Study of Geometrical Aspects of Holographic Entanglement Entropy and First Law

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

ROHIT MISHRA

List of Publications

Journal:

1. "Perturbative entanglement thermodynamics for AdS spacetime: Renormalization"

   Rohit Mishra and Harvendra Singh

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2. "Generalized geodesic deviation equations and an entanglement first law for rotating BTZ black holes"

   Avirup Ghosh and Rohit Mishra

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3. "Entanglement asymmetry for boosted black branes and the bound"

   Rohit Mishra and Harvendra Singh

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4. "Inhomogeneous Jacobi equation for minimal surfaces and perturbative change in holographic entanglement entropy"

   Avirup Ghosh and Rohit Mishra

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The following work was done during my PhD but is not included in this thesis:

1. "Entanglement entropy at higher orders for the states of a = 3 Lifshitz theory"

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ROHIT MISHRA

To My Teachers.......

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CONTENTS

Synopsis

AdS/CFT correspondence has been a very successful idea in string theory. It is a correspondence relating gravity and quantum field theory. In other words, it relates classical gravity to quantum physics of strongly correlated manybody systems in one lower dimension. It is also known as gauge/gravity duality. It was originally formulated [1–4] as a correspondence between a 4 dimensional conformal field theory(CFT) living on the conformal boundary of a 5 dimensional Anti De Sitter(AdS) spacetime with supergravity theory in the bulk.

One of the fundamental distinction between classical physics and quantum physics is the presence of entanglement. This prototypical aspects of quantum physics have led to a deep theoretical interest in this field. The subject of quantum information [5] and condensed matter physics has led to rapid development of this concept over the past decades. In the past decade, the AdS/CFT correspondence has intertwined quantum entanglement with gravitational dynamics with the natural line of thought being the emergence of geometry from QFT dynamics. The inception of this idea is sourced from the observation by Ryu-Takayanagi [6,7], who proposed that the Holographic entanglement entropy(HEE) associated with a spatial region in a holographic CFT is given by the area of a particular static, homologous, codimension-2 minimal surface in the dual bulk geometry. Later on, this was generalized to time-dependent states by Hubeny-Rangamani-Takayanagi(HRT) [8]. It is important to note that HEE satisfies the area law, strong subadditivity, and several other features. There are several checks and proof of this conjecture and its close relation with the Bekenstein Hawking entropy which is proportional to the area of black hole horizon [9–13]. This proposal has been monumental in calculating entanglement entropy for various subsystems in different asymptotically AdS spacetimes and particulary the dependence of the shape of the minimal surface on the entropy [14–18]. Direct calculation of entanglement entropy in QFT is not straightforward, exact results can only be obtained for $2d$ field theories using the replica method [19, 20]

Relative entropy is another interesting object in the CFT side. Given two states ρ and σ the relative entropy $S(\rho \mid\mid \sigma)$ [21, 22], provides a measures of distinguishability between them. It satisfies two conditions viz monotonicity and positivity. The first property implies that it decreases under inclusion i.e, tracing out the same degree of freedom from two state ρ and σ to obtain the reduced density matrices ρ_A and σ_A decreases the relative entropy $S(\rho_A \mid\mid \sigma_A) \leq S(\rho \mid\mid \sigma)$ [5]. The second property implies that it is positive for any two density matrices and is zero only when they are equal. Using this one can express the relative entropy as the difference between the change in modular hamiltonian $(\Delta\langle H_\sigma\rangle)$ and the change in entanglement entropy $(\Delta\langle S\rangle)$ w.r.t a reference state. Thus the positivity of relative entropy implies $(\Delta\langle S\rangle \leq \Delta\langle H_\sigma\rangle)$. As mentioned earlier relative entropy acts as a distance measure for neighboring states. For a one parameter family of states $\rho = \rho_0 + \epsilon\rho_1 + \epsilon^2\rho_2 \cdots$ where $\rho_0 = \sigma$ one can evaluate the relative entropy $(S(\rho \mid\mid \sigma))$ as a power series in ϵ. Since the relative entropy takes its extremum at $(\rho = \sigma)$, its first variation vanishes i.e, $(\delta S = \delta\langle H_{\rho_0}\rangle)$. This statement is similar to the first law of thermodynamics $(dE = TdS)$ hence it is called the first law of entanglement.

We know that pure AdS_{d+1} spacetime is dual to the ground state of a holographic CFT_d and asymptotically locally AdS geometries are dual to the excited states. Following [23] one can verify the first law statement from the bulk side by using the RT proposal. The change in entropy is calculated order by order by obtaining the area of a boundary anchored minimal surface in asymptotically AdS spacetime and then expanding it in a small subsystems size approximation, the AdS contribution is then subtracted from it. It is important to note that under this approximation the minimal surface is free from the IR details and only depends on the energy density of the excitations. The change in energy is calculated using the holographic stress tensor [24, 25]. There have been several check of this entanglement first for different backgrounds [26–30]. In the paper [31], we used the RT proposal and verified the first law of entanglement for boosted black brane like perturbations over pure AdS and explored proper modification of the first law to include contributions from finite chemical potential and charge. For the same background, we have also compared the effect of anisotropy due to boost on the change in entanglement entropy [32]. The asymmetry in the change in entanglement entropy was quantified by introducing a dimensionless ratio and a bound was proposed on it.

Using the RT proposal one can calculate the entropy for excitations over AdS and then compute the change by subtracting the AdS contribution from it. Following [33, 34] one can calculate the change in entanglement entropy directly from area variation. At first order, changes in the embedding functions of the minimal surface don't contribute, the only contribution comes from metric perturbations. However, the embedding change will contribute to the next order. In [35] we developed a method to incorporate these changes at second order in $2 + 1$ case. The deviation of the embeddings were obtained by solving an inhomogeneous Jacobi equation. We also extended this setup to higher dimensions in [36].

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CHAPTER 1

INTRODUCTION

String theory is one of the foremost contender for a consistent theory of quantum gravity.The interaction between both gauge and gravitational forces are incorporated in this theory. With the discovery of D-branes [37] in 1995 eventually led to the AdS/CFT correspondence [1–3]. The AdS/CFT correspondence has been a very successful idea in string theory. It relates conformal field theories living on the d-dimensional conformal boundary of $d+1$-dimensional anti De Sitter (AdS) spacetime with the supergravity theory in the bulk. More precisely certain correlation functions in the boundary CFT can be obtained by calculating certain geometrical quantities in the bulk. Entanglement entropy has been very useful in studying the correlation between nonlocal observables in quantum many body systems. The idea of entanglement entropy has also been a focus of recent studies in string theory [6, 7]. The holography has led to some understanding of entanglement entropy in strongly coupled quantum mechanical systems, particularly for quantum theories which exhibit scaling property near the critical points [38]. A significant observation has been that the smaller excitations of the subsystems in the boundary theories follow entanglement thermodynamic laws similar to the black hole thermodynamics at finite temperature [23,26,30]. In this chapter we will briefly go through each of these developments.

1.1 AdS/CFT

We know that generating functional are very essential in order to study any quantum field theory. The AdS/CFT correspondence [1–4] is a statement about the generating functional of a quantum field theory and its dual theory of quantum gravity. One can write this as [39–43]

$$\mathcal{Z}_{\text{Gravity}} = \mathcal{Z}_{\text{QFT}}, \tag{1.1}$$

One of the most authoritative and well studied examples of this duality is the Type IIB string theory on $AdS_5 \times S^5$ with N units of five form flux passing through the S^5 and maximally super conformal $\mathcal{N} = 4$ super Yang- Mills (SYM) in 4 dimensions with gauge group $U(N)$. The central idea behind this is the dual role played by the D-branes. In the context of perturbative open string theory, a D-brane can be pictured as a hyperplane on which open strings end. Now from the massless excitations of the open strings ending on the brane one can describe a gauge theory on the world volume. Another way of seeing the D-brane is to see them as nonperturbative states of the closed string spectrum. At low energy, they are described by the solitonic solution of the corresponding supergravity as their tension scales as the inverse of the string coupling constant g_s. Taking the limit $\alpha' \to 0$ (where $\sqrt{\alpha'}$ is the characteristic string length) in Type IIB string theory (in $\mathbb{R}^{1,9}$) in presence of stack of N parallel $D3$ branes, while keeping the string coupling g_s and N fixed but large. From the open string description of $D3$ branes one gets two decoupled systems viz a) The

$N = 4$, U(N) super Yang-Mills in $\mathbb{R}^{1,3}$ and b) Weakly Coupled Type IIB supergravity in $\mathbb{R}^{1,9}$. Similarly in the same limit one also gets two decoupled systems from the closed string description viz a) Full Type IIB superstring theory on $AdS_5 \times S^5$ and b) Weakly Coupled Type IIB supergravity in $\mathbb{R}^{1,9}$. Since Type IIB supergravity is common to both it is consistent to identify $N = 4$, U(N) super Yang-Mills in $3 + 1$ dimensions with Type IIB superstring theory on $AdS_5 \times S^5$. This limit is commonly called the decoupling limit. It is important to note that the isometry groups on the both sides of the duality matches with each other. The SO$(2, 4)$ conformal symmetry of the CFT is realized as isometry group of AdS_5 whereas the global $SU(4) \sim SO(6)$ R-symmetry is realized as the group of isometries of S^5. In the gauge theory side we have two dimensionless parameters the Yang-Mills coupling g_{YM} and the rank of the gauge group $U(N)$. On the gravity side we have the dimensionless string coupling and two dimension full parameters viz the string scale α' and the length scale $L(L^4 = 4\pi g_s N \alpha'^2)$ The AdS/CFT dictionary is governed by two important relations

$$g_{YM}^2 = 4\pi g_s, \quad T = \frac{\sqrt{\lambda}}{2\pi} \qquad \text{For large } N,\ g_s \ll 1 \tag{1.2}$$

where $T = \frac{L^2}{2\pi\alpha'}$ and $\lambda = g_{YM}^2 N$ is the 't Hooft coupling. Thus using this duality one can study strongly coupled QFT (at large 't Hooft coupling) using dual classical gravity description. It is interesting to note that similar examples can be obtained in the context of M-theory [44, 45]. In the next section we will study the anti De Sitter spacetime and its symmetries.

1.2 Anti De Sitter Spacetime

In this section we will start with describing the AdS spacetime [43,46–49] and its conformal structure. Let us consider a flat spacetime ($\mathbb{M}^{2,d}$) having signature $(2, d)$ i.e, two time directions and d spatial directions. The line element for this space can be written in natural coordinates $T_0, T_1, X_1, \ ..., X_d$ as

$$ds^2 = -(dT_0)^2 - (dT_1)^2 + (dX_1)^2 + \cdots + (dX_d)^2 \tag{1.3}$$

Now we consider the $(d+1)$ dimensional hyperboloid $\mathcal{H}$ of events in $\mathbb{M}^{2,d}$ satisfying

$$(T_0)^2 + (T_1)^2 - \sum_{i=1}^{d}(X_i)^2 = R^2 \tag{1.4}$$

Where R is the proper distance of the hyperboloid from the origin. The isometries of $\mathcal{H}$ is given by those isometries of $\mathbb{M}^{2,d}$ which preserves (1.4). There are $\frac{(d+1)(d+2)}{2}$ independent isometries, these isometries forms the group $SO(d, 2)$ hence $\mathcal{H}$ is maximally symmetric.

Now we choose to parametrize the hyperboloid by introducing $T_0 = \sqrt{r^2 + R^2}\cos\frac{\tau}{R}$, $T_1 = \sqrt{r^2 + R^2}\sin\frac{\tau}{R}$ where $r^2 = \sum_{i=1}^{d}(X_i)^2$. With this parametrization we can write the induced line element on $\mathcal{H}$ as

$$ds^2 = -\left(\frac{r^2}{R^2} + 1\right)d\tau^2 + \frac{dr^2}{\left(\frac{r^2}{R^2} + 1\right)} + r^2 d\Omega_{d-1}^2 \tag{1.5}$$

Where Ω_{d-1} is the volume of the $(d-1)$- dimensional unit sphere. As the coordinate τ is periodic with a period of 2π. In order to avoid closed timelike curves on $\mathcal{H}$ one can unwrap this direction by passing to the universal covering space of $\mathcal{H}$. In literature both the universal covering space and the hyperboloid are referred to as Anti De Sitter spacetime. It is important to note that any Killing field of $\mathcal{H}$ lifts to the covering space and hence, it remains maximally symmetric w.r.t a isometry group given by a covering group of $SO(d,2)$.

The coordinate used in (1.5) covers the whole of AdS space. They are called global coordinates. We will use another set of coordinates given by $z = \frac{R^2}{(T_0+X_d)}$, $t = \frac{RT_1}{(T_0+X_d)}$ and $x_i = \frac{RX_i}{(T_0+X_d)}$ for $i = 1, \ ..., d-1$. The metric then becomes

$$ds^2 = \bar{g}_{\mu\nu}dx^\mu dx^\nu = \frac{R^2}{z^2}\left(-dt^2 + \sum_{i=1}^{d-1}(dx_i)^2 + dz^2\right) \tag{1.6}$$

These coordinates are called Poincaré coordinates as they manifest a lower dimensional Poincaré symmetry (with d coordinated t, x_i). These coordinates only cover a region of AdS given by $T_0 + X_d > 0$. This region is called the Poincaré patch. We will only use the coordinates given by (1.6) in this thesis. As AdS is maximally symmetric its Riemann tensor can be written as

$$R_{\mu\nu\sigma\lambda} = \frac{-1}{R^2}(g_{\mu\sigma}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\sigma}) \tag{1.7}$$

Its scalar curvature is equal to $-\frac{d(d+1)}{R^2}$. The AdS space is a solution of the vacuum Einstein equation with negative cosmological constant $\Lambda = -\frac{d(d-1)}{2R^2}$

1.2.1 Conformal Structure of AdS Spacetime

Introducing the new radial coordinate [46] $r_* = \arctan(\frac{r}{R})$ so that the line element (1.5) becomes

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \frac{R^2}{\cos^2(r_*)}\left[-\frac{d\tau^2}{R^2} + dr_*{}^2 + \sin^2(r_*)\, d\Omega_{d-1}^2\right] \tag{1.8}$$

It is evident from the line element that $r_* = \frac{\pi}{2}$ is the conformal boundary. One can rescale metric (1.8) to define a new metric

$$\tilde{g_{\mu\nu}} = \frac{\cos^2 r_*}{R^2} g_{\mu\nu} \tag{1.9}$$

This defines a metric on a smooth manifold (M) with boundary (∂M). The induced metric at $r_* = \frac{\pi}{2}$ gives the Einstein static universe. Again from (1.6) we see that $z = 0$ is the conformal boundary. The rescaled metric

$$\tilde{g}_{\mu\nu} = \frac{z^2}{R^2}\bar{g}_{\mu\nu} \tag{1.10}$$

The induced metric at the conformal boundary $(z = 0)$ is just the d dimensional Minkowski space. Now we know that the Minkowski space $\mathbb{M}^{1,d-1}$ is conformally equivalent to a patch of Einstein static universe $\mathbb{R} \times S^{d-1}$. Hence $z = 0$ of the Poincaré patch is a diamond shape piece of the conformal boundary. It is important to note that instead of considering the rescaled metric (1.9), we could also have considered the metric given below.

$$\tilde{g}_{\mu\nu} = \frac{\cos^2 r_*}{R^2} e^{2\sigma} g_{\mu\nu} \tag{1.11}$$

Where σ is a smooth arbitrary function on M. One can check that this metric is also nonsingular at $r_* = \frac{\pi}{2}$, but the induced metric is now only conformal to $\mathbb{R} \times S^{d-1}$. The choice of a particular rescaling factor determines a representative of the corresponding conformal class of boundary metrics. This choice of rescaling factor is known as the choice of conformal frame and the particular representative is called the boundary metric. Below we state the general notion of Penrose's conformal compactification. As discussed above we consider a manifold(M), with boundary ∂M. Now we consider metrics (g) on M which are singular on ∂M, such that there exists a smooth function Ω satisfying $\Omega \mid_{\partial M} = 0,\ (d\Omega) \mid_{\partial M} \neq 0$ and $\Omega > 0$ on all of M, such that

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \tag{1.12}$$

can be extended to all of M as a sufficiently smooth non degenerate metric for which the induced metric on ∂M has a Lorentz signature. Where $\tilde{g}_{\mu\nu}$ is not unique. One can always choose a new Ω' such that

$$\Omega' = e^{\sigma}\Omega \tag{1.13}$$

for arbitrary smooth σ on M. Thus the choice of a conformal frame corresponds to a particular boundary metric. The group of transformation which preserves (1.12) are called conformal isometries. They form a group among themselves. The Poincaré group forms a subgroup of the conformal group. The conformal transformation [50] consists of four kinds of transformation viz,

$$\begin{aligned}
&\text{Translation: } x^\mu \to x^\mu + a^\mu. \\
&\text{Lorentz Transformations: } x^\mu \to \Lambda^\mu_\nu x^\nu \\
&\text{Dilatation: } x^\mu \to \alpha x^\mu \\
&\text{Special Conformal Transformation(SCT): } x^\mu \to \frac{x^\mu - b^\mu x^2}{1 - 2b.x + b^2 x^2}
\end{aligned} \tag{1.14}$$

SCT is nothing but a translation, preceded and followed by an inversion. The generators corresponding to the infinitesimal transformations are listed below.

$$\begin{aligned}
&\text{Translation: } P^\mu = -i\partial^\mu \\
&\text{Rotations: } J^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu) + S^{\mu\nu} \\
&\text{Dilatation: } D = -i(d + (x.\partial)) \\
&\text{SCT: } K^\mu = -i\left((2x^\mu x^\nu - 2g^{\mu\nu} x^2)\partial_\nu + 2dx^\mu\right) + 2x_\nu S^{\mu\nu}
\end{aligned} \tag{1.15}$$

Where $S^{\mu\nu}$ is an antisymmetric spin matrix for a given field. It satisfies the Lorentz algebra. If the underlying field is a Fermion then $d = \frac{3}{2}$ and $d = 1$ for Bosons. One can check that these generators satisfies the following commutation relation

$$\begin{gathered}
[D, D] = 0, \;\; [P^a, P^b] = 0, \;\; [D, P^a] = iP^a, \;\; [J^{ab}, P^c] = -i(g^{ac} P^b - g^{bc} P^a) \\
[J^{ab}, J^{cd}] = -i(g^{ad} J^{bc} + g^{bc} J^{ad} - g^{ac} J^{bd} - g^{bd} J^{ac}), \;\; [J^{ab}, D] = 0, \\
[D, K^a] = -iK^a, \;\; [iK^a, K^b] = 0, \;\; [K^a, P^b] = 2i(g^{ab} D - J^{ab})
\end{gathered} \tag{1.16}$$

Now we will move on to study the isometry group of AdS_4. For this we consider the Euclidean AdS_4 metric

$$ds^2 = R^2 \frac{dt^2 + dx^2 + dy^2 + dz^2}{z^2} \tag{1.17}$$

For this we write down the all the ten generators

$$\begin{gathered} J_1 = \partial_t, \;\; J_2 = \partial_x, \;\; J_3 = \partial_y, \;\; J_4 = x\partial_t - t\partial_x, \;\; J_5 = y\partial_x - x\partial_y, \;\; J_6 = t\partial_y - y\partial_t, \\ J_7 = r\partial_r - t\partial_t - x\partial_x - y\partial_y, \;\; J_8 = rt\partial_r - \frac{1}{2}t^2\partial_t - tx\partial_x - ty\partial_y, \\ J_9 = rx\partial_r - tx\partial_t - \frac{1}{2}x^2\partial_x - xy\partial_y, \;\; J_{10} = ry\partial_r - ty\partial_t - xy\partial_x - \frac{1}{2}y^2\partial_y \end{gathered} \tag{1.18}$$

After making the suitable identification $t \to it$, one can check that the generator satisfy the usual $SO(3,2)$ algebra $[J_{ab}, J_{cd}] = i(g_{ad}J_{bc} + g_{bc}J_{ad} - g_{ac}J_{bd} - g_{bd}J_{ac})$ where a, b, c, d runs from 0 to 4. One can easily check for $2+1$ dimensional flat case the generators area

$$\begin{gathered} J_a = iP_a, \;\; J_4 = -iJ_{12}, \;\; J_5 = -iJ_{13}, \;\; J_6 = -iJ_{23}, \;\; J_7 = -iD, \;\; J_8 = iK_1, \\ J_9 = iK_2, \;\; J_{10} = iK_3 \end{gathered} \tag{1.19}$$

Where a, b runs from 1 to 3 and the generators are listed in (1.15). It is easy to check that these generator satisfy the same $SO(3,2)$ algebra. So we see that in general the isometry group of AdS_{d+1} is isomorphic to the conformal group of the flat Minkowski space $\mathbb{M}^{1,d-1}$. This statement can be considered as a motivation for the AdS/CFT duality.

1.3 Holographic entanglement entropy

Although AdS/CFT has been successful in relating QFT with bulk dual gravity solution. However it is still not well understood how gravity emerges from field theory. Entanglement entropy has served as an important tool to study this aspect of the correspondence. Similar ideas can be found in the Bekenstein-Hawking(BH) formula

$$S_{BH} = \frac{\text{Area}(\Sigma)}{4G_N} \tag{1.20}$$

Where S_{BH} is the black hole entropy, Σ is the event horizon, and G_N is the Newton Constant. Similarity between black hole entropy and entanglement entropy S_A, has been pointed out before [51,52]. Where A is the space-like submanifold on a constant time slice Σ. This area law behaviour is observed in field theories. For example in d dimensional free field theories, one can show that the leading divergent term of S_A in the UV limit $\epsilon \to 0$ obey the area law.

$$S_A = c\frac{\text{Area}(\partial A)}{\epsilon^{d-2}} \tag{1.21}$$

where c is the coefficient that is independent of A and ∂A is the boundary of A in the constant time slice Σ. In QFT's the entanglement entropy is always divergent, hence one includes the UV cutoff ϵ. Thus unlike the thermal entropy, the entanglement entropy is not an extensive quantity.

In the context of AdS/CFT Ryu and Takayanagi [6,7] generalised these ideas for calculating the entanglement entropy for subsystems in a strongly coupled quantum system living on the boundary of asymptotically AdS spacetimes. In this proposal the holographic entanglement entropy (HEE) S_A of a subsystem A in the boundary CFT is given by the area of a codimension 2 minimal surface γ_A in bulk viz.

$$S_A = \frac{Area(\gamma_A)}{4G_N}. \tag{1.22}$$

Where G_N is the Newton's constant. This prescription has been verified by several non trivial checks [9, 53–55] and a direct derivation in [10]. One can find excellent discussion on this topic in [56]. Applications of this proposal to higher derivative theories can be found in [57–61]

Now we will try to describe the construction of the Ryu Takayanagi surface in details. We will follow here the discussion given in [8]. We begin with a $d+1$ dimensional asymptotically AdS spacetime M with conformal boundary ∂M. We Know that static spacetimes admits a timelike Killing field $(\frac{\partial}{\partial t})^{\mu}$. Hence the timelike Killing field is orthogonal to constant t hypersurfaces. Thus one can naturally foliate the boundary ∂M by these surfaces, such that we can write $\partial M = \prod_t \partial\mathcal{S}_t \times \mathbb{R}_t$. Now we choose a subregion A on a particular leaf $\partial\mathcal{S}$ of this foliation, such that $\partial\mathcal{S} = A \cup A^c$. Let ∂A be the boundary of this region assuming $\partial\mathcal{S}$ to be compact. It is important to note that A is $d-1$ dimensional and ∂A is $d-2$ dimensional. Now by virtue of time translation invariance in these static backgrounds, the boundary spacelike foliation naturally extends into the bulk to provide a canonical spacelike foliation $\prod_t \mathcal{S}_t$ of M. Now on a given spacelike slice M one can find a $d-1$ dimensional minimal area surface which ends on $\partial A \subset \partial\mathcal{S}$. As the bulk spacelike slice is of Euclidean signature the minimal surface $\mathcal{Y}$ is bound to exist. Thus given the minimal surface Y, the entanglement entropy associated with the region A is given by the area of the minimal surface Y in bulk Planck unit.

Now the natural question arises that what happens when the spacetime is no longer static?. In order to answer this question we need to understand the concept of a light sheet and covariant entropy bound first. Let us consider a codimension two spacelike surface B in a manifold M. Now one can construct four congruences of past and future directed null geodesics from the surface in ingoing and outgoing directions. The null geodesic congruence for which the expansion of the null geodesic is non positive definite is called a Light Sheet L_B. Due to this requirement the null geodesics along the light sheet are converging and will develop caustics, at this point the light sheet gets cut off. The covariant entropy bound or the Bousso bound [62–66] states that the entropy or amount of information S_{L_B} that can pass through a light sheet is bounded by the area of the spacelike surface B.

$$S_{L_B} \leq \frac{\text{Area}(B)}{4G_N} \tag{1.23}$$

Following [8] one gets the important result that in case of the AdS/CFT correspondence , the entanglement entropy saturates the Bousso bound. With this in mind one can describe the setup of covariant entanglement entropy proposal in the following way. As before we consider a $d+1$ dimensional asymptotically AdS spacetime M with d dimensional conformal boundary ∂M. At any given time one can choose a subregion A_t in the $d-1$ dimensional spacelike subspace of the boundary ∂M. Now the boundary ∂A_t of this subregion is a $d-2$ dimensional surface in M and a codimension two spacelike surface in ∂M. Hence one can construct upper and lower light sheets $\partial L^+_{A_t}$ and $\partial L^-_{A_t}$ for the surface ∂A_t. Now one can consider the extensions $L^{\pm}_{A_t}$ of the two light sheets $\partial L^{\pm}_{A_t}$ into the bulk such that they are also the light sheets of the same codimension two $(d-1)$ surface $Y_t = L^+_{A_t} \cap L^-_{A_t}$ in M. Now one can vary the form of the light sheets $L^{\pm}_{A_t}$ keeping $\partial L^{\pm}_{A_t}$ fixed. This give rise to a class of surfaces $\{Y_t\}$. Now the covariant entanglement entropy is given by the area of the surface $Y^{\min}$ having the least area in the class $\{Y_t\}$.

$$S_{A_t}(t) = \frac{Area(Y^{\min})}{4G_N}. \tag{1.24}$$

However we need to check whether this definition is consistent with the surface obtained by a saddle point of the area action. In order to check this we consider a codimension two surface S in a spacetime manifold M given by the following embedding functions

$$f_1(x^\nu) = 0, \;\; f_2(x^\nu) = 0 \tag{1.25}$$

Now non degeneracy requirement ensures the existence of two linearly independent normal null vectors viz

$$V_{\pm}^{\mu} = g^{\mu\nu}(\nabla_\nu f_1 + \mu_\pm \nabla_\nu f_2) \tag{1.26}$$

One can fix the normalization as

$$V_+^\mu V_-^\nu g_{\mu\nu} = -1 \tag{1.27}$$

Now one can write the null extrinsic curvature of the surface S as

$$(K_\pm)_{\mu\nu} = h_\mu^\rho h_\nu^\lambda \nabla_\rho (V_\pm)_\lambda \tag{1.28}$$

Now the expansion of an orthogonal null geodesic congruence to the surface is given by the trace of the null extrinsic curvature

$$\theta_{\pm} = (K_{\pm})^{\mu}_{\mu} \tag{1.29}$$

By definition this quantity is the mean curvature of the surface and we will see in the next section (1.3.1) that the extremal surfaces are surfaces having zero mean curvature. Thus the surfaces S with vanishing null expansions $\theta_{\pm}$ are extremal surfaces. Thus out of the class of surfaces $\{Y_t\}$, the minimal area surface $Y^{\min}$ is the surface having vanishing null expansions $\theta_{\pm}$.

Finally we need to ask when this proposal goes back to Ryu Takayanagi Proposal or in other words can we construct the covariant extremal surface in the same way as we did for the Ryu Takayanagi Case?. Now from the field theory perspective one can consider the boundary theory to be in a time varying state on a fixed background ∂M. But the bulk geometry will have an explicit time-dependence and hence no timelike Killing field. As the boundary metric is non dynamical one can chose the same equal time foliation as before by choosing a time coordinate consistent with Hamiltonian evolution of the field theory. Thus $\partial M = \prod_t \partial\mathcal{S}_t \times \mathbb{R}_t$ still holds and one can choose a region $A_t \in \partial\mathcal{S}_t$ on a given time slice and compute the entanglement entropy using path integral approach. However the equal time foliation of the boundary ∂M doesn't lead to a canonical foliation of the bulk M. Still If one can find a natural foliation and pick up a spacelike slice S_t of M given by extending the slice from ∂M and then one can find an extremal surface with the same boundary subregion. From this observation one can conclude that the maximal area slicing could be the candidates for S_t as they go to the $t =$ constant slicing for static bulk. However from [8] one can see that in order to match this construction with the covariant one the maximal slicing should also be totally geodesic. But a spacetime admitting a totally geodesic foliation must be static. Thus this method of constructing the covariant extremal surface agrees with the light sheet construction only in the trivial case of static bulk geometry.

1.3.1 Minimal Surface in AdS spacetime

The very first idea of minimal surface [67–71] comes from area minimization. We would like to obtain a condition for a surface to have minimal area. The construction is as follows, Let (M, g) be a Riemann manifold and S be a submanifold with boundary of M. Let h_{ab} be the induced metric on S. Let ∇, D be the Levi Civita connections w.r.t g and h respectively. Next we consider a variation of S in M, with fixed boundary

$$f : S \times I \to M, f_0 = id_M, f_{|\partial M} = id_{\partial M} \tag{1.30}$$

We assume that $f_t : S \to M$ are embeddings and let $S_t = f_t(S)$. Then the variational vector field is given by $N = \partial_t f \in TS_{|S_t}$ and $N_{|\partial S} \equiv 0$. The area of S is given by

$$\delta A = \int \delta\sqrt{h} \;\; d^n x. \tag{1.31}$$

Now let us consider the first variation of area of S under this flow

$$\frac{1}{\sqrt{h}}\frac{d}{dt}\sqrt{h}_{|t=0} = \frac{1}{2}h^{ab}\frac{d}{dt}h_{ab|t=0} = h^{ab}\left(g(\nabla_N T_a, T_b)\right) \tag{1.32}$$

Where $T_a = \frac{\partial}{\partial \tau^a}$ and recall that given vector fields X, Y on S we have

$$\nabla_X Y = (\nabla_X Y)^{TS} + (\nabla_X Y)^{NS} = D_X Y + K(X,Y) \tag{1.33}$$

Where $K(X,Y)$ is the Extrinsic curvature with values in normal bundle NS of S in M.

$$\begin{aligned}\frac{d}{dt}\sqrt{h}_{|t=0} &= \sqrt{h}h^{ab}g(\nabla_{T_a}N^T, T_b) + \sqrt{h}h^{ab}g(\nabla_{T_a}N^\perp, T_b) \\ &= \sqrt{h}h^{ab}h(D_{T_a}N^T, T_b) - \sqrt{h}h^{ab}g(N^\perp, \nabla_{T_a}T_b) \\ &= \sqrt{h}h^{ab}h(D_{T_a}N^T, T_b) - \sqrt{h}h^{ab}g(N, (\nabla_{T_a}T_b)^\perp) \\ &= Div_S N^T - \sqrt{h}g(N, H).\end{aligned} \tag{1.34}$$

Thus the first variation of area of S is

$$\frac{d}{dt}A_{|t=0} = \int_{\partial S}\langle \eta, N\rangle - \int_S g(N,H)\sqrt{h}d^n x \tag{1.35}$$

Where η is the outward pointing conormal along ∂S in S and $H = h^{ab}K_{ab}$ is the trace of extrinsic curvature and is called the Mean curvature . Now according to our assumption $N_{|\partial S} \equiv 0$ so the first term drops out and if we set $H = 0$ then we get $\frac{d}{dt}A_{|t=0} = 0$. Thus we see that setting mean curvature to zero gives us the minimality condition.

Now we will explore this setup for AdS_4. As AdS_4 is a static spacetime a codimension two spacelike surface in it be given by $t = const, z = z(x,y)$. We wish to solve for $z(x,y)$ using the minimality condition described above. Using standard orthonormalization technique, one can construct an orthonormal basis. These are listed below

$$
\begin{aligned}
M &= z\partial_t \\
N &= \frac{z(z_{,x}\partial_x + z_{,y}\partial_y - \partial_z)}{\sqrt{1+C}} \\
T &= \frac{z\left(z_{,y}\partial_x - z_{,x}\partial_y\right)}{\sqrt{C}} \\
S &= \frac{z\left(z_{,x}\partial_x + z_{,y}\partial_y + C\partial_z\right)}{\sqrt{C}\sqrt{1+C}}
\end{aligned}
\tag{1.36}
$$

Where $C = (z_{,x}^2 + z_{,y}^2)$. (M, N) are the two normals to the surface and (T, S) are tangent. Using this we can calculate

$$
\begin{aligned}
\nabla_T N = \frac{z^2}{\sqrt{C}\sqrt{1+C}} \Bigg[&\left((z_{,y}z_{,xx} - z_{,x}z_{,xy}) - \frac{z_{,x}\left(mz_{,y} - nz_{,x}\right)}{(1+C)} + \frac{z_{,y}}{z} \right) \partial_x \\
+ &\left((z_{,y}z_{,xy} - z_{,x}z_{,yy}) - \frac{z_{,y}\left(mz_{,y} - nz_{,x}\right)}{(1+C)} - \frac{z_{,x}}{z} \right) \partial_y + \frac{(mz_{,y} - nz_{,x})}{(1+C)}\partial_z \Bigg]
\end{aligned}
\tag{1.37}
$$

and

$$
\begin{aligned}
\nabla_S N = \frac{z^2}{\sqrt{C}\sqrt{1+C}} \Bigg[&\left(m - \frac{z_{,x}\left(mz_{,x} + nz_{,y}\right)}{(1+C)} + \frac{z_{,x}}{z} \right) \partial_x + \Bigg(n \\
- &\frac{z_{,y}\Big(mz_{,x} + nz_{,y}\Big)}{(1+C)} + \frac{z_{,y}}{z} \Bigg) \partial_y + \frac{1}{z} \left(C + \frac{z\left(mz_{,x} + nz_{,y}\right)}{(1+C)} \right) \partial_z \Bigg]
\end{aligned}
\tag{1.38}
$$

Now as the mean curvature is just the trace of extrinsic curvature, we only need to obtain the diagonal elements of the extrinsic curvature viz From here we can calculate the extrinsic curvature

$$
\begin{aligned}
-g(K(T,T),N) = g(\nabla_T N, T) &= \frac{z}{C\sqrt{1+C}}\left[z_{,y}^2 z_{,xx} + z_{,x}^2 z_{,yy} - 2z_{,y}z_{,x}z_{,xy}\right] + \frac{1}{\sqrt{1+C}} \\
-g(K(S,S),N) = g(\nabla_S N, S) &= \left[\frac{z\left(z_{,x}^2 z_{,xx} + z_{,y}^2 z_{,yy} + 2z_{,y}z_{,x}z_{,xy}\right)}{C(1+C)^{\frac{3}{2}}}\right] + \frac{1}{\sqrt{1+C}}
\end{aligned}
\tag{1.39}
$$

Thus the mean curvature $\mathcal{H}$ is given by

$$
\begin{aligned}
-\mathcal{H} &= -g(H,N) = -g(K(T,T)+K(S,S),N) \\
&= \frac{z}{(1+C)^{\frac{3}{2}}}\left((1+z_{,y}^2)z_{,xx} + (1+z_{,x}^2)z_{,yy} - 2z_{,y}z_{,x}z_{,xy} + \frac{2}{z}(1+z_{,x}^2+z_{,y}^2)\right)
\end{aligned}
$$

The minimal surface equation is given by demanding $\mathcal{H}=0$. From the above equation this equates to the following equation.

$$
(1+z_{,y}^2)z_{,xx} + (1+z_{,x}^2)z_{,yy} - 2z_{,y}z_{,x}z_{,xy} + \frac{2}{z}(1+z_{,x}^2+z_{,y}^2) = 0 \tag{1.40}
$$

One can obtain the same equation from area variation directly. Let us show it here for the embedding $t=$ constant, $z=z(x,y)$ the area functional can be written as

$$
A = \int dxdy\sqrt{h} = \int dxdy\frac{\sqrt{1+z_{,x}^2+z_{,y}^2}}{z^2} = \int dxdyS \tag{1.41}
$$

On minimizing this action we should get one equation

$$
\frac{d}{dx}\left(\frac{\partial S}{\partial z_{,x}}\right) + \frac{d}{dy}\left(\frac{\partial S}{\partial z_{,y}}\right) - \frac{\partial S}{\partial z} = 0 \tag{1.42}
$$

Where

$$S = \frac{\sqrt{1+z_{,x}^2+z_{,y}^2}}{z^2} \tag{1.43}$$

Now substituting we get

$$\begin{aligned}
&\frac{d}{dx}\left(\frac{\partial S}{\partial z_{,x}}\right)+\frac{d}{dy}\left(\frac{\partial S}{\partial z_{,y}}\right)-\frac{\partial S}{\partial z}\\
&=\frac{(z_{,xx}(1+z_{,y}^2)+z_{,yy}(1+z_{,x}^2)-2z_{,x}z_{,y}z_{,xy})}{z^2(1+z_{,x}^2+z_{,y}^2)^{\frac{3}{2}}}-\frac{2(z_{,x}^2+z_{,y}^2)}{z^3\sqrt{1+z_{,x}^2+z_{,y}^2}}+\frac{2\sqrt{1+z_{,x}^2+z_{,y}^2}}{z^3}\\
&=\frac{1}{z^2(1+z_{,x}^2+z_{,y}^2)^{\frac{3}{2}}}\left[(1+z_{,y}^2)z_{,xx}+(1+z_{,x}^2)z_{,yy}-2z_{,y}z_{,x}z_{,xy}+\frac{2}{z}(1+z_{,x}^2+z_{,y}^2)\right]=0\\
&=(1+z_{,y}^2)z_{,xx}+(1+z_{,x}^2)z_{,yy}-2z_{,y}z_{,x}z_{,xy}+\frac{2}{z}(1+z_{,x}^2+z_{,y}^2)=0
\end{aligned} \tag{1.44}$$

Which is same as (1.40).

Solutions:- It can be checked [14] that $z(x)=\sqrt{l^2-x^2-y^2}$ with $0\leq|x|\leq l, 0\leq |y|\leq l$ is a solution. If we take $z=z(y)$ then substituting $z_{,x}=0$ in (1.40) to get

$$z_{,yy}+\frac{2}{z}(1+z_{,y}^2)=0 \tag{1.45}$$

This can be solved by the boundary condition $y=0, z(0)=z_*, z_{,y}(0)=0$. Notice that z_* is the maximal height attained by the curve along the z direction. It can be checked that

$$y(z)=\frac{\sqrt{\pi}\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}z_*-\frac{z^3}{3z_*^2}F_2{}^1(\frac{1}{2},\frac{3}{4};\frac{7}{4};\frac{z^4}{z_*{}^4}),\quad z_*=\frac{\Gamma(\frac{1}{4})}{\sqrt{\pi}\Gamma(\frac{3}{4})}l \tag{1.46}$$

where Γ is the Euler gamma and F_2^1 is the hypergeometric function. For simplicity we have demonstrated this for AdS_4. But these solutions also exist in higher dimension. In this thesis we will only study these two minimal surfaces for other surface one can check [14].

Extremal surfaces in De Sitter and application to holographic entanglement entropy can be found in [72–75]

1.3.2 Minimal surfaces in asymptotically AdS spacetime

It is important to note that there are hardly any exact analytical solution of the minimal surface equation in asymptotically Anti De Sitter spacetime. Hence one has to adopt a asymptotic perturbation theory [76, 77] approach to solve the minimal surface equation. For the sake of clarity we will discuss this for the AdS_{d+1} blackbrane background. The metric of the AdS_{d+1} blackbrane is given as

$$ds^2 = \frac{1}{z^2}\left(-f(z)dt^2 + dx_1^2 + \cdots + dx_{d-1}^2 + \frac{dz^2}{f(z)}\right) \tag{1.47}$$

with

$$f(z) = 1 - \frac{z^d}{z_0^d}$$

z_0 is the horizon Now as this is a static geometry we can still take the subsystem on a constant t slice. Let us first consider the subregion to be a sphere. For the sphere we take the embedding as

$$dx_1^2 + \cdots + dx_{d-1}^2 = dr(z)^2 + r(z)^2 d\Omega_{d-2}^2 = r'(z)^2 dz^2 + r(z)^2 d\Omega_{d-2}^2$$

Where $r'(z) = \frac{dr}{dz}$. Thus the area of the spherical subsystem is given by

$$A = 2\Omega_{d-2}\int_0^l \frac{dz\; r(z)^{d-2}}{z^{d-1}}\sqrt{\frac{1}{f(z)} + r'(z)^2} \tag{1.48}$$

Where l is the radius of the disk at the boundary. The equation of motion for this action functional is

$$
\begin{aligned}
zr(z)r''(z) - (d-1)f(z)r(z)r'(z)^3 - \frac{r(z)}{2}\left((d-2) + \frac{d}{f(z)}\right)r'(z) \\
-(d-2)z\left(r'(z)^2 + \frac{1}{f(z)}\right) = 0 \qquad (1.49)
\end{aligned}
$$

Since there is no analytic closed form solution of this differential equation, we have to adopt the regularized perturbation method [76, 77] to solve this as an asymptotic series. Our first task is to identify the parameter in eqn(1.49) whose limit reduces the differential equation to a solvable one. One can check that the parameter is $\frac{1}{z_0^d}$ and taking the limit $z_0 \to \infty$ $(f(z) = 1)$ reduces the above equation to

$$
zr(z)r''(z) - \left(r'(z)^2 + 1\right)\left((d-1)r(z)r'(z) + (d-2)z\right) = 0 \qquad (1.50)
$$

This is the minimal surface equation for the same subsystem but embedded in pure AdS_{d+1}. The solution of this equation is $r(z) = \sqrt{{z_*^{(0)}}^2 - z^2}$. Where $z_*^{(0)}$ is the turning point. Thus we assume a series solution of eqn(1.49) in powers of $\frac{1}{z_0^d}$, at first order we have

$$
r(z) = h(z) + \frac{b(z)}{z_0^d}
$$

Plugging this back in eqn(1.49) retaining up to first order in $\frac{1}{z_0^d}$ gives

$$z\left(h(z)+\frac{b(z)}{z_0^d}\right)\left(h''(z)+\frac{b''(z)}{z_0^d}\right)-(d-1)\left(1-\frac{z^d}{z_0^d}\right)\left(h(z)+\frac{b(z)}{z_0^d}\right)\left(h'(z)+\frac{b'(z)}{z_0^d}\right)^3$$
$$-\frac{1}{2}\left(h(z)+\frac{b(z)}{z_0^d}\right)\left[(d-2)+d\left(1+\frac{z^d}{z_0^d}+\frac{z^{2d}}{z_0^{2d}}+\cdots\right)\right]\left(h'(z)+\frac{b'(z)}{z_0^d}\right)$$
$$-(d-2)z\left[1+\frac{z^d}{z_0^d}+\frac{z^{2d}}{z_0^{2d}}+\cdots+\left(h'(z)+\frac{b'(z)}{z_0^d}\right)^2\right]=0$$

Now retaining terms up to first order in $\frac{1}{z_0^d}$ gives

$$\left[zh(z)h''(z)-\left(h'(z)^2+1\right)\left((d-1)h(z)h'(z)+(d-2)z\right)\right]+\frac{1}{z_0^d}\Bigg[-(d-2)z$$
$$\left(2b'(z)h'(z)+z^d\right)+\frac{1}{2}h(z)\left(2zb''(z)-2(d-1)b'(z)\left(3h'(z)^2+1\right)+z^dh'(z)\right.$$
$$\left.\left(2(d-1)h'(z)^2-d\right)\right)+b(z)\left(zh''(z)-(d-1)\left(h'(z)^3+h'(z)\right)\right)\Bigg]+\cdots=0$$

If the series solution is true then the coefficient of all powers of $\frac{1}{z_0^d}$ must go to zero. Thus equating the coefficient of the zeroth order term to zero gives

$$zh(z)h''(z)-\left(h'(z)^2+1\right)\left((d-1)h(z)h'(z)+(d-2)z\right)=0$$

This is the familiar equation (1.50) and the solution is

$$h(z)=\sqrt{z_*^2-z^2}$$

Where regularity is assumed at $z = z_*$. Now plugging the value of $h(z)$ back in the coefficient of the first order and equating it to zero term gives

$$2z\left(z_*^2 - z^2\right)^2 b''(z) - z^{d+1}\sqrt{z_*^2 - z^2}\left[(d-4)z_*^2 + (d+2)z^2\right]$$
$$-2(z_*^2 - z^2)b'(z)\left[(d-1)z_*^2 + 2z^2\right] + 2(d-2)z_*^2 z b(z) = 0$$

The general solution of this equation is

$$b\left(z\right) = \frac{C1}{\sqrt{\left(z_*^2 - z^2\right)}} + z^d\left(\left(z_*^2 - z^2\right)\right)^{-d/2+1} {}_2\mathrm{F}_1(1, 3/2;\, d/2+1;\, \frac{z^2}{{z_*}^2})C2$$
$$-\frac{\left({z_*}^2 + z^2\right) z^d}{\sqrt{{z_*}^2 - z^2}\left(2\,d + 2\right)}$$

Where C_1 and C_2 are integration constant. Now setting the boundary condition $b(0) =$ finite, $b(z_*) = 0$ gives

$$b(z) = \frac{2z_*^{d+2} - z^d\left(z_*^2 + z^2\right)}{2(d+1)\sqrt{\left(z_*^2 - z^2\right)}}$$

Thus the full solution up to first order([78–80]) is

$$r(z) = \sqrt{z_*^2 - z^2} + \frac{2z_*^{d+2} - z^d\left(z_*^2 + z^2\right)}{2z_0^d(d+1)\sqrt{\left(z_*^2 - z^2\right)}} \tag{1.51}$$

Where the constant (z_*) is related to the radius of the sphere l by

$$r(0) = l = z_* + \frac{z_*^{d+1}}{z_0^d(d+1)} \tag{1.52}$$

This solution is also given in [23,81,82]. In the next section we will use the Ryu Takayanagi proposal to calculate the entanglement entropy for a strip and circular disk.

1.3.3 Entanglement entropy for a strip and circular disk using RT proposal

As our first example we will calculate the entanglement entropy for a strip of width l. The strip at the boundary is homologous to a $(d-1)$ dimensional bulk minimal surface embedded in AdS_{d+1}. The AdS_{d+1} metric in Poincaré coordinate is given by

$$ds^2 = \frac{-dt^2 + dx_1{}^2 + \cdots + dx_{d-1}{}^2 + dz^2}{z^2} \tag{1.53}$$

We embed a strip like surface in this background given by $t =$ constant, $x_1 = x_1(z), x_i = x_i$. The boundaries of the extremal bulk surface coincide with the two ends of the interval $(-\frac{l}{2} \leq x_1 \leq \frac{l}{2})$. The regulated size of the rest of the coordinates is taken large $0 \leq x_i \leq L_i$. The area of the strip like surface is given by

$$A = 2V_{d-2} \int_{\epsilon}^{z_*} \frac{dz}{z^{d-1}} \sqrt{1 + x'(z)^2} \tag{1.54}$$

Where z_* is the turning point of the surface and ϵ is the UV cutoff. The minimal surface is obtained by minimizing the area functional. On minimizing we get

$$x'(z) = \frac{1}{\sqrt{(\frac{z_*}{z})^{2(d-1)} - 1}} \tag{1.55}$$

The identification of the boundary $x^1(0) = l/2$ leads to the integral relation

$$\frac{l}{2} = \int_0^{z_*} \frac{dz}{\sqrt{(\frac{z_*}{z})^{2(d-1)} - 1}} = z_* \int_0^1 t^{d-1} \frac{dt}{\sqrt{1 - t^{2(d-1)}}} = z_* b_0 \tag{1.56}$$

where $t = \frac{z}{z_*}$ and b_0 is a precise integral Beta functions provided in the appendix(B). Using (1.55) in (1.54) we get

$$
\begin{aligned}
A &= \frac{2V_{d-2}}{{z_*}^{d-2}} \int_{\frac{\epsilon}{z_*}}^{1} \frac{dt}{t^{d-1}} \frac{1}{\sqrt{1-t^{2(d-1)}}} \\
&= \left(\frac{2V_{d-2}}{(d-2)\epsilon^{d-2}} - \frac{2V_{d-2}}{{z_*}^{d-2}} a_0 \right)
\end{aligned} \tag{1.57}
$$

Thus using (1.22) the entanglement entropy for a strip embedded in pure AdS_{d+1} is given by [6, 7].

$$
S = \frac{1}{4G_N^{d+1}} \left(\frac{2}{(d-2)} \frac{V_{d-2}}{\epsilon^{d-2}} - \frac{2^{d-1}\pi^{\frac{(d-1)}{2}}}{(d-2)} \left(\frac{\Gamma(\frac{d}{2(d-2)})}{\Gamma(\frac{1}{2(d-2)})} \right)^{(d-1)} \frac{V_{d-2}}{l^{d-2}} \right) \tag{1.58}
$$

It is important to note that the first divergent term is proportional to the area $\delta A = V_{d-2}$ and is in confirmation with the area law from field theory computations. The second term does not depend on the cutoff and hence is universal. This term can be directly compared with the field theory counterparts. Similarly one can consider a circular disk instead of the strip. The minimal surface in that case is half of a $d-1$ dimensional sphere centered at $z = 0$. Holographic entanglement entropy for a circular disk like subsystem of radius l is given by [6, 7]

$$
\begin{aligned}
S_D &= \frac{2\pi^{\frac{(d-1)}{2}}}{4G_N^{d+1}\Gamma(\frac{d-1}{2})} \int_{\frac{\epsilon}{l}}^{1} dt \frac{(1-t^2)^{\frac{(d-3)}{2}}}{t^{d-1}} \\
&= c_2(\frac{l}{\epsilon})^{d-2} + c_4(\frac{l}{\epsilon})^{d-4} + \cdots \\
&\cdots + \begin{cases} c_{d-2}(\frac{l}{\epsilon}) + c_{d-1} + \mathcal{O}(\frac{l}{\epsilon}), & d = \text{even}, \\ c_{d-3}(\frac{l}{\epsilon})^2 + q\log(\frac{l}{\epsilon}) + \mathcal{O}(1), & d = \text{odd}, \end{cases}
\end{aligned} \tag{1.59}
$$

Where the coefficients of the terms are given as

$$
\begin{aligned}
\frac{c_2}{\alpha} &= \frac{1}{(d-2)}, \quad \frac{c_4}{\alpha} = -\frac{(d-3)}{2(d-4)} \\
\frac{c_{d-1}}{\alpha} &= (\frac{2}{\sqrt{\pi}})^{-1}\Gamma(\frac{d-1}{2})\Gamma(\frac{2-d}{2}) && : d = \text{even} \\
\frac{q}{c} &= (-1)^{\frac{d-2}{2}}\frac{(d-3)!!}{(d-2)!!} && : d = \text{odd} \\
&\text{Where} \quad \alpha = \frac{\pi^{\frac{d-1}{2}}}{4G_N^{d+1}\Gamma(\frac{d-1}{2})}
\end{aligned}
\tag{1.60}
$$

In accordance with area law one can check that the leading UV divergent term $\sim \epsilon^{-d+2}$ and its coefficient is proportional to the area of the boundary ∂A. The subleading terms indicate the form of the boundary. When d is even the universal term i.e; the term independent of the cutoff is given by a constant p_{d-1}. When d is odd the universal term is given by the coefficient of the logarithmic term $\sim \log \frac{l}{\epsilon}$. It is important to note that in Lorentzian spacetime due to the presence of the time direction one can wiggle the surface in the time direction and make its area arbitrary small. Hence the notion of minimal surface is well defined in the static case where one can consider a constant time slice or one can Wick rotate and work in Euclidean setup.

1.4 Entanglement First Law

In this section we will try to formulate the entanglement first law from both CFT and bulk arguments. We know that relative entropy $S(\rho \mid \sigma)$ of two density matrices ρ and σ provides a measure of distinguishability between them. It is defined as [22,27,56]

$$S(\rho \mid \sigma) = \text{Tr}(\rho \log \rho) - \text{Tr}(\rho \log \sigma) \tag{1.61}$$

It satisfies two basic properties viz

1. Positivity: Relative entropy is non negative for any two density matrices and vanishes only when the two are equal, i.e.,

$$S(\rho \mid \sigma) \geq 0,\ S(\rho \mid \sigma) = 0 \rightarrow \rho = \sigma$$

2. Monotonicity: Relative entropy decreases under inclusion. Relative entropy decreases on tracing out the same degrees of freedom.

$$S(\rho_A \mid \sigma_A) \leq S(\rho \mid \sigma),\ \ \rho_A = \text{Tr}_{A^c}(\rho),\ \sigma_A = \text{Tr}_{A^c}(\sigma) \tag{1.62}$$

Where we trace out over a region A and its compliment A^c Now as in thermodynamic sense on can define the modular free energy as

$$F(\rho) = \text{Tr}(\rho H_\sigma) - S(\rho)$$

Where $S(\rho)$ is the von Neumann entropy of the density matrix ρ and $H_\sigma = -\log\sigma$ is the modular Hamiltonian for the density matrix σ. Now using this one can express the relative entropy as

$$\begin{aligned}
S(\rho \mid \sigma) \quad &= \text{Tr}(\rho\log\rho) - \text{Tr}(\sigma\log\sigma) + \text{Tr}(\sigma\log\sigma) - \text{Tr}(\rho\log\sigma) \\
&= -S(\rho) + S(\sigma) - \langle -\log\sigma\rangle_\sigma + \langle -\log\sigma\rangle_\rho \\
&= F(\rho) - F(\sigma) \\
&= \Delta\langle H_\sigma\rangle - \Delta\langle S\rangle \geq 0
\end{aligned} \tag{1.63}$$

Where the last inequality is guaranteed by positivity of relative entropy. For two states close to each other one can use the relative entropy as a distance measure. Let us consider a one parameter family of states $\rho = \rho_0 + \epsilon\rho_1 + \epsilon^2\rho_2 + \cdots$ in the neighbourhood of a reference state $\sigma = \rho_0$. Now one can write the relative entropy as a power series in ϵ. From the above argument this is clear that relative entropy is at least quadratic in ϵ. The contribution to relative entropy at $\mathcal{O}(\epsilon)$ is zero for any choice of ρ_0. This observation leads us to

$$\delta S = \delta\langle H_{\rho_0}\rangle \tag{1.64}$$

Thus at linear order the inequality is saturated. This statement is known as the first law of entanglement due to its close resemblance with first law of thermodyanmics [27].

We Know that the pure AdS_{d+1} spacetime is dual to a holographic CFT_d and asymptotically locally AdS spacetime would correspond to excitations over the CFT. Thus the bulk asymptotic geometry will get modified by excitations, which are viewed as specific states in the field theory Hilbert space. Now given in Poincaré coordinate one an use the scale/radius duality to observe that high energy (UV) excitations will modify the geometry near the conformal boundary i.e, small z. Similarly low energy excitations (IR) will modify the bulk geometry near the larger values of z i.e, at the center of the AdS or the Poincaré horizon. Now as we know form the Ryu Takayanagi proposal that holographic entanglement entropy of a subsystem at the boundary is given by the area of the bulk extremal surface. As the extremal surface is sensitive to the deformations in the bulk geometry so the entanglement entropy serves as a important tool to study this.Now as we saw the stationarity of relative entropy for perturbation about reference state is responsible for entanglement first law. Now from the observation of entanglement entropy one can conclude that the reference state ρ_0 is indistinguishable from the state $\rho_0 + \epsilon\rho_1$ upto $\mathcal{O}(\epsilon)$. Now our task is to encompass this from the bulk side, any asymptotically AdS spacetime is a finite change from pure AdS spacetime. So as long as we focus on macroscopic details of the excitations they would be clearly distinguishable from the ground state. Thus we have to focus our attention on reduced density matrix induced on relatively small subregions constrained to be free from the details of the IR and depend only on the energy density of the excitations. We will describe this below with an example as described in [23]. Thus one starts with an excited state which preserves spatial and time translations with rotation. Thus after fixing the radial coordinate to measure the proper size of the spatial sections one can write the bulk metric in terms of two unknown functions viz

$$ds^2 = \frac{R^2}{z^2}\left(-f(z)dt^2 + g(z)dz^2 + d\boldsymbol{x}_{d-1}^2\right) \tag{1.65}$$

Where $d\boldsymbol{x}_{d-1}^2$ is the Euclidean line element over the $(d-1)$ flat directions. Now after setting all the matter contribution to zero and and fixing the boundary metric to $\eta_{\mu\nu}$. One can express the near-boundary geometry of an excited state to be given by

$$g(z) \simeq \frac{1}{f(z)} \simeq 1 + mz^d + \mathcal{O}(z^{d+1}), \tag{1.66}$$

Where the energy density of the excitation is set by m. For this background one can compute the holographic stress energy tensor by expressing the metric in the Feffermen Graham coordinate and is given by

$$T_{\mu\nu}dx^\mu dx^\nu = \frac{R^{d-1}m}{16\pi G_N^{(d+1)}}\left((d-1)dt^2 + d\boldsymbol{x}_{d-1}^2\right) \tag{1.67}$$

As (1.66) is a near boundary geometry, the metric (1.65) can have either a blackhole or a star like configuration in the interior. As stated earlier that we will only focus on small subregion on the boundary hence interior detail of the geometry is not required. Now we know that in pure AdS_{d+1}, for a subregion(A) of size l on the boundary the bulk minimal surface probes the region within $0 < z < l$. Thus in order to make the bulk minimal surface free from the details of the IR region and make it depend on the parameter m we prescribe the following bound on the minimal surface

$$ml^d \ll 1 \Rightarrow \langle T_{\mu\nu} \rangle \tag{1.68}$$

This bound can also be rephrased as $l^{d+1} \ll \frac{R^{d-1}m}{16\pi G_N^{(d+1)}}$ i.e, we require that the energy density be much smaller than the characteristic energy scale of the subsystem A. Now given this approximation the reduced density matrix for the excited state is almost indistinguishable from that of the vacuum. Thus one will always observe

$$\rho_A(| E\rangle) \sim \rho_A(| 0\rangle) + \mathcal{O}(El^d) \tag{1.69}$$

Where E is the typical energy scale of the excited state. From here one can conclude

$$\begin{aligned} S(\rho_A(| E\rangle) \mid \rho_A(| 0\rangle)) &= \mathcal{O}(E^2\, l^{2d}) \\ \Delta S_A = \langle H_A \rangle, &\quad \text{at } \mathcal{O}(E\, l^d) \end{aligned} \tag{1.70}$$

Now let us check whether this observation agrees with bulk calculations. To check this we calculate the change in entanglement entropy when the subregion is a spherical ball with radius l. We follow our calculation in subsection(1.3.2) and (1.48) to obtain the entanglement entropy as

$$S = \frac{R^{d-1}}{16\pi G_N^{(d+1)}} 4\pi\Omega_{d-2} \int \frac{dz}{z^{d-1}} r(z)^{d-2} \sqrt{g(z) + r'(z)^2} \tag{1.71}$$

We can carry over the solution from our analysis done in subsection (1.3.2) and (1.51)

$$r(z) = \sqrt{z_*^2 - z^2} + m \frac{2z_*^{d+2} - z^d \left(z_*^2 + z^2\right)}{2(d+1)\sqrt{(z_*^2 - z^2)}} \tag{1.72}$$

Where the turning point (z_*) is related to the radius of the sphere l by

$$r(0) = l = z_* + \frac{z_*^{d+1}}{z_0^d(d+1)} \tag{1.73}$$

Now we can substitute this back in (1.71) to get

$$\begin{aligned} S &= 4\pi \frac{R^{d-1}}{16\pi G_N^{(d+1)}} \Omega_{d-2} \int_\epsilon^{z_*} dz \frac{z_*}{z^{d-1}} \left(z_*^2 - z^2\right)^{\frac{d-3}{2}} (1 + M(z)) \\ M(z) &= \frac{2(d-2)z_*^{d+2} - 2z_*^d z^2 + (d+3)z_*^2 z^d - 3(d-1)z^{d+2}}{2(d+1)(z_*^2 - z^2)} \end{aligned} \tag{1.74}$$

Expanding this integral and calculating the change in holographic entanglement entropy upto first order in m we get

$$\Delta S = \frac{2\pi}{d+1} \frac{R^{d-1}}{16\pi G_N^{(d+1)}} \Omega_{d-2} m l^d \tag{1.75}$$

Comparing this with the energy contained in the subregion A $\Delta E = \int_A d^{d-1}x \langle T_{tt} \rangle$ we get

$$\frac{\Delta S}{\Delta E} = \frac{2\pi}{d+1} l \tag{1.76}$$

A similar result is also obtained for the case of strip subregion. Thus when the subregion satisfies the bound mentioned above the change in entanglement entropy ΔS is directly proportional to the change in energy ΔE. The constant of proportionality only depends on the size of the subregion. This statement can be regarded as the first law of entanglement thermodynamics

$$\Delta S = \frac{1}{T_{ent}} \Delta E \tag{1.77}$$

Where T_{ent} can be considered as a subsystem dependent entanglement temperature for the sphere we have

$$T_{ent} \propto \frac{1}{l} : \qquad \text{For sphere} \Rightarrow \quad T_{ent} = \frac{d+1}{2\pi l} \tag{1.78}$$

1.5 Plan of the Thesis

In the present chapter we have covered some universal feature of the AdS/CFT correspondence and holographic entanglement entropy. We have covered very few selected topics in this subject which will be relevant to rest of the thesis. We have discussed two approaches to compute entanglement entropy holographicaly. The Ryu Takayanagi proposal works well in static asymptotically AdS backgrounds, while the covariant proposal is needed for stationary and time dependent backgrounds. However, both the approach requires solving for the extremal surface. As it is very difficult to obtain an exact closed form solution of the minimal surface equation, one needs to adopt a perturbative approach to obtain the solution. Another approach is to start from the area functional in pure AdS and study variations which incorporates both changes in the embedding and the background metric. Using this approach one can check that at first order both the proposal gives the same result. This is due to the fact that at first order the changes in the embedding don't contribute to the entropy. Thus, it is important to study the second order change in entanglement entropy for different backgrounds (both static and non static). This will help us to study the behavior of the minimal surface and hence the entropy, once one starts deviating from pure AdS result. The rest of the thesis is devoted to comparing these two approaches at second order.

In chapter we will use the perturbative approach to calculate the holographic entanglement entropy for the boosted black brane geometry and we also write down the entanglement first law for this background. In chapter we use the entanglement first law to quantify the asymmetry in the entanglement entropy due to the anisotropy in the boosted blackbrane background. In chapter (4) we propose a variational approach as mentioned above in $2+1$ dimensions to calculate holographic entanglement entropy upto second order. In chapter (5) we generalize the variational approach to higher dimensions. Six appendices contain necessary material to reproduce the main formulae and results presented in the main text.

CHAPTER 2

ENTANGLEMENT THERMODYNAMICS FOR ADS SPACETIME: PERTURBATIVE APPROACH

2.1 Introduction

In this chapter we adopt the perturbative approach to study the effects of IR deformations (excitations) on the change in holographic entanglement entropy for AdS spacetimes. We will start with asymptotically AdS spacetimes which carry gauge charges. We also look for modifications in the entanglement first law. In this regard we choose the boosted AdS black brane as our bulk background where the boost direction is compactified on a circle. These compactified backgrounds give rise to Kaluza Klein gauge charges. We are interested in studying dependence of the entropy on the boost and to observe its effect on the first law of entanglement. In the perturbative approach we find that first order change in entropy depend on the boost parameter. However at this order the overall form of the first law remains unchanged. At higher orders we do find that the ‘boosted’ AdS black holes give rise to a more general form of first law which includes the chemical potential and charge density. To obtain this result we have to resort to a second order perturbative calculation of the entanglement entropy. In order to express the change of Entanglement entropy at second order as a first law statement, we find that various first order thermodynamic quantities, such as entropy, energy, temperature, etc have to be suitably redefined at the second order. The effects of higher order corrections appears similar to the renormalization procedure in quantum field theories. For example the strip width (subsystem size) and entanglement temperature (T_E) have to be redefined to include corrections so that a first law like relation holds good. Since we resort to perturbative calculations, we work in the regime where the ratio $\frac{l}{z_0}$, of the strip width (l) to the horizon radius (z_0), is kept very small. This hierarchy of scales can also be thought of in terms of respective temperatures as a limit

$$T_{thermal} \ll T_E$$

We mention that the corrections to the entanglement entropy evaluated order by order in (dimensionless) quantity $\frac{T_{thermal}}{T_E}$ should not be confused with (stringy) quantum corrections to the entanglement entropy [83].

2.2 Entanglement from boosted AdS black holes

The boosted AdS_{d+1} black holes backgrounds are given by

$$ds^2 = \frac{L^2}{z^2}\left(-\frac{f dt^2}{K} + K(dy - \omega)^2 + dx_1^2 + \cdots + dx_{d-2}^2 + \frac{dz^2}{f}\right) \tag{2.1}$$

with

$$f = 1 - \frac{z^d}{z_0^d}, \qquad K = 1 + \beta^2\gamma^2\frac{z^d}{z_0^d} \tag{2.2}$$

z_0 is the horizon and $0 \leq \beta \leq 1$ is boost parameter, while $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. The boost is taken along y direction, which is compact. The Kaluza-Klein 1-form ω is given by

$$\omega = \beta^{-1}(1 - \frac{1}{K})dt \tag{2.3}$$

and L is the radius of curvature of AdS spacetime, which is very large. For example, in the $AdS_5 \times S^5$ near-horizon geometry of n coincident D3-branes, we shall have $L^4 \equiv 2\pi g_{YM}^2 n$ and the 't Hooft coupling constant $g_{YM}^2 n \gg 1$.

We study the entanglement entropy of a strip subsystem on the boundary of boosted AdS_{d+1} backgrounds in (2.1). We embed a $(d-1)$-dimensional strip-like spatial surface, in the bulk asymptotic geometry. The boundaries of the extremal bulk surface coincide with the two ends of the interval $-l/2 \leq x^1 \leq l/2$. The regulated size of the rest of the coordinates($0 \leq x^i \leq l_i$) is taken very large so that $l_i \gg l$. We shall always have coordinate y being compact, so that $0 \leq y \leq 2\pi r_y$. As per the Ryu-Takayanagi prescription [6, 7] the entanglement entropy is given in terms of the geometrical area of the extremal surface

$$\begin{aligned} S_E &\equiv \frac{[A]_{Strip}}{4G_{d+1}} \\ &= \frac{V_{d-2}L^{d-1}}{2G_{d+1}} \int_\epsilon^{z_*} \frac{dz}{z^{d-1}}\sqrt{K}\sqrt{\frac{1}{f} + (\partial_z x^1)^2} \end{aligned} \tag{2.4}$$

where G_{d+1} is $(d+1)$-dimensional Newton's constant and $V_{d-2} \equiv (2\pi r_y)l_2 l_3 \cdots l_{d-2}$ is the spatial volume of the boundary. We will be mainly working for $d > 2$. In our notation $\epsilon \sim 0$ denotes the cut-off scale near the boundary to regularize the UV divergences, and z_* is the turning point of extremal surface inside the bulk geometry. In the above $K(z)$, $f(z)$

are known functions and we only need to solve for x^1. From (2.4) it follows that a minimal surface will have to satisfy

$$\frac{dx^1}{dz} \equiv (\frac{z}{z_c})^{d-1} \frac{1}{\sqrt{f}\sqrt{K - (\frac{z}{z_c})^{2d-2}}} \tag{2.5}$$

The constant z_c is given by the turning point relation

$$K_* - (\frac{z_*}{z_c})^{2d-2} = 0 \tag{2.6}$$

where $K_* = K(z)|_{z=z_*}$. The identification of the boundary $x^1(0) = l/2$ leads to the integral relation

$$\frac{l}{2} = \int_0^{z_*} dz (\frac{z}{z_*})^{d-1} \frac{1}{\sqrt{f}\sqrt{\frac{K}{K_*} - (\frac{z}{z_*})^{2d-2}}} \tag{2.7}$$

which relates l with z_*, the turning point. The turning-point takes the mid-point value $x^1(z_*) = 0$ on the boundary. From (2.4) and (2.5) the expression of the entanglement entropy for these boosted AdS black hole solutions becomes

$$S_E = \frac{V_{d-2}L^{d-1}}{2G_{d+1}} \int_\epsilon^{z_*} \frac{dz}{z^{d-1}} \frac{K}{\sqrt{f}\sqrt{K - K_*(\frac{z}{z_*})^{2d-2}}} \tag{2.8}$$

The expression (2.8) mathematically provides the entanglement entropy for a strip-like subsystem on the boundary. For pure AdS spacetime ($z_0 \to \infty,\ f = 1 = K$) these integrals can be evaluated exactly [6, 7], but in the presence of black hole it is difficult to find analytical answers from the integral (2.8), although numerical estimates can always be made. In order to find analytical results we adopt the perturbative method in the next subsection.

2.2.1 Thin strip approximation

In the cases where the strip subsystem is a small part of a big system, so that the turning point lies in the proximity of asymptotic boundary region only ($z_* \ll z_0$), one can evaluate the entanglement entropy integral (2.8) by expanding it around its pure AdS value (treating pure AdS as a ground state). We shall take boost to be finite but small $\beta\gamma \sim 1$. Under these approximations we can expand the following expression upto first order

$$
\begin{aligned}
&\frac{K}{K_*} = \left(1+\beta^2\gamma^2\frac{z^d}{{z_0}^d}\right)\left(1+\beta^2\gamma^2\frac{{z_*}^d}{{z_0}^d}\right)^{-1} \\
&\simeq \left(1+\beta^2\gamma^2\frac{z^d}{{z_0}^d}\right)\left(1-\beta^2\gamma^2\frac{{z_*}^d}{{z_0}^d}\right) \\
&\simeq 1+\beta^2\gamma^2\left(\frac{z^d}{{z_*}^d}-1\right)\frac{{z_*}^d}{{z_0}^d}
\end{aligned}
\tag{2.9}
$$

Using this the strip width equation (2.7) can be expanded perturbatively upto first order as

$$
\begin{aligned}
l &= 2\int_0^{z_*} dz(\frac{z}{z_*})^{d-1}\frac{1}{\sqrt{f}\sqrt{\frac{K}{K_*}-(\frac{z}{z_*})^{2d-2}}} \\
&= 2\int_0^{z_*} dz(\frac{z}{z_*})^{d-1}\frac{1}{\sqrt{f}\sqrt{1-\beta^2\gamma^2\left(1-\frac{z^d}{{z_*}^d}\right)\frac{{z_*}^d}{{z_0}^d}-(\frac{z}{z_*})^{2d-2}}} \\
&= 2\int_0^{z_*} dz(\frac{z}{z_*})^{d-1}\frac{1}{\sqrt{f}\sqrt{R}\sqrt{1-\frac{\beta^2\gamma^2}{R}\left(1-\frac{z^d}{{z_*}^d}\right)\frac{{z_*}^d}{{z_0}^d}}}
\end{aligned}
$$

Denoting $R \equiv 1-\left(\dfrac{z}{z_*}\right)^{2d-2}$, we get

$$
l = 2\int_0^{z_*} dz(\frac{z}{z_*})^{d-1}\frac{1}{\sqrt{R}}\left[1-\left(\frac{z}{z_*}\right)^d\left(\frac{z_*}{z_0}\right)^d\right]^{-\frac{1}{2}}\left[1-\frac{\beta^2\gamma^2}{R}\left(1-\frac{z^d}{{z_*}^d}\right)\frac{{z_*}^d}{{z_0}^d}\right]^{-\frac{1}{2}}
\tag{2.10}
$$

$$\simeq 2\int_0^{z_*} dz(\frac{z}{z_*})^{d-1}\frac{1}{\sqrt{R}}\left[1+\frac{1}{2}\left(\frac{z}{z_*}\right)^d\left(\frac{z_*}{z_0}\right)^d+\frac{\beta^2\gamma^2}{2R}\left(1-\frac{z^d}{{z_*}^d}\right)\frac{{z_*}^d}{{z_0}^d}\right]$$

Now we will expand the above expression and retain only first order term. Denoting $\xi=\frac{z}{z_*}$

$$l = 2z_*\int_0^1 d\xi\xi^{d-1}\frac{1}{\sqrt{R}}\left[1+\frac{1}{2}\frac{z_*^d}{z_0^d}\xi^d+\frac{\beta^2\gamma^2 z_*^d}{2z_0^d}\frac{1-\xi^d}{R}+\cdots\right]$$
$$\equiv 2z_*\left(b_0+\frac{z_*^d}{2z_0^d}(b_1+\beta^2\gamma^2 I_l)\right)+\cdots \tag{2.11}$$

where the dots indicate terms of higher order in $(\frac{z_*}{z_0})^d$. The coefficients b_0, b_1, and I_l are precise integral Beta functions multiplying at various orders. These coefficients are provided in the appendix(B). Note b_0 and b_1 are positive definite quantities. Keeping only up to first order in (z_*^d/z_0^d) the equation (2.10) can be inverted to obtain

$$z_* = \frac{l/2}{b_0+\frac{z_*^d}{2z_0^d}(b_1+\beta^2\gamma^2 I_l)} \simeq \frac{\bar{z}_*}{1+\frac{\bar{z}_*^d}{2z_0^d}(\frac{b_1}{b_0}+\frac{\beta^2\gamma^2}{b_0}I_l)} \tag{2.12}$$

where $\bar{z}_*\equiv\frac{l}{2b_0}$ being the turning point value in pure AdS case(no excitations) having the same strip width l. The last equation summarizes geometrically the whole effect of IR bulk deformations (excitations), like having ‘black hole in geometry’ and boosts on the turning point value perturbatively. Having obtained the turning point expansion, a similar expansion around pure AdS can be made for the area functional also. After regularizing the area integral (2.8), in the UV limit ($\epsilon\to 0$), we find the following expansion

$$A \equiv 2\int_0^{z_*}\frac{dz}{z^{d-1}}\frac{K}{\sqrt{f}\sqrt{K-K_*(\frac{z}{z_*})^{2d-2}}}+A_{UV}$$
$$= 2\int_0^{z_*}\frac{dz}{z^{d-1}}\frac{\sqrt{K}}{\sqrt{f}\sqrt{1-\frac{K_*}{K}(\frac{z}{z_*})^{2d-2}}}+A_{UV} \tag{2.13}$$

Now

$$\frac{K_*}{K} = \left(1+\beta^2\gamma^2\frac{{z_*}^d}{{z_0}^d}\right)\left(1+\beta^2\gamma^2\frac{z^d}{{z_*}^d}\frac{{z_*}^d}{{z_0}^d}\right)^{-1}$$
$$\simeq \left[1+\beta^2\gamma^2\left(1-\frac{z^d}{{z_*}^d}\right)\frac{{z_*}^d}{{z_0}^d}\right]$$

Substituting this back into the area integral (2.13) gives

$$\begin{aligned}
A &= 2\int_0^{z_*}\frac{dz}{z^{d-1}}\frac{\sqrt{K}}{\sqrt{f}}\frac{1}{\sqrt{1-\left[1+\beta^2\gamma^2\left(1-\frac{z^d}{{z_*}^d}\right)\frac{{z_*}^d}{{z_0}^d}\right](\frac{z}{z_*})^{2d-2}}}+A_{UV} \\
&= 2\int_0^{z_*}\frac{dz}{z^{d-1}}\frac{\sqrt{K}}{\sqrt{f}\sqrt{R}}\frac{1}{\sqrt{1-\beta^2\gamma^2\frac{\left({z_*}^d-z^d\right)}{{z_0}^d}\frac{(1-R)}{R}}}+A_{UV} \\
&\simeq 2\int_0^{z_*}\frac{dz}{z^{d-1}}\frac{1}{\sqrt{R}}\left(1+\frac{z^d}{2{z_0}^d}\right)\left(1+\frac{\beta^2\gamma^2z^d}{2{z_0}^d}\right)\left(1+\beta^2\gamma^2\frac{\left({z_*}^d-z^d\right)}{2{z_0}^d}\left(\frac{1}{R}-1\right)\right)+A_{UV} \\
&\simeq 2\int_0^{z_*}\frac{dz}{z^{d-1}}\frac{1}{\sqrt{R}}\left[1+\frac{(\beta^2\gamma^2+1)z^d}{2{z_0}^d}+\beta^2\gamma^2\frac{\left({z_*}^d-z^d\right)}{2{z_0}^d}\left(\frac{1}{R}-1\right)\right]+A_{UV}
\end{aligned} \tag{2.14}$$

where we have denoted diverging UV part as $A_{UV}=\frac{2}{d-2}\frac{1}{\epsilon^{d-2}}$. The respective finite integrals can be evaluated at each order on the right hand side to give

$$\begin{aligned}
A &= \frac{2}{z_*^{d-2}}[a_0+\frac{z_*^d}{2z_0^d}(\gamma^2a_1+\beta^2\gamma^2I_l)+\cdots]+A_{UV} \\
&= \frac{2a_0}{z_*^{d-2}}[1+\frac{z_*^d}{2z_0^d}(\gamma^2\frac{a_1}{a_0}+\frac{\beta^2\gamma^2}{a_0}I_l)+\cdots]+A_{UV}
\end{aligned} \tag{2.15}$$

where new coefficients $a_0, a_1, ...$ are specific Beta-function integrals given in the appendix(B). We should note that $a_1 > 0$, but using Beta function identities we shall have $a_0 = -\frac{b_0}{d-2}$, so a_0 will be negative for all $d > 2$. Now substituting for z_* from (2.12) and only keeping terms up to first order we find that

$$
\begin{aligned}
A \qquad &= A_{UV} + \frac{2a_0}{\bar{z}_*^{d-2}}\left(1+\frac{\bar{z}_*^d}{z_0^d}\frac{d-2}{2}(\frac{b_1}{b_0}+\beta^2\gamma^2\frac{I_l}{b_0})+\frac{\bar{z}_*^d}{2z_0^d}(\gamma^2\frac{a_1}{a_0}+\beta^2\gamma^2\frac{I_l}{a_0})\right)\\
&= A_{UV} + A_0\left(1+\frac{\bar{z}_*^d}{z_0^d}\frac{d-2}{2}\frac{b_1}{b_0}+\frac{\bar{z}_*^d}{2z_0^d}\gamma^2\frac{a_1}{a_0}\right)\\
&\equiv A_{UV} + A_0 + A_1 \qquad (2.16)
\end{aligned}
$$

where in the second last line the terms involving I_l have got exactly cancelled! We have also defined

$$
A_0 = \frac{2a_0}{\bar{z}_*^{d-2}} = -\frac{(2b_0)^{d-1}}{(d-2)l^{d-2}} \qquad (2.17)
$$

as the area contribution for pure AdS_{d+1} with turning point $\bar{z}_*$ and strip width l. Thus the term A_1 contains all the first order contributions to the area. As a check, for pure AdS ($A_1 = 0$) we get the standard result [6, 7]

$$
A_{AdS} = \frac{1}{d-2}\left(\frac{2}{\epsilon^{d-2}} - \frac{2^{d-1}b_0^{d-1}}{l^{d-2}}\right). \qquad (2.18)
$$

which is a positive definite quantity. From equation (2.16) we can now find the net change in the area of extremal surface due to IR deformations (black hole with boost). It is given by

$$
\begin{aligned}
\triangle A \equiv A - A_{AdS} \quad &= \quad \frac{a_0\bar{z}_*^2}{z_0^d}\left((d-2)\frac{b_1}{b_0}+\gamma^2\frac{a_1}{a_0}\right)\\
&= \frac{a_1 l^2}{4b_0^2}\left(\frac{d-1}{d+1}+\beta^2\gamma^2\right)\frac{1}{z_0^d} \qquad (2.19)
\end{aligned}
$$

where in the second line we have used the relation between two ratios $\frac{b_1}{b_0} = -\frac{2}{(d+1)(d-2)}\frac{a_1}{a_0}$.

Below we enumerate our final observations

- It is remarkable to note that the remainder of the expression on the right hand side of eq.(2.19) is positive definite.

- This suggests that the net area of the extremal strip has effectively increased as compared to the pure AdS.

- The presence of β dependent terms precisely contain the effect of boost on the area of the extremal surface.

- In the absence of boost these terms will be absent and we shall get the result first obtained by [23].

- This suggests that the boosting of the bulk metric (which forms a type of charged excitations in the CFT$_d$) increases the strip area and hence increases the entanglement entropy for the CFT subsystem.

Following from (2.19) the change in entanglement entropy above the pure AdS ground state, up to first order is given by

$$\triangle S = \frac{L^{d-1}V_{d-2}}{16G_{d+1}}\frac{a_1 l^2}{b_0^2}\left(\frac{d-1}{d+1}+\beta^2\gamma^2\right)\frac{1}{z_0^d}\,. \tag{2.20}$$

The equation (2.20) is an important expression for the remaining part of the analysis in this section.

2.2.2 Entanglement First Law

It is left now to carefully partition the right hand side of (2.20) in terms of physical thermodynamic observables of the CFT. The physical quantities such as energy, charge and pressure can be obtained by expanding the bulk geometry (2.1) in suitable Fefferman-Graham (asymptotic) coordinates near the AdS boundary [24], given in the appendix(A). These for the subsystem of CFT$_d$ (on a circle) are summarized here. The energy and charge for the strip subsystem are

$$\begin{aligned} \triangle\mathcal{E} &= \frac{dL^{d-1}V_{d-2}l}{16\pi G_{d+1}} < t_{00} >= \frac{r_y L^{d-1}V_{d-3}l}{8G_{d+1}}(\frac{d-1}{d} + \beta^2\gamma^2)\frac{d}{z_0^d} \\ \triangle\mathcal{N} &\equiv r_y P_y = \frac{r_y L^{d-1}V_{d-2}l}{16\pi G_{d+1}}\frac{\beta\gamma^2 d}{z_0^d} \end{aligned} \tag{2.21}$$

respectively. The pressure component along the x_1 direction of the compactified CFT is

$$\triangle\mathcal{P} = \frac{2\pi r_y L^{d-1}d}{16\pi G_{d+1}} < t_{11} >= \frac{L^{d-1}r_y}{8G_{d+1}}\frac{1}{z_0^d} \tag{2.22}$$

while $V_{d-3} \equiv l_2 l_3 \cdots l_{d-2}$, and d-dimensional Newton's constant $\frac{1}{G_d} = \frac{2\pi r_y L}{G_{d+1}}$. The $\mathcal{N}$ represents integral value of (momentum) charge. In the absence of boost it would be vanishing. We note down nontrivial chemical potential in our solutions. It is given by the value of gauge potential ω at the turning point,

$$\mu = \frac{1}{r_y\beta}(1 - \frac{1}{K(z_*)}) \simeq \frac{\beta\gamma^2}{r_y}\frac{\bar{z}_*^d}{z_0^d} \tag{2.23}$$

Hence the contribution of 'entanglement chemical potential' would remain negligible in first order of approximation we are working in this section. (Note, the corresponding thermal value of chemical potential is however large $\mu_{thermal} = \frac{\beta}{r_y}$.)

Our aim is to express the right hand side of (2.20) in terms of above physical observables. From (2.21), a little guess tells us that

$$\left(\frac{d-1}{d+1} + \beta^2\gamma^2\right)\frac{1}{z_0^d} \equiv \left[\left(\frac{d-1}{d} + \beta^2\gamma^2\right) - \frac{d-1}{d+1}\frac{1}{d}\right]\frac{1}{z_0^d} \tag{2.24}$$

Using (2.21) and (2.22) we can now express eq.(2.20) as

$$\triangle S_E = \frac{1}{T_E}\left(\triangle\mathcal{E} - \frac{d-1}{d+1}\,\mathcal{V}\triangle\mathcal{P}\right) \tag{2.25}$$

where $\mathcal{V} \equiv l_2 l_3 \cdots l_{d-2}$ is the net volume of the strip subsystem. The equation (2.25) simply describes the first law of entanglement thermodynamics, which is identical to the result in [26]. An alternative first law form was first proposed by [23] for the isotropic AdS case. It leads to a difference in entanglement temperatures. If we set $\beta = 0$ in (2.25), it reduces to the known first law form obtained in [26]. Hence we can conclude that the form of the first law remains true for 'boosted' AdS black-hole case as well, even though the excitations in CFT are much different in the boosted case. For example, there are quantized charges present in these backgrounds. The entanglement temperature is given as

$$T_E = \frac{b_0^2}{a_1}\frac{d}{\pi l} = \frac{(B(\frac{d}{2d-2},\frac{1}{2}))^2}{2(d-1)B(\frac{1}{d-1},\frac{1}{2})}\frac{d}{\pi l}. \tag{2.26}$$

The temperature is inversely proportional to the width of strip. But this temperature is lower by a factor $\frac{d}{d+1}$ as compared to the isotropic unboosted case in [23]. It is evident that there is no explicit charge dependence in the first law equation (2.25). The reason for this is that the entanglement chemical potential given in (2.23) remains negligible ($\sim O(z_*^d/z_0^d)$) at the first order. The contribution of chemical potential will however become important in next higher order calculation which we perform in the following section. This contribution is expected to change the 'first order' form of the first law (2.25).

2.3 Entanglement entropy at second order

Taking similar steps as in the previous section, we now calculate the second order terms in the expansion of the area integral schematically denoted as

$$A \equiv A_{UV} + A_0 + A_1 + A_2 \tag{2.27}$$

where A_0 and all first order terms contributing to A_1 have been obtained in the previous section. Our aim is to find A_2. Now first we need to recalculate the relation between the turning point and strip width as in (2.10)and (2.12), up to second order. To begin with, we expand the ratio $\frac{K}{K_*}$ upto second order in $(\frac{z_*}{z_0})^d$.

$$\begin{aligned}
\frac{K}{K_*} &= \left(1 + \beta^2\gamma^2 \frac{z^d}{{z_0}^d}\right)\left(1 + \beta^2\gamma^2 \frac{{z_*}^d}{{z_0}^d}\right)^{-1} \\
&\simeq \left(1 + \beta^2\gamma^2 \frac{z^d}{{z_0}^d}\right)\left(1 - \beta^2\gamma^2 \frac{{z_*}^d}{{z_0}^d} + \beta^4\gamma^4 \frac{{z_*}^{2d}}{{z_0}^{2d}}\right) \\
&\simeq 1 - \beta^2\gamma^2 \left(1 - \frac{z^d}{{z_*}^d}\right)\frac{{z_*}^d}{{z_0}^d} + \beta^4\gamma^4 \left(1 - \frac{z^d}{{z_*}^d}\right)\frac{{z_*}^{2d}}{{z_0}^{2d}}
\end{aligned}$$

Using this the strip width equation (2.7) can be expanded perturbatively upto second order as

$$l = 2\int_0^{z_*} dz \left(\frac{z}{z_*}\right)^{d-1} \frac{1}{\sqrt{f}\sqrt{\frac{K}{K_*} - (\frac{z}{z_*})^{2d-2}}} \tag{2.28}$$

$$= 2\int_0^{z_*} dz(\frac{z}{z_*})^{d-1}\frac{1}{\sqrt{f}\sqrt{1-\beta^2\gamma^2\left(1-\frac{z^d}{{z_*}^d}\right)\frac{{z_*}^d}{{z_0}^d}+\beta^4\gamma^4\left(1-\frac{z^d}{{z_*}^d}\right)\frac{{z_*}^{2d}}{{z_0}^{2d}}-(\frac{z}{z_*})^{2d-2}}}$$

$$= 2z_*\int_0^1 d\xi\, \xi^{d-1}\frac{1}{\sqrt{f}\sqrt{R}\sqrt{1-\frac{\left(\beta^2\gamma^2\left(1-\xi^d\right)\frac{{z_*}^d}{{z_0}^d}-\beta^4\gamma^4\left(1-\xi^d\right)\frac{{z_*}^{2d}}{{z_0}^{2d}}\right)}{R}}}$$

Expanding and retaining upto second order terms and denoting $\xi = \dfrac{z}{z_*}$ and $R \equiv 1 - t^{2d-2}$

$$\simeq 2z_*\int_0^1 \frac{d\xi}{\sqrt{R}}\xi^{d-1}\frac{1}{\sqrt{f}}\left[1+\frac{1}{2}\beta^2\gamma^2\frac{\left(1-\xi^d\right)}{R}\frac{{z_*}^d}{{z_0}^d}+\beta^4\gamma^4\left(\frac{3}{8}\frac{\left(1-\xi^d\right)^2}{R^2}-\frac{\left(1-\xi^d\right)}{R}\right)\frac{{z_*}^{2d}}{{z_0}^{2d}}\right]$$

$$\simeq 2z_*\int_0^1 \frac{d\xi}{\sqrt{R}}\xi^{d-1}\left(1+\frac{\xi^d}{2}\frac{{z_*}^d}{{z_0}^d}+\frac{3\xi^{2d}}{8}\frac{{z_*}^{2d}}{{z_0}^{2d}}\right)\left[1+\frac{1}{2}\beta^2\gamma^2\frac{\left(1-\xi^d\right)}{R}\frac{{z_*}^d}{{z_0}^d}+\beta^4\gamma^4\left(\frac{3}{8}\frac{\left(1-\xi^d\right)^2}{R^2}\right.\right.$$
$$\left.\left.-\frac{\left(1-\xi^d\right)}{R}\right)\frac{{z_*}^{2d}}{{z_0}^{2d}}\right]$$

$$\simeq 2z_*\left[\int_0^1 \frac{d\xi}{\sqrt{R}}\xi^{d-1}+\frac{1}{2}\frac{{z_*}^d}{{z_0}^d}\int_0^1\frac{d\xi}{\sqrt{R}}\xi^{d-1}\left(\beta^2\gamma^2\frac{\left(1-\xi^d\right)}{R}+\xi^d\right)+\frac{{z_*}^{2d}}{{z_0}^{2d}}\int_0^1\frac{d\xi}{\sqrt{R}}\xi^{d-1}\right.$$
$$\left.\left(\frac{3}{8}\xi^{2d}+\frac{\beta^2\gamma^2}{4}\frac{\xi^d\left(1-\xi^d\right)}{R}+\beta^4\gamma^4\left(\frac{3}{8}\frac{\left(1-\xi^d\right)^2}{R^2}-\frac{1}{2}\frac{\left(1-\xi^d\right)}{R}\right)\right)+\cdots\right]$$

$$= 2z_*b_0\left(1+\frac{{z_*}^d}{{z_0}^d}\frac{\left(b_1+\beta^2\gamma^2 I_l\right)}{2b_0}+\frac{{z_*}^{2d}}{{z_0}^{2d}}\left(\frac{\frac{3}{8}b_2+J_l}{b_0}\right)\right) \tag{2.29}$$

Where the coefficients $b_2,\ I_l,\ J_l$ are given in the appendix(B). Now solving recursively for z_* up to second order, we get

$$z_* = \frac{\frac{l}{2b_0}}{\left(1+\frac{{z_*}^d}{{z_0}^d}\frac{(b_1+\beta^2\gamma^2 I_l)}{2b_0}+\frac{{z_*}^{2d}}{{z_0}^{2d}}\left(\frac{\bar{b_2}+J_l}{b_0}\right)\right)}$$

$$\simeq \frac{\bar{z}_*}{\left(1+\frac{{\bar{z}_*}^d}{{z_0}^d}\frac{(b_1+\beta^2\gamma^2 I_l)}{2b_0}\left(1+\frac{{\bar{z}_*}^d}{{z_0}^d}\frac{(b_1+\beta^2\gamma^2 I_l)}{2b_0}+\frac{{\bar{z}_*}^{2d}}{{z_0}^{2d}}\left(\frac{\bar{b_2}+J_l}{b_0}\right)\right)^{-d}+\frac{{\bar{z}_*}^{2d}}{{z_0}^{2d}}\left(\frac{\bar{b_2}+J_l}{b_0}\right)\right)}$$

$$\simeq \frac{\bar{z}_*}{\left(1+\frac{{\bar{z}_*}^d}{{z_0}^d}\frac{(b_1+\beta^2\gamma^2 I_l)}{2b_0}\left(1-d\frac{{\bar{z}_*}^d}{{z_0}^d}\frac{(b_1+\beta^2\gamma^2 I_l)}{2b_0}\right)+\frac{{\bar{z}_*}^{2d}}{{z_0}^{2d}}\left(\frac{\bar{b_2}+J_l}{b_0}\right)\right)}$$

$$= \bar{z}_* \left(1 + \frac{\bar{z}_*^d}{z_0^d}(\frac{b_1 + \beta^2\gamma^2 I_l}{2b_0}) + \frac{\bar{z}_*^{2d}}{z_0^{2d}}\left(\frac{\bar{b}_2 + J_l}{b_0} - d(\frac{b_1 + \beta^2\gamma^2 I_l}{2b_0})^2\right)\right)^{-1} \quad (2.30)$$

There is no need to simplify this expression any further at this step. Now our final task is to obtain the area expansion up to second order. As before, after regularizing the area integral (2.8), in the UV limit ($\epsilon \to 0$), we find the following expansion

$$A = 2\int_0^{z_*} \frac{dz}{z^{d-1}} \frac{\sqrt{K}}{\sqrt{f}\sqrt{1 - \frac{K_*}{K}(\frac{z}{z_*})^{2d-2}}} + A_{UV}$$

$$\begin{aligned} \text{Now} \qquad \frac{K_*}{K} &= \left(1 + \beta^2\gamma^2 \frac{{z_*}^d}{{z_0}^d}\right)\left(1 + \beta^2\gamma^2 \frac{z^d}{{z_*}^d}\frac{{z_*}^d}{{z_0}^d}\right)^{-1} \\ &\simeq \left(1 + \beta^2\gamma^2 \frac{{z_*}^d}{{z_0}^d}\right)\left(1 - \beta^2\gamma^2 \frac{z^d}{{z_*}^d}\frac{{z_*}^d}{{z_0}^d} + \beta^4\gamma^4 \frac{z^{2d}}{{z_*}^{2d}}\frac{{z_*}^{2d}}{{z_0}^{2d}}\right) \\ &\simeq \left[1 + \beta^2\gamma^2\left(1 - \frac{z^d}{{z_*}^d}\right)\frac{{z_*}^d}{{z_0}^d} - \beta^4\gamma^4 \frac{z^d}{{z_*}^d}\left(1 - \frac{z^d}{{z_*}^d}\right)\frac{{z_*}^{2d}}{{z_0}^{2d}}\right] \end{aligned} \quad (2.31)$$

Substituting this back into the area integral gives

$$A = 2\int_0^{z_*} \frac{dz}{z^{d-1}} \frac{\sqrt{K}}{\sqrt{f}} \frac{1}{\sqrt{1 - \left[1 + \beta^2\gamma^2\left(1 - \frac{z^d}{{z_*}^d}\right)\frac{{z_*}^d}{{z_0}^d} - \beta^4\gamma^4\frac{z^d}{{z_*}^d}\left(1 - \frac{z^d}{{z_*}^d}\right)\frac{{z_*}^{2d}}{{z_0}^{2d}}\right](\frac{z}{z_*})^{2d-2}}}$$
$$+A_{UV}$$

Denoting $\xi = \frac{z}{z_*}$ and $R \equiv 1 - \xi^{2d-2}$

$$\Delta A = \frac{2}{{z_*}^{d-2}}\int_0^1 \frac{d\xi}{\xi^{d-1}} \frac{\sqrt{K}}{\sqrt{f}\sqrt{R}} \frac{1}{\sqrt{1 - \frac{\left(\beta^2\gamma^2\left(1-\xi^d\right)\frac{{z_*}^d}{{z_0}^d} - \beta^4\gamma^4\xi^d\left(1-\xi^d\right)\frac{{z_*}^{2d}}{{z_0}^{2d}}\right)\xi^{2(d-1)}}{R}}}$$

$$
\simeq \frac{2}{{z_*}^{d-2}}\int_0^1 \frac{d\xi}{\xi^{d-1}}\frac{\sqrt{K}}{\sqrt{f}\sqrt{R}}\Bigg(1+\frac{\left(\beta^2\gamma^2\left(1-\xi^d\right)\frac{{z_*}^d}{{z_0}^d}-\beta^4\gamma^4\xi^d\left(1-\xi^d\right)\frac{{z_*}^{2d}}{{z_0}^{2d}}\right)\xi^{2(d-1)}}{2R}
$$

$$
+\frac{3}{8}\beta^4\gamma^4\frac{(1-\xi^d)^2\xi^{4(d-1)}}{R^2}\frac{{z_*}^{2d}}{{z_0}^{2d}}\Bigg)
$$

$$
=\frac{2}{{z_*}^{d-2}}\int_0^1 \frac{d\xi}{\xi^{d-1}}\frac{\sqrt{K}}{\sqrt{f}\sqrt{R}}\Bigg[1+\frac{{z_*}^d}{{z_0}^d}\beta^2\gamma^2\frac{\left(1-\xi^d\right)\xi^{2(d-1)}}{2R}+\beta^4\gamma^4\frac{{z_*}^{2d}}{{z_0}^{2d}}\Big(\frac{3}{8}\frac{(1-\xi^d)^2\xi^{4(d-1)}}{R^2}
$$

$$
-\frac{\left(\xi^d\left(1-\xi^d\right)\right)\xi^{2(d-1)}}{2R}\Big)\Bigg]
$$

$$
\simeq \frac{2}{{z_*}^{d-2}}\int_0^1 \frac{d\xi}{\xi^{d-1}}\frac{\sqrt{K}}{\sqrt{f}\sqrt{R}}\Big[1+\frac{\beta^2\gamma^2{z_*}^d}{{z_0}^d}\frac{\left(1-\xi^d\right)\xi^{2(d-1)}}{2R}+\frac{\beta^4\gamma^4{z_*}^{2d}}{{z_0}^{2d}}
$$

$$
\left(\frac{3}{8}\frac{(1-\xi^d)\xi^{2(d-1)}}{R^2}-\frac{\xi^d}{2}\right)\frac{\left(1-\xi^d\right)\xi^{2(d-1)}}{R}\Big]
$$

$$
\simeq \frac{2}{{z_*}^{d-2}}\int_0^1 \frac{d\xi}{\xi^{d-1}}\frac{1}{\sqrt{f}\sqrt{R}}\Big[1+\frac{\beta^2\gamma^2{z_*}^d}{{z_0}^d}\left(\frac{\xi^d}{2}+\frac{\left(1-\xi^d\right)\xi^{2(d-1)}}{2R}\right)+\frac{\beta^4\gamma^4{z_*}^{2d}}{{z_0}^{2d}}
$$

$$
\Big(\frac{3}{8}\frac{(1-\xi^d)^2t^{4(d-1)}}{R^2}-\frac{\xi^d\left(1-\xi^d\right)\xi^{2(d-1)}}{4R}-\frac{\xi^{2d}}{8}\Big)\Big]
$$

$$
\simeq \frac{2}{{z_*}^{d-2}}\Big[\int_0^1 \frac{d\xi}{\xi^{d-1}}\frac{1}{\sqrt{R}}+\frac{{z_*}^d}{2{z_0}^d}\left((1+\beta^2\gamma^2)\int_0^1\frac{d\xi}{\xi^{d-1}}\frac{1}{\sqrt{R}}\xi^d+\beta^2\gamma^2\int_0^1 d\xi\frac{\left(1-\xi^d\right)\xi^{(d-1)}}{R^{\frac{3}{2}}}\right)
$$

$$
+\frac{{z_*}^{2d}}{{z_0}^{2d}}\Bigg(\frac{(3+2\beta^2\gamma^2-\beta^4\gamma^4)}{8}\int_0^1\frac{d\xi}{\xi^{d-1}}\frac{1}{\sqrt{R}}\xi^{2d}+\int_0^1 d\xi\frac{\xi^{d-1}}{\sqrt{R}}\Big(\beta^2\gamma^2\xi^d\frac{(1-\xi^d)}{4R}
$$

$$
+\beta^4\gamma^4\Big(\frac{3}{8}(1-\xi^d)^2\frac{1}{R^2}-\frac{3}{8}(1-\xi^d)^2\frac{1}{R}-\frac{\xi^d\left(1-\xi^d\right)}{4R}\Big)\Big)\Bigg)\Big]
$$

$$
\simeq \frac{2}{{z_*}^{d-2}}\Bigg[\int_0^1 \frac{d\xi}{\xi^{d-1}}\frac{1}{\sqrt{R}}+\frac{{z_*}^d}{2{z_0}^d}\left((1+\beta^2\gamma^2)\int_0^1\frac{d\xi}{\xi^{d-1}}\frac{1}{\sqrt{R}}\xi^d+\beta^2\gamma^2\int_0^1 d\xi\frac{\left(1-\xi^d\right)\xi^{(d-1)}}{R^{\frac{3}{2}}}\right)
$$

$$
+\frac{{z_*}^{2d}}{{z_0}^{2d}}\Big[\frac{(3+2\beta^2\gamma^2-\beta^4\gamma^4)}{8}\int_0^1\frac{d\xi}{\xi^{d-1}}\frac{1}{\sqrt{R}}\xi^{2d}+\int_0^1 d\xi\frac{\xi^{d-1}}{\sqrt{R}}\Bigg(\beta^2\gamma^2\xi^d\frac{(1-\xi^d)}{4R}
$$

$$
+\frac{3}{8}\beta^4\gamma^4(1-\xi^d)^2\frac{1}{R^2}-\beta^4\gamma^4\frac{\left(1-\xi^d\right)}{2R}\Bigg)+\beta^4\gamma^4\int_0^1 d\xi\frac{\xi^{d-1}(1-\xi^{2d})}{8R^{\frac{3}{2}}}\Big]\Bigg]
$$

$$\simeq \frac{2a_0}{{z_*}^{d-2}}\Big[1+\frac{{z_*}^d}{2{z_0}^d}\left(\frac{a_1}{a_0}(1+\beta^2\gamma^2)+\beta^2\gamma^2\frac{I_l}{a_0}\right)+\frac{{z_*}^{2d}}{{z_0}^{2d}}\left(\frac{(3+2\beta^2\gamma^2-\beta^4\gamma^4)}{8}\frac{a_2}{a_0}\right.$$
$$\left.+\frac{J_l}{a_0}+\frac{\beta^4\gamma^4}{8}\frac{I_a}{a_0}\right)\Big] \tag{2.32}$$

Where the coefficients a_0, a_1, a_2 and I_a are precise integral Beta functions multiplying at various orders. These coefficients are provided in the appendix(B). Now our next task is to express the turning point in terms of the AdS turning point $\bar{z}_*$ and keep terms upto second order only. Now from equation (2.30) we see that

$$\begin{aligned}
{z_*}^{2-d} &= {\bar{z}_*}^{2-d}\left(1+\frac{\bar{z}_*^d}{z_0^d}(\frac{b_1+\beta^2\gamma^2 I_l}{2b_0})+\frac{\bar{z}_*^{2d}}{z_0^{2d}}\left(\frac{\bar{b}_2+J_l}{b_0}-d(\frac{b_1+\beta^2\gamma^2 I_l}{2b_0})^2\right)\right)^{d-2}\\
&\simeq {\bar{z}_*}^{2-d}\Big[1+(d-2)\frac{\bar{z}_*^d}{z_0^d}(\frac{b_1+\beta^2\gamma^2 I_l}{2b_0})+(d-2)\frac{\bar{z}_*^{2d}}{z_0^{2d}}\Big(\frac{\bar{b}_2+J_l}{b_0}-\frac{(d+3)}{2}\\
&(\frac{b_1+\beta^2\gamma^2 I_l}{2b_0})^2\Big)\Big]
\end{aligned} \tag{2.33}$$

Now substituting (2.30) and (2.33) into (2.32)

$$\begin{aligned}
\Delta A = &\frac{2a_0}{{\bar{z}_*}^{d-2}}\left[1+(d-2)\frac{\bar{z}_*^d}{z_0^d}(\frac{b_1+\beta^2\gamma^2 I_l}{2b_0})+(d-2)\frac{\bar{z}_*^{2d}}{z_0^{2d}}\left(\frac{\bar{b}_2+J_l}{b_0}-\frac{(d+3)}{2}\right.\right.\\
&\left.\left.(\frac{b_1+\beta^2\gamma^2 I_l}{2b_0})^2\right)\right]\left[1+\frac{{\bar{z}_*}^d}{{z_0}^d}\left(\frac{a_1}{2a_0}\gamma^2+\beta^2\gamma^2\frac{I_l}{2a_0}\right)+\frac{{\bar{z}_*}^{2d}}{{z_0}^{2d}}\Big(\frac{(3+2\beta^2\gamma^2-\beta^4\gamma^4)}{8}\frac{a_2}{a_0}\right.\\
&\left.+\frac{J_l}{a_0}+\frac{\beta^4\gamma^4}{8}\frac{I_a}{a_0}-d\frac{(b_1+\beta^2\gamma^2 I_l)(a_1\gamma^2+\beta^2\gamma^2 I_l)}{4a_0b_0}\Big)\right]
\end{aligned}$$

Now we will simplify the above expression and retain terms upto second order in $\frac{{\bar{z}_*}^d}{{z_0}^d}$. We will use the relation between the beta function coefficients given in Appendix(B). Thus we get

$$\Delta A \simeq \frac{2a_0}{\bar{z}_*{}^{d-2}}\Bigg[1+\frac{\bar{z}_*^d}{z_0^d}\Bigg((d-2)(\frac{b_1+\beta^2\gamma^2 I_l}{2b_0})+(\frac{a_1}{2a_0}\gamma^2+\beta^2\gamma^2\frac{I_l}{2a_0})\Bigg)+\frac{\bar{z}_*{}^{2d}}{z_0{}^{2d}}$$
$$\Bigg[\Big(\frac{(3+2\beta^2\gamma^2-\beta^4\gamma^4)}{8}\frac{a_2}{a_0}+\frac{J_l}{a_0}+\frac{\beta^4\gamma^4}{8}\frac{I_a}{a_0}-d\frac{(b_1+\beta^2\gamma^2 I_l)(a_1\gamma^2+\beta^2\gamma^2 I_l)}{4a_0b_0}\Big)$$
$$+(d-2)\Big(\frac{\bar{b}_2+J_l}{b_0}-\frac{(d+3)}{2}(\frac{b_1+\beta^2\gamma^2 I_l}{2b_0})^2\Big)+(d-2)\frac{(b_1+\beta^2\gamma^2 I_l)(a_1\gamma^2+\beta^2\gamma^2 I_l)}{4a_0b_0}\Bigg]\Bigg]$$
$$\simeq\Bigg[\frac{2a_0}{\bar{z}_*{}^{d-2}}+\frac{a_0\bar{z}_*^2}{z_0^d}\Bigg((d-2)\frac{b_1}{b_0}+\frac{a_1}{a_0}\gamma^2\Bigg)+\frac{2a_0\bar{z}_*{}^{d+2}}{z_0{}^{2d}}\Bigg[\Big(\frac{(3+2\beta^2\gamma^2-\beta^4\gamma^4)}{8}\frac{a_2}{a_0}$$
$$+\frac{J_l}{a_0}+\frac{\beta^4\gamma^4}{8}\frac{I_a}{a_0}-d\frac{(b_1+\beta^2\gamma^2 I_l)(a_1\gamma^2+\beta^2\gamma^2 I_l)}{4a_0b_0}\Big)+(d-2)\Big(\frac{\bar{b}_2+J_l}{b_0}-\frac{(d+3)}{2}(\frac{b_1+\beta^2\gamma^2 I_l}{2b_0})^2\Big)$$
$$+(d-2)\frac{(b_1+\beta^2\gamma^2 I_l)(a_1\gamma^2+\beta^2\gamma^2 I_l)}{4a_0b_0}\Bigg]\Bigg]$$
$$\simeq\Bigg[\frac{2a_0}{\bar{z}_*{}^{d-2}}+\frac{a_0\bar{z}_*^2}{z_0^d}\Bigg((d-2)\frac{b_1}{b_0}+\frac{a_1}{a_0}\gamma^2\Bigg)+\frac{2a_0\bar{z}_*{}^{d+2}}{z_0{}^{2d}}\Bigg[\Big(\frac{(3-\beta^2\gamma^2)\,\gamma^2}{8}\frac{a_2}{a_0}$$
$$+\frac{\beta^4\gamma^4}{8}\frac{I_a}{a_0}\Big)-\Big(\frac{\bar{b}_2}{a_0}-\frac{(d+3)}{2}(\frac{b_1+\beta^2\gamma^2 I_l}{2b_0})^2\Big)-\frac{(b_1+\beta^2\gamma^2 I_l)(a_1\gamma^2+\beta^2\gamma^2 I_l)}{2a_0b_0}\Bigg]\Bigg] \tag{2.34}$$

The first two terms in the above expression are same as in (2.16). The third term is the second order contribution and can be expressed as

$$A_2=\frac{2a_0}{\bar{z}_*^{d-2}}\Bigg[(d+3)\frac{(b_1+\beta^2\gamma^2 I_l)^2}{8a_0b_0}-\frac{(b_1+\beta^2\gamma^2 I_l)(\gamma^2a_1+\beta^2\gamma^2 I_l)}{2a_0b_0}$$
$$-\frac{8\bar{b}_2-(3-\beta^2\gamma^2)\gamma^2a_2}{8a_0}+\frac{1}{2}\beta^4\gamma^4\frac{I_a}{4a_0}\Bigg]\frac{\bar{z}_*^{2d}}{z_0^{2d}}$$
$$=\frac{a_1}{\bar{z}_*^{d-2}}\Bigg[(d+3)\frac{(b_1+\beta^2\gamma^2 I_l)^2}{4a_1b_0}-\frac{(b_1+\beta^2\gamma^2 I_l)(\gamma^2a_1+\beta^2\gamma^2 I_l)}{a_1b_0}$$
$$-\frac{8\bar{b}_2-(3-\beta^2\gamma^2)\gamma^2a_2}{4a_1}+\frac{1}{4}\beta^4\gamma^4(\frac{I_a}{a_1})\Bigg]\frac{\bar{z}_*^{2d}}{z_0^{2d}} \tag{2.35}$$

All parameters in the above expression are known Beta functions provided in the appendix(B). We need to further simplify the last equation. After some tedious simplifications equation (2.35) can be rearranged as

$$A_2 = a_1 \bar{z}_*^2 \left(h_0 + h_1\beta^2\gamma^2 + h_2\beta^4\gamma^4\right) \frac{\bar{z}_*^d}{z_0^{2d}} \tag{2.36}$$

where coefficients are

$$\begin{aligned} h_0 &= \frac{d-1}{d+1}\left(-\frac{b_1}{2b_0} + \frac{3}{4}\frac{d+1}{2d+1}\frac{a_2}{a_1}\right), \\ h_1 &= \left(-\frac{b_1}{b_0} + \frac{1}{2}\frac{a_2}{a_1}\right) \\ h_2 &= \frac{d+1}{d-1}\left(-\frac{b_1}{2b_0} + \frac{3}{4}\frac{1}{d+1}\frac{a_2}{a_1}\right) . \end{aligned} \tag{2.37}$$

Note the area integral (A) is expanded around the AdS (ground state value) turning point. The net change in the area of the extremal strip up to second order is given by

$$\triangle A = A_1 + A_2 = \frac{a_1 l^2}{4b_0^2}\left(\left(\frac{d-1}{d+1} + \beta^2\gamma^2\right)\frac{1}{z_0^d} + \left(h_0 + h_1\beta^2\gamma^2 + h_2\beta^4\gamma^4\right)\frac{\bar{z}_*^d}{z_0^{2d}}\right) . \tag{2.38}$$

At this point it is quite remarkable to notice that the equation (2.38) can also be written in an unique factorized form

$$\triangle A = \frac{a_1 l^2}{4b_0^2} \cdot Q \cdot \left(\left(\frac{d-1}{d+1} + \beta^2\gamma^2\right)\frac{1}{z_0^d} - \frac{q a_2}{2a_1}\beta^2\gamma^4\frac{\bar{z}_*^d}{z_0^{2d}}\right) , \tag{2.39}$$

where the factor Q (quotient) is given by

$$Q = 1 - \left(\left(1 + \frac{d+1}{d-1}\beta^2\gamma^2\right)\frac{b_1}{2b_0} - \left(p + s\frac{d+1}{d-1}\beta^2\gamma^2\right)\frac{a_2}{2a_1}\right)\frac{\bar{z}_*^d}{z_0^d} \tag{2.40}$$

with unique set of parameters $p,\ q,\ s$ taking values as

$$p = \frac{3}{2}\frac{d+1}{2d+1}, \quad s = \frac{2+8d-d^2}{4(2d+1)}, \quad q = \frac{4+6d-d^2}{4(2d+1)}\ . \tag{2.41}$$

It is important to note that the above factorization is unique. It is unique in the sense that after the factorization the remainder of the expression in (2.39) (within large bracket) precisely contains nontrivial $\beta^2\gamma^4$ term, which contributes to $\mu.\triangle\mathcal{N}$, along with usual energy and pressure terms, as we would see next. The 'Q' factor is determined by simple quotienting procedure. Crucially there is no choice of Q for which we can set $q=0$ in (2.39). Any arbitrary Q would take us back to the situation where we started from, leaving us with little or no clue. The eq.(2.39) is the complete expression representing the net change in area of the strip when calculated up to second order. From the result (2.39) we determine

$$\triangle S = \frac{L^{d-1}V_{d-2}}{16G_N}\frac{a_1 l^2 Q}{b_0^2}\left((\frac{d-1}{d+1}+\beta^2\gamma^2)\frac{1}{z_0^d} - \frac{q a_2}{2a_1}\beta^2\gamma^4\frac{\bar{z}_*^d}{z_0^{2d}}\right) \tag{2.42}$$

Which provides the complete expression representing the net change in entanglement entropy up to second order in the expansion around pure AdS (ground state) value.

2.3.1 Redefinition and Entanglement First Law

It is apparent from the expression (2.42) that we would have to define new 'redefined' quantities in order to have a first law like relation. We first introduce the redefined width of the strip as

$$l_R \equiv Q^{\frac{1}{2}} l \tag{2.43}$$

Since generally $0 < Q < 1$, the entanglement length decreases after second order corrections. This would be true so long as we work within the pertubative regime. Further we assume the principle [23] and propose that the new entanglement temperature is inversely proportional to the renormalized width

$$T_E^* = \frac{db_0^2}{\pi a_1 l_R} = \frac{T_E}{\sqrt{Q}} \tag{2.44}$$

The Q also introduces boost dependence in the entanglement temperature at the second order. Even if there is no boost ($\beta = 0$), Q would still be nontrivial. With these definitions we redefine the 'entanglement energy' and 'entanglement charge' for the subsystem (following from (2.21) and (2.22))

$$\triangle\mathcal{E}^* = \sqrt{Q} \triangle \mathcal{E}, \qquad \triangle \mathcal{N}^* = \sqrt{Q} \triangle \mathcal{N} \tag{2.45}$$

and redefine the entanglement volume as

$$\mathcal{V}^* = \sqrt{Q}\mathcal{V} = \sqrt{Q} l V_{d-3}. \tag{2.46}$$

All above would simply happen provided we realize that the actual physical size (width) of the subsystem encountered by the excitations is l_R, whereas the old l is just the 'bare' (coordinate) size of strip subsystem. Since all extensive thermodynamic quantities of the subsystem will depend on strip width, hence all expressions are redefined by the single quantity Q. Finally we shall prefer to define 'entanglement pressure' as

$$\mathcal{P}^* \equiv \frac{d-1}{d+1}\mathcal{P} \tag{2.47}$$

and the ‘entanglement chemical potential’ is

$$\mu_* = \frac{q\beta\gamma^2}{r_y}\frac{a_2}{2a_1}\frac{\bar{z}_*^d}{z_0^d} \equiv \frac{4+6d-d^2}{8(2d+1)}\frac{a_2}{a_1}\mu \tag{2.48}$$

Note μ is the turning point value given in (2.23). From (2.42) and using above expressions, we find that the changes in entanglement entropy up to second order can be expressed as

$$\triangle S_E^* = \frac{1}{T_E^*}\left(\triangle \mathcal{E}^* - \mu_* \triangle \mathcal{N}^* - \mathcal{V}_* \triangle \mathcal{P}^*\right) \tag{2.49}$$

All thermodynamic quantities in the above result quantifying excitations in the CFT subsystem are completely known.

2.3.2 The l dependent behaviour

Let us make a few comments here. The boundary CFT is a d-dimensional theory having one of its direction being compact. As there are black holes in the bulk geometry it is a finite temperature theory. The thermal temperature is given by $T_{Th} = \frac{d}{4\pi z_0\gamma}$ which is fixed. Since the size of the subsystem is taken small, so that the entanglement effects can be studied perturbatively, it leads to a hierarchy of scales

$$\frac{\bar{z}_*}{z_0} \ll 1, \qquad \frac{l}{z_0} \ll 2b_0, \qquad \frac{T_{Th}}{T_E} \ll \frac{a_1}{2b_0\gamma} \tag{2.50}$$

while we keep $\beta\gamma \sim 1$. The redefined entanglement temperature (2.44) at second order can be written as

$$T_E^* \simeq \frac{1}{\pi a_1 l}\frac{db_0^2}{\sqrt{1-\alpha_0(\frac{2\pi\gamma l T_{Th}}{db_0})^d}} \tag{2.51}$$

where $\alpha_0 \equiv \left((1+\frac{d+1}{d-1}\beta^2\gamma^2)\frac{b_1}{2b_0} - (p+s\frac{d+1}{d-1}\beta^2\gamma^2)\frac{a_2}{2a_1}\right)$ is always positive definite. This expression remains valid so long as $\frac{2\pi\gamma l T_{Th}}{b_0 d} < 1$ is maintained. The eq. (2.51) implies that the entanglement temperature has sizable corrections for large l from higher order at a given thermal temperature T_{Th}. It also tells us how the entanglement temperature will flow towards T_{Th} as l increases. From (2.51), while keeping the strip size l fixed, we can also study the flow of entanglement temperature with respect to change in (black hole) thermal temperature

$$T_E^{(2)} \simeq \frac{T_E^{(1)}}{\sqrt{1-\alpha_0(\frac{2\pi\gamma l}{db_0})^d(T_{Th}^{(2)d} - T_{Th}^{(1)d})}} \tag{2.52}$$

where $T_{Th}^{(2)}$ and $T_{Th}^{(1)}$ are two different black hole temperatures. The equation (2.52) implies that the entanglement temperature will be higher for the bigger size black hole ($T_{Th}^{(2)} > T_{Th}^{(1)}$). The '$T_E$ Vs l' graphs have been plotted in the figure (2.1) for different T_{Th} values.

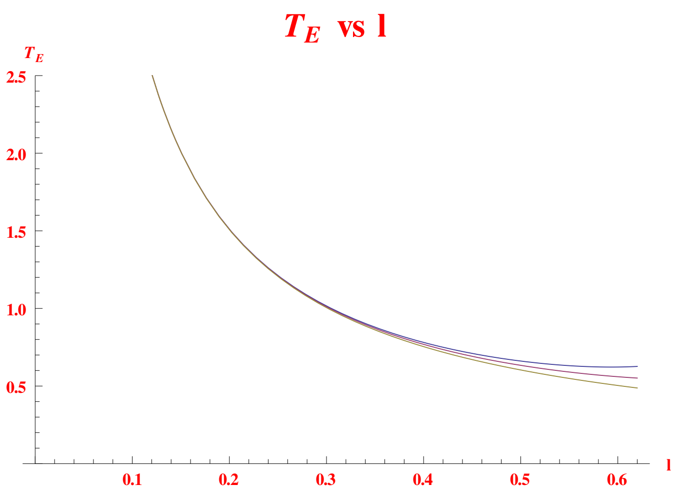


Figure 2.1: *Plots of 'T_E Vs l' for different black hole temperatures (starting from top curve) $T_{Th} = .28, .25, \&.10$ with fixed $\alpha_0 = .97$ and $(\beta\gamma)^2 = .5$ for AdS_5. The graphs split at large l showing the effect of corrections. These demonstrate that T_E is higher for higher black hole temperature.*

The entanglement energy of subsystem gets corrected as

$$\begin{aligned}
\triangle\mathcal{E}^* &= \sqrt{1-\alpha_0(\frac{2\pi\gamma l T_{Th}}{db_0})^d}\triangle\mathcal{E} \\
&= l\sqrt{1-\alpha_0(\frac{2\pi\gamma l T_{Th}}{db_0})^d}\frac{L^{d-1}V_{d-3}r_y d}{8G_{d+1}} < t_{00} >
\end{aligned} \tag{2.53}$$

From (2.48) the chemical potential up to the second order may be written

$$\begin{aligned}
\mu_* &= \frac{qa_2}{2r_y a_1}\beta\gamma^2(\frac{2\pi\gamma l T_{Th}}{db_0})^d + \text{ higher orders} \\
&\simeq \frac{qa_2}{2r_y\beta a_1}(1-\frac{1}{1+\beta^2\gamma^2(\frac{2\pi\gamma l T_{Th}}{db_0})^d})
\end{aligned} \tag{2.54}$$

where the second line merely reflects the fact that any subleading term is a higher order term which can be ignored at the second order. This will lead to following l dependence in the charge

$$\triangle\mathcal{N}^* = l\sqrt{1-\alpha_0(\frac{2\pi\gamma l T_{Th}}{db_0})^d}\frac{L^{d-1}V_{d-3}r_y^2 d}{8G_{d+1}}\frac{\beta\gamma^2}{z_0^d} \tag{2.55}$$

The large l behaviour may be predicted from here up to some value $l = l_c$, such that $l_c < \frac{db_0}{2\pi\gamma T_{Th}}$. We cannot stretch these results beyond this bound as this would lead to to the break down of perturbative regime. In large l limit we expect to see the behaviour

$$T_E^* \to T_{Th}, \quad \triangle\mathcal{E}^* \to \triangle\mathcal{E}_{Th}. \tag{2.56}$$

2.4 Conclusion

We adopted a perturbative method to calculate change in holographic entanglement entropy. This we have done by expanding the area and length integral in terms of the dimensionless parameter $\frac{z_*}{z_0}^d$ upto second order. We considered the boosted AdS black brane as our excited geometry. We found nontrivial dependence of the change in the area of the bulk minimal surface on the boost parameter (β) as well as the horizon z_0. We found that the first order change in the holographic entanglement entropy satisfies a first law like relation. We have tried to extend this relation to the next order. In order to write a first law like relation at the second order, we introduced 'redefined length' for the subsystem in order to retain the form of first law of entanglement thermodynamics. If we did not do so we will have no hope of having a first law like relation. Note that the bulk geometry is well defined and the corresponding boundary energy-momentum tensor is also fixed. Therefore, only option left for us is to look for correct subsystem size. (The length $l \equiv 2b_0\bar{z}_*$ is good for 'pure' AdS with turning point value $\bar{z}_*$). With the excitations in the CFT (z_* being new turning point) the relationship between l and z_* is known at best perturbatively (order by order), through eq. (2.30). But we can define new redefined length l_R at higher orders. With the help of given expressions, the relationship between l_R and z_* can also be fixed, perturbatively, but is not needed in our results. Thus, if l is the size at the first order, at the next order the correct size becomes l_R. Not only the length, we have to correct the chemical potential as well, remember the chemical potential is zero at the first order. Other (extensive) thermodynamic quantities depend on the length, so these also get redefined once the size becomes l_R. But, are these corrections quantum in nature? In AdS/CFT we deal with boundary CFT which is a strongly coupled quantum theory. Since we are expanding around pure AdS (describing CFT ground state), the small excitations of the CFT above the ground state will necessarily be ”quantum” in nature. These excitations for small subsystem are controlled by the smallness of the ratio $\frac{T_{th}}{T_E}$ or by the turning point to horizon ratio $\frac{\bar{z}_*}{z_0}$. For example, in $d = 4$ case, the dimensionless ratio

$$\frac{\bar{z}_*^4}{z_0^4} \propto g_{_{YM}}^4 l^4 \epsilon_0 \simeq \frac{l^4 \lambda^2 \epsilon_0}{n^2} \tag{2.57}$$

where ϵ_0 denotes energy density of the excitations. Thus the corrections to various entanglement quantities are quantum in nature and depend on perturbative Yang-Mills coupling constant $g_{_{YM}}$ (or the ’t Hooft coupling $\lambda \sim g_{_{YM}}^2 n$).

Remarks for AdS_4, AdS_5 and AdS_7:

We note that the parameter p, q, s in (2.41) are positive definite but smaller than one in string/M-theory cases with $d = 3, 4$ and $d = 6$. Also the two Beta-function ratios, $\frac{b_1}{2b_0}$ and $\frac{a_2}{2a_1}$, are both positive definite and generally smaller than one. The eq.(2.48) implies that entanglement chemical potential is positive definite for these conformal cases. Although the result in (2.48) is applicable for any d dimensions, but for $d > 6$, the parameter q changes sign, hence the chemical potential μ_* will also change sign for $d > 6$. This is a surprising result, but it simply may be an indication of the fact that we are going beyond the realm of applicability of string/M-theory.

CHAPTER 3

ENTANGLEMENT ASYMMETRY FOR BOOSTED BLACK BRANES AND THE BOUND

3.1 Introduction

In this chapter we extend our perturbative approach to study the asymmetry in the change in entanglement entropy along various directions of the CFT. We find that the boosted black branes give rise to an asymmetry in the entanglement first law. We study two types of strip subsystems one parallel to the boost and the other perpendicular to the boost direction. There is difference in the 'entanglement pressure' in two cases such that $\triangle\mathcal{P}_{\perp} \leq \triangle\mathcal{P}_{\|}$. We find that primarily the entanglement pressure is responsible for the differences in the entanglement entropies, $\triangle S_{\perp} \geq \triangle S_{\|}$, in the two cases. The entanglement asymmetry may be quantified as a dimensionless ratio

$$\begin{aligned}\mathcal{A} &\equiv \frac{\triangle S_{\perp} - \triangle S_{\parallel}}{\triangle S_{\perp} + \triangle S_{\parallel}} \\ &= \frac{\beta^2\gamma^2}{(2 + \frac{d+3}{d-1}\beta^2\gamma^2)} \leq \frac{d-1}{d+3}\end{aligned}$$

We find that the asymmetry depends only on boost and it is bounded from above. The bound is saturated only for the AdS-wave background, which is the case involving infinite boosts. To obtain these results we resort to a perturbative calculation of the entanglement entropy up to first order, where the ratio $\frac{l}{z_0}$, of the strip width (l) to the horizon size (z_0), is kept very small.

3.2 Entanglement from boosted black-branes

The boosted AdS_{d+1} backgrounds we are interested are given by

$$ds^2 = \frac{L^2}{z^2}\left(-\frac{f dt^2}{K} + K(dy - \omega)^2 + dx_1^2 + \cdots + dx_{d-2}^2 + \frac{dz^2}{f}\right) \tag{3.1}$$

with functions

$$f = 1 - \frac{z^d}{z_0^d}, \quad K = 1 + \beta^2\gamma^2\frac{z^d}{z_0^d} \tag{3.2}$$

$z = z_0$ is the horizon and $0 \leq \beta \leq 1$ is boost parameter, while $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. The boost is taken along y direction. The one-form

$$\omega = \beta^{-1}(1 - \frac{1}{K})dt \tag{3.3}$$

and L is the radius of curvature of AdS spacetime, which is taken very large in string length units. [1]

3.2.1 A thin (perpendicular) strip

We first study the entanglement entropy law for a subsystem on the boundary of the AdS_{d+1} backgrounds (3.1) where strip is perpendicular to the boost direction: the strip width is $-l/2 \leq x^1 \leq l/2$, while the boost is along y direction. Thus the steps in this section are same as in our precious work [31]. We embed the $(d-1)$-dimensional strip-like (constant t surface) inside the bulk geometry. The two boundaries of the extremal surface coincide with the two ends of the interval $\triangle x^1$. The size of the rest of the coordinates, $0 \leq y \leq l_y$, $0 \leq x^i \leq l_i$, is taken very large, such that $l_y, l_i \gg l$. As per the Ryu-Takayanagi prescription [6, 7] the entanglement entropy of the strip subsystem is given in terms of the geometrical area of the extremal surface (constant time)

$$S_{\perp} \equiv \frac{[A]_{Strip}}{4G_{d+1}} = \frac{V_{d-2}L^{d-1}}{2G_{d+1}} \int_{\epsilon}^{z_*} \frac{dz}{z^{d-1}} \sqrt{K} \sqrt{\frac{1}{f} + (\partial_z x^1)^2} \tag{3.4}$$

where G_{d+1} is $(d+1)$-dimensional Newton's constant (of bulk gravity) and $V_{d-2} \equiv l_y l_2 l_3 \cdots l_{d-2}$ is the net spatial volume of the strip on the boundary. We will be mainly working for $d > 2$ here. In our notation $z = \epsilon \sim 0$ is the cut-off scale and $z = z_*$ is the turning point of extremal surface. In the above area functional $K(z)$, and $f(z)$ are known functions, so we only need to extremize for $x^1(z)$. After extremization the entanglement entropy for

[1] For example, in the $AdS_5 \times S^5$ near-horizon geometry of n coincident D3-branes, we shall have $L^4 \equiv 2\pi g_{YM}^2 n$ and the 't Hooft coupling constant $g_{YM}^2 n \gg 1$.

perpendicular strip subsystem can be written as

$$S_{\perp}=\frac{V_{d-2}L^{d-1}}{2G_{d+1}}\int_{\epsilon}^{z_*}\frac{dz}{z^{d-1}}\frac{K}{\sqrt{f}\sqrt{K-K_*(\frac{z}{z_*})^{2d-2}}} \tag{3.5}$$

where $K_*\equiv K(z)|_{z=z_*}$. The boundary value $x^1(0)=l/2$ has the integral relation

$$\frac{l}{2}=\int_0^{z_*}dz(\frac{z}{z_*})^{d-1}\frac{1}{\sqrt{f}\sqrt{\frac{K}{K_*}-(\frac{z}{z_*})^{2d-2}}} \tag{3.6}$$

which relates l with the turning point z_*. The turning-point takes the mid-point value $x^1(z_*)=0$ on the boundary.

When strip subsystem is a small the turning point will lie in the proximity of asymptotic boundary region only ($z_*\ll z_0$). We can evaluate the entanglement entropy (3.5) by expanding it around the AdS (i.e. treating pure AdS as a ground state). We take boost to be finite such that

$$\frac{z_*^d}{z_0^d}\ll 1,\quad \frac{(\beta\gamma)^2 z_*^d}{z_0^d}\ll 1 \tag{3.7}$$

is always maintained. In this limit we can estimate the entropy perturbatively. Under these approximations, entanglement entropy contribution (above pure AdS) at first order is given by (2)

$$\triangle S_{\perp}=S_{\perp}-S_{AdS}=\frac{L^{d-1}V_{d-2}}{16G_{d+1}}\frac{a_1 l^2}{b_0^2}\left(\frac{d-1}{d+1}+\beta^2\gamma^2\right)\frac{1}{z_0^d}\,. \tag{3.8}$$

The CFT energy and pressure can be obtained by expanding the bulk geometry (3.1) in Fefferman-Graham coordinates valid near the boundary [24, 84, 85]. The energy of the excitations is given by

$$\triangle\mathcal{E} = \frac{L^{d-1}V_{d-2}l}{16\pi G_{d+1}}(\frac{d-1}{d} + \beta^2\gamma^2)\frac{d}{z_0^d} \tag{3.9}$$

The volume is $V_{d-2} \equiv l_y l_2 \cdots l_{d-2}$. The pressure along y direction (parallel to the boost direction) is

$$\triangle\mathcal{P}_{\parallel} = \triangle\mathcal{P}_y = \frac{L^{d-1}d}{16\pi G_{d+1}}(\frac{1}{d} + \beta^2\gamma^2)\frac{1}{z_0^d} \tag{3.10}$$

while the pressure along all other x_i's (perpendicular to the boost direction) is identical and is given by

$$\triangle\mathcal{P}_{\perp} = \frac{L^{d-1}}{16\pi G_{d+1}}\frac{1}{z_0^d} = \triangle\mathcal{P}_1 = \triangle\mathcal{P}_2 = \cdots \tag{3.11}$$

This pressure asymmetry is solely due to the boost. For example the pressure is more along the y (boost) direction as compared to x^i's coordinates. Using (3.9) and (3.11) we can express eq.(3.8) as

$$\triangle S_{\perp} = \frac{1}{T_E}(\triangle\mathcal{E} - \frac{d-1}{d+1}\,\mathcal{V}_{\perp} \triangle \mathcal{P}_{\perp}) \tag{3.12}$$

where $\mathcal{V}_{\perp} \equiv l[l_y l_2 \cdots l_{d-2}]$ is the net volume of the strip subsystem. The entanglement temperature is given by

$$T_E^{\perp} = \frac{(B(\frac{d}{2d-2},\frac{1}{2}))^2}{2(d-1)B(\frac{1}{d-1},\frac{1}{2})}\frac{d}{\pi l}. \tag{3.13}$$

The temperature is inversely proportional to the width of strip. The equation (3.12) simply describes the first law of entanglement thermodynamics [23,26]. Subtle changes will occur in this expression when strip is taken along the boost.

3.2.2 Strip along the boost

We now study the entanglement entropy of a strip subsystem such that its width is parallel to the boost (flow) direction. That is, we take the boundaries of the extremal surface to coincide with the two ends of $\triangle y$ interval: $-l/2 \leq y \leq l/2$. The regulated size of rest of the coordinates will be taken much larger $0 \leq x^i \leq l_i$, such that $l_i \gg l$ $(i = 1, 2, \cdots, d-2)$. It is important to note that we wish to embed the $\triangle y$ interval, but since the boost is also along y, both 'time' $t(z)$ and $y(z)$ would have to be embedded in the bulk in a covariant manner [8]. So one has to be a bit cautious while working with stationary metric cases [33] [27]. However, it can be explicitly shown that, in the perturbative expansion (for small strips) to know the entropy only upto first order (next to the pure AdS), just taking a constant t slice would suffice. The deviations in extremal surface geometry away from the constant time slice will contribute only to the second order terms in the expansion. Our aim in this work is to know only the first order terms in the expansions of z_* and strip area. Taking the constant time slice the entanglement entropy of the parallel strip becomes

$$S_{\|} = \frac{V_{d-2}L^{d-1}}{2G_{d+1}} \int_{\epsilon}^{z_*} \frac{dz}{z^{d-1}} \sqrt{\frac{1}{f} + K(\partial_z y)^2} \tag{3.14}$$

where now $V_{d-2} \equiv l_1 l_2 \cdots l_{d-2}$ is the spatial volume. The identification of the extremal strip boundary, $y(0) = l/2$, leads to the integral relation

$$\frac{l}{2} = \int_0^{z_*} dz (\frac{z}{z_*})^{d-1} \frac{1}{\sqrt{fK}\sqrt{\frac{K}{K_*} - (\frac{z}{z_*})^{2d-2}}} \tag{3.15}$$

which relates l with the turning point z_* of the strip. The turning-point takes the mid-value $y(z_*) = 0$. The final expression of the entanglement entropy for the strip subsystem parallel to the boost direction now becomes

$$S_{\parallel} = \frac{V_{d-2}L^{d-1}}{2G_{d+1}} \int_{\epsilon}^{z_*} \frac{dz}{z^{d-1}} \frac{\sqrt{K}}{\sqrt{f}\sqrt{K - K_*(\frac{z}{z_*})^{2d-2}}} \tag{3.16}$$

Since the parallel system has not been covered in (2) let us provide some essential details perturbative calculation here. In small strip cases, the equation (3.15) can be expanded perturbatively upto first order as

$$l = 2z_* \left(b_0 + \frac{z_*^d}{2z_0^d}((1 + \frac{2\beta^2\gamma^2}{d-1})b_1 - \frac{\beta^2\gamma^2}{d-1}b_0) \right) + \cdots \tag{3.17}$$

where dots indicate terms of higher powers in $(\frac{z_*}{z_0})^d$, and various coefficients are defined earlier. From here keeping only up to first order the above equation implies

$$z_* = \frac{\bar{z}_*}{1 + \frac{\bar{z}_*^d}{2z_0^d}((1 + \frac{2\beta^2\gamma^2}{d-1})\frac{b_1}{b_0} - \frac{\beta^2\gamma^2}{d-1})} \tag{3.18}$$

where $\bar{z}_* \equiv \frac{l}{2b_0}$ being the turning point of pure AdS having the same strip width as l. Having obtained the turning point expansion, a similar expansion around pure AdS can be made for the area functional also. Suppressing the details, after regularizing the area integral (3.16), the net change in the area of parallel strip (above pure AdS value) comes out to be

$$\triangle A_{\parallel} = \frac{a_0\bar{z}_*^2}{z_0^d}(\frac{a_1}{a_0} - (1 - \beta^2\gamma^2)\frac{b_1}{a_0}) \tag{3.19}$$

and thus corresponding change in the entropy for parallel strip becomes

$$\triangle S_{\parallel} = \frac{L^{d-1}V_{d-2}}{16G_{d+1}} \frac{a_1 l^2}{b_0^2} \left(\frac{d-1}{d+1} + \frac{2}{d+1}\beta^2\gamma^2 \right) \frac{1}{z_0^d}. \tag{3.20}$$

The equation (3.20) is complete expression up to the first order. The entanglement first law for a strip along the flow becomes

$$\triangle S_{\parallel} = \frac{1}{T_E^{\parallel}}(\triangle E_{\parallel} - \frac{d-1}{d+1}\mathcal{V}_{\parallel} \triangle P_{\parallel}) \tag{3.21}$$

where $\mathcal{V}_{\parallel} = lV_{d-2} = l[l_1 l_2 \cdots l_{d-2}]$, and $\triangle\mathcal{P}_{\parallel} = \triangle\mathcal{P}_y$ is defined earlier. The temperature is

$$T_E^{\parallel} = \frac{b_0^2}{a_1}\frac{d}{\pi l} = T_E^{\perp}. \tag{3.22}$$

We note that the two temperatures remain the same but the entanglement entropies differ significantly. We now go on to find this asymmetry.

3.3 Entanglement asymmetry and the bound

Following from previous section, with out any loss of generality we can always take the volume of the strip subsystems to be equal

$$\mathcal{V}_{\parallel} = \mathcal{V}_{\perp} = l.V_{d-2}. \tag{3.23}$$

This only means that regulated size of the boxes is kept the same in both the cases, along with the strip width l. It implies that

$$T_E^{\parallel} = T_E^{\perp}, \;\; \triangle\,\mathcal{E}_{\parallel} = \triangle\mathcal{E}_{\perp}. \tag{3.24}$$

Comparing the two types of entropy results, the difference is given by

$$\begin{aligned}\triangle S_\perp - \triangle S_\parallel &= \frac{L^{d-1}V_{d-2}}{16G_{d+1}}\frac{a_1 l^2}{b_0^2}\left(\frac{d-1}{d+1}\beta^2\gamma^2\right)\frac{1}{z_0^d} \\ &= \frac{d-1}{d+1}\,\mathcal{V}(\triangle\mathcal{P}_\parallel - \triangle\mathcal{P}_\perp)\,. \end{aligned} \tag{3.25}$$

The right hand side is a positive definite expression. Hence we can deduce that entanglement entropy is more for a perpendicular strip subsystem as compared to the parallel set-up, even though the energy of excitations and entanglement temperatures remain the same for both. The key to this entropy enhancement effect,

$$\triangle\, S_\perp \geq \triangle S_\parallel \tag{3.26}$$

can directly be alluded to unequal entanglement pressure;

$$\triangle\, \mathcal{P}_\perp \leq \triangle\mathcal{P}_\parallel. \tag{3.27}$$

Thus more energy is consumed by the excitations in the parallel strip (due to an increased pressure) as compared to the perpendicular strip (having a low pressure along the strip). This suggests that in the boundary CFT 'pressure' plays a vital role in determining the entanglement entropy of the subsytems. The equation (3.25) also implies that, up to first order, the net difference of the entanglement entropies is

$$S_\perp - S_\parallel = \frac{L^{d-1}V_{d-2}}{16G_{d+1}}\frac{a_1 l^2}{b_0^2}\left(\frac{d-1}{d+1}\beta^2\gamma^2\right)\frac{1}{z_0^d} \tag{3.28}$$

Thus the entropy asymmetry coexists with pressure asymmetry in the CFT.

We can now define the entanglement asymmetry as a ratio

$$\mathcal{A} \equiv \frac{\triangle S_{\perp} - \triangle S_{\parallel}}{\triangle S_{\perp} + \triangle S_{\parallel}} = \frac{\beta^2\gamma^2}{(2 + \frac{d+3}{d-1}\beta^2\gamma^2)} \tag{3.29}$$

Thus nonzero boost ($\beta \leq 1$) will always induce entanglement asymmetry in the boundary CFT. The asymmetry will however vanishes for $\beta = 0$. Note that these results have been derived in the perturbative regime described in (3.7) only up to first order. We also learn that the asymmetry will always be bounded. In the above the bound is saturated only in the large boost limit, which we shall discuss in the next section.

We could however define an entanglement entropy ratio as

$$\mathcal{R} \equiv \frac{\triangle S_{\parallel}}{\triangle S_{\perp}} = \frac{1 + \frac{2}{d-1}\beta^2\gamma^2}{1 + \frac{d+1}{d-1}\beta^2\gamma^2} \geq \frac{2}{d+1} \tag{3.30}$$

a quantity which depends on the boost only and is devoid of external factors like shape and size. Then

$$\mathcal{A} \equiv \frac{1-\mathcal{R}}{1+\mathcal{R}} \leq \frac{d-1}{d+3}. \tag{3.31}$$

We shall show that the bound is saturated in the case of AdS-wave in the next section. The maximum value $\mathcal{R}$ can take is one for which entanglement asymmetry vanishes.

3.3.1 $\beta \to 1,\ z_0 \to \infty$ limit (pressureless system)

In present examples the pressure in the CFT$_d$ can be controlled by regulating the boost. We now show that there exists a simultaneous double limit in which the pressure asymmetry of the CFT excitations becomes optimal. We take a double limit $\beta \to 1,\ z_0 \to \infty$, keeping the ratio

$$\frac{\beta^2\gamma^2}{z_0^d} = \frac{1}{z_I^d} = \text{Fixed}\ . \tag{3.32}$$

These double limits has previously been explored in [86, 87] in connection with Lifshitz type backgrounds from black Dp branes (in lightcone coordinates). Under these limits the bulk geometry (3.1) reduces to the following AdS-wave background

$$ds^2 = \frac{L^2}{z^2}\left(-K^{-1}dt^2 + K(dy - (1-K^{-1})dt)^2 + dx_1^2 + \cdots + dx_{d-2}^2 + dz^2\right) \tag{3.33}$$

with the new function $K = 1 + \frac{z^d}{z_I^d}$, where $z = z_I$ is an scale which determines momentum of the wave traveling in the y direction. (The entanglement of strip systems for AdS-waves has previously been explored by [88] also.) For this background the energy of the excitations in the CFT becomes (following from (3.9))

$$\triangle\mathcal{E} = \frac{L^{d-1}V_{d-2}l}{16\pi G_{d+1}}\frac{d}{z_I^d} \tag{3.34}$$

The pressure along the wave (y) direction becomes using (3.11)

$$\triangle\mathcal{P}_{\|} = \triangle\mathcal{P}_y = \frac{L^{d-1}d}{16\pi G_{d+1}}\frac{1}{z_I^d} \tag{3.35}$$

while the pressure along all x_i's (perpendicular to the wave direction) identically vanishes

$$\triangle\,\mathcal{P}_{\perp} = 0 \tag{3.36}$$

in the boundary CFT$_d$, which is a conformal theory with traceless energy-momentum tensor.

The double limits (3.32) can also be directly employed on the entropy results obtained in the previous section, provided we maintain $\frac{z_*^d}{z_I^d} \ll 1$. Employing the limits on the entropy expressions in eqs. (3.8) and (3.20), it gives us

$$\triangle S_{\perp} = \frac{L^{d-1}V_{d-2}}{16G_{d+1}} \frac{a_1 l^2}{b_0^2} \frac{1}{z_I^d} = \frac{1}{T_E}(\triangle \mathcal{E}) \tag{3.37}$$

while

$$\triangle S_{\parallel} = \frac{L^{d-1}V_{d-2}}{16G_{d+1}} \frac{a_1 l^2}{b_0^2} \left(\frac{2}{d+1}\right) \frac{1}{z_I^d} = \frac{1}{T_E}(\triangle \mathcal{E} - \frac{d-1}{d+1} \mathcal{V} \triangle \mathcal{P}_{\parallel}) \tag{3.38}$$

The width l is kept the same in both cases as well as the transverse volumes. Hence entanglement temperatures, $T_E = \frac{b_0^2}{a_1} \frac{d}{\pi l}$, and $\triangle \mathcal{E}$, remain the same for both the cases. Particularly in the former case there is no entanglement pressure along the strip (x^1 direction). As no 'entanglement work' seems to have been done by the excitations due to vanishing pressure ($\Delta P_{\perp} = 0$), the entropy remains maximal in the perpendicular direction. While in the latter case there is finite pressure ($\Delta P_{\parallel} \neq 0$) along the strip width, so finite energy is consumed by the excitations to work against the pressure as they take part in the entanglement. Thus the work done against entanglement pressure costs finite energy which essentially leads to a reduction in the net entanglement entropy in direction parallel to propagation of the wave.

From equations (3.37) and (3.38) for the AdS-wave case the ratio becomes

$$\mathcal{R}_{wave} = \frac{\triangle S_{\parallel}}{\triangle S_{\perp}} = \frac{2}{d+1}. \tag{3.39}$$

This is a remarkable relation and is identical to one in (3.30). It remains true at the linear order in perturbation (over and above the AdS background). At the higher orders in $\frac{z_*^d}{z_I^d}$ expansion this result might change. The entanglement asymmetry becomes

$$\mathcal{A}_{wave} \equiv \frac{\triangle S_\perp - \triangle S_\|}{\triangle S_\perp + \triangle S_\|} = \frac{d-1}{d+3}. \tag{3.40}$$

The asymmetry has optimal value and is universal in nature. The relations (3.39) and (3.40) are applicable only when $d > 2$, because for $d = 2$ (i.e. AdS_3-wave) the analogue of $\triangle S_\perp$ does not exist, but the form of entanglement first law as in (3.38) for parallel strip does hold good.

3.4 Non-conformal boosted black D-branes

The conformal cases of AdS geometries which are near horizon geometries of D3 and M2/M5 branes are covered in the previous section. In this section we wish to extend entanglement asymmetry analysis to the nonconformal Dp brane backgrounds [89]. We are interested in the boosted Dp-brane geometry so that suitable asymmetry is generated. These nonconformal backgrounds can be written as

$$\begin{aligned} ds^2 &= g_{eff}\Big[-\frac{f}{z^2K}dt^2 + \frac{K}{z^2}(dy-\omega)^2 + \frac{dx_2^2+\cdots+dx_p^2}{z^2} + \frac{4}{(5-p)^2}\frac{dz^2}{z^2 f} + d\Omega_{8-p}^2\Big] \\ e^\phi &= \frac{(2\pi)^{2-p}}{d_pN} g_{eff}^{\frac{7-p}{2}} \end{aligned} \tag{3.41}$$

along with appropriate $F_{(p+2)}$ form Ramond-Ramond flux. The strength of the string coupling depends on effective YM coupling $g_{eff} = (\lambda_p z^{3-p})^{\frac{1}{5-p}}$ and the functions are defined as

$$f = 1 - \frac{z^{\tilde{p}}}{z_0^{\tilde{p}}}, \quad K = 1 + \beta^2\gamma^2\frac{z^{\tilde{p}}}{z_0^{\tilde{p}}}$$

$$\omega = \beta^{-1}(1 - \frac{1}{K})dt \tag{3.42}$$

with $z = z_0$ being the location of horizon and $0 \leq \beta \leq 1$ is the boost. The boost is taken along the y direction and geometry along brane directions has asymmetry. The new parameters are defined as

$$\lambda_p \equiv d_p g_{YM}^2 N, \;\; \tilde{p} = \frac{14-2p}{5-p} \tag{3.43}$$

where d_p is a fixed normalization factor for a given p brane (The exact expression will not be needed here but it can be found out in [89]). The parameter λ_p is essentially the 't Hooft coupling constant and it controls the curvature of spacetime which is to be taken small in string length units ($l_s = 1$) and for which N is taken to be large enough. The boosted geometry (3.41) is conformally $AdS_{p+2} \times S^{8-p}$, a near-horizon geometry of N coincident Dp-branes. Only for $p = 3$ case the geometry becomes conformal and is discussed earlier. We are discussing the asymmetry cases for that we need $p = 2$ or $p = 4$, for them at least two asymmetric brane directions are available.

3.4.1 Entropy of thin strips

We first consider a thin strip in a perpendicular direction to the boost, say x_2. The Ryu-Takayanagi entropy functional for a strip embedded in the bulk geometry (3.41) is given by

$$
\begin{aligned}
S_{\perp} &= \frac{V_{p-1}\Theta_{8-p}Q_p}{2G_N}\int_{\epsilon}^{z_*}\frac{dz}{z^{\tilde{p}-1}}\sqrt{K}\sqrt{\frac{4}{(5-p)^2}\frac{1}{f}+(\partial_z x_2)^2} \\
&= \frac{V_{p-1}\Theta_{8-p}}{2G_N}\frac{2Q_p}{5-p}\int_{\epsilon}^{z_*}\frac{dz}{z^{\tilde{p}-1}}\sqrt{K}\sqrt{\frac{1}{f}+(\partial_z \bar{x}_2)^2}
\end{aligned}
\tag{3.44}
$$

where $Q_p \equiv \frac{(2\pi)^{2p-4}\sqrt{\lambda_p^{\tilde{p}}}}{g_{YM}^4}$ while Θ_{8-p} is the volume of unit radius S^{8-p} and G_N is the 10-dimensional Newton's. We shall consider a small legth interval $-\frac{l}{2} \le \bar{x}_2 \le \frac{l}{2}$, but due to the scaling $x_2 = \frac{2}{5-p}\bar{x}_2$ in eq.(3.44) the actual width of the strip is $\frac{2l}{5-p}$. One can see that the integrand in the second line in (3.44) is strikingly same as that for the conformal case discussed earlier, except that parameter $\tilde{p}$ can take fractional values. (For example, for D2-branes $\tilde{p} = \frac{10}{3}$, but for D4-branes $\tilde{p} = 6$.) So the rest of the calculations is straight forward: Extremizing the area and making a perturbative expansion keeping the ratio $\frac{l}{z_0} < 1$, as in previous sections. Avoiding the unnecessary details we quote the result from eq.(3.8). The entanglement entropy of the excitations above the extremality is

$$
\triangle S_{\perp} = \frac{V_{p-1}\Theta_{8-p}}{16G_N}\frac{2Q_p}{5-p}\frac{\tilde{a}_1 l^2}{\tilde{b}_0^2}\left(\frac{\tilde{p}-1}{\tilde{p}+1}+\beta^2\gamma^2\right)\frac{1}{z_0^d}
\tag{3.45}
$$

where new beta functions are given as $\tilde{b}_0 \equiv \frac{1}{2(\tilde{p}-1)}B(\frac{\tilde{p}}{2\tilde{p}-2},\frac{1}{2}))$ and $\tilde{a}_1 \equiv \frac{1}{2(\tilde{p}-1)}B(\frac{1}{\tilde{p}-1},\frac{1}{2})$. We come to conclusion that the entropy of excitations in a nonconformal $(p+1)$-dimensional theory at the first order can be written as

$$
\triangle S_{\perp} = \frac{1}{T_E^{\perp}}(\triangle\mathcal{E} - \frac{\tilde{p}-1}{\tilde{p}+1}\,\mathcal{V}_{\perp}\triangle\mathcal{P}_{\perp})
\tag{3.46}
$$

where $\mathcal{V}_\perp = \frac{2}{5-p} l V_{p-1}$ is the net volume of the strip subsystem, while the energy and pressure expressions are in appendix. The entanglement temperature is defined by

$$T_E^\perp = \frac{(B(\frac{\tilde{p}}{2\tilde{p}-2}, \frac{1}{2}))^2}{2(\tilde{p}-1)B(\frac{1}{\tilde{p}-1}, \frac{1}{2})} \frac{(7-p)}{\pi l}. \tag{3.47}$$

The temperature is inversely proportional to the width of strip. But compared to the law in (3.12) subtle changes have occured in the pressure term in (3.46). Namely the coefficient $\frac{\tilde{p}-1}{\tilde{p}+1}$ in (3.46) is different from the ratio $\frac{d-1}{d+1}$ which appears in (3.12). (Note d takes only integer values and is directly correlated with the dimensionality of AdS_{d+1}. This cannot be said about $\tilde{p}$.) Let us comment here that for unboosted nonconformal D-brane case the result (3.46) was first obtained in [30]. So it is interesting to observe that the form of first law with boost excitations remains the same as in unboosted case [30], although all physical quantities have themselves got changed.

In the next we consider an strip interval in the direction parallel to the boost, i.e. along y direction. The entropy functional is given by

$$S_\parallel = \frac{V_{p-1}\Theta_{8-p}}{2G_N g_{YM}^4} \frac{2Q_p}{5-p} \int_\epsilon^{z_*} \frac{dz}{z^{\tilde{p}-1}} \sqrt{\frac{1}{f} + K(\partial_z \bar{y})^2} \tag{3.48}$$

where now V_{p-1} is regulated volume of all the x^i coordinates. We have scaled $y = \frac{2}{5-p}\bar{y}$ and taken the width to be $-l/2 \leq \bar{y} \leq l/2$. As usual extremizing the strip area and expanding up to first order in the ratio $l/z_0 \ll 1$, we come to conclusion that the entropy of excitations above extremality for a parallel strip follows the law

$$\triangle S_\parallel = \frac{1}{T_E^\parallel}(\triangle\mathcal{E} - \frac{\tilde{p}-1}{\tilde{p}+1} \mathcal{V}_\parallel \triangle \mathcal{P}_\parallel) \tag{3.49}$$

where $\mathcal{V}_\parallel = \frac{2l}{5-p} V_{p-1}$ is the net volume of the parallel strip subsystem. Since we have kept the same strip width $\frac{2l}{5-p}$ in both the situations, the entanglement temperature are identical

$$T_E^{\parallel} = T_E^{\perp} \tag{3.50}$$

Now if we set $\mathcal{V}_{\parallel} = \mathcal{V}_{\perp}$, the excitation energies can also be made same, $\triangle\mathcal{E}_{\parallel} = \triangle\mathcal{E}_{\perp}$, however the entanglement pressures do always differ. We calculate the entanglement asymmetry, in the same way as (3.29),

$$\mathcal{A}_{nonconf} \equiv \frac{\triangle S_{\perp} - \triangle S_{\parallel}}{\triangle S_{\perp} + \triangle S_{\parallel}} = \frac{\beta^2\gamma^2}{(2 + \frac{\tilde{p}+3}{\tilde{p}-1}\beta^2\gamma^2)} \leq \frac{\tilde{p}-1}{\tilde{p}+3}. \tag{3.51}$$

As discussed in the conformal case, the bound gets saturated only in the case of Dp-branes having wave like excitations at zero temperature. For this we need to employ the same double limits $\beta \to 1, \;\; z_0 \to \infty$, given in (3.32), on the geometry (3.41). Thus for nonconformal D-branes with a wave we obtain the asymmetry ratio as

$$\mathcal{A}_{wave} \equiv \frac{\tilde{p}-1}{\tilde{p}+3}. \tag{3.52}$$

In conclusion, our results assign maximum entanglement entropy asymmetry to the wave like excitations in a zero temperature CFT. The results can be understood as we now elaborate. The wave like excitations in the CFT at zero temperature generate finite entanglement pressure along the direction of propagation of the wave, while the pressure remains vanishing in all other (transverse) directions. When we switch on finite temperature in the CFT (holographically including black hole in the bulk geometry) some entanglement pressure gets distributed along the transverse directions also. This finite temperature phenomenon reduces the net entanglement entropy asymmetry for the excitations. In the absence of a wave altogether the pressure becomes identical in all directions of the branes and hence entanglement asymmetry would also vanish. Hence the asymmetry in entanglement entropy will necessarily exist if there are uniform wave like excitations or a uniform flow in the CFT. The asymmetry only gets amplified as temperature goes to vanishing values.

3.5 Conclusion

It has been shown that the entanglement pressure plays a significant role in determining the entanglement entropy for the strip subsystems in the CFT living on the boundary of AdS_{d+1} spacetime. There is an entropy asymmetry along various directions of the CFT if their exists a pressure asymmetry. Besides the entropy asymmetry is directly proportional to the pressure asymmetry. To quantify this we have determined entanglement asymmetry ratio

$$\mathcal{A} \equiv \frac{\triangle S_{\perp} - \triangle S_{\parallel}}{\triangle S_{\perp} + \triangle S_{\parallel}} = \frac{\beta^2\gamma^2}{(2 + \frac{d+3}{d-1}\beta^2\gamma^2)} \leq \frac{d-1}{d+3} \tag{3.53}$$

which depends only on the boost parameter β and it is bounded. Interestingly the bound is saturated in the large boost limit ($\beta \to 1,\ \ z_0 \to \infty$) only (3.32). Thus a nonzero boost is simply a measure of the entanglement asymmetry. We have discussed a large boost case which is the AdS-wave case. Especially for the AdS waves there exist an optimum entanglement asymmetry

$$\mathcal{A}_{wave} = \frac{d-1}{d+3} \tag{3.54}$$

which is a universal result at the first order in perturbation analysis. It is independent of any scale such as energy of wave like excitations $\propto \frac{1}{z_I^d}$. We expect these results will get corrected by higher orders of perturbation.

In the nonconformal D-branes cases the result gets slightly modified

$$\mathcal{A}_{nonconf} = \frac{\beta^2\gamma^2}{(2+\frac{\tilde{p}+3}{\tilde{p}-1}\beta^2\gamma^2)} \leq \frac{\tilde{p}-1}{\tilde{p}+3}. \tag{3.55}$$

CHAPTER 4

GENERALIZED GEODESIC DEVIATION EQUATIONS AND ENTANGLEMENT FIRST LAW FOR ROTATING BTZ BLACK HOLES

4.1 Introduction

In the previous two chapters we adopted a perturbative approach to calculate holographic entanglement entropy. Using this approach we calculated the entanglement entropy for a strip like subsystem embedded in boosted AdS black brane background up to second order. In this chapter we adopt a variational approach to calculate change in entanglement entropy. For AdS_{d+1} the minimal surface γ_A are $(d-1)$ dimensional. The surface γ_A is an extremum of the area functional

$$Area = \int d^{d-1}\sigma\sqrt{h}, \tag{4.1}$$

where σ are the coordinates and h_{ab} is the induced metric on γ_A. Variation of the area functional depends both on metric perturbations and variation of the minimal surface itself. Change in HEE at each order can be obtained by subtracting the pure AdS contribution from the variation of the area functional. At first order, contributions from changes in the shape of the extremal surface does not appear as γ_A satisfies extremal condition on the background AdS geometry [33, 34, 90].

In this chapter we propose a way to calculate second order variations of the area functional by taking into account changes in both metric perturbation and shape of the extremal surface in $2 + 1$ dimension. This is achieved by studying geodesic deviations between geodesics in rotating BTZ black hole (seen as perturbation over pure AdS) and AdS_3. As will be clear from the construction these deviations can be obtained as solutions of a "generalized geodesic deviation equation" ([91] and references therein). Second order expressions for HEE obtained from variation of the area functional matches exactly with the second order expansion of HEE obtained by HRT proposal. We also present an alternative form of first law of entanglement thermodynamics which involves the differential change in ΔS ($d\Delta S$ for example) rather than ΔS itself. The modified first law includes contributions from angular momentum of the BTZ background and approaches the first law of black hole thermodynamics in large l (the subsystem size) limit.

It turns out that 2+1 dimensional gravity has no propagating degrees of freedom and therefore exact analytical expressions for certain quantities can be found. This is precisely the reason why 2+1 dimensional gravity can be written as Chern Simons theory. Chern Simons is topological and therefore the solutions depend only on the topology of the underlying manifold. From a geometric stand point the Weyl tensor identically vanishes in 3 dimensions and therefore the Riemann tensor is completely specified by the Ricci. As a consequence the solutions of pure gravity with negative cosmological constant are necessarily locally isometric to pure AdS_3. Despite this, there is still a rich set of solutions that differ from AdS globally. One such solution is the rotating BTZ black hole [92]. Since in the $2+1$ dimensional case exact expressions for the change in entanglement entropy and the minimal surface in BTZ is known, one might question the need of a perturbative analysis. But it turns out that the $2+1$ dimensional case is a perfect ground for checking such proposals. It is to be noted that for higher dimensional case the change in entanglement entropy might have to be calculated pertubatively and hence a precise prescription is required. We should point out that the notion of "deviations" of codimension two surfaces is well known for higher dimensions. Though algebraically difficult it is absolutely possible to find "generalized deviation equations" for codimension two surfaces for dimensions 4 or higher.

4.2 The generalized geodesic deviation equations

The generalized deviation equations have been known for quite some time, applications of which in the case of perturbed cosmological spacetimes can be found in [91] (and references therein) where perturbed null geodesics are studied for perturbations around Einstein-De Sitter universe. In the holographic context generalized deviations of null geodesics in AdS_3 has been used, only recently, in [93] however in a very different context from ours. In the holographic entanglement entropy context codimension two minimal surfaces in $2+1$ dimensions are spacelike geodesics. It is clear that the spacelike geodesics in AdS_3 anchored to the boundary subsystem are perturbed as one considers excitations over AdS_3. If we consider the variation of the area functional (HEE) $A(G, X^\mu) = \int \sqrt{\det h}\; d^n\sigma$, where $h_{ab} = g_{\mu\nu}\frac{\partial X^\mu}{\partial \sigma^a}\frac{\partial X^\nu}{\partial \sigma^b}$, the variation of the quantity is therefore,

$$A(G + \delta G, X^\mu + \delta X^\mu) - A(G, X^\mu), \tag{4.2}$$

where δX^μ is the change of the embedding functions. To first order δX^μ does not contribute. The δX^μ starts contributing only at second order. Therefore while considering second order variations, perturbations of spacelike geodesics also contribute. These changes in the embedding of the spacelike geodesics can be obtained by studying geodesic deviation between AdS_3 and the perturbed spacetime. To do this we use the following formulation.

4.2.1 First order generalized deviations

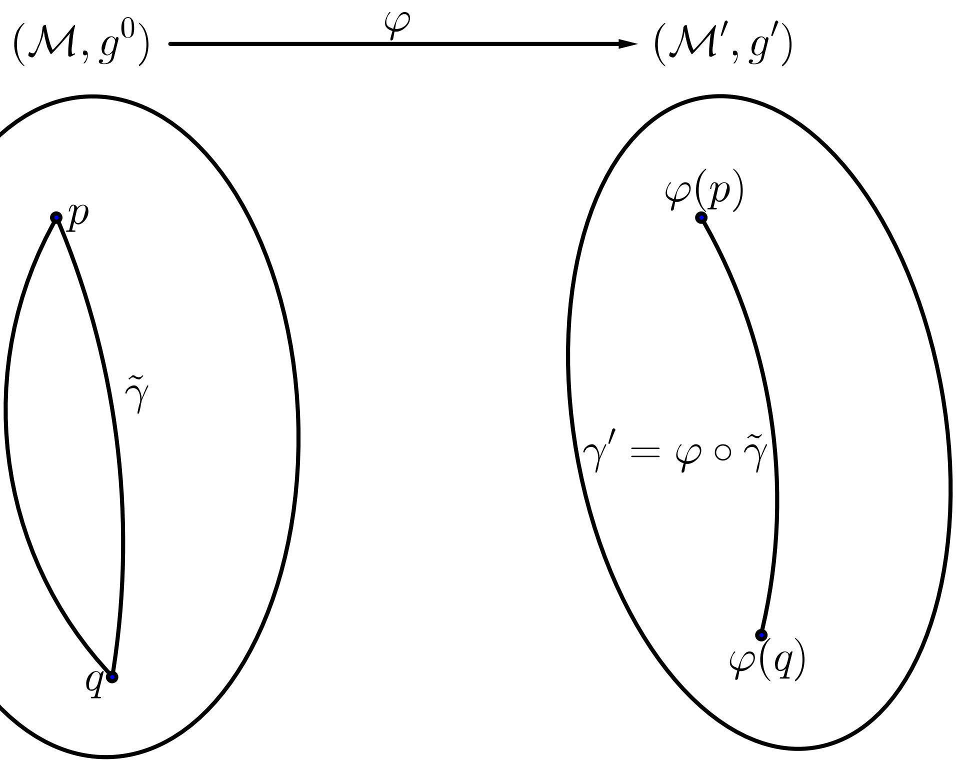
Consider an affinely parametrized geodesic parametrized by τ in a spacetime $(\mathcal{M}, \overset{0}{g})$ with end points $p, q \,\epsilon\, \mathcal{M}$.

$$\frac{d^2 x^\mu}{d\tau^2} + \overset{0}{\Gamma}{}^\mu_{\nu\rho}(x) \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0, \tag{4.3}$$

where $\overset{0}{\Gamma}{}^\mu_{\nu\rho}$ are the Christoffel symbols on $\mathcal{M}$ compatible with $\overset{0}{g}$. Consider another spacetime $(\mathcal{M}', g')$. $(\mathcal{M}', g')$ is said to be a perturbation over $(\mathcal{M}, \overset{0}{g})$ if there exists a diffeomorphism $\varphi : \mathcal{M} \to \mathcal{M}'$ such that $\varphi_* g' - \overset{0}{g} = \overset{(1)}{h}$ is a small perturbation over the unperturbed metric $\overset{0}{g}$. Let γ' be a geodesic in $\mathcal{M}'$ with parameter τ and end points $\varphi(p), \varphi(q) \,\epsilon\, \mathcal{M}'$. However it may not be affinely parametrized by τ. Let $\tilde{\gamma}$ be a curve in $\mathcal{M}$ such that $\varphi \circ \tilde{\gamma} = \gamma'$. Therefore the tangent vector to γ' in $\mathcal{M}'$ is essentially the push forward of the tangent vector of $\tilde{\gamma}$ in $\mathcal{M}$ (fig. 4.1). On $\mathcal{M}$, therefore $\tilde{\gamma}$ must satisfy,

$$\frac{d^2 \tilde{x}^\mu}{d\tau^2} + \tilde{\Gamma}^\mu_{\nu\rho}(\tilde{x}) \frac{d\tilde{x}^\nu}{d\tau} \frac{d\tilde{x}^\rho}{d\tau} = f(\tilde{x}) \frac{d\tilde{x}^\mu}{d\tau}, \tag{4.4}$$

Figure 4.1: The mapping of the geodesics



where $\tilde{\Gamma}$ are the Christoffels symbols on $\mathcal{M}$ compatible with $\varphi_* g'$. Note that $\tilde{\gamma}$ is not geodesic in $\mathcal{M}$ with respect to the initial Christoffels Γ. Let us assume that $\tilde{\gamma}$ is a small deviation about the curve γ. Therefore to first order we can write $\tilde{x}^\mu(\tau) = x^\mu(\tau) + \overset{(1)}{\eta}{}^\mu(\tau)$. We also note that to first order in metric perturbations,

$$\begin{aligned}\tilde{\Gamma}^\mu_{\nu\rho}(x) &= \overset{0}{\Gamma}{}^\mu_{\nu\rho}(x) + \frac{1}{2}\overset{0}{g}{}^{\mu\sigma}\left(\partial_\nu \overset{(1)}{h}_{\rho\sigma} + \partial_\rho \overset{(1)}{h}_{\nu\sigma} - \partial_\sigma \overset{(1)}{h}_{\nu\rho}\right) - \frac{1}{2}\overset{(1)}{h}{}^{\mu\sigma}\left(\partial_\nu \overset{0}{g}_{\rho\sigma} + \partial_\rho \overset{0}{g}_{\nu\sigma} - \partial_\sigma \overset{0}{g}_{\nu\rho}\right) \\ &= \overset{0}{\Gamma}{}^\mu_{\nu\rho}(x) + \overset{(1)}{C}{}^\mu_{\nu\rho}(x) \qquad (4.5)\end{aligned}$$

Therefore to first order,

$$\tilde{\Gamma}^{\mu}_{\nu\rho}(\tilde{x}) = \overset{0}{\Gamma}{}^{\mu}_{\nu\rho}(x) + \overset{(1)}{C}{}^{\mu}_{\nu\rho}(x) + \partial_\sigma \overset{0}{\Gamma}{}^{\mu}_{\nu\rho}(x)\overset{(1)}{\eta}{}^{\sigma} \tag{4.6}$$

Subtracting the two geodesic equations give,

$$\frac{d^2\overset{(1)}{\eta}{}^{\mu}}{d\tau^2} + \partial_\sigma \overset{0}{\Gamma}{}^{\mu}_{\nu\rho}\overset{(1)}{\eta}{}^{\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} + \overset{0}{\Gamma}{}^{\mu}_{\nu\rho}\frac{dx^\rho}{d\tau}\frac{d\overset{(1)}{\eta}{}^{\nu}}{d\tau} + \overset{0}{\Gamma}{}^{\mu}_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{d\overset{(1)}{\eta}{}^{\rho}}{d\tau} = -\overset{(1)}{C}{}^{\mu}_{\nu\rho}(x)\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} + \partial_\sigma f(x)\overset{(1)}{\eta}{}^{\sigma}\frac{dx^\mu}{d\tau} \tag{4.7}$$

and $f\Big|_\gamma = 0$, which essentially means that the initial curve is affinely parametrized. The left hand side can now be identified as just the left hand side of the Jacobi equation. Therefore,

$$\frac{\mathcal{D}^2\overset{(1)}{\eta}{}^{\mu}}{d\tau^2} + \mathcal{R}^{\mu}{}_{\nu\rho\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau}\overset{(1)}{\eta}{}^{\rho} = -\overset{(1)}{C}{}^{\mu}_{\nu\rho}(x)\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} + \partial_\sigma f(x)\overset{(1)}{\eta}{}^{\sigma}\frac{dx^\mu}{d\tau} = F^\mu + \partial_\sigma f(x)\overset{(1)}{\eta}{}^{\sigma}\frac{dx^\mu}{d\tau}$$

Where $\frac{D}{d\tau}$ is the covariant derivative along γ and $\mathcal{R}^{\mu}{}_{\nu\rho\sigma}$ is the Riemann tensor w.r.t $\overset{0}{g}$. Therefore the resulting equation is an inhomogeneous deviation equation. Note that if $\overset{(1)}{C}$ is set the zero the resulting equation is just the deviation equation for a non-affinely parametrized congruence of geodesics in a given space-time (no metric perturbations). To solve this equation the best procedure is to consider a local basis e^μ_1 which is parallely propagated along the initial geodesic and writing the deviation vector and the inhomogeneous terms, in terms of the local basis i.e $\eta^{(1)\mu} = \overset{(1)}{\eta}{}^{A} e^\mu_A$ and $F^\mu = F^A e^\mu_A$. For AdS_3 background which is maximally symmetric $\mathcal{R}_{\mu\nu\rho\lambda} = -(g_{\mu\rho}g_{\nu\lambda} - g_{\nu\rho}g_{\mu\lambda})$ and therefore for space-like geodesics, the equation reduces to,

$$\frac{d^2\overset{(1)}{\eta}{}^{A}}{d\tau^2} - \overset{(1)}{\eta}{}^{A} = F^A \qquad \text{for A=0,2} \tag{4.8}$$

$$\frac{d^2\overset{(1)}{\eta}{}^{A}}{d\tau^2} - \partial_B f(x)\overset{(1)}{\eta}{}^{B} = F^A \qquad \text{for A=1, B= 0 to 2,} \tag{4.9}$$

where $e_1^\mu = T^\mu$ is the tangent vector to the geodesic and $\partial_A = e_A^\mu \partial_\mu$. We have set the radius of the AdS space to "one" here and in all subsequent calculations. Note that the non affinity term enters only the equation for the component of the deviation vector in the direction of the tangent vector. The equations for $A = 0, 2$ can obviously be solved and the resulting solutions can be put in the equation for $A = 1$ to get a ordinary differential equation for η^1. However the equation for η^1 cannot be solved due to presence of the unknown function $f(x)$. But we will see that we actually won't be requiring a solution for η^1 for calculation of the variation of geodesic length. Note that unlike the original deviation equation where the component of the deviation vector in the direction of the geodesic can be set equal to zero, one may not be able to do the same here due to the inhomogeneous term. More precisely, the deviation along a geodesic is pure gauge and can be removed by a reparametrization of the geodesic. That is to say that the deviations along the geodesic does not affect the length of the perturbed geodesic only if the perturbed curve is a geodesic of the same space-time $(\mathcal{M}, g)$. However since in our case the perturbed curve is a geodesic in some perturbed space-time $(\mathcal{M}, g')$ these might actually become physical. But this does not happen i.e the terms containing η^1 still arise only as boundary terms evaluated at the end-points of the geodesic (section 4.3). So the only requirement is that η^1 vanishes at the endpoint of the geodesic. This provokes us to think that possibly by a different choice of gauge for the metric perturbations itself the in homogeneity F^1 can be removed. If considers the foliation induced by these geodesics, then F^1 is just the trace of the extrinsic curvature of these hypersurfaces. Therefore by choosing a maximal slicing condition (an appropriate gauge) one can actually remove this term. In this case however we are bound to work in Fefferman-Graham gauge which characterizes asymptotically AdS spacetimes, otherwise one might end up choosing a gauge that is not compatible with asymptotically AdS spacetimes. More precisely we might end up doing a gauge transformation of the metric which changes the boundary data (boundary metric or extrinsic curvature).

4.3 Variation of geodesic length

The calculation of the variation of the geodesic is prototypical of the case where there are no metric perturbations. However to our knowledge the extra terms arising due to metric perturbations have not been considered before. The action for a geodesic and the first variations is given by

$$\begin{aligned} S &= \int \underbrace{\sqrt{g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}}_{G}\, d\tau \\ \delta S &= \int \frac{1}{2\sqrt{G}}\left[2g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d\delta x^\nu}{d\tau} + \delta g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right] d\tau \end{aligned} \tag{4.10}$$

Note that G should not be confused with the gravitational constant. The geodesic equation then follows,

$$\begin{aligned} \delta S = \int \frac{1}{2\sqrt{G}} &\left[\frac{d}{d\tau}\left(2g_{\mu\nu}\frac{dx^\mu}{d\tau}\delta x^\nu\right) - 2\frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{dx^\mu}{d\tau}\frac{dx^\rho}{d\tau}\delta x^\nu \right. \\ &\left. -2g_{\mu\nu}\frac{d^2x^\mu}{d\tau^2}\delta x^\nu + \frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\rho + \tilde{\delta} g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right] d\tau \end{aligned} \tag{4.11}$$

On the back ground curve G can be set to one and therefore the geodesic equation follows if metric perturbation is zero i.e,

$$-2\frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{dx^\mu}{d\tau}\frac{dx^\rho}{d\tau} - 2g_{\mu\nu}\frac{d^2x^\mu}{d\tau^2} + \frac{\partial g_{\mu\rho}}{\partial x^\nu}\frac{dx^\mu}{d\tau}\frac{dx^\rho}{d\tau} = G_\nu = 0 \tag{4.12}$$

The second variation gives,

$$
\begin{aligned}
\delta^2 S \;=\; & \int \frac{1}{2\sqrt{G}} \left[\underbrace{2\delta g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d\delta x^\nu}{d\tau}}_{I} + \underbrace{2g_{\mu\nu}\frac{d\delta x^\mu}{d\tau}\frac{d\delta x^\nu}{d\tau}}_{II} + \underbrace{2g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d\delta^2 x^\nu}{d\tau}}_{III} \right. \\
& \left. + \underbrace{\delta^2 g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}_{IV} + \underbrace{2\delta g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d\delta x^\nu}{d\tau}}_{V} \right] d\tau \\
& - \int \frac{1}{4G^{3/2}} \underbrace{\left[2g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d\delta x^\nu}{d\tau} + \delta g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \right]^2}_{VI} d\tau
\end{aligned}
\tag{4.13}
$$

To evaluate term IV we note the following,

$$
\begin{aligned}
\delta^2 g_{\mu\nu} \;=\; & \delta\left(\frac{\partial g_{\mu\nu}}{\partial x^\rho}\delta x^\rho + \tilde{\delta} g_{\mu\nu} \right) \\
= & \underbrace{\frac{\partial^2 g_{\mu\nu}}{\partial x^\rho \partial x^\sigma}\delta x^\rho \delta x^\sigma}_{A} + 2\underbrace{\frac{\partial \tilde{\delta} g_{\mu\nu}}{\partial x^\rho}\delta x^\rho}_{B} + \underbrace{\frac{\partial g_{\mu\nu}}{\partial x^\rho}\delta^2 x^\rho}_{C} + \underbrace{\tilde{\delta}^2 g_{\mu\nu}}_{D}
\end{aligned}
$$

Note that IVC and term III together can be written as

$$
\begin{aligned}
\int \frac{1}{2\sqrt{G}} & \left[\frac{d}{d\tau}\left(2g_{\mu\nu}\frac{dx^\mu}{d\tau}\delta^2 x^\nu \right) - 2\frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{dx^\mu}{d\tau}\frac{dx^\rho}{d\tau}\delta^2 x^\nu \right. \\
& \left. - 2g_{\mu\nu}\frac{d^2 x^\mu}{d\tau^2}\delta^2 x^\nu + \frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta^2 x^\rho \right] d\tau
\end{aligned}
$$

which is nothing but,

$$
\int \frac{1}{2\sqrt{G}} \left[\frac{d}{d\tau}\left(2g_{\mu\nu}\frac{dx^\mu}{d\tau}\delta^2 x^\nu \right) + G_\nu \delta^2 x^\nu \right] d\tau
$$

Term VI is just the square of the first variation and therefore can be written as

$$\int \frac{1}{2\sqrt{G}} \left[\frac{d}{d\tau} \left(2g_{\mu\nu} \frac{dx^\mu}{d\tau} \delta x^\nu \right) + G_\nu \delta x^\nu + \tilde{\delta} g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right]^2 d\tau$$

Terms I and V together give

$$\begin{aligned}
&\int \frac{4}{2\sqrt{G}} \left[\frac{d}{d\tau} \left(\delta g_{\mu\nu} \frac{dx^\mu}{d\tau} \delta x^\nu \right) - \frac{d\delta g_{\mu\nu}}{d\tau} \frac{dx^\mu}{d\tau} \delta x^\nu - \delta g_{\mu\nu} \frac{d^2 x^\mu}{d\tau^2} \delta x^\nu \right] d\tau \\
&= \int \frac{4}{2\sqrt{G}} \left[\frac{d}{d\tau} \left(\delta g_{\mu\nu} \frac{dx^\mu}{d\tau} \delta x^\nu \right) - \frac{d}{d\tau} \left(\frac{\partial g_{\mu\nu}}{\partial x^\rho} \delta x^\rho + \tilde{\delta} g_{\mu\nu} \right) \frac{dx^\mu}{d\tau} \delta x^\nu - \delta g_{\mu\nu} \frac{d^2 x^\mu}{d\tau^2} \delta x^\nu \right] d\tau \\
&= \int \frac{4}{2\sqrt{G}} \left[\frac{d}{d\tau} \left(\delta g_{\mu\nu} \frac{dx^\mu}{d\tau} \delta x^\nu \right) - \left(\frac{\partial^2 g_{\mu\nu}}{\partial x^\rho \partial x^\sigma} \frac{dx^\sigma}{d\tau} \frac{dx^\mu}{d\tau} \delta x^\rho \delta x^\nu + \frac{\partial g_{\mu\nu}}{\partial x^\rho} \frac{d\delta x^\rho}{d\tau} \frac{dx^\mu}{d\tau} \delta x^\nu \right.\right. \\
&\left.\left. + \frac{\partial \tilde{\delta} g_{\mu\nu}}{\partial x^\rho} \frac{dx^\rho}{d\tau} \frac{dx^\mu}{d\tau} \delta x^\nu \right) - \left(\frac{\partial g_{\mu\nu}}{\partial x^\rho} \delta x^\rho + \tilde{\delta} g_{\mu\nu} \right) \frac{d^2 x^\mu}{d\tau^2} \delta x^\nu \right] d\tau \\
&= \int \frac{4}{2\sqrt{G}} \left[\frac{d}{d\tau} \left(\delta g_{\mu\nu} \frac{dx^\mu}{d\tau} \delta x^\nu \right) - \underbrace{\frac{\partial g_{\mu\nu}}{\partial x^\rho} \frac{d\delta x^\rho}{d\tau} \frac{dx^\mu}{d\tau} \delta x^\nu}_{F} - \underbrace{\frac{\partial^2 g_{\mu\nu}}{\partial x^\rho \partial x^\sigma} \frac{dx^\sigma}{d\tau} \frac{dx^\mu}{d\tau} \delta x^\rho \delta x^\nu}_{G} \right. \\
&\left. - \underbrace{\frac{\partial \tilde{\delta} g_{\mu\nu}}{\partial x^\rho} \frac{dx^\rho}{d\tau} \frac{dx^\mu}{d\tau} \delta x^\nu}_{H} - \underbrace{\frac{\partial g_{\mu\nu}}{\partial x^\rho} \frac{d^2 x^\mu}{d\tau^2} \delta x^\rho \delta x^\nu}_{J} - \tilde{\delta} g_{\mu\nu} \frac{d^2 x^\mu}{d\tau^2} \delta x^\nu \right] d\tau
\end{aligned} \tag{4.14}$$

Consider term II

$$\begin{aligned}
&\int \frac{1}{2\sqrt{G}} 2g_{\mu\nu} \frac{d\delta x^\mu}{d\tau} \frac{d\delta x^\nu}{d\tau} d\tau \\
&= \int \frac{1}{2\sqrt{G}} \left[\frac{d}{d\tau} \left(2g_{\mu\nu} \frac{d\delta x^\nu}{d\tau} \delta x^\mu \right) - 2 \underbrace{\frac{\partial g_{\mu\nu}}{\partial x^\rho} \frac{dx^\rho}{d\tau} \frac{d\delta x^\nu}{d\tau} \delta x^\mu}_{K} - \underbrace{2g_{\mu\nu} \frac{d^2 \delta x^\nu}{d\tau^2} \delta x^\mu}_{L} \right]
\end{aligned}$$

Leaving the total derivative terms the other terms A, F, G, J, K, L give,

$$\frac{1}{2\sqrt{G}}\underbrace{\left[\frac{\partial^2 g_{\mu\nu}}{\partial x^\rho \partial x^\sigma}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\rho \delta x^\sigma\right]}_{A} - \frac{4}{2\sqrt{G}}\left[\underbrace{\frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{d\delta x^\rho}{d\tau}\frac{dx^\mu}{d\tau}\delta x^\nu}_{F} + \underbrace{\frac{\partial^2 g_{\mu\nu}}{\partial x^\rho \partial x^\sigma}\frac{dx^\sigma}{d\tau}\frac{dx^\mu}{d\tau}\delta x^\rho \delta x^\nu}_{G}\right.$$
$$\left. + \underbrace{\frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{d^2 x^\mu}{d\tau^2}\delta x^\rho \delta x^\nu}_{J}\right] - \frac{1}{2\sqrt{G}}\left[2\underbrace{\frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{dx^\rho}{d\tau}\frac{d\delta x^\nu}{d\tau}\delta x^\mu}_{K} + \underbrace{2g_{\mu\nu}\frac{d^2\delta x^\nu}{d\tau^2}\delta x^\mu}_{L}\right] \tag{4.15}$$

which can be written as,

$$\frac{1}{2\sqrt{G}}\underbrace{\left[\frac{\partial^2 g_{\mu\nu}}{\partial x^\rho \partial x^\sigma}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\rho \delta x^\sigma\right]}_{A} - \frac{1}{2\sqrt{G}}\left[2\underbrace{\frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{dx^\rho}{d\tau}\frac{d\delta x^\nu}{d\tau}\delta x^\mu}_{K} + \underbrace{2g_{\mu\nu}\frac{d^2\delta x^\nu}{d\tau^2}\delta x^\mu}_{L}\right]$$
$$-\frac{2}{2\sqrt{G}}\left[\underbrace{\frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{d\delta x^\rho}{d\tau}\frac{dx^\mu}{d\tau}\delta x^\nu}_{F} + \underbrace{\frac{\partial^2 g_{\mu\nu}}{\partial x^\rho \partial x^\sigma}\frac{dx^\sigma}{d\tau}\frac{dx^\mu}{d\tau}\delta x^\rho \delta x^\nu}_{G} + \underbrace{\frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{d^2 x^\mu}{d\tau^2}\delta x^\rho \delta x^\nu}_{J}\right]$$
$$-\frac{2}{2\sqrt{G}}\left[\frac{d}{d\tau}\left(\frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{dx^\mu}{d\tau}\delta x^\rho\right)\right] - \frac{2}{2\sqrt{G}}\left[-\underbrace{\frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{d\delta x^\nu}{d\tau}\frac{dx^\mu}{d\tau}\delta x^\rho}_{F'}\right]$$

Leaving the total derivative one can write this term as,

$$\frac{1}{2\sqrt{G}}\left[\underbrace{\frac{\partial^2 g_{\mu\nu}}{\partial x^\rho \partial x^\sigma}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\rho \delta x^\sigma}_{A} - 2\underbrace{\frac{\partial^2 g_{\mu\nu}}{\partial x^\rho \partial x^\sigma}\frac{dx^\sigma}{d\tau}\frac{dx^\mu}{d\tau}\delta x^\rho \delta x^\nu}_{G} - 2\underbrace{\frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{d^2 x^\mu}{d\tau^2}\delta x^\rho \delta x^\nu}_{J}\right]$$
$$-\frac{1}{2\sqrt{G}}\left[2\underbrace{\frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{dx^\rho}{d\tau}\frac{d\delta x^\nu}{d\tau}\delta x^\mu}_{K} + 2\underbrace{\frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{d\delta x^\rho}{d\tau}\frac{dx^\mu}{d\tau}\delta x^\nu}_{F} - 2\underbrace{\frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{d\delta x^\nu}{d\tau}\frac{dx^\rho}{d\tau}\delta x^\nu}_{F'} + \underbrace{2g_{\mu\nu}\frac{d^2\delta x^\nu}{d\tau^2}\delta x^\mu}_{L}\right]$$

$$= \frac{1}{2\sqrt{G}}\left[\underbrace{\frac{\partial^2 g_{\mu\nu}}{\partial x^\rho \partial x^\sigma}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\rho \delta x^\sigma}_{A} - \underbrace{\frac{\partial^2 g_{\mu\sigma}}{\partial x^\rho \partial x^\nu}\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau}\delta x^\rho \delta x^\sigma}_{G} - \underbrace{\frac{\partial^2 g_{\nu\sigma}}{\partial x^\rho \partial x^\mu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\rho \delta x^\sigma}_{G}\right.$$

$$\left. +2\underbrace{\frac{\partial g_{\alpha\sigma}}{\partial x^\rho}\Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\rho \delta x^\sigma}_{J}\right]$$

$$-\frac{1}{2\sqrt{G}}\left[2\underbrace{\frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{dx^\rho}{d\tau}\frac{d\delta x^\nu}{d\tau}\delta x^\mu}_{K} +2\underbrace{\frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{d\delta x^\rho}{d\tau}\frac{dx^\mu}{d\tau}\delta x^\nu}_{F} -2\underbrace{\frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{d\delta x^\nu}{d\tau}\frac{dx^\rho}{d\tau}\delta x^\nu}_{F'} + \underbrace{2 g_{\mu\nu}\frac{d^2\delta x^\nu}{d\tau^2}\delta x^\mu}_{L}\right]$$

$$= \frac{1}{2\sqrt{G}}\left[-2g_{\alpha\sigma}\frac{\partial \Gamma^\alpha_{\mu\nu}}{\partial x^\rho}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\rho \delta x^\sigma - 2g_{\alpha\sigma}\frac{d^2\delta x^\alpha}{d\tau^2}\delta x^\sigma - 2g_{\alpha\sigma}\Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d\delta x^\nu}{d\tau}\delta x^\sigma \right.$$

$$\left. -2g_{\alpha\sigma}\Gamma^\alpha_{\mu\nu}\frac{d\delta x^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\sigma\right] = \frac{1}{\sqrt{G}}C^\alpha_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}g_{\alpha\sigma}\delta x^\sigma$$

The $\tilde{\delta} g_{\mu\nu}$ terms up to total derivatives give,

$$\frac{1}{\sqrt{G}}\left[\frac{\partial \tilde{\delta} g_{\mu\nu}}{\partial x^\rho}\delta x^\rho \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} - 2\frac{\partial \tilde{\delta} g_{\mu\nu}}{\partial x^\rho}\frac{dx^\rho}{d\tau}\frac{dx^\mu}{d\tau}\delta x^\nu - 2\tilde{\delta} g_{\mu\nu}\frac{d^2 x^\mu}{d\tau^2}\delta x^\nu + \frac{1}{2}\tilde{\delta}^2 g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right]$$

$$= \frac{1}{\sqrt{G}}\left[-2C^\alpha_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + \frac{1}{2}\tilde{\delta}^2 g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right]$$

The total variation is then given by,

$$\delta S = \int \frac{1}{2}\left[\frac{d}{d\tau}\left(2g_{\mu\nu}\frac{dx^\mu}{d\tau}\delta x^\nu\right) + \tilde{\delta} g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right] d\tau \tag{4.16}$$

$$\begin{aligned}\delta^2 S = &\int \frac{1}{2}\left[\frac{d}{d\tau}\left(2g_{\mu\nu}\frac{d\delta x^\nu}{d\tau}\delta x^\mu\right)\right] - \int\left[\frac{d}{d\tau}\left(\frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{dx^\mu}{d\tau}\delta x^\rho\right)\right] + \int 2\left[\frac{d}{d\tau}\left(\delta g_{\mu\nu}\frac{dx^\mu}{d\tau}\delta x^\nu\right)\right] \\ &+ \int \frac{1}{2}\left[\frac{d}{d\tau}\left(2g_{\mu\nu}\frac{dx^\mu}{d\tau}\delta^2 x^\nu\right)\right] d\tau - \int \frac{1}{4}\left[\frac{d}{d\tau}\left(2g_{\mu\nu}\frac{dx^\mu}{d\tau}\delta x^\nu\right) + \tilde{\delta} g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right]^2 d\tau \\ &+ \int\left[-C^\alpha_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}g_{\alpha\sigma}\delta x^\sigma + \frac{1}{2}\tilde{\delta}^2 g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right] d\tau\end{aligned} \tag{4.17}$$

The final expression obtained is given by,

$$\delta S = \int \frac{1}{2}\left[\overset{(1)}{h}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right] d\tau \tag{4.18}$$

$$\begin{aligned}\delta^2 S = &- \int \frac{1}{4}\left[\frac{d}{d\tau}\left(2g_{\mu\nu}\frac{dx^\mu}{d\tau}\overset{(1)}{\eta}_\mu\right) + \overset{(1)}{h}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right]^2 d\tau \\ &+ \int \left[\underbrace{-\overset{(1)}{C}{}^\alpha_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\overset{(1)}{\eta}_\alpha}_{A} - \partial_\sigma f(x)\overset{(1)}{\eta}{}^\sigma \eta^1 + \frac{1}{2}\overset{(2)}{h}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right] d\tau\end{aligned} \tag{4.19}$$

As a final step we see that the second order variation is independent of η^1. Note that the square term in the above expression can be expanded to get two $\overset{(1)}{\eta}{}^1$ dependent terms viz.,

$$-\int \left[\left(\frac{d\overset{(1)}{\eta}{}^1}{d\tau}\right)^2 + \frac{d\overset{(1)}{\eta}{}^1}{d\tau}\overset{(1)}{h}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right] \tag{4.20}$$

The above expression can be integrated by parts. Leaving a total derivative it gives $-F_1\overset{(1)}{\eta}{}^1 +$ $\partial_\sigma f(x)\overset{(1)}{\eta}{}^\sigma\overset{(1)}{\eta}{}^1$ which essentially cancels the $F_1\overset{(1)}{\eta}{}^1$ coming from term A and $-\partial_\sigma f(x)\overset{(1)}{\eta}{}^\sigma\overset{(1)}{\eta}{}^1$ term in (4.19). The final expression for $\delta^2 S$ is given by,

$$\delta^2 S = -\int \frac{1}{4}\left[\overset{(1)}{h}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right]^2 d\tau + \int \left[F^A\overset{(1)}{\eta}_A + \frac{1}{2}\overset{(2)}{h}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right] d\tau, \tag{4.21}$$

where A is summed over 0 and 2.

4.4 Solutions of “generalized deviation equation” for rotating BTZ like perturbations about AdS_3

Consider the AdS_3 metric

$$ds^2 = \frac{dz^2 - dt^2 + dx^2}{z^2} \tag{4.22}$$

The equation for a spacelike geodesic of maximal length, parametrized by τ is then given by

$$\frac{dx}{d\tau} = \frac{z^2}{z_*^{(0)}} \tag{4.23}$$

$$\frac{dz}{d\tau} = \pm z\sqrt{1 - \left(\frac{z}{z_*^{(0)}}\right)^2}, \tag{4.24}$$

where $z_*^{(0)}$ is the AdS turning point. Besides the plus sign denotes the half going into the bulk and the minus sign denotes the half of the geodesic approaching the boundary. From (4.24) we can obtain the size of the subsystem in terms of the AdS turning point

$$l = 2z_*^{(0)} \int_0^1 dk \frac{k}{\sqrt{1-k^2}} = 2z_*^{(0)} \tag{4.25}$$

In order to calculate the integrals in the previous section we need both halves of the geodesic. Though both halves of the geodesic are identical, except for a change in sign of the velocity, the deviations may undergo non-trivial changes. To account for this we continue the solution of the ingoing half of (4.24) to negative values of the affine parameter setting it equal to zero at the turning point. Hence the parameter $\tau \, \epsilon \, \left(-\infty, \infty\right)$ now covers the full geodesic. Therefore the full curve is now a map $\gamma : (-\infty, \infty) \to \mathcal{M}$. The solution given by:

$$z(\tau) = z_*^{(0)} \operatorname{sech}(\tau), \tag{4.26}$$

where we have fixed the constant of integration in such a way that $\tau = 0$ at the turning point. This solution can be substituted in (4.23) to get a solution $x(\tau)$.

The components of $C^{\mu}{}_{\nu\rho}$ can be obtained with the expressions for $h^{(1)}_{\mu\nu}$ given in Appendix D. Note that,

$$\overset{(1)}{h}_{\mu\nu} = \begin{bmatrix} \frac{(r_+^2+r_-^2)}{2} & 0 & -r_+r_- \\ 0 & 0 & 0 \\ -r_+r_- & 0 & \frac{(r_+^2+r_-^2)}{2} \end{bmatrix}$$

Denoting $a = \frac{(r_+^2+r_-^2)}{2}$ and $b = -r_+r_-$.

$$\begin{aligned} \overset{(1)}{C}{}^{t}_{z\ t} &= -z\ a,\ \overset{(1)}{C}{}^{x}_{z\ t} = z\ b,\ \overset{(1)}{C}{}^{t}_{t\ z} = -z\ a,\ \overset{(1)}{C}{}^{x}_{t\ z} = z\ b \\ \overset{(1)}{C}{}^{t}_{x\ z} &= -z\ b,\ \overset{(1)}{C}{}^{x}_{x\ z} = z\ a,\ C^{t}{}_{z\ x} = -z\ b,\ \overset{(1)}{C}{}^{x}{}_{z\ x} = z\ a \end{aligned} \tag{4.27}$$

The tetrads that are parallely propagated along the geodesic are given by,

$$e_0^{\mu} = (z,0,0),\ \ e_1^{\mu} = \left(0, \pm z\sqrt{1-\left(\frac{z}{z_*^{(0)}}\right)^2}, \frac{z^2}{z_*^{(0)}}\right),\ \ e_2^{\mu} = \left(0, \frac{z^2}{z_*^{(0)}}, \mp z\sqrt{1-\left(\frac{z}{z_*^{(0)}}\right)^2}\right) \tag{4.28}$$

We therefore only need to solve the first two of the generalized geodesic equation. The first equation can be recast as ,

$$(\eta^0)'' - \eta^0 - \frac{2}{z_*^{(0)}}\, b\, z^2\, z' = 0 \tag{4.29}$$

We have removed the (1) superscript in $\overset{(1)}{\eta}$ in this section. All η's in this section correspond to first order deviation vector. A general solution of this equation is given by:

$$\eta^0 = C_1 e^{\tau} + C_2 e^{-\tau} + \frac{2 z_*^{(0)^2} b e^{-\tau}(-1 - 2e^{2\tau} + 2e^{4\tau} + e^{6\tau})}{3(1 + e^{2\tau})^2} \tag{4.30}$$

To deal with the pathological nature of the coordinates at $z = 0$, we will put the boundary conditions $\eta^0(p) = \eta^0(-p) = 0$ for some cutoff p and take $p \to \infty$ in the integrals. With C_1, C_2 fixed in terms of p the final solution becomes,

$$\eta^0 = \frac{b}{3}[-\operatorname{sech}(p)^2 \sinh(\tau) + \operatorname{sech}(\tau)\tanh(\tau)] z_*^{(0)^2} \tag{4.31}$$

The equation for η^2 is,

$$(\eta^2)'' - \eta^2 - \frac{2}{z_*^{(0)}}\, a\, z\, z'^2 = 0. \tag{4.32}$$

Similarly as stated above the complete solution with proper boundary conditions is,

$$\eta^2 = \frac{8 z_*^{(0)^2}\ a\ e^{4p+3\tau}[-(3 + \cosh 4p)\cosh 2\tau + \cosh 2p\,(3 + \cosh 4\tau)]}{3(1 + e^{2p})^4 (1 + e^{2\tau})^3} \tag{4.33}$$

To calculate the integrals in (4.18) and (4.21) we need an expression for $\overset{(2)}{h}_{\mu\nu}$ which is obtained from $F - G$ expansion in (D).

$$\frac{\overset{(2)}{h}_{\mu\nu}}{2} = z^2 \begin{bmatrix} \frac{-(r_-^2-r_+^2)^2}{16} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{(r_-^2-r_+^2)^2}{16} \end{bmatrix}$$

and then taking the $p \to \infty$ gives,

$$\delta S = \frac{2{z_*^{(0)}}^2 a}{3} = \frac{l^2(r_+^2 + r_-^2)}{48G} \tag{4.34}$$

$$\begin{aligned} \delta^2 S &= {z_*^{(0)}}^4 \left(-\frac{1}{4}\left(\frac{32a^2}{35}\right) + \left(\frac{8b^2}{45} - \frac{8a^2}{63} - \frac{4(b^2-a^2)}{15}\right)\right) = -\frac{4{z_*^{(0)}}^4(b^2+a^2)}{45} \\ &= -\frac{l^4}{720G}\left(\frac{(r_+^2+r_-^2)^2 + 4r_+^2 r_-^2}{4}\right) \end{aligned} \tag{4.35}$$

Therefore the total change in entanglement entropy upto second order is,

$$\begin{aligned} \Delta S_E =& \frac{1}{4G}\left[\delta S + \frac{1}{2}\delta^2 S\right] \\ =& \frac{l^2(r_+^2+r_-^2)}{48G} - \frac{l^4}{1440G}\left(\frac{(r_+^2+r_-^2)^2 + 4r_+^2 r_-^2}{4}\right). \end{aligned} \tag{4.36}$$

In the next section we will verify this expression by deriving it from the expression of HEE for rotating BTZ obtained by HRT proposal.

Note: It is important to note that when the perturbed metric is static there will be no off diagonal h_{tx} like terms in the perturbation, and therefore the time component of the deviation vector will be trivial. For example in the non rotating BTZ case $r_- \to 0, r_+ \to M$ i.e $b = 0$. Hence for non 'rotating BTZ' like perturbations the time component of the deviation vector is zero. Hence the perturbed curve is still on a $t = constant$ slice. Thus the information regarding different proposals (viz RT and HRT) is already incorporated in the deviation vector and the perturbed metric. Hence both can be addressed using this construction.

4.5 Entanglement First Law

It has been shown in [28] that the change in entanglement entropy (δS_A) of a subregion A, under a small perturbations of the density matrix $\rho = \rho^0 + \delta\rho$ of a pure state in QFT, satisfies a local first law of entanglement thermodynamics viz

$$\delta S_A(x) = \beta_0 \delta E_A. \tag{4.37}$$

Where $\delta E_A = \delta < \hat{T}_{00} > Vol(A)$ is the excitation energy and $\beta_0 = \frac{\int_A \beta(x)}{Vol(A)}$ is the average inverse entanglement temperature inside A. The density matrix for a mixed state at finite temperature T and conserved charge Q_a and chemical potential μ_a has the following form

$$\rho = \frac{\exp\left(-\frac{(H-\mu_a Q_a)}{T}\right)}{Z}$$

The first law gets modified to

$$\delta S_A(x) = \beta_0 \left(\delta E_A - \mu_a \delta Q_{aA}\right). \tag{4.38}$$

Where $\delta Q_{aA} = \delta < Q_a > Vol(A)$. For rotating BTZ background corresponding density matrix is given by

$$\rho = \frac{exp - \beta \left(H - \Omega J\right)}{Z}. \tag{4.39}$$

Where H and J are the Hamiltonian and angular momentum of the CFT and Ω is the angular velocity (which is essentially a chemical potential for the conserved angular momentum). According to the above treatment a similar first law expression should also hold with μ_a replaced with Ω_A and Q_A replaced with J_A.

Now we derive the entanglement first law for rotating BTZ geometry using the expression for HEE. Holographic entanglement entropy for rotating BTZ geometry is given by [8]. A separate calculation for HEE in terms of geodesic length can be found in Appendix D.1

$$S_E = \frac{c}{6}\ln\left(\frac{\beta_+\beta_-}{\pi^2\epsilon^2}\sinh\left(\frac{\pi l}{\beta_+}\right)\sinh\left(\frac{\pi l}{\beta_-}\right)\right) \tag{4.40}$$

where $\beta_\pm = \frac{2\pi}{r_+ \pm r_-}$ are the inverse temperature for left and right moving modes and l is the size of the subsystem in the dual CFT_2. ϵ is the UV cutoff and $c = \frac{3}{2G}$ is the central charge of the dual CFT_2 and G is the 3 dimensional Newton's constant. The increase in HEE of a subsystem of size l is obtained by subtracting it from pure AdS_3 contribution given by

$$S_{AdS_3} = \frac{c}{3}\ln\left(\frac{l}{\epsilon}\right). \tag{4.41}$$

For rotating BTZ geometry the increase in HEE of a subsystem of size l is given by

$$\begin{aligned}\Delta S_E &= S_E - S_{AdS_3} \\ &= \frac{c}{6}\ln\left(\frac{\beta_+\beta_-}{\pi^2 l^2}\sinh\left(\frac{\pi l}{\beta_+}\right)\sinh\left(\frac{\pi l}{\beta_-}\right)\right).\end{aligned} \tag{4.42}$$

The physical thermodynamic observables of the dual CFT_2 can be obtained by expanding the rotating BTZ geometry in suitable Fefferman-Graham (asymptotic) coordinates near the AdS boundary [24], given in the appendix D. These are summarized here. The energy and angular momentum for the strip subsystem are

$$\Delta E = \frac{1}{8\pi G} l \frac{(r_+^2 + r_-^2)}{2} = \frac{\pi l}{8G}\left(\frac{1}{\beta_+^2} + \frac{1}{\beta_-^2}\right)$$

$$\Delta J = \frac{1}{8\pi G} l(r_+ r_-) = \frac{\pi l}{8G}\left(\frac{1}{\beta_-^2} - \frac{1}{\beta_+^2}\right). \tag{4.43}$$

The entanglement temperature has been defined in [23]. By the same argument one should be able to define an entanglement angular velocity.

$$\frac{1}{T_E} = \frac{\partial(\Delta S_E)}{\partial(\Delta E)}\bigg|_{l=fixed}, \qquad \frac{\Omega_E}{T_E} = \frac{\partial(\Delta S_E)}{\partial(\Delta J)}\bigg|_{l=fixed} \tag{4.44}$$

Using eq. 4.42, 4.43 we get

$$\frac{1}{T_E} = -\frac{(\beta_+^2 + \beta_-^2) - l\pi(\beta_+ \coth\frac{\pi l}{\beta_+} + \beta_- \coth\frac{\pi l}{\beta_-})}{2l\pi} \tag{4.45}$$

$$\frac{\Omega_E}{T_E} = \frac{\beta_+^2 - \beta_-^2 - l\pi(\beta_+ \coth\frac{\pi l}{\beta_+} - l\pi\beta_- \coth\frac{\pi l}{\beta_-})}{2l\pi} \tag{4.46}$$

Using these definition it is quite logical to write an alternative form of the first involving differential changes in ΔS. This first law is valid upto all orders in subsystem size.

$$d(\Delta S_E) = \frac{1}{T_E} d(\Delta E) - \frac{\Omega_E}{T_E} d(\Delta J) \tag{4.47}$$

In the above form ΔS must be interpreted as subtracting a ground state entropy (pure AdS)from the entropy of the excited state. The differential changes $d\Delta S$ are changes due to changes of the excited state itself. Hence the first law relates the change in ΔS due to changes in the BTZ parameters.

In the case of black hole thermodynamics the change in the entropy of a black hole is related to changes in the black hole parameters as one moves from one black hole solution to another in the phase space of solutions. Hence the above first law is closer in spirit to the first law for black hole thermodynamics. In fact we further show that it is this first law that asymptotes to the first law for the BTZ black hole in the large system size limit. As the derivatives don't act on l, (4.47) can be written in terms of mass $(M = \frac{(r_+^2+r_-^2)}{8G})$ and angular momentum $(J = \frac{r_+r_-}{4G})$ of rotating BTZ black hole as follows

$$d(\frac{2G\pi\Delta S_E}{l}) = \frac{1}{T_E}d(M) - \frac{\Omega_E}{T_E}d(J) \tag{4.48}$$

Taking the large subsystem size l limit these quantities approaches their respective thermal values

$$\lim_{l\longrightarrow\infty}\frac{2\pi\Delta S_E}{l} = \frac{\pi^2}{2G}\left(\frac{1}{\beta_+}+\frac{1}{\beta_-}\right) = \frac{\pi r_+}{2G} \tag{4.49}$$

$$\lim_{l\longrightarrow\infty} T_E = \frac{2}{(\beta_+ + \beta_-)} = \frac{r_+^2 - r_-^2}{2\pi r_+} \tag{4.50}$$

$$\lim_{l\longrightarrow\infty} \Omega_E = \frac{(\beta_+ - \beta_-)}{(\beta_+ + \beta_-)} = \frac{r_-}{r_+} \tag{4.51}$$

. Thus (4.48) approaches the first law of black hole thermodynamics in the large subsystem size (l) limit. It is important to note that we have taken a planar limit of the BTZ black hole geometry. Hence, we are calculating entropy density above as entropy is divergent in the planar case.

In the small subsystem size limit $\frac{\pi l}{\beta_\pm} \ll 1$, we can expand entanglement temperature and angular velocity up to second order in $\frac{\pi l}{\beta_\pm}$ using (4.45)

$$\frac{1}{T_E} = \frac{\pi l}{3} - \frac{\pi l^3}{90}\left(\frac{1}{\beta_+^2} + \frac{1}{\beta_-^2}\right) + \cdots \tag{4.52}$$

$$\Omega_E = \frac{\pi^2 l^2}{15}\left(\frac{1}{\beta_-^2} - \frac{1}{\beta_+^2}\right) + \cdots . \tag{4.53}$$

Thus at leading order the entanglement temperature is inversely proportional to the subsystem size. The entanglement angular velocity at leading order is proportional to $(\frac{\pi l}{\beta_\pm})^2$ these contributions to the change in HEE appears only at second order and are due to second order gravitational perturbation and first order perturbations of the extremal surface. Dependence of entanglement temperature and angular velocity on subsystem size is given in fig(2) and fig(3). Note that perturbation of the entanglement temperature in CFT has been discussed in [94] for example.

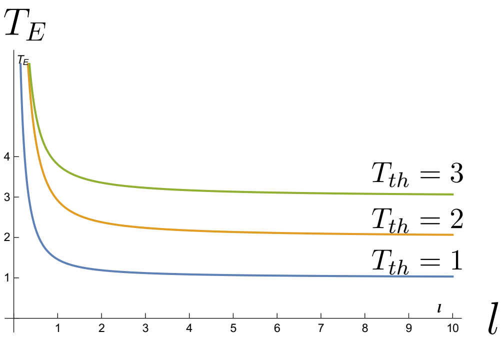


Figure 4.2: *Plot of T_E v.s 'l' for different black hole temperatures. As 'l' increases the temperature asymptotes to thermal value*

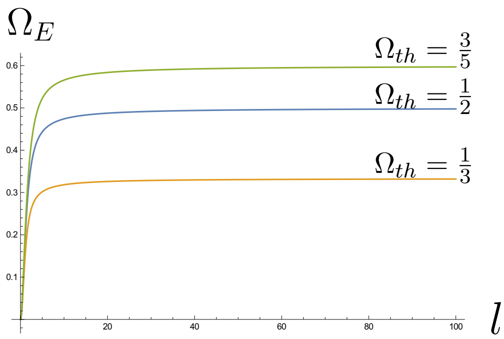


Figure 4.3: *Plot of Ω_E v.s 'l' for different blackhole angular velocity. As 'l' increases the angular velocity asymptotes to thermal value*

Similarly we can expand (4.42) in the small subsystem size limit $\frac{\pi l}{\beta_\pm} \ll 1$

$$\Delta S_E = \Delta S_1 + \Delta S_2 = \frac{l^2(r_+^2 + r_-^2)}{48G} - \frac{l^4}{1440G}\left(\frac{(r_+^2 + r_-^2)^2 + 4r_+^2 r_-^2}{4}\right) \tag{4.54}$$

It is important to note that this expression exactly matches with equation (4.36) reproduced earlier by studying geodesic deviations.It is important to note that although full expression for ΔS is known for rotating BTZ geometry, this is not the case for other backgrounds in higher dimensions. In those cases expression for ΔS is obtained perturbatively. In (4.4) we gave a prescription in $2+1$ to calculate ΔS by accounting for first order changes in the minimal surface and second order gravitational perturbations. Here we verify our result for rotating BTZ case. Further in the large l limit the ratio $\frac{2\pi\Delta S}{l}$ approaches the Bekenstein Hawking entropy as shown in fig(4).

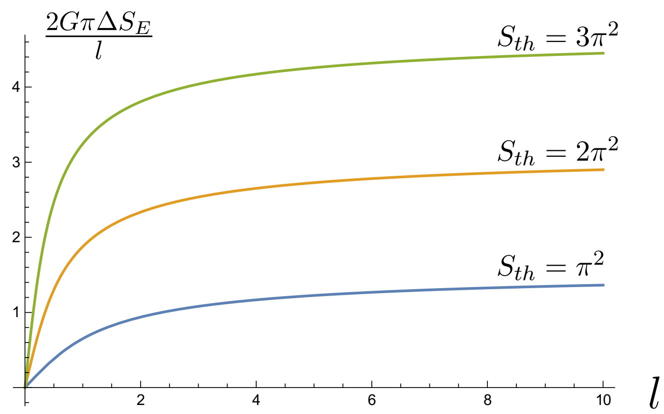


Figure 4.4: *Plot of* $\frac{2G\pi\Delta S_E}{l}$ *vs* l *for different Bekenstein Hawking entropy. As* $'l'$ *increases* $\frac{2G\pi\Delta S_E}{l}$ *asymptotes to thermal value*

Now at first order from (4.52) and (4.54) we can write the first law as

$$d(\Delta S_1) = \frac{1}{T_E^{(1)}} d(\Delta E). \tag{4.55}$$

Where $T_E^{(1)} = \frac{3}{\pi l}$ is the entanglement temperature at first order. Thus at first order (4.55) can be integrated to give

$$\Delta S_1 = \frac{1}{T_E^{(1)}} \Delta E. \tag{4.56}$$

Which is the entanglement first law obtained in [23, 95]. However at second order (4.47) can not be integrated as T_E also depends on details of excitation. Hence at second order one can at most write an inequality

$$\Delta S_E < \frac{1}{T_E}\Delta E - \frac{\Omega_E}{T_E}\Delta J. \tag{4.57}$$

It will be interesting to check whether if expression (4.57) resembles in spirit to the Bekenstein bound for rotating bodies [96–98], or the Penrose inequality for axis symmetric spacetimes [99, 100]. The QFT analogue of the Bekenstein bound for non rotating bodies was holographically verified in [27].

4.6 Discussions

It has been shown that one can calculate the covariant change in HEE in $2+1$ dimensions by calculating the metric perturbation up to second order and solving for surface deviations up to first order. Having obtained a covariant expression for the change in entanglement entropy up to second order in the perturbation series, it will be interesting to check what constraints Einstein's equation (second order linearized Einstein's equation to be precise) puts on the dynamics of ΔS as done in [33, 34, 101]. Moreover one may attempt to follow the procedure outlined in this chapter for time dependent perturbations over AdS_3, the CFT calculation of which has been per- formed in [102].

In the next chapter we will generalize this to higher dimensions. This would provide one with a definite prescription for calculation of change in entanglement entropy in higher dimension using a variational approach.

CHAPTER 5

AN INHOMOGENEOUS JACOBI EQUATION FOR MINIMAL SURFACES AND PERTURBATIVE CHANGE IN HOLOGRAPHIC ENTANGLEMENT ENTROPY

5.1 Introduction

In this chapter we intend to generalize to higher dimensions, the covariant approach to calculate change in holographic entanglement entropy presented in the previous chapter. For static geometries the timelike Killing vector (∂_t say) is hypersurface orthogonal in the bulk geometry. It can then be shown that the extremal surface must lie on $t = constant$

slice and can be shown to be minimal. Hence the proposal reduces to finding a minimal surface on a constant time slice. The proposal, initially put forward by RT was precisely this. However for non static cases, where the timelike killing vector is not hypersurface orthogonal, or for dynamical geometries, where there is no time like Killing vector, γ_A is no more minimal, and therefore RT proposal fails and one has to resort to the more general HRT proposal. *(In terms of nomenclature, in the mathematics literature, a minimal surface refers to just the critical point of the area functional and may not correspond to the minimum of the functional [103]. This is particularly the case in manifolds endowed with a Semi-Riemannian metric. We will stick to the latter nomenclature and use extremal and minimal interchangeably. Hence when we say minimal surfaces we actually mean extremal surfaces of HRT)* The equation obtained by extremizing the functional turns out to be nothing but the condition that the trace of the extrinsic curvature of the surface vanishes. The condition however yields non linear equations of motion for the embedding functions. It therefore becomes difficult to solve these equations unless the back ground geometry is highly symmetric. Consequently, though these equations for the embedding function can be obtained exactly for AdS it becomes difficult to solve them exactly even for backgrounds like the boosted black brane or the Kerr-AdS. One therefore considers doing a perturbation by treating these backgrounds as perturbations over AdS, near the asymptotic boundary. This imminently yields linear equations as the procedure involves a linearization of the minimal surface equation.

The change in HEE between AdS and excitations over it can then be calculated by considering variation of the area functional which incorporates the changes due to the change in the extremal surface γ_A and the perturbation of the bulk metric. At first order contributions only come from metric perturbations alone, while the change of the embedding of the extremal surface does not [33, 34, 90]. However at second order both first order change in the embeddings and second order metric perturbations contribute [29, 104–107]. In the chapter (4) we proposed a way to calculate the contributions to second order variations coming from the changes in the embedding, in $2+1$ dimensions. This was achieved by studying geodesic deviations between geodesics in rotating BTZ black hole (seen as perturbation over pure AdS) and pure AdS_3. These deviations were obtained as solutions of a "generalized geodesic deviation equation". In this chapter we shall generalize this to arbitrary dimensions. In order to do so one has to reproduce the above notion, but now for minimal surfaces. Simplified cases for this deformation problem can be found in [108,109].

Study of minimal surfaces in Riemannian geometries has been extensively carried out in the mathematics literature [103, 110]. In the entanglement entropy literature the plateau problem for minimal surfaces has been studied in [13]. It is known that for surfaces embedded in a a given Riemannian space the area functional of the embedded surface is stationary, that is it's first variation vanishes, when the embedded surface is minimal. Likewise when the second variation is equated to zero it gives rise to the Jacobi equation for minimal surfaces [111]. The interpretation of the solutions of the Jacobi equation is the following. The solutions of this equation gives the deviation between a minimal surface and a neighboring minimal surface. In the physics literature the Jacobi equation has been studied in the context of relativistic membranes [112] and spiky strings on a flat background [113]. However this equation is relevant only when the metric of the ambient space is fixed.

In the context of the present work one needs to modify this notion. Note that in our case one needs to study deviations between two surfaces which are minimal in two different spacetimes. The spacetimes are however related by a perturbation and not completely arbitrary. To begin with one has to ensure that all of the results obtained are manifestly gauge invariant and therefore has to be careful and precise in defining perturbations in the spirit of a covariant perturbation theory. We therefore adopt the notion introduced in [114] in the context of gravity. A priori, taking cue from the results obtained for geodesics one then expects the Jacobi equation to be modified by appearance of an inhomogeneous term. This indeed turns out to be case, as will be shown later. We also obtain an expression for the change in the area functional, in arbitrary dimensions, upto second order.

Having obtained an equation that properly mimics the situation at hand, one needs to demonstrate that the equations can indeed be solved, for the prescription to be of any relevance. We therefore solve this equation in the $3 + 1$ dimensional case for two choices of the boundary subsystem 1) Spherical subsystem and 2) Thin strip subsystem. We do this for Boosted black brane like perturbations over AdS_4. Using the solutions of the inhomogeneous Jacobi equation we obtain the change in HEE between AdS_4 and boosted black brane like perturbations over it.

5.2 Notations and conventions

Consider a $d+1$ dimensional space time $(\mathcal{M}, g)$ and another $d+1$ dimensional space time $(\mathcal{M}', g')$ which is diffeomorphic to $\mathcal{M}$. That is there is a differentiable map $\Phi : \mathcal{M} \to \mathcal{M}'$ which is however not isometric. We will call $(\mathcal{M}', g')$ to be a perturbation over $(\mathcal{M}, g)$ if $\overset{(1)}{P} = \Phi_* g' - g$ is a small perturbation over g. Consider a surface $\mathcal{S}$ isometrically embedded in $\mathcal{M}$ and given by the function $f : S \to \mathcal{M}$. It is implied that the restriction of f to the image of $\mathcal{S}$ is continuous and differentiable. In a local coordinate chart x^μ on $\mathcal{M}$ and τ^a on $\mathcal{S}$ the embedding can be represented by the embedding functions $x^\mu \circ f \circ (\tau^a)^{-1}$. This can be simply written as $x^\mu(\tau^a)$. The induced metric on $\mathcal{S}$ is the pull back of the metric g under the map f, given by $h = f_*\, g$. Again, in the local coordinates this can be written as $h_{ab} = g(\partial_a, \partial_b) = \frac{\partial x^\mu}{\partial \tau^a}\frac{\partial x^\nu}{\partial \tau^b} g(\partial_\mu, \partial_\nu)$. The quantity $\frac{\partial x^\mu}{\partial \tau^a}\partial_\mu$ is the push forward of the purely tangential vector field ∂_a to $\mathcal{M}$. 'h_{ab}' is the first fundamental form on $\mathcal{S}$. To define the second fundamental form one needs a connection or the covariant derivative on $\mathcal{M}$. The covariant derivative is a map $\nabla : T\mathcal{M} \otimes T\mathcal{M} \to T\mathcal{M}$. For two vector fields $W, Z \in T\mathcal{M}$ it is denoted as $\nabla_W Z$ and is an element of $T\mathcal{M}$. Now suppose $x \in \mathcal{S}$. One can decompose the tangent space at the point x into the tangent space of $\mathcal{S}$ and the space of normal vectors as $T_x\mathcal{M} = T_x\mathcal{S} \oplus T_x^{\perp}\mathcal{S}$. Then one defines the tangent bundle and normal bundle on $\mathcal{S}$ as $\bigcup_x T_x\mathcal{S}$ and $\bigcup_x T_x^{\perp}\mathcal{S}$ respectively. One can similarly define a covariant derivative on $\mathcal{S}$. Let it be denoted by $D : T\mathcal{S} \otimes T\mathcal{S} \to T\mathcal{S}$. Let $X, Y \in T\mathcal{S}$. Then the Gauss decomposition allows us to write,

$$\nabla_X Y = D_X Y + K(X, Y), \tag{5.1}$$

where $D_X Y$ is purely tangential and $K(X, Y)$ is a vector in the normal bundle and is the extrinsic curvature or the second fundamental form. The metric compatibility of ∇ in this notation is written as $\nabla_W g(V, U) = g(\nabla_W U, V) + g(U, \nabla_W V)$. The metric compatibility of ∇ with g will imply metric compatibility of D with h, by virtue of the above equation. One defines a connection $\nabla_X^{\perp} N^{\perp}$ in the normal bundle as $\nabla^{\perp} : T\mathcal{S} \otimes T^{\perp}\mathcal{S} \to T^{\perp}\mathcal{S}$, where $X \in T\mathcal{S}$ and $N^{\perp} \in T^{\perp}\mathcal{S}$. Then the shape operator $W_{N^{\perp}}(X)$ is defined as,

$$\nabla_X N^{\perp} = \nabla_X^{\perp} N^{\perp} - W_{N^{\perp}}(X). \tag{5.2}$$

The shape operator and the extrinsic curvatures are related by the Weingarten equation,

$$g(W_{N^{\perp}}(X), Y) = g(N^{\perp}, K(X, Y)), \tag{5.3}$$

where $X, Y \in T\mathcal{S}$ and $N^{\perp} \in T^{\perp}\mathcal{S}$. The Riemann tensor is a map $R : T\mathcal{M} \otimes T\mathcal{M} \otimes T\mathcal{M} \to T\mathcal{M}$ and is defined as,

$$R(W, U)V \equiv [\nabla_W, \nabla_U]V - \nabla_{[W,U]}V \tag{5.4}$$

Similarly one can define an intrinsic Riemann tensor by,

$$\mathcal{R}(X, Y)Z \equiv [D_X, D_Y]Z - D_{[X,Y]}Z \tag{5.5}$$

We write down the equations of Gauss and Codazzi, in this notation. Let $X, Y, Z, W \in T\mathcal{S}$ and $N^{\perp} \in T^{\perp}\mathcal{S}$. Then the Gauss equation is given as,

$$g(R(X, Y)Z, W) = g(\mathcal{R}(X, Y)Z, W) - g(K(X, Z), K(Y, W)) + g(K(X, W), K(Y, Z)), \tag{5.6}$$

and the Codazzi equation as,

$$g(R(X, Y)N^{\perp}, Z) = g((\nabla_Y K)(X, Z), N^{\perp}) - g((\nabla_X K)(Y, Z), N^{\perp}) \tag{5.7}$$

Now, we go over to notations involving perturbations. In the presence of perturbations a variation will be assumed to have have two contributions, one which is a flow along a vector $N \in T\mathcal{M}$, obtained by taking a covariant derivative ∇_N along N and another variation δ_g which is purely due to metric perturbations. Since we will be doing all the calculations in a coordinate chart in the unperturbed space time, let try to define certain quantities on $\mathcal{M}$ arising due to the perturbations, i.e due to the difference in the two metrics g and $\Phi_* \ g'$.

The metric perturbation will be given by,

$$(\delta_g\, g)(\partial_\mu, \partial_\nu) \equiv \left[\Phi_*\, g' - g\right](\partial_\mu, \partial_\nu) = \overset{(1)}{P}(\partial_\mu, \partial_\nu), \tag{5.8}$$

where $\overset{(1)}{P}$ is a symmetric bilinear form on $\mathcal{M}$. Note that δ_g only acts on the metric and does not change the vector fields ∂_μ. Now suppose there is a covariant derivative ∇' in $\mathcal{M}'$ compatible with g', then for $X, Y \in T\mathcal{M}$,

$$C(X,Y) \equiv \delta_g\Big(\nabla_X Y\Big) = \tilde{\nabla}_X Y - \nabla_X Y, \tag{5.9}$$

where $\tilde{\nabla} = \phi^* \nabla'$ is the pullback connection on M. Note that $C(X,Y)$ is a vector field in $\mathcal{M}$. When written in coordinates it has exactly the same form as $\overset{(1)}{C}{}^{\mu}_{\nu\rho}$ used in [35]. Since we will not be dealing with perturbations of further higher order, we have dropped the superscript$^{(1)}$.

We are now in a position to derive the inhomogeneous Jacobi equation for minimal surfaces. For the display of some semblance with chapter (4), a rederivation of the inhomogeneous Jacobi equation for geodesics, in this notation, is given in Appendix E.

5.3 Derivation of the Inhomogeneous Jacobi equation for surfaces

In the previous section we considered $(\mathcal{M}', g')$ to be a perturbation over $(\mathcal{M}, g)$. Let us consider a one parameter family of such perturbed spacetimes $(\mathcal{M}_\lambda, g_\lambda)$ and a one parameter family of diffeomorphism, which are not necessarily isometric, $\Phi_\lambda : \mathcal{M} \to \mathcal{M}_\lambda$ such that $\mathcal{M}_0$ corresponds to the unperturbed spacetime and Φ_0 is the identity map. Let $\mathcal{S}_\lambda$ be a family of co-dimension two minimal surfaces in $(\mathcal{M}_\lambda, g_\lambda)$ i.e the trace of their extrinsic curvatures vanishes. The surfaces can be parametrized by the embedding functions $f^\mu_\lambda(\tau^a)$, which allows one to write the tracelessness condition as $h^{ab}_\lambda K_{(\lambda)}(\partial_a, \partial_b) = 0$. Note that one would think that the coordinates τ^a may be different for different $\mathcal{S}_\lambda$. But one can always adjust the functions f^μ_λ such that the surfaces can be coordinatised by the same intrinsic coordinates. Let us construct a family of immersed submanifolds $\tilde{\mathcal{S}_\lambda}$ in $\mathcal{M}_0$, given by the embedding functions F^μ_λ such that $\Phi_\lambda \circ F^\mu_\lambda = f^\mu_\lambda$. Let's denote the deviation vector between F^μ_0 and the neighboring surface be denoted by N. Note that N can always be taken to be normal to $\tilde{\mathcal{S}_0}$, as any tangent deviation will only result in a reparametrization of the intrinsic coordinates τ^a and won't change the area of the surface. This statement is however not obvious in our case where we have metric perturbations. In this regard we take cue from the calculation done in the case of geodesic ((4)). Since we have already removed the freedom of intrinsic coordinate reparametrization, by adjusting the f_λ's, it is quite legitimate to take normal variations only. Moreover since we will ultimately be interested in area change it is sufficient for us to take normal variations only. Further N can always be chosen such that it commutes with the vectors ∂_a tangent to the submanifold i.e $[N, \partial_a] = 0\ \forall\ a$.

The condition that $\mathcal{S}_\lambda$'s are minimal in $(\mathcal{M}_\lambda, g_\lambda)$ then reduces to a condition on N in $\mathcal{M}_0$. At each order of the variation, the conditions are essentially inhomogeneous linear differential equations that N must satisfy. The equation that one obtains at linear order is the one we will be interested in, since the solutions of this will provide us with the linear deformation of the minimal surface that we are seeking. As is evident, the equation can be derived by equating the more general variation $\delta_N = \nabla_N + \delta_g$, discussed in section 5.2, of the trace of the extrinsic curvature to zero i.e

$$\delta_N H_\lambda = h^{ab}_\lambda(\delta_N(\nabla_{(\lambda)\partial_a}\partial_b)^\perp) + (\delta_N h^{ab}_\lambda)K_\lambda(\partial_a, \partial_b) = 0. \tag{5.10}$$

We will drop the λ subscript from here on, as the above variations will be calculated around the unperturbed surface i.e at $\lambda = 0$. While dropping the λ's surely will make the expressions look cleaner, one has to make sure that the minimal surface equation be used only after the derivatives have been computed. Let us first compute the first term of the above expression which involves the normal component of the covariant derivative.

$$\begin{aligned} h^{ab}\delta_N(\nabla_{\partial_a}\partial_b)^{\perp} &= h^{ab}\Bigg(\nabla_N(\nabla_{\partial_a}\partial_b) + \delta_g(\nabla_{\partial_a}\partial_b) - \nabla_N(\nabla_{\partial_a}\partial_b)^T - \delta_g(\nabla_{\partial_a}\partial_b)^T\Bigg) \\ &= h^{ab}\Bigg(\nabla_{\partial_a}\nabla_{\partial_b}N + R(N,\partial_a)\partial_b + C(\partial_a,\partial_b) - \nabla_N(\nabla_{\partial_a}\partial_b)^T - \delta_g(\nabla_{\partial_a}\partial_b)^T\Bigg) \end{aligned} \tag{5.11}$$

The action of the variation δ_N on any quantity Q on $\mathcal{M}_0$ is taken to be of the form $\delta_N(Q) = \nabla_N(Q) + \delta_g(Q)$. This notation for variation has been adopted for convenience of calculation. That this reproduces the correct result, can be seen from the derivation of the inhomogeneous Jacobi equation, obtained by adopting this notation (appendix E.1). The action of δ_g is precisely on the space of sections on a tensor bundle in $\mathcal{M}_\lambda$. If we represent a flow on $\mathcal{M}_0$ and δ_g by two parameters then a priori these two parameters are completely independent of each other, but for the perturbations to work one needs them to be equal. How the parameter of the flow ∇_N can be related to the parameter of the variation δ_g is a mathematical issue the resolution of which we will leave for some future work. Adopting the above, one obtains,

$$(\delta_N h^{ab})K(\partial_a,\partial_b) = 2h^{ab}K(\partial_a, W_N(\partial_b)) - h^{ac}h^{bd}K(\partial_a,\partial_b)\overset{(1)}{P}(\partial_c,\partial_d) \tag{5.12}$$

Substituting (5.11),(5.12) in (5.10) we get

$$\delta_N H = h^{ab}\Bigg(\nabla_{\partial_a}\nabla_{\partial_b}N + R(N,\partial_a)\partial_b + C(\partial_a,\partial_b) - \nabla_N(\nabla_{\partial_a}\partial_b)^T - \delta_P(\nabla_{\partial_a}\partial_b)^T\Bigg)$$
$$+2h^{ab}K(\partial_a, W_N(\partial_b)) - h^{ac}h^{bd}K(\partial_a,\partial_b)\overset{(1)}{P}(\partial_c,\partial_d). \tag{5.13}$$

A similar exercise with the term $h^{ab}\delta_N(\nabla_{\partial_a}\partial_b)^T$ yield the following expression,

$$h^{ab}\Bigg[(\nabla_{(\nabla_{\partial_a}\partial_b)^T}N)^\perp + (\nabla_{\partial_a}\nabla_{\partial_b}N + R(N,\partial_a)\partial_b + C(\partial_a,\partial_b))^T + h^{cd}\overset{(1)}{P}(K(\partial_a,\partial_b),\partial_c)\partial_d\Bigg]. \tag{5.14}$$

Substituting (5.14) in (5.13), we get a complete expression for $\delta_N H$,

$$\delta_N H = h^{ab}\Bigg((\nabla_{\partial_a}\nabla_{\partial_b}N + R(N,\partial_a)\partial_b + C(\partial_a,\partial_b))^\perp - (\nabla_{(\nabla_{\partial_a}\partial_b)^T}N)^\perp\Bigg) - h^{cd}\overset{(1)}{P}(H,\partial_c)\partial_d$$
$$+2h^{ab}K(\partial_a, W_N(\partial_b)) - h^{ac}h^{bd}K(\partial_a,\partial_b)\overset{(1)}{P}(\partial_c,\partial_d) \tag{5.15}$$

Noting that $(\nabla_{\partial_a}\nabla_{\partial_b}N)^\perp = -K(\partial_a, W_N(\partial_b)) + \nabla^\perp_{\partial_a}\nabla^\perp_{\partial_b}N$, the above equation, along with the minimality condition $H = 0$, can be recast in the following form, which is closer in form to the expressions known in the literature of minimal surfaces.

$$\delta_N H = \Delta^\perp N + Ric(N) + A(N) + C^\perp - \tilde{H}, \tag{5.16}$$

where we have defined $\Delta^\perp N$ to be the Laplacian on the normal bundle, given by $h^{ab}\Big(\nabla^\perp_{\partial_a}\nabla^\perp_{\partial_b}N - \nabla^\perp_{(\nabla_{\partial_a}\partial_b)^T}N\Big)$, $g(R(N,\partial_a)\partial_b, N)$ has been denoted by $Ric(N)$. $A(N) = h^{ab}K(\partial_a, W_N(\partial_b))$

is the Simon's operator whereas $C^{\perp}$ is defined as $C^{\perp} = h^{ab}C(\partial_a, \partial_b)^{\perp}$ and $\tilde{H} = \overset{(1)}{P}{}^{ab}K(\partial_a, \partial_b)$. Thus identifying the Jacobi/stability operator $(\mathcal{L})$ for minimal surfaces as

$$\mathcal{L}N = \Delta^{\perp}N + Ric(N) + A(N), \tag{5.17}$$

we can rewrite (5.16) as

$$\mathcal{L}N = -C^{\perp} + \tilde{H}. \tag{5.18}$$

This is the inhomogeneous Jacobi equation. The solutions of this equation will provide us with the deformation of a minimal surface under a perturbation of the ambient spacetime. The inhomogeneous terms in the above equation, involves perturbation of the metric and is the only term in the above equation that involves the perturbation. If there were no perturbations the equation would have corresponded to the one describing a deviation of a minimal surface to another minimal surface in the same spacetime $(\mathcal{M}_0, g_0)$. We will solve for solutions of this equation for specific cases and substitute the result in an area variation formula which we derive in the next section.

5.4 Variation of the Area functional

According to Hubeny, Rangamani, Takayanagi (HRT) proposal the area of a codimension two spacelike extremal surface(γ_A) in AdS_{d+1} whose boundary coincides with the boundary of subsystem A gives the entanglement entropy for this subsystem. Our goal therefore would be to obtain the change in area of a minimal surface up to second order with the extra constraint that the boundary of the surface remain unaltered i.e the deviations vanish at the boundary. At second order we will encounter terms which involve the deviation of the embedding functions itself. It is here that we have to use the solutions of the inhomogeneous Jacobi equation. The first variation of area of the minimal surface is given by,

$$\delta_N A = \int d^n\tau \, \frac{\sqrt{h}}{2} h^{ab}\delta_N h_{ab} = -\int d^n\tau \, \sqrt{h} g(N,H) + \frac{1}{2}\int d^n\tau \, \sqrt{h} h^{ab} \overset{(1)}{P}(\partial_a, \partial_b)$$
$$+\text{Surface terms.} \tag{5.19}$$

If the perturbations are set to zero then we get back the known expression for first variation of area. In the presence of perturbations the on-shell expression can be obtained by setting ($H = 0$).

$$\delta_N A = \frac{1}{2}\int d^n\tau \, \sqrt{h} h^{ab} \overset{(1)}{P}(\partial_a, \partial_b) \tag{5.20}$$

The second variation of area is given by

$$\delta^{(2)}_N A = -\int d^n\tau \, \delta_N(\sqrt{h} g(N,H)) + \frac{1}{2}\int d^n\tau \, \delta_N(\sqrt{h} h^{ab} \overset{(1)}{P}(\partial_a, \partial_b)) + \text{Surface terms} \tag{5.21}$$

Note that since $[N, \partial_a] = 0$ for all a, the variation of the surface term is again a surface term. From the results of the previous section 5.3, the first term in the above expression can be written in terms of the stability operator. Simplifying the second term requires a bit of algebra. Note that $\delta_N(\sqrt{h}h^{ab}\overset{(1)}{P}(\partial_a,\partial_b))$ has the following expression,

$$\sqrt{h}h^{ab}\overset{(1)}{P}(\partial_a,\partial_b)\left(-g(N,H)+\frac{1}{2}h^{cd}\overset{(1)}{P}(\partial_c,\partial_d)\right)+2\sqrt{h}h^{ac}h^{bd}g(N,K(\partial_c,\partial_d))\overset{(1)}{P}(\partial_a,\partial_b)$$
$$-\sqrt{h}h^{ac}h^{bd}\overset{(1)}{P}(\partial_c,\partial_d)\overset{(1)}{P}(\partial_a,\partial_b)+\sqrt{h}h^{ab}\left[2\overset{(1)}{P}(\nabla_{\partial_a}N,\partial_b)+2g(C(\partial_a,N),\partial_b)+\overset{(2)}{P}(\partial_a,\partial_b)\right] \tag{5.22}$$

Substituting the expression in (5.22) in (5.21) and using the conditions $H=0, \delta_N H = 0$, one arrives after a lengthy calculation at the following final expression for the second variation of the area functional [1],

$$\begin{aligned}\delta_N^{(2)}A \;=\; & \frac{1}{4}\int d^n\tau\,\sqrt{h}h^{ab}\overset{(1)}{P}(\partial_a,\partial_b)h^{cd}\overset{(1)}{P}(\partial_c,\partial_d)+\int d^n\tau\,\sqrt{h}h^{ac}h^{bd}g(N,K(\partial_c,\partial_d))\\ & \overset{(1)}{P}(\partial_a,\partial_b)-\frac{1}{2}\int d^n\tau\,\sqrt{h}h^{ac}h^{bd}\overset{(1)}{P}(\partial_c,\partial_d)\overset{(1)}{P}(\partial_a,\partial_b)+\int d^n\tau\,\sqrt{h}h^{ab}\frac{1}{2}\overset{(2)}{P}(\partial_a,\partial_b)\\ & -\int d^n\tau\,\sqrt{h}h^{ab}g(C(\partial_a,\partial_b),N)+\text{Surface terms},\end{aligned} \tag{5.23}$$

The appearance of surface terms in the above expression is not very crucial, at least in the context of our current work. Since the boundary subsystem is kept fixed, while the bulk metric is being perturbed, the boundary conditions on the deviation vector would imply that it vanishes at the boundary. Thus change in area will have no contribution from the boundary terms. If we started with a more general deviation vector which also had components tangent to the immersed surface, then the only modification of the above

[1] where we have used the following two expressions,

$$(\nabla_{\partial_a}P)(N,\partial_b)=g(C(\partial_a,\partial_b),N)+g(C(\partial_a,N),\partial_b)$$

$$\nabla_{\partial_a}[\sqrt{h}h^{ab}P(N,\partial_a)]=\sqrt{h}h^{ab}\nabla_{\partial_a}[P(N,\partial_b)]-\sqrt{h}h^{ab}P(N,(\nabla_{\partial_a}\partial_b)^T)$$

.

expression would have been through the appearance of more boundary terms. The bulk contribution still would have arised from normal variations only.

5.5 Brief outline of steps involved in obtaining Area variation upto second order

Our goal is to provide a formalism to calculate a change in the area of an extremal surface under changes of embedding and perturbation of metric. For the sake of brevity, all our calculations will be done in $3+1$ dimensions. But this can be easily generalized to higher dimensions. In this section, we provide a brief outline of this formalism

1) Our first task is to take an asymptotically AdS metric (to be considered as a perturbation over AdS) and identify the first and second order metric perturbations. In our case, this is achieved by writing the boosted AdS black brane metric in the Fefferman Graham coordinates, keeping up to second order (appendix F). From the first order metric perturbations $\overset{(1)}{P}_{\mu\nu}$ one can calculate the $(1,2)$ tensor.

$$C^{\mu}_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}\left(\partial_\nu \overset{(1)}{P}_{\rho\sigma} + \partial_\rho \overset{(1)}{P}_{\nu\sigma} - \partial_\sigma \overset{(1)}{P}_{\nu\rho}\right) - \frac{1}{2}\overset{(1)}{P}{}^{\mu\sigma}\left(\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho}\right), \qquad (5.24)$$

where $g_{\mu\nu}$ is the unperturbed AdS_4 metric. The tensor defined is nothing but $C(X,Y)$ written in a coordinate system, i.e $C(\partial_\nu, \partial_\rho) = C^{\mu}_{\nu\rho}\partial_\mu$.

2) Next we choose a free boundary extremal surface in AdS_4 [14]. We will consider two cases A) half sphere in AdS_4 which is the corresponding minimal surface for a circular disc like subsystem and B) minimal surface corresponding to a thin strip boundary subsystem. With these choices and the choice of the perturbed metric $\overset{(1)}{P}{}^{\mu\nu}$, we can now solve the inhomogeneous Jacobi equation (5.18) and obtain the deviation vector (N).

3) First and second order change in the area can be obtained by substituting the values of the deviation vector (N), first order metric perturbation ($\overset{(1)}{P}_{\mu\nu}$) and the second order metric perturbation ($\overset{(2)}{P}_{\mu\nu}, C^{\mu}_{\nu\rho}$) in the expression (5.23),(5.4) and integrating. From here the total change in area upto second order can be obtained as,

$$\Delta A = \Delta^{(1)} A + \frac{1}{2}\Delta^{(2)} A \tag{5.25}$$

In the topic of the present chapter we have selected asymptotically AdS spacetime. But this formalism can be easily applied to asymptotically flat case also. Here we have considered first order deviations of the extremal surface and second order metric perturbation to calculate the change in area up to second order. To calculate the change in area up to third order one need to consider second order deviation of the extremal surface and third order metric perturbations. Second order deviation can be obtained by extending the inhomogeneous Jacobi equation up to second order. The form of second order inhomogeneous Jacobi equation for geodesics can be found in [35]. Third order metric perturbation can be obtained by keeping third order terms in the asymptotic(Fefferman Graham) metric.

5.6 Solutions of the inhomogeneous Jacobi equations and change in area

Our choice of the asymptotic metric to be considered as a perturbation over AdS_4 is the Boosted AdS black brane metric written in the Fefferman Graham coordinates upto second order. The CFT state dual to this bulk geometry is a thermal plasma which is uniformly boosted along a certain direction and is characterized by a temperature T and boost β. This choice of a stationary spacetime is made to elucidate that our formalism can be easily applied to both static and non static spacetimes and yields expected results for the non-static case. The metric for AdS_4 in Poincaré coordinates reads as

$$ds^2 = \frac{-dt^2 + dx^2 + dy^2 + dz^2}{z^2} \tag{5.26}$$

for simplicity we have set the radius of AdS to one. Now we will solve the inhomogeneous Jacobi equation and obtain an expression for the change in area for the case of two boundary subsystems namely

5.6.1 Circular disk subsystem

In the case where the boundary subsystem is a circular disk of radius $\mathscr{R}$, it is known that the minimal surface in the AdS_{d+1} is a $d-1$ dimensional hypersphere. The embedding of such a surface in AdS_4 is given by the following embedding functions [14, 34],

$$x = \mathscr{R}\sin\theta\cos\phi + X, \;\; y = \mathscr{R}\sin\theta\sin\phi + Y, \;\; z = \mathscr{R}\cos\theta, \;\; t = constant. \tag{5.27}$$

The coordinates θ, ϕ are the coordinates intrinsic to the surface and have ranges, $0 \leq \theta \leq \frac{\pi}{2}$ and $0 \leq \phi < 2\pi$. As is evident from the above expressions in eq.(5.27), the surface of intersection of the half sphere with the AdS_4 boundary is at $\theta = \frac{\pi}{2}$. The intrinsic metric can be calculated via a pullback of the metric on the full space time and is given as,

$$ds^2_{induced} = h_{ab}\, dx^a dx^b = \frac{d\theta^2 + \sin^2\theta d\phi^2}{\cos^2\theta} \tag{5.28}$$

To facilitate our calculation we will construct a local basis adapted to this surface. To start with we first construct a local tangent basis. As is apparent from the expression for the induced metric, the tangent bases are,

$$e_2 = \cos\theta \partial_\theta, \quad e_3 = \cot\theta \partial_\phi. \tag{5.29}$$

Since the surface is purely spacelike, this set provides the space like bases for the full spacetime. The set of basis vectors spanning the normal bundle will provide us with the other two basis vectors. To obtain them we first lift the tangent vectors to the space time, by using the embedding functions and then use the orthogonality relations. As a matter of convention we mark the time like normal as e_0 and the space like normal as e_1.

$$e_0 = z\partial_t, \quad e_1 = \frac{z(x-X)}{l}\partial_x + \frac{z(y-Y)}{l}\partial_y + \frac{z^2}{l}\partial_z \tag{5.30}$$

To completely specify the embedding one also needs to find the extrinsic curvatures and the intrinsic connection. To do so we need to find the covariant derivatives between the tangent vectors. They turn out to be,

$$\nabla_{e_2} e_2 = 0, \quad \nabla_{e_3} e_3 = -\text{cosec}\,\theta\; e_2, \quad \nabla_{e_3} e_2 = \text{cosec}\,\theta\; e_3 \tag{5.31}$$

which gives the following for the intrinsic connection and the extrinsic curvature.

$$D_{e_2} e_2 = 0, \quad D_{e_3} e_3 = -\text{cosec}\,\theta\; e_2, \quad D_{e_2} e_3 = 0, \quad D_{e_2} e_2 = \text{cosec}\,\theta\; e_3, \quad D_{e_2} e_2 = \text{cosec}\,\theta e_3$$
$$K(e_2, e_2) = 0, \quad K(e_3, e_3) = 0, \quad K(e_2, e_3) = 0, \quad K(e_3, e_2) = 0 \tag{5.32}$$

The vanishing of the extrinsic curvature implies that the surface is totally geodesic i.e any curve that is a geodesic on the surface is also a geodesic of the full spacetime. Recall that the Jacobi equation involves the connection in the Normal bundle $\nabla^\perp$, which can be found by calculating the covariant derivative of a normal vector along a tangent vector.

$$\nabla_{e_2} e_0 = 0, \;\; \nabla_{e_3} e_0 = 0, \;\; \nabla_{e_2} e_1 = 0, \;\; \nabla_{e_3} e_1 = 0 \tag{5.33}$$

From this one can read off the normal connection $\nabla^{\perp}$, using the Weingarten map. The procedure involves expanding the normal connection as $\nabla^{\perp}_{e^a} e_A = \beta_A^B(e^a) e_B$ ($A, \; B$ denotes an index for basis vectors in the normal bundle) and yields,

$$\begin{aligned}\nabla^{\perp}_{e_2} e_0 = \beta_0^0(e_2)e_0 + \beta_0^1(e_2)e_1 = 0, \;\; \nabla^{\perp}_{e_3} e_0 = \beta_0^0(e_3)e_0 + \beta_0^1(e_3)e_1 = 0 \\ \nabla^{\perp}_{e_2} e_1 = \beta_1^0(e_2)e_0 + \beta_1^1(e_2)e_1 = 0, \;\; \nabla^{\perp}_{e_3} e_1 = \beta_1^0(e_3)e_0 + \beta_1^1(e_3)e_1 = 0.\end{aligned} \tag{5.34}$$

The vanishing of the $\beta' s$ is equivalent to saying that the normal bundle is flat. Using the above results, calculating the left hand side of the Jacobi equation is just a matter of algebra. We expand the deviation vector in the normal basis as $\alpha^A \, e_A$ and find the following equations for the α^A.

$$\cos^2\theta \partial_\theta^2 \alpha^A + \cos^2\theta \cot\theta \partial_\theta \alpha^A + \cot^2\theta \partial_\phi^2 \alpha^A - 2\alpha^A = F^A, \tag{5.35}$$

Where F^A has been defined for compactness of the above expression and is given as in $F^A = e^A_\mu \big(C^{\perp\mu} + \tilde{H}^\mu \big)$. Note that in this case the both the normal projections yield the one and the same equation. The source of this symmetry can be traced back as due to the symmetry of the embedding surface itself. Before proceeding to find solutions of the above equation, we need to analyze the homogeneous equations. In other words we will impose the boundary condition that the deviation vector is zero at the boundary and check if this implies that the only solution of the 'homogeneous' piece of the above equation is the trivial solution. As we will see, this knowledge would be helpful in our effort to obtain solutions of the 'inhomogeneous' equations. The homogeneous equation can be solved by the method of separation of variables $\alpha^A(\theta,\phi) = \Theta^A(\theta)\ \Phi^A(\phi)$. The equations then become ordinary differential equations.

$$\frac{d^2\Theta^A}{d\theta^2} + \cot\theta\frac{d\Theta^A}{d\theta} - (2\sec^2\theta + m^2\text{cosec}\,^2\,\theta)\Theta^A = 0 \tag{5.36}$$

and the ϕ equation is,

$$\frac{d^2\Phi^A}{d\phi^2} + m^2\Phi^A = 0 \tag{5.37}$$

For the ϕ equation the boundary condition is of course the periodic one $\Phi^A(\phi + 2\pi) = \Phi^A(\phi)$, which restricts the values of m to integers only. The most general solution of this equation is given by,

$$\begin{aligned}\Theta = C_1\ \cos^2\theta(\sin\theta)^m\ {}_2F_1\Big(1+\frac{m}{2},\frac{3}{2}+\frac{m}{2};m+1;\sin^2\theta\Big)\\ + C_2\ \cos^2\theta(\sin\theta)^{-m}\ {}_2F_1\Big(1-\frac{m}{2},\frac{3}{2}-\frac{m}{2};-m+1;\sin^2\theta\Big)\end{aligned} \tag{5.38}$$

Assuming the boundary condition $\Theta = 0$ at $\theta = \frac{\pi}{2}$ and demanding that the solution be regular at $\theta = 0$, one concludes that $C_1 = C_2 = 0$. To check this assume m to be positive (Similar arguments would hold for m negative). Note that at $\theta = 0$ the second solution diverges since ${}_2F_1\Big(1-\frac{m}{2},\frac{3}{2}-\frac{m}{2};-m+1;0\Big) = 1$, while the $\sin^{-m}(\theta)$ term diverges. This implies C_2 must be set to zero. At $\theta = \frac{\pi}{2}$ the first solution diverges. This can be argued in the following way. Note that $lim_{z\to 1^-}\frac{{}_2F_1\Big(a,b;c;z\Big)}{(1-z)^{c-a-b}} = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}$ for $\Re(c-a-b) < 0$.

Writing the first solution as,

$$\frac{z^{\frac{m}{2}}}{(1-z)^{\frac{1}{2}}}\frac{{}_2F_1\left(1+\frac{m}{2},\frac{3}{2}+\frac{m}{2};m+1;z\right)}{(1-z)^{-\frac{3}{2}}}, \tag{5.39}$$

one can realize that solution is divergent at $\theta = \frac{\pi}{2}$. Hence C_1 has to be set to zero. As expected for homogeneous spaces the only solution is the trivial one.

Now we will solve the inhomogeneous equation. By substituting $C \equiv \left(\frac{1}{3}+\beta^2\gamma^2\right)\frac{1}{z_0^3}$, $D \equiv \left(\frac{1}{3}\right)\frac{1}{z_0^3}$, $B \equiv \beta\gamma^2\frac{1}{z_0^3}$, and writing $\mathcal{R}^3 \equiv \frac{\mathscr{R}^3}{z_0^3}$, the inhomogeneous equation for e_1 turns out to be,

$$\cos^2\theta\partial_\theta^2\alpha^1 + \cos^2\theta\cot\theta\partial_\theta\alpha^1 + \cot^2\theta\partial_\phi^2\alpha^1 - 2\alpha^1 = \mathcal{R}^3\cos^4\theta\left(\frac{2}{3}+\beta^2\gamma^2\right)$$
$$+\frac{5\mathcal{R}^3\sin^2\theta\cos^4\theta}{6} + \frac{5\mathcal{R}^3\beta^2\gamma^2\sin^2\theta\cos^4\theta}{4} + \frac{5\mathcal{R}^3\beta^2\gamma^2\sin^2\theta\cos^4\theta\cos 2\phi}{4}, \tag{5.40}$$

and that for e_0 reads,

$$\cos^2\theta\partial_\theta^2\alpha^0 + \cos^2\theta\cot\theta\partial_\theta\alpha^0 + \cot^2\theta\partial_\phi^2\alpha^0 - 2\alpha^0 = 3\beta\gamma^2\mathcal{R}^3\cos^4\theta\sin\theta\cos\phi \tag{5.41}$$

Let us consider the e_1 equations first. Note that since the equation is linear one can find the solutions for individual terms in the inhomogeneous piece separately. Let us therefore consider the terms containing no function of ϕ.

$$\partial_\theta^2\alpha^1 + \cot\theta\partial_\theta\alpha^1 + \text{cosec}^2\theta\partial_\phi^2\alpha^1 - 2\sec^2\theta\,\alpha^1 = \mathcal{R}^3\cos^4\theta\left(\frac{2}{3}+\beta^2\gamma^2\right)$$
$$+\frac{5\mathcal{R}^3\sin^2\theta\cos^4\theta}{6} + \frac{5\mathcal{R}^3\sin^2\theta\cos^4\theta\beta^2\gamma^2}{4} \tag{5.42}$$

Owing to the fact that the right hand side of this equation contains no function of ϕ the only non trivial solution to this equation will come from $m = 0$. This can be understood by taking a trial solution of the form $\sum_m \left(g_m(\theta)e^{im\phi} + g_{-m}(\theta)e^{-im\phi}\right)$. If one now lists the equations for individual $m's$, then only the $m = 0$ equation will have an inhomogeneous term on the right hand side, while the other equations will be all homogeneous. But we have already shown that the solutions of the homogeneous equations are trivial. Therefore we only need to solve the $m = 0$ equation, which reads,

$$\frac{d^2\Theta^1}{d\theta^2} + \cot\theta\frac{d\Theta^1}{d\theta} - 2\ \sec^2\theta\ \Theta^1 = \mathcal{R}^3\cos^2\theta\left(\frac{2}{3} + \beta^2\gamma^2\right)$$
$$+\frac{5\mathcal{R}^3\sin^2\theta\cos^2\theta}{6} + \frac{5\mathcal{R}^3\sin^2\theta\cos^4\theta\beta^2\gamma^2}{4} + \frac{5\mathcal{R}^3\sin^2\theta\cos^4\theta\beta^2\gamma^2\cos 2\phi}{4}, \tag{5.43}$$

The solution to this equation with the conditions that it is zero at $\theta = \frac{\pi}{2}$ and regular at $\theta = 0$ is given by,

$$\Theta^1 = \frac{1}{288}\mathcal{R}^3\cos^2\theta\left(3\beta^2\gamma^2 + 2\right)\left(3\cos 2\theta - 23\right)$$

The other equation containing a $\cos 2\phi$ is equivalent to solving the θ equation for $m = 2$.

$$\partial_\theta^2\Theta^1 + \cot\theta\partial_\theta\Theta^1 - 4\ \text{cosec}\ ^2\theta\ \Theta^1 - 2\ \sec^2\theta\ \Theta^1 = \frac{5\mathcal{R}^3\beta^2\gamma^2\sin^2\theta\cos^2\theta}{4}, \tag{5.44}$$

The solution to this equation with conditions as above yields,

$$\Theta^1 = -\frac{1}{64}\mathcal{R}^3\beta^2\gamma^2\left(\sin 2\theta\right)^2 \tag{5.45}$$

The full solution is then,

$$\alpha^1 = \frac{1}{288}\mathcal{R}^3\cos^2\theta\Big(3\beta^2\gamma^2+2\Big)\Big(3\cos 2\theta - 23\Big) - \frac{1}{64}\mathcal{R}^3\beta^2\gamma^2\Big(\sin 2\theta\Big)^2\cos 2\phi \tag{5.46}$$

Now, we go over to the e_0 equation. By similar arguments, one concludes that the only contribution to the solution will come from $m = 1$ term. Therefore, the equation becomes,

$$\partial_\theta^2\alpha^0 + \cot\theta\partial_\theta\alpha^0 - \text{cosec}\,^2\theta\;\alpha^0 - 2\;\sec^2\theta\;\alpha^0 = 3\beta\gamma^2\mathcal{R}^3\cos^2\theta\sin\theta \tag{5.47}$$

Along with the usual boundary conditions, the solution to this equation is,

$$\alpha^0 = -\frac{1}{4}\beta\gamma^2\mathcal{R}^3\sin\theta\cos^2\theta\cos\phi \tag{5.48}$$

The very fact that the solution of the above e_0 equation is non trivial proves the fact that the perturbed minimal surface ceases to be on a constant t slice as was initially the case with the unperturbed minimal surface in AdS_4 background. One can also check that setting $\beta = 0$, which gives the static case of an AdS Black Brane, makes α^0 vanish.

We are now in a position to calculate the change in area. We first calculate the first order change in the area. As is known, at this order there is no contribution from deviations of the minimal surface itself, and therefore at this order the change must match with that obtained in [27]. The first order change in HEE(S) for the spherical entangling surface can be extracted from eq.(5.4) and is given by,

$$\Delta^{(1)}S = \frac{1}{4G_N}\Delta^{(1)}A = \frac{1}{8G_N}\int d^{d-1}\tau\;\sqrt{h}h^{ab}\overset{(1)}{P}(\partial_a,\partial_b) = \frac{1}{32G_N}\pi\mathscr{R}^3\left(3\beta^2\gamma^2+2\right)\frac{1}{z_0^3}$$

The second order variation has contributions from various terms. The full expression is given by eqn.(5.23),

$$\Delta^{(2)}A = \int d^{d-1}\tau\ \sqrt{h}\Big(h^{ab}h^{cd}\overset{(1)}{P}(\partial_b,\partial_d)g(N^{\perp},K(\partial_a,\partial_c)) - h^{ab}g(C(\partial_a,\partial_b),N^{\perp})\Big)$$
$$+\int d^{d-1}\tau\ \sqrt{h}\Big[\frac{h^{ab}}{2}\overset{(2)}{P}(\partial_a,\partial_b) - \frac{1}{2}h^{ac}h^{bd}\overset{(1)}{P}(\partial_a,\partial_b)\overset{(1)}{P}(\partial_c,\partial_d) + \frac{1}{4}h^{ab}h^{cd}\overset{(1)}{P}(\partial_c,\partial_d)\overset{(1)}{P}(\partial_a,\partial_b)\Big] \tag{5.49}$$

Let us analyze the above equation. The last three terms in the above equation eq.(49) are the terms coming purely from the bulk metric perturbations. The first and the second term arise from changes due change in the embedding function itself. The $N^{\perp}$ in the above equation therefore has to be substituted with the solutions of the Jacobi equation obtained before and then the integrals calculated. We therefore enumerate the results one by one. Consider the last three terms in the above expression which do not involve the deviation vector.

$$\int d^{d-1}\tau\ \sqrt{h}\frac{h^{ab}}{2}\overset{(2)}{P}(\partial_a,\partial_b) = -\frac{1}{105}\pi\mathcal{R}^6\left(6\beta^2\gamma^2-1\right) \tag{5.50}$$

The next term is a product of two metric perturbations gives,

$$\int d^{d-1}\tau\ \sqrt{h}\frac{1}{2}h^{ac}h^{bd}\overset{(1)}{P}(\partial_a,\partial_b)\overset{(1)}{P}(\partial_c,\partial_d) = \frac{2\pi\mathcal{R}^6\left(216\beta^4\gamma^4+147\beta^2\gamma^2+49\right)}{2835} \tag{5.51}$$

Finally the other term containing a product of two perturbations evaluates to,

$$\int d^{d-1}\tau\ \sqrt{h}\frac{1}{4}h^{ab}h^{cd}\overset{(1)}{P}(\partial_c,\partial_d)\overset{(1)}{P}(\partial_a,\partial_b) = \frac{2\pi\mathcal{R}^6\left(108\beta^4\gamma^4+141\beta^2\gamma^2+47\right)}{2835} \tag{5.52}$$

Note that the contribution from the first term is zero owing to the fact that the extrinsic curvature $K(\partial_a, \partial_b)$ is zero in this case of a spherical boundary subsystem. As we will see later this term does give non zero contributions for the case of a strip subsystem. While calculating the second term, the $N^\perp$ contained in the term has to be substituted with the solutions of the inhomogeneous stability equation. After substitution one obtains,

$$\int d^{d-1}\tau \sqrt{h} h^{ab} g(C(\partial_a, \partial_b), N^\perp) = \frac{\pi \mathcal{R}^6 \left(459\beta^4\gamma^4 + \beta^2 \left(81\gamma^4 + 597\gamma^2\right) + 199\right)}{1890} \tag{5.53}$$

The total second order Change in HEE is then given by,

$$\Delta^{(2)} S = \frac{1}{4G_N} \Delta^{(2)} A = -\frac{\pi \mathscr{R}^6}{4G_N} \frac{\left(1809\beta^4\gamma^4 + 3\beta^2 \left(81\gamma^2 + 713\right)\gamma^2 + 551\right)}{5670} \frac{1}{z_0^6} \tag{5.54}$$

This expression gives the second order change of HEE. Positivity of relative entropy between two states in the CFT demands that

$$\Delta H \geq \Delta S$$

Where H is the modular Hamiltonian for the spherical entangling surface, given in terms of the boundary stress tensor. One can now check that the equality is satisfied at the first order [27]. As the modular Hamiltonian remains unchanged at second order, positivity of relative entropy demands that $\Delta^{(2)} S \leq 0$ at second order. Our result eq (5.54) is therefore in agreement with this observation. The full expression for change of HEE is then given by

$$\Delta S = \Delta^{(1)}S + \frac{1}{2}\Delta^{(2)}S$$
$$= \frac{1}{32G_N}\pi\mathscr{R}^3\left(3\beta^2\gamma^2+2\right)\frac{1}{z_0^3} - \frac{\pi\mathscr{R}^6}{8G_N}\frac{\left(1809\beta^4\gamma^4+3\beta^2\left(81\gamma^2+713\right)\gamma^2+551\right)}{5670}\frac{1}{z_0^6} \tag{5.55}$$

the above expression gives the net change in HEE for spherical entangling surface upto second order over pure AdS(ground state) value.

5.6.2 Thin Strip subsystem

We now consider a two dimensional strip like subsystem on the AdS_4 boundary. The subsystem is given by the region $[-L, L] \times [-\frac{l}{2}, \frac{l}{2}]$ of the $x-y$ plane, where $L \gg l$. The minimal surface corresponding to such a subsystem [14] is characterized by the following embedding functions,

$$x = \lambda, \;\; y(\theta) = -z_* E\left(\frac{(\pi - 2\theta)}{4} \mid 2\right), \;\; z(\theta) = z_*\sqrt{\sin\theta}, \tag{5.56}$$

where z_* is the turning point of the minimal surface in AdS_4 and $E(\alpha, \beta)$ is the incomplete elliptic integral of the second kind. Note that due to the condition $L \gg l$ the effects of the sides of the minimal surface can be neglected. The embedding function clearly reflects this approximation. In intrinsic coordinates the metric takes the form

$$ds^2{}_{induced} = \frac{{z_*}^2 d\theta^2 + 4\sin\theta d\lambda^2}{4z_*^2\sin\theta^2}, \tag{5.57}$$

the range of the coordinates being $0 \leq \theta \leq \pi$ and $-L \leq \lambda \leq L$. Further the turning point z_* can be written in terms of the width l of the subsystem as $z_* = \frac{\Gamma(\frac{1}{4})l}{2\sqrt{\pi}\Gamma(\frac{3}{4})}$. We also need to calculate the extrinsic curvature and the connection in the normal bundle. We again use a local tetrad adapted to the surface. The two spacelike bases are chosen such that they are tangent to the embedded surface. In intrinsic coordinate, they have the form,

$$e_2 = 2\sin\theta\partial_\theta, \;\; e_3 = z_*\sqrt{\sin\theta}\partial_\lambda \tag{5.58}$$

These are lifted to the full spacetime coordinates and then by using orthogonality relations one can construct the bases which span the normal bundle.

$$e_1 = z(\sin\theta\partial_z - \cos\theta\partial_y), \;\; e_0 = z\partial_t \tag{5.59}$$

The covariant derivatives of the normal vectors are given by,

$$\nabla_{e_2}e_1 = \sin\theta\; e_2, \;\; \nabla_{e_3}e_1 = -\sin\theta\; e_3, \;\; \nabla_{e_2}e_0 = 0, \;\; \nabla_{e_3}e_0 = 0 \tag{5.60}$$

From these one can read of the Weingarten maps and therefore the extrinsic curvatures,

$$W_{e_1}(e_2) = -\sin\theta\; e_2, \;\; W_{e_1}(e_3) = \sin\theta\; e_3, \;\; W_{e_0}(e_2) = 0, \;\; W_{e_0}(e_3) = 0 \tag{5.61}$$

We are now in a position to calculate the left hand hand side of the Jacobi equation. We expand the deviation vector as $\alpha^A\; e_A$ and then by using the above expressions we get,

$$\begin{aligned} 4\sin^2\theta\; \partial_\theta^2\alpha^1 + 2\sin\theta\cos\theta\; \partial_\theta\alpha^1 + {z_*}^2\sin\theta\; \partial_\lambda^2\alpha^1 - 2\cos^2\theta\; \alpha^1 &= F^1 \\ 4\sin^2\theta\; \partial_\theta^2\alpha^0 + 2\sin\theta\cos\theta\; \partial_\theta\alpha^0 + {z_*}^2\sin\theta\; \partial_\lambda^2\alpha^0 - 2\alpha^0 &= F^0 \end{aligned} \tag{5.62}$$

As before, we first analyze the homogeneous equations by solving them using separation of variables.

$$\begin{aligned}
\frac{d^2\Theta^1}{d\theta^2} + \frac{1}{2}\cot\theta\frac{d\Theta^1}{d\theta} - \left(\frac{1}{2}\cot^2\theta + \frac{k^2}{4\sin\theta}\right)\Theta^1 = 0 \\
\frac{d^2\Theta^0}{d\theta^2} + \frac{1}{2}\cot\theta\frac{d\Theta^0}{d\theta} - \left(\frac{1}{2}\text{cosec }^2\theta + \frac{k^2}{4\sin\theta}\right)\Theta^0 = 0 \\
\frac{d^2\Phi^{(0,1)}}{d\lambda^2} + \left(\frac{k}{z_*}\right)^2\Phi^{(0,1)} = 0
\end{aligned} \tag{5.63}$$

The solution to the θ part is given in terms of the generalized Heun's function, and can be shown to yield trivial solutions under the boundary conditions assumed. We will now solve the inhomogeneous Jacobi equation for the strip subsystem for two separate cases ,

1. Strip along $'x'$ boost along $'x'$: In this case we consider the width of the strip to be along the y direction and length along the x direction in bounday of AdS_4. The inhomogeneous term for the Jacobi equation in this case is calculated for the asymptotic Boosted AdS blackbrane geometry (appendix F) where the boost is along the x direction.

2. Strip along $'x'$ boost along $'y'$: In this case the direction of the strip remains unchanged but the inhomogeneous term is now calculated for the same geometry but with the boost being along y direction.

Changing the boost direction results in different deformations of the minimal surface. In the first case the surface remains on the same constant time (t) slice while in the second case there is a deviation of the surface along the time direction.

Strip along ‘x’ boost along ‘x’

In this case the e_0 equation turns out to be trivial i,e the inhomogeneous term is zero in the e_0 equation. Hence the surface remains on the same time slice. The e_1 equation is however non trivial. Note that since the right hand side is not a function of λ, only the $k = 0$ solution will be non trivial, which can be recast into,

$$\frac{d^2\Theta^1}{d\theta^2} + \frac{1}{2}\cot\theta\frac{d\Theta^1}{d\theta} - \left(\frac{1}{2}\cot^2\theta\right)\Theta^1 = \frac{1}{4}\left(3D + \frac{3C}{2}\right)z_*^3\left(\sin\theta\right)^{\frac{1}{2}} - \frac{7}{8}Dz_*^3\left(\sin\theta\right)^{\frac{5}{2}}, \tag{5.64}$$

where expressions for C, D can be found in appendix F. The homogeneous solutions for this is,

$$\Theta^1(\theta) = \frac{C_1\cos\theta}{\sqrt{\sin\theta}} + C_2\sin\theta\ {}_2F_1\left(\frac{1}{4}, 1; \frac{1}{2}, \cos^2\theta\right), \tag{5.65}$$

and the Wronskian is $W(\theta) = e^{-\frac{1}{2}\int\cot(\theta)d\theta} = \frac{1}{\sqrt{\sin\theta}}$. The full solution is then $\Theta^1_c + \Theta^1_p$.

$$\begin{aligned}\Theta^1 &= \frac{C_1\cos(\theta)}{\sqrt{\sin(\theta)}} + C_2\sin\theta\ {}_2F_1\left(\frac{1}{4}, 1; \frac{1}{2}, \cos^2\theta\right)\\ -\frac{\cos(\theta)}{\sqrt{\sin(\theta)}}&\int^\theta\left[\frac{1}{4}\left(3D + \frac{3C}{2}\right)z_*^3\left(\sin\theta\right)^{\frac{1}{2}} - \frac{7}{8}Dz_*^3\left(\sin\theta\right)^{\frac{5}{2}}\right]\left(\sin\theta'\right)^{\frac{3}{2}}\ {}_2F_1\left(\frac{1}{4}, 1; \frac{1}{2}, \cos^2\theta'\right)d\theta'\\ +\sin\theta\ {}_2F_1&\left(\frac{1}{4}, 1; \frac{1}{2}, \cos^2\theta\right)\int^\theta\left[\frac{1}{4}\left(3D + \frac{3C}{2}\right)z_*^3\left(\sin\theta\right)^{\frac{1}{2}} - \frac{7}{8}Dz_*^3\left(\sin\theta\right)^{\frac{5}{2}}\right]\cos(\theta')d\theta'\end{aligned} \tag{5.66}$$

It is not possible to get an analytical form of the integral involving the hypergeometric function. However since certain definite integrals are known for hypergeometric function, we hope that the final integral involving the change in area can be obtained by doing an integration by parts. To evaluate the integration constants we put the boundary condition $\Theta = 0$ at $\theta = 0$ and $\theta = \pi$. On demanding these the values of the constants turn out to be $C_1 = \frac{\pi z_*^3}{16}\left(2C + D\right)$ and $C_2 = -\frac{\Gamma\left(\frac{1}{4}\right)^2 z_*^3(2C+D)}{16\sqrt{2\pi}}$.

We now go over to the calculation of the integrals for calculating the change of area.

Before calculating the terms involving the deviation vector, we first evaluate the ones involving the metric perturbations only. The first order change in HEE is,

$$\Delta^{(1)}S = \frac{1}{4G_N}\Delta^{(1)}A = \frac{1}{8G_N}\int d^{d-1}\tau\ \sqrt{h}h^{ab}\overset{(1)}{P}(\partial_a,\partial_b) = \frac{2L}{32G_N}\pi z_*^2(2C+D)$$
$$= \frac{2L\times l^2}{4G_N z_0^3}\frac{(1+2\beta^2\gamma^2)\Gamma\left(\frac{1}{4}\right)^2}{32\,\Gamma\left(\frac{3}{4}\right)^2}, \tag{5.67}$$

which again matched with the results obtained in [31, 32]. As before the last three terms in the second variation formula are,

$$\int d^{d-1}\tau\ \sqrt{h}\frac{h^{ab}}{2}\overset{(2)}{P}(\partial_a,\partial_b) = \frac{2L\times\pi^{3/2}c^5(7C'+5D')}{21\sqrt{2}\Gamma\left(\frac{3}{4}\right)^2} \tag{5.68}$$

The next term which involves the product of perturbations is,

$$\int d^{d-1}\tau\ \sqrt{h}\frac{1}{2}h^{ac}h^{bd}\overset{(1)}{P}(\partial_a,\partial_b)\overset{(1)}{P}(\partial_c,\partial_d) = \frac{2L\times z_*^5 K\left(\frac{1}{2}\right)(77C^2+45D^2)}{231\sqrt{2}} \tag{5.69}$$

Finally we have the term

$$\int d^{d-1}\tau\ \sqrt{h}\frac{1}{4}h^{ab}h^{cd}\overset{(1)}{P}(\partial_c,\partial_d)\overset{(1)}{P}(\partial_a,\partial_b) = \frac{2L\times\sqrt{\pi}z_*^5\Gamma\left(\frac{5}{4}\right)(77C^2+110CD+45D^2)}{462\Gamma\left(\frac{3}{4}\right)} \tag{5.70}$$

Now we go over to the other integrals. Consider the term,

$$\int d^{d-1}\tau\ \sqrt{h}\Big(h^{ab}h^{cd}\overset{(1)}{P}(\partial_b,\partial_d)g(N^\perp,K(\partial_a,\partial_c)) - h^{ab}g(C(\partial_a,\partial_b),N^\perp)\Big)$$
$$= \int_{-L}^{L}\int_0^\pi \frac{1}{2z_*\sin^{\frac{3}{2}}\theta}\left[z_*^3\left(\frac{3C}{2}+3D\right)\sin^{\frac{5}{2}}\theta - \frac{7}{2}z_*^3 D\sin^{\frac{9}{2}}\theta\right]\Theta^1\, d\theta\, d\lambda \tag{5.71}$$

Note that Θ^1 contains two terms. One that does not have an analytical form and the other which does. Lets write these as $\Theta^1 = -\frac{\cos(\theta)}{\sqrt{2\sin(\theta)}}\int_0^\theta f(\theta')\,d\theta' + G(\theta) + \Theta_c^1(\theta)$. Therefore the above integral becomes,

$$\int_{-L}^{L}\int_0^\pi \qquad \frac{1}{2z_*\sin^{\frac{3}{2}}\theta}\left[z_*^3\left(\frac{3C}{2}+3D\right)\sin^{\frac{5}{2}}\theta - \frac{7}{2}z_*^3 D\sin^{\frac{9}{2}}\theta\right] \times\left(-\frac{\cos(\theta)}{\sqrt{\sin(\theta)}}\int_0^\theta f(\theta')\,d\theta' + G(\theta) + \Theta_c^1\right)d\theta\,d\lambda \tag{5.72}$$

Note that the $G(\theta)$ can be obtained easily and the value evaluates to,

$$\frac{2L\times\sqrt{\pi}z_*^5\Gamma\left(\frac{9}{4}\right)(77C^2+110CD+29D^2)}{352\Gamma\left(\frac{11}{4}\right)} \tag{5.73}$$

The complementary part of the solution gives,

$$-\frac{2L}{64}\sqrt{\frac{\pi}{2}}z_*^5\Gamma\left(\frac{1}{4}\right)^2(2C+D)^2 \tag{5.74}$$

The other integral is of the form $\int_0^\pi\left(g(\theta)\int_0^\theta f(\theta')d\theta'\right)d\theta$ and can be evaluated by parts,

$$\int_0^\pi g(x)\int_0^x f(x')dx'dx = \left[\left(\int_0^x f(x')dx'\right)\left(\int g(x)dx\right)\right]_0^\pi - \int_0^\pi f(x)\int g(x')dx'dx \tag{5.75}$$

The first term in the above expression does not contribute, while the second term reproduces the number obtained for $G(\theta)$. The total variation $\Delta^{(2)}S$ is then given as,

$$\Delta^{(2)}S = \frac{1}{4G_N}\Delta^{(2)}A = \frac{2L\times l^5}{z_0^6}\frac{\Gamma\left(\frac{1}{4}\right)^5}{\Gamma\left(\frac{3}{4}\right)^7}\frac{(-84(\pi-1)\beta^4\gamma^4+28(4-3\pi)\beta^2\gamma^2+(48-21\pi))}{21504\times 4G_N\sqrt{2}\pi} \tag{5.76}$$

This expression gives the second order change of HEE. As in the case of circular disk, the positivity of relative entropy demands that $\Delta^{(2)}A \leq 0$. This can be checked through a plot of $\Delta^{(2)}A$ against β (See Fig-1). The whole expression is negative (at $\beta = 0$) and monotonically decreasing as β. The change ΔS or the plot cannot however be trusted for too large values of β, since one needs to add further higher order corrections to the change for large β.

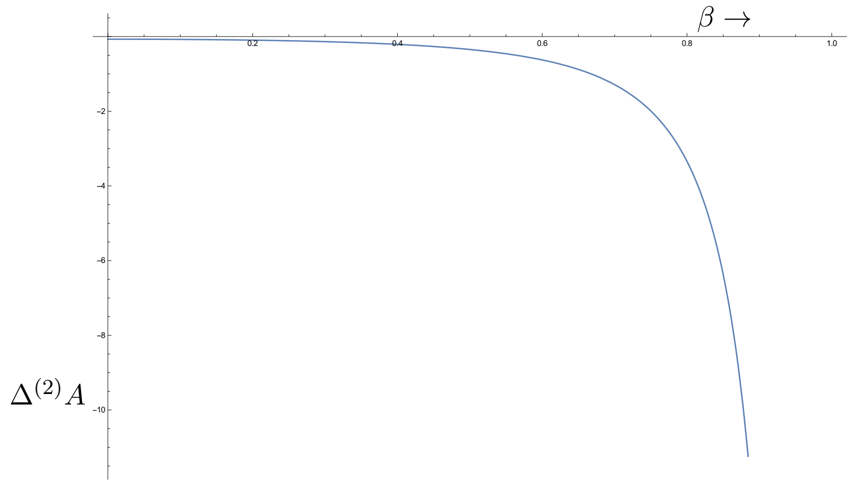


Figure 5.1: *Plot of $\Delta^{(2)}A$ vs β for strip along x boost along y*

The full expression for change of HEE is then given by

$$\Delta S = \Delta^{(1)}S + \frac{1}{2}\Delta^{(2)}S$$
$$= \frac{2L \times l^2}{4G_N z_0^3} \frac{(1+2\beta^2\gamma^2)\Gamma\left(\frac{1}{4}\right)^2}{32\,\Gamma\left(\frac{3}{4}\right)^2} + \frac{2L \times l^5}{2 \times z_0^6} \frac{\Gamma\left(\frac{1}{4}\right)^5}{\Gamma\left(\frac{3}{4}\right)^7} \frac{(-84(\pi-1)\beta^4\gamma^4 + 28(4-3\pi)\beta^2\gamma^2 + (48-21\pi))}{4G_N \times 21504\,\sqrt{2}\pi} \tag{5.77}$$

the above expression gives the net change in HEE for strip entangling surface upto second order over pure AdS(ground state) value. This is the most important result of this chapter. One can easily check that the above result exactly matches with the (2.38) for $d = 3$ case.

It is also important to note that in this case the e_0 component of the deviation vector is zero. This shows that the perturbed minimal surface for this choice of subsystem still remains on the $t =$ constant slice. This is the reason why this result agrees with that in (2.38). Although the boosted black brane background is not static, but our solution tells us that the surface remains on the same slice as pure AdS at first order deviation.

Strip along 'x' boost along 'y'

In this case all the integrals for e_1 are same as that of the previous case with $C,\ D$ replaced by $\tilde{C},\ \tilde{D}$ and $C',\ D'$ replaced by $\tilde{C}',\ \tilde{D}'$ (see appendix F). However in this case the non homogeneous part of the e_0 equation is non trivial. Hence the extremal surface doesn't remain on the same time slice . The equation is,

$$4\sin^2\theta\ \partial_\theta^2\alpha^0 + 2\sin\theta\cos\theta\ \partial_\theta\alpha^0 + {z_*}^2\sin\theta\ \partial_\lambda^2\alpha^0 - 2\alpha^0 = -3z_*^3(\sin\theta)^{\frac{5}{2}}B\cos\theta, \quad (5.78)$$

which following the previous arguments reduces to solving only the equation,

$$\frac{d^2\Theta^0}{d\theta^2} + \frac{1}{2}\cot\theta\frac{d\Theta^0}{d\theta} - \frac{1}{2}\text{cosec}\,^2\theta\Theta^0 = -\frac{3}{4}z_*^3(\sin\theta)^{\frac{1}{2}}B\cos\theta \quad (5.79)$$

The solutions of this can be obtained in a straightforward manner and therefore we do not have to resort to efforts made in the previous section. The full solutions turns out to be of the form,

$$\Theta^0 = \frac{-\tilde{B}z_*^3\theta}{4\sqrt{\sin\theta}} + \frac{\tilde{B}z_*^3\sin 2\theta}{8\sqrt{\sin\theta}} - \frac{2C_1E\Big(\frac{1}{4}(\pi-2\theta)\big|\,2\Big)}{\sqrt{\sin(\theta)}} + \frac{C_2}{\sqrt{\sin(\theta)}} \quad (5.80)$$

Imposing the conditions $\Theta = 0$ at $\theta = 0$ and $\theta = \pi$, fixes C_1 and C_2 to, the solutions of which are,

$$C_1 = \frac{\pi \tilde{B} z_*^3}{8\sqrt{2}\left(2E\left(\frac{1}{2}\right) - K\left(\frac{1}{2}\right)\right)}, \quad C_2 = \frac{1}{8}\pi \tilde{B} z_*^3, \tag{5.81}$$

where $K(\alpha)$ and $E(\alpha)$ are the complete elliptic integral of the first and second kind respectively. The contributions coming from the component α^1 of the deviation vector turns out to be same as that in the previous section with $C,\ D$ replaced by $\tilde{C},\ \tilde{D}$ and $C',\ D'$ replaced by $\tilde{C}',\ \tilde{D}'$. The only other contribution different from the previous case comes from $-Tr(C)$ for the component α^0 of the deviation vector and evaluates to,

$$\frac{2L\, \pi^{3/2}(21\pi - 80)B^2 z_*^5}{336\sqrt{2}\Gamma\left(\frac{3}{4}\right)^2} \tag{5.82}$$

Total variation $\Delta^{(2)}S$ without the previous term is then given by,

$$\Delta^{(2)}S = \frac{2L \times l^5}{4G_N z_0^6}\ \frac{\Gamma\left(\frac{1}{4}\right)^5}{\Gamma\left(\frac{3}{4}\right)^7}\left(\frac{(20-21\pi)\beta^4\gamma^4 + 2(40-21\pi)\beta^2\gamma^2 + 2(21\pi-80)\beta\gamma^4 + (48-21\pi)}{21504\sqrt{2}\pi}\right) \tag{5.83}$$

As in the previous case $\Delta^{(2)}A \leq 0$. This can be checked by plotting $\Delta^{(2)}S$ against β(see Fig-2). It is negative and monotonically decreasing as a function of β. It is important to note that the boost independent term in the expression for $\Delta^{(2)}S$ for both the cases is same. Setting boost to zero makes both the cases identical to AdS black brane geometry.

The first order change in HEE is given by

$$\Delta^{(1)}S = \frac{1}{4G_N}\Delta^{(1)}A = \frac{2L \times l^2}{4G_N z_0^3}\frac{(1+\beta^2\gamma^2)\Gamma\left(\frac{1}{4}\right)^2}{32\,\Gamma\left(\frac{3}{4}\right)^2} \tag{5.84}$$

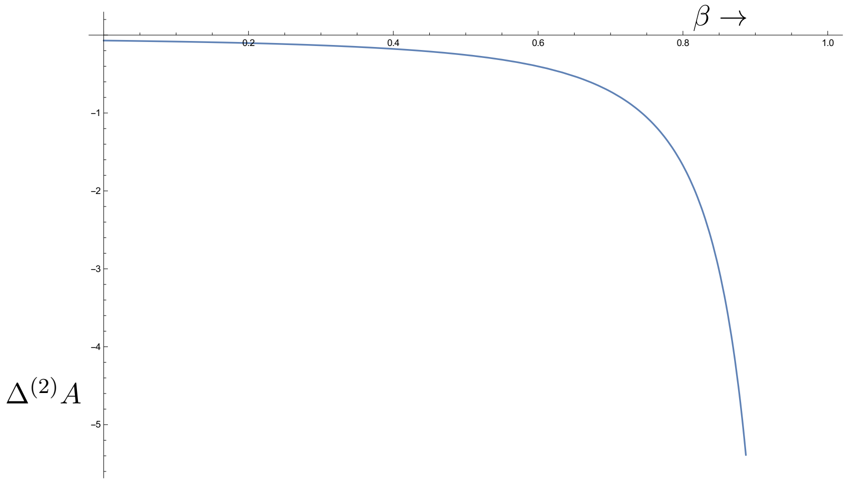


Figure 5.2: *Plot of* $\Delta^{(2)}A$ *vs* β *for strip along* x *boost along* y

Thus the full expression for change in HEE is then given by

$$\begin{aligned}\Delta S &= \Delta^{(1)}S + \frac{1}{2}\Delta^{(2)}S \\ &= \frac{2L \times l^2}{4G_N z_0^3}\frac{(1+\beta^2\gamma^2)\Gamma\left(\frac{1}{4}\right)^2}{32\,\Gamma\left(\frac{3}{4}\right)^2}\end{aligned}$$

$$+\frac{2L \times l^5}{8G_N z_0^6}\ \frac{\Gamma\left(\frac{1}{4}\right)^5}{\Gamma\left(\frac{3}{4}\right)^7}\left(\frac{(20-21\pi)\beta^4\gamma^4 + 2(40-21\pi)\beta^2\gamma^2 + 2(21\pi-80)\beta\gamma^4 + (48-21\pi)}{21504\sqrt{2}\pi}\right) \tag{5.85}$$

the above expression gives the net change in HEE for strip entangling surface upto second order over pure AdS(ground state) value.

5.7 Issues of Gauge dependence

The Φ_λ's in section 5.3 are called the identification maps. It encodes the information about how points in the perturbed and the unperturbed space times are to be identified. The notion of gauge transformation can be shown to arise due to different choices of the Φ_λ's.

It is evident that the identification maps can be so chosen that the location of the perturbed minimal surface in the unperturbed spacetime is same as that of the unperturbed minimal surface. This is precisely the interpretation of the Hollands-Wald gauge [115] used in [106, 116, 117]. But it seems that this in general can be done at any order of perturbation and not just at the linear order. Further, it seems that by choice of such gauge one renders the inhomogeneous term, in the Jacobi equation obtained, trivial and therefore irrelevant. We must emphasize that this is not the case. In order to find the Hollands Wald gauge (at linear order) one has to solve a linear second order differential equation which is precisely the inhomogeneous Jacobi equation. This has also been pointed out in [118]. Therefore choosing the Hollands-Wald gauge does not trivialize the problem of finding the change in area. However, it is absolutely possible that the Holland- Wald gauge is a convenient choice if one tries to find identities that the higher order perturbations of the area functional satisfy or finding relations between two gauge independent quantities like the 'Fisher information' and the canonical energy [106].

Having discussed this it is quite viable to state that the inhomogeneous equation is gauge covariant. In other words any gauge transformation of the metric perturbation can be absorbed in a shift of the deviation vector itself. This is a quite plausible conclusion that follows from the following lemma due to [114]. The linear perturbation Q_1 of a quantity Q_0 on (M, g) is gauge invariant if and only if one of the following holds: (i) Q_0 vanishes, (ii) Q_0 is a constant scalar, (iii) Q_0 is a constant linear combination of products of Kronecker deltas. In our case Q_0 is the mean curvature (H) of the extremal surface in the background spacetime and hence is identically zero. However there is a subtle issue in application of the above lemma in our case. The quantities Q defined in the lemma are globally defined while H is locally defined on a codimension two surface. The expression for the second variation of the area functional is however invariant under different choices of $\Phi_\lambda : M \to M_\lambda$.

5.8 Conclusion

A few comments about higher order perturbations are in order. As is usual with any perturbation theory, the homogeneous part of the second order perturbation equation would be same as the Jacobi equation. However the inhomogeneous term will now depend both on second order perturbations as well as first order deviations. Note the second order deviation vector M (say), can always be taken to commute with N owing to the fact that they represent independent variations. Since the normal bundle is two dimensional one can have at most two mutually commuting directions. Hence it seems that the perturbation will terminate at second order and the complete change of entanglement entropy can be obtained by exponentiating this change upto second order. However this is speculative and requires further investigation. We have presented a systematic approach to obtain the change in HEE up to second order. For simplicity we have calculated this in 4-dimensions but the approach remains unchanged in higher dimensions. The inhomogeneous Jacobi equation and second variation of the area functional presented here can be applied to non AdS geometries also. In fact the Jacobi operator simplifies for the asymptotically flat case. We have seen that second order change receives contributions from first order changes in the embeddings and second order change of the bulk metric. In this approach the nature of the flow of the extremal surface can be understood by looking at the components of the deviation vector. Further, having obtained the second variation one can check if more general entropy bounds [96–98, 100] or has any relation with geometric inequalities [99] in general.

CHAPTER 6

SUMMARY

In chapter(2) We found that the first law of entanglement thermodynamics for 'boosted' AdS_{d+1} having black hole in the IR region is given by

$$\triangle\mathcal{E}^* = T_E^* \triangle S_E^* + \mu_* \triangle \mathcal{N}^* + \mathcal{V}_* \triangle \mathcal{P}^*$$

Our result emphasizes the fact that the form of the first law changes under higher order corrections to the entanglement entropy. It is apparent when the entanglement law (2.25) at the first order is compared with the second order result in (6.1). We find that even in the absence of boosts the renormalization of the thermodynamic quantities like entropy, energy, subsystem size (all extensive quantities) and entanglement temperature (intensive quantity) becomes essential at the second order. The chemical potential which is negligible at the first order becomes relevant at next order. We expect no further changes in the form of the first law for the AdS background (2.1), so the first law form (6.1) will remain unchanged at higher orders provided we renormalize/redefine the thermodynamic quantities appropriately. Also, we have determined that the entanglement temperature of the subsystem will be higher for a bigger size black hole. Finally, as we have studied (IR) excitations in AdS spacetime, and since AdS background is an universal solution of (gauged) supergravities with negative cosmological constant, we expect these results will be holding true quite generally.

The physical relevance of our results in chapter (3) is indicated by the fact that the entanglement entropy of subsytems is affected in the presence of boost, or a flow. It is not entirely an unexpected result as the boost indeed represents an asymmetric excitation of the system. It means subsystems along the flow and perpedicular to it get differently entangled as we have determined, $\triangle S_{\perp} > \triangle S_{\|}$. Upto first order this asymmetry is proportional to β^2 (for small velocities). These result however will change at the second order perturbative calculations. Our results however imply more generic situations. Even in the absence of a flow, provided there exists pressure asymmetry in the CFT due to some other reason, the entanglement asymmetry will always arise. The boosted black brane systems are used here only as the known examples to study asymmetric systems. It would be worthwhile to explore other systems like Bianchi models having more generic asymmetry.

In chapter (4) we found that second order changes in ΔS_E for $2+1$ dimensions correspond to

1. Second order gravitational perturbations and

2. First order changes in the shape of extremal surface.

The second order gravitational perturbations can be obtained by solving the perturbed Einstein's equation. Alternatively when the bulk metric is known, this corresponds to the $\mathcal{O}(z^4)$ (in $2+1$ dimensions) terms in the Fefferman Graham expansion. However a systematic approach for finding the change in shape of the extremal surface is not known. We propose that these changes can be systematically calculated by solving the "generalized deviation equation". We further write an alternative form of the first law for entanglement thermodynamics given by,

$$d(\Delta S_E) = \frac{1}{T_E} d(\Delta E) - \frac{\Omega_E}{T_E} d(\Delta J) \tag{6.1}$$

This has been shown to asymptote exactly to the black hole first law for BTZ in the large system size limit.

In Chapter (5) we generalized our variational approach to higher dimensions. We were able to generalize the geodesic deviation equation to a surface deviation equation(Inhomogeneous Jacobi equation). We have obtained the area variation upto second order metric perturbations and first order surface deviation. This gave us a covariant and coordinate independent expression. We have checked our results with those in chapter(2) and the results are identical. It is important to note that our approach is independent of background geometry. One can use this approach in asymptotically flat spacetimes as well.

Thus in this thesis we have seen two approaches of calculating second order change in holographic entanglement entropy. The perturbative or passive approach discussed in (2) and (3) starts with solving the minimal surface equation perturbatively in some asymptotically AdS background. This perturbative or asymptotic approach take into consideration only the asymptotic region of the perturbed background. The details of IR does not contribute to the change in entanglement entropy. This approach is passive in the sense it starts from the perturbed geometry and slowly reaches AdS near the asymptotic region as perturbation.

The variational approach is active in the sense it starts from the pure AdS background and reaches the asymptotically AdS geometry perturbatively. In this approach we don't solve for the minimal surface equation, instead we solve for the inhomogeneous Jacobi equation and try to get the minimal surface order by order in deviation. In this respect both the approaches are perturbative. We conclude by saying that both the approach merge at solving the minimal surface asymptotically or obtaining it order by order in the deviation.

Appendix A

Conventions for Boosted Black Brane

The physical observables such as energy, momentum and pressure can be obtained by expanding the bulk AdS geometry (3.1) in suitable Feffermann-Graham asymptotic coordinates [24, 84, 85]

$$ds^2 = \frac{L^2}{u^2}\left(du^2 + G^{\frac{4}{d}}[\frac{-f dt^2}{K} + K(dy-\omega)^2 + dx_1^2 + \cdots + dx_{d-2}^2]\right)$$
$$G = 1 + \frac{u^d}{u_0^d}, \quad f \simeq (1 - \frac{4u^d}{u_0^d}), \quad K \simeq 1 + 4\beta^2\gamma^2\frac{u^d}{u_0^d} \tag{A.1}$$

In u coordinate the boundary is at $u = 0$, and $u_0^d = 4z_0^d$. The Kaluza-Klein gauge form is

$$\omega = \beta^{-1}(1 - \frac{1}{K})dt. \tag{A.2}$$

In these asymptotic coordinates, the coefficients of u^d terms in the metric expansion give rise to the energy-momentum tensor of the boundary CFT. From (A.1) these coefficients of the metric are

$$< t_{00} >= (\frac{d-1}{d} + \beta^2\gamma^2)\frac{4}{u_0^d}, \qquad < t_{0y} >= \beta\gamma^2\frac{4}{u_0^d}$$
$$< t_{11} >= \frac{1}{d}\frac{4}{u_0^d} =< t_{22} >= \ \cdots \tag{A.3}$$

The boundary energy-momentum tensor, $< T_{ab} >= \frac{dL^{d-1}}{16\pi G} < t_{ab} >$, is traceless as we have conformal theory. The energy of excitations and the momentum for the boosted CFT$_d$ will be

$$E = \frac{dL^{d-1}v_{d-1}}{16\pi G_{d+1}} < t_{00} >= \frac{dL^{d-1}v_{d-2}r_y}{8G_{d+1}}(\frac{d-1}{d} + \beta^2\gamma^2)z_0^{-d}$$
$$P_y = \frac{dL^{d-1}v_{d-2}r_y}{8G_{d+1}}\beta\gamma^2 z_0^{-d} \tag{A.4}$$

where volume $v_{d-2} = l_1 l_2 \cdots l_{d-2}$, and we have compactified y on a circle of radius r_y. Note the momentum (charge) $P_y = \frac{N}{r_y}$ is quantized and N would have integral values. In the absence of boost the charge would be vanishing. We note down the nontrivial chemical potential which is defined by the value of gauge potential at the horizon

$$\mu_{Th} = \frac{\beta}{r_y} \tag{A.5}$$

Corresponding thermal entropy and temperature can be obtained from (3.1). These are given by

$$S_{Th} \equiv \frac{[Area]_{horizon}}{4G_{d+1}} = \frac{\pi L^{d-1}v_{d-2}r_y}{2G_{d+1}}\frac{\gamma}{z_0^{d-1}}$$
$$T_{Th} = \frac{d}{4\pi z_0\gamma} \tag{A.6}$$

These thermal quantities satisfy the following first law of black hole mechanics

$$\delta E_{Th} = T_{Th}\delta S_{Th} + \mu_{_{Th}}\delta N \ . \tag{A.7}$$

But if we allow small volume changes, say $\delta v = (\delta l_1)l_2 l_3 \cdots l_{d-2}$, the black hole thermodynamic law would be

$$\delta E_{Th} = T_{Th}\delta S_{Th} + \mu_{_{Th}}\delta N - \mathcal{P}_1 \delta v \ . \tag{A.8}$$

where pressure component is $\mathcal{P}_1 = \frac{L^{d-1} r_y}{8G_{d+1}} z_0^{-d}$.

APPENDIX B

SOME USEFUL BETA FUNCTION

Some useful Beta function integrals we have used are given here

$$
\begin{aligned}
b_0 &= \int_0^1 d\xi \xi^{d-1} \frac{1}{\sqrt{R}} = \frac{1}{2(d-1)} B(\frac{d}{2d-2}, \frac{1}{2}) \\
b_1 &= \int_0^1 d\xi \xi^{2d-1} \frac{1}{\sqrt{R}} = \frac{1}{2(d-1)} B(\frac{d}{d-1}, \frac{1}{2}) \\
b_2 &= \int_0^1 d\xi \xi^{3d-1} \frac{1}{\sqrt{R}} = \frac{1}{2(d-1)} B(\frac{3d}{2d-2}, \frac{1}{2}) \\
I_l &= \int_0^1 d\xi \xi^{d-1} (1-\xi^d) \frac{1}{R^{\frac{3}{2}}} = \frac{d+1}{d-1} b_1 - \frac{1}{d-1} b_0 \\
J_l &= \int_0^1 dt \;\; t^{d-1} \left[\frac{\beta^2 \gamma^2}{4} t^d - \frac{\beta^4 \gamma^4}{8} \left(4 - 3 \frac{\left(1-t^d\right)}{R} \right) \right] \frac{\left(1-t^d\right)}{R^{\frac{3}{2}}}
\end{aligned} \tag{B.1}
$$

where $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ are the Beta-functions. Further integrals are

$$
\begin{aligned}
a_0 &= \int_0^1 d\xi \xi^{-d+1} \frac{1}{\sqrt{R}} = \frac{1}{2(d-1)} B(\frac{1-d/2}{d-1}, \frac{1}{2}) \\
a_1 &= \int_0^1 d\xi \xi^{-d+1} \frac{\xi^d}{\sqrt{R}} = \frac{1}{2(d-1)} B(\frac{1}{d-1}, \frac{1}{2}) \\
a_2 &= \int_0^1 d\xi \xi^{-d+1} \frac{\xi^{2d}}{\sqrt{R}} = \frac{1}{2(d-1)} B(\frac{1+d/2}{d-1}, \frac{1}{2}) \\
I_a &= \int_0^1 d\xi \xi^{d-1} (1-\xi^{2d}) \frac{1}{R^{3/2}} = \frac{2d+1}{d-1} b_2 - \frac{1}{d-1} b_0
\end{aligned}
\tag{B.2}
$$

Some identities we have used are

$$
b_0 = (2-d) a_0, \quad b_1 = \frac{2}{d+1} a_1, \quad b_2 = \frac{2+d}{2d+1} a_2 \, . \tag{B.3}
$$

APPENDIX C

THE ASYMPTOTIC EXPANSION FOR NONCONFORMAL BLACK D-BRANES

The asymptotic expansion in the Fefferman-Graham coordinates is required to find the energy-momentum tensor of the boundary field theory. The relevant details on holographic renormalization can be found in [24, 84, 85, 119]. Let us define a new holographic coordinate u through

$$z^2 = F^{-\frac{4}{\tilde{p}}} u^2, \qquad F = 1 + \frac{u^{\tilde{p}}}{u_0^{\tilde{p}}}, \qquad u_0^{\tilde{p}} \equiv 4 z_0^{\tilde{p}} \tag{C.1}$$

In these u coordinates an expansion of (3.41) in the neighborhood of UV boundary $(u = 0)$ becomes

$$\begin{aligned} ds^2 \simeq g_{eff} \Bigg[\frac{1}{u^2} \big[(-1 + 4(\frac{\tilde{p}-1}{\tilde{p}} + \beta^2\gamma^2) \frac{u^{\tilde{p}}}{u_0^{\tilde{p}}} + \cdots) dt^2 + (1 + 4(\frac{1}{\tilde{p}} + \beta^2\gamma^2) \frac{u^{\tilde{p}}}{u_0^{\tilde{p}}} + \cdots) dy^2 \\ - \frac{8\beta\gamma^2 u^4}{u_0^{\tilde{p}}} dt dy + (1 + \frac{4}{\tilde{p}} \frac{u^{\tilde{p}}}{u_0^{\tilde{p}}} + \cdots)(dx_2^2 + \cdots + dx_p^2) \big] + \frac{4}{(5-p)^2} \frac{du^2}{u^2} + d\Omega_{8-p}^2 \Bigg] \\ \equiv g_{eff} \left[\frac{1}{u^2} (\eta_{\alpha\beta} + t_{\alpha\beta} u^{\tilde{p}} + \cdots) dx^\alpha dx^\beta + \frac{4}{(5-p)^2} \frac{du^2}{u^2} + d\Omega_{8-p}^2 \right] \end{aligned} \tag{C.2}$$

The last line in the above equation indicates that the spacetime geometry is expanded in asymptotic neighborhood of conformally $AdS_{p+2} \times S^{8-p}$ spacetime. Besides in these coordinates, u coincides with the energy scale of the AdS_{p+2} geometry. The $\eta_{\alpha\beta}$ is flat Minkowski metric with index $\alpha = 0, 1, 2, \cdots, p$. The effective coupling has the FG expansion (RG flow) given by

$$g_{eff} = \frac{(\lambda_p u^{3-p})^{\frac{1}{5-p}}}{F^{\frac{3-p}{7-p}}} \simeq (\lambda_p u^{3-p})^{\frac{1}{5-p}}(1 - \frac{3-p}{7-p}\frac{u^{\tilde{p}}}{u_0^{\tilde{p}}} + \cdots) \tag{C.3}$$

In $p = 3$ (conformal) case the g_{eff} however remains fixed. The important point to notice from the FG expansion is that the overall conformal factor of the string metric (C.2) and the string coupling e^{ϕ} (given in (3.41)) are both governed by the fluctuations of the single quantity g_{eff}. The fluctuations of the dilaton field, $\delta\phi$, can also be obtained from the expression

$$\begin{aligned} e^{\phi} &= \frac{(2\pi)^{2-p}}{d_p N}(\lambda_p u^{3-p})^{\frac{\tilde{p}}{4}}(1 - \frac{3-p}{2}\frac{u^{\tilde{p}}}{u_0^{\tilde{p}}} + \mathcal{O}(u^{2\tilde{p}})) \\ &\equiv e^{\phi_0}(1 + \delta\phi_{(\tilde{p})}u^{\tilde{p}} + \cdots) \end{aligned} \tag{C.4}$$

where ϕ_0 represents the dilaton field in the absence of the excitations. The first order fluctuation of dilaton are thus $\delta\phi_{(\tilde{p})} = -\frac{3-p}{2}\frac{1}{u_0^{\tilde{p}}}$. Obviously $\delta\phi_{(\tilde{p})}$ has opposite signs for $p > 3$ and $p < 3$ branes. (For D3 brane $\delta\phi_{(\tilde{p})}$ vanishes as it should be for $4D$ conformal field theory.) The nonvanishing components of stress-energy tensor of the boundary theory can now be obtained from the expression within the angular brackets in asymptotic expansion

(C.2)

$$
\begin{aligned}
t_{00} &= (\frac{\tilde{p}-1}{\tilde{p}} + \beta^2\gamma^2)\frac{4}{u_0^{\tilde{p}}}, \quad t_{yy} = (\frac{1}{\tilde{p}} + \beta^2\gamma^2)\frac{4}{u_0^{\tilde{p}}} \\
t_{0y} &= \beta\gamma^2\frac{4}{u_0^{\tilde{p}}}, \quad t_{ii} = \frac{1}{\tilde{p}}\frac{4}{u_0^{\tilde{p}}}, \quad (i = 2, 3, ..., p)
\end{aligned}
\tag{C.5}
$$

The tensor $t_{\alpha\beta}$ has a nonvanishing trace. It is worthwhile to observe that the trace, $t_\alpha^{\ \alpha}$, and $\delta\phi$ have a relationship

$$
\frac{1}{4}t_\alpha^{\ \alpha} - \frac{3-p}{7-p}\delta\phi_{(\tilde{p})} = 0 \tag{C.6}
$$

as they both depend on single deformation parameter u_0. Actually this relation follows from Ward identities in holographic renormalization of the boundary theory [120]. Also $\nabla_\alpha t^{\alpha\beta} = 0$ trivially. We should not be checking them over here as these are automatic in the FG expansion (C.2) of nonextremal geometry. The energy of the excitations above the extremality for the boosted solutions is then given by

$$
\triangle\mathcal{E} = \frac{V_p\Theta_{8-p}Q_p}{16\pi G_N}(\frac{\tilde{p}-1}{\tilde{p}} + \beta^2\gamma^2)\frac{7-p}{z_0^{\tilde{p}}} \tag{C.7}
$$

where V_p is the p-dimensional spatial volume of all x_i's and Q_p is a combinatoric factor defined earlier. Θ_{8-p} is unit volume of the S^{8-p}, and G_N is the Newton's constant in ten dimensions. Similarly pressure components along the boost and in perpendicular directions are

$$
\begin{aligned}
\triangle\mathcal{P}_\parallel &= \triangle\mathcal{P}_y = \frac{\Theta_{8-p}Q_p}{16\pi G_N}(\frac{1}{\tilde{p}} + \beta^2\gamma^2)\frac{7-p}{z_0^{\tilde{p}}} \\
\triangle\mathcal{P}_\perp &= \triangle\mathcal{P}_{x_2} = \frac{\Theta_{8-p}Q_p}{16\pi G_N}\frac{7-p}{\tilde{p}z_0^{\tilde{p}}} = \triangle\mathcal{P}_{x_3} = \cdots .
\end{aligned}
\tag{C.8}
$$

APPENDIX D

FEFFERMAN GRAHAM EXPANSION, BOUNDARY STRESS TENSOR AND PERTURBATIONS ABOUT AdS_3

The rotating BTZ metric is given as,

$$ds^2 = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2}\, dt^2 + \frac{r^2}{(r^2 - r_+^2)(r^2 - r_-^2)}\, dr^2 + r^2 \left(d\phi - \frac{r_+ r_-}{r^2}\, dt\right)^2 , \tag{D.1}$$

where r_+, r_- are the radii of the outer and inner horizon respectively. The physical observables like energy and angular momentum can be obtained by expanding the above metric in suitable Fefferman Graham(Asymptotic) coordinates near the AdS_3 boundary. This can be realized by defining a new coordinate ρ through $\frac{d\rho}{\rho} = \frac{dr}{r\sqrt{f(r)}}$. In terms of ρ this metric becomes

$$ds^2 = \frac{d\rho^2}{\rho^2} + \rho^2\Bigg[(-dt^2 + d\phi^2) + \frac{1}{\rho^2}\left(\frac{(r_+^2 + r_-^2)}{2}dt^2 - 2r_+r_-dtd\phi + \frac{(r_+^2 + r_-^2)}{2}d\phi^2\right) + \frac{1}{\rho^4}\left(\frac{-(r_-^2 - r_+^2)^2}{16}dt^2 + \frac{(r_-^2 - r_+^2)^2}{16}d\phi^2\right)\Bigg] \tag{D.2}$$

In coordinates ($\rho = \frac{1}{z}$) the metric becomes

$$ds^2 = \frac{dz^2 + (\eta_{\mu\nu} + z^2\gamma^{(2)}_{\mu\nu} + z^4\gamma^{(4)}_{\mu\nu} + ...)dx^\mu dx^\nu}{z^2} \tag{D.3}$$

The above metric is now in Fefferman Graham form.

Where the boundary energy momentum tensor ($\langle T_{\mu\nu}\rangle = \frac{d}{16\pi G}\gamma^{(d)}_{\mu\nu}$ in $d+1$ dimensions) [25, 121] is given by

$$8\pi G\, T_{\mu\nu} = \gamma^{(2)}_{\mu\nu} = \begin{bmatrix} \frac{(r_+^2+r_-^2)}{2} & -r_+r_- \\ -r_+r_- & \frac{(r_+^2+r_-^2)}{2} \end{bmatrix}$$

In $2+1$ dimensions there are no conformal anomalies in the stress tensor [121] and $\gamma^{(4)}_{\mu\nu}$ is given by

$$\gamma^{(4)}_{\mu\nu} = \begin{bmatrix} \frac{-(r_-^2-r_+^2)^2}{16} & 0 \\ 0 & \frac{(r_-^2-r_+^2)^2}{16} \end{bmatrix}$$

D.1 Length of space-like geodesic for rotating BTZ

According to HRT proposal extremal surfaces in 2+1 dimensions are given by spacelike geodesics. Here we obtain HEE for rotating BTZ black hole by calculating geodesic length without using the fact that BTZ is locally isometric to AdS_3. In the rotating BTZ metric

(D.1), we will introduce the following notations [122],

$$\mathscr{M} = r_+^2 + r_-^2, \;\; \mathscr{J} = 2r_+r_-, \;\; \beta_\pm = \frac{2\pi}{r_+ \pm r_-}, \tag{D.4}$$

These notations should be read independently of those introduced in the body of the chapter. However the final expression is obtained in terms of quantities which have been introduced earlier. For a general curve parametrized by λ (say) such that the tangent vector (v^a) is given by $v^a = \left(\frac{dt}{d\lambda}, \frac{dr}{d\lambda}, \frac{d\phi}{d\lambda}\right)$, one has,

$$-m^2 = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2}\,\dot{t}^2 + \frac{r^2}{(r^2 - r_+^2)(r^2 - r_-^2)}\dot{r}^2 + r^2\left(\dot{\phi} - \frac{r_+r_-}{r^2}\,\dot{t}\right)^2, \tag{D.5}$$

where dots imply derivative with respect to λ. If the curve is a geodesic and ∂_t and ∂_ϕ being Killing vectors, one has the following constants of motion.

$$\mathscr{E} = -g_{ab}v^a\xi^b = \left(-\mathscr{M} + r^2\right)\dot{t} + \frac{\mathscr{J}}{2}\dot{\phi}, \tag{D.6}$$

where $\xi^a = \partial_t^a$, and

$$\mathscr{L} = g_{ab}v^a\Phi^b = r^2\dot{\phi} - \frac{\mathscr{J}}{2}\dot{t}, \tag{D.7}$$

where $\Phi^a = \partial_\phi^a$. We define the following dimensionless coordinates and parameters for brevity.

$$\begin{aligned} \hat{r} &= \frac{r}{\sqrt{\mathscr{M}}}, \quad \hat{\phi} = \phi\sqrt{\mathscr{M}}, \quad \hat{t} = t\sqrt{\mathscr{M}} \\ \hat{\mathscr{E}} &= \frac{\mathscr{E}}{\sqrt{\mathscr{M}}}, \quad \hat{\mathscr{L}} = \frac{\mathscr{L}}{\sqrt{\mathscr{M}}}, \quad \hat{\mathscr{J}} = \frac{\mathscr{J}}{\mathscr{M}} \end{aligned} \tag{D.8}$$

If the parameter λ is taken to be the length along the geodesic, then following equations of motion follow for space-like geodesics ($m^2 = -1$).

$$
\begin{aligned}
r^2\dot{r}^2 &= \left(r^4 - r^2 + \frac{\mathscr{J}^2}{4}\right) + \left(\mathscr{E}^2 - \mathscr{L}^2\right) r^2 + \mathscr{L}^2 - \mathscr{J}\mathscr{E}\mathscr{L} \\
\dot{\phi} &= \frac{(r^2-1)L - \frac{1}{2}\mathscr{J}\mathscr{E}}{(r^2 - r_+^2)(r^2 - r_-^2)} \\
\dot{t} &= \frac{\mathscr{E}r^2 - \frac{1}{2}\mathscr{J}\mathscr{L}}{(r^2 - r_+^2)(r^2 - r_-^2)}
\end{aligned}
\tag{D.9}
$$

where we have omitted the hat from the quantities.

It is easy to note that in the limit $r \to \infty$,

$$
\frac{dt}{d\phi} \approx \frac{\mathscr{E}}{\mathscr{L}}
\tag{D.10}
$$

The geodesic will penetrate most into the bulk if this is zero. This precisely implies that $\mathcal{E} = 0$ [123]. Therefore with the substitution $u = r^2$ the radial equation reduces to

$$
\begin{aligned}
\frac{1}{4}\left(\frac{du}{d\lambda}\right)^2 &= u^2 - (1+\mathscr{L}^2)u + \left(\mathscr{L}^2 + \frac{\mathscr{J}^2}{4}\right) \\
&= (u-a)(u-b),
\end{aligned}
\tag{D.11}
$$

where the following relations hold for a and b.

$$
a + b = (1+\mathscr{L}^2), \; ab = \mathscr{L}^2 + \frac{\mathscr{J}^2}{4}
\tag{D.12}
$$

Without loss of generality, we can take a to be the greater of the two roots and therefore the turning point of the geodesic. Then the geodesic length can be obtained as,

$$\begin{aligned}\Lambda &= \int_a^{u_\infty} \frac{du}{\sqrt{(u-a)(u-b)}} \\ &= \log\left(\frac{4r_\infty^2}{a-b}\right)\end{aligned} \tag{D.13}$$

To express the above in terms of the subsystem size one has to relate $a-b$ to the subsystem size. We note that,

$$\begin{aligned}\frac{d\phi}{du} &= \frac{1}{2}\frac{(u-1)\mathscr{L}}{(u-u_+)(u-u_-)\sqrt{(u-a)(u-b)}} \\ &= \frac{1}{2}\left[\frac{A}{u-u_+}+\frac{B}{u-u_-}\right]\frac{1}{\sqrt{(u-a)(u-b)}},\end{aligned} \tag{D.14}$$

where $A = -\frac{u_-\mathscr{L}}{u_+-u_-},\ B = \frac{u_+\mathscr{L}}{u_+-u_-}$ [1].

Now each of the integrals are of the form,

$$\int \frac{dx}{(x-c)\sqrt{(x-a)(x-b)}} = \frac{1}{\sqrt{(a-c)(b-c)}} \log\left[\frac{\sqrt{(a-c)(b-c)}(c-x)}{a(-2b+c+x)-2\sqrt{(a-c)(b-c)(x-a)(x-b)}-2cx+b(c+x)}\right] \tag{D.15}$$

Putting the limits,

$$\int_a^{\infty} \frac{dx}{(x-c)\sqrt{(x-a)(x-b)}} = \frac{1}{\sqrt{(a-c)(b-c)}}\log\left[\frac{\sqrt{(a-c)}+\sqrt{(b-c)}}{\sqrt{(a-c)}-\sqrt{(b-c)}}\right] \tag{D.16}$$

[1]Note that in these coordinates $u_+ + u_- = 1$ and $u_+u_- = \frac{\mathscr{L}^2}{4}$

Therefore the $\dot{\phi}$ integral gives,

$$l = \frac{A}{u_+ - u_-}\log\left(\frac{\sqrt{(a-u_+)}+\sqrt{(b-u_+)}}{\sqrt{(a-u_+)}-\sqrt{(b-u_+)}}\right)+\frac{B}{u_+ - u_-}\log\left(\frac{\sqrt{(a-u_-)}+\sqrt{(b-u_-)}}{\sqrt{(a-u_-)}-\sqrt{(b-u_-)}}\right), \tag{D.17}$$

where l is the subsystem size. We need to impose a further restriction. Note that,

$$\begin{aligned} \frac{dt}{du} &= -\frac{1}{4}\frac{\mathscr{J}\mathscr{L}}{(u-u_+)(u-u_-)\sqrt{(u-a)(u-b)}} \\ &= \frac{1}{4}\left[\frac{C}{u-u_+}+\frac{D}{u-u_-}\right]\frac{1}{\sqrt{(u-a)(u-b)}}, \end{aligned} \tag{D.18}$$

where $C = \frac{1}{u_+ - u_-}$ and $D = -\frac{1}{u_+ - u_-}$. Therefore the interval of time elapsed is given by,

$$\begin{aligned} \Delta T = \frac{\mathscr{J}}{2\sqrt{u_-}(u_+ - u_-)}\log\left(\frac{\sqrt{(a-u_+)}+\sqrt{(b-u_+)}}{\sqrt{(a-u_+)}-\sqrt{(b-u_+)}}\right) \\ - \frac{\mathscr{J}}{2\sqrt{u_+}(u_+ - u_-)}\log\left(\frac{\sqrt{(a-u_-)}+\sqrt{(b-u_-)}}{\sqrt{(a-u_-)}-\sqrt{(b-u_-)}}\right) \end{aligned} \tag{D.19}$$

Since the subsystem is on a constant t slice on the boundary, the total elapsed time must be zero [123]. Therefore,

$$(\beta_+ + \beta_-)\tanh^{-1}\sqrt{\frac{b-u_+}{a-u_+}} = (\beta_+ - \beta_-)\tanh^{-1}\sqrt{\frac{b-u_-}{a-u_-}} \tag{D.20}$$

which can be re-ordered to give,

$$\beta_+\left[\tanh^{-1}\sqrt{\frac{b-u_-}{a-u_-}} - \tanh^{-1}\sqrt{\frac{b-u_+}{a-u_+}}\right] = \beta_-\left[\tanh^{-1}\sqrt{\frac{b-u_-}{a-u_-}} - \tanh^{-1}\sqrt{\frac{b-u_+}{a-u_+}}\right] \tag{D.21}$$

Therefore using (D.21) one has the following conditions on the solutions (D.17),

$$
\begin{aligned}
\tanh\frac{\pi l}{\beta_+} &= \frac{\sqrt{(b-u_-)(a-u_+)}-\sqrt{(b-u_+)(a-u_-)}}{\sqrt{(a-u_+)(a-u_-)}-\sqrt{(b-u_+)(b-u_-)}} \\
\tanh\frac{\pi l}{\beta_-} &= \frac{\sqrt{(b-u_-)(a-u_+)}+\sqrt{(b-u_+)(a-u_-)}}{\sqrt{(a-u_+)(a-u_-)}+\sqrt{(b-u_+)(b-u_-)}}
\end{aligned} \tag{D.22}
$$

From these one can get the following expression for $\sinh\frac{\pi l}{\beta_+}$ and $\sinh\frac{\pi l}{\beta_-}$.

$$
\begin{aligned}
\sinh^2\frac{\pi l}{\beta_+} &= \frac{\left(\sqrt{(b-u_-)(a-u_+)}-\sqrt{(b-u_+)(a-u_-)}\right)^2}{(a-b)^2} \\
\sinh^2\frac{\pi l}{\beta_-} &= \frac{\left(\sqrt{(b-u_-)(a-u_+)}+\sqrt{(b-u_+)(a-u_-)}\right)^2}{(a-b)^2}
\end{aligned} \tag{D.23}
$$

Therefore,

$$
\sinh^2\frac{\pi l}{\beta_+}\sinh^2\frac{\pi l}{\beta_-} = \frac{(u_+-u_-)^2}{(a-b)^2} \tag{D.24}
$$

Taking the positive square root we get,

$$
\sinh\frac{\pi l}{\beta_+}\sinh\frac{\pi l}{\beta_-} = \frac{4\pi^2}{\beta_+\beta_-(a-b)} \tag{D.25}
$$

Therefore one can has the desired result for the geodesic length in terms of the subsystem size,

$$
\Lambda = \log\left(\frac{\beta_+\beta_-}{\pi^2\epsilon^2}\sinh\left(\frac{\pi l}{\beta_+}\right)\sinh\left(\frac{\pi l}{\beta_-}\right)\right) \tag{D.26}
$$

where we have put $r_\infty = \frac{1}{\epsilon}$

To find the turning point in terms of the subsystem size, note that

$$\tanh \frac{\pi l}{\beta_+} \tanh \frac{\pi l}{\beta_-} = \frac{4\pi^2}{\beta_+\beta_-(a+b-1)} \tag{D.27}$$

In the small subsystem size approximation,

$$\begin{aligned} a+b &\approx \frac{4}{l^2} \\ a-b &\approx \frac{4}{l^2} \\ a &\approx \frac{4}{l^2} \end{aligned} \tag{D.28}$$

So the turning point in the r coordinate upto leading order is given by

$$r_* \approx \frac{2}{l} \tag{D.29}$$

APPENDIX E

REVISITING THE DERIVATION OF THE INHOMOGENEOUS JACOBI EQUATION FOR GEODESICS

To make sure that in the used notation the equation we have obtained is indeed the correct equation we are looking for, we will derive the inhomogeneous Jacobi equation derived in [35].

Note that the geodesic equation can be written as $(\nabla_T T)^\perp = 0$ (where T is the tangent vector to the geodesic and satisfies $\nabla_T T = fT$). We will consider a variation of the geodesic under δ_N. The variation accounts for both change of embeddings and metric perturbations.

$$\delta_N(\nabla_T T)^\perp = \delta_N(\nabla_T T) - \delta_N(\nabla_T T)^T \tag{E.1}$$

$$= \nabla_T^2 N + R(N,T)T + C(T,T) - \nabla_N(fT) - \delta_g(f)T$$

Our convention implies that $\delta_P(\nabla_X Y) = C(X,Y)$. Noting that $f = \frac{g(\nabla_T T,T)}{g(T,T)}$, we can find the variation $\delta_g f$. After a few algebraic steps one gets the following expression,

$$\delta_P f = \frac{g(C(T,T),T)}{g(T,T)} \tag{E.2}$$

Also note that,

$$\nabla_N f = \frac{g(\nabla_T^2 N,T)}{g(T,T)} - \frac{fg(\nabla_T N,T)}{g(T,T)} \tag{E.3}$$

Substituting (E.2) and (E.3) in (E.1) we get

$$\delta_N(\nabla_T T)^\perp = \nabla_T^2 N - (\nabla_T^2 N)^T + R(N^\perp,T)T - (f\nabla_T N - f(\nabla_T N)^T) + C(T,T)^\perp. \tag{E.4}$$

Equating the above to zero gives the inhomogeneous equation,

$$\nabla_T^2 N^\perp + R(N^\perp,T)T - f\nabla_T N^\perp + C(T,T)^\perp = 0. \tag{E.5}$$

In [35] the unperturbed geodesic was taken to be affinely parametrised. Therefore putting $f = 0$ in the above equation reproduces the equation obtained.

APPENDIX F

BOOSTED BLACK BRANE AS A PERTURBATION OVER ADS

The boosted black brane metric in holographic coordinates is of the following form

$$ds^2 = \frac{R^2}{z^2}\left[-\mathcal{A}(z)dt^2 + \mathcal{B}(z)dx^2 + \mathcal{C}(z)dtdx + dx^2 + \frac{dz^2}{f(z)}\right], \tag{F.1}$$

where,

$$\mathcal{A}(z) = 1 - \gamma^2(\frac{z}{z_0})^3, \;\; \mathcal{B}(z) = 1 + \beta^2\gamma^2(\frac{z}{z_0})^3,$$
$$\mathcal{C}(z) = 2\beta\gamma^2(\frac{z}{z_0})^3, \;\; f(z) = 1 - (\frac{z}{z_0})^3$$

z_0 is the location of the horizon and $0 \leq \beta \leq 1$ is the boost parameter, while $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. With the boost along x direction. The boosted black brane is a finite change from AdS and hence cannot be observed as a perturbation over it. In order to see it as a perturbation over AdS, we have to write it in suitable asymptotic (Fefferman Graham) coordinates. The

Fefferman Graham coordinates are obtained by demanding [25, 121]

$$\frac{dz}{z\sqrt{f(z)}} = \frac{d\rho}{\rho} \tag{F.2}$$

Integrating this and setting the integration constant to $({\rho_0}^3 = 4{z_0}^3)$ we get

$$\frac{1}{z^2} = \frac{1}{\rho^2}(1+(\frac{\rho}{\rho_0})^3)^{\frac{4}{3}} = \frac{1}{\rho^2}g(\rho)^{\frac{4}{3}} \tag{F.3}$$

Now we expand the metric coefficient upto second order in $(\frac{\rho}{\rho_0})^3$, Substituting this back in the metric we get

$$ds^2 = \frac{R^2}{\rho^2}\left[d\rho^2 + \left(\eta_{\mu\nu} + \rho^3\gamma^{(3)}_{\mu\nu} + \rho^6\gamma^{(6)}_{\mu\nu}\right)dx^\mu dx^\nu\right] \tag{F.4}$$

Where

$$\gamma^{(3)}_{\mu\nu} = \begin{bmatrix} -(\frac{1}{3}-\gamma^2)(\frac{1}{z_0})^3 & \beta\gamma^2(\frac{1}{z_0})^3 & 0 \\ \beta\gamma^2(\frac{1}{z_0})^3 & \left(\frac{1}{3}+\beta^2\gamma^2\right)(\frac{1}{z_0})^3 & 0 \\ 0 & 0 & \frac{1}{3}(\frac{1}{z_0})^3 \end{bmatrix} \tag{F.5}$$

One can check that $Tr(\gamma^{(3)}_{\mu\nu}) = 0$ and

$$\gamma^{(6)}_{\mu\nu} = \begin{bmatrix} -\left(\frac{2}{9}+\frac{8}{3}\gamma^2\right)\frac{1}{16{z_0}^6} & -\frac{1}{6}\beta\gamma^2(\frac{1}{z_0})^6 & 0 \\ -\frac{1}{6}\beta\gamma^2(\frac{1}{z_0})^6 & \left(\frac{2}{9}-\frac{8}{3}\beta^2\gamma^2\right)\frac{1}{16{z_0}^6} & 0 \\ 0 & 0 & \frac{2}{9}\frac{1}{16{z_0}^6} \end{bmatrix} \tag{F.6}$$

The perturbation $\overset{(1)}{P}_{\mu\nu}$ and $\overset{(2)}{P}_{\mu\nu}$ can be read off as, $\overset{(1)}{P}_{\mu\nu} = \gamma^{(3)}_{\mu\nu}z$ and $\frac{1}{2}\overset{(2)}{P}_{\mu\nu} = \gamma^{(6)}_{\mu\nu}z^4$ respectively. To calculate the non homogeneous term in the Jacobi equation, we need the expression for $C(\partial_\mu, \partial_\nu)$, which in a given coordinate system can be written as,

$$C^{\mu}_{\nu\rho}(x) = \frac{1}{2} g^{\mu\sigma} \left(\partial_\nu \overset{(1)}{P}_{\rho\sigma} + \partial_\rho \overset{(1)}{P}_{\nu\sigma} - \partial_\sigma \overset{(1)}{P}_{\nu\rho} \right) - \frac{1}{2} \overset{(1)}{P}{}^{\mu\sigma} \left(\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho} \right) \tag{F.7}$$

Note that this quantity is a vector field in the tangent bundle and therefore it's coordinate expression has three indices. We will calculate this for boosts both in the x direction and the y direction. Note that though the direction of the boost does not affect the results for a spherical boundary subsystem, it does so for the strip subsystem. In the Fefferman graham gauge the expression for $C(\partial_\mu, \partial_\nu)$.

For boost along the x axis, the expression for $\overset{(1)}{P}(\partial_\mu, \partial_\nu)$ and $\overset{(2)}{P}(\partial_\mu, \partial_\nu)$ is of the following form.

$$\overset{(1)}{P}_{\mu\nu} = \begin{pmatrix} A\,z & B\,z & 0 & 0 \\ B\,z & C\,z & 0 & 0 \\ 0 & 0 & D\,z & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \frac{1}{2}\overset{(2)}{P}_{\mu\nu} = \begin{pmatrix} A'\,z^4 & B'\,z^4 & 0 & 0 \\ B'\,z^4 & C'\,z^4 & 0 & 0 \\ 0 & 0 & D'\,z^4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The quantity $C^{\mu}_{\nu\rho}$ can be calculated from eqn (F.7),

$$\begin{gathered} C^{z}{}_{t\,t} = -\frac{1}{2}\,z^2\,A,\;\; C^{z}{}_{x\,t} = -\frac{1}{2}\,z^2\,B,\;\; C^{t}{}_{z\,t} = -\frac{3}{2}\,z^2\,A,\;\; C^{x}{}_{z\,t} = \frac{3}{2}\,z^2\,B \\ C^{z}{}_{t\,x} = -\frac{1}{2}\,z^2\,B,\;\; C^{z}{}_{x\,x} = -\frac{1}{2}\,z^2\,C,\;\; C^{t}{}_{z\,x} = -\frac{3}{2}\,z^2\,B,\;\; C^{x}{}_{z\,x} = \frac{3}{2}\,z^2\,C \\ C^{z}{}_{y\,y} = -\frac{1}{2}\,z^2\,D,\;\; C^{y}{}_{z\,y} = \frac{3}{2}\,z^2\,D,\;\; C^{t}{}_{t\,z} = -\frac{3}{2}\,z^2\,A,\;\; C^{x}{}_{t\,z} = \frac{3}{2}\,z^2\,B \\ C^{t}{}_{x\,z} = -\frac{3}{2}\,z^2\,B,\;\; C^{x}{}_{x\,z} = \frac{3}{2}\,z^2\,C,\;\; C^{y}{}_{y\,z} = \frac{3}{2}\,z^2\,D, \end{gathered} \tag{F.8}$$

where C, D can be read off from the previous expression for P's and γ's eqn. (F.5) and is given as $C = \left(\frac{1}{3} + \beta^2\gamma^2\right)\frac{1}{z_0^3},\ D = \frac{1}{3}\frac{1}{z_0^3}$. The components of $\frac{1}{2}\overset{(2)}{P}_{\mu\nu}$ will be C', D' and is given as $C' = \left(\frac{2}{9} - \frac{8}{3}\beta^2\gamma^2\right)\frac{1}{16{z_0}^6},\ D' = \frac{2}{9}\frac{1}{16{z_0}^6}$.

For boost along the y axis, $\overset{(1)}{P}(\partial_\mu, \partial_\nu)$ and $\overset{(2)}{P}(\partial_\mu, \partial_\nu)$ is of the form,

$$\overset{(1)}{P}_{\mu\nu} = \begin{pmatrix} \tilde{A}\, z & 0 & \tilde{B}\, z & 0 \\ 0 & \tilde{C}\, z & 0 & 0 \\ \tilde{B}\, z & 0 & \tilde{D}\, z & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \frac{1}{2}\overset{(2)}{P}_{\mu\nu} = \begin{pmatrix} \tilde{A}'\, z^4 & 0 & \tilde{B}'\, z^4 & 0 \\ 0 & \tilde{C}'\, z^4 & 0 & 0 \\ \tilde{B}'\, z^4 & 0 & \tilde{D}'\, z^4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The quantity $C^\mu_{\nu\rho}$ is therefore,

$$\begin{gathered} C^z{}_{t\,t} = -\frac{1}{2}\, z^2\, \tilde{A},\ C^z{}_{y\,t} = -\frac{1}{2}\, z^2\, \tilde{B},\ C^t{}_{z\,t} = -\frac{3}{2}\, z^2\, \tilde{A},\ C^y{}_{z\,t} = \frac{3}{2}\, z^2\, \tilde{B} \\ C^z{}_{x\,x} = -\frac{1}{2}\, z^2\, \tilde{C},\ C^x{}_{z\,x} = \frac{3}{2}\, z^2\, \tilde{C},\ C^z{}_{t\,y} = -\frac{1}{2}\, z^2\, \tilde{B},\ C^z{}_{y\,y} = -\frac{1}{2}\, z^2\, \tilde{D} \\ C^t{}_{z\,y} = -\frac{3}{2}\, z^2\, \tilde{B},\ C^y{}_{z\,y} = \frac{3}{2}\, z^2\, \tilde{D},\ C^t{}_{t\,z} = -\frac{3}{2}\, z^2\, \tilde{A},\ C^y{}_{t\,z} = \frac{3}{2}\, z^2\, \tilde{B} \\ C^x{}_{x\,z} = \frac{3}{2}\, z^2\, \tilde{C},\ C^t{}_{y\,z} = -\frac{3}{2}\, z^2\, \tilde{B},\ C^y{}_{y\,z} = \frac{3}{2}\, z^2\, \tilde{D} \end{gathered} \tag{F.9}$$

where $\tilde{C} = \frac{1}{3}(\frac{1}{z_0})^3,\ \tilde{D} = \left(\frac{1}{3} + \beta^2\gamma^2\right)(\frac{1}{z_0})^3,\ \tilde{C}' = \frac{2}{9}\frac{1}{16{z_0}^6},\ \tilde{D}' = \left(\frac{2}{9} - \frac{8}{3}\beta^2\gamma^2\right)\frac{1}{16{z_0}^6},\ B = \tilde{B} = \beta\gamma^2(\frac{1}{z_0})^3$. This completes our first step in calculation of area, now we can proceed with solving the inhomogeneous Jacobi equation.

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