ASPECTS OF SUPERSYMMETRIC LOCALIZATION AND EXACT RESULTS IN N=2 SUPERSYMMETRIC THEORIES

By

AUGNIVA RAY

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Saha Institute of Nuclear Physics, Kolkata

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____ Date: April 13, 2021

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List of Publications

Publications included in the thesis

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3. "Supersymmetric Graphene on squashed Hemisphere", R. Gupta, A. Ray, K. Sil, [arXiv: 2012.01990]

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Journals & Pre-prints:

1. "Scale invariance with fundamental matters and anomaly: A holographic description", A. Banerjee, A. Kundu and A. Ray, , *JHEP* 1806 **(2018)** 144, [arXiv: 1802.05069].

Augniva Ray

Dedicated to Aloo & Patch,
"Always" . . .

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∼∼ *It was written I should be loyal to the nightmare of my choice.* ∼∼

Joseph Conrad

Penning this section at the very end of writing a thesis proved epiphanic. Nothing screamed louder that this part of the journey is long bygone than authoring this section now. Time rolls coldly on and yet, one must pause against flowing tides and say, “Thank you for the experiences” before being dragged elsewhere.

A journey of half a decade should never be experienced within the solitary confines of one’s paltry existence - rather, it is to be cherished by acknowledging the presence of all as a collection creating an environment which fosters growth. Each plays a nurturing role - some do so explicitly, and some, by staying as reassuring figures in the backdrop. And in the dusk, it behoves me that I let them know that this journey would not have been possible without them.

I should first mention my guide Dr. Arnab Kundu and thank him. We often, as students, wonder how a supervisor should be. Should s/he be a tough ring-master, cracking whips to ensure the cold conformity of machines. Should s/he be, instead, more a friend discussing physics over evening tea? I feel he has never conformed to either of this binary. On the contrary, the only word that comes to my mind when I talk about him is *empowering*. He has been empowering. He has encouraged me to grow, to explore, to fall if need be and then to stand up again (probably after a few days of whining). Such encouragement naturally transgresses the boundaries of academic activities and one learns life lessons. I can safely say that I feel, today, more than the discussions regarding physics we have had, I hold

more dearly the lessons I carry with me about the need to accept failures as a part of one's growth. I think that's his gift to me - and I can hardly ask for anything more.

I do not know if he recalls our first encounter, way back in end 2013 - when he possibly was in Barcelona. I had naively mailed him saying, " I want to write papers on string theory." Absurd as the mail sounds to even me now, he did not say "No" and, instead, gave me a paper written by Ooguri, Maldacena et al to read. And although, I can safely say, I have hardly read that paper, I can equally safely say that that day, over and above the paper he gave me hope. And that was his greatest teaching to me - that a fresh graduate student, with no formal training in string theory, can hope to write papers on string theory. He lent me *hope.*

I would also single out Rudra da. I think with him I have shared a more personal journey, where we discussed about failures and hardships. I recall the endless letters of recommendation he wrote on my behalf to the subtle pep-talks he gave me when the chips were down. Thank you - for teaching me that it is only humane to stand up for young struggling students. For teaching me that with time, they can and will flourish - that often all it takes is some moral support and they are up on their feet again. When my day comes, I hope to pass this light on to a new Augniva trying to find his way about.

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But should physics be devoid of its moments of guffaws? What is research without its share of comedy of errors? And here, I have been blessed with an excellent environment at the Theory Room, SINP. If only there were recordings of the theories we came up with on the white board every other day! And my "co-authors" here have been Avik (Choto), Avik (Boro), Udit, Avirup da, Aritra da, Chiru da, Kumar da, Mugdha da, Aranya, Bithika, Pritam, Maulik Shahib, Ayan, Arunima, Ritesh et al. I have lost count of the times Pradyut da said, " *Acha, tomar eto pen laage kano? Ki koro board a ?*". If only he knew, eh? Fun aside, though, I think I could submit my thesis on time because Choto was around, pushing me just at the right moments.

I am grateful to my friends too, who, with their mirth made my life less lost. I feel, though, that tragedy brought us closer. Whenever I turned unsurely around, I saw them grappling with their own existential angst and negotiating a space, just as I was. Nothing binds like stories of shared misery, ironically.

At long last, however, I arrive at the question which I have dreaded - how do I thank our families? How do I tell them what their efforts mean to me when I am only a product of their belief, compassion and love? How will any such an effort not reduce to an act of silly ostentation? The only answer I ever get is that it perhaps will. But, perhaps, I should just accept that words will never be enough to thank them but say it nonetheless.

Perhaps I should take a moment to thank Pepe. I go back to June 2011, when, confused as I was and working as an engineer for an IT company, to me she said, “ If you want to pursue physics, do it. Do it now. *Na khete peye morbi na. Bhaat ami kinei debo shara jibon.* Don’t be afraid.” I resigned after that call, on that day without telling anyone else. Roop, this is for you and your belief in me.

Perhaps I should take a moment to thank my sister who was more a mother to me when I was an infant. I love you.

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Perhaps I should take a moment to thank Aloo, Patch, Manku and their mother Deeaa. You tried to teach me nothing, yet from you I learnt everything that I value. Should a day come when all lights fail me, I hope I carry with me the audacity of love that fills your hearts. To Remy (2021) and Jane Doe (2019) - in our hearts, you will always have names.

Perhaps, now, I should take an eternity to thank Ma and yet fail. Perhaps, I should not even try lest my words make a mockery of the sacrifices that mothers make, that you made. I am scared, Ma, that there is nothing I can write here that holds a pale candle to what you mean to me. How can words ever represent a mother’s love? Thank you. Everything that I am, or will be, is because of you.

Lastly, and for ever, my gratitude is to all the nameless, faceless denizens whose toil enable my position of privilege. You will forever drive my pen.

∼∼ *May the light that I borrow help others shine too.* ∼∼

Augniva Ray

Contents

List of Figures

List of Tables

CHAPTER 5

CONCLUSION AND OUTLOOK

Historically, supersymmetric field theories have carved out a niche for themselves in theoretical physics. From taming loop divergences via effective cancellations between bosonic and fermionic loops to being open to non perturbative calculations, theories are more amenable to extracting meaningful physical results when one adds an additional spacetime symmetry which flips particle statistics. Although, its signature still remains elusive at the energy scales accessible to present day experiments, notably, the LHC, leading many to conjecture that the symmetry is broken at a much higher energy scale, supersymmetry remains an elegant symmetry for theories to have. From purely a mathematical and theoretical perspective, setting aside valid criticisms stemming from a lack of experimental validation, a shot in the arm was received when Nekrasov in his seminal work in Ref. [125] used principles of supersymmetric localization, discussed in **Introduction** in Ch. 1. The technique allows exact calculations of BPS observables and since the days of Nekrasov and Pestun, it has been used profitably to extract results, often even in the case of strongly coupled theories, which, otherwise would have been opaque to manipulations. In this thesis, following this direction of theoretical physics, we have calculated certain observables in $3d\,\mathcal{N}=2$ theory placed on various manifolds and extracted physical results.

Counting black hole microstates has been an open and active area of theoretical physics, especially in the context of string theory following the works of Sen et al in Ref. [126]. A direction in this regard has been the proposal to use holography and get a measure of the entropy from some observable of the dual field theory, following the works of Kim et al in Ref. [127] and Zaffaroni et al in Ref. [47]. We take up the latter prescription in **Ch.** 2 for a large class of quiver gauge theories. It has been suggested that topologically twisted indices, defined as the partition function on product spaces of the kind $\Sigma_{\mathfrak{g}} \times S^1$ with $\Sigma_{\mathfrak{g}}$ being a $2d$ manifold of genus $\mathfrak{g}$, capture the entropy of the dual (magnetic) black holes. This has also been explicitly verified for the so-called A_2 quiver gauge theory, also known in the literature as the ABJM Model. Then, an interesting way forward is to calculate the same observable for a much broader class of theories known as quiver gauge theories, specially the A_n quivers for $n > 2$ and the D_n quivers for theories preserving $\mathcal{N}=2$ supersymmetry. We have further calculated the free energy for such theories on S^3 and shown the relationship that exists between them. Our work then predicts the expected black hole entropy (at the large rank limit) of the dual black holes. Future gravity side calculations of entropy of dual black holes (in $4d$) would provide necessary consistency checks and we leave this is a future programme.

Witten showed in Ref. [70] that gravity in three dimensions may be reinterpreted as a topological field theory. Following his idea in **Ch.** 3, we express $3d$ gravity with a positive cosmological constant as a Chern Simons theory of two gauge fields each of which are $\mathfrak{su}(2)$ Lie algebra valued. Such a formulation of gravity is immediately helpful because we can supersymmetrize the bosonic theory, by adding non-dynamical fermions and auxiliary scalars (the remaining fields of the $\mathcal{N}=2$ Vector Multiplet). Such a supersymmetric

Chern Simons theory forms the standard kinetic term for the vector multiplet and we are free to use the full machinery of localization and we calculate the quantum gravity partition function as the partition function of a Chern Simons theory with a gauge group $G = SU(2) \otimes SU(2)$. A question remains, of course, as to which saddles contribute to this partition function. We obtain the saddles as orbifolds of S^3, called the Lens spaces $L(p,q)$, the global topology of which is given $S^3/\mathbb{Z}_p$. This countably infinite number of saddles contribute to the partition function, leading to a divergence as $p \to \infty$. We express the result in terms of the Kloosterman Zeta functions and analyse the divergences present in such theories.

A natural extension of analysis of theories placed on compact manifolds would be to consider theories placed on manifolds with a boundary. With appropriate boundary conditions, such theories can preserve some supersymmetry - most often, however, supersymmetry is reduced as translation invariance in the direction perpendicular to the boundary is broken. In **Ch.** 4, we study a "mixed dimensional QED$_3$", which is an (abelian) $\mathcal{N} = 2$ theory on $4d$ interacting with matter placed on a boundary. This theory is of interest in itself as a supersymmetric model, as well as in condensed matter systems, as a simple model for graphene. Here, for concreteness, we have chosen the "bulk" to be a squashed hemisphere HS^4 with chiral matter at the boundary placed at $r = \frac{\pi}{2}$. In this chapter, we discuss two separate types of squashing preserving different isometries. We have calculated the boundary two point stress tensor $\langle TT \rangle$, which depends on the second derivative of the free energy with respect to the squashing parameter. We see that the final expression depends on the complexified gauge coupling τ and R charges (which extremise the free energy). We consider two separate cases of one and two (oppositely charged) chiral matter and state the results for both at strong and weak coupling numerically. We also present analytical results as a perturbative series in $|\tau|$ (valid strictly in the regimes $|\tau| >> 1$ and $|\tau| << 1$). A graphical comparison between the two results is also presented to validate the calculations.

The details of some lengthy calculations have been shown in the Appendix section of this thesis for clarity. **Appendix** A contains further details supplementing **Ch.** 2. Similarly, **Appendix** B contains details relevant to **Ch.** 3 and **Appendix** C contains details relevant to **Ch.** 4. Finally, some γ matrix identities which were liberally used for all the calculations relevant to this thesis have been presented in **Appendix** D.

SUMMARY

Supersymmetric theories on curved spaces provide an interesting arena for physics. Observables like the partition function, two point functions of BPS observables etc in such theories are often cured of IR divergences when they are placed on spaces that are compact. Furthermore, supersymmetric theories on compact manifolds are often amenable to non-perturbative techniques like supersymmetric localization. This tool gives powerful results for different theories placed on various manifolds preserving various amounts of supersymmetry. Using the well-established AdS/CFT Correspondence, such exact calculations often provide strong coupling results of the dual theory, where, otherwise such calculations would not have been possible. They further provide evidence of the correspondence when results in the weakly coupled gauge or gravity sector (obtained with or without localization) compares with results in the strongly coupled dual sector (obtained most likely using localization).

In the literature, there has been evidence that the partition function on $\Sigma_{\mathfrak{g}} \times S^1$, where, $\Sigma_{\mathfrak{g}}$ is a two dimensional Riemann surface of genus $\mathfrak{g}$ captures the degrees of freedom of the dual (magnetic) black holes on spacetimes that are asymptotically anti de-Sitter. It has been explicitly shown to reproduce known results for black holes dual to the ABJM theory

by Zaffaroni et al. Calculation of this partition function for a much larger class of field theories, known as ADE quiver theories, then, becomes an important exercise as these observables are expected to predict the entropy of a large class of black holes placed in the dual geometries.

Furthermore, one can duly exploit the fact that $3d$ gravity is topological and express de-Sitter gravity as a topological field theory and supersymmetrize it. Such manipulations help one to use the machinery of localization and calculate the quantum gravity partition function as the partition function of (two copies of) supersymmetric Chern Simons theories. There too, we are free to use supersymmetric localization desirably.

One can also imagine a theory placed on a manifold with a boundary - i.e., bulk degrees of freedom interacting with those residing at the boundary. Carefully chosen boundary conditions help preserve some amount of supersymmetry and such theories are the natural extension of supersymmetric theories placed on compact spaces. Here, we have calculated two point correlation function of a local operator - the stress tensor both at strong and weak coupling for a mixed dimensional QED theory, which has bulk photons on a $4d$ hemisphere interacting with matter at the three dimensional boundary placed at $r = \frac{\pi}{2}$.

In this thesis, I have primarily aimed at using the machinery of supersymmetric localization specifically to $\mathcal{N} = 2$ theories defined on various 3-manifolds with and without boundary and extracted non-perturbative results for various BPS observables.

CHAPTER 1

INTRODUCTION

1.1 Historical Development of Supersymmetry

Supersymmetry as a new *spacetime* symmetry connecting the bosonic and fermionic degrees of freedom of a theory first made its appearance in early string theory literature. In the late 1960's, in an effort to construct the S matrix for strong interactions, a new idea was proposed that visualised hadrons, not as particles, but rather as vibration modes of fundamental strings.

A two dimensional field theory of d bosonic fields was suggested with an action $\Big($ a functional of $X^{\mu}(\tau, \sigma)\Big)$ given by

$$L_1[X] = \frac{T}{2} \int d\sigma \, d\tau \left[\frac{\partial X^{\mu}}{\partial \tau} \frac{\partial X_{\mu}}{\partial \tau} - \frac{\partial X^{\mu}}{\partial \sigma} \frac{\partial X_{\mu}}{\partial \sigma} \right] \tag{1.1.1}$$

where, $\{\tau, \sigma\}$ are the coordinates on the 2-dimensional "worldsheet", formed as the string propagates across a fixed d dimensional Minkowski background.

Later, a proposal to introduce fermions to the theory defined by the action in Eq. (1.1.1) were given by Ramond in Ref. [1] and Neveu and Schwarz in Ref. [2] for a complete spectrum. Finally, Gervais and Sakita proposed an action for this modified theory in Ref. [3] as

$$L_2[X,\psi] = \int d\sigma^+ d\sigma^- \left[T \frac{\partial X^\mu}{\partial\sigma^+}\frac{\partial X_\mu}{\partial\sigma^-} + i\left(\psi_1^\mu \frac{\partial}{\partial\sigma^-}\psi_{1\mu} + \psi_2^\mu \frac{\partial}{\partial\sigma^+}\psi_{2\mu}\right)\right] \tag{1.1.2}$$

where, $\{\sigma^+,\sigma^-\}$ are related to the string worldsheet coordinates as $\sigma^\pm = \tau \pm \sigma$. Gervais and Sakita noted, that in additional to conformal symmetry on the worldsheet and d dimensional Lorentz symmetry on the target space, the Lagrangian has additional symmetry relating particles of two different statistics, namely

$$\delta\text{boson} \sim \text{fermion}\,, \qquad \delta\text{fermion} \sim \text{boson}\,,$$

$$\delta\psi_1^\mu(\sigma^+,\sigma^-) = iT\epsilon^+(\sigma^+)\frac{\partial}{\partial\sigma^+}X^\mu\,, \tag{1.1.3a}$$

$$\delta\psi_2^\mu(\sigma^+,\sigma^-) = iT\epsilon^-(\sigma^-)\frac{\partial}{\partial\sigma^-}X^\mu\,, \tag{1.1.3b}$$

$$\delta X^\mu(\sigma^+,\sigma^-) = \epsilon^+(\sigma^+)\delta\psi_1^\mu(\sigma^+,\sigma^-) + \epsilon^-(\sigma^-)\delta\psi_2^\mu(\sigma^+,\sigma^-)\,. \tag{1.1.3c}$$

where, $\{\epsilon^+,\epsilon^-\}$ are arbitrary Grassmann odd functions. This kind of spacetime symmetry interchanging particle statistics was later called *supersymmetry*.

Eq. (1.1.3) furnished an example of what is known in string theory as world sheet supersymmetry - supersymmetry in two dimensions. It was later extended to quantum field theories in $4d$ first by Wess and Zumino in Ref. [4] giving rise to the celebrated Wess Zumino models. Simultaneously, another group working separately in Russia, Gol'fand and Likhtman in Ref. [5] extended the Poincare algebra to include graded algebras, as discussed in Sec. 1.3.1 and constructed field theories that are invariant under these super-algebras. Thus, independently, they discovered supersymmetric field theories in $4d$ as a theory whose symmetry generators together form a closed graded lie algebra, of which the Poincare algebra is the bosonic sub-algebra. We review the concept of graded algebras in Sec. 1.3.1 to shed more light on generators of supersymmetry.

1.2 Conventions

In this thesis, the Latin letters $\{a, b, c, \dots, m, n, \dots\}$ denote flat (tangent space) indices, the middle Latin letters $\{i, j\}$ will also denote the R symmetry index in Ch. 4, the early Greek letters $\{\alpha, \beta, \dots\}$ denote spinorial indices, whereas the later Greek letters $\{\mu, \nu, \dots\}$ denote vector indices.

In the literature, various different yet consistent conventions are followed by different authors. Here, at the onset, we define our conventions for disambiguation. Unless otherwise specified, we follow the "van der Waerden" notation for two component (Weyl) spinors. Further details, which are skipped in this section for brevity, are to be found in Ref. [6].

We define spinors as those furnishing the basic representation of $SL(2, C)$ group - the group of unimodular complex 2×2 matrices. Since, for a given $M \in SL(2, C)$, M^* (complex conjugate), $M^{T\,-1}$ (transpose inverse), and $M^{\dagger\,-1}$ (Hermitian conjugate inverse) also belong to $SL(2, C)$, we define four objects below, defined by their transformation, as

$$\psi_\alpha \to \psi'_\alpha = M_\alpha{}^\beta \psi_\beta \,, \qquad \bar{\psi}^{\dot{\alpha}} \to \bar{\psi}'^{\,\dot{\alpha}} = (M^{*-1})_{\dot{\beta}}{}^{\dot{\alpha}} \bar{\psi}^{\dot{\beta}} \,, \tag{1.2.1a}$$

$$\psi^\alpha \to \psi'^{\,\alpha} = (M^{-1})_\beta{}^\alpha \psi^\beta \,, \qquad \bar{\psi}_{\dot{\alpha}} \to \bar{\psi}'_{\dot{\alpha}} = (M^*)_{\dot{\alpha}}{}^{\dot{\beta}} \bar{\psi}^{\dot{\alpha}} \,. \tag{1.2.1b}$$

where,

$$\psi^\alpha \equiv \epsilon^{\alpha\beta}\psi_\beta \ , \qquad \bar{\psi}_{\dot{\alpha}} \equiv \bar{\epsilon}_{\dot{\alpha}\dot{\beta}}\bar{\psi}^{\dot{\beta}} \ . \tag{1.2.2}$$

and the $SL(2,C)$ invariant totally anti-symmetric tensors are defined as $\epsilon^{12} = \bar{\epsilon}^{\,\dot{1}\dot{2}} = -\epsilon_{12} = -\bar{\epsilon}_{\dot{1}\dot{2}} = 1$. Using the known homomorphism between $SL(2,C)$ and $SO(3,1)$ groups, we define that ψ_α furnishes the $(\frac{1}{2},0)$ representation of the Lorentz algebra, where as the $\bar{\psi}_{\dot{\alpha}}$ furnishes the $(0,\frac{1}{2})$ representation. With $\{\sigma^j\}$ as the j-th Pauli matrix, we define the basis for the $SL(2,C)$ group as

$$(\sigma^m)_{\alpha\dot{\alpha}} = i\left(\mathbb{I}_2, -\sigma^j\right)_{\alpha\dot{\alpha}} \ , \qquad (\bar{\sigma}^m)^{\dot{\alpha}\alpha} = i\left(\mathbb{I}_2, \sigma^j\right)^{\dot{\alpha}\alpha} \qquad \text{for } \ j = 1,2,3 \ . \tag{1.2.3}$$

We further define the $4d$ γ matrices (in the Weyl representation) as

$$\gamma^m = \begin{pmatrix} 0 & \sigma^m \\ \bar{\sigma}^m & 0 \end{pmatrix} \ , \quad \{\gamma^m, \gamma^n\} = 2\eta^{mn}\mathbb{I}_4 \ , \quad \eta^{mn} = \text{diag}(-1,1,1,1) \ . \tag{1.2.4}$$

When working on Riemannian manifolds, we will use the following definitions.

$$(\sigma^m)_{\alpha\dot{\alpha}} = \left(\mathbb{I}_2, -i\sigma^j\right)_{\alpha\dot{\alpha}} \ , \qquad (\bar{\sigma}^m)^{\dot{\alpha}\alpha} = \left(\mathbb{I}_2, i\sigma^j\right)^{\dot{\alpha}\alpha} \qquad \text{for } \ j = 1,2,3 \ , \tag{1.2.5}$$

$$\gamma^m = \begin{pmatrix} 0 & \sigma^m \\ \bar{\sigma}^m & 0 \end{pmatrix} \ , \quad \{\gamma^m, \gamma^n\} = 2\delta^{mn}\mathbb{I}_4 \ . \tag{1.2.6}$$

We work with the "NW-SE" contraction for the undotted indices (and the reverse for the dotted indices) as

$$\psi\psi \equiv \psi^\alpha\psi_\alpha \,, \quad \bar{\psi}\bar{\psi} \equiv \bar{\psi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} \,, \quad \chi\sigma^\mu\bar{\phi} \equiv \chi^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\bar{\phi}^{\dot{\alpha}} \,, \quad \bar{\chi}\bar{\sigma}^\mu\phi \equiv \bar{\chi}_{\dot{\alpha}}\bar{\sigma}^{\mu\,\dot{\alpha}\alpha}\phi_\alpha \,. \tag{1.2.7}$$

This defines bilinears like $\psi^\alpha\psi_\alpha$ and $\psi_{\dot{\alpha}}\psi^{\dot{\alpha}}$ as $SL(2,C)$ scalars.

1.3 Supersymmetry Algebra Revisited

1.3.1 Graded Lie Algebra

Under very general assumptions like the existence of a finite number of particles below a given mass, locality, unitarity and analyticity of S matrix, Coleman and Mandula proved in Ref. [7] that the most general symmetry group G of a theory can be a direct sum of Poincare group $ISO(3,1)$ and some internal symmetry group T which are independent of the momenta and spins of the particle states on which they act - that is,

$$\left[\mathfrak{g}_{Poincare}\,,\mathfrak{t}^A\right] = 0\,, \quad G = ISO(3,1)\times T \tag{1.3.1}$$

where, $\mathfrak{g}_{Poincare}$ and $\mathfrak{t}^A$ are the generators of $ISO(3,1)$ and T respectively.

A key assumption to their celebrated theorem was that the symmetry generators are bosonic and therefore, they form a Lie algebra. Soon after, Haag, Lopuszanski and Sohnius (HLS) extended the ambit of the Coleman-Mandula theorem by relaxing the condition that the algebra of the symmetry generators forms only a “Lie Algebra” to include “Graded Lie Algebras”. In Ref. [8], they showed that such an extension to a Graded Lie algebra enables us to accommodate fermionic generators of symmetry which, unlike the Poincare generators, have specific *anti*-commutation relations.

Mathematically, an algebra is a vector space L acting on some real or complex fields (denoted by $\mathbb{R}$ and $\mathbb{C}$ respectively) with an operator $\mathfrak{O}$ such that $\forall v_i \in L$ and $\forall \alpha_i \in \mathbb{R}$ or $\mathbb{C}$,

$$\mathfrak{O} : [L, L] \to L \tag{1.3.2}$$

$$\text{Linearity :} \quad [v_1, v_2 + v_3] = [v_1, v_2] + [v_1, v_3] \tag{1.3.3}$$

$$\text{Anti-Commutativity :} \quad [v_1, v_2] = -[v_2, v_1] \tag{1.3.4}$$

$$\text{Jacobi Identity :} \quad [v_1, [v_2, v_3]] + [v_3, [v_1, v_2]] + [v_2, [v_3, v_1]] = 0 \tag{1.3.5}$$

We can then, further, define a graded Lie algebra (of grade n) as the direct sum of vector spaces (L_i) and an operator $\mathfrak{O}$ such that for

$$L = \oplus_{i=0}^{i=n} L_i \tag{1.3.6}$$

$$\mathfrak{O} : \big[L_i, L_j\big\} \to L \qquad \forall\, i, j \tag{1.3.7}$$

$$\big[v_p, v_q\big\} \in v_r \ \text{ where, } \ r = p + q \mod (n+1) \tag{1.3.8}$$

$$\big[v_p, v_q\big\} = -(-)^{pq} \big[v_q, v_p\big\} \tag{1.3.9}$$

$$(-1)^{ik} \Big[v_i, \big[v_j, v_k\big\}\Big\} + (-1)^{ji} \big[v_j, [v_k, v_i\}\big\} + (-1)^{kj} \Big[v_k, \big[v_i, v_j\big\}\Big\} = 0 \tag{1.3.10}$$

where $\{v_i\} \in L_i$. Supersymmetry algebra, then, is the simplest graded Lie algebra - a graded Lie algebra of grade 1, i.e.,

$$L = L_0 \oplus L_1 \tag{1.3.11}$$

where, L_0 is the standard Poincare Lie algebra and the generators $(Q^I_\alpha, \bar{Q}^{I\,\dot{\alpha}}) \in L_1$, for $I = 1, 2, \ldots \mathcal{N}$.

1.3.2 $3d\ \mathcal{N}=2$ supersymmetry algebra

Here, we explicitly state the $3d\ \mathcal{N}=2$ supersymmetry algebra as this is the relevant case for this thesis. Many more details of the theory may be found in Ref. [9], but in this section we will have an occasion to present just the algebra.

$3d\ \mathcal{N}=2$ theories may obtained as a dimensional reduction of $4d\ \mathcal{N}=1$ theories to $3d$. They have, therefore, the same number of supercharges. The supersymmetry algebra is given by

$$\Big[P_\mu, Q^I_\alpha\Big] = 0\ , \qquad \Big[P_\mu, \bar{Q}^I_{\dot\alpha}\Big] = 0\ , \tag{1.3.12a}$$

$$\Big[M_{\mu\nu}, Q^I_\alpha\Big] = i(\sigma_{\mu\nu})_\alpha^{\ \beta} Q^I_\beta\ , \qquad \Big[M_{\mu\nu}, \bar{Q}^{I\dot\alpha}\Big] = i(\bar\sigma_{\mu\nu})^{\dot\alpha}_{\ \dot\beta}\bar{Q}^{I\dot\beta}\ , \tag{1.3.12b}$$

$$\Big\{Q^I_\alpha, \bar{Q}^J_\beta\Big\} = 2(\sigma^\mu)_{\alpha\beta}P_\mu\delta^{IJ} + 2i\epsilon_{\alpha\beta}Z^{IJ}\ , \quad \Big\{Q^I_\alpha, Q^J_\beta\Big\} = 0\ , \quad \Big\{\bar{Q}^I_\alpha, \bar{Q}^J_\beta\Big\} = 0\ . \tag{1.3.12c}$$

Here, the notations are standard, i.e., $\{M_{\mu\nu}, P_\rho\}$ are the Poincare generators, $\{Q^I_\alpha, Q^{I\alpha}\}$ are the supercharges and Z is the real central charge of the theory. Furthermore, the theory enjoys an automorphism among the supercharges, which gives rise to a non-trivial $U(1)_R$ R symmetry. The (bosonic) R symmetry generator ($\mathcal{R}$) has the following commutation relations with the supercharges and the Poincare generators, collectively denoted by $\mathfrak{g}_{Poincare}$

$$\Big[\mathcal{R}, Q_\alpha\Big] = -Q_\alpha\ , \qquad [\mathcal{R}, \bar{Q}^\alpha] = \bar{Q}^\alpha\ , \qquad \Big[\mathcal{R}\ , \mathfrak{g}_{Poincare}\Big] = 0\ . \tag{1.3.13}$$

In Eq. (1.3.12), Z is a central charge. This is manifestly seen by the commutation relations

$$\Big[Z, Q_\alpha\Big] = 0\ , \qquad [Z, \bar{Q}^\alpha] = 0\ , \qquad \Big[Z\ , \mathcal{R}\Big] = 0\ . \tag{1.3.14}$$

1.3.3 $\mathcal{N} = 1$ and $\mathcal{N} = 2$ multiplets

Here, we briefly review the field contents of $\mathcal{N} = 1$ and $\mathcal{N} = 2$ theories (on $3d$). Fields in supersymmetry appear in multiplets. In each multiplet, the constituent fields vary in their statistics, being formed by the action of supercharges on the Clifford vacuum. But, fields of the same multiplet have the same mass. However, the theoretical prediction of this mass degeneracy among a pair of bosons and fermions has not yet been observed empirically, suggesting that supersymmetry is broken at some scale higher than that probed by present day experiments.

$\mathcal{N} = 1$ Multiplets

The field content of the $\mathcal{N} = 1$ chiral multiplet is a Weyl fermion (ψ) and a complex scalar (ϕ). For invariance under CPT transformations, one adds its CPT conjugate fields and the multiplet is obtained as shown in Eq. (1.3.15a), where, the numbers in brackets denote the spins of the constituent fields. Historically, this multiplet is also known as the Wess Zumino (WZ) multiplet as they first wrote a Lagrangian involving a Majorana fermion and scalars (in $4d$) in Ref. [4]. For the $\mathcal{N} = 1$ vector multiplet, one has a gauge field (A_μ) and a Weyl fermion (transforming in the adjoint representation of the gauge group) and their CPT conjugates. The field content is succinctly represented in Eq. (1.3.15b). Similarly, in the gravitino and the graviton multiplet, one has a graviton and a gauge field, and a graviton and its superpartner, a gravitino respectively, as shown in Eq. (1.3.15c) and Eq. (1.3.15d).

$$\text{Chiral Multiplet} \;:\; \left(0, \tfrac{1}{2}\right) \oplus \left(-\tfrac{1}{2}, 0\right) \tag{1.3.15a}$$

$$\text{Vector Multiplet} \;:\; \left(\tfrac{1}{2}, 1\right) \oplus \left(-1, -\tfrac{1}{2}\right) \tag{1.3.15b}$$

$$\text{Gravitino Multiplet} \;:\; \left(1, \tfrac{3}{2}\right) \oplus \left(-\tfrac{3}{2}, -1\right) \tag{1.3.15c}$$

$$\text{Graviton Multiplet} \;:\; \left(\tfrac{3}{2}, 2\right) \oplus \left(-2, -\tfrac{3}{2}\right) \tag{1.3.15d}$$

$\mathcal{N} = 2$ Multiplets

The field content of $\mathcal{N} = 2$ hypermultiplet consists of two copies of the WZ multiplets, as shown in Eq. (1.3.16a). This multiplet forms the matter sector of the $\mathcal{N} = 2$ theory. The vector multiplet consists of one vector, two Weyl fermions and a complex scalar, all of which necessarily transform in the adjoint representation of the gauge group. The multiplet is shown in Eq. (1.3.16b). The $\mathcal{N} = 2$ gravitino and the graviton multiplet consist of respectively, a gravitino, two vectors called gravi-photons and a Weyl fermion, and a graviton, two gravitini and a gravi-photon. The multiplets are shown in Eq. (1.3.16c) and Eq. (1.3.16d) respectively.

$$\text{Hypermultiplet} \;:\; \left(-\tfrac{1}{2}, 0, 0, \tfrac{1}{2}\right) \oplus \left(-\tfrac{1}{2}, 0, 0, \tfrac{1}{2}\right) \tag{1.3.16a}$$

$$\text{Vector Multiplet} \;:\; \left(0, \tfrac{1}{2}, \tfrac{1}{2}, 1\right) \oplus \left(-1, -\tfrac{1}{2}, -\tfrac{1}{2}, 0\right) \tag{1.3.16b}$$

$$\text{Gravitino Multiplet} \;:\; \left(\tfrac{1}{2}, 1, 1, \tfrac{3}{2}\right) \oplus \left(-\tfrac{3}{2}, -1, -1, -\tfrac{1}{2}\right) \tag{1.3.16c}$$

$$\text{Graviton Multiplet} \;:\; \left(1, \tfrac{3}{2}, \tfrac{3}{2}, 2\right) \oplus \left(-2, -\tfrac{3}{2}, -\tfrac{3}{2}, -1\right) \tag{1.3.16d}$$

1.4 Supersymmetry on curved spaces

1.4.1 General principle

A general prescription for placing field theories on curved spaces is as follows - we first introduce gravitational interaction to the theory, so that the metric is made dynamical. Then, we send the Newton's constant G_N to zero, which defines for us the *rigid limit*. This effectively *freezes* the dynamics of the metric to a value g different from the Minkowski metric η, i.e., $\eta \to g$ and we define then the theory on a curved manifold $(\mathcal{M}, g)$ instead of the original Minkowski spacetime $(\mathbb{R}^{d-1,1}, \eta)$.

Festuccia and Seiberg in Ref. [10] followed a similar strategy to place supersymmetric field theories on curved spaces. The idea was that since, now, we wish to define a supersymmetric theory on curved spaces (or spacetimes), we need to couple the theory to supergravity theories, instead of introducing just gravitational dynamics, as was the requirement for the case of ordinary QFTs. However, unlike the case of purely bosonic gravity, the supergravity multiplet contains fields over and above the metric - viz., the gravitino and other fields of spin 1. For example, the supergravity multiplet of the $\mathcal{N} = 2$ theory, given in Eq. (1.3.16d), shows the presence of the graviton, the gravitini and the gravi-photon which is a spin-1 field. These fields are off-shell in the rigid limit and are not determined in terms of the other fields of the multiplet. Festuccia et al showed that to ensure that the supersymmetric theory, upon placing on a curved manifold $(\mathcal{M}, g)$ retains some amount of supersymmetry, it suffices to impose that the fermionic fields of the supergravity multiplet and their variations becomes zero in the rigid limit. That is, we look for the values of the supergravity backgrounds that satisfy

$$\psi_{\mu\alpha} = 0 \;, \qquad \delta\psi_{\mu\alpha} = 0 \;. \tag{1.4.1}$$

Eq. (1.4.1) gives a set of partial differential equation, called the generalized Killing spinor equation, which generically involves the bosonic fields of the supergravity multiplet. Should a solution to the generalized Killing spinor equation exist on $(\mathcal{M}, g)$, we can place a theory there and have some supersymmetry.

1.4.2 $3d$ Case

Here, let us briefly investigate the Festuccia Seiberg prescription in a bit more detail. For further details, we refers to Refs. [10, 11].

We recall the principle of minimal coupling in gauge theories. If any global symmetry is gauged, we promote the partial derivatives to gauge covariant derivatives. This is equivalent to coupling the theory to a gauge field via some conserved current $\mathcal{J}^\mu$. Explicitly, for generic fields denoted by $\{\Phi\}$,

$$\mathcal{L}\big[\, \{\Phi, \partial_\mu \Phi\} \,\big] \stackrel{gauging}{\longrightarrow} \mathcal{L}\big[\, \{\Phi, D_\mu \Phi\} \,\big] = \mathcal{L}\big[\, \{\Phi, \partial_\mu \Phi\} \,\big] - \mathcal{J}^\mu A_\mu \;. \tag{1.4.2}$$

One may introduce dynamics for the gauge field via a Maxwell term, or one may also be interested in keeping it as a fixed background, which has no equation of motion. In our case, we will take this latter approach.

A similar story prevails when working with spacetime symmetries of the theory. Corresponding to the translation symmetries, generated by P_μ, there exists a conserved symmetric stress tensor $T_{\mu\nu}$. This plays the same role as the conserved current $\mathcal{J}^\mu$ for the global symmetry. One can, then, work with linearised gravity models, by perturbing the Minkowski metric η slightly such that $\eta \to g = \eta + h$. The prescription for minimal coupling then suggests that

$$\mathcal{L}\big[\,\{\Phi, \partial_\mu \Phi\}\,;\eta\big] \longrightarrow \mathcal{L}\big[\,\{\Phi, \nabla_\mu \Phi\}\,;g\big] = \mathcal{L}\big[\,\{\Phi, \partial_\mu \Phi\}\,\big] + \frac{1}{2} T^{\mu\nu} h_{\mu\nu}\,. \tag{1.4.3a}$$

$$\text{where}\,, \qquad T^{\mu\nu} = -\frac{2}{\sqrt{|g|}} \frac{\delta S}{\delta g_{\mu\nu}}\,. \tag{1.4.3b}$$

where ∇_μ is the general covariant derivative. However, for our case of supersymmetric field theories on curved spaces, we have further fields constituting the supergravity multiplet. Especially, every supersymmetric theory will have *at least* one conserved supercharge Q_α. Defining the corresponding supercurrent as S^μ_α, we have couplings of the kind $S^{\alpha\mu}\psi_{\alpha\mu}$. Depending on the fields in the supercurrent multiplet, we will have similar such interactions.

We now focus on our specific case of $3d$ $\mathcal{N} = 2$ theory with a $U(1)_R$ R symmetry. The story will be similar and we will find interactions given by the fields in background supergravity multiplet coupling to currents in the supercurrent multiplet. Following Ref. [12], the appropriate supercurrent multiplet containing the charges generating the algebra in Eqs. (1.3.12), (1.3.13) and (1.3.14) is given by the $\mathfrak{R}$ multiplet. The $\mathfrak{R}$ multiplet is given by

$$\mathfrak{R} = \left\{ T^{\mu\nu}, S^\mu_\alpha, \tilde{S}^\mu_\alpha, j^\mu_R, j^\mu_Z, j_Z \right\} \tag{1.4.4}$$

Here, $T_{\mu\nu}$ is the symmetric stress tensor, $S^\mu_\alpha, \tilde{S}^\mu_\alpha$ are the currents corresponding to the supercharges, j^μ_R is the R current corresponding to the R charge, j^μ_Z is the Z current corresponding to the central charge and j_Z is a topological current which is present in $3d$.

The conjugate supergravity multiplet $(\mathfrak{H})$ is called the "new minimal supergravity", the field content of which is given by

$$\mathfrak{H} = \left\{ h_{\mu\nu}, \psi_{\mu\alpha}, \tilde{\psi}_{\mu\alpha}, A_\mu^{(R)}, C_\mu, B_{\mu\nu} \right\} . \tag{1.4.5}$$

Here, $h_{\mu\nu}$ is the graviton, $\psi_{\mu\alpha}, \tilde{\psi}_{\mu\alpha}$ are the gravitini, $A_\mu^{(R)}$ and C_μ are 1-form gauge fields and a 2-form gauge field $B_{\mu\nu}$. We define the dual quantities via the equations

$$V^\rho = -i\epsilon^{\rho\mu\nu}\partial_{[\mu}C_{\nu]} , \qquad H = \frac{i}{2}\epsilon^{\rho\mu\nu}\partial_{[\rho}B_{\mu\nu]} . \tag{1.4.6}$$

Once we have the background supergravity multiplet, we follow the ideas of Sec. 1.4.1, set the gaugino and its variation to zero, as in Eq. (1.4.1) and find the generalised Killing spinor equation. We state the result as

$$\left(\nabla_\mu - iA_\mu^{(R)}\right)\xi_+ = -\frac{H}{2}\gamma_\mu\xi_+ - iV_\mu\xi_+ - \frac{1}{2}\epsilon_{\mu\nu\rho}V^\nu\gamma^\rho\xi_+ , \tag{1.4.7a}$$

$$\left(\nabla_\mu + iA_\mu^{(R)}\right)\xi_- = -\frac{H}{2}\gamma_\mu\xi_- + iV_\mu\xi_- + \frac{1}{2}\epsilon_{\mu\nu\rho}V^\nu\gamma^\rho\xi_- . \tag{1.4.7b}$$

where, $\{\xi_+, \xi_-\}$ are two Killing spinors (with opposite R charges), and the background bosonic supergravity fields are defined following Eq. (1.4.5) and in Eq. (1.4.6). Now, the Killing spinors are functions of the spacetime coordinates - this is a signature of supersymmetric theories on curved spaces that their Killing spinors are functions of the spacetime coordinates - and one finds specific solutions to Eq. (1.4.7) for the manifold $(\mathcal{M}, g)$. The metric functions enter the equation via the gauge covariant derivate and the Levi-civita tensor density $\epsilon_{\mu\nu\rho}$. Should we find solutions to Eq. (1.4.7) on $\mathcal{M}$, then we can place supersymmetric theories on $\mathcal{M}$. In Ch. 4, we will follow this strategy and solve Eq. (1.4.7) for our case of squashed sphere.

1.5 Supersymmetric Localization

Exact or non-perturbative calculation of the partition function in Quantum Field Theories (QFTs) has been the holy grail of theoretical physics for over three quarters of a century now. Yet, the efforts have met only partial success. Till recently, non-perturbative results were accessible only for the trivial case of free theories, where exact gaussian integrations may be performed on the space of fields, or for a limited and special class of topological theories defined on compact manifolds, as was worked out in Ref. [13] and for the supersymmetric case in Ref. [14]. Witten's work in supersymmetric quantum mechanics in Ref. [15] provides a precursor to these works. A wide range of results soon followed in Refs. [16, 17]. Further progress has been made by Pestun in Ref. [18] where he explicitly calculated the Wilson Loop for the $\mathcal{N} = 4$ theory on S^4. Since then, a plethora of supersymmetry protected local and non-local operators[1] as well as surface operators have been calculated for various theories in various dimensions. To instantiate, supersymmetric observables have been calculated exactly for theories placed on S^2 in Ref. [19, 20], on S^3 in Ref. [21], its orbifolds in Ref. [22], on S^5 in Ref. [23, 24], on squashed S^3 in Ref. [25], on squashed S^5 in Ref. [26].

[1]Specifically, Wilson Loops and 't Hooft Loops.

1.5.1 General principle

The basic ingredients of supersymmetric Localization consist of the following :

$$\text{A conserved supercharge } Q \,, \qquad \text{Square of the supercharge } B \equiv Q^2 \,,$$
$$\text{BPS observables } \mathcal{O} \ \ (\text{with } \ \{\mathcal{O}|Q\mathcal{O}=0\}) \,,$$
$$\text{A } Q\text{-exact term } Q\Psi \ \ (\text{with } \ \{\Psi|B\Psi=0\}) \,, \tag{1.5.1}$$

Let us imagine we have a quantum field theory which enjoys a fermionic symmetry. The generators of this symmetry are denoted by the Grassmann odd charge Q. The symmetry is assumed non-anomalous. Further, as explained in Eq. (1.5.1), let us have a bosonic charge B which may generate some combinations spacetime symmetries or internal symmetries. The operators which we would like to consider are denoted by $\mathcal{O}$ which are Q-closed, i.e.,

$$B \equiv Q^2 \,, \qquad Q\mathcal{O} = 0 \,. \tag{1.5.2}$$

Generally, they entail a broad class of interesting operators, like local operators or their products, surface operators and even non-local operators like Wilson Loops. With this background, let us try and evaluate $\langle\mathcal{O}\rangle$ given by[2]

$$\langle\mathcal{O}\rangle \equiv \int_{\mathfrak{F}} [\mathcal{D}\Phi]\, \mathcal{O}\, e^{-S[\Phi]} \tag{1.5.3}$$

Under the assumptions that the symmetry generated by the fermionic charge Q is non-

[2]Here, $\{\Phi\}$ denotes the generic fields present in the theory and $\mathfrak{F}$ denotes the space of fields.

anomalous and that there are no boundary terms,

$$\langle Q\Psi\rangle \equiv \int_{\mathfrak{F}} [\mathcal{D}\Phi]\,(Q\Psi)\; e^{-S[\Phi]} = \int_{\mathfrak{F}} [\mathcal{D}\Phi]\, Q(\Psi e^{-S[\Phi]}) = 0 \tag{1.5.4}$$

$$\text{and ,} \qquad \langle \mathcal{O} + Q\Psi\rangle = \langle \mathcal{O}\rangle. \tag{1.5.5}$$

This implies that the under these assumptions that the expectation values of operators are insensitive to insertions of Q-closed observables and expectation value of Q-closed observables O depends on its Q-cohomology class. This allows us to calculate the expectation value of O by choosing a representative element of its equivalence class instead. Often, this simplifies the problem and allows one to do exact calculations.

With this observation, let us now focus on a *perturbed* path integral given by[3],

$$\langle \mathcal{O}^{(t)}\rangle \equiv \int_{\mathfrak{F}} [\mathcal{D}\Phi]\; \mathcal{O}\; e^{-S[\Phi]-tQ\Psi} \quad \forall t > 0 \tag{1.5.6}$$

for some fictitious parameter t. It stands to immediate checks that

$$\begin{aligned} \frac{d}{dt}\langle \mathcal{O}^{(t)}\rangle &= \frac{d}{dt}\int_{\mathfrak{F}} [\mathcal{D}\Phi]\; \mathcal{O}\; e^{-S[\Phi]-tQ\Psi[\Phi]} \\ &= -\int_{\mathfrak{F}} [\mathcal{D}\Phi]\; Q\Psi[\Phi]\; \mathcal{O}\; e^{-S[\Phi]-tQ\Psi[\Phi]} \\ &= -\int_{\mathfrak{F}} [\mathcal{D}\Phi]\; Q\Big(\Psi[\Phi]\; \mathcal{O}\; e^{-S[\Phi]-tQ\Psi[\Phi]}\Big) \\ &= 0. \end{aligned} \tag{1.5.7}$$

again, on the assumption that there is a fast decay of the fields at the asymptotes.

Thus, the perturbed integral is actually independent of the parameter t and we can evaluate the integral for any judiciously chosen value of t. In practice, the saddle point

[3] In actual calculations, one finds a $Q\Psi[\Phi]$ such that its bosonic part $(Q\Psi[\Phi])|_{\text{bosonic}} \geq 0$.

analysis at the large t limit, $t \to \infty$ becomes exact and we can evaluate Eq. (1.5.3) non-perturbatively. Mathematically,

$$\begin{aligned}\langle \mathcal{O} \rangle &\equiv \lim_{t\to 0} \int_{\mathfrak{F}} [\mathcal{D}\Phi]\, \mathcal{O}\, e^{-S[\Phi]-tQ\Psi[\Phi]} \\ &= \lim_{t\to \infty} \int_{\mathfrak{F}} [\mathcal{D}\Phi]\, \mathcal{O}\, e^{-S[\Phi]-tQ\Psi[\Phi]} \end{aligned} \tag{1.5.8}$$

In the large t limit, the integral is dominated by the saddle points of the localizing part of the action $S_{loc}[\Phi]$, where $S_{loc}[\Phi] \equiv Q\Psi[\Phi]$.

One canonical choice[4] for $\mathcal{L}_{loc}[\Phi]$ is

$$\mathcal{L}_{loc}[\Phi] = Q \sum_{\substack{i \\ \psi_i \in \text{Fermions}}} \left((Q\psi_i)^\dagger \psi_i + \psi_i^\dagger (Q\psi_i^\dagger)^\dagger \right) \tag{1.5.9}$$

where, $\psi_i \in \Phi$ such that they are the fermions of the theory. Then,

$$\begin{aligned}\mathcal{L}_{loc}[\Phi]\, |_{\text{bosonic}} &= \sum_{\substack{i \\ \psi_i \in \text{Fermions}}} \left((Q\psi_i)^\dagger Q\psi_i + Q\psi_i^\dagger (Q\psi_i^\dagger)^\dagger \right) \\ &= \sum_{\substack{i \\ \psi_i \in \text{Fermions}}} \left(\|Q\psi_i\|^2 + \|Q\psi_i^\dagger\|^2 \right) \end{aligned} \tag{1.5.10}$$

From Eq. (1.5.9) and Eq. (1.5.10) we see the saddles coincide with the BPS configurations given by

$$\psi_i = \psi_i^\dagger = 0\,, \qquad Q\psi_i = Q\psi_i^\dagger = 0\,. \tag{1.5.11}$$

Let us collectively denote the BPS locus by $\Phi_0 = \mathfrak{F}_Q \subset \mathfrak{F}$. We see that the infinite dimen-

[4]There exists other choices as well.

sional integral over field space $\mathfrak{F}$ reduces to a subspace which, often, is finite dimensional and the integral proves tractable. To evaluate the integral, let us expand the fields Φ about the classical saddles of $S_{loc}[\Phi]$ as

$$\Phi = \Phi_0 + \frac{1}{\sqrt{t}}\delta\Phi \,. \tag{1.5.12}$$

The above normalization of the fluctuations is just a convenient choice which sets the quadratic fluctuations t independent.

Therefore, Eq. (1.5.8) simplifies to

$$\begin{aligned}\langle\mathcal{O}\rangle &= \lim_{t\to\infty}\left[\int_{\mathfrak{F}}[\mathcal{D}\Phi]\,\mathcal{O}\,e^{-S[\Phi]-tQ\Psi[\Phi]}\right] \\ &= \lim_{t\to\infty}\left[\int_{\mathfrak{F}_Q}[\mathcal{D}\Phi_0]\,\mathcal{O}|_{\Phi=\Phi_0}\,e^{-S[\Phi_0]}\frac{1}{\mathrm{SDet}\left[\frac{\delta^2 S_{loc}[\Phi]}{\delta\Phi^2}\right]_{\Phi=\Phi_0}} + O(\tfrac{1}{\sqrt{t}}\text{terms})\right] \\ &= \int_{\mathfrak{F}_Q}[\mathcal{D}\Phi_0]\,\mathcal{O}|_{\Phi=\Phi_0}\,e^{-S[\Phi_0]}\frac{1}{\mathrm{SDet}\left[\frac{\delta^2 S_{loc}[\Phi]}{\delta\Phi^2}\right]_{\Phi=\Phi_0}}\,.\end{aligned} \tag{1.5.13}$$

This result is 1-Loop exact as we are allowed to take the formal limit $t \to \infty$ which suppresses any further corrections to exactly zero.

1.5.2 $3d$ Case

Let us briefly see how this works out in our specific case of $3d$. For clarity as well as brevity, we will just turn on gauge sector. For localization of the three dimensional theory with matter fields in some arbitrary representation $\mathfrak{R}_i$, we refer to Refs. [21, 27, 28].

The vector multiplet of the $3d$ $\mathcal{N}=2$ theory has been explored in Eq. (1.3.16b). We have two standard choices for Lagrangian of the gauge field - the super Yang Mills (SYM)

Lagrangian and the topological supersymmetric Chern Simons (SCS) Lagrangian. Both actions are Q-closed (i.e., they preserve some supersymmetry), however, the former is *also* Q-exact. Hence, following the argument in Sec. 1.5.1, the SYM action can be made use of, in deforming the path integral in Eq. (1.5.6). However, in this brief section, we will choose the topological action as the one providing dynamics to the gauge field. The action is given by

$$S_{SCS} = \frac{\kappa}{4\pi} \int d^3x \sqrt{|g|} \,\mathrm{Tr} \Big(\epsilon^{\mu\nu\rho} \big(A_\mu \partial_\nu A_\rho + \frac{2i}{3} A_\mu A_\nu A_\rho \big) + \lambda^\dagger \lambda + 2iD\sigma \Big) \tag{1.5.14}$$

We state the transformations that are a symmetry of the action given Eq. (1.5.14). They are given by[5]

$$\delta A_\mu = -\tfrac{i}{2} \lambda^\dagger \gamma_\mu \xi_+ \,, \tag{1.5.15a}$$

$$\delta\sigma = -\tfrac{1}{2} \lambda^\dagger \xi_+ \,, \tag{1.5.15b}$$

$$\delta D = -\tfrac{i}{2} \big(D_\mu \lambda^\dagger \gamma^\mu \big) \xi_+ + \tfrac{1}{4} \lambda^\dagger \xi_+ + \tfrac{i}{2} [\lambda^\dagger, \sigma] \xi_+ \,, \tag{1.5.15c}$$

$$\delta\lambda = \big(-\tfrac{1}{2} \gamma^{\mu\nu} F_{\mu\nu} - D - \not{D}\sigma - \sigma \big) \xi_+ \,, \tag{1.5.15d}$$

$$\delta\lambda^\dagger = 0 \,. \tag{1.5.15e}$$

On S^3, the generalised Killing spinor equations given in Eq. (1.4.7) reduce to[6]

$$\nabla_\mu \xi_\pm = \pm \frac{i}{2} \gamma_\mu \xi_\pm \tag{1.5.16}$$

This is the equation ξ_+ in Eq. (1.5.15) must satisfy.

[5]In euclidean signature, λ and $\lambda^\dagger$ are independent degrees of freedom. The transformation of one is independent of the other.

[6]On S^3, this equation implies that a solution for the Killing spinor exists for trivial values of the bosonic fields of the background supergravity multiplet. This is not true if S^3 is deformed.

Localization As discussed in Sec. 1.5.1, to localize the theory one needs to choose a deforming part of the action, which is Q-exact. One such choice is

$$Q\Psi = \delta\, \mathrm{Tr}\left((\delta\lambda)^\dagger \lambda\right) \tag{1.5.17}$$

This is a judicious choice because

$$\delta\, \mathrm{Tr}\left((\delta\lambda)^\dagger \lambda\right)|_{\text{bosonic}} = \mathrm{Tr}\left((\delta\lambda)^\dagger \delta\lambda\right) \geq 0 \tag{1.5.18}$$

and the path integral localizes to the BPS locus given by the saddles of this expression.

Saddles From Eq. (1.5.15d), Eq. (1.5.17) and Eq. (1.5.18), we evaluate

$$Q\Psi = \int d^3x\, \sqrt{|g|}\ \mathrm{Tr}\left(\tfrac{1}{2}F_{\mu\nu}F^{\mu\nu} + D_\mu\sigma D^\mu\sigma + (D+\sigma)^2 + i\lambda^\dagger\gamma^\mu\nabla_\mu\lambda + i[\lambda^\dagger,\sigma]\lambda - \tfrac{1}{2}\lambda^\dagger\lambda\right) \tag{1.5.19}$$

Therefore, the saddles $\{\Phi_0\}$ are given by

$$F^{\mu\nu} = 0\,, \quad D_\mu\sigma = 0\,, \quad (D+\sigma) = 0\,. \tag{1.5.20a}$$

$$\text{i.e.,}\quad \Phi_0:\ A = 0\,, \quad D = -\sigma\,, \quad \sigma = \sigma_0\ \text{(constant)}\,. \tag{1.5.20b}$$

From Eq. (1.5.13), with $\mathcal{O} = \mathbb{I}$,

$$Z = \langle \mathbb{I} \rangle = \int d\sigma_0\, e^{S_{classical}[\Phi_0]}\ Z_{1-Loop}\,. \tag{1.5.21}$$

From Eq. (1.5.14) with $\kappa = 4\pi$, $A_\mu = 0$ and vol(S^3) = $2\pi^2$,

$$S_{classical}[\phi_0] = i \int d^3x \sqrt{|g|}\, 2\,\mathrm{Tr}(D\sigma) = -4i\pi^2\,\mathrm{Tr}(\sigma_0^2)\,. \tag{1.5.22}$$

Plugging Eq. (1.5.22) back into Eq. (1.5.21),

$$Z = \int d\sigma_0\, e^{-4i\pi^2\,\mathrm{Tr}(\sigma_0^2)}\;\; Z_{1-Loop}\,. \tag{1.5.23}$$

$1 - Loop$ **Determinant** The last ingredient to the exact calculation using localization is calculating the $1 - Loop$ determinant coming from the quadratic fluctuations. Here, the fluctuations of the field about their classical values are parameterised as follows, i.e., $\Phi \to \Phi_0 + \frac{1}{\sqrt{t}}\Phi'$. Explicitly,

$$\sigma \to \sigma_0 + \frac{1}{\sqrt{t}}\sigma'\,, \quad D \to -\sigma_0 + \frac{1}{\sqrt{t}}D'\,, \quad \Theta \to \frac{1}{\sqrt{t}}\Theta \tag{1.5.24}$$

where, Θ denotes the rest of the fields of the vector multiplet[7].

Plugging Eq. (1.5.24) in Eq. (1.5.19), and integrating the action by parts, we obtain as the leading order term

$$t\,Q\Psi = \int d^3x\,\sqrt{|g|}\;\;\mathrm{Tr}\;\;\Big(-A^\mu\Delta A_\mu - [A_\mu,\sigma_0]^2 + \partial_\mu\sigma'\partial^\mu\sigma' + i\lambda^\dagger\nabla\!\!\!\!/\,\lambda + i[\lambda^\dagger,\sigma_0]\lambda$$
$$-\;\; \tfrac{1}{2}\lambda^\dagger\lambda\Big) + O(\tfrac{1}{\sqrt{t}}) \tag{1.5.25}$$

where, Δ is the Laplacian operator and $\nabla\!\!\!\!/$ is the Dirac operator on S^3. Following the

[7]The fluctuations are normalized such that the quadratic fluctuations are t independent. Any other normalization does not change the conclusion.

strategy outlined in Ref. [29] and separating the gauge field into its divergence A^d and divergence free part A^{df}_μ, contribution of Eq. (1.5.25) to the path integral can be shown to reduce to

$$t\,Q\Psi = \int d^3x\,\sqrt{|g|}\,\mathrm{Tr}\left(\underbrace{-A^{df}_\mu \Delta A^{df\,\mu} - [A^{df}_\mu, \sigma_0]^2}_{\text{bosonic sector}} + \underbrace{i\lambda^\dagger \not{\lambda} + i[\lambda^\dagger, \sigma_0]\lambda - \frac{1}{2}\lambda^\dagger\lambda}_{\text{fermionic sector}}\right) \tag{1.5.26a}$$

$$\text{where,} \qquad A_\mu = \partial_\mu A^d + A^{df}_\mu \tag{1.5.26b}$$

So, Eq. (1.5.26a) suggests that we have the final result of the path integral as a ratio of the eigenvalues of the Laplacian and the Dirac operator on S^3 . The spectrum of these operators on S^3 have been well studied over the years and we can directly borrow those results.

As a last technical detail, we can use the Weyl integration formula, which reduces the integral over the Lie algebra valued element σ_0 to a sub-space spanned by the Cartan sub-algebra $\mathfrak{h} \subset \mathfrak{g}$ at the cost of introducing a Vandermonde determinant $\prod \alpha(\sigma_{\mathfrak{h}})$ and we can divide the integral by the order of the Coxeter group $(\mathcal{W})$ to take of the residual symmetry. Finally, in short, we obtain for the partition function

$$Z = \frac{1}{|\mathcal{W}|}\int d\sigma_{\mathfrak{h}} \prod_{\text{roots}} \alpha(\sigma_{\mathfrak{h}}) \left(\exp\big(-4i\pi^2\,\mathrm{Tr}(\sigma_{\mathfrak{h}}^2)\big)\, Z_{1-Loop}[\sigma_{\mathfrak{h}}] \right) \tag{1.5.27}$$

To determine $Z_{1-Loop}[\sigma_{\mathfrak{h}}]$, we need to project Eq. (1.5.26a) to the Cartan subspace. For the chosen normalization of the ladder operators $\{E_\alpha\}$ given by $\mathrm{Tr}(E_\alpha E_\beta) = \delta_{\alpha,-\beta}$, we obtain

$$t\,Q\Psi = \int d^3x\,\sqrt{|g|}\sum_{\text{roots}}\Big((A^{df\,\mu})_{-\beta}\,\underbrace{(-\Delta+\beta(\sigma_{\mathfrak{h}})^2)}_{I}\,(A^{df}_{\mu})_{\beta}+\lambda^{\dagger}_{-\beta}\,\underbrace{(i\nabla\!\!\!/ + i\beta(\sigma_{\mathfrak{h}})-\tfrac{1}{2})}_{II}\,\lambda_{\beta}\Big) \tag{1.5.28}$$

where, for clarity, we show a standard Lie algebra element calculation as

$$[\sigma_{\mathfrak{h}}, A^{df\,\mu}] =$$

$$\sum_{\text{roots}}[\sigma^i_{\mathfrak{h}}h_i, A^{df\,\mu}_{\beta}E_\beta] = \sum_{\text{roots}}\sigma^i_{\mathfrak{h}}A^{df\,\mu}_{\beta}[h_i,E_\beta] = \sigma^i_{\mathfrak{h}}A^{df\,\mu}_{\beta}\beta_i E_\beta = \sum_{\text{roots}}\beta(\sigma_{\mathfrak{h}})A^{df\,\mu}_{\beta}E_\beta \tag{1.5.29a}$$

$$\text{where,}\qquad i=1,\ldots,r\,,\quad \{h_i\}\in\mathfrak{h}\,,\quad \text{and}\,,\quad \beta:\mathfrak{h}\to\mathfrak{F}\,. \tag{1.5.29b}$$

So, from Eq. (1.5.28), we conclude the problem finally reduces to finding the spectrum of the operators defined in I, II on S^3. Using the eigenvalues of the operators I, II on S^3, we not a significant cancellation between the eigenvalues of the fermionic operator and the bosonic operators[8] and we obtain a final result as

$$Z_{1-Loop} = \prod_{\text{roots}}\Big(\frac{2\sinh(\pi\beta(\sigma_{\mathfrak{h}}))}{\pi\beta(\sigma_{\mathfrak{h}})}\Big)\,. \tag{1.5.30}$$

From Eq. (1.5.27) and the denominator of Eq. (1.5.30), we see that Vandermonde determinant drops out and we are left with

$$Z = \frac{1}{|\mathcal{W}|}\int d\sigma_{\mathfrak{h}}\exp\big(-4i\pi^2\,\mathrm{Tr}(\sigma^2_{\mathfrak{h}})\big)\prod_{\text{roots}}\Big(2\sinh(\pi\beta(\sigma_{\mathfrak{h}}))\Big)\,. \tag{1.5.31}$$

This shows the tremendous simplification supersymmetric Localization can bring about -

[8]This is almost always the case in such calculations.

the infinite dimensional path integral has reduced to the Cartan subspace of the field space. Often, given a gauge group, Eq. (1.5.31) is tractable, giving us tremendously powerful non-perturbative results of certain theories on compact manifolds. In the following chapters, we will, directly or indirectly, exploit the machinery of supersymmetric localization to calculate supersymmetric observables in different contexts.

CHAPTER 2

3D $\mathcal{N} = 2$ $\widehat{ADE}$ CHERN-SIMONS QUIVERS

This chapter is based on

1. "$3d$ $\mathcal{N} = 2$ $\widehat{ADE}$ Chern-Simons quivers ", D. Jain, A. Ray, *Phys. Rev.* D100 **(2019)** 4, 046007, [arXiv: 1902.10498].

2.1 Introduction and Outline

Supersymmetric localization has made a whole host of theories accessible to non-perturbative analysis. It provides a powerful framework to construct and compute quantities along the RG flow non-perturbatively. One of them is the exact partition function Z for supersymmetric gauge theories put on various curved manifolds in different dimensions. This has led to a deeper understanding, checks and / or discovery of various dualities among field theories, even across dimensions. Since one gets access to exact results, one can test AdS/CFT correspondence in the regime relating weak gravity results (may or may not have been obtained via localization) to strong coupling results in the field theory (highly likely to have been obtained via localization). We will focus here on this latter possibility with the study of supersymmetric quiver gauge theories in three-dimensions, i.e., an example of $\text{AdS}_4/\text{CFT}_3$.

Free Energy. Localization was successfully applied to compute the partition function of $3d$ Chern-Simons-matter (CSm) theories placed on 3-sphere S^3 in Refs. [21, 30, 31]. The first explicit construction of a $3d$ $\mathcal{N} = 6$ gauge theory with M-theory dual was by ABJM in Ref. [32]. It involved two $U(N)$ gauge groups with CS terms at levels $\pm k$ and four bifundamental chiral multiplets (in terms of $\mathcal{N} = 2$ multiplets). The dual geometry involved placing N M2-branes at the tip of a $\mathbb{C}^4/\mathbb{Z}_k$ singularity such that in large N limit, the $\text{AdS}_4 \times S^7/\mathbb{Z}_k$ vacuum solution of M-theory was obtained. Following this, a large number of $\mathcal{N} \geq 2$ dual pairs have been identified, with the M-theory dual of the form $\text{AdS}_4 \times Y_7$, where Y_7 is a (tri-)Sasaki-Einstein manifold given by the base of a certain 8d (hyper)kähler cone, as worked out variously in Refs. [33–39]. The AdS/CFT dictionary relates $\text{Vol}(Y_7)$ to the free energy F_{S^3} of the dual gauge theories in the large N limit, as explained in Refs. [34, 40]

$$F_{S^3} = -\log |Z_{S^3}| = N^{3/2} \sqrt{\frac{2\pi^6}{27\,\text{Vol}(Y_7)}} \,. \tag{2.1.1}$$

This provides an important tool to compute the volumes via computations in the dual field theory.

We will consider general $\mathcal{N} = 2$ quiver gauge theories on S^3 involving matter multiplets with arbitrary R-charges Δ's in the large N limit. This will lead us to a large class of quiver theories whose free energy scales as $N^{3/2}$ from requiring that the long-range forces in the resulting matrix model cancel (or equivalently, that the matrix model be local) along with a constraint on the R-charges of bifundamental multiplets given by Eq. (2.2.6). A subset of this constraint leads to the $\widehat{ADE}$ classification via a simple constraint on the bifundamental R-charges:

$$\Delta_{(a,b)} + \Delta_{(b,a)} = 1\,. \tag{2.1.2}$$

We note that for $\mathcal{N} \geq 3$ case, this condition is automatic since the supersymmetry enhancement fixes the R-charges to be $\frac{1}{2}$ and $\widehat{ADE}$ classification was presented in Ref. [41]. We will then explicitly solve the large N matrix model of the $\mathcal{N} = 2$ $\widehat{D}$ quiver theories[1], whose dual geometry involves certain 7-dimensional Sasaki-Einstein manifolds Y_7. Computation of their volumes directly does not necessarily give the volume for the Calabi-Yau metric necessary for the AdS/CFT correspondence[2]. This can be circumvented by using the geometrical result of volume minimization that fixes the Reeb vector and gives the correct volume of the Ricci-flat Kähler manifold, which corresponds in the dual field theory to F-maximization that fixes the R-charges at the IR fixed point, as discussed in Refs. [35, 45, 46]. We will leave the check of this correspondence in the case of $\widehat{D}$ quivers for future work and treat the F_{S^3} computed in Section 2.4 as predicting the volumes of the relevant Sasaki-Einstein Y_7's.

Twisted Index. Localization has also been used to compute the partition function of $3d$ CSm theories on $\Sigma_{\mathfrak{g}} \times S^1$ with a partial topological twist of Ref. [13] on the Riemann sur-

[1]The $\mathcal{N} = 2$ $\widehat{A}$ quivers have been discussed in detail in [42] and $\widehat{E}$ quivers can be solved using the approach discussed in this chapter, but due to increasing complexity (and decreasing clarity) of the expressions, we refrain from giving the explicit results here.

[2]It was not the case for $\mathcal{N} \geq 3$ theories where the hyperkähler structure guarantees the CY condition, which was used to calculate explicit volumes for toric quivers like $\widehat{A}$ in Ref. [43] and nontoric ones like $\widehat{D}$ in Ref. [44].

face ($\Sigma_{\mathfrak{g}}$) of genus $\mathfrak{g}$ in Refs. [47–49]. This partition function is usually called topologically twisted index and depends on chemical potentials $\nu \equiv A_t^{bg} + i\sigma^{bg}$ (complex mass parameters constructed from the background vector multiplets coupled to the flavour symmetries) as well as background magnetic fluxes $\mathfrak{n}$ through $\Sigma_{\mathfrak{g}}$ for the flavour and R-symmetry. It was shown in Ref. [50] that the large N limit of $\Re \log Z_{S^2 \times S^1}$ for ABJM theory reproduces the macroscopic entropy S_{BH} of supersymmetric magnetic AdS_4 black holes discussed in Ref. [51]. The large N limit for many other theories has been considered in Refs. [52, 53], which revealed a connection of Bethe potential $\mathcal{V}$ – obtained as an intermediate step while computing the twisted index – to the F_{S^3} discussed above. In addition, an "index theorem" was proven which showed that the twisted index could be written directly in terms of the $\mathcal{V}$ and its derivatives with respect to the chemical potentials.

We will again consider general $\mathcal{N} = 2$ quiver gauge theories on $\Sigma_{\mathfrak{g}} \times S^1$ in large N limit and find that $\widehat{ADE}$ classification [as a subset of quiver theories which satisfy Eqs. (2.3.7) and (2.3.12)] follows from the requirement that the matrix model is local and the following set of constraints is satisfied:

$$\nu_{(a,b)} + \nu_{(b,a)} = \tfrac{1}{2} \qquad \text{and} \qquad \mathfrak{n}_{(a,b)} + \mathfrak{n}_{(b,a)} = 1 \,. \tag{2.1.3}$$

We will then compute the large N limit of the topologically twisted index for $\widehat{AD}$ quivers. Abusing the terminology slightly, we will denote $\mathcal{I} = \log |Z_{\Sigma_{\mathfrak{g}} \times S^1}|$ and call it the twisted index most of the time[3]. Along the way, we will extend (and simplify) the proof of the relation between the Bethe potential and the twisted index (i.e., the index theorem) to cover not just the $\widehat{A}$-type quiver theories in [52] but a large class of theories including the $\widehat{DE}$ quivers. Once again, we will not construct the dual AdS_4 black hole solutions to compute

[3] We will consider here field theories having M-theory duals only. Theories with type IIA duals can also be similarly considered as have been done in Refs. [54–56].

the entropy S_{BH} explicitly - a recent review [57] and references therein give more details for on twisted index and entropy matching. Assuming AdS/CFT correspondence to hold, we can conjecture that the twisted index computed in Section 2.5 for $\widehat{AD}$ quivers is the entropy for the corresponding dual black holes (after extremization with respect to the chemical potentials), leaving an explicit check for future. However, for the specific case of the universal twist in Refs. [58, 59], we provide further evidence for the AdS/CFT correspondence. In this case, due to holographic RG flow from AdS_4 to AdS_2, the black hole entropy follows a simple relation ($\mathfrak{g} > 1$):

$$S_{BH} = (\mathfrak{g} - 1) F_{S^3} \, . \tag{2.1.4}$$

The twisted index is also proportional to the free energy and a simple relation between various quantities introduced till now follows

$$S_{BH}[\tfrac{\Delta}{2}] = \mathcal{I}[\tfrac{\Delta}{2}] = (\mathfrak{g} - 1)\left[4\mathcal{V}[\tfrac{\Delta}{2}] = F_{S^3}[\Delta] = \frac{4\pi N^{3/2}}{3}\mu[\Delta]\right] \quad \text{with} \quad \frac{1}{8\mu^2} = \frac{\text{Vol}(Y_7)}{\text{Vol}(S^7)} \, . \tag{2.1.5}$$

Here, Δ's are the R-charges of the bifundamental fields appearing in the $\widehat{ADE}$ quiver at a superconformal fixed point where F_{S^3} is extremized, i.e., $\frac{\partial F_{S3}}{\partial \Delta_{(a,b)}} = 0$.

Outline. In **Section** 2.2 we review the computation of free energy on S^3 in large N limit. In **Section** 2.3 we revisit the twisted index computation in large N limit and set up our notation consistent with the previous section. Along the way, we provide some new results including a simple proof of the index theorem. In **Section** 2.4 we specialize to the free energy computation: we review the result for $\widehat{A}_m$ quivers; provide an explicit example of $\widehat{D}_4$ quiver and conjecture the result for $\widehat{D}_n$ quivers. In **Section** 2.5 we move on to the twisted index computation: we provide explicit computations for $\widehat{A}_3$ and $\widehat{D}_4$ quivers, and present the general results for $\widehat{A}_m$[4] and $\widehat{D}_n$ quivers based on the previous section. We discuss a few possible future directions arising from this work in **Section** 2.6. We present a brief summary of the chapter in **Section** 2.7 and in the **Appendix** A.1 we collect some derivations and proofs to make this chapter self-contained.

2.2 S^3 and Free Energy

We consider $\mathcal{N} = 2$ quiver CS gauge theories involving vector multiplets (VM) with gauge group $G = \otimes_a U(N_a)$ and matter multiplets (MM) in representation $\otimes_i R_i$ of G. We will deal with (anti-)bifundamental and (anti-)fundamental representations only. VM consists of a gauge field A_μ, an auxiliary complex fermion λ_α ($\alpha = 1, 2$) and two auxiliary real scalars σ and D. MM consists of a complex scalar ϕ, a complex fermion ψ_α and an auxiliary complex scalar F.

The theories in consideration have been localized on S^3 in Ref. [21, 30, 31]. According to them, Z_{S^3} gets localized on configurations where σ_a in the $\mathcal{N} = 2$ VMs are constant $N_a \times N_a$ matrices and thus the original path integral reduces to a matrix model:

$$Z_{S^3} = \frac{1}{|\mathcal{W}|} \int \left(\prod_{a} \prod_{\text{Cartan}} d\sigma_a \right) e^{i\pi \sum_a k_a \operatorname{tr}(\sigma_a^2)} \prod_{a} \det_{Ad} \left(2 \sinh(\pi \alpha(\sigma_a)) \right) \prod_{\substack{\text{MM in} \\ \text{rep } R_i}} \det_{R_i} \left(e^{\ell(1-\Delta_i + i\rho_i(\sigma))} \right) , \tag{2.2.1}$$

$$\ell(z) = \frac{i}{2\pi} \operatorname{Li}_2 \left(e^{2\pi i z} \right) + \frac{i\pi}{2} z^2 - z \log \left(1 - e^{2\pi i z} \right) - \frac{i\pi}{12} ; \qquad \ell'(z) = -\pi z \cot(\pi z) , \tag{2.2.2}$$

where k_a are the CS levels of the VM corresponding to $U(N_a)$, Δ_i are the R-charges of the

[4] To our knowledge, the general result for twisted index of $\widehat{A}_m$ quivers presented here is new and only certain limits of that result are available in the literature.

corresponding MM in representation R_i, and $\alpha(\sigma)$, $\rho(\sigma)$ are the roots and weights of the appropriate matter representations. Denoting the eigenvalues of σ_a matrices by $\lambda_{a,i}$ with $i = 1, \cdots, N_a$ leads to a simple expression for free energy:

$$F_{S^3} = -\log|Z_{S^3}| \Rightarrow Z_{S^3} = \int \prod_{a,i} d\lambda_{a,i} e^{-F_{S^3}(\{\lambda_{a,i}\})}$$

$$\Rightarrow F_{S^3} \approx -i\pi \sum_{a,i} k_a \lambda_{a,i}^2 - 2\sum_a \sum_{i>j} \log\left|2\sinh\left(\pi\lambda_{a,i} - \pi\lambda_{a,j}\right)\right|$$
$$- \sum_{(a,b)\in E} \sum_{i,j} \ell\big(1 - \Delta_{(a,b)} + i(\lambda_{a,i} - \lambda_{b,j})\big) - \sum_a \sum_{\{f^a\}} \sum_i \ell\big(1 - \Delta_{f^a} + i\lambda_{a,i}\big). \tag{2.2.3}$$

Here, we have included only bifundamental and fundamental representations explicitly; the (anti-)reps can be similarly added and will be added below as required. We are concerned mostly with the above expression's large rank limit, keeping the CS levels fixed. For that purpose, we rewrite $N_a \to n_a N$ for some integers $n_a (\geq 1)$ and then take $N \to \infty$ by going to a continuum limit. We will mostly follow Refs. [34, 37, 41, 42, 60] in our saddle point analysis of F_{S^3} so most of this section has appeared before in the literature in one form or the other, apart from the explicit identification of $\mathcal{N} = 2\ \widehat{ADE}$ quivers.

The saddle point equation following from Eq. (2.2.3) for $\lambda_{a,i}$ is:

$$0 = \frac{\partial F_{S^3}}{\partial \lambda_{a,i}} \propto 2\sum_{j\neq i} \coth[\pi(\lambda_{a,i} - \lambda_{a,j})]$$
$$- \sum_{b|(a,b)\in E, j} (1 - \Delta_{(a,b)} + i(\lambda_{a,i} - \lambda_{b,j})) \coth[\pi(\lambda_{a,i} - \lambda_{b,j} + i\Delta_{(a,b)})]$$
$$- \sum_{b|(a,b)\in E, j} (1 - \Delta_{(b,a)} - i(\lambda_{a,i} - \lambda_{b,j})) \coth[\pi(\lambda_{a,i} - \lambda_{b,j} - i\Delta_{(b,a)})]. \tag{2.2.4}$$

The CS term and terms from fundamental matter are subleading compared to the vector and bifundamental contribution so we do not write them above. To take the continuum limit, we assume the eigenvalue distribution for $U(n_a N)$ to be

$$\lambda_{a,i} \to \lambda_{a,I}(x) = N^\alpha x + i y_{a,I}(x) \qquad \text{(with } I = 1, \cdots, n_a\text{)}, \tag{2.2.5}$$

and introduce an eigenvalue density $\rho(x) = \frac{1}{N}\sum_i \delta(x - x_i)$ such that $\int dx \rho(x) = 1$. This allows us to use the large argument approximation for $\coth[\pi(\lambda_{a,i} - \lambda_{b,j})] \approx \mathrm{sgn}(x - x')$ and convert $\sum_i \to N\int dx\rho(x)\sum_I$. We note that if we demand the same number of bifundamental and anti-bifundamental matters at each edge, then no contributions arise at $\mathcal{O}(N^{1+\alpha})$. The contribution at $\mathcal{O}(N)$ then gives a constraint on n_a's and R-charges as follows:

$$\begin{aligned} 0 = \frac{\partial F_{S^3}}{\partial \lambda_{a,i}} &\propto \Bigg(2n_a - \sum_{b|(a,b)\in E}(2 - \Delta_{(a,b)} - \Delta_{(b,a)})n_b\Bigg) N \int dx' \rho(x')\,\mathrm{sgn}(x - x') \\ &\Rightarrow 2n_a = \sum_{b|(a,b)\in E}(2 - \Delta_{(a,b)} - \Delta_{(b,a)})n_b\,. \end{aligned} \tag{2.2.6}$$

This constraint originating from the saddle point analysis guarantees the cancellation of long-range forces and the expression for free energy will turn out to be local. We will present the off-shell expression for free energy with generic R-charges but for explicit computation of free energy, we will consider a stricter constraint: $\Delta_{(a,b)} + \Delta_{(b,a)} = 1$. It is easy to see that this gives us an $\widehat{ADE}$ classification (see Figure 2.1) for these $\mathcal{N} = 2$ quivers just like in the $\mathcal{N} = 3$ case[5]. This condition can also be motivated from the analysis

[5]This is not the only simple solution of Eq. (2.2.6). For example, ABJM ($\widehat{A}_1$) and other odd $\widehat{A}$ quivers can still be constructed with the less strict condition: $\Delta_{(a-1,a)} + \Delta_{(a,a-1)} + \Delta_{(a,a+1)} + \Delta_{(a+1,a)} = 2$. It would be interesting to study generic non-$\widehat{ADE}$ theories with non-trivial constraints on R-charges compatible with Eq. (2.2.6).

of superpotential as discussed in Ref. [42].

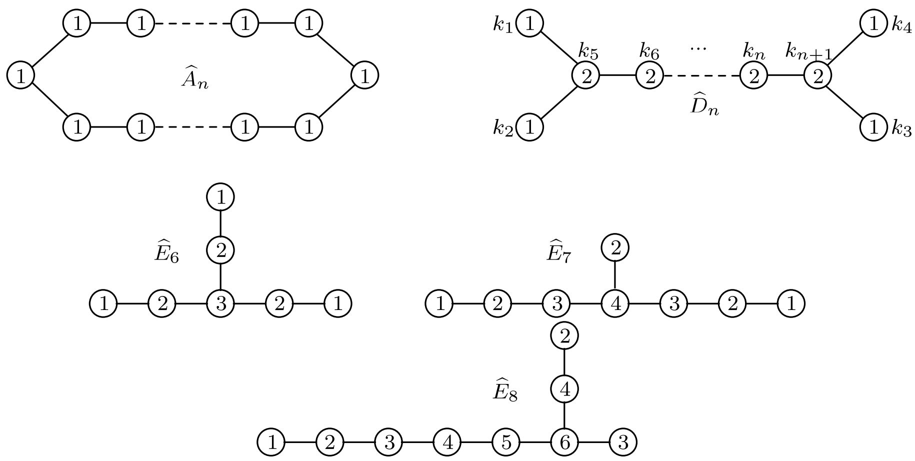

Figure 2.1: $\widehat{ADE}$ quivers with comarks n_a written inside the nodes. (For $\widehat{D}_n$ quivers, CS levels are also marked.)

Now moving to F_{S^3}, we have from Eq. (2.2.3)

$$\begin{aligned}
F_{S^3} \approx & -i\pi N \int dx\rho(x) \sum_{a,I} k_a \big(N^\alpha x + iy_{a,I}(x)\big)^2 \\
& - N^2 \int dxdx'\rho(x)\rho(x') \sum_{a,I,J} \log \Big| 2\sinh\big(\pi N^\alpha(x-x') + i\pi(y_{a,I}(x) - y_{a,J}(x'))\big)\Big| \\
& - N^2 \int dxdx'\rho(x)\rho(x') \sum_{(a,b)\in E} \sum_{I,J} \ell\big(1 - \Delta_{(a,b)} + iN^\alpha(x-x') - (y_{a,I}(x) - y_{b,J}(x'))\big) \\
& - N \int dx\rho(x) \sum_{a,\{f^a\},I} \ell\big(1 - \Delta_{f^a} + iN^\alpha x - y_{a,I}\big) - N \int dx\rho(x) \sum_{a,\{\bar{f}^a\},I} \ell\big(1 - \bar{\Delta}_{f^a} - iN^\alpha x + y_{a,I}\big).
\end{aligned}$$

We change variables from $N^\alpha(x - x') \to \xi$ where required and keep at most two highest orders of N in each term to get[6]

$$
\begin{aligned}
F_{S^3} \approx& -i\pi N^{1+2\alpha}\sum_a (n_a k_a)\int dx\rho(x)x^2 + 2\pi N^{1+\alpha}\int dx\rho(x)\sum_{a,I} k_a x y_{a,I}(x) \\
&+ \frac{1}{4\pi}N^{2-\alpha}\int dx\rho(x)^2 \sum_{a,I,J}\arg\left(e^{2\pi i(y_{a,I}-y_{a,J}-1/2)}\right)^2 \\
&- \frac{1}{4\pi}N^{2-\alpha}\int dx\rho(x)^2 \sum_{(a,b)\in E}\sum_{I,J}\Big[\big(1-\Delta_{(a,b)}-(y_{a,I}-y_{b,J})\big)\arg\left(e^{2\pi i(1/2-\Delta_{(a,b)}-(y_{a,I}-y_{b,J}))}\right)^2 \\
&+ \frac{1}{3\pi}\arg\left(e^{2\pi i(1/2-\Delta_{(a,b)}-(y_{a,I}-y_{b,J}))}\right)\left(\pi^2 - \arg\left(e^{2\pi i(1/2-\Delta_{(a,b)}-(y_{a,I}-y_{b,J}))}\right)^2\right) + (\Delta_{(b,a)}\text{ terms})\Big] \\
&+ \frac{i\pi}{2}N^{1+2\alpha}\sum_a (f^a n_a - \bar{f}^a n_a)\int dx\rho(x)x^2 \\
&+ \pi N^{1+\alpha}\sum_{a,I}\int dx\rho(x)|x|\Big[\sum_{\{f^a\}}\big(1-\Delta_{f^a}-y_{a,I}\big) + \sum_{\{\bar{f}^a\}}\big(1-\bar{\Delta}_{f^a}+y_{a,I}\big)\Big].
\end{aligned}
\tag{2.2.7}
$$

Here, $f^a(\bar{f}^a)$ are the total number of (anti-)fundamental fields at node a. We see 3 powers of N so let us assume $\sum_a n_a k_a = 0$ and $\sum_a \left(f^a - \bar{f}^a\right) n_a = 0$ so that we can match $N^{1+\alpha} = N^{2-\alpha}$ giving us the expected $\alpha = \frac{1}{2}$. We also point out that to get non-trivial solutions, a much stricter equality $f^a = \bar{f}^a$ needs to be imposed[7] leading us to the final

[6]We use $\arg\left(e^{2\pi i x}\right) = 2\pi x + 2\pi\left\lfloor \frac{1}{2} - x\right\rfloor$ later. We have omitted divergent (as well as constant) terms that cancel due to Eq. (2.2.6). Appendix A.1 has further details.

[7]While solving the matrix models explicitly, we will set $f^a = 0$ since non-zero f^a modify the resulting expressions in a well-known (and trivial) way (for example Refs. [60, 61]).

expression to be extremized:

$$
\begin{aligned}
F_{S^3} = N^{3/2} \int dx \rho(x) \Bigg[& 2\pi x \sum_{a,I} k_a y_{a,I}(x) + \frac{1}{4\pi}\rho(x) \Bigg(\sum_{a,I,J} \arg\left(e^{2\pi i (y_{a,I} - y_{a,J} - 1/2)}\right)^2 \\
& - \sum_{(a,b)\in E} \sum_{I,J} \Big[\left(1 - \Delta_{(a,b)} - (y_{a,I} - y_{b,J})\right) \arg\left(e^{2\pi i (1/2 - \Delta_{(a,b)} - (y_{a,I} - y_{b,J}))}\right)^2 \\
& + \frac{1}{3\pi} \arg\left(e^{2\pi i (1/2 - \Delta_{(a,b)} - (y_{a,I} - y_{b,J}))}\right) \left(\pi^2 - \arg\left(e^{2\pi i (1/2 - \Delta_{(a,b)} - (y_{a,I} - y_{b,J}))}\right)^2\right) + (\Delta_{(b,a)} \text{ terms}) \Big] \Bigg) \\
& + \pi |x| \left(2n_F - \Delta_F\right) \Bigg] - 2\pi \mu N^{3/2} \left(\int dx\, \rho(x) - 1 \right). \qquad (2.2.8)
\end{aligned}
$$

We defined $n_F = \sum_a f^a n_a = \sum_a \bar{f}^a n_a$, $\Delta_F = \sum_a \sum_{\{f^a\}} n_a \left(\Delta_{f^a} + \bar{\Delta}_{f^a}\right)$ and have added a Lagrange multiplier (μ) term to enforce the normalizability of the eigenvalue density. On general grounds described in Ref. [38], extremizing F_{S^3} gives

$$
\bar{F}_{S^3} = \frac{4\pi N^{3/2}}{3}\mu. \qquad (2.2.9)
$$

We will sometimes use a bar to denote an on-shell quantity as in Eq. (2.2.9) above, when compared to the off-shell quantity given by an integral expression as in Eq. (2.2.8).

This completes the review of the free energy F_{S^3}. Let us now turn to computation of the twisted index.

2.3 $\Sigma_{\mathfrak{g}} \times S^1$ and Twisted Index

The topologically twisted index is the $\Sigma_{\mathfrak{g}} \times S^1$ partition function with a topological twist along the Riemann surface of genus $\mathfrak{g}$, $\Sigma_{\mathfrak{g}}$. It was derived for $\Sigma_{\mathfrak{g}} = S^2$ in Ref. [47] and extended to generic $\mathfrak{g}$ in Ref. [48]. The main result reads (we choose unit radius for the circle S^1):

$$Z_{\Sigma_{\mathfrak{g}}\times S^1} =$$

$$\frac{1}{|\mathcal{W}|}\sum_{\mathfrak{m}_a}\oint\Big(\prod_a\prod_{\text{Cartan}}du_a\Big)\mathcal{B}^{\mathfrak{g}}\,e^{2\pi\sum_a k_a u_a\cdot\mathfrak{m}_a}\prod_a\Big(\prod_{\alpha\in G}\Big(1-e^{2\pi\alpha(u_a)}\Big)^{1-\mathfrak{g}}\prod_{\alpha>0}(-1)^{\alpha(\mathfrak{m}_a)}\Big)$$
$$\times\prod_I\prod_{\rho\in R_I}\left(\frac{e^{\pi\rho(u_I)+\pi i\nu_I}}{1-e^{2\pi\rho(u_I)+2\pi i\nu_I}}\right)^{\rho_I(\mathfrak{m})+(\mathfrak{g}-1)\left(\mathfrak{n}_I+(\Delta_I-1)\right)}, \tag{2.3.1}$$

where $u = i\left(\int_{S^1} A + i\sigma\right)$ are the holonomies and $\mathfrak{m} = \frac{1}{2\pi}\int_{\Sigma_{\mathfrak{g}}} F$ are the magnetic fluxes corresponding to the gauge group[8], $\nu = \left(\int_{S^1} A^{bg} + i\sigma^{bg}\right)$ are the holonomies (or chemical potentials) and $\mathfrak{n}$ are the fluxes for the background vector multiplet coupled to flavour symmetry such that $\mathfrak{n}(\mathfrak{g}-1)$ is integer-quantized.[9] The real part of ν is defined modulo 1 so we choose ν to satisfy $0 < \nu < 1$. The Hessian $\mathcal{B}$ is a contribution due to fermionic zero-modes and (up to some constant factors) is given by $\mathcal{B} \approx \det_{ai,bj}\frac{\partial^2 Z_{\text{cl+1-loop}}}{\partial u_a^i \partial m_b^j}$, where $Z_{\text{cl+1-loop}}$ is the full integrand appearing in Eq. (2.3.1) except for the $\mathcal{B}^{\mathfrak{g}}$ factor.

As with Z_{S^3}, we are interested in the large N limit of the above expression. It was studied for ABJM $(\widehat{A}_1)$ theory in Ref. [50] and for $\widehat{A}$-type quiver theories in Ref. [52]. As discussed there, due to the sum over magnetic fluxes, evaluating this limit becomes a two-step process: (1) Sum over magnetic fluxes, $\mathfrak{m}_a$; (2) Integrate over the holonomies, u_a. The first step involves summing a geometric series, which generates factors like $\frac{1}{1-e^{i\mathcal{B}_a^i(u_a)}}$ leading to poles at $\hat{u}_a$ such that $e^{i\mathcal{B}_a^i(\hat{u}_a)} = 1$. We solve for $\hat{u}_a$ by constructing an auxiliary object called the 'Bethe potential' $\mathcal{V}$ defined as $\frac{\partial\mathcal{V}}{\partial u_a^i} = \mathcal{B}_a^i$ such that extremizing $\mathcal{V}$ gives the Bethe ansatz equations (BAEs): $\frac{\partial\mathcal{V}}{\partial u_a^i}\Big|_{u=\hat{u}} = \mathcal{B}_a^i(\hat{u}) = 0.$[10] $\mathcal{V}$ once again turns out to be

[8] We have kept the $(-1)^{\alpha(\mathfrak{m})}$ contribution of the vector multiplet explicitly as it contributes to the Bethe potential and is required for a consistent result, the way we take the large N limit (see Appendix A.1).

[9] The different definitions for the same quantities corresponding to gauge and flavour groups are chosen for later convenience when comparing the large N results for twisted index to those for free energy on S^3.

[10] $\mathcal{B}_a^i(\hat{u}) = 0$ is stricter than $e^{i\mathcal{B}_a^i(\hat{u}_a)} = 1$ but we will see that the solution obtained is consistent with

related to F_{S^3} so we can easily solve it in the large N limit. The second step then involves substituting this solution back in $Z_{\Sigma_{\mathfrak{g}}\times S^1}$ and using the residue theorem to get the final result:

$$Z_{\Sigma_{\mathfrak{g}}\times S^1} =$$
$$\sum_{\hat{u}\in\text{BAE}} \left(\mathcal{B}(\hat{u})\right)^{\mathfrak{g}-1} \prod_a \prod_{\alpha\in G} \left(1 - e^{2\pi\alpha(\hat{u}_a)}\right)^{1-\mathfrak{g}} \prod_I \prod_{\rho\in R_I} \left(\frac{e^{\pi\rho(\hat{u}_I)+\pi i\nu_I}}{1 - e^{2\pi\rho(\hat{u}_I)+2\pi i\nu_I}}\right)^{(\mathfrak{g}-1)(\hat{\mathfrak{n}}_I-1)}, \tag{2.3.2}$$

where the Hessian can now be rewritten as $\mathcal{B} = \det_{ai,bj} \frac{\partial^2\mathcal{V}}{\partial u_a^i \partial u_b^j}$ and we have shifted the flavour flux with the R-charge $\hat{\mathfrak{n}} = (\mathfrak{n} + \Delta)$ but we will suppress the $\hat{}$ over $\mathfrak{n}$ in what follows. Again, we will evaluate this final expression only in the large N limit.

2.3.1 Summing Fluxes

We consider $\mathcal{N} = 2$ quiver theories with gauge group $\otimes_a U(N_a)$ now so most expressions below have a non-trivial summation $\sum_{b|(a,b)\in E}$ accompanying the vector and bifundamental matter contributions when compared to similar expressions in the literature.

We begin with the BAEs which are obtained as coefficients of $\mathfrak{m}_a^i$ from the exponentiated form of the integrand in Eq. (2.3.1):

$$0 = i\mathcal{B}_a^i =$$
$$2\pi k_a u_a^i + \sum_j \text{sgn}(j-i)i\pi + \sum_{b|(a,b)\in E} \sum_j \left(v'(u_a^i - u_b^j + i\nu_{(a,b)}) - v'(u_b^j - u_a^i + i\nu_{(b,a)})\right)$$
$$+ \sum_{f^a} v'(u_a^i + i\nu_{f^a}) - \sum_{\bar{f}^a} v'(-u_a^i + i\bar{\nu}_{f^a}). \tag{2.3.3}$$

known results and has expected behaviour in simplifying limits of ν's.

These can be derived from the following Bethe potential $\mathcal{V}$ via $\mathcal{B}_a^i = \frac{\partial \mathcal{V}}{\partial u_a^i}$:

$$\mathcal{V} = -i\sum_{a,i} \pi k_a (u_a^i)^2 + \frac{1}{2}\sum_{a,i,j} \pi \,\mathrm{sgn}(j-i)(u_a^i - u_a^j) - i \sum_{(a,b)\in E} \sum_{i,j} \Big(v(u_a^i - u_b^j + i\nu_{(a,b)}) + v(u_b^j - u_a^i + i\nu_{(b,a)}) \Big) - i\sum_{a,i}\sum_{f^a} v(u_a^i + i\nu_{f^a}) - i\sum_{a,i}\sum_{\bar{f}^a} v(-u_a^i + i\bar{\nu}_{f^a}), \quad (2.3.4)$$

where we defined, in analogy to $\ell(z)$,

$$v(z) = \tfrac{1}{2\pi}\,\mathrm{Li}_2\left(e^{2\pi z}\right) + \tfrac{\pi}{2}z^2 - \tfrac{\pi}{12} \quad \Rightarrow \quad v'(z) = \mathrm{Li}_1(e^{2\pi z}) + \pi z\,. \quad (2.3.5)$$

We chose $v(z)$ such that $v(0)=0$, however, $v'(z)$ is divergent at $z=0$.

To take the continuum limit, we again denote the eigenvalues of the u_a matrices by $\lambda_{a,i}$ and assume the eigenvalue distribution for a node with $U(n_a N)$ group to be the same as before:

$$\lambda_{a,i} \to \lambda_{a,I}(x) = N^\alpha x + i y_{a,I}(x) \quad \text{(with } I = 1,\cdots,n_a\text{)}, \quad (2.3.6)$$

with an associated eigenvalue density $\rho(x)$ normalized as $\int dx\rho(x) = 1$. We convert $\sum_i \to N\int dx\rho(x)\sum_I$ and note that we again need the same number of bifundamental and anti-bifundamental matters at each edge to cancel higher order terms. To cancel potential divergent terms (as before), we are led to a constraint relating the comarks n_a's and chemical potentials ν's as follows (see Appendix A.1 for details):

$$\sum_a \frac{n_a^2}{2} = \sum_{(a,b)\in E} (1 - \nu_{(a,b)} - \nu_{(b,a)}) n_a n_b\,. \quad (2.3.7)$$

This leads to a larger class of theories than those considered in the literature whose twisted index turns out to scale as $N^{3/2}$ in the large N limit. We note that for $\nu_{(a,b)} + \nu_{(b,a)} = \frac{1}{2}$, we get an $\widehat{ADE}$ classification just like the F_{S^3} as the above equation becomes equivalent to $2n_a = \sum_{b|(a,b)\in E} n_b$.[11] This condition can also be derived from the analysis of possible superpotential terms as discussed in Refs. [50, 52]. Thus, we are led to the same constraint on α as before $(1+\alpha = 2-\alpha)$ implying $\alpha = \frac{1}{2}$ and the Bethe potential in large N limit reads

$$\mathcal{V} \approx N^{3/2} \int dx \rho(x) \Bigg[2\pi x \sum_{a,I} k_a y_{a,I}(x) - \frac{1}{24\pi^2}\rho(x) \sum_{(a,b)\in E} \sum_{I,J} \Big[\arg\left(e^{2\pi i (y_{a,I} - y_{b,J} + \nu_{(a,b)} - 1/2)}\right) \times \Big(\pi^2 - \arg\left(e^{2\pi i (y_{a,I} - y_{b,J} + \nu_{(a,b)} - 1/2)}\right)^2\Big) + (\nu_{(b,a)} \text{ term})\Big] + \pi |x| (n_F - \nu_F) \Bigg] - 2\pi \tilde{\mu} N^{3/2} \Big(\int dx\, \rho(x) - 1 \Big). \tag{2.3.8}$$

Here, $\nu_F = \sum_a \sum_{\{f^a\}} n_a \left(\nu_{f^a} + \bar{\nu}_{f^a}\right)$ and we have again set $\sum_a n_a k_a = 0$, $f^a = \bar{f}^a$. We have also added a Lagrange multiplier ($\tilde{\mu}$) term to enforce the normalizability of the eigenvalue density. We can also simplify the exponent by using the constraint $\nu_{(a,b)} + \nu_{(b,a)} = \frac{1}{2}$, which will be employed below to derive the twisted index in terms of the Bethe potential. We notice the similarities and differences of the above expression with the expression for F_{S^3} in Eq. (2.2.8), especially the scaling $N^{3/2}$ and missing vector contributions. This naïvely seems to suggest that $\mathcal{V} \approx F_{S^3}$ may not hold for the larger class of theories being considered here. We will see later that it is not so. On general grounds given in Ref. [38], extremizing $\mathcal{V}$ gives (just like the free energy)

$$\bar{\mathcal{V}} = \frac{4\pi N^{3/2}}{3} \tilde{\mu}\,. \tag{2.3.9}$$

It turns out that the large N limit of $\mathcal{V}$ is not enough to compute the twisted index be-

[11] As discussed in footnote 5, for ABJM and other odd $\widehat{A}$ quivers, the condition can be made less strict: $\nu_{(a-1,a)} + \nu_{(a,a-1)} + \nu_{(a,a+1)} + \nu_{(a+1,a)} = 1$. However, we will not discuss non-$\widehat{ADE}$ constraints in detail.

cause $\mathcal{V}$ has no divergences at leading order whereas the original BAEs display divergent behaviour. This behaviour follows due to bifundamental contributions involving $v'(z)$ being divergent at $z = 0$, discussed in Ref. [50]. Let us separate out the divergent part of Eq. (2.3.3) but continue to denote rest of the finite terms as $\mathcal{B}_a^I$ and schematically introduce exponentially small corrections as follows:

$$\begin{aligned} 0 = \mathcal{B}_a^I + \sum_{b|(a,b)\in E} \sum_J \Big[v'\Big(i \left(y_{a,I}(x) - y_{b,J}(x) + \nu_{(a,b)} \right) + e^{-N^{1/2} Y^+_{(a,I;b,J)}(x)} \Big) \\ - v'\Big(i \left(y_{b,J}(x) - y_{a,I}(x) + \nu_{(b,a)} \right) + e^{-N^{1/2} Y^-_{(a,I;b,J)}(x)} \Big) \Big] \\ \Rightarrow \mathcal{B}_a^I \approx -N^{1/2} \sum_{b|(a,b)\in E} \sum_J \Big[\delta_{(\delta y_{ab,IJ}(x) + \nu_{(a,b)},0)} Y^+_{(a,I;b,J)}(x) - \delta_{(\delta y_{ab,IJ}(x) - \nu_{(b,a)},0)} Y^-_{(a,I;b,J)}(x) \Big] \,, \end{aligned} \tag{2.3.10}$$

where $\delta_{(f(x),0)}$ is the Kronecker delta symbol that equals 1 when $f(x) = 0$ and 0 otherwise. We used the following large N limit:

$$\begin{aligned} \mathrm{Li}_1\big(\exp\big(2\pi e^{-N^{1/2} Y(x)}\big)\big) = -\log\big(1 - \exp\big(2\pi e^{-N^{1/2} Y(x)}\big)\big) \approx -\log(-2\pi e^{-N^{1/2} Y(x)}) \\ \approx +N^{1/2} Y(x). \end{aligned} \tag{2.3.11}$$

We note that $Y^{\pm}(x) \geq 0$ for all x so that the exponential term is subleading and is a consistency check for explicit computations. We stress that the above equation is used to extract the $Y^{\pm}(x)$ functions (while keeping track of the sign) from (naïve) equations of motion $\mathcal{B}_a^I$ evaluated at the saturation values of the $y(x)$'s as denoted by the $\delta_{(\delta y(x) \pm \nu, 0)}$.

2.3.2 Integrating Holonomies

Moving back to $Z_{\Sigma_{\mathfrak{g}}\times S^1}$, we now have to derive the large N limit of Eq. (2.3.2). This limit can be taken in a similar way to the Bethe potential (see Appendix A.1 for some details) but with fixed $\alpha = \frac{1}{2}$ such that the overall scaling of the index turns out to be $N^{3/2}$ as expected. To cancel the divergent terms in order to get local integrands as in the case of Bethe potential, we are led to a constraint on the flavour fluxes:

$$\sum_a n_a^2 = \sum_{(a,b)\in E} (2 - \mathfrak{n}_{(a,b)} - \mathfrak{n}_{(b,a)}) n_a n_b \,. \tag{2.3.12}$$

This general constraint goes together with Eq. (2.3.7) to define a larger class of theories with $N^{3/2}$ scaling of their twisted index. For $\mathfrak{n}_{(a,b)} + \mathfrak{n}_{(b,a)} = 1$, we recover the $\widehat{ADE}$ classification which we will impose for evaluating examples explicitly[12]. Finally, the large N limit of the twisted index reads (see appendix A.1 for some details):

$$\begin{aligned}\mathcal{I} = \log |Z_{\Sigma_{\mathfrak{g}}\times S^1}| \approx (\mathfrak{g}-1)N^{3/2}\int dx\rho(x)\Bigg[\frac{1}{4\pi}\rho(x)\Bigg(\sum_{a,I,J}\arg\left(e^{2\pi i(y_{a,I}(x)-y_{a,J}(x)-1/2)}\right)^2 \\ - \sum_{(a,b)\in E}\sum_{I,J}\left[\mathfrak{n}_{(b,a)}\arg\left(e^{2\pi i(y_{a,I}(x)-y_{b,J}(x)-\nu_{(b,a)})}\right)^2\right] - (\mathfrak{n}_{(a,b)}\text{ term})\Bigg) \\ + \sum_{(a,b)\in E}\sum_{I,J}\delta_{(\delta y_{ab,IJ}(x)\pm\nu_{(\cdot,\cdot)},0)}\mathfrak{n}_{(\cdot,\cdot)}Y^{\pm}_{(a,I;b,J)}(x) + \pi|x|(2n_F - \mathfrak{n}_F)\Bigg],\end{aligned} \tag{2.3.13}$$

where $\mathfrak{n}_F$ is defined similar to ν_F and the conditions on $\mathfrak{n}_{(a,b)}$'s and $\nu_{(a,b)}$'s have been used. The above expression is to be evaluated by substituting $\{\rho(x), y_{a,I}(x), Y^{\pm}_{(a,I;b,J)}(x)\}$ obtained from extremizing the Bethe potential.

[12] For ABJM and other odd $\widehat{A}$ quivers, the condition is less strict: $\mathfrak{n}_{(a-1,a)} + \mathfrak{n}_{(a,a-1)} + \mathfrak{n}_{(a,a+1)} + \mathfrak{n}_{(a+1,a)} = 2$, as expected by now.

2.3.3 Index Theorem

For the statement and a slick proof of the index theorem, Ref. [52] is a good reference. Here we present an enlightening proof by trying to relate $\mathcal{I}$ to the Bethe potential $\mathcal{V}$. First, $\mathcal{V}$ in Eq. (2.3.8) is augmented by terms similar to the bifundamental contributions that look like adjoint contributions parameterized with ν_a such that for $\nu_a = 0$, these adjoint terms vanish (basically we add 0 to $\mathcal{V}$). Then, we can write off-shell:

$$\mathcal{I} = (\mathfrak{g} - 1) \sum_I \mathfrak{n}_I \frac{\partial \mathcal{V}}{\partial \nu_I}\bigg|_{\mathfrak{n}_a=-1,\nu_a=0}, \tag{2.3.14}$$

where I runs over all multiplets and it is understood that for vectors we set $\mathfrak{n}_a = -1$ and $\nu_a = 0$ at the end of the differentiation. This is true simply because $\mathcal{V}$ depends on $v(z)$ functions and $\mathcal{I}$ on $v'(z)$ multiplied with $(\mathfrak{g} - 1)\mathfrak{n}$, though Eq. (2.3.12) has to be used to cancel some $\frac{\pi}{12}$'s. The Kronecker δ contributions are also included in this form, which can be shown by using the equations of motion Eq. (2.3.10) and chain rule for differentiation, for example,

$$\mathfrak{n}_{(a,b)} \frac{\partial \mathcal{V}}{\partial y_{a,I}} \frac{\partial y_{a,I}}{\partial \nu_{(a,b)}} = \mathfrak{n}_{(a,b)} \left(\delta_{(\delta y_{ab,IJ}(x)+\nu_{(a,b)},0)} Y^+_{(a,I;b,J)}(x) \right) (+1)\,, \tag{2.3.15}$$

which is what appears in Eq. (2.3.13). The $\mathfrak{n}_{(b,a)} Y^-(x)$ term with proper sign also similarly follows. Thus, we have proven that the twisted index can be obtained from the Bethe potential and this relation is valid for a larger class of theories than considered in Ref. [52].

The above formula focusses on the integrands and under certain conditions (for example, whenever definite integration and differentiation commutes), it is valid even after the integration is done (i.e., on-shell):

$$\bar{\mathcal{I}} = (\mathfrak{g}-1)\sum_I \mathfrak{n}_I \frac{\partial \bar{\mathcal{V}}}{\partial \nu_I}\,. \tag{2.3.16}$$

It is understood that the index I now runs only over the matter multiplets since vector ν_a's are set to zero already at the level of the integrand. For (anti-)fundamental matter contributions, we can take $\nu_I = n_F - \nu_F$ and $\mathfrak{n}_I = 2n_F - \mathfrak{n}_F$ and the above relation continues to hold. We note that we are allowed to choose a suitable basis for the $\mathfrak{n}$'s and ν's by including even redundant combinations. Thus, to keep the expression for $\bar{\mathcal{V}}$ tractable, constraints on $\nu_{(a,b)}$ and $\mathfrak{n}_{(a,b)}$ may be imposed and that makes the sum over I for all bifundamentals ill-defined leading to violation of Eq. (2.3.16). To understand this better, let us compare what happens to the sum $\sum_{(a,b)\oplus(b,a)}$ if the two constraints $\nu_{(a,b)} + \nu_{(b,a)} = \frac{1}{2}$ and $\mathfrak{n}_{(a,b)} + \mathfrak{n}_{(b,a)} = 1$ are imposed after and before the differentiation:

$$\begin{aligned}\text{After:}\quad \sum_I \mathfrak{n}_I \frac{\partial \bar{\mathcal{V}}}{\partial \nu_I} &= \mathfrak{n}_{(a,b)} \frac{\partial \bar{\mathcal{V}}(\nu_{(a,b)},\cdots)}{\partial \nu_{(a,b)}} + \mathfrak{n}_{(b,a)} \frac{\partial \bar{\mathcal{V}}(\nu_{(b,a)},\cdots)}{\partial \nu_{(b,a)}} + \cdots \\ &= \mathfrak{n}_{(a,b)} \bar{\mathcal{V}}'(\nu_{(a,b)},\cdots) + (1-\mathfrak{n}_{(a,b)})\bar{\mathcal{V}}'(\tfrac{1}{2}-\nu_{(a,b)},\cdots) + \cdots \end{aligned} \tag{2.3.17}$$

$$\begin{aligned}\text{Before:}\quad {\sum_I}' \mathfrak{n}_I \frac{\partial \bar{\mathcal{V}}}{\partial \nu_I} &= \mathfrak{n}_{(a,b)} \left(\bar{\mathcal{V}}'(\nu_{(a,b)},\cdots) - \bar{\mathcal{V}}'(\tfrac{1}{2}-\nu_{(a,b)},\cdots)\right) + \cdots \\ &= \sum_I \mathfrak{n}_I \frac{\partial \bar{\mathcal{V}}}{\partial \nu_I} - \bar{\mathcal{V}}'(\tfrac{1}{2}-\nu_{(a,b)},\cdots), \end{aligned} \tag{2.3.18}$$

where $\sum'$ denotes sum over independent set of ν's, which seems to be missing a term when compared to the full $\sum$. Let us look at the following expression now:

$$\sum_I{}' \nu_I \frac{\partial \bar{\mathcal{V}}}{\partial \nu_I} = \nu_{(a,b)} \left(\bar{\mathcal{V}}'(\nu_{(a,b)}, \cdots) - \bar{\mathcal{V}}'(\tfrac{1}{2} - \nu_{(a,b)}, \cdots)\right) + \cdots$$

$$= \nu_{(a,b)}\bar{\mathcal{V}}'(\nu_{(a,b)}, \cdots) + (\tfrac{1}{2} - \nu_{(a,b)})\bar{\mathcal{V}}'(\tfrac{1}{2} - \nu_{(a,b)}, \cdots) - \tfrac{1}{2}\bar{\mathcal{V}}'(\tfrac{1}{2} - \nu_{(a,b)}, \cdots) + \cdots$$

$$= \sum_I \nu_I \frac{\partial \bar{\mathcal{V}}}{\partial \nu_I} - \frac{1}{2}\bar{\mathcal{V}}'(\tfrac{1}{2} - \nu_{(a,b)}, \cdots), \tag{2.3.19}$$

where the last term is half of the extra term found in Eq. (2.3.18). Now, the index theorem follows:

$$\bar{\mathcal{I}} = (\mathfrak{g} - 1)\sum_I \mathfrak{n}_I \frac{\partial \bar{\mathcal{V}}}{\partial \nu_I} = (\mathfrak{g} - 1)\left[\sum_I{}' \mathfrak{n}_I \frac{\partial \bar{\mathcal{V}}}{\partial \nu_I} + 2\left(\sum_I \nu_I \frac{\partial \bar{\mathcal{V}}}{\partial \nu_I} - \sum_I{}' \nu_I \frac{\partial \bar{\mathcal{V}}}{\partial \nu_I}\right)\right]$$

$$= (\mathfrak{g} - 1)\left[2\sum_I \nu_I \frac{\partial \bar{\mathcal{V}}}{\partial \nu_I} + \sum_I{}' (\mathfrak{n}_I - 2\nu_I)\frac{\partial \bar{\mathcal{V}}}{\partial \nu_I}\right]$$

$$\Rightarrow \quad \bar{\mathcal{I}} = (\mathfrak{g} - 1)\left[4\bar{\mathcal{V}} + \sum_I{}' (\mathfrak{n}_I - 2\nu_I)\frac{\partial \bar{\mathcal{V}}}{\partial \nu_I}\right]. \tag{2.3.20}$$

We used the 'homogeneous' property of $\bar{\mathcal{V}}$ such that $\sum_I \nu_I \frac{\partial \bar{\mathcal{V}}}{\partial \nu_I} = 2\bar{\mathcal{V}}$ (proven in appendix A.1) to write the first term. We will also see later that $4\bar{\mathcal{V}}[\nu] = \bar{F}_{S^3}[2\nu]$ for the $\widehat{ADE}$ quivers. The 2ν's here become the R-charges Δ's in F_{S^3} for this comparison, as can be expected from the constraints imposed on them to get $\widehat{ADE}$ classification. In general, $\mathcal{I}$ needs to be extremized with respect to ν's and critical values for ν's are obtained in terms of the flavour fluxes $\mathfrak{n}$'s. The resulting expression $\bar{\mathcal{I}}(\nu(\mathfrak{n}), \mathfrak{n})$ is supposed to match the corresponding black hole entropy S_{BH} as discussed in Section 2.1. However, for the case of universal twist, $\mathfrak{n}_I = 2\nu_I$, dealt with in Refs. [58, 59] leading to the expected simple relation for $\widehat{ADE}$ quiver theories and their duals:

$$\text{Universal twist:} \qquad S_{BH} = \bar{\mathcal{I}} = (\mathfrak{g}-1)4\bar{\mathcal{V}} = (\mathfrak{g}-1)\bar{F}_{S^3} \qquad \text{given that } \frac{\partial\bar{\mathcal{V}}}{\partial\nu_I} \equiv \frac{\partial\bar{F}_{S3}}{\partial\nu_I} = 0\,. \tag{2.3.21}$$

This completes the setup for the twisted index $\mathcal{I}$. Let us now turn to explicit computation of the free energy of $\widehat{AD}$ quivers.

2.4 Free Energy and Volume

In this section, we consider the $\widehat{AD}$ quivers and evaluate their free energy, or equivalently the $\mathrm{Vol}(Y_7)$. We will follow the algorithm developed in Ref. [60] but suitably modified for the case of general R-charges. We briefly review it here to introduce the terminology we use when writing down the explicit solutions.

Algorithm. We take the principle value for the $\arg()$ functions leading to the inequalities:

$$\begin{gathered} 0 < y_{a,I} - y_{a,J} < 1\,; \qquad 0 < y_{a,I} - y_{b,J} + \Delta_{(a,b)} < 1\,, \quad -1 < y_{a,I} - y_{b,J} - \Delta_{(b,a)} < 0\,. \\ \Rightarrow |y_{a,I} - y_{a,J}| < 1\,; \qquad -\Delta_{(a,b)} < y_{a,I} - y_{b,J} < \Delta_{(b,a)}\,. \end{gathered} \tag{2.4.1}$$

As discussed in previous section, we will insist $\Delta_{(a,b)} + \Delta_{(b,a)} = 1$. Since we have pairing up of bifundamentals, while the inequalities are not violated, the contribution from these fields to Eq. (2.2.8) simplifies:

$$-\sum_{(a,b)\oplus(b,a)}\pi\left(2-\Delta^{+}_{(a,b)}\right)\int dx\rho(x)^2\sum_{I,J}\left[\left(y_{a,I}-y_{b,J}+\frac{\Delta^{-}_{(a,b)}}{2}\right)^2+\frac{1}{12}\left(3-\Delta^{+}_{(a,b)}\right)\left(1-\Delta^{+}_{(a,b)}\right)\right]$$
$$=-\sum_{(a,b)\oplus(b,a)}\pi\int dx\rho(x)^2\sum_{I,J}\left(y_{a,I}-y_{b,J}+\frac{\Delta^{-}_{(a,b)}}{2}\right)^2, \quad (2.4.2)$$

where $\Delta^{\pm}_{(a,b)}=\Delta_{(a,b)}\pm\Delta_{(b,a)}$. We will also insist that all $y_{a,I}(x)-y_{a,J}(x)=0$ initially, which simplifies the vector contribution to just $\sum_a\int dx\rho(x)^2\sum_{I,J}\frac{\pi}{4}$.

Extremizing F_{S^3} now with respect to $y(x)$'s and $\rho(x)$, we find a solution which is consistent only in a bounded region around the origin ($x=0$). This is because as $|x|$ increases, the differences $y_{a,I}(x)-y_{b,J}(x)\equiv\delta y_{ab,IJ}(x)$ monotonically increase (or decrease), saturating at least one of the inequalities given above at some point on either side of $x=0$, which we label as $x_1^{\pm}$. This saturation is maintained beyond these points, requiring the corresponding $y_{a,I}(x)$'s to either bifurcate (for $n_a>1$) or develop a kink. Once an inequality is saturated, we have to remove one of the $y(x)$'s from the integral expression Eq. (2.2.8) by using the saturation value and solve the revised equations of motion separately on both positive and negative side of the x-axis until new saturation points are encountered on both sides. This leads to pair of regions on either side of the central region (or region 1), which we will label as "region $2^{\pm}$" bounded by $x_2^{\pm}$ for obvious reason. This procedure needs to be iterated until either all $y(x)$'s get related or $\rho(x)=0$, determining a maximum of $\sum_a n_a$ regions for $\widehat{A}$ quivers and $\sum_a n_a-1$ regions for $\widehat{DE}$ quivers[13]. Once the eigenvalue density $\rho(x)$ is determined in all the regions, the value of μ is found from the normalization

[13] We count disjointed $n^{\pm}$ regions as one single region so $\widehat{A}_1$ has **two** regions, even though there are four saturation points bounding *three* apparent regions $\{\cdot 2^-\cdot 1\cdot 2^+\cdot\}$.

condition of $\rho(x)$, which gives the quantities we want via the following relations:

$$\bar{F}_{S^3} = \frac{4\pi N^{3/2}}{3}\mu\,; \qquad \frac{\text{Vol}(Y_7)}{\text{Vol}(S^7)} = \frac{1}{8\mu^2}\,. \tag{2.4.3}$$

We combined the former equation with Eq. (2.1.1) to get the latter.

2.4.1 $\widehat{A}_m$ Revisited

We review the $\widehat{A}$ quivers dealt succinctly in Ref. [42]. The above discussion applies to this case just by setting the values of $I, J = 1$. The contribution from bifundamentals in Eq. (2.4.2) can be rewritten as $\tilde{F}$ given by eq (4.2) of Ref. [42]. The solution for free energy is given in terms of the area of the following polygon:

$$\mathcal{P} = \left\{(s,t) \in \mathbb{R}^2 \middle| \textstyle\sum_{a=1}^{m+1} |t + q_a s| + c_1 t + c_2 s \leq 1\right\}; \quad c_1 \equiv \sum_{(a,b)\in E} \Delta^-_{(a,b)}, \quad c_2 \equiv \sum_{(a,b)\in E} q_a \Delta^-_{(a,b)}. \tag{2.4.4}$$

The redefined CS levels q_a are constrained parameters obeying $\sum_{a=1}^{m+1} q_a = 0$ and are related to k_a's as follows:

$$q_a = k_a - k_{a+1}\,,\ a = 1, \cdots, m\,; \quad q_{m+1} = k_{m+1} - k_1\,. \tag{2.4.5}$$

The $\text{Area}(\mathcal{P})$ is related to $\int dx \rho(x)$ such that we get[14]

$$\frac{\text{Vol}(Y_7)}{\text{Vol}(S^7)} = \frac{1}{8\mu^2} = \frac{1}{2}\,\text{Area}(\mathcal{P}) = \frac{1}{4}\sum_{a=1}^{m+1}\left[\frac{|\gamma_{a,a+1}|}{\sigma_a\sigma_{a+1}} + \frac{|\gamma_{a,a+1}|}{\sigma_{a+m+1}\sigma_{a+m+2}}\right]. \tag{2.4.6}$$

[14]It is a fun exercise to show that the definition of $\mathcal{P}$ as given in Eq. (2.4.4) can be 'integrated' to get precisely the area of $\mathcal{P}$ as given in Eq. (2.4.6) in Ref. [38].

This reduces to the correct $\mathcal{N} = 3$ expression when all $\Delta_{(a,b)} = \frac{1}{2}$ as can be directly checked from the definition of σ's:

$$\beta_a = \begin{pmatrix} 1 \\ q_a \end{pmatrix} \text{ for } a = 1, \cdots, m+1\,, \ \beta_{m+2} = -\beta_1\ ; \qquad \gamma_{a,b} = \beta_a \wedge \beta_b\,;$$
$$\sigma_a = \sum_{b=1}^{m+1} \left[|\gamma_{a,b}| + \gamma_{b,a} \Delta^{-}_{(b,b+1)} \right]\,, \quad \sigma_{a+m+1} = \sum_{b=1}^{m+1} \left[|\gamma_{a,b}| - \gamma_{b,a} \Delta^{-}_{(b,b+1)} \right]. \tag{2.4.7}$$

A trivial example to check the above formulas is $\widehat{A}_1$ quiver (consider the ordering $q_1 \geq 0 \geq q_2$ and $q_1 = \frac{k}{2}$):

$$\frac{\mathrm{Vol}(Y_7)}{\mathrm{Vol}(S^7)} = \frac{1}{4} \begin{pmatrix} \frac{2q_1}{(4q_1\Delta^A_{(2,1)})(4q_1\Delta^B_{(1,2)})} + \frac{2q_1}{(4q_1\Delta^B_{(1,2)})(4q_1\Delta^B_{(2,1)})} \\ + \frac{2q_1}{(4q_1\Delta^B_{(2,1)})(4q_1\Delta^A_{(1,2)})} + \frac{2q_1}{(4q_1\Delta^A_{(1,2)})(4q_1\Delta^A_{(2,1)})} \end{pmatrix} = \frac{1}{32\Delta^A_{(1,2)}\Delta^B_{(1,2)}\Delta^A_{(2,1)}\Delta^B_{(2,1)}\,q_1}\,. \tag{2.4.8}$$

This expression appears in literature a lot and it can be straightforwardly checked that it reproduces the correct $\frac{1}{k}$ for ABJM theory when all Δ's equal $\frac{1}{2}$. A slightly non-trivial example is $\widehat{A}_3$ but we will discuss it for twisted index in the next section.

Let us move on to the $\widehat{D}$ quivers now (specifically $\widehat{D}_4$ which is related to $\widehat{A}_3$ via unfolding procedure in the $\mathcal{N} = 3$ case of Refs. [60, 62]).

2.4.2 $\widehat{D}_4$ Solved

We give the detailed solution for the $\widehat{D}_4$ quiver here and to make the expressions easier to read, we do a bit of housekeeping first. Let us redefine the five constrained CS levels k's to four unconstrained variables p's as follows:

$$k_1 = -(p_1+p_2), \quad k_2 = p_1-p_2, \quad k_3 = p_3-p_4, \quad k_4 = p_3+p_4, \quad k_5 = p_2-p_3\,. \tag{2.4.9}$$

We will also suppress the second index on the four $y_{a,1}$ with $a = 1, \cdots, 4$. Furthermore, we introduce a 'vector' of R-charges:

$$\alpha_b(\Delta^-) = \Big\{\tfrac{1}{2}(\Delta^-_{(1,5)}-\Delta^-_{(2,5)}), \tfrac{1}{2}(\Delta^-_{(1,5)}+\Delta^-_{(2,5)}), -\tfrac{1}{2}(\Delta^-_{(4,5)}+\Delta^-_{(3,5)}), -\tfrac{1}{2}(\Delta^-_{(4,5)}-\Delta^-_{(3,5)})\Big\}, \tag{2.4.10}$$

which will appear in a combination $\sum_{b=1}^{4} p_b\alpha_b(\Delta^-) \equiv p\cdot\alpha(\Delta^-)$ below. For generic p's, there are going to be 5 regions consisting of one central region spanning both negative and positive side of the x-axis and 4 pairs of disjointed regions beyond the central one as explained in the algorithm above. Let us now enumerate the solution in each region for a particular ordering $p_1 \geq p_2 \geq p_3 \geq p_4 \geq 0$.

Region 1: $-\frac{2\mu}{4(p_1+p_2)-2p\cdot\alpha(\Delta^-)} \leq x \leq \frac{2\mu}{4(p_1+p_2)+2p\cdot\alpha(\Delta^-)}$

$$\rho(x) = \tfrac{1}{2}\mu - \tfrac{1}{2}xp\cdot\alpha(\Delta^-)\,;$$

$$y_1 - y_{5,2} = -\tfrac{1}{2}\Delta^-_{(1,5)} + \tfrac{x(p_1+p_2)}{-\mu+xp\cdot\alpha(\Delta^-)}\,, \qquad y_2 - y_{5,2} = -\tfrac{1}{2}\Delta^-_{(2,5)} + \tfrac{x(-p_1+p_2)}{-\mu+xp\cdot\alpha(\Delta^-)}\,,$$

$$y_3 - y_{5,2} = -\tfrac{1}{2}\Delta^-_{(3,5)} + \tfrac{x(-p_3+p_4)}{-\mu+xp\cdot\alpha(\Delta^-)}\,, \qquad y_4 - y_{5,2} = -\tfrac{1}{2}\Delta^-_{(4,5)} - \tfrac{x(p_3+p_4)}{-\mu+xp\cdot\alpha(\Delta^-)}\,, \qquad y_{5,1} - y_{5,2} = 0\,.$$

Region 2$^-$: $-\frac{2\mu}{4p_1-2p\cdot\alpha(\Delta^-)} \le x \le -\frac{2\mu}{4(p_1+p_2)-2p\cdot\alpha(\Delta^-)}$

$$\rho(x) = \tfrac{1}{2}\mu - \tfrac{1}{2}xp\cdot\alpha(\Delta^-)\,;$$

$$y_1 - y_{5,2} = 1 - \Delta_{(1,5)}\,, \qquad y_2 - y_{5,2} = \tfrac{1}{2}\big(1 - \Delta^-_{(2,5)}\big) - \tfrac{2xp_1}{-\mu+xp\cdot\alpha(\Delta^-)}\,,$$

$$y_3 - y_{5,2} = \tfrac{1}{2}\big(1 - \Delta^-_{(3,5)}\big) - \tfrac{x(p_1+p_2+p_3-p_4)}{-\mu+xp\cdot\alpha(\Delta^-)}\,, \qquad y_4 - y_{5,2} = \tfrac{1}{2}\big(1 - \Delta^-_{(4,5)}\big) - \tfrac{x(p_1+p_2+p_3+p_4)}{-\mu+xp\cdot\alpha(\Delta^-)}\,,$$

$$y_{5,1} - y_{5,2} = 1 - \tfrac{2x(p_1+p_2)}{-\mu+xp\cdot\alpha(\Delta^-)}\,.$$

Region 2$^+$: $\frac{2\mu}{4(p_1+p_2)+2p\cdot\alpha(\Delta^-)} \le x \le \frac{2\mu}{4p_1+2p\cdot\alpha(\Delta^-)}$

$$\rho(x) = \tfrac{1}{2}\mu - \tfrac{1}{2}xp\cdot\alpha(\Delta^-)\,;$$

$$y_1 - y_{5,2} = -\Delta_{(1,5)}\,, \qquad y_2 - y_{5,2} = -\tfrac{1}{2}\big(1 + \Delta^-_{(2,5)}\big) - \tfrac{2xp_1}{-\mu+xp\cdot\alpha(\Delta^-)}\,,$$

$$y_3 - y_{5,2} = \tfrac{1}{2}\big(1 - \Delta^-_{(3,5)}\big) - \tfrac{x(p_1+p_2+p_3-p_4)}{-\mu+xp\cdot\alpha(\Delta^-)}\,, \qquad y_4 - y_{5,2} = \tfrac{1}{2}\big(1 - \Delta^-_{(4,5)}\big) - \tfrac{x(p_1+p_2+p_3+p_4)}{-\mu+xp\cdot\alpha(\Delta^-)}\,,$$

$$y_{5,1} - y_{5,2} = 1 - \tfrac{2x(p_1+p_2)}{-\mu+xp\cdot\alpha(\Delta^-)}\,.$$

Region 3$^-$: $-\frac{2\mu}{2(p_1+p_2+p_3+p_4)-2p\cdot\alpha(\Delta^-)} \le x \le -\frac{2\mu}{4p_1-2p\cdot\alpha(\Delta^-)}$

$$\rho(x) = \mu + xp_1 - xp\cdot\alpha(\Delta^-)\,;$$

$$y_1 - y_{5,2} = 1 - \Delta_{(1,5)}\,, \qquad y_2 - y_{5,2} = -\Delta_{(2,5)}\,, \qquad y_3 - y_{5,2} = -\tfrac{1}{2}\Delta^-_{(3,5)} - \tfrac{x(p_2+p_3-p_4)}{-2\mu-2xp_1+2xp\cdot\alpha(\Delta^-)}\,,$$

$$y_4 - y_{5,2} = -\tfrac{1}{2}\Delta^-_{(4,5)} - \tfrac{x(p_2+p_3+p_4)}{-2\mu-2xp_1+2xp\cdot\alpha(\Delta^-)}\,, \qquad y_{5,1} - y_{5,2} = -\tfrac{2xp_2}{-2\mu-2xp_1+2xp\cdot\alpha(\Delta^-)}\,.$$

Region 3$^+$: $\frac{2\mu}{4p_1+2p\cdot\alpha(\Delta^-)} \le x \le \frac{2\mu}{2(p_1+p_2+p_3+p_4)+2p\cdot\alpha(\Delta^-)}$

$$\rho(x) = \mu - xp_1 - xp\cdot\alpha(\Delta^-)\,;$$

$$y_1 - y_{5,2} = -\Delta_{(1,5)}\,, \qquad y_2 - y_{5,2} = 1 - \Delta_{(2,5)}\,, \qquad y_3 - y_{5,2} = -\tfrac{1}{2}\Delta^-_{(3,5)} - \tfrac{x(p_2+p_3-p_4)}{-2\mu+2xp_1+2xp\cdot\alpha(\Delta^-)}\,,$$

$$y_4 - y_{5,2} = -\tfrac{1}{2}\Delta^-_{(4,5)} - \tfrac{x(p_2+p_3+p_4)}{-2\mu+2xp_1+2xp\cdot\alpha(\Delta^-)}\,, \qquad y_{5,1} - y_{5,2} = -\tfrac{2xp_2}{-2\mu+2xp_1+2xp\cdot\alpha(\Delta^-)}\,.$$

Region 4^-: $-\frac{2\mu}{2(p_1+p_2+p_3-p_4)-2p\cdot\alpha(\Delta^-)} \leq x \leq -\frac{2\mu}{2(p_1+p_2+p_3+p_4)-2p\cdot\alpha(\Delta^-)}$

$$\rho(x) = \tfrac{3}{2}\mu + \tfrac{1}{2}x(3p_1 + p_2 + p_3 + p_4) - \tfrac{3}{2}xp \cdot \alpha(\Delta^-)\,;$$

$$y_1 - y_{5,2} = 1 - \Delta_{(1,5)}\,, \qquad y_2 - y_{5,2} = -\Delta_{(2,5)}\,,$$

$$y_3 - y_{5,2} = -\tfrac{1}{6} - \tfrac{1}{2}\Delta^-_{(3,5)} + \tfrac{2x(2p_4-p_2-p_3)}{-9\mu-3x(3p_1+p_2+p_3+p_4)+9xp\cdot\alpha(\Delta^-)}\,, \qquad y_4 - y_{5,2} = -\Delta_{(4,5)}\,,$$

$$y_{5,1} - y_{5,2} = -\tfrac{1}{3} + \tfrac{2x(-2p_2+p_3+p_4)}{-9\mu-3x(3p_1+p_2+p_3+p_4)+9xp\cdot\alpha(\Delta^-)}\,.$$

Region 4^+: $\frac{2\mu}{2(p_1+p_2+p_3+p_4)+2p\cdot\alpha(\Delta^-)} \leq x \leq \frac{2\mu}{2(p_1+p_2+p_3-p_4)+2p\cdot\alpha(\Delta^-)}$

$$\rho(x) = \tfrac{3}{2}\mu - \tfrac{1}{2}x(3p_1 + p_2 + p_3 + p_4) - \tfrac{3}{2}xp \cdot \alpha(\Delta^-)\,;$$

$$y_1 - y_{5,2} = -\Delta_{(1,5)}\,, \qquad y_2 - y_{5,2} = 1 - \Delta_{(2,5)}\,,$$

$$y_3 - y_{5,2} = \tfrac{1}{6} - \tfrac{1}{2}\Delta^-_{(3,5)} + \tfrac{2x(2p_4-p_2-p_3)}{-9\mu+3x(3p_1+p_2+p_3+p_4)+9xp\cdot\alpha(\Delta^-)}\,, \qquad y_4 - y_{5,2} = 1 - \Delta_{(4,5)}\,,$$

$$y_{5,1} - y_{5,2} = \tfrac{1}{3} + \tfrac{2x(-2p_2+p_3+p_4)}{-9\mu+3x(3p_1+p_2+p_3+p_4)+9xp\cdot\alpha(\Delta^-)}\,.$$

Region 5^-: $-\frac{2\mu}{2(p_1+p_2)-2p\cdot\alpha(\Delta^-)} \leq x \leq -\frac{2\mu}{2(p_1+p_2+p_3-p_4)-2p\cdot\alpha(\Delta^-)}$

$$\rho(x) = 2\mu + x(2p_1 + p_2 + p_3) - 2xp \cdot \alpha(\Delta^-)\,;$$

$$y_1 - y_{5,2} = 1 - \Delta_{(1,5)}\,, \qquad y_2 - y_{5,2} = -\Delta_{(2,5)}\,, \qquad y_3 - y_{5,2} = -\Delta_{(3,5)}\,,$$

$$y_4 - y_{5,2} = -\Delta_{(4,5)}\,, \qquad y_{5,1} - y_{5,2} = -\tfrac{1}{2} - \tfrac{x(p_2-p_3)}{-4\mu-2x(2p_1+p_2+p_3)+4xp\cdot\alpha(\Delta^-)}\,.$$

Finally, the last saturation occurs at the end of this region with $y_{5,1} - y_{5,2} = -1$.

Region 5^+: $\frac{2\mu}{2(p_1+p_2+p_3-p_4)+2p\cdot\alpha(\Delta^-)} \leq x \leq \frac{2\mu}{2(p_1+p_2)+2p\cdot\alpha(\Delta^-)}$

$$\rho(x) = 2\mu - x(2p_1 + p_2 + p_3) - 2xp \cdot \alpha(\Delta^-)\,;$$

$$y_1 - y_{5,2} = -\Delta_{(1,5)}\,, \qquad y_2 - y_{5,2} = 1 - \Delta_{(2,5)}\,, \qquad y_3 - y_{5,2} = 1 - \Delta_{(3,5)}\,,$$

$$y_4 - y_{5,2} = 1 - \Delta_{(4,5)}\,, \qquad y_{5,1} - y_{5,2} = \tfrac{1}{2} - \tfrac{x(p_2-p_3)}{-4\mu+2x(2p_1+p_2+p_3)+4xp\cdot\alpha(\Delta^-)}\,.$$

Finally, the last saturation occurs at the end of this region with $y_{5,1} - y_{5,2} = 1$.

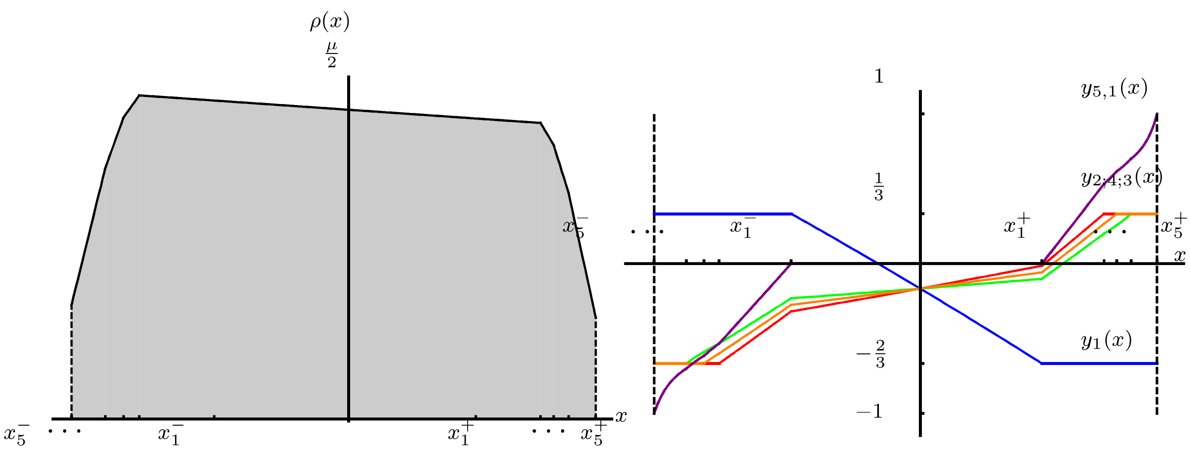

Figure 2.2: Eigenvalue density $\rho(x)$ and distributions $y_{a,I}(x)$ for $\widehat{D}_4$ quiver ($y_{5,2}(x) = 0$).

To get a feel of these expressions for $\rho(x)$ and $y(x)$'s, we have plotted them in Figure 2.2 using the numerical values: $p_1 = 15$, $p_2 = 8$, $p_3 = 4$, $p_4 = 1$ and all Δ's equal to $\frac{2}{3}$. With the $\rho(x)$ known in all the regions, we can just use the normalization condition $\int dx \rho(x) = 1$ to get $\frac{1}{\mu^2}$, which is directly related to the $\mathrm{Vol}(Y_7)$. As with the $\widehat{A}$ quiver, this volume can be recast as a polygon's area and for $\widehat{D}_4$ quiver, this polygon turns out to be

$$\mathcal{P} = \left\{ (s,t) \in \mathbb{R}^2 \middle| \textstyle\sum_{a=1}^{4} \left(|t + p_a s| + |t - p_a s|\right) - 4|t| + 2\,p\cdot\alpha(\Delta^-) s \leq 1 \right\}, \quad \text{with} \quad \frac{\mathrm{Vol}(Y_7)}{\mathrm{Vol}(S^7)} = \frac{1}{4}\,\mathrm{Area}(\mathcal{P})\,. \tag{2.4.11}$$

For the above mentioned numerical values, the polygon is shown in Figure 2.3 with $\frac{1}{4}\,\text{Area}(\mathcal{P}) = \frac{1992856091659101}{388764834312025600} \approx 0.005$. This value matches $\frac{\text{Vol}(Y_7)}{\text{Vol}(S^7)} = \frac{1}{8\mu^2}$ exactly. Also, we note that this construction is valid for all possible orderings and signs of p's.

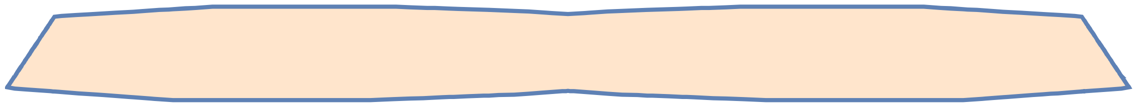

Figure 2.3: Polygon $\mathcal{P}$ for $\widehat{D}_4$ quiver. ($s - t$ coordinate system rotated by $\frac{\pi}{2}$.)

We can, of course, write the explicit volume for $\widehat{D}_4$ here but instead we prefer to give the explicit expression for general $\widehat{D}_n$ quiver directly.

2.4.3 $\widehat{D}_n$ Result

Given the result for $\widehat{D}_4$ quiver above and the known result for $\mathcal{N} = 3$ $\widehat{D}_n$ quivers in Ref. [60], we conjecture the polygon for $\mathcal{N} = 2$ $\widehat{D}_n$ quivers to be:

$$\mathcal{P} = \left\{(s,t) \in \mathbb{R}^2 \middle| \sum_{a=1}^{n} \big(|t + p_a s| + |t - p_a s|\big) - 4|t| + c\, s \leq 1\right\}; \quad c \equiv \sum_{b=1}^{n} (2p_b)\alpha_b(\Delta^-)\,, \tag{2.4.12}$$

$$\begin{aligned}\alpha_b(\Delta^-) = \Big\{&\tfrac{1}{2}\big(\Delta^-_{(1,5)} - \Delta^-_{(2,5)}\big), \tfrac{1}{2}\big(\Delta^-_{(1,5)} + \Delta^-_{(2,5)}\big), \Delta^-_{(5,6)}, \cdots, \Delta^-_{(n,n+1)},\\ &-\tfrac{1}{2}\big(\Delta^-_{(4,n+1)} + \Delta^-_{(3,n+1)}\big), -\tfrac{1}{2}\big(\Delta^-_{(4,n+1)} - \Delta^-_{(3,n+1)}\big)\Big\}.\end{aligned} \tag{2.4.13}$$

For generic n, the p's are related to the CS levels as follows:

$$\begin{aligned}k_1 = -(p_1 + p_2), \quad k_2 = p_1 - p_2, \quad k_3 = p_{n-1} - p_n, \quad k_4 = p_{n-1} + p_n,\\ k_i = p_{i-3} - p_{i-2}; \quad i = 5, \cdots, n+1\,.\end{aligned} \tag{2.4.14}$$

We note the difference with Eq. (2.4.4) for $\widehat{A}$ quivers which requires two constants. As explained in Ref. [42], this is due to the two $U(1)$ isometries of the toric $\widehat{A}$ quivers so it makes sense that for the case of non-toric $\widehat{D}$ quivers which has only one $U(1)$ isometry, we see only one constant in the polygon formula in Eq. (2.4.12).

One can verify that this polygon's area gives the general volume formula corresponding to $\widehat{D}_n$ quivers:

$$\frac{\mathrm{Vol}(Y_7)}{\mathrm{Vol}(S^7)} = \frac{1}{8\mu^2} = \frac{1}{4}\mathrm{Area}(\mathcal{P}) = \frac{1}{4}\sum_{a=0}^{n}\left[\frac{|\gamma_{a,a+1}|}{\bar{\sigma}_a^+\bar{\sigma}_{a+1}^+} + \frac{|\gamma_{a,a+1}|}{\bar{\sigma}_a^-\bar{\sigma}_{a+1}^-}\right], \tag{2.4.15}$$

which we have explicitly checked for $\widehat{D}_5, \cdots, \widehat{D}_{10}$.[15] The definitions of various quantities are slightly elaborate here:

$$\beta_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \beta_{\pm a} = \begin{pmatrix} 1 \\ \pm p_a \end{pmatrix} \text{ for } a = 1, \cdots, n,\ \beta_{n+1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \qquad \gamma_{a,b} = \beta_a \wedge \beta_b\,;$$
$$\bar{\sigma}_a^{\pm} = \sum_{b=1}^{n}\left[|\gamma_{a,b}| + |\gamma_{a,-b}| \pm (\gamma_{a,b} - \gamma_{a,-b})\alpha_b(\Delta^-)\right] - 4|\gamma_{a,n+1}|\,. \tag{2.4.16}$$

The combination $(\gamma_{a,b} - \gamma_{a,-b}) = 2p_b$ for $a \neq 0$ is used to show similarity with the definitions for $\widehat{A}$ quivers in Eq. (2.4.7), otherwise it is a simple factor defining c in Eq. (2.4.12).

This completes the free energy or dual volume computation of $\widehat{AD}$ quivers. Let us now continue with the computation of their twisted indices.

[15] It is interesting to note that the $\pm$ structure in Eq. (2.4.15) produces independent terms, which is in contrast to the expression in Eq. (2.4.6) of $\widehat{A}$ quivers, where the analogous σ^+ and σ^- terms produce one mixed term. However, that is just an artifact of the way we have defined β's. β_0 is quite redundant if we realize $\frac{1}{\bar{\sigma}_0^+\bar{\sigma}_1^+} + \frac{1}{\bar{\sigma}_0^-\bar{\sigma}_1^-} = \frac{|\gamma_{-1,1}|}{\bar{\sigma}_1^-\bar{\sigma}_1^+}$.

2.5 Twisted Index and Entropy

We will again work on $\widehat{AD}$ quivers and first evaluate the Bethe potential in Eq. (2.3.8) and then the index in Eq. (2.3.13) (equivalently, dual black hole entropy). We will follow the same algorithm used to evaluate F_{S^3} but start with the reduced set of inequalities as VM do not contribute to $\mathcal{V}$:

$$0 < y_{a,I} - y_{b,J} + \nu_{(a,b)} < 1\,; \qquad -1 < y_{a,I} - y_{b,J} - \nu_{(b,a)} < 0\,. \tag{2.5.1}$$

As discussed before, we will insist $\nu_{(a,b)} + \nu_{(b,a)} = \frac{1}{2}$. Since we have pairing up of bifundamentals, while the inequalities are not violated, the contribution from these fields simplify to

$$\begin{aligned} -\sum_{(a,b)\oplus(b,a)} \pi\left(1 - \nu^+_{(a,b)}\right) \int dx \rho(x)^2 \sum_{I,J} \left[\left(y_{a,I} - y_{b,J} + \frac{\nu^-_{(a,b)}}{2}\right)^2 - \frac{1}{12}\nu^+_{(a,b)}\left(2 - \nu^+_{(a,b)}\right)\right] \\ = -\sum_{(a,b)\oplus(b,a)} \frac{\pi}{2} \int dx \rho(x)^2 \sum_{I,J} \left[\left(y_{a,I} - y_{b,J} + \frac{\nu^-_{(a,b)}}{2}\right)^2 - \frac{1}{16}\right], \end{aligned} \tag{2.5.2}$$

where $\nu^{\pm}_{(a,b)} = \nu_{(a,b)} \pm \nu_{(b,a)}$. Comparing Eqs. (2.2.8) and (2.4.2) in the central region (i.e., $y_{a,I} = y_{a,J}$) with Eqs. (2.3.8) and (2.5.2), we find that the two expressions (whether off-shell or on-shell) are same up to the scalings given in Table 2.1.

Once an inequality is saturated, we have to use the general expression involving $\arg()$ functions. This step is to be taken much more seriously here than the case of free energy because moving away from the central region generates terms like $(y_{a,I} - y_{a,J})^2$ leading to new inequalities:

$$-\tfrac{1}{2} < y_{a,I} - y_{a,J} < 0 \quad \text{or} \quad 0 < y_{a,I} - y_{a,J} < \tfrac{1}{2}\,, \tag{2.5.3}$$

Free Energy	$\to$	Bethe Potential
F_{S^3}	$\to$	$4\mathcal{V}$
μ	$\to$	$4\tilde{\mu}$
Δ	$\to$	2ν
y	$\to$	$2y$
x	$\to$	$2x$
ρ	$\to$	$\frac{1}{2}\rho$

Table 2.1: Scaling different parameters to relate F_{S^3} and $\mathcal{V}$.

which can drastically affect the evaluation of $\mathcal{V}$ in the new regions. This process of generation of new terms and inequalities that look like coming from vector contributions of F_{S^3} means that $\mathcal{V}$ can indeed be related to F_{S^3} in all the regions, not just in the central region, even though these two expressions seemed very different for $\widehat{DE}$ quivers in subsection 2.3.1. In fact, using the scalings given in Table 2.1, we can verify that it is indeed so allowing us to use the results for F_{S^3} to write down $\mathcal{V}$ for the same quiver.

As far as saturation points, $\rho(x)$ and $y_{a,I}(x)$ are concerned, we can get them from the similar computations already done for F_{S^3} but to get the divergent contributions $Y^{\pm}(x)$, we need to perform one more step during extremization of $\mathcal{V}$ in different regions. This step is to substitute the solutions of each region $n^{\pm}$ in the equations of motion $\mathcal{B}_a^I$ found in the region 1. Of course, $\mathcal{B}_a^I \neq 0$ in other regions but provide the divergent contributions $Y^{\pm}(x)$'s via Eq. (2.3.10). One technicality is that the $\mathcal{B}_a^I$ of Eq. (2.3.10) are related to the equations of motion obtained from $\mathcal{V}$ via $\frac{\partial \mathcal{V}}{\partial y_{a,I}} = N\rho(x)\mathcal{B}_a^I$. This step needs a slight modification as discussed in subsection 2.5.3.

2.5.1 $\widehat{A}_3$ Solved

As far as we know, only theories like $\widehat{A}_1$ quiver (ABJM) whose matrix models involve just 2 regions have been discussed in the literature. So we improve the situation by considering a non-trivial example explicitly for $\widehat{A}$ quivers: $\widehat{A}_3$, whose matrix model involves 4 regions. Let us set up some notation before presenting the explicit solution. We use the redefined CS variables following Eq. (2.4.5) with the given ordering: $q_1 > q_2 > 0 > q_3$ and $q_4 = -\sum_{a=1}^{3} q_a$. We will again suppress the second index on the four $y_{a,1}$ with $a = 1, \cdots, 4$ and introduce two short-hand notations:

$$\begin{aligned} \Sigma_\nu &= \nu^-_{(1,2)} + \nu^-_{(2,3)} + \nu^-_{(3,4)} + \nu^-_{(4,1)}\,, \\ \alpha_b(\nu^-) &= \Big\{ \big(\nu^-_{(1,2)} - \nu^-_{(4,1)}\big), \big(\nu^-_{(2,3)} - \nu^-_{(4,1)}\big), \big(\nu^-_{(3,4)} - \nu^-_{(4,1)}\big) \Big\}\,, \end{aligned} \tag{2.5.4}$$

which will appear in a combination $\sum_{b=1}^{3} q_b \alpha_b(\nu^-) \equiv q \cdot \alpha(\nu^-)$ below.

Region 1: $-\frac{2\tilde{\mu}}{2q_1+q_1\Sigma_\nu - q\cdot\alpha(\nu^-)} \le x \le \frac{2\tilde{\mu}}{2q_1 - q_1\Sigma_\nu + q\cdot\alpha(\nu^-)}$

$$\rho(x) = -\frac{32\tilde{\mu} - 16xq\cdot\alpha(\nu^-)}{(\Sigma_\nu - 2)(\Sigma_\nu+2)}\,;$$

$$y_a - y_{a+1} = \frac{2\tilde{\mu}(\Sigma_\nu - 4\nu^-_{(a,a+1)}) + x[q_a(\Sigma_\nu-2)(\Sigma_\nu+2) - q\cdot\alpha(\nu^-)(\Sigma_\nu - 4\nu^-_{(a,a+1)})]}{16\tilde{\mu} - 8xq\cdot\alpha(\nu^-)}\,, \qquad a = 1,2,3\,.$$

Region 2^-: $-\frac{2\tilde{\mu}}{2(q_1+q_2+q_3) - (q_1+q_2+q_3)\Sigma_\nu - q\cdot\alpha(\nu^-)} \le x \le -\frac{2\tilde{\mu}}{2q_1+q_1\Sigma_\nu - q\cdot\alpha(\nu^-)}$

$$\rho(x) = -\frac{24\tilde{\mu} - 4x[q_1(\Sigma_\nu-2) + 3q\cdot\alpha(\nu^-)]}{(\Sigma_\nu-2)(\Sigma_\nu+1)}\,; \qquad y_1 - y_2 = \frac{1}{2} - \nu_{(1,2)}\,,$$

$$y_a - y_{a+1} = \frac{2\tilde{\mu}(2\Sigma_\nu - 6\nu^-_{(a,a+1)} - 1) + x[(q_1(1+2\nu^-_{(a,a+1)}) + 2q_a(\Sigma_\nu+1))(\Sigma_\nu-2) - q\cdot\alpha(\nu^-)(2\Sigma_\nu - 6\nu^-_{(a,a+1)} - 1)]}{24\tilde{\mu} - 4xq_1(\Sigma_\nu-2) - 12xq\cdot\alpha(\nu^-)}\,, \quad a = 2,3\,;$$

$$Y^-_{(1;2)} = -\frac{4\pi\tilde{\mu} + 2\pi x[q_1(\Sigma_\nu+2) - q\cdot\alpha(\nu^-)]}{\Sigma_\nu + 1}\,.$$

Region 2^+: $\frac{2\tilde{\mu}}{2q_1-q_1\Sigma_\nu+q\cdot\alpha(\nu^-)} \leq x \leq \frac{2\tilde{\mu}}{2(q_1+q_2+q_3)+(q_1+q_2+q_3)\Sigma_\nu+q\cdot\alpha(\nu^-)}$

$$\rho(x) = -\frac{24\tilde{\mu}-4x[q_1(\Sigma_\nu+2)+3q\cdot\alpha(\nu^-)]}{(\Sigma_\nu+2)(\Sigma_\nu-1)}\,; \qquad y_1-y_2=-\nu_{(1,2)}\,,$$

$$y_a-y_{a+1}=\frac{2\tilde{\mu}(2\Sigma_\nu-6\nu^-_{(a,a+1)}+1)+x[(q_1(-1+2\nu^-_{(2,3)})+2q_a(\Sigma_\nu-1))(\Sigma_\nu+2)-q\cdot\alpha(\nu^-)(2\Sigma_\nu-6\nu^-_{(a,a+1)}+1)]}{24\tilde{\mu}-4xq_1(\Sigma_\nu+2)-12xq\cdot\alpha(\nu^-)}\,, \quad a=2,3\,;$$

$$Y^+_{(1;2)}=\frac{4\pi\tilde{\mu}+2\pi x[q_1(\Sigma_\nu-2)-q\cdot\alpha(\nu^-)]}{\Sigma_\nu-1}\,.$$

Region 3^-: $-\frac{2\tilde{\mu}}{q_1+q_2+q_3\Sigma_\nu-q\cdot\alpha(\nu^-)} \leq x \leq -\frac{2\tilde{\mu}}{2(q_1+q_2+q_3)-(q_1+q_2+q_3)\Sigma_\nu-q\cdot\alpha(\nu^-)}$

$$\rho(x)=\frac{-16\tilde{\mu}-4x[2q_1+(q_2+q_3)(1+\Sigma_\nu)-2q\cdot\alpha(\nu^-)]}{(\Sigma_\nu+1)(\Sigma_\nu-1)}\,; \qquad y_1-y_2=\tfrac{1}{2}-\nu_{(1,2)}\,, \qquad y_4-y_1=-\nu_{(4,1)}\,,$$

$$y_3-y_4=\frac{4\tilde{\mu}(\Sigma_\nu-2\nu^-_{(3,4)})+x[2q_1(\Sigma_\nu-2\nu^-_{(3,4)})+(q_2(1-2\nu^-_{(3,4)})-q_3(1-2\Sigma_\nu+2\nu^-_{(3,4)}))(\Sigma_\nu+1)-2q\cdot\alpha(\nu^-)(\Sigma_\nu-2\nu^-_{(3,4)})]}{16\tilde{\mu}+8xq_1+4x(q_2+q_3)(\Sigma_\nu+1)-8xq\cdot\alpha(\nu^-)}\,;$$

$$Y^-_{(1;2)}=\frac{-4\pi\tilde{\mu}-2\pi x[q_1(\Sigma_\nu+2)-q\cdot\alpha(\nu^-)]}{\Sigma_\nu+1}\,, \qquad Y^+_{(4;1)}=\frac{4\pi\tilde{\mu}-2\pi x[(q_1+q_2+q_3)(\Sigma_\nu-2)+q\cdot\alpha(\nu^-)]}{\Sigma_\nu-1}\,.$$

Region 3^+: $\frac{2\tilde{\mu}}{2(q_1+q_2+q_3)+(q_1+q_2+q_3)\Sigma_\nu+q\cdot\alpha(\nu^-)} \leq x \leq \frac{2\tilde{\mu}}{q_1+q_2-q_2\Sigma_\nu+q\cdot\alpha(\nu^-)}$

$$\rho(x)=\frac{-16\tilde{\mu}+4x[2q_1-(q_2+q_3)(\Sigma_\nu-1)+2q\cdot\alpha(\nu^-)]}{(\Sigma_\nu+1)(\Sigma_\nu-1)}\,; \qquad y_1-y_2=-\nu_{(1,2)}\,, \qquad y_4-y_1=\tfrac{1}{2}-\nu_{(4,1)}\,,$$

$$y_3-y_4=\frac{4\tilde{\mu}(\Sigma_\nu-2\nu^-_{(3,4)})-x[2q_1(\Sigma_\nu-2\nu^-_{(3,4)})+(q_2(1+2\nu^-_{(3,4)})-q_3(1+2\Sigma_\nu-2\nu^-_{(3,4)}))(\Sigma_\nu-1)+2q\cdot\alpha(\nu^-)(\Sigma_\nu-2\nu^-_{(3,4)})]}{16\tilde{\mu}-8xq_1+4x(q_2+q_3)(\Sigma_\nu-1)-8xq\cdot\alpha(\nu^-)}\,;$$

$$Y^+_{(1;2)}=\frac{4\pi\tilde{\mu}+2\pi x[q_1(\Sigma_\nu-2)-q\cdot\alpha(\nu^-)]}{\Sigma_\nu-1}\,, \qquad Y^-_{(4;1)}=\frac{-4\pi\tilde{\mu}+2\pi x[(q_1+q_2+q_3)(\Sigma_\nu+2)+q\cdot\alpha(\nu^-)]}{\Sigma_\nu+1}\,.$$

Region 4^-: $-\frac{2\tilde{\mu}}{q_1+q_2+q_2\Sigma_\nu-q\cdot\alpha(\nu^-)} \leq x \leq -\frac{2\tilde{\mu}}{q_1+q_2+q_3\Sigma_\nu-q\cdot\alpha(\nu^-)}$

$$\rho(x)=\frac{-8\tilde{\mu}-4x[q_1+q_2(1+\Sigma_\nu)-q\cdot\alpha(\nu^-)]}{\Sigma_\nu(\Sigma_\nu+1)}\,;$$

$$y_1-y_2=\tfrac{1}{2}-\nu_{(1,2)}\,, \qquad y_3-y_4=-\nu_{(3,4)}\,, \qquad y_4-y_1=-\nu_{(4,1)}\,;$$

$$Y^-_{(1;2)}=\frac{-4\pi\tilde{\mu}-2\pi x[q_1(\Sigma_\nu+2)-q\cdot\alpha(\nu^-)]}{\Sigma_\nu+1}\,, \qquad Y^+_{(3;4)}=\frac{4\pi\tilde{\mu}+2\pi x[q_1+q_2+q_3\Sigma_\nu-q\cdot\alpha(\nu^-)]}{\Sigma_\nu}\,,$$

$$Y^+_{(4;1)}=\frac{4\pi\tilde{\mu}+2\pi x[q_1+q_2-(q_1+q_2+q_3)\Sigma_\nu-q\cdot\alpha(\nu^-)]}{\Sigma_\nu}\,.$$

Region 4^+: $\frac{2\tilde{\mu}}{q_1+q_2-q_2\Sigma_\nu+q\cdot\alpha(\nu^-)} \leq x \leq \frac{2\tilde{\mu}}{q_1+q_2-q_3\Sigma_\nu+q\cdot\alpha(\nu^-)}$

$$\rho(x) = \frac{-8\tilde{\mu}+4x[q_1+q_2-q_3\Sigma_\nu+q\cdot\alpha(\nu^-)]}{\Sigma_\nu(\Sigma_\nu+1)}\,;$$

$$y_1 - y_2 = -\nu_{(1,2)}\,, \qquad y_2 - y_3 = -\nu_{(2,3)}\,, \qquad y_4 - y_1 = \frac{1}{2} - \nu_{(4,1)}\,;$$

$$Y^+_{(1;2)} = \frac{4\pi\tilde{\mu}+2\pi x[q_1\Sigma_\nu-(q_1+q_2)-q\cdot\alpha(\nu^-)]}{\Sigma_\nu}\,, \qquad Y^+_{(2;3)} = \frac{4\pi\tilde{\mu}-2\pi x[q_1-q_2(\Sigma_\nu-1)+q\cdot\alpha(\nu^-)]}{\Sigma_\nu}\,,$$

$$Y^-_{(4;1)} = \frac{-4\pi\tilde{\mu}+2\pi x[(q_1+q_2+q_3)(\Sigma_\nu+2)+q\cdot\alpha(\nu^-)]}{\Sigma_\nu+1}\,.$$

Let us visualize all these expressions for $\rho(x)$ and $y(x)$'s in Figure 2.4 using the numerical values: $q_1 = 78$, $q_2 = 2$, $q_3 = -29$ and all Δ's equal to $\frac{1}{5}$. With the $\rho(x)$ known in

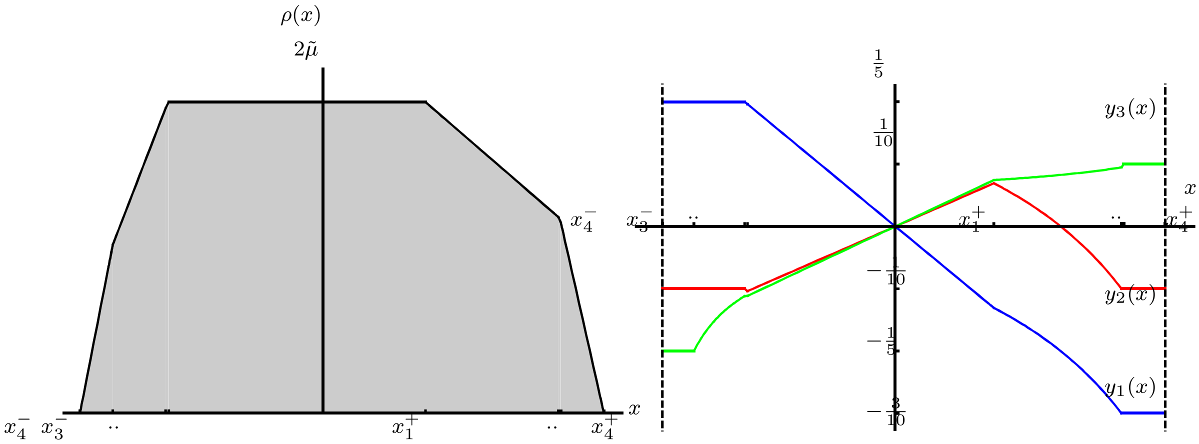

Figure 2.4: Eigenvalue density $\rho(x)$ and distributions $y_a(x)$ for $\widehat{A}_3$ quiver ($y_4(x) = 0$).

all the regions, we can just use the normalization condition $\int dx\rho(x) = 1$ to get $\tilde{\mu}$, which gives the Bethe potential $\mathcal{V} \propto \tilde{\mu}$. Next, we plot the divergent contributions in Figure 2.5, which are crucial to get the correct twisted index. We note that all the $Y^\pm(x)$'s are in the upper half plane as required by consistency.

Finally, we have to integrate the expression given in Eq. (2.3.13) with all the $\left\{\rho(x), y_{a,I}(x), Y^\pm_{(a,I;b,J)}\right\}$ obtained here in each region carefully. The result is a huge expression and unless we take

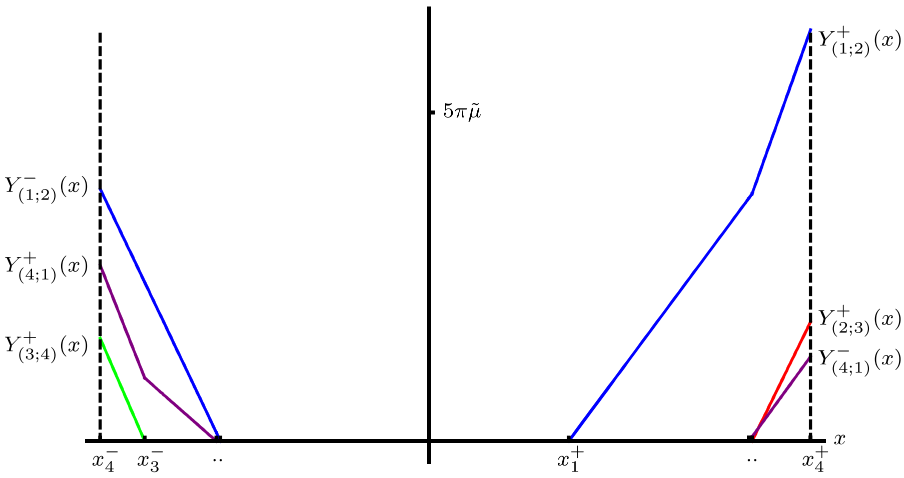

Figure 2.5: Divergent contributions $Y^{\pm}_{(a;b)}(x)$ for $\widehat{A}_3$ quiver.

the help of index theorem, it is hard to make sense of it. Though, we can make sure that the integrated expression and the one obtained via the index theorem are identical, which we have done for both $\widehat{A}_2$ and $\widehat{A}_3$ to check that Eq. (2.3.10) does give the correct $Y^{\pm}(x)$'s. Thus, instead of writing the full expression for $\widehat{A}_3$ here, we present the explicit general result for $\widehat{A}_m$ quiver directly.

2.5.2 $\widehat{A}_m$ Result

Having discussed the non-trivial case of $\widehat{A}_3$ quiver of this class explicitly, we write down the generalization of (well-known) $\widehat{A}_1$ and (above-mentioned) $\widehat{A}_3$ results quite straightforwardly:

$$\mathcal{V} = \frac{4\pi N^{3/2}}{3}\tilde{\mu} \quad \text{with} \quad \frac{1}{\tilde{\mu}^2} = 32\sum_{a=1}^{m+1}\left[\frac{|\gamma_{a,a+1}|}{\sigma_a\sigma_{a+1}} + \frac{|\gamma_{a,a+1}|}{\sigma_{a+m+1}\sigma_{a+m+2}}\right], \tag{2.5.5}$$

where only the σ's definitions slightly changes compared to Eq. (2.4.7)

$$\sigma_a = \sum_{b=1}^{m+1} \left[|\gamma_{a,b}| + 2\gamma_{b,a}\nu^-_{(b,b+1)} \right] , \quad \sigma_{a+m+1} = \sum_{b=1}^{m+1} \left[|\gamma_{a,b}| - 2\gamma_{b,a}\nu^-_{(b,b+1)} \right] . \tag{2.5.6}$$

Thus, we see that if we substitute $\Delta \to 2\nu$ in Eq. (2.4.6), we get for $\widehat{A}$ quivers:

$$\frac{1}{\mu[2\nu]^2} = \frac{1}{16\tilde{\mu}[\nu]^2} \qquad \Rightarrow \qquad 4\mathcal{V}[\nu] = F_{S^3}[2\nu] \tag{2.5.7}$$

as promised earlier.

The index is implicitly given by the relation in Eq. (2.3.20) but massaging it a little bit, we can give an explicit expression in terms of $\tilde{\mu}$ that facilitates checking with the expression given by the integral in Eq. (2.3.13):

$$\bar{\mathcal{I}} = (\mathfrak{g}-1)\frac{4\pi N^{3/2}}{3}\tilde{\mu}^3 \left[\frac{4}{\tilde{\mu}^2} - \frac{1}{2}\sum_I (\mathfrak{n}_I - 2\nu_I)\frac{\partial\left(\frac{1}{\tilde{\mu}^2}\right)}{\partial \nu_I} \right]. \tag{2.5.8}$$

The derivative term reads explicitly (after some tedious algebra) as follows:

$$\sum_I (\mathfrak{n}_I - \nu_I)\frac{\partial\left(\frac{1}{\tilde{\mu}^2}\right)}{\partial \nu_I} = -64\sum_{a=1}^{m+1} |\gamma_{a,a+1}| \Bigg[\frac{\sigma_a\left(f_{a+1}(\mathfrak{n}) - 2f_{a+1}(\nu)\right) + \left(f_a(\mathfrak{n}) - 2f_a(\nu)\right)\sigma_{a+1}}{\sigma_a^2\sigma_{a+1}^2} + \frac{\sigma_{a+m+1}\left(f_{a+m+2}(\mathfrak{n}) - 2f_{a+m+2}(\nu)\right) + \left(f_{a+m+1}(\mathfrak{n}) - 2f_{a+m+1}(\nu)\right)\sigma_{a+m+2}}{\sigma_{a+m+1}^2\sigma_{a+m+2}^2} \Bigg],$$

where $f_{a(+m+1)}(\mathfrak{n}) = (-)\sum_{b=1}^{m+1} 2\gamma_{b,a}\mathfrak{n}_{(b,b+1)}$, and similarly for $f_a(\nu)$.

Let us move on to the $\widehat{D}$ quivers now with an explicit solution for $\widehat{D}_4$ quiver first.

2.5.3 $\widehat{D}_4$ Solved

Given the scalings of Table 2.1, the boundary x-values, $\rho(x)$ and $y_{a,I}(x)$ follow straightforwardly from Section 2.4.2 so we do not repeat them here. Only the divergent contributions $Y^{\pm}(x)$'s are new and we enumerate them region-wise below (we again suppress the $I = 1$ index for $a = 1, \cdots, 4$ and $J = 2$ for $b = 5$). It turns out that there are no kinks in $Y^{\pm}(x)$'s here so we write only the new ones appearing in each given region.

Region 1: $-\frac{2\tilde{\mu}}{2(p_1+p_2)-2p\cdot\alpha(\nu^-)} \leq x \leq \frac{2\tilde{\mu}}{2(p_1+p_2)+2p\cdot\alpha(\nu^-)}$

No $Y^{\pm}(x)$'s yet.

Region 2$^-$: $-\frac{2\tilde{\mu}}{2p_1-2p\cdot\alpha(\nu^-)} \leq x \leq -\frac{2\tilde{\mu}}{2(p_1+p_2)-2p\cdot\alpha(\nu^-)}$

$$Y^-_{(1;5)} = -4\pi\big(\tilde{\mu} + x(p_1+p_2) - xp\cdot\alpha(\nu^-)\big).$$

Region 2$^+$: $\frac{2\tilde{\mu}}{2(p_1+p_2)+2p\cdot\alpha(\nu^-)} \leq x \leq \frac{2\tilde{\mu}}{2p_1+2p\cdot\alpha(\nu^-)}$

$$Y^+_{(1;5)} = -4\pi\big(\tilde{\mu} - x(p_1+p_2) - xp\cdot\alpha(\nu^-)\big).$$

Region 3$^-$: $-\frac{2\tilde{\mu}}{p_1+p_2+p_3+p_4-2p\cdot\alpha(\nu^-)} \leq x \leq -\frac{2\tilde{\mu}}{2p_1-2p\cdot\alpha(\nu^-)}$

$$Y^+_{(2;5)} = -4\pi\big(\tilde{\mu} + xp_1 - xp\cdot\alpha(\nu^-)\big).$$

Region 3$^+$: $\frac{2\tilde{\mu}}{2p_1+2p\cdot\alpha(\nu^-)} \leq x \leq \frac{2\tilde{\mu}}{p_1+p_2+p_3+p_4+2p\cdot\alpha(\nu^-)}$

$$Y^-_{(2;5)} = -4\pi\big(\tilde{\mu} - xp_1 - xp\cdot\alpha(\nu^-)\big).$$

Region 4^-: $-\frac{2\tilde{\mu}}{p_1+p_2+p_3-p_4-2p\cdot\alpha(\nu^-)} \leq x \leq -\frac{2\tilde{\mu}}{p_1+p_2+p_3+p_4-2p\cdot\alpha(\nu^-)}$

$$Y^+_{(4;5)} = -2\pi\big(2\tilde{\mu} + x(p_1+p_2+p_3+p_4) - 2xp\cdot\alpha(\nu^-)\big).$$

Region 4^+: $\frac{2\tilde{\mu}}{p_1+p_2+p_3+p_4+2p\cdot\alpha(\nu^-)} \leq x \leq \frac{2\tilde{\mu}}{p_1+p_2+p_3-p_4+2p\cdot\alpha(\nu^-)}$

$$Y^-_{(4;5)} = -2\pi\big(2\tilde{\mu} - x(p_1+p_2+p_3+p_4) - 2xp\cdot\alpha(\nu^-)\big).$$

Region 5^-: $-\frac{2\tilde{\mu}}{p_1+p_2-2p\cdot\alpha(\nu^-)} \leq x \leq -\frac{2\tilde{\mu}}{p_1+p_2+p_3-p_4-2p\cdot\alpha(\nu^-)}$

$$Y^+_{(3;5)} = -2\pi\big(2\tilde{\mu} + x(p_1+p_2+p_3-p_4) - 2xp\cdot\alpha(\nu^-)\big).$$

Region 5^+: $\frac{2\tilde{\mu}}{p_1+p_2+p_3-p_4+2p\cdot\alpha(\nu^-)} \leq x \leq \frac{2\tilde{\mu}}{p_1+p_2+2p\cdot\alpha(\nu^-)}$

$$Y^-_{(3;5)} = -2\pi\big(2\tilde{\mu} - x(p_1+p_2+p_3-p_4) - 2xp\cdot\alpha(\nu^-)\big).$$

These $Y^{\pm}(x)$'s are plotted in Figure 2.6 using the numerical values: $p_1 = 15$, $p_2 = 8$, $p_3 = 4$, $p_4 = 1$ and all ν's equal to $\frac{1}{3}$ and we see that all of them are in the upper half plane as expected.

Finally, we integrate the expression given in Eq. (2.3.13) by substituting the $\{\rho(x), y_{a,I}(x), Y^{\pm}_{(a,I;b,J)}\}$ in each region carefully. The result is again a huge expression and we take help of the index theorem to write it concisely. Before we do that, a comment about insufficiency of Eq. (2.3.10) for $\widehat{D}_n$ with $n > 4$ is in order, after which, we will present the explicit general result for $\widehat{D}_n$ quiver.

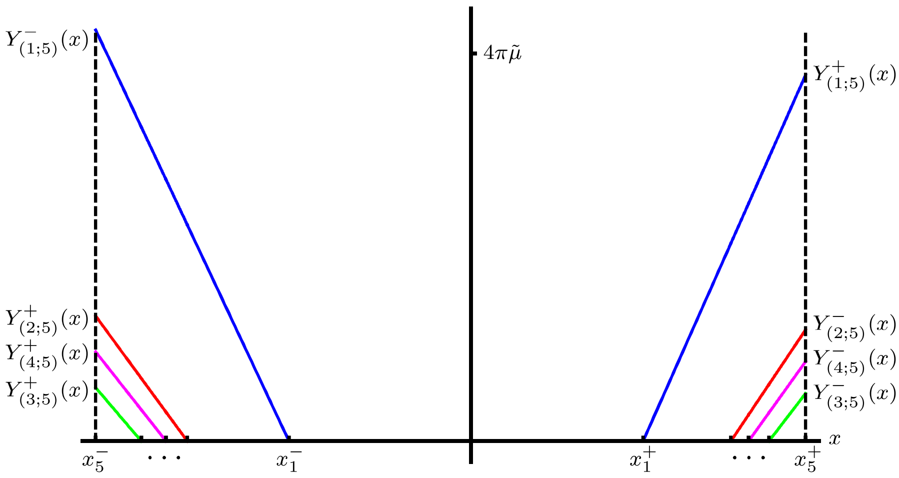

Figure 2.6: Divergent contributions $Y^{\pm}_{(a,1;5,2)}(x)$ for $\widehat{D}_4$ quiver.

2.5.4 $\widehat{D}_n$ Result

It should be no surprise that (after calculational dust settles) the result for $\widehat{D}$ quivers will look similar to that for $\widehat{A}$ quivers:

$$\mathcal{V} = \frac{4\pi N^{3/2}}{3}\tilde{\mu} \quad \text{with} \quad \frac{1}{\tilde{\mu}^2} = 32\sum_{a=0}^{n}\left[\frac{|\gamma_{a,a+1}|}{\bar{\sigma}_a^+\bar{\sigma}_{a+1}^+} + \frac{|\gamma_{a,a+1}|}{\bar{\sigma}_a^-\bar{\sigma}_{a+1}^-}\right], \tag{2.5.9}$$

where only the $\bar{\sigma}$'s definitions slightly changes compared to Eq. (2.4.16)

$$\bar{\sigma}_a^{\pm} = \sum_{b=1}^{n}\left[|\gamma_{a,b}| + |\gamma_{a,-b}| \pm 2(\gamma_{a,b} - \gamma_{a,-b})\alpha_b(\nu^-)\right] - 4|\gamma_{a,n+1}|\,. \tag{2.5.10}$$

Thus, we again see that upon substituting $\Delta \to 2\nu$ in Eq. (2.4.15), we get for $\widehat{D}$ quivers

$$\frac{1}{\mu[2\nu]^2} = \frac{1}{16\tilde{\mu}[\nu]^2} \qquad \Rightarrow \qquad 4\mathcal{V}[\nu] = F_{S^3}[2\nu] \tag{2.5.11}$$

as expected.

One caveat here is that the result for $\widehat{A}_m$ quivers is an exact result whereas that for $\widehat{D}_n$ quivers is a conjecture. This boils down to the polygon formulas Eqs. (2.4.4) and (2.4.12). While the former is a proven solution to the $\widehat{A}_m$ matrix model Ref. [42], the latter is a conjecture that we have checked for $\widehat{D}_n$ matrix model up to $n = 10$.

Finally, the index can be written explicitly as follows:

$$\mathcal{I} = (\mathfrak{g}-1)\frac{4\pi N^{3/2}}{3}\tilde{\mu}^3\left[\frac{4}{\tilde{\mu}^2} - \frac{1}{2}\sum_I (\mathfrak{n}_I - 2\nu_I)\frac{\partial\left(\frac{1}{\tilde{\mu}^2}\right)}{\partial\nu_I}\right]; \tag{2.5.12}$$

$$\text{with} \quad \sum_I (\mathfrak{n}_I - 2\nu_I)\frac{\partial\left(\frac{1}{\tilde{\mu}^2}\right)}{\partial\nu_I} =$$

$$-64\left(f(\mathfrak{n}) - 2f(\nu)\right)\left(\frac{|\gamma_{-1,1}|\left(\bar{\sigma}_1^- + \bar{\sigma}_1^+\right)}{(\bar{\sigma}_1^-)^2(\bar{\sigma}_1^+)^2} + \sum_{\pm,a=1}^{n}\frac{|\gamma_{a,a+1}|\left(\bar{\sigma}_a^\pm + \bar{\sigma}_{a+1}^\pm\right)}{(\bar{\sigma}_a^\pm)^2(\bar{\sigma}_{a+1}^\pm)^2}\right),$$

where $f(\mathfrak{n}) = \sum_{b=1}^n 2(\gamma_{a,b} - \gamma_{a,-b})\alpha_b(\mathfrak{n})$, and similarly for $f(\nu)$. Due to the fact that $(\gamma_{a,b} - \gamma_{a,-b}) = 2p_b$ does not depend on the subscript a, these $f(\cdot)$'s become an overall factor and the explicit expression for $\widehat{D}$ quivers' index simplifies considerably compared to the analogous expression for $\widehat{A}$ quivers.

2.6 Discussions and future directions

We note that one could study these theories on more general Seifert manifolds as discussed in Ref. [63–65]. The $\mathcal{M}_{\mathfrak{g},p}$ manifolds include both the manifolds studied here as in $\mathcal{M}_{0,1} = S^3$ and $\mathcal{M}_{\mathfrak{g},0} = \Sigma_\mathfrak{g} \times S^1$. In this framework, the observation $4\mathcal{V}[\nu] = F_{S^3}[2\nu]$ in the present context may be easily explained following the logic of Ref. [64]. In addition, it should be possible to generalize the results presented here straightforwardly to these manifolds.

An elephant in the room is the fact that expressions for free energies of $\widehat{E}_{6,7,8}$ are missing in this chapter. As is well-known, even in the $\mathcal{N} = 3$ case in Ref. [60] the known expressions are valid only for a subset of CS levels. An all-encompassing formula in terms of roots or graphs as in the case of $\widehat{AD}$ quivers is not known for them. So we refrained from giving the $\mathcal{N} = 2$ extensions of the $\mathcal{N} = 3$ formulas but comment that it would be much more interesting to figure out the fully general volume formula for $\widehat{E}$ quivers. The Fermi-gas formalism, Refs. [66–69] could be a helpful tool in this quest, given that the polygon formula appears naturally as a Fermi surface in this formalism.

Finally, it goes without saying that computing volumes of the Sasaki-Einstein 7-manifolds explicitly and constructing explicit M-theory duals for $\widehat{ADE}$ quivers with non-universal flavour fluxes would be an interesting exercise to test the AdS/CFT correspondence.

2.7 Chapter summary

This chapter contains two interconnected results:

Volume: We computed the explicit free energy F_{S^3} for $\widehat{D}$ quivers in terms of the R-charges $\Delta_{(a,b)}$ of the bifundamentals, obtained by combining Eqs. (2.2.9) and (2.4.15). According to AdS/CFT correspondence, the formula Eq. (2.4.15) provides a prediction for the volumes of certain Sasaki-Einstein 7-manifolds Y_7, which describe the $\mathrm{AdS}_4 \times Y_7$ M-theory duals.

Entropy: We computed the explicit twisted index $\mathcal{I}$ for $\widehat{AD}$ quivers, Eq. (2.5.8) and Eq. (2.5.12), in terms of the chemical potentials $\nu_{(a,b)}$ and flavour fluxes $\mathfrak{n}_{(a,b)}$. We expect that the extremization of these formulas with respect to ν's leading to the expression $\mathcal{I}(\nu(\mathfrak{n}), \mathfrak{n})$ reproduces the macroscopic entropy of the dual black hole solutions in the 4d gauged supergravity uplifted to M-theory with the above-mentioned Y_7's. In the simplifying case of universal twist, the extremization procedure is automatic, leading to $\mathfrak{n}_I = 2\nu_I$ and $S_{BH} = \mathcal{I} = (g-1)F_{S^3}$ follows via the index theorem given in Eq. (2.3.20) for $\widehat{ADE}$ quivers as shown holographically in Ref. [58, 59].

Along the way, we computed the large N limit of the partition functions for $3d$ $\mathcal{N} = 2$ quiver theories on S^3 and $\Sigma_{\mathfrak{g}} \times S^1$ involving bifundamental and fundamental matters. We obtained constraints on relevant parameters (Δ for F_{S^3} and $\{\nu, \mathfrak{n}\}$ for $\mathcal{I}$) under the requirement that the resulting matrix model be local, leading to a large class of CSm quiver theories including the $\widehat{ADE}$ quivers. The fundamental matters contribute in a trivial way and that contribution can be included in the results presented here following Ref. [60, 61]. An intermediate construction to obtain the twisted index is that of the Bethe potential $\mathcal{V}$, which we find is related to the free energy via $F_{S^3}[2\nu] = 4\mathcal{V}[\nu]$ with an explicit matching of the matrix model. It was shown in Ref. [52] that for $\widehat{A}$ quivers and related theories, this relation is true off-shell too but with a *different* numerical factor. We extended this result to $\widehat{DE}$ quivers and showed that the relation holds true in all the integration regions with the *same* numerical factor of 4. This fact fits nicely with the simpler proof of the index theorem provided in the main text.

CHAPTER 3

SUPERSYMMETRIC LOCALIZATION ON DE SITTER: SUM OVER TOPOLOGIES

This chapter is based on

1. "Supersymmetric Localization on dS: Sum over topologies", R. Basu, A. Ray, *Eur. Phys. J. C* 80 85 **(2020)**, [arXiv: 1911.07480].

3.1 Introduction

Quantum theory of gravity in 3 space-time dimensions does not cease to surprise us, owing to the richness of physical and mathematical structures that are being continually revealed for more than 3 decades starting from Ref. [70]. It is interesting that, gravity in $3d$ is devoid of local degrees of freedom. One of the main causes of non-triviality in $3d$ gravity is the BTZ black hole solution, as reported in Ref. [71] for negative cosmological constant. The most interesting sector of solutions for the case of negative cosmological constant is asymptotically AdS. A huge body of work has stemmed from the seminal work of Brown and Henneaux, in Ref. [72], which showed that the asymptotic symmetries of asymptotically AdS space-time in $3d$ form two copies of Virasoro algebra; thereby hinting at a plausible conformal field theory (CFT) at the two dimensional asymptotic boundary. As an example of low dimensional holography, this generated a great deal of physical and mathematical curiosities; motivated just from the question of calculating partition function for quantum gravity and arriving at black hole entropy from it. Refs. [73, 74] are directions in this direction in recent times.

Analogous progress in the case for zero cosmological constant is being pursued recently, specially in the works of Refs. [75, 76]. In this sector, one attempts at a quantum gravity for asymptotically flat space-time, now equipped with the BMS_3 algebra. Ref. [77] contains a relatively extensive discussion of quantum gravity in $3d$ from the perspective of asymptotic symmetries for asymptotically non-AdS space-time, even including higher spin degrees of freedom.

Whereas these aspects of quantum gravity are under focus of intensive studies in recent times, one might be curious for the case of positive cosmological constant. Vacuum solution to the corresponding Einstein equation is the dS_3 space-time. However unlike Minkowski space-time, here exists a horizon at thermal equilibrium. As argued in Ref. [78], correlation function of any quantum degree of freedom with respect to a time-like observer is a thermal correlator. The corresponding vacuum state, as discussed in Ref. [79] and named as the Hartle Hawking state, is the Euclidean partition function.

The choice of Hartle Hawking state as a candidate for vacuum state circumvents an otherwise conceptually difficult problem in the following manner. Standard wisdom says that isometries of a maximally symmetric space-time like de Sitter should fix the vacuum state. But if one wishes to incorporate effects from quantum gravity, one has to incorporate all possible quantum fluctuations on the de Sitter background, from a perturbative viewpoint. Hartle Hawking state is however defined as the Euclidean path integral considering all possible geometries with some fixed boundary data and therefore, captures quantum gravity effects.

Now in de Sitter space, a time-like observer is in causal contact with what is known as the static patch, defined in Euclidean time as:

$$ds^2 = dr^2 + \cos^2 r d\tau^2 + \sin^2 r d\phi^2. \tag{3.1.1}$$

Euclideanizing has been done by setting $t = -i\tau$ and it makes the static patch geometry identical to that of S^3 with $\tau \in [0, 2\pi], \phi \in [0, 2\pi], r \in [0, \pi/2]$.

It would therefore be natural to consider fluctuations over round S^3 background geometry to construct the Hartle Hawking state. However, as pointed out in Ref. [78], there is an infinite class of topologically distinct manifolds which allow smooth local geometry as Eq. (3.1.1). These are of the form S^3/Γ, where Γ is a discrete subgroup of the isometry group of S^3. In terms of the coordinates in Eq. (3.1.1), these quotient spaces with smooth local dS geometry are understood by the following identifications:

$$(\tau, \phi) \sim (\tau, \phi) + 2\pi\,(\frac{m}{p}, \frac{m\,q}{p} + n) \quad \text{for} \quad m, n \in \mathbb{Z}. \tag{3.1.2}$$

Here q, p are coprime positive integers with p always being the greater of the pair. That this identification indeed results into the topological quotient space $S^3/\mathbb{Z}_p$ can be easily understood by first defining

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \Big| |z_1|^2 + |z_2|^2 = 1\}. \tag{3.1.3}$$

Then the $\mathbb{Z}_p$ action on it is:

$$(z_1, z_2) \to \left(e^{\frac{2\pi i}{p}} z_1, e^{\frac{2\pi i q}{p}} z_2\right). \tag{3.1.4}$$

Finally defining $(z_1, z_2) = \left(\cos r\, e^{i\tau}, \sin r\, e^{i\phi}\right)$ makes the identification Eq. (3.1.2) clear. The resultant manifold is named as a Lens space $L(p,q)$, now equipped with the smooth geometry given by Eq. (3.1.1). All of these manifolds are therefore valid classical smooth saddles of Einstein equation.

Since S^3 as well as all the quotients $L(p,q)$ are closed, Hartle Hawking state, considering all quantum gravity effects, would simply be given by:

$$Z = \sum_{L(p,q)} \int [Dg] e^{-S_E[g]} \tag{3.1.5}$$

where S_E is the Euclidean action for the theory of gravity. Interestingly as reported in Ref. [78], the functional integral, when summed over all Lens space saddles, diverges as a harmonic series in the integer p: $\sum_{p=1}^{\infty} \frac{1}{p} = \zeta(1)$, which cannot be regularized. The computation for a single Lens space (before summing up) was performed in a perturbative one-loop calculation in metric variables and cross-checked with results from a non-perturbative computation in first order formulation of gravity (Chern Simons (CS) theory), in Ref. [80].

The divergence seems to be tamed after including further degrees of freedom, like topological massive modes, as discussed in Ref. [81] making the Hartle Hawking vacuum state normalizable. This was later established in Ref. [82] using a twisted first order theory of gravity (again CS formulation) and a dimensionless parameter, which can be tuned to get rid of the divergent piece. Interestingly, using results from $SU(N)$ topological invariants in Ref. [83] in 3-manifolds, one can repeat the calculations for higher spin cases. For this, one introduces a consistently truncated tower of higher spins over gravitational degrees of freedom, to see that the sum over all Lens spaces become finite, for spins ≥ 4, as worked out in Ref. [84]. Even with these attempts, it is still not clear, which deformations or coupling of newer excitations on top of gravity can make the Hartle-Hawking vacuum state normalizable. It therefore calls for further attempts to make a classification scheme for such well behaving excitations, in a fashion analogous to classifying renormalizable quantum field theories.

One further motivation towards a definition of Hartle Hawking state in $3d$ quantum de Sitter comes from an analogous question in AdS_3. Euclidean AdS has a topology of solid torus. The two dimensional toric boundary serves as the asymptotics. Using the fact that asymptotic symmetry in AdS_3 is given by $2d$ conformal algebra, one may come up with speculations [85] regarding a candidate $2d$ CFT at the boundary. An exact non-perturbative calculation for the bulk partition function (corresponding to fixed boundary modular parameter) can lead one a long way towards a definite answer regarding the dual field theory.

A series of recent remarkable results in AdS viz., Refs. [86, 87] have taken the approach of supersymmetrizing the gravity theory (CS formulation) and exploiting the elegant methods of supersymmetric localization. Although the original theory is not supersymmetric, modelled as a purely bosonic theory of gauge fields, the localization procedure brings in fermionic degrees of freedom in the dynamics. However, it remains guaranteed, as we will later review in the present article that the computation of the partition function for the localized theory is same as the one, if one could evaluate the one for the original purely bosonic one. In the AdS case it is believed that the non-perturbative result after localization would constrain completely the CFT dual to the original bosonic gravity theory. For further progress in localization in low-dimensional AdS space times, the interested reader is referred to Refs. [88, 89]. These references focus on the program of localization on non-compact manifolds.

It is, in the passing, to be noted that in our present perspective we don't aim at the holographic point of view. Rather, we take cue from the above analysis as far as exact partition function is concerned. We want to investigate if the divergence in Hartle-Hawking state, previously found in purely bosonic theory, while summing over all saddles can be tamed or modified by the introduction of supersymmetry. To this end we use the first order CS formulation here, and supersymmetrize it to write down the exact partition function.

To put this point properly in context of our present work, let us digress a bit on the meaning of partition function. As long as the quantum theory of a classical Euclidean action $S_E[\Phi]$ is renormalizable, one is generally interested in the functional integral:

$$Z[J] = \int D\Phi \, e^{-S_E[\Phi] + \int J \cdot \Phi} \tag{3.1.6}$$

in presence of a probe background field J. Correlations of local operators

$$\langle O_1(x_1) \cdots O_n(x_n) \rangle = \frac{\int D\Phi \, O_1(x_1) \cdots O_n(x_n) \, e^{-S_E[\Phi]}}{Z[0]} \tag{3.1.7}$$

generally are found as appropriate derivatives of $Z[J]$ with respect to J at the point $J = 0$, while the presence of 'normalizing' factor $Z[0]$ in the denominator of Eq. (3.1.7) is also standard.

In contrast, the goal of the present series of works in Refs. [78, 81, 82, 84] is to investigate the Hartle-Hawking vacuum via evaluating the partition functions of Chern Simons theory considering all saddles relevant to gravity on locally de Sitter background and then sum over all geometries. These saddles being Lens spaces, each such partition function is a topological invariant, as was discussed in Ref. [83] and for each Lens space $L(p, q)$, the CS partition function is a functions of p, q and the CS level k (possible shifted by quantum correction). We will see in this chapter, how this quantity for each Lens space, has a spin-dependent multiplicative power law dependence on the volume of that particular Lens space. We will further see that fermions brought in by supersymmetric localization basically does the job of altering the volume prefactor's power, keeping the rest of the functional dependence of k, p, q unchanged with respect to the bosonic case. This alteration introduced by fermions, makes the previously encountered divergence worse.

However, it is fair to assume that, had we been interested in a *fixed* (i.e., defined by a particular value of p and q) background question of calculating correlators[1] via the definition given in Eq. (3.1.7) on a *fixed* $L(p, q)$, the prefactor would have gotten cancelled due to normalization and the results would remain same as in the purely bosonic theory.

Furthermore, an investigation of whether inclusion of higher bosonic spins and the corresponding supersymmetrization would change the behaviour of the proposed partition function is also due. We here realize a better insight into the interplay between the spin content in the theory and the divergence structure. In previous analysis in Ref. [84], it was encountered that bosonic higher spin contributions make product of volume prefactors suppress the divergent contributions. We will elaborate quite the opposite feature here brought in by the fermionic degrees of freedom.

As choice of newer degrees of freedom, higher spins are obvious, as these in $3d$ are much more tractable than in the case of higher dimensions, because of an allowed consistent truncation of the higher spin tower at any finite spin > 2. Effect of finite number of higher spin fluctuations coupled with the background spin-2 fluctuations has been as found

[1]Since pure Einstein gravity in 3 dimensions is devoid of local dynamics, it is hard to define physical, non-trivial correlation functions, particularly in the bulk.

in numerous AdS and flat-space calculations. Analysis in the presence of higher spin fields in AdS spacetimes has been worked out in the seminal works by Gopakumar et al in Refs. [90–94]. In flat spacetimes, similar such work has been carried out in Refs. [95–97]. In this chapter we comment very briefly on the volume prefactor, as we are more interested in the divergence structure of such theories.

The chapter is organized as follows. In **Section** 3.2, we introduce the CS formulation of $3d$ bosonic gravity. In **Subsection** 3.2.1, we obtain the supersymmetric extension of bosonic CS theory. In **Subsection** 3.3.1, we discuss the technique of supersymmetric localization of our theory. In **Subsection** 3.3.2, we explicitly evaluate the partition function, obtained as a matrix model, for our case of spin-2 gravity. We also explicitly identify the divergent pieces in the partition function. In the following **Section** 3.4 , we evaluate the same for higher spin cases and comment on the divergences observed. In **Section** 3.5 , we comment on some future directions that may be explored. Finally, **Section** 3.6 presents a lightning summary of the chapter. **Appendix** B.1 carries a small note on our conventions.

3.2 Chern Simons formulation for 3d gravity and its supersymmetrization

Since the proposal by Witten in Ref. [70], $3d$ gravity is known to be equivalent to a pure CS theory. Let us first briefly take a detour through this equivalence, particularly for the case of positive cosmological constant in Euclidean setting. One can start with a CS functional on a 3-manifold M out of a $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ valued 1-form (gauge field). Also the Lie algebra is equipped with an Ad invariant symmetric bilinear quadratic form $\mathrm{Tr} \equiv \langle \cdot, \cdot \rangle$ valued to be $\mathrm{diag}(k, k, k)$ and $\mathrm{diag}(-k, -k, -k)$ respectively on the first and the second $\mathfrak{su}(2)$. The CS functional then can be written as difference of two $\mathfrak{su}(2)$ CS functional, Tr now evaluating $\mathrm{diag}(1,1,1)$:

$$\begin{aligned} S_{CS}[\mathcal{A}^+, \mathcal{A}^-] &= \frac{k}{4\pi} \mathrm{Tr} \int_M (\mathcal{A}^+ \wedge d\mathcal{A}^+ + \frac{2}{3}\mathcal{A}^+ \wedge \mathcal{A}^+ \wedge \mathcal{A}^+) \\ &- \frac{k}{4\pi} \mathrm{Tr} \int_M (\mathcal{A}^- \wedge d\mathcal{A}^- + \frac{2}{3}\mathcal{A}^- \wedge \mathcal{A}^- \wedge \mathcal{A}^-) \end{aligned} \tag{3.2.1}$$

with,

$$\mathcal{A}^{\pm} = \omega \pm e \,, \qquad k = \frac{1}{4G} \tag{3.2.2}$$

keeping the positive cosmological constant $\Lambda = 1$, G is the Newton's constant in 3 dimensions and e and ω are the $\mathfrak{su}(2)$ triad and connection respectively. Then, we see that Eq. (3.2.1) is actually the action for first order gravity:

$$S_{CS}[\mathcal{A}^+, \mathcal{A}^-] = \frac{k}{4\pi} \int_M \left(e^I \wedge \left(2d\omega_I + i\, \epsilon_{IJK}\omega^J \omega^K \right) + \frac{i}{3}\epsilon_{IJK} e^I \wedge e^J \wedge e^K \right) \tag{3.2.3}$$

A simple derivation[2]:

$$\begin{aligned} S^+_{CS}[\mathcal{A}^+] &= \frac{k}{4\pi} \mathrm{Tr} \int_M (\mathcal{A}^+ \wedge d\mathcal{A}^+ + \frac{2}{3}\mathcal{A}^+ \wedge \mathcal{A}^+ \wedge \mathcal{A}^+) \\ &= \frac{k}{8\pi} \int_M \Bigg[\left(e^I + \omega^I \right) \wedge (de_I + d\omega_I) \\ &+ \frac{i}{3}\epsilon_{IJK} \left(e^I + \omega^I \right) \wedge \left(e^J + \omega^J \right) \wedge \left(e^K + \omega^K \right) \Bigg] \end{aligned} \tag{3.2.4}$$

[2]Normalization: $\mathrm{Tr}\, T_I T_J = \frac{1}{2}\delta_{IJ}$

Let us focus just on the terms in the integrand of the above equation.

$$\text{Integrand:} \quad \Bigg[e^I \wedge de_I + \underbrace{e^I \wedge d\omega_I + \omega^I \wedge de_I}_{I} + \omega^I \wedge d\omega_I \; + \\ \frac{i}{3}\Big(\omega^I \wedge \omega^J \wedge \omega^K + \underbrace{\omega^I \wedge \omega^J \wedge e^K + \omega^I \wedge e^J \wedge \omega^K + e^I \wedge \omega^J \wedge \omega^K}_{II} + \\ \underbrace{\omega^I \wedge e^J \wedge e^K + e^I \wedge \omega^J \wedge e^K + e^I \wedge e^J \wedge \omega^K}_{III} + e^I \wedge e^J \wedge e^K \Big) \Bigg] \tag{3.2.5}$$

On using Eq. (B.2.3), II and III are simplified. Now,

$$\text{Integrand:} \quad \Bigg[e^I \wedge de_I + \underbrace{2e^I \wedge d\omega_I + d(e^I \wedge \omega_I)}_{I} + \omega^I \wedge d\omega_I + \\ \frac{i}{3}\Big(\omega^I \wedge \omega^J \wedge \omega^K + 3e^I \wedge \omega^J \wedge \omega^K + 3\omega^I \wedge e^J \wedge e^K + e^I \wedge e^J \wedge e^K \Big) \Bigg] \tag{3.2.6}$$

On compact manifolds, the integral over closed forms vanishes and I simplifies. That is,

$$\text{Integrand} \;\sim \Bigg[\underbrace{e^I \wedge de_I} + 2e^I \wedge d\omega_I + \underbrace{\omega^I \wedge d\omega_I} \; + \\ \frac{i}{3}\Big(\underbrace{\omega^I \wedge \omega^J \wedge \omega^K} + 3e^I \wedge \omega^J \wedge \omega^K + \underbrace{3\omega^I \wedge e^J \wedge e^K} + e^I \wedge e^J \wedge e^K \Big) \Bigg] \tag{3.2.7}$$

Further, on evaluating $S^-_{CS}[\mathcal{A}^-]$, we would obtain an equivalent expression given in Eq. (3.2.7), with the substitution $e^I \to -e^I$. Taking their difference to evaluate Eq. (3.2.3), we would be left with only those terms with an odd number of frame fields twice over, and those with an even number cancels out. So, in Eq. (3.2.7), the terms with underbraces drop out and Eq. (3.2.3) follows.

If M is closed (for example the manifolds we will be dealing with in this chapter, i.e., the static patch of Euclidean $\mathrm{dS}_3 \sim S^3$ or S^3/Γ), i.e., $\partial M = \emptyset$, the variational principle holds for the action Eq. (3.2.1) without any concern for boundary terms. Equations of motion are flatness conditions of the CS connections, discussed in Eq. (B.2.4), which translate into

$$\text{torsionless condition :} \qquad de^I + \epsilon^{IJK} e_J \wedge \omega_K = 0 \;\text{ and} \tag{3.2.8a}$$

$$\text{curvature equation :} \qquad 2d\omega^I + i\,\epsilon^{IJK}\omega_J \wedge \omega_K = -i\,\epsilon^{IJK} e_J \wedge e_K \tag{3.2.8b}$$

for gravity variables. Interestingly, the following action

$$\begin{aligned} \tilde{S}[\mathcal{A}^+, \mathcal{A}^-] &= \frac{k_+}{4\pi}\,\mathrm{Tr}\int_M (\mathcal{A}^+ \wedge d\mathcal{A}^+ + \frac{2i}{3}\mathcal{A}^+ \wedge \mathcal{A}^+ \wedge \mathcal{A}^+) \\ &+ \frac{k_-}{4\pi}\,\mathrm{Tr}\int_M (\mathcal{A}^- \wedge d\mathcal{A}^- + \frac{2i}{3}\mathcal{A}^- \wedge \mathcal{A}^- \wedge \mathcal{A}^-) \end{aligned} \tag{3.2.9}$$

with modified levels $k_\pm$ also gives the same equations of motion Eq. (3.2.8) for gravity variables. For sake of convenience we introduce a parameter γ such that, $k_\pm = \frac{a(1/\gamma \pm 1)}{4G}$ and Eq. (3.2.9) gives back Eq. (3.2.1) at the limit $\gamma \to \infty$, as observed in Ref. [82]. The equations of motion are independent of γ. This applies to the space solutions as well. On the other hand, other aspects of the dynamics of the theory, i.e., canonical structures are parametrized by γ. For example, the pre-symplectic structure on the space of solutions given in Eq. (3.2.8) is[3]:

$$\begin{aligned} \Omega(\delta_1, \delta_2) &= \frac{k_+}{2\pi}\,\mathrm{Tr}\int_\Sigma \delta_1\mathcal{A}^+ \wedge \delta_2\mathcal{A}^+ - \frac{k_-}{2\pi}\,\mathrm{Tr}\int_\Sigma \delta_1\mathcal{A}^- \wedge \delta_2\mathcal{A}^- \\ &= \frac{2a}{8\pi G}\left(\int_\Sigma (\delta_1\omega^I \wedge \delta_2\omega^I + \delta_1 e^I \wedge \delta_2 e^I) + \frac{2}{\gamma}\int_\Sigma \delta_{[1}\omega^I \wedge \delta_{2]} e^I\right) \end{aligned} \tag{3.2.10}$$

[3] In our definition : $A_{[a}B_{b]} \equiv \frac{1}{2}(A_a B_b - A_b B_a)$.

3.2.1 Supersymmetrization

To evaluate the partition function given by Eq. (3.1.5) exactly, we would use the recently developed supersymmetric Localization techniques of Pestun et al, adapted to our purpose. Towards this, we start by supersymmetrizing a CS gauge field $\mathcal{A}$ valued in some semi-simple Lie algebra. Later we will specialize to mainly $\mathfrak{su}(2)$, the case of relevance to $3d$ gravity. We construct the $3d$ $\mathcal{N}=2$ vector multiplet, discussed in Eq. (1.3.16b), as $\mathcal{V}=(\mathcal{A},\sigma,D,\lambda,\bar{\lambda})$.

The supersymmetric CS Theory action is written as

$$S_{SCS}[\mathcal{V}] = S_{CS}[\mathcal{A}] + \frac{i\kappa}{4\pi}\int d^3x\sqrt{|g|}\ \mathrm{Tr}(\bar{\lambda}\lambda + 2iD\sigma) \tag{3.2.11}$$

We note that in the $3d$ $\mathcal{N}=2$ vector multiplet, the additional fields $(\sigma,D,\lambda,\bar{\lambda})$ are not dynamical and have no kinetic terms in the action.

3.3 Localization of the 3d Supersymmetric Chern Simons Theory on Lens Spaces

With the connection between $3d$ Euclidean gravity and the Supersymmetric CS Theory made explicit in Eqs. (3.2.1) and (3.2.3), we will now evaluate the partition function of the $3d$ Supersymmetric CS Theory via supersymmetric Localization techniques. Since, we are interested in evaluating gravity partition function on Lens Spaces, we would we would try localizing the CS Theory on Lens Spaces $L(p,q)$.

3.3.1 Principle of Localization

Suppose we have a theory on a compact manifold $\mathcal{M}$, defined by an action S[Φ][4], which has a Grassmann-odd symmetry[5] δ. Let us further assume that there exists an operator V which is invariant under the transformation δ^2, *i.e.* $\delta^2 V = 0$. Once we have established the existence of such a special V, let us now consider not the original partition function, but rather a perturbed one, viz.

$$Z(t) = \int_{\mathcal{M}} \mathcal{D}\phi \, e^{-S[\Phi]-t\delta V} \tag{3.3.1}$$

Note that this function is independent of t as[6]

$$\frac{dZ(t)}{dt} = -\int_{\mathcal{M}} \mathcal{D}\phi \, \delta V e^{-S[\Phi]-t\delta V} = -\int_{\mathcal{M}} \mathcal{D}\phi \, \delta(V e^{-S[\Phi]-t\delta V}) = 0 \tag{3.3.2}$$

This means that the original unperturbed partition function maybe evaluated by evaluating the perturbed partition function $Z(t)$ for any value of t (that is dictated by convenience) and especially, for $t \to \infty$. This is immediately useful. If the perturbing operator has a positive definite bosonic part, the integral localizes to a sub-space, often even a finite dimensional one, of field spaces $\{\Phi_0\}$ where we have $(\delta V)_B|_{\{\Phi_0\}} = 0$.

With this motivation, we will try and evaluate the partition function of Supersymmetric CS theory on $L(p, q)$. Now, to have some supersymmetric actions on some curved 3-manifold, we need to find some background, off-shell supergravity theories that preserve some rigid supersymmetry. These theories can then be made to couple to some supersymmetric field theory. This is done via the stress tensor multiplet.

[4] $\{\Phi\}$ stands for whatever the fields of the theory are.
[5] δ is assumed non anomalous. This is a crucial and non-trivial point.
[6] recalling $\delta S = 0$ and $\delta^2 V = 0$.

For our specific case of $3d\ \mathcal{N}=2$ theory, this supergravity theory was called the "new minimal supergravity" which has the following field content

$$\text{Field Content: } \Big\{\text{Metric } g_{\mu\nu},\ \text{R Symmetry Gauge Field } \mathrm{A}_\mu^{(R)},\ \text{2-Form Gauge Field } \mathrm{B}_{\mu\nu}$$
$$\text{Central Charge Symmetry Gauge Field } \mathrm{C}_\mu, \text{Gravitini } (\psi^\mu, \tilde{\psi}^\mu)\Big\} \tag{3.3.3}$$

We define the (dualized) field strengths

$$H \equiv \frac{i}{2}\epsilon^{\mu\nu\rho}\partial_\mu B_{\nu\rho}, \qquad V^\mu \equiv -i\epsilon^{\mu\nu\rho}\partial_\nu C_\rho \tag{3.3.4}$$

To ensure that we have rigid supersymmetry, we need to find Killing spinors $(\zeta, \tilde{\zeta})$ which satisfy the Killing spinor equations, given in terms of these fields, as

$$\begin{aligned}(\nabla_\mu - i\mathrm{A}_\mu^{(R)})\zeta &= -\frac{1}{2}H\gamma_\mu\zeta - iV_\mu\zeta - \frac{1}{2}\epsilon_{\mu\nu\rho}V^\nu\gamma^\rho\zeta \\ (\nabla_\mu + i\mathrm{A}_\mu^{(R)})\tilde{\zeta} &= -\frac{1}{2}H\gamma_\mu\tilde{\zeta} + iV_\mu\tilde{\zeta} + \frac{1}{2}\epsilon_{\mu\nu\rho}V^\nu\gamma^\rho\tilde{\zeta}\end{aligned} \tag{3.3.5}$$

In terms of these Killing spinors, the general Supersymmetric variations of the fields in the gauge multiplet for the $3d\ \mathcal{N}=2$ theory placed on S^3 (or its orbifolds), which is our case of interest, are given by Eq. (1.5.15), which we reproduce here as

$$\delta A_\mu = -\tfrac{i}{2}\bar{\lambda}\gamma_\mu\xi_+\ , \tag{3.3.6a}$$

$$\delta\sigma = -\tfrac{1}{2}\bar{\lambda}\xi_+\ , \tag{3.3.6b}$$

$$\delta D = -\tfrac{i}{2}\big(D_\mu\bar{\lambda}\gamma^\mu\big)\xi_+ + \tfrac{1}{4}\bar{\lambda}\xi_+ + \tfrac{i}{2}[\bar{\lambda},\sigma]\xi_+\ , \tag{3.3.6c}$$

$$\delta\lambda = \big(-\tfrac{1}{2}\gamma^{\mu\nu}F_{\mu\nu} - D - \not{D}\sigma - \sigma\big)\xi_+\ , \tag{3.3.6d}$$

$$\delta\bar{\lambda} = 0\ . \tag{3.3.6e}$$

The corresponding Killing spinor equation is given in Eq. (1.5.16).

We also recall that the $3d\,\mathcal{N}=2$ super Yang-Mills (SYM) action on S^3, given by[7]

$$S_{SYM} = \int d^3x\sqrt{|g|}\,\mathrm{Tr}\left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\sigma D^\mu\sigma + \frac{1}{2}\left(D+\frac{i}{l}\sigma\right)^2 - i\bar{\lambda}\gamma^\mu D_\mu\lambda + i\bar{\lambda}[\sigma,\lambda] - \frac{1}{2l}\bar{\lambda}\lambda\right]. \tag{3.3.7}$$

The action is invariant under the transformations given by Eq. (3.3.6).

Eq. (3.3.7) hands us a prime candidate for the operator δV mentioned in the preceding paragraph, viz. , $\delta V = S_{SYM}$. Explicitly, its variation under Grassmann odd symmetry δ_ξ is zero and has manifestly positive definite bosonic part.

So, we would like to evaluate

$$Z(t) \equiv \int_{\mathcal{M}} \mathcal{D}\phi\, e^{-S_{SCS}[\mathcal{V}]-tS_{SYM}} \tag{3.3.8}$$

with $\mathcal{M} = L(p,q)$ and in the limit $t\to\infty$ where the partition function localises to a finite dimensional integral and the evaluation is exact. The bosonic part of Eq. (3.3.7) is manifestly positive, being expressed as the sum of squares, immediately gives us the BPS configurations. They are viz. ,

$$F_{\mu\nu}=0\,,\qquad D_\mu\sigma=0\,,\qquad D+\frac{i}{l}\sigma=0 \tag{3.3.9}$$

Here, solving the equations in Eq. (3.3.9), we face non trivialities due to difference in global topology of $L(p,q)$ when compared to S^3. We need the classical saddles corresponding to Eq. (3.3.8) on orbifolds $L(p,q)$ on which the localized partition function will

[7]Actually this is the action given not just on S^3 but also on quotient spaces of the kind $S^3/\mathbb{Z}_p$. This is understood as such spaces are locally equivalent to 3-spheres and transformations generated by supercharges are local.

be supported. Non-triviality of this statement arises from the fact that the flat connections on a manifold are characterized by holonomies around non-contractible loops on the base manifold, modulo a homogeneous adjoint group action at the base point of the loop. These loops form the fundamental group of the base manifold. Hence the moduli space of space of flat connections modulo gauge transformation is given by

$$\mathrm{hom}\left(\pi_1(M) \to G\right)/\mathrm{Ad}_G. \tag{3.3.10}$$

For the present case, $L(p,q)$ is a free $\mathbb{Z}_p$ quotient of the simply connected manifold S^3. Therefore we have the first homotopy group as $\pi_1\big(L(p,q)\big) = \mathbb{Z}_p$. This implies that the CS saddles i.e., the flat connections are labelled by $g \in G$, with $g^p = 1$. If we take g to lie in the maximal torus (this can be always be done for simply connected lie groups by the Ad action), we have

$$g = e^{\frac{2\pi i}{p}\mathfrak{m}}, \qquad \mathfrak{m} \in \Lambda/(p\Lambda) \tag{3.3.11}$$

where, Λ is the co-weight lattice of the group G and $\mathfrak{m}$ is N dimensional vector, where N is the rank of group G.

Note that Eq. (3.3.11) would then imply that $\mathfrak{m}_j \in \mathbb{Z}_p$. For example, for $G = SU(N)$, we have

$$g = \mathrm{diag}\Big(\ e^{\frac{2\pi i \mathfrak{m}_1}{p}}\ ,\ e^{\frac{2\pi i \mathfrak{m}_2}{p}}\ ,\ \dots\ ,\ e^{\frac{2\pi i \mathfrak{m}_N}{p}}\ \Big) \tag{3.3.12}$$

with $\sum_i \mathfrak{m}_i = 0$. The rest of the equations in the Eq. (3.3.9), imply

$$\sigma = ilD \equiv \frac{\hat{\sigma}_0}{l} = \text{constant}\ , \qquad [\hat{\sigma}_0, \mathfrak{m}] = 0 \tag{3.3.13}$$

We will take $\hat{\sigma}_0$ to lie in the Cartan sub-algebra $\mathfrak{h}$ of the Lie Algebra $\mathfrak{g}$ of the group G. Note that, the second equation of Eq. (3.3.9) motivates why we can expand $\mathfrak{m}$ in the same Cartan basis.

Classical Contribution : The classical (tree level) contribution to the action is obtained by plugging in the BPS configurations in S_{SCS}.

There will be two such contributions, one coming from the scalars, σ and D, which have been shown to be constant in Eq. (3.3.13) and the flat gauge fields. Since the volume of $S^3/\mathbb{Z}_p$ is $(\frac{2\pi^2 l^3}{p})$, the contribution from the scalars is

$$S^{I}_{SCS}(\hat{\sigma}_0) = \frac{i\,\kappa\,\mathrm{vol}(S^3/\mathbb{Z}_p)}{2\pi l^3}\,\mathrm{Tr}(\hat{\sigma}_0^2) = \frac{i\kappa\pi}{p}\,\mathrm{Tr}(\hat{\sigma}_0^2) \tag{3.3.14}$$

where q^* is defined via the relation $qq^* = 1 \bmod p$.

The contribution from the flat gauge fields is

$$S^{II}_{CS}(\mathfrak{m}) = -\frac{i\kappa\pi}{p}\,\mathrm{Tr}(q^*\,\mathfrak{m}^2) \tag{3.3.15}$$

The total classical contribution is then

$$S_{SCS}(\hat{\sigma}_0, \mathfrak{m}) = S^{I}_{SCS}(\hat{\sigma}_0) + S^{II}_{CS}(\mathfrak{m}) = \frac{i\kappa\pi}{p}\,\mathrm{Tr}(\hat{\sigma}_0^2 - q^*\,\mathfrak{m}^2) \tag{3.3.16}$$

.

1-Loop Determinants : We calculate the 1-Loop determinants from the quadratic fluctuations of the fields about their BPS configurations. Specifically,

$$A_\mu = t^{-\frac{1}{2}} A'_\mu \ , \ \sigma = \frac{\hat{\sigma}_0}{l} + t^{-\frac{1}{2}}\sigma' \ , \ D = -\frac{i}{l^2}\hat{\sigma}_0 + t^{-\frac{1}{2}} D' \ , \ \lambda = t^{-\frac{1}{2}}\lambda' \ , \ \bar{\lambda} = t^{-\frac{1}{2}}\bar{\lambda}' \tag{3.3.17}$$

Plugging these values in Eq.(3.3.7), we obtain the terms in the action proportional to t^{-1} as

$$S'_{SYM} = t^{-1}\int d^3x\sqrt{|g|}\,\mathrm{Tr}\left[\frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + \frac{1}{2}\partial_\mu\sigma'\partial^\mu\sigma' - \frac{1}{2l^2}[A'_\mu,\hat{\sigma}_0]^2 \right.$$
$$\left. + \frac{1}{2}\left(D' + \frac{i}{l}\sigma'\right)^2 - i\bar{\lambda}'\gamma^\mu D_\mu\lambda' + \frac{i}{l}\bar{\lambda}[\hat{\sigma}'_0,\lambda'] - \frac{1}{2l}\bar{\lambda}'\lambda'\right] + \mathcal{O}(t^{-2}) \tag{3.3.18}$$

The integration over D' can trivially be done and it just alters the overall normalization constant sitting in front. To deal with the Vector and Fermionic fields, we decompose the gauge field into a divergenceless part (X) and the rest as

$$A'_\mu = X_\mu + \partial_\mu\phi \tag{3.3.19}$$

The integrals over ϕ and σ' give determinants that cancel and we are left with a divergenceless Vector field and Fermionic fields. Next, we expand them in the Γ_a of the Lie Algebra with the definition

$$[\hat{\sigma}_0,\Gamma_\alpha] = \alpha(\hat{\sigma}_0) \tag{3.3.20}$$

The action then becomes

$$\int d^3x\sqrt{|g|}\sum_{\alpha\in Ad(G)}\Big(\frac{1}{2}X^{\mu}_{-\alpha}\big(-\nabla^2+\frac{1}{l^2}\alpha(\hat{\sigma}_0)^2\big)X_{\alpha,\mu}+\tilde{\lambda}'_{-\alpha}\big(-i\gamma^{\mu}\nabla_{\mu}+\frac{i}{l}\alpha(\hat{\sigma_0})-\frac{1}{2l}\big)\lambda'_{\alpha}\Big) \tag{3.3.21}$$

The 1-Loop Determinant is then, simply

$$\mathcal{Z}^{1-Loop}_{gauge}(\hat{\sigma}_0,\mathfrak{m};p,q)=\frac{\det\big(-i\gamma^{\mu}\nabla_{\mu}+\frac{i}{l}\alpha(\hat{\sigma_0})-\frac{1}{2l}\big)}{\det(-\nabla^2+\frac{1}{l^2}\alpha(\hat{\sigma_0})^2)^{\frac{1}{2}}} \tag{3.3.22}$$

Following Refs. [22, 98], on Lens Spaces $L(p,q)$ this contribution may be calculated as :

$$\mathcal{Z}^{1-Loop}_{gauge}(\hat{\sigma}_0,\mathfrak{m};p,q)=\frac{4\ \prod_{\alpha>0}\sinh\frac{\pi}{p}\alpha(\hat{\sigma}_0+i\mathfrak{m})\sinh\frac{\pi}{p}\alpha(\hat{\sigma}_0-iq^*\mathfrak{m})}{\prod_{\alpha>0}\alpha(\hat{\sigma_0})^2} \tag{3.3.23}$$

where, α are the roots of G and q^* is defined as $q^*q=1$ mod(p).

Finally, we integrate over the BPS configurations, here, denoted by σ_i's and sum over the holonomies, identified by the components of the vector $\mathfrak{m}$. Using Weyl Integration formula, as always, the integral reduces from the vector space spanned by the entire Lie Algebra to a vector subspace spanned by just the Cartan Sub-Algebra ($\mathfrak{h}$). This, however, introduces a Vandermonde Determinant $\prod_{\alpha>0}\alpha(\hat{\sigma_0})^2$. This is exactly cancelled by the denominator in Eq. (3.3.23). Also, to take into account the residual Weyl symmetry of the gauge group, we divide the final expression by the order of the Weyl group of the gauge group.

Explicitly, the expression for the partition function becomes

$$\mathcal{Z}(\hat{\sigma}_0, \mathfrak{m}; p, q) = \frac{1}{|\mathcal{W}|} \sum_{\mathfrak{m}} \int_{\mathfrak{h}} d\hat{\sigma}_0 \prod_{\alpha>0} \alpha(\hat{\sigma}_0)^2 \, e^{-S_{SCS}} \, \mathcal{Z}^{1-Loop}_{gauge}(\hat{\sigma}_0, \mathfrak{m}; p, q)$$
$$= \frac{4}{|\mathcal{W}|} \sum_{\mathfrak{m}} \int_{\mathfrak{h}} d\hat{\sigma}_0 \; e^{-\frac{i\kappa\pi}{p} \mathrm{Tr}_{\mathrm{CS}}(\hat{\sigma}_0^2 - q^* \mathfrak{m}^2)} \prod_{\alpha>0} \sinh \frac{\pi}{p} \alpha(\hat{\sigma}_0 + i\mathfrak{m}) \sinh \frac{\pi}{p} \alpha(\hat{\sigma}_0 - iq^* \mathfrak{m}) \tag{3.3.24}$$

We will evaluate the RHS of Eq. (3.3.24) explicitly next.

3.3.2 Partition Function : Evaluation of the Matrix Model for spin-2 Gravity

As described in the section 3.2, the CS version of the spin-2 gravity we are interested in is based on the gauge group $G = SU(2) \times SU(2)$ for the gauge fields in Eq. Eq. (3.2.9). Here we would perform the localized integral Eq. (3.3.24) and choose those CS saddles that correspond to smooth gravity background solutions.

Let us, then, evaluate the partition function given by Eq. (3.3.24) for $G = SU(2) \times SU(2)$.

The Weyl Group for $SU(N)$ is the permutation group S_N , the order of which is $N!$. The rank of $SU(N)$ group is $(N-1)$, which, for our case of $SU(2)$ is simply 1. Hence, flat connections are identified by the component of a one component vector $\mathfrak{m}$, denoted by $m_\pm$ for the two gauge fields $\mathcal{A}_\pm$ corresponding to the two $SU(2)$ groups of the gauge group G.

The partition function, for each saddle, identified by a value of p, receive contribution from two values of $\mathrm{m}_{\pm}$. Following Ref. [78], they are calculated to be,

$$\mathrm{m}_{\pm} = \frac{(q \pm 1)}{2} \,. \tag{3.3.25}$$

With the values of $\mathrm{m}_{\pm}$'s in our hand, we can directly proceed to calculate the integral given in the RHS of Eq. (3.3.24) explicitly

As discussed, since the rank of $SU(2)$ group is 1, the evaluation of the partition function reduces to the problem of solving a one dimensional integral, viz. :

$$\mathcal{Z}_{+}(\hat{\sigma}_0, \mathfrak{m}; p, q) = \frac{4}{2!} \int d\lambda_{+} \;\; e^{-\frac{ik_{+}\pi}{p}(\lambda_{+}^2 - q^* \mathrm{m}_{+}^2)} \sinh \frac{\pi}{p}(\lambda_{+} + i\mathrm{m}_{+}) \sinh \frac{\pi}{p}(\lambda_{+} - iq^* \mathrm{m}_{+}) \tag{3.3.26}$$

Fortunately, the integral given in Eq. (3.3.26) is tractable.

Since our chosen gauge group is a product group we have another flat connection, identified by m_{-}. The corresponding CS level is denoted by k_{-} and we obtain an equivalent expression for the second flat connection. Explicitly,

$$\mathcal{Z}_{-}(\hat{\sigma}_0, \mathfrak{m}; p, q) = -\frac{4}{2!} \int d\lambda_{-} \;\; e^{-\frac{ik_{-}\pi}{p}(\lambda_{-}^2 - q^* \mathrm{m}_{-}^2)} \sinh \frac{\pi}{p}(\lambda_{-} + i\mathrm{m}_{-}) \sinh \frac{\pi}{p}(\lambda_{-} - iq^* \mathrm{m}_{-}) \tag{3.3.27}$$

As yet, the CS levels are arbitrary but we will choose a special parametrization, viz. ,

$$k_{+} = a(\frac{1}{\gamma} + 1) \,, \qquad k_{-} = a(\frac{1}{\gamma} - 1) \tag{3.3.28}$$

Here, γ is a tunable parameter, whose large limit, for *e.g.*, reproduces $k_{+} + k_{-} = 0$. However, we would focus on the small γ regime for the purpose of divergence structure of the partition function.

The total contribution to the partition function is their product. Explicitly,

$$\underbrace{\mathcal{Z}(\hat{\sigma}_0, \mathfrak{m}; p, q)}_{\mathfrak{su}(2)\oplus\mathfrak{su}(2)} = \underbrace{\mathcal{Z}_+(\hat{\sigma}_0, \mathfrak{m}; p, q)}_{\mathfrak{su}(2)} \times \underbrace{\mathcal{Z}_-(\hat{\sigma}_0, \mathfrak{m}; p, q)}_{\mathfrak{su}(2)} \tag{3.3.29}$$

Using the the values of m_+ and m_- from Eq. (3.3.25), the RHS of Eq. (3.3.29) gives

$$\mathcal{Z}(\hat{\sigma}_0, \mathfrak{m}; p, q) = \frac{ip\gamma}{(2!)^2 a\sqrt{1-\gamma^2}} e^{\frac{i\pi(a(q+q^*+2\gamma)-4(1+q)\gamma)}{2p\gamma}} \Bigg(1 + e^{\frac{4i\pi(1+q)}{p}} + e^{\frac{2i\pi(q-q^*)}{p}} + e^{\frac{2i\pi(2+q+q^*)}{p}} - e^{\frac{2i\pi(a(1-q)(1-\gamma)+2\gamma)}{ap(\gamma-1)}} - e^{\frac{2i\pi(a(3+q)(-1+\gamma)+2\gamma)}{ap(\gamma-1)}} - e^{\frac{2i\pi(a(q^*-1)(1-\gamma)+2\gamma)}{ap(\gamma-1)}} - 2e^{\frac{2i\pi(a(1+q)(1+\gamma)-2\gamma)}{ap(\gamma+1)}} - e^{\frac{2i\pi(a(1+2q-q^*)(1+\gamma)-2\gamma)}{ap(\gamma+1)}} - e^{\frac{2i\pi(a(1+q^*)(1+\gamma)-2\gamma)}{ap(\gamma+1)}} + e^{\frac{4i\pi(a(\gamma^2-1)+2\gamma)}{ap(1+\gamma)(1-\gamma)}} + e^{\frac{4i\pi(aq(\gamma^2-1)+2\gamma)}{ap(\gamma+1)(\gamma-1)}} + e^{\frac{2i\pi(a(2+q-q^*)(\gamma^2-1)+4\gamma)}{ap(\gamma+1)(\gamma-1)}} + e^{\frac{2i\pi(a(q+q^*)(\gamma^2-1)+4\gamma)}{ap(\gamma+1)(\gamma-1)}} - e^{\frac{2i\pi(a(1+2q+q^*)(\gamma-1)+2\gamma}{ap(\gamma-1)}} \Bigg) \tag{3.3.30}$$

This is one of the most striking points in our analysis, which requires further attention. We should note that, the above expression is same as that appearing in the purely bosonic analysis of Ref. [82] or the one in Ref. [78] (for $\gamma \to \infty$), apart from the overall pre-factor p. For this purpose we take $\gamma \to \infty$ and large a in Eq. (3.3.30) with an analytical continuations $a \to i\,a$. For large $|a|$ (which means assuming large dS radius in comparison to Planck length), i.e., where we expect the CS theory to be describe quantum gravity, Eq. (3.3.30) yields:

$$\mathcal{Z}(a, p, q) = \frac{8\pi^2}{a\,V_{L(p,q)}} F(a, q, p). \tag{3.3.31}$$

Whereas the result for pure bosonic gravity, as found in Ref. [78] [8], which also is a special

[8]There lingers a typo in Ref. [78], particularly in eq. (4.32), involving extra factors of 2 in the cosine

case for higher spin result of Ref. [84] is:

$$\mathcal{Z}'(a,p,q) = \frac{V_{L(p,q)}}{4a\pi^2}\, F(a,q,p), \tag{3.3.32}$$
$$\text{where } F(a,q,p) = e^{\frac{2\pi k}{p}} \left(\left(\cos\tfrac{\pi}{p} - \cos\tfrac{\pi q}{p}\right) \left(\cos\tfrac{\pi}{p} - \cos\tfrac{\pi q^*}{p}\right)\right).$$

Here, $V_{L(p,q)} = 2\pi^2/p$ is the volume of $L(p,q)$, measured in units of dS length cubed. This clearly shows that inclusion of fermionic modes basically introduced an altered power law volume dependence. This factor, as explained also in the introduction, would cancel if one is interested in local physics i.e., calculate correlation functions on a particular Lens space. However as already motivated, we defer that analysis here and go on finding an answer to a question rather topological in nature. We sum over all possible gravity saddles, i.e., Lens spaces. In short, the overall contribution to the gravity partition function $\mathcal{Z}_{\text{gravity}}$, we will have a sum over p and sum over q to accommodate the various contributions of all the saddles. In short, the gravity partition function will be obtained by :

$$\mathcal{Z}_{\text{gravity}} = \sum_{p=1}^{\infty} \sum_{\substack{q=1 \\ (p,q)=1}}^{p} \mathcal{Z}(\hat{\sigma}_0, \mathfrak{m}; p, q). \tag{3.3.33}$$

We observe an overall positive power of p multiplying the trigonometric terms. When summed over all topologies, i.e., lens spaces, this p dependence might be a serious cause of divergence. Interestingly, for the pure bosonic theory (for $\gamma \to \infty$), in Ref. [78] and (finite γ), in Ref. [82] the overall p dependence was $1/p$. Therefore the supersymmetric theory does not reproduce exactly the same answer as that of the purely bosonic theory. We will shortly come back to the detailed analytic structure of the sum and explore deeper in this aspect. We will express our result, after the sum over q's in terms of Kloosterman Sums

phases

$S(x, y; p)$, which are tailor made for such sums. The Kloosterman sums are defined as

$$S(x, y; p) \equiv \sum_{\substack{q=1 \\ (p,q)=1}}^{p} e^{\frac{2i\pi}{p}(xq+yq^*)}$$

In terms of these Kloosterman sums, the q sum in Eq. (3.3.33) gives[9] :

$$\mathcal{Z}_{\text{gravity}} = \frac{i}{(2!)^2} \sum_{p=1}^{\infty} \frac{p\gamma}{a\sqrt{1-\gamma^2}} e^{\frac{i\pi a}{p}} \Big[4\cos\tfrac{2\pi}{p} \Big(S(\alpha-1, \alpha; p) + S(\alpha+1, \alpha; p) \Big) - 2\big(1+\cos\tfrac{4\pi}{p}\big) S(\alpha, \alpha; p) - \Big(S(\alpha-1, \alpha-1; p) + 2S(\alpha+1, \alpha-1; p) + S(\alpha+1, \alpha+1; p) \Big) \Big] \tag{3.3.34}$$

To carry out the summation over p, we expand the cosine and the exponential function in their respective Maclaurin series. We obtain an infinite series of Kloosterman Zeta function, defined as

$$L(x, y; s) = \sum_{p=1}^{\infty} p^{-2s} S(x, y; p) \tag{3.3.35}$$

The Kloosterman Zeta functions are analytic for $\mathfrak{Re}(s) > \frac{1}{2}$. Writing our result explicitly, in terms of these functions, will also help us isolate the divergent pieces in the gravity partition function, as explicitly those terms with $\mathfrak{Re}(s) \leq \frac{1}{2}$. The final expression for

[9] $\alpha \equiv \dfrac{a}{4\gamma}$

$\mathcal{Z}_{\text{gravity}}$ is then obtained as :

$$\mathcal{Z}_{\text{gravity}} = \frac{i\,\gamma}{(2!)^2 a\sqrt{1-\gamma^2}} \sum_{m=0}^{\infty} \frac{(i\pi a)^m}{m!} \Bigg[\sum_{n=0}^{\infty} (-1)^n \frac{4\,\pi^{2n}}{(2n)!} \Bigg(L\big(\alpha - \frac{1}{2}, \alpha, \frac{m+2n-1}{2}\big) + L\big(\alpha + \frac{1}{2}, \alpha, \frac{m+2n-1}{2}\big) - 2^{2n-1} L\big(\alpha, \alpha, \frac{m+2n-1}{2}\big) \Bigg) - 2L\big(\alpha, \alpha, \frac{m-1}{2}\big) - L\big(\alpha - \frac{1}{2}, \alpha, \frac{m-1}{2}\big) - 2L\big(\alpha - \frac{1}{2}, \alpha + \frac{1}{2}, \frac{m-1}{2}\big) - L\big(\alpha, \alpha - \frac{1}{2}, \frac{m-1}{2}\big) \Bigg] . \tag{3.3.36}$$

Let us investigate the analytic structure of the partition function summed over all Lens spaces Eq. (3.3.36). From Eq. (3.3.35), i.e., the analyticity of the Kloosterman zeta function, it is easy to see a set of divergence is sourced from the terms for which $m + 2n \leq 2$ in Eq. (3.3.36) and another set being originated from $m \leq 2$ for n independent terms.

It might be instructive to review the divergence properties in semi-classical regime along with $\gamma \to \infty$, so that a direct comparison with the $\zeta(1)$ divergence appearing in Ref. [78] can be made. This is actually made apparent by comparing Eqs. (3.3.32) and (3.3.31). Even in the milder case of purely bosonic theory, which depends linearly on volume as an over-all factor, a divergence occurs when one considers sum over all Lens spaces as a harmonic series in p, since $V_{L(p,q)} \sim 1/p$, i.e., very slowly while accumulating up smaller Lens space volumes. However, we should keep in mind that, this divergence is completely different in nature to those commonly seen in local QFTs while probing short length-scales, i.e., the UV divergences. Those originate from integrating high energy modes. For a renormalizable theory these divergences can be absorbed into local counterterms. We don't have any such mechanism here, as also commented in Ref. [78].

In contrast, our analysis shows existence of more such divergent pieces in Eq. (3.3.36), due to dynamical fermions due to localization. Individual Lens space contributions are finite as before but summing over Lens spaces makes the divergence worse. As the prior motivation for this sum over saddles was to inspect the Hartle-Hawking state for the static patch of de Sitter space, the present result summarizes that quantum Hartle-Hawking is not a good choice of vacuum for 3 dimensional dS, even in the supersymmetrized version.

3.4 Higher Spin Case

Linearized higher spin fields can be coupled consistently to gravity in 3 dimensions with finite height of the higher spin tower, which is nicely captured by the Fronsdal action of symmetric traceless tensor fields. In principle, we imagine a (finite) tower of higher spins, namely s = 3, 4, 5, ... , N over and above the spin-2 cases. This construction is possible only in three dimensions where we can have a consistent truncation to arbitrary spins. For higher dimensions $(d > 3)$ we must include all the higher spin fields. In three dimensions, however, we have the added advantage where we can have a truncated tower of higher spin fields.

These higher spin fields are all minimally coupled to the spin-2 field which forms a background. Following the analysis put forward in Ref. [94], we would include higher spins in our analysis and evaluate the partition function and see the nature of divergence, if any. We would explicitly work out the effect of adding a spin-3 fields as that is the most tractable case in these theories of higher finite spins. As explained, the background is still furnished by the spin-2 field such that $g_{\mu\nu}$ remains the metric of the static patch of Euclidean de Sitter spacetime, given by (3.1.1). We define a metric compatible connection ∇ on the manifold such that

$$\left[\nabla_\mu, \nabla_\nu\right] A^\rho = R^\rho{}_{\sigma\mu\nu} A^\sigma \tag{3.4.1}$$

which defines the Riemann Curvature tensor on the manifold for a probe field A^μ.

Spin-3 case To introduce a massless spin-3 field $\phi_{(\mu\nu\rho)}$ which is minimally coupled to pure gravity in 3 dimensions, following Ref. [99], we introduce the linearised Fronsdal action given by

$$S[\phi] = \int d^3x \sqrt{g}\, \phi^{\alpha_1\alpha_2\alpha_3} \left(\mathcal{H}_{\alpha_1\alpha_2\alpha_3} - \frac{1}{2} g_{(\alpha_1\alpha_2} \mathcal{H}_{\alpha_3)\mu}{}^{\mu} \right) \tag{3.4.2}$$

where the definitions are as follows,

$$\mathcal{H}_{\alpha_1\alpha_2\alpha_3} \equiv \Delta\phi_{\alpha_1\alpha_2\alpha_3} - \nabla_{(\alpha_1}\nabla^{\lambda}\phi_{\alpha_2\alpha_3)\lambda} + \frac{1}{2}\nabla_{(\alpha_1}\nabla_{\alpha_2}\phi_{\alpha_3)\lambda}{}^{\lambda} + 2g_{(\alpha_1\alpha_2}\phi_{\alpha_3)\lambda}{}^{\lambda} \tag{3.4.3}$$

We also note, in passing, that the linearised theory enjoys a gauge symmetry given by

$$\delta\phi_{\alpha_1\alpha_2\alpha_3} = \nabla_{(\alpha_1}\xi_{\alpha_2\alpha_3)}$$

where, $\xi_{\alpha\beta}$ is symmetric and traceless.

Interestingly, a first order version of this theory can also be formulated in terms of CS gauge fields. Ref. [100] gives an elaborate AdS counterpart of that exposition. Our work is similar in spirit but with a positive cosmological constant, which, has not been explored before. At the level of corresponding Lie algebra for CS theory, going from AdS to dS background amounts to changing the structure constants. The CS theory that describes spin 3 fields on the backdround of (euclideanised) $3d$ dS spacetime, has a gauge group $SU(3) \times SU(3)$ as discussed in Ref. [84].

For the ease of generalizing to spin-3 case, in spirit of the Eq. (3.2.2), we define

$$(j_{\pm})_{\mu}^{\ p} \equiv (\omega \pm e)_{\mu}^{\ p} \tag{3.4.4}$$

Let us further define higher tensorial objects obtained similarly as a linear combinations gauge potentials

$$(t_{\pm})_{\mu}^{\ p_1 p_2 \dots p_{s-1}} \equiv (\omega \pm e)_{\mu}^{\ p_1 p_2 \dots p_{s-1}} \tag{3.4.5}$$

We then define the one form gauge fields as

$$\mathcal{A}^{\pm} \equiv ((j_{\pm})_{\mu}^{\ p} J_p + (t_{\pm})_{\mu}^{\ p_1 p_2 \dots p_{s-1}} T_{p_1 p_2 \dots p_{s-1}}) dx^{\mu} \tag{3.4.6}$$

where $\{T_{p_1 p_2 \dots p_{s-1}}\}$ are higher spin generators which are to be added to $\{j_p\}$.

Here too, there are no local degrees of freedom, and we associate the equations of motion with the condition for flatness for these gauge fields. This is, again, similar in spirit to the $d = 3$ Einstein gravity we explored earlier. Thus, we arrive at the Chern Simons formulation of higher spin gravity.

Explicitly, we need to state the algebra of these higher spin generators $\{T_{p_1 p_2 \dots p_{s-1}}\}$. Firstly, we note that, from Eqs. (3.4.4), (3.4.5) & (3.4.6), the generators must transform in some irreducible representation of $su(2)$. This implies that they are symmetric and traceless. Furthermore, similar to the $\{J_p\}$, they satisfy

$$\begin{aligned} \left[J_q, J_r\right] &= \epsilon_{qrt} J^t \\ \left[J_r, T_{\ p_1 p_2 \dots p_{s-1}}\right] &= \epsilon^{q}_{\ \ r(p_1} T_{p_2 p_3 p_{s-1})q} \end{aligned} \tag{3.4.7}$$

Particularly, for the case of $s = 3$, Eq. (3.4.7) allows for a non-trivial algebra of the higher spin generators, namely,

$$\begin{aligned} \left[J_q, J_r\right] &= \epsilon_{qrt} J^t \\ \left[J_r, T_{p_1 p_2}\right] &= \epsilon^q{}_{r(p_1} T_{p_2)q} \\ \left[T_{p_1 p_2}, T_{p_3 p_4}\right] &= \left(\delta_{p_1(p_3} \epsilon_{p_4)p_2 r} + \delta_{p_2(p_3} \epsilon_{p_4)p_1 r}\right) J^r \end{aligned} \tag{3.4.8}$$

One can further show that the algebra given by Eq. (3.4.8) is isomorphic to $su(3)$. That we are working on a Riemannian manifold is made explicit by the appearance of the Kronecker delta as opposed to the Minkowski metric in the algebra Eq. (3.4.8) .

With the set of generators $\left\{J_p, T_{p_1 p_2 \cdots p_{s-1}}\right\}$ which generate a Lie Algebra $\mathfrak{g}$, assumed to admit a non-degenerate bilinear form Tr, we define a Chern Simons action

$$\begin{aligned} S[\mathcal{A}^+, \mathcal{A}^-] &= \frac{k_+}{4\pi} \mathrm{Tr} \int_M (\mathcal{A}^+ \wedge d\mathcal{A}^+ + \frac{2i}{3} \mathcal{A}^+ \wedge \mathcal{A}^+ \wedge \mathcal{A}^+) \\ &- \frac{k_-}{4\pi} \mathrm{Tr} \int_M (\mathcal{A}^- \wedge d\mathcal{A}^- + \frac{2i}{3} \mathcal{A}^- \wedge \mathcal{A}^- \wedge \mathcal{A}^-). \end{aligned} \tag{3.4.9}$$

We would like to calculate the exact partition function in this case so as to check whether supersymmetric version of the higher spins make the sum over topologies better in terms of convergence properties. Let us now evaluate the partition function given by Eq. (3.3.24) for $G = SU(3) \times SU(3)$. As the rank of the group $SU(3)$ is 2, the flat connections are identified by the component of a two component vector $\mathfrak{m}$, denoted by $\mathrm{m}_{\pm}^{(\mathrm{i})}$, where, i running from 1 to 2, denotes the two components of $\mathfrak{m}$ and $\pm$, as before, denote the two gauge fields $\mathcal{A}_{\pm}$ corresponding to the two $SU(3)$ groups of the gauge group G.

At this point, as in the case for spin-2 in Eq. (3.3.25), we will have to choose a pair of elements from the corresponding A_2 co-weight lattice. This choice is physically motivated by the fact that quantum fluctuations are considered over the background that describes dS geometry in terms of gravitons and zero excitations for higher spin degrees of freedom. The exact co-weight points are thus found by a principal embedding of $\mathfrak{su}(2)$ in $\mathfrak{su}(3)$. Thus the two components of $\mathfrak{m}_\pm$ as

$$\mathrm{m}_+^{(i)} = \{q+1, 0\}\,, \qquad \mathrm{m}_-^{(i)} = \{q-1, 0\} \tag{3.4.10}$$

With the values of $\mathrm{m}_\pm^{(i)}$'s in our hand, we can directly proceed to calculate the integral given in the RHS of Eq. (3.3.24) explicitly.

$$\begin{aligned}
&\mathcal{Z}_\pm(\hat{\sigma}_0, \mathfrak{m}; p, q) = \\
&\pm\frac{4}{3!}\int d\lambda_\pm^{(1)} d\lambda_\pm^{(2)}\; e^{-\frac{ik_\pm\pi}{p}(\lambda_\pm^{(1)2}+\lambda_\pm^{(2)2}-q^*(\mathrm{m}_\pm^{(1)2}+\mathrm{m}_\pm^{(2)2}))}\sinh\frac{\pi}{p}(2\lambda_\pm^{(1)}-\lambda_\pm^{(2)}+i(2\mathrm{m}_\pm^{(1)}-\mathrm{m}_\pm^{(2)}))\times\\
&\sinh\frac{\pi}{p}(2\lambda_\pm^{(1)}-\lambda_\pm^{(2)}-iq^*(2\mathrm{m}_\pm^{(1)}-\mathrm{m}_\pm^{(2)}))\sinh\frac{\pi}{p}(2\lambda_\pm^{(2)}-\lambda_\pm^{(1)}+i(\mathrm{m}_\pm^{(2)}-\mathrm{m}_\pm^{(1)}))\times\\
&\sinh\frac{\pi}{p}(2\lambda_\pm^{(2)}-\lambda_\pm^{(1)}-iq^*(\mathrm{m}_\pm^{(2)}-\mathrm{m}_\pm^{(1)}))\sinh\frac{\pi}{p}(\lambda_\pm^{(1)}+\lambda_\pm^{(2)}+i(\mathrm{m}_\pm^{(1)}+\mathrm{m}_\pm^{(2)}))\times\\
&\sinh\frac{\pi}{p}(\lambda_\pm^{(1)}+\lambda_\pm^{(2)}-iq^*(\mathrm{m}_\pm^{(1)}+\mathrm{m}_\pm^{(2)}))
\end{aligned} \tag{3.4.11}$$

The integral in Eq. (3.4.11) is Gaussian and therefore, tractable. The argument preceding Eq. (3.3.29) holds in this case too, and we have

$$\underbrace{\mathcal{Z}(\hat{\sigma}_0, \mathfrak{m}; p, q)}_{\mathfrak{su}(3)\oplus\mathfrak{su}(3)} = \underbrace{\mathcal{Z}_+(\hat{\sigma}_0, \mathfrak{m}; p, q)}_{\mathfrak{su}(3)} \times \underbrace{\mathcal{Z}_-(\hat{\sigma}_0, \mathfrak{m}; p, q)}_{\mathfrak{su}(3)}, \tag{3.4.12}$$

with $k_\pm$ being parameterized similarly as in the $SU(2)$ case, via Eq. (3.3.28), and the values of $\mathrm{m}_\pm$ obtained in Eq. (3.4.10), obtained in the preceding section, the RHS of Eq. (3.4.12) gives

$$\mathcal{Z}(\hat{\sigma}_0, \mathfrak{m}; p, q) = \frac{(p\gamma)^2}{(3!)^2 a^2(\gamma^2 - 1)}\, e^{\frac{2i\pi}{p\gamma}(a(q+q^*+2\gamma)-4(1+q)\gamma)}\, G(\gamma, a, p, q). \tag{3.4.13}$$

Here the function G is a linear combination of 824 phase factors, similar in form, to those appearing inside Eq. (3.3.30). Due to the cumbersome appearance and of less significance of these terms, they have been omitted here.

Again, following similar arguments as before, the gravity partition function is given by a sum over the topologies, which classify the various saddles, and is obtained as

$$\mathcal{Z}_{\text{gravity}} = \sum_{p=1}^{\infty} \sum_{\substack{q=1 \\ (p,q)=1}}^{p} \mathcal{Z}(\hat{\sigma}_0, \mathfrak{m}; p, q) \tag{3.4.14}$$

Even without knowing the explicit structure of the terms in the right hand side of Eq. (3.4.13), just from the pre-factor p^2 we can conclude as in the spin-2 case that Eq. (3.4.14) will diverge because of terms appearing in the non-analytic domain of Kloosterman zeta function.

We conclude by a comparative remark with the purely bosonic theory. For example, Ref. [84] states that the partition function for a purely bosonic theory of higher spins truncated at a tower of spin N on a Lens space is given by:

$$\sim (V_{L(p,q)})^{N-1}\, e^{2\pi k/p} \prod_{\pm} \prod_{s=2}^{N-1} \prod_{r=1}^{s-1} \sin\left(r\pi \frac{q \pm 1}{p}\right) \sin\left(r\pi \frac{q^* \pm 1}{p}\right). \tag{3.4.15}$$

This makes the sum over topologies more convergent for higher spins. In the supersymmetrized version however:

$$\mathcal{Z}_{\text{spin}-N} \sim p^{N-1} \sim \frac{1}{V_{L(p,q)})^{N-1}}.$$

Due to the opposite statistics of the fermions and opposite power of fermionic determinants in partition function calculations, the divergence in the full partition function gets only worse. However, as already discussed before, this divergence only tells about stability of quantum gravity fluctuations on de Sitter background, but goes away while calculating correlators of non-gravitational interactions on fixed Lens space backgrounds.

3.5 Discussions and future directions

We have calculated, as the definition of Hartle-Hawking vacuum state, the exact quantum gravity partition function on the static patch of Euclidean de Sitter space-time. In trying to do so, we have argued that the quantum gravity path integral receives contributions from all the classical saddles, which we have obtained as the quotient spaces of S^3 by the abelian group, $\mathbb{Z}_p$. This have been identified with the Lens Space $L(p, q)$ and we expect a formal sum over p and q, the parameters of the space to capture the contributions from the saddles.

To evaluate the quantum gravity partition function exactly, we have worked in the CS formulation of $3d$ gravity. This has proved immediately helpful in calculating the exact partition function by supersymmetric localization technique. We have calculated the partition function for both spin-2 gravity and higher spin cases. We observe that the Kloosterman zeta functions arise naturally in the result of the partition functions from where we identify explicitly the divergent pieces. We also observe that our result, being exact, reproduces the known result in large k limit, apart from an overall volume factor. That contribution has been ascribed to the effect of introduction of dynamical fermionic degrees of freedom. Due to the presence of this change in the prefactor, the analytic properties of the sum over all Lens space does change. It becomes divergent even for those ranges of parameters, for which the bosonic theory was finite.

To explore further, let us focus that the divergence is caused basically from the prefactor volume contributions from Lens space of higher p. As one goes on incorporating higher values of p, smaller volumes contribute as $p/(2\pi^2)$ as per Eq. (3.3.31). Therefore one of the most natural yet brute-force regularization would be to consider only those Lens space whose volumes are greater than some particular volume V_Λ, similar in spirit to putting a UV cut-off. One obvious choice for V_Λ is of course the Planck volume. This gives a seemingly plausible regulator. However the ultimate physicality of this scheme would be to first compute expectation value of local operators or correlators and then take $V_\Lambda \to 0$ and check that the results converge uniformly to a finite limiting value. That would make a very clear sense of the Hartle-Hawking vacuum for 3 dimensional dS space, with all quantum gravity effects included. In fact the above scheme is planned for an immediate future check, which we would like to perform by including local degrees of freedom in the form of scalar fields. The present results of this article from the localized gravitational part of the path integral would make that calculation relatively more tractable.

For the higher spin cases, we have proposed a set of saddles which are points in the A_2 co-root lattice. With this prescription for $\mathfrak{m}$, we calculate the partition function and observe that the divergence is indeed worse. Observing the trend of the divergence reflected on the the volume prefactor, we have also predicted a conjectural form for arbitrary higher spin cases. The dependence of the individual partition function on each Lens space scales as positive spin dependent power law. In contrast, in the purely bosonic theory this dependence was a negative power law, which made a concrete proof of convergence result for spins greater than 3, possible.

Apart from the immediate future problem as pointed above, as a further direction worth exploring we set aside the task of evaluating the quantum gravity partition function for the $\mathcal{N} = 2$ supergravity theory, instead of the above purely Einstein gravity using the CS formulation. In that case, the fermions would be dynamical and we expect non-trivial contributions to the partition function, coming directly from the fermionic sector. It would also be interesting to see if the addition of dynamical fermions takes care of the divergences in the partition function, as one might expect from boson-fermion loop contribution cancellations. It would be interesting to study how the fermionic contributions from the supergravity theory differ from the present case. That eventually will be a valuable progress in classifying all possible excitations consistent with quantum gravity on de Sitter static patch.

3.6 Chapter summary

The objective of the present work was the exact evaluation of quantum gravity partition function with a positive cosmological constant in $3d$. We focus on the space of solutions which constitutes the dS_3 spacetime, the static patch of which has been euclideanised for the rigorous definition of the partition function. This leads one to the S^3 metric. Since we are dealing with quantum gravity partition functions, we let the metric fluctuate and the integral receives non trivial contributions from all the saddles. The saddles have been identified as the orbifolds of S^3, denoted by the Lens spaces $L(p, q)$. Thus, we expect a formal sum over all the contributions coming from each Lens space, denoted by individual values of p and q. For the purpose of exact evaluation, we take help of the CS formulation of gravity, and we supersymmetrize the (two copies of) $\mathfrak{su}(2)$ Lie algebra valued bosonic

CS theory to obtain the $\mathcal{N} = 2$ VM. We use the technique of supersymmetric localization to evaluate the partition function of the $\mathcal{N} = 2$ theory as the partition function of the gravity theory. Eq. (3.3.36) is the main result of this chapter for the case of $s = 2$. We see the presence of fermions, albeit non-dynamical, modifies the purely bosonic result by a p dependent factor, which diverges as $p \to \infty$. We further identity the divergent pieces in terms of the Kloosterman Zeta functions. They are finite in number and these divergences have been well studied in the mathematical literature. For the higher spin case $s > 2$, we indicate the nature of divergence as $p \to \infty$ in Eq. (3.4.13) . We also show that since the divergence appears as an overall multiplicative factor, in calculating correlation functions of well defined operators, it would cancel out and we expect finite and physical results.

CHAPTER 4

SUPERSYMMETRIC GRAPHENE ON SQUASHED HEMISPHERE

This chapter is based on

1. "Supersymmetric Graphene on squashed Hemisphere", R. Gupta, A. Ray, K. Sil, [arXiv:2012.01990].

4.1 Introduction

Quantum field theory on a manifold with boundary finds diverse applications ranging from string theory to condensed matter physics such as D-branes, topological insulators and graphene. The fixed point in the renormalization group flow in quantum field theory is particularly interesting as the physics is described by a conformal field theory. These theories also find a useful application in condensed matter systems, for example, in describing the second-order phase transitions.

Our focus here will be on 4-dimensional conformal field theories in the presence of a boundary with conformally invariant boundary conditions. One such example of 4-dimensional boundary conformal field theories (bCFT) is mixed dimensional quantum electrodynamics where 4-dimensional electromagnetic field interacts with charged matter fields on the 3-dimensional boundary. These theories exhibit many novel properties, as described for example in Refs. [101–104]. Interestingly, the mixed dimensional quantum electrodynamics belongs to a more general class of field theories called reduced or pseudo-QED, as described in Refs. [105–108].

One of the characteristic features of a conformal field theory is the presence of a quantitative measure of the number of degrees of freedom that decreases along the RG flow connecting two CFTs. In $2d$ and $4d$, it coincides with the central charge c and a, respectively, while in $3d$ CFTs, the free energy of the theory computed on S^3 plays the similar role. Monotonicity theorem also exists in d-dimensional bCFTs, as discussed in Refs. [109–113]. The boundary free energy defined from the hemisphere partition function as

$$\frac{|Z_{HS^d}|^2}{Z_{S^d}} = e^{\text{div.} - 2F_\partial} \,, \tag{4.1.1}$$

where terms in "div." are divergent terms which have $(d-1)$-dimensional origin, conjectured to decrease along the RG flow triggered by the boundary relevant operator. Our goal would be to compute the boundary free energy in the 4-dimensional supersymmetric bCFTs.

The localization computation of the partition function for 4-dimensional $\mathcal{N}=2$ supersymmetric theories with Neumann and Dirichlet boundary conditions was first performed in Ref. [114]. Subsequently, the partition function of $\mathcal{N}=2$ supersymmetric graphene-like theories with general boundary conditions appeared in Ref. [115]. Here, the authors computed the partition function as a function of trial R-charge of the charged matter fields at the boundary. The partition function takes the form of real one-dimensional integral with integrand given in terms of Jafferis ℓ-function. One of the novel features of the partition function is that it depends on the complexified gauge coupling

$$\tau = \frac{\theta}{2\pi} + \frac{2\pi i}{g^2}\,, \tag{4.1.2}$$

which is exactly marginal. This gives rise to a boundary conformal field theory for every value of the complex coupling τ. Thus, the partition function computed using the method of localization is a function of the complex coupling τ and the choice of R-symmetry. The R-symmetry is determined using the 3-dimensional F-maximization as reported in Ref. [30]. Moreover, the partition function, as a function of the background sources for gauge and flavor currents, was used to compute boundary transport coefficients that appear in the 2-point function of the corresponding currents.

In the present article, we will generalize the above computation to include the metric background deformations. More specifically, we will consider $\mathcal{N} = 2$ supersymmetric mixed dimensional QED on a squashed hemisphere. Supersymmetric theories on a squashed sphere have been studied in various dimensions. Provided the squashing deformations preserve some supersymmetry, the partition function as the function of the squashing parameters can be computed using the localization technique, for example in Refs. [25, 28, 116, 117]. The free energy as a function of squashing deformations allows us to compute correlation functions that contain the insertion of the energy-momentum tensor. We will be interested here to compute the transport coefficient that appears in the 2-point function of the energy-momentum tensor in 3-dimensional boundary. Conformal symmetry and the conservation law fix the 2-point function of energy-momentum tensor in

3-dimensional flat space up to a constant.

The two point correlator of the stress tensor is given by

$$\langle T_{\mu\nu}(x)T_{\rho\sigma}(0)\rangle = -\frac{\tau_R}{64\pi^2}(\delta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)(\delta_{\rho\sigma}\partial^2 - \partial_\rho\partial_\sigma)\frac{1}{x^2}$$
$$+\frac{\tau_R}{64\pi^2}\Big((\delta_{\mu\rho}\partial^2 - \partial_\mu\partial_\rho)(\delta_{\nu\sigma}\partial^2 - \partial_\nu\partial_\sigma) + (\mu \leftrightarrow \nu)\Big)\frac{1}{x^2} \tag{4.1.3}$$

The quantity of interest τ_R can be computed by placing the conformal field theory on S^3_b. It is given by the second derivative of the squashed free energy with respect to the squashing parameter b, as was first discussed in Ref. [11]. That is,

$$\tau_R = \frac{2}{\pi^2}\text{Re}\frac{\partial^2 F_b}{\partial b^2}\Big|_{b=1}, \qquad \text{where} \quad F_b = -\ln Z_b\,, \tag{4.1.4}$$

where the free energy is evaluated using the superconformal R-charge.

We extend the above computation to 2-point function in bCFT. In particular, we compute the 2-point function of the *boundary* energy-momentum tensor by differentiating the *boundary* free energy as

$$\tau_R = \frac{2}{\pi^2}\frac{\partial^2 F_{\partial,b}}{\partial b^2}\Big|_{b=1}. \tag{4.1.5}$$

Basically, the idea is that after integrating over the bulk degrees of freedom for a given conformal boundary condition (and dividing by the sphere partition function), we can think of the F_∂ as the free energy of some effective CFT at the boundary. At the perturbative level, the effect of bulk degrees of freedom can be mimicked by introducing interactions in the boundary theory involving auxiliary fields. These were the original arguments in Ref.

[118] that provided evidence in support of the decrease of boundary-free energy along the boundary RG flow.

To compute τ_R, we follow the strategy outlined in Ref. [119]. We

1. Evaluate the partition function on S_b^3 and determine the R-charge that maximizes the real part of the free energy,

2. Use the R-charge thus obtained to evaluate the free energy on S_b^3, and

3. Compute the second derivative of the free energy, which is the function of squashing parameter b, to obtain the expression for τ_R.

The outline of the rest of the chapter is as follows. In **Section** 4.2, we review two different ways of squashing a S^4. We then discuss supersymmetry on the squashed sphere. After that, we discuss the condition the background fields need to satisfy to have supersymmetry on the squashed hemisphere. In **Section** 4.3, we find the partition function of mixed dimensional QED coupled to charged matter fields at the boundary. In **Section** 4.4, we compute the 2-point function of the energy-momentum tensor. In **Section** 4.5 we discuss the possible future directions. Finally, **Section** 4.6 presents a lightning summary of the chapter. In **Appendix** C.1, we present our conventions for gamma matrices and reality condition on fermions. In **Appendix** C.2, we discuss the supersymmetry on the squashed hemisphere. Here we also discuss the requirement on the supergravity background fields. In **Appendices** C.3 and C.4, we present the supersymmetric action and background fields on the squashed sphere. In **Appendix** C.5, we give explicit expressions for functions that appear in the subleading computations of τ_R. Finally, in **Appendix** D.1, we state various Pauli and γ matrices identities needed for various manipulations in simplifying supersymmetric transformations.

4.2 Supersymmetry on squashed hemisphere

In this section, we will introduce the supersymmetric background on a squashed hemisphere. The squashed hemisphere is obtained by considering the squashed sphere and placing the boundary at the equator. We will be considering here two different kinds of squashing depending on the isometry preserved by the deformation of S^4. The first kind of squashing preserves $SU(2) \times U(1)$ isometry. The metric is given by

$$ds_I^2 = dr^2 + \frac{1}{4}\sin^2 r\Big(d\theta^2 + \sin^2\theta\, d\phi^2 + h(r)^2(d\psi + \cos\theta\, d\phi)^2\Big) \tag{4.2.1}$$

where $0 \leq r \leq \pi$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$ and $0 \leq \psi \leq 4\pi$ and $h(r)$ is an arbitrary smooth function. The regularity of the metric near the north pole (i.e., at $r = 0$) and south pole (i.e., at $r = \pi$) requires the following behaviour the function

$$\text{North pole:} \qquad h(r) = 1 + c_2 r^2 + \mathcal{O}(r^3)\,, \tag{4.2.2}$$

$$\text{South pole:} \qquad h(r) = 1 + d_2(\pi - r)^2 + \mathcal{O}((\pi - r)^3)\,. \tag{4.2.3}$$

The squashed hemisphere is obtained by the requirement that $0 \leq r \leq \frac{\pi}{2}$. The induced metric at the boundary is given by

$$ds_{\partial I}^2 = \frac{1}{4}\Big(d\theta^2 + \sin^2\theta\, d\phi^2 + h\Big(\frac{\pi}{2}\Big)^2(d\psi + \cos\theta\, d\phi)^2\Big) \tag{4.2.4}$$

The above metric on the squashed S^3 preserves $SU(2) \times U(1)$ symmetry. Thus, any arbitrary smooth function satisfying the regularity condition given in Eq. (4.2.2) gives rise to a smooth deformation of the HS^4. The deformation can then be used to compute the 2-point function of energy momentum tensor of the boundary conformal field theory. For the convenience of the later computations, will choose the following form of the function

$$h(r) = 1 + \frac{2\alpha}{\pi^2} \sin^3 r \,. \tag{4.2.5}$$

In the above α is a positive real parameter with $\alpha = 0$ correspond to unsquashed background. The above form of $h(r)$ satisfies the smoothness criteria given in Eq. (4.2.2) with $c_2 = 0$.

The second kind of the deformation we will consider preserves $U(1) \times U(1)$ isometry. The squashed metric is given by

$$ds^2_{II} = \delta_{ab} E^a E^b \,, \tag{4.2.6}$$

where the vielbeins are

$$E^1 = \ell \, \sin r \, \cos\theta \, d\phi, \quad E^2 = \tilde{\ell} \, \sin r \, \sin\theta \, d\chi, \quad E^3 = f(\theta) \, \sin r \, d\theta + h(r,\theta) dr,$$
$$E^4 = g(r,\theta) \, dr \,. \tag{4.2.7}$$

In the above, the range of the coordinates are $0 \leq r \leq \pi$, $0 \leq \theta \leq \frac{\pi}{2}$ and $0 \leq \phi, \chi \leq 2\pi$. The parameters ℓ and $\tilde{\ell}$ are constants and the functions appearing in the metric are given by

$$f(\theta) = \sqrt{\ell^2 \sin^2\theta + \tilde{\ell}^2 \cos^2\theta}, \quad g(r,\theta) = \sqrt{\rho^2 \sin^2 r + \frac{\ell^2 \tilde{\ell}^2}{f(\theta)^2} \cos^2 r}\,,$$
$$h(r,\theta) = \frac{\tilde{\ell}^2 - \ell^2}{f(\theta)} \cos r \, \sin\theta \, \cos\theta \,. \tag{4.2.8}$$

The induced metric at the boundary of the squashed hemisphere i.e., at $r = \frac{\pi}{2}$ is

$$ds^2_{\partial II} = \ell^2 \cos^2\theta \, d\phi^2 + \tilde{\ell}^2 \sin^2\theta \, d\chi^2 + (\tilde{\ell}^2 \cos^2\theta + \ell^2 \sin^2\theta) \, d\theta^2 \, . \tag{4.2.9}$$

Next, we will discuss the background which needs to be turned on in order to preserve supersymmetry on the squashed hemisphere.

4.2.1 Supersymmetric background on squashed hemisphere

We will be interested in $\mathcal{N} = 2$ supersymmetric theory on the squashed hemisphere backgrounds given in Eq. (4.2.1) and Eq. (4.2.6). The squashed metric background does not admit any rigid supersymmetry itself. However, the supersymmetric theory can be put on the squashed hemisphere if we turn on some non-dynamical supergravity background fields. The theory is invariant under the rigid supersymmetry transformations that are generated by the solution of the Killing spinor equation

$$\mathcal{D}_\mu \xi^i + T^{ab}\gamma_{ab}\gamma_\mu \xi^i = \gamma_\mu \xi'^i \, . \tag{4.2.10}$$

In addition to above, the Killing spinor also satisfies an auxiliary equation

$$\not{D}\not{D}\xi^i + 4\mathcal{D}_c T_{ab}\gamma^{ab}\gamma^c \xi^i = M\xi^i \, . \tag{4.2.11}$$

The fields T^{ab} and M are auxiliary background fields. Also, the $SU(2)_R$ connection appears in the Killing spinor equation through the covariant derivative given by

$$\mathcal{D}_\mu \xi^i = (\partial_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^{ab})\xi^i + \mathcal{V}^i_{\mu\, j}\xi^j \,. \tag{4.2.12}$$

These Killing spinors also satisfy the symplectic Majorana condition

$$\bar{\xi}_i \equiv (\xi^i)^\dagger = \epsilon_{ij}\xi^{j\,T} C \,, \tag{4.2.13}$$

where C is the charge conjugation matrix.

Now, the presence of the boundary at $r = \frac{\pi}{2}$ breaks half of the supersymmetry transformations i.e., the boundary in our case preserves 4 out of 8 supercharges. Given a solution ξ^i to the Killing spinor equations (4.2.10) and (4.2.11), we define the following projected spinors following Ref. [115]

$$\xi^i_\pm = \Pi^i_{\pm\, j}\xi^j \,, \tag{4.2.14}$$

where the projector is given by

$$\Pi^i_{\pm\, j} = \frac{1}{2}(\delta^i_j \pm i\tau^i_{3\, j}\gamma_5\gamma_n) \,. \tag{4.2.15}$$

Here, γ_n is the flat space gamma matrix corresponding to the direction perpendicular to the boundary i.e., n. We require that at $r = \frac{\pi}{2}$, $\Pi^i_{+\, j}\xi^j\Big|_{r=\frac{\pi}{2}} = 0$, and $\xi^i_-\Big|_{r=\frac{\pi}{2}}$ generates the supersymmetry on the boundary. It is important to note that $\xi^i_\pm$ are not the solution of the 4-dimensional Killing spinor equations.

The spinor $\xi^i_-\Big|_{r=\frac{\pi}{2}}$ being the Killing spinor on the boundary, i.e., a solution to the boundary

Killing spinor equation, requires the background to satisfy certain conditions. Using the fact that the components of spin connections corresponding to the metrics given by Eqs. (4.2.1) and (4.2.6) satisfy

$$\omega_A^{Bn} = 0, \qquad \text{for} \quad A, B = 1, 2, 3\,, \tag{4.2.16}$$

it turns out that $\xi^i_-\Big|_{r=\frac{\pi}{2}}$ solves 3-dimensional Killing spinor equation if the background satisfies the following condition (see the Appendix (C.2) for more details)

$$T_{Bn}|_{r=\frac{\pi}{2}} = 0\,. \tag{4.2.17}$$

Note that the above condition is the requirement for having $\mathcal{N} = 2$ supersymmetry with $U(1)_R$ symmetry at the boundary. Thus, the spinor ξ^i_- at the boundary satisfies the Killing spinor equation

$$\mathcal{D}_A^{3d}\xi^i_- + T_{BC}(\gamma_{BC}\gamma_A + \frac{1}{3}\gamma_A\gamma_{BC})\xi^i_- = \frac{1}{3}\gamma_A\gamma^B\mathcal{D}_B^{3d}\xi^i_-\,. \tag{4.2.18}$$

In particular, one can write the above as

$$\nabla_A^{3d}\psi^i_- = \frac{i}{2}H\gamma_A\psi^i_- + V_A\gamma_3\psi^i_-\,, \tag{4.2.19}$$

for some choice of H and V_A and $\psi^i_- = (\xi_-, \xi'_-)$. The Appendix C.1 sets the convention. This is the 3-dimensional Killing spinor equation with H and V_A being the auxiliary fields in 3-dimensional supergravity.

In the following, we will try to find a possible solution for the supergravity background fields T, V and M that solves the sets of Killing spinor equation as given in Eqs. (4.2.10), (4.2.11) and also at the same time satisfies the condition as given in Eq. (4.2.17) at the boundary. For this we have considered two different kinds of squashing of the four dimensional manifold as already mentioned, namely the one which preserve $SU(2) \times U(1)$ isometry and other with $U(1) \times U(1)$ isometry. The supersymmetric backgrounds for the two cases are separately discussed below.

Squashed sphere with $SU(2) \times U(1)$ isometry

In this section, we will consider supersymmetric theories on the squashed hemisphere described by the metric in Eq. (4.2.1). The squashing was first discussed in Ref. [120] in the context of supersymmetric localization. The presentation below follows their analysis closely. Our choice of vielbeins are

$$e^1 = -\frac{\sin r}{2}\big(\cos\psi d\theta + \sin\psi\sin\theta d\phi\big)\,, \qquad e^2 = \frac{\sin r}{2}\big(\sin\psi d\theta - \cos\psi\sin\theta d\phi\big)\,,$$
$$e^3 = -\frac{\sin r}{2}h(r)\big(d\psi + \cos\theta d\phi\big)\,, \qquad e^4 = dr\,. \tag{4.2.20}$$

Following Ref. [120], we choose an ansatz for the Killing spinors that is compatible with the symplectic Majorana reality condition, given in Eq. (4.2.13). The ansatz is

$$\xi^1 = \begin{pmatrix} s(r) \\ 0 \\ ic\frac{h(r)}{s(r)}\sin r \\ 0 \end{pmatrix}, \quad \xi^2 = \begin{pmatrix} 0 \\ s(r) \\ 0 \\ -ic\frac{h(r)}{s(r)}\sin r \end{pmatrix} \tag{4.2.21}$$

with c being a real constant. Solving the Killing spinor equation with the above ansatz, we find

$$s(r) = h(r) \cos \frac{r}{2} \,. \tag{4.2.22}$$

Note that in the limit of vanishing squashing parameter ($\alpha = 0$), the above ansatz for the Killing spinor satisfies Killing spinor equation on the round sphere. However for non zero α, the above spinors solve the Killing spinor equation provided one turns on appropriate background supergravity fields. These background fields, the antisymmetric matrix T^{ab} and the $SU(2)_R$ gauge field $\mathcal{V}_\mu$, defined in Eq. (4.2.10) have the following form,

$$T^{ab}\gamma_{ab} = \begin{pmatrix} it_3^+ & i(t_1^+ - it_2^+) & 0 & 0 \\ i(t_1^+ + it_2^+) & -it_3^+ & 0 & 0 \\ 0 & 0 & it_3^- & i(t_1^- - it_2^-) \\ 0 & 0 & i(t_1^- + it_2^-) & -it_3^- \end{pmatrix} , \tag{4.2.23}$$

and

$$V_a \equiv \begin{pmatrix} iv_{3,a} & i(v_{1,a} + iv_{2,a}) \\ i(v_{1,a} - iv_{2,a}) & -iv_{3,a} \end{pmatrix} , \tag{4.2.24}$$

with, $\mathcal{V}_\mu = e^a_\mu V_a$. The only non zero components of the background fields, $v_{3,3}$, t_3^+, t_3^- and also M are given in Appendix (C.3). In presence of the boundary at $r = \pi/2$, we also require to impose the boundary condition as given in Eq. (4.2.17) on the antisymmetric tensor. Given the explicit results for the background field components in Eq. (C.3.1), the above condition on the antisymmetric tensor at the boundary translates into the following relation between t_3^+ and t_3^-,

$$(t_3^+ - t_3^-)\,|_{r=\pi/2} = 0 \Rightarrow c = \pm\left(\frac{1}{2} + \frac{\alpha}{\pi^2}\right). \tag{4.2.25}$$

In other words, the real constant c is fixed by considering the above boundary condition. Moreover, due to the boundary, we expect only half of the supersymmetry to be present compared to the case with no boundary. We define a projected spinor $\xi^i_\pm$ as

$$\xi^i_\pm = \Pi^i_{\pm j}\xi^j. \tag{4.2.26}$$

Imposing the condition that $\xi^i_+|_{r=\pi/2} = 0$ sets the value $c = +\left(\frac{1}{2} + \frac{\alpha}{\pi^2}\right) = \frac{1}{2}h(\frac{\pi}{2})$. The boundary supersymmetry is generated by the Killing spinor $\xi^i_-\Big|_{r=\frac{\pi}{2}}$.
The corresponding Killing vector is given by

$$K = -4h(\frac{\pi}{2})\frac{\partial}{\partial\psi}. \tag{4.2.27}$$

The boundary Killing spinor equation is given by Eq. (4.2.19) and the supergravity fields are

$$H = h(\frac{\pi}{2}), \quad V_A = -i\delta_{A3}\Big(h(\frac{\pi}{2}) - \frac{1}{h(\frac{\pi}{2})}\Big). \tag{4.2.28}$$

Squashed sphere with $U(1)\times U(1)$ isometry

In this section, we will discuss the squashing described by the metric Eq. (4.2.6). This particular kind of squashed geometry for S^4 was considered in Ref. [28]. The solution of Killing spinor equations and the exact results for the background fields were obtained in

Ref. [28]. The Killing spinors are given by,

$$\xi^1 = \begin{pmatrix} \frac{1}{2}e^{\frac{i}{2}(-\theta+\phi+\chi)}\sin\frac{r}{2} \\ -\frac{1}{2}e^{\frac{i}{2}(\theta+\phi+\chi)}\sin\frac{r}{2} \\ \frac{i}{2}e^{\frac{i}{2}(-\theta+\phi+\chi)}\cos\frac{r}{2} \\ -\frac{i}{2}e^{\frac{i}{2}(\theta+\phi+\chi)}\cos\frac{r}{2} \end{pmatrix}, \quad \xi^2 = \begin{pmatrix} \frac{1}{2}e^{\frac{i}{2}(-\theta-\phi-\chi)}\sin\frac{r}{2} \\ \frac{1}{2}e^{\frac{i}{2}(\theta-\phi-\chi)}\sin\frac{r}{2} \\ -\frac{i}{2}e^{\frac{i}{2}(-\theta-\phi-\chi)}\cos\frac{r}{2} \\ -\frac{i}{2}e^{\frac{i}{2}(\theta-\phi-\chi)}\cos\frac{r}{2} \end{pmatrix} \tag{4.2.29}$$

Given this Killing spinor one can easily obtain the projected Killing spinors ξ^i_+ and ξ^i_- on the three dimensional boundary. It turns out that ξ^i_+ being proportional to $\left(\cos\frac{r}{2} - \sin\frac{r}{2}\right)$ goes to zero at $r = \pi/2$. Moreover, for the Killing spinor equation Eq. (4.2.10) to be satisfied at the boundary with the remaining spinor ξ^i_-, the antisymmetric tensor must obey equation Eq. (4.2.17). The components of T_{Bn} which are nonzero turn out to be $T_{\phi r}$ and $T_{\chi r}$, given by,

$$T_{\phi r} = \frac{h(r,\theta)\sin\theta}{8f(\theta)g(r,\theta)}, \quad T_{\chi r} = -\frac{h(r,\theta)\cos\theta}{8f(\theta)g(r,\theta)}. \tag{4.2.30}$$

These two components vanish at $r = \pi/2$, since $h(\frac{\pi}{2},\theta) = 0$.

The corresponding Killing vector is given by

$$K = -\frac{1}{\ell}\frac{\partial}{\partial\phi} - \frac{1}{\tilde{\ell}}\frac{\partial}{\partial\chi}\,. \tag{4.2.31}$$

At the boundary, the Killing spinors in Eq. (4.2.29) satisfy the 3-dimensional Killing spinor equation (4.2.19). The background supergravity fields are given by

$$H = -\frac{1}{f(\theta)}, \quad V_1 = -\frac{i}{2\cos\theta}\Big(\frac{1}{f(\theta)} - \frac{1}{\ell}\Big), \quad V_2 = -\frac{i}{2\sin\theta}\Big(\frac{1}{f(\theta)} - \frac{1}{\tilde{\ell}}\Big), \quad V_3 = 0\,. \tag{4.2.32}$$

4.3 Supersymmetric partition function

In this section, we will compute the partition function of a supersymmetric theory of interest on the squashed hemisphere using localization. In particular, we will be interested in the supersymmetric theory of n number of chiral multiplets on the boundary of the squashed hemisphere coupled to an abelian vector multiplet having propagating degrees of freedom in bulk. We will further denote by n_+ and n_-, with $n = n_+ + n_-$, the number of positively and negatively charged chiral multiplets, respectively. The Lagrangian of the theory on a squashed background is available in the literature Refs. [25, 28], however, for the convenience of a reader, we have given the Lagrangian for vector multiplet in the Appendix C.4.

4.3.1 Squashing preserving $SU(2) \times U(1)$

We begin with the partition function computation on the squashed hemisphere in Eq. (4.2.1). The supersymmetric locus is obtained by solving the bosonic equations $\delta\lambda^i = 0$. To state the solution explicitly, we choose the deformation to be Eq. (4.2.5). With this we have a Killing spinor with

$$s(r) = \cos\frac{r}{2}\left(1 + \frac{2\alpha}{\pi^2}\sin^3 r\right). \tag{4.3.1}$$

The solution to the variation $\delta\lambda^i = 0$ is given by (in perturbative expansion in α)

$$\begin{aligned} S &= s_0 - \frac{s_0\alpha}{\pi^2}\sin^2\frac{r}{2}(4 + 5\sin r + 4\sin 2r + \sin 3r) + ... \\ D_3 &= s_0 + \frac{s_0\alpha}{4\pi^2}(-8 + 16\cos r + 8\sin r + 2\sin 2r + 8\sin 3r + 5\sin 4r) + ... \end{aligned} \tag{4.3.2}$$

In the above s_0 is constant and all other fields are set to zero. Evaluating the classical action (see Appendix C.4) on the localization background we obtain (to order α^2)[1]

$$I = -is_0^2\Big(\pi - \frac{4\alpha}{\pi} + \frac{12\alpha^2}{\pi^3} + \ldots\Big)\bar{\tau}\,, \tag{4.3.3}$$

where

$$\bar{\tau} = \frac{\theta}{2\pi} - \frac{2\pi i}{g_{YM}^2}\,. \tag{4.3.4}$$

The α expansion of the action agrees with the closed form expression of the action given as (we have checked it for a quite a few order)

$$I = -is_0^2\frac{\pi}{(1+\frac{2\alpha}{\pi^2})^2}\bar{\tau} = -is_0^2\frac{\pi}{h(\frac{\pi}{2})^2}\bar{\tau}. \tag{4.3.5}$$

Thus, the partition function is given by

$$Z^{\partial I} = \int d\sigma\, e^{-i\frac{\pi\sigma^2}{h(\frac{\pi}{2})^2}\bar{\tau}}\, Z^I_{1-\text{loop}}(\sigma,\alpha)\,, \tag{4.3.6}$$

where the integration contour is chosen along the imaginary direction of s_0 i.e., $s_0 = i\sigma$ with $\sigma \in \mathbb{R}$. The one loop determinant for each chiral multiplet is given by Ref. [25][2]

$$Z^I_{1-\text{loop}}(\sigma,\alpha) = \prod_{n>0}\frac{n+1-q+i\frac{\sigma}{h(\frac{\pi}{2})}}{n-1+q-i\frac{\sigma}{h(\frac{\pi}{2})}}\,. \tag{4.3.7}$$

[1]Note that the partition function depends on $\bar{\tau}$. This reflects the choice of the Killing spinor. It is possible to find the Killing spinor which gives rise to the classical action on the localization background depending on τ. However, we will not be using the partition function for the future computations, and hence we will not proceed to find such Killing spinor.

[2]Note that our Killing spinor is normalized differently than Ref. [25]. In particular, at $r = \frac{\pi}{2}$ it is given by $\bar{\xi}_i\xi^i = 2h(\frac{\pi}{2})^2$. Also, the localization background at $r = \frac{\pi}{2}$ is $S = \frac{s_0}{h(\frac{\pi}{2})^2}$ and $D_3 = \frac{s_0}{h(\frac{\pi}{2})}$.

Thus, we see that the partition function depends trivially on $h(\frac{\pi}{2})$.

4.3.2 Squashing preserving $U(1) \times U(1)$

Next, we consider the deformation given by Eq. (4.2.6). The localization background is obtained by solving the bosonic equations $\delta\lambda^i = 0$ and is given by Ref. [28]

$$D_1 = -\frac{h(r,\theta)s_0}{f(\theta)g(r,\theta)}\sin(\phi+\chi), \quad D_2 = -\frac{h(r,\theta)s_0}{f(\theta)g(r,\theta)}\cos(\phi+\chi), \quad D_3 = -\frac{s_0}{f(\theta)} \tag{4.3.8}$$

and s_0 is the constant value of the scalar field S. Rest all other fields are zero. The bulk action (see Appendix C.4) evaluated on the localization background is given by

$$I_g = -\frac{2\pi^2}{g_{YM}^2}\ell\tilde{\ell}\, s_0^2\,. \tag{4.3.9}$$

Similarly, the boundary action evaluated on the localization background is

$$I_{\partial g} = \frac{i\theta}{2} s_0^2\,\ell\tilde{\ell}\,. \tag{4.3.10}$$

Thus, the complete action on the localization background is given by

$$I = I_g + I_{\partial g} = i\ell\tilde{\ell}s_0^2\pi\left(\frac{\theta}{2\pi} + \frac{2\pi i}{g_{YM}^2}\right) = i\ell\tilde{\ell}\pi\tau s_0^2\,. \tag{4.3.11}$$

The partition function is given by

$$Z^{\partial II} = \int d\sigma\, e^{i\ell\tilde{\ell}\pi\tau\sigma^2} Z^{II}_{1-\text{loop}}\,, \tag{4.3.12}$$

where the integration contour is chosen along the imaginary direction of s_0 i.e., $s_0 = i\sigma$ with $\sigma \in \mathbb{R}$ and $Z^{II}_{1-\text{loop}}$ is the one loop contribution from the matter multiplets at the boundary. The one-loop contribution from the n_+ and n_- chiral multiplets at the boundary with R-charge q_+ and q_-, respectively is given in terms of hyperbolic Gamma function in Ref. [25, 116] as

$$Z^{II}_{1-\text{loop}} = \Big(\Gamma_h(\ell\tilde{\ell}\sigma + i\omega q_+; i\omega_1, i\omega_2)\Big)^{n_+}\Big(\Gamma_h(-\ell\tilde{\ell}\sigma + i\omega q_-; i\omega_1, i\omega_2)\Big)^{n_-}, \qquad (4.3.13)$$

where $\omega_1 = b, \omega_2 = \frac{1}{b}$ and $\omega = \frac{1}{2}(\omega_1 + \omega_2)$. The parameter b is the squashing parameter given by $b = \sqrt{\frac{\tilde{\ell}}{\ell}}$. For our purposes, it is convenient to use the integral representation of the hyperbolic Gamma function. It is given by

$$\Gamma_h(z; w_1, w_2) = e^{i\int_0^\infty \frac{dy}{y}\left(\frac{z-w}{w_1 w_2 y} - \frac{\sin 2y(z-w)}{2\sin w_1 y \, \sin w_2 y}\right)}. \qquad (4.3.14)$$

The above integral is well defined for $0 < \text{Im}\, z < 2\text{Im}\, w$, where $w = \frac{1}{2}(w_1 + w_2)$. In the case when there is no squashing, the hyperbolic Gamma function reduces to Jafferis ℓ-function. The precise relation is

$$\Gamma_h(z; i, i) = e^{\ell(1+iz)}. \qquad (4.3.15)$$

Next, we will use the above partition function to compute the 2-point correlation function of boundary energy-momentum tensor.

4.4 τ_R computation

In this section, we will compute the 2-point function of boundary energy-momentum tensor for various matter degrees of freedom at the boundary interacting with a bulk photon. The 2-point function of energy momentum tensor in flat space is given by

$$\begin{aligned}\langle T_{\mu\nu}(x)T_{\rho\sigma}(0)\rangle &= -\frac{\tau_R}{64\pi^2}(\delta_{\mu\nu}\partial^2-\partial_\mu\partial_\nu)(\delta_{\rho\sigma}\partial^2-\partial_\rho\partial_\sigma)\frac{1}{x^2}\\ &+\frac{\tau_R}{64\pi^2}\Big((\delta_{\mu\rho}\partial^2-\partial_\mu\partial_\rho)(\delta_{\nu\sigma}\partial^2-\partial_\nu\partial_\sigma)+(\mu\leftrightarrow\nu)\Big)\frac{1}{x^2}\,. \end{aligned} \tag{4.4.1}$$

As discussed previously, the quantity of interest, τ_R can be computed by placing the theory on a squashed sphere. It is given by

$$\tau_R=\frac{2}{\pi^2}\text{Re}\frac{\partial^2 F}{\partial b^2}\Big|_{b=1},\qquad F=-\ln Z^{\partial}\,. \tag{4.4.2}$$

4.4.1 Free matter at the boundary

We begin with the case of the free conformal matter at the boundary i.e., there is no interaction with the bulk photon. This case will be important for our future discussion when we will consider the interaction to be very small. For a conformal chiral matter, we have $q=\frac{1}{2}$. Thus, the one-loop determinant is given by

$$\begin{aligned}Z'^{\text{chiral}}_{1-\text{loop}} &= e^{-F}=\Gamma_h(\omega(iq);i\omega_1,i\omega_2)\\ &=.\text{exp}\Big(i\int_0^\infty\frac{dx}{x}\Big[-\frac{\omega(iq)-i\omega}{\omega_1\omega_2 x}+\frac{\sin(2x(iq\omega-\omega))}{2\sinh\omega_1 x\,\sinh\omega_2 x}\Big]\Big)\,. \end{aligned} \tag{4.4.3}$$

In the above $\omega_1 = b, \omega_2 = \frac{1}{b}$ and $\omega = \frac{1}{2}(\omega_1 + \omega_2)$. Thus, the free energy is given as

$$F = \int_0^\infty \frac{dx}{x}\Big[-\frac{(b+b^{-1})(q-1)}{2x} + \frac{\sinh(x(b+b^{-1})(q-1))}{2\sinh bx\,\sinh b^{-1}x}\Big]\,. \tag{4.4.4}$$

Substituting the R-charge $q = \frac{1}{2}$ in the above, we get the free energy

$$F = -\int_0^\infty \frac{dx}{x}\Big[-\frac{(b+b^{-1})}{4x} + \frac{\sinh(\frac{x}{2}(b+b^{-1}))}{2\sinh bx\,\sinh b^{-1}x}\Big]\,. \tag{4.4.5}$$

Calculating the second derivative w.r.t b and then setting $b = 1$, we obtain

$$\frac{\partial^2 F}{\partial b^2}\Big|_{b=1} = -\int_0^\infty \frac{dx}{x}\Big(-\frac{1}{2x} - \frac{x}{2}\frac{\cosh x}{\sinh^2 x} + \frac{x^2}{\sinh^3 x}\Big) = \frac{\pi^2}{8}\,. \tag{4.4.6}$$

Thus, the coefficient τ_R for a free chiral multiplet is

$$\tau_R = \frac{1}{4}\,. \tag{4.4.7}$$

4.4.2 Interacting case: Non chiral theory

Next, we will consider the partition function of non-chiral matter at the boundary interacting with the bulk photon. It consists of an equal number of chiral multiplets of positively and negatively charged coupled to a single $U(1)$ gauge field. In this case, the squashing dependent partition function is given by[3]

$$Z^\partial = \int d\sigma\, e^{i\sigma^2\pi\tau}\,(\Gamma_h(\sigma + i\omega q_+; i\omega_1, i\omega_2))^n(\Gamma_h(-\sigma + i\omega q_-; i\omega_1, i\omega_2))^n\,. \tag{4.4.8}$$

[3]Here onwards we will consider only the squashing dependent part of the partition function, ignoring the overall factor of $\ell\tilde{\ell}$.

In the above expression, we have not included the monopole charge q_t since it vanishes at the extremum in the present case.

We first consider the case of $n = 1$ i.e., of two oppositely charged chiral multiplets. Using the exponential form of the hyperbolic gamma function, the partition function is given by

$$Z^{\partial} = \int d\sigma\, e^{i\sigma^2\pi\tau + G(\sigma)}\,, \tag{4.4.9}$$

where

$$G(\sigma) = \int_0^\infty \frac{dy}{y}\Big[\frac{2\omega(q_f - 1)}{y} \\ +\frac{i\sin(2y(\sigma + i\omega q_+ - i\omega)) + i\sin(2y(-\sigma + i\omega q_- - i\omega))}{2\sinh by\, \sinh b^{-1}y}\Big]\,. \tag{4.4.10}$$

We will further write $q_\pm$ as $q_\pm = q_f \pm q_g$.

Large τ analysis: We will start with saddle point calculation in the weak coupling limit i.e., $|\tau| >> 1$. In this limit, the matter degrees of freedom at the boundary interact weakly with the bulk photon, and as a result, one would expect $\tau_R \sim \frac{2}{4} = \frac{1}{2}$ to leading order in large $|\tau|$ expansion. We will see this explicitly below.

In the large $|\tau|$ limit, the partition function on S^3 is extremum at the value $q_\pm$ given by Ref. [115]

$$q_g = 0, \quad \text{and} \quad q_f = \frac{1}{2} - \frac{\sin\alpha}{\pi|\tau|} - \frac{\pi^2 - 4 + (4 + (1+2n)\pi^2)\cos 2\alpha}{4\pi^2|\tau|^2} + \mathcal{O}(|\tau|^{-3})\,, \tag{4.4.11}$$

where $\tau = |\tau| e^{i\alpha}$. We will do saddle point calculation with the above choice of q_f and q_g. We see that $\sigma = 0$ is the saddle point, since the function $G(\sigma)$ satisfies

$$\frac{\partial G(\sigma)}{\partial \sigma}\Big|_{\sigma=0} = 0\,. \tag{4.4.12}$$

Let the function has the following Taylor series expansion about $\sigma = 0$

$$G(\sigma) = G_0 + \frac{1}{2} G_2 \sigma^2 + \mathcal{O}(\sigma^3) \tag{4.4.13}$$

The values of the above functions are

$$G_0 = \int_0^\infty \frac{dy}{y}\Big[\frac{2\omega(q_f-1)}{y} - \frac{\sinh(2y\omega(q_f-1))}{\sinh by\, \sinh b^{-1}y}\Big]\,, \tag{4.4.14}$$

and

$$G_2 = \int_0^\infty dy \Big[\frac{4y \sinh(2y\omega(q_f-1))}{\sinh by\, \sinh b^{-1}y}\Big]\,. \tag{4.4.15}$$

Note that both the integrals are entirely convergent since $q_f < 2$. Thus the partition function to a leading order is given by

$$Z^{\partial} = e^{G_0}\Big(\frac{1}{\sqrt{-i\tau}} + \frac{1}{2} G_2 \frac{1}{2\pi(-i\tau)^{3/2}}\Big) + \mathcal{O}(|\tau|^{-5/2})\,. \tag{4.4.16}$$

Thus the real part of free energy is

$$|Z^{\partial}|^2 = e^{-2\mathrm{Re}F_{\partial}} = \frac{e^{\bar{G}_0+G_0}}{|\tau|}\Big(1 + \frac{i}{4\pi|\tau|^2}(G_2\bar{\tau} - \bar{G}_2\tau)\Big) + \mathcal{O}(|\tau|^{-3})\,. \tag{4.4.17}$$

Thus the expression for τ_R is

$$\tau_R = \frac{2}{\pi^2}\text{Re}\frac{\partial^2 F_\partial}{\partial b^2}\Big|_{b=1} = -\frac{1}{\pi^2}\frac{\partial^2}{\partial b^2}(\bar{G}_0 + G_0 + \frac{i}{4\pi|\tau|^2}(\bar{\tau}G_2 - \tau\bar{G}_2))\Big|_{b=1} \tag{4.4.18}$$

We will calculate the above one by one

$$\frac{\partial^2}{\partial b^2}(\bar{G}_0 + G_0)\Big|_{b=1} = -\frac{\pi^2}{2} - \frac{\pi}{3}\frac{\sin\alpha}{|\tau|} - \frac{\pi}{3}\frac{\sin\alpha}{|\tau|} + \mathcal{O}(\tau^{-2}) = -\frac{\pi^2}{2} - \frac{2\pi}{3}\frac{\sin\alpha}{|\tau|} + \mathcal{O}(\tau^{-2}) \tag{4.4.19}$$

Next we calculate

$$\frac{\partial^2}{\partial b^2}(\frac{i}{4\pi|\tau|^2}(\bar{\tau}G_2 - \tau\bar{G}_2))\Big|_{b=1} = \frac{\text{Im}\tau}{2\pi|\tau|^2}\frac{\partial^2}{\partial b^2}G_2\Big|_{b=1} = \frac{\text{Im}\tau}{2\pi|\tau|^2}\frac{\pi^2}{2}(-8+\pi^2) = \frac{\pi\text{Im}\tau}{4|\tau|^2}(-8+\pi^2) \tag{4.4.20}$$

Thus we have

$$\tau_R = \frac{1}{2} + \frac{\sin\alpha}{12\pi|\tau|}(32 - 3\pi^2) + \mathcal{O}(\tau^{-2})\,, \tag{4.4.21}$$

where $\tau = |\tau|e^{i\alpha}$.

We can easily extend the above analysis for the case of $n_\pm = n$. In this case, the partition function is given by

$$Z^\partial = \int d\sigma\, e^{i\sigma^2\pi\tau + \tilde{F}(\sigma)}\,, \tag{4.4.22}$$

where

$$\tilde{F}(\sigma) = n \int_0^\infty \frac{dy}{y} \Big[\frac{2\omega(q_f - 1)}{y} + \frac{i\sin(2y(\sigma + i\omega q_+ - i\omega)) + i\sin(2y(-\sigma + i\omega q_- - i\omega))}{2\sinh by \, \sinh b^{-1}y} \Big] \tag{4.4.23}$$

Since n is an overall factor, one, therefore, has

$$\tau_R = \frac{n}{2} + \frac{n\sin\alpha}{12\pi|\tau|}(32 - 3\pi^2) + \mathcal{O}(\tau^{-2}) \,. \tag{4.4.24}$$

Small τ analysis: Next, we consider the large coupling expansion, i.e., $|\tau| << 1$, for the coefficient τ_R. We will focus here the case of $n_+ = n_- = 1$. The partition function is

$$Z^\partial = \int d\sigma \, e^{i\sigma^2\pi\tau} \, \Gamma_h(\sigma + i\omega q_+; i\omega_1, i\omega_2)\Gamma_h(-\sigma + i\omega q_-; i\omega_1, i\omega_2) \,. \tag{4.4.25}$$

We will evaluate the above partition function for the value of q_f and q_g which extremizes the corresponding S^3 partition function in the limit of $|\tau| << 1$. To the leading order, the extremum occurs at $q_g = 0$ and

$$q_f = \frac{1}{3} + \frac{54\sqrt{3} - 90\pi + 8\sqrt{3}\pi^2}{9\Big(8\pi(3\sqrt{3} - 2\pi) - 27\Big)} \, |\tau| \sin\alpha + \mathcal{O}(\tau^2) \,. \tag{4.4.26}$$

Thus, we have

$$\tau_R = \frac{2}{\pi^2} \mathrm{Re} \frac{\partial^2 F}{\partial b^2}\Big|_{b=1} = -\frac{2}{\pi^2} \mathrm{Re}\Big(\frac{1}{Z^\partial} \partial_b^2 Z^\partial \Big|_{b=1} \Big) \,, \tag{4.4.27}$$

where the RHS is evaluated at Eq. (4.4.26).

In the above we have used the relation

$$\partial_b \Gamma_h(\pm\sigma + iq_\pm\omega; ib, \frac{i}{b})\Big|_{b=1} = 0\,. \tag{4.4.28}$$

Calculating the second derivative w.r.t b, we obtain

$$\partial_b^2 Z^\partial\Big|_{b=1} = \int d\sigma\, e^{i\sigma^2\pi\tau}\, e^{\ell(1-i\sigma-q_-)+\ell(1+i\sigma-q_+)}\Big(X^+(\sigma,q_+) + X^-(\sigma,q_-)\Big)\,, \tag{4.4.29}$$

where

$$\begin{aligned} X^\pm(\sigma,q_\pm) &= \partial_b^2\Big(i\int_0^\infty \frac{dx}{x}\Big[\frac{\sin(2x(\pm\sigma+iq_\pm\omega-i\omega))}{2\sinh bx\,\sinh\frac{x}{b}} - \frac{\sigma+i\omega q_\pm - i\omega}{x}\Big]\Big)\Big|_{b=1}\,, \\ &= \int_0^\infty dx\,\Big[(q_\pm-1)\Big(\frac{1}{x^2} - \frac{\cosh(2x(1-q_\pm\mp i\sigma))}{\sinh^2 x}\Big) - \\ &\qquad \frac{\sinh(2x)-2x}{2\sinh^4 x}\sinh(2x(1-q_\pm+\pm 2ix\sigma))\Big] \end{aligned} \tag{4.4.30}$$

In the above we have used the relation

$$\Gamma_h(z;i,i) = e^{\ell(1+iz)}\,, \tag{4.4.31}$$

where $\ell(z)$ is the Jafferis ℓ-function. To evaluate Eq. (4.4.29) at strong coupling, we go to the dual frame where the computation reduces to the weak coupling computation. The dual frame is obtained by using the Fourier transform, as described in Ref. [115]

$$\int d\sigma\, e^{\ell(1-q_++i\sigma)+\ell(1-q_--i\sigma)+2\pi i\kappa\sigma} = e^{\ell(1-q_+-q_-)+\ell(\frac{q_++q_-}{2}+i\kappa)+\ell(\frac{q_++q_-}{2}-i\kappa)+\pi\kappa(q_+-q_-)}\,. \tag{4.4.32}$$

Using the above identity, we obtain

$$\partial_b^2 Z^\partial\Big|_{b=1} = \int d\sigma\; d\kappa\, e^{-2\pi i\kappa\sigma} e^{i\sigma^2\pi\tau}\, e^{\ell(1-q_+-q_-)+\ell(i\kappa+\frac{q_-+q_+}{2})+\ell(-i\kappa+\frac{q_-+q_+}{2})} e^{\pi\kappa(q_+-q_-)}$$
$$\times\Big(X^+(\sigma,q_+)+X^-(\sigma,q_-)\Big)\,,$$
$$= \int d\kappa\, e^{-\frac{i\pi\kappa^2}{\tau}}\, e^{\ell(1-q_+-q_-)+\ell(i\kappa+\frac{q_-+q_+}{2})+\ell(-i\kappa+\frac{q_-+q_+}{2})} e^{\pi\kappa(q_+-q_-)}$$
$$\times\int_{\mathrm{Im}\tilde\sigma=\frac{\kappa\mathrm{Im}\tau}{|\tau|^2}} d\tilde\sigma\, e^{i\tilde\sigma^2\pi\tau}\Big(X^+(\tilde\sigma+\frac{\kappa}{\tau},q_+)+X^-(\tilde\sigma+\frac{\kappa}{\tau},q_-)\Big)\,,$$
$$= \int d\kappa\, e^{-\frac{i\pi\kappa^2}{\tau}}\, e^{\ell(1-q_+-q_-)+\ell(i\kappa+\frac{q_-+q_+}{2})+\ell(-i\kappa+\frac{q_-+q_+}{2})} e^{\pi\kappa(q_+-q_-)}$$
$$\times\int_{-\infty}^{\infty} d\tilde\sigma\, e^{i\tilde\sigma^2\pi\tau}\Big(X^+(\tilde\sigma+\frac{p}{\tau},q_+)+X^-(\tilde\sigma+\frac{p}{\tau},q_-)\Big)\,,$$
$$= \int d\kappa\, e^{-\frac{i\pi\kappa^2}{\tau}}\, e^{\ell(1-2q_f)+\ell(i\kappa+q_f)+\ell(-i\kappa+q_f)}$$
$$\times\int_0^\infty \frac{dx}{\sqrt{-i\tau}} e^{-\frac{ix^2}{\pi\tau}}\Big[e^{\frac{ix^2}{\pi\tau}}\frac{2(q_f-1)}{x^2} - \frac{2(q_f-1)}{\sinh^2 x}\cos\frac{2px}{\tau}\cosh(2x(q_f-1))$$
$$-\frac{\sinh(2x)-2x}{\sinh^4 x}\cos\frac{2px}{\tau}\sinh(2x(1-q_f))\Big]\,. \tag{4.4.33}$$

In the above we have deformed the contour back to the real axis and also substituted $q_\pm = q_f$, since $q_g = 0$. In the limit $|\tau| \sim 0$, we evaluate the above integral in the saddle point approximation. In this approximation, we get

$$\frac{1}{Z^\partial}\partial_b^2 Z^\partial\Big|_{b=1} = \int_0^\infty dx\Big[\frac{2(q_f-1)}{x^2} + \frac{1}{2}e^{-2\ell(\frac{1}{3})+\ell(i\frac{x}{\pi}+\frac{1}{3})+\ell(-i\frac{x}{\pi}+\frac{1}{3})}\Big(2+|\tau|f_1(x)+|\tau|f_1(-x)-$$
$$2|\tau|f_1(0)\Big)\Big(-\frac{2(q_f-1)}{\sinh^2 x}\cosh(2x(q_f-1)) - \frac{\sinh(2x)-2x}{\sinh^4 x}\sinh(2x(1-q_f))\Big)\Big] + \mathcal{O}(|\tau|^2)\,,$$
$$\tag{4.4.34}$$

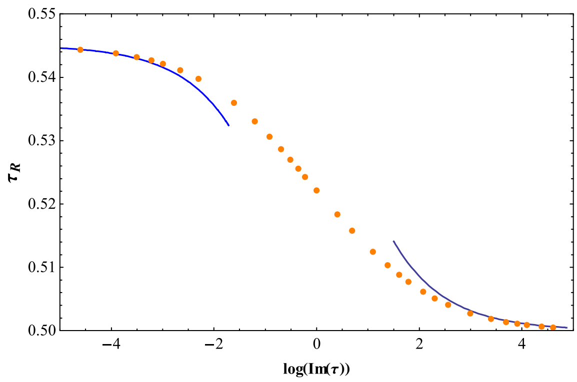

Figure 4.1: The plot of τ_R vs. Imτ for the $n_+ = n_- = 1$ theory. The dots represent the numerical computations and the solid lines are the saddle point approximation.

where the function $f_1(x)$ is given in the Appendix C.5. Thus, we have

$$\tau_R = -\frac{2}{\pi^2}\mathrm{Re}\int_0^\infty dx\Big[\frac{2(q_f-1)}{x^2} + \frac{1}{2}e^{-2\ell(\frac{1}{3})+\ell(i\frac{x}{\pi}+\frac{1}{3})+\ell(-i\frac{x}{\pi}+\frac{1}{3})}\Big(2+|\tau|f_1(x)+|\tau|f_1(-x) -2|\tau|f_1(0)\Big)\Big(-\frac{2(q_f-1)}{\sinh^2 x}\cosh(2x(q_f-1)) - \frac{\sinh(2x)-2x}{\sinh^4 x}\sinh(2x(1-q_f))\Big)\Big] + \mathcal{O}(|\tau|^2)\,, \tag{4.4.35}$$

with the R-charge q_f to the first order in $|\tau|$ given in Eq. (4.4.26). We could not evaluate the above integral explicitly; however, numerical evaluation is possible for the various values of the angle α. In particular, for $\alpha = \frac{\pi}{2}$, we obtain

$$\tau_R = 0.545 - 0.07035|\tau| + \mathcal{O}(|\tau|^2)\,. \tag{4.4.36}$$

The full behaviour of τ_R as a function of Imτ can be seen in Fig. 4.1.

Large n-analysis: We can also find the expression for τ_R in the large n-limit. In this case, we consider the partition function given by

$$Z^{\partial} = \int d\sigma \, e^{i\sigma^2\pi\tau} \, (\Gamma_h(\sigma + i\omega q_+; i\omega_1, i\omega_2))^n (\Gamma_h(-\sigma + i\omega q_-; i\omega_1, i\omega_2))^n \,. \tag{4.4.37}$$

The computation of τ_R in $\frac{1}{n}$-expansion proceeds in a similar manner as in previous cases and we will not repeat here. To compute τ_R, we need to know the R-charge which maximize the free energy on S^3. In the $\frac{1}{n}$-expansion it is given by Ref. [115]

$$q_f = \frac{1}{2} - \frac{2}{\pi^2 n} + \mathcal{O}(n^{-2}), \quad \text{and} \quad q_g = 0\,. \tag{4.4.38}$$

Using the above expression for the R-charge, the explicit computation then gives

$$\tau_R = \frac{n}{2} - 2\Big(1 - \frac{8}{3\pi^2}\Big) + \mathcal{O}(n^{-1})\,. \tag{4.4.39}$$

4.4.3 Interacting case: Chiral theory

Next, we consider the chiral theory with a positively charged matter field at the boundary i.e., $n_+ = 1, n_- = 0$. The squashing dependent partition function is given by

$$Z^{\partial}_{\text{chiral}} = \int d\sigma \, e^{i\sigma^2\pi\tau} \, \Gamma_h(\sigma + i\omega q_g; i\omega_1, i\omega_2) e^{2\pi\omega q_t \sigma} \,, \tag{4.4.40}$$

where q_t is the monopole charge. Note that since $\partial_b \omega\Big|_{b=1} = 0$, we have a vanishing one point function of energy momentum tensor

$$\partial_b Z^{\partial}_{\text{chiral}}\Big|_{b=1} = 0\,. \tag{4.4.41}$$

For the computation of τ_R, we need the second derivative of the partition function with respect to the squashing parameter and is given by

$$\partial_b^2 Z^{\partial}_{\text{chiral}}\Big|_{b=1} = \int d\sigma \, e^{i\pi\tau\sigma^2+\ell(1-q_g+i\sigma)+2\pi q_t\sigma} X(\sigma)\,, \tag{4.4.42}$$

where

$$X(\sigma) = 2\pi\sigma q_t + \int_0^\infty dx \left[\frac{q_g-1}{x^2} + \frac{(1-q_g)\cosh(2x(q_g-1)-2ix\sigma)}{\sinh^2 x} - \frac{1}{2\sinh^4 x}(\sinh(2x)-2x)\sinh(2x(1-q_g)+2ix\sigma)\right] \tag{4.4.43}$$

We first calculate the τ_R in the weak coupling limit i.e., $|\tau| \to \infty$. The superconformal R-symmetry is determined by the value of q_g and q_t that extremize the partition function on S^3. In the saddle point approximation, these are calculated in Ref. [115] as

$$q_g = \frac{1}{2} - \frac{\text{Im}\tau}{\pi|\tau|^2} + \mathcal{O}(|\tau|^{-2}), \quad q_t = -\frac{\text{Re}\tau}{4|\tau|^2} + \mathcal{O}(|\tau|^{-2})\,. \tag{4.4.44}$$

The computation of τ_R proceeds similarly as in the non-chiral case and we will not repeat here. The saddle point approximation in the limit $|\tau| \to \infty$ gives

$$\tau_R = \frac{1}{4} + \frac{\sin\alpha}{24\pi|\tau|}(32-3\pi^2) + \mathcal{O}(\tau^{-2})\,. \tag{4.4.45}$$

Note that to the order $\mathcal{O}(\frac{1}{|\tau|})$, the result of τ_R is exactly half of the non chiral case as shown in Eq. (4.4.21). For the computation of τ_R at the strong coupling $|\tau| < 1$, we will use the

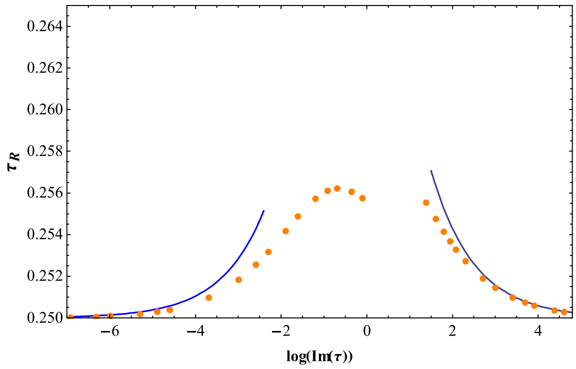

Figure 4.2: The plot of τ_R vs. Imτ for the $n_+ = 1, n_- = 0$ theory. The dots represent the numerical computations and the solid lines are the saddle point approximation.

following identity

$$\int d\sigma\, e^{\frac{i\pi}{2}\sigma^2+2\pi i\sigma\kappa+\ell(1-q_g+i\sigma)} = e^{-2\pi i\kappa^2+\ell(\frac{q_g+1}{2}+i\kappa)-\frac{3\pi i}{8}(q_g+2i\kappa-\frac{1}{3})^2+\frac{i\pi}{12}}\,. \tag{4.4.46}$$

Using the above identity, the derivative of the partition function can be written as

$$\partial_b^2 Z^{\partial}_{\text{chiral}}\Big|_{b=1} = \int d\kappa\, dp\, e^{-2\pi i p\kappa} e^{2\pi q_t\kappa} e^{i\pi(\tau-\frac{1}{2})\kappa^2} e^{-2\pi i p^2} e^{\ell(\frac{\tilde{q}+1}{2})-\frac{3\pi i}{8}(\tilde{q}-\frac{1}{3})^2+\frac{i\pi}{12}} X(\kappa)\,, \tag{4.4.47}$$

where the function $X(\kappa)$ is given in Eq. (4.4.43). After some simplifications and performing the integration over κ, the integral on the RHS can be written as

$$\partial_b^2 Z^{\partial}_{\text{chiral}}\Big|_{b=1} = e^{\frac{\pi i}{2}q_t^2-\frac{3\pi i}{8}(q_g-\frac{1}{3})^2-\frac{3\pi i}{2}q_t(q_g-\frac{1}{3})+\frac{i\pi}{12}} \int dp\, e^{-\frac{\pi i}{2}p^2} e^{\ell(\frac{q_g+1}{2}+q_t+ik)+2\pi p(\frac{3q_g-1}{4}-\frac{q_t}{2})}\tilde{Y}(p)\,. \tag{4.4.48}$$

with

$$\tilde{Y}(k) = 2\pi q_t \frac{e^{i\frac{\pi}{4}}}{\sqrt{-i(\tau-\frac{1}{2})^3}} e^{-\frac{i\pi}{\tau-\frac{1}{2}}k^2} k + \int_{-\infty}^{\infty} dx \, \frac{1}{2\sqrt{-i(\tau-\frac{1}{2})}} \Bigg[\frac{q_g-1}{x^2} e^{-\frac{i\pi}{\tau-\frac{1}{2}}k^2}$$
$$+ \frac{(1-q_g)}{\sinh^2 x} e^{2x(q_g-1)} e^{-\frac{i}{\pi(\tau-\frac{1}{2})}(\pi k+x)^2} + \frac{(\sinh(2x)-2x)}{2\sinh^4 x} e^{2x(q_g-1)} e^{-\frac{i}{\pi(\tau-\frac{1}{2})}(\pi k+x)^2} \Bigg] . \tag{4.4.49}$$

Next, we compute the integral given in Eq. (4.4.48) in saddle point approximation when the coupling $\tau \sim \frac{1}{2}$. Let us define a complex coupling τ' by

$$\tau' = -\frac{1}{2} - \frac{1}{\tau - \frac{1}{2}} \, . \tag{4.4.50}$$

In the limit $|\tau'| >> 1$, the superconformal R-symmetry is given by

$$q_g = \frac{1}{4} + \mathcal{O}(|\tau'|^{-2}), \quad q_g + 2q_t = \frac{2\sin\alpha'}{\pi|\tau'|} + \frac{1}{|\tau'|^2}\Big(\frac{3}{4} - \frac{3}{\pi^2} + (\frac{5}{4} + \frac{3}{\pi^2})\cos 2\alpha'\Big) + \mathcal{O}(|\tau'|^{-3}) \, , \tag{4.4.51}$$

where $\tau' = |\tau'|e^{i\alpha'}$. Evaluating the integral in Eq. (4.4.48) in the saddle point approximation, using the above R-charge, we find to the first subleading order

$$\tau_R = \frac{1}{4} - \frac{2}{\pi^2}\text{Re}\frac{1}{\tau'}\int_{-\infty}^{\infty} dx \, g(x,\alpha') + \mathcal{O}(\tau'^{-2}) \, , \tag{4.4.52}$$

where the explicit form the function $g(x, \alpha')$ is given in the Appendix C.5. For $\alpha' = \frac{\pi}{2}$, we have

$$\tau_R = \frac{1}{4} + \frac{0.0570}{|\tau'|} + \mathcal{O}(|\tau'|^{-2})\,. \tag{4.4.53}$$

The full behaviour of τ_R as a function of Imτ can be seen in the Fig. 4.2.

4.5 Discussions and future directions

As a future direction, it would be interesting to extend the above computation for the case when the bulk involves non-abelian gauge fields, in particular for the $\mathcal{N} = 4$ SYM. Aspects of Ref. [121] also deal with this direction - however, not in the context of squashed geometries. Another interesting analysis would be to compute the anomalous contributions to the trace of the energy-momentum tensor. These contributions depend on the extrinsic curvature of the boundary. For the computation of these contributions, we need to consider the manifold where the boundary has non zero extrinsic curvature. The squashed hemisphere has a vanishing extrinsic curvature. It would be interesting to see if it is possible to find a squashing deformation that preserves some supersymmetry so that the localization computation can be done, and also has non zero extrinsic curvature. For example, one way to obtain non zero extrinsic curvature is to start with a 4-dimensional sphere and put the boundary at $r \neq \frac{\pi}{2}$.

4.6 Chapter summary

Our original motivation for the present work is to compute the correlation function of the energy-momentum tensor for the 4-dimensional bCFT. The most general form of the 2-

point function of the energy-momentum tensor requiring the conservation law and conformal invariance is known of which further details may be found in Refs. [122–124]. It is given in terms of 3 arbitrary functions that encode the dynamics of the bCFT. Our goal was to compute these coefficients in a supersymmetric bCFT. In this direction, we computed the partition function of the mixed dimensional supersymmetric dimensional QED on squashed hemispheres. We considered two different squashings that preserve either $SU(2) \times U(1)$ or $U(1) \times U(1)$ isometry of the original sphere. The boundary free energy depends on the deformation parameter. In the case of the squashing that preserves $SU(2) \times U(1)$ isometry of the original sphere, the free energy is trivial as a function of the deformation. In contrast, the boundary-free energy depends on the squashing parameter in the case of the squashing that preserves $U(1) \times U(1)$ isometry. We then computed the coefficient τ_R by differentiating the free energy twice with respect to the squashing parameter. An important feature of τ_R is that it depends on the bulk marginal coupling τ. We then find the behaviour of τ_R as we change the coupling. For the non-chiral case, we find that the coefficient τ_R decreases from strong coupling to weak coupling. This is the main result for the non chiral theory which is presented graphically in Fig. 4.1. In contrast, the computation in the chiral case reveals that it first increases and then decreases as we change the coupling. Fig. 4.2 highlights this behaviour for the chiral theory. We present both the numerical and analytic results for the boundary stress tensor two point correlator, both at strong and weak coupling. We see excellent convergence between the analytic results and numeric results in the perturbative regions.

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