driven modes that are slowed down. Actually, we have seen that the "isolated particle-vacant neighbor" mode of 10 is non-driven since $E_{10} = 0$ whereas "isolated particle-single or multi particle neighbor" mode, e.g., mode of 13 is driven by the bias $E_{13} = \varepsilon$- and the hop rate ψ of the non-driven mode is slowed down by a decreasing function of the bias ε which is $\psi(\varepsilon) = 1/(1+\varepsilon)$- so that finally we obtain negative differential mobility of the particles. So, indeed we see that slowing down a non-driven mode results in NDM.

This behavior of current can be understood more rigorously from the exact steady state weights of AMAP. As discussed earlier, AMAP has a factorized form $P(\{n_i\}) \sim \prod_i f(n_i)$ when the rate functions satisfy a certain constraint expressed by Eq. (4.33). Using the dynamics (4.47) in (4.33), we find that an FSS is guaranteed for the following functional form of ψ, In the present case, these conditions require

$$\psi(\varepsilon) = \frac{2 - e^{\varepsilon} + 2\delta(1 - e^{-\varepsilon})}{3e^{\varepsilon} - 4} \tag{4.48}$$

with $\delta = \frac{1}{4}(e^{\varepsilon} - 2 + \sqrt{4 + 12e^{\varepsilon} + e^{2\varepsilon}})$, when $f(n) = \delta^{n-1}\ \forall\ n > 0$ and $f(0) = 1$. Note that $\psi(\varepsilon)$ in Eq. (4.48) is a decreasing function for all ε, but the model is well defined only in the regime $\varepsilon > \ln\frac{4}{3}$ where $\psi > 0$. The grand canonical partition function is $Z_L = F(z)^L$ with $F(z) = \sum_n f(n)z^n = 1 + \frac{z}{1-\delta z}$, where fugacity z controls the particle density through $\rho(z) = zF'(z)/F(z) = z[(1-\delta z)(1+z-\delta z)]^{-1}$.

Finally, the current is,

$$J = \frac{1}{2F(z)^2}[(F(z)+2\psi-2e^{-\varepsilon}-1)(F(z)-1-z) + 2z(1-\psi)(F(z)-1)]. \quad (4.49)$$

Figure 4.3 (solid line) shows J as a function of ε for density $\rho = 0.15$; NDM is observed for $\varepsilon \gtrsim 0.9$.

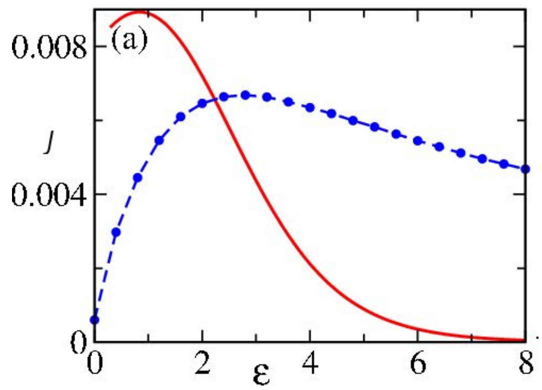

Figure 4.3: Current j versus ε for AMP dynamics (4.47) for density $\rho = 0.15$. Circle: $\psi = 1/(1+\varepsilon)$ (simulations), solid line: exact results for ψ given by Eq. (4.48)

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4.2.5 Condensation in AMAP

In this section, we turn our attention to AMAP models which give rise to condensation transition. A typical example of such asymmetric rate functions in AMAP that lead to condensation is the following, where we consider rates $u_{R,L}(m,n)$ that fall in the special class of AMAP hop rates represented by Eq. (4.34) with $w(m) = \frac{1}{1+b}(1+\frac{b}{m})$ (for $m \geq 1$),

$$u_L(m,n) = u_R(n+1,m-1)\,\frac{1+\frac{b}{m}}{1+\frac{b}{n+1}}. \quad (4.50)$$

This model would result in a FSS given by Eq. (4.11) along with the single site steady state weight

$$f(n) \;=\; \frac{n!(b+1)^n}{(b+1)_n}, \tag{4.51}$$

where $(c)_n = c(c+1)\ldots(c+n-1)$is the Pochhammer symbol. Now, we can calculate the grand canonical partition function $Z = F(z)^L$ where $F(z) = \sum_{n=0}^{\infty} \frac{n!(1+b)^n}{(1+b)_n} z^n$. Thus z varies in the range $(0, z_c)$ where $z_c = (1+b)^{-1}$ is the radius of convergence of $F(z)$. The density of the system is now $\rho(z) = z\frac{F'(z)}{F(z)}$; the critical density above which condensation takes place is

$$\rho_c = \rho(z_c) = \begin{cases} \infty & b \leq 2 \\ \frac{1}{b-2} & b > 2. \end{cases} \tag{4.52}$$

Thus, for AMAP with dynamics (4.50), the system under consideration can macroscopically distribute any number of particles if $b \leqslant 2$. However, for $b > 2$, the maximum allowed density is $\rho_c = \frac{1}{b-2}$ and if ρ is larger than ρ_c, a macroscopic number, $(\rho - \rho_c)L$, of particles gather on some particular site resulting in the formation of a single site condensate.

Let us consider another example that belong to the class of models (4.37), where both the left and right hop rates are in product form. As discussed earlier, in this case one can obtain a factorized steady state as in (4.32) with any arbitrary choice of $w(m)$ when the the functions $c_{R,L}(.)$ appearing in the hop rates follow the recursion relation (4.38). Now for a specific choice $w(m) = \frac{m^\gamma d}{(d+(m-1)^\gamma)}$ ($d > 0$ and $\gamma > 0$), the recursion relation can be solved,

$$c_{R,L}(m) = c_{R,L}(0) \left[1 + \sum_{k=1}^{m} \frac{d^k}{((m-k)!)^\gamma} \left(\frac{c_{R,L}(1)}{c_{R,L}(0)} - 1 \right)^k \prod_{p=m-k}^{m-1} \frac{1}{(d+p^\gamma)} \right]. \tag{4.53}$$

Thus, with above choice of $c_{R,L}(.)$ the rate functions (4.37) gives rise to the stationary state

$$P(\{m_i\}) = \frac{1}{Z_{L,N}} f(m_i)\delta\left(\sum_{i=1}^{L} m_i - N\right) \quad \text{with} \quad f(m) = \frac{f(0)}{d^m (m!)^\gamma} \prod_{k=0}^{m-1} (d + k^\gamma). \qquad (4.54)$$

It is interesting to note, even though the dynamics of these asymmetric misanthrope process is different, the steady state weight of the model for $\gamma = 1$ is identical to the steady state of the inclusion process studied in [86]. For general $\gamma > 2$, the steady state is also same as that of the misanthrope process studied in [179], which exhibit explosive condensation transition.

4.3 Asymmetric finite range process (AFRP)

Factorized steady states are a very special type of stationary measure but it is not a generic feature of systems out of equilibrium. Stochastic processes like ZRP, AZRP, MAP, AMAP constitute a specific class of non-equilibrium processes that enjoy the simplicity of FSS. But one can also have pair factorized steady state (PFSS) [72] and cluster factorized steady state (CFSS) [34] for generic models where particle interaction extends beyond departure and arrival sites. Such finite range processes (FRP) introduce spatial correlations among occupation at different sites leading to condensates spreading over a finite region. Shape and size of these extended condensates has been extensively studied in these systems [181]. In this section, we would like to focus on asymmetric FRP in $1d$ where the rate functions depend on occupation of K-nearest neighbors both to right and left of the departure site but the functional form of the hop rates now depend on the direction (left or right) of hopping. We would like to find out specific and sufficient conditions that must be obeyed by an asymmetric finite range process (AFRP) to achieve a cluster factorized steady state (CFSS).

4.3.1 Model

Consider a one dimensional periodic lattice with L sites labeled by $i = 1, 2, \ldots, L$. Each site i contains an integer number of particles $n_i (\geqslant 0)$. A particle from each site i (with $n_i > 0$) can hop either to its nearest right neighbor $(i + 1)$ with rate $u_R(n_{i-K}, n_{i-K+1}, \ldots, n_i, n_{i+1} \ldots, n_{i+K-1}, n_{i+K})$ or it can hop to left nearest neighbor $(i - 1)$ at a rate $u_L(n_{i-K} \ldots, n_{i-1}, n_i \ldots n_{i+K})$. So both the right and left rate functions depend on $(2K + 1)$ terms, namely the departure site and its K nearest neighbors in both right and left directions. The $(2K + 1)$ arguments of $u_{R,L}(\ldots)$ are spatially ordered, i.e., arguments 1 to $(2K + 1)$ correspond to occupancy of site $i - K$ to $i + K$ respectively. Thus, the argument $(K + 1)$ corresponds to the occupancy of the departure site i, and the arguments $(K + 2)$ and K are the occupancy of the arrival site for right and left moves respectively. We assume that a cluster factorized steady state is possible for AFRP, as given below, and derive consistently the constraint required on the rate functions to obtain such a state.

4.3.2 Criterion on the rates for CFSS: h-balance scheme

A cluster factorized steady state is represented by

$$P(\{n_i\}) \sim \prod_{i=1}^{L} g(n_i, n_{i+1}, \ldots, n_{i+K}) \delta(\sum_{i=1}^{L} n_i - N), \tag{4.55}$$

where we call $g(.)$ the cluster weight function that depends on $(K + 1)$ variables. In the steady state, with suitable rearrangement of terms, the Master equation of AFRP can be written as a sum of L terms, each one being a unique function $F(.)$ of $(2K + 3)$ arguments $(n_{i-K-1}, \ldots, n_{i-1}, n_i, n_{i+1}, \ldots, n_{i+K+1})$. So, in the steady state,

$$\frac{d}{dt}P(\{n_i\}) = \sum_{i=1}^{L} F(n_{i-K-1},\ldots,n_{i-1},n_i,n_{i+1},\ldots,n_{i+K+1}) \;=\; 0. \tag{4.56}$$

A sufficient condition to satisfy the above equation (4.56) is when each of the L terms in the right hand side individually vanish, i.e., $F(n_{i-K-1},\ldots n_i\ldots,n_{i+K+1}) = 0$ for every i ($i = 1,2,\ldots,L$). Clearly this condition is too restrictive and it is *not* a necessary condition for having CFSS. We restrict ourselves to this simple case which effectively leads to,

$$\begin{aligned} &u_R(n_{i-K},\ldots,n_i,n_{i+1}\ldots,n_{i+K}) + u_L(n_{i-K},\ldots,n_{i-1},n_i\ldots,n_{i+K}) \\ &= u_R(n_{i-K-1}\ldots,n_{i-1}+1,n_i-1\ldots n_{i+K-1}) \prod_{j=i-K-1}^{i} \frac{g(\tilde{n}_j,\tilde{n}_{j+1},\ldots,\tilde{n}_{j+K})}{g(n_j,n_{j+1},\ldots,n_{j+K})} \\ &+ u_L(n_{i-K+1}\ldots,n_i-1,n_{i+1}+1\ldots n_{i+K+1}) \prod_{j=i-K}^{i+1} \frac{g(\hat{n}_j,\hat{n}_{j+1},\ldots,\hat{n}_{j+K})}{g(n_j,n_{j+1},\ldots,n_{j+K})} \\ &. \end{aligned} \tag{4.57}$$

Here $\tilde{n}_j = n_j + \delta_{j,i-1} - \delta_{j,i}$ and $\hat{n}_j = n_j - \delta_{j,i} + \delta_{j,i+1}$. This constraint (4.57) on the rate functions can be satisfied by a family of hop rates, parametrized by $\delta > 0$ and $\gamma > 0$,

$$\begin{aligned} u_R(n_{i-K},\ldots,n_i,n_{i+1}\ldots,n_{i+K}) \;&=\; \delta \frac{g(n_{i-K},n_{i-K+1},\ldots,n_i-1)}{g(n_{i-K},n_{i-K+1},\ldots,n_i)} \\ &\times \prod_{j=i-K+1}^{i} g(\hat{n}_j,\hat{n}_{j+1},\ldots,\hat{n}_{j+K}) + \gamma \prod_{j=i-K}^{i} \frac{g(\bar{n}_j,\bar{n}_{j+1},\ldots,\bar{n}_{j+K})}{g(n_j,n_{j+1},\ldots,n_{j+K})} \\ u_L(n_{i-K},\ldots,n_{i-1},n_i\ldots,n_{i+K}) \;&=\; \delta \prod_{j=i-K}^{i-1} g(\tilde{n}_j,\tilde{n}_{j+1},\ldots,\tilde{n}_{j+K}) \frac{g(n_i-1,n_{i+1}\ldots n_{i+K})}{g(n_i,n_{i+1}\ldots n_{i+K})} \end{aligned} \tag{4.58}$$

where the newly introduced $\bar{n}_j = n_j - \delta_{j,i}$ and δ,γ are constant parameters.

Let us consider the simplest case of AFRP, where particle interaction extends over a range $K = 1$. In this case, we expect a pair factorized steady state $P(\{n_i\}) \sim$

$\prod_i g(n_i, n_{i+1})\delta(\sum_{i=1}^{L} n_i - N)$ when hop rates are,

$$
\begin{aligned}
u_R(k,m,n) = \frac{g(k,m-1)}{g(k,m)}\left[\delta g(m-1,n+1) \,+\, \gamma\frac{g(m-1,n)}{g(m,n)}\right] \\
u_L(k,m,n) = \delta g(k+1,m-1)\frac{g(m-1,n)}{g(m,n)}.
\end{aligned}
\tag{4.59}
$$

Note that for $\gamma = 0$, the hop rates satisfy detailed balance, and for $\gamma = 1, \delta = 0$, we recover the usual condition required for pair factorized states discussed in [72].

4.3.3 Discussion on the possibility of current reversal and NDM in AFRP

We observe that, current reversal is not possible for these particular set of rate functions in Eq. (4.59) which result in pair factorized steady states. This is because, the current in these models turns out to be $J = \gamma z$, which is just proportional to the fugacity z and since density $\rho(z)$ is a monotonic function of z, it is not possible to reverse the direction of the current by changing $z(\geq 0)$ or equivalently the density $\rho(z)$. In fact, for $K > 1$ also the rate functions in Eq. (4.58) give the same average current $J = \gamma z$, meaning that there is no current reversal by tuning of the fugacity or density for these class of models. However, the possibility of current reversal with a CFSS produced by asymmetric right-left rate functions in one dimension is still not ruled out, because, to satisfy the Master equation in the steady state, one may find a balance condition different from the one used here; then J may not take such a simple form. Also, though it seems technically tedious, one might try from Eq. (4.58) to choose suitably the rates such that there are non-driven and driven modes present in the system and then may try to obtain negative differential mobility with the corresponding rates by slowing down the non-driven modes.

4.3.4 Condensation in AFRP

Another common feature of AZRP and AMAP is the formation of condensates which, unlike current reversal, can also be observed in case of AFRP within the framework of rate functions given by Eq. (4.58). We illustrate this briefly with a simple example. For $K = 1$, let us choose $g(m,n) = \frac{m+n+1}{(m+1)^b}$, where b is a tunable parameter indicating the onset of condensation. The corresponding right-left hop rates are

$$u_R(k,m,n) = \frac{k+m}{k+m+1}\left[\delta\frac{m+n+1}{m^b} + \gamma(1+\frac{1}{m})^b\frac{m+n}{m+n+1}\right]$$

$$u_L(k,m,n) = \delta\frac{k+m+1}{(k+2)^b}\,(1+\frac{1}{m})^b\frac{m+n}{m+n+1}. \tag{4.60}$$

Using the transfer matrix formalism developed in [34], one can calculate the partition function $Q_L(z)$ in the grand canonical ensemble, where z is the fugacity associated with a particle in GCE and subsequently one can also obtain the density $\rho(z)$. Now if we proceed to calculate the critical density $\rho_c = \lim_{z\to 1}\rho(z)$, we find that for $b \le 4$, ρ_c diverges indicating that the system remains in the fluid phase for $b \le 4$ at any density. Whereas, when $b > 4$, we have a finite value of the critical density given by

$$\rho_c = \frac{\xi_1(b-1) - 2\xi_2(b) + \xi_3(b)}{2\xi_2(b) + 2\zeta(b-1)\sqrt{\xi_2(b)}} + \frac{\zeta(b-2) - \zeta(b-1)}{\sqrt{\xi_2(b)} + \zeta(b-1)}. \tag{4.61}$$

where $\xi_k(b) = \zeta(b)\zeta(b-k)$ and $\zeta(b)$ are Riemann zeta functions. So, for $b > 4$, if the density of the system is greater than the critical density, i.e., $\rho > \rho_c$, one can observe a macroscopic number of particles $(\rho - \rho_c)L$ gathering at a single but arbitrary lattice site forming a single site condensate.

One can also observe spatially extended condensates in AFRP like the one discussed in [72], only this time with asymmetric rate functions given by

$$u_R(k,m,n) = \begin{cases} e^{U\delta_{m,1}}[e^{-\lambda(n-m+3)} + e^{-2\lambda}\theta(m-n) + e^{2\lambda}(1-\theta(m-n))] & m \le k, n+2 \\ e^{U\delta_{m,1}}[e^{-\lambda(m-n-3)} + e^{2\lambda}] & m > k, n+2 \\ e^{U\delta_{m,1}}[e^{-\lambda(n-m+1)} + \theta(m-n) + e^{2\lambda}(1-\theta(m-n))] & m > k, m \le n+2 \\ e^{U\delta_{m,1}}[e^{-\lambda(m-n-1)} + 1] & m \le k, m > n+2 \end{cases}$$

and

$$u_L(k,m,n) = \begin{cases} e^{-\lambda(k-m+3)+U\delta_{m,1}} & m \le k+2, n \\ e^{-\lambda(m-k-3)+U\delta_{m,1}} & m > k+2, n \\ e^{-\lambda(k-m+1)+U\delta_{m,1}} & m \le k+2, m > n \\ e^{-\lambda(m-k-1)+U\delta_{m,1}} & m > k+2, m \le n. \end{cases}$$

These rate functions lead to a PFSS with $g(m,n) = e^{-\lambda|m-n|+\frac{U}{2}(\delta_{m,0}+\delta_{n,0})}$. Here λ, U are the parameters that can be tuned to study the possibility of a condensation transition. As discussed in [72], if $\lambda > \lambda_c = U - \ln(e^U - 1)$, a macroscopic number of particles condensate over a spatial extent $O(L^{1/2})$ when the density ρ exceeds a critical value $\rho_c = \frac{1}{e^{2(\lambda-\lambda_c)}-1}$. Since this asymmetric FRP shares the same steady state, we expect a similar condensation transition here.

In brief, we have discussed the possibility of formation of both single site and extended condensates in case of AFRP with $K = 1$.

4.4 Summary

In this chapter, we generalize FRP with *unidirectional* motion of particles studied in the previous chapter by introducing *asymmetric* transfer of particles to their neighbors. In these models, both right and left hop rates depend on the occupation of the departure site and their neighbors, but their functional forms are different. In usual driven diffusive systems the asymmetric rate appears from spatial inhomogeneity created by an external potential, which does not depend on the microscopic occupation. However it is not difficult to imagine, in fact actually has been shown recently, through simulations [169] and in biological systems [82, 110, 122], that geometric irregularity can result in asymmetric diffusion of particles. It is interesting to ask what kind of rate functions are realistic for a particular geometry and the answer to this question is not understood well. So, here we focus on generic asymmetric rate functions and derive a sufficient condition on them so that the steady state measure can be calculated exactly. Various stochastic processes that belong to the class of such asymmetric models are asymmetric zero-range process (AZRP), asymmetric misanthrope process (AMAP) and the most generic case, asymmetric finite range process (AFRP).

Unlike ZRP, which has a factorized steady state (FSS) for *any* hop rate $u(n)$, AZRP with rate functions $u_{R,L}(n)$ lead to FSS only when the rate-functions obey a specific condition, Eq. (4.7). On the other hand, a desired FSS as in Eq. (4.11) can *always* be obtained from a two parameter family of AZRP having left and right hop rates described by Eq. (4.10).

It is well known [34, 77] that misanthrope processes cannot have a cluster-factorized steady state and its steady state has a factorized form only for certain hop rates $u(m,n)$ which satisfy Eq. (4.28). AMAP shares the same feature but with a different constraint on the rate functions; it leads to a FSS only when the hop rates $u_{R,L}(m,n)$ follow Eq. (4.33). Both AZRP and AMAP show a condensation transition similar to other models having an FSS. Interestingly in the case of AZRP, the condensation transition can be induced or broken by tuning the relative choice of $u_{R,L}(n)$, i.e., by changing the factor that decides how often a right move occurs with respect to a left move.

The important role of asymmetric dynamics, both in AZRP and AMAP, appears in the particle current. Unlike ZRP or MAP where the direction of current is fixed by the external bias, here the direction can get reversed by changing only the particle density with the parameters in the dynamics kept fixed- resulting in the phenomena of density dependent current reversal. Another interesting feature exhibited by AMAP with specific choice of rates is the negative differential mobility when the particle current surprisingly decreases with increasing bias.

We also extend this idea of asymmetry between right-left hop rates to obtain a cluster-factorized steady state in AFRP. In particular, we describe specific examples where the rate functions depend on the occupation of departure site and its two nearest neighbors (right and left), but the functional form for the right hop is different from that of the left; in this case we have obtained a sufficient condition required for a pair factorized state. Also, these examples include the formation of both localized and extended condensates. The general condition required for AFRP to have CFSS is much more complicated and we could not obtain the most generic class of rates which satisfy this constraint. However, we have discussed a specific family of models parametrized by two constants although they do not show density dependent current reversal or negative differential mobility.

An interesting point to note in context of the phenomena of current reversal is that, at the point of reversal, the current becomes zero, now at this particular density we have a non-equilibrium steady state where the dynamics does not satisfy detailed balance but still the particle current is zero. So, it would be interesting to study how to differentiate these zero-current non-equilibrium systems from equilibrium ones which are also zero-current states.

In the next chapter we generalize finite range processes further where the hop rates may depend on the occupation of the departure site and that of K_r neighbors to right and K_l ($\neq K_r$ in general) neighbors to left, i.e., the number of right neighbors and left neighbors are not the same unlike the rates of the FRP. This helps us in considering misanthrope process and its generalizations to be a part of the generic class of finite range processes. Also, the study of these finite range processes with different range of neighbors in the following chapter gives us a new method of solving the steady state measure which is the *matrix product ansatz for interacting particle systems without hardcore constraints.*

Chapter 5

AFRP with different range of neighbors

In this chapter, we are going to generalize the dynamics of the finite range process (FRP) and asymmetric finite range process (AFRP) studied in the previous chapters in such a manner that in the hop rates, the range of neighboring lattice sites (with respect to the departure site) in different directions, for example number of right neighbors and number of left neighbors of the departure site on a one dimensional lattice, are different. Note that these ranges, i.e., the number of neighboring lattice sites along different directions with respect to the departure site, are the same for the FRP and AFRP discussed in chapters 3. and 4 respectively. Interestingly, with these modifications change the steady state structures substantially. Also, as we will see soon, other than the flux cancellation schemes used up to now, there might be other ways which can be handy to obtain the steady states of these modified models. In particular, in the present chapter, we are going to introduce matrix product ansatz for interacting particle systems without hardcore repulsion with dynamics that in general represents AFRP with different range of neighbors. Apart from examples of this general process, we will also show that for specific choice of the hop rates, some of the steady states obtained in the previous chapters 3. and 4 can be recast in the form of the matrix product states that will be introduced soon in this chapter.

Matrix product ansatz (MPA) [27] for interacting particle systems following hardcore constraint is known to be one of the most useful and elegant analytical tool for finding NESS. Soon after being introduced in context of totally asymmetric simple exclusion process (TASEP) [56], MPA has found enormous applications in different branches of physics. MPA has been very helpful in calculating spatial correlation functions for exclusion processes with point objects [17] as well as for extended objects [36] in one dimension. Study of relations between algebraic Bethe ansatz [121] and matrix product states for stochastic Markovian models in $1d$ [89] and the same for spin$-\frac{1}{2}$ Heisenberg chains [117] brought calculational convenience and also gave good physical insight to the problems. In connection to correlated non-equilibrium systems, MPA can describe asymptotic distributions of the sum of correlated random variables [5]. Moreover, as a natural extension of MPA on discrete lattice, continuous matrix Product States(cMPS) have been introduced as variational states for $1d$ continuum models [178] and cMPS have already proved to be convenient in studying Bose gas in $1d$ [147] , interacting spin$-\frac{1}{2}$ systems [45] etc. In a nutshell, MPA is attracting interest in vast research areas starting from condensed matter physics to quantum information [156].

In matrix product ansatz (MPA), any configuration $\{n_i\}$ ($n_i = 0$ or 1 for exclusion processes) in the configuration space is represented by a matrix string $\{A_i^\alpha\}$, where each matrix A_i^α represents either a vacancy($\alpha = 0$) or a particle of any one of the species($\alpha = 1, 2, \dots$) present at site i. Generally the representation of the matrices A_i^α do not depend on the site index i. But, notably, particles of different species and vacancies are denoted by matrices that are in general *non-commuting* ($[A^\alpha, A^\beta] \neq 0$ for $\alpha \neq \beta$). If the system is open one needs additional vectors (say $\langle W|$ and $|V\rangle$) to represent the boundaries. The MPA assumes that the steady state weights of the configuration $\{n_i\}$ to be,

$$P(\{n_i\} \propto \begin{cases} \mathrm{Tr}[\prod_{i=1}^L A_i^\alpha] & \text{periodic} \\ \langle W| \prod_{i=1}^L A_i^\alpha |V\rangle & \text{open.} \end{cases} \tag{5.1}$$

A specific dynamics on the lattice insist the matrices and vectors (if present) to satisfy a set of equations, commonly known as *matrix algebra*. Any representation (of the matrices) that satisfy this matrix algebra provide a steady state solution of the respective dynamics. An important point to note is, in all these systems particles are constrained by *hard-core interaction*s, that lead to *a finite number* of algebraic equations to be satisfied.

In this chapter, we study interacting particle systems *without hardcore interaction*, where each lattice site can be occupied by any number of particles. To form a matrix product state (MPS), thus, we require infinitely many matrices; any given dynamics of the system would then insist on a algebra containing infinitely many matrix equations. It is not a-priori clear whether such a steady state in matrix product form is at all possible. Here we show that, *if the matrices are function of the occupation numbers*, the matrix algebra for a class of models reduce to a single functional relation which is easier to deal with. In fact, a solution to *this* functional relation eventually leads to an exact steady state weights of the model. We demonstrate this in a class of interacting particle systems where particles hop to one of the nearest neighbor with rates that depend on the occupation of the departure site and its neighbors within a finite range.

5.1 Matrix Product Ansatz in absence of exclusion

In this section we introduce the matrix product formulation for interacting particle systems in absence of hardcore exclusion, i.e., the systems allow multiple occupancy at any lattice site. We first consider a generic stochastic process where particles execute directed motion on a periodic lattice in one dimension ($1d$). The dynamics of the model is totally asymmetric in a sense that particles here hop along a specified direction with hop rates depending on the occupancy of several sites- namely the departure site, its left neighbors within a range K_l and right neighbors within a range K_r. Below we describe the model in details.

Let the sites of the periodic lattice be labeled by $i = 1, 2, ..., L$. With each site i, is associated a non- negative integer variable $n_i (\geqslant 0)$ representing the number of particles at that site (for a vacant site $n_i = 0$). The dynamics is as follows. A particle from a randomly chosen site i hops to its right neighbor $i + 1$ with rate $u(n_{i-K_l}, .., n_{i-1}, n_i, n_{i+1}, .., n_{i+K_r})$:

$$
\begin{aligned}
(\dots, n_{i-1}, n_i, n_{i+1}, \dots) \quad & \longrightarrow (\dots, n_{i-1}, n_i - 1, n_{i+1} + 1, \dots) \\
& \text{with } \textbf{rate } u(n_{i-K_l}, \dots, n_i, \dots, n_{i+K_r}). \qquad (5.2)
\end{aligned}
$$

Clearly, this driven non-equilibrium dynamics conserves the total number of particles (N) in the system. The Master equation dictating the evolution of probability $P(\{n_i\})$ of every configuration $\{n_i\}$ of the system reads

$$
\frac{d}{dt} P(\{n_i\}) \quad = \quad \sum_{i=1}^{L} u(n_{i-K_l}, \dots, n_i, \dots, n_{i+K_r}) P(\{n_i\})
$$

$$- \sum_{i=1}^{L} u(n_{i-K_l}, \ldots, n_i + 1, n_{i+1} - 1, .., n_{i+K_r}) \\ \times P(\ldots, n_i + 1, n_{i+1} - 1, \ldots). \tag{5.3}$$

In steady state, the net probability flux must vanish for each configuration $\{n_i\}$, i.e., the total out-flux (the first sum on the right hand side of Eq. (5.3)) must balance the in-flux (the second sum). This cancellation may occur in several different ways, with detailed balance being one of the *special* cases which, if exists, guarantees equilibrium. Pairwise balance is another special condition giving rise to non-equilibrium steady states.

For the dynamics in Eq. (5.2), to ensure that the in-flux is balanced by the out-flux we first make an ansatz that the steady state weight $P(\{n_i\})$ can be written in the matrix product form

$$P(\{n_i\}) = \frac{1}{Q_{L,N}} \mathrm{Tr}\,[\prod_{i=1}^{L} A(n_i)]\,\delta(\sum_i n_i - N), \tag{5.4}$$

where any configuration is represented by a string of L matrices, $A(n_k)$ being the matrix associated with k-th site containing n_k particles. The δfunction here ensures the particle number conservation and $Q_{L,N}$ is the canonical partition function.

5.1.1 flux cancellation scheme: auxiliary matrices

Now for the ansatz to be a valid one, we must ensure that the matrices in Eq. (5.4) satisfy Eq. (5.3) in steady state. This can be achieved by constructing a suitable cancellation scheme involving additional *auxiliary matrices* [56]. In this context, we propose the following cancellation scheme,

$$u(n_{i-K_l}, \ldots, n_i, n_{i+1} \ldots, n_{i+K_r}) \quad A(n_{i-K_l}) \ldots A(n_i) A(n_{i+1}) \ldots A(n_{i+K_r}) -$$

$$u(n_{i-K_l},..n_i+1,n_{i+1}-1..n_{i+K_r})A(n_{i-K_l})..A(n_i+1)A(n_{i+1}-1)..A(n_{i+K_r})$$

$$= A(n_{i-K_l})..\widetilde{A}(n_i)A(n_{i+1})..A(n_{i+K_r}) - A(n_{i-K_l})..A(n_i)\widetilde{A}(n_{i+1})..A(n_{i+K_r})$$

$$= A(n_{i-K_l})\dots A(n_{i-1})[\widetilde{A}(n_i)A(n_{i+1}) - A(n_i)\widetilde{A}(n_{i+1})]A(n_{i+2})\dots A(n_{i+K_r}) \tag{5.5}$$

where we have introduced a new set of *auxiliary matrices* matrices $\tilde{A}(n)$. It is straightforward to check that the above cancellation-scheme satisfies the Master equation Eq. (5.3) in steady state. What remains, is to find suitable representation of the set of matrices $\{A(n_i)\}$ and the *auxiliary matrices* $\{\widetilde{A}(n_i)\}$ which follow the *matrix-algebra* given by Eq. (5.5).

A sufficient condition (though not necessary) that satisfies Eq. (5.5) is

$$\begin{aligned} u(n_{i-K_l},\dots,n_i,n_{i+1},\dots,n_{i+K_r}) \quad & A(n_{i-K_l})\dots A(n_i)A(n_{i+1})\dots A(n_{i+K_r}) \\ &= A(n_{i-K_l})\dots A(n_{i-1})\widetilde{A}(n_i)A(n_{i+1})\dots A(n_{i+K_r}) \end{aligned} \tag{5.6}$$

$$\begin{aligned} u(n_{i-K_l},..n_i+1,n_{i+1}-1..,n_{i+K_r})&A(n_{i-K_l})..A(n_i+1)A(n_{i+1}-1)..A(n_{i+K_r}) \\ &= A(n_{i-K_l})\dots A(n_{i-1})A(n_i)\widetilde{A}(n_{i+1})\dots A(n_{i+K_r}). \end{aligned} \tag{5.7}$$

Now both of the above equations (5.6) and (5.7) are satisfied consistently if we choose the auxiliary matrix $\widetilde{A}(n)$ to be

$$\widetilde{A}(n) = A(n-1) \quad \text{for} \quad n \geqslant 0 \tag{5.8}$$

along with $\widetilde{A}(0) = 0$. With this choice, Eqs. (5.6) and (5.7) reduces to

$$\begin{aligned} u(n_{i-K_l},\dots,n_i,n_{i+1},\dots,n_{i+K_r}) \quad & A(n_{i-K_l})\dots A(n_i)A(n_{i+1})\dots A(n_{i+K_r}) \\ &= A(n_{i-K_l})\dots A(n_{i-1})A(n_i-1)A(n_{i+1})\dots A(n_{i+K_r}). \end{aligned} \tag{5.9}$$

Thus the matrix product ansatz, formulated here for systems with multiple site occupancy, finally leads to a unique set of equations, as above. For a given model, with totally asymmetric hop rate $u(.)$, we have to solve the matrix algebra (5.9) to find a possible representation of $\{A(0), A(1), \dots\}$. Practically this is a difficult task as we need to solve infinitely many matrix equations to be satisfied by an infinitely large set of matrices $\{A(0), A(1), \dots\}$ which are non commuting and, in principle, independent and unrelated to each other. However for a generic class of models, which we discuss in the following sections, it is possible to find a matrix representation where $A(n)$ is a function of n; i.e., all elements of the matrix $A(n)$ are specific functions of n represented by $A(n)_{i,j} = f_{i,j}(n)$. In that case, we don't need to solve the matrix algebra to obtain $A(0), A(1), A(2), \dots$ separately, rather we should obtain the general matrix function $A(n)$ by treating the algebra (5.9) as a single equation of the matrix function $A(n)$. Once we find such a matrix function $A(n)$ any desired matrix $A(k)$ can be obtained just by putting the desired value $n = k$. This is indeed possible for a large class of hop rates which we are going to discuss in details in the following sections. Also, since $A(n)$ is now a general matrix function of the site occupation variable n, we call it *the site occupation matrix.*

In order to proceed further, we need to be specific about the dynamics as the matrix algebra (5.9) explicitly depend on the hop rates. In the next section we will discuss the formulation of matrix product ansatz for the generic finite range process, which is an interacting particle system where particles do not obey hardcore constraints.

5.2 MPA for Finite Range Process

In this section we consider finite range process(FRP) where particles hop to right with a hop rate that depends on occupation of the departure site and its neighbor within a range K_l to left and K_r to right. In particular we discuss different cases, $K_l = 0 = K_r$ (ZRP), $K_l = 0, K_r = 1$ (namely misanthrope process), $K_l = 1 = K_r$ (systems having PFSS) and the generic scenario with $K_l = K = K_r$. For $K_l = K_r = K$, the hop rate $u(n_{i-K}, \dots, n_i, \dots, n_{i+K})$ in FRP depends on *same* number (K) of neighbors in both directions with respect to the departure site i. This special case was studied earlier [34] and it was shown that, when the hop-rates obey certain conditions, FRP leads to a $(K+1)$-cluster factorized steady state (CFSS), $P(\{n_i\}) \sim \prod_{i=1}^{L} g(n_i, n_{i+1}, \dots n_{i+K})$. For $K = 1$ we have a 2-cluster factorized state, commonly known as the pair factorized state $P(\{n_i\}) \sim \prod_{i=1}^{L} g(n_i, n_i + 1)$. Clearly, a pair factorized state can equivalently be written as a matrix product state as $g(n_i, n_i + 1)$ can directly be considered as the elements of an infinite matrix T, i.e., $T_{n_i,n_{i+1}} = g(n_i, n_{i+1})$. In the following we show that, whenever a PFSS is possible, the matrix product ansatz also naturally lead to the same. For $K > 1$, however, existence of a cluster factorized state does not ensure that it can also be written as matrix product state. In this section, we show that, even for $K > 1$, one can construct matrix product states through the matrix formulation developed in the previous section. Below, some examples of totally asymmetric finite range processes for which one obtains steady states in matrix product form, are discussed in details.

5.2.1 Zero range process ($K_l = 0 = K_r$)

Zero range process (ZRP) is a very familiar stochastic process where hop rate of particles depends only on the occupation of the departure site; thus FRP reduces to ZRP when $K_l = K_r = 0$. One important feature of ZRP is that its steady state has a simple factorized form irrespective of the functional form of the hop rates, lattice geometry or spatial dimension. In spite of having a rather simple dynamics, ZRP shows condensation transition for specific choice of rates - the condensation transition can be mapped to a phase separation transition in an equivalent exclusion process. Interestingly related phenomena like wealth condensation [31] in agent based models, jamming in traffic flow [43] can be related to condensation transition in ZRP.

Clearly ZRP fits into the generic matrix product formulation discussed in previous section, as the hop rate $u(n_{i-K_l}, \ldots, n_i, \ldots, n_{i+K_r})$ here is equivalent to $u(n_i)$. Thus for ZRP, the matrix algebra in Eq. (5.9) reduces to

$$\begin{aligned} u(n_i) \quad & A(n_{i-K_l}) \ldots A(n_i) A(n_{i+1}) \ldots A(n_{i+K_r}) \\ = & A(n_{i-K_l}) \ldots A(n_{i-1}) A(n_i - 1) A(n_{i+1}) \ldots A(n_{i+K_r}), \end{aligned} \tag{5.10}$$

along with the auxiliary matrix $\widetilde{A}(n) = A(n-1)$, as in Eq. (5.8). First we try for a scalar solution to $A(n)$ by setting $A(n) = a(n)$, a positive function for $n \geq 0$. This particular choice implies that the auxiliary matrices for ZRP are also scalar, $\widetilde{A}(n) = a(n-1)$. So Eq. (5.10) simplifies to

$$u(n_i) a(n_i) = a(n_i - 1) \qquad \Rightarrow \qquad a(n) = \frac{a(n-1)}{u(n)} = a(0) \prod_{j=1}^{n} \frac{1}{u(j)} \tag{5.11}$$

Thus, the steady state weight is

$$P(\{n_i\}) \sim \mathrm{Tr}[\prod_{i=1}^{L} A(n_i)] = \prod_{i=1}^{L} a(n_i) \tag{5.12}$$

which is the familiar factorized steady state we know for ZRP [71]. The matrix $A(n)$ being scalar, there are no spatial correlations between occupation of particles at different lattice sites, apart from the global conservation of the total number of particles.

5.2.2 Misanthrope Process ($K_l = 0, K_r = 1$)

Misanthrope process is a special case of FRP with $K_l = 0$ and $K_r = 1$, i.e., the hop rate $u(n_i, n_{i+1})$ here is no longer departure-site symmetric, it depends on the occupation number of the departure site(i) and arrival site($i+1$) only. For certain choice of hop rates, misanthrope process is known to have a factorized steady state as studied in [47, 77]. Here we will show that the same factorized state can be obtained starting from matrix product ansatz. Note, that the generic choice of auxiliary matrices $\widetilde{A}(n) = a(n-1)$ along with $A(n) = a(n)$, given by Eqs. (5.8) and (5.9), would result in $u(n_i, n_{i+1}) = \frac{a(n_i-1)}{a(n_i)}$ which is inconsistent as this choice does not allow the hop rate to depend on the occupation of the arrival site. We now proceed with a scalar choice $A(n) = a(n), \widetilde{A}(n) = \widetilde{a}(n)$, where both functions $a(n)$ and $\widetilde{a}(n)$ are yet to be determined. The cancellation scheme in Eq. (5.5) now becomes

$$u(n_i, n_{i+1}) - u(n_i+1, n_{i+1}-1)\frac{a(n_i+1)a(n_{i+1}-1)}{a(n_i)a(n_{i+1})} = \frac{\widetilde{a}(n_i)}{a(n_i)} - \frac{\widetilde{a}(n_{i+1})}{a(n_{i+1})}. \quad (5.13)$$

Since the hop rates $u(m, n) = 0$ for $m < 1$ or for $n < 0$, the above equation, for $n_{i+1} = 0$ and for $n_i = 0$ reduces to,

$$u(n, 0) = \frac{\widetilde{a}(n)}{a(n)} - \frac{\widetilde{a}(0)}{a(0)} = u(1, n-1)\frac{a(1)}{a(0)}\frac{a(n-1)}{a(n)}. \quad (5.14)$$

These equations further results in,

$$a(n) = \frac{a(1)}{a(0)}\frac{u(1,n-1)}{u(n,0)}a(n-1) = a(0)\left(\frac{a(1)}{a(0)}\right)^n \prod_{k=1}^{n} \frac{u(1,k-1)}{u(k,0)} \tag{5.15}$$

$$\widetilde{a}(n) = \left[\frac{\widetilde{a}(0)}{a(0)} + u(n,0)\right] a(n) \tag{5.16}$$

It appears from Eq. (5.15) that we have a factorized steady state $P(\{n_i\}) \propto \prod_{i=1}^{L} a(n_i)$ for any hop rate $u(m,n)$ which is certainly not true, because the equations (5.14) and (5.16), derived by using specific boundary conditions ($n_i = 0$ or $n_{i+1} = 0$) must also respect Eq. (5.13) for all $n_i > 0, n_{i+1} > 0$. Using Eq. (5.16) in Eq.(5.13) we get

$$u(n_i, n_{i+1}) - u(n_i+1, n_{i+1}-1)\frac{a(n_i+1)a(n_{i+1}-1)}{a(n_i)a(n_{i+1})} = u(n_i,0) - u(n_{i+1},0). \tag{5.17}$$

Thus, in MAP, we have a factorized steady state $P(\{n_i\}) \propto \prod_{i=1}^{L} a(n_i)$ only when the hop rate $u(n_i, n_{i+1})$ satisfy Eq. (5.17), which is the familiar constraint that has been reported earlier [77]. The steady state weights, given by Eq. (5.15) is also identical to the one which is already known for MAP [77]. Thus, clearly the matrix product formulation leads to the correct steady state measure and the condition on hop rates for its validity.

5.2.3 FRP with $K_l = K_r = K = 1$

If $K_l = K_r = K = 1$, particle from a randomly chosen site i hops to site $(i+1)$ with rate $u(n_{i-1}, n_i, n_{i+1})$ that depends on the occupancies of the departure site (i) , its *left nearest* and *right nearest* neighbors ($i-1$ and $i+1$ respectively). For a special class of hop rates, this finite range process has a pair factorized steady state(PFSS) [72] given by

$$P(\{n_i\}) = \frac{1}{Z_{L,N}} \prod_{i=1}^{L} g(n_i, n_{i+1}) \delta(\sum_i n_i - N) \tag{5.18}$$

Obviously $g(n_i, n_{i+1})$ itself can be considered as the elements of an infinite dimensional matrix as n_i and n_{i+1} can take arbitrarily large positive integer values. In other words, PFSS is a matrix product state, represented by infinite dimensional matrices. We show below that for a class of models one can obtain finite dimensional representation.

Let us consider hop rates of the form

$$u(n_{i-1}, n_i, n_{i+1}) = \frac{\langle \alpha(n_{i-1}) \mid \beta(n_i - 1)\rangle \langle \alpha(n_i - 1) \mid \beta(n_{i+1})\rangle}{\langle \alpha(n_{i-1}) \mid \beta(n_i)\rangle \langle \alpha(n_i) \mid \beta(n_{i+1})\rangle}, \tag{5.19}$$

$\alpha(n)$ and $\beta(n)$ are arbitrary positive functions with $\alpha(-1) = \beta(-1) = 0$. It is easy to see that the matrix algebra (5.9), along with the choice of auxiliary $\widetilde{A}(n) = A(n-1)$ as in Eq. (5.8), can be satisfied if,

$$A(n) = |\beta(n)\rangle \langle \alpha(n)|. \tag{5.20}$$

Correspondingly, the steady state probability of configurations are $P(\{n_i\}) \sim \mathrm{Tr}(\prod_i A(n_i))\delta(\sum_i n_i - N)$. The grand canonical partition function is then

$$Z_L(z) = \sum_{\{n_i\}} z^{n_i} \mathrm{Tr}(\prod_i^L A(n_i)) = \mathrm{Tr}[T(z)^L]; \quad T(z) = \sum_n z^n |\beta(n)\rangle\langle\alpha(n)| \tag{5.21}$$

Now one can conveniently calculate steady state average of desired observables in the steady state, like spatial correlations, density fluctuations, particle current etc.. For example, since the particle hop towards right only, the average steady state current of the system is

$$J = \langle u(n_{i-1}, n_i, n_{i+1})\rangle = \frac{1}{Z_L(z)} \sum_{\{n_i\}} u(n_1, n_2, n_3) z^{n_i} \mathrm{Tr}[\prod_i^L A(n_i)] = z. \tag{5.22}$$

To find the dependence of J on the average particle density ρ, one can calculate $\rho(z) = \frac{z}{Z_L}\frac{d}{dz}Z_L$ and then invert this relation.

5.2.4 Finite range process with $K_l = K_r = K > 1$

For a more general finite range process (FRP) corresponding to $K_l = K_r = K > 1$ the hop rate $u(n_{i-K}, \ldots, n_i, \ldots, n_{i+K})$ is a function of $(2K+1)$ site variables, namely the occupation number of the departure site and that of K neighbors to its left and to right. This model was introduced earlier in Ref. [34], where it has been shown that the steady state of the system is cluster factorized when the hop rates $u(.)$ satisfy certain specific conditions. For a cluster factorized steady state (CFSS), the probability of configurations are given by,

$$P(\{n_i\}) = \frac{1}{Z_{L,N}}\prod_{i=1}^{L} g(n_i, n_{i+1}, \ldots, n_{i+K})\delta(\sum_i n_i - N), \tag{5.23}$$

where $g(n_i, n_{i+1}, \ldots, n_{i+K})$, a function of $(K+1)$ variables, is called to be the cluster weight function and $Z_{L,N}$ is the canonical partition function. The authors in [34] have restricted their study to FRP where the cluster weight function has a 'sum-form' $g(n_i, n_{i+1}, \ldots, n_{i+K}) = \sum_{k=0}^{K} f_k(n_{i+k})$. For example, when $K = 2$, FRP has a 3-cluster factorized steady state with weight function $g(n_i, n_{i+1}, n_{i+2}) = \gamma_0(n_i) + \gamma_1(n_{i+1}) + \gamma_2(n_{i+2})$ if the hop rate (that satisfies the required condition) is

$$u(n_{i-2}, n_{i-1}, n_i, n_{i+1}, n_{i+2}) = \prod_{k=0}^{2} \frac{\gamma_0(n_{i-2+k}) + \gamma_1(n_{i-1+k}) + \gamma_2(n_{i+k} - 1)}{\gamma_0(n_{i-2+k}) + \gamma_1(n_{i-1+k}) + \gamma_2(n_{i+k})}. \tag{5.24}$$

Clearly $g(.)$ being a function of $(K+1)$ variables, unlike for $K = 1$ case, it can not be considered directly as matrix. Thus, for $K > 1$, rewriting a cluster factorized steady state as a matrix product state is already challenging. Moreover, here we will discuss more generalized forms of the hop rates which does not necessarily lead to the 'sum-form' of the cluster weight function. Let us consider an example

$K_l = K_r = K = 2$, where a particle from a randomly chosen site i hops to its right neighbor $(i+1)$ with a rate

$$u(n_{i-2}, n_{i-1}, n_i, n_{i+1}, n_{i+2}) = \prod_{k=0}^{2} \frac{\langle f_0(n_{i-2+k})|f_1(n_{i-1+k})\rangle + \langle f_2(n_{i-1+k})|f_3(n_{i+k}-1)\rangle}{\langle f_0(n_{i-2+k})|f_1(n_{i-1+k})\rangle + \langle f_2(n_{i-1+k})|f_3(n_{i+k})\rangle} \quad (5.25)$$

where $\langle f_\nu(n)| = (h^1_\nu(n), h^2_\nu(n), h^2_\nu(n), \dots h^d_\nu(n))$ are d-dimensional row-vectors and $|f_\nu(n)\rangle = \langle f_\nu(n)|^T$ (here $\nu = 0, 1, 2, 3$). In fact the rates here satisfy the conditions required for a system to have 3-cluster factorized steady state $P(\{n_i\}) \sim \prod_i g(n_i, n_{i+1}, n_{i+2})$ with

$$g(l, m, n) = \langle f_0(l)|f_1(m)\rangle + \langle f_2(m)|f_3(n)\rangle. \quad (5.26)$$

Although we have exact steady state weights for these rates, it is not very useful in calculating the partition function or other physical observables. This is because, any occupation variable n_i appears thrice in the cluster factorized state and carrying out the sum over the all possible values of n_i is not straightforward. In this regards, the matrix formulation, where the matrices are parametrized by the *local* occupation number, is very helpful. In the following we proceed with the MPA and use the auxiliary matrices $\widetilde{A}(n) = A(n-1)$, as in Eq. (5.8). The matrices $A(n)$ should then follow the matrix algebra given by Eq. (5.9) with hop rate there replaced by Eq. (5.25). We find that this algebra is satisfied by the following representation of matrices,

$$A(n) = (|\beta(n)\rangle \otimes I)\, \Gamma(n)\, (I \otimes \langle\alpha(n)|) \quad (5.27)$$

where,

$$|\beta(n)\rangle = \begin{pmatrix} 1 \\ |f_3(n)\rangle \end{pmatrix}; \quad \langle\alpha(n)| = \begin{pmatrix} \langle f_0(n)| & 1 \end{pmatrix} \quad (5.28)$$

are $(d+1)$-dimensional vectors and

$$\Gamma(n) = \begin{pmatrix} |f_1(n)\rangle & 0_{d\times d} \\ 0 & \langle f_2(n)| \end{pmatrix} \tag{5.29}$$

is a $(d+1)$-dimensional matrix. Also, I is the identity matrix in $(d+1)$ dimension. The operation $\otimes$ is the familiar *direct product.* Note that $|\beta(n)\rangle \otimes I$ and $I \otimes \langle\alpha(n)|$ are not square matrices; their dimensions are respectively $(d+1)^2 \times (d+1)$ and $(d+1) \times (d+1)^2$.

Thus, the dimension of the matrices $A(n)$ that represent the steady state weights are $(d+1)^2$. In the next section, we have discussed how to generate the matrix representation systematically for a dynamics (5.25) or equivalently for a model which has a cluster factorized steady state with weight factor $g(.)$ given by Eq. (5.26).

Let us illustrate the dynamics and the steady state weights for a specific example where the hop rates are given by Eq. (5.25) with scalar choice of $\langle f_\nu(n)|$, i.e, $\langle f_\nu(n)| = f_\nu(n) = |f_\nu(n)\rangle$. Explicitly, the hop rates are now

$$u(n_{i-2}, n_{i-1}, n_i, n_{i+1}, n_{i+2}) = \prod_{k=0}^{2} \frac{f_0(n_{i-2+k})\ f_1(n_{i-1+k}) + f_2(n_{i-1+k})\ f_3(n_{i+k}-1)}{f_0(n_{i-2+k})\ f_1(n_{i-1+k}) + f_2(n_{i-1+k})\ f_3(n_{i+k})}. \tag{5.30}$$

For this simple choice of hop rate,

$$|\beta(n)\rangle = \begin{pmatrix} 1 \\ f_3(n) \end{pmatrix};\ \langle\alpha(n)| = \begin{pmatrix} f_0(n) & 1 \end{pmatrix};\ \Gamma(n) = \begin{pmatrix} f_1(n) & 0 \\ 0 & f_2(n) \end{pmatrix},$$

and correspondingly the steady state matrix $A(n)$, from Eq. (5.27), reduces to a 4-dimensional matrix

$$A(n) = \begin{pmatrix} f_0(n)f_1(n) & f_1(n) & 0 & 0 \\ 0 & 0 & f_0(n)f_2(n) & f_2(n) \\ f_0(n)f_1(n)f_3(n) & f_1(n)f_3(n) & 0 & 0 \\ 0 & 0 & f_0(n)f_2(n)f_3(n) & f_2(n)f_3(n) \end{pmatrix}.$$

Thus, we obtain the matrix product steady state $P(\{n_i\}) \sim \mathrm{Tr}[\prod_{i=1}^{L} A(n_i)]$ for the dynamics (5.30). As we have already mentioned, the steady state of this dynamics has 3-cluster factorized form $P(\{n_i\}) \sim \prod_i g(n_i, n_{i+1}, n_{i+2})$. Finally, once the representation of matrices $A(n)$ as in (5.27) are known, it is quite straightforward to calculate the partition function and any desired observable.

5.2.5 Matrix product form of cluster factorized steady states

We have seen in this section that the matrix product ansatz naturally leads to a cluster factorized steady state if the dynamics of the system allows one. Depending on the dynamics of the model, MPA results in a specific matrix-algebra, but there are no systematic methods for obtaining matrix representation from a given algebra. Thus for models that have a cluster factorized steady state, it is useful to construct the matrices from the known steady state whenever possible. We must recall that, for FRP with $K \geq 2$, calculating the partition function or average value of observables is not straightforward even when the exact steady state weights are known in cluster factorized form; in such situations the matrix formulation is certainly a relief.

To this end we construct the matrices from a 3-cluster factorized steady state; it is straightforward to generalize this for larger clusters. Let us consider a specific CFSS, $P(\{n_i\}) \sim \prod_i g(n_{i-1}, n_i, n_{i+1})$ with

$$g(k, l, m) = \langle f_0(k)|f_1(l)\rangle + \langle f_2(l)|f_3(m)\rangle. \tag{5.31}$$

where $\langle f_\nu(n)| = (h_\nu^1(n), h_\nu^2(n), h_\nu^2(n), \dots h_\nu^d(n))$ are d-dimensional row-vectors and $|f_\nu(n)\rangle = \langle f_\nu(n)|^T$ (here $\nu = 0, 1, 2, 3$). This form of the cluster weight function can be rewritten as inner product of vectors and matrices each of which now depends on *a single* individual occupation number. More precisely,

$$g(k,l,m) = \langle\alpha(k)|\Gamma(l)|\beta(m)\rangle, \tag{5.32}$$

$$\text{where } \langle\alpha(k)| = \begin{pmatrix} \langle f_0(k)| & 1 \end{pmatrix} \; ; \; |\beta(m)\rangle = \begin{pmatrix} 1 \\ |f_3(m)\rangle \end{pmatrix}$$
$$\text{and } \Gamma(l) = \begin{pmatrix} |f_1(l)\rangle & 0_{d\times d} \\ 0 & \langle f_2(l)| \end{pmatrix}. \tag{5.33}$$

Now the steady state weights can be written as

$$\begin{aligned} P(\{n_i\}) &\sim \prod_i g(n_{i-1}, n_i, n_{i+1}) \\ &= \langle\alpha(k)|\Gamma(l)|\beta(m)\rangle \; \langle\alpha(l)|\Gamma(m)|\beta(n)\rangle \; \langle\alpha(m)|\Gamma(n)|\beta(p)\rangle \; \langle\alpha(n)|\Gamma(p)|\beta(q)\rangle \ldots \\ &= \mathrm{Tr}\left[\Gamma(l)|\beta(m)\rangle\langle\alpha(l)| \;\; \Gamma(m)|\beta(n)\rangle\langle\alpha(m)| \;\; \Gamma(n)|\beta(p)\rangle\langle\alpha(n)|\ldots\right] \\ &= \mathrm{Tr}\left[G(l,m) \;\; G(m,n) \;\; G(n,p)\ldots\right] \end{aligned} \tag{5.34}$$

Thus we have transformed the 3-cluster weight functions to a matrix product form with matrices $G(l,m) = \Gamma(l)|\beta(m)\rangle\langle\alpha(l)|$ depending on occupancy of two neighboring sites. To get matrices $A(n)$ which depend *only on a single site occupation number,* as in the matrix product ansatz (5.4), we proceed as follows. Since the direct product of any two vectors $|b\rangle$ and $\langle a|$ can be written as

$$|b\rangle\langle a| = (I \otimes \langle a|)(|b\rangle \otimes I) \tag{5.35}$$

with I being the identity matrix of same dimension as that of $|b\rangle$ and $\langle a|$, we rewrite $G(l,m)$ as

$$G(l,m) = \Gamma(l)|\beta(m)\rangle\langle\alpha(l)| = \Gamma(l) \; (I \otimes \langle\alpha(l)|) \; (|\beta(m)\rangle \otimes I). \tag{5.36}$$

Using this in Eq. (5.34) we get

$$P(\{n_i\}) \sim \mathrm{Tr}[\prod_i A(n_i)]$$

$$\text{with } A(n) = (|\beta(n)\rangle \otimes I)\, \Gamma(n)\, (I \otimes \langle\alpha(n)|) \tag{5.37}$$

So, we have demonstrated how to obtain a matrix product form from a known 3-cluster factorized steady state. There is no particular difficulty in extending this formulation to systems with larger cluster factorized steady state (like the FRP with $K > 2$).

5.3 MPA for finite range process with asymmetric rate functions

In the previous sections, we have studied finite range processes where particles hop only to the right. In fact, if we introduce a parameter p, the probability that a particle chooses the right neighbor as a target site and moves there with rate $u(.)$ or other wise (i.e., with probability $1 - p$ it decides to hop to left and moves there with the same rate $u(.)$), the steady state measure of FRP remains invariant. A non-trivial situation is when the functional form of rate functions for right hop is different from that of the left hop. A class of such asymmetric motion of particles without hardcore constraints has recently been introduced and studied in [33] in context of asymmetric zero range process (AZRP), asymmetric misanthrope process (AMAP) and asymmetric finite range process (AFRP)- each of them having exactly solvable non-equilibrium invariant measures (factorized steady states (FSS) for AZRP, AMAP and cluster factorized steady states (CFSS) for AFRP). AZRP, AMAP show interesting features like density dependent current reversal (keeping the external bias fixed), condensation (tuned by the proportion of right and left moves executed by the particles)- phenomena solely induced by different functional forms of the left and right rates. Now, AZRP and AMAP having FSS, would not be of much interest in context of matrix product states since in previous sections we have already discussed how the matrices and the auxiliaries reduce to scalars for a steady state to have a factorized form. So we would like to explore only the possibility of obtaining a matrix product state for AFRP. In this section we will first introduce a very general dynamics for asymmetric hopping process in one dimension which includes AFRP as a special case. We will then illustrate the matrix formulation with some examples.

5.3.1 General asymmetric hopping dynamics

Let us consider an interacting particle system on a one dimensional periodic lattice where particles (without hardcore exclusion) can hop in both directions with respective forward and backward rates; the rate functions depend on the occupation of several lattice sites as well as on the direction of motion of the particles, i.e., the right and left hop rates can have different functional forms. The model is defined on a one dimensional periodic lattice with L sites where each site i contains n_i particles with $n_i(\geqslant 0)$ being a nonnegative integer. A particle from a randomly chosen site i (with $n_i > 0$), can move either to its immediate right neighbor $(i+1)$ with rate $u_R(n_{i-K_l}, \dots, n_i, \dots, n_{i+K_r})$ or it can hop to its immediate left neighbor $(i-1)$ with rate $u_L(n_{i-K'_l}, \dots, n_i, \dots, n_{i+K'_r})$. Note that, the model is different from the one discussed in [38] as not only the forward and backward rates have different functional forms $u_R(.)$ and $u_L(.)$, also, they may have different number of arguments; the right hop rate depends on K_l left neighbors and K_r right neighbors in contrast to K'_l left and K'_r right neighbors for the left hop rate. In general, all four numbers K_l, K_r, K'_l, K'_r can be different. We ask if this stochastic process can lead to a non-equilibrium steady state, particularly in matrix product form. Below we study two specific examples.

5.3.1.1 Example 1

Solving the matrix algebra to find out a matrix product state for arbitrary values of K_l, K_r, K'_l, K'_r appears to be quite complex. We restrict our selves to some special cases. Our first example is $K_l = 1 \neq K'_l = 2$ and $K_r = 2 \neq K'_r = 0$, i.e., a particle from site i hops to the right neighbor with rate $u_R(n_{i-1}, n_i, n_{i+1}, n_{i+2})$ and it hops to the left with rate $u_L(n_{i-2}, n_{i-1}, n_i)$. This dynamics has not been studied earlier in context of particle or mass transfer processes and clearly the criteria for having a factorized or cluster-factorized steady state is not known. In the following we show, using a specific example, that one can use MPA to obtain an exact steady state weights of these models in some special cases.

Let us choose the rate functions in the following form

$$\begin{aligned} u_R(n_{i-1}, n_i, n_{i+1}, n_{i+2}) &= u(n_{i-1}, n_i, n_{i+1}) + v(n_i, n_{i+1}, n_{i+2}) \\ u_L(n_{i-2}, n_{i-1}, n_i) &= v(n_{i-2}, n_{i-1}, n_i). \end{aligned} \tag{5.38}$$

Here, the right hop rate $u_R(.)$ is a sum of two independent functions $-$the first part $u(.)$ is symmetric with respect to the departure site and the rest $v(.)$ is arrival-site symmetric. On the other hand, the left hop rate $u_L(.) \equiv v(.)$ is purely arrival-site symmetric. Assuming that the steady state of the model can be written as a matrix product form $P(\{n_i\}) \sim \mathrm{Tr}(\prod_i A(n_i))\delta(\sum_i n_i - N)$, the Master equation for dynamics (5.38) in steady state reduces to,

$$\sum_{i=1}^{L} [u(n_{i-1}, n_i, n_{i+1}) + v(n_i, n_{i+1}, n_{i+2}) + v(n_{i-2}, n_{i-1}, n_i)] \\ \mathrm{Tr}[\dots A(n_{i-2})A(n_{i-1})A(n_i)A(n_{i+1})A(n_{i+2})\dots]$$

$$-\sum_{i=1}^{L}[u(n_{i-2},n_{i-1}+1,n_i-1)\mathrm{Tr}[..A(n_{i-2})A(n_{i-1}+1)A(n_i-1)..]$$
$$+v(n_{i-1}+1,n_i-1,n_{i+1})\mathrm{Tr}[..A(n_{i-1}+1)A(n_i-1)A(n_{i+1})..]$$

$$+v(n_{i-1},n_i-1,n_{i+1}+1)\mathrm{Tr}[..A(n_{i-1})A(n_i-1)A(n_{i+1}+1)..]]=0 \tag{5.39}$$

The above equation can be equivalently written as

$$\sum_{i=1}^{L}\mathrm{Tr}[\dots A(n_{i-2})\mathbf{F}(n_{i-1},n_i,n_{i+1})A(n_{i+2})\dots] \;=\; 0, \tag{5.40}$$

where,

$$\begin{aligned}
&\mathbf{F}(n_{i-1},n_i,n_{i+1}) \;= \\
&[u(n_{i-1},n_i,n_{i+1})A(n_{i-1})A(n_i)A(n_{i+1})-u(n_{i-1},n_i+1,n_{i+1}-1)A(n_{i-1})A(n_i+1)A(n_{i+1}-1)] \\
&+[v(n_{i-1},n_i,n_{i+1})A(n_{i-1})A(n_i)A(n_{i+1})-v(n_{i-1}+1,n_i-1,n_{i+1})A(n_{i-1}+1)A(n_i-1)A(n_{i+1})] \\
&+[v(n_{i-1},n_i,n_{i+1})A(n_{i-1})A(n_i)A(n_{i+1})-v(n_{i-1},n_i-1,n_{i+1}+1)A(n_{i-1})A(n_i-1)A(n_{i+1}+1)].
\end{aligned}$$

Equation (5.40) is a sum of L similar terms where each term carries a three site function $\mathbf{F}(x,y,z)$ that contains the relevant information about the dynamics, i.e., the in-flux and out-flux for a given configuration. So it would be reasonable to find a local three site cancellation scheme for $F(x,y,z)$ that would make the sum of L terms in Eq. (5.40) equal to zero.

5.3.1.2 Flux cancellation scheme

A cancellation scheme, we propose is the following.

$$\begin{aligned}
\mathbf{F}(n_{i-1},n_i,n_{i+1}) \;=\; & [A(n_{i-1})\widetilde{A}(n_i)A(n_{i+1})-A(n_{i-1})A(n_i)\widetilde{A}(n_{i+1})] \\
& +\; [\widehat{A}(n_{i-1})\bar{A}(n_i)A(n_{i+1})-A(n_{i-1})\widehat{A}(n_i)\bar{A}(n_{i+1})] \\
& +\; [A(n_{i-1})\bar{A}(n_i)\widehat{A}(n_{i+1})-\bar{A}(n_{i-1})\widehat{A}(n_i)A(n_{i+1})].
\end{aligned} \tag{5.41}$$

It is easy to check that this form of $\mathbf{F}(n_{i-1},n_i,n_{i+1})$ indeed serves the purpose. Note that, unlike the previous cases where we had only one kind of auxiliary matrix $\widetilde{A}(n)$, here we have used three different auxiliary matrices $\widetilde{A}(n), \widehat{A}(n), \bar{A}(n)$. In fact, if all three auxiliaries were same, i.e., $\widetilde{A}(n) = \widehat{A}(n) = \bar{A}(n)$, then (5.41) reduces to the familiar cancellation scheme studied here in (5.5) with $K_l = K_r = 1$ and correspondingly one obtains a matrix product steady state for totally asymmetric hoping model with hop rate $u_R = u(n_{i-1}, n_i, n_{i+1})$ and $u_L = 0$, a model which we have already discussed in the previous section.

To proceed further with the asymmetric hopping model we need to be more specific about the dynamics, that is one has to choose possible functional forms of $u(.)$ and $v(.)$. If we consider the functions $u(.), v(.)$ to be in the following form

$$
\begin{aligned}
u(n_{i-1}, n_i, n_{i+1}) &= \frac{\langle \alpha(n_{i-1}) \mid \beta(n_i - 1)\rangle\langle \alpha(n_i - 1) \mid \beta(n_{i+1})\rangle}{\langle \alpha(n_{i-1}) \mid \beta(n_i)\rangle\langle \alpha(n_i) \mid \beta(n_{i+1})\rangle} \\
v(n_{i-1}, n_i, n_{i+1}) &= \frac{\langle \alpha(n_{i-1}) \mid \beta(n_i + 1)\rangle\langle \alpha(n_i + 1) \mid \beta(n_{i+1})\rangle}{\langle \alpha(n_{i-1}) \mid \beta(n_i)\rangle\langle \alpha(n_i) \mid \beta(n_{i+1})\rangle},
\end{aligned} \tag{5.42}
$$

then, Eq. (5.41) results in the following solution:

$$
\begin{aligned}
\widetilde{A}(n) &= A(n-1); \bar{A}(n) = A(n+1); \widehat{A}(n) = \theta(n)A(n) \\
A(n) &= |\beta(n)\rangle\langle\alpha(n)|,
\end{aligned} \tag{5.43}
$$

where $\theta(n)$ is the Heaviside step function. So, to summarize, if particles on a one dimensional periodic lattice undergo asymmetric hopping with different right and left rate functions (constructed below by substituting Eq.(5.42) in (5.38))

$$
\begin{aligned}
u_L(n_{i-2}, n_{i-1}, n_i) &= \frac{\langle \alpha(n_{i-2}) \mid \beta(n_{i-1} + 1)\rangle\langle \alpha(n_{i-1} + 1) \mid \beta(n_i)\rangle}{\langle \alpha(n_{i-2}) \mid \beta(n_{i-1})\rangle\langle \alpha(n_{i-1}) \mid \beta(n_i)\rangle} \\
u_R(n_{i-1}, n_i, n_{i+1}, n_{i+2}) &= \frac{\langle \alpha(n_{i-1}) \mid \beta(n_i - 1)\rangle\langle \alpha(n_i - 1) \mid \beta(n_{i+1})\rangle}{\langle \alpha(n_{i-1}) \mid \beta(n_i)\rangle\langle \alpha(n_i) \mid \beta(n_{i+1})\rangle}, \\
&+ u_L(n_i, n_{i+1}, n_{i+2})
\end{aligned} \tag{5.44}
$$

along with $u_R(x, 0, z, w) = 0$ and $u_L(x, y, 0) = 0$, the steady state of the model has a matrix product form $P(\{n_i\}) \sim \mathrm{Tr}(\prod_i A(n_i))\delta(\sum_i n_i - N)$ with matrices

$A(n) = |\beta(n)\rangle\langle\alpha(n)|$ and the auxiliary matrices $\widetilde{A}(.)$,$\widehat{A}(.)$ and $\bar{A}(.)$ given by Eq. (5.43).

We conclude this subsection with the following remark. Matrices $A(n)$ we obtain for the asymmetric hopping dynamics (5.44) are same as those we obtain for dynamics (5.19). The auxiliary matrices in two cases are different, but they do not explicitly appear in the steady state weights. This indicates that these two very different dynamics lead to the same steady state measure.

5.3.1.3 Example 2

In this example we study an asymmetric finite range process where $K_l = K_r = K'_l = K'_r = 1$. In details, we consider a one dimensional periodic lattice with L sites with each site i containing $n_i(\geq 0)$ particles and a particle from a randomly chosen site i (if not vacant) jumps either to its right neighbor $(i+1)$ with a hop rate $u_R(n_{i-1}, n_i, n_{i+1})$ or to its left neighbor $(i-1)$ with rate $u_L(n_{i-1}, n_i, n_{i+1})$. In this model both the right and left rate functions are symmetric with respect to the the departure site (i). Let us assume that the steady state probability of any configuration $\{n_i\}$ of this stochastic process can be expressed as a product of matrices in the form $P(\{n_i\}) \sim \mathrm{Tr}(\prod_i A(n_i))\delta(\sum_i n_i - N)$ where $A(n_i)$ is the site occupation matrix corresponding to site i containing n_i particles. The steady state Master equation for this interacting particle system reads as

$$\begin{aligned}
\sum_{i=1}^{L}[u_R(n_{i-1}, n_i, n_{i+1}) + u_L(n_{i-1}, n_i, n_{i+1})]\mathrm{Tr}[\ldots A(n_{i-1})A(n_i)A(n_{i+1})\ldots] \\
-\sum_{i=1}^{L}[u_R(n_{i-2}, n_{i-1}+1, n_i-1)\mathrm{Tr}[\ldots A(n_{i-2})A(n_{i-1}+1)A(n_i-1)\ldots] \\
+u_L(n_i-1, n_{i+1}+1, n_{i+2})\mathrm{Tr}[\ldots A(n_i-1)A(n_{i+1}+1)A(n_{i+2})\ldots]] &= 0
\end{aligned} \tag{5.45}$$

Shifting the sum indexes in Eq. (5.45) and rearranging them suitably, we arrive at $\sum_{i=1}^{L} \mathrm{Tr}[\ldots A(n_{i-2})\ \mathbf{F}(n_{i-1}, n_i, n_{i+1})\ A(n_{i+2}) \ldots] \ = \ 0$, where

$$\begin{aligned}\mathbf{F}(n_{i-1}, n_i, n_{i+1}) \ = \ & [u_R(n_{i-1}, n_i, n_{i+1}) + u_L(n_{i-1}, n_i, n_{i+1})]\, A(n_{i-1})A(n_i)A(n_{i+1}) \\ & - u_R(n_{i-1}, n_i + 1, n_{i+1} - 1)\ A(n_{i-1})A(n_i + 1)A(n_{i+1} - 1) \\ & - u_L(n_{i-1} - 1, n_i + 1, n_{i+1})\ A(n_{i-1} - 1)A(n_i + 1)A(n_{i+1}) \qquad (5.46)\end{aligned}$$

So, just like the previous example, the Master equation in steady state has been written as a sum of L terms each containing a three site function $F(x, y, z)$, which we must write in a way using auxiliaries so that the terms within the sum cancel with each other. To this end, we further specify the rate functions $u_{R,L}(.)$ as

$$\begin{aligned}u_R(n_{i-1}, n_i, n_{i+1}) &= \gamma\ \frac{\langle \alpha(n_{i-1}) \mid \beta(n_i - 1)\rangle\langle \alpha(n_i - 1) \mid \beta(n_{i+1})\rangle}{\langle \alpha(n_{i-1}) \mid \beta(n_i)\rangle\langle \alpha(n_i) \mid \beta(n_{i+1})\rangle} \\ &+ \ \delta\ \frac{\langle \alpha(n_{i-1}) \mid \beta(n_i - 1)\rangle}{\langle \alpha(n_{i-1}) \mid \beta(n_i)\rangle}\langle \alpha(n_i - 1) \mid \beta(n_{i+1} + 1)\rangle \\ u_L(n_{i-1}, n_i, n_{i+1}) &= \delta\ \langle \alpha(n_{i-1} + 1) \mid \beta(n_i - 1)\rangle \frac{\langle \alpha(n_i - 1) \mid \beta(n_{i+1})\rangle}{\langle \alpha(n_i) \mid \beta(n_{i+1})\rangle}. \qquad (5.47)\end{aligned}$$

These hop rates resemble the rate functions considered by the authors in [38] in context of asymmetric finite range process.

5.3.1.4 Flux cancellation scheme:

Here too, we use three auxiliary matrices $\widetilde{A}$, $\widehat{A}$ and $\bar{A}$ but now the last two auxiliary matrices are functions of two arguments whereas $\widetilde{A}$ has one argument as in earlier cases. Explicitly, the cancellation scheme reads as,

$$\begin{aligned}\mathbf{F}(n_{i-1}, n_i, n_{i+1}) \ = \ & [A(n_{i-1})\widetilde{A}(n_i)A(n_{i+1}) - A(n_{i-1})A(n_i)\widetilde{A}(n_{i+1})] \\ & + \ [A(n_{i-1})\widehat{A}(n_i, n_{i+1})A(n_{i+1}) - \widehat{A}(n_{i-1}, n_i)A(n_i)A(n_{i+1})] \\ & + \ [A(n_{i-1})\bar{A}(n_{i-1}, n_i)A(n_{i+1}) - A(n_{i-1})A(n_i)\bar{A}(n_i, n_{i+1})].\end{aligned} \qquad (5.48)$$

One can easily check that Eq. (5.48) satisfies the steady state condition (5.46) and it results in a matrix product state with matrices $A(n)$ in the familiar form

$$A(n) \;=\; |\beta(n)\rangle\langle\alpha(n)|. \tag{5.49}$$

The corresponding choice of auxiliary matrices are then

$$\widetilde{A}(n) = \gamma\ A(n-1), \quad \widehat{A}(m,n) = \delta\ A(m-1)|\beta(n+1)\rangle\langle\alpha(m)|,$$
$$\bar{A}(m,n) = \delta\ |\beta(n)\rangle\langle\alpha(m+1)|A(n-1). \tag{5.50}$$

So, if we have an asymmetric particle transfer process with right and left rate functions expressed by (5.47) we have a matrix product steady state, same as the one obtained for dynamics (5.44) or for (5.19).

However, it should be mentioned that the cancellation scheme used here in Eq. (5.48) is again very different from the schemes used in the previous examples.

5.4 Summary

In this chapter, we have introduced matrix product ansatz for systems of interacting particles *without any hardcore constraints.* In these models, particles on a one dimensional lattice jump to their neighboring sites with some rate that depends on the occupation of the departure site and its neighbors within a specified range. In case of MPA for exclusion processes, where particles obey hard core constraints, we need only a few matrices, each representing one species (of particle). Here, the sites can either be vacant or occupied by arbitrary number of particles and thus a matrix product state that describe these systems would require infinite number of matrices (in contrast to the hardcore exclusion processes), each corresponding to a specific occupation number. Again, any given dynamics would insist the matrices to follow an *algebra*, consisting of infinitely many matrix-relations. Finding specific representation of these infinite set of matrices appears to be complex, but here, in this chapter, for a generic class of models, we show that the matrices can be parametrized by the occupation number (which essentially leads to the name *site occupancy matrices* of the matrices $A(n)$), i.e., the elements of the matrix are functions of the occupation number. This parametrization actually helps to treat the infinite set of matrix algebra as a single equation of the matrix function $A(n)$- which can be solved once and for all for any general n, so that one no more has to solve for the matrices $A(0), A(1), A(2)\dots$ separately.

The class of hopping models we studied here is very general; many well known models, like zero range process, misanthrope process, models with pair factorized steady state, and finite range processes are only some of the special cases, for which the exact steady state weights are already known. Here, first we re-derive the steady state weights for these familiar stochastic processes using matrix product formulation.

We also study FRP for very general rates which has not been studied in the previous chapters, and show that their steady state can be expressed as matrix product states. A specific example is FRP with $K = 2$, which leads to a 3-cluster factorized steady state with weights $P(\{n_i\}) \sim \prod_{i=1}^{L} g(n_{i-1}, n_i, n_{i+1})$ when the hop-rates satisfy a specific condition. Even when the steady state is known exactly, for a genetic form of weight function $g(.)$ in $K = 2$ case, there are practical difficulties in calculating the partition function or average steady state values of the observables; this is because any particular occupation variable n_i appears thrice in the product and carrying out sum of n_i for all possible values is non-trivial. For some special cases, like when the weight function has a sum form

$g(k,l,m) = f_0(k) + f_1(l) + f_2(m)$ one can write the steady state in a matrix product form, where matrices depend on only a single occupation variable n_i which enables us to carry out the corresponding sum over n_i. In [34], re-writing the 3-cluster factorized steady state in a matrix product form was only a mathematical trick, a relation between the matrices and dynamics of the system were not established. When $g(k,l,m)$ has a 'sum-form', the matrix product ansatz formulated here leads to a matrix algebra which is naturally satisfied by the matrices constructed in [34]. Moreover we explicitly derive matrix representations for certain other class of weight functions $g(k,l,m) = f_0(k)f_1(l) + f_2(l)f_3(m)$ and more generally for $g(l,m,n) = \langle f_0(l)|f_1(m)\rangle + \langle f_2(m)|f_3(n)\rangle$. However there are no well defined methods to obtain matrix representation from a given matrix algebra. Fortunately for systems having a cluster factorized steady state, the matrix representations can be derived systematically which we have discussed this in details.

We further study asymmetric finite range processes where the rate functions for right and left hops are different in the sense that they may have different number of arguments and/or different functional forms. In particular, we introduce a model where the hop rate for right move $u_R(.)$ depends on occupation of departure sites, K_l neighbors to its left and K_r neighbors to the right. Whereas the left hop rate $u_L(.)$ depends on the departure site and $K'_{l,r}$ sites to its left and right respectively. We obtain matrix product steady state for two specific cases (i)$K_l = 1 \neq K'_l = 2$ and $K_r = 2 \neq K'_r = 0$, (ii) $K_l = K_r = K'_l = K'_r = 1$. Interestingly, both models lead to same matrix product steady state, but the auxiliaries, used in the cancellation scheme to satisfy the Master equation in steady state are very different.

There are many other interesting directions to pursue in the study of matrix product formulation for interacting particles in absence of hardcore constraints. One important direction is to investigate the open systems, where particle can enter (say from left boundary) and exit from the system (from right boundary). It is well known that open exclusion processes (EP), where particles obey hardcore constraints, give rise to interesting results; even the simplest case, namely totally asymmetric simple exclusion process (TASEP) which is exactly solved through MPA, shows rich variety of phases and transitions among them as the entry and exit rate of particles are varied [56]. One can also study exclusion processes that can be mapped to a particular finite range process. It is well known that, steady weight of exclusion processes can always be written in matrix product form if they can be mapped to zero-range process; in this situation explicit representations can be obtained from the known steady state weights of the corresponding zero range process, which helps in finding spatial correlation in EP. In a similar fashion, using matrix product formulation, one can study the spatial correlation functions in exclusion processes which can be mapped to finite range processes.

We conclude with a general comment that chapters 3., 4. and the present chapter 5.- all three of them deal with interacting particle systems *without hardcore exclusion*, the corresponding steady state measures have been obtained exactly using different flux cancellation schemes that include pairwise balance condition, h-balance scheme and matrix algebra involving auxiliary matrices following the matrix product ansatz. In the next chapter, we would like to turn our attention towards another class of exactly solvable models *with the particles obeying hardcore exclusion* and we will see how the flux cancellation schemes we have used up to now in context of systems without hardcore repulsion, also prove to be helpful in the next chapter.

Chapter 6

Exclusion models: Multi species assisted exchange models

In the present chapter, we consider an interacting particle system where the particles do feel hardcore repulsions and thus a single lattice site can not be occupied by more than one particle. A typical example of the exclusion processes is the totally asymmetric simple exclusion process (TASEP) [171]. In particular, the steady state of TASEP with open boundaries through which particles can enter or exit the system has been solved exactly [55], and it is well known that TASEP exhibits novel phase transitions as the particle entrance and exit rates vary. Among many generalizations of TASEP, a few are the asymmetric simple exclusion process (ASEP) [139] considering the motion of the particles in both directions in one dimension, multi-species [69] models with particles of several species, restricted exclusion process [19] with restricted motion of the particles etc. Among these, the restricted asymmetric simple exclusion process (RASEP) exhibits absorbing phase transition due to the fact that the particles can move only when they are assisted by other particles so that isolated particles cannot move. So, RASEP can be considered as a single species assisted hopping model where the hopping of a particle is assisted by others. The aim of this chapter is to study the steady state and observables in a system with multiple species of particles undergoing an assisted exchange dynamics with each other in a way such that the rates depend on the neighboring particles other than the pair of particles that are being exchanged.

More elaborately, in this chapter, we introduce a class of assisted exchange models on a one dimensional periodic lattice where each site can be occupied by exactly one particle of any of the types $k = 0, 1, \dots K$. In these models, a particle at any site can exchange its position with one of its nearest neighbors with a rate that depends on both, the type of particle pair which are exchanged, and the type of particle present at the left most neighbor of the exchanging-pair. We primarily address two questions about the $(K+1)$- species assisted exchange models (AEM). The first one is to find the exact steady state measure of this non-equilibrium system $-$in particular we derive the conditions under which the steady state has a pair-factorized form using the h-balance scheme. We argue that for any finite K, a pair factorized steady state can not give rise to phase separation transition; in other words the systems in this case remains in a mixed phase exhibiting nontrivial spatial correlations. We also aim at obtaining the exact steady state current of each particle species. It turns out that AEM exhibits density dependent current reversal and negative differential mobility of particle current, which have been subjects of interest in recent years [35, 38].

Multi-species models with simple exchange dynamics, where exchange of different type of particle pairs occur with different rates, have been introduced earlier

[106]. It turns out that steady state of these models can not be written in pair-factorized form, but there can be a matrix product steady state with matrices satisfying a *diffusion algebra*. Some explicit examples of these models with $K = 1, 2$ have been discussed in Refs. [71, 106]. In fact, asymmetric or symmetric exclusion process ($K = 1$) [139] belong to these class of models with $K = 1$. Some other examples, with $K = 2$ are two species exclusion models [71], ABC model [9, 46] and extended AHR model [8]; some of these models, like ABC model, exhibits phase separation transition in one dimension.

6.1 Model

Consider a system of $(K + 1)$species of particles on a one dimensional periodic lattice with L sites represented by $i = 1, 2, \ldots, L$. Each site i can be occupied by exactly one particle of any of the types $k = 0, 1, \ldots, K$; accordingly the site variable s_i takes an non-negative integer value smaller than $(K+1)$. The dynamics of the model is given by

$$XIJ \underset{u(X,J,I)}{\overset{u(X,I,J)}{\leftrightharpoons}} XJI, \tag{6.1}$$

where $u(X, I, J)$ are the exchange rates. Clearly $u(X, I, J) = 0$ when $I = J$ The exchange dynamics (6.1), by definition, conserves the particle number N_k of each kind - the model has K conservation laws along with the trivial one $\sum_{k=0}^{K} N_k = L$.

In some examples we discuss here, sites with $s_i = k = 0$ are considered as vacant sites; in that case there are only K-species of particles of type $k = 1, 2, \ldots, K$; the exchange of a particle with 0 present in the left (or right) neighbor will then represent hopping of that particle to left (or right) - the density conservation of each species $\rho_k = \frac{N_k}{L}$, where N_k is the number of particles of type k, remains unaltered.

6.1.1 Steady state: h-balance scheme

The Master Equation describing the time evolution of the probabilities of different configurations following the dynamics (6.1) is as follows

$$\begin{aligned}\frac{d}{dt}P(\{s_i\}) &= \sum_{i=1}^{L} u(s_{i-1}, s_{i+1}, s_i) P(\ldots s_{i-1}, s_{i+1}, s_i, s_{i+2} \ldots) \\ &- \sum_{i=1}^{L} u(s_{i-1}, s_i, s_{i+1}) P(...s_{i-1}, s_i, s_{i+1}, s_{i+2}...)\ . \qquad (6.2)\end{aligned}$$

Our first goal is to find the steady state by setting the left hand side of Eq.(6.2) equal to zero. It seems to be quite complex to obtain the steady state for the general dynamics (6.1). Instead we look for class of rates for which we can have a pair factorized steady state (PFSS) for this assisted hopping and exchange model. In case of PFSS (which by construction is a spatially correlated state in contrast to factorized steady states that may be simpler to achieve but do not contain spatial correlations among its constituents), the steady state weight of any possible configuration is expressed as a product of pairs of a function of successive neighbors on the lattice. Explicitly, the steady state weight of any configuration $\{s_i\}$ is given by

$$P(\{s_i\}) \sim \prod_{i=1}^{L} g(s_i, s_{i+1}) \prod_{k=1}^{K} \delta\left(\sum_{i=1}^{L} \delta_{s_i,k} - N_k\right)\ . \qquad (6.3)$$

The right hand side of (6.2) contains the sum of numerous in-flux and out-flux terms, which must equal to zero in the steady state; this cancellation can happen in several ways. To achieve the PFSS described in (6.3), it is sufficient to follow the condition

$$u(s_{i-1}, s_{i+1}, s_i)\frac{g(s_{i-1}, s_{i+1})g(s_{i+1}, s_i)g(s_i, s_{i+2})}{g(s_{i-1}, s_i)g(s_i, s_{i+1})g(s_{i+1}, s_{i+2})} - u(s_{i-1}, s_i, s_{i+1})$$

$$= h(s_{i-1}, s_i, s_{i+1}) - h(s_i, s_{i+1}, s_{i+2}) \tag{6.4}$$

where the function $h(.)$ is yet to be determined suitably. Note the right hand side of the condition Eq. (6.4), when summed over all lattice sites gives zero and ensures stationary, $\frac{d}{dt}P(\{s_i\}) = 0$. Any arbitrary rate $u(s_{i-1}, s_i, s_{i+1})$ does not obey Eq. (6.4). However for a class of rates

$$u(s_{i-1}, s_i, s_{i+1}) \;\; = \;\; \frac{g(s_{i-1}, s_{i+1})}{g(s_{i-1}, s_i)g(s_i, s_{i+1})} \tag{6.5}$$

with $u(s_{i-1}, s_i = k, s_{i+1} = k) = 0 \; \forall \; k$, it is straightforward to check that Eq. (6.4) is satisfied by

$$h(s_{i-1}, s_i, s_{i+1}) \;\; = \;\; -u(s_{i-1}, s_i, s_{i+1}) \; . \tag{6.6}$$

Thus a desired factorized steady state can always be obtained in AEM with dynamics (6.5).

6.1.2 Transfer matrix formulation

The next task would be to calculate the partition function of the $(K+1)$-species exchange model, which is,

$$Q(\{N_k\}) \;\; = \;\; \sum_{\{s_i\}} \prod_{i=1}^{L} g(s_i, s_{i+1}) \prod_{k=0}^{K} \delta\left(\sum_i \delta_{s_i,k} - N_k\right)$$

The δ-functions here ensure that the particle numbers N_ks are conserved. We now work in the grand canonical ensemble (GCE) and associate fugacities $\{z_k\}$, one for each species, which will control the particle densities $\{\rho_k\}$. Also we set $z_0 = 1$, without loss of generality. Hence the partition function in the GCE is

$$Z(\{z_k\}) \;\; = \;\; \sum_{\{N_k\}}^{\infty} Q(\{N_k\}) \prod_k (z_k)^{N_k} \tag{6.7}$$

$$= \;\; \sum_{\{s_i\}} \prod_i z_{s_{i+1}} g(s_i, s_{i+1}) = \mathrm{Tr}[T^L] \tag{6.8}$$

where T is a $(K+1)$ dimensional square matrix

$$T = \sum_{k',k=0}^{K} g(k',k) z_k |k'\rangle\langle k| \tag{6.9}$$

which is formally known as the transfer matrix. Here $\{|k\rangle\}$ with $k = 0, 1, \ldots, K$ are the standard basis vectors in $(K+1)$-dimension. The transfer matrix T can also be written as

$$T = \sum_{k=0}^{K} z_k D_k \quad \text{with} \quad D_k = \sum_{k'=0}^{K} g(k',k) |k'\rangle\langle k| \tag{6.10}$$

where the matrix D_k represents a particle of the species k. With these set of matrices $\{D_k\}$ we write the steady state weights of the system in matrix product form

$$P(\{s_i\}) \sim \prod_i g(s_i, s_{i+1}) = \mathrm{Tr}[\prod_i D_{s_i}] \ . \tag{6.11}$$

In matrix product form, the correspondence of particles by a representing matrix, helps in calculating expectation values of several observables, which is discussed below.

With Eq.(6.10) in hand, we can proceed to calculate different observables analytically. Let us start with density ρ_k of the particles of species k.

$$\rho_k = \langle k \rangle = \frac{\mathrm{Tr}[z_k D_k T^{L-1}]}{\mathrm{Tr}[T^L]} = \frac{\langle k|T^L|k\rangle}{\mathrm{Tr}[T^L]}, \tag{6.12}$$

Let the eigenvalues of T are $\lambda, \lambda_1, \lambda_2, \ldots, \lambda_K$ with corresponding right and left eigenvectors (normalized) $\{|\psi\rangle, |\psi_1\rangle, |\psi_2\rangle \ldots |\psi_K\rangle\}$ and $\{\langle\phi|, \langle\phi_1|, \langle\phi_2| \ldots \langle\phi_K|\rangle\}$ respectively, with λ having the largest absolute value. Since T is a positive matrix (as $g(i,j) > 0$), the largest eigenvalue λ is non-degenerate and the corresponding eigenvector $|\psi\rangle$ can be chosen positive. Thus,

$$T^n = \lambda^n |\psi\rangle\langle\phi| + \sum_{k=1}^{K} \lambda_k^n |\psi_k\rangle\langle\phi_k| \ . \tag{6.13}$$

Using this in Eq. (6.12) we calculate the density of each species in the thermodynamic limit $L \to \infty$ (i.e., $\lambda^L \gg \lambda_k^L \; \forall 1 \leq k \leq K$),

$$\rho_k = \langle k|\psi\rangle\langle\phi|k\rangle. \tag{6.14}$$

In a similar way one can calculate expectation values, that species k is a right neighbor of m in steady state,

$$\begin{aligned}\langle mk\rangle &= \frac{\mathrm{Tr}[z_m z_k D_m D_k T^{L-2}]}{\mathrm{Tr}[T^L]} \\ &= \frac{z_k g(m,k)}{\lambda}\langle k|\psi\rangle\langle\phi|m\rangle,\end{aligned} \tag{6.15}$$

where in the last step we have used the thermodynamic limit. In a similar manner, one can obtain the two-point spatial correlation functions between any two species of particles (say k and k') at a distance r,

$$\begin{aligned}C_{k,k'}(r) &= \frac{\mathrm{Tr}[z_k D_k T^{r-1} z_{k'} D_{k'} T^{L-r-1}]}{\mathrm{Tr}[T^L]} - \rho_k\rho_{k'} \\ &= \frac{\langle k|T^r|k'\rangle\langle k'|T^{L-r}|k\rangle}{\mathrm{Tr}[T^L]} - \rho_k\rho_{k'} \\ &= \langle k'|\psi\rangle\langle\phi|k\rangle\sum_{m=1}^{K}\langle k|\psi_m\rangle\langle\phi_m|k'\rangle\left(\frac{\lambda_m}{\lambda}\right)^r.\end{aligned} \tag{6.16}$$

Here, to obtain the last step, we used Eq. (6.14) and taken the thermodynamic limit. For large r, the dominant contribution to the correlation function comes from the first term $m = 1$ (in the sum), i.e., $C_{k,k'}(r) \sim (\frac{\lambda_1}{\lambda})^r = e^{-r/\xi}$ where the correlation length $\xi = |\ln\frac{\lambda_1}{\lambda}|^{-1}$.

From the study of correlation functions it is clear that a pair factorized steady state can not give rise to diverging correlation length if the number of species are finite. For $(K+1)$ species model, one gets $(K+1)$- dimensional transfer matrix with elements $\langle m|T|n\rangle = z_n g(m,n) > 0$; the largest eigenvalue λ then remains non-degenerate following Perron-Frobenius theorem [154]. Thus, the correlation length

$\xi = |\ln \frac{\lambda_1}{\lambda}|^{-1}$ is finite and possibility of phase transition is ruled out in $(K+1)$-component systems with PFSS. To get out of this situation, one may think of setting some matrix elements to zero so that the transfer matrix become reducible and then, Perron-Frobenius theorem does not apply. However, it is easy to check that a reducible form of $g(m,n)$ force the steady state weight of all configurations to be zero.

There are numerous examples of exchange models which exhibit phase transition; like extended KLS models with ferro [112] or anti-ferromagnetic [128] interactions, ABC models in an interval [9] or a ring [46]. However, the steady state of these models are not pair factorized. In some models, the steady state can be written in matrix product form [27] with matrices having dimensions larger than the number of components $(K+1)$, then the matrix elements of corresponding transfer matrix can not be treated as the weight factors, as in PFSS. In this case one can have a reducible transfer matrix which does not restrict the phase space even though a block of matrix elements are zero.

The assisted exchange models, with pair factorized steady state, does not give rise to phase separation transitions in general, but they exhibit interesting steady-state current behavior which will be discussed in the next section.

6.1.3 Average current of particles

The average current is an entity of interest since the non-equilibrium phenomena are characterized by a net flow or current in the system comparing to their equilibrium counterparts which are dictated by zero average current. In this section we will focus on calculating exactly the average particle current in the pair factorized steady states of $(K+1)$species assisted exchange models. In particular, the average current for the species k would be

$$J_k = \sum_{k' \neq k}^{K} \sum_{m=0}^{K} u(m,k,k')\langle mkk'\rangle - u(m,k',k)\langle mk'k\rangle.$$

Using the matrix representations in Eq. (6.10), it is quite straightforward to obtain

$$\begin{aligned} \langle mkk'\rangle &= \frac{z_k z_{k'}}{\lambda^2} g(m,k)g(k,k')\langle k'|\psi\rangle\langle\phi|m\rangle \\ &= \frac{\langle mk\rangle\langle kk'\rangle}{\rho_k} . \end{aligned} \tag{6.17}$$

Then, for the exchange dynamics (6.5), the current of species k is

$$\begin{aligned} J_k &= \sum_{k' \neq k}^{K} \sum_{m=0}^{K} \frac{z_k z_{k'}}{\lambda^2} \langle\phi|m\rangle \left(g(m,k')\langle k'|\psi\rangle - g(m,k)\langle k|\psi\rangle \right) \\ &= \frac{1}{\lambda} \sum_{k' \neq k}^{K} \sum_{m=0}^{K} \left(z_k\langle mk'\rangle - z_{k'}\langle mk\rangle \right). \end{aligned} \tag{6.18}$$

To proceed further we need to be more specific about the dynamics. In the following section we choose a specific form of weight function $g(m,n)$ for which the steady state results for current and other observables can be obtained exactly for any arbitrary K.

6.2 Exact results for a class of AEM

In this section we choose a specific form of weight function $g(m,n)$ for which the steady state calculations of current can be done explicitly for any arbitrary K. Let us consider the weight function to be

$$g(m,n) = \frac{g(m,0)g(0,n)}{\gamma g(0,0)} \qquad m,n > 0, \tag{6.19}$$

where $g(m,0)$ and $g(0,n)$ are $(2K+1)$ independent parameters; once these parameters are fixed, the rest of the elements of $g(m,n)$ can be calculated using Eq. (6.19). In the following, for convenience, we set a short-hand notation

$$v_m = g(m,0)g(0,m). \tag{6.20}$$

It is easy to check that the transfer matrix T, with elements $\langle m|T|n\rangle = z_m g(m,n)$ along with Eq. (6.19), has the following properties,

$$I. \qquad \mathrm{Tr}[T] = \sum_{k=0}^{K} z_k g(k,k) \equiv 2\tau \tag{6.21}$$

$$II. \qquad \mathrm{Det}[T] = 0 \tag{6.22}$$

$$\begin{aligned} III. \qquad & \text{Nonzero eigenvalues} : \lambda_\pm = \tau \pm \sqrt{\tau^2 - \delta} \\ & \delta \equiv (\gamma^{-1} - 1) \sum_{k=1}^{K} z_k v_k \end{aligned} \tag{6.23}$$

Note that the other eigenvalue 0 is $(K-1)$-fold degenerate. Let $|\psi\rangle$, and $\langle\phi|$ be the left and right eigenvectors (normalized), corresponding to the largest eigenvalue $\lambda \equiv \lambda_+$ with elements, $\langle m|\psi\rangle$, and $\langle\phi|m\rangle$ with $m = 0, 1, 2, \ldots K$. For $m > 0$ we have,

$$\begin{aligned} \langle m|\psi\rangle &= \eta g(m,0)\langle 0|\psi\rangle; \quad \langle\phi|m\rangle = \eta z_m g(0,m)\langle\phi|0\rangle; \\ & \text{with} \ \ \eta = \frac{1}{\lambda - 2\tau + g(0,0)}. \end{aligned} \tag{6.24}$$

For $m = 0$, one can determine $\langle 0|\psi\rangle$ and $\langle\phi|0\rangle$ from the normalization condition $\langle\phi|\psi\rangle = 1$,

$$\begin{aligned} \langle\phi|0\rangle\langle 0|\psi\rangle &= \left(1 + \frac{\lambda - g(0,0)}{\lambda - 2\tau + g(0,0)}\right)^{-1} \\ &= \left(1 + \gamma g(0,0)\eta^2(2\tau - g(0,0))\right)^{-1} . \end{aligned} \tag{6.25}$$

Using these properties of T we will now calculate the observables. Let us use the grand canonical ensemble and remind ourselves that, without loss of generality, we can set the fugacity $z_0 = 1$. The particle densities are then

$$\rho_m = \frac{\gamma z_m v_m g^2(1,1)}{2\alpha'\alpha}\left[\alpha' + \alpha - v_1^2\right] \tag{6.26}$$

where,

$$\alpha' = \sqrt{(\alpha + v_1^2)^2 + 4(\gamma - 1)\alpha v_1^2}$$

$$\text{and } \alpha = \gamma g^2(1,1) \sum_{k=1}^{K} z_k v_k. \tag{6.27}$$

Similarly, the correlation function is

$$\begin{aligned} C_{m,m'}(r) &= \frac{\gamma z_{m'} g(m,0) g(0,m') g^2(1,1)}{2\alpha'\alpha} \\ &\quad \times (\alpha' - \alpha + v_1^2) \left(\frac{\lambda_-}{\lambda}\right)^r. \end{aligned} \tag{6.28}$$

This being stated, we proceed to obtain the particle current given by (6.18). From now onwards, we will be concerned with the average current of the species $k = 0$. As we have already mentioned that $k = 0$ can be considered as vacant sites and $k > 0$ as the K-particle species; the particles $k = 1, 2, \ldots, K$ exchange with each other whereas the exchange of any species k with 0 will now represent diffusion, i.e., hopping of that species to right or left vacant neighbor. Clearly in this case the total current including that of 0 or vacant sites will be $J + J_0 = 0$ where J is the total particle current of K-species of particles $k = 1, 2, \ldots, K$. Then, $J = -J_0$. Current J_0 can be calculated from Eq. (6.18), which gives the total particle current

$$\begin{aligned} J &= \Bigg[\left((1-\gamma)\frac{g(0,0)}{\lambda^2} + \frac{1 - g(0,0)/\lambda}{2\tau - g(0,0)} \right) \sum_{i=1}^{K} z_i \\ &\quad - \eta \left(1 - \frac{g(0,0)}{\lambda}\right) \Bigg] \langle \phi | 0 \rangle \langle 0 | \psi \rangle, \end{aligned} \tag{6.29}$$

where $\langle \phi | 0 \rangle \langle 0 | \psi \rangle$ is given by Eq. (6.25). Inverting the density fugacity relations, we arrive at the following equation

$$\begin{aligned} z_i &= \frac{\rho_i}{v_i} \frac{(v_1)^K}{2g^2(1,1)\rho_0(1-\rho_0)^2} [(1-2\rho_0)^2 + 2\gamma^{-1}\rho_0(1-\rho_0) \\ &\quad + (1-2\rho_0)\sqrt{(1-2\rho_0)^2 + 4\gamma^{-1}\rho_0(1-\rho_0)}], \end{aligned} \tag{6.30}$$

that expresses the fugacities z_i $(i = 1, 2, \ldots, k)$ in terms of the densities ρ_i so that z_i can be replaced in Eq. (6.29). Note the difference between the expressions of the average particle current J in (6.29) compare to (6.18); it became possible to write down the current in (6.29) in terms of the input parameters like the densities and hop rates (after replacing z_is using (6.30)), which was not that straightforward for (6.18).

Now, as we have obtained the exact average current for the K-species assisted exchange model, our next move would be to investigate possible interesting properties of the current. First we consider $K = 1$ case where a single species of particles undergo assisted hopping on a $1d$ periodic lattice; we will discuss how these simple models exhibit density dependent current reversal and negative differential mobility.

6.3 Assisted exchange model ($K = 1$)

Consider a $1d$ periodic lattice with L sites $i = 1, 2, \ldots, L$ where each site i can be occupied by at most one particle(*i.e*, the particles are hard core) represented by 1 or it can be vacant, represented by 0. A particle from a randomly chosen site i can move to its right neighbor $(i + 1)$, if vacant, with a rate that depends on the left neighbor $(i - 1)$ of the departure site. Whereas, the particle from i can also move to its left neighbor $(i - 1)$, if vacant, with a rate that depends on the left neighbor $(i - 2)$ of the arrival site $(i - 1)$. More precisely, in a nutshell, the motion of the particles are assisted by their neighbors- isolated particles and crowded particles (particles with occupied neighbor(s)) hop in different manner. The dynamics is represented as

$$010 \underset{q}{\overset{p}{\rightleftharpoons}} 001 \qquad\qquad 110 \underset{\gamma_2 q}{\overset{\gamma_1 p}{\rightleftharpoons}} 101 \tag{6.31}$$

For this dynamics we have pair factorized steady state, following (6.5) only if

$\gamma_1 = \frac{1}{\gamma_2} \equiv \gamma$; the pair-weight functions are then

$$g(0,0) = \frac{1}{q};\ g(1,0)g(0,1) = v_1 = \frac{1}{pq}; g(1,1) = \frac{1}{\gamma p} \tag{6.32}$$

and the corresponding dynamics,

$$010 \underset{q}{\overset{p}{\rightleftharpoons}} 001 \qquad\qquad 110 \underset{q\gamma^{-1}}{\overset{\gamma p}{\rightleftharpoons}} 101 \tag{6.33}$$

has three parameters p, q and γ. It is easy to show that the particle current for this single species assisted exchange model, in terms of particle density ρ or corresponding fugacity z, is

$$\begin{aligned}
J &= p(\langle 010\rangle + \gamma\langle 110\rangle) - q(\langle 001\rangle + \frac{1}{\gamma}\langle 101\rangle) \\
&= \frac{\langle\phi|0\rangle\langle 0|\psi\rangle}{\gamma\lambda^2}\left[(1-\gamma)\left(\lambda - \frac{1}{q} + \frac{z}{\gamma p}\right) + \lambda z\left(\gamma - \frac{q}{\gamma p}\right)\right] \\
&\text{where } \lambda = \frac{1}{2}\left(\frac{1}{q} + \frac{z}{\gamma p} + \sqrt{(\frac{1}{q} - \frac{z}{\gamma p})^2 + 4\frac{z}{pq}}\right) \\
&\text{and } \langle\phi|0\rangle\langle 0|\psi\rangle = \frac{pq\lambda - qz/\gamma}{2pq\lambda - p - qz/\gamma}.
\end{aligned} \tag{6.34}$$

In the above equation, we have calculated the current in the grand canonical ensemble by associating a fugacity z to the particles. In order to express the average current J in terms of particle or vacancy densities, one can replace z in (6.34) by inverting the density-fugacity relation as follows

$$\begin{aligned}
z &= \frac{\gamma p}{q} + \frac{\gamma p(1-2\rho_0)}{2q\rho_0(1-\rho_0)} \\
&\times \left[(1-2\rho_0)\gamma + \sqrt{(1-2\rho_0)^2\gamma^2 + 4\gamma\rho_0(1-\rho_0)}\right].
\end{aligned} \tag{6.35}$$

Now we would like to discuss two interesting features viz. density dependent current reversal and negative differential mobility of particles emerging from the expression of particle current in Eq. (6.34) that resulted from the stochastic process (6.31) along with (6.32).

6.3.1 Current reversal in AEM with $K = 1$

Let us discuss a more specific example with, $q = \frac{1}{p}$ and $\gamma = \frac{1}{p^2}$; the dynamics is then,

$$010 \underset{\frac{1}{p}}{\overset{p}{\rightleftharpoons}} 001 \ ; \ 110 \underset{p}{\overset{\frac{1}{p}}{\rightleftharpoons}} 101 \quad . \tag{6.36}$$

Now comparing the rates in Eq. (6.36) with that in (6.31) and further using Eq. (6.32), we conclude that the dynamics in (6.36) indeed gives rise to a pair factorized steady state with the pair weight factors obtained as $g(1,0) = g(0,1) = 1$ and $g(0,0) = g(1,1) = p$. Correspondingly, the particle current from Eq. (6.34) will be

$$\begin{aligned} J &= \frac{(z-1)(\lambda-p)(1-p^2)}{\lambda^2}\langle\phi|0\rangle\langle 0|\psi\rangle, \\ &\quad \text{where } \lambda = \frac{1}{2}(p(1+z) + \sqrt{p^2(1-z)^2+4z}) \\ &\quad \text{and } \langle\phi|0\rangle\langle 0|\psi\rangle = \frac{\lambda - pz}{2\lambda - p - pz}. \end{aligned} \tag{6.37}$$

Thus the particle current J vanishes when $p = 1, \lambda$ or when $z = 1$ (note that $\lambda > pz$ and thus $\langle\phi|0\rangle\langle 0|\psi\rangle$ can not be zero). Since $\lambda = p$ corresponds to $z = 0$ (i.e., the particle density $\rho = 0$) and $p = 1$ corresponds to equilibrium, the only nontrivial condition for a zero current state is $z = 1$. The density fugacity relation can be obtained using $\rho_0 = 1 - \rho$ in Eq. (6.35),

$$z = 1 + \frac{2\rho - 1}{2p^2\rho(1-\rho)}[2\rho - 1 + \sqrt{(2\rho-1)^2 + 4p^2\rho(1-\rho)}] \ . \tag{6.38}$$

This indicates that for any arbitrary $p \neq 1$ the current $J \equiv J$ vanishes at density $\rho = \frac{1}{2}$.

In Fig. 6.1 we plot the particle current J as a function of density for different values of p. For all values of p, the current is antisymmetric about density $\rho = \frac{1}{2}$, i.e., $J(\rho) = -J(1-\rho)$. This is evident from the fact that dynamics (6.36) is invariant under particle-hole exchange $0 \to 1$ and $1 \to 0$; thus, the magnitude of

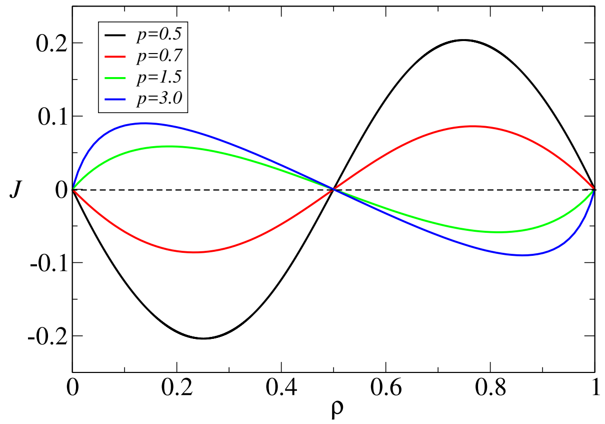

Figure 6.1: Current as a function of density for $K = 1$. Density dependent current reversal for single species assisted hopping model obeying dynamics (6.36) for $p = 0.5, 0.7, 1.5, 3$- the current becomes zero at $\rho^* = \frac{1}{2}$ (point of reversal) and then reverses its direction as density is increased.

current experienced by particles must be same as the current experienced by holes after particle-hole exchange, but the direction of hole current is opposite to that of particles. Also, for all values of p, when particle density is small ($\rho \approx 0$), the average current flows towards right if $p > 1$ (or towards left, if $p < 1$), it reaches a maximum (minimum) value and then decreases (increases) continuously until J becomes zero at density $\rho^* = \frac{1}{2}$, which we call to be the *point of reversal*; further increase of the particle density results in flow of current in opposite direction. The density-dependent current reversal is uncommon in single species of particles. In fact, for systems with two or more species of particles, each species can experience the external field differently (say, positive and negatively charged particles in an electric field) and the particle current can change direction when the relative density is varied. The density-dependent current reversal we observed here is due to inter-particle-interaction. Although, we have only one kind of particle, the current gets contribution from two current carrying modes which experience the external bias differently: particle in isolation experience a bias $2\ln p$ (bias is defined for dynamics (6.36), following local detailed balance [118]) whereas particles in crowded environment experience an opposite bias $-2\ln p$. With increasing density, the relative density of these two current carrying modes changes, resulting in current reversal at some threshold density $\rho = \rho^*$.

6.3.2 Negative differential mobility in AEM with $K = 1$

In this section, we consider yet another specific case of dynamics (6.33), by setting $p = 1, q = e^{-\epsilon}$ without loss of generality and choosing γ as a function of ε, i.e., $\gamma = \frac{1}{1+\sigma\epsilon}$. The dynamics is then

$$010 \underset{e^{-\epsilon}}{\overset{1}{\rightleftharpoons}} 001 \quad 110 \underset{e^{-\epsilon}\gamma^{-1}}{\overset{\gamma}{\rightleftharpoons}} 101 \quad ; \gamma = \frac{1}{1+\sigma\epsilon} \tag{6.39}$$

The above hop rates, when compared to Eq. (6.31) along with (6.32), imply that the model has a pair factorized steady state with $g(1,0) = g(0,1) = e^{\epsilon/2}$,$g(0,0) = e^{\epsilon}$ and $g(1,1) = (1+\sigma\epsilon)$. The parameter σ introduced here measures the strength of the particle current for a given density and more importantly, tunes the dependence of γ on ϵ. Note that, in (6.39), for $\sigma = 0$, the exchange dynamics is insensitive to the occupancy of the left neighbor of the exchanging-pair and the model reduces to asymmetric exclusion process on a ring [139] where all configurations are equally likely in steady state and the particle current $J = (p-q)\rho(1-\rho) = (1-e^{-\varepsilon})\rho(1-\rho)$ is an increasing function of the bias ε.

For dynamics (6.39), it is easy to see following local detailed balance condition[118] that the bias on the isolated particles is simply ϵ whereas the crowded particle experiences a rightward bias given by $(\epsilon - 2\ln(1+\sigma\epsilon))$. For any $0 \le \sigma \le 1/2$, both these biases increase monotonically with ϵ and one expects the particle current to increase as ϵ is increased. This is however not true. We find that the particle current J exhibits non-monotonic behavior.

Let us calculate the particle current, for the present dynamics (6.39), using Eq. (6.34),

$$\begin{aligned} J &= \frac{(\lambda - e^{\epsilon})}{\lambda^2}\left(\lambda\left(1 - e^{-\epsilon}\left(1+\sigma\epsilon\right)\right) - \sigma\epsilon(z-1)\right)\langle\phi|0\rangle\langle 0|\psi\rangle, \\ & \text{where } \lambda = \frac{1}{2}\left(\mathrm{e}^{\epsilon} + \mathrm{z}(1+\sigma\epsilon) + \sqrt{(\mathrm{e}^{\epsilon} - \mathrm{z}(1+\sigma\epsilon))^2 + 4\mathrm{z}\mathrm{e}^{\epsilon}}\right) \\ & \text{and } \langle\phi|0\rangle\langle 0|\psi\rangle = \frac{\gamma\lambda - \mathrm{z}}{2\gamma\lambda - \gamma\mathrm{e}^{\epsilon} - \mathrm{z}}. \end{aligned} \tag{6.40}$$

From the density-fugacity relation, one can also express z in terms of the density,

$$\begin{aligned} z &= \frac{e^{\epsilon}}{(1+\sigma\epsilon)} + \frac{(1-2\rho_0)}{2(1+\sigma\epsilon)^2\rho_0(1-\rho_0)} \times \left[e^{\epsilon}(1-2\rho_0)\right. \\ & \left. + \sqrt{e^{\epsilon}(e^{\epsilon}(1-2\rho_0)^2 + 4e^{\epsilon}(1+\sigma\epsilon)\rho_0(1-\rho_0))}\right]. \end{aligned}$$

Let us study the system at a specific density $\rho = \frac{1}{2} = \rho_0$; then the fugacity z in the above equation simply relates to the bias ϵ as

$$z = \frac{e^{\epsilon}}{(1+\sigma\epsilon)}. \tag{6.41}$$

Correspondingly, substituting this value of z in Eq. (6.40), the particle current at $\rho = \frac{1}{2} = \rho_0$ becomes

$$J = \frac{1 - e^{-\epsilon}(1+\sigma\epsilon)}{2(1+\sigma\epsilon + \sqrt{1+\sigma\epsilon})}. \tag{6.42}$$

In Fig. 6.2 (a) we plot the current J as a function of the bias ϵ for $\rho = \frac{1}{2}$ with three different parameter values $\sigma = 0, \frac{1}{3}, \frac{1}{2}$. As expected from dynamics (6.39), irrespective of the value of σ and ρ, J vanishes for $\epsilon = 0$ (corresponds to equilibrium) and it increases as the bias is increased. It reaches its maximum value at some finite $\epsilon = \epsilon^{\star}(\sigma)$ (e.g., $\epsilon^{\star}(1/3) \approxeq 2.516, \epsilon^{\star}(1/2) \approxeq 2.487$), after which, *as we*

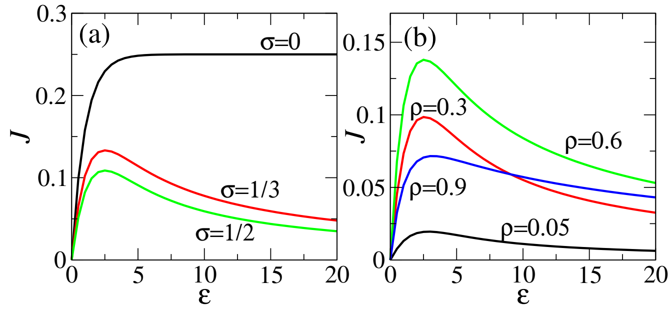

Figure 6.2: The current J as a function of bias ε for AEM with $K = 1$ following dynamics (6.39). (a) J versus ε for $\rho = \frac{1}{2}$ and $\sigma = 0, \frac{1}{3}, \frac{1}{2}$. (b) J versus ε for $\sigma = \frac{1}{3}$ and $\rho = 0.05, 0.3, 0.6, 0.9$. Negative differential mobility is observed for all cases except for $\sigma = 0$ which corresponds to asymmetric exclusion process on a ring [139].

increase the bias the current decreases gradually giving rise to negative differential mobility (NDM) (i.e., $\frac{\mathrm{d}J}{\mathrm{d}\epsilon} < 0$ in the parameter region $\epsilon^\star < \epsilon < \infty$). The occurrence of NDM has been evident from Eq. (6.42); as J vanishes in the extreme limits $\epsilon \to 0$ and $\epsilon \to \infty$, there must be a finite threshold bias $\epsilon = \epsilon^\star$ at which the current reaches to an extremum. Current J as a function of ε for $\sigma = \frac{1}{3}$, and different ρ are shown in Fig. 6.2 (b).

In this model, both current carrying modes experience a positive external bias ϵ; decrease of current with increasing bias is due to inter-particle interaction. Possibility of NDM in interacting systems has been explored recently in Ref. [35]; it was argued that the slowing down of certain non-driven modes could generate NDM. Here, both isolated and crowded particles are driven by external bias in the same direction but the asymmetry factor $\gamma = 1/(1 + \sigma\epsilon)$ decreases with increasing ϵ (for any value $0 \leq \sigma \leq 1/2$ considered here), leading to NDM. To emphasize this point we also study the model with $\sigma = 0$ (i.e. $\gamma = 1$); the corresponding current $J = (1 - e^{-\epsilon})\rho(1 - \rho)$ is plotted in Fig. 6.2. Clearly non-monotonic current behavior disappears when dependence of γ on ϵ is switched off (i.e., $\sigma = 0$) and current settles to a finite constant value $\rho(1-\rho)$ as $\epsilon \to \infty$.

To understand the mechanism of negative response better, let us re-write the second part of the dynamics (6.39), i.e., the part involving parameter γ,

$$1101 \underset{e^{-\epsilon}\gamma^{-1}}{\overset{\gamma}{\rightleftharpoons}} 1011; \quad 1100 \underset{e^{-\epsilon}\gamma^{-1}}{\overset{\gamma}{\rightleftharpoons}} 1010 \tag{6.43}$$

In the first step, hopping of a particle to the left or right neighbor does not change the number of domains (which are uninterrupted sequence of 1s) as as the particle leave one domain and join immediately another existing domain. Whereas in second step, rightward hopping of the particle (with rate γ) increases the number of domains and the reverse hop (with rate $e^{-\epsilon}\gamma^{-1}$) merge the domains and thus, decrease the total number. The factor $\gamma^{-1} = (1+\sigma\epsilon)$, being an increasing function of ϵ, favors small number of domains for large ϵ. This in turn enhances formation of larger domains and there by decreases the particle current. Consequently we have negative differential mobility of particles for the dynamics (6.39) when $\gamma^{-1} = (1+\sigma\epsilon)$ in the parameter region $0 \leq \sigma \leq \frac{1}{2}$.

This concludes our discussion about the single species assisted hopping model with a generic dynamics of the form (6.31)- which possess pair factorized steady state when the rates are specified by the condition (6.32). We have shown explicitly that this model, with some specific choice of rates, as given by Eq. (6.36) and (6.39) can exhibit density dependent current reversal (Fig. 6.1) and negative differential mobility (Fig. 6.2) respectively.

6.4 Assisted exchange model with $K = 2$

In this section we describe a simple example of the dynamics in Eq. (6.1) for $K = 2$. We start by considering a specific rate,

$$\begin{aligned} g(0,0) &= \frac{3}{10}, g(0,1) = \frac{1}{5}, g(0,2) = \frac{2}{5}, g(1,0) = \frac{1}{2} = g(1,1), \\ g(1,2) &= 1, g(2,0) = \frac{1}{10} = g(2,2), g(2,1) = \frac{1}{8}. \end{aligned} \tag{6.44}$$

The exchange rates, that gives rise to a pair-factorized state with the above pair weight functions, can be constructed in a straightforward way using Eq. (6.5). The fugacities z_1 and z_2 corresponding to the particles of the two different species, can now be expressed as a function of the corresponding particle densities ρ_1, ρ_2 and $\rho_0 = 1 - \rho_1 - \rho_2$ as (using Eq. (6.30)),

$$\begin{aligned} & z_1 = 2 s_1 \rho_1; \quad z_2 = \frac{1}{2} s_1 \rho_2 \qquad\qquad (6.45) \\ & \text{where } s_1 = \frac{1 - \rho_0(1-\rho_0) + (1-2\rho_0)\sqrt{1 + 2\rho_0(1-\rho_0)}}{\rho_0(1-\rho_0)^2}. \end{aligned}$$

The particle current, as given by (6.29), takes the following form

$$\begin{aligned} J &= \frac{1}{\lambda^2}\left[\left(\frac{3}{10}\rho_0 + \sqrt{\rho_0} s_2\right)(z_1 + z_2) - (\sqrt{\rho_0} + 5 s_2) s_2\right] \\ \text{where} \quad & s_2 = s_1\left(\frac{\rho_1}{\sqrt{5}} + \frac{\rho_2}{5\sqrt{2}}\right). \end{aligned} \tag{6.46}$$

Finally, by replacing $z_{1,2}$ from (6.45) into (6.46), we obtain the expression of current J as a function of the particle densities $\rho_{1,2}$.

Figure 6.3 shows the steady state particle current as a function of the second species particle density ρ_2 for a fixed value $\rho_1 = 0.2$; here the direction of current is changed multiple times. For small density $\rho_2 \simeq 0$, current flows towards left (here almost the whole of the current is carried by the species $k = 1$ only) and it changes it direction and flows towards right once density crosses a threshold value

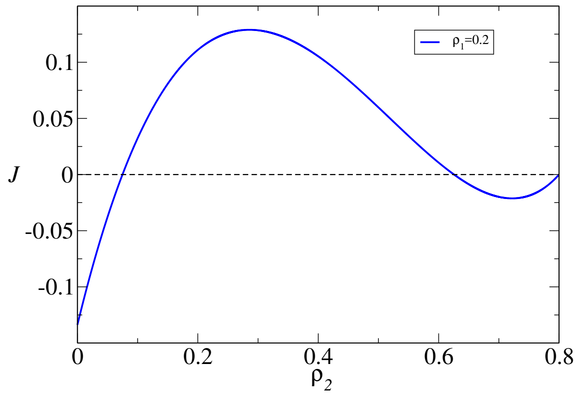

Figure 6.3: Density dependent current reversal for a two species ($K = 2$) assisted exchange model (see Eq. (6.44)). The direction of current is reversed twice when ρ_2 is increased keeping $\rho_1 = 0.2$ fixed. The reversal points are $\rho_2^* = 0.07, 0.62$

$\rho_2^* \approx 0.07$. Again the direction of current changes at $\rho_2 \approx 0.62$. Note that multiple current reversal with density was not possible for one species, $K = 1$ exchange models. One should keep in mind that this current reversal can also be observed by tuning ρ_1 with a fixed value of ρ_2. In fact, in general, J is a function of (ρ_1, ρ_2) and one expects a line of reversal in the ρ_1-ρ_2 plane, where, along this threshold line the the current becomes zero before changing its direction.

6.4.1 Negative differential mobility in AEM with $K = 2$

Just like current reversal, in this section, we would like to show an example where the assisted exchange model exhibits negative differential mobility of particle current. In particular, we consider $K = 2$ with pair-weight functions,

$$
\begin{aligned}
&g(0,0) = e^{\epsilon}, g(1,0) = g(0,1) = e^{\epsilon/2}, g(1,1) = (1 + \frac{\epsilon}{2}),\\
&g(2,0) = e^{\epsilon/5}, g(0,2) = e^{7\epsilon/10}, g(1,2) = (1 + \frac{\epsilon}{2})e^{\epsilon/5},\\
&g(2,1) = (1 + \frac{\epsilon}{2})e^{-3\epsilon/10}, g(2,2) = (1 + \frac{\epsilon}{2})e^{-\epsilon/10}.
\end{aligned} \tag{6.47}
$$

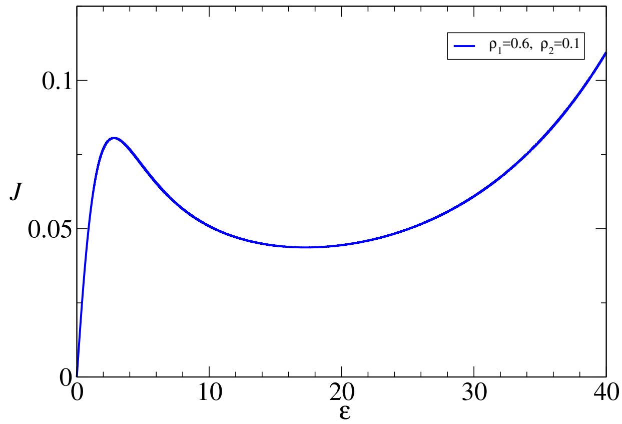

Figure 6.4: Negative differential mobility exhibited by the total particle current in the region $3 \lessapprox \epsilon \lessapprox 17$ for a two species ($(K = 2)$ assisted exchange model at densities $\rho_1 = 0.6, \rho_2 = 0.1$

Corresponding two species assisted exchange dynamics can be derived from Eq. (6.5). Since, here we focus on the current as a function of the bias ϵ, we consider the particle densities of the two species to be fixed at $\rho_1 = 0.6$ and $\rho_2 = 0.1$ respectively. It is quite straightforward to check that all the relevant biases in the total particle current are increasing functions of the bias ϵ under consideration. Now, the exact form of the current in this case can be calculated directly from Eq. (6.18). We have not provided the expression of exact steady state current here as the expression is long and complicated - it does not provide any useful physical insight. Instead, we show the plot of the total particle current as a function of the bias in Fig. 6.4.

We observe that the current increases as bias ϵ is increased and beyond a threshold value $\epsilon^* \simeq 3$ it starts decreasing, exhibiting negative differential mobility. The differential mobility remains negative in a finite region $3 \lessapprox \epsilon \lessapprox 17$ and then the current starts showing the usual positive response. Observation of negative mobility for a finite range of bias is quite interesting; at present we don't have a better understanding of this effect. Further study, in this direction will be reported elsewhere.

6.5 Summary

We have introduced in this chapter an assisted exchange model on a one dimensional periodic lattice with hardcore particles of $(K+1)$ (K being any finite positive integer) different species, where the dynamics conserves the total number of particles as well as the number of particles of each species. We refer to it as *assisted* exchange model (AEM) because the particle exchange rate, between two different species of particles, depends on both, the pair of particles taking part in exchange and their immediate left neighbor. Firstly, we investigate the possible steady state measure of this stochastic process and using the h-balance scheme, we obtained an explicit, sufficient condition, for which the steady state has a pair-factorized form. We provide a transfer matrix formalism that helps us to calculate the spatial correlation function and steady state averages of several other observables. We have been mostly interested in the particle current that characterizes a non-equilibrium state. The $(K+1)$-species exchange model can also be interpreted as a hopping model of K-species particles by denoting one of the species as vacancy; exchange with the vacancy is then equivalent to the hopping and exchange with any other species will remain as exchange dynamics. In this case the total particle current J is same as $-J_0$, where J_0 is the current of species 0 (vacancy).

We also provide exact and rigorous results for a specific class of pair-weight functions; the steady state current and density-fugacity relations are calculated explicitly for any arbitrary $(K+1)$-species of particles. Some specific examples of

$K = 1$ and 2 are also discussed elaborately. To illustrate intriguing features like density dependent current reversal for a fixed set of rates and negative differential response of the particle current with increasing bias, we have extensively discussed a single species assisted hopping model where isolated particles (particles with both neighbors vacant) and crowded particles (particles with at least one neighbor occupied) hop with different rates. This model resembles the partially asymmetric conserved lattice gas [19]. Moreover, these interesting behavior of current, i.e, current reversal and negative differential mobility has also been described briefly for two species assisted hopping and exchange model with specified rates; for $K = 2$ models we observe additional features like *multiple* points of reversal in context of current reversal and an unusual response in particle current where the negative mobility is restricted to a *finite* region of bias.

We conclude that the phenomena like reversal of current with increased density, and negative differential mobility of particles can generally occur in multi-species assisted hopping and exchange models for suitable choice of the dynamical rates. One can in general ask if AEM can have a steady state measure, other than PFSS and explore the possibility of phase separated states. It would be interesting to explore these exactly solvable models to study the characterization of non-equilibrium states in terms of current and its higher order cumulants.

Chapter 7

Conclusion

We have introduced several classes of driven interacting particle systems and solved exactly the corresponding steady state measures using different schemes. As it is a well-known fact that the non-equilibrium systems, unlike their equilibrium counterparts characterized by Gibbs-Boltzmann distribution, do not have any unique steady state measure, we have tried to shed some light on this issue by considering certain class of non-equilibrium models which could be solved exactly. Since the steady state of non-equilibrium systems are very much dependent on the complexity of the dynamics, *it is difficult to track down a path how to systematically proceed to obtain the steady state of a system with a given dynamics*. In this regard, starting from the Master equation that governs the time evolution of a many particle system in the configuration space, several schemes (more precisely, flux cancellation schemes) have been used throughout the thesis that help us to find steady state measures. These schemes include matrix product ansatz, h-balance scheme and pairwise balance condition.

In context of stochastic processes driven out of equilibrium, we have introduced the finite range process on periodic lattices in one dimension where particles are free of hard core repulsion and can stay in multiple number on individual lattice sites. The dynamics of this process is such that a particle can hop to its nearest neighbor along a given direction with a *rate that depends on the occupation number of a finite number of neighboring lattice sites* (including the departure one) making the process to be *finite range process.* Using pairwise balance condition, we found that the corresponding steady states can be written in a cluster-factorized form meaning the steady state probabilities are expressed as a product of functions that depend on a cluster of neighboring site occupation variables. Since the finite range dynamics invokes a spatial correlation between different lattice sites, we have calculated analytically the spatial correlations in some of these models with the help of a transfer matrix formulation. In subsequent chapters, we have generalized this finite range process by considering motion of the components in all possible directions (asymmetric finite range process) and also by including different number of neighbors in different directions in the hop rates (asymmetric finite range process with different range of neighbors). Although these generalizations are simple to state, their steady state structures are quite intriguing. A simple modification in the dynamics may render inoperative the cancellation scheme used earlier to obtain the exact steady state measure, then one has to invent a new scheme. This makes study of non-equilibrium steady states a challenging task on one hand but also interesting on the other. In asymmetric finite range processes, the steady state is achieved through the h-balance scheme which involves the introduction of an undetermined function $h(.)$ in a suitable way that necessarily satisfies the steady state Master equation, whereas for the asymmetric FRP with different range of neighbors, we have extensively devised matrix product ansatz for particles without hard core exclusions and the flux balance scheme used there involves the representation of an algebra of infinite set of matrices as a single functional relation of the site occupation variables and also the introduction of some auxiliary undetermined matrices in a suitable way that essentially leads to the steady state Master equation. Apart from these non-equilibrium models, we have also studied multi-species assisted exchange models where the particles obey hard core exclusion and the corresponding steady state for particular choice of exchange rates have been found exactly using the h-balance scheme.

Along with the exact solution of the steady state measures of several non-equilibrium processes using different flux cancellation schemes, we have calculated analytically physically relevant observables like spatial correlations and particle current in many of these models. In particular, to differentiate non-equilibrium systems from the equilibrium ones which are zero current systems, the particle current in non-equilibrium cases is always an interesting quantity to cultivate. The particle current, for specific choice of the dynamical rates, exhibits several special features like density dependent current reversal and negative differential mobility. For systems with multiple current carrying modes that feel net bias in different directions, for fixed values of the rate parameters, only by tuning the particle density one can observe that the current reverses its direction. We have observed such phenomena in asymmetric zero range process and asymmetric misanthrope process. Moreover, in the assisted exchange models with multiple species studied here, we have observed the existence of multiple points of reversal, i.e., the particle current reverses its direction more than once with the change of density. The negative differential mobility of particles has been observed in several processes, e.g., asymmetric misanthrope processes, where by slowing down the activity of some current carrying modes through the biasing of other modes, the current has been found to be decreasing with increasing bias. We have also studied the possibility of condensation transitions in one dimensions in these processes. In fact, for special choice of hop rates in finite range and asymmetric finite range processes, we have found that a macroscopic fraction of particles accumulate at a single lattice site leading to the formation of condensates.

The present thesis on the exactly solvable driven interacting particle systems also raises some open questions. For example, it would be interesting to study the finite range process and their generalizations with *open boundaries*. The steady states of a system on periodic lattice and with open boundaries can be very different from the dynamics on a periodic system. For example, in case of the totally asymmetric simple exclusion process- the steady state on a one dimensional periodic lattice consists of all equally likely configurations whereas the same bulk system with open boundaries has a matrix product state and exhibit novel phase transitions. In this context, it would be interesting to explore the steady state measures and possibility of phase transitions in open boundary finite range process. Also, remembering the fact the exactly solving higher dimensional models may be very different from that of one dimensional problems, one may study finite range process and their generalizations in higher dimensional lattices to see how to obtain the corresponding steady states. Another interesting point revealed from the study of current reversal is the existence of a special point, namely the *point of reversal* where the particle current vanishes even when the system is far from equilibrium. It is interesting to ask how to differentiate these zero current non-equilibrium situations from equilibrium system which is naturally a zero current states. In general, the schemes used here to exactly solve the steady states and to analytically calculate observables, may help in rigorous study of characterization of non-equilibrium systems in general.

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